# Continuous Random Variables

Reuy-Lin Sheu

Department of Mathematics, National Cheng Kung University,
Tainan, Taiwan

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# Recall the Definition of a discrete random variable

- Let $(\Omega, \mathcal{F}, P)$ be a probability space that corresponds to a random experiment and suppose $X$ is a real-valued function from $(\Omega, \mathcal{F}, P)$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.
- We say that $X$ is a discrete random variable if $X$ ONLY takes "*finitely or countably infinite*" many values $x_i$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ so that it has a discrete distribution (mass) function

$$p_X(x_i) = P(X = x_i).$$

- A discrete random variable is often a count. For example, count the number of heads in $n$ tosses (Bernoulli$(n, p)$); count the number of occurrences over a time interval (Poisson$(\lambda)$); or count the number of tosses before the first head comes up (Geometric).

# Definition of a continuous random variable

- If the image of $X$ is an uncountable set on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ (usually an interval $[a, b]$ on $\mathbb{R}$ or the entire $\mathbb{R}$), then $X$ is called a continuous random variable.
- A continuous random variable $X$ is often a measurement. For example, $X$ denotes the measurement of the length of a bar; or $X$ is the length of time before the first occurrence if it occurs according to a Poisson distribution.
- The most important special case of a continuous random variable is the so-called "absolute continuous" random variable which assigns the probability of a Borel set by a probability density function and which must assign the probability of a singleton set to the value 0.

# (Review) Induced measure on Borel sets by a random variable

- Let $(\Omega, \mathcal{F}, P)$ be probability space and $X: \ (\Omega, \mathcal{F}, P) \ \mapsto (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a r.v. on $\Omega$, discrete or continuous.
- We can utilize $P$ on $\mathcal{F}$ to define an induced measure $\mathcal{L}_X$ by $X$ on any Borel set $B \in \mathcal{B}(\mathbb{R})$ by

$$\mathcal{L}_X(B) = P(X^{-1}(B)) = P(X \in B).$$

- If $X$ is a discrete r.v. taking values $x_i, \ i = 1, 2, \ldots,$ since each singleton set $\{x_i\}$ is Borel, the induced measure $\mathcal{L}_X$ on each $x_i$ : (the "*probability mass function*" of $X$,)

$$\begin{aligned} p_X(x_i) &\triangleq \mathcal{L}_X(\{x_i\}) = P(X^{-1}(\{x_i\})) \\ &= P(X = x_i) = P(\omega \in \Omega : X(\omega) = x_i). \end{aligned}$$

- For a continuous r.v. $X \in [a, b]$, however, the more important thing on each $x \in [a, b]$ is the "probability density" at $x$.

## Definition of an absolutely continuous random variable taking values on the entire $\mathbb{R}$ (page 58 in the textbook)

- Let $X$ be an absolutely continuous r.v. defined on a probability space $(\Omega, \mathcal{F}, P)$ which assigns $\Omega$ to the entire $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.
- There must exist a probability density function

$$f_X : \mathbb{R} \longrightarrow \mathbb{R}_+$$

so that, for any Borel set $I \in \mathcal{B}(\mathbb{R})$, the induced measure $\mathcal{L}_X$ on $I$ is computed through the integration of $f_X$ over $I$. That is,

$$\mathcal{L}_X(I) = P(X^{-1}(I)) = P(X \in I) = P(\omega \in \Omega : \ X(\omega) \in I) = \int_I f_X(s)ds.$$

- In case that $I = [a, b]$, we have

$$P(X \in [a, b]) = P(a \leq X \leq b) = \int_{[a,b]} f_X(s)ds = \int_a^b f_X(s)ds.$$

- In case that $I = \{a\} = [a, a]$,

$$P(X \in \{a\}) = P(X = a) = \int_{[a,a]} f_X(s)ds = \int_a^a f_X(s)ds = 0.$$

# Absolutely Continuous Random Variable

- In case that $I = (-\infty, x]$, we have the cumulative distribution function of $X$ as

$$F_X(x) = P(X \in (-\infty, x]) = P(X \leq x) = \int_{-\infty}^{x} f_X(s)ds. \qquad (1)$$

- For $I = (-\infty, \infty)$,

$$P(X \in (-\infty, \infty)) = P(\omega \in \Omega) = \int_{-\infty}^{\infty} f_X(s)ds = 1.$$

- Since $P(X = a) = 0$ for an absolute continuous random variable $X$, we have

$$P(X < a) = P(X \leq a) = \int_{-\infty}^{a} f_X(s)ds.$$

# Absolutely Continuous Random Variable

- Let $X$ be an absolutely Continuous Random Variable with density $f_X(x)$ and the cumulative distribution function $F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(s)ds$.
- By Fundamental Theorem of Calculus, on an open interval where $f$ is continuous, $F_X'(x) = f_X(x)$ for $x$ in that open interval.
- The reason we say that $f(x)$ is the probability "density" (rate of change of probability at a particular $x \in \mathbb{R}$ with respect to the unit Borel length) is because

$$f_X(x) = F_X'(x) = \lim_{h \to 0} \frac{F_X(x+h) - F_X(x)}{h} = \lim_{h \to 0} \frac{P(X \in [x, x+h])}{h}$$

