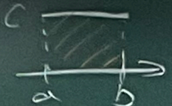
Properties:

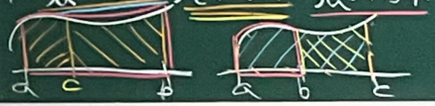
1° $\int_a^b c\,dx = c(b-a)$



2° $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

3° $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$

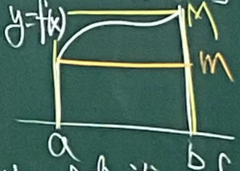
4° $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$



5° If $f(x) \geq 0$ on $[a,b]$, $\Rightarrow \int_a^b f(x)\,dx \geq 0$

6° $f(x) \geq g(x)$ on $[a,b]$, $\Longrightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

7° $m \leq f(x) \leq M$, on $[a,b] \Rightarrow m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$



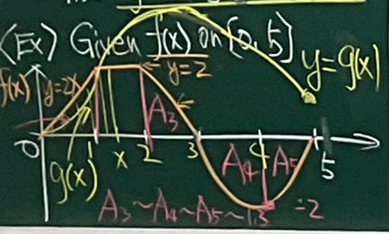
pf: Use definition of integration. ※

§4.3 Fundamental Theorem of Calculus. F.T.C.

Let $f(x)$ be a conti. fn on $[a,b]$, $x \in [a,b]$

Also $g(x) = \int_0^x f(t)\,dt$

<Ex> Given $f(x)$ on $[0,5]$



Compute $g(0) = 0$

$g(1) = \int_0^1 2x\,dx = A_1 = 1$

$g(2) = \int_0^2 f(t)\,dt = \int_0^1 2t\,dt + \int_1^2 2\,dt = A_1 + A_2 = 3$

$g(3) = \int_0^1 + \int_1^2 + \int_2^3 f(t)\,dt = A_1 + A_2 + A_3 \approx 4.3$

$g(4) = \int_0^1 + \int_1^2 + \int_2^3 + \int_3^4 f(t)\,dt = A_1 + A_2 + A_3 - A_4 \approx 4.3 - 1.3 \approx 3$

$g(5) =$ 〃 $+ \int_4^5 f(t)\,dt = A_1 + A_2 + A_3 - A_4 - A_5 \approx 3 - 1.3 \approx 1.7$

F.T.C. part 1

If f = conti. on $[a,b]$, then $g(x) = \int_a^x f(t)\,dt$ (*)

is conti. on $[a,b]$ and differentiable on (a,b)

and $g'(x) = f(x)$

pf: For $x \in (a,b)$, show g = diff.

$\Leftrightarrow \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$ exists $(= f(x))$

Check $g(x+h) - g(x) \overset{(*)}{=} \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$

(*) $= \int_x^{x+h} f(t)\,dt$

$\Rightarrow \frac{g(x+h)-g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt$

Set $m \equiv \min f(t)$, $t \in [x, x+h]$

$M \equiv \max f(t)$, $t \in [x, x+h]$

(**) $\Rightarrow m = \frac{mh}{h} \le \boxed{\frac{1}{h}\int_x^{x+h} f(t)\,dt} \le \frac{Mh}{h} = M$

(*) $h \to 0$ $m, M \to f(x)$ ($\because f$ = conti)

$\Rightarrow f(x) \le \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt \le f(x)$

$\therefore \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x)$

i.e. $g'(x) = f(x)$

For $x = a$, $\lim_{h\to 0^+} \frac{g(x+h)-g(x)}{h} = f(a)$ one-sided limit

For $x = b$, $\lim_{h\to 0^-} \frac{g(b+h)-g(b)}{h} = f(b)$ //

$\xrightarrow{\text{Thm 2.2.4}} \lim_{h\to 0^+} g(a+h) = g(a)$ & $\lim_{h\to 0^-} g(b+h) = g(b)$

i.e. g = conti on $[a,b]$ ※

FTC part 2 If f = conti. on $[a,b]$

$\Rightarrow \int_a^b f(x)\,dx = F(b) - F(a)$, F = antiderivative of f

pf Let $g(x) = \int_a^x f(t)\,dt$ / (antiderivative of f) $= F(x) + C$

$F(b) - F(a) = (g(b) - \cancel{C}) - (\underset{\to 0}{g(a)} - \cancel{C})$ Corollary 3.2.7 $= \int_a^b f(t)\,dt$ ※

