8.4 HW4

Theorem 8.4.1. Let $B_n = \sum_{k=1}^n b_n$. Given

$$a_n \searrow 0 \text{ or } a_n \nearrow 0 \text{ as } n \to \infty$$

and

$$\exists M, \forall n, |B_n| < M$$

We have

$$\sum_{n=1}^{\infty} a_n b_n \text{ converge}$$

Proof. Let N satisfy

$$a_{n+N} \nearrow 0 \text{ or } a_{n+N} \searrow 0$$

We see

$$\sum_{k=N+1}^{n} b_n = B_n - B_N \text{ is also bounded}$$

If $a_{n+N} \searrow 0$, then by Dirichlet's test we see

$$\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{N} a_k b_k + \sum_{n=N+1}^{\infty} a_k b_k \text{ converge}$$

If $a_{n+N} \nearrow 0$, we see $-a_{n+N} \searrow 0$. Then by Dirichlet's test we see

$$\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{N} a_k b_k - \sum_{k=N+1}^{\infty} (-a_k) b_k \text{ converges}$$

∎

Question 45

Suppose that $\sum_{k=1}^{\infty} a_k$ converges and that $b_k \searrow b$ as $k \to \infty$. Prove that

$$\sum_{k=1}^{\infty} a_k b_k \text{ converges.}$$

Proof. Define

$$b'_n := b_n - b$$

Deduce

$$\begin{aligned}\sum_{k=1}^{\infty} a_k b_k &= \sum_{k=1}^{\infty} a_k(b + b'_k) \\ &= b\sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} a_k b'_k\end{aligned}$$

Notice

$$b_k \searrow b \text{ as } k \to \infty \implies b'_k \searrow 0 \text{ as } k \to \infty$$

Then by Dirichlet's test, we know

$$\sum_{k=1}^{\infty} a_k b'_k \text{ converge}$$

Because by premise $\sum_{k=1}^{\infty} a_k$ converge, we know $\sum_{k=1}^{\infty} a_k b_k = b\sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} a_k b'_k$ converge. ■

Question 46

Show that under the hypotheses of Dirichlet's test,

$$\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty} s_k(b_k - b_{k+1}),$$

where

$$s_k = \sum_{j=1}^{k} a_j.$$

Proof. Abel's formula give us

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k &= \sum_{k=1}^{n}(s_k - s_{k-1})b_k \\ &= \sum_{k=1}^{n} s_k b_k - \sum_{k=1}^{n-1} s_k b_{k+1} \\ &= \sum_{k=1}^{n-1} s_k(b_k - b_{k+1}) + s_n b_n\end{aligned}$$

Define

$$X_n := \sum_{k=1}^{n} a_k b_k \text{ and } Y_n := \sum_{k=1}^{n} s_k(b_k - b_{k+1})$$

Abel's formula then can be rewritten into

$$\forall n, Y_n = X_{n+1} - s_n b_n$$

Define

$$\alpha := \lim_{n\to\infty} X_n$$

The question ask us to prove

$$\lim_{n\to\infty} Y_n = \alpha$$

The hypothesis of Dirichlet's test are

$$s_n = \sum_{k=1}^{n} a_k \text{ are bounded and } b_n \searrow 0$$

Let $M \in \mathbb{R}^+$ satisfy

$$\forall n, |s_n| < M$$

Arbitrarily pick $\epsilon \in \mathbb{R}^+$. We know there exists N_1 such that

$$\forall n > N_1, 0 \le b_n < \frac{\epsilon}{2M}$$

Because $\alpha = \lim_{n\to\infty} X_n$, we also know there exists N_2 such that

$$\forall n > N_2, |X_n - \alpha| < \frac{\epsilon}{2}$$

We then see

$$\begin{aligned}\forall n > \max\{N_1, N_2\}, |Y_n - \alpha| &= |X_{n+1} - \alpha - s_n b_n| \\ &\le |X_{n+1} - \alpha| + |s_n b_n| \\ &< \frac{\epsilon}{2} + M \times \frac{\epsilon}{2M} = \epsilon\end{aligned}$$

Because ϵ is arbitrarily picked from $\mathbb{R}^+$. We have proved $\lim_{n\to\infty} Y_n = \alpha$ ■

