HWs

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Contents

Chapter 1

General Analysis HW

1.1 HW1

Question 1

Show $\mathbb{R}^n$ is complete.

Proof. Let $\mathbf{x}_k$ be an arbitrary Cauchy sequence in $\mathbb{R}^n$. We are required to show $\mathbf{x}_k$ converge in $\mathbb{R}^n$. For each k, denote $\mathbf{x}_k$ by $(x_{(1,k)}, \dots, x_{(n,k)})$. We claim that for each $i \in \{1, \dots, n\}$

$$x_{(i,k)} \text{ is a Cauchy sequence}$$

Fix i and $\epsilon > 0$. To show $x_{(i,k)}$ is a Cauchy sequence, we are required to find $N \in \mathbb{N}$ such that for all $r, m \geq N$ we have

$$\left|x_{(i,r)} - x_{(i,m)}\right| \leq \epsilon$$

Because $\mathbf{x}_k$ is a Cauchy sequence in $\mathbb{R}^n$, we know there exists $N \in \mathbb{N}$ such that for all $r, m \geq N$, we have

$$|\mathbf{x}_r - \mathbf{x}_m| < \epsilon$$

Fix such N and arbitrary $r, m \geq M$. Observe

$$\left|x_{(i,r)} - x_{(i,m)}\right| \leq \sqrt{\sum_{j=1}^{n} \left|x_{(j,r)} - x_{(j,m)}\right|^2} = |\mathbf{x}_r - \mathbf{x}_m| < \epsilon$$

We have proved that for each $i \in \{1, \dots, n\}$, the real sequence $x_{(i,k)}$ is Cauchy. We now claim that for each $i \in \{1, \dots, n\}$, we have

$$\limsup_{r\to\infty} x_{(i,r)} \in \mathbb{R} \text{ and } \lim_{k\to\infty} x_{(i,k)} = \limsup_{r\to\infty} x_{(i,r)}$$

Again fix i. Because $x_{(i,k)}$ is a Cauchy sequence, we know there exists some N such that for all $r, m \geq N$, we have

$$\left|x_{(i,r)} - x_{(i,m)}\right| < 1$$

This implies that for all $r \geq N$, we have

$$x_{(i,r)} < x_{(i,N)} + 1 \tag{1.1}$$

Equation 1.1 then tell us

$$x_{(i,N)} + 1 \text{ is an upper bound of } \{x_{(i,r)} : r \geq N\}$$

Then by definition of sup, we have

$$\sup\{x_{(i,r)} : r \geq N\} \leq x_{(i,N)} + 1 \in \mathbb{R}$$

This then implies $\limsup_{r\to\infty} x_{(i,r)} \in \mathbb{R}$. We now prove

$$\lim_{k\to\infty} x_{(i,k)} = \limsup_{r\to\infty} x_{(i,r)} \tag{1.2}$$

Fix $\epsilon > 0$. We are required to find N such that

$$\forall k \geq N, \left|x_{(i,k)} - \limsup_{r\to\infty} x_{(i,r)}\right| \leq \epsilon$$

Because $\{x_{(i,k)}\}_{k\in\mathbb{N}}$ is a Cauchy sequence, we can let N_0 satisfy

$$\forall k, m \geq N_0, \left|x_{(i,k)} - x_{(i,m)}\right| < \frac{\epsilon}{2}$$

Because $\sup\{x_{(i,k)} : k \geq N'\} \searrow \limsup_{r\to\infty} x_{(i,r)}$ as $N' \to \infty$, we know there exists $N_1 > N_0$ such that

$$\limsup_{r\to\infty} x_{(i,r)} - \frac{\epsilon}{2} < \sup\{x_{(i,k)} : k \geq N_0\} \leq \limsup_{r\to\infty} x_{(i,r)} + \frac{\epsilon}{2}$$

Then because $\limsup_{r\to\infty} x_{(i,r)} - \frac{\epsilon}{2}$ is strictly smaller than the smallest upper bound of $\{x_{(i,k)} : k \geq N_1\}$, we see $\limsup_{n\to\infty} x_{(i,r)} - \frac{\epsilon}{2}$ is not an upper bound of $\{x_{(i,k)} : k \geq N_1\}$. This implies the existence of some N such that $N \geq N_1$ and

$$\limsup_{r\to\infty} x_{(i,r)} - \frac{\epsilon}{2} < x_{(i,N)} \leq \limsup_{r\to\infty} x_{(i,r)} + \frac{\epsilon}{2}$$

Now observe that for all $k \geq N$, because $N \geq N_1 \geq N_0$

$$\limsup_{r\to\infty} x_{(i,r)} - \epsilon < x_{(i,N)} - \frac{\epsilon}{2} < x_{(i,k)} < x_{(i,N)} + \frac{\epsilon}{2} \leq \limsup_{r\to\infty} x_{(i,r)} + \epsilon$$

This implies for all $k \geq N$, we have

$$\left| x_{(i,k)} - \limsup_{r\to\infty} x_{(i,r)} \right| \leq \epsilon$$

We have just proved Equation 1.2. Lastly, to close out the proof, we show

$$\lim_{k\to\infty} \mathbf{x}_k = \left(\lim_{k\to\infty} x_{(1,k)}, \dots, \lim_{k\to\infty} x_{(n,k)} \right) \tag{1.3}$$

Fix $\epsilon > 0$. For each $i \in \{1, \dots, n\}$, let N_i satisfy

$$\forall r \geq N_i, \left| x_{(i,r)} - \lim_{k\to\infty} x_{(i,k)} \right| \leq \frac{\epsilon}{\sqrt{n}}$$

Observe that for all $r \geq \max_{i\in\{1,\dots,n\}} N_i$, we have

$$\begin{aligned} \left| \mathbf{x}_r - \left(\lim_{k\to\infty} x_{(1,k)}, \dots, \lim_{k\to\infty} x_{(n,k)} \right) \right| &= \sqrt{\sum_{i=1}^{n} \left| x_{(i,r)} - \lim_{k\to\infty} x_{(i,k)} \right|^2} \\ &\leq \sqrt{\sum_{i=1}^{n} \frac{\epsilon^2}{n}} = \epsilon \end{aligned}$$

We have proved Equation 1.3.

■

Question 2

Show $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. Fix $x \in \mathbb{R}$ and $\epsilon > 0$. To show $\mathbb{Q}$ is dense in $\mathbb{R}$, we have to find $q \in \mathbb{Q}$ such that $|x - q| < \epsilon$.

