and so (14.13) implies that

$$\langle \chi, \psi \rangle = \sum_{i=1}^{k} c_i d_i.$$

The result follows.

■

Decomposing $\mathbb{C}$G-modules

It is sometimes of practical importance to be able to decompose a given $\mathbb{C}G$-module into a direct sum of $\mathbb{C}G$-submodules, and we now describe a process for doing this.

Once more we adopt Hypothesis 14.7:

> $\mathbb{C}G = W_1 \oplus W_2$, where the $\mathbb{C}G$-modules W_1 and W_2 have no common composition factor; and $1 = e_1 + e_2$ with $e_1 \in W_1$, $e_2 \in W_2$.

Let V be any $\mathbb{C}G$-module. We can write $V = V_1 \oplus V_2$, where every composition factor of V_1 is a composition factor of W_1 and every composition factor of V_2 is a composition factor of W_2.

14.25 Proposition

With the above notation, for all $v_1 \in V_1$ and $v_2 \in V_2$ we have

$$v_1 e_1 = v_1, \quad v_2 e_1 = 0,$$
$$v_1 e_2 = 0, \quad v_2 e_2 = v_2.$$

Proof If $v_1 \in V_1$ then the function $w_2 \to v_1 w_2$ $(w_2 \in W_2)$ is clearly a $\mathbb{C}G$-homomorphism from W_2 to V_1. Since W_2 and V_1 have no common composition factor, we deduce the stated results just as in the proof of Proposition 14.8.

■

14.26 Proposition
If χ is an irreducible character of G, and V is any $\mathbb{C}G$-module, then

$$V\left(\sum_{g\in G}\chi(g^{-1})g\right)$$

is equal to the sum of those $\mathbb{C}G$-submodules of V which have character χ (where for $r \in \mathbb{C}G$, we define $Vr = \{vr: v \in V\}$).

Proof Write

$$\mathbb{C}G = U_1 \oplus \ldots \oplus U_r,$$

a direct sum of irreducible $\mathbb{C}G$-submodules U_i. Let W_1 be the sum of those $\mathbb{C}G$-submodules U_i which have character χ, and let W_2 be the sum of the remaining $\mathbb{C}G$-submodules U_i. Then the character of W_1 is $m\chi$ where $m = \chi(1)$, by Theorem 11.9. Also W_1 and W_2 satisfy Hypothesis 14.7, and by Proposition 14.10, the element e_1 of W_1 is given by

$$e_1 = \frac{m}{|G|}\sum_{g\in G}\chi(g^{-1})g.$$

Let V_1 be the sum of those $\mathbb{C}G$-submodules of V which have character χ. Then Proposition 14.25 shows that $Ve_1 = V_1$. Clearly we may omit the constant multiplier $m/|G|$, so

$$V_1 = V\left(\sum_{g\in G}\chi(g^{-1})g\right).$$

■

Once the irreducible characters of our group G are known, Proposition 14.26 provides a useful practical tool for finding $\mathbb{C}G$-submodules of a given $\mathbb{C}G$-module V. The procedure is as follows:

(14.27) (1) Choose a basis $v_1, \ldots, v_n$ of V.

(2) For each irreducible character χ of G, calculate the vectors $v_i(\sum_{g\in G}\chi(g^{-1})g)$ for $1 \leqslant i \leqslant n$, and let V_χ be the subspace of V spanned by these vectors.

(3) Then V is the direct sum of the $\mathbb{C}G$-modules V_χ as χ runs over the irreducible characters of G. The character of V_χ is a multiple of χ.

We illustrate this method with a couple of simple examples. Some more complicated uses of the method can be found in Chapter 32.

14.28 Examples

(1) Let G be any finite group and let V be any non-zero $\mathbb{C}G$-module. Taking χ to be the trivial character of G in Proposition 14.26, we see that

$$V\left(\sum_{g\in G} g\right)$$

is the sum of all the trivial $\mathbb{C}G$-submodules of V. For example, let $G = S_n$ and let V be the permutation module, with basis $v_1, \ldots, v_n$ such that $v_i g = v_{ig}$ for all i and all $g \in G$. Then

$$V\left(\sum_{g\in G} g\right) = \mathrm{sp}\,(v_1 + \ldots + v_n).$$

Hence V has a unique trivial $\mathbb{C}G$-submodule.

(2) Let G be the subgroup of S_4 which is generated by

$$a = (1\,2\,3\,4) \text{ and } b = (1\,2)(3\,4).$$

Then $G \cong D_8$ (compare Example 1.5). Here is a list of the irreducible characters $\chi_1, \ldots, \chi_5$ of D_8 (see Example 16.3(3)):

	1	a	a^2	a^3	b	ab	a^2b	a^3b
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	-1	-1
χ_3	1	-1	1	-1	1	-1	1	-1
χ_4	1	-1	1	-1	-1	1	-1	1
χ_5	2	0	-2	0	0	0	0	0

Let V be the permutation module for G, with basis v_1, v_2, v_3, v_4 such that $v_i g = v_{ig}$ for all i and all $g \in G$.

