**6a.** Consider the alternating series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$

$:= \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(a_{2n-1} - a_{2n}) \boxed{\neq} \sum_{n=1}^{\infty} a_{2n-1} - \sum_{n=1}^{\infty} a_{2n}$. Explain!

$$\sum_{n=1}^{\infty} a_{2n-1} = 1 + \frac{1}{3} + \frac{1}{5} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2n-1} = \infty$$

$$\sum_{n=1}^{\infty} a_{2n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2n} = \infty$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \ln 2$$

$$\therefore \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \neq \sum_{n=1}^{\infty} a_{2n-1} - \sum_{n=1}^{\infty} a_{2n} .$$

**6b.** Let $\sum_{n=1}^{\infty} a_n$ conditionally converge to $S$. $\sum_{a_n>0} a_n = \boxed{\infty}$ and $\sum_{a_n<0} a_n = \boxed{-\infty}$. Explain!

**7.** Suppose that $f(x) = \dfrac{4}{x^2+2x+2}$, $g(x) = 2e^{-x^2}$, and $h(x) := f(x) + g(x)$. Prove that

(**a**) the function $f(x)$ is uniformly continuous on $\mathbb{R}$.

$\because$ $f$ is continuous on $\mathbb{R}$ and differentiable on $\mathbb{R}$.

Let $x, y \in \mathbb{R}$, by MVT, $\exists\, c \in (x,y)$ s.t. $|f(x) - f(y)| = |f'(c)| \cdot |x-y|$.

$\left( |f'(c)| = \left| \dfrac{-8c-8}{c^2+2c+2} \right| \le \left| \dfrac{-8c-8}{2c+2} \right| = 4 \right)$ $\quad \le 4 \cdot |x-y| < 4\delta = \varepsilon$.

$\forall \varepsilon > 0$, take $\delta = \frac{\varepsilon}{4}$ s.t. $|x-y| < \delta$, $\forall x, y \in \mathbb{R}$

$\Rightarrow |f(x) - f(y)| < \varepsilon$. $\Rightarrow f(x)$ is uniformly continuous on $\mathbb{R}$.

(**b**) the Gaussian $g(x)$ is uniformly continuous on $\mathbb{R}$.

$\because$ $g$ is continuous and differentiable on $\mathbb{R}$.

Let $x, y \in \mathbb{R}$, by MVT, $\exists\, c \in (x,y)$ s.t. $|g(x) - g(y)| = |g'(c)| \cdot |x-y|$

$\left( |g'(c)| = |-4ce^{-c^2}| \le |2e^{-\frac{1}{4}}| = 2e^{-\frac{1}{4}} \right)$ $\quad \le 2e^{-\frac{1}{4}} \cdot \delta = \varepsilon$.

$\forall \varepsilon > 0$, take $\delta = \frac{1}{2} e^{\frac{1}{4}} \cdot \varepsilon$ s.t. $|x-y| < \delta$, $\forall x, y \in \mathbb{R}$

$\Rightarrow |g(x) - g(y)| < \varepsilon$. $\Rightarrow g(x)$ is uniformly continuous on $\mathbb{R}$.

(**c**) the function $h(x)$ is uniformly continuous on $\mathbb{R}$.

**8.** Let $\langle X, d_X\rangle$ be a metric space. Let $A \subset X$. Define the boundary of $A$ as $\partial A := \overline{A} \cap \overline{X - A}$. Show that

(a) $A^\circ = A - \partial A$.

$A^\circ \subseteq A$ since the interior is always contained within the set itself.

$A^\circ \cap \partial A = \phi$ since the interior and the boundary are disjoint sets.

Therefore, $A^\circ \subseteq A - \partial A$.

$A - \partial A \subseteq A$ since removing the boundary does not go beyond the original set.

$A - \partial A$ is open since removing the boundary points ensures that every point in $A - \partial A$ has an open ball around it contained entirely within $A - \partial A$. Therefore, $A - \partial A \subseteq A^\circ$ $\therefore A^\circ = A - \partial A$ $\square$

(b) $\overline{A} = A \cup \partial A$.

$A \subseteq \overline{A}$ since $\overline{A}$ contains the set $A$ itself.

$\partial A \subseteq \overline{A}$ since $\overline{A}$ includes the boundary points.

Therefore, $A \cup \partial A \subseteq \overline{A}$

$A \subseteq \overline{A}$. And $\overline{A} - A \subseteq \partial A$ since $\overline{A}$ contains the boundary points, and any point in $\overline{A}$ but not in $A$ must be a boundary point.

Therefore, $\overline{A} \subseteq A \cup \partial A$.

$\therefore \overline{A} = A \cup \partial A$ $\square$

**9a.** Prove that $f(x) = \sin\frac{1}{x}$ for $x \neq 0$; 0 for $x = 0$ is not continuous at $x = 0$ and continuous on $\{x \neq 0\}$.

