

HWs

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# Chapter 1

## General Analysis HW

### 1.1 HW1

#### Question 1

Show  $\mathbb{R}^n$  is complete.

*Proof.* Let  $\mathbf{x}_k$  be an arbitrary Cauchy sequence in  $\mathbb{R}^n$ . We are required to show  $\mathbf{x}_k$  converge in  $\mathbb{R}^n$ . For each  $k$ , denote  $\mathbf{x}_k$  by  $(x_{(1,k)}, \dots, x_{(n,k)})$ . We claim that for each  $i \in \{1, \dots, n\}$

$x_{(i,k)}$  is a Cauchy sequence

Fix  $i$  and  $\epsilon > 0$ . To show  $x_{(i,k)}$  is a Cauchy sequence, we are required to find  $N \in \mathbb{N}$  such that for all  $r, m \geq N$  we have

$$|x_{(i,r)} - x_{(i,m)}| \leq \epsilon$$

Because  $\mathbf{x}_k$  is a Cauchy sequence in  $\mathbb{R}^n$ , we know there exists  $N \in \mathbb{N}$  such that for all  $r, m \geq N$ , we have

$$|\mathbf{x}_r - \mathbf{x}_m| < \epsilon$$

Fix such  $N$  and arbitrary  $r, m \geq M$ . Observe

$$|x_{(i,r)} - x_{(i,m)}| \leq \sqrt{\sum_{j=1}^n |x_{(j,r)} - x_{(j,m)}|^2} = |\mathbf{x}_r - \mathbf{x}_m| < \epsilon$$

We have proved that for each  $i \in \{1, \dots, n\}$ , the real sequence  $x_{(i,k)}$  is Cauchy. We now claim that for each  $i \in \{1, \dots, n\}$ , we have

$$\limsup_{r \rightarrow \infty} x_{(i,r)} \in \mathbb{R} \text{ and } \lim_{k \rightarrow \infty} x_{(i,k)} = \limsup_{r \rightarrow \infty} x_{(i,r)}$$

Again fix  $i$ . Because  $x_{(i,k)}$  is a Cauchy sequence, we know there exists some  $N$  such that for all  $r, m \geq N$ , we have

$$|x_{(i,r)} - x_{(i,m)}| < 1$$

This implies that for all  $r \geq N$ , we have

$$x_{(i,r)} < x_{(i,N)} + 1 \quad (1.1)$$

Equation 1.1 then tell us

$$x_{(i,N)} + 1 \text{ is an upper bound of } \{x_{(i,r)} : r \geq N\}$$

Then by definition of sup, we have

$$\sup\{x_{(i,r)} : r \geq N\} \leq x_{(i,N)} + 1 \in \mathbb{R}$$

This then implies  $\limsup_{r \rightarrow \infty} x_{(i,r)} \in \mathbb{R}$ . We now prove

$$\lim_{k \rightarrow \infty} x_{(i,k)} = \limsup_{r \rightarrow \infty} x_{(i,r)} \quad (1.2)$$

Fix  $\epsilon > 0$ . We are required to find  $N$  such that

$$\forall k \geq N, \left| x_{(i,k)} - \limsup_{r \rightarrow \infty} x_{(i,r)} \right| \leq \epsilon$$

Because  $\{x_{(i,k)}\}_{k \in \mathbb{N}}$  is a Cauchy sequence, we can let  $N_0$  satisfy

$$\forall k, m \geq N_0, |x_{(i,k)} - x_{(i,m)}| < \frac{\epsilon}{2}$$

Because  $\sup\{x_{(i,k)} : k \geq N'\} \searrow \limsup_{r \rightarrow \infty} x_{(i,r)}$  as  $N' \rightarrow \infty$ , we know there exists  $N_1 > N_0$  such that

$$\limsup_{r \rightarrow \infty} x_{(i,r)} - \frac{\epsilon}{2} < \sup\{x_{(i,k)} : k \geq N_0\} \leq \limsup_{r \rightarrow \infty} x_{(i,r)} + \frac{\epsilon}{2}$$

Then because  $\limsup_{r \rightarrow \infty} x_{(i,r)} - \frac{\epsilon}{2}$  is strictly smaller than the smallest upper bound of  $\{x_{(i,k)} : k \geq N_1\}$ , we see  $\limsup_{n \rightarrow \infty} x_{(i,r)} - \frac{\epsilon}{2}$  is not an upper bound of  $\{x_{(i,k)} : k \geq N_1\}$ . This implies the existence of some  $N$  such that  $N \geq N_1$  and

$$\limsup_{r \rightarrow \infty} x_{(i,r)} - \frac{\epsilon}{2} < x_{(i,N)} \leq \limsup_{r \rightarrow \infty} x_{(i,r)} + \frac{\epsilon}{2}$$

Now observe that for all  $k \geq N$ , because  $N \geq N_1 \geq N_0$

$$\limsup_{r \rightarrow \infty} x_{(i,r)} - \epsilon < x_{(i,N)} - \frac{\epsilon}{2} < x_{(i,k)} < x_{(i,N)} + \frac{\epsilon}{2} \leq \limsup_{r \rightarrow \infty} x_{(i,r)} + \epsilon$$

This implies for all  $k \geq N$ , we have

$$\left| x_{(i,k)} - \limsup_{r \rightarrow \infty} x_{(i,r)} \right| \leq \epsilon$$

We have just proved [Equation 1.2](#). Lastly, to close out the proof, we show

$$\lim_{k \rightarrow \infty} \mathbf{x}_k = \left( \lim_{k \rightarrow \infty} x_{(1,k)}, \dots, \lim_{k \rightarrow \infty} x_{(n,k)} \right) \quad (1.3)$$

Fix  $\epsilon > 0$ . For each  $i \in \{1, \dots, n\}$ , let  $N_i$  satisfy

$$\forall r \geq N_i, \left| x_{(i,r)} - \lim_{k \rightarrow \infty} x_{(i,k)} \right| \leq \frac{\epsilon}{\sqrt{n}}$$

Observe that for all  $r \geq \max_{i \in \{1, \dots, n\}} N_i$ , we have

$$\begin{aligned} \left| \mathbf{x}_r - \left( \lim_{k \rightarrow \infty} x_{(1,k)}, \dots, \lim_{k \rightarrow \infty} x_{(n,k)} \right) \right| &= \sqrt{\sum_{i=1}^n \left| x_{(i,r)} - \lim_{k \rightarrow \infty} x_{(i,k)} \right|^2} \\ &\leq \sqrt{\sum_{i=1}^n \frac{\epsilon^2}{n}} = \epsilon \end{aligned}$$

We have proved [Equation 1.3](#). ■

## Question 2

Show  $\mathbb{Q}$  is dense in  $\mathbb{R}$ .

*Proof.* Fix  $x \in \mathbb{R}$  and  $\epsilon > 0$ . To show  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , we have to find  $q \in \mathbb{Q}$  such that  $|x - q| < \epsilon$ .

Let  $m \in \mathbb{N}$  satisfy  $\frac{1}{m} < \epsilon$ . Let  $n$  be the largest integer such that  $n \leq mx$ . Because  $n$  is the largest integer such that  $n \leq mx$ , we know  $mx - n < 1$ , otherwise we can deduce  $n + 1 \leq mx$ , which is impossible, since  $n + 1$  is an integer and  $n$  is the largest integer such that  $n \leq mx$ . We now see that

$$\frac{n}{m} \in \mathbb{Q} \text{ and } \left| x - \frac{n}{m} \right| = \frac{mx - n}{m} < \frac{1}{m} < \epsilon$$
■

**Theorem 1.1.1. (Distance Formula)** Given two subsets  $A, B$  of a metric space, we have

$$d(A, B) = \inf_{b \in B} d(A, b)$$

*Proof.* Fix arbitrary  $b \in B$ . It is clear that

$$d(A, B) \leq d(A, b)$$

It then follows  $d(A, B) \leq \inf_{b \in B} d(A, b)$ . Fix arbitrary  $a \in A$  and  $b_0 \in B$ . Observe that

$$d(a, b_0) \geq d(A, b_0) \geq \inf_{b \in B} d(A, b)$$

It then follows  $\inf_{b \in B} d(A, b) \leq d(A, B)$ . ■

### Question 3

Let  $E_1, E_2$  be non-empty sets in  $\mathbb{R}^n$  with  $E_1$  closed and  $E_2$  compact. Show that there are points  $x_1 \in E_1$  and  $x_2 \in E_2$  such that

$$d(E_1, E_2) = |x_1 - x_2|$$

Deduce that  $d(E_1, E_2)$  is positive if such  $E_1, E_2$  are disjoint.

*Proof.* Because

(a)  $f(x) \triangleq d(E_1, x)$  is a continuous function on  $\mathbb{R}^n$ .

(b)  $E_2$  is compact.

It now follows by EVT there exists some  $x_2 \in E_2$  such that

$$d(E_1, x_2) = \min_{x \in E_2} d(E_1, x) = \inf_{x \in E_2} d(E_1, x) = d(E_1, E_2)$$

where the last equality is proved above. We can now reduce the problem into finding  $x_1$  in  $E_1$  such that

$$d(x_1, x_2) = d(E_1, x_2)$$

For each  $n \in \mathbb{N}$ , let  $t_n$  satisfy

$$t_n \in E_1 \text{ and } d(t_n, x_2) < d(E_1, x_2) + \frac{1}{n}$$

Clearly,  $t_n$  is a bounded sequence. Then by Bolzano-Weierstrass Theorem, there exists a convergence subsequence  $t_{n_k}$ . Now, because  $E_1$  is closed, we know

$$x_1 \triangleq \lim_{k \rightarrow \infty} t_{n_k} \in E_1$$

It then follows from the function  $f(x) \triangleq d(x, x_2)$  being continuous on  $\mathbb{R}^n$  such that

$$d(x_1, x_2) = \lim_{k \rightarrow \infty} d(t_{n_k}, x_2) = d(E_1, x_2)$$
■

#### Question 4

Prove that the distance between two nonempty, compact, disjoint sets in  $\mathbb{R}^n$  is positive.

*Proof.* The proof follows from the result in last question while acknowledging compact is closed. ■

#### Question 5

Prove that if  $f$  is continuous on  $[a, b]$ , then  $f$  is Riemann-integrable on  $[a, b]$ .

*Proof.* Let  $\overline{\int_a^b} f dx$  and  $\underline{\int_a^b} f dx$  respectively denote the upper and lower Darboux sums. We prove that

$$\overline{\int_a^b} f dx = \underline{\int_a^b} f dx$$

Fix  $\epsilon$ . We reduce the problem into proving the existence of some partition  $\{a = x_0, x_1, \dots, x_n = b\}$  such that

$$\sum_{i=1}^n [M_i - m_i] (x_i - x_{i-1}) \leq \epsilon$$

where

$$M_i \triangleq \sup_{t \in [x_{i-1}, x_i]} f(t) \text{ and } m_i \triangleq \inf_{t \in [x_{i-1}, x_i]} f(t)$$

Because  $f$  is continuous on the compact interval  $[a, b]$ , we know  $f$  is uniformly continuous on  $[a, b]$ . Let  $\delta$  satisfy

$$|x - y| < \delta \text{ and } x, y \in [a, b] \implies |f(x) - f(y)| < \frac{\epsilon}{b - a}$$

Let  $n$  satisfy  $\frac{b-a}{n} < \delta$ . We claim the partition

$$\{a = x_0, x_1, \dots, x_n = b\} \text{ where } x_i \triangleq a + \frac{i(b-a)}{n} \text{ suffices}$$

Now, by EVT, we know that for each  $i$ , there exists some  $t_{i,M}, t_{i,m} \in [x_{i-1}, x_i]$  such that

$$f(t_{i,m}) = m_i \text{ and } f(t_{i,M}) = M_i$$

Then because

$$|t_{i,m} - t_{i,M}| \leq x_i - x_{i-1} \leq \frac{b-a}{n} < \delta$$



We know  $M_i - m_i < \frac{\epsilon}{b-a}$ . This now give us

$$\begin{aligned} \sum_{i=1}^n [M_i - m_i] (x_i - x_{i-1}) &< \sum_{i=1}^n \frac{\epsilon}{(b-a)} (x_i - x_{i-1}) \\ &= \frac{\epsilon}{b-a} \sum_{i=1}^n (x_i - x_{i-1}) \\ &= \frac{\epsilon}{b-a} (b-a) = \epsilon \end{aligned}$$

■

### Question 6

Find  $\limsup_{n \rightarrow \infty} E_n$  and  $\liminf_{n \rightarrow \infty} E_n$  where

$$E_n \triangleq \begin{cases} [\frac{-1}{n}, 1] & \text{if } n \text{ is odd} \\ [-1, \frac{1}{n}] & \text{if } n \text{ is even} \end{cases}$$

*Proof.* Fix arbitrary  $n \in \mathbb{N}$ . Let  $p, q \geq n$  respectively be odd and even. We see

$$[0, 1] \subseteq E_p \text{ and } [-1, 0] \subseteq E_q$$

This now implies

$$[-1, 1] \subseteq \bigcup_{k \geq n} E_k$$

Then because  $n$  is arbitrary, it follows

$$\limsup_{n \rightarrow \infty} E_n = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k = [-1, 1]$$

Again, fix arbitrary  $n \in \mathbb{N}$  and  $\epsilon > 0$ . Let  $p, q$  respectively be even and odd integers greater than  $\max\{n, \frac{1}{\epsilon}\}$ . We now see

$$\epsilon \notin [-1, \frac{1}{p}] = E_p \text{ and } -\epsilon \notin [-\frac{1}{q}, 1] = E_q$$

Because  $\epsilon$  is arbitrary and clearly  $0 \in E_k$  for all  $k$ , we now see

$$\bigcap_{k \geq n} E_k = \{0\}$$

Then because  $n$  is arbitrary, we see

$$\liminf_{n \rightarrow \infty} E_n = \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} E_k = \{0\}$$

■

### Question 7

Show that

$$(\limsup_{n \rightarrow \infty} E_n)^c = \liminf_{n \rightarrow \infty} (E_n)^c$$

and

$$E_n \searrow E \text{ or } E_n \nearrow E \implies \limsup_{n \rightarrow \infty} E_n = \liminf_{n \rightarrow \infty} E_n = E$$

*Proof.* Fix arbitrary  $x \in (\limsup_{n \rightarrow \infty} E_n)^c$ . We can deduce

$$\exists n, x \notin \bigcup_{k \geq n} E_k$$

This implies

$$\exists n, x \in \bigcap_{k \geq n} E_k^c$$

Then we see

$$x \in \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} E_k^c = \liminf_{n \rightarrow \infty} E_n^c$$

We have proved  $(\limsup_{n \rightarrow \infty} E_n)^c \subseteq \liminf_{n \rightarrow \infty} E_n^c$ . We now prove the converse. Fix arbitrary  $x \in \liminf_{n \rightarrow \infty} E_n^c$ . We can deduce

$$\exists n, x \in \bigcap_{k \geq n} E_k^c$$

This implies

$$\exists n, x \notin \bigcup_{k \geq n} E_k$$

Then we see

$$x \notin \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k = \limsup_{n \rightarrow \infty} E_n$$

