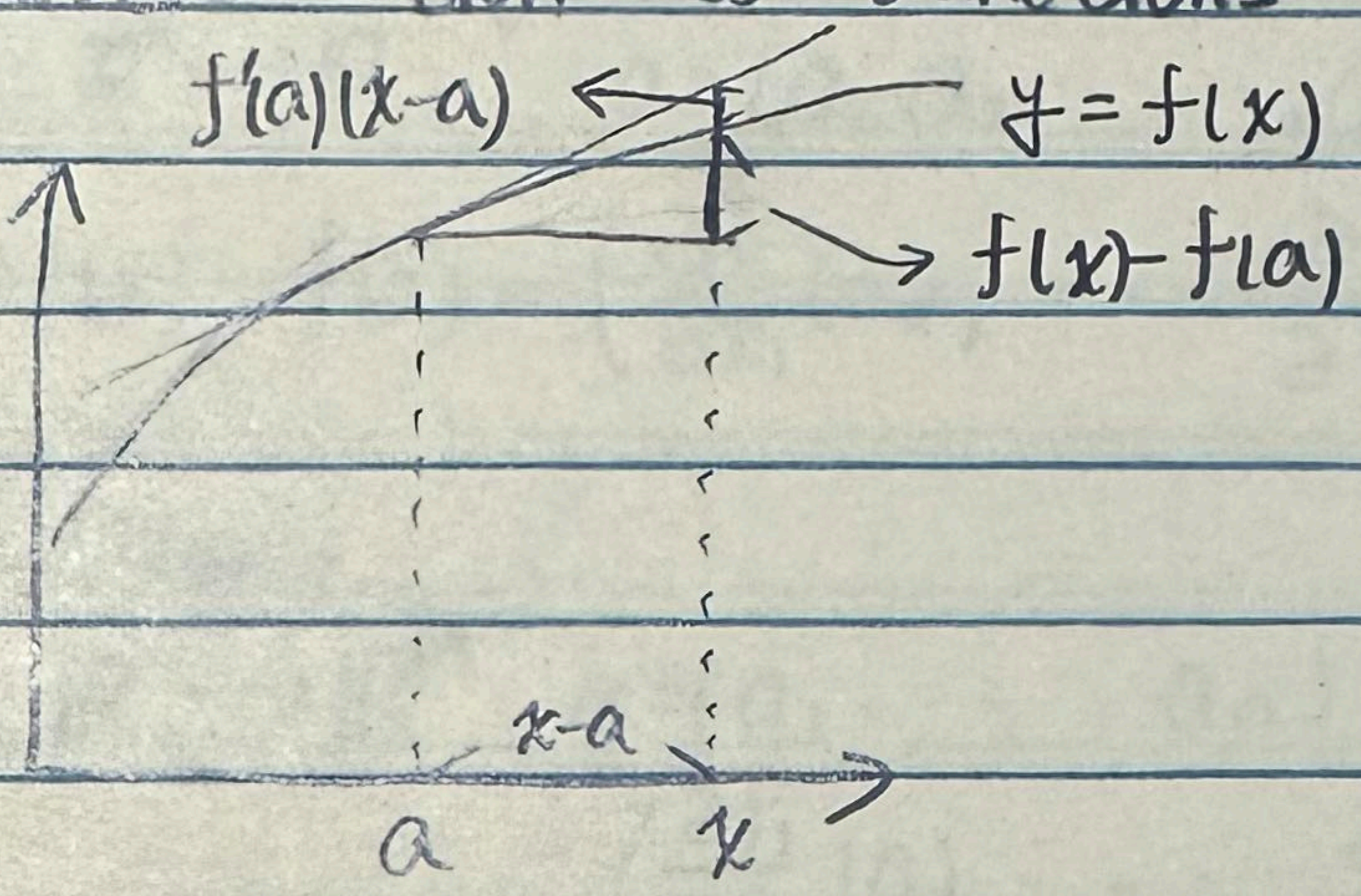
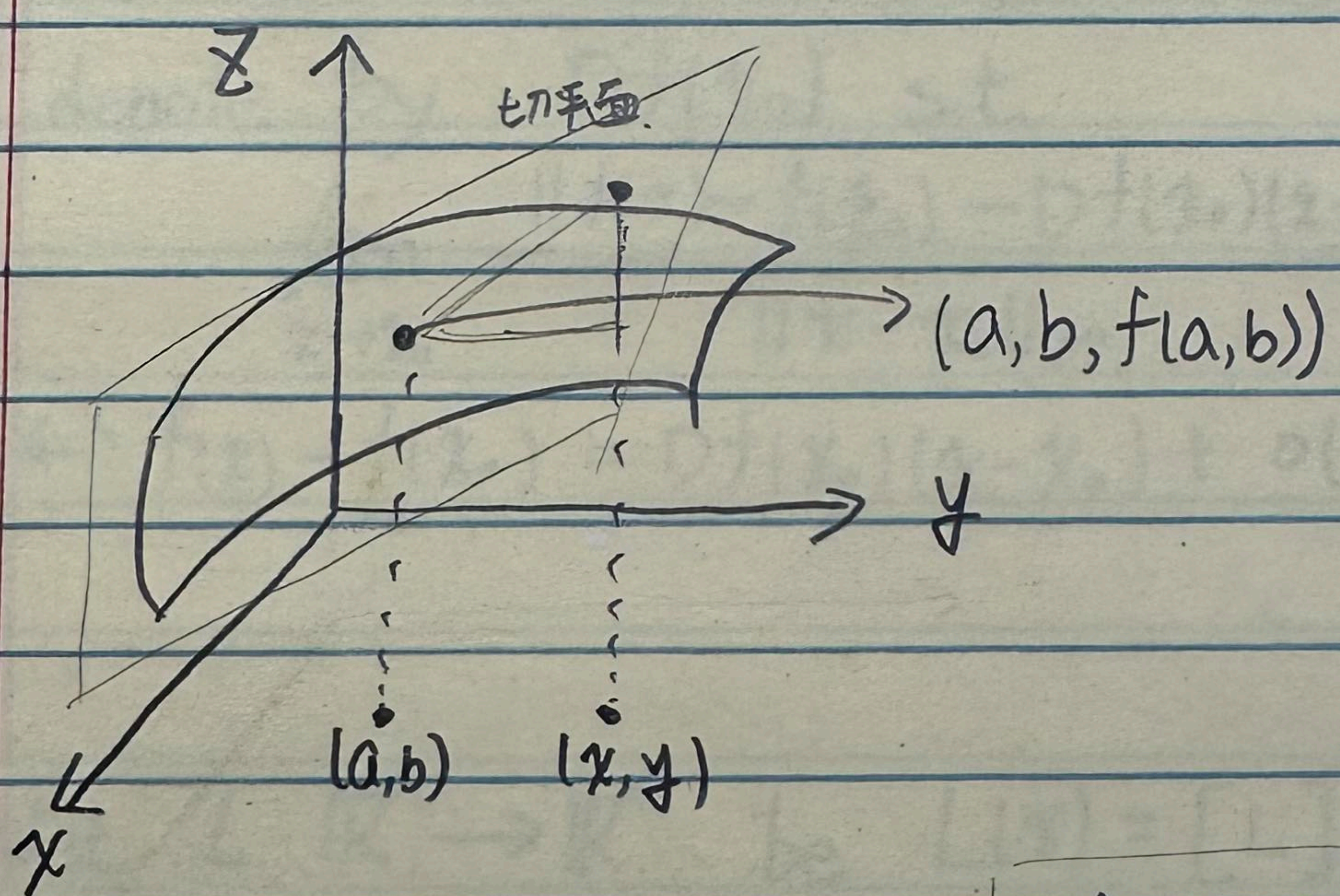
differentiation of functions of several variables.




