Suns

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Contents

Chapter 1

HWs

1.1 HW1

For question 1, recall that by class equation, p-group has nontrivial center.

Question 1

Show that

(i) If $H \diagup Z(H)$ is cyclic, then H is abelian.

(ii) If H is of order p^2, then H is abelian.

From now on, suppose G is non-abelian with order p^3.

(iii) $o(Z(G)) = p$.

(iv) $Z(G) = G^{(1)}$.

Proof. Let $a, b \in H$ and $H \diagup Z(H) \triangleq \langle hZ(H) \rangle$. Write $a \triangleq h^n z_1$ and $b \triangleq h^m z_2$. Because $z_1, z_2 \in Z(H)$, we may compute:

$$ab = h^n z_1 h^m z_2 = h^{n+m} z_1 z_2 = ba$$

as desired. Let $o(H) = p^2$. Because H is a p-group, we know the center of H is nontrivial. Therefore the order of its center is $\in \left\{p, p^2, \right\}$. To see that its order isn't p, just observe that if so, then by (i), H is abelian, which contradicts to $o(Z(H)) = p$. We have shown $o(Z(H)) = p^2 = o(H)$, so H is abelian.

Because G is non-abelian with order p^3, we know $o(Z(G)) \in \left\{p, p^2\right\}$. Part (i) tell us that $o(Z(G)) \neq p^2$, so $o(Z(G)) = p$.

We now prove $Z(G) = G^{(1)}$. Because $o(Z(G)) = p$, by part (ii) we know $G/Z(G)$ is abelian, which implies $G^{(1)} \leq Z(G)$, which implies $G^{(1)}$ is either trivial or equal to $Z(G)$. Because G is non-abelian, we know $G^{(1)}$ is nontrivial. Therefore $G^{(1)} = Z(G)$, as desired. ∎

Question 2

(i) Let M, N be two normal subgroups of G with $MN = G$. Prove that

$$G/(M \cap N) \cong (G/M) \times (G/N)$$

(ii) Let H, K be two distinct subgroups of G of index 2. Prove that $H \cap K$ is a normal subgroup with index 4 and $G/(H \cap K)$ is not cyclic.

Proof. The map $G/(M \cap N) \to (G/M) \times (G/N)$ defined by

$$g(M \cap N) \mapsto (gM, gN) \tag{1.1}$$

is clearly a well-defined group homomorphism, since if $gM = hM$ and $gN = hN$, then $gh^{-1} \in M$ and $gh^{-1} \in N$, which implies $gh^{-1} \in M \cap N$, which implies $g(M \cap N) = h(M \cap N)$. Let $gM = M$ and $gN = N$. Then $g \in M \cap N$ and $g(M \cap N) = M \cap N$. Therefore map 1.1 is also injective. It remains to show map 1.1 is surjective. Fix $g, h \in G$. Write $g = mn$ and $h = \widetilde{m}\widetilde{n}$. Clearly $gM = nM = \widetilde{m}nM$ and $hN = \widetilde{m}N = \widetilde{m}nN$. This implies that mapping 1.1 maps $\widetilde{m}n$ to (gM, hN), as desired.

Because H, K are both of index 2 in G, we know they are both normal in G. This by second isomorphism theorem implies HK forms a subgroup of G. Because $H \neq K$, we know HK properly contains H, which by finiteness of G implies the index of HK is strictly less than H, i.e., $HK = G$. Note that $H \cap K$ is normal since it is the intersection of normal subgroups. By part (i), we now have $G/(H \cap K) \cong (G/H) \times (G/K) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, which shows that $H \cap K$ has index 4 and $G/(H \cap K)$ is cyclic. ∎

Question 3

Let G be a group of order pq, where $p > q$ are prime.

(i) Show that there exists a unique subgroup of order p.

(ii) Suppose $a \in G$ with $o(a) = p$. Show that $\langle a \rangle \subseteq G$ is normal and for all $x \in G$, we have $x^{-1}ax = a^i$ for some $0 < i < p$.

Proof. The third Sylow theorem stated that the number n_p of Sylow p-subgroups satisfies

$$n_p \equiv 1 \pmod{p} \quad \text{and} \quad n_p \mid q$$

Because $p > q$, together they implies $n_p = 1$. Since Sylow p-subgroups of G are exactly subgroups of order p, we have proved (i).

