臺灣大學數學系113學年度碩士班甄試筆試試題

科目：線性代數 2023.11.02

1. Let $A \in M(3, \mathbb{R})$ be given by

$$A = \begin{pmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) (**10 points.**) Find the Jordan-Chevalley decomposition of A.

(b) (**10 points.**) Compute

$$\exp A := I_3 + \sum_{k=1}^{\infty} \frac{A^k}{k!}.$$

2. Let V be the space of all polynomials in x over $\mathbb{R}$ of degree ≤ 2. Let an inner product on V be defined by

$$\langle f, g \rangle := \int_{-1}^{1} f(x)g(x)\,dx.$$

(a) (**10 points.**) Find a polynomial $k(x,t)$ in x and t such that

$$f(x) = \int_{-1}^{1} k(x,t)f(t)\,dt$$

for all $f \in V$.

(b) (**10 points.**) Let $T : V \to V$ be the linear transformation defined by $T(a_2x^2 + a_1x + a_0) = 2a_2x + a_1$. Find the linear transformation $T^* : V \to V$ such that $\langle T(f), g\rangle = \langle f, T^*(g)\rangle$ for all $f, g \in V$.

3. (**20 points.**) Let $V = M(n, \mathbb{R})$ be the vector space of all $n \times n$ matrices over $\mathbb{R}$ and $f : V \to \mathbb{R}$ be a linear transformation. Assume that $f(AB) = f(BA)$ for all $A, B \in V$ and $f(I_n) = n$, where I_n is the identity matrix in V. Prove that f is the trace function. (*Hint*: Consider the cases $A = E_{ij}$ and $B = E_{k\ell}$ for various E_{ij} and $E_{k\ell}$. Here E_{ij} denotes the matrix whose (i,j)-entry is 1 and whose other entries are 0.)

4. Let U and V be finite-dimensional vector spaces, and U^* and V^* be their dual spaces, respectively. For a linear transformation $T : U \to V$, define $T^* : V^* \to U^*$ by $(T^*f)(u) = f(Tu)$ for $f \in V^*$ and $u \in U$.

(a) (**10 points.**) Prove that T is injective if and only if T^* is surjective.

(b) (**10 points.**) Prove that T is surjective if and only if T^* is injective.

5. (**20 points.**) Let V a finite-dimensional vector space over a field F and $T : V \to V$ be a linear transformation. Assume that $f(x)$ and $g(x)$ are two relatively prime polynomials in $F[x]$. Prove that $\ker(f(T)g(T)) = \ker f(T) \oplus \ker g(T)$. (Here for a linear transformation S, we let $\ker S$ denote the kernel of S.)

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科目：線性代數

2022.10.20

Notation: $\mathbb{R}$ is the set of real numbers and $\mathbb{C}$ is the set of complex numbers. If $F = \mathbb{R}$ or $\mathbb{C}$ and n is a positive integer, we denote by $\mathrm{M}_n(F)$ the set of $n \times n$ matrices with entries in F and by I_n the identity matrix in $\mathrm{M}_n(F)$.

Problem 1 (15 pts)**.** Let

$$\mathbf{v}_1 = (1, 2, 0, 4),\ \mathbf{v}_2 = (-1, 1, 3, -3),\ \mathbf{v}_3 = (0, 1, -5, -2),\ \mathbf{v}_4 = (-1, -9, -1, -4)$$

be vectors in $\mathbb{R}^4$. Let W_1 be the subspace spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ and let W_2 be the subspace spanned by $\mathbf{v}_3$ and $\mathbf{v}_4$. Find the dimension and a basis of $W_1 \cap W_2$.

Problem 2. Let

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 3 & -4 & 1 \\ 3 & -8 & 5 \end{pmatrix}.$$

(1) (10pts) Find an invertible matrix $Q \in \mathrm{M}_3(\mathbb{C})$ such that

$$Q^{-1}AQ = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 12 \\ 0 & 1 & 1 \end{pmatrix}.$$

(2) (15pts) Find an invertible matrix $P \in \mathrm{M}_3(\mathbb{C})$ such that $P^{-1}AP$ is a diagonal matrix.

Problem 3. For any $A \in \mathrm{M}_2(\mathbb{C})$, define

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.$$

(1) (5pts) Evaluate $\sin \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$.

(2) (15pts) Prove or disprove: There exists $A \in \mathrm{M}_2(\mathbb{R})$ such that

$$\sin A = \begin{pmatrix} 1 & 2022 \\ 0 & 1 \end{pmatrix}.$$

Problem 4 (20pts)**.** Let $A = (a_{ij}) \in \mathrm{M}_n(\mathbb{C})$ and let $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ be roots of characteristic polynomial of A (counted with multiplicity). Show that

$$AA^* = A^*A \text{ if and only if } \sum_{1 \le i,j \le n} |a_{ij}|^2 = \sum_{k=1}^{n} |\lambda_k|^2.$$

Problem 5 (20pts)**.** Let $A, B \in \mathrm{M}_n(\mathbb{C})$. Suppose that all of the eigenvalues of A and B are positive real numbers. If $A^4 = B^4$, prove that $A = B$.

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科目：線性代數

2020.10.23

In the following, for a linear map $f : V \to V$, $\ker f$ and $\operatorname{im} f$ denote the kernel and the image of f, respectively.

1. Let V be a finite-dimensional complex inner product space. Let $d : V \to V$ be a linear map satisfying $d^2 = 0$. Let $\delta : V \to V$ be the adjoint of d and $\Delta = d\delta + \delta d$. Prove the following.
   (a) [5%] $d\delta x = 0$ implies that $\delta x = 0$, and $\delta d x = 0$ implies that $dx = 0$, for all $x \in V$.
   (b) [10%] $\ker \Delta = \ker d \cap \ker \delta$.
   (c) [10%] There is the orthogonal decomposition $V = \ker \Delta \oplus \operatorname{im} d \oplus \operatorname{im} \delta$.
   (d) [5%] There is the orthogonal decomposition $\ker d = \ker \Delta \oplus \operatorname{im} d$.

2. [10%] Let $V = \mathbb{R}^n$ be the space of column vectors, and M a positive definite symmetric $n \times n$ real matrix. Suppose the matrix $A \in M_n(\mathbb{R})$ satisfies $MAM^{-1} = A^t$. Show that there exists $P \in M_n(\mathbb{R})$ satisfying $P^t M P = I_n$ such that $P^{-1}AP$ is diagonal. (Here B^t denotes the transpose of the matrix B.)

3. (a) [10%] Let M be an invertible $n \times n$ complex matrix. Prove that there exists an invertible matrix A such that $A^2 = M$.
   (b) [10%] Let $n \geq 2$ and N be an $n \times n$ matrix over a field such that $N^n = 0$ but $N^{n-1} \neq 0$. Prove that there is no square matrix B such that $B^2 = N$.

4. [20%] Let V be a vector space over a field F and $u_1, \cdots, u_n \in V$ are linearly independent. Show that, for any $v_1, \cdots, v_n \in V$, $u_1 + \alpha v_1, \cdots, u_n + \alpha v_n$ are linearly independent for all but finitely many values of $\alpha \in F$.

5. [20%] Let P be an $n \times n$ matrix with coefficients in a field. Suppose $\operatorname{rank}(P) + \operatorname{rank}(I_n - P) = n$. Prove that $P^2 = P$.

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科目：線性代數

2019.10.18

1. Let A be a 4×4 real symmetric matrix. Suppose that 1 and 2 are eigenvalues of A and the eigenspace for the eigenvalue 2 is 3-dimensional. Assume that $(1, -1, -1, 1)^t$ is an eigenvector for the eigenvalue 1. (Here v^t denotes the transpose of v.)

(a) Find an orthonormal basis for the eigenspace for the eigenvalue 2 of A. **(10 points.)**

(b) Find Av, where $v = (1, 0, 0, 0)^t$. **(10 points.)**

2. Let A be a real $n \times n$ matrix. Prove that

$$\operatorname{rank}(A^2) - \operatorname{rank}(A^3) \leq \operatorname{rank}(A) - \operatorname{rank}(A^2).$$

(10 points.)

3. Let V be a vector space of finite dimension over $\mathbb{R}$ and S, T, and U be subspaces of V. Prove or disprove (by giving counterexamples) the following statements:

(a) $\dim(S + T) = \dim S + \dim T - \dim(S \cap T)$. **(10 points.)**

(b) $\dim(S + T + U) = \dim S + \dim T + \dim U - \dim(S \cap T) - \dim(T \cap U) - \dim(U \cap S) + \dim(S \cap T \cap U)$. **(10 points.)**

4. (a) Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$. Compute $\exp A$. **(10 points.)**

(b) Prove that $\det(\exp A) = \exp(\operatorname{tr} A)$ for $A \in M(n, \mathbb{C})$. **(10 points.)**

(c) Prove or disprove (by giving counterexamples) that if A is nilpotent, then so is $\exp A - I_n$. Here a matrix M is said to be nilpotent if $M^k = 0$ for some positive integer k and I_n is the identity matrix of size n. **(10 points.)**

5. Let U and V be finite-dimensional vector spaces, and U^* and V^* be their dual spaces, respectively. For a linear transformation $T : U \to V$, define $T^* : V^* \to U^*$ by $(T^* f)(u) = f(Tu)$ for $f \in V^*$ and $u \in U$.

(a) Prove that T is injective if and only if T^* is surjective. **(10 points.)**

(b) Prove that T is surjective if and only if T^* is injective. **(10 points.)**

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科目：線性代數

2018. 10. 19

1. Find all possible Jordan forms for 8×8 real matrices having $x^2(x-2)^3$ as minimal polynomial. (**20 points.**)

2. Let V be a vector space over a field $\mathbb{F}$ of infinite elements, and let $v_1, \ldots, v_n$ be vectors in V, where n is a positive integer. Suppose that $v_0 + zv_1 + \ldots + z^n v_n = 0$ for infinitely many z in $\mathbb{F}$. Prove that all v_i's are zero. (**20 points.**)

3. Let $V = M(n, \mathbb{R})$ be the vector space of all $n \times n$ matrices and $f : V \to \mathbb{R}$ be a linear transformation. Assume that $f(AB) = f(BA)$ for all $A, B \in V$ and $f(I_n) = n$, where I_n is the identity matrix in V. Prove that f is the trace function. (**20 points.** *Hint*: Consider the cases $A = E_{ij}$ and $B = E_{k\ell}$ for various E_{ij} and $E_{k\ell}$. Here E_{ij} denotes the matrix whose (i, j)-entry is 1 and whose other entries are 0.)

4. Let V be an n-dimensional vector space over $\mathbb{R}$ and $B : V \times V \to \mathbb{R}$ be a symmetric bilinear form on V. (*Symmetric* means $B(u, v) = B(v, u)$ for all $u, v \in V$. *Bilinear* means that B is linear in each of the two variables.)

(a) Let W be a vector subspace of V and let

$$W^\perp = \{u \in V : B(u, v) = 0 \text{ for all } v \in W\}.$$

Prove that if $\dim W = m$, then $\dim W^\perp \geq n - m$. (**10 points.** *Hint*: Choose a basis $\{v_1, \ldots, v_m\}$ for W and consider the map

$$u \longmapsto (B(u, v_1), \ldots, B(u, v_m))$$

from V into $\mathbb{R}^m$.)

(b) Prove that $V = W \oplus W^\perp$ if and only if the restriction of B to W is nondegenerate. (*Nondegenerate* means that $v = 0$ is the only vector of W such that $B(u, v) = 0$ for all $u \in W$.) (**15 points.**)

(c) Prove that if B is nondegenerate on V, then there is a nonnegative integer p with $p \leq n$ and a basis $\{v_1, \ldots, v_n\}$ such that

$$B(v_i, v_j) = \begin{cases} 1, & \text{if } 1 \leq i = j \leq p, \\ -1, & \text{if } p+1 \leq i = j \leq n, \\ 0, & \text{if } i \neq j. \end{cases}$$

(**15 points.**)

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科目：線性代數

2017.10.20

(1) (20 points) Let $A = \begin{pmatrix} -1 & 3 & -2 \\ 2 & 3 & 0 \\ 11 & -6 & 7 \end{pmatrix}$. Find the lower triangular Jordan canonical form of A. Please compute $\exp(tA)$ and derive the general solution to $x'(t) = A\ x(t)$, where $x(t)$ is a 3-dimensional column vector.

(2) (20 points) Let V be an n-dimensional complex vector space, and $T : V \to V$ be an invertible linear map such that $T^2 = 1$. (a) Show that T is diagonalizable, (b) Let S be the vector space of linear transformations from V to V that commute with T. Please express $\dim_{\mathbb{C}} S$ in terms of n and the trace of T.