Lemma 8.4.2. Given n

$$\sum_{k=1}^{\infty} x_k \text{ converge } \implies \sum_{k=n}^{\infty} x_k = \sum_{k=1}^{\infty} x_k - \sum_{k=1}^{n-1} x_k$$

Proof. For all $u \in \mathbb{N}$, define

$$s_u := \sum_{k=n}^{n+u} x_k \text{ and } v_u := \sum_{k=1}^{u} x_k$$

From definition, we have

$$\forall u \in \mathbb{N}, s_u = v_{n+u} - \sum_{k=1}^{n-1} x_k \text{ and } \lim_{u\to\infty} v_u \text{ converge}$$

Notice

$$\lim_{u\to\infty} v_{n+u} = \lim_{u\to\infty} v_u$$

Then we have

$$\lim_{u\to\infty} s_u = \lim_{u\to\infty}(v_{n+u} - \sum_{k=1}^{n-1} x_k) = (\lim_{u\to\infty} v_{n+u}) - \sum_{k=1}^{n-1} x_k = \lim_{u\to\infty} v_u - \sum_{k=1}^{n-1} x_k$$

Which written in series is

$$\sum_{k=n}^{\infty} x_k = \sum_{k=1}^{\infty} x_k - \sum_{k=1}^{n-1} x_k$$

■

Lemma 8.4.3.

$$\sum_{k=1}^{\infty} x_k \text{ converge} \implies \lim_{n\to\infty} \sum_{k=n}^{\infty} x_k = 0$$

Proof. By Lemma 8.4.2, we have

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=n}^{\infty} x_k &= \lim_{n\to\infty} \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n-1} a_k \\
&= \sum_{k=1}^{\infty} a_k - \lim_{n\to\infty} \sum_{k=1}^{n-1} a_k \\
&= \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{\infty} a_k = 0
\end{aligned}$$

■

Question 47

Suppose that $\sum_{k=1}^{\infty} a_k$ converges, that $b_k \nearrow \infty$ and that $\sum_{k=1}^{\infty} a_k b_k$ converge. Prove

$$\lim_{m\to\infty} b_m \sum_{k=m}^{\infty} a_k = 0$$

Proof. Define

$$c_k = a_k b_k \text{ and } C_k = \sum_{j=k}^{\infty} c_j$$

By Lemma 8.4.3, we have

$$\lim_{k\to\infty} C_k = 0$$

Observe

$$\begin{aligned}
\sum_{k=n}^{\infty} a_k &= \sum_{k=n}^{\infty} \frac{c_k}{b_k} \\
&= \sum_{k=n}^{\infty} \frac{C_k - C_{k+1}}{b_k} \\
&= \lim_{N\to\infty} \sum_{k=n}^{N} \frac{C_k - C_{k+1}}{b_k} \\
&= \lim_{N\to\infty} \sum_{k=n}^{N} \frac{C_k}{b_k} - \sum_{k=n+1}^{N+1} \frac{C_k}{b_{k-1}} \\
&= \lim_{N\to\infty} \sum_{k=n+1}^{N} C_k\left(\frac{1}{b_k} - \frac{1}{b_{k-1}}\right) - \frac{C_{N+1}}{b_N} + \frac{C_n}{b_n}
\end{aligned}$$

Because $C_{N+1} \to 0$ and $b_N \to \infty$ as $N \to \infty$, we have

$$\begin{aligned}
\sum_{k=n}^{\infty} a_k &= \lim_{N\to\infty} \sum_{k=n+1}^{N} C_k\left(\frac{1}{b_k} - \frac{1}{b_{k-1}}\right) + \frac{C_n}{b_n} \\
&= \sum_{k=n+1}^{\infty} C_k\left(\frac{1}{b_k} - \frac{1}{b_{k-1}}\right) + \frac{C_n}{b_n}
\end{aligned}$$

This give us

$$\begin{aligned}
\left| b_n \sum_{k=n}^{\infty} a_n \right| &= \left| b_n \sum_{k=n+1}^{\infty} C_k\left(\frac{1}{b_k} - \frac{1}{b_{k-1}}\right) + b_n \times \frac{C_n}{b_n} \right| \\
&\leq \left| b_n \sum_{k=n+1}^{\infty} C_k\left(\frac{1}{b_k} - \frac{1}{b_{k-1}}\right) \right| + |C_n|
\end{aligned}$$

Arbitrarily pick $\epsilon \in \mathbb{R}^+$. Because $C_n \to 0$ and $b_n \nearrow \infty$ as $n \to \infty$, we know there exists N such that $|C_n| < \frac{\epsilon}{2}$ and $b_n > 0$ when $n > N$.

