Analysis (Honors) I, Fall 2023
Midterm Exam Solutions

1. Let M be a nonempty set and let d_1, d_2 be metrics on M. Define

$$d_{\mathrm{M}}(x,y) = \max\{d_1(x,y), d_2(x,y)\} \quad \text{for all } x, y \in M.$$

(a) Show that d_{M} is also a metric on M.

Solution. • Note that $d_{\mathrm{M}} \geq 0$ since both d_1, $d_2 \geq 0$. Also, $d_{\mathrm{M}}(x,y) = 0$ if and only if $d_1(x,y) = d_2(x,y) = 0$ if and only if $x = y$. Symmetry is clear. Finally, for the triangle inequality, for $x, y, z \in M$,

$$\begin{aligned} d_{\mathrm{M}}(x,z) &= \max\{d_1(x,z), d_2(x,z)\} \\ &\leq \max\{d_1(x,y) + d_1(y,z), d_2(x,y) + d_2(y,z)\} \\ &\leq \max\{d_1(x,y), d_2(x,y)\} + \max\{d_1(y,z), d_2(y,z)\} \\ &= d_{\mathrm{M}}(x,y) + d_{\mathrm{M}}(y,z). \end{aligned}$$

□

(b) Are the following functions continuous? If yes, prove it; otherwise, give a counterexample.

i. $\mathrm{id} : (M, d_{\mathrm{M}}) \to (M, d_1)$, where $\mathrm{id}(x) = x$ for all $x \in M$.

Solution. It is continuous. Fix $x \in X$. Given any $\varepsilon > 0$, we pick $\delta = \varepsilon$. Then whenever $d_{\mathrm{M}}(x,y) < \delta$, we have $d_1(\mathrm{id}(x), \mathrm{id}(y)) = d_1(x,y) \leq d_{\mathrm{M}}(x,y) < \varepsilon$, showing that $\mathrm{id} : (M, d_{\mathrm{M}}) \to (M, d_1)$ is continuous. □

ii. $\mathrm{id} : (M, d_1) \to (M, d_{\mathrm{M}})$, where $\mathrm{id}(x) = x$ for all $x \in M$.

Solution. It is not continuous in general. Take $M = \mathbb{R}$, d_1 to be the standard metric, and d_2 to be the discrete metric. Then for any $x \in \mathbb{R}$,

$$B_{d_{\mathrm{M}}}(x, 1/2) = \{x\}$$

is open in (M, d_{M}), but $\mathrm{id}^{-1}(\{x\}) = \{x\}$ is not open in (M, d_1). □

(c) Let $d_{\mathrm{m}} = \min\{d_1, d_2\}$. Is d_{m} a metric on M? Justify your answer.

Solution. d_{m} is not a metric in general. For example, take $M = \{x, y, z\}$, and define

$$d_1(x,y) = 3, \quad d_1(y,z) = 1, \quad d_1(x,z) = 2,$$

$$d_2(x,y) = 3, \quad d_2(y,z) = 2, \quad d_2(x,z) = 1,$$

and extend the definitions of d_1 and d_2 such that they are symmetric and positive definite. It can be easily checked that d_1, d_2 are metrics on M. However,

$$d_{\mathrm{m}}(x,y) = 3, \quad d_{\mathrm{m}}(y,z) = 1, \quad d_{\mathrm{m}}(x,z) = 1,$$

and d_{m} does not satisfy the triangle inequality. □

2. Define a topology $\mathcal{T}$ on $\mathbb{R}^2$ by $\mathcal{T} = \{\varnothing, \mathbb{R}^2\} \cup \{P_r : r \in \mathbb{R}\}$, where

$$P_r = \{(x,y) \in \mathbb{R}^2 : x^2 - y > r\}.$$

(a) Show that $(\mathbb{R}^2, \mathcal{T})$ is a topological space.

Solution.

