§4. Complex manifolds

Def: A complex structure on a differentiable manifold M is a collection of atlas $\{\varphi_i : U_i \xrightarrow{\sim} \varphi_i(U) \subset \mathbb{C}^n\}_i$ such that $\forall i, j$.

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \longrightarrow \varphi_i(U_i \cap U_j)$$

is holomorphic. A complex manifold X is a differentiable manifold together with a complex structure.

Def: Let X be a complex manifold. A function $f : X \longrightarrow \mathbb{C}$ is said to be holomorphic if $f \circ \varphi_i^{-1} : \varphi_i(U_i) \subset \mathbb{C}^n \longrightarrow \mathbb{C}$ is holomorphic for any chart $\varphi_i : U_i \longrightarrow \varphi_i(U_i) \subset \mathbb{C}^n$.

For each open $U \subset X$, define

$$\Gamma(U, \mathcal{O}_X) := \{ f : U \longrightarrow \mathbb{C} \mid f \text{ is holomorphic} \}$$

Prop: If X is a compact connected complex manifold, then $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$.

Pf: Let $f \in \Gamma(X, \mathcal{O}_X)$. By compactness, $|f|$ attends its maximum value at some point $p \in X$. Let $\varphi : U \longrightarrow \mathbb{C}^n$ be a chart for X around p. Then $f \circ \varphi^{-1} : \varphi(U) \longrightarrow \mathbb{C}$ is a holomorphic function so that $|f \circ \varphi^{-1}|$ attend its maximum in the open set $\varphi(U)$. By maximal principle, $f \circ \varphi^{-1}$ is constant.

This proves that the set

$$S := \{ x \in X : f(x) = f(p) \}$$

is open and non-empty. Clearly, S is closed. By connectedness, $S = M$ and so f is constant.

Rmk: By Hartogs' theorem, when $\dim_{\mathbb{C}}(X) \geq 2$, we have

$$\Gamma(X \setminus \{x\}, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)$$

Def: Let X, Y be complex manifolds. A continuous map $F : X \to Y$ is holomorphic if for any charts (U, φ) of X, (U', φ') of Y, the map $\varphi' \circ F \circ \varphi^{-1} : \varphi(F^{-1}(U') \cap U) \longrightarrow \varphi'(U')$ is holomorphic. Two complex manifolds X, Y is said to be biholomorphic if there exists a holomorphic homeomorphism $F : X \longrightarrow Y$.

Example: $\mathbb{C}^n$ is of course a complex manifold.

Example: Define an equivalence relation on $\mathbb{C}^{n+1} \setminus \{0\}$ by

$$z_1 \sim z_2 \quad \text{iff} \quad z_2 = \lambda z_1 \text{ for some } \lambda \in \mathbb{C}^\times$$

The quotient

$$\mathbb{P}^n := \frac{\mathbb{C}^{n+1} \setminus \{0\}}{\sim}$$

is called the complex projective space. It can be covered by the following charts:

$$U_i := \{ [z_0 : z_1 : \cdots : z_n] \in \mathbb{P}^n : z_i \neq 0 \} \quad i = 0, 1, \cdots, n$$

Define $\varphi_i : U_i \longrightarrow \mathbb{C}^n$ by

$$[z_0 : z_1 : \cdots : z_n] \longmapsto \left(\frac{z_0}{z_i}, \frac{z_1}{z_i}, \cdots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \cdots, \frac{z_n}{z_i}\right).$$

φ_i has inverse

$$(w_1, \cdots, w_n) \longmapsto [w_1 : \cdots : w_{i-1} : 1 : w_{i+1} : \cdots : w_n] \in U_i$$

For $i<j$, the composition $\varphi_j \circ \varphi_i^{-1}$ is given by

$$(w_1, \cdots, w_n) \longmapsto [w_1 : \cdots : \underset{(i+1)\text{-th}}{1} : \cdots w_n]$$

$$\longmapsto \left(\frac{w_1}{w_j}, \frac{w_2}{w_j}, \cdots, \frac{w_i}{w_j}, \frac{1}{w_j}, \frac{w_{i+1}}{w_j}, \cdots, \frac{w_{j-1}}{w_j}, \frac{w_{j+1}}{w_j}, \cdots, \frac{w_n}{w_j}\right)$$

which is holomorphic. We can identify $\mathbb{P}^n$ with the set of all lines through the origin:

$$\mathbb{P}^n \cong \{\ell \subset \mathbb{C}^{n+1} : \dim_{\mathbb{C}} \ell = 1, 0 \in \ell\}$$

by sending a point $[z_0 : z_1 : \cdots : z_n] \in \mathbb{P}^n$ to the line passing through 0 and $(z_0, z_1, \cdots, z_n)$.

