Deadline : 2023/05/31, 17:00.

1. Let S be the triangular region with vertices $(0,0)$, $(1,1)$, $(0,1)$. Find the image of S under the transformation $x = u^2$, $y = v$.

2. Evaluate the integral by making an appropriate change of variables.

   (a) $\displaystyle\iint_R \sin(9x^2+4y^2)\,dA$, where R is the region in the first quadrant bounded by the ellipse $9x^2+4y^2=1$.

   (b) $\displaystyle\iint_R e^{x+y}\,dA$, where R is given by the inequality $|x|+|y|\le 1$.

3. (a) Evaluate
$$\iint_D \frac{1}{(x^2+y^2)^{\frac{n}{2}}}\,dA$$
   where n is an integer and D is the region bounded by the circles with center the origin and radii r and R, $0<r<R$.

   (b) For what values of n does the integral in part (a) have a limit as $r\to 0^+$?

   (c) Find
$$\iiint_E \frac{1}{(x^2+y^2+z^2)^{\frac{n}{2}}}\,dV$$
   where E is the region bounded by the spheres with center the origin and radii r and R, $0<r<R$.

   (d) For what values of n does the integral in part (c) have a limit as $r\to 0^+$?

4. If $[[x]]$ denotes the greatest integer in x, evaluate the integral
$$\iint_R [[x+y]]\,dA$$
   where $R=\{(x,y)\mid 1\le x\le 3,\ 2\le y\le 5\}$.

5. Evaluate the integral
$$\int_0^1\int_0^1 e^{\max\{x^2,\ y^2\}}\,dy\,dx$$
   where $\max\{x^2,\ y^2\}$ means the larger of the number x^2 and y^2.

6. If f is continuous, show that
$$\int_0^t\int_0^x\int_0^y f(z)\,dz\,dy\,dx=\frac{1}{2}\int_0^t (t-z)^2 f(z)\,dz.$$

7. Prove that the order of the integral in

$$I = \int_0^1 \left[\int_0^1 \frac{y - x}{(x+y)^3}\, dx \right] dy$$

can not be reversed.

8. Prove that if $f(x, y)$ is a continuous function on a domain D in the xy-plane and if every region R contained in that domain $\iint_R f(x, y)\, dx\, dy = 0$, then $f(x, y)$ is identically 0.

9. (a) If C is a smooth curve given by a vector function $\mathbf{r}(t)$, $a \leq t \leq b$, and $\mathbf{v}$ is a constant vector, show that

$$\int_C \mathbf{v} \cdot d\mathbf{r} = \mathbf{v} \cdot [\mathbf{r}(b) - \mathbf{r}(a)].$$

(b) If C is a smooth curve given by a vector function $\mathbf{r}(t)$, $a \leq t \leq b$, show that

$$\int_C \mathbf{r} \cdot d\mathbf{r} = \frac{1}{2} \left[|\mathbf{r}(b)|^2 - |\mathbf{r}(a)|^2 \right].$$

10. Find a function f such that $\mathbf{F} = \nabla f$ and evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

(a) $\mathbf{F}(x, y) = \langle 3 + 2xy^2, 2x^2y \rangle$, C is the arc of the hyperbola $y = \frac{1}{x}$ from $(1, 1)$ to $(4, \frac{1}{4})$.

(b) $\mathbf{F}(x, y) = \langle (1 + xy)e^{xy}, x^2e^{xy} \rangle$, $C : \mathbf{r}(t) = \langle \cos t, 2 \sin t \rangle$, $0 \leq t \leq \frac{\pi}{2}$.

(c) $\mathbf{F}(x, y, z) = \langle \sin y, x \cos y + \cos z, -y \sin z \rangle$, $C : \mathbf{r}(t) = \langle \sin t, t, 2t \rangle$, $0 \leq t \leq \frac{\pi}{2}$.