NCKU 112.2
Geometry 1

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Contents

Chapter 1

Curve

1.1 Frenet Trihedron

In this section, we will use I to denote an **bounded open interval**. By a **curve** in $\mathbb{R}^n$, we mean a function form an open interval I to $\mathbb{R}^n$. We say a curve is **differentiable** if it is infinitely differentiable (smooth). That is, $\gamma^{(n)}(t)$ exists and are continuous for all $n \in \mathbb{N}$ and $t \in I$.

We say a differentiable curve $\gamma : I \to \mathbb{R}^n$ is **regular** if $\gamma'(t) \neq 0$ for all $t \in I$. We say a differentiable curve $\gamma : I \to \mathbb{R}^n$ is a **parametrized by arc-length** if $|\gamma'(t)| = 1$ for all $t \in I$.

For a regular curve γ, we say $\gamma'(t)$ is the **tangent vector** of γ at t, and we define the **unit tangent vector** T by

$$T(t) \triangleq \frac{\gamma'(t)}{|\gamma'(t)|}$$

We say $\gamma''(t)$ is the **oriented curvature** (normal vector) of γ at t, and we define the **unit normal vector** N by

$$N(t) \triangleq \frac{T'(t)}{|T'(t)|} = \frac{\gamma''(t)}{|\gamma''(t)|}$$

If the curve is parametrized by arc-length, with some messy computation, we can observe

$$T'(t) = \frac{\gamma''(t)}{|\gamma'(t)|}$$

Some interesting facts can be observed from what we have deduced.

(a) γ', γ'' always exists.

(b) γ is parametrized by arc-length $\implies \gamma' \perp \gamma''$

(c) γ is parametrized by arc-length $\implies \gamma$ is regular

(d) T and T' exists at $t \iff \gamma$ is regular at t

(e) $T = \gamma' \iff \gamma$ is parametrized by arc-length

(f) N exists at $t \iff \gamma''(t) \neq 0 \iff \kappa(t) \neq 0$

(g) N and T' point to the same direction γ''.

(h) $|T'| = \kappa \iff \gamma$ is paramterized by arc-length

(i) $\gamma \perp \gamma'$ and $\gamma'' \perp \gamma'''$ are generally false even for curve γ paramterized by arc-length.

(j) Given a curve γ parametrized by arc-length

$$\begin{aligned}
\gamma \text{ is a straight line on } [a,b] &\iff \gamma' \text{ and } T \text{ are constant on } (a,b) \\
&\iff \gamma''(t) = 0 \text{ on } (a,b) \\
&\iff \kappa(t) = 0 \text{ on } (a,b) \\
&\iff T'(t) = 0 \text{ on } (a,b)
\end{aligned}$$

Notice that the last fact is false if γ is not paramtetrized by arc-length, since γ can move in the straight line with changing speed γ'.

Given a curve γ, if $T(t)$ and $N(t)$ exists (regular and non-zero curvature), we define its **binormal** vector by

$$B(t) = T(t) \times N(t)$$

Fix t. We say

$$\{T(t), N(t), B(t)\} \text{ form a } \textbf{positively oriented orthonormal basis} \text{ of } \mathbb{R}^3$$

This basis in general is constantly changing, yet always form an orthonormal basis.

Also, we say

$$\text{span}\Big(T(t), N(t)\Big) \text{ is the } \textbf{osculating plane} \text{ of } \gamma \text{ at } t$$

Suppose γ is parametrized by arc-length and always has non-zero curvature. With some geometric intuition, one shall note that $|T'|$ measure how curved γ is and that $|B'|$ measure how fast γ leave the osculating plane.

Because $|B| = 1$ is a constant, we can deduce

$$B' \perp B$$

and the computation

$$B' = T' \times N + T \times N' = T \times N'$$

give us

$$B' \perp T$$

This ultimately show us

$$B', N, T' \text{ are all parallel where } N, T' \text{ even point to the same direction}$$

Notice that if we parametrize the curve with opposite direction, then

(a) T, γ' change direction

(b) N, γ'' keep the same direction

(c) B change direction

(d) B' keep the same direction

Now, for a curve γ parametrized by arc-length, we define the **curvature** κ and **torsion** τ of γ by

$$\kappa(t) = |\gamma''(t)| \text{ and } \tau(t) = \frac{B'(t)}{N(t)}$$

With unfortunately heavy computation, we can verify that the definition of curvature must stay in the framework of curve parametrized by arc-length, otherwise we will be given two different values of curvature of two curves that are equivalent in the sense of sets.

Now, notice that we already have $T' = \kappa N$ and $B' = \tau N$, and by basic identity, we have $N = B \times T$.

Then with some computation, we have the **Frenet Formula**

$$\begin{cases} T' = \kappa N \\ N' = B' \times T + B \times T' = -\tau B - \kappa T \\ B' = \tau N \end{cases}$$

Given two vectors $u, v \in \mathbb{R}^n$, we use **dot product**

$$u \cdot v = u_1 v_1 + \cdots + u_n v_n$$

to denote the Euclidean inner product, and we use **length**

$$|u| = \sqrt{\sum_{k=1}^{n} u_k^2}$$

to denote the Euclidean norm. Note that we clearly have

$$|u| = \sqrt{u \cdot u}$$

Given three vectors $u, v, w \in \mathbb{R}^3$, we define **cross product** by

$$\begin{aligned} u \times v &\triangleq \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1) \end{aligned}$$

With some simple computation, we have the following identity

(a) $u \times v = -v \times u$ (anti-commutative)

(b) $(au + w) \times v = a(u \times v) + w \times v$ (Linearity)

(c) $u \times (aw + v) = a(u \times w) + u \times v$

(d) $u \times v = 0 \iff u = cv$ for some $c \in \mathbb{R}$

(e) $(u \times v) \cdot w = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

(f) $(u \times v) \cdot v = 0 = (u \times v) \cdot u$

(g) $u \times v \perp u$ and $u \times v \perp v$

(h) $u \perp v \implies |u \times v| = |u| \cdot |v|$

(i) $(u \times v) \times w = (u \cdot w)v - (v \cdot w)u$

All proofs except that of the last identity are merely manipulation of determinant. A simple proof of the last identity follows from the fact both side are linear in all u, v, w, and the observation

$$(e_1 \times e_2) \times e_3 = 0 = (e_1 \cdot e_3)e_2 - (e_2 \cdot e_3)e_1$$

Theorem 1.1.1. (Differentiate the Dot Product) Given two parametrized curves $u, v : (a, b) \to \mathbb{R}^n$, such that u, v are differentiable at $t \in (a, b)$. We have

$$\frac{d}{dt}\big(u(t) \cdot v(t)\big) = u'(t) \cdot v(t) + u(t) \cdot v'(t)$$

Proof.

$$\begin{aligned}
\frac{d}{dt}\big(u(t) \cdot v(t)\big) &= \frac{d}{dt} \sum_{k=1}^{n} u_k(t)v_k(t) \\
&= \sum_{k=1}^{n} \frac{d}{dt} u_k(t)v_k(t) \\
&= \sum_{k=1}^{n} u_k'(t)v_k(t) + u_k(t)v_k'(t) \\
&= \sum_{k=1}^{n} u_k'(t)v_k(t) + \sum_{k=1}^{n} u_k(t)v_k'(t) \\
&= u'(t) \cdot v(t) + u(t) \cdot v'(t)
\end{aligned}$$

■

Theorem 1.1.2. (Differentiate the Cross Product) Given two curves $u, v : (a, b) \to \mathbb{R}^3$, such that u, v are differentiable at $t \in (a, b)$. We have

$$\frac{d}{dt}\big(u(t) \times v(t)\big) = u'(t) \times v(t) + u(t) \times v'(t)$$

Proof.

$$\begin{aligned}\frac{d}{dt}\big(u(t)\times v(t)\big) =&\frac{d}{dt}(u_2v_3-u_3v_2,u_3v_1-u_1v_3,u_1v_2-u_2v_1)\\ =&(u_2'v_3+u_2v_3'-u_3'v_2-u_3v_2',\\ &u_3'v_1+u_3v_1'-u_1'v_3-u_1v_3',\\ &u_1'v_2+u_1v_2'-u_2'v_1-u_1v_2')\\ =&(u_2'v_3-u_3'v_2,u_3'v_1-u_1'v_3,u_1'v_2-u_2'v_1)\\ +&(u_2v_3'-u_3v_2',u_3v_1'-u_1v_3',u_1v_2'-u_1v_2')\\ =&u'\times v+u\times v'\end{aligned}$$

■

Theorem 1.1.3. (Integrating the Dot Product) Given a curve $u:[a,b]\to\mathbb{R}^n$ and a vector $v\in\mathbb{R}^n$, suppose

(a) u is differentiable on (a,b)

