臺灣大學數學系113學年度碩士班甄試筆試試題

科目：線性代數

2023.11.02

1. Let $A \in M(3, \mathbb{R})$ be given by

$$A = \begin{pmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) (**10 points.**) Find the Jordan-Chevalley decomposition of A.

(b) (**10 points.**) Compute

$$\exp A := I_3 + \sum_{k=1}^{\infty} \frac{A^k}{k!}.$$

2. Let V be the space of all polynomials in x over $\mathbb{R}$ of degree ≤ 2. Let an inner product on V be defined by

$$\langle f, g \rangle := \int_{-1}^{1} f(x)g(x)\,dx.$$

(a) (**10 points.**) Find a polynomial $k(x,t)$ in x and t such that

$$f(x) = \int_{-1}^{1} k(x,t)f(t)\,dt$$

for all $f \in V$.

(b) (**10 points.**) Let $T : V \to V$ be the linear transformation defined by $T(a_2x^2 + a_1x + a_0) = 2a_2x + a_1$. Find the linear transformation $T^* : V \to V$ such that $\langle T(f), g \rangle = \langle f, T^*(g) \rangle$ for all $f, g \in V$.

3. (**20 points.**) Let $V = M(n, \mathbb{R})$ be the vector space of all $n \times n$ matrices over $\mathbb{R}$ and $f : V \to \mathbb{R}$ be a linear transformation. Assume that $f(AB) = f(BA)$ for all $A, B \in V$ and $f(I_n) = n$, where I_n is the identity matrix in V. Prove that f is the trace function. (*Hint*: Consider the cases $A = E_{ij}$ and $B = E_{k\ell}$ for various E_{ij} and $E_{k\ell}$. Here E_{ij} denotes the matrix whose (i,j)-entry is 1 and whose other entries are 0.)

4. Let U and V be finite-dimensional vector spaces, and U^* and V^* be their dual spaces, respectively. For a linear transformation $T : U \to V$, define $T^* : V^* \to U^*$ by $(T^*f)(u) = f(Tu)$ for $f \in V^*$ and $u \in U$.

(a) (**10 points.**) Prove that T is injective if and only if T^* is surjective.

(b) (**10 points.**) Prove that T is surjective if and only if T^* is injective.

5. (**20 points.**) Let V a finite-dimensional vector space over a field F and $T : V \to V$ be a linear transformation. Assume that $f(x)$ and $g(x)$ are two relatively prime polynomials in $F[x]$. Prove that $\ker(f(T)g(T)) = \ker f(T) \oplus \ker g(T)$. (Here for a linear transformation S, we let $\ker S$ denote the kernel of S.)

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科目：線性代數

2022.10.20

Notation: $\mathbb{R}$ is the set of real numbers and $\mathbb{C}$ is the set of complex numbers. If $F = \mathbb{R}$ or $\mathbb{C}$ and n is a positive integer, we denote by $\mathrm{M}_n(F)$ the set of $n \times n$ matrices with entries in F and by I_n the identity matrix in $\mathrm{M}_n(F)$.

Problem 1 (15 pts)**.** Let

$$\mathbf{v}_1 = (1, 2, 0, 4),\ \mathbf{v}_2 = (-1, 1, 3, -3),\ \mathbf{v}_3 = (0, 1, -5, -2),\ \mathbf{v}_4 = (-1, -9, -1, -4)$$

be vectors in $\mathbb{R}^4$. Let W_1 be the subspace spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ and let W_2 be the subspace spanned by $\mathbf{v}_3$ and $\mathbf{v}_4$. Find the dimension and a basis of $W_1 \cap W_2$.

Problem 2. Let

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 3 & -4 & 1 \\ 3 & -8 & 5 \end{pmatrix}.$$

(1) (10pts) Find an invertible matrix $Q \in \mathrm{M}_3(\mathbb{C})$ such that

$$Q^{-1}AQ = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 12 \\ 0 & 1 & 1 \end{pmatrix}.$$

(2) (15pts) Find an invertible matrix $P \in \mathrm{M}_3(\mathbb{C})$ such that $P^{-1}AP$ is a diagonal matrix.

Problem 3. For any $A \in \mathrm{M}_2(\mathbb{C})$, define

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.$$

(1) (5pts) Evaluate $\sin \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$.