Let $m \in \mathbb{N}$ satisfy $\frac{1}{m} < \epsilon$. Let n be the largest integer such that $n \leq mx$. Because n is the largest integer such that $n \leq mx$, we know $mx - n < 1$, otherwise we can deduce $n + 1 \leq mx$, which is impossible, since $n + 1$ is an integer and n is the largest integer such that $n \leq mx$. We now see that

$$\frac{n}{m} \in \mathbb{Q} \text{ and } \left| x - \frac{n}{m} \right| = \frac{mx - n}{m} < \frac{1}{m} < \epsilon$$

■

Theorem 1.1.1. (Distance Formula) Given two subsets A, B of a metric space, we have

$$d(A, B) = \inf_{b\in B} d(A, b)$$

Proof. Fix arbitrary $b \in B$. It is clear that

$$d(A,B) \le d(A,b)$$

It then follows $d(A,B) \le \inf_{b\in B} d(A,b)$. Fix arbitrary $a \in A$ and $b_0 \in B$. Observe that

$$d(a,b_0) \ge d(A,b_0) \ge \inf_{b\in B} d(A,b)$$

It then follows $\inf_{b\in B} d(A,b) \le d(A,B)$. ∎

Question 3

Let E_1, E_2 be non-empty sets in $\mathbb{R}^n$ with E_1 closed and E_2 compact. Show that there are points $x_1 \in E_1$ and $x_2 \in E_2$ such that

$$d(E_1,E_2) = |x_1 - x_2|$$

Deduce that $d(E_1,E_2)$ is positive if such E_1, E_2 are disjoint.

Proof. Because

(a) $f(x) \triangleq d(E_1,x)$ is a continuous function on $\mathbb{R}^n$.

(b) E_2 is compact.

It now follows by EVT there exists some $x_2 \in E_2$ such that

$$d(E_1,x_2) = \min_{x\in E_2} d(E_1,x) = \inf_{x\in E_2} d(E_1,x) = d(E_1,E_2)$$

where the last equality is proved above. We can now reduce the problem into finding x_1 in E_1 such that

$$d(x_1,x_2) = d(E_1,x_2)$$

For each $n \in \mathbb{N}$, let t_n satisfy

$$t_n \in E_1 \text{ and } d(t_n,x_2) < d(E_1,x_2) + \frac{1}{n}$$

Clearly, t_n is a bounded sequence. Then by Bolzano-Weierstrass Theorem, there exists a convergence subsequence t_{n_k}. Now, because E_1 is closed, we know

$$x_1 \triangleq \lim_{k\to\infty} t_{n_k} \in E_1$$

It then follows from the function $f(x) \triangleq d(x,x_2)$ being continuous on $\mathbb{R}^n$ such that

$$d(x_1,x_2) = \lim_{k\to\infty} d(t_{n,k},x_2) = d(E_1,x_2)$$

∎

Question 4

Prove that the distance between two nonempty, compact, disjoint sets in $\mathbb{R}^n$ is positive.

Proof. The proof follows from the result in last question while acknowledging compact is closed. ■

Question 5

Prove that if f is continuous on $[a,b]$, then f is Riemann-integrable on$[a,b]$.

Proof. Let $\overline{\int_a^b} f\,dx$ and $\underline{\int_a^b} f\,dx$ respectively denote the upper and lower Darboux sums. We prove that

$$\overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx$$

Fix ϵ. We reduce the problem into proving the existence of some partition $\{a = x_0, x_1, \dots, x_n = b\}$ such that

$$\sum_{i=1}^n \Big[M_i - m_i\Big](x_i - x_{i-1}) \le \epsilon$$

where

$$M_i \triangleq \sup_{t\in[x_{i-1},x_i]} f(t) \text{ and } m_i \triangleq \inf_{t\in[x_{i-1},x_i]} f(t)$$

Because f is continuous on the compact interval $[a,b]$, we know f is uniformly continuous on $[a,b]$. Let δ satisfy

$$|x-y| < \delta \text{ and } x,y \in [a,b] \implies |f(x)-f(y)| < \frac{\epsilon}{b-a}$$

Let n satisfy $\frac{b-a}{n} < \delta$. We claim the partition

$$\{a = x_0, x_1, \dots, x_n = b\} \text{ where } x_i \triangleq a + \frac{i(b-a)}{n} \text{ suffices}$$

Now, by EVT, we know that for each i, there exists some $t_{i,M}, t_{i,m} \in [x_{i-1}, x_i]$ such that

$$f(t_{i,m}) = m_i \text{ and } f(t_{i,M}) = M_i$$

Then because

$$|t_{i,m} - t_{i,M}| \le x_i - x_{i-1} \le \frac{b-a}{n} < \delta$$

We know $M_i - m_i < \frac{\epsilon}{b-a}$. This now give us

$$\begin{aligned}\sum_{i=1}^{n} \Big[M_i - m_i\Big](x_i - x_{i-1}) &< \sum_{i=1}^{n} \frac{\epsilon}{(b-a)}(x_i - x_{i-1}) \\ &= \frac{\epsilon}{b-a}\sum_{i=1}^{n}(x_i - x_{i-1}) \\ &= \frac{\epsilon}{b-a}(b-a) = \epsilon\end{aligned}$$

■

Question 6

Find $\limsup_{n\to\infty} E_n$ and $\liminf_{n\to\infty} E_n$ where

$$E_n \triangleq \begin{cases} [\frac{-1}{n}, 1] & \text{if } n \text{ is odd} \\ [-1, \frac{1}{n}] & \text{if } n \text{ is even} \end{cases}$$

Proof. Fix arbitrary $n \in \mathbb{N}$. Let $p, q \geq n$ respectively be odd and even. We see

$$[0,1] \subseteq E_p \text{ and } [-1,0] \subseteq E_q$$

This now implies

$$[-1,1] \subseteq \bigcup_{k\geq n} E_k$$

Then because n is arbitrary, it follows

$$\limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty}\bigcup_{k\geq n} E_k = [-1,1]$$

Again, fix arbitrary $n \in \mathbb{N}$ and $\epsilon > 0$. Let p, q respectively be even and odd integers greater than $\max\{n, \frac{1}{\epsilon}\}$. We now see

$$\epsilon \notin [-1, \frac{1}{p}] = E_p \text{ and } -\epsilon \notin [\frac{-1}{q}, 1] = E_q$$

Because ϵ is arbitrary and clearly $0 \in E_k$ for all k, we now see

$$\bigcap_{k\geq n} E_k = \{0\}$$

Then because n is arbitrary, we see

$$\liminf_{n\to\infty} E_n = \bigcup_{n=1}^{\infty}\bigcap_{k\geq n} E_k = \{0\}$$

$\blacksquare$

Question 7

Show that

$$(\limsup_{n\to\infty} E_n)^c = \liminf_{n\to\infty}(E_n)^c$$

and

$$E_n \searrow E \text{ or } E_n \nearrow E \implies \limsup_{n\to\infty} E_n = \liminf_{n\to\infty} E_n = E$$

Proof. Fix arbitrary $x \in (\limsup_{n\to\infty} E_n)^c$. We can deduce

$$\exists n, x \notin \bigcup_{k\geq n} E_k$$

This implies

$$\exists n, x \in \bigcap_{k\geq n} E_k^c$$

Then we see

$$x \in \bigcup_{n=1}^{\infty}\bigcap_{k\geq n} E_k^c = \liminf_{n\to\infty} E_n^c$$

We have proved $(\limsup_{n\to\infty} E_n)^c \subseteq \liminf_{n\to\infty} E_n^c$. We now prove the converse. Fix arbitrary $x \in \liminf_{n\to\infty} E_n^c$. We can deduce

$$\exists n, x \in \bigcap_{k\geq n} E_k^c$$

This implies

$$\exists n, x \notin \bigcup_{k\geq n} E_k$$

Then we see

$$x \notin \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k = \limsup_{n\to\infty} E_n$$