■

**Theorem 1.1.2. (Equivalent Definition for Limit Superior)** If we let  $E$  be the set of subsequential limits of  $a_n$

$$E \triangleq \{L \in \overline{\mathbb{R}} : L = \lim_{k \rightarrow \infty} a_{n_k} \text{ for some } n_k\}$$

The set  $E$  is non-empty and

$$\max E = \limsup_{n \rightarrow \infty} a_n$$

*Proof.* Let  $n_1 \triangleq 1$ . Recursively, because

$$\sup_{j \geq n_k} a_j \geq \limsup_{n \rightarrow \infty} a_n > \limsup_{n \rightarrow \infty} a_n - \frac{1}{k} \text{ for each } k$$

We can let  $n_{k+1}$  be the smallest number such that

$$a_{n_{k+1}} > \limsup_{n \rightarrow \infty} a_n - \frac{1}{k}$$

It is straightforward to check  $a_{n_k} \rightarrow \limsup_{n \rightarrow \infty} a_n$  as  $k \rightarrow \infty$ . Note that no subsequence can converge to  $\limsup_{n \rightarrow \infty} a_n + \epsilon$  because there exists  $N$  such that  $\sup_{k \geq N} a_k < \limsup_{n \rightarrow \infty} a_n + \epsilon$ . ■

### Question 8

Show that

$$\limsup_{n \rightarrow \infty} (-a_n) = -\liminf_{n \rightarrow \infty} a_n$$

*Proof.* Note that  $-a_{n_k}$  converge if and only if  $a_{n_k}$  converge. Then if we respectively define  $E$  and  $E^-$  to be the set of subsequential limits of  $a_n$  and  $-a_n$ , we see

$$E^- = \{-L \in \mathbb{R} : L \in E\}$$

We now see

$$\limsup_{n \rightarrow \infty} (-a_n) = \max E^- = -\min E = -\liminf_{n \rightarrow \infty} a_n$$

■

### Question 9

Show that

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n \quad (1.4)$$

*Proof.* Fix arbitrary  $\epsilon$ . Let  $N_a, N_b$  respectively satisfy

$$\sup_{n \geq N_a} a_n \leq \limsup_{n \rightarrow \infty} a_n + \frac{\epsilon}{2} \text{ and } \sup_{n \geq N_b} b_n \leq \limsup_{n \rightarrow \infty} b_n + \frac{\epsilon}{2}$$

Let  $N \triangleq \max\{N_a, N_b\}$ . We now see that

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \sup_{n \geq N} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n + \epsilon$$

The result then follows from  $\epsilon$  being arbitrary. ■

### Question 10

$$a_n, b_n \text{ is bounded non-negative} \implies \limsup_{n \rightarrow \infty} (a_n b_n) \leq (\limsup_{n \rightarrow \infty} a_n)(\limsup_{n \rightarrow \infty} b_n) \quad (1.5)$$

*Proof.* There are three cases we should consider

- (a) Both  $\limsup_{n \rightarrow \infty} a_n$  and  $\limsup_{n \rightarrow \infty} b_n$  equal 0.
- (b) Between  $\limsup_{n \rightarrow \infty} a_n$  and  $\limsup_{n \rightarrow \infty} b_n$ , only one of them equals 0.
- (c) Neither  $\limsup_{n \rightarrow \infty} a_n$  nor  $\limsup_{n \rightarrow \infty} b_n$  equals to 0.

In the first case, because  $a_n, b_n$  are both non-negative, we can deduce

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = 0$$

which implies

$$\limsup_{n \rightarrow \infty} (a_n b_n) = \lim_{n \rightarrow \infty} a_n b_n = 0 = \lim_{n \rightarrow \infty} a_n \lim_{n \rightarrow \infty} b_n$$

For second case, WOLG, suppose  $\limsup_{n \rightarrow \infty} a_n = 0$ . Fix arbitrary  $\epsilon$ . We can let  $N$  satisfy

$$\sup_{n \geq N} a_n < \frac{\epsilon}{\sup_{n \in \mathbb{N}} b_n}$$

Since for all  $n \geq N$ , we have

$$a_n b_n \leq \frac{b_n \epsilon}{\sup_{k \in \mathbb{N}} b_k} \leq \epsilon$$

We now see

$$\limsup_{n \rightarrow \infty} (a_n b_n) \leq \sup_{n \geq N} a_n b_n \leq \epsilon$$

The result

$$\limsup_{n \rightarrow \infty} a_n b_n = 0 = \limsup_{n \rightarrow \infty} a_n \limsup_{n \rightarrow \infty} b_n$$

then follows from  $\epsilon$  being arbitrary.

Lastly, for the last case, let  $N_a, N_b$  respectively satisfy

$$\sup_{n \geq N_a} a_n \leq \limsup_{n \rightarrow \infty} a_n \sqrt{1 + \epsilon} \text{ and } \sup_{n \geq N_b} b_n \leq \limsup_{n \rightarrow \infty} b_n \sqrt{1 + \epsilon}$$

Let  $N \triangleq \max\{N_a, N_b\}$ , because for each  $n \geq N$ , we have

$$a_n b_n \leq \left( \sup_{k \geq N_a} a_k \right) \left( \sup_{k \geq N_b} b_k \right) \leq (1 + \epsilon) \left( \limsup_{n \rightarrow \infty} a_n \right) \left( \limsup_{n \rightarrow \infty} b_n \right)$$

It then follows that

$$\limsup_{n \rightarrow \infty} (a_n b_n) \leq \sup_{n \geq N} (a_n b_n) \leq (1 + \epsilon) \left( \limsup_{n \rightarrow \infty} a_n \right) \left( \limsup_{n \rightarrow \infty} b_n \right)$$

The result then follows from  $\epsilon$  being arbitrary. ■

### Question 11

Show that if either  $a_n$  or  $b_n$  converge, the equalities in [Equation 1.4](#) and [Equation 1.5](#) both hold true.

*Proof.* WOLG, suppose  $\lim_{n \rightarrow \infty} a_n = L \in \mathbb{R}$ . We then see

$$(a_{n_k} + b_{n_k}) \text{ converge} \iff b_{n,k} \text{ converge}$$

Let  $E_{a,b}$  and  $E_b$  respectively be the set of subsequential limits of  $(a_n + b_n)$  and  $b_n$ . We now have

$$E_{a,b} = \{L + L_b \in \mathbb{R} : L_b \in E_b\}$$

This give us

$$\limsup_{n \rightarrow \infty} (a_n + b_n) = \max E_{a,b} = L + \max E_b = \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$$

Now, additionally, suppose  $a_n, b_n$  are both bounded and nonnegative. Again because

$$a_{n_k} b_{n_k} \text{ converge} \iff b_{n_k} \text{ converge}$$

We see

$$E_{a,b} = \{L(L_b) \in \mathbb{R} : L_b \in E_b\}$$

This give us

$$\limsup_{n \rightarrow \infty} (a_n b_n) = \max E_{a,b} = L \max E_b = (\limsup_{n \rightarrow \infty} a_n)(\limsup_{n \rightarrow \infty} b_n)$$

■

## Question 12

Give example for which inequality in [Equation 1.4](#) and [Equation 1.5](#) are not equalities.

*Proof.* If

$$a_n \triangleq \begin{cases} 1 & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ is even} \end{cases} \quad \text{and} \quad b_n \triangleq \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

we have

$$\limsup_{n \rightarrow \infty} (a_n + b_n) = 0 < 2 = \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$$

Let  $L > 1$  and

$$a_n \triangleq \begin{cases} L - \frac{1}{k} & \text{if } n = 2k - 1 \\ (L - \frac{1}{k})^{-1} & \text{if } n = 2k \end{cases} \quad \text{and} \quad b_n \triangleq \begin{cases} (L - \frac{1}{k})^{-1} & \text{if } n = 2k - 1 \\ (L - \frac{1}{k}) & \text{if } n = 2k \end{cases}$$

We have

$$\limsup_{n \rightarrow \infty} a_n b_n = 1 < L^2 = \limsup_{n \rightarrow \infty} a_n \limsup_{n \rightarrow \infty} b_n$$

■

### Question 13

Give an example of a decreasing sequence of nonempty closed sets in  $\mathbb{R}^n$  whose intersection is empty.

*Proof.*

$$F_n \triangleq [n, \infty) \text{ suffices}$$

■

### Question 14

Given an example of two disjoint, nonempty closed sets in  $E_1$  and  $E_2$  in  $\mathbb{R}^n$  for which  $d(E_1, E_2) = 0$ .

*Proof.* Let

$$E_1 \triangleq \left\{n - \frac{1}{n} \in \mathbb{R} : n \in \mathbb{N} \text{ and } n \geq 2\right\} \text{ and } E_2 \triangleq \left\{n - \frac{1}{2n} \in \mathbb{R} : n \in \mathbb{N} \text{ and } n \geq 2\right\}$$

To see  $E_1 \cap E_2 = \emptyset$ , suppose  $n - \frac{1}{n} = k - \frac{1}{2k}$  where  $n, k$  are two natural numbers greater than 2. We then see  $\frac{1}{n} - \frac{1}{2k} = n - k$ , which is impossible, since

$$\left| \frac{1}{n} - \frac{1}{2k} \right| < \max\left\{\frac{1}{2k}, \frac{1}{n}\right\} < 1$$

The fact  $E_1, E_2$  are closed follows from both of them being totally disconnected. Now observe that for all  $\epsilon$ , there exists large enough  $n$  such that

$$\left(n + \frac{1}{n}\right) - \left(n + \frac{1}{2n}\right) < \frac{1}{n} < \epsilon$$

This implies  $d(E_1, E_2) = 0$ .

■

### Question 15

If  $f$  is defined and uniformly continuous on  $E$ , show there is a function  $\bar{f}$  defined and continuous on  $\bar{E}$  such that  $\bar{f} = f$  on  $E$ .

*Proof.* Define  $\bar{f}$  on  $E$  by  $\bar{f} = f$ . For each  $x \in \bar{E} \setminus E$ , associate  $x$  with a sequence  $t_{n,x}$  in  $E$  converging to  $x$ . We now claim that for each  $x \in \bar{E} \setminus E$  the limit

$$\lim_{n \rightarrow \infty} f(t_{n,x}) \text{ converge in } \mathbb{R}$$

Fix  $\epsilon$ . Because  $f$  is uniformly continuous on  $E$ , we know there exists  $\delta$  such that

$$a, b \in E \text{ and } |a - b| \leq \delta \implies |f(a) - f(b)| < \epsilon$$

Because  $t_{n,x}$  converge, we know  $t_{n,x}$  is Cauchy, then we know there exists  $N$  such that  $|t_{n,x} - t_{m,x}| < \delta$  for all  $n, m > N$ , we then see that for all  $n, m > N$ , we have

$$|f(t_{n,x}) - f(t_{m,x})| < \epsilon$$

This implies  $\{f(t_{n,x})\}_{n \in \mathbb{N}}$  is a Cauchy sequence in  $\mathbb{R}$ , thus converge in  $\mathbb{R}$ .

Define

$$\bar{f}(x) \triangleq \lim_{n \rightarrow \infty} f(t_{n,x}) \text{ for all } x \in \bar{E} \setminus E$$

We are required to show  $\bar{f}$  is also continuous on  $\bar{E} \setminus E$ . Fix  $\epsilon$  and  $x \in \bar{E} \setminus E$ . Let  $\delta$  satisfy

$$a, b \in E \text{ and } |a - b| \leq \delta \implies |f(a) - f(b)| < \frac{\epsilon}{3}$$

We claim

$$\sup_{t \in B_{\frac{\delta}{2}}(x) \cap \bar{E}} |\bar{f}(t) - \bar{f}(x)| \leq \epsilon$$

Fix  $t \in B_{\frac{\delta}{2}}(x) \cap \bar{E}$ . There are two possibilities

(a)  $t \in E$

(b)  $t \in \bar{E} \setminus E$

If  $t \in E$ , let  $n$  satisfy

$$|f(t_{n,x}) - \bar{f}(x)| < \frac{\epsilon}{3} \text{ and } |t_{n,x} - x| < \frac{\delta}{2}$$

Because

$$|t_{n,x} - t| \leq |t_{n,x} - x| + |t - x| < \delta$$

we can deduce  $|f(t_{n,x}) - f(t)| < \frac{\epsilon}{3}$ . This now give us

$$|f(t) - \bar{f}(x)| \leq |f(t_{n,x}) - f(t)| + |f(t_{n,x}) - \bar{f}(x)| < \epsilon$$

If  $t \in \bar{E} \setminus E$ . Write  $y = t$  and let  $t_{n,y}$  be the associated sequence in  $E$ . Because  $y \in B_{\frac{\delta}{2}}(x)$ , we know there exists  $t_{n,y}$  such that

$$t_{n,y} \in B_{\frac{\delta}{2}}(x) \text{ and } |f(t_{n,y}) - \bar{f}(y)| < \frac{\epsilon}{3}$$



Again, let  $m$  satisfy

$$t_{m,x} \in B_{\frac{\delta}{2}}(x) \text{ and } |f(t_{m,x}) - \bar{f}(x)| < \frac{\epsilon}{3}$$

We know  $|t_{n,y} - t_{m,x}| \leq \delta$  because they both belong to  $B_{\frac{\delta}{2}}(x)$ . We can now deduce

$$|\bar{f}(y) - \bar{f}(x)| = |\bar{f}(y) - f(t_{n,y})| + |f(t_{n,y}) - f(t_{m,x})| + |f(t_{m,x}) - \bar{f}(x)| < \epsilon$$

which finish the proof. ■

### Question 16

If  $f$  is defined and uniformly continuous on a bounded set  $E$ , show that  $f$  is bounded on  $E$ .

*Proof.* By last question, we can extend  $f$  to a continuous  $\bar{f}$  onto  $\bar{E}$ . Now because  $\bar{E}$  is compact and  $|\bar{f}|$  is continuous on  $\bar{E}$ , by EVT, there exists  $a \in \bar{E}$  such that

$$\sup_{x \in E} |f(x)| \leq \max_{x \in \bar{E}} |\bar{f}(x)| = \bar{f}(a)$$
■

## 1.2 HW2

### Question 17

Construct a two-dimensional Cantor set in the unit square  $[0, 1]^2$  as follows. Subdivide the square into nine equal parts and keep only the four closed corner squares, removing the remaining region (which form a cross). Then repeat this process in a suitably scaled version for the remaining squares, ad infinitum. Show that the resulting set is perfect, has plane measure zero, and equals  $\mathcal{C} \times \mathcal{C}$ .

