11.2 Dirichlet's Problem

Let $D(R) \subseteq \mathbb{R}^2$ denote the unit open disk, and let $g : \partial D(R) \to \mathbb{R}$ be a continuous function defined on the boundary. Dirichlet's problem asked:

> Is there continuous $u : \overline{D(R)} \to \mathbb{R}$ such that u is harmonic on the interior and agree with g on boundary?

Let's first suppose such u exists. If we define $f : D(R) \to \mathbb{C}$ by

$$f = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

then f is holomorphic according to Cauchy-Riemann Theorem. Because f is holomorphic, we may let $F : D(R) \to \mathbb{C}$ be its anti-derivative. It is clear that by computation that F has the form

$$F = u + iv + C$$

where C is some constant. Let $C = 0$. By Cauchy Integral Formula, for all $r < R$, we have

$$F(z) = \sum_{n=0}^{\infty} c_n z^n \text{ for } |z| < r$$

for some c_n. By Identity Theorem, we can just write

$$F(z) = \sum_{n=0}^{\infty} c_n z^n \text{ for } |z| < R$$

If we write

$$c_n = \alpha_n + i\beta_n$$

because $u = \operatorname{Re} F$, we may compute

$$u = \sum_{n=0}^{\infty} r^n (a_n \cos(n\theta) + b_n \sin(n\theta)) \text{ on } D(R)$$

where $a_n = \alpha_n$ and $b_n = -\beta_n$. Because u is continuous, we know

$$g(\theta) = \lim_{r\to 1}\Big(\sum_{n=0}^{\infty} r^n(a_n\cos(n\theta) + b_n\sin(n\theta))\Big) \text{ exists} \tag{11.4}$$

It then follows from a non-trivial estimation (you may do this by yourself) that

$$g(\theta) = \sum_{n=0}^{\infty} a_n\cos(n\theta) + b_n\sin(n\theta)$$

Because g is continuous, we know

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\cos(n\theta)d\theta \text{ and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\sin(n\theta)d\theta \text{ for } n \geq 1$$

and

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)d\theta$$

We thus may write

$$\begin{aligned}
u &= \sum_{n=0}^{\infty} r^n(a_n\cos(n\theta) + b_n\sin(n\theta)) \text{ on } D(R) \qquad (11.5)\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\varphi)d\varphi + \sum_{n=1}^{\infty}\frac{r^n}{\pi}\int_{-\pi}^{\pi} g(\varphi)\cos(n\varphi)\cos(n\theta) + g(\varphi)\sin(n\varphi)\sin(n\theta)d\varphi\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\varphi)d\varphi + \sum_{n=1}^{\infty}\frac{r^n}{\pi}\int_{-\pi}^{\pi} g(\varphi)\cos\big(n(\theta-\varphi)\big)d\varphi\\
&= \int_{-\pi}^{\pi} g(\varphi)\Big[\frac{1}{2\pi} + \sum_{n=1}^{\infty}\frac{r^n}{\pi}\cos\big(n(\theta-\varphi)\big)\Big]d\varphi \text{ on } D(R)
\end{aligned}$$

Note that the last equality holds since the series is dominated by some constant multiples of g. In fact, we can simplify the expression using the next lemma.

Lemma 11.2.1. For $0 \leq r < 1$ and $\xi \in \mathbb{R}$,

$$\frac{1}{2} + \sum_{n=1}^{\infty} r^n\cos(n\xi) = \frac{1}{2}\cdot\frac{1-r^2}{1-2r\cos(\xi)+r^2}$$

Proof. Let $w = re^{in\xi}$. Compute $|w| = |r| < 1$. Therefore,

$$\begin{aligned}
\frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos(n\xi) &= \frac{1}{2} + \frac{1}{2}\Big(\sum_{n=1}^{\infty} w^n + \overline{w^n}\Big) \\
&= \frac{1}{2}\Big(\sum_{n=0}^{\infty} w^n + \sum_{n=1}^{\infty} \overline{w^n}\Big) \\
&= \frac{1}{2}\Big(\frac{1}{1-w} + \frac{\overline{w}}{1-\overline{w}}\Big) \\
&= \frac{1}{2} \cdot \frac{1-|w|^2}{|1-w|^2} = \frac{1}{2} \cdot \frac{1-r^2}{1-2r\cos\theta + r^2}
\end{aligned}$$

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In conclusion,

$$u = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\varphi)\frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}d\varphi \text{ on } D(R)$$

The fact that this expression have boundary limit g follows from Equation 11.4 and Equation 11.5.