Chapter 1

General Topology

1.1 Equivalent Characterizations of basic notions

Abstract

This section give a compact and comprehensive development of some of the most basic notions in the study of topology. In this section, $(X, \mathscr{T})$ is a topological space.

Given an arbitrary subset $E \subseteq X$, we

(a) say $x \in X$ is a **limit point of** E if every open O containing x contain a point $y \in E$ such that $y \neq x$.

(b) say $x \in E$ is an **interior point of** E if there exists $O \in \mathscr{T}$ such that $x \in O \subseteq E$.

(c) define the **interior** E° **of** E to be the union of all open sets contained by E.

(d) say $E \subseteq X$ is a **closed set** if $E^c \in \mathscr{T}$.

(e) define the **closure** $\overline{E}$ **of** E by $\overline{E} \triangleq E \cup E'$ where E' is the set of limit points of E.

(f) define the **boundary** ∂E **of** E by $\partial E \triangleq \overline{E} \setminus E^\circ$

Theorem 1.1.1. (Equivalent Definitions of Interior) The following sets are equivalent

(a) E°

(b) The largest open set contained by E.

(c) The set of interior points of E.

Proof. Check straightforward. ■

Theorem 1.1.2. (Equivalent Definitions of Closed) The following are equivalent.

(a) E is closed.

(b) the set of limit points of E is contained by E.

(c) $\overline{E} = E$.

Proof. The proof of (a) $\implies$ (b) $\implies$ (c) are straight forward. The proof of (c) $\implies$ (a) follows from first noting no $x \in E^c$ is a limit point of E. Then shows $E^c = \bigcup_{x \notin E} O_x$ where O_x is an open set containing x and disjoint with E. ■

Theorem 1.1.3. (Equivalent Definitions of Closure) The following sets are equivalent.

(a) $\overline{E}$

(b) $((E^c)^\circ)^c$

(c) The smallest closed set containing E.

(d) $\{x \in X : \text{ every open } O \text{ containing } x \text{ intersect with } E\ \}$

Proof. It is straightforward to check $\overline{E}$ is closed. For each closed F containing E, it is straightforward to check $E' \subseteq F' \subseteq F$. This established (a) = (c). It is straightforward to check $(\overline{E})^c = (E^c)^\circ$ using the largest open set and the smallest closed set characterization of interior and closure. This established (b) = (c). It is straightforward to check (a) = (d). ■

Theorem 1.1.4. (Equivalent Definitions of Boundary) The following sets are equivalent.

(a) ∂E

(b) $\overline{E} \cap \overline{E^c}$

(c) $\{x \in X : \text{ every open } O \text{ containing } x \text{ intersect with both } E \text{ and } E^c\ \}$

Proof. It is straightforward to check $\partial E = \overline{E} \cap \overline{E^c}$ using $(E^\circ)^c = \overline{E^c}$. This established (a) = (b). It is straightforward to check (b) = (c). ■

We now develop some concepts one may not used in the study of metric space. Given a collection $\mathcal{B} \subseteq \mathscr{T}$ of open sets, we say $\mathcal{B}$ is a

(a) **basis** of $\mathscr{T}$ if for each $O \in \mathscr{T}$, there exists a subcollection $\mathcal{B}_O \subseteq \mathcal{B}$ such that $O = \bigcup \mathcal{B}_O$.

(b) **subbase** of $\mathscr{T}$ if $\mathcal{B}$ cover X and $\mathscr{T}$ is the collection of unions of finite intersections of element of $\mathcal{B}$. In a more formal language, $\mathcal{B}$ has to satisfy $\bigcup \mathcal{B} = X$ and $\mathscr{T} =$

$\left\{\bigcup E : E \subseteq \mathcal{A}\right\}$ where $\mathcal{A} = \left\{\bigcap \mathcal{S} : \mathcal{S} \subseteq \mathcal{B}, |\mathcal{S}| \in \mathbb{N}\right\}$.