(b) u is continuous on $[a,b]$

We have

$$\int_a^b u'(t)\cdot v dt=\Big(\int_a^b u'(t)dt\Big)\cdot v=\big(u(b)-u(a)\big)\cdot v$$

Proof.

$$\begin{aligned}\int_a^b u'(t)\cdot v dt&=\int_a^b\sum_{k=1}^n u_k'(t)\cdot v_k dt\\ &=\sum_{k=1}^n\int_a^b u_k'(t)\cdot v_k dt\\ &=\sum_{k=1}^n v_k\int_a^b u_k'(t)dt\\ &=v\cdot\Big(\int_a^b u'(t)dt\Big)\end{aligned}$$

■

For the result above, sometimes we write

$$(u\cdot v)'=u'\cdot v+u\cdot v' \text{ and } (u\times v)'=u'\times v+u\times v'$$

Theorem 1.1.4. (MVT for curve) Given a curve $\alpha : [a,b] \to \mathbb{R}^n$ such that

(a) α is differentaible on (a,b)

(b) α is continuous on $[a,b]$

there exists $\xi \in (a,b)$ such that

$$|\alpha(b) - \alpha(a)| \leq |\alpha'(\xi)| (b-a)$$

Proof. Define $\phi : [a,b] \to \mathbb{R}$ by

$$\phi(t) = \alpha(t) \cdot \big(\alpha(b) - \alpha(a)\big)$$

Clearly ϕ satisfy the hypothesis of Lagrange's MVT, then we know there exists $\xi \in (a,b)$ such that

$$\phi(b) - \phi(a) = \phi'(\xi) \cdot (b-a)$$

Written the equation in α, we have

$$|\alpha(b) - \alpha(a)|^2 = (b-a)\alpha'(\xi) \cdot \big(\alpha(b) - \alpha(a)\big)$$

Notice that Cauchy-Schwarz inequality give us

$$\begin{aligned} (b-a) |\alpha'(\xi)| \cdot |\alpha(b) - \alpha(a)| &\geq (b-a) \left|\alpha'(\xi) \cdot \big(\alpha(b) - \alpha(a)\big)\right| \\ &= |\alpha(b) - \alpha(a)|^2 \end{aligned}$$

This then implies

$$(b-a) |\alpha'(\xi)| \geq |\alpha(b) - \alpha(a)|$$

■

Corollary 1.1.5. (Mean Value Inequality) Given a curve $\alpha : [a,b] \to \mathbb{R}^n$ such that

(a) α is differentiable on (a,b)

(b) α is continuous on $[a,b]$

we have

$$|\alpha(b) - \alpha(a)| \leq (b-a) \sup_{(a,b)} |\alpha'|$$

Trick to parametrize by arc-length.

Given a regular curve $\gamma : I \to \mathbb{R}^n$ and fix $t_0 \in I$. We use

$$s(t) = \int_{t_0}^{t} |\gamma'(x)| \, dx$$

to define the arc-length of γ from $\gamma(t_0)$ to $\gamma(t)$. Because γ is regular, by FTC, it is clear that s is one-to-one.

Let $t(s)$ be the inverse of s. Define

$$\beta(s) \triangleq \alpha(t(s))$$

We have by Chain rule

$$\begin{aligned}\beta'(s) &= t'(s)\alpha'(t(s)) \\ &= \frac{\alpha'(t(s))}{s'(t)} \\ &= \frac{\alpha'(t(s))}{|\alpha'(t(s))|}\end{aligned}$$

(Frenet Formula Summary)

By definition, we are given

$$\begin{cases} T' = \kappa N \\ B' = \tau N \end{cases}$$

To compute N', an identity should be first given

$$N = B \times T$$

We can now complete the Frenet Formula

$$\begin{aligned}N' &= B' \times T + B \times T' \\ &= \tau N \times T + B \times \kappa N \\ &= -\tau B - \kappa T\end{aligned}$$

In conclusion

$$\begin{cases} T' = \kappa N B' = \tau N \\ N' = -\tau B - \kappa T \end{cases}$$

Give very close attention to the fact the two definitions of curvature

$$\kappa = \frac{T'}{N} \text{ and } \kappa = |\gamma''|$$

coincides only when γ is parametrzied by arc-length. The first definition remain same for all parametrizaiton of the same curve, while the latter doesn't.

Some comment should be dropped for the computation of torsion. If you overlook the fact α is parametrized by arc-length and disregard Frenet Formula, it is very likely you will get a result that you can not even sure if it is valid (the nominator and denominator may end up not seem explicitly parallel), let alone an identity beautiful as below.

Theorem 1.1.6. (Identity of Torsion) Given a parametrized by arc-length cruve $\alpha : I \to \mathbb{R}^3$, we have

$$\tau(s) = -\frac{\big(\alpha'(s) \times \alpha''(s)\big) \cdot \alpha'''(s)}{\kappa^2(s)}$$

Proof. Because α is parametrized by arc-length, we have

$$\alpha'(s) = T(s)$$

We first show

$$\alpha''(s) = \kappa(s) N(s) \tag{1.1}$$

Compute

$$\begin{aligned} N(s) &= \frac{T'(s)}{|T'(s)|} \\ &= \frac{\alpha''(s)}{|\alpha''(s)|} = \frac{\alpha''(s)}{\kappa(s)} \text{ (done)} \end{aligned}$$

We now show

$$\alpha'''(s) = \kappa(s)\big(- (\tau B)(s) - (\kappa T)(s)\big) + \kappa'(s) N(s)$$

By Equation 1.1 and Frenet Formula, we have

$$\begin{aligned} \alpha'''(s) &= \kappa'(s) N(s) + \kappa(s) N'(s) \\ &= \kappa'(s) N(s) + \kappa(s)\big(- (\tau B)(s) - (\kappa T)(s)\big) \text{ (done)} \end{aligned}$$

Lastly, we verify

$$
\begin{aligned}
-\frac{\big(\alpha'(s) \times \alpha''(s)\big) \cdot \alpha'''(s)}{\kappa^2(s)} &= -\frac{(T \times \kappa N) \cdot \Big(\kappa\big(-\tau B - \kappa T\big) + \kappa' N\Big)}{\kappa^2} \\
&= -\frac{-\kappa^2 \tau (T \times N) \cdot B}{\kappa^2} \qquad (\because T \times N \cdot (T \text{ or } N) = 0) \\
&= \tau
\end{aligned}
$$

■

1.2 HW1

Question 1: 1-2: 2

Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is the point of the trace of α closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

Proof. Define $g : I \to \mathbb{R}$ by

$$g(t) \triangleq |\alpha(t)|^2 = (\alpha \cdot \alpha)(t)$$

Notice that

$$g'(t) = (2\alpha' \cdot \alpha)(t) \text{ if exists}$$

From premise, we know g attains minimum at t_0. This tell us

$$0 = g'(t_0) = (2\alpha' \cdot \alpha)(t_0)$$

Then, we can deduce

$$\alpha'(t_0) \cdot \alpha(t_0) = 0$$

This implies $\alpha(t_0) \perp \alpha'(t_0)$. ■

Question 2: 1-2: 5

Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve, with $\alpha''(t) \neq 0$ for all $t \in I$. Show that $|\alpha(t)|$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

Proof. We wish to prove

$$\exists \beta \in \mathbb{R}^*, \forall t \in I, |\alpha(t)| = \beta \iff \forall t \in I, (\alpha \cdot \alpha')(t) = 0$$

Define $g : I \to \mathbb{R}$ by

$$g(t) \triangleq |\alpha(t)|^2 = (\alpha \cdot \alpha)(t)$$

Notice that

$$g'(t) = (2\alpha' \cdot \alpha)(t) \tag{1.2}$$

$(\longrightarrow)$

From premise, g is a constant on I. This implies $g'(t) = 0$ for all $t \in I$. Then, from Equation 1.2, we see

$$(\alpha \cdot \alpha')(t) = 0 \text{ for all } t \in I$$

$(\longleftarrow)$

Again, from Equation 1.2, we deduce

$$\forall t \in I, (\alpha \cdot \alpha')(t) = 0 \implies \forall t \in I, g'(t) = 0$$

This implies g is a constant, which implies $|\alpha|$ is a constant, that is

$$\exists \beta \in \mathbb{R}, |\alpha(t)| = \beta$$

Lastly, we have to show

$$\beta \neq 0$$

Assume $\beta = 0$. Then, we see $\alpha(t) = 0$ for all $t \in I$. This implies $\alpha''(t) = 0$ for all $t \in I$, which CaC to the premise. (done) ∎

Question 3: 1-3:2

2. A circular disk of radius 1 in the plane xy rolls without slipping along the x axis. The figure described by a point of the circumference of the disk is called a *cycloid* (Fig. 1-7).