- By the differential form, we also have the Leibnitz notation connecting the cumulative distribution function $F_X(x)$ and the density function $f_X(x)$ of $X$ by

$$dF_X = dF_X(x, dx) = F_X'(x) \cdot dx = f_X(x) \cdot dx.$$

# Absolutely Continuous Random Variable

- Not every continuous r.v. is absolutely continuous.
- A continuous random variable $X$ could be "singular." That is, $X'(x) = 0, a.e.$. For example, the Cantor-Lebesgue function. We are not going to discuss singular r.v.'s in this course.
- For an absolutely continuous random variable $X$ taking values on $[a, b]$ with the density $f_X(x)$, we can define its expectation as (where $a$ could be $-\infty$, and $b$ could be $\infty$ in which case the improper integral is used.):

$$E(X) = \int_a^b x \cdot f_X(x)dx = \int_a^b x \cdot dF_X$$

- By partition $[a, b]$ into $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, the expectation of $X$ can be approximated by Riemann Sum:

$$\begin{aligned}
& x_1 \cdot P(X \in [x_0, x_1]) + x_2 \cdot P(X \in [x_1, x_2]) + \cdots + x_n \cdot P(X \in [x_{n-1}, x_n]) \\
= \quad & x_1 \cdot \frac{P(X \in [x_0, x_1])}{\Delta x_1}\Delta x_1 + x_2 \cdot \frac{P(X \in [x_1, x_2])}{\Delta x_2}\Delta x_2 + \cdots + x_n \cdot \frac{P(X \in [x_{n-1}, x_n])}{\Delta x_n}\Delta x_n \\
\xrightarrow{\max \|\Delta x_i\| \to 0} \quad & \int_a^b x \cdot f_X(x)dx.
\end{aligned}$$

# Expectation and Variance of an absolutely continuous random variable

- For a discrete r.v., we have

$$E(X) = \underbrace{\sum_{\omega\in\Omega} X(\omega)P(\omega)}_{\text{sum over sample space}} = \underbrace{\sum_{i=1}^{\infty} x_i p_X(x_i)}_{\text{sum over foreground space}}$$

- and $X$ is a discrete r.v.,

$$E(g(X)) = \underbrace{\sum_{\omega\in\Omega} g(X)(\omega)P(\omega)}_{\text{sum over sample space}} = \underbrace{\sum_{i=1}^{\infty} g(x_i) p_X(x_i)}_{\text{sum over foreground space}}$$

- For a continuous r,v, $X$, the formula for $E(g(X))$ can be proved to be the integration of $Y = g(X)$ w.r.t. the distribution function of $X$ as

$$E(g(X)) = \int_{g(a)}^{g(b)} y \cdot dF_Y(y) = \int_a^b g(x) \cdot dF_X(x) = \int_a^b g(x) \cdot f_X(x)dx$$

- Variance of $X$ is computed by the same formula:

$$\mathrm{Var}(X) = E(X-\mu_X)^2 = E(X^2) - \mu_X^2 = E(X^2) - (EX)^2.$$

# Example 6.1 (page 59 in the textbook)

- Let $X$ be a (absolutely) continuous random variable with the following density function:

$$f_X(x) = \begin{cases} cx, & \text{if } x \in (0,4); \\ 0, & \text{otherwise.} \end{cases}$$

- By the fact that

$$P(-\infty < X < \infty) = P(\Omega) = 1 = \int_0^4 f_X(x) \cdot dx = \int_0^4 cx \cdot dx = c\frac{4^2}{2} = 8c,$$

  it implies that $c = \frac{1}{8}$ and the density of $X$ is $f_X(x) = \frac{x}{8}, \ x \in (0,4)$.

- The probability $P(X \in [1,2]) = \int_1^2 \frac{x}{8} dx = \frac{3}{16}$.
- The expectation of $X$ is $E(X) = \int_0^4 x \cdot f_X(x) dx = \int_0^4 \frac{x^2}{8} dx = \frac{8}{3}$
- To compute the variance of $X$, we first do

$$E(X^2) = \int_0^4 x^2 \cdot f_X(x) dx = \int_0^4 \frac{x^3}{8} dx = 8.$$

  Then, $\mathrm{Var}(X) = E(X^2) - (EX)^2 = 8 - \frac{8^2}{3^2} = \frac{8}{9}$.

# Example 6.2 (page 59 in the textbook)

- Let $X$ be an (absolutely) continuous random variable with the following density function:

$$f_X(x) = \begin{cases} 3x^2, & \text{if } x \in [0,1]; \\ 0, & \text{otherwise.} \end{cases}$$

  Compute the density $f_Y$ for $Y = 1 - X^4$.

- We first determine the range of $Y$ to be also between 0 and 1. Then, we compute the accumulative distribution function $F_Y(y)$ for $y \in [0,1]$. That is,

$$\begin{aligned} F_Y(y) &= P(Y \le y) = \int_{-\infty}^{y} f_Y(x)dx \\ &= P(1 - X^4 \le y) = P(\sqrt[4]{1-y} \le X) \\ &= \int_{\sqrt[4]{1-y}}^{1} 3x^2 dx. \end{aligned}$$

- Then,

$$f_Y(y) = \frac{d}{dy}F_Y(y) = -3\left(\sqrt[4]{1-y}\right)^2 \frac{1}{4}(1-y)^{-\frac{3}{4}}(-1) = \frac{3}{4\sqrt[4]{1-y}}.$$

