臺灣大學數學系 112 學年度碩士班甄試試題

科目：高等微積分 2022.10.20

1. (15 points)
Let $M = \{f : [0, \infty) \to [0, \infty); \int_0^\infty f(x)^2 dx \le 1\}$. Evaluate the following:

$$\sup_{f \in M} \int_0^\infty f(x) e^{-x} dx.$$

2. (15 points)
Assume $A \subset \mathbb{R}^n$ is compact and let $a \in A$. Suppose $\{a_n\}$ is a sequence in A such that every convergent subsequence of $\{a_n\}$ converges to a.
(1) Does the sequence $\{a_n\}$ also converge to a ? Justify your result. → Yes, $\limsup a_n = \liminf a_n = \lim a_n$
(2) Now assume A is not compact and suppose $\{a_n\}$ is a sequence in A such that every convergent subsequence of $\{a_n\}$ converges to $a \in A$. Does the sequence $\{a_n\}$ also converge to a ? Justify your result. No! In this problem, $\infty / -\infty$ is not in conv. case.

3. (20 points)
Let M be a metric space and $A \subset M$ a compact subset. Suppose $f : A \to A$ is continuous and satisfies $d(f(x), f(y)) \ge d(x, y)$ for all x, y. Prove that f is onto A, i.e $f(A) = A$.

4. (20 points)
Define a sequence of functions $\{f_n(x)\}$ on $[0, 1]$ as:

$$f_n(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1 & \text{if } x \in (\frac{2k}{2^n}, \frac{2k+1}{2^n}], k = 0, 1, \cdots, 2^{n-1} - 1 \\ -1 & \text{if } x \in (\frac{2k+1}{2^n}, \frac{2k+2}{2^n}], k = 0, 1, \cdots, 2^{n-1} - 1 \end{cases}$$

Prove or disprove that we always have $\lim_{n \to \infty} \int_0^1 f_n(x) g(x) dx = 0$ as long as g is a continuous function.

5. (20 points)
Denote P_2 the set of all polynomials with real coefficients and degree ≤ 2. Consider the function $G : P_2 \to \mathbb{R}$ by

$$G(p) = \int_0^1 p(x)^2 dx.$$

Let $S = \{p \in P_2; p(1) = 1\}$. Does G attain any extremal value on S ? If yes, find $p \in S$ such that G attains an extremal value at p.

6. (10 points)
Suppose $f(x) : [a, b] \to \mathbb{R}$ is Riemann integrable, and $g(x) : \mathbb{R} \to \mathbb{R}$ satisfies $|g(x) - g(y)| \le C|x - y|$ for all x, y. Prove that $g(f(x))$ is Riemann integrable.

Prob 1, Hölder inequality

$$\int_0^\infty f(x)e^{-x}\,dx \le \left(\int_0^\infty f^2(x)\,dx\right)^{\frac{1}{2}}\left(\int_0^\infty e^{-2x}\,dx\right)^{1/2} \le \left(\int_0^\infty e^{-2x}\,dx\right)^{1/2}$$

$$= \frac{1}{\sqrt{2}}$$

when $f(x) = e^{-x}$, $\int_0^\infty f^2(x)\,dx = \int_0^\infty e^{-2x}\,dx = \frac{1}{\sqrt{2}} \le 1$

so $\sup_{f\in M} \int_0^\infty f(x)e^{-x}\,dx = \frac{1}{\sqrt{2}}$ #

Prob. 2 limsup / liminf

(1) Yes, since A is compact, liminf / limsup of seq $\{a_k\}_{k=1}$ is finite,

Note that there exist subseq s.t. $a_{k_j} \xrightarrow{j\to\infty} \limsup a_k$ / $a_{k_j} \xrightarrow{j\to\infty} \liminf a_k$.

Thus, by hypothesis, $\liminf a_k = \limsup a_k = a$, That implies $\lim a_k = a$

(2) No!! Let $\{a_k\}_{k=1}$ define by $a_k = \begin{cases} k, & k \text{ even} \\ 0, & k\in \text{odd} \end{cases}$, a_k diverge but All conv. subseq conv. to 0. (参考自 Phoebe Chen

Prob. 3 Compact, Continuous map

Suppose not, $\exists\ a\in A$, $a\notin f(A)$, Note that Both A, $f(A)$ are compact,

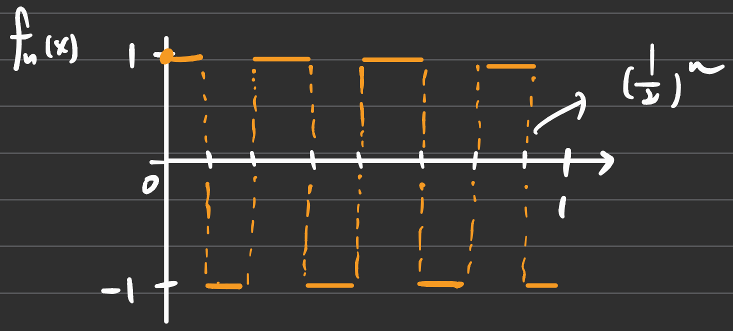
so $d(a, f(A)) > 0$, let $r := \frac{1}{2}d(a, f(A))$ and consider open cover

$\bigsqcup_{a_j\in f(A)} B_r(a_j)$, Since $f(A)$ is compact, $\exists\ N\in\mathbb{N}$ s.t. $f(A) \subseteq \bigsqcup_{j=1}^{N} B_r(a_j)$,

But $f(a)\in f(A)$, $d(f(a), f(a_j)) \ge d(a, a_j) > r$, so $f(a) \notin \bigsqcup_{j=1}^{N} B_r(a_j)$

That mean $f(a)\notin f(A)$, $\Rightarrow\Leftarrow$, so $f(A) = A$.

Prob. 4 Stoke - weierstrass theorem



Note that $\exists\{P_n(x)\}_{n=1}^{\infty} \subseteq$ polynomial such that $P_n \xrightarrow{u.c.} f$ for any f conti on $[a,b]$, so we test

$$\int_0^1 P_m(x) f_n(x)\,dx \overset{?}{=} 0 \quad \forall m\in\mathbb{N} \quad \text{as } n\to\infty$$

$$\lim_{n\to\infty}\int_0^1 f_n(x)\, 1\, dx \equiv 0 \qquad \forall\, n\in\mathbb{N}$$

$$\lim_{n\to\infty}\int_0^1 f_n(x)\, x^m dx = \lim_{n\to\infty}\sum_{k=0}^{2^{n-1}-1}\left(\int_{2k/2^n}^{2k+1/2^n} x^m dx - \int_{2k+1/2^n}^{2k+2/2^n} x^m dx\right)$$

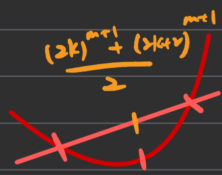
$$= \lim_{n\to\infty}\sum_{k=0}^{2^{n-1}-1}\left(\frac{1}{m+1}x^{m+1}\Big|_{\frac{2k}{2^n}}^{\frac{2k+1}{2^n}} - \frac{x^{m+1}}{m+1}\Big|_{\frac{2k+1}{2^n}}^{\frac{2k+2}{2^n}}\right)$$

$$= \lim_{n\to\infty}\sum_{k=0}^{2^{n-1}-1}\frac{1}{m+1}\times\frac{1}{2^{n(m+1)}}\times\left((2k+1)^{m+1}-(2k)^{m+1}-(2k+2)^{m+1}+(2k+1)^{m+1}\right)$$

$$= \frac{1}{(m+1)\,2^{n(m+1)}}\sum_{k=0}^{2^{n-1}-1}\left[2(2k+1)^{m+1}-(2k)^{m+1}-(2k+2)^{m+1}\right]$$

$$= \frac{1}{(m+1)\,2^{\underset{2n}{n(m+1)}}}\times\sum_{k=0}^{2^{n-1}-1}\left[2(2k+1)^{m+1}-(2k)^{m+1}-(2k+2)^{m+1}\right]$$

when $m=1$, $\lim_{n\to\infty}(\sim) \approx \lim_{n\to\infty}\frac{-2\times(2^{n-1}-1)}{2\times 2^{2n}} \longrightarrow 0$



For general

$$2(2k+1)^{m+1} - (2k)^{m+1} - (2k+2)^{m+1} \qquad (2k+1)^{m+1}$$

$$2\left((2k)^{m+1}\right) - (2k)^{m+1} - (2k)^{m+1} = 0 \qquad \deg \downarrow 1$$

$$2\times C_1^{m+1}(2k)^m - C_1^{m+1}(2k)^m\, 2 = 0 \qquad \deg \downarrow 1$$

$$\deg \text{ of } 2(2k+1)^{m+1} - (2k)^{m+1} - (2k+2)^{m+1} = m-1 \ \#$$

$$\sum_{k=0}^{2^{n-1}-1}(\sim) \approx \left(2^{n-1}\right)^m \quad \text{so} \quad \lim_{n\to\infty}\frac{-1}{(m+1)\,2^{n(m+1)}}(\sim)$$

$$\cancel{nm} - m - \cancel{mn} - n$$

$$\approx \lim_{n\to\infty}\frac{-2^{m(n-1)}}{(m+1)\,2^{n(m+1)}} \qquad = -m-n$$

$$\approx \lim_{n\to\infty}\frac{-1}{(m+1)\,2^{m+n}} \longrightarrow 0$$

$\forall f\in C[0,1]$ $\exists \{P_m\}_{m=1}^{\infty}$ s.t. $P_m \xrightarrow{u.c.} f$, Given $\epsilon>0$

$$\lim_{n\to\infty}\left|\int_0^1 f_n f\right| \le \lim_{n\to\infty}\underbrace{\left|\int_0^1 f_n(f-P_m)\right|}_{<\ \underbrace{\|f-P_m\|_\infty}_{=\epsilon}\ \overset{\to =1}{\|f_n\|}} + \left|\int_0^1 f_n P_m\right|$$

$\leq \epsilon \times 1 + \lim_{n\to\infty} \left| \int_0^1 t_n P_m \right| \leq \epsilon$,

Since ϵ is arbitrary , $\lim_{n\to\infty} \left| \int_0^1 f_n f \right| = 0 \Rightarrow \lim_{n\to\infty} \int_0^1 f_n f = 0$

Prob. 5 Lagrange Multiplers

Let $P(x) = ax^2 + bx + c$, $S = a+b+c = 1$

$$G(p) = \int_0^1 p^2(x)\,dx = \int_0^1 a^2x^4 + b^2x^2 + c^2 + 2abx^3 + 2bcx + 2acx^2 \,dx$$

$$= \frac{1}{5}a^2 + \frac{1}{3}b^2 + c^2 + \frac{1}{2}ab + bc + \frac{2}{3}ac$$

G may atain the max on S, use L.M. , Let $g(a,b,c) = a+b+c-1$

$$d_a g = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \quad , \quad \nabla G(a,b,c) = \begin{pmatrix} \frac{2}{5}a + \frac{1}{2}b + \frac{2}{3}c \\ \frac{2}{3}b + \frac{1}{2}a + c \\ 2c + b + \frac{2}{3}a \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} , \quad a+b+c = 1$$

$$\Rightarrow \begin{pmatrix} \frac{2}{5}a + \frac{1}{2}b + \frac{2}{3}c - \lambda = 0 \\ \frac{2}{3}b + \frac{1}{2}a + c - \lambda = 0 \\ 2c + b + \frac{2}{3}a - \lambda = 0 \\ a+b+c = 1 \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{2}{5}a + \frac{1}{2}b + \frac{2}{3}c - \lambda = 0 \\ \frac{1}{2}a + \frac{2}{3}b + c - \lambda = 0 \\ \frac{2}{3}a + b + 2c - \lambda = 0 \\ a + b + c \qquad = 1 \end{pmatrix}$$

$\Rightarrow$ Solve this linear system , $(a,b,c,\lambda) = \left(\frac{10}{3}, \frac{-8}{3}, \frac{1}{3}, \frac{2}{9}\right)$ may be a extreme point.

Another way ,

Take $G(p)$ as a length of p, $\langle f,g \rangle \equiv \int_0^1 fg\,dx$ is a inner product.