(3) (20 points) Let $A = (A_{ij})$ be a real invertible skew-symmetric $2n \times 2n$ matrix.
(a) Show that all eigenvalues of A are pure imaginary.
(b) Define the Pfaffian $Pf(A)$ of A by

$$Pf(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) A_{\sigma(1),\sigma(2)} A_{\sigma(3),\sigma(4)} \cdots A_{\sigma(2n-1),\sigma(2n)} \ .$$

Let B be any real $2n \times 2n$ matrix. Show that $Pf(BAB^T) = Pf(A)\det(B)$.
(c) Assuming the fact that there exists a real orthogonal $2n \times 2n$ matrix O such that

$$OAO^T = \operatorname{diag}\left\{ \begin{pmatrix} 0 & m_1 \\ -m_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & m_2 \\ -m_2 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & m_n \\ -m_n & 0 \end{pmatrix} \right\},$$

where $m_i \in \mathbb{R}$ for $i = 1, \cdots, n$. Show that $\det(A) = Pf(A)^2$.

(4) (20 points) Let $A, B \in M_n(\mathbb{C})$ be $n \times n$ complex matrices. Show that A and B are simultaneously triangularizable (*i.e.* there exists an invertible matrix $P \in GL_n(\mathbb{C})$ such that PAP^{-1} and PBP^{-1} are both upper triangular) if A and B commute.
Hint: Let λ be one of the eigenvalues of A. Try to show $B(\ker(A - \lambda I)) \subset \ker(A - \lambda I)$.

(5) (20 points) Show that

$$\begin{vmatrix} X_0 & X_1 & X_2 & \dots & X_{n-1} \\ X_{n-1} & X_0 & X_1 & \dots & X_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_1 & X_2 & X_3 & \dots & X_0 \end{vmatrix} = \prod_{j=0}^{n-1} \left(\sum_{k=0}^{n-1} \zeta^{jk} X_k \right)$$

where ζ is a primitive n-th root of unity.

Hint: You may first compute, for example, $\begin{pmatrix} X_0 & X_1 & X_2 & X_3 \\ X_3 & X_0 & X_1 & X_2 \\ X_2 & X_3 & X_0 & X_1 \\ X_1 & X_2 & X_3 & X_0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 & \zeta^3 \\ 1 & \zeta^2 & \zeta^4 & \zeta^6 \\ 1 & \zeta^3 & \zeta^6 & \zeta^9 \end{pmatrix}$.

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科目：線性代數

2016.10.21

1. (20%) Let $A \in M_{n\times n}(F)$ where F is a field.
   (a) Show that if k is the largest integer such that some $k \times k$ submatrix of A has a nonzero determinant, then $\text{rank}(A) = k$.
   (b) If A is nilpotent of index m (that is, $A^m = 0$ but $A^{m-1} \neq 0$), and if, for each vector v in F^n, W_v is defined to be the subspace generated by $v, Av, \ldots, A^{m-1}v$, how large can the dimension of W_v be? (Justify your answer.)

2. (30%)
   (a) Let $A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$. Find the general solution to the system of differential equations
   $$\frac{dX}{dt} = AX, \text{ where } X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$
   where for each i, $x_i(t)$ is a differentiable real-valued function of the real variable t.
   (b) Let V be the space of all real polynomials having degree less than 4 with the inner product $\langle f(x), g(x)\rangle = \int_{-1}^{1} f(t)g(t)dt$. Let T be a linear operator on V defined by $T(f(x)) = f'(x) + 3f(x)$.
   Use the Gram-Schmidt process to replace $\beta = \{1, 1+x, x+x^2, x^2+x^3\}$ by an orthonormal basis for V and find the matrix representation of the adjoint T^* of T in this orthonormal basis.

3. (30%)
   (a) Let $A \in M_{n\times n}(\mathbb{R})$. Show that there exists an orthogonal matrix Q and a positive semi-definite symmetric matrix P such that $A = QP$.
   (b) Let V be a finite-dimensional vector space over $\mathbb{C}$ and T be a linear operator on V. Show that T is normal if and only if its adjoint $T^* = g(T)$ for some polynomial $g(x) \in \mathbb{C}[x]$.

4. (20 %) Let $T \in \text{End}_{\mathbb{C}}(V)$ for a finite-dimensional $\mathbb{C}$-vector space V.
   (a) Show that we have an expression of T as $T = S + N$ with $S, N \in \text{End}_{\mathbb{C}}(V)$, such that S is diagonalisable, N is nilpotent and $SN = NS$.
   (b) Show that both S and N are uniquely defined by these conditions.
   (c) Show that there is a polynomial $p(x) \in \mathbb{C}[x]$ with $p(0) = 0$ such that $S = p(T)$.

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科目：線性代數

2015. 10. 23

There are five problems $1 \sim 5$ in total; some problems contain sub-problems, indexed by (a), (b), etc.

1. [20%] Prove the Cayley-Hamilton theorem: Let V be a finite-dimensional vector space and $T : V \to V$ a linear transformation with characteristic polynomial $p(x)$. Then $p(T) = 0$.

2. Let F be a field and $A \in M_{m\times n}(F)$ be an m by n matrix.

   (a) [10%] Show that the row rank of A equals the column rank of A.

   (b) [10%] Denote by A^t the transport of A and let r be the row rank of A. Show that AA^t is of rank r.

3. [20%] Let F be a field and $\{A_i \in M_n(F) \,|\, i \in I\}$ be a collection of n by n matrices. (I is a set; it might be finite or infinite.) Suppose that A_i are diagonalizable for all $i \in I$ and $A_iA_j = A_jA_i$ for any $i, j \in I$. Show that there exists an invertible matrix $P \in M_n(F)$ such that PA_iP^{-1} are diagonal for all $i \in I$.

4. [20%] Let $A \in M_n(\mathbb{C})$ be an n by n matrix over the field of complex numbers. Denote by A^* the conjugate transport of A. Suppose $AA^* = A^*A$. Show that there exists a matrix P such that (i) $PP^* = I$, the identity matrix, and (ii) PAP^* is a diagonal matrix. (You may do the case $A = A^*$ for half credit.)

5. [20%] Let V be a finite-dimensional vector space over the field of complex numbers and $T : V \to V$ a linear transformation with characteristic polynomial $p(x)$. Suppose that $p(x) = q_1(x)q_2(x)$ for two polynomials $q_1(x)$ and $q_2(x)$ which do not have a common root. Show that there are two subspaces W_1 and W_2 of V satisfying that (i) $W_1 \cap W_2 = \{0\}$ and $V = W_1 + W_2$, (ii) $T(W_i) \subset W_i$ for each $i = 1, 2$, and (iii) regarding T as a linear transformation on W_i, it has characteristic polynomial $q_i(x)$ for $i = 1, 2$.

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科目：線性代數

2014.10.24

Notation: $\mathbb{Q}$ is the set of rational numbers, $\mathbb{R}$ is the set of real numbers and $\mathbb{C}$ is the set of complex numbers. Let n be a positive integer and I_n be the identity matrix in $M_2(\mathbb{Q})$.

Problem 1 (20 pts).

(a) For each $x \in \mathbb{R}$, let V_x be the subspace of $\mathbb{R}^4$ generated by
$$(x,1,1,1),(1,x,1,1),(1,1,x,1),(1,1,1,x).$$
Determine all x such that $\dim_{\mathbb{R}} V_x \leq 3$.

(b) Find the dimension and a basis for the space of $\mathbb{R}$-linear maps $L : \mathbb{R}^5 \to \mathbb{R}^3$ whose kernels contain $(0,2,-3,0,1)$.

Problem 2 (20 pts). Let
$$A = \begin{pmatrix} -1 & 4 & -2 \\ -2 & 5 & -2 \\ -1 & 2 & 0 \end{pmatrix}.$$

(1) Compute the characteristic polynomial of A.
(2) If $f(x) = (x-3)^2 + 5$, find the eigenvalues of $f(A)$.
(3) Find an orthogonal matrix $P \in M_3(\mathbb{R})$ such that $P^{-1}AP$ is diagonal.

Problem 3 (15 pts). Let $A, B \in M_n(\mathbb{R})$ be invertible matrices. Show that

(1) If $ABA^{-1}B^{-1} = c \cdot I_n$, then $c = \pm 1$;
(2) If $AB - BA = c \cdot I_n$, then $c = 0$.

Problem 4 (10pts). Let $A \in M_n(\mathbb{R})$ such that $A^3 = A$. Show that $\operatorname{rank} A = \operatorname{trace} A^2$.

Problem 5 (15pts). Let $A \in M_n(\mathbb{R})$ such that $\operatorname{rank} A + \operatorname{rank}(I_n - A) = n$. Show that $A^2 = A$.

Problem 6 (20 pts). Let $A \in M_n(\mathbb{Q})$ with $A^n = 0$ but $A^{n-1} \neq 0$. Show that if $B \in M_n(\mathbb{Q})$ commutes with A ($\iff$ $BA = AB$), then
$$B = a_1 + a_2 A + \ldots + a_n A^{n-1} \text{ for some } a_1, \ldots, a_n \in \mathbb{Q}.$$

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科目：線性代數

2013. 10. 18

1. (20%)
   (a) Let $A \in M_{m\times n}(F)$, $B \in M_{n\times p}(F)$ where F is a field.
   Show that $\operatorname{rank}(AB) \le \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$.
   Moreover, if $n = p$ and B is invertible, show that $\operatorname{rank}(AB) = \operatorname{rank}(A)$.
   (b) Let $A \in M_{m\times n}(\mathbb{C})$. Show that $\operatorname{rank}(A^*A) = \operatorname{rank}(A)$ where A^* is the conjugate transpose of A.

2. (20%)
   (a) Let A be an $n \times n$ real symmetric matrix. Show that if λ is an eigenvalue of A in $\mathbb{C}$, then λ is real.
   (b) Let A be an $n \times n$ real symmetric matrix. Show that one can find an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

3. (20%)
   (a) Which $n \times n$ real matrices B have the property that $AB = BA$ for all $n \times n$ real matrices A? Justify your answer.
   (b) Let A, B be two $n \times n$ real symmetric matrices. Show that A and B are simultaneously diagonalizable if and only if $AB = BA$.

4. (20 %) For all $x \in \mathbb{R}^n$, we define the norm of x by $\|x\| = \sqrt{\langle x, x\rangle}$ where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^n$.
   Let $A = \begin{pmatrix} 0 & 4 & 0 & 4 \\ 1 & 2 & 1 & 2 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.
   (a) Find a vector p in the column space of A (the subspace of $\mathbb{R}^3$ spanned by the column vectors of A) such that $\|p - b\| \le \|A \cdot x - b\|$ for all $x \in \mathbb{R}^4$
   (b) Find $s \in \mathbb{R}^4$ such that $A \cdot s = p$ and s has the minimum norm, that is, $\|s\| \le \|v\|$ for all solutions v (in $\mathbb{R}^4$) of $A \cdot x = p$.
   (Justify your answers.)

5. (20 %) Find the Jordan form B of
$$A = \begin{pmatrix} 3 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 0 & 5 & -3 \\ 4 & -1 & 3 & -1 \end{pmatrix}$$
   and the matrix P such that $B = P^{-1}AP$.
   (Notice! Show your works in details. No points will be assigned for non-substantial answers.)

臺灣大學數學系 102 學年度碩士班甄試試題

科目：線性代數

2012.10.19

1. [20%] Let V be an n-dimensional vector space over a field. Let $T : V \to V$ be a linear map. Show that the degree of the minimal polynomial of T equals

$$\max_{v \in V} \left\{ \dim \langle v, T(v), T^2(v), \cdots, T^{n-1}(v) \rangle \right\}.$$

(Here $\langle w_1, \cdots, w_r \rangle$ denotes the subspace spanned by $w_1, \cdots, w_r$.)

2. [20%] Consider the real $n \times n$ matrix $A = (a_{ij})$ satisfying

- $a_{ij} \geq 0$ for all i, j,
- $a_{ii} = 0$ for all i, and
- $\sum_{j=1}^{n} a_{ij} = \gamma$ for all $i = 1, 2, \cdots, n$ for some constant $\gamma \neq 0$.

Show that

(a) If $\lambda \in \mathbb{R}$ is a real eigenvalue of A, then $-\gamma \leq \lambda \leq \gamma$.

(b) γ is an eigenvalue of A and the corresponding eigenspace has dimension one.

(c) The eigenspace corresponding to $-\gamma$ has dimension either zero or one.

3. [20%] Let $A = (a_{ij})$ be a real $n \times n$ symmetric matrix. Show that A is *positive definite* (meaning: $v^t A v > 0$ for any non-zero $v \in \mathbb{R}^n$ where v^t is the transport of v) if and only if, for any $r = 1, 2, \cdots, n$, we have

$$\det A_r > 0 \quad \text{where } A_r = (a_{ij})_{1 \leq i,j \leq r} \in M_r(\mathbb{R}).$$

4. [20%] Let $T : V \to W$ be a linear map between two finite dimensional vector spaces. Let V^* and W^* be the dual spaces of V and W, respectively. Prove that

(a) T is injective if and only if the transport $T^* : W^* \to V^*$ is surjective.

(b) T is surjective if and only if the transport $T^* : W^* \to V^*$ is injective.

(Recall that the transport T^* is defined by $\big(T^*(f)\big)(v) = f\big(T(v)\big)$ for $f \in W^*, v \in V$.)