*Proof.* Let  $\mathcal{C}'_n \subseteq \mathbb{R}^2$  be the result after the  $n$ th stage of removal, and let  $\mathcal{C}_n \subseteq \mathbb{R}$  be the result after the  $n$ th stage of removal in the construction of the classical ternary Cantor set. It is clear that

$$\mathcal{C}'_n = \mathcal{C}_n \times \mathcal{C}_n \text{ for all } n$$

It then follows

$$\bigcap_n \mathcal{C}'_n = \bigcap_n \mathcal{C}_n \times \mathcal{C}_n = \mathcal{C} \times \mathcal{C}$$

The fact that  $\mathcal{C} \times \mathcal{C}$  has plane measure zero follows from [Lemma 1.2.1](#). Fix  $(a, b) \in \mathcal{C} \times \mathcal{C}$ . Because  $\mathcal{C}$  is perfect, there exists some  $b' \in \mathcal{C}$  such that

$$0 < |b' - b| < \epsilon$$

To see that  $\mathcal{C}'$  is perfect, one see that

$$(a, b) \neq (a, b') \text{ and } (a, b') \in \mathcal{C}' \times \mathcal{C}' \text{ and } |(a, b) - (a, b')| = |b' - b| < \epsilon$$

■

### Question 18

Construct a subset of  $[0, 1]$  in the same manner as the Cantor set, except that at the  $k$ th stage, each interval removed has length  $\delta 3^{-k}$ ,  $0 < \delta < 1$ . Show that the resulting set is perfect, has measure  $1 - \delta$ , and contains no intervals.

*Proof.* Let  $\mathcal{C}'_n \subseteq \mathbb{R}$  be the result after the  $n$ th stage of removal according to the description. Clearly, each  $\mathcal{C}'_n$  has  $2^n$  amount of connected component, we then can compute the length of  $\mathcal{C}' \triangleq \bigcap \mathcal{C}'_n$  to be

$$1 - \sum_{k=1}^{\infty} 2^{k-1} \delta 3^{-k} = 1 - \frac{\delta \frac{2}{3}}{1 - \frac{2}{3}} = 1 - \delta$$

Since each  $\mathcal{C}'_n$  has  $2^n$  amount of connected component of equal length and  $\mathcal{C}'_n \subseteq [0, 1]$ , we know the length of each connected component of  $\mathcal{C}'_n$  must not be greater than  $\frac{1}{2^n}$ . It then follows that no interval  $[a, a + h]$  can be contained by all  $\mathcal{C}'_n$  because if  $[a, a + h]$  is a subset of some connected component of  $\mathcal{C}'_k$  of some  $k$ , then the measure  $h = |[a, a + h]|$  must be smaller than  $\frac{1}{2^k}$ , which is false when  $k$  is large enough. ■

### Question 19

If  $E_k$  is a sequence of sets with  $\sum |E_k|_e < \infty$ , show that  $\limsup_{n \rightarrow \infty} E_n$  has measure zero.

*Proof.* Note that

$$\sum_{k=N}^{\infty} |E_k|_e = \left( \sum_{k=1}^{\infty} |E_k|_e - \sum_{k=1}^{N-1} |E_k|_e \right) \rightarrow 0 \text{ as } N \rightarrow \infty$$

and note for all  $N$  we have

$$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \subseteq \bigcup_{k=N}^{\infty} E_k$$

Then for arbitrary  $\epsilon$ , if we let  $N$  satisfy  $\sum_{k=N}^{\infty} |E_k|_e < \epsilon$ , we see that

$$\left| \limsup_{n \rightarrow \infty} E_n \right|_e = \left| \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \right|_e \leq \left| \bigcup_{k=N}^{\infty} E_k \right|_e \leq \sum_{k=N}^{\infty} |E_k|_e < \epsilon$$

■

### Question 20

If  $E_1, E_2$  are measurable, show that

$$|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|$$

*Proof.* Observe the following expression of each set in disjoint union

- (a)  $E_1 = (E_1 \setminus E_2) \sqcup (E_1 \cap E_2)$
- (b)  $E_2 = (E_2 \setminus E_1) \sqcup (E_1 \cap E_2)$
- (c)  $E_1 \cup E_2 = (E_1 \setminus E_2) \sqcup (E_1 \cap E_2) \sqcup (E_2 \setminus E_1)$

It now follows

$$\begin{aligned} |E_1 \cup E_2| + |E_1 \cap E_2| &= |E_1 \setminus E_2| + |E_1 \cap E_2| + |E_1 \cap E_2| + |E_2 \setminus E_1| \\ &= |E_1| + |E_2| \end{aligned}$$

■

**Lemma 1.2.1.** Given two subsets  $Z_1, Z_2$  of  $\mathbb{R}$ , if  $|Z_1| = 0$ , then  $|Z_1 \times Z_2| = 0$ .

*Proof.* Let  $A_n \triangleq [n, n+1)$ . Because

$$Z_1 \times Z_2 = \bigsqcup_{n \in \mathbb{Z}} Z_1 \times (A_n \cap Z_2)$$

To show  $|Z_1 \times Z_2| = 0$ , we only have to  $|Z_1 \times (A_n \cap Z_2)| = 0$  for all  $n \in \mathbb{Z}$ . In other words, we can WOLG suppose  $Z_2$  is bounded.

Now, fix  $\epsilon$ . We are required to find a countable closed cube cover  $Q_n \times C_n$  for  $Z_1 \times Z_2$  such that  $\sum_n |Q_n \times C_n| < \epsilon$ . Let  $C_n = C$  for all  $n$  where  $C$  is a compact interval containing  $Z_2$ , and let  $Q_n$  be a countable compact interval cover for  $Z_1$  such that  $\sum |Q_n| < \frac{\epsilon}{|C|}$ . It then follows  $\sum_n |Q_n \times C_n| = \sum_n |Q_n| |C| < \epsilon$ . ■

**Theorem 1.2.2. (Product of Finite Measure Set)** If  $E_1$  and  $E_2$  are measurable subset of  $\mathbb{R}$  and  $|E_1|, |E_2| < \infty$ , then  $E_1 \times E_2$  is measurable in  $\mathbb{R}^2$  and

$$|E_1 \times E_2| = |E_1| |E_2|$$

*Proof.* Write  $E_1 \triangleq H_1 \sqcup Z_1$  and  $E_2 \triangleq H_2 \sqcup Z_2$  where  $H_1, H_2 \in F_\sigma$  and  $|H_1| = |E_1|$  and  $|H_2| = |E_2|$ . Now observe

$$E_1 \times E_2 = (H_1 \times H_2) \cup (Z_1 \times E_2) \cup (E_1 \times Z_2)$$

Note that if we write  $H_1 = \bigcap F_{1,n}$  and  $H_2 = \bigcap F_{2,n}$ , we see  $H_1 \times H_2 = \bigcap F_{1,n} \times F_{2,n} \in F_\sigma$  in  $\mathbb{R}^2$ , it now follows from **Lemma 1.2.1** that  $E_1 \times E_2$  is measurable.

Now, let  $S_n$  be a decreasing sequence of open set containing  $E_1$  such that  $|S_n \setminus E_1| < \frac{1}{n}$ , and let  $T_n$  be a decreasing sequence of open set containing  $E_2$  such that  $|T_n \setminus E_2| < \frac{1}{n}$ . In other words,

$$E_1 = S \setminus Z_1 \text{ and } E_2 = T \setminus Z_2 \text{ where } \begin{cases} S \triangleq \bigcap S_n \\ T \triangleq \bigcap T_n \\ |Z_1| = |Z_2| = 0 \end{cases}$$

We now have

$$S \times T = (E_1 \times E_2) \cup (Z_1 \times E_2) \cup (E_1 \times Z_2)$$

This then implies  $|S \times T| = |S \times T|_e \leq |E_1 \times E_2|_e + |Z_1 \times E_2|_e + |E_1 \times Z_2|_e = |E_1 \times E_2|$ , where the last inequality follows from [Lemma 1.2.1](#). The reverse inequality is clear, since  $E_1 \times E_2 \subseteq S \times T$ . We have proved  $|E_1 \times E_2| = |S \times T|$ .

Now, for each  $n$ , write

$$S_n = \bigcup_{k \in \mathbb{N}} I_{k, S_n} \text{ and } T_n = \bigcup_{k \in \mathbb{N}} I_{k, T_n}$$

where  $(I_{k, S_n})_k$  and  $(I_{k, T_n})_k$  are non-overlapping compact interval. It then follows that

$$|S_n \times T_n| = \sum_{i, j} |I_{i, S_n} \times I_{j, T_n}| = \sum_{i, j} |I_{i, S_n}| \times |I_{j, T_n}| = \sum_i |I_{i, S_n}| \sum_j |I_{j, T_n}| = |S_n| |T_n|$$

Write  $S \triangleq \bigcap_{n \in \mathbb{N}} S_n$  and  $T \triangleq \bigcap_{n \in \mathbb{N}} T_n$ . Because

- (a) Each  $S_n \times T_n$  is open.
- (b)  $|S_n \times T_n| = |S_n| |T_n|$  is bounded ( $\because |S_n| \searrow |E_1| < \infty$ ).
- (c)  $S_n \times T_n \searrow S \times T$

We can now deduce

$$\begin{aligned} |E_1 \times E_2| &= |S \times T| = \lim_{n \rightarrow \infty} |S_n \times T_n| \\ &= \lim_{n \rightarrow \infty} |S_n| |T_n| \\ &= |E_1| |E_2| \end{aligned}$$

■

### Question 21

If  $E_1$  and  $E_2$  are measurable subset of  $\mathbb{R}$ , then  $E_1 \times E_2$  is a measurable subset of  $\mathbb{R}^2$  and  $|E_1 \times E_2| = |E_1| |E_2|$

*Proof.* Define

$$A_n \triangleq \{x \in \mathbb{R} : n \leq x < n + 1\}$$

Because

$$E_1 = \bigcup_{n \in \mathbb{Z}} E_1 \cap A_n \text{ and } E_2 = \bigcup_{k \in \mathbb{Z}} E_2 \cap A_k$$

We can deduce

$$E_1 \times E_2 = \bigcup_{n,k \in \mathbb{Z}} (E_1 \cap A_n) \times (E_2 \cap A_k)$$

Note that **Theorem 1.2.2** tell us  $(E_1 \cap A_n) \times (E_2 \cap A_k)$  is measurable, which implies  $E_1 \times E_2$  is measurable. **Theorem 1.2.2** also tell us  $|(E_1 \cap A_n) \times (E_2 \cap A_k)| = |E_1 \cap A_n| |E_2 \cap A_k|$ , which allow us to deduce

$$\begin{aligned} |E_1 \times E_2| &= \sum_{n,k \in \mathbb{Z}} |(E_1 \cap A_n) \times (E_2 \cap A_k)| = \sum_{n,k \in \mathbb{Z}} |E_1 \cap A_n| |E_2 \cap A_k| \\ &= \sum_{n \in \mathbb{Z}} |E_1 \cap A_n| \sum_{k \in \mathbb{Z}} |E_2 \cap A_k| = |E_1| |E_2| \end{aligned}$$

■

### Question 22

Give an example that shows that the image of a measurable set under a continuous transformation may not be measurable. See also Exercise 10 of Chapter 7.

*Proof.* Consider the Cantor-Lebesgue function denoted by  $f : [0, 1] \rightarrow [0, 1]$  and denote the classical ternary Cantor set by  $\mathcal{C}$ . Let  $V$  be a Vitali set contained by  $[0, 1]$ . Because  $f(\mathcal{C}) = [0, 1]$ , we know there exists  $E \subseteq \mathcal{C}$  such that  $f(E) = V$ . Such  $E$  is measurable since  $|E|_e \leq |\mathcal{C}| = 0$ , yet its continuous image  $V = f(E)$  is by definition non-measurable. ■

### Question 23

Show that there exists disjoint  $E_1, E_2, \dots$  such that  $|\bigcup E_k|_e < \sum |E_k|_e$  with strict inequality.

*Proof.* Let  $V$  be a Vitali Set contained by  $[0, 1]$ . Enumerate  $[0, 1] \cap \mathbb{Q}$  by  $x_n$ . Define

$$E_n \triangleq \{v + x_n \in \mathbb{R} : v \in V\} \text{ for all } n$$

The sequence  $E_n$  is disjoint, since if  $p \in E_n \cap E_m$ , then there exists some pair  $v_n, v_m$  belong to  $V$  such that

$$v_n + x_n = p = v_m + x_m \tag{1.6}$$

which is impossible, since Equation 1.6 implies that  $v_n \neq v_m$  and  $v_n, v_m$  are of difference of a rational number.

Now, note that for arbitrary  $n$  and  $v \in V$ , because  $v \in V \subseteq [0, 1]$  and  $x_n \in [0, 1]$ , we have  $v + x_n \in [0, 2]$ . This implies

$$\bigsqcup_n E_n \subseteq [0, 2] \text{ and } \left| \bigsqcup_n E_n \right|_e \leq 2$$

Because  $V$  is non-measurable by definition, we know  $|V|_e > 0$ , and since outer measure is translation invariant, we can now deduce

$$\sum_n |E_n|_e = \sum_n |V|_e = \infty > 2 \geq \left| \bigsqcup_n E_n \right|_e$$

■

### Question 24

Show that there exists decreasing sequence  $E_k$  of sets such that

- (a)  $E_k \searrow E$ .
- (b)  $|E_k|_e < \infty$ .
- (c)  $\lim_{k \rightarrow \infty} |E_k|_e > |E|_e$

*Proof.* Let  $V$  be a Vitali Set contained by  $[0, 1]$ . Enumerate  $[0, 1] \cap \mathbb{Q}$  by  $x_n$ . Define

$$V + x_n \triangleq \{v + x_n \in \mathbb{R} : v \in V\}$$

In last question, we have proved that  $V + x_n$  is pairwise disjoint. Define for all  $n \in \mathbb{N}$

$$E_n \triangleq \bigsqcup_{k \geq n} V + x_k$$

Observe that

$$E_k \searrow \bigcap E_n = \emptyset$$

which implies  $|\bigcap E_n|_e = 0$ , but

$$\lim_{n \rightarrow \infty} |E_n|_e = \lim_{n \rightarrow \infty} \left| \bigsqcup_{k \geq n} V + x_k \right| \geq \lim_{n \rightarrow \infty} |V + x_n| = |V| > 0$$

■

### Question 25

Let  $Z$  be a subset of  $\mathbb{R}$  with measure zero. Show that the set  $\{x^2 : x \in Z\}$  also has measure zero.

*Proof.* Fix  $Z_n \triangleq Z \cap [-n, n]$ . Since

$$|\{x^2 : x \in Z\}| \leq \sum_{n=1}^{\infty} |\{x^2 : x \in Z_n\}|_e$$

We only have to prove

$$|\{x^2 : x \in Z_n\}|_e = 0 \text{ for all } n$$

Fix  $\epsilon, n$ . Let  $I_k$  be a compact interval cover of  $Z_n$  such that  $\sum |I_k| < \frac{\epsilon}{2n}$ . We shall suppose  $I_k \subseteq [-n, n]$ , since if not, we can just let  $I'_k \triangleq I_k \cap [-n, n]$ .

Now, define

$$I_k^2 \triangleq \{x^2 : x \in I_k\}$$

Clearly,  $I_k^2$  are all compact intervals, and if we write  $I_k \triangleq [a_k, b_k]$ , we have the following inequalities

$$\begin{cases} 0 \leq a_k \leq b_k \implies |I_k^2| = b_k^2 - a_k^2 = |I_k| (b_k + a_k) \leq 2n |I_k| \\ a_k \leq 0 \leq b_k \implies |I_k^2| = \max\{a_k^2, b_k^2\} \leq (b_k - a_k)^2 = |I_k| (b_k - a_k) \leq 2n |I_k| \\ a_k \leq b_k \leq 0 \implies |I_k^2| = a_k^2 - b_k^2 = |I_k| (-a_k - b_k) \leq 2n |I_k| \end{cases}$$

Note that  $\{I_k^2\}_{k \in \mathbb{N}}$  is a compact interval cover of  $\{x^2 : x \in Z_n\}$ , we now see

$$|\{x^2 : x \in Z_n\}|_e \leq \sum_k |I_k^2| \leq 2n \sum |I_k| < \epsilon$$

■



## 1.3 Brunn-Minkowski Inequality

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### Abstract

This HW assignment require us to prove Brunn-Minkowski Inequality.