Let $n > N$. We now have

$$\begin{aligned}\left|b_n \sum_{k=n}^{\infty} a_n\right| &\leq b_n \left|\sum_{k=n+1}^{\infty} C_k(\frac{1}{b_k} - \frac{1}{b_{k-1}})\right| + \frac{\epsilon}{2} \\ &\leq b_n \sum_{k=n+1}^{\infty} \left|C_k(\frac{1}{b_k} - \frac{1}{b_{k-1}})\right| + \frac{\epsilon}{2} \\ &\leq b_n \sum_{k=n+1}^{\infty} |C_k| \times \left|\frac{1}{b_k} - \frac{1}{b_{k-1}}\right| + \frac{\epsilon}{2} \\ &= b_n \sum_{k=n+1}^{\infty} |C_k| \times (\frac{1}{b_{k-1}} - \frac{1}{b_k}) + \frac{\epsilon}{2}\end{aligned}$$

Notice that because $C_n \to 0$, we know $\{C_n : n \in \mathbb{N}\}$ is bounded. Then we know for each n, the supremum $\sup_{k\geq n} |C_k|$ exists. Define

$$X_n = \sup_{k\geq n+1} |C_k|$$

For $n > N$, we now have

$$\begin{aligned}\left|b_n \sum_{k=n}^{\infty} a_n\right| &\leq b_n \sum_{k=n+1}^{\infty} |C_k| \times (\frac{1}{b_{k-1}} - \frac{1}{b_k}) + \frac{\epsilon}{2} \\ &\leq b_n \sum_{k=n+1}^{\infty} X_n \times (\frac{1}{b_{k-1}} - \frac{1}{b_k}) + \frac{\epsilon}{2} \\ &= (X_n b_n \sum_{k=n+1}^{\infty} \frac{1}{b_{k-1}} - \frac{1}{b_k}) + \frac{\epsilon}{2} \\ &= \lim_{u\to\infty} X_n b_n (\frac{1}{b_n} - \frac{1}{b_u}) + \frac{\epsilon}{2} \\ &= X_n \frac{b_n}{b_n} + \frac{\epsilon}{2} \\ &= X_n + \frac{\epsilon}{2}\end{aligned}$$

Notice that

$$\lim_{n\to\infty} X_n = \lim_{n\to\infty} X_{n-1} = \lim_{n\to\infty} \sup_{k\geq n} |C_k| = \limsup_{n\to\infty} |C_n|$$

Then because $C_n \to 0$, we know

$$\lim_{n\to\infty} X_n = \limsup_{n\to\infty} |C_n| = \lim_{n\to\infty} |C_n| = 0$$

This tell us there exists N_1 such that $X_n < \frac{\epsilon}{2}$. Then for $n > \max\{N, N_1\}$, we have

$$\left| b_n \sum_{k=n}^{\infty} a_n \right| \leq X_n + \frac{\epsilon}{2} = \epsilon$$

This finish the proof. ■

Question 48

Suppose that $a_k > 0$ and

$$\sum_{k=1}^{\infty} a_k$$

converges. Prove that there exist b_k such that

$$\lim_{k \to \infty} \frac{b_k}{a_k} = \infty$$

and

$$\sum_{k=1}^{\infty} b_k$$

converges.

Proof. Define

$$r_n := \sum_{k=n}^{\infty} a_k$$

Because $\{a_k\}$ are positive, by Lemma 8.4.2, we know r_n monotonically decrease.

Moreover, by Lemma 8.4.3, we know $r_n \searrow 0$.

