2.3 HW3

Question 29

12. **Show that x: $U \subset R^2 \to R^3$ given by**

$$\mathbf{x}(u, v) = (a \sin u \cos v, b \sin u \sin v, c \cos u), \quad a, b, c \neq 0,$$

where $0 < u < \pi, 0 < v < 2\pi$, is a parametrization for the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Describe geometrically the curves $u =$ const. on the ellipsoid.

Proof. We are required to show

(a) range of $\mathbf{x}$ lies in the ellipsoid

(b) $\mathbf{x}$ is smooth

(c) $d\mathbf{x}$ is one-to-one everywhere on $U \triangleq (0, \pi) \times (0, 2\pi)$

(d) $\mathbf{x}$ is a homeomorphism

Compute

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \cos^2 u = 1$$

This shows that the range of $\mathbf{x}$ indeed lies in the ellipsoid.

It is clear that $\mathbf{x}$ is smooth.

Compute

$$d\mathbf{x} = \begin{bmatrix} a \cos u \cos v & -a \sin u \sin v \\ b \cos u \sin v & b \sin u \cos v \\ -c \sin u & 0 \end{bmatrix}$$

Then compute

$$\frac{\partial(y, z)}{\partial(u, v)} = bc \sin^2 u \cos v \text{ and } \frac{\partial(x, z)}{\partial(u, v)} = -ac \sin^2 u \sin v$$

Because $u \in (0,\pi)$ and $v \in (0,2\pi)$, and $b,c \neq 0$, we now can deduce

$$\frac{\partial(y,z)}{\partial(u,v)} = 0 \iff v \in \{\frac{\pi}{2}, \frac{3}{2}\pi\}$$
$$\frac{\partial(y,z)}{\partial(u,v)} = 0 \iff v = \pi$$

This then let us deduce

$$d\mathbf{x} \text{ is one-to-one everywhere on } (0,\pi)\times(0,2\pi)$$

Traditionally, the function arctan is defined on $\mathbb{R}$ and have codomaiin $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Deduce first from the z-component of $\mathbf{x}$. We see

$$\mathbf{x}^{-1}(x,y,z) = \Big(\arccos\frac{z}{c}, \begin{cases} \arctan\frac{ay}{bx} & \text{if } x,y \in \mathbb{R}^+ \text{ (first quadrant)} \\ \frac{\pi}{2} & \text{if } x=0, y\in\mathbb{R}^+ \\ \frac{3\pi}{2} + \arctan\frac{ay}{bx} & \text{if } x\in\mathbb{R}^-, y\in\mathbb{R}^+ \text{ (second quadrant)} \\ \pi + \arctan\frac{ay}{bx} & \text{if } x\in\mathbb{R}^-, y\in\mathbb{R}^-_0 \text{ (third quadrant)} \\ \frac{3\pi}{2} & \text{if } x=0, y\in\mathbb{R}^- \\ \frac{4\pi}{2} + \arctan\frac{ay}{bx} & \text{if } x\in\mathbb{R}^+, y\in\mathbb{R}^- \text{ (forth quadrant)} \end{cases}\Big)$$

Now it follows that $\mathbf{x}$ is indeed a homeomorphism.

When u is fixed, the image is an oval missing a point $(a\sin u, 0, c\cos u)$ floating in air (contained by $\{(x,y,c_0)\in\mathbb{R}^3 : (x,y)\in\mathbb{R}^2\}$ where $c_0 = c\cos u$ is fixed) ■

Definition 2.3.1. (Definition of regular plane curve) We say $C \subseteq \mathbb{R}^2$ is a regular plane curve if for all $p \in C$ there exists

(a) an open neighborhood $p \in V \subseteq \mathbb{R}^2$

(b) an open set $U \subseteq \mathbb{R}$

(c) a function $\mathbf{x} : U \to V \cap C$

such that $\mathbf{x}$ satisfy

(a) $\mathbf{x}$ is smooth

(b) $\mathbf{x}$ is a homoeomorphism between U and $V \cap C$

(c) $d\mathbf{x}_q \in L(\mathbb{R}, \mathbb{R}^2)$ is one-to-one for all $q \in U$

Definition 2.3.2. (Definition of regular space curve) We say $C \subseteq \mathbb{R}^3$ is a regular space curve if for all $p \in C$ there exists

