Rule:

1° To find f_x, regard y as a const. and diff. f w.r.t. x

2° To find f_y, ... x ...

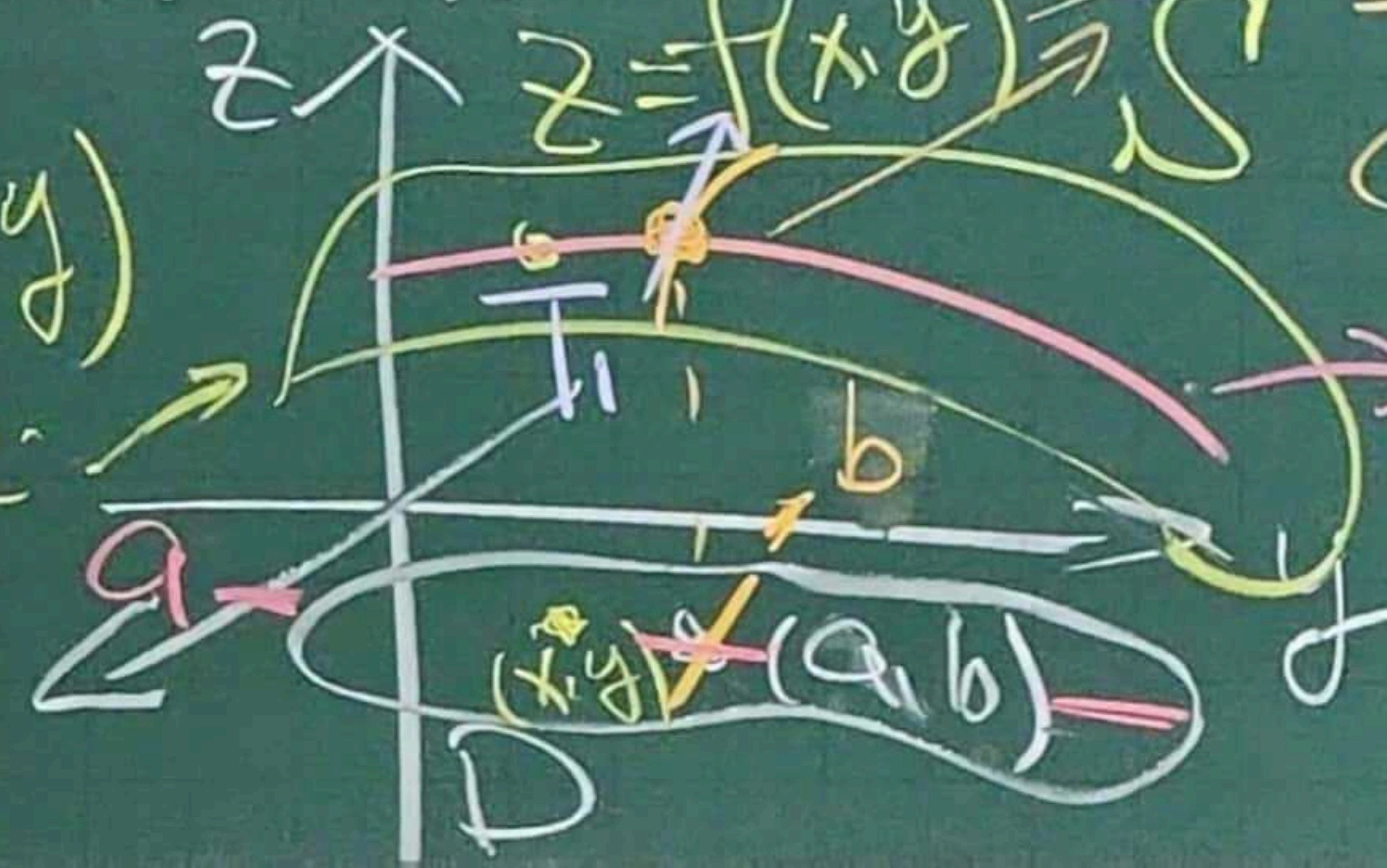
Interpretation: $P(a,b,f(a,b))$

$z=f(x,y)$ surface

$z=f(x,y)$ → S

$f(x,b)\equiv g(x)$, curve C_1

$f(a,y)\equiv G(y)$ curve C_2



$g'(a)=f_x(a,b)$. slope of tangent line

fixed T_1 of curve C_1 at P

$G'(b)=f_y(a,b)$, //

T_2 of curve C_2 at P

$w=f(x,y,z)$ at (a,b,c)

$f_x(a,b,c)$ = slope of tangent line T_1

fixed of curve C_1 at P.