■

Theorem 1.1.2. (Equivalent Definition for Limit Superior) If we let E be the set of subsequential limits of a_n

$$E \triangleq \{L \in \overline{\mathbb{R}} : L = \lim_{k\to\infty} a_{n_k} \text{ for some } n_k\}$$

The set E is non-empty and

$$\max E = \limsup_{n\to\infty} a_n$$

Proof. Let $n_1 \triangleq 1$. Recursively, because

$$\sup_{j\geq n_k} a_k \geq \limsup_{n\to\infty} a_n > \limsup_{n\to\infty} a_n - \frac{1}{k} \text{ for each } k$$

We can let n_{k+1} be the smallest number such that

$$a_{n_{k+1}} > \limsup_{n\to\infty} a_n - \frac{1}{k}$$

It is straightforward to check $a_{n_k} \to \limsup_{n\to\infty} a_n$ as $k \to \infty$. Note that no subsequence can converge to $\limsup_{n\to\infty} a_n + \epsilon$ because there exists N such that $\sup_{k\geq N} a_k < \limsup_{n\to\infty} a_n + \epsilon$. ■

Question 8

Show that

$$\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n$$

Proof. Note that $-a_{n_k}$ converge if and only if a_{n_k} converge. Then if we respectively define E and E^- to be the set of subsequential limits of a_n and $-a_n$, we see

$$E^- = \{-L \in \mathbb{R} : L \in E\}$$

We now see

$$\limsup_{n\to\infty}(-a_n) = \max E^- = -\min E = -\liminf_{n\to\infty} a_n$$

■

Question 9

Show that

$$\limsup_{n\to\infty}(a_n+b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \tag{1.4}$$

Proof. Fix arbitrary ϵ. Let N_a, N_b respectively satisfy

$$\sup_{n\ge N_a} a_n \le \limsup_{n\to\infty} a_n + \frac{\epsilon}{2} \text{ and } \sup_{n\ge N_b} b_n \le \limsup_{n\to\infty} b_n + \frac{\epsilon}{2}$$

Let $N \triangleq \max\{N_a, N_b\}$. We now see that

$$\limsup_{n\to\infty}(a_n+b_n) \le \sup_{n\ge N}(a_n+b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n + \epsilon$$

The result then follows from ϵ being arbitrary. ■

Question 10

$$a_n, b_n \text{ is bounded non-negative } \implies \limsup_{n\to\infty}(a_n b_n) \le (\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n) \tag{1.5}$$

Proof. There are three cases we should consider

(a) Both $\limsup_{n\to\infty} a_n$ and $\limsup_{n\to\infty} b_n$ equal 0.

(b) Between $\limsup_{n\to\infty} a_n$ and $\limsup_{n\to\infty} b_n$, only one of them equals 0.

(c) Neither $\limsup_{n\to\infty} a_n$ nor $\limsup_{n\to\infty} b_n$ equals to 0.

In the first case, because a_n, b_n are both non-negative, we can deduce

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$$

which implies

$$\limsup_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n b_n = 0 = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$$

For second case, WOLG, suppose $\limsup_{n\to\infty} a_n = 0$. Fix arbitrary ϵ. We can let N satisfy

$$\sup_{n\ge N} a_n < \frac{\epsilon}{\sup_{n\in\mathbb{N}} b_n}$$

Since for all $n \geq N$, we have

$$a_n b_n \leq \frac{b_n \epsilon}{\sup_{k\in\mathbb{N}} b_k} \leq \epsilon$$

We now see

$$\limsup_{n\to\infty}(a_n b_n) \leq \sup_{n\geq N} a_n b_n \leq \epsilon$$

The result

$$\limsup_{n\to\infty} a_n b_n = 0 = \limsup_{n\to\infty} a_n \limsup_{n\to\infty} b_n$$

then follows from ϵ being arbitrary.

Lastly, for the last case, let N_a, N_b respectively satisfy

$$\sup_{n\geq N_a} a_n \leq \limsup_{n\to\infty} a_n \sqrt{1+\epsilon} \text{ and } \sup_{n\geq N_b} b_n \leq \limsup_{n\to\infty} b_n \sqrt{1+\epsilon}$$

Let $N \triangleq \max\{N_a, N_b\}$, because for each $n \geq N$, we have

$$a_n b_n \leq (\sup_{k\geq N_a} a_k)(\sup_{k\geq N_b} b_k) \leq (1+\epsilon)(\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n)$$

It then follows that

$$\limsup_{n\to\infty}(a_n b_n) \leq \sup_{n\geq N}(a_n b_n) \leq (1+\epsilon)(\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n)$$

The result then follows from ϵ being arbitrary. ■

Question 11

Show that if either a_n or b_n converge, the equalities in Equation 1.4 and Equation 1.5 both hold true.

Proof. WOLG, suppose $\lim_{n\to\infty} a_n = L \in \mathbb{R}$. We then see

$$(a_{n_k} + b_{n_k}) \text{ converge } \iff b_{n,k} \text{ converge}$$

Let $E_{a,b}$ and E_b respectively be the set of subsequential limits of $(a_n + b_n)$ and b_n. We now have

$$E_{a,b} = \{L + L_b \in \mathbb{R} : L_b \in E_b\}$$

This give us

$$\limsup_{n\to\infty}(a_n+b_n) = \max E_{a,b} = L + \max E_b = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

Now, additionally, suppose a_n, b_n are both bounded and nonnegative. Again because

$$a_{n_k} b_{n,k} \text{ converge } \iff b_{n,k} \text{ converge}$$

We see

$$E_{a,b} = \{L(L_b) \in \mathbb{R} : L_b \in E_b\}$$

This give us

$$\limsup_{n\to\infty}(a_n b_n) = \max E_{a,b} = L \max E_b = (\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n)$$

$\blacksquare$

Question 12

Give example for which inequality in Equation 1.4 and Equation 1.5 are not equalities.

Proof. If

$$a_n \triangleq \begin{cases} 1 & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ is even} \end{cases} \text{ and } b_n \triangleq \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

we have

$$\limsup_{n\to\infty}(a_n+b_n) = 0 < 2 = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

Let $L > 1$ and

$$a_n \triangleq \begin{cases} L - \frac{1}{k} & \text{if } n = 2k-1 \\ (L-\frac{1}{k})^{-1} & \text{if } n = 2k \end{cases} \text{ and } b_n \triangleq \begin{cases} (L-\frac{1}{k})^{-1} & \text{if } n = 2k-1 \\ (L-\frac{1}{k}) & \text{if } n = 2k \end{cases}$$

We have

$$\limsup_{n\to\infty} a_n b_n = 1 < L^2 = \limsup_{n\to\infty} a_n \limsup_{n\to\infty} b_n$$

$\blacksquare$

Question 13

Give an example of a decreasing sequence of nonempty closed sets in $\mathbb{R}^n$ whose intersection is empty.