$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=f'(a)$$

$$\Leftrightarrow \lim_{x\to a}\frac{f(x)-f(a)-f'(a)(x-a)}{x-a}=0$$

$$\Leftrightarrow f(x)-f(a)-f'(a)(x-a)=o(|x-a|)$$




各方向的高加起来.

$$\lim_{(x,y)\to(a,b)}\frac{f(x,y)-f(a,b)-\left[\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)\begin{pmatrix}x-a\\ y-b\end{pmatrix}\right]}{\|(x,y)-(a,b)\|}$$

$$\Leftrightarrow f(x,y)-f(a,b)-\boxed{\left(\frac{\partial f}{\partial x}(a,b)(x-a)+\frac{\partial f}{\partial y}(a,b)(y-b)\right)}$$

$$=o(\|(x,y)-(a,b)\|)$$

Thm.

let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be open sets.
$f: U \to \mathbb{R}^m$, $g: V \to \mathbb{R}^\ell$ and $f(U) \subseteq V$
If $f$ is diff at $x_0 \in U$ and
$g$ is diff at $f(x_0)$
then $F(x) = g \circ f(x)$ is diff at $x_0$ and
$(DF)(x_0) = (Dg)(f(x_0))(Df)(x_0)$
※ $(DF)_{\ell \times n} = (Dg)_{\ell \times m} (Df)_{m \times n}$

eg.

Consider the polar coordinate $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

then $g: \mathbb{R}^2 \to \mathbb{R}$ is associated with a function
$F: [0,\infty) \times [0, 2\pi) \to \mathbb{R}$ s.t
$F(r,\theta) = g(r\cos\theta, r\sin\theta) = g \circ f(r,\theta)$
$\Rightarrow (DF) = (Dg)(Df)$
$\Rightarrow \left(\frac{\partial F}{\partial r}, \frac{\partial F}{\partial \theta}\right) = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}$

$\Rightarrow \begin{cases} \frac{\partial F}{\partial r} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial r} \\ \frac{\partial F}{\partial \theta} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial \theta} \end{cases}$

eg.

let $\gamma: (0,1) \to \mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}$ be differentiable
let $F(t) = f(\gamma(t))$
then $F'(t) = (Df)(D\gamma)$
$= (Df)_{1\times n} \cdot \gamma'(t)_{n\times 1}$

eg.

$$\int_0^\infty e^{-x^2}\,dx$$

let $I=\int_0^\infty e^{-x^2}\,dx$

$$I^2=\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

$$=\int_0^\infty\int_0^{\frac{\pi}{2}} e^{-r^2}\,r\,d\theta\,dr$$

$$=\frac{\pi}{2^2}\int_0^\infty 2re^{-r^2}\,dr$$

$$=\frac{\pi}{4} \qquad \Rightarrow I=\frac{\sqrt{\pi}}{2}$$

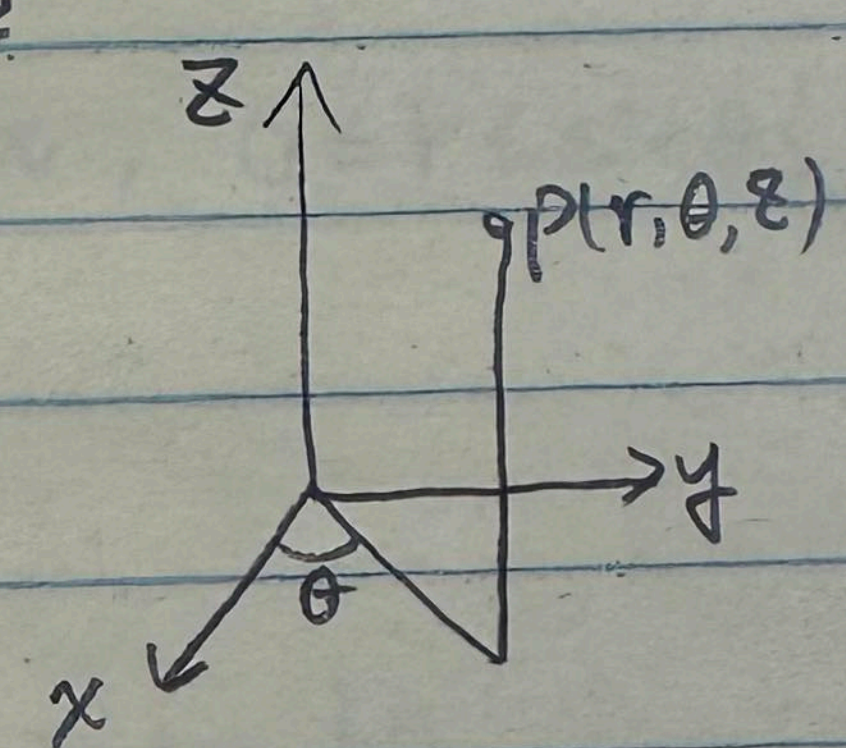
def. Cylinderical coordinate

$P(x,y,z)=P(r,\theta,z)$

$x=r\cos\theta$

$y=r\sin\theta$

$z=z$




$$\Rightarrow \iiint_T f(x,y,z)\,dx\,dy\,dz$$

$$=\iiint_{S_{r,\theta,z}} \tilde{f}(r,\theta,z)\cdot r\,dr\,d\theta\,dz$$

eg.