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We first introduce some notation. Given two sets  $A, B \subseteq \mathbb{R}^d$  and a point  $p \in \mathbb{R}^d$ , we write

$$A + p \triangleq \{a + p \in \mathbb{R}^d : a \in A\}$$

and write

$$A + B \triangleq \{a + b \in \mathbb{R}^d : a \in A \text{ and } b \in B\}$$

Note that elementary set theory tell us

$$(A + p) + (B + q) = (A + B) + (p + q) \tag{1.7}$$

**Theorem 1.3.1. (Brunn-Minkowski Inequality for Bricks)** Suppose  $A, B$  are two **bricks**, i.e.,  $A$  is of the form  $\prod_{j=1}^d [x_j, y_j]$  and so is  $B$ , then we have

$$|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}} \leq |A + B|^{\frac{1}{d}}$$

*Proof.* Because Lebesgue measure is translation invariant, by [Equation 1.7](#), we can WOLG suppose

$$A = \prod_{j=1}^d [0, a_j] \text{ and } B = \prod_{j=1}^d [0, b_j]$$

It is clear that

$$A + B = \prod_{j=1}^d [0, a_j + b_j]$$

Now, by direct computation, we know that

$$|A + B| = \prod_{j=1}^d (a_j + b_j) \text{ and } |A| = \prod_{j=1}^d a_j \text{ and } |B| = \prod_{j=1}^d b_j$$

Note that by AM-GM inequality, we have

$$\frac{|A|^{\frac{1}{d}}}{|A+B|^{\frac{1}{d}}} = \left( \prod_{j=1}^n \frac{a_j}{a_j + b_j} \right)^{\frac{1}{n}} \leq \frac{1}{d} \sum_{j=1}^d \frac{a_j}{a_j + b_j}$$

Similarly by AM-GM inequality, we have

$$\frac{|B|^{\frac{1}{d}}}{|A+B|^{\frac{1}{d}}} = \left( \prod_{j=1}^n \frac{b_j}{a_j + b_j} \right)^{\frac{1}{n}} \leq \frac{1}{d} \sum_{j=1}^d \frac{b_j}{a_j + b_j}$$

Adding two inequalities together, now have

$$\frac{|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}}{|A+B|^{\frac{1}{d}}} \leq \frac{1}{d} \sum_{j=1}^d \frac{a_j + b_j}{a_j + b_j} = 1$$

The result then follows from multiplying both side with  $|A+B|^{\frac{1}{d}}$ . ■

**Theorem 1.3.2. (Brunn-Minkowski Inequality for finite union of non-overlapping of bricks)** Suppose  $A$  is a union of a finite collection of non-overlapping brick and so is  $B$ . We have

$$|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}} \leq |A+B|^{\frac{1}{d}}$$

*Proof.* We prove by induction on the sum  $k$  of the amount of bricks consisting  $A$  and the amount of bricks consisting  $B$ . The base case  $k = 2$  have been proved by [Theorem 1.3.1](#). Suppose the proposition hold true when  $k \leq r$ . Let  $k = r + 1$ . Because the bricks consisting of  $A$  are non-overlapping, by a translation (and renaming axis if necessary), we can suppose

$A^+ \triangleq A \cap \{\mathbf{x} \in \mathbb{R}^d : x_1 \geq 0\}$  is a union of a collection of non-overlapping bricks with amount at least one fewer than  $A$ .

This is true since if we write  $A = A_1 \cup \dots \cup A_m$ , then by translation and remaining axis, we can suppose  $A_1$  lies in only one of the closed subspaces  $\{\mathbf{x} \in \mathbb{R}^d : x_1 \geq 0\}$ ,  $\{\mathbf{x} \in \mathbb{R}^d : x_1 \leq 0\}$  and  $A_2$  lies in another, and since for all  $n \geq 3$ ,  $A_n \cap \{\mathbf{x} \in \mathbb{R}^d : x_1 \geq 0\}$  is either also a brick or empty. With similar reason, we now see

$A^- \triangleq A \cap \{\mathbf{x} \in \mathbb{R}^d : x_1 \leq 0\}$  is a union of a collection of non-overlapping bricks with amount at least one fewer than  $A$ .

Now, note that  $h : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$h(t) \triangleq \left| \left( B + (t, 0, \dots, 0) \right) \cap \{\mathbf{x} \in \mathbb{R}^d : x_1 \geq 0\} \right|$$

is clearly a continuous function. ( $h$  can be expressed as a finite sum of continuous function  $\sum_{k=1}^p h_k^p$  if  $B$  consist of  $p$ -amount of bricks)

We then can translate  $B$  to let  $B$  satisfy

$$\frac{|A^+|}{|A|} = \frac{|B^+|}{|B|} \text{ where } B^+ \triangleq B \cap \{\mathbf{x} \in \mathbb{R}^d : x_1 \geq 0\} \quad (1.8)$$

Define  $B^- \triangleq B \cap \{\mathbf{x} \in \mathbb{R}^d : x_1 \leq 0\}$ . With similar reason stated above, we know  $B^+$  and  $B^-$  are both union of collection of non-overlapping bricks with amount not greater than  $B$ .

At this point, one should note that the sum of the amount of bricks consisting  $A^+$  (resp.  $A^-$ ) and the amount bricks consisting  $B^+$  (resp.  $B^-$ ) is at least one fewer than the sum  $r + 1$  of the sum of the amount of bricks consisting  $A$  and the amount bricks consisting  $B$ . Then because the proposition hold true when  $k \leq r$ , we now have

$$|A^+|^{\frac{1}{d}} + |B^+|^{\frac{1}{d}} \leq |A^+ + B^+|^{\frac{1}{d}} \text{ and } |A^-|^{\frac{1}{d}} + |B^-|^{\frac{1}{d}} \leq |A^- + B^-|^{\frac{1}{d}}$$

Note that for each  $\mathbf{x}$  in the interior of  $A^+ + B^+$ , we must have  $x_1 > 0$ , and for each  $\mathbf{y}$  in the interior of  $A^- + B^-$ , we must have  $y_1 < 0$ . This implies that  $(A^+ + B^+)$  and  $(A^- + B^-)$  are non-overlapping. Now, because

$$A + B = (A^+ + B^+) \cup (A^- + B^-)$$

if we define  $\rho \triangleq \frac{|A^+|}{|A|}$ , from [Equation 1.8](#) we can finally deduce

$$\begin{aligned} |A + B| &= |A^+ + B^+| + |A^- + B^-| \\ &\geq \left(|A^+|^{\frac{1}{d}} + |B^+|^{\frac{1}{d}}\right)^d + \left(|A^-|^{\frac{1}{d}} + |B^-|^{\frac{1}{d}}\right)^d \\ (\because \frac{|A^+|}{|A|} = \frac{|B^+|}{|B|} = 1 - \rho) \quad &= \left((\rho|A|)^{\frac{1}{d}} + (\rho|B|)^{\frac{1}{d}}\right)^d + \left(((1 - \rho)|A|)^{\frac{1}{d}} + ((1 - \rho)|B|)^{\frac{1}{d}}\right)^d \\ &= \left(\rho^{\frac{1}{d}}(|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}})\right)^d + \left((1 - \rho)^{\frac{1}{d}}(|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}})\right)^d \\ &= (\rho + 1 - \rho)(|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}})^d \\ &= (|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}})^d \end{aligned}$$

This then give us the desired

$$|A + B|^{\frac{1}{d}} \geq |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}$$

■

# Chapter 2

## Complex Analysis HW

### 2.1 HW1

**Theorem 2.1.1.**

$$(1+i)^n, \frac{(1+i)^n}{n}, \frac{n!}{(1+i)^n} \text{ all diverge as } n \rightarrow \infty$$

*Proof.* Note that

$$|(1+i)^n| = 2^{\frac{n}{2}} \rightarrow \infty \text{ as } n \rightarrow \infty$$

This implies  $(1+i)$  is unbounded, thus diverge.

Note that

$$\left| \frac{(1+i)^n}{n} \right| = \frac{(\sqrt{2})^n}{n}$$

Observe

$$\begin{aligned} \frac{(\sqrt{2})^n}{n} &= \frac{[(\sqrt{2}-1)+1]^n}{n} = \frac{\sum_{k=0}^n \binom{n}{k} (\sqrt{2}-1)^k}{n} \\ &\geq \frac{\binom{n}{2} (\sqrt{2}-1)^2}{n} = (n-1) \left[ \frac{(\sqrt{2}-1)^2}{2} \right] \rightarrow \infty \text{ as } n \rightarrow \infty \end{aligned}$$

This implies  $\frac{(1+i)^n}{n}$  is unbounded, thus diverge.

Note that

$$\left| \frac{n!}{(1+i)^n} \right| = \frac{n!}{(\sqrt{2})^n}$$

Note that for all  $k \geq 8$ , we have

$$\frac{k}{\sqrt{2}} \geq \frac{\sqrt{8}}{\sqrt{2}} = 2$$

This implies

$$\frac{n!}{(\sqrt{2})^n} = \prod_{k=1}^n \frac{k}{\sqrt{2}} = \frac{7!}{(\sqrt{2})^7} \prod_{k=8}^n \frac{k}{\sqrt{2}} \geq \frac{7!}{(\sqrt{2})^7} \prod_{k=8}^n 2 \geq \frac{7!2^{n-8+1}}{(\sqrt{2})^7} \rightarrow \infty$$

which implies  $\frac{n!}{(1+i)^n}$  is unbounded, thus diverge. ■

**Theorem 2.1.2.**

$$n!z^n \text{ converge} \iff z = 0$$

*Proof.* If  $z = 0$ , then  $n!z^n = 0$  for all  $n$ , which implies  $n!z^n \rightarrow 0$ . Now, suppose  $z \neq 0$ . Let  $M \in \mathbb{N}$  satisfy  $|z| > \frac{1}{M}$ . Observe

$$|n!z^n| = \left| \prod_{k=1}^n kz \right| = \left| \prod_{k=1}^{2M-1} kz \right| \left| \prod_{k=2M}^n kz \right| \geq \left| \prod_{k=1}^{2M-1} kz \right| \left| \prod_{k=2M}^n 2Mz \right| \geq \left| \prod_{k=1}^{2M-1} kz \right| 2^{n-2M+1} \rightarrow \infty$$

This implies  $n!z^n$  is unbounded, thus diverge. ■

**Theorem 2.1.3.**

$$u_n \rightarrow u \implies v_n \triangleq \sum_{k=1}^n \frac{u_k}{n} \rightarrow u$$

*Proof.* Because

$$\sum_{k=1}^n \frac{u_k}{n} = \sum_{k \leq \sqrt{n}} \frac{u_k}{n} + \sum_{\sqrt{n} < k \leq n} \frac{u_k}{n}$$

It suffices to prove

$$\sum_{k \leq \sqrt{n}} \frac{u_k}{n} \rightarrow 0 \text{ and } \sum_{\sqrt{n} < k \leq n} \frac{u_k}{n} \rightarrow u \text{ as } n \rightarrow \infty$$

Because  $u_n$  converge, we can let  $M$  bound  $|u_n|$ . Observe

$$\left| \sum_{k \leq \sqrt{n}} \frac{u_k}{n} \right| \leq \sum_{k \leq \sqrt{n}} \left| \frac{u_k}{n} \right| \leq \sum_{k \leq \sqrt{n}} \frac{M}{n} \leq \frac{M\sqrt{n}}{n} = \frac{M}{\sqrt{n}} \rightarrow 0 \text{ as } n \rightarrow \infty \text{ (done)}$$

Because

$$\sum_{\sqrt{n} < k \leq n} \frac{u_k}{n} = \frac{n - \lceil \sqrt{n} \rceil + 1}{n} \sum_{\sqrt{n} < k \leq n} \frac{u_k}{n - \lceil \sqrt{n} \rceil + 1}$$

and

$$\lim_{n \rightarrow \infty} \frac{n - \lceil \sqrt{n} \rceil + 1}{n} = 1$$

We can reduce the problem into proving

$$\lim_{n \rightarrow \infty} \sum_{\sqrt{n} < k \leq n} \frac{u_k}{n - \lceil \sqrt{n} \rceil + 1} = u$$

Fix  $\epsilon$ . Let  $N$  satisfy that for all  $n \geq N$ , we have  $|u_n - u| < \epsilon$ . Then for all  $n \geq N^2$ , we have

$$\begin{aligned} \left| \left( \sum_{\sqrt{n} < k \leq n} \frac{u_k}{n - \lceil \sqrt{n} \rceil + 1} \right) - u \right| &= \left| \sum_{\sqrt{n} < k \leq n} \frac{u_k - u}{n - \lceil \sqrt{n} \rceil + 1} \right| \\ &\leq \sum_{\sqrt{n} < k \leq n} \frac{|u_k - u|}{n - \lceil \sqrt{n} \rceil + 1} \\ &\leq \sum_{\sqrt{n} < k \leq n} \frac{\epsilon}{n - \lceil \sqrt{n} \rceil + 1} = \epsilon \text{ (done)} \end{aligned}$$

■

## 2.2 Exercise 1

Let  $R$  be a complex algebra with  $1_A$  and  $a \in R$ . Given a complex polynomial

$$f(Z) = a_0 + a_1Z + \cdots + a_nZ^n,$$

we define the evaluation of  $f$  at  $a$  by

$$f(a) = a_01_A + a_1a + \cdots + a_na^n.$$

### Question 26

Let  $R = \mathbb{C}$  and  $a = 1 + i$ . Given  $f(Z) = Z^3$ . Evaluate  $f(a)$ .

*Proof.*  $f(a) = (1 + i)^3 = 2i(1 + i) = -2 + 2i$  ■

### Question 27

Let  $R = M_{2 \times 2}(\mathbb{C})$  be the algebra of  $2 \times 2$  complex matrices. Take

$$a = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

and  $g(Z) = 3 + 2Z$ . Evaluate  $g(a)$ .

*Proof.*

$$g(a) = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 2 & 5 \end{bmatrix}$$
 ■

### Question 28

Let  $R$  be the algebra of complex valued periodic functions of period  $2\pi$ , i.e.,  $a \in R$  is a continuous function  $a : \mathbb{R} \rightarrow \mathbb{C}$  so that  $a(x + 2\pi) = a(x)$ . Let  $e(x) = \cos x + i \sin x$  and

$$h(Z) = 1 + Z + Z^2 + \cdots + Z^9.$$

Find  $h(e)$ .

