NCKU 112.2
Miscellaneous Facts

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Contents

Chapter 1

General Topology

1.1 Directed Sets

Axiom 1.1.1. (Axioms in Order Theory) Given an relation $(X, \leq)$, and suppose $x, y, z \in X$.

(a) $x \leq x$ (Reflexive)

(b) $x \leq y \leq z \implies x \leq z$ (Transitive)

(c) $x \leq y$ and $y \leq x \implies x = y$ (Antisymmetric)

(d) $x \leq y$ or $y \leq x$ (Connected)

(e) $\forall x, y \in X, \exists z \in X, x \leq z$ and $y \leq z$ (Directed)

We say $(X, \leq)$ form a

(a) total order if it is reflexive, transitive, antisymmetric and connected.

(b) partial order if it is reflexive, transitive and antisymmetric.

(c) preorder if it is reflexive and transitive.

(d) directed set if it is reflexive, transitive and directed.

Theorem 1.1.2. (Why is it called Preorder) Given a preorder $(X, \leq)$, the relation $\sim$ defined by

$$x \sim y \iff x \leq y \text{ and } y \leq x$$

is an equivalence relation and if we define $\leq^e$ on the equivalence class by

$$\exists x \in A, y \in B, x \leq y \implies A \leq^e B$$

Then $\leq^e$ is a partial order. Moreover, if the preorder $\leq$ is directed, then $\leq^e$ is also directed.

Proof. We first show $\sim$ is an equivalence relation. Because preoder is reflexive, we see

$$\forall x \in X, x \leq x \text{ which implies } \forall x \in X, x \sim x$$

For symmetry, it is easy to see

$$x \sim y \implies x \leq y \text{ and } y \leq x \implies y \sim x$$

For transitive, see

$$\begin{aligned} x \sim y \text{ and } y \sim z &\implies x \leq y \text{ and } y \leq x \text{ and } y \leq z \text{ and } z \leq y \\ &\implies x \leq z \text{ and } z \leq x \implies x \sim z \text{ (done)} \end{aligned}$$

We now show $\leq^e$ is a partial order. Reflexive property and Transitive property of $\leq^e$ follow from that of $\leq$. Suppose $A \leq^e B$ and $B \leq^e A$, where $x_1, x_2 \in A, y_1, y_2 \in B$ satisfy $x_1 \leq y_1$ and $y_2 \leq x_2$. Because $x_1, x_2 \in A$ and $y_1, y_2 \in B$, we have

$$x_1 \leq x_2 \text{ and } x_2 \leq x_1 \text{ and } y_1 \leq y_2 \text{ and } y_2 \leq y_1$$

Then because $\leq$ satisfy transitive, we have

$$\begin{cases} x_2 \leq x_1 \leq y_1 \implies x_2 \leq y_1 \\ y_1 \leq y_2 \leq x_2 \implies y_1 \leq x_2 \end{cases}$$

This tell us

$$x_2 \sim y_1$$

which implies $A = B$, thus proving $\leq^e$ is antisymmetirc. (done)

Lastly, we show $\leq$ is directed $\implies$ $\leq^e$ is directed. Let A, B be two arbitrary equivalence class. We wish to find an equivalence class T such that

$$A \leq^e T \text{ and } B \leq^e T$$

Let a, b respectively be an arbitrary element of A, B. Because $\leq$ is directed, we know there exists $c \in X$ such that

$$a \leq c \text{ and } b \leq c$$

We immediately see

$$A \leq^e [c] \text{ and } B \leq^e [c] \text{ (done)}$$

$\blacksquare$

Corollary 1.1.3. (Chunk Structure of Preorder) Given two equivalence class A, B, we have

$$A \leq^e B \implies \forall x \in A, y \in B, x \leq y$$

Proof. Because $A \leq^e B$, we know

$$\exists x_0 \in A, y_0 \in B, x_0 \leq y_0$$

Then by definition of $\sim$, we have

$$x \leq x_0 \leq y_0 \leq y$$

This give us

$$x \leq y$$

■

Definition 1.1.4. (Definition of Maximal element in Preorder) Let $(I, \leq)$ be a preorder. We say $m \in I$ is a maximal element if

$$\forall y \in I, m \leq y \implies y \leq m$$

Theorem 1.1.5. (In Preorder, Maximal element form an Equivalence class) Let $(I, \leq)$ be a preorder, and $m \in I$ be a maximal element. Then

$$\forall x \in [m], x \text{ is a maximal element}$$

Proof. Arbitrarily pick an element x in $[m]$. Suppose

$$x \leq y$$

By definition of $\sim$, we have

$$m \leq x \leq y$$

Thus $m \leq y$. Then because m is maximal, we know $y \leq m$. This now give us

$$y \leq m \leq x$$

■

Notice that in partially ordered set, where anti-symmetric property is true, the definition of maximal element $m \in I$ falls into

$$\forall y \in I, m \leq y \implies y = m$$

Definition 1.1.6. (Definition of Greatest element in Preorder) Let $(I, \leq)$ be a preorder. We say $x \in I$ is a greatest element if

$$\forall y \in I, y \leq x$$

Theorem 1.1.7. (In Directed Set, Maximal element is the Greatest) Suppose $(I, \leq)$ is a directed set.

$$x \in I \text{ is a maximal element } \implies x \in I \text{ is the greatest element}$$

Proof. Arbitrarily pick an element $y \in I$. Because I is directed, we see there exists an element z such that

$$y \leq z \text{ and } x \leq z$$

Then because x is maximal, we know

$$y \leq z \leq x$$

This shows

$$y \leq x$$

■

Theorem 1.1.8. (Sufficient Condition for Preorder to become Directed)

$$(I, \leq) \text{ is a preorder and has a greatest element } x \implies I \text{ is a directed set}$$

Proof. Given arbitrary two element $y, z \in I$, we see $y \leq x$ and $z \leq x$. ■

Example 1 (Partial Order that is Directed)

$$X = \{a, b, c\} \text{ and } a \leq c \text{ and } b \leq c$$

Example 2 (Partial Order that is Not Directed)

$$X = \{a, b, c\} \text{ and } a \leq b \text{ and } a \leq c$$

Example 3 (Partial Order that is Directed)

$$X = \mathbb{Z}_0^+ \text{ and } \forall x, y \in \mathbb{N}, x \leq y \iff y - x | 2 \text{ and } x \leq y$$
$$\text{and } \forall x \in \mathbb{N}, x \leq 0$$

Example 4 (Partial Order that is not Directed)

$$X = \mathbb{N} \text{ and } \forall x, y \in \mathbb{N}, x \leq y \iff y - x | 2 \text{ and } x \leq y$$

Example 5 (Directed Set that is not Partially Ordered)

$$\begin{aligned} X = \{a, b, c\} &\text{ and } a \leq b \text{ and } b \leq a \\ &\text{ and } a \leq c \text{ and } b \leq c \end{aligned}$$

Example 6 (Preorder that is Neither Directed nor Partially Ordered)

$$\begin{aligned} X = \{a, b, c, d\} &\text{ and } a \leq b \text{ and } b \leq a \\ &\text{ and } a \leq c \text{ and } b \leq c \\ &\text{ and } a \leq d \text{ and } b \leq d \end{aligned}$$

Example 7 (Directed Sets)

X is a metric space and $x \leq y \iff d(y, x_0) \leq d(x, x_0)$ where x_0 is a fixed point in X

Notice that this directed set is generally not antisymmetric, meaning it generally isn't a partial order. Also, notice that x_0 is the greatest element. Also, this order is connected, meaning if we take equivalence class on it, it become a total order.

Lastly, notice that if we remove x_0, X can still be directed, say if $X = \mathbb{R}^2$ and x_0 is the origin.

Example 8 (Directed Sets)

Suppose X, Y are both directed sets. We see $X \times Y$ is a directed set if we define

$$(x, y) \leq (a, b) \iff x \leq a \text{ and } y \leq b$$

Example 9 (Partial Order)

Every collection of sets is a partial order if we define

$$A \le B \iff A \subseteq B$$

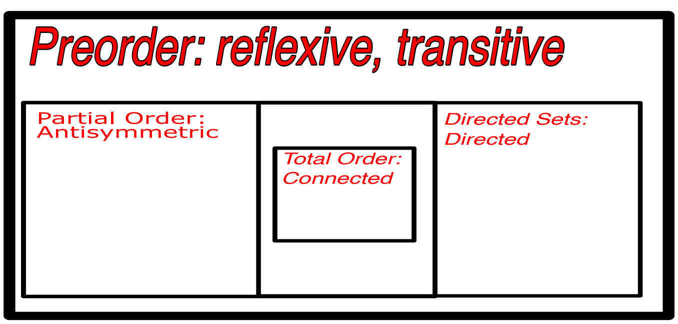
Also, every collection of sets form a partial order if we define

$$A \le B \iff A \supseteq B$$

Example 10 (Directed Sets)

Suppose (X, τ) is a topological space and $x \in X$. Then all of τ, neighborhoods of x and open neighborhoods of x form directed sets under $\subseteq$, since X is open.

Also, τ, neighborhoods of x and open neighborhoods of x form directed sets under $\supseteq$, because intersection of two open set is again an open set and intersection of two neighborhood is again a neighborhood.



Definition 1.1.9. (Definition of Cofinal) Given a directed set $\mathcal{D}$, a subset $\mathcal{D}' \subseteq \mathcal{D}$ is called cofinal if

$$\forall d \in \mathcal{D}, \exists e \in \mathcal{D}', d \le e$$

Theorem 1.1.10. (Cofinal Subset is a Directed Set with Original Order) Given a directed set $\mathcal{D}$

$$\mathcal{D}' \subseteq \mathcal{D} \text{ is cofinal } \implies \mathcal{D}' \text{ is a directed set}$$

Proof. Arbitrarily pick two $a, b \in \mathcal{D}'$. Because $\mathcal{D} \ni a, b$ is directed, we know

$$\exists c \in \mathcal{D}, a \leq c \text{ and } b \leq c$$

Then because $\mathcal{D}'$ is cofinal in $\mathcal{D}$, we know

$$\exists d \in \mathcal{D}', c \leq d$$

Then because transitivity of directed set, our proof is finished, as we have found an element d in $\mathcal{D}'$ that is greater than the arbitrary picked elements $a, b \in \mathcal{D}'$. ■

1.2 Net

Definition 1.2.1. (Subnet) Given a net $w : \mathcal{D} \to X$ and $v : \mathcal{E} \to X$ and a function $h : \mathcal{E} \to \mathcal{D}$ we say v is a subnet of w if

$$\begin{cases} \forall e, e' \in \mathcal{E}, e \leq e' \implies h(e) \leq h(e')(\text{monotone}) \\ h[E] \text{ is cofinal in } \mathcal{D} \\ v = w \circ h \end{cases}$$

Definition 1.2.2. (Net convergence) We say the net $w : \mathcal{D} \to X$ converge to x, $w \to x$ if

Theorem 1.2.3. ($w \to x \implies v \to x$) Suppose v is a subnet of w, we have

$$w \to x \implies v \to x$$

Proof. ∎

Theorem 1.2.4. ()

Definition 1.2.5. ()

Chapter 2

Metric Space

2.1

Chapter 3

Calculus

3.1 Examples for uniform convergence

Theorem 3.1.1. (Test Example) The sequence

$$f_n(x) = \frac{x^2}{x^2 + (1 - nx)^2} \text{ is not equicontinuous on } [0, 1]$$

Proof. Notice that

$$f_n(\frac{1}{n}) = 1 \text{ and } f_n(0) = 0$$

Then for all δ, we see that if n is large enough

$$\text{then } \left|\frac{1}{n} - 0\right| < \delta \text{ and } \left|f_n(\frac{1}{n}) - f_n(0)\right| = 1$$

■

Theorem 3.1.2. (Test Example) Prove

$$\frac{x}{1 + nx^2} \text{ uniformly converge on } \mathbb{R}$$

Proof. It is clear that $\frac{x}{1+nx^2}$ pointwise converge to 0. Because $\frac{x}{1+nx^2}$ is an odd function, fixing ϵ, we only wish to find N such that

$$\forall x > 0, \forall n > N, \frac{x}{1 + nx^2} < \epsilon$$

Observe

$$\begin{aligned} \frac{x}{1 + nx^2} < \epsilon &\iff x < \epsilon(1 + nx^2) \\ &\iff \frac{x - \epsilon}{\epsilon x^2} < n \end{aligned}$$

Notice that $\frac{x-\epsilon}{\epsilon x^2}$ is bounded since it is continuous and converge to 0 as $x \to \infty$. ■

3.2 Test Example

Theorem 3.2.1. (Cauchy-Schwarz Inequality for Integral) Let $\mathscr{R}\Big([a,b]\Big)$ be the space of Riemann-Integrable functions on $[a,b]$. It is clear that $\mathscr{R}\Big([a,b]\Big)$ is a vector space over $\mathbb{R}$. Define $\langle\cdot,\cdot\rangle$ on $\mathscr{R}\Big([a,b]\Big)$ by

$$\langle f,g\rangle=\int_a^b f(x)g(x)dx$$

It is easy to show

(a) $\forall f\in\mathscr{R}\Big([a,b]\Big),\langle f,f\rangle\geq 0$ (non-negativity)

(b) $\forall f,g\in\mathscr{R}\Big([a,b]\Big),\langle f,g\rangle=\langle g,f\rangle$ (Symmetry)

(c) $\forall f,g,h\in\mathscr{R}\Big([a,b]\Big),\forall c\in\mathbb{R},\langle cf+g,h\rangle=c\langle f,h\rangle+\langle g,h\rangle$ (Linearity in first argument)

This make $\langle\cdot,\cdot\rangle$ a **positive semi-definite Hermitian form**. We shall prove Cauchy-Schwarz Inequality hold for positive semi-definite Hermitian form. That is, we shall prove

$$\forall f,g\in\mathscr{R}\Big([a,b]\Big),\langle f,g\rangle\leq\|f\|\cdot\|g\|$$

Proof. ∎

Theorem 3.2.2. (Application) Given $f\in\mathscr{R}\Big([a,b]\Big)$ such that

(a) $f(a)=0=f(b)$

(b) $\int_a^b f^2(x)dx=1$

(c) f is continuously differentiable on (a,b)

(d) $f'\in\mathscr{R}\Big([a,b]\Big)$

We have

$$\int_a^b xf(x)f'(x)=\frac{-1}{2}$$

and have

$$\int_a^b\big(f'(x)\big)^2dx\cdot\int_a^b\big(xf(x)\big)^2dx>\frac{1}{4}$$

Proof. Notice that

$$\frac{d}{dx}xf^2(x) = f^2(x) + 2xf(x)f'(x)$$

Then by Integral by Part (We have to check $\big(xf^2(x)\big)'(t) = f^2(t) + 2tf(t)f'(t)$ for all $t \in (a,b)$, and we have to check $xf^2(x)$ is continuous on $[a,b]$), we have

$$1 = \int_a^b f^2(x)dx = xf^2(x)\Big|_a^b - \int_a^b 2xf(x)f'(x)dx$$

Then because $f(b) = f(a) = 0$, we see

$$2\int_a^b xf(x)f'(x)dx = -1$$

We wish to show

$$\|f'\|^2 \cdot \|xf(x)\|^2 > \frac{1}{4} = \Big(\langle f', xf(x)\rangle\Big)^2$$

It is clear that $\geq$ is valid from Cauchy-Schwarz Inequality. We have to prove $\neq$. In other words, we have to prove

$$f' \text{ and } xf(x) \text{ are linearly independent}$$

Assume f' and $xf(x)$ are linearly dependent. Then

$$\exists c \in \mathbb{R}, \forall x \in [a,b], f'(x) = cxf(x)$$

The solution for this first order linear homogeneous ODE is

$$f(x) = Ae^{\frac{cx^2}{2}} \text{ where } A \in \mathbb{R} \text{ depends on } f(a) \text{ and } f(b)$$

Then because $f(a) = f(b) = 0$, we see $A = 0$. Then $\int_a^b f^2(x)dx = 0$ CaC ∎

Theorem 3.2.3. (Example) Given $G, g, \alpha : [a,b] \to \mathbb{R}$, suppose

(a) $G'(x) = g(x)$ for all $x \in (a,b)$ (G is differentiable on (a,b))

(b) G is continuous on $[a,b]$

(c) α increase on $[a,b]$

(d) g is properly Riemann-Integrable on $[a,b]$

Prove

$$\int_a^b \alpha(x)g(x)dx = \alpha G\Big|_a^b - \int_a^b G(x)d\alpha$$

Proof. ∎

3.3 Dini's Theroem

Theorem 3.3.1. **(Dini's Theorem)** Given a topological space X and a sequence of functions $f_n : X \to \mathbb{R}$, suppose

(a) X is compact

(b) f_n is continuous

(c) $f_n \to f$ pointwise

(d) f is continuous

(e) $f_n(x) \leq f_{n+1}(x)$ for all $x \in X$

Then

$$f_n \to f \text{ uniformly}$$

Proof. Define $g_n : X \to \mathbb{R}$

$$g_n = f - f_n$$

We reduce the problem into

$$\text{proving } g_n \to 0 \text{ uniformly}$$

Notice that we have the property

(a) $g_n(x) \geq g_{n+1}(x)$ for all $x \in X$

(b) g_n is continuous

(c) $g_n \to 0$ pointwise

Fix ϵ. We wish

$$\text{to find } N \text{ such that } \forall n > N, \forall x \in X, g_n(x) < \epsilon$$

Define $E_n \subseteq X$ by

$$E_n = \{x \in X : g_n(x) < \epsilon\}$$

Because g_n is continuous and $E_n = g_n^{-1}\Big[(-\infty, \epsilon)\Big]$, we know

$$E_n \text{ is open for all } n \in \mathbb{N}$$

We first prove

$$\{E_n\}_{n\in\mathbb{N}} \text{ is an open cover of } X$$

Fix $y \in X$. We wish

$$\text{to find } n \text{ such that } y \in E_n$$

Because $g_n(y) \to 0$, this is clear. (done)

We now prove

$$\{E_n\}_{n\in\mathbb{N}} \text{ is ascending}$$

Fix $n \in \mathbb{N}$. We wish

$$\text{to prove } E_n \subseteq E_{n+1}$$

Because $g_n(x) \geq g_{n+1}(x)$ for all $x \in X$ and $E_n = g_n^{-1}\Big[(-\infty, \epsilon)\Big]$ by definition, we see

$$y \in E_n \implies g_{n+1}(y) < g_n(y) < \epsilon \implies y \in E_{n+1} \text{ (done)}$$

Now, because X is compact and $\{E_n\}_{n\in\mathbb{N}}$ is an open cover of X, we know

$$\text{there exists } N \text{ such that } X \subseteq \bigcup_{k=1}^{N} E_k = E_N \tag{3.1}$$

It is clear such N works. (done) ■

3.4 Exam Shit

$$\begin{aligned}
A_{n\times n} \text{ is invertible } &\iff \det(A) \neq 0 \\
&\iff L_A(x) \triangleq [A_{n\times n}][x] \text{ is invertible} \\
&\iff \text{ Columns space is of dimension } n \\
&\iff \text{ Row spaces is of dimension } n \\
&\iff L_{A^t}(x) \triangleq [A^t][x] \text{ is invertible} \\
&\iff L_A \text{ is one-to-one} \\
&\iff L_A \text{ is onto} \\
&\iff N(L_A) \text{ is of dimension } 0 \\
&\iff R(L_A) \text{ is of dimension } n \\
&\iff \text{ the above four same things but change } L_A \text{ to } L_{A^t}
\end{aligned}$$

Proof. Recall that A^{-1} is the matrix defined to satisfy

$$A^{-1}A = I$$

We now see

$$y = Ax \implies A^{-1}(y) = A^{-1}(Ax) = (A^{-1}A)x = Ix = x$$

Using $\det(A^{-1})\det(A) = \det(A^{-1}A) = \det(I) = 1$, it is to see that $\det(A^{-1}) \neq 0$. This implies the row space of A^{-1} is full rank. Now, because

$$(AA^{-1})A = A(A^{-1}A) = AI = A$$

By distribution law, we can deduce

$$(AA^{-1} - I)A = 0$$

Arbitrarily pick $x \in M_{1\times n}(\mathbb{R})$. We see

$$[x(AA^{-1} - I)]A = 0$$

Because the row space of A^{-1} is full rank, this give us

$$x(AA^{-1} - I) = 0$$

Now, because x is arbitrary, we can deduce $AA^{-1} - I = 0$, which implies

$$AA^{-1} = I$$

This give us

$$A^{-1}x = y \implies x = Ix = (AA^{-1})x = A(A^{-1}x) = Ay$$

$\blacksquare$

Chapter 4

Multi-Variable Calculus

4.1

Chapter 5

HW

5.1 HW1

Question 1

1. Let $f_k : [0,1] \to \mathbb{R}$ be given by

$$f_k(x) = \begin{cases} 0, & \text{if } \frac{1}{k} \leq x \leq 1, \\ -kx+1, & \text{if } 0 \leq x < \frac{1}{k}. \end{cases}$$

   (a) Does $\{f_k\}_{k=1}^{\infty}$ converge pointwise on $[0,1]$? If so, find f such that $f_k \to f$ pointwise on $[0,1]$.

   (b) Does f_k converge uniformly on $[0,1]$?

Proof. **(a)** We claim

$$f_k \to f \text{ pointwise on } [0,1] \text{ where } f(x) = \begin{cases} 0 & \text{if } x \in (0,1] \\ 1 & \text{if } x = 0 \end{cases}$$

Because $\forall k \in \mathbb{N}, f_k(0) = 1$, it is clear $f_k(0) \to f(0)$. Now, let $x \in (0,1]$. We reduce our problem into proving

$$f_k(x) \to 0 \text{ as } k \to \infty$$

By definition, we have

$$\forall n > \frac{1}{x}, f_n(x) = 0 \text{ (done)}$$

Above is true since $n > \frac{1}{x} \implies \frac{1}{n} < x$.

b No. It is easy to show that f_k are all continuous and that f is discontinuous at 0. This let us deduce that the convergence is not uniform, since if it is, the function f should have been continuous. ■

Question 2

2. Let $f_k : [0,1] \to \mathbb{R}$ be given by $f_k(x) = x^k$.

   (a) Does $\{f_k\}_{k=1}^{\infty}$ converge pointwise on $[0,1]$? If so, find f such that $f_k \to f$ pointwise on $[0,1]$.

   (b) Does f_k converge uniformly on $[0,1]$?

   (c) For any $a \in (0,1)$, Does f_k converge uniformly on $[0,a]$?

Proof. **(a)** We claim

$$f_k \to f \text{ pointwise on } [0,1] \text{ where } f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \in [0,1) \end{cases}$$

Because $f_k(1) = 1$ for all $k \in \mathbb{N}$, it is clear $f_k(1) \to f(1)$. Now, let $x \in (0,1]$. We reduce our problem into proving

$$f_k(x) \to 0 \text{ as } k \to \infty$$

Fix ϵ. We wish

$$\text{to find } N \text{ such that } \forall n > N, f_n(x) < \epsilon$$

We claim

$$N > \log_x \epsilon \text{ works}$$

Fix $n > N$. Because $x < 1$, we see

$$f_n(x) = x^n < x^N < \epsilon \text{ (done)}$$

b No. It is easy to show that f_k are all continuous and that f is discontinuous at 1. This let us deduce that the convergence is not uniform, since if it is, the function f should have been continuous.

(c) Yes. Fix ϵ and $a \in (0,1)$. We wish to

$$\text{find } N \text{ such that } \forall n > N, \forall x \in [0,a], f_n(x) \leq \epsilon$$

We claim

$$N > \log_a \epsilon \text{ works}$$

Observe

$$\forall n > N, \forall x \in [0, a], f_n(x) = x^n \leq a^n \leq a^N < \epsilon vdon$$

■

Question 3

3. Let $f_k : \mathbb{R} \to \mathbb{R}$ be given by $f_k(x) = \dfrac{\sin x}{k}$.

   (a) Does $\{f_k\}_{k=1}^{\infty}$ converge pointwise on $\mathbb{R}$? If so, find f such that $f_k \to f$ pointwise on $\mathbb{R}$.

   (b) Does f_k converge uniformly on $\mathbb{R}$?

Proof. We show

$$f_k \to 0 \text{ uniformly}$$

Remark: Notice that the 0 above is the function that map all reals to 0.

Fix ϵ.

$$\text{find } N \text{ such that } \forall n > N, \|f_n - 0\|_\infty \leq \epsilon$$

We claim

$$N > \frac{1}{\epsilon} \text{ works}$$

Using the fact $|\sin x| \leq 1$ for all $x \in \mathbb{R}$, we can deduce

$$\forall n > N, \forall x \in \mathbb{R}, |f_n(x)| = \left|\frac{\sin x}{n}\right| \leq \frac{1}{n} < \frac{1}{N} < \epsilon$$

This then implies $\|f_n - 0\|_\infty \leq \epsilon$ (done) .

Remark: Notice that it is of course possible that $\|f_N\|_\infty = \epsilon$. This is why you shouldn't always set the goal by proving strict inequality when proving convergence. That maybe "technically cool" if you catch my drift, but it is just unnecessary and stupid. ■

Question 4

4. Let f_n be integrable on $[0,1]$ and $f_n \to f$ uniformly on $[0,1]$. Show that if $b_n \nearrow 1$ as $n \to \infty$, then

$$\lim_{n\to\infty} \int_0^{b_n} f_n(x)\,dx = \int_0^1 f(x)\,dx$$

Proof. Because f_n is Riemann-integrable on $(0,1)$ and $f_n \to f$ uniformly on $(0,1)$. We know f is Riemann-integrable on $(0,1)$ and

$$\int_0^1 f_n dx \to \int_0^1 f(x)dx \text{ as } n \to \infty$$

Then

$$\lim_{n\to\infty} \int_0^{b_n} f_n dx = \lim_{n\to\infty} \Big(\int_0^1 f_n dx - \int_{b_n}^1 f_n dx \Big) = \int_0^1 f dx - \lim_{n\to\infty} \int_{b_n}^1 f_n dx$$

This let us reduce the problem into proving

$$\int_{b_n}^1 f_n dx \to 0 \text{ as } n \to \infty$$

Fix ϵ. We wish

$$\text{to find } N \text{ such that } \forall n > N, \left| \int_{b_n}^1 f_n dx \right| \leq \epsilon$$

Because each $f_n : [0,1] \to \mathbb{R}$ is bounded (f_n is integrable), and $f_n \to f$ uniformly. We know f_n are uniformly bounded (This will be *fully* justified in the proof for Question 7). Then, we know there exists M such that

$$M > \sup_n (\sup_{[0,1]} |f_n|)$$

Because $b_n \nearrow 1$. We know

$$\exists N, \forall n > N, |b_n - 1| < \frac{\epsilon}{M}$$

We claim

$$\text{such } N \text{ works}$$

Let $n > N$. See

$$\begin{aligned}
\left|\int_{b_n}^1 f_n dx\right| &\le \int_{b_n}^1 |f_n|\, dx \\
&\le \int_{1-\frac{\epsilon}{M}}^1 |f_n|\, dx \\
&\le \int_{1-\frac{\epsilon}{M}}^1 M dx = \epsilon \text{ (done)}
\end{aligned}$$

■

Lemma 5.1.1. (product of uniformly convergent sequence is uniformly convergent on bounded domain) Given

(a) $f_n \to f$ and $g_n \to g$ uniformly on I

(b) f, g are bounded on I

Then

$$f_n g_n \to fg \text{ on } I$$

Proof. Observe

$$\begin{aligned}
|(f_n g_n)(x) - (fg)(x)| &= \left|\big((f_n - f)g_n\big)(x) + \big(f(g_n - g)\big)(x)\right| \\
&\le |(f_n - f)(x)| \cdot |g_n(x)| + |f(x)| \cdot |(g_n - g)(x)|
\end{aligned}$$

Notice that there exists M globally greater than both $|g_n|$ and $|f|$, and that $(f_n - f)(x)$ and $(g_n - g)(x)$ both uniformly converge to 0 and we are done. ■

Question 5

5. If f is continuous on $[0,1]$ and if

$$\int_0^1 f(x)\,x^n\,dx = 0 \;\; (n = 0, 1, 2, ...)$$

Prove that $f(x) = 0$ on $[0,1]$. Hint: The integral of the product of f with any polynomial is zero. Use the Weierstrass theorem to show that $\int_0^1 f^2(x)\,dx = 0$

Proof. By Stone-Weierstrass Theorem, there exists a sequence of polynomial $P_k \to f$ uniformly. Because each polynomial is an finite linear combination of $x^n \quad (n = 0, 1, 2, \dots)$, from premise we can deduce

$$\int_0^1 fP_k dx = 0 \text{ for all } k \in \mathbb{N}$$

Because f is continuous on the compact domain $[0,1]$ and $P_n \to f$. It is easy to see that f and P_n satisfy the hypothesis of Lemma 5.1.1. Then, we see

$$fP_n \to f^2 \text{ uniformly}$$

This then let us deduce

$$\int_0^1 f^2 dx = 0$$

Assume $f(x) \neq 0$ for some $x \in [0,1]$, in the aiming for a contradiction. Because f^2 is continuous at x ($\because f$ is continuous at x). We know there exists δ such that

$$\inf_{[x-\delta, x+\delta]} f^2 = \alpha > 0$$

for some appropriate α, says, $\alpha = \frac{f^2(x)}{2}$.

Now, because $f^2 \geq 0$, we have

$$\int_0^1 f^2 dt \geq \int_{x-\delta}^{x+\delta} f^2 dt \geq 2\delta\alpha > 0 \text{ CaC } \text{ to } \int_0^1 f^2 dt = 0$$

■

Question 6

6. Show that if $\{f_n\}$ is a sequence of continuous functions on E such that converges uniformly to f, then f is continuous on E.

Proof. Click the following hyperlink (Theorem 5.3.1) ■

Question 7

7. Prove that if f_n is bounded on E, $\forall n \in \mathbf{N}$ and f_n converges uniformly to a bounded function f on E, then $\{f_n\}$ is uniformly bounded on E.

Proof. We first prove

$$f \text{ is bounded}$$

Assume f is not bounded. Let $p \in E$, we know there exists sequence $x_n \subseteq E$ such that $d(f(x_n), p) \to \infty$. Now, for arbitrary $k \in \mathbb{N}$, we see

$$d(f(x_n), p) \leq d(f_k(x_n), f(x_n)) + d(f_k(x_n), p)$$

Then because $f_k(x_n) \to f(x_n)$ uniformly, this give us

$$d(f_k(x_n), p) \geq d(f(x_n, p)) - d(f_k(x_n), f(x_n)) \to \infty$$

This implies f_k is unbounded CaC . (done)

We now prove

$$f_n \text{ is uniformly bounded}$$

Let $p \in E$ and $M \in \mathbb{R}^+$ satisfy

$$f[E] \subseteq B_M(p)$$

Because $\|f_n - f\|_\infty \to 0$, we know there exists $L \in \mathbb{R}^+$ such that $\|f_n - f\|_\infty < L$ for all $n \in \mathbb{N}$. We claim

$$\bigcup_{n \in \mathbb{N}} f[E] \subseteq B_{M+L}(p)$$

Fix $n \in \mathbb{N}$ and $x \in E$. We wish to show

$$d(f_n(x), p) < M + L$$

Observe

$$d(f_n(x), p) \leq d(f_n(x), f(x)) + d(f(x), p) < L + M \text{ (done)}$$

$\blacksquare$

Question 8

8. Let $f_k : [0, 1] \to \mathbb{R}$ be a sequence of functions such that

   (1) $|f_k(x)| \leq M_1$ for all $k \in \mathbb{N}$ and $x \in [0, 1]$,

   (2) $|f_k'(x)| \leq M_2$ for all $k \in \mathbb{N}$ and $x \in [0, 1]$.

   for some positive M_1, M_2.

   (a) Prove that there exists a subsequence of $\{f_k\}_{k=1}^{\infty}$ which converges uniformly on $[0, 1]$.

   (b) If the assumption (1) is omitted, can $\{f_k\}_{k=1}^{\infty}$ still have a convergent subsequence? If yes, prove it; If not, give an counterexample.

   (c) Show that the assumption (1) can be replaced by $f_k(0) = 0$ for all $k \in \mathbb{N}$.

Proof. **(a)** The assumption (1) implies f_k is pointwise bounded. We first show

$$f_k \text{ are equicontinuous}$$

Fix ϵ. We wish to find δ such that

$$\forall n \in \mathbb{N}, \forall x, y \in [0, 1], |x - y| < \delta \implies |f_n(x) - f_n(y)| \leq \epsilon$$

We claim

$$\delta < \frac{\epsilon}{M_2} \text{ works}$$

Fix $n \in \mathbb{N}$ and $x, y \in [0, 1]$ such that $|x - y| < \delta$. By Lagrange's MVT, we see

$$\frac{|f_k(x) - f_k(y)|}{|x - y|} \leq M_2$$

Then

$$|f_k(x) - f_k(y)| \leq M_2 \cdot |x - y| \leq M_2 \cdot \delta = \epsilon \text{ (done)}$$

(b). No. Consider $f_k(x) = x + k$. It is clear that $f_k(x)$ has no even pointwise convergent sequence, as for all x_0, the sequence $f_k(x_0)$ diverge.

