1.6 Chain Condition

Give some collection Σ of sets, we say Σ satisfies the **ascending chain condition, a.c.c.**, if for each chain $x_1 \subseteq x_2 \subseteq \cdots$ there exists n such that $x_n = x_{n+1} = \cdots$, and we say Σ satisfies the **descending chain condition, d.c.c.**, if for each chain $x_1 \supseteq x_2 \supseteq \cdots$ there exists n such that $x_n = x_{n+1} = \cdots$. Let M be some module. We say M is **Noetherian** if the collection of submodules of M satisfies a.c.c., and we say M is **Artinian** if the collection of submodules satisfies d.c.c. Thanks to axiom of choice, module M is Noetherian if and only if every nonempty collection of submodules of M has a maximal element if and only if every submodule of M is finitely generated.

Given a finite **chain** of submodules

$$M_0 \subset M_1 \subset \cdots \subset M_n$$

we say this chain is of **length** n. Under the obvious assignment of order on the collection of all finite chains of submodules of M, by a **composition series** of M, we mean a maximal finite chain. Clearly, a finite chain

$$0 = M_0 \subset \cdots \subset M_n = M$$

is maximal if and only if $M_k \diagup M_{k-1}$ are simple.

Theorem 1.6.1. (Length of modules is well defined) Every composition series of a module M have the same length.

Proof. Suppose M has a composition series, and let $l(M)$ denote the least length of a composition series of M. We wish to show every chain has length smaller than $l(M)$. Before such, we first prove

$$N \subset M \implies l(N) < l(M) \tag{1.1}$$

Let $M_0 \subset \cdots \subset M_n = M$ be a composition series of least length. Define $N_k \triangleq N \cap M_k$ for all $k \in \{0, \dots, n\}$. Consider the obvious homomorphism $N_k \diagup N_{k-1} \to M_k \diagup M_{k-1}$. We see that either $N_k \diagup N_{k-1} \cong M_k \diagup M_{k-1}$ or $N_k = N_{k-1}$. This implies that the chain $N_0 \subset \cdots \subset N_n$ will be a composition series of N after the unnecessary terms are removed. It remains to show there are unnecessary terms in $N_0 \subset \cdots \subset N_n$. Assume not for a contradiction. Because $N_1 \subseteq M_1$ and $N_1 \diagup \{0\} \cong M_1 \diagup \{0\}$, we have $N_1 = M_1$. Repeating the same argument, we have $N = N_n = M_n = M$, a contradiction. We have proved statement 1.1.

Now, let $M'_0 \subset \cdots \subset M'_r$ be some composition series of M. The proof then follows from using statement 1.1 to deduce

$$l(M) = l(M'_r) > \cdots > l(M'_0) = 0 \implies r \leq l(M)$$

■

Because of Theorem 1.6.1, we may well define the **length** $l(M)$ of module. For obvious reason, if module M has no composition series, we say M has infinite length and write $l(M) = \infty$. Clearly, if M is of finite length, then M is both Noetherian and Artinian. Conversely, if M is both Noetherian and Artinian, then by the maximal element definition of Noetherian, there exists a decreasing sequence $M = M_0 \supset M_1 \supset M_2 \supset \cdots$, which by d.c.c. must be finite.

Chapter 3

Big Theorems

3.1 Hilbert's Basis Theorem

Before we prove the Hilbert's Basis Theorem, we must first show that finitely generated modules over Noetherian rings is also Noetherian.

Proposition 3.1.1. (Formal properties of Noetherian modules) Given a short exact sequence of A-modules:

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

M is Noetherian if and only if M' and M'' are both Noetherian.

Proof. Consider the ascending chain condition definition. For the "if" part, let L_n be an ascending chain of submodules of M, and use short five lemma on

$$\begin{tikzcd}
0 \arrow[r] & \alpha^{-1}(L_n) \arrow[r, "\alpha"] \arrow[d, hook] & L_n \arrow[r, "\beta"] \arrow[d, hook] & \beta(L_n) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \alpha^{-1}(L_{n+1}) \arrow[r, "\alpha"] & L_{n+1} \arrow[r, "\beta"] & \beta(L_{n+1}) \arrow[r] & 0
\end{tikzcd}$$

to conclude that L_n must stop at some point. ■

Suppose A is a Noetherian ring. Applying Proposition 3.1.1 inductively to

$$0 \longrightarrow A \longrightarrow A^n \longrightarrow A^{n-1} \longrightarrow 0$$

we see the module A^n is also Noetherian, and so any finitely generated module over A, isomorphic to some quotient of A^n, is also Noetherian. We may now give a simple proof to Hilbert's Basis Theorem.

Theorem 3.1.2. (Hilbert's Basis Theorem) If A is Noetherian, than the polynomial ring $A[x]$ is also Noetherian.

Proof. Let X be an ideal in $A[x]$. We are required to show that X is finitely generated. Let I be the ideal in A that contains exactly the leading coefficients of elements of X. Because A is Noetherian, we may let $I = \langle a_1, \dots, a_n \rangle$ and let $f_1, \dots, f_n \in X$ have leading coefficients $a_1, \dots, a_n$. Let $X' \triangleq \langle f_1, \dots, f_n \rangle \subseteq X$ and let $r \triangleq \max\{\deg(f_1), \dots, \deg(f_n)\}$.

We first show

$$X = \Big(X \cap \langle 1, x, \dots, x^{r-1} \rangle \Big) + X' \tag{3.1}$$

Let $f \in X$ with $\deg(f) = m$ and leading coefficients a. We wish to show $f \in (X \cap \langle 1, x, \dots, x^{r-1} \rangle) + X'$. Because $a \in I$, we may find some $u_i \in A$ such that $a = \sum u_i a_i$. Clearly, these u_i satisfy

$$f - \sum u_i f_i x^{m-\deg(f_i)} \in X, \quad \text{and } \sum u_i f_i x^{m-\deg(f_i)} \in X'$$

and satisfy

$$\deg \Big(f - \sum u_i f_i x^{m-\deg(f_i)} \Big) < m$$

Proceeding this way, we end up with $f - g = h$ where $g \in X'$ and $h \in X \cap \langle 1, x, \dots, x^{r-1} \rangle$. We have proved Equation 3.1. Now, because X' is finitely generated, to show X is finitely generated, it only remains to show the ideal $X \cap \langle 1, x, \dots, x^{r-1} \rangle$ is finitely generated, which follows immediately from noting $\langle 1, x, \dots, x^{r-1} \rangle$ as a module is Noetherian. ■

We close this section by giving a cute corollary of Hilbert's Basis Theorem in classical algebraic geometry. Suppose $E \subseteq R[x_0, \dots, x_{n-1}]$ is an infinite collection of polynomials. Let V be the set of common roots of these polynomials, i.e.,

$$V \triangleq \{ x \in R^n : f(x) = 0 \text{ for all } f \in E \}$$

Clearly,

$$V = \{ x \in R^n : f(x) = 0 \text{ for all } f \in \langle E \rangle \}$$

Induction with Hilbert's Basis Theorem shows that $R[x_0, \dots, x_{n-1}]$ is Noetherian, so $\langle E \rangle$ is finitely generated. This allow us to write $\langle E \rangle = \langle f_1, \dots, f_n \rangle$ for some finite set of polynomials $f_1, \dots, f_n \in R[x_0, \dots, x_{n-1}]$. We now see that the locus V of an infinite collection of polynomials can always be written as a locus of some finite collection of polynomials.