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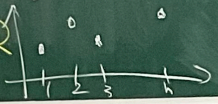
§11.5 Alternating Series

Recall: seq. $a_1, a_2, \cdots a_n \cdots$

$\Longleftrightarrow$ fn $f(n) = a_n$, $n \in \mathbb{N}$

$f: \mathbb{N} \to \mathbb{R}$



$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n \left(= \sum a_n\right), \quad b_n \geq 0 \quad \forall n$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + \cdots = \begin{matrix} \text{Div.} \\ \tfrac{1}{2} \end{matrix} \ ?$$

partial sum $S_n = a_1 + a_2 + \cdots + a_n$

$$S = \sum_{n=1}^{\infty} a_n \equiv \lim_{n \to \infty} S_n$$

$$S \sim a_1 + a_2 + a_3 + a_4 + a_5 + \cdots\cdots$$

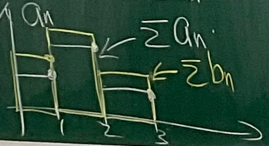
Recall:

Integral Test: $f(n) = a_n \geq 0$. $\int_1^{\infty} f(x)\,dx$ conv. $\Rightarrow \sum a_n$ conv.

Comparison test $\sum a_n, \sum b_n$. $0 \leq a_n \leq b_n$.

$\Rightarrow \sum a_n = \text{div.} \Rightarrow \underline{\sum b_n} = \text{div.}$

$\sum b_n = \text{conv.} \Rightarrow \sum a_n = \text{conv.}$



Limit comparison Test

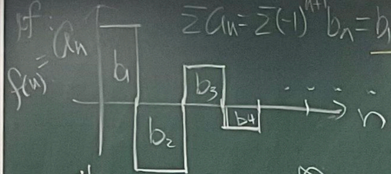
$0 \leq a_n, b_n$ and $\lim \frac{a_n}{b_n} = L > 0$

$\Rightarrow \sum a_n \text{ conv.} \Longleftrightarrow \sum b_n = \text{conv.}$

Alternating Series Test

If $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$, $b_n > 0$, $b_n \searrow$

$\Rightarrow \sum (-1)^{n+1} b_n$ conv.



$\sum a_n = \sum (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$

$\sum (-1)^{n+1} b_n = \sum_{k=1}^{\infty} b_{2k-1} - \sum_{k=1}^{\infty} b_{2k}$ $\quad \{b_{2k-1}\} \searrow$ $\quad \{b_{2k}\} \searrow$

$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} (-1)^{k+1} b_k$

$(-1)^{2\ell-1}$

$= \sum_{\ell=1} (-1)^{2\ell+1} b_{2\ell} + \sum_{\ell=1} (-1)^{2\ell} b_{2\ell-1}$

$S_{2n-1} = \sum_{k=1}^{2n-1} (-1)^{k+1} b_k$ $\quad k = 2n,\ 2n+1$

$S_{2n+1} = \sum_{k=1}^{2n+1} (-1)^{k+1} b_k = \sum_{k=1}^{2n-1} (-1)^{k+1} b_k + (-1)^{2n+1} b_{2n} + (-1)^{2n+2} b_{2n+1}$

$= S_{2n-1} \underbrace{- b_{2n} + b_{2n+1}}_{(*) < 0} < S_{2n-1}$

$\overset{(*)}{\Longrightarrow} \{S_{2n-1}\}_{n=1}^{\infty} \searrow$

$S_{2n+2} = S_{2n} + \underbrace{b_{2n+1} - b_{2n+2}}_{(*) > 0} > S_{2n}, \quad \{S_{2n}\} \nearrow$

$S_2 < S_4 < S_6 < \cdots < S_{2n} \cdots \cdots S_{2n-1} < \cdots < S_5 < S_3 < S_1$

$S_{2n+1} = S_{2n} + \underbrace{b_{2n+1}}_{>0} \Rightarrow S_{2n+1} > S_{2n}$

$S_{2n+2} = S_{2n+1} \underbrace{- b_{2n+2}}_{<0} \Rightarrow S_{2n+1} > S_{2n+2} > S_{2n}$

$S_6 < S_7 = \underbrace{b_1 - b_2 + b_3 - b_4 + b_5 - b_6}_{"S_6"} + b_7 = S_6 + b_7$

$S_6 < S_9 = S_6 + \underbrace{b_7 - b_8 + b_9}_{>0}$

$S_{11} = S_6 + \underbrace{b_7 - b_8}_{>0} + \underbrace{b_9 - b_{10}}_{>0} + b_{11}$

$\Rightarrow S_{11} > S_6$

$\cdots$

$S_{2n-1} > S_6 \quad \forall n \in \mathbb{N}$

$\Rightarrow S_{2n-1} > S_8 \quad \forall n \in \mathbb{N}$

$\vdots$

by Monotonic Seq Thm $\{S_{2n}\} \xrightarrow{(**)} S_e$

$\{S_{2n+1}\} \xrightarrow{(*)} S_o$

$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} b_{2n+1}$ = 0

$\Rightarrow S_o = S_e \equiv S$

$\therefore \lim_{n\to\infty} S_n = S$ ※

Similarly $|S - S_{2n+1}| < b_{2n}$

Alternating Series Estimation Thm

If $S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$, $b_{n+1} \le b_n$. $\lim_{n\to\infty} b_n = 0$

(Alternating Series).

