**Question 24**

Let $A \subseteq \mathbb{R}^n$ be an open set and $B \subset \mathbb{R}^n$ be any set. Then the set

$$A + B \equiv \{a + b : a \in A \text{ and } b \in B\}$$

is open.

*Proof.* Notice

$$A + B = \bigcup \{\{a + b : a \in A\} : b \in B\} \tag{6.19}$$

We only have to prove for all $b \in B$, the set $A + b := \{a + b : a \in A\}$ is open.

Fix $b$. Arbitrarily pick $a + b \in A + b$. Because $A$ is open, we know there exists $r \in \mathbb{R}^+$ such that

$$B_r(a) \subseteq A \tag{6.20}$$

Let

$$B_r(a) + b := \{x + b : x \in B_r(a)\} \tag{6.21}$$

We now prove

$$B_r(a) + b = B_r(a + b) \tag{6.22}$$

Arbitrarily pick $x + b \in B_r(a) + b$. We have

$$|(x + b) - (a + b)| = |x - a| < r \tag{6.23}$$

We have proved $B_r(a) + b \subseteq B_r(a + b)$. Arbitrarily pick $y \in B_r(a + b)$. Let $z = y - b$. We have

$$y = z + b \text{ and } |z - a| = |y - (a + b)| < r \tag{6.24}$$

The latter tell us $z \in B_r(a)$, so we have

$$y = z + b \in B_r(a) + b \tag{6.25}$$

Because $y$ is arbitrarily picked from $B_r(a+b)$, we have proved $B_r(a+b) \subseteq B_r(a)+b$ (done)

Notice that $r$ is selected to satisfy

$$B_r(a) \subseteq A \tag{6.26}$$

and it is clear that

$$B_r(a) + b \subseteq A + b \tag{6.27}$$

So we have

$$B_r(a + b) = B_r(a) + b \subseteq A + b \tag{6.28}$$

Notice $a + b$ is arbitrarily picked from $A + b$. Our proof is done (done) ■

## Question 25

Show that every open set in $\mathbb{R}$ is the union of at most countable collection of disjoint open intervals.

*Proof.* Because $\mathbb{Q}$ is countable and dense in $\mathbb{R}$, so in Theorem 2.9.4, we have proved the set

$$\mathcal{O} := \{B_r(p) : p \in \mathbb{Q} \text{ and } r \in \mathbb{Q}^+\} \tag{6.29}$$

is a countable base. Notice that every open ball $B_r(p)$ can be expressed as $(p-r, p+r)$, so we know $\mathcal{O}$ is a countable collection of open interval.

Let $A$ be an open set. We wish to prove $A$ can be expressed as union of a countable collection of disjoint open interval.

Because $\mathcal{O}$ is a base, for each $a \in A$, we can find $B_{r_a}(p_a)$, such that

$$a \in B_{r_a}(p_a) \subseteq A \tag{6.30}$$

Collect all such open ball for each points in $A$ and call this collection $\mathcal{O}'$.

Define a relation $\sim$ on $\mathcal{O}'$ by

$$(a,b) \sim (c,d) \text{ if } \exists S \in \mathcal{P}(\mathcal{O}') \text{ such that} \tag{6.31}$$

$$\bigcup S \text{ is an open interval and } (a,b) \subseteq \bigcup S \text{ and } (c,d) \subseteq \bigcup S \tag{6.32}$$

We wish to prove $\sim$ is an equivalence relation.

To show $(a_1,b_1) \sim (a_1,b_1)$, use $S = \{(a_1,b_1)\}$. To show $(a_1,b_1) \sim (a_2,b_2) \implies (a_2,b_2) \sim (a_1,b_1)$, use the same $S$. It is left to prove

$$(a_1,b_1) \sim (a_2,b_2) \text{ and } (a_2,b_2) \sim (a_3,b_3) \implies (a_1,b_1) \sim (a_3,b_3) \tag{6.33}$$

Let $S_1 \in \mathcal{P}(\mathcal{O}')$ be from $(a_1,b_1) \sim (a_2,b_2)$, and let $S_2 \in \mathcal{P}(\mathcal{O}')$ be from $(a_2,b_2) \sim (a_3,b_3)$.

Define

$$S_3 := S_1 \cup S_2 \tag{6.34}$$

Because $(a_2,b_2) \subseteq \bigcup S_1 \cap \bigcup S_2$, we know $\bigcup S_1 \cap \bigcup S_2$ are not disjoint. Then because union of two intersecting open interval is again an open interval, we know $\bigcup S_3$ is an open interval.

Deduce

$$(a_1,b_1) \subseteq \bigcup S_1 \subseteq \bigcup S_3 \tag{6.35}$$

and deduce

$$(a_3, b_3) \subseteq \bigcup S_2 \subseteq \bigcup S_3 \text{ (done)} \tag{6.36}$$

Let $E$ be the collection of union of each equivalent class of $\sim$ on $\mathcal{O}'$. Because $\mathcal{O}'$ is countable, we know $E$ is countable. Every element of $\mathcal{O}'$ is a subset of $A$ by definition, so we know $\bigcup E = \bigcup \mathcal{O}' \subseteq A$. Every element of $A$ is in some $O \in \mathcal{O}'$ by definition, so we have $A \subseteq \bigcup \mathcal{O}' = \bigcup E$. It is only left to prove each member in $E$ is an open interval and each two members of $E$ are disjoint.

We first prove disjoint.