5. [20%] Let A be a real $n \times n$ matrix such that $A^t = -A$ (where A^t denotes the transport of A). Let $\lambda = a + bi$ be a complex eigenvalue of A where $a, b \in \mathbb{R}$ and $i^2 = -1$. Show that $a = 0$.

臺灣大學數學系
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科目：線性代數

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You should include in your answer every piece of computation and every piece of reasoning so that the corresponding partial credit could be gained.

(20%) **1.** (a) Prove that if A is a symmetric matrix then A^2 is symmetric. Is the converse true? Justify your answer.

(b) Determine all real $m \times n$ matrices A for which $A^{\mathrm{T}}A = 0$. Justify your answer.

(c) Suppose that K is a square matrix with $K = -K^{\mathrm{T}}$ and that $I - K$ is nonsingular. Let $B = (I + K)(I - K)^{-1}$. Prove that $B^{\mathrm{T}}B = BB^{\mathrm{T}} = I$.

(20%) **2.** Let V be the real vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$.

(a) For any integer n, define $f_n(x) = x + n$. Determine the dimension of the subspace of V generated by $\{f_n(x) : n \in \mathbb{Z}\}$. Justify your answer.

(b) Define $g: \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 4i + 1 - x, & \text{if } 4i \le x < 4i + 2 \text{ for } i \in \mathbb{Z}; \\ x - 4i - 3, & \text{if } 4i + 2 \le x < 4i + 4 \text{ for } i \in \mathbb{Z}. \end{cases}$$

For any integer n, define $g_n(x) = g(x + n)$. Determine the dimension of the subspace of V generated by $\{g_n(x) : n \in \mathbb{Z}\}$. Justify your answer.

(20%) **3.** (a) Suppose I is the $n \times n$ identity matrix and J is the $n \times n$ matrix whose entries are all 1. Determine the ranks of J and $J - I$. Justify your answer.

(b) Prove that the rank of an $n \times n$ $(0,1)$-matrix A with $A_{ij} + A_{ji} = 1$ for $1 \le i < j \le n$ is either n or $n - 1$.

(20%) **4.** A square matrix is called *unimodular* if its determinant is 0 or ± 1. A matrix is called *totally unimodular* if all of its square submatrices are unimodular. It is easy to see that any entry of a totally unimodular matrix is 0 or ± 1.

(a) For any $n \ge 3$, give an $n \times n$ $(0,1)$-matrix which is not unimodular. Justify your answer.

(b) Prove that any $m \times n$ matrix in which every column has exactly one 1, exactly one -1 and all other entries 0 is totally unimodular.

(20%) **5.** (a) Prove that all eignevalues of a real symmetric matrix are real.

(b) Suppose S is an $m \times m$ real symmetric matrix whose eigenvalues are $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_m$. Recall that there is an orthonormal basis $v_1, v_2, \ldots, v_m$ for which each v_i is a corresponding eigenvector of λ_i. Prove that $\lambda_1 \ge \frac{x^{\mathrm{T}}Sx}{x^{\mathrm{T}}x} \ge \lambda_m$ for any nonzero m-vector x.

(c) Suppose A is an $n \times n$ real symmetric matrix whose eigenvalues are $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n$. Let B be the matrix obtained from A by deleting the last row and the last column, and its eigenvalues are $\mu_1 \ge \mu_2 \ge \ldots \ge \mu_{n-1}$. Prove that these eigenvalues are interlacing, that is $\lambda_1 \ge \mu_1 \ge \lambda_2 \ge \mu_2 \cdots \ge \lambda_{n-1} \ge \mu_{n-1} \ge \lambda_n$.

※ 注意：請於試卷上「非選擇題作答區」內依序作答，並應註明作答之大題及其題號。

Instructions.

- There are two problems in two pages.
- In a problem, if an exercise depends on the conclusions of other exercises that precede it, you may assume these conclusions without solving them.

Problem 1 (80 points). Let m and n be two positive integers. The $\mathbf{C}$-vector space of matrices of size $m \times n$ with coefficients in $\mathbf{C}$ is denoted by $M_{m,n}(\mathbf{C})$. We also set $M_n(\mathbf{C}) = M_{n,n}(\mathbf{C})$.

The aim of this problem is to prove the following statement.

Theorem. *Let m, n and r be positive integers with $r \le m \le n$. Let $V \subset M_{m,n}(\mathbf{C})$ be a $\mathbf{C}$-linear subspace. Assume that every matrix A in V satisfies $\operatorname{rank} A \le r$. Then*

$$\dim V \le nr.$$

(1) Show that it suffices to prove the theorem for $m = n$.

(2) Assume that $m = n$. Show that we can assume that V contains the block matrix

$$R = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

where I_r is the identity matrix of rank r.

From now on, we assume that $m = n$, and that $R \in V$.

(3) Let

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \in V$$

be a block matrix in V with $M_{11} \in M_r(\mathbf{C})$. Show that

$$M_{22} = 0 \quad \text{and} \quad M_{21}M_{12} = 0.$$

(Hint: you may consider the $(r+1) \times (r+1)$ minors of $M + tR$ for $t \in \mathbf{C}$.)

(4) Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{pmatrix} \in V, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & 0 \end{pmatrix} \in V$$

be two block matrices with $A_{11}, B_{11} \in M_r(\mathbf{C})$. Show that

$$A_{21}B_{12} + B_{21}A_{12} = 0.$$

(5) Let $\phi : V \to M_{r,n}(\mathbf{C})$ be the map sending a matrix $M \in V$ to its first r rows. Define the $\mathbf{C}$-linear subspace

$$W = \left\{ \begin{pmatrix} 0 & 0 \\ A_{21} & 0 \end{pmatrix} \in V \;\middle|\; A_{21} \in M_{n-r,r}(\mathbf{C}) \right\} \subset V,$$

and let $s = \dim W$. Show that

$$\dim \phi(V) \leq nr - s,$$

by considering the map

$$\psi : W \to M_{r,n}(\mathbb{C})^{\vee}$$
$$\begin{pmatrix} 0 & 0 \\ A_{21} & 0 \end{pmatrix} \mapsto T_{A_{21}}$$

to the dual of $M_{r,n}(\mathbb{C})$, where $T_{A_{21}}$ is the linear form defined by

$$T_{A_{21}}(B_{11}, B_{12}) = \operatorname{Tr}(A_{21}B_{12})$$

for every block matrix $(B_{11}, B_{12}) \in M_{r,n}(\mathbb{C})$ with $B_{11} \in M_r(\mathbb{C})$.

(6) Conclude that

$$\dim V \leq nr.$$

(7) Show that the inequality in the theorem is optimal. More precisely, for all positive integers m, n and r with $r \leq m \leq n$, construct $V \subset M_{m,n}(\mathbb{C})$ as in the theorem such that

$$\dim V = nr.$$

Problem 2 (20 points). Let V be a nonzero vector space over a field F. Let

$$B : V \times V \to F$$

be a non-degenerate symmetric bilinear form on V, and let

$$q : V \to F$$
$$v \mapsto B(v, v)$$

be the associated quadratic form. For every $x \in F$, we say that q represents x if $q(v) = x$ for some nonzero $v \in V$.

(1) Suppose that q represents 0. Show that q represents every element of F. (Hint: Consider $q(cv + w)$ with $c \in F$ and some suitable $w \in V$.)

(2) Show that B extends to a non-degenerate symmetric bilinear form on $V \oplus F$ whose associated quadratic form represents every element of F.

Notation: $\mathbf{R}$ is the set of real numbers, and $\mathbf{C}$ is the set of complex numbers. If $F = \mathbf{R}$ or $\mathbf{C}$, denote by $M_n(F)$ the $n \times n$ matrices with entries in F. If $A \in M_{m\times n}(F)$, denote by $A^t \in M_{n\times m}(F)$ the transpose of A. Denote by I_n the $n \times n$ identity matrix and 0_n the $n \times n$ zero matrix.

Problem 1 (10pts). Let $\mathbf{i} = \sqrt{-1} \in \mathbf{C}$ be a root of $X^2 + 1$. Let

$$v_1 = (1, 0, -\mathbf{i}), \quad v_2 = (1 + \mathbf{i}, 1 - \mathbf{i}, 1), \quad v_3 = (\mathbf{i}, \mathbf{i}, \mathbf{i}).$$

Show that $\{v_1, v_2, v_3\}$ is a basis of $\mathbf{C}^3$ and express the vector $v_4 = (1, 0, 1)$ as a linear combination of v_1, v_2 and v_3, namely find $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{C}$ such that $v_4 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$.

Problem 2 (15 pts). Let

$$\mathbf{v}_1 = (0, 3, 3, 1),\ \mathbf{v}_2 = (2, 1, -3, 7),\ \mathbf{v}_3 = (1, 8, 6, 6),\ \mathbf{v}_4 = (1, 10, -4, 2)$$

be vectors in $\mathbf{R}^4$. Let $W_1 = \operatorname{span}_{\mathbf{R}}\{\mathbf{v}_1, \mathbf{v}_2\}$ and let $W_2 = \operatorname{span}_{\mathbf{R}}\{\mathbf{v}_3, \mathbf{v}_4\}$. Find the dimension and a basis of $W_1 \cap W_2$.

Problem 3 (25 pts). Let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix} \in M_{2\times 3}(\mathbf{R}).$$

(1) (15pts) Find an orthogonal matrix $P \in M_3(\mathbf{R})$ such that $P^{-1}A^tAP$ is a diagonal matrix.

(2) (10pts) Find the singular value decomposition of A. In other words, factorize $A = U\Sigma V^t$, where $U \in M_3(\mathbf{R})$ and $V \in M_3(\mathbf{R})$ are orthogonal matrices and $\Sigma \in M_{2\times 3}(\mathbf{R})$ is of the form

$$\Sigma = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \end{pmatrix}, \quad \lambda_1 \geq \lambda_2 \geq 0$$

Problem 4 (15pts). Let $V = M_3(\mathbf{C})$ be a 9-dimension vector space over $\mathbf{C}$ and let

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}.$$

Define the linear transformation $T : V \to V$ by

$$T(B) = AB - BA.$$

(1) (5pts) Find the dimension of $\operatorname{Ker} T$.

(2) (10pts) Show that T is diagonalizable.

Problem 5 (15pts). Let $A, B \in M_n(\mathbf{R})$. Prove that $\operatorname{rank} A + \operatorname{rank} B \leq n$ if and only if there exists an invertible matrix $X \in M_n(\mathbf{R})$ such that $AXB = 0_n$.

Problem 6 (20pts). Let A and B be elements in $M_n(\mathbf{C})$. Suppose that

$$AB - BA = c \cdot (A - B)$$

for some non-zero $c \in \mathbf{C}$. Prove that there exists an invertible matrix $P \in M_n(\mathbf{C})$ such that $P^{-1}AP$ and $P^{-1}BP$ are upper-triangular matrices with the same diagonal entries.

Notation: We denote by $\mathbf{C}$ the set of complex numbers. For any positive integer n, we denote by $\mathbf{C}^n$ the n-dimensional column vector spaces over $\mathbf{C}$; let I_n be the identity matrix in $\mathrm{M}_n(\mathbf{C})$.

Problem 1 (15pts). Let $T : \mathbf{C}^4 \to \mathbf{C}^3$ be the linear transformation defined by $T(v) = A \cdot v$, where

$$A = \begin{pmatrix} 5 & -3 & 1 & 2 \\ -1 & 3 & 3 & -2 \\ 1 & 0 & 1 & 0 \end{pmatrix} \in \mathrm{M}_{3\times 4}(\mathbf{C}).$$

(1) (5 pts) Find the rank and the nullity of T.
(2) (10pts) Find a base of $\mathrm{Ker}\, T$ (the kernel of T).

Problem 2 (15pts). For any complex number $a \in \mathbf{C}$, let V_a be the subspace spanned by the row vectors

$$(2, -5, a),\ (1, a, -4),\ (a, -1, -2).$$

Determine all possible values $a \in \mathbf{C}$ such that $\dim_{\mathbf{C}} V_a = 2$.

Problem 3 (25pts). Let

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 2 & -1 & 2 \\ 2 & -2 & 5 \end{pmatrix}.$$

(1) (15pts) Find an invertible matrix $P \in \mathrm{M}_3(\mathbf{C})$ such that $P^{-1}AP$ is a diagonal matrix.
(2) (10pts) Find an invertible matrix $Q \in \mathrm{M}_3(\mathbf{C})$ such that

$$Q^{-1}AQ = \begin{pmatrix} 0 & 0 & -4 \\ 1 & 0 & 1 \\ 0 & 1 & 4 \end{pmatrix}.$$

Problem 4 (15pts). Let $A \in \mathrm{M}_n(\mathbf{C})$ be a Hermitian matrix $\iff A = A^*$.
(1) (5 pts) Show that $\mathrm{Ker}\, A \cap \mathrm{Im}\, A = \{0\}$.
(2) (10pts) If $A^3 = 2A^2 + 2A$, show that $A = 0$.