$\{1, x, x^2\} \longrightarrow \{1, 2\sqrt{3}x - \sqrt{3}, 6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}\}$ (orthogonal basis)

$\Rightarrow p(x) = a + b(2\sqrt{3}x - \sqrt{3}) + c(6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5})$

$\Rightarrow p(1) = 1 \longrightarrow a + \sqrt{3}b + \sqrt{5}c = 1 \qquad G(p) = a^2 + b^2 + c^2$

$\Rightarrow G(p)$ is square of length of $p^{\nearrow (a,b,c)}$, length of $p^{\nearrow (a,b,c)}$ has min on the constrain

$a+\sqrt{3}b+\sqrt{5}c=1$, so $G(p)$ attain a extreme point (minimum)

$$\text{Min } G(p) = \left(\frac{1}{\sqrt{1^2+\sqrt{3}^2+\sqrt{5}^2}} \right)^2 = \frac{1}{9}$$

$(a,b,c) \parallel (1,\sqrt{3},\sqrt{5})$ and (a,b,c) on $a+\sqrt{3}b+\sqrt{5}c=1$

$$\Rightarrow (a,b,c) = \left(\frac{1}{9}, \frac{\sqrt{3}}{9}, \frac{\sqrt{5}}{9}\right), \quad p(x) = \frac{1}{9} + \frac{\sqrt{3}}{9}(2\sqrt{3}x-\sqrt{3}) + \frac{\sqrt{5}}{9}(6\sqrt{5}x^2-6\sqrt{5}x+\sqrt{5})$$

$$= \frac{1}{3} - \frac{8}{3}x + \frac{10}{3}x^2 \quad \#$$

Prob. 6 Riemann integral

5.9 Definition.

Let $a, b \in \mathbf{R}$ with $a < b$. A function $f : [a, b] \to \mathbf{R}$ is said to be (*Riemann*) *integrable* on $[a, b]$ if and only if f is bounded on $[a, b]$, and for every $\varepsilon > 0$ there is a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$.

We need to test that $\boxed{g\circ f}$ h is bd. and $\forall \epsilon>0, \exists P$ s.t. $U(f,P) - L(f,P) < \epsilon$.

First $|f| < M$ and g is conti on $[-M, M]$, so $g\circ f$ is bounded.

Given $\epsilon > 0$, Choose $\delta > 0$ such that $|g(x) - g(y)| < \epsilon$ when $|x-y| < \delta$, $\delta < \epsilon$ (Since g is Lipchiz, we can do it). Choose a partition P such that $U(f,P) - L(f,P) < \delta^2$. Then separate partition in two set, A 、B,

$$A = \{ [x_i, x_{i+1}] : |M_f - m_f| < \delta \} ; \quad B = \{ [x_i, x_{i+1}] : |M_f - m_f| \geq \delta \}$$

For $[x_i, x_{i+1}] \in A$, $\quad M_h - m_h < \epsilon$

For $[x_i, x_{i+1}] \in B$, $\quad \delta \sum_{i \in B} \Delta x_i \leq \sum_{i\in B} |M_f - m_f| \Delta x_i \leq U(f,P) - L(f,P) < \delta^2$

$$\Rightarrow \sum_{i\in B} \Delta x_i \leq \delta$$

Thus,
$$\begin{aligned} U(h,P) - L(h,P) &= \sum_{i\in A} |M_h - m_h| \Delta x_i + \sum_{i\in B} |M_h - m_h| \Delta x_i \\ &\leq \epsilon \times (b-a) + 2K\delta \qquad K = \sup g \text{ on } [-M, M] \\ &\leq \epsilon (b - a + 2K) \ \# \end{aligned}$$

Hence, h is Riemann integrable.

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1. (10 points)
Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable. Assume that its Jacobian matrix $(\frac{\partial f_i}{\partial x_j})$ has rank n everywhere and $f^{-1}(K)$ is compact whenever K is compact. Show that $f(\mathbb{R}^n) = \mathbb{R}^n$

2. (15 points)
Let $F(x, y, z) = (x, y, z)$, and S be the boundary of the region $x^2 + y^2 \leq z \leq \sqrt{2 - x^2 - y^2}$, oriented so that the normal points out of the region. Compute $\int\int_S F \cdot n dA$.

$|M| = 1 \Rightarrow$ connect

3. (20 points)
Let M be a metric space with countable elements. Prove or disprove that M is disconnected.

$|M| \geq 2 \Rightarrow$ disconnect

4. (20 points)
Let $f(x) = \frac{1}{4} + x - x^2$. For a real number x, define $x_{n+1} = f(x_n)$, where $x_0 = x$. (1) Given $x = 0$, show that the sequence $\{x_n\}$ converges and find its limit L. (2) Find all real numbers x such that their corresponding sequences all converge to L.

5. (20 points)
Let X consist of all real valued functions f on $[0, 1]$ such that
(1)$f(0) = 0$
(2)$||f|| = \sup\{\frac{|f(x)-f(y)|}{|x-y|^{1/3}}; x \neq y\}$ is finite.
Prove that $||\cdot||$ is a norm for X and X is complete with respect to this norm.

6. (15 points)
Assume $f : (a, b) \to \mathbb{R}$ is differentiable. Consider its derivative $f'(x)$, show that $f'(x)$ never has a jump discontinuity.

Prob. 1 Inverse func. theorem & proper map

Note 1, f is open map, that is, $\forall\ \mathcal{O}$ is open, $f(\mathcal{O})$ is open,

$\Rightarrow$ Let $\mathcal{O}$ be a open set, then for any $f(a) \in f(\mathcal{O})$, since $|J_f(a)| \neq 0$, by Inverse func. theorem, $\exists\ W_a$ is open s.t. f^{-1} is C' on $f(W_a)$, Consider $\mathcal{O} \cap W_a$, a open set contain a, then $f^{-1}(\mathcal{O} \cap W_a)$ is a open set contain $f(a)$

Then $f(\mathcal{O}) \subseteq \bigsqcup_{f(a) \in f(\mathcal{O})} f^{-1}(\mathcal{O} \cap W_a)$, Moreover, $f(\mathcal{O}) = \bigsqcup_{f(a) \in f(\mathcal{O})} f^{-1}(\mathcal{O} \cap W_a)$

That mean f is a open map since $\bigsqcup f^{-1}(\mathcal{O} \cap W_a)$ is open.

Note 2, Every open proper map is surjective, that is $f(\mathbb{R}^n) = \mathbb{R}^n$

We claim that $f(\mathbb{R}^n)$ is a open / closed set, since $\mathbb{R}^n$ is connected, $f(\mathbb{R}^n) = \mathbb{R}^n$. $f(\mathbb{R}^n)$ is open since f is a open map. To show $f(\mathbb{R}^n)$ is closed, Let y_0 be a limit point of $f(\mathbb{R}^n)$, consider $B_1(y_0)$, a compact set, since y_0 is a limit point, so $\exists\ \{y_k\}_{k=1}^{\infty}$ s.t. $y_k \to y_0$. Take attension on $\{y_{k_j}\} \subseteq B_1(y_0)$ and consider $\{x_{k_j} = f^{-1}(y_{k_j})\}_{j=1}^{\infty} \equiv A$

then $A \subseteq f^{-1}(B_1(y))$ $\rightarrow$ compact, so A has a conv. subseq $\hookrightarrow$ choose one x_{k_j}

s.t. $x_{k_j} \longrightarrow x$, then

$$y = \lim_{j \to \infty} y_{k_j} = \lim_{j \to \infty} f(x_{k_j}) = f(\lim_{j \to \infty} x_{k_j}) = f(x) \in f(\mathbb{R}^n)$$

so $y \in f(\mathbb{R}^n)$, $f(\mathbb{R}^n)$ is closed #

Prob. 2 Surface integration, stoke、Gauss、divergent

$$S = \partial \{ (x,y,z) : x^2 + y^2 \leq z \leq \sqrt{2 - (x^2 + y^2)} \} = \partial A$$

$S =$ 

$F(x,y,z) = (x,y,z)$, $\int_S F\cdot n\,dA = \int_A \operatorname{div} F\,dA = 3\int_A dA$

$$= 3\times \underset{\theta}{\int_0^{2\pi}} \underset{r}{\int_0^1} \underset{z}{\int_{r^2}^2} r\,dr\,dz\,d\theta = 3\times 2\pi \times \int_0^1\int_{r^2}^2 r\,dz\,dr = \frac{9}{2}\pi$$

Prob. 3 Connect.

If $|M| = 1$, then M is connect obviously,

If $|M| \geq 2$, then M is not connect, Choose two element of M, x, y

Let $r = d(x,y) > 0$ and $\tilde{r} = \alpha r$, $\alpha \in [0,1]$, Then $\exists\, \alpha \in [0,1]$ such that

$A \equiv B_{\alpha r}(x) = \{ m \in M : d(m,x) \leq \alpha r \}$ and $\tilde{A} \equiv B_{\alpha r}(x) = \{ m \in M : d(m,x) \geq \alpha r \}$

has no intersection ($A \cap \tilde{A}$) (if not, M will not be countable)

Then consider $A^\circ = \{ m \in M : d(m,x) < \alpha r \}$ and $\tilde{A}^\circ = \{ m \in M : d(m,x) > \alpha r \}$

A° and $\tilde{A}^\circ$ is open obviously, and $M \subseteq A^\circ \sqcup \tilde{A}^\circ$, so M is not conneeted.

Prob. 4 Contraction mapping

$f'(x) = 0$ only when $x = \frac{1}{2}$

(1) Note that $f(x) = -(x-\frac{1}{2})^2 + \frac{1}{2} \leq \frac{1}{2}$ and $f'(x) = 1 - 2x$,

'=' only when $x = \frac{1}{2}$

Moreover, $f(x) - x = \frac{1}{4} - x^2 \geq 0$ for $x \in [0, \frac{1}{2}]$, Thus we just need to

discuss $x \in [0, \frac{1}{2}]$, $x_0 = 0$, $x_1 = \frac{1}{4}$, $x_2 = \frac{7}{16}$, since $f(x) - x \geq$ on $[0, \frac{1}{2}]$,

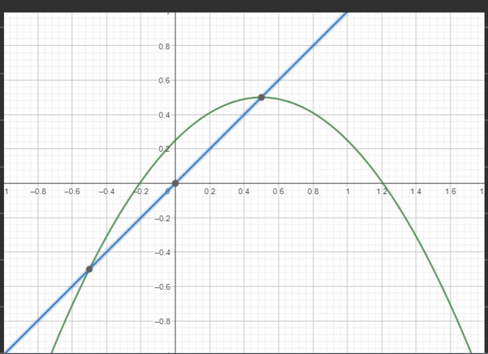
$\{x_k\}_{k=1}^\infty$ is increasing (by induction), Moreover, by M.V.T, $|f(x_n) - f(x_{n-1})| \leq |f'(\xi_n)|\,|x_n - x_{n-1}|$

$\xi_n \in (x_{n-1}, x_n)$, since $\{x_k\}_{k=1}^\infty$ increasing, $\xi_n > \xi_2$ $\forall n$ $\xi_2 \in (\frac{1}{4}, \frac{7}{16})$. So $f'(\xi_n) < f'(\xi_2)$

$\forall n \in \mathbb{N}$, Then $|f(x_n) - f(x_{n-1})| \leq \underbrace{|f'(\xi_2)|}_{<1}\,|f(x_{n-1}) - f(x_{n-2})| \cdots \leq |f'(\xi_2)|^{n-3}\,|f(x_2) - f(x_1)|$

which is contractive, so $x_n \longrightarrow x$ for some x, and $x = f(x) \Rightarrow x = \frac{1}{2}$.

(2)



Note that if some $x_n \in (-\frac{1}{2}, \frac{3}{2})$, Then x_n will converge to $\frac{1}{2}$ as $n \to \infty$ (left picture) since $f(x) - x < 0$ as $x < -\frac{1}{2} \Rightarrow x_n \searrow -\infty$ as $n \to \infty$

use the symmetric, we conclude that $x \in (-\frac{1}{2}, \frac{3}{2})$

Prob. 5

Let $X \equiv \{ f : |f(x) - f(y)| \le M |x-y|^{\frac{1}{3}}$ for some $M > 0 \}$ $\|f\| \equiv M$

if $M = 0$, $|f(x) - f(y)| \le 0 \Rightarrow f(x) = f(y)$, Since $f(0) = 0$ $f \equiv 0$

Let $\alpha \in \mathbb{R}$, $\sup\limits_{x,y} \dfrac{|\alpha f(x) - \alpha f(y)|}{|x-y|^{1/3}} = \sup\limits_{x,y} |\alpha| \dfrac{|f(x) - f(y)|}{|x-y|^{1/3}} = |\alpha| \|f\|$

Let $f, g \in X$ $\dfrac{|(f+g)(x) - (f+g)(y)|}{|x-y|^{1/3}} \le \dfrac{|f(x) - f(y)|}{|x-y|^{1/3}} + \dfrac{|g(x) - g(y)|}{|x-y|^{1/3}}$

$$\Rightarrow \|f+g\| \le \|f\| + \|g\|$$

To show that X is complete, Chose a Cauchy seq $\{f_n\}_{n=1}^{\infty}$,

Given $\epsilon > 0$, fixed $x \in [0,1]$

$$|f_n(x) - f_m(x)| = |(f_n - f_m)(x) - (f_n - f_m)(0)| \le |x| \times \epsilon$$

Since $\{f_n\}$ is cauchy, Then $\{f_n(x)\}_{n=1}^{\infty}$ is a cauch seq in $\mathbb{R}$, so

it conv. to $\hat{x} \in \mathbb{R}$, We define $f(x) = \lim\limits_{n\to\infty} f_n(x) = \hat{x}$, we claim that $f_n \to f$

First, $f \overset{?}{\in} X$, Note that $\{f_n\}$ is bounded, i.e. $\exists$ M such that $\|f_n\| \le M$

so $|f(x) - f(y)| = |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)|$

$$\le |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$$
$$\le 2\epsilon + M|x-y|^{1/3}$$

Since ϵ is arbitrary, $|f(x) - f(y)| \le M|x-y|^{1/3}$. Thus $\|f\| < M$, $f \in X$

Moreover, Choose $n, m > N \in \mathbb{N}$ s.t. $\|f_n - f_m\| < \epsilon/2$, since $\|\cdot\|$ is conti,

$$\lim_{m\to\infty} \|f_n - f_m\| = \|f_n - \lim_{m\to\infty} f_m\| = \|f_n - f\| \le \epsilon/2 < \epsilon$$

Hence $f_n \to f$ and X is complete.

