12

Conjugacy classes

We take a break from representation theory to discuss some topics in group theory which will be relevant in our further study of representations. After defining conjugacy classes, we develop enough theory to determine the conjugacy classes of dihedral, symmetric and alternating groups. At the end of the chapter we prove a result linking the conjugacy classes of a group to the structure of its group algebra.

Throughout the chapter, G is a finite group.

Conjugacy classes

12.1 Definition

Let $x, y \in G$. We say that x is *conjugate to y in G* if

$$y = g^{-1}xg \quad \text{for some } g \in G.$$

The set of all elements conjugate to x in G is

$$x^G = \{g^{-1}xg\colon g \in G\},$$

and is called the *conjugacy class* of x in G.

Our first result shows that two distinct conjugacy classes have no elements in common.

12.2 Proposition

If $x, y \in G$, then either $x^G = y^G$ or $x^G \cap y^G$ is empty.

Proof Suppose that $x^G \cap y^G$ is not empty, and pick $z \in x^G \cap y^G$. Then there exist $g, h \in G$ such that

$$z = g^{-1}xg = h^{-1}yh.$$

Hence $x = gh^{-1}yhg^{-1} = k^{-1}yk$, where $k = hg^{-1}$. So

$$\begin{aligned} a \in x^G &\Rightarrow a = b^{-1}xb \quad \text{for some } b \in G \\ &\Rightarrow a = b^{-1}k^{-1}ykb \\ &\Rightarrow a = c^{-1}yc \quad \text{where } c = kb \\ &\Rightarrow a \in y^G. \end{aligned}$$

Therefore $x^G \subseteq y^G$. Similarly $y^G \subseteq x^G$ (using $y = kxk^{-1}$), and so $x^G = y^G$.

■

Since every element x of G lies in the conjugacy class x^G (as $x = 1^{-1}x1$ with $1 \in G$), G is the union of its conjugacy classes and so we deduce immediately

12.3 Corollary
Every group is a union of conjugacy classes, and distinct conjugacy classes are disjoint.

Another way of seeing this Corollary 12.3 is to observe that conjugacy is an equivalence relation, and that the conjugacy classes are the equivalence classes.

12.4 Definition
If $G = x_1^G \cup \ldots \cup x_l^G$, where the conjugacy classes $x_1^G, \ldots, x_l^G$ are distinct, then we call $x_1, \ldots, x_l$ *representatives* of the conjugacy classes of G.

12.5 Examples
(1) For every group G, $1^G = \{1\}$ is a conjugacy class of G.

(2) Let $G = D_6 = \langle a, b\colon a^3 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$. The elements of G are 1, a, a^2, b, ab, a^2b. Since $g^{-1}ag$ is a or a^2 for every $g \in G$, and $b^{-1}ab = a^2$, we

have

$$a^G = \{a, a^2\}.$$

Also, $a^{-i}ba^i = a^{-2i}b$ for all integers i, so

$$b^G = \{b, ab, a^2 b\}.$$

Thus the conjugacy classes of G are

$$\{1\}, \{a, a^2\}, \{b, ab, a^2 b\}.$$

(3) If G is abelian then $g^{-1}xg = x$ for all $x, g \in G$, and so $x^G = \{x\}$. Hence every conjugacy class of G consists of just one element.

The next proposition is often useful when calculating conjugacy classes.

12.6 Proposition
Let $x, y \in G$. If x is conjugate to y in G, then x^n is conjugate to y^n in G for every integer n, and x and y have the same order.

Proof Observe that for $a, b \in G$, we have

$$g^{-1}abg = (g^{-1}ag)(g^{-1}bg).$$

Hence $g^{-1}x^n g = (g^{-1}xg)^n$. Suppose that x is conjugate to y in G, so that $y = g^{-1}xg$ for some $g \in G$. Then $y^n = g^{-1}x^n g$ and therefore x^n is conjugate to y^n in G. Let x have order m. Then $y^m = g^{-1}x^m g = 1$, and for $0 < r < m < y^r = g^{-1}x^r g \neq 1$, so y also has order m.

