Quiz 3, Advanced Calculus I, Yung Fu Fang

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1. Let $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b, \in \mathbb{Q}\}$ Prove that $\mathbb{Q}(\sqrt{2})$ is a vector space and a countable set.

(i) If $\vec{x}, \vec{y} \in Q(\sqrt{2})$, $\vec{x} + \vec{y} \in Q(\sqrt{2})$

(ii) If $a \in Q$ and $\vec{x} \in Q(\sqrt{2})$, $a\vec{x} \in Q(\sqrt{2})$

(1) $\forall \vec{x}, \vec{y} \in Q(\sqrt{2})$, $\vec{x} + \vec{y} = \vec{y} + \vec{x}$

(2) $\forall \vec{x}, \vec{y}, \vec{z} \in Q(\sqrt{2})$, $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$

(3) $\exists\, 0 \in Q(\sqrt{2})$ s.t. $\vec{x} + 0 = \vec{x}$, $\forall \vec{x} \in Q(\sqrt{2})$

(4) $\exists\, \vec{y} \in Q(\sqrt{2})$ s.t. $\vec{x} + \vec{y} = 0$, $\forall \vec{x} \in Q(\sqrt{2})$

(5) $\exists\, 1 \in Q$ s.t. $1 \cdot \vec{x} = \vec{x}$, $\forall \vec{x} \in Q(\sqrt{2})$

(6) $\forall a, b \in Q$, $(ab)\vec{x} = a(b\vec{x})$, $\forall \vec{x} \in Q(\sqrt{2})$

(7) $\forall a \in Q$, $\vec{x}, \vec{y} \in Q(\sqrt{2})$, $a(\vec{x} + \vec{y}) = a\vec{x} + a\vec{y}$

(8) $\forall a, b \in Q$, $\vec{x} \in Q(\sqrt{2})$, $(a+b)\vec{x} = a\vec{x} + b\vec{x}$

Since Q is countable, let $C_{ij} = a_i + b_j\sqrt{2} \in Q(\sqrt{2})$, $a_i, b_j \in Q$.

$\Rightarrow C_{ij}$ is countable $\left(\{a_i\} = \{\frac{1}{1}, \frac{-1}{1}, \frac{1}{2}, \frac{-1}{2}, \frac{2}{1}, \frac{-2}{1}, \dots\} = \{b_j\}\right)$.

2.

A **neighborhood** of a point p, $N_r(p) =$ $\{q \mid d(p,q) < r\}$. The number r is called the radius of p

p is a **limit point** of the set E if every neighborhood of p contains a point $q \neq p$ s.t. $q \in E$.

p is an **isolated point** of the set E if $p \in E$ and p isn't a limit point of E.

E is a **closed set** if every limit point of E is a point of E.

p is an **interior point** of the set E if there is a neighborhood N of p s.t. $N \subset E$.

E is an **open set** if every point of E is an interior point of E.

The **complement set** of E, $E^c :=$ $\{p \mid p \in X,\ p \notin E\}$.

E is an **perfect set** if E is closed and if every point of E is a limit point of E

E is a **bounded set** if there is a real number M and a point $q \in X$ s.t. $d(p,q) < M$, $\forall p \in E$

E is **dense** in X if every point of X is a limit point of E, or a point of E.

3. (Green's Theorem) Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

4. (Stokes' Theorem) Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\vec{F}(x, y, z)$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C F \cdot dr = \iint_S \operatorname{curl} \vec{F} \cdot dS.$$