國立臺灣大學數學系114學年度碩士班甄試入學筆試

高等微積分

I. Let $x = (x_1, x_2, x_3)$ be coordinates of $\mathbb{R}^3$. Let

$$f(x) = \frac{x_1^3 + x_2^2 + i\ln(1 + x_2^2 + x_3^2)}{x_1^2 + x_2^2 + ix_3^2\sin(\frac{x_1}{x_3})} \quad \text{if } x_1x_2x_3 \neq 0,$$
$$f(x) = 0 \quad \text{if } x_1x_2x_3 = 0,$$

be a function on $\mathbb{R}^3$, where ln is the natural logarithm, that is the inverse function of e^x and $i = \sqrt{-1}$. Is f a continuous function on $\mathbb{R}^3$? If not, find all $(a, b, c) \in \mathbb{R}^3$ such that $f(x)$ is not continuous at (a, b, c). (15 pts)

II. Let $x = (x_1, x_2, x_3)$ be coordinates of $\mathbb{R}^3$. Let

$$f(x) = \frac{x_1^3 + e^{-\frac{1}{x_2^2}} + i\sin^2 x_3^2}{x_1^2 + x_2^2 + ix_3^3} \quad \text{if } x_1x_2x_3 \neq 0,$$
$$f(x) = 0 \quad \text{if } x_1x_2x_3 = 0,$$

be a function on $\mathbb{R}^3$. Is f differentiable at $(0, 0, 0)$? (10 pts)

In the following, we will use the following notations: let U be an open set of $\mathbb{R}^n$. Let $\mathcal{C}^k(U)$ be the space of k-times continuously differentiable functions on U, $k \in \mathbb{N} \cup \{0\}$. Let $\mathcal{C}^\infty(U) := \cap_k \mathcal{C}^k(U)$.

III. Let $f(x, y) \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^m)$, where $x = (x_1, \ldots, x_n)$ denotes the coordinates in $\mathbb{R}^n$, $y = (y_1, \ldots, y_n)$ denotes the coordinates in $\mathbb{R}^m$. Suppose that $f(0, 0) = 0$, $\frac{\partial f}{\partial x_j}(0, 0) = 0$, $j = 1, \ldots, n$ and the matrix $\left(\frac{\partial^2 f}{\partial x_j \partial x_\ell}(0, 0)\right)_{j,\ell=1}^n$ is invertible.

(a) Show that there is an open set V of $0 \in \mathbb{R}^m$ and a smooth function $g : V \to \mathbb{R}^n$, such that $\frac{\partial f}{\partial x_j}(g(y), y) = 0$, for every $y \in V$. (10 pts)

(b) Show that there are open sets Ω, Ω_1 of $0 \in \mathbb{R}^n$ in $\mathbb{R}^n$ and a smooth function $H : \Omega_1 \to \mathbb{R}$, such that

$$\{(x_1, \ldots, x_n, \frac{\partial f}{\partial x_1}(x, 0), \ldots, \frac{\partial f}{\partial x_n}(x, 0)); x \in \Omega\}$$
$$= \{(\frac{\partial H}{\partial \xi_1}(\xi), \ldots, \frac{\partial H}{\partial \xi_n}(\xi), \xi_1, \ldots, \xi_n); \xi \in \Omega_1\}.$$

(10 pts)

IV. Let U be an open set in $\mathbb{R}^n$. Let $f_k : U \to \mathbb{R}$, $k = 1, 2, \ldots$, be functions on U. Let $f : U \to \mathbb{R}$ be a function on U.

(a) Please give a precise meaning of "f_k converges uniformly to f on U as $k \to +\infty$". (5 pts)

(b) Let

$$f_k(x) = k^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-k(|x-y|^2+|x-y|^4)+i\sum_{j=1}^n x_j^2 y_j} dy,$$

where $|x-y|^2 = \sum_{j=1}^n |x_j - y_j|^2$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$. Show that there is a function $f(x) : \mathbb{R}^n \to \mathbb{R}$ such that for every compact set $K \subset \mathbb{R}^n$, f_k converges uniformly to f on K as $k \to +\infty$. Can you find $f(x)$? (15 pts)

(c) Let

$$f_k(x) = k^{\frac{n}{2}} \int_{|y| \le M} e^{-k(|x-y|^2+|x-y|^4)+i\sum_{j=1}^n x_j^2 y_j} dy,$$

where $M > 0$ is a constant, $M < +\infty$, $|x-y|^2 = \sum_{j=1}^n |x_j - y_j|^2$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$. Show that f_k converges uniformly to a function $f(x) : \mathbb{R}^n \to \mathbb{R}$ on $\mathbb{R}^n$ as $k \to +\infty$. (15 pts)

V. Let U be an open set of $\mathbb{R}$. Let $f_k : U \to \mathbb{R}$ be smooth function on U, $k = 1, 2, \ldots$. Assume that for every compact set $K \subset U$ and every $m \in \mathbb{N} \cup \{0\}$, there is a constant $C_{K,m} > 0$ such that

$$\sup \left\{ \left| (\frac{d^m f_k}{dx^m})(x) \right| ; x \in K \right\} \le C_{K,m},$$

for every $k = 1, 2, \ldots$. Suppose that $\lim_{k \to +\infty} f_k(x) = f(x)$, for every $x \in U$, where $f : U \to \mathbb{R}$ is a function on $\mathbb{R}$. Show that

(a) $f(x)$ is a smooth function on U, i.e. $f(x) \in \mathcal{C}^\infty(U)$. (10 pts)

(b) For every $m \in \mathbb{N} \cup \{0\}$, every compact subset $K \subset U$, $\frac{d^m f_k}{dx^m}(x)$ converges to $\frac{d^m f}{dx^m}(x)$ uniformly on K. (10 pts)

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科目：高等微積分 2023.11.02

1. (15%) If every closed and bounded set of a metric space (M, d) is compact, does it follow that (M, d) is complete? If your answer is "yes", prove it; if your answer is "no", give a counter-example.

2. (20%) Determine the values of h for which the following series converges uniformly on $I_h = \{x \in \mathbb{R} : |x| \le h\}$:
$$\sum_{n=1}^{\infty} \frac{(n!)^2\, x^n}{(2n)!} \ .$$
Show your work.

3. (10%+10%+5%) Consider
$$(\star) \qquad F(x) = \int_0^{\infty} \frac{e^{-xt} - e^{-t}}{t}\, \mathrm{d}t$$
on $I = \{x \in \mathbb{R} : \frac{1}{2} \le x \le 2\}$.
   (a) Show that $(\star)$ converges on I, and $F(x)$ is continuous on I.
   (b) Show that
$$F'(x) = \int_0^{\infty} -e^{-xt}\, \mathrm{d}t \ .$$
   (c) Evaluate $F(x)$.

4. (5%+15%) Let f be a smooth function on $\mathbb{R}^n$ with $\det\big(\big[\frac{\partial^2 f}{\partial x_i \partial x_j}\big]_{1\le i,j\le n}\big) = 2$ everywhere.
   (a) Show that there exist an open neighborhood $U \subset \mathbb{R}^n$ of the origin and an open set $V \subset \mathbb{R}^n$ such that $\mathbf{x} = (x_1, \ldots, x_n) \mapsto Df(\mathbf{x}) = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$ is a bijection map from U and V, and its inverse is also a smooth map.
   (b) Denote the inverse map in part (a) by $\boldsymbol{\xi}(\mathbf{y}) = (\xi_1(\mathbf{y}), \ldots, \xi_n(\mathbf{y}))$. For any $\mathbf{y} \in V$, let
$$f^*(\mathbf{y}) = -f(\boldsymbol{\xi}(\mathbf{y})) + \sum_{i=1}^{n} y_i\, \xi_i(\mathbf{y}) \ .$$
   Compute $\det\big(\big[\frac{\partial^2 f^*}{\partial y_i \partial y_j}\big]_{1\le i,j\le n}\big)$.

5. (20%) Let $f(x)$ be a C^1 function for $x \in [0, \infty)$. Suppose that $f(x) \ge 0$ and $f'(x) \le 1$ for every $x \ge 0$, and $\int_0^{\infty} f(x)\, \mathrm{d}x$ converges. Does $\lim_{x\to\infty} f(x)$ exist? If your answer is "yes", determine the limit and prove it; if your answer is "no", give a counter-example.

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科目：高等微積分

2022.10.20

1. (15 points)
Let $M = \{f : [0,\infty) \to [0,\infty); \int_0^\infty f(x)^2 dx \le 1\}$. Evaluate the following:

$$\sup_{f\in M} \int_0^\infty f(x)e^{-x}dx.$$

2. (15 points)
Assume $A \subset \mathbb{R}^n$ is compact and let $a \in A$. Suppose $\{a_n\}$ is a sequence in A such that every convergent subsequence of $\{a_n\}$ converges to a.
(1) Does the sequence $\{a_n\}$ also converge to a ? Justify your result.
(2) Now assume A is not compact and suppose $\{a_n\}$ is a sequence in A such that every convergent subsequence of $\{a_n\}$ converges to $a \in A$. Does the sequence $\{a_n\}$ also converge to a ? Justify your result.

3. (20 points)
Let M be a metric space and $A \subset M$ a compact subset. Suppose $f : A \to A$ is continuous and satisfies $d(f(x), f(y)) \ge d(x, y)$ for all x, y. Prove that f is onto A, i.e $f(A) = A$.

4. (20 points)
Define a sequence of functions $\{f_n(x)\}$ on $[0,1]$ as:

$$f_n(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1 & \text{if } x \in (\frac{2k}{2^n}, \frac{2k+1}{2^n}], k = 0, 1, \cdots, 2^{n-1} - 1 \\ -1 & \text{if } x \in (\frac{2k+1}{2^n}, \frac{2k+2}{2^n}], k = 0, 1, \cdots, 2^{n-1} - 1 \end{cases}$$

Prove or disprove that we always have $\lim_{n\to\infty} \int_0^1 f_n(x)g(x)dx = 0$ as long as g is a continuous function.

