

Notes on Linear Algebra

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CONTENTS

CHAPTER 1 LINEAR ALGEBRA DONE OUTRAGEOUS _ PAGE 2_____

1.1	Quick Recap	2
1.2	Jordan-Chevalley Decomposition	6
1.3	Spectral	11
1.4	Matrix Exponential	13

CHAPTER 2 NTU MATH M.A. PROGRAM ENTRANCE EXAM PAGE 14_____

2.1	Year 113	14
2.2	Year 112	19
2.3	Year 111	23

CHAPTER 3 ARCHIVED _____ PAGE 24_____

3.1	Tensor Algebra	24
3.2	Operator Norm	32
3.3	Cauchy-Schwarz for Positive semi-definite Hermitian form	37

Chapter 1

Linear Algebra Done Outrageous

1.1 Quick Recap

Abstract

In this section, V and W are two vector spaces over the same field \mathbb{F} , \mathbb{F} can be interpreted as either the field of real numbers or the field of complex numbers, and S is subset of V .

We say S is

- (a) **linearly independent** if to write 0 as finite linear combination of S , the coefficients must all be zero.
- (b) S **spans** V if all $v \in V$ can be written as finite linear combination of S .

We say S is a **basis** if any of the following equivalent conditions hold true

- (a) S is a maximum linearly independent subset of V .
- (b) S is linearly independent and spans V .
- (c) All elements of V can be uniquely written as some finite linear combination of S .

Suppose V has a basis of cardinality n . Using **Gauss elimination**, one see that all linear independent subset of V must have cardinality not greater than n , and all basis must have cardinality n . Therefore, for vector space with some basis of finite cardinality, it make sense to refer to its **dimension**. Let $\dim(V) = n$. Each linearly independent subset S of V of cardinality n must be a basis, otherwise we may construct a linearly independent set of cardinality greater than n . By an algorithm, one see that

- (a) If S spans V and has cardinality greater than n , then there exists a subset of S that is a basis of V .
- (b) If B is a basis of V and S is linearly independent, then there exists some subset U of B such that $S \cup U$ is a basis of V .

Let $F : V \rightarrow W$ be a linear map. It is clear that both image and kernel of F form some vector space. Therefore, it make sense to define the **rank** of F by

$$\text{rank}(F) \triangleq \dim(\text{Im } F)$$

By extending a basis of the kernel of F to a basis of the whole V , one can prove the **Rank-Nullity Theorem**

$$\dim(\text{Ker } F) + \text{rank}(F) = \dim(V)$$

We use V^* to denote the vector space of linear map from V to \mathbb{F} , the **dual space** of V . Let $F : V \rightarrow W$ be a linear map. Its **dual map** $F^* : W^* \rightarrow V^*$ is defined by

$$F^*(\varphi) \triangleq \varphi \circ F$$

If V is finite-dimensional with basis $\{v_1, \dots, v_n\}$, then its **dual basis** $\{\varphi_1, \dots, \varphi_n\}$ is defined by

$$\varphi_i(v_j) \triangleq \delta_j^i$$

And there exists a natural isomorphism between V and its double dual $(V^*)^*$ by identifying $v \in (V^*)^*$ as

$$v(\varphi) \triangleq \varphi(v)$$

Let $\{w_1, \dots, w_m\}$ be a basis of W with dual basis $\{\xi_i, \dots, \xi_m\}$. It is clear that

The matrices of F and F^* are transpose of each other.

This together with the observation that for finite dimensional V, W , we have

$$\text{rank}(F) + \dim(\text{ker } F^*) = \dim(W)$$

gain us a quick proof that the dimensions of column space and row space of a fixed matrix are always equal.

Given some n -by- n square matrix A , its **determinant** is by definition

$$\det A \triangleq \sum_{\sigma \in S_n} \left(\operatorname{sgn} \sigma \prod_{k=1}^n A_{\sigma(k),k} \right)$$

If one identify the space of n -by- n square matrix over \mathbb{F} with $(\mathbb{F}^n)^n$, determinant can be equivalently defined to be the unique alternating multilinear map from $(\mathbb{F}^n)^n$ to \mathbb{F} such that

$$\det(I) = 1$$

By an algorithm, one see that, for each alternating multilinear map $F : (\mathbb{F}^n)^n \rightarrow \mathbb{F}$, we have

$$F(B) = (\det B) \cdot F(I) \text{ for all } B.$$

Therefore, if we observe for each n -by- n square matrix A over \mathbb{F} that the map $B \mapsto \det AB$ is indeed alternating multilinear, then we immediately have the celebrated result $\det AB = \det A \det B$ as a corollary. This multiplicative property of determinant allow us to well define determinant for each linear epimorphism F over some finite-dimensional vector space by

$$\det F \triangleq \det([F])$$

where $[F]$ is the matrix representation of F with respect to some basis. Moreover, because the inverse of a linear map is linear if exists, we see that for linear map F and square matrix A ,

$$F \text{ is invertible} \iff \det F \neq 0$$

And

$$A \text{ is invertible} \iff \det A \neq 0$$

Given some linear epimorphism F over V , we say $v \in V$ is an **eigenvector** with respect to the **eigenvalue** $\lambda \in \mathbb{F}$ if $v \neq 0$ and $F(v) = \lambda v$. If there exists some basis of V consisting eigenvectors of F , we say F is **diagonalizable**. Let A be some square matrix, if there exists some invertible square matrix P such that PAP^{-1} diagonal, we

also say A is **diagonalizable**. Immediately, we see that if V is finite dimensional, then with respect to any basis of V

$$[F] \text{ is diagonalizable} \iff F \text{ is diagonalizable}$$

Let V be finite dimensional. We define the **characteristic polynomial** of F and A by

$$\det(tI - F) \text{ and } \det(tI - A)$$

Obviously, λ is an eigenvalue of F if and only if λ is a root of the characteristic polynomial of F .

1.2 Jordan-Chevalley Decomposition

Abstract

In this section, all vector spaces are finite dimensional, and we say that V is a **direct sum** of some collection $\{U_i\}_{i \in I}$ of subspaces of V and writes $V = \bigoplus_{i \in I} U_i$ if for each $v \in V$ there exists some unique tuple $(u_i)_{i \in I}$ such that $u_i \neq 0$ for finite number of $i \in I$ and $v = \sum u_i$.

Given $F \in \text{End}(V)$, we know the kernels of its powers is increasing

$$\{0\} = \text{Ker } F^0 \subseteq \text{Ker } F^1 \subseteq \text{Ker } F^2 \subseteq \text{Ker } F^3 \subseteq \dots$$

This sequence grows in good manner. If it stops growing at some points, say, $\text{Ker } F^n = \text{Ker } F^{n+1}$ for some $n \in \mathbb{Z}_0^+$, then it stops forever

$$\text{Ker } F^n = \text{Ker } F^{n+1} = \text{Ker } F^{n+2} = \text{Ker } F^{n+3} = \dots$$

In particular, by counting dimensions, we know that this sequence must stop before reaching to the dimension of V in the sense that

$$\text{Ker}(F^{\dim V}) = \text{Ker}(F^{\dim V+1}) \tag{1.1}$$

Equation 1.1 allows us to elegantly unify distinct notions within a single framework. For instance, for each eigenvalue λ , we define the **generalized eigenspace** $G(\lambda, F)$ as

$$G(\lambda, F) \triangleq \text{Ker}(F - \lambda I)^{\dim V}$$

Similarly, we say that F is **nilpotent** if

$$\text{Ker } F^{\dim V} = V$$

Theorem 1.2.1. (Linear Independence of Generalized Eigenspaces) Let $F \in \text{End}(V)$. If v_1, \dots, v_n are generalized eigenvectors with respect to distinct eigenvalues $\lambda_1, \dots, \lambda_n$, then they are linearly independent.

Proof. Suppose

$$a_1 v_1 + \dots + a_n v_n = 0 \tag{1.2}$$

Because v_1 is a generalized eigenvector, we may let k be the largest non-negative integer such that $(F - \lambda_1 I)^k v_1 \neq 0$, so that

$$w \triangleq (F - \lambda_1 I)^k v_1 \text{ is an eigenvector with eigenvalue } \lambda_1.$$

Now, noting that the set of endomorphism $\{(F - \lambda_1 I)^k, (F - \lambda_2 I)^{\dim V}, \dots, (F - \lambda_n I)^{\dim V}\}$ indeed commutes, we may apply the endomorphism $(F - \lambda_1 I)^k (F - \lambda_2 I)^{\dim V} \dots (F - \lambda_n I)^{\dim V}$ onto both sides of [Equation 1.2](#) and get

$$\begin{aligned} 0 &= a_1 (F - \lambda_2 I)^{\dim V} \dots (F - \lambda_n I)^{\dim V} w \\ &= a_1 (\lambda_1 - \lambda_2)^{\dim V} \dots (\lambda_1 - \lambda_n)^{\dim V} w \end{aligned}$$

Which implies $a_1 = 0$. WLOG, we have shown $a_1 = \dots = a_n = 0$. ■

[Equation 1.1](#) together with Rank-Nullity Theorem give us a theoretically crucial decomposition of V

$$V = \text{Ker } F^{\dim V} \oplus \text{Im } F^{\dim V} \quad (1.3)$$

As we will later see, [Decomposition 1.3](#) plays a central role in the proof for [Theorem 1.2.2](#).