Theorem 1.1.5. (Equivalent Definition of Basis) The following are equivalent.

(a) $\mathcal{B}$ is a basis.

(b) For all $O \in \mathscr{T}$ and $x \in O$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq O$.

Proof. Check straightforward. ■

Theorem 1.1.6. (Equivalent Definitions of Subbase) The following are equivalent

(a) $\mathcal{B}$ is a subbase of $\mathscr{T}$.

(b) $\mathcal{B}$ cover X and $\mathscr{T}$ is the smallest topology containing $\mathcal{B}$.

Proof. Check straightforward. ■

Immediately, with the equivalent definitions, one should check

(a) Given any cover $\mathcal{B}$ of X, there always exists a unique topology $\mathscr{T}$ containing $\mathcal{B}$ as a subbase. We say $\mathscr{T}$ is the **topology generated by $\mathcal{B}$**.

(b) The set $\mathcal{A} \triangleq \left\{\bigcap \mathcal{S} : \mathcal{S} \subseteq \mathcal{B}, |\mathcal{S}| \in \mathbb{N}\right\}$ of finite intersections of cover $\mathcal{B}$ is a basis of the topology generated by $\mathcal{B}$.

(c) Not every cover $\mathcal{B}$ of X has some topology $\mathscr{T}$ containing $\mathcal{B}$ as a basis. Consider $\mathcal{B} \triangleq \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(b, \infty) : b \in \mathbb{R}\}$. Even the smallest topology containing $\mathcal{B}$, i.e. the standard topology, does not have $\mathcal{B}$ as a basis.

(d) However, cover $\mathcal{B}$ is the basis of the topology $\mathscr{T}$ generated by itself if for all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$.

(e) If $\mathcal{B}$ is a basis of $\mathscr{T}$, then $\mathcal{B}$ is also a subbase of $\mathscr{T}$.

(f) Basis is not necessarily closed under finite intersection. Consider the basis $\{(a, a + \frac{1}{n}) : a \in \mathbb{R}, n \in \mathbb{N}\}$ for $\mathbb{R}$'s standard topology.

Note that in (a), to check the generated $\mathscr{T}$ is indeed a topology, one may need to utilize the identity

$$\left(\bigcup_{i \in I} A_i\right) \cap \left(\bigcup_{j \in J} B_j\right) = \bigcup_{i \in I, j \in J} A_i \cap B_j$$

We now develop the theory of continuity by first giving a pointwise definition. Given another topological space $(Y, \mathscr{S})$ and a function $f : X \to Y$, we say f is **continuous at** $x \in X$ if for all open O containing $f(x)$, there exists open E containing x such that $f(E) \subseteq O$. We say f is a **continuous (or** $(\mathscr{T}, \mathscr{S})$**-continuous, if necessary) function** if f is continuous at all $x \in X$.

It is easy to see the composition of two continuous function must be continuous. However, one should notice that the composition of a continuous function and a discontinuous function can be continuous. Just let one of them be a constant function.

Theorem 1.1.7. (Equivalent Definitions of Continuous function) The following are equivalent

(a) f is continuous.

(b) $f^{-1}(O) \in \mathscr{T}$ for all $O \in \mathscr{S}$.

(c) $f^{-1}(F)$ is closed for all closed F in Y.

(d) For all $B \subseteq Y$, $f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$

(e) For all $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.

(f) For all $B \subseteq Y, \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$

(g) For all subbase $\mathcal{B}$ of Y, $f^{-1}(B) \in \mathscr{T}$ for all $B \in \mathcal{B}$.

