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Source: *Annals of Mathematics*, Vol. 104, No. 2 (Sep., 1976), pp. 357-387
Published by: Mathematics Department, Princeton University
Stable URL: https://www.jstor.org/stable/1971050
Accessed: 14-07-2025 06:00 UTC

Annals of Mathematics, **104** (1976), 357–387

Algebraic surfaces of general type with small c_1^2, I

By EIJI HORIKAWA*

Introduction

Let S be a minimal algebraic surface of general type defined over C and let K denote the canonical bundle of S. As usual p_g and c_1 denote, respectively, the geometric genus and the first Chern class of S. In general we have the inequality $c_1^2 \geqq 2p_g - 4$.

The purpose of the present paper is to study minimal algebraic surfaces of general type with the equality $c_1^2 = 2p_g - 4$ (hence $p_g \geqq 3$). Enriques has already called attention to these surfaces in [6] (see also [7], Chapter VIII, 11). Moishezon has studied surfaces with $p_g = 3$ and $c_1^2 = 2$ in detail ([29], Chapter VI).

In Section 1 we shall prove that the canonical map Φ_K induces a holomorphic map of degree 2 onto a surface of degree $n - 1$ in $\mathbf{P}^n$, $n = p_g - 1$. We shall classify our surfaces according to their canonical images.

The other sections are devoted to a study of deformations of such surfaces. We say that a surface S is a deformation of a surface S_0 if there exists a finite number of surfaces $S_0, S_1, \cdots, S_k, \cdots, S_m = S$ such that, for any k, S_k and S_{k-1} belong to one and the same complex analytic family of surfaces. An equivalence class with respect to this equivalence relation will be called a *deformation type*. Since two surfaces with the same deformation type are diffeomorphic, it makes sense to speak of the diffeomorphic type or the homotopy type of a deformation type.

In Section 2 we shall calculate the number of moduli $m(S)$ of "generic" surfaces S. We have $m(S) = \dim H^1(S, \Theta_S)$ (Theorem 2.1).

In Sections 3 and 5 we shall prove that minimal algebraic surfaces with given p_g and c_1^2 satisfying $c_1^2 = 2p_g - 4$ and $p_g \geqq 3$ have one and the same deformation type provided that c_1^2 is not divisible by 8.

Sections 4 and 6 are devoted to a study of surfaces $p_g = 6$, $c_1^2 = 8$ and $p_g = 4$, $c_1^2 = 4$, respectively. Surfaces with $p_g = 6$ and $c_1^2 = 8$ are divided into two deformation types which are not homotopically equivalent. Some of the surfaces with $p_g = 4$ and $c_1^2 = 4$ give an example of obstructions for

* Partially supported by NSF grant MPS72-05055A02.

deformations.

In Section 7 we shall study the case in which c_1^2 is divisible by 8. If we fix c_1^2, these surfaces are divided into two deformation types. They are homotopically equivalent or not according to whether c_1^2 is divisible by 16 or not. It is not known whether these two deformation types are diffeomorphic or not when c_1^2 is divisible by 16.

The appendix will provide a proposition on deformations of compositions of holomorphic maps.

1. Surfaces with $c_1^2 = 2p_g - 4$ and $p_g \geqq 3$

Let S be a minimal algebraic surface of general type for which the geometric genus p_g and the Chern number c_1^2 satisfy the conditions $c_1^2 = 2p_g - 4$ and $p_g \geqq 3$. We note that this is the minimal possible value of c_1^2 when $p_g(\geqq 3)$ is given (see [13], Lemmas 1 and 2 or [2], Theorem 9). We also note that the irregularity q vanishes ([2], Theorem 10). In this section we shall prove that S is a (ramified) double covering of a rational surface or of some degenerate form of such double coverings.

LEMMA 1.1. *Let S be a minimal algebraic surface with $c_1^2 = 2p_g - 4$ and $p_g \geqq 3$. Then the canonical system $|K|$ has no base point. Moreover, if $\Phi_K\colon S \to \mathbf{P}^n$, $n = p_g - 1$, denotes the holomorphic map defined by $|K|$, Φ_K induces a holomorphic map of degree 2 onto a surface W of degree $n-1$ in $\mathbf{P}^n$.*

Proof. In [13], Lemmas 1 and 2, we have proved that $|K|$ has no base point and any general member C of $|K|$ is a nonsingular hyperelliptic curve. Let W be the image of Φ_K and $f\colon S \to W$ the induced holomorphic map. We have

$$\deg W \cdot \deg f = K^2 = 2n - 2.$$

Since $2K$ induces the canonical bundle of C, we have $\deg f \geqq 2$. On the other hand, since W is not contained in any hyperplane, we have $\deg W \geqq n - 1$. Therefore, we conclude that $\deg f = 2$ and $\deg W = n - 1$. Q.E.D.

In order to describe the structure of W, we introduce the following notation. For a nonnegative integer d, we let Σ_d denote the Hirzebruch surface of degree d; that is Σ_d is a $\mathbf{P}^1$-bundle over $\mathbf{P}^1$ which has a section Δ_0 such that $\Delta_0^2 = -d$. If d is positive, Δ_0 is unique and will be referred to as the 0-section of Σ_d. We let Γ denote a fibre of the projection $\Sigma_d \to \mathbf{P}^1$ and let Δ be an irreducible curve in $|\Delta_0 + d\Gamma|$. Δ is called the ∞-section of Σ_d.

We refer the reader to [27] for the following lemma.

LEMMA 1.2. *Let W be an irreducible surface of degree $n-1$ in $\mathbf{P}^n$*

which is not contained in any hyperplane. Then W is one of the following:

(i) $n = 2$ *and* $W = \mathbf{P}^2$;

(ii) $n = 5$ *and* $W = \mathbf{P}^2$ *embedded in* $\mathbf{P}^5$ *by* $|2H|$ *where* H *denotes a line on* $\mathbf{P}^2$;

(iii) $n = 3, 4, \cdots$, $W = \Sigma_d$ *where* $n - d - 3$ *is a nonnegative even integer.* W *is embedded in* $\mathbf{P}^n$ *by* $|\Delta_0 + (n - 1 + d)/2\Gamma|$;

(iv) $n = 3, 4, \cdots$, *and* W *is a cone over a rational curve of degree* $n - 1$ *in* $\mathbf{P}^{n-1}$.

We now recall some of the results in [13], Section 2. Let $f: S \to W$ be a surjective holomorphic map of degree 2 between nonsingular algebraic surfaces. We assume that there exists no exceptional curve in any fibre of f. Let K and $\mathfrak{k}$ denote, respectively, the canonical bundles of S and W and let R be the ramification divisor of f. Then we have $K = f^*\mathfrak{k} + [R]$, where $[R]$ denotes the line bundle associated with the divisor R. We define the branch locus B to be the direct image f_*R. Then B is an effective divisor on W which has no multiple component. Such a divisor will be referred to as a *curve*. We say that a curve is nonsingular if it is a disjoint sum of irreducible nonsingular curves.

We assume that there exists a line bundle F on W such that $[B] = 2F$. We take a sufficiently fine open covering $\{U_i\}$ of W. Let $b_i = 0$ be local equations of B on U_i and let $\{f_{ij}\}$ be a system of transition functions of F. We may assume that $b_i = f_{ij}^2 b_j$ on $U_i \cap U_j$. We let w_i denote fibre coordinates on F over U_i. We define a subvariety S' of F by the equations $w_i^2 - b_i = 0$. Then S' is a normal surface with isolated singular points. We shall call S' the *double covering of* W *with branch locus* B.

We say that a curve B *has no infinitely near triple points* if the following conditions are satisfied:

(1) B has no singular points of multiplicity $\geqq 4$.

(2) Every triple point s of B (if any) decomposes into a singularity of multiplicity $\leqq 2$ after a quadric transformation with center at s.

LEMMA 1.3. *Let $f: S \to W$ be a surjective holomorphic map of degree* 2 *as above and let R and B denote, respectively, the ramification divisor and the branch locus of f. Assume that there exists a line bundle F on W such that $[B] = 2F$ and such that $R \in |f^*F|$. Then B has no infinitely near triple points. Moreover, the double covering S' of W with branch locus B has only rational double points as its singularities and S is the minimal resolution of singularity of S'.*

For the proof, see [13], Lemmas 4 and 5.

We shall discuss the four cases in Lemma 1.2.

(i) The case $n = 2$, $W = \mathbf{P}^2$: In this case, we have a surjective map $f: S \to \mathbf{P}^2$ of degree 2. Let R and B denote, respectively, the ramification divisor and the branch locus of f. If H denotes a line on $\mathbf{P}^2$, R is linearly equivalent to $4f^*H$. This implies that B is of degree 8. Moreover, B has no infinitely near triple points by Lemma 1.3.

Conversely, let B be a curve of degree 8 which has no infinitely near triple points and let S' be the double covering of W with branch locus B. Furthermore, let S be the minimal resolution of singularity of S'. Then the canonical bundle K of S is induced by $[H]$. From this fact, we infer that S is a minimal algebraic surface with $p_g = 3$, $q = 0$ and $c_1^2 = 2$ (see [13], Lemma 6).

Remark. These surfaces are known as an example for which the tri-canonical maps Φ_{3K} are not birational (see [29], [19]).

(ii) The case $n = 5$, $W = \mathbf{P}^2$: Let $f: S \to \mathbf{P}^2$ denote the holomorphic map of degree 2 induced by Φ_K. Then the ramification divisor R of f is linearly equivalent to $5f^*H$. It follows that the branch locus B is of degree 10 and has no infinitely near triple points.

Conversely, we can construct, from a curve of degree 10 which has no infinitely near triple points, a minimal algebraic surface with $p_g = 6$, $q = 0$, and $c_1^2 = 8$.

(iii) The case $W = \Sigma_d$ embedded in $\mathbf{P}^n$: Let $f: S \to \Sigma_d$ denote the holomorphic map of degree 2 induced by Φ_K. Since the canonical bundle of W is $[-2\Delta_0 - (d+2)\Gamma]$, the ramification divisor R of f is linearly equivalent to $f^*(3\Delta_0 + (n+3+3d)/2\Gamma)$. This implies that the branch locus B of f satisfies $\Gamma B = 6$ and $\Delta_0 B = n + 3 - 3d$. It follows that B is linearly equivalent to $6\Delta_0 + (n+3+3d)\Gamma$ and has no infinitely near triple points.

Conversely if there exists a curve B on Σ_d which satisfies the above two conditions, we can construct a minimal algebraic surface with $p_g = n + 1$, $q = 0$ and $c_1^2 = 2n - 2$. Since B has no multiple component, we should have $\Delta_0 B \geqq -d$, i.e., $n \geqq 2d - 3$. The following lemma proves that the condition is also sufficient.

LEMMA 1.4. (1) *Assume $n \geqq 3d - 3$. Then generic curves in $|6\Delta_0 + (n+3+3d)\Gamma|$ on Σ_d are irreducible and nonsingular.*

(2) *Assume that $3d - 3 > n \geqq 2d - 3$. Then generic curves in $|6\Delta_0 + (n+3+3d)\Gamma|$ on Σ_d are of the form $\Delta_0 + B_0$ where the B_0 are irreducible nonsingular curves. Moreover, the B_0 intersect transversally at $n + 3 - 2d$ points with Δ_0.*

Proof. (1) We note that $|\Delta|$ has no base point. Hence $|6\Delta_0+(n+3+3d)\Gamma|$, which is nothing but $|6\Delta + (n + 3 - 3d)\Gamma|$, has no base point by the assumption. Moreover, it is not composite with a pencil. Hence the assertion follows by Bertini's theorem.

