§6. Almost complex manifolds

Def: Let X be a smooth manifold. An almost complex structure on X is a bundle map $J: TX \longrightarrow TX$ s.t. $J^2 = -\mathrm{Id}_{TX}$.

Rmk: If X carries an almost complex structure, then $\dim_{\mathbb{R}}(X)$ is even.

Prop: Any complex manifold X induces an almost complex structure on the underlying smooth manifold.

Pf: Let $z_i = x_i + \sqrt{-1}\, y_i$ be complex coordinates. Define

$$J\left(\frac{\partial}{\partial x_i}\right) := \frac{\partial}{\partial y_i}$$

$$J\left(\frac{\partial}{\partial y_i}\right) := -\frac{\partial}{\partial x_i}$$

We only need to check that J is well-defined.

Suppose $\tilde{z}_i = x_i + \sqrt{-1}\, y_i$ are another set of complex coordinates.

Then

$$\frac{\partial}{\partial \tilde{x}_j} = \sum_{i=1}^{n} \frac{\partial x_i}{\partial \tilde{x}_j} \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} \frac{\partial y_i}{\partial \tilde{x}_j} \frac{\partial}{\partial y_i}$$

$$\frac{\partial}{\partial \tilde{y}_j} = \sum_{i=1}^{n} \frac{\partial y_i}{\partial \tilde{y}_j} \frac{\partial}{\partial y_i} + \sum_{i=1}^{n} \frac{\partial x_i}{\partial \tilde{y}_j} \frac{\partial}{\partial x_i}$$

and so

$$J\left(\frac{\partial}{\partial \tilde{x}_j}\right) = \sum_{i=1}^{n} \frac{\partial x_i}{\partial \tilde{x}_j} \frac{\partial}{\partial y_i} - \sum_{i=1}^{n} \frac{\partial y_i}{\partial \tilde{x}_j} \frac{\partial}{\partial x_i}$$

$\square$

$$= \sum_{i=1}^{n} \frac{\partial y_i}{\partial \tilde{y}_j} \frac{\partial}{\partial y_i} + \sum_{i=1}^{n} \frac{\partial x_i}{\partial \tilde{y}_j} \frac{\partial}{\partial y_i}$$

$$= \frac{\partial}{\partial \tilde{y}_j}$$

by the Cauchy-Riemann equations. Similarly, we have

$$J\left(\frac{\partial}{\partial \tilde{y}_j}\right) = -\frac{\partial}{\partial \tilde{x}_j}$$

Hence J is well-defined and by construction, $J^2 = -Id$. $\square$

Given (X, J), we decompose $T_{\mathbb{C}}X := TX \otimes \mathbb{C}$ as

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$$

as direct sum of $\pm\sqrt{-1}$-eigenspaces of J.

As in the linear case, we define

$$\Lambda^k_{\mathbb{C}} X := \Lambda^k (T_{\mathbb{C}} X)^*$$

$$\Lambda^{p,q} X := \Lambda^p (T^{1,0}X)^* \otimes \Lambda^q (T^{0,1}X)^*$$

which are complex vector bundles on X. Denote

$$A^k_{\mathbb{C}}(X) := \{\alpha : X \xrightarrow{C^\infty} \Lambda^k_{\mathbb{C}} X : \pi \circ \alpha = id_X\}$$

$$A^{p,q}(X) := \{\alpha : X \xrightarrow{C^\infty} \Lambda^{p,q}_{\mathbb{C}} X : \pi \circ \alpha = id_X\}$$

the space of smooth complex k-forms and (p,q)-forms on X.

<u>Prop</u>: We have

$$\Lambda^k_{\mathbb{C}} X = \bigoplus_{p+q=k} \Lambda^{p,q} X \qquad \text{and} \qquad A^k_{\mathbb{C}}(X) = \bigoplus_{p+q=k} A^{p,q}(X)$$

$$\overline{\Lambda^{p,q} X} = \Lambda^{q,p} X \qquad\qquad \overline{A^{p,q}(X)} = A^{q,p}(X)$$

<u>Pf</u> : Trivial. $\square$

Let $\pi^{p,q} : A^{p+q}_{\mathbb{C}}(X) \longrightarrow A^{p,q}(X)$ be the projection and define

$$\partial := \pi^{p+1,q} \circ d : A^{p,q}(X) \longrightarrow A^{p+1,q}(X)$$

$$\bar{\partial} := \pi^{p,q+1} \circ d : A^{p,q}(X) \longrightarrow A^{p,q+1}(X)$$

We continue to have

$$\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge \partial\beta \ , \ \bar{\partial}(\alpha \wedge \beta) = \bar{\partial}\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge \bar{\partial}\beta$$

