Friedberg Ch6.3

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In this note, V always stand for a vector space over $\mathbb{F}$, V^- stands for a finite dimensional vector space over $\mathbb{F}$, and T is always a linear operator on V^-

Definition and Theorem

Theorem 1. *Let $g : V^- \to \mathbb{F}$ be a linear transformation*

$$\exists! z \in V^-, \forall x \in V^-, \langle x, z\rangle = g(x)$$

Proof. Let $\beta = \{v_1, \dots v_n\}$ be an orthonormal basis of V^-

Let $z = \overline{g(v_1)}v_1 + \cdots + \overline{g(v_n)}v_n$

Then $\forall 1 \leq i \leq n, \langle v_i, z\rangle = \langle v_i, \overline{g(v_1)}v_1 + \cdots + \overline{g(v_n)}v_n\rangle = \sum_{j=1}^n g(v_j)\langle v_i, v_j\rangle = g(v_j)$

Let $z' \in V^-, \forall x \in V^-, \langle x, z'\rangle = g(x)$

$\langle z - z', z'\rangle = g(z - z') = \langle z - z', z\rangle \implies \langle z - z', z' - z\rangle = 0 \implies z - z' = 0 \implies z = z'$ ■

Definition 1. *The **adjoint** T^* of T is the linear operator satisfy*

$$\forall x, y \in V^-, \langle T^*(x), y\rangle = \langle x, T(y)\rangle$$

Theorem 2. *T^* is well defined and indeed linear*

Proof. Let $g_x : V^- \to \mathbb{F}$ be defined by $g_x(y) = \langle T(y), x\rangle$

We now prove g_x is linear

Let $y, y' \in V^-$ and $c \in \mathbb{F}$

$g_x(y+cy') = \langle T(y+cy'), x\rangle = \langle T(y)+cT(y'), x\rangle = \langle T(y), x\rangle + c\langle T(y'), x\rangle = g_x(y) + cg_x(y')$ (done)

We now prove T^* can be defined by at least one wey

For each $x \in V^-$, pick $z_x \in V^-$, so that $\forall y \in V^-, \langle y, z_x\rangle = g_x(y) = \langle T(y), x\rangle$

Define $T^*(x) = z_x$

Notice $\forall x, y \in V^-, \langle y, T^*(x)\rangle = \langle y, z_x\rangle = g_x(y) = \langle T(y), x\rangle$

This implies $\forall x, y \in V^-, \langle T^*(x), y\rangle = \langle x, T(y)\rangle$ (done)

We now prove This is the only way to define adjoint T^*

Assume there exists another adjoint $(T^*)'$ of T different from T^*

Pick $x \in V^-$, such that $(T^*)'(x) \neq T^*(x)$

Let $y = T^*(x) - (T^*)'(x)$

Notice $\langle T^*(x), y\rangle = \langle x, T(y)\rangle = \langle (T^*)'(x), y\rangle$

So $\langle T^*(x) - (T^*)'(x), T^*(x) - (T^*)'(x)\rangle = \langle T^*(x) - (T^*)'(x), y\rangle = 0$

Which give us $T^*(x) - (T^*)'(x) = 0$, so $T^*(x) = (T^*)'(x)$ CaC (done)

We now prove T^* is linear

Let $c \in \mathbb{F}$ and $u \in V^-$

$\forall y \in V^-, \langle y, z_{cx+u}\rangle = \langle T(y), cx+u\rangle = \overline{c}\langle T(y), x\rangle + \langle T(y), u\rangle = \overline{c}\langle y, z_x\rangle + \langle y, z_u\rangle = \langle y, cz_x + z_u\rangle$

This give us $z_{cx+u} = cz_x + z_u$

So $T^*(cx+u) = cT^*(x) + T^*(u)$ (done)

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Corollary 2.1. *The adjoint of a linear operator on an infinite vector space may not exist, but if it exists, it is unique and linear following the proof above*

Lemma 3. $\forall x, y \in V, \langle T^*(x), y\rangle = \langle x, T(y)\rangle$
$\forall x, y \in V, \langle T(x), y\rangle = \langle x, T^*(y)\rangle$

Proof. The first one follows the definition

$\langle T^*(y), x\rangle = \langle y, T(x)\rangle \implies \overline{\langle T^*(y), x\rangle} = \overline{\langle y, T(x)\rangle} \implies \langle x, T^*(y)\rangle = \langle T(x), y\rangle$