(a) an open neighborhood $p \in V \subseteq \mathbb{R}^3$

(b) an open set $U \subseteq \mathbb{R}$

(c) a function $\mathbf{x} : U \to V \cap C$

such that $\mathbf{x}$ satisfy

(a) $\mathbf{x}$ is smooth

(b) $\mathbf{x}$ is a homoeomorphism between U and $V \cap C$

(c) $d\mathbf{x}_q \in L(\mathbb{R}, \mathbb{R}^3)$ is one-to-one for all $q \in U$

Question 30

17. Define a regular curve in analogy with a regular surface. Prove that

a. The inverse image of a regular value of a differentiable function

$$f : U \subset R^2 \to R$$

is a regular plane curve. Give an example of such a curve which is not connected.

b. The inverse image of a regular value of a differentiable map

$$F : U \subset R^3 \to R^2$$

is a regular curve in R^3. Show the relationship between this proposition and the classical way of defining a curve in R^3 as the intersection of two surfaces.

***c. The set $C = \{(x, y) \in R^2; x^2 = y^3\}$ is not a regular curve.**

Proof. **(a)**

Suppose $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a smooth function and c is a regular value. We wish to prove

$$C \triangleq f^{-1}[c] \text{ is a regular plane curve}$$

Fix $p \in f^{-1}[c]$. We wish

$$\text{to find a local parametrization } \mathbf{x} : I \subseteq \mathbb{R} \to C \text{ around } p$$

Because c is a regular value, we know df_p is one-to-one. Then, WOLG, we can let $\partial_y F(p) \neq c$. Define $F : U \to \mathbb{R}^2$ by

$$F(x,y) \triangleq \Big(x, f(x,y)\Big)$$

Compute

$$dF = \begin{bmatrix} 1 & 0 \\ \partial_x f & \partial_y f \end{bmatrix}$$

It is now clear that $\det(dF_p) \neq 0$. Now, because f is smooth, we can use inverse function Theorem and obtain a diffeoemorphism F between open neighborhood around p and open neighborhoood around $f(p)$. Now, note that $f[C] = \{c\}$. This tell us

$$F[C] \subseteq \{(x,c) \in \mathbb{R}^2 : x \in \mathbb{R}\}$$

we now claim

$$\mathbf{x}(u) \triangleq F^{-1}(u,c) \text{ is the desired local parametrization around } p$$

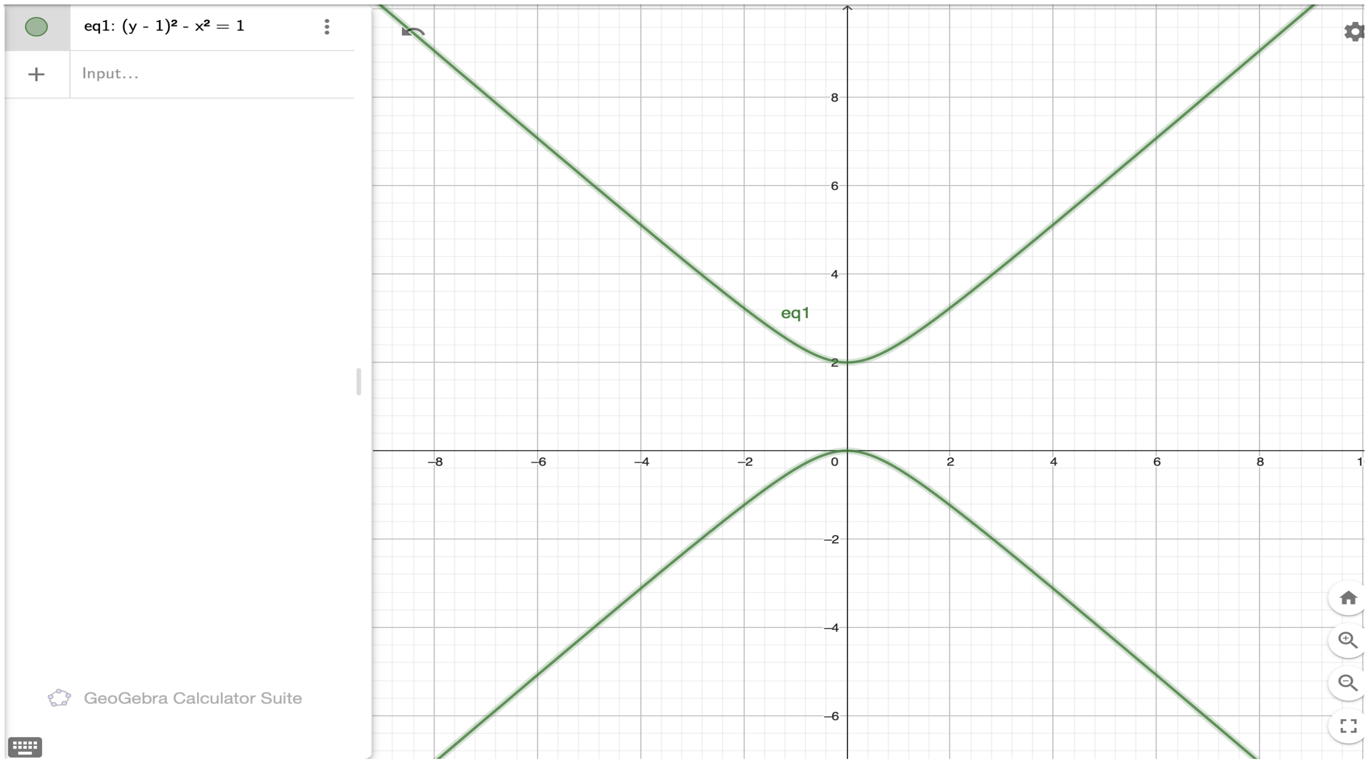
The fact that $\mathbf{x}$ is smooth and homeomorphism follows from

