

pdftitle=Assignment, colorlinks=true, linkcolor=doc!90, bookmarksnumbered=true,  
bookmarksopen=true

NCKU 112.1  
Rudin

Eric Liu

# CONTENTS

CHAPTER 1	THE REAL AND COMPLEX NUMBER SYSTEM
	PAGE 2_____

1.1 Introduction	2
1.2 Ordered Fields	5
1.3 Real Numbers Field	8
1.4 Irrational Power	15
1.5 Exercises	22

# Chapter 1

## The Real and Complex Number System

### 1.1 Introduction

In this section, we will define the concept of ordered sets, and give a close look of the completeness property of real numbers, by showing the "uncompleteness" of rational numbers. First, we prove an elementary and classic theorem of rational numbers.

**Theorem 1.1.1.** There exists no rational  $p$  such that  $p^2 = 2$

*Proof.* Assume **there is, and we write  $p$  in the form  $p = \frac{a}{b}$ , where  $a, b \in \mathbb{Z}$  and one of  $a, b$  is odd.** Observe that  $p^2 = 2 \implies a^2 = 2b^2 \implies 2 \text{ divides } a \implies 2 \text{ divides } b$  **CaC** ■

For the sake of our discussion below, we will use  $\mathbb{Q}^+$  instead of  $\mathbb{Q}$  as our universal. Now, we divide the "rational numbers line" in half at the point  $\sqrt{2}$ , and have two subdivisions  $A = \{x \in \mathbb{Q}^+ : x^2 < 2\}$ ,  $B = \{x \in \mathbb{Q}^+ : x^2 > 2\}$

**Axiom 1.1.2.** Let  $S$  be an ordered set and  $X$  be a subset of  $S$ . Axiomatically define

$$\max X \in X \text{ and } \forall x \in X, x \leq \max X \quad (1.1)$$

$$\min X \in X \text{ and } \forall x \in X, \min X \leq x \quad (1.2)$$

**Theorem 1.1.3.**  $\max A$  and  $\min B$  both doesn't exist.

*Proof.* We wish to construct a function  $q(p)$  on  $\mathbb{Q}^+$  such that for all  $p \in \mathbb{Q}^+$ , we have  $p^2 < 2 \implies p^2 < q^2 < 2$  and have  $2 < p^2 \implies 2 < q^2 < p^2$ . Notice  $p < q \iff p^2 < q^2$ , so we can translate the wanted property of  $q$  into

$$p^2 < 2 \implies p < q \text{ and } q^2 < 2 \quad (1.3)$$

$$p^2 > 2 \implies q < p \text{ and } 2 < q^2 \quad (1.4)$$

Let  $a, b$  be two function of  $p$  on  $\mathbb{Q}^+$ . To satisfy the above properties, we can let the below equation first be true and then solve for  $a, b$ .

$$q - p = \frac{2 - p^2}{a} \quad (1.5)$$

$$q^2 - 2 = \frac{p^2 - 2}{b} \quad (1.6)$$

Now we solve for  $a, b$

$$q - p = \frac{2 - p^2}{a} \text{ and } q^2 - 2 = \frac{p^2 - 2}{b} \implies q = \frac{2 - p^2}{a} + p \text{ and } q^2 = \frac{p^2 - 2}{b} + 2 \quad (1.7)$$

$$\iff \left[\frac{2 - p^2}{a} + p\right]^2 = \frac{p^2 - 2}{b} + 2 \quad (1.8)$$

$$\iff \frac{(2 - p^2)^2}{a^2} + \frac{2p(2 - p^2)}{a} + p^2 = \frac{p^2 - 2}{b} + 2 \quad (1.9)$$

$$\iff (p^2 - 2)^2\left(\frac{1}{a^2}\right) + (p^2 - 2)\left(-\frac{2p}{a} + 1 - \frac{1}{b}\right) = 0 \quad (1.10)$$

$$\iff \frac{p^2 - 2}{a^2} - \frac{2p}{a} + 1 - \frac{1}{b} = 0 \text{ (because } p^2 - 2 \neq 0) \quad (1.11)$$

$$\iff \frac{p^2 - 2 - 2ap + a^2}{a^2} = \frac{1}{b} \quad (1.12)$$

$$\iff b = \frac{a^2}{p^2 - 2 - 2ap + a^2} = \frac{a^2}{(p - a)^2 - 2} \quad (1.13)$$

Define  $a := p + c$  where  $c^2 > 2$ , and we are finished. ■

Now, we come back to define the concept of ordered set.

**Definition 1.1.4. (Ordered Set Axioms)**  $S$  is an ordered set if there is a relation  $\sim$  on it that satisfy

$$\forall x \in S, x \not\sim x \quad (1.14)$$

$$\forall x, y \in S, x \neq y \implies x \sim y \text{ exclusively or } y \sim x \quad (1.15)$$

$$\forall x, y, z \in S, x \sim y \text{ and } y \sim z \implies x \sim z \quad (1.16)$$

From now on, we write this relation as  $<$  and if we write  $x > y$ , we mean  $y < x$ .

To discuss the uncompleteness of rational numbers, which is an ordered set, we first define a few concepts brought by concept of ordered set.

**Definition 1.1.5.** Let  $S$  be an ordered set and  $E \subseteq S$ .  $E$  is bounded above if

$$\exists a \in S, \forall b \in E, a \geq b \quad (1.17)$$

In this case, we say  $a$  is an upper bound of  $E$  and  $E$  is bounded above by  $a$ . On the other hand,  $E$  is bounded below by  $c$  if

$$\forall b \in E, c \leq b \quad (1.18)$$

Before we give the definition of supremum, aka least upper bound, we first prove a theorem about it.

**Theorem 1.1.6.** If  $x$  is the smallest upper bound of a bounded above nonempty set  $E$ , then any number smaller than  $x$  is not an upper bound of  $E$ , and every upper bound of  $E$  is greater than or equal to  $x$ .

*Proof.* Let  $A$  be the set of upper bounds of  $E$ . Arbitrarily pick an  $m$  such that  $m < x$ . Assume  $\forall z \in E, m > z$ . We see  $m \in A$ , and because  $x = \min A$ , we see  $x \leq m$  **CaC**. Assume  $\exists n \in A, n < x$ . Then  $\exists z \in E, n < z$ , which implies  $n \notin A$  **CaC** ■

**Corollary 1.1.7.** Any number smaller than the greatest lower bound is a lower bound, and any number greater than the greatest lower bound is not a lower bound.

**Definition 1.1.8. (Definition of Supremum and Infimum)** Let  $A, B$  respectively be the set of upper bounds of  $E$  and the set of lower bounds of  $E$ . We define

$$\sup E := \min A \quad (1.19)$$

and

$$\inf E := \max B \quad (1.20)$$

if they ever exist.

**Theorem 1.1.9.** Let  $A$  be a subset of ordered set  $S$ . If  $\max A$  exists, then  $\max A = \sup A$ . Similarly, if  $\min A$  exists, then  $\min A = \inf A$

*Proof.*  $\max A$  is an upper bound and  $\min A$  is a lower bound hold true by definition. Any number smaller than  $\max A$  can not be an upper bound since  $\max A \in A$ . Similarly, and number greater than  $\min A$  can not be a lower bound since  $\min A \in A$  ■

Now, we look back to our subdivisions  $A = \{x \in \mathbb{Q}^+ : x^2 < 2\}, B = \{x \in \mathbb{Q}^+ : x^2 > 2\}$ . We first show that  $B$  is exactly the set of all upper bounds of  $A$ .

**Theorem 1.1.10.** Given  $A = \{x \in \mathbb{Q}^+ : x^2 < 2\}, B = \{x \in \mathbb{Q}^+ : x^2 > 2\}$ . We have

$$B = \{x \in \mathbb{Q}^+ : \forall y \in A, y \leq x\} \quad (1.21)$$

*Proof.* Arbitrarily pick  $x \in B$ , and we see  $\forall y \in A, y^2 < 2 < x^2$ . Then  $\forall y \in A, y < x$ . This implies  $B \subseteq \{x \in \mathbb{Q}^+ : \forall y \in A, y \leq x\}$ . Let  $x$  satisfy  $\forall y \in A, y \leq x$ . Assume  $x^2 < 2$ . We immediately see  $x = \max A$ , but  $\max A$  doesn't exist **CaC** ■

By Theorem 1.1.3, we then can see that  $\sup A$  does not exist. Now, we give this idea a name.