(c) Suppose we are given assumption (2). It suffice to show that

$$\forall k \in \mathbb{N}, f_k(0) = 0 \implies \exists M_1 \in \mathbb{R}^+, \forall k \in \mathbb{N}, \forall x \in [0,1], |f_k(x)| \leq M_1$$

We claim

$$M_1 = M_2 \text{ works}$$

Fix $k \in \mathbb{N}$ and $x \in [0,1]$. By FTC and assumption two, we see

$$|f_k(x)| = \left| \int_0^x f_k' dt \right| \leq \int_0^x |f_k'| \, dt \leq \int_0^1 |f_k'| \, dt \leq \int_0^1 M_2 dt = M_2 = M_1 \text{ (done)}$$

∎

5.2 Limit Interchange

Given an arbitrary set X and a complete metric space $(\overline{Y}, d)$, in Section **??**, we have proved that the set of functions with the following properties

(a) boundedness

(b) unboundedness

are respectively closed under uniform convergence. In next section (Section 5.3), we will prove that the following three properties

(a) continuity

(b) uniform continuity

(c) K-Lipschitz continuity

are again all closed under uniform convergence, where the proof for continuity is closed under uniform convergence use Theorem 5.2.1 as a lemma.

Here, we prove

(a) convergent of sequences

in, of course, complete metric space, is also closed under uniform convergence.

The reason we require the codomain $\overline{Y}$ of sequence to be complete is explained in the last paragraph of Section **??**. An example of such beautiful closure is lost if the codmain (Y, d) is not complete is $Y = \mathbb{R}^*$ and $a_{n,k} = \frac{1}{n} + \frac{1}{k}$.

Theorem 5.2.1. (Change Order of Limit Operations: Part 1) Given a double sequence $a_{n,k}$ whose codomain is (Y, d). Suppose

(a) $a_{n,k} \to a_{\bullet,k}$ uniformly as $n \to \infty$

(b) $a_{n,k} \to A_n$ pointwise as $k \to \infty$.

(c) $A_n \to A$

Then we can deduce

$$\lim_{k\to\infty} a_{\bullet,k} \text{ exists and } \lim_{k\to\infty} a_{\bullet,k} = A$$

In other words, we can switch the order of limit operations

$$\lim_{k\to\infty} \lim_{n\to\infty} a_{n,k} = \lim_{n\to\infty} \lim_{k\to\infty} a_{n,k}$$

Proof. We wish to prove

$$a_{\bullet,k} \to A \text{ as } k \to \infty$$

Fix ϵ. Because $a_{n,k} \to a_{\bullet,k}$ uniformly and $A_n \to A$ as $n \to \infty$, we know there exists m such that

$$d(A_m, A) < \frac{\epsilon}{3} \text{ and } \forall k \in \mathbb{N}, d(a_{m,k}, a_{\bullet,k}) < \frac{\epsilon}{3} \tag{5.1}$$

Then because $a_{m,k} \to A_m$ as $k \to \infty$, we know there exists K such that

$$\forall k > K, d(a_{m,k}, A_m) < \frac{\epsilon}{3} \tag{5.2}$$

We now claim

$$\forall k > K, d(a_{\bullet,k}, A) < \epsilon$$

The claim is true since by Equation 5.1 and Equation 5.2, we have

$$\forall k > K, d(a_{\bullet,k}, A) \leq d(a_{\bullet,k}, a_{m,k}) + d(a_{m,k}, A_m) + d(A_m, A) < \epsilon \text{ (done)}$$

■

Theorem 5.2.2. (Change Order of Limit Operations: Part 2) Given a double sequence $a_{n,k}$ whose codomain is (Y, d). Suppose

(a) $a_{n,k} \to a_{\bullet,k}$ uniformly as $n \to \infty$

(b) $a_{n,k} \to A_n$ pointwise as $k \to \infty$

(c) $a_{\bullet,k} \to A$ as $k \to \infty$

Then we can deduce

$$A_n \text{ converge and } A_n \to A$$

Proof. Fix ϵ. We wish to find N such that

$$\forall n > N, d(A_n, A) < \epsilon$$

Because $a_{n,k} \to a_{\bullet,k}$ uniformly as $n \to \infty$, we can let N satisfy

$$\forall n > N, \forall k \in \mathbb{N}, d(a_{n,k}, a_{\bullet,k}) < \frac{\epsilon}{3} \tag{5.3}$$

We claim

$$\text{such } N \text{ works}$$

Arbitrarily pick $n > N$. Because $a_{\bullet,k} \to A$, and because $a_{n,k} \to A_n$, we know there exists j such that

$$d(a_{\bullet,j}, A) < \frac{\epsilon}{3} \text{ and } d(a_{n,j}, A_n) < \frac{\epsilon}{3} \tag{5.4}$$

From Equation 5.3 and Equation 5.4, we now have

$$d(A_n, A) \leq d(A_n, a_{n,j}) + d(a_{n,j}, a_{\bullet,j}) + d(a_{\bullet,j}, A) < \epsilon \text{ (done)}$$

■

In summary of Theorem 5.2.1 and Theorem 5.2.2, given a double sequence $a_{n,k}$ converging both side

(a) $a_{n,k} \to a_{\bullet,k}$ pointwise as $n \to \infty$

(b) $a_{n,k} \to a_{n,\bullet}$ pointwise as $k \to \infty$

As long as

(a) one side of convergence is uniform

(b) between two sequence $\{a_{\bullet,k}\}_{k\in\mathbb{N}}$ and $\{a_{n,\bullet}\}_{n\in\mathbb{N}}$, one of them converge, say, to A

Then the other sequence also converge, and the limit is also A.

It is at this point, we shall introduce two other terminologies. Suppose f_n is a sequence of functions from an arbitrary set X to a metric space Y. We say f_n is **pointwise Cauchy** if for all fixed $x \in X$, the sequence $f_n(x)$ is Cauchy. We say f_n is **uniformly Cauchy** if for all ϵ, there exists $N \in \mathbb{N}$ such that

$$\forall n, m > N, \forall x \in X, d\big(f_n(x), f_m(x)\big) < \epsilon$$

In last Section (Section **??**), we define the **uniform metric** d_∞ on X^Y by

$$d_\infty(f, g) = \sup_{x\in X} d\big(f(x), g(x)\big)$$

and say that $f_n \to f$ uniformly if and only if $f_n \to f$ in (X^Y, d_∞). Similar to this clear fact, we have

$$f_n \text{ is uniformly Cauchy} \iff f_n \text{ is Cauchy in } (X^Y, d_\infty)$$

It should be very easy to verify that if f_n uniformly converge, then f_n is uniformly Cauchy, and just like sequences in metric space, the converse hold true if and only if the space $\big(X^Y, d_\infty\big)$ is complete. In Theorem 5.2.3, we give a necessary and sufficient condition for $\big(X^Y, d_\infty\big)$ to be complete.

Theorem 5.2.3. (Space of functions $\big(X^Y, d_\infty\big)$ is Complete iff Y is Complete) Given an arbitrary set X and a metric space (Y, d), we have

$$\text{the extended metric space } \big(X^Y, d_\infty\big) \text{ is complete } \iff Y \text{ is complete}$$

Proof. $(\longleftarrow)$

Suppose f_n is uniformly Cauchy. We wish

$$\text{to construct a } f : X \to Y \text{ such that } f_n \to f \text{ uniformly}$$

Because f_n is uniformly Cauchy, we know that for all $x \in X$, the sequence $f_n(x)$ is Cauchy in (Y, d). Then because Y is complete, we can define $f : X \to Y$ by

$$f(x) = \lim_{n\to\infty} f_n(x)$$

We claim

$$\text{such } f \text{ works, i.e. } f_n \to f \text{ uniformly}$$

Fix ϵ. We wish

$$\text{to find } N \in \mathbb{N} \text{ such that for all } n > N \text{ and } x \in X \text{ we have } d\big(f_n(x), f(x)\big) < \epsilon$$

Because f_n is uniformly Cauchy, we know there exists N such that

$$\forall n, m > N, \forall x \in X, d\big(f_n(x), f_m(x)\big) < \frac{\epsilon}{2} \tag{5.5}$$

We claim

$$\text{such } N \text{ works}$$

Assume there exists $n > N$ and $x \in X$ such that $d\big(f_n(x), f(x)\big) \geq \epsilon$. Because $f_k(x) \to f(x)$ as $k \to \infty$, we know

$$\exists m \in \mathbb{N}, d\big(f_m(x), f(x)\big) < \frac{\epsilon}{2} \tag{5.6}$$

Then from Equation 5.5 and Equation 5.6, we can deduce

$$\epsilon \leq d\big(f_n(x), f(x)\big) \leq d\big(f(x), f_m(x)\big) + d\big(f_n(x), f_m(x)\big) < \epsilon \text{ CaC (done)}$$

$(\longrightarrow)$

Let K be the set of constant functions in X^Y. We first prove

$$K \text{ is closed}$$

Arbitrarily pick $f \in K^c$. We wish

$$\text{to find } \epsilon \in \mathbb{R}^+ \text{ such that } B_\epsilon(f) \in K^c$$

Because f is not a constant function, we know there exists $x_1, x_2 \in X$ such that

$$d\big(f(x_1), f(x_2)\big) > 0$$

We claim that

$$\epsilon = \frac{d\big(f(x_1), f(x_2)\big)}{3} \text{ works}$$

Arbitrarily pick $g \in B_\epsilon(f)$. We wish

$$\text{to show } g \in K^c$$

Notice the triangle inequality

$$3\epsilon = d\big(f(x_1), f(x_2)\big) \leq d\big(f(x_1), g(x_1)\big) + d\big(g(x_1), g(x_2)\big) + d\big(g(x_2), f(x_2)\big) \tag{5.7}$$

Also, because $g \in B_\epsilon(f)$, we have

$$\forall x \in X, d\big(f(x), g(x)\big) < \epsilon \tag{5.8}$$

Then by Equation 5.7 and Equation 5.8, we see

$$d\big(g(x_1), g(x_2)\big) > \epsilon$$

This then implies g is not a constant function. (done)

Now, Because by premise (X^Y, d_∞) is complete, and we have proved K is closed in (X^Y, d_∞), we know K is complete. Then, we resolve the whole problem into proving

$$Y \text{ is isometric to } K$$

Define $\sigma : Y \to K$ by

$$y \mapsto \tilde{y} \text{ where } \forall x \in X, \tilde{y}(x) = y$$

It is easy to verify σ is an isometry. (done) ∎

Corollary 5.2.4. (Space of Bounded functions $\big(B(X,Y),d_\infty\big)$ is Complete iff Y is Complete)

$$\big(B(X,Y),d_\infty\big) \text{ is complete } \iff Y \text{ is complete}$$

Proof. $(\longleftarrow)$

By Theorem 5.2.3, the space (X^Y,d_∞) is complete. Then because $B(X,Y)$ is closed in (X^Y,d_∞), we know $B(X,Y)$ is complete.

$(\longrightarrow)$

Notice that the set of constant function K is a subset of the galaxy $B(X,Y)$. The whole proof in Theorem 5.2.3 works in here too. ∎

Remember in the beginning of this section we say we will prove convergent sequences in Y is closed under uniform convergence if Y is complete. The proof of this result relies on Theorem 5.2.3.

Now, before we actually prove convergence sequences are closed under uniform convergence if codomain (Y,d) is complete (Theorem 5.2.6), we will state and prove Weierstrass M-test (Theorem 5.2.5), which concerns the uniform convergence of series of complex functions.

Theorem 5.2.5. (Weierstrass M-test) Given sequences $f_n : X \to \mathbb{C}$, and suppose

$$\forall n \in \mathbb{N}, \forall x \in X, |f_n(x)| \leq M_n \tag{5.9}$$

Then

$$\sum_{n=1}^{\infty} M_n \text{ converge} \implies \sum_{n=1}^{\infty} f_n \text{ uniformly converge}$$

Proof. Because $(\mathbb{C},\|\cdot\|_2)$ is complete, by Corollary 5.2.4, we only wish to prove

$$\left\langle \sum_{k=1}^{n} f_k \right\rangle_{n\in\mathbb{N}} \text{ is uniformly Cauchy}$$

Fix ϵ. We wish

$$\text{to find } N \text{ such that } \forall n,m > N, \forall x \in X, \left|\sum_{k=n}^{m} f_k(x)\right| < \epsilon$$

Because $\sum_{n=1}^{\infty} M_n$ converge, we know there exists N such that

$$\forall n, m > N, \sum_{k=n}^{m} M_k < \epsilon$$

We claim

$$\text{such } N \text{ works}$$

By Premise 5.9, we have

$$\forall n, m > N, \forall x \in X, \left| \sum_{k=n}^{m} f_k(x) \right| \leq \sum_{k=n}^{m} |f_k(x)| \leq \sum_{k=n}^{m} M_k < \epsilon$$

■

Theorem 5.2.6. (Convergent Sequences are Closed under Uniform Convergence if Codomain $\big(Y, d\big)$ is Complete) Given a complete metric space $\big(Y, d\big)$, let $\mathcal{C}_{\mathbb{N}}^{Y}$ be the set of convergent sequences in Y.

$$Y \text{ is complete } \implies \mathcal{C}_{\mathbb{N}}^{Y} \text{ is closed under uniform convergent}$$

Proof. Let $a_{n,k} \to a_{\bullet,k}$ uniformly as $n \to \infty$ where for all $n, k \in \mathbb{N}, a_{n,k} \in Y$ and let $A_n = \lim_{k\to\infty} a_{n,k}$ for all $n \in \mathbb{N}$.

$$\text{to prove } a_{\bullet,k} \text{ converge}$$

By Theorem 5.2.2, we can reduce the problem to

$$\text{proving } A_n \text{ converge}$$

Then because Y is complete, we can then reduce the problem into proving

$$A_n \text{ is Cauchy}$$

Fix ϵ. We wish to find N such that

$$\forall n, m > N, d(A_n, A_m) < \epsilon$$

Because $a_{n,k} \to a_{\bullet,k}$ uniformly, we can find N such that

$$\forall n, m > N, d_{\infty}(\{a_{n,k}\}_{k\in\mathbb{N}}, \{a_{m,k}\}_{k\in\mathbb{N}}) < \frac{\epsilon}{3} \tag{5.10}$$

We claim

$$\text{such } N \text{ works}$$

Arbitrarily pick $n, m > N$. We wish to prove

$$d(A_n, A_m) < \epsilon$$

Because $a_{n,k} \to A_n$ and $a_{m,k} \to A_m$ as $k \to \infty$, we can find j such that

$$d(a_{n,j}, A_n) < \frac{\epsilon}{3} \text{ and } d(a_{m,j}, A_m) < \frac{\epsilon}{3} \tag{5.11}$$

Then from Equation 5.10 and Equation 5.11, we can deduce

$$d(A_n, A_m) \leq d(A_n, a_{n,j}) + d(a_{n,j}, a_{m,j}) + d(a_{m,j}, A_m) < \epsilon \text{ (done)}$$

■

5.3 Closed under Uniform Convergence

The end goal for this section is to prove that the following properties

(a) continuity

(b) uniform continuity

(c) K-Lipschitz Continuity

Theorem 5.3.1. (Uniform Limit Theorem) Given a sequence of function f_n from a topological space (X, τ) to a metric space (Y, d), suppose

(a) $f_n \to f$ uniformly as $n \to \infty$

(b) f_n is continuous for all $n \in \mathbb{N}$

Then f is also continuous.

Proof. Fix $x \in X$, and let $x_k \to x$. We wish to prove

$$f(x_k) \to f(x)$$

Because $f_n \to f$ uniformly as $n \to \infty$, we know

$$f_n(x_k)_{k\in\mathbb{N}} \to f(x_k)_{k\in\mathbb{N}} \text{ uniformly as } n \to \infty \tag{5.12}$$

Also, because for each $n \in \mathbb{N}$, the function f_n is continuous at x, we know

$$\forall n \in \mathbb{N}, f_n(x_k) \to f_n(x) \text{ as } k \to \infty \tag{5.13}$$

Then because $f_n \to f$ pointwise, we know

$$f_n(x) \to f(x) \tag{5.14}$$

Now, because Equation 5.12, Equation 5.13 and Equation 5.14, by Theorem 5.2.1, we have

$$\lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} \lim_{n\to\infty} f_n(x_k) = \lim_{n\to\infty} \lim_{k\to\infty} f_n(x_k) = \lim_{n\to\infty} f_n(x) = f(x) \text{ (done)}$$

∎

Suppose X is a compact Hausdroff space, with Theorem **??**, we can now say that the set $\mathcal{C}(X)$ of complex-valued continuous functions on X

Theorem 5.3.2. (Uniformly Continuous functions are Closed under Uniform Convergence) Given a sequence of functions f_n from a metric space (X, d_X) to metric space (Y, d_Y), suppose

(a) $f_n \to f$ uniformly

(b) f_n is uniformly continuous for all $n \in \mathbb{N}$

Then f is also uniformly continuous

Proof. Fix ϵ. We wish

$$\text{to find } \delta \text{ such that } \forall x, y \in X, d_X(x,y) < \delta \implies d_Y\big(f(x), f(y)\big) < \epsilon$$

Because $f_n \to f$ uniformly, we know there exists $m \in \mathbb{N}$ such that

$$\forall x \in X, d_Y\big(f_m(x), f(x)\big) < \frac{\epsilon}{3} \tag{5.15}$$

Because f_m is uniformly continuous, we know

$$\exists \delta, \forall x, y \in X, d_X(x,y) < \delta \implies d_Y\big(f_m(x), f_m(y)\big) < \frac{\epsilon}{3} \tag{5.16}$$

We claim

$$\text{such } \delta \text{ works}$$

Let $x, y \in X$ satisfy $d_X(x,y) < \delta$. We wish

$$\text{to prove } d_Y\big(f(x), f(y)\big) < \epsilon$$

From Equation 5.15 and Equation 5.16, we have

$$d_Y\big(f(x), f(y)\big) \leq d_Y\big(f(x), f_m(x)\big) + d_Y\big(f_m(x), f_m(y)\big) + d_Y\big(f_m(y), f(y)\big) = \epsilon \text{ (done)}$$

■

Theorem 5.3.3. **(K-Lipschitz functions are Closed under Uniform Convergence)** Given a sequence of functions f_n from metric space (X, d_X) to metric space (Y, d_Y), suppose

(a) $f_n \to f$ uniformly as $n \to \infty$

(b) f_n is K-Lipschtize continuous for all $n \in \mathbb{N}$

Then f is also K-Lipschtize continuous.

Proof. Arbitrarily pick $x, y \in X$, to show f is K-Lipschtize continuous, we wish

$$\text{to show } d_Y\big(f(x), f(y)\big) \leq K d_X(x,y)$$

Fix ϵ. We reduce the problem into proving

$$d_Y\big(f(x), f(y)\big) < K d_X(x,y) + \epsilon$$

Because $f_n \to f$ uniformly as $n \to \infty$, we know there exists m such that

$$\forall z \in X, d_Y\big(f(z), f_m(z)\big) < \frac{\epsilon}{2} \tag{5.17}$$

Because f_m is K-Lispchitz continuous, we know

$$d_Y\big(f_m(x), f_m(y)\big) \leq K d_X(x, y) \tag{5.18}$$

Now, from Equation 5.18 and Equation 5.17, we now see

$$d_Y\big(f(x), f(y)\big) \leq d_Y\big(f(x), f_m(x)\big) + d_Y\big(f_m(x), f_m(y)\big) + d_Y\big(f_m(y), f(y)\big) < K d_X(x, y) + \epsilon$$

$\blacksquare$

An example of sequences of Lipschitz continuous functions with unbounded Lipschitz constant can uniformly converge to a non-Lipschitz continuous function is given below

Example 11 (Lipschitz functions with Unbounded Lipschitz constant Uniformly Converge to a non-Lipschitz function)

$$X = [0, 1] \text{ and } f_n(x) = \sqrt{x + \frac{1}{n}}$$

5.4 HW2

Question 9

1. Suppose f is Riemann integrable on $[0, A]$ for all $A < \infty$, and $f(x) \to 1$ as $x \to \infty$. Prove that
$$\lim_{t\to 0^+} t\int_0^\infty e^{-tx} f(x)\,dx = 1$$

Proof. We can reduce the problem into proving

$$\lim_{t\to 0^+}\int_0^\infty te^{-tx}f(x)dx - 1 = 0$$

Notice that for each $t > 0$, we have

$$1 = \int_0^\infty te^{-tx}dx$$

This then give us

$$\begin{aligned}\int_0^\infty e^{-tx}f(x)dx - 1 &= \int_0^\infty te^{-tx}f(x)dx - \int_0^\infty te^{-tx}dx \\ &= \int_0^\infty te^{-tx}\big[f(x)-1\big]dx\end{aligned}$$

Define $g(x) \triangleq f(x) - 1$. Because $f \to 1$ at ∞, we know $g \to 0$ at infinity. We now reduce the problem into proving

$$\lim_{t\to 0^+}\int_0^\infty te^{-tx}g(x)dx = 0$$

Note that with simple computation

$$\int_0^\infty te^{-tx}dx \text{ exists for all } t \in \mathbb{R}^+$$

Then because we have $te^{-tx} \sim te^{-tx}f(x)$ as $x \to \infty$, we see

$$\int_0^\infty te^{-tx}g(x)dx \text{ exists for all } t \in \mathbb{R}^+ \text{ by Integral Test and Limit Comparison Test}$$

Fix ϵ. We now reduce the problem into proving

$$\text{finding } \delta \text{ such that } \left|\int_0^\infty te^{-tx}g(x)dx\right| \le \epsilon \text{ for all } t \in (0,\delta)$$

Let A be large enough such that $g(x)$ is $\frac{\epsilon}{2}$-close to 0 whenever $x \geq A$. Note that g is bounded on $[A, \infty)$ and bounded on $[0, A]$ because g is integrable on $[0, A]$. Now, let $M > \sup_{\mathbb{R}^+} |g|$. We claim

$$\delta = \frac{-\ln(1 - \frac{\epsilon}{2M})}{A} \text{ works}$$

Observe

$$\begin{aligned}
\left| \int_0^\infty te^{-tx} g(x) dx \right| &\leq \left| \int_0^A te^{-tx} g(x) dx \right| + \left| \int_A^\infty te^{-tx} g(x) dx \right| \\
&\leq \int_0^A te^{-tx} \left| g(x) \right| dx + \int_A^\infty te^{-tx} \left| g(x) \right| dx \\
&\leq M \int_0^A te^{-tx} dx + \frac{\epsilon}{2} \int_A^\infty te^{-tx} dx \\
&\leq -Me^{-tx} \Big|_{x=0}^A + \frac{\epsilon}{2} \int_0^\infty te^{-tx} dx \\
&\leq M(1 - e^{-tA}) + \frac{\epsilon}{2} \\
&\leq M(1 - e^{-\delta A}) + \frac{\epsilon}{2} \\
&= M(1 - e^{\ln(1 - \frac{\epsilon}{2M})}) + \frac{\epsilon}{2} = \epsilon \text{ (done)}
\end{aligned}$$

■

Question 10

2. For $\delta \in (0, \pi)$ we define $f(x) = \begin{cases} 1 & , |x| \leq \delta \\ 0 & , \delta < |x| \leq \pi \end{cases}$, also $f(x + 2\pi) = f(x)$ for all x.

   (a) Compute the Fourier coefficients of f.

   (b) Conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2}$.

   (c) Deduce from Parseval's theorem that $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} = \frac{\pi - \delta}{2}$.

   (d) Let $\delta \to 0$ and prove that $\displaystyle\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$.

   (e) Put $\delta = \dfrac{\pi}{2}$ in (c), what do you discover?

Proof. **(a)**

For $n \neq 0$, compute

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\ &= \frac{1}{2\pi} \int_{-\delta}^{\delta} \cos(-nx) + i \sin(nx) dx \\ &= \frac{1}{2\pi} \int_{-\delta}^{\delta} \cos(-nx) dx \quad (\because \text{ sin is odd function }) \\ &= \frac{1}{2\pi} \cdot \frac{\sin(-nx)}{-n} \Big|_{x=-\delta}^{\delta} = \frac{\sin(n\delta)}{n\pi} \end{aligned}$$

Compute

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{\delta}{\pi}$$

(b)

Note that $c_{-n} = c_n$. We then deduce that the Fourier Series of f is

$$\frac{\delta}{\pi} + \sum_{n=1}^{\infty} \frac{2\sin(n\delta)}{n\pi} e^{-inx}$$

Because f is constant around 0, it is clearly Lipschitz at 0. We now deduce

$$1 = f(0) = \frac{\delta}{\pi} + \sum_{n=1}^{\infty} \frac{2\sin(n\delta)}{n\pi}$$

This implies

$$\sum_{n=1}^{\infty} \frac{2\sin(n\delta)}{n\pi} = 1 - \frac{\delta}{\pi}$$

which implies

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = (1 - \frac{\delta}{\pi}) \cdot \frac{\pi}{2} = \frac{\pi - \delta}{\pi} \cdot \frac{\pi}{2} = \frac{\pi - \delta}{2}$$

(c)

Parseval's Theorem says

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \sum_{-\infty}^{\infty} |c_n|^2$$

Clearly f is Riemann-Integrable. Plugin our setting, we see

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2 \, dx = \frac{\delta}{\pi}$$

and

$$\begin{aligned}\sum_{-\infty}^{\infty} |c_n|^2 &= (\frac{\delta}{\pi})^2 + \sum_{n=1}^{\infty} 2\Big(\frac{\sin(n\delta)}{n\pi}\Big)^2 \\ &= (\frac{\delta}{\pi})^2 + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2}\end{aligned}$$

This let us deduce

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2} = \frac{\pi^2}{2} \cdot (\frac{\delta}{\pi} - \frac{\delta^2}{\pi^2}) = \frac{\pi\delta - \delta^2}{2}$$

So

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} = \frac{\pi - \delta}{2}$$

(d)

Fix ϵ. Because $\int_0^\infty \frac{\sin^2}{x^2} dx$ absolutely converge, we can find R satisfying

$$\left| \int_R^\infty \frac{\sin^2 x}{x^2} dx \right| < \frac{\epsilon}{3} \text{ and } R > \frac{3}{\epsilon}$$

Define $\delta_N \triangleq \frac{R}{N}$. Partition $[0, R]$ by $\{0, R(\frac{1}{N}), R(\frac{2}{N}), \ldots, R\}$. We see

$$\sum_{n=1}^{N} \frac{\sin^2(n\delta_N)}{(n\delta_N)^2} \delta_N \text{ is a Riemann Sum of Norm } |\delta_N|$$

This implies that there exists N_0 such that

$$\forall N > N_0, \left| \sum_{n=1}^{N} \frac{\sin^2(n\delta_N)}{(n\delta_N)^2} \delta_N - \int_0^R \frac{\sin^2 x}{x^2} dx \right| < \frac{\epsilon}{3}$$

Fix $N > N_0$. Observe that

$$\begin{aligned}
\left| \sum_{n=N+1}^{\infty} \frac{\sin^2(n\delta_N)}{(n\delta_N)^2} \delta_N \right| &\leq \sum_{n=N+1}^{\infty} \frac{\sin^2(n\delta_N)}{n^2\delta_N} \\
&\leq \sum_{n=N+1}^{\infty} \frac{1}{n^2\delta_N} \\
&= \frac{1}{\delta_N} \sum_{n=N+1}^{\infty} \frac{1}{n^2} \\
&\leq \frac{1}{\delta_N} \int_N^\infty \frac{1}{x^2} dx \quad (\because \frac{1}{x^2} \searrow) \\
&= \frac{1}{N\delta_N} = \frac{1}{R} < \frac{\epsilon}{3}
\end{aligned}$$

We now see

$$\begin{aligned}
&\left|\sum_{n=1}^{\infty}\frac{\sin^2(n\delta_N)}{(n\delta_N)^2}\delta_N-\int_0^{\infty}\frac{\sin^2 x}{x^2}dx\right| \\
\leq&\left|\sum_{n=1}^{N}\frac{\sin^2(n\delta_N)}{(n\delta_N)^2}\delta_N-\int_0^{R}\frac{\sin^2 x}{x^2}dx\right|+\left|\sum_{n=N+1}^{\infty}\frac{\sin^2(n\delta_N)}{(n\delta_N)^2}\delta_N\right|+\left|\int_R^{\infty}\frac{\sin^2 x}{x^2}dx\right| \\
\leq&\frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon
\end{aligned}$$

This now implies, for all ϵ, we can find R and a threshold N_0 corresponding to R such that

$$\forall N>N_0,\left|\sum_{n=1}^{\infty}\frac{\sin^2(n\delta_N)}{(n\delta_N)^2}\delta_N-\int_0^{\infty}\frac{\sin^2 x}{x^2}dx\right|\leq\epsilon$$

Then for $\epsilon_k=\frac{1}{k}$, we can find a sequence of real number $\delta_k\triangleq\frac{R_k}{N_k}\to 0$ such that

$$\left|\sum_{n=1}^{\infty}\frac{\sin^2(n\delta_k)}{(n\delta_k)^2}\delta_k-\int_0^{\infty}\frac{\sin^2 x}{x^2}dx\right|\leq\frac{1}{k}$$

Because we know

$$\sum_{n=1}^{\infty}\frac{\sin^2(n\delta_k)}{(n\delta_k)^2}\delta_k=\frac{\pi-\delta_k}{2}$$

We now see for each ϵ', because $\delta_k\to 0$, we can find k large enough such that

$$\begin{aligned}
\left|\int_0^{\infty}\frac{\sin^2 x}{x^2}dx-\frac{\pi}{2}\right|&=\left|\int_0^{\infty}\frac{\sin^2 x}{x^2}dx-\frac{\pi-\delta_k}{2}-\frac{\delta_k}{2}\right| \\
&\leq\left|\int_0^{\infty}\frac{\sin^2 x}{x^2}dx-\sum_{n=1}^{\infty}\frac{\sin^2(n\delta_k)}{(n\delta_k)^2}\delta_k\right|+\frac{\delta_k}{2} \\
&\leq\frac{1}{k}+\frac{\delta_k}{2}<\epsilon'
\end{aligned}$$

(e)

Put $\delta=\frac{\pi}{2}$. We have

$$\begin{aligned}
\frac{\pi}{4}&=\frac{\pi-\delta}{2} \\
&=\frac{2}{\pi}\cdot\sum_{n=1}^{\infty}\frac{\sin^2\frac{\pi}{2}n}{n^2}
\end{aligned}$$

This then implies

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

$\blacksquare$

Question 11

3. If $f(x) = (\pi - |x|)^2$ on $[-\pi, \pi]$, prove that $f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$, and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad , \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Proof. Compute Fourier coefficient

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - |x|)^2 dx = \frac{\pi^2}{3}$$

and

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - |x|)^2 e^{-inx} dx \\ &= \frac{2}{n^2} \end{aligned}$$

Note that f is an even function, that $f'(x) = 2(x - \pi)$ on $(0, \pi]$ and that

$$f'(0) = \lim_{x \to 0^+} \frac{(\pi - |x|)^2 - \pi^2}{x} = \lim_{x \to 0^+} \frac{-2\pi x + x^2}{x} = -2\pi$$

This now let us deduce

$$|f'| \leq 2\pi \text{ on } [-\pi, \pi]$$

Which implies f is 2π-Lipschitz on $[-\pi, \pi]$. This tell us that the Fourier Series $s_N(f; x)$ converge to f, meaning

$$\begin{aligned} f(x) = \sum_{-\infty}^{\infty} c_n e^{-inx} &= \frac{\pi^2}{3} + \sum_{-\infty, n \neq 0}^{\infty} \frac{2}{n^2} \cdot \big(\cos(-nx) + i \sin(-nx)\big) \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4 \cos nx}{n^2} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} \end{aligned}$$

We now can deduce

$$\begin{aligned} f(0) &= \pi^2 \\ &= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} \end{aligned}$$

This then implies

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Because f is continuous on $[-\pi, \pi]$, we know f is Riemann-Integrable. Then Parseval's Theorem assert

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2 \, dx = \sum_{-\infty}^{\infty} |c_n|^2$$

Now compute

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2 \, dx &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - |x|)^4 dx \\ &= \frac{1}{\pi} \int_{0}^{\pi} (\pi - x)^4 dx \quad (\because (\pi - |x|)^4 \text{ is even }) \\ &= \frac{1}{\pi} \cdot \frac{(\pi - x)^5}{-5} \Big|_{x=0}^{\pi} = \frac{\pi^4}{5} \end{aligned}$$

and

$$\sum_{-\infty}^{\infty} |c_n|^2 = \frac{\pi^4}{9} + 2\sum_{n=1}^{\infty} \frac{4}{n^4}$$

This now implies

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{8} \cdot (\frac{\pi^4}{5} - \frac{\pi^4}{9}) = \frac{\pi^4}{90}$$

■

Question 12

4. Let $K_N(x) = \dfrac{1}{N+1}\displaystyle\sum_{n=0}^{N} \dfrac{\sin(n+\frac{1}{2})x}{\sin(x/2)}$, show that

   (a) $K_N(x) = \dfrac{1}{N+1} \cdot \dfrac{1-\cos(N+1)x}{1-\cos x}$.

