NCKU 112.2
Miscellaneous Facts

Eric Liu

Contents

Chapter 1

General Topology

1.1 Directed Sets

Axiom 1.1.1. (Axioms in Order Theory) Given an relation $(X, \leq)$, and suppose $x, y, z \in X$.

(a) $x \leq x$ (Reflexive)

(b) $x \leq y \leq z \implies x \leq z$ (Transitive)

(c) $x \leq y$ and $y \leq x \implies x = y$ (Antisymmetric)

(d) $x \leq y$ or $y \leq x$ (Connected)

(e) $\forall x, y \in X, \exists z \in X, x \leq z$ and $y \leq z$ (Directed)

We say $(X, \leq)$ form a

(a) total order if it is reflexive, transitive, antisymmetric and connected.

(b) partial order if it is reflexive, transitive and antisymmetric.

(c) preorder if it is reflexive and transitive.

(d) directed set if it is reflexive, transitive and directed.

Theorem 1.1.2. (Why is it called Preorder) Given a preorder $(X, \leq)$, the relation $\sim$ defined by

$$x \sim y \iff x \leq y \text{ and } y \leq x$$

is an equivalence relation and if we define $\leq^e$ on the equivalence class by

$$\exists x \in A, y \in B, x \leq y \implies A \leq^e B$$

Then $\leq^e$ is a partial order. Moreover, if the preorder $\leq$ is directed, then $\leq^e$ is also directed.

Proof. We first show $\sim$ is an equivalence relation. Because preoder is reflexive, we see

$$\forall x \in X, x \leq x \text{ which implies } \forall x \in X, x \sim x$$

For symmetry, it is easy to see

$$x \sim y \implies x \leq y \text{ and } y \leq x \implies y \sim x$$

For transitive, see

$$\begin{aligned} x \sim y \text{ and } y \sim z &\implies x \leq y \text{ and } y \leq x \text{ and } y \leq z \text{ and } z \leq y \\ &\implies x \leq z \text{ and } z \leq x \implies x \sim z \text{ (done)} \end{aligned}$$

We now show $\leq^e$ is a partial order. Reflexive property and Transitive property of $\leq^e$ follow from that of $\leq$. Suppose $A \leq^e B$ and $B \leq^e A$, where $x_1, x_2 \in A, y_1, y_2 \in B$ satisfy $x_1 \leq y_1$ and $y_2 \leq x_2$. Because $x_1, x_2 \in A$ and $y_1, y_2 \in B$, we have

$$x_1 \leq x_2 \text{ and } x_2 \leq x_1 \text{ and } y_1 \leq y_2 \text{ and } y_2 \leq y_1$$

Then because $\leq$ satisfy transitive, we have

$$\begin{cases} x_2 \leq x_1 \leq y_1 \implies x_2 \leq y_1 \\ y_1 \leq y_2 \leq x_2 \implies y_1 \leq x_2 \end{cases}$$

This tell us

$$x_2 \sim y_1$$

which implies $A = B$, thus proving $\leq^e$ is antisymmetirc. (done)

Lastly, we show $\leq$ is directed $\implies$ $\leq^e$ is directed. Let A, B be two arbitrary equivalence class. We wish to find an equivalence class T such that

$$A \leq^e T \text{ and } B \leq^e T$$

Let a, b respectively be an arbitrary element of A, B. Because $\leq$ is directed, we know there exists $c \in X$ such that

$$a \leq c \text{ and } b \leq c$$

We immediately see

$$A \leq^e [c] \text{ and } B \leq^e [c] \text{ (done)}$$

■

Corollary 1.1.3. (Chunk Structure of Preorder) Given two equivalence class A, B, we have

$$A \leq^e B \implies \forall x \in A, y \in B, x \leq y$$

Proof. Because $A \leq^e B$, we know

$$\exists x_0 \in A, y_0 \in B, x_0 \leq y_0$$

Then by definition of $\sim$, we have

$$x \leq x_0 \leq y_0 \leq y$$

This give us

$$x \leq y$$

∎

Definition 1.1.4. (Definition of Maximal element in Preorder) Let $(I, \leq)$ be a preorder. We say $m \in I$ is a maximal element if