Observe

$$\sqrt{r_n} - \sqrt{r_{n+1}} = \frac{r_n - r_{n+1}}{\sqrt{r_n} + \sqrt{r_{n+1}}} = \frac{a_n}{\sqrt{r_n} + \sqrt{r_{n+1}}}$$

Define $\{b_n\}$ by

$$b_n := \frac{a_n}{\sqrt{r_n} + \sqrt{r_{n+1}}} = \sqrt{r_n} - \sqrt{r_{n+1}}$$

Observe

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \sqrt{r_k} - \sqrt{r_{k+1}} = \sqrt{r_1}$$

Notice

$$\frac{b_n}{a_n} = \frac{1}{\sqrt{r_n} + \sqrt{r_{n+1}}}$$

And deduce

$$r_n \searrow 0 \implies \sqrt{r_n} \searrow 0 \implies \frac{b_n}{a_n} = \frac{1}{\sqrt{r_n} + \sqrt{r_{n+1}}} \nearrow \infty$$

■

Question 49

Suppose that $a_k > 0$ and

$$\sum_{k=1}^{\infty} a_k$$

diverges. Prove that there exist b_k such that

$$\lim_{k\to\infty} \frac{b_k}{a_k} = 0$$

and

$$\sum_{k=1}^{\infty} b_k$$

diverges.

Proof. Define

$$A_n := \sum_{k=1}^{n} a_k$$

And define

$$b_n := \frac{a_n}{A_n}$$

Notice that $\{a_n\}$ are positive and $\sum_{k=1}^{\infty} a_k$ diverges implies

$$\lim_{n\to\infty} A_n = \sum_{k=1}^{\infty} a_k = \infty$$

So we have

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{1}{A_n} = 0$$

We now show the following, which will be essential for proving $\sum_{k=1}^{\infty} b_k$ diverges.

Define

$$X_p^q := \sum_{n=p}^{q} b_n$$

Let $q > p$. Because $\{a_n\}$ are positive, we have

$$A_p < A_q$$

This give us

$$X_p^q = \sum_{n=p}^{q} \frac{a_n}{A_n} \geq \sum_{n=p}^{q} \frac{a_n}{A_q} = \frac{A_q - A_{p-1}}{A_q} = 1 - \frac{A_{p-1}}{A_q}$$

Then because $\lim_{n\to\infty} A_n = \infty$, we have

$$\forall p, \liminf_{q\to\infty} X_p^q \geq \liminf_{q\to\infty} 1 - \frac{A_{p-1}}{A_q} = \lim_{q\to\infty} 1 - \frac{A_{p-1}}{A_q} = 1$$

Define $p_1 = 1$. We now know there exists $p_2 > p_1$ such that

$$\sum_{n=p_1}^{p_2-1} b_n = X_{p_1}^{p_2-1} > \frac{1}{2}$$

Again, because $\forall p, \liminf_{q\to\infty} X_p^q = 1$, we know there exists $p_3 > p_2$ such that

$$\sum_{n=p_2}^{p_3-1} b_n = X_{p_2}^{p_3-1} > \frac{1}{2}$$

Proceeding the argument above, we know there exists an increasing index sequence $\{p_k\}_{k\in\mathbb{N}}$ such that

$$\forall k, \sum_{n=p_k}^{p_{k+1}-1} b_k > \frac{1}{2}$$

Then we have

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \sum_{n=p_k}^{p_{k+1}-1} b_k \geq \sum_{k=1}^{\infty} \frac{1}{2} = \infty$$

■

Question 50

Prove that

$$\sum_{k=1}^{\infty} a_k \cos(kx)$$

converges for every $x \in (0, 2\pi)$ and every $a_k \searrow 0$. What happens when $x = 0$?

Proof. Let $x \in (0, 2\pi)$, and let

$$B_n = \sum_{k=1}^{n} \cos(kx)$$

We wish to prove

$$\{B_n\} \text{ is bounded}$$

Deduce

$$\begin{aligned}
\forall n, |B_n| = \left|\sum_{k=1}^{n} \cos(kx)\right| &= \left|\sum_{k=1}^{n} \text{ Re } e^{i(kx)}\right| \\
&= \left|\sum_{k=1}^{n} \text{ Re } (e^{ix})^k\right| \\
&= \left| \text{ Re } \sum_{k=1}^{n} (e^{ix})^k\right| \\
&= \left| \text{ Re } \frac{e^{ix}(1 - e^{nix})}{1 - e^{ix}}\right| \\
&\leq \left|\frac{e^{ix}(1 - e^{nix})}{1 - e^{ix}}\right| \\
&= \frac{\left|e^{ix} - e^{(n+1)ix}\right|}{|1 - e^{ix}|} \\
&\leq \frac{\left|e^{ix}\right| + \left|e^{(n+1)ix}\right|}{|1 - e^{ix}|} \\
&\leq \frac{2}{|1 - e^{ix}|}
\end{aligned}$$

Notice that because $x \in (0, 2\pi)$, we have

$$e^{ix} \neq 1$$

So we know $\frac{2}{|1-e^{ix}|} \in \mathbb{R}^+$. Let $M = \frac{2}{|1-e^{ix}|}$. By the long inequality above, we have

$$\forall n, |B_n| \le M \text{ (done)}$$

Because $\{B_n\}$ is bounded, by Dirichlet's Test, we are done.