(a) F is a diffeomorphism around p

(b) $\mathbf{x}$ can be identified as restriction of F^{-1}

Note that $d(F^{-1})_p = \big(dF_{F^{-1}(p)}\big)^{-1} \neq 0$. Now, because $\mathbf{x}$ is restriction of F^{-1}, we see $d\mathbf{x}$ must not be 0 around p. (done)

An example is $f(x,y) = (y-1)^2 - x^2$.



(b)

Suppose $F : U \subseteq \mathbb{R}^3 \to \mathbb{R}^2$ is a smooth function and (c_0, c_1) is a regular value. We wish to prove

$$C \triangleq F^{-1}[(c_0, c_1)] \text{ is a regular space curve}$$

Fix $p \in F^{-1}[(c_0, c_1)]$. We wish

$$\text{to find a local parametrization } \mathbf{x} : I \subseteq \mathbb{R} \to C \text{ around } p$$

Define $G : U \to \mathbb{R}^3$ by

$$G(x, y, z) \triangleq \Big(x, F(x, y, z)\Big)$$

Compute

$$dG = \begin{bmatrix} 1 & 0 & 0 \\ \partial_x F_1 & \partial_y F_1 & \partial_z F_1 \\ \partial_x F_2 & \partial_y F_2 & \partial_z F_2 \end{bmatrix}$$

Because p is a regular point of F, we can WOLG, suppose

$$\det\big(dG_p\big) = \det\left(\frac{\partial(F_1, F_2)}{\partial(y, z)}\Big|_p\right) \neq 0$$

This Then, by Inverse function Theorem, G is locally a diffeomorphism around p. We now see

$$\mathbf{x}(t) \triangleq G^{-1}(t, c_0, c_1) \text{ is the desired local parametrization around } p \text{ (done)}$$

Suppose we are given two function $A, B : \mathbb{R}^3 \to \mathbb{R}$, and suppose $A^{-1}[c_0], B^{-1}[c_1]$ are two surfaces. Define $F : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$F(p) \triangleq (A(p), B(p))$$

We see that

$$\text{the intersection } A^{-1}[c_0] \cap B^{-1}[c_1] \text{ is exactly } F^{-1}[(c_0, c_1)]$$

(c)

Assume for a contradiction, C is a regular curve. Note that $(0,0) \in C$. We know there exists an open-neighborhood $N \subseteq \mathbb{R}^2$ around $(0,0)$ such that $N \cap C$ is the graph of some differentiable function in x or y. However, this is impossible, since if one view $N \cap C$ as a function in x, the function $y = x^{\frac{2}{3}}$ is not differetaible at $x = 0$, and one can not even view $N \cap C$ as a function in y as each y correspond to two x, namely $x = \pm y^{\frac{3}{2}}$. CaC

■

Question 31

2. Let $S \subset R^3$ be a regular surface and $\pi : S \to R^2$ be the map which takes each $p \in S$ into its orthogonal projection over $R^2 = \{(x, y, z) \in R^3; z = 0\}$. Is π differentiable?