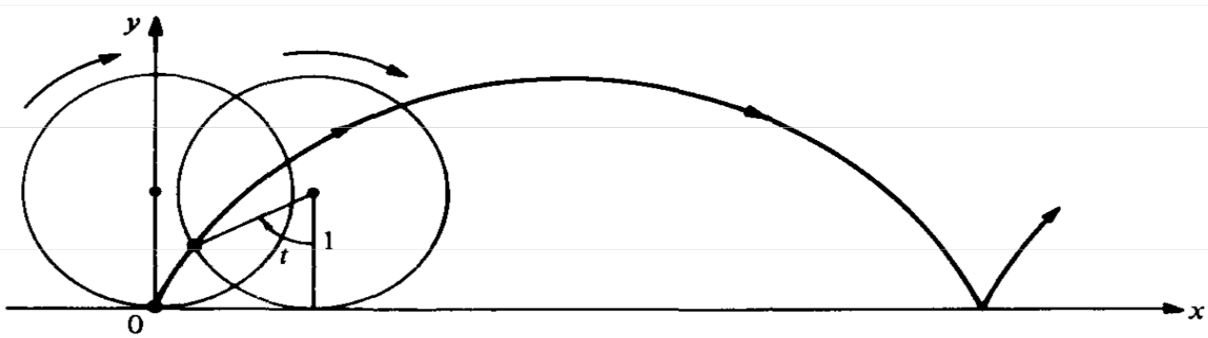


Figure 1-7. The cycloid.

***a.** Obtain a parametrized curve $\alpha: R \longrightarrow R^2$ the trace of which is the cycloid, and determine its singular points.

b. Compute the arc length of the cycloid corresponding to a complete rotation of the disk.

Proof. The solution of the question **a** is

$$\alpha(t) = (t - \sin t, 1 - \cos t)$$

Compute

$$\alpha'(t) = (1 - \cos t, \sin t)$$

and compute

$$|\alpha'(t)| = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} = \sqrt{2} \cdot \sqrt{1 - \cos t}$$

This implies the singular points are

$$\{2n\pi : n \in \mathbb{Z}\}$$

The solution of the question **b** is then

$$\begin{aligned}
\int_0^{2\pi} |\alpha'(t)|\, dt &= \sqrt{2} \int_0^{2\pi} \sqrt{1 - \cos t} dt \\
&= \sqrt{2} \int_0^{2\pi} \sqrt{2} \left| \sin \frac{t}{2} \right| dt \\
&= 2 \int_0^{2\pi} \left| \sin \frac{t}{2} \right| dt \\
&= 4 \int_0^{\pi} \sin(\frac{t}{2}) dt \\
&= -8 \cos \frac{t}{2} \Big|_0^{\pi}
\end{aligned}$$

■

Question 4: 1-3:4

4. Let $\alpha: (0, \pi) \longrightarrow R^2$ be given by

$$\alpha(t) = \left(\cos t, \cos t + \log \tan \frac{t}{2} \right),$$

where t is the angle that the y axis makes with the vector $\alpha(t)$. The trace of α is called the *tractrix* (Fig. 1-9). Show that

19:

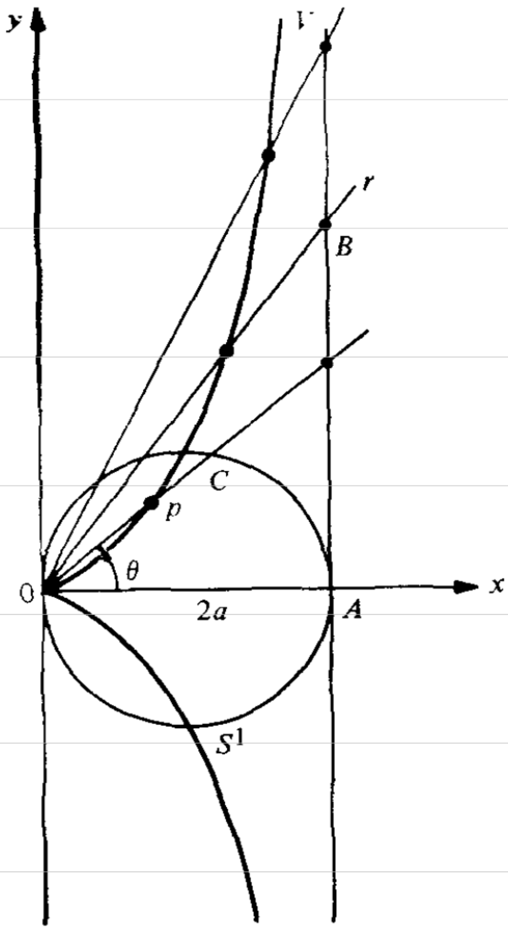


Figure 1-8. The cissoid of Diocles.

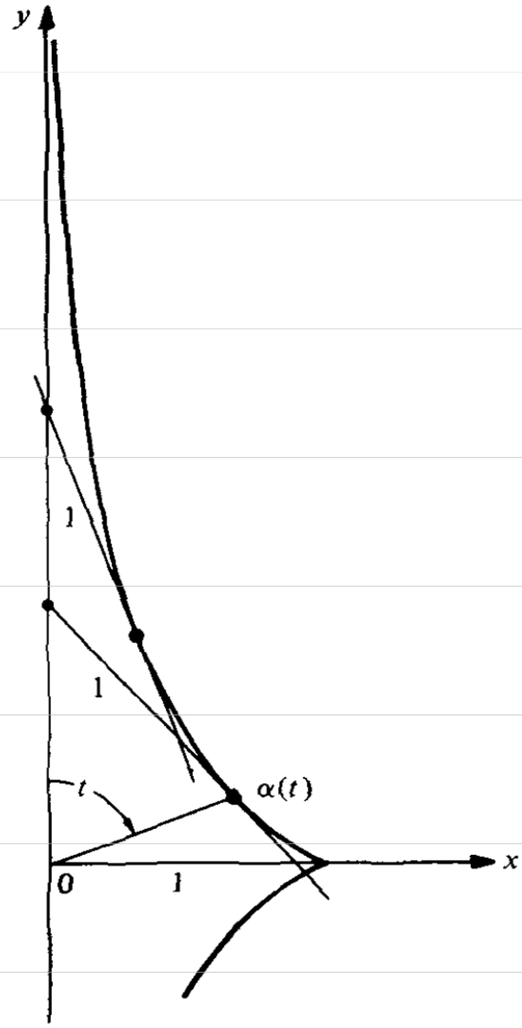


Figure 1-9. The tractrix.

a. α is a differentiable parametrized curve, regular except at $t = \pi/2$.

b. The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.

Typo correction: $\alpha(t) = (\sin t, \cos t + \ln \tan \frac{t}{2})$

Proof. **(a)**
Notice that the interval I is $(0, \pi)$. It is clear that

(a) $\sin t$ is smooth on $\mathbb{R}$

(b) $\cos t$ is smooth on $\mathbb{R}$

(c) $\ln t$ is smooth on $\mathbb{R}$ $\tan \frac{t}{2}$ is smooth on I

Then it follows that α is a differentiable curve.

Compute

$$\alpha'(t) = (\cos t, -\sin t + \frac{1}{\tan \frac{t}{2}} \cdot \sec^2 \frac{t}{2} \cdot \frac{1}{2})$$

Because $\cos t = \alpha_1'(t)$ is 0 on I only when $t = \frac{\pi}{2}$, we know α is regular on I except possibly at $t = \frac{\pi}{2}$.

Compute

$$\alpha'(\frac{\pi}{2}) = (0, -1 + \frac{1}{1} \cdot 2 \cdot \frac{1}{2}) = (0, 0)$$

We now conclude α is regular on I except $\frac{\pi}{2}$.

(b)
A useful Identity give us

$$\alpha'(t) = (\cos t, -\sin t + \csc t)$$

From the following facts

(a) the first argument of the segment is from 0 to $\sin t = \alpha(t)$

(b) $\alpha_x'(t) = \cos t$

(c) $\frac{\sin t}{\cos t} = \tan t$

We conclude that the length of the segment is

$$\begin{aligned} |\tan t| \cdot |\alpha'(t)| &= |\tan t| \cdot \sqrt{\cos^2 t + \sin^2 t - 2 \sin t \csc t + \csc^2 t} \\ &= |\tan t| \cdot \sqrt{1 - 2 + \csc^2 t} \\ &= |\tan t| \cdot \sqrt{\csc^2 t - 1} = |\tan t| \cdot \sqrt{\cot^2 t} = 1 \end{aligned}$$

■

Question 5

7. A map $\alpha: I \longrightarrow R^3$ is *called a curve of class* C^k if each of the coordinate functions in the expression $\alpha(t) = (x(t), y(t), z(t))$ has continuous derivatives up to order k. If α is merely continuous, we say that α is of class C^0. A curve α is called *simple* if the map α is one-to-one. Thus, the curve in Example 3 of Sec. 1-2 is not simple.

