18

Some elementary character tables

We now illustrate the techniques we have presented so far by constructing the character tables of several groups, including the groups S_4 and A_4, and all dihedral groups.

18.1 The group S_4

In Example 17.4, we produced three irreducible characters χ_1, χ_2, χ_3 of S_4 by lifting characters of the factor group S_4/V_4. We shall now use Proposition 17.14, which deals with the product of a character with a linear character, to complete the character table of S_4.

Let χ_4 be the character

$$\chi_4(g) = |\text{fix}\,(g)| - 1 \quad (g \in S_4)$$

which is given in Proposition 13.24. By Proposition 17.14, the product $\chi_4\chi_2$ is also a character of S_4. The values of χ_2, χ_4 and $\chi_4\chi_2$ are as follows:

g_i $\|C_G(g_i)\|$	1 24	(1 2) 4	(1 2 3) 3	(1 2)(3 4) 8	(1 2 3 4) 4
χ_2	1	-1	1	1	-1
χ_4	3	1	0	-1	-1
$\chi_4\chi_2$	3	-1	0	-1	1

Note that

$$\langle \chi_4, \chi_4 \rangle = \frac{9}{24} + \frac{1}{4} + \frac{1}{8} + \frac{1}{4} = 1,$$

so χ_4 is irreducible. The character $\chi_4\chi_2$ is also irreducible, either by using the same calculation or by quoting the result of Proposition 17.14. Let $\chi_5 = \chi_4\chi_2$. Since S_4 has five conjugacy classes, and we have produced five irreducible characters, we have now found the complete character table of S_4, as shown.

Character table of S_4

g_i $\lvert C_G(g_i)\rvert$	1 24	(1 2) 4	(1 2 3) 3	(1 2)(3 4) 8	(1 2 3 4) 4
χ_1	1	1	1	1	1
χ_2	1	−1	1	1	−1
χ_3	2	0	−1	2	0
χ_4	3	1	0	−1	−1
χ_5	3	−1	0	−1	1

18.2 The group A_4

Let $G = A_4$, the alternating group of degree 4. Then $|G| = 12$, and G has four conjugacy classes, with representatives

$$1,\ (1\,2)(3\,4),\ (1\,2\,3),\ (1\,3\,2)$$

(see Example 12.18(1)).

Let ν be the character of A_4 given by Proposition 13.24, so that $\nu(g) = |\mathrm{fix}(g)| - 1$ for all $g \in A_4$. The values of ν are as follows:

g_i $\lvert C_G(g_i)\rvert$	1 12	(1 2)(3 4) 4	(1 2 3) 3	(1 3 2) 3
ν	3	−1	0	0

Note that

$$\langle \nu, \nu \rangle = \frac{9}{12} + \frac{1}{4} = 1,$$

so ν is an irreducible character of G of degree 3.

Since G has four irreducible characters, and the sum of the squares of their degrees is 12, there must be exactly three linear characters of G. Thus $|G/G'| = 3$ by Theorem 17.11. It is not difficult to confirm this by showing that

$$G' = V_4 = \{1, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}.$$

Now $G/G' = \{G', G'(1\ 2\ 3), G'(1\ 3\ 2)\} \cong C_3$, and the character table of G/G' is

	G'	$G'(1\ 2\ 3)$	$G'(1\ 3\ 2)$
$\tilde{\chi}_1$	1	1	1
$\tilde{\chi}_2$	1	ω	ω^2
$\tilde{\chi}_3$	1	ω^2	ω

(where $\omega = e^{2\pi i/3}$). The lifts of $\tilde{\chi}_1, \tilde{\chi}_2, \tilde{\chi}_3$ to G, together with the character $\chi_4 = \nu$, give the complete character table of A_4:

Character table of A_4

g_i $\lvert C_G(g_i)\rvert$	1 12	$(1\,2)(3\,4)$ 4	$(1\,2\,3)$ 3	$(1\,3\,2)$ 3
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

18.3 The dihedral groups

Let G be the dihedral group D_{2n} of order $2n$, with $n \geqslant 3$, so that

$$G = \langle a, b \colon a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle.$$

We shall derive the character table of G.