Proof.

$$F_n \triangleq [n, \infty) \text{ suffices}$$

■

Question 14

Given an example of two disjoint, nonempty closed sets in E_1 and E_2 in $\mathbb{R}^n$ for which $d(E_1, E_2) = 0$.

Proof. Let

$$E_1 \triangleq \{n - \frac{1}{n} \in \mathbb{R} : n \in \mathbb{N} \text{ and } n \geq 2\} \text{ and } E_2 \triangleq \{n - \frac{1}{2n} \in \mathbb{R} : n \in \mathbb{N} \text{ and } n \geq 2\}$$

To see $E_1 \cap E_2 = \varnothing$, suppose $n - \frac{1}{n} = k - \frac{1}{2k}$ where n, k are two natural numbers greater than 2. We then see $\frac{1}{n} - \frac{1}{2k} = n - k$, which is impossible, since

$$\left| \frac{1}{n} - \frac{1}{2k} \right| < \max\{\frac{1}{2k}, \frac{1}{n}\} < 1$$

The fact E_1, E_2 are closed follows from both of them being totally disconnected. Now observe that for all ϵ, there exists large enough n such that

$$(n + \frac{1}{n}) - (n + \frac{1}{2n}) < \frac{1}{n} < \epsilon$$

This implies $d(E_1, E_2) = 0$. ■

Question 15

If f is defined and uniformly continuous on E, show there is a function $\overline{f}$ defined and continuous on $\overline{E}$ such that $\overline{f} = f$ on E.

Proof. Define $\overline{f}$ on E by $\overline{f} = f$. For each $x \in \overline{E} \setminus E$, associate x with a sequence $t_{n,x}$ in E converging to x. We now claim that for each $x \in \overline{E} \setminus E$ the limit

$$\lim_{n \to \infty} f(t_{n,x}) \text{ converge in } \mathbb{R}$$

Fix ϵ. Because f is uniformly continuous on E, we know there exists δ such that

$$a, b \in E \text{ and } |a-b| \leq \delta \implies |f(a) - f(b)| < \epsilon$$

Because $t_{n,x}$ converge, we know $t_{n,x}$ is Cauchy, then we know there exists N such that $|t_{n,x} - t_{m,x}| < \delta$ for all $n, m > N$, we then see that for all $n, m > N$, we have

$$|f(t_{n,x}) - f(t_{m,x})| < \epsilon$$

This implies $\{f(t_{n,x})\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, thus converge in $\mathbb{R}$.

Define

$$\overline{f}(x) \triangleq \lim_{n\to\infty} f(t_{n,x}) \text{ for all } x \in \overline{E} \setminus E$$

We are required to show $\overline{f}$ is also continuous on $\overline{E} \setminus E$. Fix ϵ and $x \in \overline{E} \setminus E$. Let δ satisfy

$$a, b \in E \text{ and } |a-b| \leq \delta \implies |f(a) - f(b)| < \frac{\epsilon}{3}$$

We claim

$$\sup_{t \in B_{\frac{\delta}{2}}(x) \cap \overline{E}} \left|\overline{f}(t) - \overline{f}(x)\right| \leq \epsilon$$

Fix $t \in B_{\frac{\delta}{2}}(x) \cap \overline{E}$. There are two possibilities

(a) $t \in E$

(b) $t \in \overline{E} \setminus E$

If $t \in E$, let n satisfy

$$\left|f(t_{n,x}) - \overline{f}(x)\right| < \frac{\epsilon}{3} \text{ and } |t_{n,x} - x| < \frac{\delta}{2}$$

Because

$$|t_{n,x} - t| \leq |t_{n,x} - x| + |t - x| < \delta$$

we can deduce $|f(t_{n,x}) - f(t)| < \frac{\epsilon}{3}$. This now give us

$$\left|f(t) - \overline{f}(x)\right| \leq |f(t_{n,x}) - f(t)| + \left|f(t_{n,x}) - \overline{f}(x)\right| < \epsilon$$

If $t \in \overline{E} \setminus E$. Write $y = t$ and let $t_{n,y}$ be the associated sequence in E. Because $y \in B_{\frac{\delta}{2}}(x)$, we know there exists $t_{n,y}$ such that

$$t_{n,y} \in B_{\frac{\delta}{2}}(x) \text{ and } \left|f(t_{n,y}) - \overline{f}(y)\right| < \frac{\epsilon}{3}$$

Again, let m satisfy

$$t_{m,x} \in B_{\frac{\delta}{2}}(x) \text{ and } \left|f(t_{m,x}) - \overline{f}(x)\right| < \frac{\epsilon}{3}$$

We know $|t_{n,y} - t_{m,x}| \leq \delta$ because they both belong to $B_{\frac{\delta}{2}}(x)$. We can now deduce

$$\left|\overline{f}(y) - \overline{f}(x)\right| = \left|\overline{f}(y) - f(t_{n,y})\right| + |f(t_{n,y}) - f(t_{m,x})| + \left|f(t_{m,x}) - \overline{f}(x)\right| < \epsilon$$

which finish the proof. ■

Question 16

If f is defined and uniformly continuous on a bounded set E, show that f is bounded on E.

Proof. By last question, we can extend f to a continuous $\overline{f}$ onto $\overline{E}$. Now because $\overline{E}$ is compact and $\left|\overline{f}\right|$ is continuous on $\overline{E}$, by EVT, there exists $a \in \overline{E}$ such that

$$\sup_{x\in E} |f(x)| \leq \max_{x\in\overline{E}} |f(x)| = f(a)$$

■

1.2 HW2

Question 17

Construct a two-dimensional Cantor set in the unit square $[0,1]^2$ as follows. Subdivide the square into nine equal parts and keep only the four closed corner squares, removing the remaining region (which form a cross). Then repeat this process in a suitably scaled version for the remaining squares, ad infinitum. Show that the resulting set is perfect, has plane measure zero, and equals $C \times C$.

Question 18

Construct a subset of $[0,1]$ in the same manner as the Cantor set, except that at the kth stage, each interval removed has length $\delta 3^{-k}, 0 < \delta < 1$. Show that the resulting set is perfect, has measure $1-\delta$, and contains no intervals.

Question 19

If E_k is a sequence of sets with $\sum |E_k|_e < \infty$, show that $\limsup_{n\to\infty} E_n$ has measure zero.

