1.2 HW2

Question 17

Construct a two-dimensional Cantor set in the unit square $[0,1]^2$ as follows. Subdivide the square into nine equal parts and keep only the four closed corner squares, removing the remaining region (which form a cross). Then repeat this process in a suitably scaled version for the remaining squares, ad infinitum. Show that the resulting set is perfect, has plane measure zero, and equals $\mathcal{C} \times \mathcal{C}$.

Proof. Let $\mathcal{C}'_n \subseteq \mathbb{R}^2$ be the result after the nth stage of removal, and let $\mathcal{C}_n \subseteq \mathbb{R}$ be the result after the nth stage of removal in the construction of the classical ternary Cantor set. It is clear that

$$\mathcal{C}'_n = \mathcal{C}_n \times \mathcal{C}_n \text{ for all } n$$

It then follows

$$\bigcap_n \mathcal{C}'_n = \bigcap_n \mathcal{C}_n \times \mathcal{C}_n = \mathcal{C} \times \mathcal{C}$$

The fact that $\mathcal{C} \times \mathcal{C}$ has plane measure zero follows from Lemma 1.2.1. Fix $(a,b) \in \mathcal{C} \times \mathcal{C}$. Because $\mathcal{C}$ is perfect, there exists some $b' \in \mathcal{C}$ such that

$$0 < |b' - b| < \epsilon$$

To see that $\mathcal{C}'$ is perfect, one see that

$$(a,b) \neq (a,b') \text{ and } (a,b') \in \mathcal{C}' \times \mathcal{C}' \text{ and } |(a,b) - (a,b')| = |b' - b| < \epsilon$$

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Question 18

Construct a subset of $[0,1]$ in the same manner as the Cantor set, except that at the kth stage, each interval removed has length $\delta 3^{-k}, 0 < \delta < 1$. Show that the resulting set is perfect, has measure $1 - \delta$, and contains no intervals.

Proof. Let $\mathcal{C}'_n \subseteq \mathbb{R}$ be the result after the nth stage of removal according to the description. Clearly, each $\mathcal{C}'_n$ has 2^n amount of connected component, we then can compute the length of $\mathcal{C}' \triangleq \bigcap \mathcal{C}'_n$ to be

$$1 - \sum_{k=1}^{\infty} 2^{k-1} \delta 3^{-k} = 1 - \frac{\frac{\delta}{3}}{1 - \frac{2}{3}} = 1 - \delta$$

Since each $\mathcal{C}'_n$ has 2^n amount of connected component of equal length and $\mathcal{C}'_n \subseteq [0,1]$, we know the length of each connected component of $\mathcal{C}'_n$ must not be greater than $\frac{1}{2^n}$. It then follows that no interval $[a, a+h]$ can be contained by all $\mathcal{C}'_n$ because if $[a, a+h]$ is a subset of some connected component of $\mathcal{C}'_k$ of some k, then the measure $h = |[a, a+h]|$ must be smaller than $\frac{1}{2^k}$, which is false when k is large enough. ■

Question 19

If E_k is a sequence of sets with $\sum |E_k|_e < \infty$, show that $\limsup_{n\to\infty} E_n$ has measure zero.

Proof. Note that

$$\sum_{k=N}^{\infty} |E_k|_e = \left(\sum_{k=1}^{\infty} |E_k|_e - \sum_{k=1}^{N-1} |E_k|_e \right) \to 0 \text{ as } N \to \infty$$

and note for all N we have

$$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \subseteq \bigcup_{k=N}^{\infty} E_k$$

Then for arbitrary ϵ, if we let N satisfy $\sum_{k=N}^{\infty} |E_k|_e < \epsilon$, we see that

$$\left| \limsup_{n\to\infty} E_n \right|_e = \left| \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \right|_e \leq \left| \bigcup_{k=N}^{\infty} E_k \right|_e \leq \sum_{k=N}^{\infty} |E_k|_e < \epsilon$$