# Law of the Unconscious Statistician

- Example 6.2 above can be generalized to prove a special case of Law of the Unconscious Statistician.
- Let $X$ be an (absolutely) continuous r.v. defined on defined on a probability space $(\Omega, \mathcal{F}, P)$ with the range set $\mathcal{X}$ having the density function $f_X(x)$ defined on $\mathcal{X}$; and $g : \mathbb{R} \to \mathbb{R}$ is a Borel function so that $Y = g(X)$ is a r.v. also with the range set $\mathcal{Y}$.
- Then, Law of the Unconscious Statistician says that

$$
\begin{aligned}
E(Y) &= \int_{\mathcal{Y}} y \cdot f_Y(y) \cdot dy \\
&= E(g(X)) = < g(x), f_X(x) > = \int_{\mathcal{X}} g(x) \cdot f_X(x) \cdot dx.
\end{aligned}
$$

- Here, we only prove for a special case that $g$ is a differentiable monotonic function so that $g^{-1}$ exists and is also monotone. Then, a general Borel function can be approximated a.e. by a sequence of monotonic increase functions.

# Law of the Unconscious Statistician

- Since $y = g(x)$ is assumed to be monotonic and differentiable, its inverse function $x = g^{-1}(y)$ exists and also differentiable. In fact, the differential form gives $dx = \frac{d}{dy}g^{-1}(y) \cdot dy$.
- On the other hand, we have

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(g(X) \leq y) \\ &= P(X \leq g^{-1}(y)) = F_X(g^{-1}(y)) \end{aligned}$$

- Then,

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X(g^{-1}(y)) = f_X(g^{-1}(y)) \cdot \frac{d}{dy}g^{-1}(y)$$

- Therefore, by change of variable $y = g(x)$ in integration,

$$\begin{aligned} E(Y) = \int_{\mathcal{Y}} y \cdot f_Y(y) \cdot dy &= \int_{\mathcal{Y}} y \cdot f_X(g^{-1}(y)) \cdot \frac{d}{dy}g^{-1}(y) \cdot dy \\ &= \int_{\mathcal{X}} g(x) \cdot f_X(x) \cdot dx. \end{aligned}$$

# Uniform Random Variable (page 60 in the textbook)

- A continuous random variable $X$ on a probability space $(\Omega, \mathcal{F}, P)$ is called a uniform random variable, denoted by $X \sim \mathrm{Unif}[\alpha, \beta]$, if $X$ defined on $(\Omega, \mathcal{F}, P)$ takes values on $[\alpha, \beta], \alpha < \beta \in \mathbb{R}$ with the following density

$$f_X(x) = \begin{cases} \frac{1}{\beta-\alpha}, & \text{if } x \in [\alpha, \beta]; \\ 0, & \text{otherwise.} \end{cases}$$

- That is, the probability density is a constant for all points on $[\alpha, \beta]$, thus the name "uniform."
- Suppose $X \sim \mathrm{Unif}[\alpha, \beta]$. For $\alpha < a < b < \beta$, the probability for the event that $X$ takes some value on $[a, b]$ to happen, is the portion of length of $[a, b]$ in terms of the entire $[\alpha, \beta]$.

$$P(a \leq X \leq b) = \int_a^b f_X(x)dx = \frac{1}{\beta - \alpha}\int_a^b dx = \frac{b-a}{\beta-\alpha}.$$

- For example, if $X$ is the time at which an event occurred and $X \sim \mathrm{Unif}[\alpha, \beta]$. Then, each interval in $[\alpha, \beta]$ of equal length should have the same probability of containing the event.
- The expectation $EX = \frac{1}{\beta-\alpha}\int_\alpha^\beta x \cdot dx = \frac{\alpha+\beta}{2}$. The variance $\mathrm{Var}(X) = \frac{(\beta-\alpha)^2}{12}$.

# Uniform Random Variable (page 60 in the textbook)

- For $[\alpha, \beta] = [0,1]$, $f_X(x) = 1, \forall x \in [0,1]$. In this case, $X$ models an ideal random number generator on a computer[1].
- Assume that $X \sim \mathrm{Unif}[0,1)$. The probability for $X$ to take a value in $\mathbb{Q}$ (let $\{q_1, q_2, \ldots, q_n, \ldots\} \subset [0,1)$ be an enumeration of $\mathbb{Q}$) is $P(X \in \mathbb{Q}) = P(\cup_i \{X = q_i\}) = \sum_{i=1}^{n} P(X = q_i) = 0.$[2]
- Each point $x \in [0,1)$ has the binary expression

$$x = (0.x_1x_2x_3\cdots)_2 = \sum_{i=1}^{\infty} \frac{x_i}{2^i}, \ x_i = 0 \vee 1.$$

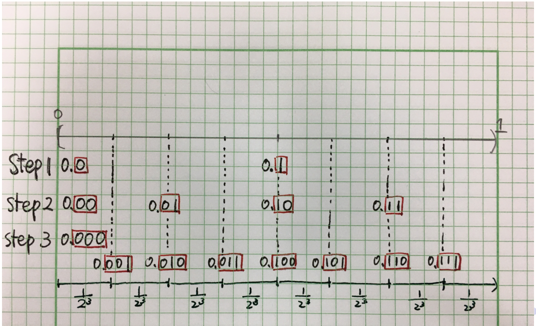
- The set $\{x = (0.0x_2x_3\cdots)_2\}$ has the smallest number 0, the largest one $(0.1)_2 = 0.5$.
- The set $\{x = (0.1x_2x_3\cdots)_2\}$ has the smallest number $(0.1)_2 = 0.5$, but no largest one because $(0.11111\ldots)_2 = 1 \notin [0,1)$.