- By definition, $\varnothing, \mathbb{R}^2 \in \mathcal{T}$.
- Let (U_α) be a family in $\mathcal{T}$. We may assume none of the U_α is $\varnothing$ or $\mathbb{R}^2$. Then $U_\alpha = P_{r_\alpha}$ for some $r_\alpha \in \mathbb{R}$. Define $r = \inf_\alpha r_\alpha$. Then

$$\bigcup_\alpha U_\alpha = \bigcup_\alpha P_{r_\alpha} = \begin{cases} P_r & \text{if } r > -\infty, \\ \mathbb{R}^2 & \text{if } r = -\infty. \end{cases}$$

  Thus, $\bigcup_\alpha U_\alpha \in \mathcal{T}$.
- Let $U_1, U_2 \in \mathcal{T}$. Again, we may assume both are not $\varnothing$ or $\mathbb{R}^2$. Write $U_i = P_{r_i}$ for some $r_i \in \mathbb{R}$. Then $U_1 \cap U_2 = P_{\max\{r_1.r_2\}} \in \mathcal{T}$.

Therefore $\mathcal{T}$ is a topology. □

(b) Does $(\mathbb{R}^2, \mathcal{T})$ have a countable basis? Justify your answer.

Solution. Yes, we can take $\mathcal{B} = \{P_r : r \in \mathbb{Q}\}$. This is countable, and, moreover,

$$\mathbb{R}^2 = \bigcup_{r \in \mathbb{Q}} P_r$$

(this follows from our argument in (a)), and whenever $P_r, P_s \in \mathcal{B}$ and $x \in P_r \cap P_s = P_{\max\{r,s\}}$, we can find $P_q \in \mathcal{B}$ (where $q = \max\{r,s\} \in \mathbb{Q}$) such that $x \in P_q \subseteq P_r \cap P_s$. This shows that $\mathcal{B}$ is a basis. □

(c) Is $(\mathbb{R}^2, \mathcal{T})$ compact? Justify your answer.

Solution. No, it is not compact. We can cover $\mathbb{R}^2$ by $\bigcup_{n \in \mathbb{Z}} P_n$, but the union of any finite subcollection of $(P_n)_{n \in \mathbb{Z}}$ cannot be the whole $\mathbb{R}^2$. □

(d) Recall that $A \subseteq X$ (where X is a topological space) is dense in X if $\overline{A} = X$. Are the following sets dense in $\mathbb{R}^2$ under the topology $\mathcal{T}$? Justify your answers.

i. $\{(x,0) : x \in \mathbb{R}\}$.

ii. $\{(0,y) : y \in \mathbb{R}\}$.

Solution. Both are dense in $\mathbb{R}^2$. It suffices to show that $P_r \cap \{(x,0) : x \in \mathbb{R}\} \neq \varnothing$ and $P_r \cap \{(0,y) : y \in \mathbb{R}\} \neq \varnothing$ for all $r \in \mathbb{R}$.

Fix $r \in \mathbb{R}$. Pick any $x_0 \in \mathbb{R}$ such that $x_0^2 > r$. Then $(x_0, 0) \in P_r \cap \{(x,0) : x \in \mathbb{R}\}$. For ii., one has $(0, -r-1) \in P_r \cap \{(0,y) : y \in \mathbb{R}\}$. □

(e) Find the limit(s) of the sequence (x_n) in $(\mathbb{R}^2, \mathcal{T})$, where $x_n = (n, 0)$ for $n \geq 1$.

Solution. (x_n) converges to every point in $\mathbb{R}^2$. Pick any $(x, y) \in \mathbb{R}^2$, and let U be a neighborhood of (x, y). If $U = \mathbb{R}^2$, then $x_n \in U$ for all n. If $U = P_r$ for some $r \in \mathbb{R}$ instead, pick $N \in \mathbb{N}$ such that $N^2 > r$. Then whenever $n \geq N$, $(n, 0) \in U$. In either case, we have $x_n \in U$ for all large n. Therefore, (x_n) converges to (x, y). $\square$

3. (a) Prove that there is no continuous injective function $f : \mathbb{R}^2 \to \mathbb{R}$, where both $\mathbb{R}^2$ and $\mathbb{R}$ are equipped with the standard topologies. (**Hint**: Use connectedness.)