5. (20 points)
Denote P_2 the set of all polynomials with real coefficients and degree ≤ 2. Consider the function $G : P_2 \to \mathbb{R}$ by

$$G(p) = \int_0^1 p(x)^2 dx.$$

Let $S = \{p \in P_2; p(1) = 1\}$. Does G attain any extremal value on S ? If yes, find $p \in S$ such that G attains an extremal value at p.

6. (10 points)
Suppose $f(x) : [a,b] \to \mathbb{R}$ is Riemann integrable, and $g(x) : \mathbb{R} \to \mathbb{R}$ satisfies $|g(x) - g(y)| \le C|x - y|$ for all x, y. Prove that $g(f(x))$ is Riemann integrable.

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科目：高等微積分

2020.10.23

1. (10 points)
Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable. Assume that its Jacobian matrix $(\frac{\partial f_i}{\partial x_j})$ has rank n everywhere and $f^{-1}(K)$ is compact whenever K is compact. Show that $f(\mathbb{R}^n) = \mathbb{R}^n$

2. (15 points)
Let $F(x, y, z) = (x, y, z)$, and S be the boundary of the region $x^2 + y^2 \leq z \leq \sqrt{2 - x^2 - y^2}$, oriented so that the normal points out of the region. Compute $\int\int_S F \cdot n dA$.

3. (20 points)
Let M be a metric space with countable elements. Prove or disprove that M is disconnected.

4. (20 points)
Let $f(x) = \frac{1}{4} + x - x^2$. For a real number x, define $x_{n+1} = f(x_n)$, where $x_0 = x$. (1) Given $x = 0$, show that the sequence $\{x_n\}$ converges and find its limit L. (2) Find all real numbers x such that their corresponding sequences all converge to L.

5. (20 points)
Let X consist of all real valued functions f on $[0, 1]$ such that
(1)$f(0) = 0$
(2)$||f|| = \sup\{\frac{|f(x)-f(y)|}{|x-y|^{1/3}}; x \neq y\}$ is finite.
Prove that $|| \cdot ||$ is a norm for X and X is complete with respect to this norm.

6. (15 points)
Assume $f : (a, b) \to \mathbb{R}$ is differentiable. Consider its derivative $f'(x)$, show that $f'(x)$ never has a jump discontinuity.

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科目：高等微積分

2019.10.18

1. (15 points)
Consider a series $\sum_{k=2}^{\infty} a_k \sin(kx)$, where $\{a_k\}$ is a sequence of real numbers. Is it possible to construct a sequence $\{a_k\}$ such that $\sum_{k=2}^{\infty} a_k \sin(kx)$ converges uniformly to $\sin(x)$ on $[0, \pi]$? Justify your result.

2. (15 points)
Suppose $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous. Show that f must be bounded by a linear function, i.e, there exist constants A, B such that $|f(x)| \leq A + B|x|$ for all x.

3. (20 points)
(1): Let $f : M \to N$ be a map from metric space M to another metric space N with the property that if a sequence $\{p_n\}$ in M converges, then the sequence $\{f(p_n)\}$ in N also converges. Is f continuous ? Justify your result.
(2): Let $f : M \to \mathbb{R}$ where M is a metric space, and define $G = \{(x, y) \in M \times \mathbb{R}; y = f(x)\}$. If G is compact (in product space $M \times \mathbb{R}$). Is f continuous ? Justify your result.

4. (20 points)
Suppose $L : \mathbb{R}^3 \to \mathbb{R}^3$ is an invertible linear map and $g : \mathbb{R}^3 \to \mathbb{R}^3$ has continuous first order partial derivatives and satisfies $||g(x)|| \leq C||x||^{1+\epsilon}$ for some $\epsilon > 0$ and all $x \in \mathbb{R}^3$ ($||\cdot||$ is the usual Euclidean norm). Show that $f(x) = L(x) + g(x)$ is invertible near the origin 0.

5. (20 points)
(1): Find the volume of the ellipsoid $(x + 2y)^2 + (x - 2y + z)^2 + 3z^2 \leq 1$.
(2): Let C be a positively oriented simple closed curve. Find the curve C that maximizes the integral $\int_C y^3 dx + (3x - x^3) dy$.

6. (10 points)
Let $f(x)$ be a real valued continuous function on $[0, 1]$. Find the limit

$$\lim_{n \to \infty} (n + 1) \int_0^1 x^n f(x) dx.$$

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科目：高等微積分

2018.10.19

(1) [10 分] Does the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n \log(n+1)}{n}$$

converge? Does it converge absolutely? Justify your answer.

(2) [10+10 分] Consider the function

$$f(x,y) = \frac{1}{(1-xy)^2}$$

defined on $\Omega = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1\ ,\ 0 \le y \le 1\ ,\ (x,y) \neq (1,1)\}$.

(a) For any $\kappa \in (0,1)$, let $U_\kappa = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le \kappa\ ,\ 0 \le y \le \kappa\}$. Is $f(x,y)$ uniformly continuous on U_κ? Justify your answer.

(b) Is $f(x,y)$ uniformly continuous on Ω? Justify your answer.

(3) [10+15 分] For any $n \in \mathbb{N}$, consider $f_n(x) = n\,x^n\,(1-x)$ on $I = \{x \in \mathbb{R} \mid 0 \le x \le 1\}$.

(a) Determine $\lim_{n\to\infty} f_n(x)$ for every $x \in I$.

(b) Is the convergence uniform on I? Give your reason.

(4) [15+10 分] Let

$$F(x) = \int_0^\infty \frac{1-\cos(xt)}{t^2 e^t}\,\mathrm{d}t\ .$$

(a) Can you switch the order of integration and differentiation to obtain the formulae for $F'(x)$ and $F''(x)$? Explain the reason.

(b) Find the explicit[1] formula for $F'(x)$ and $F(x)$.

(5) [10+10 分] Consider

$$\begin{array}{rccl} F: & \mathbb{R}^4 & \to & \mathbb{R}^2 \\ & (x,y,u,v) & \mapsto & \left(\displaystyle\int_{x-y^2}^{x^2+y} (e^{t^2}+u)\mathrm{d}t, x^3+v \right) . \end{array}$$

(a) Prove that near $(1,1,0,0)$, the two equations $F(x,y,u,v) = (\int_0^2 e^{t^2}\mathrm{d}t, 1)$ can be solved for u, v as continuously differentiable functions of x, y.

(b) For the functions $u(x,y)$ and $v(x,y)$ in part (a), find all their first order partial derivatives at $(x,y) = (1,1)$.

[1]Not an improper integral.

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科目：高等微積分

2017. 10. 21

1. (15 points. **No partial credit will be given if the answer is wrong.**) Evaluate the integral

$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \frac{1}{1+3\sin^2 x}\,dx. \quad (\textbf{Hint.}\ \text{Let } u = \tan x.)$$

2. (15 points) State and prove Leibniz's criterion for convergence of alternating series.

3. (15 points) Let F be an $\mathbf{R}$-valued C^∞ function on $\mathbf{R}^2$ such that at a point $p = (a, b)$ we have

$$\frac{\partial F}{\partial x}(p) = \frac{\partial F}{\partial y}(p) = 0,\ \frac{\partial^2 F}{\partial x^2}(p) > 0, \text{ and } \frac{\partial^2 F}{\partial x^2}(p)\frac{\partial^2 F}{\partial y^2}(p) - \left(\frac{\partial^2 F}{\partial x \partial y}(p)\right)^2 > 0.$$

Show that there exists $R > 0$ such that $F(p) < F(q)$ for $q \in \{(x, y) \in \mathbf{R}^2 \,|\, (x-a)^2 + (y-b)^2 < R^2\}$.

4. We adopt the following definitions.

> Let (X, d) be a metric space. (i) A family $\mathcal{F}$ of $\mathbf{R}$-valued functions on X is *equicontinuous at a point* $x_0 \in X$ (with respect to d) if
>
> $$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, f \in \mathcal{F} \quad \forall x \in X \quad d(x.x_0) < \delta \ \Rightarrow\ |f(x) - f(x_0)| < \varepsilon.$$
>
> (ii) A sequence f_n of $\mathbf{R}$-valued functions on X *converges compactly* to some function f on X if for every compact subset K of the metric space (X, d) the sequence $f_n|_K$ converges to $f|_K$ uniformly.

(15 points) Let f_n be a sequence of $\mathbf{R}$-valued functions on a metric space (X, d) which converges *pointwise* to a *continuous* function f on (X, d). Suppose that $\{f_n \,|\, n \in \mathbf{N}\}$ is equicontinuous at every point of X. Show that f_n converges compactly to f on X.

5. (15 points.) Compute the <u>outward</u> flux of the vector field $(x + ye^z,\ e^x \sin(yz),\ ye^{zx})$ through the boundary of the region $D = \left\{(x, y, z) \in \mathbf{R}^3 \,\middle|\, \left(\sqrt{x^2+y^2} - 3\right)^2 + z^2 < 1\right\}$.

6. (25 points) Show that the function $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{\cos(ne)}{n^x}$ (where $e = \sum_{m=0}^{\infty} \frac{1}{m!}$) is well-defined (i. e., the series converges) on $(0, \infty)$ and is continuous.

7. (30 points. In your argument if any theorems are used you have to clearly verify that their conditions are fulfilled.) Let f and g be $\mathbf{R}$-valued C^∞ functions on $\mathbf{R}^2$ and let $S = \{(x, y) \in \mathbf{R}^2 \,|\, f(x, y) = 0\}$. Suppose that at some point $p = (a, b) \in S$ we have $\dfrac{\partial f}{\partial x}(p) = -1$, $\dfrac{\partial f}{\partial y}(p) = 2$, $\dfrac{\partial g}{\partial x}(p) = 3$, $\dfrac{\partial g}{\partial y}(p) = -6$,

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(p) & \frac{\partial^2 f}{\partial x \partial y}(p) \\ \frac{\partial^2 f}{\partial y \partial x}(p) & \frac{\partial^2 f}{\partial y^2}(p) \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} \frac{\partial^2 g}{\partial x^2}(p) & \frac{\partial^2 g}{\partial x \partial y}(p) \\ \frac{\partial^2 g}{\partial y \partial x}(p) & \frac{\partial^2 g}{\partial y^2}(p) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}.$$

Show that there exists $R > 0$ such that $g(p) < g(q)$ for $q \in S \cap \{(x, y) \in \mathbf{R}^2 \,|\, (x-a)^2 + (y-b)^2 < R^2\}$.