Theorem 1.2.2. (Decomposition into Generalized Eigenspaces) Let V be a finite dimensional complex vector space, let $F \in \text{End}(V)$, and let $\{\lambda_1, \dots, \lambda_m\}$ be the set of eigenvalues of F . We have

$$V = G(\lambda_1, F) \oplus \dots \oplus G(\lambda_m, F)$$

Proof. This is proved by induction on the dimension of V . The two base cases correspond to the 0-dimensional and 1-dimensional cases, both of which are trivial. Now, suppose that such a decomposition always exists for complex vector spaces of strictly smaller dimension than V and their endomorphisms. By [Equation 1.3](#), we may decompose V into

$$V = G(\lambda_1, F) \oplus U, \quad \text{where } U = \text{Im}(F - \lambda_1 I)^{\dim V}.$$

Noting that F and $F - \lambda_1 I$ commutes, we conclude that U is stable under F . Therefore, the restriction $F|_U$ defines an endomorphism $F|_U \in \text{End}(U)$. By inductive hypothesis, we may decompose U into

$$U = G(\lambda_2, F|_U) \oplus \dots \oplus G(\lambda_m, F|_U)$$

WLOG, it remains to show

$$G(\lambda_2, F|_U) = G(\lambda_2, F)$$

Arbitrary select $v \in G(\lambda_2, F)$. We may decompose $v = a_1 v_1 + \dots + a_m v_m$, where $v_1 \in G(\lambda_1, F)$ and $v_n \in G(\lambda_n, F|_U)$ for $2 \leq n \leq m$. Because $(F - \lambda_2 I)^{\dim V} v_2 = 0$, we know

$$(F - \lambda_2 I)^{\dim V} (a_1 v_1 + a_3 v_3 + \dots + a_m v_m) = 0$$

This implies $a_1 v_1 + a_3 v_3 + \dots + a_m v_m \in G(\lambda_2, F)$. It now follows from [Theorem 1.2.1](#) that $a_1 = a_3 = \dots = a_m = 0$. We have shown $v \in G(\lambda_2, F|_U)$. ■

Given some finite dimensional vector space V and some $F \in \text{End}(V)$, we are particularly concerned with the existence and uniqueness of the **Jordan-Chevalley decomposition** of F , i.e, some diagonalizable $S \in \text{End}(V)$ such that

- (a) $N \triangleq F - S$ is nilpotent.
- (b) S and N commute.

If V is over \mathbb{C} , then **Theorem 1.2.2** assert the existence of such decomposition by letting S maps $v \in G(\lambda_i, F)$ to $\lambda_i v$. To see such decomposition is unique, let $S \in \text{End}(V)$ be some Jordan-Chevalley decomposition of F and decompose V into

$$V = V_1 \oplus \cdots \oplus V_k$$

where V_i are the eigenspaces of S corresponding to distinct eigenvalues λ_i . Because F, S commute and $F - S$ is nilpotent, we may conclude that λ_i are indeed eigenvalues of F and

$$V_i \subseteq G(\lambda_i, F) \text{ for all } i$$

This together with **Theorem 1.2.2** shows the uniqueness of Jordan-Chevalley decomposition of endomorphisms of finite dimensional vector spaces over \mathbb{C} .

Theorem 1.2.3. (Jordan forms of Nilpotent Operators) Let V be a finite-dimensional complex vector space, and let $N \in \text{End}(V)$ be nilpotent. There exists some $v_1, \dots, v_m \in V$ and nonnegative integer n_1, \dots, n_m such that

- (a) $N^{n_1}v_1, \dots, Nv_1, v_1, \dots, N^{n_m}v_m, \dots, Nv_m, v_m$ form a basis for V .
- (b) $N^{n_1+1}v_1 = \cdots = N^{n_m+1}v_m = 0$.

Proof. This is also proved by induction on the dimension of V . The two base cases correspond to the 0-dimensional and 1-dimensional cases, both of which are trivial. We now prove the inductive case.

Because N is nilpotent, we know $\dim(\text{Im } N) < \dim V$. Noting that $\text{Im } N$ is stable under N , we see $N|_{\text{Im } N} \in \text{End}(\text{Im } N)$. From inductive hypothesis, we have $v_1, \dots, v_n \in \text{Im } N$ and nonnegative integer k_1, \dots, k_n such that

$$\{N^{k_1}v_1, \dots, Nv_1, v_1, \dots, N^{k_n}v_n, \dots, Nv_n, v_n\} \text{ form a basis for } \text{Im } N.$$

and

$$N^{k_1+1}v_1 = \cdots = N^{k_n+1}v_n = 0.$$

Because $v_1, \dots, v_n \in \text{Im } N$, we may let $u_1, \dots, u_n \in V$ satisfy $v_j = Nu_j$ for all j . To see

$$\{N^{k_1+1}u_1, \dots, Nu_1, u_1, \dots, N^{k_n}u_n, \dots, Nu_n, u_n\} \text{ is linearly independent,} \quad (1.4)$$

suppose some finite linear combination equals to 0. By applying N to this finite linear combination, we see the only possible nonzero coefficients are those of $N^{k_j+1}u_j$, otherwise

$$\{N^{k_1}v_1, \dots, Nv_1, v_1, \dots, N^{k_n}v_n, \dots, Nv_n, v_n\} \text{ would not be linearly independent.}$$

Knowing that the only possible nonzero coefficients are those of $N^{k_j+1}u_j = N^{k_j}v_j$, we may conclude that even these coefficients are zero, since $\{N^{k_1}v_1, \dots, N^{k_n}v_n\}$ is linearly independent in the first place. We may now expand [Set 1.4](#) to a basis

$$\{N^{k_1+1}u_1, \dots, Nu_1, u_1, \dots, N^{k_n+1}u_n, \dots, Nu_n, u_n, w_1, \dots, w_p\} \quad (1.5)$$

for V . Because $\{N^{k_1+1}u_1, \dots, N^2u_1, Nu_1, \dots, N^{k_n+1}u_n, \dots, N^2u_n, Nu_n\}$ is a basis for $\text{Im } N$, we may subtract each w_i with some element of

$$\text{span}\{N^{k_1}u_1, \dots, Nu_1, u_1, \dots, N^{k_n+1}u_n, \dots, Nu_n, u_n\}$$

so that [Set 1.5](#) form a desired basis for V . ■

Let V be some finite-dimensional complex vector space and $F \in \text{End}(V)$. Let $S + N$ be the Jordan-Chevalley decomposition of F . The basis for V from [Theorem 1.2.3](#) is called a **Jordan basis**. If we express F as a matrix with respect to this basis, we say that matrix is in its **Jordan form**. The Jordan form looks like

$$\begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_r \end{bmatrix}$$

where each block matrix looks like

$$J = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of F , and let r_i be the dimension of the largest Jordan block with eigenvalue λ_i . It is clear that the **minimal polynomial** of F , some polynomial m of smallest degree such that $m(F) = 0$, take the form

$$m(x) = (x - \lambda_1)^{r_1} \cdots (x - \lambda_m)^{r_m}$$

It is also clear that even if V is over \mathbb{R} instead of \mathbb{C} , we have the **Cayley-Hamilton**

Theorem, that is, $p(F) = 0$, where p is the characteristic polynomial of F .

Interestingly, we may also take a completely algebraic approach to construct the Jordan-Chevalley decomposition S, N and show that they are polynomials in F , without ever invoking [Theorem 1.2.1](#), [Theorem 1.2.2](#), nor any result that depends on them. The argument proceeds as follows:

Because $\text{End}(V)$ is finite-dimensional, the minimal polynomial m exists. By the Fundamental Theorem of Algebra, we can factor m into linear terms:

$$m(x) = (x - \xi_1)^{r_1} \cdots (x - \xi_k)^{r_k}.$$

Consider the polynomials f_i defined by

$$f_i(x) \triangleq \prod_{j=1; j \neq i}^k (x - \xi_j)^{r_j}.$$

Since $\{f_1, \dots, f_k\}$ are coprime, by Bézout's identity, we may find polynomials $q_i \in \mathbb{C}[x]$ such that

$$1 = \sum_{i=1}^k q_i f_i.$$

Define $\pi_i \in \text{End}(V)$ by $\pi_i \triangleq (q_i f_i)(F)$. Because m divides $f_i f_j$ for $i \neq j$, we conclude that

$$\pi_i \pi_j = 0 \quad \text{for } i \neq j.$$

This, together with $\sum \pi_j = 1$, shows that π_i are **projections**:

$$\pi_i^2 = \pi_i \quad \text{for all } i.$$

It is clear that V is a direct sum of the images of these projections. Define

$$S \triangleq \sum_{i=1}^k \xi_i \pi_i.$$

Since S is a polynomial in F , we conclude that S and N commute. The fact that N is nilpotent follows from the definition of f_i .

