1.4 HW3

Question 26

Let f be a simple function, taking its distinct values on disjoint sets $E_1, \dots, E_N$. Show that f is measurable if and only if $E_1, \dots, E_N$ are measurable.

Proof. WOLG, let f take value a_n on E_n and

$$a_1 < a_2 < \cdots < a_N$$

If $E_1, \dots, E_N$ are all measurable, we see that for each $a \in \mathbb{R}$

$$\{f \geq a\} = \{f \geq a_n\} = E_n \sqcup \cdots \sqcup E_N \text{ is measurable}$$

where n is the smallest integer such that $a_n \geq a$. We have prove the if part. To see the only if part hold true, observe that for all $n \in \{1, \dots, N-1\}$

$$E_n = \{f \geq a_n\} \setminus \{f \geq a_{n+1}\} \text{ is measurable}$$

and

$$E_N = \{f \geq a_N\} \text{ is measurable}$$

■

Question 27

Let f be defined and measurable on $\mathbb{R}^n$. If T is a nonsingular linear transformation of $\mathbb{R}^n$, show that $f(T\mathbf{x})$ is measurable. (If $E_1 = \{\mathbf{x} : f(\mathbf{x}) > a\}$, and $E_2 = \{\mathbf{x} : f(T\mathbf{x}) > a\}$, show that $E_2 = T^{-1}E_1$)

Proof. Fix $a \in \mathbb{R}$. We are required to show

$$\{\mathbf{x} : f(T\mathbf{x}) > a\} \text{ is measurable}$$

Because f is measurable, we know $\{\mathbf{x} : f(\mathbf{x}) > a\}$ is measurable. The proof then follows from noting

$$\{\mathbf{x} : f(T\mathbf{x}) > a\} = T^{-1}\Big(\{\mathbf{x} : f(\mathbf{x}) > a\}\Big)$$

and the fact that $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ as a linear transformation preserve measurability. ■

Question 28

Give an example to show that $\varphi \circ f$ may not be measurable if $\varphi, f : \mathbb{R} \to \mathbb{R}$ are measurable and finite. (Let F be the Cantor-Lebesgue function and let f be its inverse suitably defined. Let φ be the characteristic function of a set of measure zero whose image under F is not measurable.) Show that the same may be true even if f is continuous. (Let $g(x) = x + F(x)$ and consider $f = g^{-1}$)

Proof. Let $F : [0,1] \to [0,1]$ be the Cantor-Lebesgue function, $\mathcal{C} \subseteq [0,1]$ be the classical ternary Cantor set. Note that $F(\mathcal{C}) = [0,1]$. By axiom of choice, we can let $\mathcal{C}'$ be some subset of $\mathcal{C}$ such that $F|_{\mathcal{C}'} : \mathcal{C}' \to [0,1]$ is a bijection. We can now define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) \triangleq \begin{cases} (F|_{\mathcal{C}'})^{-1}(x) & \text{if } x \in [0,1] \\ x & \text{if } x \notin [0,1] \end{cases}$$

$f : \mathbb{R} \to \mathbb{R}$ is measurable because f is increasing. Let V be a non-measurable set contained by $[0,1]$, and let $E \triangleq f(V)$. Define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(x) \triangleq \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$$

Note that E is measurable because

$$V \subseteq [0,1] \implies E = f(V) = (F|_{\mathcal{C}'})^{-1}(V) \subseteq \mathcal{C}'$$

It then follows that $\varphi : \mathbb{R} \to \mathbb{R}$ is measurable. Lastly, to see $\varphi \circ f : \mathbb{R} \to \mathbb{R}$ is not measurable, observe that

$$(\varphi \circ f)^{-1}(\{1\}) = f^{-1}(E) = V \text{ is not measurable}$$

where the last inequality follows since $f|_V : V \to E$ is a bijection.

