13.3 Script

Question 1

There are three types of isolated singularities: removable singularity, poles and essential singularity. Provide the definition of each type and give an example.

Proof. Let $f : D_\epsilon(z_0) \setminus \{z_0\} \to \mathbb{C}$ be holomorphic so that f has an isolated singularity at z_0. By Laurent's Theorem, we may express f as a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

If $c_n = 0$ for all negative n, say

$$f(z) = (z - z_0)^2 + 7 \text{ on } D_\epsilon(z_0) \setminus \{z_0\}$$

we say f has a removable singularity at z_0. If there are only finite numbers of negative n such that $c_n \neq 0$, and m is the smallest integer such that

$$c_n \neq 0 \text{ for } n = m$$

we say f has a pole of order m at z_0. For example, f can be

$$f(z) = \frac{1}{z - z_0} \text{ on } D_\epsilon(z_0) \setminus \{z_0\}$$

If there are infinite number of negative n such that $c_n \neq 0$, for example,

$$f(z) = e^{\frac{1}{(z-z_0)}} = \sum_{n=0}^{\infty} \frac{1}{n!}(z - z_0)^{-n} \text{ on } D_\epsilon(z_0) \setminus \{z_0\}$$

we say f has an essential singularity at z_0. ∎

Question 2

State the definition of the residue of a function at an isolated singularity.

Proof. Let $f : D_\epsilon(z_0) \setminus \{z_0\} \to \mathbb{C}$ be holomorphic so that f has an isolated singularity at z_0. By Laurent's Theorem, we may express f as a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

The residue of f at z_0 is defined to be c_{-1}.

$$\text{Res}(f, z_0) \triangleq c_{-1}$$

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Question 3

If z_0 is a simple pole of f, prove that

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

Proof. Because z_0 is a simple pole of f, we may write

$$f(z) = [\operatorname{Res}(f, z_0)](z - z_0)^{-1} + \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

This give us

$$\begin{aligned}
\lim_{z \to z_0} (z - z_0) f(z) &= \lim_{z \to z_0} \operatorname{Res}(f, z_0) + \sum_{n=0}^{\infty} c_n (z - z_0)^{n+1} \\
&= \operatorname{Res}(f, z_0) + \sum_{n=0}^{\infty} \lim_{z \to z_0} c_n (z - z_0)^{n+1} \\
&= \operatorname{Res}(f, z_0)
\end{aligned}$$

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Question 4

If $f(z) = \frac{q(z)}{p(z)}$ where p and q are holomorphic, and z_0 is a simple zero of p, prove that

$$\operatorname{Res}(f, z_0) = \frac{q(z_0)}{p'(z_0)}$$

Proof. Because z_0 is a simple zero of p, we know

$$p'(z_0) \neq 0$$

If $q(z_0) \neq 0$, then f has a simple pole at z_0, and from result of last question we may compute

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} \frac{(z - z_0) q(z)}{p(z) - p(z_0)} = \frac{q(z_0)}{p'(z_0)}$$

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Question 5

If z_0 is a pole of f of order m, prove that

$$\text{Res}(f, z_0) = \lim_{z\to z_0} \frac{1}{(m-1)!}\Big(\frac{d}{dz}\Big)^{m-1} g(z)$$

where $g(z) = (z-z_0)^m f(z)$

Proof. Because z_0 is a pole of f of order m, we may write

$$f(z) = \sum_{n=-m}^{\infty} c_n(z-z_0)^n \text{ and } g(z) = \sum_{n=0}^{\infty} c_{n-m}(z-z_0)^n$$

Or write

$$g(z) = c_{-m} + c_{-m+1}(z-z_0) + \cdots + c_{-1}(z-z_0)^{m-1} + c_0(z-z_0)^m + \cdots$$

Compute

$$g^{(m-1)}(z) = \frac{(m-1)!}{0!}c_{-1} + \frac{m!}{1!}c_0(z-z_0) + \frac{(m+1)!}{2!}c_1(z-z_0)^2 + \cdots$$

This give us

$$\lim_{z\to z_0} \frac{1}{(m-1)!} g^{(m-1)}(z) = c_{-1} = \text{Res}(f, z_0)$$

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Question 6

Find the residue of the following function at the indicated points:

(a) $\frac{z}{(2-3z)(4z+3)}$ at $z = \frac{2}{3}$.

(b) $\frac{z-\frac{1}{6}z^3-\sin z}{z^8}$ at $z = 0$.