The third Sylow theorem also stated that $n_p = |G : N_G(P)|$ for any Sylow p-subgroup $P \leq G$. Therefore, $N_G(\langle a \rangle) = G$, i.e., $\langle a \rangle$ is normal in G. Fix $x \in G$. It remains to prove $xax^{-1} \neq e$, which is a consequence of the fact that conjugacy (automorphism) preserves order. ■

Question 4

Let H, K be two subgroups of G of finite indices m, n. Show that

$$\text{lcm}(m, n) \leq |G : H \cap K| \leq mn$$

Proof. Let $\Omega_{H\cap K}, \Omega_H$, and Ω_K respectively denote the set of left cosets of $H \cap K, H$, and K. The map $\Omega_{H\cap K} \to \Omega_H \times \Omega_K$ defined by

$$g(H \cap K) \mapsto (gH, gK) \tag{1.2}$$

is well defined since

$$g(H \cap K) = l(H \cap K) \implies g^{-1}l \in H \cap K \implies gH = lH \text{ and } gK = lK$$

such set map is injective since if $gH = lH$ and $gK = lK$, then $g^{-1}l \in H$ and $g^{-1}l \in K$, which implies $g(H \cap K) = l(H \cap K)$, as desired. From the injectivity of map 1.2, we have shown index of $H \cap K$ indeed have upper bound mn.

Because

$$|G : H \cap K| = |G : H| \cdot |H : H \cap K| = |G : K| \cdot |K : H \cap K|$$

we know both n and m divides $|G : H \cap K|$, which gives the desired lower bound $\text{lcm}(m, n)$. ■

Question 5

(i) Let G be a group, $H \leq G$, and $x \in G$ of finite order. Prove that if k is the smallest natural number that makes $x^k \in H$, then $k \mid o(x)$.

(ii) Let G be a group and N a normal subgroup of G. Prove that

$$o(gN) = \inf \left\{ k \in \mathbb{N} : g^k \in N \right\}, \quad \text{where } \inf \varnothing = \infty$$

(iii) Let G be a finite group, H, N two subgroups of G with N normal. Show that if $o(H)$ and $|G : N|$ are coprime, then $H \leq N$.

Proof. (i): Let $a = qk + r \in \mathbb{N}$ with $0 \leq r < k$. If $x^a \in H$, then $x^r = x^a \cdot (x^k)^{-q} \in H$, which implies $r = 0$. We have shown that k divides all natural numbers a that makes $x^a \in H$, which includes $o(x)$.

(ii): This is a simple observation that $(gN)^k = g^k N \in N \iff g^k \in N$.

(iii): By second isomorphism theorem, we know $|HN : N| = |H : H \cap N|$ which divides both $o(H)$ and $|G : N|$. This by coprimality implies $|H : H \cap N| = 1$, which shows that $H \leq N$. ■

Question 6

Let G be a finite group with Sylow p-subgroup P and normal subgroup N. Show that $P \cap N$ forms a Sylow p-subgroup of N, and use such to deduce N have index $p^{\nu_p(o(PN)) - \nu_p(o(N))}$ in PN.

Proof. By second isomorphism theorem, we have

$$o(PN) \cdot o(P \cap N) = o(P) \cdot o(N)$$

Because P is Sylow with $P \subseteq PN$, we know

$$\nu_p(o(PN)) = \nu_p(o(P))$$

This shows that, indeed, $P \cap N$ forms a Sylow p-subgroup of N:

$$\nu_p(o(P \cap N)) = \nu_p(o(N))$$

as desired. Because $P \cap N \leq P$ and because P is Sylow, we know $o(P \cap N)$ is a power of p. It then follows that:

$$|PN : N| = \frac{o(PN)}{o(N)} = \frac{o(P)}{o(P \cap N)} = p^{\nu_p(o(P)) - \nu_p(o(P \cap N))} = p^{\nu_p(o(PN)) - \nu_p(o(P))}$$

■

Question 7

Prove that if H is a Hall subgroup of G and $N \trianglelefteq G$, then $H \cap N$ is a Hall subgroup of N and HN/N is a Hall subgroup of G/N.

Proof. The facts that:

(i) By second isomorphism theorem, we have $|N : H \cap N| = |HN : H|$, which divides $|G : H|$.

(ii) $o(H \cap N) \mid o(H)$.

(iii) $o(H)$ and $|G : H|$ are coprime.

implies $o(H \cap N)$ and $|N : H \cap N|$ is coprime, i.e., $H \cap N$ is Hall in N.

