1.5 HW4

Theorem 1.5.1. (Monotone Convergence Theorem) Let f_n be a sequence of measurable functions on E:

(a) If $f_n \nearrow f$ a.e. on E and there exists $\varphi \in L(E)$ such that $f_n \geq \varphi$ a.e. on E for all n, then $\int_E f_n \to \int_E f$.

(b) If $f_n \searrow f$ a.e. on E and there exists $\varphi \in L(E)$ such that $f_n \leq \varphi$ a.e. on E for all n, then $\int_E f_n \to \int_E f$.

Question 33

Show that Monotone convergence Theorem may fail if we drop the hypothesis that f_n is dominated by φ. Show that Uniform convergence Theorem may fail if we drop the hypothesis that domain is of finite measure.

Proof. For failure of first part of monotone convergence Theorem, define $f_n : (0, \infty) \to \mathbb{R}$ by

$$f_n(x) \triangleq \begin{cases} 1 & \text{if } x < n \\ -\infty & \text{if } x \geq n \end{cases}$$

Observe that $f_n \nearrow 1$ but $\int_0^\infty f_n = -\infty$ does not converge to $\int_0^\infty 1 = \infty$. For failure of the second part of monotone convergence Theorem, define $f_n : (0, \infty) \to \mathbb{R}$ by

$$f_n \triangleq \mathbf{1}_{(n,\infty)}$$

Observe that $f_n \searrow 0$ but $\int_0^\infty f_n = \infty$ does not converge to 0. For failure of uniform convergence Theorem, consider $\frac{1}{n} \to 0$ uniformly on $\mathbb{R}$ but

$$\int_{\mathbb{R}} \frac{1}{n} dx = \infty \text{ does not converge to } \int_{\mathbb{R}} 0 dx = 0$$

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Question 34

If $f \in L(0, 1)$, show that $x^n f(x) \in L(0, 1)$ for all $n \in \mathbb{N}$ and $\int_0^1 x^n f(x) dx \to 0$.

Proof. Because $f(x)$ and x^n are both measurable on $(0, 1)$, we know $x^n f(x)$ is measurable on $(0, 1)$. Because $0 < x < 1$, if $f(x)$ is finite, then $x^n f(x) \to 0$ as $n \to \infty$. Because f is integrable on $(0, 1)$, we know that f is finite almost everywhere on $(0, 1)$. We now see that

$x^n f(x)$ converge to 0 almost everywhere on $(0,1)$. Again because $0<x<1$, we see that $|x^n f(x)| \leq |f(x)|$. In other words, $x^n f(x)$ is dominated by $|f| \in L(0,1)$. We now can use dominated convergence Theorem to deduce $\int_0^1 x^n f(x) \to 0$. ■

Question 35

Let $f:(0,1)^2 \to \mathbb{R}$ satisfy

(a) $f(x,y)$ is always integrable in y on $(0,1)$.

(b) $\frac{\partial f}{\partial x}$ exists and is bounded on $(0,1)^2$.

Show that $\frac{\partial f}{\partial x}$ is a measurable function in y for all x and

$$\frac{d}{dx}\int_0^1 f(x,y)dy = \int_0^1 \frac{\partial}{\partial x} f(x,y)dy$$

Proof. For all $n \geq 2$, define $g_n : (0, 1-\frac{1}{n}) \times (0,1)$ by

$$g_n(x,y) \triangleq \frac{f(x+\frac{1}{n},y)-f(x,y)}{\frac{1}{n}}$$

It is clear that for all $n \geq 2$

$$\frac{\partial f}{\partial x}\Big|_{(0,1-\frac{1}{n})\times(0,1)} = \lim_{k\to\infty}\left(g_{n+k}\Big|_{(0,1-\frac{1}{n})\times(0,1)}\right)$$

This implies that

$$\frac{\partial f}{\partial x} \text{ is measurable on } (0,1-\frac{1}{n})\times(0,1) \text{ for all } n \geq 2$$

It then follows that $\frac{\partial f}{\partial x}$ is measurable on $(0,1)^2$. Observe that for all $x_0 \in (0,1)$, we have

$$\{y\in(0,1) : \frac{\partial f}{\partial x}(x_0,y) > a\} = \{(x,y)\in(0,1)^2 : \frac{\partial f}{\partial x}(x,y) > a\} \cap (\{x_0\}\times(0,1))$$

