Chapter 1

Complex Analysis Final

1.1 Complex Analysis Final

(1) (20 pts) Let $S^2 \subseteq \mathbb{R}^3$ be the closed unit sphere defined by the equation $x^2+y^2+z^2 = 1$. Let $N = (0,0,1)$ be the north pole of S^2 and $S = (0,0,-1)$ be the south pole of S^2. Define

$$U_N \triangleq S^2 \setminus \{N\} \text{ and } U_S = S^2 \setminus \{S\}$$

Question 1

(a) (5pts) Prove that U_N and U_S are open subsets of S^2. (Here S^2 is equipped with the subspace topology induced by $\mathbb{R}^3$.)

Proof. Fix $p \in U_N$. Because $N \notin U_N$, we know $|p-N| > 0$. Let $\epsilon < |p-N|$, so that the open ball $B_\epsilon(p)$ in $\mathbb{R}^3$ does not contain N. This give us

$$p \in B_\epsilon(p) \cap S^2 \subseteq U_N$$

Where $B_\epsilon(p) \cap S^2$ is open in S^2 by definition of subspace topology. We have shown that for each $p \in U_N$ there exists some subset $M_p \subseteq S^2$ open in S^2, containing p and contained by U_N. This implies U_N is open in S^2.

Fix $p \in U_S$. Because $S \notin U_S$, we know $|p-S| > 0$. Let $\epsilon < |p-S|$, so that the open ball $B_\epsilon(p)$ in $\mathbb{R}^3$ does not contain S. This give us

$$p \in B_\epsilon(p) \cap S^2 \subseteq U_S$$

Where $B_\epsilon(p) \cap S^2$ is open in S^2 by definition of subspace topology. We have shown that for each $p \in U_S$ there exists some subset $M_p \subseteq S^2$ open in S^2, containing p and contained by U_S. This implies U_S is open in S^2. ■

Question 2

(b) (5pts) Define $\varphi_N : U_N \to \mathbb{C}$ and $\varphi_S : U_S \to \mathbb{C}$ by

$$\varphi_N(a,b,c) \triangleq \frac{a+bi}{1-c} \text{ and } \varphi_S(a,b,c) \triangleq \frac{a-bi}{1+c}$$

Prove that both φ_N and φ_S are homeomorphisms.

Proof. The continuity of φ_N and φ_S is obvious. Suppose

$$x+yi = \frac{a+bi}{1-c} = \varphi_N(a,b,c)$$

Multiply both side by $1-c$

$$(1-c)(x+yi) = a+bi \tag{1.1}$$

This give us

$$(1-c)^2(x^2+y^2)+c^2 = a^2+b^2+c^2 = 1$$

Which give us

$$(x^2+y^2+1)c^2 - 2(x^2+y^2)c + (x^2+y^2-1) = 0$$

By quadratic formula,

$$\begin{aligned} c &= \frac{2(x^2+y^2) \pm \sqrt{4(x^2+y^2)^2 - 4(x^2+y^2+1)(x^2+y^2-1)}}{2(x^2+y^2+1)} \\ &= \frac{x^2+y^2 \pm 1}{x^2+y^2+1} = \frac{x^2+y^2-1}{x^2+y^2+1} \end{aligned}$$

Note that the last equality hold because $(a,b,c) \in U_N \implies c \neq 1$. From Equation 1.1, we may now compute

$$a = (1-c)x = \frac{2x}{x^2+y^2+1} \text{ and } b = (1-c)y = \frac{2y}{x^2+y^2+1}$$

We have shown that φ_N is a bijection between U_N and $\mathbb{C}$, and its inverse is exactly

$$\varphi_N^{-1}(x+yi) = \frac{(2x, 2y, x^2+y^2-1)}{x^2+y^2+1} \tag{1.2}$$

The continuity of $\varphi_N^{-1} : \mathbb{C} \to U_N$ is obvious. We have shown φ_N is indeed a homeomorphism. Now, suppose

