Calculus HW5

Date: Mar 13 Made by Eric

1.

Proof. a) open, simply connected

b) open

c) closed, connected

d) open, connected ■

2.

2.(a)

Proof.

$$\frac{\partial P}{\partial y} = \frac{d}{dy}\frac{-y}{x^2+y^2} = \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \tag{1}$$

$$\frac{\partial Q}{\partial x} = \frac{d}{dx}\frac{x}{x^2+y^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \tag{2}$$

■

2.(b)

Proof. Define

$$r_1(t) := (\cos t, \sin t) \text{ and } r_2(t) := (\cos -t, \sin -t) \tag{3}$$

Define

$$C_1 = r_1(t), 0 \le t \le \pi \text{ and } C_2 = r_2(t), 0 \le t \le \pi \tag{4}$$

Notice that C_1 and C_2 both starts at $(0,0)$ and $(-1,0)$, so if F is conservative, then we should see $\int_{C_1} F \cdot dr_1 = \int_{C_2} F \cdot dr_2$.

Compute

$$F(r_1(t)) = (-\sin t, \cos t) \text{ and } F(r_2(t)) = (\sin t, \cos t) \tag{5}$$

$$r_1'(t) = (-\sin t, \cos t) \text{ and } r_2'(t) = (-\sin t, -\cos t) \tag{6}$$

$$F(r_1(t)) \cdot r_1'(t) = 1 \text{ and } F(r_2(t)) \cdot r_2(t) = -1 \tag{7}$$

$$\int_{C_1} F \cdot dr_1 = \int_0^\pi F(r_1(t)) \cdot r_1'(t) dt = \int_0^\pi dt = \pi \neq -\pi = \int_0^\pi -dt = \int_{C_2} F \cdot dr_2 \tag{8}$$

Notice that the domain of F is not simply-connected, so it does not contracted to the theorem. ■

3.

3.(a)

Proof. Define

$$r(t) := \langle x_1, y_1\rangle(1-t) + t\langle x_2, y_2\rangle \tag{9}$$

So we can express C in the form

$$C = \{r(t), 0 \le r \le 1\} \tag{10}$$

Define

$$F := \langle -y, x\rangle \tag{11}$$

So we can compute

$$I := \int_C xdy - ydx = \int_0^1 F(r(t)) \cdot r'(t)dt \tag{12}$$

Observe

$$r'(t) = \langle x_2 - x_1, y_2 - y_1\rangle \text{ and } F(r(t)) = \langle -y_1 - t(y_2 - y_1), x_1 + t(x_2 - x_1)\rangle \tag{13}$$

Then

$$I = \int_0^1 -(x_2 - x_1)[y_1 + t(y_2 - y_1)] + (y_2 - y_1)[x_1 + t(x_2 - x_1)]dt \tag{14}$$

$$= \int_0^1 -(x_2 - x_1)y_1 + (y_2 - y_1)x_1 dt = \int_0^1 -x_2y_1 + x_1y_2 dt = -x_2y_1 + x_1y_2 \tag{15}$$

■

3.(b)

Proof.

$$I := A = \frac{1}{2}\int\int_D 2dA = \frac{1}{2}\int\int_D \frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}dA = \frac{1}{2}\int_C -ydx + xdy \tag{16}$$

$$I = \frac{1}{2}(\int_{C_1} -ydx + xdy + \int_{C_2} -ydx + xdy + \cdots + \int_{C_n} -ydx + xdy) \tag{17}$$

$$I = \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_ny_1 - x_1y_n)] \tag{18}$$

■

3.(c)

Proof. Notice the counterclockwise order is $(0,0) \to (2,1) \to (1,3) \to (0,2) \to (-1,1)$. Use the formula from part (b) we have $A = 4$

■

4.