Let $B, C \in E$ and let $B', C'$ be equivalent class that satisfy

$$B = \bigcup B' \text{ and } C = \bigcup C' \tag{6.37}$$

Assume $B, C$ is not disjoint, say, $x \in B \cap C$, we have

$$\exists (a, b) \in B', x \in (a, b) \subseteq B \tag{6.38}$$

and have

$$\exists (c, d) \in C', x \in (c, d) \subseteq C \tag{6.39}$$

We then can see $(a, b)$ and $(c, d)$ intersect, then we can use $S = \{(a, b), (c, d)\}$ to show $(a, b) \sim (c, d)$ CaC (done) .

We now prove open interval.

Let $B \in E$, and let $B'$ be equivalent class that satisfy $B = \bigcup B'$. Because $B'$ contain only open sets, we know $B$ is open. Then we know $\sup B$ and $\inf B$ if exists, is not in $B$. We only have to prove every point in $(\inf B, \sup B)$ is in $B$, where infimum and supremum can be negative or positive infinity. Notice that after we prove such, the proof is done, since $(\inf B, \sup B) \subseteq B$ is trivial.

Assume $\exists x \in (\inf B, \sup B), x \notin B$. We have

$$\exists (y_1, z_1) \in B, (y_1, z_1) \subseteq (\inf B, x) \text{ and } \exists (y_2, z_2) \in B, (y_2, z_2) \subseteq (x, \sup B) \tag{6.40}$$

Because $(y_1, z_1) \sim (y_2, z_2)$ there exists $S \in \mathcal{O}'$ such that $(y_1, z_2) \subseteq \bigcup S$ and $\bigcup S$ is an open interval. If $S$ contain any element $(v, w)$ not in $B$, we can use $S$ to show $(v, w) \sim (y_2, z_2)$, causing a contradiction. We have proved $S$ is a subset of $B'$.

Now, Because $x \in \bigcup S$, we know there must exists some interval in $S \subseteq B'$ containing $x$ CaC to $x \notin B$ (done) (done) ■

## Question 26

Let $A \subseteq B \subseteq \mathbb{R}$. Suppose that $A$ is a dense subset of $B$.
1. Prove that $B \subseteq \overline{A}$.
2. If $B$ is closed, determine whether $B = \overline{A}$.

*Proof.* $A$ is a dense subset of $B$ means that every point $b \in B \setminus A$ is a limit point of $A$ in the scope of $B$. Notice that $b$ is a limit point of $A$ in the scope of $B$ also means $b$ is a limit point in the scope of $\mathbb{R}$. Then we have proved $B \setminus A \subseteq A'$. It follows

$$B = A \cup (B \setminus A) \subseteq A \cup A' = \overline{A} \tag{6.41}$$

If $B$ is closed, we have

$$\overline{A} \subseteq \overline{B} = B \tag{6.42}$$

Then because $B \subseteq \overline{A}$, we have $B = \overline{A}$

■

**Definition 0.1.** A metric space $X$ is *sequentially compact* if every sequence of points in $X$ has a convergent sub-sequence converging to a point in $X$.

## Question 27

Let $A$ and $B$ be subsets of a metric space $(M, d)$ and denote $\text{cl}(A) = \overline{A}$. Show that
1. $\text{cl}(\text{cl}(A)) = \text{cl}(A)$.
2. $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$.
3. $\text{cl}(A \cap B) \subseteq \text{cl}(A) \cap \text{cl}(B)$. Find an example such that $\text{cl}(A \cap B) \subsetneq \text{cl}(A) \cap \text{cl}(B)$.

*Proof.* Notice that closure is the smallest closed set containing the set.

Because $\overline{A}$ is a closed set, we know the smallest closed set containing $\overline{A}$ is $\overline{A}$ itself.

Deduce

$$A \subseteq A \cup B \implies \overline{A} \subseteq \overline{A \cup B} \tag{6.43}$$

and deduce

$$B \subseteq A \cup B \implies \overline{B} \subseteq \overline{A \cup B} \tag{6.44}$$

so deduce

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cup B} \tag{6.45}$$

Notice that $\overline{A} \cup \overline{B}$ is a closed set containing both $A$ and $B$, thus containing $A \cup B$, so because $\overline{A \cup B}$ by definition is the smallest closed set containing $A \cup B$, we have

$$\overline{A \cup B} \subseteq \overline{A} \cup \overline{B} \tag{6.46}$$

Notice that $\overline{A}$ and $\overline{B}$ are both closed set containing $A \cap B$. Then because $\overline{A \cap B}$ is the smallest closed set containing $A \cap B$. We have

$$\overline{A \cap B} \subseteq \overline{A} \text{ and } \overline{A \cap B} \subseteq \overline{B} \tag{6.47}$$

Then have

$$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B} \tag{6.48}$$

Let $A = (0, 1)$ and $B = (1, 2)$. We have

$$\overline{A \cap B} = \overline{\varnothing} = \varnothing \text{ and } \overline{A} \cap \overline{B} = [0, 1] \cap [1, 2] = \{1\} \tag{6.49}$$

■

**Question 28**

Let $K$ be a sequentially compact set in a metric space $(M, d)$ and let $F \subseteq K$ be closed. Prove that $F$ is sequentially compact.

*Proof.* Let $\{x_i\}_{i=1}^{\infty}$ be a sequence in $F$

$$\{x_i\}_{i=1}^{\infty} \subseteq F \subseteq K \tag{6.50}$$

Because $K$ is sequentially compact, we know there exists a sub-sequence $\{x_{n_i}\}_{i=1}^{\infty}$ converge to some point $a$

$$\lim_{i \to \infty} \{x_{n_i}\} = a \tag{6.51}$$

We only have to prove $a$ is in $F$.

Assume $a \notin F$. Because $F$ is closed, we know $F^c$ is open. Then $a \in F^c$ tell us there exists an open ball $B_r(a)$ contain no point in $F$, thus disjoint to $\{x_{n_i}\}$ CaC

■

**Definition 0.2.** The discrete metric $d$ on a set $X$ is defined by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

for any $x, y \in X$. In this case $(X, d)$ is called a discrete metric space.

