HWs

Eric Liu

Chapter 1

Differential Geometry HW

1.1 HW 3

Question 1

Let V be a finite dimensional vector space over $\mathbb{R}$. Show that for

$$\dim(V) < 4$$

Every non-zero element of $\bigwedge^2(V)$ can be expressed as a wedge product of two vectors in V. Give an example to show that this is not true if $\dim(V) = 4$.

Theorem 1.1.1. (Case of Zero and One Dimension) If

$$\dim(V) \leq 1$$

Then every non-zero element of $\bigwedge^2(V)$ can be expressed as a wedge product of two vectors in V.

Proof. Recall

$$\dim\left(\bigwedge^2(V)\right) = \binom{\dim(V)}{2} = 0$$

This implies $\bigwedge^2(V) = 0$. There does not exists non-zero element of $\bigwedge^2(V)$, rendering the proposition viciously true. ■

Theorem 1.1.2. (Case of Two Dimension) If

$$\dim(V) = 2$$

Then every non-zero element of $\bigwedge^2(V)$ can be expressed as a wedge product of two vectors in V.

Proof. Let $\{e_1, e_2\}$ be a basis for V. We have

$$\bigwedge^2(V) = \text{span}\{e_1 \wedge e_2\}$$

Therefore, for all $\omega \in \bigwedge^2(V)$, we have

$$\omega = c(e_1 \wedge e_2) = (ce_1) \wedge e_2 \text{ for some } c \in \mathbb{R}$$

■

Theorem 1.1.3. (Case of Three Dimensions) If

$$\{e_1, e_2, e_3\} \text{ is a basis for } V$$

Then every non-zero element of $\bigwedge^2(V)$ can be expressed as a wedge product of two vectors in V.

Proof. We know $\bigwedge^2(V)$ have the following basis

$$\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$$

Therefore, for arbitrary $\omega \in \bigwedge^2(V)$, we may express

$$\omega = \omega_1(e_1 \wedge e_2) + \omega_2(e_1 \wedge e_3) + \omega_3(e_2 \wedge e_3) \text{ for some } \omega_1, \omega_2, \omega_3 \in \mathbb{R}$$

Write $\mathbf{x} = (\omega_3, -\omega_2, \omega_1) \in \mathbb{R}^3$. By premise, $\mathbf{x} \neq \mathbf{0}$. Using Gram-Schmidt algorithm, we know there exists some $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$ such that

$$|\mathbf{a}| = |\mathbf{b}| = 1 \text{ and } \{\mathbf{x}, \mathbf{a}, \mathbf{b}\} \text{ are orthogonal}$$

The orthogonality of $\{\mathbf{x}, \mathbf{a}, \mathbf{b}\}$ implies

$$\mathbf{x} = c\mathbf{a} \times \mathbf{b} \text{ for some } c \in \mathbb{R}$$

Explicitly,

$$\begin{cases} \omega_1 = \mathbf{x}_3 = c(a_1b_2 - a_2b_1) \\ \omega_2 = -\mathbf{x}_2 = c(a_1b_3 - a_3b_1) \\ \omega_3 = \mathbf{x}_1 = c(a_2b_3 - a_3b_2) \end{cases}$$

We now see

$$\begin{aligned} &[c(a_1e_1 + a_2e_2 + a_3e_3)] \wedge (b_1e_1 + b_2e_2 + b_3e_3) \\ &= c(a_1b_2 - a_2b_1)(e_1 \wedge e_2) + c(a_1b_3 - a_3b_1)(e_1 \wedge e_3) + c(a_2b_3 - a_3b_2)(e_2 \wedge e_3) \\ &= \omega_1(e_1 \wedge e_2) + \omega_2(e_1 \wedge e_3) + \omega_3(e_2 \wedge e_3) = \omega \end{aligned}$$

We have shown

$$\omega = (ca_1e_2 + ca_2e_2 + ca_3e_3) \wedge (b_1e_1 + b_2e_2 + b_3e_3)$$

That is, ω can indeed be expressed as a wedge product of two vectors in V.

■

Theorem 1.1.4. (Case of Four Dimensions) If

$$\{e_1, e_2, e_3, e_4\} \text{ is a basis for } V$$

Then $e_1 \wedge e_2 + e_3 \wedge e_4$ can not be expressed as a wedge product of two vectors in V.

