Calculus Done Taiwan

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2.1 Compact

In this section, if we say a topological space X is **compact**, we mean its every open cover has a finite subcover. If we say a topological space X is **sequentially compact**, we mean that every sequence inside X has a subsequence that converges to some $x \in X$. If we say a topological space X is **limit point compact**, we mean that every infinite subset has a limit point in X. If we say a metric space M is **totally bounded**, we mean that for any ϵ, there exists some finite cover of M consisting of only open balls of radius ϵ. If we say a metric space M is **complete**, we mean every **Cauchy sequence** in M converge to some point in M.

Equivalent Definition 2.1.1. (Compactness of metric space) Let M be a metric space. The followings are equivalent:

(i) M is compact.

(ii) M is sequentially compact.

(iii) M is limit point compact.

(iv) M is totally bounded and complete.

Proof. ■

Chapter 3

What belongs to Differential Geometry

Chapter 4

Convergent Divergent

4.1 Tests

Abstract

This section prove some basic result on sequence and series, which will be heavily used in next section on analytic functions and Chapter: Beauty. Although written in an almost glossary form, we present the Theorems in a structural order based on the necessity of notion of absolute convergence and limit superior. Note that in this section, z, v, w always represent complex numbers, and a, b, c always represent real numbers.

Theorem 4.1.1. (Weierstrass M-test) Given sequences $f_n : X \to \mathbb{C}$, and suppose

$$\forall n \in \mathbb{N}, \forall x \in X, |f_n(x)| \leq M_n$$

Then

$$\sum_{n=1}^{\infty} M_n \text{ converge } \implies \sum_{n=1}^{\infty} f_n \text{ uniformly converge}$$

Proof. The proof follows from noting

$$\forall x \in X, \left| \sum_{k=m}^{n} f_k(x) \right| \leq \sum_{k=m}^{n} |f_k(x)| \leq \sum_{k=m}^{n} M_k$$

∎

Note that in our proof of Weierstrass M-test, we reduce the proof for uniform convergence into uniform Cauchy, which is a technique we shall also use later in Abel's test for uniform convergence. We now prove summation by part, which is a result hold in all fields, and is the essence of the proof of Dirichlet's test and Abel's test for uniform convergence.

Theorem 4.1.2. (Summation by Part)

$$\begin{aligned} f_n g_n - f_m g_m &= \sum_{k=m}^{n-1} f_k \Delta g_k + g_k \Delta f_k + \Delta f_k \Delta g_k \\ &= \sum_{k=m}^{n-1} f_k \Delta g_k + g_{k+1} \Delta f_k \end{aligned}$$

Proof. The proof follows induction which is based on

$$f_m g_m + f_m \Delta g_m + g_m \Delta f_m + \Delta f_m \Delta g_m = f_{m+1} g_{m+1}$$

∎

Theorem 4.1.3. (Dirichlet's Test) Suppose

(a) $a_n \to 0$ monotonically.

(b) $\sum_{n=1}^{N} z_n$ is bounded.

We have

$$\sum a_n z_n \text{ converge}$$

Proof. Define $Z_n \triangleq \sum_{n=1}^{N} z_n$ and let M bound $|Z_n|$. Using summation by part by letting $f_k = a_k$ and $g_k = Z_{k-1}$, we have

$$\begin{aligned} \left| \sum_{k=m}^{n} a_k z_k \right| &= \left| a_{n+1} Z_n - a_m Z_{m-1} - \sum_{k=m}^{n} Z_k (a_{k+1} - a_k) \right| \\ &\leq |a_{n+1} Z_n| + |a_m Z_{m-1}| + \left| \sum_{k=m}^{n} Z_k (a_{k+1} - a_k) \right| \\ (\because a_n \text{ is monotone }) \qquad &\leq M \Big(|a_{n+1}| + |a_m| + |a_{n+1} - a_m| \Big) \end{aligned}$$

∎

Theorem 4.1.4. (Abel's Test for Uniform Convergence) Suppose $g_n : X \to \mathbb{R}$ is a uniformly bounded pointwise monotone sequence. Then given a sequence $f_n : X \to \mathbb{R}$,

$$\sum f_n \text{ uniformly converge } \implies \sum f_n g_n \text{ uniformly converge}$$

Proof. Define $R_n \triangleq \sum_{k=n}^{\infty} f_k$. Let M uniformly bound g_n. Because $R_n \to 0$ uniformly, we can let N satisfy

$$\forall n \geq N, \forall x \in X, |R_n(x)| < \frac{\epsilon}{6M}$$

Then for all $n, m \geq N$, using summation by part, we have

$$\begin{aligned}
\left| \sum_{k=m}^{n} f_k g_k \right| &= \left| \sum_{k=m}^{n} g_k \Delta R_k \right| \\
&\leq |R_{n+1} g_{n+1}| + |R_{m+1} g_{m+1}| + \sum_{k=m}^{n} |R_{k+1} \Delta g_k| \\
(\because g_n \text{ is pointwise monotone }) \quad &\leq |R_{n+1} g_{n+1}| + |R_{m+1} g_{m+1}| + \frac{\epsilon}{6M} |g_{n+1} - g_m| \leq \epsilon
\end{aligned}$$

■

Although the proofs of Dirichlet's test and Abel's test for uniform convergence are quite similar, one should note that the "ways" summation by part is applied are slightly different, as one use $R_n \triangleq \sum_{k=n}^{\infty} f_k$ instead of $\sum_{k=1}^{n} f_k$, like $Z_n \triangleq \sum_{j=1}^{n} z_j$. As corollaries of Dirichlet's test, one have the famous **alternating series test** and Abel's test for complex series.

Theorem 4.1.5. (Abel's Test for Complex Series) Suppose

(a) $\sum z_n$ converge.

(b) b_n is a bounded monotone sequence.

We have

$$\sum z_n b_n \text{ converge}$$

Proof. Denote $B \triangleq \lim_{n\to\infty} b_n$. By Dirichlet's Test, we know $\sum z_n(b_n - B)$ converge. The proof now follows form noting

$$\sum z_n b_n = \sum z_n(b_n - B) + B \sum z_n$$

■

We now introduce the idea of absolute convergence, which we shall use throughout the remaining of the section. By a **permutation** $\sigma : E \to E$ on some set E, we merely mean σ is a bijective function. We say $\sum z_n$ **absolutely converge** if $\sum |z_n|$ converge, and say $\sum z_n$ **unconditionally converge** if for all permutation $\sigma : \mathbb{N} \to \mathbb{N}$, the series

$\sum z_{\sigma(n)}$ converge and converge to the same value.