Solution. Suppose there is continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ which is injective. By the intermediate value theorem, $f(\mathbb{R}^2)$ has to be an interval (since intervals are the only subsets of $\mathbb{R}$ that satisfies the intermediate value property). This interval has to be nondegenerate, otherwise f cannot be injective. Pick any y in the interior of $f(\mathbb{R}^2)$. Then $f(\mathbb{R}^2) \setminus \{y\}$ is disconnected. On the other hand, $\mathbb{R}^2 \setminus f^{-1}(\{y\})$ is connected, since $f^{-1}(\{y\})$ is a singleton (by injectivity), and it is rather easy to see that $\mathbb{R}^2$ removing a point is path connected. However, $f(\mathbb{R}^2 \setminus f^{-1}(\{y\}))$ is disconnected, a contradiction. $\square$

(b) Does the statement in (a) still hold if we equip $\mathbb{R}^2$ with the topology $\mathcal{T}$ in problem 2 instead (where $\mathbb{R}$ is still equipped with the standard topology)? Justify your answer.

Solution. The statement still holds; that is, there is no continuous injection from $\mathbb{R}^2$ to $\mathbb{R}$. If such a continuous injective exists, then it has to be continuous from $(\mathbb{R}^2, \mathcal{T}')$ to $\mathbb{R}$, where $\mathcal{T}'$ is the standard topology on $\mathbb{R}^2$, since every open set in $\mathcal{T}$ is open in $\mathcal{T}'$, but we have shown in (a) that just a map cannot be injective.

Alternatively, one can also argue directly that such a map does not exist. By following the argument in (a), it suffices to show that $\mathbb{R}^2 \setminus \{x\}$ is connected in $\mathcal{T}$ for any $x \in \mathbb{R}^2$. Note that the open sets in $\mathcal{T}$ are nested, in the sense that if $s < r$ then P_r is properly contained in P_s. In particular, any P_r cannot be clopen. (If P_r were clopen, then $P_r = P_s^c$ for some $s \in \mathbb{R}$. Clearly, $s \neq r$. If $s < r$, then $P_s^c = P_r \subseteq P_s$, which would be absurd. The case that $r < s$ is similar.) Therefore, there does not exist a nontrivial separation for any subset of $\mathbb{R}^2$ under $\mathcal{T}$. So every subset of $\mathbb{R}^2$ is connected; in particular, $\mathbb{R}^2 \setminus \{x\}$ is connected $\square$

4. Let E be the set of all functions $u : [0, 1] \to \mathbb{R}$ such that $u(0) = 0$ and $|u(x) - u(y)| \leq 3|x - y|$ for all $x, y \in [0, 1]$. Define $\phi : E \to \mathbb{R}$ by

$$\phi(u) = \int_0^1 (2u(x)^3 + 3u(x)^2) \, \mathrm{d}x.$$

Prove that there exists $u_0 \in E$ such that $\phi(u_0) = \inf_{u \in E} \phi(u)$.

Solution. Let $u \in E$. Note that $|u(x)| = |u(x) - u(0)| \leq 3|x|$ for all $x \in [0, 1]$. So

$$\begin{aligned} |\phi(u)| &\leq \int_0^1 2|u(x)|^3 \, \mathrm{d}x + \int_0^1 3u(x)^2 \, \mathrm{d}x \\ &\leq \int_0^1 2 \cdot 27x^3 \, \mathrm{d}x + \int_0^1 3 \cdot 9x^2 \, \mathrm{d}x = \frac{27}{2} + 9. \end{aligned}$$