Proof. Note that

$$\sum_{k=N}^{\infty} |E_k|_e = \left(\sum_{k=1}^{\infty} |E_k|_e - \sum_{k=1}^{N-1} |E_k|_e \right) \to 0 \text{ as } N \to \infty$$

and note for all N we have

$$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \subseteq \bigcup_{k=N}^{\infty} E_k$$

Then for arbitrary ϵ, if we let N satisfy $\sum_{k=N}^{\infty} |E_k|_e < \epsilon$, we see that

$$\left| \limsup_{n\to\infty} E_n \right|_e = \left| \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \right|_e \leq \left| \bigcup_{k=N}^{\infty} E_k \right|_e \leq \sum_{k=N}^{\infty} |E_k|_e < \epsilon$$

■

Theorem 1.2.1. (Product of Finite Measure Set) If E_1 and E_2 are measurable subset of $\mathbb{R}$ and $|E_1|, |E_2| < \infty$, then $E_1 \times E_2$ is measurable in $\mathbb{R}^2$ and

$$|E_1 \times E_2| = |E_1| \, |E_2|$$

Proof. Fix ϵ. We are required to find open set $O \subseteq \mathbb{R}^2$ such that

$$E_1 \times E_2 \subseteq O \text{ and } \left|O \setminus (E_1 \times E_2)\right|_e \leq \epsilon \tag{1.6}$$

Let O_1, O_2 be two open subset of $\mathbb{R}$ respectively containing E_1, E_2 and satisfy the following four conditions

(a) $|O_2| < |E_2| + 1$

(b) $|O_1| < |E_1| + 1$

$$|O_1 \setminus E_1| \leq \frac{\epsilon}{2(|E_2|+1)} \text{ and } |O_2 \setminus E_2| \leq \frac{\epsilon}{2(|E_1|+1)}$$

We claim

$$O \triangleq O_1 \times O_2 \text{ suffices for Equation 1.6}$$

Observe that

$$\Big(O_1 \times O_2\Big) \setminus \Big(E_1 \times E_2\Big) = \Big((O_1 \setminus E_1) \times O_2\Big) \sqcup \Big(E_1 \times (O_2 \setminus E_2)\Big)$$

∎

Question 20

If E_1 and E_2 are measurable subset of $\mathbb{R}$, then $E_1 \times E_2$ is a measurable subset of $\mathbb{R}^2$ and $|E_1 \times E_2| = |E_1|\,|E_2|$

Proof. Define

$$A_n \triangleq \{x \in \mathbb{R} : n \leq x < n+1\}$$

Because

$$E_1 = \bigcup_{n \in \mathbb{Z}} E_1 \cap A_n \text{ and } E_2 = \bigcup_{k \in \mathbb{Z}} E_2 \cap A_k$$

We can deduce

$$E_1 \times E_2 = \bigcup_{n,k \in \mathbb{Z}} (E_1 \cap A_n) \times (E_2 \cap A_k)$$

and deduce

$$|E_1| = \sum_{n \in \mathbb{Z}} |E_1 \cap A_n| \text{ and } |E_2| = \sum_{k \in \mathbb{Z}} |E_2 \cap A_k|$$

which give us

$$|E_1|\,|E_2| = \sum_{n,k\in\mathbb{Z}} |E_1 \cap A_n|\,|E_2 \cap A_k|$$

We then can reduce the problem into proving for all $n, k \in \mathbb{Z}$

$$H \triangleq (E_1 \cap A_n) \times (E_2 \cap A_k) \text{ is measurable and } |H| = |E_1 \cap A_n|\,|E_2 \cap A_k|$$

■

Question 21

Give an example that shows that the image of a measurable set under a continuous transformation may not be measurable. See also Exercise 10 of Chapter 7.

Question 22

Show that there exists disjoint $E_1, E_2, \dots$ such that $|\bigcup E_k|_e < \sum |E_k|_e$ with strict inequality.

Question 23

Show that there exists decreasing sequence E_k of sets such that

(a) $E_k \searrow E$.

(b) $|E_k|_e < \infty$.

(c) $\lim_{k\to\infty} |E_k|_e > |E|_e$

Question 24

Let Z be a subset of $\mathbb{R}$ with measure zero. Show that the set $\{x^2 : x \in Z\}$ also has measure zero.

1.3 Brunn-Minkowski Inequality

Chapter 2

Complex Analysis HW

2.1 HW1

Theorem 2.1.1.

$$(1+i)^n, \frac{(1+i)^n}{n}, \frac{n!}{(1+i)^n} \text{ all diverge as } n \to \infty$$

Proof. Note that

$$|(1+i)^n| = 2^{\frac{n}{2}} \to \infty \text{ as } n \to \infty$$

This implies $(1+i)$ is unbounded, thus diverge.

Note that

$$\left|\frac{(1+i)^n}{n}\right| = \frac{(\sqrt{2})^n}{n}$$

Observe

$$\begin{aligned}\frac{(\sqrt{2})^n}{n} = \frac{[(\sqrt{2}-1)+1]^n}{n} &= \frac{\sum_{k=0}^{n}\binom{n}{k}(\sqrt{2}-1)^k}{n} \\ &\geq \frac{\binom{n}{2}(\sqrt{2}-1)^2}{n} = (n-1)[\frac{(\sqrt{2}-1)^2}{2}] \to \infty \text{ as } n \to \infty\end{aligned}$$

This implies $\frac{(1+i)^n}{n}$ is unbounded, thus diverge.

Note that

$$\left|\frac{n!}{(1+i)^n}\right| = \frac{n!}{(\sqrt{2})^n}$$

Note that for all $k \geq 8$, we have

$$\frac{k}{\sqrt{2}} \geq \frac{\sqrt{8}}{\sqrt{2}} = 2$$

This implies

$$\frac{n!}{(\sqrt{2})^n} = \prod_{k=1}^{n} \frac{k}{\sqrt{2}} = \frac{7!}{(\sqrt{2})^7} \prod_{k=8}^{n} \frac{k}{\sqrt{2}} \geq \frac{7!}{(\sqrt{2})^7} \prod_{k=8}^{n} 2 \geq \frac{7!2^{n-8+1}}{(\sqrt{2})^7} \to \infty$$

which implies $\frac{n!}{(1+i)^n}$ is unbounded, thus diverge. ∎

Theorem 2.1.2.

$$n!z^n \text{ converge} \iff z = 0$$

Proof. If $z = 0$, then $n!z^n = 0$ for all n, which implies $n!z^n \to 0$. Now, suppose $z \neq 0$. Let $M \in \mathbb{N}$ satisfy $|z| > \frac{1}{M}$. Observe

$$|n!z^n| = \left|\prod_{k=1}^{n} kz\right| = \left|\prod_{k=1}^{2M-1} kz\right| \left|\prod_{k=2M}^{n} kz\right| \geq \left|\prod_{k=1}^{2M-1} kz\right| \left|\prod_{k=2M}^{n} 2Mz\right| \geq \left|\prod_{k=1}^{2M-1} kz\right| 2^{n-2M+1} \to \infty$$

This implies $n!z^n$ is unbounded, thus diverge. ∎

Theorem 2.1.3.

$$u_n \to u \implies v_n \triangleq \sum_{k=1}^{n} \frac{u_k}{n} \to u$$

Proof. Because

$$\sum_{k=1}^{n} \frac{u_k}{n} = \sum_{k \leq \sqrt{n}} \frac{u_k}{n} + \sum_{\sqrt{n} < k \leq n} \frac{u_k}{n}$$

It suffices to prove

$$\sum_{k \leq \sqrt{n}} \frac{u_k}{n} \to 0 \text{ and } \sum_{\sqrt{n} < k \leq n} \frac{u_k}{n} \to u \text{ as } n \to \infty$$

