HWs

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Contents

Chapter 1

General Analysis HW

1.1 HW1

Theorem 1.1.1. $\mathbb{R}^n$ is complete.

Proof. Let $\mathbf{x}_k$ be an arbitrary Cauchy sequence in $\mathbb{R}^n$. We are required to show $\mathbf{x}_k$ converge in $\mathbb{R}^n$. For each k, denote $\mathbf{x}_k$ by $(x_{(1,k)}, \dots, x_{(n,k)})$. We claim that for each $i \in \{1, \dots, n\}$

$$x_{(i,k)} \text{ is a Cauchy sequence}$$

Fix i and $\epsilon > 0$. To show $x_{(i,k)}$ is a Cauchy sequence, we are required to find $N \in \mathbb{N}$ such that for all $r, m \geq N$ we have

$$\left| x_{(i,r)} - x_{(i,m)} \right| \leq \epsilon$$

Because $\mathbf{x}_k$ is a Cauchy sequence in $\mathbb{R}^n$, we know there exists $N \in \mathbb{N}$ such that for all $r, m \geq N$, we have

$$|\mathbf{x}_r - \mathbf{x}_m| < \epsilon$$

Fix such N and arbitrary $r, m \geq M$. Observe

$$\left| x_{(i,r)} - x_{(i,m)} \right| \leq \sqrt{\sum_{j=1}^{n} \left| x_{(j,r)} - x_{(j,m)} \right|^2} = |\mathbf{x}_r - \mathbf{x}_m| < \epsilon$$

We have proved that for each $i \in \{1, \dots, n\}$, the real sequence $x_{(i,k)}$ is Cauchy. We now claim that for each $i \in \{1, \dots, n\}$, we have

$$\limsup_{r \to \infty} x_{(i,r)} \in \mathbb{R} \text{ and } \lim_{k \to \infty} x_{(i,k)} = \limsup_{r \to \infty} x_{(i,r)}$$

Again fix i. Because $x_{(i,k)}$ is a Cauchy sequence, we know there exists some N such that for all $r, m \geq N$, we have

$$\left|x_{(i,r)} - x_{(i,m)}\right| < 1$$

This implies that for all $r \geq N$, we have

$$x_{(i,r)} < x_{(i,N)} + 1 \tag{1.1}$$

Equation 1.1 then tell us

$$x_{(i,N)} + 1 \text{ is an upper bound of } \{x_{(i,r)} : r \geq N\}$$

Then by definition of sup, we have

$$\sup\{x_{(i,r)} : r \geq N\} \leq x_{(i,N)} + 1 \in \mathbb{R}$$

This then implies $\limsup_{r\to\infty} x_{(i,r)} \in \mathbb{R}$. We now prove

$$\lim_{k\to\infty} x_{(i,k)} = \limsup_{r\to\infty} x_{(i,r)} \tag{1.2}$$

Fix $\epsilon > 0$. We are required to find N such that

$$\forall k \geq N, \left|x_{(i,k)} - \limsup_{r\to\infty} x_{(i,r)}\right| \leq \epsilon$$

Because $\{x_{(i,k)}\}_{k\in\mathbb{N}}$ is a Cauchy sequence, we can let N_0 satisfy

$$\forall k, m \geq N_0, \left|x_{(i,k)} - x_{(i,m)}\right| < \frac{\epsilon}{2}$$

Because $\sup\{x_{(i,k)} : k \geq N'\} \searrow \limsup_{r\to\infty} x_{(i,r)}$ as $N' \to \infty$, we know there exists $N_1 > N_0$ such that

$$\limsup_{r\to\infty} x_{(i,r)} - \frac{\epsilon}{2} < \sup\{x_{(i,k)} : k \geq N_0\} \leq \limsup_{r\to\infty} x_{(i,r)} + \frac{\epsilon}{2}$$

Then because $\limsup_{r\to\infty} x_{(i,r)} - \frac{\epsilon}{2}$ is strictly smaller than the smallest upper bound of $\{x_{(i,k)} : k \geq N_1\}$, we see $\limsup_{n\to\infty} x_{(i,r)} - \frac{\epsilon}{2}$ is not an upper bound of $\{x_{(i,k)} : k \geq N_1\}$. This implies the existence of some N such that $N \geq N_1$ and

$$\limsup_{r\to\infty} x_{(i,r)} - \frac{\epsilon}{2} < x_{(i,N)} \leq \limsup_{r\to\infty} x_{(i,r)} + \frac{\epsilon}{2}$$