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Question 20

If E_1, E_2 are measurable, show that

$$|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|$$

Proof. Observe the following expression of each set in disjoint union

(a) $E_1 = (E_1 \setminus E_2) \sqcup (E_1 \cap E_2)$

(b) $E_2 = (E_2 \setminus E_1) \sqcup (E_1 \cap E_2)$

(c) $E_1 \cup E_2 = (E_1 \setminus E_2) \sqcup (E_1 \cap E_2) \sqcup (E_2 \setminus E_1)$

It now follows

$$\begin{aligned}|E_1 \cup E_2| + |E_1 \cap E_2| &= |E_1 \setminus E_2| + |E_1 \cap E_2| + |E_1 \cap E_2| + |E_2 \setminus E_1| \\ &= |E_1| + |E_2|\end{aligned}$$

■

Lemma 1.2.1. Given two subsets Z_1, Z_2 of $\mathbb{R}$, if $|Z_1| = 0$, then $|Z_1 \times Z_2| = 0$.

Proof. Let $A_n \triangleq [n, n+1)$. Because

$$Z_1 \times Z_2 = \bigsqcup_{n \in \mathbb{Z}} Z_1 \times (A_n \cap Z_2)$$

To show $|Z_1 \times Z_2| = 0$, we only have to $|Z_1 \times (A_n \cap Z_2)| = 0$ for all $n \in \mathbb{Z}$. In other words, we can WOLG suppose Z_2 is bounded.

Now, fix ϵ. We are required to find an countable closed cube cover $Q_n \times C_n$ for $Z_1 \times Z_2$ such that $\sum_n |Q_n \times C_n| < \epsilon$. Let $C_n = C$ for all n where C is a compact interval containing Z_2, and let Q_n be a countable compact interval cover for Z_1 such that $\sum |Q_n| < \frac{\epsilon}{|C|}$. It then follows $\sum_n |Q_n \times C_n| = \sum_n |Q_n| \, |C| < \epsilon$. ■

Theorem 1.2.2. (Product of Finite Measure Set) If E_1 and E_2 are measurable subset of $\mathbb{R}$ and $|E_1|, |E_2| < \infty$, then $E_1 \times E_2$ is measurable in $\mathbb{R}^2$ and

$$|E_1 \times E_2| = |E_1| \, |E_2|$$

Proof. Write $E_1 \triangleq H_1 \sqcup Z_1$ and $E_2 \triangleq H_2 \sqcup Z_2$ where $H_1, H_2 \in F_\sigma$ and $|H_1| = |E_1|$ and $|H_2| = |E_2|$. Now observe

$$E_1 \times E_2 = (H_1 \times H_2) \cup (Z_1 \times E_2) \cup (E_1 \times Z_2)$$

Note that if we write $H_1 = \bigcap F_{1,n}$ and $H_2 = \bigcap F_{2,n}$, we see $H_1 \times H_2 = \bigcap F_{1,n} \times F_{2,n} \in F_\sigma$ in $\mathbb{R}^2$, it now follows from Lemma 1.2.1 that $E_1 \times E_2$ is measurable.

Now, let S_n be a decreasing sequence of open set containing E_1 such that $|S_n \setminus E_1| < \frac{1}{n}$, and let T_n be a decreasing sequence of open set containing E_2 such that $|T_n \setminus E_2| < \frac{1}{n}$. In other words,

$$E_1 = S \setminus Z_1 \text{ and } E_2 = T \setminus Z_2 \text{ where } \begin{cases} S \triangleq \bigcap S_n \\ T \triangleq \bigcap T_n \\ |Z_1| = |Z_2| = 0 \end{cases}$$

We now have

$$S \times T = (E_1 \times E_2) \cup (Z_1 \times E_2) \cup (E_1 \times Z_2)$$

This then implies $|S \times T| = |S \times T|_e \leq |E_1 \times E_2|_e + |Z_1 \times E_2|_e + |E_1 \times Z_2|_e = |E_1 \times E_2|$, where the last inequality follows from Lemma 1.2.1. The reverse inequality is clear, since $E_1 \times E_2 \subseteq S \times T$. We have proved $|E_1 \times E_2| = |S \times T|$.