$A=\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 1,\ \sqrt{x}\le y\le 1\}$

$$\int_A e^{y^3}\,dA=\int_0^1\int_{\sqrt{x}}^1 e^{y^3}\,dy\,dx$$

$$=\int_0^1\int_0^{y^2} e^{y^3}\,dx\,dy$$

$$=\int_0^1 y^2 e^{y^3}\,dy=\frac{e-1}{3}$$

eg.

$A=\{(x_1,x_2,x_3) \mid x_1\ge 0,\ x_2\ge 0,\ x_3\ge 0,\ x_1+x_2+x_3\le 1\}$

$S=[0,1]\times[0,1]\times[0,1]$

$$\int_A (x_1+x_2+x_3)^2\,dx_1\,dx_2\,dx_3$$

$$=\int_0^1\int_0^{1-x_3}\int_0^{1-x_2-x_3}(x_1+x_2+x_3)^2\,dx_1\,dx_2\,dx_3$$

$= \cdots\cdots$

eg.

find the volume $\omega_n$ of the n-dimensional unit ball.

$$\omega_n=\int_{-1}^{1}\int_{-\sqrt{1-x_n^2}}^{\sqrt{1-x_n^2}}\int_{-\sqrt{1-x_n^2-x_{n-1}^2}}^{\sqrt{1-x_n^2-x_{n-1}^2}}\cdots\int_{-\sqrt{1-x_n^2-\cdots-x_2^2}}^{\sqrt{1-x_n^2-\cdots-x_2^2}}dx_1\cdots dx_n$$

= 不会考，之後再補.

eg. (bartle 有)

let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$$Jf(0,0) = \left( \frac{\partial f(0,0)}{\partial x}, \frac{\partial f(0,0)}{\partial y} \right) = (0,0) \to \text{exists}$$

but lim not exists $((x,y) \to 0$ from $(t,-t), (t,t))$

eg.

let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$f(x,y) = \begin{cases} x & \text{if } y=1 \\ y & \text{if } x=0 \\ 1 & \text{otherwise} \end{cases}$$

$$Jf(0,0) = \left( \frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0) \right) = (1,1) \longrightarrow \text{exists}$$

$$f(x,y) - f(0,0) - Df(0,0) \begin{pmatrix} x-0 \\ y-0 \end{pmatrix}$$

$= 1 - x - y$ if $xy \neq 0$

$\longrightarrow 1$ as $(x,y) \to (0,0)$

Thm.

let $U \subseteq \mathbb{R}^n$ be open, $a \in U$ and $f: U \to \mathbb{R}^m$ if the Jacobian matrix of $f$ exists in a nbhd of $a$ and continuous at $a$

$\Rightarrow f$ is differentiable at $a$

eg.

let $\begin{cases} f(u,v,w) = u^2v + wv^2 \\ g(x,y) = (xy, \sin x, e^x) \end{cases}$

let $h = f\circ g$ $\quad \mathbb{R}^2 \to \mathbb{R}$

find $\frac{\partial h}{\partial x}$

$$\Rightarrow \frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}$$

$$= 2uvy + (u^2 + 2wv)\cos x + v^2 e^x$$

Thm 均值定理

let $f: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ with $f = (f_1, \cdots f_m)$

Suppose that $f$ is diff on $U$ and $x, y \in U$, then $\exists\, c_1, \cdots c_m$ on the segment joining $x$ and $y$ s.t

$f_i(y) - f_i(x) = (Df_i)(c_i)(y - x)$ $\quad (i = 1, 2, \cdots, m)$

eg.

let $f: \mathbb{R} \to \mathbb{R}^2$ be given by $f(x) = (\cos x, \sin x)$

is there exists $c$ s.t $f(2\pi) - f(0) = f'(c)(2\pi - 0)$

$$\Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\pi \begin{pmatrix} -\sin c \\ \cos c \end{pmatrix} \quad \nexists\, c$$

eg.

$f: U \subseteq \mathbb{R}^n \xrightarrow{\text{diff}} \mathbb{R}^m$

if $Df(x) = 0$ $\forall x \in U$, then $f$ is a constant vector.

eg.

let $f(x,y) = x^2 + 2y$

$\Rightarrow Df(a,b) = (2a, 2)$

we can verify:

$f(x,y) - f(a,b) - [2a(x-a) + 2(y-b)]$

$= (x^2+2y) - (a^2+2b) - [2a(x-a) + 2(y-b)]$

$= (x-a)^2$

$$\lim_{(x,y)\to(a,b)} \frac{(x-a)^2}{\|(x,y)-(a,b)\|} = \lim_{(x,y)\to(a,b)} \frac{(x-a)^2}{\sqrt{(x-a)^2+(y-b)^2}} = 0$$

$\Rightarrow (x-a)^2 = o(\|(x,y)-(a,b)\|)$

def.

let $u$ be an openset in $\mathbb{R}^n$ and $f: U \to \mathbb{R}^m$, then

$$(Jf)(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}, & \frac{\partial f_1}{\partial x_2}, & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial f_m}{\partial x_1}, & \frac{\partial f_m}{\partial x_2}, & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

is call the Jacabian matrix of $f$ at $x$

note:

A function $f$ might not be differentiable even if the Jacabian matrix $(Jf)$ exists.

