3.6 PDE HW 3

Question 37

Find a solution of

$$\begin{cases} u_t = u_{xx} \\ u(x,0) = x^2 \end{cases}$$

Proof. Clearly $u = x^2 + 2t$ suffices. ■

Question 38

Consider the ODE

$$\begin{cases} u'' + u = 0 \\ u(0) = 0 \text{ and } u(L) = 0 \end{cases}$$

Is the solution unique? Does the answer depend on L?

Proof. We know the general solution space is exactly spanned by $\cos x$ and $\sin x$. Because

(a) $u(0) = 0$.

(b) $\sin 0 = 0$

(c) $\cos 0 = 1$

we know the solution of our original ODE must be of the form

$$u(x) = C \sin x$$

This implies that the solution is unique if and only if $2\pi \not\equiv L \pmod{2\pi}$ ■

Question 39

Find the regions in the xy plane where the equation

$$(1 + x)u_{xx} + 2xyu_{xy} - y^2 u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch them.

Proof. The discriminant is exactly

$$\begin{aligned}(xy)^2-(1+x)(-y^2)&=x^2y^2+xy^2+y^2\\&=y^2(x^2+x+1)\\&=y^2[(x+\frac{1}{2})^2+\frac{3}{4}]\end{aligned}$$

It then follows that the equation is parabolic if and only if $y=0$, and elliptic if and only if $y\neq 0$.

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