**Question 29**

Let $(M, d)$ be a metric space with discrete metric $d$. Prove that every compact set in $M$ is finite.

*Proof.* Notice we have

$$\forall p \in M, B_1(p) = \{p\} \tag{6.52}$$

So we know no set in $M$ has a limit point. Let $K \subseteq M$ be compact. By Theorem 2.7.5, we know $K$ is limit point compact. Assume $K$ is infinite. Then $K$ has a limit point CaC . ■

**Question 30**

Let $\{x_n\}_{n=1}^\infty \subseteq \mathbb{R}$ be a convergent sequence with the limit $x$. Prove that the set $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact.

*Proof.* Let

$$E \subseteq \{x_n : n \in \mathbb{N}\} \cup \{x\} \tag{6.53}$$

If $E$ is infinite, we can regard $E \setminus \{x\}$ as a sub-sequence of $\{x_n\}$. Then because $\{x_n\}_{n=1}^\infty$ converge to $x$, we know $E \setminus \{x\}$ converge to $x$. Then we know $x$ is a limit point of $E \setminus \{x\}$ thus a limit point of $E$. We have proved every infinite subset of $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ has a limit point $x$ in $\{x_n : n \in \mathbb{N}\} \cup \{x\}$. By Theorem 2.7.5, $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact. ■

**Question 31**

1. Let $A$ and $B$ be two subsets of a metric space $(M, d)$. The distance between $A$ and $B$ is defined by

$$d(A, B) \equiv \inf\{d(x, y) : x \in A, y \in B\}.$$

   (a) Give an example of two disjoint, nonempty, closed sets $A$ and $B$ in $\mathbb{R}^n$ for which $d(A, B) = 0$.

   (b) Let $A, B$ be nonempty sets in $\mathbb{R}^n$ with $A$ closed and $B$ compact. Show that there are points $a \in A$ and $b \in B$ such that $d(a, b) = |a - b|$. Deduce that $d(A, B)$ is positive if such $A, B$ are disjoint.

*Proof.* Let $n = 1$. Let

$$A = \{a_k = k - 1 + \sum_{i=1}^{k-1} 2^{-i-1} : k \in \mathbb{N}\} \tag{6.54}$$

Let

$$B = \{b_k = k - \sum_{i=1}^{k-1} 2^{-i-1} : u \in \mathbb{N}\} \tag{6.55}$$

Notice the pattern

$$a_1 = 0 < \frac{1}{2} < 1 = b_1 \tag{6.56}$$

$$a_2 = 1 + \frac{1}{4} < 1 + \frac{1}{2} < 2 - \frac{1}{4} = b_2 \tag{6.57}$$

$$a_3 = 2 + \frac{1}{4} + \frac{1}{8} < 2 + \frac{1}{2} < 3 - \frac{1}{4} - \frac{1}{8} = b_3 \tag{6.58}$$

$$\vdots \tag{6.59}$$

and compute

$$b_k - a_k = 2^{1-k} \tag{6.60}$$

First observe for each non-negative integer $m$m the interval $[m, m+1]$ contain exactly one point from $A$ and contain exactly one point from $B$. To see $A$ and $B$ are disjoint, observe that because $A, B$ contain only non-negative numbers, so every point is in $[m, m+1]$ for some non-negative integer $m$, and we know each interval only contain $a_{m+1}$ from $A$ and $b_{m+1}$ from $B$ where $b_{m+1} - a_{m+1} = 2^{-m}$.

We now show $A, B$ are both closed. For negative $-x \in \mathbb{R}$, observe $B_x(-x)$ intersect with neither $A$ nor $B$. We know $B_1(0)$ does not intersect with $B$ and $0 \in A$. For positive $x$ not in $A$, let $m = \lfloor x \rfloor$. We know point closest to $x$ is either $a_m, a_{m+1}$ or $a_{m+2}$. Then we can let $r = \min\{|x - a_m|, |x - a_{m+1}|, |x - a_{m+2}|\}$, and see $B_r(x)$ does not intersect with $A$.For positive $x$ not in $B$, let $m = \lfloor x \rfloor$. We know point closest to $x$ is either $b_m, b_{m+1}$ or $b_{m+2}$. Then we can let $r = \min\{|x-b_m|, |x-b_{m+1}|, |x-b_{m+2}|\}$, and see $B_r(x)$ does not intersect with $B$.

Lastly, to prove $d(A, B) = 0$ is to prove for each positive real $r$, there exists a pair of $a, b$ such that $|a - b| < r$. Observe

$$b_k - a_k = 2^{1-k} < r \iff 1 - k < \log_2 r \iff k > 1 - \log_2 r \tag{6.61}$$

And we are done, by picking great enough $k$.

We first prove the triangle inequality

$$d(X, Z) \leq d(X, Y) + d(Y, Z) \tag{6.62}$$

Assume $d(X, Z) > d(X, Y) + d(Y, Z)$. We have

$$d(X, Z) - d(X, Y) > d(Y, Z) = \inf\{d(y, z) : y \in Y, z \in Z\} \tag{6.63}$$

Then we know there exists $y' \in Y, z' \in Z$ such that $d(y', z') < d(X, Z) - d(X, Y)$. Then we have

$$\inf\{d(x, y) : x \in X, y \in Y\} = d(X, Y) < d(X, Z) - d(y', z') \tag{6.64}$$

Then we know there exists $x' \in X, y'' \in Y$ such that

$$d(x', y'') < d(X, Z) - d(y', z') \tag{6.65}$$

in other words

$$d(x', y'') + d(y', z') < d(X, Z) \tag{6.66}$$

Notice that

$$d(x', z') \leq \min\{d(x', y'') + d(y'', z'), d(x', y') + d(y', z')\} \leq d(x', y'') + d(y', z') \text{ CaC (done)} \tag{6.67}$$

Now we do (b).