   (b) $K_N \geq 0$.

   (c) $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} K_N(x)\,dx = 1$.

   (d) If $0 < \delta \leq |x| \leq \pi$ then $K_N(x) \leq \dfrac{1}{N+1} \cdot \dfrac{2}{1-\cos\delta}$.

   (e) Let $s_N(f;\, x)$ be the N-th partial sum of the Fourier series of f, consider the arithmedic means

   $$\sigma_N = \frac{s_0 + s_1 + \ldots + s_N}{N+1}$$

   Prove that $\sigma_N(f;\, x) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x-t)K_N(t)\,dt$.

   (f) Use (e)'s result to prove the Fejer's Theorem:

   If f is continuous with period 2π, then $\sigma_N(f;\, x) \to f(x)$ uniformly on $[-\pi,\, \pi]$.

Proof. Proving (a) can be reduced to proving

$$\sum_{n=0}^{N} \frac{\sin(n+\frac{1}{2})x}{\sin(\frac{x}{2})} = \frac{1-\cos(N+1)x}{1-\cos x}$$

Using $\sin\alpha\sin\beta = \frac{1}{2}\Big(\cos(\alpha-\beta) - \cos(\alpha+\beta)\Big)$ to compute

$$\begin{aligned}
\sum_{n=0}^{N} \frac{\sin(n+\frac{1}{2})x}{\sin(\frac{x}{2})} &= \frac{1-\cos x}{1-\cos x}\sum_{n=0}^{N}\frac{\sin(n+\frac{1}{2})x}{\sin(\frac{x}{2})} \\
&= \frac{2\sin^2(\frac{x}{2})}{1-\cos x}\sum_{n=0}^{N}\frac{\sin(n+\frac{1}{2})x}{\sin(\frac{x}{2})} \\
&= \frac{1}{1-\cos x}\sum_{n=0}^{N} 2\sin(\frac{x}{2})\sin(n+\frac{1}{2})x \\
&= \frac{1}{1-\cos x}\sum_{n=0}^{N} \cos(-nx) - \cos(n+1)x \\
&= \frac{1}{1-\cos x}\sum_{n=0}^{N} \cos(nx) - \cos(n+1)x = \frac{1-\cos(N+1)x}{1-\cos x} \text{ (done)}
\end{aligned}$$

(b)

Notice that $\cos x < 1$ and $\cos(N+1)x \le 1 \quad (\because K_N$ is only well defined on $(0,2\pi)$). This then implies

$$1-\cos x > 0 \text{ and } 1-\cos(N+1)x \ge 0$$

Then we can deduce

$$K_N(x) = \frac{1}{N+1}\cdot\frac{1-\cos(N+1)x}{1-\cos x} \ge 0$$

(c)

We first compute the Dirchlet Kernel D_N

$$\begin{aligned}
D_N(x) &= \sum_{-N}^{N} e^{-inx} \\
&= \frac{e^{i(-N)x} - e^{i(N+1)x}}{1-e^{ix}} \\
&= \frac{e^{i(-N-\frac{1}{2})x} - e^{i(N+\frac{1}{2})x}}{e^{i\frac{-1}{2}x} - e^{i\frac{1}{2}x}} \\
&= \frac{2i\sin((-N-\frac{1}{2})x)}{2i\sin(\frac{-1}{2}x)} = \frac{\sin(N+\frac{1}{2})x}{\sin(\frac{x}{2})}
\end{aligned}$$

and

$$D_N(x) = \sum_{-N}^{N} e^{-inx} = 1 + 2\sum_{n=1}^{N} \cos nx$$

Now we can compute

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)dx &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{N+1}\sum_{n=0}^{N} \frac{\sin(n+\frac{1}{2})x}{\sin\frac{x}{2}}dx \\
&= \frac{1}{2\pi(N+1)}\sum_{n=0}^{N}\int_{-\pi}^{\pi} D_n(x)dx \\
&= \frac{1}{2\pi(N+1)}\sum_{n=0}^{N}\int_{-\pi}^{\pi}(1+2\sum_{k=1}^{n}\cos kx)dx \\
&= \frac{1}{2\pi(N+1)}\sum_{n=0}^{N} 2\pi = 1 \quad (\because \int_{-\pi}^{\pi}\cos kx = 0)
\end{aligned}$$

(d)

Suppose $0 < \delta \le |x| \le \pi$. Observe

$$\begin{aligned}
K_N(x) &= \frac{1}{N+1}\cdot\frac{1-\cos(N+1)x}{1-\cos x} \\
&\le \frac{1}{N+1}\cdot\frac{2}{1-\cos x} \quad (\because \cos x < 1) \\
&\le \frac{1}{N+1}\cdot\frac{2}{1-\cos\delta} \quad (\because 0 < \delta \le |x| \le \pi \implies \cos x \le \cos\delta < 1)
\end{aligned}$$

(e)

Compute

$$
\begin{aligned}
\sigma_N(f;x) &= \frac{(s_0 + \cdots + s_N)}{N+1}(f;x) \\
&= \frac{1}{N+1}\sum_{k=0}^{N} s_k(f;x) \\
&= \frac{1}{N+1}\sum_{k=0}^{N}\sum_{n=-k}^{k} c_n e^{inx} \\
&= \frac{1}{N+1}\sum_{k=0}^{N}\sum_{n=-k}^{k} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dte^{inx} \\
&= \frac{1}{N+1}\sum_{k=0}^{N}\frac{1}{2\pi}\sum_{n=-k}^{k}\int_{-\pi}^{\pi} f(t)e^{in(x-t)}dt \\
&= \frac{1}{N+1}\sum_{k=0}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\sum_{n=-k}^{k} e^{in(x-t)}dt \\
&= \frac{1}{N+1}\sum_{k=0}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_k(x-t)dt \\
&= \frac{1}{(N+1)2\pi}\sum_{k=0}^{N}\int_{x+\pi}^{x-\pi} -f(x-u)D_k(u)du \quad (\because u = x-t) \\
&= \frac{1}{(N+1)2\pi}\sum_{k=0}^{N}\int_{x-\pi}^{x+\pi} f(x-u)D_k(u)du \\
&= \frac{1}{(N+1)2\pi}\sum_{k=0}^{N}\int_{-\pi}^{\pi} f(x-u)D_k(u)du \quad (\because \text{ periodicity of } D_k \text{ and } f\ ) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\cdot\Big(\frac{1}{N+1}\sum_{k=0}^{N} D_k(u)\Big)du \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)K_N(u)du
\end{aligned}
$$

(e)

Fix ϵ. We wish

$$\text{to find } N' \text{ such that for all } N > N' \text{ and } x \in \mathbb{R} \text{ we have } |\sigma_N(f;x) - f(x)| \leq \epsilon$$

Because f is continuous with period 2π, we know f is uniformly continuous on $\mathbb{R}$. We then can fix δ small enough such that

$$\sup_{|t|\leq\delta} |f(x-t)-f(x)| < \frac{\epsilon}{2}$$

Also, we can fix $M > \sup_{[-\pi,\pi]} |f|$. Define $Q_\delta = \frac{4M(\pi-\delta)}{\pi(1-\cos\delta)}$. We claim

$$N' > \frac{2Q_\delta}{\epsilon} \text{ works}$$

Fix $N > N'$ and $x \in \mathbb{R}$. Using $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)dt = 1$ and $K_N \geq 0$, see

(a) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)dt = 1$

(b) $K_N \geq 0$

(c) $\pi \geq |x| \geq \delta \implies K_N(x) \leq \frac{1}{N+1} \cdot \frac{2}{1-\cos\delta}$

$$\begin{aligned}
|\sigma_N(f;x) - f(x)| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)dt - f(x)\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)dt\right| \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-t)-f(x)|\, K_N(t)dt \\
&= \frac{1}{2\pi}\int_{-\delta}^{\delta} |f(x-t)-f(x)|\, K_N(t)dt + \frac{2(\pi-\delta)2M}{2\pi} \cdot \frac{1}{N+1} \cdot \frac{2}{1-\cos\delta} \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\epsilon}{2} K_N(t)dt + \frac{4M(\pi-\delta)}{(N+1)\pi(1-\cos\delta)} \\
&= \frac{\epsilon}{2} + \frac{4M(\pi-\delta)}{\frac{2Q_\delta}{\epsilon}\pi(1-\cos\delta)} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ (done)}
\end{aligned}$$

$\blacksquare$

Question 13

5. If $f(x) = \sum_{k=0}^{\infty} a_k(x - x_0)^k$ is a power series with positive radius of convergence R, show that

$$f'(x) = \sum_{k=1}^{\infty} k a_k (x - x_0)^{k-1}$$

for $x \in (x_0 - R, x_0 + R)$.

Theorem 5.4.1. (Power Series are Smooth) Given a power series (a, c_n) of convergence radius R, if we define $f : D_R(a) \to \mathbb{C}$ by

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$$

Then

$$f \text{ is of class } C^\infty \text{ on } D_R(a) \text{ and } f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} c_n (z - a)^{n-k} \text{ on } D_R(a)$$

Proof. We prove by induction. Base case $k = 0$ is trivial. Fix $k \geq 0$. Suppose we have

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} c_n (z - a)^{n-k} \text{ on } D_R(a)$$

We are required to prove

$$f^{(k+1)}(z) = \sum_{n=k+1}^{\infty} \frac{n!}{(n-k-1)!} c_n (z - a)^{n-k-1} \text{ on } D_R(a)$$

Set f_m

$$f_m(z) \triangleq \sum_{n=k}^{k+m} \frac{n!}{(n-k)!} c_n (z - a)^{n-k}$$

We have

$$f_m \to f^{(k)} \text{ pointwise on } D_R(a) \text{ and } f'_m(z) = \sum_{n=k+1}^{k+m} \frac{n!}{(n-k-1)!} c_n (z - a)^{n-k-1} \tag{5.19}$$

We abstract our problem into proving

$$f_m' \to f^{(k+1)} \text{ pointwise on } D_R(a)$$

Fix $z_0 \in D_R(a)$. We only wish to prove

$$(f^{(k)})'(z_0) = \lim_{m\to\infty} f_m'(z_0)$$

Fix ϵ such that $|z_0 - a| < R - \epsilon$. By Equation 5.19, using Theorem 5.5.2 (Uniform Convergence and Differentiaiton). We only have to prove

$$f_m' \text{ uniformly converge on } \overline{D}_{R-\epsilon}$$

Note that

$$f_m'(z) = \sum_{n=0}^{m-1} \frac{(n+k+1)!}{n!} c_{n+k+1}(z-a)^n$$

so we can compute the radius of convergence for f_m'

$$\begin{aligned}
\limsup_{n\to\infty} \sqrt[n]{\frac{(n+k+1)!}{n!}\,|c_{n+k+1}|} &= \limsup_{n\to\infty} \sqrt[n]{|c_{n+k+1}|} \\
&= \limsup_{n\to\infty} \sqrt[n]{|c_n|} = R
\end{aligned}$$

Together by Cauchy-Hadamrd (absolute convergent on $a + R - \epsilon$) and M-test show that

$$\sum_{n=0}^{\infty} \frac{(n+k+1)!}{n!} c_{n+k+1}(z-a)^n \text{ uniformly converge on } \overline{D}_{R-\epsilon}(a) \quad \text{(done)}$$

$\blacksquare$

Question 14

6. Let $I \subseteq \mathbb{R}$ be a finite interval.

(a) Let $f_k : I \to \mathbb{R}$ be differentiable for all $k \in \mathbb{N}$, and $\{f_k'\}$ converges uniformly on I. Determine whether $\{f_k\}$ converges?

(b) Let $f_k : I \to \mathbb{R}$ be differentiable for all $k \in \mathbb{N}$, and $\{f_k\}$ converges uniformly on I. Determine whether f is differentiable?

Proof. **(a)** No. Let $f_k = k$. It is then a trivial counter example.

(b) No. Consider $|x|$. The function $|x|$ is continuous on $[-1,1]$ but not differentiable on $x = 0$. By Weierstrass approximation Theorem, we know there exists a sequence of polynomials on $[-1,1]$ uniformly converge to $|x|$, and they clearly all are differentiable. ■

Question 15

7. Let $f_k : [0,1] \to \mathbb{R}$ be differentiable on $(0,1)$, and f_k converges uniformly to f on $[0,1]$ for some $f : [0,1] \to \mathbb{R}$. Determine whether f_k' converges uniformly?

Proof. No. Consider

Example 12 (Derivative won't necessarily converge to the right place)

$$X = \mathbb{R} \text{ and } f_n(x) = \frac{\sin nx}{\sqrt{n}}$$

Compute

$$f'(x) = 0 \text{ and } f_n'(x) = \sqrt{n} \cos nx$$

$f_n'(0) \to \infty$ shows that f_n' doesn't even have to be pointwise convergence. Note that the fact f_k uniformly converge can be easily proved by choosing $n > \frac{1}{\epsilon^2}$ ■

Question 16

8. Let $f_k : I \to \mathbb{R}$ be Riemann integrable where $I \subseteq \mathbb{R}$ be a finite interval. Suppose f_k converges pointwise to a function $f : I \to \mathbb{R}$. Determine whether f is Riemann integrable on I?

Proof. No. Consider

Example 13 (Riemann-integrable functions Pointwise Converge to a Non-Riemann-integrable function)

$$X = [-1, 1] \text{ and } f_m(x) = \lim_{n\to\infty} (\cos m!\pi x)^{2n}$$

Because cos has range $[-1, 1]$, we know that

$$m!x \in \mathbb{Z} \iff f_m(x) \neq 0$$

This tell us that for all $x \in \mathbb{R} \setminus \mathbb{Q}$, $f_m(x) = 0$, and that for all $x \in \mathbb{Q}$, we have $f_n(x) = 1$ for large enough n, with some simple computation.

Then, we see that

$$f_n \to \mathbf{1}_{\mathbb{Q}} \text{ pointwise}$$

Now, notice that for all fixed m, if $m!x \in \mathbb{Z}$, we must have

$$x = \frac{p}{m!} \text{ for some } p \in \mathbb{Z}$$

Such x in bounded domain must then happen only finite amount of time. This show f_n are all continuous almost everywhere and thus integrable, while $\mathbf{1}_{\mathbb{Q}}$, the function to which they converge, is not, as it is discontinuous almost everywhere. ■

5.5 Uniform Convergence on Integration and Differentiation

Theorem 5.5.1. (Riemann-Integration and Uniform Convergence) Given a function $\alpha : [a,b] \to \mathbb{R}$ and a sequence of functions $f_n : [a,b] \to \mathbb{R}$ such that

(a) α increase on $[a,b]$

(b) $\int_a^b f_n d\alpha$ exists for all $n \in \mathbb{N}$

(c) $f_n \to f$ uniformly on $[a,b]$

Then

$$\lim_{n\to\infty} \int_a^b f_n d\alpha \text{ exists and } \int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha$$

Proof. We first prove

$$\int_a^b f d\alpha \text{ exists}$$

Fix ϵ. We wish to prove

$$\overline{\int_a^b} f d\alpha - \underline{\int_a^b} f d\alpha < \epsilon$$

Let $\epsilon_n = \|f_n - f\|_\infty$. Because $f_n \to f$ uniformly, we know

$$\text{there exists } n \in \mathbb{N} \text{ such that } \epsilon_n = \|f_n - f\|_\infty < \frac{\epsilon}{2\big[\alpha(b) - \alpha(a)\big]}$$

Because α increase, by definition of ϵ_n, we see

$$\int_a^b (f_n - \epsilon_n) d\alpha \leq \underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha \leq \int_a^b (f_n + \epsilon_n) d\alpha$$

Because $\epsilon_n < \frac{\epsilon}{2\big[\alpha(b)-\alpha(a)\big]}$, we now see

$$\begin{aligned}\overline{\int_a^b} f d\alpha - \underline{\int_a^b} f d\alpha &\leq \int_a^b (f_n + \epsilon_n) d\alpha - \int_a^b (f_n - \epsilon_n) d\alpha \\ &= \int_a^b (2\epsilon_n) d\alpha < 2\epsilon_n \cdot \big[\alpha(b) - \alpha(a)\big] = \epsilon \text{ (done)}\end{aligned}$$

We now prove

$$\int_a^b f_n d\alpha \to \int_a^b f d\alpha \text{ as } n \to \infty$$

Fix ϵ. We wish

$$\text{to find } N \text{ such that } \forall n > N, \left| \int_a^b f_n d\alpha - \int_a^b f d\alpha \right| < \epsilon$$

Recall the definition $\epsilon_n = \|f_n - f\|_\infty$. Because $\epsilon_n \to 0$, we know

$$\text{there exists } N \text{ such that } \forall n > N, \epsilon_n < \frac{\epsilon}{\alpha(b) - \alpha(a)} \tag{5.20}$$

We claim

$$\text{such } N \text{ works}$$

Fix $n > N$. From Equation 5.20, we see

$$\begin{aligned}
\left| \int_a^b f_n d\alpha - \int_a^b f d\alpha \right| &= \left| \int_a^b (f_n - f) d\alpha \right| \\
&\leq \int_a^b |f_n - f| \, d\alpha \\
&\leq \int_a^b \epsilon_n d\alpha = \epsilon_n \big[\alpha(b) - \alpha(a)\big] < \epsilon \text{ (done)}
\end{aligned}$$

■

Before the next Theorem, let's see three examples why this time we don't (can't) use the hypothesis: $f_n \to f$ uniformly.

Example 14 (Differentiable functions are NOT closed under uniform convergence)

$$X = [-1, 1] \text{ and } f(x) = |x|$$

By Weierstrass approximation Theorem, there is a sequence of polynomials (differentiable) that uniformly converge to f, which is not differntiable at 0.

Example 15 (Derivative won't necessarily converge to the right place)

$$X = \mathbb{R} \text{ and } f_n(x) = \frac{\sin nx}{\sqrt{n}}$$

Compute

$$f'(x) = 0 \text{ and } f_n'(x) = \sqrt{n} \cos nx$$

Example 16 (Derivative won't necessarily converge to the right place)

$$X = \mathbb{R} \text{ and } f_n(x) = \frac{x}{1 + nx^2}$$

Compute

$$f = \tilde{0} \text{ and } f_n'(0) = 1$$

Informally speaking, these examples together with the fact integral are closed under uniform convergence (Theorem 5.5.1) should give you some ideas that differentiation and integration although are operations inverse to each other, are NOT symmetric. There is a certain hierarchy on continuous functions on a fixed compact interval. Thus, we have the next Theorem in its form.

Theorem 5.5.2. (Uniform Convergence and Differentiation) Given a sequence of function $f_n : [a, b] \to \mathbb{R}$ such that

(a) $f_n(x_0) \to L$ for some $x_0 \in [a, b]$

(b) f_n are differentiable on (a, b)

(c) f_n are continuous on $[a, b]$

(d) f_n' uniformly converge on (a, b)

Then there exists a function $f : [a, b] \to \mathbb{R}$ such that

$$\begin{aligned} f &\text{ is differentiable on } (a, b) \\ \text{and } f_n &\to f \text{ uniformly on } [a, b] \\ \text{and } f_n' &\to f' \text{ uniformly on } (a, b) \end{aligned}$$

Proof. We first prove

$$f_n \text{ uniformly converge on } [a, b] \tag{5.21}$$

Fix ϵ. We wish

$$\text{to find } N \text{ such that } \|f_n - f_m\|_\infty \leq \epsilon \text{ for all } n, m > N$$

Because $f_n(x_0)$ converge, and f_n' uniformly converge, we know there exists N such that

$$\begin{cases} |f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2} \\ \|f_n' - f_m'\|_\infty < \frac{\epsilon}{2(b-a)} \end{cases} \quad \text{for all } n, m > N \tag{5.22}$$

We claim

$$\text{such } N \text{ works}$$

Fix $x \in [a, b]$ and $n, m > N$. We need

$$\text{to show } |f_n(x) - f_m(x)| \leq \epsilon$$

We first prove

$$|f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| \leq \frac{\epsilon}{2}$$

Because $(f_n - f_m)' = f_n' - f_m'$, by MVT (Theorem **??**) and Equation 5.22, we can deduce

$$\begin{aligned} |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| &= |(f_n - f_m)(x) - (f_n - f_m)(x_0)| \\ &= \Big|\big[(f_n - f_m)'(t)\big](x - x_0)\Big| \text{ for some } t \text{ between } x, x_0 \\ &< \frac{\epsilon}{2(b-a)} \cdot |x - x_0| \\ &\leq \frac{\epsilon}{2(b-a)} \cdot (b-a) = \frac{\epsilon}{2} \ \ \big(\because x, x_0 \in [a, b]\big) \text{ (done)} \end{aligned}$$

Now, by Equation 5.22, we have

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ (done)} \end{aligned}$$

We claim

$$f(x) \triangleq \lim_{n \to \infty} f_n(x) \text{ for all } x \in [a, b] \text{ works} \tag{5.23}$$

We first show

$$f \text{ is differentiable on } (a, b)$$

Fix $x \in (a,b)$. We wish to prove

$$\lim_{t \to x} \frac{f(t)-f(x)}{t-x} \text{ exists}$$

Define $\phi : [a,b] \setminus x \to \mathbb{R}$ by

$$\phi(t) \triangleq \frac{f(t)-f(x)}{t-x}$$

We reduce our problem into proving

$$\lim_{t \to x} \phi(t) \text{ exists}$$

Set $\phi_n : [a,b] \setminus x \to \mathbb{R}$ by

$$\phi_n(t) \triangleq \frac{f_n(t)-f_n(x)}{t-x}$$

We first show

$$\phi_n \text{ uniformly converge on } [a,b] \setminus x \tag{5.24}$$

Fix ϵ. We have

$$\text{to find } N \text{ such that } |\phi_n(t) - \phi_m(t)| \leq \epsilon \text{ for all } n,m > N \text{ and } t \in [a,b] \setminus x$$

Because f'_n uniformly converge on $[a,b]$, we know there exists N such that

$$\|f'_n - f'_m\|_\infty \leq \epsilon \text{ for all } n,m > N \tag{5.25}$$

We claim

$$\text{such } N \text{ works}$$

Fix $n,m > N$ and $t \in [a,b] \setminus x$. We wish to prove

$$|\phi_n(t) - \phi_m(t)| \leq \epsilon$$

Because $(f_n - f_m)' = f'_n - f'_m$, by MVT (Theorem **??**) and Equation 5.25, we can deduce

$$\begin{aligned}
|\phi_n(t) - \phi_m(t)| &\leq \left| \frac{f_n(t)-f_n(x)}{t-x} - \frac{f_m(t)-f_m(x)}{t-x} \right| \\
&= \left| \frac{\big(f_n - f_m\big)(t) - \big(f_n - f_m\big)(x)}{t-x} \right| \\
&= \left| \big(f'_n - f'_m\big)(t_0) \right| \text{ for some } t_0 \text{ between } t,x \\
&\leq \epsilon \text{ (done)}
\end{aligned}$$

We now show

$$\phi_n \to \phi \text{ pointwise on } [a,b] \setminus x \tag{5.26}$$

Because $f_n \to f$ on $[a,b]$ by definition (Equation 5.23), (the convergence is in fact uniform as we have shown. This doesn't matter here tho), for each $t \in [a,b] \setminus x$, we can deduce

$$\lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty} \frac{f_n(t) - f_n(x)}{t - x} = \frac{f(t) - f(x)}{t - x} = \phi(t) \text{ (done)}$$

Now, by Equation 5.24 and Equation 5.26, we know

$$\phi_n \to \phi \text{ uniformly on } [a,b] \setminus x$$

Notice that because $f_n'(x)$ converge, we know

$$\lim_{n\to\infty} \lim_{t\to x} \phi_n(t) = \lim_{n\to\infty} f_n'(x) \text{ exists}$$

Then (Notice that the second equality below hold true because we have known $\lim_{n\to\infty} \lim_{t\to x} \phi_n(t)$ exists), we can finally deduce

$$\begin{aligned} \lim_{t\to x} \phi(t) &= \lim_{t\to x} \lim_{n\to\infty} \phi_n(t) \\ &= \lim_{n\to\infty} \lim_{t\to x} \phi_n(t) \\ &= \lim_{n\to\infty} f_n'(x) \text{ exists} \quad \text{(done)} \end{aligned}$$

Now, notice that $f'(x) = \lim_{t\to x} \phi(t)$, so in fact, we have just proved $f_n' \to f'$, and the convergence is uniform by premise. Also, the statement

$$f_n \to f \text{ uniformly on } [a,b]$$

has been proved, since we already have $f_n \to f$ by our setting (Equation 5.23) and we have proved such convergence is uniform (Equation 5.21). The proof is now completed. (done) ■

As Rudin remarked, a much shorter (and much more intuitive) proof can be given, if we require f' to be continuous on $[a,b]$.

Theorem 5.5.3. (Uniform Convergence and Differentiation: Weaker Version) Given a sequence of function $f_n : [a,b] \to \mathbb{R}$ such that

(a) $f_n(x_0) \to L$ for some $x_0 \in [a,b]$

(b) f_n are differentiable on (a,b)

(c) f_n' are continuous on $[a,b]$ (f_n' at a,b are one-sided)

(d) f_n are continuous on $[a,b]$

(e) f_n' uniformly converge on $[a,b]$

Then there exists a function $f:[a,b]\to\mathbb{R}$ such that

$$\begin{aligned} f &\text{ is differentiable on } (a,b) \\ \text{and } f_n &\to f \text{ uniformly on } [a,b] \\ \text{and } f_n' &\to f' \text{ uniformly on } (a,b) \end{aligned}$$

Proof. We claim

$$f(x)=\lim_{n\to\infty}\int_{x_0}^{x} f_n'(t)dt+L \text{ works}$$

Note that $\lim_{n\to\infty}\int_{x_0}^{x} f_n'(t)dt$ exists because f_n' uniformly converge (Theorem 5.5.1).

Because f_n' uniformly converge and are continuous on $[a,b]$, by ULT, we know

$$\int_{x_0}^{x}\lim_{n\to\infty} f_n'(t)dt+L \text{ exists}$$

and know

$$f(x)=\int_{x_0}^{x}\lim_{n\to\infty} f_n'(t)dt+L$$

By FTC, we see

$$f'(x)=\lim_{n\to\infty} f_n'(x) \text{ on } (a,b)$$

Such convergence is uniform by premise. To finish the proof, we now only have to prove

$$f_n\to f \text{ uniformly on } [a,b]$$

Fix ϵ. We wish

$$\text{to find } N \text{ such that } |f_n(x)-f(x)|\leq\epsilon \text{ for all } n>N \text{ and } x\in[a,b]$$

Because $f_n'\to f$ uniformly, and $f_n(x_0)\to L=f(x_0)$ (Check $L=f(x_0)$), we know there exists N such that

$$\begin{cases} \|f_n'-f\|_\infty<\frac{\epsilon}{2(b-a)} \\ |f_n(x_0)-f(x_0)|<\frac{\epsilon}{2} \end{cases} \quad \text{for all } n>N$$

We claim

$$\text{such } N \text{ works}$$

Fix $n > N$ and $x \in [a, b]$. Observe

$$\begin{aligned}
|f(x) - f_n(x)| &= \left| \int_{x_0}^{x} \big(f'(t) - f_n'(t)\big) dt + f(x_0) - f_n(x_0) \right| \\
&\leq \int_{x_0}^{x} |f'(t) - f_n'(t)| \, dt + |f(x_0) - f_n(x_0)| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ (done)}
\end{aligned}$$

■

5.6 HW3

Question 17

Definition:

(i) The Fourier transform of f on $\mathbb{R}$ is defined by $\widehat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\, dx$.

(ii) The Fourier inverse transform of f on $\mathbb{R}$ is defined by $f(x) = \mathcal{F}^{-1}[\widehat{f}] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{ix\xi}\, d\xi$.

1. Show that $\widehat{f'} = i\xi\widehat{f}$ and $\widehat{xf} = i\frac{d}{d\xi}\widehat{f}$. (You may assume $f \to 0$ as $x \to \pm\infty$)

Proof. Compute

$$\begin{aligned}\hat{f}' - i\xi\hat{f} &= \int_{-\infty}^{\infty} \left(f'(x)e^{-i\xi x} - i\xi f(x)e^{-i\xi x}\right) dx \\ &= f(x)e^{-i\xi x}\Big|_{x=-\infty}^{\infty}\end{aligned}$$

Note that

$$\left|f(x)e^{-i\xi x}\right| = |f(x)|$$

Compute

$$\left|f(M)e^{-i\xi M} - f(-M)e^{i\xi M}\right| \leq |f(M)| + |f(-M)| \to 0 \text{ as } M \to \infty$$

This now implies

$$\hat{f}' - i\xi\hat{f} = \lim_{M\to\infty} f(x)e^{-i\xi x}\Big|_{x=-M}^{M} = 0$$

Define

$$\phi(x,\xi) \triangleq f(x)e^{-i\xi x}$$

It is clear that

$$\partial_\xi \phi(x,\xi) = -ixf(x)e^{-i\xi x} \text{ is continuous every where}$$

Then, we can apply Feynman's Trick to compute

$$\begin{aligned} i\frac{d}{d\xi}\hat{f} &= i\frac{d}{d\xi}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx \\ &= i\int_{-\infty}^{\infty} -ixf(x)e^{-i\xi x}dx \\ &= \int_{-\infty}^{\infty} xf(x)e^{-i\xi x}dx = \widehat{xf} \end{aligned}$$

■

Theorem 5.6.1. (Gaussian Integral)

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

Proof. Fix $I = \int_{-\infty}^{\infty} e^{-x^2}dx$. Compute using Fubini's Theorem

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy \\ &= \int_{0}^{\infty}\int_{0}^{2\pi} re^{-r^2}d\theta dr \\ &= 2\pi\int_{0}^{\infty} re^{-r^2}dr \\ &= -\pi e^{-r^2}\Big|_{r=0}^{\infty} = \pi \end{aligned}$$

Because e^{-x^2} is a positive function, we now have

$$\int_{-\infty}^{\infty} e^{-x^2}dx = I = \sqrt{I^2} = \sqrt{\pi}$$

■

Theorem 5.6.2. (Gaussian Integral)

$$\int_{-\infty}^{\infty} e^{-\frac{(x-a)^2}{b}}dx$$

Proof. Fix

$$y \triangleq \frac{x-a}{\sqrt{b}} \text{ and } \frac{dy}{dx} = \frac{1}{\sqrt{b}}$$

Compute using Theorem 5.6.1

$$\int_{-\infty}^{\infty} e^{-\frac{(x-a)^2}{b}} dx = \int_{-\infty}^{\infty} e^{-y^2} \sqrt{b} dy$$
$$= \sqrt{b\pi}$$

∎

Question 18

2. Let $g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$, $\sigma \neq 0$. $g(x)$ is called the normalized Gaussian function in $\mathbb{R}$. Find the Fourier transform of g on $\mathbb{R}$.

Proof. Compute

$$g'(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot \frac{-2x}{2\sigma^2} e^{\frac{-x^2}{2\sigma^2}} = -\frac{1}{\sigma^2} x g(x)$$

Using the statement of the first question, which we have proved, we now have

$$i\xi \widehat{g} = \widehat{g'} = -\frac{1}{\sigma^2} \widehat{xg} = \frac{-i}{\sigma^2} \frac{d\widehat{g}}{d\xi}$$

This give us the first order homogenoeus ODE

$$\frac{d}{d\xi} \widehat{g} + \sigma^2 \xi \widehat{g} = 0$$

Compute the general solution

$$\widehat{g}(\xi) = C e^{\frac{-\sigma^2 \xi^2}{2}}$$

Compute using Theorem 5.6.2

$$C = \widehat{g}(0) = \int_{-\infty}^{\infty} g(x) dx = 1$$

We now have the answer

$$\widehat{g}(\xi) = e^{\frac{-\sigma^2 \xi^2}{2}}$$

∎

Question 19

3. The convolution of two functions f and g is defined by $f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy$. Show that $\widehat{f * g} = \widehat{f}\widehat{g}$. (You may assume the Fubini's Theorem always holds.)

