Recall $\sum a_n$ A.C. $\Rightarrow \sum a_{p(n)}$ conv. $\left(=\sum a_n\right)$.

permutation

$\sum a_n = \sum (-1)^{n-1} b_n$ C.C. $\quad \sum b_{2n} = \infty$ ✓ $\quad \sum b_{2n-1} = \infty$ ✓

⟨Ex⟩ $\sum (-1)^{n-1}\frac{1}{n}$ conv. $(= \ln 2)$

$\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \cdots\}$

$\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \cdots\}$

$A_1 - \frac{1}{2} < \pi < 1 + \frac{1}{3} + \cdots + \frac{1}{151} \equiv A_1$

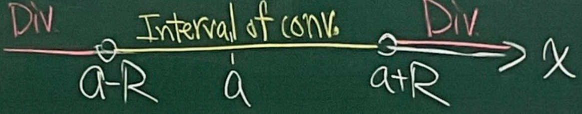
$A_2 - \frac{1}{4} < \pi < A_1 - \frac{1}{2} + \frac{1}{153} + \cdots + \frac{1}{2n_1 - 1} \equiv A_2$

$A_3 - \frac{1}{6} < \pi < A_2 - \frac{1}{4} + \frac{1}{2n_1+1} + \cdots + \frac{1}{2n_2 - 1} \equiv A_3$

$\vdots$

$A_n \to \pi$

R = radius of convergence



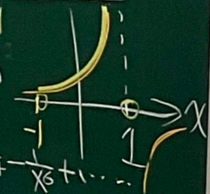
power series $\sum c_n (x-a)^n$

⟨Ex⟩ series	radius of conv.	Interval of conv.
$\sum x^n$	$R=1$	$(-1, 1)$
$\sum n!\,x^n$	$R=0$,	$\{0\}$
$\sum \frac{(x-3)^n}{n}$	$R=1$,	$[2, 4)$
$\sum \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$	$R=\infty$	$(-\infty, \infty)$

§11.9 Representation of fns as a Power Series

For $|x|<1$ $\quad \underbrace{1 + x + x^2 + \cdots}_{<\infty \text{ only for } |x|<1} = \frac{1}{1-x}$ $\quad x \neq 1$

$\sum x^n = \lim S_n$, $S_n = 1 + x + \cdots + x^n$



⟨Ex⟩ $\frac{1}{1+x^2} = \sum (-x^2)^n = 1 - \frac{1}{x^2} + \frac{1}{x^4} - \frac{1}{x^6} + \cdots$

$\forall x$ $\quad |x^2| < 1$

Differentiation and Integration of Power Series (term-by-term).

Thm: If $\sum c_n(x-a)^n$ has R.C., $R>0$.

$\Rightarrow f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum c_n(x-a)^n$

is diff. on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$

$\left(\sum c_n(x-a)^n\right)' =$ term-by-term differentiation

(ii) $\int f(x)\,dx = C + c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \frac{c_2}{3}(x-a)^3 + \cdots = C + \sum_{n=0} \frac{c_n(x-a)^{n+1}}{n+1}$ term-by-term integration

R.C. of (i) and (ii) are both R.

Note $\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx} c_n(x-a)^n$

$\int \left[\sum c_n(x-a)^n\right] = \sum \int c_n(x-a)^n\,dx$

⟨EX⟩ Find a power series of $\frac{1}{(1-x)^2}$.

Sol: $\left(\frac{1}{1-x}\right)^2 = (1+x+x^2+\cdots)^2 \quad |x|<1$

(A)

	1	x	x^2
1	1	x	x^2
x	x	x^2	x^3
x^2	x^2	x^3	x^4

$= 1 + 2x + 3x^2 + 4x^3 + \cdots$

(B) $\frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}(1+x+x^2+\cdots)$

$\frac{(-1)(-1)}{(x-1)^2} = 1 + 2x + 3x^2 + \cdots$ ※

⟨EX⟩ Find a power series of $\ln(1+x)$.

Sol: $\int \frac{1}{1+x}dx = \int 1 - x + x^2 - x^3 + \cdots\,dx$

$\ln|x+1| = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots + C$

$x=0 \quad \ln|0+1| = 0 = C$

$\therefore \ln(x+1) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}$

$R=1$ ※

⟨EX⟩ (a) find a power series for $f(x)=\tan^{-1}x$.

Sol: $\tan^{-1}x = \int \frac{1}{1+x^2}\,dx = \int (1-x^2+x^4-x^6+\cdots)\,dx$

$= x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\cdots+\underset{(x=0):\ 0}{C}$

(b) Approximate $\int_0^{\frac{1}{2}} \frac{1}{1+x^2}\,dx$ correct to within 10^{-7}

$S \equiv \int_0^{\frac{1}{2}} \frac{1}{1+x^2}\,dx = \left(x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\cdots\right)\Big|_{x=0}^{x=\frac{1}{2}}$

$= \frac{1}{2}-\frac{1}{3}\left(\frac{1}{2}\right)^3+\frac{1}{5}\left(\frac{1}{2}\right)^5-\frac{1}{7}\left(\frac{1}{2}\right)^7+\cdots$

1 →	1
2 →	3
3 →	5
4 →	7

By Alternating series test.