1.3 Spectral

By a **norm**, we mean some **positive-definite** functional $\|\cdot\| : V \rightarrow \mathbb{R}$ that satisfies **absolute homogeneity** and **triangle inequality**. In this context, for $\|\cdot\|$ to be positive-definite, it must satisfy

$$\|v\| = 0 \implies v = 0$$

Observing that

$$\|0\| = \|0 + v\| \leq \|0\| + \|v\| \text{ for all } v \in V$$

we see norm must also be nonnegative.

By an **inner product**, we mean some **positive-definite** map $\langle \cdot, \cdot \rangle : V^2 \rightarrow \mathbb{F}$ that is linear in the first argument and satisfies **conjugate symmetry**:

$$\langle v, w \rangle = \overline{\langle w, v \rangle} \text{ for all } v, w \in V$$

In this context, for $\langle \cdot, \cdot \rangle : V^2 \rightarrow \mathbb{F}$ to be positive-definite, it has to satisfy

$$\langle v, v \rangle > 0 \text{ for all } v \neq 0$$

With an inner product equipped on V , we may discuss the **orthogonality** of vectors. Given a countable set $\{v_1, v_2, v_3, \dots\}$ of non-zero vectors, we use the term **Gram-Schmidt process** to refer to the process of defining

$$e_n \triangleq v_n - \frac{\langle v_n, e_{n-1} \rangle}{\|e_{n-1}\|^2} e_{n-1} - \dots - \frac{\langle v_n, e_1 \rangle}{\|e_1\|^2} e_1$$

so that $\{e_1, e_2, e_3, \dots\}$ become orthogonal while

$$\text{span}(v_1, \dots, v_n) = \text{span}(e_1, \dots, e_n) \text{ for all } n.$$

Although implicit, Gram-Schmidt process may be considered one of the most important notions in inner product space. For example, the Gram-Schmidt process, together with **Pythagorean Theorem**, can be used to prove the **Cauchy-Schwarz Inequality**, which in term shows that the functional $f : V \rightarrow \mathbb{R}$ induced by

$$f(v) \triangleq \sqrt{\langle v, v \rangle}$$

indeed satisfies triangle inequality, thus forming a norm. In addition, given an endomorphism T of some finite-dimensional complex vector space V , by applying Gram-Schmidt process to a Jordan basis, we get an orthogonal basis under which T becomes an upper triangular matrix, called its **Schur's form**.

Let V, W be two inner product space over \mathbb{F} , and let $T : V \rightarrow W$ be some linear map. If some linear map $T^\dagger : W \rightarrow V$ satisfies

$$\langle Tv, w \rangle = \langle v, T^\dagger w \rangle \quad \text{for all } v \in V, w \in W,$$

we say T^\dagger is an **adjoint** of T . If both V and W are finite-dimensional, then T^\dagger exists uniquely, and its matrix representation is always the complex conjugate of that of T , regardless of the choices of bases. It is obvious that the double adjoint of a linear map between two finite-dimensional inner product spaces over \mathbb{F} is itself.

If T is **normal**, then by direct computation, we see that its Schur's form must be diagonal. This argument is called **Spectral Theorem for complex finite-dimensional vector space**.

1.4 Matrix Exponential

Given some square matrix A over \mathbb{F} , we define its **matrix exponential** by

$$e^A \triangleq \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

It

Chapter 2

NTU Math M.A. Program Entrance Exam

2.1 Year 113

Question 1

Let

$$A \triangleq \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Find the Jordan-Chevalley decomposition of A and compute e^A .

Proof. Routine computation give us

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \text{ and } P = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } A = PJP^{-1}$$

Therefore, the Jordan-Chevalley decomposition of A is

$$A = D + N \text{ where } D = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} P^{-1} \text{ and } N = P \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} P^{-1}$$

And

$$e^A = P \begin{bmatrix} e & 0 & 0 \\ 0 & e^{-1} & e^{-1} \\ 0 & 0 & e^{-1} \end{bmatrix} P^{-1}$$

Question 2

Let V be the space of polynomial in x over \mathbb{R} of degree not higher than 2. Define an inner product for V by

$$\langle f, g \rangle \triangleq \int_{-1}^1 fg$$

Find a polynomial $k(x, t)$ such that

$$f(x) = \int_{-1}^1 k(x, t)f(t)dt \quad \text{for all } f \in V \quad (2.1)$$

Define $T \in \text{End}(V)$ by

$$T(a_2x^2 + a_1x + a_0) \triangleq 2a_2x + a_1$$

That is, $Tf \triangleq f'$. Find the adjoint of T .

Proof. Because of the linearity, for k to satisfy [Equation 2.1](#) for all $f \in V$, it only has to satisfies

$$1 = \int_{-1}^1 kdt \text{ and } x = \int_{-1}^1 ktdt \text{ and } x^2 = \int_{-1}^1 kt^2dt \quad (2.2)$$

If we write k in the form

$$k = f_0(x) + f_1(x)t + f_2(x)t^2 + \dots$$

Then [Equation 2.2](#) becomes

$$\begin{aligned} 1 &= \frac{2f_0(x)}{1} + \frac{2f_2(x)}{3} + \frac{2f_4(x)}{5} + \dots \\ x &= \frac{2f_1(x)}{3} + \frac{2f_3(x)}{5} + \frac{2f_5(x)}{7} + \dots \\ x^2 &= \frac{2f_0(x)}{3} + \frac{2f_2(x)}{5} + \frac{2f_4(x)}{7} + \dots \end{aligned}$$

Thus, k can be

$$k(x, t) = \left(\frac{9}{8} + \frac{-15}{8}x^2\right) + \left(\frac{3x}{2}\right)t + \left(\frac{-15}{8} + \frac{45}{8}x^2\right)t^2$$

Routine computation give us an orthonormal basis

$$\left\{ \sqrt{\frac{1}{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) \right\} \text{ for } V$$

With respect to this ordered basis, the matrix representation of T is

$$\begin{bmatrix} 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{15} \\ 0 & 0 & 0 \end{bmatrix}$$

Because the matrix representation of T^\dagger with respect to the same basis is the conjugate transpose of that of T , we may now compute

$$T^\dagger(a_2x^2 + a_1x + a_0) = \frac{15a_1}{2}\left(x^2 - \frac{1}{3}\right) + (3a_0 + a_2)x$$

■

Question 3

Let V be the vector space of all n -by- n square matrices over \mathbb{R} , and let $F : V \rightarrow \mathbb{R}$ be linear. Suppose

$$F(AB) = F(BA) \text{ and } F(I) = n$$

for all $A, B \in V$. Show that F is the trace function.

Proof. Let $E_{i,j} \in V$ be the matrix whose only nonzero entry is 1, located in the i -th row and j -th column. If $i \neq j$, then

$$E_{i,i}E_{i,j} = E_{i,j} \text{ and } E_{i,j}E_{i,i} = 0.$$

This, together with the linearity of F , shows that F depends only on the diagonal entries. For each permutation $\sigma \in S_n$, there exists a **permutation matrix** C such that

The $\sigma(i)$ -th row of CA is identical to the i -th row of A .

And

The $\sigma(i)$ -th column of AC^{-1} is identical to the i -th column of A .