Theorem 4.1.6. (Absolutely Convergent Series Unconditionally Converge)

$$\sum z_n \text{ absolutely converge } \implies \sum z_n \text{ unconditionally converge}$$

Proof. The fact $\sum z_n$ converge follows from noting

$$\left|\sum_{k=n}^{m} z_k\right| \leq \sum_{k=n}^{m} |z_k| \leq \sum_{k=n}^{\infty} |z_k|$$

Now, fix ϵ and permutation σ. Let N_1 and N_2 satisfy

$$\sum_{n=N_1}^{\infty} |z_n| < \frac{\epsilon}{2} \text{ and } \left|\sum_{n=N}^{\infty} z_n\right| < \frac{\epsilon}{2} \text{ for all } N > N_2$$

Let $M \triangleq \max\{N_1, N_2\}$. Observe that for all $N > \max_{1\leq r\leq M} \sigma^{-1}(r)$, we have

$$\left|\sum z_n - \sum_{n=1}^{N} z_{\sigma(n)}\right| \leq \left|\sum_{n=M+1}^{\infty} z_n\right| + \sum_{n=M+1}^{\infty} |z_n| < \epsilon$$

■

Theorem 4.1.7. (Riemann Rearrangement Theorem) If $\sum a_n$ converge but not absolutely, then for each $L \in \overline{\mathbb{R}}$, there exists a permutation σ such that

$$\sum a_{\sigma(n)} = L$$

Proof. Define a_n^+ and a_n^- by

$$a_n^+ \triangleq \max\{a_n, 0\} \text{ and } a_n^- \triangleq \min\{a_n, 0\}$$

Because

$$\sum (a_n^+ + a_n^-) \text{ converge but } \sum (a_n^+ - a_n^-) = \infty$$

We know

$$\sum a_n^+ = \sum (-a_n^-) = \infty$$

WOLG, (why?), fix $L \in \mathbb{R}$ and suppose $a_n \neq 0$ for all n. Let $A = B = L$, and let two increasing sequence $\sigma^+, \sigma^- : \mathbb{N} \to \mathbb{N}$ satisfy

$$\sigma^+(k+1) = \min\{n \in \mathbb{N} : a_n > 0 \text{ and } n > \sigma^+(k)\}$$

and similar for σ^-. Now, recursively define p_k, q_k by

$$p_1 \text{ is the smallest number such that } \sum_{n=1}^{p_1} a_{\sigma^+(n)} \geq A \tag{4.1}$$

$$q_1 \text{ is the smallest number such that } \sum_{n=1}^{p_1} a_{\sigma^+(n)} + \sum_{n=1}^{q_1} a_{\sigma^-(n)} \leq B \tag{4.2}$$

$$p_{k+1} \text{ is the smallest number such that } \sum_{n=1}^{p_{k+1}} a_{\sigma^+(n)} + \sum_{n=1}^{q_k} a_{\sigma^-(n)} \geq A \tag{4.3}$$

$$q_{k+1} \text{ is the smallest number such that } \sum_{n=1}^{p_{k+1}} a_{\sigma^+(n)} + \sum_{n=1}^{q_{k+1}} a_{\sigma^-(n)} \leq B \tag{4.4}$$

We then define σ by

$$\sigma^+(1), \ldots, \sigma^+(p_1), \sigma^-(1), \ldots, \sigma^-(q_1), \sigma^+(p_1+1), \ldots, \sigma^+(p_2), \sigma^-(q_1+1), \ldots, \sigma^-(q_2), \ldots$$

It then follows from

$$\left| \sum_{n=1}^{p} a_{\sigma^+}(n) + \sum_{n=1}^{q_k} a_{\sigma^-}(n) - L \right| \leq \min\{a_{\sigma^+(p_{k+1})}, |a_{\sigma^-(q_k)}|\} \text{ for all } p_k \leq p \leq p_{k+1}$$

and $a_n \to 0$ that $\sum a_{\sigma(n)} = L$. ■

Note that the method we deploy in the proof of Riemann rearrangement Theorem can be used to control the sequence to have arbitrary large set of subsequential limits by modifying the number of A, B in Equation (4.1), (4.2), (4.3) and (4.4).

Using Riemann rearrangement Theorem and equation

$$\max_{1 \leq r \leq d} |x_n| \leq |\mathbf{x}| \leq \sum_{r=1}^{d} |x_r|$$

we can now generalize and strengthen Theorem 4.1.6 to

$$\begin{aligned}\sum \mathbf{x}_n \text{ absolutely converge } &\iff \sum_n x_{n,r} \text{ absolutely converge for all } r\\ &\iff \sum_n x_{n,r} \text{ unconditionally converge for all } r\\ &\iff \sum \mathbf{x}_n \text{ unconditionally converge}\end{aligned}$$

With this in mind, we can now well state the Fubini's Theorem for Double Series.

Theorem 4.1.8. (Fubini's Theorem for Double Series) If

$$\sum_n \sum_k |z_{n,k}| \text{ converge}$$

Then

$$\sum_{n,k} |z_{n,k}| \text{ converge and } \sum_{n,k} z_{n,k} = \sum_n \sum_k z_{n,k} = \sum_k \sum_n z_{n,k}$$

Proof. The fact $\sum z_{n,k}$ absolutely converge follow from

$$\sum_{n=1}^N \sum_{k=1}^N |z_{n,k}| \leq \sum_n \sum_k |z_{n,k}| \text{ for all } N$$

WOLG, it remains to prove

$$\sum_{n,k} z_{n,k} = \sum_n \sum_k z_{n,k}$$

Because $\sum_n \sum_k |z_{n,k}|$ converge, we can reduce the problem into proving the same statement for nonnegative series $a_{n,k}$. (why?)

$$\sum_n \sum_k |a_{n,k}| \text{ converge } \implies \sum_{n,k} a_{n,k} = \sum_n \sum_k a_{n,k}$$

Because

$$\sum_{n=1}^N \sum_{k=1}^N a_{n,k} \leq \sum_{n=1}^N \sum_k a_{n,k} \leq \sum_n \sum_k a_{n,k} \text{ for all } N$$

we see

$$\sum_{n,k} a_{n,k} \leq \sum_n \sum_k a_{n,k}$$

It remains to prove

$$\sum_{n,k} a_{n,k} \geq \sum_n \sum_k a_{n,k}$$

Fix N and ϵ. We reduce the problem into proving

$$\sum_{n,k} a_{n,k} \geq \sum_{n=1}^{N} \sum_k a_{n,k} - \epsilon$$

Let K satisfy

$$\text{For all } 1 \leq n \leq N, \ \sum_{k=K+1}^{\infty} a_{n,k} < \frac{\epsilon}{N}$$

It then follows

$$\sum_{n,k} a_{n,k} \geq \sum_{n=1}^{N} \sum_{k=1}^{K} a_{n,k} \geq \sum_{n=1}^{N} \sum_k a_{n,k} - \epsilon \text{ (done)}$$

■

Example 1 (Counter-Example for Fubini's Theorem for Double Series)

$$a_{n,k} \triangleq \begin{cases} 1 & \text{if } n = k \\ -1 & \text{if } n = k+1 \\ 0 & \text{if otherwise} \end{cases}$$

$$\sum |a_{n,k}| = \infty \text{ and } \sum_n \sum_k a_{n,k} = 1 \text{ and } \sum_k \sum_n a_{n,k} = 0$$

Theorem 4.1.9. (Merten's Theorem for Cauchy Product) Suppose

(a) $\sum_{n=0}^{\infty} z_n$ converge absolutely

(b) $\sum_{n=0}^{\infty} z_n = Z$

(c) $\sum_{n=0}^{\infty} v_n = V$

(d) $w_n = \sum_{k=0}^{n} z_k v_{n-k}$

Then we have

$$\sum_{n=0}^{\infty} w_n = ZV$$

Proof. We prove

$$\left|V\sum_{n=0}^{N} z_n - \sum_{n=0}^{N} w_n\right| \to 0 \text{ as } N \to \infty$$

Compute

$$\begin{aligned}V\sum_{n=0}^{N} z_n - \sum_{n=0}^{N} w_n &= \sum_{n=0}^{N} z_n (V - \sum_{k=0}^{N-n} v_k) \\ &= \sum_{n=0}^{N} z_n \sum_{k=N-n+1}^{\infty} v_k\end{aligned}$$