Proof. Yes. Fix p in S. We wish to prove

$$\pi \text{ is differentiable at } p \text{ in the sense of manifold}$$

Let $\mathbf{x}_1 : U_1 \subseteq \mathbb{R}^2 \to V_1 \cap S \subseteq \mathbb{R}^3$ be a local parametrization around p. Define a local parametrization $\mathbf{x}_2 : U_2 \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ around $\pi(p)$ by

$$\mathbf{x}_2 \triangleq \mathbf{id}_{U_2}$$

We are require to prove

$$\mathbf{x}_2^{-1} \circ \pi \circ \mathbf{x}_1 \text{ is differentiable at } \mathbf{x}_1^{-1}(p)$$

Notice that

(a) $\mathbf{x}_1 : U_1 \to \mathbb{R}^3$ is differentiable at p by definition

(b) $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is clearly differntiable, with derivative $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(c) $\mathbf{x}_2^{-1} = \mathbf{id}_{U_2} : U_2 \to \mathbb{R}^2$ is clearly differntiable.

This shows that $\mathbf{x}_2^{-1} \circ \pi \circ \mathbf{x}_1$ is differentiable at p. (done) ■

Question 32

3. Show that the paraboloid $z = x^2 + y^2$ is diffeomorphic to a plane.

Proof. Let S be the paraboloid. We show

$$S \text{ is diffeomorphic to } \{(x,y,0) : x,y \in \mathbb{R}\}$$

Define

$$\pi : S \to \{(x,y,0) : x,y \in \mathbb{R}\} \text{ by } \pi(x,y,z) \triangleq (x,y,0)$$

We wish to show

$$\pi \text{ and } \pi^{-1} \text{ is differentiable everywhere in the sense of manifold}$$

Define global parametrizaitons $\mathbf{x}_1$ of S and global parametrization $\mathbf{x}_2$ of $\{(x,y,0) : x,y \in \mathbb{R}\}$ by

(a) $\mathbf{x}_1 : \mathbb{R}^2 \to S$ and $\mathbf{x}_1(x,y) \triangleq (x,y,x^2+y^2)$

(b) $\mathbf{x}_2 : \mathbb{R}^2 \to \{(x,y,0) : x,y \in \mathbb{R}\}$ and $\mathbf{x}_2 \triangleq \mathbf{id}_{\mathbb{R}^2}$

We now reword the problem into proving

$$\mathbf{x}_2^{-1} \circ \pi \circ \mathbf{x}_1 : \mathbb{R}^2 \to \mathbb{R}^2 \text{ and } \mathbf{x}_1^{-1} \circ \pi^{-1} \circ \mathbf{x}_2 : \mathbb{R}^2 \to \mathbb{R}^2 \text{ both are differentiable}$$

Because $\pi, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_2^{-1}$ are clearly differentiable, we only have to prove

$$\pi^{-1} : \{(x,y,0) : x,y \in \mathbb{R}\} \to \mathbb{R}^3 \text{ is differentiable and } \mathbf{x}_1^{-1} : \mathbb{R}^3 \cap S \to \mathbb{R}^2 \text{ is differnetiable on } S$$

Observe

$$\pi^{-1}(x,y,0) \equiv (x,y,x^2+y^2) \text{ and } \mathbf{x}_1(x,y,z) \equiv (x,y)$$

It is now clear that π^{-1} and $\mathbf{x}_1^{-1}$ are both differentiable. (done) ■

Question 33

6. Prove that the definition of a differentiable map between surfaces does not depend on the parametrizations chosen.

Proof. Suppose we are given a map $\pi : S_1 \to S_2$ differentiable in the sense of manifold. We know for arbitrary p in S_1, there exists

(a) $\mathbf{x}_1 : U_1 \to S_1$ (a local parametrization around p)

(b) $\mathbf{x}_2 : U_2 \to S_2$ (a local parametrization around $\pi(p)$)

such that

$$\left(\mathbf{x}_2^{-1} \circ \pi \circ \mathbf{x}_1\right) : U_1 \subseteq \mathbb{R}^2 \to U_2 \subseteq \mathbb{R}^2 \text{ is a diffeomorphism} \tag{2.8}$$

Now, fix two arbitrary

(a) $\mathbf{x}_1' : U_1' \to S_1$ (a local parametrization around p)