**Definition 1.1.11. (Definition of Completed Ordered Set)** An ordered set  $S$  satisfy least-upper-bound property if

$$E \subseteq S \text{ and } E \neq \emptyset \implies \sup E \text{ exists} \quad (1.22)$$

Also, we say  $S$  is completed.

$\sup A$  does not exists indicate that  $\mathbb{Q}$  as an ordered set doesn't satisfy the least-upper-bound property. Before we close this section, we reveal the face of the twin brother of the least-upper-bound property. In fact, it is more like property itself in the mirror, since they are equivalent.

**Theorem 1.1.12. (LUB  $\iff$  GLB)**  $S$  satisfy the least-upper-bound property if and only if  $S$  satisfy the greatest-lower-bound property.

*Proof.* From left to right, consider a bounded below set  $E$ . Let  $L$  be the set of lower bounds of  $E$ . We know  $\sup L$  exists and every elements of  $E$  is an upper bound of  $L$ , so  $\sup L$  is an lower bound of  $E$ . Then  $\sup L = \inf E$ . The other direction use the same method. ■

## 1.2 Ordered Fields

In this section, we first give the definition of ordered fields, and prove basic result concerning positivity. Notice in this section that  $x, y, z$  are all in  $\mathbb{F}$ .

**Axiom 1.2.1. (Ordered Field Axioms)** An ordered field  $\mathbb{F}$  is a field that is not only a ordered set, but also satisfy the following axioms

$$y < z \implies x + y < x + z \quad (1.23)$$

$$x > 0 \text{ and } y > 0 \implies xy > 0 \quad (1.24)$$

**Theorem 1.2.2. (Negate reverse positivity)**  $(x > 0 \iff -x < 0)$  and  $(x < 0 \iff -x > 0)$

*Proof.* Observe  $x > 0 \implies x + (-x) > 0 + (-x) \implies -x < 0$ , and  $x < 0 \implies x + (-x) < 0 + (-x) \implies -x > 0$ . Clearly,  $x = 0 \implies -x = 0$ . Then the Theorem follows. ■

**Theorem 1.2.3. (Multiply a negative number reverse positivity)** Given  $y < 0$ , we have

$$(x > 0 \iff xy < 0) \text{ and } (x < 0 \iff xy > 0) \quad (1.25)$$

*Proof.* Observe  $x > 0 \implies x(-y) > 0 \implies -xy > 0 \implies xy < 0$ , and  $x < 0 \implies (-x)(-y) > 0 \implies xy > 0$ . Clearly,  $x = 0 \implies xy = 0$ . Then the Theorem follows. ■

**Theorem 1.2.4. (Multiply on both side)** Given  $y < z$ , we have

$$(x > 0 \iff xy < xz) \text{ and } (x < 0 \iff xy > xz) \quad (1.26)$$

*Proof.* First observe  $y < z \implies z - y > 0$ . Now observe  $x > 0 \implies x(z - y) > 0 \implies xz - xy > 0 \implies xz > xy$ . Observe  $x < 0 \implies x(z - y) < 0 \implies xz < xy$ . Obviously  $x = 0 \implies xy = xz$ . Then the Theorem follows. ■

**Theorem 1.2.5. (Squares are nonnegative)**  $x \neq 0 \implies x^2 > 0$

*Proof.* If  $x > 0$ , then  $x^2 > 0$  follows from the axiom. If  $x < 0$ , then  $x^2 > 0$  follows from Theorem 1.2.2 ■

**Corollary 1.2.6.**  $1 > 0$

**Theorem 1.2.7. (Inverse preserve positivity)**  $(x > 0 \iff \frac{1}{x} > 0)$  and  $(x < 0 \iff \frac{1}{x} < 0)$

*Proof.* If  $x > 0$  but  $\frac{1}{x} < 0$ , then  $1 < 0$ . The same logic applies to  $\frac{1}{x} > 0 \implies x > 0$ . Because  $\frac{1}{0}$  does not exist, the theorem follows. ■

If we were to define  $x^{-2} := x^{-1}x^{-1}$ , we must realize  $0^{-1}$  does not exist and realize we have not yet prove some common inequalities concerning integer powers. Notice that the following inequalities require base to be positive.

**Definition 1.2.8. (Definition of Inverse)** For all nonzero  $x$  and naturals  $p$ , we define  $x^{-p} := (x^{-1})^p$ , and define  $x^0 := 1$

**Theorem 1.2.9. (Inequality when base is fixed)** Given a positive  $a$  and two integer  $x, y$  where  $x < y$ , we have

$$\begin{cases} a^x < a^y & \iff a > 1 \\ a^x = a^y & \iff a = 1 \\ a^x > a^y & \iff 0 < a < 1 \end{cases} \quad (1.27)$$

*Proof.* By Theorem 1.2.4, observe  $1 < a \iff 1 < a < a^2 \iff 1 < a < a^2 < a^3 \iff \dots \iff 1 < a < \dots < a^{y-x} \iff a^x < a^y$ . If  $a = 1$ , we know  $a^x = 1 = a^y$ . Again by Theorem 1.2.4, observe  $0 < a < 1 \iff a^2 < a < 1 \iff \dots \iff a^{y-x} < \dots < 1 \iff a^y < a^x$ . ■



**Theorem 1.2.10. (Inequality when integer power is fixed)** Given  $0 < b < c$  and  $z \in \mathbb{Z}$ , we have

$$\begin{cases} b^z < c^z & \iff 0 < z \\ b^z = c^z & \iff 0 = z \\ b^z > c^z & \iff 0 > z \end{cases} \quad (1.28)$$

*Proof.* If  $z > 0$ , then by Theorem 1.2.4, observe  $0 < b < c \implies 0 < b^2 < bc < c^2 \implies b^3 < b^2c < bc^2 < c^3 \implies \dots \implies b^z < c^z$ . Obviously,  $0 = z \implies b^z = 1 = c^z$ . If  $z < 0$ , then  $b^z = (\frac{1}{b})^{-z}$  and  $c^z = (\frac{1}{c})^{-z}$  by definition. Observe  $b < c \implies 1 = \frac{1}{b}b < \frac{1}{b}c = \frac{c}{b} \implies \frac{1}{c} < \frac{1}{b}$ . By the logic above, we deduce  $c^z = (\frac{1}{c})^{-z} < (\frac{1}{b})^{-z} = b^z$ . Then the Theorem follows. ■

**Theorem 1.2.11. (Positivity of integer power)** If  $a > 0$ , then  $\forall x \in \mathbb{Z}, a^x > 0$ . If  $a < 0$ , then  $a^x > 0 \iff 2|x$

*Proof.* The former result is a direct consequence of Axiom 1.2.1 and Theorem 1.2.7. The latter result is a direct consequence of Theorem 1.2.3 and Theorem 1.2.5 ■

Notice that if the base is negative, the corresponding inequalities can all be deduced by the above theorems with a little effort, although the results are quite messy.

Now we prove some arithmetic properties concerning nonzero base, unlike the above inequalities concerning only positive base. Notice that those properties of natural power inherited by integer power can be proven inductively, and that the base  $x$  can be any nonzero number.