[1] The existing random number generator on a computer is "far from random" though!

[2] This can be interpreted as the "length" of $\mathbb{Q}$ accounts for 0% of the total length 1 of $[0,1]$, indicating that in $\mathbb{R}$ they are essentially all irrational numbers.

# Uniform Random Variable (page 60 in the textbook)

- As the same pattern repeats, the set $\{x = (0.00x_3x_4\cdots)_2\}$ has the smallest number 0, while the largest one $(0.01)_2 = 0.25$.
- The set $\{x = (0.01x_2x_3\cdots)_2\}$ has the smallest number $(0.01)_2 = 0.25$, while the largest one $(0.1)_2 = 0.5$.
- The set $\{x = (0.10x_2x_3\cdots)_2\}$ has the smallest number $(0.1)_2 = 0.5$, while the largest one $(0.11)_2 = 0.75$..
- The set $\{x = (0.11x_2x_3\cdots)_2\}$ has the smallest number $(0.11)_2 = 0.75$, while there is no largest one in the set.



# Uniform Random Variable (page 60 in the textbook)

- In general, at the $n^{th}$ step, the interval $[0, 1)$ is divided into $2^n$ subintervals, each of the length $\frac{1}{2^n}$. The first $n$ binary digits of $x$ determine which of the $2^n$ subintervals $x$ belongs to.
- If $X$ is a uniform random variable on $[0, 1)$, any of the $2^n$ subintervals are equally likely, each with the probability of $\frac{1}{2^n}$ to happen.
- In other words, the binary digits of a uniformly distributed $X \sim \mathrm{Unif}[0, 1)$

$$X(\omega) = (0.x_1x_2x_3\cdots)_2 = \sum_{i=1}^{\infty} \frac{x_i}{2^i}, \;\; x_i = 0 \vee 1$$

  are the result of an infinite sequence of independent fair coin tosses.

# Infinite coin tossing space

- Let $\Omega_\infty = \{\omega = (\omega_1, \omega_2, \omega_3, \ldots) : \ \omega_i = H \vee T, \ \forall i = 1, 2, \ldots\}$ be the set of all nonterminating sequences of $H$ and $T$, modeling the situation that a coin can be tossed repeatedly without stopping.
- $\Omega_\infty$ is an uncountably infinite space.
- For each integer $n$, we define $\mathcal{F}_n$ to be the $\sigma$-algebra containing information up to the first $n$ tosses.
- For example,

$$\begin{aligned}\mathcal{F}_2 = \quad \{ \quad & \emptyset, \Omega, A_{HH}, A_{HT}, A_{TH}, A_{TT}, \\ & A_H, A_T, A_{HH} \cup A_{TH}, A_{HT} \cup A_{TT}, A_{HH} \cup A_{TT}, A_{HT} \cup A_{TH}, \\ & A_{HH}^c, A_{HT}^c, A_{TH}^c, A_{TT}^c\}.\end{aligned}$$

  where

$$A_{HH} = \{\omega = (H, H, \omega_3, \omega_4, \ldots) : \ \omega_i = H \vee T, \ \forall i = 3, 4, \ldots\},$$

$$A_{HT} = \{\omega = (H, T, \omega_3, \omega_4, \ldots) : \ \omega_i = H \vee T, \ \forall i = 3, 4, \ldots\},$$

  and so forth.
- Each $A_{HH}, A_{HT}, A_{TH}, A_{TT}$ consists of an uncountable number of sample points, so do their unions.

# Infinite coin tossing space

- We define the $\sigma$-algebra $\mathcal{F}_\infty$ on $\Omega_\infty$ to be the smallest $\sigma$-algebra generated by the union of all $\mathcal{F}_n$'s, denoted by $\mathcal{F}_\infty = \sigma(\bigcup_{n=1}^\infty \mathcal{F}_n)$.
- Notice that $\mathcal{F}_\infty$ contains sets not belonging to $\bigcup_{n=1}^\infty \mathcal{F}_n$.
- For example, the set containing the single sequence

$$\{(H, H, H, \cdots)\} = \{H \text{ on every toss}\} = \bigcap_{n=1}^{\infty} \{H \text{ on the first } n \text{ tosses}\}$$

  is in $\mathcal{F}_\infty$ because the singleton set $\{(H, H, H, \cdots)\}$ is formed by countable intersections of $A_H \in \mathcal{F}_1, A_{HH} \in \mathcal{F}_2, A_{HHH} \in \mathcal{F}_3, \cdots$
- Another example is

$$\{(H, T, H, T, H \cdots)\} = A_H \cap A_{HT} \cap A_{HTH} \cap \cdots$$

- However, either the "singleton" events $\{(H, H, H, \cdots)\}$ or $\{(H, T, H, T, H \cdots)\}$ are not in any of the $\mathcal{F}_n$'s because each element in $\mathcal{F}_n, \ \forall n \in \mathbb{N}$ consisting of an uncountable number of sample points (except for $\emptyset$).

