國立清華大學 113 學年度碩士班考試入學試題

系所班組別：數學系碩士班

考試科目（代碼）：線性代數（0102）

　　　*請在【答案卷】作答

Notation.

- $\mathbb{R}$ = the set of all real numbers;
- $M_{m\times n}(\mathbb{R})$ = the set of all real $m \times n$ matrices;
- $P_n(\mathbb{R})$ = the set of all polynomials of degrees at most n with real coefficients;
- $C^\infty(\mathbb{R})$ = the set of all infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$;
- If f is a differentiable function, we write f' for its derivative;
- If v is a vector in an inner product space, we write $\|v\|$ for its norm.

1. Let $a \in \mathbb{R}$, and let $T\colon P_3(\mathbb{R}) \to P_2(\mathbb{R})$ be the function defined by

$$T\big(f(x)\big) = f'(x-a) + (ax+1)f''(x)$$

for all $f(x) \in P_3(\mathbb{R})$.

(a) **(8 points)** For which real numbers a is the function T linear? Prove your answer.

(b) **(10 points)** Find all real numbers a such that T is not surjective.

2. Let V be an $\mathbb{R}$-vector space, and let $T\colon V \to V$ be a linear operator on V.

(a) **(8 points)** Show that if $V = C^\infty(\mathbb{R})$ and $T(f) = f'$ for all $f \in C^\infty(\mathbb{R})$, then T has infinitely many eigenvalues.

(b) **(10 points)** Give a **detailed** proof that if $V = P_n(\mathbb{R})$, then any linear operator T on V has only finitely many eigenvalues. Your proof should contain enough details so that the grader can see clearly why it does not work for $V = C^\infty(\mathbb{R})$.

3. Let V be a (not necessarily finite-dimensional) real inner product space. Let u_1 and u_2 be two distinct vectors in V, and let

$$S = \big\{v \in V \;\big|\; \|v-u_1\| = \|v-u_2\|\big\}.$$

(a) **(10 points)** What is the necessary and sufficient condition on u_1 and u_2 so that S is a subspace of V? Prove your answer.

(b) **(10 points)** When V is finite-dimensional and S is a subspace, what is the relation between $\dim V$ and $\dim S$? Prove your answer.

4. Let $A \in M_{m\times n}(\mathbb{R})$ be a matrix and $b \in \mathbb{R}^m$ be a column vector such that the system of linear equations $Ax = b$ has no solutions for $\|x\| > 2024$, but has at least one solution for $\|x\| \leq 2024$. Give a proof or an explicit counterexample for each of the following statements.

   (a) **(10 points)** The system of linear equations $Ax = b$ has only one solution for $x \in \mathbb{R}^n$.

   (b) **(10 points)** $m \geq n$.

5. **(12 points)** Let u_1, u_2, v_1, v_2 be column vectors in $\mathbb{R}^n$ such that u_1, u_2 are linearly independent and v_1, v_2 are linearly independent. Prove that the following two conditions are equivalent.

   (a) There exists an $n \times n$ orthogonal matrix A such that $Au_1 = v_1$ and $Au_2 = v_2$.

   (b) $\|u_1\| = \|v_1\|$, $\|u_2\| = \|v_2\|$, and $\|u_1 - u_2\| = \|v_1 - v_2\|$.

6. **(12 points)** Does there exist a matrix $A \in M_{3\times 3}(\mathbb{R})$ such that

$$A^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}?$$

Prove your answer.

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(1) (10%) Let

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix}.$$

For which triples $C^t = (c_1, c_2, c_3)$ does the system $AX = C$ have a solution? And find the solutions, if any. Here C^t is the transpose of C.

(2) Let

$$A = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) (5%) Prove that the left-multiplication transformation L_A is a reflection.

(b) (10%) Find the axis in R^2 about which L_A reflects.

(c) (5%) Prove that L_{AB} and L_{BA} are rotations.

(3) (20%) Let $V = P_2(R)$ be the space of all polynomials with coefficients in R, having degree at most 2. Define a linear operator T on V by

$$T(f(x)) = -xf''(x) + f'(x) + 2f(x).$$

Find the minimal polynomial of T.