$f_y(a,b,c)$, $f_z(a,b,c)$

Def Let $w = f(x, y, z)$ 蕭祐傑

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y, z) \equiv f_x(x, y, z) = \lim_{\Delta h \to 0} \frac{f(x+\Delta h, y, z) - f(x, y, z)}{\Delta h}$$

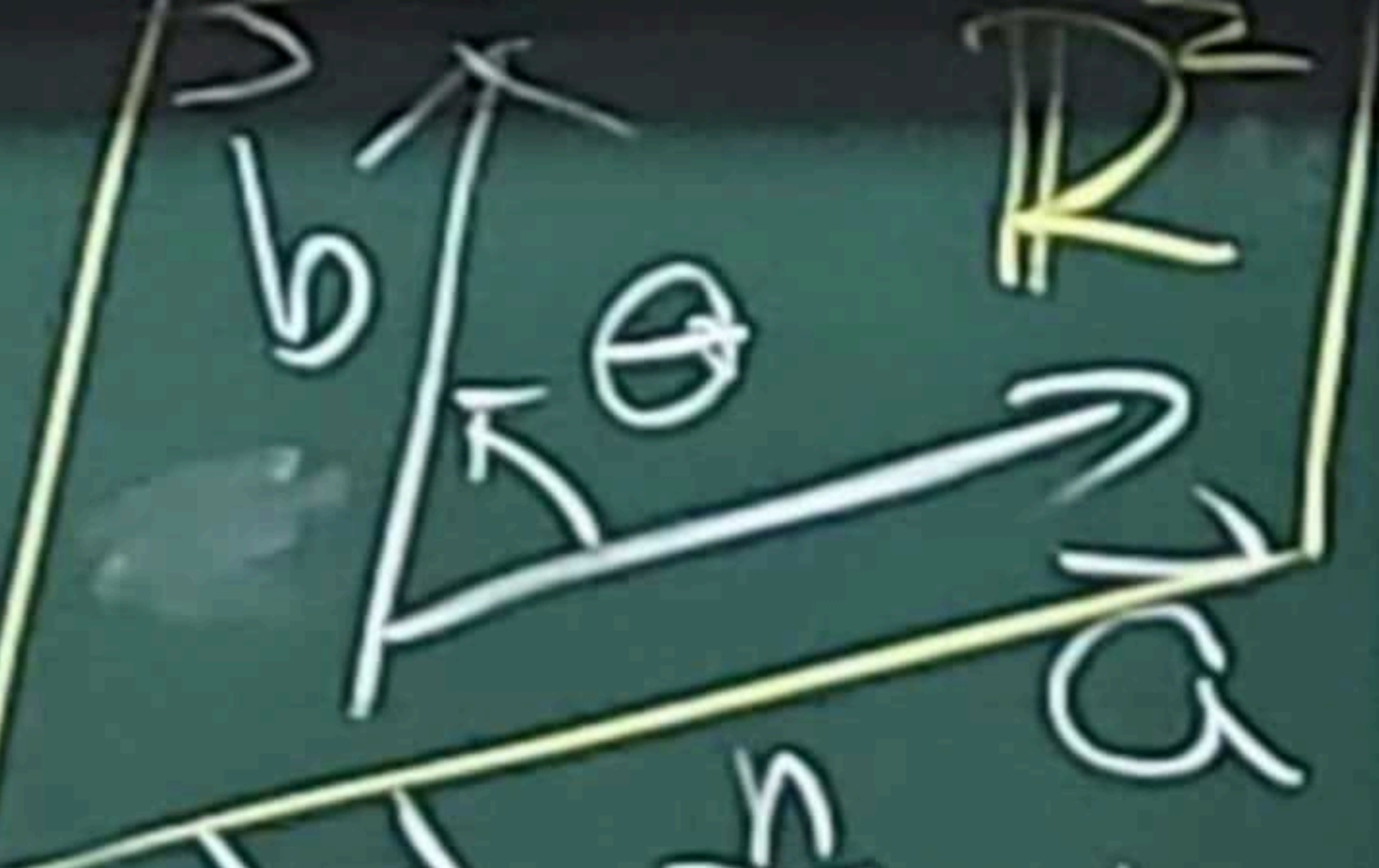
$$\frac{\partial w}{\partial y} = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y, z) \equiv f_y(x, y, z) = \lim_{\Delta k \to 0} \frac{f(x, y+\Delta k, z) - f(x, y, z)}{\Delta k}$$

$$\frac{\partial w}{\partial z} = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z} f(x, y, z) \equiv f_z(x, y, z) = \lim_{\Delta h \to 0} \frac{f(x, y, z+\Delta h) - f(x, y, z)}{\Delta h}$$

Def let $u = f(x_1 \cdots x_n)$

$$D_i f = f_{x_i} = \frac{\partial f}{\partial x_i} = \frac{\partial u}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1 \cdots x_i + h, \cdots x_n) - f(x_1 \cdots x_i \cdots x_n)}{h}$$

Recall $\vec{a}, \vec{b} \in \mathbb{R}^n$



$$\vec{a} \cdot \vec{b} \equiv \sum_{i=1}^{n} a_i b_i$$

$$= |\vec{a}||\vec{b}| \cos\theta$$

It is sufficient to assume $\vec{a}, \vec{b} \in \mathbb{R}^2$.

