Ch8

(Feynmann)

(EX) Compute $\int_0^\infty \frac{\sin x}{x}\,dx = I(0)$

Note $f(x)=\int_0^x \frac{\sin y}{y}\,dy$ not elementary fn

Sol: set $I'(b) \equiv \frac{d}{db}\int_0^\infty \frac{\sin x}{x} e^{-bx}\,dx$

$$\overset{\text{Advaced Calulus}}{=} \int_0^\infty \frac{\sin x}{x} \frac{d}{db} e^{-bx}\,dx$$

$$= \int_0^\infty \frac{\sin x}{x}(-\cancel{x}) e^{-bx}\,dx$$

$$= -\int_0^\infty \underbrace{e^{-bx}}_{f(x)} \underbrace{\sin x}_{g'(x)}\,dx$$ integration by parts twice

$$= -1 + b^2 \int_0^\infty e^{-bx} \sin x\,dx = -1 + b^2(-I'(b))$$

$$\Rightarrow I'(b) = \frac{-1}{1+b^2}$$

$$\Rightarrow I(b) = -\tan^{-1} b + C$$

$$\Rightarrow \lim_{b\to\infty} I(b) = -\frac{\pi}{2} + C$$

$$= \lim_{b\to\infty}\int_0^\infty \frac{\sin x}{x} e^{-bx}\,dx = \int_0^\infty \frac{\sin x}{x} \lim_{b\to\infty} e^{-bx}\,dx = 0$$

$$\Rightarrow C = \frac{\pi}{2}$$

Hence $\int_0^\infty \frac{\sin x}{x}\,dx \overset{\text{Advanced Calculus}}{=} I(0) = \lim_{b\to 0} I(b) = \lim_{b\to 0}\left(-\tan^{-1} b + \frac{\pi}{2}\right)$

$$= \frac{\pi}{2}$$

//

Per-Duet

(Laplace)

〈Ex〉 Evaluate $\int_0^\infty e^{-x^2}dx$

Note $f(x)=\int_0^x e^{-y^2}dy$ not elementary fn

Sol: Set $I \equiv \int_0^\infty e^{-x^2}dx$

$$\Rightarrow I^2 = \int_0^\infty e^{-x^2}dx \cdot \int_0^\infty e^{-y^2}dy$$

$$\overset{\text{Double integral}}{=} \int_0^\infty \int_0^\infty e^{-x^2-y^2}\,dy\,dx$$

$$\overset{y=tx}{=} \int_0^\infty \int_0^\infty e^{-(1+t^2)x^2}x\,dt\,dx$$

$$\overset{\text{Fubini}}{=} \int_0^\infty \frac{-1}{2(1+t^2)}\int_0^\infty -2(1+t^2)x\,e^{-(1+t^2)x^2}dx\,dt$$

$$= \int_0^\infty \frac{-1}{2(1+t^2)}\int_0^\infty \underline{\frac{d}{dx}e^{-(1+t^2)x^2}dx}\,dt$$

$$\overset{\text{FTC}}{=} \int_0^\infty \frac{-1}{2(1+t^2)}\, e^{-(1+t^2)x^2}\Big|_{x=0}^{x=\infty}dt$$

$$= \int_0^\infty \frac{+1}{2(1+t^2)}(\cancel{0-}1)\,dt$$

$$= \frac{1}{2}\int_0^\infty \frac{1}{1+t^2}dt$$

$$= \frac{1}{2}\tan^{-1}t\Big|_0^\infty = \frac{\pi}{4}$$

$$\therefore\ I = \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

※