Therefore,

$$A_{i,i} = (CAC^{-1})_{\sigma(i),\sigma(i)}.$$

Observing that

$$F(CAC^{-1}) = F(A(CC^{-1})) = F(A),$$

we have shown that F is stable under the permutation of diagonal entries. Define $\sigma \in S_n$ by

$$\sigma(i) \triangleq \begin{cases} i+1 & \text{if } 1 \leq i < n, \\ 1 & \text{if } i = n. \end{cases}$$

Let C_k be the permutation matrix corresponding to σ^k for all $1 \leq k \leq n$. The proof now follows from computing

$$nF(A) = \sum_{k=1}^n F(C_k A C_k^{-1}) = F\left(\sum_{k=1}^n C_k A C_k^{-1}\right) = F((\text{tr } A)I) = n \text{tr } A.$$

■

Question 4

Let U, V be two finite dimensional space over the same field. Let $T : U \rightarrow V$ be a linear map, and let $T^* : U^* \rightarrow V^*$ be its dual map. Prove

$$T \text{ is injective} \iff T^* \text{ is surjective}$$

And

$$T \text{ is surjective} \iff T^* \text{ is injective}$$

Proof. Let $\{T(u_1), \dots, T(u_n)\}$ be a basis for the image of T . Extend this to a basis $\{T(u_1), \dots, T(u_n), v_1, \dots, v_m\}$ for V . Let $\{\xi_1, \dots, \xi_{n+m}\}$ be its dual basis. It is clear that $\{\xi_{n+1}, \dots, \xi_{n+m}\}$ belongs to the kernel of T^* . Observe

$$T^* \xi_1 v_1 = 1$$

to conclude that $\xi_i \notin \text{Ker } T^*$ for all $1 \leq i \leq n$. We have shown $\{\xi_{n+1}, \dots, \xi_{n+m}\}$ is a basis for the kernel of T^* . In other words,

$$\text{rank } T + \text{Dim}(\text{Ker } T^*) = \dim V.$$

This, together with Rank-Nullity Theorem, proves both propositions. ■

Question 5

Let V be some finite-dimensional vector space over some field \mathbb{F} and let $T \in \text{End}(V)$. Let f, g be two relatively prime polynomials. Prove

$$\text{Ker}(f(T)g(T)) = \text{Ker } f(T) \oplus \text{Ker } g(T).$$

Proof. To show that the two kernels form a direct sum, assume for contradiction that there exists a nonzero vector $v \in V$ such that v belongs to both kernels. Since f is a polynomial satisfying $f(T)v = 0$, there must exist a polynomial p of minimal degree such that $p(T)v = 0$. By polynomial division, we can write

$$f = pq + r, \quad \text{where } q, r \text{ are polynomials and } \deg r < \deg p.$$

Since r has a strictly smaller degree than p , the minimality of p implies that $r = 0$. Thus, f is divisible by p . WLOG, we may assume that g is also divisible by p . This contradicts the assumption that f and g are relatively prime polynomials. We have shown the two kernels indeed form a direct sum. It is clear that

$$\text{Ker } f(T) \oplus \text{Ker } g(T) \subseteq \text{Ker}(f(T)g(T))$$

We now prove the opposite. Let $v \in \text{Ker}(f(T)g(T))$. Because f, g are relatively prime, by **Bezout's identity**, there exists some polynomials a, b such that

$$af + bg = 1$$

The proof then follows from noting

$$(af)(T)v \in \text{Ker } g(T) \text{ and } (bg)(T)v \in \text{Ker } f(T)$$

■

2.2 Year 112

Question 6

Let

$$\mathbf{v}_1 = (1, 2, 0, 4), \mathbf{v}_2 = (-1, 1, 3, -3), \mathbf{v}_3 = (0, 1, -5, -2), \mathbf{v}_4 = (-1, -9, -1, -4)$$

be vectors in \mathbb{R}^4 . Let W_1 be the subspace spanned by \mathbf{v}_1 and \mathbf{v}_2 , and let W_2 be the subspace spanned by \mathbf{v}_3 and \mathbf{v}_4 . Find a basis for $W_1 \cap W_2$.

Proof. Use Gauss elimination to show

$$\begin{cases} \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \text{ is linearly independent.} \\ \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\} \text{ is linearly independent.} \end{cases}$$

Use Gauss elimination to show

$$3\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = 0$$

And conclude

$$W_1 \cap W_2 = \text{span}\{\mathbf{v}_3 + \mathbf{v}_4\}$$

■

Question 7

Let

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 3 & -4 & 1 \\ 3 & -8 & 5 \end{bmatrix}$$

Find an invertible matrix $Q \in M_3(\mathbb{C})$ such that

$$Q A Q^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 12 \\ 0 & 1 & 1 \end{bmatrix}$$

Find an invertible matrix $P \in M_3(\mathbb{C})$ such that $P A P^{-1}$ is diagonal. *Note that my wording differs slightly with the original.*

Proof. Noting that characteristic polynomial is defined uniquely up to change of basis,

$$\det(tI - Q A Q^{-1}) = \det(Q(tI - A)Q^{-1}) = \det(tI - A)$$

We may quickly compute that the characteristic polynomial of A is $t(t-4)(t+3)$. Routine computation now give us

$$P = \begin{bmatrix} 1 & 7 & 0 \\ 1 & -1 & 1 \\ 1 & -29 & 1 \end{bmatrix} \text{ and } PAP^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Note that one may find Q by solving a 9-by-9 linear equation. Here we give a smarter approach. By looking at QAQ^{-1} , we may reduce the problem into finding $v \in \mathbb{C}^3$ such that

$$A^3v = 12Av + A^2v \text{ and } \{v, Av, A^2v\} \text{ is linear independent}$$

So that

$$Q \triangleq [v \quad Av \quad A^2v] \text{ suffices.}$$

Because the characteristic polynomial of A is $t(t-4)(t+3)$, by Cayley-Hamilton Theorem, we know all $v \in \mathbb{C}^3$ satisfy the first condition. To satisfy the linear independence, we write

$$v = c_1e_1 + c_2e_2 + c_3e_3$$

where e_1, e_2, e_3 are the eigenvectors

$$e_1 \triangleq \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } e_2 \triangleq \begin{bmatrix} 7 \\ -1 \\ -29 \end{bmatrix} \text{ and } e_3 \triangleq \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

So that we may just looks for $c_1, c_2, c_3 \in \mathbb{C}$ that makes

$$\begin{bmatrix} c_1 & c_2 & c_3 \\ 0 & 4c_2 & -3c_3 \\ 0 & 16c_3 & 9c_3 \end{bmatrix} \text{ invertible.}$$

By computing the determinant, we see that an obvious choice is $c_1 = c_2 = c_3 = 1$. That is,

$$v = \begin{bmatrix} 8 \\ 1 \\ -27 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 8 & 28 & 112 \\ 1 & -7 & -7 \\ -27 & -119 & -455 \end{bmatrix}$$

■

Question 8

Define **matrix trigonometry** by

$$\sin A \triangleq \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}$$

Compute

$$\sin \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

Show that there exist no matrix $A \in M_2(\mathbb{R})$ such that

$$\sin A = \begin{pmatrix} 1 & 2022 \\ 0 & 1 \end{pmatrix}$$

Proof. Using the identity

$$\sin A = \frac{e^{iA} - e^{-iA}}{2i}$$

we may compute

$$\sin \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sin(1) & 3 \cos(1) \\ 0 & \sin(1) \end{pmatrix}$$

Because $\sin(PBP^{-1}) = P(\sin B)P^{-1}$ for all B and all invertible P , and because

$$\sin \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \sin \lambda_1 & 0 \\ 0 & \sin \lambda_2 \end{pmatrix} \text{ and } \sin \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \sin \lambda & \cos \lambda \\ 0 & \sin \lambda \end{pmatrix}$$

We may finish the proof by computing the Jordan form

$$\begin{pmatrix} 1 & 2022 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & \frac{1}{2022} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & \frac{1}{2022} \end{pmatrix}^{-1}$$

that there exists no matrix $A \in M_2(\mathbb{R})$ such that

$$\sin A = \begin{pmatrix} 1 & 2022 \\ 0 & 1 \end{pmatrix}$$

■

Question 9

Let $A = (a_{ij}) \in M_n(\mathbb{C})$, and let $\lambda_1, \dots, \lambda_n$ be roots of characteristic polynomial of A counted with multiplicity. Show that

$$A \text{ is normal} \iff \|A\|_F = \sum_{k=1}^n |\lambda_k|^2$$

where $\|A\|_F$ is the Frobenius norm of A .

Proof. Because the underlying field is \mathbb{C} , we may apply **Schur's decomposition** to get an upper triangular matrix D and a unitary matrix Q such that

$$A = QDQ^{-1}$$

Because A and D has the same characteristic polynomial, we know the eigenvalues of A lie on the diagonal line of D counted with multiplicity. The proof now follows from computing

$$\|A\|_F = \text{tr}(A^*A) = \text{tr}(D^*D) = \|D\|_F$$

and **Spectral Theorem for complex finite-dimensional vector space**. ■

Question 10

Let $A, B \in M_n(\mathbb{C})$. Suppose that all of the eigenvalues of A and B are positive real numbers. Prove

$$A^4 = B^4 \implies A = B$$

Proof. ■

2.3 Year 111

Question 11

Let V be a finite-dimensional complex inner product space. Let $d \in \text{End}(V)$ satisfy

$$d^2 = 0$$

Let δ be the adjoint of d . Define $\Delta \in \text{End}(V)$ by

$$\Delta \triangleq d\delta + \delta d$$

Prove

- (a) $\text{Ker } d\delta \subseteq \text{Ker } \delta$ and $\text{Ker } \delta d \subseteq \text{Ker } d$.
- (b) $\text{Ker } \Delta = \text{Ker } d \cap \text{Ker } \delta$.
- (c) $V = \text{Ker } \Delta \oplus \text{Im } d \oplus \text{Im } \delta$ is an orthogonal decomposition.
- (d) $\text{Ker } d = \text{Ker } \Delta \oplus \text{Im } d$ is an orthogonal decomposition.