(2) If $3d - 3 > n \geqq 2d - 3$, Δ_0 is a fixed component of $|6\Delta_0+(n+3+3d)\Gamma|$. As above we can see that any generic curve B_0 in $|5\Delta_0 + (n + 3 + 3d)\Gamma|$ is irreducible and nonsingular. In order to prove the last assertion, we consider the exact sequence

$$0 \longrightarrow \mathcal{O}(B_0 - \Delta_0) \longrightarrow \mathcal{O}(B_0) \longrightarrow \mathcal{O}_{\Delta_0}(n + 3 - 2d) \longrightarrow 0 .$$

Since we have $H^1(\Sigma_d, \mathcal{O}(B_0 - \Delta_0)) = 0$, it follows that the restriction map

$$H^0(\Sigma_d, \mathcal{O}(B_0)) \longrightarrow H^0(\Delta_0, \mathcal{O}_{\Delta_0}(n + 3 - 2d))$$

is surjective. Hence B_0 intersects transversally at $n + 3 - 2d$ points with Δ_0, provided that B_0 is generic. Q.E.D.

Thus we have seen that the third case occurs if and only if $p_g - 1 \geqq \max(d + 3, 2d - 3)$ and $p_g - d$ is even.

(iv) The case in which W is a cone: Let W denote the image of the canonical map $\Phi_K: S \to \mathbf{P}^n$, which is a cone over a rational curve of degree $n - 1$ in $\mathbf{P}^{n-1}$. We consider the linear system $|\Delta|$ on Σ_{n-1}. This gives rise to a holomorphic map $q: \Sigma_{n-1} \to \mathbf{P}^n$ whose image coincides with W up to a projective transformation. We note that Σ_{n-1} is the minimal resolution of singularity of W.

LEMMA 1.5. *Φ_K factors through $q: \Sigma_{n-1} \to \mathbf{P}^n$.*

Proof. Let $(x_0, x_1, \cdots, x_n)$ be a system of homogeneous coordinates on $\mathbf{P}^n$. We may assume that W is defined by

$$\operatorname{rank}\begin{pmatrix} x_1 & x_2 & \cdots & x_{n-1} \\ x_2 & x_3 & \cdots & x_n \end{pmatrix} < 2 .$$

Then the ratio

$$\frac{x_1}{x_2} = \frac{x_2}{x_3} = \cdots = \frac{x_{n-1}}{x_n}$$

induces a rational function g on S. We set $(g) = D - D_1$, where D and D_1 are two effective divisors without common components. Then we can write

$$(x_1) = D + G_1,\ (x_2) = D_1 + G_1 = D + G_2,\ \cdots,$$
$$(x_{n-1}) = D_1 + G_{n-2} = D + G_{n-1},\ (x_n) = D_1 + G_{n-1}$$

where $G_1, G_2, \cdots, G_{n-1}$ are nonnegative divisors on S. It follows that

$$D - D_1 = G_1 - G_2 = \cdots = G_{n-2} - G_{n-1} .$$

Since D and D_1 have no common component, we can write

$$G_1 = D + G'_1,\ G_2 = D_1 + G'_1 = D + G'_2,\ \cdots,$$
$$G_{n-2} = D_1 + G'_{n-3} = D + G'_{n-2},\ G_{n-1} = D_1 + G'_{n-2}\ ,$$

with nonnegative divisors $G'_1, G'_2, \cdots, G'_{n-2}$. This process leads us to the following:

$$(x_k) = (n-k)D + (k-1)D_1 + G \qquad \text{for } k = 1, 2, \cdots, n$$

with a nonnegative divisor G.

Since $|K|$ has no base point, the support of (x_0) does not meet G. In particular, $KG = 0$. Hence $KD = 2$ and $2 = (n-1)D^2 + DG$. Since D^2 is even by the adjunction formula, it follows that $D^2 = 0$ and $DG = 2$. This implies that the rational map $g\colon S \to \mathbf{P}^1$ defined by $|D|$ is holomorphic and $G \neq 0$. Let ζ be a section of the line bundle $[G]$ over S whose divisor is G. Then ζ/x_0 induces a meromorphic section of $[-(n-1)D]$, hence defines a rational map $f\colon S \to \Sigma_{n-1}$. Since (x_0) does not meet G, f is holomorphic. Clearly, Φ_K is the composition $q \circ f$ up to a projective transformation. Q.E.D.

Let $f\colon S \to \Sigma_{n-1}$ denote the holomorphic map induced by Φ_K as in the above proof. Then the branch locus B of f is linearly equivalent to $6\Delta_0 + 4n\Gamma$ and has no infinitely near triple points. Such a curve B exists if and only if $n = 3$, 4 or 5. In fact, since B has no multiple component, we should have $\Delta_0 B \leqq -(n-1)$, i.e., $n \leqq 5$. Conversely, if $n = 3$, the linear system $|6\Delta_0 + 12\Gamma|$ has no base point on Σ_2. If $n = 4$ or 5, Δ_0 is a fixed component of $|6\Delta_0 + 4n\Gamma|$. However, a generic curve in $|6\Delta_0 + 4n\Gamma|$ is of the form $\Delta_0 + B_0$ where B_0 is an irreducible nonsingular curve intersecting transversally at $5 - n$ points with Δ_0.

Conversely, let B be a curve on Σ_{n-1} which is linearly equivalent to $6\Delta_0 + 4n\Gamma$ and which has no infinitely near triple points, and let S' be the double covering of Σ_{n-1} with branch locus B. Then the minimal resolution of singularity of S' is a minimal algebraic surface with $p_g = n + 1$, $q = 0$ and $c_1^2 = 2n - 2$.

We summarize the above results in the following theorem.

THEOREM 1.6. *Let n be an integer $\geqq 2$ and let S be a minimal algebraic surface with $p_g = n + 1$ and $c_1^2 = 2n - 2$. Then S is the minimal resolution of singularity of one of the following normal surfaces*:

(i) *double coverings of $\mathbf{P}^2$ with branch loci of degree* 8 ($n = 2$);

(ii) *double coverings of $\mathbf{P}^2$ with branch loci of degree* 10 ($n = 5$);

(iii) *double coverings of Σ_d whose branch loci are linearly equivalent to* $6\Delta_0 + (n + 3 + 3d)\Gamma$, *where* $n \geqq \max(d + 3, 2d - 3)$ *and $n - d$ is odd*;

(iv) *double coverings of* Σ_{n-1} *whose branch loci are linearly equivalent to* $6\mathbf{\Delta}_0 + 4n\Gamma$, *where* $n = 3$, 4 *or* 5.

Moreover, in any case, the branch locus has no infinitely near triple points.

For later use we introduce the following definition. We say that a surface S as above is of type (∞), of type (d), or of type (d') according to whether the canonical image $\Phi_K(S)$ is biholomorphically equivalent to $\mathbf{P}^2$, Σ_d, or a cone over a rational curve of degree d in $\mathbf{P}^d$. We shall say that S is generic if the branch locus B of the holomorphic map $f\colon S \to W$, $W = \mathbf{P}^2$ or Σ_d, has only the simplest possible singularities (cf. Lemma 1.4).

We conclude this section with the following:

COROLLARY 1.7. *Let S be a minimal algebraic surface with* $c_1^2 = 2p_g - 4$ *and* $p_g \geqq 3$. *Assume that S is not of type* (∞). *Then there exists a surjective holomorphic map* $g\colon S \to \mathbf{P}^1$ *whose general fibre is an irreducible nonsingular curve of genus* 2. *Such a fibre structure is unique up to an automorphism of* $\mathbf{P}^1$ *except for the case*: $p_g = 4$, $c_1^2 = 4$ *and is of type* (0).

Proof. It remains to prove the uniqueness. Let $g'\colon S \to \mathbf{P}^1$ be a surjective holomorphic map with general fibre D' nonsingular of genus 2. Then we have

$$2 = D'f^*\Delta_0 + \frac{n-1+d}{2}\,D'f^*\Gamma\ .$$

We claim that $D'f^*\Gamma = 0$. If this is true, $(D')^2 = 0$ implies that D' is a rational multiple of a fibre of g ([30], p. 92). Since we have $KD' = Kf^*\Gamma = 2$, it follows that D' is linearly equivalent to $f^*\Gamma$.

We now prove the equality $D'f^*\Gamma = 0$. Suppose that $D'f^*\Gamma > 0$. If $D'f^*\Gamma = 1$, g induces a birational map $D' \to \mathbf{P}^1$ which is a contradiction. If $D'f^*\Gamma \geqq 2$, we have $n \leqq 3 - d$. This is possible only if $n = 3$, $d = 0$. Q.E.D.

Remark. If $p_g = 4$, $c_1^2 = 4$ and is of type (0), we have two fibre structures $g\colon S \to \mathbf{P}^1$ with general fibre of genus 2, corresponding to the projections $\Sigma_0 \to \mathbf{P}^1$. However, these are the only possible ones.

2. Number of moduli

Throughout this section, S will denote a minimal algebraic surface with $c_1^2 = 2p_g - 4$ and $p_g \geqq 3$. We set $n = p_g - 1$. The purpose of this section is to prove the following theorem.

THEOREM 2.1. *Let S be a generic surface of type* (∞) *or of type* (d). *Then the number of moduli* $m(S)$ *of S is defined* (see [20], §11) *and we have* *the equalities*

$$m(S) = \dim H^1(S, \Theta_S) = \begin{cases} 7n + 22 & \textit{if } S \textit{ is of type } (\infty) \\ 7n + 21 & \textit{if } S \textit{ is of type } (d) \end{cases}$$

where Θ_S denotes the sheaf of germs of holomorphic vector fields on S.

Remark. We have $\dim H^2(S, \Theta_S) = n - 2$ or $n - 3$ according to whether S is of type (∞) or of type (d).

First we introduce some notations. For any sheaf $\mathcal{F}$ on S, we set $h^i(\mathcal{F}) = \dim H^i(S, \mathcal{F})$ and $\chi(\mathcal{F}) = \sum_{i=0}^{2}(-1)^i h^i(\mathcal{F})$. If $\mathcal{F} = \mathcal{O}(D)$ with a divisor D, we also write $h^i(D) = h^i(\mathcal{O}(D))$ and $\chi(D) = \chi(\mathcal{O}(D))$. For any complex manifold X, we let Θ_X denote the sheaf of germs of holomorphic vector fields on X. If $f: X \to Y$ is a holomorphic map of complex manifolds, $\Theta_{X/Y}$ denotes the sheaf of germs of relative vector fields and $\mathcal{T}_{X/Y}$ denotes the cokernel of the canonical homomorphism $F: \Theta_X \to f^*\Theta_Y$ induced by f.

If S is of type (∞), Theorem 2.1 is included in a result of Wavrik [32]. Hence we assume that S is of type (d). We note that $n \geqq d + 3$.

Let $W = \Sigma_d$ and let $f: S \to W$ be the holomorphic map of degree 2 induced by the canonical map Φ_K. We let $g: S \to \mathbf{P}^1$ denote the composition of f and the natural projection $W \to \mathbf{P}^1$. As in Section 1, we let Γ and $\mathbf{\Delta}_0$ denote, respectively, a fibre of $W \to \mathbf{P}^1$ and the 0-section of W. We set $D = f^*\Gamma$.

In the following two lemmas, we do not have to assume that S is generic.

LEMMA 2.2. $h^0(2D) = 3$, $h^1(2D) = 0$ *and* $h^2(2D) = n - 3$.

LEMMA 2.3. *The natural maps*

$$H^i(W, \Theta_W) \longrightarrow H^i(S, f^*\Theta_W)$$

are bijective for $i = 0, 1$, and $h^2(f^*\Theta_W) = n - 3$.

Proof of Lemma 2.2. Since $|D|$ is an irreducible pencil, we have $h^0(2D) = 3$, and, by the Riemann-Roch theorem, $\chi(2D) = n$. Also by the Serre duality, we have

$$h^2(2D) = h^0\left(f^*\left(\Delta_0 + \frac{n - 5 + d}{2}\,\Gamma\right)\right).$$

We note that the natural map

$$H^0\left(W, \mathcal{O}\left(\left[\Delta_0 + \frac{n - 1 + d}{2}\,\Gamma\right]\right)\right) \longrightarrow H^0(S, \mathcal{O}(K))$$

is bijective ([13], Lemma 6). It follows that

$$h^0\left(f^*\left(\Delta_0 + \frac{n-5+d}{2}\Gamma\right)\right) = h^0\left(\Delta_0 + \frac{n-5+d}{2}\Gamma\right) = n-3\,.$$

Thus we obtain also $h^1(2D) = 0$. Q.E.D.