$$x+yi = \frac{a-bi}{1+c} = \varphi_S(a,b,c) \tag{1.3}$$

Multiply both side by $1+c$

$$(1+c)(x+yi) = a - bi$$

This give us

$$(1+c)^2(x^2+y^2) + c^2 = a^2 + b^2 + c^2 = 1$$

Which give us

$$(x^2+y^2+1)c^2 + 2(x^2+y^2)c + (x^2+y^2-1) = 0$$

By quadratic formula

$$\begin{aligned} c &= \frac{-2(x^2+y^2) \pm \sqrt{4(x^2+y^2)^2 - 4(x^2+y^2+1)(x^2+y^2-1)}}{2(x^2+y^2+1)} \\ &= \frac{-x^2-y^2 \pm 1}{x^2+y^2+1} = \frac{-x^2-y^2+1}{x^2+y^2+1} \end{aligned}$$

Note that the last equality hold because $(a,b,c) \in U_S \implies c \neq -1$. From Equation 1.3, we may now compute

$$a = (1+c)x = \frac{2x}{x^2+y^2+1} \text{ and } b = (1+c)y = \frac{2y}{x^2+y^2+1}$$

We have shown φ_S is a bijection between U_S and $\mathbb{C}$, and its inverse is exactly

$$\varphi_S^{-1}(x+yi) = \frac{(2x, 2y, 1-x^2-y^2)}{x^2+y^2+1}$$

The continuity of $\varphi_S^{-1} : \mathbb{C} \to U_S$ is obvious. We have shown φ_S is indeed a homeomorphism.

∎

Question 3

(c) (5pts) Prove that

$$\varphi_N(S) = \varphi_S(N) = 0 \text{ and } \varphi_N(U_N \cap U_S) = \varphi_S(U_N \cap U_S) = \mathbb{C}^*$$

Proof. Compute

$$\varphi_N(S) = \varphi_N(0,0,-1) = \frac{0+0i}{2} = 0$$

Compute

$$\varphi_S(N) = \varphi_S(0,0,1) = \frac{0-0i}{2} = 0$$

Compute

$$U_N \cap U_S = U_N \setminus \{S\} = U_S \setminus \{N\}$$

It then follows from the fact φ_N maps U_N into $\mathbb{C}$ bijectively that

$$\varphi_N(U_N \cap U_S) = \varphi_N(U_N \setminus \{S\}) = \mathbb{C} \setminus \{\varphi_N(S)\} = \mathbb{C} \setminus \{0\} = \mathbb{C}^*$$

Similarly, it follows from the fact φ_S maps U_S into $\mathbb{C}$ bijectively that

$$\varphi_N(U_N \cap U_S) = \varphi_N(U_S \setminus \{N\}) = \mathbb{C} \setminus \{\varphi_S(N)\} = \mathbb{C} \setminus \{0\} = \mathbb{C}^*$$

■

Question 4

(d) (5pts) Show that

$$f = \varphi_S \circ \varphi_N^{-1} : \mathbb{C}^* \to \mathbb{C}^*$$

is a holomorphic function.

Proof. Using Equation 1.2, we may compute for all $x + yi \in \mathbb{C}^*$

$$\begin{aligned}
f(x+yi) &= \varphi_S(\varphi_N^{-1}(x+yi)) \\
&= \varphi_S\Big(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\Big) \\
&= \frac{\frac{2x}{x^2+y^2+1} - \frac{2iy}{x^2+y^2+1}}{1 + \frac{x^2+y^2-1}{x^2+y^2+1}} \\
&= \frac{2x - 2iy}{2(x^2+y^2)} = \frac{x - iy}{x^2+y^2}
\end{aligned}$$

Compute

$$\begin{aligned}
\frac{\partial}{\partial x}\operatorname{Re} f &= \frac{\partial}{\partial x}\frac{x}{x^2+y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \\
\frac{\partial}{\partial y}\operatorname{Im} f &= \frac{\partial}{\partial y}\frac{-y}{x^2+y^2} = \frac{y^2-x^2}{(x^2+y^2)^2}
\end{aligned}$$

Compute

$$\begin{aligned}
\frac{\partial}{\partial x}\operatorname{Im} f &= \frac{\partial}{\partial x}\frac{-y}{x^2+y^2} = \frac{2xy}{(x^2+y^2)^2} \\
\frac{\partial}{\partial y}\operatorname{Re} f &= \frac{\partial}{\partial y}\frac{x}{x^2+y^2} = \frac{-2xy}{(x^2+y^2)^2}
\end{aligned}$$