Because u_n converge, we can let M bound $|u_n|$. Observe

$$\left|\sum_{k \leq \sqrt{n}} \frac{u_k}{n}\right| \leq \sum_{k \leq \sqrt{n}} \left|\frac{u_k}{n}\right| \leq \sum_{k \leq \sqrt{n}} \frac{M}{n} \leq \frac{M\sqrt{n}}{n} = \frac{M}{\sqrt{n}} \to 0 \text{ as } n \to 0 \text{ (done)}$$

Because

$$\sum_{\sqrt{n}<k\leq n}\frac{u_k}{n}=\frac{n-\lceil\sqrt{n}\rceil+1}{n}\sum_{\sqrt{n}<k\leq n}\frac{u_k}{n-\lceil\sqrt{n}\rceil+1}$$

and

$$\lim_{n\to\infty}\frac{n-\lceil\sqrt{n}\rceil+1}{n}=1$$

We can reduce the problem into proving

$$\lim_{n\to\infty}\sum_{\sqrt{n}<k\leq n}\frac{u_k}{n-\lceil\sqrt{n}\rceil+1}=u$$

Fix ϵ. Let N satisfy that for all $n\geq N$, we have $|u_n-u|<\epsilon$. Then for all $n\geq N^2$, we have

$$\begin{aligned}\left|\left(\sum_{\sqrt{n}<k\leq n}\frac{u_k}{n-\lceil\sqrt{n}\rceil+1}\right)-u\right|&=\left|\sum_{\sqrt{n}<k\leq n}\frac{u_k-u}{n-\lceil\sqrt{n}\rceil+1}\right|\\&\leq\sum_{\sqrt{n}<k\leq n}\frac{|u_k-u|}{n-\lceil\sqrt{n}\rceil+1}\\&\leq\sum_{\sqrt{n}<k\leq n}\frac{\epsilon}{n-\lceil\sqrt{n}\rceil+1}=\epsilon\text{ (done)}\end{aligned}$$

■

Chapter 3

PDE intro HW

3.1 HW1

Theorem 3.1.1.

$$\text{Show } u \mapsto u_x + uu_y \text{ is non-linear}$$

Proof. See that

$$2u \mapsto 2u_x + 4uu_y \neq 2(u_x + uu_y) \tag{3.1}$$

■

Theorem 3.1.2.

$$\text{Solve } (1 + x^2)u_x + u_y = 0$$

Proof. The characteristic curve has the derivative

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

The solution to this ODE is

$$y = \arctan x + C$$

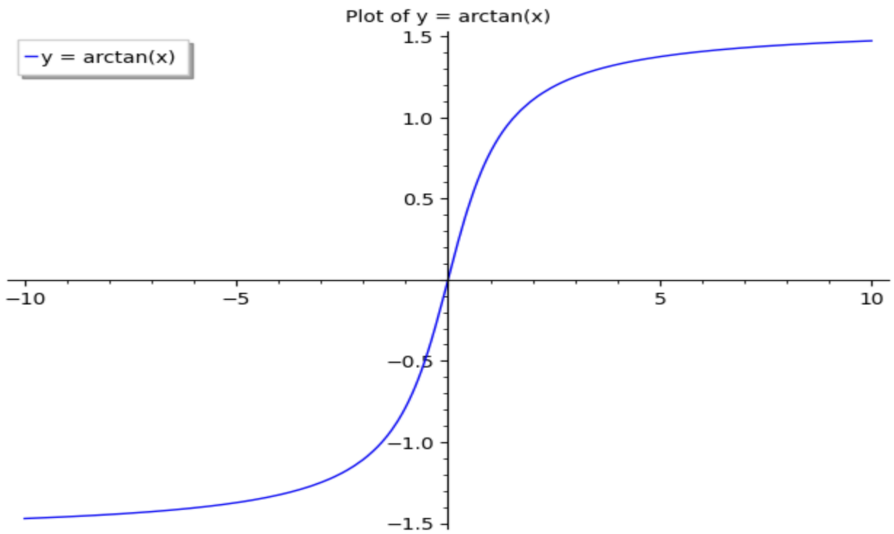
We now see that the solution to the PDE in Equation 3.1 is

$$u = f\big((\arctan x) - y\big) \text{ where } f : \mathbb{R} \to \mathbb{R} \text{ is an arbitrary smooth function}$$

A characteristic curve is as followed.

```
[3]: plot(arctan(x), (x, -10, 10), title='Plot of y = arctan(x)', legend_label='y = arctan(x)')
```

[3]:



■

Theorem 3.1.3.

$$\text{Solve } au_x + bu_y + cu = 0 \tag{3.2}$$

Proof. Fix

$$\begin{cases} x' \triangleq ax + by \\ y' \triangleq bx - ay \end{cases}$$

This map is clearly a diffeomorphism. Compute

$$\begin{cases} u_x = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = au_{x'} + bu_{y'} \\ u_y = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} = bu_{x'} - au_{y'} \end{cases}$$

Plugging it back into the PDE in Equation 3.2, we have

$$cu + (a^2 + b^2)u_{x'} = 0 \tag{3.3}$$

If $c = a^2 + b^2 = 0$, then all smooth functions are solution. If $a^2 + b^2 = 0$ but $c \neq 0$, then clearly the only solution is $u = \tilde{0}$. If $a^2 + b^2 \neq 0$ but $c = 0$, then $u_{x'} = \tilde{0}$, which implies $u = f(y')$ where $y' = bx - ay$ and f can be arbitrary smooth function.

Now, suppose $a^2 + b^2 \neq 0 \neq c$, note that the PDE in Equation 3.3 is just an ODE of the form

$$y + \frac{a^2 + b^2}{c}y' = 0$$

The general solution to this ODE is

$$y = Ce^{\frac{-ct}{a^2+b^2}}$$

In other words, the general solution of the PDE in Equation 3.3 is

$$u = Ce^{\frac{-cx'}{a^2+b^2}} = Ce^{\frac{-c(ax+by)}{a^2+b^2}}$$

■

3.2 HW2

Question 25

Consider hear flow in a long circular cylinder where the temperature depends only on t and on the distance r to the axis of the cylinder. Here $r = \sqrt{x^2 + y^2}$ is the cylindrical coordinate. From the three dimensional hear equation derive the equation $u_t = k(u_{rr} + \frac{u_r}{r})$

Proof. Write the three dimensional hear equation by

$$u_t = k\Delta u$$

Note that the Laplacian Δu when written in cylindrical coordinate is

$$\Delta u = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} + u_{zz}$$

Because the premise says that u is constant in z and θ, we know $u_{\theta\theta} = u_{zz} = 0$

$$\Delta u = u_{rr} + \frac{u_r}{r}$$

This give us

$$u_t = k(u_{rr} + \frac{u_r}{r})$$

■

Chapter 4

Differential Geometry HW

4.1 HW1

Abstract

In this HW, we give precise definition to $\mathbb{P}^n$ and $\mathbb{R}P^n$, and we rigorously show

(a) $\mathbb{R}P^n$ has a smooth structure.

(b) $\mathbb{P}^n$ is homeomorphic to $\mathbb{R}P^n$

(c) $\mathbb{P}^n$ has a smooth structure.

