1.5 Exercises

For question 1, recall that by class equation, p-group can not have trivial center, and recall that G/N is abelian if and only if $[G,G] \leq N$.

Question 1

Show that

(i) If $H/Z(H)$ is cyclic, then H is abelian.

(ii) If H is of order p^2, then H is abelian.

From now on, suppose G is non-abelian with order p^3.

(iii) $|Z(G)| = p$.

(iv) $Z(G) = [G,G]$.

Proof. Let $a, b \in H$ and $H/Z(H) = \langle hZ \rangle$. Write $a = h^n z_1$ and $b = h^m z_2$. Because $z_1, z_2 \in Z(H)$, we may compute:

$$ab = h^n z_1 h^m z_2 = h^{n+m} z_1 z_2 = ba$$

as desired.

Let $|H| = p^2$. Because H is a p-group, we know $Z(H)$ is nontrivial, therefore either $|Z(H)| = p$ or $|Z(H)| = p^2$. To see the former is impossible, just observe that if so, then $|H/Z(H)| = p$, which implies $H/Z(H)$ is cyclic, which by part (i) implies $Z(H) = H$.

Because G is non-abelian, we know $|Z(G)| \neq p^3$. Because G is a p-group, we know $|Z(G)| \neq 1$. Therefore, either $|Z(G)| = p$ or $|Z(G)| = p^2$. Part (i) tell us that $|Z(G)| \neq p^2$, otherwise G is abelian, a contradiction. We have shown $|Z(G)| = p$, as desired.

We now prove $Z(G) = [G,G]$. Because $|Z(G)| = p$, by part (ii) we know $G/Z(G)$ is abelian. This implies $[G,G] \leq Z(G)$, which implies $[G,G]$ is either trivial or equal to $Z(G)$. Because G is non-abelian, we know $[G,G]$ can not be trivial. This implies $Z(G) = [G,G]$, as desired. ■

Question 2

(i) Let M, N be two normal subgroups of G with $MN = G$. Prove that

$$G/(M \cap N) \cong (G/M) \times (G/N)$$

(ii) Let H, K be two distinct subgroups of G of index 2. Prove that $H \cap K$ is a normal subgroup with index 4 and $G/(H \cap K)$ is not cyclic.

Proof. The map $G/(M \cap N) \to (G/M) \times (G/N)$ defined by

$$g(M \cap N) \mapsto (gM, gN) \tag{1.1}$$

is clearly a well-defined group homomorphism, since if $gM = hM$ and $gN = hN$, then $gh^{-1} \in M$ and $gh^{-1} \in N$, which implies $gh^{-1} \in M \cap N$, which implies $g(M \cap N) = h(M \cap N)$. Let $gM = M$ and $gN = N$. Then $g \in M \cap N$ and $g(M \cap N) = M \cap N$. Therefore map 1.1 is also injective. It remains to show map 1.1 is surjective. Fix $g, h \in G$. Write $g = mn$ and $h = \widetilde{m}\widetilde{n}$. Clearly $gM = nM = \widetilde{m}nM$ and $hN = \widetilde{m}N = \widetilde{m}nN$. This implies that mapping 1.1 maps $\widetilde{m}n$ to (gM, hN), as desired.

Because H, K are both of index 2 in G, we know they are both normal in G. This by second isomorphism theorem implies HK forms a subgroup of G. Because $H \neq K$, we know HK properly contains H, which by finiteness of G implies the index of HK is strictly less than H, i.e., $HK = G$. Note that $H \cap K$ is normal since it is the intersection of normal subgroups. By part (i), we now have $G/(H \cap K) \cong (G/H) \times (G/K) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, which shows that $H \cap K$ has index 4 and $G/(H \cap K)$ is cyclic. ∎

Question 3

Let G be a group of order pq, where $p > q$ are prime.

(i) Show that there exists a unique subgroup of order p.

(ii) Suppose $a \in G$ with $o(a) = p$. Show that $\langle a \rangle \subseteq G$ is normal and for all $x \in G$, we have $x^{-1}ax = a^i$ for some $0 < i < p$.

Proof. The third Sylow theorem stated that the number n_p of Sylow p-subgroups satisfies

$$n_p \equiv 1 \pmod{p} \quad \text{and} \quad n_p \mid q$$

Because $p > q$, together they implies $n_p = 1$. Since Sylow p-subgroups of G are exactly subgroups of order p, we have proved (i).