Write $\varepsilon = e^{2\pi i/n}$. For each integer j with $1 \leqslant j < n/2$, define

$$A_j = \begin{pmatrix} \varepsilon^j & 0 \\ 0 & \varepsilon^{-j} \end{pmatrix}, \ B_j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Check that

$$A_j^n = B_j^2 = I, \ B_j^{-1} A_j B_j = A_j^{-1}.$$

It follows that by defining $\rho_j \colon G \to \mathrm{GL}(2, \mathbb{C})$ by

$$(a^r b^s)\rho_j = (A_j)^r (B_j)^s \quad (r, s \in \mathbb{Z}),$$

we obtain a representation ρ_j of G for each j with $1 \leqslant j < n/2$.

Each ρ_j is an irreducible representation, either by the proof of Example 5.5(2) or by applying the result of Exercise 8.4.

If i and j are distinct integers with $1 \leqslant i < n/2$ and $1 \leqslant j < n/2$, then $\varepsilon^i \neq \varepsilon^j$ and $\varepsilon^i \neq \varepsilon^{-j}$, so $a\rho_i$ and $a\rho_j$ have different eigenvalues. Therefore there is no matrix T with $a\rho_i = T^{-1}(a\rho_j)T$, and so ρ_i and ρ_j are not equivalent.

Let ψ_j be the character of ρ_j. We have now constructed distinct irreducible characters ψ_j of G, one for each j which satisfies $1 \leqslant j < n/2$.

At this point it is convenient to consider separately the cases where n is odd and where n is even.

Case 1: *n* odd

By (12.11) the conjugacy classes of D_{2n} (n odd) are

$$\{1\}, \{a^r, a^{-r}\} (1 \leqslant r \leqslant (n-1)/2), \{a^s b \colon 0 \leqslant s \leqslant n-1\}.$$

Thus there are $(n + 3)/2$ conjugacy classes.

The $(n - 1)/2$ irreducible characters

$$\psi_1, \psi_2, \ldots, \psi_{(n-1)/2}$$

each have degree 2. As G has $(n + 3)/2$ irreducible characters in all, there are two more to be found.

Since $\langle a \rangle \lhd G$ and $G/\langle a \rangle \cong C_2$, we obtain two linear characters χ_1, χ_2 of G by lifting the irreducible characters of $G/\langle a \rangle$ to G. These characters χ_1 and χ_2 are given by $\chi_1 = 1_G$ and

$$\chi_2(g) = \begin{cases} 1 & \text{if } g = a^r \text{ for some } r, \\ -1 & \text{if } g = a^r b \text{ for some } r. \end{cases}$$

We have now found all the irreducible characters of D_{2n} (n odd). (Incidentally, we have proved that $D'_{2n} = \langle a \rangle$ for n odd, in view of Theorem 17.11.)

The character table of D_{2n} (n odd) is therefore as follows (where $\varepsilon = e^{2\pi i/n}$):

g_i $\lvert C_G(g_i)\rvert$	1 $2n$	a^r $(1 \leqslant r \leqslant (n-1)/2)$ n	b 2
χ_1	1	1	1
χ_2	1	1	-1
ψ_j $(1 \leqslant j \leqslant (n-1)/2)$	2	$\varepsilon^{jr} + \varepsilon^{-jr}$	0

Case 2: n even

If n is even, say $n = 2m$, then the conjugacy classes of D_{2n}, as supplied by (12.12), are

$$\{1\}, \{a^m\}, \{a^r, a^{-r}\} (1 \leqslant r \leqslant m-1), \{a^s b: s \text{ even}\}, \{a^s b: s \text{ odd}\}.$$

Hence G has $m + 3$ irreducible characters, of which $m - 1$ are given by

$$\psi_1, \psi_2, \ldots, \psi_{m-1}.$$

To find the remaining four irreducible characters, we first note that $\langle a^2 \rangle = \{a^j : j \text{ even}\}$ is a normal subgroup of G and

$$\begin{aligned} G/\langle a^2 \rangle &= \{\langle a^2 \rangle, \langle a^2 \rangle a, \langle a^2 \rangle b, \langle a^2 \rangle ab\} \\ &\cong C_2 \times C_2. \end{aligned}$$