# Infinite coin tossing space

- We next construct a probability measure $P$ on $(\Omega_\infty, \mathcal{F}_\infty)$ which corresponds to probability $p \in [0,1]$ for a single toss $H$ and $q = 1 - p$ for $T$.
- First, for $A \in \mathcal{F}_n$, since it depends on only the first $n$ tosses, $P(A)$ can be defined to be the product of the $p$'s and $q$'s corresponding to the $n$ tosses. For example, we define $P(A_{HH}) = p^2, P(A_{TH}) = qp$ so that $P(A_{HH} \cup A_{TH}) = p^2 + qp = p$.
- In other words, the probability of the event for a $H$ on the second toss (in tossing a coin infinitely many times) is $p$, the same as the probability to get a $H$ in a single toss.
- For sets $A \in \mathcal{F}_\infty \setminus \bigcup_{n=1}^\infty \mathcal{F}_n$, we define $P(A)$ by the limit.
- For example, we can define $P(\{(H, H, H, \cdots)\}) = \lim_{n\to\infty} p^n$ since $\{(H, H, H, \cdots)\}$ can be represented as the intersection of a sequence of decreasing sets: $A_H, A_{HH}, A_{HHH}, \cdots$
- When $p = 1$, $P(\{(H, H, H, \cdots)\}) = 1$. Otherwise, $P(\{(H, H, H, \cdots)\}) = 0$ for $0 \le p < 1$.

# Infinite coin tossing space

- On $\Omega_\infty$, let us define a sequence of random variables $Y_1, Y_2, \ldots$ by
$$Y_k(\omega) = \begin{cases} 1, & \omega_k = H, \\ 0, & \omega_k = T. \end{cases}$$
- With $\{Y_k\}_{k=1}^{\infty}$, let us define $X(\omega) = \sum_{k=1}^{\infty} \frac{Y_k(\omega)}{2^k}$.
- By this way, the random variable $X$ sends a sample point $\omega \in \Omega_\infty$ into a value in $[0,1]$ which has the binary expression $X(\omega) = (0.Y_1(\omega)Y_2(\omega)Y_3(\omega)\cdots)_2$
- At the first step, we toss a fair coin to determine which of the two subintervals $[0, 0.5]$, $[0.5, 1)$ the number $X(\omega)$ belongs to.
- Suppose $Y_1(\omega) = T$, $X(\omega)$ belongs to $[0, 0.5]$.
- The event that the infinite coin tossing with the first trial to be tail is $A_T = \{\omega = (T, \omega_2, \omega_3, \omega_4, \ldots) : \ \omega_i = H \vee T, \ \forall i = 2, 4, \ldots\}$, which is sent by the r.v. $X$ to $[0, 0.5]$.

# Uniform Random Variable (page 60 in the textbook)

- At the second step, we toss a fair coin again to determine which of the two subintervals $[0, 0.25]$, $[0.25, 0.5]$ the number $X(\omega)$ belongs to.
- Otherwise, if $Y_1(\omega) = H$, $X(\omega)$ belongs to $[0.5, 1)$, at the second step, we toss a fair coin again to determine which of the two subintervals $[0.5, 0.75]$, $[0.75, 1)$ the number $X(\omega)$ belongs to.
- Continue the experiment for infinitely many times. We can then obtain a real number in $[0, 1)$ in an equally likely manner.
- However, since a computer cannot execute a random experiment for infinitely many times, the random number generator is difficulty to achieve.

# Homework Exercise

A "dyadic rational number" is a real number of the form $\frac{m}{2^k}$ where $k$ and $m$ are integers. Suppose we set $p = q = \frac{1}{2}$ in the construction for a probability measure on $\Omega_\infty$ and $X(\omega) = \sum_{k=1}^{\infty} \frac{Y_k(\omega)}{2^k}$ is a random variable on $\Omega_\infty$.

- Show that, the induced measure $\mathcal{L}_X$ by the random variable $X$ on $\Omega$ satisfies that, for any positive integers $k$ and $m$ such that $0 \leq \frac{m-1}{2^k} < \frac{m}{2^k} \leq 1$, we have

$$\mathcal{L}_X[\frac{m-1}{2^k}, \frac{m}{2^k}] = \frac{1}{2^k}.$$

  In other words, the induced measure $\mathcal{L}_X$ on all intervals in $[0, 1]$ whose endpoints are dyadic rational numbers is the same as the Lebesgue measure of these intervals. The only possible way is that $\mathcal{L}_X$ is indeed the Lebesque measure.
- Show that, in this case ($p = \frac{1}{2}$), the random variable $X(\omega) = \sum_{k=1}^{\infty} \frac{Y_k(\omega)}{2^k}$ is uniformly distributed on $[0, 1]$.

# Exponential Random Variable (page 61 in the textbook)

- An exponential random variable, denoted by $X \sim \mathrm{Exp}(\lambda)$, is a continuous random variable taking non-negative values on $x \in [0, \infty)$ while having the following density function with parameter $\lambda > 0$:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \in [0, \infty); \\ 0, & \text{if } x < 0. \end{cases}$$

- The expectation

$$EX = \int_0^\infty \lambda x e^{-\lambda x} dx = \frac{1}{\lambda} \int_0^\infty t e^{-t} dt = \frac{1}{\lambda} \int_0^\infty -t d e^{-t} = \frac{1}{\lambda}.$$

- $\mathrm{Var}(X) = \frac{1}{\lambda^2}$. (This is left as an exercise)
- $P(X \geq x) = \int_x^\infty \lambda e^{-\lambda t} dt = e^{-\lambda x}$; $P(X \leq x) = \int_0^x \lambda e^{-\lambda t} dt = 1 - e^{-\lambda x}$.
- (Example 6.5) Let $X \sim \mathrm{Exp}(\lambda)$ be the lifespan of a lightbulb which is random. Assuming that the lightbulbs last on average 100 hours. What is the probability that it lasts less than 50 hours?
- We first note that $\lambda = \frac{1}{\mu_X} = 0.01$. Then, $P(X < 50) = 1 - e^{-0.01 \cdot 50} \approx 0.3935$.

