Friedberg Ch6.2

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In this note, V always stand for an inner product vector space over $\mathbb{F}$

Definition and Theorem

Definition 1. *If $\langle w, v\rangle = 0$, then $w \perp v$*

Definition 2. *Let $S \subseteq V$*

$$S^\perp = \{w \in V | \forall v \in S, w \perp v\}$$

Definition 3. *Let $S \subseteq V$*

$$S \text{ is an } \textbf{orthogonal} \text{ set if } \forall v, v' \in S, v \perp v'$$

Theorem 1. *Let S be an orthogonal set*

$$S \text{ is linearly independent}$$

Proof. Write $S = \{v_1, v_2, \dots, v_n\}$

We prove by induction

$$\text{Base step: } \{v_1, v_2\} \text{ is linearly independent}$$

Assume $\{v_1, v_2\}$ is linearly dependent

Write $v_2 = cv_1, \exists c \neq 0 \in \mathbb{F}$

$\langle v_1, v_2\rangle = \overline{c}\langle v_1, v_1\rangle \neq 0$ CaC

$$\text{Induction step: } \{v_1, \dots, v_k\} \text{ is independent} \implies \{v_1, \dots, v_{k+1}\} \text{ is independent}$$

Assume $\{v_1, \dots, v_{k+1}\}$ is linearly dependent

Write $v_{k+1} = a_1v_1 + \cdots + a_kv_k$

Pick $i : 1 \leq i \leq k$, such that $a_i \neq 0$

$\langle v_{k+1}, v_i\rangle = \langle a_1v_1 + \cdots + a_kv_k, v_i\rangle = a_i\langle v_i, v_i\rangle \neq 0$ CaC ∎

Theorem 2. *Let $S \subseteq V$*

$$S^\perp \text{ is a subspace of } V$$

Proof. Let $w, w' \in S^\perp$

$\forall v \in S, \langle w + w', v\rangle = \langle w, v\rangle + \langle w,' v\rangle = 0 + 0 = 0 \implies w + w' \in S^\perp$

$\forall c \in \mathbb{F}, \forall v \in V, \langle cw, v\rangle = c\langle w, v\rangle = 0 \implies \forall c \in \mathbb{F}, cw \in S^\perp$ ∎

Theorem 3. *Let $S = \{w_1, w_2, \dots, w_n\}$ be a linearly independent subset of V. Define $S' = \{v_1, \dots, v_n\}$, where $v_1 = w_1$ and*

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j \text{ for } 2 \leq k \leq n$$

Then S' is an orthogonal basis of $span(S)$

Proof. We prove by induction

Base step: $span(v_1, v_2) = span(w_1, w_2)$

$v_1 = w_1 \in W$

$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1 \in W$

$c_1 v_1 + c_2 v_2 = 0 \implies (c_1 - c_2 \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2}) w_1 + c_2 w_2 = 0$

So $c_2 = 0$

$c_1 = 0$

Then v_1, v_2 is linearly independent

Induction step: $span(v_1, \dots, v_k) = span(w_1, \dots, w_k) \implies span(v_1, \dots, v_{k+1}) = span(w_1, \dots, w_{k+1})$

$v_{k+1} = w_{k+1} - \sum_{j=1}^{k} \frac{\langle w_{k+1}, v_j \rangle}{\|v_j\|^2} v_j \in W$

Let $1 \leq i \leq k$

$\langle v_{k+1}, v_i \rangle = \langle w_{k+1} - \sum_{j=1}^{k} \frac{\langle w_{k+1}, v_j \rangle}{\|v_j\|^2} v_j, v_i \rangle = \langle w_{k+1}, v_i \rangle - \sum_{j=1}^{k} \frac{\langle w_{k+1}, v_j \rangle}{\|v_j\|^2} \langle v_j, v_i \rangle = \langle w_{k+1}, v_i \rangle - \frac{\langle w_{k+1}, v_i \rangle}{\|v_i\|^2} \langle v_i, v_i \rangle = \langle w_{k+1}, v_i \rangle - \langle w_{k+1}, v_i \rangle = 0$

Then $\{v_1, \dots, v_{k+1}\}$ consist an orthogonal set, thus linearly independent

REMARK: Notice the process is $v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j$, but not $v_k = w_k - \sum_{j-1}^{k-1} \frac{\langle v_k, w_{k+1} \rangle}{\|v_j\|^2} \langle v_j, v_i \rangle$ ■

Corollary 3.1. *Let S' be an orthogonal subset of V*

We can extend S' to be an orthogonal basis containing S'