Problem 5 (15pts). Let $A \in \mathrm{M}_n(\mathbf{C})$ such that $A^n = 0$ but $A^{n-1} \neq 0$.
(1) (7pts) Show that there exists $v \in \mathbf{C}^n$ such that $\{v, Av, A^2v, \ldots, A^{n-1}v\}$ is a basis of $\mathbf{C}^n$.
(2) (8pts) If $B \in \mathrm{M}_n(\mathbf{C})$ such that $AB = BA$, prove that

$$B = a_0 + a_1A + a_2A^2 + \cdots a_{n-1}A^{n-1}$$

for some $a_0, \ldots, a_{n-1} \in \mathbf{C}$.

Problem 6 (15pts). Let $A, B \in \mathrm{M}_n(\mathbf{C})$. Suppose that the eigenvalues of A, B are all non-negative real numbers and that $\mathrm{null}(A) = \mathrm{null}(A^2)$ and $\mathrm{null}(B) = \mathrm{null}(B^2)$. If $A^4 = B^4$, prove that $A = B$.
(Recall that $\mathrm{null}(A)$:=the nullity of A = the dimension of the kernel of A)

Linear Algebra

1. **(20 points.)** Let $A, B \in M_{n\times n}(F)$ be two $n \times n$ matrices over a field F.
   (a) Prove that $\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$.
   (b) Prove that $\operatorname{rank}(A) + \operatorname{rank}(B) \leq \operatorname{rank}(AB) + n$.
2. **(15 points.)** Let A be an $n \times n$ matrix over $\mathbb{C}$ of the form

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_0 \end{pmatrix}.$$

   Define $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1}$ and $\omega = e^{2\pi i/n}$. Prove that

$$\det A = \prod_{j=0}^{n-1} f(\omega^j).$$

3. **(15 points.)** Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V over $\mathbb{C}$ and $f(x) \in \mathbb{C}[x]$ be a polynomial. Prove that the linear transformation $f(T)$ is invertible if and only if $f(x)$ and the minimal polynomial T have no common roots.
4. **(15 points.)** Let $v_1, \ldots, v_k$ be eigenvectors corresponding to k distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ of a linear operator T on a vector space V. Prove that the T-cyclic subspace generated by $v = v_1 + \cdots + v_k$ has dimension k.
5. **(15 points.)** Let $T : V \to V$ be a linear operator on a finite-dimensional inner product space V over $\mathbb{R}$ and T^* be its adjoint. Suppose that $T^* = T^3$. Prove that T^2 is diagonalizable over $\mathbb{R}$.
6. **(20 points.)** Let V be a vector space of dimension n over a field F. Determine the dimension over F of the vector space of multilinear alternating functions $f : V \times \cdots \times V \to F$ (k copies of V).

試題隨卷繳回

- Unless otherwise specified, everything is over $\mathbb{R}$.
- The ordinary inner product of $\mathbb{R}^n$ is denoted by $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$.
- $\mathcal{S}_n$ is the space of $n \times n$ square matrices.
- $\mathcal{P}$ is the vector space of polynomials of one variable x with real coefficients.
- Dual space V^* of real vector space V is $\{\alpha \mid \alpha : V \to \mathbb{R}, \alpha \text{ is linear}\}$.

(1) [16%] $V \subset \mathbb{R}^4$ is a subspace span by $\vec{\mathbf{u}} = [1\ \ -4\ 8\ 3]^{\mathrm{t}}$ and $\vec{\mathbf{v}} = [2\ \ -2\ 10\ 3]^{\mathrm{t}}$. Define a linear transformation $T : V \to V$ by

$$\begin{aligned} T(\vec{\mathbf{u}}) &= 5\,\vec{\mathbf{u}} + 2\,\vec{\mathbf{v}} \\ T(\vec{\mathbf{v}}) &= 7\,\vec{\mathbf{u}} + \vec{\mathbf{v}} \end{aligned}$$

The induced inner product of V from $\mathbb{R}^4$ is defined by $< \vec{\mathbf{x}}, \vec{\mathbf{y}} >= \vec{\mathbf{x}} \cdot \vec{\mathbf{y}}$, $\vec{\mathbf{x}}, \vec{\mathbf{y}} \in V$. Is T self-adjoint with respect to $<\ ,\ >$? Demonstrate your answer.

(2) [16%] $\mathcal{P}_3 \equiv \{f(x) \in \mathcal{P} \mid \deg(f(x)) \leq 3\}$. Let $\mathcal{P}_3^*$ be the dual space of $\mathcal{P}_3$. For any $a \in \mathbb{R}$, define $\widehat{a} \in \mathcal{P}_3^*$ by $\widehat{a}(f(x)) = f(a)$ and $\mathrm{d}\hat{a} \in \mathcal{P}_3^*$ by $\mathrm{d}\hat{a}(f(x)) = f'(a)$.

a. Find the basis $\phi_{-1}(x)$, $\phi_0(x)$, $\phi_d(x)$, $\phi_1(x)$ of $\mathcal{P}_3$ such that $\widehat{-1}$, $\widehat{0}$, $\mathrm{d}\hat{0}$, $\widehat{1}$ are their corresponding dual basis.

b. Define $I \in \mathcal{P}_3^*$ by $I(f(x)) = \int_{-1}^{1} f(x)dx$. Find α, β, γ, $\epsilon \in \mathbb{R}$ such that
$$I = \alpha\,\widehat{-1} + \beta\,\widehat{0} + \gamma\,\mathrm{d}\hat{0} + \epsilon\,\widehat{1}$$

c. If there is $f(x) \in \mathcal{P}_3$ such that $f(-1) = -2$, $f(0) = 2$, $f'(0) = \pi$, $f(1) = -6$, evaluate $\int_{-1}^{1} f(x)dx$.

(3) [16%] $\Gamma = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \in \mathcal{S}_n$. $\mathcal{C}_n = \{X \mid X\Gamma = \Gamma X\}$ is a subspace of $\mathcal{S}_n$.
Determine $\dim \mathcal{C}_n$ and find a basis of $\mathcal{C}_n$.

(4) [16%] $A \in \mathcal{S}_n$. Define m_{ij} to be the determinant of the submatrix formed by deleting the i-th row and j-th column of A. Define the classical adjont matrix $\operatorname{adj} A = [(-1)^{i+j} m_{ji}]$. Suppose A is not invertible, show that rank of $\operatorname{adj} A$ is ≤ 1. When is the rank of $\operatorname{adj} A = 1$?

(5) [16%] If $A = [a_{ij}] \in \mathcal{S}_n$ is positive definite, show that $\det A \leq a_{11} a_{22} \cdots a_{nn}$.

(6) [20%] $A \in \mathcal{S}_n(\mathbb{C})$. Over $\mathbb{C}$, show the following two statements are equivalent.

a. The characteristic polynomial of A is equal to minimal polynomial of A.

b. For any $X \in \mathcal{S}_n(\mathbb{C})$ satisfies $XA = AX$, X is a polynomial of A.

(1) (20 points) Let V_1 be the $\mathbb{R}$-*linear span* of functions: $\sin^i x \cdot \cos^j x$, $i, j = 0, ..., n$. Let V_2 be the $\mathbb{R}$-*linear span* of functions: $\sin kx.\ \cos kx$, $k = 0, ..., n$. Determine the dimensions of V_1 and V_2 and prove your assertion. Is it true that $V_1 = V_2$? Prove or disprove it.

(2) (15 points) Let $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a *linear transformation* and let id be the identity map sending every $v \in \mathbb{R}^n$ to v. Prove that there exist $C > 0$ such that for all $t \in \mathbb{R}$, $|t| > C$, the map $id + t \cdot \varphi$ is surjective.

(2) (15 points) Let $A := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $B := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$,

$$V = \{v \in \mathbb{C}^4 \mid A \cdot v = \lambda_a \cdot v. B \cdot v = \lambda_b \cdot v, \text{ for some } \lambda_a, \lambda_v \in \mathbb{C}\}.$$

Find a basis of V.

(4) (15 points) Let A be an $n \times n$ *diagonal* matrix with diagonal entries $A_{11}, ..., A_{nn}$. Show that the *linear span* W of A^k, $k = 0, 1, ...$, is of dimension n if and only if $A_{ii} \neq A_{jj}$ for different i and j.

(5) (15 points) Suppose φ and g are $\mathbb{R}$-linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $g \circ \varphi = \varphi^2 \circ g$ and g is *injective*. Show that φ and φ^2 have the same kernel (null-space), image, *eigenvalues* and *eigenspaces*.

(6) Prove or disprove the following statements (10 points for each).
Let $Q : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a quadratic form.

(a) Let $\mathbb{Z}^n \subset \mathbb{R}^n$ denote the subset consisting of vectors with integer coordinates. Then Q is *positive definite* if and only if $Q(v) > 0$ for all $v \in \mathbb{Z}^n$.

(b) There is some $n \times n$ matrix A such that $Q(v) = v^t \cdot A^t \cdot A \cdot v$, for all $v \in \mathbb{R}^n$. Here, B^t denotes the *transpose* of B.

試題隨卷繳回

※ 注意：全部題目均請作答於試卷內之「非選擇題作答區」，請標明題號依序作答。

- Unless otherwise specified, everything is over $\mathbb{R}$.
- The ordinary inner product of $\mathbb{R}^n$ is denoted by $\vec{u}\cdot\vec{v}$.
- $\mathcal{M}_{m\times n}$ is the space of $m\times n$ matrices; $f_M(t)=\det(tI_n-M)$ is the characteristic polynomial of M; $\operatorname{im} A$ is the image of A; $\ker A$ is the kernel of A; $V^\perp$ is the normal space of V. Parallelepiped = 平行六面體.
- Duel space V^* of real vector space V is $\{\alpha \mid \alpha : V \to \mathbb{R}, \alpha \text{ is linear}\}$.

A. [15%] 是非題. 若錯誤, 需說明原因或給出反例. <u>本題答案須寫在答案簿最前面</u>.

1. There is a linear transformation $A:\mathbb{R}^3 \to \mathbb{R}^3$ such that $\operatorname{im} A=\ker A$.
2. $A\in\mathcal{M}_{n\times n}$. Suppose $A^2=A$ then $\ker A=(\operatorname{im} A)^\perp$.
3. For any A, B, $C\in\mathcal{M}_{n\times n}$, $\operatorname{tr}(ABC)=\operatorname{tr}(CBA)$.
4. The matrix representation A of an adjoint transformation satisfies $A^t=A$.
5. Symmetric matrix A is positive definite if and only if all its diagonal elements are positive.

B. [85%] 計算/證明題。(6A) 和 (6B) 只選擇一題作答，兩題皆答，以先寫者計算。

(1) [15%] Find all Jordan canonical forms for square matrices in $\mathcal{M}_{n\times n}$, $n\le 6$, with minimal polynomial $(t-1)^2(t+1)^2$.

(2) [15%] For A, $B\in\mathcal{M}_{m\times n}$, show that $f_{BA^t}(t)=f_{B^tA}(t)\cdot t^{m-n}$.

(3) [15%] Consider $V=\{A \mid AX=XA, \text{for any } X\in\mathcal{M}_{n\times n}\}\subset\mathcal{M}_{n\times n}$. Show that V is an one dimensional subspace of $\mathcal{M}_{n\times n}$.

(4) [15%] $A\in\mathcal{M}_{n\times n}$. Suppose $(t^2+1)|f_A(t)$, are there $\vec{u}$, $\vec{v}\in\mathbb{R}^n$ such that $A\vec{u}=\vec{v}$ and $A\vec{v}=-\vec{u}$? Prove or disprove it.

(5) [15%] U is a subspace of a finite dimensional vector space V. Consider $D_U\subset V^*$ defined by $\{\alpha\in V^* \mid U \text{ is a subspace of } \ker\alpha\}$. Show that D_U is a subspace of dimension $\dim V-\dim U$.

(6A) [10%] Show the volume V of the parallelepiped span by $\vec{u}$, $\vec{v}$, $\vec{w}\in\mathbb{R}^n$ satisfies

$$\begin{aligned} V^2 =& \|\vec{u}\|^2\,\|\vec{v}\|^2\,\|\vec{w}\|^2 + 2\,(\vec{u}\cdot\vec{v})\,(\vec{v}\cdot\vec{w})\,(\vec{w}\cdot\vec{u}) \\ & - \|\vec{u}\|^2\,(\vec{v}\cdot\vec{w})^2 - \|\vec{v}\|^2\,(\vec{w}\cdot\vec{u})^2 - \|\vec{w}\|^2\,(\vec{u}\cdot\vec{v})^2 \end{aligned}$$

(6B) [10%] Following diagram of vector spaces and linear transformations satisfies

(a) $\ker f_{i+1}=\operatorname{im} f_i$, $\ker g_{i+1}=\operatorname{im} g_i$, $i=0, 1, 2, 3, 4$.