(b) $\mathbf{x}_2' : U_2' \to S_2$ (a local parametrization around $\pi(p)$)

We are required to prove (Note that the domain of each composited function may be smaller, but this does not undermine the validity of our argument, since we only care about the differentiablity at p)

$$\left((\mathbf{x}_2')^{-1} \circ \pi \circ \mathbf{x}_1'\right) : U_1' \subseteq \mathbb{R}^2 \to U_2' \subseteq \mathbb{R}^2 \text{ is a diffeomorphism}$$

Note that

$$\begin{aligned}
(\mathbf{x}_2')^{-1} \circ \pi \circ \mathbf{x}_1' &= (\mathbf{x}_2')^{-1} \circ \left(\mathbf{x}_2 \circ \mathbf{x}_2^{-1}\right) \circ \pi \circ \left(\mathbf{x}_1 \circ \mathbf{x}_1^{-1}\right) \circ \mathbf{x}_1' \\
&= (\mathbf{x}_2')^{-1} \circ \mathbf{x}_2 \circ \mathbf{x}_2^{-1} \circ \pi \circ \mathbf{x}_1 \circ \mathbf{x}_1^{-1} \circ \mathbf{x}_1' \\
&= h_2 \circ \mathbf{x}_2^{-1} \circ \pi \circ \mathbf{x}_1 \circ h_1
\end{aligned}$$

where

$$\begin{cases} h_1 \equiv \mathbf{x}_1^{-1} \circ \mathbf{x}_1' : U_1' \to U_1 \\ h_2 \equiv (\mathbf{x}_2')^{-1} \circ \mathbf{x}_2 : U_2 \to U_2' \end{cases} \quad \text{are changes of coordinate}$$

Now, because changes of coordinates are diffeomorhphism, and $\mathbf{x}_2^{-1} \circ \pi \circ \mathbf{x}_1$ is diffeomorphism by Equation 2.8 (definition), we see that

$$(\mathbf{x}_2')^{-1} \circ \pi \circ \mathbf{x}_1' = h_2 \circ \left(\mathbf{x}_2^{-1} \circ \pi \circ \mathbf{x}_1\right) \circ h_1 \text{ is a diffeomorphism at } (\mathbf{x}_1')^{-1}(p)$$

This then conclude the proof, since p is arbitrary picked from S_1. (done) ∎

Definition 2.3.3. **(Definition of Differentiable function on a regular curve)** Given two regular curve C_1, C_2, we say the function $f : C_1 \to C_2$ is differentiable at p if for all local parametrizations $\mathbf{x}_1 : I \to C_1 \ni p, \mathbf{x}_2 : I \to C_2 \ni f(p)$, we have

$$\mathbf{x}_2^{-1} \circ f \circ \mathbf{x}_1 \text{ is differentaible as a real to real function}$$

Question 34

9. a. **Define the notion of differentiable function on a regular curve. What does one need to prove for the definition to make sense? Do not prove it now. If you have not omitted the proofs in this section, you will be asked to do it in Exercise 15.**

b. **Show that the map $E: R \to S^1 = \{(x, y) \in R^2; x^2 + y^2 = 1\}$ given by**

$$E(t) = (\cos t, \sin t), \quad t \in R,$$

is differentiable (geometrically, E "wraps" R around S^1).

Proof. Fix $t_0 \in \mathbb{R}$. We wish to prove

$$E \text{ is differentaible at } t_0 \text{ in the sense of manifold}$$

Locally parametize t_0 and $E(t_0)$ by

$$\mathbf{x}_1(t) \triangleq t \text{ and } \mathbf{x}_2(t) \triangleq (\cos t, \sin t)$$

Now, see that

$$\mathbf{x}_2^{-1} \circ E \circ \mathbf{x}_1(t) = t \text{ is clearly differentiable} \quad \text{(done)}$$

∎

Question 35

14. Let $A \subset S$ be a subset of a regular surface S. Prove that A is itself a regular surface if and only if A is open in S; that is, $A = U \cap S$, where U is an open set in R^3.