Proof. For the first function, observe

$$\lim_{z\to\frac{2}{3}} (z-\frac{2}{3}) \cdot \frac{z}{(2-3z)(4z+3)} = \lim_{z\to\frac{2}{3}} \frac{z}{-3(4z+3)} = \frac{2}{-51} \neq 0$$

which implies $z = \frac{2}{3}$ is a simple pole. Therefore, from the question earlier, we may deduce its residue is exactly $\frac{2}{-51}$. For the second function, we can just compute its Laurent series by

$$\frac{z-\frac{1}{6}z^3-\sin z}{z^8}=\frac{1}{z^7}-\frac{1}{6z^5}-\frac{1}{z^8}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}$$
$$=\frac{1}{5!}z^{-3}+\frac{-1}{7!}z^{-1}+\frac{1}{9!}z+\cdots$$

Therefore, its residue is $\frac{-1}{7!}$. ■

Question 7

Evaluate

$$\int_{-\pi}^{\pi}\frac{d\theta}{5+3\cos\theta}$$

Proof. Define $\gamma:[-\pi,\pi]\to\mathbb{C}$ by

$$\gamma(t)=e^{it}$$

We have

$$I\triangleq -2i\int_\gamma\frac{dz}{3z^2+10z+3}=\int_\gamma\frac{dz}{\frac{3i}{2}z^2+5iz+\frac{3}{2}i}$$
$$=\int_\gamma\frac{1}{5+\frac{3}{2}(z+\frac{1}{z})}\cdot\frac{1}{iz}dz$$
$$=\int_{-\pi}^{\pi}\frac{1}{5+\frac{3}{2}(e^{it}+e^{-it})}\cdot\frac{ie^{it}}{ie^{it}}dt$$
$$=\int_{-\pi}^{\pi}\frac{1}{5+3\cos t}dt$$

In summary

$$\int_{-\pi}^{\pi}\frac{d\theta}{5+3\cos\theta}=-2i\int_\gamma\frac{dz}{3z^2+10z+3}$$

Note that

$$\frac{1}{3z^2+10+3}=\frac{1}{(3z+1)(z+3)}$$

have two simple poles $z=-3$ and $z=\frac{1}{-3}$, and have residue $\frac{1}{8}$ at pole $z=\frac{1}{-3}$. It then follows from residue theorem that

$$\int_{-\pi}^{\pi}\frac{d\theta}{5+3\cos\theta}=-2i\cdot 2\pi i\cdot\frac{1}{8}=\frac{\pi}{2}$$

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Question 8

Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$$

Proof. Let $S_R : [0,\pi] \to \mathbb{C}$ be the semi-circle $S_R(t) \triangleq Re^{it}$, $L_R : [-R,R] \to \mathbb{C}$ be the line segment $L_R(t) \triangleq t$, and γ the wedge $\gamma = S_R + L_R$. By residue theorem,

$$\int_\gamma \frac{dz}{z^2+2z+2} = 2\pi i \cdot \frac{1}{2i} = \pi$$

for large enough R. Observe

$$\left| \int_{S_R} \frac{dz}{z^2+2z+2} \right| \leq \pi R \cdot \max_{S_R} \frac{1}{|z+1-i| \cdot |z+1+i|} \leq \frac{\pi R}{(R-\sqrt{2})^2}$$

This implies

$$\lim_{R\to\infty} \int_{S_R} \frac{dz}{z^2+2z+2} = 0$$

Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} &= \lim_{R\to\infty} \int_{L_R} \frac{dz}{z^2+2z+2} \\ &= \lim_{R\to\infty} \pi - \int_{S_R} \frac{dz}{z^2+2z+2} = \pi\end{aligned}$$

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Question 9

Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+2x^2+1}$$

Proof. Let $S_R : [0,\pi] \to \mathbb{C}$ be the semi-circle $S_R(t) \triangleq Re^{it}$, $L_R : [-R,R] \to \mathbb{C}$ be the line segment $L_R(t) \triangleq t$, and γ the wedge $\gamma = S_R + L_R$. By residue theorem,

$$\int_\gamma \frac{dz}{z^4+2z^2+1} = 2\pi i \lim_{z\to i} \frac{d}{dz} \frac{1}{(z+i)^2} = \frac{\pi}{2}$$

for large enough R. Observe

$$\left|\int_{S_R} \frac{dz}{z^4+2z^2+1}\right| \leq \pi R \cdot \max_{S_R} \frac{1}{|z+i|^2 \cdot |z-i|^2} \leq \frac{\pi R}{(R-1)^4}$$

This implies

$$\lim_{R\to\infty} \int_{S_R} \frac{dz}{z^4+2z^2+1} = 0$$

Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dx}{x^4+2x^2+1} &= \lim_{R\to\infty} \int_{L_R} \frac{dz}{z^4+2z^2+1} \\ &= \lim_{R\to\infty} \frac{\pi}{2} - \int_{S_R} \frac{dz}{z^4+2z^2+1} = \frac{\pi}{2}\end{aligned}$$

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