*Proof.* Note that

$$\begin{aligned} (\cos x + i \sin x)(\cos y + i \sin y) &= (\cos x \cos y - \sin x \sin y) + i(\sin x \cos y + \cos x \sin y) \\ &= \cos(x + y) + i \sin(x + y) \end{aligned}$$

This give us

$$h(e) = \sum_{k=0}^9 \cos(kx) + i \sin(kx)$$

■



## 2.3 HW2

### Theorem 2.3.1. (Root Test is Stronger Than Ratio Test)

$$\liminf_{n \rightarrow \infty} \left| \frac{z_{n+1}}{z_n} \right| \leq \liminf_{n \rightarrow \infty} \sqrt[n]{|z_n|} \leq \limsup_{n \rightarrow \infty} \sqrt[n]{|z_n|} \leq \limsup_{n \rightarrow \infty} \left| \frac{z_{n+1}}{z_n} \right|$$

*Proof.* Fix  $\epsilon$  and WOLG suppose  $\liminf_{n \rightarrow \infty} \left| \frac{z_{n+1}}{z_n} \right| > 0$ . We prove

$$\liminf_{n \rightarrow \infty} \sqrt[n]{|z_n|} \geq \liminf_{n \rightarrow \infty} \left| \frac{z_{n+1}}{z_n} \right| - \epsilon$$

Let  $\alpha \in \mathbb{R}$  satisfy

$$\liminf_{n \rightarrow \infty} \left| \frac{z_{n+1}}{z_n} \right| - \epsilon < \alpha < \liminf_{n \rightarrow \infty} \left| \frac{z_{n+1}}{z_n} \right|$$

Let  $N$  satisfy

$$\text{For all } n \geq N, \left| \frac{z_{n+1}}{z_n} \right| > \alpha$$

We then see

$$\sqrt[N+n]{|z_{N+n}|} \geq \sqrt[N+n]{|z_N|} \alpha^n = \alpha \left( \frac{|z_N|^{\frac{1}{N+n}}}{\alpha^{\frac{N}{N+n}}} \right) \rightarrow \alpha \text{ as } n \rightarrow \infty \text{ (done)}$$

The proof for the other side is similar. ■

#### Question 29

Find the radius of convergence of the following series:

- (a)  $\sum \frac{z^n}{n}$ .
- (b)  $\sum \frac{z^n}{n!}$ .
- (c)  $\sum n! z^n$ .
- (d)  $\sum n^k z^n$  where  $k$  is a positive integer.
- (e)  $\sum z^{n!}$ .

*Proof.* We know

$$n^{\frac{1}{n}} = e^{\frac{\ln n}{n}} \rightarrow 1 \text{ as } n \rightarrow \infty \tag{2.1}$$

Equation 2.1 implies  $n^{\frac{-1}{n}} \rightarrow 1$  as  $n \rightarrow \infty$  and that  $\sum \frac{z^n}{n}$  has radius of convergence 1. Equation 2.1 also implies  $n^{\frac{k}{n}} \rightarrow 1$  and  $\sum n^k z^n$  has radius of convergence 1.

We know

$$\lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} = \infty$$

Then Theorem 2.3.1 tell us

$$\lim_{n \rightarrow \infty} (n!)^{\frac{1}{n}} = \infty \quad (2.2)$$

which implies that  $\sum n! z^n$  has radius of convergence 0 and  $\sum \frac{z^n}{n!}$  has radius of convergence  $\infty$ . Note that

$$\sum z^{n!} = \sum a_n z^n \text{ where } a_n = \begin{cases} 1 & \text{if } n = k! \text{ for some } k \\ 0 & \text{otherwise} \end{cases}$$

It is then clear that

$$\limsup_{n \rightarrow \infty} (a_n)^{\frac{1}{n}} = 1$$

and the radius is 1. ■

### Question 30

The 0th order Bessel function  $J_0(z)$  is defined by the power series

$$J_0(z) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2} \frac{z^n}{2n}$$

Find its radius of convergence.

*Proof.* Equation 2.1 and Equation 2.2 tell us

$$\lim_{n \rightarrow \infty} (2n(n!)^2)^{\frac{1}{n}} = \infty$$

which implies that the radius of convergence of  $J_0(z)$  is  $\infty$ . ■

**Theorem 2.3.2. (Abel's Test for Power Series)** Suppose  $a_n \rightarrow 0$  monotonically and  $\sum a_n z^n$  has radius of convergence  $R$ .

The power series  $\sum a_n z^n$  at least converge on  $\overline{D_R(0)} \setminus \{R\}$

*Proof.* Note that

$$\sum \frac{a_n}{R^n} z^n \text{ has radius of convergence } R$$

Fix  $z \in \overline{D_R(0)} \setminus \{R\}$ . Note that

$$\left| \sum_{n=0}^N \frac{z^n}{R^n} \right| = \left| \frac{1 - (\frac{z}{R})^{N+1}}{1 - \frac{z}{R}} \right| \leq \frac{2}{|1 - \frac{z}{R}|} \text{ for all } N$$

It then follows from Dirichlet's Test that  $\sum a_n (\frac{z}{R})^n$  converge. ■

### Question 31

Suppose that  $\sum a_n z^n$  has radius of convergence  $R$  and let  $C$  be the circle  $\{z \in \mathbb{C} : |z| = R\}$ . Prove or disprove

- (a) If  $\sum a_n z^n$  converge at every point on  $C$ , except possibly one, then it converges absolutely everywhere on  $C$

*Proof.* Consider  $a_n \triangleq \frac{1}{n}$  for all  $n \in \mathbb{N}$  and  $a_0 \triangleq 1$ . Then  $\sum a_n z^n$  has convergence radius 1. Since  $a_n \searrow 0$ , it follows from [Theorem 2.3.2](#),  $\sum a_n z^n$  converge everywhere on  $C \setminus \{1\}$ . Observe that when  $z = 1$ , the series is just harmonic series, which diverge. ■

### Question 32

If  $\sum a_n z^n$  has radius of convergence  $R$ , find the radius of convergence of

- (a)  $\sum n^3 a_n z^n$ .  
 (b)  $\sum a_n z^{3n}$ .  
 (c)  $\sum a_n^3 z^n$

*Proof.* Since  $(n^3)^{\frac{1}{n}} \rightarrow 1$ , we know  $\sum n^3 a_n z^n$  also had radius of convergence  $R$ . We claim that the series  $\sum a_n z^{3n}$  has convergence radius  $R^{\frac{1}{3}}$ . If  $|z| < R^{\frac{1}{3}}$ , then  $|z^3| < R$  and thus

$$\sum a_n (z^3)^n \text{ converge}$$

and if  $|z| > R^{\frac{1}{3}}$ , then  $|z^3| > R$  and

$$\sum a_n (z^3)^n \text{ diverge}$$

We have proved that  $\sum a_n z^{3n}$  has convergence radius  $R^{\frac{1}{3}}$ .

Note that given a sub-sequence  $|a_{n_k}|^{\frac{1}{n_k}}$ ,

$|a_{n_k}|^{\frac{1}{n_k}}$  converge in extended reals if and only if  $|a_{n_k}|^{\frac{3}{n_k}}$  converge in extended reals and if the former converge to  $L$ , then the latter converge to  $L^3$ . It now follows that

$$\limsup_{n \rightarrow \infty} |a_n^3| = (\limsup_{n \rightarrow \infty} |a_n|)^3 = \frac{1}{R^3}$$

It now follows that  $\sum a_n^3 z^n$  has convergence radius  $R^3$ . ■

### Theorem 2.3.3. (Summation by Part)

$$\begin{aligned} f_n g_n - f_m g_m &= \sum_{k=m}^{n-1} f_k \Delta g_k + g_k \Delta f_k + \Delta f_k \Delta g_k \\ &= \sum_{k=m}^{n-1} f_k \Delta g_k + g_{k+1} \Delta f_k \end{aligned}$$

*Proof.* The proof follows induction which is based on

$$f_m g_m + f_m \Delta g_m + g_m \Delta f_m + \Delta f_m \Delta g_m = f_{m+1} g_{m+1}$$
■

### Question 33

Prove that, for  $z \neq 1$

$$\sum_{n=1}^k \frac{z^n}{n} = \frac{z}{1-z} \left( \sum_{n=1}^{k-1} \frac{1}{n(n+1)} - \sum_{n=1}^{k-1} \frac{z^n}{n(n+1)} + \frac{1-z^k}{k} \right)$$

Show that the series  $\sum \frac{z^n}{n}$  and  $\sum \frac{z^n}{n(n+1)}$  have radius of convergence 1; that the latter series converge everywhere on  $|z| = 1$ , while the former converges everywhere on  $|z| = 1$  except  $z = 1$ .

*Proof.* We prove by induction. The base case  $k = 1$  is trivial. Suppose the equality hold when  $k = m$ . The difference of the left hand side is clearly  $\frac{z^{m+1}}{m+1}$ , and the difference of the

right hand side is

$$\begin{aligned}
& \frac{z}{1-z} \left( \frac{1}{m(m+1)} - \frac{z^m}{m(m+1)} + \frac{1-z^{m+1}}{m+1} - \frac{1-z^m}{m} \right) \\
&= \frac{z}{1-z} \cdot \frac{1 - z^m + m - mz^{m+1} + (m+1)z^m - (m+1)}{m(m+1)} \\
&= \frac{z}{1-z} \cdot \frac{-mz^{m+1} + mz^m}{m(m+1)} = \frac{z(z^m - z^{m+1})}{(1-z)(m+1)} = \frac{z^{m+1}}{m+1}
\end{aligned}$$

The fact that both series have radius of convergence 1 follows from  $n^{\frac{1}{n}} \rightarrow 1$ . Both of them converge on  $\overline{D_1(0)} \setminus \{1\}$  by [Theorem 2.3.2](#). The former clearly diverge at  $z = 1$ , since it would be a harmonic series, and the latter converge at  $z = 1$  by comparison test with  $\frac{1}{n(n+1)} \leq \frac{1}{n^2}$  for all  $n \in \mathbb{N}$ . ■

### Question 34

Suppose that the power series  $\sum a_n z^n$  has a recurring sequence of coefficients; that is  $a_{n+k} = a_n$  for some fixed positive integer  $k$  and all  $n$ . Prove that the series converge for  $|z| < 1$  to a rational function  $\frac{p(z)}{q(z)}$  where  $p, q$  are polynomials, and the roots of  $q$  are all on the unit circle. What happens if  $a_{n+k} = \frac{a_n}{k}$  instead?

*Proof.* Let

$$L^- \triangleq \min\{|a_0|, \dots, |a_{k-1}|\} \text{ and } L^+ \triangleq \max\{|a_0|, \dots, |a_{k-1}|\}$$

We see that

$$1 = \liminf_{n \rightarrow \infty} (L^-)^{\frac{1}{n}} \leq \lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} \leq \limsup_{n \rightarrow \infty} (L^+)^{\frac{1}{n}} = 1$$

It then follows that  $\sum a_n z^n$  has convergence radius 1. Now observe that for  $|z| < 1$ , we have

$$z^k \sum_{n=0}^{\infty} a_n z^n = \sum_{n=k}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{k-1} a_n z^n$$

This now implies

$$\sum_{n=0}^{\infty} a_n z^n = \frac{\sum_{n=0}^{k-1} a_n z^n}{1 - z^k}$$

Since  $q(z) = 1 - z^k$ , clearly the roots are all on the unit circle. Suppose now  $b_n \triangleq a_n$  for all  $n < k$  and  $b_{n+k} \triangleq \frac{b_n}{k}$  for all  $n \geq k$ . We then have

$$b_n = \frac{a_n}{k^{q(n)}} \text{ where } q \text{ is the largest integer such that } qk \leq n$$

Note that  $n - q(n)$  is always smaller than  $k$ . It then follows that

$$(k^{q(n)})^{\frac{1}{n}} = k^{\frac{q(n)}{n}} = k \cdot k^{\frac{q(n)-n}{n}} \rightarrow k$$

We then see that

$$\limsup_{n \rightarrow \infty} |b_n|^{\frac{1}{n}} = \frac{\limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}}{k} = \frac{1}{k}$$

It then follows that  $\sum b_n z^n$  has convergence radius  $k$ . Now observe that for  $|z| < k$ , we have

$$z^k \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} b_n z^{n+k} = \frac{1}{k} \sum_{n=k}^{\infty} b_n z^n = \frac{1}{k} \left( \sum_{n=0}^{\infty} b_n z^n - \sum_{n=0}^{k-1} b_n z^n \right)$$

This now implies

$$\sum_{n=0}^{\infty} b_n z^n = \frac{\sum_{n=0}^{k-1} b_n z^n}{k \left( \frac{1}{k} - z^k \right)}$$

■

## 2.4 Exercises 2

Let  $(M, d)$  be a metric space,  $x \in M$  and  $F$  a subset of  $M$ .

### Question 35

Prove that the following statements are equivalent

- (a) There exists a sequence  $\{x_n\}$  in  $F$  with  $x_n \neq x$  so that  $\lim_{n \rightarrow \infty} x_n = x$ .
- (b) For any  $\epsilon$ , the intersection of  $B'_\epsilon(x) \triangleq \{y \in M : 0 < d(x, y) < \epsilon\}$  and  $F$  are non-empty.

*Proof.* If (a) is true, then for all  $\epsilon$  there exists some  $x_n \in F$  such that  $d(x_n, x) < \epsilon$ . Because  $x_n \neq x$ , we know that  $0 < d(x_n, x)$ . This now implies  $x_n \in B'_\epsilon(x) \cap F$ .

If (b) is true, then for all  $n$ , we simply select a point in  $x_n \in B'_{\frac{1}{n}}(x) \cap F$ . After such selection, we see that  $x_n \neq x$  and for all  $\epsilon$ , if  $n > \frac{1}{\epsilon}$ , then  $x_n \in B'_\epsilon(x) \cap F$ . ■

### Question 36

Prove that the following statements are equivalent

- (a)  $F$  contain all its limit point.
- (b)  $U = M \setminus F$  is open.

*Proof.* If (a) is true, then for all  $p \in U$ , we know that  $p$  is not a limit point of  $F$ , then from the first question, we know that there exists  $\epsilon$  such that  $B'_\epsilon(x) \cap F = \emptyset$ . Because  $x \in U = M \setminus F$  also does not belong  $x$ , we also know that  $B_\epsilon(x) \cap F = \emptyset$ . This then implies that  $B_\epsilon(x) \subseteq U$ , since  $U = M \setminus F$ . We have proved that  $U$  is open.

If (b) is true, then for arbitrary  $p \notin F$ , we know there exists some  $\epsilon$  such that  $B_\epsilon(x)$  is disjoint with  $F$ . Because  $B'_\epsilon(x)$  is a subset of  $B_\epsilon(x)$ , we can deduce that  $B_\epsilon(x) \cap F = \emptyset$ , which from the first question implies that  $p$  is not a limit point of  $F$ . Because  $p$  is arbitrary selected from  $M \setminus F$ , we have proved that none of the points in  $M \setminus F$  is a limit point of  $F$ . This implies that if  $F$  has any limit point, then  $F$  must contain that limit point. ■

### Question 37

Prove the following statements

- (a)  $M$  and  $\emptyset$  are closed.

- (b) The intersection of any family of closed subsets of  $M$  is closed.
- (c) The union of finitely many closed subsets of  $M$  is closed.