The facts that:

(i) $o(HN/N) = \frac{o(HN)}{o(N)} = \frac{o(H)}{o(H \cap N)}$ divides $o(H)$. (second isomorphism theorem)

(ii) $|(G/N) : (HN/N)| = |G : HN|$ divides $|G : H|$.

(iii) $o(H)$ and $|G : H|$ are coprime.

implies $o(HN/N)$ and $|(G/N) : (HN/N)|$ are coprime, i.e., HN/N is Hall in G/N.

■

1.2 HW2

Question 1: subgroup of p-group of index p is normal

Prove that if p is a prime and $o(G) = p^\alpha$ with $\alpha \in \mathbb{N}$, then every subgroup H of index p is normal.

Deduce that every group of order p^2 has a normal subgroup of order p.

Proof. Let G acts on the left cosets spaces Ω of H. We have a group homomorphism $\varphi : G \to \operatorname{Sym}(\Omega)$. Clearly we have $\ker \varphi \subseteq H$. By first isomorphism theorem, we know

$$|G : \ker \varphi| = o(\operatorname{Im} \varphi) \mid \operatorname{Sym}(\Omega)$$

Noting that $|\operatorname{Sym} \Omega| = p!$, we see $\ker \varphi$ has index $\leq p$, which when combined with the fact $\ker \varphi \subseteq H$ shows that $H = \ker \varphi$, as desired.

Suppose $\alpha = 2$. By first Sylow theorem, there is a subgroup of G of order p. This subgroup is normal from what we have just proved. ∎

Question 2

Let G be a group of odd order. Prove that for any $x \neq e \in G$, we have $\operatorname{Cl}(x) \neq \operatorname{Cl}(x^{-1})$.

Proof. Assume for a contradiction that $\operatorname{Cl}(x) = \operatorname{Cl}(x^{-1})$. Because $(gxg^{-1})^{-1} = gx^{-1}g^{-1} \in \operatorname{Cl}(x^{-1}) = \operatorname{Cl}(x)$, the inversion is well defined on $\operatorname{Cl}(x)$, and moreover clearly bijective. Because $o(G)$ is odd, we may pair up the elements of $\operatorname{Cl}(x)$ via inversion to see $|\operatorname{Cl}(x)|$ is even. This is impossible since by orbit-stabilizer theorem, $|\operatorname{Cl}(x)|$ is the index of some subgroup of G . ∎

Question 3

Let $o(G) = p^n$ with $n \geq 3$ and $o(Z(G)) = p$. Prove that G has a conjugacy class of size p.

Proof. Class equation stated that

$$o(G) = o(Z(G)) + \sum |\operatorname{Cl}(x)| \tag{1.3}$$

and the orbit stabilizer theorem shows that $|\operatorname{Cl}(x)|$ is of order powers of p. If they are of p-powers ≥ 2, then we see

$$0 \equiv o(G) \equiv p \equiv o(Z(G)) + \sum |\operatorname{Cl}(x)| \pmod p$$

a contradiction. ■

Question 4

Prove that if the center of G is of index n, then every conjugacy class has at most n elements.

Proof. Let $x \in G$. Because $Z(G) \subseteq C_G(x)$, by orbit-stabilizer theorem, we have:

$$|\mathrm{Cl}(a)| = |G : C_G(a)| \leq |G : Z(G)| = n$$

■

Question 5

Let $H, K \subseteq G$ be two finite subgroups. Show that

$$|HK| = \frac{o(H)o(K)}{o(H \cap K)}$$

Remark: The hint give a rigorous proof, but I prefer a heuristic one.

Proof. Consider the right coset spaces $\Omega \triangleq \{Hx : x \in G\}$, and let K acts on Ω by right multiplication. Because $Hk = H$ if and only if $k \in H$, we know the stabilizer subgroup K_H is identical to $K \cap H$. Therefore, by orbit-stabilizer theorem, we have

$$\frac{o(K)}{o(H \cap K)} = |\{Hk : k \in K\}|$$

Define an equivalence class in K by setting $k \sim \widetilde{k} \overset{\triangle}{\iff} Hk = H\widetilde{k}$. Pick a representative element our of each class and collect them into a set T. Clearly

$$|T| = |\{Hk : k \in K\}|$$

and we have a natural bijection $H \times T \to HK$. This finishes the proof. ■

Question 6

Let G be a non-abelian group of order 21. Prove that $Z(G) = 1$.

Proof. If $o(Z(G)) = 3$ or 7, then because $G/Z(G)$ is cylic ■

Question 7

Find all finite groups which have exactly two conjugacy classes.

