Deadline : 2023/05/10, 17:00.

1. Use Lagrange multipliers to prove that the rectangle with maximum area that has a given perimeter p is a square.

2. Consider the problem of minimizing the function $f(x, y) = x$ on the curve $y^2+x^4-x^3 = 0$.

   (a) Try using Lagrange multipliers to solve the problem.

   (b) Show that the minimum value is $f(0, 0) = 0$ but the Lagrange condition $\nabla f(0, 0) = \lambda\nabla g(0, 0)$ is not satisfied for any value of λ.

   (c) Explain why Lagrange multipliers fail to find the minimum value in this case.

3. (a) Find the maximum value of

$$f(x_1, x_2, \cdots, x_n) = \sqrt[n]{x_1x_2\cdots x_n}$$

   given that $x_1, x_2, \cdots, x_n$ are positive numbers and $x_1 + x_2 + \cdots + x_n = c$, where c is a constant.

   (b) Deduce from part (a) that if $x_1, x_2, \cdots, x_n$ are positive numbers, then

$$\sqrt[n]{x_1x_2\cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

   This inequality says that the geometric mean of n numbers is no larger than the arithmetic mean of the numbers. Under what circumstances are these two means equal?

4. (a) Maximize $\displaystyle\sum_{i=1}^{n} x_iy_i$ subject to the constraint $\displaystyle\sum_{i=1}^{n} x_i^2 = 1$ and $\displaystyle\sum_{i=1}^{n} y_i^2 = 1$.

   (b) Put

$$x_i = \frac{a_i}{\sqrt{\sum a_j^2}} \text{ and } y_i = \frac{b_i}{\sqrt{\sum b_j^2}}.$$

   Show that

$$\sum a_ib_i \leq \sqrt{\sum a_j^2}\sqrt{\sum b_j^2}$$

   for any numbers $a_1, \cdots, a_n, b_1, \cdots, b_n$. This inequality is known as the Cauchy-Schwarz Inequality.

5. Let $D_1 = (0,1) \times (0,2)$, $D_2 = (0,2) \times (0,1)$ and $D = D_1 \cup D_2$. Suppose that f is differentiable on D. Determine whether the following statement is true.

   "For every $\mathbf{a}$, $\mathbf{b} \in D$, there exists $\mathbf{c} \in D$ such that

$$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{c}) \cdot (\mathbf{b} - \mathbf{a})."$$

6. Find the volume of the given solid.

   (a) Under the surface $z = 1 + x^2y^2$ and above the region enclosed by $x = y^2$ and $x = 4$.

   (b) Bounded by the cylinder $y^2 + z^2 = 4$ and the planes $x = 2y$, $x = 0$, $z = 0$ in the first octant.

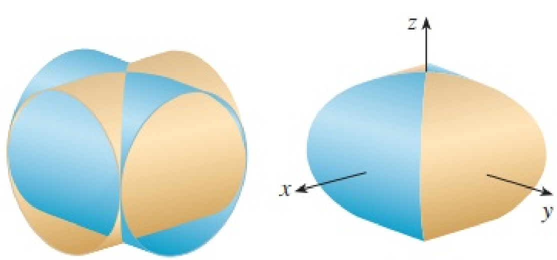
7. Suppose we can use Fubini Theorem in the following cases.

   (a) Evaluate $\displaystyle\int_{\mathbb{R}} e^{-x^2}\, dx$.

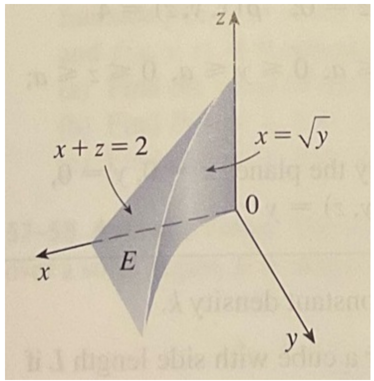
   (b) Let $\mathbf{x} = (x_1, \cdots, x_n)$. Evaluate $\displaystyle\int_{\mathbb{R}^n} e^{-|\mathbf{x}|^2}\, d\mathbf{x}$

   (c) Let $\mathbf{x} = (x_1, \cdots, x_n)$. Evaluate $\displaystyle\int_{\mathbb{R}^n} x_i e^{-|\mathbf{x}|^2}\, d\mathbf{x}$ for $i = 1, 2, \cdots, n$.

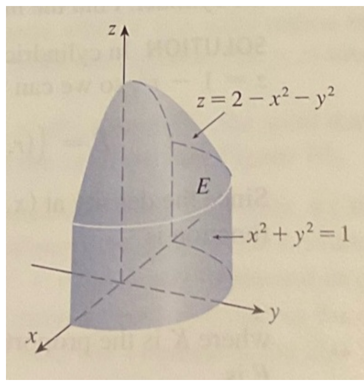
8. The figure shows the surface created when the cylinder $y^2 + z^2 = 1$ intersects the cylinder $x^2 + z^2 = 1$. Find the area of this surface.



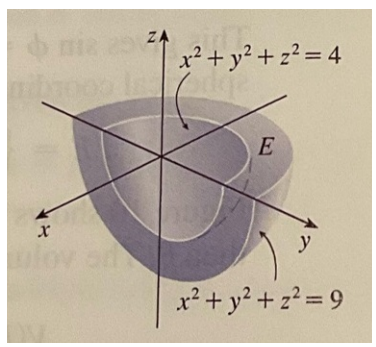
9. Evaluate the triple integral $\iiint_E f(x,y,z)\,dV$ for the given function f and solid region E.



   (a) $f(x,y,z) = x + y$



   (b) $f(x,y,z) = x^2 + y^2$



   (c) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

10. (a) The double integral $\displaystyle\int_0^1\int_0^1 \frac{1}{1-xy}\,dxdy$ is an improper integral and could be defined as the limit of double integrals over the rectangle $[0,t]\times[0,t]$ as $t \to 1^-$. But if we expand the integrand as a geometric series, we can express the integral as the sum of an infinite series. Show that

$$\int_0^1\int_0^1 \frac{1}{1-xy}\,dxdy = \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

   (b) Leonhard Euler was able to find the exact sum of the series in part (a). In 1736 he proved that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

In this part we ask you to prove this fact by evaluating the double integral in part (a). Start by making the change of variables

$$x = \frac{u-v}{\sqrt{2}} \qquad y = \frac{u+v}{\sqrt{2}}.$$

This gives a rotation about the origin through the angle $\frac{\pi}{4}$. You will need to sketch the corresponding region in the uv-plane.

Hint: If, in evaluating the integral, you encounter either of the expressions $\dfrac{1-\sin\theta}{\cos\theta}$ or $\dfrac{\cos\theta}{1+\sin\theta}$, you might like to use the identity $\cos\theta = \sin(\frac{\pi}{2} - \theta)$ and the corresponding identity for $\sin\theta$.