1.6 HW2

Question 13

1. Given the parametrized curve (helix)

$$\alpha(s) = \left(a\cos\frac{s}{c}, a\sin\frac{s}{c}, b\frac{s}{c}\right), \qquad s \in R,$$

where $c^2 = a^2 + b^2$,

a. Show that the parameter s is the arc length.

b. Determine the curvature and the torsion of α.

c. Determine the osculating plane of α.

d. Show that the lines containing $n(s)$ and passing through $\alpha(s)$ meet the z axis under a constant angle equal to $\pi/2$.

e. Show that the tangent lines to α make a constant angle with the z axis.

Proof. **(a)** By computation

$$\alpha'(s) = (\frac{-a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c})$$

So

$$|\alpha'(s)| = \sqrt{\frac{a^2+b^2}{c^2}} = 1 \qquad\qquad (\because\ \sin^2+\cos^2 = 1\ )$$

This shows α is parametrized by arc-length.

(b) By computation

$$\alpha''(s) = \left(\frac{-a}{c^2}\cos\frac{s}{c}, \frac{-a}{c^2}\sin\frac{s}{c}, 0\right)$$

Then because α is parametrized by arc-length, we have

$$\begin{aligned}\kappa(s) = |\alpha''(s)| &= \sqrt{\frac{a^2}{c^4}} \\ &= \frac{|a|}{c^2}\end{aligned}$$

By computation

$$\alpha'''(s) = \Big(\frac{a}{c^3}\sin\frac{s}{c}, \frac{-a}{c^3}\cos\frac{s}{c}, 0\Big)$$

Then using the identity of torsion, we have

$$\begin{aligned}\tau(s) &= -\frac{-\big(\alpha'(s)\times\alpha''(s)\big)\cdot\alpha'''(s)}{\left|\kappa(s)\right|^2}\\ &= -\frac{\frac{a^2b}{c^6}}{\frac{a^2}{c^4}}\\ &= \frac{b}{-c^2}\end{aligned}$$

(c) Fix s. Define a set A by

$$A = \operatorname{span}\Big(\alpha'(s), \alpha''(s)\Big)$$

The osculating plane of α at s is then exactly

$$\{a + \alpha(s) : a \in A\}$$

(d) Because $\alpha''(s)$ by our computation is valued 0 in z-opponent, we know if the line containing N and passing through α meet the z axis, it must be under a constant angle equal to $\frac{\pi}{2}$. (use dot product to check this fact.).

Now, we only have to prove that the line does meet the z-axis. See that

$$\alpha + c^2\alpha'' = \Big(0, 0, b\frac{s}{c}\Big)$$

and we are done.
(e) Observe that

$$\alpha' \cdot \Big(0, 0, 1\Big) = \frac{b}{c} \text{ is a constant}$$

This together with the fact $|\alpha'|$ is a constant show that the angle between the tangent to α and z-axis is a constant.

$\blacksquare$

Question 14

*2. Show that the torsion τ of α is given by

$$\tau(s) = -\frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{|k(s)|^2}.$$

Theorem 1.6.1. (Identity of Torsion) Given a parametrized by arc-length cruve $\alpha : I \to \mathbb{R}^3$, we have

$$\tau(s) = -\frac{\big(\alpha'(s) \times \alpha''(s)\big) \cdot \alpha'''(s)}{\kappa^2(s)}$$

Proof. Because α is parametrized by arc-length, we have

$$\alpha'(s) = T(s)$$

We first show

$$\alpha''(s) = \kappa(s)N(s) \tag{1.13}$$

Compute

$$\begin{aligned} N(s) &= \frac{T'(s)}{|T'(s)|} \\ &= \frac{\alpha''(s)}{|\alpha''(s)|} = \frac{\alpha''(s)}{\kappa(s)} \text{ (done)} \end{aligned}$$

We now show

$$\alpha'''(s) = \kappa(s)\big(-(\tau B)(s) - (\kappa T)(s)\big) + \kappa'(s)N(s)$$

By Equation 1.13 and Frenet Formula, we have

$$\begin{aligned} \alpha'''(s) &= \kappa'(s)N(s) + \kappa(s)N'(s) \\ &= \kappa'(s)N(s) + \kappa(s)\big(-(\tau B)(s) - (\kappa T)(s)\big) \text{ (done)} \end{aligned}$$

Lastly, we verify

$$\begin{aligned} -\frac{\big(\alpha'(s) \times \alpha''(s)\big) \cdot \alpha'''(s)}{\kappa^2(s)} &= -\frac{(T \times \kappa N) \cdot \Big(\kappa\big(-\tau B - \kappa T\big) + \kappa' N\Big)}{\kappa^2} \\ &= -\frac{-\kappa^2 \tau (T \times N) \cdot B}{\kappa^2} \qquad (\because T \times N \cdot (T \text{ or } N) = 0) \\ &= \tau \end{aligned}$$

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Question 15

3. Assume that $\alpha(I) \subset R^2$ (i.e., α is a plane curve) and give k a sign as in the text. Transport the vectors $t(s)$ parallel to themselves in such a way that the origins of $t(s)$ agree with the origin of R^2; the end points of $t(s)$ then describe a parametrized curve $s \longrightarrow t(s)$ called the *indicatrix of tangents* of α. Let $\theta(s)$ be the angle from e_1 to $t(s)$ in the orientation of R^2. Prove (a) and (b) (notice that we are assuming that $k \neq 0$).

a. The indicatrix of tangents is a regular parametrized curve.

b. $dt/ds = (d\theta/ds)n$, that is, $k = d\theta/ds$.

