Calculus HW2

Date: Mar 13 Made by Eric

1.

Proof. $(\longleftarrow)$

$\lim_{t\to t_0}\mathbf{f}(t) = \lim_{t\to t_0}\langle f_1(t), f_2(t), f_3(t)\rangle = \langle \lim_{t\to t_0} f_1(t), \lim_{t\to t_0} f_2(t), \lim_{t\to t_0} f_3(t)\rangle = \langle L_1, L_2, L_3\rangle = \mathbf{L}$

$(\longrightarrow)$

$\lim_{t\to t_0}\mathbf{f}(t) = \mathbf{L} \implies \langle \lim_{t\to t_0} f_1(t), \lim_{t\to t_0} f_2(t), \lim_{t\to t_0} f_3(t)\rangle = \langle L_1, L_2, L_3\rangle \implies \forall 1 \le i \le 3, \lim_{t\to t_0} f_i(t) = L_i$

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2.

Proof. Let p, q be two vector valued function defined by $p(t) = r(t) - L$ and $q(t) = s(t) - M$

From now on, we use ϵ, δ to denote positive real number.

We know $\forall\epsilon, \exists\delta, \forall t \in (a-\delta, a+\delta), |p(t) = r(t) - L| < \epsilon$

And we know $\forall\epsilon, \exists\delta, \forall t \in (a-\delta, a+\delta), |q(t) = s(t) - M| < \epsilon$

(a)
$|L\cdot M - r(t)\cdot s(t)| = |L\cdot M - [p(t)+L]\cdot[q(t)+M]| = |L\cdot M - p(t)\cdot q(t) - L\cdot q(t) - M\cdot p(t) - L\cdot M| = |p(t)\cdot q(t) + L\cdot q(t) + M\cdot p(t)| \le |p(t)\cdot q(t)| + |L\cdot q(t)| + |M\cdot p(t)| \le |p(t)||q(t)| + |L||q(t)| + |M||p(t)|$

$\forall\epsilon, \exists\delta_0, \forall t \in (a-\delta_0, a+\delta_0), |p(t)| < \sqrt{\frac{\epsilon}{3}}$ **(Because $\sqrt{\frac{\epsilon}{3}} \in \mathbb{R}^+$)**

$\forall\epsilon, \exists\delta_1, \forall t \in (a-\delta_0, a+\delta_0), |q(t)| < \sqrt{\frac{\epsilon}{3}}$ **(Because $\sqrt{\frac{\epsilon}{3}} \in \mathbb{R}^+$)**

$\forall\epsilon, \exists\delta_2, \forall t \in (a-\delta_0, a+\delta_0), |p(t)| < \frac{\epsilon}{3|L|}$ **(Because $\frac{\epsilon}{3|L|} \in \mathbb{R}^+$)**

$\forall\epsilon, \exists\delta_3, \forall t \in (a-\delta_0, a+\delta_0), |q(t)| < \frac{\epsilon}{3|M|}$ **(Because $\frac{\epsilon}{3|M|} \in \mathbb{R}^+$)**

For all ϵ, we can let $\delta = min(\delta_0, \delta_1, \delta_2, \delta_3)$

So $\forall t \in (a-\delta, a+\delta), |L\cdot M - r(t)\cdot s(t)| \le |p(t)||q(t)| + |L||q(t)| + |M||p(t)| < (\sqrt{\frac{\epsilon}{3}})^2 + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$

(b)

$|L \times M - r(t) \times s(t)| = |L \times M - [p(t) + L] \times [q(t) + M]| = |L \times M - p(t) \times q(t) - L \times q(t) - p(t) \times M - L \times M| = |p(t) \times q(t) + L \times q(t) + p(t) \times M| \leq |p(t) \times q(t)| + |L \times q(t)| + |p(t) \times M| \leq |p(t)||q(t)| + |L||q(t)| + |M||p(t)|$

$\forall \epsilon, \exists \delta_0, \forall t \in (a - \delta_0, a + \delta_0), |p(t)| < \sqrt{\frac{\epsilon}{3}}$ **(Because $\sqrt{\frac{\epsilon}{3}} \in \mathbb{R}^+$)**

$\forall \epsilon, \exists \delta_1, \forall t \in (a - \delta_0, a + \delta_0), |q(t)| < \sqrt{\frac{\epsilon}{3}}$ **(Because $\sqrt{\frac{\epsilon}{3}} \in \mathbb{R}^+$)**

$\forall \epsilon, \exists \delta_2, \forall t \in (a - \delta_0, a + \delta_0), |p(t)| < \frac{\epsilon}{3|L|}$ **(Because $\frac{\epsilon}{3|L|} \in \mathbb{R}^+$)**

$\forall \epsilon, \exists \delta_3, \forall t \in (a - \delta_0, a + \delta_0), |q(t)| < \frac{\epsilon}{3|M|}$ **(Because $\frac{\epsilon}{3|M|} \in \mathbb{R}^+$)**

For all ϵ, we can let $\delta = min(\delta_0, \delta_1, \delta_2, \delta_3)$

So $\forall t \in (a-\delta, a+\delta), |L \times M - r(t) \times s(t)| \leq |p(t)||q(t)| + |L||q(t)| + |M||p(t)| < (\sqrt{\frac{\epsilon}{3}})^2 + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$

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3.