Proof. Assume for a contradiction that for some $(a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4) \in \mathbb{R}^4$, we have

$$e_1 \wedge e_2 + e_3 \wedge e_4 = (a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4) \wedge (b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4) \tag{1.1}$$

Equating the coefficients of $e_1 \wedge e_2$, we have

$$a_1b_2 - a_2b_1 = 1$$

This implies that one of a_1, b_1 is non-zero. WLOG, suppose $a_1 \neq 0$. Now, equating the coefficients of $e_1 \wedge e_3$ and $e_1 \wedge e_4$, we have

$$a_1b_3 - a_3b_1 = 0 = a_1b_4 - a_4b_1$$

Dividing a_1, we may deduce

$$b_3 = \frac{a_3b_1}{a_1} \text{ and } b_4 = \frac{a_4b_1}{a_1}$$

Therefore, the coefficients of $e_3 \wedge e_4$ in the right side expression of Equation 1.1 is

$$a_3b_4 - a_4b_3 = \frac{a_3a_4b_1}{a_1} - \frac{a_4a_3b_1}{a_1} = 0$$

which does not equals to 1, the coefficient of $e_3 \wedge e_4$ in the left side expression of Equation 1.1. This cause a contradiction. ■

Question 2

Let α be the 1-form $dz + xdy$ on $\mathbb{R}^3$.

(a) Find a basis for $\operatorname{Ker} \alpha$.

(b) Compute $\alpha \wedge d\alpha$.

(c) Find the vector field R that satisfies $\alpha(R) = 1$ and $\iota_R d\alpha = 0$.

(d) Let R be the same vector filed in (c), and let $\varphi_t : \mathbb{R}^3 \to \mathbb{R}^3$ denote its flows. Compute $\mathcal{L}_R\alpha$ and $\varphi_t^*\alpha$ for all fixed t.

Theorem 1.1.5. (a) For all $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$, the kernel of $\alpha_{\mathbf{x}} : T_{\mathbf{x}}\mathbb{R}^3 \to \mathbb{R}$ has the basis

$$\left\{ \frac{\partial}{\partial x}\Big|_{\mathbf{x}}, \frac{\partial}{\partial y}\Big|_{\mathbf{x}} - x\frac{\partial}{\partial z}\Big|_{\mathbf{x}} \right\}$$

Proof. Let

$$c_1\frac{\partial}{\partial x}\Big|_{\mathbf{x}} + c_2\frac{\partial}{\partial y}\Big|_{\mathbf{x}} + c_3\frac{\partial}{\partial z}\Big|_{\mathbf{x}} \in \operatorname{Ker}\alpha_{\mathbf{x}}$$

Compute

$$\begin{aligned} 0 = \alpha_{\mathbf{x}}\Big(c_1\frac{\partial}{\partial x}\Big|_{\mathbf{x}} + c_2\frac{}{\partial y}\Big|_{\mathbf{x}} + c_3\frac{\partial}{\partial z}\Big|_{\mathbf{x}}\Big) &= (dz + xdy)\Big(c_1\frac{\partial}{\partial x}\Big|_{\mathbf{x}} + c_2\frac{\partial}{\partial y}\Big|_{\mathbf{x}} + c_3\frac{\partial}{\partial z}\Big|_{\mathbf{x}}\Big) \\ &= c_3 + xc_2 \end{aligned}$$

This implies

$$\frac{\partial}{\partial x}\Big|_{\mathbf{x}}, \frac{\partial}{\partial y}\Big|_{\mathbf{x}} - x\frac{\partial}{\partial z}\Big|_{\mathbf{x}} \in \operatorname{Ker}\alpha_{\mathbf{x}}$$

Because

$$\alpha_{\mathbf{x}}\frac{\partial}{\partial z}\Big|_{\mathbf{x}} = 1$$

We know $\operatorname{Im}(\alpha_{\mathbf{x}}) = \mathbb{R}$. Therefore,

$$\dim(\operatorname{Ker}\alpha_{\mathbf{x}}) = 3 - \dim(\operatorname{Im}\alpha_{\mathbf{x}}) = 2$$

It is clear that

$$\left\{ \frac{\partial}{\partial x}\Big|_{\mathbf{x}}, \frac{\partial}{\partial y}\Big|_{\mathbf{x}} - x\frac{\partial}{\partial z}\Big|_{\mathbf{x}} \right\} \subseteq \operatorname{Ker}\alpha_{\mathbf{x}} \text{ is linearly independent}$$