Proof. **(a)**

The indicatrix of tangents $\gamma : I \to \mathbb{R}^2$ is defined by

$$\gamma = \frac{\alpha'(s)}{|\alpha'(s)|}$$

Express $\alpha'(s)$ by

$$\alpha' \triangleq (x, y)$$

To show γ is regular. We wish to show

$$\gamma'(s) \neq 0 \text{ for all } s \in I$$

Express γ by

$$\gamma = \frac{(x, y)}{\sqrt{x^2 + y^2}}$$

Then, we see the x-component of $\gamma'(s)$ is

$$\gamma'(s)\Big|_x = \frac{x'y^2}{(x^2 + y^2)^{\frac{3}{2}}}$$

With similar computation on the y-component, we now arrive at

$$\gamma'(s) = \frac{\left(x'y^2, y'x^2\right)}{(x^2 + y^2)^{\frac{3}{2}}}$$

Now, for a contradiction, Assume $\gamma'(s) = 0$ for some s. Then one of the three things below must happen

(a) $x' = y' = 0$

(b) $y^2 = x^2 = 0$

(c) $x' = x^2 = 0$ WLOG

Because $(x, y) = \alpha'$ and α is parametrized by arc-length and curvature is non-zero by premise, we know it can not happen $\alpha'' = (x', y') = 0$.

Because $(x, y) = \alpha'$ and α is parametrized by arc-length, we also know it can not happen $\alpha' = (x, y) = 0$.

Now, we are given the hypothesis $x' = x^2 = 0$. Because α is parametrized by arc-length, from $x = 0$, we know $y = \pm 1$. Then because $|\alpha'|$ is constant, we can deduce

$$\begin{aligned} 0 &= (x', y') \cdot (x, y) \\ &= (0, y') \cdot (0, \pm 1) \end{aligned}$$

This show us $y' = 0$, which is impossible, since if $(x', y') = 0$ then the curvature is 0 CaC (done)

(b) The functions $\theta : [0, l] \to \mathbb{R}$, is defined by

$$T = (x, y) \triangleq (\cos\theta, \sin\theta)$$

By Frenet Formula, we have

$$\kappa N = T' = \theta'(-\sin\theta, \cos\theta) \tag{1.14}$$

Because $|(-\sin\theta, \cos\theta)| = 1$ and $(-\sin\theta, \cos\theta) \cdot T = 0$ and

$$\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = 1$$

we can identify $(-\sin\theta, \cos\theta) = N$. Then from Equation 1.14, we now can deduce

$$\kappa = \theta'$$

$\blacksquare$

Question 16

6. A *translation* by a vector v in R^3 is the map $A: R^3 \longrightarrow R^3$ that is given by $A(p) = p + v$, $p \in R^3$. A linear map $\rho: R^3 \longrightarrow R^3$ is an *orthogonal transformation* when $\rho u \cdot \rho v = u \cdot v$ for all vectors $u, v \in R^3$. A *rigid motion* in R^3 is the result of composing a translation with an orthogonal transformation with positive determinant (this last condition is included because we expect rigid motions to preserve orientation).

a. Demonstrate that the norm of a vector and the angle θ between two vectors, $0 \leq \theta \leq \pi$, are invariant under orthogonal transformations with positive determinant.

b. Show that the vector product of two vectors is invariant under orthogonal transformations with positive determinant. Is the assertion still true if we drop the condition on the determinant?

c. Show that the arc length, the curvature, and the torsion of a parametrized curve are (whenever defined) invariant under rigid motions.

Proof. Let A be a translation and ρ be an orthogonal transformation.

(a) Observe

$$\begin{aligned}\|v\| &= \sqrt{v \cdot v} \\ &= \sqrt{\rho v \cdot \rho v} \\ &= \|\rho v\|\end{aligned}$$

Because θ is given by

$$\theta = \arccos \frac{v \cdot w}{|v| \cdot |w|}$$

and because norm is invariant under orthogonal transformation, from the definition of orthogonal transformation, we now see

$$\begin{aligned}\theta &= \arccos \frac{v \cdot w}{|v| \cdot |w|} \\ &= \arccos \frac{\rho v \cdot \rho w}{|v| \cdot |w|} \\ &= \arccos \frac{\rho v \cdot \rho w}{|\rho v| \cdot |\rho w|} = \theta_\rho\end{aligned}$$

where θ_ρ is the angle between ρv and ρw.