Proof. Compute using Fubini's Theorem

$$\begin{aligned}
\widehat{f * g}(\xi) &= \int_{-\infty}^{\infty} (f * g)(u)e^{-i\xi u}du \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u-y)g(y)e^{-i\xi u}dudy
\end{aligned}$$

Compute using Fubini's Theorem

$$\begin{aligned}
\widehat{f}\cdot\widehat{g}(\xi) &= \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx \int_{-\infty}^{\infty} g(y)e^{-\xi y}dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)e^{-i\xi(x+y)}dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u-y)g(y)e^{-i\xi u}dudy \qquad \text{where } u = x + y \text{ and } \frac{du}{dx} = 1 \\
&= \widehat{f * g}(\xi)
\end{aligned}$$

■

Question 20

4. For $0 < \alpha < 1$, define $C_\alpha := \Gamma(\frac{\alpha}{2})$, where $\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\,dt$ is the gamma function. Show that

$$C_\alpha \mathcal{F}^{-1}[\pi^{\frac{1}{2}} 2^\alpha |\xi|^{-\alpha}\widehat{f}](x) = C_{1-\alpha}\int_{\mathbb{R}} \frac{f(y)}{|x-y|^{1-\alpha}}\,dy.$$

(You may assume the Fubini's Theorem always holds.)

Proof. Define

$$g(x) \triangleq \frac{1}{|x|^{1-\alpha}}$$

We see

$$\begin{aligned}\int_{\mathbb{R}} \frac{f(y)}{|x-y|^{1-\alpha}} dy &= \int_{\mathbb{R}} \frac{f(x-u)}{|u|^{1-\alpha}} du \quad (\because u = x-y) \\ &= \int_{\mathbb{R}} f(x-u) g(u) du = f * g(x)\end{aligned}$$

Compute

$$C_\alpha \mathcal{F}^{-1}[\pi^{\frac{1}{2}} 2^\alpha |\xi|^{-\alpha} \widehat{f}](x) = C_\alpha \pi^{\frac{1}{2}} 2^\alpha \mathcal{F}^{-1}[|\xi|^{-\alpha} \widehat{f}](x)$$

We now can reduce the problem into proving

$$C_\alpha \pi^{\frac{1}{2}} 2^\alpha \mathcal{F}^{-1}[|\xi|^{-\alpha} \widehat{f}](x) = C_{1-\alpha} f * g(x)$$

Using Fourier Inversion Theorem, and Convolution Theorem, we then can reduce the problem into proving

$$C_\alpha \pi^{\frac{1}{2}} 2^\alpha \frac{1}{|\xi|^\alpha} \widehat{f}(\xi) = C_{1-\alpha} \widehat{g}(\xi) \cdot \widehat{f}(\xi)$$

Then, we reduce the problem into

$$C_\alpha \pi^{\frac{1}{2}} 2^\alpha \frac{1}{|\xi|^\alpha} = C_{1-\alpha} \widehat{g}(\xi)$$

Compute

$$\begin{aligned}
\widehat{g}(\xi) &= \int_{-\infty}^{\infty} |x|^{\alpha-1} e^{-i\xi x} dx \\
&= \int_{-\infty}^{\infty} |x|^{\alpha-1} \big(\cos(\xi x) - i\sin(\xi x)\big) dx \\
&= \int_{-\infty}^{\infty} |x|^{\alpha-1} \cos(\xi x) dx \quad (\because |x|^{\alpha-1}\sin(\xi x) \text{ is odd in } x\ ) \\
&= 2\int_{0}^{\infty} |x|^{\alpha-1} \cos(\xi x) dx \quad (\because |x|^{\alpha-1}\cos(\xi x) \text{ is even in } x\ ) \\
&= 2\int_{0}^{\infty} |x|^{\alpha-1} \operatorname{Re} e^{i\xi x} dx \\
&= \operatorname{Re} 2\int_{0}^{\infty} |x|^{\alpha-1} e^{i\xi x} dx \\
&= \operatorname{Re} 2\int_{0}^{\infty} \left|\frac{u}{\xi}\right|^{\alpha-1} e^{iu}\frac{du}{\xi} \quad (u \equiv \xi x) \\
&= \operatorname{Re} \frac{2}{|\xi|^{\alpha}}\int_{0}^{\infty} u^{\alpha-1}e^{iu} du \\
&= \operatorname{Re} \frac{2}{|\xi|^{\alpha}}e^{i\frac{\alpha\pi}{2}}\int_{0}^{\infty} x^{\alpha-1}e^{-x} dx \quad (\because \text{ Cauchy Integral Theorem })\\
&= \operatorname{Re} \frac{2}{|\xi|^{\alpha}}e^{i\frac{\alpha\pi}{2}}\Gamma(\alpha) \\
&= \frac{2\cos\frac{\alpha\pi}{2}\Gamma(\alpha)}{|\xi|^{\alpha}}
\end{aligned}$$

We can reduce our problem into proving

$$\frac{\Gamma(\frac{\alpha}{2})\sqrt{\pi}2^{\alpha}}{|\xi|^{\alpha}} = \frac{2\cos\frac{\alpha\pi}{2}\Gamma(\alpha)\Gamma(\frac{1-\alpha}{2})}{|\xi|^{\alpha}}$$

Reduce to

$$\Gamma(\frac{\alpha}{2})\sqrt{\pi}2^{\alpha-1} = \cos\frac{\alpha\pi}{2}\Gamma(\alpha)\Gamma(\frac{1-\alpha}{2})$$

Note that the Legendre Duplication Formula give us

$$\Gamma(\frac{\alpha}{2})\Gamma(\frac{\alpha+1}{2}) = 2^{1-\alpha}\Gamma(\alpha)\sqrt{\pi}$$

This give us

$$\Gamma(\frac{\alpha}{2})\sqrt{\pi}2^{\alpha-1} = \frac{2^{1-\alpha}\Gamma(\alpha)\sqrt{\pi}}{\Gamma(\frac{\alpha+1}{2})}\sqrt{\pi}2^{\alpha-1}$$

$$= \frac{\Gamma(\alpha)\pi}{\Gamma(\frac{\alpha+1}{2})} \tag{5.27}$$

Note that Euler Reflection Formula give us

$$\Gamma(\frac{1-\alpha}{2})\Gamma(\frac{1+\alpha}{2}) = \frac{\pi}{\sin(\pi\frac{1+\alpha}{2})} = \frac{\pi}{\cos\frac{\alpha\pi}{2}}$$

This give us

$$\cos\frac{\alpha\pi}{2}\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha) = \cos\frac{\alpha\pi}{2}\Gamma(\alpha)\frac{\pi}{\cos\frac{\alpha\pi}{2}\Gamma(\frac{1+\alpha}{2})}$$

$$= \frac{\Gamma(\alpha)\pi}{\Gamma(\frac{\alpha+1}{2})} \tag{5.28}$$

Note that Equation 5.27 and Equation 5.28 are identical, and we are done. (done) ■

Theorem 5.6.3. (Remainder of Taylor's Theorem in Mean Values Form) Given

$$f : I \subseteq \mathbb{R} \to \mathbb{R} \text{ is } n \text{ time continuously differentiable at } a \in I$$

Define

(a) $P_n(x) \triangleq \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!}$

(b) $R_n(x) \triangleq f(x) - P_n(x)$

If

(a) G is continuous on $[a, x]$

(b) G' exists and not equals to 0 on (a, x)

We have

$$\exists \xi \in (a, x), R_n(x) = \frac{f^{(n+1)}(\xi)(x-\xi)^n}{n!}\frac{G(x) - G(a)}{G'(\xi)}$$

Proof. WLOG suppose $x > a$. Define $F : (a, x) \to \mathbb{R}$ by

$$F(t) = \sum_{k=0}^{n} \frac{f^{(k)}(t)(x-t)^k}{k!}$$

By Cauchy's MVT, we know

$$\exists \xi \in (a, x), \frac{F'(\xi)}{G'(\xi)} = \frac{F(x) - F(a)}{G(x) - G(a)}$$

Compute

$$F(x) = f(x)$$

Compute

$$F(a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!} = P_n(x)$$

Compute

$$\begin{aligned} F'(\xi) &= \sum_{k=0}^{n} \frac{f^{(k+1)}(\xi)(x-\xi)^k - kf^{(k)}(\xi)(x-\xi)^{k-1}}{k!} \\ &= \frac{f^{(n+1)}(\xi)(x-\xi)^n}{n!} \end{aligned}$$

We now have

$$\frac{\frac{f^{(n+1)}(\xi)(x-\xi)^n}{n!}}{G'(\xi)} = \frac{R_n(x)}{G(x) - G(a)}$$

Then we can deduce

$$R_n(x) = \frac{f^{(n+1)}(\xi)(x-\xi)^n}{n!} \frac{G(x) - G(a)}{G'(\xi)}$$

■

Corollary 5.6.4. (Lagarange Form of Remainders in Taylor's Theorem) Let

$$G(t) = (x-t)^{n+1}$$

We have

$$\exists \xi \in (a, x), R_n(x) = \frac{f^{(n+1)}(\xi)(x-a)^{n+1}}{(n+1)!}$$

Proof. Compute

$$\begin{aligned} G'(\xi) &= -(n+1)(x-\xi)^n \\ G(x) &= 0 \\ G(a) &= (x-a)^{n+1} \end{aligned}$$

The result now follows from Theorem 5.6.3. ■

Theorem 5.6.5. ($\sin x \le x$)

$$|\sin x| \le |x| \qquad (x \in [\frac{-\pi}{2}, \frac{\pi}{2}])$$

Proof. Because $|\sin x|$ and $|x|$ are both odd and positive, WOLG, we only have to prove when $x \in (0, \frac{\pi}{2}]$. Compute the Taylor polynomials to second degree and its remainder.

$$\sin x = x - \cos(\xi)\frac{x^3}{3!} \text{ for some } \xi \in (0, x)$$

Because $0 < \xi < x$, it is now clear that

$$0 < \sin x = x - \cos(\xi)\frac{x^3}{3!} \le x$$

This then implies

$$|\sin x| \le |x|$$

■

Question 21

5. Determine whether the Dirichlet kernel $D_N(x) = \sum_{n=-N}^{N} e^{-inx} = \frac{\sin(N+\frac{1}{2})x}{\sin(\frac{x}{2})}$ is a good kernel?

Proof. No. Compute using Theorem 5.6.5

$$\begin{aligned}\int_{-\pi}^{\pi} |D_N(x)|\,dx &= \int_{-\pi}^{\pi} \left|\frac{\sin(N+\frac{1}{2})x}{\sin(\frac{x}{2})}\right| dx \\ &\ge 2\int_{-\pi}^{\pi} \frac{|\sin(N+\frac{1}{2})x|}{|x|} dx\end{aligned}$$

Using $u = (N+\frac{1}{2})x, dx = \frac{du}{N+\frac{1}{2}}$, we have the approximation

$$\begin{aligned}
2\int_{-(N+\frac{1}{2})\pi}^{(N+\frac{1}{2})\pi} \frac{|\sin u|}{\left|\frac{u}{N+\frac{1}{2}}\right|}\frac{1}{N+\frac{1}{2}}du &= 4\int_0^{(N+\frac{1}{2})\pi}\frac{|\sin u|}{|u|}du \\
&\geq 4\Big(\int_0^\pi \frac{\sin u}{u}du + \int_\pi^{N\pi}\frac{|\sin u|}{|u|}du + \int_{N\pi}^{(N+\frac{1}{2})\pi}\frac{|\sin u|}{|u|}du\Big) \\
&\geq 4\Big(\int_0^\pi \frac{\sin u}{\pi}du + \int_\pi^{N\pi}\frac{|\sin u|}{|u|}du + \int_{N\pi}^{(N+\frac{1}{2})\pi}\frac{|\sin u|}{(N+\frac{1}{2})\pi}du\Big) \\
&= 4\int_\pi^{N\pi}\frac{|\sin u|}{|u|}du + \frac{8}{\pi} + \frac{4}{(N+\frac{1}{2})\pi} \\
&= 4\sum_{k=1}^{N-1}\int_{k\pi}^{(k+1)\pi}\frac{|\sin u|}{|u|}du + \frac{8}{\pi} + \frac{4}{(N+\frac{1}{2})\pi} \\
&\geq 4\sum_{k=1}^{N-1}\frac{1}{(k+1)\pi}\int_{k\pi}^{(k+1)\pi}|\sin u|\,du + \frac{8}{\pi} + \frac{4}{(N+\frac{1}{2})\pi} \\
&= 4\sum_{k=1}^{N-1}\frac{2}{(k+1)\pi} + \frac{8}{\pi} + \frac{4}{(N+\frac{1}{2})\pi} \\
&= \frac{8}{\pi}\sum_{k=1}^{N-1}\frac{1}{k+1} + \frac{8}{\pi} + \frac{4}{(N+\frac{1}{2})\pi} \to \infty
\end{aligned}$$

where the last expression tends to infinity because $\sum_{k=1}^{N}\frac{1}{k}$ tends to infinity and the other two terms stay bounded.

We have now seen

$$\begin{aligned}
\int_{-\pi}^{\pi}|D_N(x)|\,dx &\geq 2\int_{-\pi}^{\pi}\frac{\left|\sin(N+\frac{1}{2})x\right|}{|x|}dx \\
&= 2\int_{-(N+\frac{1}{2})\pi}^{(N+\frac{1}{2})\pi}\frac{|\sin u|}{\left|\frac{u}{N+\frac{1}{2}}\right|}\frac{1}{N+\frac{1}{2}}du \to \infty \text{ as } N \to \infty
\end{aligned}$$

This shows that the Dirichlet's Kernel $D_N(x)$ does NOT satisfy the second criterion. ■

Lemma 5.6.6.

$$D_N(x) \triangleq \sum_{n=-N}^{N} e^{-inx} = 1 + 2\sum_{n=1}^{N}\cos nx = \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}$$

Proof.

$$\begin{aligned}\sum_{n=-N}^{N} e^{-inx} &= 1 + 2\sum_{n=1}^{N}\big(\cos nx + i\sin nx + \cos nx - i\sin nx\big)\\ &= 1 + 2\sum_{n=1}^{N}\cos nx\end{aligned}$$

■

Lemma 5.6.7.

$$|\sin x| \geq \frac{|x|}{2} \qquad (x \in [-\frac{\pi}{2}, \frac{\pi}{2}])$$

Proof. Because both $|\sin x|$ and $\frac{|x|}{2}$ are both odd and positive, WOLG, it suffices to just prove for $x \in (0, \frac{\pi}{2}]$.

Notice that $\sin x$ is concave on $[0, \frac{\pi}{2}]$ by computing second derivative.

Then, for all $x \in [0, \frac{\pi}{2}]$, we have

$$\sin x \geq \sin 0 + x\frac{\sin\frac{\pi}{2} - \sin 0}{\frac{\pi}{2} - 0}$$

This give us

$$\sin x \geq \frac{2x}{\pi} \geq \frac{x}{2} \quad (\because 2 \geq \frac{\pi}{2})$$

■

Question 22

6. Determine whether the Fejér kernel $F_N(x) = \frac{1}{N}\sum_{n=0}^{N-1} D_n(x) = \frac{1}{N}\frac{\sin^2(\frac{Nx}{2})}{\sin^2(\frac{x}{2})}$ is a good kernel?

Proof. Yes. For first condition, compute

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{N+1}\sum_{n=0}^{N}\frac{\sin(n+\frac{1}{2})x}{\sin\frac{x}{2}}dx$$
$$= \frac{1}{2\pi(N+1)}\sum_{n=0}^{N}\int_{-\pi}^{\pi} D_n(x)dx$$
$$= \frac{1}{2\pi(N+1)}\sum_{n=0}^{N}\int_{-\pi}^{\pi}(1+2\sum_{k=1}^{n}\cos kx)dx \quad \text{(Lemma 5.6.6)}$$
$$= \frac{1}{2\pi(N+1)}\sum_{n=0}^{N} 2\pi = 1 \quad (\because \int_{-\pi}^{\pi}\cos kx = 0)$$

For second condition, just not that F_N is positive, so

$$\int_{-\pi}^{\pi} |F_n(x)|\, dx = \int_{-\pi}^{\pi} F_n(x) = 2\pi$$

For third condition, suppose $0 < \delta \le |x| \le \pi$.

Using Lemma 5.6.7 to compute

$$0 \le F_n(x) = \frac{\sin^2\frac{nx}{2}}{n\sin^2\frac{x}{2}} \le \frac{1}{n\sin^2\frac{x}{2}} \le \frac{1}{n(\frac{x}{4})^2} \le \frac{1}{n(\frac{\delta}{4})^2} \searrow 0 \text{ as } n\to\infty$$

Then

$$\int_{\delta\le|x|\le\pi} F_n(x)dx \le \int_{\delta\le|x|\le\pi}\frac{16}{n\delta^2}dx = \frac{32(\pi-\delta)}{n\delta^2} \searrow 0 \text{ as } n\to\infty$$

■

Question 23

7. The **Poisson kernel** is given by $P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}$, $-\pi \le \theta \le \pi$. Show that if $0 \le r < 1$, then $P_r(\theta) = \dfrac{1-r^2}{1-2r\cos\theta + r^2}$.

Proof. Compute

$$
\begin{aligned}
P_r(\theta) &= \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} \\
&= 1 + \sum_{n=1}^{\infty} r^n e^{in\theta} + \sum_{n=1}^{\infty} r^n e^{-in\theta} \\
&= 1 + \frac{re^{i\theta}}{1 - re^{i\theta}} + \frac{re^{-i\theta}}{1 - re^{-i\theta}} \\
&= 1 + \frac{re^{i\theta}(1 - re^{-i\theta}) + re^{-i\theta}(1 - re^{i\theta})}{(1 - re^{i\theta})(1 - re^{-i\theta})} \\
&= 1 + \frac{re^{i\theta} + re^{-i\theta} - 2r^2}{1 - re^{i\theta} - re^{-i\theta} + r^2} \\
&= 1 + \frac{2r\cos\theta - 2r^2}{1 - 2r\cos\theta + r^2} = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}
\end{aligned}
$$

■

Question 24

8. If $0 \leq r < 1$, Determine whether the Poisson kernel kernel is a good kernel?

Proof. Yes. For first condition, compute

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} d\theta \\
&= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} \int_{-\pi}^{\pi} e^{in\theta} d\theta \\
&= \frac{1}{2\pi} r^0 2\pi = 1
\end{aligned}
$$

For second condition, note that

$$
1 - 2r\cos\theta + r^2 \geq 1 - 2r + r^2 = (1-r)^2 \in \mathbb{R}^+
$$

Then because $1 - r^2 \in \mathbb{R}^+$, we see

$$
P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2} \in \mathbb{R}^+
$$

We now have

$$\int_{-\pi}^{\pi} |P_r(\theta)| \, d\theta = \int_{-\pi}^{\pi} P_r(\theta) d\theta = 1$$

Note that P_r is even, we then can reduce proving the third critirion into proving

$$P_r(\theta) \to 0 \text{ uniformly on } [\delta, \pi] \text{ as } r \nearrow 1$$

Compute

$$P_r'(\theta) = \frac{-2r \sin\theta(1-r^2)}{(1-2r\cos\theta + r^2)^2} < 0 \text{ on } [\delta, \pi]$$

This then give us

$$P_r(\theta) \leq P_r(\delta) \text{ on } [\delta, \pi]$$

Compute

$$P_r(\delta) = \frac{1-r^2}{(1-r)^2 + 2r(1-\cos\delta)} \to 0 \text{ as } r \nearrow 1$$

and we are done (done) ■

5.7 HW4

Question 25

1. If $\mathcal{X}$ and $\mathcal{Y}$ are normed vector space, we denote the space of all bounded linear maps from $\mathcal{X}$ to $\mathcal{Y}$ by $L(\mathcal{X},\mathcal{Y})$. Show that if $\mathcal{Y}$ is complete, then so is $L(\mathcal{X},\mathcal{Y})$.

Proof. Suppose $\mathcal{Y}$ is complete. Let $BL(\mathcal{X},\mathcal{Y})$ be the space of bounded linear transformation from $\mathcal{X}$ to $\mathcal{Y}$. We wish to prove

$$\Big(BL(\mathcal{X},\mathcal{Y}), \|\cdot\|_{\text{op}}\Big) \text{ is complete}$$

Fix a Cauchy-sequence $\{T_n\}_{n\in\mathbb{N}}$ in $\Big(BL(\mathcal{X},\mathcal{Y}), \|\cdot\|_{\text{op}}\Big)$. We reduce the problem into proving

$$T_n \text{ converge to some bounded linear operator with respect to } \|\cdot\|_{\text{op}}$$

We first show

$$\text{for all } x \in \mathcal{X}, \text{ the sequence } \{T_n x\}_{n\in\mathbb{N}} \text{ converge in } \mathcal{Y}$$

Fix x. Because $\mathcal{Y}$ is complete, we can reduce the problem into showing

$$\{T_n x\}_{n\in\mathbb{N}} \text{ is Cauchy}$$

Fix ϵ. We wish

$$\text{to find } N \text{ such that for all } n > m > N \text{ we have } \|T_n x - T_m x\|_{\mathcal{Y}} \leq \epsilon$$

Because $\{T_n\}_{n\in\mathbb{N}}$ is a Cauchy-sequence in $\Big(BL(\mathcal{X},\mathcal{Y}), \|\cdot\|_{\text{op}}\Big)$, we know there exists N' such that

$$\|T_n - T_m\|_{\text{op}} < \frac{\epsilon}{\|x\|_{\mathcal{X}}} \text{ for all } n > m > N'$$

Note that if $\|x\|_{\mathcal{X}} = 0$, then $x = 0$ and the proof become trivial.

We claim

$$\text{such } N' \text{ works}$$

Observe

$$\begin{aligned}\|T_n x - T_m x\|_{\mathcal{Y}} &= \|(T_n - T_m)x\|_{\mathcal{Y}} \\ &\leq \|T_n - T_m\|_{\text{op}} \|x\|_{\mathcal{X}} < \epsilon \text{ (done)}\end{aligned}$$

Now, we can define a function $S : \mathcal{X} \to \mathcal{Y}$ by

$$S(x) \triangleq \lim_{n\to\infty} T_n(x)$$

We claim

$$S \in BL(\mathcal{X}, \mathcal{Y}) \text{ and } T_n \to S \text{ with respect to } \|\cdot\|_{\text{op}}$$

Observe

$$\begin{aligned}S(x + cy) &= \lim_{n\to\infty} T_n(x + cy) \\ &= \lim_{n\to\infty} T_n(x) + cT_n(y) \\ &= \lim_{n\to\infty} T_n(x) + c \lim_{n\to\infty} T_n(y) = S(x) + cS(y)\end{aligned}$$

This show S is indeed linear. Now, we show

$$S \text{ is indeed bounded}$$

In other words, we wish to show

$$\{\|Sx\|_{\mathcal{Y}} : x \in \mathcal{X} \text{ and } \|x\|_{\mathcal{X}} = 1\} \text{ is bounded}$$

Because T_n is Cauchy with respect to $\|\cdot\|_{\text{op}}$, we know $\{\|T_n\|_{\text{op}}\}$ is bounded by some $M \in \mathbb{R}^+$. We claim

$$\sup_{\|x\|_{\mathcal{X}}=1} \|Sx\|_{\mathcal{Y}} \leq M + 1$$

Fix $\|x\|_{\mathcal{X}} = 1$. We reduce the problem into proving

$$\|Sx\|_{\mathcal{Y}} \leq M + 1$$

Because $T_n x \to Sx$ by definition of S, we know there exists some $k \in \mathbb{N}$ such that $\|T_k x - Sx\|_{\mathcal{Y}} < 1$. Now, observe

$$\begin{aligned}\|Sx\|_{\mathcal{Y}} &\leq \|(S - T_k)x\|_{\mathcal{Y}} + \|T_k(x)\|_{\mathcal{Y}} \\ &< 1 + \|T_k\|_{\text{op}} \quad (\because \|x\|_{\mathcal{X}} = 1) \\ &\leq 1 + M \text{ (done)}\end{aligned}$$

It remains to prove

$$\|T_n - S\|_{\text{op}} \to 0 \text{ as } n \to \infty$$

Fix ϵ. We wish

$$\text{to find } N \text{ such that for all } n > N \text{ we have } \|T_n - S\|_{\text{op}} \leq \epsilon$$

Because T_n is Cauchy with respect to $\|\cdot\|_{\text{op}}$, we know there exists N' such that for all $m > n > N'$, we have

$$\|T_m - T_n\|_{\text{op}} < \epsilon$$

We claim

$$\text{such } N' \text{ works}$$

Fix $\|x\|_{\mathcal{X}} = 1$ and $n > N'$. We reduce the problem into proving

$$\|T_n(x) - S(x)\|_{\mathcal{Y}} \leq \epsilon$$

Observe

$$\begin{aligned}
\|T_n(x) - S(x)\|_{\mathcal{Y}} &= \|T_n(x) - \lim_{m\to\infty} T_m(x)\|_{\mathcal{Y}} \\
&= \|\lim_{m\to\infty} \big((T_n - T_m)(x)\big)\|_{\mathcal{Y}} \\
&= \lim_{m\to\infty} \|(T_n - T_m)(x)\|_{\mathcal{Y}} \quad \Big(\because \lim_{m\to\infty} \big((T_n - T_m)(x)\big) = T_n(x) - S(x) \text{ exists } \Big) \\
&\leq \limsup_{m\to\infty} \|T_n - T_m\|_{\text{op}} \leq \epsilon \text{ (done)}
\end{aligned}$$

■

Question 26

2. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed vector space and $T : \mathcal{X} \to \mathcal{Y}$ be a linear map. Then show that T is bounded if and only if it is continuous.

Proof. See Theorem **??** ■

Question 27

3. (Refer problem 10 of ch6) Let $1 \le p, q \le +\infty$ such that $1/p + 1/q = 1$. For $f \in \mathcal{R}[a,b]$, we define

$$\|f\|_p = \left\{ \int_a^b |f(x)|^p \, dx \right\}^{1/p}, \quad \|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$$

You may assume Young's inequality $ab \le a^p/p + b^q/q$ is true, where $a, b \ge 0$. Then show that $\forall\, f, g \in \mathcal{R}[a,b]$, we have

(a) Holder's inequality : $\|fg\|_1 \le \|f\|_p \|g\|_q$

(b) Minkowski inequality : $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

Proof. **(Proof of Holder's Inequality)**

We first prove

$$\text{when } p = 1 \text{ and when } q = 1$$

WOLG, we

$$\text{only have to prove when } p = 1$$

Because $|g(x)| \le \sup_{t \in [a,b]} |g(t)|$ for all $x \in [a,b]$, we have

$$\int_a^b |fg| \, dx \le \int_a^b |f| \sup_{t \in [a,b]} |g(t)| \, dx = \|f\|_1 \cdot \|g\|_\infty \text{ (done)}$$

We now prove

$$\text{when } p \in (1, \infty)$$

We first prove

$$\text{the special case } \|f\|_p = 0 \text{ or } \|g\|_q = 0$$

WOLG, suppose $\|f\|_p = 0$. From $\|f\|_p = 0$, we can deduce

$$\int_a^b |f|^p \, dx = 0$$

This tell us $|f|$ is 0 almost everywhere, and give us

$$\|fg\|_1 = \int_a^b |fg|\, dx = 0 = \|f\|_p \|g\|_q \text{(done)}$$

Now, we come back

to prove the general case where $\|f\|_p \neq 0 \neq \|g\|_q$

Applying young's inequality to $\frac{|f|}{\|f\|_p}$ and $\frac{|g|}{\|g\|_q}$, we have

$$\frac{|fg|}{\|f\|_p \|g\|_q} \leq \frac{1}{p}(\frac{|f|}{\|f\|_p})^p + \frac{1}{q}(\frac{|g|}{\|g\|_q})^q$$

Integrating both side

$$\begin{aligned}
\frac{1}{\|f\|_p \|g\|_q} \int_a^b |fg|\, dx &\leq \frac{1}{p\|f\|_p^p} \int_a^b |f|^p\, dx + \frac{1}{q\|g\|_q^q} \int_a^b |g|^q\, dx \\
&= \frac{1}{p\|f\|_p^p} \|f\|_p^p + \frac{1}{q\|g\|_q^q} \|g\|_q^q \\
&= \frac{1}{p} + \frac{1}{q} = 1
\end{aligned}$$

Multiplying $\|f\|_p \|g\|_q$ to both side, we now have

$$\|fg\|_1 = \int_a^b |fg|\, dx \leq \|f\|_p \|g\|_q \text{ (done)}$$

(Proof of Minkowski inequality) Compute $\|f+g\|_p^p$

$$\begin{aligned}
\|f+g\|_p^p &= \int_a^b |f+g|^p\, dx \\
&\leq \int_a^b |f+g|^{p-1}\, (|f| + |g|) dx \\
&= \int_a^b |f+g|^{p-1} \cdot |f|\, dx + \int_a^b |f+g|^{p-1} \cdot |g|\, dx \\
&= \|\, |f+g|^{p-1}\, f\|_1 + \|\, |f+g|^{p-1}\, g\|_1
\end{aligned}$$

Check that $\frac{p}{p-1}$ and p form a holder conjugate. Now use Holder's Inequality

$$\begin{aligned}
\|f+g\|_p^p &\leq \|\, |f+g|^{p-1}\, f\|_1 + \|\, |f+g|^{p-1}\, g\|_1 \\
&\leq \|\, |f+g|^{p-1}\, \|_{\frac{p}{p-1}} \|f\|_p + \|\, |f+g|\, \|_{\frac{p}{p-1}} \|g\|_p \\
&= (\int_a^b |f+g|^p\, dx)^{\frac{p-1}{p}} (\|f\|_p + \|g\|_p) \\
&= \|f+g\|_p^{p-1} (\|f\|_p + \|g\|_p)
\end{aligned}$$

Dividing both side by $\|f+g\|_p^{p-1}$ (note that if $\|f+g\|_p^{p-1}=0$, then the proof become trivial),

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

■

Question 28

4. Let E be a compact set and K be a real valued function continuous on E. Define a linear map $A:\mathcal{R}(E)\to\mathcal{R}(E)$ by $(Af)(t)=K(t)f(t),\ \forall t\in E$. Show that

   (a) A is bounded, i.e. $\exists M\geq 0$ such that $\|Af\|_2\leq M\|f\|_2,\ \forall f\in\mathcal{R}(E)$

   (b) If we define operator norm $\|A\|=\sup\{\|Af\|_2:\|f\|_2=1\}$, then $\|A\|=\|a\|_\infty$.

Proof. **(a)**

Because E is compact and K is continuous on E, we know

$$M' = \sup_E |K|^2 \text{ exists}$$

We claim

$$M=\sqrt{M'} \text{ suffices}$$

See

$$\begin{aligned}\|Af\|_2 &= \left(\int_E |Kf|^2\,dx\right)^{\frac{1}{2}}\\ &= \left(\int_E |K|^2|f|^2\,dx\right)^{\frac{1}{2}}\\ &= \left(M'\int_E |f|^2\,dx\right)^{\frac{1}{2}} = M\|f\|_2 \text{ (done)}\end{aligned}$$

(b)

We wish to prove

$$\|A\| \overset{\text{def}}{=} \sup_{\|f\|_2=1}\|Kf\|_2 = \|K\|_\infty$$

We first prove

$$\sup_{\|f\|_2=1} \|Kf\|_2 \leq \|K\|_\infty$$

Fix $\|f\|_2 = 1$. We reduce the problem into

$$\text{proving } \|Kf\|_2 \leq \|K\|_\infty$$

Compute

$$\begin{aligned}\|Kf\|_2 &= (\int_E |K|^2 |f|^2 \, dx)^{\frac{1}{2}} \\ &\leq (\|K\|_\infty^2 \int_E |f|^2 \, dx)^{\frac{1}{2}} \\ &= \|K\|_\infty \|f\|_2 = \|K\|_\infty \text{ (done)}\end{aligned}$$

We now prove

$$\sup_{\|f\|_2=1} \|Kf\|_2 \geq \|K\|_\infty$$

Fix ϵ. We reduce the problem into

$$\text{finding } f \text{ such that } \|f\|_2 = 1 \text{ and } \|Kf\|_2 \geq \|K\|_\infty - \epsilon$$

Because of EVT and the fact $|K|$ is continuous on the compact E, we know there exists a compact interval $I \subseteq E$ such that

(a) $|K| > \|K\|_\infty - \epsilon$ on I

We claim

$$f(t) = \begin{cases} (\mu(I))^{\frac{-1}{2}} & \text{if } t \in I \\ 0 & \text{if } t \notin I \end{cases} \quad \text{suffices}$$

Compute

$$\|f\|_2 = (\int_I ((\mu(I))^{\frac{-1}{2}})^2 dt)^{\frac{1}{2}} = 1$$

Compute using the fact $|K| > \|K\|_\infty - \epsilon$ on I, we have

$$\begin{aligned}\|Kf\|_2 &= \Big(\int_E |K|^2 \cdot |f|^2 \, dx\Big)^{\frac{1}{2}} \\ &= \Big(\int_I |K|^2 \cdot |f|^2 \, dx\Big)^{\frac{1}{2}} \\ &\geq \Big((\|K\|_\infty - \epsilon)^2 \int_I |f|^2 \, dx\Big)^{\frac{1}{2}} \\ &= (\|K\|_\infty - \epsilon)\|f\|_2 = \|K\|_\infty - \epsilon \text{ (done)} \quad \text{(done)}\end{aligned}$$

■

Question 29

5. Let $\mathcal{C}[0,1]$ be a normed vector space with sup-norm. Define $T : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ by

$$(Tf)(x) = \int_0^x f(t)\,dt.$$

Show that T is linear, continuous, and find $\|T\|$

Proof. It is quite clear that T is linear. See

$$\begin{aligned}\big(T(f+cg)\big)(x) &= \int_0^x (f+cg)(t)dt \\ &= \int_0^x f(t)dt + c\int_0^x g(t)dt \\ &= (Tf + cTg)(x)\end{aligned}$$

We now show

$$\|T\|_{\text{op}} = 1$$

In other words, we wish to show

$$\sup_{\|f\|_\infty \leq 1} \|Tf\|_\infty = 1$$

Fix

$$g(x) \triangleq 1$$

We see that $\|g\|_\infty \leq 1$ and

$$(Tg)(x) = \int_0^x dt = x$$

which implies $\|Tg\|_\infty = 1$. This then implies

$$\sup_{\|f\|_\infty \leq 1} \|Tf\|_\infty \geq 1$$

We can now reduce the problem into proving

$$\sup_{\|f\|_\infty \leq 1} \|Tf\|_\infty \leq 1$$

Fix $f \in \mathcal{C}[0,1]$ such that $\|f\|_\infty \leq 1$. We reduce the problem into proving

$$\|Tf\|_\infty \leq 1$$

Fix $x \in [0,1]$. We reduce the problem into proving

$$|Tf(x)| \leq 1$$

Observe

$$\begin{aligned}
|Tf(x)| &= \left| \int_0^x f(t)dt \right| \\
&\leq \int_0^x |f(t)|\, dt \\
&\leq \int_0^x 1dt \quad (\because \|f\|_\infty \leq 1) \\
&= x \leq 1 \text{ (done)}
\end{aligned}$$

Note that we have shown T is bounded. This implies T is continuous, since T is a linear transformation. (See Question 2) ∎

Question 30

6. Let $T(x,y) = (2x+y, x+2y)$ be a map on $\mathbb{R}^2$. Show T linear, bounded, and find $\|T\|$.

Proof. Let $\alpha = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. We have

$$[T]_\alpha = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Compute the diagonalization

$$[T]_\alpha = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

Let $P, V \in L(\mathbb{R}^2, \mathbb{R}^2)$ satisfy

$$[P]_\alpha = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \text{ and } [V]_\alpha = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

Note that $\frac{1}{\sqrt{2}}P$ is an orthogonal transformation. Because orthogonal transformation preserve distance, we see that $\|\frac{1}{\sqrt{2}}P\|_{\text{op}} = 1$. Because $(\frac{1}{\sqrt{2}}P)^{-1} = \sqrt{2}P^{-1}$ and the invert of an orthogonal transformation is again an orthogonal transformation, we see that $\|\sqrt{2}P^{-1}\|_{\text{op}} = 1$.