Notice

$$d(A, B) = \inf \bigcup \{\{d(a, b) : a \in A\} : b \in B\} \tag{6.68}$$

and notice

$$d(A, B) = \inf\{\inf\{d(a, b) : a \in A\} : b \in B\} = \inf\{d(A, \{b\}) : b \in B\} \tag{6.69}$$

We first prove if $d(A, B) = 0$, then $A$ and $B$ intersect.

If $B$ is finite, then $0 = d(A, B) = \min\{d(A, \{b\} : b \in B)\}$, so we know there exists $b \in B$ such that $0 = d(A, \{b\}) = \inf\{d(a, b) : a \in A\}$. Then we can deduce every open ball $B_r(b)$ contain a point $a \in A$, since there exists $a$ such that $d(a, b) < r$. We have proved $b$ is a limit point of $A$, then because $A$ is closed, we know $b \in A$.

If $B$ is infinite, then $0 = \inf\{d(A, \{b\}) : b \in B\}$. If there exists $b$ such that $d(A, \{b\}) = 0$, we can use the same argument above to finish the proof. We only have to consider when $\forall b \in B, d(A, \{b\}) > 0$. For each $n$, we then know there exists $b_n$ such that $d(A, \{b_n\}) < \frac{1}{n}$.

Construct an infinite sequence $\{b_n\}_{n \in \mathbb{N}}$ by picking $b_n$ such that $d(A, \{b_n\}) < \frac{1}{n}$. We can see our method for picking $b_n$ will give us an infinite sequence as every $b$ satisfy $d(A, \{b\}) > 0$. By Theorem 2.7.5, we know $B$ is limit point compact, so we know $\{b_n\}_{n \in \mathbb{N}}$ has a limit point in $B$.

Denote the limit point for $\{b_n\}_{n \in \mathbb{N}}$ as $p$. We now show $p \in A$. Assume $p \in A^c$. Because $A$ is closed, we know there exists an open ball $B_r(p)$ disjoint with $A$. Then we know $d(A, \{p\}) \geq r$.

Then from

$$\forall b \in B, r \leq d(A, \{p\}) \leq d(A, \{b\}) + d(\{b\}, \{p\}) \tag{6.70}$$

we have

$$\forall b \in B[d(A,\{b\}) < \frac{r}{2} \implies d(\{b\},\{p\}) > \frac{r}{2}] \tag{6.71}$$

Then we see if $n > \frac{2}{r}$, then $b_n$ is not contained by the open ball $B_{\frac{r}{2}}(p)$. In other words, the open ball $B_{\frac{r}{2}}(p)$ that center a limit point $p$ contain at most $\lfloor \frac{2}{r} \rfloor$ amount of point in $\{b_n\}_{n\in\mathbb{N}}$ CaC .

Lastly, we prove there exists $a \in A, b \in B$ such that $d(A,B) = d(a,b)$. If $A$ and $B$ are not disjoint, the fact $\exists a \in A, \exists b \in B, d(A,B) = |a-b|$ is clear by picking $a = b$. We only have to consider when $A$ and $B$ are disjoint.

Let $d(A,B) = u$. We know $u > 0$, because of the result we have proved.

We have

$$\inf\{d(A,\{b\}) : b \in B\} = d(A,B) = u \tag{6.72}$$

At this stage, we wish to prove $\exists b \in B$ such that $d(A,\{b\}) = u$. Assume there does not exist such $b$

We know, for each $n \in \mathbb{N}$, there exists a point $b_n$ such that

$$d(A,\{b_n\}) < u + \frac{1}{n} \tag{6.73}$$

Because $\forall b \in B, d(A,\{b\}) > u$, we know if we collect one $b_n$ that satisfy $d(A,\{b_n\}) < u + \frac{1}{n}$ for each $n \in \mathbb{N}$, the sequence $\{b_n\}_{n\in\mathbb{N}}$ is infinite.

Then by Theorem 2.7.5, we know there exists a limit point $p \in B$ for $\{b_n\}_n \in \mathbb{N}$.

We know $d(A,\{p\}) > u$ by our assumption.

Let $m = d(A,\{p\}) > u$. Then from

$$\forall b \in B, m = d(A,\{p\}) < d(A,\{b\}) + d(b,p) \tag{6.74}$$

we have

$$\forall b \in B, d(A,\{b\}) < u + \frac{1}{n} \implies d(b,p) > m - u - \frac{1}{n} \tag{6.75}$$

Let $k$ be a natural such that $u + \frac{1}{k} < m$. For each natural $t$ greater than $k$, we have

$$d(b_t,p) > m - u - \frac{1}{k} \tag{6.76}$$

In other words, the open ball $B_{m-u-\frac{1}{k}}(p)$ centering a limit point contain at most $k$ amount of point in $\{b_n\}_{n\in\mathbb{N}}$ CaC (done)

Let $b \in B$ satisfy $d(A,\{b\}) = u$. At the final stage, we wish to prove $\exists a \in A, d(a,b) = u$.

Assume $\forall a \in A, d(a,b) > u$. We know, for each $n \in \mathbb{N}$, there exists a point $a_n$ such that

$$d(a_n, b) < u + \frac{1}{n} \tag{6.77}$$

Because $\forall a \in A, d(a,b) > u$, we know if we we collect one $a_n$ that satisfy $d(a_n,b) < u + \frac{1}{n}$ for each $n \in \mathbb{N}$, the sequence $\{a_n\}_{n\in\mathbb{N}}$ is infinite.