<u>Example</u>: Let $f : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ be holomorphic s.t.

$$\{z \mid df(z) = 0\} \cap f^{-1}(0) = \phi$$

Then by implicit function theorem, $f^{-1}(0)$ is a complex submanifold of $\dim_{\mathbb{C}}$ n.

<u>Example</u>: Let $f : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ be homogeneous polynomial of degree d, i.e.

$$f(\lambda \cdot z) = \lambda^d f(z) \quad \forall \lambda \in \mathbb{C}^\times, z \in \mathbb{C}^{n+1}$$

This gives a well-defined subset

$$V(f) = \{[z] \in \mathbb{P}^n : f(z) = 0\} = f^{-1}(0)/\sim$$

Using the chart $\varphi_i : U_i \longrightarrow \mathbb{C}^n$, we define

$$f_i : (w_1, \cdots, w_n) \longmapsto f(w_1, \cdots, w_i, 1, w_{i+1}, \cdots, w_n)$$

whose zero set is $\varphi_i(V(f))$. If

$$\{z \mid df(z) = 0\} \cap f^{-1}(0) = \phi$$

Then

$$\{w \mid df_i(w) = 0\} \cap f_i^{-1}(0) = \phi$$

Then by implicit function theorem again, $V(f)$ is an $(n-1)$-dim complex submanifold of $\mathbb{P}^n$.

<u>Example</u>: More general, one can consider k holomorphic function

$$f_1, \cdots, f_k : \mathbb{C}^n \longrightarrow \mathbb{C}$$

If $0 \in \mathbb{C}^k$ is a regular value of $F := (f_1, \cdots, f_k)$, then $F^{-1}(0) = Z(f_1) \cap \cdots \cap Z(f_k)$ is an $(n-k)$-dim complex manifold. Similarly, if

$$f_1, \cdots, f_k : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$$

are homogeneous polynomials such that 0 is a regular value of $F = (f_1, \cdots, f_k) : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{C}^k$.

Then $F^{-1}(0)/\mathbb{C}^\times \subset \mathbb{P}^n$ is also a complex manifold of dim $n-k$. These type of complex manifolds are called <u>complete intersections</u>.

Example: Let $\Gamma \subset \mathbb{C}^n$ be a lattice, i.e. a free abelian group of rank $2n$. Then $\Gamma \curvearrowright \mathbb{C}^n$ by translation:

$$a : (z_1, \cdots, z_n) \longmapsto (z_1 + a_1, \cdots, z_n + a_n)$$

Then $X := \mathbb{C}^n/\Gamma$ is a complex manifold, called a complex torus as it is diffeomorphic to $(S^1)^{2n}$.

However, for a general pair of lattices Γ_1, Γ_2, $\mathbb{C}^n/\Gamma_1$ may not be isomorphic to $\mathbb{C}^n/\Gamma_2$ as complex manifolds. For example, when $n=1$, up to a coordinate change, we can write any $\Gamma \subset \mathbb{C}$ as

$$\Gamma_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}, \quad \mathrm{Im}(\tau) > 0.$$

There is a natural $SL_2(\mathbb{Z})$ action on

$$\mathbb{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$$

given by

$$A : \tau \longmapsto \frac{a\tau + b}{c\tau + d} \quad \text{for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Then $\mathbb{C}/\Gamma_\tau \cong \mathbb{C}/\Gamma_{\tau'}$ if and only if $\exists\, A \in SL_2(\mathbb{Z})$ such that $\tau' = A\tau$ (proof omitted).

We usually call $\mathbb{C}/\Gamma$ an elliptic curve.

Example: Fix $\lambda \in (0,1)$ and let $\mathbb{Z} \curvearrowright \mathbb{C}^n \setminus \{0\}$ by

$$k \cdot (z_1, \cdots, z_n) := (\lambda^k z_1, \cdots, \lambda^k z_n)$$

We get a complex manifold $X := (\mathbb{C}^n \setminus \{0\})/\mathbb{Z}$, called the Hopf manifold. It is diffeomorphic to $S^1 \times S^{2n-1}$. Indeed, regrading $S^{2n-1} \subset \mathbb{R}^{2n} \cong \mathbb{C}^n$, we define $S^1 \times S^{2n-1} \longrightarrow X$ by

$$(e^{2\pi\sqrt{-1}t}, z_1, \cdots, z_n) \longmapsto [\lambda^t(z_1, \cdots, z_n)]$$

This map has inverse

$$[(z_1, \cdots, z_n)] \longmapsto \left(e^{2\pi\sqrt{-1}\log_\lambda |z|}, \frac{z_1}{|z|}, \cdots, \frac{z_n}{|z|}\right)$$

where $|z|^2 = \sum_{i=1}^{n} |z_i|^2$.