Because $\sum_{k=n}^{\infty} v_k \to 0$ as $n \to \infty$, we know there exists M such that

$$\left|\sum_{k=n}^{\infty} v_k\right| < M \text{ for all } n$$

Let N_0 satisfy

$$\sum_{n=N_0+1}^{\infty} |z_n| < \frac{\epsilon}{2M}$$

Let $N_1 > N_0$ satisfy

$$\left|\sum_{k=N-N_0+1}^{\infty} v_k\right| < \frac{\epsilon}{2(N_0+1)\sum_n |z_n|} \text{ for all } N > N_1$$

Now observe that for all $N > N_1$

$$\left|\sum_{n=0}^{N} z_n \Big(\sum_{k=N-n+1}^{\infty} v_k\Big)\right| \le \sum_{n=0}^{N_0} |z_n| \left|\sum_{k=N-n+1}^{\infty} v_k\right| + \sum_{n=N_0+1}^{N} |z_n| \left|\sum_{k=N-n+1}^{\infty} v_k\right| < \epsilon \text{ (done)}$$

■

We first define the **limit superior** by

$$\limsup_{n\to\infty} a_n \triangleq \lim_{n\to\infty} (\sup_{k\ge n} a_k)$$

Note that $\limsup_{n\to\infty} a_n$ must exists because $(\sup_{k\ge n} a_k)_n$ is a decreasing sequence.

Theorem 4.1.10. (Equivalent Definition for Limit Superior) If we let E be the set of subsequential limits of a_n

$$E \triangleq \{L \in \overline{\mathbb{R}} : L = \lim_{k\to\infty} a_{n_k} \text{ for some } n_k\}$$

The set E is non-empty and

$$\max E = \limsup_{n\to\infty} a_n$$

Proof. Let $n_1 \triangleq 1$. Recursively, because

$$\sup_{j\geq n_k} a_k \geq \limsup_{n\to\infty} a_n > \limsup_{n\to\infty} a_n - \frac{1}{k} \text{ for each } k$$

We can let n_{k+1} be the smallest number such that

$$a_{n_{k+1}} > \limsup_{n\to\infty} a_n - \frac{1}{k}$$

It is straightforward to check $a_{n_k} \to \limsup_{n\to\infty} a_n$ as $k \to \infty$. Note that no subsequence can converge to $\limsup_{n\to\infty} a_n + \epsilon$ because there exists N such that $\sup_{k\geq N} a_k < \limsup_{n\to\infty} a_n + \epsilon$. ∎

We can now state the **limit comparison test** as follows. Given a positive sequence b_n,

$$\limsup_{n\to\infty} \frac{|z_n|}{b_n} \in \mathbb{R} \text{ and } \sum b_n \text{ converge} \implies \sum z_n \text{ absolutely converge}$$
$$\liminf_{n\to\infty} \frac{b_n}{|z_n|} > 0 \text{ and } \sum z_n \text{ diverge} \implies \sum b_n \text{ diverge}$$

Theorem 4.1.11. (Geometric Series)

$$|z| < 1 \implies \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

Proof. The proof follows from noting

$$(1-z)\sum_{n=0}^{N} z^n = 1 - z^{N+1} \to 1 \text{ as } N \to \infty$$

∎

Theorem 4.1.12. (Ratio and Root Test)

$$\limsup_{n\to\infty} \sqrt[n]{|z_n|} < 1 \text{ or } \limsup_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| < 1 \implies \sum z_n \text{ absolutely converge}$$
$$\liminf_{n\to\infty} \sqrt[n]{|z_n|} > 1 \text{ or } \liminf_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| > 1 \implies \sum z_n \text{ diverge}$$

Proof. The convergent part follows from comparison to an appropriate geometric series and the diverge part follows from noting $|z_n|$ does not converge to 0. ∎

Theorem 4.1.13. (Root Test is Stronger Than Ratio Test)

$$\liminf_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| \leq \liminf_{n\to\infty} \sqrt[n]{|z_n|} \leq \limsup_{n\to\infty} \sqrt[n]{|z_n|} \leq \limsup_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right|$$

Proof. Fix ϵ and WLOG suppose $\liminf_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| > 0$. We prove

$$\liminf_{n\to\infty} \sqrt[n]{|z_n|} \geq \liminf_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| - \epsilon$$

Let $\alpha \in \mathbb{R}$ satisfy

$$\liminf_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| - \epsilon < \alpha < \liminf_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right|$$

Let N satisfy

$$\text{For all } n \geq N, \left|\frac{z_{n+1}}{z_n}\right| > \alpha$$

We then see

$$\sqrt[N+n]{|z_{N+n}|} \geq \sqrt[N+n]{|z_N|\,\alpha^n} = \alpha\left(\frac{|z_N|^{\frac{1}{N+n}}}{\alpha^{\frac{N}{N+n}}}\right) \to \alpha \text{ as } n \to \infty \text{ (done)}$$

The proof for the other side is similar. ∎

Theorem 4.1.14. (Root Test Trick) For all $k \in \mathbb{N}$

$$\limsup_{n\to\infty} |z_{n+k}|^{\frac{1}{n}} = \limsup_{n\to\infty} |z_n|^{\frac{1}{n}}$$

Proof. This is a direct corollary of equivalent definition for limit superior. ∎

Lastly, we prove Cauchy's condensation Test, whose existence is almost solely for investigating p-Series.

Theorem 4.1.15. (Cauchy's Condensation Test) Suppose $a_n \searrow 0$. We have

$$\sum_{n=0}^{\infty} 2^n a_{2^n} \text{ converge } \iff \sum_{n=1}^{\infty} a_n \text{ converge}$$

Proof. Observe that for all $N \in \mathbb{N}$

$$\sum_{n=0}^{N} 2^n a_{2^n} \geq \sum_{n=0}^{N} \sum_{k=1}^{2^n} a_{2^n+k-1} = \sum_{n=1}^{2^{N+1}-1} a_n$$

and

$$2\sum_{n=1}^{2^N-1} a_n = 2\sum_{n=1}^{N} \sum_{k=0}^{2^{n-1}-1} a_{2^{n-1}+k} \geq 2\sum_{n=1}^{N} 2^{n-1} a_{2^n} = \sum_{n=1}^{N} 2^n a_{2^n}$$

∎

Theorem 4.1.16. (p-Series)

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converge } \iff p > 1$$

Proof. Observe that

$$\sum_{n=0}^{\infty} 2^n \frac{1}{(2^n)^p} = \sum_{n=0}^{\infty} (2^{1-p})^n$$

The result then follows from Cauchy's Condensation Test and geometric series. ∎

4.2 Analytic Functions

Abstract

This section introduces the concept of analytic functions and proves some of their basic properties, including the Identity Theorem. We will rely on the tools developed in the previous section on sequences and series. Note that throughout this section, z will always denote a complex number.

