Chapter 1

Final

1.1 Main Body

Abstract

Note that $\mathfrak{X}(M)$ always denote the space of smooth vector field on M, and $\Omega^k(M)$ always denote the space of smooth k-forms on M.

Question 1

Let X be a vector field on a compact manifold M. Let θ_t be the global flow generated by X. Let $\alpha \in \Omega^k(M)$. Show

$$\theta_t^* \alpha = \alpha \text{ for all } t \iff \mathcal{L}_X \alpha = 0$$

Proof. Left to right follows from computing

$$(\mathcal{L}_X \alpha)_p = \lim_{t \to 0} \frac{(\theta_t^* \alpha)_p - \alpha_p}{t} = \lim_{t \to 0} \frac{\alpha_p - \alpha_p}{t} = 0 \text{ for all } p \in M$$

Suppose $\mathcal{L}_X \alpha = 0$. Fix arbitrary $p \in M$ and arbitrary $V_1, \dots, V_k \in \mathfrak{X}(M)$. Define $\beta : \mathbb{R} \to \mathbb{R}$ by

$$\beta(t) \triangleq (\theta_t^* \alpha)_p(V_1, \dots, V_k)$$

Fix arbitrary $t \in \mathbb{R}$. By definition,

$$(\theta_t^* \alpha)_p(V_1, \dots, V_k) = \alpha_{\theta_t(p)}\big((\theta_t)_{*,p} V_1, \dots, (\theta_t)_{*,p} V_k\big) \tag{1.1}$$

Because

$$\theta_s = \theta_{s-t} \circ \theta_t$$

We know

$$\theta_s^*\alpha = \theta_t^*\theta_{s-t}^*\alpha$$

This give us

$$\begin{aligned}(\theta_s^*\alpha)_p(V_1,\dots,V_k) &= (\theta_t^*\theta_{s-t}^*\alpha)_p(V_1,\dots,V_k)\\ &= (\theta_{s-t}^*\alpha)_{\theta_t(p)}\big((\theta_t)_{*,p}V_1,\dots,(\theta_t)_{*,p}V_k\big)\end{aligned} \tag{1.2}$$

We may now use Equation 1.1 and Equation 1.2 to compute

$$\begin{aligned}\beta'(t) &= \lim_{s\to t}\frac{(\theta_s^*\alpha)_p-(\theta_t^*\alpha)_p}{s-t}(V_1,\dots,V_k)\\ &= \lim_{s\to t}\frac{(\theta_{s-t}^*\alpha)_{\theta_t(p)}-\alpha_{\theta_t(p)}}{s-t}\big((\theta_t)_{*,p}V_1,\dots,(\theta_t)_{*,p}V_k\big)\\ &= \lim_{h\to 0}\frac{(\theta_h^*\alpha)_{\theta_t(p)}-\alpha_{\theta_t(p)}}{h}\big((\theta_t)_{*,p}V_1,\dots,(\theta_t)_{*,p}V_k\big)\\ &= (\mathcal{L}_X\alpha)_{\theta_t(p)}\big((\theta_t)_{*,p}V_1,\dots,(\theta_t)_{*,p}V_k\big) = 0\end{aligned}$$

Because t is arbitrary, we have shown $\beta' = 0$ on $\mathbb{R}$. This implies β is a constant, i.e.,

$$(\theta_t^*\alpha)_p(V_1,\dots,V_k) = \alpha_p(V_1,\dots,V_k) \text{ for all } t$$

Because $V_1,\dots,V_k$ are arbitrary, this implies

$$(\theta_t^*\alpha)_p = \alpha_p \text{ for all } t$$

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Because p is arbitrary, this implies

$$\theta_t^*\alpha = \alpha \text{ for all } t$$

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Question 2

Show that the total space T^*M of the cotangent bundle of any manifold M^n is orientable.