<u>Example</u>: Let V be an n-dim $\mathbb{C}$-vector space. The (k,n)-Grassmanian is define to be

$$Gr(k,n) := \{W \subset V : \dim_{\mathbb{C}}(W) = k\}$$

Given $W \in Gr(k,n)$, let $W' \subset V$ be s.t. $W \oplus W' = V$.

Define

$$\mathcal{U}(W) := \{U \subset V : U \oplus W' = V\}$$

For each $U \in \mathcal{U}(W)$, let $\pi|_U : U \longrightarrow W$ be the restriction of the projection $\pi : V \longrightarrow W$ to U.

We define

$$\varphi_W : \mathrm{Hom}(W, W') \longrightarrow \mathcal{U}(W)$$
$$\phi \longmapsto \mathrm{Graph}(\phi)$$

<u>Check</u>: If $w + \phi(w) \in W'$, then $w \in W'$, so $w \in W \cap W' = \{0\}$.

Hence $\mathrm{Graph}(\phi) \cap W' = 0$. Moreover, we have

$$\dim_{\mathbb{C}}(\mathrm{Graph}(\phi)) = \dim_{\mathbb{C}}(W) = k,$$

so $(\mathrm{Graph}(\phi)) \oplus W' = V$.

Claim: φ_W is bijective.

Injectivity: $\mathrm{Graph}(\phi) = \mathrm{Graph}(\phi')$

$\Leftrightarrow \quad w + \phi(w) = w' + \phi'(w')$

$\Leftrightarrow \quad w = w'$ and $\phi(w) = \phi(w')$

Surjectivity: $\forall\, U \in \mathcal{U}(W)$, we want find ϕ s.t.

$\forall\, u \in U$, $\exists\, w \in W$ s.t. $w + \phi(w) = u$

Write $u = (w, w') \in W \oplus W'$. We simply define $\phi(w) := w'$. One checks that ϕ is well-defined and linear.

Since $\mathrm{Hom}(W, W')$ is a $\mathbb{C}$-vector space, this gives a chart of $Gr(k,n)$. We leave it as an exercise for the reader to check that

$$\varphi_{W_2}^{-1} \circ \varphi_{W_1} : \varphi_{W_1}^{-1}\big(\mathcal{U}(W_1) \cap \mathcal{U}(W_2)\big) \longrightarrow \varphi_{W_2}\big(\mathcal{U}(W_1) \cap \mathcal{U}(W_2)\big)$$

is holomorphic.

Example: Let V be an n-$\dim_{\mathbb{C}}$ vector space. Given $0 < k_1 < k_2 < \cdots < k_\ell < n$. The set

$$\mathrm{Flag}(k_1, \cdots, k_\ell; n) = \{ W_1 \subset W_2 \subset \cdots \subset W_\ell \subset V \mid \dim_{\mathbb{C}}(W_i) = k_i \}$$

carries a complex manifold structure. Such complex manifold is called a flag variety. The special case

$$\mathrm{Flag}(1, \cdots, n-1; n)$$

is called a complete flag variety. Note that

$$\mathrm{Flag}(1;n) = \mathbb{P}^n$$

$$\mathrm{Flag}(k;n) = \mathrm{Gr}(k,n).$$

Def: Let X be a complex manifold of $\dim_{\mathbb{C}} n$ and $Y \subset X$ be a smooth submanifold of X of $\dim_{\mathbb{R}} 2k$. We say Y is a complex submanifold of X if there a holomorphic atlas $\{(U_i, \varphi_i)\}$ of X such that $\varphi_i|_{U_i \cap Y} : U_i \cap Y \xrightarrow{\sim} \varphi_i(U_i) \cap \mathbb{C}^k$.

Def: A complex manifold is said be projective if it is a complex submanifold of $\mathbb{P}^n$ for some n.

Since a compact complex manifold supports no non-constant holomorphic functions, we obtain the following

Prop: $\mathbb{C}^n$ contains no compact complex manifolds of positive dimension. □

For submanifolds in $\mathbb{P}^n$, we have the following

Thm: [Chow]

If X is a complex submanifold of $\mathbb{P}^n$, then there exists homogeneous polynomials $f_1, \cdots, f_k : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ such that $X = V(f_1) \cap \cdots \cap V(f_k)$. □