We have shown that f satisfy the Cauchy-Riemann criteria. Because both $\frac{\partial}{\partial x}\operatorname{Re} f, \frac{\partial}{\partial y}\operatorname{Re} f : \mathbb{R}^2 \setminus \{\mathbf{0}\} \to \mathbb{R}$ are continuous, we know $\operatorname{Re} f : \mathbb{R}^2 \setminus \{\mathbf{0}\} \to \mathbb{R}$ is differentiable. Because both $\frac{\partial}{\partial x}\operatorname{Im} f, \frac{\partial}{\partial y}\operatorname{Im} f : \mathbb{R}^2 \setminus \{\mathbf{0}\} \to \mathbb{R}$ are continuous, we know $\operatorname{Im} f : \mathbb{R}^2 \setminus \{\mathbf{0}\} \to \mathbb{R}$ is differentiable. It now follows from the Cauchy-Riemann Theorem that $f : \mathbb{C}^* \to \mathbb{C}$ is indeed holomorphic. ∎

(2) (20 pts) We identify $\mathbb{C}$ with $S^2 \setminus \{N\}$. Denote N by ∞. Denote S^2 by $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. From the previous problem, we know that N is the point of U_S corresponding to $w = 0$.

Let f be a function defined on $|z| > R$ for some $R > 0$. We say that ∞ is a pole of f of order m if the function $g : D_\epsilon(0) \setminus 0 \to \mathbb{C}$ defined by

$$g(z) \triangleq f(\frac{1}{z})$$

has a pole of order m at $z = 0$.

Question 5

(a) Prove that $f(z) = -3z + z^2$ has a pole at ∞. Find the residue and order of the pole of f at ∞.

Proof. Compute

$$g(z) = f(\frac{1}{z}) = \frac{-3}{z} + \frac{1}{z^2} = \frac{-3z+1}{z^2}$$

Because

$$\lim_{z\to 0} z^2 g(z) = \lim_{z\to 0} -3z + 1 = 1 \in \mathbb{C}^*$$

We see g has a pole of order 2 at 0. Therefore, f has a pole of order 2 at ∞. Note that the residue of f at ∞ is defined to be

$$\begin{aligned}\operatorname{Res}(f, \infty) &= -\operatorname{Res}\left(\frac{1}{z^2} g(z), 0\right) \\ &= -\operatorname{Res}\left(\frac{-3z+1}{z^4}, 0\right) = 0\end{aligned}$$

∎

Question 6

(b) Prove that a function on $\mathbb{C}_\infty$ is meromorphic if and only if it is a rational function, i.e., f is meromorphic if and only if

$$f(z) = \frac{Q(z)}{P(z)}$$

for some complex polynomial P, Q such that $P \neq 0 \in \mathbb{C}[z]$ and P, Q share no roots.

Proof. For the 'if' part, by fundamental theorem of algebra, we may write

$$f(z) = \frac{(z - z_0') \cdots (z - z_q')}{(z - z_0) \cdots (z - z_p)}$$

This implies that f has at most $p + 2$ numbers of poles, and f is differentiable everywhere except for points at which f has a pole. That is, f is meromorphic on $\mathbb{C}_\infty$.

For the 'only if' part, suppose $f : \mathbb{C}_\infty \setminus Z \to \mathbb{C}$ is meromorphic, where

$$Z \text{ is the set of poles of } f$$

Because f is either differentiable or has a pole at ∞, we know there exists $R > 0$ such that f is defined on $\{z \in \mathbb{C} : |z| > R\}$. By letting R be larger if necessary, we may WLOG suppose $Z \setminus \{\infty\}$ is contained by $D_R(0)$, the open disk centering origin with radius R. Define $g : D_R(0) \to \mathbb{C}$ by

$$g(z) \triangleq \begin{cases} \frac{1}{f(z)} & \text{if } z \notin Z \\ 0 & \text{if } z \in Z \end{cases}$$

Because Z is the set of poles of f, we know g is holomorphic. Obviously, g can not vanish identically, otherwise f has a pole at p for all $p \in D_R(0)$. It then follows from Identity Theorem that $Z \setminus \{\infty\}$, the set on which g vanish, has no limit points in $D_R(0)$. By repeating the same procedure for similarly defined $g : D_{R+\epsilon}(0) \to \mathbb{C}$, we may WLOG suppose $Z \setminus \{\infty\}$ has no limit points in $\mathbb{C}$. It then follows from $Z \setminus \{\infty\}$ is bounded that Z is finite, since otherwise Z has a limit point in $\mathbb{C}$, by Hiene-Borel and the fact limit point compact and compact are equivalent for $\mathbb{C}$.