Prob. 6

We claim that for any $f'(c) < y_0 < f'(d)$, $\exists$ $x_0 \in (c,d)$ s.t. $f'(x_0) = y_0$, Consider $(c,d) \subset (a,b)$

$F(x) = f(x) - y_0 x$, then $F(x)$ has absolutely min, Note that $F'(c) = f'(c) - y_0 < 0$ and $F'(d) = f'(d) - y_0 > 0$, That mean $F(c+h) - F(c) < 0$ for $h > 0$ enough small, similar $F(d+h) - F(d) < 0$ for $h < 0$ enough small, so a, b is not min $\Rightarrow x_0 \in (c,d) \Rightarrow F'(x_0) = f'(x_0) - y_0 = 0$ #

臺灣大學數學系 109 學年度碩士班甄試試題
科目：高等微積分

2019.10.18

1. (15 points)
Consider a series $\sum_{k=2}^{\infty} a_k \sin(kx)$, where $\{a_k\}$ is a sequence of real numbers. Is it possible to construct a sequence $\{a_k\}$ such that $\sum_{k=2}^{\infty} a_k \sin(kx)$ converges uniformly to $\sin(x)$ on $[0,\pi]$? Justify your result.

2. (15 points)
Suppose $f:\mathbb{R}\to\mathbb{R}$ is uniformly continuous. Show that f must be bounded by a linear function, i.e, there exist constants A, B such that $|f(x)| \le A + B|x|$ for all x.

3. (20 points)
(1): Let $f: M \to N$ be a map from metric space M to another metric space N with the property that if a sequence $\{p_n\}$ in M converges, then the sequence $\{f(p_n)\}$ in N also converges. Is f continuous ? Justify your result. Yes
(2): Let $f: M \to \mathbb{R}$ where M is a metric space, and define $G = \{(x,y) \in M \times \mathbb{R}; y = f(x)\}$. If G is compact (in product space $M \times \mathbb{R}$). Is f continuous ? Justify your result.
Yes

4. (20 points)
Suppose $L: \mathbb{R}^3 \to \mathbb{R}^3$ is an invertible linear map and $g: \mathbb{R}^3 \to \mathbb{R}^3$ has continuous first order partial derivatives and satisfies $||g(x)|| \le C||x||^{1+\epsilon}$ for some $\epsilon > 0$ and all $x \in \mathbb{R}^3$ ($||\cdot||$ is the usual Euclidean norm). Show that $f(x) = L(x) + g(x)$ is invertible near the origin 0.

5. (20 points)
(1): Find the volume of the ellipsoid $(x+2y)^2 + (x-2y+z)^2 + 3z^2 \le 1$.
(2): Let C be a positively oriented simple closed curve. Find the curve C that maximizes the integral $\int_C y^3 dx + (3x - x^3) dy$.

6. (10 points)
Let $f(x)$ be a real valued continuous function on $[0,1]$. Find the limit

$$\lim_{n\to\infty} (n+1) \int_0^1 x^n f(x) dx.$$

Prob. 1 orthogonal basis of L^2, $\{\cos(kx), \sin(kx)\}_{k=1}^{\infty}$

No !! Assume that there exist $S_n = \sum_{k=2}^{n} a_k \sin(kx) \xrightarrow{u.c.} \sin(kx)$

Note that $\int_0^{\pi} \sin^2 x \, dx = \int_0^{\pi} \frac{1}{2} - \frac{1}{2}\cos 2x \, dx = \frac{\pi}{2} \neq 0$, but

$$\int_0^{\pi} \sin x \sin kx \, dx = \int_0^{\pi} \frac{1}{2}\Big(\cos\big((k-1)x\big) - \cos\big((k+1)x\big)\Big) dx$$

$$= \frac{1}{2}\int_0^{\pi} \cos(k-1)x \, dx - \frac{1}{2}\int_0^{\pi} \cos(k+1)x \, dx = 0$$

Thus, $\frac{\pi}{2} = \int_0^{\pi} \sin^2 x = \int_0^{\pi} (\sin x) \lim_{n\to\infty} S_n(x) \, dx = \lim_{n\to\infty} \int_0^{\pi} \sin(x) S_n(x) dx$

$$= \lim_{n\to\infty} \sum_{k=2}^{n} \int_0^{\pi} \sin(x) \times a_n \sin(kx) \, dx$$

$$= \lim_{n\to\infty} 0$$

$$= 0$$

That is a contradiction.

Prob. 2 Uniformly Continuous

Since f is uniformly continuous, Given $\epsilon = 1$, $\exists\, \delta_0$ s.t. $|x-y| < \delta_0$ implies that $|f(x) - f(y)| < 1$, Let $x \in \mathbb{R}$ and $|x| = r$, Choose $n \in \mathbb{N}$ such that $\delta_0 n > r$ but $\delta_0 (n-1) < r$ $\left(n = \left\lceil \frac{r}{\delta_0} \right\rceil \right)$, Then

$$|f(x)| \leq |f(x) - f(0)| + |f(0)| \leq |f(x) - f(x_{n-1})| + |f(x_{n-1}) - f(x_{n-2})| + \cdots + |f(x_1) - f(0)| + |f(0)|$$

$$\leq \sum_{k=1}^{n} 1 + |f(0)| = n + |f(0)| \leq \frac{r}{\delta_0} + 1 + |f(0)|$$

$$\leq \frac{1}{\delta_0}|x| + |f(0)| + 1 \quad \#$$

Prob. 3 Continuous

(1) If not, $\exists\, p \in M$ such that f is not conti at p, that is,

Yes

given $\epsilon > 0$, $\forall\, n \in \mathbb{N}$, $\exists\, x_n \in B_{1/n}(p)$ s.t. $|f(p) - f(x_n)| > \epsilon$

we define $\{x_j\}_{j=1}^{\infty}$ by $x_j = \begin{cases} x_n & \text{if } j = \text{odd},\ 2n-1 = j \\ p & \text{if } j \in \text{even} \end{cases}$

Then $x_j \longrightarrow p$ as $j \to \infty$ but $f(x_j)$ don't converge since

$$|f(x_j) - f(x_{j+1})| = |f(x_j) - f(p)| > \epsilon \quad \forall j \in \mathbb{N} \ \#$$

(2) Suppose not, $\exists$ a such that f is not conti on a, then there exist $\epsilon > 0$ and $\{x_n\}_{n=1}^{\infty}$ s.t. $x_n \to a$ but $|f(x_n) - f(x)| > \epsilon$,

Yes

Consider $\{(x_n, f(x_n))\}_{n=1}^{\infty} \subseteq G(f)$, then exist a subseq $\{(x_{n_k}, f(x_{n_k})\}_{k=1}^{\infty}$ such that $(x_{n_k}, f(x_{n_k})) \longrightarrow (\hat{x}, f(\hat{x}))$, but since $x_n \to a$, $\hat{x} = a$, $f(\hat{x}) = f(a)$, that is contradictive,

Prob. 4 Inverse func. theorem

Note that $\|g(x)\| \le C\|x\|^{1+\epsilon}$ implies $g(0) = 0$ and $J_g(x) = [\,0\,]$,

Note: $J_L(x) = L(x)$, Thus, $J_f(x) = L(x)$ which is invertible at 0,

By inverse func. theorem, f is invertible near the origin 0.

Prob. 5 Change variable

(1) Let $V = \{(x,y,z) : (x+2y)^2 + (x-2y+z)^2 + 3z^2 \le 1\}$

Let $a = x+2y$, $b = (x-2y+z)$ $c = \sqrt{3}z$

$$\begin{array}{c|ccc|} & a & b & c \\ x & 1 & 1 & 0 \\ y & 2 & -2 & 0 \\ z & 0 & 1 & \sqrt{3} \end{array} = 4\sqrt{3}$$

$$\int_V dv = \int_{\bar{V}} 4\sqrt{3}\, d\bar{v} = 4\sqrt{3} \times \frac{4}{3}\pi = \frac{16\sqrt{3}}{3}\pi$$

(2) Green theorem, divergent theorem

$$\int_C (y^3, 3x - x^3) \cdot ds = \int_V \frac{\partial}{\partial x}(3x - x^3) - \frac{\partial}{\partial y} y^3 \, dv = \int_V 3 - 3x^2 - 3y^2 \, dv$$

Let $3 - 3x^2 - 3y > 0$ on V, $\Rightarrow$ $1 > x^2 + y^2$ $\Rightarrow$ $C: (\cos t, \sin t)$ $t \in [0, 2\pi)$

Prob. 6 stoke weierstruss theorem

Note that $x^n f(x) \le f(x)$ $\forall x \in [0,1]$, By LDCT, we have

$$\lim_{n\to\infty} \int_0^1 x^n f(x)\, dx = \int_0^1 \lim_{n\to\infty} x^n f(x)\, dx = \int_0^1 0\, dx = 0$$

Note that $\forall m \in \mathbb{N}$ $\int_0^1 n x^n \cdot x^m\, dx = \frac{n}{n+m+1}$, as $n\to\infty$, $\frac{n}{n+m+1} \to 1$

$P_n(x) \to f$

so we gauss $\lim_{n\to\infty} \int_0^1 n x^n f(x)\, dx = f(1)$ since $P_n(1) \to f(1)$ as $n\to\infty$

Choose $\{P_m(x)\}_{m=1}^{\infty}$ s.t. $P_m(x) \xrightarrow{u.c.} f$. Given $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $m > N$ implies $|P_m(x) - f(x)| < \epsilon$ $\forall x \in [0,1]$. Then we have

$$\left| \int_0^1 n x^n f(x)\, dx - f(1) \right| \le \left| \int_0^1 n x^n f(x) - P_m(x) \right| + \left| \int_0^1 n x^n P_m(x) - P_m(1)\, dx \right|$$

$$+ \left| P_m(1) - f(1) \right|$$

$$\le \epsilon \frac{1}{n+1} + \left| \sum_{k=0}^{m} \frac{n}{n+k+1} a_k - P_m(1) \right|$$

$$+ \epsilon$$

Take $n\to\infty$, $\lim_{n\to\infty} \left| \int_0^1 n x^n f(x)\, dx - f(1) \right| \le 2\epsilon$. Since ϵ is arbitary, $\lim_{n\to\infty} \int_0^1 n x^n f(x)\, dx = f(1)$, Thus $\lim_{n\to\infty} \int_0^1 (n+1) x^n f(x)\, dx = f(1)$.

臺灣大學數學系 108 學年度碩士班甄試試題
科目：高等微積分

2018.10.19

(1) [10 分] Does the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n \log(n+1)}{n}$$

converge? Does it converge absolutely? Justify your answer.

(2) [10+10 分] Consider the function

$$f(x,y) = \frac{1}{(1-xy)^2}$$

defined on $\Omega = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1\ ,\ 0 \le y \le 1\ ,\ (x,y) \neq (1,1)\}$.

(a) For any $\kappa \in (0,1)$, let $U_\kappa = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le \kappa\ ,\ 0 \le y \le \kappa\}$. Is $f(x,y)$ uniformly continuous on U_κ? Justify your answer.

(b) Is $f(x,y)$ uniformly continuous on Ω? Justify your answer.

(3) [10+15 分] For any $n \in \mathbb{N}$, consider $f_n(x) = n\,x^n\,(1-x)$ on $I = \{x \in \mathbb{R} \mid 0 \le x \le 1\}$.

