11

More on the group algebra

We now go further into the structure of the group algebra $\mathbb{C}G$ of a finite group G. As in Chapter 10, we write

$$\mathbb{C}G = U_1 \oplus \ldots \oplus U_r,$$

a direct sum of irreducible $\mathbb{C}G$-modules U_i. In Theorem 10.5 we proved that *every* irreducible $\mathbb{C}G$-module U is isomorphic to one of the U_i. The question arises: how many of the U_i are isomorphic to U? There is an elegant and significant answer to this question: the number is precisely dim U (see Theorem 11.9).

Our proof of Theorem 11.9 is based on a study of the vector space of $\mathbb{C}G$-homomorphisms from one $\mathbb{C}G$-module to another.

The space of $\mathbb{C}$G-homomorphisms

11.1 Definition

Let V and W be $\mathbb{C}G$-modules. We write $\mathrm{Hom}_{\mathbb{C}G}(V, W)$ for the set of all $\mathbb{C}G$-homomorphisms from V to W.

Define addition and scalar multiplication on $\mathrm{Hom}_{\mathbb{C}G}(V, W)$ as follows: for $\vartheta, \phi \in \mathrm{Hom}_{\mathbb{C}G}(V, W)$ and $\lambda \in \mathbb{C}$, define $\vartheta + \phi$ and $\lambda\vartheta$ by

$$v(\vartheta + \phi) = v\vartheta + v\phi,$$

$$v(\lambda\vartheta) = \lambda(v\vartheta)$$

for all $v \in V$. Then $\vartheta + \phi, \lambda\vartheta \in \mathrm{Hom}_{\mathbb{C}G}(V, W)$. With these definitions, it is easily checked that $\mathrm{Hom}_{\mathbb{C}G}(V, W)$ is a vector space over $\mathbb{C}$.

We begin our study of the vector space $\mathrm{Hom}_{\mathbb{C}G}(V, W)$ with an easy consequence of Schur's Lemma.

11.2 Proposition

Suppose that V and W are irreducible $\mathbb{C}G$-modules. Then

$$\dim(\operatorname{Hom}_{\mathbb{C}G}(V, W)) = \begin{cases} 1, & \textit{if } V \cong W, \\ 0, & \textit{if } V \not\cong W. \end{cases}$$

Proof If $V \not\cong W$ then this is immediate from Schur's Lemma 9.1(1).

Now suppose that $V \cong W$, and let $\vartheta: V \to W$ be a $\mathbb{C}G$-isomorphism. If $\phi \in \operatorname{Hom}_{\mathbb{C}G}(V, W)$, then $\phi\vartheta^{-1}$ is a $\mathbb{C}G$-isomorphism from V to V, so by Schur's Lemma 9.1(2), there exists $\lambda \in \mathbb{C}$ such that

$$\phi\vartheta^{-1} = \lambda 1_V.$$

Then $\phi = \lambda\vartheta$, and so $\operatorname{Hom}_{\mathbb{C}G}(V, W) = \{\lambda\vartheta: \lambda \in \mathbb{C}\}$, a 1-dimensional space.

■

For the next result, recall the definition of a composition factor of a $\mathbb{C}G$-module from 10.4.

11.3 Proposition

Let V and W be $\mathbb{C}G$-modules, and suppose that $\operatorname{Hom}_{\mathbb{C}G}(V, W) \neq \{0\}$. Then V and W have a common composition factor.

Proof Let ϑ be a non-zero element of $\operatorname{Hom}_{\mathbb{C}G}(V, W)$. Then $V = \operatorname{Ker}\vartheta \oplus U$ for some non-zero $\mathbb{C}G$-module U, by Maschke's Theorem. Let X be an irreducible $\mathbb{C}G$-submodule of U. Since $X\vartheta \neq \{0\}$, Schur's Lemma 9.1(1) implies that $X\vartheta \cong X$. Therefore X is a common composition factor of V and W.

■

The next few results show how to calculate the dimension of $\operatorname{Hom}_{\mathbb{C}G}(V, W)$ in general. The key step is the following proposition.