Proof of Lemma 2.3. First we note the exact sequence

$$0 \longrightarrow \Theta_{W/\mathbf{P}^1} \longrightarrow \Theta_W \longrightarrow \mathcal{O}(2\Gamma) \longrightarrow 0\,.$$

By virtue of Lemma 2.2, it suffices to prove that the natural maps

$$H^i(W, \Theta_{W/\mathbf{P}^1}) \longrightarrow H^i(S, f^*\Theta_{W/\mathbf{P}^1})$$

are bijective for $i = 0, 1, 2$.

Let B be the branch locus of f and let S' be the double covering of W with branch locus B. Then S' has only rational double points as its singularities and S is the minimal resolution of singularities of S'. It follows that $R^1f_*\mathcal{O}_S = 0$. Therefore, we have natural isomorphisms

$$H^i(W, f_*f^*\Theta_{W/\mathbf{P}^1}) \cong H^i(S, f^*\Theta_{W/\mathbf{P}^1})\,.$$

Thus it suffices to prove that the natural maps

$$H^i(W, \Theta_{W/\mathbf{P}^1}) \longrightarrow H^i(W, f_*f^*\Theta_{W/\mathbf{P}^1}) \tag{2.1}$$

are bijective for $i = 0, 1, 2$.

Let $f'\colon S' \to W$ denote the covering map. Then we have $f_*\mathcal{O}_S = f'_*\mathcal{O}_{S'}$. Therefore, by the construction of S', we have the following exact sequence

$$0 \longrightarrow \mathcal{O}_W \longrightarrow f_*\mathcal{O}_S \longrightarrow \mathcal{O}_W(-F) \longrightarrow 0$$

where $F = [3\Delta_0 + (n+3+3d)/2\Gamma]$, i.e., $2F = [B]$. Tensoring $\Theta_{W/\mathbf{P}^1}$, we get the exact sequence

$$0 \longrightarrow \Theta_{W/\mathbf{P}^1} \longrightarrow f_*f^*\Theta_{W/\mathbf{P}^1} \longrightarrow \Theta_{W/\mathbf{P}^1}(-F) \longrightarrow 0\,,$$

while we have an isomorphism

$$\Theta_{W/\mathbf{P}^1}(-F) \cong \mathcal{O}_W\left(-\Delta_0 - \frac{n+3+d}{2}\Gamma\right).$$

This implies that $H^i\big(W, \Theta_{W/\mathbf{P}^1}(-F)\big) = 0$ for $i = 0, 1, 2$. This proves that (2.1) is bijective for $i = 0, 1, 2$. Q.E.D.

We now prove Theorem 2.1 in the case when $n \geqq 3d - 3$.

LEMMA 2.4. *Assume that S is generic and that $n \geqq 3d-3$. Then we have $h^0(\mathcal{T}_{S/W}) = 7n + 27$ and $h^1(\mathcal{T}_{S/W}) = 0$.*

Proof. We note that B is irreducible and nonsingular (Lemma 1.4). Hence $\mathcal{T}_{S/W}$ is isomorphic to $f^*\mathfrak{N}_B$ where $\mathfrak{N}_B$ denotes the sheaf of germs of sections of the normal bundle of B in W (see [13], Lemma 10). By the Riemann-Roch theorem we have $h^0(\mathcal{T}_{S/W}) = 7n + 27$ and $h^1(\mathcal{T}_{S/W}) = 0$.

Q.E.D.

Let $\pi\colon \mathcal{W} \to N$ be a complete family of deformations of $W = \pi^{-1}(0)$ with $0 \in N$ such that the infinitesimal deformation map $\rho'\colon T_0(N) \to H^1(W, \Theta_W)$ is bijective ([22], or [31]), where $T_0(N)$ denotes the tangent space of N at 0. By Lemma 2.4 we can apply a theorem of existence ([11], Theorem 5.4) to $S \to W \to \mathcal{W}$. Thus we can construct a family $p\colon \mathcal{S} \to M$ of deformations of $S = p^{-1}(0)$, $0 \in M$, and holomorphic maps $s\colon M \to N$ with $s(0) = 0$ and $\Phi\colon \mathcal{S} \to \mathcal{W}$ over s such that the characteristic map $\tau\colon T_0(M) \to D_{S/\mathcal{W}}$ is bijective, where $D_{S/\mathcal{W}}$ is the space of infinitesimal deformations of $S \to \mathcal{W}$. By Lemma 2.3 and by the Corollary to Lemma 5.1 of [11], we see that the infinitesimal deformation map $\rho\colon T_0(M) \to H^1(S, \Theta_S)$ is surjective.

Since we have $H^0(S, \Theta_S) = 0$ (see [25]), it follows that the number of moduli $m(S)$ is defined and equals $h^1(\Theta_S)$. Finally, from the exact sequence

$$0 \longrightarrow \Theta_S \longrightarrow f^*\Theta_W \longrightarrow \mathcal{T}_{S/W} \longrightarrow 0$$

and Lemmas 2.3 and 2.4, we infer that $h^1(\Theta_S) = 7n + 21$ and $h^2(\Theta_S) = n - 3$.

Next we shall prove Theorem 2.1 in the case in which $3d - 3 > n \geqq 2d - 3$. In this case, the branch locus B is of the form $\Delta_0 + B_0$ where B_0 is an irreducible nonsingular curve intersecting transversally at $r = n - 2d + 3$ points $s_1, s_2, \cdots, s_r$ with Δ_0. Let $q\colon \widetilde{W} \to W$ be the composition of quadric transformations with centers at $s_1, s_2, \cdots, s_r$. Then $f\colon S \to W$ factors as $f = q \circ \tilde{f}$ with $\tilde{f}\colon S \to \widetilde{W}$ (see [13], Lemma 5). The branch locus of $\tilde{f}$ is $G + \widetilde{B}_0$ where G and $\widetilde{B}_0$ denote, respectively, the proper transforms of Δ_0 and B_0 by q. We set $E_i = q^{-1}(s_i)$ and $\widetilde{E}_i = \tilde{f}^{-1}(E_i)$. Then the $\widetilde{E}_i$ are nonsingular rational curves on S.

LEMMA 2.5. *The composition*

$$P \circ f^*\colon H^1(W, \Theta_W) \longrightarrow H^1(S, \mathcal{T}_{S/W})$$

of $f^*\colon H^1(W, \Theta_W) \to H^1(S, f^*\Theta_W)$ *and* $P\colon H^1(S, f^*\Theta_W) \to H^1(S, \mathcal{T}_{S/W})$ *is surjective.*

Proof. We have an exact sequence

$$0 \longrightarrow \mathcal{T}_{S/\widetilde{W}} \longrightarrow \mathcal{T}_{S/W} \longrightarrow \tilde{f}^*\mathcal{T}_{\widetilde{W}/W} \longrightarrow 0\ ,$$

and $\tilde{f}^*\mathcal{T}_{\widetilde{W}/W}$ is isomorphic to $\bigoplus_{i=1}^r \mathcal{O}_{\widetilde{E}_i}(2)$, where each $\mathcal{O}_{\widetilde{E}_i}(2)$ denotes the invertible sheaf of degree 2 on $\widetilde{E}_i$ (see [13], Lemma 12). It follows that the natural map

$$H^1(S, \mathcal{T}_{S/\widetilde{W}}) \longrightarrow H^1(S, \mathcal{T}_{S/W}) \tag{2.2}$$

is surjective.

On the other hand, we have the following commutative diagram:

$$\begin{array}{ccccc} H^1(\widetilde{W}, \Theta_{\widetilde{W}}) & \xrightarrow{f^*} & H^1(S, \widetilde{f}^*\Theta_{\widetilde{W}}) & \xrightarrow{\tilde{P}} & H^1(S, \mathcal{T}_{S/\widetilde{W}}) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(\widetilde{W}, q^*\Theta_W) & \xrightarrow{f^*} & H^1(S, f^*\Theta_W) & \xrightarrow{P} & H^1(S, \mathcal{T}_{S/W}) \end{array}$$

where vertical maps are those induced by q. In view of a canonical isomorphism $H^1(W, \Theta_W) \cong H^1(\widetilde{W}, q^*\Theta_W)$ and the surjectivity of (2.2), it suffices to prove that

$$\widetilde{P}\circ\widetilde{f}^*: H^1(\widetilde{W}, \Theta_{\widetilde{W}}) \longrightarrow H^1(S, \mathcal{T}_{S/\widetilde{W}})$$

is surjective.

We note that $H^1(S, \mathcal{T}_{S/\widetilde{W}})$ is isomorphic to $H^1(G, \mathfrak{N}_G)$ where $\mathfrak{N}_G$ denotes the sheaf of germs of sections of the normal bundle of G in $\widetilde{W}$. Moreover, by this isomorphism, $\widetilde{P}\circ\widetilde{f}^*$ corresponds to a natural map

$$H^1(\widetilde{W}, \Theta_{\widetilde{W}}) \longrightarrow H^1(G, \mathfrak{N}_G) \tag{2.3}$$

induced by the projection $\Theta_{\widetilde{W}} \to \mathfrak{N}_G$. We now recall the following two exact sequences:

$$0 \longrightarrow \Theta_{\widetilde{W}}(-G) \longrightarrow \Theta_{\widetilde{W}} \longrightarrow \Theta_{\widetilde{W}} \mid G \longrightarrow 0 \ ,$$
$$0 \longrightarrow \Theta_G \longrightarrow \Theta_{\widetilde{W}} \mid G \longrightarrow \mathfrak{N}_G \longrightarrow 0 \ .$$

In view of these exact sequences, to prove that (2.3) is surjective, it suffices to show $H^2(\widetilde{W}, \Theta_{\widetilde{W}}(-G)) = 0$.

For this purpose, we use the exact sequence

$$0 \longrightarrow \Theta_{\widetilde{W}} \longrightarrow q^*\Theta_W \longrightarrow \textstyle\bigoplus_{i=1}^r \mathcal{O}_{E_i}(1) \longrightarrow 0$$

(see [13], Lemma 12). From this we get the exact sequence

$$0 \longrightarrow \Theta_{\widetilde{W}}(-G) \longrightarrow q^*\Theta_W(-G) \longrightarrow \textstyle\bigoplus_{i=1}^r \mathcal{O}_{E_i} \longrightarrow 0 \ .$$

Hence it suffices to show $H^2(\widetilde{W}, q^*\Theta_W(-G)) = 0$. We have another exact sequence

$$0 \longrightarrow q^*\Theta_W(-\Delta_0) \longrightarrow q^*\Theta_W(-G) \longrightarrow \textstyle\bigoplus_{i=1}^r \mathcal{O}_{E_i}(-1)^2 \longrightarrow 0 \ .$$

From this we infer that

$$H^2(\widetilde{W}, q^*\Theta_W(-G)) \cong H^2(W, \Theta_W(-\Delta_0)) = 0 \ .$$

In fact, the last equality follows from the exact sequence

$$0 \longrightarrow \mathcal{O}_W(\Delta) \longrightarrow \Theta_W(-\Delta_0) \longrightarrow \mathcal{O}_W(2\Gamma - \Delta_0) \longrightarrow 0$$

and the Serre duality. This completes the proof of Lemma 2.5.

Let $\pi: \mathcal{W} \to N$ be a complete family of deformations of $W = q^{-1}(0)$ with $0 \in N$ such that the infinitesimal deformation map $\rho': T_0(N) \to H^1(W, \Theta_W)$ is bijective. By Lemma 2.5 we can apply a theorem of existence ([11], Theorem 5.4) to $S \to W \to \mathcal{W}$. Thus, by the same argument as in the case when $n \geqq 3d-3$, we can prove that $m(S)$ is defined and equals $h^1(\Theta_S)$.