**Theorem 1.2.12. (Arithmetic property of integer power)** For all nonzero  $x$  and integers  $p, q$ , we have

$$x^{p+q} = x^p x^q \quad (1.29)$$

*Proof.* If  $p, q$  are both positive, the Theorem is proven by induction. If  $p, q$  are both negative, observe  $x^p x^q = (x^{-1})^{-p} (x^{-1})^{-q} = (x^{-1})^{-(p+q)} = x^{p+q}$ , where the last equality hold true because  $p+q < 0$ . If  $p > 0 > q$  and  $p+q > 0$ , observe  $x^p x^q = x^p (x^{-1})^{-q} = x^{p-(-q)} = x^{p+q}$ . If  $p > 0 > q$  and  $p+q < 0$ , observe  $x^p x^q = x^p (x^{-1})^{-q} = (x^{-1})^{-(q+p)} = x^{p+q}$ , where the last equality hold true because  $p+q < 0$ . ■

**Theorem 1.2.13. (Arithmetic property of integer power)** For all nonzero  $x$  and integers  $p, q$ , we have

$$x^{pq} = (x^p)^q = (x^q)^p \quad (1.30)$$

*Proof.* If  $p, q$  are both positive or if any of  $p, q$  are zero, the proof is trivial. If  $p < 0 < q$ , observe  $(x^p)^q = ((x^{-1})^{-p})^q = (x^{-1})^{-pq} = x^{pq}$ , where the last equality hold true because  $pq < 0$ , and observe  $(x^q)^p = ((x^q)^{-1})^{-p} = ((x^{-1})^q)^{-p} = (x^{-1})^{-qp} = x^{qp}$ , where the second equality hold true can be proven by induction. ■

**Theorem 1.2.14. (Arithmetic property of integer power)** For all nonzero  $x, y$  and integer  $p$ , we have

$$x^p y^p = (xy)^p \quad (1.31)$$

*Proof.* If  $p \geq 0$ , the proof is trivial. If  $p < 0$ , then  $x^p y^p = (x^{-1})^{-p} (y^{-1})^{-p} = (x^{-1} y^{-1})^{-p} = ((xy)^{-1})^{-p} = (xy)^p$ . ■

## 1.3 Real Numbers Field

Although the title of this section is "Real Numbers Field", here, we will not construct the real numbers field, nor use any common property of real numbers. In fact, we will not even use the symbol  $\mathbb{R}$  in this section, since we are merely proving theorems about an ordered field with least-upper-bound property. We don't know if there exists any ordered field with least-upper-property. Let's say there does; yet, we don't know if such structure is unique. Let's say it is unique; yet, we don't know if that structure have relation with  $\mathbb{R}$ . Here, we will use the symbol  $\mathbb{F}$  to denote an ordered field with least-upper-bound property. One should realize that we can use algorithm to define a subset containing  $1 \in \mathbb{F}$  that is isomorphic to  $\mathbb{N}$ , and thereby we abuse the notation to denote that subset  $\mathbb{N}$ . A subfield of  $\mathbb{F}$  isomorphic to  $\mathbb{Q}$  can also be defined after we define  $\mathbb{Z}$ , so we also thereby abuse the notation to denote that subfield  $\mathbb{Q}$ .

**Theorem 1.3.1.**  $\mathbb{N}$  is unbounded above.

*Proof.* Assume  $\mathbb{N}$  is bounded above. Because  $1 > 0$ , we know  $\sup \mathbb{N} - 1 < \sup \mathbb{N}$ . Then  $\sup \mathbb{N} - 1$  is not an upper bound of  $\mathbb{N}$ . Arbitrarily pick any  $m \in \mathbb{N}$  greater than  $\sup \mathbb{N} - 1$ . We see  $m > \sup \mathbb{N} - 1 \implies m + 1 > \sup \mathbb{N}$ , where  $m + 1 \in \mathbb{N}$  CaC ■

**Corollary 1.3.2.** Both  $\mathbb{Z}$  and  $\mathbb{Q}$  are unbounded both above and below.

**Corollary 1.3.3. (Divided by 1)** Given any  $x \in \mathbb{F}$ , there exists  $n \in \mathbb{Z}$  such that  $n \leq x < n + 1$

*Proof.* If  $x > 0$ , let  $S = \{n \in \mathbb{N} : n > x\}$ . Notice  $S = \emptyset$  implies  $\mathbb{N}$  is bounded above by  $x$ , so  $S$  is nonempty. Then by well-ordering principle, we know  $\min S$  exists. We now show  $\min S - 1 \leq x < \min S$ . Observe that  $\min S \in S \implies x < \min S$ . Assume  $\min S - 1 > x$ . We immediately see  $\min S - 1 \in S$  CaC (done) .

If  $x < 0$ , let  $S = \{n \in \mathbb{N} : n \geq -x\}$ . Again,  $S = \emptyset$  implies  $\mathbb{N}$  is bounded above by  $-x$ , so  $S$  is nonempty. Then by well-ordering principle, we know  $\min S$  exists. We now show  $-\min S \leq x < -\min S + 1$ . Observe that  $\min S \in S \implies \min S \geq -x \implies x \geq -\min S$ . Assume  $-\min S + 1 \leq x$ . Then  $\min S - 1 \geq -x > 0$ ; thus

$\min S - 1 \in S$  CaC (done)

If  $x = 0$ , then we let  $n = 0$ . ■

**Theorem 1.3.4. (Archimedean Property)** Given  $x, y \in \mathbb{F}$  and  $0 < x$ , there exists  $n \in \mathbb{N}$  such that  $nx > y$

*Proof.* Because  $\mathbb{N}$  is unbounded above, we know  $\frac{y}{x}$  can not be an upper bound of  $\mathbb{N}$ , so we know  $\exists n \in \mathbb{N}, n > \frac{y}{x}$ . Then because  $x > 0$ , we can deduce  $nx > y$ . ■

**Theorem 1.3.5. ( $\mathbb{Q}$  is dense in  $\mathbb{F}$ )** Given  $x, y \in \mathbb{F}$  and  $x < y$ , we know there exists  $p \in \mathbb{Q}$  such that  $x < p < y$

*Proof.* Every rational, positive or negative, can be expressed in the form  $\frac{m}{n}$  for some integer  $m$  and natural  $n$ . We seek to find some integer  $m$  and  $n$  such that  $x < \frac{m}{n} < y$ . Notice that  $x < \frac{m}{n} < y \iff nx < m < ny$ . Because  $m$  has to be an integer, we know for  $nx < m < ny$  to hold true, we must first have  $ny - nx > 1$ . Because  $y - x > 0$ , by Archimedean Property, there exists  $n \in \mathbb{N}$  such that  $ny - nx = n(y - x) > 1$ . By Corollary 1.3.3, we know there exists  $m \in \mathbb{Z}$  such that  $m \leq ny < m + 1$ .

Notice  $m = ny$  if and only if  $y \in \mathbb{Q}$ . So we can split the proof into two cases.

Case 1:  $y \in \mathbb{Q}$

We see that the set  $\{r \in \mathbb{Q} : r < y\}$  have supremum  $y$ , since  $y \in \mathbb{Q}$ . Then  $x < y$  tell us  $x$  is not an upper bound of the set, then we can pick some rational  $r$  in the set greater than  $x$ , so  $x < r < y$  (done) .

Case 2:  $y \notin \mathbb{Q}$

We know  $m < ny < m + 1$ .  $ny < m + 1$  tell us  $nx < ny - 1 < m$ , so  $nx < m < ny$  (done) ■

**Theorem 1.3.6. (Positive root of power uniquely exists)** For all natural  $n$  and  $y > 0$ , there exists a unique positive  $x$  such that  $x^n = y$

*Proof.* By Theorem 1.2.10, we know two different positive numbers  $0 < x < x'$  are different when raised to the power of  $n$ , being  $0 < x^n < (x')^n$ , so if such positive power exists, it must be unique.

We have handled the uniqueness part of the proof. Denote  $E := \{m \in \mathbb{F}^+ : m^n < y\}$  and  $x := \sup E$ . Now we do the existence part by proving  $x$  exists and  $x^n = y$ .

To show  $x = \sup\{m \in \mathbb{F}^+ : m^n < y\}$  exists, we only have to show the set  $\{m \in \mathbb{F}^+ : m^n < y\}$  is nonempty and bounded above. In other word, we wish to construct function  $a \in \mathbb{F}^+$  and  $b$  of  $y$  such that for all positive input  $y > 0$ , we have  $a^n < y$  and  $(0 < m^n < y \longrightarrow m < b)$ . In the followings, the domain of  $a$  and  $b$  are only positives.