Let $\alpha: I \longrightarrow R^3$ be a simple curve of class C^0. We say that α has a *weak tangent* at $t = t_0 \in I$ if the line determined by $\alpha(t_0 + h)$ and $\alpha(t_0)$ has a limit position when $h \longrightarrow 0$. We say that α has a *strong tangent* at $t = t_0$ if the line determined by $\alpha(t_0 + h)$ and $\alpha(t_0 + k)$ has a limit position when $h, k \longrightarrow 0$. Show that

a. $\alpha(t) = (t^3, t^2)$, $t \in R$, has a weak tangent but not a strong tangent at $t = 0$.

***b.** If $\alpha: I \longrightarrow R^3$ is of class C^1 and regular at $t = t_0$, then it has a strong tangent at $t = t_0$.

c. The curve given by

$$\alpha(t) = \begin{cases} (t^2, t^2), & t \geq 0, \\ (t^2, -t^2), & t \leq 0, \end{cases}$$

is of class C^1 but not of class C^2. Draw a sketch of the curve and its tangent vectors.

Proof. **(a)** Let $v = (0, 1)$. Compute

$$\frac{\alpha(t) - \alpha(0)}{|\alpha(t) - \alpha(0)|} \cdot v = \frac{t^2}{\sqrt{t^6 + t^4}} = \frac{1}{\sqrt{t^2 + 1}} \to 1 \text{ as } t \to 1$$

This implies α has a weak tangent at $t = 0$. Now, if α has a strong tangent, we must have

$$\frac{\alpha(h) - \alpha(-h)}{2h} \cdot v \to 1 \text{ or } \to -1$$

But this is clearly not the case as

$$\frac{\alpha(h) - \alpha(-h)}{2h} \cdot v = 0 \text{ for all } h > 0$$

So we have the conclusion that α has no strong tangent at 0.

(b) By MVT, for each h, k there exists a set of real numbers $\{c_x, c_y, c_z\}$ between $t+h$ and $t+k$ such that

$$\frac{\alpha(t_0+h)-\alpha(t_0+k)}{h-k} = \Big(x'(c_x), y'(c_y), z'(c_z)\Big)$$

Then because

$$h, k \to 0 \implies t_0+h, t_0+k \to t_0 \implies c_x, c_y, c_z \to t_0$$

Then from the fact α is of class C^1 (x', y', z' are all continuous), we can now deduce

$$\frac{\alpha(t_0+h)-\alpha(t_0+k)}{h-k} \to \alpha'(t_0) \text{ as } h, k \to 0 \tag{1.3}$$

Now, because $\alpha'(t_0) \neq 0$ as α is regular, we see

$$\lim_{h,k\to 0} \frac{\alpha(t_0+h)-\alpha(t_0+k)}{h-k} \cdot \alpha'(t_0) = |\alpha'(t_0)|^2$$

This then implies

$$\lim_{h,k\to 0} \frac{\alpha(t_0+h)-\alpha(t_0+k)}{|\alpha(t_0+h)-\alpha(t_0+k)|} \cdot \frac{\alpha'(t_0)}{|\alpha'(t_0)|} = 1$$

which implies the "strong tangent" must always converge to $\alpha'(t_0)$.

Notice that the last implication is backed by Equation 1.3

(c)

From

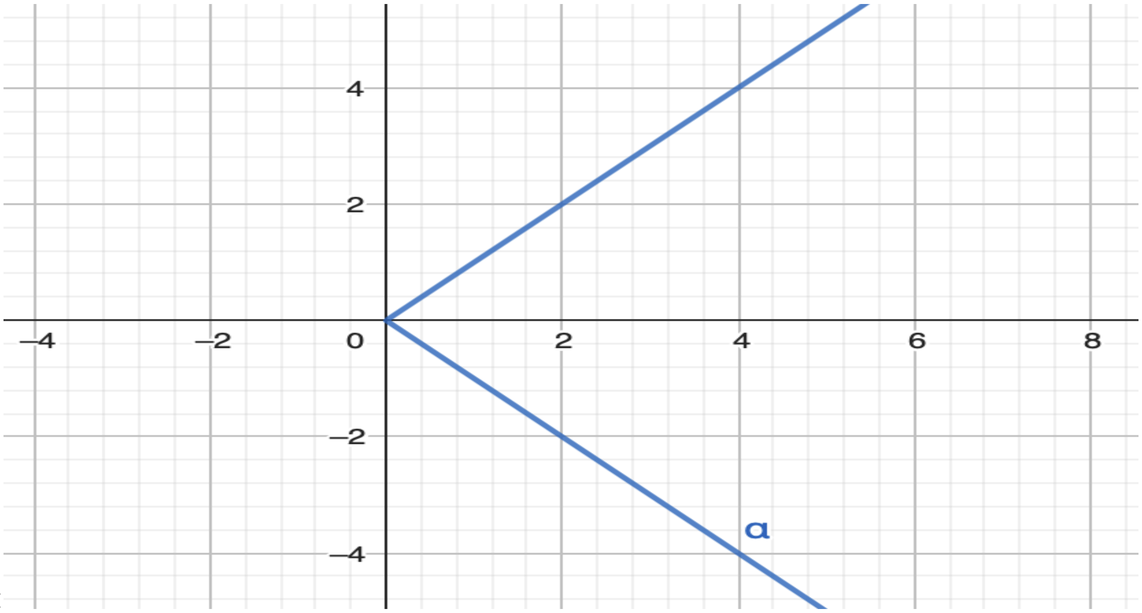
$$\alpha(t) = \left(t^2, \begin{cases} t^2 & \text{if } t \geq 0 \\ -t^2 & \text{if } t \leq 0 \end{cases}\right)$$

Compute

$$\alpha'(t) = \left(2t, \begin{cases} 2t & \text{if } t \geq 0 \\ -2t & \text{if } t \leq 0 \end{cases}\right)$$

Notice that the derivative at $t = 0$ is computed from definition instead of product rule.

Now, it is clear that x', y' are continuous. This implies $\alpha \in C^1$. Yet, we see y' is not differentiable at $t = 0$. This implies $\alpha \notin C^2$.

The sketch:  ■

Question 6

***8.** Let $\alpha: I \longrightarrow R^3$ be a differentiable curve and let $[a, b] \subset I$ be a closed interval. For every *partition*

$$a = t_0 < t_1 < \cdots < t_n = b$$

of $[a, b]$, consider the sum $\sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})| = l(\alpha, P)$, where P stands for the given partition. The norm $|P|$ of a partition P is defined as

$$|P| = \max(t_i - t_{i-1}),\ i = 1, \ldots, n.$$

Geometrically, $l(\alpha, P)$ is the length of a polygon inscribed in $\alpha([a, b])$ with vertices in $\alpha(t_i)$ (see Fig. 1-12). The point of the exercise is to show that the arc length of $\alpha([a, b])$ is, in some sense, a limit of lengths of inscribed polygons.

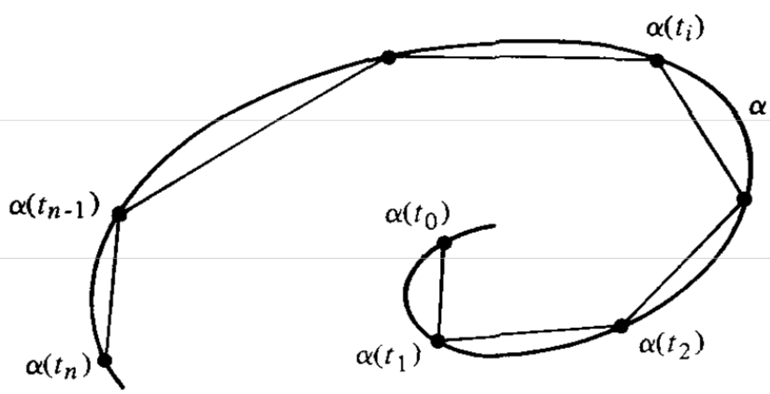


Figure 1-12

Prove that given $\epsilon > 0$ there exists $\delta > 0$ such that if $|P| < \delta$ then

$$\left| \int_a^b |\alpha'(t)|\, dt - l(\alpha, P) \right| < \epsilon.$$

Proof. We first prove

$$\int_a^b |\alpha'(t)|\, dt \geq l(\alpha, P)$$

By FTC, we have

$$\begin{aligned} |\alpha(t_i) - \alpha(t_{i-1})| &= \left| \int_{t_{i-1}}^{t_i} \alpha'(t) dt \right| \\ &\leq \int_{t_{i-1}}^{t_i} |\alpha'(t)| \, dt \end{aligned}$$