Proof. $|\mathbf{r} \cdot \mathbf{f}(t)| \leq \|r\|\|f(t)\|$

$\iff |\mathbf{r} \cdot \mathbf{f}(t)|^2 \leq \|r\|^2\|f(t)\|^2$

$\iff (r_1 f_1(t) + r_2 f_2(t) + \cdots + r_n f_n(t))^2 \leq (r_1^2 + r_2^2 + \cdots + r_n^2)(f_1(t)^2 + f_2(t)^2 + \cdots + f_n(t)^2)$

$\iff (r_1 f_1(t) + r_2 f_2(t) + \cdots + r_n f_n(t))^2 - (r_1^2 f_1(t)^2 + \cdots r_n^2 f_n(t)^2) \leq (r_1^2 + r_2^2 + \cdots + r_n^2)(f_1(t)^2 + f_2(t)^2 + \cdots + f_n(t)^2) - (r_1^2 f_1(t)^2 + \cdots r_n^2 f_n(t)^2)$

$\iff \sum_{1 \leq i < j \leq n} 2 r_i f_i(t) r_j f_j(t) = \sum_{1 \leq i < j \leq n} r_i f_i(t) r_j f_j(t) + \sum_{1 \leq i < j \leq n} r_j f_j(t) r_i f_i(t) = \sum_{1 \leq i < j \leq n} r_i f_i(t) r_j f_j(t) + \sum_{1 \leq j < i \leq n} r_i f_i(t) r_j f_j(t) = \sum_{1 \leq i \neq j \leq n} r_i f_i(t) r_j f_j(t) = (r_1 f_1(t) + r_2 f_2(t) + \cdots + r_n f_n(t))^2 - (r_1^2 f_1(t)^2 + \cdots r_n^2 f_n(t)^2) \leq (r_1^2 + r_2^2 + \cdots + r_n^2)(f_1(t)^2 + f_2(t)^2 + \cdots + f_n(t)^2) - (r_1^2 f_1(t)^2 + \cdots r_n^2 f_n(t)^2) = \sum_{1 \leq i \neq j \leq n} r_i^2 f_j(t)^2 = \sum_{1 \leq i < j \leq n} r_i^2 f_j(t)^2 + \sum_{1 \leq j < i \leq n} r_i^2 f_j(t)^2 = \sum_{1 \leq i < j \leq n} r_i^2 f_j(t)^2 + \sum_{1 \leq i < j \leq n} r_j^2 f_i(t)^2 = \sum_{1 \leq i < j \leq n} r_i^2 f_j(t)^2 + r_j^2 f_i(t)^2$

$\iff 0 \leq \sum_{1 \leq i \neq j \leq n} r_i^2 f_j(t)^2 + r_j^2 f_i(t)^2 - \sum_{1 \leq i \neq j \leq n} r_i f_j(t) r_j f_i(t)$ **Notice in the last term, we only change the order of multiplication**

$\iff 0 \leq \sum_{1 \leq i \neq j \leq n} r_i^2 f_j(t)^2 - 2 r_i f_j(t) r_j f_i(t) + r_j^2 f_i(t)^2$

$\iff 0 \leq \sum_{1 \leq i \neq j \leq n} [r_i f_j(t) - r_j f_i(t)]^2$, which is obviously true.

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4.

NO. We raise a counter example

Proof. Let $f(t) = \langle 1+t^2, t^3 \rangle$

$f'(t) = \langle 2t, 3t^2 \rangle$, so f is differentiable on $[0,2]$

$\frac{1}{2-0}[f(2) - f(0)] = \frac{1}{2}(\langle 1+2^2, 2^3 \rangle - \langle 1+0^2, 0^3 \rangle) = \langle 2, 4 \rangle$

If $f'(c) = \langle 2, 4 \rangle$, then $2c = 2 \iff c = 1$ by the first cordinate

But we immediately see $f'(c) = \langle 2, 3 \rangle \neq \langle 2, 4 \rangle$ CaC ■

5.

