Deadline : 2023/10/30, 17:00.

1. Let S_1 and S_2 be two nonempty subsets in a metric space with $S_1 \cap \overline{S_2} = S_2 \cap \overline{S_1} = \emptyset$. If $A \subseteq S_1 \cup S_2$ is a connected set, then either $A \subseteq S_1$ or $A \subseteq S_2$.

2. If A_1 and A_2 are two nonempty and connected sets with $A_1 \cap A_2 \neq \emptyset$. Prove or disprove that

   1. $A_1 \cap A_2$ is connected.
   2. $A_1 \cup A_2$ is connected.

3. Let $\{A_k\}_{k=1}^{\infty}$ be a family of connected subsets of M, and suppose that A is a connected subset of M such that $A_k \cap A \neq \emptyset$ for all $k \in \mathbb{N}$. Show that the union $(\cup_{k\in\mathbb{N}} A_k) \cup A$ is also connected.

4. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence and define $s_n = \frac{1}{n}\sum_{k=1}^{n} a_k$. Prove or disprove that

   1. If a_k converge, then s_n converge.
   2. If s_n converge, then a_k converge.
   3. Let $t_n = \frac{(2n-1)a_1 + (2n-3)a_2 + \cdots + 3a_{n-1} + x_n}{n^2}$. Assume a_k converge to a. Does t_n also converge to a ?

5. If $a_k > 0$ for all $k \in \mathbb{N}$, prove that

$$\liminf_{k\to\infty} \frac{a_{k+1}}{a_k} \le \liminf_{k\to\infty} \sqrt[k]{a_k} \le \limsup_{k\to\infty} \sqrt[k]{a_k} \le \limsup_{k\to\infty} \frac{a_{k+1}}{a_k}$$

   Moreover, find a $\{a_k\}_{k=1}^{\infty}$ such that $\limsup\limits_{k\to\infty} \sqrt[k]{a_k} < \limsup\limits_{k\to\infty} \frac{a_{k+1}}{a_k}$

6. If $s_1 = \sqrt{2}$, and

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \quad (n = 1, 2, 3, \cdots),$$

   prove that s_n converges, and that $s_n < 2$ for $n = 1, 2, 3, \cdots$.

7. Suppose $a_n > 0$ and $s_n = \sum_{k=1}^{n} a_k$. If s_n diverge. Prove or disprove that $t_n = \sum_{k=1}^{n} \frac{a_k}{1+a_k}$ diverges. What can be said about

   1. $S_n = \sum_{k=1}^{n} \frac{a_k}{1+ka_k}$.
   2. $T_n = \sum_{k=1}^{n} \frac{a_k}{1+k^2 a_k}$.
   3. If $s_n = \sum_{k=1}^{n} a_k$ converge. Does $J_n = \sum_{k=1}^{n} k a_k$ converge.

8. Assume $A \subset \mathbb{R}$ is compact and let $a \in A$. Suppose $\{a_n\}$ is a sequence in A such that every convergent sub-sequence of $\{a_n\}$ converges to a.

   1. Does the sequence $\{a_n\}$ also converge to a ?
   2. Without the assumption of A is compact. Does the sequence $\{a_n\}$ converge to a ?

9. Suppose that $a_k \neq 0$ for large k and that

$$p = \lim_{k\to\infty} \frac{\ln(1/|a_k|)}{\ln(k)}$$

exists as an extended real number. If $p > 1$, then $\sum_{k=1}^{\infty}$ converges absolutely. If $p < 1$, then $\sum_{k=1}^{\infty}$ diverges.

10. Suppose that $f : \mathbb{R} \to (0, \infty)$ is differentiable, that $f(x) \to 0$ as $x \to \infty$, and that

$$\alpha \equiv \lim_{x\to\infty} \frac{xf'(x)}{f(x)}$$

exists. If $\alpha < -1$, prove that $\sum_{k=1}^{\infty} f(k)$ converges.

11. Suppose that $\{a_n\}$ is a sequence of nonzero real numbers and that

$$p = \lim_{x\to\infty} k\left(1 - \left|\frac{a_{k+1}}{a_k}\right|\right)$$

exists as an extended real number. Prove that $\sum_{k=1}^{\infty} a_k$ converges absolutely when $p > 1$.

Extra question
(If you finish there problems and want to obtain extra points, please email
symmetrickelly@gmail.com)

12. Please read, state and prove following theorem from William R Wade's "An Introduction to Analysis" P.209 $\sim$ P.211
    1. Abel's Formula
    2. Dirichlet's Test
    3. Leibniz's criterion (Alternating series test)

13. Use Abel's Formula directly prove Leibniz's criterion (Consider S_{2n+1} and S_{2n} and show that they are monotone).

14. Show that $\sum_{k=1}^{\infty} \frac{sin(k)}{k}$ and $\sum_{k=1}^{\infty} \frac{cos(k)}{k}$

15. Understand what is the sequence of function and what is the definition of the sequence of function point-wise converge and uniformly converge.