§3. Local theory

Let $U \subset \mathbb{C}^n$ be an open subset and x_i, y_i, $i=1,\cdots,n$ be real coordinate on $\mathbb{C}^n = \mathbb{R}^n \oplus \sqrt{-1}\mathbb{R}^n$. Define $J : TU \to TU$ by

$$\begin{cases} J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i} \\ J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i} \end{cases}$$

$\rightsquigarrow$ almost complex structure J on U

Dually, we have

$$\begin{cases} J^*(dx_i) = -dy_i \\ J^*(dy_i) = dx_i \end{cases}$$

We complexify TU and decompose it into $\pm\sqrt{-1}$-eigenspaces:

$$T_{\mathbb{C}}U = T^{1,0}U \oplus T^{0,1}U$$

Then

$$T^{1,0}U = \mathrm{Span}_{\mathbb{C}}\left\{ \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right) \,\middle|\, i=1,\cdots,n \right\}$$

$$T^{0,1}U = \mathrm{Span}_{\mathbb{C}}\left\{ \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right) \,\middle|\, i=1,\cdots,n \right\}$$

Similarly, the complexified cotangent bundle $T^*_{\mathbb{C}}U$ satisfies

$$T^*_{\mathbb{C}}U = (T^*U)^{1,0} \oplus (T^*U)^{0,1}$$

and

$$(T^*U)^{1,0} = \mathrm{Span}_{\mathbb{C}}\left\{ dx_i + \sqrt{-1}\,dy_i \,\middle|\, i=1,\cdots,n \right\}$$

$$(T^*U)^{0,1} = \mathrm{Span}_{\mathbb{C}}\left\{ dx_i - \sqrt{-1}\,dy_i \,\middle|\, i=1,\cdots,n \right\}$$

Note that $(T^*U)^{1,0} = (T^{1,0}U)^*$ and $(T^*U)^{0,1} = (T^{0,1}U)^*$.

We put

$$dz_i = dx_i + \sqrt{-1}\, dy_i$$
$$d\bar{z}_i = dx_i - \sqrt{-1}\, dy_i$$

Then we have

$$dz_i\left(\frac{\partial}{\partial z_j}\right) = \delta_{ij} = d\bar{z}_i\left(\frac{\partial}{\partial \bar{z}_j}\right)$$
$$dz_i\left(\frac{\partial}{\partial \bar{z}_j}\right) = 0 = d\bar{z}_i\left(\frac{\partial}{\partial z_j}\right)$$

for all $i, j = 1, \cdots, n$.

<u>Prop</u>: Let $f : U \to V$ be a holomorphic map. Then

$$df_x(T_x^{1,0} U) \subset T_{f(x)}^{1,0} V$$
$$df_x(T_x^{0,1} U) \subset T_{f(x)}^{0,1} V$$

<u>Pf</u>: Exercise.

Define the complex vector bundle over U:

$$\Lambda^{p,q} T^* U := \Lambda^p (T^*U)^{1,0} \otimes \Lambda^q (T^*U)^{0,1}$$

and $A^{p,q}(U)$ the space of smooth sections of $\Lambda^{p,q} T^* U$.

They are called (p,q)-forms. We continue to have the decomposition

$$\Lambda^k T^*_{\mathbb{C}} U = \bigoplus_{p+q=k} \Lambda^{p,q} T^* U$$

$$A^k_{\mathbb{C}}(U) = \bigoplus_{p+q=k} A^{p,q}(U)$$

space of complex-valued k-forms on U.

We have the projection : $\pi^{p,q} : A^k_{\mathbb{C}}(U) \longrightarrow A^{p,q}(U)$

Def: Let $d: A^k_{\mathbb{C}}(U) \to A^{k+1}_{\mathbb{C}}(U)$ be the complex linear extension of the exterior differential. We define

$$\partial := \Pi^{p+1,q} \circ d\big|_{A^{p,q}(U)} \quad , \quad \bar{\partial} := \Pi^{p,q+1} \circ d\big|_{A^{p,q}(U)}$$

For any $f \in C^\infty(U)$, we have

$$df = \sum_{i=1}^n \frac{\partial f}{\partial z_i} dz_i + \sum_{i=1}^n \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i$$

Then $\partial f = \sum_{i=1}^n \frac{\partial f}{\partial z_i} dz_i$ and $\bar{\partial} f = \sum_{i=1}^n \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i$. In particular,

$$f \text{ is holomorphic} \Leftrightarrow \bar{\partial} f = 0.$$

In general,

$$\partial(f\, dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}) = \sum_{i=1}^n \frac{\partial f}{\partial z_i} dz_i \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$$

$$\bar{\partial}(f\, dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}) = \sum_{i=1}^n \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$$

Lemma: On $A^\bullet_{\mathbb{C}}(U)$, we have

① $d = \partial + \bar{\partial}$

② $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$

③ $\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge \partial\beta$

④ $\bar{\partial}(\alpha \wedge \beta) = \bar{\partial}\alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge \bar{\partial}\beta$

Pf: ① follows from the formula of d, ∂ and $\bar{\partial}$.

② Recall that $d^2 = 0$, so

$$0 = d^2 = \partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2$$

We consider the following diagram:

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & & A^{p+2,q}(U) \\
 & A^{p+1,q}(U) \arrow[urr, "\partial"] \arrow[drr, "\bar\partial"] & & \\
A^{p,q}(U) \arrow[ur, "\partial"] \arrow[dr, "\bar\partial"'] & & & A^{p+1,q+1}(U) \\
 & A^{p,q+1}(U) \arrow[urr, "\partial"] \arrow[drr, "\bar\partial"'] & & \\
 & & & A^{p,q+2}(U)
\end{tikzcd}$$

By degree reason, we get ②.

③ We have

$$d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^{\deg(\alpha)}\alpha\wedge d\beta$$

Suppose $\alpha \in A^{p,q}(U)$ and $\beta \in A^{r,s}(U)$. Then

$$\partial(\alpha\wedge\beta),\ \partial\alpha\wedge\beta,\ \alpha\wedge\partial\beta \in A^{p+r+1,q+s}(U)$$
$$\bar\partial(\alpha\wedge\beta),\ \bar\partial\alpha\wedge\beta,\ \alpha\wedge\bar\partial\beta \in A^{p+r,q+s+1}(U)$$

Hence ③ $\square$

Recall the Poincaré lemma.

Thm: Let U be a contractible open subset of $\mathbb{R}^n$ and $\alpha\in A^k(U)$ be such that $d\alpha = 0$ (d-closed). Then there exists $\beta\in A^{k-1}(U)$ such that $\alpha = d\beta$ (d-exact).

A $\bar\partial$-analog of the Poincaré lemma holds.

Thm: [$\bar{\partial}$-Poincaré lemma]

Let B be a polydisk and $\alpha \in A^{p,q}(B)$ be such that $\bar{\partial}\alpha = 0$. Then there exists $\beta \in A^{p,q-1}(B)$ such that $\alpha = \bar{\partial}\beta$

Pf: Omitted $\square$