For the second part. Define $g : [0,1] \to [0,2]$ by

$$g(x) \triangleq x + F(x)$$

Because $F : [0,1] \to [0,1]$ is increasing, we may deduce

$$x < y \text{ and } x, y \in [0,1] \implies x + F(x) < y + F(y)$$

This implies g is strictly increasing. Note that g is continuous because g is the addition of two continuous function, and note that $g(0) = 0, g(1) = 2$. This allow us to deduce $g : [0,1] \to [0,2]$ is a bijection. Now, observe that $[0,1] \setminus \mathcal{C}$ is a countable union of disjoint open interval. For each connected components $I \subseteq [0,1] \setminus \mathcal{C}$, because F maps I to some constant, we see $g(I)$ is also an interval with the same length $|g(I)| = I$. Then from $|[0,1] \setminus \mathcal{C}| = 1$, we can deduce $|g([0,1] \setminus \mathcal{C})| = 1$, which implies $g(\mathcal{C}) = 1$. We then can let V be some non-measurable set contained by $g(\mathcal{C})$. Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) \triangleq \begin{cases} g^{-1}(x) & \text{if } x \in [0,2] \\ \frac{x}{2} & \text{if } x \notin [0,2] \end{cases}$$

$h : \mathbb{R} \to \mathbb{R}$ is measurable because it is increasing. Let $E \triangleq h(V)$. We see $E \subseteq \mathcal{C}$, which implies E is measurable, so when we define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(x) \triangleq \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

we see $\varphi : \mathbb{R} \to \mathbb{R}$ is also measurable. Lastly, to see $\varphi \circ h : \mathbb{R} \to \mathbb{R}$ is not measurable, observe

$$(\varphi \circ h)^{-1}(\{1\}) = h^{-1}(E) = V$$

■

Question 29

(a) Show that the limit of a decreasing (increasing) sequence of functions upper (lower) semicontinuous at $\mathbf{x}_0$ is upper (lower) semicontinuous at $\mathbf{x}_0$. In particular, the limit of a decreasing (increasing) sequence of functions continuous at $\mathbf{x}_0$ is upper (lower) semicontinuous at $\mathbf{x}_0$.

(b) Let f be upper semicontinuous and less than ∞ on $[a,b]$. Show that there exists continuous f_k on $[a,b]$ such that $f_k \searrow f$. (First show that there exist continuous f_k on $[a,b]$ such that $f_k \searrow f$)

Proof. **(a)** Let $f_n \searrow f$ and f_n be upper semicontinuous at $\mathbf{x}_0$. Fix ϵ. Because $f_n(\mathbf{x}_0) \searrow f(\mathbf{x}_0)$, we know there exists some N such that $f_N(\mathbf{x}_0) < f(\mathbf{x}_0) + \epsilon$. Because $f \leq f_N$ everywhere and f_N is upper semicontinuous at $\mathbf{x}_0$, we have

$$\limsup_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) \leq \limsup_{\mathbf{x} \to \mathbf{x}_0} f_N(\mathbf{x}) \leq f_N(\mathbf{x}_0) < f(\mathbf{x}_0) + \epsilon$$

Because ϵ can be arbitrary small, we have shown

$$\limsup_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) \leq f(\mathbf{x}_0)$$

i.e., f is also upper semicontinuous. The second part of question (a) ask us to prove the same thing with stronger hypothesis that f_n are continuous at $\mathbf{x}_0$, which implies f_n are upper semicontinuous at $\mathbf{x}_0$, so it immediately follows from what we have proved.

(b) Define $f_n : [a,b] \to [-\infty, \infty]$ by

$$f_n(x) \triangleq \sup_{p\in[a,b]} \Big(f(p) - n\,|x-p| \Big)$$

Fix n. We first show that $f_n : [a,b] \to [-\infty,\infty]$ is continuous. Fix ϵ, and let $x, y \in [a,b]$ satisfy $|x-y| < \frac{\epsilon}{n}$. For all $p \in [a,b]$, we may use reverse triangle inequality to show that

$$\Big| \Big(f(p) - n\,|x-p| \Big) - \Big(f(p) - n\,|y-p| \Big) \Big| = n \Big| \Big(|x-p| - |y-p| \Big) \Big| \leq n\,|x-y| < \epsilon$$