Notice that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is also bounded as one can check $B_{2(u+1)}(a_1)$ contain all $\{a_n\}_{n\in\mathbb{N}}$, using $b$ as a "pivot".

Then by Corollary 2.9.5, we know $\{a_n\}_{n\in\mathbb{N}}$ is compact, then by Theorem 2.7.5, we know $\{a_n\}_{n\in\mathbb{N}}$ has a limit point in itself.

Let $a_k$ be a limit point of $\{a_n\}_{n\in\mathbb{N}}$.

Let $m = d(a_k, b) > u$. Observe that

$$\forall a_n, m = d(a_k, b) < d(b, a_n) + d(a_n, a_k) \tag{6.78}$$

give us

$$\forall a_n, d(a_n, b) < u + \frac{1}{n} \implies d(a_n, a_k) > m - u - \frac{1}{n} \tag{6.79}$$

Let $s \in \mathbb{N}$ be great enough so that $m - u - \frac{1}{s} > 0$. Then we see for each natural $w$ greater than $s$, we have

$$d(a_w, b) < u + \frac{1}{s} \tag{6.80}$$

so we have

$$d(a_w, a_k) > m - u - \frac{1}{s} \tag{6.81}$$

Then we see the open ball $B_{m-u-\frac{1}{s}}(a_k)$ contain at most $s$ amount of point in $\{a_n\}_{n\in\mathbb{N}}$ CaC (done)

$\blacksquare$

### Question 32

9. Let $(M, d)$ be a metric space.
   (a) Show that the union of a finite number of compact subsets of $M$ is compact.
   (b) Show that the intersection of an arbitrary collection of compact subsets of $M$ is compact.

*Proof.* Let $\mathcal{K} = \{K_1, \ldots, K_n\}$ be a finite collection of compact subset of $M$. We wish to show $\bigcup \mathcal{K}$ is compact. Let $\mathcal{G}$ be an open cover for $\mathcal{K}$. For each $m \in \mathbb{N} : 1 \le m \le n$, because $K_m \subseteq \bigcup \mathcal{K}$, we know $\mathcal{G}$ is also an open cover for $K_m$. Then, we can pick a finite sub-cover $\mathcal{G}_m \subseteq \mathcal{G}$ for $K_m$. Collect all such finite sub-cover $\mathcal{G}_1, \ldots, \mathcal{G}_m$. The union $\bigcup\{\mathcal{G}_1, \ldots, \mathcal{G}_m\}$ is a finite open cover for $\bigcup \mathcal{K}$.

Notice that a compact set must be closed, and arbitrary collection of closed sets are closed. Then, we know the intersection of an arbitrary collection of compact set must be a closed set and a subset of every compact set in the collection. Then because closed subset of compact set is compact, we know the intersection is compact. ■

### Question 33

10. A metric space $(M, d)$ is said to be *separable* if there is a countable subset $A$ which is dense in $M$. Show that every compact set is separable.

*Proof.* See the proof in Theorem 2.7.5, there is a summary where you can find the numbering of rigorous proof for each step. Some proof for corollary is omitted because they are immediate consequences of theorem before.

Because I use a program written by artificial intelligence to update my reference in latex code whenever I edit my latex code, some reference may be directed to a wrong theorem, but all the tools are there in Chapter 2. ■

## 2.6 General Topology: Three Non-Equivalent Compactness and Separable

In this section, we will first give two equivalent definition of base and prove they are equivalent.

**Definition 2.6.1. (Definition of Base)** We say a sub-collection $\mathcal{O}$ of topology $\tau$ on $X$ is a base if every open neighborhood $O$ around $x$ contain an open set $U$ in $\mathcal{O}$ containing $U$. That is

$$\forall x \in X, \forall O \in \tau : x \in O, \exists U \in \mathcal{O}, x \in U \subseteq O \tag{2.119}$$

We call open sets in a base basic open sets.

**Theorem 2.6.2. (Equivalent Definition of Base)** A sub-collection of topology $\tau$ is a base if and only if every open set can be expressed as a union of a set of basic open sets. That is, the necessary and sufficient condition for $\mathcal{O}$ to be a base is

$$\forall O \in \tau, \exists \mathcal{U} \subseteq \mathcal{O}, O = \bigcup \mathcal{U} \tag{2.120}$$

*Proof.* From left to right, for each point $x \in O$, we know there exists $U_x \in \mathcal{O}$ such that $x \in U_x \subseteq O$. Collect $U_x$ for each $x \in O$ as $\mathcal{U}$, and we are partially done.

From right to left, let $O$ be an open neighborhood around $x$. If there exists no $U \in \mathcal{O}$ such that $x \in U \subseteq O$, then every union of sub-collection $\mathcal{U} \subseteq \mathcal{O}$ that contain $x$ contain some points not in $O$, which is impossible. ■

Then, we will define separable, and give a sufficient condition of separable.

**Definition 2.6.3. (Definition of Separable)** We say $X$ is separable if

$$X \text{ has a countable dense subset} \tag{2.121}$$

**Theorem 2.6.4. (Base and Dense Set)**

$$\text{A set intersecting with all basic open set is dense} \tag{2.122}$$

*Proof.* For each point $p$ not in $A$ and for each open neighborhood $N_p$ around $p$, there exists an basic open set $O_p$ contained by $N_p$. The fact $O_p$ intersect $A$ implies $N_p$ intersect with $A$. We have proved every point $p$ not in $A$ is a limit point of $A$. ■

**Corollary 2.6.5. (Countable Base Implies Separable)**

$$X \text{ has a countable base } \implies X \text{ is separable} \tag{2.123}$$

*Proof.* Let $\mathcal{O}$ be a countable base for $X$. For each $O \in \mathcal{O}$, we can collect a point $x_O \in O$, and have $A := \{x_O : O \in \mathcal{O}\}$. Because $\mathcal{O}$ is countable, we know $A$ is countable, and by Theorem 2.6.4, we know $A$ is dense. ∎

The above let us draw a Venn diagram, where existence of countable base is a small circle in the big circle separable.