$$\forall y \in I, m \leq y \implies y \leq m$$

Theorem 1.1.5. (In Preorder, Maximal element form an Equivalence class) Let $(I, \leq)$ be a preorder, and $m \in I$ be a maximal element. Then

$$\forall x \in [m], x \text{ is a maximal element}$$

Proof. Arbitrarily pick an element x in $[m]$. Suppose

$$x \leq y$$

By definition of $\sim$, we have

$$m \leq x \leq y$$

Thus $m \leq y$. Then because m is maximal, we know $y \leq m$. This now give us

$$y \leq m \leq x$$

∎

Notice that in partially ordered set, where anti-symmetric property is true, the definition of maximal element $m \in I$ falls into

$$\forall y \in I, m \leq y \implies y = m$$

Definition 1.1.6. (Definition of Greatest element in Preorder) Let $(I, \leq)$ be a preorder. We say $x \in I$ is a greatest element if

$$\forall y \in I, y \leq x$$

Theorem 1.1.7. (In Directed Set, Maximal element is the Greatest) Suppose $(I, \leq)$ is a directed set.

$$x \in I \text{ is a maximal element } \implies x \in I \text{ is the greatest element}$$

Proof. Arbitrarily pick an element $y \in I$. Because I is directed, we see there exists an element z such that

$$y \leq z \text{ and } x \leq z$$

Then because x is maximal, we know

$$y \leq z \leq x$$

This shows

$$y \leq x$$

■

Theorem 1.1.8. (Sufficient Condition for Preorder to become Directed)

$$(I, \leq) \text{ is a preorder and has a greatest element } x \implies I \text{ is a directed set}$$

Proof. Given arbitrary two element $y, z \in I$, we see $y \leq x$ and $z \leq x$. ■

Example 1 (Partial Order that is Directed)

$$X = \{a, b, c\} \text{ and } a \leq c \text{ and } b \leq c$$

Example 2 (Partial Order that is Not Directed)

$$X = \{a, b, c\} \text{ and } a \leq b \text{ and } a \leq c$$

Example 3 (Partial Order that is Directed)

$$X = \mathbb{Z}_0^+ \text{ and } \forall x, y \in \mathbb{N}, x \leq y \iff y - x | 2 \text{ and } x \leq y$$
$$\text{and } \forall x \in \mathbb{N}, x \leq 0$$

Example 4 (Partial Order that is not Directed)

$$X = \mathbb{N} \text{ and } \forall x, y \in \mathbb{N}, x \leq y \iff y - x | 2 \text{ and } x \leq y$$

Example 5 (Directed Set that is not Partially Ordered)

$$X = \{a, b, c\} \text{ and } a \leq b \text{ and } b \leq a$$
$$\text{and } a \leq c \text{ and } b \leq c$$

Example 6 (Preorder that is Neither Directed nor Partially Ordered)

$$X = \{a, b, c, d\} \text{ and } a \leq b \text{ and } b \leq a$$
$$\text{and } a \leq c \text{ and } b \leq c$$
$$\text{and } a \leq d \text{ and } b \leq d$$

Example 7 (Directed Sets)

X is a metric space and $x \leq y \iff d(y, x_0) \leq d(x, x_0)$ where x_0 is a fixed point in X

Notice that this directed set is generally not antisymmetric, meaning it generally isn't a partial order. Also, notice that x_0 is the greatest element. Also, this order is connected, meaning if we take equivalence class on it, it become a total order.

Lastly, notice that if we remove x_0, X can still be directed, say if $X = \mathbb{R}^2$ and x_0 is the origin.

Example 8 (Directed Sets)

Suppose X, Y are both directed sets. We see $X \times Y$ is a directed set if we define

$$(x, y) \leq (a, b) \iff x \leq a \text{ and } y \leq b$$

Example 9 (Partial Order)

Every collection of sets is a partial order if we define

$$A \leq B \iff A \subseteq B$$

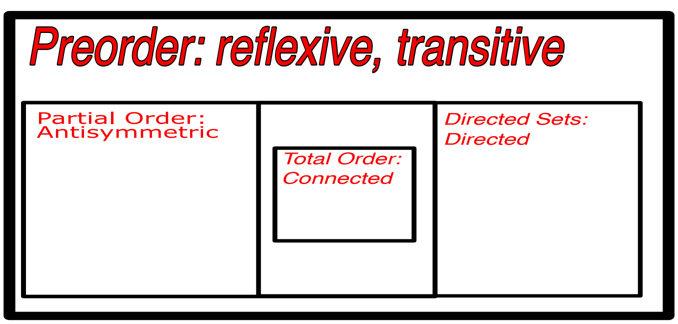
Also, every collection of sets form a partial order if we define

$$A \leq B \iff A \supseteq B$$

Example 10 (Directed Sets)