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Conjugacy class sizes

The next theorem determines the sizes of the conjugacy classes in G in terms of certain subgroups which we now define.

12.7 Definition

Let $x \in G$. The *centralizer of x in G*, written $C_G(x)$, is the set of elements of G which commute with x; that is,

$$C_G(x) = \{g \in G\colon xg = gx\}.$$

(So also $C_G(x) = \{g \in G\colon g^{-1}xg = x\}$.)

It is easy to check that $C_G(x)$ is a subgroup of G (Exercise 12.1). Observe that $x \in C_G(x)$ and indeed, $\langle x\rangle \subseteq C_G(x)$ for all $x \in G$.

12.8 Theorem

Let $x \in G$. Then the size of the conjugacy class x^G is given by

$$|x^G| = |G\colon C_G(x)| = |G|/|C_G(x)|.$$

In particular, $|x^G|$ divides $|G|$.

Proof Observe first that for $g, h \in G$, we have

$$\begin{aligned} g^{-1}xg = h^{-1}xh &\Leftrightarrow hg^{-1}x = xhg^{-1} \\ &\Leftrightarrow hg^{-1} \in C_G(x) \\ &\Leftrightarrow C_G(x)g = C_G(x)h. \end{aligned}$$

By dint of this, we may define an injective function f from x^G to the set of right cosets of $C_G(x)$ in G by

$$f\colon g^{-1}xg \to C_G(x)g \quad (g \in G).$$

Clearly f is surjective. Hence f is a bijection, proving that $|x^G| = |G{:}C_G(x)|$.

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Before summarizing our results on conjugacy classes, we make the observation that

(12.9)
$$\begin{aligned}|x^G| = 1 &\Leftrightarrow g^{-1}xg = x \quad \text{for all } g \in G \\ &\Leftrightarrow x \in Z(G),\end{aligned}$$

where $Z(G)$ is the centre of G, as defined in 9.15.

We have now proved all parts of the following result.

12.10 The Class Equation

Let $x_1, \ldots, x_l$ be representatives of the conjugacy classes of G. Then

$$|G| = |Z(G)| + \sum_{x_i \notin Z(G)} |x_i^G|,$$

where $|x_i^G| = |G{:}C_G(x_i)|$, *and both* $|Z(G)|$ *and* $|x_i^G|$ *divide* $|G|$.

Conjugacy classes of dihedral groups

We illustrate the use of Theorem 12.8 by finding the conjugacy classes of all dihedral groups.

Let $G = D_{2n}$, the dihedral group of order $2n$. Thus

$$G = \langle a, b{:}\ a^n = b^2 = 1,\ b^{-1}ab = a^{-1}\rangle.$$

In finding the conjugacy classes of G, it is convenient to consider separately the cases where n is odd and where n is even.

(1) *n odd*

First consider a^i $(1 \leqslant i \leqslant n-1)$. Since $C_G(a^i)$ contains $\langle a\rangle$,

$$|G{:}\ C_G(a^i)| \leqslant |G{:}\ \langle a\rangle| = 2.$$

Also $b^{-1}a^ib = a^{-i}$, so $\{a^i, a^{-i}\} \subseteq (a^i)^G$. As n is odd, $a^i \neq a^{-i}$, and so $|(a^i)^G| \geqslant 2$. Using Theorem 12.8, we have

$$2 \geqslant |G: C_G(a^i)| = |(a^i)^G| \geqslant 2.$$

Hence equality holds here, and

$$C_G(a^i) = \langle a \rangle, (a^i)^G = \{a^i, a^{-i}\}.$$

Next, $C_G(b)$ contains $\{1, b\}$; and as $b^{-1}a^ib = a^{-i}$, no element a^i or a^ib (with $1 \leqslant i \leqslant n-1$) commutes with b. Thus

$$C_G(b) = \{1, b\}.$$

Therefore by Theorem 12.8, $|b^G| = n$. Since all the elements a^i have been accounted for, b^G must consist of the remaining n elements of G. That is,

$$b^G = \{b, ab, \ldots, a^{n-1}b\}.$$

We have shown

(12.11) *The dihedral group D_{2n} (n odd) has precisely $\frac{1}{2}(n+3)$ conjugacy classes*:

$$\{1\}, \{a, a^{-1}\}, \ldots, \{a^{(n-1)/2}, a^{-(n-1)/2}\}, \{b, ab, \ldots, a^{n-1}b\}.$$