2nd partial derivatives of $z = f(x,y)$

陳彥佑

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \lim_{h\to 0} \frac{f_x(x+h,y,z) - f_x(x,y,z)}{h}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \lim_{k\to 0} \frac{f_x(x,y+k,z) - f_y(x,y,z)}{k}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \lim_{p\to 0} \frac{f_y(x+p,y,z) - f_y(x,y,z)}{p}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \lim_{q\to 0} \frac{f_y(x,y+q,z) - f_y(x,y,z)}{q}$$

Clairaut Thm: $f_{xy} = f_{yx}$

Def Partial derivatives of order 3

$z = f(x,y)$

$$f_{xyy} = (f_{xy})_y = \cdots\cdots$$

$$= \lim_{h\to 0} \frac{f_{xy}(x,y+h) - f_{xy}(x,y)}{h}$$

Partial Differential Eq'ns ($z = u(x,y)$)

(Ex) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ (*) (Laplace eq'n)

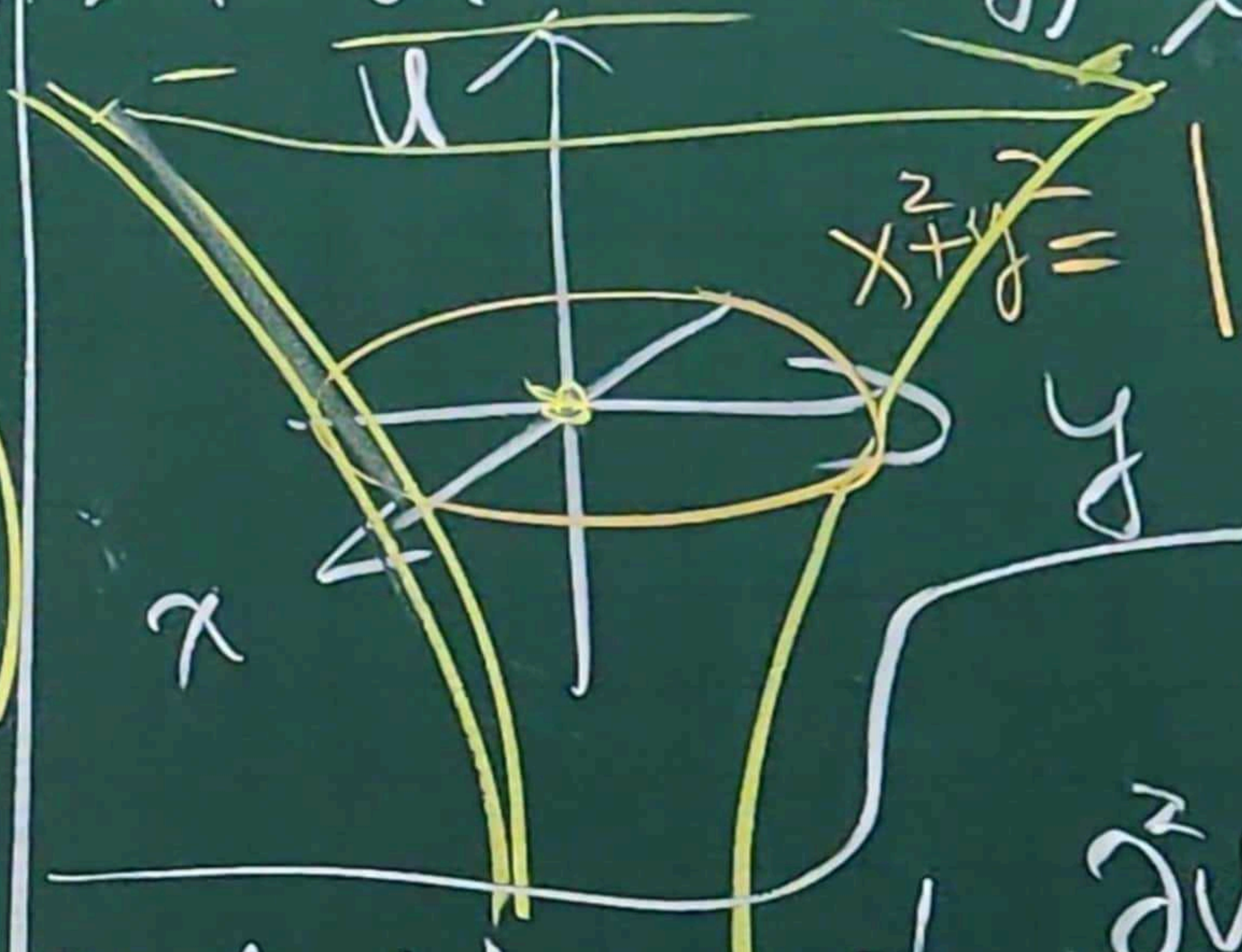
Def u = <u>harmonic fn</u> if u satisfies (*)