<u>Prop</u> : Let (X, J) be a complex manifold. Then

1. $d = \partial + \bar{\partial}$
2. $\partial^2 = \bar{\partial}^2 = 0$

<u>Pf</u> : As we have local complex coordinates, the proof follows from local theory. $\square$

However, for a general almost complex manifold (X, J), it's not necessarily that $d = \partial + \bar{\partial}$ on $A^k_{\mathbb{C}}(X)$ when $k \geq 1$

$$\begin{tikzcd}
 & A^{2,0}(X) \\
d : A^{1,0}(X) \arrow[ur, "\partial"] \arrow[r, "\bar{\partial}"] \arrow[dr, "N"'] & A^{1,1}(X) \\
 & A^{0,2}(X)
\end{tikzcd}$$

<u>Def</u>: An almost complex structure J on X is said to be integrable if $[T^{0,1}, T^{0,1}] \subset T^{0,1}X$, where $[-,-]$ denotes the Lie bracket on $T_{\mathbb{C}}X$.

<u>Prop</u>: The followings are equivalent:

① J is integrable.

② $N = 0$

③ $d = \partial + \bar{\partial}$ on $A^k_{\mathbb{C}}(X)$

④ $\bar{\partial}^2 = 0$

<u>Pf</u>: We prove that ① ⇒ ② ⇒ ③ ⇒ ④ ⇒ ①

① ⇒ ②: $\forall u, v \in T^{0,1}X$ and $\alpha \in A^{1,0}(X)$

$$N(u,v)\alpha = d\alpha(u,v) = u\,\alpha(v) - v\,\alpha(u) - \alpha([u,v]) = 0$$

since, $u, v, [u,v] \in T^{0,1}X$.

② ⇒ ③: It is trivial for $\alpha \in A^1_{\mathbb{C}}(X)$. For general $\alpha \in A^k_{\mathbb{C}}(X)$ write α as

$$\alpha = \sum_{p+q=k} \alpha_{i_1 \cdots i_p j_1 \cdots j_q} w_{i_1} \wedge \cdots \wedge w_{i_p} \otimes \bar{w}_{j_1} \wedge \cdots \wedge \bar{w}_{j_q}$$

Then

$$\begin{aligned} d\alpha = & \sum d\alpha_{i_1 \cdots i_p j_1 \cdots j_q} w_{i_1} \wedge \cdots \wedge w_{i_p} \otimes \bar{w}_{j_1} \wedge \cdots \wedge \bar{w}_{j_q} \\ & + \sum (-1)^{\ell} \alpha_{i_1 \cdots i_p j_1 \cdots j_q} w_{i_1} \wedge \cdots \wedge dw_{i_\ell} \wedge \cdots \wedge w_{i_p} \otimes \bar{w}_{j_1} \wedge \cdots \wedge \bar{w}_{j_q} \\ & + \sum (-1)^{p+\ell} \alpha_{i_1 \cdots i_p j_1 \cdots j_q} w_{i_1} \wedge \cdots \wedge w_{i_p} \otimes \bar{w}_{j_1} \wedge \cdots \wedge d\bar{w}_{j_\ell} \wedge \cdots \wedge \bar{w}_{j_q} \end{aligned}$$

Since $w_i \in A^{1,0}(X)$ and $\bar{w}_j \in A^{0,1}(X)$, we have

$$dw_i = \partial w_i + \bar{\partial} w_i, \quad d\bar{w}_j = \partial \bar{w}_j + \bar{\partial} \bar{w}_j$$

Moreover, $\alpha_{i_1\cdots i_p j_1\cdots j_q}$ is just a function, so we also have

$$d\alpha_{i_1\cdots i_p j_1\cdots j_q} = \partial\alpha_{i_1\cdots i_p j_1\cdots j_q} + \bar\partial\alpha_{i_1\cdots i_p j_1\cdots j_q}$$

As a whole, we have $d\alpha = \partial\alpha + \bar\partial\alpha$.

③ ⇒ ④ : It follows from $d^2 = 0$.

④ ⇒ ① : Given $\alpha \in A^{0,1}(X)$ and $u, v \in T^{0,1}X$, we have

$$d\alpha(u,v) = u\alpha(v) - v\alpha(u) - \alpha([u,v]) = \bar\partial\alpha(u,v)$$

Lets take $\alpha = \bar\partial f$, for $f \in A^0_{\mathbb{C}}(X)$:

$$\begin{aligned} 0 = \bar\partial^2 f(u,v) &= u\bar\partial f(v) - v\bar\partial f(u) - \bar\partial f([u,v]) \\ &= u(vf) - v(uf) - \bar\partial f([u,v]) \\ &= df([u,v]) - \bar\partial f([u,v]) = \partial f([u,v]) \end{aligned}$$

As ∂f generates $A^{1,0}(X)$ locally, we must have $[u,v] \in T^{0,1}X$. □

Thm: [Newlander - Nierenberg]

An almost complex structure is integrable if and only if it is induced by complex coordinates.