But, if $f$ is differentiable at $x_0$, then $Df(x_0) = (Jf)(x_0)$

eg.

let $f(x_1, x_2) = (x_1^2, x_1^2 x_2, x_1^4 x_2^2)$

then $(Df(x)) = (Jf)(x) = \begin{pmatrix} 2x_1, & 0 \\ 2x_1x_2, & x_1^2 \\ 4x_1^3x_2^2, & 2x_1^4x_2 \end{pmatrix}$

Thm (inverse function thm).

let $D \subseteq \mathbb{R}^n$ be open, $x_0 \in D$

$f: D \to \mathbb{R}^n$ be of class $C'$ and $\boxed{Df(x_0)}$ be invertible $\to$ Jacobian $J_f(x_0) \neq 0$

then $\exists$ an open nbhd $u$ of $x_0$ and an open nbhd $V$ of $f(x_0)$ s.t.

(I) $f: u \to V$ is 1-1 and onto (local open mapping at $x_0$)

(II) the inverse function $f^{-1}: V \to u$ is of class $C'$

(III) if $x = f^{-1}(y)$, then $(Df)^{-1}(y) = [(Df)(x)]^{-1}$

pf.

in 高微，略

eg.

let $f: \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x + 2x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$\Rightarrow f'(x) = 1 - 2\cos\frac{1}{x} + 4x\sin\frac{1}{x} \qquad x \neq 0$$

$$f'(0) = \frac{f(h) - f(0)}{h} = 1$$

$\Rightarrow$ 找不到反函數，因 $f'(x)$ 在 0 附近不連續

def

let $f$ be real-valued and defined on a nbhd of $x_0 \in \mathbb{R}^n$ and let $V \in \mathbb{R}^n$ with $^{*}\boxed{\|V\| = 1}$ then

$$(D_V f)(x_0) = \frac{d}{dt}\bigg|_{t=t_0} f(x_0 + tV) = \lim_{t \to 0} \frac{f(x_0 + tV) - f(x_0)}{t}$$

**prop.**

if $(\nabla f)(x_0) \neq 0$, the vector $\dfrac{\nabla f(x_0)}{\|\nabla f(x_0)\|_{\mathbb{R}^n}}$ is the unit normal to the level set $\{x \in U \mid f(x) = f(x_0)\}$ at $x_0$

**eg.**

$S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 3\}$, find the normal to $S$ at $(1,1,1)$

$$\Rightarrow \frac{\nabla f(1,1,1)}{\|\nabla f(1,1,1)\|} = \frac{(2,2,2)}{\sqrt{2^2+2^2+2^2}}$$

**eg.**

$S = \{(x, y, z) \mid x^2 - y^2 + xyz = 1\}$, find the tangent plane of $S$ at $(1,0,1)$

$\Rightarrow N_{normal} = \nabla f(1,0,1) = (2,1,0)$

$T_{angent} = 2(x-1) + 1(y-0) + 0(z-1) = 0.$

**thm.**

let $\varphi_1(x) \leq \varphi_2(x) \quad \forall x \in S$

$A = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} \mid x \in S,\ \varphi_1(x) \leq y \leq \varphi_2(x)\}$

let $f: A \rightarrow \mathbb{R}$ be cont, then

$$\int_A f(x,y)\, d(x,y) = \int_S \left( \int_{\varphi_1(y)}^{\varphi_2(y)} f(x,y)\, dy \right) dx$$