(2) *n even*

Write $n = 2m$. As $b^{-1}a^mb = a^{-m} = a^m$, the centralizer of a^m in G contains both a and b, and hence $C_G(a^m) = G$. Therefore the conjugacy class of a^m in G is just $\{a^m\}$. As in case (1), $(a^i)^G = \{a^i, a^{-i}\}$ for $1 \leqslant i \leqslant m-1$.

For every integer j,

$$a^jba^{-j} = a^{2j}b, a^j(ab)a^{-j} = a^{2j+1}b.$$

It follows that

$$b^G = \{a^{2j}\text{b}: 0 \leqslant j \leqslant m-1\}, (ab)^G = \{a^{2j+1}b: 0 \leqslant j \leqslant m-1\}.$$

Hence

(12.12) *The dihedral group* D_{2n} *(n even, n = 2m) has precisely* $m + 3$ *conjugacy classes*:

$$\{1\},\ \{a^m\},\ \{a,\ a^{-1}\},\ \ldots,\ \{a^{m-1},\ a^{-m+1}\},$$
$$\{a^{2j}b\colon 0 \leqslant j \leqslant m-1\},\ \{a^{2j+1}b\colon 0 \leqslant j \leqslant m-1\}.$$

Conjugacy classes of S_n

We shall later need to know the conjugacy classes of the symmetric group S_n. Our first observation is simple but crucial.

12.13 Proposition

Let x be a k-cycle $(i_1\ i_2\ \ldots\ i_k)$ *in* S_n*, and let* $g \in S_n$*. Then* $g^{-1}xg$ *is the k-cycle* $(i_1 g\ i_2 g\ \ldots\ i_k g)$.

Proof Write $A = \{i_1, \ldots, i_k\}$. For $i_r \in A$,

$$i_r g(g^{-1}xg) = i_r xg = i_{r+1} g \ \ (\text{or } i_1 g \text{ if } r = k).$$

Also, for $1 \leqslant i \leqslant n$ and $i \notin A$,

$$ig(g^{-1}xg) = ixg = ig.$$

Hence $g^{-1}(i_1\ i_2\ \ldots\ i_k)\, g = (i_1 g\ i_2 g\ \ldots\ i_k g)$, as required.

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Now consider an arbitrary permutation $x \in S_n$. Write

$$x = (a_1\ \ldots\ a_{k_1})(b_1\ \ldots\ b_{k_2})\ \ldots\ (c_1\ \ldots\ c_{k_s}),$$

a product of disjoint cycles, with $k_1 \geqslant k_2 \geqslant \ldots \geqslant k_s$. By Proposition 12.13, for $g \in S_n$ we have

(12.14)

$$g^{-1}xg = g^{-1}(a_1 \ldots a_{k_1})gg^{-1}(b_1 \ldots b_{k_2})g \ldots g^{-1}(c_1 \ldots c_{k_s})g$$
$$= (a_1g \ldots a_{k_1}g)(b_1g \ldots b_{k_2}g) \ldots (c_1g \ldots c_{k_s}g).$$

We call $(k_1, \ldots, k_s)$ the *cycle-shape* of x, and note that x and $g^{-1}xg$ have the same cycle-shape. On the other hand, given any two permutations x, y of the same cycle-shape, say

$$x = (a_1 \ldots a_{k_1}) \ldots (c_1 \ldots c_{k_s}),$$
$$y = (a'_1 \ldots a'_{k_1}) \ldots (c'_1 \ldots c'_{k_s}),$$

(products of disjoint cycles), there exists $g \in S_n$ sending $a_1 \to a'_1, \ldots, c_{k_s} \to c'_{k_s}$, and so by (12.14),

$$g^{-1}xg = y.$$

We have proved the following result.