(Ex) $u(x,y) = e^x \sin y$ is a sol'n of (*)

$\dfrac{\partial^2}{\partial x^2}(e^x \sin y) + \dfrac{\partial^2}{\partial y^2}(e^x \sin y) \;(=0)$

$= \dfrac{\partial}{\partial x}\left(\dfrac{\partial}{\partial x}(e^x \sin y)\right) + \dfrac{\partial}{\partial y}\left(\dfrac{\partial}{\partial y}(e^x \sin y)\right) = \cdots = 0$

(Ex) Check $u(x,y) = \ln\sqrt{x^2+y^2}$ satisfies (*) if $x^2+y^2 \neq 0$



Def: Wave eq'n $\dfrac{\partial^2 u}{\partial t^2} - a^2 \dfrac{\partial^2 u}{\partial x^2} = 0$ (*)

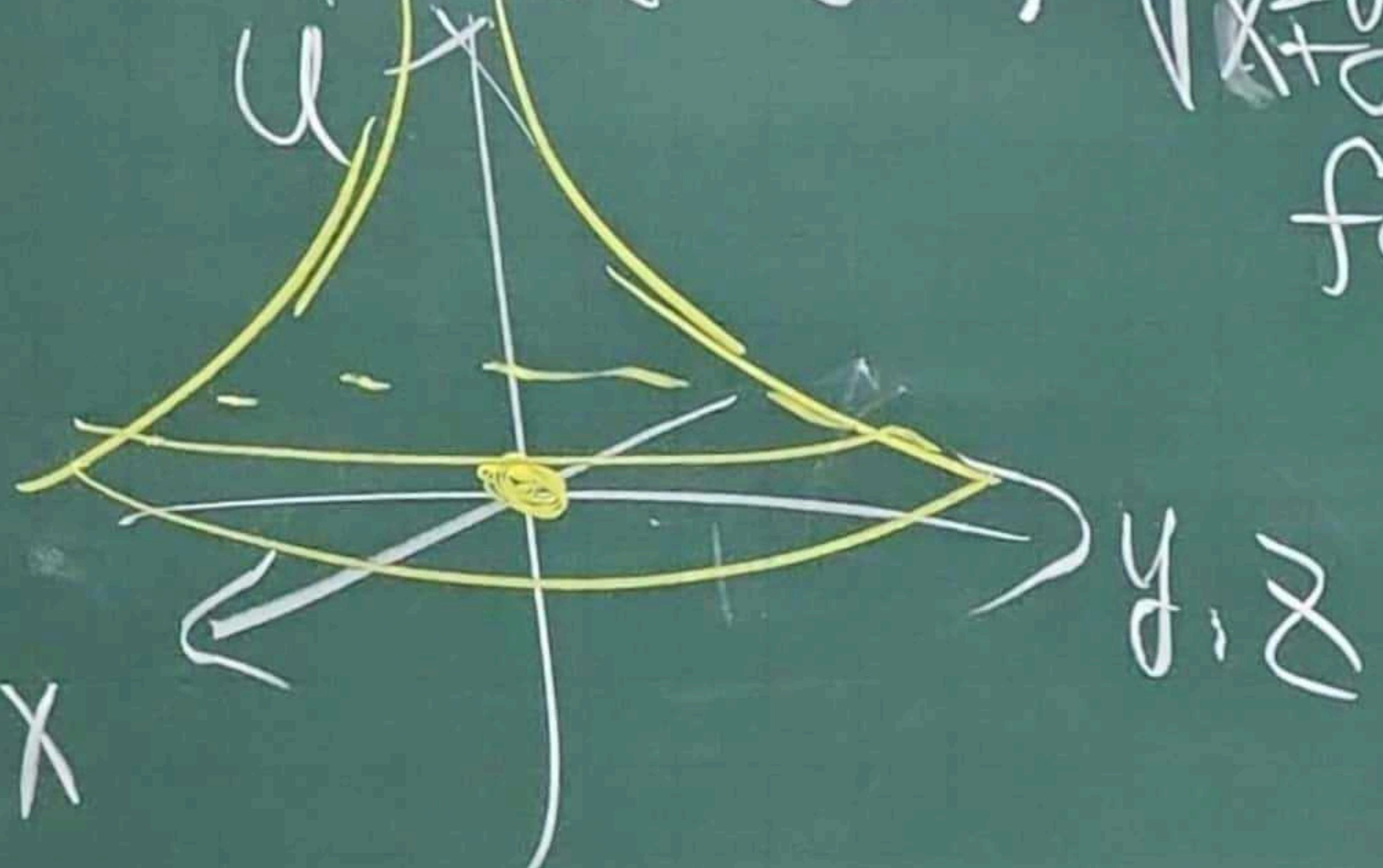
(Ex) check $u(x,t) = \sin(x - at)$ is a sol'n of (*)