Now, from

$$T = \frac{1}{\sqrt{2}}P \circ V \circ \sqrt{2}P^{-1}$$

We can deduce

$$\|T\|_{\text{op}} \leq \|\frac{1}{\sqrt{2}}P\|_{\text{op}}\|V\|_{\text{op}}\|\sqrt{2}P^{-1}\|_{\text{op}} = \|V\|_{\text{op}}$$

and deduce

$$\|V\|_{\text{op}} \leq \|\sqrt{2}P^{-1}\|_{\text{op}}\|T\|_{\text{op}}\|\frac{1}{\sqrt{2}}P\|_{\text{op}} = \|T\|_{\text{op}}$$

This give us

$$\|T\|_{\text{op}} = \|V\|_{\text{op}}$$

Now consider $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $x^2 + y^2 = 1$. We wish to find the maximum of

$$\left| V \begin{bmatrix} x \\ y \end{bmatrix} \right|$$

Compute

$$\left| V \begin{bmatrix} x \\ y \end{bmatrix} \right| = \sqrt{9x^2 + y^2}$$

In other words, we wish to find the maximum of

$$\sqrt{9x^2 + y^2} \text{ when } x^2 + y^2 = 1$$

Compute

$$\sqrt{9x^2 + y^2} = \sqrt{1 + 8x^2}$$

This implies $\sqrt{9x^2 + y^2}$ is maximum when $x = \pm 1$ and the value is 3.

In conclusion, $\|T\|_{\text{op}} = 3$. This implies that T is bounded, and further implies that T is continuous. See Question 2. ■

Question 31

7. Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be a linear operator with $\|T\| < 1$. Show that $T_k = 1+T+\ldots+T^{k-1}$ converges to a linear operator S and $S \circ (1-T) = (1-T) \circ S = 1$.

Proof. We first show

$$T_n \text{ is Cauchy in } \Big(L(\mathbb{R}^n,\mathbb{R}^n), \|\cdot\|_{\text{op}}\Big)$$

Note that this suffices, since $\Big(L(\mathbb{R}^n,\mathbb{R}^n), \|\cdot\|_{\text{op}}\Big)$ is complete, by Question 1.

Fix ϵ. We wish

$$\text{to find } N \text{ such that for all } n > m > N \text{ we have } \|T_n - T_m\|_{\text{op}} \leq \epsilon$$

Because $\|T\|_{\text{op}} < 1$, we see the geometric series

$$\sum_{k=0}^{\infty} \|T\|_{\text{op}}^k \text{ converge}$$

Then by direct comparison test ($\|T^k\|_{\text{op}} \leq \|T\|_{\text{op}}^k$), we know

$$\sum_{k=0}^{\infty} \|T^k\|_{\text{op}} \text{ converges, thus Cauchy}$$

Then we know

$$\text{there exists } N' \text{ such that for all } n > m > N', \text{ we have } \sum_{k=m}^{n-1} \|T^k\|_{\text{op}} < \epsilon$$

We claim

$$\text{such } N' \text{ works}$$

Fix $n > m > N'$. Observe

$$\begin{aligned}\|T_n - T_m\|_{\text{op}} &= \|\sum_{k=m}^{n-1} T^k\|_{\text{op}} \\ &\leq \sum_{k=m}^{n-1} \|T^k\|_{\text{op}} \\ &\leq \sum_{k=m}^{n-1} \|T\|_{\text{op}}^k \leq \epsilon \text{ (done)}\end{aligned}$$

Now, let $S \triangleq \lim_{k\to\infty} T_k$. Note that $(1-T)\circ S = 1 \implies S = (1-T)^{-1} \implies S\circ(1-T) = 1$, so it suffices to just prove

$$(1 - T) \circ S = 1$$

Fix $x \in \mathbb{R}^n$. We wish to prove

$$(1 - T) \lim_{k\to\infty} T_k(x) = x$$

Because we know $\lim_{k\to\infty} T_k(x)$ exists, and $1 - T$ is continuous, (all linear transformation in $\mathbb{R}^n$ is continuous, see Theorem **??**), we have

$$(1 - T) \lim_{k\to\infty} T_k(x) = \lim_{k\to\infty} (1 - T)T_k(x)$$

We can now reduce the problem into proving

$$\lim_{k\to\infty} (1 - T)T_k(x) = x$$

Compute

$$\begin{aligned}(1 - T)T_k &= T_k - TT_k \\ &= \sum_{n=0}^{k-1} T^n - \sum_{n=1}^{k} T^n \\ &= 1 - T^k\end{aligned}$$

This let us reduce the problem into

$$\lim_{k\to\infty} T^k(x) = 0$$

Fix $r = \|T\|_{\text{op}} < 1$. Observe

$$\|T^k(x)\|_{\text{op}} \leq \|T\|_{\text{op}}^k |x| = r^k |x|$$

Because $r^k |x| \to 0$ as $k \to \infty$, this implies

$$\lim_{k\to\infty} \|T^k(x)\| = 0$$

and implies

$$\lim_{k\to\infty} T^k(x) = 0 \text{ (done)}$$

■

Question 32

8. Two norms $\|\cdot\|$ and $\|\cdot\|'$ on $\mathcal{X}$ are said to be equivalent if $\exists c_1, c_2 > 0$ such that $c_1\|x\| \le \|x\|' \le c_2\|x\|, \forall x \in X$. Show that if $\mathcal{X}$ is a finite-dimensional vector space, then all norm on $\mathcal{X}$ are equivalent. Hint : Use basis, and the fact that unit ball in $\mathcal{X}$ isometric to unit ball in $\mathbb{R}^n$.

Proof. See Theorem **??** ■

5.8 Operator Norm

In this section, and particularly in functional analysis, we say a function T between two metric space is a **bounded operator** if T always map bounded set to bounded set. In particular, if T is a linear transformation between two normed space, we say T is a **bounded linear operator**.

Suppose $\mathcal{X}, \mathcal{Y}$ are two normed space over $\mathbb{R}$ or $\mathbb{C}$. In space $L(\mathcal{X}, \mathcal{Y})$, alternatively, we can define the boundedness for each linear transformation T by

$$T \text{ is bounded } \overset{\triangle}{\iff} \exists M \in \mathbb{R}, \forall x \in \mathcal{X}, \|Tx\| \leq M\|x\|$$

The proof of equivalency is simple. For $(\longrightarrow)$, $E \triangleq \{y \in \mathcal{X} : \|y\| = 1\}$ is non-empty. Clearly, E is bounded. Let $M = \sup_{y \in E} \|Ty\|$. We now have

$$\|Tx\| = \|x\| \cdot \|T\frac{x}{\|x\|}\| \leq M\|x\|$$

For $(\longleftarrow)$, just observe $\|Tx - Ty\| = \|T(x-y)\| \leq M\|x-y\|$.

Here, we first show a linear transformation is continuous if and only if it is bounded. (Theorem **??**)

Theorem 5.8.1. (Liner Operator is Bounded if and only if it is Continuous) Given two normed space $\mathcal{X}, \mathcal{Y}$ over $\mathbb{R}$ or $\mathbb{C}$ and $T \in L(\mathcal{X}, \mathcal{Y})$, we have

$$T \text{ is a bounded operator } \iff T \text{ is continuous on } \mathcal{X}$$

Proof. $(\longrightarrow)$ We show

$$T \text{ is Lipschitz continuous on } V$$

Because T is bounded, we can let $M \in \mathbb{R}^+$ satisfy $\|Tx\| \leq M\|x\|$. We see

$$\|Tx - Ty\| \leq \|T(x-y)\| \leq M\|x-y\| \text{ (done)}$$

$(\longleftarrow)$

Because T is linear and continuous at 0, we know there exists ϵ such that

$$\sup_{\|y\| \leq \epsilon} \|Ty\| \leq 1$$

We claim

$$\|Tx\| \leq \frac{1}{\epsilon}\|x\| \qquad (x \in \mathcal{X})$$

Fix $x \in V$. Compute

$$\|Tx\| = \frac{\|x\|}{\epsilon} T \frac{\epsilon x}{\|x\|} \leq \frac{\|x\|}{\epsilon} \text{ (done)}$$

∎

Here, we introduce a new terminology, which shall later show its value. Given a set X, we say two metrics d_1, d_2 on X are **equivalent**, and write $d_1 \sim d_2$, if we have

$$\exists m, M \in \mathbb{R}^+, \forall x, y \in X, md_1(x,y) \leq d_2(x,y) \leq Md_1(x,y)$$

Now, given a fixed vector space V, naturally, we say two norms $\|\cdot\|_1, \|\cdot\|_2$ on V are **equivalent** if

$$\exists m, M \in \mathbb{R}^+, \forall x \in X, m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1$$

We say two metric d_1, d_2 on X are **topologically equivalent** if the topology they induce on X are identical.

A few properties can be immediately spotted.

(a) Our definition of $\sim$ between metrics of a fixed X is an equivalence relation.

(b) Our definition of $\sim$ between norms on a fixed V is an equivalence relation.

(c) Equivalent norms induce equivalent metrics.

(d) Equivalent metrics are topologically equivalent.

We now prove that if V is finite-dimensional, then all norms on V are equivalent (Theorem **??**). This property will later show its value, as used to prove that linear map of finite-dimensional domain is alwasy continuous (Theorem **??**)

Theorem 5.8.2. (All Norms on Finite-dimensional space are Equivalent) Suppose V is a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. Then

$$\text{all norms on } V \text{ are equivalent}$$

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis of V. Define ∞-norm $\|\cdot\|_\infty$ on V by

$$\|\sum \alpha_i e_i\|_\infty \triangleq \max |\alpha_i|$$

It is easily checked that $\|\cdot\|_\infty$ is indeed a norm. Fix a norm $\|\cdot\|$ on V. We reduce the problem into

$$\text{finding } m, M \in \mathbb{R}^+ \text{ such that } m\|x\|_\infty \leq \|x\| \leq M\|x\|_\infty$$

We first claim

$$M = \sum \|e_i\| \text{ suffices}$$

Compute

$$\|x\| = \sum \alpha_i e_i \leq \sum |\alpha_i| \, \|e_i\| \leq \|x\|_\infty \sum \|e_i\| = M\|x\|_\infty \text{ (done)}$$

Reverse triangle inequality give us

$$\Big| \|x\| - \|y\| \Big| \leq \|x - y\| \leq M\|x - y\|_\infty$$

This implies that $\|\cdot\| : \Big(V, \|\cdot\|_\infty\Big) \to \mathbb{R}$ is Lipschitz continuous.

Define $S \triangleq \{y \in V : \|y\|_\infty = 1\}$ is non-empty. Check that S is compact in $\|\cdot\|_\infty$ by checking S is sequentially compact using the fact $\mathbb{R}^{n-1}$ is locally compact.

Now, by EVT, we know $\min_{y \in S} \|y\|$ exists. Note that $\min_{y \in S} \|y\| > 0$, since $0 \notin S$.

We claim

$$m = \min_{y \in S} \|y\| \text{ suffices}$$

Fix $x \in V$. Compute

$$m\|x\|_\infty = \|x\|_\infty (\min_{y \in S} \|y\|) \leq \|x\|_\infty \cdot \frac{x}{\|x\|_\infty} = \|x\| \text{ (done)} \quad \text{(done)}$$

∎

Theorem 5.8.3. (Linear map of Finite-dimensional Domain is always Continuous) Given a finite-dimensional normed space $\mathcal{X}$ over $\mathbb{R}$ or $\mathbb{C}$, an arbitrary normed space $\mathcal{Y}$ over $\mathbb{R}$ or $\mathbb{C}$ and a linear transformation $T : \mathcal{X} \to \mathcal{Y}$, we have

$$T \text{ is continuous}$$

Proof. Fix $x \in \mathcal{X}, \epsilon$. We wish

$$\text{to find } \delta \text{ such that } \forall h \in \mathcal{X} : \|h\| \leq \delta, \|T(x + h) - Tx\| \leq \epsilon$$

Let $\{e_1, \dots, e_n\}$ be a basis of $\mathcal{X}$. Note that $\|\sum \alpha_i e_i\|_1 := \sum |\alpha_i|$ is a norm. By Theorem **??**, we know $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent. Then, we can fix $M \in \mathbb{R}^+$ such that

$$\|x\|_1 \leq M\|x\| \quad (x \in V)$$

We claim

$$\delta = \frac{\epsilon}{M(\max \|Te_i\|)} \text{ suffices}$$

Fix $\|h\| \leq \delta$ and express $h = \sum \alpha_i e_i$. Compute using linearity of T

$$\begin{aligned}\|T(x+h) - Tx\| &= \|\sum \alpha_i Te_i\| \\ &\leq \sum |\alpha_i| \, \|Te_i\| \\ &\leq \|h\|_1 (\max \|Te_i\|) \\ &\leq M\|h\|(\max \|Te_i\|) \leq \epsilon \text{ (done)}\end{aligned}$$

■

As a corollary of Theorem **??** and Theorem **??**, we now see that, if $\mathcal{X}$ is finite-dimensional, then all linear map of domain $\mathcal{X}$ are bounded. A counter example to the generalization of this statement is followed.

Example 17 (Differentiation is an Unbounded Linear Operator)

$$\mathcal{X} = \Big(\mathbb{R}[x]|_{[0,1]}, \|\cdot\|_\infty\Big), D(P) \triangleq P'$$

Note that $\{x^n\}_{n \in \mathbb{N}}$ is bounded in $\mathcal{X}$ and $\{D(x^n)\}_{n \in \mathbb{N}}$ is not.

Now, suppose $\mathcal{X}, \mathcal{Y}$ are two fixed normed spaces over $\mathbb{R}$ or $\mathbb{C}$. We can easily check that the set $BL(\mathcal{X}, \mathcal{Y})$ of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ form a vector space over whichever field $\mathcal{Y}$ is over.

Naturally, our definition of boundedness of linear operator derive us a norm on $BL(\mathcal{X}, \mathcal{Y})$, as followed

$$\|T\|_{\text{op}} \triangleq \inf\{M \in \mathbb{R}^+ : \forall x \in \mathcal{X}, \|Tx\| \leq M\|x\|\} \tag{5.29}$$

Before we show that our definition is indeed a norm, we first give some equivalent definitions and prove their equivalency.

Theorem 5.8.4. (Equivalent Definitions of Operator Norm) Given two fixed normed space $\mathcal{X}, \mathcal{Y}$ over $\mathbb{R}$ or $\mathbb{C}$, a bounded linear operator $T : \mathcal{X} \to \mathcal{Y}$, and define $\|T\|_{\text{op}}$ as in

(**??**), we have

$$\|T\|_{\text{op}} = \sup_{x \in \mathcal{X}, x \neq 0} \frac{\|Tx\|}{\|x\|}$$

Proof. Define $J = \{M \in \mathbb{R}^+ : \forall x \in \mathcal{X}, \|Tx\| \leq M\|x\|\}$, so that we have $\|T\|_{\text{op}} = \inf J$. Now, observe

$$\begin{aligned} J &= \{M \in \mathbb{R}^+ : M \geq \frac{\|Tx\|}{\|x\|}, \forall x \neq 0 \in \mathcal{X}\} \\ &= M \in \mathbb{R}^+ : M \geq \sup_{x \in \mathcal{X}, x \neq 0} \frac{\|Tx\|}{\|x\|} \end{aligned}$$

This let us conclude

$$\|T\|_{\text{op}} = \inf J = \sup_{x \in \mathcal{X}, x \neq 0} \frac{\|Tx\|}{\|x\|}$$

■

It is now easy to see

$$\begin{aligned} \|T\|_{\text{op}} &= \sup_{x \in \mathcal{X}, x \neq 0} \frac{\|Tx\|}{\|x\|} && (5.30) \\ &= \sup_{x \in \mathcal{X}, \|x\|=1} \|Tx\| && (5.31) \end{aligned}$$

It is not all in vain to introduce the equivalent definitions. See that the verification of $\|\cdot\|_{\text{op}}$ being a norm on $BL(\mathcal{X}, \mathcal{Y})$ become simple by utilizing the equivalent definitions.

(a) For positive-definiteness, fix non-trivial T and fix $x \in \mathcal{X} \setminus N(T)$. Use (**??**) to show $\|T\|_{\text{op}} \geq \frac{\|Tx\|}{\|x\|} > 0$.

(b) For absolute homogenity, use (**??**) and $\|Tcx\| = |c| \cdot \|Tx\|$.

(c) For triangle inequality, use (**??**) and $\|(T_1 + T_2)x\| \leq \|T_1 x\| + \|T_2 x\|$.

Naturally, and very very importantly, (**??**) give us

$$\|Tx\| \leq \|T\|_{\text{op}} \cdot \|x\| \qquad (x \in \mathcal{X})$$

This inequality will later be the best tool to help analyze the derivatives of functions between Euclidean spaces, and perhaps better, it immediately give us

$$\frac{\|T_1 T_2 x\|}{\|x\|} \leq \frac{\|T_1\|_{\text{op}} \cdot \|T_2\|_{\text{op}} \cdot \|x\|}{\|x\|} = \|T_1\|_{\text{op}} \cdot \|T_2\|_{\text{op}}$$

Then (**??**) give us

$$\|T_1 T_2\|_{\text{op}} \leq \|T_1\|_{\text{op}} \cdot \|T_2\|_{\text{op}}$$

5.9 HW5

Question 33

1. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x, y) = (xy, x\cos y, x\sin y) \quad , \quad g(u, v, w) = uv + vw + wu$$

Calculate $(Df)(1, 0)$, $(Dg)(0, 1, 0)$, and $[D(g \circ f)](1, 0)$.

Proof. Compute

$$[df_{(x,y)}] = \begin{bmatrix} y & x \\ \cos y & -x\sin y \\ \sin y & x\cos y \end{bmatrix}$$

This give us

$$(Df)(1, 0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Compute

$$[dg_{(0,1,0)}] = \begin{bmatrix} v + w & u + w & u + v \end{bmatrix}$$

This give us

$$(Dg)(0, 1, 0) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

Compute

$$f(1, 0) = (0, 1, 0)$$

Then Chain rule give us

$$[D(g \circ f)](1, 0) = [dg_{(0,1,0)}][df_{(1,0)}] = \begin{bmatrix} 0 & 2 \end{bmatrix}$$

■

Question 34

2. (a) Let $(X, \|-\|_X)$, $(Y, \|-\|_Y)$ be normed spaces and U be an open subset of X. If $f : U \to Y$ is differentiable at some $x_0 \in U$, prove that $(Df)(x_0)$ is **uniquely** determined by f.

   (Hint: If L_1 and L_2 are derivatives of f at x_0, write out the definition. Also U is open means you can find a small ball $B(x_0; r) \subset U$ with some $r > 0$, this r can do some work in your above definition.)

   (b) If U is **NOT** supposed to be open, does the result of (a) remain true? Prove it or give a counterexample.

Proof. Given two normed space $\mathcal{X}, \mathcal{Y}$, suppose f maps an open neighborhood O around x in $\mathcal{X}$ into $\mathcal{Y}$. We say f is **differentiable at** x if there exists a bounded linear transformation $A_x : \mathcal{X} \to \mathcal{Y}$ (from now, A_x will be denoted df_x) such that

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - df_x(h)\|_{\mathcal{Y}}}{\|h\|_{\mathcal{X}}} = 0 \tag{5.32}$$

Immediately, we should check that the linear approximation is unique. Suppose df_x and df'_x both satisfy (**??**). We are required to show $(df_x - df'_x)h = 0$ for all $\|h\|_{\mathcal{X}} = 1$. Fix $h \in \mathcal{X}$ such that $\|h\|_{\mathcal{X}} = 1$. Note that

$$\frac{(df_x - df'_x)th}{t} \text{ is a constant in } t \text{ for } t \neq 0$$

This then reduced the problem into showing

$$\frac{(df_x - df'_x)th}{t\|h\|_{\mathcal{X}}} \to 0 \text{ as } t \to 0 \tag{5.33}$$

Observe

$$(df_x - df'_x)th = \Big(f(x+th) - f(x) - df'_x(th)\Big) - \Big(f(x+th) - f(x) - df_x(th)\Big)$$

which implies

$$\|(df_x - df'_x)th\|_{\mathcal{Y}} \leq \|f(x+th) - f(x) - df'_x(th)\|_{\mathcal{Y}} + \|f(x+th) - f(x) - df_x(th)\|_{\mathcal{Y}}$$

and thus implies (**??**). ■

Question 35

3. For a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, prove that $\pm \dfrac{\nabla f}{\|\nabla f\|_{\mathbb{R}^n}}$ is the direction in which f increases/decreases most rapidly.

Proof. Because

$$\nabla f(p) = \begin{bmatrix} \partial_1 f(p) \\ \vdots \\ \partial_n f(p) \end{bmatrix} \text{ and } [df_p] = \begin{bmatrix} \partial_1 f(p) & \cdots & \partial_n f(p) \end{bmatrix}$$

We see $\nabla f(p)$ satisfy

$$\nabla f(p) \cdot v = df_p(v) \text{ for all } v \in \mathbb{R}^n$$

Fix a direction (a unit vector) $v \in \mathbb{R}^n$. The direction derivative is defined by

$$\partial_v f(p) \triangleq \lim_{t \to 0} \frac{f(p+tv) - f(p)}{t}$$

The Frechet definition of differentaibility of f give us

$$\lim_{t \to 0} \frac{f(p+tv) - f(p) - df_p(tv)}{t} = 0$$

Using the linearity of df_p, we then have

$$\partial_v f(p) = df_p(v) = \nabla f(p) \cdot v$$

Now, using C-S Inequality, we see $\partial_v f(p)$ respectively reach to maximum and minium when $v = \frac{\nabla f(p)}{|\nabla f(p)|}$ and $v = \frac{-\nabla f(p)}{|\nabla f(p)|}$. ■

Question 36

4. (a) If E is an open and connected subset in $\mathbb{R}^n$, prove that E is path-connected.

(b) Let E be a open and connected subset of $\mathbb{R}^n$ and $f : E \to \mathbb{R}^m$ is differentiable. Show that if $(Df)(x)$ is a zero matrix for all $x \in E$, then f can only be a constant on E.

(c) If E is changed to be open and convex, does the result of (b) remain true? Prove it or give a counterexample.

Proof. **(a)** Let U, K be two different path-connected components of E. Suppose $U \neq \varnothing$. We are required to prove

$$K = \varnothing$$

Because E is connected, we can reduce the problem into proving

$$U, K \text{ are both open}$$

Note that we didn't mention if the ambient space is E or $\mathbb{R}^n$. This doesn't matter since E is open in $\mathbb{R}^n$, which implies $\tau_E = \mathcal{P}(E) \cap \tau_{\mathbb{R}^n}$.

WOLG, we only have to prove

$$U \text{ is open}$$

Fix $x \in U$. Because E is open. We know there exists ϵ such that

$$B_\epsilon(x) \subseteq E$$

It is clear that every point p in $B_\epsilon(x)$ can be joined with x by a straight line

$$f(t) \triangleq (p - x)t + x$$

This then implies $B_\epsilon(x) \subseteq U$ (done) .

(b) Let $c_0 \in f(E)$ and define

$$A \triangleq \{p \in E : f(p) = c_0\}$$

We are required to prove

$$A = E$$

Because E is connected, we can reduce the problem into proving

$$A \text{ is both open and close}$$

Fix a convergent sequence $x_n \in A$. Because f is continuous, we see

$$f\Big(\lim_{n\to\infty} x_n\Big) = \lim_{n\to\infty} f(x_n) = c_0$$

Let $x = \lim_{n\to\infty} x_n$. We now see $f(x) = c_0$, which implies $x \in A$. This implies A is closed, since x_n is arbitrary.

Fix $a \in A \subseteq E$. Because E is open, there exists

$$B_\epsilon(a) \subseteq E$$

Clearly, for all $x \in B_\epsilon(a)$, there exists a differetiable straight path γ that starts from a and ends at x. Now, observe

$$f(x) - f(a) = f \circ \gamma(t)\Big|_{t=0}^{1} = \int_0^1 (f \circ \gamma)'(t)dt = 0$$

The last inequality follows from the fact $df = 0$ and Chain Rule.

Now, we have $f(x) = f(a)$, and of course $x \in A$. Because x is arbitrarily picked from $B_\epsilon(a)$. We can conclude $B_\epsilon(a) \subseteq A$. (done)

(c)

Yes, the result remains true. A convex subset is path-connected (every two points are joined by a straight line), thus connected. The rest of the proof follows from that of **(b)**.

■

Question 37

5. (Mean Value Theorem in multivariable Calculus)

   (a) Let U be an open subset of $\mathbb{R}^n$ and $f = (f_1, ..., f_n) : U \to \mathbb{R}^m$ be a differentiable function. If for $x, y \in U$, the line segment L joining x and y lies in U, prove that there exists $c_1, ..., c_m \in L$ such that

   $$f_i(y) - f_i(x) = [(Df_i)(c_i)](y - x),\ i = 1, ..., m$$

   (b) If for $x, y \in U$, L does **NOT** lie in U, the above theorem may not hold. Give a counterexample.

Proof. **(a)**

Define $\gamma : [0,1] \to \mathbb{R}^n$ by

$$\gamma(t) = x + (y - x)t$$

We see

$$f_i(y) = (f_i \circ \gamma)(1) \text{ and } f_i(x) = (f_i \circ \gamma)(0)$$

Using the fact f is differentiable and Chain Rule, we see

$$f_i \circ \gamma : [0,1] \to \mathbb{R} \text{ is differenetiable on } (0,1)$$

Then by MVT, we know there exists t_i such that

$$\begin{aligned} f_i(y) - f_i(x) &= (f_i \circ \gamma)'(t_i) \\ &= d(f_i)_{\gamma(t_i)}(\gamma'(t_i)) \\ &= [(Df_i)(c_i)](y - x) \end{aligned}$$

where $c_i = \gamma(t_i)$.

(b) Let $U = (-1,0) \cup (1,2)$. Define $f : U \to \mathbb{R}$ by

$$f(t) = \begin{cases} 0 & \text{if } t \in (-1,0) \\ 1 & \text{if } t \in (1,2) \end{cases}$$

It is clear that $df = 0$ everywhere, and if x, y does not belong to the same connected component, then the result of **(a)** does not hold true. ■

Question 38

6. (a) Let $f(x,y)=\begin{cases}\dfrac{x^4+2x^2y^2-y^4}{x^2+y^2} & ,(x,y)\neq(0,0)\\ 0 & ,(x,y)=(0,0)\end{cases}$, use $\varepsilon-\delta$ definition to check that f is differentiable at $(0,0)$.

(b) Let $g(x,y)=\begin{cases}\dfrac{2x^3+x^2y+2xy^2-y^3}{x^2+y^2} & ,(x,y)\neq(0,0)\\ 0 & ,(x,y)=(0,0)\end{cases}$, prove that g is **NOT** differentiable at $(0,0)$.

(c) Calculate the value of $\dfrac{\partial}{\partial y}\left(\dfrac{\partial g}{\partial x}\right)$ and $\dfrac{\partial}{\partial x}\left(\dfrac{\partial g}{\partial y}\right)$ at $(0,0)$.

Proof. **(a)**

We claim

$$df_0 = 0 \text{ suffices}$$

We wish to prove

$$\lim_{(x,y)\to 0}\frac{(x^2+y^2)^2-2y^4}{(x^2+y^2)^{\frac{3}{2}}}=0$$

We can reduce the problem into proving

$$\lim_{(x,y)\to 0}\frac{y^4}{(x^2+y^2)^{\frac{3}{2}}}=0$$

Observe

$$0\le\frac{y^4}{(x^2+y^2)^{\frac{3}{2}}}\le\frac{(x^2+y^2)^2}{(x^2+y^2)^{\frac{3}{2}}}=\sqrt{x^2+y^2}\to 0 \text{ as } (x,y)\to 0 \text{ (done)}$$

(b) Observe

$$g(h,h)=\begin{cases}2 & \text{if } h\neq 0\\ 0 & \text{if } h=0\end{cases}$$

This implies g is not even continuous at 0, let alone differentiable.

Compute

$$g(x,0) = 2x \text{ and } g(0,y) = -y$$

This implies

$$\partial_x g(0,0) = 2 \text{ and } \partial_y g(0,0) = -1$$

and implies

$$\partial_y \partial_x g(0,0) = \partial_x \partial_y g(0,0) = 0$$

■

5.10 HW6

Question 39

1. Let $u(x,y)=\dfrac{x^4+y^4}{x}$, $v(x,y)=\sin x+\cos y$ and f be a function that maps (x,y) to (u,v). Find the point (x,y) where we can solve for x,y in terms of u,v. Also, find $\dfrac{\partial x}{\partial u},\dfrac{\partial x}{\partial v},\dfrac{\partial y}{\partial u},\dfrac{\partial y}{\partial v}$ at $f\left(\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$.

Proof. Compute

$$[df_{(x,y)}]=\begin{bmatrix}3x^2 & \frac{4y^3}{x}\\ \cos x & -\sin y\end{bmatrix}$$

By Differentiability Theorem, it is now clear that f is continuously differentiable everywhere on its domain $\{(x,y)\in\mathbb{R}^2 : x\neq 0\}$.

Compute the determinant

$$\det\Big(df_{(x,y)}\Big)=-3x^2\sin y-\frac{4y^3\cos x}{x}$$

Now by Inverse Function Theorem, the set of points (x,y) where we can solve for x,y in terms of u,v is exactly

$$\{(x,y)\in\mathbb{R}^2 : 3x^2\sin y+\frac{4y^3}{x}\cos x\neq 0\text{ and }x\neq 0\}$$

Observe

$$[df_{(\frac{\pi}{2},\frac{\pi}{2})}]=\begin{bmatrix}\frac{3\pi^2}{4} & \pi^2\\ 0 & -1\end{bmatrix}$$

This implies the local inverse is

$$[d(f^{-1})_{f(\frac{\pi}{2},\frac{\pi}{2})}]=[df_{(\frac{\pi}{2},\frac{\pi}{2})}]^{-1}=\begin{bmatrix}\frac{4}{3\pi^2} & \frac{4}{3}\\ 0 & -1\end{bmatrix}$$

We now have

$$\frac{\partial u}{\partial x}=\frac{4}{3\pi^2}\text{ and }\frac{\partial u}{\partial y}=\frac{4}{3}\text{ and }\frac{\partial v}{\partial x}=0\text{ and }\frac{\partial v}{\partial y}=-1$$

■

Question 40

2. Let $f : \mathbb{R}^4 \to \mathbb{R}^2$ be given by $f(x, y, u, v) = (xu + yv^2, xv^3 + y^2u^6)$.

   (a) Show that x, y can be solved in terms of u, v for (u, v) near $(1, -1)$ and (x, y) near $(1, -1)$.

