Linear Algebra – Second Midterm
May 4, 2022

1. (10%) Answer true or false for the following statement. If true, explain or prove tour answer. If false, give an example to show that the statement is not always true.
   Statement: If U, V, W are subspaces of R^3 and if $U \perp V$ and $V \perp W$, then $U \perp W$.

2. (10%) Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be nonzero vectors in R^m and R^n, respectively, and let $A = \boldsymbol{x}\boldsymbol{y}^T$. Show that $\{\boldsymbol{x}\}$ is a basis for the column space of A and that $\{\boldsymbol{y}^T\}$ is a basis for the row space of A.

3. Let L be the linear operator on R^3 defined by
   $$L(\boldsymbol{x}) = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \end{bmatrix}$$
   and let $S = Span((1,0,1)^T)$.
   (a) (5%) Find the kernel of L.
   (b) (5%) Determine $L(S)$

4. (10%) Show that if $\boldsymbol{v}$ is orthogonal to both $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$, then $\boldsymbol{v}$ is orthogonal to $k_1\boldsymbol{w}_1 + k_2\boldsymbol{w}_2$ for all scalars k_1 and k_2.

5. (10%) Consider the matrix
   $$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

P is the transition matrix from the standard basis $S = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ to what basis B for R^3?

6. Let A be a 5 by 7 matrix with rank 4.
   (a) (5%) What is the dimension of the solution space of $A\boldsymbol{x} = \boldsymbol{0}$?
   (b) (5%) Is $A\boldsymbol{x} = \boldsymbol{b}$ consistent for all vectors $\boldsymbol{b}$ in R^5? Explain.

7. Suppose A is the sum of two matrices of rank one: $A = \boldsymbol{u}\boldsymbol{v}^T + \boldsymbol{w}\boldsymbol{z}^T$
   (a) (5%) Which vectors span the column space of A?
   (b) (5%) Which vectors span the row space of A?

8. (30%) True of false. You can get 5 points for each correct answer. However, you will be deducted 5 points for each wrong answer.

(a) All solution vectors of the linear system $A\boldsymbol{x} = \boldsymbol{b}$ are orthogonal to the row vectors of the matrix A if and only if $\boldsymbol{b} = \boldsymbol{0}$.

(b) If B_1 and B_2 are bases for a vector space V, then there exists a transition matrix from B_1 to B_2.

(c) There is an invertible matrix A and a singular matrix B such that the row spaces of A and B are the same.

(d) The nullity of a square matrix with linearly dependent rows is at least one.

(e) If $\boldsymbol{v}_0$ is a nonzero vector in V, then $T(\boldsymbol{v}) = \boldsymbol{v}_0 + \boldsymbol{v}$ defines a linear operator on V.

(f) If two matrices A and B are invertible and similar, then A^{-1} and B^{-1} are similar.