Deadline : 2023/10/02, 17:00.

1. Prove that the following statements are equivalent: for a given sequence $\{x_n\}$,
    1. if for every $0 < \epsilon \in \mathbb{Q}$, there exists $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ whenever $n \geq N$.
    2. if for every $0 < \epsilon \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ whenever $n \geq N$.
2. Let $\{x_n\}_{n=1}^{\infty}$ be a monotone increasing sequence such that $x_{n+1} - x_n \leq 1/n$. Determine whether the sequence converges. (If yes, prove it; if not, disprove it or give a counterexample.)
3. Let $M_{n\times m}$ be the collection of all $n \times m$ matrices with real entries. Define a func. $\|\cdot\| : M_{n\times m} \to \mathbb{R}$ by

$$\|A\| = \sup_{x\in\mathbb{R}^m, x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

here we recall that $\|\cdot\|_2$ is the 2-norm on Euclidean space given by

$$\|x\|_2 = \left(\sum_{i=1}^{k} x_i^2\right)^{1/2} \text{ if } x \in \mathbb{R}^k.$$

Show that

    1. $\|A\| = \sup_{x\in\mathbb{R}^m, x=1} \|Ax\|_2 = \inf\{M \in \mathbb{R} : \|Ax\|_2 \leq M \|x\|_2 \, \forall x \in \mathbb{R}^m\}$.
    2. $\|Ax\|_2 \leq \|A\| \|x\|_2$ for all x $\mathbb{R}^m$.
    3. $\|\cdot\|$ defines a norm on $M_{n\times m}$.
4. Suppose that $S_1, S_2, \cdots, S_n$ are sets in $\mathbb{R}$ and $S = \cup_{i=1}^{n} S_i$. Define $B_i = \sup S_i$ for $i = 1, \cdots, n$.
    1. Show that $\sup S = \max\{B_1, B_2, \cdots, B_n\}$.
    2. If S is the union of an infinite collection of S_i, find the relation between $\sup S$ and B_i.
5. Let A be non-empty set of $\mathbb{R}$ which is bounded below. Define the set $-A$ by $-A \equiv \{-x \in \mathbb{R} : x \in A\}$. Prove that

$$\inf(A) = -\sup(-A)$$

6. Let A, B be non-empty subset of $\mathbb{R}$. Define $A + B = \{x + y : x \in A, y \in B\}$. Justify if the following statement are true of false by providing a proof for the true statement and giving a counter-example for the false ones.
    1. $\sup(A + B) = \sup A + \sup B$.
    2. $\inf(A + B) = \inf A + \inf B$.
    3. $\sup(A \cap B) \leq \min\{\sup A, \sup B\}$.
    4. $\sup(A \cap B) = \min\{\sup A, \sup B\}$.
    5. $\sup(A \cup B) \geq \max\{\sup A, \sup B\}$.

6. $\sup(A \cup B) = \max\{\sup A, \sup B\}$

7. Let $S \subseteq \mathbb{R}$ be bounded below and non-empty. Show that

$$\inf S = \sup\{x \in \mathbb{R} : \text{x is a lower bound for S}\}.$$

8. Let f be a continuous function on $\mathbb{R}$ and D is a dense subset in $\mathbb{R}$. Prove that

   1. $\sup_{x \in D} f(x) = \sup_{x \in \mathbb{R}} f(x)$.
   2. there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in D such that $\lim\limits_{n \to \infty} f(x_n) = \sup\limits_{x \in \mathbb{R}} f(x)$

Extra question for group B

9. Let $\{x_n\}_{n=1}^{\infty}$ be a monotone increasing sequence such that $x_{n+1} - x_n \leq (\frac{1}{2})^n$. Determine whether the sequence converges. (If yes, prove it; if not, disprove it or give a counterexample.)

10. Understand the Baire Category Theorem and use it to show $\mathbb{R}$ is not countable.