(2) (15pts) Prove or disprove: There exists $A \in \mathrm{M}_2(\mathbb{R})$ such that

$$\sin A = \begin{pmatrix} 1 & 2022 \\ 0 & 1 \end{pmatrix}.$$

Problem 4 (20pts)**.** Let $A = (a_{ij}) \in \mathrm{M}_n(\mathbb{C})$ and let $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ be roots of characteristic polynomial of A (counted with multiplicity). Show that

$$AA^* = A^*A \text{ if and only if } \sum_{1 \le i,j \le n} |a_{ij}|^2 = \sum_{k=1}^{n} |\lambda_k|^2.$$

Problem 5 (20pts)**.** Let $A, B \in \mathrm{M}_n(\mathbb{C})$. Suppose that all of the eigenvalues of A and B are positive real numbers. If $A^4 = B^4$, prove that $A = B$.

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科目：線性代數

2020.10.23

In the following, for a linear map $f: V \to V$, $\ker f$ and $\operatorname{im} f$ denote the kernel and the image of f, respectively.

1. Let V be a finite-dimensional complex inner product space. Let $d: V \to V$ be a linear map satisfying $d^2 = 0$. Let $\delta: V \to V$ be the adjoint of d and $\Delta = d\delta + \delta d$. Prove the following.

   (a) [5%] $d\delta x = 0$ implies that $\delta x = 0$, and $\delta dx = 0$ implies that $dx = 0$, for all $x \in V$.

   (b) [10%] $\ker \Delta = \ker d \cap \ker \delta$.

   (c) [10%] There is the orthogonal decomposition $V = \ker \Delta \oplus \operatorname{im} d \oplus \operatorname{im} \delta$.

   (d) [5%] There is the orthogonal decomposition $\ker d = \ker \Delta \oplus \operatorname{im} d$.

2. [10%] Let $V = \mathbb{R}^n$ be the space of column vectors, and M a positive definite symmetric $n \times n$ real matrix. Suppose the matrix $A \in M_n(\mathbb{R})$ satisfies $MAM^{-1} = A^t$. Show that there exists $P \in M_n(\mathbb{R})$ satisfying $P^t M P = I_n$ such that $P^{-1}AP$ is diagonal. (Here B^t denotes the transpose of the matrix B.)

3. (a) [10%] Let M be an invertible $n \times n$ complex matrix. Prove that there exists an invertible matrix A such that $A^2 = M$.

   (b) [10%] Let $n \geq 2$ and N be an $n \times n$ matrix over a field such that $N^n = 0$ but $N^{n-1} \neq 0$. Prove that there is no square matrix B such that $B^2 = N$.

4. [20%] Let V be a vector space over a field F and $u_1, \cdots, u_n \in V$ are linearly independent. Show that, for any $v_1, \cdots, v_n \in V$, $u_1 + \alpha v_1, \cdots, u_n + \alpha v_n$ are linearly independent for all but finitely many values of $\alpha \in F$.

5. [20%] Let P be an $n \times n$ matrix with coefficients in a field. Suppose $\operatorname{rank}(P) + \operatorname{rank}(I_n - P) = n$. Prove that $P^2 = P$.

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科目：線性代數

2019.10.18

1. Let A be a 4×4 real symmetric matrix. Suppose that 1 and 2 are eigenvalues of A and the eigenspace for the eigenvalue 2 is 3-dimensional. Assume that $(1,-1,-1,1)^t$ is an eigenvector for the eigenvalue 1. (Here v^t denotes the transpose of v.)

(a) Find an orthonormal basis for the eigenspace for the eigenvalue 2 of A. **(10 points.)**

(b) Find Av, where $v = (1,0,0,0)^t$. **(10 points.)**

2. Let A be a real $n \times n$ matrix. Prove that

$$\operatorname{rank}(A^2) - \operatorname{rank}(A^3) \leq \operatorname{rank}(A) - \operatorname{rank}(A^2).$$

(10 points.)