In this section, by a **power series**, we mean a pair (z_0, c_n) where $z_0 \in \mathbb{C}$ is called the **center** of power series, and $c_n \in \mathbb{C}$ are the coefficients sequence. By **radius of convergence**, we mean a unique $R \in \mathbb{R}_0^+ \cup \infty$ such that

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n \begin{cases} \text{converge absolutely} & \text{if } |z - z_0| < R \\ \text{diverge} & \text{if } |z - z_0| > R \end{cases}$$

Such R always exist (and is unique, the uniqueness can be checked without computing the actual value of R) and is exactly

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{c_n}} \tag{4.5}$$

This result is called **Cauchy-Hadamard Theorem** and is proved by applying Root Test to $\sum c_n (z - z_0)^n$. Note that Cauchy-Hadamard Theorem does not tell us whether a power series converges at points of boundary of disk of convergence. It require extra works to determine if the power series converge at boundary.

Theorem 4.2.1. (Abel's Test for Power Series) Suppose $a_n \to 0$ monotonically and $\sum a_n z^n$ has radius of convergence R .

$$\text{The power series } \sum a_n z^n \text{ at least converge on } \overline{D_R(0)} \setminus \{R\}$$

Proof. Note that

$$\sum \frac{a_n}{R^n} z^n \text{ has radius of convergence } R$$

Fix $z \in \overline{D_R(0)} \setminus \{R\}$. Note that

$$\left| \sum_{n=0} \frac{z^n}{R^n} \right| = \left| \frac{1 - (\frac{z}{R})^{N+1}}{1 - \frac{z}{R}} \right| \leq \frac{2}{|1 - \frac{z}{R}|} \text{ for all } N$$

It then follows from Dirichlet's Test that $\sum a_n (\frac{z}{R})^n$ converge. ∎

Example 2 (Discussion of Convergence on Boundary)

$$f_q(z) = \sum_{n=0}^{\infty} n^q z^n \text{ provided } q \in \mathbb{R}$$

It is clear that f_q has convergence radius 1 for all $q \in \mathbb{R}$. For boundary, we have

$$\begin{cases} q < -1 \implies f_q \text{ converge on } S^1 \\ -1 \leq q < 0 \implies f_q \text{ converge on } S^1 \setminus \{1\} \\ 0 \leq q \implies f_q \text{ diverge on } S^1 \end{cases}$$

Note that

(a) At $z = 1$, the discussion is just p-series.

(b) $n^q \searrow 0$ if and only if $q < 0$; and if $n^q \searrow 0$, then the series converge by Abel's test for power series.

(c) If $q \geq 0$, $n^q z^n$ does not converge to 0 on $S^1 \setminus \{1\}$

Notice that the fact $\sum c_n(z-z_0)^n$ absolutely converge in $D_R(z_0)$ implies the convergence is uniform on all $\overline{D_{R-\epsilon}(z_0)}$ by M-Test. However, on $D_R(z_0)$, the convergence is not always uniform.

Example 3 (Failure of Uniform Convergence on $D_R(z_0)$)

$$f(z) = \sum_{n=0}^{\infty} z^n$$

Note $R = 1$. Use Geometric series formula to show $f(z) = \frac{1}{1-z}$ on $D_1(0)$. It is then clear that f is unbounded on $D_1(0)$ while all partial sums $\sum_{k=0}^{n} z^k$ is bounded on $D_1(0)$.

We now introduce some terminologies. We say a complex function f is **analytic at** $z_0 \in \mathbb{C}$ if f there exists a power series (z_0, c_n) whose convergence radius is greater than 0 and f agrees with $\sum_{n=0}^{\infty} c_n(z - z_0)^n$ on $D_R(z_0)$ for some R (of course, such R must not be strictly greater than the radius of convergence of (a, c_n)). It shall be quite clear that if f, g are both analytic at $z \in \mathbb{C}$ with radius $R_f \leq R_g$, then by Merten's Theorem for Cauchy product, $f + g$ and fg are analytic at z with radius at least R_f. We now investigate deeper into analytic functions.

Theorem 4.2.2. (Term by Term Differentiation) Given a power series (z_0, c_n) of convergence radius $R > 0$, if we define $f : D_R(z_0) \to \mathbb{C}$ by

$$f(z) \triangleq \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

Then f is holomorphic on $D_R(z_0)$ and its derivative at z_0 is also a power series with radius of convergence R

$$f'(z) = \sum_{n=0}^{\infty} (n+1) c_{n+1} (z - z_0)^n$$

Proof. Because $(n+1)^{\frac{1}{n}} \to 1$, we can use Theorem 4.1.14 to deduce

$$\limsup_{n\to\infty} \left((n+1) \left| c_{n+1} \right| \right)^{\frac{1}{n}} = \limsup_{n\to\infty} \left| c_{n+1} \right|^{\frac{1}{n}} = \limsup_{n\to\infty} \left| c_n \right|^{\frac{1}{n}}$$

which implies that the power series $\sum_{n=0}^{\infty} (n+1) c_{n+1} (z - z_0)^n$ is of radius of convergence R. We now prove

$$f'(z) = \sum_{n=0}^{\infty} (n+1) c_{n+1} (z - z_0)^n \text{ on } D_R(z_0)$$

Define $f_m : D_R(z_0) \to \mathbb{C}$ by

$$f_m(z) \triangleq \sum_{n=0}^{m} c_n (z - z_0)^n$$

Observe

(a) $f_m \to f$ pointwise on $D_R(a)$

(b) $f'_m(z) = \sum_{n=0}^{m-1} (n+1) c_{n+1} (z - z_0)^n$ for all m

Fix $z \in D_R(z_0)$. Proposition (b) allow us to reduce the problem into proving

$$f'(z) = \lim_{m\to\infty} f'_m(z) \text{ on } D_R(a) \tag{4.6}$$

Let $z \in D_r(z_0)$ where $r < R$. With proposition (a) in mind, to show Equation 4.6, by Theorem **??**, we only have to prove f'_m uniformly converge on $D_r(z_0)$, which follows from M-Test and the fact that $\sum_{n=0}^{\infty} (n+1) c_{n+1} (z - z_0)^n$ absolutely converge on $D_R(z_0)$. (done) ■

Suppose

$$f(z) \triangleq \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

Now by repeatedly applying Theorem 4.2.2, we see

$$f^{(k)}(z) = \sum_{n=0}^{\infty} (n+k) \cdots (n+1) c_{n+k} (z - z_0)^n \text{ for all } k \in \mathbb{Z}_0^+ \tag{4.7}$$

This then give us

$$c_k = \frac{f^{(k)}(z_0)}{k!} \text{ for all } k \in \mathbb{Z}_0^+ \tag{4.8}$$

and

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots \text{ on } D_R(z_0) \tag{4.9}$$

Equation 4.9 is often called the **Taylor expansion of f at z_0**. Notably, Equation 4.8 tell us that if f is constant 0, then $c_n = 0$ for all n.

Example 4 (Smooth but not Analytic Function)

$$f(x) = \begin{cases} e^{\frac{-1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Use induction to show that

$$f^{(k)}(x) = P_k(\frac{1}{x}) e^{-(\frac{1}{x})^2} \quad \exists P_k \in \mathbb{R}[x^{-1}], \forall k \in \mathbb{Z}_0^+, \forall x \in \mathbb{R}^*$$

and again use induction to show that

$$f^{(k)}(0) = 0 \quad \forall k \in \mathbb{Z}_0^+$$

The trick to show $f^{(k)}(0) = 0$ is let $u = \frac{1}{x}$.

Now, with Theorem 4.2.2, we see that f is not analytic at 0.

Example 5 (Bump Function)

$$f(x) = \begin{cases} e^{\frac{-1}{1-x^2}} & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

Use the same trick (but more advanced) to show f is smooth, and note that f is not analytic at ± 1.

