RTFT Ch12 Conjugacy Class

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In this note, G is always a group.

In this note, V is always a vector space.

Definitions

Definition 1. *Let $x \in G$. The **conjugacy class** of x in G, written x^G is defined by*

$$x^G = \{gxg^{-1} | g \in G\}$$

Definition 2. *Let $x \in G$. The **centralizer** of x in G, written $C_G(x)$, is the set of elements of G that commute with x*

$$C_G(x) = \{g \in G | gxg^{-1} = x\}$$

Theorem 1. *For all $x \in G$, $C_G(x)$ is a subgorup of G*

Proof. Let $g, h \in C_G(x)$

$(gh)x(gh)^{-1} = ghxh^{-1}g^{-1} = x \implies gh \in C_G(x)$

$exe^{-1} = x \implies e \in G$

$g^{-1}xg = g^{-1}(gxg^{-1})g = x \implies g^{-1} \in C_G(x)$ ■

Theorems

Theorem 2. $|x^G| = (G : C_G(x))$

Proof. Let $\psi : G/C_G(x) \to x^G$ be defined by $hC_G(x) \mapsto hxh^{-1}$

We first prove that ψ is well defined

Let $hC_G(x) = gC_G(x)$

$h^{-1}g \in C_G(x) \implies h^{-1}gx(h^{-1}g)^{-1} = x \implies h^{-1}gxg^{-1}h = x \implies gxg^{-1} = hxh^{-1}$ (done)

We now prove ψ is bijective

Clearly ψ is onto

$gxg^{-1} = hxh^{-1} \implies h^{-1}gxg^{-1}h = x \implies h^{-1}gx(h^{-1}g)^{-1} = x \implies h^{-1}g \in C_G(x) \implies hC_G(x) = gC_G(x)$ (done) ■