12.15 Theorem
For $x \in S_n$, the conjugacy class x^{S_n} of x in S_n consists of all permutations in S_n which have the same cycle-shape as x.

12.16 Examples
(1) The conjugacy classes of S_3 are

Class	*Cycle-shape*
{1}	(1)
{(1 2), (1 3), (2 3)}	(2)
{(1 2 3), (1 3 2)}	(3)

(2) The conjugacy class of (1 2)(3 4) in S_4 consists of all the elements of cycle-shape (2, 2) and is

$$\{(1\ 2)(3\ 4),\ (1\ 3)(2\ 4),\ (1\ 4)(2\ 3)\}.$$

(3) There are precisely five conjugacy classes of S_4, with representatives (see Definition 12.4):

$$1,\ (1\ 2),\ (1\ 2\ 3),\ (1\ 2)(3\ 4),\ (1\ 2\ 3\ 4).$$

To calculate the sizes of the conjugacy classes, we simply count the number of 2-cycles, 3-cycles, and so on. The number of 2-cycles is equal to the number of pairs that can be chosen from {1, 2, 3, 4}, which is $\binom{4}{2} = 6$. (The notation $\binom{n}{r}$ means the binomial coefficient $n!/(r!(n-r)!)$.) The number of 3-cycles is 4 × 2 (4 for the choice of fixed point and 2 because there are two 3-cycles fixing a given point). Similarly, there are three elements of cycle-shape (2, 2) and there are six 4-cycles. Thus for $G = S_4$, the conjugacy class representatives g, the conjugacy class sizes $|g^G|$ and the centralizer orders $|C_G(g)|$ (obtained using Theorem 12.8) are as follows:

Representative	g	1	(1 2)	(1 2 3)	(1 2)(3 4)	(1 2 3 4)
Class size	$\lvert g^G\rvert$	1	6	8	3	6
	$\lvert C_G(g)\rvert$	24	4	3	8	4

We check our arithmetic by noting that

$$|S_4| = 1 + 6 + 8 + 3 + 6.$$

(4) Similarly, the corresponding table for $G = S_5$ is

Rep.	g	1	(1 2)	(1 2 3)	(1 2)(3 4)	(1 2 3 4)	(1 2 3)(4 5)	(1 2 3 4 5)
	$\lvert g^G\rvert$	1	10	20	15	30	20	24
	$\lvert C_G(g)\rvert$	120	12	6	8	4	6	5

Conjugacy classes of A_n

Given an even permutation $x \in A_n$, we have seen in Theorem 12.15 that the conjugacy class x^{S_n} consists of all permutations in S_n which have the same cycle-shape as x. The conjugacy class x^{A_n} of x in A_n, given by

$$x^{A_n} = \{g^{-1}xg\colon g \in A_n\},$$

is of course contained in x^{S_n}; however, x^{A_n} might not be equal to x^{S_n}. For an easy example where equality does not hold, consider $x = (1\ 2\ 3) \in A_3$; here $x^{A_3} = \{x\}$, while $x^{S_3} = \{x, x^{-1}\}$.

The next result determines precisely when x^{A_n} and x^{S_n} are equal, and what happens when equality fails.

12.17 Proposition
Let $x \in A_n$ with $n > 1$.

(1) *If x commutes with some odd permutation in S_n, then $x^{S_n} = x^{A_n}$.*

(2) *If x does not commute with any odd permutation in S_n then x^{S_n} splits into two conjugacy classes in A_n of equal size, with representatives x and* $(1\ 2)^{-1}x(1\ 2)$.