Proof. Let $\{\varphi_\alpha\}$ be an atlas for M, and use $\{\lambda^i\}$ to denote the dual basis of $\{\frac{\partial}{\partial x^i}\}$. The smooth structure of T^*M is by definition given by the atlas $\{\Phi_\alpha\}$ defined by

$$\Phi_\alpha(p, \sum_j \xi^j\lambda^j) \triangleq \Big(\varphi_\alpha(p), \xi^1,\dots,\xi^n\Big)$$

We first show

$$\widetilde{\lambda}^i = \sum_j \frac{\partial \widetilde{\mathbf{x}}^i}{\partial \mathbf{x}^j} \lambda^j \text{ for all } i \tag{1.3}$$

For all $\omega \in \Omega^1(M)$, we may write in local coordinate

$$\omega = \sum_j \omega^j \lambda^j = \sum_i \widetilde{\omega}^i \widetilde{\lambda}^i \tag{1.4}$$

Compute

$$\omega^j = \omega\Big(\frac{\partial}{\partial \mathbf{x}^j}\Big|_p\Big) = \omega\Big(\sum_i \frac{\partial \widetilde{\mathbf{x}}^i}{\partial \mathbf{x}^j}(p) \frac{\partial}{\partial \widetilde{\mathbf{x}}^i}\Big|_p\Big) = \sum_i \frac{\partial \widetilde{\mathbf{x}}^i}{\partial \mathbf{x}^j}(p) \widetilde{\omega}^i \tag{1.5}$$

Because λ^j is a basis, for all i, we may write

$$\widetilde{\lambda}^i = \sum_j c_{i,j} \lambda^j$$

It then follows from Equation 1.4 and Equation 1.5 that

$$\sum_{i,j} \widetilde{\omega}^i c_{i,j} \lambda^j = \sum_i \widetilde{\omega}^i \widetilde{\lambda}^i = \sum_j \omega^j \lambda^j = \sum_{i,j} \frac{\partial \widetilde{\mathbf{x}}^i}{\partial \mathbf{x}^j} \widetilde{\omega}^i \lambda^j$$

Because $\{\lambda^j\}$ is linearly independent, we may now deduce for all fixed j

$$\sum_i \widetilde{\omega}^i c_{i,j} = \sum_i \frac{\partial \widetilde{\mathbf{x}}^i}{\partial \mathbf{x}^j} \widetilde{\omega}^i \tag{1.6}$$

Fix i. If we let $\widetilde{\omega}^k = \delta_i^k$, Equation 1.6 becomes

$$c_{i,j} = \frac{\partial \widetilde{\mathbf{x}}^i}{\partial \mathbf{x}^j}$$

Which implies

$$\widetilde{\lambda}^i = \sum_j \frac{\partial \widetilde{\mathbf{x}}^i}{\partial \mathbf{x}^j} \lambda^j \text{ (done)}$$

We may now compute the transition function between $\widetilde{\Phi}, \Phi$ by

$$\begin{aligned}
\widetilde{\Phi} \circ \Phi^{-1}(\mathbf{x}^1, \ldots, \mathbf{x}^n, \xi^1, \ldots, \xi^n) &= \widetilde{\Phi}\Big(\varphi^{-1}(\mathbf{x}^1, \ldots, \mathbf{x}^n), \sum_i \xi^i \lambda^i\Big) \\
&= \widetilde{\Phi}\Big(\varphi^{-1}(\mathbf{x}^1, \ldots, \mathbf{x}^n), \sum_i \xi^i \sum_j \frac{\partial \widetilde{\mathbf{x}}^j}{\partial \mathbf{x}^i} \widetilde{\lambda}^j\Big) \\
&= \widetilde{\Phi}\Big(\varphi^{-1}(\mathbf{x}^1, \ldots, \mathbf{x}^n), \sum_j \sum_i \frac{\partial \widetilde{\mathbf{x}}^j}{\partial \mathbf{x}^i} \xi^i \widetilde{\lambda}^j\Big) \\
&= \Big(\widetilde{\mathbf{x}}^1, \ldots, \widetilde{\mathbf{x}}^n, \sum_i \frac{\partial \widetilde{\mathbf{x}}^1}{\partial \mathbf{x}^i}(\mathbf{x})\xi^i, \ldots, \sum_i \frac{\partial \widetilde{\mathbf{x}}^n}{\partial \mathbf{x}^i}(\mathbf{x})\xi^i\Big)
\end{aligned}$$