**9b.** Prove that $g(x) = x\sin\frac{1}{x}$ for $x \neq 0$; 0 for $x = 0$ is continuous on $\mathbb{R}$. uniformly continuous on $\mathbb{R}$?

$\because$ $g(x) = x \sin\frac{1}{x}$ is continuous on $\mathbb{R} - \{0\}$.

$\therefore$ Need to show that $g$ is continuous at $0$.

$\left| x \cdot \sin\frac{1}{x} - 0 \right| = \left| x \sin\frac{1}{x} \right| \leq |x| = |x - 0| < \delta = \varepsilon$.

$\forall \varepsilon > 0$, take $\delta = \varepsilon$ s.t. $|x - 0| < \delta$, then $\left| x \sin\frac{1}{x} - 0 \right| < \varepsilon$

$\Rightarrow$ $g(x)$ is continuous at $0$ $\Rightarrow$ $g$ is continuous on $\mathbb{R}$.

Partition $\mathbb{R} = [-1,1] \cup [-1,1]^c$

$[-1,1]$: $\because$ $g$ is continuous on $\mathbb{R}$ $\Rightarrow$ $g$ is continuous on $[-1,1]$

$\Rightarrow$ $g$ is uniformly continuous on $[-1,1]$.

$[-1,1]^c$: $g'(x) = \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}$

$\Rightarrow |g'(x)| = \left| \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x} \right| \leq \left| \sin\frac{1}{x} \right| + \left| \frac{1}{x}\cos\frac{1}{x} \right| \leq 2, \forall x \in [-1,1]^c$

(9b). $\because \forall x, y \in [-1,1]^c$, by MVT $\Rightarrow |g(x) - g(y)| \le 2|x-y| < 2\delta = \varepsilon$.

$\forall \varepsilon > 0$, take $\delta = \frac{\varepsilon}{2}$ s.t. $|x-y| < \delta$ $\forall x, y \in [-1,1]^c$.

$\Rightarrow |g(x) - g(y)| < \varepsilon$. $\Rightarrow$ $g(x)$ is uniformly continuous on $\mathbb{R}$.

(9c). Partion $\mathbb{R} = [-1,1] \cup [-1,1]^c$

$[-1,1]$: $\because$ $h$ is continuous on $\mathbb{R} \Rightarrow h$ is continuous on $[-1,1]$

$\Rightarrow$ $h$ is uniformly continuous on $[-1,1]$.

$[-1,1]^c$ : $h'(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x}$

$\Rightarrow |h'(x)| = |2x \sin\frac{1}{x} - \cos\frac{1}{x}| \le 2|x|\,|\sin\frac{1}{x}| + |\cos\frac{1}{x}| \le 3$, $\forall x \in [-1,1]^c$

Note that for $0 \le y \le 1 \Rightarrow \left|\frac{\sin y}{y}\right| \le 1$.

let $y = \frac{1}{x} \Rightarrow x \ge 1 \Rightarrow |x \sin\frac{1}{x}| \le 1$

let $x, y \in [-1,1]^c$, by MVT $\Rightarrow |h(x) - h(y)| \le 3 \cdot |x-y| < 3\delta = \varepsilon$

$\forall \varepsilon > 0$, take $\delta = \frac{\varepsilon}{3}$ s.t. $|x-y| < \delta$, $\forall x, y \in [-1,1]^c$.

$\Rightarrow |h(x) - h(y)| < \varepsilon$ $\Rightarrow$ $h(x)$ is uniformly continuous on $\mathbb{R}$.

**9c.** Prove that $h(x) = x^2 \sin \frac{1}{x}$ for $x \neq 0$; 0 for $x = 0$ is continuous on $\mathbb{R}$. uniformly continuous on $\mathbb{R}$?

$\because h(x) = x^2 \sin \frac{1}{x}$ is continuous on $\mathbb{R} - \{0\}$

$\therefore$ Need to show that $h$ is continuous at $0$.

$$\left| x^2 \sin \frac{1}{x} - 0 \right| = \left| x^2 \sin \frac{1}{x} \right| \le |x|^2 = |x-0| < \delta^2 = \varepsilon.$$

$\forall \varepsilon > 0$, take $\delta = \sqrt{\varepsilon}$ s.t. $|x-0| < \delta$, then $\left| x^2 \sin \frac{1}{x} - 0 \right| < \varepsilon$

$\Rightarrow h(x)$ is continuous at $0$

$\Rightarrow h(x)$ is continuous on $\mathbb{R}$.

**9d.** Compute $f'(x) := \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$, $g'(x) := \lim_{t \to x} \frac{g(t) - g(x)}{t - x}$, and $h'(x) := \lim_{t \to x} \frac{h(t) - h(x)}{t - x}$.

Also graph the functions $f$, $g$, $h$, $f'$, $g'$, and $h'$.