Now, it comes an interesting question. Given a complex-valued function f analytic at z_0 with radius R, and suppose $z_1 \in D_R(z_0)$.

(a) Is f also analytic at z_1?

(b) What do we know about the radius of convergence of f at z_1?

(c) Suppose f is indeed analytic at z_1. It is trivial to see that the power series $(z_0, c_{0;n})$ and $(z_1, c_{1;n})$ must agree in the intersection of their convergence disks, and because f is given, we by Theorem 4.2.2 and Equation 4.8, have already known the value of $c_{1;n}$. Can we verify that the power series $(z_0, c_{0;n})$ and $(z_1, c_{1;n})$ do indeed agree with each other on the common convergence interval?

Taylor's Theorem for power series give satisfying answers to these problems.

Theorem 4.2.3. (Taylor's Theorem for Power Series) Given a function f analytic at z_0 with radius R, and suppose $z_1 \in D_R(z_0)$. Then

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_1)}{k!}(z - z_1)^k \text{ on } D_{R-|z_1-z_0|}(z_1)$$

Proof. WOLG, let $z_0 = 0$. Suppose z satisfy $|z - z_1| < R - |z_1|$. By Equation 4.8, we can compute

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(z - z_1 + z_1)^k \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \sum_{n=0}^{k} \binom{k}{n} (z - z_1)^n z_1^{k-n} \\ &= \sum_{k=0}^{\infty} \sum_{n=0}^{k} \frac{f^{(k)}(0)}{k!} \binom{k}{n} (z - z_1)^n z_1^{k-n} \end{aligned}$$

Note that

$$\begin{aligned}
\sum_{k=0}^{\infty}\left|\sum_{n=0}^{\infty}\frac{f^{(k)}(0)}{k!}\binom{k}{n}(z-z_1)^n z_1^{k-n}\right| &\leq \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\left|\frac{f^{(k)}(0)}{k!}\right|\binom{k}{n}|z-z_1|^n\cdot|z_1|^{k-n} \\
&= \sum_{k=0}^{\infty}\left|\frac{f^{(k)}(0)}{k!}\right|\sum_{n=0}^{\infty}\binom{k}{n}|z-z_1|^n\cdot|z_1|^{k-n} \\
&= \sum_{k=0}^{\infty}\left|\frac{f^{(k)}(0)}{k!}\right|\big(|z-z_1|+|z_1|\big)^k
\end{aligned}$$

is a convergent series, by Cauchy-Hadamard Theorem and $|z-z_1|+|z_1|<R$; thus, we can use Fubini's Theorem for double series to deduce

$$\begin{aligned}
\sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{f^{(k)}(0)}{k!}\binom{k}{n}(z-z_1)^n z_1^{k-n} &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{f^{(k)}(0)}{k!}\binom{k}{n}(z-z_1)^n z_1^{k-n} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}\binom{k}{n}(z-z_1)^n z_1^{k-n} \\
&= \sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{f^{(k)}(0)}{k!}\binom{k}{n}(z-z_1)^n z_1^{k-n} \\
&= \sum_{n=0}^{\infty}\Big[\sum_{k=n}^{\infty}\frac{f^{(k)}(0)}{k!}\binom{k}{n}z_1^{k-n}\Big](z-z_1)^n
\end{aligned}$$

We have reduced the problem into proving

$$\sum_{k=n}^{\infty}\frac{f^{(k)}(0)}{k!}\binom{k}{n}z_1^{k-n} = \frac{f^{(n)}(z_1)}{n!}$$

Because z_1 is in $D_R(0)$, by Equation 4.7 and Equation 4.8, we can compute

$$\begin{aligned}
f^{(n)}(z_1) &= \sum_{k=0}^{\infty}(k+n)\cdots(k+1)\cdot\frac{f^{(n+k)}(0)}{(n+k)!}z_1^k \\
&= \sum_{k=n}^{\infty}(k)\cdots(k-n+1)\cdot\frac{f^{(k)}(0)}{k!}\cdot z_1^{k-n} \\
&= \sum_{k=n}^{\infty}\frac{f^{(k)}(0)}{(k-n)!}z_1^{k-n}
\end{aligned}$$

We now have

$$\frac{f^{(n)}(z_1)}{n!} = \sum_{k=n}^{\infty} \frac{f^{(k)}(0)}{n!(k-n)!} z_1^{k-n} = \sum_{k=n}^{\infty} \frac{f^{(k)}(0)}{k!} \binom{k}{n} z_1^{k-n} \text{ (done)}$$

■

Lastly, to close this section, we prove the Identity Theorem, which is extremely useful in complex analysis.

Theorem 4.2.4. (Identity Theorem) Given two analytic complex-valued function $f, g : D \to \mathbb{C}$ defined on some open connected $D \subseteq \mathbb{C}$, if f, g agree on some subset $S \subseteq D$ such that S has a limit point in D, then f, g agree on the whole region D.

Proof. Define

$$T \triangleq \{z \in D : f^{(k)}(z) = g^{(k)}(z) \text{ for all } k \geq 0\}$$

Since D is connected, we can reduce the problem into proving T is non-empty, open and closed in D. Let c be a limit point of S in D. We first show

$$c \in T$$

Assume $c \notin T$. Let m be the smallest integer such that $f^{(m)}(c) \neq g^{(m)}(c)$. We can write the Taylor expansion of $f - g$ at c by

$$\begin{aligned}(f-g)(z) &= (z-c)^m \Big[\frac{(f-g)^{(m)}(c)}{m!} + \frac{(f-g)^{(m+1)}(c)}{(m+1)!}(z-c) + \cdots \Big] \\ &\triangleq (z-c)^m h(z)\end{aligned}$$

Clearly, $h(c) \neq 0$. Now, because h is continuous at c (h is a well-defined power series at c with radius greater than 0), we see h is non-zero on some $B_\epsilon(c)$, which is impossible, since $(f-g) \equiv 0$ on $S \setminus \{c\}$ implies $h = 0$ on $S \setminus \{c\}$. CaC (done)

Fix $z \in T$. Because f, g are analytic at z and $f^{(k)}(z) = g^{(k)}(z)$ for all k, we see $f - g$ is constant 0 on some open disk $B_\epsilon(z)$. We have proved that T is open. To see T is closed in D, one simply observe that

$$T = \bigcap_{k \geq 0} \{z \in D : (f-g)^{(k)}(z) = 0\}$$

and $(f-g)^{(k)}$ is continuous on D. (done) ■

4.3 L'Hospital Rule

Abstract

This section state and prove the L'Hospital Rule, and provide examples to show the necessity of each hypotheses of L'Hospital Rule. Note that although L'Hospital Rule is not really directly used in most results in Theory of Calculus, it is used in the proof of Taylor's Theorem.

Theorem 4.3.1. (L'Hospital Rule) Let $I \subseteq \mathbb{R}$ be an open interval containing c and let $f, g : I \to \mathbb{R}$ be two function continuous on I and differentiable on I everywhere except possibly at c, where

$$g'(x) \neq 0 \text{ for all } x \in I \setminus \{c\}$$

If $\frac{f}{g}$ is indeterminate form, i.e.