$\Rightarrow |R_n| \equiv |S - S_n| \le b_{n+1}$

error approximation

pf:

$|S - S_{2n}| = |b_{2n+1} \underbrace{- b_{2n+2} + b_{2n+3}}_{<0} \underbrace{- b_{2n+4} + b_{2n+5}}_{<0} - \cdots|$

$< |b_{2n+1}| = b_{2n+1}$ ※

§11.6 Absolutely Convergence & Ratio Test & Root Test

Consider $\sum |a_n| = |a_1| + |a_2| + \cdots$

Def: $\sum a_n$ = conv. if $\lim S_n$ exists.

Def $\sum a_n$ = absolutely convergence A.C. if $\sum |a_n|$ conv.

<Ex> $\sum (-1)^{k+1} \frac{1}{k^2} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

$\sum \left|(-1)^{k+1} \frac{1}{k^2}\right| = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

conv. (p-series)

$\therefore \sum (-1)^{k+1} \frac{1}{k^2}$ A.C.

Note: A.C. $\Rightarrow$ conv.

<EX> $\sum (-1)^{k-1}\frac{1}{k} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\cdots$

$\sum \left|(-1)^{k-1}\frac{1}{k}\right| = \sum \frac{1}{k} = \text{div.}$

By Alternating Series Test

$\frac{1}{k}>0$, $\{\frac{1}{k}\}\searrow 0$

$\Rightarrow \sum(-1)^{k-1}\frac{1}{k}$ conv.

Def: $\sum(-1)^{k+1}b_k$ = conditional convergence

if $\sum(-1)^{k+1}b_k$ = conv. but $\sum b_k$ = div.

conv. $\sum a_n$: A.C. | C.C.

Thm $\sum a_n = \text{A.C.} \Rightarrow \sum a_n = \text{conv.}$

pf: $\because$ $0 \le \sum_{n=1}^{\infty} a_n+|a_n| \le \sum_{n=1}^{\infty} 2|a_n|$ conv.

$\Rightarrow$ by Comparison Test $\sum a_n+|a_n|$ conv.

$a_n = \underline{a_n+|a_n|} - \underline{|a_n|}$.

$\because \underline{\sum a_n+|a_n|}$ conv. $\underline{\sum|a_n|}$ conv.

$\Rightarrow \sum a_n = \sum(a_n+|a_n|) - \sum|a_n|$ conv. ※

Ratio Test: Set $\lim\left|\frac{a_{n+1}}{a_n}\right| = L$

(i) If $L<1 \Rightarrow \sum a_n = \text{A.C.}$

(ii) If $L>1 \Rightarrow \sum a_n = \text{div.}$

(iii) $L=1$, $\Rightarrow$ inconclusive.

pf: (i) $\because \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < r < 1$

$\exists N$ s.t. $\left|\frac{a_{n+1}}{a_n}\right| < r \quad \forall n>N$

$\Rightarrow |a_{n+1}| < r|a_n| \quad \forall n \ge N$

$\Rightarrow |a_{N+1}| < r|a_N|$

$|a_{N+2}| < r|a_{N+1}| < r^2|a_N|$ ...... $|a_n| < r^{n-N}|a_N|$

$$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{N-1}|a_n| + \sum_{n=N}^{\infty}|a_n|$$

$$< \sum_{n=1}^{N-1}|a_n| + \underbrace{\sum_{n=N}^{\infty} r^{n-N}}_{\substack{\text{Geometric series} \\ \text{conv.}}} |a_N|$$

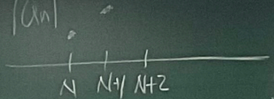
(ii) $\lim \left|\frac{a_{n+1}}{a_n}\right| = L > 1$

$\exists N$ s.t. $\left|\frac{a_{n+1}}{a_n}\right| > 1 \ \forall n \geq N$

$\Rightarrow |a_{n+1}| > |a_n| > \cdots > |a_N|$

$\Rightarrow \lim a_n \neq 0$

$\Rightarrow \sum a_n = \text{div.}$



$|a_n| > \underbrace{|a_N|}_{>0} \ \forall n > N$

(Recall $\sum a_n = \text{conv.} \Rightarrow \lim a_n = 0$)

(iii) $\sum \frac{1}{n}$ div. and $\lim \left|\frac{a_{n+1}}{a_n}\right| = 1$

$\sum \frac{1}{n^2}$ conv. and $\lim \left|\frac{a_{n+1}}{a_n}\right| = 1$

Root Test: Set $\lim_{n\to\infty}\sqrt[n]{|a_n|} \equiv L$.

(i) $L < 1 \Rightarrow \sum a_n$ A.C.

(ii) $L > 1 \Rightarrow \sum a_n = \text{div.}$

(iii) $L = 1 \Rightarrow$ inconclusive.

pf: (iii)

$\sum \frac{1}{n} = \text{div.}$ $\sqrt[n]{\frac{1}{n}} = \frac{1}{\sqrt[n]{n}} \to 1$

$\sum \frac{1}{n^2} = \text{conv.}$ $\sqrt[n]{\frac{1}{n^2}} = \frac{1}{\sqrt[n]{n^2}} \to 1$

<EX>

By

$\Rightarrow$

Def:

a

$\Rightarrow$

Ratio

(i)

(ii)