Proof (1) Assume that x commutes with an odd permutation g. Let $y \in x^{S_n}$, so that $y = h^{-1}xh$ for some $h \in S_n$. If h is even then $y \in x^{A_n}$; and if h is odd then $gh \in A_n$ and

$$y = h^{-1}xh = h^{-1}g^{-1}xgh = (gh)^{-1}x(gh),$$

so again $y \in x^{A_n}$. Thus $x^{S_n} \subseteq x^{A_n}$, and so $x^{S_n} = x^{A_n}$.

(2) Assume that x does not commute with any odd permutation. Then

$$C_{S_n}(x) = C_{A_n}(x).$$

Hence by Theorem 12.8,

$$|x^{A_n}| = |A_n\colon C_{A_n}(x)| = \tfrac{1}{2}|S_n\colon C_{A_n}(x)| \quad (\text{as } |A_n| = \tfrac{1}{2}|S_n|)$$
$$= \tfrac{1}{2}|S_n\colon C_{S_n}(x)| = \tfrac{1}{2}|x^{S_n}|.$$

Next, we observe that

$$\{h^{-1}xh\colon h \text{ is odd}\} = ((1\ 2)^{-1}x(1\ 2))^{A_n}$$

since every odd permutation has the form (1 2)a for some $a \in A_n$. Now

$$x^{S_n} = \{h^{-1}xh\colon h \text{ is even}\} \cup \{h^{-1}xh\colon h \text{ is odd}\}$$
$$= x^{A_n} \cup ((1\ 2)^{-1}x(1\ 2))^{A_n}.$$

Since $|x^{A_n}| = \tfrac{1}{2}|x^{S_n}|$, the conjugacy classes x^{A_n} and $((1\ 2)^{-1}x(1\ 2))^{A_n}$ must be disjoint and of equal size, as we wished to show.

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12.18 Examples

(1) We find the conjugacy classes of A_4. The elements of A_4 are the identity, together with the permutations of cycle-shapes (2, 2) and (3). Since (1 2)(3 4) commutes with the odd permutation (1 2), Proposition 12.17 implies that

$$(1\ 2)(3\ 4)^{A_4} = (1\ 2)(3\ 4)^{S_4} = \{(1\ 2)(3\ 4),\ (1\ 3)(2\ 4),\ (1\ 4)(2\ 3)\}.$$

However, the 3-cycle (1 2 3) commutes with no odd permutation: for if $g^{-1}(1\ 2\ 3)\,g = (1\ 2\ 3)$ then $(1\ 2\ 3) = (1\ g\ 2\ g\ 3\ g)$ by Proposition 12.13, so g is 1, (1 2 3) or (1 3 2), an even permutation. Hence by Proposition 12.17, $(1\ 2\ 3)^{S_4}$ splits into two conjugacy classes in A_4 of size 4, with representatives (1 2 3) and $(1\ 2)^{-1}(1\ 2\ 3)(1\ 2) = (1\ 3\ 2)$.

Thus the conjugacy classes of A_4 are

Representative	1	(1 2)(3 4)	(1 2 3)	(1 3 2)
Class size	1	3	4	4
Centralizer order	12	4	3	3

(2) We find the conjugacy classes of A_5. The non-identity even permutations in S_5 are those of cycle-shapes (3), (2, 2) and (5). The elements (1 2 3) and (2 3)(4 5) commute with the odd permutation (4 5); but (1 2 3 4 5) commutes with no odd permutation. (Check this by using the argument in (1) above.) Hence by Proposition 12.17, the conjugacy classes of A_5 are represented by 1, (1 2 3), (1 2)(3 4), (1 2 3 4 5) and $(1\ 2)^{-1}(1\ 2\ 3\ 4\ 5)(1\ 2) = (1\ 3\ 4\ 5\ 2)$. Using Proposition 12.17(2), we see that the class sizes and centralizer orders are as follows:

Representative	1	(1 2 3)	(1 2)(3 4)	(1 2 3 4 5)	(1 3 4 5 2)
Class size	1	20	15	12	12
Centralizer order	60	3	4	5	5

Normal subgroups

Normal subgroups are related to conjugacy classes by the following elementary result.