3. Let V be a vector space of finite dimension over $\mathbb{R}$ and S, T, and U be subspaces of V. Prove or disprove (by giving counterexamples) the following statements:

(a) $\dim(S+T) = \dim S + \dim T - \dim(S \cap T)$. **(10 points.)**

(b) $\dim(S+T+U) = \dim S + \dim T + \dim U - \dim(S \cap T) - \dim(T \cap U) - \dim(U \cap S) + \dim(S \cap T \cap U)$. **(10 points.)**

4. (a) Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$. Compute $\exp A$. **(10 points.)**

(b) Prove that $\det(\exp A) = \exp(\operatorname{tr} A)$ for $A \in M(n, \mathbb{C})$. **(10 points.)**

(c) Prove or disprove (by giving counterexamples) that if A is nilpotent, then so is $\exp A - I_n$. Here a matrix M is said to be nilpotent if $M^k = 0$ for some positive integer k and I_n is the identity matrix of size n. **(10 points.)**

5. Let U and V be finite-dimensional vector spaces, and U^* and V^* be their dual spaces, respectively. For a linear transformation $T : U \to V$, define $T^* : V^* \to U^*$ by $(T^*f)(u) = f(Tu)$ for $f \in V^*$ and $u \in U$.

(a) Prove that T is injective if and only if T^* is surjective. **(10 points.)**

(b) Prove that T is surjective if and only if T^* is injective. **(10 points.)**

臺灣大學數學系 108 學年度碩士班甄試試題

科目：線性代數

2018.10.19

1. Find all possible Jordan forms for 8×8 real matrices having $x^2(x-2)^3$ as minimal polynomial. (**20 points.**)

2. Let V be a vector space over a field $\mathbb{F}$ of infinite elements, and let $v_1, \ldots, v_n$ be vectors in V, where n is a positive integer. Suppose that $v_0 + zv_1 + \ldots + z^n v_n = 0$ for infinitely many z in $\mathbb{F}$. Prove that all v_i's are zero. (**20 points.**)

3. Let $V = M(n, \mathbb{R})$ be the vector space of all $n \times n$ matrices and $f : V \to \mathbb{R}$ be a linear transformation. Assume that $f(AB) = f(BA)$ for all $A, B \in V$ and $f(I_n) = n$, where I_n is the identity matrix in V. Prove that f is the trace function. (**20 points.** *Hint*: Consider the cases $A = E_{ij}$ and $B = E_{k\ell}$ for various E_{ij} and $E_{k\ell}$. Here E_{ij} denotes the matrix whose (i,j)-entry is 1 and whose other entries are 0.)

4. Let V be an n-dimensional vector space over $\mathbb{R}$ and $B : V \times V \to \mathbb{R}$ be a symmetric bilinear form on V. (*Symmetric* means $B(u,v) = B(v,u)$ for all $u, v \in V$. *Bilinear* means that B is linear in each of the two variables.)

(a) Let W be a vector subspace of V and let

$$W^{\perp} = \{u \in V : B(u,v) = 0 \text{ for all } v \in W\}.$$

Prove that if $\dim W = m$, then $\dim W^{\perp} \geq n - m$. (**10 points.** *Hint*: Choose a basis $\{v_1, \ldots, v_m\}$ for W and consider the map

$$u \longmapsto (B(u, v_1), \ldots, B(u, v_m))$$

from V into $\mathbb{R}^m$.)

(b) Prove that $V = W \oplus W^{\perp}$ if and only if the restriction of B to W is nondegenerate. (*Nondegenerate* means that $v = 0$ is the only vector of W such that $B(u,v) = 0$ for all $u \in W$.) (**15 points.**)

(c) Prove that if B is nondegenerate on V, then there is a nonnegative integer p with $p \leq n$ and a basis $\{v_1, \ldots, v_n\}$ such that

$$B(v_i, v_j) = \begin{cases} 1, & \text{if } 1 \leq i = j \leq p, \\ -1, & \text{if } p+1 \leq i = j \leq n, \\ 0, & \text{if } i \neq j. \end{cases}$$

(**15 points.**)

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科目：線性代數

2017.10.20

(1) (20 points) Let $A = \begin{pmatrix} -1 & 3 & -2 \\ 2 & 3 & 0 \\ 11 & -6 & 7 \end{pmatrix}$. Find the lower triangular Jordan canonical form of A. Please compute $\exp(tA)$ and derive the general solution to $x'(t) = A\, x(t)$, where $x(t)$ is a 3-dimensional column vector.

(2) (20 points) Let V be an n-dimensional complex vector space, and $T : V \to V$ be an invertible linear map such that $T^2 = 1$. (a) Show that T is diagonalizable, (b) Let S be the vector space of linear transformations from V to V that commute with T. Please express $\dim_{\mathbb{C}} S$ in terms of n and the trace of T.