(a) Determine $\lim_{n\to\infty} f_n(x)$ for every $x \in I$.

(b) Is the convergence uniform on I? Give your reason.

(4) [15+10 分] Let

$$F(x) = \int_0^\infty \frac{1-\cos(xt)}{t^2 e^t}\,\mathrm{d}t\ .$$

(a) Can you switch the order of integration and differentiation to obtain the formulae for $F'(x)$ and $F''(x)$? Explain the reason.

(b) Find the explicit[1] formula for $F'(x)$ and $F(x)$.

(5) [10+10 分] Consider

$$\begin{array}{rccl} F: & \mathbb{R}^4 & \to & \mathbb{R}^2 \\ & (x,y,u,v) & \mapsto & \left(\displaystyle\int_{x-y^2}^{x^2+y} (e^{t^2}+u)\mathrm{d}t,\ x^3+v\right) \end{array}.$$

(a) Prove that near $(1,1,0,0)$, the two equations $F(x,y,u,v) = (\int_0^2 e^{t^2}\mathrm{d}t, 1)$ can be solved for u, v as continuously differentiable functions of x, y.

(b) For the functions $u(x,y)$ and $v(x,y)$ in part (a), find all their first order partial derivatives at $(x,y) = (1,1)$.

[1]Not an improper integral.

Prob. 1 Dirichlet test

$\forall m \in \mathbb{N}$, $\sum_{n=1}^{m} (-1)^n$ is bounded, let $f(x) = \frac{\log(x+1)}{x}$, then $f'(x) = \frac{\frac{x}{x+1} - \log(x+1)}{x^2}$, as $x > 2$, $f'(x) < 0$ $f(x)$ is decreasing, use L.H can obtain that $\lim_{x\to\infty} f(x) = 0$, By Dirichlet test, $\sum_{n=1}^{\infty} \frac{(-1)^n \log(n+1)}{n}$ converge, But $\frac{\log(n+1)}{n} > \frac{1}{n}$ for $n > 2$, so it don't absolutely conv.

Prob. 2 Uniformly continuous

(a) Since $f(x,y) = \frac{1}{(1-xy)^2}$ is continuous on $\mathcal{U}_k$ and $\mathcal{U}_k$ is compact, (closed and bounded) f is uniform conti on $\mathcal{U}_k$

(b) Not!! Note that if f is uniformly on E and a cauchy seq $x_n \in E$, then $\{f(x_n)\}_{n=1}^{\infty}$ is a cauchy seq. Let $x_n = (1-\frac{1}{n}, 1-\frac{1}{n})$, $x_n \in \Omega$ and

$$\|X_n - X_m\|_2 = \sqrt{2}\left|\frac{1}{n} - \frac{1}{m}\right|.$$

So x_n is a Cauchy seq, but $f(x_n) = \frac{1}{(1-(1-\frac{1}{n})^2)^2} = \frac{n^2}{(2-\frac{1}{n})^2}$

$$|f(x_n) - f(x_m)| = \left|\frac{n^2}{(2-1/n)^2} - \frac{m^2}{(2-1/m)^2}\right| \quad (\text{W.L.O.G}, n \gg m)$$
$$> \left|\frac{n^2}{4} - m^2\right|$$

which is not cauchy, so f is not uniformly continuous on Ω.

Prob. 3 Uniformly conv.

(a) For $x=0$ or $x=1$ $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 0 = 0$, For $x \in (0,1)$, Since $\lim_{n\to\infty} \frac{n}{(\frac{1}{x})^n} = 0$, so $\lim_{n\to\infty} f_n(x) \to 0$ when $x \in (0,1)$.

(b) Note that $f_n \xrightarrow{u.c.} f$ iff $\sup_x |f_n(x) - f(x)| \to 0$ as $n \to \infty$

But for any f_n, $f_n' = n^2 x^{n-1}(1-x) - nx^n$, $f_n'(x) = 0 \Leftrightarrow x = \frac{n}{n+1}$

at $x = \frac{n}{n+1}$, $f_n(x) = n\left(\frac{n}{n+1}\right)^n \left(\frac{1}{n+1}\right) = \frac{1}{\left(1+\frac{1}{n}\right)^n} \times \frac{n}{n+1}$

Then $\lim_{n\to\infty} \sup_x |f_n(x)| \geq \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} \times \frac{n}{n+1} = \frac{1}{e}$

Thus $f_n \longrightarrow f$ is not uniform.

Prob.4 LDCT

Since $\cos(-xt) = \cos(xt)$, so we discuss $x_0 \geq 0$, and similarly, $x_0 < 0$ holds. Fixed $x_0 \geq 0$, Choose $\delta > 0$ and h_k such that $h_k \to 0$ as $k \to \infty$ and $\delta > h_k$ $\forall k \in \mathbb{N}$. Then find $t > 0$ s.t. $(x_0 + h_k)t < (x_0 + \delta)t < \pi/2$

Then, By M.V.T

$\xi_k \in (x_0, x_0 + h_k)$

$$\left(\frac{1-\cos(x_0+h_k)t}{t^2 e^t} - \frac{1-\cos x_0 t}{t^2 e^t}\right) \times \frac{1}{h_k} = \left(\frac{\sin(\xi_k t)}{t e^t}\right) := f_k(x,t)$$

Then $$f_k(t) := \left|\frac{\sin \xi_k t}{t e^t}\right| \leq \frac{\sin(x_0+\delta)t}{t e^t} \quad \text{when } t < \frac{\pi/2}{x_0+\delta}$$

Since $\varphi(t) := \frac{\sin(x_0+\delta)t}{t e^t}$ is $\mathcal{L}^1$ on $[0, \pi/2 / x_0+\delta]$ and

$f_k(t) \leq \frac{1}{t e^t} := \psi$ is $\mathcal{L}^1$ on $(\pi/2 / x_0+\delta, \infty)$

By LDCT

$$F'(x) = \lim_{k\to\infty} \int_0^\infty f_k(t)\,dt = \int_0^\infty \lim_{k\to\infty} f_k(t)\,dt = \int_0^\infty \frac{\partial f}{\partial x}\Big|_{x=x_0} dt$$

So $$F'(x) = \int_0^\infty \frac{\partial}{\partial x} \frac{1-\cos xt}{t^2 e^t}\,dt = \int_0^\infty \frac{\sin xt}{t e^t} \quad \text{and}$$

$$F''(x) = \int_0^\infty \frac{\cos xt}{e^t}\,dt \quad \left(\text{Similar discuss on } \left|\frac{\cos xt}{e^t}\right| < e^{-t}\right)$$

Then $$F'(x) = F'(0) + \int_0^x F''(s)\,ds = 0 + \int_0^x \int_0^\infty e^{-t} \cos st\,dt\,ds$$

$$= \int_0^x \frac{1}{1+s^2}\,ds = \tan^{-1} x$$

$$F(x) = F(0) + \int_0^x F'(x)\,dx = 0 + \int_0^x \tan^{-1} s\,ds = s\tan^{-1} s\Big|_0^x - \int_0^x \frac{s}{s^2+1}\,ds$$

$$= x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) \quad \#$$

Prob. 6. Implict func. Theorem

$$F(x,y,u,v) = \Big(\underbrace{\int_{x-y^2}^{x^2+y} (e^{t^2}+u)\,dt}_{F_1},\ \underbrace{x^3+v}_{F_2}\Big)$$

at $(1,1,0,0)$, $F(1,1,0,0) = \left(\int_0^2 e^{t^2}dt,\ 1\right)$, test $\left.\frac{\partial(F_1,F_2)}{\partial(x,y)}\right|_{1100}$

$$\frac{\partial(F_1,F_2)}{\partial(x,y)} = \begin{pmatrix} x^2+y-x+y^2, & 0 \\ 0, & 1 \end{pmatrix}$$

$$\Rightarrow \left.\frac{\partial(F_1,F_2)}{\partial(x,y)}\right|_{(1,1,0,0)} = \begin{pmatrix} 2, & 0 \\ 0, & 1 \end{pmatrix} \Rightarrow \text{invertible.}$$

$\Rightarrow$ By I.F.T, we done.

(b) Note that $J_g = -\left(\frac{\partial(F_1,\cdots,F_n)}{\partial(x_1,\cdots,x_n)}\right)^{-1}\left(\frac{\partial(F_1,\cdots,F_n)}{\partial(x_{n+1},\cdots,x_{n+m})}\right)$

$$\Rightarrow \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = -\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 2e^4-1 & e^4+1 \\ 3 & 0 \end{pmatrix}$$

$$= -\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2e^4-1 & 3 \\ e^4+1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1-2e^4}{2} & \frac{3}{2} \\ -(1+e^4) & 0 \end{pmatrix} \#$$

臺灣大學數學系107學年度碩士班甄試試題
科目：高等微積分

2017.10.21

1. (15 points. **No partial credit will be given if the answer is wrong.**) Evaluate the integral
$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \frac{1}{1+3\sin^2 x}dx. \quad (\textbf{Hint.}\ \text{Let } u = \tan x.)$$

2. (15 points) State and prove Leibniz's criterion for convergence of alternating series.

3. (15 points) Let F be an $\mathbf{R}$-valued C^∞ function on $\mathbf{R}^2$ such that at a point $p=(a,b)$ we have
$$\frac{\partial F}{\partial x}(p) = \frac{\partial F}{\partial y}(p) = 0,\ \frac{\partial^2 F}{\partial x^2}(p) > 0, \text{ and } \frac{\partial^2 F}{\partial x^2}(p)\frac{\partial^2 F}{\partial y^2}(p) - \left(\frac{\partial^2 F}{\partial x \partial y}(p)\right)^2 > 0.$$
Show that there exists $R>0$ such that $F(p) < F(q)$ for $q \in \{(x,y) \in \mathbf{R}^2 \,|\, (x-a)^2 + (y-b)^2 < R^2\}$.

4. We adopt the following definitions.

   Let (X,d) be a metric space. (i) A family $\mathcal{F}$ of $\mathbf{R}$-valued functions on X is *equicontinuous at a point* $x_0 \in X$ (with respect to d) if
$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, f \in \mathcal{F} \quad \forall x \in X \quad d(x.x_0) < \delta \ \Rightarrow\ |f(x) - f(x_0)| < \varepsilon.$$
   (ii) A sequence f_n of $\mathbf{R}$-valued functions on X *converges compactly* to some function f on X if for every compact subset K of the metric space (X,d) the sequence $f_n|_K$ converges to $f|_K$ uniformly.

(15 points) Let f_n be a sequence of $\mathbf{R}$-valued functions on a metric space (X,d) which converges *pointwise* to a *continuous* function f on (X,d). Suppose that $\{f_n \,|\, n \in \mathbf{N}\}$ is equicontinuous at every point of X. Show that f_n converges compactly to f on X.

5. (15 points.) Compute the <u>outward</u> flux of the vector field $(x + ye^z,\ e^x \sin(yz),\ ye^{zx})$ through the boundary of the region $D = \left\{(x,y,z) \in \mathbf{R}^3 \,\middle|\, \left(\sqrt{x^2+y^2} - 3\right)^2 + z^2 < 1\right\}$.

6. (25 points) Show that the function $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{\cos(ne)}{n^x}$ (where $e = \sum_{m=0}^{\infty} \frac{1}{m!}$) is well-defined (i. e., the series converges) on $(0,\infty)$ and is continuous.

7. (30 points. In your argument if any theorems are used you have to clearly verify that their conditions are fulfilled.) Let f and g be $\mathbf{R}$-valued C^∞ functions on $\mathbf{R}^2$ and let $S = \{(x,y) \in \mathbf{R}^2 \,|\, f(x,y) = 0\}$. Suppose that at some point $p = (a,b) \in S$ we have $\dfrac{\partial f}{\partial x}(p) = -1$, $\dfrac{\partial f}{\partial y}(p) = 2$, $\dfrac{\partial g}{\partial x}(p) = 3$, $\dfrac{\partial g}{\partial y}(p) = -6$,
$$\begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2}(p) & \dfrac{\partial^2 f}{\partial x \partial y}(p) \\ \dfrac{\partial^2 f}{\partial y \partial x}(p) & \dfrac{\partial^2 f}{\partial y^2}(p) \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} \dfrac{\partial^2 g}{\partial x^2}(p) & \dfrac{\partial^2 g}{\partial x \partial y}(p) \\ \dfrac{\partial^2 g}{\partial y \partial x}(p) & \dfrac{\partial^2 g}{\partial y^2}(p) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}.$$
Show that there exists $R > 0$ such that $g(p) < g(q)$ for $q \in S \cap \{(x,y) \in \mathbf{R}^2 \,|\, (x-a)^2 + (y-b)^2 < R^2\}$.