Now observe that for all $k \geq N$, because $N \geq N_1 \geq N_0$

$$\limsup_{r\to\infty} x_{(i,r)} - \epsilon < x_{(i,N)} - \frac{\epsilon}{2} < x_{(i,k)} < x_{(i,N)} + \frac{\epsilon}{2} \leq \limsup_{r\to\infty} x_{(i,r)} + \epsilon$$

This implies for all $k \geq N$, we have

$$\left| x_{(i,k)} - \limsup_{r\to\infty} x_{(i,r)} \right| \leq \epsilon$$

We have just proved Equation 1.2. Lastly, to close out the proof, we show

$$\lim_{k\to\infty} \mathbf{x}_k = \left(\lim_{k\to\infty} x_{(1,k)}, \dots, \lim_{k\to\infty} x_{(n,k)} \right) \tag{1.3}$$

Fix $\epsilon > 0$. For each $i \in \{1, \dots, n\}$, let N_i satisfy

$$\forall r \geq N_i, \left| x_{(i,r)} - \lim_{k\to\infty} x_{(i,k)} \right| \leq \frac{\epsilon}{\sqrt{n}}$$

Observe that for all $r \geq \max_{i\in\{1,\dots,n\}} N_i$, we have

$$\begin{aligned} \left| \mathbf{x}_r - \left(\lim_{k\to\infty} x_{(1,k)}, \dots, \lim_{k\to\infty} x_{(n,k)} \right) \right| &= \sqrt{\sum_{i=1}^{n} \left| x_{(i,r)} - \lim_{k\to\infty} x_{(i,k)} \right|^2} \\ &\leq \sqrt{\sum_{i=1}^{n} \frac{\epsilon^2}{n}} = \epsilon \end{aligned}$$

We have proved Equation 1.3.

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Theorem 1.1.2. ($\mathbb{Q}$ is dense in $\mathbb{R}$)

Proof. Fix $x \in \mathbb{R}$ and $\epsilon > 0$. To show $\mathbb{Q}$ is dense in $\mathbb{R}$, we have to find $q \in \mathbb{Q}$ such that $|x - q| < \epsilon$.

Let $m \in \mathbb{N}$ satisfy $\frac{1}{m} < \epsilon$. Let n be the largest integer such that $n \leq mx$. Because n is the largest integer such that $n \leq mx$, we know $mx - n < 1$, otherwise we can deduce $n + 1 \leq mx$, which is impossible, since $n + 1$ is an integer and n is the largest integer such that $n \leq mx$. We now see that

$$\frac{n}{m} \in \mathbb{Q} \text{ and } \left| x - \frac{n}{m} \right| = \frac{mx - n}{m} < \frac{1}{m} < \epsilon$$

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Chapter 2

Complex Analysis HW

2.1 HW1

Theorem 2.1.1.

$$(1+i)^n, \frac{(1+i)^n}{n}, \frac{n!}{(1+i)^n} \text{ all diverge as } n \to \infty$$

Proof. Note that

$$|(1+i)^n| = 2^{\frac{n}{2}} \to \infty \text{ as } n \to \infty$$

This implies $(1+i)$ is unbounded, thus diverge.

Note that

$$\left|\frac{(1+i)^n}{n}\right| = \frac{(\sqrt{2})^n}{n}$$

Observe

$$\begin{aligned}\frac{(\sqrt{2})^n}{n} = \frac{[(\sqrt{2}-1)+1]^n}{n} &= \frac{\sum_{k=0}^{n}\binom{n}{k}(\sqrt{2}-1)^k}{n} \\ &\geq \frac{\binom{n}{2}(\sqrt{2}-1)^2}{n} = (n-1)[\frac{(\sqrt{2}-1)^2}{2}] \to \infty \text{ as } n \to \infty\end{aligned}$$

This implies $\frac{(1+i)^n}{n}$ is unbounded, thus diverge.