The third Sylow theorem also stated that $n_p = |G : N_G(P)|$ for any Sylow p-subgroup $P \leq G$. Therefore, $N_G(\langle a \rangle) = G$, i.e., $\langle a \rangle$ is normal in G. Fix $x \in G$. It remains to prove $xax^{-1} \neq e$, which is a consequence of the fact that conjugacy (automorphism) preserves order. ∎

Question 4

Let H, K be two subgroups of G of coprime finite indices m, n. Show that

$$\operatorname{lcm}(m,n) \leq |G : H \cap K| \leq mn$$

Proof. Let $\Omega_{H\cap K}, \Omega$, and Ω_K respectively denote the set of left cosets of $H \cap K, H$, and K. The map $\Omega_{H\cap K} \to \Omega_H \times \Omega_K$ defined by

$$g(H\cap K) \mapsto (gH, gK) \tag{1.2}$$

is well defined since

$$g(H\cap K) = l(H\cap K) \implies g^{-1}l \in H\cap K \implies gH = lH \text{ and } gK = lK$$

such set map is injective since if $gH = lH$ and $gK = lK$, then $g^{-1}l \in H$ and $g^{-1}l \in K$, which implies $g(H\cap K) = l(H\cap K)$, as desired. From the injectivity of map 1.2, we have shown index of $H \cap K$ indeed have upper bound mn.

Because

$$|G : H\cap K| = |G : H| \cdot |H : H \cap K| = |G : K| \cdot |K : H \cap K|$$

we know both n and m divides $|G : H \cap K|$, which gives the desired lower bound $\operatorname{lcm}(m, n)$. ■

Question 5

(i) Let G be a group, $H \leq G$, and $x \in G$ of finite order. Prove that if k is the smallest natural number that makes $x^k \in H$, then $k \mid o(x)$.

(ii) Let G be a group and N a normal subgroup of G. Prove that

$$o(gN) = \inf\left\{k \in \mathbb{N} : g^k \in N\right\}, \quad \text{where } \inf \varnothing = \infty$$

(iii) Let G be a finite group, H, N two subgroups of G with N normal. Show that if $o(H)$ and $|G : N|$ are coprime, then $H \leq N$.

Proof. (i): Let $a = qk + r \in \mathbb{N}$ with $0 \leq r < k$. If $x^a \in H$, then $x^r = x^a \cdot (x^k)^{-q} \in H$, which implies $r = 0$. We have shown that k divides all natural numbers a that makes $x^a \in H$, which includes $o(x)$.

(ii): This is a simple observation that $(gN)^k = g^k N \in N \iff g^k \in N$.

(iii): By second isomorphism theorem, we know $|HN : N| = |H : H \cap N|$ which divides both $o(H)$ and $|G : N|$. This by coprimality implies $|H : H \cap N| = 1$, which shows that $H \leq N$. ■

Question 6

Let G be a finite group with Sylow p-subgroup P and normal subgroup N. Show that $P \cap N$ forms a Sylow p-subgroup of N, and use such to deduce N have index $p^{\nu_p(o(PN))-\nu_p(o(N))}$ in PN.

Proof. By second isomorphism theorem, we have

$$o(PN) \cdot o(P \cap N) = o(P) \cdot o(N)$$

Because P is Sylow with $P \subseteq PN$, we know

$$\nu_p(o(PN)) = \nu_p(o(P))$$

This shows that, indeed, $P \cap N$ forms a Sylow p-subgroup of N:

$$\nu_p(o(P \cap N)) = \nu_p(o(N))$$

as desired. Because $P \cap N \leq P$ and because P is Sylow, we know $o(P \cap N)$ is a power of p. It then follows that:

$$|PN : N| = \frac{o(PN)}{o(N)} = \frac{o(P)}{o(P \cap N)} = p^{\nu_p(o(P))-\nu_p(o(P\cap N))} = p^{\nu_p(o(PN))-\nu_p(o(P))}$$

■

Question 7

Prove that if H is a Hall subgroup of G and $N \trianglelefteq G$, then $H \cap N$ is a Hall subgroup of N and HN/N is a Hall subgroup of G/N.

Proof. The facts that:

(i) By second isomorphism theorem, we have $|N : H \cap N| = |HN : H|$, which divides $|G : H|$.

(ii) $o(H \cap N) \mid o(H)$.

(iii) $o(H)$ and $|G : H|$ are coprime.

implies $o(H \cap N)$ and $|N : H \cap N|$ is coprime, i.e., $H \cap N$ is Hall in N.

The facts that:

(i) $o(HN/N) = \frac{o(HN)}{o(N)} = \frac{o(H)}{o(H \cap N)}$ divides $o(H)$. (second isomorphism theorem)

(ii) $|(G/N) : (HN/N)| = |G : HN|$ divides $|G : H|$.

(iii) $o(H)$ and $|G : H|$ are coprime.

implies $o(HN/N)$ and $|(G/N) : (HN/N)|$ are coprime, i.e., HN/N is Hall in G/N.

■