(4) (20%) Describe all linear operators T on R^2 such that T is diagonalizable and $T^3 - 2T^2 + T = T_0$, where T_0 is the zero transformation.

(5) (15%) Let g be a non-degenerate form on a finite-dimensional space V. Show that each linear operator T has an operator T' such that

$$g(Tv, w) = g(v, T'w)$$

for all v, w.

(6) (a) (5%) If N is a nilpotent 3×3 matrix over C, prove that $A = I + \frac{1}{2}N - \frac{1}{8}N^2$ satisfies $A^2 = I + N$,i.e., A is a square root of $I + N$.

(b) (10%) If N is a nilpotent $n \times n$ matrix over C, find a square root of $I + N$.

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1 (10%) Let V be a finite dimensional vector space over $\mathbb{R}$, and v_1, v_2 be two distinct vectors in V. Show that there is an $\mathbb{R}$-linear transofrmation $f : V \to \mathbb{R}$ for which

$$f(v_1) \neq f(v_2).$$

2 (18%) Let $V := \mathrm{Mat}_{1\times 3}(\mathbb{R})$ and define $f : V \to \mathbb{R}$ by

$$f(x, y, z) := \det \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \\ x & y & z \end{pmatrix}.$$

(i) Show that f is a linear transformation over $\mathbb{R}$.

(ii) Put $W := \mathrm{Ker} f$. Find an $\mathbb{R}$-basis of W.

(iii) Let V/W be the quotient space of V by W, and elements in V/W are denoted by $\bar{v}$ for $v \in V$. Show that the map $\bar{f} : V/W \to \mathbb{R}$ given by

$$\bar{f}\left(\overline{(a, b, c)}\right) := \det \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \\ a & b & c \end{pmatrix}$$

is well-defined and is an isomorphism of vector spaces.

3 (14%) Find the Journal canonical form of the following matrix

$$\begin{pmatrix} 2 & -4 & 2 & 2 \\ -2 & 0 & 1 & 3 \\ -2 & -2 & 3 & 3 \\ -2 & -6 & 3 & 7 \end{pmatrix}.$$

4 (12%) Suppose that V is finite dimensional inner product space over $\mathbb{C}$, and T is a normal linear operator on V such that $T^9 = T^8$. Prove that T is self-adjoint and $T^2 = T$.

5 (20%) Let V be a finite dimensional vector space over $\mathbb{C}$ with two inner products $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle'$. Prove the following.

(i) There exists a unique linear operator T on V so that $\langle x, y \rangle' = \langle T(x), y \rangle$ for all $x, y \in V$.

(ii) The linear operator T in (i) is positive definite with respect to both inner products.

6 (14%) Let $V := \mathbb{R}^n$ and let $W \subset V$ be the vector subspace defined as the set of solutions of $x_1 + \cdots + x_n = 0$. Define $W^0 := \{f \in V^* | f(w) = 0 \text{ for all } w \in W\}$, where

$$V^* := \{f : V \to \mathbb{R} | f \text{ is a linear transformation over } \mathbb{R}\},$$

the dual space of V. Show that W^0 is equal to the set of all f of the form

$$f\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \lambda(a_1 + \cdots + a_n) \text{ for some } \lambda \in \mathbb{R}.$$

7. (12%) Given a nonzero matrix $A \in \mathrm{Mat}_n(\mathbb{R})$ and a nonzero vector $\mathbf{b} \in \mathrm{Mat}_{n\times 1}(\mathbb{R})$, show that if there exists a row vector $C \in \mathrm{Mat}_{1\times n}(\mathbb{R})$ for which $CA = 0$ and $C\mathbf{b} = 1$, then $A\mathbf{x} = \mathbf{b}$ has no solution.

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Notation: $\mathbb{R}$ denotes the field of real numbers; $\mathbb{C}$ denotes the field of complex numbers. F denotes an arbitrary field; $M_{m\times n}(F)$ denotes the set of all $m \times n$ matrices with entries in F. If T is a linear transformation, $R(T)$ denotes the range of T, and $N(T)$ denotes the null space of T. If $A \in M_{m\times n}(F)$, A^t denotes the transpose of A, and L_A denotes the linear transformation from F^n to F^m that sends each vector $v \in F^n$ to $Av \in F^m$.