Proof. Let G be a finite group that has exactly two conjugacy classes. One of the conjugacy class is $\{e\}$. Let a be an element of the other class. By class equation and orbit-stabilizer theorem, we have

$$|G| - 1 = |\mathrm{Cl}(a)| \mid o(G)$$

This implies $|G| = 2$, which implies $G = \mathbb{Z}_2$. ■

Question 8

Let H be a subgroup of G and let

$$\bigcup_{g \in G} gHg^{=1} = G$$

Show that $H = G$.

1.3 Exercises III

Question 9

Let $o(G) = 60$. Show that if G is simple, then G must have exactly 24 elements of order 5 and 20 elements of order 3.

Proof. By sylow, we have

$$n_5 \equiv 1 \pmod 5 \quad \text{and} \quad n_5 \mid 12$$

which by simplicity of G implies $n_5 = 6$. The same argument gives us $n_3 \in \{4, 10\}$. To see $n_3 \neq 4$, just recall that second sylow stated that conjugacy action $G \longrightarrow \text{Sym}(\text{Syl}_3(G)) \cong S_{n_3}$ is nontrivial, and is therefore injective by simplicity of G. We now see that $n_3 = 4$ is too small to satisfies

$$o(G) = 60 \mid n_3!$$

■

Question 10

Let $o(G) = pqr$ with $p < q < r$ prime. Prove that G has a normal Sylow r-subgroup H.

Proof. By sylow and counting arguments we know $1 \in \{n_p, n_q, n_r\}$. Therefore, if neither of n_p and n_q is 1, we are done. Suppose $1 \in \{n_p, n_q\}$. Either way, we get a normal subgroup N such that $o(G \diagup N) \in \{qr, pr\}$. We also get a normal $H \diagup N \in \text{Syl}_r(G \diagup N)$. This give us a characteristic $K \in \text{Syl}_r(H)$, which is normal in G. ■

Question 11

Let $o(G) = p^3 q$ with p, q prime. Show that one of the followings statement is true:

(i) G has a normal Sylow p-subgroup.

(ii) G has a normal Sylow q-subgroup.

(iii) $p = 2$, $q = 3$.

Proof. Suppose (i) and (ii) are both false. Then by sylow we have $n_p = q$ and $p < q$. Because $p < q$, applying sylow again we have $n_q \in \{p^2, p^3\}$. Because $n_p > 1$, by counting we see that $n_q \neq p^3$. Therefore $n_q = p^2$. Then by sylow, $p^2 = n_q \equiv 1 \pmod q$, which implies $q \mid (p-1)(p+1)$. Because $p < q$ and q is prime, we now see $q = p + 1$, which can only happens if $p = 2$ and $q = 3$. ■

Question 12

Show that no group of order 30 is simple.

Proof. Consider n_3 and n_5. We have $n_5 \in \{1, 6\}$ and $n_3 \in \{1, 10\}$. If $n_5 \neq 1 \neq n_3$, then there are 24 elements of order 5 and 20 elements of order 3, impossible for a group of order 30. ∎

Question 13

Let G be a finite group with sylow p-subgroup P and normal subgroup N. Show that $P \cap N$ is p-sylow in N and that PN/N is p-sylow in G/N.

Proof. Second isomorphism theorem implies that

$$o(PN) \cdot o(P \cap N) = o(P) \cdot o(N)$$

Because P is p-sylow, we know $o(P)$ and $o(PN)$ has the same p-power, which implies that $o(P \cap N)$ and $o(N)$ has the same p-power, as desired. Again counting the p-power of $PN/N \subseteq G/N$, we see PN/N is p-sylow. ∎

Question 14

Let G be a finite group, $H \leq G$ a subgroup with $[G : H] = n$. Show that:

(i) For all subgroup $K \leq G$, we have $[H : H \cap K] \leq [G : K]$.

(ii) $[H : H \cap H^g] \leq n$ for all $g \in G$.

(iii) If H is a maximal proper subgroup of G and H is abelian, show that $H \cap H^g \trianglelefteq G$ for all $g \notin H$.

(iv) Suppose that G is simple. If H is abelian and n is prime, then $H = 1$.

Proof. Let $H/H \cap K$ and G/K denote left coset spaces. (i) is a consequence of verifying that the function

$$H/H \cap K \longrightarrow G/K; \quad h(H \cap K) \mapsto hK$$

is well-defined and injective. (ii) is then a corollary of (i).