And compute the derivative of $\widetilde{\Phi} \circ \Phi^{-1}$

$$[d(\widetilde{\Phi} \circ \Phi^{-1})] = \begin{bmatrix}
\frac{\partial \widetilde{\mathbf{x}}^1}{\partial \mathbf{x}^1} & \cdots & \frac{\partial \widetilde{\mathbf{x}}^1}{\partial \mathbf{x}^n} & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial \widetilde{\mathbf{x}}^n}{\partial \mathbf{x}^1} & \cdots & \frac{\partial \widetilde{\mathbf{x}}^n}{\partial \mathbf{x}^n} & 0 & \cdots & 0 \\
A_{1,1} & \cdots & A_{1,n} & \frac{\partial \widetilde{\mathbf{x}}^1}{\partial \mathbf{x}^1} & \cdots & \frac{\partial \widetilde{\mathbf{x}}^1}{\partial \mathbf{x}^n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
A_{n,1} & \cdots & A_{n,n} & \frac{\partial \widetilde{\mathbf{x}}^n}{\partial \mathbf{x}^1} & \cdots & \frac{\partial \widetilde{\mathbf{x}}^n}{\partial \mathbf{x}^n}
\end{bmatrix} \tag{1.7}$$

Where

$$A_{i,j} = \sum_k \frac{\partial \widetilde{\mathbf{x}}^i}{\partial \mathbf{x}^k \partial \mathbf{x}^j} \xi^k$$

Using Equation 1.7, we may conclude

$$\det d(\widetilde{\Phi} \circ \Phi^{-1}) = [\det d(\widetilde{\varphi} \circ \varphi^{-1})]^2 > 0$$

Note that the last inequality hold true because the fact $\widetilde{\varphi} \circ \varphi^{-1}$ is a diffeomorphism between open subsets of $\mathbb{R}^n$ implies $d(\widetilde{\varphi} \circ \varphi^{-1})$ is invertible, which implies $\det d(\widetilde{\varphi} \circ \varphi^{-1})$ is non-zero. We have shown $\{\Phi_\alpha\}$ is an orientable atlas, which implies T^*M is orientable. ∎

Note that one can give a quick proof for Equation 1.3 by changing the notation

$$\widetilde{\lambda}^i = d\widetilde{\mathbf{x}}^i = \sum_j \frac{\partial \widetilde{\mathbf{x}}^i}{\partial \mathbf{x}^j} d\mathbf{x}^j = \sum_j \frac{\partial \widetilde{\mathbf{x}}^i}{\partial \mathbf{x}^j} \lambda^j \tag{1.8}$$

I submitted Equation 1.8 as one of my supplementary argument. Its core lies in the standard extension of functions $\mathbf{x}^i$ from a single chart to the whole manifold using bump function.

Question 3

Show that the following are special cases of Stoke's Theorem for manifold with boundary.

(a) Let C be the image of a smooth embedding $\mathbf{r} : S^1 \to \mathbb{R}^2$ and let D be the region in $\mathbb{R}^2$ bounded by C. If $P, Q : \mathbb{R}^2 \to \mathbb{R}$ are smooth functions.

$$\int_C P dx + Q dy = \int_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \Big) dx dy$$

(b) Let S be a compact oriented surface with smooth boundary C. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be smooth.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$$

(c) Let E be the compact closure of some open subset of $\mathbb{R}^3$ with smooth boundary S. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be smooth.

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_E \nabla \cdot \mathbf{F} dx dy dz$$

Proof. Because $P, Q : \mathbb{R}^2 \to \mathbb{R}$ are smooth, we know

$$\omega \triangleq P dx + Q dy \text{ is a smooth 1-form on } \mathbb{R}^2$$

Note that all interpretations (Riemann, Riemann-Stieltjes, Lebesgue or Lebesgue-Stieltjes integral) equal to

$$\int_C P dx + Q dy \triangleq \int_{\partial D} \omega \text{ and } \int_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \Big) dx dy \triangleq \int_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \Big) dx \wedge dy$$

Compute

$$\begin{aligned} d\omega &= \Big(\frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy \Big) \wedge dx + \Big(\frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial y} dy \Big) \wedge dy \\ &= \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \Big) dx \wedge dy \end{aligned}$$

Green's Theorem now follows from Stoke's Theorem,

$$\int_C P dx + Q dy = \int_{\partial D} \omega = \int_D d\omega = \int_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \Big) dx \wedge dy = \int_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \Big) dx dy$$