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科目：高等微積分

2016.10.21

1. (15 points) Let A be the unit ball $B_1(0)$ in $\mathbf{R}^3$. Compute

$$\int_A \cos(x+y+z)\,dx\,dy\,dz.$$

2. Let f be a real-valued function on $\mathbf{R}$ which has period 2π and is Riemann integrable on $[-\pi,\pi]$. We define its Fourier coefficients

$$a_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx\ (n=0,1,2,\dots) \text{ and } b_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx\ (n=1,2,\dots).$$

(1) (10 points) Show that $f(x)^2$ is Riemann integrable on $[-\pi,\pi]$.

(2) (15 points) Show that the series $\frac{1}{2}a_0^2 + \sum\limits_{n=1}^{\infty}(a_n^2+b_n^2)$ converges.

3. (1) (15 points) Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers and let $B_n = b_1 + \cdots + b_n$ $(n \in \mathbf{N})$. Suppose that $a_n \searrow 0$ as $n \to \infty$ and that there exists $M > 0$ such that $|B_n| \leqslant M$ for every $n \in \mathbf{N}$. Show that the series $\sum\limits_{n=1}^{\infty} a_n b_n$ converges.

(2) (10 points) Show that the function series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\sin(nx)$$

converges uniformly on $[-K,K]$ if $|K| < \pi$.

4. Definition. Let $\mathcal{F}$ be a set of real-valued functions on a set X. $\mathcal{F}$ is *uniformly bounded* if there exists $M > 0$ such that $|f(x)| \leqslant M$ for every $x \in X$ and every $f \in \mathcal{F}$.

(20 points) Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of convex functions on $[-2,2]$ and let $f_n = F_n|_{[-1,1]}$ $(n \in \mathbf{N})$. Suppose that $\{F_n \mid n \in \mathbf{N}\}$ is uniformly bounded. Show that there exists a subsequence of $\{f_n\}_{n=1}^{\infty}$ which is uniformly convergent on $[-1,1]$.

5. (15 points) Let $f = (f_1,\dots,f_n) : U \to \mathbf{R}^n$ be a C^1 map from an open set U in $\mathbf{R}^n$, and let $g : V \to U$ be a continuous map from an open set V in $\mathbf{R}^n$. Suppose that

$$\det\left(\frac{\partial f_j}{\partial x_k}(x)\right) \neq 0 \ \text{ for every } x \in U,$$

and that $f(g(x)) = x$ for every $x \in V$. Show that g is C^1.

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科目：高等微積分

2015. 10. 22

1. 30% (each 10%)

(i) Calculate $\sup_{x\neq 0}\frac{x^2-\ln\left(1+x^2\right)}{x^4}=?$

(ii) Calculate $\iint_A xy\sin\left(x^2-y^2\right)dxdy$, where

$A=\left\{(x,y):0<y<1, x>y \text{ and } x^2-y^2<1\right\}$

(iii) Prove that $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}$ (Hint: use Fourier series)

2. 30% (each 10%)

Let $C=\left\{\sum_{n=1}^{\infty}\frac{a_n}{3^n}:a_n\in\{0,2\},n\in\mathbb{N}\right\}$.

(i) Prove that $C\subset[0,1]$.

(ii) Does C have interior point? Justify your answer.

(iii) Is C a compact subset of $\mathbb{R}$? Justify your answer.

3. 20% (each 10%)

Let $A=\left\{f\in C^0[0,1]:\|f\|_{C^0}\leq 1\right\}$ and $B=\left\{f\in C^1[0,1]:\|f\|_{C^1}\leq 1\right\}$,

where $C^0[0,1]=\left\{f:[0,1]\to\mathbb{R}\mid f \text{ is continuous on } [0,1]\right\}$,

$C^1[0,1]=\left\{f:[0,1]\to\mathbb{R}\mid f \text{ is of } C^1 \text{ on } [0,1]\right\}$, $\|\cdot\|_{C^0}$ is the standard sup-norm, and $\|\cdot\|_{C^1}$ is the standard C^1-norm.

(i) Is A (sequentially) compact? Justify your answer.

(ii) Prove that any sequence of B has a convergent subsequence in A.

4. 20%

Let $f:[0,1]\to\mathbb{R}$ be a continuous function. Let $F(x)=\int_0^x f(t)\,dt$ for $x\in[0,1]$. Suppose $F(1)=0$. Determine if the value

$$\sup_{g\in C^0[0,1]} \frac{\int_0^1 F(x)g(x)\,dx}{\left(\int_0^1 G^2(x)\,dx\right)^{1/2}}$$

exists, where $G(x)=\int_0^x g(t)\,dt$ for $x\in[0,1]$, and $C^0[0,1]=\{g:[0,1]\to\mathbb{R} \mid g \text{ is continuous on } [0,1]\}$.

Justify your answer.

臺灣大學數學系 104 學年度碩士班甄試試題
科目：高等微積分

2014. 10. 24

1. (10%) Determine whether $\int_0^\infty x^{-\frac{1}{3}} e^{-x}\,\mathrm{d}x$ *converges* or not. Justify your answer.

2. (20% = 10% + 10%) For the following functions, determine whether they are *uniformly continuous* or not. Justify your answer.

   (a) $f(x) = \log(2014 + x^{10})$ for $x \in \mathbb{R}$, where log is the natural logarithm function.

   (b) For $(x, y) \in [-1, 1] \times [-1, 1]$,

$$g(x,y) = \begin{cases} x \sin(\dfrac{y}{x}) & \text{when } x \neq 0\ , \\ 0 & \text{when } x = 0\ . \end{cases}$$

3. (20% = 10% + 10%) For the following sequences of continuous functions on $\mathbb{R}^1$, determine whether they have a *uniformly convergent* subsequence or not. Justify your answer.

   (a) $f_n(x) = \exp(-(x-n)^2)$, $n \in \mathbb{N}$.

   (b) $g_n(x) = \sin(x - n)$, $n \in \mathbb{N}$.

4. (20% = 10% + 10%) Consider the system of equations:

$$x + yu + z^2 + x^3 = 0\ ,$$
$$z + x^2 + z^2 u - y^3 = 0\ .$$

   It is clear that $x = y = z = u = 0$ is a solution.

   (a) Near $(0, 0, 0, 0)$, can the system be solved for x, z as continuously differentiable functions of $y, u \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$? Justify your answer.

   (b) Near $(0, 0, 0, 0)$, can the system be solved for y, u as continuously differentiable functions of $x, z \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$? Justify your answer.

5. (20% = 10% + 10%) Suppose that $\{a_n\}_{n=1}^\infty$ is a sequence such that $\sum_{n=1}^\infty n|a_n|$ converges.

   (a) Show that both $\sum_{n=1}^\infty a_n \sin(nx)$ and $\sum_{n=1}^\infty n a_n \cos(nx)$ *converges uniformly* for all $x \in \mathbb{R}$.

   (b) Let $f(x) = \sum_{n=1}^\infty a_n \sin(nx)$, and $g(x) = \sum_{n=1}^\infty n a_n \cos(nx)$. Prove that the derivative of $f(x)$ is $g(x)$.

6. (10%) Let $f(x, y, z)$ be a smooth function on $\mathbb{R}^3$, and κ be a constant within $(0, 3)$. Prove that

$$(3-\kappa)\iiint_{|\mathbf{x}|\le r} f^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \le r\iint_{|\mathbf{x}|=r} f^2\,\mathrm{d}S + \frac{1}{\kappa}\iiint_{|\mathbf{x}|\le r} |\mathbf{x}|^2|\nabla f|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

   for any $r > 0$. Here, $\mathbf{x} = (x, y, z)$ and ∇f is the gradient of f.

   Hint: Apply the divergence theorem and the Cauchy–Schwarz inequality.

臺灣大學數學系 103 學年度碩士班甄試試題
科目：高等微積分

2013. 10. 18

請為每一題預留充分書寫空間，依題號次序 1. (a)(b)(c)，2. (a)(b)，…作答。若未依題目次序，其跳題作答之部份不予批閱、計分。

Answer the following questions in order.
All functions and sequences are real-valued.

1. (25%=5+5+15)

   (a) State the definition that a sequence (a_n) is a *Cauchy sequence.*

   (b) State the definition that a function $F : D \subset \mathbb{R} \to \mathbb{R}$ is *uniformly continuous* on D.

   (c) Let $f(x)$ be a continuous function on $\mathbb{R}$ and (x_n) be a Cauchy sequence. Use an $\epsilon - \delta/\epsilon - N$ type argument to show that $(f(x_n))$ is also a Cauchy sequence.

2. (25%=5+20)

   (a) State the definition that a sequence of functions $(h_n), h_n : D \subset \mathbb{R} \to \mathbb{R}$, *converges to* H *uniformly* on D.

   (b) Let (g_n) be a sequence of differentiable functions on (a, b) such that $\lim_{n\to\infty} g_n(x) = G(x)$ exists for all $x \in (a, b)$. Suppose that there exists a constant $M > 0$ such that $\sup_{x\in(a,b)} |g_n'(x)| < M$ for all n. Show that (g_n) converges to G uniformly on (a, b).

3. (25%=7+8+10) Let $\lambda > 0, J(\lambda) = \displaystyle\int_0^\infty \frac{dx}{(1+x)x^{2\lambda}}$ and $\Gamma(\lambda) = \displaystyle\int_0^\infty x^{\lambda-1}e^{-x}\,dx.$

   (a) Show that for $\lambda > 0$ the improper integral $\Gamma(\lambda)$ converges.