(b) Fix $v, w \in \mathbb{R}^3$ and a positive determinant orthogonal transformation ρ. We wish to show

$$\rho v \times \rho w = \rho(v \times w)$$

We can reduce the problem into proving

$$\rho v \times \rho w \cdot z = \rho(v \times w) \cdot z \text{ for all } z \in \mathbb{R}^3$$

Fix $z \in \mathbb{R}^3$. Because ρ has non-zero determinant, we know there exists $z' \in \mathbb{R}^3$ such that

$$\rho z' = z$$

Now, because orthogonal transformation has determinant ± 1 and we have know ρ has positive determinant, we know

$$\begin{aligned}
\rho v \times \rho w \cdot z &= \rho v \times \rho w \cdot \rho z' \\
&= \begin{vmatrix} \rho v \\ \rho w \\ \rho z' \end{vmatrix} \\
&= \begin{vmatrix} \rho v & \rho w & \rho z' \end{vmatrix} \qquad (\because \det A = \det A^t) \\
&= |\rho| \cdot \begin{vmatrix} v & w & z' \end{vmatrix} \qquad (\because \det A \det B = \det AB) \\
&= \begin{vmatrix} v \\ w \\ z' \end{vmatrix} \qquad (\because \det \rho = 1) \\
&= v \times w \cdot z' \\
&= \rho(v \times w) \cdot \rho z' \\
&= \rho(v \times w) \cdot z \text{ (done)}
\end{aligned}$$

The assertion is clearly false if the determinant is negative. One can check $v = (1, 0, 0)$ and $w = (0, 1, 0)$ and $\rho(x, y, z) = (-x, y, z)$.

(c) We first show arc length is invariant under rigid motion. We first show

arc length is invariant under orthogonal transformation

To show such, we only have to show

$$|(\rho \circ \gamma)'| = |\gamma'|$$

■

Fix $y \in I$. We have

$$|\gamma'(y)| = \left|\lim_{t\to y} \frac{\gamma(t)-\gamma(y)}{t-y}\right| = \lim_{t\to y}\left|\frac{\gamma(t)-\gamma(y)}{t-y}\right|$$

Notice that in above deduction, we exchange limit and norm. Such exchange hold true because the function inside is continuous.

Similarly, we have

$$|(\rho\circ\gamma)'(y)| = \lim_{t\to y}\left|\frac{\rho\circ\gamma(t)-\rho\circ\gamma(y)}{t-y}\right|$$

Then, we can reduce the problem into

$$\text{proving } |\gamma(t)-\gamma(y)| = |\rho\circ\gamma(t)-\rho\circ\gamma(y)|$$

Because $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ is an orthogonal transformation (a linear transformation too), we can deduce

$$\begin{aligned} |\rho\circ\gamma(t)-\rho\circ\gamma(y)| &= \big|\rho\big(\gamma(t)-\gamma(y)\big)\big| \\ &= |\gamma(t)-\gamma(y)| \ \text{ (done)} \end{aligned}$$

We have proved arc-length is invariant under orthogonal transformation. With some simple computation, it is clear that arc-length is invariant under translation. This let us conclude arc length is invariant under rigid motion.

Now, to show curvature and torsion are also invariant under rigid motion. We first recall the following identities for curve parametrzied by arc-length

$$\kappa = |\gamma''| \ \text{ and } \tau = -\frac{\gamma'\times\gamma''\cdot\gamma'''}{\kappa^2}$$

We now prove

curvature is invariant under rigid motion

Notice that γ' is invariant under translation, so in fact, we only have to prove

curvature is invariant under orthogonal transformation

Observe

$$|\gamma''(y)| = \left|\lim_{t\to y} \frac{\gamma'(t)-\gamma'(y)}{t-y}\right| = \lim_{t\to y}\left|\frac{\gamma'(t)-\gamma'(y)}{t-y}\right|$$

and

$$|(\rho\circ\gamma)''(y)| = \lim_{t\to y}\left|\frac{(\rho\circ\gamma)'(t)-(\rho\circ\gamma)'(y)}{t-y}\right|$$

We can now reduce the problem into proving

$$|\gamma'(t)-\gamma'(y)| = |(\rho\circ\gamma)'(t)-(\rho\circ\gamma)'(y)|$$

Because ρ is a linear transformation, we can compute

$$\begin{aligned}(\rho\circ\gamma)'(t) &= \lim_{u\to t}\frac{\rho\circ\gamma(u)-\rho\circ\gamma(t)}{u-t}\\ &= \lim_{u\to t}\rho\Big(\frac{\gamma(u)-\gamma(t)}{u-t}\Big)\\ &= \rho\lim_{u\to t}\Big(\frac{\gamma(u)-\gamma(t)}{u-t}\Big) = \rho\circ\gamma'(t)\end{aligned} \tag{1.15}$$

We now using the fact norm is invariant under orthogonal transformation to compute

$$\begin{aligned}|(\rho\circ\gamma)'(t)-(\rho\circ\gamma)'(y)| &= |\rho\circ\gamma'(t)-\rho\circ\gamma'(y)|\\ &= \big|\rho\big(\gamma'(t)-\gamma'(y)\big)\big|\\ &= |\gamma'(t)-\gamma'(y)| \ \ \text{(done)}\end{aligned}$$

Now, notice that in Equation 1.15, we just proved

$$(\rho\gamma)' = \rho\gamma'$$

Iterating the same argument, we can show

$$\begin{aligned}(\rho\gamma)'' &= \big((\rho\gamma)'\big)'\\ &= (\rho\gamma')'\\ &= \rho\gamma''\end{aligned}$$

and also show

$$\begin{aligned}(\rho\gamma)''' &= \big((\rho\gamma)''\big)'\\ &= (\rho\gamma'')'\\ &= \rho\gamma'''\end{aligned}$$