First we construct  $a$ . By Theorem 1.2.9, we know if  $a < \min\{1, y\}$ , then  $a^n < a < y$ , so we construct  $a$  such that  $0 < a < \min\{1, y\}$ . Notice that  $a$  must be positive because we are constructing a number in  $E$ , where  $E$  contain only positives. Express  $a$  in the form  $a = \frac{p}{q}$  where  $p, q$  are both function of  $y$ . In the process of construction, We must be careful to make sure  $a$  exists for all positive  $y$ .

To satisfy  $0 < a$ , we need only guarantee  $p, q$  are always of the same sign for all positive  $y$ . If such  $p, q$  exists, we can change both sign of  $p, q$  when they are negative, and get two positive function. So, we can just require  $p, q$  to be positive for all positive  $y$ .

To satisfy  $a < 1$ , observe  $a < 1 \iff \frac{p}{q} < 1 \iff p < q$ . The easiest construction is to let  $q = p + c$  where  $c$  is positive.

To satisfy  $a < y$ , observe  $a < y \iff \frac{p}{q} < y \iff p(y - 1) + cy = qy - p > 0$ . The easiest construction is to let  $p(y - 1) + cy = y^2$ , which is possible, if we let  $p = y$  and  $c = 1$ . In this case  $p = y > 0$  and  $q = y + 1 > 0$ , and  $a = \frac{y}{y+1}$ . We finished proving  $E$  is nonempty. **(Notice  $c = y^2, p = y^3, q = y^3 + y^2, a = \frac{y^3}{y^3+y^2}$  also do the trick)**

Now we construct  $b$ . By Theorem 1.2.10, we know if  $0 < b$  and  $0 < m^n < b^n$ , then  $m < b$ , so we construct  $b$  such that  $y < b^n$  which lead to  $0 < m^n < y < b^n$  if  $m^n < y$ . Because  $y > 0$ , this is fairly easy. Simply let  $b = y + 1$ , so we have  $b > 1$  and  $b > y$ ; thus by Theorem 1.2.9, we have  $b^n > b > y$ , finishing proving  $E$  is bounded above, where  $b = y + 1$  is an upper bound. **(done)**

To show  $x^n = y$ , we show  $x^n \geq y$  and  $x^n \leq y$ . We will assume that  $x^n < y$  or  $x^n > y$ , but before we do such, let's see what property from which can we possibly draw contradiction. Notice that because we just prove the existence of the supremum of  $E$ , we haven't use the fact that  $x = \sup E$  in anywhere of our proof. We know

$$x = \sup E \iff \forall d > 0, \begin{cases} x + d \notin E \text{ (} x \text{ is an upper bound)} \\ \text{and} \\ x - d \text{ is not an upper bound of } E \text{ (the least upper bound)} \end{cases} \quad (1.32)$$

So, you see, we wish to construct a small and positive  $h$  and  $k$  such that if we assume

$x^n < y$  or  $x^n > y$  we can draw  $x + h \in E$  or  $x - k$  is an upper bound of  $E$ . **(We are going to assume  $\sup E$  is smaller or greater than  $\sqrt[n]{y}$ )**

Observe  $x + h \in E \iff (x + h)^n < y \iff (x + h)^n - x^n < y - x^n$ , and observe  $x - k$  is an upper bound of  $E \iff (m^n < y \implies m < x - k) \iff (m \geq x - k \implies m^n \geq y) \iff (m \geq x - k \implies x^n - m^n \leq x^n - y)$ .

Notice that the act of subtracting  $x^n$  at the both side of the inequality play an important role in our proof: not only does the act allow us to use the identity  $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \dots + b^{n-1})$ , and the act also tell us between  $x + h \in E$  and  $x - k$  is an upper bound of  $E$ , which contradiction statement should we draw from  $x^n > y$ . If  $y - x^n < 0$ , then  $y - x^n < 0 < (x + h)^n - x^n$ , so we can not possibly draw  $x + h \in E$  from  $x^n > y$ . **(If  $\sup E > \sqrt[n]{y}$ , then it is too big, we can find a smaller upper bound of  $E$ )**

Assume  $x^n > y$ . We wish to construct positive  $k$  such that  $m \geq x - k \implies x^n - m^n \leq x^n - y$ , so we can draw the contradiction  $x - k$  is an upper bound of  $E$ .

Notice that  $m \geq x - k \implies x^n - m^n \leq x^n - (x - k)^n$ , so if  $x^n - (x - k)^n \leq x^n - y$ , our proof at this part is finished. Now our job is to single out the  $k$  in the inequality to give an condition such that  $x^n - (x - k)^n \leq x^n - y$  hold if the condition hold. It is easy to see that computing the polynomial of  $k$  on left hand side of inequality and to prove such positive  $k$  exists for all  $n$  is almost impossible. Thus, we take a possible but actually non-existing risk in our next step of the proof. Use the  $a^n - b^n$  identity and that  $x - k < x$  to deduce

$$x^n - (x - k)^n = k(x^{n-1} + x^{n-2}(x - k) + \dots + (x - k)^{n-1}) \leq knx^{n-1} \quad (1.33)$$

So, if we have  $knx^{n-1} \leq x^n - y$ , which is equivalent to  $k \leq \frac{x^n - y}{nx^{n-1}}$ , our proof is partially finished. Notice that  $x^n - y > 0$  and  $nx^{n-1} > 0$ , so  $\frac{x^n - y}{nx^{n-1}} > 0$ ; thus the positive  $k$  exists. **(done) CaC (The reason I use the word "possible risk" is that if we use an identity that show us  $x^n - (x - k)^n$  smaller than a quantity greater than  $x^n - y$ , the proof can not be done.)**

Assume  $x^n < y$ . We wish to construct positive  $h$  such that  $(x + h)^n - x^n < y - x^n$ , so we can draw the contradiction  $x + h \in E$ .

Again, we use the same identity to deduce

$$(x + h)^n - x^n = h((x + h)^{n-1} + (x + h)^{n-2}x + \dots + x^{n-1}) < hn(x + h)^{n-1} \quad (1.34)$$

To single out the  $h$  in the  $hn(x + h)^{n-1}$ , notice that we can take the risk to add the constraint  $h < 1$  at the end of our construction to have  $(x + h)^n - x^n < hn(x + h)^{n-1} < hn(x + 1)^{n-1}$ .

Then, if we have  $hn(x+1)^{n-1} < y - x^n$ , which is equivalent to  $h < \frac{y-x^n}{n(x+1)^{n-1}}$ , our proof is finished. To sum up, any  $h$  satisfy  $h < \min\{1, \frac{y-x^n}{n(x+1)^{n-1}}\}$  does the trick, and such  $h$  exists, since  $0 < \min\{y - x^n, n(x+1)^{n-1}\}$ . (done) CaC (done) ■

If you want a proof with less of my commentary, here you go.

*Proof.* Observe  $\min\{1, \frac{y}{2}\} \in E$  and  $\max\{1, y\}$  is an upper bound of  $E$ , so  $\sup E$  exists. Denote  $x := \sup E$ . Assume  $x^n > y$ . Observe  $x - k$  is an upper bound of  $E \iff m^n < y \implies m < x - k \iff m \geq x - k \implies m^n \geq y$ . We know  $(x - k)^n \geq y \iff x^n - (x - k)^n \leq x^n - y$ , and know  $x^n - (x - k)^n \leq knx^{n-1}$ . If we let  $0 < k \leq \frac{x^n - y}{nx^{n-1}}$ , then  $x - k$  is an upper bound of  $E$  CaC. Assume  $x^n < y$ . Observe  $x + h \in E \iff (x + h)^n - x^n < y - x^n$ . We know that  $(x + h)^n - x^n < hn(x + h)^{n-1}$  and that if  $h < 1$ , we have  $hn(x + h)^{n-1} < hn(x + 1)^{n-1}$ . So we let  $h = \min\{1, \frac{y - x^n}{n(x+1)^{n-1}}\}$ , and we can see  $x + h \in E$  CaC. ■

**Definition 1.3.7.** The number  $x$  in Theorem 1.3.6 is written  $x = \sqrt[n]{y}$  or  $x = y^{\frac{1}{n}}$ .

The above theorem is by far the trickiest we have seen. Often the theorem is proven only in special case  $\sqrt{2}$  as a classical example in the first class of analysis. Here, we prove a general result. The following theorem is to make sure the definition of rational power make sense. In last section we prove some inequalities concerning integer power. Here, we prove those inequalities are inherited by rational power. Of course, the arithmetic properties of rational power, also inherited from those of integer power, will be proven after these inequalities.