Note that in this PDF, brown text is always a clickable hyperlink reference.

Define an equivalence relation on $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ by

$$\mathbf{x} \sim \mathbf{y} \overset{\triangle}{\iff} \mathbf{y} = \lambda \mathbf{x} \text{ for some } \lambda \in \mathbb{R}^*$$

Let $\mathbb{R}P^n \triangleq (\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}) \setminus \sim$ be the quotient space and let

$$V_i \triangleq \{\mathbf{x} \in \mathbb{R}^{n+1} : \mathbf{x}^i \neq 0\} \text{ for each } 1 \leq i \leq n+1$$

By definition, it is clear that

$$\text{either } \pi^{-1}(\pi(\mathbf{x})) \subseteq V_i \text{ or } \pi^{-1}(\pi(\mathbf{x})) \subseteq V_i^c$$

Then if we define $\phi_i : V_i \to \mathbb{R}^n$ by

$$\phi_i(\mathbf{x}) \triangleq \left(\frac{\mathbf{x}^1}{\mathbf{x}^i}, \dots, \frac{\mathbf{x}^{i-1}}{\mathbf{x}^i}, \frac{\mathbf{x}^{i+1}}{\mathbf{x}^i}, \dots, \frac{\mathbf{x}^{n+1}}{\mathbf{x}^i} \right)$$

because $\pi(\mathbf{x}) = \pi(\mathbf{y}) \implies \phi_i(\mathbf{x}) = \phi_i(\mathbf{y})$, we can well induce a map

$$\Phi_i : U_i \triangleq \pi(V_i) \subseteq \mathbb{R}P^n \to \mathbb{R}^n ; \pi(\mathbf{x}) \mapsto \phi_i(\mathbf{x})$$

Note that one has the equation

$$\Phi_i(\pi(\mathbf{x})) = \phi_i(\mathbf{x}) \text{ for all } \mathbf{x} \in V_i$$

Theorem 4.1.1. (Real Projective Space with a differentiable atlas) We have

$$\mathbb{R}P^n \text{ with atlas } \{(U_i, \Phi_i) : 1 \leq i \leq n+1\} \text{ is a differentiable manifold}$$

Proof. We are required to prove

(a) (U_i, Φ_i) are all charts.

(b) $\{(U_i, \Phi_i) : 1 \leq i \leq n+1\}$ form a differentiable atlas.

(c) $\mathbb{R}P^n$ is Hausdorff.

(d) $\mathbb{R}P^n$ is second-countable.

Because $\pi^{-1}(U_i) = V_i$ and V_i is clearly open in $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$, we know $U_i \subseteq \mathbb{R}P^n$ is open. Note that clearly, $\Phi_i(U_i) = \mathbb{R}^n$. To show (U_i, Φ_i) is a chart, it remains to show that Φ_i is a homeomorphism between U_i and $\mathbb{R}^n$. It is straightforward to check Φ_i is one-to-one on U_i. This implies Φ_i is a bijective between U_i and $\mathbb{R}^n$.

Fix open $E \subseteq \mathbb{R}^n$. We see

$$\pi^{-1}(\Phi_i^{-1}(E)) = \phi_i^{-1}(E)$$

Then because $\phi_i : V_i \to \mathbb{R}^n$ is clearly continuous, we see $\phi_i^{-1}(E)$ is open in $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$, and it follows from definition of quotient topology $\Phi_i^{-1}(E) \subseteq \mathbb{R}P^n$ is open. Then because U_i is open in $\mathbb{R}P^n$, we see $\Phi_i^{-1}(E)$ is open in U_i. We have proved $\Phi_i : U_i \to \mathbb{R}^n$ is continuous.

Define $\Psi_i : \mathbb{R}^n \to V_i$ by

$$\Psi(\mathbf{x}^1, \dots, \mathbf{x}^n) = (\mathbf{x}^1, \dots, \mathbf{x}^{i-1}, 1, \mathbf{x}^i, \dots, \mathbf{x}^n)$$

Observe that for all $\mathbf{x} \in \Phi_i(U_i)$, we have

$$\Phi_i^{-1}(\mathbf{x}) = \pi(\Psi_i(\mathbf{x}))$$

It then follows from $\Psi_i : \mathbb{R}^n \to V_i$ and $\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{R}P^n$ are continuous that $\Phi_i^{-1} : \mathbb{R}^n \to \mathbb{R}P^n$ is continuous.

We have proved that (Ψ_i, U_i) are all charts. Now, because V_i clearly cover $\mathbb{R}^{n+1}$, we know U_i also cover $\mathbb{R}P^n$. We have proved $\{(U_i, \Phi) : 1 \leq i \leq n+1\}$ form an atlas. The fact $\mathbb{R}P^n$ is second-countable follows.

Fix $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \Phi_i(U_i \cap U_j)$. We compute

$$\Phi_j \circ \Phi_i^{-1}(\mathbf{x}^1, \dots, \mathbf{x}^n) = \Phi_j\Big([(\mathbf{x}^1, \dots, \mathbf{x}^{i-1}, 1, \mathbf{x}^i, \mathbf{x}^{i+1}, \dots, \mathbf{x}^n)]\Big)$$
$$= \begin{cases} (\frac{\mathbf{x}^1}{\mathbf{x}^j}, \dots, \frac{\mathbf{x}^{j-1}}{\mathbf{x}^j}, \frac{\mathbf{x}^{j+1}}{\mathbf{x}^j}, \dots, \frac{\mathbf{x}^{i-1}}{\mathbf{x}^j}, \frac{1}{\mathbf{x}^j}, \frac{\mathbf{x}^i}{\mathbf{x}^j}, \dots, \frac{\mathbf{x}^n}{\mathbf{x}^j}) & \text{if } j < i \\ (\frac{\mathbf{x}^1}{\mathbf{x}^{j-1}}, \dots, \frac{\mathbf{x}^{i-1}}{\mathbf{x}^{j-1}}, \frac{1}{\mathbf{x}^{j-1}}, \frac{\mathbf{x}^i}{\mathbf{x}^{j-1}}, \dots, \frac{\mathbf{x}^{j-2}}{\mathbf{x}^{j-1}}, \frac{\mathbf{x}^j}{\mathbf{x}^{j-1}}, \dots, \frac{\mathbf{x}^n}{\mathbf{x}^{j-1}}) & \text{if } j > i \end{cases}$$

This implies our atlas is indeed differentiable.

