NCKU 112.1
Discrete Math

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Contents

Chapter 0

General Counting Methods for Arrangements and Selection

0.1 Practical Identity

Theorem 0.1.1. (Fundamental Identity) We have

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \tag{1}$$

Proof. Notice that $\binom{n}{k}$ represent amount of ways to pick k numbers from the set $\{1, \ldots, n\}$. There are two possibilities:

$$\begin{cases} 1 \text{ is picked} \\ 1 \text{ is not picked} \end{cases} \tag{2}$$

The amount of ways to pick k numbers from the set $\{1, \ldots, n\}$ when 1 is mandatory to pick is $\binom{n-1}{k-1}$, and the amount of ways when 1 is mandatory not to pick is $\binom{n-1}{k}$ ∎

The above identity is the most important in the sense that it allow us to deduce other identities with induction.

Theorem 0.1.2. (First Identity) We have

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$$

Proof. Consider possibility of picking subsets $I \subseteq \{1, \ldots, n\}$ of cardinality k and subsets $I_1 \subseteq I$ of cardinality m. The amount of possibilities equals to $\binom{n}{k}\binom{k}{m}$.

We can first pick the subset I_1 which has $\binom{n}{m}$ possibilities. We can then pick the subset I by picking $k-m$ amount of numbers in $\{1,\ldots,n\} \setminus I_1$ and add I_1 to have I. There are $\binom{n-m}{k-m}$ ways to do such, as $|\{1,\ldots,n\} \setminus I_1| = n-m$. ■

Theorem 0.1.3. (Second Identity) We have

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof. We have

$$\begin{aligned} 2^n &= (1+1)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} 1^k 1^n \\ &= \sum_{k=0}^{n} \binom{n}{k} \end{aligned}$$

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The first identity rely on the usage of intuition of picking r objects from n distinct objects. The second identity use binomial theorem.

Theorem 0.1.4. (Identity When Both Arguments are Growing) We have

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$$

Proof. Base case: $r = 0$

$$\begin{aligned} \sum_{k=0}^{r} \binom{n+k}{k} &= \sum_{k=0}^{0} \binom{n}{0} \\ &= 1 \\ &= \binom{n+1}{0} \\ &= \binom{n+r+1}{r} \end{aligned}$$

Induction case: suppose

$$\sum_{k=0}^{s} \binom{n+k}{k} = \binom{n+s+1}{s}$$

Observe

$$\begin{aligned}\binom{n+s+2}{s+1} &= \binom{n+s+1}{s+1} + \binom{n+s+1}{s} \\ &= \sum_{k=0}^{s} \binom{n+k}{k} + \binom{n+s+1}{s+1} \\ &= \sum_{k=0}^{s+1} \binom{n+k}{k}\end{aligned}$$

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Corollary 0.1.5. (Putting at most r different things in n barrels) We have

$$\sum_{k=0}^{r} H_k^n = \sum_{k=0}^{r} \binom{(n-1)+k}{k} = \binom{n+r}{r} = H_r^{n+1}$$

Theorem 0.1.6. (Identity When Only The Larger Argument is Growing) We have

$$\sum_{k=0}^{n-r} \binom{r+k}{r} = \binom{n+1}{r+1}$$

Proof. Base case: $n = r$

$$\sum_{k=0}^{n-r} \binom{r+k}{r} = \binom{r}{r} = 1 = \binom{r+1}{r+1} = \binom{n+1}{r+1}$$

Induction case: Suppose

$$\sum_{k=0}^{n-s} \binom{s+k}{s} = \binom{n+1}{s+1}$$

Observe

$$
\begin{aligned}
\binom{n+2}{s+1} &= \binom{n+1}{s+1} + \binom{n+1}{s} \\
&= \sum_{k=0}^{n-s} \binom{s+k}{s} + \binom{n+1}{s} \\
&= \sum_{k=0}^{n+1-s} \binom{s+k}{s}
\end{aligned}
$$

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The above two identities can be applied when the arguments are growing, notice that the second identity is an identity when the smaller argument is growing.

Theorem 0.1.7. (Fifth Identity) We have

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Proof. Imagine an $n \times n$ grid. Imagine we wish to go from the lower left corner to top right corner without going left or low.