About the coverage of definition, notice that we didn't and won't define  $y^{\frac{1}{n}}$  when  $y$  is negative. Also, notice that for all nonzero rational  $s$ , we can define  $0^s = 0$ .

**Theorem 1.3.8.** Given  $m, p \in \mathbb{Z}$  and  $n, q \in \mathbb{N}$  and  $a > 0$  and  $\frac{m}{n} = \frac{p}{q}$ , we have

$$(a^m)^{\frac{1}{n}} = (a^p)^{\frac{1}{q}} \quad (1.35)$$

*Proof.* Observe

$$x = (a^m)^{\frac{1}{n}} \quad (1.36)$$

$$\iff x^n = a^m \quad (1.37)$$

$$\iff (x^n)^q = (a^m)^q \quad (1.38)$$

$$\iff x^{nq} = a^{mq} = a^{np} \text{ (because } n, m, q \in \mathbb{Z}) \quad (1.39)$$

$$\iff (x^q)^n = (a^p)^n \text{ (again, because } n, p, q \in \mathbb{Z}) \quad (1.40)$$

$$\iff x^q = a^p \text{ (by Theorem 1.3.6)} \quad (1.41)$$

$$\iff x = (a^p)^{\frac{1}{q}} \text{ (by Theorem 1.3.6)} \quad (1.42)$$

$$\iff (a^m)^{\frac{1}{n}} = x = (a^p)^{\frac{1}{q}} \quad (1.43)$$

■

**Definition 1.3.9. (Definition of Rational Powers)** Given a rational  $r = \frac{p}{q}$ , where  $q \in \mathbb{N}$ . For all  $a$ , we define  $a^r = (a^p)^{\frac{1}{q}}$

**Theorem 1.3.10. (Inequality when base is fixed)** Given a positive  $a$  and two rational  $x, y$  where  $x < y$ , we have

$$\begin{cases} a^x < a^y & \iff a > 1 \\ a^x = a^y & \iff a = 1 \\ a^x > a^y & \iff 0 < a < 1 \end{cases} \quad (1.44)$$

*Proof.* Express  $x, y$  in the form  $x = \frac{q}{p}, y = \frac{n}{m}$  where  $q, n \in \mathbb{Z}$  and  $m, p \in \mathbb{N}$ . Notice  $x < y \implies \frac{q}{p} < \frac{n}{m} \implies mq < np$ . Observe

$$a^x < a^y \implies a^{\frac{q}{p}} < a^{\frac{n}{m}} \quad (1.45)$$

$$\implies a^q = (a^{\frac{q}{p}})^p < (a^{\frac{n}{m}})^p \text{ (by Theorem 1.2.10)} \quad (1.46)$$

$$\implies a^{mq} = (a^q)^m < ((a^{\frac{n}{m}})^p)^m = ((a^{\frac{n}{m}})^m)^p = (a^n)^p = a^{np} \quad (1.47)$$

$$\implies a > 1 \text{ (by Theorem 1.2.9)} \quad (1.48)$$

Notice that the above implication still hold true if  $<$  is replaced by  $>$  and  $>$  is replaced by  $<$ , and notice that the above implication still hold true if  $<$  and  $>$  are all replaced by  $=$ , so we in fact have three implications. The Theorem follows from the three implication.

■

**Theorem 1.3.11. (Inequality when rational power is fixed)** Given  $0 < b < c$  and  $z \in \mathbb{Q}$ , we have

$$\begin{cases} b^z < c^z & \iff 0 < z \\ b^z = c^z & \iff 0 = z \\ b^z > c^z & \iff 0 > z \end{cases} \quad (1.49)$$

*Proof.* Express  $z = \frac{q}{p}$ , where  $q$  is an integer and  $p$  is a natural. Notice  $q$  and  $z$  are of the same sign. Observe by Theorem 1.2.10, we have  $b^{\frac{q}{p}} < c^{\frac{q}{p}} \implies b^q = (b^{\frac{q}{p}})^p < (c^{\frac{q}{p}})^p = c^q \implies 0 < q$ .

Notice that the above implication still hold true if  $<$  is replaced by  $>$  and  $>$  is replaced by  $<$ , and notice that the above implication still hold true if  $<$  and  $>$  are all replaced by  $=$ , so we in fact have three implications. The Theorem follows from the three implication. ■

Now we prove the arithmetic properties of rational power concerning only positive base. Notice there is no definition of a negative raised to the power of a rational. From Here

**Theorem 1.3.12. (Arithmetic property of rational power)** Given  $r, s \in \mathbb{Q}$  and  $a > 0$ , we have

$$a^{r+s} = a^r a^s \quad (1.50)$$

*Proof.* Express  $r, s$  in the form  $r = \frac{p}{q}, s = \frac{m}{n}$  where  $q, n \in \mathbb{N}$  and  $p, m \in \mathbb{Z}$ . Observe

$$(a^{r+s})^{nq} = (a^{\frac{np+mq}{nq}})^{nq} \quad (1.51)$$

$$= a^{np+mq} \quad (1.52)$$

Then observe

$$(a^r a^s)^{nq} = (a^{\frac{p}{q}} a^{\frac{m}{n}})^{nq} \quad (1.53)$$

$$= (a^{\frac{p}{q}})^{nq} (a^{\frac{m}{n}})^{nq} \text{ (by Theorem 1.2.14)} \quad (1.54)$$

$$= ((a^{\frac{p}{q}})^q)^n ((a^{\frac{m}{n}})^n)^q \quad (1.55)$$

$$= (a^p)^n (a^m)^q \quad (1.56)$$

$$= a^{np+mq} = (a^{r+s})^{nq} \text{ (Theorem 1.2.12 and Theorem 1.2.13)} \quad (1.57)$$

Then by Theorem 1.3.6, we can deduce  $a^r a^s = a^{r+s}$  ■

**Theorem 1.3.13. (Arithmetic property of rational power)** Given  $r, s \in \mathbb{Q}$  and  $a > 0$ , we have

$$(a^r)^s = a^{rs} \quad (1.58)$$

*Proof.* Express  $r, s$  in the form  $r = \frac{p}{q}, s = \frac{m}{n}$  where  $q, n \in \mathbb{N}$  and  $p, m \in \mathbb{Z}$ . Observe

$$(a^{rs})^{nq} = (a^{\frac{mp}{nq}})^{nq} \quad (1.59)$$

$$= a^{mp} \quad (1.60)$$



Then observe

$$((a^r)^s)^{nq} = (((a^{\frac{p}{q}})^{\frac{m}{n}})^n)^q \quad (1.61)$$

$$= ((a^{\frac{p}{q}})^m)^q \quad (1.62)$$

$$= (a^{\frac{p}{q}})^{mq} \quad (1.63)$$

$$= ((a^{\frac{p}{q}})^q)^m \quad (1.64)$$

$$= (a^p)^m = a^{mp} = (a^{rs})^{nq} \quad (1.65)$$

■

**Corollary 1.3.14.**  $a^{\frac{q}{p}} = (a^{\frac{1}{p}})^q$

**Theorem 1.3.15. (Arithmetic property of rational power)** Given  $r \in \mathbb{Q}$  and  $a, b > 0$ , we have

$$a^r b^r = (ab)^r \quad (1.66)$$

*Proof.* Express  $r$  in the form  $r = \frac{q}{p}$ , and observe  $(a^r b^r)^p = (a^r)^p (b^r)^p = (a^{\frac{q}{p}})^p (b^{\frac{q}{p}})^p = a^q b^q = (ab)^q = ((ab)^r)^p$ . ■

## 1.4 Irrational Power

After the rational power, we now try to define irrational power. In most of the textbooks, irrational power as a rigorous definition often come after the definition of Euler number, but here, we define the irrational power with a technique not so advanced and to be fair quite cumbersome compared to the approaches in most textbooks. Of course, the common inequalities and arithmetic properties of irrational power will be presented, although their order of presentation is different to how we present in the section before.