Now, for each n, write

$$S_n = \bigcup_{k \in \mathbb{N}} I_{k,S_n} \text{ and } T_n = \bigcup_{k \in \mathbb{N}} I_{k,T_n}$$

where $(I_{k,S_n})_k$ and $(I_{k,T_n})_k$ are non-overlapping compact interval. It then follows that

$$|S_n \times T_n| = \sum_{i,j} |I_{i,S_n} \times I_{j,T_n}| = \sum_{i,j} |I_{i,S_n}| \times |I_{j,T_n}| = \sum_{i} |I_{i,S_n}| \sum_{j} |I_{j,T_n}| = |S_n| \, |T_n|$$

Write $S \triangleq \bigcap_{n \in \mathbb{N}} S_n$ and $T \triangleq \bigcap_{n \in \mathbb{N}} T_n$. Because

(a) Each $S_n \times T_n$ is open.

(b) $|S_n \times T_n| = |S_n| \, |T_n|$ is bounded ($\because |S_n| \searrow |E_1| < \infty$).

(c) $S_n \times T_n \searrow S \times T$

We can now deduce

$$\begin{aligned} |E_1 \times E_2| = |S \times T| &= \lim_{n \to \infty} |S_n \times T_n| \\ &= \lim_{n \to \infty} |S_n| \, |T_n| \\ &= |E_1| \, |E_2| \end{aligned}$$

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Question 21

If E_1 and E_2 are measurable subset of $\mathbb{R}$, then $E_1 \times E_2$ is a measurable subset of $\mathbb{R}^2$ and $|E_1 \times E_2| = |E_1| \, |E_2|$

Proof. Define

$$A_n \triangleq \{x \in \mathbb{R} : n \leq x < n + 1\}$$

Because

$$E_1 = \bigcup_{n \in \mathbb{Z}} E_1 \cap A_n \text{ and } E_2 = \bigcup_{k \in \mathbb{Z}} E_2 \cap A_k$$

We can deduce

$$E_1 \times E_2 = \bigcup_{n,k \in \mathbb{Z}} (E_1 \cap A_n) \times (E_2 \cap A_k)$$

Note that Theorem 1.2.2 tell us $(E_1 \cap A_n) \times (E_2 \cap A_k)$ is measurable, which implies $E_1 \times E_2$ is measurable. Theorem 1.2.2 also tell us $|(E_1 \cap A_n) \times (E_2 \cap A_k)| = |E_1 \cap A_n| \, |E_2 \cap A_k|$, which allow us to deduce

$$\begin{aligned}|E_1 \times E_2| = \sum_{n,k \in \mathbb{Z}} |(E_1 \cap A_n) \times (E_2 \cap A_k)| &= \sum_{n,k \in \mathbb{Z}} |E_1 \cap A_n| \, |E_2 \cap A_k| \\ &= \sum_{n \in \mathbb{Z}} |E_1 \cap A_n| \sum_{k \in \mathbb{Z}} |E_2 \cap A_k| = |E_1| \, |E_2|\end{aligned}$$

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Question 22

Give an example that shows that the image of a measurable set under a continuous transformation may not be measurable. See also Exercise 10 of Chapter 7.

Proof. Consider the Cantor-Lebesgue function denoted by $f : [0,1] \to [0,1]$ and denote the classical ternary Cantor set by $\mathcal{C}$. Let V be a Vitali set contained by $[0,1]$. Because $f(\mathcal{C}) = [0,1]$, we know there exists $E \subseteq \mathcal{C}$ such that $f(E) = V$. Such E is measurable since $|E|_e \leq |\mathcal{C}| = 0$, yet its continuous image $V = f(E)$ is by definition non-measurable. ∎

Question 23

Show that there exists disjoint $E_1, E_2, \dots$ such that $|\bigcup E_k|_e < \sum |E_k|_e$ with strict inequality.