In order to calculate $h^1(\Theta_S)$, we use the exact sequence

$$0 \longrightarrow \Theta_S \longrightarrow f^*\Theta_W \longrightarrow \mathcal{T}_{S/W} \longrightarrow 0 .$$

The above Lemma 2.5 implies that $P: H^1(S, f^*\Theta_W) \to H^1(S, \mathcal{T}_{S/W})$ is surjective. Therefore, by Lemma 2.3, we have $h^2(\Theta_S) = h^2(f^*\Theta_W) = n-3$. On the other hand, the Riemann-Roch theorem yields $h^1(\Theta_S) - h^2(\Theta_S) = 6n + 24$. Hence we obtain $h^1(\Theta_S) = 7n + 21$.

3. Deformations

In this section we shall study deformations of minimal algebraic surfaces with $c_1^2 = 2p_g - 4$ and $p_g \geqq 3$. First of all we note that any deformation is a minimal algebraic surface with the same numerical characters as the original one ([17], Theorem 23). Our purpose is to determine the number of deformation types (for definition, see Introduction) of these surfaces.

First we shall prove the following

THEOREM 3.1. *Minimal algebraic surfaces with given p_g and c_1^2 satisfying $c_1^2 = 2p_g - 4$, $p_g \geqq 3$ of the same type* (∞), (d), *or* (d') *have the same deformation type.*

Proof. Let S be a minimal algebraic surface with $c_1^2 = 2p_g - 4$, $p_g \geqq 3$ and of type (d). Then we have a holomorphic map $f: S \to \Sigma_d$ of degree 2 whose branch locus B is linearly equivalent to $6\Delta_0 + (n+3+3d)\Gamma$. By a result of Brieskorn ([3], [4]), S is a deformation of a generic surface of the same type.

On the other hand, all generic surfaces of type (d) are parametrized by a connected open subset of the projective space $|6\Delta_0 + (n+3+3d)\Gamma|$. This completes the proof of the assertion for surfaces of type (d). The proofs are the same for surfaces of type (d') or (∞).

Next we shall study variations of types under deformations.

THEOREM 3.2. *Let S be a minimal algebraic surface with $c_1^2 = 2p_g - 4$ and $p_g \geqq 3$. Assume that S is of type (d) with $d \geqq 2$. Then S is a deformation of a surface of type (d_0) with some integer $d_0 < d$, $d_0 \equiv d \bmod 2$, except for the case: $c_1^2 \equiv 0 \bmod 8$ and $p_g = 2d - 2$.*

Proof. Let $W = \Sigma_d$ and let $f: S \to W$ be the holomorphic map of degree 2 induced by the canonical map. By virtue of Theorem 3.1, we may assume that S is generic.

First we assume that $p_g \geqq 3d - 2$. In this case we have $H^1(S, \mathcal{T}_{S/W}) = 0$ (Lemma 2.4). Hence the assertion is a consequence of a theorem of stability ([11], Theorem 6.1). In this case d_0 may be any nonnegative integer which satisfies the above conditions.

Next we consider the case in which $3d - 2 > p_g \geqq 2d - 2$. In this case the branch locus B of f is of the form $\Delta_0 + B_0$ where B_0 is an irreducible nonsingular curve intersecting transversally at $p_g - 2d + 2$ points with Δ_0.

Let $q: \mathcal{W} \rightarrow N$ be a complete family of deformations of $W = q^{-1}(0)$, $0 \in N$. We may assume that, for $s \neq 0$, $W_s = q^{-1}(s)$ are respectively biholomorphically equivalent to $\Sigma_{d(s)}$ with some $d(s)$ satisfying $d(s) < d$ and $d(s) \equiv d \mod 2$ (see [31]). As we have seen in the proof of Theorem 2.1, there exists a family $p: \mathcal{S} \rightarrow M$ of deformations of $S = p^{-1}(0)$, $0 \in M$, a holomorphic map $s: M \rightarrow N$ with $s(0) = 0$ and a holomorphic map $\Phi: \mathcal{S} \rightarrow \mathcal{W}$ over s which induces f on S such that the characteristic map $\tau: T_0(M) \rightarrow D_{S/\mathcal{W}}$ is bijective. We note that the infinitesimal deformation map $\rho: T_0(M) \rightarrow H^1(S, \Theta_S)$ is surjective.

Let $f_t: S_t \rightarrow W_{s(t)}$ denote the holomorphic map induced by Φ on S_t. Since we have $H^1(S, \mathcal{O}_S) = 0$, f_t coincides with the holomorphic map induced by the canonical map. Therefore, if $s(t) \neq 0$ for some t, S_t is a surface of type (d_0) with $d_0 < d$.

We now assume that $s: M \rightarrow N$ is a constant map, which will lead us to a contradiction. We note that Φ induces a holomorphic map $\mathcal{S} \rightarrow W \times M$ over M, which is still denoted by Φ. Let $\mathcal{R}$ denote the ramification divisor of Φ. Then, on each S_t, $\mathcal{R}$ induces the ramification divisor R_t of the induced map $f_t: S_t \rightarrow W$.

We claim that the branch loci B_t of f_t form a flat family of divisors on W. In order to prove this claim, we consider the line bundle $F = [3\Delta_0 + (n + 3 + 3d)/2\Gamma]$ and the exact sequence

$$0 \longrightarrow \mathcal{O}_W \longrightarrow (f_t)_*\mathcal{O}_{S_t} \longrightarrow \mathcal{O}(-F) \longrightarrow 0$$

on W (see the proof of Lemma 2.3). From this we obtain the two exact sequences

$$0 \longrightarrow \mathcal{O}(F) \longrightarrow (f_t)_*\mathcal{O}\left([R_t]\right) \longrightarrow \mathcal{O}_W \longrightarrow 0\ ,$$
$$0 \longrightarrow \mathcal{O}(2F) \longrightarrow (f_t)_*\mathcal{O}\left([2R_t]\right) \longrightarrow \mathcal{O}(F) \longrightarrow 0\ .$$

Since $H^0(W, (f_t)_*\mathcal{O}([R_t])) \rightarrow H^0(W, \mathcal{O}_W)$ are surjective, we have $\dim H^0(S_t, \mathcal{O}([R_t])) = \dim H^0(W, \mathcal{O}(F)) + 1$. We take a section $\psi_t \in H^0(S_t, \mathcal{O}([R_t]))$ with $(\psi_t) = R_t$, which depends holomorphically on t. The above equality implies that $H^0(S_t, \mathcal{O}([R_t]))$ is a direct sum of $H^0(W, \mathcal{O}(F))$ and $\psi_t\mathbf{C}$. From the second exact sequence it follows that

$$\dim H^0(S_t, \mathcal{O}([2R_t])) \leqq \dim H^0(W, \mathcal{O}(2F)) + \dim H^0(W, \mathcal{O}(F))\ .$$

On the other hand $H^0(S_t, \mathcal{O}([2R_t]))$ contains $H^0(W, \mathcal{O}(2F)) \oplus \psi_t H^0(W, \mathcal{O}(F))$. Hence these two spaces coincide.

By a theorem of Grauert ([9]), $p_*\mathcal{O}([2\mathcal{R}])$ is a locally free sheaf on M. Moreover, we can find $b_t \in H^0(W, \mathcal{O}(F))$ and $c_t \in H^0(W, \mathcal{O}(2F))$ depending holomorphically on t such that

$$\psi_t^2 = 2b_t\psi_t + c_t\ .$$

Thus the branch loci B_t are defined by the equations $b_t^2 - c_t = 0$. This proves our claim.

As we have seen in Section 1, B_t are of the form $\Delta_0 + B_{0t}$. Here $\{B_{0t} \mid t \in M\}$ describes a flat family of curves on W. Assume for a while that $\Delta_0 \cap B_{0t} = \varnothing$. Then we have $p_g = 2d - 2$. This implies that d is even and that $c_1^2 = 4d - 8$ is divisible by 8. This is the case which we have excluded. Thus, for each t, we can find a point $x_t \in \Delta_0 \cap B_{0t}$, which depends holomorphically on t. On each S_t, $\widetilde{E}_t = f_t^{-1}(x_t)$ is a nonsingular rational curve with $\widetilde{E}_t^2 = -2$. This implies that the curve $\widetilde{E} = f^{-1}(x_0)$ on S extends to a family $\{\widetilde{E}_t \mid t \in M\}$ of curves on S_t. We recall that the infinitesimal deformation map $\rho\colon T_0(M) \to H^1(S, \Theta_S)$ is surjective. Hence the above fact implies that the natural map

$$H^1(S, \Theta_S) \longrightarrow H^1(\widetilde{E}, \mathfrak{N}_{\widetilde{E}}) \tag{3.1}$$

is a 0-map where $\mathfrak{N}_{\widetilde{E}}$ denotes the sheaf of germs of sections of the normal bundle of $\widetilde{E}$ in S. This contradicts a theorem in [5] and [15] to the effect that (3.1) is surjective. This completes the proof of Theorem 3.2.

THEOREM 3.3. *Assume that $c_1^2 \not\equiv 0 \bmod 8$. Then minimal algebraic surfaces with given p_g and c_1^2 satisfying $c_1^2 = 2p_g - 4$ and $p_g \geqq 3$ have one and the same deformation type.*

This theorem is a consequence of Theorem 3.1 and Theorem 3.2, provided that p_g is different from 4 or 5.

In the case in which $p_g = 4$ and $c_1^2 = 4$, any surface of type (2′) is a deformation of a surface of type (0). The proof is the same as that of Theorem 3.2 for the case $p_g \geqq 3d - 2$.

The proof of Theorem 3.3 for surfaces with $p_g = 5$ and $c_1^2 = 6$ will be postponed until Section 5, Theorem 5.1.

THEOREM 3.4. *Every minimal algebraic surface S with $c_1^2 = 2p_g - 4$ and $p_g \geqq 3$ is simply connected.*

Proof. First we note that S admits no finite unramified covering. In fact, if S has an m-sheeted unramified covering, we have $c_1^2 \geqq 2p_g + 2 - (6/m)$ (see [2], Theorem 14). This implies that $m = 1$.

In view of the above fact it suffices to prove that the fundamental group $\pi_1(S)$ is abelian. Moreover by Theorem 3.2, we may restrict ourselves to the following three cases:

(i) S is of type (∞),

(ii) S is of type (d) with $p_g \geqq 3d - 2$,

(iii) S is of type (d) or (d') with $p_g = 2d - 2$.

In addition we may assume that S is generic. Therefore in either case the branch locus B is nonsingular. In the first two cases B is irreducible, while in the last case B is a disjoint union of two irreducible curves Δ_0 and B_0. For our purpose it suffices to prove that $\pi_1(\mathbf{P}^2 - B)$ or $\pi_1(\Sigma_d - B)$ is abelian. This follows immediately from [33], Proposition 3. In fact, employing the notation in that proposition, we apply it to $V = \mathbf{P}^2$ or Σ_d, $A = B$, and $W = \varnothing$ in case (i) or (ii) and to $V = \Sigma_d$, $A = B_0$ and $W = \Delta_0$ in case (iii).

4. Surfaces with $p_g = 6$ and $c_1^2 = 8$

As we have proved in Section 1, minimal algebraic surfaces with $p_g = 6$ and $c_1^2 = 8$ are classified into four types: type (∞), type (0), type (2) and type (4′). In this section, we shall study deformations of these surfaces. Our main theorem is the following.

THEOREM 4.1. *Minimal algebraic surfaces with $p_g = 6$ and $c_1^2 = 8$ are classified into two deformation types. One consists of surfaces of type* (0) *and surfaces of type* (2). *The other consists of surfaces of type* (∞) *and surfaces of type* (4′). *These two deformation types are not homotopically equivalent.*

Let S be a surface of type (4′) and let $f: S \to \Sigma_4$ be the holomorphic map of degree 2 induced by the canonical map (see Lemma 1.5). Let B denote the branch locus of f. Then B is a disjoint sum $\Delta_0 + B_0$ where B_0 is a curve on Σ_4 which is linearly equivalent to $5\Delta_0 + 20\Gamma$. It follows that $f^*\Delta_0 = 2F$ where F is a nonsingular rational curve with $F^2 = -2$. This implies that the canonical bundle K of S is divisible by 2, i.e., $K = 2L$ with $L = [F + 2f^*\Gamma]$.