$|S-S_n| < b_{n+1} = \frac{1}{2n+1}\left(\frac{1}{2}\right)^{2n+1} < 10^{-7}$

$\frac{1}{(2n+1)}2^{-(2n+1)} < 10^{-7}$

Use math software.

Find the smallest n

$2n+1=21 \Leftrightarrow n=10$ ※

$\boxed{\text{Midterm I Mar. 08. 12:10–15:00}}$

§11.10 Taylor and Maclaurin Series

Suppose $f(x)$ can be represented by a power series

$f(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \cdots \quad |x-a|<R$

$= \sum_{n=0}^{\infty} C_n(x-a)^n$ Find C_n

$\Rightarrow f(a) = C_0$

$(f(x))' = \left(\sum_{n=0}^{\infty} C_n(x-a)^n\right)'$

$\Rightarrow f'(x)\big|_{x=a} = \sum_{n=1}^{\infty} nC_n(x-a)^{n-1}\Big|_{x=a}$

$\Rightarrow f'(a) = C_1$

$f''(a) = f''(x)\big|_{x=a} = \left(\sum C_n(x-a)^n\right)''\Big|_{x=a} = 2C_2$

$\Rightarrow f'''(a) = 3!C_3 \cdots f^{(n)}(a) = n!C_n \Rightarrow C_n = \frac{f^{(n)}(a)}{n!}$

Thm. If f has a power series expansion at a

i.e. $f(x)=\sum_{n=0}^{\infty} C_n (x-a)^n$, $|x-a|<R$

$\Rightarrow$ Coefficients $C_n = \frac{f^{(n)}(a)}{n!}$, Thus

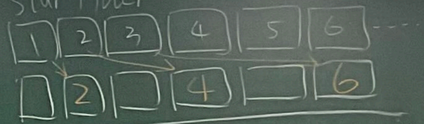
$$\boxed{f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots}$$

Taylor Series

$a=0$, $\Rightarrow$ $\boxed{f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n}$ Maclaurin Series.

<EX> $1 = 0.9+0.09+0.009+\cdots$

Star Hotel



<EX> Find Maclarin series of $e^x = f(x)$

$\Rightarrow f^{(n)}(x)\Big|_{x=0} = e^x\Big|_{x=0} = 1$ $\quad 1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\cdots$

$f^{(n)}(0)=$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{1}{n!}x^n =$$

Find R.C. R by ratio test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right| = \left|\frac{1}{n+1}\right| \xrightarrow[n\to\infty]{} 0 = L < 1$$

$\therefore e^x = \sum_{n=0}^{\infty}\frac{1}{n!}x^n$ A.C. $\forall x$, $R=\infty$

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k+\sum_{k=n+1}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

$\divideontimes$ true value $= \underbrace{T_n}_{\text{approximation}} + \underbrace{R_n}_{\text{error}}$

<EX> $T_1(x)=1+x$

$T_2(x)=1+x+\frac{1}{2!}x^2$

$T_3(x)=1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3$

: Graph

$T_1, T_2, T_3, \dots\, e^x$

$$\lim_{n\to\infty} R_n = 0 \Leftrightarrow \lim_{k\to\infty} T_n(x) = f(x)$$

Thm: If $f(x) = T_n(x) + R_n(x)$, where $T_n(x) = n$-th degree polynomial of f at a and $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a|<R$

$\Rightarrow \lim_{n\to\infty} T_n(x) = f(x)$ on $|x-a|<R$.

$=$ sum of its Taylor series

Taylor's Inequality : (T.I.)

If $|f^{(n+1)}(x)| \le M$ (*) for $|x-a|<R \Rightarrow$

The remainder of the Taylor Series satisfies

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \text{ for } |x-a|<R \quad \forall n$$

$n=1$, $a<x<a+R$,

$$R_1(x) = f(x) - T_1(x) = f(x) - f(a) - \frac{f'(a)}{1!}(x-a).$$

FTC $\int_a^x f'(t)\,dt - \int_a^x f'(a)\,dt = \int_a^x [f'(t) - f'(a)]\,dt$

MVT $\int_a^x \underset{\max}{f''(c)}(t-a)\,dt \le M_1 \int_a^x (t-a)\,dt = \frac{1}{2}M_1(x-a)^2$

$$|R_2(x)| \le M \int_a^x \int_a^t (s-a)\,ds\,dt = \frac{M}{2\cdot 3}(x-a)^3.$$

For $n=2$, $|R_2(x)| = |f(x) - T_2(x)| =$

$|f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2!}(x-a)^2|$

$\underset{\text{MVT}}{\overset{\text{FTC}}{\le}} \int_a^x \int_a^t [\underbrace{f'''(c)}_{(*) \le M}(s-a)]\,ds\,dt$

By induction $|R_n(x)| \le \frac{M}{(n+1)!}(x-a)^{n+1} \le M\frac{R^{n+1}}{(n+1)!}$

Note $\lim_{n\to\infty} |R_n(x)| \le \lim_{n\to\infty} M\frac{R^{n+1}}{(n+1)!} = 0 \Leftrightarrow \lim_{n\to\infty} T_n(x) = f(x)$