Since $|\phi(u)|$ is bounded, the infimum exists. Call the infimum α. By the definition of infimum, we can find a sequence (u_n) in E such that $\phi(u_n) \to \alpha$ as $n \to \infty$. Since $|u_n(x)| \leq 3|x| \leq 3$ for all $x \in [0,1]$, (u_n) is a bounded sequence in $C^0([0,1],\mathbb{R})$. Moreover, since (u_n) is 3-Lipschitz, the sequence is also equicontinuous. Hence, by Arzelá–Ascoli, there is a subsequence (u_{n_k}) of (u_n) that converges uniformly on $[0,1]$. Call the limit u_0. Note that $u_0 \in E$, because $u_0(0) = \lim_{k\to\infty} u_{n_k}(0) = 0$, and $|u_0(x) - u_0(y)| = \lim_{k\to\infty} |u_{n_k}(x) - u_{n_k}(y)| \leq 3|x-y|$ for any $x, y \in [0,1]$. Finally, by uniform convergence, we can interchange the limit and integral, which yields

$$\phi(u_0) = \int_0^1 (2u_0(x)^3 + 3u_0(x)^2)\ \mathrm{d}x = \lim_{k\to\infty} \int_0^1 (2u_{n_k}(x)^3 + 3u_{n_k}(x)^2)\ \mathrm{d}x = \phi(u_{n_k}) = \alpha. \qquad \square$$

5. (a) Let $a < b$ and let $g : [a,b] \to \mathbb{R}$ be a bounded function. Prove that g is Riemann integrable if and only if for each $0 < \varepsilon < \frac{b-a}{2}$, $g|_{[a+\varepsilon, b-\varepsilon]}$ is Riemann integrable (recall that $g|_{[a+\varepsilon,b-\varepsilon]}$ is the restriction of g to $[a+\varepsilon, b-\varepsilon]$).

Solution. We knew that if g is Riemann integrable then its restriction to a subinterval is also Riemann integrable.

On the other hand, suppose that for each $0 < \varepsilon < \frac{b-a}{2}$, $g|_{[a+\varepsilon,b-\varepsilon]}$ is Riemann integrable. We fix $0 < \varepsilon < \frac{b-a}{2}$, and let $M > 1$ be an upper bound for $|g|$ (on $[a,b]$). By assumption, $g|_{[a+\frac{\varepsilon}{6M}, b-\frac{\varepsilon}{6M}]}$ is Riemann integrable, so there exists a partition P of $[a+\frac{\varepsilon}{6M}, b-\frac{\varepsilon}{6M}]$ such that

$$U(g|_{[a+\frac{\varepsilon}{6M}, b-\frac{\varepsilon}{6M}]}, P) - L(g|_{[a+\frac{\varepsilon}{6M}, b-\frac{\varepsilon}{6M}]}, P) < \frac{\varepsilon}{3}.$$

Consider the partition $P' = \{a,b\} \cup P$ of $[a,b]$. Then

$$U(g,P') - L(g,P') \leq 4 \cdot M \cdot \frac{\varepsilon}{6M} + U(g|_{[a+\frac{\varepsilon}{6M}, b-\frac{\varepsilon}{6M}]}, P) - L(g|_{[a+\frac{\varepsilon}{6M}, b-\frac{\varepsilon}{6M}]}, P) < \varepsilon.$$

Here, in the first inequality, each $M \cdot \frac{\varepsilon}{6M}$ comes from upper bounding M_i or m_i by M, and the fact that the lengths of the first and last intervals in P' is of length $\frac{\varepsilon}{6M}$. Therefore, g is Riemann integrable. $\square$

(b) Define

$$\psi(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$$

Let $f : [-1,1] \to [0,1]$ be Riemann integrable.

i. Show that $\psi \circ f$ is Riemann integrable.

Solution. Note that ψ is continuous, and we know that $\psi \circ f$ is Riemann integrable if ψ is continuous and f is Riemann integrable. $\square$

ii. Is $f \circ \psi$ Riemann integrable? Prove your assertion.