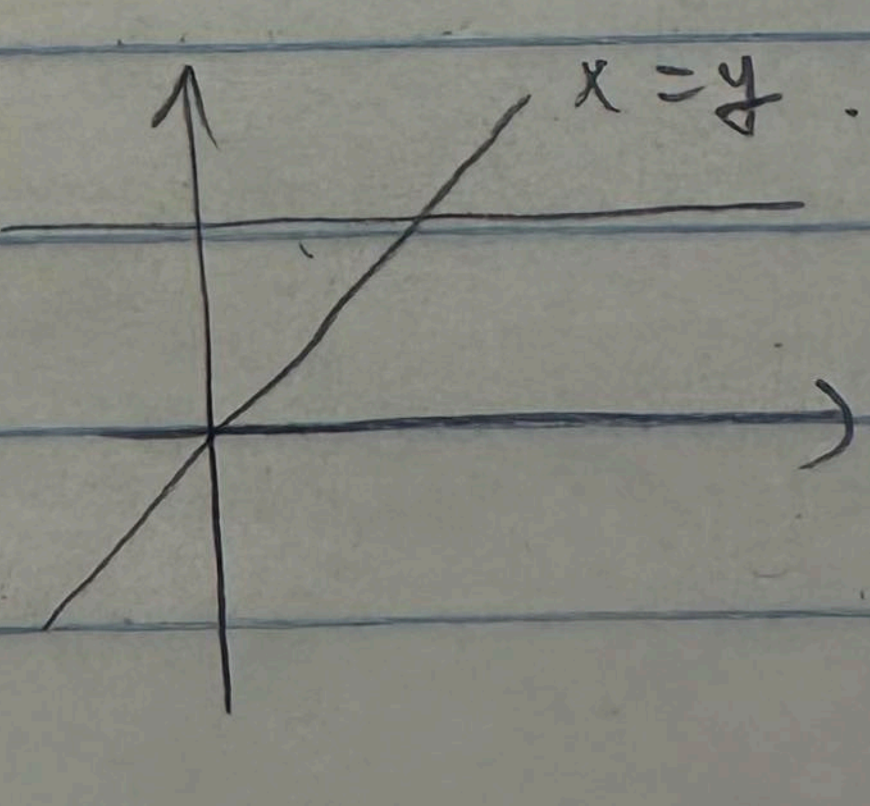
**eg.**

$A = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1,\ x \leq y \leq 1)$

$f(x,y) = xy$

$$\Rightarrow \int_A f(x,y)\, dA = \int_0^1 \int_x^1 xy\, dy\, dx$$

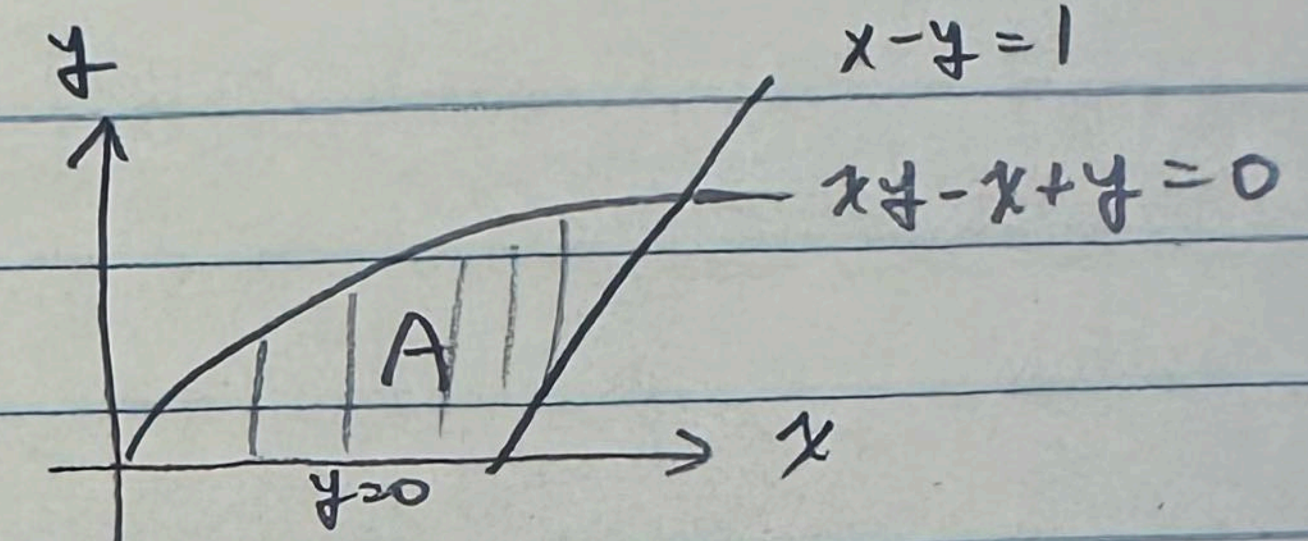
$$= \int_0^1 \int_0^y xy\, dx\, dy$$



eg.

$$f(x,y) = x^2y^2(x+y)e^{-(x-y)^2}$$

$$\int_A f(x,y)\, d(x,y)$$

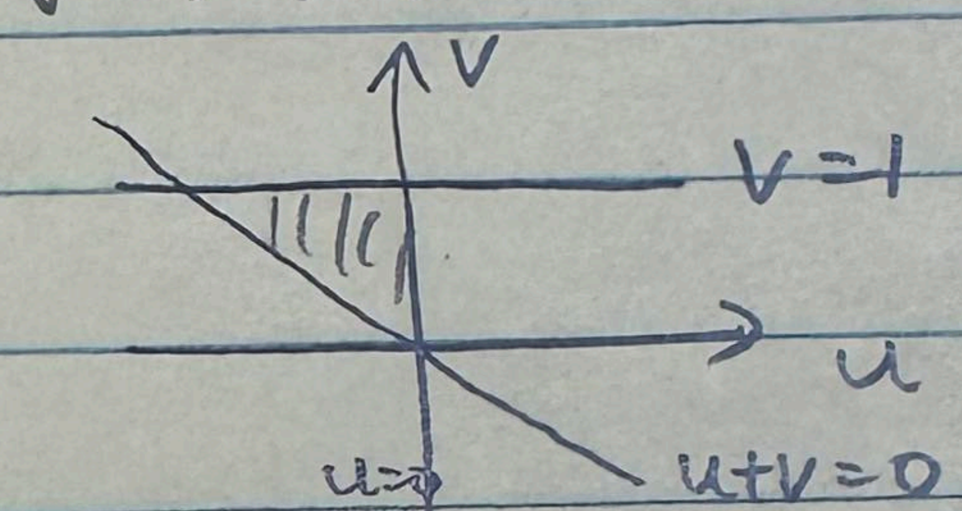



$$\Rightarrow \int_{\Omega_{u,v}} (u+v)^2 (x+y) e^{-v^2} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

let $u = xy - x + y$ , $v = x - y$

$u + v = xy$.

$y=0 \Rightarrow u+v=0$




其中 $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

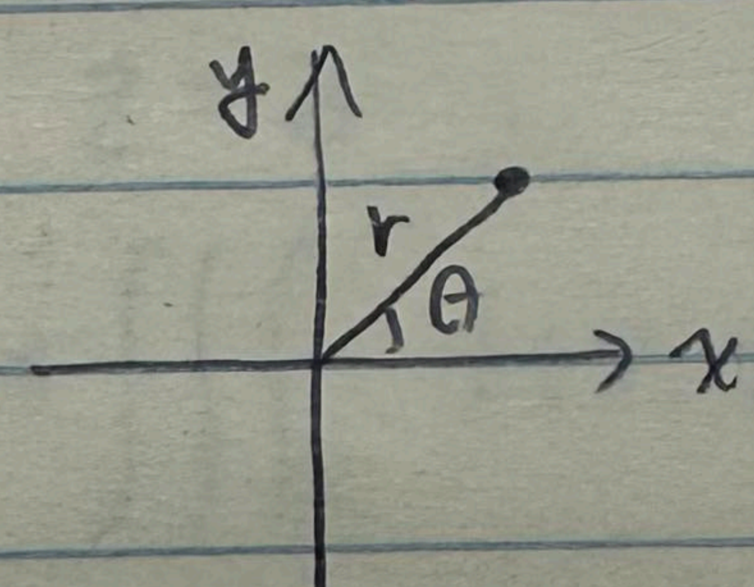
$$= \left| \frac{\partial(u,v)}{\partial(x,y)} \right|^{-1}$$

$$= \begin{vmatrix} y-1 & x+1 \\ 1 & -1 \end{vmatrix}^{-1}$$

$$= |x+y|^{-1}$$

$$= \int_0^1 \int_{-v}^0 (u+v)^2 e^{-v^2}\, du\, dv \quad = -\frac{1}{6}\left(\frac{2}{e} - 1\right)$$

def. Polar coordinate




$x = r\cos\theta$

$y = r\sin\theta$

$$\left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r$$

$$\int_A f(x,y)\, dx\, dy$$

$$= \int_{\Omega_{r,\theta}} \hat{f}(r,\theta)\, r\, dr\, d\theta$$

So we can get

$f: \mathbb{R}^n \to \mathbb{R}$, $a = (a_1, a_2, \cdots, a_n)$, $x = (x_1, x_2, \cdots, x_n)$

$$\Rightarrow f(x) - f(a) - \left(\frac{\partial f}{\partial x_1}(a), \cdots \frac{\partial f}{\partial x_n}(a)\right)\begin{pmatrix} x_1 - a_1 \\ \vdots \\ x_n - a_n \end{pmatrix} = o(\|x-a\|)$$