Proof. It is straightforward to check (a) $\implies$ (b) $\implies$ (c). Fix $x \in X$ and $O \in \mathscr{S}$ containing $f(x)$. Check $x \in (f^{-1}(O^c))^c$ and $f((f^{-1}(O^c))^c) \subseteq O$. This give us (c) $\implies$ (a). Because, respectively, $f^{-1}(B^\circ) \subseteq f^{-1}(B), A \subseteq f^{-1}(\overline{f(A)})$ and $f^{-1}(B) \subseteq f^{-1}(\overline{B})$, we have, respectively, (b) $\implies$ (d), (c) $\implies$ (e) and (c) $\implies$ (f). It is straightforward to check (d) $\implies$ (b). To check (e) $\implies$ (f), let $A = f^{-1}(B)$. (f) $\implies$ (c) is straightforward. Finally, (b) $\iff$ (g) is straightforward. ■

One may wonder: Why isn't "For all $A \subseteq X, f(A)^\circ \subseteq f(A^\circ)$" a characterization of f being continuous? Consider function that maps some topological space with some subset that has empty interior into the topological space Y having only a single point.

Theorem 1.1.8. (Equivalent Definition of Finer/Coarser) Given another topology $\mathscr{T}'$ on X, the following are equivalent.

(a) $\mathscr{T} \subseteq \mathscr{T}'$

(b) $\mathbf{id} : (X, \mathscr{T}') \to (X, \mathscr{T})$ is continuous.

(c) Given any basis $\mathcal{B}$ of $\mathscr{T}$ and any basis $\mathcal{B}'$ of $\mathscr{T}'$, for all $x \in X$ and basic open $B \in \mathcal{B}$ containing x, there exists basic open $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$

(d) There exists subbase $\mathcal{B}, \mathcal{B}'$ of $\mathscr{T}, \mathscr{T}'$ such that $\mathcal{B} \subseteq \mathcal{B}'$

Proof. (d) $\iff$ (a) $\iff$ (b) is straightforward. From (a) to (c), one find B' by noting $B \in \mathscr{T}'$ and utilizing the definition of basis. From (c) to (a), one show $O \in \mathscr{T}$ belongs to $\mathscr{T}'$ by taking $O = \bigcup_{x \in O} B'_x$. ∎

Now, given a collection $(X_\alpha, \mathscr{T}_\alpha)_{\alpha \in J}$ of topological space, we define the **product topology** on $X \triangleq \prod_{\alpha \in J} X_\alpha$ to be the smallest topology such that for all $\alpha \in J$, the projection $\pi_\alpha : X \to X_\alpha$ that maps $(y_\alpha)_{\alpha \in J}$ to y_α is continuous.

Theorem 1.1.9. (Equivalent Definition of Product Topology) The following topologies are equivalent.

(a) The product topology on X

(b) The topology on X generated by basis $\left(\prod U_\alpha \right)_{U_\alpha \neq X_\alpha \text{ for finitely many } \alpha \text{ and } U_\alpha \in \mathscr{T}_\alpha}$

(c) The smallest topology on X satisfying the statement: For all topological space $(Z, \mathscr{T}_Z)$, a function $f : Z \to X$ is continuous if and only if for all $\alpha \in J$, the function $f \circ \pi_\alpha : Z \to X_\alpha$ is continuous.

Proof. By definition, we know X has the subbase $\bigcup_{\alpha \in J} \mathcal{G}_\alpha$ where $\mathcal{G}_\alpha \triangleq \{\pi_\alpha^{-1}(U) : U \in \mathscr{T}_\alpha\}$. This give (a) = (b). For (a) = (c), one have to prove that: Given a topology on X, the following two statements are equivalent.

(i) All projection π_α are continuous.

(ii) For all topological space $(Z, \mathscr{T}_Z)$ and function $f : Z \to X$, the function $f : Z \to X$ is continuous if and only if for all $\alpha \in J$, the function $f \circ \pi_\alpha : Z \to X_\alpha$ is continuous.

Take $Z \triangleq X$ and $f \triangleq \mathbf{id}$ for (ii) $\implies$ (i). For (i) $\implies$ (ii), if f is continuous, then $f \circ \pi_\alpha$ are clearly all continuous, and conversely, if $f \circ \pi_\alpha$ are all continuous, use the pointwise definition of continuity and the basis given in (b) to show f is continuous. ∎

Immediately, one should check

(a) In X, a sequence p_n converge to p if and only if $\pi_\alpha(p_n)$ converge to $\pi_\alpha(p)$ for all $\alpha \in J$.