We now prove (iii). Fix $g \notin H$. There are two cases: Either $H = H^g$ or $H \neq H^g$. For the first case, just observe that by maximality of H, we will have $N_G(H) = G$. We now claim that $H \neq H^g \implies H \cap H^g \subseteq Z(G)$. Because H is abelian, we know $H \cap H^g \leq Z(H)$. Clearly we also have $H \cap H^g \leq Z(H^g)$. We now have $H \cap H^g \leq Z(\langle H, H^g \rangle)$, where

$\langle H, H^g \rangle = G$ by maximality of H, as desired.

We now prove (iv). Clearly the primality of n forces H to be a maximal proper subgroup of G. Therefore by (iii), $H \cap H^g = 1$ for all $g \notin H$. This by (ii) implies $n \leq o(G) \leq n^2$. Write $o(G) \triangleq nk$ so $k \in \{1, \dots, n\}$. We wish to show $k = 1$. To see $k \neq n$, just recall that if so, then G would be abelian, contradicting to its simplicity. To see $k \notin \{2, \dots, n-1\}$, just observe that if so, then the unique n-Sylow subgroup would be proper, contradicting to simplicity of G. ■

Question 15

Let G be a finite group with $P \in \mathrm{Syl}_p(G)$. Suppose that N is a normal subgroup of G with $[G : N] = o(P) > 1$. Show that

(i) N is the subset of G consisting of all elements of order not divisible by p.

(ii) If the elements of $G - N$ all has p-power order, then $P = N_G(P)$.

Proof. Because P is p-sylow and $[G : N] = o(P)$, we know $p \nmid o(N)$. This implies that no element of N has order divisible by p. Let $g \in G$ with $p \nmid o(g)$. To see that $g \in N$, just observe that because $o(gN) \mid o(g)$ and $o(gN)$ is a power of p, we have $o(gN) = 1$.

Assume for a contradiction that $P < N_G(P)$. Then there exists some nontrivial sylow q-subgroup Q of $N_G(P)$ with $q \neq p$. By definition we have $[Q, P] \leq P$. By (i), $Q \leq N$. Therefore we also have $[Q, P] \leq N$. Coprimality of orders of N and P now tell us that $[Q, P] = 1$. We now see that the product of two nontrivial elements $x \in Q, y \in P$ has order divisible by pq, a contradiction to the premise. ■

1.4 Exercises IV

Question 16

Show that the center of products is a product of centers:

$$Z(G_1) \times \cdots \times Z(G_n) = Z(G_1 \times \cdots \times G_n)$$

Deduce that a direct product of groups is abelian if and only if each of its factor is abelian.

Proof. The "$\subseteq$" is clear. To see that

$$g_1 \times \cdots \times g_n \in Z(G_1 \times \cdots \times G_n) \implies g_i \in Z(G_i)$$

just observe that if not, then

$$[g_1 \times \cdots \times g_n, e_1 \times \cdots \times x_i \times \cdots \times e_n] \neq e \in \prod G_j$$

The second part then follows from noting

$$Z(G_1 \times \cdots \times G_n) = G_1 \times \cdots \times G_n \iff Z(G_i) = G_i, \quad \text{for all } i$$

■

Question 17

Let $G \triangleq A_1 \times \cdots \times A_n$ and $B_i \trianglelefteq A_i$ for all i. Prove that $B_1 \times \cdots \times B_n \trianglelefteq G$ and that

$$\frac{A_1 \times \cdots \times A_n}{B_1 \times \cdots \times B_n} = \frac{A_1}{B_1} \times \cdots \times \frac{A_n}{B_n}$$

Proof.

$$(g_1, \dots, g_n)(b_1, \dots, b_n)(g_1, \dots, g_n)^{-1} = (g_1 b_1 g_1^{-1}, \dots, g_n b_n g_n^{-1}) \in \prod B_i$$

The second part require us to show that

$$\prod \left(\frac{A_i}{B_i}\right) \longrightarrow \frac{\prod A_i}{\prod B_i}; \quad \prod \left(\frac{a_i}{B_i}\right) \mapsto \frac{\prod a_i}{\prod B_i}$$

is a well-defined group isomorphism, which boils down to showing that it is (i) well-defined, (ii) actually a homomorphism, (iii) injective, and (iv) surjective. To see it is injective, just observe that if $\prod a_i \in \prod B_i$, then $a_i \in B_i$ for all i, and therefore $\prod \frac{a_i}{B_i} = e$. The rest are clear. ■

Question 18

Let G be a finite abelian group with $m \mid o(G)$. Show that G has a subgroup of order m.