As a converse we have the following lemma.

LEMMA 4.2. *Assume that there exists a line bundle L on S such that $K = 2L$. Then S is either of type (∞) or of type* (4′).

Proof. By the Riemann-Roch theorem and the Serre duality, we have

$$2h^0(L) - h^1(L) = 6 .$$

On the other hand, we have

$$2h^0(L) - 1 \leqq h^0(2L) = 6 .$$

Therefore, we obtain $h^0(L) = 3$ and $h^1(L) = 0$.

First we assume that $|L|$ is not composite with a pencil. Let $\{\varphi_0, \varphi_1, \varphi_2\}$ be a basis of $H^0(S, \mathcal{O}(L))$. Then, by our assumption, the map $z \longrightarrow (\varphi_0(z), \varphi_1(z), \varphi_2(z))$ is a generically surjective rational map $S \longrightarrow \mathbf{P}^2$. It follows that the products $\varphi_i\varphi_j$ $(0 \leqq i \leqq j \leqq 2)$ are linearly independent. Hence these products form a basis of $H^0(S, \mathcal{O}(K))$. Since $|K|$ has no base point, this implies that $|L|$ has no base point. This proves that S is of type (∞).

Next we assume that $|L|$ is composite with a pencil. Since we have $h^0(L) = 3$ and since the irregularity vanishes ([2], Theorem 10), we can write $|L| = |2D| + F$ where D is an irreducible pencil possibly with base points and F is the fixed part of $|L|$. It follows that $K = [4D + 2F]$. Let $\{g_0, g_1\}$ be a basis of $H^0(S, \mathcal{O}(D))$ and let $\zeta \in H^0(S, \mathcal{O}([2F])$ satisfy $(\zeta) = 2F$. Then $g_0^k g_1^{4-k}\zeta$, $k = 0, 1, \cdots, 4$, are linearly independent sections of $H^0(S, \mathcal{O}(K))$. This implies that the canonical map $\Phi_K\colon S \longrightarrow \mathbf{P}^5$ factors through a cone over a rational curve of degree 4 in $\mathbf{P}^4$. This proves that S is of type $(4')$. Q.E.D.

Concerning the hypothesis of Lemma 4.2, we have the following lemma.

LEMMA 4.3. *There exists a line bundle L on S such that $K = 2L$ if and only if the second Stiefel-Whitney class W_2 vanishes.*

Proof. From the exact sequence

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 0$$

we get the exact sequence

$$H^1(S, \mathcal{O}) \longrightarrow H^1(S, \mathcal{O}^*) \longrightarrow H^2(S, \mathbf{Z}) \longrightarrow H^2(S, \mathcal{O}) .$$

It follows that there exists a line bundle L such that $K = 2L$ if and only if the first Chern class $c_1 \in H^2(S, \mathbf{Z})$ is divisible by 2. The last condition is equivalent to the vanishing of W_2. Q.E.D.

Since W_2 is a homotopy invariant this proves the last assertion of Theorem 4.1.

Now we shall prove the first assertion of Theorem 4.1. By virtue of Theorems 3.1 and 3.3, any two surfaces of type (0) or (2) have the same deformation type. Furthermore, by Theorem 3.1, it suffices to prove the following lemma.

LEMMA 4.4. *Let S be a generic surface of type* $(4')$. *Then there exists a family $p\colon \mathcal{S} \longrightarrow M$ of deformations of $S = p^{-1}(0)$, $0 \in M$ such that $S_t = p^{-1}(t)$, $t \neq 0$, are of type* (∞).

Proof. We represent S as a double covering $f\colon S \to \Sigma_4$ and let $B = \Delta_0 + B_0$ denote the branch locus of f. We note that $f^*\Delta_0 = 2F$ where F is a nonsingular rational curve with $F^2 = -2$.

We construct a family as follows. Let Q be a nonsingular quadric curve in $\mathbf{P}^2$. We cover $\mathbf{P}^2$ by sufficiently small coordinate neighborhoods U_i. Let $q_i(z) = 0$ be the equations of Q on U_i and set $e_{ij}(z) = q_i(z)/q_j(z)$ on $U_i \cap U_j$. Then the 1-cocycle $\{e_{ij}\}$ defines the line bundle of degree 2 over $\mathbf{P}^2$.

Let V be the $\mathbf{P}^1$-bundle over $\mathbf{P}^2$ which is the completion of the line bundle of degree -2. That is, V is covered by $U_i \times \mathbf{P}^1$ and $(z, w_i) \in U_i \times \mathbf{P}^1$ coincides with $(z, w_j) \in U_j \times \mathbf{P}^1$ if and only if $w_i = e_{ij}(z)^{-1}w_j$, where w^i denote inhomogeneous coordinates on $\mathbf{P}^1$.

We set $N = \{s \in \mathbf{C} \mid |s| < \varepsilon\}$ with a sufficiently small positive number ε. With each point s of N we associate a subvariety W_s of V defined by the equations

$$w_i q_i(z) = s \quad \text{on} \quad U_i \times \mathbf{P}^1 .$$

If $s \neq 0$, the projection $\pi\colon V \to \mathbf{P}^2$ induces a biholomorphic map $W_s \to \mathbf{P}^2$. Let V_0 denote a section of π defined by $w_i = 0$. Then W_0 consists of $\pi^{-1}(Q)$ and V_0. We note that $\pi^{-1}(Q)$ is biholomorphically equivalent to Σ_4 and that $\pi^{-1}(Q)$ intersects V_0 along its 0-section Δ_0. We define a 3-dimensional submanifold $\mathcal{W}$ of $V \times N$ to be the union $\bigcup_{s \in N} W_s \times s$.

Let V_∞ denote a section of π defined by $w_i = \infty$. Then the line bundle $[5V_\infty]$ induces $[B_0]$ on $\pi^{-1}(Q)$. Therefore, we get the exact sequence

$$0 \longrightarrow \mathcal{O}(5V_\infty - \pi^*Q) \longrightarrow \mathcal{O}(5V_\infty) \longrightarrow \mathcal{O}_{\pi^{-1}(Q)}(B_0) \longrightarrow 0 .$$

By [32], Proposition 2.1, we have

$$\begin{aligned} H^1\big(V, \mathcal{O}(5V_\infty - \pi^*Q)\big) &= \textstyle\bigoplus_{k=0}^{5} H^1\big(\mathbf{P}^2, \mathcal{O}((k-1)Q)\big) \\ &= 0 . \end{aligned}$$

Since $|5V_\infty|$ has no base point on V, we can find a nonsingular divisor $\mathcal{B}_0 \in |5V_\infty|$ which induces B_0 on $\pi^{-1}(Q)$. We set $\mathcal{B} = V_0 + \mathcal{B}_0$. Then $\mathcal{B}$ induces B on $\pi^{-1}(Q)$, while, since V_0 is disjoint from W_s for $s \neq 0$, $\mathcal{B}$ induces a divisor B_s of degree 10 on each W_s, $s \neq 0$.

Let $\varpi\colon \mathfrak{M} \to \mathcal{W}$ be the double covering of $\mathcal{W}$ with branch locus $\mathcal{B}$ and let $\tilde{S}_s$ denote the fibre of $\mathfrak{M} \to N$ over $s \in N$. For each $s \neq 0$, $\tilde{S}_s$ is the double covering of $\mathbf{P}^2$ with branch locus B_s.

Since V_0 is a component of the branch locus $\mathcal{B}$, $\varpi^* V_0$ is of the form $2E$ where E is a divisor on $\mathfrak{M}$. We note that E is biholomorphically equivalent to $\mathbf{P}^2$ and that $\tilde{S}_0$ has two components E and S; moreover, S intersects E along the curve F.

We claim that the normal bundle $\mathfrak{N}_E$ of E in $\mathfrak{M}$ is of degree -1. In order to prove this claim, let $\mathfrak{N}_{V_0}$ be the normal bundle of V_0 in V. Then $\mathfrak{N}_{V_0}$ is of degree -2. Hence we have

$$2 \deg \mathfrak{N}_E = \deg \varpi^* \mathfrak{N}_{V_0} = -2 .$$

This proves our claim.

By a theorem of Nakano (see [28], [8]) we can contract E into a non-singular point and obtain a complex manifold $\mathfrak{M}'$. We let S'_s denote the fibre of the natural projection $\mathfrak{M}' \to N$ over $s \in N$. For each $s \neq 0$, S'_s is biholomorphically equivalent to $\tilde{S}_s$, while S'_0 is the surface obtained from S by contracting F into an ordinary double point.

From the family $\{S'_s \mid s \in N\}$ we can construct a family $\{S_t \mid |t| < \varepsilon^{1/2}\}$ such that $S_0 = S$ and $S_t = S'_s$ with $s = t^2$ for $t \neq 0$ (see [1] or [3]). This completes the proof of Lemma 4.4.

As a corollary to Theorem 4.1, we have the following theorem.

THEOREM 4.5. *Let X be the underlying differentiable manifold of a surface S_0 of type* (∞) *or* (4′). *Then any complex structure on X is obtained from S_0 by deformation.*

Proof. Let S be a complex structure on X. Since the second Stiefel-Whitney class vanishes, S is minimal. Therefore S is a deformation of S_0 by Theorem 4.1. Q.E.D.

5. Surfaces with $p_g = 5$ and $c_1^2 = 6$

In this section we shall study minimal algebraic surfaces with $p_g = 5$ and $c_1^2 = 6$. If S is such a surface S is either of type (1) or of type (3′). Our purpose is to prove the following theorem.

THEOREM 5.1. *Any two minimal algebraic surfaces with $p_g = 5$ and $c_1^2 = 6$ have the same deformation type.*

In order to prove this theorem, it suffices, by Theorems 3.1 and 3.3, to prove the following lemma.

LEMMA 5.2. *Let S be a generic surface of type* (3′). *Then there exists a family $p: \mathcal{S} \to M$ of deformations of $S = p^{-1}(0)$, $0 \in M$ such that $S_t = p^{-1}(t)$, $t \neq 0$ are of type* (1).

Proof. We consider the holomorphic map $f: S \to \Sigma_3$ of degree 2 defined in Lemma 1.5. Let $B = \Delta_0 + B_0$ denote the branch locus of f. Then B_0 intersects at a point b with Δ_0. Let $\tilde{W} \to \Sigma_3$ be the quadric transformation with center at b. We may identify $\tilde{W}$ with a quadric transform of Σ_4 with center at a point x which is on the ∞-section Δ of Σ_4. Let $q: \tilde{W} \to \Sigma_4$ denote the quadric transformation and set $E = q^{-1}(x)$.

Since B has no infinitely near triple points, it follows that f factors through $\tilde{W}$ ([13], Lemma 5). Let $\tilde{f}\colon S \to \tilde{W}$ be the holomorphic map induced by f and let $\tilde{B}$ denote the branch locus of $\tilde{f}$. We let Δ_0 denote the 0-section of Σ_4. Then $\tilde{B}$ is a disjoint sum $q^*\Delta_0 + \tilde{B}_0$ where $\tilde{B}_0$ is an irreducible non-singular curve linearly equivalent to $5q^*\Delta - 4E$.

Let $\pi\colon V \to \mathbf{P}^2$, Q, V_0, V_∞ and $\{W_s \mid s \in N\}$ be the same as in the proof of Lemma 4.4. We identify $\pi^{-1}(Q)$ with Σ_4. Then x is identified with a point on V_∞ which belongs to any W_s.