(b) If $f : X \times Y \to Z$ is continuous, then for all $x \in X$, the function $f_{(x,\cdot)} : Y \to Z$ defined by $f_{(x,\cdot)}(y) \triangleq f(x, y)$ is continuous. The converse is not true thought, in the sense that f can be discontinuous even if $f_{(x,\cdot)}, f_{(\cdot,y)}$ are all continuous. Elementary counterexample can be constructed in Euclidean Space.

(c) The third characterization of product topology shows that product is independent of expression. For example, given an enumeration $(X_\alpha)_{\alpha\leq\gamma}$ of topological space, $X_1 \times \prod_{1<\alpha\leq\gamma} X_\alpha$ is homeomorphic with $\prod_{1\leq\alpha\leq\gamma} X_\alpha$.

Because of the first property, some people also call the product topology "topology of pointwise convergence".

We will later introduce an "inferior" alternative for assigning topology to Cartesian product of topological spaces. Now, given a topological space $(X, \mathscr{T})$ and a subset $E \subseteq X$, we define the **subspace topology** $\mathscr{T}_E$ on E by $\mathscr{T}_E \triangleq \{O \cap E : O \in \mathscr{T}\}$. Immediately, one can check $\mathscr{T}_E$ is indeed a topology and

(a) Given a subset $F \subseteq E$, viewing F as a subspace of E or X make no difference.

(b) A set $F \subseteq E$ open in X remains open in $(E, \mathscr{T}_E)$.

(c) The collection of closed sets in $(E, \mathscr{T}_E)$ are $\{F \cap E : F \text{ is a closed set of } (X, \mathscr{T}_X)\}$.

(d) For all $F \subseteq X$, $\mathrm{cl}_E(F \cap E) \subseteq \mathrm{cl}_X(F) \cap E$. The equality hold when $F \subseteq E$.

(e) Given a function $f : X \to Y$ and $p \in E \subseteq X$, f is continuous at p implies $f|_E : E \to Y$ is continuous at p. The converse is true only when E is open in X.

(f) Given a function $f : X \to Y$ and a subset F of Y containing $f(X)$, $f : X \to Y$ is continuous at p if and only if $f : X \to F$ is continuous at p.

(g) $\mathscr{T}_X = \mathscr{T}$

Theorem 1.1.10. (Equivalent Definition of Subspace Topology) Given a basis $\mathcal{B}$ a subbase $\mathcal{B}'$ of $\mathscr{T}$, the following sets are equivalent.

(a) $\mathscr{T}_E$

(b) The topology on E generated by basis $\mathcal{B}_E \triangleq \{B \cap E : B \in \mathcal{B}\}$

(c) The topology on E generated by subbase $\mathcal{B}'_E \triangleq \{B' \cap E : B' \in \mathcal{B}'\}$

(d) The smallest topology on E such that the inclusion map $\iota : E \to X$ is continuous.

Proof. Check straightforward. ∎

At this point, one should check the compatibility between the definitions of subspace topology and product topology. Given a collection $(X_\alpha)_{\alpha\in J}$ of topological spaces and a subspace $(A_\alpha)_{\alpha\in J}$ of each X_α, one can view $A \triangleq \prod A_\alpha$ either as a subspace of the product $X \triangleq \prod X_\alpha$ or the product of subspaces $(A_\alpha)_{\alpha\in J}$. The two topologies are identical, and the proof goes like

(a) Showing the product topology have the subbase $(\pi_{\alpha,A}^{-1}(U_\alpha))_{\alpha\in J, U_\alpha\in\mathscr{T}_{A_\alpha}}$ where $\pi_{\alpha,A} : A \to A_\alpha$ is the projection mapping.