We now using the fact that $|\rho| = 1$ to compute

$$\begin{aligned}(\rho\gamma)'\times(\rho\gamma)''\cdot(\rho\gamma)''' &= \big|(\rho\gamma)' \quad (\rho\gamma)'' \quad (\rho\gamma)'''\big|\\ &= \big|\rho\gamma' \quad \rho\gamma'' \quad \rho\gamma'''\big|\\ &= \big|\rho\left[\gamma' \quad \gamma'' \quad \gamma'''\right]\big|\\ &= |\rho|\cdot\big|\gamma' \quad \gamma'' \quad \gamma'''\big|\\ &= \gamma'\times\gamma''\cdot\gamma'''\end{aligned}$$

Above computation with identity of torsion and the fact curvature is invariant under orthogonal transformation with positive determinant then show that torsion is also invariant under orthogonal transformation with positive determinant.

Because $(\gamma + c)' = \gamma'$, together with what we have proved, it is easy to check torsion is also invariant under rigid motion.

Question 17

9. Given a differentiable function $k(s)$, $s \in I$, show that the parametrized plane curve having $k(s) = k$ as curvature is given by

$$\alpha(s) = \left(\int \cos\theta(s)\, ds + a, \int \sin\theta(s)\, ds + b \right),$$

where

$$\theta(s) = \int k(s)\, ds + \varphi,$$

and that the curve is determined up to a translation of the vector (a, b) and a rotation of the angle φ.

Proof. By Fundamental Theorem of Local Curves (you can think of our application as identifying $\tau = 0$), we know if such curve exists then it is unique up to translation and rotation. This reduced our proof into showing

$$\alpha \text{ has curvature } \kappa$$

Compute

$$\alpha' = (\cos\theta, \sin\theta)$$

This shows that α is parametrized by arc-length, and shows that we can compute

$$\begin{aligned} |\alpha''| &= |\theta'(-\sin\theta, \cos\theta)| \\ &= |\theta'| = |\kappa| = \kappa \text{ (done)} \end{aligned}$$

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Question 18

11. One often gives a plane curve in polar coordinates by $\rho = \rho(\theta)$, $a \le \theta \le b$.

a. Show that the arc length is

$$\int_a^b \sqrt{\rho^2 + (\rho')^2}\, d\theta,$$

where the prime denotes the derivative relative to θ.

b. Show that the curvature is

$$k(\theta) = \frac{2(\rho')^2 - \rho\rho'' + \rho^2}{\{(\rho')^2 + \rho^2\}^{3/2}}.$$

Proof. **(a)**
Parametrize by

$$\alpha(\theta) \triangleq (x, y) \triangleq (r\cos\theta, r\sin\theta)$$

where $r(\theta)$ is a function. With respect to θ, we compute

$$\begin{aligned}
(x')^2 + (y')^2 &= (r'\cos\theta - r\sin\theta)^2 + (r'\sin\theta + r\cos\theta)^2 \\
&= (r')^2\cos^2\theta + r^2\sin^2\theta + (r')^2\sin^2\theta + r^2\cos^2\theta \qquad (\because \text{ elimination }) \\
&= r^2 + (r')^2
\end{aligned}$$

We now see that the arc-length can be computed by

$$\begin{aligned}
\int_a^b |\alpha'(\theta)|\, d\theta &= \int_a^b \sqrt{(x')^2 + (y')^2} d\theta \\
&= \int_a^b \sqrt{r^2 + (r')^2} d\theta
\end{aligned}$$

(b)

Recall that

$$\kappa(t) = \frac{x'y'' - x''y'}{\left((x')^2 + (y')^2\right)^{\frac{3}{2}}}$$

plugin

$$\begin{aligned}
(x', y') &= (r'\cos\theta - r\sin\theta, r'\sin\theta + r\cos\theta) \\
(x'', y'') &= (r''\cos\theta - 2r'\sin\theta - r\cos\theta, r''\sin\theta + 2r'\cos\theta - r\sin\theta)
\end{aligned}$$

To compute

$$x'y'' = r'r''\cos\theta\sin\theta + 2(r')^2\cos^2\theta - rr'\cos\theta\sin\theta - rr''\sin^2\theta - 2rr'\cos\theta\sin\theta + r^2\sin^2\theta$$
$$x''y' = r'r''\sin\theta\cos\theta - 2(r')^2\sin^2\theta - rr'\cos\theta\sin\theta + rr''\cos^2\theta - 2rr'\cos\theta\sin\theta - r^2\cos^2\theta$$

Eliminating the odd terms and using $\cos^2 + \sin^2 = 1$, we now compute

$$\begin{aligned}\kappa &= \frac{x'y'' - x''y'}{\left((x')^2 + (y')^2\right)^{\frac{3}{2}}} \\ &= \frac{2(r')^2 - 2rr'' + r^2}{\left(r^2 + (r')^2\right)^{\frac{3}{2}}}\end{aligned}$$

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Question 19

17. In general, a curve α is called a *helix* if the tangent lines of α make a constant angle with a fixed direction. Assume that $\tau(s) \neq 0$, $s \in I$, and prove that:

***a.** α is a helix if and only if $k/\tau = \text{const}$.

***b.** α is a helix if and only if the lines containing $n(s)$ and passing through $\alpha(s)$ are parallel to a fixed plane.