(b) $\alpha_{i+1}\circ f_i=g_i\circ\alpha_i$, $i=1, 2, 3, 4$.

$$\begin{array}{ccccccccccccc}
\{0_{A_1}\} & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{f_2} & C_1 & \xrightarrow{f_3} & D_1 & \xrightarrow{f_4} & E_1 & \xrightarrow{f_5} & \{0_{E_1}\} \\
 & & \alpha_1\downarrow\cong & & \alpha_2\downarrow\cong & & \alpha_3\downarrow & & \alpha_4\downarrow\cong & & \alpha_5\downarrow\cong & & \\
\{0_{A_2}\} & \xrightarrow{g_0} & A_2 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & C_2 & \xrightarrow{g_3} & D_2 & \xrightarrow{g_4} & E_2 & \xrightarrow{g_5} & \{0_{E_1}\}
\end{array}$$

Show that if α_1, α_2, α_4, α_5 are isomorphisms, then α_3 is an isomorphism.

- Unless otherwise specified, everything is over $\mathbb{R}$.
- The ordinary inner product of $\mathbb{R}^n$ is denoted by $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$.
- $\mathcal{M}_{m\times n}$ is the space of $m \times n$ matrices; $f_M(t) = \mathsf{det}(tI_n - M)$ is the characteristic polynomial of M; $\mathrm{im}\, A$ is the image of A; $\ker A$ is the kernel of A; $V^\perp$ is the normal space of V. Parallelepiped = 平行六面體.
- Duel space V^* of real vector space V is $\{\alpha \mid \alpha : V \to \mathbb{R}, \alpha \text{ is linear}\}$.

A. [15%] 是非題. 若錯誤, 需說明原因或給出反例. <u>本題答案須寫在答案簿最前面</u>.

1. There is a linear transformation $A : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\mathrm{im}\, A = \ker A$.
2. $A \in \mathcal{M}_{n\times n}$. Suppose $A^2 = A$ then $\ker A = (\mathrm{im}\, A)^\perp$.
3. For any A, B, $C \in \mathcal{M}_{n\times n}$, $\mathsf{tr}(ABC) = \mathsf{tr}(CBA)$.
4. The matrix representation A of an self-adjoint transformation satisfies $A^{\mathsf{t}} = A$.
5. Symmetric matrix A is positive definite if and only if all its diagonal elements are positive.

B. [85%] 計算/證明題。(6A) 和 (6B) 只選擇一題作答，兩題皆答，以先寫者計算。

(1) [15%] Find all Jordan canonical forms for square matrices in $\mathcal{M}_{n\times n}$, $n \le 6$, with minimal polynomial $(t-1)^2(t+1)^2$.

(2) [15%] For A, $B \in \mathcal{M}_{m\times n}$, show that $f_{BA^{\mathsf{t}}}(t) = f_{B^{\mathsf{t}}A}(t) \cdot t^{m-n}$.

(3) [15%] Consider $V = \{A \mid AX = XA, \text{for any } X \in \mathcal{M}_{n\times n}\} \subset \mathcal{M}_{n\times n}$. Show that V is an one dimensional subspace of $\mathcal{M}_{n\times n}$.

(4) [15%] $A \in \mathcal{M}_{n\times n}$. Suppose $(t^2+1) | f_A(t)$, are there nonzero $\vec{\mathbf{u}}$, $\vec{\mathbf{v}} \in \mathbb{R}^n$ such that $A\vec{\mathbf{u}} = \vec{\mathbf{v}}$ and $A\vec{\mathbf{v}} = -\vec{\mathbf{u}}$? Prove or disprove it.

(5) [15%] U is a subspace of a finite dimensional vector space V. Consider $D_U \subset V^*$ defined by $\{\alpha \in V^* \mid U \text{ is a subspace of } \ker \alpha\}$. Show that D_U is a subspace of dimension $\dim V - \dim U$.

(6A) [10%] Show the volume V of the parallelepiped span by $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{w}} \in \mathbb{R}^n$ satisfies

$$\begin{aligned} V^2 =& ||\vec{\mathbf{u}}||^2\, ||\vec{\mathbf{v}}||^2\, ||\vec{\mathbf{w}}||^2 \;+\; 2\,(\vec{\mathbf{u}}\cdot\vec{\mathbf{v}})\,(\vec{\mathbf{v}}\cdot\vec{\mathbf{w}})\,(\vec{\mathbf{w}}\cdot\vec{\mathbf{u}}) \\ & - ||\vec{\mathbf{u}}||^2\,(\vec{\mathbf{v}}\cdot\vec{\mathbf{w}})^2 - ||\vec{\mathbf{v}}||^2\,(\vec{\mathbf{w}}\cdot\vec{\mathbf{u}})^2 - ||\vec{\mathbf{w}}||^2\,(\vec{\mathbf{u}}\cdot\vec{\mathbf{v}})^2 \end{aligned}$$

(6B) [10%] Following diagram of vector spaces and linear transformations satisfies

(a) $\ker f_{i+1} = \mathrm{im}\, f_i$, $\ker g_{i+1} = \mathrm{im}\, g_i$, $i = 0, 1, 2, 3, 4$.

(b) $\alpha_{i+1} \circ f_i = g_i \circ \alpha_i$, $i = 1, 2, 3, 4$.

$$\begin{array}{ccccccccccccc} \{0_{A_1}\} & \stackrel{f_0}{\to} & A_1 & \stackrel{f_1}{\to} & B_1 & \stackrel{f_2}{\to} & C_1 & \stackrel{f_3}{\to} & D_1 & \stackrel{f_4}{\to} & E_1 & \stackrel{f_5}{\to} & \{0_{E_1}\} \\ & & \alpha_1 \downarrow\cong & & \alpha_2 \downarrow\cong & & \alpha_3 \downarrow & & \alpha_4 \downarrow\cong & & \alpha_5 \downarrow\cong & & \\ \{0_{A_2}\} & \stackrel{g_0}{\to} & A_2 & \stackrel{g_1}{\to} & B_2 & \stackrel{g_2}{\to} & C_2 & \stackrel{g_3}{\to} & D_2 & \stackrel{g_4}{\to} & E_2 & \stackrel{g_5}{\to} & \{0_{E_1}\} \end{array}$$

Show that if α_1, α_2, α_4, α_5 are isomorphisms, then α_3 is an isomorphism.

Notice: You *must* show all your *work* in order to receive full credit.

(1) (20 points) Show that if $\mathbb{R}^n = W_1 \cup W_2 \cup \cdots \cup W_k \cup \cdots$, where each W_k is a subspace, then $\mathbb{R}^n = W_i$ holds for some i.

(2) (20 points) Let a, b, c, d, e, f be real numbers such that the quadratic form $Q(x,y,z) := ax^2 + by^2 + cz^2 + 2dxy + 2eyz + 2fxz$ is positive definite. Then the region bounded by the surface $Q(x,y,z) = 1$ has volume equals

$$\frac{4\pi}{3\sqrt{abc + 2def - ae^2 - bf^2 - cd^2}}.$$

(3) (15 points) The are infinitely many t in $\mathbb{R}$ such that the vectors $(t, 2t^2, 3t^3, 4t^4)$, $(t^2, 2t^3, 3t^4, 4)$, $(t^3, 2t^4, 3t, 4t^2)$, $(t^4, 2t, 3t^2, 4t^3)$ form a basis of $\mathbb{R}^4$.

(4) (15 points) Determine all values of $a, b, c, d, e, f \in \mathbb{R}$ such that the matrix $A := \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 2 & f \\ 0 & 0 & 0 & 2 \end{pmatrix}$ is *not* diagonalizable.

(5) (10 points) If a 5×5 matrix $A \in M_5(\mathbb{R})$ satisfies $A^7 = I_5$ (the identity matrix), then 1 is an eigenvalue of A.

(6) Prove or disprove the following statements (10 points for each).

(a) If $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear transformation with the null space (kernel) of dimension $n-1$, then there exists some $v \in \mathbb{R}^n$ and a non-zero $\lambda \in \mathbb{R}$ such that $\psi(v) = \lambda \cdot v$.

(b) If W_1 and W_2 are 8-dimensional subspaces of $\mathbb{R}^{10}$, then there exist $a_1, a_2, \cdots, a_{10}, b_1, b_2, \cdots, b_{10}, c_1, c_2, \cdots, c_{10}, d_1, d_2, \cdots, d_{10}$ in $\mathbb{R}$ such that the intersection $W_1 \cap W_2$ is the set of all vectors $(x_1, ..., x_{10})$ with $x_1, ..., x_{10}$ a solution to the system of equations

$$\begin{cases} a_1x_1 + a_2x_2 + \cdots + a_ix_i + \cdots + a_{10}x_{10} &= 0 \\ b_1x_1 + b_2x_2 + \cdots + b_ix_i + \cdots + b_{10}x_{10} &= 0 \\ c_1x_1 + c_2x_2 + \cdots + c_ix_i + \cdots + c_{10}x_{10} &= 0 \\ d_1x_1 + d_2x_2 + \cdots + d_ix_i + \cdots + d_{10}x_{10} &= 0. \end{cases}$$

試題隨卷繳回

GRADUATE ENTRANCE EXAM 2016: LINEAR ALGEBRA

Notation: R is the set of real numbers, and C is the set of complex numbers. If $F = \mathrm{R}$ or C, denote by $M_n(F)$ the $n \times n$ matrices with entries in F.

Problem 1 (10pts). Find all possible $a \in \mathrm{R}$ such that the vectors

$$(1,3,a),\ (a,4,3),\ (0,a,1) \in \mathrm{R}^3$$

are linearly dependent.

Problem 2 (10pts). Find a set of polynomials $p_0(t) = a$, $p_1(t) = b + ct$ and $p_2(t) = d + et + ft^2$ with coefficients $a, b, c, d, e, f \in \mathrm{R}$ so that $\{p_0, p_1, p_2\}$ is an orthonormal set of polynomials with respect to the inner product $\langle f, g\rangle = \int_0^2 f(t)g(t)dt$.

Problem 3 (20pts). Let

$$A = \begin{pmatrix} 1 & -3 & 0 \\ 3 & 4 & -3 \\ 3 & 3 & -2 \end{pmatrix} \in M_3(\mathrm{R}).$$

Find an invertible $P \in M_3(\mathrm{R})$ such that

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & -3 & 1 \end{pmatrix}.$$

Problem 4 (15pts). Let $V = M_3(\mathrm{C})$ be a 9-dimension vector space over C and let

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}.$$

Define the linear transformation $T : V \to V$ by

$$T(B) = ABA^{-1}.$$

Show that T is also diagonalizable.

Problem 5 (20pts). Let $A, B \in M_n(\mathrm{C})$. Suppose that eigenvalues of A and B are all real numbers and that $\operatorname{rank} A = \operatorname{rank} A^2$ and $\operatorname{rank} B = \operatorname{rank} B^2$. If A^3 is similar to B^3 (namely there exists an invertible $P \in M_n(\mathrm{C})$ such that $P^{-1}A^3P = B^3$), prove that A is similar to B.

Problem 6 (25pts). Let A and B be elements in $M_n(\mathrm{C})$. If $A^2B + BA^2 = 2ABA$, show that $(AB - BA)^n = 0$.

試題隨卷繳回

(1) (15%) Let $V = \mathbb{R}^6$. Let W_1 be the subspace of V spanned by

$$(1,2,3,4,5,6), (3,4,6,7,9,10), (0,1,0,2,0,3), (1,-2,3,-4,5,-6),$$

and W_2 be the subspace of V spanned by

$$(1,1,1,2,2,3), (-2,0,-1,0,1,2), (1,0,1,0,2,0), (0,0,1,0,-2,-2).$$

Find the dimension of the subspace $W_1 \cap W_2$ and find a basis for this subspace.

(2) (15%) Let

$$C = \begin{bmatrix} -x & 1 & 3 & 1 & 2 \\ -2 & 0 & x & 2 & 2 \\ x & 0 & -2 & -3 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & x & -2 \end{bmatrix}.$$

Find an integer x such that all entries of the inverse of C are integers. For such x, find C^{-1}.

(3) (15%) Let V be the vector space of all $n \times n$ matrices over F. Let T be the linear operator on V defined by $T(A) = A^t$. Test T for diagonalizability, and if T is diagonalizable, find a basis for V such that the matrix representation of T is diagonal.

(4) (15%) Let V and W be F-vector spaces, and V^* and W^* be the dual space of V and W, respectively. Let $T : V \to W$ be a linear transformation. Define $T^* : W^* \to V^*$ by $T^*(f) = f \circ T$ for all $f \in W^*$. Show that T is onto if and only if T^* is one to one.

(5) (10%) Let A and B be $n \times n$ matrices over a field F. Show that if A is invertible, there are at most n scalars c in F such that $cA + B$ is not invertible.

(6) (15%)

(a) Let S and T be linear operators on a finite-dimensional vector space. If $p(t)$ is a polynomial such that $p(ST) = 0$, and if $q(t) = tp(t)$, show that $q(TS) = 0$.

(b) What is the relation between the minimal polynomials of ST and TS.

(7) (15%) Let V be a vector space with a basis $\{u_1, u_2, \ldots, u_n\}$. Let $\langle\ ,\ \rangle$ be an inner product on V. If $c_1, c_2, \ldots, c_n$ are any n scalars, show that there is exactly one vector v in V such that $\langle v, u_j \rangle = c_j$, $j = 1, 2, \ldots, n$.