   (b) Find the range of $\lambda > 0$ on which the improper integral $J(\lambda)$ converges.

   (c) Show that $J(\lambda) = \Gamma(2\lambda)\Gamma(1 - 2\lambda)$ when both sides are meaningful. Hint. Express $(1 + x)^{-1}$ as an integral.

4. (25%=10+15) Let $u_k, p_k, k = 1, \ldots, n$, be positive numbers and $p_1 + \cdots + p_n = 1$.

   (a) Evaluate the limit
   $$\lim_{t\to\infty} \left(\sum_{k=1}^n p_k u_k^t \right)^{\frac{1}{t}}.$$

   (b) Evaluate the limit
   $$\lim_{t\to 0} \left(\sum_{k=1}^n p_k u_k^t \right)^{\frac{1}{t}}.$$

   Hint. Use $u^t = \exp(t \ln u)$ and Taylor expansions.

臺灣大學數學系 102 學年度碩士班甄試試題

科目：高等微積分 2012.10.19

Advanced Calculus

1. (40 points) True or False. Prove or disprove each of the following statements.

   a) If the real series $\sum_{j=1}^{\infty} b_j = 1$ is conditional convergent (which is not absolutely convergent), then there exist a rearrangement of $\{b_j\}$ such that $\sum_{j=1}^{\infty} b_{\sigma(j)} = 2012$, where $\{\sigma(j)\}_{j=1}^{\infty}$ is a permutation of $\{j\}_{j=1}^{\infty}$.

   b) If, for some positive number $M > 0$, the partial sum $|\sum_{j=1}^{n} b_j| \leq M$ for all $n \in \mathbb{N}$ and $\{a_j\}_{j=1}^{\infty}$ is a positive decreasing sequence that tends to 0, then $\sum_{j=1}^{\infty} a_j b_j$ converges.

   c) If $f(x)$ is a continuous function defined on the closed interval $[-1, 1]$ such that $f'(0) = 1$, then there exists a $\delta > 0$ such that $f(x)$ is an increasing function for all $x \in [-\delta, \delta]$.

   d) Suppose $g(x)$ is a nonnegative continuous function defined on the closed interval $[0, 1]$ and $f(x)$ is a positive monotone continuous function, then there exists a number $\xi \in [0, 1]$ such that
   $$\int_0^1 f(t)g(t)dt = f(\xi)\int_0^1 g(t)dt.$$

2. (50 points) Evaluate

$$\int_0^1 x^9(10\ln x + 1)dx, \qquad \int_2^4 \int_{4/x}^{(20-4x)/(8-x)} (y-4)dydx,$$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 dydx, \qquad \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n+i},$$

$$\int_A xdxdydz,$$

where A is the region in $\mathbb{R}^3$ bounded by the planes $x = 0$, $y = 0$, $z = 2$ and the surface $z = x^2 + y^2$.

3. (10 points) Find the maximum of $f(x, y, z) = x + y + 2z$ under the constraints
$$x^2 - xy + y^2 + z^2 = 2.$$
At which point does $f(x, y, z)$ achieve its maximum ?

臺灣大學數學系
101 學年度碩士班甄試試題
科目：高等微積分 2011.10.21

1. (30 points) Calculate

   (1)
$$\int\int_{\mathbb{R}^2} e^{-(13x^2+16xy+5y^2)}dxdy$$

   (2)
$$\int_0^\infty \frac{\sin x}{x}dx.$$

2. (30 points)

   (1) Suppose n is a positive integer, $a_i \geq 0$, $b_i \geq 0$ for $i = 1, 2, \cdots, n$, and p and q are two positive numbers such that $1/p + 1/q = 1$. Prove the following Holder's inequality.
$$\sum_{i=1}^n a_i b_i \leq (\sum_{i=1}^n a_i^p)^{1/p} (\sum_{i=1}^n b_i^q)^{1/q}.$$

   (2) Suppose $a_1 \geq a_2 \geq a_3 \geq a_4$ and $b_1 \geq b_2 \geq b_3 \geq b_4$. Prove the following inequality.
$$a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 \geq a_1b_2 + a_2b_4 + a_3b_1 + a_4b_3.$$

3. (30 points) Prove or disprove the following statements.

   (1) Let $f(x, y)$ be a real-valued function on $\mathbb{R}^2$ such that both $\frac{\partial^2 f(x_0,y_0)}{\partial x \partial y}$ and $\frac{\partial^2 f(x_0,y_0)}{\partial y \partial x}$ exist. Then $\frac{\partial^2 f(x_0,y_0)}{\partial x \partial y} = \frac{\partial^2 f(x_0,y_0)}{\partial y \partial x}$.

   (2) Let $f_n(x)$ be a sequence of continuous functions defined on $[0, 1]$. If $f_n(x)$ converges uniformly on $[0, 1]$ as $n \to \infty$, then $\lim_{n\to\infty} f_n(x)$ is a continuous function on $[0, 1]$.

4. (10 points) Suppose $f(x)$ is a continuous function defined on $[0, 1]$. Prove that
$$\lim_{n\to\infty} \int_0^1 x^n f(x)dx = 0.$$

臺灣大學數學系
100 學年度碩士班甄試試題
科目：高等微積分

2010.10.22

(1) (25 pts) Suppose the series $f(x)=\sum_{n=0}^{\infty} a_n x^n$ converges for $|x|<R$. Show that f is continuous and differentiable on $(-R,R)$ and

$$f'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1} \text{ for } |x|<R.$$

(2) (25 pts) Let $\{f_n\}$ and f be defined on $[0,2)$. Suppose that $\{f_n\}$ are continuous and

$$\lim_{n\to\infty} f_n(x_n)=f(x)$$

for every sequence $\{x_n\}\subset[0,2)$ such that $\lim_{n\to\infty} x_n = x$ and $x\in[0,2)$.
(a) Is it true that $\{f_n\}$ converges uniformly to f on $[0,2)$?
(b) Is it true that f is continuous on $[0,2)$?

(3) (25 pts)
(a) Prove that

$$\left|\int_0^1 f(x)g(x)\,dx\right| \le \left(\int_0^1 f^2(x)\,dx\right)^{\frac{1}{2}}\left(\int_0^1 g^2(x)\,dx\right)^{\frac{1}{2}}.$$

(b) Let $h(x)$ be a continuous function on $[0,1]$. Show that

$$\lim_{n\to\infty}\int_0^1 h(x)\sin(nx)\,dx=0.$$

(4) (25 pts) Let $\mathbb{N}=\{1,2,3,\ldots\}$ denote the natural numbers and E be defined as follows: $A\in E$ if and only if A is a subset of $\mathbb{N}$. Show that there is a one-to-one and onto mapping from E to the open interval $(0,1)$.

臺灣大學數學系

九十九學年度碩士班甄試試題

科目：高等微積分　　2009.10.30

(1) (25 pts) Define $f(0,0)=0$ and $f(x,y)=\dfrac{x^3}{3x^2+y^2}$, $(x,y)\neq(0,0)$.

(a) Show that $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist at $(0,0)$ and f is continuous at $(0,0)$.

(b) Prove that f is not differentiable at $(0,0)$.

(c) Is f^2 differential at $(0,0)$?

(2) (25 pts) Assume $a>0$ and $b>0$. Find all (a,b) such that $\dfrac{\sin(x^a)}{1+x^b}$ is uniformly continuous on $\{x : x>0\}$.

(3) (25 pts)

(a) Let $\{b_n\}$ be defined by $b_1=1$; $b_{2m}=\dfrac{b_{2m-1}}{4}$; $b_{2m+1}=1+b_{2m}$. Find $\limsup_{n\to\infty} b_n$ and $\liminf_{n\to\infty} b_n$.

(b) Let $\{c_n\}$ and $\{d_n\}$ be two strictly increasing sequences of positive integers satisfying $c_n+d_n<1.5n$. Define $p(m)=1$ if $m\in\{c_n\}$, $p(m)=0$ if $m\notin\{c_n\}$; $q(m)=1$ if $m\in\{d_n\}$, $q(m)=0$ if $m\notin\{d_n\}$, where m is a positive integer. Find $\limsup_{m\to\infty}(p(m)+q(m))$.

(4) (25 pts)

(a) Let $f(x)$ be an increasing function on $[0,1]$. Prove that f is Riemann integrable.

(b) Assume $g(x,y)<g(x,z)$ if $y<z$ and $g(x,y)<g(t,y)$ if $x<t$. Prove or disprove that the assumptions above imply $g(x,y)$ is Riemann integrable on $[0,1]\times[0,1]$.

注意：考試開始鈴響前，不得翻閱試題，

並不得書寫、畫記、作答。

國立清華大學 113 學年度碩士班考試入學試題

系所班組別：數學系

科 目 代 碼 ：0101

考 試 科 目 ：高等微積分

—作答注意事項—

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
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6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「國立清華大學試場規則及違規處理辦法」，無法因本試題封面作答注意事項中未列明而稱未知悉。

國立清華大學 113 學年度碩士班考試入學試題

系所班組別：數學系碩士班

考試科目（代碼）：高等微積分（0101）

共__2__頁，第__1__頁　　*請在【答案卷】作答

You need to explain how you get your answers in each problem. Please mark your answers clearly.

Problem 1: (10 points) Let $P = x - y^3 - y$ and $Q = x^3 + y^2$. Find the line integral of $\oint_C Pdx + Qdy$ where C is the simple closed curve $x^2 + y^2 = 1$. The line integral is taken counter-clockwisely.

Problem 2: (10 points) Find $\int_2^{24} \frac{d[\sqrt{x}]}{\sqrt{x}}$ where $[\cdot]$ is the greatest integer function.

Problem 3: Let $f_n(x) = nx^n(1-x)$, $n = 1, 2, 3, \cdots$.
(a) (5 points) Determine if the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwisely on $[0,1]$.
(b) (10 points) Determine if the family of functions $\{f_n\}_{n=1}^{\infty}$ is equicontinuous on $[0,1]$.