It then follows that

$$\left\{ \frac{\partial}{\partial x}\Big|_{\mathbf{x}}, \frac{\partial}{\partial y}\Big|_{\mathbf{x}} - x\frac{\partial}{\partial z}\Big|_{\mathbf{x}} \right\} \text{ is indeed a basis for } \operatorname{Ker}\alpha_{\mathbf{x}}$$

■

Theorem 1.1.6. (b)

$$\alpha \wedge d\alpha = dx \wedge dy \wedge dz$$

Proof. Compute

$$\begin{aligned} d\alpha &= d(dz + xdy) \\ &= d^2 z + dx \wedge dy + xd^2 y \\ &= dx \wedge dy \end{aligned}$$

Compute

$$\begin{aligned} \alpha \wedge d\alpha &= (dz + xdy) \wedge (dx \wedge dy) \\ &= dz \wedge dx \wedge dy + xdy \wedge dx \wedge dy \\ &= dx \wedge dy \wedge dz \end{aligned}$$

∎

Theorem 1.1.7. (c)

$$R \triangleq \frac{\partial}{\partial z} \text{ is the unique vector filed that satisfies } \alpha(R) = 1 \text{ and } \iota_R d\alpha = 0$$

Proof. Suppose

$$R \triangleq R^1 \frac{\partial}{\partial x} + R^2 \frac{\partial}{\partial y} + R^3 \frac{\partial}{\partial z} \text{ satisfies } \alpha(R) = 1 \text{ and } \iota_R d\alpha = 0$$

For all $V \in \mathfrak{X}(\mathbb{R}^3)$, if we write

$$V = V^1 \frac{\partial}{\partial x} + V^2 \frac{\partial}{\partial y} + V^3 \frac{\partial}{\partial z}$$

Then

$$\begin{aligned} \begin{vmatrix} R^1 & V^1 \\ R^2 & V^2 \end{vmatrix} = \begin{vmatrix} dxR & dxV \\ dyR & dyV \end{vmatrix} &= (dx \wedge dy)(R, V) \\ &= d\alpha(R, V) = \iota_R d\alpha(V) = 0 \end{aligned} \tag{1.2}$$

If any of R^1 or R^2 is non-zero at some point $p \in \mathbb{R}^3$, by setting $V^1 = -R^2$ and $V_2 = R^1$ at p we have

$$\begin{vmatrix} R^1 & V^1 \\ R^2 & V^2 \end{vmatrix} \text{ is non-zero at } p$$

which contradicts to Equation 1.2. Therefore, we must have $R^1 = R^2 = 0$ on $\mathbb{R}^3$. We may now compute

$$1 = \alpha(R) = (dz + xdy)(R^3 \frac{\partial}{\partial z}) = R^3$$

We may now conclude

$$R = \frac{\partial}{\partial z}$$

■

Theorem 1.1.8. (d) For all fixed t,

$$\varphi_t^* \alpha = \alpha$$

And

$$\mathcal{L}_R \alpha = 0$$

Proof. Fix t. Obviously,

$$\varphi_t(x, y, z) = (x, y, z + t)$$

Let $p = (x_0, y_0, z_0) \in \mathbb{R}^3$ and

$$v = v^1 \frac{\partial}{\partial x}\Big|_p + v^2 \frac{\partial}{\partial y}\Big|_p + v^3 \frac{\partial}{\partial z}\Big|_p \in T_p\mathbb{R}^3$$

Denote $(x_0, y_0, z_0 + t)$ by q. Compute

$$\begin{aligned}
(\varphi_t^* \alpha)_p(v) &= \alpha_q((\varphi_t)_{*,p} v) \\
&= \alpha_q\Big(v^1 \frac{\partial}{\partial x}\Big|_q + v^2 \frac{\partial}{\partial y}\Big|_q + v^3 \frac{\partial}{\partial z}\Big|_q\Big) \\
&= (dz + x_0 dy)\Big(v^1 \frac{\partial}{\partial x}\Big|_q + v^2 \frac{\partial}{\partial y}\Big|_q + v^3 \frac{\partial}{\partial z}\Big|_q\Big) \\
&= v^3 + x_0 v^2 \\
&= (dz + x_0 dy)\Big(v^1 \frac{\partial}{\partial x}\Big|_p + v^2 \frac{\partial}{\partial y}\Big|_p + v^3 \frac{\partial}{\partial z}\Big|_p\Big) \\
&= \alpha_p(v)
\end{aligned}$$

We have shown $(\varphi_t^* \alpha)_p = \alpha_p$. Because p is arbitrary, this implies $\varphi_t^* \alpha = \alpha$. We may now compute

$$\mathcal{L}_R \alpha = \lim_{t \to 0} \frac{(\varphi_t^* \alpha)_p - \alpha_p}{t} = \lim_{t \to 0} \frac{0}{t} = 0$$

■

Question 3

Orient S^n in $\mathbb{R}^{n+1}$ as the boundary of the unit closed ball.