Proof. Because the underlying field is \mathbb{C} , we may express d in its **Jordan form**. Because $d^2 = 0$, we know its Jordan blocks all have eigenvalue 0 and have size no greater than 2. Knowing that the matrix representation of δ with respect to the same Jordan basis is just the conjugate transpose of d , we may easily prove all four propositions with computation, except

$$\text{Ker } \Delta \perp \text{Im } d \quad \text{and} \quad \text{Im } d \perp \text{Im } \delta \quad \text{and} \quad \text{Im } \delta \perp \text{Ker } \Delta$$

Direct computation with the hint $\delta = d^\dagger$ and $d^2 = 0$ shows $\text{Im } d \perp \text{Im } \delta$. Direct computation with hint $\text{Ker } \Delta = \text{Ker } d \cap \text{Ker } \delta$ shows the other two orthogonal relationships. ■

Question 12

Let $V = \mathbb{R}^n$ be the space of column vectors, and M a positive definite symmetric $n \times n$ real matrix. Suppose $A \in M_n(\mathbb{R})$ satisfies

$$MAM^{-1} = A^t$$

Show that there exists some $P \in M_n(\mathbb{R})$ such that

$$P^t MP = I \text{ and } P^{-1}AP \text{ is diagonal}$$

Chapter 3

Archived

3.1 Tensor Algebra

Abstract

In this section, by the term **ring**, we mean a ring with a multiplication identity, and by the term **real algebra**, we mean a real vector space equipped with a vector multiplication compatible with both scalar multiplication and addition. In this definition, for a real algebra A to be a ring, A must be associative. By the term **ideal**, we mean a 2-sided ideal. If we say a multi-linear map $M : V^k \rightarrow Z$ is **alternating**, we mean that M maps (v_1, \dots, v_n) to 0 if two arguments coincide.

Given a finite collection (V_1, \dots, V_n) of finite dimensional real vector space, by the term **tensor product of V_1, \dots, V_n** , we mean a real vector space usually denoted by $V_1 \otimes \dots \otimes V_n$ and a multilinear map $\otimes : V_1 \times \dots \times V_n \rightarrow V_1 \otimes \dots \otimes V_n$ satisfying the universal property: If $B : V_1 \times \dots \times V_n \rightarrow Z$ is a multilinear map, then there exists a unique linear map $\beta : V_1 \otimes \dots \otimes V_n$ such that

$$B(v_1, \dots, v_n) = \beta(v_1 \otimes \dots \otimes v_n)$$

In other words, we have the commutative diagram

$$\begin{array}{ccc} V_1 \times \dots \times V_n & \xrightarrow{B} & U \\ \otimes \downarrow & \nearrow \exists! \beta & \\ V_1 \otimes \dots \otimes V_n & & \end{array}$$

This approach indeed define a pair of vector space and multilinear map uniquely up to isomorphism, in the sense of [Theorem 3.1.1](#), where we define the isomorphism between tensor product.

Theorem 3.1.1. (Uniqueness of Tensor product) Given a finite collection (V_1, \dots, V_n) of finite dimensional real vector space, if $V_1 \otimes \dots \otimes V_n, V_1 \otimes' \dots \otimes' V_n$ both satisfy the universal property, then there exists an linear isomorphism $T : V_1 \otimes \dots \otimes V_n \rightarrow V_1 \otimes' \dots \otimes' V_n$ such that

$$T(v_1 \otimes \dots \otimes v_n) = v_1 \otimes' \dots \otimes' v_n$$

Proof. Because $V_1 \otimes \dots \otimes V_n$ satisfies the universal property, there exists a linear map $T : V_1 \otimes \dots \otimes V_n \rightarrow V_1 \otimes' \dots \otimes' V_n$ such that

$$\otimes' = T \circ \otimes$$

It remains to show T is bijective. Similarly, because $V_1 \otimes' \dots \otimes' V_n$ satisfies the universal property, there exists a linear map $T' : V_1 \otimes' \dots \otimes' V_n \rightarrow V_1 \otimes \dots \otimes V_n$ such that

$$\otimes = T' \circ \otimes'$$

Composing the two equations, we have

$$\otimes' = T \circ T' \circ \otimes'$$

It then follows from uniqueness of the induced linear map in universal property that $T \circ T' = \text{id} : V_1 \otimes' \dots \otimes' V_n \rightarrow V_1 \otimes' \dots \otimes' V_n$. This implies T is indeed bijective. ■

We have shown that tensor products is unique up to isomorphism. A construction further shows that if B_i are bases for V_i , then

$$\{v_1 \otimes \dots \otimes v_n : v_i \in B_i \text{ for all } 1 \leq i \leq n\} \text{ form a basis for } V_1 \otimes \dots \otimes V_n$$

Theorem 3.1.2. (Associativity of the Tensor product) Given three finite-dimensional real vector spaces X, Y, Z , there exists a unique linear isomorphism $F : X \otimes Y \otimes Z \rightarrow (X \otimes Y) \otimes Z$ that satisfy

$$F(x \otimes y \otimes z) = (x \otimes y) \otimes z$$

Proof. Define $f : X \times Y \times Z \rightarrow (X \otimes Y) \otimes Z$ by

$$f(x, y, z) \triangleq (x \otimes y) \otimes z$$

It follows from the universal property that there exists a unique linear map $F : X \otimes Y \otimes Z \rightarrow (X \otimes Y) \otimes Z$ such that

$$F(x \otimes y \otimes z) = f(x, y, z) = (x \otimes y) \otimes z$$

It remains to show F is bijective. For all $z \in Z$, define $h_z : X \times Y \rightarrow X \otimes Y \otimes Z$ by

$$h_z(x, y) \triangleq x \otimes y \otimes z$$

It follows from the universal property that there exists a unique linear map $H_z : X \otimes Y \rightarrow X \otimes Y \otimes Z$ such that

$$H_z(x \otimes y) = h_z(x, y) = x \otimes y \otimes z$$

Define $h : (X \otimes Y) \times Z \rightarrow X \otimes Y \otimes Z$ by

$$h(v, z) \triangleq H_z(v)$$

It is clear that h is linear in $(X \otimes Y)$. We now show h is linear in Z , that is

$$H_{c_1 z_1 + z_2} = c_1 H_{z_1} + H_{z_2}$$

By definition,

$$\begin{aligned} (c_1 H_{z_1} + H_{z_2})(x \otimes y) &= c_1 x \otimes y \otimes z_1 + x \otimes y \otimes z_2 \\ &= x \otimes y \otimes (c_1 z_1 + z_2) = h_{c_1 z_1 + z_2}(x, y) \end{aligned}$$

It then follows from the uniqueness part of the universal property that $H_{c_1 z_1 + z_2} = c_1 H_{z_1} + H_{z_2}$. (done)

We have shown h is indeed bilinear. It follows from the universal property that there exists a unique linear map $H : (X \otimes Y) \otimes Z \rightarrow X \otimes Y \otimes Z$ such that

$$H((x \otimes y) \otimes z) = h(x \otimes y, z) = H_z(x \otimes y) = x \otimes y \otimes z$$

Let $\otimes : X \times Y \times Z \rightarrow X \otimes Y \otimes Z$ denotes the tensor product, we now have

$$\otimes = H \circ F \circ \otimes$$

It then follows from universal property that $H \circ F = \text{id} : X \otimes Y \otimes Z \rightarrow X \otimes Y \otimes Z$. This implies F is indeed bijective. (done) ■

Let V be a finite-dimensional real vector space. By its **tensor algebra**, we mean any real associative algebra $T(V)$ with an injective linear map $i : V \rightarrow T(V)$ that satisfies the universal property: If A is a real associative algebra and $f : V \rightarrow A$ is a linear map, then there exists a unique algebra homomorphism $F : T(V) \rightarrow A$ such that the

diagram

$$\begin{array}{ccc}
 V & \xrightarrow{i} & T(V) \\
 & \searrow f & \downarrow F \\
 & & A
 \end{array}$$

commutes. The proof that such definition is indeed unique up to isomorphism is similar to that of [Theorem 3.1.1](#) and thus omitted. We now give the most useful construction.