If $x = 0$, it is possible that $a_k = \frac{1}{k}$, and we will have

$$\sum_{k=1}^{\infty} a_k \cos(kx) = \sum_{k=1}^{\infty} a_k \text{ diverge when } a_k \searrow 0$$

$\blacksquare$

Question 51

Prove that

$$\sum_{k=1}^{\infty} a_k \sin((2k+1)x)$$

converges for every $x \in \mathbb{R}$ and every $a_k \searrow 0$.

Proof. Notice that

$$x \equiv 0 \bmod \pi \implies \forall k, \sin((2k+1)x) = 0$$

This make our proof trivial. We only have to consider when

$$x \not\equiv 0 \bmod \pi$$

Observe

$$\begin{aligned}
\forall n, \left|\sum_{k=1}^{n} \sin((2k+1)x)\right| &= \left|\sum_{k=1}^{n} \text{ Im } e^{i(2k+1)x}\right| \\
&= \left| \text{ Im } \sum_{k=1}^{n} e^{i(2k+1)x}\right| \\
&\le \left|\sum_{k=1}^{n} e^{i(2k+1)x}\right| \\
&= \left|\frac{e^{i3x}(1-e^{i2nx})}{1-e^{i2x}}\right| \\
&\le \frac{\left|e^{i3x}\right| + \left|e^{i(3x+2nx)}\right|}{|1-e^{i2x}|} \\
&\le \frac{2}{|1-e^{i2x}|}
\end{aligned}$$

Because $x \not\equiv 0 \mod \pi$, we know $\frac{2}{|1-e^{i2x}|} \in \mathbb{R}^+$. We have proved the partial sums $\sum_{k=1}^{n} \sin((2k+1)x)$ is bounded. The proof for this problems follows from Dirichlet's Test.

■

Question 52

Suppose that

$$\sum_{k=1}^{\infty} a_k^2$$

and

$$\sum_{k=1}^{\infty} b_k^2$$

converges. Prove that the following series

1.
$$\sum_{k=1}^{\infty} |a_k b_k|$$

2.
$$\sum_{k=1}^{\infty} (a_k + b_k)^2$$

3.
$$\sum_{k=1}^{\infty} \frac{|a_k|}{k}$$

converge.

Proof. Deduce

$$\begin{aligned} 0 \leq (a_k - b_k)^2 &\implies 0 \leq a_k^2 - 2a_k b_k + b_k^2 \\ &\implies a_k b_k \leq \frac{a_k^2 + b_k^2}{2} \end{aligned}$$

Deduce

$$\begin{aligned} 0 \leq (a_k + b_k)^2 &\implies 0 \leq a_k^2 + 2a_k b_k + b_k^2 \\ &\implies a_k b_k \geq -\frac{a_k^2 + b_k^2}{2} \end{aligned}$$

Above give us

$$|a_k b_k| \leq \frac{a_k^2 + b_k^2}{2}$$

Clearly

$$\sum_{k=1}^{\infty} \frac{a_k^2 + b_k^2}{2} \text{ converges}$$

Then by comparison test, we know $\sum_{k=1}^{\infty} |a_k b_k|$ converges.

Notice that

$$0 \leq (a_k + b_k)^2 = a_k^2 + 2a_k b_k + b_k^2 \leq a_k^2 + 2\,|a_k b_k| + b_k^2$$

Then, because

$$\sum_{k=1}^{\infty} a_k^2 + 2\,|a_k b_k| + b_k^2 \text{ converges}$$

By comparison test, we now know

$$\sum_{k=1}^{\infty} (a_k + b_k)^2 \text{ converges}$$

Notice

$$\sum_{k=1}^{\infty} a_k^2 \text{ converge} \implies \lim_{k\to\infty} a_k^2 = 0$$

Then we know there exists N such that

$$\forall n > N, a_n^2 < 1$$

Because $a_n^2 < 1 \implies |a_n| < 1$. This give us

$$\forall n, |a_n| \leq \max\{|a_1|, |a_2|, \dots, |a_N|, 1\}$$

We have proved $|a_n|$ is bounded. Notice that

$$\frac{1}{k} \searrow 0$$

Then by Dirichlet's Test, we are done.