(a) Show that a volume form on S^n is

$$\omega = \sum_{i=1}^{n+1} (-1)^{i-1} \mathbf{x}^i d\mathbf{x}^1 \wedge \cdots \wedge \widehat{d\mathbf{x}^i} \wedge \cdots \wedge d\mathbf{x}^{n+1}$$

where the caret $\widehat{}$ over $d\mathbf{x}^i$ indicates that $d\mathbf{x}^i$ is to be omitted.

(b) Show that on S^2

$$\omega = \begin{cases} \frac{dy \wedge dz}{x} & \text{for } x \neq 0 \\ \frac{dz \wedge dx}{y} & \text{for } y \neq 0 \\ \frac{dx \wedge dy}{z} & \text{for } z \neq 0 \end{cases}$$

(c) Calculate $\int_{S_2} \omega$

Theorem 1.1.9. (a) Show that a volume form on S^n is

$$\omega = \sum_{i=1}^{n+1} (-1)^{i-1} \mathbf{x}^i d\mathbf{x}^1 \wedge \cdots \wedge \widehat{d\mathbf{x}^i} \wedge \cdots \wedge d\mathbf{x}^{n+1}$$

where the caret $\widehat{}$ over $d\mathbf{x}^i$ indicates that $d\mathbf{x}^i$ is to be omitted.

Proof. Let $i : S^n \to \mathbb{R}^{n+1}$ be the inclusion map and define $V \in \mathfrak{X}(\mathbb{R}^{n+1})$ by

$$V_{\mathbf{y}} \triangleq \sum_{i=1}^{n+1} \mathbf{y}^i \frac{\partial}{\partial \mathbf{x}^i}\Big|_{\mathbf{y}}$$

So that V is nowhere tangent to S^n. By Proposition 15.21 of "Introduction to Smooth Manifold" by John Lee, we know

$$i^*(\iota_V d\mathbf{x}^1 \wedge \cdots \wedge d\mathbf{x}^{n+1}) \text{ is a volume form on } S^n$$

Compute

$$\begin{aligned}
i^*(\iota_V d\mathbf{x}^1 \wedge \cdots \wedge d\mathbf{x}^{n+1}) &= i^*\Big(\sum_{i=1}^{n+1}(-1)^{i-1}V^i d\mathbf{x}^1 \wedge \cdots \wedge \widehat{d\mathbf{x}^i} \wedge \cdots \wedge d\mathbf{x}^{n+1}\Big) \\
&= \sum_{i=1}^{n+1}(-1)^{i-1}(V^i \circ i) d(\mathbf{x}^1 \circ i) \wedge \cdots \wedge \widehat{d\mathbf{x}^i} \wedge \cdots \wedge d(\mathbf{x}^{n+1} \circ i) \\
&= \sum_{i=1}^{n+1}(-1)^{i-1}V^i d\mathbf{x}^1 \wedge \cdots \wedge \widehat{d\mathbf{x}^i} \wedge \cdots \wedge d\mathbf{x}^{n+1} \\
&= \sum_{i=1}^{n+1}(-1)^{i-1}\mathbf{x}^i d\mathbf{x}^1 \wedge \cdots \wedge \widehat{d\mathbf{x}^i} \wedge \cdots \wedge d\mathbf{x}^{n+1} = \omega
\end{aligned}$$

We have shown

$$\omega = i^*(\iota_V d\mathbf{x}^1 \wedge \cdots \wedge d\mathbf{x}^{n+1})$$

This implies ω is indeed a volume form on S^n. ∎

Theorem 1.1.10. (b) Show that on S^2,

$$\omega = \begin{cases} \frac{dy\wedge dz}{x} & \text{for } x \neq 0 \\ \frac{dz\wedge dx}{y} & \text{for } y \neq 0 \\ \frac{dx\wedge dy}{z} & \text{for } z \neq 0 \end{cases}$$