Let V be finite-dimensional real vector space. We use the notation

$$T^n(V) \triangleq \overbrace{V \otimes \cdots \otimes V}^{n \text{ copies}}$$

and call $T^n(V)$ the **n -th tensor power of V** or the **n -fold tensor product of V** . Define

$$\begin{aligned}
 T(V) &\triangleq \bigoplus_{n=0}^{\infty} T^n(V) \\
 &= \mathbb{R} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots
 \end{aligned}$$

and define for all $f, g \in T(V)$ the multiplication

$$(fg)(n) \triangleq \sum_{k=0}^n f(k)g(n-k)$$

where

$$\begin{aligned}
 &\left(\sum_I a_I v_{I(1)} \otimes \cdots \otimes v_{I(k)} \right) \left(\sum_J b_J v_{J(1)} \otimes \cdots \otimes v_{J(l)} \right) \\
 &\triangleq \sum_{I,J} a_I b_J v_{I(1)} \otimes \cdots \otimes v_{I(k)} \otimes v_{J(1)} \otimes \cdots \otimes v_{J(l)}
 \end{aligned}$$

where $\{v_1, \dots, v_m\}$ is some basis for V , I run through the set of function that maps $\{1, \dots, k\}$ into $\{1, \dots, m\}$ and J run through the set of function that maps $\{1, \dots, l\}$ into $\{1, \dots, m\}$. For example, given two elements

$$(5, 0, v_1 \otimes v_2, 0, 0, \dots) \text{ and } (7, v_3, 0, 0, \dots)$$

of $T(V)$, their product is defined to be

$$(35, 5v_3, 7v_1 \otimes v_2, v_1 \otimes v_2 \otimes v_3, 0, 0, \dots)$$

Tedious effort shows that our multiplication is consistent with abuse of notation in the sense that if $f, g \in T(V)$ is defined by

$$f(k) \triangleq \begin{cases} w_1 \otimes \cdots \otimes w_n & \text{if } k = n \\ 0 & \text{if otherwise} \end{cases} \quad \text{and } g(k) \triangleq \begin{cases} w_{n+1} \otimes \cdots \otimes w_{n+l} & \text{if } k = l \\ 0 & \text{if otherwise} \end{cases}$$

then

$$(fg)(k) = \begin{cases} w_1 \otimes \cdots \otimes w_{n+l} & \text{if } n = k + l \\ 0 & \text{if otherwise} \end{cases}$$

does form an associative algebra with multiplication identity $1 \in \mathbb{R}$. Thus, $T(V)$ is in fact a ring. Let $I(V) \subseteq T(V)$ be the ideal generated by $\{v \otimes v : v \in V\}$. By definition, ideal $I(V)$ is a subgroup of $T(V)$. To see that $I(V)$ is closed under scalar multiplication, observe that for all $t \in \mathbb{R}$ and $x \in T(V)$, the scalar multiplication tx is identical to tx where t is treated as an element of $T(V)$, so it follows from definition of ideal that $I(V)$ is also a vector subspace of $T(V)$. Let $\{v_1, \dots, v_n\}$ be a basis for V , and let S be the set of function that maps $\{1, \dots, n\}$ into $\{1, \dots, k\}$. We know for a fact that

$$T^k(V) = \text{span}\{v_{I(1)} \otimes \cdots \otimes v_{I(k)} : I \in S\}$$

If we define $I^k(V) \triangleq I(V) \cap T^k(V)$, one then have

$$I^k(V) = \text{span}\{v_{I(1)} \otimes \cdots \otimes v_{I(k)} : I(j) = I(j+1) \text{ for some } j\} \quad (3.1)$$

This is proved by showing $I^0(V) \oplus I^1(V) \oplus I^2(V) \oplus \cdots$ is indeed the smallest ideal containing $\{v \otimes v : v \in V\}$. Define an equivalence class on $T(V)$ by

$$x \sim y \iff x - y \in I(V)$$

Because ideal form a subgroup, we see that our definition indeed give an equivalence relation. We then can define on the set of equivalence class $T(V) \setminus I(V)$ addition, scalar multiplication and vector multiplication

$$[x] + [y] \triangleq [x + y] \text{ and } [x] \wedge [y] \triangleq [xy] \text{ and } c[x] \triangleq [cx]$$

which is well defined and form an algebra as one can check. We call this algebra $T(V) \setminus I(V)$ the **exterior algebra** $\wedge^*(V)$. Note that if we refer to $v, w \in T^k(V)$ as elements of $\wedge^*(V)$, we mean $[v], [w]$. Immediately, we see that the wedge product is **alternating** in the sense that if $v \in V$, then

$$v \wedge v = 0$$

and is **anti-symmetric** in the sense that if $v, w \in V$, then

$$v \wedge w = -w \wedge v$$

We use the notation

$$\wedge^k(V) \triangleq \left\{ [x] \in \wedge^*(V) : x \in \overbrace{V \otimes \cdots \otimes V}^{k \text{ copies}} \right\}$$

Immediately from [Equation 3.1](#), we see that $\wedge^k(V)$ is the vector space

$$\text{span}\{v_{I(1)} \wedge \cdots \wedge v_{I(k)} : I \in S\}$$

where $\{v_1, \dots, v_n\}$ is a basis for V and S is the set of function that maps $\{1, \dots, k\}$ into $\{1, \dots, n\}$. If we define the vector subspace $I^k(V) \triangleq T^k(V) \cap I(V)$, there exists a natural vector space isomorphism

$$\wedge^k(V) \underset{\text{v.s.}}{\cong} T^k(V) \setminus I^k(V); [x] \leftrightarrow [x]$$

where $T^k(V) \setminus I^k(V)$ is the quotient vector space.

Theorem 3.1.3. (Universal mapping property for alternating k -linear map) For any vector space Z over \mathbb{R} and any alternating k -linear map $f : V^k \rightarrow Z$, there is a unique linear map $F : \wedge^k V \rightarrow Z$ such that the diagram

$$\begin{array}{ccc} \wedge^k V & & \\ \uparrow \wedge & \searrow F & \\ V^k & \xrightarrow{f} & Z \end{array}$$

commute, i.e.,

$$F(v_1 \wedge \cdots \wedge v_k) = f(v_1, \dots, v_k) \text{ for all } v_1, \dots, v_k \in V$$

Proof. By universal property of tensor product, there exists unique linear map $h : T^k(V) \rightarrow Z$ such that

$$h(v_1 \otimes \cdots \otimes v_k) = f(v_1, \dots, v_k)$$

Because f is alternating, we see from the characterization of $I^k(V)$ given in [Equation 3.1](#) that h vanishes on $I^k(V)$. We then can induce a linear map

$$F : \wedge^k(V) \cong \frac{T^k(V)}{I^k(V)} \rightarrow Z$$

by $F([x]) \triangleq h(x)$. This then give us the desired

$$F(v_1 \wedge \cdots \wedge v_k) = h(v_1 \otimes \cdots \otimes v_k) = f(v_1, \dots, v_k)$$

Note that F is unique because all such linear map take the same values on $\{v_{I(1)} \wedge \cdots \wedge v_{I(k)} : I \in S\}$ which spans $\wedge^k(V)$. ■

Let $\{w_1, \dots, w_l\} \subseteq V$ be linear independent. An immediate consequence of the [universal mapping property for alternating \$k\$ -linear map](#) is that one may define alternating multilinear $f : V^l \rightarrow \mathbb{R}$ by

$$B(v_1, \dots, v_l) \triangleq \det(M) \text{ where } v_i = \sum_j M_{i,j} w_j$$

and see that $F : \wedge^l(V) \rightarrow \mathbb{R}$ take $w_1 \wedge \cdots \wedge w_l$ to 1. This implies that

$$w_1 \wedge \cdots \wedge w_l \neq 0$$

Theorem 3.1.4. (Anti-symmetry of wedge product) If $\alpha \in \wedge^k(V), \beta \in \wedge^l(V)$, then $\alpha \wedge \beta = (-1)^{kl}(\beta \wedge \alpha)$.

Proof. Let v_1, \dots, v_n be a basis of V . Let S_k be the space of function that maps $\{1, \dots, k\}$ into $\{1, \dots, n\}$, S_l be the space of function that maps $\{1, \dots, l\}$ into $\{1, \dots, n\}$. We may then write

$$\alpha = \sum_{I \in S_k} a_I (v_{I(1)} \wedge \cdots \wedge v_{I(k)}) \text{ and } \beta = \sum_{J \in S_l} b_J (v_{J(1)} \wedge \cdots \wedge v_{J(l)})$$

and compute

$$\begin{aligned}
\alpha \wedge \beta &= \sum_{I \in S_k, J \in S_l} a_I b_J (v_{I(1)} \wedge \cdots \wedge v_{I(k)} \wedge v_{J(1)} \wedge \cdots \wedge v_{J(l)}) \\
&= \sum_{I \in S_k, J \in S_l} (-1) a_I b_J (v_{I(1)} \wedge \cdots \wedge v_{J(1)} \wedge v_{I(k)} \wedge \cdots \wedge v_{J(l)}) \\
&= \sum_{I \in S_k, J \in S_l} (-1)^k a_I b_J (v_{J(1)} \wedge v_{I(1)} \wedge \cdots \wedge v_{I(k)} \wedge v_{J(2)} \wedge \cdots \wedge v_{J(l)}) \\
&= \sum_{I \in S_k, J \in S_l} (-1)^{kl} a_I b_J (v_{J(1)} \wedge \cdots \wedge v_{J(l)} \wedge v_{I(1)} \wedge \cdots \wedge v_{I(k)}) = (-1)^{kl} \beta \wedge \alpha
\end{aligned}$$