For $1 \leqslant i \leqslant 5$, let

$$e_i = \frac{\chi_i(1)}{8} \sum_{g \in G} \chi_i(g^{-1})g.$$

For example, $e_5 = \frac{1}{2}(1 - a^2)$. Then

$$\begin{aligned}
Ve_1 &= \text{sp}\,(v_1 + v_2 + v_3 + v_4), \\
Ve_2 &= 0, \\
Ve_3 &= 0, \\
Ve_4 &= \text{sp}\,(v_1 - v_2 + v_3 - v_4), \\
Ve_5 &= \text{sp}\,(v_1 - v_3,\, v_2 - v_4).
\end{aligned}$$

We have

$$V = Ve_1 \oplus Ve_4 \oplus Ve_5,$$

and so we have expressed V as a direct sum of irreducible $\mathbb{C}G$-submodules whose characters are χ_1, χ_4 and χ_5, respectively.

You might like to check that

$$e_1 + \ldots + e_5 = 1,$$

$$e_i^2 = e_i \text{ for } 1 \leqslant i \leqslant 5,$$

$$e_i e_j = 0 \text{ for } i \neq j.$$

Compare these results with Corollary 14.9.

Note that the procedure described in (14.27) does *not* in general enable us to write a given $\mathbb{C}G$-module as a direct sum of irreducible $\mathbb{C}G$-submodules (since V_χ is not in general irreducible).

Summary of Chapter 14

1. The inner product of two functions ϑ, ϕ from G to $\mathbb{C}$ is given by

$$\langle \vartheta, \phi \rangle = \frac{1}{|G|} \sum_{g \in G} \vartheta(g) \overline{\phi(g)}.$$

2. The irreducible characters $\chi_1, \ldots, \chi_k$ of G form an orthonormal set; that is, $\langle \chi_i, \chi_j \rangle = \delta_{ij}$ for all i, j.
3. Every $\mathbb{C}G$-module is determined by its character.
4. If $\chi_1, \ldots, \chi_k$ are the irreducible characters of G, and ψ is any character, then

$$\psi = d_1\chi_1 + \ldots + d_k\chi_k \text{ where } d_i = \langle \psi, \chi_i \rangle.$$

   Each d_i is a non-negative integer. Also, ψ is irreducible if and only if $\langle \psi, \psi \rangle = 1$.

Exercises for Chapter 14

1. Let $G = S_4$. We shall see in Chapter 18 that G has characters χ and ψ which take the following values on the conjugacy classes:

Class representative	1	(1 2)	(1 2 3)	(1 2)(3 4)	(1 2 3 4)
$\lvert C_G(g_i)\rvert$	24	4	3	8	4
χ	3	-1	0	3	-1
ψ	3	1	0	-1	-1

Calculate $\langle \chi, \chi \rangle$, $\langle \chi, \psi \rangle$ and $\langle \psi, \psi \rangle$. Which of χ and ψ is irreducible?

2. Let $G = Q_8 = \langle a, b\colon a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1} \rangle$, and let ρ_1, ρ_2, ρ_3 be the representations of G over $\mathbb{C}$ for which

$$a\rho_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \; b\rho_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$a\rho_2 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \; b\rho_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$a\rho_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \; b\rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show that ρ_1 and ρ_2 are equivalent, but ρ_3 is not equivalent to ρ_1 or ρ_2.

3. Suppose that ρ and σ are representations of G, and that for each g in G there is an invertible matrix T_g such that

$$g\sigma = T_g^{-1}(g\rho)T_g.$$

Prove that there is an invertible matrix T such that for all g in G,

$$g\sigma = T^{-1}(g\rho)T.$$

4. Suppose that χ is a non-zero, non-trivial character of G, and that $\chi(g)$ is a non-negative real number for all g in G. Prove that χ is reducible.
5. If χ is a character of G, show that

$$\langle \chi_{\mathrm{reg}}, \chi \rangle = \chi(1).$$

6. If π is the permutation character of S_n, prove that

$$\langle \pi, 1_{S_n} \rangle = 1.$$

   (Hint: you may find Exercise 11.4 relevant.)

7. Let $\chi_1, \ldots, \chi_k$ be the irreducible characters of the group G, and suppose that

$$\psi = d_1\chi_1 + \ldots d_k\chi_k$$

   is a character of G. What can you say about the integers d_i in the cases $\langle \psi, \psi \rangle = 1, 2, 3$ or 4?

8. Suppose that χ is a character of G and that for every $g \in G$, $\chi(g)$ is an even integer. Does it follow that $\chi = 2\phi$ for some character ϕ?