Now, we give definition of three different compactness.

**Definition 2.6.6. (Definition of Open Cover)** We say a collection $\{G_\alpha\}$ of open sets is an open cover of $E$ if

$$E \subseteq \bigcup \{G_\alpha\} \tag{2.124}$$

**Definition 2.6.7. (Definition of Compact Space)** We say a topological space $X$ is compact if

$$\text{every open cover for } X \text{ has a finite open sub-cover} \tag{2.125}$$

**Definition 2.6.8. (Definition of Countably Compact Space)** We say a topological space $X$ is countably compact if

$$\text{every countable open cover for } X \text{ has a finite open sub-cover} \tag{2.126}$$

**Definition 2.6.9. (Definition of Limit Point Compact Space)** We say a topological space $X$ is limit point compact if

$$\text{every infinite subset of } X \text{ has a limit point} \tag{2.127}$$

It is clear that compact implies countably compact, as every countable cover in compact space has a finite sub-cover. We now explore the relation of the three compactness.

**Theorem 2.6.10. (Compact Implies Countably Compact)**

$$X \text{ is compact } \implies X \text{ is countably compact} \tag{2.128}$$

**Theorem 2.6.11. (Countably Compact Implies Limit Point Compact)**

$$X \text{ is countably compact } \implies X \text{ is limit point compact} \tag{2.129}$$

*Proof.* Arbitrarily pick an infinite $A$. We know $A$ is either countable or uncountable. We first prove if $A$ is countably infinite, then $A$ has a limit point.

Assume $A$ has no limit point. For each $a \in A$, because $a$ is not a limit point of $A$, we know there exists an open neighborhood $O_a$ around $a$ such that $O_a \cap A = \{a\}$. Notice that $A$ has no limit point let us deduce

$$A' = \varnothing \implies \overline{A} = A \implies A^c \text{ is open} \tag{2.130}$$

For each $a \in A$, collect one $O_a$ into the collection $\mathcal{O} = \{O_a : a \in A\}$.

Then, because $A$ is countable, we see $\mathcal{O} \cup \{A^c\}$ is an countable cover for $X$, and every finite open sub-cover contain only finite amount of point of $A$ when $A$ is infinite. CaC (done)

If $A$ is uncountable, then $A$ contain a countable set $B$. We know $B$ has a limit point $p$, and we know $p$ is also a limit point for $A$. ∎

**Corollary 2.6.12. (Compact Implies Limit Point Compact)** If $X$ is compact, then $X$ is limit point compact.

Lastly, we give a theorem involving countable base.

**Theorem 2.6.13. (Existence of Countable Base Implies Existence of Countable Sub-Cover)**

$$X \text{ has a countable base} \implies \text{every open cover for } X \text{ has a countable sub-cover} \tag{2.131}$$

*Proof.* Let $\mathcal{O}$ be a countable base for $X$ and let $\mathcal{G}$ be an open cover of $X$. We wish to find an countable open sub-cover of $\mathcal{G}$.

Because $\mathcal{O}$ is a base, for each $G \in \mathcal{G}$, there exists basic open set $O \in \mathcal{O}$ contained by $G$. Then because $\mathcal{G}$ is an open cover, for each $p \in X$, there exists $G_p, O_p$ such that

$$p \in O_p \subseteq G_p \tag{2.132}$$

Collect all $G_p$ from above so we have a sub-cover $\mathcal{G}'$. Because $\mathcal{O}$ is countable, we know $\mathcal{G}'$ is also countable. ∎

Now, if we want to draw a conclusion for this section by drawing a Venn diagram, we should first draw compact inside countably compact inside limit point compact.

## 2.7 Metric Space: Compact, Countably Compact and Limit Point Compact

There are three tools we need to prove the MEGA TFAE.

**Theorem 2.7.1. (In Metric Space, Existence of Countable Base is Equivalent to Separable)** Let $(X, d)$ be a metric space.

$$X \text{ has a countable base} \iff X \text{ is separable} \tag{2.133}$$

*Proof.* Because by Corollary 2.6.5, $X$ is separable if $X$ has a countable base, we only have to prove separable implies existence of countable base. Let $X$ be separable. Then we have a countable dense subset $E$ of $X$. We now prove

$$\mathcal{O} := \{B_r(p) : p \in E \text{ and } r \in \mathbb{Q}\} \tag{2.134}$$

is a countable base.

Notice $\mathcal{O} = \bigcup\{\{B_r(p) : r \in \mathbb{Q}\} : p \in E\}$ is a countable union of countable sets, so $\mathcal{O}$ is countable. We only have to prove $\mathcal{O}$ is a base.

