6.8 PDE HW 8

Question 126

Solve the Schrodinger equation

$$\begin{cases} u_t = iku_{xx} \text{ for } 0 < x < l \textbf{ (Homogeneous DE)} \\ u_x(0,t) = u(l,t) = 0 \textbf{ (BC)} \end{cases}$$

for real $k \in (0, l)$.

Proof. Again we do the separation of the variables

$$u \triangleq T(t)X(x)$$

Some tedious efforts shows that u is a solution of this original question as long as X, T satisfy the following ODE

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) = X(l) = 0 \end{cases} \quad \text{and } T'(t) + \lambda ikT(t) = 0$$

where $\lambda \in \mathbb{C}$ is arbitrary constant. The solution of the second ODE is obviously

$$T(t) \triangleq Ce^{-\lambda ikt}$$

where $C \in \mathbb{C}$ is arbitrary constant. It remains to find what values can λ take so that X has non-trivial solutions. If $\lambda = 0$, then to satisfy the ordinary differential equation, the solution must take the forms $X = C + Dx$, where $C, D \in \mathbb{C}$ are arbitrary constant. Plugging the initial conditions, we see that $C = D = 0$. In other words, if $\lambda = 0$, then X can only be trivial. If $\lambda \neq 0$, ODE of X suggest that X must take the form

$$X \triangleq Ae^{\gamma x} + Be^{-\gamma x}$$

where $\gamma \in \mathbb{C}$ satisfy $\gamma^2 = -\lambda$. Plug in $X'(0) = 0$, we see

$$0 = \gamma(A - B)$$

which implies $A = B$. Plug in $X(l) = 0$, we see

$$0 = Ae^{\gamma l} + Be^{-\gamma l} = A(e^{\gamma l} + e^{-\gamma l})$$

Then for X to be non-trivial, we must have

$$e^{\gamma l} + e^{-\gamma l} = 0$$

By periodicity property of exponential function, we then can deduce

$$\gamma = \frac{i\pi(2n+1)}{l} \text{ and } \lambda = \frac{(2n+1)^2\pi^2}{l^2}$$

It then follows from $X = Ae^{\gamma x} + Be^{-\gamma x}$ that

$$\begin{aligned} X &= (A+B)\cos\left(\frac{\pi(2n+1)x}{l}\right) + (A-B)i\sin\left(\frac{\pi(2n+1)x}{l}\right) \\ &= (A+B)\cos\left(\frac{\pi(2n+1)x}{l}\right) \quad (\because A = B) \end{aligned}$$

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