This implies

$$f(p) - n\,|x-p| + \epsilon > f(p) - n\,|y-p| > f(p) - n\,|x-p| - \epsilon$$

Taking supremum on both side, we have

$$f_n(x) + \epsilon = \sup_{p\in[a,b]} \Big(f(p) - n\,|x-p| \Big) + \epsilon \geq \sup_{p\in[a,b]} \Big(f(p) - n\,|y-p| \Big) = f_n(y)$$

In summary,

$$f_n(x) + \epsilon \geq f_n(y) \geq f_n(x) - \epsilon$$

Because ϵ is arbitrary, we have shown f_n is (uniform) continuous. We now show f_n is decreasing. Fix $x \in [a,b]$ and $n < m$. Observe that for all $p \in [a,b]$

$$f(p) - m\,|x-p| < f(p) - n\,|x-p|$$

Taking supremum on both side, we have

$$f_m(x) = \sup_{p\in[a,b]} \Big(f(p) - m\,|x-p| \Big) \leq \sup_{p\in[a,b]} \Big(f(p) - n\,|x-p| \Big) = f_n(x)$$

Because x is arbitrary, this implies f_n is indeed decreasing. Lastly, we show $f_n \to f$. Fix $x_0 \in [a,b]$ and ϵ. Because f is finite and upper semicontinuous on $[a,b]$, we may let $M \in \mathbb{R}$ be the upper bound of f on $[a,b]$ and let δ satisfy

$$\sup_{[x_0-\delta, x_0+\delta]} f(x) \leq f(x_0) + \epsilon$$

Let $N > \frac{M-(f(x_0)+\epsilon)}{\delta}$. If $p \in [x_0 - \delta, x_0 + \delta]$, then $f(p) - N|p - x_0| \leq f(p) \leq f(x_0) + \epsilon$. If $p \notin [x_0 - \delta, x_0 + \delta]$, then

$$f(p) - N|p - x_0| \leq M - N|p - x_0| < f(x_0) + \epsilon$$

Letting p run through $[a, b]$, we now see

$$f_N(x_0) = \sup_{p \in [a,b]} \Big(f(p) - N|p - x_0| \Big) \leq f(x_0) + \epsilon$$

Because ϵ and x_0 are arbitrary, and f_n is decreasing, we have shown $f_n \to f$. ■

Question 30

Let f_k be a sequence of measurable function defined on a measurable set E with finite measure. If $|f_k(\mathbf{x})| \leq M_{\mathbf{x}} < \infty$ for all k and for each $\mathbf{x} \in E$, show that given $\epsilon > 0$, there exists closed $F \subseteq E$ and finite M such that $|E - F| < \epsilon$ and $|f_k(\mathbf{x})| \leq M$ for all k and $\mathbf{x} \in F$.

Proof. Define for all $n \in \mathbb{N}$

$$E_n \triangleq \bigcap_{k=1}^{\infty} \{f_k \leq n\}$$

Because f_k are measurable on E, we know E_n are measurable. Because for all $\mathbf{x} \in E, \sup_{n \in \mathbb{N}} |f_n(\mathbf{x})| < \infty$, we see that $E_n \nearrow E$. Then because E is of finite measure, we know there exists some N such that

$$|E \setminus E_N| < \frac{\epsilon}{2}$$

Because E_N is measurable, we know there exists some closed $F \subseteq E_N$ such that

$$|E_N \setminus F| < \frac{\epsilon}{2}$$

It then follows that

$$|E \setminus F| < \epsilon$$

and for all $\mathbf{x} \in F$,

$$\mathbf{x} \in F \implies \mathbf{x} \in E_N \implies |f_k(\mathbf{x})| < N \text{ for all } k \in \mathbb{N}$$

■

Question 31

If f is measurable on E, define $\omega_f(a) \triangleq |\{f > a\}|$ for $a \in \mathbb{R}$. If $f_k \nearrow f$, show $\omega_{f_k} \nearrow \omega_f$. If $f_k \xrightarrow{m} f$, show that $\omega_{f_k} \to \omega_f$ at each point of continuity of ω_f. (For the second part, show that if $f_k \xrightarrow{m} f$, then $\limsup_{k\to\infty} \omega_{f_k}(a) \leq \omega_f(a-\epsilon)$ and $\liminf_{k\to\infty} \omega_{f_k}(a) \geq \omega_f(a+\epsilon)$ for all $\epsilon > 0$).

