(1) Let X be a vector field on a compact manifold M (without boundary). Let φ_t be the one-parameter group of diffeomorphisms generated by X. Suppose α is a k-form on M. Show that $\varphi_t^*\alpha = \alpha$ for all t if and only if the Lie derivative $\mathcal{L}_X\alpha = 0$.

(2) Show that the total space T^*M of the cotangent bundle of any manifold M is orientable.

(3) Show that the following are special cases of Stokes's Theorem for manifolds with boundary. (Justify the notations on your own.)

(a) Let C be the image of a smooth embedding $\mathbf{r} : S^1 \to \mathbb{R}^2$ and let D be the region in $\mathbb{R}^2$ bounded by C. If $P, Q : \mathbb{R}^2 \to \mathbb{R}$ are smooth functions, then

$$\int_C P\,dx + Q\,dy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

(b) Let S be a compact oriented surface in $\mathbb{R}^3$ with smooth boundary C. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be smooth. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.$$

(c) Let E be the compact closure of an open subset of $\mathbb{R}^3$ with smooth boundary surface S. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be smooth. Then

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_E \operatorname{div} \mathbf{F}\, dV.$$

(4) Consider a smooth map $f : S^3 \to S^2$. Let $\alpha \in \Omega^2(S^2)$ be a form representing a non-trivial de Rham cohomology class $a \in H^2(S^2)$. Show that there exists a 1-form θ on S^3 such that $f^*\alpha = d\theta$. Moreover, show that the de Rham cohomology class in $H^3(S^3)$ of the 3-form $\theta \wedge f^*\alpha$ is independent of the choice of θ and of α representing a.

(5) Suppose M, N, P are compact connect orientable manifolds without boundary of the same dimension and $f : M \to N$ and $g : N \to P$ are smooth maps. Show that $\deg(g \circ f) = \deg(g)\deg(f)$. Then show that the antipodal map $\phi(p) = -p$ on the unit sphere in $\mathbb{R}^m$ has degree $(-1)^m$.