**Lemma 1.4.1.** For  $b, x \in \mathbb{F}$ , define  $B(x) := \{b^t : t \in \mathbb{Q}, t \leq x\}$ . Then for all  $x \in \mathbb{F}$ ,  $b > 1$  implies  $\sup B(x)$  exists, and  $0 < b \leq 1$  implies  $\inf B(x)$  exists.

*Proof.* For all  $x$ , we know  $B(x)$  is nonempty, since if not,  $\mathbb{Q}$  is bounded below. Because  $\mathbb{Q}$  is nonbounded above, we can pick a rational  $y$  greater than  $x$ . If  $b > 1$ , we deduce  $\forall b^t \in B(x), b^t \leq b^y$ ; thus  $b^y$  is an upper bound of  $B(x)$ . If  $0 < b < 1$ , we deduce  $\forall b^t \in B(x), b^y \leq b^t$ ; thus  $b^y$  is a lower bound of  $B(x)$ . If  $b = 1$ , just notice  $B(x) = \{1\}$ . ■

**Definition 1.4.2. (Irrational Power Definition)** For all  $b, x \in \mathbb{F}$ , define  $B(x)$  as in Lemma 1.4.1, and for all  $x \in \mathbb{F}$ , if  $b > 1$ , define  $b^x := \sup B(x)$ , and if  $0 < b \leq 1$ , define  $b^x := \inf B(x)$

The above definition is in fact not very appropriate, since we have already define  $b^x$  where  $x \in \mathbb{Q}$ . We don't know if the above definition is consistent with our old definition. The next theorem is to show it is.

**Theorem 1.4.3. (Consistency of power definitions)** Given  $r \in \mathbb{Q}$ , if  $b > 1$ , then  $b^r = \sup B(r)$ , and if  $b < 1$ , then  $b^r = \inf B(r)$

*Proof.* If  $b > 1$ , deduce  $b^r = \max B(r) = \sup B(r)$ . If  $b < 1$ , deduce  $b^r = \min B(r) = \inf B(r)$ . ■

Now, we are going to prove one common inequalities. The other will be proven after some work.

**Theorem 1.4.4. (Inequality when base is fixed)** Given a positive  $b$  and  $x < y$ , we have

$$\begin{cases} b^x < b^y & \Longleftrightarrow b > 1 \\ b^x = b^y & \Longleftrightarrow b = 1 \\ b^x > b^y & \Longleftrightarrow 0 < b < 1 \end{cases} \quad (1.67)$$

*Proof.* By Theorem 1.3.5, we can pick a rational  $q$  such that  $x < q < y$ . We have  $b^q \in B(y)$ . We now consider three possible situations.

If  $b > 1$ , then  $b^q \leq \sup B(y)$ . Because  $\forall b^t \in B(x), t \leq x < q$ , we can deduce  $\forall b^t \in B(x), b^t < b^q$ ; that is,  $b^q$  is an upper bound of  $B(x)$ . Notice that there exists rational between  $x < q$ , so  $b^q$  can not be the least upper bound of  $B(x)$ . Then we can deduce  $\sup B(x) < b^q \leq \sup B(y)$ .

If  $b = 1$ , then  $b^x = 1 = b^y$ .

If  $b < 1$ , then  $\inf B(y) \leq b^q$ . Because  $\forall b^t \in B(x), t \leq x < q$ , we can deduce  $\forall b^t \in B(x), b^t > b^q$ ; that is,  $b^q$  is an lower bound of  $B(x)$ . Notice that there exists rational between  $x < q$ , so  $b^q$  can not be the greatest lower bound of  $B(x)$ . Then we can deduce  $\inf B(y) \leq b^q < \inf B(x)$ .

The Theorem follows from the three implications of the three situation. ■

**Lemma 1.4.5.** Let  $S, U$  and be two bounded above subset of  $\mathbb{F}$ , containing only nonnegative numbers. Define  $SU := \{su : s \in S, u \in U\}$ . We have

$$\sup SU = \sup S \sup U \quad (1.68)$$

and have

$$\inf SU = \inf S \inf U \quad (1.69)$$

where the infimum part hold true even if  $S, U$  are not bounded above

*Proof.* Because  $S, U$  contain only nonnegative numbers, we know  $s \leq \sup S$  and  $u \leq \sup U$  implies  $su \leq \sup S \sup U$ , so  $\sup S \sup U$  is an upper bound of  $SU$ . Now we show  $\sup S \sup U$  is the *least* upper bound of  $SU$ .

Assume there is an upper bound  $x$  of  $AB$  smaller than  $\sup A \sup B$ . Express  $x$  in the form  $x = \sup S \frac{x}{\sup S}$ . Because  $x < \sup S \sup U$ , we know  $\frac{x}{\sup S} < \sup U$ , which implies there is a number  $u \in U$  greater than  $\frac{x}{\sup S}$ . Observe  $u > \frac{x}{\sup S} \implies \frac{x}{u} < \sup S$ , which implies that there is a number  $s \in S$  greater than  $\frac{x}{u}$ . Observe  $\frac{x}{u} < s \implies x < su$  CaC (done)

Clearly, we know  $s \geq \inf S$  and  $u \geq \inf U$  implies  $su \geq \inf S \inf U$ , so  $\inf S \inf U$  is a lower bound of  $SU$ . Now we show  $\sup S \sup U$  is the *greatest* lower bound of  $SU$ .

Assume there is a lower bound  $x$  of  $AB$  greater than  $\sup A \sup B$ . Express  $x$  in the form  $x = \inf S \frac{x}{\inf S}$ . Because  $x > \inf S \inf U$ , we know  $\frac{x}{\inf S} > \inf U$ , which implies there is a number  $u \in U$  less than  $\frac{x}{\inf S}$ . Observe  $u < \frac{x}{\inf S} \implies \frac{x}{u} > \inf S$ , which implies that there is a number  $s \in S$  less than  $\frac{x}{u}$ . Observe  $\frac{x}{u} > s \implies x > su$  CaC (done) ■

**Theorem 1.4.6. (Arithmetic property)** Given  $r, s \in \mathbb{F}$  and  $0 < a$ , we have

$$a^{r+s} = a^r a^s \quad (1.70)$$

*Proof.* We prove  $A(r+s) = \{xy : x \in A(r) \text{ and } y \in A(s)\}$ . Denote the set on right side  $E$ . Observe that  $xy \in E \implies \exists q \leq r \in \mathbb{Q}, x = a^q$  and  $\exists m \leq s \in \mathbb{Q}, y = a^m \implies xy = a^{q+m}$  where  $q+m \leq r+s \implies xy \in A(r+s)$ . Then we know  $E \subseteq A(r+s)$ . Given  $a^d \in A(r+s)$ , we know  $d \leq r+s$ , so if we express  $a^d = a^r a^{d-r}$ , we are sure  $a^{d-r} \in A(s)$  and  $a^r \in A(r)$ , which implies  $a^d \in E$ . Then we can deduce  $A(r+s) \subseteq E$ . (done)

By Lemma 1.4.5, our proof is finished. ■

Now, we introduce an idea to aid our proof of inequality.

**Definition 1.4.7. (Definition of order homomorphism)** Let  $S$  be a subset of  $\mathbb{F}$ , the function  $\phi : S \rightarrow \mathbb{F}$  is an *order homomorphism*, if for all  $a, b \in S$ , we have

$$a < b \implies \phi(a) < \phi(b) \quad (1.71)$$

Two subset  $U, V$  are said to be *order isomorphic*, if there exists an bijective order homomorphism from one to another.

**Lemma 1.4.8.** Let  $S, U$  be two order isomorphic bounded above subset of  $\mathbb{F}$ , containing only nonnegative numbers, where  $\phi : S \rightarrow U$  is an order isomorphism. Let  $US_\phi = \{s\phi(s) : s \in S\}$ . We have

$$\sup S \sup U = \sup US_\phi \quad (1.72)$$

and have

$$\inf S \inf U = \inf US_\phi \quad (1.73)$$

where the infimum part hold true even if  $S, U$  are not bounded above.