*Proof.* It is clear that  $M$  is open and trivially true that  $\emptyset$  is open. It then follows from the second question that  $M$  and  $\emptyset$  are both closed.

Let  $(F_\alpha)$  be a collection of closed subsets of  $M$ . Arbitrary select a limit point  $x$  of  $\bigcap F_\alpha$ . Let  $\{x_n\}$  be a sequence in  $\bigcap F_\alpha$  with  $x_n \neq x$  so that  $\lim_{n \rightarrow \infty} x_n = x$ . Arbitrary select  $\beta$ . Note that  $\{x_n\}$  is also a sequence in  $F_\beta$  that converge to  $x$  with  $x_n \neq x$ . This now implies that  $x$  is a limit point of  $F_\beta$ . Then because  $F_\beta$  is closed, we see that  $x \in F_\beta$ . Now, since  $\beta$  is arbitrary selected, we see  $x \in \bigcap_\alpha F_\alpha$ . Because  $x$  is arbitrary, we have proved  $\bigcap F_\alpha$  contained all its limit points.

Let  $\{F_1, \dots, F_N\}$  be a collection of closed subsets of  $M$ . Let  $x$  be an arbitrary limit point of  $\bigcup_{n=1}^N F_n$ . Let  $\{x_n\}$  be a sequence in  $\bigcup_{n=1}^N F_n$  with  $x_n \neq x$  converging to  $x$ . It is clear that there must exists some  $j \in \{1, \dots, N\}$  such that  $F_j$  contain infinite terms of  $\{x_n\}$ , i.e., there exists a subsequence  $x_{n_k}$  such that  $x_{n_k} \in F_j$  for all  $k$ . Because  $\lim_{k \rightarrow \infty} x_{n_k} = \lim_{n \rightarrow \infty} x_n = x$ , we now see that  $x$  is a limit point of  $F_j$ . It then follows from  $F_j$  being closed that  $x \in F_j \subseteq \bigcup_{n=1}^N F_n$ . Because  $x$  is arbitrary, we have proved that  $\bigcup_{n=1}^N F_n$  is closed. ■



## 2.5 Exercise 3

### Question 38

Provide a counterexample to the following statement: Suppose

$$f(z) = u(x, y) + iv(x, y)$$

is defined in a neighborhood of  $z_0 = a + ib$ . If the partial derivatives of  $u$  and  $v$  exist at  $(a, b)$  and satisfy the Cauchy-Riemann equations  $u_x(a, b) = v_y(a, b)$  and  $u_y(a, b) = -v_x(a, b)$ , then  $f$  is holomorphic at  $z_0$ .

*Proof.* WOLG, let  $a = b = 0$  and define

$$u(x, y) \triangleq \begin{cases} x & \text{if } y = 0 \\ y & \text{if } x = 0 \\ 0 & \text{if otherwise} \end{cases} \quad \text{and } v(x, y) \triangleq \begin{cases} -x & \text{if } y = 0 \\ y & \text{if } x = 0 \\ 0 & \text{if otherwise} \end{cases}$$

It is clear that

$$u_x = 1 = v_y \text{ and } u_y = 1 = -v_x \text{ at } (0, 0)$$

but

$$\lim_{t \rightarrow 0; t \in \mathbb{R}} \frac{f(t) - f(0)}{t - 0} = \lim_{t \rightarrow 0; t \in \mathbb{R}} \frac{t - it}{t} = 1 - i$$

together with

$$\lim_{t \rightarrow 0; t \in \mathbb{R}} \frac{f(t + it) - f(0)}{t + it} = \lim_{t \rightarrow 0; t \in \mathbb{R}} \frac{0}{t + it} = 0$$

shows that  $f$  is not holomorphic at  $(0, 0)$ . ■

### Question 39

Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function. Suppose that  $f$  is differentiable at  $(a, b)$  and that  $f'(x) = 0$  for all  $x \in (a, b)$ . Prove that  $f$  is a constant function.

*Proof.* Assume  $f(x) \neq f(y)$  for some  $x \neq y \in [a, b]$ . By MVT, we then see there exists some  $t$  between  $x, y$  (thus  $t \in (a, b)$ ) such that  $f'(t) = \frac{f(y) - f(x)}{y - x} \neq 0$ , which is impossible.

CaC ■

### Question 40

Let  $B = B_R(x_0)$  be the open ball in  $\mathbb{R}^n$  centered at  $x_0$  with radius  $R > 0$ . Prove that if  $f : B \rightarrow \mathbb{R}$  is a differentiable function such that  $\nabla f = 0$  on  $B$ , then  $f$  is a constant function.

*Proof.* Let  $\mathbf{x}, \mathbf{y}$  be two points in  $B$ . We are required to show  $f(\mathbf{x}) = f(\mathbf{y})$ . Define  $g : [0, 1] \rightarrow B$  by

$$g(t) \triangleq f(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))$$

Note that  $g$  is well-defined since  $B$  is convex. Because  $f$  is differentiable, we have

$$g'(t) = \nabla f(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) \cdot (\mathbf{y} - \mathbf{x}) = 0$$

It then follows from the last question that

$$f(\mathbf{y}) = g(1) = g(0) = f(\mathbf{x})$$

■

### Question 41

Let  $U$  be an open subset of  $\mathbb{R}^n$ . A function  $f : U \rightarrow \mathbb{R}$  is called **locally constant** if, for each  $x \in U$ , there exists an open neighborhood  $W$  of  $x$  such that  $W \subseteq U$  and  $f : W \rightarrow \mathbb{R}$  is constant on  $W$ . Prove that  $f$  is locally constant function if and only if  $\nabla f = 0$  on  $U$ .

*Proof.* The if part follows from the last question by taking some small enough  $r$  such that  $B_r(x) \subseteq W$ . We now prove the only if part. Fix arbitrary  $x \in U$ . Because  $f$  is locally constant at  $x$ , we know there exists some  $B_r(x)$  such that  $f$  is constant on  $B_r(x)$ . Therefore, we can let  $c \in \mathbb{R}$  satisfy

$$f(y) = c \text{ for all } y \in B_r(x)$$

To see  $\nabla f(x) = 0$ , just observe that for arbitrary axis  $\mathbf{j}$

$$f_{\mathbf{j}}(x) = \lim_{t \rightarrow 0} \frac{f(x + t\mathbf{j}) - f(x)}{t} = 0$$

since  $f(x + t\mathbf{j}) = c = f(x)$  as long as  $|t| < r$ . Because  $\mathbf{j}$  is arbitrary, it then follows that  $\nabla f(x) = 0$ , and because  $x$  is arbitrary selected from  $U$ , we have proved  $\nabla f$  is 0 on  $U$ . ■

### Question 42

Let  $D$  be an open, connected subset of  $\mathbb{R}^n$ . Prove that if  $f : D \rightarrow \mathbb{R}$  is a locally constant function, then  $f$  is a constant function.

# Chapter 3

## PDE intro HW

### 3.1 1.2 First Order Linear Equations

**(Principle of Geometric Method)** Given a first order homogeneous linear PDE with the form

$$u_x + g(x, y)u_y = 0$$

We know if a curve  $\gamma(x) = (x, y)$  satisfy

$$\gamma'(x) = c_x(1, g(x, y)) \text{ for some } c_x$$

Then

$$(u \circ \gamma)'(x) = 0 \text{ for all } x$$

Since

$$\gamma'(x) = (1, \frac{dy}{dx})$$

To find  $\gamma$ , we only wish to solve

$$\frac{dy}{dx} = g(x, y)$$

#### Question 43

Solve

$$(1 + x^2)u_x + u_y = 0$$

*Proof.* The ODE

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

has general solution  $y = \arctan x + C$ , so

$$u(x, y) = f(y - \arctan x)$$

■

### Question 44

Solve

$$\begin{cases} yu_x + xu_y = 0 \\ u(0, y) = e^{-y^2} \end{cases}$$

In which region of the  $xy$  plane is the solution uniquely determined?

*Proof.* We are required to solve the ODE

$$\frac{dy}{dx} = \frac{x}{y}$$

Separating the variables and integrate

$$\int yy' dx = \int x dx \implies \frac{y^2}{2} = \frac{x^2}{2} + C$$

This now implies

$$u(x, y) = f(y^2 - x^2)$$

Initial Condition then give us

$$u = e^{x^2 - y^2}$$

■

### Question 45

Solve the equation

$$u_x + u_y = 1$$

*Proof.* Clearly  $u = \frac{x}{2} + \frac{y}{2}$  is a solution. We now solve the PDE

$$v_x + v_y = 0$$

Solving the ODE

$$\frac{dy}{dx} = 1$$

we have

$$y = x + C$$

Thus the general solution is

$$u = \frac{x}{2} + \frac{y}{2} + f(y - x)$$

■

### Question 46

Solve

$$\begin{cases} u_x + u_y + u = e^{x+2y} \\ u(x, 0) = 0 \end{cases}$$

*Proof.* Let  $\gamma(x) = x + C$ , we have

$$(u \circ \gamma)' + (u \circ \gamma) = e^{3x+2C}$$

We now solve the ODE

$$y' + y = e^{3x+2C}$$

The particular solution is clearly

$$y = \frac{1}{4}e^{3x+2C}$$

Thus the general solution is

$$y = \frac{e^{3x+2C}}{4} + \tilde{C}e^{-x}$$

We then now know

$$(u \circ \gamma)(x) = \frac{e^{3x+2C}}{4} + \tilde{C}e^{-x} \tag{3.1}$$

In other words,

$$u(x, x + C) = \frac{e^{3x+2C}}{4} + \tilde{C}e^{-x}$$

Putting in the initial conditions, we have

$$0 = u(-C, 0) = \frac{e^{-C}}{4} + \tilde{C}e^C$$

This implies

$$\tilde{C} = \frac{-e^{-2C}}{4}$$

Putting the back into Equation 3.1, we have

$$u(x, x + C) = \frac{e^{3x+2C} - e^{-x-2C}}{4}$$

So

$$u(x, y) = \frac{e^{3x+2(y-x)} - e^{-x-2(y-x)}}{4}$$

■

### Question 47

Use the coordinate method to solve the equation

$$u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2$$

*Proof.* Let

$$\begin{cases} \xi \triangleq 2x - y \\ \eta \triangleq x + 2y \end{cases}$$

We see

$$\begin{cases} u_\xi = \frac{2}{5}u_x - \frac{1}{5}u_y \\ u_\eta = \frac{1}{5}u_x + \frac{2}{5}u_y \end{cases}$$

We then can rewrite

$$5u_\eta + \xi u = \xi \eta \tag{3.2}$$

which clearly have particular solution

$$u = \eta - \frac{5}{\xi}$$

To solve the linear homogeneous PDE

$$5u_\eta + \xi u = 0$$

Observe that for all fixed  $\xi$ , the PDE is just an ODE whose solution is exactly  $u = C_\xi e^{\frac{-\xi\eta}{5}}$ . We now know the general solution for **PDE 3.2** is exactly

$$u = \eta - \frac{5}{\xi} + f(\xi)e^{\frac{-\xi\eta}{5}}$$

Then

$$u = x + 2y - \frac{5}{2x - y} + e^{\frac{-(2x-y)(x+2y)}{5}} f(2x - y)$$

■



## 3.2 1.4 Initial and Boundary Condition

### Question 48

Find a solution of

$$\begin{cases} u_t = u_{xx} \\ u(x, 0) = x^2 \end{cases}$$

*Proof.* Clearly  $u = x^2 + 2t$  suffices. ■

## 3.3 1.5 Well Posed Problems

Given a vector field  $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ , Divergence Theorem shows

$$\iiint_D \nabla \cdot F dV = \iint_{\text{bdy } D} F \cdot \mathbf{n} dS$$

Then if  $F$  is the gradient of some scalar field  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ , we have

$$\iiint_D \Delta f dV = \iint_{\text{bdy } D} \frac{\partial f}{\partial \mathbf{n}} dS$$

### Question 49

Consider the ODE

$$\begin{cases} u'' + u = 0 \\ u(0) = 0 \text{ and } u(L) = 0 \end{cases}$$

Is the solution unique? Does the answer depend on  $L$ ?

*Proof.* We know the general solution space is exactly spanned by  $\cos x$  and  $\sin x$ . Because

(a)  $u(0) = 0$ .

(b)  $\sin 0 = 0$

(c)  $\cos 0 = 1$

we know the solution of our original ODE must be of the form

$$u(x) = C \sin x$$

This implies that the solution is unique if and only if  $2\pi \not\equiv L \pmod{2\pi}$  ■

### Question 50

Consider the ODE

$$\begin{cases} u'' + u' = f \\ u'(0) = u(0) = \frac{1}{2}(u'(l) + u(l)) \end{cases}$$

where  $f$  is given.

(a) Is the solution unique?

(b) Does a solution necessarily exist, or is there a condition that  $f$  must satisfy for

existence?

*Proof.* The solution space of linear homogeneous ODE  $u'' + u' = 0$  is spanned by  $e^{-x}$  and constant. If we add in the initial condition  $u'(0) = u(0)$ , then the solution space become the subspace spanned by  $e^{-x} - 2$ . One can check that if  $u \in \text{span}(e^{-x} - 2)$ , then

$$u(0) = \frac{1}{2}(u'(l) + u(l)) \text{ for all } l \in \mathbb{R}$$

We now know the solution of the original ODE is not unique, since any solution added by  $e^{-x} - 2$  is again a solution.

Integrating both side on  $[0, l]$ , we see that given the boundary conditions,  $f$  must satisfy

$$\begin{aligned} \int_0^l f(x)dx &= \int_0^l u'' + u'dx \\ &= u(l) + u'(l) - u(0) - u'(0) = 0 \end{aligned}$$

■

### Question 51

Consider the Neumann problem

$$\Delta u = f(x, y, z) \text{ in } D \text{ and } \frac{\partial u}{\partial n} = 0 \text{ on } \text{bdy } D$$

- (a) What can we add to any solution to get another solution?
- (b) Use the divergence Theorem and the PDE to show that

$$\iiint_D f(x, y, z) dxdydz = 0$$

is a necessary condition for the Neumann problem to have a solution.

*Proof.* Clearly, constants suffices, and observe

$$\iiint_D f dxdydz = \iiint_D \Delta u dxdydz = \iiint_D \nabla \cdot (\nabla u) dxdydz = \iint_{\text{bdy } D} \nabla u \cdot \mathbf{n} dS = 0$$

■

### Question 52

Consider the equation

$$u_x + yu_y = 0$$

with the boundary condition  $u(x, 0) = \phi(x)$ .

- (a)  $\phi(x) = x \implies$  no solution exists
- (b)  $\phi(x) = 1 \implies$  multiple solutions exist.