11.4 Proposition

Let V, V_1, V_2 and W, W_1, W_2 be $\mathbb{C}G$-modules. Then

(1) $\dim(\mathrm{Hom}_{\mathbb{C}G}(V, W_1 \oplus W_2)) =$

$\dim(\mathrm{Hom}_{\mathbb{C}G}(V, W_1)) + \dim(\mathrm{Hom}_{\mathbb{C}G}(V, W_2))$,

(2) $\dim(\mathrm{Hom}_{\mathbb{C}G}(V_1 \oplus V_2, W)) =$

$\dim(\mathrm{Hom}_{\mathbb{C}G}(V_1, W)) + \dim(\mathrm{Hom}_{\mathbb{C}G}(V_2, W))$.

Proof (1) Define the functions $\pi_1: W_1 \oplus W_2 \to W_1$ and $\pi_2: W_1 \oplus W_2 \to W_2$ by

$$(w_1 + w_2)\pi_1 = w_1, \quad (w_1 + w_2)\pi_2 = w_2$$

for all $w_1 \in W_1$, $w_2 \in W_2$. By Proposition 7.11, π_1 and π_2 are $\mathbb{C}G$-homomorphisms. If $\vartheta \in \mathrm{Hom}_{\mathbb{C}G}(V, W_1 \oplus W_2)$, then $\vartheta\pi_1 \in \mathrm{Hom}_{\mathbb{C}G}(V, W_1)$ and $\vartheta\pi_2 \in \mathrm{Hom}_{\mathbb{C}G}(V, W_2)$ (see Exercise 7.1).

We now define a function f from $\mathrm{Hom}_{\mathbb{C}G}(V, W_1 \oplus W_2)$ to the (external) direct sum of $\mathrm{Hom}_{\mathbb{C}G}(V, W_1)$ and $\mathrm{Hom}_{\mathbb{C}G}(V, W_2)$ by

$$f: \vartheta \to (\vartheta\pi_1, \vartheta\pi_2) \quad (\vartheta \in \mathrm{Hom}_{\mathbb{C}G}(V, W_1 \oplus W_2)).$$

Clearly f is a linear transformation. We show that f is invertible.

Given $\phi_i \in \mathrm{Hom}_{\mathbb{C}G}(V, W_i)$ $(i = 1, 2)$, the function

$$\phi: v \to v\phi_1 + v\phi_2 \quad (v \in V)$$

lies in $\mathrm{Hom}_{\mathbb{C}G}(V, W_1 \oplus W_2)$, and the image of ϕ under f is (ϕ_1, ϕ_2). Hence f is surjective.

If $\vartheta \in \mathrm{Ker}\, f$, then $v\vartheta\pi_1 = 0$ and $v\vartheta\pi_2 = 0$ for all $v \in V$, so $v\vartheta = v\vartheta(\pi_1 + \pi_2) = 0$. Therefore $\vartheta = 0$, so $\mathrm{Ker}\, f = \{0\}$ and f is injective.

We have established that f is an invertible linear transformation from $\mathrm{Hom}_{\mathbb{C}G}(V, W_1 \oplus W_2)$ to $\mathrm{Hom}_{\mathbb{C}G}(V, W_1) \oplus \mathrm{Hom}_{\mathbb{C}G}(V, W_2)$. Consequently these two vector spaces have equal dimensions, and (1) follows.

(2) For $\vartheta \in \mathrm{Hom}_{\mathbb{C}G}(V_1 \oplus V_2, W)$, define $\vartheta_{V_i}: V_i \to W$ ($i = 1, 2$) to be the restriction of ϑ to V_i; that is, ϑ_{V_i} is the function

$$v_i\vartheta_{V_i} = v_i\vartheta \quad (v_i \in V_i).$$

Then $\vartheta_{V_i} \in \mathrm{Hom}_{\mathbb{C}G}(V_i, W)$ for $i = 1, 2$.

Now let h be the function from $\mathrm{Hom}_{\mathbb{C}G}(V_1 \oplus V_2, W)$ to $\mathrm{Hom}_{\mathbb{C}G}(V_1, W) \oplus \mathrm{Hom}_{\mathbb{C}G}(V_2, W)$ which is given by

$$h\colon \vartheta \to (\vartheta_{V_1}, \vartheta_{V_2}) \quad (\vartheta \in \mathrm{Hom}_{\mathbb{C}G}(V_1 \oplus V_2, W)).$$

Clearly h is an injective linear transformation. Given $\phi_i \in \mathrm{Hom}_{\mathbb{C}G}(V_i, W)$ ($i = 1, 2$), the function

$$\phi\colon v_1 + v_2 \to v_1\phi_1 + v_2\phi_2 \quad (v_i \in V_i \text{ for } i = 1, 2)$$

lies in $\mathrm{Hom}_{\mathbb{C}G}(V_1 \oplus V_2, W)$ and has image (ϕ_1, ϕ_2) under h. Hence h is surjective. We have shown that h is an invertible linear transformation, and (2) follows.