This then implies

$$l(\alpha, P) = \sum |\alpha(t_i) - \alpha(t_{i-1})| \leq \sum \int_{t_{i-1}}^{t_i} |\alpha'(t)| \, dt = \int_a^b |\alpha'(t)| \, dt \text{ (done)}$$

We have reduced the problem into

$$\text{finding } \delta \text{ such that } \forall P : |P| < \delta, \int_a^b |\alpha'(t)| \, dt - l(\alpha, P) < \epsilon$$

Because α' is uniformly continuous on $[a, b]$ ($\because$ continuous function on compact domain is uniformly continuous), we know there exists δ' such that

$$|\alpha'(s) - \alpha'(t)| < \frac{\epsilon}{2(b-a)} \text{ if } |s - t| < \delta'$$

We claim

$$\text{such } \delta' \text{ works}$$

Let $|P| < \delta$, and let $s_i \in [t_{i-1}, t_i]$. Because $|s_i - t_i| < \delta$, we have

$$|\alpha'(s_i) - \alpha'(t_i)| < \frac{\epsilon}{2(b-a)} \tag{1.4}$$

This give us

$$|\alpha'(s_i)| < |\alpha'(t_i)| + \frac{\epsilon}{2(b-a)}$$

Now, we can deduce

$$\begin{aligned}
\int_{t_{i-1}}^{t_i} |\alpha'(s)|\, ds &\leq |\alpha'(t_i)|\, \Delta t_i + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&= \int_{t_{i-1}}^{t_i} |\alpha'(t_i)|\, dt + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&= \left| \int_{t_{i-1}}^{t_i} \alpha'(t_i) dt \right| + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&= \left| \int_{t_{i-1}}^{t_i} \alpha'(t_i) - \alpha'(t) dt + \int_{t_{i-1}}^{t_i} \alpha'(t) dt \right| + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&\leq \left| \int_{t_{i-1}}^{t_i} \alpha'(t_i) - \alpha'(t) dt \right| + \left| \int_{t_{i-1}}^{t_i} \alpha'(t) dt \right| + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&\leq \frac{\epsilon}{2(b-a)}\Delta t_i + |\alpha(t_i) - \alpha(t_{i-1})| + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&= |\alpha(t_i) - \alpha(t_{i-1})| + \frac{\epsilon}{b-a}\Delta t_i
\end{aligned}$$

Notice that the last inequality follows from Equation 1.4. The long deduction above then give us

$$\begin{aligned}
\int_a^b |\alpha'(t)|\, dt &\leq \sum |\alpha(t_i) - \alpha(t_{i-1})| + \frac{\epsilon}{b-a}(b-a) \\
&= l(\alpha, P) + \epsilon
\end{aligned}$$

Then we have

$$\int_a^b |\alpha'(t)|\, dt - l(\alpha, P) \leq \epsilon \text{ (done)}$$

■

Question 7

9. a. Let $\alpha : I \longrightarrow R^3$ be a curve of class C^0 (cf. Exercise 7). Use the approximation by polygons described in Exercise 8 to give a reasonable definition of arc length of α.

b. (*A Nonrectifiable Curve.*) The following example shows that, with any reasonable definition, the arc length of a C^0 curve in a closed interval may be unbounded. Let $\alpha : [0, 1] \longrightarrow R^2$ be given as $\alpha(t) = (t, t\sin(\pi/t))$ if $t \neq 0$, and $\alpha(0) = (0, 0)$. Show, geometrically, that the arc length of the portion of the curve corresponding to $1/(n+1) \leq t \leq 1/n$ is at least $2/(n+\frac{1}{2})$. Use this to show that the length of the curve in the interval $1/N \leq t \leq 1$ is greater than $2\sum_{n=1}^{N} 1/(n+1)$, and thus it tends to infinity as $N \longrightarrow \infty$.

Proof. **(a)** Suppose $I = [a, b]$. Define arc length by

$$\sup_P l(P, \alpha) \text{ where sup runs over all partition } P \text{ of } [a, b]$$

(b)

Geometrically, we know the arc length of the portion of the curve corresponding to $t \in [\frac{1}{n+1}, \frac{1}{n}]$ must be greater than

$$\left|\alpha(\frac{1}{n}) - \alpha(\frac{1}{n+\frac{1}{2}})\right| + \left|\alpha(\frac{1}{n+1}) - \alpha(\frac{1}{n+\frac{1}{2}})\right| \tag{1.5}$$

WOLG of n being odd or even, Compute

$$
\begin{aligned}
\left|\alpha(\frac{1}{n}) - \alpha(\frac{1}{n+\frac{1}{2}})\right| &= \left|(\frac{1}{n}, 0) - (\frac{1}{n+\frac{1}{2}}, \frac{1}{n+\frac{1}{2}})\right| \\
&= \sqrt{(\frac{1}{n} - \frac{1}{n+\frac{1}{2}})^2 + (\frac{1}{n+\frac{1}{2}})^2} \\
&= \sqrt{\frac{1}{n^2} - \frac{4}{n(2n+1)} + \frac{8}{(2n+1)^2}} \\
&= \sqrt{\frac{(2n+1)^2 - 4n(2n+1) + 8n^2}{n^2(2n+1)^2}} \\
&= \sqrt{\frac{4n^2+1}{n^2(2n+1)^2}} \\
&= \frac{\sqrt{4n^2+1}}{n(2n+1)} \geq \frac{\sqrt{4n^2}}{n(2n+1)} = \frac{2}{2n+1}
\end{aligned}
$$

and compute

$$
\begin{aligned}
\left|\alpha(\frac{1}{n+\frac{1}{2}}) - \alpha(\frac{1}{n})\right| &= \left|(\frac{1}{n+\frac{1}{2}}, \frac{1}{n+\frac{1}{2}}) - (\frac{1}{n+1}, 0)\right| \\
&= \sqrt{(\frac{1}{n+1} - \frac{1}{n+\frac{1}{2}})^2 + (\frac{1}{n+\frac{1}{2}})^2} \\
&= \sqrt{\frac{1}{(n+1)^2} - \frac{4}{(n+1)(2n+1)} + \frac{8}{(2n+1)^2}} \\
&= \sqrt{\frac{(2n+1)^2 - 4(n+1)(2n+1) + 8(n+1)^2}{(n+1)^2(2n+1)^2}} \\
&= \sqrt{\frac{4n^2+8n+5}{(n+1)^2(2n+1)^2}} \\
&\geq \frac{\sqrt{4n^2+8n+4}}{(n+1)(2n+1)} = \frac{2}{2n+1}
\end{aligned}
$$

From the computation and Equation 1.5, it is now clear that the arc length of the portion of the curve corresponding to $t \in [\frac{1}{n+1}, \frac{1}{n}]$ is at least $\frac{2}{n+\frac{1}{2}}$. With simple addition, this then implies the arc length of the curve in the interval $[\frac{1}{N}, 1]$ is at least

$$\sum_{n=1}^{N-1} \frac{2}{2n+1} = 2\sum_{n=1}^{N-1} \frac{1}{2n+1}$$

The number is clearly greater than

$$2\sum_{n=1}^{N-1} \frac{1}{2n+2}$$

which equals to

$$\sum_{n=1}^{N-1} \frac{1}{n+1}$$

The series diverge to $+\infty$ as N to ∞. ∎

Theorem 1.2.1. (Integrating the Dot Product) Given a curve $u : [a,b] \to \mathbb{R}^n$ and a vector $v \in \mathbb{R}^n$, suppose

(a) u is differentiable on (a,b)

(b) u is continuous on $[a,b]$

We have

$$\int_a^b u'(t) \cdot v dt = \Big(\int_a^b u'(t) dt \Big) \cdot v = \big(u(b) - u(a) \big) \cdot v$$

Proof.

$$\begin{aligned}
\int_a^b u'(t) \cdot v dt &= \int_a^b \sum_{k=1}^n u_k'(t) \cdot v_k dt \\
&= \sum_{k=1}^n \int_a^b u_k'(t) \cdot v_k dt \\
&= \sum_{k=1}^n v_k \int_a^b u_k'(t) dt \\
&= v \cdot \Big(\int_a^b u'(t) dt \Big)
\end{aligned}$$

∎

Question 8

10. (*Straight Lines as Shortest.*) Let $\alpha: I \longrightarrow R^3$ be a parametrized curve. Let $[a, b] \subset I$ and set $\alpha(a) = p$, $\alpha(b) = q$.

a. Show that, for any constant vector v, $|v| = 1$,

$$(q - p)\cdot v = \int_a^b \alpha'(t)\cdot v\, dt \leq \int_a^b |\alpha'(t)|\, dt.$$

b. Set

$$v = \frac{q - p}{|q - p|}$$

and show that

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)|\, dt;$$

that is, the curve of shortest length from $\alpha(a)$ to $\alpha(b)$ is the straight line joining these points.