(3) (20 points) Let $A = (A_{ij})$ be a real invertible skew-symmetric $2n \times 2n$ matrix.
(a) Show that all eigenvalues of A are pure imaginary.
(b) Define the Pfaffian $Pf(A)$ of A by

$$Pf(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) A_{\sigma(1),\sigma(2)} A_{\sigma(3),\sigma(4)} \cdots A_{\sigma(2n-1),\sigma(2n)} \ .$$

Let B be any real $2n \times 2n$ matrix. Show that $Pf(BAB^T) = Pf(A)\det(B)$.
(c) Assuming the fact that there exists a real orthogonal $2n \times 2n$ matrix O such that

$$OAO^T = \mathrm{diag}\left\{ \begin{pmatrix} 0 & m_1 \\ -m_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & m_2 \\ -m_2 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & m_n \\ -m_n & 0 \end{pmatrix} \right\},$$

where $m_i \in \mathbb{R}$ for $i = 1, \cdots, n$. Show that $\det(A) = Pf(A)^2$.

(4) (20 points) Let $A, B \in M_n(\mathbb{C})$ be $n \times n$ complex matrices. Show that A and B are simultaneously triangularizable (*i.e.* there exists an invertible matrix $P \in GL_n(\mathbb{C})$ such that PAP^{-1} and PBP^{-1} are both upper triangular) if A and B commute.
Hint: Let λ be one of the eigenvalues of A. Try to show $B(\ker(A - \lambda I)) \subset \ker(A - \lambda I)$.

(5) (20 points) Show that

$$\begin{vmatrix} X_0 & X_1 & X_2 & \dots & X_{n-1} \\ X_{n-1} & X_0 & X_1 & \dots & X_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_1 & X_2 & X_3 & \dots & X_0 \end{vmatrix} = \prod_{j=0}^{n-1} \left(\sum_{k=0}^{n-1} \zeta^{jk} X_k \right)$$

where ζ is a primitive n-th root of unity.

Hint: You may first compute, for example, $\begin{pmatrix} X_0 & X_1 & X_2 & X_3 \\ X_3 & X_0 & X_1 & X_2 \\ X_2 & X_3 & X_0 & X_1 \\ X_1 & X_2 & X_3 & X_0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 & \zeta^3 \\ 1 & \zeta^2 & \zeta^4 & \zeta^6 \\ 1 & \zeta^3 & \zeta^6 & \zeta^9 \end{pmatrix}$.

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科目：線性代數

2016.10.21

1. (20%) Let $A \in M_{n\times n}(F)$ where F is a field.
   (a) Show that if k is the largest integer such that some $k \times k$ submatrix of A has a nonzero determinant, then $\operatorname{rank}(A) = k$.
   (b) If A is nilpotent of index m (that is, $A^m = 0$ but $A^{m-1} \neq 0$), and if, for each vector v in F^n, W_v is defined to be the subspace generated by $v, Av, \ldots, A^{m-1}v$, how large can the dimension of W_v be? (Justify your answer.)

2. (30%)
   (a) Let $A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$. Find the general solution to the system of differential equations

$$\frac{dX}{dt} = AX, \text{ where } X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

   where for each i, $x_i(t)$ is a differentiable real-valued function of the real variable t.
   (b) Let V be the space of all real polynomials having degree less than 4 with the inner product $\langle f(x), g(x)\rangle = \int_{-1}^{1} f(t)g(t)dt$. Let T be a linear operator on V defined by $T(f(x)) = f'(x) + 3f(x)$.
   Use the Gram-Schmidt process to replace $\beta = \{1, 1+x, x+x^2, x^2+x^3\}$ by an orthonormal basis for V and find the matrix representation of the adjoint T^* of T in this orthonormal basis.

3. (30%)
   (a) Let $A \in M_{n\times n}(\mathbb{R})$. Show that there exists an orthogonal matrix Q and a positive semi-definite symmetric matrix P such that $A = QP$.
   (b) Let V be a finite-dimensional vector space over $\mathbb{C}$ and T be a linear operator on V. Show that T is normal if and only if its adjoint $T^* = g(T)$ for some polynomial $g(x) \in \mathbb{C}[x]$.