Solution. Yes, $f \circ \psi$ is Riemann integrable. Fix $\varepsilon \in (0, \frac{1}{2}$. By (a), it suffices to show that $f \circ \psi$ is Riemann integrable when it is restricted on $[\varepsilon, 1-\varepsilon]$. ψ has at most finitely many critical points in $[\varepsilon, 1-\varepsilon]$. Call these critical points $x_1 < x_2 < \cdots < x_n$. Take

$$0 < \delta < \frac{1}{2}(x_i - x_{i-1})$$

(if $i = 0$, we define $x_{-1} = \varepsilon$). Fix $i = 1, \ldots, n$. Then there exists $K > 0$ such that $|\psi'(x)| \geq K$ whenever $x \in [x_{i-1} + \delta, x_i - \delta]$. So ψ is a homeomorphism on $[x_{i-1} + \delta, x_i - \delta]$, and its inverse is Lipschitz (with Lipschitz constant K^{-1}). Hence, restricted on $[x_{i-1} + \delta, x_i - \delta]$, $f \circ \psi$ is Riemann integrable. Since δ is arbitrary, by (a), $f \circ \psi$ is Riemann integrable when restricted on $[x_{i-1}, x_i]$. Hence $f \circ \psi$ is Riemann integrable when restricted on $[\varepsilon, x_n]$. Similarly one can show that $f \circ \psi$ is Riemann integrable on $[x_n, 1-\varepsilon]$. Thus, $f \circ \psi$ is Riemann integrable on $[\varepsilon, 1-\varepsilon]$. □

6. Let $n \geq 1$ and let $S \subseteq \mathbb{R}^n$ (under the standard topology) be uncountable. Let T be the set of condensation points of S (the set of x such that each neighborhood of x contains uncountably many points of S).

   (a) Show that $S \cap T^c$ is countable and $S \cap T$ is uncountable.

   Solution. For any $x \in S \cap T^c$, x is not a condensation point, so there exists a neighborhood U_x of x such that $U_x \cap S$ is at most countable. Note that $\mathcal{U} := \{U_x : x \in S \cap T^c\}$ is an open cover of $S \cap T^c$. Since $\mathbb{R}^n$ is second countable (it has a countable basis), $\mathcal{U}$ has a countable subcover $\mathcal{U}'$ of $S \cap T^c$.

   Now,

$$S \cap T^c = \bigcup_{U \in \mathcal{U}'} (U \cap (S \cap T^c)),$$

   and the right side is a countable union of countable sets, which is countable. This proves the first part. For the second part, note that $S = (S \cap T^c) \cup (S \cap T)$. Since S is uncountable and $S \cap T^c$ is countable, $S \cap T$ has to be uncountable. □

   (b) Show that T is closed and contains no isolated points. ($x \in A$ is isolated if there exists a neighborhood U of x such that $U \cap A = \{x\}$.)

   Solution. We first show that T is closed, or T^c is open. Let $x \in T^c$. Then x is not a condensation point, and there exists a neighborhood U_x of x such that $U_x \cap S$ is countable. For each $y \in U_x$, there exists a neighborhood V_y of y such that $V_y \subseteq U_x$. Then $V_y \cap S \subseteq U_x \cap S$, which means that $V_y \cap S$ is also countable. In other words, if $y \in U_x$, then $y \in T^c$. That is, $U_x \subseteq T^c$. By definition, T^c is open.

   Next, we claim that T has no isolated points. Suppose that x is an isolated point of T. Then there exists a neighborhood Y of x such that $U \cap T = \{x\}$. In particular, $x \in T$, so any neighborhood of x, particularly U, has an uncountable intersection with S. All such points cannot be in T, so $S \cap T^c$ is uncountable, contradicts the first part of (a). Therefore T has no isolated points. □

7. Let X, Y be topological spaces and $f : X \to Y$ be continuous. Consider the following conditions.

(C1) For each $y \in Y$, $f^{-1}(\{y\})$ is compact in X.