Proof. $f(t)$ is parallel to $f''(t) \iff f(t) \times f''(t) = 0$

$f'(t) \times f'(t) = 0 \implies f(t) \times f''(t) + f'(t) \times f'(t) = 0 \implies \frac{d}{dt}[f(t) \times f'(t)] = 0 \implies f(t) \times f'(t) = c, \exists c \in F^n$ ■

6.

6.(a)

Proof. $r(t) : \mathbb{R} \to \mathbb{R}^n$ is a continuous function $\implies \forall t_0 \in \mathbb{R}, \forall 1 \leq i \leq n, \lim_{t \to t_0} r_i(t) = r_i(t_0) \implies \forall t_0 \in \mathbb{R}, \lim_{t \to t_0} \|r(t)\| = \lim_{t \to t_0} \sqrt{\sum_{i=1}^n r_i(t)^2} = \sqrt{\sum_{i=1}^n \lim_{t \to t_0} r_i(t)^2} = \sqrt{\sum_{i=1}^n r_i(t_0)^2} = \|r(t_0)\|$ (**We can put limit inside the sqrt and sum because the right hand side of equation exists, implicating we can put it back to left and obtain an equation**) ■

6.(b)

Proof. $r(t) : \mathbb{R} \to \mathbb{R}^n$ is a differentiable function $\implies \forall t_0 \in \mathbb{R}, \forall 1 \leq i \leq n, \exists L_i \in \mathbb{R}, r_i'(t_0) = L_i \implies \forall t_0 \in \mathbb{R}, \langle L_1, L_2, \ldots L_n \rangle = \langle r_1'(t_0), r_2'(t_0), \ldots r_n'(t_0) \rangle = r'(t_0)$ ■

6.(c)

No.

Proof. Let $r(t) : \mathbb{R} \to \mathbb{R}^2$ be defined by $r(t) = \begin{cases} \langle 1, 0 \rangle & \text{if } x \in \mathbb{Q} \\ \langle 0, 1 \rangle & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$, so $r(t)$ is not a continuous function.

$\forall t \in \mathbb{R}, \|r(t)\| = 1$, so $\|r(t)\|$ is a continuous function. ■

7.

Proof. Let θ be the angle between $r(t) = \langle e^t cost, e^t sint \rangle$ and $r'(t)$

Integrating by part, we have $r'(t) = \langle e^t(-sint + cost), e^t(sint + cost) \rangle$

$|r(t)||r'(t)|cos\theta = r(t) \cdot r'(t) \implies cos\theta = \frac{r(t) \cdot r'(t)}{|r(t)||r'(t)|}$
$= \frac{e^{2t}(cos^2t - costsint) + e^{2t}(sin^2t + sintcost)}{\sqrt{e^{2t}(cos^2t+sin^2t)e^{2t}(sin^2t - 2sintcost + cos^2t + sin^2t + 2sintcost + cos^2t)}} = \frac{e^{2t}(cos^2t + sin^2t)}{\sqrt{e^{4t}(2sin^2t + 2cos^2t)}} =$
$\frac{1}{\sqrt{2}} \implies \theta = \frac{1}{4}\pi$ ∎

8.

8.(a)

Proof. $|x(t)| = \sqrt{cos^2t + cos^2tsin^2t + sin^4t} = \sqrt{cos^2t + sin^2t(cos^2t + sin^2t)} =$
$\sqrt{cos^2t + sin^2t} = 1$ ∎

8.(b)

Proof. $x'(t) = \langle -sint, -sin^2t + cos^2t, 2sintcost \rangle$

$x \cdot x'(t) = -costsint - sin^2tcostsint + cos^2tcostsint + 2sintcostsin^2t =$
$costsint(-1 - sin^2t + cos^2t + 2sin^2t) = costsint(cos^2t + sin^2t - 1) = 0$ ∎

9.

Proof. Notice $r(x) = r(y) \implies \sqrt{2}x = \sqrt{2}y \implies x = y$

$(\frac{d\frac{2}{3}(1+t)^{\frac{3}{2}}}{dt})^2 = 1 + t$

$(\frac{d\frac{2}{3}(1-t)^{\frac{3}{2}}}{dt})^2 = 1 - t$

$(\frac{d\sqrt{2}t}{dt})^2 = 2$

$L = \int_{\frac{1}{-2}}^{\frac{1}{2}} \sqrt{(\frac{d\frac{2}{3}(1+t)^{\frac{3}{2}}}{dt})^2 + (\frac{d\frac{2}{3}(1-t)^{\frac{3}{2}}}{dt})^2 + (\frac{d\frac{2}{3}(1-t)^{\frac{3}{2}}}{dt})^2 dt} = \int_{\frac{1}{2}}^{\frac{1}{2}} 4dt = 4$

∎