Problem 4: (10 points) Let $f(x,y) = \frac{x^5 - y^3}{x^4 + y^2}$ when $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Determine if f is differentiable at $(0,0)$.

Problem 5: (15 points) Let $S := \{(x, sin(\frac{1}{x}))|\ 0 < x < \pi\} \cup \{(0,y)|\ 0 \leq y \leq 1\ , y \in \mathbb{Q}\}$. Determine if S is connected. You can use the fact that $\{(x, sin(\frac{1}{x}))|\ 0 < x < \pi\}$ is connected.

Problem 6: (15 points) Let f be an Riemann integrable function on $[0,1]$. Suppose that for any $0 < a < b < 1$, there is at least one number $c \in (a,b)$ such that $f(c) = 1$ or $f(c) = 0$. If $\int_0^1 f(x)dx = 1/2$, prove that there is an uncountable subset $S \subseteq [0,1]$ such that $f(s) = 0$ for any $s \in S$. Note that f may not be continuous.

Problem 7: (10 points) Prove that $\sqrt[3]{x}$ is uniformly continuous on $\mathbb{R}$.

Problem 8: (15points) Let $s_n(x) = sinx + \frac{1}{2}sin2x + \frac{1}{3}sin3x + \cdots + \frac{1}{n}sinnx$. Prove that the sequence $\{s_n\}_{n=1}^{\infty}$ converges uniformly on $[1, 2]$.

注意：考試開始鈴響前，不得翻閱試題，
並不得書寫、畫記、作答。

國立清華大學 112 學年度碩士班考試入學試題

系所班組別：數學系

科 目 代 碼 ：0101

考 試 科 目 ：高等微積分

—作答注意事項—

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5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
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國立清華大學 112 學年度碩士班考試入學試題

系所班組別：數學系碩士班

考試科目 (代碼)：高等微積分 (0101)

1.(10 pts) Let A be a nonempty set of real numbers which is bounded below. Let $-A = \{-x | \, x \in A\}$. Prove that $\inf A = -\sup(-A)$.

2. (15 pts) Let E be a nonempty subset of $\mathbb{R}^n$. Let E' be the set of all limit points (accumulation point). (i) Is E' a closed set ? (ii) Does E and E' always have the same limit points? Prove the statements or give counterexamples for (i) and (ii). (iii) If $E = \{(x, \sin \frac{1}{x}) | \, x \in (0, 1)\}$. What is E' ?

3. (10 pts) Let a_1 and $a_{n+1} = 1 + \sqrt{a_n}$, $n \in \mathbb{N}$. Find

$$\limsup_{n \to \infty} a_n.$$

4. (15 pts) (i) Suppose $a_n \geq 0$, $n \in \mathbb{N}$ and $\sum a_n$ diverges. Prove $\sum a_n/(1 + a_n)$ also diverges. (ii) Suppose $\sum a_n$ is a series of real numbers which converges absolutely. Prove that every rearrangement of $\sum a_n$ converges to the same sum.

5. (10 pts) Use the finite open covering property of the compactness to show the following. If f is a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X.

國立清華大學 112 學年度碩士班考試入學試題

系所班組別：數學系碩士班

考試科目 (代碼)：高等微積分 (0101)

6.(15 pts) Give the reasons in your computation. (i) Let $a > 0$. Find

$$\lim_{n\to\infty} \int_a^\infty \frac{\sin(nx)}{nx} dx.$$

(ii) What is the answer for (i) if $a = 0$?

(iii) Let $[x]$ be the greatest integer function. That is $[x] = \sup\{n \mid n \le x,\ n \in \mathbb{Z}\}$. Find

$$\int_0^2 x d[x].$$

7. (15 pts) Let $f : \mathbb{R}^5 \to \mathbb{R}^2$ defined by $f = (f_1, f_2)$ with

$$f_1(x_1, x_2, y_1, y_2, y_3) = 2e^{x_1} + x_2 y_1 - 4y_2 + 3$$
$$f_2(x_1, x_2, y_1, y_2, y_3) = x_2 \cos x_1 - 6x_1 + 2y_1 - y_3.$$

Note that $f(0, 1, 3, 2, 7) = (0, 0)$. (i) Prove that there exists a function $g : \mathbb{R}^3 \to \mathbb{R}^2$ defined on a neighborhood of $(3, 2, 7)$ so that $f(g(y_1, y_2, y_3), y_1, y_2, y_3) = 0$. (ii) Find the derivative of g at $(3, 2, 7)$.

8.(10 pts) (i) Let $(x, y) \in \mathbb{R}^2 \setminus (0, 0)$, $P(x, y) = \frac{-y}{x^2+y^2}$ and $Q(x, y) = \frac{x}{x^2+y^2}$. Let C be the curve goes from $(a, 0), a > 0$ to itself once along $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, b > 0$. Find

$$\oint_C P(x, y)dx + Q(x, y)dy.$$

(ii) Can we find a function $f : \mathbb{R}^2 \setminus (0, 0) \to \mathbb{R}$ such that

$$\frac{\partial f}{\partial x} = P(x, y) \text{ and } \frac{\partial f}{\partial y} = Q(x, y)?$$

Give the reason for your answer.

注意：考試開始鈴響前，不得翻閱試題，

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國立清華大學 111 學年度碩士班考試入學試題

系所班組別：數學系

科目代碼：0101

考試科目：高等微積分

—作答注意事項—

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國立清華大學 111 學年度碩士班考試入學試題

系所班組別：數學系碩士班

考試科目（代碼）：高等微積分（0101）

共__1__頁，第__1__頁　　　*請在【答案卷、卡】作答

1. (10%) Find
$$\lim_{n\to\infty}\int_3^n (1-\frac{x}{n})^n dx$$

2. (10%) Let $f:\mathbb{R}-\{0\}\to\mathbb{R}$ be defined by
$$f(x)=\frac{1}{x}$$
Is f a continuous function? Prove your claim.

3. (15%) Consider $\mathbb{Q}$ as a topological subspace of $\mathbb{R}$ where $\mathbb{R}$ is given the Euclidean topology. Let $f:\mathbb{Q}\to\mathbb{R}$ be a continuous function and $A\subset\mathbb{Q}$ be a compact subset. Must $f(A)$ be compact in $\mathbb{R}$? Prove or give a counterexample.

4. (15%) Construct a sequence of Riemann integrable functions $f_n:[0,1]\to\mathbb{R}$ which converges pointwise to a function $f:[0,1]\to\mathbb{R}$ but
$$\lim_{n\to\infty}\int_0^1 f_n(x)dx\neq\int_0^1 f(x)dx$$

5. (15%) Let $f:\mathbb{R}^2\to\mathbb{R}^2$ be defined by
$$f(x,y):=\begin{cases}(\frac{\sin(x^2+y^2-1)}{x^2+y^2-1},\ \cos(x^2+y^2-1)), & \text{if } x^2+y^2\neq 1\\ (1,1), & \text{if } x^2+y^2=1\end{cases}$$
Is f differentiable at $(1,0)$? Prove your claim.

6. Let
$$E=\{\sum_{k=0}^{n}a_kx^{2k+1}|n\in\mathbb{N}\cup\{0\}\}$$
be the set of polynomials of odd degree in each term defined on $[1,2]$.
   (a) (10%) Show that E is not closed in $\mathscr{C}^0([1,2])$ where $\mathscr{C}^0([1,2])$ is the space of continuous functions from $[1,2]$ to $\mathbb{R}$ with the sup norm.
   (b) (10%) Is E dense in $\mathscr{C}^0([1,2])$? Prove your claim.

7. (15%) Given a continuous function $h:[0,1]\times[0,1]\to\mathbb{R}$ such that
$$\sup_{x\in[0,1]}\{\int_0^1|h(x,y)|dy\}<1$$
Suppose that $g:[0,1]\to\mathbb{R}$ is a continuous function. Show that there is a unique continuous function $f:[0,1]\to\mathbb{R}$ such that
$$f(x)-\int_0^1 h(x,y)f(y)dy=g(x)$$

注意：考試開始鈴響前，不得翻閱試題，

並不得書寫、畫記、作答。

國立清華大學 110 學年度碩士班考試入學試題

系所班組別：數學系

科 目 代 碼 ：0101

考 試 科 目 ：高等微積分

—作答注意事項—

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
2. 考試開始後，請於作答前先翻閱整份試題，是否有污損或試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。
3. 考生限在答案卷上標記「由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。
4. 答案卷用盡不得要求加頁。
5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「**國立清華大學試場規則及違規處理辦法**」，無法因本試題封面作答注意事項中未列明而稱未知悉。

國立清華大學 110 學年度碩士班考試入學試題

系所班組別：數學系碩士班

考試科目（代碼）：高等微積分（0101）

1. (10%) Calculate

$$\int_0^2 x\sin(3x+1)\sin(4x+1)dx$$

2. (10%) Calculate the line integral

$$\int_C \frac{xdy - ydx}{x^2+y^2}$$

where C is the positive oriented curve in $\mathbb{R}^2 - \{(0,0)\}$ parametrized by

$$\varphi(\theta) = \begin{cases} (\cos 4\pi\theta,\ \sin 4\pi\theta), & \text{if } \theta \in [0, \frac{1}{2}]; \\ (2-\cos 4\pi\theta,\ -\sin 4\pi\theta), & \text{if } \theta \in [\frac{1}{2}, 1] \end{cases}$$

3. (10%) Determine if

$$\sum_{n=1}^{\infty} n\cos\frac{x}{n}$$

converges uniformly on $[-\pi,\pi]$.

4. (10%) Show that the function $f:\mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x,y) = \frac{1}{10}(\sin^2(3x+y),\ \cos(4x+2y))$$

has a fixed point.

5. (10%) Show that the set

$$X = \{(x_1,x_2,x_3) \in \mathbb{R}^3 | x_1^2 + 2x_2^4 + 3(e^{x_1}+x_3^2)^6 = 2021\}$$

is compact.

6. (10%) Let X be a topological space and $A, B \subset X$ be compact subsets. Is $A \cup B$ always a compact set? Prove or give a counterexample.