Therefore G has four linear characters $\chi_1, \chi_2, \chi_3, \chi_4$ (and $G' = \langle a^2 \rangle$). Since these linear characters are the lifts of the irreducible characters of $G/\langle a^2 \rangle$, they are easy to calculate, and their values appear in the following complete character table of D_{2n} (n even, $n = 2m$, $\varepsilon = e^{2\pi i/n}$).

g_i $\lvert C_G(g_i)\rvert$	1 $2n$	a^m $2n$	a^r $(1 \leqslant r \leqslant m-1)$ n	b 4	ab 4
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	$(-1)^m$	$(-1)^r$	1	-1
χ_4	1	$(-1)^m$	$(-1)^r$	-1	1
ψ_j $(1 \leqslant j \leqslant m-1)$	2	$2(-1)^j$	$\varepsilon^{jr} + \varepsilon^{-jr}$	0	0

18.4 Another group of order 12

We shall now describe a non-abelian group G of order 12 which is not isomorphic to either A_4 or D_{12}, and we shall construct the character table of G. It is in fact known that every non-abelian group of order 12 is isomorphic to A_4, D_{12} or G, but we shall not prove this result here.

Let a and b be the following permutations in S_{12}:

$$a = (1\ 2\ 3\ 4\ 5\ 6)(7\ 8\ 9\ 10\ 11\ 12),$$

$$b = (1\ 7\ 4\ 10)(2\ 12\ 5\ 9)(3\ 11\ 6\ 8),$$

and let $G = \langle a, b\rangle$, a subgroup of S_{12}. Since a has order 6 and $b \notin \langle a\rangle$, the group G has at least 12 elements, namely

$$a^r,\ a^r b \quad (0 \leqslant r \leqslant 5).$$

Check that a and b satisfy

$$a^6 = 1,\ a^3 = b^2,\ b^{-1}ab = a^{-1}.$$

It follows from these relations that every element of G has the form $a^r b^s$ with $0 \leqslant r \leqslant 5$, $0 \leqslant s \leqslant 1$ as given above, and so $|G| = 12$.

The relations further imply that

$$C_G(a) = \langle a\rangle,\ C_G(a^3) = G,\ C_G(b) = \{1, a^3, b, a^3 b\}.$$

These, and similar facts, help us to find the conjugacy classes of G, which are tabulated below:

Conjugacy class	Representative g_i	$\lvert C_G(g_i)\rvert$
$\{1\}$	1	12
$\{a^3\}$	a^3	12
$\{a, a^{-1}\}$	a	6
$\{a^2, a^{-2}\}$	a^2	6
$\{b, a^2b, a^4b\}$	b	4
$\{ab, a^3b, a^5b\}$	ab	4

Therefore G has six irreducible characters.

Observe that $\langle a^2\rangle = \{1, a^2, a^4\} \lhd G$, and

$$G/\langle a^2\rangle = \{\langle a^2\rangle, \langle a^2\rangle a, \langle a^2\rangle b, \langle a^2\rangle ab\}.$$

Since $\langle a^2\rangle a = \langle a^2\rangle b^2$, we have $G/\langle a^2\rangle \cong C_4$. By lifting the irreducible characters of C_4 to G, we obtain the linear characters $\chi_1, \chi_2, \chi_3, \chi_4$ of G given below:

g_i $\lvert C_G(g_i)\rvert$	1 12	a^3 12	a 6	a^2 6	b 4	ab 4
χ_1	1	1	1	1	1	1
χ_2	1	-1	-1	1	i	$-\mathrm{i}$
χ_3	1	1	1	1	-1	-1
χ_4	1	-1	-1	1	$-\mathrm{i}$	i
χ_5	α_1	α_2	α_3	α_4	α_5	α_6
χ_6	β_1	β_2	β_3	β_4	β_5	β_6

It remains to find the values α_r, β_r taken by the last two irreducible characters χ_5, χ_6. For this, we shall use the column orthogonality relations, Theorem 16.4(2).