Notation: $\mathbb{R}$ is the set of real numbers and $\mathbb{C}$ is the set of complex numbers.

Problem 1 (15 pts). Let $<,>$ be the standard inner product on $\mathbb{R}^3$ given by $< v, w >= a_1a_2 + b_1b_2 + c_1c_2$ if $v = (a_1, b_1, c_1)$ and $w = (a_2, b_2, c_2)$. Let W be the subspace in $\mathbb{R}^3$ given by

$$W = \{(x, y, z) \in \mathbb{R} \mid 2x + 7y = 0,\ x - 2y + z = 0\}.$$

Find an orthonormal basis of W. Namely, find a basis $\{w_1, w_2\}$ of W such that $< w_1, w_1 >=< w_2, w_2 >= 1$ and $< w_1, w_2 >= 0$.

Problem 2 (20 pts). Let

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix}.$$

(1) Compute the characteristic polynomial of A.
(2) Find an invertible $P \in M_3(\mathbb{R})$ such that $P^{-1}AP$ is diagonal.

Problem 3 (25pts). Let V be a finite dimensional vector space over $\mathbb{R}$ and let $A : V \to V$ be a $\mathbb{R}$-linear transformation. Prove that

(1) (10 pts) if $A^k = 0$ for some positive integer k, then $I - A$ is invertible, where I is the identity map.
(2) (15 pts) V is generated by kernel of A^k and the image of A^k for some k. In other words, prove $V = \operatorname{Ker} A^k + \operatorname{Im} A^k$ for some k.

Problem 4 (20pts). Let $\mathbf{L} : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ be the linear transformation defined by

$$\mathbf{L}(X) = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix} X - X \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$$

(1) Find the dimension of the kernel of $\mathbf{L}$.
(2) Find a basis for the image of $\mathbf{L}$.

Problem 5 (20 pts). If $A \in M_n(\mathbb{C})$ such that $AA^* = A^*A$ and $v \in \mathbb{C}^n$ is a column vector, prove that

(1) $A^2v = 0$, then $Av = 0$.
(2) If $A^kv = 0$ for some $k \geq 1$, then $Av = 0$.
(3) Show that the minimal polynomial of A has distinct roots.

試題隨卷繳回

國立清華大學 113 學年度碩士班考試入學試題

系所班組別：數學系碩士班

考試科目（代碼）：線性代數（0102）

 *請在【答案卷】作答

Notation.

- $\mathbb{R}$ = the set of all real numbers;
- $M_{m\times n}(\mathbb{R})$ = the set of all real $m \times n$ matrices;
- $P_n(\mathbb{R})$ = the set of all polynomials of degrees at most n with real coefficients;
- $C^\infty(\mathbb{R})$ = the set of all infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$;
- If f is a differentiable function, we write f' for its derivative;
- If v is a vector in an inner product space, we write $\|v\|$ for its norm.

1. Let $a \in \mathbb{R}$, and let $T\colon P_3(\mathbb{R}) \to P_2(\mathbb{R})$ be the function defined by

$$T\big(f(x)\big) = f'(x-a) + (ax+1)f''(x)$$

for all $f(x) \in P_3(\mathbb{R})$.

(a) **(8 points)** For which real numbers a is the function T linear? Prove your answer.

(b) **(10 points)** Find all real numbers a such that T is not surjective.

2. Let V be an $\mathbb{R}$-vector space, and let $T\colon V \to V$ be a linear operator on V.

(a) **(8 points)** Show that if $V = C^\infty(\mathbb{R})$ and $T(f) = f'$ for all $f \in C^\infty(\mathbb{R})$, then T has infinitely many eigenvalues.

(b) **(10 points)** Give a **detailed** proof that if $V = P_n(\mathbb{R})$, then any linear operator T on V has only finitely many eigenvalues. Your proof should contain enough details so that the grader can see clearly why it does not work for $V = C^\infty(\mathbb{R})$.

3. Let V be a (not necessarily finite-dimensional) real inner product space. Let u_1 and u_2 be two distinct vectors in V, and let

$$S = \big\{v \in V \bigm| \|v-u_1\| = \|v-u_2\|\big\}.$$

(a) **(10 points)** What is the necessary and sufficient condition on u_1 and u_2 so that S is a subspace of V? Prove your answer.

(b) **(10 points)** When V is finite-dimensional and S is a subspace, what is the relation between $\dim V$ and $\dim S$? Prove your answer.

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 *請在【答案卷】作答

4. Let $A \in M_{m\times n}(\mathbb{R})$ be a matrix and $b \in \mathbb{R}^m$ be a column vector such that the system of linear equations $Ax = b$ has no solutions for $\|x\| > 2024$, but has at least one solution for $\|x\| \le 2024$. Give a proof or an explicit counterexample for each of the following statements.

   (a) **(10 points)** The system of linear equations $Ax = b$ has only one solution for $x \in \mathbb{R}^n$.

   (b) **(10 points)** $m \ge n$.

5. **(12 points)** Let u_1, u_2, v_1, v_2 be column vectors in $\mathbb{R}^n$ such that u_1, u_2 are linearly independent and v_1, v_2 are linearly independent. Prove that the following two conditions are equivalent.

   (a) There exists an $n \times n$ orthogonal matrix A such that $Au_1 = v_1$ and $Au_2 = v_2$.

   (b) $\|u_1\| = \|v_1\|$, $\|u_2\| = \|v_2\|$, and $\|u_1 - u_2\| = \|v_1 - v_2\|$.

6. **(12 points)** Does there exist a matrix $A \in M_{3\times 3}(\mathbb{R})$ such that

$$A^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}?$$

Prove your answer.

國立清華大學 112 學年度碩士班考試入學試題

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考試科目（代碼）：線性代數（0102）

 *請在【答案卷、卡】作答

(1) (10%) Let

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix}.$$

For which triples $C^t = (c_1, c_2, c_3)$ does the system $AX = C$ have a solution? And find the solutions, if any. Here C^t is the transpose of C.

(2) Let

$$A = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) (5%) Prove that the left-multiplication transformation L_A is a reflection.

(b) (10%) Find the axis in R^2 about which L_A reflects.

(c) (5%) Prove that L_{AB} and L_{BA} are rotations.

(3) (20%) Let $V = P_2(R)$ be the space of all polynomials with coefficients in R, having degree at most 2. Define a linear operator T on V by

$$T(f(x)) = -xf''(x) + f'(x) + 2f(x).$$

Find the minimal polynomial of T.

(4) (20%) Describe all linear operators T on R^2 such that T is diagonalizable and $T^3 - 2T^2 + T = T_0$, where T_0 is the zero transformation.

(5) (15%) Let g be a non-degenerate form on a finite-dimensional space V. Show that each linear operator T has an operator T' such that

$$g(Tv, w) = g(v, T'w)$$

for all v, w.

(6) (a) (5%) If N is a nilpotent 3×3 matrix over C, prove that $A = I + \frac{1}{2}N - \frac{1}{8}N^2$ satisfies $A^2 = I + N$,i.e., A is a square root of $I + N$.

(b) (10%) If N is a nilpotent $n \times n$ matrix over C, find a square root of $I + N$.

國立清華大學 111 學年度碩士班考試入學試題

系所班組別：數學系碩士班

考試科目（代碼）：線性代數（0102）

　　*請在【答案卷、卡】作答

1 (10%) Let V be a finite dimensional vector space over $\mathbb{R}$, and v_1, v_2 be two distinct vectors in V. Show that there is an $\mathbb{R}$-linear transofrmation $f : V \to \mathbb{R}$ for which

$$f(v_1) \neq f(v_2).$$

2 (18%) Let $V := \mathrm{Mat}_{1\times 3}(\mathbb{R})$ and define $f : V \to \mathbb{R}$ by

$$f(x, y, z) := \det \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \\ x & y & z \end{pmatrix}.$$

(i) Show that f is a linear transformation over $\mathbb{R}$.

(ii) Put $W := \mathrm{Ker} f$. Find an $\mathbb{R}$-basis of W.

(iii) Let V/W be the quotient space of V by W, and elements in V/W are denoted by $\bar{v}$ for $v \in V$. Show that the map $\bar{f} : V/W \to \mathbb{R}$ given by

$$\bar{f}\left(\overline{(a, b, c)}\right) := \det \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \\ a & b & c \end{pmatrix}$$

is well-defined and is an isomorphism of vector spaces.

3 (14%) Find the Journal canonical form of the following matrix

$$\begin{pmatrix} 2 & -4 & 2 & 2 \\ -2 & 0 & 1 & 3 \\ -2 & -2 & 3 & 3 \\ -2 & -6 & 3 & 7 \end{pmatrix}.$$

4 (12%) Suppose that V is finite dimensional inner product space over $\mathbb{C}$, and T is a normal linear operator on V such that $T^9 = T^8$. Prove that T is self-adjoint and $T^2 = T$.

5 (20%) Let V be a finite dimensional vector space over $\mathbb{C}$ with two inner products $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle'$. Prove the following.

(i) There exists a unique linear operator T on V so that $\langle x, y \rangle' = \langle T(x), y \rangle$ for all $x, y \in V$.

(ii) The linear operator T in (i) is positive definite with respect to both inner products.

6 (14%) Let $V := \mathbb{R}^n$ and let $W \subset V$ be the vector subspace defined as the set of solutions of $x_1 + \cdots + x_n = 0$. Define $W^0 := \{f \in V^* | f(w) = 0 \text{ for all } w \in W\}$, where

$$V^* := \{f : V \to \mathbb{R} | f \text{ is a linear transformation over } \mathbb{R}\},$$

the dual space of V. Show that W^0 is equal to the set of all f of the form

$$f\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \lambda(a_1 + \cdots + a_n) \text{ for some } \lambda \in \mathbb{R}.$$

7. (12%) Given a nonzero matrix $A \in \mathrm{Mat}_n(\mathbb{R})$ and a nonzero vector $\mathbf{b} \in \mathrm{Mat}_{n\times 1}(\mathbb{R})$, show that if there exists a row vector $C \in \mathrm{Mat}_{1\times n}(\mathbb{R})$ for which $CA = 0$ and $C\mathbf{b} = 1$, then $A\mathbf{x} = \mathbf{b}$ has no solution.

國立清華大學 110 學年度碩士班考試入學試題

系所班組別：數學系碩士班

考試科目（代碼）：線性代數（0102）

共__1__頁，第__1__頁　　*請在【答案卷、卡】作答

Notation: $\mathbb{R}$ denotes the field of real numbers; $\mathbb{C}$ denotes the field of complex numbers. F denotes an arbitrary field; $M_{m\times n}(F)$ denotes the set of all $m \times n$ matrices with entries in F. If T is a linear transformation, $R(T)$ denotes the range of T, and $N(T)$ denotes the null space of T. If $A \in M_{m\times n}(F)$, A^t denotes the transpose of A, and L_A denotes the linear transformation from F^n to F^m that sends each vector $v \in F^n$ to $Av \in F^m$.

1. **(12 points)** Let V and W be F-vector spaces, and let $T\colon V \to W$ be a linear transformation. Prove that $\dim R(T)+\dim N(T) = \dim V$ if V is finite-dimensional.

2. **(10 points)** Find a matrix $A \in M_{3\times 3}(\mathbb{R})$ such that

$$R(L_A) = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; 3a-2b+4c=0 \right\} \text{ and } N(L_A) = \left\{ \begin{pmatrix} 2t \\ 3t \\ -t \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; t \in \mathbb{R} \right\}.$$

   You need to show that the matrix you find has the required properties.

3. **(12 points)** Let $A \in M_{m\times n}(F)$. Show that the system of linear equations $Ax = b$ has a solution for all $b \in F^m$ if and only if the system of linear equations $A^t x = 0$ has no nonzero solutions.

4. **(12 points)** Let $A \in M_{m\times n}(F)$ and $B \in M_{n\times m}(F)$. Show that if $\operatorname{rank}(AB) = m$, then $\operatorname{rank}(BA) = m$.

5. Let $T\colon V \to V$ be a linear operator on a finite-dimensional F-vector space V.

   (a) **(6 points)** State the definition of eigenvectors of T.

   (b) **(6 points)** Give an explicit example of T that has no eigenvectors.

   (c) **(8 points)** Prove that T has an eigenvector if $F = \mathbb{C}$.

6. **(10 points)** Let $A \in M_{n\times n}(F)$. Show that if $Q \in M_{n\times n}(F)$ is an invertible matrix such that $Q^{-1}AQ$ is diagonal, then each column vector of Q is an eigenvector of L_A.

7. **(12 points)** Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Show that if T preserves the Euclidean distance between any two points, that is, $\|T(u) - T(v)\| = \|u - v\|$ for any $u, v \in \mathbb{R}^n$, then the matrix representation of T relative to the standard basis is an orthogonal matrix.

8. **(12 points)** Let $A \in M_{n\times n}(\mathbb{R})$ be a real symmetric matrix. Show that there exists a real symmetric matrix B such that $B^3 = A$.

