國立成功大學 114 學年度「碩士班」甄試入學考試

高等微積分

1. (15%) Determine the following limit by δ-ϵ definition.

$$\lim_{x\to 1}(2x^3 + 7x^2 + 5x + 8)$$

2. (15%) Show by definition that the following set is open:

$$\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 < x + y\}$$

3. (15%) Assume function f is increasing on interval $[20, 24]$.

   (a) Show that the set of discontinuity of f is countable.

   (b) Show that f is Riemann integrable.

4. (15%) Denote x_n the unique positive root of the polynomial

$$f_n(x) = x^n + x^{n-1} + \cdots + x - 114, \quad n = 1, 2, 3, \cdots.$$

   Show that the sequence $\{x_n\}$ is convergent. Also find the limit.

5. (20%) Yes or No. Justify your answer.

   (a) Determine if $f(x) = \sqrt{1 + x^2}$ is uniformly continuous on $(-\infty, \infty)$.

   (b) Determine if $f(x) = \sqrt{1 + x^4}$ is uniformly continuous on $(-\infty, \infty)$.

6. (20%) Find the value of the limit. Justify your answer.

$$\lim_{n\to\infty} \int_0^1 \frac{nx^n}{1 + e^x} dx$$

國立成功大學 114 學年度「碩士班」甄試入學考試

線性代數

Notation:

- $M_{n\times n}(\mathbb{F})$: the set of all $n \times n$ matrices over the field $\mathbb{F}$
- I_n: the $n \times n$ identity matrix
- $\mathrm{End}(V)$: the set of all linear transformations from V to itself
- A^*: the conjugate transpose of the matrix A
- $\ker(\alpha)/\operatorname{im}(\alpha)/\operatorname{tr}(\alpha)$: kernel/image/trace of α

(1) Let $S = \{2, 3, 5, 6, 7\}$. Let V be the vector space of all functions $S \to \mathbb{R}$ with $(f+g)(x) = f(x) + g(x)$ and $(cf)(x) = cf(x)$ for $f, g \in V, c \in \mathbb{R}$.

(a) (8%) V is a k-dimensional vector space over $\mathbb{R}$, $k =$?

(b) (8%) Let $f_i(x) = x^i$. Is $\{f_1, f_2, \dots, f_k\}$ a basis for V?

(2) Let V be the vector space of all polynomials with real coefficients satisfying $\deg(f(x)) < n$. Let $T \in \mathrm{End}(V)$ defined by $T(f(x)) = x^2(f(x+1) - f(x) - f'(x))$.

(a) (10%) In the case $n = 5$, find all eigenvectors of T.

(b) (10%) In the general case, find all eigenvalues of T. Is T diagonalizable?

(3) (10%) Let $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 0 & 1 & 2 \end{bmatrix}$, compute A^{100}.

(4) (12%) Let V be a 2-dimensional vector space over $\mathbb{F}$ and $\alpha \in \mathrm{End}(V)$, $\alpha^2 \neq 0$. Show that $V = \ker(\alpha) \oplus \operatorname{im}(\alpha)$. (Hint: consider minimal polynomial)

(5) Let $V = M_{4\times 4}(\mathbb{C})$, define $\langle A, B\rangle = \operatorname{tr}(AB^*)$.

(a) (10%) Show that $\langle \cdot, \cdot \rangle$ defines an inner product on V over $\mathbb{C}$.

(b) (10%) Let W be the subspace of V consisting of all skew-symmmetric matrices (i.e. $A = -A^T$). Find an orthonormal basis for W.

(6) (10%) Let V be an n-dimensional vector space over $\mathbb{F}$. Let $\alpha \in \mathrm{End}(V)$ for which there exists a set S of $n+1$ eigenvectors satisfying the condition that every subset of size n is a basis for V. Show that $\alpha = cI_n$ for some constant c.

(7) (12%) Let $A \in M_{n\times n}(\mathbb{R})$ satisfy $A^2 + I_n = 0$. Show that n is even, and there exists $P \in M_{n\times n}(\mathbb{R})$ such that $P^{-1}AP = \begin{bmatrix} 0 & -I_{n/2} \\ I_{n/2} & 0 \end{bmatrix}$.