Since $\partial^2 = \bar\partial^2 = 0$, we can define the cohomology groups

$$H^{p,q}_{\partial}(X) := \frac{\ker(\partial : A^{p,q}(X) \longrightarrow A^{p+1,q}(X))}{\operatorname{Im}(\partial : A^{p-1,q}(X) \longrightarrow A^{p,q}(X))}$$

$$H^{p,q}_{\bar\partial}(X) := \frac{\ker(\bar\partial_E : A^{p,q}(X) \longrightarrow A^{p,q+1}(X))}{\operatorname{Im}(\bar\partial_E : A^{p,q-1}(X) \longrightarrow A^{p,q}(X))}$$

called the Hodge cohomology groups.

A version of the Newlander-Nierenberg theorem holds for complex vector bundles. To state the theorem, for a complex vector bundle $E \longrightarrow X$, we write $A^k(E)$ and $A^{p,q}(E)$ for the space of smooth sections of $\Lambda^k_{\mathbb{C}} X \otimes E$ and $\Lambda^{p,q} X \otimes E$, respectively.

<u>Thm</u>: Let X be complex manifold. A complex vector bundle $E \to X$ carries a holomorphic structure if and only if there exists $\bar{\partial}_E : A^{p,q}(E) \longrightarrow A^{p,q+1}(E)$ s.t.

$$\bar{\partial}_E(\alpha \otimes s) = \bar{\partial}\alpha \otimes s + (-1)^{p+q} \alpha \wedge \bar{\partial}_E s$$

$\forall \alpha \in A^{p,q}(X)$, $s \in A^0(E)$, and $\bar{\partial}_E^2 = 0$.

<u>Pf</u>: We only prove the existence of $\bar{\partial}_E$ for a holomorphic vector bundle $E \to X$. Let ψ_i be a local trivialization of E and (ψ_{ij}) be the corresponding cocycle. For a smooth section $s \in A^0(E)$, write

$$s|_{U_i} = \sum_{\alpha=1}^{r} s_i^{(\alpha)} \psi_i^{-1}(1_\alpha)$$

for some smooth $s_i^{(\alpha)} : U_i \to \mathbb{C}$. Define

$$\bar{\partial}_E s|_{U_i} = \sum_{\alpha=1}^{r} \bar{\partial} s_i^{(\alpha)} \otimes \psi_i^{-1}(1_\alpha)$$

We prove that $\bar{\partial}_E$ is well-defined. Recall that

$$s_i^{(\alpha)} = \sum_{\beta=1}^{r} s_j^{(\beta)} \psi_{ij}^{(\beta\alpha)}$$

Then

$$\begin{aligned}
\bar{\partial}_E s|_{U_i} &= \sum_{\alpha=1}^{r} \bar{\partial} s_i^{(\alpha)} \otimes \psi_i^{-1}(1_\alpha) \\
&= \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} \bar{\partial}(s_j^{(\beta)} \psi_{ij}^{(\beta\alpha)}) \otimes \psi_i^{-1}(1_\alpha)
\end{aligned}$$

$$= \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} \overline{\partial} s_j^{(\beta)} \psi_{ij}^{(\beta\alpha)} \otimes \psi_i^{-1}(1_\alpha) \quad (\because \psi_{ij}^{(\alpha\beta)} \text{ is holomorphic})$$

$$= \sum_{\beta=1}^{r} \overline{\partial} s_j^{(\beta)} \otimes \psi_i^{-1}(1_\beta)$$

$$= \overline{\partial}_E s|_{U_j}$$

The Leibniz's rule and $\overline{\partial}_E^2 = 0$ follow from the one for $\overline{\partial}$. □

Def: We call $\overline{\partial}_E : A^{p,q}(E) \longrightarrow A^{p,q+1}(E)$ the Dolbeault operator of the holomorphic vector bundle $E \longrightarrow X$.

The Dolbeault operator induces a cohomology:

$$H^{p,q}(X,E) := \frac{\ker(\overline{\partial}_E : A^{p,q}(E) \longrightarrow A^{p,q+1}(E))}{\operatorname{Im}(\overline{\partial}_E : A^{p,q-1}(E) \longrightarrow A^{p,q}(E))}$$

called the Dolbeault cohomology groups.

Rmk: When $p = q = 0$, we have $H^{0,0}(X,E) = \Gamma(X,E)$.

Rmk: When $E = \mathcal{O}_X$, we have $H^{p,q}(X,E) = H^{p,q}_{\overline{\partial}}(X)$.