Representation Theory of finite groups

Assignment Set 1

Due Day: Mar 8 (or Mar 11 if you are using latex)

Problem A. In this problem, we establish another presentation for the dihedral group by two non-commuting reflections. Let $n \geq 3$ and let F_n be the abstract group defined by

$$F_n = \left\langle\ \alpha, \beta \,|\, \alpha^2 = \beta^2 = 1, \quad (\alpha\beta)^n = 1 \right\rangle$$

Show that $F_n \cong D_n$, the dihedral group of order $2n$.
(Hint: Construct first a homomorphism from F_n to D_n by the presentation, then explain that it is surjective. Comparing the orders of $|F_n|$ and $|D_n|$ shows that they are isomorphic. You may also use the presentation of D_n given in our lecture.)

Problem B. Recall the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, where i, j, k are the corresponding elements in the quaternion algebra $\mathbb{H}$, is a non-abelian group of order 8 by multiplication in $\mathbb{H}$. Show that the abstract group $P =< a, b \,|\, a^4 = b^4 = 1, a^2 = b^2, ba = a^{-1}b >$ is isomorphic to Q_8; that is, the above gives a presentation of Q_8.
(Hint: Construct a homomorphism from P to Q_8 by the presentation and then show that P has exactly 8 elements.)

Problem C. In this problem, we realize D_n as a matrix group. Consider $M_2(\mathbb{Z}_n)$, the matrix ring with entries in $\mathbb{Z}_n$, which is exactly the same matrix ring except that all operations in those entries are modulo n now. For $n \geq 3$, consider the following set

$$\tilde{D}_n = \left\{ \begin{pmatrix} \pm 1 & k \\ 0 & 1 \end{pmatrix} \mid k \in \mathbb{Z}_n \right\}$$

Note that elements in $\tilde{D}_n$ are all invertible (with determinants ± 1) and they form a group by matrix multiplication. Show that $\tilde{D}_n \cong D_n$.
(Hint: Find a matrix to play the role of a (=rotation) and a matrix to play the role of b (=reflection) in our presentation of D_n.)