Proof. Define $f \in \Omega^0(\mathbb{R}^3)$ by

$$f(x,y,z) \triangleq \sqrt{x^2+y^2+z^2}$$

So we have

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

Let $i : S^2 \to \mathbb{R}^3$ be the inclusion map. Because $f \circ i : S^2 \to \mathbb{R}^3$ is constant 1, we may compute

$$\begin{aligned}
0 = d(f\circ i) = d(i^* f) = i^*(df) &= i^*\Big(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\Big) \\
&= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \\
&= \frac{xdx+ydy+zdz}{\sqrt{x^2+y^2+z^2}} = xdx+ydy+zdz
\end{aligned}$$

This give us

$$\begin{cases} dx = \frac{ydy+zdz}{-x} \text{ for } x \neq 0 \\ dy = \frac{xdx+zdz}{-y} \text{ for } y \neq 0 \\ dz = \frac{xdx+ydy}{-z} \text{ for } z \neq 0 \end{cases}$$

Therefore, for $x \neq 0$

$$\begin{aligned} \omega &= xdy \wedge dz - ydx \wedge dz + zdx \wedge dy \\ &= xdy \wedge dz - y(\frac{ydy + zdz}{-x}) \wedge dz + z(\frac{ydy + zdz}{-x}) \wedge dy \\ &= (x + \frac{y^2}{x} + \frac{z^2}{x})dy \wedge dz \\ &= \frac{(x^2 + y^2 + z^2)dy \wedge dz}{x} = \frac{dy \wedge dz}{x} \end{aligned}$$

Similarly, for $y \neq 0$

$$\begin{aligned} \omega &= xdy \wedge dz - ydx \wedge dz + zdx \wedge dy \\ &= x(\frac{xdx + zdz}{-y}) \wedge dz - ydx \wedge dz + zdx \wedge (\frac{xdx + zdz}{-y}) \\ &= (\frac{x^2}{y} + y + \frac{z^2}{y})dz \wedge dx \\ &= \frac{(x^2 + y^2 + z^2)dz \wedge dx}{y} = \frac{dz \wedge dx}{y} \end{aligned}$$

Lastly, for $z \neq 0$

$$\begin{aligned} \omega &= xdy \wedge dz - ydx \wedge dz + zdx \wedge dy \\ &= xdy \wedge (\frac{xdx + ydy}{-z}) - ydx \wedge (\frac{xdx + ydy}{-z}) + zdx \wedge dy \\ &= (\frac{x^2}{z} + \frac{y^2}{z} + z)dx \wedge dy \\ &= \frac{(x^2 + y^2 + z^2)dx \wedge dy}{z} = \frac{dx \wedge dy}{z} \end{aligned}$$

■

Theorem 1.1.11. (c) If we orient S^n in $\mathbb{R}^{n+1}$ as the boundary of the unit closed ball, then

$$\int_{S^2} \omega = 4\pi$$

Proof. Because

$$\omega = i^*(\iota_V d\mathbf{x}^1 \wedge \cdots \wedge d\mathbf{x}^{n+1})$$

And because

$$d\mathbf{x}^1 \wedge \cdots \wedge d\mathbf{x}^{n+1} \text{ is a positively oriented volume form on the unit closed ball}$$

We know ω as a volume form of S^n is also positively oriented. Therefore, when we consider the chart

$$U \triangleq \{(x,y,z) \in S^2 : z > 0\} \text{ and } \varphi(x,y,z) \triangleq (x,y)$$

And the chart

$$V \triangleq \{(x,y,z) \in S^2 : z < 0\} \text{ and } \psi(x,y,z) \triangleq (x,y)$$

According to our computation in part 2, we may integrate

$$\begin{aligned}\int_U \omega &= \int_{\varphi(U)} \frac{1}{\sqrt{1-x^2-y^2}} dxdy \\ &= \int_0^1 \int_0^{2\pi} \frac{r}{\sqrt{1-r^2}} d\theta dr = 2\pi \int_0^1 \frac{r}{\sqrt{1-r^2}} dr = 2\pi(-\sqrt{1-r^2})\Big|_{r=0}^1 = 2\pi\end{aligned}$$