■

Now suppose that we have $\mathbb{C}G$-modules V, W, V_i, W_j ($1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$). By an obvious induction using Proposition 11.4, we have

(11.5) (1) $\dim(\mathrm{Hom}_{\mathbb{C}G}(V, W_1 \oplus \ldots \oplus W_s))$

$$= \sum_{j=1}^{s} \dim(\mathrm{Hom}_{\mathbb{C}G}(V, W_j)),$$

(2) $\dim(\mathrm{Hom}_{\mathbb{C}G}(V_1 \oplus \cdots \oplus V_r, W))$

$$= \sum_{i=1}^{r} \dim(\mathrm{Hom}_{\mathbb{C}G}(V_i, W)).$$

These in turn imply

(3) $\dim(\mathrm{Hom}_{\mathbb{C}G}(V_1 \oplus \ldots \oplus V_r, W_1 \oplus \ldots \oplus W_s))$

$$= \sum_{i=1}^{r} \sum_{j=1}^{s} \dim(\mathrm{Hom}_{\mathbb{C}G}(V_i\ W_j)).$$

By applying (3) when all V_i and W_j are irreducible, and using Proposition 11.2, we can find $\dim(\mathrm{Hom}_{\mathbb{C}G}(V, W))$ in general. In the following corollary we single out the case where one of the $\mathbb{C}G$-modules is irreducible.

11.6 Corollary
Let V be a $\mathbb{C}G$-module with

$$V = U_1 \oplus \ldots \oplus U_s,$$

where each U_i is an irreducible $\mathbb{C}G$-module. Let W be any irreducible $\mathbb{C}G$-module. Then the dimensions of $\mathrm{Hom}_{\mathbb{C}G}(V, W)$ and $\mathrm{Hom}_{\mathbb{C}G}(W, V)$ are both equal to the number of $\mathbb{C}G$-modules U_i such that $U_i \cong W$.

Proof By (11.5),

$$\dim(\mathrm{Hom}_{\mathbb{C}G}(V, W)) = \sum_{i=1}^{s} \dim(\mathrm{Hom}_{\mathbb{C}G}(U_i, W)), \text{ and}$$

$$\dim(\mathrm{Hom}_{\mathbb{C}G}(W, V)) = \sum_{i=1}^{s} \dim(\mathrm{Hom}_{\mathbb{C}G}(W, U_i)).$$

And by Proposition 11.2,

$$\dim(\mathrm{Hom}_{\mathbb{C}G}(U_i, W)) = \dim(\mathrm{Hom}_{\mathbb{C}G}(W, U_i)) = \begin{cases} 1, & \textit{if } U_i \cong W, \\ 0, & \textit{if } U_i \not\cong W. \end{cases}$$

The result follows.

■

11.7 Example

For $G = D_6$, we saw in Example 10.8(2) that

$$\mathbb{C}G = U_1 \oplus U_2 \oplus U_3 \oplus U_4,$$

a direct sum of irreducible $\mathbb{C}G$-submodules, with $U_3 \cong U_4$ but U_3 not isomorphic to U_1 or U_2. Thus by Corollary 11.6, we have

$$\dim(\mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}G, U_3)) = \dim(\mathrm{Hom}_{\mathbb{C}G}(U_3, \mathbb{C}G)) = 2.$$

You are asked in Exercise 11.5 to find bases for these two vector spaces of $\mathbb{C}G$-homomorphisms.

The next proposition investigates the space of $\mathbb{C}G$-homomorphisms from the regular $\mathbb{C}G$-module to any other $\mathbb{C}G$-module. When combined with Corollary 11.6, it will give the main result of this chapter.