4.(a)

Proof.

$$\frac{1}{2A}\int_C x^2 dy = \frac{1}{2A}\int\int_D 2xdA = \frac{1}{A}\int\int_D xdA = \overline{x} \tag{19}$$

$$\frac{1}{2A}\int_C y^2 dx = \frac{1}{2A}\int\int_D 2ydA = \frac{1}{A}\int\int_D ydA = \overline{y} \tag{20}$$

■

4.(b)

Proof. Define

$$D := \{(x,y) : 0 \le x^2 + y^2 \le a^2, 0 \le x, y \le a\} \tag{21}$$

Let $(\overline{x}, \overline{y})$ be the centroid of D, and let C be the boundary of D, so

$$C = \{(x,0) : 0 \le x \le a\} \cup \{(a\cos\theta, a\sin\theta) : 0 \le \theta \le \frac{\pi}{2}\} \cup \{(0,y) : 0 \le y \le a\} \tag{22}$$

Divided the simply close piecewise-smooth curve C into three smooth sub-curve C_1, C_2, C_3, where C_1, C_2, C_3 are respectively the three subsets of C above. Notice the direction we set is countercolockwise.

$$\overline{x} = \frac{1}{2A}\int_C x^2 dy = \frac{1}{2A}(\int_{C_1} x^2 dy + \int_{C_2} x^2 dy + \int_{C_3} x^2 dy) \tag{23}$$

Notice that $dy = 0$ in C_1 and $x = 0$ in C_3, so we can eliminate the integral over C_1 and the integral over C_3. Then, we parametrize with θ

$$\overline{x} = \frac{1}{2A}\int_0^{\frac{\pi}{2}} a^3 \cos^3\theta d\theta = \frac{1}{2A}\frac{2a^3}{3} = \frac{4}{3}a^3 \tag{24}$$

Notice D is symmetric about the line $x = y$, so we can deduce $\overline{y} = \overline{x}$ ■

4.(c)

Proof. We compute by the formula $(\overline{x}, \overline{y}) = (\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}) = (\frac{2a}{3}, \frac{b}{3})$ ■

5.

Let $F := (P, Q, R)$ and $G := (W, S, T)$

5.(a)

$$\nabla \cdot (F+G) = (\frac{\partial P+W}{\partial x}, \frac{\partial Q+S}{\partial y}, \frac{\partial R+T}{\partial z}) \tag{25}$$

$$= (\frac{\partial P}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial R}{\partial z}) + (\frac{\partial W}{\partial x}, \frac{\partial S}{\partial y}, \frac{\partial T}{\partial z}) = \nabla \cdot F + \nabla \cdot G \tag{26}$$

5.(b)

$$\nabla \times (F+G) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (P+W) & (Q+S) & (R+T) \end{vmatrix} \tag{27}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ W & S & T \end{vmatrix} = \nabla \times F + \nabla \times G \tag{28}$$

5.(c)

$$\nabla \cdot (fF) = \frac{\partial f(x,y,z)P}{\partial x} + \frac{\partial f(x,y,z)Q}{\partial y} + \frac{\partial f(x,y,z)R}{\partial z} \tag{29}$$

$$= f_x P + f_y Q + f_z R + f(x,y,z)(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}) = \nabla f \cdot F + f(\nabla \cdot F) \tag{30}$$

5.(d)

$$\nabla \times (fF) = (\frac{\partial fR}{\partial y} - \frac{\partial fQ}{\partial z}, \frac{\partial fP}{\partial z} - \frac{\partial fR}{\partial x}, \frac{\partial fQ}{\partial x} - \frac{\partial fP}{\partial y}) \tag{31}$$

$$= (f_y R + fR_y - f_z Q - fQ_z, f_z P + fP_z - f_x R - fR_x, f_x Q + fQ_x - f_y P - fP_y) \tag{32}$$

$$= f(R_y - Q_z, P_z - R_x, Q_x - P_y) + (f_x, f_y, f_z) \times (P, Q, R) \tag{33}$$

$$= f(\nabla \times F) + \nabla f \times F \tag{34}$$

5.(e)

$$\nabla \cdot (F \times G) = \frac{\partial QT - SR}{\partial x} + \frac{\partial RW - PT}{\partial y} + \frac{\partial PS - QW}{\partial z} \tag{35}$$

$$= Q_x T + QT_x - S_x R - SR_x + R_y W + RW_y - P_y T - PT_y + P_z S + PS_z - Q_z W - QW_z \tag{36}$$

$$G \cdot (\nabla \times F) = W(R_y - Q_z) + S(P_z - R_x) + T(Q_x - P_y) \tag{37}$$

$$F \cdot (\nabla \times G) = P(T_y - S_z) + Q(W_z - T_x) + R(S_x - W_y) \tag{38}$$

$$\implies \nabla \cdot (F \times G) = G \cdot (\nabla \times F) - F \cdot (\nabla \times G) \tag{39}$$

