Exercise 1

Let R be a complex algebra with 1_A and $a \in R$. Given a complex polynomial $f(Z) = a_0 + a_1 Z + \cdots + a_n Z^n$, we define the evaluation of f at a by

$$f(a) = a_0 1_A + a_1 a + \cdots + a_n a^n.$$

(1) Let $R = \mathbb{C}$ and $a = 1 + i$. Given $f(Z) = Z^3$. Evaluate $f(a)$.

(2) Let $R = M_{2\times 2}(\mathbb{C})$ be the algebra of 2×2 complex matrices. Take $a = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $g(Z) = 3 + 2Z$ Evaluate $g(a)$.

(3) Let R be the algebra of complex valued periodic functions of period 2π, i.e. $a \in \mathbb{R}$ is a continuous function $a : \mathbb{R} \to \mathbb{C}$ so that $a(x + 2\pi) = a(x)$. Let $e(x) = \cos x + i \sin x$ and $h(Z) = 1 + Z + Z^2 + \cdots + Z^9$. Find $h(e)$.