***c.** α is a helix if and only if the lines containing $b(s)$ and passing through $\alpha(s)$ make a constant angle with a fixed direction.

d. The curve

$$\alpha(s) = \left(\frac{a}{c}\int \sin\theta(s)\,ds, \frac{a}{c}\int \cos\theta(s)\,ds, \frac{b}{c}s\right),$$

where $a^2 = b^2 + c^2$, is a helix, and that $k/\tau = b/a$.

Proof. **(a)** $(\longrightarrow)$

Because α is a helix, we know there exists fixed unit $a \in \mathbb{R}^3$ and $b \in \mathbb{R}$ such that

$$\alpha' \cdot a = b \text{ for all } s$$

This then implies

$$\alpha'' \cdot a = 0 \text{ for all } s$$

which implies

$$N \cdot a = 0$$

since N is parallel with α''. Because $\{T, N, B\}$ is an orthonormal basis, this $(N \cdot a = 0)$ together with a being unit then tell us we can express a by

$$a = T\cos\theta + B\sin\theta \text{ for some fixed } \theta \in \mathbb{R}$$

We now have the information $T\cos\theta + B\sin\theta$ is a constant function in s. Then, using Frenet Formula, we can deduce

$$0 = (T\cos\theta + B\sin\theta)' = \kappa N\cos\theta + \tau N\sin\theta$$

This them implies

$$\frac{\kappa}{\tau} = \frac{-\sin\theta}{\cos\theta} \text{ is a constant since } \theta \text{ is fixed.}$$

Notice that $\cos\theta \neq 0$ because $\tau \neq 0$ for all s.

$(\longleftarrow)$

Define $\theta \in \mathbb{R}$ by

$$\theta = \arctan\frac{-\kappa}{\tau}$$

We wish to show

$$a = T\cos\theta + B\sin\theta \text{ suffice}$$

Because we have

$$T \cdot a = \cos\theta$$

We only wish to show

$$a \text{ is a constant function in } s$$

Because $\theta = \arctan\frac{-\kappa}{\tau}$, we know

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{-\kappa}{\tau}$$

This then tell us

$$\tau\sin\theta + \kappa\cos\theta = 0$$

and implies

$$\tau N\sin\theta + \kappa N\cos\theta = 0$$

Then

$$
\begin{aligned}
a' &= (T\cos\theta + N\sin\theta)' \\
&= \kappa N\cos\theta + \tau N\sin\theta = 0
\end{aligned}
$$

This implies a is indeed a constant. (done)

(b)

$(\longrightarrow)$

Let $a \in \mathbb{R}^3$ be the unit vector such that

$$T \cdot a \text{ is fixed}$$

We now see

$$N \cdot a = (T \cdot a)' = 0$$

This implies

$$\text{the plane } \{a\}^\perp \text{ suffice}$$

$(\longleftarrow)$

Observe that

$$
\begin{aligned}
0 = N \cdot a &= \frac{T'}{|T'|} \cdot a \\
&\implies T' \cdot a = 0 \\
&\implies T \cdot a \text{ is fixed}
\end{aligned}
$$

(c)

$(\longrightarrow)$

Because $T \cdot a$ is fixed, we can deduce

$$\kappa N \cdot a = T' \cdot a = 0$$

Now observe from Frenet Formula that

$$(B \cdot a)' = -\tau N \cdot a = 0$$

This implies $B \cdot a$ is fixed.

$(\longleftarrow)$

Because $B \cdot a$ is fixed, we can deduce

$$\begin{aligned} 0 = (B \cdot a)' &= -\tau N \cdot a \\ \implies N \cdot a &= 0 \end{aligned}$$

The proof then now follows from the result of **(b)**.

(d)

First we have to notice the fucking typo correction $\frac{\kappa}{\tau} = \frac{a}{b}$.

Compute

$$\begin{aligned} \alpha'(s) &= \Big(\frac{a}{c}\sin\theta, \frac{a}{c}\cos\theta, \frac{b}{c}\Big) \\ \alpha''(s) &= \Big(\theta'\frac{a}{c}\cos\theta, \theta'\frac{-a}{c}\sin\theta, 0\Big) \\ \alpha'''(s) &= \Big(\theta''\frac{a}{c}\cos\theta + (\theta')^2\frac{-a}{c}\sin\theta, \theta''\frac{-a}{c}\sin\theta + (\theta')^2\frac{-a}{c}\cos\theta, 0\Big) \end{aligned}$$

This give us

$$\begin{aligned} &\alpha' \times \alpha'' \cdot \alpha''' \\ &= \frac{b}{c}\Big[\theta'\theta''\frac{-a^2}{c^2}\cos\theta\sin\theta + (\theta')^3\frac{-a^2}{c^2}\cos^2\theta - \theta'\theta''\frac{-a^2}{c}\sin\theta\cos\theta - (\theta')^3\frac{a^2}{c^2}\sin^2\theta\Big] \\ &= \frac{b}{c}\Big((\theta')^3\frac{-a^2}{c^2}\Big) = \frac{-a^2b}{c^3}(\theta')^3 \end{aligned}$$

And give us

$$\kappa = \theta'\frac{a}{c}$$

We now compute

$$\begin{aligned} \frac{\kappa}{\tau} &= \frac{\kappa}{-\frac{\alpha' \times \alpha'' \cdot \alpha'''}{\kappa^2}} \\ &= \frac{-\kappa^3}{\alpha' \times \alpha'' \cdot \alpha'''} \\ &= \frac{-(\theta')^3\frac{a^3}{c^3}}{\frac{-a^2b}{c^3}(\theta')^3} = \frac{a}{b} \end{aligned}$$

■

Question 20

3. **Compute the curvature of the ellipse**

$$x = a\cos t, \qquad y = b\sin t, \qquad t \in [0, 2\pi],\ a \neq b,$$

and show that it has exactly four vertices, namely, the points $(a, 0)$, $(-a, 0)$, $(0, b)$, $(0, -b)$.