(b) Showing the subspace topology have the subbase $(\pi_{\alpha,X}^{-1}(U_\alpha)\cap A)_{\alpha\in J, U_\alpha\in\mathscr{T}_{X_\alpha}}$ where $\pi_{\alpha,X} : X \to X_\alpha$ where $\pi_{\alpha,X} : X \to X_\alpha$ is the projection mapping.

(c) Showing $\{\pi_{\alpha,A}^{-1}(U_\alpha) \subseteq A : \alpha\in J, U_\alpha\in\mathscr{T}_{A_\alpha}\} = \{\pi_{\alpha,X}^{-1}(U_\alpha)\cap A : \alpha\in J, U_\alpha\in\mathscr{T}_{X_\alpha}\}$

We now give definition to three notions so important that drive us to study Topology in the first place. Given a topological space $(X,\mathscr{T})$, we say nonempty $E\subseteq X$ is **connected** if E can not be written as $E = A\cup B$ so that $\overline{A}\cap B = A\cap\overline{B} = \varnothing$ and $A\neq\varnothing\neq B$. We say $E\subseteq X$ is **path-connected** if for each $p,q\in E$, there exists a continuous function $f:[0,1]\to E$ such that $f(0)=p, f(1)=q$. We say $E\subseteq X$ is **compact** if every open cover has a finite subcover.

The three properties are often called **topological properties**, since they are invariant under continuous function, the "morphism" between topological space. Put more precisely, If $E\subseteq X$ satisfy a topological property and $f:X\to Y$ is continuous, then $f(E)$ also satisfy the topological property.

Immediately, one should again check the "natural" behaviors of subspace topology: Whether a set E is connected, path-connected or compact is independent of the choices of ambient space. In other words, given $E\subseteq X$, E is connected, path-connected or compact in $(X,\mathscr{T})$ if and only if E is connected, path-connected or compact in $(E,\mathscr{T}_E)$.

Theorem 1.1.11. (Equivalent Definitions of Connected) The following statements are equivalent.

(a) E is connected in $(X,\mathscr{T})$.

(b) E is connected in $(E,\mathscr{T}_E)$.

(c) The only clopen sets in $(E,\mathscr{T}_E)$ are E and $\varnothing$.

(d) In $(E,\mathscr{T}_E)$, the only set that has empty boundary are E and $\varnothing$.

(e) All continuous function from $(E,\mathscr{T}_E)$ to $\{0,1\}$ with discrete topology is constant.

Proof. For (a) $\iff$ (b), use the identity $\forall A \subseteq E, \mathrm{cl}_X(A) \cap E = \mathrm{cl}_E(A)$. Check straightforward for (b) $\iff$ (c) and (d) $\iff$ (c) $\iff$ (e). ■

Three things to note here

(a) If $E \subseteq X$ is connected, E can not be covered by any two disjoint open sets intersecting with E in $(X, \mathscr{T}_X)$. The converse is not true. Consider finite subset of an infinite set with cofinite topology.

(b) Union of collection $(A_\alpha)_{\alpha \in J}$ of connected sets with non-empty intersection is connected. Prove this by a proof of contradiction. Path-connectedness has the same property, and the proof is much easier.

(c) If E is connected and $E \subseteq F \subseteq \overline{E}$, then F is connected. Use the fifth equivalent definitions for connectedness to prove this. Path-connectedness doesn't have the same property this time. Consider the "fattened" Topologist's sine curve $\{(x, y) \in \mathbb{R}^2 : \left|y - \sin \frac{1}{x}\right| < x\}$. It's closure can be easily proved to be $\{(x, y) \in \mathbb{R}^2 : \left|y - \sin \frac{1}{x}\right| \leq x\} \cup \{(0, y) \in \mathbb{R}^2 : y \in [-1, 1]\}$.

(d) Path-connectedness is strictly stronger than connectedness. This can be proved using a proof of contradiction and supremum. Two famous counterexamples of the converse are Topologist's sine curve and Long line.

Theorem 1.1.12. (Equivalent Definitions of Compact) The following statements are equivalent.

(a) E is compact in $(X, \mathscr{T})$.

