1. In class 3

(1) Provide a counterexample to the following statement: Suppose

$$f(z) = u(x, y) + iv(x, y)$$

is defined in a neighborhood of $z_0 = a + ib$. If the partial derivatives of u and v exist at (a, b) and satisfy the Cauchy-Riemann equations $u_x(a, b) = v_y(a, b)$ and $u_y(a, b) = -v_x(a, b)$, then f is complex differentiable at z_0.

(2) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose that f is differentiable on (a, b) and that $f'(x) = 0$ for all $x \in (a, b)$. Prove that f is a constant function.

(3) Let $B = B_R(x_0)$ be the open ball in $\mathbb{R}^n$ centered at x_0 with radius $R > 0$. Prove that if $f : B \to \mathbb{R}$ is a differentiable function such that $\nabla f = 0$ on B, then f is a constant function. **Hint**: Given $x \in B$, define $g_x(t) = f((1 - t)x_0 + tx)$. Prove that g_x is continuous on $[0, 1]$ and differentiable on $(0, 1)$ and find $g_x'(t)$ for $0 < t < 1$.

(4) Let U be an open subset of $\mathbb{R}^n$. A function $f : U \to \mathbb{R}$ is called **locally constant** if, for each $x \in U$, there exists an open neighborhood W of x such that $W \subset U$ and $f : W \to \mathbb{R}$ is constant on W. Prove that f is a locally constant function and only if $\nabla f = 0$ on U.

(5) Let D be an open, connected subset of $\mathbb{R}^n$. Prove that if $f : D \to \mathbb{R}$ is a locally constant function, then f is a constant function.