■

Following from [Theorem 3.1.4](#), [Equation 3.1](#) and tedious effort, one can see that if $\{v_1, \dots, v_n\}$ is a basis for V , then

$$\{v_{i_1} \wedge \cdots \wedge v_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}$$

form a basis for $\wedge^k(V)$. If $A : V \rightarrow W$ is a linear map, we define linear map $\wedge^k A : \wedge^k(V) \rightarrow \wedge^k(W)$ by linear extension of

$$\wedge^k(A)(v_1 \wedge \cdots \wedge v_n) = Av_1 \wedge \cdots \wedge Av_n$$

Note that if $A : V \rightarrow V$ and $\dim(V) = n$, then $\wedge^n A : \wedge^n(V) \rightarrow \wedge^n(V)$ is given by the determinant since given basis $\{v_1, \dots, v_n\}$, we have

$$\begin{aligned}
\wedge^n A(v_1 \wedge \cdots \wedge v_n) &= \left(\sum_j A_{j,1} v_j \right) \wedge \cdots \wedge \left(\sum_j A_{j,n} v_j \right) \\
&= \sum_{\sigma \in S_n} A_{\sigma(1),1} \cdots A_{\sigma(n),n} v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)} \\
&= \sum_{\sigma \in S_n} \text{sgn}(\sigma) A_{\sigma(1),1} \cdots A_{\sigma(n),n} v_1 \wedge \cdots \wedge v_n
\end{aligned}$$

3.2 Operator Norm

Abstract

This section introduces the concept of the operator norm and proves some fundamental results related operator norm and finite-dimensional normed spaces. For example, we establish results such as **a linear operator being bounded if and only if it is continuous** and **the equivalence of all norms on finite-dimensional vector spaces**.

In this section, and particularly in functional analysis, we say a function T between two metric space is a **bounded operator** if T always map bounded set to bounded set. In particular, if T is a linear transformation between two normed space, we say T is a **bounded linear operator**. Now, suppose \mathcal{X}, \mathcal{Y} are two normed space over \mathbb{R} or \mathbb{C} . In space $L(\mathcal{X}, \mathcal{Y})$, alternatively, we can define

$$T \text{ is bounded} \iff \exists M \in \mathbb{R}, \forall x \in \mathcal{X}, \|Tx\| \leq M\|x\|$$

The proof of equivalency is simple. For (\longrightarrow) , observe

$$\|Tx\| = \|x\| \cdot \left\| T \frac{x}{\|x\|} \right\| \leq \left(\sup\{\|Ty\| : \|y\| = 1\} \right) \|x\|$$

For (\longleftarrow) , observe

$$\|Tx - Ty\| = \|T(x - y)\| \leq M\|x - y\|$$

We first show that **a linear transformation is continuous if and only if it is bounded**.

Theorem 3.2.1. (Linear Operator is Bounded if and only if it is Continuous)

Given two normed space \mathcal{X}, \mathcal{Y} over \mathbb{R} or \mathbb{C} and $T \in L(\mathcal{X}, \mathcal{Y})$, we have

$$T \text{ is a bounded operator} \iff T \text{ is continuous on } \mathcal{X}$$

Proof. If T is bounded, we see that T is Lipschitz.

$$\|Tx - Ty\| \leq M\|x - y\|$$

Now, suppose T is linear and continuous at 0. Let ϵ satisfy

$$\sup_{\|y\| \leq \epsilon} \|Ty\| \leq 1$$

Observe that for all $x \in \mathcal{X}$, we have

$$\|Tx\| = \frac{\|x\|}{\epsilon} \left\| T \frac{\epsilon x}{\|x\|} \right\| \leq \frac{\|x\|}{\epsilon}$$

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Here, we introduce a new terminology, which shall later show its value. Given a set X , we say two metrics d_1, d_2 on X are **equivalent**, and write $d_1 \sim d_2$, if we have

$$\exists m, M \in \mathbb{R}^+, \forall x, y \in X, md_1(x, y) \leq d_2(x, y) \leq Md_1(x, y)$$

Now, given a fixed vector space V , naturally, we say two norms $\|\cdot\|_1, \|\cdot\|_2$ on V are **equivalent** if

$$\exists m, M \in \mathbb{R}^+, \forall x \in X, m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1$$

We say two metric d_1, d_2 on X are **topologically equivalent** if the topology they induce on X are identical.

A few properties can be immediately spotted.

- (a) Our definition of \sim between metrics of a fixed X is an equivalence relation.
- (b) Our definition of \sim between norms on a fixed V is an equivalence relation.
- (c) Equivalent norms induce equivalent metrics.
- (d) Equivalent metrics are topologically equivalent.

We now prove **if V is finite-dimensional, then all norms on V are equivalent**. This property will later show its value, as used to prove **linear map of finite-dimensional domain is always continuous**

Theorem 3.2.2. (All Norms on Finite-dimensional space are Equivalent) Suppose V is a finite-dimensional vector space over \mathbb{R} or \mathbb{C} . Then

all norms on V are equivalent

Proof. Let $\{e_1, \dots, e_n\}$ be a basis of V . Define ∞ -norm $\|\cdot\|_\infty$ on V by

$$\left\| \sum \alpha_i e_i \right\|_\infty \triangleq \max |\alpha_i|$$

It is easily checked that $\|\cdot\|_\infty$ is indeed a norm. Fix a norm $\|\cdot\|$ on V . We reduce the problem into

finding $m, M \in \mathbb{R}^+$ such that $m\|x\|_\infty \leq \|x\| \leq M\|x\|_\infty$

We first claim

$$M = \sum \|e_i\| \text{ suffices}$$

Compute

$$\|x\| = \left\| \sum \alpha_i e_i \right\| \leq \sum |\alpha_i| \|e_i\| \leq \|x\|_\infty \sum \|e_i\| = M \|x\|_\infty \text{ (done)}$$

Note that reverse triangle inequality give us

$$\left| \|x\| - \|y\| \right| \leq \|x - y\| \leq M \|x - y\|_\infty \quad (3.2)$$

Then we can check that

(a) $\|\cdot\| : (V, \|\cdot\|_\infty) \rightarrow \mathbb{R}$ is Lipschitz continuous because of **Equation 3.2**.

(b) $S \triangleq \{y \in V : \|y\|_\infty = 1\}$ is sequentially compact in $\|\cdot\|$ and non-empty.

Now, by EVT, we know $\min_{y \in S} \|y\|$ exists. Note that $\min_{y \in S} \|y\| > 0$, since $0 \notin S$. We claim

$$m = \min_{y \in S} \|y\| \text{ suffices}$$

Fix $x \in V$ and compute

$$m \|x\|_\infty = \|x\|_\infty (\min_{y \in S} \|y\|) \leq \|x\|_\infty \cdot \left\| \frac{x}{\|x\|_\infty} \right\| = \|x\| \text{ (done) (done)}$$

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Theorem 3.2.3. (Linear map of Finite-dimensional Domain is always Continuous) Given a finite-dimensional normed space \mathcal{X} over \mathbb{R} or \mathbb{C} , an arbitrary normed space \mathcal{Y} over \mathbb{R} or \mathbb{C} and a linear transformation $T : \mathcal{X} \rightarrow \mathcal{Y}$, we have

T is continuous

Proof. Fix $x \in \mathcal{X}, \epsilon$. We wish

to find δ such that $\forall h \in \mathcal{X} : \|h\| \leq \delta, \|T(x+h) - Tx\| \leq \epsilon$

Let $\{e_1, \dots, e_n\}$ be a basis of \mathcal{X} . Note that $\|\sum \alpha_i e_i\|_1 \triangleq \sum |\alpha_i|$ is a norm. **Because \mathcal{X} is finite-dimensional, we know $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent.** Then, we can fix $M \in \mathbb{R}^+$ such that

$$\|x\|_1 \leq M \|x\| \quad (x \in V)$$

We claim

$$\delta = \frac{\epsilon}{M(\max \|Te_i\|)} \text{ suffices}$$

Fix $\|h\| \leq \delta$ and express $h = \sum \alpha_i e_i$. Compute using linearity of T

$$\begin{aligned} \|T(x+h) - Tx\| &= \left\| \sum \alpha_i Te_i \right\| \\ &\leq \sum |\alpha_i| \|Te_i\| \\ &\leq \|h\|_1 (\max \|Te_i\|) \\ &\leq M \|h\| (\max \|Te_i\|) = \epsilon \text{ (done)} \end{aligned}$$

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We now see that, because **Linear transformation is bounded if and only if it is continuous** and **Linear map of finite-dimensional domain is always continuous**, if \mathcal{X} is finite-dimensional, then all linear map of domain \mathcal{X} are bounded. A counter example to the generalization of this statement is followed.

