Deadline : 2023/03/22, 17:00.

1. Let $\mathbf{f}(t) = \langle f_1(t), f_2(t), \cdots, f_n(t)\rangle$ be a vector valued function and $\mathbf{L} = \langle L_1, L_2, \cdots, L_n\rangle$ be a vector. Prove that

$$\lim_{t\to t_0} \mathbf{f}(t) = \mathbf{L} \quad \text{if and only if} \quad \lim_{t\to t_0} f_i(t) = L_i \quad \text{for } i = 1, \cdots, n.$$

2. Let $\mathbf{r}(t)$ and $\mathbf{s}(t)$ be 3 dimensional vector valued functions defined on I. Suppose that

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{L} \quad \text{and} \quad \lim_{t\to a} \mathbf{s}(t) = \mathbf{M}.$$

Use the definition of limit of vector valued functions to prove

(a) $\lim_{t\to a} \mathbf{r}(t) \cdot \mathbf{s}(t) = \mathbf{L} \cdot \mathbf{M}$.

(b) $\lim_{t\to a} \mathbf{r}(t) \times \mathbf{s}(t) = \mathbf{L} \times \mathbf{M}$.

(Do not express $\mathbf{r}(t)$ and $\mathbf{s}(t)$ as the forms of component functions and compute these exercises directly)

3. Let $\mathbf{r} \in \mathbb{R}^n$ be a vector and $\mathbf{f}(t)$ be a n dimensional vector valued function. Show that

$$|\mathbf{r} \cdot \mathbf{f}(t)| \leq \|\mathbf{r}\|\|\mathbf{f}(t)\|.$$

4. Determine whether the following statement is true or false.

"If $\mathbf{f}(t) : [a, b] \to \mathbb{R}^n$ is differentiable, then there exists $c \in (a, b)$ such that

$$\mathbf{f}'(c) = \frac{1}{b-a}[\mathbf{f}(b) - \mathbf{f}(a)]."$$

5. Prove that if $\mathbf{f}(t)$ is parallel to $\mathbf{f}''(t)$ for all t in some interval, then $\mathbf{f}(t) \times \mathbf{f}'(t)$ is constant on that interval

6. (a) If $\mathbf{r}(t) : \mathbb{R}^1 \to \mathbb{R}^n$ is a continuous vector valued function, prove that $\|\mathbf{r}(t)\|$ is also a continuous function.

   (b) If $\mathbf{r}(t) : \mathbb{R}^1 \to \mathbb{R}^n$ is a differentiable vector valued function with $\mathbf{r}(t) \neq 0$, prove that $\|\mathbf{r}(t)\|$ is also a differentiable function.

   (c) Is the converse of (a) still true?

7. Let $\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t \rangle$. Show that the angle between $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ remains constant. What is the angle?

8. Let $\mathbf{x}(t) = \langle \cos t, \cos t \sin t, \sin^2 t \rangle$.

   (a) Show that for any $t \in \mathbb{R}$, $\mathbf{x}(t)$ is a unit vector.

   (b) Show that $\mathbf{x}(t)$ is always perpendicular to $\mathbf{x}'(t)$ for any $t \in \mathbb{R}$.

9. Find the arc length of the curve

$$\mathbf{r}(t) = \langle \frac{2}{3}(1+t)^{\frac{3}{2}}, \frac{2}{3}(1-t)^{\frac{3}{2}}, \sqrt{2}t \rangle$$

   from $t = -\frac{1}{2}$ to $t = \frac{1}{2}$.