Knowing that $Z \setminus \{\infty\}$ is finite, we may write $Z \setminus \{\infty\} = \{z_1, \dots, z_k\}$. Let $n_1, \dots, n_k$ be the order of these poles. If we define

$$g(z) \triangleq (z - z_1)^{n_1} \cdots (z - z_k)^{n_k} f(z)$$

We know on $\mathbb{C}$, g can only have removable singularity, so g is in fact an entire function. Write

$$g(z) = \sum_{n=0}^{\infty} a_n z^n \text{ for all } z \in \mathbb{C} \tag{1.4}$$

Now, let N be the order of the pole of f at ∞, where $N = 0$ if f is differentiable at ∞. By definition of g,

$$g \text{ has a pole at } \infty \text{ of order } N + n_1 + \cdots + n_k \triangleq M$$

It now follows from

$$g(\frac{1}{z}) = \sum_{n=0}^{\infty} a_n z^{-n} \text{ for all } |z| > 0$$

that $a_{M+n} = 0$ for all $n > 0$. Then from Equation 1.4, we see g is in fact a polynomial

$$g(z) = a_M z^M + \cdots + a_0$$

It follows from definition of g that

$$\begin{aligned} f(z) &= \frac{g(z)}{(z - z_1)^{n_1} \cdots (z - z_k)^{n_k}} \\ &= \frac{a_M z^M + \cdots + a_0}{(z - z_1)^{n_1} \cdots (z - z_k)^{n_k}} \text{ is indeed rational} \end{aligned}$$

Two things to note here: First, because

$$g(z) = (z - z_1)^{n_1} \cdots (z - z_k)^{n_k} f(z) \text{ and } z_j \text{ is of order } n_j \text{ for } f$$

We have

$$\lim_{z \to z_j} g(z) \neq 0$$

That is, g indeed shares no roots with $(z - z - 1)^{n_1} \cdots (z - z_k)^{n_k}$. Second, the argument holds true even if $Z \setminus \{\infty\} = \varnothing$. In such case, $P = 1$ and f is just g. ■

Question 7

(3) (15 pts) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Suppose that there exist two nonzero complex numbers ω_1 and ω_2 such that

(a) $\{\omega_1, \omega_2\}$ form a basis for $\mathbb{C}$ if $\mathbb{C}$ is viewed as a vector space over $\mathbb{R}$.

(b) $f(z+\omega_1)=f(z+\omega_2)=f(z)$ for all $z\in\mathbb{C}$

Show that f is constant.

Proof. Let

$$F\triangleq\{c_1\omega_1+c_2\omega_2:0\leq c_1,c_2\leq 1\}$$

Because F is by Hiene-Borel compact (F is the closed parallelogram with vertices being $\{0,\omega_1,\omega_2,\omega_1+\omega_2\}$), and f is continuous on F, we know by EVT $|f|$ is bounded by some $M>0$ on F. Now, for all $z\in\mathbb{C}$, because $\{\omega_1,\omega_2\}$ form a basis

$$z=a_1\omega_1+a_2\omega_2 \text{ for some unique pair } a_1,a_2\in\mathbb{R}$$

By Euclidean algorithm, we may further write

$$z=(n_1+c_1)\omega_1+(n_2+c_2)\omega_2$$

For some $n_1,n_2\in\mathbb{Z}$ and $c_1,c_2\in[0,1]$. It then follows from the premise that

$$\begin{aligned}|f(z)|&=|f((n_1+c_1)\omega_1+(n_2+c_2)\omega_2)|\\&=|f(c_1\omega_1+c_2\omega_2)|\leq M\end{aligned}$$