(C2) If $A \subseteq X$ is closed in X then $f(A)$ is closed in Y.

(a) Show that if f satisfies both (C1) and (C2) then $f^{-1}(K)$ is compact in X for every compact set $K \subseteq Y$.

Solution. Let $K \subseteq Y$ be compact, and let (U_α) be an open cover of $f^{-1}(K)$. By (C1), for any given $y \in K$, $f^{-1}(\{y\})$ is compact. Since (U_α) covers $f^{-1}(\{y\})$ also, it has a finite subcover $\{U_{y,i} : i = 1, \ldots, n_y\}$. Define

$$U_y = \bigcup_{i=1}^{n_y} U_{y,i} \quad \text{and} \quad V_y = f(U_y^c)^c.$$

Then (C2) implies that V_y is open in Y. On the other hand, $U_y \supseteq f^{-1}(\{y\})$.

We claim that $y \in V_y$. Suppose not. Then $y \in f(U_y^c)$. By definition, there exists $x \in U_y^c$ such that $y = f(x)$. Then $x \in U_y^c \cap f^{-1}(\{y\})$, a contradiction. So $y \in V_y$.

Next, we claim that $f^{-1}(V_y) \subseteq U_y$. If $x \in f^{-1}(V_y)$, then $f(x) \in V_y = f(U_y^c)^c$. That is, $f(x) \notin f(U_y^c)$. So $x \notin U_y^c$, or $x \in U_y$.

By the first claim, $\{V_y : y \in K\}$ forms an open cover of K, so by compactness of K it has a finite subcover $\{V_{y_1}, \ldots, V_{y_n}\}$. By the second claim,

$$f^{-1}(K) \subseteq f^{-1}\left(\bigcup_{i=1}^{n} V_{y_i}\right) = \bigcup_{i=1}^{n} f^{-1}(V_{y_i}) \subseteq \bigcup_{i=1}^{n} U_{y_i} = \bigcup_{i=1}^{n} \bigcup_{j=1}^{n_{y_i}} U_{y_i,j}.$$

Note that each $U_{y_i,j}$ is from (U_α). This shows that (U_α) has a finite subcover of $f^{-1}(K)$. Therefore, $f^{-1}(K)$ is compact.

Alternatively, one may also use the finite intersection property to solve this problem. Let $K \subseteq Y$ be compact, and let $\mathcal{C}$ be the collection of all closed subsets of $f^{-1}(K)$ that satisfies the finite intersection property. Let $\mathcal{B}$ be the collection of finite intersections of elements in $\mathcal{C}$. Then $\mathcal{B}$ also has the finite intersection property. Also, for finitely many elements $B_1, \ldots, B_n \in \mathcal{B}$, one has $f(B_1) \cap \cdots \cap f(B_n) \supseteq f(B_1 \cap \cdots \cap B_n) \neq \varnothing$. By (C2), each $f(B_i)$ is closed. Since K is compact, this implies

$$\bigcap_{B \in \mathcal{B}} f(B) \neq \varnothing.$$

Pick y in the above intersection. Then $f^{-1}(\{y\}) \cap B \neq \varnothing$ for all $B \in \mathcal{B}$. In other words, for any finite collection $C_1, \ldots, C_n \in \mathcal{C}$, one has $f^{-1}(\{y\}) \cap (C_1 \cap \cdots \cap C_n) \neq \varnothing$. By (C1), $f^{-1}(\{y\})$ is compact. Also, the collection $\{f^{-1}(\{y\}) \cap C : C \in \mathcal{C}\}$ satisfies the finite intersection property. Moreover, each $f^{-1}(\{y\}) \cap C$ is a closed subset of $f^{-1}(\{y\})$ under the subspace topology. Therefore,

$$f^{-1}(\{y\}) \cap \bigcap_{C \in \mathcal{C}} C \neq \varnothing.$$

In particular, $\bigcap_{C \in \mathcal{C}} C \neq \varnothing$. Therefore, $f^{-1}(K)$ is compact. $\square$

Remark: Actually we do not need the assumption that f is continuous. It is not used anywhere.