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = L \text{ where } L \in \{0, \infty, -\infty\}$$

and

$$\lim_{x\to c} \frac{f'(x)}{g'(x)} \in \mathbb{R}$$

Then we have

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} \tag{4.10}$$

Proof. Suppose $I = (a, b)$. Note that since $g'(x) \neq 0$ on (c, b), by MVT, we know there exists at most one $x \in (c, b)$ such that $g(x) = 0$. With similar argument for (a, c), we see that

$$g(x) \neq 0 \text{ on } (c - \epsilon, c + \epsilon) \setminus \{c\} \text{ for some } \epsilon$$

We now see that the expression $\lim_{x\to c} \frac{f(x)}{g(x)}$ is at least well-defined, and WOLG, we can suppose $g(x) \neq 0$ on $I \setminus \{c\}$. Define $m, M : I \setminus \{c\} \to \mathbb{R}$ by

$$m(x) \triangleq \inf \frac{f'(t)}{g'(t)} \text{ and } M(x) \triangleq \sup \frac{f'(t)}{g'(t)} \text{ where } t \text{ ranges over values between } x \text{ and } c$$

Because the value $\frac{f'(t)}{g'(t)}$ converge at c, we can deduce

$$\lim_{x\to c} m(x) = \lim_{x\to c} M(x) = \lim_{x\to c} \frac{f'(x)}{g'(x)} \tag{4.11}$$

We now prove

$$\text{the case when } \lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$$

Fix $x \in I \setminus \{c\}$, and WOLG suppose $x < c$. By Cauchy's MVT, we know that

$$m(x) \leq \frac{f(x) - f(y)}{g(x) - g(y)} \leq M(x) \text{ for all } y \in (x, c)$$

Note that $g(x) \neq g(y)$ by MVT, since $g' \neq 0$ on $I \setminus \{c\}$. It now follows from

$$\lim_{y\to c^-} \frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f(x)}{g(x)}$$

that

$$m(x) \leq \frac{f(x)}{g(x)} \leq M(x)$$

The proof of Equation 4.10 then follows from Equation 4.11. (done)

We now prove

$$\text{the case when } \lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \infty$$

Again, fix $x \in I \setminus \{c\}$, and WOLG suppose $x < c$. By Cauchy's MVT, we know that

$$m(x) \leq \frac{f(y) - f(x)}{g(y) - g(x)} \leq M(x) \text{ for all } y \in (x, c)$$

Note that $g(x) \neq g(y)$ by MVT, since $g' \neq 0$ on $I \setminus \{c\}$. It now follows

$$\frac{f(y) - f(x)}{g(y) - g(x)} = \frac{\frac{f(y)}{g(y)} - \frac{f(x)}{g(y)}}{1 - \frac{g(x)}{g(y)}} \to \lim_{x\to c} \frac{f(x)}{g(x)} \text{ as } y \to c^-$$

The proof of Equation 4.10 then follows form Equation 4.11. (done) ∎

Chapter 5

NTU Math M.A. Program Entrance Exam

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Question 1

If every closed and bounded set of a metric space M is compact, does it follows that (M, d) is complete?

Proof. Yes. Let $\{x_n\}$ be a Cauchy sequence in M. To prove $\{x_n\}$ converge in M, one let E be the closure of $\{x_n\}$, and prove that E is indeed bounded. This by premise implies E is compact. Therefore we may apply **Hiene-Borel Theorem** to show that $\{x_n\}$ converge in $E \subseteq M$. ∎

Question 2

Determine the values of h for which the following series converges uniformly on $I_h = \{x \in \mathbb{R} : |x| \leq h\}$.

$$\sum_{n=1}^{\infty} \frac{(n!)^2 x^n}{(2n)!} \tag{5.1}$$

Proof. Defining $c_n \triangleq (n!)^2 \diagup (2n)!$, we may write Series 5.1 as $\sum c_n x^n$. Because

$$\frac{c_{n+1}}{c_n} = \frac{(n+1)^2}{(2n+2)(2n+1)} \text{ converges to } \frac{1}{4}$$

We may apply Theorem 4.1.13 to deduce $\sqrt[n]{c_n} \to \frac{1}{4}$. This together with Weierstrass M-test and Cauchy-Hadamard Theorem implies Series 5.1 converges uniformly on I_h at least for any $h < 4$.

$\blacksquare$

Question 3

Consider

$$F(x) \triangleq \int_0^\infty \frac{e^{-xt} - e^{-t}}{t} dt$$

on $I \triangleq \{x \in \mathbb{R} : \frac{1}{2} \leq x \leq 2\}$.

(i) Show that F is defined on I and F is continuous on I.

(ii) Show that

$$F'(x) = \int_0^\infty -e^{-xt} dt$$

(iii) Evaluate $F(x)$

Proof. Fix $x \in [\frac{1}{2}, 2]$, and define $f(t) \triangleq (e^{-xt} - e^{-t})\big/t$. We first show:

(i) f is integrable as $t \to 0$.

(ii) f is integrable as $t \to \infty$.

The former is easy, as applying L'Hospital rule, we see f moreover converges at 0^+. For the latter, we have to observe that because $x \geq \frac{1}{2}$, we have:

$$\left|e^{-xt} - e^{-t}\right| \leq \left|e^{-xt}\right| + \left|e^{-t}\right| \leq e^{\frac{-t}{2}} + e^{-t} \leq 2e^{\frac{-t}{2}} \tag{5.2}$$

for any large t. The latter now follows from **Comparison test** and the fact e^{-ct} integrable as $t \to \infty$ for positive c. We now prove (ii) using **measure-theoretical Feymann's trick**, and the continuity of F will follow. To perform Feymann's trick, we have to establish the existence of some L^1 function $g : (0, \infty) \to \mathbb{R}$ that dominate $f : (0, \infty) \to \mathbb{R}$ for all $x \in [\frac{1}{2}, 2]$. As bound 5.2 have shown,

$$g(t) \triangleq \begin{cases} M & \text{if } t \text{ is small} \\ \frac{2e^{\frac{-t}{2}}}{t} & \text{if } t \text{ is large} \end{cases}$$

suffices. Lastly, we compute $F(x)$ on $[1/2, 2]$ using **FTC**:

$$F(1) = 0 \quad \text{and} \quad F'(x) = \int_0^\infty -e^{-xt}dt = \frac{1}{-x} \implies F(x) = \ln\left(\frac{1}{x}\right)$$

■

Question 4

Consider smooth $f : \mathbb{R}^n \to \mathbb{R}$ whose **Hessian Determinant** is 2 everywhere. We denote its **gradient** $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ by F.

(i) Show that there exists neighborhood $U \subseteq \mathbb{R}^n$ of origin such that the restriction of F forms a smooth diffeomorphism from U to the image of U. Note that the image is guaranteed to be open by **invariance of domain** if we really prove that the action is injective on U.

(ii) Denote the inverse map in part (i) by $\boldsymbol{\xi}(\mathbf{y}) \triangleq (\xi_1(\mathbf{y}), \dots, \xi_n(\mathbf{y}))$. For any $\mathbf{y}$ in the image of U, define

$$f^*(\mathbf{y}) \triangleq -f(\boldsymbol{\xi}(\mathbf{y})) + \sum_{i=1}^n y_i \xi_i(\mathbf{y})$$

Compute the Hessian Determinant of f^*.