Proof. Let V be a Vitali Set contained by $[0,1]$. Enumerate $[0,1] \cap \mathbb{Q}$ by x_n. Define

$$E_n \triangleq \{v + x_n \in \mathbb{R} : v \in V\} \text{ for all } n$$

The sequence E_n is disjoint, since if $p \in E_n \cap E_m$, then there exists some pair v_n, v_m belong to V such that

$$v_n + x_n = p = v_m + x_m \tag{1.6}$$

which is impossible, since Equation 1.6 implies that $v_n \neq v_m$ and v_n, v_m are of difference of a rational number.

Now, note that for arbitrary n and $v \in V$, because $v \in V \subseteq [0,1]$ and $x_n \in [0,1]$, we have $v + x_n \in [0,2]$. This implies

$$\bigsqcup_n E_n \subseteq [0,2] \text{ and } \left|\bigsqcup_n E_n\right|_e \leq 2$$

Because V is non-measurable by definition, we know $|V|_e > 0$, and since outer measure is translation invariant, we can now deduce

$$\sum_n |E_n|_e = \sum_n |V|_e = \infty > 2 \geq \left|\bigsqcup_n E_n\right|_e$$

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Question 24

Show that there exists decreasing sequence E_k of sets such that

(a) $E_k \searrow E$.

(b) $|E_k|_e < \infty$.

(c) $\lim_{k\to\infty} |E_k|_e > |E|_e$

Proof. Let V be a Vitali Set contained by $[0,1]$. Enumerate $[0,1] \cap \mathbb{Q}$ by x_n. Define

$$V + x_n \triangleq \{v + x_n \in \mathbb{R} : v \in V\}$$

In last question, we have proved that $V + x_n$ is pairwise disjoint. Define for all $n \in \mathbb{N}$

$$E_n \triangleq \bigsqcup_{k \geq n} V + x_k$$

Observe that

$$E_k \searrow \bigcap E_n = \varnothing$$

which implies $|\bigcap E_n|_e = 0$, but

$$\lim_{n\to\infty} |E_n|_e = \lim_{n\to\infty} \left|\bigsqcup_{k\geq n} V + x_k\right| \geq \lim_{n\to\infty} |V + x_n| = |V| > 0$$

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Question 25

Let Z be a subset of $\mathbb{R}$ with measure zero. Show that the set $\{x^2 : x \in Z\}$ also has measure zero.

Proof. Fix $Z_n \triangleq Z \cap [-n, n]$. Since

$$\left|\{x^2 : x \in Z\}\right| \leq \sum_{n=1}^{\infty} \left|\{x^2 : x \in Z_n\}\right|_e$$

We only have to prove

$$\left|\{x^2 : x \in Z_n\}\right|_e = 0 \text{ for all } n$$

Fix ϵ, n. Let I_k be a compact interval cover of Z_n such that $\sum |I_k| < \frac{\epsilon}{2n}$. We shall suppose $I_k \subseteq [-n, n]$, since if not, we can just let $I_k' \triangleq I_k \cap [-n, n]$.

Now, define

$$I_k^2 \triangleq \{x^2 : x \in I_k\}$$

Clearly, I_k^2 are all compact intervals, and if we write $I_k \triangleq [a_k, b_k]$, we have the following inequalities

$$\begin{cases} 0 \leq a_k \leq b_k \implies \left|I_k^2\right| = b_k^2 - a_k^2 = |I_k| \left(b_k + a_k\right) \leq 2n \left|I_k\right| \\ a_k \leq 0 \leq b_k \implies \left|I_k^2\right| = \max\{a_k^2, b_k^2\} \leq (b_k - a_k)^2 = |I_k| \left(b_k - a_k\right) \leq 2n \left|I_k\right| \\ a_k \leq b_k \leq 0 \implies \left|I_k^2\right| = a_k^2 - b_k^2 = |I_k| \left(-a_k - b_k\right) \leq 2n \left|I_k\right| \end{cases}$$

Note that $\{I_k^2\}_{k \in \mathbb{N}}$ is a compact interval cover of $\{x^2 : x \in Z_n\}$, we now see

$$\left|\{x^2 : x \in Z_n\}\right|_e \leq \sum_k \left|I_k^2\right| \leq 2n \sum |I_k| < \epsilon$$

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