Before we prove $\mathbb{R}P^n$ is Hausdorff, we first prove that $\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{R}P^n$ is an open mapping. Let $U \subseteq \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ be open. Observe that

$$\pi^{-1}(\pi(U)) = \{t\mathbf{x} \in \mathbb{R}^{n+1} : t \neq 0 \text{ and } \mathbf{x} \in U\}$$

Fix $t_0\mathbf{x} \in \pi^{-1}(\pi(U))$. Let $B_\epsilon(\mathbf{x}) \subseteq U$. Observe that

$$B_{|t_0|\epsilon}(t_0\mathbf{x}) \subseteq t_0 B_\epsilon(\mathbf{x}) \subseteq t_0 U \subseteq \pi^{-1}(\pi(U))$$

This implies $\pi^{-1}(\pi(U))$ is open. (done)

Now, because π is open, to show $\mathbb{R}P^n$ is Hausdorff, we only have to show

$$R_\pi \triangleq \{(\mathbf{x}, \mathbf{y}) \in (\mathbb{R}^{n+1} \setminus \{\mathbf{0}\})^2 : \pi(\mathbf{x}) = \pi(\mathbf{y})\} \text{ is closed}$$

Define $f : (\mathbb{R}^{n+1} \setminus \{\mathbf{0}\})^2 \to \mathbb{R}$ by

$$f(\mathbf{x}, \mathbf{y}) \triangleq \sum_{i \neq j} (\mathbf{x}^i \mathbf{y}^j - \mathbf{x}^j \mathbf{y}^i)^2$$

Note that f is clearly continuous and $f^{-1}(0) = R_\pi$, which finish the proof. ■

Alternatively, we can characterize $\mathbb{R}P^n$ by identifying the antipodal pints on $S^n \triangleq \{\mathbf{x} \in \mathbb{R}^{n+1} : |\mathbf{x}| = 1\}$ as one point

$$\mathbf{x} \sim \mathbf{y} \overset{\triangle}{\iff} \mathbf{x} = \mathbf{y} \text{ or } \mathbf{x} = -\mathbf{y}$$

and let $\mathbb{P}^n \triangleq S^n \setminus \sim$ be the quotient space.

Theorem 4.1.2. (Equivalent Definitions of Real Projective Space)

$$\mathbb{R}P^n \text{ and } \mathbb{P}^n \text{ are homeomorphic}$$

Proof. Define $F : \mathbb{P}^n \to \mathbb{R}P^n$ by

$$\{\mathbf{x}, -\mathbf{x}\} \mapsto \{\lambda \mathbf{x} : \lambda \in \mathbb{R}^*\}$$

It is straightforward to check that F is well-defined and bijective. Define $f : S^n \to \mathbb{R}P^n$ by

$$f = \pi \circ \mathbf{id}$$

where $\mathbf{id} : S^n \to \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ and $\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{R}P^n$ are continuous. Check that

$$f = F \circ p$$

where $p : S^n \to \mathbb{P}^n$ is the quotient mapping. It now follows from the universal property that F is continuous, and since $\mathbb{P}^n$ is compact and $\mathbb{R}P^n$ is Hausdorff, it also follows that F is a homeomorphism between $\mathbb{R}P^n$ and $\mathbb{P}^n$. ■

Knowing that $F : \mathbb{P}^n \to \mathbb{R}P^n$ is a homeomorphism and $\mathbb{R}P^n$ is a smooth manifold, we see that $\mathbb{P}^n$ is Hausdorff and second-countable, and if we define the atlas

$$\{(F^{-1}(U_i), \Phi_i \circ F) : 1 \leq i \leq n+1\}$$

We see this atlas is indeed smooth, since

$$(\Phi_i \circ F) \circ (\Phi_j \circ F)^{-1} = \Phi_i \circ \Phi_j^{-1}$$

4.2 Appendix

Theorem 4.2.1. (Homeomorphism between Compact Space and Hausdorff Space) Suppose

(a) X is compact.

(b) Y is Hausdorff.

(c) $f : X \to Y$ is a continuous bijective function.

Then

$$f \text{ is a homeomorphism between } X \text{ and } Y$$

Proof. Because closed subset of compact set is compact and continuous function send compact set to compact set, we see for each closed $E \subseteq X$, $f(E) \subseteq Y$ is compact. The result then follows from $f(E) \subseteq Y$ being closed since Y is Hausdorff. ■

Theorem 4.2.2. (Hausdorff and Quotient) If $\pi : X \to Y$ is an open mapping, and we define

$$R_\pi \triangleq \{(x, y) \in X^2 : \pi(x) = \pi(y)\}$$

Then

$$R_\pi \text{ is closed} \iff Y \text{ is Hausdorff}$$

Proof. Suppose R_π is closed. Fix some x, y such that $\pi(x) \neq \pi(y)$. Because R_π is closed, we know there exists open neighborhood U_x, U_y such that $U_x \times U_y \subseteq (R_\pi)^c$. It is clear that $\pi(U_x), \pi(U_y)$ are respectively open neighborhood of $\pi(x)$ and $\pi(y)$. To see $\pi(U_x)$ and $\pi(U_y)$ are disjoint, assume that $\pi(a) \in \pi(U_x) \cap \pi(U_y)$. Let $a_x \in U_x$ and $a_y \in U_y$ satisfy $\pi(a_x) = \pi(a) = \pi(a_y)$, which is impossible because $(a_x, a_y) \in (R_\pi)^c$. CaC

Suppose Y is Hausdorff. Fix some x, y such that $\pi(x) \neq \pi(y)$. Let U_x, U_y be open neighborhoods of $\pi(x), \pi(y)$ separating them. Observe that $(x, y) \in \pi^{-1}(U_x) \times \pi^{-1}(U_y) \subseteq (R_\pi)^c$ ■

4.3 Example: $S^1, \mathbb{R} \setminus \mathbb{Z}$ diffeomorphism

Equip $S^1 \triangleq \{(x,y) \in \mathbb{R}^2 : |(x,y)| = 1\}$ with the standard four projection chart smooth atlas as one of them being

$$V \triangleq \{(x,y) \in \mathbb{R}^2 : y > 0\} \text{ and } \phi_V : V \to \mathbb{R}; (x,y) \mapsto x$$

Let $p : \mathbb{R} \to \mathbb{R} \setminus \mathbb{Z}$ be the quotient map and let

$$U_0 \triangleq p\Big((0,1)\Big) \text{ and } U_1 \triangleq p\Big((\frac{-1}{2}, \frac{1}{2})\Big)$$

which are both open as one can readily check. Define $\phi_0 : U_0 \to (0,1)$ by

$$\phi_0(p(t)) \triangleq t_0 \text{ where } t_0 \in (0,1) \text{ and } p(t_0) = p(t)$$

and $\phi_1 : U_1 \to (-\frac{1}{2}, \frac{1}{2})$ by

$$\phi_1(p(t)) \triangleq t_0 \text{ where } t_0 \in (-\frac{1}{2}, \frac{1}{2}) \text{ and } p(t_0) = p(t)$$

Clearly, the function $G : \mathbb{R} \setminus \mathbb{Z} \to S^1$ well-defined by $G(p(x)) \triangleq (\cos 2\pi x, \sin 2\pi x)$ is a homeomorphism, as one can check that

(a) G is a continuous bijection. (Using Universal property of quotient map)

(b) $\mathbb{R} \setminus \mathbb{Z}$ is compact. (by finite sub-cover definition)

(c) S^1 is Hausdorff.

We now compute that $\phi_V \circ G \circ \phi_0^{-1}$ is defined on whole $(0,1)$, and is exactly

$$\phi_V \circ G \circ \phi_0^{-1}(t) \triangleq \cos 2\pi t \text{ smooth}$$