Proof. Suppose $f_n \nearrow f$. We know that for all $a \in \mathbb{R}$

$$\{f_n > a\} \nearrow \{f > a\}$$

This implies

$$\omega_{f_n}(a) = |\{f_n > a\}| \nearrow |\{f > a\}| = \omega_f(a)$$

Because a is arbitrary, we have shown $\omega_{f_n} \nearrow \omega_f$.

Suppose $f_n \xrightarrow{m} f$. Fix ϵ. Observe that for all $a \in \mathbb{R}$ and $n \in \mathbb{N}$

$$\{f_n > a\} \subseteq \{|f_n - f| > \epsilon\} \cup \{f > a - \epsilon\}$$

Measuring both side, we get

$$\omega_{f_n}(a) = |\{f_n > a\}| \leq |\{|f_n - f| > \epsilon\}| + |\{f > a - \epsilon\}| = |\{|f_n - f| < \epsilon\}| + \omega_f(a - \epsilon)$$

Then because $f_n \xrightarrow{m} f$ implies $|\{|f_n - f| < \epsilon\}| \to 0$, if we take limit superior on both side, we get

$$\limsup_{n\to\infty} \omega_{f_n}(a) \leq \omega_f(a - \epsilon)$$

Again, observe that for all $a \in \mathbb{R}$ and $n \in \mathbb{N}$

$$\{f > a + \epsilon\} \subseteq \{|f_n - f| > \epsilon\} \cup \{f_n > a\}$$

Measuring both side, we get

$$\omega_f(a + \epsilon) = |\{f > a + \epsilon\}| \leq |\{|f_n - f| > \epsilon\}| + |\{f_n > a\}| = |\{|f_n - f| > \epsilon\}| + \omega_{f_n}(a)$$

Then because $f_n \xrightarrow{m} f$ implies $|\{|f_n - f| < \epsilon\}| \to 0$, if we take limit inferior on both side, we get

$$\omega_f(a + \epsilon) \leq \liminf_{n\to\infty} \omega_{f_n}(a)$$

Now, if ω_f is continuous at a, we have

$$\omega_f(a+\epsilon) \searrow \omega_f(a) \text{ and } \omega_f(a-\epsilon) \nearrow \omega_f(a) \text{ as } \epsilon \searrow 0$$

This then implies

$$\liminf_{n\to\infty} \omega_{f_n}(a) = \limsup_{n\to\infty} \omega_{f_n}(a) = \omega_f(a)$$

as we wished. ■

Question 32

If f is measurable and finite almost everywhere on $[a,b]$, show that given $\epsilon > 0$, there is a continuous g on $[a,b]$ such that $|\{f \neq g\}| < \epsilon$. Formulate and prove a similar result in $\mathbb{R}^n$ by combining Lusin's Theorem with the Tietze extension Theorem.

Proof. Let $E \triangleq \{x \in [a,b] : f(x) \in \mathbb{R}\}$. E is measurable because f is measurable on $[a,b]$. It is clear that f is indeed measurable on E. By Lusin's Theorem, there exists some closed set $F \subseteq E$ such that $|E \setminus F| < \epsilon$ and $f|_F : F \to \mathbb{R}$ is continuous. Because F is compact, (bounded by $[a,b]$), Tietze extension Theorem give us some continuous $g : [a,b] \to \mathbb{R}$ such that $g = f$ on F. It then follows that

$$\{f \neq g\} \subseteq [a,b] \setminus F$$

which give us the desired estimation

$$|\{f \neq g\}| \leq (b-a) - |F| = |E| - |F| < \epsilon$$

We may formulate the same result by

> If f is measurable and finite almost everywhere on some compact $K \subseteq \mathbb{R}^d$, then for all ϵ, there exists continuous g on K such that $|\{f \neq g\}| < \epsilon$.

and give exactly the same argument to prove it. ■