2.6 HW 3

Question 50

Let $\mathbb{C}_\pi \triangleq \{z \in \mathbb{C} : z \notin \mathbb{R}_0^-\}$. Prove that $\mathbb{C}_\pi$ is a domain. Define $r : \mathbb{C}_\pi \to \mathbb{C}$ by $(r(z))^2 = z$ and $\operatorname{Re} r(z) > 0$. Prove that r is continuous on $\mathbb{C}_\pi$ and $r'(z) = \frac{1}{2r(z)}$.

Proof. It is clear that $\mathbb{C}_\pi$ is non-empty and open. To see $\mathbb{C}_\pi$ is path-connected, observe that for all point $x + iy \in \mathbb{C}_\pi$, we can join $x + iy$ with 1 linearly by defining $\gamma : [0, 1] \to \mathbb{C}_\pi$ by

$$\gamma(t) \triangleq 1 + t(x + iy - 1)$$

We have proved $\mathbb{C}_\pi$ is a domain. Note that

$$\mathbb{C}_\pi = \{a + bi \in \mathbb{C} : a > 0 \text{ and } b \in (-\pi,)\}$$

and the exact definition of $r : \mathbb{C}_\pi \to \mathbb{C}$ is

$$r(e^{a+bi}) \triangleq e^{\frac{a+bi}{2}}$$

This implies r is continuous. Compute

$$1 = \frac{d}{dz} z = \frac{d}{dz}(r(z))^2 = 2r(z)r'(z)$$

This give us $r'(z) = \frac{1}{2r(z)}$. ∎

Theorem 2.6.1. (Conjugated Polynomial)

$$\overline{z^n} \text{ is holomorphic at 0 for all } n > 1$$

Proof. If we write

$$u + iv = \overline{(x + iy)^n}$$

Because $n > 1$, we see from binomial Theorem that $u \in \mathbb{R}[x, y]$ is a polynomial with two indeterminate x, y whose terms all have degree greater than 1. Thus, both $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ are polynomial with two indeterminate x, y whose terms all have degree greater than 0. This implies

$$\frac{\partial u}{\partial x}(0) = \frac{\partial u}{\partial y}(0) = 0$$

Similar arguments shows that

$$\frac{\partial v}{\partial x}(0) = \frac{\partial v}{\partial y}(0) = 0$$

Because $u, v \in \mathbb{R}[x, y]$ are real multivariate polynomial, they are obviously real differentiable. Finally, we can use Cauchy-Riemann criteria to conclude that $\overline{z^n} = u + iv$ is holomorphic at 0. ∎

Question 51

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial. Prove that the function $g : \mathbb{C} \to \mathbb{C}$ defined by

$$g(z) \triangleq \overline{f(\overline{z})}$$

is holomorphic everywhere, but the function $h : \mathbb{C} \to \mathbb{C}$ defined by

$$h(z) \triangleq \overline{f(z)}$$

is holomorphic at 0 if and only if $f'(0) = 0$.

Proof. We can write

$$f(z) \triangleq \sum_{n=0}^{N} c_n z^n$$

and compute

$$g(z) = \sum_{n=0}^{N} \overline{c_n} z^n$$

We have shown $g : \mathbb{C} \to \mathbb{C}$ is a polynomial. It follows that g is holomorphic on $\mathbb{C}$. Compute

$$h(z) = \sum_{n=2}^{N} \overline{c_n z^n} + \overline{c_1 z} + \overline{c_0} \text{ and } f'(0) = c_1$$

Theorem 2.6.1 shows that

$$\sum_{n=2}^{N} \overline{c_n z^n} + \overline{c_0} \text{ is holomorphic at } 0$$

Note that $\overline{z}$ is not holomorphic at 0 since if we write $u + iv = \overline{z}$, then

$$\frac{\partial u}{\partial x} = 1 \neq -1 = \frac{\partial v}{\partial y}$$

The claim "h is holomorphic at 0 if and only if $f'(0) = 0$" then follows. ■

Question 52

Define

(a) $u, v : \mathbb{R}^2 \to \mathbb{R}$ by

$$u(x, y) = x^3 - 3xy^2 \text{ and } v(x, y) = 3x^2y - y^3$$

(b) $u, v : \{(x, y) \in \mathbb{R}^2 : x > 0\} \to \mathbb{R}$ by

$$u(x, y) = \frac{\ln(x^2 + y^2)}{2} \text{ and } v(x, y) = \sin^{-1}(\frac{y}{\sqrt{x^2 + y^2}})$$

Verify the Cauchy-Riemann Equation for these functions and state a complex function shows real and imaginary parts are u, v.

Proof. For (a), compute

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 3x^2 - 2y^2 \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -6xy$$

and observe

$$u + iv = (x + iy)^3$$

For (b), compute

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{y}{x^2 + y^2}$$

and observe

$$x + iy = e^{2u+iv}$$

which implies the function map z to $\log(z) - \frac{\ln|z|}{2}$. ■

Question 53

Let $f(z) = \sqrt{|xy|}$. Show that f satisfy the Cauchy-Riemann equation at 0, yet $f'(0)$ does not exists. Explain why.

Proof. Observe that

$$f(x) = f(iy) = 0 \text{ for all } x, y \in \mathbb{R}$$

We then have Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$$

Then if f is holomorphic at 0, we should have $f'(0) = 0$, but we can compute

$$\lim_{t\to 0; t\in\mathbb{R}^+} \frac{f(t+ti) - f(0)}{t+ti} = \lim_{t\to 0; t\in\mathbb{R}^+} \frac{t}{t+ti} = \frac{1}{1+i} \neq 0$$

which implies f is not holomorphic at 0. The reason that f satisfy the Cauchy-Riemann equation yet isn't holomorphic is that its real part when treated as a function from $\mathbb{R}^2$ to $\mathbb{R}$ is not differentiable at 0, as we have shown. (Note that $f = \operatorname{Re} f$) ∎

Question 54

Suppose that $f(z) = \sum a_n z^n$ is entire and satisfy

$$f' = f \text{ and } f(0) = 1$$

Find a_n. Show that

$$f(a+b) = f(a)f(b) \text{ for all } a, b \in \mathbb{C}$$

and compute $f(1)$ to five decimal points.

Proof. $f(0) = 1$ implies $a_0 = 1$. $f' = f$ implies $(n+1)a_{n+1} = a_n$, which give us

$$a_n = \frac{1}{n!} \text{ for all } n \geq 0$$

Fix $a, b \in \mathbb{C}$. Define $g : \mathbb{C} \to \mathbb{C}$ by

$$g(z) \triangleq f(a+b-z)f(z)$$

Compute

$$\begin{aligned} g'(z) &= -f'(a+b-z)f(z) + f(a+b-z)f'(z) \\ &= -f(a+b-z)f(z) + f(a+b-z)f(z) = 0 \end{aligned}$$

This implies g is constant. We then can deduce

$$f(a)f(b) = g(a) = g(0) = f(a+b)f(0) = f(a+b)$$

We know

$$f(1) = \sum_{n=0} \frac{1}{n!} = 2.7182818.....$$

∎