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系所班組別：數學系碩士班

考試科目（代碼）：線性代數（0102）

共__1__頁，第__1__頁　　　*請在【答案卷、卡】作答

In the following, $\mathbb{F}$ denotes a field with infinitely many elements.

1. (15%)Express

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 4 & 5 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

as a product of elementary matrices.

2. (10%) Show that eigenvectors from different eigenspaces of a matrix are linearly independent.

3. Let

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \end{pmatrix}$$

Let $\beta := \{(1,1),(1,2)\}$ be an ordered basis for $\mathbb{R}^2$ and $\gamma := \{(0,0,1),(0,1,1),(1,1,1)\}$ be an ordered basis for $\mathbb{R}^3$.

(a) (10%) Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that the matrix representation

$$[T]_\beta^\gamma = A$$

(b) (5%) Find rank(T).

4. (10%) Prove the following theorem: For $A \in M_{n\times n}(\mathbb{F}), \mathrm{b} \in \mathbb{F}^n$, if the system $A\mathbb{X} = \mathrm{b}$ has exactly one solution, then A is invertible.

5. (15%) Let $L : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be given by

$$L[p(t)] = p(t) + p(1)(t-3) - 2p'(1)(2t-1)$$

Find the eigenvalues and corresponding eigenvectors of L where $P_3(\mathbb{R})$ is the vector space of real polynomials of degree ≤ 3.

6. (a) (10%) Show that if $A \in M_{m\times n}(\mathbb{F})$ is of rank m, there exists $B \in M_{n\times m}(\mathbb{F})$ such that

$$BA = I_n$$

(b) (5%) What is the rank of B?

7. (10%) Give $A \in M_{2\times 2}(\mathbb{Q})$ which is not diagonalizable over $\mathbb{Q}$, but A is diagonalizable over $\mathbb{R}$.

8. (10%) Prove or give a counterexample: any $A \in M_{n\times n}(\mathbb{C})$ is similar to A^t.

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Notation:

- $M_{n\times n}(\mathbb{F})$: the set of all $n \times n$ matrices over the field $\mathbb{F}$
- I_n: the $n \times n$ identity matrix
- $\mathrm{End}(V)$: the set of all linear transformations from V to itself
- A^*: the conjugate transpose of the matrix A
- $\ker(\alpha)/\operatorname{im}(\alpha)/\operatorname{tr}(\alpha)$: kernel/image/trace of α

(1) Let $S = \{2, 3, 5, 6, 7\}$. Let V be the vector space of all functions $S \to \mathbb{R}$ with $(f+g)(x) = f(x) + g(x)$ and $(cf)(x) = cf(x)$ for $f, g \in V, c \in \mathbb{R}$.

(a) (8%) V is a k-dimensional vector space over $\mathbb{R}$, $k =$?

(b) (8%) Let $f_i(x) = x^i$. Is $\{f_1, f_2, ..., f_k\}$ a basis for V?

(2) Let V be the vector space of all polynomials with real coefficients satisfying $\deg(f(x)) < n$. Let $T \in \mathrm{End}(V)$ defined by $T(f(x)) = x^2(f(x+1) - f(x) - f'(x))$.

(a) (10%) In the case $n = 5$, find all eigenvectors of T.

(b) (10%) In the general case, find all eigenvalues of T. Is T diagonalizable?

(3) (10%) Let $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 0 & 1 & 2 \end{bmatrix}$, compute A^{100}.

(4) (12%) Let V be a 2-dimensional vector space over $\mathbb{F}$ and $\alpha \in \mathrm{End}(V)$, $\alpha^2 \neq 0$. Show that $V = \ker(\alpha) \oplus \operatorname{im}(\alpha)$. (Hint: consider minimal polynomial)

(5) Let $V = M_{4\times 4}(\mathbb{C})$, define $\langle A, B\rangle = \operatorname{tr}(AB^*)$.

(a) (10%) Show that $\langle \cdot, \cdot \rangle$ defines an inner product on V over $\mathbb{C}$.

(b) (10%) Let W be the subspace of V consisting of all skew-symmmetric matrices (i.e. $A = -A^T$). Find an orthonormal basis for W.

(6) (10%) Let V be an n-dimensional vector space over $\mathbb{F}$. Let $\alpha \in \mathrm{End}(V)$ for which there exists a set S of $n+1$ eigenvectors satisfying the condition that every subset of size n is a basis for V. Show that $\alpha = cI_n$ for some constant c.

(7) (12%) Let $A \in M_{n\times n}(\mathbb{R})$ satisfy $A^2 + I_n = 0$. Show that n is even, and there exists $P \in M_{n\times n}(\mathbb{R})$ such that $P^{-1}AP = \begin{bmatrix} 0 & -I_{n/2} \\ I_{n/2} & 0 \end{bmatrix}$.

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- Show all your work and justify all your answers.
- $\mathbb{R}$ denotes the field of real numbers, and n denotes a positive integer.

1. (12 points) Let A be an $n \times n$ real matrix whose (i,j) entry is

$$A_{ij} = \begin{cases} j, & \text{if } i \leq j, \\ 0, & \text{otherwise,} \end{cases}$$

where $i, j = 1, \ldots, n$. Find the inverse of A.

2. (12 points) Let $V = \{(x,y) \mid x, y \in \mathbb{R}\}$. For $(x_1, y_1), (x_2, y_2) \in V$ and $a \in \mathbb{R}$, define

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 y_2) \quad \text{and} \quad a(x_1, y_1) = (ax_1, y_1).$$

Is V a vector space over $\mathbb{R}$ with these operations?

3. (15 points) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the function defined by

$$T(x, y, z) = (2x - y, 3y - 2z, x + y - z)$$

for $(x, y, z) \in \mathbb{R}^3$. Prove that T is a linear transformation. Is T one-to-one?

4. (15 points) Let $V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid 2x_1 - x_2 + 3x_3 - x_4 = 0\}$. Find a basis β for V such that $(2, 1, -1, 0) \in \beta$.

5. (15 points) Let V be the real vector space of all $n \times n$ real matrices, and let $A \in V$. Suppose that W is the subspace of V spanned by the set $\{A^i \mid i \text{ is a non-negative integer}\}$, where A^0 is defined to be the $n \times n$ identity matrix. Prove that $\dim(W) \leq n$.

6. (15 points) Let V be a finite-dimensional complex inner product space, and let T be a positive definite linear operator on V. Prove that $T = S^*S$ for some invertible linear operator S on V. Here S^* denotes the adjoint of S.

7. (16 points) Find the Jordan canonical form of the real matrix $\begin{bmatrix} 4 & -3 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

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1. (10 points) Let A, B be two $m \times n$ matrix. Show that $|\operatorname{rank}(A) - \operatorname{rank}(B)| \leq \operatorname{rank}(A + B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$.

2. (16 points) Let A be an $n \times n$ matrix and $r_k = \operatorname{rank}(A^k)$.

   (a). Show that $\lim_{k\to\infty} r_k$ exist.

   (b). If $r_3 \neq r_4$, Is A diagonalizable? Show your answer.

3. (8 points) Let $A = [a_{ij}]$ be an $n \times n$ matrix satisfying the condition that each a_{ij} is either equal to 1 or to -1. Show that $\det(A)$ is an integer multiple of 2^{n-1}.

4. (16 points) Let S, T be linear operator on V such that $S^2 = S$. Show that the range of S is invariant under T if and only if $STS = TS$. Show that both the range and null space of S are invariant under T if and only if $ST = TS$.

5. (20 points) Define a real vector space $V = \{f(x) \mid f(x) = ax^2 + bx + c, a, b, c \in \mathbb{R}\}$, with inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$.

   (a). Find an orthonormal basis for V.

   (b). Using (a), find $f \in V$ to maximize $f(\frac{1}{2})$ subject to the constraint $\langle f, f\rangle = 1$.

6. (16 points) Let $A = \begin{bmatrix} -2 & 3 & 1 \\ 0 & a & 3 \\ 0 & -3 & 4-a \end{bmatrix}$. Find the condition of a such that A is diagonalizable over real number.

7. (14 points) Let A be an $n \times n$ real symmetric matrix. Show that the matrix $A^2 + A + I$ is positive-definite.

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In this test, all vector spaces are finite dimensional over $\mathbb{C}$.

1. (15 points) Let T be a linear operator on a vector space V. Prove that T is diagonalizable if and only if its minimal polynomial is square-free.

2. (15 points) Let V be a vector space. A linear operator S on V is semisimple if for every S-invariant subspace W of V there exists an S-invariant subspace W' of V such that $V = W \oplus W'$. Prove that every diagonalizable operator on V is semisimple, and deduce that every linear operator T on V can be decomposed uniquely as $T = S + N$, where S is semisimple, N is nilpotent, and $SN = NS$.

3. (15 points) Let T and U be normal operators on an inner product space V such that $TU = UT$. Prove that $UT^* = T^*U$, where T^* is the adjoint of T.

4. (15 points) Let T and U be Hermitian operators on an inner product space $(V, \langle \cdot, \cdot \rangle)$ such that $\langle T(x), x \rangle > 0$ for all nonzero $x \in V$. Prove that UT is diagonalizable and has only real eigenvalues.

5. (15 points) Find the total number of distinct equivalence classes of congruent $n \times n$ real symmetric matrices and justify your answer.

6. (15 points) Let A be an $n \times n$ complex matrix, t be a variable, and I be the identity matrix. Prove that
$$\det(I - tA) = \exp\left(-\sum_{i \geq 1} \frac{\operatorname{tr}(A^i)t^i}{i} \right).$$

7. (10 points) Let $A = (a_{i,j})$ be a $2n \times 2n$ matrix such that $A^T = -A$. The Phaffian of A is defined as
$$\operatorname{pf}(A) := \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)},$$
where S_{2n} is the symmetric group of order $2n$ and $\operatorname{sgn}(\sigma)$ is the signature of σ. Prove that for any $2n \times 2n$ matrix B,
$$\operatorname{pf}(BAB^T) = \det(B)\operatorname{pf}(A).$$

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1. Find the inverse of

$$\begin{pmatrix} 2 & 0 & 0 & 3 & 0 \\ 0 & 2 & 0 & 0 & 8 \\ 0 & 0 & 9 & 0 & 0 \\ 3 & 0 & 0 & 5 & 0 \\ 0 & 4 & 0 & 0 & 17 \end{pmatrix}$$

(15 points)

2. Show that for any $A \in \mathbb{R}^{m\times n}$, $\operatorname{rank}(A^TA) = \operatorname{rank}(A)$. (15 points)

3. Let $\mathcal{P}_2$ be the real vector space of real quadratic polynomials (polynomials of degree at most 2). Find an orthonormal basis for $\mathcal{P}_2$ with respect to the inner product $\langle f, g\rangle = f(-1)g(-1) + f(0)g(0) + f(1)g(1)$. (You do not need to show that it is truely an inner product.) (15 points)

4. For real t show that

$$e^{\begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

(15 points)

5. (a) Find the matrix $P \in \mathbb{R}^{3\times 3}$ such that $x \mapsto Px$ is the orthogonal projection of $\mathbb{R}^3$ onto $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. (10 points)

   (b) Find $\min_{x\in\mathbb{R}^2} \left\| \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} x \right\|$. (10 points)

6. (a) Let $A \in \mathbb{C}^{n\times n}$. Prove that if $x^*Ax \geq 0$ for all $x \in \mathbb{C}^n$, then A is Hermitian. (10 points)

   (b) Let $A \in \mathbb{R}^{n\times n}$. Is it true that $x^TAx \geq 0$ for all $x \in \mathbb{R}^n$ implies A is symmetric? (10 points)

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1. Find e^A, where $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$. (15 points)

2. Let $T_j : \mathbb{R}^2 \to \mathbb{R}^2$, $j = 1, 2$, be a rotation by some angle θ_j about some point $x_j \in \mathbb{R}^2$. Show that if $\theta_1 + \theta_2 \notin \{2k\pi \mid k \in \mathbb{Z}\}$, then the composition T_2T_1, is also a rotation about some point. (15 points)

3. Let A be a real skew-symmetric matrix, that is, $A^t = -A$. Prove the following statements.

   (a) Each eigenvalue of A is either 0 or a purely imaginary number. (10 points)

   (b) The rank of A is even. (10 points)

4. Let $C([-\pi, \pi])$ be the space of real continuous functions on $[-\pi, \pi]$ with inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$. Find an orthonormal basis for the subspace $W = \operatorname{span}(1, x, \sin x)$. (15 points)

5. Let $M_{2\times 2}$ be the space of 2×2 real matrices. Consider the linear operator S on $M_{2\times 2}$ defined by

$$S(X) = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} X + X \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix},$$

   where $a, b, c, d \in \mathbb{R}$.