7. (10%) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function defined by

$$f(x,y,z) = \frac{x^2 \sin x}{x^2 + y^2 + z^2}$$

for $(x,y,z) \in \mathbb{R}^3 - \{0\}$. Can f be extended to a differentiable function on $\mathbb{R}^3$?

8. Let $\mathscr{C}^0([0,\pi]) := \{f : [0,\pi] \to \mathbb{R} | f \text{ is continuous }\}$ with the sup norm.

   (a) (10%) Is the family $\{\cos(nx)\}_{n=1}^{\infty}$ equicotinuous on $[0,\pi]$?

   (b) (10%) Does the sequence $\{\cos(nx)\}_{n=1}^{\infty}$ in $\mathscr{C}^0([0,\pi])$ have a convergent subsequence?

   (c) (10%) Is the family $\{\frac{1}{n}\sin(nx)\}_{n=1}^{\infty}$ in $\mathscr{C}^0([0,\pi])$ compact?

注意：考試開始鈴響前，不得翻閱試題，

並不得書寫、畫記、作答。

國立清華大學 109 學年度碩士班考試入學試題

系所班組別：數學系

科 目 代 碼 ：0101

考 試 科 目 ：高等微積分

－作答注意事項－

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國立清華大學 109 學年度碩士班考試入學試題

系所班組別：數學系碩士班

考試科目（代碼）：高等微積分（0101）

共__1__頁，第__1__頁　　*請在【答案卷、卡】作答

1. (10 pts) Let $f:(0,\infty)\to\mathbb{R}$ be differentiable. If $f(x)\to 5$ and $f'(x)\to\lambda$ as $x\to\infty$, prove that $\lambda=0$.

2. (10 pts) Find the limit
$$\lim_{n\to\infty}\frac{n}{\log n}\left(n^{1/n}-1\right).$$

3. (12 pts) Prove that the function
$$f(x)=\sum_{n=1}^{\infty}\frac{x^2}{x^2+n^2}$$
is continuous on $\mathbb{R}$.

4. (12 pts) Suppose that $\{p_n\}$ is a sequence of polynomials, and that $p_n\to f$ uniformly on the interval $[0,1]$. Must f be differentiable?

5. (12 pts) Is there a simple closed curve C in the xy-plane which maximizes the value of
$$\oint_C y^3\,dx+\left(3x-x^3\right)dy\ ?$$
If so, find the maximum value.

6. (12 pts) Let $B=\{x\in\mathbb{R}^n/\ \|x\|\le r\}$, and suppose $f:B\to\mathbb{R}^n$ satisfying $\|f(0)\|\le\frac{2}{3}r$ and
$$\|f(x)-f(y)\|\le\frac{1}{3}\|x-y\|\quad\text{for all } x,y\in B.$$
Prove that there exists a unique $x\in B$ such that $f(x)=x$.

7. (16 pts) Let $F:\mathbb{R}^n\to\mathbb{R}^n$ be defined by $F(x)=L(x)+G(x)$, where L is a linear isomorphism and G is a C^1-function. Suppose that there are positive constants M and ε such that $\|G(x)\|\le M\|x\|^{1+\varepsilon}$ for all x in a neighborhood of the origin. Prove that F is locally invertible near the origin.

8. (16 pts) Let $f:(a,b)\to\mathbb{R}$ be a C^n-function, and suppose for some $c\in(a,b)$,
$$f'(c)=f''(c)=\cdots=f^{(n-1)}(c)=0,\qquad\text{but}\quad f^{(n)}(c)\ne 0.$$
Prove that

   (a) For n even, f has a local minimum at c if $f^{(n)}(c)>0$, and a local maximum at c if $f^{(n)}(c)<0$.

   (b) For n odd, there is neither a local maximum nor a local minimum at c.

國立成功大學 114 學年度「碩士班」甄試入學考試

高等微積分

1. (15%) Determine the following limit by δ-ϵ definition.

$$\lim_{x\to 1}(2x^3+7x^2+5x+8)$$

2. (15%) Show by definition that the following set is open:

$$\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2<1<x+y\}$$

3. (15%) Assume function f is increasing on interval $[20,24]$.

   (a) Show that the set of discontinuity of f is countable.

   (b) Show that f is Riemann integrable.

4. (15%) Denote x_n the unique positive root of the polynomial

$$f_n(x)=x^n+x^{n-1}+\cdots+x-114,\quad n=1,2,3,\cdots.$$

   Show that the sequence $\{x_n\}$ is convergent. Also find the limit.

5. (20%) Yes or No. Justify your answer.

   (a) Determine if $f(x)=\sqrt{1+x^2}$ is uniformly continuous on $(-\infty,\infty)$.

   (b) Determine if $f(x)=\sqrt{1+x^4}$ is uniformly continuous on $(-\infty,\infty)$.

6. (20%) Find the value of the limit. Justify your answer.

$$\lim_{n\to\infty}\int_0^1 \frac{nx^n}{1+e^x}dx$$

國立成功大學 113 學年度「碩士班」甄試入學考試

高等微積分

Advanced Calculus

October 19, 2023

1. *(15 points)* Let $f : [0,1] \to \mathbb{R}$ be a function defined by

$$f(x) := \begin{cases} \dfrac{1}{k} & \text{when } x \in \mathbb{Q} \text{ and } x = \dfrac{q}{2^k} \text{ for some } q, k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f is continuous at every irrational point.

2. **(a)**. *(10 points)* Determined whether the following series is convergent or not:

$$\sum_{n=1}^{\infty} 5^{-n+(-1)^n}.$$

(b). *(10 points)* Prove that the power series $\sum_{n=0}^{\infty} a_n x^n$ converges on $(-R, R)$ when $\sum_{n=0}^{\infty} R^{2n} a_n^2$ converges.

3. *(15 points)* Let $X \subset \mathbb{R}^2$ be an open set and $f : X \to \mathbb{R}$ be a continuous function with $f(x) \neq 0$ for all $x \in X$. Prove that either $f(x) > 0$ for all $x \in X$, $f(x) < 0$ for all $x \in X$ or X is disconnected[1].

4. *(20 points)* Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of positive, Riemann integrable functions defined on $[0,1]$ and $g_n(x) \leq 1$ for all $x \in [0,1]$, $n \in \mathbb{N}$. Prove that the sequence

$$\left\{ f_n(x) := \int_0^x g_n(s)ds \right\}_{n\in\mathbb{N}}$$

has a subsequence which is uniformly convergent on $[0,1]$.

5. *(15 points)* Let $f : [0,1] \to \mathbb{R}$ be a Riemann integrable function. Prove that f^3 is also Riemann integrable.

6. *(15 points)* Let $S := \{(x,y,z) \in \mathbb{R}^3 | z = 7 - x^2 - y^2, z \geq 3\}$, $\vec{n}$ be the outer normal of S and

$$V := (z^2, -3xy, x^3y^3)$$

be a vector field defined on $\mathbb{R}^3$. Find

$$\int_S curl(V) \cdot \vec{n} ds.$$

[1] A set X is disconnected iff there exist two disjoint nonempty open sets A, B cover X and $A \cap X \neq \emptyset$, $B \cap X \neq \emptyset$.

國立成功大學 112 學年度「碩士班」甄試入學考試

高等微積分

1. (a) Determine the limit of $f(x) = \dfrac{3x^2 - 5x + 4}{x + 1}$ at $x = 2$ by δ-ϵ definition. (10%)

   (b) Determine whether $f(x)$ is uniformly continuous on $[0, \infty)$. (10%)

2. Show by definition that the interior of triangular region with vertices $(1, 1)$, $(5, 2)$, $(3, 4)$ is an open set. (10%)

3. Denote x_n the positive root of the polynomial $f_n(x) = x^n + \cdots + x^2 + x - 1$. Show that the sequence $\{x_n\}$ is convergent and find the limit. (15%)

4. Given the fact that the following integral is convergent for all $p > 0$.

$$\int_1^\infty \frac{\sin x}{x^p}\,dx$$

   For what values of $p > 0$ is the integral convergent absolutely/conditionally? (15%)

5. (a) By observing the graph of $y = \frac{n}{1+n^2x^2}$ as n increases and evaluating its integral on $(-\infty, \infty)$, find the value the following limit. (10%)

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{n\,e^{\cos x}}{1 + n^2x^2}\,dx$$

   (b) Justify the convergence by N-ϵ definition. (10%)

6. (a) Evaluate the limit. (10%)

$$\lim_{n\to\infty} \left(\frac{3^n + 4^n + 5^n}{3}\right)^{\frac{1}{n}} + \left(\frac{3^{\frac{1}{n}} + 4^{\frac{1}{n}} + 5^{\frac{1}{n}}}{3}\right)^n$$

   (b) Evaluate the limit. (10%)

$$\lim_{n\to\infty} \left(\int_0^1 e^{nx(1-x)}\,dx\right)^{\frac{1}{n}} + \left(\int_0^1 e^{\frac{x(1-x)}{n}}\,dx\right)^n$$

國立成功大學 111 學年度「碩士班」研究生甄試入學考試

高等微積分

Throughout the exam, the Euclidean spaces $\mathbb{R}^n$ are all equipped with usual Euclidean metric $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.

1. (10 points) Evaluate the line integral

$$\int_C \mathbf{V} \cdot d\mathbf{r},$$

where the vector field is given by

$$\mathbf{V} = \langle 2y + xe^{x^2}, 4x + e^{y^2}\cos y \rangle,$$

and C is the path staring from $(1, 0)$ to $(0, -1)$ by a *counterclockwise* circular path, and then from $(0, -1)$ to $(0, 0)$ by a straight line.

2. For a continuous function $f : [a, b] \to \mathbb{R}$ that is differentiable on (a, b),
   (a) (8 points) State and prove the *Rolle's Theorem* for f.
   (b) (7 points) State and prove the *Mean Value Theorem* for f.

3. (10 points) Given the fact that $\mathbb{R}$ is complete, prove the *Monotone Convergence Theorem:*

Any bounded monotonic sequence in $\mathbb{R}$ is convergent.