*Proof.* For all  $s\phi(s)$  in  $US_\phi$ , we know  $0 \leq s \leq \sup S$  and  $0 \leq \phi(s) \leq \sup U$ , so we can deduce  $s\phi(s) \leq \sup S \sup U$ . We have proven that  $\sup S \sup U$  is an upper bound of  $US_\phi$ . We now prove it is the least. Assume **there exists an upper bound  $x$  of  $US_\phi$  smaller than  $\sup S \sup U$** . Because  $\frac{x}{\sup U} < \sup S$ , we know there exists  $s \in S$  such that  $x < (\sup U)s$ . Then because  $\frac{x}{s} < \sup U$ , we know there exists  $\phi(s') \in U$  such that  $x < s\phi(s')$ . If  $s < s'$ , then we see  $x < s\phi(s') < s'\phi(s') \in US_\phi$ . If  $s > s'$ , then we see  $x < s\phi(s') < s\phi(s)$ . If  $s = s'$ , then we see  $x < s\phi(s') = s\phi(s)$  **CaC**

For all  $s\phi(s)$  in  $US_\phi$ , we know  $s \geq \inf S$  and  $\phi(s) \geq \inf U$ , so we can deduce  $s\phi(s) \geq \inf S \inf U$ . We have proven that  $\inf S \inf U$  is a lower bound of  $US_\phi$ . We now prove it is the greatest. Assume **there exists a lower bound  $x$  of  $US_\phi$  greater than  $\inf S \inf U$** . Because  $\frac{x}{\inf U} > \inf S$ , we know there exists  $s \in S$  such that  $x > (\inf U)s$ . Then because  $\frac{x}{s} > \inf U$ , we know there exists  $\phi(s') \in U$  such that  $x > s\phi(s')$ . If  $s < s'$ , then we see  $x > s\phi(s') > s\phi(s) \in US_\phi$ . If  $s > s'$ , then we see  $x > s\phi(s') > s'\phi(s')$ . If  $s = s'$ , then we see  $x > s\phi(s') = s\phi(s)$  **CaC** ■

**Theorem 1.4.9. (Arithmetic property)** Given  $b, c > 0$  and  $z \in \mathbb{F}$ , we have

$$b^z c^z = (bc)^z \quad (1.74)$$

*Proof.* From now, we will use  $C(z)$  to denote  $\{c^u : u \in \mathbb{Q}, u \leq z\}$ . Define  $\phi : B(z) \rightarrow C(z)$  as  $b^t \mapsto c^t$ . Observe that if  $b, c$  are both greater than 1, then  $b^t < b^m \implies t < m \implies c^t < c^m$ , and that if  $b, c$  are both less than 1, then  $b^t < b^m \implies t > m \implies c^t < c^m$ . So if  $b, c$  are both greater than 1 or both less than 1, then  $\phi$  is an order isomorphism. Notice that  $\{(bc)^u : u \in \mathbb{Q}, u \leq z\} = \{b^t \phi(b^t) : t \in \mathbb{Q}, t \leq z\} = C(z)B(z)_\phi$ . Then by Lemma 1.4.8, our proof is finished. We now show  $(b^{-1})^z b^z = 1$ .

WOLG, let  $b < 1$ . Denote  $H := \{b^t : t \in \mathbb{Q}, t \leq z\}$ ,  $E := \{b^{-t} : t \in \mathbb{Q}, t \leq z\}$ , so  $b^z = \inf H$  and  $(b^{-1})^z = \sup E$ . Assume **(sup  $E$ )(inf  $H$ ) > 1**. Notice  $\sup E > \frac{1}{\inf H}$  implies that there exists  $t \leq z$  such that  $b^{-t} > \frac{1}{\inf H}$ , which leads to  $\inf H > b^t$  **CaC**. Assume **(sup  $E$ )(inf  $H$ ) < 1**. Notice  $\inf H < \frac{1}{\sup E}$  implies that there exists  $t \leq z$  such that  $b^{-t} < \frac{1}{\sup E}$ , which leads to  $\sup E < b^t$  **CaC** **(done)**

If  $b < 1 < c$  and  $bc > 1$ , observe that  $(bc)^z (b^{-1})^z = c^z \implies (bc)^z = (bc)^z (b^{-1})^z b^z = b^z c^z$ . If  $b < 1 < c$  and  $bc < 1$ , observe that  $(bc)^z (c^{-1})^z = b^z \implies (bc)^z = (bc)^z (c^{-1})^z c^z = b^z c^z$  ■

**Theorem 1.4.10. (Inequality when power is fixed)** Given  $0 < b < c$  and  $z \in \mathbb{F}$ , we have

$$\begin{cases} b^z < c^z & \iff 0 < z \\ b^z = c^z & \iff 0 = z \\ b^z > c^z & \iff 0 > z \end{cases} \quad (1.75)$$

*Proof.* By Theorem 1.4.4,  $z > 0 \implies (\frac{c}{b})^z > (\frac{c}{b})^0 = 1 \implies c^z > b^z$ , and  $z = 0 \implies b^z = 1 = c^z$ , and  $z < 0 \implies (\frac{c}{b})^z < (\frac{c}{b})^0 = 1$  ■

Before the last arithmetic property, we first prove the existence of logarithm, which play an important role in the proof of last arithmetic property.

**Theorem 1.4.11. (Existence of logarithm)** For all positive  $b, y$  where  $b \neq 1$ , there exists a unique  $x \in \mathbb{F}$  such that  $b^x = y$

*Proof.* The uniqueness part have been handled by Theorem 1.4.4. Now we split the proof into two cases:  $b > 1$  and  $b < 1$

Case 1:  $b > 1$

Define  $A := \{w : b^w < y\}$  and  $x = \sup A$ .

We first prove the identity for all positive integer  $m$  and  $c > 1$ , we have  $c - 1 \geq (c^{\frac{1}{m}} - 1)m$

Let  $d = c^{\frac{1}{m}}$ , so we have  $c - 1 = d^m - 1 = (d - 1)(d^{m-1} + d^{m-2} + \dots + 1) \geq (d - 1)m = (c^{\frac{1}{m}} - 1)m$  (done)

Assume  $b^x > y$ . Let  $t = \frac{b^x}{y}$  and  $\frac{b-1}{t-1} < n = m$  and  $b = c$  to use the identity, so we have  $b - 1 \geq n(b^{\frac{1}{n}} - 1)$ . Then we have  $\frac{b-1}{n} \geq b^{\frac{1}{n}} - 1$ . Notice that  $b^x > y \implies t - 1 > 0$ , so because  $n > \frac{b-1}{t-1}$ , we then have  $t - 1 > \frac{b-1}{n} \geq b^{\frac{1}{n}} - 1$ , which implies  $t > b^{\frac{1}{n}}$ .

Observe  $b^{x-\frac{1}{n}} = \frac{b^x}{b^{\frac{1}{n}}} > \frac{b^x}{t} = y$ . Then  $x - \frac{1}{n}$  is an upper bound of  $A$  CaC .

Assume  $b^x < y$ . Let  $t = \frac{y}{b^x}$  and let  $\frac{b-1}{t-1} < n = m$  and  $b = c$  to use the identity, so we have  $b - 1 \geq n(b^{\frac{1}{n}} - 1)$ . Then we have  $\frac{b-1}{n} \geq b^{\frac{1}{n}} - 1$ . Notice that  $b^x < y \implies t - 1 > 0$ , so because  $n > \frac{b-1}{t-1}$ , we then have  $t - 1 > \frac{b-1}{n} \geq b^{\frac{1}{n}} - 1$ , which implies  $t > b^{\frac{1}{n}}$ .

Observe  $b^{x+\frac{1}{n}} = b^x b^{\frac{1}{n}} < b^x t = y$ . Then  $x + \frac{1}{n} \in A$  CaC (done)

Case 2:  $b < 1$

Define  $B := \{w : b^w < y\}$  and  $x := \inf B$ .

We first prove the identity for all positive integer  $m$  and  $0 < c < 1$ , we have  $c-1 \leq (c^{\frac{1}{m}}-1)m$

Let  $d = c^{\frac{1}{m}}$ , so we have  $c - 1 = d^m - 1 = (d - 1)(d^{m-1} + d^{m-2} + \dots + 1) \leq (d - 1)m = (c^{\frac{1}{m}} - 1)m$  (done) .