# Normal Random Variable (page 61-62 in the textbook)

- A normal random variable, denoted by $X \sim N(\mu, \sigma^2)$, is a continuous random variable taking all real values on $\mathbb{R}$ while having the following density function with parameter $\mu, \sigma^2$:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \ x \in (-\infty, \infty).$$

- Certainly, for any $\mu$ and $\sigma^2$, there is

$$\int_{-\infty}^{\infty} f_X(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1.$$

- The expectation

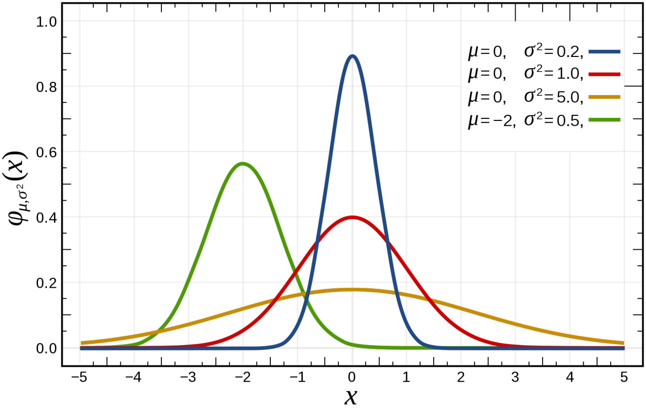
$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} (x-\mu) e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx + \mu \\ &= \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sigma\sqrt{2\pi}} z e^{-\frac{z^2}{2\sigma^2}}}_{\text{odd function}} dz + \mu = \mu \end{aligned}$$

- Variance (calculation omitted): $\mathrm{Var}(X) = \sigma^2$.

# Normal Random Variable (page 62 in the textbook)

- Density functions of $X \sim N(\mu, \sigma^2)$ with different parameters.

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \ x \in (-\infty, \infty).$$



# Normal Random Variable (page 62 in the textbook)

- Let $X \sim N(\mu, \sigma^2)$ be a normal r.v. and let $Y = \alpha X + \beta$, with $\alpha > 0$, which is a linear transformation on the value of a normal r.v.
- We start by computing the cumulative distribution of $Y$:

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(\alpha X + \beta \leq y) \\ &= P(X \leq \frac{y-\beta}{\alpha}) \\ &= \int_{-\infty}^{\frac{y-\beta}{\alpha}} f_X(x)dx \end{aligned}$$

- The density of $Y$

$$f_Y(y) = F_Y'(y) = f_X(\frac{y-\beta}{\alpha})\frac{1}{\alpha} = \frac{1}{\alpha\sigma\sqrt{2\pi}} e^{-\frac{(y-\beta-\alpha\mu)^2}{2\alpha^2\sigma^2}}$$

- Then, $Y \sim N(\alpha\mu + \beta, (\alpha\sigma)^2)$ is normal with $EY = \alpha\mu + \beta$ and variance $\mathrm{Var}(Y) = (\alpha\sigma)^2$.

# Normal Random Variable (page 63 in the textbook)

- In particular, if $X \sim N(\mu, \sigma^2)$ and let $Z = \frac{X-\mu}{\sigma}$, then $Z$ is also normal with

$$EZ = \frac{EX - \mu}{\sigma} = 0 \text{ and } \mathrm{Var}(Z) = (\frac{1}{\sigma} \cdot \sigma)^2 = 1.$$

- Such a $N(0, 1^2)$ random variable is called *standard* Normal. It has density:

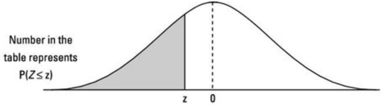
$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \ z \in (-\infty, \infty).$$

- The cumulative distribution of $Z$ is denoted by $\Phi(z)$ with

$$\Phi(z) = F_Z(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx$$

- The integral for $\Phi(z)$ cannot be computed as an elementary function, so approximate values are given in tables.
- By the fact that $f_Z(z)$ is even, we have $\Phi(-z) = 1 - \Phi(z)$.