We have shown f is bounded on the whole $\mathbb{C}$. Because f is entire, it follows from the Liouville's Theorem that f is a constant. ■

Theorem 1.1.1. (Truncated Laurent Series of $\cot(\pi z)$) Near 0, we have

$$\cot(\pi z)=\frac{1}{\pi z}-\frac{\pi z}{3}-\frac{(\pi z)^3}{45}-\frac{2(\pi z)^5}{945}-\frac{(\pi z)^7}{4725}+O(z^9)$$

Near n, we have

$$\cot(\pi z)=\frac{1}{\pi(z-n)}-\frac{\pi(z-n)}{3}-\frac{(\pi(z-n))^3}{45}-\frac{2(\pi(z-n))^5}{945}-\frac{(\pi(z-n))^7}{4725}+O((z-n)^9)$$

Proof. Direct computation allow us to expand the Taylor series of tan at 0

$$\tan(\pi z)=z+\frac{(\pi z)^3}{3}+\frac{2(\pi z)^5}{15}+\frac{17(\pi z)^7}{315}+\frac{62(\pi z)^9}{2835}+O\left(z^{11}\right)$$

This implies

$$\cot(\pi z)=\frac{1}{\pi z+\frac{(\pi z)^3}{3}+\frac{2(\pi z)^5}{15}+\frac{17(\pi z)^7}{315}+\frac{62(\pi z)^9}{2835}+O(z^{11})}$$

Then by long division, we may compute

$$\cot(\pi z) = \frac{1}{\pi z} - \frac{\pi z}{3} - \frac{(\pi z)^3}{45} - \frac{2(\pi z)^5}{945} - \frac{(\pi z)^7}{4725} + O(z^9)$$

The truncated Laurent series around n follows from the fact $\cot(\pi z)$ is a function with period 1. ■

Theorem 1.1.2. (Two damned facts I can't prove) Let $\xi \notin \mathbb{Z}$ and $f(z) \triangleq \frac{1}{z-\xi} + \frac{1}{z}$. There exists some sequence $(C_N)_{N=1}^{\infty}$ of closed contours such that the region enclosed by C_N converge to $\mathbb{C}$ and

$$\lim_{N\to\infty} \int_{C_N} f(z)\cot(\pi z)dz = 0$$

Also,

$$\log\left(\frac{\pi}{2}\right) + \sum_{n=1}^{\infty} \log(1 - \frac{1}{4n^2}) = 0$$

Question 8

(4) (20 pts) Prove the formula

$$\sin \pi z = \pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

using the following steps:

(a) Consider $f(z) = \frac{1}{z-\xi} + \frac{1}{z}$, and show that when $\xi \notin \mathbb{Z}$,

$$\pi\cot(\pi\xi) = \frac{1}{\xi} + \sum_{n=1}^{\infty} \frac{2\xi}{\xi^2 - n^2}$$

(b) Integrate $\pi \cot \pi z$ along a suitable contour to show that

$$\log \sin \pi z = \log \pi z + \sum_{n=1}^{\infty} \log(1 - \frac{z^2}{n^2})$$

where log is chosen such that $\log 1 = 0$ in each term.

Proof. Fix $\xi \notin \mathbb{Z}$ and define

$$f(z) \triangleq \frac{1}{z-\xi} + \frac{1}{z} = \frac{2z-\xi}{z(z-\xi)}$$

Because $\cot(\pi z)$ has poles at $z = n \in \mathbb{Z}$ and they are all simple, we know $f(z)\cot(\pi z)$ have simple poles at $z = n \in \mathbb{Z}$'$z = \xi$, and have a double pole at 0. Using Theorem 1.1.1, we may compute

$$\begin{aligned}
\operatorname{Res}(f(z)\cot(\pi z), 0) &= \operatorname{Res}\left(\frac{\cot(\pi z)}{z-\xi}, 0\right) + \operatorname{Res}\left(\frac{\cot(\pi z)}{z}, 0\right) \\
&= \operatorname{Res}\left(\frac{\cot(\pi z)}{z-\xi}, 0\right) \\
&= \lim_{z\to 0} \frac{z\cot(\pi z)}{z-\xi} = \frac{\frac{1}{\pi}}{-\xi} = \frac{1}{-\pi\xi}
\end{aligned}$$