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Proof. Note that the correct interpretation of surface integral is

$$\int_S \mathbf{G} \cdot d\mathbf{S} \triangleq \int_S \mathbf{G}^1 dy \wedge dz + \mathbf{G}^2 dz \wedge dx + \mathbf{G}^3 dx \wedge dy \tag{1.9}$$

And the correct interpretation of line integral is

$$\int_C \mathbf{G} \cdot d\mathbf{r} \triangleq \int_C \mathbf{G}^1 dx + \mathbf{G}^2 dy + \mathbf{G}^3 dz \tag{1.10}$$

Because $\mathbf{F}$ is smooth, we know

$$\omega \triangleq \mathbf{F}^1 dx + \mathbf{F}^2 dy + \mathbf{F}^3 dz \text{ is a smooth 1-form on } \mathbb{R}^3$$

Compute

$$\begin{aligned}
d\omega &= \Big(\frac{\partial \mathbf{F}^1}{\partial x}dx + \frac{\partial \mathbf{F}^1}{\partial y}dy + \frac{\partial \mathbf{F}^1}{\partial z}dz\Big) \wedge dx + \Big(\frac{\partial \mathbf{F}^2}{\partial x}dx + \frac{\partial \mathbf{F}^2}{\partial y}dy + \frac{\partial \mathbf{F}^2}{\partial z}dz\Big) \wedge dy \\
&\quad + \Big(\frac{\partial \mathbf{F}^3}{\partial x}dx + \frac{\partial \mathbf{F}^3}{\partial y}dy + \frac{\partial \mathbf{F}^3}{\partial z}dz\Big) \wedge dz \\
&= \Big(\frac{\partial \mathbf{F}^3}{\partial y} - \frac{\partial \mathbf{F}^2}{\partial z}\Big)dy \wedge dz + \Big(\frac{\partial \mathbf{F}^1}{\partial z} - \frac{\partial \mathbf{F}^3}{\partial x}\Big)dz \wedge dx + \Big(\frac{\partial \mathbf{F}^2}{\partial x} - \frac{\partial \mathbf{F}^1}{\partial y}\Big)dx \wedge dy \\
&= (\nabla \times \mathbf{F})^1 dy \wedge dz + (\nabla \times \mathbf{F})^2 dz \wedge dx + (\nabla \times \mathbf{F})^3 dx \wedge dy
\end{aligned}$$

Because $C = \partial S$, by Equation 1.9, Equation 1.10 and Stoke's Theorem, we now have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \omega = \int_S d\omega = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

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Proof. Note that all interpretations (Riemann, Riemann-Stieltjes, Lebesgue or Lebesgue-Stieltjes integral) equal to

$$\int_E f dx dy dz = \int_E f dx \wedge dy \wedge dz \tag{1.11}$$

Identifying $\mathbf{F} \in \mathfrak{X}(\mathbb{R}^3)$ as the vector field

$$\mathbf{F} \simeq \mathbf{F}^1 \frac{\partial}{\partial x} + \mathbf{F}^2 \frac{\partial}{\partial y} + \mathbf{F}^3 \frac{\partial}{\partial z}$$

We may compute

$$\begin{aligned}
\iota_{\mathbf{F}}(dx \wedge dy \wedge dz) &= \mathbf{F}^1 dy \wedge dz - \mathbf{F}^2 dx \wedge dz + \mathbf{F}^3 dx \wedge dy \\
&= \mathbf{F}^1 dy \wedge dz + \mathbf{F}^2 dz \wedge dx + \mathbf{F}^3 dx \wedge dy
\end{aligned}$$

And compute

$$
\begin{aligned}
d\iota_{\mathbf{F}}(dx \wedge dy \wedge dz) &= d(\mathbf{F}^1 dy \wedge dz - \mathbf{F}^2 dx \wedge dz + \mathbf{F}^3 dx \wedge dy) \\
&= \Big(\frac{\partial \mathbf{F}^1}{\partial x} + \frac{\partial \mathbf{F}^2}{\partial y} + \frac{\partial \mathbf{F}^3}{\partial z}\Big) dx \wedge dy \wedge dz \\
&= \nabla \cdot \mathbf{F} dx \wedge dy \wedge dz
\end{aligned}
$$