5.(f)

$$\nabla f\times\nabla g=(f_y g_z-f_z g_y, f_z g_x-f_x g_z, f_x g_y-f_y g_x) \tag{40}$$
$$\nabla\cdot(\nabla f\times\nabla g)=\frac{\partial}{\partial x}(f_y g_z-f_z g_y)+\frac{\partial}{\partial y}(f_z g_x-f_x g_z)+\frac{\partial}{\partial z}(f_x g_y-f_y g_x) \tag{41}$$
$$=(f_{xy}g_z+f_y g_{xz}-f_{xz}g_y-f_z g_{xy})+(f_{yz}g_x+f_z g_{xy}-f_{xy}g_z-f_x g_{yz}) \tag{42}$$
$$+(f_{xz}g_y+f_x g_{yz}-f_{yz}g_x-f_y g_{xz}) \tag{43}$$
$$=0 \tag{44}$$

5.(g)

$$\nabla\times(\nabla\times F)=\nabla\times(R_y-Q_z, P_z-R_x, Q_x-P_y) \tag{45}$$
$$=(Q_{xy}-P_{yy}-P_{zz}+R_{xz}, P_{xy}-Q_{xx}+R_{yz}-Q_{zz}, P_{xz}-R_{xx}-R_{yy}+Q_{yz}) \tag{46}$$
$$\nabla(\nabla\cdot F)=\nabla(P_x+Q_y+R_z) \tag{47}$$
$$=(P_{xx}+Q_{xy}+R_{xz}, P_{xy}+Q_{yy}+R_{yz}, P_{xz}+Q_{yz}+R_{zz}) \tag{48}$$
$$\Delta F=(\Delta P,\Delta Q,\Delta R)=(P_{xx}+P_{yy}+P_{zz}, Q_{xx}+Q_{yy}+Q_{zz}, R_{xx}+R_{yy}+R_{zz}) \tag{49}$$
$$\implies \nabla(\nabla\cdot F)+\Delta F=\nabla\times(\nabla\times F) \tag{50}$$

6.

6.(a)

Proof.

$$\int_C f(\nabla g)\cdot n ds=\int\int_D \nabla\cdot(f\nabla g)dA=\int\int_D f(\nabla\cdot\nabla g)+\nabla g\cdot\nabla f dA \tag{51}$$
$$\implies \int\int_D f(\Delta g)dA=\int_C f(\nabla g)\cdot n ds-\int\int_D \nabla g\cdot\nabla f dA \tag{52}$$

∎

6.(b)

Proof.

$$\int\int_D (f\Delta g-g\Delta f)dA=\int\int_D f\Delta g dA-\int\int_D g\Delta f dA \tag{53}$$
$$=\int_C f(\nabla g)\cdot n ds-\int\int_D \nabla g\cdot\nabla f dA-\int_C g(\nabla f)\cdot n ds+\int\int_D \nabla f\cdot\nabla g dA \tag{54}$$
$$=\int_C (f\nabla g-g\nabla f)\cdot n ds \tag{55}$$

∎

7.

Proof. We take advantage of the symmetry of the surface and the surface formula. We compute **(Notcie** $D = \{(r\cos\theta, r\sin\theta) : 0 \le r \le a\cos\theta, \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}\}$**)**

$$I := A(S) = 2\int\int_D \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 dA} \tag{56}$$

$$I = 2\int\int_D \sqrt{1 + \frac{x^2+y^2}{a^2-x^2-y^2}} dA = 2a\int\int_D \frac{1}{\sqrt{a^2-x^2-y^2}} dA \tag{57}$$

$$I = 2a\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_0^{a\cos\theta} \frac{r}{\sqrt{a^2-r^2}} dr d\theta = -2a\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sqrt{a^2-r^2}|_0^{r=a\cos\theta} d\theta \tag{58}$$

$$I = -2a^2\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} |\sin\theta| - 1 d\theta = -2a^2(2-\pi) \tag{59}$$

■

8.