*Proof.* Using the geometric method, we see the characteristic curve is exactly  $y = \tilde{C}e^x$ . Thus the general solution is of the form

$$u(x, y) = f(e^{-x}y)$$

The boundary condition implies

$$\phi(x) = u(x, 0) = f(0)$$

The result then follows. ■

## 3.4 1.6 Types of Second-Order Equations

Consider the constant coefficient Linear PDE

$$u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0$$

Ignoring the term with order less than 2, we have

$$u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} = 0$$

Or equivalently

$$(\partial_x + a_{12}\partial_y)^2u + (a_{22} - a_{12}^2)\partial_{yy}u = 0$$

Now, if we set

$$x \triangleq \xi \text{ and } y \triangleq a_{12}\xi + (|a_{22} - a_{12}^2|)^{\frac{1}{2}}\eta$$

we see that

$$\begin{cases} a_{22} - a_{12}^2 > 0 \implies u_{\xi\xi} + u_{\eta\eta} = 0 & (\text{Elliptic}) \\ a_{22} - a_{12}^2 = 0 \implies u_{\xi\xi} = 0 & (\text{Parabolic}) \\ a_{22} - a_{12}^2 < 0 \implies u_{\xi\xi} - u_{\eta\eta} = 0 & (\text{Hyperbolic}) \end{cases}$$

More generally, if we are given

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} = 0$$

then the discriminant is exactly

$$a_{11}a_{22} - a_{12}^2$$

### Question 53

What is the type of each of the following equations.

(a)  $u_{xx} - u_{xy} + u_{yy} + \cdots + u = 0$ .

(b)  $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$

*Proof.* The discriminant for (a) and (b) are respectively  $\frac{3}{4}$  and 0, thus elliptic and parabolic. ■

### Question 54

Find the regions in the  $xy$  plane where the equation

$$(1+x)u_{xx} + 2xyu_{xy} - y^2u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch them.

*Proof.* The discriminant is exactly

$$\begin{aligned}(xy)^2 - (1+x)(-y^2) &= x^2y^2 + xy^2 + y^2 \\ &= y^2(x^2 + x + 1) \\ &= y^2\left[\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\right]\end{aligned}$$

It then follows that the equation is parabolic if and only if  $y = 0$ , and hyperbolic if and only if  $y \neq 0$ . ■

### Question 55

Reduce the elliptic equation

$$u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u = 0$$

to the form

$$v_{xx} + v_{yy} + cv = 0$$

by a change of dependent variable

$$u \triangleq ve^{\alpha x + \beta y}$$

then a change of scale

$$y' = \gamma y$$

*Proof.* The original elliptic equation can be written in the form

$$e^{\alpha x + \beta y} \left[ v_{xx} + 2\alpha v_x + \alpha^2 v + 3(v_{yy} + 2\beta v_y + \beta^2 v) - 2(v_x + \alpha v) + 24(v_y + \beta v) + 5v \right] = 0$$

which implies

$$v_{xx} + 3v_{yy} + (2\alpha - 2)v_x + (6\beta + 24)v_y + (\alpha^2 + 3\beta^2 - 2\alpha + 24\beta + 5)v = 0$$

Letting  $\alpha \triangleq 1$  and  $\beta \triangleq -4$ , we see

$$v_{xx} + 3v_{yy} + \tilde{c}v = 0$$

Letting  $y \triangleq \sqrt{3}y'$ , we then see

$$v_{xx} + v_{y'y'} + \tilde{c}v = 0$$

■

### Question 56

Consider the equation  $3u_y + u_{xy} = 0$ .

- (a) What is its type?
- (b) Find the general solution. (Hint: Substitute  $v = u_y$ ).
- (c) With the auxiliary conditions  $u(x, 0) = e^{-3x}$  and  $u_y(x, 0) = 0$ , does a solution exist? Is it unique?

*Proof.* Since the discriminant is exactly  $\frac{-1}{4}$ , the type is hyperbolic. Letting  $v \triangleq u_y$ , we see

$$3v + v_x = 0$$

This PDE on each horizontal line is an ODE and thus have the solution

$$u_y = v = f(y)e^{-3x}$$

Then  $u$  must be of the form

$$u = F(y)e^{-3x} + g(x)$$

Applying the initial condition  $u_y(x, 0) = 0$ , we see

$$f(0)e^{-3x} = u_y(x, 0) = 0$$

which implies  $f(0) = 0$ . Now apply another initial condition  $u(x, 0) = e^{-3x}$ .

$$F(0)e^{-3x} + g(x) = u(x, 0) = e^{-3x}$$

We then see the solutions exist and are not unique, since

$$\begin{cases} F(y) = 1 \\ g(x) = 0 \end{cases} \quad \text{and} \quad \begin{cases} F(y) = y^2 \\ g(x) = e^{-3x} \end{cases}$$

are both solutions.

■

## 3.5 2.1 The Wave Equation

---

### Abstract

In this section,  $c \in \mathbb{R}^*$ .

---

**Theorem 3.5.1. (General Solution of The Wave Equation)** The general solution of the wave equation

$$u_{tt} = c^2 u_{xx}$$

is of the form

$$u = f(x + ct) + g(x - ct)$$

*Proof.* Observe that

$$0 = u_{tt} - c^2 u_{xx} = (\partial_t + c\partial_x)(\partial_t - c\partial_x)u$$

If we let  $v = u_t - cu_x$ , then we must have  $v_t + cv_x = 0$ . We know the general solution of  $v$  is  $v = g(x - ct)$ . We have reduce the problem into solving

$$u_t - cu_x = g(x - ct) \quad (3.3)$$

Now observe that for all  $w : \mathbb{R} \rightarrow \mathbb{R}$

$$(\partial_t - c\partial_x)(w(x - ct)) = 2cw'(x - ct)$$

We then see the particular solution for [Equation 3.3](#) is

$$u = \frac{1}{2c}G(x - ct)$$

and the general solution is then

$$u = f(x + ct) + \frac{1}{2c}G(x - ct)$$

■

**Theorem 3.5.2. (IVP for The Wave Equation)** The Initial Value Problem

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x, 0) = \phi(x) \\ u_t(x, 0) = \psi(x) \end{cases}$$

has exactly one solution

$$u(x, t) = \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds$$



*Proof.* Write  $u(x, t) = f(x + ct) + g(x - ct)$ . By initial condition, we know

$$f(x) + g(x) = \phi(x) \text{ and } f'(x) - g'(x) = \frac{\psi(x)}{c}$$

Differentiating the former, we also have

$$f'(x) + g'(x) = \phi'(x)$$

This then give us

$$f'(x) = \frac{\phi'(x)}{2} + \frac{\psi(x)}{2c} \text{ and } g'(x) = \frac{\phi'(x)}{2} - \frac{\psi(x)}{2c}$$

It now follows that

$$f(s) = \frac{\phi(s)}{2} + \frac{1}{2c} \int_0^s \psi(x) dx + A \text{ and } g(s) = \frac{\phi(s)}{2} - \frac{1}{2c} \int_0^s \psi(x) dx + B$$

Note that since  $f(x) + g(x) = \phi(x)$ , we know  $B = -A$ .

We now have

$$\begin{aligned} u(x, t) &= f(x + ct) + g(x - ct) \\ &= \frac{\phi(x + ct) + \phi(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(x) dx \end{aligned}$$

■

### Question 57

Solve

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x, 0) = e^x \\ u_t(x, 0) = \sin x \end{cases}$$

*Proof.* Let

$$u \triangleq f(x + ct) + g(x - ct)$$

Plugging the initial conditions, we know

$$\begin{cases} f(x) + g(x) = e^x \\ f'(x) - g'(x) = \frac{\sin x}{c} \end{cases}$$

This give us

$$f'(x) = \frac{e^x + \frac{\sin x}{c}}{2} \text{ and } g'(x) = \frac{e^x - \frac{\sin x}{c}}{2}$$

Then by FTC, we have

$$\begin{cases} f(x) = \frac{e^x - \frac{\cos x}{c}}{2} + f(0) - \frac{1}{2} + \frac{1}{2c} \\ g(x) = \frac{e^x + \frac{\cos x}{c}}{2} + g(0) - \frac{1}{2} - \frac{1}{2c} \end{cases}$$

Note that  $f(0) + g(0) = e^0 = 1$ , which cancel the constant terms in  $u$ , i.e.

$$\begin{aligned} u &= f(x + ct) + g(x - ct) \\ &= \frac{e^{x+ct} + e^{x-ct} + \frac{-\cos(x+ct) + \cos(x-ct)}{c}}{2} \end{aligned}$$

■

### Question 58

If both  $\phi$  and  $\psi$  are odd functions of  $x$ , show that the solution of  $u(x, t)$  of the wave equation is also odd in  $x$  for all  $t$ .

*Proof.* Suppose

$$u = f(x + ct) + g(x - ct)$$

We have

$$\begin{cases} f + g = \phi \\ f' - g' = \frac{\psi}{c} \end{cases}$$

This give us

$$f' = \frac{\phi' + \frac{\psi}{c}}{2} \text{ and } g' = \frac{\phi' - \frac{\psi}{c}}{2}$$

and

$$\begin{cases} f(x) = \frac{1}{2} \int_0^x (\phi' + \frac{\psi}{c}) ds + f(0) \\ g(x) = \frac{1}{2} \int_0^x (\phi' - \frac{\psi}{c}) ds + g(0) \end{cases}$$

that is

$$\begin{cases} f(x) = f(0) + \frac{1}{2} \left[ \phi(x) - \phi(0) \right] + \frac{1}{2c} \int_0^x \psi(s) ds \\ g(x) = g(0) + \frac{1}{2} \left[ \phi(x) - \phi(0) \right] - \frac{1}{2c} \int_0^x \psi(s) ds \end{cases}$$

Noting  $f + g = \phi$ , we now have

$$\begin{aligned} u(x, t) &= f(x + ct) + g(x - ct) \\ &= \frac{\phi(x + ct) + \phi(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds \end{aligned}$$

and

$$\begin{aligned} u(-x, t) &= \frac{\phi(-x + ct) + \phi(-x - ct)}{2} + \frac{1}{2c} \int_{-x-ct}^{-x+ct} \psi(s) ds \\ &= \frac{-\phi(x - ct) - \phi(x + ct)}{2} - \frac{1}{2c} \int_{x+ct}^{x-ct} \psi(-u) du \quad (\because u = -s) \\ &= \frac{-\phi(x - ct) - \phi(x + ct)}{2} + \frac{1}{2c} \int_{x+ct}^{x-ct} \psi(u) du \quad (\because \psi \text{ is odd}) \\ &= \frac{-\phi(x - ct) - \phi(x + ct)}{2} - \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(u) du = -u(x, t) \end{aligned}$$

■

### Question 59

A spherical wave is a solution of the three-dimensional wave equation of the form  $u(r, t)$ , where  $r$  is the distance to the origin (the spherical coordinate). The wave equation takes the form

$$u_{tt} = c^2 \left( u_{rr} + \frac{2}{r} u_r \right)$$

- (a) Change variables  $v = ru$  to get the equation for  $v$  :  $v_{tt} = c^2 v_{rr}$ .
- (b) Solve for  $v$  and thereby solve the spherical wave equation.
- (c) Solve it with the initial condition  $u(r, 0) = \phi(r)$ ,  $u_t(r, 0) = \psi(r)$ , taking both  $\phi(r)$  and  $\psi(r)$  to be even functions of  $r$ .

*Proof.* If we let  $v = ru$ , then

$$v_{tt} = ru_{tt} \text{ and } v_{rr} = ru_{rr} + 2u_r$$

This then give us

$$v_{tt} = ru_{tt} = rc^2 \left( u_{rr} + \frac{2}{r} u_r \right) = c^2 (ru_{rr} + 2u_r) = c^2 v_{rr}$$

The general solution of  $v$  is

$$v = f(r + ct) + g(r - ct)$$

and thus

$$u(r, t) = \frac{f(ct + r) + g(r - ct)}{r}$$

Plugging the initial conditions, we have

$$\frac{f(r) + g(r)}{r} = u(r, 0) = \phi(r) \text{ and } \frac{c[f'(r) - g'(r)]}{r} = u_t(r, 0) = \psi(r)$$

In other words,

$$\begin{cases} f(r) + g(r) = r\phi(r) \\ f'(r) - g'(r) = \frac{r\psi(r)}{c} \end{cases}$$

Differentiating the first equation, we have

$$f'(r) + g'(r) = \phi(r) + r\phi'(r)$$

We now can solve  $f', g'$

$$f'(r) = \frac{\phi(r) + r\phi'(r) + \frac{r\psi(r)}{c}}{2} \text{ and } g'(r) = \frac{\phi(r) + r\phi'(r) - \frac{r\psi(r)}{c}}{2}$$

We now have

$$\begin{aligned} f(r) &= f(1) + \int_1^r f'(s)ds \\ &= f(1) + \left[ \frac{s\phi(s)}{2} \right] \Big|_{s=1}^r + \frac{1}{2c} \int_1^r s\psi(s)ds \end{aligned}$$

and

$$\begin{aligned} g(r) &= g(1) + \int_1^r g'(s)ds \\ &= g(1) + \left[ \frac{s\phi(s)}{2} \right] \Big|_{s=1}^r - \frac{1}{2c} \int_1^r s\psi(s)ds \end{aligned}$$

Noting that  $f(1) + g(1) = 1\phi(1)$ , we can cancel these terms and get

$$\begin{aligned} u(r, t) &= \frac{f(r + ct) + g(r - ct)}{r} \\ &= \frac{(r + ct)\phi(r + ct) + (r - ct)\phi(r - ct)}{2r} + \frac{1}{2cr} \int_1^{r+ct} s\phi(s)ds - \frac{1}{2cr} \int_1^{r-ct} s\phi(s)ds \\ &= \frac{(r + ct)\phi(r + ct) + (r - ct)\phi(r - ct)}{2r} + \frac{1}{2cr} \int_{r-ct}^{r+ct} s\phi(s)ds \end{aligned}$$

■

### Question 60

Solve

$$\begin{cases} u_{xx} + u_{xt} - 20u_{tt} = 0 \\ u(x, 0) = \phi(x) \\ u_t(x, 0) = \psi(x) \end{cases}$$

*Proof.* The PDE can be write in the form of

$$(\partial_x + 5\partial_t)(\partial_x - 4\partial_t)u = 0$$

which have the general solution

$$u(x, t) = f(5x - t) + g(4x + t)$$

We now see

$$f(5x) + g(4x) = \phi(x) \text{ and } -f'(5x) + g'(4x) = \psi(x)$$

This then give us

$$f'(5x) = \frac{(\phi' - 4\psi)(x)}{9} \text{ and } g'(4x) = \frac{(\phi' + 5\psi)(x)}{9}$$

and thus

$$\begin{aligned} f(x) &= f(0) + \int_0^x f'(s)ds \\ &= f(0) + \int_0^x \frac{(\phi' - 4\psi)(\frac{s}{5})}{9} ds \\ &= f(0) + \frac{5}{9} \left[ \phi\left(\frac{x}{5}\right) - \phi(0) \right] - \frac{4}{9} \int_0^x \psi\left(\frac{s}{5}\right) ds \end{aligned}$$

and similarly

$$\begin{aligned} g(x) &= g(0) + \int_0^x g'(s)ds \\ &= g(0) + \int_0^x \frac{(\phi' + 5\psi)(\frac{s}{4})}{9} ds \\ &= g(0) + \frac{4}{9} \left[ \phi\left(\frac{x}{4}\right) - \phi(0) \right] + \frac{5}{9} \int_0^x \psi\left(\frac{s}{4}\right) ds \end{aligned}$$