Proof. This follows from noting that if $o(a) = p^n$, then $o(a^{p^{n-d}}) = p^d$. (Ans also structure theorem for finite abelian group) ∎

Question 19

Show that the subgroups and quotients of a nilpotent group G are also nilpotent.

Proof. Let H be a subgroup of G, and write

$$0 = G_{(n)} \trianglelefteq \cdots \trianglelefteq G_{(1)} \trianglelefteq G_{(0)} = G, \qquad \text{with } G_{(k)} \triangleq [G, G_{(k-1)}]$$

To see that

$$0 \leq H_n \leq \cdots \leq H_1 \leq H$$

form a central series, where $H_k \triangleq H \cap G_{(k)}$, just observe that

$$[H, H \cap G_{(k)}] \leq H \text{ and } [H, H \cap G_{(k)}] \leq [G, G_{(k)}] \leq G_{(k-1)}$$

together implies

$$[H, H_k] \leq H \cap G_{(k-1)} = H_{k-1}$$

Let N be a normal subgroup of G, and let $m \leq n$ be the largest number such that $N \leq G_{(m)}$. It is clear that

$$\frac{N}{N} \leq \frac{G_{(m)}}{N} \leq \cdots \leq \frac{G_{(1)}}{N} \leq \frac{G}{N}$$

form a central series. ∎

Question 20

Show that if $G/Z(G)$ is nilpotent, then G is nilpotent.

Proof. Consider the central series

$$\frac{Z(G)}{Z(G)} \trianglelefteq \frac{G_1}{Z(G)} \trianglelefteq \cdots \trianglelefteq \frac{G_n}{Z(G)} = \frac{G}{Z(G)}$$

Clearly we have the central series

$$0 \trianglelefteq Z(G) \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$$

∎

Question 21

Let $o(G) = pqr$ with $p < q < r$ prime. Show that G is solvable.

Proof. Recall that we have a normal subgroup $M \in \mathrm{Syl}_r(G)$. Then we have a normal subgroup $\frac{H}{M} \in \mathrm{Syl}_q(G)$. Then $1 \trianglelefteq M \trianglelefteq H \trianglelefteq G$ forms the desired series. ■

Question 22

Show that a finite group G is nilpotent if and only if every $a, b \in G$ that makes $\gcd(o(a), o(b)) = 1$ also makes $ab = ba$.

Proof. ($\implies$): Write $G = P_1 \times \cdots \times P_n$ with P_i sylow. Clearly if the orders of $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are coprime to each other, then for all i, we must have either $x_i = e$ or $y_i = e$. This implies the commutativity.

($\impliedby$): We need to show that Sylow subgroups of G are normal. Let $P_1, \ldots, P_n$ each be a Sylow subgroup of G with distinct p. By premise, we see that $P_k \subseteq N_G(P_1)$ for all $k \geq 2$. This then implies $G = N_G(P_1)$, as desired. ■

Question 23

Let $G = HK$ be finite and $S \leq G$ be a p-subgroup that contains some p-Sylow subgroup P of H and some p-Sylow subgroup Q of K. Show that

(i) S is p-Sylow in G.

(ii) $S = (S \cap H)(S \cap K)$

Proof. Because $P \cap Q \leq H \cap K$, we know p-part of

$$o(G) = \frac{o(H)o(K)}{o(H \cap K)}$$

is smaller than

$$\frac{o(P)o(Q)}{o(P \cap Q)} = |PQ| \leq o(S)$$

which can only happen if S is Sylow with $|PQ| = o(S)$. By definition, $P \leq S \cap H \leq H$. Because S is a p-group, we know $S \cap H$ is also a p-group. Sylowness of $P \leq H$ then forces $S \cap H = P$. Similarly, we have $S \cap K = Q$. Now, to see $S = PQ$, just recall that $|PQ| = o(S)$ ■

Question 24

Let $M \trianglelefteq G$ and $N \trianglelefteq G$ with M, N finite and nilpotent. Prove that MN is nilpotent.

Proof. The proof follows form noting that if $S \in \mathrm{Syl}_p(MN)$, then by earlier questions, S is uniquely determined by $S = (M \cap S)(N \cap S)$ with $M \cap S \in \mathrm{Syl}_p(M)$ and $N \cap S \in \mathrm{Syl}_p(N)$ uniquely determined. ∎

Question 25

Let G be finite with $A, B \trianglelefteq G$ and $G/A, G/B$ solvable. Prove that $G/(A \cap B)$ is solvable.

Proof. ∎