■

Theorem 4. *Let $\beta = \{v_1, \dots, v_n\}$ be an orthonormal basis of V*

$$[T^*]_\beta = ([T]_\beta)^*$$

Proof. Let $A = [T^*]_\beta$ and $B = ([T_\beta])^*$

Write $T^*(v_j) = \sum_{k=1}^n c_k v_k$

Write $T(v_i) = \sum_{k=1}^n d_k v_k$

$A_{i,j} = c_i = \langle T^*(v_j), v_i\rangle = \langle v_j, T(v_i)\rangle = \overline{\langle T(v_i), v_j\rangle} = \overline{d_j} = \overline{([T]_\beta)_{j,i}} = B_{i,j}$ ■

Corollary 4.1. *Let A be an $n \times n$ matrix*

$$L_{A^*} = (L_A)^*$$

Proof. Let β be the standard ordered basis of $\mathbb{F}^n$

$$[(L_A)^*]_\beta = ([L_A]_\beta)^* = A^* = [L_{A^*}]_\beta$$

This give us $(L_A)^* = L_{A^*}$ ■

Theorem 5. *Let T and U be linear operators on V*

(i) $(T + U)^* = T^* + U^*$
(ii) $(cT)^* = \overline{c}T^*$
(iii) $(T \circ U)^* = U^* \circ T^*$
(iv) $(T^*)^* = T$
(v) $I_V^* = I_V$

Proof. **(i)**

$\forall x, y \in V, \langle (T+U)^*(x), y\rangle = \langle x, (T+U)(y)\rangle = \langle x, T(y)\rangle + \langle x, U(y)\rangle = \langle T^*(x), y\rangle + \langle U^*(x), y\rangle = \langle T^*(x) + U^*(x), y\rangle$

(ii)

$\forall x, y \in V, \langle (cT)^*(x), y\rangle = \langle x, (cT)(y)\rangle = \langle x, cT(y)\rangle = \overline{c}\langle x, T(y)\rangle = \overline{c}\langle T^*(x), y\rangle = \langle \overline{c}T^*(x), y\rangle$

(iii)

$\forall x, y \in V, \langle (T \circ U)^*(x), y\rangle = \langle x, (T \circ U)(y)\rangle = \langle x, T(U(y))\rangle = \langle T^*(x), U(y)\rangle = \langle U^*(T^*(x)), y\rangle = \langle U^* \circ T^*(x), y\rangle$

(iv)

$\forall x, y \in V, \langle (T^*)^*(x), y\rangle = \langle x, T^*(y)\rangle = \langle T(x), y\rangle$

(v)

$\forall x, y \in V, \langle I_V^*(x), y\rangle = \langle x, I_V(y)\rangle = \langle x, y\rangle = \langle I_V(x), y\rangle$

■

Exercises

2.

2.(c)

Proof. Let $\{v_1, v_2, v_3\}$ be an orthonormal basis

Write $v_1 = 1$, $v_2 = 2\sqrt{3}(x - \frac{1}{2})$, and $v_3 = 6\sqrt{5}(x^2 - x + \frac{1}{6})$

Let $y = v_1 + \sqrt{3}v_2 + 7\sqrt{5}v_3$

Write $f = a_1 v_1 + a_2 v_2 + a_3 v_3$

$g(f) = \langle f, y \rangle = a_1 + \sqrt{3}a_2 + 7\sqrt{5}a_3$

$v_1(0) = 1$

$v_2(0) = -\sqrt{3}$

$v_3(0) = \sqrt{5}$

$v_1'(1) = 0$

$v_2'(1) = 2\sqrt{3}$

$v_3'(1) = 6\sqrt{5}$

$g(f) = f(0) + f'(1) = (a_1 v_1 + a_2 v_2 + a_3 v_3)(0) + (a_1 v_1 + a_2 v_2 + a_3 v_3)'(1) = a_1 - \sqrt{3}a_2 + \sqrt{5}a_3 + 2\sqrt{3}a_2 + 6\sqrt{5}a_3 = a_1 + \sqrt{3}a_2 + 7\sqrt{5}a_3$ ■

3.

3.(a)

Proof. $\begin{bmatrix} 11 \\ -12 \end{bmatrix}$ ■

3.(c)

Proof. $T^*(-2t + 4) = 6t + 12$ ■

6.