Let $\sigma\colon \tilde{V} \to V$ be the monoidal transformation with center at x. On $\tilde{V}$, we have a family $\{\tilde{W}_s\}$ of proper transforms $\tilde{W}_s$ of W_s. We note that $\tilde{W}_s$, $s \neq 0$, are biholomorphically equivalent to Σ_1 and that $\tilde{W}_0$ is identified with $\tilde{W} \cup \sigma^{-1}(V_0)$.

We let $\mathcal{E}$ denote the exceptional divisor $\sigma^{-1}(x)$. Then $[5\sigma^* V_\infty - 4\mathcal{E}]$ induces $[\tilde{B}_0]$ on $\tilde{W}$.

LEMMA 5.3. *There exists a divisor* $\mathcal{B}_0 \in |5\sigma^* V_\infty - 4\mathcal{E}|$ *on* $\tilde{V}$ *which induces* $\tilde{B}_0$ *on* $\tilde{W}$.

Proof. From the exact sequence

$$0 \longrightarrow \mathcal{O}(5\sigma^* V_\infty - 3\mathcal{E} - \sigma^*\pi^* Q) \longrightarrow \mathcal{O}(5\sigma^* V_\infty - 4\mathcal{E}) \longrightarrow \mathcal{O}_{\tilde{W}}(\tilde{B}_0) \longrightarrow 0 ,$$

we get the exact sequence

$$\begin{aligned} 0 \longrightarrow H^0\big(\tilde{V}, \mathcal{O}(5\sigma^* V_\infty - 3\mathcal{E} - \sigma^*\pi^* Q)\big) \\ \longrightarrow H^0\big(\tilde{V}, \mathcal{O}(5\sigma^* V_\infty - 4\mathcal{E})\big) \xrightarrow{r} H^0\big(\tilde{W}, \mathcal{O}_{\tilde{W}}(\tilde{B}_0)\big) \longrightarrow \cdots . \end{aligned}$$

We claim that

$$\begin{aligned} \dim H^0\big(\tilde{W}, \mathcal{O}_{\tilde{W}}(\tilde{B}_0)\big) &= 56 , \\ \dim H^0\big(\tilde{V}, \mathcal{O}(5\sigma^* V_\infty - 4\mathcal{E})\big) &= 135 , \\ \dim H^0\big(\tilde{V}, \mathcal{O}(5\sigma^* V_\infty - 3\mathcal{E} - \sigma^*\pi^* Q)\big) &= 79 . \end{aligned}$$

The first equality can be easily proved. In order to prove the second equality we let $\tilde{V}_\infty$ denote the proper transform of V_∞ by σ. Then we can identify $\tilde{V}_\infty$ with Σ_1 and we have the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{O}\big((\nu - 1)\sigma^* V_\infty - (\nu - 2)\mathcal{E}\big) \longrightarrow \mathcal{O}\big(\nu\sigma^* V_\infty - (\nu - 1)\mathcal{E}\big) \\ \longrightarrow \mathcal{O}_{\Sigma_1}\big(2\nu\Delta - (\nu - 1)\Delta_0\big) \longrightarrow 0 \end{aligned}$$

for any integer ν. From this we infer that

$$\begin{aligned} &\dim H^0\big(\tilde{V}, \mathcal{O}(5\sigma^* V_\infty - 4\mathcal{E})\big) \\ &\quad = \dim H^0\big(\tilde{V}, \mathcal{O}(\sigma^* V_\infty)\big) + \textstyle\sum_{\nu=2}^{5} \dim H^0\big(\Sigma_1, \mathcal{O}(2\nu\Delta - (\nu - 1)\Delta_0)\big) \\ &\quad = 135 . \end{aligned}$$

To prove the last equality we note that

$$|5\sigma^* V_\infty - 3\mathfrak{S} - \sigma^*\pi^* Q| = \sigma^* V_0 + |4\sigma^* V_\infty - 3\mathfrak{S}| .$$

We have $\dim H^0(\tilde{V}, \mathcal{O}(4\sigma^* \tilde{V}_\infty - 3\mathfrak{S})) = 79$ by a similar argument.

These three equalities imply that the restriction map r is surjective.

Q.E.D.

Now let $\varpi: \mathfrak{M} \to \tilde{\mathfrak{W}}$ be the double covering of $\tilde{\mathfrak{W}} = \bigcup_{s \in N} \tilde{W}_s \times s$ with branch locus $\mathfrak{B}_0 + \sigma^* V_0$ and let $\tilde{S}_s$ denote the fibre of $\mathfrak{M} \to N$ over s. Then for $s \neq 0$, $\tilde{S}_s$ are surfaces of type (1) and S_0 is a union of S and $G = \varpi^{-1}\sigma^{-1}(V_0)$. As in the proof of Lemma 4.4, G is an exceptional divisor. It follows that we can construct a family $p: \mathfrak{S} \to M$ with desired property. This completes the proof of Theorem 5.1.

6. Surfaces with $p_g = 4$ and $c_1^2 = 4$

Let S be a minimal algebraic surface with $p_g = 4$ and $c_1^2 = 4$. We assume that S is a generic surface of type (2′). Then the canonical map induces a map $f: S \to \Sigma_2$ of degree 2. We note that the branch locus B of f is disjoint from the 0-section Δ_0. It follows that $f^*\Delta_0$ is a disjoint sum $F_1 + F_2$ where F_i, $i = 1, 2$, are nonsingular rational curves with $F_i^2 = -2$. If we deform the complex structure of S, F_1 and F_2 either remain or disappear simultaneously. This phenomenon causes obstructions for deformations of S (cf. [5], [15]). We shall prove the following theorem.

THEOREM 6.1. *Let S be a minimal algebraic surface with $p_g = 4$ and $c_1^2 = 4$ and let $p: \mathfrak{S} \to M$ be the Kuranishi family of deformations of $S = p^{-1}(0)$, $0 \in M$. Assume that S is generic of type* (2′). *Then M is a union of two 42-dimensional manifolds M_0 and M_1, which intersect transversally in a 41-dimensional manifold N. The surface S_t, $t \in M$, is of type* (0) *or of type* (2′) *according to whether $t \in M - N$ or $t \in N$.*

The proof is analogous to that of [13], Theorem 3. We set $W = \Sigma_2$ and let $g: S \to \mathbf{P}^1$ denote the composition of f with the natural projection $W \to \mathbf{P}^1$. D will denote a general fibre of g. Furthermore, we shall employ the notation of Section 2.

LEMMA 6.2. $h^0(2D) = 3$ *and* $h^1(2D) = h^2(2D) = 1$.

Proof. By the Riemann-Roch theorem we have $\chi(2D) = 3$, while we have $h^2(2D) = h^0(F_1 + F_2) = 1$ by the Serre duality. Combined with $h^0(2D) = 3$ this shows that $h^1(2D) = 1$. Q.E.D.

LEMMA 6.3. $h^1(D) = h^1(D + F_1 + F_2) = 0$.

Proof. We have $h^0(D) = 2$, $\chi(D) = 4$ and $h^2(D) = h^0(D + F_1 + F_2) = 2$. The assertion follows from these equalities. Q.E.D.

LEMMA 6.4. $h^0(\mathcal{T}_{S/W}) = 48$, $h^1(\mathcal{T}_{S/W}) = 0$.

This follows immediately from [13], Lemma 10 and the Riemann-Roch theorem for a curve.

LEMMA 6.5. $h^0(f^*\Theta_W) = 7$, $h^1(f^*\Theta_W) = 2$ *and* $h^2(f^*\Theta_W) = 1$.

Proof. From the equality $K = [f^*\Delta]$, we get the exact sequence

$$0 \longrightarrow \mathcal{O}(K + F_1 + F_2) \longrightarrow f^*\Theta_W \longrightarrow \mathcal{O}(2D) \longrightarrow 0 .$$

We have $h^0(K + F_1 + F_2) = 4$, $h^1(K + F_1 + F_2) = 1$ and $h^2(K + F_1 + F_2) = 0$ (see [16], Theorem 2.2). On the other hand, $H^0(W, \Theta_W) \to H^0(W, \mathcal{O}(2\Gamma))$ is surjective and $H^0(W, \mathcal{O}(2\Gamma)) \to H^0(S, \mathcal{O}(2D))$ is bijective. It follows that $H^0(S, f^*\Theta_W) \to H^0(S, \mathcal{O}(2D))$ is surjective. Combined with Lemma 6.2, this proves Lemma 6.5. Q.E.D.

LEMMA 6.6. $h^0(\Theta_S) = 0$, $h^1(\Theta_S) = 43$ *and* $h^2(\Theta_S) = 1$.

Proof. The first equality has been proved in [25]. The others follow from the exact sequence

$$0 \longrightarrow \Theta_S \longrightarrow f^*\Theta_W \longrightarrow \mathcal{T}_{S/W} \longrightarrow 0 ,$$

and with the aid of Lemmas 6.4 and 6.5. Q.E.D.

LEMMA 6.7. *The canonical homomorphism*

$$G^q\colon H^q(S, \Theta_S) \longrightarrow H^q(S, g^*\Theta_{\mathbf{P}^1})$$

is surjective for $q = 1$ *and is bijective for* $q = 2$.

This follows immediately from the above lemmas.

By Lemma 6.7, we can apply, to $g\colon S \to \mathbf{P}^1$, a theorem of existence of deformations of holomorphic maps ([11], Theorem 4.3). Thus we can construct a family $p_1\colon \mathcal{S}_1 \to M_1$ of deformations of $S = p_1^{-1}(0)$ with $0 \in M_1$ and a holomorphic map $\Psi\colon \mathcal{S}_1 \to \mathbf{P}^1 \times M_1$ over M_1 which induces g on S. Moreover, we may assume that the infinitesimal deformation map

$$\rho\colon T_0(M_1) \longrightarrow H^1(S, \Theta_S)$$

induces a bijection onto Ker G^1.

Let $p\colon \mathcal{S} \to M$ be the Kuranishi family of deformations of $S = p^{-1}(0)$, $0 \in M$ (see [23]). By Lemma 6.6, M is an analytic subset of

$$\mathbf{D} = \{t \in \mathbf{C}^{43} \mid |t| < \varepsilon\}$$

where $\varepsilon > 0$ is sufficiently small. Let $\varphi(t)$ be a (0, 1)-form with coefficient in Θ_S which determines the Kuranishi family. Then M is defined by the equation

$$\mathbf{H}[\varphi(t), \varphi(t)] = 0$$

where $\mathbf{H}$ denotes the projection onto the space of harmonic forms for a fixed Hermitian metric on S. We may regard $\mathbf{H}[\varphi(t), \varphi(t)]$ as a holomorphic function on $\mathbf{D}$.

We shall prove the following lemma.

LEMMA 6.8. $\mathbf{H}[\varphi(t), \varphi(t)] = t_1 t_2 +$ (*higher terms*) *for some appropriate choice of a system of coordinates* $(t_1, t_2, \cdots, t_{43})$ *on* $\mathbf{D}$.

The proof is quite similar to that of [13], Lemma 32 and consists of several lemmas.

LEMMA 6.9. *The natural homomorphism*

$$\zeta_*: H^1(S, \Theta_S) \longrightarrow H^1(F_1, \mathfrak{N}_{F_1}) \oplus H^1(F_2, \mathfrak{N}_{F_2})$$

is surjective, where $\mathfrak{N}_{F_i}$ *denote the sheaves of germs of sections of the normal bundles of* F_i *in* S *for* $i = 1, 2$.

Proof. We first note that $\dim H^1(F_i, \mathfrak{N}_{F_i}) = 1$ for $i = 1, 2$. Hence it suffices to prove that $\dim \operatorname{Ker} \zeta_* \leqq 41$.