Proof. Compute

$$\begin{cases} x' = -a\sin t \text{ and } x'' = -a\cos t \\ y' = b\cos t \text{ and } y'' = -b\sin t \end{cases}$$

Plugging the curvature formula

$$\kappa = \frac{x'y'' - x''y'}{\left((x')^2 + (y')^2\right)^{\frac{3}{2}}}$$

We now have

$$\kappa = \frac{ab}{\left(a^2\sin^2 t + b^2\cos^2 t\right)^{\frac{3}{2}}}$$

Compute

$$\kappa' = (2a^2\sin t\cos t - 2b^2\sin t\cos t)(a^2\sin^2 t + b^2\cos^2 t)^{\frac{-5}{2}} \cdot (ab)$$

We see that

$$\kappa' = 0 \iff \sin 2t = 0$$

This only happens when $t \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$ where 2π is just 0 in the sense of parametrizeation of closed curve. We have shown there are exactly four vertices

$$(a, 0), (-a, 0), (0, b), (0, -b)$$

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Question 21

***4.** Let C be a plane curve and let T be the tangent line at a point $p \in C$. Draw a line L parallel to the normal line at p and at a distance d of p (Fig. 1-36). Let h be the length of the segment determined on L by C and T (thus, h is the "height" of C relative to T). Prove that

$$|k(p)| = \lim_{d \to 0} \frac{2h}{d^2},$$

where $k(p)$ is the curvature of C at p.

Proof. WOLG, let $p = (0,0)$, T be the x-axis and some neighborhood around p be above T. Positively oriented parametrize C by arc-length using (x,y) and $(x,y)(0) = (0,0)$. Using Taylor Theorem about $y(0)$, we see

$$y(s) = y(0) + y'(0)s + \frac{y''(0)}{2}s^2 + R_y \text{ where } \frac{R_y}{s^2} \to 0 \text{ as } s \to 0$$

Because T the tangent is the x-axis, we know $x''(0) = 0 \quad (\because N = (0,1))$. This tell us

$$\begin{aligned} |\kappa(0)| &= \sqrt{(x'')^2(0) + (y'')^2(0)} \\ &= y''(0) \qquad (\because N = (0,1)) \end{aligned}$$

■

By our setting $(x,y)(0) = (0,0)$, we see

$$y(0) = y'(0) = 0 \qquad (\because (x', y') = (1,0))$$

We now see

$$y''(0) = \frac{2\big(y(s) - R_y\big)}{s^2} \text{ for all } s \neq 0$$

This tell us

$$y''(0) = \lim_{s \to 0} \frac{2\big(y(s) - R_y\big)}{s^2} = \lim_{s \to 0} \frac{2y(s)}{s^2}$$

Using Taylor Theorem about $x(0)$, we see

$$x(s) = x(0) + x'(0)s + R_x \text{ where } \frac{R_x}{s} \to 0 \text{ as } x \to 0$$

Because $x(0) = 0$ and $x'(0) = 1$, we see

$$\lim_{s\to 0} \frac{x(s)}{s} = \lim_{s\to 0} \frac{s + R_x}{s} = 1$$

This now give us

$$|\kappa(0)| = y''(0) = \lim_{s\to 0} \frac{2y(s)}{s^2} = \lim_{s\to 0} \frac{2y(s)}{x^2(s)} = \lim_{d\to 0} \frac{2h}{d^2}$$

Question 22

6. Let $\alpha(s)$, $s \in [0, l]$ be a closed convex plane curve positively oriented. The curve

$$\beta(s) = \alpha(s) - rn(s),$$

where r is a positive constant and n is the normal vector, is called a *parallel* curve to α (Fig. 1-37). Show that

a. Length of β = length of $\alpha + 2\pi r$.

b. $A(\beta) = A(\alpha) + rl + \pi r^2$.

c. $k_\beta(s) = k_\alpha(s)/(1 + r)$.