FTC: Suppose f = conti. on [a,b]

(1) If $g(x) \equiv \int_a^x f(t)\,dt \Rightarrow g'(x) = f(x)$

(2) If $F'(x) = f(x)$, $\Rightarrow \int_a^b f(x)\,dx = F(b) - F(a)$

(1)' $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

(2)' $\int_a^b F'(x)\,dx = F(b) - F(a)$

§4.4 Indefinite Integrals and the Net change Theorem

Note $\int f(x)\,dx \equiv$ indefinite integral and (= an antiderivative of f)

$\int f(x)\,dx = F(x) + C$ means $F'(x) = f(x)$

$(F(x)+C)' = f(x)$

⟨EX⟩ $\int x^2\,dx = \frac{1}{3}x^3 + C$ for any const. C

Check $(\frac{1}{3}x^3 + C)' = x^2$

Table of indefinite integrals.

$\int c f(x)\,dx = c\int f(x)\,dx$

$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

$\int k\,dx = kx + C$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$, $n+1 \neq 0$

$\int \sin x\,dx = -\cos x + C$

$\int \cos x\,dx = \sin x + C$

$\int \sec^2 x\,dx = \tan x + C$

$\int \csc^2 x\,dx = -\cot x + C$

$\int \sec x \tan x\,dx = \sec x + C$

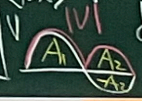
$\int \csc x \cot x\,dx = -\csc x + C$

Net change Thm

The integral of a rate of change is the net change

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

⟨EX⟩ v = velocity



displacement $\int_{t_1}^{t_2} v(t)\,dt = A_1 - A_2$

distance $\int_{t_1}^{t_2} |v(t)|\,dt = A_1 + A_2$

§4.5 Substitution Rule

(EX) $\int \underline{2x}\sqrt{1+x^2}\,\underline{dx} = \int \sqrt{1+x^2}\,\underline{dx^2}$

$\underset{u=1+x^2}{=} \int \sqrt{u}\,du$ $\quad dx^2 = 2x\,dx$, $\quad d(1+x^2) = 2x\,dx$

$= \int u^{\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{1}{2}+1} + C$

$= \frac{2}{3}(1+x^2)^{\frac{3}{2}} + C$ #

In general, $\int F'(g(x))g'(x)\,dx = F(g(x)) + C$

or $\int f(g(x))g'(x)\,dx \overset{F'(t)=f(t)}{=\!=} F(g(x)) + C$

(Substitution Rule)

$\int f(g(x))g'(x)\,dx \overset{u=g(x)}{=} \int f(u)\,du$

Def $F(b) - F(a) \equiv F(x)\Big|_a^b$

Substitution Rule for definite integrals

If $g' =$ conti. on $[a,b]$, $f =$ conti. on $g([a,b])$ (Dom, range of g)

$$\boxed{\int_a^b f(g(x))g'(x)\,dx \overset{u=g(x)}{=} \int_{g(a)}^{g(b)} f(u)\,du} \quad (*)$$

Chain Rule

Pf: Let $F' = f \Rightarrow \int_a^b (F(g(x)))'\,dx \overset{?}{=} \int_a^b F'(g(x))\cdot g'(x)\,dx$ // $(*)$

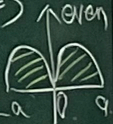
$\overset{FTC}{\Longrightarrow} F(g(x))\Big|_a^b = F(u)\Big|_{g(a)}^{g(b)}$ #

Integrals of Symmetric fns

Suppose $f =$ conti. on $[-a, a]$

(a) $f = $ even $(\Leftrightarrow f(-x) = f(x)) \Longrightarrow$

$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$



(b) $f = $ odd $(\Leftrightarrow f(-x) = -f(x))$

$\int_{-a}^{a} f(t)\,dt = 0$