And compute

$$\operatorname{Res}(f(z)\cot(\pi z), n) = \lim_{z\to n} f(z)z\cot(\pi z) = \frac{2n-\xi}{\pi n(n-\xi)}$$

And compute

$$\operatorname{Res}(f(z)\cot(\pi z), \xi) = \lim_{z\to\xi}(1 + \frac{z-\xi}{z})\cot(\pi z) = \cot(\pi\xi)$$

It now follows from Theorem 1.1.2 and Residue Theorem that

$$\begin{aligned}
0 &= \lim_{N\to\infty} \int_{C_N} f(z)\cot(\pi z)dz \\
&= \lim_{N\to\infty} \frac{1}{-\pi\xi} + \cot(\pi\xi) + \sum_{0<|n|\le N} \frac{2n-\xi}{\pi n(n-\xi)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\cot(\pi\xi) &= \frac{1}{\pi\xi} + \sum_{|n|>0} \frac{2n-\xi}{\pi n(\xi-n)} \\
&= \frac{1}{\pi\xi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{2n-\xi}{n(\xi-n)} + \frac{-2n-\xi}{-n(\xi+n)}\right) \\
&= \frac{1}{\pi\xi} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(2n-\xi)(\xi+n) + (2n+\xi)(\xi-n)}{n(\xi^2-n^2)} \\
&= \frac{1}{\pi\xi} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{2n\xi}{n(\xi^2-n^2)} \\
&= \frac{1}{\pi\xi} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{2\xi}{\xi^2-n^2}
\end{aligned}$$

Multiplying both side with π, we now have

$$\pi\cot(\pi\xi) = \frac{1}{\xi} + \sum_{n=1}^{\infty}\frac{2\xi}{\xi^2 - n^2} \tag{1.5}$$

Now, note that

$$\begin{aligned}
\frac{d}{dz}\Big(\log\sin(\pi z)\Big) &= \frac{\pi\cos(\pi z)}{\sin(\pi z)} = \pi\cot(\pi z) \\
\frac{d}{dz}\Big(\log \pi z\Big) &= \frac{1}{z} \\
\frac{d}{dz}\Big(\sum_{n=1}^{\infty}\log(1-\frac{z^2}{n^2})\Big) &= \sum_{n=1}^{\infty}\frac{d}{dz}\log(1-\frac{z^2}{n^2}) = \sum_{n=1}^{\infty}\frac{\frac{-2z}{n^2}}{1-\frac{z^2}{n^2}} = \sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}
\end{aligned}$$

Fix $z \notin \mathbb{Z}$, and let γ be some contour that starts at $\frac{1}{2}$ and ends at z without touching any integer. It follows from Theorem 1.1.2 and fundamental theorem of calculus for complex function that

$$\begin{aligned}
\log(\sin(\pi z)) &= \int_\gamma \pi\cot(\pi z)dz \\
&= \int_\gamma \Big(\frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}\Big)dz \\
&= \log(\pi z) + \sum_{n=1}^{\infty}\log(1-\frac{z^2}{n^2})
\end{aligned}$$

Taking exponential on both side, we finally have

$$\sin(\pi z) = \pi z\prod_{n=1}^{\infty}(1-\frac{z^2}{n^2})$$

■

Question 9

(5) (15pts) If $|a| > e$, use Rouche's Theorem to prove that the equation

$$e^z = az^n$$

has n roots with $|z| < 1$

Proof. Let $\mathbf{D}$ be the unit disk centered at origin. Define entire $f, g : \mathbb{C} \to \mathbb{C}$ by

$$f(z) \triangleq az^n \text{ and } g(z) \triangleq -e^z$$

Because $e < |a|$, on $\partial \mathbf{D}$, we have

$$|g(z)| = |e^z| = e^{\operatorname{Re} z} \leq e < |a| = |az^n| = |f(z)|$$

Therefore, by Rouche's Theorem, $az^n - e^z = f + g$ has the same number of zeros in $\mathbf{D}$ as $f = az^n$. It is clear that az^n only has zero $z = 0$ with multiplicity n. Therefore,

$$az^n - e^z \text{ has } n \text{ zeros in } \mathbf{D}$$

We have shown

$$e^z = az^n \text{ has } n \text{ roots in } \mathbf{D}$$

■

Question 10

(6) (10pts) Let $f : \mathbf{D} \to \mathbb{C}$ be a holomorphic function, where $\mathbf{D}$ is the unit open disk centering $\mathbf{D}$. If $f(0) = 0$ and $|f(z)| \leq 1$ on $\mathbf{D}$, prove that

(a) $|f(z)| \leq |z|$ for all $z \in \mathbf{D}$.