(b) Give an example of f which only satisfies (C1) and a compact $K \subseteq Y$ but $f^{-1}(K)$ is not compact.

Solution. Take $X = \mathbb{R}$ with the discrete topology, and take $Y = \mathbb{R}$ with the standard topology. Then $f : X \to Y$ defined by $f(x) = x$ is continuous. (C1) is clearly satisfied. (C2) fails because any open interval (a, b) is closed in X but $f((a, b)) = (a, b)$ is not closed in Y. Also, $K = [a, b]$ is compact in Y, but $f^{-1}(K) = [a, b]$ is not compact in X. $\square$

(c) Give an example of f which only satisfies (C2) and a compact $K \subseteq Y$ but $f^{-1}(K)$ is not compact.

Solution. Take $X = \mathbb{R}$ with the standard topology, and $Y = \{y\}$ to be a singleton. Take f to be the constant function y on X. Then f is continuous, (C1) fails, and (C2) holds. The compact set Y does not have a compact preimage. $\square$

8. Let X be a topological space and let $f : X \to X$ be continuous.

(a) Suppose that X is compact. Show that there exists $x \in X$ such that for any neighborhood V of x, there exists $n \geq 1$ such that $f^{\circ n}(x) \in V$. Here, $f^{\circ 1} = f$ and $f^{\circ n} = f \circ f^{\circ n-1}$ for $n \geq 2$.

Solution. Consider the family $\mathcal{F}$ of nonempty closed subsets Y of X satisfying $f(Y) \subseteq Y$, ordered by set inclusion. We claim that $\mathcal{F}$ has a minimal element. If we have a totally ordered subset $\mathcal{G}$ of $\mathcal{F}$ then since X is compact, by the finite intersection property $\bigcap_{G \in \mathcal{G}} G \neq \varnothing$; of course $\bigcap_{G \in \mathcal{G}} G$ also closed. Moreover,

$$f\left(\bigcap_{G \in \mathcal{G}} G\right) \subseteq \bigcap_{G \in \mathcal{G}} f(G) \subseteq \bigcap_{G \in \mathcal{G}} G,$$

since each $G \in \mathcal{G}$ satisfies $f(G) \subseteq G$. Therefore, $\bigcap_{G \in \mathcal{G}} G$ is a lower bound in $\mathcal{F}$ for $\mathcal{G}$. By Zorn's lemma, $\mathcal{F}$ has a minimal element Y_0.

Let $x \in Y_0$. We claim that x satisfies the desired property. Consider $Y = \overline{\{f^{\circ n}(x) : n \geq 1\}}$. Then $Y \subseteq Y_0$ because Y_0 is closed and $f(Y_0) \subseteq Y_0$. But so is Y, by minimality $Y = Y_0$. Therefore each neighborhood of x contains some $f^{\circ n}(x)$. $\square$

Remark: This is also known as the Birkhoff recurrence theorem. The set $\{x, f(x), f^{\circ 2}(x), \ldots\}$ can be seen as the orbit of x. The integer n can be seen as the time, and $f(x)$ can be thought of as the position x moves to after one time unit. The recurrence theorem tells us that if X is compact, then eventually we get can back to a point very close to x after a certain amount of time. We can actually prove it without using Zorn's lemma, but this is not something easy (and might be out of our scope somehow).

(b) Does the conclusion in (a) still hold if X is locally compact instead? Justify your answer.

Solution. No, it does not hold in general. Take $X = \mathbb{Z}$ with the discrete topology, and consider $f : X \to X$ defined by $f(x) = x + 1$. Then X is locally compact, f is continuous, but it is clear that for each $x \in X$, $f^{\circ n}(x) = x + n \notin \{x\}$ for all $n \geq 1$, while $\{x\}$ is a neighborhood of x. □