Assume  $b^x > y$ . Let  $t = \frac{y}{b^x}$  and  $\frac{b-1}{t-1} < n = m$  and  $b = c$  to use the identity, so we have  $b - 1 \leq (b^{\frac{1}{n}} - 1)n$ . Then we have  $\frac{b-1}{n} \leq b^{\frac{1}{n}} - 1$ . Notice that  $b^x > y \implies t - 1 < 0$ , so because  $n > \frac{b-1}{t-1}$ , we then have  $t - 1 < \frac{b-1}{n} \leq b^{\frac{1}{n}} - 1$ , which implies  $t < b^{\frac{1}{n}}$ .

Observe  $b^{x+\frac{1}{n}} = b^x b^{\frac{1}{n}} > b^x t = y$ . Then  $x + \frac{1}{n}$  is an lower bound of  $B$  CaC .

Assume  $b^x < y$ . Let  $t = \frac{b^x}{y}$  and  $\frac{b-1}{t-1} < n = m$  and  $b = c$  to use the identity, so we have  $b - 1 \leq (b^{\frac{1}{n}} - 1)n$ . Then we have  $\frac{b-1}{n} \leq b^{\frac{1}{n}} - 1$ . Notice that  $b^x < y \implies t - 1 < 0$ , so because  $n > \frac{b-1}{t-1}$ , we then have  $t - 1 < \frac{b-1}{n} \leq b^{\frac{1}{n}} - 1$ , which implies  $t < b^{\frac{1}{n}}$ .

Observe  $b^{x-\frac{1}{n}} = \frac{b^x}{b^{\frac{1}{n}}} < \frac{b^x}{t} = y$ . Then  $x - \frac{1}{n} \in B$  CaC (done) ■

**Lemma 1.4.12.** Given  $0 < x < rs$  and  $0 < r$  and  $0 < s$ , there exist positive rational  $u < s$  and positive rational  $t < r$  such that  $x < tu < rs$ .

*Proof.*  $0 < x < rs \implies \frac{x}{s} < r$ . Then there exists rational  $t$  such that  $\frac{x}{s} < t < r$ , which implies  $\frac{x}{t} < s$ . Then there exists rational  $u$  such that  $\frac{x}{t} < u < s$ . Then  $x < tu < rs$ . ■

**Theorem 1.4.13. (Arithmetic property)** Given  $r, s \in \mathbb{F}$  and  $a > 0$ , we have

$$(a^r)^s = a^{rs} \tag{1.76}$$

*Proof.* The proof is very very long. We first denote three important sets:

$$T := \{a^t : t \in \mathbb{Q}, t \leq r\} \tag{1.77}$$

$$U := \{(a^r)^u : u \in \mathbb{Q}, u \leq s\} \tag{1.78}$$

$$X := \{a^x : x \in \mathbb{Q}, x \leq rs\} \tag{1.79}$$

So, we have

$$a^r = (\sup \text{ or } \inf) T \tag{1.80}$$

$$(a^r)^s = (\sup \text{ or } \inf) U \tag{1.81}$$

$$a^{rs} = (\sup \text{ or } \inf)X \quad (1.82)$$

Notice that from now, when variables  $t, u, x$  and their variants  $t', u', x'$  are mentioned, they are rational and respectively less than or equal to  $r, s, rs$ .

Now, we split the proof into sixteen cases:  $a$  may be smaller or greater than 1;  $r, rs$  may be positive or negative;  $rs$  is rational or irrational.

We first do all 8 cases of  $rs \notin \mathbb{Q}$ . Notice that  $rs \notin \mathbb{Q} \implies x < rs$ .

The easiest case first:  $a > 1$  and  $r > 0$  and  $s > 0$  and  $rs \notin \mathbb{Q}$ .

Assume  $a^{rs} > (a^r)^s$ ; that is,  $\sup X > \sup U$ . Then  $\exists x, \forall u, a^x > (a^r)^u = (\sup T)^u$ . Let  $u > 0$ , so we have  $\forall t, a^{\frac{x}{u}} > a^t$ . By Lemma 1.4.12 **CaC**. Assume  $a^{rs} < (a^r)^s$ ; that is,  $\sup X < \sup U$ . Then  $\exists u, \forall x, (\sup T)^u = (a^r)^u > a^x$ . Because  $a^r > 1$ , we know  $u > 0$ , since  $u < 0 \implies (a^r)^u < 1$ . Then we have  $\exists u, \forall x, \sup T > a^{\frac{x}{u}}$ , so we have  $\exists u, \forall x, \exists t, a^t > a^{\frac{x}{u}}$ . But we see that for all  $u$ , we can let  $x > ru$ , so  $a^{\frac{x}{u}} > a^r$ . **CaC (done)**

The second case:  $a > 1$  and  $r > 0$  and  $s < 0$  and  $rs \notin \mathbb{Q}$ .

Assume  $a^{rs} > (a^r)^s$ ; that is,  $\sup X > \sup U$ . Then  $\exists x, \forall u, a^x > (a^r)^u = (\sup T)^u$ . Because  $u \leq s < 0$ , we have  $\exists x, \forall u, a^{\frac{x}{u}} < \sup T$ . Then, we have  $\exists x, \forall u, \exists t, a^{\frac{x}{u}} < a^t$ . Express  $x = rm$ , and let  $0 > s > u > m$ , so we have  $\frac{x}{u} = \frac{rm}{u} > r$  **CaC**. Assume  $a^{rs} < (a^r)^s$ ; that is,  $\sup X < \sup U$ . Then  $\exists u, \forall x, a^x < (a^r)^u = (\sup T)^u$ . Because  $u \leq s < 0$ , we have  $\exists u, \forall x, a^{\frac{x}{u}} > \sup T$ . Then, we have  $\exists u, \forall x, \forall t, a^{\frac{x}{u}} > a^t$ . If  $s \notin \mathbb{Q}$ , then we notice that because  $u < s < 0$ , we have  $r(\frac{s}{u}) < r$ . Pick  $t$  such that  $r(\frac{s}{u}) < t < r$ , so we have  $rs > tu$ . Then pick  $x$  such that  $0 > rs > x > tu$ , and observe  $a^{\frac{x}{u}} < a^t$  **CaC**. If  $s \in \mathbb{Q}$ , then it is possible to happen  $u = s$ . Then  $a^{\frac{x}{u}} = a^{\frac{x}{s}} > a^{\frac{rs}{s}} = a^r \geq a^t$  **CaC (done)**

The third case:  $a > 1$  and  $r < 0$  and  $s > 0$  and  $rs \notin \mathbb{Q}$

Assume  $a^{rs} > (a^r)^s$ ; that is,  $\sup X > \inf U$ . Then  $\exists x, \exists u, a^x > (a^r)^u = (\sup T)^u$ . Notice that  $u \leq 0 \implies (a^r)^u \geq 1 > a^x$ , so  $u$  must be positive. Then we have  $\exists x, \exists u, a^{\frac{x}{u}} > \sup T$ . Then, we have  $\exists x, \exists u, \forall t, a^{\frac{x}{u}} > a^t$ . Notice that  $\frac{x}{u} < \frac{rs}{u} < r$ , so there exists  $t$  such that  $a^{\frac{x}{u}} < a^t$  **CaC**. Assume  $a^{rs} < (a^r)^s$ ; that is,  $\sup X < \inf U$ . ■

## 1.5 Exercises

### Question 1

Given a nonzero rational  $r$ , prove that  $r + x$  and  $rx$  are irrational.

### Question 2

Prove that no rational  $r$  satisfy  $r^2 = 12$

### Question 3

Let  $E$  be a nonempty subset of an ordered set; suppose  $\alpha$  is a lower bound and  $\beta$  is an upper bound of  $E$ . Prove  $\alpha < \beta$

### Question 4

Let  $A$  be a nonempty subset of real numbers which is bounded below. Define  $-A := \{-x : x \in A\}$ . Prove that

$$\inf A = -\sup(-A) \tag{1.83}$$