And integrate

$$\int_V \omega = -\int_{\psi(V)} \frac{1}{-\sqrt{1-x^2-y^2}} dxdy = 2\pi$$

Therefore,

$$\int_{S^2} \omega = \int_U \omega + \int_V \omega = 4\pi$$

■

Question 4

Let M be a manifold of dimension n, and $\{U_i\}_{i\in I}$ be a countable open cover. Suppose that each U_i is diffeomorphic to $\mathbb{R}^n$ and all $U_{ij} \triangleq U_i \cap U_j$ and $U_{ijk} \triangleq U_i \cap U_j \cap U_k$ are either diffeomorphic to $\mathbb{R}^n$ or empty. Choose a total order $<$ on I, and consider the following sequence of real vector space

$$\mathcal{W}_1 = \prod_{i\in I} \mathbb{R} \xrightarrow{\lambda} \mathcal{W}_2 = \prod_{i<j\in I; U_{ij}\neq\varnothing} \mathbb{R} \xrightarrow{\mu} \mathcal{W}_3 = \prod_{i<j<k\in I; U_{ijk}\neq\varnothing} \mathbb{R}$$

where the linear maps are defined by

$$\lambda : (c_i)_{i\in I} \mapsto (c_i - c_j)_{i<j\in I; U_{ij}\neq\varnothing}$$
$$\mu : (c_{ij})_{i<j\in I; U_{ij}\neq\varnothing} \mapsto (c_{ij} + c_{jk} - c_{ik})_{i<j<k\in I; I_{ijk}\neq\varnothing}$$

which satisfies

$$\mu \circ \lambda = 0$$

(a) Let α be a closed 1-form. Show that for each $i \in I$, we have $\alpha|_{U_i} = df_i$ for some smooth function $f_i : U_i \to \mathbb{R}$. Show that there exists a unique element (c_{ij}) in $\mathcal{W}_2$ with $f_i|_{U_{ij}} - f_j|_{U_{ij}} = c_{ij}$ for all $i < j, U_{ij} \neq \varnothing$. Show that $\mu((c_{ij})) = 0$.

(b) Show that in Part (a), the element $(c_{ij}) + \operatorname{Im}\lambda \in \operatorname{Ker}\mu \big/ \operatorname{Im}\lambda$ is independent of the choice of f_i, and depend only on the cohomology class $[\alpha] \in H^1(M)$.

From part (a) and (b), one define a linear map $\Phi : H^1(M) \to \operatorname{Ker}\mu \big/ \operatorname{Im}\lambda$.

(c) Show that Φ is injective.

(d) Suppose $(c_{ij})_{i<j\in I; U_{ij}\neq\varnothing}$ lies in $\operatorname{Ker}\mu \subseteq \mathcal{W}_2$. Choose a partition of unity $\{\rho_i\}_{i\in I}$ subordinate to $\{U_i\}_{i\in I}$. Define function $f_i : U_i \to \mathbb{R}$ by

$$f_i \triangleq \sum_{j\in I; i<j, U_{ij}\neq\varnothing} c_{ij}\rho_j|_{U_i} - \sum_{j\in I, j<i, U_{ij}\neq\varnothing} c_{ji}\rho_j|_{U_i}$$

Show that there exists a closed one-form α such that these f_i and c_{ij} are possible choices in part (a). Deduce that Φ is surjective.

(e) Show that if M is compact, then $H^1(M)$ is finite-dimensional.

For part (a), note that because

(i) U_i is diffeomorphic to $\mathbb{R}^n$.

(ii) α is closed.

(iii) $H^1(\mathbb{R}^n) = 0$ by Poincare Lemma.

There indeed exists smooth $f_i : U_i \to \mathbb{R}$ such that $\alpha|_{U_i} = df_i$. Fix such $(f_i)_{i\in I}$. Observe that for all fixed $i < j, U_{ij} \neq \varnothing$, we may compute

$$d(f_i - f_j) = df_i - df_j = \alpha - \alpha = 0 \text{ on } U_{ij}$$

This implies

$$f_i|_{U_{ij}} - f_j|_{U_{ij}} \text{ is some unique constant } c_{ij} \text{ on } U_{ij}$$

Fix such $(c_{ij}) \in \mathcal{W}_2$. To see $\mu((c_{ij})) = 0$, fix $i < j < k, p \in U_{ijk}$, and compute

$$\begin{aligned} c_{ij} + c_{jk} - c_{ik} &= (f_i - f_j)(p) + (f_j - f_k)(p) - (f_i - f_k)(p) \\ &= (f_i - f_j - f_k + f_k - f_i)(p) = 0 \end{aligned}$$

Theorem 1.1.12. (b) The map

$$\alpha \mapsto (c_{ij}) + \operatorname{Im} \lambda \in \frac{\operatorname{Ker} \mu}{\operatorname{Im} \lambda}$$

is well-defined and sends closed one-forms within the same cohomology class to the same element.