⟨Ex⟩ $u_{xx}+u_{yy}+u_{zz}=0$ (*) Laplace eq'n in 3D

$\Leftrightarrow \Delta u = 0$

Check $u(x,y,z)=\dfrac{1}{\sqrt{x^2+y^2+z^2}}$ satisfies (*)

for $x^2+y^2+z^2\neq 0$



§14.4 Tangent Planes and Linear Approximations

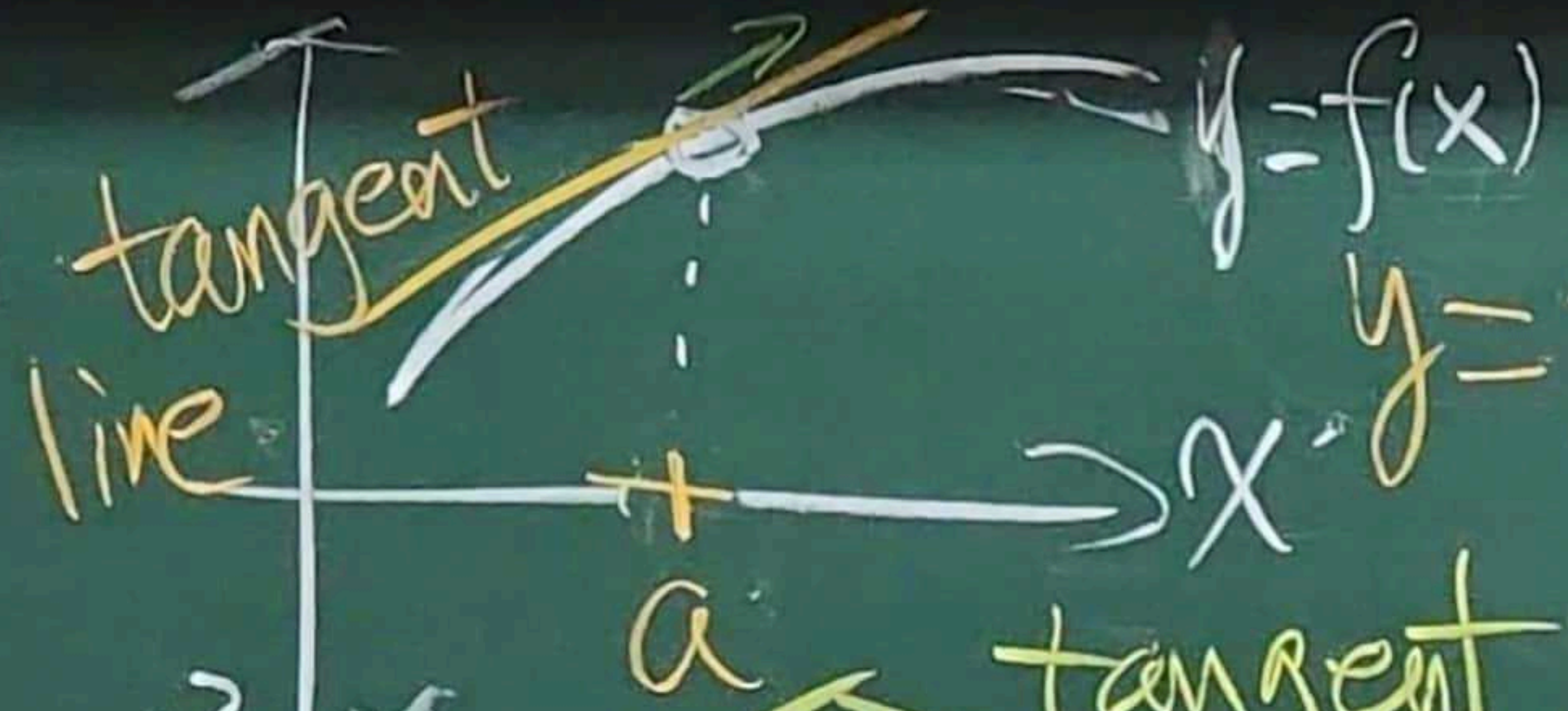
黃偵數

Recall $f(x) \approx f(a)+f'(a)(x-a)$

Taylor $\underbrace{f(a)+\dfrac{f'(a)}{1!}(x-a)}_{\text{Linear approximation}}+\underbrace{\dfrac{f^{(2)}(a)}{2!}(x-a)^2+\cdots}_{\text{error}}$

$$f(x,y) \overset{\text{Tayor}}{=} \underbrace{f(a,b)+\dfrac{f_x(a,b)}{1!}(x-a)+\dfrac{f_y(a,b)}{1!}(y-b)}_{\text{Linear approximation (plane)}}+\underbrace{\cdots}_{\text{error}}$$