Example 1 (Differentiation is an Unbounded Linear Operator)

$$\mathcal{X} = (\mathbb{R}[x]_{[0,1]}, \|\cdot\|_\infty), D(P) \triangleq P'$$

Note that $\{x^n\}_{n \in \mathbb{N}}$ is bounded in \mathcal{X} and $\{D(x^n)\}_{n \in \mathbb{N}}$ is not.

Now, suppose \mathcal{X}, \mathcal{Y} are two fixed normed spaces over \mathbb{R} or \mathbb{C} . We can easily check that the set $BL(\mathcal{X}, \mathcal{Y})$ of bounded linear operators from \mathcal{X} to \mathcal{Y} form a vector space over whichever field \mathcal{Y} is over.

Naturally, our definition of boundedness of linear operator derive us a norm on $BL(\mathcal{X}, \mathcal{Y})$, as followed

$$\|T\|_{\text{op}} \triangleq \inf \{M \in \mathbb{R}^+ : \forall x \in \mathcal{X}, \|Tx\| \leq M\|x\|\} \quad (3.3)$$

Before we show that our definition is indeed a norm, we first give some equivalent definitions and prove their equivalency.

Theorem 3.2.4. (Equivalent Definitions of Operator Norm) Given two fixed normed space \mathcal{X}, \mathcal{Y} over \mathbb{R} or \mathbb{C} , a bounded linear operator $T : \mathcal{X} \rightarrow \mathcal{Y}$, and define $\|T\|_{\text{op}}$ as in **Equation 3.3**, we have

$$\|T\|_{\text{op}} = \sup_{x \in \mathcal{X}, x \neq 0} \frac{\|Tx\|}{\|x\|}$$

Proof. Define $J \triangleq \{M \in \mathbb{R}^+ : \forall x \in \mathcal{X}, \|Tx\| \leq M\|x\|\}$ and observe

$$J = \{M \in \mathbb{R}^+ : M \geq \frac{\|Tx\|}{\|x\|}, \forall x \neq 0 \in \mathcal{X}\}$$

This let us conclude

$$\sup_{x \in \mathcal{X}, x \neq 0} \frac{\|Tx\|}{\|x\|} = \min J = \|T\|_{\text{op}}$$

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It is now easy to see

$$\|T\|_{\text{op}} = \sup_{x \in \mathcal{X}, x \neq 0} \frac{\|Tx\|}{\|x\|} \tag{3.4}$$

$$= \sup_{x \in \mathcal{X}, \|x\|=1} \|Tx\| \tag{3.5}$$

It is not all in vain to introduce the equivalent definitions. See that the verification of $\|\cdot\|_{\text{op}}$ being a norm on $BL(\mathcal{X}, \mathcal{Y})$ become simple by utilizing the equivalent definitions.

- (a) For positive-definiteness, fix non-trivial T and fix $x \in \mathcal{X} \setminus N(T)$. Use [Equation 3.4](#) to show $\|T\|_{\text{op}} \geq \frac{\|Tx\|}{\|x\|} > 0$.
- (b) For absolute homogeneity, use [Equation 3.5](#) and $\|Tcx\| = |c| \cdot \|Tx\|$.
- (c) For triangle inequality, use [Equation 3.5](#) and $\|(T_1 + T_2)x\| \leq \|T_1x\| + \|T_2x\|$.

Naturally, and very very importantly, [Equation 3.4](#) give us

$$\|Tx\| \leq \|T\|_{\text{op}} \cdot \|x\| \quad (x \in \mathcal{X})$$

This inequality will later be the best tool to help analyze the derivatives of functions between Euclidean spaces, and perhaps better, it immediately give us

$$\frac{\|T_1T_2x\|}{\|x\|} \leq \frac{\|T_1\|_{\text{op}} \cdot \|T_2\|_{\text{op}} \cdot \|x\|}{\|x\|} = \|T_1\|_{\text{op}} \cdot \|T_2\|_{\text{op}}$$

Then [Equation 3.4](#) give us

$$\|T_1T_2\|_{\text{op}} \leq \|T_1\|_{\text{op}} \cdot \|T_2\|_{\text{op}}$$

3.3 Cauchy-Schwarz for Positive semi-definite Hermitian form

Sometimes, we do not require $\langle \cdot, \cdot \rangle : V^2 \rightarrow \mathbb{F}$ to be positive-definite, but only **positive semi-definite**, i.e. $\langle v, v \rangle \geq 0$ for all $v \in V$. In this case, we say $\langle \cdot, \cdot \rangle : V^2 \rightarrow \mathbb{F}$ is a **positive semi-definite Hermitian form**. If we again induce a functional $\|\cdot\| : V \rightarrow \mathbb{R}$ from some positive semi-definite Hermitian form using Equation ??, then the functional $\|\cdot\| : V \rightarrow \mathbb{R}$ in general is not positive-definite, and is called a **semi norm**, since, as we later show, it still satisfies triangle inequality and absolute homogeneity.

Theorem 3.3.1. (Basic Property of Positive semi-definite Hermitian form) Given a positive semi-definite Hermitian form $\langle \cdot, \cdot \rangle : V^2 \rightarrow \mathbb{F}$ and $x, y \in V$, we have

$$\langle x, x \rangle = 0 \implies \langle x, y \rangle = 0$$

Proof. Assume $\langle x, y \rangle \neq 0$. Fix $t > \frac{\|y\|^2}{2|\langle x, y \rangle|^2}$. Compute

$$\begin{aligned} \|y - t\langle y, x \rangle x\|^2 &= \|y\|^2 + \|(-t)\langle y, x \rangle x\|^2 + \langle -t\langle y, x \rangle x, y \rangle + \langle y, -t\langle y, x \rangle x \rangle \\ &= \|y\|^2 + t^2 |\langle x, y \rangle|^2 \|x\|^2 - t\langle y, x \rangle \langle x, y \rangle - t\langle x, y \rangle \langle y, x \rangle \\ &= \|y\|^2 - 2t |\langle x, y \rangle|^2 < 0 \text{ CaC} \end{aligned}$$

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Theorem 3.3.2. (Cauchy-Schwarz Inequality) Given a positive semi-definite Hermitian form $\langle \cdot, \cdot \rangle : V^2 \rightarrow \mathbb{C}$ on vector space V over \mathbb{C} , we have

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

Moreover, the equality hold true if x, y are linearly independent. In addition, if $\langle \cdot, \cdot \rangle : V^2 \rightarrow \mathbb{F}$ is an inner product, then the equality hold true only if x, y are linearly independent.

Proof. We first prove

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\| \quad (x, y \in V)$$

Fix $x, y \in V$. **Theorem 3.3.1** tell us $\|x\| = 0 \implies \langle x, y \rangle = 0$. Then we can reduce the problem into proving

$$\frac{|\langle x, y \rangle|^2}{\|x\|^2} \leq \|y\|^2$$

Set $z \triangleq y - \frac{\langle y, x \rangle}{\|x\|^2} x$. We then have

$$\langle z, x \rangle = \langle y - \frac{\langle y, x \rangle}{\|x\|^2} x, x \rangle = \langle y, x \rangle - \frac{\langle y, x \rangle}{\|x\|^2} \langle x, x \rangle = 0$$

Then from $y = z + \frac{\langle y, x \rangle}{\|x\|^2} x$, we can now deduce

$$\begin{aligned} \langle y, y \rangle &= \langle z + \frac{\langle y, x \rangle}{\|x\|^2} x, z + \frac{\langle y, x \rangle}{\|x\|^2} x \rangle \\ &= \langle z, z \rangle + \left| \frac{\langle y, x \rangle}{\langle x, x \rangle} \right|^2 \langle x, x \rangle \\ &= \langle z, z \rangle + \frac{|\langle x, y \rangle|^2}{\langle x, x \rangle} \end{aligned}$$

Because $\langle z, z \rangle \geq 0$, we now have

$$\langle y, y \rangle = \langle z, z \rangle + \frac{|\langle x, y \rangle|^2}{\langle x, x \rangle} \geq \frac{|\langle x, y \rangle|^2}{\langle x, x \rangle} \text{ (done)}$$

The equality hold true if and only if $\langle z, z \rangle = 0$. This explains the other two statements regarding the equality. ■

Now, with Cauchy-Schwarz Inequality, we can check the triangle inequality

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &= \langle x, x \rangle + \langle y, y \rangle + 2 \operatorname{Re} \langle x, y \rangle \\ &\leq \|x\|^2 + \|y\|^2 + 2 |\langle x, y \rangle| \\ &\leq \|x\|^2 + \|y\|^2 + 2 \|x\| \cdot \|y\| = (\|x\| + \|y\|)^2 \end{aligned}$$

indeed holds true for space equipped with only positive semi-definite Hermitian form.