Note that

$$\left|\frac{n!}{(1+i)^n}\right| = \frac{n!}{(\sqrt{2})^n}$$

Note that for all $k \geq 8$, we have

$$\frac{k}{\sqrt{2}} \geq \frac{\sqrt{8}}{\sqrt{2}} = 2$$

This implies

$$\frac{n!}{(\sqrt{2})^n} = \prod_{k=1}^{n} \frac{k}{\sqrt{2}} = \frac{7!}{(\sqrt{2})^7} \prod_{k=8}^{n} \frac{k}{\sqrt{2}} \geq \frac{7!}{(\sqrt{2})^7} \prod_{k=8}^{n} 2 \geq \frac{7! 2^{n-8+1}}{(\sqrt{2})^7} \to \infty$$

which implies $\frac{n!}{(1+i)^n}$ is unbounded, thus diverge. ■

Theorem 2.1.2.

$$n! z^n \text{ converge } \iff z = 0$$

Proof. If $z = 0$, then $n!z^n = 0$ for all n, which implies $n!z^n \to 0$. Now, suppose $z \neq 0$. Let $M \in \mathbb{N}$ satisfy $|z| > \frac{1}{M}$. Observe

$$|n!z^n| = \left| \prod_{k=1}^{n} kz \right| = \left| \prod_{k=1}^{2M-1} kz \right| \left| \prod_{k=2M}^{n} kz \right| \geq \left| \prod_{k=1}^{2M-1} kz \right| \left| \prod_{k=2M}^{n} 2Mz \right| \geq \left| \prod_{k=1}^{2M-1} kz \right| 2^{n-2M+1} \to \infty$$

This implies $n!z^n$ is unbounded, thus diverge. ■

Theorem 2.1.3.

$$u_n \to u \implies v_n \triangleq \sum_{k=1}^{n} \frac{u_k}{n} \to u$$

Proof. Because

$$\sum_{k=1}^{n} \frac{u_k}{n} = \sum_{k \leq \sqrt{n}} \frac{u_k}{n} + \sum_{\sqrt{n} < k \leq n} \frac{u_k}{n}$$

It suffices to prove

$$\sum_{k \leq \sqrt{n}} \frac{u_k}{n} \to 0 \text{ and } \sum_{\sqrt{n} < k \leq n} \frac{u_k}{n} \to u \text{ as } n \to \infty$$

Because u_n converge, we can let M bound $|u_n|$. Observe

$$\left| \sum_{k \leq \sqrt{n}} \frac{u_k}{n} \right| \leq \sum_{k \leq \sqrt{n}} \left| \frac{u_k}{n} \right| \leq \sum_{k \leq \sqrt{n}} \frac{M}{n} \leq \frac{M\sqrt{n}}{n} = \frac{M}{\sqrt{n}} \to 0 \text{ as } n \to 0 \text{ (done)}$$

Because

$$\sum_{\sqrt{n}<k\leq n} \frac{u_k}{n} = \frac{n - \lceil \sqrt{n} \rceil + 1}{n} \sum_{\sqrt{n}<k\leq n} \frac{u_k}{n - \lceil \sqrt{n} \rceil + 1}$$

and

$$\lim_{n\to\infty} \frac{n - \lceil \sqrt{n} \rceil + 1}{n} = 1$$

We can reduce the problem into proving

$$\lim_{n\to\infty} \sum_{\sqrt{n}<k\leq n} \frac{u_k}{n - \lceil \sqrt{n} \rceil + 1} = u$$

Fix ϵ. Let N satisfy that for all $n \geq N$, we have $|u_n - u| < \epsilon$. Then for all $n \geq N^2$, we have

$$\begin{aligned}
\left| \left(\sum_{\sqrt{n}<k\leq n} \frac{u_k}{n - \lceil \sqrt{n} \rceil + 1} \right) - u \right| &= \left| \sum_{\sqrt{n}<k\leq n} \frac{u_k - u}{n - \lceil \sqrt{n} \rceil + 1} \right| \\
&\leq \sum_{\sqrt{n}<k\leq n} \frac{|u_k - u|}{n - \lceil \sqrt{n} \rceil + 1} \\
&\leq \sum_{\sqrt{n}<k\leq n} \frac{\epsilon}{n - \lceil \sqrt{n} \rceil + 1} = \epsilon \text{ (done)}
\end{aligned}$$