$\blacksquare$

Question 53

Does the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k \ln(k+1)}{k}$$

converge? Does it converge absolutely? Justify your answer.

Proof. First notice

$$\frac{d}{dt}\frac{\ln(t+1)}{t} = \frac{\frac{t}{t+1} - \ln(t+1)}{t^2}$$

Because

$$\frac{t}{t+1} \nearrow 1 \text{ and } \ln(t+1) \nearrow \infty$$

We know for large k,

$$\frac{d}{dt}\frac{\ln(t+1)}{t}\bigg|_{t=k} = \frac{1}{k^2}(\frac{k}{k+1} - \ln(k+1)) < 0$$

By fundamental theorem of calculus,

$$\frac{\ln(k+2)}{k+1} - \frac{\ln(k+1)}{k} = \int_k^{k+1} \frac{d}{dt}\frac{\ln(t+1)}{t}dt < 0 \text{ for large } k$$

We now know

$$\frac{\ln(k+1)}{k} \text{ monotonically decrease for large } k$$

Notice that

$$\lim_{k\to\infty} \frac{\ln(k+1)}{k} \overset{\text{H}}{=} \lim_{k\to\infty} \frac{\frac{1}{k+1}}{1} = 0$$

We have proved $\frac{\ln(k+1)}{k} \searrow 0$ as $k \to \infty$. Then by Alternating Series Test, we know

$$\sum_{k=1}^{\infty}(-1)^k \frac{\ln(k+1)}{k} \text{ converges}$$

Notice

$$\int \frac{\ln k}{k} dk = \frac{(\ln k)^2}{2}$$

Then we have

$$\int_1^{\infty} \frac{\ln k}{k} dk = \frac{(\ln k)^2}{2}\bigg|_{k=1}^{\infty} = \infty$$

Notice that

$$\frac{\ln k}{k} \leq \frac{\ln(k+1)}{k} \text{ for large } k$$

Then we know

$$\int_1^{\infty} \frac{\ln(k+1)}{k} dk = \infty$$

By integral test, we now know

$$\sum_{k=1}^{\infty}\left|(-1)^k\frac{\ln(k+1)}{k}\right| = \ln 2 + \sum_{k=2}^{\infty}\frac{\ln(k+1)}{k} \text{ diverges}$$

∎

Question 54

Find all values of $p \in \mathbb{R}$ that make the following series converge absolutely:

$$\sum_{k=1}^{\infty}(-1)^k\frac{1}{\ln^p(k)}$$

Proof. Suppose $p \leq 0$. Let $q = -p$. We have

$$\left|(-1)^k\frac{1}{\ln^p(k)}\right| = \ln^q(k) \text{ where } q \geq 0$$

Then we have

$$\lim_{k\to\infty}\left|(-1)^k\frac{1}{\ln^p(k)}\right| = \lim_{k\to\infty}\ln^q(k) \neq 0$$

This tell us if $p \leq 0$, the series do not converge.

Suppose $p > 0$. We see

$$\ln(x^{\frac{1}{p}}) \leq x^{\frac{1}{p}} \text{ for large } x$$

Then for large x we have

$$\frac{1}{p}\ln x \leq x^{\frac{1}{p}}$$

This give us

$$\ln x \leq px^{\frac{1}{p}}$$

This give us

$$(\ln x)^p \leq p^p x$$

This give us

$$\frac{1}{(\ln x)^p} \geq \frac{1}{p^p x}$$

Then because

$$\frac{1}{p^p}\sum_{x=1}^{\infty}\frac{1}{x} \text{ diverge}$$

By Comparison Test, we see

$$\sum_{x=1}^{\infty} \left| (-1)^x \frac{1}{\ln^p(x)} \right| = \sum_{x=1}^{\infty} \frac{1}{(\ln x)^p} \text{ diverge}$$

In conclusion, for all p, the series diverge. ■

Question 55

Let a_k and b_k be real sequences. Decide which of the following statements are true and which are false. Prove the true ones and give counterexamples to the false ones.