Proof. $(\longleftarrow)$

Fix $a \in A$. Because $a \in S$, we know there exists a local parametrization $\mathbf{x}_1 : E \to V \cap S$ around a where V is open in . Suppose

$$E' \triangleq \mathbf{x}_1^{-1}[U \cap V \cap S]$$

Now, from

(a) $U \cap V \cap S$ is open in $V \cap S$ $\qquad (\because U \text{ is open in } \mathbb{R}^3)$

(b) $\mathbf{x}_1$ is a homeomorphism between E and $V \cap S$

(c) $\mathbf{x}_1$ is smooth

(d) $d\mathbf{x}_1$ is one-to-one for all $p \in E$

(e) $E' \subseteq E$

We see

(a) E' is open in $\mathbb{R}^2$

(b) the restriction $\mathbf{x}_1|_{E'}$ is a local parametrizaiton around a contained by A

Because a is arbitrary, this established that A is a regular surface.

$(\longrightarrow)$

Suppose A is a regular surface. Fix arbitrary $a \in A$. Using Proposition 2.2.3 in Do Carmo, WOLG, we can suppose there exists a chart $\mathbf{x} : U \to \overline{V} \cap S$ around a where $\overline{V}$ is open in $\mathbb{R}^3$ such that

$$\mathbf{x}(x, y) = (x, y, f(x, y)) \text{ for some smooth } f$$

Because each curve in A lies in S and A is itself a regular surface, we see that

$$T_a(A) = T_a(S)$$

This tell us the restriction

$$\mathbf{x}|_A \text{ is one-to-one}$$

Note that $\mathbf{x}$ is smooth. Now by inverse function theorem, it follows that there exists V open in $\mathbb{R}^3$ such that

$$\mathbf{x}|_{A \cap V} \text{ is a chart around } a$$

Now, define $W \triangleq V \cap \overline{V}$. Now, identify

$$\mathbf{x}|_{A\cap W} \text{ is a chart in both } A \text{ and } S$$

We now see

$$A \cap W \text{ is an open neighborhood around } a \text{ in topology of } S$$

This shows that $a \in A^\circ$ in topology of S. Because a is arbitrary, it follows that A is open in S. ∎

Question 36

***16.** Let $R^2 = \{(x, y, z) \in R^3; z = -1\}$ be identified with the complex plane $\mathbb{C}$ by setting $(x, y, -1) = x + iy = \zeta \in \mathbb{C}$. Let $P\colon \mathbb{C} \to \mathbb{C}$ be the complex polynomial

$$P(\zeta) = a_0\zeta^n + a_1\zeta^{n-1} + \cdots + a_n, \quad a_0 \neq 0, a_i \in \mathbb{C}, i = 0, \ldots, n.$$

Denote by π_N the stereographic projection of $S^2 = \{(x, y, z) \in R^3; x^2 + y^2 + z^2 = 1\}$ from the north pole $N = (0, 0, 1)$ onto R^2. Prove that the map $F\colon S^2 \to S^2$ given by

$$\begin{aligned} &F(p) = \pi_N^{-1} \circ P \circ \pi_N(p), \quad \text{if } p \in S^2 - \{N\}, \\ &F(N) = N \end{aligned}$$

is differentiable.

Proof. Note that

$$\pi_N(x,y,z) = (\frac{2x}{1-z}, \frac{2y}{1-z}) \text{ and } \pi_N^{-1}(u,v) = (\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{u^2+v^2-4}{u^2+v^2+4})$$

and that

$$P(x,y,-1) = \Big(Q(x,y), T(x,y), -1\Big) \text{ for some } Q, T \in \mathbb{R}[x,y]$$

It is then now clear that

$$F \text{ is differentiable on } S^2 \setminus N$$

Note that

$$\pi_S(x,y,z) = (\frac{2x}{1+z}, \frac{2y}{1+z}) \text{ and } \pi_S^{-1}(u,v) = (\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{-u^2-v^2+4}{u^2+v^2+4})$$

Note that

$$\pi_S^{-1} \circ \pi_S \circ F \circ \pi_S^{-1} \circ \pi_S \text{ coincide with } F \text{ on } S^2 \setminus S$$

Then, we can reduce proving F is differentiable at N into

$$\text{proving } \pi_S^{-1} \circ \pi_S \circ F \circ \pi_S^{-1} \circ \pi_S \text{ is differentiable at } N$$

Compute

$$\begin{aligned}
\pi_N \circ \pi_S^{-1}(u,v) &= \pi_N(\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{-u^2-v^2+4}{u^2+v^2+4}) \\
&= (\frac{4u}{u^2+v^2}, \frac{4v}{u^2+v^2})
\end{aligned}$$