Proof. **(a)**

The first equality

$$(q-p)\cdot v = \int_a^b \alpha'(t)\cdot v dt$$

follows directly from Theorem 1.2.1.

Now, by Cauchy-Schwarz inequality, we have

$$|\alpha'(t)\cdot v| \leq |\alpha'(t)|\cdot|v|$$

This then give us

$$\alpha'(t)\cdot v \leq |\alpha'(t)\cdot v| \leq |\alpha'(t)|\cdot|v| = |\alpha'(t)|$$

We now have

$$\int_a^b \alpha'(t)\cdot v \leq |\alpha'(t)|\, dt$$

as desired.

(b)
The first inequality tell us that if v is a constant and $|v| = 1$, we have

$$(q-p) \cdot v \leq \int_a^b |\alpha'(t)| \, dt$$

If $v = \frac{q-p}{|q-p|}$, it is clear that v is a constant and $|v| = 1$, and at the same time, we have

$$(q-p) \cdot v = \frac{(q-p) \cdot (q-p)}{|q-p|} = \frac{|q-p|^2}{|q-p|} = |q-p|$$

We now have

$$|q-p| = (q-p) \cdot v \leq \int_a^b |\alpha'(t)| \, dt$$

from the first inequality ∎

Question 9

1. Check whether the following bases are positive:
a. The basis $\{(1, 3), (4, 2)\}$ in R^2.
b. The basis $\{(1, 3, 5), (2, 3, 7), (4, 8, 3)\}$ in R^3.

Proof. Compute

$$\begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} = -10$$

and compute

$$\begin{vmatrix} 1 & 2 & 4 \\ 3 & 3 & 8 \\ 5 & 7 & 3 \end{vmatrix} = -9$$

Both bases are negatively oriented. ∎

Question 10

***2.** A plane P contained in R^3 is given by the equation $ax + by + cz + d = 0$. Show that the vector $v = (a, b, c)$ is perpendicular to the plane and that $|d|/\sqrt{a^2 + b^2 + c^2}$ measures the distance from the plane to the origin $(0, 0, 0)$.

Proof. Arbitrarily pick two points u, w in P. We wish to show

$$v \cdot (u - w) = 0$$

Because $v = (a, b, c)$ and

$$\begin{cases} au_1 + bu_2 + cu_3 = -d aw_1 + bw_2 + cw_3 = -d \end{cases}$$

We see

$$\begin{aligned} v \cdot (u - w) &= a(u_1 - w_2) + b(u_2 - w_2) + c(u_3 - w_2) \\ &= (-d) - (-d) = 0 \text{ (done)} \end{aligned}$$

To measure the distance between P and the origin, we wish to find a vector u such that $u \perp P$ and $u \in P$. We know that u must be linearly dependent with $v = (a, b, c)$, since the dimension of $P^\perp$ is 1. Then, we can write

$$u = c_0(a, b, c) \text{ for some } c_0 \in \mathbb{R}$$

Because $u \in P$, we know

$$c_0 a^2 + c_0 b^2 + c_0 c^2 + d = 0$$

This tell us

$$c_0 = \frac{-d}{a^2 + b^2 + c^2}$$

We now see that the distance $|u|$ between P and origin is

$$|u| = |c_0| \cdot \sqrt{a^2 + b^2 + c^2} = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$$

■

Question 11

***3. Determine the angle of intersection of the two planes $5x + 3y + 2z - 4 = 0$ and $3x + 4y - 7z = 0$.**

Proof. From last question, we know the two vectors u, v that are respectively perpendicular to $P : 5x + 3y + 2z - 4 = 0$ and $Q : 3x + 4y - 7z = 0$ respectively have the direction

$$(5, 3, 2) \text{ and } (3, 4, -7)$$

Then, we see the angle of the intersection are

$$\arccos \frac{5 \cdot 3 + 3 \cdot 4 + 2 \cdot (-7)}{\sqrt{5^2 + 3^2 + 2^2}\sqrt{3^2 + 4^2 + 7^2}} = \arccos \frac{13}{\sqrt{38}\sqrt{71}}$$

Notice that this angle is smaller than $\frac{\pi}{2}$ as we intend it to be. ■

Question 12

***6.** **Given two nonparallel planes $a_i x + b_i y + c_i z + d_i = 0$, $i = 1, 2$, show that their line of intersection may be parametrized as**

$$x - x_0 = u_1 t, \qquad y - y_0 = u_2 t, \qquad z - z_0 = u_3 t,$$

where (x_0, y_0, z_0) belongs to the intersection and $u = (u_1, u_2, u_3)$ is the vector product $u = v_1 \wedge v_2$, $v_i = (a_i, b_i, c_i)$, $i = 1, 2$.

Proof. Let $v = (x, y, z)$ be a point on the line of intersection. We see the vector $v - (x_0, y_0, z_0)$ lies on both planes, and thus must be perpendicular to $(a_1, b_1, c_1) = v_1$ and $(a_2, b_2, c_2) = v_2$ thus satisfying

$$v - (x_0, y_0, z_0) = tv_1 \times v_2 = tu \text{ for some } t \in \mathbb{R}$$

sine in $\mathbb{R}^3$, the only direction perpendicular to both v_1, v_2 is $v_1 \times v_2$. We can rewrite the above equation of course into

$$x - x_0 = u_1 t, y - y_0 = u_2 t, z - z_0 = u_3 t$$

■

1.3 Fundamental Theorem of Local Curves

In this section, by an **orthogonal transformation** we mean a linear transformation M from $\big(V, \langle\cdot,\cdot\rangle_V\big)$ to $\big(W, \langle\cdot,\cdot\rangle_W\big)$ such that

$$\langle v, w\rangle_V = \langle Mv, Mw\rangle_W \qquad\qquad (v, w \in V)$$

By a **rigid motion** M, we mean an orthogonal transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that

$$\det\big([M]_{\{e_1,e_2,e_3\}}\big) > 0$$

Theorem 1.3.1. (Fundamental Theorem of Local Curves: Uniqueness Part 1) Given an <u>open</u> interval $I \subseteq \mathbb{R}$, a parametrized by arc-length curve $\alpha : I \to \mathbb{R}^3$ with positive curvature, a rigid motion M and a vector $c \in \mathbb{R}^3$, we see that the function $\beta : I \to \mathbb{R}^3$ defined by

$$\beta(s) = (M \circ \gamma)(s) + c$$

is a curve parametrized by arc-length such that

$$\alpha \text{ and } \beta \text{ has the same curvature and torsion on all } s \in I$$

Proof. We first have to prove

$$\beta : I \to \mathbb{R}^3 \text{ is parametrized by arc-length}$$

Fix $s \in I$. We have to prove

$$|\beta'(s)| = 1$$

Compute

$$\begin{aligned} |\beta'(s)| &= |(M \circ \gamma)'(s)| \\ &= || \;\; \text{(done)} \end{aligned}$$

We now prove

$$\beta'(s)$$

Because we have the identity

$$\kappa(s) = |\alpha''(s)| \text{ and } \tau(s) = -\frac{(\alpha' \times \alpha'') \cdot \alpha'''}{\kappa^2}$$

■

Theorem 1.3.2. (Fundamental Theorem of Local Curves: Uniqueness Part 2) Given an open interval $I \subseteq \mathbb{R}$ and two parametrized by arc-length curves $\alpha, \overline{\alpha} : I \to \mathbb{R}^3$ such that

$$\kappa(s) = \overline{\kappa}(s) \text{ and } \tau(s) = \overline{\tau}(s) \qquad\qquad (s \in I)$$

Then there exists an rigid motion M and a vector $c \in \mathbb{R}^3$ such that

$$\alpha(s) = M\big(\overline{\alpha}(s)\big) + c \qquad\qquad (s \in I)$$

Proof. Fix distinct $s_0 \in I$. We first have to

$$\text{find a rigid motion } M : \mathbb{R}^3 \to \mathbb{R}^3 \text{ and some vector } c \in \mathbb{R}^3$$
$$\text{such that } \begin{cases} \alpha(s_0) = (M \circ \overline{\alpha})(s_0) + c \\ T(s_0) = M \circ \overline{T}(s_0) \\ N(s_0) = M \circ \overline{N}(s_0) \\ B(s_0) = M \circ \overline{B}(s_0) \end{cases} \qquad \text{(done)}$$

Now, express

$$T_M(s) = \text{ normal tangent of } M \circ \overline{\alpha} + c \text{ at } s$$