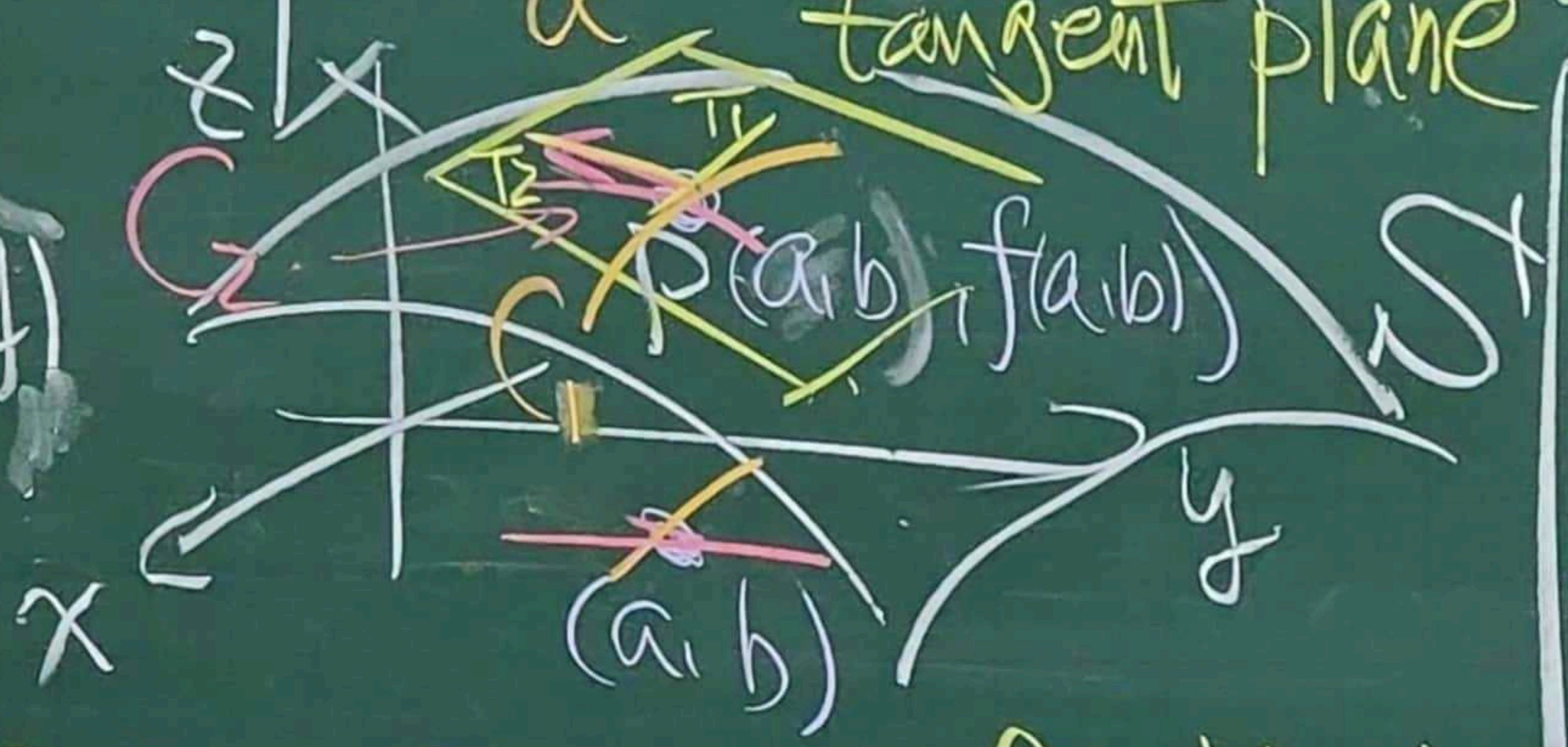
$y=f(x)$ curve



$y=f(a)+\underbrace{f'(a)}_{\text{slope}}(x-a) \Leftrightarrow y-y_0=f'(x_0)(x-x_0)$

$P(x_0,y_0,z_0)\in S$

$z=f(x,y)$ surface



Surface $S: z=f(x,y)$,

$\underline{C_1}: (x, y_0, f(x,y_0))$

$\underline{C_2}: (x_0, y, f(x_0,y))$

Tangent plane $P: z-\overset{f(x_0,y_0)}{\underline{z_0}}=\overset{a}{a}(x-\underline{x_0})+\overset{b}{b}(y-\underline{y_0})$

$\underline{T_1}$ = tangent line of C_1

$\overset{y=y_0}{z-z_0=a(x-x_0)}$

$z=\underbrace{f(a,b)}_{=z_0}+f_x(\underset{x_0}{a},\underset{y_0}{b})(x-a)+f_y(a,b)(y-b)$

$\Leftrightarrow z-z_0=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$

T_2 = tangent line of C $z - z_0 \overset{x=x_0}{=} b(y-y_0)$

slope of $T_1 = a = f_x(x_0, y_0)$

slope of $T_2 = b = f_y(x_0, y_0)$

$$\therefore \; \underbrace{z - \underbrace{f(x_0, y_0)}_{=z_0} = f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0)}_{\text{Tangent plane}}$$

A surface S: $z = f(x, y)$ (for good fn)