Theorem 4. *Let $S = \{v_1, \dots, v_k\}$ be an orthogonal subset of V. Let $y \in span(S)$*

$$y = \sum_{i=1}^{k} \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i$$

Proof. S is linearly indepenedent, so S is a basis of $span(S)$

Write $y = \sum_{i=1}^{k} a_i v_i, \exists a_i \in \mathbb{F}$

Then for each $1 \le j \le k$, we have $\langle y, v_j \rangle = \sum_{i=1}^{k} a_i \langle v_i, v_j \rangle = a_j \langle v_j, v_j \rangle$

So $a_j = \frac{\langle y, v_j \rangle}{\|v_j\|^2}$ ■

Corollary 4.1. *Let V be a finite dimensional inner product space with an orthonormal basis $\beta = \{v_1, \dots, v_n\}$. Let T be a linear operator on β. Let $A = [T]_\beta$*

$$A_{i,j} = \langle T(v_j), v_i \rangle$$

Definition 4. *Let β be an orthonormal subset of an inner product space V, and let $x \in V$*

*The **Fourier coefficients** of x relative to β is $\langle x, y \rangle$, where $y \in \beta$*

Theorem 5. *Let W be a finite-dimensional subspace of V, and let $y \in V$*

there exists unique $u \in W$ and $z \in W^\perp$, such that $y = u + z$

Proof. Let $\{w_1, \dots, w_n\}$ be a basis of W

Let $u = \sum_{i=1}^{n} \frac{\langle y, w_i \rangle}{\|w_i\|^2} w_i$

Let $z = y - u$

$\forall 1 \le i \le n, \langle z, w_i \rangle = \langle y - u, w_i \rangle = \langle y, w_i \rangle - \langle u, w_i \rangle = \langle y, w_i \rangle - \sum_{j=1}^{n} \frac{\langle y, w_j \rangle}{\|w_j\|^2} \langle w_j, w_i \rangle = \langle y, w_i \rangle - \frac{\langle y, w_i \rangle}{\|w_i\|^2} \langle w_i, w_i \rangle = \langle y, w_i \rangle - \langle y, w_i \rangle = 0$

So $z \in W^\perp$, such pair of u, z at least exists

Let $u + z = u' + z'$, where $u' \in W$ and $z' \in W^\perp$

$u - u' = z' - z \in W \cap W^\perp \implies u - u' = z' - z = 0 \implies u = u'$ and $z = z'$

■

Theorem 6. *Let W be a finite-dimensional subspace of V*

$$V = W \oplus W^\perp$$

Proof. Let $\{w_1, \dots, w_n\}$ be a basis of W

Let $\{v_1, \dots\}$ be a basis of $W^\perp$

We now prove $\{w_1, \dots, w_n\} \cup \{v_1, \dots\}$ is a basis of V

Assume $\{w_1, \dots, w_n\} \cup \{v_1, \dots\}$ is linearly dependent

Let $\sum_I c_i w_i + \sum_J c_j v_j = 0, \exists \{c_i \neq 0 | i \in I\}, \{c_j \neq 0 | j \in J\}, I, J$

Such non-empty J must exists, otherwise $\{w_1, \ldots, w_n\}$ is linearly dependent, CaC

Then we see $0 = \sum_I c_i w_i + \sum_J c_j v_j = 2\sum_I c_i w_i + 2\sum_J c_j v_j$, where $\sum_I c_i w_i \neq 0$ (otherwise, $\{w_1, \ldots, w_n\}$ is linearly indepenedent) CaC to the uniqueness of Theorem 5

■

Exercises

2.(a)

Proof. $v_1 = \frac{1}{\sqrt{2}}(1, 0, 1)$

$v_2 = \sqrt{\frac{4}{6}}(\frac{1}{-2}, 1, \frac{1}{2})$

$v_3 = \sqrt{3}(\frac{1}{3}, \frac{1}{3}, \frac{-1}{3})$

$\langle x, v_1 \rangle = \frac{1}{\sqrt{2}}(3)$

$\langle x, v_2 \rangle = \sqrt{\frac{3}{2}}$

$\langle x, v_3 \rangle = 0$ ■

2.(c)

Proof. $v_1 = 1$

$v_2 = \sqrt{12}(x - \frac{1}{2})$

$v_3 = \sqrt{180}(x^2 - x + \frac{1}{6})$

$h(x) = \frac{3}{2}v_1 + \frac{\sqrt{12}}{12}v_2$ ■

4.

Proof. $S^\perp = span((2, -1 + i, -2i))$ ■

6.

