Algebra Assignment

Date: Feb 27 Made by F74102276

27.6

Proof. Let $f_c(x) = x^3 + x^2 + c$

If $c = 0$, $f_c(0) = 0 \implies x^3 + x^2 + c = x(x^2 + 1) \implies x^3 + x^2 + c$ is reducible $\implies \langle x^3 + x^2 + c\rangle$ is not maximal $\implies \mathbb{Z}_3[x]/\langle x^3 + x^2 + c\rangle$ is not a field.

If $c = 1$, $f_c(1) = 0 \implies x^3 + x^2 + c = (x - 1)(x^2 + x + 2) \implies \langle x^3 + x^2 + c\rangle$ is not maximal $\implies \mathbb{Z}_3[x]/\langle x^3 + x^2 + c\rangle$ is not a field.

If $c = 2$, $f_c(0) = 2, f_c(1) = 1, f_c(2) = 2 \implies x^3 + x^2 + c$ is irreducible $\implies \langle x^3 + x^2 + c\rangle$ is maximal $\implies \mathbb{Z}_3[x]/\langle x^3 + x^2 + c\rangle$ is a field.

The answer is $\{2\}$ ■

27.18

Proof. No.

$x^2 - 5x + 6 = (x - 2)(x - 3) \implies x^2 - 5x + 6$ is reducible $\implies \langle x^2 - 5x + 6\rangle$ is not maximal $\implies \mathbb{Q}[x]/\langle x^2 - 5x + 6\rangle$ is not a field. ■

34.2

(a)

Proof. $\phi(a) = 0 \iff 12|10a \iff 6|a \implies ker(\phi) = \{0, 6, 12\}$ ■

(b)

Proof. $K + 0 = \{0, 6, 12\}$
$K + 1 = \{1, 7, 13\}$
$K + 2 = \{2, 8, 14\}$
$K + 3 = \{3, 9, 15\}$
$K + 4 = \{4, 10, 16\}$
$K + 5 = \{5, 11, 17\}$

■

(c)

Proof. $\phi(0) = 0$
$\phi(1) = 10$
$\phi(2) = 8$
$\phi(3) = 6$

$\phi(4) = 4$
$\phi(5) = 2$

$\phi[\mathbb{Z}_{18}] = \{0, 10, 8, 6, 4, 2\}$ ■

(d)

Proof. Let H be a coset of K, where $h \in H$.

$\mu(H) = \mu(\gamma_k(K + h)) = \phi(h)$

$\mu(K + 0) = \phi(0) = 0$
$\mu(K + 1) = \phi(1) = 10$
$\mu(K + 2) = \phi(2) = 8$
$\mu(K + 3) = \phi(3) = 6$
$\mu(K + 4) = \phi(4) = 4$
$\mu(K + 5) = \phi(5) = 2$

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