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-a)$$

$L(x,y) \equiv$ Linearization of f at (a,b)

Recall for $y = f(x) \Rightarrow \Delta y = f(a+\Delta x) - f(a)$

$$= f'(a)\Delta x + \varepsilon \Delta x$$

For $z = f(x, y) \Rightarrow \Delta z = \boxed{f(a,b) + f_x(a+\Delta x, b)\Delta x + f_y(a, b+\Delta y)\Delta y}$

$$+ \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

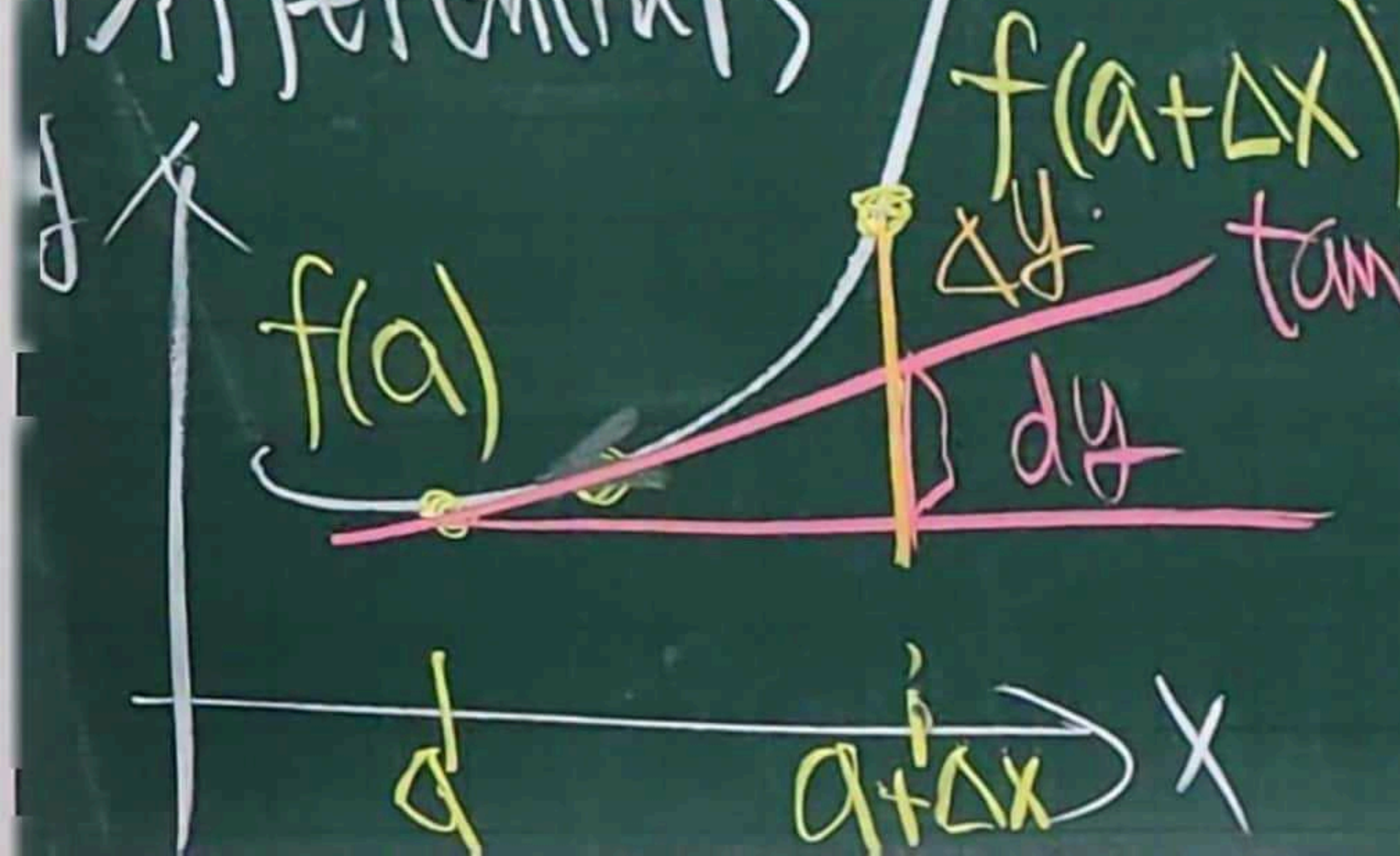
$(\Delta x, \Delta y) \to (0,0) \Rightarrow \varepsilon_1, \varepsilon_2 \to 0$

Rmk f = diff. $\Rightarrow L(x,y) \sim f(x,y)$ is good

Thm: If f_x, f_y exist near (a,b), conti. at (a,b)

$\Rightarrow f$ is differentiable at (a,b)