Because $S = \partial E$, by Equation 1.9, Equation 1.11 and Stoke's Theorem, we now have

$$
\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \iota_{\mathbf{F}}(dx \wedge dy \wedge dz) = \int_E d\iota_{\mathbf{F}}(dx \wedge dy \wedge dz) = \int_E \nabla \cdot \mathbf{F} dxdydz
$$

∎

Note that our interpretation of line integral in Equation 1.10 is correct because for all parametrization $\gamma : [a, b] \to C$, we have

$$
\begin{aligned}
\int_a^b \mathbf{G}(\gamma(t)) \cdot \gamma'(t) dt &= \int_{\gamma^{-1}(C)} \gamma^*(\mathbf{G}^1 dx + \mathbf{G}^2 dy + \mathbf{G}^3 dz) \\
&= \int_C \mathbf{G}^1 dx + \mathbf{G}^2 dy + \mathbf{G}^3 dz
\end{aligned}
$$

Also note that our interpretation of surface integral in Equation 1.9 is correct because for all parametrization $\mathbf{u} : D \subseteq \mathbb{R}^2 \to S$, we have

$$
\begin{aligned}
\int_D \mathbf{G}(\mathbf{u}(s,t)) \cdot \Big(\frac{\partial \mathbf{u}}{\partial s} \times \frac{\partial \mathbf{u}}{\partial t}\Big) d(s,t) &= \int_{\mathbf{u}^{-1}(S)} \mathbf{u}^*(\mathbf{G}^1 dy \wedge dz + \mathbf{G}^2 dz \wedge dx + \mathbf{G}^3 dx \wedge dy) \\
&= \int_S \mathbf{G}^1 dy \wedge dz + \mathbf{G}^2 dz \wedge dx + \mathbf{G}^3 dx \wedge dy
\end{aligned}
$$

Question 4

Consider a smooth map $F : S^3 \to S^2$. Let $\alpha \in \Omega^2(S^2)$ be a form representing a non-trivial De Rham cohomology class $a \in H^2(S^2)$. Show that there exists a 1-form θ on S^3 such that $F^*\alpha = d\theta$. Moreover show that the De Rham cohomology class in $H^3(S^3)$ of the 3-form $\theta \wedge F^*\alpha$ is independent of the choice of θ and of α representing a.

Proof. Because α is a 2-form and S^2 has dimension 2, we may compute

$$
d(F^*\alpha) = F^*(d\alpha) = F^*(0) = 0
$$

Therefore, $F^*\alpha$ is a closed 2-form on S^3. It then follows from the fact $H^2(S^3) \cong 0$ that $F^*\alpha$ is exact. That is, there exists some $\theta \in \Omega^1(S^3)$ such that

$$F^*\alpha = d\theta$$

To see that the cohomology class of the 3-form $\theta \wedge F^*\alpha = \theta \wedge d\theta$ is independent of the choice of θ, let $\theta' \in \Omega^1(S^3)$ also satisfy

$$d\theta' = F^*\alpha = d\theta$$

Because

$$d(\theta' - \theta) = 0 \text{ and } H^1(S^3) \cong 0$$

We know $\theta' - \theta = d\delta$ for some $\delta \in \Omega^0(S^3)$. Therefore, we may compute

$$\begin{aligned} \theta' \wedge d\theta' - \theta \wedge d\theta &= (\theta' - \theta) \wedge d\theta \\ &= d\delta \wedge d\theta = d(\delta \wedge d\theta) \end{aligned}$$

Showing that $[\theta' \wedge d\theta'] = [\theta \wedge d\theta]$. ■

The following is a failed attempt to prove the independence of choice of α.