Similarly

$f: \mathbb{R}^n \to \mathbb{R}^m$ $a = (a_1, \cdots, a_n)$ $x = (x_1, \cdots, x_n)$

$$\Rightarrow f(x) - f(a) - \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix}\begin{pmatrix} x_1 - a_1 \\ \vdots \\ x_n - a_n \end{pmatrix} = o(\|x-a\|)$$

def.

let $u$ be an open set A function $u \to \mathbb{R}^m$ is differentiable. at $x_0 \in A$. if there is a linear transformation from $\mathbb{R}^n \to \mathbb{R}^m$ denote by $Df(x_0)$ s.t

$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - Df(x_0)(x - x_0)\|_{\mathbb{R}^m}}{\|x - x_0\|_{\mathbb{R}^n}} = 0$$

$$\Leftrightarrow f(x) - f(x_0) = Df(x_0)(x - x_0) + o(x - x_0) \quad \text{as } x \to x_0$$

eg.

let $L: \mathbb{R}^n \to \mathbb{R}^m$ be $L(x) = [L]_{m \times n}[x]_n$ $\forall x \in \mathbb{R}^n$

then $L$ is differentiable and $L(x) - L(x_0) - L(x - x_0) = 0$

$\Rightarrow Df(x_0) = [L]_{m \times n}$

So we can get

$f: \mathbb{R}^n \to \mathbb{R}$ , $a=(a_1, a_2, \cdots, a_n)$ , $x=(x_1, x_2, \cdots, x_n)$

$$\Rightarrow f(x) - f(a) - \left(\frac{\partial f}{\partial x_1}(a), \cdots \frac{\partial f}{\partial x_n}(a)\right)\begin{pmatrix} x_1-a_1 \\ \vdots \\ x_n-a_n \end{pmatrix} = o(\|x-a\|)$$

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def.

let $u$ be an open set A function $u \to \mathbb{R}^m$ is differentiable at $x_0 \in A$. if there is a linear transformation from $\mathbb{R}^n \to \mathbb{R}^m$ denote by $Df(x_0)$ s.t

$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - Df(x_0)(x-x_0)\|_{\mathbb{R}^m}}{\|x - x_0\|_{\mathbb{R}^n}} = 0$$

$\Leftrightarrow f(x) - f(x_0) = Df(x_0)(x-x_0) + o(x-x_0)$ as $x \to x_0$

eg.

let $L: \mathbb{R}^n \to \mathbb{R}^m$ be $L(x) = [L]_{m\times n}[x]_n$ $\forall x \in \mathbb{R}^n$

then $L$ is differentiable and $L(x) - L(x_0) - L(x-x_0) = 0$

$\Rightarrow Df(x_0) = [L]_{m\times n}$

Rmk.

let $\gamma:(-\delta,\delta)\longrightarrow\mathbb{R}^n$ satify $\gamma(0)=x_0$ and $\gamma'(0)=V$

then $(f\circ\gamma)'(0)=[Df(x_0)]_{1\times n}\,\gamma'(0)_{n\times 1}$

$=Df(x_0)V$

$$\begin{tikzcd} (-\delta,\delta) \arrow[r, "\gamma"] \arrow[rr, bend right, "f\circ\gamma"'] & \mathbb{R}^n \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

thm.

let $u$ be open and $f:u\longrightarrow\mathbb{R}^n$ be diff at $x_0$, then

$D_v f(x_0)=Df(x_0)(V)$

pf.

$f(x)-f(x_0)-Df(x_0)(x-x_0)=o(\|x-x_0\|)$

let $x=x_0+tV$

$$\Rightarrow \frac{f(x_0+tV)-f(x_0)}{t}-\frac{Df(x_0)(tV)}{t}=o\left(\frac{\|x-x_0\|}{t}\right)$$

$\Rightarrow$ let $t\to 0$ $\Rightarrow$ $D_v f(x_0)=Df(x_0)(V)$

eg.

$$f(x,y)=\begin{cases}\dfrac{x^3}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

$$\Rightarrow D_v f(0)=\lim_{t\to 0}\frac{f(tV_1,tV_2)-f(0,0)}{t}$$

$$=\lim_{t\to 0}\frac{V_1^3}{V_1^2+V_2^2}=V_1^3\neq \underbrace{Df(0)(V)}_{=0}$$

def.

let $f:U\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}$, the derivative of $f$ is called the gradient of $f$ denote by $\nabla f=\left(\frac{\partial f}{\partial x_1},\cdots\cdots\frac{\partial f}{\partial x_n}\right)$

eg.

$$\iiint_T (x^2+y^2)\,dx\,dy\,dz$$

$T:\ -2 \le x \le 2$

$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$

$0 \le z \le 4-x^2-y^2$

$$= \int_0^2 \int_0^{2\pi} \int_0^{4-r^2} r^2 \cdot r\,dz\,d\theta\,dr$$

$$= 2\pi \int_0^2 (4-r^2) r^3\,dr$$

$$= 2\pi\left(16 - \frac{64}{6}\right) = \frac{32}{3}\pi$$

eg.

find the volume of the solid T bounded below by $z=y$ and above $z=x^2+y^2$

$\Rightarrow x^2+y^2=y \Rightarrow r^2 = r\sin\theta$

$\Rightarrow r = \sin\theta$

$\Omega_{r,\theta} = \{(r,\theta) \mid 0 \le \theta \le \pi,\ 0 \le r \le \sin\theta\}$

$$\Rightarrow \iiint_T 1\,dx\,dy\,dz$$

$$= \int_0^{\pi} \int_0^{\sin\theta} \int_{r^2}^{r\sin\theta} r\,dz\,dr\,d\theta$$

$$= \frac{\pi}{32}$$

Thm.