   (b) From (a), If we write $x = g_1(u, v)$, $y = g_2(u, v)$ for (u, v) near $(1, -1)$ and let $g = (g_1, g_2)$, Find $Dg(u, v)$. (You don't need to calculate explicitly.)

Proof. Compute

$$[df] = \begin{bmatrix} u & v^2 & x & 2yv \\ v^3 & 2yu^6 & 6u^5y^2 & 3xv^2 \end{bmatrix}$$

By Differentiability Theorem, it is now clear that f is continuously differentiable on $\mathbb{R}^4$.

Compute

$$[df_{(1,-1,1,-1)}] = \begin{bmatrix} 1 & 1 & 1 & 2 \\ -1 & -2 & 6 & 3 \end{bmatrix}$$

It is clear that the left Jacobian $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ is invertible. Now by Implicit Function Theorem, we can conclude (a), and if we write the implicit function by g we have

$$[dg_{(u,v)}] = -\begin{bmatrix} u & v^2 \\ v^3 & 2yu^6 \end{bmatrix}^{-1} \circ \begin{bmatrix} x & 2yv \\ 6u^5y^2 & 3xv^2 \end{bmatrix} = \begin{bmatrix} \frac{6u^5v^2y^2 - 2u^6xy}{v^5 - 2u^7y} & \frac{3v^4x - 4u^6vy^2}{v^5 - 2u^7y} \\ \frac{v^3x - 6u^6y^2}{v^5 - 2u^7y} & \frac{-3uv^2x + 2v^4y}{v^5 - 2u^7y} \end{bmatrix}$$

∎

Question 41

3. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(x, y, z) = (xe^y + ye^z, xe^z + ze^y)$.

   (a) Show that y, z can be solved in terms of x for x near -1 and (y, z) near $(1, 1)$.

   (b) From (a), If we write $(y, z) = g(x)$ for x near -1, Find $g'(x)$. (You don't need to calculate explicitly.)

Proof. Compute

$$[df] = \begin{bmatrix} e^y & xe^y + e^z & ye^z \\ e^z & ze^y & xe^z + e^y \end{bmatrix}$$

By Differentiability Theorem, it is now clear that f is continuously differentiable on $\mathbb{R}^3$. Compute

$$[df_{(-1,1,1)}] = \begin{bmatrix} e & 0 & e \\ e & e & 0 \end{bmatrix}$$

It is clear that the right Jacobian $\begin{bmatrix} 0 & e \\ e & 0 \end{bmatrix}$ is invertible. Now, by Implicit Function Theorem, we can conclude (a), and if we write the implicit function by g, we have

$$[dg_x] = -\begin{bmatrix} xe^y + e^z & ye^z \\ ze^y & xe^z \end{bmatrix}^{-1} \circ \begin{bmatrix} e^y \\ e^z \end{bmatrix}$$

■

Question 42

4. Suppose $f : \mathbb{R} \to \mathbb{R}^n$ is differentiable, such that $|f(t)| = 1$ for every $t \in \mathbb{R}$.

   (a) Prove that $f'(t) \cdot f(t) = 0$. (Here "$\cdot$" is the standard inner product in $\mathbb{R}^n$)

   (b) In fact, this result has a geometric interpretation. For example, in $\mathbb{R}^2$, the function $f(t) = (\cos t, \sin t)$ satisfies $|f(t)| = 1$. Draw the graph of $f(t)$ and $f'(t)$ on $\mathbb{R}^2$, what do you discover?

Proof. If we define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(t) \triangleq f(t) \cdot f(t)$$

We have

$$g \text{ is constant } 1 \text{ and } g' = 0$$

It remains to prove

$$g'(t) = 2f'(t) \cdot f(t)$$

Observe

$$g(t) = \sum_{k=1}^{n} f_k^2(t)$$

This implies

$$g'(t) = \sum_{k=1}^{n} 2f_k(t)f_k'(t) = 2f'(t) \cdot f(t) \text{ (done)}$$

From the graph below, we see that $f(t)$ and $f'(t)$ orthogonal. ∎

Question 43

5. (Second Derivative Test) Let V be an open subset of $\mathbb{R}^2$ and $(a, b) \in V$, and suppose that $f : V \to \mathbb{R}$ satisfy $\nabla f(a, b) = 0$. Suppose also that $f \in \mathcal{C}^2(V)$, and set $D = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2$. Prove the following statements:

   (a) If $D > 0$ and $f_{xx}(a, b) > 0$, then f admits a local minimum at (a, b).

   (b) If $D > 0$ and $f_{xx}(a, b) < 0$, then f admits a local maximum at (a, b).

   (c) If $D < 0$, then f is a saddle point at (a, b).

Definition 5.10.1. (Definition of Hessian) For all $p \in V$, we define

$$H(p) \triangleq \begin{bmatrix} f_{xx}(p) & f_{xy}(p) \\ f_{yx}(p) & f_{yy}(p) \end{bmatrix} \text{ and } D(p) \triangleq (f_{xx} \cdot f_{yy} - f_{xy}^2)(p)$$

Note that $f \in \mathcal{C}^2(V) \implies f_{xy} = f_{yx}$ on $V \implies H$ is symmetric on V.

Then by Spectral Theorem, we know

$$H \text{ has an orthonormal basis consisting of eigenvectors}$$

Definition 5.10.2. (Definition of Saddle) Given real-valued function f exists on an open set $V \subseteq \mathbb{R}^2$ containing p and is differentiable at p, we say f is a saddle point at p if

$$df_p = 0 \text{ and } f \text{ is neither a local minimum nor a local maximum at } p$$

Lemma 5.10.3. Given a $\alpha(t)$ that lies in V and satisfy $\alpha'' = 0$. We have

$$(f \circ \alpha)'' = \alpha' \cdot H\alpha'$$

Proof. Using Chain Rule and the property of gradient, we can write

$$(f \circ \alpha)' = \nabla f(\alpha) \cdot \alpha'$$

This give us

$$\begin{aligned}(f \circ \alpha)'' &= (\nabla f(\alpha))' \cdot \alpha' + \nabla f(\alpha) \cdot \alpha'' \\ &= (\nabla f(\alpha))' \cdot \alpha'\end{aligned}$$

This reduce the problem into proving

$$(\nabla f(\alpha))' \cdot \alpha' = \alpha' \cdot H\alpha'$$

Express

$$\nabla f(\alpha) = \Big((\partial_x f)(\alpha), (\partial_y f)(\alpha)\Big)$$

Compute

$$\big(\nabla f(\alpha)\big)' = \Big(\nabla(\partial_x f)(\alpha) \cdot \alpha', \nabla(\partial_y f)(\alpha) \cdot \alpha'\Big)$$

Note that

$$\nabla(\partial_x f) = (\partial_{xx} f, \partial_{xy} f) \text{ and } \nabla(\partial_y f) = (\partial_{yx} f, \partial_{yy} f)$$

Express $\alpha = (\alpha_1, \alpha_2)$. Compute

$$\begin{aligned}\nabla(\partial_x f)(\alpha) \cdot \alpha' &= \partial_{xx} f(\alpha) \cdot \alpha_1' + \partial_{xy} f(\alpha) \cdot \alpha_2' \\ \nabla(\partial_y f)(\alpha) \cdot \alpha' &= \partial_{yx} f(\alpha) \cdot \alpha_1' + \partial_{yy} f(\alpha) \cdot \alpha_2'\end{aligned}$$

We can now compute

$$\begin{aligned}\big(\nabla f(\alpha)\big)' \cdot \alpha' &= \Big((\alpha_1')\partial_{xx} f(\alpha) + (\alpha_2')\partial_{xy} f(\alpha), (\alpha_1)'\partial_{yx} f(\alpha) + (\alpha_2')\partial_{yy} f(\alpha)\Big) \cdot \alpha' \\ &= (\alpha_1')^2 \partial_{xx} f(\alpha) + 2(\alpha_1'\alpha_2')\partial_{xy} f(\alpha) + (\alpha_2')^2 \partial_{yy} f(\alpha) \\ &= \alpha' \cdot H\alpha' \text{ (done)}\end{aligned}$$

∎

Lemma 5.10.4. (Property of H)

(a) If $D > 0$ and $f_{xx} > 0$, then H is positive definite.

(b) If $D > 0$ and $f_{xx} < 0$, then H is negative definite.

(c) If $D < 0$, then H has a positive and a negative eigenvalue.

Proof. Observe that $\det(H) = D$. This let us write the characteristic polynomial of H by

$$t^2 - (\partial_{xx} f + \partial_{yy} f)t + D$$

Then we know the eigenvalues are

$$t = \frac{(\partial_{xx} f + \partial_{yy} f) \pm \sqrt{(\partial_{xx} f + \partial_{yy} f)^2 - 4D}}{2}$$

Now, if

$$\partial_{xx} f \partial_{yy} f - (\partial_{xy} f)^2 = D > 0$$

Then we have

$$\partial_{xx} f \partial_{yy} f > 0$$

Moreover, if $\partial_{xx} f > 0$, we can deduce

$$\partial_{xx} f + \partial_{yy} f > 0$$

Then from the formula of the eigenvalues t, because $D > 0$, regardless of the sign of the square-root term, we have

$$t > 0$$

Note that $t \in \mathbb{R}$ by Spectral Theorem.

Similarly, if $D > 0$ and $\partial_{xx} f < 0$, then $t < 0$, and if $D < 0$, then the two eigenvalues are of opposite signs and non-zero.

Now, if $D > 0$ and $f_{xx} > 0$, the two eigenvalues of H are all positive. Fix the orthonormal basis $\{e_1, e_2\}$ corresponding to eigenvalues $t_1, t_2 > 0$. We see that for all $c_1 e_1 + c_2 e_2 \neq 0 \in \mathbb{R}^2$

$$(c_1 e_1 + c_2 e_2) \cdot H(c_1 e_2 + c_2 e_2) = t_1 c_1^2 + t_2 c_2^2 > 0$$

Similarly, if $D > 0$ and $f_{xx} < 0$, the two eigenvalues of H are negative, which leads to

$$(c_1 e_1 + c_2 e_2) \cdot H(c_1 e_1 + c_2 e_2) = t_1 c_1^2 + t_2 c_2^2 < 0$$

$\blacksquare$

Proof. **($D > 0$ and $\partial_{xx} f(a,b) > 0$ Case)** Because $f \in \mathcal{C}^2(V)$, we know there exists $B_\epsilon(a,b)$ such that D and $\partial_{xx} f$ are positive on $B_\epsilon(a,b)$. We claim

$$f(a,b) = \min_{(x,y)\in B_\epsilon(a,b)} f(x,y)$$

Fix $(c,d) \in B_\epsilon(a,b)$. We reduce the problem into proving

$$f(a,b) \leq f(c,d)$$

Let $\gamma : [-1,1] \to B_\epsilon(a,b)$ be the line joining $(2a-c, 2b-d)$ and (c,d)

$$\gamma(t) \triangleq \Big(a + (c-a)t, b + (d-b)t\Big)$$

Note that $\gamma'' = 0$ and $\gamma(0) = (a,b)$. Now by Lemma **??** and Lemma **??**, we have

$$(f \circ \gamma)'' > 0 \text{ on } [-1,1]$$

This implies $f \circ \gamma$ is convex on $[-1,1]$. Compute using $\nabla f(a,b) = 0$

$$(f \circ \gamma)'(0) = df_{(a,b)}(c-a, d-b) = \nabla f(a,b) \cdot (c-a, d-b) = 0$$

This then implies

$$(f \circ \gamma)(0) \text{ is minimum on } [-1,1] \quad \text{(done)}$$

∎

Proof. **($D > 0$ and $\partial_{xx} f(a,b) < 0$ Case)** Because $f \in \mathcal{C}^2(V)$, we know there exists $B_\epsilon(a,b)$ such that D is positive and $\partial_{xx} f$ is negative on $B_\epsilon(a,b)$. We claim

$$f(a,b) = \max_{(x,y)\in B_\epsilon(a,b)} f(x,y)$$

Fix $(c,d) \in B_\epsilon(a,b)$. We reduce the problem into proving

$$f(a,b) \geq f(c,d)$$

Let $\gamma : [-1,1] \to B_\epsilon(a,b)$ be the line joining $(2a-c, 2b-d)$ and (c,d)

$$\gamma(t) \triangleq \Big(a + (c-a)t, b + (d-b)t\Big)$$

Note that $\gamma'' = 0$ and $\gamma(0) = (a,b)$. Now by Lemma **??** and Lemma **??**, we have

$$(f \circ \gamma)'' < 0 \text{ on } [-1,1]$$

This implies $f \circ \gamma$ is concave on $[-1,1]$. Compute using $\nabla f(a,b) = 0$

$$(f \circ \gamma)'(0) = df_{(a,b)}(c-a, d-b) = \nabla f(a,b) \cdot (c-a, d-b) = 0$$

This then implies

$$(f \circ \gamma)(0) \text{ is maximum on } [-1,1] \quad \text{(done)}$$

∎

Proof. **($D < 0$ Case)** From $\nabla f(a,b) = 0$, we know $df_{(a,b)} = 0$. This reduce the problem into proving

$$(a,b) \text{ is not a local extremum}$$

Fix ϵ such that $D < 0$ on $B_\epsilon(a,b)$. We prove

$$\min_{(x,y)\in B_\epsilon(a,b)} f(x,y) < f(a,b) < \max_{(x,y)\in B_\epsilon(a,b)} f(x,y)$$

Fix two curve $\alpha, \beta : (-\epsilon, \epsilon) \to \mathbb{R}^2$ by

$$\alpha(t) \triangleq (a,b) + te_1 \text{ and } \beta(t) \triangleq (a,b) + te_2$$

where e_1, e_2 are the eigenvectors of $H(p)$, e_1 correspond to the positive eigenvalue and e_2 correspond to the negative eigenvalue.

Compute using Lemma **??**

$$(f \circ \alpha)''(0) > 0 \text{ and } (f \circ \beta)''(0) < 0$$

Because $(f \circ \alpha)''$ and $(f \circ \beta)''$ is continuous, we know $(f \circ \alpha)'' > 0$ and $(f \circ \beta)'' < 0$ on a small enough interval I containing 0. In other words

$$(f \circ \alpha) \text{ is strictly convex and } (f \circ \beta) \text{ is strictly concave on } I$$

Again, we know $(f \circ \alpha)'(0) = \nabla f(a,b) \cdot e_1 = 0$ and, similarly, $(f \circ \beta)'(0) = 0$. We can now deduce

$$\begin{cases} f \circ \alpha \\ f \circ \beta \end{cases} \text{reach } \begin{cases} \text{strict minimum} \\ \text{strict maximum} \end{cases} \text{at } p \text{ on } I$$

The former implies $f(a,b) < \max_{(x,y)\in B_\epsilon(a,b)} f(x,y)$, and the latter implies $f(a,b) > \min_{(x,y)\in B_\epsilon(a,b)} f(x,y)$. (done) ∎

Question 44

6. Let f be a nonnegative and measurable function on E. Prove that

$$\int_E f(x)\,dx = \sup\left[\sum_j \left(\inf_{x\in E_j} f(x)\right)|E_j|\right]$$

where the supremum is taken over all decompositions $E = \bigcup_j E_j$ of E into the union of a finite number of disjoint measurable sets E_j.

Proof. We first show

$$\int_E f(x)dx \geq \sup\left[\sum_j \left(\inf_{x\in E_j} f(x)\right)|E_j|\right]$$

Fix a finite disjoint measurable decomposition E_j of E. We reduce the problem into proving

$$\int_E f(x)dx \geq \sum_j \left(\inf_{x\in E_j} f(x)\right)|E_j|$$

Define

$$s_0 \triangleq \sum_j \left(\inf_{x\in E_j} f(x)\right)\mathbf{1}_{E_j}$$

Because E_j are disjoint, we know

$$s_0(x) = \inf_{t\in E_j} f(t) \text{ if } x \in E_j$$

This implies s_0 is simple. Because E_j, are all measurable, we can deduce s_0 is measurable. In conclusion, $s_0 : E \to \mathbb{R}$ is a simple measurable function.

Given arbitrary $x \in E$, we know x must belong to one and only one of the decomposition. Express $x \in E_i$. This give us

$$s_0(x) = \inf_{t\in E_i} f(t) \leq f(x)$$

Because x is arbitrary, we can conclude $s \leq f$ on E.

Now by definition of Lebesgue Integral, we see

$$\sum_j \Big(\inf_{x\in E_j} f(x)\Big) |E_j| = \int_E s_0 dx \leq \int_E f dx \text{ (done)}$$

We now show

$$\int_E f(x)dx \leq \sup \Big[\sum_j \Big(\inf_{x\in E_j} f(x)\Big) |E_j|\Big]$$

Fix an arbitrary measurable simple $s : E \to [0,\infty]$ such that $f \geq s$ on E. Because s is arbitrary, by definition of Lebesgue Integral, we can reduce the problem into proving

$$\int_E s dx \leq \sup \Big[\sum_j \Big(\inf_{x\in E_j} f(x)\Big) |E_j|\Big]$$

Let c_j be the set of range of s. For each j, define $E_j \triangleq \{x \in E : s(x) = c_j\}$. Because s is measurable, we know E_j are measurable. Express s in the form

$$s = \sum_j c_j \mathbf{1}_{E_j}$$

We claim

$$\int_E s dx \leq \sum_j \Big(\inf_{x\in E_j} f(x)\Big) |E_j|$$

In other words, we wish to prove

$$\sum_j c_j |E_j| \leq \sum_j \Big(\inf_{x\in E_j} f(x)\Big) |E_j|$$

Fix j. We reduce the problem into proving

$$c_j \leq \inf_{x\in E_j} f(x)$$

Because $s \leq f$ on E, we know

$$c_j = s(x) \leq f(x) \text{ for all } x \in E_j$$

This conclude $c_j \leq \inf_{x\in E_j} f(x)$. (done) ■

Question 45

7. If $\{f_k\}_{k\in\mathbb{N}}$ is a sequence of nonnegative and measurable functions on E, prove that

$$\int_E \left(\sum_{k=1}^{\infty} f_k(x)\right) dx = \sum_{k=1}^{\infty}\left(\int_E f_k(x)\,dx\right)$$

Proof. Fix

$$f_n \triangleq \sum_{k=1}^{n} f_k \text{ and } f \triangleq \sum_{k=1}^{\infty} f_k$$

Because f_k are non-negative, we know f_n are pointwise increasing. Clearly $f_n \to f$ pointwise. Note that f_n are all measurable, since measurable functions are closed under addition.

Now by Lebesgue monotone convergence Theorem, we have

$$\begin{aligned}\int_E \Big(\sum_{k=1}^{\infty} f_k\Big) dx = \int_E f dx = \lim_{n\to\infty}\int_E f_n dx = \lim_{n\to\infty}\int_E \sum_{k=1}^{n} f_k dx &= \lim_{n\to\infty}\sum_{k=1}^{n}\int_E f_k dx \\ &= \sum_{k=1}^{\infty}\int_E f_k(x) dx\end{aligned}$$

■

Question 46

8. Let f be nonnegative and measurable on E, prove that

$$\int_E f(x)\,dx = 0 \iff \exists Z \subset E \text{ such that } |Z| = 0 \text{ and } f(x) = 0 \text{ on } E\setminus Z$$

Proof. Note that if $|E| = 0$, then $\int_E f dx = 0$, and we can just let $Z = E$. We suppose $|E| > 0$ from now.

$(\longleftarrow)$

Decompose E into arbitrary disjoint finite measurable union E_j. Because the decomposition is arbitrary, using the result of the sixth question, we only have to prove

$$\sum_j (\inf_{E_j} f) |E_j| = 0$$

We reduce the problem into proving

$$\forall j, |E_j| > 0 \implies \inf_{E_j} f = 0$$

Fix $|E_j| > 0$. Notice that $f = 0$ on Z^c. This allow us to reduce the problem into proving

$$|E_j \cap Z^c| > 0$$

Note that $|Z^c| = |E| - |Z| = |E|$. Observe

$$|E| \geq |E_j \cup Z^c| \geq |Z^c| = |E|$$

This implies $|E_j \cup Z^c| = |E|$. This give us

$$|E_j| + |Z^c| - |E_j \cap Z^c| = |E_j \cup Z^c| = |E|$$

Now because $|Z^c| = |E|$, we can deduce

$$|E_j \cap Z^c| = |E_j| > 0 \text{ (done)}$$

$(\longrightarrow)$

Fix

$$B_n \triangleq \{x \in E : f(x) \geq \frac{1}{n}\}$$

Because f is measurable, we know B_n are measurable. Note that each B_n must be of zero-measure, otherwise $\int_E f \geq \int_{B_n} f \geq \frac{|B_n|}{n} > 0$.

Now let

$$Z \triangleq \bigcup_{n \in \mathbb{N}} B_n$$

Observe

$$|Z| \leq \sum_{n \in \mathbb{N}} |B_n| = 0$$

and observe

$$x \in Z^c \implies \forall n \in \mathbb{N}, f(x) \notin B_n \implies \forall n \in \mathbb{N}, 0 \leq f(x) < \frac{1}{n} \implies f(x) = 0$$

$\blacksquare$

Question 47

9. Let $g(x) = \begin{cases} 0 & , 0 \le x \le 1/2 \\ 1 & , 1/2 < x \le 1 \end{cases}$ and $f_{2k}(x) = g(x)$, $f_{2k-1}(x) = g(1-x)$, $x \in [0, 1]$, $k \in \mathbb{N}$. Show that

$$\liminf_{n\to\infty} f_n(x) = 0 \text{ on } [0, 1]$$

but $\displaystyle\int_0^1 f_n(x)\, dx = \frac{1}{2}$. (This gives an example of strict inequality in **Fatou lemma**)

Proof. Compute

$$f_{2k}(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}] \\ 1 & \text{if } x \in (\frac{1}{2}, 1] \end{cases} \text{ and } f_{2k-1}(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}) \\ 0 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

Observe

$$\text{For all } x \in [0, \frac{1}{2}), \{f_{2k}(x)\}_{k\in\mathbb{N}} = 0 \implies \liminf_{n\to\infty} f_n = 0 \text{ on } [0, \frac{1}{2})$$

Observe

$$\{f_k(x)\}_{k\in\mathbb{N}} = 0 \implies \liminf_{n\to\infty} f_n = 0 \text{ on } \{\frac{1}{2}\}$$

Observe

$$\text{For all } x \in (\frac{1}{2}, 1], \{f_{2k-1}(x)\}_{k\in\mathbb{N}} = 0 \implies \liminf_{n\to\infty} f_n = 0 \text{ on } (\frac{1}{2}, 1]$$

This now conclude

$$\liminf_{n\to\infty} f_n = 0 \text{ on } [0, 1]$$

Compute

$$\int_0^1 f_{2k} dx = \int_{\frac{1}{2}}^1 1 dx = \frac{1}{2}$$

Compute

$$\int_0^1 f_{2k-1} dx = \int_0^{\frac{1}{2}} 1 dx = \frac{1}{2}$$

This conclude $\int_0^1 f_n dx = \frac{1}{2}$ for all $n \in \mathbb{N}$. ∎

Question 48

10. Let $f(x) = \begin{cases} 1/n & , |x| \le n \\ 0 & , |x| > n \end{cases}$. Show that $f_n(x)$ uniformly converges to 0 on $\mathbb{R}$ but $\displaystyle\int_{-\infty}^{\infty} f_n(x)\, dx = 2$ for all $n \in \mathbb{N}$.

Proof. Fix ϵ. Let $N > \frac{1}{\epsilon}$. We now see that for each $n > N$, we have

$$\sup_{\mathbb{R}} |f| = \frac{1}{n} < \epsilon$$

This conclude that f_n uniformly converge to $\mathbb{R}$.

Compute

$$\int_{\mathbb{R}} f_n(x)dx = \int_{-n}^{n} \frac{1}{n} dx = 2$$

$\blacksquare$

5.11 Taylor's Formula

Theorem 5.11.1. (Taylor's Formula) For each nice function $g : [0,1] \to \mathbb{R}$ and $m \in \mathbb{N}$, we have

$$g(1) - g(0) = \sum_{k=1}^{m-1} \frac{g^{(k)}(0)}{k!} + \frac{g^{(m)}(c)}{m!} \text{ for some } c \in (0,1)$$

and

$$g(1) - g(0) - \sum_{k=1}^{m-1} \frac{g^{(k)}(0)}{k!} = \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{m-1}} g^{(m)}(t_m) dt_m dt_{m-1} \cdots dt_1$$

Proof. If $m = 1$, then the proof is trivial by MVT and FTC.

Suppose the Theorem hold true for m, we prove the Theorem hold true for $m+1$.

We first prove

$$g(1) - g(0) - \sum_{k=1}^{m} \frac{g^{(k)}(0)}{k!} = \int_0^1 \int_0^{t_1} \cdots \int_0^{t_m} g^{(m+1)}(t_{m+1}) dt_{m+1} \cdots dt_1$$

By induction hypothesis,

$$\begin{aligned}
g(1) - g(0) - \sum_{k=1}^{m} \frac{g^{(k)}(0)}{k!} &= g(1) - g(0) - \sum_{k=1}^{m-1} \frac{g^{(k)}(0)}{k!} - \frac{g^{(m)}(0)}{m!} \\
&= \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{m-1}} g^{(m)}(t_m) dt_m dt_{m-1} \cdots dt_1 - \frac{g^{(m)}(0)}{m!} \\
&= \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{m-1}} \big(g^{(m)}(t_m) - g^{(m)}(0)\big) dt_m dt_{m-1} \cdots dt_1 \\
&\overset{\text{FTC}}{=} \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{m-1}} \int_0^{t_m} g^{(m+1)}(t_{m+1}) dt_{m+1} dt_m dt_{m-1} \cdots dt_1 \text{ (done)}
\end{aligned}$$

We now prove

$$g(1) - g(0) - \sum_{k=1}^{m} \frac{g^{(k)}(0)}{k!} = \frac{g^{(m+1)}(c)}{(m+1)!}$$

By induction hypothesis,

$$\begin{aligned}
g(1)-g(0)-\sum_{k=1}^{m}\frac{g^{(k)}(0)}{k!} &= \int_0^1\int_0^{t_1}\cdots\int_0^{t_m} g^{(m+1)}(t_{m+1})dt_{m+1}\cdots dt_1 \\
&\overset{\text{Fubini}}{=} \int_0^1\int_{t_{m+1}}^1\int_{t_{m+1}}^{t_1}\cdots\int_{t_{m+1}}^1 g^{(m+1)}(t_{m+1})dt_m\cdots dt_1 dt_{m+1} \\
&= \int_0^1 g^{(m+1)}(t_{m+1})\int_{t_{m+1}}^1\int_{t_{m+1}}^{t_1}\cdots\int_{t_{m+1}}^{t_{m-1}} dt_m\cdots dt_1 dt_{m+1} \\
&= \int_0^1 g^{(m+1)}(t_{m+1})F(t_{m+1})dt_{m+1} \\
&\overset{\text{MVT}}{=} g^{(m+1)}(c)\int_0^1 F(t_{m+1})dt_{m+1} = \frac{g^{(m+1)}(c)}{(m+1)!} \text{ (done)}
\end{aligned}$$

■

Theorem 5.11.2. (Taylor's Formula) Given $f:\mathbb{R}^n\to\mathbb{R}$ such that $f\in\mathcal{C}^m(S)$ for some $S\overset{\text{open,convex}}{=}\mathbb{R}^n$, for each $a,b\in S$, if we define $g:[0,1]\to\mathbb{R}$ by

$$g(t)\triangleq f(a+(b-a)t)$$

then there exists $\theta\in(0,1)$ such that

$$f(b)-f(a)=\sum_{k=1}^{m-1}\frac{1}{k!}f^{(k)}(a;b-a)+\frac{1}{m!}f^{(m)}(g(\theta);b-a)$$

Proof. Observe

$$\begin{aligned}
f(b)-f(a) &= g(1)-g(0) \\
&= \sum_{k=1}^{m-1}\frac{1}{k!}g^{(k)}(0)+g^m(\theta) \text{ for some } \theta
\end{aligned}$$

The problem can now be reduced into proving

$$g^{(k)}(t)=f^{(k)}(g(t);b-a) \text{ for all } k\in\{1,\dots,m\}$$

We prove by induction, for base case, compute

$$\begin{aligned}
g'(t) &= \sum_{k=1}^{n}[(\partial_k f)(a+(b-a)t)]\cdot(b-a)_k \\
&= f'(a;b-a)
\end{aligned}$$

Suppose

$$
\begin{aligned}
g^{(k)}(t) &= f^{(k)}(g(t); b-a) \\
&= \sum_{j_1,\dots,j_k \in \{1,\dots,n\}} \Big[[(\partial_{j_1,\dots,j_k} f)\big(a + (b-a)\mathbf{i})] \cdot \prod_{i=1}^{k} (b-a)_{j_i} \Big]
\end{aligned}
$$

This give us

$$
\begin{aligned}
g^{(k+1)}(t) &= \sum_{j_1 \dots, j_k \in \{1,\dots,n\}} \frac{d}{dt} \Big[[(\partial_{j_1,\dots,j_k} f)\big(a + (b-a)t\big)] \cdot \prod_{i=1}^{k} (b-a)_{j_i} \Big] \\
&= \sum_{j_1 \dots, j_k \in \{1,\dots,n\}} \Big[[\frac{d}{dt} (\partial_{j_1,\dots,j_k} f)\big(a + (b-a)t\big)] \cdot \prod_{i=1}^{k} (b-a)_{j_i} \Big] \\
&= \sum_{j_1 \dots, j_k \in \{1,\dots,n\}} \Big[[\sum_{j_{k+1} \in \{1,\dots,n\}} (\partial_{j_1,\dots,j_k,j_{k+1}} f)\big(a + (b-a)t\big) \cdot (b-a)_{j_{k+1}}] \cdot \prod_{i=1}^{k} (b-a)_{j_i} \Big] \\
&= \sum_{j_1 \dots, j_k, j_{k+1} \in \{1,\dots,n\}} \Big[[(\partial_{j_1,\dots,j_k,j_{k+1}} f)\big(a + (b-a)t\big)] \cdot \prod_{i=1}^{k+1} (b-a)_{j_i} \Big] \\
&= f^{(k+1)}(g(t); b-a)
\end{aligned}
$$

■

Note that our definition is

$$
f^{(k)}(z; c) = \sum_{j_1,\dots,j_k \in \{1,\dots,n\}} \Big[[\partial_{j_1,\dots,j_k} f(z)] \cdot \prod_{i=1}^{k} (c)_{j_i} \Big]
$$

When $k = 1$, then

$$
f^{1}(z; c) = \sum_{j=1}^{n} \partial_j f(z) \cdot c_j = \nabla f(z) \cdot c = df_z(c)
$$

5.12 Fourier Stuff

Definition 5.12.1. (Definition of Trigonometric Polynomials) By a trigonometric polynomials, we mean a function $P : \mathbb{R} \to \mathbb{C}$ of the form

$$P(x) = a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx) \text{ where } a_n, b_n \text{ are complex numbers}$$

Note that

$$P(x) = \sum_{-N}^{N} c_n e^{inx} \text{ where } c_n = \frac{a_n}{2} + \frac{b_n}{2i} \text{ and } c_{-n} = \frac{a_n}{2} - \frac{b_n}{2i} \text{ for each } n \in \mathbb{N}$$

Definition 5.12.2. (Definition of inner products and norms) Given a function f, g defined on $[-\pi, \pi]$, we say

$$\langle f, g \rangle \triangleq \frac{1}{2\pi} \int_{-\pi}^{\pi} f\overline{g}dx$$

and

$$\|f\|_2 \triangleq \sqrt{\langle f, f \rangle}$$

It is clear that $\{e^{inx}\}_{n\in\mathbb{Z}}$ is orthonormal in $L^2(\mu)$. From now, we will write $\phi_n \triangleq e^{inx}$.