# Normal Random Variable ($P(Z \leq -2.67) = 0.0038$)




| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| –3.6 | .0002 | .0002 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 |
| –3.5 | .0002 | .0002 | .0002 | .0002 | .0002 | .0002 | .0002 | .0002 | .0002 | .0002 |
| –3.4 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0002 |
| –3.3 | .0005 | .0005 | .0005 | .0004 | .0004 | .0004 | .0004 | .0004 | .0004 | .0003 |
| –3.2 | .0007 | .0007 | .0006 | .0006 | .0006 | .0006 | .0006 | .0005 | .0005 | .0005 |
| –3.1 | .0010 | .0009 | .0009 | .0009 | .0008 | .0008 | .0008 | .0008 | .0007 | .0007 |
| –3.0 | .0013 | .0013 | .0013 | .0012 | .0012 | .0011 | .0011 | .0011 | .0010 | .0010 |
| –2.9 | .0019 | .0018 | .0018 | .0017 | .0016 | .0016 | .0015 | .0015 | .0014 | .0014 |
| –2.8 | .0026 | .0025 | .0024 | .0023 | .0023 | .0022 | .0021 | .0021 | .0020 | .0019 |
| –2.7 | .0035 | .0034 | .0033 | .0032 | .0031 | .0030 | .0029 | .0028 | .0027 | .0026 |
| –2.6 | .0047 | .0045 | .0044 | .0043 | .0041 | .0040 | .0039 | .0038 | .0037 | .0036 |
| –2.5 | .0062 | .0060 | .0059 | .0057 | .0055 | .0054 | .0052 | .0051 | .0049 | .0048 |
| –2.4 | .0082 | .0080 | .0078 | .0075 | .0073 | .0071 | .0069 | .0068 | .0066 | .0064 |
| –2.3 | .0107 | .0104 | .0102 | .0099 | .0096 | .0094 | .0091 | .0089 | .0087 | .0084 |
| –2.2 | .0139 | .0136 | .0132 | .0129 | .0125 | .0122 | .0119 | .0116 | .0113 | .0110 |
| –2.1 | .0179 | .0174 | .0170 | .0166 | .0162 | .0158 | .0154 | .0150 | .0146 | .0143 |
| –2.0 | .0228 | .0222 | .0217 | .0212 | .0207 | .0202 | .0197 | .0192 | .0188 | .0183 |
| –1.9 | .0287 | .0281 | .0274 | .0268 | .0262 | .0256 | .0250 | .0244 | .0239 | .0233 |
| –1.8 | .0359 | .0351 | .0344 | .0336 | .0329 | .0322 | .0314 | .0307 | .0301 | .0294 |
| –1.7 | .0446 | .0436 | .0427 | .0418 | .0409 | .0401 | .0392 | .0384 | .0375 | .0367 |
| –1.6 | .0548 | .0537 | .0526 | .0516 | .0505 | .0495 | .0485 | .0475 | .0465 | .0455 |
| –1.5 | .0668 | .0655 | .0643 | .0630 | .0618 | .0606 | .0594 | .0582 | .0571 | .0559 |
| –1.4 | .0808 | .0793 | .0778 | .0764 | .0749 | .0735 | .0721 | .0708 | .0694 | .0681 |
| –1.3 | .0968 | .0951 | .0934 | .0918 | .0901 | .0885 | .0869 | .0853 | .0838 | .0823 |
| –1.2 | .1151 | .1131 | .1112 | .1093 | .1075 | .1056 | .1038 | .1020 | .1003 | .0985 |
| –1.1 | .1357 | .1335 | .1314 | .1292 | .1271 | .1251 | .1230 | .1210 | .1190 | .1170 |
| –1.0 | .1587 | .1562 | .1539 | .1515 | .1492 | .1469 | .1446 | .1423 | .1401 | .1379 |
| –0.9 | .1841 | .1814 | .1788 | .1762 | .1736 | .1711 | .1685 | .1660 | .1635 | .1611 |
| –0.8 | .2119 | .2090 | .2061 | .2033 | .2005 | .1977 | .1949 | .1922 | .1894 | .1867 |

## Normal Random Variable (Example 6.7 page 63 in the textbook)

- What is the probability that a Normal random variable differs from its mean $\mu$ by more than $\sigma$? more than $2\sigma$? more than $3\sigma$?
- In mathematical symbols, if $X \sim N(\mu, \sigma^2)$, we need to compute $P(|X-\mu| \geq \sigma)$, $P(|X-\mu| \geq 2\sigma)$, and $P(|X-\mu| \geq 3\sigma)$.
- The computation is easier through transforming to a standard normal random variable $Z = \frac{X-\mu}{\sigma} \sim N(0, 1^2)$. That is,

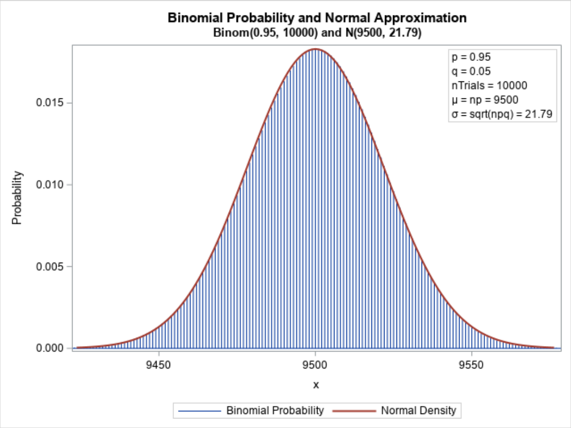
$$
\begin{aligned}
P(|X-\mu| \geq \sigma) &= P(|\frac{X-\mu}{\sigma}| \geq 1) \\
&= 2P(Z \leq -1) \\
&\approx 2 \cdot 0.1587 = 0.3174.
\end{aligned}
$$

- Similarly, $P(|X-\mu| \geq 2\sigma) = P(|Z| \geq 2) = 2P(Z \leq -2) = 2 \cdot (0.0228) = 0.0456$.