The amount of path that cross $(0, n)$ is $\binom{n}{0}^2$ and the amount of path that cross $(1, n-1)$ is $\binom{n}{1}^2, \ldots$. ■

Theorem 0.1.8. (Identity When Multiplying Two Binomial where the Sum of the Smaller Arguments is Fixed)

$$\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}$$

Proof. The right hand side is to pick r objects from $m+n$ distinct object. The left hand side is also the same, but done so by first picking from m then from n ■

Corollary 0.1.9. (identity When Multiplying Two Binomials where the Gap of the Smaller Arguments is Fixed)

$$\sum_{k=0}^{m} \binom{m}{k} \binom{n}{r+k} = \binom{m+n}{m+r}$$

Proof.

$$\begin{aligned}
\sum_{k=0}^{m} \binom{m}{k}\binom{n}{r+k} &= \sum_{k=0}^{m} \binom{m}{m-k}\binom{n}{r+k} \\
&= \sum_{u=0}^{m} \binom{m}{u}\binom{n}{m+r-u} \text{ where } u = m-k \\
&= \binom{m+n}{m+r}
\end{aligned}$$

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Chapter 1

Generating Function

1.1 Modeling of Generating Function

Chapter 6 has 3 question, 6.4 has 1 question.

Theorem 1.1.1. (Putting Same Object into Distinct Barrels) Given

$$\sum_{i=1}^{n} e_i = r$$

There are

$$H_r^n := \binom{n+r-1}{n-1} = \binom{n+r-1}{r} \text{ amount of solutions}$$

1.2 Calculation of Generating Function

Theorem 1.2.1. (Important Identity) We have

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$$\frac{1}{(1-x)^n} = (\sum_{k=0}^{\infty} x^k)^n = \sum_{k=0}^{\infty} H_k^n x^k = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$$

1.3 Exponential Generating Function

Theorem 1.3.1. **(r Arrangement with and without Repetition of n objects)**

This means the numbers of ways of selecting r object from distinct n object and rearrange then

Theorem 1.3.2. **(Taylor Expansion)**

Chapter 2

HW

2.1 HW3

Question 1

How many numbers between 0 and 10000 have a sum of digit

(a) Equal to 7?

(b) Less than or equal to 7?

(c) Equal to 13

Proof. Represent the number by

$$e_0e_1e_2e_3e_4$$

where if the number is 237, we have

$$e_0 = 0 \text{ and } e_1 = 0 \text{ and } e_2 = 2 \text{ and } e_3 = 3 \text{ and } e_4 = 7$$

And of course we have the constraint

$$0 \leq e_1, e_2, e_3, e_4 \leq 9 \text{ and } e_0 \in \{0, 1\} \text{ and } e_0 = 1 \implies e_1 = e_2 = e_3 = e_4 = 0$$

Then the first question is equivalent to asking the amount of solution of

$$e_0 + e_1 + e_2 + e_3 + e_4 = 7$$

If $e_1 = 1$, clearly there is no solution. If $e_0 = 0$, there exists $H_7^4 = \binom{10}{7} = 120$ amount of solutions.

The second question is equivalent to asking the amount of solution of

$$e_0 + e_1 + e_2 + e_3 + e_4 \leq 7$$

If $e_0 = 1$, clearly there exists only one solution. If $e_0 = 0$, there exists

$$\sum_{k=0}^{7} H_k^4 = \sum_{k=0}^{7} \binom{k+3}{k} = \binom{11}{7} = 330$$

So

$$\text{The amount of solutions are } 331$$

The third question is equivalent to asking the amount of solution of

$$e_0 + e_1 + e_2 + e_3 + e_4 = 13$$

Clearly, we can not have $e_0 = 1$, so our question has become

$$e_1 + e_2 + e_3 + e_4 = 13$$

where the constrain is

$$0 \leq e_1, e_2, e_3, e_4 \leq 9$$

If we remove the 9 upper bound constrain, the amount of solutions is then

$$H_{13}^4 = 560$$

Adding the constrain back, we need to remove those solutions that doesn't satisfy the constrain, i.e. $e_j > 9$ for some $j \in \{1, 2, 3, 4\}$.

