Deadline : 2024/04/15, 17:00.

Definition:

(i) The Fourier transform of f on $\mathbb{R}$ is defined by $\widehat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx$.

(ii) The Fourier inverse transform of f on $\mathbb{R}$ is defined by $f(x) = \mathcal{F}^{-1}[\widehat{f}] = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{ix\xi}\,d\xi$.

1. Show that $\widehat{f'} = i\xi\widehat{f}$ and $\widehat{xf} = i\dfrac{d}{d\xi}\widehat{f}$. (You may assume $f \to 0$ as $x \to \pm\infty$)

2. Let $g(x) = \dfrac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}}$, $\sigma \neq 0$. $g(x)$ is called the normalized Gaussian function in $\mathbb{R}$. Find the Fourier transform of g on $\mathbb{R}$.

3. The convolution of two functions f and g is defined by $f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy$. Show that $\widehat{f * g} = \widehat{f}\widehat{g}$. (You may assume the Fubini's Theorem always holds.)

4. For $0 < \alpha < 1$, define $C_\alpha := \Gamma(\dfrac{\alpha}{2})$, where $\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\,dt$ is the gamma function. Show that

$$C_\alpha\mathcal{F}^{-1}[\pi^{\frac{1}{2}}2^\alpha|\xi|^{-\alpha}\widehat{f}](x) = C_{1-\alpha}\int_{\mathbb{R}} \frac{f(y)}{|x-y|^{1-\alpha}}\,dy.$$

(You may assume the Fubini's Theorem always holds.)

Remark : If we consider in n dimension, one can show that for $0 < \alpha < n$,

$$C_\alpha\mathcal{F}^{-1}[\pi^{\frac{n}{2}}2^\alpha|\xi|^{-\alpha}\widehat{f}](x) = C_{n-\alpha}\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy.$$

Definition: A family of kernels $\{K_n(x)\}_{n=1}^{\infty}$ on $[-\pi,\pi]$ is said to be a family of **good kernels** (or **summability kernels**) if it satisfies the following properties:

(a) For all $n \geq 1$,

$$\frac{1}{2\pi}\int_{\pi}^{\pi} K_n(x)\,dx = 1.$$

(b) There exists $M > 0$ such that for all $n \geq 1$,

$$\int_{\pi}^{\pi} |K_n(x)|\, dx \leq M.$$

(c) For every $\delta > 0$,

$$\int_{\delta \leq |x| \leq \pi} |K_n(x)|\, dx \to 0, \quad \text{as } n \to \infty.$$

5. Determine whether the Dirichlet kernel $D_N(x) = \sum_{n=-N}^{N} e^{-inx} = \dfrac{\sin(N+\frac{1}{2})x}{\sin(\frac{x}{2})}$ is a good kernel?

6. Determine whether the Fejér kernel $F_N(x) = \dfrac{1}{N}\sum_{n=0}^{N-1} D_n(x) = \dfrac{1}{N}\dfrac{\sin^2(\frac{Nx}{2})}{\sin^2(\frac{x}{2})}$ is a good kernel?

7. The **Poisson kernel** is given by $P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$, $-\pi \leq \theta \leq \pi$. Show that if $0 \leq r < 1$, then $P_r(\theta) = \dfrac{1-r^2}{1-2r\cos\theta + r^2}$.

8. If $0 \leq r < 1$, Determine whether the Poisson kernel kernel is a good kernel?