Proof. (i) is a direct consequence of the **Inverse function theorem** and the observation that $dF = \operatorname{Hess} f$. Fix k, and compute:

$$\begin{aligned}
\left.\frac{\partial f^*}{\partial y_k}\right|_{\mathbf{y}} &= -\left(\sum_{j=1}^n \left.\frac{\partial f}{\partial x_j}\right|_{\boldsymbol{\xi}(\mathbf{y})} \left.\frac{\partial \xi_j}{\partial y_k}\right|_{\mathbf{y}}\right) + \left(\sum_{i \neq k} y_i \left.\frac{\partial \xi_i}{\partial y_k}\right|_{\mathbf{y}}\right) + \xi_k(\mathbf{y}) + y_k \left.\frac{\partial \xi_k}{\partial y_k}\right|_{\mathbf{y}} \\
&= -\left(\sum_{j=1}^n y_j \left.\frac{\partial \xi_j}{\partial y_k}\right|_{\mathbf{y}}\right) + \left(\sum_{i \neq k} y_i \left.\frac{\partial \xi_i}{\partial y_k}\right|_{\mathbf{y}}\right) + \xi_k(\mathbf{y}) + y_k \left.\frac{\partial \xi_k}{\partial y_k}\right|_{\mathbf{y}} \\
&= \xi_k(\mathbf{y})
\end{aligned}$$

Note that the second equality hold true by definition of $\boldsymbol{\xi}$. We now see that the matrix

$$\operatorname{Hess} f^* = (d\boldsymbol{\xi})^t = d\boldsymbol{\xi} = (dF)^{-1} = (\operatorname{Hess} f)^{-1}$$

always have determinant $\frac{1}{2}$. ■

Question 5

Let C^1 function $f : [0, \infty) \to \mathbb{R}$ satisfies:

(i) $f(x) \geq 0$ and $f'(x) \leq 1$ for all $x \geq 0$.

(ii) $\int_0^\infty f(x)dx$ converges.

Does $f(x)$ converge as $x \to \infty$?

Proof. Yes. We will moreover show that $f(x)$ converge to 0 as $x \to \infty$. Assume for a contradiction that there exists some ϵ and $x_n \nearrow \infty$ such that $f(x_n) \geq \epsilon$ for all n and WLOG, $x_{n+1} - x_n \geq \epsilon$. Fix n. Because f is C^1 with derivative always smaller than 1, by **MVT**, every $x \in [x_n - \epsilon, x_n)$ must satisfies $[f(x_n) - f(x)] \big/ (x_n - x) \leq 1$. In other words, every $x \in [x_n - \epsilon, x_n)$ satisfies

$$\epsilon - (x_n - x) \leq f(x_n) - (x_n - x) \leq f(x)$$

We may now estimate

$$\int_{x_n - \epsilon}^{x_n} f(x)dx \geq \int_{x_n - \epsilon}^{x_n} \epsilon - (x_n - x)dx = \frac{\epsilon^2}{2}$$

This cause a contradiction:

$$\int_0^\infty f(x)dx \geq \sum_n \int_{x_n - \epsilon}^{x_n} f(x)dx = \infty$$

■

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Question 6

Let

$$M \triangleq \left\{ f : [0,\infty] \to [0,\infty) \middle| \int_0^\infty f^2(x)dx \le 1 \right\}$$

Evaluate

$$\sup_{f\in M} \int_0^\infty f(x)e^{-x}dx \tag{5.3}$$

Proof. Note that M is the space of nonnegative functions whose L^2 norm is smaller than 1, and note that value 5.3 equals to $\sup \|e^{-x}f\|_1$. By **Holder inequality**, definition of M, and direct computation, we see that for all $f \in M$,

$$\left\|e^{-x}f\right\|_1 \le \|f\|_2 \cdot \left\|e^{-x}\right\|_2 \le \left\|e^{-x}\right\|_2 = \frac{1}{\sqrt{2}}$$

Because $\|e^{-x}\|_2 = 1/\sqrt{2}$, we know $\sqrt{2}e^{-x} \in M$. It is easy to compute that when $f = \sqrt{2}e^{-x}$, we have $\|e^{-x}f\|_1 = 1/\sqrt{2}$. Therefore value 5.3 equals to $1/\sqrt{2}$.

Remark: The solution is (only) obvious if one have learned some real analysis. ■

Question 7

Let $a \in A \subseteq \mathbb{R}^n$, set A be compact, and all convergent subsequences of the sequence $(a_n) \subseteq A$ converge to a.

(i) Does (a_n) converge to a?

(ii) If we remove the hypothesis of A being compact, does (a_n) converge to a?

Proof. For (i), yes. Assume not for a contradiction. There exists some ϵ and subsequence (a_{n_k}) such that every a_{n_k} is ϵ-away from a. By definition of compact metric space, there exists some convergent subsequence of (a_{n_k}), while being a subsequence of (a_n), converges to a point other than a, a contradiction. For (ii), no. Let $A \triangleq [0,1)$ and a_n be $\frac{1}{n}$ if n is even and $1 - \frac{1}{n}$ if n is odd. ■

Question 8

Let A be some compact metric space, and let continuous $f : A \to A$ never maps two points strictly closer to each other. Show that f is onto.

Proof. Assume $a_0 \in A - f(A)$ for a contradiction, and define $a_n \triangleq f^n(a_0)$. Because f is continuous and A is compact, the image $f(A)$ is compact. This implies the existence of some positive real r smaller than the distance between a_0 and p for any $p \in f(A)$. Because $a_n \in f(A)$ for all positive n, we know $d(a_n, a_0) \geq r$ for every positive n. Note that by induction,

$$d(a_n, a_{n+1}) = d(f(a_{n-1}), f(a_n)) \geq d(a_{n-1}, a_n) \geq r, \quad \text{for all } n$$

which by induction implies

$$d(a_n, a_m) = d(f(a_{n-1}), f(a_{m-1})) \geq d(a_{n-1}, a_{m-1}) \geq r, \quad \text{for all } 0 \leq n < m. \tag{5.4}$$

The fact that $\{a_n\}$ is a sequence in $f(A)$ that satisfies inequality 5.4 contradicts to the sequential compactness of $f(A)$. ∎

Question 9

Define a sequence of function $\{f_n(x)\}$ on $[0, 1]$ as:

$$f_n(x) \triangleq \begin{cases} 1 & \text{if } x = 0 \\ 1 & \text{if } x \in (\frac{2k}{2^n}, \frac{2k+1}{2^n}], k \in \{0, 1, \dots, 2^{n-1} - 1\} \\ -1 & \text{if } x \in (\frac{2k+1}{2^n}, \frac{2k+2}{2^n}], k \in \{0, 1, \dots, 2^{n-1} - 1\} \end{cases}$$

Let g be a continuous function. Prove or disprove that $\lim_{n\to\infty} \int_0^1 f_n g dx$ always converge to 0.

Remark. The original question, as stated, lacks precision. It doesn't specify the domain of the function g. If g is only defined on $(0, 1)$, then the product $f_n g$ may fail to be integrable on $(0, 1)$, making the expression $\int_0^1 f_n g \, dx$ ill-defined—for instance, if $g(x) = \frac{1}{x}$. At a minimum, the question should require $g \in L^1(0, 1)$, so that each $f_n g$ is integrable on $(0, 1)$. With this added hypothesis, one can show that the sequence of integrals $\int_0^1 f_n g \, dx$ always converges to zero, regardless of whether g has singularities at the endpoints.

I will just suppose g is continuous on $[0, 1]$.