Proof. Let $\widehat{f}_i : U_i \to \mathbb{R}$ also satisfy $\alpha|_{U_i} = d\widehat{f}_i$, and again induce

$$\widehat{c_{ij}} \triangleq \widehat{f}_i - \widehat{f}_j$$

Because

$$d(f_i - \widehat{f}_i) = df_i - d\widehat{f}_i = \alpha - \alpha = 0$$

We know $f_i, \widehat{f}_i$ differ by some constant, which we denote

$$c_i \triangleq f_i - \widehat{f}_i$$

Now, compute

$$\begin{aligned} \lambda(c_i) &= (c_i - c_j)_{i<j\in I; U_{ij}\neq\varnothing} \\ &= (f_i - \widehat{f}_i - f_j + \widehat{f}_j)_{i<j\in I; U_{ij}\neq\varnothing} \\ &= (c_{ij} - \widehat{c_{ij}})_{i<j\in I; U_{ij}\neq\varnothing} \end{aligned}$$

We have shown $(\widehat{c_{ij}})_{i<j\in I; U_{ij}\neq\varnothing}, (c_{ij})_{i<j\in I; U_{ij}\neq\varnothing}$ differ by $\lambda(c_i)$. That is, the map

$$\alpha \mapsto (c_{ij}) + \operatorname{Im} \lambda \in \frac{\operatorname{Ker} \mu}{\operatorname{Im} \lambda} \text{ is well-defined}$$

Let $\gamma \in \Omega^1(M)$ be some exact one-form. It remains to show

$$(c_{ij}) \in \operatorname{Im} \lambda \text{ where } (c_{ij}) \text{ is induced by } \gamma$$

Write $\gamma = dg$, where $g \in \Omega^0(M)$. Let $f_i : U_i \to \mathbb{R}$ satisfy

$$\gamma|_{U_i} = df_i$$

Because

$$d(g - f_i) = dg - df_i = \gamma - \gamma = 0 \text{ on } U_i$$

We know g, f_i on U_i differ by some constant, which we denote

$$c_i \triangleq f_i - g \text{ on } U_i$$

Now, to close the proof, compute

$$\begin{aligned}\lambda((c_i)_{i\in I}) &= (c_i - c_j)_{i<j\in I;U_{ij}\neq\varnothing} \\ &= (f_i - g - f_j + g)_{i<j\in I;U_{ij}\neq\varnothing} \\ &= (f_i - f_j)_{i<j\in I;U_{ij}\neq\varnothing} = (c_{ij})_{i<j\in I:U_{ij}\neq\varnothing}\end{aligned}$$

where the last inequality hold because the (c_{ij}) we are referring to is induced by γ. ■

Theorem 1.1.13. (c) Φ is injective.

Proof. Fix $[\alpha] \in H^1(M)$, and let $(f_i : U_i \to \mathbb{R}), (c_{ij}) \in \mathcal{W}_2$ be induced by α. Suppose $(c_{ij}) = \lambda((c_i))$ for some $(c_i) \in \mathcal{W}_1$. We are required to show

$$\alpha \text{ is exact}$$

Define $g_i : U_i \to \mathbb{R}$ by

$$g_i \triangleq f_i - c_i$$

So if $i < j$ satisfy $U_{ij} \neq \varnothing$, we see

$$g_i - g_j = (f_i - c_i) - (f_j - c_j) = (f_i - f_j) - c_{ij} = 0 \text{ on } U_{ij}$$