Prob. 1

$$\int_{\pi/4}^{5\pi/4} \frac{1}{1+3\sin^2 x}\,dx = \int_{\pi/4}^{5\pi/4} \frac{\sec^2 x}{\sec^2 x + 3\tan^2 x}\,dx$$

$$= \int_{\pi/4}^{5\pi/4} \frac{\sec^2 x}{1+4\tan^2 x}\,dx \quad — (*)$$

Let $u = \tan x \Rightarrow du = \sec^2 x\, dx$; $v = 2u$

$$(*) = \int_1^{-1} \frac{du}{1+4u^2} = \int_2^{-2} \frac{1}{1+v^2} \times \frac{1}{2}\,dv = \frac{1}{2}\left[\tan^{-1}(-2) - \tan^{-1}(2)\right]$$

Prob. 2 Leibniz's criterion

State : An Alternate series converge if absolute value of its term is monotone decreasing to zero as $n \to \infty$.

Prove : Let $S_n = \sum_{k=1}^{n} (-1)^k a_k$, $a_n \geq 0$; Note that

(1) For $n \in$ even, $n = 2j$

$$S_{2j+2} = \sum_{k=1}^{2j+2} (-1)^k a_k = \sum_{k=1}^{2j} (-1)^k a_k + (-1)^{2j+1} a_{2j+1} + (-1)^{2j+2} a_{2j+2}$$

$$\geq \sum_{k=1}^{2j} (-1)^k a_k = S_{2j}$$

(2) For $n \in$ odd, $n = 2j+1$

$$S_{2j+1} = \sum_{k=1}^{2j+2} (-1)^k a_k = \sum_{k=1}^{2j-1} (-1)^k a_k + (-1)^{2j+1} a_{2j+1} + (-1)^{2j} a_{2j}$$

$$\leq \sum_{k=1}^{2j-1} (-1)^k a_k = S_{2j-1}$$

(3)

$$a_1 - a_2 = S_2 \leq S_{2j+2} \leq S_{2j+1} \leq a_1$$

So $S_{2j} \nearrow L_1$ as $j \to \infty$ and $S_{2j+1} \searrow L_2$ as $j \to \infty$.

Since $\lim_{j\to\infty} S_{2j+1} - S_{2j} = \lim_{j\to\infty} a_{2j+1} = 0 \Rightarrow L_1 = L_2$.

Hence, we claim $\lim_{k\to\infty} S_k = L$: $k \in$ even $|S_k - L| \leq |S_{k+1} - S_k|$ $= a_{k+1} \to 0$ as $k \to \infty$; $k \in$ odd $|S_k - L| \leq |S_k - S_{k-1}| = a_k \to 0$ as $k \to \infty$

Prob. 3 n-dim Taylor expression of C^∞ func.

Use Taylor of f at p, we have

$$f(p+h) - f(p) = \frac{1}{2} D^2 f(p)(h) + \varepsilon(h)\|h\|^2,$$

where $\varepsilon(h) \to 0$ as $\|h\| \to 0$, Then since $f \in C^\infty \Rightarrow D^2 f(p)$ is conti

$\because$ $D^2 f(p) > 0$, consider $D^2 f(p)$ on $B_1(p)$, it has a min $m \neq 0$

$\therefore$ Chose $\|h\| < \delta$ such $|\varepsilon(h)| < \frac{m}{2}$, then

$$f(p+h) - f(p) < \|h\|^2 \left(\frac{1}{2} D^2 f(p)(h) + \varepsilon(h)\right) > 0$$

Then on $B_\delta(p)$, $f(g) = f(p+g-p) = f(p+h) > f(p)$ #

Prob. 4 Uniform conv. subseq on compact set

Since $\overset{= f_0}{f}$ is continuous and f_n is equicontinuous, Given $\epsilon > 0$, Chose $\delta > 0$ such that

$$|f_i(x) - f_i(y)| < \epsilon \text{ if } |x-y| < \delta \quad \forall i$$

Since K is compact, $\exists$ $x_1, \cdots, x_n$ such that $K = \bigcup_{j=1}^n B_\delta(x_j)$

Chose N_j s.t. $|f_m(x_j) - f(x_j)| < \epsilon$ as $m > N_j$, let $N = \max\{N_j\}_{j=1}^n$

Hence, $\forall x \in K$, $x \in B_\delta(x_j)$ for some j, let $m > N$

$$|f(x) - f_m(x)| \le |f(x) - f(x_j)| + |f(x_j) - f_m(x_j)| + |f_m(x_j) - f_m(x)|$$
$$\le 3\epsilon \ \#$$

Therefore, $f_n \to f$ unf conv.

Prob. 5 Diverge theorem / Change variable / Torus.

$$\int_{\partial V} \vec{F} \cdot \vec{n} \, dv = \int_V \operatorname{div} F \, dv = \int_V 1 + \underbrace{\cancel{ze^x \cos(yz)}}_{\text{odd func. for } z} + \overbrace{\cancel{xye^{zx}}}^{\text{odd func for } y} \, dv$$

$$= \int_V 1\, dv = \int_{(r-3)^2+z^2<1} r\, dr\, d\theta\, dz = 2\pi \int_{(r-3)^2+z^2<1} r\, dr\, dz$$

$$= 2\pi \int_{r^2+z^2<1} r+3\, dr\, dz = 2\pi \int_{0}^{1}\int_{0}^{2\pi} (\rho \cos\theta + 3)\, \rho\, d\rho\, d\theta$$

(lower limits labelled ρ and θ)

$$= \frac{3}{2} \times 2\pi \times 2\pi = 6\pi^2 \#$$

Prob. 6 Dirichlet test

State : Let $\sum a_k = \sum f_k g_k$, if $|\sum^n f_k| < M \ \forall n$ and $g_k \searrow 0$

as $k \to \infty$, then $\sum a_k$ is conv.

$$\sum_{k=1}^{n} \cos(k\theta) = \cos\theta + \cos 2\theta + \cdots + \cos n\theta$$

$$\sin\theta \sum_{k=1}^{n} \cos(k\theta) = \sum_{k=1}^{n} \frac{1}{2}\left(\sin[k+1]\theta - \sin[k-1]\theta\right) = \frac{1}{2}\sin[n+1]\theta$$

$$\Rightarrow \left|\sum_{k=1}^{n} \cos(k\theta)\right| = \left|\frac{\sin(n+1)\theta}{2\sin\theta}\right| \le \frac{1}{2\sin\theta}$$

Thus, $f(x) = \sum_{k=1}^{\infty} \dfrac{\cos k\theta}{k^x}$ conv. for any $x \in (0, \infty)$

Then for any $x \in (0,\infty)$, find $\delta > 0$ such that $\delta \in (0, x)$

Note that when $x \in [2,\infty)$, $\left|\sum \dfrac{\cos(n\theta)}{n^x}\right| < \sum \dfrac{1}{n^2} < \infty$ By M-test,

$\sum \dfrac{\cos(n\theta)}{n^x} \xrightarrow{u.t} f$. Since $\sum \dfrac{\cos(n\theta)}{n^x}$ is conti, f is conti on $[2,\infty)$

For $x \in [\delta, 2]$, $|\sum \cos(n\theta)|$ is bound and $g_k(x) = \dfrac{1}{k^x} < \dfrac{1}{k^\delta} \ \forall x \in [\delta, \infty)$

$g_k(x) \xrightarrow{u.t} 0$ (Choose k such that $1/k^\delta < \epsilon$), By dirichlet test,

$\sum \dfrac{\cos(n\theta)}{n^x} \xrightarrow{u.t.} f$, f is conti on $[\delta, \infty)$, Discuss

$$a = \inf\{\delta \in (0,\infty) : f \text{ is conti on } (\delta, \infty)\}$$

Then we obtain $a = 0$. (by above process with $x = a > 0$, $\delta = \frac{a}{2}$.)

Prob. 7 Implicit func. Theorem , Lagrange theorem

At p, $\dfrac{\partial f(p)}{\partial x} = -1 \ne 0$, by IFT, $\exists\ h(t)$ and $W \overset{\text{open}}{\subset} \mathbb{R}$ s.t

$f(h(t), t) \in S$ for $t \in W$, Moreover, $h'(t) = -(-1) \times 2 = 2$

At $p = (h(t_1), t_1) \in W$, Let $\varphi(t) = g(h(t), t)$ on $t \in W$ $\nearrow \in C^\infty$

$$\frac{d\varphi(t)}{dt} = \nabla g(h(t_1), t_1) \cdot (h'(t_1), 1) = 0 \quad \text{at } t_1$$

By Taylar expresion

$$\varphi(t_1+h) - \varphi(t_1) = \frac{\varphi''(t_1)}{2} h^2 + \epsilon(h) h^2$$

$\epsilon(h) \to 0$ as $h \to 0$,

$$\varphi''(t_1) = \frac{d}{dt^2} g(h(t), t) = \frac{d}{dt} \nabla g(h(t), t) \cdot \left(\frac{dh}{dt}, 1\right)$$

$$= \frac{d}{dt}[g_x \times d_t h + g_y \times 1]$$

$$= g_{xx}(d_t h)^2 + g_{xy}(d_t h) + g_x(d_t^2 h) + g_{yx} d_t h + g_{yy}$$

$f(h(t), t) = 0 \Rightarrow f_x h_t + f_y = 0$

$\Rightarrow f_{xx} h_t^2 + f_{xy} h_t + f_x h_{tt} + f_{yx} h_t + f_{yy} = 0$

$$h_{tt} = \frac{1 \times 2^2 + 3 \times 2 + 3 \times 2 + 0}{1} = 16$$

$$\varphi''(t_1) = 3 \times 2^2 + (-1) \times 2 + 3 \times 16 + (-1) \times 2 + 2$$

$$= 58 > 0$$

As h enough smal, $\varphi(t_1+h) - \varphi(t_1) = h^2 \underbrace{(29 + \epsilon(h))}_{>0}$

Then we choose $r > 0$ s.t. $B_r(p) \cap S \subseteq W$ and $B_r(p) \cap S$ can be parametered by $(h(t), t)$, $t \in [t_1, t_1+h]$. This r satisfies $g(p) < g(q)$ when $q \in S \cap B_r(p)$.

臺灣大學數學系 106 學年度碩士班甄試試題
科目：高等微積分

2016.10.21

1. (15 points) Let A be the unit ball $B_1(0)$ in $\mathbf{R}^3$. Compute

$$\int_A \cos(x+y+z)\,dx\,dy\,dz.$$

2. Let f be a real-valued function on $\mathbf{R}$ which has period 2π and is Riemann integrable on $[-\pi,\pi]$. We define its Fourier coefficients

$$a_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx\ (n=0,1,2,\dots) \text{ and } b_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx\ (n=1,2,\dots).$$

(1) (10 points) Show that $f(x)^2$ is Riemann integrable on $[-\pi,\pi]$.

(2) (15 points) Show that the series $\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2+b_n^2)$ converges.

3. (1) (15 points) Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers and let $B_n = b_1 + \cdots + b_n$ ($n \in \mathbf{N}$). Suppose that $a_n \searrow 0$ as $n \to \infty$ and that there exists $M > 0$ such that $|B_n| \leqslant M$ for every $n \in \mathbf{N}$. Show that the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

(2) (10 points) Show that the function series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\sin(nx)$$

converges uniformly on $[-K,K]$ if $|K|<\pi$.

4. **Definition.** Let $\mathcal{F}$ be a set of real-valued functions on a set X. $\mathcal{F}$ is *uniformly bounded* if there exists $M>0$ such that $|f(x)| \leqslant M$ for every $x \in X$ and every $f \in \mathcal{F}$.

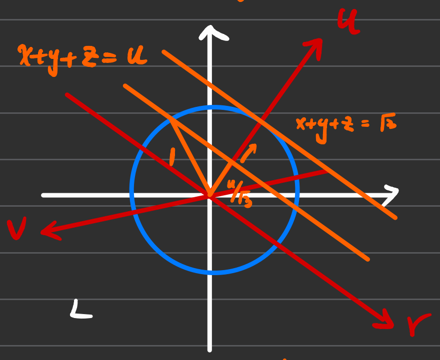
(20 points) Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of convex functions on $[-2,2]$ and let $f_n = F_n|_{[-1,1]}$ ($n \in \mathbf{N}$). Suppose that $\{F_n \mid n \in \mathbf{N}\}$ is uniformly bounded. Show that there exists a subsequence of $\{f_n\}_{n=1}^{\infty}$ which is uniformly convergent on $[-1,1]$.