Proof. **(a)**

Using Frenet Formula to compute

$$\beta'(s) = \alpha'(s) + r\kappa T(s)$$

Because α is parametrized by arc-length, we now know

$$|\beta'| = |(1 + r\kappa)\alpha'| = |1 + r\kappa| = 1 + r\kappa$$

This now give us

$$\int_0^l |\beta'|\, ds = l + r \int_0^l \kappa ds$$

Because a closed convex curve must also be simple (Sec. 5-7, Prop. 1), we now can deduce

$$\begin{aligned} \text{Length of } \beta &= l + r \int_0^l \kappa ds \\ &= \text{Length of } \alpha + r(2\pi) \end{aligned}$$

(b)

Set

$$\alpha = (x, y) \text{ and } \beta = (x - rN_1, y - rN_2)$$

Because

$$\beta' = (1 + r\kappa)\alpha'$$

We know

$$\beta' = \Big((1 + r\kappa)x', (1 + r\kappa)y'\Big)$$

Now, we use Green's Theorem to compute the Area

$$\begin{aligned}
A(\beta) &= \frac{1}{2}\int_0^l (x - rN_1)(1 + r\kappa)y' - (y - rN_2)(1 + r\kappa)x' ds \\
&= \frac{1}{2}\int_0^l (xy' - yx')ds + \frac{r}{2}\int_0^l (\kappa xy' + \kappa x'y)ds \\
&+ \frac{r}{2}\int_0^l -(N_1 y' + N_2 x')ds + \frac{r^2}{2}\int_0^l (-N_1 y'\kappa + N_2 x'\kappa)ds
\end{aligned}$$

Notice that by Frenet Formula, we have

$$N' = -\kappa(x', y')$$

so in fact we know

$$\kappa xy' + \kappa x'y = N' \cdot (-y, x)$$

Now using integral by part and the fact $\alpha = (x, y)$ is closed, we know

$$\begin{aligned}
\int_0^l (\kappa xy' + \kappa x'y)ds &= \int_0^l N' \cdot (-y, x)ds \\
&= \int_0^l N \cdot (-y', x')ds
\end{aligned}$$

Then now we have

$$\frac{r}{2}\int_0^l (\kappa xy' + \kappa x'y)ds + \frac{r}{2}\int_0^l -N_1 y' + N_2 x' ds = \frac{r}{2}\int_0^l 2N \cdot (-y', x')ds$$

Using positive orientation and the fact $|N| = 1 = |(-y', x')|$ to identify that $N = (-y', x')$, we now have

$$\frac{r}{2}\int_0^l 2N \cdot (-y', x')ds = rl$$

and have

$$\frac{r^2}{2}\int_0^l (-N_1 y'\kappa + N_2 x'\kappa)ds = \frac{r^2}{2}\int_0^l \kappa ds = r^2\pi$$

since (x, y) is simple closed. This finishes the proof.

(c)

Recall that

$$\kappa(a, b) = \frac{a'b'' - a''b'}{\big((a')^2 + (b')^2\big)^{\frac{3}{2}}}$$

We use this formula on β to compute

$$\begin{aligned}
\kappa_\beta &= \frac{(1 + r\kappa)x'\big((1 + r\kappa)y'\big)' - \big((1 + r\kappa)x'\big)'(1 + r\kappa)y'}{(1 + r\kappa)^3} \\
&= \frac{(1 + r\kappa)^2(x'y'' - x''y')}{(1 + r\kappa)^3} \\
&= \frac{x'y'' - x''y'}{1 + r\kappa} = \frac{\kappa}{1 + r\kappa} \qquad (\because (x')^2 + (y')^2 = 1)
\end{aligned}$$

■

Question 23

8. ***a.** Let $\alpha(s)$, $s \in [0, l]$, be a plane simple closed curve. Assume that the curvature $k(s)$ satisfies $0 < k(s) \leq c$, where c is a constant (thus, α is less curved than a circle of radius $1/c$). Prove that

$$\text{length of } \alpha \geq \frac{2\pi}{c}.$$

b. In part a replace the assumption of being simple by "α has rotation index N." Prove that

$$\text{length of } \alpha \geq \frac{2\pi N}{c}.$$

Proof. **(a)**

Because α is simple closed and $\kappa \leq c$, we know

$$cl = \int_0^l c ds \geq \int_0^l \kappa ds = 2\pi$$

This then implies

$$\text{Length of } \alpha = l \geq \frac{2\pi}{c}$$

(b)

Because α has rotation index N and $\kappa \leq c$, we know

$$cl = \int_0^l c ds \geq \int_0^l \kappa ds = N2\pi$$

This the implies

$$\text{Length of } \alpha = l \geq \frac{2\pi N}{c}$$

■

Question 24

***11.** Given a nonconvex simple closed plane curve C, we can consider its *convex hull* H (Fig. 1-39), that is, the boundary of the smallest convex set containing the interior of C. The curve H is formed by arcs of C and by the segments of the tangents to C that bridge "the nonconvex gaps" (Fig. 1-39). It can be proved that H is a C^1 closed convex curve. Use this to show that, in the isoperimetric problem, we can restrict ourselves to convex curves.

Figure 1-39

Proof. Suppose we have proved that a convex closed curve must satisfy the isoperimetric inequality. Let C be an arbitrary closed plane curve, and let H be its convex hull. Now, because straight line is the shortest curve between two point and because we know H, a convex curve, must satisfy isoperimetric inequality, we now see

$$4\pi A(C) \leq 4\pi A(H) \leq l_H^2 \leq l_C^2$$

If the equality hold true, we can deduce from $l_H = l_C$ that $H = C$ and use the argument for isoperimetric inequality of convex curve to argue that $C = H$ must be a circle. ■

Question 25

3. Show that the two-sheeted cone, with its vertex at the origin, that is, the set $\{(x, y, z) \in R^3; x^2 + y^2 - z^2 = 0\}$, is not a regular surface.