1. If $a_k \searrow 0$ as $k \to \infty$, and $\sum_{k=1}^{\infty} b_k$ converges conditionally, then
$$\sum_{k=1}^{\infty} a_k b_k$$
converges.
2. If $a_k \to 0$ as $k \to \infty$, then
$$\sum_{k=1}^{\infty} (-1)^k a_k$$
converges.
3. If $a_k \to 0$ as $k \to \infty$, and $a_k \geq 0$ for all $k \in \mathbb{N}$, then
$$\sum_{k=1}^{\infty} (-1)^k a_k$$
converges.
4. If $a_k \to 0$ as $k \to \infty$, and
$$\sum_{k=1}^{\infty} (-1)^k a_k$$
converges, then $a_k \searrow 0$ as $k \to \infty$.

Proof. From now, we let

$$A_n = \sum_{k=1}^{n} a_n \text{ and } B_n = \sum_{k=1}^{n} b_n$$

We claim

$$a_k \searrow 0 \text{ as } k \to \infty \text{ and } \sum_{k=1}^{\infty} b_k \text{ converges } \implies \sum_{k=1}^{\infty} a_k b_k \text{ converges}$$

Notice

$$\sum_{k=1}^{\infty} b_k \text{ converges } \implies \exists L \in \mathbb{R}, \lim_{n\to\infty} B_n = L$$

Let $\epsilon = 1$, we know there exists N such that

$$\forall n > N, B_n < L + 1$$

Then we see

$$\forall k, B_k < \max\{B_1 + 1, B_2 + 1, \dots, B_N + 1, L + 1\}$$

Then by Dirichlet's Test, we are done. (done)

Define

$$a_k = (-1)^k \frac{1}{k}$$

We have

$$\lim_{k\to\infty} a_k = 0 \text{ and } \sum_{k=1}^{\infty} (-1)^k a_k = \sum_{k=1}^{\infty} (-1)^{2k} \frac{1}{k} = \sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges}$$

Define

$$a_k = \begin{cases} \frac{1}{\sqrt{\lceil \frac{k}{2} \rceil + 1} - 1} & \text{if } k \text{ is odd} \\ \frac{1}{\sqrt{\lceil \frac{k}{2} \rceil + 1} + 1} & \text{if } k \text{ is even} \end{cases}$$

The sequence we have is

$$a_1 = \frac{1}{\sqrt{2} - 1}, a_2 = \frac{1}{\sqrt{2} + 1}, a_3 = \frac{1}{\sqrt{3} - 1}, a_4 = \frac{1}{\sqrt{3} + 1}, a_5 = \frac{1}{\sqrt{4} - 1}, \dots$$

Because

$$\lim_{k\to\infty} \lceil \frac{k}{2} \rceil = \infty$$

We have

$$\lim_{k\to\infty} a_k = 0$$

Clearly, $\forall k, a_k \geq 0$. Define

$$B_n := \sum_{k=1}^{n} (-1)^k a_k$$

Notice

$$B_{2n} = \sum_{k=1}^{n} \frac{-1}{\sqrt{k+1}-1} + \frac{1}{\sqrt{k+1}+1} = \sum_{k=1}^{n} \frac{-2}{k}$$

Then we have

$$\lim_{n\to\infty} B_{2n} = \sum_{k=1}^{n} \frac{-2}{k} \text{ diverge}$$

This tell us

$$\sum_{n=1}^{\infty} (-1)^n a_n = \lim_{n\to\infty} B_n \text{ diverge}$$

Otherwise the sub-sequence $\lim_{n\to\infty} B_{2n}$ would have converge.

Define

$$a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{1}{(\frac{n}{2})^2} & \text{if } n \text{ is even} \end{cases}$$

The form of this sequence is

$$a_1 = 0, a_2 = \frac{1}{1^2}, a_3 = 0, a_4 = \frac{1}{2^2}, a_5 = 0, a_6 = \frac{1}{3^2}, a_7 = 0, \dots$$

Clearly, $a_n \to 0$. We see

$$\sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} a_{2n+1} + \sum_{n=1}^{\infty} a_{2n} = \sum_{n=0}^{\infty} 0 + \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converge}$$

Yet, for all even n, we have

$$a_n = \frac{1}{(\frac{n}{2})^2} > 0 = a_{n+1}$$

So we don't have $a_n \searrow 0$ as $n \to \infty$. This serve as a counter example for the last question. $\blacksquare$