4. (20 %) Let $T \in \operatorname{End}_{\mathbb{C}}(V)$ for a finite-dimensional $\mathbb{C}$-vector space V.
   (a) Show that we have an expression of T as $T = S + N$ with $S, N \in \operatorname{End}_{\mathbb{C}}(V)$, such that S is diagonalisable, N is nilpotent and $SN = NS$.
   (b) Show that both S and N are uniquely defined by these conditions.
   (c) Show that there is a polynomial $p(x) \in \mathbb{C}[x]$ with $p(0) = 0$ such that $S = p(T)$.

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科目：線性代數

2015. 10. 23

There are five problems 1 $\sim$ 5 in total; some problems contain sub-problems, indexed by (a), (b), etc.

1. [20%] Prove the Cayley-Hamilton theorem: Let V be a finite-dimensional vector space and $T : V \to V$ a linear transformation with characteristic polynomial $p(x)$. Then $p(T) = 0$.

2. Let F be a field and $A \in M_{m\times n}(F)$ be an m by n matrix.

   (a) [10%] Show that the row rank of A equals the column rank of A.

   (b) [10%] Denote by A^t the transport of A and let r be the row rank of A. Show that AA^t is of rank r.

3. [20%] Let F be a field and $\{A_i \in M_n(F) \,|\, i \in I\}$ be a collection of n by n matrices. (I is a set; it might be finite or infinite.) Suppose that A_i are diagonalizable for all $i \in I$ and $A_iA_j = A_jA_i$ for any $i, j \in I$. Show that there exists an invertible matrix $P \in M_n(F)$ such that PA_iP^{-1} are diagonal for all $i \in I$.

4. [20%] Let $A \in M_n(\mathbb{C})$ be an n by n matrix over the field of complex numbers. Denote by A^* the conjugate transport of A. Suppose $AA^* = A^*A$. Show that there exists a matrix P such that (i) $PP^* = I$, the identity matrix, and (ii) PAP^* is a diagonal matrix. (You may do the case $A = A^*$ for half credit.)

5. [20%] Let V be a finite-dimensional vector space over the field of complex numbers and $T : V \to V$ a linear transformation with characteristic polynomial $p(x)$. Suppose that $p(x) = q_1(x)q_2(x)$ for two polynomials $q_1(x)$ and $q_2(x)$ which do not have a common root. Show that there are two subspaces W_1 and W_2 of V satisfying that (i) $W_1 \cap W_2 = \{0\}$ and $V = W_1 + W_2$, (ii) $T(W_i) \subset W_i$ for each $i = 1, 2$, and (iii) regarding T as a linear transformation on W_i, it has characteristic polynomial $q_i(x)$ for $i = 1, 2$.

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科目：線性代數

2014.10.24

Notation: $\mathbb{Q}$ is the set of rational numbers, $\mathbb{R}$ is the set of real numbers and $\mathbb{C}$ is the set of complex numbers. Let n be a positive integer and I_n be the identity matrix in $M_2(\mathbb{Q})$.

Problem 1 (20 pts).

(a) For each $x \in \mathbb{R}$, let V_x be the subspace of $\mathbb{R}^4$ generated by

$$(x, 1, 1, 1), (1, x, 1, 1), (1, 1, x, 1), (1, 1, 1, x).$$

Determine all x such that $\dim_{\mathbb{R}} V_x \leq 3$.

(b) Find the dimension and a basis for the space of $\mathbb{R}$-linear maps $L : \mathbb{R}^5 \to \mathbb{R}^3$ whose kernels contain $(0, 2, -3, 0, 1)$.

Problem 2 (20 pts). Let

$$A = \begin{pmatrix} -1 & 4 & -2 \\ -2 & 5 & -2 \\ -1 & 2 & 0 \end{pmatrix}.$$

(1) Compute the characteristic polynomial of A.
(2) If $f(x) = (x-3)^2 + 5$, find the eigenvalues of $f(A)$.
(3) Find an orthogonal matrix $P \in M_3(\mathbb{R})$ such that $P^{-1}AP$ is diagonal.

Problem 3 (15 pts). Let $A, B \in M_n(\mathbb{R})$ be invertible matrices. Show that

(1) If $ABA^{-1}B^{-1} = c \cdot I_n$, then $c = \pm 1$;
(2) If $AB - BA = c \cdot I_n$, then $c = 0$.

Problem 4 (10pts). Let $A \in M_n(\mathbb{R})$ such that $A^3 = A$. Show that $\operatorname{rank} A = \operatorname{trace} A^2$.

