Friedberg Ch6.6

Date: Mar 28 Made by Eric

In this note, V always stand for a vector space over $\mathbb{F}$, V^- stands for a finite dimensional vector space over $\mathbb{F}$, and T is always a linear operator on V^-

Definitions

Definition 1. *Let $V = W_1 \oplus W_2$. T is a **projection** on W_1 if for all $w_1 \in W_1$ and $w_2 \in W_2$*

$$T(w_1 + w_2) = w_1$$

(Notice the definition of T solely depends on W_1 and not on W_2)

Theorem 1. *T is a projection if and only if $T = T^2$*

Proof. $(\longrightarrow)$

Let T be a projection on W_1 along W_2

Arbitrarily pick v from V and express v in the form of $v = w_1 + w_2$

$T^2(v) = T^2(w_1 + w_2) = T(w_1) = w_1 = T(v)$

$(\longleftarrow)$

We first prove $N(T) \oplus R(T)$

Let $v \in N(T) \cap R(T)$

Because $v \in R(T)$, we know there exists $w \in V$ such that $v = T(w)$

Because $v \in N(T)$, we deduce $v = T(w) = T^2(w) = T(v) = 0$

Now we have concluded, $N(T) \cap R(T) = \{0\}$

By Rank-Nullity Theorem, $dim(N(T)) + dim(R(T)) = dim(V)$ (done)

We now prove T is a projection on $R(T)$ along $N(T)$

Arbitrarily pick v from V and express v in the form of $v = v_1 + T(v_2)$, where $v_1 \in N(T)$

$T(v) = T(v_1 + T(v_2)) = T(v_1) + T^2(v_2) = T(v_2)$ (done)

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Corollary 1.1. *If T is a projection, $V = N(T) \oplus R(T)$*

Definition 2. *T is an **orthogonal projection** if T is a projection and $N(T) \perp R(T)$*

Theorems

Theorem 2. *T is an orthogonal projection if and only if*

$$T^* \text{ exists and } T^2 = T = T^*$$

Proof. $(\longrightarrow)$

Notice the definition of adjoint is $\forall x, y \in V, \langle T(x), y\rangle = \langle x, T^*(y)\rangle$

We show such linear transformation T^* exists and $T^* = T$ by showing $\forall x, y \in V, \langle T(x), y\rangle = \langle x, T(y)\rangle$

Arbitrarily pick $x, y \in V$

Because T is a projection, so we know $V = N(T) \oplus R(T)$

We express x, y in the form of $x = v_1 + v_2, y = w_1 + w_2$ where $v_1, w_1 \in N(T)$ and $v_2, w_2 \in R(T)$

Because T is an orthogonal projection, we know $\langle v_1, w_2\rangle = 0 = \langle v_2, w_1\rangle$

Then we can deduce $\langle T(x), y\rangle = \langle v_2, w_1 + w_2\rangle = \langle v_2, w_2\rangle = \langle v_1 + v_2, w_2\rangle = \langle x, T(y)\rangle$ (done)

$(\longleftarrow)$

We show $N(T) \perp R(T)$

Arbitrarily pick $v \in N(T)$, and arbitrarily pick $T(w) \in R(T)$

$\langle v, T(w)\rangle = \langle T^*(v), w\rangle = \langle T(v), w\rangle = \langle 0, w\rangle = 0$ (done)

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Theorem 3. *Suppose that T is orthogonally diagonalizable on V^- with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. For each i, let W_i be the eigenspace of T corresponding to the eigenvalue λ_i and let T_i be the orthogonal projection on W_i*

(a) $W_i^\perp = W_1 \oplus \cdots \oplus W_{i-1} \oplus W_{i+1} \oplus \cdots \oplus W_k$
(b) $I = T_1 + \cdots + T_k$
(c) $T = \lambda_1 T_1 + \cdots + \lambda_k T_k$

Proof. **(a)**

Notice that because T is orthogonally diagonalizable, so $V = W_1 \oplus \cdots \oplus W_k$

Arbitrarily pick $v \in W_i^\perp$ and express v in the form of $v = w_1 + \cdots + w_k$ where $w_j \in W_j$

$0 = \langle v, w_i \rangle = \langle w_1 + \cdots + w_k, w_i \rangle = \langle w_i, w_i \rangle \implies w_i = 0 \implies v \in W_1 \oplus \cdots \oplus W_{i-1} \oplus W_{i+1} \oplus \cdots \oplus W_k$

So we have concluded $W_i^{\perp} \subseteq W_1 \oplus \cdots \oplus W_{i-1} \oplus W_{i+1} \oplus \cdots \oplus W_k$

Arbitrarily pick $w_i \in W_i$ and arbitrarily pick $w_1 + \cdots + w_{i-1} + w_{i+1} + \cdots + w_k \in W_1 \oplus \cdots \oplus W_{i-1} \oplus W_{i+1} \oplus \cdots \oplus W_k$