11.8 Proposition

If U is a $\mathbb{C}G$-module, then

$$\dim(\mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}G, U)) = \dim U.$$

Proof Let $d = \dim U$. Choose a basis $u_1, \ldots, u_d$ of U. For $1 \leqslant i \leqslant d$, define ϕ_i: $\mathbb{C}G \to U$ by

$$r\phi_i = u_i r \quad (r \in \mathbb{C}G).$$

Then $\phi_i \in \mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}G, U)$ since for all $r, s \in \mathbb{C}G$,

$$(rs)\phi_i = u_i(rs) = (u_i r)s = (r\phi_i)s.$$

We shall prove that $\phi_1, \ldots, \phi_d$ is a basis of $\mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}G, U)$.

Suppose that $\phi \in \mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}G, U)$. Then

$$1\phi = \lambda_1 u_1 + \ldots + \lambda_d u_d$$

for some $\lambda_i \in \mathbb{C}$. Since ϕ is a $\mathbb{C}G$-homomorphism, for all $r \in \mathbb{C}G$ we have

$$\begin{aligned} r\phi = (1r)\phi = (1\phi)r & \\ &= \lambda_1 u_1 r + \ldots + \lambda_d u_d r \\ &= r(\lambda_1 \phi_1 + \ldots + \lambda_d \phi_d). \end{aligned}$$

Hence $\phi = \lambda_1\phi_1 + \ldots + \lambda_d\phi_d$. Therefore $\phi_1, \ldots, \phi_d$ span $\mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}G, U)$.

Now assume that

$$\lambda_1\phi_1 + \ldots + \lambda_d\phi_d = 0 \quad (\lambda_i \in \mathbb{C}).$$

Evaluating both sides at the identity 1, we have

$$\begin{aligned} 0 &= 1(\lambda_1\phi_1 + \ldots + \lambda_d\phi_d) \\ &= \lambda_1 u_1 + \ldots + \lambda_d u_d, \end{aligned}$$

which forces $\lambda_i = 0$ for all i. Hence $\phi_1, \ldots, \phi_d$ is a basis of $\mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}G, U)$, which therefore has dimension d.

■

We now come to the main theorem of the chapter, which tells us how often each irreducible $\mathbb{C}G$-module occurs in the regular $\mathbb{C}G$-module.

11.9 Theorem
Suppose that

$$\mathbb{C}G = U_1 \oplus \ldots \oplus U_r,$$

a direct sum of irreducible $\mathbb{C}G$-submodules. If U is any irreducible $\mathbb{C}G$-module, then the number of $\mathbb{C}G$-modules U_i with $U_i \cong U$ is equal to dim U.

Proof By Proposition 11.8,

$$\dim U = \dim(\mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}G, U)),$$

and by Corollary 11.6, this is equal to the number of U_i with $U_i \cong U$.

■

11.10 Example

Recall again from Example 10.8(2) that if $G = D_6$ then

$$\mathbb{C}G = U_1 \oplus U_2 \oplus U_3 \oplus U_4,$$

where U_1, U_2 are non-isomorphic 1-dimensional $\mathbb{C}G$-modules, and U_3, U_4 are isomorphic irreducible 2-dimensional $\mathbb{C}G$-modules. This illustrates Theorem 11.9:

$$U_1 \text{ occurs once, } \dim U_1 = 1;$$

$$U_2 \text{ occurs once, } \dim U_2 = 1;$$

$$U_3 \text{ occurs twice, } \dim U_3 = 2.$$

We conclude the chapter with a significant consequence of Theorem 11.9 concerning the dimensions of all irreducible $\mathbb{C}G$-modules.

11.11 Definition

We say that the irreducible $\mathbb{C}G$-modules $V_1, \cdots, V_k$ form a *complete set of non-isomorphic irreducible $\mathbb{C}G$-modules* if every irreducible $\mathbb{C}G$-module is isomorphic to some V_i, and no two of $V_1, \cdots, V_k$ are isomorphic. (By Corollary 10.7, for any finite group G there exists a complete set of non-isomorphic irreducible $\mathbb{C}G$-modules.)

11.12 Theorem

Let $V_1, \cdots, V_k$ form a complete set of non-isomorphic irreducible $\mathbb{C}G$-modules. Then

$$\sum_{i=1}^{k} (\dim V_i)^2 = |G|.$$

Proof Let $\mathbb{C}G = U_1 \oplus \ldots \oplus U_r$, a direct sum of irreducible $\mathbb{C}G$-submodules. For $1 \leqslant i \leqslant k$, write $d_i = \dim V_i$. By Theorem 11.9, for each i, the number of $\mathbb{C}G$-modules U_j with $U_j \cong V_i$ is equal to d_i. Therefore

$$\begin{aligned}\dim \mathbb{C}G &= \dim U_1 + \ldots + \dim U_r \\ &= \sum_{i=1}^{k} d_i(\dim V_i) = \sum_{i=1}^{k} d_i^2.\end{aligned}$$

As $\dim \mathbb{C}G = |G|$, the result follows.