Definition 5.12.3. (Definition of Fourier coefficients) For each $n \in \mathbb{Z}$, the n-th Fourier coefficients of f is

$$c_n \triangleq \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx}dx = \langle f, \phi_n \rangle$$

and we write

$$s_N(f) = \sum_{-N}^{N} c_n e^{inx} = \sum_{-N}^{N} c_n \phi_n$$

Theorem 5.12.4. (Special-case of Riesz-Fischer's Theorem) If f is Riemann-integrable on $[-\pi, \pi]$, then

$$\lim_{N\to\infty} \|f - s_N\|_2 = 0$$

Theorem 5.12.5. (Parseval's Identity) Suppose f, g are Riemann-integrable and $f \sim c_n$ and $g \sim \gamma_n$. We have

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\overline{g}dx = \sum_{-\infty}^{\infty} c_n\overline{\gamma_n}$$

Theorem 5.12.6. (Bounded Variation)

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \text{ is of bounded variation on } [0, 2\pi]$$

Proof. Observe

$$\begin{aligned} |f'(x)| = \left|2x\sin\frac{1}{x} - \cos\frac{1}{x}\right| &\leq \left|2x\sin\frac{1}{x}\right| + \left|\cos\frac{1}{x}\right| \\ &\leq |2x| + 1 \text{ is bounded on } (0, 2\pi) \end{aligned}$$

Then there exists positive M such that $M > |f'(x)|$ for all $x \in (0, 2\pi)$. Now, consider

$$g(x) \triangleq f(x) + Mx \text{ and } h(x) \triangleq Mx$$

Observe

$$g' = f' + M > 0 \text{ on } (0, 2\pi)$$

This implies g is increasing on $[0, 2\pi]$. It is clear that h is also increasing on $[0, 2\pi]$. Then because

(a) $f = g - h$ on $[0, 2\pi]$

(b) $BV\big([0, 2\pi]\big)$ is closed under addition

(c) Monotone functions are of bounded variation

We see $f \in BV\big([0, 2\pi]\big)$. ∎

Theorem 5.12.7. (Test)

$$\frac{\sin x}{x} \text{ is Riemann-integrable on } [a, \infty) \text{ for all } a \in \mathbb{R}^+$$

Proof. Because $\frac{\sin x}{x}$ is continuous on $(0, \infty)$, we know

$$\int_{\pi}^{a} \frac{\sin x}{x} dx \in \mathbb{R} \text{ for all } a \in \mathbb{R}^+$$

This reduce the problem into proving

$$\int_{\pi}^{\infty} \frac{\sin x}{x} dx \in \mathbb{R}$$

Suppose $x \in [k\pi, (k+1)\pi]$. It is clear that

$$\frac{\sin x}{x} \begin{cases} \geq 0 & \text{if } k \text{ is even} \\ \leq 0 & \text{if } k \text{ is odd} \end{cases}$$

This give us

$$\begin{aligned} \int_{\pi}^{\infty} \frac{\sin x}{x} dx &= \sum_{k=1}^{\infty} \left| \int_{k\pi}^{(k+1)\pi} \frac{\sin x}{x} dx \right| (-1)^k \\ &= \sum_{k=1}^{\infty} \int_{k\pi}^{(k+1)\pi} \left| \frac{\sin x}{x} \right| dx (-1)^k \end{aligned}$$

Alternating Series Test now allow us to reduce the problem into proving

$$\int_{k\pi}^{(k+1)\pi} \left| \frac{\sin x}{x} \right| dx \searrow 0 \text{ as } k \to \infty$$

For each $k \in \mathbb{N}$ and $x \in [k\pi, (k+1)\pi]$, we see

$$\left| \frac{\sin x}{x} \right| = \frac{|\sin(x+\pi)|}{x} \geq \frac{|\sin(x+\pi)|}{x+\pi} \geq \left| \frac{\sin(x+\pi)}{x+\pi} \right|$$

This implies

$$\int_{k\pi}^{(k+1)\pi} \left| \frac{\sin x}{x} \right| dx \geq \int_{(k+1)\pi}^{(k+1)\pi} \left| \frac{\sin x}{x} \right| dx$$

and reduce the problem into proving

$$\int_{k\pi}^{(k+1)\pi} \left| \frac{\sin x}{x} \right| dx \to 0 \text{ as } k \to 0$$

Observe

$$\begin{aligned} \int_{k\pi}^{(k+1)\pi} \left| \frac{\sin x}{x} \right| dx &\leq \int_{k\pi}^{(k+1)\pi} \frac{1}{x} dx \\ &= \ln x \Big|_{x=k\pi}^{(k+1)\pi} \\ &= \ln(k+1)\pi - \ln k\pi = \ln \frac{k+1}{k} = \ln(1+\frac{1}{k}) \to 0 \text{ (done)} \end{aligned}$$

■

Theorem 5.12.8. (Test) For all $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^*$

$$\left|\frac{\cos(\alpha+\beta)-\cos\alpha}{\beta}+\sin\alpha\right| \leq |\beta|$$

Proof. Compute

$$\begin{aligned}\frac{1}{\beta}\int_{\alpha}^{\alpha+\beta}(\sin\alpha-\sin t)dt &= \sin\alpha+\left(\frac{\cos t}{\beta}\right)\Big|_{t=\alpha}^{\alpha+\beta}\\ &= \frac{\cos(\alpha+\beta)-\cos\alpha}{\beta}+\sin\alpha\end{aligned}$$

The problem is then reduced into proving

$$\left|\int_{\alpha}^{\alpha+\beta}(\sin\alpha-\sin t)dt\right| \leq |\beta^2|$$

■

Chapter 6

ODE

6.1 Solution 2

1.(a)

$$x^2 + 2x + 2$$

1.(b)

$$x$$

2.(a)

$$y_P(t) = 9te^{2t} + \frac{-1}{5}\cos t + \frac{3}{5}\sin t$$

2.(b)

$$y_P(t) = \frac{t^3}{6}e^t + t^2 + 4t + 6$$

2.(c)

$$y_1 = e^t, y_2 = e^{-t}, y_3 = t, y_4 = 1, y_P = \frac{-t^3}{6} + \frac{t}{2}e^t$$

2.(d)

$$\begin{cases} y_1 = e^t, y_2 = te^t, y_3 = 1, y_4 = \cos t, y_5 = \sin t \\ y_P(t) = \frac{t^2}{2} + 2t + \frac{t^2}{4}e^t - \frac{t\cos t}{4} \end{cases}$$

3.(a)

$$y_P(t) = \left(\frac{\sin 2t}{2} - t\right)\cos t + \left(\frac{-\cos 2t}{2}\right)\sin t$$

3.(b)

$$y_P(t) = te^t + (\ln t)te^t$$

3.(c)

$$y_P(x) = \left(\frac{x}{2} - 1\right)e^{2x}$$

Summary of the information I gathered: 7 questions.

(a) 3 questions are solving second order homogeneous linear ODE with constant coefficients. i.e. $ay'' + by' + cy = 0, y(t_0) = y_0, y'(t_0) = y_0'$. The Characteristic polynomials of these 3 questions will respectively have distinct roots, complex roots, and repeated root.

(b) 2 questions are solving second order non-homogeneous linear ODE. Probably, one of them shall be solved by "undetermined coefficients", and another shall be solved by "variation of parameter"

(c) The last 2 questions are third order (linear) ODE. The method wasn't revealed.

6.2 ODE2

Theorem 6.2.1. (Basic Laplace Transformation: Polynomial) We have

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

where $s > 0$

Proof. We use induction. Observe

$$\mathcal{L}(1) = \int_{\mathbb{R}^+} e^{-ts}dt = \frac{e^{-ts}}{s}\Big|_{t=\infty}^{0} = \frac{1}{s}$$

Observe for all $n \in \mathbb{N}$

$$\mathcal{L}(t^n) = \int_{\mathbb{R}^+} e^{-ts}t^n dt \overset{\text{i.b.p}}{=} \frac{n}{s}\int_{\mathbb{R}^+} e^{-ts}t^{n-1}dt = \frac{n}{s}\mathcal{L}(t^{n-1})$$

■

Theorem 6.2.2. (Basic Laplace Transformation: exponential) We have

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

where $s > a$

Proof. Observe

$$\mathcal{L}(e^{at}) = \int_{\mathbb{R}^+} e^{t(a-s)}dt = \frac{e^{t(a-s)}}{a-s}\Big|_{t=0}^{\infty} = \frac{1}{s-a}$$

■

Theorem 6.2.3. (Basic Laplace transformation: trigonometric) We have

$$\mathcal{L}(\cos at) = \frac{s}{s^2+a^2} \text{ and } \mathcal{L}(\sin at) = \frac{a}{s^2+a^2}$$

where $s > 0$

Proof. Observe

$$\int_{\mathbb{R}^+} e^{t(-s+ia)}dt = \int_{\mathbb{R}^+} e^{-ts}\cos at + ie^{-ts}\sin at dt = \mathcal{L}(\cos at) + i\mathcal{L}(\sin at)$$

Compute

$$\int_{\mathbb{R}^+} e^{t(-s+ia)}dt = \frac{e^{t(-s+ia)}}{-s+ia}\Big|_{t=0}^{\infty} = \frac{1}{s-ia} = \frac{s+ia}{s^2+a^2}$$

This conclude the desired result. ■

Theorem 6.2.4. (Basic property of Laplace Transformation) Given nice f, we have

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

Proof. Compute

$$\begin{aligned}\mathcal{L}(f') &= \int_{\mathbb{R}^+} e^{-st} f'(t)dt \\ &= e^{-st} f(t)\Big|_{t=0}^{\infty} + s\int_{\mathbb{R}^+} e^{-st} f(t)dt \\ &= s\mathcal{L}(f) - f(0)\end{aligned}$$

■

Corollary 6.2.5. For all $n \in \mathbb{N}$

$$\begin{aligned}\mathcal{L}(f^{(n)}) &= s^n \mathcal{L}(f) - \sum_{k=1}^{n} s^{(n-k)} f^{(k-1)}(0) \\ &= s^n \mathcal{L}(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)\end{aligned}$$

Theorem 6.2.6. (Inverse Laplace)

$$\mathcal{L}(u_c(t) f(t-c)) = e^{-cs} \mathcal{L}(f)$$

Proof. Observe

$$\begin{aligned}\mathcal{L}(u_c(t) f(t-c)) &= \int_c^{\infty} e^{-ts} f(t-c)dt \\ &= \int_0^{\infty} e^{-s(u+c)} f(u)du \quad (\because u = t-c) \\ &= e^{-sc}\mathcal{L}(f)\end{aligned}$$

■

Corollary 6.2.7. (Inverse Laplace) If

$$F = Ge^{-cs}$$

then

$$f = u_c(t) g(t-c)$$

Theorem 6.2.8. (Inverse Laplace)

$$\mathcal{L}(e^{ct} f(t))(s) = \mathcal{L}(f)(s-c)$$

Proof. Observe

$$\begin{aligned}\mathcal{L}(e^{ct}f(t))(s) &= \int_0^\infty e^{t(c-s)}f(t)dt \\ &= \mathcal{L}(f)(s-c)\end{aligned}$$

■

Corollary 6.2.9. (Inverse Laplace) If

$$F(s) = G(s-c)$$

then

$$f(t) = e^{ct}g(t)$$

Summary of the trick

$$\begin{aligned}F = Ge^{-cs} &\iff f(t) = u_c(t)g(t-c) \\ F(s) = G(s-c) &\iff f(t) = e^{ct}g(t)\end{aligned}$$

The table: for all $n \in \mathbb{Z}_0^+$

$$\begin{aligned}\mathcal{L}(t^n) = \frac{n!}{s^{n+1}} &\text{ and } \mathcal{L}(e^{at}) = \frac{1}{s-a} \\ \mathcal{L}(\cos at) = \frac{s}{s^2+a^2} &\text{ and } \mathcal{L}(\sin at) = \frac{a}{s^2+a^2}\end{aligned}$$

The formula

$$\mathcal{L}(y^{(n)}) = s^n\mathcal{L}(y) - s^{n-1}y(0) - s^{n-2}y^{(1)}(0) - \cdots - sy^{(n-1)}(0) - y^{(n-1)}(0)$$

The formula

$$h = f * g \iff H = FG$$

Theorem 6.2.10. (Forcing Functions) Solve

$$\begin{cases} 2y'' + y' + 2y = u_5 - u_{20} \\ y(0) = 0, y'(0) = 0 \end{cases}$$

Proof. Taking Laplace transformation

$$\frac{e^{-5s} - e^{-20s}}{s} = (2s^2 + s + 2)\mathcal{L}(y)$$

This give us

$$
\begin{aligned}
\mathcal{L}(y) &= (e^{-5s}+e^{-20s})\frac{1}{s(2s^2+s+2)} \\
&= (e^{-5s}+e^{-20s})\Big(\frac{1}{2s}+\frac{-2s-1}{2(2s^2+s+2)}\Big) \\
&= (e^{-5s}+e^{-20s})\Big(G(s)+H(s)\Big) \text{ where } G(s)=\frac{1}{2s} \text{ and } H(s)=\frac{-s-\frac{1}{2}}{2s^2+s+2}
\end{aligned}
$$

and if we let

$$\mathcal{L}(f_1) = (e^{-5s}+e^{-20s})G(s) \text{ and } \mathcal{L}(f_2) = (e^{-5s}+e^{-20s})H(s)$$

we would have $y = f_1 + f_2$. We now reduce the problem into finding f_1, f_2.

It is easily deduced that

$$\begin{cases} f_1(t) = u_5(t)g(t-5) + u_{20}(t)g(t-20) \\ f_2(t) = u_5(t)h(t-5) + u_{20}(t)h(t-20) \end{cases}$$

We now reduce the problem into finding g and h. It is easily seen that $g(t) = \frac{1}{2}$. It remains to find h.

Write

$$
\begin{aligned}
H(s) &= \frac{\frac{-s}{2}-\frac{1}{4}}{(s+\frac{1}{4})^2+\frac{15}{16}} \\
&= \frac{\frac{-1}{2}(s+\frac{1}{4})+\frac{-1}{8}}{(s+\frac{1}{4})^2+\frac{15}{16}} \\
&= A(s+\frac{1}{4}) \text{ where } A(s) = \frac{\frac{-1}{2}s+\frac{-1}{8}}{s^2+\frac{15}{16}}
\end{aligned}
$$

Because $h(t) = e^{\frac{-1}{4}t}a(t)$, it remains to find a.

By the table, we have

$$a(t) = (\frac{-1}{2})\cos(\frac{\sqrt{15}t}{4}) + (\frac{-1}{2\sqrt{15}})\sin(\frac{\sqrt{15}t}{4})$$

■

Chapter 7

Geometry Archived

7.1 Prerequiste

In this section, we will use I to denote an **bounded open interval**. By a **curve** in $\mathbb{R}^n$, we mean a function form an open interval I to $\mathbb{R}^n$. We say a curve is **differentiable** if it is infinitely differentiable (smooth). That is, $\gamma^{(n)}(t)$ exists and are continuous for all $n \in \mathbb{N}$ and $t \in I$.

We say a differentiable curve $\gamma : I \to \mathbb{R}^n$ is **regular** if $\gamma'(t) \neq 0$ for all $t \in I$. We say a differentiable curve $\gamma : I \to \mathbb{R}^n$ is a **parametrized by arc-length** if $|\gamma'(t)| = 1$ for all $t \in I$.

For a regular curve γ, we say $\gamma'(t)$ is the **tangent vector** of γ at t, and we define the **unit tangent vector** T by

$$T(t) \triangleq \frac{\gamma'(t)}{|\gamma'(t)|}$$

We say $\gamma''(t)$ is the **oriented curvature** (normal vector) of γ at t, and we define the **unit normal vector** N by

$$N(t) \triangleq \frac{T'(t)}{|T'(t)|}$$

Some interesting facts can be observed from what we have deduced.

(a) γ', γ'' always exists.

(b) γ is parametrized by arc-length $\implies \gamma' \perp \gamma''$

(c) γ is parametrized by arc-length $\implies$ γ is regular

(d) T and T' exists at t $\iff$ γ is regular at t

(e) $T = \gamma'$ $\iff$ γ is parametrized by arc-length

(f) N exists at t $\iff$ $\gamma''(t) \neq 0$ $\iff$ $\kappa(t) \neq 0$

(g) N and T' point to the same direction γ''.

(h) $|T'| = \kappa$ $\iff$ γ is paramterized by arc-length

(i) $\gamma \perp \gamma'$ and $\gamma'' \perp \gamma'''$ are generally false even for curve γ paramterized by arc-length.

(j) Given a curve γ parametrized by arc-length

$$\begin{aligned}\gamma \text{ is a straight line on } [a,b] &\iff \gamma' \text{ and } T \text{ are constant on } (a,b) \\ &\iff \gamma''(t) = 0 \text{ on } (a,b) \\ &\iff \kappa(t) = 0 \text{ on } (a,b) \\ &\iff T'(t) = 0 \text{ on } (a,b)\end{aligned}$$

Notice that the last fact is false if γ is not paramtetrized by arc-length, since γ can move in the straight line with changing speed γ'.

Given a curve γ, if $T(t)$ and $N(t)$ exists (regular and non-zero curvature), we define its **binormal** vector by

$$B(t) = T(t) \times N(t)$$

Fix t. We say

$$\{T(t), N(t), B(t)\} \text{ form a } \textbf{positively oriented orthonormal basis} \text{ of } \mathbb{R}^3$$

This basis in general is constantly changing, yet always form an orthonormal basis.

Also, we say

$$\text{span}\Big(T(t), N(t)\Big) \text{ is the } \textbf{osculating plane} \text{ of } \gamma \text{ at } t$$

Suppose γ is parametrized by arc-length and always has non-zero curvature. With some geometric intuition, one shall note that $|T'|$ measure how curved γ is and that $|B'|$ measure how fast γ leave the osculating plane.

Because $|B| = 1$ is a constant, we can deduce

$$B' \perp B$$

and the computation

$$B' = T' \times N + T \times N' = T \times N'$$

give us

$$B' \perp T$$

This ultimately show us

$$B', N, T' \text{ are all parallel where } N, T' \text{ even point to the same direction}$$

Notice that if we parametrize the curve with opposite direction, then

(a) T, γ' change direction

(b) N, γ'' keep the same direction

(c) B change direction

(d) B' keep the same direction

Now, for a curve γ parametrized by arc-length, we define the **curvature** κ and **torsion** τ of γ by

$$\kappa(t) = |\gamma''(t)| \text{ and } \tau(t) = \frac{B'(t)}{N(t)}$$

With unfortunately heavy computation, we can verify that the definition of curvature must stay in the framework of curve parametrized by arc-length, otherwise we will be given two different values of curvature of two curves that are equivalent in the sense of sets.

Now, notice that we already have $T' = \kappa N$ and $B' = \tau N$, and by basic identity, we have $N = B \times T$.

Then with some computation, we have the **Frenet Formula**

$$\begin{cases} T' = \kappa N \\ N' = B' \times T + B \times T' = -\tau B - \kappa T \\ B' = \tau N \end{cases}$$

Given two vectors $u, v \in \mathbb{R}^n$, we use **dot product**

$$u \cdot v = u_1 v_1 + \cdots + u_n v_n$$

to denote the Euclidean inner product, and we use **length**

$$|u| = \sqrt{\sum_{k=1}^{n} u_k^2}$$

to denote the Euclidean norm. Note that we clearly have

$$|u| = \sqrt{u \cdot u}$$

Given three vectors $u, v, w \in \mathbb{R}^3$, we define **cross product** by

$$\begin{aligned} u \times v &\triangleq \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1) \end{aligned}$$

With some simple computation, we have the following identity

(a) $u \times v = -v \times u$ (anti-commutative)

(b) $(au + w) \times v = a(u \times v) + w \times v$ (Linearity)

(c) $u \times (aw + v) = a(u \times w) + u \times v$

(d) $u \times v = 0 \iff u = cv$ for some $c \in \mathbb{R}$

(e) $(u \times v) \cdot w = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

(f) $(u \times v) \cdot v = 0 = (u \times v) \cdot u$

(g) $u \times v \perp u$ and $u \times v \perp v$

(h) $u \perp v \implies |u \times v| = |u| \cdot |v|$

(i) $(u \times v) \times w = (u \cdot w)v - (v \cdot w)u$

All proofs except that of the last identity are merely manipulation of determinant. A simple proof of the last identity follows from the fact both side are linear in all u, v, w, and the observation

$$(e_1 \times e_2) \times e_3 = 0 = (e_1 \cdot e_3)e_2 - (e_2 \cdot e_3)e_1$$

Theorem 7.1.1. (Differentiate the Dot Product) Given two parametrized curves $u, v : (a, b) \to \mathbb{R}^n$, such that u, v are differentiable at $t \in (a, b)$. We have

$$\frac{d}{dt}\big(u(t) \cdot v(t)\big) = u'(t) \cdot v(t) + u(t) \cdot v'(t)$$

Proof.

$$\begin{aligned}
\frac{d}{dt}\big(u(t) \cdot v(t)\big) &= \frac{d}{dt}\sum_{k=1}^{n} u_k(t)v_k(t) \\
&= \sum_{k=1}^{n} \frac{d}{dt} u_k(t)v_k(t) \\
&= \sum_{k=1}^{n} u_k'(t)v_k(t) + u_k(t)v_k'(t) \\
&= \sum_{k=1}^{n} u_k'(t)v_k(t) + \sum_{k=1}^{n} u_k(t)v_k'(t) \\
&= u'(t) \cdot v(t) + u(t) \cdot v'(t)
\end{aligned}$$

■

Theorem 7.1.2. (Differentiate the Cross Product) Given two curves $u, v : (a, b) \to \mathbb{R}^3$, such that u, v are differentiable at $t \in (a, b)$. We have

$$\frac{d}{dt}\big(u(t) \times v(t)\big) = u'(t) \times v(t) + u(t) \times v'(t)$$

Proof.

$$
\begin{aligned}
\frac{d}{dt}\big(u(t)\times v(t)\big) =& \frac{d}{dt}(u_2v_3-u_3v_2, u_3v_1-u_1v_3, u_1v_2-u_2v_1)\\
=&(u_2'v_3+u_2v_3'-u_3'v_2-u_3v_2',\\
&u_3'v_1+u_3v_1'-u_1'v_3-u_1v_3',\\
&u_1'v_2+u_1v_2'-u_2'v_1-u_1v_2')\\
=&(u_2'v_3-u_3'v_2, u_3'v_1-u_1'v_3, u_1'v_2-u_2'v_1)\\
+&(u_2v_3'-u_3v_2', u_3v_1'-u_1v_3', u_1v_2'-u_1v_2')\\
=&u'\times v+u\times v'
\end{aligned}
$$

■

Theorem 7.1.3. (Integrating the Dot Product) Given a curve $u:[a,b]\to\mathbb{R}^n$ and a vector $v\in\mathbb{R}^n$, suppose

(a) u is differentiable on (a,b)

(b) u is continuous on $[a,b]$

We have

$$
\int_a^b u'(t)\cdot v dt = \Big(\int_a^b u'(t)dt\Big)\cdot v = \big(u(b)-u(a)\big)\cdot v
$$

Proof.

$$
\begin{aligned}
\int_a^b u'(t)\cdot v dt &= \int_a^b \sum_{k=1}^n u_k'(t)\cdot v_k dt\\
&= \sum_{k=1}^n \int_a^b u_k'(t)\cdot v_k dt\\
&= \sum_{k=1}^n v_k \int_a^b u_k'(t) dt\\
&= v\cdot\Big(\int_a^b u'(t)dt\Big)
\end{aligned}
$$

■

For the result above, sometimes we write

$$
(u\cdot v)' = u'\cdot v+u\cdot v' \text{ and } (u\times v)'=u'\times v+u\times v'
$$

Theorem 7.1.4. (MVT for curve) Given a curve $\alpha : [a, b] \to \mathbb{R}^n$ such that

(a) α is differentaible on (a, b)

(b) α is continuous on $[a, b]$

there exists $\xi \in (a, b)$ such that

$$|\alpha(b) - \alpha(a)| \leq |\alpha'(\xi)| (b - a)$$

Proof. Define $\phi : [a, b] \to \mathbb{R}$ by

$$\phi(t) = \alpha(t) \cdot \big(\alpha(b) - \alpha(a)\big)$$

Clearly ϕ satisfy the hypothesis of Lagrange's MVT, then we know there exists $\xi \in (a, b)$ such that

$$\phi(b) - \phi(a) = \phi'(\xi) \cdot (b - a)$$

Written the equation in α, we have

$$|\alpha(b) - \alpha(a)|^2 = (b - a)\alpha'(\xi) \cdot \big(\alpha(b) - \alpha(a)\big)$$

Notice that Cauchy-Schwarz inequality give us

$$\begin{aligned}(b - a) |\alpha'(\xi)| \cdot |\alpha(b) - \alpha(a)| &\geq (b - a) \left|\alpha'(\xi) \cdot \big(\alpha(b) - \alpha(a)\big)\right| \\ &= |\alpha(b) - \alpha(a)|^2\end{aligned}$$

This then implies

$$(b - a) |\alpha'(\xi)| \geq |\alpha(b) - \alpha(a)|$$

■

Corollary 7.1.5. (Mean Value Inequality) Given a curve $\alpha : [a, b] \to \mathbb{R}^n$ such that

(a) α is differentiable on (a, b)

(b) α is continuous on $[a, b]$

we have

$$|\alpha(b) - \alpha(a)| \leq (b - a) \sup_{(a,b)} |\alpha'|$$

Trick to parametrize by arc-length.

Given a regular curve $\gamma : I \to \mathbb{R}^n$ and fix $t_0 \in I$. We use

$$s(t) = \int_{t_0}^{t} |\gamma'(x)|\,dx$$

to define the arc-length of γ from $\gamma(t_0)$ to $\gamma(t)$. Because γ is regular, by FTC, it is clear that s is one-to-one.

Let $t(s)$ be the inverse of s. Define

$$\beta(s) \triangleq \alpha(t(s))$$

We have by Chain rule

$$\begin{aligned}\beta'(s) &= t'(s)\alpha'(t(s)) \\ &= \frac{\alpha'(t(s))}{s'(t)} \\ &= \frac{\alpha'(t(s))}{|\alpha'(t(s))|}\end{aligned}$$

(Frenet Formula Summary)

By definition, we are given

$$\begin{cases} T' = \kappa N \\ B' = \tau N \end{cases}$$

To compute N', an identity should be first given

$$N = B \times T$$

We can now complete the Frenet Formula

$$\begin{aligned}N' &= B' \times T + B \times T' \\ &= \tau N \times T + B \times \kappa N \\ &= -\tau B - \kappa T\end{aligned}$$

In conclusion

$$\begin{cases} T' = \kappa N B' = \tau N \\ N' = -\tau B - \kappa T \end{cases}$$

Give very close attention to the fact the two definitions of curvature

$$\kappa = \frac{T'}{N} \text{ and } \kappa = |\gamma''|$$

coincides only when γ is parametrzied by arc-length. The first definition remain same for all parametrizaiton of the same curve, while the latter doesn't.

Some comment should be dropped for the computation of torsion. If you overlook the fact α is parametrized by arc-length and disregard Frenet Formula, it is very likely you will get a result that you can not even sure if it is valid (the nominator and denominator may end up not seem explicitly parallel), let alone an identity beautiful as below.

7.2 Examples of Regular Surfaces

Among all regular surfaces, the most classic one is perhaps the S^2. Here, we show some local parametriatoin of S^2.

Note that because $S^2 = F^{-1}[0]$, where $F(x,y,z) = x^2+y^2+z^2-1$ clearly has non-zero derivative everywhere on $\mathbb{R}^3 \setminus 0$, we know S^2 is a regular surface.

Example 18 (Graph Coordinates of S^2)

$$U = \{(u,v) : u^2+v^2 < 1\} \text{ and } S^2 = \{(x,y,z) : x^2+y^2+z^2 = 1\}$$

(a) $f_1 : B_1(0) \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ by $f_1(x,y) = (x, y, \sqrt{1-x^2-y^2})$

(b) $f_2 : B_1(0) \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ by $f_2(x,y) = (x, y, -\sqrt{1-x^2-y^2})$

(c) $f_3 : B_1(0) \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ by $f_3(x,z) = (x, \sqrt{1-x^2-z^2}, z)$

(d) $f_4 : B_1(0) \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ by $f_4(x,z) = (x, -\sqrt{1-x^2-z^2}, z)$

(e) $f_5 : B_1(0) \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ by $f_5(y,z) = (\sqrt{1-y^2-z^2}, y, z)$

(f) $f_6 : B_1(0) \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ by $f_6(y,z) = (-\sqrt{1-y^2-z^2}, y, z)$

Example 19 (Spherical Coordinates of S^2)

$$S^2 = \{(x,y,z) : x^2+y^2+z^2 = 1\}$$

(a) $\mathbf{x}_1 : (0,\pi) \times (0,2\pi) \to S^2$ by $\mathbf{x}_1(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

(b) $\mathbf{x}_2 : (0,\pi) \times (0,2\pi) \to S^2$ by $\mathbf{x}_2(\theta,\phi) = (-\sin\theta\cos\phi, \cos\theta, \sin\theta\sin\phi)$

Note that

(a) $\mathbf{x}_1$ does not contain $\{(x,0,z) \in S^2 : \begin{cases} x^2+z^2 = 1 \\ x \geq 0 \end{cases} \}$

(b) $\mathbf{x}_2$ does not contain $\{(x,y,0) \in S^2 : \begin{cases} x^2+y^2 = 1 \\ x \leq 0 \end{cases} \}$

Example 20 (Stereographical Coordinates of S^2: Projection Plane be the Equator)

$$U = \mathbb{R}^2 \text{ and } S^2 = \{(x,y,z) : x^2+y^2+z^2 = 1\}$$

Note that

(a) $\mathbf{x}_N^{-1}(x,y,z) \equiv (\frac{x}{1-z}, \frac{y}{1-z})$

(b) $\mathbf{x}_S^{-1}(x,y,z) \equiv (\frac{x}{z+1}, \frac{y}{z+1})$

For explicit expression of $\mathbf{x}_N$, Use the trick

$$u^2+v^2 = \frac{x^2+y^2}{(1-z)^2} = \frac{1-z^2}{(1-z)^2} \text{ and } x = u(1-z), y = v(1-z)$$

to first solve for z, then solve for x, y.

Now, we have

(a) $\mathbf{x}_N(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$

(b) $\mathbf{x}_S(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{1-u^2-v^2}{1+u^2+v^2}\right)$

Let

$$\mathbf{x}_N(u,v) = (x,y,z)$$

Compute

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{4(-u^2-v^2+1)}{u^6+3u^4v^2+3u^4+3u^2v^4+6u^2v^2+3u^2+v^6+3v^4+3v^2+1}$$

$$\frac{\partial(y,z)}{\partial(u,v)} = \frac{-8u}{u^6+3u^4v^2+3u^4+3u^2v^4+6u^2v^2+3u^2+v^6+3v^4+3v^2+1}$$

$$\frac{\partial(x,z)}{\partial(u,v)} = \frac{8v}{u^6+3u^4v^2+3u^4+3u^2v^4+6u^2v^2+3u^2+v^6+3v^4+3v^2+1}$$

This shows $\mathbf{x}_N$ is indeed a local parametrization.

Note that

$$\mathbf{x}_S^{-1} \circ \mathbf{x}_N(u,v) = (\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2})$$

is a diffeomorphism on $\mathbb{R}^2 \setminus 0$, since $\det\Big(d(\mathbf{x}_S^{-1} \circ \mathbf{x}_N)\Big) = \frac{-1}{(u^2+v^2)^2}$.

Also, note that if we identify $(u, v) \equiv u + iv \triangleq \xi$, we have

$$\mathbf{x}_S^{-1} \circ \mathbf{x}_N(\xi) = \frac{\xi}{|\xi|^2}$$

Example 21 (Stereographical Coordinates of S^2: Projection Plane at the Bottom)

$$U = \mathbb{R}^2 \text{ and } S^2 = \{(x, y, z) : x^2 + y^2 + z^2\}$$

Chapter 8

Archived Materials

8.1 Equivalent Definitions for Riemann Integral

In this section, we will give a principal Theorem (Theorem **??**) that can serve as a lemma to prove the equivalency of multiple different definitions of Riemann integral on a compact interval. With this approach, we shall diminish the trouble of getting through miscellaneous minor definitions, where they are all equivalent, with only the difference of taking different tags and partitions of certain pattern, which solely serve as a pedagogical tool to give students a concrete idea of integration.

A caveat will be made clear here: this section concern only the proper Riemann Integral. That is, we only consider the integration of a function bounded on a compact interval. For a treatment of inproper integral, see Section **??**.

In this section, by a **partition** P of $[a,b]$, we mean a finite set of values $P = \{a = x_0 \le x_1 \le \cdots \le x_{n_P - 1} \le x_{n_P} = b\}$. We say the partition P' is **finer** than P if $P \subseteq P'$. Given a partition P, we put

$$\begin{cases} M_i = \sup_{[x_{i-1},x_i]} f(x) \\ m_i = \inf_{[x_{i-1},x_i]} f(x) \end{cases} \quad \text{and} \quad \begin{cases} U(P,f) = \sum_{i=1}^{n_P} M_i \Delta x_i \\ L(P,f) = \sum_{i=1}^{n_P} m_i \Delta x_i \end{cases} \quad \text{where } \Delta x_i = x_i - x_{i-1}$$

We shall write n instead of n_P if no confusion will be made.