5. (15 points) Let $f = (f_1,\dots,f_n) : U \to \mathbf{R}^n$ be a C^1 map from an open set U in $\mathbf{R}^n$, and let $g : V \to U$ be a continuous map from an open set V in $\mathbf{R}^n$. Suppose that

$$\det\left(\frac{\partial f_j}{\partial x_k}(x)\right) \neq 0 \quad \text{for every } x \in U,$$

and that $f(g(x)) = x$ for every $x \in V$. Show that g is C^1.

Prob. Integral



$x+y+z=$ Constant , $u = x+y+z$

$(x,y,z) \xrightarrow{\text{rotation}} (u,v,r)$

$$\int_{B_1(0)} \cos(x+y+z)\, dS_{xyz} = \int_{B_1(0)} \cos(x+y+z)\, dS_{uvr}$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \cos(u) \times \pi \times \left(1 - \frac{u^2}{3}\right) du$$

$$= \frac{4}{3} \sin\sqrt{3} - \frac{4\sqrt{3}}{3} \cos\sqrt{3}$$

$\hat{u} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$

$x+y+z=0$

$d(\hat{u}, x+y+z=0) = \dfrac{\sqrt{3}}{\sqrt{3}} = 1 \quad d(x+y+z=u,\ x+y+z=0) = \dfrac{|u|}{\sqrt{3}}$

Prob. 2 Fourier series

(1) Given $\epsilon > 0$, $\exists$ partition P s.t. $U(f,P) - L(f,P) < \epsilon$

Note that if f is Riemann integrable, so is $|f|$, since

$$M_{\Delta x}|f| - m_{\Delta x}|f| \leq M_{\Delta x} f - m_{\Delta x} f$$

$\hookrightarrow$ max on Δx $\quad \hookrightarrow$ min on Δx

$\forall\ \Delta x \in P$ $\left(P = \{x_0 = a, \cdots, x_n = b\},\ \Delta x = [x_{i-1}, x_i]\ \ i = 1, \cdots, n \right)$

Then $M_{\Delta x}(f^2) - m_{\Delta x}(f^2) = M_{\Delta x}(|f|^2) - m_{\Delta x}(|f|^2)$

$$= \left[M_{\Delta x}(|f|) + m_{\Delta x}(|f|) \right] \times \left[M_{\Delta x}(|f|) - m_{\Delta x}(|f|) \right]$$

$$= 2M\left(M_{\Delta x}(|f|) - m_{\Delta x}(|f|) \right), \quad \text{where } |f| < M$$

Hence f^2 also be Riemann integrable.

(2) Note that (1) $\displaystyle\int_{-\pi}^{\pi} \cos(kx)\cos(jx)\,dx = \begin{cases} 0 & \text{if } k \neq j \\ \pi & \text{if } k = j \end{cases}$ $\Rightarrow \|\cos kx\| = 1$

(2) $\displaystyle\int_{-\pi}^{\pi} \sin(kx)\sin(jx)\,dx = \begin{cases} 0 & \text{if } k \neq j \\ \pi & \text{if } k = j \end{cases}$ $\Rightarrow \|\sin kx\| = 1$

(3) $\displaystyle\int_{-\pi}^{\pi} \cos(kx)\sin(jx) = 0$

Let $f_n(x) = \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx) \qquad b_0 = 0$

$$0 \le \| f - f_n(x) \|^2 = \langle f - f_n(x), f - f_n(x) \rangle \quad \to \langle \cdot, \cdot \rangle \equiv \frac{1}{\pi}\int_{-\pi}^{\pi} f \times g \, dx$$

$$= \langle f, f \rangle - 2\langle f, f_n \rangle + \langle f_n, f_n \rangle$$

$$= \langle f, f \rangle - \langle f_n, f_n \rangle$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f^2 dx - \left(\Sigma_{k=1}^{n} a_k^2 + b_k^2 \right)$$

$$\Rightarrow \frac{1}{2}a_0^2 + \Sigma_{k=1}^{n} a_k^2 + b_k^2 \le \frac{1}{2}a_0^2 + \frac{1}{\pi}\int_{-\pi}^{\pi} f^2 dx < \infty$$

$$\Rightarrow \frac{1}{2}a_0^2 + \Sigma_{k=1}^{n} a_k^2 + b_k^2 \text{ conv.}$$

Prob. 3 Dirichlet test, summation by part,

(1) Let $S_n = \Sigma_{k=1}^{n} a_k b_k$, $B_n = \Sigma_{k=1}^{n} b_k$

$$S_n = a_n B_n + \Sigma_{k=1}^{n-1} B_k (a_k - a_{k+1}), \text{ so we have}$$

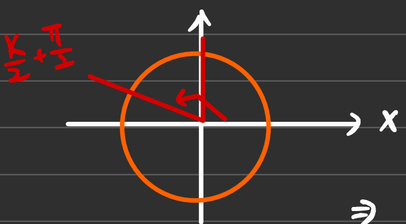
$$|S_n| \le |a_n B_n| + \left| \Sigma_{k=1}^{n-1} B_k (a_k - a_{k+1}) \right|$$

$$\le M |a_n| + M \Sigma_{k=1}^{n-1} |a_k - a_{k+1}| \le M|a_n| + M(a_1 - a_n)$$

As $n \to \infty$ we discover that $a_n B_n$ converge and $\Sigma_{k=1}^{n-1} B_k (a_k - a_{k+1})$ conv., so S_n converge.

(2) First, note that $-f_n(x) = \Sigma_{k=1}^{n} \frac{(-1)^n \sin(nx)}{n}$ is odd on $[-k, k]$, so we just need show $-f_n \xrightarrow{u.t.} -f$ on $[0, k]$, $k < \pi$.

Second, $-f_n(x) = \Sigma_{k=1}^{n} \frac{(-1)^n \sin(kx)}{k} = \Sigma_{k=1}^{n} \frac{\sin(k(x+\pi))}{k}$



Note that

$$\sin(k(x+\pi))\sin\left(\frac{x+\pi}{2}\right) = \frac{1}{2}\left[\cos[k-\tfrac{1}{2}](x+\pi) - \cos(k+\tfrac{1}{2})(x+\pi)\right]$$

$$\Rightarrow \Sigma_{k=1}^{n} \sin(n(x+\pi)) = \frac{\cos(\frac{1}{2}(x+\pi)) - \cos(n+\frac{1}{2})(x+\pi)}{2\sin(\frac{x}{2}+\frac{\pi}{2})}$$

$$= \frac{\sin\left(\frac{n+1}{2}\right)(x+\pi)\sin\frac{n}{2}(x+\pi)}{\sin\left(\frac{x}{2}+\frac{\pi}{2}\right)} \le \frac{1}{\left|\sin\left(\frac{k}{2}+\frac{\pi}{2}\right)\right|}$$

***7.16 Theorem.** [DIRICHLET'S TEST FOR UNIFORM CONVERGENCE]. *Let E be a nonempty subset of $\mathbf{R}$ and suppose that $f_k, g_k : E \to \mathbf{R}$, $k \in \mathbf{N}$. If*

$$\left|\sum_{k=1}^{n} f_k(x)\right| \le M < \infty$$

for $n \in \mathbf{N}$ and $x \in E$, and if $g_k \downarrow 0$ uniformly on E as $k \to \infty$, then $\sum_{k=1}^{\infty} f_k g_k$ converges uniformly on E.

$\Rightarrow$ Take $g_k(x) = \frac{1}{k}$ and $f_k = \sin k(x+\pi)$,

$\sum_{k=1}^{\infty} f_k < 1/\sin\left(\frac{K}{2}+\frac{\pi}{2}\right)$, we done.

Prob. 4

7.25 Theorem *If K is compact, if $f_n \in \mathscr{C}(K)$ for $n = 1, 2, 3, \ldots$, and if $\{f_n\}$ is pointwise bounded and equicontinuous on K, then*

(a) $\{f_n\}$ is uniformly bounded on K,

(b) $\{f_n\}$ contains a uniformly convergent subsequence.

$|F_n(x)| < M \quad \forall x \in [-2,2]$

Since $\{F_n \mid n \in \mathbb{N}\}$ is uniformly bounded $\Rightarrow \{f_n\}$ is uniformly bounded on $[-1,1]$

Note that since F_n is convex, $\forall s,t \in [-1,1]$

$$\frac{F_n(-1)-F_n(-2)}{(-1)-(-2)} \le \underbrace{\frac{F_n(s)-F_n(t)}{s-t}}_{f(s)-f(t)/s-t} \le \frac{F_n(2)-F_n(1)}{2-1}$$

$\Rightarrow |f_n(s)-f_n(t)| \le |F_n(-1)-F_n(-2)| \times |s-t| \le 2M|s-t|$

$\Rightarrow \{f_n\}_{n=1}^{\infty}$ is equicontinuous $\Rightarrow \{f_n\}_{n=1}^{\infty}$ has a u.f conv. subseq.

Prob. 5 Inverse func. Theorem, Contractive map

Let Ω be the set of all inversible linear operators on $\mathbb{R}^n$.

(a) If $A \in \Omega$, $B \in L(\mathbb{R}^n)$, and $\|B-A\| \cdot \|A^{-1}\| < 1$,

then $B \in \Omega$

(b) Ω is an open set and the map $\tau : \Omega \to \Omega$; $A \to A^{-1}$ is continuous.

p.f. Put $\|A^{-1}\| = 1/\alpha$, $\|B-A\| = \beta \Rightarrow \beta < \alpha$. $\forall x \in \mathbb{R}^n$

$$\alpha|x| = \alpha|AA^{-1}x| \le \alpha\|A^{-1}\| \cdot |Ax| = |Ax|$$

$$\le |(A-B)x| + |Bx| \le \beta|x| + |B(x)|$$

$\Rightarrow |\alpha-\beta||x| \le |B(x)| \Rightarrow B(x) \neq 0$ if $|x| \neq 0 \Rightarrow B^{-1}$ exist.

$\Rightarrow$ This hold for all $\|B-A\| < \alpha \Rightarrow \Omega$ is open.

(b) replace x by $B^{-1}y$ $\Rightarrow$ $(\alpha-\beta)|B^{-1}y| \le |BB^{-1}y| = |y|$

$\Rightarrow$ $\|B^{-1}y\| \le 1/\alpha-\beta$, Note that $B^{-1}-A^{-1} = B^{-1}(B-A)A^{-1}$,

So $\|B^{-1}-A^{-1}\| \le \|B^{-1}\|\cdot\|B-A\|\cdot\|A^{-1}\| \le \dfrac{\beta}{\alpha(\alpha-\beta)}$

($\beta = \|B-A\|$ $\Rightarrow$ τ is continuous)

matrix norm ↑ Given exist

Note $f \in C'(E) \iff \partial_x f$ are all conti $\iff \|f'(y)-f'(x)\| < \epsilon$ if $|x-y|<\delta$

$\Rightarrow$ Thus, we first show that g is differentiable and $g'(x) = (f'(y))^{-1}$

For any $x \in U$ ($y=g(x)$). Then $g'(y) = \tau(f'(x)) = \underbrace{\tau}_{\text{conti}} \circ \underbrace{f'(x)}_{\text{conti by hypothesis}}$ is continuous.

Δ By hypothesis, $\forall\, y \in U$ $f'(y)$ is invertible, so $T = (f'(y))^{-1}$ exist

Let $y+h = g(x+k)$, $y = g(x)$, $TT^{-1}h - Tk = -T(k-T^{-1}h)$

$$\frac{|g(x+k)-g(x)-T(k)|}{|k|} = \frac{|h-T(k)|}{|k|} = \frac{|-T[f(y+h)-f(y)-f'(y)h]|}{|k|} \quad \text{—— } (*)$$

$\Rightarrow$ Find relation between h, k

$$\begin{tikzcd} \boxed{x} \arrow[r, shift left, "g"] & \boxed{y} \arrow[l, shift left, "f"] \end{tikzcd}$$

Let $\|f'(y)^{-1}\| = \tilde{\lambda}$, Choose λ such that $2\lambda\tilde{\lambda} < 1$, Since f' is conti.

$\exists\, U \subseteq \mathbb{R}^n$ with center at y s.t. $\|f'(\xi)-f'(y)\| < \lambda$ $\xi \in U$

$$\varphi(\xi) = \xi + (f'(y))^{-1}(x - f(\xi))$$

$$\varphi'(\xi) = I + (f'(y))^{-1} f'(\xi) = (f'(y))^{-1}(f'(y) - f'(\xi))$$

$\Rightarrow$ $\|\varphi'(\xi)\| \le \|f'(y)^{-1}\|\,\|f'(y)-f'(\xi)\| \le \frac{1}{2}$ $\Rightarrow$ φ is a contractive map.