We show

$$\begin{cases} T_M = T \\ N_M = N \qquad \text{on } I \\ B_M = B \end{cases}$$

By Frenet Formula, compute

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\Big(|T-T_M|^2+|N-N_M|^2+|B-B_M|^2 \Big) \\
=&\frac{1}{2}\frac{d}{ds}\Big(\sum_{X=T,N,B}(X-X_M)\cdot(X-X_M)\Big) \\
=&\sum_{X=T,N,B}(X-X_M)'\cdot(X-X_M) \\
=&\sum_{X=T,N,B}(X'-X_M')\cdot(X-X_M) \\
=&(T'-T_M')\cdot(T-T_M)+(N'-N_M')\cdot(N-N_M)+(B'-B_M')\cdot(B-B_M) \\
\\
=&(\kappa N'-\kappa N_M')\cdot(T-T_M) \\
+&\Big)\Big(-\kappa T-\tau B+\kappa T_M'+\tau BT_M'\Big)\cdot(N-N_M) \\
+&\Big)\Big(\tau N-\tau N_M\Big)\cdot(B-B_M) \qquad (\because \text{ Frenet Formula and } \alpha,\alpha_M \text{ same curvature and torsion}) \\
=&0 \qquad (\because \text{ Elimination })
\end{aligned}
$$

We now know

$$|T-T_M|^2+|N-N_M|^2+|B-B_M|^2 \text{ is a constant}$$

Moreover, because by our setting

$$\Big(|T-T_M|^2+|N-N_M|^2+|B-B_M|^2 \Big)(s_0)=0$$

We know

$$|T-T_M|^2+|N-N_M|^2+|B-B_M|^2=0 \qquad (s\in I)$$

This implies

$$\begin{cases} T=T_M \\ N=N_M \\ B=B_M \end{cases} \quad \text{on } I \text{ (done)}$$

Because both α and α_M are parametrized by arc-length and $\alpha(s_0)=\alpha_M(s_0)$, we now see

$$\alpha(s)=\int_{s_0}^{s}T(x)dx+\alpha(s_0)=\int_{s_0}^{s}T_M(x)dx+\alpha_M(s_0)=\alpha_M(s) \qquad (s\in I)$$

This finish the proof. ∎

1.4 Isoperimetric Inequality

In this section, by **a closed plane curve**, we mean a regular parametrized curve $\alpha : [a, b] \to \mathbb{R}^2$ such that

$$\alpha^{(n)}(a) = \alpha^{(n)}(b) \text{ for all } n \in \mathbb{Z}_0^+$$

If we say a closed plane curve $\alpha : [a, b] \to \mathbb{R}^2$ is **simple**, we mean

$$\alpha(t_1) \neq \alpha(t_2) \text{ for all distinct pair } (t_1, t_2) \subseteq [a, b] \text{ except } (a, b)$$

A closed plane curve must divide $\mathbb{R}^2$ into two separate subsets, in the sense that $\mathbb{R}^2 \setminus \alpha\big[[a, b]\big]$ has two connected component. The one connected component that has finite area in the sense of Lebesgue outer measure is called the **interior** of the α. If the interior is always on the left side of α, we say α is **positively oriented**, in other words, α runs counter clockwise.

Theorem 1.4.1. (Green's Theorem) Given a positively oriented, piecewise smooth, simple closed plane curve $\alpha : [a, b] \to \mathbb{R}^2$, where C is the image of α and D is the region bounded by C, and two function $L, M : D \to \mathbb{R}$ that has continuous partial derivative, we have

$$\oint_C L dx + M dy = \iint_D (M_x - L_y) dA$$

If we define area for bounded **region** D by

$$A(D) \triangleq \iint_D 1 dA$$

Green's Theorem give us

$$\begin{aligned} A(D) &= \oint_C x dy = \oint_C -y dx = \oint_C \frac{-y}{2} dx + \frac{x}{2} dy \\ &= \int_a^b x(t) y'(t) dt = \int_a^b -x'(t) y(t) dt = \frac{1}{2} \int_a^b (xy' - y'x) dt \end{aligned}$$

Theorem 1.4.2. (Isoperimetric Inequality: Part 1) Let C be a piece-wise C^1 simple closed curve with length l, and let A be the area of the region bounded by C. Then

$$A \leq \frac{l^2}{4\pi}$$

Proof. Parametrize C with $\big(x(t), y(t)\big) : [a,b] \to \mathbb{R}^2$. Because $x : [a,b] \to \mathbb{R}$ is a continuous function, by EVT, we know there exists $c', d \in [a,b]$ such that

$$x(c') = \min_{t\in[a,b]} x(t) \text{ and } x(d) = \max_{t\in[a,b]} x(t)$$

Now, let $\gamma : [0,l]$ be a positively oriented arc-length parametrization such that

$$\gamma(l) = \gamma(0) \triangleq x(d)$$

Let $c \in [0,l]$ satisfy

$$\gamma(c) \triangleq x(c')$$

Let S be a circle such that

$$S \text{ has the radius } r = \frac{\gamma(0) - \gamma(c)}{2}$$

We first show

$$A + \pi r^2 \leq lr \tag{1.6}$$

We translate S so that S centers at origin, and translate C so that $\gamma(0)$ has value $(r,0)$. Note that such translation does not change area, which can be verified using change of variable.

Now, we know $S = \{(x,y) : x^2 + y^2 = r^2\}$. If we parametrize S by $(r\cos t, r\sin t)$, with Green's Theorem, we see

$$\begin{aligned} A(S) &= \oint (r\cos t)(dr\sin t) \\ &= \int_0^{2\pi} (r^2\cos^2 t)dt \\ &= \int_0^{2\pi} r^2 \cdot \frac{\cos 2t + 1}{2} = r^2\pi \end{aligned}$$

Now, express $\gamma(s)$ by

$$\gamma(s) = (x(s), y(s))$$

We positively oriented parametrize S by

$$\alpha(t) \triangleq (x(t), \overline{y}(t))$$

We now prove

$$\pi r^2 = -\int_0^l \overline{y} x' ds \tag{1.7}$$

By Green's Theorem

$$\pi r^2 = A(S) = \oint -\overline{y} dx = \int_0^l -\overline{y} x' ds \text{ (done)}$$

We now prove

$$(xy' - \overline{y}x')^2 \leq \left(x^2 + (\overline{y})^2\right)\left((x')^2 + (y')^2\right) \tag{1.8}$$

Using Cauchy-Schwarz Inequality on $(x, \overline{y})$ and $(y', -x')$. We see

$$\begin{aligned}(xy' - \overline{y}x')^2 &= \left|(x, \overline{y}) \cdot (y', -x')\right|^2 \\ &\leq \left|(x, \overline{y})\right|^2 \cdot \left|(y', -x')\right|^2 \\ &= \left(x^2 + (\overline{y})^2\right)\left((x')^2 + (y')^2\right) \text{ (done)}\end{aligned} \tag{1.9}$$

Now, because

(a) Green's Theorem

(b) Equation 1.7

(c) Equation 1.8

(d) $C = (x, y)$ is parametrized by arc-length

(e) $S = (x, \overline{y})$ is a circle of radius r

we have

$$\begin{aligned}A + \pi r^2 &= \int_0^l x dy - \int_0^l \overline{y} x' ds \\ &= \int_0^l xy' - \overline{y} x' ds \\ &\leq \int_0^l \sqrt{(xy' - \overline{y}x')^2} ds \\ &\leq \int_0^l \sqrt{\left(x^2 + (\overline{y})^2\right)\left((x')^2 + (y')^2\right)} ds \\ &\leq \int_0^l \sqrt{x^2 + (\overline{y})^2} ds \\ &\leq \int_0^l r ds = rl \text{ (done)}\end{aligned} \tag{1.10}$$

Lastly, we show

$$A \leq \frac{l^2}{4\pi}$$

By AM-GM inequality and Equation 1.10, we now can deduce

$$\sqrt{A\pi r^2} \leq \frac{A + \pi r^2}{2} \leq \frac{rl}{2} \tag{1.11}$$

This let us deduce

$$A \leq \frac{l^2}{4\pi} \text{ (done)}$$

$\blacksquare$

Theorem 1.4.3. (Isoperimetric Inequality: Part 2) Let C be a piece-wise C^1 simple closed curve with length l, and let A be the area of the region bounded by C. We have

$$A = \frac{l^2}{4\pi} \implies C \text{ is a circle}$$

Proof. Do exactly the same thing in the proof of First part of Isoperimetric Inequality 1.4.2 on C.