The word **norm of partition** $\|P\|$ is defined by $\max_{1\le i\le n} \Delta x_i$. We say $U(P,f)$ is an **upper sum** of f. We say the **upper integral** $\overline{\int_a^b} f\,dx$ of f on $[a,b]$ is $\inf_P U(P,f)$ where the infimum run through all partitions P of $[a,b]$. The **lower integral** $\underline{\int_a^b} f\,dx$ is defined similarly. We say a function f is **integrable** on $[a,b]$ if $\overline{\int_a^b} f dx = \underline{\int_a^b} f dx$.

Give close attention to the setting that P is finite. This is crucial for making the integration operation possible, since if P is countable and we define $U(P,f)$ by taking limits for sums, the order of addition can make a difference if the sum does not converge absolutely. This fact is backed by Riemann Rearrangement Theorem (Theorem **??**), of which we will later give a proof.

Theorem 8.1.1. (Principal for Proving Equivalency of Definitions for Riemann Integral)

$$\int_a^b f dx \in \mathbb{R} \iff \forall \{P_k\} : \|P_k\| \to 0, U(P_k,f) - L(P_k,f) \to 0$$

Proof. From right to left is obvious. We prove only

$$\int_a^b f dx \in \mathbb{R} \implies \forall \{P_k\} : \|P_k\| \to 0, U(P_k,f) - L(P_k,f) \to 0$$

Fix ϵ. We wish to find a positive number $\beta \in \mathbb{R}^+$ such that $\forall P : \|P\| \leq \beta, U(P,f) - L(P,f) < \epsilon$. Because $\int_a^b f dx \in \mathbb{R}$, we can let W be a partition such that

$$U(W,f) - L(W,f) < \frac{\epsilon}{2}$$

Let $W = \{a = x_0^*, x_1^*, \dots, x_{n_W}^* = b\}$, and let $J = \{1, \dots, n_W\}$ be the set of indices of W. Suppose

$$L = \max_{1 \leq j \leq n_W - 1} \Big(\sup_{[x_{j-1}^*, x_{j+1}^*]} f(x) - \inf_{[x_{j-1}^*, x_{j+1}^*]} f(x) \Big) \tag{8.1}$$

Notice that if $L = 0$, then f must be constant and the proof become trivial, so we can assume $L > 0$. We claim that

$$L\beta n_W \leq \frac{\epsilon}{2} \text{ and } \beta < \min_{j \in J} \Delta x_j$$

suffice so that $\forall P : \|P\| \leq \beta, U(P,f) - L(P,f) < \epsilon$. Let $C = \min_{j \in J} \Delta x_j$. In other words, we now reduce the problem into proving

$$\|P\| \leq \min\{\frac{\epsilon}{2Ln_W}, C\} \implies U(P,f) - L(P,f) < \epsilon$$

Let $I = \{1, \dots, n_P\}$ be the set of indices for P. Suppose

$$P = \{a = x_0, x_1, \dots, x_{n_P} = b\}$$

We partition I into

$$\begin{cases} A = i \in I : \exists j \in J, [x_{i-1}, x_i] \subseteq [x^*_{j-1}, x^*_j] \\ B = I \setminus A \end{cases}$$

We now have

$$U(P,f) - L(P,f) = \sum_{i \in A} \big(M_i - m_i\big)\Delta x_i + \sum_{i \in B} \big(M_i - m_i\big)\Delta x_i \tag{8.2}$$

Because for each $i \in A$, there is a unique corresponding $j \in J$ such that $[x_{i-1}, x_i] \subseteq [x^*_{j-1}, x^*_j]$, we have

$$\sum_{i \in A} \big(M_i - m_i\big)\Delta x_i \leq \sum_{j \in J} \big(M^W_j - m^W_j\big)\Delta x^*_j = U(W,f) - L(W,f) < \frac{\epsilon}{2} \tag{8.3}$$

Because $\|P\| \leq C = \min_{j \in J} \Delta x_j$, we know for each distinct $i \in B$, there exists a distinct $j \in J$ such that $[x_{i-1}, x_i] \subseteq [x^*_{j-1}, x^*_{j+1}]$, so by definition of L (Equation **??**), we have

$$\sum_{i \in B} \big(M_i - m_i\big)\Delta x_i \leq L \sum_{i \in B} \Delta x_i \leq L n_W \|P\| \leq \frac{\epsilon}{2} \tag{8.4}$$

Combining Equation **??**, Equation **??** and Equation **??**, we now see

$$U(P,f) - L(P,f) < \epsilon \text{ (done)}$$

■

Recall that we say a series $\sum_{n=1}^{\infty} a_n$ **absolutely converge** if $\sum_{n=1}^{\infty} |a_n|$ converge. We can show that a series converges if it absolutely converges by proving it is Cauchy. In this section, by a **permutation on** $\mathbb{N}$, we mean a bijective function σ from $\mathbb{N}$ to $\mathbb{N}$. Another two important terminologies are the followings. We say that $\sum_{n=1}^{\infty} a_n$ **unconditionally converge** if for all permutation $\sigma : \mathbb{N} \to \mathbb{N}$, the series $\sum_{n=1}^{\infty} a_{\sigma(n)}$ converge to the same number. We say $\sum_{n=1}^{\infty} a_n$ **conditionally converge** if it converge but not unconditionally.

In our treatment, Riemann Rearrangement Theorem will be split into 4 parts. The summary is at Theorem **??**. The first part (Theorem **??**) states that the limit an absolutely convergent series remain the same under any permutations. The proof for the first part (Theorem **??**) may seem technical, but the essence is quite easy to remember.

Just "see" that

$$\left|\sum_{k=1}^{\infty} a_{\sigma(k)} - L\right| \le \left|\sum_{i<M} a_i - L + \sum_{i\ge M} a_i\right|$$
$$\le \left|\sum_{i<M} a_i - L\right| + \sum_{i\ge M} |a_i| \to 0 \text{ as } M \to 0$$

Theorem 8.1.2. (Riemann Rearrangement Theorem, Part 1)

$$\sum_{k=1}^{\infty} a_k \text{ absolutely converge } \implies \sum_{k=1}^{\infty} a_k \text{ unconditionally converge}$$

Proof. Suppose $\sum_{k=1}^{\infty} |x_k|$ converge. Let $\sum_{k=1}^{\infty} x_k = L$. Fix permutation $\sigma : \mathbb{N} \to \mathbb{N}$. We wish to prove

$$\sum_{k=1}^{\infty} x_{\sigma(k)} = L$$

Fix ϵ. We reduce the problem into

$$\text{finding } N \text{ such that } \forall n > N, \left|\sum_{k=1}^{n} x_{\sigma(k)} - L\right| < \epsilon$$

Because both $\sum_{k=1}^{\infty} x_k$ and $\sum_{k=1}^{\infty} |x_k|$ converge by premise. We know there exists M such that

$$\forall n > M, \left|\sum_{k=1}^{n} x_k - L\right| < \frac{\epsilon}{2} \text{ and } \sum_{k=n}^{\infty} |x_k| < \frac{\epsilon}{2} \tag{8.5}$$

Let

$$I = \sigma^{-1}\big(\{1, \dots, M\}\big) \text{ and } N = \max I$$

We claim

$$\text{such } N \text{ works}$$

To prove our claim, fix $n > N$. We wish to show

$$\left|\sum_{k=1}^{n} x_{\sigma(k)} - L\right| < \epsilon$$

Let $I_n = \{1, \dots, n\}$. Observe that

$$\left| \sum_{k=1}^{n} x_{\sigma(k)} - L \right| = \left| \sum_{k \in I_n \setminus I} x_{\sigma(k)} + \sum_{k \in I} x_{\sigma(k)} - L \right| \tag{8.6}$$

Notice that $k \notin I \implies \sigma(k) > M$. Then by definition of M (Equation **??**), we have

$$\left| \sum_{k \in I_n \setminus I} x_{\sigma(k)} \right| \leq \sum_{k \in I_n \setminus I} \left| x_{\sigma(k)} \right| \leq \sum_{j > M}^{\infty} |x_j| < \frac{\epsilon}{2} \tag{8.7}$$

Notice that $\sigma[I] = \{1, \dots, M\}$. Then also by definition of M (Equation **??**), we have

$$\left| \sum_{k \in I} x_{\sigma(k)} - L \right| = \left| \sum_{j=1}^{M} x_j - L \right| < \frac{\epsilon}{2} \tag{8.8}$$

Then by inequalities Equation **??**, Equation **??** and Equation **??**, we now have

$$\begin{aligned} \left| \sum_{k=1}^{n} a_{\sigma(k)} - L \right| &= \left| \sum_{k \in I_n \setminus I} x_{\sigma(k)} + \sum_{k \in I} x_{\sigma(k)} - L \right| \\ &\leq \left| \sum_{k \in I_n \setminus I} x_{\sigma(k)} \right| + \left| \sum_{k \in I} x_{\sigma(k)} - L \right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ (done)} \end{aligned}$$

■

The second, third and forth parts (Theorem **??**, Theorem **??** and Theorem **??**), respectively states that if a series converge but not absolutely, then the limit value can be changed to any real number, infinite, negative infinite and even jumping-ly diverges.

The detail of the proof is very tedious and cumbersome, while the essence is easy to understand. The only two tools for proving Theorem **??**, Theorem **??** and Corollary **??**, is Lemma **??** and the fact $\sum a_k \to L \implies a_k \to 0$. If any part of the proof can be considered interesting, I believe it lies in that of Lemma **??**, where one split the series $\sum a_k$ into two $\sum a_k^+, \sum a_k^-$, and shows that they must both diverge.

Lemma 8.1.3. (Intrinsic Structure of Series that Converge but not Absolutely) Let $f^+ : \mathbb{N} \to \mathbb{N}$ and $f^- : \mathbb{N} \to \mathbb{N}$ satisfy that $\{a_{f^+(n)}\}$ contain all and only positive terms of $\{a_n\}$ and $\{a_{f^-(n)}\}$ contain all and only negative terms. If $\sum_{k=1}^{\infty} a_k$ converge but not absolutely, then for each $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{N}$, there exists $u_n > n$ and $t_n > n$ such that

$$\sum_{n \le k \le u_n} a_{f^+(k)} > \alpha \text{ and } \sum_{n \le k \le t_n} a_{f^-(k)} < -\alpha$$

Proof. Let $a_n^+ = \max\{0, a_n\}$ and $a_n^- = \min\{0, a_n\}$. It is easy to check $\forall n \in \mathbb{N}, a_n = a_n^+ + a_n^-$. Because $\sum_{k=1}^{\infty} a_k$ converge but not absolutely, we know

$$\begin{cases} \sum_{k=1}^{\infty} a_k^+ \to \infty \\ \sum_{k=1}^{\infty} a_k^- \to -\infty \end{cases} \tag{8.9}$$

This is true because if both of them converge then $\sum_{k=1}^{\infty} |a_k|$ converges and if only one of them converge them $\sum_{k=1}^{\infty} a_k$ diverges.

Because of Equation **??**, we know

$$\begin{cases} \sum_{k=1}^{\infty} a_{f^+(k)} \to \infty \\ \sum_{k=1}^{\infty} a_{f^-(k)} \to -\infty \end{cases}$$

The result then follow, since

$$\forall n \in \mathbb{N}, \sum_{k \ge n} a_{f^+(k)} \nearrow \infty \text{ and } \sum_{k \ge n} a_{f^-(k)} \searrow -\infty$$

■

Theorem 8.1.4. (Riemann Rearrangement Theorem, Part 2) If $\sum_{k=1}^{\infty} a_k$ converge but not absolutely, then there exists permutations $\sigma_\infty, \sigma_{-\infty} : \mathbb{N} \to \mathbb{N}$ such that $\sum_{k=1}^{\infty} a_{\sigma_\infty(k)} \to \infty$ and $\sum_{k=1}^{\infty} a_{\sigma_{-\infty}(k)} \to -\infty$.

Proof. We wish

$$\text{to construct } \sigma_\infty : \mathbb{N} \to \mathbb{N} \text{ such that } \sum_{k=1}^{\infty} a_{\sigma_\infty(k)} \to \infty$$

Using Lemma **??**, construct $\sigma_\infty : \mathbb{N} \to \mathbb{N}$ as follows. Let p_n be a sequence of natural number such that for each $n \in \mathbb{N}$, p_{n+1} is the smallest natural number such that

$$\sum_{k=p_n+1}^{p_{n+1}} a_{f^+(k)} > 3 \text{ and } p_1 = 0 \tag{8.10}$$

Similarly, let q_n be a sequence of natural number such that for each $n \in \mathbb{N}, q_{n+1}$ is the smallest natural number such that

$$\sum_{k=q_n+1}^{q_{n+1}} a_{f^-(k)} < -1 \text{ and } q_1 = 0 \tag{8.11}$$

Notice that the definition of p_n and q_n (Equation **??**, Equation **??**) are done recursively. Now, recursively define σ_∞ to follow the order

$$\begin{aligned} &f^+(p_1+1), \dots, f^+(p_2), f^-(q_1+1), \dots, f^-(q_2) \\ \longrightarrow &f^+(p_2+1), \dots, f^+(p_3), f^-(q_2+1), \dots, f^-(q_3), f^+(p_3+1), \dots \end{aligned}$$

If there exists $k \in \mathbb{N}$ such that $a_k = 0$, which is not in the range $f^+[\mathbb{N}] \cup f^-[\mathbb{N}]$, we can merge these zero term into our σ_∞ by putting them in terms of even order. This way, our σ_∞ then become bijecetive, a permutation.

We claim

$$\text{such } \sigma_\infty \text{ works}$$

Recall the definition of p_n (Equation **??**) is that for each $n \in \mathbb{N}, p_{n+1}$ is the smallest natural number such that

$$\sum_{k=p_n+1}^{p_{n+1}} a_{f^+(k)} > 3$$

Also recall the similarly defined q_n. This tell us

$$\sum_{k=p_n+q_n+1}^{p_{n+1}+q_{n+1}} a_{\sigma_\infty(k)} \to 2 \text{ as } n \to \infty$$

where

$$\sum_{k=p_n+q_n+1}^{p_{n+1}+q_n} a_{\sigma_\infty(k)} \to 3 \text{ and } \sum_{k=p_{n+1}+q_n+1}^{p_{n+1}+q_{n+1}} a_{\sigma_\infty(k)} \to -1 \text{ as } n \to \infty$$

With this, it is easy to verify $\sum_{k=1}^{\infty} a_{\sigma_\infty(k)} \to \infty$ (done) . The construction of $\sigma_{-\infty}$ and the proof for its validity is done similarly. ■

Theorem 8.1.5. (Riemann Rearrangement Theorem, Part 3) If $\sum_{k=1}^{\infty} a_k$ converges but not absolutely, then for all $[L, M] \subseteq \mathbb{R}$, there exists a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\liminf_{n\to\infty} \sum_{k=1}^{n} a_{\sigma(k)} = L$ and $\limsup_{n\to\infty} \sum_{k=1}^{n} a_{\sigma(k)} = M$.

Proof. We wish

to construct a working σ

The construction of σ is similar to that of σ_∞ in Theorem **??**. WOLG, let $M > 0$. Let $p_1 = 0$, and let p_2 be the smallest natural number such that

$$\sum_{k=1}^{p_2} a_{f^+(k)} > M \text{ and } p_1 = 0$$

Next, define $q_1 = 0$ and let q_2 be the smallest natural number such that

$$\sum_{k=1}^{p_2} a_{f^+(k)} + \sum_{k=1}^{q_2} a_{f^-(k)} < L$$

Then, let p_3 be the smallest natural number such that

$$\sum_{k=1}^{p_3} a_{f^+(k)} + \sum_{k=1}^{q_2} a_{f^-(k)} > M$$

Recursively do such. We get two sequences $\{p_n\}, \{q_n\}$ of natural number such that for all $n \in \mathbb{N}, p_{n+1}$ is the smallest natural number such that

$$\sum_{k=1}^{p_{n+1}} a_{f^+(k)} + \sum_{k=1}^{q_n} a_{f^-(k)} > M$$

and for all $n \in \mathbb{N}, q_{n+1}$ is the smallest natural number such that

$$\sum_{k=1}^{p_{n+1}} a_{f^+(k)} + \sum_{k=1}^{q_{n+1}} a_{f^-(k)} < L$$

Them, recursively define σ to follow the order

$$\begin{aligned} &f^+(p_1+1), \dots, f^+(p_2), f^-(q_1+1), \dots, f^-(q_2) \\ \longrightarrow &f^+(p_2+1), \dots, f^+(p_3), f^-(q_2+1), \dots, f^-(q_3), f^+(p_3+1), \dots \end{aligned}$$

Again, merge in the zero terms like in Theorem **??**. The proof for the claim such σ works is easy to verify knowing $a_{\sigma(k)} \to 0$ (done) ■

Corollary 8.1.6. (Riemann Rearrangement Theorem, Part 4) If $\sum_{k=1}^\infty a_k$ converges but not absolutely, then for all $L \in \mathbb{R}$, there exists a permutation σ such that $\sum_{k=1}^\infty a_{\sigma(k)} = L$

Theorem 8.1.7. (Summary of Riemann Rearrangement Theorem) If $\sum_{k=1}^{\infty} a_k$ converge, then

$$\sum_{k=1}^{\infty} a_k \text{ absolutely converges} \iff \sum_{k=1}^{\infty} a_k \text{ unconditionally converges}$$

Proof. $(\longrightarrow)$

This is Theorem **??**.

$(\longleftarrow)$

The fact that the contraposition of this statement is true is implied by any of Theorem **??**, Theorem **??** and Corollary **??**. ∎

8.2 Equivalent axiomazations

In the first lecture of Topology, we learned that a topological space (X, τ) is a space X with a topology $\tau \subseteq \mathcal{P}(X)$ such that τ satisfy the following three axioms.

$$\begin{cases} X, \varnothing \in \tau \\ \forall O, Y \in \tau, O \cap Y \in \tau \text{ (Closed under finite intersection)} \\ \forall T \subseteq \mathcal{P}(\tau), \bigcup T \in \tau \text{ (Closed under arbitrary union)} \end{cases} \tag{8.12}$$

It is only after the explicit listing of the above three of open sets, we then start defining "closed sets", "neighborhoods", "continuous functions", "compact sets" or "connected sets" based on open sets.

Although this approach, axiomatization via open sets, is mathematically sufficient, in history, there are other axiomatization proved to be equivalent to the traditional axiomaization via open sets.

In this section, we will give other three axiomatizations of topology, via neighborhood systems, via nets and via filters, and show they are equivalent with each other.

In this note, a *neighborhood system* is a function $\mathcal{N} : X \to \mathcal{P}(\mathcal{P}(X))$. We shall first show that there exists an explicit one-to-one correspondence between collection of topologies on X and collection of neighborhood systems that satisfy the following axioms.

Axiom 8.2.1. (Axioms of neighborhood systems)

$$\begin{cases} \forall x \in X, \mathcal{N}(x) \neq \varnothing \text{ (Non empty)} \\ \forall x \in X, \forall M \in \mathcal{N}(x), x \in M \text{ (Around)} \\ \forall x \in X, \forall M \subseteq X, \exists N \in \mathcal{N}(x), N \subseteq M \implies M \in \mathcal{N}(x) \text{(Closed under superset)} \\ \forall x \in X, \forall M, N \in \mathcal{N}(x), M \cap N \in \mathcal{N}(x) \text{(Closed under finite intersection)} \\ \forall x \in X, \forall M \in \mathcal{N}(x), \exists N \in \mathcal{N}(x), N \subseteq M \text{ and } \forall y \in N, M \in \mathcal{N}(y) \text{(Open neighborhood)} \end{cases}$$

Suppose C is the collection of topologies on X and D is the collection of neighborhood systems that satisfy Axioms 1.1.1. Now we wish to prove that the function $f : C \to D$

$$\tau \mapsto \mathcal{N}_\tau \text{ where } \mathcal{N}_\tau(x) = \{A \in \mathcal{P}(X) : \exists O \in \tau, x \in O \subseteq A\} \tag{8.13}$$

is bijective. In order to prove this, we first have to prove that f is well-defined, meaning for each topology τ, the function $\mathcal{N}_\tau$ is indeed a neighborhood system that satisfy the above axioms. This is easy, which we will omit here. It remains that we have to prove two statements: f is one-to-one, and f is onto.

Theorem 8.2.2. (f in Equation ?? is one-to-one) As titled.

Proof. Suppose τ and τ' are two different topologies on X. WOLG, suppose $O \in \tau \setminus \tau'$. By definition of f (Equation **??**), we know

$$\forall x \in O, O \in \mathcal{N}_\tau(x)$$

This means, to prove $\mathcal{N}_\tau \neq \mathcal{N}_{\tau'}$, we only have to prove the existence of some $x \in O$ such that $O \notin \mathcal{N}_\tau(x)$. Assume $\forall x \in O, O \in \mathcal{N}_{\tau'}(x)$. By definition of f (Equation **??**), we can deduce $\forall x \in O, \exists A_x \in \tau', x \in A_x \subseteq O$. It is easy to verify that $O = \bigcup_{x \in O} A_x$. Then by Axiom **??** of open sets , we $O = \bigcup_{x \in O} A_x \in \tau'$ CaC to that $O \in \tau \setminus \tau'$. ∎

To prove f is onto is a little bit complicated.

Theorem 8.2.3. (f in Equation ?? is onto) As titled.

Proof. Define $g : D \to C$ by

$$\mathcal{N} \mapsto \sigma_{\mathcal{N}} \text{ where } \sigma_{\mathcal{N}} = \{O \in \mathcal{P}(X) : \forall x \in O, O \in \mathcal{N}(x)\} \tag{8.14}$$

It is easy to check for each neighborhood system $\mathcal{N} : X \to \mathcal{P}(\mathcal{P}(X))$ satisfying Axioms 1.1.1, the collection $\sigma_{\mathcal{N}}$ is a topology.

Then, to prove f is onto, it suffice to prove

$$\forall \mathcal{N} \in D, \mathcal{N} = f(\sigma_{\mathcal{N}})$$

In other words, we wish to prove

$$\forall \mathcal{N} \in D, \forall x \in X, \mathcal{N}(x) = \big(f(\sigma_{\mathcal{N}})\big)(x)$$

By definition of f (Equation **??**), we know

$$\big(f(\sigma_{\mathcal{N}})\big)(x) = \{A \in \mathcal{P}(X) : \exists O \in \sigma_{\mathcal{N}}, x \in O \subseteq A\} \tag{8.15}$$

Suppose $A \in \big(f(\sigma_{\mathcal{N}})\big)(x)$. Let $O \in \sigma_{\mathcal{N}}$ satisfy $x \in O \subseteq A$. By definition of g (Equation **??**), we know $O \in \mathcal{N}(x)$. Then because Axiom 1.1.1 says that $\mathcal{N}(x)$ is closed under superset, we now see $A \in \mathcal{N}(x)$. We have proved $\big(f(\sigma_{\mathcal{N}})\big)(x) \subseteq \mathcal{N}(x)$.

Suppose $M \in \mathcal{N}(x)$. By the fifth axiom listed in Axiom 1.1.1, we know there exists some $N \in \mathcal{N}(x)$ such that $N \subseteq M$ and $\forall y \in N, M \in \mathcal{N}(y)$. Notice that the union of such N is still a subset of M and M is a neighborhood of each point in the union. This means that there exists a maximum N that satisfy

$$\begin{cases} N \subseteq M \\ \forall y \in N, M \in \mathcal{N}(y) \end{cases}$$

We now prove $N \in \sigma_{\mathcal{N}}$. Arbitrarily pick $z \in N$, by definition of N above, we can deduce $M \in \mathcal{N}(z)$. Then by the fifth axiom listed in Axiom 1.1.1, we know there exists some $N' \in \mathcal{N}(z)$ satisfying $N' \subseteq M$ and $\forall y \in N', M \in \mathcal{N}(y)$. Because N by definition is the maximum of such neighborhood, we see $N' \subseteq N$. Then because Axiom 1.1.1 says that $\mathcal{N}(z)$ is closed under superset, we see $N \in \mathcal{N}(z)$.

We have shown $\forall z \in N, N \in \mathcal{N}(z)$. Then by definition of g (Equation **??**), we see $N \in \sigma_{\mathcal{N}}$ (done) .

Then because $N \subseteq M$ by definition of N and because $N \in \sigma_{\mathcal{N}}$, we see $M \in \big(f(\sigma_{\mathcal{N}})\big)(x)$, according to Equation **??**. We have proved $\mathcal{N}(x) \subseteq \big(f(\sigma_{\mathcal{N}})\big)(x)$ (done) ■

Before embarking on the axiomatization via nets, we first have to settle the terminologies. Recall that a set D is **directed** if there exists an ordering $\leq$ on D such that $\leq$ is reflexive, transitive, and we have $\forall i, j \in D, \exists k \in D, i \leq k$ and $j \leq k$. By a **net**, we mean a function w whose domain is directed. We say a subset D' of a directed set is **cofinal** if $\forall d \in D, \exists d' \in D', d \leq d'$. By a **subnet** of $w : D \to X$, we mean a net $v : E \to X$ such that there exists $h : E \to D$ such that

$$\begin{cases} \forall e, e' \in E, e \leq e' \implies h(e) \leq h(e') (\text{Monotone}) \\ h[E] \text{ is cofinal} \\ v = w \circ h \end{cases}$$

One can check that when w is a sequence x_n and v is the sub-sequence x_{n_k}, the corresponding h is just n_k.

By a **tail** T_d of a directed set D, we mean $T_d = \{e \in D : d \leq e\}$. We say $w : D \to X$ **is eventually in** $A \subseteq X$ if $\exists d \in D, w[T_d] \subseteq A$. We say $w : D \to X$ **is frequently in** $A \subseteq X$ if $\forall d \in D, \exists e \in T_d, w(e) \in A$. Given a topological space (X, τ), we say w **converge** to a point x, if for all neighborhood O around x there exists $d \in D$ such that $w[T_d] \subseteq O$. Notice that if we wish to prove $w \to x$ we only have to verify for all open neighborhoods O around x. Also notice that w can converge to multiple points. A trivial example is when two point are topologically indistinguishable.

8.3 archived

If we are given a pair $(X, \mathcal{A})$ where $\mathcal{A}$ is a σ-algebra on X, we say $(X, \mathcal{A})$ is a **measurable space**. We call the elements of $\mathcal{A}$ **measurable sets**. Furthermore, suppose X has a topology τ. We say $(X, \sigma(\tau))$ is a **Borel space**, and we say $\sigma(\tau)$ is a **Borel σ-algebra**. Also, if we are given a family $\{(X_i, \mathcal{A}_i)\}$ of measurable space, we say $\sigma(\prod \mathcal{A}_i)$ is the **product σ-algebra**.

Theorem 8.3.1. (Product Borel σ-algebra and Borel σ-algebra of Product Space) Given a family $\{(X_i, \mathcal{A}_i)\}_{i\in I}$ of Borel space, we have

$$\sigma(\prod \mathcal{A}_i) \subseteq \sigma(\prod X_i)$$

The equality hold true when I is countable and X_i are all second countable.

Suppose that we are given a Borel space $(X, \sigma(\tau))$. Let S be a subset of X, and let τ_S be the subspace topology. It immediately comes the question, Is it true that

$$\sigma(\tau_S) \subseteq \sigma(\tau) \text{ ?}$$

where $\sigma(\tau_S)$ is the Borel σ-algebra of S. This is true when S is open or closed in X (Theorem **??**), but not true in general.

Theorem 8.3.2. (Sub-Borel space) Given a Borel space $(X, \tau, \mathcal{A})$ and an open or close $S \subseteq X$, denote the sub-space topology by τ_S, and let $\sigma(\tau_S)$ be the Borel σ-algebra of S. We have

$$\sigma(\tau_S) \subseteq \sigma(\tau)$$

Proof. If S is open, then $\tau_S \subseteq \tau$. This implies $\tau_S \subseteq \sigma(\tau)$ ∎

8.4 measurable archived

Theorem 8.4.1. (Measurable $f : X \to [0,\infty]$ is pointwise limit of monotone sequence of real-valued measurable simple function) Given $f : X \to [-\infty,\infty]$, there exists a sequence $s_n : X \to \mathbb{R}$ of simple functions such that

$$s_n \to f \text{ pointwise}$$

If f is measurable, then s_n can be taken to be measurable. If f is non-negative, then s_n can be taken to be increasing.

Proof. Because $f = f^+ - f^-$, WOLG, we can assume f is non-negative.

Fix

$$\delta_n \triangleq 2^{-n}$$

For each $n \in \mathbb{N}$ and $t \in \mathbb{R}$, we know there exists a unique $k(n,t) \in \mathbb{R}$ such that

$$k\delta_n \leq t < (k+1)\delta_n$$

Define $\phi_n : [0,\infty] \to \mathbb{R}$ by

$$\phi_n(t) \triangleq \begin{cases} k\delta_n & \text{if } t \in [0,n) \\ n & \text{if } t \in [n,\infty] \end{cases}$$

We claim

$$s_n \triangleq \phi_n \circ f \text{ suffices}$$

It is straightforward to check ϕ_n are measurable and simple. It then follows that s_n are simple, and s_n are measurable if f is. Because $\phi_n(f(x)) = s_n(x)$, we can now reduce the problem into proving

$$\text{For all } t \in [0,\infty]\text{, we have } \phi_n(t) \nearrow t \text{ as } n \to \infty$$

If $t = \infty$, then $\phi_n(t) = n \nearrow \infty = t$ as $n \to \infty$. Fix $t \in \mathbb{R}_0^+$. We reduce the problem into proving

$$\phi_n(t) \nearrow t$$

For large enough n, we have

$$|\phi_n(t) - t| \leq \delta_n \to 0$$

We can now reduce the problem into proving

$$\phi_n(t) \text{ increase as } n \to \infty$$

Fix $m \in \mathbb{Z}_0^+$ such that $m \le t < m+1$. Observe that

$$\phi_n(t) = \begin{cases} k_n\delta_n & \text{if } n > m \\ n & \text{if } 0 \le n \le m \end{cases}$$

It is now clear that

$$1 = \phi_1(t) \le \phi_2(t) \le \cdots \le \phi_m(t) = m$$

It remains to prove

$$\phi_m(t) \le \phi_{m+1}(t) \le \cdots$$

Observe that for each n

$$k_n = \max\{j \in \mathbb{Z}_0^+ : j \le 2^n t\}$$

This give us

$$\phi_m(t) = m = (m2^{m+1})\delta_{m+1} \le k_{m+1}\delta_{m+1} = \phi_{m+1}(t)$$

Fix $n \in \mathbb{N}$, by induction, we can now reduce the problem into proving

$$\phi_{m+n}(t) \le \phi_{m+n+1}(t)$$

which is just proving

$$2k_{m+n} \le k_{m+n+1}$$

Using definition of k_n

$$2k_{m+n} \le 2^{m+n+1}t$$

The proof now follows from the definition of k_{m+n+1}. (done) ■

Theorem 8.4.2. (Basic property of pre-measure) Given an increasing sequence of measurable set A_n

$$\mu(A_n) \nearrow \mu(A)$$

where $A = \bigcup A_n$. If A_n is decreasing, then

$$\mu(A_n) \searrow \mu(A)$$

where $A = \bigcap A_n$.

Proof. If A_n is increasing, fix $B_n \triangleq A_n \setminus A_{n-1}$ for each $n \ge 2$ and $B_1 = A_1$. The proof then follows from checking

$$\mu(A_n) = \sum_{k=1}^{n} \mu(B_k) \text{ and } \mu(A) = \sum_{k=1}^{\infty} \mu(B_k)$$

The proof for another statement is similar. ■

8.5 RPS

Two player α, β are playing rock-paper-scissor. Let $\Lambda \triangleq \{R, P, S\}$, and let $\mathcal{P}(\Lambda)$ denote the power set of Λ. Define the relation $\geq$ on Λ by

$$R > P > S > R$$

Note that R, P, S respectively represent rock, paper and scissor. Let M be the set of all probability measure on the measurable $(\Lambda, \mathcal{P}(\Lambda))$.

Note that given two probability measure $\mu, \mu' \in M$ on $(\Lambda, \mathcal{P}(\Lambda))$, we can define a probability pre-measure $\nu^{\text{pre}}_{(\mu,\mu')} : \Lambda^2 \to [0,1]$ by

$$\nu\big((A, B)\big) = \mu(\{A\}) \cdot \mu'(\{B\})$$

Denote the extension of $\nu^{\text{pre}}_{(\mu,\mu')} : \Lambda^2 \to [0,1]$ onto $\mathcal{P}(\Lambda^2)$ by $\nu_{(\mu,\mu')} : \mathcal{P}(\Lambda) \to [0,1]$.

Theorem 8.5.1. (The "unbeatable" strategy for Rock-Paper-Scissor is uniform random) Let $\mu, \mu' \in M$ and $\Omega \triangleq \{W, L\}$ where μ satisfy

$$\mu(\{R\}) = \mu(\{P\}) = \mu(\{S\}) = \frac{1}{3}$$

8.6 General Analysis HW1

Theorem 8.6.1. **($\mathbb{R}^n$ is complete)**

Theorem 8.6.2. **($\mathbb{Q}$ is dense in $\mathbb{R}$)**