Clearly if $e_j > 9$, then no other digit would be greater than 9. The amount of solutions that should be removed are

$$4(H_{13-10}^3 + H_{13-11}^3 + H_{13-12}^3 + H_{13-13}^3) = 4(\sum_{k=0}^{3} H_k^3) = 4(\sum_{k=0}^{3} \binom{k+2}{k}) = 4\binom{6}{3} = 80$$

Then the amount of solutions are

$$560 - 80 = 480 \text{ ways}$$

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Question 2

Evaluate

$$\sum_{k=1}^{n} \binom{n}{k} \binom{n}{k-1}$$

Proof.

$$\begin{aligned}
\sum_{k=1}^{n} \binom{n}{k} \binom{n}{k-1} &= \sum_{k=1}^{n} \binom{n+1}{k} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} - \binom{n+1}{0} - \binom{n+1}{n+1} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} - 2 \\
&= 2^{n+1} - 2
\end{aligned}$$

∎

Question 3

Show that the generating function for the number of integer solutions to

$$e_1 + e_2 + e_3 + e_4 = r, 0 \leq e_1 \leq e_2 \leq e_3 \leq e_4$$

is

$$\left(1 + x + x^2 + \cdots\right)\left(1 + x^2 + x^4 + \cdots\right)\left(1 + x^3 + x^6 + \cdots\right)\left(1 + x^4 + x^8 + \cdots\right)$$

Proof. Define

$$d_1 := e_2 - e_1 \text{ and } d_2 := e_3 - e_2 \text{ and } d_3 := e_4 - e_3$$

Then we have

$$e_2 = e_1 + d_1 \text{ and } e_3 = e_1 + d_1 + d_2 \text{ and } e_4 = e_1 + d_1 + d_2 + d_3$$

The question is thus reduced to finding the generating function for

$$4e_1 + 3d_1 + 2d_2 + d_3 = r, \{e_1, d_1, d_2, d_3\} \in \mathbb{N} \cup \{0\}$$

Which is

$$\left(1 + x^4 + x^8 + \cdots\right)\left(1 + x^3 + x^6 + \cdots\right)\left(1 + x^2 + x^4 + \cdots\right)\left(1 + x + x^2 + \cdots\right)$$

∎

Question 4

Use the equation

$$\frac{(1-x^2)^n}{(1-x)^n} = (1+x)^n$$

to show that

$$\sum_{k=0}^{\frac{m}{2}} (-1)^k \binom{n}{k}\binom{n+m-2k-1}{n-1} = \binom{n}{m} \quad m \leq n \text{ and } m \text{ even}$$

Proof. Observe that

$$\binom{n}{m} \text{ is the coefficient of } x^m \text{ in } (1+x)^n$$

So we only have to prove that

$$\sum_{k=0}^{\frac{m}{2}} (-1)^k \binom{n}{k}\binom{n+m-2k-1}{n-1} \text{ is the coefficient of } x^m \text{ in } \frac{(1-x^2)^n}{(1-x)^n}$$

Observe that

$$(1-x^2)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{2k}$$

and that

$$\frac{1}{(1-x)^n} = \sum_{u=0}^{\infty} \binom{u+n-1}{u} x^u$$

Then because m is even, we can compute the coefficient by summing $k \in \left[0, \frac{m}{2}\right]$ and $u = m - 2k$ from $(2k + u = m)$, which tell us that the coefficient is

$$\sum_{k=0}^{\frac{m}{2}} (-1)^k \binom{n}{k}\binom{m-2k+n-1}{m-2k}$$

which equals to

$$\sum_{k=0}^{\frac{m}{2}} (-1)^k \binom{n}{k}\binom{n+m-2k-1}{n-1}$$

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Question 5

Show that

$$2\left(1-x\right)^{-3}\left[\left(1-x\right)^{-3}+\left(1+x\right)^{-3}\right]$$

is the generating function for the number of ways to toss r identical dice and obtain even sum.

Proof. The generating function of ways to toss r identical dice is

$$\left(\sum_{k=1}^{6} x^k\right)^r$$

which equals to

$$\left(\frac{1-x^7}{1-x}\right)^r$$

Then the generating function for the number of ways to toss r identical dice and obtain even sum is

$$\frac{1}{2}\left[\left(\frac{1-x^7}{1-x}\right)^r-\left(\frac{1+x^7}{1+x}\right)^r\right]$$

■