$$\int_{U_y} f(y)\,dy = \int f(y(x)) \left| \frac{\partial(y_1, \cdots, y_n)}{\partial(x_1, \cdots, x_n)} \right| dx$$

eg.

$$\int_0^1 (1-x) f(x)\,dx = 5 \quad , \quad \text{find } \int_0^1 \int_0^x f(x-y)\,dy\,dx$$

$$\int_0^1 \int_0^x f(x-y)\,dy\,dx$$

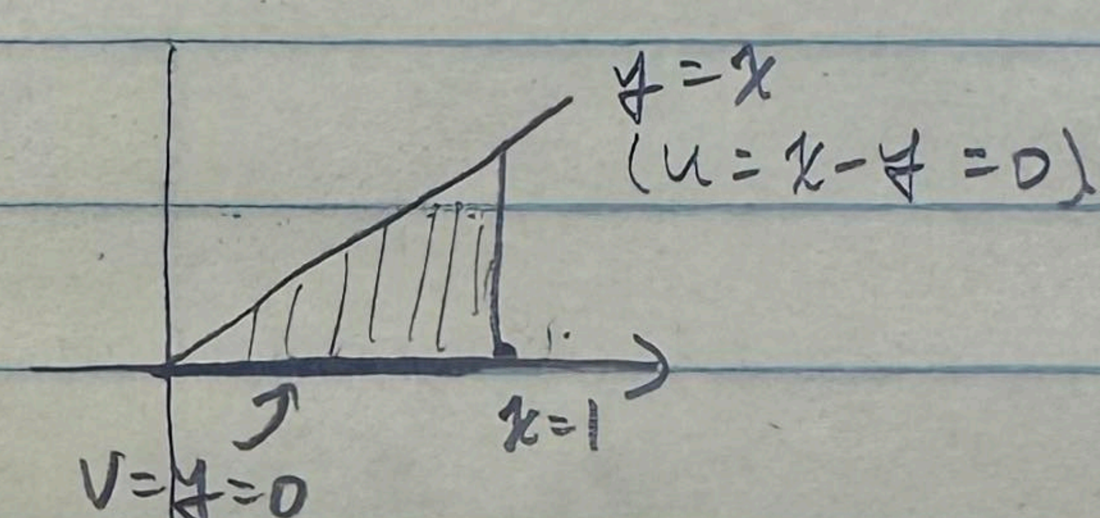
let $u = x-y \quad v = y$

$x = u+v \quad y = v$

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 1$$

$$= \iint f(u) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$= \iint f(u)\,du\,dv$$

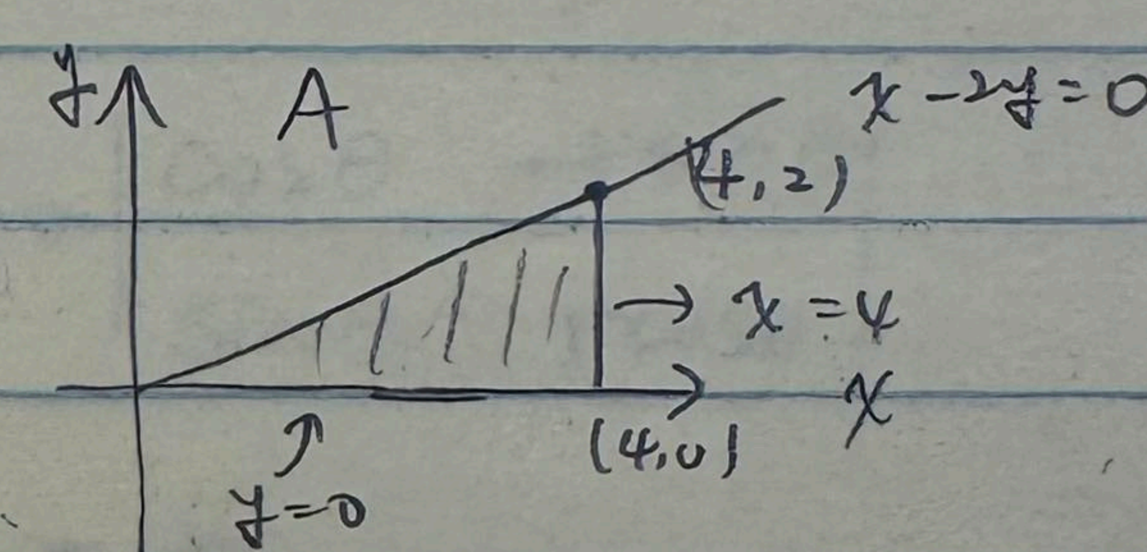



$$= \int_0^1 f(u)(1-u)\,du$$

$$= 5.$$

eg.

$f(x,y) = y\sqrt{x-2y}$ find $\int_A f(x,y)\,dx\,dy$



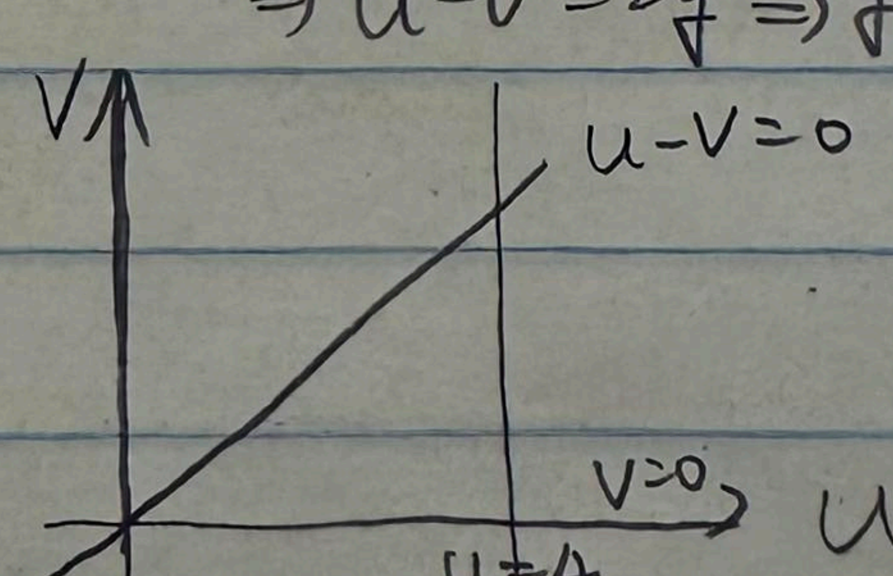

$$\int_A f(x,y)\,dx\,dy =$$

$$= \int_{\Omega(u,v)} \left(\frac{u-v}{2}\right)\sqrt{v} \underbrace{\left| \frac{\partial(x,y)}{\partial(u,v)} \right|}^{=\frac{1}{2}} du\,dv$$