$\langle w_i, w_1 + \cdots + w_{i-1} + w_{i+1} + \cdots + w_k \rangle = \langle w_i, 0 \rangle = 0$

So we have concluded $W_1 \oplus \cdots \oplus W_{i-1} \oplus W_{i+1} \oplus \cdots \oplus W_k \subseteq W_i^{\perp}$

(b)

Arbitrarily pick $v \in V$ and express v in the form of $v = w_1 + \cdots + w_k$, where $w_j \in W_j$

$(T_1 + \cdots + T_k)(v) = (T_1 + \cdots + T_k)(v_1 + \cdots + v_k) = T_1 v_1 + \cdots + T_k v_k = v_1 + \cdots + v_k = v$

(c)

Arbitrarily pick $v \in V$ and express v in the form of $v = w_1 + \cdots + w_k$, where $w_j \in W_j$

$T(v) = T(v_1 + \cdots + v_k) = T(v_1) + \cdots + T(v_k) = \lambda_1 v_1 + \cdots + \lambda_k v_k = \lambda_1 T_1(v) + \cdots + \lambda_k T_k(v) = (\lambda_1 T_1 + \cdots + \lambda_k T_k)(v)$

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Exercises

2.

Proof. $\frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$\frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ ■

4.

Let W be a finite dimensional subspace of an inner product space V. Show that if T is orthogonal projection of V on W, then $I - T$ is the orthogonal projection of V on $W^{\perp}$

Proof. Because T is an orthogonal projection of V on W, we know $V = N(T) \oplus W$ and $N(T) \perp W$

We first show $N(T) = W^\perp$

Because $N(T) \perp W$, so we know $N(T) \subseteq W^\perp$

Arbitrarily pick $v \in W^\perp$ and express v in the form of $v_1 + w$, where $v_1 \in N(T)$ and $w \in W$

$0 = \langle v, w \rangle = \langle v_1 + w, w \rangle = \langle w, w \rangle \implies w = 0 \implies v = v_1$

$T(v) = T(v_1) = 0 \implies v \in N(T)$

We have concluded $W^\perp \subseteq N(T)$ (done)

With Violet, we now know T project V on W along $W^\perp$

Arbitrarily pick $v \in V$ and express v in the form of $v_1 + w$, where $v_1 \in W^\perp$ and $w \in W$

$(I - T)(v) = v - T(v) = v_1 + w - w = v_1$

So $I - T$ is a projection on $W^\perp$

Lastly, to show $I - T$ is not just a projection, but an orthogonal projection, we prove $N(I - T) \perp W^\perp$

Arbitrarily pick $v \in N(I - T)$ and $v' \in W^\perp$

Because $W = R(T)$, so we deduce $v \in N(I - T) \implies v - T(v) = 0 \implies v = T(v) \implies \langle v, v' \rangle = \langle T(v), v' \rangle = 0$ (done)

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7.

Let T be a normal operator on a finite-dimensional complex inner product space V. Express T in the form of spectral decomposition $T = \lambda_1 T_1 + \cdots + \lambda_k T_k$

7.(a)

Let g be a polynomial

Show $g(T) = g(\lambda_1)T_1 + \cdots + g(\lambda_k)T_k$

Proof. Let $n \in \mathbb{Z}_0^+$

We first prove $T^n = \lambda_1^n T_1 + \cdots + \lambda_k^n T_k$

From Theorem 3, we have already proven $T^0 = \lambda_1^0 T_1 + \cdots + \lambda_k^0 T_k$ and $T^1 = \lambda_1^1 T_1 + \cdots + \lambda_k^1 T_k$

Let $m \in \mathbb{N}$

We only have to prove $T^m = \lambda_1^m T_1 + \cdots + \lambda_k^m T_k \implies T^{m+1} = \lambda_1^{m+1} T_1 + \cdots + \lambda_k^{m+1} T_k$

Notice $T_i T_j = \delta_{i,j} T_i$, so we deduce

$T^{m+1} = TT^m = (\lambda_1 T_1 + \cdots + \lambda_k T_k)(\lambda_1^m T_1 + \cdots + \lambda_k^m T_k) = \lambda_1^{m+1} T_1 + \cdots + \lambda_k^{m+1} T_k$ (done)

Express $g(x)$ in the form of $g(x) = \sum_{i=0}^r c_i x^i$

$g(T) = \sum_{i=0}^r c_i T^i = \sum_{i=0}^r c_i (\lambda_1^i T_1 + \cdots + \lambda_k^i T_k) = \sum_{i=0}^r c_i \lambda_1^i T_1 + \cdots + \sum_{i=0}^r c_i \lambda_k^i T_k = g(\lambda_1) T_1 + \cdots + g(\lambda_k) T_k$

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