Proof. Pick a basis $\{w_1, \ldots, w_n\}$ of W

Do Gram-Schmidt on $\{w_1, \ldots, w_n\}$ and we have an orthogonal basis $\{w_1', \ldots, w_n'\}$ of W

Extend $\{w_1', \dots, w_n'\}$ to a basis $\{w_1', \dots, w_n', v_{n+1}, \dots, v_k, \dots\}$ of V

Do Gram-Schmidt on $\{w_1', \dots, w_n', v_{n+1}, \dots, v_k\}$ and we have an orthogonal basis $\{w_1', \dots, w_n', v_{n+1}', \dots, v_k', \dots\}$ of V

Express $x = a_1 w_1' + \cdots + a_n w_n' + a_{n+1} v_{n+1}' + \cdots + a_k v_k' + \cdots$

We know there exists $i : n+1 \leq i \leq k$, such that $a_i \neq 0$, otherwise, $x \in W$

$\langle x, v_i' \rangle = \langle a_i v_i', v_i' \rangle = a_i \langle v_i', v_i' \rangle \neq 0$

■

11.

Proof. Let $Row : M_{n \times n}(\mathbb{F}) \times i \to \mathbb{F}^n$ maps (A, i) to (the i-th row of $A)^t$

Let $Col : M_{n \times n}(\mathbb{F}) \times i \to \mathbb{F}^n$ maps (A, i) to (the i-th column of A)

Let $\overline{(x_1, x_2, \dots, x_n)}$ be defined by $(\overline{x_1}, \overline{x_2}, \dots, \overline{x_n})$

Notice $Col(A^*, j) = \overline{Row(A, j)}$

From $AA^* = I$, we know that $\langle Row(A, i), Col(A^*, j) \rangle = 0$, if $i \neq j$, and that $\langle Row(A, i), Col(A^*, j) \rangle = 1$, if $i = j$

So $\forall i \neq j$, $\langle Row(A, i), Row(A, j) \rangle = 0$, and $\forall i$, $\langle Row(A, i), Row(A, i) \rangle = 1$

This implies that the rows of A form an orthonormal basis for $\mathbb{C}^n$

Conversely, the rows of A form an orthonormal basis for $\mathbb{C}^n$ implies that $\forall i \neq j$, $\langle Row(A, i), Row(A, j) \rangle = 0$, and $\forall i$, $\langle Row(A, i), Row(A, i) \rangle = 1$

And this implies that $\langle Row(A, i), Col(A^*, j) \rangle = 0$, if $i \neq j$, and that $\langle Row(A, i), Col(A^*, j) \rangle = 1$, if $i = j$

This tell us $AA^* = I$

■

13.

13.(a)

Proof. Let $x \in S^\perp$

$\forall s \in S, x \perp s \implies \forall s \in S_0, x \perp s_0$

So $x \in S_0^\perp$ ■

13.(b)

Proof. Let $s \in S$

$\forall x \in S^{\perp}, x \perp s \implies s \in (S^{\perp})^{\perp}$

■

13.(c)

Proof. $W \subseteq (W^{\perp})^{\perp}$ by 13.(b)

Let $x \in (W^{\perp})^{\perp}$

Let $\{v_1, \dots\}$ be a basis of $W^{\perp}$

and $\{w_1, \dots, w_n\}$ be an orthogonal basis of W

Assume $x \notin W$

Write $x = \sum_{i=1}^{n} c_i w_i + \sum_J c_j v_j, \exists\{c_1, \dots, c_n\}, \{c_j \neq 0 | j \in J\}, J$

There must exists such non-empty J , otherwise $x \in W$

Arbitrarily pick k from J

$\langle x, v_k\rangle = \langle \sum_{i=1}^{n} c_i w_i + \sum_J c_j v_j, v_k\rangle = c_k\langle v_k, v_k\rangle \neq 0$ CaC to that $x \in (W^{\perp})^{\perp}$ ■

13.(d)

Proof. This is Theorem 6 ■

18.

Proof. We now prove $W_o \subseteq W_e^{\perp}$

Let $f \in W_o$

Let $g \in W_e$

fg is an odd function

So $\int_{-1}^{1} fg dt = 0$

That is $\langle f, g\rangle = 0$ (done)

We now prove $W_e^{\perp} \subseteq W_o$

Let $f \in W_e^{\perp}$

We know $1 \in W_e$

So we know $\langle f, 1 \rangle = 0$

That is $\int_{-1}^{1} f 1 dt = 0$

So f is an odd function, an element of W_o (done) ■

19.(b)

Proof. $(2, 1, 3) = (\frac{29}{14}, \frac{17}{14}, \frac{20}{7}) + (\frac{-1}{14}, \frac{-3}{14}, \frac{1}{7})$ ■

19.(c)

Proof. $x + \frac{13}{3}$ ■