Observe that α_1, β_1 are the degrees of χ_5, χ_6, so they are positive integers; also a^3 is an element of order 2, so α_2 and β_2 are integers by Corollary 13.10. By the column orthogonality relations applied to columns 1 and 2, we have

$$4 + \alpha_1^2 + \beta_1^2 = 12,$$
$$4 + \alpha_2^2 + \beta_2^2 = 12,$$
$$\alpha_1\alpha_2 + \beta_1\beta_2 = 0.$$

Since α_1, β_1 are positive integers, the first equation gives $\alpha_1 = \beta_1 = 2$. The other two equations then imply that $\alpha_2 = -\beta_2 = \pm 2$. Since we have not yet distinguished between χ_5 and χ_6, we may take $\alpha_2 = 2$ and $\beta_2 = -2$.

For $r > 2$, the column orthogonality relations

$$\sum_{i=1}^{6} \chi_i(g_r)\overline{\chi_i(g_1)} = 0 \text{ and } \sum_{i=1}^{6} \chi_i(g_r)\overline{\chi_i(g_2)} = 0$$

now give us two equations involving $2\alpha_r + 2\beta_r$ and $2\alpha_r - 2\beta_r$, respectively, so we can solve them for α_r and β_r. Explicitly:

$$
\begin{aligned}
r = 3{:}\quad & 2\alpha_3 + 2\beta_3 = 0,\quad 4 + 2\alpha_3 - 2\beta_3 = 0;\\
r = 4{:}\quad & 4 + 2\alpha_4 + 2\beta_4 = 0,\quad 2\alpha_4 - 2\beta_4 = 0;\\
r = 5{:}\quad & 2\alpha_5 + 2\beta_5 = 0,\quad 2\alpha_5 - 2\beta_5 = 0;\\
r = 6{:}\quad & 2\alpha_6 + 2\beta_6 = 0,\quad 2\alpha_6 - 2\beta_6 = 0.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\alpha_3 &= -1, & \beta_3 &= 1,\\
\alpha_4 &= -1, & \beta_4 &= -1,\\
\alpha_5 &= 0, & \beta_5 &= 0,\\
\alpha_6 &= 0, & \beta_6 &= 0.
\end{aligned}
$$

The complete character table of G is therefore as follows:

g_i $\lvert C_G(g_i)\rvert$	1 12	a^3 12	a 6	a^2 6	b 4	ab 4
χ_1	1	1	1	1	1	1
χ_2	1	-1	-1	1	i	$-$i
χ_3	1	1	1	1	-1	-1
χ_4	1	-1	-1	1	$-$i	i
χ_5	α_1	α_2	α_3	α_4	α_5	α_6
χ_6	β_1	β_2	β_3	β_4	β_5	β_6

We can deduce that G is not isomorphic to A_4 or D_{12} from the fact that the character table of G is different from those of A_4 and D_{12}.

It is instructive to note that we produced the last two irreducible characters of G by simply using the orthogonality relations, without constructing the corresponding $\mathbb{C}G$-modules. This is typical of more advanced calculations, and illustrates the fact that it is usually much easier to construct an irreducible character of a group than to obtain an irreducible representation. (In fact, it is not hard to construct the representations of the above group G with characters χ_5 and χ_6 – see Exercise 17.6.)

Summary of Chapter 18

In this chapter we gave the character tables of various groups, as follows.

1. Section 18.1: the group S_4.
2. Section 18.2: the group A_4.
3. Section 18.3: the dihedral groups.

Exercises for Chapter 18

1. Regard D_8 as a subgroup of S_4 permuting the four corners of a square, as in Example 1.1(3). Let π be the corresponding permutation character of D_8. Find the values of π on the elements of D_8, and express π as a sum of irreducible characters.

2. Write down explicitly the character table of D_{12}, and show that all its entries are integers.

   Use the character table to find seven distinct normal subgroups of D_{12}. (Hint: use Proposition 17.5.)

3. Let $G = T_{4n} = \langle a, b\colon a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1}\rangle$, as in Exercise 17.6. Find the character table of G.

   (Hint: use the result of Exercise 17.6. It is a good idea to do the cases n odd and n even separately.)

4. Let $G = U_{6n} = \langle a, b\colon a^{2n} = b^3 = 1, a^{-1}ba = b^{-1}\rangle$, as in Exercise 17.7. Find the character table of G.

5. Let $G = V_{8n} = \langle a, b\colon a^{2n} = b^4 = 1,\ ba = a^{-1}b^{-1},\ b^{-1}a = a^{-1}b\rangle$, with n odd, as in Exercise 17.8. Find the character table of G.