Proof. $U_1^* = (T + T^*)^* = T^* + T = U_1$

$U_2^* = (T \circ T^*)^* = (T^*)^* \circ T^* = T \circ T^* = U_2$ ■

8.

Proof. $I_V^* = I_V \implies (T^{-1}T)^* = I_V \implies T^*(T^{-1})^* = I_V \implies (T^*)^{-1} = (T^{-1})^*$ ■

10.

Proof. $(\longleftarrow)$

$\forall x \in V, \|T(x)\| = \sqrt{\langle T(x), T(x)\rangle} = \sqrt{\langle x, x\rangle} = \|x\|$

$(\longrightarrow)$

Let $x, y \in V$

$\langle x, y\rangle$

$= \frac{1}{4}(i\|x+iy\|^2 - \|x-y\|^2 - i\|x-iy\|^2 + \|x+y\|^2)$

$= \frac{1}{4}(i\|T(x+iy)\|^2 - \|T(x-y)\|^2 - i\|T(x-iy)\|^2 + \|T(x+y)\|^2)$

$= \frac{1}{4}(i\|T(x)+iT(y)\|^2 - \|T(x)-T(y)\|^2 - i\|T(x)-iT(y)\|^2 + \|T(x)+T(y)\|^2)$

$= \langle T(x), T(y)\rangle$ ∎

12.

12.(a)

Proof. $T(x) = 0 \iff \forall y \in V, 0 = \langle T(x), y\rangle = \langle x, T^*(y)\rangle$ ∎

12.(b)

Proof. $V = R(T^*) \oplus N(T)$

Let $\{v_1, \dots v_n\}$ be an orthonormal basis of $R(T^*)$ and $\{v_{n+1}, \dots, v_k\}$ be an orthonormal basis of $N(T)$

Let $x \in N(T)^\perp$

Write $x = \sum_{i=1}^{k} c_i v_i$

$\forall j : n+1 \leq j \leq k, 0 = \langle x, v_j\rangle = c_j$

So $x \in R(T^*)$

Let $y \in R(T^*)$

Write $y = \sum_{i=1}^{n} d_i v_i$

$\forall j : n+1 \leq j \leq k, \langle y, v_j\rangle = 0 \implies \forall j : n+1 \leq j \leq k, y \perp v_j \implies y \in N(T)^\perp$ ∎

13.

13.(a)

Proof. $T^*(T(x)) = 0 \iff \forall y \in V, 0 = \langle T^*(T(x)), y\rangle = \langle T(x), T(y)\rangle \iff \forall y \in V, T(x) \perp T(y) \iff T(x) \in R(T)^\perp \iff T(x) \in N(T)$ ■

13.(b)

Proof. Let $\{v_1, \dots, v_n\}$ be an orthonormal basis of $N(T)$ and $\{v_{n+1}, \dots v_k\}$ be an orthonormal basis of $R(T)$

We know $R(T^*)^\perp = N(T)$

Then $R(T^*) \oplus N(T)$

So $rank(T^*) = dim(V) - dim(N(T)) = rank(T)$

REMARK: Perpendicular must direct sume the whole space, but direct sum the whole space may not perpendicular ■

13.(c)

Proof. $rank(A^*A) = rank(L_{A^*A}) = rank(L_{A^*} \circ L_A) = dim(V) - N(L_{A^*} \circ L_A) = dim(V) - dim(N(L_A)) = rank(L_A) = rank(A) = rank(A^t) = rank(A^*) = rank(L_{A^*}) = dim(V) - dim(N(L_{A^*})) = dim(V) - dim(N((L_{A^*})^* \circ L_{A^*})) = dim(V) - dim(N(L_A \circ L_{A^*})) = rank(L_A \circ L_{A^*}) = rank(L_{AA^*}) = rank(AA^*)$ ■

14.

Proof. Let $x, u \in V$, and $c \in \mathbb{F}$

$T(x + cu) = \langle x + cu, y\rangle z = (\langle x, y\rangle + c\langle u, y\rangle)z = \langle x, y\rangle z + c\langle u, y\rangle z = T(x) + cT(u)$

Let $T^*(x) = \langle x, z\rangle y$

Observe $\forall x, v \in V, \langle T(x), v\rangle = \langle\langle x, y\rangle z, v\rangle = \langle x, y\rangle\langle z, v\rangle = \langle z, v\rangle\langle x, y\rangle = \langle x, \langle v, z\rangle y\rangle = \langle x, T^*(v)\rangle$ ■