4. Consider the usual Euclidean space $\mathbb{R}^n$,
   (a) (8 points) If $E \subset K \subset \mathbb{R}^n$, where E is closed and K is compact, then E is compact.
   (b) (8 points) Prove that if $\{K_\alpha\}_{\alpha \in A}$ is any family of compact subsets of $\mathbb{R}^n$, then

$$\bigcap_{\alpha \in A} K_\alpha$$

   is also compact.

5. (15 points) Prove that the sequence of functions $\{f_n\}$ on $[0, 1]$ given by

$$f_n(x) = nx(1 - x^2)^n$$

pointwise converges to a function $f(x)$ but does not converge uniformly.

6. Given two metric spaces (E, d_E) and (F, d_F),

   (a) (8 points) If $f : E \to F$ is continuous and E is compact, then f is uniformly continuous.

   (b) (8 points) Use part (a) to prove that

   $$f(x) = e^{\cos^2 x \sin x}$$

   is uniformly continuous on $\mathbb{R}$.

7. (a) (4 points) State the *Inverse Function Theorem* on $\mathbb{R}^n$.

   (b) (4 points) State the *Implicit Function Theorem* on $\mathbb{R}^{n+m}$.

   (c) (10 points) These two theorems are in fact equivalent. Prove *either* one of the implications. (i.e. prove either (a) $\Rightarrow$ (b) or (b) $\Rightarrow$ (a)).

國立成功大學 110 學年度「碩士班」研究生甄試入學考試

高等微積分

1. (10%) Show that $f(x) = \sqrt[3]{x}$ is uniformly continuous on $\mathbb{R}$.

   Hint: find a constant $c > 0$ such that $|a-b|^3 \le c|a^3 - b^3|$ for all $a, b \in \mathbb{R}$.

2. (15%) Denote $\mathbf{x} = (x, y) \in \mathbb{R}^2$ and $\|\mathbf{x}\| = \sqrt{x^2 + y^2}$.

   Let $G \subseteq \mathbb{R}^2$ be an open set such that the closed disk $\{\|\mathbf{x}\| \le 1\} \subseteq G$.

   Show that there exists $\epsilon > 0$ such that $\{\|\mathbf{x}\| \le 1 + \epsilon\} \subseteq G$.

3. (15%) Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence such that $x_n \longrightarrow x$. Show that

$$\frac{(2n-1)x_1 + (2n-3)x_2 + \cdots + 3x_{n-1} + x_n}{n^2} \longrightarrow x.$$

4. (15%) Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers. Show that

$$\limsup_{n\to\infty} \sqrt[n]{x_n} \le \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

   Also provide an example such that $\limsup_{n\to\infty} \sqrt[n]{x_n} < 1 < \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}$.

5. (15%) Let $f(x)$ be a Riemann integrable function on $[0, 1]$. Show that

$$\lim_{n\to\infty} \int_0^1 f(x)\cos(nx)dx = 0.$$

   Hint: first let $f(x)$ be a piecewise constant function.

6. (15%) Determine the values of $p \ge 0$ for which the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{(1+n^2)^p}$$

   is divergent, conditionally convergent or absolutely convergent.

7. (15%) Evaluate the limit. Justify your calculation.

$$\lim_{n\to\infty} \int_0^1 n \cdot \sin\left(\frac{x^2}{n}\right) dx$$

國立成功大學 109 學年度「碩士班」研究生甄試入學考試

高等微積分

1. (15%) Suppose that E is a compact subset of $\mathbb{R}$ and that $f : \mathbb{R} \to \mathbb{R}$. Prove that if f is continuous on E, then f is uniformly continuous on E.

2. (a) (5%) State the Implicit Function Theorem.

   (b) (10%) Prove that there exist functions $u(x,y), v(x,y)$, and $w(x,y)$, and an $r > 0$ such that u, v, w are continuously differentiable and satisfy the equations

$$\begin{aligned} u^5 + xv^2 - y + w &= 0 \\ v^5 + yu^2 - x + w &= 0 \\ w^4 + y^5 - x^4 &= 1 \end{aligned}$$

   on $B_r(1,1)$, and $u(1,1) = 1, v(1,1) = 1, w(1,1) = -1$.

3. (a) (10%) Prove that if $f : [0,1] \to \mathbb{R}$ is continuous and $\int_0^1 |f(x)|dx = 0$, then $f(x) = 0$ for all $x \in [0,1]$.

   (b) (5%) State the Weierstrass Approximation Theorem.

   (c) (10%) Prove that if $f : [0,1] \to \mathbb{R}$ is continuous and

$$\int_0^1 f(x)x^k dx = 0 \quad \text{for } k = 0, 1, \cdots,$$

   then $f(x) = 0$ for all $x \in [0,1]$.

4. (a) (5%) State the Arzela-Ascoli Theorem.

   (b) (10%) Let $f_n : [0,1] \to \mathbb{R}$ be continuous and be such that $f_n(0) = 0$ for every $n \in \mathbb{N}$. Suppose that the derivatives f_n' exist and are uniformly bounded on $(0,1)$. Prove or disprove that f_n has a uniformly convergent subsequence.

5. (15%) Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \le \frac{1}{2}|x - y|$ for all $x, y \in \mathbb{R}$. Prove that f has a unique fixed point.

6. (15%) Let $f_n : [0,1] \to \mathbb{R}$ be continuous for every $n \in \mathbb{N}$. Suppose that $\{f_n\}$ converges uniformly to f on $[0,1]$ and that $\{x_n\}$ is a sequence in $[0,1]$ converging to a point $x \in [0,1]$. Prove or disprove that $\lim_{n\to\infty} f_n(x_n) = f(x)$.

【基礎數學】：Part II. 高等微積分

1. (10 points) Evaluate

$$\int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\, dy dx$$

by interchanging limits of integrations.

2. (10 points) A function $f : \mathbb{R} \to \mathbb{R}$ is called *convex* if for every $s < t$ and $\lambda \in [0, 1]$,

$$f(\lambda s + (1 - \lambda)t) \leq \lambda f(s) + (1 - \lambda) f(t).$$

Prove that if a convex function is differentiable, its derivative f' is an increasing function:

$$s \leq t \Rightarrow f'(s) \leq f'(t).$$

Note: Here we do *NOT* assume the existence of f''. It is useful to consider the fact that for $s < t$, the graph of f over $[s, t]$ always lies below the line through $(s, f(s))$ and $(t, f(t))$.

3. For a subset E of a metric space (X, d), let E^o be the set of interior points of E. Prove that

   (a) (10 points) E^o is open.

   (b) (10 points) If $G \subset E$ and G is open, then $G \subset E^o$

   Note: $x \in E$ is an *interior point* if there exists $r > 0$ so that

$$B_r(x) = \{y \in X \mid d(x, y) < r\} \subset E.$$

4. (10 points) Given a sequence of Riemann integrable functions $\{f_n\}$ on $[a, b]$ and assume that f_n converges uniformly to a function f, it is known that f is also Riemann integrable on $[a, b]$. With this fact, prove that

$$\int_a^b f\, dx = \lim_{n\to\infty} \int_a^b f_n\, dx.$$

5. Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Define the *operator norm* of A by

$$\|A\| := \sup_{|\mathrm{x}|_n \le 1} \{|A\mathrm{x}|_m\}.$$

Here, $|\cdot|_n$ and $|\cdot|_m$ are the usual Euclidean lengths in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

(a) (10 points) Prove that

$$\|A\| = \sup_{|\mathrm{x}|_n = 1} \{|A\mathrm{x}|_m\}.$$

(b) (10 points) Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & -2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

$\|A\| = ?$

6. (10 points) Prove that the function $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f(x, y, z) = (e^{x^3+2x}, e^{y-z} + \cos z, e^z)$$

is locally invertible. That is, for every $P \in \mathbb{R}^3$, there exists a neighborhood U of P so that $f|_U : U \to f(U)$ is invertible.

7. Consider the sequence of functions $\{f_n\}_{n=1}^{\infty}$ on $[0, 1]$ given by

$$f_n(x) = \frac{\sin^4(nx)}{\sin^2(nx) + (1 - nx)^2}.$$

(a) (10 points) Prove that $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded. That is, there exists $M > 0$ so that $|f_n(x)| \le M$ for all n, and all $x \in [0, 1]$.

(b) (10 points) Use Arzela-Ascoli Theorem to prove that $\{f_n\}_{n=1}^{\infty}$ is *NOT* equicontinuous.

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【基礎數學】: Part II. 高等微積分

1. (25 points) Given a set X,
   (a) (5 points) State the definition for a function $d : X \times X \to \mathbb{R}$ to be a *metric*.
   (b) (5 points) State the definition for a subset $E \subset X$ to be *open* with respect to d.
   (c) (5 points) State the definition for a subset $E \subset X$ to be *closed* with respect to d.
   (d) (5 points) Let $X = \mathbb{R}$ and $E = \{x\} \subset \mathbb{R}$ for some $x \in \mathbb{R}$. Prove that E is closed with respect to the metric $d(x, y) = |x - y|$.
   (e) (5 points) Define a metric d on $\mathbb{R}$ so that the subset E in part (d) is open. Explain your answer.

2. (15 points) Assume *Bolzano-Weirstrass Theorem* on $\mathbb{R}$ with the usual Euclidean metric:

   Any bounded sequence has a convergent subsequence,

   prove the same theorem for $\mathbb{R}^2$, also with standard Euclidean metric

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

3. (15 points) Given a function $f : \mathbb{R}^n \to \mathbb{R}^m$,
   (a) (5 points) State the definition for f to be *differentiable* at $\mathbf{p} \in \mathbb{R}^n$.
   (b) (10 points) Show, *directly* (ie. without using big theorem) from the definition in part (a), that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x, y) = (xy + x, y^2 + y)$$

   is differentiable at $(0, 0)$.