Proof. Write

$$\int_0^1 f_n g dx = \sum_{k=0}^{2^{n-1}-1} \int_{\frac{2k}{2^n}}^{\frac{2k+1}{2^n}} g(x) - g(x + \frac{1}{2^n}) dx$$

Because **continuous function on a compact domain is uniformly continuous**, for all ϵ, there exists some N such that

$$\left| g(x) - g(x + \frac{1}{2^n}) \right| \leq \epsilon, \quad \text{for all } n \geq N \text{ and } x \in [0, 1 - \frac{1}{2^n}]$$

We may now estimate:

$$\left| \int_0^1 f_n g dx \right| \leq \sum_{k=0}^{2^{n-1}-1} \int_{\frac{2k}{2^n}}^{\frac{2k+1}{2^n}} \left| g(x) - g(x + \frac{1}{2^n}) \right| dx \leq \sum_{k=0}^{2^{n-1}-1} \frac{\epsilon}{2^n} = \frac{\epsilon}{2}$$

for all $n \geq N$. ∎

Question 10

Let P_2 be the set of real polynomial with degree no greater than 2, and let $S \triangleq \{p \in P_2 : p(1) = 1\}$. Define $G : P_2 \to \mathbb{R}$ by

$$G(p) \triangleq \int_0^1 p^2(x) dx$$

Does $G|_S$ have any extreme value? If it does, find them.

Remark: The original phrasing of the question is misleading.

Proof. Because $F : \mathbb{R}^3 \to P_2, (a, b, c) \mapsto ax^2 + bx + c$ is surjective, the set on which $G|_S$ attains extreme values is exactly the image of the set on which $G \circ F|_{F^{-1}(S)}$ attains extreme values under F. We have transformed our **optimization problem** to the form in which the **objective function** is:

$$G \circ F(a, b, c) = \frac{a^2}{5} + \frac{ab}{2} + \frac{2ac + b^2}{3} + bc + c^2$$

and there is only one **constraint function**, being:

$$g(a, b, c) \triangleq a + b + c - 1$$

This is best solved using **Lagrange multiplier theorem for single constraint**. Note that there are lots of versions of Lagrange multiplier theorem for single constraint, and one of them requires:

(i) Both objective and constraint functions are C^1.

(ii) The dimension of codomain of the single constraint function is no greater than the dimension of its domain.

Moreover, Lagrange multiplier theorem can only detect local extremum on which the Jacobian of the constraint function is full rank. As harsh as these conditions seems, note that our case clearly satisfies both hypotheses and note that the Jacobian of our constraint function g is full rank globally, so we don't need to worry about any of these conditions.

Compute:

$$\nabla(G \circ F)(a,b,c) = \left(\frac{2a}{5} + \frac{b}{2} + \frac{2c}{3}, \frac{a}{2} + \frac{2b}{3} + c, \frac{2a}{3} + b + 2c\right)$$

Compute:

$$\nabla g(a,b,c) = (1,1,1)$$

We are now required to solve a system of linear equations with four unknown:

$$\begin{cases} \frac{2}{5}a + \frac{1}{2}b + \frac{2}{3}c & = \lambda \\ \frac{1}{2}a + \frac{2}{3}b + c & = \lambda \\ \frac{2}{3}a + b + 2c & = \lambda \\ a + b + c & = 1 \end{cases}$$

One may verify that there is only one solution, i.e.,

$$(a,b,c) = \left(\frac{10}{3}, -\frac{8}{3}, \frac{1}{3}\right) \text{ and } \lambda = \frac{2}{9}$$

It is easy to see that this is a local minimum. Because the Jacobian of our constraint function g is full rank globally, we know this local minimum is the only local extremum and also a global minimum. ■

Question 11

Let $C \in \mathbb{R}^+$, function $f : [a,b] \to \mathbb{R}$ be proper Riemann integrable, and $g : \mathbb{R} \to \mathbb{R}$ be C-Lipschitz continuous. Prove that $g \circ f : [a,b] \to \mathbb{R}$ is proper Riemann integrable.

Remark: The original question write "Riemann integrable" instead of "proper Riemann integrable." The statement doesn't hold true if we include improper Riemann integrable f. For example, let $[a,b] \triangleq [0,1]$, $f(x) \triangleq (\sin(1/x))/x$, and $g(x) \triangleq |x|$.

Proof. There are essentially two way to "answer" this question. One is to construct an honest bound. Another is to cheat by quoting **Lebesgue's criteria for proper Riemann integrability**. The honest bound proof for this question and the proof for Lebesgue's criteria for proper Riemann integrability are morally the same. To quote Lebesgue's criteria for proper Riemann integrability, one only have to prove that $g \circ f$ is continuous at $x \in [a, b]$ if f is continuous at x. ∎

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Question 12

Show that C^1 **proper** map $f : \mathbb{R}^n \to \mathbb{R}^n$ (proper as preimage of compact subspace is compact) whose Jacobian is globally full rank is surjective.

Proof. Because $\mathbb{R}^n$ is connected, to prove f is surjective we only have to prove image of f is clopen in $\mathbb{R}^n$. The image of f is open in $\mathbb{R}^n$ by **Inverse function theorem**. The image of f is closed in $\mathbb{R}^n$ is morally a topological result. In particular, we prove Theorem 5.3.1, which finish the proof. ∎

Theorem 5.3.1. (Proper continuous map from locally compact Hausdorff space to Hausdorff space is closed.) Given two locally compact Hausdorff topological spaces X and Y, if preimage of compact set under continuous $f : X \to Y$ is always compact, then f as a map from X to Y is a closed map.

Proof. Fix closed $A \subseteq X$ and $y \in \overline{f(A)}$. We are required to find $x \in A$ mapped to y. Because $y \in \overline{f(A)}$ and Y is locally compact, WLOG, there exists some sequence $\{x_n\} \subseteq A$ and compact set $K \subseteq Y$ such that

(i) y and $\{f(x_n)\}$ are all contained by K.

(ii) $f(x_n)$ converges to y.

By premise, $f^{-1}(K)$ is compact. Because $A \cap f^{-1}(K)$ is closed in the compact space $f^{-1}(K)$, we know $A \cap f^{-1}(K)$ is compact[1]. It now follows from X being Hausdorff that $A \cap f^{-1}(K)$ is closed[2]. Because $\{x_n\} \subseteq A \cap f^{-1}(K)$ and $A \cap f^{-1}(K)$ is compact Hausdorff, we know there exists some subsequence $\{x_{n_k}\}$ that converges to some $x' \in A \cap f^{-1}(K)$.[3] Because f is continuous, we know $f(x_{n_k})$ converges to $f(x')$. It now follows from Y being Hausdorff that $y = \lim f(x_{n_k}) = f(x')$. We have shown this x' suffices. ∎

Question 13

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be the identity map, and define $M \triangleq \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq z \leq \sqrt{2 - x^2 + y^2}\}$. Let the oriented atlas of M contains identity, so $dx \wedge dy \wedge dz \in \Omega^3(M)$ is a positively oriented volume form. Denote $dA \triangleq \iota(dx \wedge dy \wedge dz) \in \Omega^2(\partial M)$. Compute

$$\int_{\partial M} F \cdot \mathbf{n} dA$$

[1] Closed subset of compact space is compact. This is a standard topological proposition

[2] Compact subspace of Hausdorff space is closed. This is a standard topological proposition.

[3] This is a standard topological proposition, generalized of **Bolzano-Weierstrass Theorem**.