Note that the second equality follows from $(c_{ij}) = \lambda((c_i))$ and the last equality follows from definition of (c_{ij}). In summary, we have shown

$$g_i = g_j \text{ on } U_{ij}$$

Therefore, we may well define $g : M \to \mathbb{R}$ by

$$g(p) \triangleq g_i(p) \text{ if } p \in U_i$$

To close out the proof, observe on each U_i,

$$\alpha = df_i = dg_i = dg$$

where the second equality hold because g_i, f_i differ by a constant. This implies

$$\alpha = dg \text{ on } M$$

We have shown α is exact. That is, $[\alpha] = 0$. ■

Theorem 1.1.14. (d) Let $(c_{ij}) \in \operatorname{Ker}\mu \subseteq \mathcal{W}_2$ and $\{\rho_i\}_{i\in I}$ be a partition of unity subordinate to $\{U_i\}_{i\in I}$. If we define $f_i : U_i \to \mathbb{R}$ by

$$f_i \triangleq \sum_{j\in I; i<j, U_{ij}\neq\varnothing} c_{ij}\rho_j|_{U_i} - \sum_{j\in I, j<i, U_{ij}\neq\varnothing} c_{ji}\rho_j|_{U_i}$$

there exists some closed one-form α such that $\alpha = df_i$ on each U_i and

$$f_i - f_j = c_{ij} \text{ on } U_{ij} \text{ for all } i<j, U_{ij}\neq\varnothing$$

Proof. Fix $i<j, U_{ij}\neq\varnothing$. Because

$$f_i = \sum_{i<k} c_{ik}\rho_k - \sum_{k<i} c_{ki}\rho_k$$
$$f_j = \sum_{j<k} c_{jk}\rho_k - \sum_{k<j} c_{kj}\rho_k$$

We may compute

$$f_i - f_j = \sum_{j<k}(c_{ik}-c_{jk})\rho_k + \sum_{i<k<j}(c_{ik}+c_{kj})\rho_k + \sum_{k<i}(-c_{ki}+c_{kj})\rho_k + c_{ij}\rho_j + c_{ij}\rho_i \tag{1.3}$$

Because $(c_{ij}) \in \operatorname{Ker}\mu$, for all $k>j$, we may deduce

$$c_{ij} + c_{jk} - c_{ik} = 0 \implies c_{ik} - c_{jk} = c_{ij}$$

For all $k : i<k<j$, we may deduce

$$c_{ik} + c_{kj} - c_{ij} = 0 \implies c_{ik} + c_{kj} = c_{ij}$$

For all $k<i$, we may deduce

$$c_{ki} + c_{ij} - c_{kj} = 0 \implies -c_{ki} + c_{kj} = c_{ij}$$

Thus, we may continue the computation from Equation 1.3 and get

$$f_i - f_j = \sum_{k\in I} c_{ij}\rho_k = c_{ij} \text{ on } U_{ij}$$

where the last equality hold true because $\{\rho_k\}_{k\in I}$ is a partition of unity. We have established that for each $i<j, U_{ij}\neq\varnothing$, the functions f_i, f_j differ by a constant on where they overlap. Therefore, we may well define a closed one form α on M by

$$\alpha|_{U_i} \triangleq df_i \text{ for all } i\in I$$

Note that α is indeed closed, since

$$(d\alpha)|_{U_i} = d(\alpha|_{U_i}) = d(df_i) = 0 \text{ for all } i\in I \implies d\alpha = 0 \text{ on } M$$

■

Now, for all element X of $\operatorname{Ker}\mu \diagup \operatorname{Im}\lambda$, when we pick a representative element $(c_{ij}) \in X \subseteq \operatorname{Ker}\mu$, using Theorem 1.1.14, we may find some closed one-from α such that α can induce (c_{ij}), which give us

$$\Phi([\alpha]) = X$$

In other words, Φ is surjective. For part (e), suppose M is compact. Because M is compact, we may let I be finite, which allow us to deduce

$$\operatorname{Dim}(\mathcal{W}_2) \le (\operatorname{card} I)^2 \in \mathbb{N}$$

and deduce

$$\operatorname{Dim}(\operatorname{Ker}\mu \diagup \operatorname{Im}\lambda) \le \operatorname{Dim}(\operatorname{Ker}\mu) \le \operatorname{Dim}(\mathcal{W}_2) \in \mathbb{Z}_0^+$$

Lastly, because $\Phi : H^1(M) \to \operatorname{Ker}\mu \diagup \operatorname{Im}\lambda$ is injective, we can moreover deduce

$$\operatorname{Dim}(H^1(M)) \le \operatorname{Dim}(\operatorname{Ker}\mu \diagup \operatorname{Im}\lambda) \in \mathbb{Z}_0^+$$

That is, $H^1(M)$ is finite dimensional.