Deadline : 2023/04/26, 17:00.

1. **Ideal Gas Law** The gas law for a fixed mass $m$ of an ideal gas at absolute temperature $T$, pressure $P$, and volume $V$ is $PV = mRT$, where $R$ is the gas constant.

   (a) Evaluate $\dfrac{\partial P}{\partial V}\dfrac{\partial V}{\partial T}\dfrac{\partial T}{\partial P}$.

   (b) Show that $T\dfrac{\partial P}{\partial T}\dfrac{\partial V}{\partial T} = mR$.

2. Find all the first order partial derivatives of the function.

   (a) $f(x,t) = \tan^{-1}(x\sqrt{t})$

   (b) $f(x,y) = \displaystyle\int_y^x \cos(t^2)\,dt$

   (c) $f(\mathbf{x}) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

   (d) $u(x,y) = f(\dfrac{x}{y})$

3. If $u(x_1, x_2, \cdots, x_n) = e^{a_1x_1+\cdots+a_nx_n}$, where $a_1^2 + \cdots + a_n^2 = 1$. Show that

$$\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = u.$$

4. Let

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{x^2+y^2}\right), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

   (a) Show that $f_x$ and $f_y$ exist on $\mathbb{R}^2$ but not continuous at $(0,0)$.

   (b) Compute $f_{xy}(0,0)$ and $f_{yx}(0,0)$.

5. Let $\mathbf{x} \in \mathbb{R}^n$, $r(\mathbf{x}) = \|\mathbf{x}\|$ and $\mathbf{r}(\mathbf{x}) = \langle x_1, \cdots, x_n\rangle$. Show that

   (a) $\nabla\left(\dfrac{1}{r(\mathbf{x})}\right) = -\dfrac{\mathbf{r}(\mathbf{x})}{r^3(\mathbf{x})}$ for $r(\mathbf{x}) \neq 0$.

   (b) $\nabla(r^m(\mathbf{x})) = (mr^{m-2}(\mathbf{x}))\mathbf{r}(\mathbf{x})$ for $r(\mathbf{x}) \neq 0$.

6. **Chain Rule** Let $g : \mathbb{R} \to \mathbb{R}$ and $h : D \subset \mathbb{R}^n \to \mathbb{R}$ be two functions and $\mathbf{a} \in D$. Define $f(\mathbf{x}) := g(h(\mathbf{x})) : D \to \mathbb{R}$. Suppose that $h$ has the partial derivative $\dfrac{\partial h}{\partial x_i}$ at $\mathbf{a}$ and $g$ is differentiable at $h(\mathbf{a})$. Prove that $f$ has the partial derivative $\dfrac{\partial f}{\partial x_i}$ at $\mathbf{a}$ and

$$\frac{\partial f}{\partial x_i}(\mathbf{a}) = g'\big(h(\mathbf{a})\big)\frac{\partial h}{\partial x_i}(\mathbf{a}).$$

7. Let $D = [-1, 1] \times (-1, 1) \subset \mathbb{R}^2$ and $f(x, y) : D \to \mathbb{R}$ be continuous. If $|f_y(x, y)| \le M$ for all $(x, y) \in D$, prove that $f(x, y)$ is bounded on $D$.

8. Use the chain rule to find the derivatives.
   (a) $u = x + 4\sqrt{xy} - 3y$ ; $x = t^3$, $y = t^{-1} (t > 0)$. Find $\dfrac{du}{dt}$.
   (b) $u = xy + yz + zx$ ; $x = t^2$, $y = t(1 - t)$, $z = (1 - t)^2$. Find $\dfrac{du}{dt}$.
   (c) $u = z^2 \sec(xy)$ ; $x = 2st$, $y = s - t^2$, $z = s^2 t$. Find $\dfrac{\partial u}{\partial s}$, $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial^2 u}{\partial t \partial s}$.
   (d) $u = u(x, y, z)$ where

$$x = x(w, t), \quad y = y(w, t), \quad z = z(w, t)$$

$$w = w(r, s), \quad t = t(r, s)$$

   Find $\dfrac{\partial u}{\partial r}$.

9. Let $u = u(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$. Assume that $u$ has continuous second partial derivatives. Show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r}.$$

10. Find an equation for the tangent plane of the given surface at the point $P$.
    (a) $\sqrt{x} + \sqrt{y} + \sqrt{z} = 4$ ; $P(1, 4, 1)$.
    (b) $z = \sin(x \cos y)$ ; $P(1, \frac{1}{2}\pi, 0)$.

11. Find the direction derivatives of $f(x, y, z) = x^2 + 2xyz - yz^2$ at $(1, 1, 2)$ in the directions parallel to the line

$$\frac{x - 1}{2} = y - 1 = \frac{z - 2}{-3}.$$

12. Use "Second Derivative Test" to find the local maximum and minimum points with their values of the function $f(x, y) = e^y(y^2 - x^2)$, also find the saddle point if it exists.