For each open neighborhood $O$ around $x \in X$, by definition of open in metric space, we know there exists an open ball $B_r(x)$ contained by $O$. Because $E$ is dense, we know $x$ either in $E$ or in $E'$. If $x$ is in $E$, we can pick a positive rational $n$ smaller than $r$, so we have

$$x \in B_n(x) \subseteq B_r(x) \subseteq O \text{ and } B_n(x) \in \mathcal{O} \tag{2.135}$$

and we are done. If $x \in E'$, we know there exists a point $q$ in $E \cap B_{\frac{r}{2}}(x)$. Then we can pick a rational $n$ between $d(q,x)$ and $\frac{r}{2}$. Now, observe

$$x \in B_n(q) \subseteq B_r(x) \subseteq O \text{ and } B_n(q) \in \mathcal{O} \tag{2.136}$$

and we are done. (done) ∎

**Lemma 2.7.2. (Existence of Intuitively Small Enough Open Ball)** Let $p \in B_r(q)$. There exists an open ball $B_{r'}(p)$ small enough to be contained by $B_r(q)$ and not containing $q$

*Proof.* If $d(p,q) \leq \frac{r}{2}$, we can let $r' = d(p,q)$. Clearly $B_{r'}(p)$ doesn't contain $q$, and for each $x \in B_{r'}(p)$ we have

$$d(x,q) \leq d(x,p) + d(p,q) \leq r' + \frac{r}{2} \leq r \tag{2.137}$$

If $d(p,q) \geq \frac{r}{2}$, we can let $r' = r - d(p,q)$. Clearly $B_{r'}(p) \subseteq B_r(q)$, and we can deduce

$$2d(p,q) \geq r \implies d(p,q) \leq r - d(p,q) = r' \tag{2.138}$$

so we have $q \in B_{r'}(p)$. In short, we use the open ball $B_{\min\{d(p,q), r-d(p,q)\}}(p)$ ∎

**Theorem 2.7.3. (In Metric Space, Limit Point Compact Implies Separable)** Let $K$ be a metric space.

$$K \text{ is limit point compact} \implies K \text{ is separable} \tag{2.139}$$

*Proof.* Let $\delta > 0$ and let $a_{\delta,1} \in K$. We now prove there exists a finite set $A_\delta = \{a_{\delta,1}, a_{\delta,2}, \dots, a_{\delta,n}\}$ such that $K \subseteq \bigcup\{B_\delta(a_{\delta,i}) : a_{\delta,i} \in A_\delta\}$.

To prove such, we construct $A_\delta$ by adding element $a_{\delta,i}$ such that $\forall j < i, d(a_{\delta,j}, a_{\delta,i}) \geq \delta$ to $\{a_{\delta,1}\}$ until impossible. Assume such finite set $A_\delta$ can not be constructed using this method. In other words, we assume our method of adding points can go on infinitely. Then we have set $A^{\mathbb{N}} = \{a_{\delta,i} : i \in \mathbb{N}\}$ such that

$$\forall a_{\delta,i}, a_{\delta,j} \in A^{\mathbb{N}}, d(a_{\delta,i}, a_{\delta,j}) \geq \delta \tag{2.140}$$

Arbitrarily pick $p \in K$, we wish to find an open ball $B_r(p)$ small enough so that

$$B_r(p) \cap A^{\mathbb{N}} \subseteq \{p\} \tag{2.141}$$

Let

$$E = \bigcup\{B_\delta(a_{\delta,i}) : a_{\delta,i} \in A^{\mathbb{N}}\} \tag{2.142}$$

Notice that each ball $B_\delta(a_{\delta,i})$ must not contain any other $a_{\delta,j}$.

If $p$ is in $E$, we know $p$ is contained by some ball $B_\delta(a_{\delta,i})$. If $p = a_{\delta,i}$ then $B_\delta(p)$ is small enough. If $p \neq a_{\delta,i}$, we wish to find an open ball $B_r(p)$ small enough so that it neither contain $a_{\delta,i}$ nor any other $a_{\delta,j}$. Notice that if the ball $B_r(p)$ is contained by $B_\delta(a_{\delta,i})$, then $B_r(p)$ contain no $a_{\delta,j}$. By Lemma 2.7.2, we have found an open ball small enough to satisfy our need.

If $p$ is not in $E$, clearly $B_\delta(p)$ is small enough.

We have proved $A^{\mathbb{N}}$ has no limit point CaC to $K$ is limit point compact. (done)

Let $A_\delta$ be the finite set from violet part. Consider the collection

$$A = \bigcup\{A_1, A_{\frac{1}{2}}, A_{\frac{1}{3}}, \dots\} \tag{2.143}$$

Because $A$ is a countable union of finite sets, we know $A$ is countable. We now prove $A$ is dense.

Arbitrarily pick $p \in K \setminus A$. For each open ball $B_r(p)$, there exists $\frac{1}{n}$ small enough to be smaller than $r$ to give us $p \in B_{\frac{1}{n}}(a)$ for some $a \in A_{\frac{1}{n}}$, and give us $a \in B_{\frac{1}{n}}(p) \subseteq B_r(p)$ (done) ∎

**Theorem 2.7.4. (In Metric Space, Limit Point Compact Implies Compact)**

*Proof.* Let $\mathcal{G}$ be an open cover for $K$. By Theorem 2.7.3, we know $K$ is separable. Then by Theorem 2.7.1, we know $K$ has a countable base. Then by Theorem 2.6.13, we know $\mathcal{G}$ contain a countable sub-cover

$$\{G_i\}_{i\in\mathbb{N}} \tag{2.144}$$

We now prove there exists $m \in \mathbb{N}$ such that $\{G_i\}_{i=1}^m$ cover $K$.

For each $n \in \mathbb{N}$, denote

$$E_n = \bigcup \{G_i\}_{i=1}^n \tag{2.145}$$

Obviously we have

$$E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots \tag{2.146}$$

We know $E_n$ converge to $K$, that is

$$\bigcup \{E_i\}_{i=1}^\infty = K \tag{2.147}$$

Assume $\forall m \in \mathbb{N}, E_m \neq K$. Then we can construct an infinite sequence

$$\{x_i : x_i \in E_i^c\}_{i=1}^\infty \tag{2.148}$$

Because $K$ is limit point compact, we know $\{x_i\}_{i=1}^\infty$ has a limit point $p$.