Problem 5 (15pts). Let $A \in M_n(\mathbb{R})$ such that $\operatorname{rank} A + \operatorname{rank}(I_n - A) = n$. Show that $A^2 = A$.

Problem 6 (20 pts). Let $A \in M_n(\mathbb{Q})$ with $A^n = 0$ but $A^{n-1} \neq 0$. Show that if $B \in M_n(\mathbb{Q})$ commutes with A ($\Longleftrightarrow$ $BA = AB$), then

$$B = a_1 + a_2 A + \ldots + a_n A^{n-1} \text{ for some } a_1, \ldots, a_n \in \mathbb{Q}.$$

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科目：線性代數

2013.10.18

1. (20%)
   (a) Let $A \in M_{m\times n}(F)$, $B \in M_{n\times p}(F)$ where F is a field.
   Show that $\text{rank}(AB) \le \min\{\text{rank}(A), \text{rank}(B)\}$.
   Moreover, if $n = p$ and B is invertible, show that $\text{rank}(AB) = \text{rank}(A)$.
   (b) Let $A \in M_{m\times n}(\mathbb{C})$. Show that $\text{rank}(A^*A) = \text{rank}(A)$ where A^* is the conjugate transpose of A.

2. (20%)
   (a) Let A be an $n \times n$ real symmetric matrix. Show that if λ is an eigenvalue of A in $\mathbb{C}$, then λ is real.
   (b) Let A be an $n \times n$ real symmetric matrix. Show that one can find an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

3. (20%)
   (a) Which $n \times n$ real matrices B have the property that $AB = BA$ for all $n \times n$ real matrices A? Justify your answer.
   (b) Let A, B be two $n \times n$ real symmetric matrices. Show that A and B are simultaneously diagonalizable if and only if $AB = BA$.

4. (20 %) For all $x \in \mathbb{R}^n$, we define the norm of x by $\|x\| = \sqrt{\langle x, x\rangle}$ where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^n$.
   Let $A = \begin{pmatrix} 0 & 4 & 0 & 4 \\ 1 & 2 & 1 & 2 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.
   (a) Find a vector p in the column space of A (the subspace of $\mathbb{R}^3$ spanned by the column vectors of A) such that $\|p - b\| \le \|A \cdot x - b\|$ for all $x \in \mathbb{R}^4$
   (b) Find $s \in \mathbb{R}^4$ such that $A \cdot s = p$ and s has the minimum norm, that is, $\|s\| \le \|v\|$ for all solutions v (in $\mathbb{R}^4$) of $A \cdot x = p$.
   (Justify your answers.)

5. (20 %) Find the Jordan form B of

$$A = \begin{pmatrix} 3 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 0 & 5 & -3 \\ 4 & -1 & 3 & -1 \end{pmatrix}$$

   and the matrix P such that $B = P^{-1}AP$.
   (Notice! Show your works in details. No points will be assigned for non-substantial answers.)

臺灣大學數學系 102 學年度碩士班甄試試題
科目：線性代數

2012.10.19

1. [20%] Let V be an n-dimensional vector space over a field. Let $T : V \to V$ be a linear map. Show that the degree of the minimal polynomial of T equals

$$\max_{v\in V}\left\{\dim\langle v, T(v), T^2(v), \cdots, T^{n-1}(v)\rangle\right\}.$$

(Here $\langle w_1, \cdots, w_r\rangle$ denotes the subspace spanned by $w_1, \cdots, w_r$.)

2. [20%] Consider the real $n \times n$ matrix $A = (a_{ij})$ satisfying

- $a_{ij} \geq 0$ for all i, j,
- $a_{ii} = 0$ for all i, and
- $\sum_{j=1}^{n} a_{ij} = \gamma$ for all $i = 1, 2, \cdots, n$ for some constant $\gamma \neq 0$.

Show that

(a) If $\lambda \in \mathbb{R}$ is a real eigenvalue of A, then $-\gamma \leq \lambda \leq \gamma$.

(b) γ is an eigenvalue of A and the corresponding eigenspace has dimension one.

(c) The eigenspace corresponding to $-\gamma$ has dimension either zero or one.