Suppose (X, τ) is a topological space and $x \in X$. Then all of τ, neighborhoods of x and open neighborhoods of x form directed sets under $\subseteq$, since X is open.

Also, τ, neighborhoods of x and open neighborhoods of x form directed sets under $\supseteq$, because intersection of two open set is again an open set and intersection of two neighborhood is again a neighborhood.



Definition 1.1.9. (Definition of Cofinal) Given a directed set $\mathcal{D}$, a subset $\mathcal{D}' \subseteq \mathcal{D}$ is called cofinal if

$$\forall d \in \mathcal{D}, \exists e \in \mathcal{D}', d \leq e$$

Theorem 1.1.10. (Cofinal Subset is a Directed Set with Original Order) Given a directed set $\mathcal{D}$

$$\mathcal{D}' \subseteq \mathcal{D} \text{ is cofinal } \implies \mathcal{D}' \text{ is a directed set}$$

Proof. Arbitrarily pick two $a, b \in \mathcal{D}'$. Because $\mathcal{D} \ni a, b$ is directed, we know

$$\exists c \in \mathcal{D}, a \leq c \text{ and } b \leq c$$

Then because $\mathcal{D}'$ is cofinal in $\mathcal{D}$, we know

$$\exists d \in \mathcal{D}', c \leq d$$

Then because transitivity of directed set, our proof is finished, as we have found an element d in $\mathcal{D}'$ that is greater than the arbitrary picked elements $a, b \in \mathcal{D}'$. ■

1.2 Net

Definition 1.2.1. (Subnet) Given a net $w : \mathcal{D} \to X$ and $v : \mathcal{E} \to X$ and a function $h : \mathcal{E} \to \mathcal{D}$ we say v is a subnet of w if

$$\begin{cases} \forall e, e' \in \mathcal{E}, e \leq e' \implies h(e) \leq h(e')(\text{monotone}) \\ h[E] \text{ is cofinal in } \mathcal{D} \\ v = w \circ h \end{cases}$$

Definition 1.2.2. (Net convergence) We say the net $w : \mathcal{D} \to X$ converge to x, $w \to x$ if

Theorem 1.2.3. **($w \to x \implies v \to x$)** Suppose v is a subnet of w, we have

$$w \to x \implies v \to x$$

Proof. ■

Theorem 1.2.4. ()

Definition 1.2.5. ()

Chapter 2

Metric Space

2.1

Chapter 3

Calculus

3.1 Examples for uniform convergence

Theorem 3.1.1. (Test Example) The sequence

$$f_n(x) = \frac{x^2}{x^2 + (1 - nx)^2} \text{ is not equicontinuous on } [0, 1]$$

Proof. Notice that

$$f_n(\frac{1}{n}) = 1 \text{ and } f_n(0) = 0$$

Then for all δ, we see that if n is large enough

$$\text{then } \left|\frac{1}{n} - 0\right| < \delta \text{ and } \left|f_n(\frac{1}{n}) - f_n(0)\right| = 1$$

∎

Theorem 3.1.2. (Test Example) Prove

$$\frac{x}{1 + nx^2} \text{ uniformly converge on } \mathbb{R}$$

Proof. It is clear that $\frac{x}{1+nx^2}$ pointwise converge to 0. Because $\frac{x}{1+nx^2}$ is an odd function, fixing ϵ, we only wish to find N such that

$$\forall x > 0, \forall n > N, \frac{x}{1 + nx^2} < \epsilon$$

Observe

$$\begin{aligned} \frac{x}{1 + nx^2} < \epsilon &\iff x < \epsilon(1 + nx^2) \\ &\iff \frac{x - \epsilon}{\epsilon x^2} < n \end{aligned}$$