It follows that $\frac{\partial f}{\partial x}$ is measurable in y for all x. Fix $x \in (0,1)$. By MVT, we know that for all y and h (small enough to make the following express make sense) we have

$$\frac{f(x+h,y)-f(x,y)}{h} = \frac{\partial f}{\partial x}(x+t,y) < M \text{ for some } t$$

where M is the constant that bounds $\frac{\partial f}{\partial x}$ on $(0,1)^2$. It then follows from DCT (dominated by M on $(0,1)$) that

$$\begin{aligned}
\frac{d}{dx}\int_0^1 f(x,y)dy &= \lim_{h\to 0}\frac{\int_0^1 f(x+h,y)dy - \int_0^1 f(x,y)dy}{h} \\
&= \lim_{h\to 0}\int_0^1 \frac{f(x+h,y)-f(x,y)}{h}dy \\
&= \int_0^1 \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}dy \\
&= \int_0^1 \frac{\partial f}{\partial x}(x,y)dy
\end{aligned}$$

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Question 36

Suppose $p > 0$ and $\int_E |f - f_n|^p \to 0$. Show that $f_n \xrightarrow{m} f$ (and thus have a almost everywhere convergent subsequence).

Proof. Fix ϵ. The proof follows from Tchebysheff's inequality,

$$\left|\left\{\, |f_n - f| > \epsilon \right\}\right| = \left|\left\{\, |f_n - f|^p > \epsilon^p \right\}\right| \geq \frac{\int_E |f - f_n|^p}{\epsilon^p} \to 0$$

■

Question 37

If $p > 0$, $\int_E |f - f_n|^p \to 0$ and $\int_E |f_n|^p \leq M$ for all n, show that $\int_E |f|^p \leq M$.

Proof. By the last question, there exist some subsequence f_{n_k} converge to f almost everywhere. It then follows that $|f_{n_k}|^p \to |f|^p$ almost everywhere. We then have from Fatou's Lemma that

$$\int_E |f|^p \leq \liminf_{k\to\infty}\int_E |f_{n_k}|^p \leq M$$

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Question 38

Give an example of a bounded continuous f on $(0,\infty)$ such that $\lim_{x\to\infty} f(x) = 0$ but $f \notin L^p((0,\infty))$.

Proof. Let $f(x) \triangleq \frac{1}{\ln(x+2)}$. It is clear that f is bounded continuous on $(0,\infty)$ and converge to 0 at infinity. Note that doing a change of variables $t = \ln(x+2)$, we have $dt = \frac{dx}{x+2} = \frac{dx}{e^t}$. Then for all $p > 0$, we have

$$\begin{aligned}\int_0^\infty f^p(x)dx &= \int_0^\infty \frac{1}{(\ln(x+2))^p}dx \\ &= \int_{\ln 2}^\infty \frac{e^t}{t^p}dt\end{aligned}$$

which diverge since the integrand itself converge to ∞. ■

Question 39

If $\int_A f = 0$ for every measurable subset A of measurable set E, show that $f = 0$ almost everywhere on E.

Proof. Observe that for all $n \in \mathbb{N}$

$$0 = \int_{\{f>\frac{1}{n}\}} f dx \geq \int_{\{f>\frac{1}{n}\}} \frac{1}{n}dx = \frac{\left|\{f > \frac{1}{n}\}\right|}{n}$$

This implies that $\left|\{f > \frac{1}{n}\}\right| = 0$ for all $n \in \mathbb{N}$. Again observe for all $n \in \mathbb{N}$

$$0 = \int_{\{f<\frac{-1}{n}\}} f dx \leq \int_{\{f<\frac{-1}{n}\}} \frac{-1}{n}dx = \frac{-\left|\{f < \frac{-1}{n}\}\right|}{n}$$

This implies that $\left|\{f < \frac{-1}{n}\}\right| = 0$ for all $n \in \mathbb{N}$. It then follows from $\{f > \frac{1}{n}\} \cup \{f < \frac{-1}{n}\} \nearrow \{f \neq 0\}$ that $|\{f \neq 0\}| = 0$, i.e., $f = 0$ almost everywhere on E. ■