Let $\rho \in \operatorname{Ker} \zeta_*$, which corresponds to an infinitesimal deformation $S_\rho \to \operatorname{Spec}(\mathbf{C}[t]/t^2)$ of S. The vanishing of $\zeta_*\rho$ implies that F_1 and F_2 both extend to divisors on S_ρ. This implies that the line bundle $[D]$ extends to a line bundle over S_ρ (cf. the proof of [13], Lemma 25). Since we have $h^1(D) = 0$ by Lemma 6.3, it follows that g extends to a holomorphic map $\Psi: S_\rho \to \mathbf{P}^1$. Finally from $\zeta_*\rho = 0$ and $h^1(K) = 0$, we infer that $f: S \to W$ extends to a holomorphic map $\Phi: S_\rho \to W$. Therefore ρ is killed by the canonical homomorphism $H^1(S, \Theta_S) \to H^1(S, f^*\Theta_W)$. By Lemmas 6.4 and 6.5, this implies that $\dim \operatorname{Ker} \zeta_* \leqq 41$. Q.E.D.

COROLLARY. $\operatorname{Ker} \zeta_* = \operatorname{Ker}\big(H^1(S, \Theta_S) \to H^1(S, f^*\Theta_W)\big)$.

Remark. Lemma 6.9 is a consequence of a result of Burns-Wahl [5]. However, we shall use the corollary later.

Now we have reached the same situation as in [13], Section 5, after the proof of Lemma 24. Hence, by the same argument as in [13], we can define a linear map

$$\gamma: H^1(S, \Theta_S) \longrightarrow H^0(F_1, \mathcal{O}_{F_1}) \oplus H^0(F_2, \mathcal{O}_{F_2})$$

which satisfies the following Lemma 6.10. Before stating Lemma 6.10, we prepare the following definition. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}(2D) \longrightarrow \mathcal{O}(K) \longrightarrow \mathcal{O}_{F_1} \oplus \mathcal{O}_{F_2} \longrightarrow 0 .$$

It follows that the image of the natural map

$$(*) \qquad H^0(S, \mathcal{O}(K)) \longrightarrow H^0(F_1, \mathcal{O}_{F_1}) \oplus H^0(F_2, \mathcal{O}_{F_2})$$

is 1-dimensional. Moreover, this image is not identically zero either on F_1 or F_2. We say that an element

$$\gamma = (\gamma_1, \gamma_2) \in H^0(F_1, \mathcal{O}_{F_1}) \oplus H^0(F_2, \mathcal{O}_{F_2})$$

satisfies $\gamma_1 = \gamma_2$ if and only if γ is in the image of (*).

LEMMA 6.10. *Let* $\rho \in H^1(S, \Theta_S)$ *and let* $\gamma(\rho) = (\gamma_1(\rho), \gamma_2(\rho))$ *with* $\gamma_i(\rho) \in H^0(F_i, \mathcal{O}_{F_i})$, $i = 1, 2$. *Then*:

1) $G^1\rho = 0$ *if and only if* $\gamma_1(\rho) = \gamma_2(\rho)$.
2) $\zeta_*\rho = 0$ *if and only if* $\gamma(\rho) = 0$.

For the proof, see [13], Lemma 29.

In view of the above lemmas, we can choose a basis $\{\rho_1, \rho_2, \cdots, \rho_{43}\}$ of $H^1(S, \Theta_S)$ such that

$$\begin{cases} \gamma_1(\rho_1) \neq 0,\ \gamma_2(\rho_1) = 0 \\ \gamma_1(\rho_2) = \gamma_2(\rho_2) \neq 0 \\ \gamma_1(\rho_\nu) = \gamma_2(\rho_\nu) = 0 \quad \text{for} \quad \nu \geqq 3. \end{cases}$$

LEMMA 6.11. $[\rho_\lambda, \rho_\nu] = 0$ *if* $\lambda, \nu \geqq 2$.

Proof. We have $G^2[\rho_\lambda, \rho_\nu] = 0$ (cf. [11], Theorem 4.4). Therefore $[\rho_\lambda, \rho_\nu]$ vanishes by Lemma 6.7. Q.E.D.

LEMMA 6.12. *If* ρ_λ *and* ρ_ν ($1 \leqq \lambda, \nu \leqq 43$) *satisfy* $[\rho_\lambda, \rho_\nu] = 0$, *then we have*

$$\gamma_1(\rho_\lambda)\gamma_1(\rho_\nu) = \gamma_2(\rho_\lambda)\gamma_2(\rho_\nu) .$$

Proof. By the same arguments as in the proof of [13], Lemma 31, the assumption implies that $\gamma(\rho_\lambda)\gamma(\rho_\nu)$ is contained in the image of the restriction map

$$H^0(S, \mathcal{O}(2K)) \longrightarrow H^0(F_1, \mathcal{O}_{F_1}) \oplus H^0(F_2, \mathcal{O}_{F_2}) .$$

This proves our assertion.

Let $(t_1, t_2, \cdots, t_{43})$ be a system of coordinates on $\mathbf{D}$ such that $\partial/\partial t_\lambda$ corresponds respectively to ρ_λ. Then Lemmas 6.11 and 6.12 imply that

$$\mathbf{H}[\varphi(t), \varphi(t)] = \textstyle\sum_{\lambda=1}^{43} a_\lambda t_1 t_\lambda + (\text{higher terms})$$

with $a_2 \neq 0$. By a linear change of coordinates, we may assume that $a_\lambda = 0$ for $\lambda \neq 2$. This completes the proof of Lemma 6.8.

Now we can prove Theorem 6.1 in a similar way to [13], Theorem 3. By the construction after Lemma 6.7, we have a family $p_1\colon \mathcal{S}_1 \to M_1$ of deformations of S. We may identify M_1 with a submanifold of M. Moreover,

$T_0(M_1)$ is generated by $\partial/\partial t_2, \cdots, \partial/\partial t_{43}$. Hence M_1 is defined in $\mathbf{D}$ by the equation of the form $q_1(t) = 0$ with

$$q_1(t) = t_1 + (\text{higher terms}) .$$

Since $\mathbf{H}[\varphi(t), \varphi(t)]$ vanishes on M_1, it follows that

$$\mathbf{H}[\varphi(t), \varphi(t)] = q_0(t)q_1(t)$$

with

$$q_0(t) = t_2 + (\text{higher terms}) .$$

This proves that M is a union of $M_0 = \{t \in M \mid q_0(t) = 0\}$ and M_1. Clearly, M_0 and M_1 intersect transversally in a 41-dimensional manifold N.

In order to prove the last assertion of Theorem 6.1, we may assume that the family $p: \mathcal{S} \to M$ is complete at each point $t \in M$. Hence we have $\dim H^1(S_t, \Theta_{S_t}) \leqq 42$ for $t \in M - N$. While if S_t is of type (2′) we have $\dim H^1(S_t, \Theta_{S_t}) \geqq 43$ by Lemma 6.6 and the upper semi-continuity. This proves that S_t is of type (0) for each $t \in M - N$.

Next we shall prove that S_t is of type (2′) for $t \in N$. For this purpose we apply to $f: S \to W$ a theorem of existence of deformations of holomorphic maps ([10], Theorem 3.1). This is possible by Lemma 6.4. Thus we obtain a family $p_2: \mathcal{S}_2 \to M_2$ of deformations of $S = p_2^{-1}(0)$, $0 \in M_2$ such that the infinitesimal deformation map $\rho: T_0(M_2) \to H^1(S, \Theta_S)$ gives a bijection onto $\operatorname{Ker}(H^1(S, \Theta_S) \to H^1(S, f^*\Theta_W))$. Since any member of $\mathcal{S}_2$ is of type (2′) we may identify M_2 with a submanifold of N. Moreover, by the Corollary to Lemma 6.9, we have $T_0(M_2) = T_0(N)$. This proves that $M_2 = N$ in a neighborhood of 0. This completes the proof of Theorem 6.1.

The following construction may explain how two different families occur. For simplicity we shall construct 1-parameter families.

Let Z denote the diagonal of $\Sigma_0 = \mathbf{P}^1 \times \mathbf{P}^1$ and let $\pi: V \to \Sigma_0$ be the $\mathbf{P}^1$-bundle which is the completion of $[-Z]$. We cover Σ_0 by sufficiently small coordinate neighborhoods U_i and let $q_i = 0$ be the local equations of Z on U_i. We set $e_{ij} = q_i/q_j$ on $U_i \cap U_j$. Then V is covered by $U_i \times \mathbf{P}^1$ and $(z, w_i) \in U_i \times \mathbf{P}^1$ coincides with $(z, w_j) \in U_j \times \mathbf{P}^1$ if and only if $w_i = e_{ij}(z)^{-1}w_j$ where w_i denote inhomogeneous coordinates on $\mathbf{P}^1$.

Let t vary in a neighborhood of 0 in $\mathbf{C}$. We define divisors W_t of V by the equations

$$w_i q_i(z) = t \quad on \quad U_i \times \mathbf{P}^1 .$$

For $t \neq 0$, W_t are biholomorphically equivalent to Σ_0 via the projection π, while W_0 is a union $\pi^{-1}(Z) \cup V_0$ where V_0 denotes the section of π defined by $w_i = 0$. We note that $\pi^{-1}(Z)$ is biholomorphically equivalent to Σ_2 and that the 0-section of $\pi^{-1}(Z)$ is identified with the diagonal of V_0.

Next we take a nonsingular divisor $\mathcal{B}$ on V which is linearly equivalent to $6V_\infty$. Here V_∞ denotes the section of π defined by $w_i = \infty$. We assume that $\mathcal{B}$ is disjoint from V_0. Let X_t be the double covering of W_t with branch locus $\mathcal{B} \cap W_t$ for each t. For $t \neq 0$, X_t is a minimal algebraic surface with $p_g = 4$ and $c_1^2 = 4$ of type (0). While X_0 consists of 3 components S, E_1 and E_2, where S is the double covering of $\pi^{-1}(Z)$ with branch locus $\mathcal{B} \cap \pi^{-1}(Z)$ and E_1 and E_2 are components of the preimage of V_0.

We identify each E_i with Σ_0. Then the normal bundle of E_i in $\bigcup_{t \in M} X_t \times t$ is nothing but $-[Z]$. Therefore (see [28], [8]) we can contract each E_i to a line in two different directions.

Thus we obtain families $\mathcal{S} = \{S_t\}$ of $S_0 = S$ and $S_t = X_t$, $t \neq 0$. If we contract E_1 and E_2 in the same direction, then the holomorphic map $g: S \to \mathbf{P}^1$ extends over the family. However, if we contract E_1 and E_2 in two different directions, then g does not extend.

Remark. The linear system $|D + F_1|$ or $|D + F_2|$ defines a *rational map* $g': S \to \mathbf{P}^1$ with a fixed component F_1 or F_2. Over the family $\mathcal{S}_0 \to M_0$ of Theorem 6.1, g' extends to a family of rational maps $g'_t: S_t \to \mathbf{P}^1$ and for $t \in M_0 - N$, the g'_t are holomorphic. This is a consequence of a theorem of existence of deformations of rational maps (see [14]).

7. Surfaces with $c_1^2 = 2p_g - 4 \equiv 0 \bmod 8$

Let S be a minimal algebraic surface with $c_1^2 = 2p_g - 4$ and $p_g \geqq 3$. In this section we shall study the case in which $c_1^2 \equiv 0 \bmod 8$. We note that this is the only case which is excluded in Theorem 3.4. Since we have already studied the case: $p_g = 6$ and $c_1^2 = 8$ in Section 4, we assume that $p_g = 4k + 2$ and $c_1^2 = 8k$ with an integer $k \geqq 2$. By Theorem 1.5, we can classify S into the following types: type (0), type (2), $\cdots$, type $(2k + 2)$.

The main theorem of this section is the following.

THEOREM 7.1. *Let k be an integer $\geqq 2$. Then minimal algebraic surfaces with $p_g = 4k + 2$ and $c_1^2 = 8k$ are classified into two deformation types. One consists of surfaces of type* (0), *type* (2), $\cdots$, *type* $(2k)$. *The other consists of surfaces of type* $(2k + 2)$. *These two deformation types are homotopically equivalent to each other if k is even. They are not homotopically equivalent if k is odd.*

First we note that surfaces of type (0), type (2), $\cdots$, type $(2k)$ have one and the same deformation type (Theorems 3.1 and 3.3), while surfaces of type $(2k + 2)$ have one and the same deformation type (Theorem 3.1). In order to prove that these two deformation types are distinct, it suffices to establish the following two lemmas.