Because $\{G_i\}_{i=1}^\infty$ is an open cover, we know $p \in G_n$ for some $n$. Then we know there exist open ball $B_r(p)$ small enough to be contained by $G_n$. Notice $B_r(p)$ is contained by $G_n$ implies $B_r(p)$ is contained by $E_n$, so we know $B_r(p)$ at best contain all $x_j$ where $j \leq n$ CaC to Theorem 2.5.6 (done) ■

| |
|---|
| We now prove the MEGA TFAE. |

**Theorem 2.7.5. (In Metric Space, Limit Point Compact is Equivalent to Compact)** For metric space $K$, the followings three are equivalent

(a) $K$ is a limit point compact space

(b) $K$ is a compact space

(c) $K$ is a countably compact space

Above three implies the following two equivalent statement ($\mathbb{R}$ is an example the above statements are stronger)

(1) $K$ is separable

(2) $K$ has a countable base

and all above implies

(i) every open cover for $K$ has a countable sub-cover.

*Proof.* In Theorem 2.6.5, we prove that if $K$ has a count base then $K$ is separable by collecting one point from each basic open set, and in Theorem 2.7.1, we construct a countable base by collecting all open balls of rational radius around each point in the countable dense subset.

In Theorem 2.7.3, we prove if $K$ is limit point compact then $K$ is separable. We proved such by covering $K$ with open ball of radius $\frac{1}{n}$ for each $n \in \mathbb{N}$, and argue the set of centers $A$ is countable by proving to cover $K$ always require only finite amount of point (which is easy to see if one know $K$ is bounded, but our method here is to show if require infinite centers, then the set of infinite centers has no limit point, which is also easy to see if one use geometric intuition), and argue that each point $p$ that isn't in $A$ is a limit point of $A$ by arguing for each distance $r$, there exists an $a \in A$ closed enough to be closer to $p$ than $r$.

One may observe, at the end of the proof for Theorem 2.7.3, we also construct a countable base for $K$.

In Theorem 2.6.13, we prove if $K$ has a countable base then every open cover for $K$ has a countable sub-cover by filtering the open cover with countable base.

Notice that we have proved that if $K$ is limit point compact, then $K$ is separable, has a countable base and every open cover for $K$ has a countable sub-cover.

Then, in Theorem 2.7.4, we proved that if $K$ is limit point compact, the countable sub-cover $\{G_i\}_{i\in\mathbb{N}}$ for $K$ must has a finite sub-cover $\{G_i\}_{i=1}^{m}$, otherwise we can construct an infinite set $\{x_i\}_{i=1}^{\infty}$ by picking a point from area that isn't covered by $\{G_i\}_{i=1}^{n}$ for each natural $n$, and show that the limit point of $\{x_i\}_{i=1}^{\infty}$, being in some $G_n$, must contain at most of $n$ amount of points of $\{x_i\}_{i=1}^{\infty}$, which is impossible.

It is clear that $K$ is compact implies $K$ is countably compact. In Theorem 2.6.11, we proved if $K$ is countably compact, then $K$ is limit point compact by showing if no point is a limit point of a countably infinite set $A$, then for each point $a \in A$, we can find an open neighborhood around $a$ small enough to contain only one point in $A$, i.e. $a$ itself, and the collection of such small enough open neighborhood and $A^c$ form an open cover that has no finite sub-cover, as any finite sub-cover contain only finite amount of point in $A$, and observe every uncountable set must contain some countable set thus also have a limit point.

In summary

$$(2) \implies (1) \text{ by Corollary 2.6.5} \tag{2.149}$$
$$(1) \implies (2) \text{ by Theorem 2.7.1} \tag{2.150}$$
$$(a) \implies (1) \text{ and } (2) \text{ by Theorem 2.7.3} \tag{2.151}$$
$$(2) \implies (i) \text{ by Theorem 2.6.13} \tag{2.152}$$
$$(a) \implies (b) \text{ by Theorem 2.7.4, which use (i)} \tag{2.153}$$
$$(b) \implies (c) \text{ by Theorem 2.6.10, which is trivial} \tag{2.154}$$
$$(c) \implies (a) \text{ by Theorem 2.6.11} \tag{2.155}$$

■

Above is a very important result. One must know how to prove such.

## 2.8 Metric Space: Some Properties Implied by Compact

Notice there is a difference between compact space and compact set

**Definition 2.8.1. (Definition of Compact Subset)** Let $X$ be a metric space, we say $K \subseteq X$ is a compact subset if

$$\text{every open cover for } K \text{ has a finite sub-cover} \tag{2.156}$$

**Theorem 2.8.2. (Compact Subset is itself a Compact Space)**

$$\text{A compact subset regarded as a metric space is compact} \tag{2.157}$$

*Proof.* Let $K$ be a compact subset, and regard $K$ as a metric space. Let $\{G_\lambda : \lambda \in \Lambda\}$ be an open cover for $K$ with respect to $K$. We know there exists a collection of open set $\{G_i : i \in I\}$ such that

$$\{G_\lambda : \lambda \in \Lambda\} = \{G_i \cap K : i \in I\} \tag{2.158}$$

Because $\{G_\lambda\}$ is an open cover for $K$ with respect to $K$, we know $\{G_i\}$ is an open cover for $K$. Then because $K$ is compact, there exists a finite sub-cover $\{G_j\}$. Notice $\{G_j \cap K\}$ is a finite sub-cover for $K$ with respect to $K$. ■

**Theorem 2.8.3. (Compact Sets Are Closed)** Let $X$ be a metric space, and let $K \subseteq X$

$$K \text{ is compact} \implies K \text{ is closed} \tag{2.159}$$