$\Rightarrow$ Then $\varphi(y+h) - \varphi(y) = h + (f'(y))^{-1}(f(y) - f(y+h))$
$= h - (f'(y))^{-1}k$

$\Rightarrow$ $|\varphi(y+h) - \varphi(y)| = |h - (f'(y))^{-1}k| < \frac{1}{2}|h|$ $\quad (|f'(y)^{-1}k| < \frac{3}{2}|h|)$

$$\Rightarrow |(f'(y))^{-1}k| \geq \frac{1}{2}|h| \Rightarrow |k| \geq \frac{1}{2\lambda}|h| \Rightarrow \frac{1}{2\lambda} \times \frac{1}{|k|} \leq \frac{1}{|h|}$$

$$\Rightarrow (*) = \frac{|g(x+k) - g(x) - T(k)|}{|k|} \leq \|T\| 2\lambda \frac{|f(y+h) - f(y) - T^{-1}h|}{|h|}$$

$$\leq \frac{|f(y+h) - f(y) - (f'(y))h|}{|h|} \longrightarrow 0 \text{ as } h \to 0 \quad (k \to 0 \Rightarrow h \to 0)$$

So g is differentiable and $g'(x) = (f'(y))^{-1}$, $y = g(x)$.

Hence $g \in C^1$ by our previous discuss $g'(x) = \tau \circ f'$ #

國立成功大學 112 學年度「碩士班」甄試入學考試

高等微積分

1. (a) Determine the limit of $f(x) = \dfrac{3x^2 - 5x + 4}{x+1}$ at $x = 2$ by δ-ϵ definition. (10%)

   (b) Determine whether $f(x)$ is uniformly continuous on $[0, \infty)$. (10%)

2. Show by definition that the interior of triangular region with vertices $(1,1)$, $(5,2)$, $(3,4)$ is an open set. (10%)

3. Denote x_n the positive root of the polynomial $f_n(x) = x^n + \cdots + x^2 + x - 1$. Show that the sequence $\{x_n\}$ is convergent and find the limit. (15%)

4. Given the fact that the following integral is convergent for all $p > 0$.

$$\int_1^{\infty} \frac{\sin x}{x^p}\, dx$$

   For what values of $p > 0$ is the integral convergent absolutely/conditionally? (15%)

5. (a) By observing the graph of $y = \frac{n}{1+n^2x^2}$ as n increases and evaluating its integral on $(-\infty, \infty)$, find the value the following limit. (10%)

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{n\, e^{\cos x}}{1 + n^2x^2}\, dx$$

   (b) Justify the convergence by N-ϵ definition. (10%)

6. (a) Evaluate the limit. (10%)

$$\lim_{n\to\infty} \left(\frac{3^n + 4^n + 5^n}{3}\right)^{\frac{1}{n}} + \left(\frac{3^{\frac{1}{n}} + 4^{\frac{1}{n}} + 5^{\frac{1}{n}}}{3}\right)^{n}$$

   (b) Evaluate the limit. (10%)

$$\lim_{n\to\infty} \left(\int_0^1 e^{nx(1-x)}\, dx\right)^{\frac{1}{n}} + \left(\int_0^1 e^{\frac{x(1-x)}{n}}\, dx\right)^{n}$$

Prob. 1 Definition & uniformly continuous

(a) $f(2) = 2$, Given $\epsilon > 0$

$$|f(x) - f(2)| = \left| \frac{3x^2 - 7x + 2}{x+1} \right| = \left| \frac{3x-1}{x+1} \right| |x-2|$$

Note that if $|x-2| < \min\{\epsilon, 1\}$, $|3x-1| < 7$

Choose $\delta = \min\{\epsilon/7, 1\}$ $|x-2| < \delta$ implies $|f(x) - f(2)| < \epsilon$ ∗

(b) $$f'(x) = \frac{(6x-5)(x+1) - (3x^2 - 5x + 4)}{(x+1)^2} = \frac{3(x+3)(x-1)}{(x+1)^2}$$

Note that For x enough large, $f'(x) < 4$ and f' is conti on $[0, a]$, when $f'(x) < 4$ for $x > a$, so $\max_{[0,a]} f'(x)$ exist and we denote it by M, then by M.V.T

$$|f(x) - f(y)| \leq \max\{4, M\} |x-y|$$

$\Rightarrow f$ is uniform continuous.

Prob. 2 Def of open set

Let L_1 = line pass $(1,1)$ & $(5,2)$; L_2 = line pass $(5,2)$ & $(3,4)$

L_3 = line pass $(1,1)$ & $(3,4)$, A be the interior of tria-region

If $p \in A$, define $d = \min\{d(p, L_1), d(p, L_2), d(p, L_3)\}$,

then $B_{d/2}(p) \subseteq A$. (if $m \in B_{d/2}(p)$ but $m \notin A$), Consider

$d/2 > d(p,m) > d(p, L_i) + d(m, L_i) > d$, $\Rightarrow \Leftarrow$ ∗

Prob. 3 Axiom of $\mathbb{R}$ (Every bounded monotone seq conv.)

Note $f_n(0) = -1 < 0$ and if $f_n(x_n) = 0$, $f_{n+1}(x_n) = x_n^{n+1} > 0$.

Note another thing that $f_n'(x) = n x^{n-1} + (n-1) x^{n-2} + \cdots + 1$

$f_n'(x) > 0 \quad \forall\, n$ and $x > 0$, Thus, $x_{n+1} \leq x_n$ and bounded below. Thus $\{x_n\}_{n=1}^{\infty}$ is convergent by axiom of $\mathbb{R}$.

Notice that $f_n(x) = \dfrac{x^n - 1}{x-1} - 2$

$$\Rightarrow \lim_{n\to\infty} \frac{x_n^n - 1}{x_n - 1} - 2 = \frac{-1}{\lim_{n\to\infty} x_n - 1} - 2 = 0$$

$$\Rightarrow \lim_{n\to\infty} x_n - 1 = \frac{-1}{2} \Rightarrow \lim_{n\to\infty} x_n = \frac{1}{2} \quad \#$$

Prob. 4 Convergent absolutely / conditionally

(a)
$$\int_1^{\infty} \frac{\sin x}{x^p}\,dx = -\frac{\cos x}{x^p}\Big|_1^{\infty} + \int_1^{\infty} \frac{-1}{p}\frac{\cos x}{x^{p+1}}\,dx$$

$$= \cos 1 - \int_1^{\infty} \frac{1}{p}\frac{\cos x}{x^{p+1}}\,dx$$

Since $\displaystyle\int_1^{\infty} \frac{1}{p}\frac{\cos x}{x^{p+1}}\,dx \le \int_1^{\infty} \frac{1}{x^{p+1}}\,dx < \infty$, we done.

(b)
$$\int_1^{\infty} \left|\frac{\sin x}{x^p}\right| dx = \overbrace{\int_1^{2\pi} \left|\frac{\sin x}{x^p}\right| dx}^{A} + \sum_{k=1}^{\infty} \int_{2k\pi}^{2(k+1)\pi} \frac{|\sin x|}{x^p}\,dx$$

$$\ge A + \sum_{k=1}^{\infty} \frac{1}{(2(k+1)\pi)^p} \underbrace{\int_{2k\pi}^{2(k+1)\pi} |\sin x|\,dx}_{\text{Constant, } B}$$

$$\ge A + \frac{B}{(2\pi)^p} \sum_{k=1}^{\infty} \frac{1}{(k+1)^p}$$

Similar
$$\int_1^{\infty} \left|\frac{\sin x}{x^p}\right| dx \le A + \frac{B}{(2\pi)^p} \sum_{k=1}^{\infty} \frac{1}{k^p}$$

Thus, $\displaystyle\int_1^{\infty} \frac{\sin x}{x^p}\,dx$ conv. absolutely iff $p > 1$

Prob. 5 Convolution

Let $y = \dfrac{1}{1+x^2}$, $\underbrace{\dfrac{1}{\epsilon} y\left(\dfrac{x}{\epsilon}\right)}_{y_\epsilon(x)} = \dfrac{1}{\epsilon}\dfrac{1}{1+(x/\epsilon)^2} = \dfrac{n}{1+n^2x^2}$

when we look at $\epsilon \to 0$ as $n \to \infty$, $n = 1/\epsilon$.

Note that $f * \underbrace{\varphi_\epsilon(x)}_{e} \longrightarrow f(x)$ as $\epsilon \to 0$

poisson kernel

so $$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{n e^{\cos x}}{1+n^2x^2} dx = \lim_{\epsilon \to 0} e^{\cos x} * \frac{1}{1+x^2} = \pi e$$

$\Rightarrow p(x) = \frac{1}{\pi} \frac{1}{1+x^2} \in o\left(\frac{1}{x}\right)$, $\int_{-\infty}^{+\infty} p(x)\, dx = 1$

$f \in L^\infty(\mathbb{R}) \Rightarrow f * p \to f$ at continuous point of f.

Prob. 6 ℓ_p norm, ℓ_∞ norm,

(a) $$\lim_{n\to\infty} \left(\frac{3^n+4^n+5^n}{3}\right)^{\frac{1}{n}} \le \lim_{n\to\infty} 3^{\frac{1}{n}} \times 5 = 5$$

$$\lim_{n\to\infty} \left(\frac{3^n+4^n+5^n}{3}\right)^{\frac{1}{n}} \ge \lim_{n\to\infty} \left(\frac{5^n}{3}\right)^{\frac{1}{n}} = 5$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{3^n+4^n+5^n}{3}\right)^{\frac{1}{n}} = 5$$

$$\lim_{n\to\infty} \left(\frac{3^{1/n}+4^{1/n}+5^{1/n}}{3}\right)^n = \lim_{\epsilon\to 0} \left(\frac{3^\epsilon+4^\epsilon+5^\epsilon}{3}\right)^{\frac{1}{\epsilon}} = \lim_{\epsilon\to 0} e^{\frac{\ln(3^\epsilon+4^\epsilon+5^\epsilon)}{\epsilon}}$$

$$\overset{L.H}{\Leftarrow} \lim_{\epsilon\to 0} e^{\frac{1}{3^\epsilon+4^\epsilon+5^\epsilon} \times \left((\ln 3)3^\epsilon + (\ln 4)4^\epsilon + (\ln 5)5^\epsilon\right)}$$

$$\Leftarrow \exp\left(\frac{\ln 3 + \ln 4 + \ln 5}{3}\right)$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{3^n+4^n+5^n}{3}\right)^{\frac{1}{n}} + \left(\frac{3^{1/n}+4^{1/n}+5^{1/n}}{3}\right)^n = 5 + \exp\left(\frac{\ln 3+\ln 4+\ln 5}{3}\right)$$

(b) We claim that $\|\cdot\|_{L^p} \to \|\cdot\|_\infty$ as $p \to \infty$ when $|E| < \infty$

$$\|f\|_p^p = \int_E |f|^p dx \le \|f\|_\infty^p |E| \Rightarrow \|f\|_p \le \|f\|_\infty |E|^{\frac{1}{p}}$$

$$\Rightarrow \lim_{p\to\infty} \|f\|_p \le \|f\|_\infty$$

Given $\epsilon > 0$ $|A = \{ |f| > \|f\|_\infty - \epsilon \}| > 0$

$$\|f\|_p^p = \int_E |f|^p dx \ge \int_A |f|^p \ge \int_A (\|f\|_\infty - \epsilon)^p = |A| \times (\|f\|_\infty - \epsilon)^p$$

$$\Rightarrow \|f\|_p \ge |A|^{\frac{1}{p}} (\|f\|_\infty - \epsilon) \Rightarrow \lim_{p\to\infty} \|f\|_p \ge \|f\|_\infty - \epsilon$$

$\epsilon \to 0 \Rightarrow \lim_{p\to\infty} \|f\|_p \ge \|f\|_\infty$. Thus $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$

Let E be a measurable set in $\mathbf{R}^n$ with $|E| < \infty$. Suppose that $f > 0$ a.e. in E and $f, \log f \in L^1(E)$. Prove that

$$\lim_{p \to 0+} \left(\frac{1}{|E|} \int_E f^p \right)^{1/p} = \exp\left(\frac{1}{|E|} \int_E \log f \right).$$

or $\displaystyle \lim_{n\to\infty} \left(\frac{1}{|E|} \int_E |f|^{\frac{1}{n}} \right)^n = \lim_{\epsilon \to 0} \left(\frac{1}{|E|} \int_E |f|^{\epsilon} \right)^{1/\epsilon}$

$$= \lim_{\epsilon\to 0} \exp\left(\frac{\ln\left(\frac{1}{|E|} \int_E |f|^{\epsilon} \right)}{\epsilon} \right) \overset{L.H}{\Leftarrow} \lim_{\epsilon \to 0} \exp\left(\frac{\frac{1}{|E|} \int_E (\ln|f|)|f|^{\epsilon}}{\frac{1}{|E|} \int_E |f|^{\epsilon} dx} \right)$$

$$= \exp\left(\frac{1}{|E|} \int_E \ln|f| \, dx \right)$$

So $\displaystyle \lim_{n\to\infty} \left(\int_0^1 \left(e^{x(1-x)} \right)^n dx \right)^{\frac{1}{n}} + \lim_{n\to\infty} \left(\int_0^1 \left(e^{x(1-x)} \right)^{\frac{1}{n}} \right)^n$

$$= \| e^{x(1-x)} \|_\infty + \exp\left(\int_0^1 x(1-x)\, dx \right)$$

$$= \max_{[0,1]} \exp\left(\frac{1}{4} - \left(x - \frac{1}{2}\right)^2 \right) + \exp\left(\frac{1}{2}x^2 - \frac{1}{3}x^3 \Big|_0^1 \right)$$

$$= e^{\frac{1}{4}} + e^{\frac{1}{6}} \quad \#$$

國立成功大學 111 學年度「碩士班」研究生甄試入學考試

高等微積分

Throughout the exam, the Euclidean spaces $\mathbb{R}^n$ are all equipped with usual Euclidean metric $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.