## de Moivre-Laplace Central Limit Theorem (1783) (page 64 in the textbook)

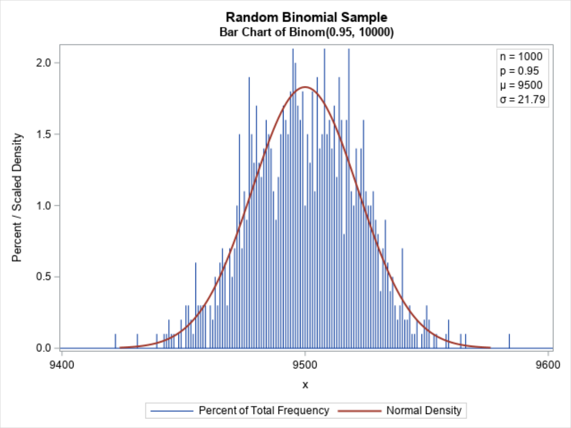
- Let $S_n \sim \mathrm{Binomial}(n, p)$. Recall that its mean it $np$ and its variance is $np(1-p) = npq$.
- If we pretend that $S_n$ is Normal with mean $np$ and variance $npq$, then
$$\frac{S_n - np}{\sqrt{npq}} \sim N(0, 1).$$
- de Moivre-Laplace Central Limit Theorem assures that such pretending is indeed “quite real” when $p$ is fixed and $n$ is large. That is, the normal distribution can be used to approximate the binomial distribution “under certain conditions.”
- For example, if $k$ is very close to $np$, we can directly comptue
$$\binom{n}{k} p^k q^{n-k} \simeq \frac{1}{\sqrt{2\pi npq}} e^{-\frac{(k-np)^2}{2npq}}$$
by Stirling's formula $n! \simeq n^n e^{-n}\sqrt{2\pi n}$ and $\ln(1+x) \simeq x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

# de Moivre-Laplace Central Limit Theorem (1783)




- Notice that, the binomial density is discrete, which is defined only for positive integers, whereas the normal density is defined for all real numbers.

# de Moivre-Laplace Central Limit Theorem (1783)




- If we take a sample size of 1000 from the binomial distribution $\mathrm{Binomial}(10000, 0.95)$, the distribution of the sample (percent) looks, at first glance, a bit alike to the density curve of normal, but quite different at a closer look.

## Central Limit Theorem (Theorem 8.9 at page 98 in the textbook)

- Assume that $\{X, X_1, X_2, \ldots\}$ is a sequence of independent, identically distributed ($i.i.d.$) r.v's with finite mean $\mu_X = EX$, and variance $\sigma_X^2 = VAR(X)$. Then,

$$P(\frac{X_1 + X_2 + \cdots + X_n - \mu n}{\sigma\sqrt{n}} \leq x) \to P(Z \leq n), \quad as \quad n \to \infty,$$

  where $Z$ is standard normal.
- Observe that, the r.v. $X$ has an arbitrary distribution with given expectation and variance. The central limit theorem says that their sum approximates to a random variable of a very particular normal distribution.
- Adding many independent copies of a r.v. erases all information about its distribution other than expectation and variance.

## de Moivre-Laplace Central Limit Theorem (1783) (page 64 in the textbook)

- Let $S_n \sim \mathrm{Binomial}(n, p)$ and $m_1, m_2$ be two positive integers. The probability that the number of successes between $m_1 < m_2$ has a precise formula:

$$P(m_1 \leq S_n \leq m_2) = \sum_{i=m_1}^{m_2} \binom{n}{i} p^i q^{n-i}.$$

- For large number of $m_1$ and $m_2$, the computation of the precise formula could be tedious.
- However, according to de Moivre-Laplace Central Limit Theorem, $\frac{S_n - np}{\sqrt{npq}} \sim N(0, 1)$ so that

$$\begin{aligned} P(m_1 \leq S_n \leq m_2) &= P(\underbrace{\frac{m_1 - np}{\sqrt{npq}}}_{=\alpha} \leq \frac{S_n - np}{\sqrt{npq}} \leq \underbrace{\frac{m_2 - np}{\sqrt{npq}}}_{=\beta}) \\ &= \int_{\alpha}^{\beta} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \end{aligned}$$

# de Moivre-Laplace Central Limit Theorem

- A die is threw 12000 times. What is the probability that there will be exactly 1800 rolls of 6?
- This is a Binomial trial for $n = 12000$, $p = \frac{1}{6}$. The exact probability is $\binom{12000}{1800}\left(\frac{1}{6}\right)^{1800}\left(\frac{5}{6}\right)^{10200}$, whose exact value is difficult to compute.
- The probability can be approximated by Poisson($np$)=Poisson(2000) $=\frac{e^{-2000}2000^{1800}}{1800!}$, which is still very difficult to compute.
- However, if we approximate by de Moivre-Laplace Central Limit Theorem,

$$\binom{12000}{1800}\left(\frac{1}{6}\right)^{1800}\left(\frac{5}{6}\right)^{10200} \approx \underbrace{\frac{1}{\sqrt{2\pi \cdot 1666.67}}}_{=9.772\times 10^{-3}} \cdot \underbrace{e^{-\frac{(1800-2000)^2}{2\cdot 1666.67}}}_{=6.144\times 10^{-6}} \approx 6.004\times 10^{-8}.$$

- (EXERCISE) What is the approximate probability for the number of $6's$ lies in the interval $[1950, 2100]$?