RTFT Ch9 Schuer's Lemma

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In this note, G is always a group.

In this note, V is always a vector space.

Definitions

Definition 1. *Let $\mathbb{C}G$ be a group algebra*

$$\mathbb{Z}(\mathbb{C}G) = \{z \in \mathbb{C}G | \forall v \in \mathbb{C}G, vz = zv\}$$

Theorems

Theorem 1. *Let V and W be two irreducible $\mathbb{C}G$-module. Let ϕ be a $\mathbb{C}G$-homomorphism from V to W*

$$\phi \textit{ is either an } \mathbb{C}G \textit{ isomorphism or a trivial } \mathbb{C}G \textit{ homomorphism}$$

Proof. Because $N(\phi)$ is a submodule of V, and V is irreducible, we know either $N(\phi) = \{0\}$ or $N(\phi) = V$

Case 1: $N(\phi) = V$

ϕ is a trivial $\mathbb{C}G$-homomorphism

Case 2:$N(\phi) = \{0\}$

Because $R(\phi)$ is a submodule of W, we know either $R(\phi) = \{0\}$ or $R(\phi) = V$

If $R(\phi) = \{0\}$, then $N(\phi) = V \neq \{0\}$ CaC

So $R(\phi) = W$

Then ϕ is an $\mathbb{C}G$ isomorphism ∎

Theorem 2. ***(Schur's Lemma)*** *Let V and W be two irreducible $\mathbb{C}G$-module. Let ϕ be a $\mathbb{C}G$ isomorphism from V to W*

$$\phi = \lambda I_V$$

Proof. Solve the characteristic polynomial of ϕ

We have an eigenvalue λ of ϕ

So $N(\phi - \lambda I_V) \neq \{0\}$

Then $N(\phi - \lambda I_V) = V$

This give us $\forall v \in V, \phi v = \lambda I_V v = \lambda v$

$\blacksquare$

Corollary 2.1. *Let V e a non-trivial $\mathbb{C}G$-module, where every $\mathbb{C}G$-homomorphism ϕ from V to V satisfy $\phi = \lambda I_V$*

V is irreducible

Proof. Assume V is reducible

Write $V = U \oplus W$, where U and W are both submodule

Let $\phi : V \to W$ be defined by $u + w \mapsto w$

$\phi u = 0$ and $\phi w = w \implies \phi \neq \lambda I_V, \forall \lambda \in \mathbb{C}$ CaC

$\blacksquare$

Corollary 2.2. *Let $\rho : G \to GL(n, \mathbb{C})$ be a representation of G. Let $S = \{A \in M_n(\mathbb{C}) | \forall g \in G, A\rho(g) = \rho(g)A\}$*

ρ is irreducible if and only if $\forall A \in S, A = \lambda I_n$

Proof. Let $V = \mathbb{C}^n$ be the $\mathbb{C}G$-module defined by $gv = \rho(g)v$

$(\longleftarrow)$

Let $\pi : V \to V$ be a $\mathbb{C}G$-homomorphism and $g \in G$

$\pi\rho(g)v = \rho(g)\pi v$

Let E be the standard ordered basis of V, and $v \in V$

$[\pi]_E \rho(g)v = [\pi]_E[\rho(g)v]_E = [\pi\rho(g)v]_E = [\rho(g)\pi v]_E = \rho(g)[\pi]_E v$

So $[\pi]_E \rho(g) = \rho(g)[\pi]_E$

Then $[\pi]_E \in S$

So $\pi = \lambda I_V$

$(\longrightarrow)$

Let $A \in S$

$A(gv) = A\rho(g)v = \rho(g)Av = g(Av)$

So L_A is a $\mathbb{C}G$-homomorphism on V

Because V is irreducible, $R(L_A) = \{0\}$ or V

If $R(L_A) = 0$, then $L_A = 0I_V$ and $A = O$

If $R(L_A) = V$, then L_A is an $\mathbb{C}G$-isomorphism from V to V

By theorem 2, $L_A = \lambda I_V$, so $A = \lambda I_n$

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Theorem 3. *Let G be finite abelian, and V be an irreducible $\mathbb{C}G$-module*

$$dim(V) = 1$$

Proof. Assume $dim(V) = n > 2$

Let $g \in G$

$\forall h \in G, hgv = (hg)v = (gh)v = ghv$

This give us that g is a $\mathbb{C}G$ homomorphism from V to V, so by Schur's Lemma, we know that $\exists \lambda_g \in \mathbb{C}, g = \lambda_g I_V$

Let $\alpha = \{v_1, v_2\}$ be a basis of V

$span(v_1)$ is a submodule of V, since $\forall g \in G, gv_1 = \lambda_g v_1 \in span(v_1)$

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Theorem 4. *Let V be a irreducible $\mathbb{C}G$-module, and $\mathbb{C}G$ be a group algebra, and $z \in Z(\mathbb{C}G)$*

$$\phi : v \mapsto zv \text{ is a } \mathbb{C}G\text{-homomorphism, satisfy } \exists \lambda \in \mathbb{C}, \forall v \in V, \phi v = \lambda v$$

Proof. Let $g \in G$

$g\phi v = gzv = (gz)v = (zg)v = zgv = \phi gv$

So ϕ is a $\mathbb{C}G$-homomorphism

Because V is irreducible, by Schur's Lemma, $\exists \lambda \in \mathbb{C}, \forall v \in V, \phi v = \lambda v$ ■

Theorem 5. *If there exists a faithful irreducible $\mathbb{C}G$-module, then $Z(G)$ is cyclic*

Proof. Let $z \in Z(G)$

Because $\forall g \in G, zg = gz$, so $z \in \mathbb{Z}(\mathbb{C}G)$

We know there exists $\lambda \in \mathbb{C}$ s.t. $\forall v \in V, zv = \lambda v$

Let $\phi : Z(G) \to \mathbb{C}$ defined by $z \mapsto \lambda_z$, where $\forall v \in V, zv = \lambda_z v$ and $v \in V$

Let $z, l \in Z(G)$

$zlv = \lambda_z lv = \lambda_z \lambda_l v$, so $\phi(zl) = \phi(z)\phi(l)$

This give us ϕ is a group homomorphism

$\phi(z) = \phi(l) \implies \lambda_z = \lambda_l \implies lv = \lambda_l v = \lambda_z v = zv \implies z^{-1}lv = z^{-1}zv = v \implies z^{-1}l = e \implies z = l$

ϕ is one-to-one

Then $Z[G] \simeq \phi[Z(G)] \leq \mathbb{C}^*$

Finite subgroup of $\mathbb{C}^*$ is cyclic

REMARK:

$Z(G)$ is not cyclic implies that no $\mathbb{C}G$-module is both faithful and irreducible

But $Z(G)$ is cyclic don't imply anything

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Summary

1.Between two irreducible $\mathbb{C}G$-module, a $\mathbb{C}G$-homomorphism either be trivial or an isomorphism that have eigenspace as the whole space

2.To tell that one $\mathbb{C}G$-module is irreducible, prove that every $\mathbb{C}G$-homomorphism on it is a scaler multiplier

3. To tell that a $\mathbb{F}G$-module is reducible, show that there exists one submodule

4. A Regular $\mathbb{F}G$-module is faithful

5. A Permutation $\mathbb{F}G$-module is faithful

6. A permutation $\mathbb{F}G$-module that can be expreesed as a direct sum of two permutation is reducible

Exercises

1.

Write down the irreducible representation over $\mathbb{C}$ of the group $C_2, C_3, C_2 \times C_2$

Proof. ■