4.5 PDE HW5

Question 96

Solve

$$\begin{aligned} &u_t = ku_{xx} \\ &u(x, 0+) = e^{-x} \\ &u(0+, t) = 0 \end{aligned}$$

on the half line $0 < x < \infty$

Proof. Extend the initial condition to

$$\phi_{\text{odd}}(x) \triangleq \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -e^{x} & \text{if } x < 0 \end{cases}$$

We then solve

$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{\pi kt}} \int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4kt}} \phi_{\text{odd}}(y)dy \\ &= \frac{1}{2\sqrt{\pi kt}} \Big[\int_0^\infty e^{\frac{-(x-y)^2}{4kt}} e^{-y} dy - \int_{-\infty}^0 e^{\frac{-(x-y)^2}{4kt}} e^{y} dy \Big] \\ &= \frac{1}{2\sqrt{\pi kt}} \Big[\int_0^\infty e^{\frac{-(y-(x-2kt))^2 - 4ktx + 4k^2t^2}{4kt}} dy - \int_{-\infty}^0 e^{\frac{-(y-(x+2kt))^2 + 4ktx + 4k^2t^2}{4kt}} dy \Big] \\ &= \frac{1}{2\sqrt{\pi kt}} \Big[e^{-x+kt} \int_0^\infty e^{\frac{-(y-(x-2kt))^2}{4kt}} dy - e^{x+kt} \int_{-\infty}^0 e^{\frac{-(y-(x+2kt))^2}{4kt}} dy \Big] \\ &= \frac{1}{\sqrt{\pi}} \Big[e^{-x+kt} \int_{\frac{2kt-x}{2\sqrt{kt}}}^\infty e^{-p^2} dp - e^{x+kt} \int_{-\infty}^{\frac{-2kt-x}{2\sqrt{kt}}} e^{-p^2} dp \Big] \\ &= \frac{1}{\sqrt{\pi}} \Big[e^{-x+kt} \Big(\frac{\sqrt{\pi}}{2} - \frac{2}{\sqrt{\pi}} \operatorname{erf}(\frac{2kt-x}{2\sqrt{kt}}) \Big) - e^{x+kt} \Big(\frac{\sqrt{\pi}}{2} - \frac{2}{\sqrt{\pi}} \operatorname{erf}(\frac{2kt+x}{2\sqrt{kt}}) \Big) \Big] \end{aligned}$$

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