1.8 Exercises 3

Question 15

Let $o(G) = 60$. Show that if G is simple, then G must have exactly 24 elements of order 5 and 20 elements of order 3.

Proof. By sylow, we have

$$n_5 \equiv 1 \pmod 5 \quad \text{and} \quad n_5 \mid 12$$

which by simplicity of G implies $n_5 = 6$. The same argument gives us $n_3 \in \{4, 10\}$. To see $n_3 \neq 4$, just recall that second sylow stated that conjugacy action $G \longrightarrow \text{Sym}(\text{Syl}_3(G)) \cong S_{n_3}$ is nontrivial, and is therefore injective by simplicity of G. We now see that $n_3 = 4$ is too small to satisfies

$$o(G) = 60 \mid n_3!$$

■

Question 16

Let $o(G) = pqr$ with $p < q < r$ prime. Prove that G has a normal subgroup H, where $o(H) \in \{p, q, r\}$.

Proof. We show either $n_q = 1$ or $n_r = 1$. By sylow, if $n_r \neq 1$, then $n_r = pq$ and there remains

$$pqr - (r-1)pq = pq$$

number of elements of order that isn't r. Because $q > p$, by sylow, $n_q \in \{1, r, pr\}$. Noting that $(q-1)r > pq$, we see we must have $n_q = 1$. ■

Question 17

Let $o(G) = p^3 q$ with p, q prime. Show that one of the followings statement is true:

(i) G has a normal Sylow p-subgroup.

(ii) G has a normal Sylow q-subgroup.

(iii) $p = 2$, $q = 3$.

Proof. Suppose (i) and (ii) are both false. Then by sylow we have $n_p = q$ and $p < q$. Because $p < q$, applying sylow again we have $n_q \in \{p^2, p^3\}$. Because $n_p > 1$, by counting we see that $n_q \neq p^3$. Therefore $n_q = p^2$. Then by sylow, $p^2 = n_q \equiv 1 \pmod{q}$, which implies $q \mid (p-1)(p+1)$. Because $p < q$ and q is prime, we now see $q = p+1$, which can only happen if $p = 2$ and $q = 3$. ■

Question 18

Show that no group of order 30 is simple.

Proof. Consider n_3 and n_5. We have $n_5 \in \{1, 6\}$ and $n_3 \in \{1, 10\}$. If $n_5 \neq 1 \neq n_3$, then there are 24 elements of order 5 and 20 elements of order 3, impossible for a group of order 30. ■

Question 19

Let G be a finite group with sylow p-subgroup P and normal subgroup N. Show that $P \cap N$ is p-sylow in N and that PN/N is p-sylow in G/N.

Proof. Second isomorphism theorem implies that

$$o(PN) \cdot o(P \cap N) = o(P) \cdot o(N)$$

Because P is p-sylow, we know $o(P)$ and $o(PN)$ has the same p-power, which implies that $o(P \cap N)$ and $o(N)$ has the same p-power, as desired. Again counting the p-power of $PN/N \subseteq G/N$, we see PN/N is p-sylow. ■

Question 20

Let G be a finite group, $H \leq G$ a subgroup with $[G : H] = n$. Show that:

(i) For all subgroup $K \leq G$, we have $[H : H \cap K] \leq [G : K]$.

(ii) $[H : H \cap H^g] \leq n$ for all $g \in G$.

(iii) If H is a maximal proper subgroup of G and H is abelian, show that $H \cap H^g \trianglelefteq G$ for all $g \notin H$.

(iv) Suppose that G is simple. If H is abelian and n is prime, then $H = 1$.

Proof. Let $H/H \cap K$ and G/K denote left coset spaces. (i) is a consequence of verifying that the function

$$H/H \cap K \longrightarrow G/K; \quad h(H \cap K) \mapsto hK$$

is well-defined and injective. (ii) is then a corollary of (i).

We now prove (iii). Fix $g \notin H$. There are two cases: Either $H = H^g$ or $H \neq H^g$. For the first case, just observe that by maximality of H, we will have $N_G(H) = G$. We now claim that $H \neq H^g \implies H \cap H^g \subseteq Z(G)$. Because H is abelian, we know $H \cap H^g \leq Z(H)$. Clearly we also have $H \cap H^g \leq Z(H^g)$. We now have $H \cap H^g \leq Z(\langle H, H^g \rangle)$, where $\langle H, H^g \rangle = G$ by maximality of H, as desired.

We now prove (iv). Clearly the primality of n forces H to be a maximal proper subgroup of G. Therefore by (iii), $H \cap H^g = 1$ for all $g \notin H$. This by (ii) implies $n \leq o(G) \leq n^2$. Write $o(G) \triangleq nk$ so $k \in \{1, \ldots, n\}$. We wish to show $k = 1$. To see $k \neq n$, just recall that if so, then G would be abelian, contradicting to its simplicity. To see $k \notin \{2, \ldots, n-1\}$, just observe that if so, then the unique n-Sylow subgroup would be proper, contradicting to simplicity of G. ■

Question 21

Let G be a finite group with $P \in \mathrm{Syl}_p(G)$. Suppose that N is a normal subgroup of G with $[G : N] = o(P) > 1$. Show that

(i) N is the subset of G consisting of all elements of order not divisible by p.

(ii) If the elements of $G - N$ all has p-power order, then $P = N_G(P)$.

Proof. Because P is p-sylow and $[G : N] = o(P)$, we know $p \nmid o(N)$. This implies that no element of N has order divisible by p. Let $g \in G$ with $p \nmid o(g)$. To see that $g \in N$, just observe that because $o(gN) \mid o(g)$ and $o(gN)$ is a power of p, we have $o(gN) = 1$.

Assume for a contradiction that $P < N_G(P)$. Then there exists some nontrivial sylow q-subgroup Q of $N_G(P)$ with $q \neq p$. By definition we have $[Q, P] \leq P$. By (i), $Q \leq N$. Therefore we also have $[Q, P] \leq N$. Coprimality of orders of N and P now tell us that $[Q, P] = 1$. We now see that the product of two nontrivial elements $x \in Q, y \in P$ has order divisible by pq, a contradiction to the premise. ■