Proof. Fix arbitrary $\beta \in a$, and let $\gamma \in \Omega^1(S^2), \theta' \in \Omega^1(S^3)$ satisfy

$$\beta = \alpha + d\gamma \text{ and } F^*\beta = d\theta'$$

So that

$$d(\theta' - \theta - F^*\gamma) = 0$$

Because $H^1(S^3) \cong 0$, we now have

$$\theta' - \theta - F^*\gamma = d\varphi \text{ for some } \varphi \in \Omega^0(S^3)$$

Compute

$$\theta' = \theta + F^*\gamma + d\varphi \text{ and } d\theta' = d\theta + dF^*\gamma$$

Compute

$$\begin{aligned} \theta' \wedge d\theta' &= (\theta + F^*\gamma + d\varphi) \wedge (d\theta + dF^*\gamma) \\ &= \theta \wedge d\theta + \theta \wedge dF^*\gamma + F^*\gamma \wedge d\theta + F^*\gamma \wedge dF^*\gamma + d\varphi \wedge d\theta + d\varphi \wedge dF^*\gamma \\ &= \theta \wedge d\theta + d(\varphi \wedge d\theta + dF^*\gamma) + \theta \wedge dF^*\gamma + F^*\gamma \wedge d\theta + F^*\gamma \wedge dF^*\gamma \end{aligned}$$

This implies

$$[\theta' \wedge d\theta' - \theta \wedge d\theta] = [\theta \wedge dF^*\gamma + F^*\gamma \wedge d\theta + F^*\gamma \wedge dF^*\gamma]$$

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Question 5

Suppose M, N, P are compact connected orientable manifolds without boundary of the same dimension n and $F : M \to N$ and $G : N \to P$ are smooth maps. Shows that

$$\deg(G \circ F) = \deg G \cdot \deg F$$

Then show that the antipodal map $\varphi(p) \triangleq -p$ on the unit sphere in $\mathbb{R}^m$ has degree $(-1)^m$.

Proof. Because P is compact connected orientable and without boundary, there exists a top degree smooth form ω on P such that

$$\int_P \omega = 1$$

Compute

$$\deg(G \circ F) = \int_M (G \circ F)^* \omega = \int_M F^*(G^* \omega) = \deg F \int_N G^* \omega = (\deg F)(\deg G)$$

For each $1 \leq i \leq m$, define $\varphi_i : S^{m-1} \to S^{m-1}$ by

$$\varphi_i(\mathbf{x}^1, \dots, \mathbf{x}^m) \triangleq (\mathbf{x}^1, \dots, -\mathbf{x}^i, \dots, \mathbf{x}^m)$$

So that the antipodal map φ can be expressed as the product

$$\varphi = \varphi_1 \circ \dots \circ \varphi_m \tag{1.12}$$

Fix i. Define

$$U_i \triangleq \{\mathbf{x} \in S^{m-1} : \mathbf{x}^j > 0\} \text{ for some } j \neq i$$

And define $\psi_i : U_i \to \mathbb{R}^{m-1}$ by

$$\psi_i(\mathbf{x}) \triangleq (\mathbf{x}^1, \dots, \mathbf{x}^{j-1}, \mathbf{x}^{j+1}, \dots, \mathbf{x}^{m-1})$$

Compute

$$\psi_i \circ \varphi_i \circ \psi_i^{-1}(\mathbf{x}^1, \dots, \mathbf{x}^{j-1}, \mathbf{x}^{j+1}, \dots, \mathbf{x}^{m-1}) = (\mathbf{x}^1, \dots, -\mathbf{x}^i, \dots, \mathbf{x}^{j-1}, \mathbf{x}^{j+1}, \dots, \mathbf{x}^{m-1})$$

This implies the derivative matrix of $d(\psi_i \circ \varphi_i \circ \psi_i^{-1})_{\mathbf{x}}$ has only non-zero entry in diagonal line, and all the diagonal entries, except the i-th one being -1, are 1. It then follows that

$\det d(\psi_i \circ \varphi_i \circ \psi_i^{-1})_{\mathbf{x}} = -1$. We have shown that all points in U_i are regular points of φ_i. Fix $\mathbf{x} \in U_i$. Because φ_i is bijective, we may now conclude

$$\deg \varphi_i = \sum_{\mathbf{y} \in \varphi^{-1}(\varphi(\mathbf{x}))} \operatorname{sgn}(\det(\psi_i \circ \varphi_i \circ \psi_i^{-1})_{\mathbf{y}}) = -1$$

It then follows from Equation 1.12 that

$$\deg \varphi = \prod_{i=1}^{m} \det(\varphi_i) = (-1)^m$$

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$\blacksquare$