RTFT HW2

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Problem A. Suppose that ϕ and τ are equivalent representations of the same group G

Show

(a) If ϕ is faithful, then so is τ.
(b) If ϕ is irreducible, then so is τ.
(c) If ϕ is of degree 1, then $\phi = \tau$. (They are exactly the SAME, not just being equivalent)

Definition 1. *V is the FG-module defined by $\forall v \in V, g(v) = \phi(g)v$*

Definition 2. *V' is the FG-module defined by $\forall v' \in V', g(v') = \tau(g)v'$*

Definition 3. *$n = dim(V) = dim(V')$* ***(V and V' have the same dimension since ϕ and τ are equivalent, so this is well defined)***

Definition 4. *E is the standard ordered basis of V*

Definition 5. *T is the matrix such $\forall g \in G, T\phi(g)T^{-1} = \tau(g)$*

Definition 6. *$A_g = \tau(g)$ and $\phi(g) = T^{-1}A_gT$*

Definition 7. *Let $B \in M(n, F)$. $(B)_i$ is the i-th column vector of B*

(a)

Proof. We know $\phi(g) = I \iff g = e$

$\tau(g) = I \iff T\phi(g)T^{-1} = \tau(g) = I \iff T\phi(g) = T \iff \phi(g) = T^{-1}T = I \iff g = e$

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(b)

Proof. Assume V' have a submodule W' of dimension k

Let W' have a basis $\alpha' = \{w_1', w_2', \dots, w_k'\}$ and extend it to a basis $\beta' = \{w_1', \dots, w_n'\}$ of V'

So $\forall g \in G, \forall 1 \leq i \leq k, A_gw_i' = \tau(g)w_i' = g(w_i') \in W' \implies \forall g \in G, \forall 1 \leq i \leq k$, $A_gw_i' = \sum_{i=1}^{k} c_iw_i', \exists\{c_i \in F | i \leq 1 \leq k\}$

Let $M', M \in GL(n, F)$ respectively be defined by $\forall 1 \leq i \leq k, M_i' = w_i'$ and $M = T^{-1}M'$

M' is an invertible matrix since the set of all column vectors $\{M_1', M_2', \ldots, M_n'\}$ of M' is a basis of V'. This give us $M = T^{-1}M'$ is invertible, thus the set of all column vectors $\{M_1, M_2, \ldots M_k\}$ of M, which we now denote as $\{M_1, M_2, \ldots M_k\} = \beta = \{w_1, w_2, \ldots, w_n\}$, is a basis of V

We now prove $span(w_1, w_2, \ldots w_k)$ is a submodule of V, which CaC

$M = T^{-1}M' \implies TM = M' \implies \forall 1 \leq i \leq n, TM_i = M_i' \implies \forall 1 \leq i \leq n, Tw_i = w_i' \implies \forall 1 \leq i \leq n, w_i = T^{-1}w_i'$

$\forall 1 \leq i \leq k, g(w_i) = \phi(g)w_i = T^{-1}A_gTw_i = T^{-1}A_g{w_i}' = T^{-1}\sum_{i=1}^{k} c_iw_i' = \sum_{i=1}^{k} c_iT^{-1}w_i' = \sum_{i=1}^{k} c_iw_i \in span(w_1, w_2, \ldots, w_k), \exists\{c_i \in F|1 \leq i \leq k\}$

So $\forall w \in span(w_1, w_2, \ldots, w_k), g(w) = \sum_{i=1}^{k} d_ig(w_i) \in span(w_1, w_2, \ldots, w_k), \exists\{d_i \in F|1 \leq i \leq k\}$ (done)

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(c)

Proof. $\forall g \in G, \phi(g) \in GL(1, F)$

Notice a one-dimensional matrix satisfy commutative law by triviality.

So $\forall g \in G, \tau(g) = T\phi(g)T^{-1} = \phi(g)TT^{-1} = \phi(g)$

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Problem B. Let V be a H-module, and $\theta : G \to H$ be a homomorphism. Show V is also a G-module by the following action

$$g(v) := \theta(g)(v)$$

Proof. $\forall g, l \in G, \forall v \in V, g(l(v)) = g(\theta(l)(v)) = \theta(g)(\theta(l)(v)) = (\theta(g)\theta(l))(v) = (\theta(gl))(v) = (gl)(v)$

Let e' be the identity element of H, $\forall v \in V, e(v) = \theta(e)v = e'v = v$

V is at least a G-set

$\forall g \in G, \forall v, w \in V, g(v+w) = \theta(g)(v+w) = \theta(g)(v)+\theta(g)(w) = g(v)+g(w)$

$\forall g \in G, \forall v \in V, \forall c \in F, cg(v) = c\theta(g)(v) = \theta(g)(cv) = g(cv)$

Every g have action that is a linear transformation. ■

Problem C. Recall the quaternion group $Q_8 = \langle a, b | a^4 = b^4 = 1, a^2 = b^2, ba = a^{-1}b\rangle$. Define the map $\psi : Q_8 \to GL_4(\mathbb{C})$ by

$$\psi(a) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \psi(b) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(1) Show that ψ give a representation of Q_8

(2) Show that ψ is faithful

(1)

Proof. $\psi(a)^4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^4 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$

$$\psi(b)^4 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}^4 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$$

$$\psi(a)^2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \psi(b)^2$$

$$\psi(b)\psi(a) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\psi(a)^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\psi(a)^{-1}\psi(b) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} = \psi(b)\psi(a)$$

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(2)

Proof. Let $S = \{e, a, b, ab, a^3, b^3, ab^3, a^2\} \subseteq Q_8$

We show $\psi[S]$ is a set of 8 distinct matrices, so $\psi : Q_8 \to \psi[Q_8]$ must be onto, then by the fact that $|Q_8| = 8 < \infty$, ψ is faithful.

$\psi(e) = I_4$

$$\psi(a) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\psi(b) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\psi(ab) = \psi(a)\psi(b) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\psi(a^3) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\psi(b^3) = \psi(b)^3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\psi(ab^3) = \psi(a)\psi(b)^3 \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\psi(a^2) = \psi(a)^2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

They are obviously all distinct to each other

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