12.19 Proposition
Let H be a subgroup of G. Then $H \lhd G$ if and only if H is a union of conjugacy classes of G.

Proof If H is a union of conjugacy classes, then

$$h \in H, g \in G \Rightarrow g^{-1}hg \in H,$$

so $g^{-1}Hg \subseteq H$. Thus $H \lhd G$.

Conversely, if $H \lhd G$ then for all $h \in H$, $g \in G$, we have $g^{-1}hg \in H$, and so $h^G \subseteq H$. Therefore

$$H = \bigcup_{h \in H} h^G,$$

and so H is a union of conjugacy classes of G.

■

12.20 Example

We find all the normal subgroups of S_4. Let $H \triangleleft S_4$. Then by Proposition 12.19, H is a union of conjugacy classes of S_4. As we saw in Example 12.16(3), these conjugacy classes have sizes 1, 6, 8, 3, 6. Since $|H|$ divides 24 by Lagrange's Theorem, and $1 \in H$, there are just four possibilities:

$$|H| = 1,\ 1+3,\ 1+8+3 \text{ or } 1+6+8+3+6.$$

In the first case $H = \{1\}$, in the last case $H = S_4$, and in the third case $H = A_4$. In the case where $|H| = 1 + 3$, we have

$$H = 1^{S_4} \cup (1\ 2)(3\ 4)^{S_4} = \{1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}.$$

This is easily checked to be a subgroup of S_4; we write it as V_4 (V stands for 'Viergruppe', meaning 'four-group').

We have now shown that S_4 has exactly four normal subgroups:

$$\{1\},\ S_4,\ A_4 \text{ and } V_4 = \{1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}.$$

The centre of a group algebra

In this final section we link the conjugacy classes of the group G to the centre of the group algebra $\mathbb{C}G$. Recall from Definition 9.12 that the centre of $\mathbb{C}G$ is

$$Z(\mathbb{C}G) = \{z \in \mathbb{C}G\colon zr = rz \text{ for all } r \in \mathbb{C}G\}.$$

We know that $Z(\mathbb{C}G)$ is a subspace of the vector space $\mathbb{C}G$. There is a convenient basis for this subspace which can be described in terms of the conjugacy classes of G.

12.21 Definition
Let $C_1, \ldots, C_l$ be the distinct conjugacy classes of G. For $1 \leqslant i \leqslant l$, define

$$\overline{C}_i = \sum_{g \in C_i} g \in \mathbb{C}G.$$

The elements $\overline{C}_1, \ldots, \overline{C}_l$ of $\mathbb{C}G$ are called *class sums*.

12.22 Proposition
The class sums $\overline{C}_1, \ldots, \overline{C}_l$ form a basis of $Z(\mathbb{C}G)$.

Proof First we show that each $\overline{C}_i$ belongs to $Z(\mathbb{C}G)$. Let C_i consist of the r distinct conjugates $y_1^{-1} g y_1, \ldots, y_r^{-1} g y_r$ of an element g, so

$$\overline{C}_i = \sum_{j=1}^{r} y_j^{-1} g y_j.$$

For all $h \in G$,

$$h^{-1} \overline{C}_i h = \sum_{j=1}^{r} h^{-1} y_j^{-1} g y_j h.$$

As j runs from 1 to r, the elements $h^{-1} y_j^{-1} g y_j h$ run through C_i, since

$$h^{-1} y_j^{-1} g y_j h = h^{-1} y_k^{-1} g y_k h \Leftrightarrow y_j^{-1} g y_j = y_k^{-1} g y_k.$$

Hence

$$\sum_{j=1}^{r} h^{-1} y_j^{-1} g y_j h = \overline{C}_i,$$

and so $h^{-1} \overline{C}_i h = \overline{C}_i$. That is,

$$\bar{C}_i h = h\bar{C}_i.$$

Therefore each $\bar{C}_i$ commutes with all $h \in G$, hence with all $\sum_{h\in G}\lambda_h h \in \mathbb{C}G$, and so $\bar{C}_i \in Z(\mathbb{C}G)$.