Differentials



$$\Delta y = f(a+\Delta x) - f(a)$$

$$dy = f'(a)\Delta x$$

$z = f(x,y)$, $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$

$$dz = f_x(a,b)\Delta x + f_y(a,b)\Delta y$$
$$= f_x(a,b)\,dx + f_y(a,b)\,dy$$

For $w = f(x,y,z)$:

$$f(x,y,z) \simeq f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$$

$$\Delta w = f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z)$$

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

$z = f(x, y),\ x = g(s, t),\ y = h(s, t)$

$$\Rightarrow \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

林湘淩

$$\frac{\partial z}{\partial s} = \frac{\partial}{\partial s}\left[\, z(g(s,t), h(s,t)) \,\right]$$

$$= \lim_{\Delta s \to 0} \frac{z(g(s+\Delta s, t), h(s+\Delta s, t)) - z(g(s,t), h(s,t))}{\Delta s}$$

$$\Rightarrow = \lim_{\Delta s \to 0} \frac{z(g(s+\Delta s, t), h(s+\Delta s, t)) - z(g(s,t), h(s+\Delta s, t))}{\Delta s}$$

$$\lim_{\Delta s \to 0} \frac{z(g(s,t), h(s+\Delta s, t)) - z(g(s,t), h(s,t))}{\Delta s}$$

$$= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$\#$

§14.5 Chain Rule

翁瑄涵

$$y=f(x),\ x=g(t) \Rightarrow \frac{dy}{dt}=\frac{dy}{du}\cdot\frac{du}{dt}$$

$$z=f(x,y).\ x=g(t),\ y=h(t)$$

$$\Rightarrow \frac{dz}{dt}=\frac{d}{dt}\left(f(\underline{g(t)},\underline{\underline{h(t)}})\right)$$

$$=\frac{\partial}{\partial x}f(g(t)\ h(t))\cdot\frac{\partial g}{\partial t}+\frac{\partial f}{\partial y}(g(t)\ h(t))\cdot\frac{\partial h}{\partial t}$$

$$\frac{dz}{dt}=\frac{d}{dt}\left[f(g(t),h(t))\right]$$

$$=\lim_{\Delta h\to 0}\frac{f(g(t+\Delta h),h(t+\Delta h))-f(g(t),h(t))}{\Delta h}$$

$$=\lim_{\Delta h\to 0}\frac{f(g(t+\Delta h),h(t+\Delta h))-f(g(t),h(t+\Delta h))+f(g(t),h(t+\Delta h))-f(g(t),h(t))}{\Delta h}$$

$$=\lim_{\Delta h\to 0}\left[\frac{f_x(x,y)\left[h(t+\Delta h)-h(t)\right]}{\Delta h}+\frac{f_x(x,y)\left[g(t+\Delta h)-g(t)\right]}{\Delta h}\right]$$

$$=f_x(x,y)\,h'(t)+f_y(x,y)\,g'(t)$$

蕭祐傑

$\Delta s \to 0$

$$= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} \quad \#$$

$t), h(s,t))$

$$\frac{dz}{dt} = \frac{d}{dt}\left[f(g(t), h(t))\right]$$

$$= \lim_{\Delta h \to 0} \frac{f(g(t+\Delta h), h(t+\Delta h)) - f(g(t), h(t))}{\Delta h}$$

$$= \lim_{\Delta h \to 0} \frac{f(g(t+\Delta h), h(t+\Delta h)) - f(g(t), h(t+\Delta h)) + f(g(t), h(t+\Delta h)) - f(g(t), h(t))}{\Delta h}$$

$$= \lim_{h \to 0} \frac{f(g(t+\Delta h), h(t+\Delta h)) - f(g(t), h(t+\Delta h))}{h}$$

$$+ \lim_{h \to 0} \frac{f(g(t), h(t+\Delta h)) - f(g(t), h(t))}{h}$$

$$= \frac{\partial f}{\partial x} \cdot \frac{d}{dt} g(t) + \frac{\partial f}{\partial y} \cdot \frac{d}{dt} h(t)$$

蕭祐傑

$$= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$t), h(s,t))$

$$\frac{dz}{dt} = \frac{d}{dt}\left[f(g(t), h(t))\right]$$

$$= \lim_{\Delta h \to 0} \frac{f(g(t+\Delta h), h(t+\Delta h)) - f(g(t), h(t))}{\Delta h}$$

$$= \lim_{\Delta h \to 0} \frac{f(g(t+\Delta h), h(t+\Delta h)) - f(g(t), h(t+\Delta h)) + f(g(t), h(t+\Delta h)) - f(g(t), h(t))}{\Delta h}$$

$$= \lim_{h \to 0} \frac{f(g(t+\Delta h), h(t+\Delta h)) - f(g(t), h(t+\Delta h))}{h}$$

$$+ \lim_{h \to 0} \frac{f(g(t), h(t+\Delta h)) - f(g(t), h(t))}{h}$$

$$= \frac{\partial f}{\partial x} \cdot \frac{d}{dt} g(t) + \frac{\partial f}{\partial y} \cdot \frac{d}{dt} h(t)$$

蕭祐傑