4. (15 points) Consider the sequence of functions $\{f_n\}$ defined on $[0, 1]$ given by

$$f_n(x) = \log nx^2(1 - x^3)^n.$$

   (a) (5 points) Find $f(x) := \lim_{n\to+\infty} f_n(x)$.
   (b) (10 points) Prove that f_n are not uniformly convergent.

5. (15 points) If a function $\phi : \mathbb{R} \to \mathbb{R}$ satisfies

$$|\phi(x) - \phi(y)| \leq \frac{1}{2}|x - y|,$$

   prove that
   (a) (5 points) ϕ is continuous (with respect to the usual metric $d(x, y) = |x - y|$).
   (b) (10 points) $\phi(x) = x$ for some $x \in \mathbb{R}$.

6. (15 points) Consider $C([0,1])$, the space of continuous complex valued functions on $[0,1]$.

(a) (5 points) State the *Stone-Weirstrass Theorem* on this space.

(b) (10 points) Prove that if $f \in C([0,1])$ satisfies the fact that

$$\int_0^1 x^n f(x)dx = 0 \ \ \forall n \in \mathbb{N},$$

then $f = 0$. (Hint: first show that $\int_0^1 (f(x))^2 \ dx = 0$ using part (a).)

【基礎數學】: Part I. 線性代數

Note: $\mathbb{R}$ denotes the field of real numbers, and n denotes a positive integer.

1. (10%) Is there a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1,-2,0) = (1,1)$, $T(3,-5,1) = (2,3)$, and $T(-1,3,1) = (3,0)$? *Justify your answer.*

2. (15%) Let V be the vector space of all polynomials of degree at most n with real coefficients. For $i = 0, 1, \ldots, n$, let $p_i(x) = x^i + x^{i+1} + \cdots + x^n \in V$. Show that $\{p_0(x), p_1(x), \ldots, p_n(x)\}$ is a basis for V.

3. (20%) Let A be an $n \times n$ real matrix such that $A^2 = A$. Show that the trace of A is equal to the rank of A. Is A similar over $\mathbb{R}$ to a diagonal matrix? *Justify your answer.*

4. (20%) Let T be a linear operator on a finite-dimensional vector space such that $\operatorname{rank}(T^2) = \operatorname{rank}(T)$. Show that $\mathrm{N}(T) \cap \mathrm{R}(T) = \{0\}$. (Here $\mathrm{N}(T)$ and $\mathrm{R}(T)$ are the null space and the range of T respectively.)

5. (15%) Let V be the vector space of all polynomials of degree at most 3 with real coefficients. Let D be the linear operator on V defined by $D(p) = p'$ for $p \in V$. Find the Jordan form of D.

6. (20%) Let T and U be linear operators on an n-dimensional vector space V. Suppose that $\{v, T(v), \ldots, T^{n-1}(v)\}$ is a basis for V for some $v \in V$, and that $TU = UT$. Show that $U = p(T)$ for some polynomial p.

【基礎數學】: Part I. 高等微積分

1. Entrance Exam

(1) (15 Points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$f(x,y) = \begin{cases} \dfrac{x}{y} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}.$$

(a) (5 Points) Use definition to show that $f_x(0,0)$ and $f_y(0,0)$ exist.
(b) (5 Points) Prove that f is not continuous at $(0,0)$.
(c) (5 Points) Is f differentiable at $(0,0)$? Explain.

(2) (15 Points) Let $a > 0$. The following iterated integral can be rewritten as a double integral:

$$\int_{-a}^{a} \left(\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)} dy \right) dx = \iint_D e^{-(x^2+y^2)} dA.$$

Sketch the region D and evaluate the iterated integral by computing the double integral.

(3) (15 Points)

(a) (7 Points) State Rolle's Theorem.
(b) (8 Points) Let $f : (0,\infty) \to \mathbb{R}$ be a continuously differentiable function. Suppose that the equation

$$\frac{x^2}{2} + \ln x = f(x)$$

has at least two distinct solutions (on $(0,\infty)$). Show that there exists a positive real number t such that $f'(t) \geq 2$.

(4) (15 Points)

(a) (7 Points) State the Bolzano-Weierstrass Theorem (in $\mathbb{R}^n$.)
(b) (8 Points) Prove or disprove that the sequence of real numbers $\{a_n\}$ defined by

$$a_n = e^{\sin n}, \quad n \in \mathbb{N}$$

has a convergent subsequence.

(5) (15 Points) For each vector $x \in \mathbb{R}^n$, we define its Euclidean norm to be

$$\|x\|_{\mathbb{R}^n} = \sqrt{x_1^2 + \cdots + x_n^2},$$

where $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. For any two vectors $x, y \in \mathbb{R}^n$, we define their inner product to be

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i.$$

Let $a_1, \cdots, a_m$ be unit vectors in $\mathbb{R}^n$, i.e. $\|a_i\|_{\mathbb{R}^n} = 1$ for all $1 \leq i \leq n$. Define a function $T : \mathbb{R}^n \to \mathbb{R}^m$ by

$$T(x) = (a_1 \cdot x, \cdots, a_m \cdot x), \quad x \in \mathbb{R}^n.$$

(a) (7 Points) Prove that

$$\|T(x)\|_{\mathbb{R}^m} \leq \sqrt{m}\|x\|_{\mathbb{R}^n} \text{ for any } x \in \mathbb{R}^n$$

(b) (8 Points) Prove that T is uniformly continuous on $\mathbb{R}^n$ via $\epsilon - \delta$ language.

(6) (25 Points) Let $X = C([0,1])$ be the space of real valued continuous functions on $[0,1]$ equipped with the norm

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|.$$

For each $f, g \in X$, we set $d(f,g) = \|f - g\|_\infty$. It is well known that (X, d) is a complete metric space, i.e every Cauchy sequence in X is convergent.

For each $n \in \mathbb{N} \cup \{0\}$, define $f_n : [0,1] \to \mathbb{R}$ inductively by $f_0 = 0$ and

$$f_{n+1}(x) = 1 + \int_0^x t f_n(t)dt, \quad n \geq 0.$$

Then $f_n \in X$ for all $n \in \mathbb{N}$.

(a) (7 Points) Prove that

$$\|f_{n+1} - f_n\|_\infty \leq \frac{1}{2}\|f_n - f_{n-1}\|_\infty, \quad \text{for all } n \in \mathbb{N}.$$

(b) (9 Points) Use completeness of X to show that $\{f_n : n \in \mathbb{N}\}$ is uniformly convergent. (Hint: use (a) to show that $\{f_n\}$ is a uniform Cauchy sequence.)

(c) (9 Points) Let f be the uniform limit of $\{f_n : n \in \mathbb{N}\}$ in X i.e.

$$f = \lim_{n\to\infty} f_n \text{ in } X.$$

Show that f is continuously differentiable on $[0,1]$ and solve for f.

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【基礎數學】：Part I. 高等微積分

1. (10 %) Suppose f is a real, three times differentiable function on $[-1,1]$, such that

$$f(-1)=0, \quad f(0)=0, \quad f(1)=1, \quad f'(0)=0.$$

Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1,1)$.

2. (12 %) Let

$$f_m(x) = \lim_{n\to\infty} (\cos m!\pi x)^{2n} \quad (x \in \mathbb{R},\ m \in \mathbb{N}).$$

Show that $\{f_m\}$ converges to a function f in $\mathbb{R}$, but not uniformly.

3. (a) (10 %) Show that $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.

   (b) (8 %) Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, where $\Gamma(\alpha) = \displaystyle\int_0^{\infty} e^{-x}x^{\alpha-1}dx$ is the gamma function, $0 < \alpha < \infty$.

4. (15 %) Let $f : [0,1] \to [0,1]$ be continuous on $[0,1]$ and differentiable on $(0,1)$. Assume that $f(0) = 0$ and $f(1) = 1$. Show that for each $n \in \mathbb{N}$, there are n distinct points $a_1,\ a_2, \cdots,\ a_n \in [0,1]$ such that

$$\frac{1}{f'(a_1)} + \frac{1}{f'(a_2)} + \cdots + \frac{1}{f'(a_n)} = n.$$

5. (a) (5 %) State the Weierstrass approximation theorem.

   (b) (10 %) Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $\int_0^1 f(x)x^n dx = 0$ for all nonnegative integers. Prove or disprove that $f(x) = 0$ for all $x \in [0,1]$.

6. (a) (7 %) State the Arzela-Ascoli theorem.

   (b) (8 %) Let

$$f_n(x) = \frac{x^2}{x^2 + (1-nx)^2} \quad (0 \leq x \leq 1,\ n = 1,\ 2,\ 3,\ \ldots).$$

   Show that $\{f_n\}$ is not equicontinuous on $[0,1]$.

7. Define

$$f(x,y) = \begin{cases} \sin\left(\frac{y^2}{x}\right) \cdot \sqrt{x^2+y^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

   (a) (7 %) Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous at $(0,0)$ and has directional derivatives in every direction at $(0,0)$.

   (b) (8 %) Show that there is no plane that is tangent to the graph of $f : \mathbb{R}^2 \to \mathbb{R}$ at the point $(0, 0, f(0,0))$.

國立成功大學 103 學年度「碩士班」研究生甄試入學考試

【基礎數學】：Part I. 高等微積分

advanced calculus

1. Prove that $\lim_{x\to\infty}(x^5 - 3x^4 - x^2 + 1)^{-1} = 0$.   (20 points)

2. Define a function f by

$$f(x) = \begin{cases} x \cdot \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Prove or disprove that f is uniformly continuous on $\mathbb{R}$.   (20 points)

3. Let f be a smooth function on $(-1,1)$ and $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $x \in (-1,1)$. Let $\{x_m\}$ be a sequence with $x_m \neq 0$ for all $m \in \mathbb{N}$. Assume that $\{x_m\}$ converges to zero with $f(x_m) = 0$. Show that $f = 0$ on $(-1,1)$.   (20 points)

4. Let f be continuous on $[0,1]$ and suppose that $f(0) = 0$. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x^n) = 0.$$

(20 points)

5. Evaluate the integral $\int_0^\infty e^{-x^2} dx$.   (20 points)