(b) E is compact in $(E, \mathscr{T}_E)$.

(c) Given subbase $\mathcal{B}$ of $(X, \mathscr{T})$, every cover of E consisting of the elements of $\mathcal{B}$ has a finite subcover.

(d) Every infinite subset M of E has a complete limit point in E, that is, a point $x \in E$ such that all open set O containing x satisfy $|O \cap M| = |M|$.

(e) Every collection of closed sets of $(E, \mathscr{T}_E)$ that has finite intersection property has non-empty intersection.

(f) For all topological space Y, the projection $\pi_Y : E \times Y \to Y$ is a closed mapping.

Proof. For (b) $\iff$ (e), use proofs by contradiction. (b) $\iff$ (a) $\implies$ (c) is clear. We now prove

$$(c) \implies (b)$$

Fix F and $\mathcal{B}$. Assume E is not compact. Then the collection $\mathbb{S}$ of all open cover that has no finite subcover is non-empty. Let $\mathcal{C}$ be a maximal element of $\mathbb{S}$. It is clear that $\mathcal{C} \cap \mathcal{B}$ is not a cover of E by premise. Let $x \in E \setminus \bigcup(\mathcal{C} \cap \mathcal{B})$. Let U be an element of $\mathcal{C} \setminus \mathcal{B}$ containing x. Because $\mathcal{B}$ is a subbase, we know there exists finite $B_1, \dots, B_n \in \mathcal{B}$ such that $x \in B_1 \cap \cdots \cap B_n \subseteq U$. Because $\mathcal{C}$ is a maximal element of $\mathbb{S}$, we know $\mathcal{C} \cup \{B_j\}$ does not belong to $\mathbb{S}$. This implies that for each $j \in \{1, \dots, n\}$, there exists a finite sub-collection $\mathcal{C}_j \subseteq \mathcal{C}$ such that $\mathcal{C}_j \cup \{B_j\}$ cover E. Let $\mathcal{C}_F \triangleq \bigcup_{j=1}^n \mathcal{C}_j$. Because $\mathcal{C}_j \cup \{B_j\}$ are covers of E, we know $\mathcal{C}_F \cup \{B_1, \dots, B_n\}$ is a cover of E. This implies $\mathcal{C}_F \cup \{U\} \subseteq \mathcal{C}$ is a finite subcover. CaC (done)

We now prove

$$(a) \implies (d)$$

Assume there exists infinite $M \subseteq E$ that has no complete limit point. By premise, for each $x \in E$, there exists an open set O_x containing x such that $|M \cap O_x| < |M|$. Because $(O_x)_{x \in E}$ is an open cover of E, there exists a finite sub-cover $(O_x)_{x \in I}$. Note that M is infinite, so we can deduce

$$|M| = \left| \bigcup_{x \in I} M \cap O_x \right| \leq \sum_{x \in I} |M \cap O_x| < |M| \quad \text{CaC} \quad \text{(done)}$$

We now prove

$$(d) \implies (a)$$

Assume E is not compact. Let $\mathcal{O}$ be an open cover of E that has no finite sub-cover with smallest cardinality c. Well order $\mathcal{O}$ by $\mathcal{O} \triangleq \{O_\alpha\}_{\alpha < c}$. Use transfinite recursion to build $M \triangleq \{x_\alpha : \alpha < c\}$ where $x_\alpha \in E \setminus \bigcup_{\beta < \alpha} O_\beta$. Such x_α always exists, otherwise there exists an open cover of E that has no finite sub-cover with cardinality smaller than c. To cause a contradiction, it remains to show

$$M \text{ has no complete limit point in } E$$

Because $\mathcal{O}$ is an open cover of E, for all x, there exists some O_α containing x. Observe using definition of M

$$|O_\alpha \cap M| \leq |\{x_\gamma : \gamma \leq \alpha\}| \leq |\alpha| < c = |M| \quad \text{CaC} \quad \text{(done)}$$

Before we prove (a) $\implies$ (f), we first prove the Generalized Tube Lemma. That is,

Given a product space $X \times Y$, compact $A \subseteq X$, compact $B \subseteq Y$ and $N \overset{\text{open}}{\subseteq} X \times Y$ containing $A \times B$, there exists $U \overset{\text{open}}{\subseteq} X, V \overset{\text{open}}{\subseteq} Y$ such that $A \times B \subseteq U \times V \subseteq N$.

