Deadline : 2024/1/3, 17:00.

1. Let f be a real-valued function defined on an open interval E such that

$$|f(x) - f(y)| \leq |x - y|^{\alpha} \ , \ \forall \, x, y \in E$$

Then show that (a) f is continuous on E, if $\alpha > 0$ (b) f is constant on E, if $\alpha > 1$.

2. Let f be a differentiable real function defined in (a, b). Prove that f is convex if and only if f' is monotonically increasing. Assume next that $f''(x)$ exists for every $x \in (a, b)$, and prove that f is convex if and only if $f''(x) \geq 0$ for $x \in (a, b)$.

3. Let f be a real-valued function defined on (a, ∞) for some $a \in \mathbf{R}$ such that f' and f'' exist on (a, ∞). Further, if M_0, M_1, M_2 are the least upper bounds of f, f', f'', on (a, ∞), respectively. Prove that $M_1^2 \leq 4M_0M_2$.

4. Let f be a real-valued function defined on $[-1, 1]$ such that $f^{(n)}$ exists on $[-1, 1]$ for $n = 1, 2, 3$, and

$$f(-1) = 0, \ f(0) = 0, \ f(1) = 1, \ f'(0) = 0.$$

Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1, 1)$.

5. (a) Let $f(x) = \exp\left(-\frac{1}{|x|}\right)$, $x \neq 0$ and $f(0) = 0$. Then calculate $f^{(n)}(0)$ by definition of derivative $\forall n \in \mathbf{N}$.

   (b) Show that $f(x)$ can NOT have its Taylor expansion at $x = 0$ (or, say the expand radius of $f(x)$ at $x = 0$ is zero).

6. Let f have a continuous derivative in the interval $[a, b]$, and let $f''(x) \geq 0$ for every value of x. Then if ξ is any point in the interval, the curve nowhere falls below its tangent at the point $x = \xi$, $y = f(\xi)$. (Hint: It suffices to show $f(x) \geq f(\xi) + f'(\xi)(x - \xi)$ on $[a, b]$, i.e. consider the Taylor expansion of $f(x)$ at $x = \xi$).

7. Let f be a continuous function on $[a, b]$, then prove that $f(x) = 0$ for all $x \in [a, b]$ if and only if (a) $\displaystyle\int_a^b |f(x)| \, dx = 0$, (b) $\displaystyle\int_a^c f(x) \, dx = 0$, $\forall \, c \in [a, b]$.

8. Prove that if $\displaystyle\lim_{b \to \infty} \int_1^b |f(x)| \, dx$ exists and finite, then $\displaystyle\lim_{n \to \infty} \int_1^\infty f(x^n) \, dx = 0$.

9. Prove that if f is continuous on $[a, b]$, and ϕ is monotone increasing on $[a, b]$, then $\exists\, \xi \in [a, b]$ such that

$$\int_a^b f d\phi = f(\xi)(\phi(b) - \phi(a))$$

10. Let $\{g_n\}$ be a sequence of non-negative and Stieltjes integrable with respect to an increasing function ϕ on $[a, b]$ such that

$$\lim_{n\to\infty} \int_a^b g_n \, d\phi = 0$$

Prove that if f is also integrable, then

$$\lim_{n\to\infty} \int_a^b f \, g_n \, d\phi = 0$$

11. Let ϕ be a strictly monotone increasing function on $[a, b]$.

    (a) Show that if f is Riemann-Stieltjes integrable with respect to ϕ, then the following quantities all exist and finite for $p > 0$.

$$\|f\|_\infty = \sup\{|f(x)| : x \in [a, b]\}$$
$$\|f\|_p = (\int_a^b |f|^p \; d\phi)^{1/p}$$

    (b) Prove that if f is continuous on $[a, b]$, then

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$$