Next, observe that $\bar{C}_1, \ldots, \bar{C}_l$ are linearly independent: for if $\sum_{i=1}^{l}\lambda_i\bar{C}_i = 0$ $(\lambda_i \in \mathbb{C})$, then all $\lambda_i = 0$ as the classes $C_1, \ldots, C_l$ are pairwise disjoint by Corollary 12.3.

It remains to show that $\bar{C}_1, \ldots, \bar{C}_l$ span $Z(\mathbb{C}G)$. Let $r = \sum_{g\in G}\lambda_g g \in Z(\mathbb{C}G)$. For $h \in G$, we have $rh = hr$, so $h^{-1}rh = r$. That is,

$$\sum_{g\in G}\lambda_g h^{-1}gh = \sum_{g\in G}\lambda_g g.$$

So for every $h \in G$, the coefficient λ_g of g is equal to the coefficient $\lambda_{h^{-1}gh}$ of $h^{-1}gh$. That is to say, the function $g \to \lambda_g$ is constant on conjugacy classes of G. It follows that $r = \sum_{i=1}^{l}\lambda_i\bar{C}_i$ where λ_i is the coefficient λ_{g_i} for some $g_i \in C_i$. This completes the proof.

■

12.23 Examples

(1) From Example 12.16(1), a basis for $Z(\mathbb{C}S_3)$ is

$$1, (1\ 2) + (1\ 3) + (2\ 3), (1\ 2\ 3) + (1\ 3\ 2).$$

(2) From (12.12), a basis for $Z(\mathbb{C}D_8)$ is

$$1, a^2, a + a^3, b + a^2b, ab + a^3b.$$

Summary of Chapter 12

1. Every group is a union of conjugacy classes, and distinct conjugacy classes are disjoint.
2. For an element x of a group G, the centralizer $C_G(x)$ is the set of elements of G which commute with x. It is a subgroup of G, and the number of

elements in the conjugacy class x^G is equal to $|G:C_G(x)|$.

3. The conjugacy classes of S_n correspond to the cycle-shapes of permutations in S_n.
4. If $x \in A_n$ then $x^{S_n} = x^{A_n}$ if and only if x commutes with some odd permutation in S_n.
5. The class sums in $\mathbb{C}G$ form a basis for the centre of $\mathbb{C}G$.

Exercises for Chapter 12

1. If G is a group and $x \in G$, show that $C_G(x)$ is a subgroup of G which contains $Z(G)$.
2. Let G be a finite group and suppose that $g \in G$ and $z \in Z(G)$. Prove that the conjugacy classes g^G and $(gz)^G$ have the same size.
3. Let $G = S_n$.

   (a) Prove that $|(1\ 2)^G| = \binom{n}{2}$ and find $C_G((1\ 2))$. Verify that your solution satisfies Theorem 12.8.

   (b) Show that $|(1\ 2\ 3)^G| = 2\binom{n}{3}$ and $|(1\ 2)(3\ 4)^G| = 3\binom{n}{4}$.

   (c) Now let $n = 6$. Show that

   $$|(1\ 2\ 3)(4\ 5\ 6)^G| = 40 \text{ and } |(1\ 2)(3\ 4)(5\ 6)^G| = 15,$$

   and find the sizes of the other conjugacy classes of S_6. (There are 11 conjugacy classes in all.)
4. What are the cycle-shapes of those permutations $x \in A_6$ for which $x^{A_6} \neq x^{S_6}$?
5. Show that A_5 is a simple group. (Hint: use the method of Example 12.20.)
6. Find the conjugacy classes of the quaternion group Q_8. Give a basis of the centre of the group algebra $\mathbb{C}Q_8$.
7. Let p be a prime number, and let n be a positive integer. Suppose that G is a group of order p^n.

   (a) Use the Class Equation 12.10 to show that $Z(G) \neq \{1\}$.

(b) Suppose that $n \geqslant 3$ and that $|Z(G)| = p$. Prove that G has a conjugacy class of size p.