3. [20%] Let $A = (a_{ij})$ be a real $n \times n$ symmetric matrix. Show that A is *positive definite* (meaning: $v^tAv > 0$ for any non-zero $v \in \mathbb{R}^n$ where v^t is the transport of v) if and only if, for any $r = 1, 2, \cdots, n$, we have

$$\det A_r > 0 \quad \text{where } A_r = (a_{ij})_{1\leq i,j\leq r} \in M_r(\mathbb{R}).$$

4. [20%] Let $T : V \to W$ be a linear map between two finite dimensional vector spaces. Let V^* and W^* be the dual spaces of V and W, respectively. Prove that

(a) T is injective if and only if the transport $T^* : W^* \to V^*$ is surjective.

(b) T is surjective if and only if the transport $T^* : W^* \to V^*$ is injective.

(Recall that the transport T^* is defined by $\big(T^*(f)\big)(v) = f\big(T(v)\big)$ for $f \in W^*, v \in V$.)

5. [20%] Let A be a real $n \times n$ matrix such that $A^t = -A$ (where A^t denotes the transport of A). Let $\lambda = a + bi$ be a complex eigenvalue of A where $a, b \in \mathbb{R}$ and $i^2 = -1$. Show that $a = 0$.

臺灣大學數學系

101 學年度碩士班甄試試題

科目：線性代數

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You should include in your answer every piece of computation and every piece of reasoning so that the corresponding partial credit could be gained.

(20%) **1.** (a) Prove that if A is a symmetric matrix then A^2 is symmetric. Is the converse true? Justify your answer.

(b) Determine all real $m \times n$ matrices A for which $A^{\mathrm{T}}A = 0$. Justify your answer.

(c) Suppose that K is a square matrix with $K = -K^{\mathrm{T}}$ and that $I - K$ is nonsingular. Let $B = (I + K)(I - K)^{-1}$. Prove that $B^{\mathrm{T}}B = BB^{\mathrm{T}} = I$.

(20%) **2.** Let V be the real vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$.

(a) For any integer n, define $f_n(x) = x + n$. Determine the dimension of the subspace of V generated by $\{f_n(x): n \in \mathbb{Z}\}$. Justify your answer.

(b) Define $g: \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 4i + 1 - x, & \text{if } 4i \le x < 4i + 2 \text{ for } i \in \mathbb{Z}; \\ x - 4i - 3, & \text{if } 4i + 2 \le x < 4i + 4 \text{ for } i \in \mathbb{Z}. \end{cases}$$

For any integer n, define $g_n(x) = g(x + n)$. Determine the dimension of the subspace of V generated by $\{g_n(x): n \in \mathbb{Z}\}$. Justify your answer.

(20%) **3.** (a) Suppose I is the $n \times n$ identity matrix and J is the $n \times n$ matrix whose entries are all 1. Determine the ranks of J and $J - I$. Justify your answer.

(b) Prove that the rank of an $n \times n$ $(0,1)$-matrix A with $A_{ij} + A_{ji} = 1$ for $1 \le i < j \le n$ is either n or $n - 1$.

(20%) **4.** A square matrix is called *unimodular* if its determinant is 0 or ± 1. A matrix is called *totally unimodular* if all of its square submatrices are unimodular. It is easy to see that any entry of a totally unimodular matrix is 0 or ± 1.

(a) For any $n \ge 3$, give an $n \times n$ $(0,1)$-matrix which is not unimodular. Justify your answer.

(b) Prove that any $m \times n$ matrix in which every column has exactly one 1, exactly one -1 and all other entries 0 is totally unimodular.

(20%) **5.** (a) Prove that all eignevalues of a real symmetric matrix are real.

(b) Suppose S is an $m \times m$ real symmetric matrix whose eigenvalues are $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_m$. Recall that there is an orthonormal basis $v_1, v_2, \ldots, v_m$ for which each v_i is a corresponding eigenvector of λ_i. Prove that $\lambda_1 \ge \frac{x^{\mathrm{T}}Sx}{x^{\mathrm{T}}x} \ge \lambda_m$ for any nonzero m-vector x.

(c) Suppose A is an $n \times n$ real symmetric matrix whose eigenvalues are $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n$. Let B be the matrix obtained from A by deleting the last row and the last column, and its eigenvalues are $\mu_1 \ge \mu_2 \ge \ldots \ge \mu_{n-1}$. Prove that these eigenvalues are interlacing, that is $\lambda_1 \ge \mu_1 \ge \lambda_2 \ge \mu_2 \cdots \ge \lambda_{n-1} \ge \mu_{n-1} \ge \lambda_n$.