(b) $|f'(0)| \leq 1$

Proof. Because the proposition is trivial for constant f, suppose f is non-constant. Define $g : \mathbf{D} \to \mathbb{C}$ by

$$g(z) \triangleq \begin{cases} \frac{f(z)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0 \end{cases}$$

Applying maximum modulus principle to g on open disk centered at origin with radius $r < 1$, we have

$$|f'(0)| = |g(0)| \leq |g(z)| = \frac{|f(z)|}{|z|} \leq \frac{1}{r} \text{ for some } z \text{ such that } |z| = r$$

Letting $r \to 1$, this implies

$$|f'(0)| \leq 1$$

Note that when $z = 0$,

$$|f(z)| = |f(0)| = 0 \leq 0 = |z|$$

Fix $z \neq 0 \in \mathbf{D}$. The maximum modulus principle implies that for each open disk D_r centered at origin with radius $r < 1$ that contains z, there exists some $z_r \in \partial D_r$ such that

$$|g(z)| \leq |g(z_r)| = \frac{|f(z_r)|}{|z_r|} \leq \frac{1}{r}$$

Letting $r \to 1$, we now have

$$\frac{|f(z)|}{|z|} = |g(z)| \leq 1$$

Multiplying both side with $|z|$, we now have

$$|f(z)| \leq |z| \text{ for all } z \neq 0 \in \mathbf{D}$$

$\blacksquare$

Theorem 1.1.3. (Open Mapping Theorem for Disk) Let U be an open disk and $f : U \to \mathbb{C}$ be some non-constant holomorphic function.

$$f(U) \text{ is open}$$

Proof. Fix arbitrary $w_0 \in f(U)$, and let $z_0 \in U$ satisfy $f(z_0) = w_0$. Define holomorphic $g : U \to \mathbb{C}$ by $g(z) \triangleq f(z) - w_0$. Because g is non-constant holomorphic, by Identity Theorem, the zeros of g are isolated. Note that z_0 is a zero of g. We may now let B be a closed disk centering z_0 such that B is contained by U and contains no other zero of g. Because $\partial B \subseteq U$ is compact, by EVT, we may let

$$a \triangleq \min_{\partial B} |g|$$

Note that $a > 0$ because g has no zeros in $\partial B \subseteq B$. Let D be the open disk centering w_0 with radius a. Fix $w_1 \neq w_0 \in D$. If we define $h : U \to \mathbb{C}$ by $h(z) \triangleq f(z) - w_1$, we see that for all $z \in \partial B$, we have

$$|g(z) - h(z)| = |w_0 - w_1| < a \leq |g(z)|$$

Therefore, by Rocuhe's Theorem, h has some zero in B°. That is, $f(z_1) = w_1$ for some $z_1 \in B^\circ$. Because w_1 is arbitrarily picked from D, we have shown $D \subseteq f(U)$. That is, w_0 is an interior point of $f(U)$. Because w_0 is arbitrarily picked from $f(U)$. We have shown $f(U)$ is open. $\blacksquare$

Question 11

(7) (20pts) Let $f : D \to \mathbb{C}$ be a non-constant holomorphic function defined on a domain D contained in $\mathbb{C}$. Prove that f is an open mapping.

Proof. Let $E \subseteq D$ be open. We are required to show $f(E)$ is open. Because the set of open disk form a basis for $\mathbb{C}$, we may let

$$E = \bigcup_{i \in I} U_i$$

where $\{U_i\}_{i\in I}$ is a collection of open disk. It now follows from Theorem 1.1.3 that

$$f(E) = \bigcup_{i\in I} f(U_i) \text{ is open}$$

because $f(E)$ is a union of open sets. ■