1. (10 points) Evaluate the line integral

$$\int_C \mathbf{V} \cdot d\mathbf{r},$$

where the vector field is given by

$$\mathbf{V} = \langle 2y + xe^{x^2}, 4x + e^{y^2} \cos y \rangle,$$

and C is the path staring from $(1,0)$ to $(0,-1)$ by a *counterclockwise* circular path, and then from $(0,-1)$ to $(0,0)$ by a straight line.

2. For a continuous function $f : [a,b] \to \mathbb{R}$ that is differentiable on (a,b),
   (a) (8 points) State and prove the *Rolle's Theorem* for f.
   (b) (7 points) State and prove the *Mean Value Theorem* for f.

3. (10 points) Given the fact that $\mathbb{R}$ is complete, prove the *Monotone Convergence Theorem*:

Any bounded monotonic sequence in $\mathbb{R}$ is convergent.

4. Consider the usual Euclidean space $\mathbb{R}^n$,
   (a) (8 points) If $E \subset K \subset \mathbb{R}^n$, where E is closed and K is compact, then E is compact.
   (b) (8 points) Prove that if $\{K_\alpha\}_{\alpha \in A}$ is any family of compact subsets of $\mathbb{R}^n$, then

$$\bigcap_{\alpha \in A} K_\alpha$$

is also compact.

5. (15 points) Prove that the sequence of functions $\{f_n\}$ on $[0,1]$ given by

$$f_n(x) = nx(1-x^2)^n$$

pointwise converges to a function $f(x)$ but does not converge uniformly.

6. Given two metric spaces (E, d_E) and (F, d_F),

(a) (8 points) If $f : E \to F$ is continuous and E is compact, then f is uniformly continuous.

(b) (8 points) Use part (a) to prove that

$$f(x) = e^{\cos^2 x \sin x}$$

is uniformly continuous on $\mathbb{R}$.

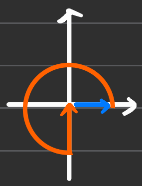
7. (a) (4 points) State the *Inverse Function Theorem* on $\mathbb{R}^n$.

(b) (4 points) State the *Implicit Function Theorem* on $\mathbb{R}^{n+m}$.

(c) (10 points) These two theorems are in fact equivalent. Prove *either* one of the implications. (i.e. prove either (a) $\Rightarrow$ (b) or (b) $\Rightarrow$ (a)).

Prob. 1 **Vector calculus**

$V = \langle 2y + xe^{x^2},\ 4x + e^{y^2}\cos y \rangle = \langle P(x,y),\ Q(x,y) \rangle$



$\int_C V \cdot dr = \int_A \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA - \int_{C_3} V \cdot dr$, where

$A = \frac{3}{4}$ circle

$\Rightarrow \int_A \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \frac{3}{4}\pi \times 1^2 \times 2$

$C_3 = (t, 0)\quad t \in [0,1]$

$$\int_0^1 te^{t^2}\,dt = \frac{1}{2}e^{t^2}\Big|_0^1 = \frac{e}{2} - \frac{1}{2}$$

$$\int_C V \cdot dr = 2 - \left(\frac{e}{2} - \frac{1}{2}\right) = \frac{5}{2} - \frac{e}{2} \quad \#$$

Prob. 2 State : If $f(a) = f(b)$, $\exists\ c \in (a,b)$, $f'(c) = 0$

State : $\exists\ c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$

general : if f, g conti on $[a,b]$ and diff on (a,b)

$\exists\ c \in (a,b)$ s.t. $f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a))$

$\Rightarrow$ consider $h(x) = (f(x) - f(a))(g(b) - g(a)) - (g(x) - g(a))(f(b) - f(a))$

Then $h(a) = h(b) = 0$, by Rolle's Theorem, $\exists\ c \in (a,b)$ $h'(c) = 0$.

so $f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a))$ #

Prob. 3 Consider $\sup\{x_n\}_{n=1}^{\infty}$, Since $\{x_n\}_{n=1}^{\infty}$ is bounded, $\sup\{x_n\}_{n=1}^{\infty} < \infty$

That is easy to show $x_n \to \sup\{x_n\}_{n=1}^{\infty}$ (Assume $x_n \uparrow$)

Prob. 4 (a) Let $\{O_\alpha\}_{\alpha \in A}$ be a open cover of E, then $\{E^c\} \cup \{O_\alpha\}_{\alpha \in A}$ is a open cover of K, Since K is compact, $\exists\ I$, $|I| < \infty$ such that

$K \subseteq \left(\bigsqcup_{\alpha \in I} O_\alpha\right) \cup E^c$, so $E \subseteq \left(\bigsqcup_{\alpha \in I} O_\alpha\right) \cup E^c \Rightarrow E \subseteq \bigsqcup_{\alpha \in I} O_\alpha$ #

(b) K_α is compact $\Rightarrow$ K_α is closed and bounded $\Rightarrow$ $\prod_\alpha K_\alpha$ is closed

$\Rightarrow \prod_{\alpha \in A} K_\alpha$ is a closed set of K_α $\Rightarrow \prod_\alpha K_\alpha$ is compact.

Prob. 5

(a) $\lim_{n\to\infty} nx(1-x^2)^n = 0$ since if $x=0,1 \Rightarrow nx(1-x^2)^n \equiv 0$

if $x \in (0,1) \Rightarrow (1-x^2) \in (0,1)$, $(1-x^2) = 1/p$ $p>1$

$\lim_{n\to\infty} nx(1-x^2)^n = \frac{nx}{p^n} \overset{L.H}{\Longleftarrow} \lim_{n\to\infty} \frac{x}{\ln p(p^n)} = 0$ so $f_n \to f(x)=0$ on $[0,1]$

Note that $\sup_x f_n(x) = \frac{n}{\sqrt{2n+1}} \times \left(\frac{2n}{2n+1}\right)^n \to \infty \cdot e^{\frac{1}{2}}$

so f_n does not converge uniformly.

Prob. 6

Given $\epsilon > 0$, For each $x \in E$, $\exists\ \delta_x > 0$ such that $|f(y) - f(x)| < \epsilon$

as $|x-y| < \delta_x$, Consider $\{ \mathcal{O}_x = B_{\frac{\delta_x}{2}}(x) \}_{x\in E}$, Since E is compact

$\exists\ N \in \mathbb{N}$ s.t. $E \subseteq \bigcup_{x_i, i=1}^{N} B_{\delta_{x_i}}(x)$, let $\delta_0 = \frac{1}{3}\min\{\delta_{x_i}\}_{i=1}^{N}$

if $|x_0 - y_0| < \delta_0$, note that $x_0 \in B_{\delta_{x_i}}(x_i)$ for some x_i

then $|y_0 - x_i| \le |x_i - x_0| + |x_0 - y_0| \le \frac{1}{2}\delta_x + \frac{1}{3}\delta_x < \delta_x$

so $|f(x_0) - f(y_0)| \le |f(x_0) - f(x_i)| + |f(x_i) - f(y_0)| < 2\epsilon$ #

(b)

Note that f has a period 2π and f is conti on $[0,2\pi]$,

by (a) f is uniform conti on $[0,2\pi]$, moreover, f is u.c. on $[2k\pi, 2(k+1)\pi]$

$k \in \mathbb{Z}$, Then f is u.f conti on $\mathbb{R}$ (Given $\epsilon > 0$, $\forall\ x,y \in \mathbb{R}$ $\exists\ \delta > 0$

such that if $|x-y| < \delta \Rightarrow |f(x-y)| < \epsilon$ when $x,y \in$ some $[2k\pi, 2(k+1)\pi]$

$(y<x)$

Then take $\delta_0 = \frac{1}{3}\delta$, $\forall\ x,y \in \mathbb{R}$, if $x \in [2k\pi, 2(k+1)\pi]$ but

$y \in [2k-1\pi, 2k\pi] \Rightarrow |x-y| < |x-2k\pi| + |y-2k\pi| < \delta \Rightarrow |f(x)-f(y)| < \epsilon$

(a)
Prob. 7

Let $V \overset{open}{\subseteq} \mathbb{R}^n$ and $f: V \xrightarrow{C^1} \mathbb{R}^n$, if $a \in V$ such that

$$Jf(a) = \begin{pmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 & \cdots & \partial_{x_n} f_1 \\ \vdots & & & \\ \partial_{x_1} f_n & \partial_{x_2} f_n & \cdots & \partial_{x_n} f_n \end{pmatrix}$$ full rank ($\det Jf(a) \neq 0$)

Then $\exists\ W \overset{open}{\in} \mathbb{R}^n$, $a \in W$, s.t. (1) f is 1-1 (2) f^{-1} is C^1

(3) $Jf^{-1}(f(a)) = (Jf(a))^{-1}$

(b) Let $V \overset{open}{\subseteq} \mathbb{R}^{n+m}$, $f : V \xrightarrow{C^1} \mathbb{R}^n$, Let $v \in \mathbb{R}^{n+m}$ be rewritten by

(x,t), $x \in \mathbb{R}^n$ and $t \in \mathbb{R}^m$. If $J_{f,\mathbb{R}^n} = \dfrac{\partial(f_1,\cdots,f_n)}{\partial(x_1,\cdots,x_n)}$ at some (x_0,t_0) full rank,

$f(x_0,t_0) = 0$

Then exist $W \in \mathbb{R}^m$ $t_0 \in W$ and $g(t) : W \to V$ s.t.

(1) $f(g(t),t) = 0$ on $t \in W$ & $g(t_0) = x_0$

(2) g is unique & C^1

(3) $J_g(t_0) = -\left(\dfrac{\partial(f_1,\cdots,f_n)}{\partial(x_1,\cdots,x_n)}\right)^{-1}\left(\dfrac{\partial(f_1,\cdots,f_n)}{\partial(t_1,\cdots,t_{m})}\right)$ $\to \begin{pmatrix} \partial_{x_1}f_1 & \cdots & \partial_{x_n}f_1 \\ \vdots & & \\ \partial_{x_1}f_n & \cdots & \partial_{x_n}f_n \end{pmatrix}^{-1} \begin{pmatrix} \partial_{t_1}f_1 & \cdots & \partial_{t_{m+p}}f_1 \\ \vdots & & \\ \partial_{t_1}f_n & \cdots & \partial_{t_p}f_n \end{pmatrix}$

(c) Let $\tilde{F}(x,t) = (F(x,t), t)$, $J_{\tilde{F}(x_0,t_0)} = \begin{array}{cc} n\times n & n\times m \\ \left(\begin{matrix} J_{F,\mathbb{R}^n}(x_0,t_0) & X \\ 0 & I \end{matrix}\right) \\ m\times n & m\times m \end{array}$

$\det(J_{\tilde{F}}(x_0,t_0)) = \det(J_{F,\mathbb{R}^n}(x_0,t_0)) \neq 0$ $\quad \tilde{F}(x_0,t_0) = (0,t_0)$

$(x_0,t_0) \in \Omega_1$, $(0,t_0) \in \Omega_2$

$\Rightarrow$ IFT $\Rightarrow \exists\ G(x,t) \in C^1$ s.t. $\begin{cases} \tilde{F}(x,t) \circ G(x,t) = (x,t) \\ G(x,t) \circ \tilde{F}(x,t) = (x,t) \end{cases}$

Let $\phi(x,t) = (G_1,\cdots,G_n)$ and $g(t) = \phi(0,t)$ on

$$W : \{ t \in \mathbb{R}^p : (0,t) \in \Omega_2 \}$$

Then $g(t)$ is a func. we want.