7.12 PDE HW 12

Question 191

Prove that the eigenfunctions $\{\sin my \sin nz\}$ are orthogonal on the square $\{0 < y < \pi, 0 < z < \pi\}$.

Proof. Let p, q be a pair of integer such that $p \neq m$ and $q \neq n$. Because $\{\sin(ix)\}_{i\in\mathbb{Z}}$ is an orthogonal system on $(0, \pi)$, we may compute

$$\begin{aligned}
&\int_0^\pi \int_0^\pi \sin(my)\sin(nz)\sin(py)\sin(qz)dydz \\
&= \int_0^\pi \sin(my)\sin(py)dy \int_0^\pi \sin(nz)\sin(qz)dz \\
&= 0 \cdot 0 = 0
\end{aligned}$$

This shows that $\{\sin(my)\sin(nz)\}$ are indeed orthogonal on the square $\{0 < y < \pi, 0 < z < \pi\}$. Note that we used the Fubini's theorem in our first equality, and the reason we can use the Fubini's theorem is that the function $\sin(my)\sin(nz)\sin(py)\sin(qz)$ is bounded by 1 on the bounded domain $[0, \pi]^2$. ■