Notice that $\frac{x-\epsilon}{\epsilon x^2}$ is bounded since it is continuous and converge to 0 as $x \to \infty$. ∎

3.2 Test Example

Theorem 3.2.1. (Cauchy-Schwarz Inequality for Integral) Let $\mathscr{R}\Big([a,b]\Big)$ be the space of Riemann-Integrable functions on $[a,b]$. It is clear that $\mathscr{R}\Big([a,b]\Big)$ is a vector space over $\mathbb{R}$. Define $\langle \cdot, \cdot \rangle$ on $\mathscr{R}\Big([a,b]\Big)$ by

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx$$

It is easy to show

(a) $\forall f \in \mathscr{R}\Big([a,b]\Big), \langle f, f \rangle \geq 0$ (non-negativity)

(b) $\forall f, g \in \mathscr{R}\Big([a,b]\Big), \langle f, g \rangle = \langle g, f \rangle$ (Symmetry)

(c) $\forall f, g, h \in \mathscr{R}\Big([a,b]\Big), \forall c \in \mathbb{R}, \langle cf + g, h \rangle = c\langle f, h \rangle + \langle g, h \rangle$ (Linearity in first argument)

This make $\langle \cdot, \cdot \rangle$ a **positive semi-definite Hermitian form**. We shall prove Cauchy-Schwarz Inequality hold for positive semi-definite Hermitian form. That is, we shall prove

$$\forall f, g \in \mathscr{R}\Big([a,b]\Big), \langle f, g \rangle \leq \|f\| \cdot \|g\|$$

Proof. ∎

Theorem 3.2.2. (Application) Given $f \in \mathscr{R}\Big([a,b]\Big)$ such that

(a) $f(a) = 0 = f(b)$

(b) $\int_a^b f^2(x)dx = 1$

(c) f is continuously differentiable on (a,b)

(d) $f' \in \mathscr{R}\Big([a,b]\Big)$

We have

$$\int_a^b xf(x)f'(x) = \frac{-1}{2}$$

and have

$$\int_a^b \big(f'(x)\big)^2 dx \cdot \int_a^b \big(xf(x)\big)^2 dx > \frac{1}{4}$$

Proof. Notice that

$$\frac{d}{dx}xf^2(x) = f^2(x) + 2xf(x)f'(x)$$

Then by Integral by Part (We have to check $\big(xf^2(x)\big)'(t) = f^2(t) + 2tf(t)f'(t)$ for all $t \in (a,b)$, and we have to check $xf^2(x)$ is continuous on $[a,b]$), we have

$$1 = \int_a^b f^2(x)dx = xf^2(x)\Big|_a^b - \int_a^b 2xf(x)f'(x)dx$$

Then because $f(b) = f(a) = 0$, we see

$$2\int_a^b xf(x)f'(x)dx = -1$$

We wish to show

$$\|f'\|^2 \cdot \|xf(x)\|^2 > \frac{1}{4} = \Big(\langle f', xf(x)\rangle\Big)^2$$

It is clear that $\geq$ is valid from Cauchy-Schwarz Inequality. We have to prove $\neq$. In other words, we have to prove

$$f' \text{ and } xf(x) \text{ are linearly independent}$$

Assume f' and $xf(x)$ are linearly dependent. Then

$$\exists c \in \mathbb{R}, \forall x \in [a,b], f'(x) = cxf(x)$$

The solution for this first order linear homogeneous ODE is

$$f(x) = Ae^{\frac{cx^2}{2}} \text{ where } A \in \mathbb{R} \text{ depends on } f(a) \text{ and } f(b)$$

Then because $f(a) = f(b) = 0$, we see $A = 0$. Then $\int_a^b f^2(x)dx = 0$ CaC ■

Theorem 3.2.3. (Example) Given $G, g, \alpha : [a,b] \to \mathbb{R}$, suppose

(a) $G'(x) = g(x)$ for all $x \in (a,b)$ (G is differentiable on (a,b))

(b) G is continuous on $[a,b]$

(c) α increase on $[a,b]$

(d) g is properly Riemann-Integrable on $[a,b]$

Prove

$$\int_a^b \alpha(x)g(x)dx = \alpha G\Big|_a^b - \int_a^b G(x)d\alpha$$

Proof. ■

Chapter 4

Multi-Variable Calculus

4.1