Proof. Let $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 0\}$. It is clear S contain $(0, 0, 0)$. To show S is not regular, we only wish to find a neighborhood V around $(0, 0, 0)$ in S such that V can not expressed as graph of differentiable functions from $\mathbb{R}^2$ to $\mathbb{R}$. This is trivially true, as all neighborhood ought to contain some open ball $B_\epsilon(0)$, and in this open ball, if we fix, say $(x, y) \in B_\epsilon(0)$ such that $(x, y, \sqrt{x^2 + y^2}) \in B_\epsilon(0)$, we see that $z = -\sqrt{x^2 + y^2}$ is also in $B_\epsilon(0)$ and S. The same argument applies to when (x, z) and (y, z) are fixed. ■

Question 26

6. Give another proof of Prop. 1 by applying Prop. 2 to $h(x, y, z) = f(x, y) - z$.

Proof. Because f is differentiable, we see f_x, f_y are all continuous on U. This then implies

$$h_x(x,y,z) = f_x(x,y), h_y(x,y,z) = f_y(x,y), h_z = -1 \text{ are all continuous on } U$$

We have shown h is differentiable. Now that observe

$$h(x,y,z) = 0 \implies (x,y,z) = (x,y,f(x,y))$$

The converse of course hold true. This then implies

$$f[U] = h^{-1}[0]$$

Fix arbitrary $(x,y) \in U$. We see

$$\mathbf{d}h\big(x,y,f(x,y)\big) = \begin{bmatrix} h_x & h_y & h_z \end{bmatrix}\Big|_{(x,y,f(x,y))} = \begin{bmatrix} f_x(x,y) & f_y(x,y) & -1 \end{bmatrix}$$

which is clearly not onto. This show

$$(x,y,f(x,y)) \text{ is not a critical point}$$

Because $(x,y) \in U$ is arbitrary, we have shown $f[U]$ contain no critical point. Now it follows 0 is a regular value and $f[U] = h^{-1}[0]$ is a regular surface. ∎

Question 27

7. Let $f(x, y, z) = (x + y + z - 1)^2$.

a. Locate the critical points and critical values of f.

b. For what values of c is the set $f(x, y, z) = c$ a regular surface?

c. Answer the questions of parts a and b for the function $f(x, y, z) = xyz^2$.

Proof. **(a)**

Compute

$$f_x = f_y = f_z = 2(x+y+z-1)$$

This implies the set of critical points are

$$\{(x,y,z) \in \mathbb{R}^3 : x+y+z = 1\}$$

Then it follows from simple computation the set of critical values is exactly

$$\{0\}$$

(b)

For all $c > 0$, the set $f^{-1}[c]$ is a regular surface, and for all $c < 0$, the set $f^{-1}[c]$ is empty (thus trivially regular).

(c)

Compute

$$f_x = yz^2 \text{ and } f_y = xz^2 \text{ and } f_z = 2xyz$$

This implies the set of critical points is

$$\{(x, y, z) : z = 0 \text{ or } x = y = 0\}$$

With simple computation, we see the set of critical values is exactly

$$\{0\}$$

The set of regular values are exactly $\mathbb{R}^*$, so all $c \neq 0$ suffice. ■

Question 28

8. Let $\mathbf{x}(u, v)$ be as in Def. 1. Verify that $d\mathbf{x}_q \colon R^2 \to R^3$ is one-to-one if and only if

$$\frac{\partial \mathbf{x}}{\partial u} \wedge \frac{\partial \mathbf{x}}{\partial v} \neq 0.$$

Proof. Note that

$$dx_q = \begin{bmatrix} \partial_u x & \partial_v x \end{bmatrix}$$

This give us

$$dx_q : \mathbb{R}^2 \to \mathbb{R}^3 \text{ is one-to-one} \iff \partial_u x, \partial_v x \in \mathbb{R}^3 \text{ is linearly independent everywhere}$$

Then we can reduce the problem into proving

$$\partial_u x, \partial_v x \in \mathbb{R}^3 \text{ is linearly independent everywhere} \iff \partial_u x \times \partial_v x \neq 0 \text{ everywhere}$$

This then follows from Theorem 1.6.2 at the next page, as one can see that each component of the output of cross product is exactly the three determinant. ■

Theorem 1.6.2. (Computation to check Linearly Independence)

$$\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \end{bmatrix} \text{ is linearly independent } \iff \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} \neq 0 \text{ or } \begin{vmatrix} v_2 & w_2 \\ v_3 & w_3 \end{vmatrix} \neq 0 \text{ or } \begin{vmatrix} v_1 & w_1 \\ v_3 & w_3 \end{vmatrix} \neq 0$$

Proof. $(\longleftarrow)$

Assume v, w are linearly dependent. Fix $w_k = cv_k$. We see

$$\begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} = cv_1v_2 - cv_1v_2 = 0 \text{ CaC}$$

$(\longrightarrow)$

Assume all determinant are 0. Pick k such that v_k is non-zero. Define

$$c \triangleq \frac{w_k}{v_k}$$

WOLG, suppose

$$w_1 = cv_1 \text{ and } v_1 \neq 0$$

We then can deduce

$$\begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} \implies cv_1v_2 = v_1w_2 \implies w_2 = cv_2$$

The same arguement implies $w_3 = cv_3$ CaC ∎