   (a) Write down the representative matrix of S with respect to the basis

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.\ \text{(10 points)}$$

   (b) Give the necessary and sufficient condition on a, b, c, d so that S is invertible. (10 points)

6. Let A be an $n \times n$ (real or complex) matrix. Show that if A is nilpotent (i.e. $A^k = 0$ for some $k \in \mathbb{N}$), then $I - A$ is invertible, where I is the $n \times n$ identity matrix. (15 points)

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【基礎數學】: Part I. 線性代數

Linear Algebra

In the following, $F^{m\times n}$ denotes the class of all $m \times n$ matrices with entries in the field F, where $F = \mathbb{R}$ or $\mathbb{C}$. Vectors in F^n will be regarded as column vectors. We say a matrix $A \in \mathbb{R}^{n\times n}$ is symmetry if $A^T = A$, a matrix $A \in \mathbb{C}^{n\times n}$ is Hermitian if $A^* = \overline{A}^T = A$. A matrix $A \in \mathbb{R}^{n\times n}$ is positive definite if $x^T Ax > 0$ for all nonzero $x \in \mathbb{R}^n$.

(1) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $U = \{X \in \mathbb{R}^{2\times 2} : AX = XA\}$, find the dimension of U. (20 points)

(2) Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $\theta \in [0, 2\pi]$.

a. Show that $A^n = \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix}$ for all $n \in \mathbb{N}$. (10 points)

b. Calculate A^{-n} for all $n \in \mathbb{N}$. (10 points)

(3) Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix.

a. Show that all eigenvalues of A are real. (10 points)

b. If (λ_1, y_1) and (λ_2, y_2) are two eigenpairs of A with $\lambda_1 \neq \lambda_2$, show that $\langle y_1, y_2\rangle = 0$. (10 points)

(4) Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be symmetric positive definite matrix, show that

a. $a_{ii} > 0$ for all $1 \le i \le n$. (10 points)

b. $a_{ii}a_{jj} > a_{ij}^2$ for all $i \neq j$. (10 points)

(5) Let $w_i \in \mathbb{R}$, $1 \le i \le 4$ and $A = \begin{bmatrix} w_1w_1 & w_1w_2 & w_1w_3 & w_1w_4 \\ w_2w_1 & w_2w_2 & w_2w_3 & w_2w_4 \\ w_3w_1 & w_3w_2 & w_3w_3 & w_3w_4 \\ w_4w_1 & w_4w_2 & w_4w_3 & w_4w_4 \end{bmatrix}$ with $w_1^2 + w_2^2 + w_3^2 + w_4^2 = 1$.

a. Find all eigenvalues of A and its algebraic multiplicity. (10 points)

b. Calculate $\det(I_4 - 2A)$. (10 points)

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【基礎數學】：Part I. 線性代數

In the following, $F^{m\times n}$ denotes the class of all $m \times n$ matrices with entries in the field F, where $F = \mathbb{R}$ or $\mathbb{C}$. Vectors in F^n will be regarded as column vectors. $F^{m\times n}$ and F^n are vector spaces over F in the canonical way.

Justify all your answers for the problems below.

1. Let $W \subset \mathbb{R}^4$ be the space of solutions of the system of linear equations $AX = 0$, where $A = \begin{bmatrix} 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{bmatrix}$. Find a basis for W. (15 points)

2. Let L be the line $y = mx$ in $\mathbb{R}^2$, where $m \in \mathbb{R}$. Find the matrix $A \in \mathbb{R}^{2\times 2}$ so that $x \mapsto Ax$ is the orthogonal projection onto L. (15 points)

3. Compute $\det(M)$, where M is the following $n \times n$ tridiagonal matrix:
$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}. \text{ (15 points)}$$

4. Suppose $n \geq m$. Let $v_1, \ldots, v_m$ be linearly independent vectors in $\mathbb{C}^n$, and $K_1, \ldots, K_m$ be linear subspaces of $\mathbb{C}^n$. Let A be the subspace of $\mathbb{C}^{n\times n}$ containing all matrices M such that $Mv_j \in K_j$ for $j = 1, 2, \ldots, m$. Find $\dim(A)$ (in terms of $n, \dim(K_1), \ldots, \dim(K_m)$). (20 points)

5. Let $P = \begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix} \in \mathbb{R}^{2\times 2}$. Give the necessary and sufficient condition on a, b such that $\lim\limits_{n\to\infty} P^n$ exists. (20 points)

6. Find a nonsingular $Q \in \mathbb{C}^{3\times 3}$ such that $A = QJQ^{-1}$, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, and J is the Jordan form of A. (15 points)

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【基礎數學】: Part I. 線性代數

Note: $\mathbb{R}$ denotes the field of real numbers, and n denotes a positive integer.

1. (10%) Is there a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1,-2,0) = (1,1)$, $T(3,-5,1) = (2,3)$, and $T(-1,3,1) = (3,0)$? *Justify your answer.*

2. (15%) Let V be the vector space of all polynomials of degree at most n with real coefficients. For $i = 0, 1, \ldots, n$, let $p_i(x) = x^i + x^{i+1} + \cdots + x^n \in V$. Show that $\{p_0(x), p_1(x), \ldots, p_n(x)\}$ is a basis for V.

3. (20%) Let A be an $n \times n$ real matrix such that $A^2 = A$. Show that the trace of A is equal to the rank of A. Is A similar over $\mathbb{R}$ to a diagonal matrix? *Justify your answer.*

4. (20%) Let T be a linear operator on a finite-dimensional vector space such that $\operatorname{rank}(T^2) = \operatorname{rank}(T)$. Show that $\mathrm{N}(T) \cap \mathrm{R}(T) = \{0\}$. (Here $\mathrm{N}(T)$ and $\mathrm{R}(T)$ are the null space and the range of T respectively.)

5. (15%) Let V be the vector space of all polynomials of degree at most 3 with real coefficients. Let D be the linear operator on V defined by $D(p) = p'$ for $p \in V$. Find the Jordan form of D.

6. (20%) Let T and U be linear operators on an n-dimensional vector space V. Suppose that $\{v, T(v), \ldots, T^{n-1}(v)\}$ is a basis for V for some $v \in V$, and that $TU = UT$. Show that $U = p(T)$ for some polynomial p.

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【基礎數學】：Part II. 線性代數

Entrance exam for master degree program: Linear Algebra

1. (20 points) Consider the 5×5 real matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and the following problems concerning A.

(a) Find an invertible matrix P such that PA is a row-reduced echelon matrix.

(b) Find a basis for the row space W of A.

(c) Find a basis for the vector space V of all 5×1 column matrices X such that $AX = \mathbf{0}$.

(d) For what 5×1 column matrices Y does the equation $AX = Y$ has solutions?

2. (10 points) Let A be an $n \times n$ real matrix with transpose A^T. Prove that $\operatorname{rank}(A^T A) = \operatorname{rank} A$.

3. (10 points) Let V be the vector space of all real 3×3 matrices and let A be the diagonal matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Calculate the determinant of the linear transformation T on V defined by $T(X) = AX + XA$.

4. (20 points) Let A be an $n \times n$ orthogonal matrix, that is, A is a real $n \times n$ matrix with $A^T A = I$ where I is the $n \times n$ identity matrix.

(a) Show that $\det A = \pm 1$.

(b) Show that x and Ax have the same length for all $x \in \mathbb{R}^n$.

(c) If λ is an eigenvalue of A, Prove that $|\lambda| = 1$.

(d) If $n = 3$ and $\det A = 1$, prove that 1 is an eigenvalue of A.

5. (15 points) Show that all the eigenvalues of a real symmetric matrix are real, and that the eigenvectors are perpendicular to each other when they correspond to different eigenvalues.

6. (15 points) Consider the matrix

$$A = \begin{pmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix}.$$

Find a Jordan form J of A and an invertible matrix Q such that $A = QJQ^{-1}$.

7. (10 points) Show that every matrix is similar to its transpose.

This exam has 7 questions, for a total of 100 points.

國立成功大學 104 學年度「碩士班」研究生甄試入學考試

【基礎數學】：Part II. 線性代數

In what follows, $\mathbb{R}$ denotes the field of all real numbers, and $\mathrm{M}_{n\times n}(\mathbb{R})$ denotes the vector space of all $n \times n$ real matrices.

1. Let $P_2(\mathbb{R})$ be the vector space of all polynomials of degree at most 2 with real coefficients. Suppose $T\colon \mathrm{M}_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ is the linear transformation defined by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 3c + (a-2b)x + dx^2.$$

   (a) (10%) Find a basis for the null space of T and determine the dimension of the range of T.

   (b) (10%) Let $\gamma = \{1, x, x^2\}$, which is the standard ordered basis for $P_2(\mathbb{R})$. Find an ordered basis β for $\mathrm{M}_{2\times 2}(\mathbb{R})$ such that the matrix representation $[T]_\beta^\gamma$ of T in β and γ is

$$\begin{pmatrix} 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

2. Let n be a positive integer, and let $S_i = \{A \in \mathrm{M}_{n\times n}(\mathbb{R}) \mid A^t = (-1)^i A\}$ for $i = 1, 2$. Here A^t denotes the transpose of A.

   (a) (6%) Prove that S_i is a subspace of $\mathrm{M}_{n\times n}(\mathbb{R})$ for $i = 1, 2$.

   (b) (12%) Prove that $\mathrm{M}_{n\times n}(\mathbb{R})$ is the direct sum of S_1 and S_2.

3. Consider the real matrix $A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & -1 & -1 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$.

   (a) (8%) Find the characteristic polynomial for A.

   (b) (12%) Find the minimal polynomial for A. Is A similar to a diagonal matrix? *Justify your answer.*

4. (12%) Let V be a finite-dimensional inner product space whose inner product is denoted by $\langle \cdot\,, \cdot \rangle$, and let T be a self-adjoint operator on V (that is, T is equal to its adjoint T^*). Prove that if $\langle v, T(v)\rangle = 0$ for all $v \in V$, then T is the zero linear operator.

5. (18%) Let T be a linear operator on a finite-dimensional complex vector space V. Suppose W is a T-invariant subspace of V and $W \neq V$. Prove that there exists a vector $v \in V \setminus W$ such that $T(v) - \lambda v \in W$ for some eigenvalue λ of T.

6. (12%) Let A be a 9×9 real matrix such that $A^6 + A^3 = A^5 + A^4$. Is A similar over $\mathbb{R}$ to a upper triangular matrix? *Justify your answer.*

國立成功大學 103 學年度「碩士班」研究生甄試入學考試

【基礎數學】: Part II. 線性代數

Notation

- n: a positive integer
- $\mathrm{M}_{n\times n}(F)$: the set of all $n \times n$ matrices over the field F
- $\mathbb{R}$: the field of all real numbers
- $\mathbb{C}$: the field of all complex numbers
- A^*: the conjugate transpose of the matrix A

1. (12%) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the $\mathbb{R}$-linear map defined by
$$T(a,b,c) = (a - 3b - 2c, a + b, 3a + 5b + c).$$
Find the rank of T, and find a basis for the null space of T.

2. (12%) Suppose W_1 and W_2 are the following subspaces of the real vector space $\mathrm{M}_{3\times 3}(\mathbb{R})$:
$$W_1 = \left\{ \begin{pmatrix} a & 2a & b \\ b & c & 0 \\ 0 & 0 & d \end{pmatrix} : a,b,c,d \in \mathbb{R} \right\}, \quad W_2 = \left\{ \begin{pmatrix} a & b & 2a \\ b & 2c & d \\ 0 & d & 0 \end{pmatrix} : a,b,c,d \in \mathbb{R} \right\}.$$
Find the dimension of the subspace $W_1 + W_2$.

3. Consider the real matrix $A = \begin{pmatrix} 12 & -5 & -5 & 3 \\ 20 & -8 & -10 & 0 \\ 7 & -3 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.
   (a) (10%) Find the characteristic polynomial of A.
   (b) (5%) Is A similar to a 4×4 diagonal matrix over $\mathbb{R}$? Justify your answer.
   (c) (5%) Is A similar to a 4×4 diagonal matrix over $\mathbb{C}$? Justify your answer.

4. (12%) Show that if A is a 3×3 real matrix, then A is similar to
$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} 0 & \lambda & 0 \\ 1 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 0 & 0 & \lambda \\ 1 & 0 & \mu \\ 0 & 1 & \nu \end{pmatrix}$$
for some $\lambda, \mu, \nu \in \mathbb{R}$.

5. (12%) Let A be a 6×6 complex matrix such that $A^3 = 0$. Find all possible Jordan canonical forms of A.

6. (12%) Suppose $N \in \mathrm{M}_{n\times n}(\mathbb{C})$ is normal, i.e., $N^*N = NN^*$. Show that N is self-adjoint if and only if all eigenvalues of N are real.

7. Let $\langle A, B\rangle$ be the trace of AB^* for all $A, B \in \mathrm{M}_{n\times n}(\mathbb{C})$.
   (a) (10%) Show that $\langle\cdot,\cdot\rangle$ is an inner product on $\mathrm{M}_{n\times n}(\mathbb{C})$.
   (b) (10%) Let $P \in \mathrm{M}_{n\times n}(\mathbb{C})$ be invertible, and let T be the linear operator on $\mathrm{M}_{n\times n}(\mathbb{C})$ defined by $T(A) = P^{-1}AP$. Find the adjoint of T with respect to the inner product $\langle\cdot,\cdot\rangle$.