First note that for all $(a,b) \in A \times B$, there exists $U_{(a,b)} \overset{\text{open}}{\subseteq} X$ and $V_{(a,b)} \overset{\text{open}}{\subseteq} Y$ such that $(a,b) \in U_{(a,b)} \times V_{(a,b)} \subseteq N$. Because A is compact and for all b, the collection $(U_{(a,b)})_{a\in A}$ is an open cover of A, there exists a finite subset $A_b \subseteq A$ for all b such that $A \subseteq \bigcup_{a\in A_b} U_{(a,b)}$. Now, let $U_b \triangleq \bigcup_{a\in A_b} U_{(a,b)}$ and $V_b \triangleq \bigcap_{a\in A_b} V_{(a,b)}$. It is clear that U_b, V_b are open, and it straightforward to check $A \times \{b\} \subseteq U_b \times V_b \subseteq N$. Again, because B is compact and $(V_b)_{b\in B}$ is an open cover of B, there exists an finite subset $B_0 \subseteq B$ such that $B \subseteq \bigcup_{b\in B_0} V_b$. Let $V \triangleq \bigcup_{b\in B_0} V_b$ and $U \triangleq \bigcap_{b\in B_0} U_b$. It is straightforward to check U, V suffice. (done)

We now prove

$$\text{(a)} \implies \text{(f)}$$

Given $A \overset{\text{close}}{\subseteq} X \times Y$, we are required to prove $\pi_Y(A)$ is closed. We should assume $\pi_Y(A) \neq Y$. Fix $y \in Y \setminus \pi_Y(A)$. Because $X, \{y\}$ are compact and $X \times \{y\}$ is a subset of the open set A^c, by Generalized Tube Lemma, there exists open $V \subseteq Y$ such that $X \times \{y\} \subseteq X \times V \subseteq A^c$. It is straightforward to check $V \cap \pi_Y(A) = \varnothing$. (done)

We now prove

$$\text{(f)} \implies \text{(a)}$$

Assume X is not compact. Let $(O_\alpha)_{\alpha\in J}$ be an open cover with no finite subcover. Check that

(a) $\mathcal{U} \triangleq \left\{ \bigcup_{\alpha\in I} O_\alpha : I \text{ is a finite subset of } J \right\}$ is an open cover with no finite subcover.

(b) $\mathcal{U}$ is closed under finite union.

(c) $\mathcal{F} \triangleq \{U^c : U \in \mathcal{U}\}$ is a collection of non-empty closed sets closed under finite intersection.

(d) If we let $Y \triangleq X \cup \{p\}$ where $p \notin X$, then $\mathscr{T}_Y \triangleq \mathcal{P}(X) \cup \{\{p\} \cup A : \exists F \in \mathcal{F}, F \subseteq A \subseteq X\}$ is a topology on Y, where $\mathcal{P}(X)$ is collection of all subsets of X.

(e) Let $C \triangleq \text{cl}_{X\times Y}\{(x,x) \in X \times Y : x \in X\}$.

(f) Fix $x \in X$. Because $\mathcal{U}$ is an open cover of X, there exists $U \in \mathcal{U}$ containing x. Note that $\{p\} \cup U^c$ is open in Y. This implies $U \times (\{p\} \cup U^c)$ is an open subset of $X \times Y$ containing (x,p). We have proved $C \subseteq X \times X$.