LEMMA 7.2. *Let S be a surface of type* (d) *and let S' be a sufficiently small deformation of S. Then S' is of type* (d') *with* $d' \leqq d$, $d' \equiv d \bmod 2$.

LEMMA 7.3. *Let S be a minimal algebraic surface with* $p_g = 4k + 2$ *and* $c_1^2 = 8k$ *with* $k \geqq 2$. *Assume that S is of type* $(2k+2)$. *Then any sufficiently small deformation of S is of type* $(2k+2)$.

The following lemma is useful.

LEMMA 7.4. *Let S be a surface of type* (d) *and let* $f\colon S \longrightarrow \Sigma_d$ *be the holomorphic map induced by the canonical map. Then for any family* $p\colon \mathcal{S} \longrightarrow M$ *of deformations of* $S = p^{-1}(0)$, $0 \in M$, *there exist a family* $q\colon \mathcal{W} \longrightarrow M$ *of deformations of* $\Sigma_d = q^{-1}(0)$ *and a holomorphic map* $\Phi\colon \mathcal{S} \longrightarrow \mathcal{W}$ *over M which induces f on S, provided M is sufficiently small.*

This is an immediate consequence of a theorem of costability ([12], Theorem 8.1) in view of Lemma 2.3. However, since Σ_d is unobstructed, the proof is much easier. It suffices to use a proposition in the appendix which is a generalization of [11], Proposition 7.1.

Proof of Lemma 7.2. It is well known that, for each $t \in M$, $q^{-1}(t)$ is biholomorphically equivalent to $\Sigma_{d'}$ for some d' with $d' \leqq d$, $d' \equiv d \bmod 2$ (see[31]). This implies that $S_t = p^{-1}(t)$ is of type (d'). Q.E.D.

Proof of Lemma 7.3. Let $\Phi\colon \mathcal{S} \longrightarrow \mathcal{W}$ be an extension of $f\colon S \longrightarrow W$ and let $f_t\colon S_t \longrightarrow W_t$ denote the holomorphic maps of $S_t = p^{-1}(t)$ onto $W_t = q^{-1}(t)$ induced by Φ. Let $\mathcal{R}$ be the ramification divisor of Φ and let $\mathcal{B} = \Phi(\mathcal{R})$. Then $\mathcal{B}$ is a divisor of $\mathcal{W}$ which induces the branch locus B_t of f_t on each W_t. Since B_0 is disconnected, it follows that B_t is disconnected provided t is sufficiently near to 0. This proves that S_t is of type $(2k+2)$. Q.E.D.

By a result of Milnor [26], the second assertion of Theorem 7.1 will follow from the following theorem.

THEOREM 7.5. *Let S be a minimal algebric surface with* $c_1^2 = 2p_g - 4$ *and* $p_g \geqq 3$. *Then the second Stiefel-Whitney class* W_2 *of S vanishes if and only if S is one of the following*:

(i) $p_g = 6$, $c_1^2 = 8$, *of type* (∞),

(ii) $p_g = 6$, $c_1^2 = 8$, *of type* $(4')$,

(iii) $p_g = 4k + 2$, $c_1^2 = 8k$ *with an odd integer* $k \geqq 3$ *and of type* $(2k+2)$.

Proof. Assume that $W_2 = 0$. Then there exists a line bundle L on S such that $K = 2L$. By the adjunction formula L^2 is even, which we call $2k$.

It follows that $c_1^2 = 8k$ and $p_g = 4k + 2$.

We insert here the following lemma.

LEMMA 7.6. *Let S be an algebraic surface with $p_g \geqq 1$ and let L be a line bundle on S. Assume that L is not composite with a pencil and that $LD \geqq 0$ for any effective divisor D on S. Then we have*

$$L^2 \geqq 2h^0(L) - 4 .$$

Proof. (Compare [13], Lemma 2). Let $\pi\colon \tilde{S} \rightarrow S$ be a composition of quadric transformations such that $|\pi^*L|$ has no base point. We write

$$|\pi^*L| = |M| + F$$

where $|M|$ and F denote, respectively, the variable part and the fixed part of $|\pi^*L|$. Let C be a general member of $|M|$. By assumption, C is irreducible and nonsingular. Let M_C denote the restriction of M to C. Then we have

$$h^0(M_C) \geqq h^0(M) - 1 = h^0(L) - 1 .$$

We let $\tilde{K}$ denote the canonical bundle of $\tilde{S}$ and let $\tilde{K}_C$ be its restriction to C. We have

$$h^1(M_C) = h^0(\tilde{K}_C) \geqq p_g - h^0(\tilde{K} - C) .$$

Suppose that $h^0(\tilde{K} - C) = p_g$. Then C is a fixed component of $|\tilde{K}|$. This implies that $h^0(C) = 1$, which is a contradiction. Thus we have $h^1(M_C) \geqq 1$. Hence, by a theorem of Clifford (see [24], 2.3), we have

$$M^2 \geqq 2h^0(M_C) - 2 .$$

On the other hand, we have $L^2 = M^2 + (M + \pi^*L)F$. We note that $(\pi^*L)F$, as well as MF, is nonnegative. Therefore we obtain $L^2 \geqq M^2$.

Combining these three inequalities, we conclude that $L^2 \geqq 2h^0(L) - 4$.

Q.E.D.

Returning to the proof of Theorem 7.5, we assume that $2L = K$. Then, by the Riemann-Roch theorem, we have

$$2h^0(L) - h^1(L) = 3k + 3 . \tag{7.1}$$

Suppose that $|L|$ is not composite with a pencil. Then by Lemma 7.6 we have

$$2k = L^2 \geqq 3k - 1 .$$

Therefore, we have $k = 1$. That is, if $k \geqq 2$, $|L|$ is composite with a pencil.

Since we have already settled the case $k = 1$ in Section 4, we hereafter assume that $k \geqq 2$. Then we have

$$L = [rD + F]$$

where D is an irreducible pencil possibly with base points, $r = h^0(L) - 1$, and F is the fixed part of L. It follows that

$$2k \geqq rLD .$$

We claim that $LD = 1$. In fact, if $LD = 0$ we have $D^2 \leqq 0$ by Hodge's index theorem. This would imply $D^2 = 0$, hence $D = 0$. On the other hand, $LD \geqq 2$ would imply

$$2k \geqq 2r \geqq 3k + 1 ,$$

which is a contradiction.

From the equality $LD = 1$, we infer that $D^2 = 0$ and $DF = 1$. This implies that $|D|$ defines a surjective holomorphic map $g: S \to \mathbf{P}^1$ whose general fibre is a nonsingular curve of genus 2.

On the other hand, let $f: S \to \Sigma_d$ be the holomorphic map of degree 2 induced by the canonical map. Employing the same notation as in Section 1, we have

$$K = f^*\left[\Delta_0 + \frac{n - 1 + d}{2}\Gamma\right]$$

where $n = p_g - 1$. By Corollary 1.7, D is linearly equivalent to f^*L. Therefore, $f^*\Delta_0$ is linearly equivalent to $2F + (2r - (n - 1 + d)/2)D$. From this fact we infer that $2r \leqq (n - 1 + d)/2$. Combining with (7.1) we obtain $d \geqq 2k + 2$. By Theorem 1.6 we conclude that $d = 2k + 2$. Thus we have reduced the proof to the following.

LEMMA 7.7. *Let S be a minimal algebraic surface with $p_g = 4k + 2$ and $c_1^2 = 8k$ of type $(2k + 2)$. Then the second Stiefel-Whitney class W_2 of S vanishes if and only if k is odd.*

Proof. Let $f: S \to \Sigma_{2k+2}$ be the holomorphic map of degree 2 induced by the canonical map. We have $f^*\Delta_0 = 2F$ with $F^2 = -k - 1$. Therefore, we have $W_2 \neq 0$ if k is even.

On the other hand, the canonical bundle K of S is given by $2F + (3k+1)f^*\Gamma$. Therefore, we have $W_2 = 0$ if k is odd. Q.E.D.

Appendix. Deformations of compositions of holomorphic maps

In this appendix we shall generalize [11], Proposition 7.1. First we recall some definitions. Let g: $Y \to Z$ be a holomorphic map of a complex manifold Y into a complex manifold Z. We let Θ_Y and Θ_Z denote, respectively, the sheaves of germs of holomorphic vector fields on Y and Z. Then we have a canonical homomorphism $G: \Theta_Y \to g^*\Theta_Z$. We regard this homomorphism as a *complex*

$$(1) \qquad \cdots \longrightarrow E^0 \xrightarrow{d_0} E^1 \longrightarrow \cdots$$

with $E^0 = \Theta_Y$, $E^1 = g^*\Theta_Z$, $E^i = 0$ for $i \neq 0, 1$, and $d_0 = G$. Let $D_{Y/Z}$ be the first hypercohomology group of the complex (1). This definition coincides with the definition in [11], Section 4.

If $f: X \to Y$ is a holomorphic map of a complex manifold X into Y, then we can pull back the complex (1) into a complex

$$(2) \qquad \cdots \longrightarrow 0 \longrightarrow f^*\Theta_Y \longrightarrow f^*g^*\Theta_Z \longrightarrow 0 \longrightarrow \cdots$$

on X. We let $f^*D_{Y/Z}$ denote the first hypercohomology group of the complex (2). Clearly, we have a natural homomorphism

$$(3) \qquad f^*: D_{Y/Z} \longrightarrow f^*D_{Y/Z} .$$

If g is nondegenerate (i.e., max rank $dg = \dim Y$), and if $f^*G: f^*\Theta_Y \to f^*g^*\Theta_Z$ is injective, then we have

$$D_{Y/Z} = H^0(Y, \mathcal{T}_{Y/Z}),\ f^*D_{Y/Z} = H^0(X, f^*\mathcal{T}_{Y/Z}) ,$$

where $\mathcal{T}_{Y/Z}$ denotes the cokernel of $G: \Theta_Y \to g^*\Theta_Z$.

Whereas if g is smooth, we have

$$D_{Y/Z} = H^1(Y, \Theta_{Y/Z}),\ f^*D_{Y/Z} = H^1(X, f^*\Theta_{Y/Z})$$

where $\Theta_{Y/Z}$ denotes the sheaf of germs of holomorphic vector fields along fibres of g.

In both cases, (3) coincides with usual pull-back homomorphisms.

PROPOSITION. *Let $p: \mathcal{X} \to M$, $q: \mathcal{Y} \to N$ be families of compact complex manifolds, Z a complex manifold and let $\Upsilon: \mathcal{X} \to Z \times M$ and $\Psi: \mathcal{Y} \to Z \times N$ be holomorphic maps over M and N, respectively. Let $0 \in M$, $0' \in N$, $X = p^{-1}(0)$, $Y = q^{-1}(0')$ and let $h: X \to Z$ and $g: Y \to Z$ be holomorphic maps induced by Υ and Ψ, respectively. Furthermore, let $f: X \to Y$ be a holomorphic map such that $h = g \circ f$. Assume that the composition*

$$f^* \circ \tau: T_{0'}(N) \longrightarrow f^*D_{Y/Z}$$

is surjective, where $\tau: T_{0'}(N) \to D_{Y/Z}$ is the characteristic map of the family $(\mathcal{Y}, \Psi, q, N)$ at $0'$. Then there exist an open neighborhood M' of 0 in M, a holomorphic map $s: M' \to N$ satisfying $s(0) = 0'$ and a holomorphic map $\Phi: \mathcal{X} \mid M' \to \mathcal{Y}$ over s such that $(\mathrm{id} \times s) \circ (\Upsilon \mid M') = \Psi \circ \Phi$.

The proof is quite similar to that of [11], Proposition 7.1.

We note that if Z is a point and if Y is unobstructed, this proposition proves the assertion of [12], Theorem 8.1.

Finally, if Z is a point and if f is the identity, this proposition gives a theorem of completeness [21].

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(Received November 7, 1975)