let $u = x \quad v = x-2y$

$\Rightarrow u - v = 2y \Rightarrow y = \frac{u-v}{2} = 0.$

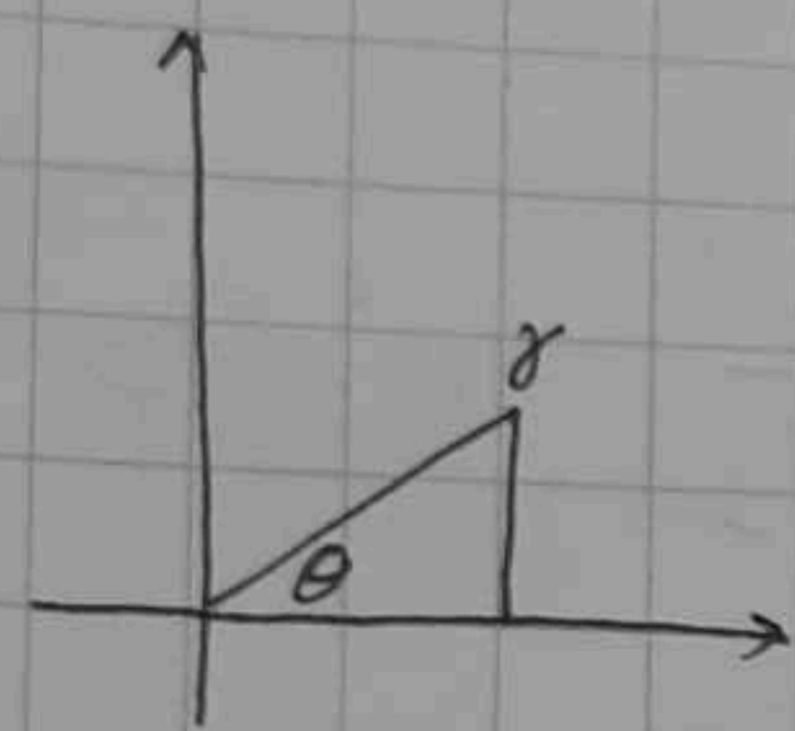
$$= \frac{1}{4} \int_0^4 \int_0^u (u-v)\sqrt{v}\,dv\,du$$




$$= \frac{256}{105}$$

$$\begin{vmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

# Polar coordinate




$x = r\cos\theta$

$y = r\sin\theta$

$$\int_A f(x,y)\,dx\,dy = \int_{\Omega_{r,\theta}} \hat{f}(r,\theta)\cdot r\,dr\,d\theta$$

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

eg. $\int_0^\infty e^{-x^2}\,dx =$

會考 15分

let $I = \int_0^\infty e^{-x^2}\,dx$, $I^2 = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$

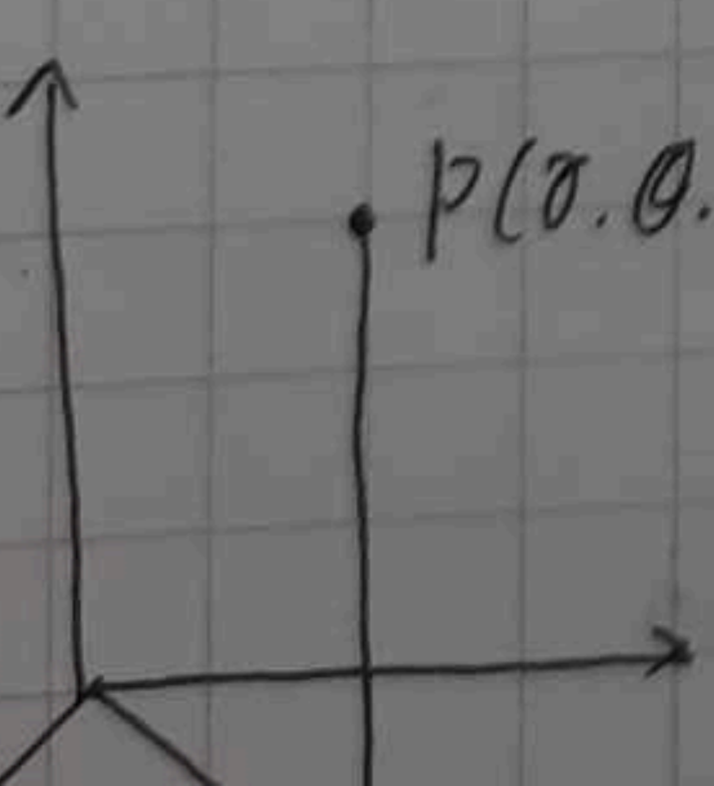
$$= \int_0^\infty\int_0^{2\pi} e^{-r^2}\,d\theta\,dr = \int_0^\infty\int_0^{\frac{\pi}{2}} e^{-r^2} r\,d\theta\,dr = \frac{\pi}{2^2}\int_0^\infty 2r\,e^{-r^2}\,dr = \frac{\pi}{4}, \quad I = \frac{\sqrt{\pi}}{2}$$

Tangent plane

1題方向導數

2題3選6

# Cylinderical coordinate




$p(x.y.z) = p(r.\theta.z)$.

$x = r\cos\theta$. $y = r\sin\theta$. $z = z$

$$\left|\frac{\partial(x.y.z)}{\ldots}\right| = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\sin\theta & 0 \\ \ldots \end{vmatrix} = r$$