Compute

$$\pi_S \circ \pi_N^{-1}(u,v) = (\frac{4u}{u^2+v^2}, \frac{4v}{u^2+v^2})$$

Note that if we identify $\mathbb{R}^2 \setminus 0$ by $\mathbb{C}^*$, we see

$$\pi_S \circ \pi_N^{-1}(z) = \pi_N \circ \pi_S^{-1}(z) = \frac{4z}{|z^2|} = \frac{4z}{z\overline{z}} = \frac{4}{\overline{z}} \qquad (z \in \mathbb{C}^*)$$

Fix $z \in \mathbb{C}^*$. Compute

$$\begin{aligned}
\pi_S \circ F \circ \pi_S^{-1}(z) &= \pi_S \circ \pi_N^{-1} \circ P \circ \pi_N \circ \pi_S^{-1}(z) \\
&= \pi_S \circ \pi_N^{-1} \circ P(\frac{4}{\overline{z}}) \\
&= \pi_S \circ \pi_N^{-1}\Big(a_0(\frac{4}{\overline{z}})^n + \cdots + a_n\Big) \\
&= \frac{4}{\overline{a_0(\frac{4}{\overline{z}})^n + \cdots + a_n}} \\
&= \frac{4}{\overline{a_0}(\frac{4}{z})^n + \cdots + \overline{a_n}} \\
&= \frac{4z^n}{\overline{a_n}z^n + \cdots + \overline{a_0}4^n}
\end{aligned}$$

Compute

$$\pi_S \circ F \circ \pi_S^{-1}(0) = \pi_S \circ F(N) = \pi_S(N) = 0$$

Then because $a_0 \neq 0$. It is now clear that

$$\pi_S \circ F \circ \pi_S^{-1}(z) = \frac{4z^n}{\overline{a_n} z^n + \cdots + \overline{a_0} 4^n} \qquad (z \in \mathbb{C})$$

which is differentiable at 0. Now, we have

(a) $\pi_S : S^2 \setminus S \to \mathbb{C}$ is differentiable on $S^2 \setminus S$

(b) $\pi_S \circ F \circ \pi_S^{-1} : \mathbb{C} \to \mathbb{C}$ is differnetiable on $\mathbb{C}$

(c) $\pi_S^{-1} : \mathbb{C} \to S^2 \setminus S$ is differntiable on $\mathbb{C}$

We can deduce

$$\pi_S^{-1} \circ \left(\pi_S \circ F \circ \pi_S^{-1} \right) \circ \pi_S \text{ is differentiable on } S^2 \setminus S \text{ (done)}$$

Note that we used Theorem 2.3.4 in our proof. ∎

Theorem 2.3.4. (Composition of Differentiable functions is differentiable) Given three regular surfaces $\{S_1, S_2, S_3\}$, two differentiable functions $f_1 : S_1 \to S_2$ and $f_2 : S_2 \to S_3$, we see

$$f_2 \circ f_1 \text{ is differentiable on } S_1$$

Proof. Fix $p_1 \in S_1$. We wish to prove

$$f_2 \circ f_1 \text{ is differentiable at } p_1$$

Set

(a) $p_2 \triangleq f_1(p_1)$

(b) $p_3 \triangleq f_2(p_2)$

Let

(a) $\mathbf{x}_1 : U_1 \to V_1 \cap S_1 \ni p_1$ be a local parametrzaiton

(b) $\mathbf{x}_2 : U_2 \to V_2 \cap S_2 \ni p_2$ be a local parametrizaiton

(c) $\mathbf{x}_3 : U_3 \to V_3 \cap S_3 \ni p_3$ be a local parametrizaiton

We wish to prove

$$\mathbf{x}_3^{-1} \circ f_2 \circ f_1 \circ \mathbf{x}_1 \text{ is differentiable at } p_1$$

Observe that

$$\begin{aligned}
\mathbf{x}_3^{-1} \circ f_2 \circ f_1 \circ \mathbf{x}_1 &= \mathbf{x}_3^{-1} \circ f_2 \circ \mathbf{x}_2 \circ \mathbf{x}_2^{-1} \circ f_1 \circ \mathbf{x}_1 \\
&= \left(\mathbf{x}_3^{-1} \circ f_2 \circ \mathbf{x}_2\right) \circ \left(\mathbf{x}_2^{-1} \circ f_1 \circ \mathbf{x}_1\right) \text{ is differentiable } \ \text{(done)}
\end{aligned}$$

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