(g) It is clear that X is not closed in Y. Now observe that π_Y maps closed set C to the open set $X \subseteq Y$. CaC (done)

$\blacksquare$

In later sections, we will discuss more about compactness. Here, we give a general fact

(a) Closed subsets F of compact set E are compact. Prove this by noting every open cover of F adding F^c is an open cover of E.

At last, we come to fulfill our earlier promise of introducing the "inferior" alternative for assigning topology to Cartesian product of topological space. We define the **box topology** on $X \triangleq \prod_{\alpha \in J} X_\alpha$ to be the topology on X generated by basis $\left\{ \prod_{\alpha \in J} O_\alpha : O_\alpha \in \mathscr{T}_\alpha \right\}$. Immediately, one can see that the box topology is always larger than the product topology, and the two coincide when the product are finite.

Theorem 1.1.13. (Equivalent Definition of Box Topology) Suppose for each α, $\mathcal{B}_\alpha$ is a basis of $\mathscr{T}_\alpha$. The following topology on X are equivalent.

(a) The box topology.

(b) The topology generated by basis $\left\{ \prod_\alpha B_\alpha : B_\alpha \in \mathcal{B}_\alpha \right\}$.

Proof. Check straightforward. ∎

In mathematical texts, when one treat a product X of topological space as a topological space without specifying the topology on X, the topology referred are mostly often the product topology instead of the box topology. The reason we take this "convention" is because the product topology behave better, in the sense that product topology preserve more qualities we wish "product" to preserve than box topology, as we shall show.

Given a collection $(X_\alpha)_{\alpha \in J}$ of topological spaces, one can prove

$$\prod_{\alpha \in J} X_\alpha \text{ is connected} \iff \text{For all } \alpha \in J,\ X_\alpha \text{ is connected}$$

From right to left, one approach is to use transfinite induction, the fifth equivalent definition of connected and the (b) property of product topology. Left to right follows from projection being continuous.

Of course, one can also prove

$$\prod_{\alpha \in J} X_\alpha \text{ is path-connected} \iff \text{For all } \alpha \in J,\ X_\alpha \text{ is path-connected}$$

Right to left follows from the third equivalent definition of product topology. Left to right also follows from projection being continuous.

Most crucially, one can prove the **Tychonoff's Theorem**

$$\prod_{\alpha\in J} X_\alpha \text{ is compact} \iff \text{For all } \alpha \in J,\ X_\alpha \text{ is compact}$$

Right to left follows from a proof by contradiction using Alexander Subbase Theorem (The third equivalent definition for compact) and the fact $\{\pi_\alpha^{-1}(O_\alpha) : \alpha \in J, O_\alpha \in \mathscr{T}_\alpha\}$ is a subbase of $\prod_{\alpha\in J} X_\alpha$, and left to right follows, again, from projection being continuous.

Example 1 (Box topology does not preserve connectedness)

$$X \triangleq \mathbb{R}^\omega$$

Note that by $\mathbb{R}^\omega$, we meant the space of real sequences. Let A be the set of bounded sequence and B be the set of bounded sequence. Check straightforward that A, B is a separation of X. Note that $\mathbb{R}$ is path-connected, and $\mathbb{R}^\omega$ is not even connected.

Example 2 (Box topology does not preserve Compactness)

$$X \triangleq \{0,1\}^\omega$$

Note that by $\{0,1\}^\omega$, we meant the set of sequences consisting of element that are only 0 or 1, and each $\{0,1\}$ is equipped with discrete topology. Check straightforward that the box topology on X is identical to the discrete topology on X.

Example 3 (Box topology sucks in functional analysis)

$$X \triangleq \mathbb{R}^{[0,1]}$$

Note that by $\mathbb{R}^{[0,1]}$, we meant the space of function from $[0,1]$ to $\mathbb{R}$, and each $\mathbb{R}$ is equipped with the standard topology. In product topology, we have the amazing property

$$f_n \text{ converge to } f \text{ if and only if } f_n \text{ pointwise converge to } f$$

but in box topology, $f_n(t) \triangleq \frac{1}{n}$ does not even converge to $f(t) \triangleq 0$.