Proof. Gauss' law stated, $Q = \phi_E \epsilon_0$ where $\phi_E = \int\int_S E \cdot n dS$.

Notice that $E \cdot n = 3$ at the vertices, $E \cdot n = 2$ at the sides, and $E \cdot n = 1$ at the surfaces. So we can compute $\int\int_S E \cdot n dS = A$ where A is the surface area of S.

$A = 24$, so $Q = 24\epsilon_0$. ■

9.

Proof.

$$I := \int_C (y + \sin x)dx + (z^2 + \cos y)dy + x^3 dz = \int_C F \cdot dr \tag{60}$$

where

$$F = (y + \sin x, z^2 + \cos y, x^3) \tag{61}$$

By Stokes' theorem

$$I = \int_C F \cdot dr = -\int\int_S \nabla \times F \cdot dA = -\int\int_S (-2z, -3x^2, -1) \cdot dA \tag{62}$$

Notice r travel clockwise, so S "face downward", thus the negate before the integral.

Because $\sin 2t = 2\sin\cos t$, and S lies on the surface that C lies on, which is $z = 2xy$, we deduce

$$S = \{(x, y, 2xy) : 0 \leq x^2 + y^2 \leq 1\} \tag{63}$$

Then

$$I = -\int\int_S (-2z, -3x^2, -1) \cdot dA = \int\int_D -8xy^2 - 6x^3 + 1dA \tag{64}$$

where D is the unit circle.

Compute

$$I = \int_0^{2\pi}\int_0^1 (-8r^3\cos\theta\sin^2\theta - 6r^3\cos^3\theta + 1)rdrd\theta \tag{65}$$

$$I = \int_0^{2\pi} \frac{-8}{5}\cos\theta\sin^2\theta + \frac{-6}{5}\cos^3\theta + \frac{1}{2}d\theta \tag{66}$$

$$I = \frac{-8}{15}\sin^3\theta + \frac{-6}{5}\sin\theta + \frac{6}{5}\sin^3\theta + \frac{1}{2}\theta|_0^{2\pi} = \pi \tag{67}$$

■

10.

Proof.

$$\int\int_S (2x+2y+z^2)dS = \int\int_S (2,2,z)\cdot(x,y,z)dS = \int\int_S (2,2,z)\cdot ndS \tag{68}$$

$$\int\int\int_E \nabla\cdot(2,2,z)dV = \int\int\int_E 1dV = \frac{4\pi}{3} \tag{69}$$

■

11.

Proof.

$$I := \int_C (y^3 - y)dx - 2x^3dy = \int\int_D -6x^2 - 3y^2 + 1dA \tag{70}$$

Notice that $-6x^2 - 3y^2 + 1 \geq 0 \iff (x, y) \in A$, where A is ellipse, which is connected.

Then the maximum of I occur when $D = A$, where $A = \{(x, y) : 0 \leq 6x^2 + 3y^2 \leq 1\}$, that is, when $C = r(\sqrt{6}\cos\theta, \sqrt{3}\sin\theta), 0 \leq \theta \leq 2\pi$. Notice C is counterclockwise. ■

12.

Proof. Define

$$F := \frac{1}{2}(bz - cy, cx - ax, ay - bx) \tag{71}$$

Then we can express

$$I = \frac{1}{2}\int_C (bz - cy)dx + (cx - az)dy + (ay - bx)dz = \int_C F \cdot dr \tag{72}$$

We know

$$I = \int_C F \cdot dr = \int\int_D (\nabla \times F) \cdot ndS \tag{73}$$

Compute

$$\nabla \times F = (a, b, c) = n \tag{74}$$

$$I = \int\int_D n \cdot ndS = \int\int_D dS = A \tag{75}$$

■