■

11.13 Example

Let G be a group of order 8, and let $d_1, \ldots, d_k$ be the dimensions of all the irreducible $\mathbb{C}G$-modules. By Theorem 11.12,

$$\sum_{i=1}^{k} d_i^2 = 8.$$

Observe that the trivial $\mathbb{C}G$-module is irreducible of dimension 1, and so $d_i = 1$ for some i. Hence the possibilities for $d_1, \ldots, d_k$ are

$$\begin{aligned}&1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1 \quad \text{and}\\ &1,\ 1,\ 1,\ 1,\ 2.\end{aligned}$$

Both these possibilities do occur: the first holds when G is an abelian group (see Proposition 9.5), and the second when $G = D_8$ (see Exercise 10.4).

We shall see later that $\dim V_i$ divides $|G|$ for all i, and this fact, combined with Theorem 11.12, is quite a powerful tool in finding the dimensions of irreducible $\mathbb{C}G$-modules.

Summary of Chapter 11

1. $\dim (\mathrm{Hom}_{\mathbb{C}G} (V_1 \oplus \ldots \oplus V_r, W_1 \oplus \ldots \oplus W_s))$

$$= \sum_{i=1}^{r} \sum_{j=1}^{s} \dim (\mathrm{Hom}_{\mathbb{C}G} (V_i, W_j)).$$

2. $\dim (\mathrm{Hom}_{\mathbb{C}G} (\mathbb{C}G, U)) = \dim U$.
3. Let $\mathbb{C}G = U_1 \oplus \ldots \oplus U_r$, a direct sum of irreducible $\mathbb{C}G$-modules, and let U be any irreducible $\mathbb{C}G$-module. Then the number of U_i with $U_i \cong U$ is equal to $\dim U$.
4. If $V_1, \ldots, V_k$ is a complete set of non-isomorphic irreducible $\mathbb{C}G$-modules, then

$$\sum_{i=1}^{k} (\dim V_i)^2 = |G|.$$

Exercises for Chapter 11

1. If G is a non-abelian group of order 6, find the dimensions of all the irreducible $\mathbb{C}G$-modules.
2. If G is a group of order 12, what are the possible degrees of all the irreducible representations of G?

   Find the degrees of the irreducible representations of D_{12}.
   (Hint: use Exercise 5.3.)

3. Let G be a finite group. Find a basis for $\mathrm{Hom}_{\mathbb{C}G} (\mathbb{C}G, \mathbb{C}G)$.
4. Suppose that $G = S_n$ and V is the n-dimensional permutation module for G over $\mathbb{C}$, as defined in 4.10. If U is the trivial $\mathbb{C}G$-module, show that $\mathrm{Hom}_{\mathbb{C}G} (V, U)$ has dimension 1.
5. Let $G = D_6$ and let $\mathbb{C}G = U_1 \oplus U_2 \oplus U_3 \oplus U_4$, a direct sum of irreducible $\mathbb{C}G$-modules, as in Example 10.8(2). Find a basis for $\mathrm{Hom}_{\mathbb{C}G}$ $(\mathbb{C}G, U_3)$ and a basis for $\mathrm{Hom}_{\mathbb{C}G} (U_3, \mathbb{C}G)$.
6. Let $V_1, \ldots, V_k$ be a complete set of non-isomorphic irreducible $\mathbb{C}G$-modules, and let V, W be arbitrary $\mathbb{C}G$-modules. Assume that for $1 \leqslant i \leqslant k$,

$$d_i = \dim\left(\mathrm{Hom}_{\mathbb{C}G}(V, V_i)\right) \text{ and } e_i = \dim\left(\mathrm{Hom}_{\mathbb{C}G}(W, V_i)\right).$$

Show that $\dim\left(\mathrm{Hom}_{\mathbb{C}G}(V, W)\right) = \sum_{i=1}^{k} d_i e_i$.