We wish to prove

$$x^2 + y^2 = r^2$$

Because

$$A = \frac{l^2}{4\pi} \implies A\pi r^2 = (\frac{rl}{2})^2 \implies \sqrt{A\pi r^2} = \frac{rl}{2}$$

Then from Equation 1.11, we deduce

$$\sqrt{A\pi r^2} = \frac{A + \pi r^2}{2}$$

Then because AM-GM inequality become an equality only when two sides equals, we now have

$$A = \pi r^2 \text{ and } l = 2\pi r$$

This let us deduce

$$A + \pi r^2 = 2\pi r^2 = 2rl$$

Then by Equation 1.10, we can deduce

$$\begin{aligned}\left|(x,\overline{y})\cdot(y',-x')\right|^2 &= (xy'-\overline{y}x')^2\\ &= \big(x^2+(\overline{y})^2\big)\big((x')^2+(y')^2\big)\\ &= \left|(x,\overline{y})\right|^2\cdot\left|(x',y')\right|^2\end{aligned}$$

Because Cauchy-Schwarz inequality become an equality only when two vectors are linearly independent, we know there exist $\lambda\in\mathbb{R}$ such that

$$(x,\overline{y})=\lambda(y',-x')$$

This let us deduce

$$\lambda=\frac{x}{y'}=\frac{\overline{y}}{-x'}\text{ and }\lambda=\frac{\sqrt{x^2+(\overline{y})^2}}{\sqrt{(y')^2+(x')^2}} \tag{1.12}$$

Now, because $\gamma=(x,y)$ is parametrized by arc-length and $(x,\overline{y})$ form the circle S with radius r, from Equation 1.12, we have

$$\lambda=\sqrt{x^2+(\overline{y})^2}=r \tag{1.13}$$

Then from Equation 1.12 and Equation 1.13, we can deduce

$$\frac{x}{y'}=\frac{\overline{y}}{-x'}=\lambda=r$$

This then give us

$$x=ry'$$

Now, do exactly the same thing in the proof of First part of Isoperimetric Inequality 1.4.2, except, at this time, we parametrize S by $(\overline{x},y)$. The similar argument then went on and give us

$$y=rx'$$

Finally, because $\gamma=(x,y)$ is parametrized by arc-length, we have

$$x^2+y^2=r^2\big((y')^2+(x')^2\big)=r^2\text{ (done)}$$

$\blacksquare$

1.5 Four Vertex Theorem

1.6 HW2

Question 13

1. Given the parametrized curve (helix)

$$\alpha(s) = \left(a\cos\frac{s}{c}, a\sin\frac{s}{c}, b\frac{s}{c}\right), \qquad s \in R,$$

where $c^2 = a^2 + b^2$,

a. Show that the parameter s is the arc length.

b. Determine the curvature and the torsion of α.

c. Determine the osculating plane of α.

d. Show that the lines containing $n(s)$ and passing through $\alpha(s)$ meet the z axis under a constant angle equal to $\pi/2$.

e. Show that the tangent lines to α make a constant angle with the z axis.

Question 14

***2.** Show that the torsion τ of α is given by

$$\tau(s) = -\frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{|k(s)|^2}.$$

Question 15

3. Assume that $\alpha(I) \subset R^2$ (i.e., α is a plane curve) and give k a sign as in the text. Transport the vectors $t(s)$ parallel to themselves in such a way that the origins of

$t(s)$ agree with the origin of R^2; the end points of $t(s)$ then describe a parametrized curve $s \longrightarrow t(s)$ called the *indicatrix of tangents* of α. Let $\theta(s)$ be the angle from e_1 to $t(s)$ in the orientation of R^2. Prove (a) and (b) (notice that we are assuming that $k \neq 0$).

a. The indicatrix of tangents is a regular parametrized curve.

b. $dt/ds = (d\theta/ds)n$, that is, $k = d\theta/ds$.

Question 16

6. A *translation* by a vector v in R^3 is the map $A: R^3 \longrightarrow R^3$ that is given by $A(p) = p + v$, $p \in R^3$. A linear map $\rho: R^3 \longrightarrow R^3$ is an *orthogonal transformation* when $\rho u \cdot \rho v = u \cdot v$ for all vectors $u, v \in R^3$. A *rigid motion* in R^3 is the result of composing a translation with an orthogonal transformation with positive determinant (this last condition is included because we expect rigid motions to preserve orientation).

a. Demonstrate that the norm of a vector and the angle θ between two vectors, $0 \leq \theta \leq \pi$, are invariant under orthogonal transformations with positive determinant.

b. Show that the vector product of two vectors is invariant under orthogonal transformations with positive determinant. Is the assertion still true if we drop the condition on the determinant?

c. Show that the arc length, the curvature, and the torsion of a parametrized curve are (whenever defined) invariant under rigid motions.

Question 17

9. Given a differentiable function $k(s)$, $s \in I$, show that the parametrized plane curve having $k(s) = k$ as curvature is given by

$$\alpha(s) = \left(\int \cos\theta(s)\, ds + a, \int \sin\theta(s)\, ds + b\right),$$

where

$$\theta(s) = \int k(s)\, ds + \varphi,$$

and that the curve is determined up to a translation of the vector (a, b) and a rotation of the angle φ.

Question 18

11. One often gives a plane curve in polar coordinates by $\rho = \rho(\theta)$, $a \leq \theta \leq b$.

a. Show that the arc length is

$$\int_a^b \sqrt{\rho^2 + (\rho')^2}\, d\theta,$$

where the prime denotes the derivative relative to θ.

b. Show that the curvature is

$$k(\theta) = \frac{2(\rho')^2 - \rho\rho'' + \rho^2}{\{(\rho')^2 + \rho^2\}^{3/2}}.$$

Question 19

17. In general, a curve α is called a *helix* if the tangent lines of α make a constant angle with a fixed direction. Assume that $\tau(s) \neq 0$, $s \in I$, and prove that:

***a.** α is a helix if and only if $k/\tau = \text{const.}$

***b.** α is a helix if and only if the lines containing $n(s)$ and passing through $\alpha(s)$ are parallel to a fixed plane.

***c.** α is a helix if and only if the lines containing $b(s)$ and passing through $\alpha(s)$ make a constant angle with a fixed direction.

d. The curve

$$\alpha(s) = \left(\frac{a}{c}\int \sin\theta(s)\,ds, \frac{a}{c}\int \cos\theta(s)\,ds, \frac{b}{c}s\right),$$

where $a^2 = b^2 + c^2$, is a helix, and that $k/\tau = b/a$.

Question 20

3. Compute the curvature of the ellipse

$$x = a\cos t, \qquad y = b\sin t, \qquad t \in [0, 2\pi],\ a \neq b,$$

and show that it has exactly four vertices, namely, the points $(a, 0)$, $(-a, 0)$, $(0, b)$, $(0, -b)$.

Question 21

*4. Let C be a plane curve and let T be the tangent line at a point $p \in C$. Draw a line L parallel to the normal line at p and at a distance d of p (Fig. 1-36). Let h be the length of the segment determined on L by C and T (thus, h is the "height" of C relative to T). Prove that

$$|k(p)| = \lim_{d \to 0} \frac{2h}{d^2},$$

where $k(p)$ is the curvature of C at p.

Question 22

6. Let $\alpha(s)$, $s \in [0, l]$ be a closed convex plane curve positively oriented. The curve

$$\beta(s) = \alpha(s) - rn(s),$$

where r is a positive constant and n is the normal vector, is called a *parallel* curve to α (Fig. 1-37). Show that

a. Length of β = length of $\alpha + 2\pi r$.

b. $A(\beta) = A(\alpha) + rl + \pi r^2$.

c. $k_\beta(s) = k_\alpha(s)/(1 + r)$.

Question 23

8. ***a.** Let $\alpha(s)$, $s \in [0, l]$, be a plane simple closed curve. Assume that the curvature $k(s)$ satisfies $0 < k(s) \leq c$, where c is a constant (thus, α is less curved than a circle of radius $1/c$). Prove that

$$\text{length of } \alpha \geq \frac{2\pi}{c}.$$

b. In part a replace the assumption of being simple by "α has rotation index N." Prove that

$$\text{length of } \alpha \geq \frac{2\pi N}{c}.$$

Question 24

***11.** Given a nonconvex simple closed plane curve C, we can consider its *convex hull* H (Fig. 1-39), that is, the boundary of the smallest convex set containing the interior of C. The curve H is formed by arcs of C and by the segments of the tangents to C that bridge "the nonconvex gaps" (Fig. 1-39). It can be proved that H is a C^1 closed convex curve. Use this to show that, in the isoperimetric problem, we can restrict ourselves to convex curves.

Figure 1-39

Question 25

3. Show that the two-sheeted cone, with its vertex at the origin, that is, the set $\{(x, y, z) \in R^3; x^2 + y^2 - z^2 = 0\}$, is not a regular surface.

Question 26

6. Give another proof of Prop. 1 by applying Prop. 2 to $h(x, y, z) = f(x, y) - z$.

Question 27

7. Let $f(x, y, z) = (x + y + z - 1)^2$.
   a. Locate the critical points and critical values of f.
   b. For what values of c is the set $f(x, y, z) = c$ a regular surface?
   c. Answer the questions of parts a and b for the function $f(x, y, z) = xyz^2$.