

§3. Local theory

Let $U \subset \mathbb{C}^n$ be an open subset and $x_i, y_i, i=1, \dots, n$ be real coordinate on $\mathbb{C}^n = \mathbb{R}^n \oplus \sqrt{-1}\mathbb{R}^n$. Define $J: TU \rightarrow TU$ by

$$\begin{cases} J(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i} \\ J(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i} \end{cases}$$

\leadsto almost complex structure J on U

Dually, we have

$$\begin{cases} J^*(dx_i) = -dy_i \\ J^*(dy_i) = dx_i \end{cases}$$

We complexify TU and decompose it into $\pm\sqrt{-1}$ -eigenspaces:

$$T_{\mathbb{C}}U = T^{1,0}U \oplus T^{0,1}U$$

Then

$$T^{1,0}U = \text{Span}_{\mathbb{C}} \left\{ \frac{1}{2} \left(\frac{\partial}{\partial x_i} - \sqrt{-1} \frac{\partial}{\partial y_i} \right) \mid i=1, \dots, n \right\}$$

$$T^{0,1}U = \text{Span}_{\mathbb{C}} \left\{ \frac{1}{2} \left(\frac{\partial}{\partial x_i} + \sqrt{-1} \frac{\partial}{\partial y_i} \right) \mid i=1, \dots, n \right\}$$

Similarly, the complexified cotangent bundle $T_{\mathbb{C}}^*U$ satisfies

$$T_{\mathbb{C}}^*U = (T^*U)^{1,0} \oplus (T^*U)^{0,1}$$

and

$$(T^*U)^{1,0} = \text{Span}_{\mathbb{C}} \left\{ dx_i + \sqrt{-1} dy_i \mid i=1, \dots, n \right\}$$

$$(T^*U)^{0,1} = \text{Span}_{\mathbb{C}} \left\{ dx_i - \sqrt{-1} dy_i \mid i=1, \dots, n \right\}$$

Note that $(T^*U)^{1,0} = (T^{1,0}U)^*$ and $(T^*U)^{0,1} = (T^{0,1}U)^*$.

We put

$$dz_i = dx_i + \sqrt{-1} dy_i$$

$$d\bar{z}_i = dx_i - \sqrt{-1} dy_i$$

Then we have

$$dz_i \left(\frac{\partial}{\partial z_j} \right) = \delta_{ij} = d\bar{z}_i \left(\frac{\partial}{\partial \bar{z}_j} \right)$$

$$dz_i \left(\frac{\partial}{\partial \bar{z}_j} \right) = 0 = d\bar{z}_i \left(\frac{\partial}{\partial z_j} \right)$$

for all $i, j = 1, \dots, n$.

Prop: Let $f: U \rightarrow V$ be a holomorphic map. Then

$$df_x(T'_x U) \subset T'_{f(x)} V$$

$$df_x(T^{0,1}_x U) \subset T^{0,1}_{f(x)} V$$

Pf: Exercise.

Define the complex vector bundle over U :

$$\Lambda^{p,q} T^* U := \Lambda^p (T^* U)^{1,0} \otimes \Lambda^q (T^* U)^{0,1}$$

and $A^{p,q}(U)$ the space of smooth sections of $\Lambda^{p,q} T^* U$.

They are called (p,q) -forms. We continue to have the decomposition

$$\Lambda^k T^* U = \bigoplus_{p+q=k} \Lambda^{p,q} T^* U$$

$$A^k_C(U) = \bigoplus_{p+q=k} A^{p,q}(U)$$

\uparrow
space of complex-valued k -forms on U .

We have the projection: $\pi^{p,q}: A^k_C(U) \rightarrow A^{p,q}(U)$

Def: Let $d: A_{\mathbb{C}}^k(U) \rightarrow A_{\mathbb{C}}^{k+1}(U)$ be the complex linear extension of the exterior differential. We define

$$\partial := \pi^{p+1, q} \circ d|_{A^{p, q}(U)}, \quad \bar{\partial} := \pi^{p, q+1} \circ d|_{A^{p, q}(U)}$$

For any $f \in C^{\infty}(U)$, we have

$$df = \sum_{i=1}^n \frac{\partial f}{\partial z_i} dz_i + \sum_{i=1}^n \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i$$

Then $\partial f = \sum_{i=1}^n \frac{\partial f}{\partial z_i} dz_i$ and $\bar{\partial} f = \sum_{i=1}^n \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i$. In particular,

$$f \text{ is holomorphic} \Leftrightarrow \bar{\partial} f = 0.$$

In general,

$$\partial(f dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}) = \sum_{i=1}^n \frac{\partial f}{\partial z_i} dz_i \wedge dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$$

$$\bar{\partial}(f dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}) = \sum_{i=1}^n \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i \wedge dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$$

Lemma: On $A_{\mathbb{C}}^{\bullet}(U)$, we have

$$\textcircled{1} \quad d = \partial + \bar{\partial}$$

$$\textcircled{2} \quad \partial^2 = \bar{\partial}^2 = 0 \quad \text{and} \quad \partial \bar{\partial} = -\bar{\partial} \partial$$

$$\textcircled{3} \quad \partial(\alpha \wedge \beta) = \partial \alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge \partial \beta$$

$$\textcircled{4} \quad \bar{\partial}(\alpha \wedge \beta) = \bar{\partial} \alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge \bar{\partial} \beta$$

Pf: $\textcircled{1}$ follows from the formula of d , ∂ and $\bar{\partial}$.

$\textcircled{2}$ Recall that $d^2 = 0$, so

$$0 = d^2 = \partial^2 + \partial \bar{\partial} + \bar{\partial} \partial + \bar{\partial}^2$$

We consider the following diagram:

$$\begin{array}{ccccc}
 & & & \partial & A^{p+2,q}(u) \\
 & & & \nearrow & \\
 & & A^{p+1,q}(u) & \xrightarrow{\partial} & \\
 A^{p,q}(u) & \xrightarrow{\partial} & & \searrow & \\
 & & A^{p,q+1}(u) & \xrightarrow{\partial} & A^{p+1,q+1}(u) \\
 & \searrow & & \nearrow & \\
 & & A^{p,q+2}(u) & &
 \end{array}$$

By degree reason, we get ②.

③ We have

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge d\beta$$

Suppose $\alpha \in A^{p,q}(u)$ and $\beta \in A^{r,s}(u)$. Then

$$\partial(\alpha \wedge \beta), \partial\alpha \wedge \beta, \alpha \wedge \partial\beta \in A^{p+r+1,q+s}(u)$$

$$\bar{\partial}(\alpha \wedge \beta), \bar{\partial}\alpha \wedge \beta, \alpha \wedge \bar{\partial}\beta \in A^{p+r,q+s+1}(u)$$

Hence ③

□

Recall the Poincaré lemma.

Thm: Let U be a contractible open subset of \mathbb{R}^n and $\alpha \in A^k(u)$ be such that $d\alpha = 0$ (d -closed). Then there exists $\beta \in A^{k-1}(u)$ such that $\alpha = d\beta$ (d -exact).

A $\bar{\partial}$ -analog of the Poincaré lemma holds.

Thm: $\bar{\partial}$ -Poincaré lemma

Let B be a polydisk and $\alpha \in A^{p,q}(B)$ be such that

$\bar{\partial}\alpha = 0$. Then there exists $\beta \in A^{p,q-1}(B)$ such that $\alpha = \bar{\partial}\beta$

Pf: Omitted

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