1. **(12 points)** Let V and W be F-vector spaces, and let $T\colon V \to W$ be a linear transformation. Prove that $\dim R(T)+\dim N(T) = \dim V$ if V is finite-dimensional.

2. **(10 points)** Find a matrix $A \in M_{3\times 3}(\mathbb{R})$ such that

$$R(L_A) = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; 3a-2b+4c=0 \right\} \text{ and } N(L_A) = \left\{ \begin{pmatrix} 2t \\ 3t \\ -t \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; t \in \mathbb{R} \right\}.$$

   You need to show that the matrix you find has the required properties.

3. **(12 points)** Let $A \in M_{m\times n}(F)$. Show that the system of linear equations $Ax = b$ has a solution for all $b \in F^m$ if and only if the system of linear equations $A^t x = 0$ has no nonzero solutions.

4. **(12 points)** Let $A \in M_{m\times n}(F)$ and $B \in M_{n\times m}(F)$. Show that if $\operatorname{rank}(AB) = m$, then $\operatorname{rank}(BA) = m$.

5. Let $T\colon V \to V$ be a linear operator on a finite-dimensional F-vector space V.

   (a) **(6 points)** State the definition of eigenvectors of T.

   (b) **(6 points)** Give an explicit example of T that has no eigenvectors.

   (c) **(8 points)** Prove that T has an eigenvector if $F = \mathbb{C}$.

6. **(10 points)** Let $A \in M_{n\times n}(F)$. Show that if $Q \in M_{n\times n}(F)$ is an invertible matrix such that $Q^{-1}AQ$ is diagonal, then each column vector of Q is an eigenvector of L_A.

7. **(12 points)** Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Show that if T preserves the Euclidean distance between any two points, that is, $\|T(u) - T(v)\| = \|u - v\|$ for any $u, v \in \mathbb{R}^n$, then the matrix representation of T relative to the standard basis is an orthogonal matrix.

8. **(12 points)** Let $A \in M_{n\times n}(\mathbb{R})$ be a real symmetric matrix. Show that there exists a real symmetric matrix B such that $B^3 = A$.

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In the following, $\mathbb{F}$ denotes a field with infinitely many elements.

1. (15%) Express

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 4 & 5 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

as a product of elementary matrices.

2. (10%) Show that eigenvectors from different eigenspaces of a matrix are linearly independent.

3. Let

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \end{pmatrix}$$

Let $\beta := \{(1,1),(1,2)\}$ be an ordered basis for $\mathbb{R}^2$ and $\gamma := \{(0,0,1),(0,1,1),(1,1,1)\}$ be an ordered basis for $\mathbb{R}^3$.

   (a) (10%) Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that the matrix representation

$$[T]_\beta^\gamma = A$$

   (b) (5%) Find rank(T).

4. (10%) Prove the following theorem: For $A \in M_{n\times n}(\mathbb{F}), \mathrm{b} \in \mathbb{F}^n$, if the system $A\mathbb{X} = \mathrm{b}$ has exactly one solution, then A is invertible.

5. (15%) Let $L : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be given by

$$L[p(t)] = p(t) + p(1)(t-3) - 2p'(1)(2t-1)$$

Find the eigenvalues and corresponding eigenvectors of L where $P_3(\mathbb{R})$ is the vector space of real polynomials of degree ≤ 3.

6. (a) (10%) Show that if $A \in M_{m\times n}(\mathbb{F})$ is of rank m, there exists $B \in M_{n\times m}(\mathbb{F})$ such that

$$BA = I_n$$

   (b) (5%) What is the rank of B?

7. (10%) Give $A \in M_{2\times 2}(\mathbb{Q})$ which is not diagonalizable over $\mathbb{Q}$, but A is diagonalizable over $\mathbb{R}$.

8. (10%) Prove or give a counterexample: any $A \in M_{n\times n}(\mathbb{C})$ is similar to A^t.