Deadline : 2024/6/3, 17:00.

In this homework, "measurable" means **Lebesgue measurable**.

1. Let $u(x,y) = \dfrac{x^4+y^4}{x}$, $v(x,y) = \sin x + \cos y$ and $f$ be a function that maps $(x,y)$ to $(u,v)$. Find the point $(x,y)$ where we can solve for $x,y$ in terms of $u,v$. Also, find $\dfrac{\partial x}{\partial u}, \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial u}, \dfrac{\partial y}{\partial v}$ at $f\left(\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

2. Let $f : \mathbb{R}^4 \to \mathbb{R}^2$ be given by $f(x,y,u,v) = (xu + yv^2, xv^3 + y^2u^6)$.

   (a) Show that $x,y$ can be solved in terms of $u,v$ for $(u,v)$ near $(1,-1)$ and $(x,y)$ near $(1,-1)$.

   (b) From (a), If we write $x = g_1(u,v)$, $y = g_2(u,v)$ for $(u,v)$ near $(1,-1)$ and let $g = (g_1, g_2)$, Find $Dg(u,v)$. (You don't need to calculate explicitly.)

3. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(x,y,z) = (xe^y + ye^z, xe^z + ze^y)$.

   (a) Show that $y,z$ can be solved in terms of $x$ for $x$ near $-1$ and $(y,z)$ near $(1,1)$.

   (b) From (a), If we write $(y,z) = g(x)$ for $x$ near $-1$, Find $g'(x)$. (You don't need to calculate explicitly.)

4. Suppose $f : \mathbb{R} \to \mathbb{R}^n$ is differentiable, such that $|f(t)| = 1$ for every $t \in \mathbb{R}$.

   (a) Prove that $f'(t) \cdot f(t) = 0$. (Here "$\cdot$" is the standard inner product in $\mathbb{R}^n$)

   (b) In fact, this result has a geometric interpretation. For example, in $\mathbb{R}^2$, the function $f(t) = (\cos t, \sin t)$ satisfies $|f(t)| = 1$. Draw the graph of $f(t)$ and $f'(t)$ on $\mathbb{R}^2$, what do you discover?

5. (Second Derivative Test) Let $V$ be an open subset of $\mathbb{R}^2$ and $(a, b) \in V$, and suppose that $f : V \to \mathbb{R}$ satisfy $\nabla f(a, b) = 0$. Suppose also that $f \in \mathcal{C}^2(V)$, and set $D = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2$. Prove the following statements:

   (a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f$ admits a local minimum at $(a, b)$.

   (b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f$ admits a local maximum at $(a, b)$.

   (c) If $D < 0$, then $f$ is a saddle point at $(a, b)$.

**Remark.** In fact $D$ is the determinant of **Hessian** at $(a, b)$.

$$\mathrm{Hess}(f) := \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

The geometric meaning of Hessian matrix is to represent the **curvature** at $(a, b)$. The eigenvalues of $\mathrm{Hess}(f)$ is, in fact, the maximal and minimal value of all second directional derivatives at $(a, b)$.[1] From this we can see that when $D = \det \mathrm{Hess}(f) = \lambda_1\lambda_2 > 0$, there're two cases: either $\lambda_1, \lambda_2 > 0$ or $\lambda_1, \lambda_2 < 0$. In the first case, the maximal and minimal value of second directional derivative are both positive, so the second directional derivative at $(a, b)$ is always positive. In particular, $f_{xx}(a, b) > 0$. This is why we need to check "$f_{xx}(a, b) > 0$". Or, if you want, you can check $f_{yy}(a, b) > 0$ instead.

6. Let $f$ be a nonnegative and measurable function on $E$. Prove that

$$\int_E f(x)\,dx = \sup \left[ \sum_j \left( \inf_{x \in E_j} f(x) \right) |E_j| \right]$$

where the supremum is taken over all decompositions $E = \bigcup_j E_j$ of $E$ into the union of a finite number of disjoint measurable sets $E_j$.

7. If $\{f_k\}_{k\in\mathbb{N}}$ is a sequence of nonnegative and measurable functions on $E$, prove that

$$\int_E \left( \sum_{k=1}^{\infty} f_k(x) \right) dx = \sum_{k=1}^{\infty} \left( \int_E f_k(x)\,dx \right)$$

8. Let $f$ be nonnegative and measurable on $E$, prove that

$$\int_E f(x)\,dx = 0 \iff \exists\, Z \subset E \text{ such that } |Z| = 0 \text{ and } f(x) = 0 \text{ on } E \backslash Z$$ [2]

[1] When $f \in \mathcal{C}^2(V)$, $f_{xy} = f_{yx}$, so $\mathrm{Hess}(f)$ is a symmetric matrix. A real symmetric matrix can always be diagonalized with two real eigenvalues.

[2] We also say $f(x) = 0$ **almost everywhere** on $E$.

9. Let $g(x) = \begin{cases} 0 & , 0 \le x \le 1/2 \\ 1 & , 1/2 < x \le 1 \end{cases}$ and $f_{2k}(x) = g(x)$, $f_{2k-1}(x) = g(1-x)$, $x \in [0,\, 1]$, $k \in \mathbb{N}$. Show that

$$\liminf_{n\to\infty} f_n(x) = 0 \text{ on } [0,\, 1]$$

but $\displaystyle\int_0^1 f_n(x)\,dx = \frac{1}{2}$. (This gives an example of strict inequality in **Fatou lemma**)

10. Let $f(x) = \begin{cases} 1/n & , |x| \le n \\ 0 & , |x| > n \end{cases}$. Show that $f_n(x)$ uniformly converges to 0 on $\mathbb{R}$ but $\displaystyle\int_{-\infty}^{\infty} f_n(x)\,dx = 2$ for all $n \in \mathbb{N}$.

    **Remark.** This problem tells you that uniform convergence doesn't imply dominate convergence, i.e. $\displaystyle\lim_{n\to\infty}\int_E f_n(x)\,dx = \int_E f(x)\,dx$. However, if $E$ is of finite measure, then uniformly convergent sequence of bounded functions implies dominate convergence.