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Chapter 3

PDE intro HW

3.1 HW1

Theorem 3.1.1.

$$\text{Show } u \mapsto u_x + uu_y \text{ is non-linear}$$

Proof. See that

$$2u \mapsto 2u_x + 4uu_y \neq 2(u_x + uu_y) \tag{3.1}$$

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Theorem 3.1.2.

$$\text{Solve } (1 + x^2)u_x + u_y = 0$$

Proof. The characteristic curve has the derivative

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

The solution to this ODE is

$$y = \arctan x + C$$

We now see that the solution to the PDE in Equation 3.1 is

$$u = f\big((\arctan x) - y\big) \text{ where } f : \mathbb{R} \to \mathbb{R} \text{ is an arbitrary smooth function}$$

A characteristic curve is as followed.

[3]:
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plot(arctan(x), (x, -10, 10), title='Plot of y = arctan(x)', legend_label='y = arctan(x)')
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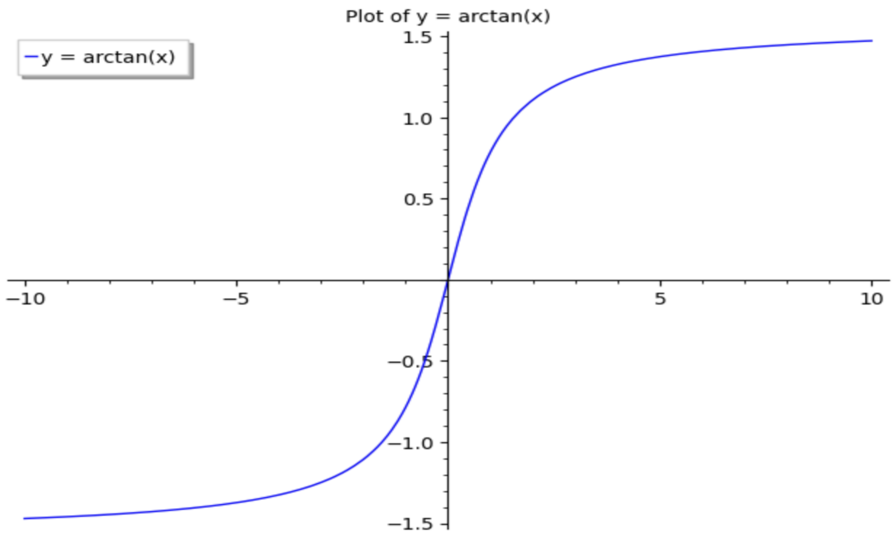
[3]:



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Theorem 3.1.3.

$$\text{Solve } au_x + bu_y + cu = 0 \tag{3.2}$$

Proof. Fix

$$\begin{cases} x' \triangleq ax + by \\ y' \triangleq bx - ay \end{cases}$$

This map is clearly a diffeomorphism. Compute

$$\begin{cases} u_x = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = au_{x'} + bu_{y'} \\ u_y = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} = bu_{x'} - au_{y'} \end{cases}$$

Plugging it back into the PDE in Equation 3.2, we have

$$cu + (a^2 + b^2)u_{x'} = 0 \tag{3.3}$$

If $c = a^2 + b^2 = 0$, then all smooth functions are solution. If $a^2 + b^2 = 0$ but $c \neq 0$, then clearly the only solution is $u = \tilde{0}$. If $a^2 + b^2 \neq 0$ but $c = 0$, then $u_{x'} = \tilde{0}$, which implies $u = f(y')$ where $y' = bx - ay$ and f can be arbitrary smooth function.

Now, suppose $a^2 + b^2 \neq 0 \neq c$, note that the PDE in Equation 3.3 is just an ODE of the form

$$y + \frac{a^2 + b^2}{c}y' = 0$$

The general solution to this ODE is

$$y = Ce^{\frac{-ct}{a^2+b^2}}$$

In other words, the general solution of the PDE in Equation 3.3 is

$$u = Ce^{\frac{-cx'}{a^2+b^2}} = Ce^{\frac{-c(ax+by)}{a^2+b^2}}$$

$\blacksquare$