Noting that  $f(0) + g(0) = u(0, 0) = \psi(0)$ , we now have

$$\begin{aligned} u(x, t) &= f(5x - t) + g(4x + t) \\ &= \frac{5\phi(\frac{5x-t}{5}) + 4\phi(\frac{4x+t}{4})}{9} - \frac{4}{9} \int_0^{5x-t} \psi(\frac{s}{5}) ds + \frac{5}{9} \int_0^{4x+t} \psi(\frac{s}{4}) ds \end{aligned}$$

■

### Question 61

Find the general solution of

$$3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x + t)$$

*Proof.* Clearly, we can rewrite the equation to be

$$(3\partial_x + \partial_t)(\partial_x + 3\partial_t)u = \sin(x + t)$$

If we let  $v = u_x + 3u_t$ , then we have

$$3v_x + v_t = \sin(x + t)$$

Define

$$\gamma(x) \triangleq (x, \frac{x}{3} + c)$$

We then see

$$(v \circ \gamma)'(x) = \frac{\sin(x + \frac{x}{3} + c)}{3}$$

which implies

$$v(x, \frac{x}{3} + c) = v \circ \gamma(x) = \frac{\cos(\frac{4x}{3} + c)}{-4} + C_c$$

and thus

$$\begin{aligned} v(x, t) &= \frac{\cos(\frac{4x}{3} + t - \frac{x}{3})}{-4} + f(t - \frac{x}{3}) \\ &= \frac{\cos(x + t)}{-4} + f(3t - x) \end{aligned}$$

It remains to solve

$$u_x + 3u_t = \frac{\cos(x + t)}{-4} + f(3t - x)$$

Now define

$$\gamma(x) \triangleq (x, 3x + c)$$

We then see

$$(u \circ \gamma)'(x) = \frac{\cos(4x + c)}{-4} + f(8x + 3c)$$

which implies

$$u(x, 3x + c) = \frac{\sin(4x + c)}{-16} + \frac{F(8x + 3c)}{8} + u(0, c) + f(c)$$

and thus

$$u(x, t) = \frac{\sin(x + t)}{-16} + \tilde{F}(-x + 3t) + g(t - 3x)$$

where  $g$  is the initial condition. ■

## 3.6 2.2 Causality and Energy

### Question 62

Show that the wave equation has the following invariant properties

- (a) Any translate  $u(x - y, t)$  where  $y$  is fixed, is also a solution.
- (b) Any derivative, say  $u_x$ , is also a solution.
- (c) The dilated function  $u(ax, at)$  is also a solution.

*Proof.* The first property follows from direct computation, the second property follows from  $0_x = 0$  and the third property follows from observing  $v \triangleq u(ax, at)$  satisfy  $v_{tt} = a^2 u_{tt} = a^2 c^2 u_{xx} = v_{xx}$ . ■

### Question 63

If  $u(x, t)$  satisfy the wave equation  $u_{tt} = u_{xx}$ , prove the identity

$$u(x + h, t + k) + u(x - h, t - k) = u(x + k, t + h) + u(x - k, t - h)$$

*Proof.* Define  $\phi, \psi : \mathbb{R} \rightarrow \mathbb{R}$  by

$$\phi(x) \triangleq u(x, 0) \text{ and } \psi(x) \triangleq u_t(x, 0)$$

We then know that

$$\begin{aligned} u(x, t) &= \frac{\phi(x + t) + \phi(x - t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} \psi(s) ds \\ &\triangleq \frac{A(x, t) + B(x, t) + C(x, t)}{2} \end{aligned}$$

where

$$\begin{cases} A(x, t) \triangleq \phi(x + t) \\ B(x, t) \triangleq \phi(x - t) \\ C(x, t) \triangleq \int_{x-t}^{x+t} \psi(s) ds \end{cases}$$

The proof then follows from

$$\begin{aligned} A(x + h, t + k) &= A(x + k, t + h) \text{ and } A(x - h, t - k) = A(x - k, t - h) \\ B(x + h, t + k) &= B(x - k, t - h) \text{ and } B(x - h, t - k) = B(x + k, t + h) \\ C(x + h, t + k) &= C(x + k, t + h) \text{ and } C(x - h, t - k) = C(x - k, t - h) \end{aligned}$$



### Question 64

Suppose that we have the PDE of damped string

$$\begin{cases} \rho u_{tt} - Tu_{xx} + ru_t = 0 & \text{where } r > 0 \\ u(x, 0) = 0 & \text{if } |x| > N \end{cases}$$

Show that if we define the energy  $E(t)$  of this system by

$$E(t) = \int_{-\infty}^{\infty} (\rho u_t^2 + Tu_x^2) dx$$

Then the energy decrease as time goes.

*Proof.* Because  $u$  is smooth, we have

$$\begin{aligned} E'(t) &= \int_{-\infty}^{\infty} (\rho u_t^2 + Tu_x^2)_t dx \\ &= \int_{-\infty}^{\infty} (2\rho u_t u_{tt} + 2Tu_x u_{xt}) dx \\ &= \int_{-\infty}^{\infty} [2u_t(Tu_{xx} - ru_t) + 2Tu_x u_{xt}] dx \\ &= \int_{-\infty}^{\infty} [2T(u_t u_x)_x - 2ru_t^2] dx \\ &= 2Tu_t u_x \Big|_{x=0}^{\infty} - \int_{-\infty}^{\infty} 2ru_t^2 dx \\ &= - \int_{-\infty}^{\infty} 2ru_t^2 dx \leq 0 \end{aligned}$$

■

## 3.7 2.3 The Diffusion Equation

In this section, we shall first set

$$\Omega \triangleq (0, l) \text{ and } \Gamma \triangleq \Omega \times \{0\} \cup \partial\Omega \times [0, T]$$

and write

$$\Omega_T \triangleq \Omega \times (0, T)$$

We suppose  $u : \overline{\Omega_T} \rightarrow \mathbb{R}$  satisfy

$$u \in C^2(\Omega \times (0, T])$$

If  $u$  achieve a maximum on  $\Omega \times (0, T]$ , then at that point  $u$  must have

$$u_t \geq 0 \text{ and } u_{xx} \leq 0$$

**Theorem 3.7.1. (Weak Maximum Principle)** If

$$u_t - ku_{xx} \leq 0 \text{ on } \Omega \times (0, T] \tag{3.4}$$

then  $u$  must achieve its maximum at  $\Gamma$ .

*Proof.* Because  $\Gamma$  is compact, we know there exists a maximum  $M$  of  $u$  on  $\Gamma$ . Fix  $\epsilon$  and define  $v : \overline{\Omega_T} \rightarrow \mathbb{R}$

$$v(x, t) \triangleq u(x, t) + \epsilon x^2$$

Because

$$u(x, t) \leq \max_{\overline{\Omega_T}} v - \epsilon x^2 \text{ for all } (x, t) \in \overline{\Omega_T}$$

we can reduce the problem into proving

$$\max_{\overline{\Omega_T}} v \leq M + \epsilon l^2 \text{ for all } p \in F$$

From Equation 3.4, we can deduce

$$v_t - kv_{xx} = u_t - ku_{xx} - 2k\epsilon < 0$$

Because  $v$  is continuous, we know  $v$  attain its maximum at some point. Now, with diffusion inequality, we can deduce

- (a) The maximum of  $v$  must not be in  $\Omega_T$ , otherwise at that point  $v_t = 0$  and  $v_{xx} \leq 0$  yield a contradiction.

- (b) The maximum of  $v$  must also not be in the top edge  $\partial\Omega_T \setminus \Gamma$ , otherwise  $v_t \geq 0$  and  $v_{xx} \leq 0$  yield a contradiction.

We have proved that  $v$  can only attain maximum at some point  $(x_0, t_0) \in F_0$ , and it follows that

$$\max_{(x,t) \in F} v(x, t) = v(x_0, t_0) = u(x_0, t_0) + \epsilon x_0^2 \leq M + \epsilon l^2 \text{ (done)}$$

■

**Corollary 3.7.2. (Weak Minimum Principle)** The minimum of  $u$  must also happen on  $F_0$ .

Now, consider the Dirichlet's problem

$$\begin{cases} u_t - ku_{xx} = f(x, t) \text{ for } 0 < x < l \text{ and } t > 0 \\ u(x, 0) = \phi(x) \text{ for } 0 \leq x \leq l \\ u(0, t) = g(t) \text{ and } u(l, t) = h(t) \text{ for } t \geq 0 \end{cases} \quad (3.5)$$

Note that for all  $T$ , because the difference  $w$  of two solution  $u_1, u_2$  for Dirichlet's function must satisfy

$$\begin{cases} w_t = kw_{xx} \text{ on } \overline{\Omega_T} \setminus \Gamma \\ w(x, 0) = w(0, t) = 0 \text{ for any } 0 \leq x \leq l \text{ and } 0 \leq t \leq T \end{cases}$$

By minimum and maximum principle we can deduce  $w = 0$  on  $\Omega$ , and thus  $u_1 = u_2$  on  $F$ . It then follows that  $u_1 = u_2$  on  $[0, l] \times [0, \infty)$ .

**Theorem 3.7.3. (Uniqueness for the Dirichlet's problem for the diffusion equation with Energy Method)** If  $u_1, u_2 : [0, l] \times [0, \infty)$  are both solution of the Dirichlet's problem, then  $u_1 = u_2$ .

*Proof.* Define  $w : [0, l] \times [0, \infty) \rightarrow \mathbb{R}$  by  $w = u_1 - u_2$ . Multiplying  $w$  with  $(w_t - kw_{xx})$ , we see that for all  $x \in (0, l)$  and  $t > 0$ ,

$$0 = (w_t - kw_{xx})w = \left(\frac{w^2}{2}\right)_t + (-kw_x w)_x + kw_x^2$$

Because  $w(0, t) = w(l, t) = 0$  for all  $t$ , it follows that for all  $t > 0$

$$\begin{aligned} 0 &= \int_0^l \left[ \left(\frac{w^2}{2}\right)_t + (-kw_x w)_x + kw_x^2 \right] dx \\ &= \int_0^l \left[ \left(\frac{w^2}{2}\right)_t + kw_x^2 \right] dx \end{aligned}$$

which implies

$$I'(t) \leq 0 \text{ if we define } I : [0, \infty) \rightarrow \mathbb{R} \text{ by } I(t) \triangleq \int_0^l \left( \frac{w^2}{2} \right) dx$$

Because  $I(0) = 0$  by definition and  $I(t)$  are integrals of non-negative functions, we can deduce  $I$  is identically 0. The desired result  $w(x, t) = 0$  for all  $x, t \in [0, l] \times [0, \infty)$  then follows. ■

Now, consider **Dirichlet's problem** with different initial conditions  $\phi_1, \phi_2 : [0, l] \rightarrow \mathbb{R}$ , and suppose  $u_1, u_2 : [0, l] \times [0, \infty)$  are corresponding solutions. The maximum and minimum principle give us a  $L^\infty$  estimation for stability

$$\max_{[0, l] \times [0, \infty)} |u_1 - u_2| \leq \max_{[0, l]} |\phi_1 - \phi_2|$$

While the energy method give us a  $L^2$  estimation for stability: For all  $t \geq 0$ ,

$$\int_0^l \left( \frac{w^2(x, t)}{2} \right) dx = I(t) \leq I(0) = \int_0^l \left( \frac{w^2(x, 0)}{2} \right) dx = \int_0^l \frac{(\phi_1 - \phi_2)^2}{2} dx$$

### Question 65

Consider the diffusion equation

$$\begin{cases} u_t = u_{xx} \text{ on } (0, 1) \times (0, \infty) \\ u(0, t) = u(1, t) = 0 \text{ for all } t > 0 \\ u(x, 0) = 1 - x^2 \end{cases}$$

- (a) Show that  $u(x, t) > 0$  for all  $(x, t) \in (0, 1) \times (0, \infty)$ .
- (b) Define  $\mu : (0, \infty) \rightarrow \mathbb{R}$  by  $\mu(t) \triangleq \max_{x \in [0, 1]} u(x, t)$ . Show that  $\mu$  is a decreasing function.

*Proof.* The proof of (a) follows from a loose application of the strong minimum principle.

The proof of (b) follows from noting  $v(x, t) \triangleq u(x, t + t_0) : [0, 1] \times [0, \infty)$  also is a solution of the diffusion equation and application of maximum principle on  $v$ . ■

### Question 66

Consider the Dirichlet's problem

$$\begin{cases} u_t = u_{xx} \text{ on } (0, 1) \times (0, \infty) \\ u(0, t) = u(1, t) = 0 \text{ for all } t \geq 0 \\ u(x, 0) = 4x(1 - x) \end{cases}$$

Show that

- (a)  $0 < u(x, t) < 1$  for all  $t > 0$  and  $0 < x < 1$ .
- (b)  $u(x, t) = u(1 - x, t)$  for all  $t \geq 0$  and  $0 \leq x \leq 1$ .
- (c) Use the energy method to show that  $\int_0^1 u^2 dx$  is a strictly decreasing function of  $t$ .

*Proof.* (a) follows from application of strong maximum and minimum principle. (b) follows from uniqueness of Dirichlet's problem and the observation that  $u(1 - x, t)$  is also a solution.

The energy method give us

$$\frac{d}{dt} \int_0^1 \frac{u^2}{4} dx = - \int_0^1 u_x^2 dx \leq 0 \text{ for all } t > 0$$

and (c) follows. ■

### Question 67

Verify that

$$u = -2xt - x^2 \text{ is a solution of } u_t = xu_{xx}$$

and find the location of maximum of  $t$  in the close rectangle  $\{-2 \leq x \leq 2, 0 \leq t \leq 1\}$ .

*Proof.* Write

$$u = -(x + t)^2 + t^2$$

It follows that the maximum occurs at  $t = -x = 1$ . ■

### Question 68

Prove the comparison principle for the diffusion equation: If  $u$  and  $v$  are two solutions

and

$$u \leq v \text{ for } t = 0, x = 0, x = l$$

then

$$u \leq v \text{ on } [0, l] \times [0, \infty)$$

*Proof.* This follows from application of the minimum principle on  $v - u$ . ■

### Question 69

Suppose

$$\begin{cases} u_t - ku_{xx} = f \\ v_t - kv_{xx} = g \end{cases} \quad \text{and } f \leq g$$

and suppose

$$u \leq v \text{ at } x = 0, x = l \text{ and } t = 0$$

Prove that

$$u \leq v \text{ on } [0, l] \times [0, \infty)$$

*Proof.* Let  $w \triangleq u - v : \overline{\Omega_T} \rightarrow \mathbb{R}$ . It is clear that

$$w_t - kw_{xx} \leq 0 \text{ on } \overline{\Omega_T} \setminus \Gamma$$

It then follows that  $w$  attains its maximum on  $\Gamma$ , which must not be greater than 0. ■

## 3.8 2.4 Diffusion on the whole line

In this section, we are concerned with solving the following initial value problem (**Cauchy problem**)

$$\begin{cases} u_t = ku_{xx} \text{ on } \mathbb{R} \times (0, \infty) \\ \lim_{t \rightarrow 0} u(x, t) = \phi(x) \text{ for all specified } x \end{cases}$$

We shall mostly express our answer with function  $\text{erf} : \mathbb{R} \rightarrow \mathbb{R}$

$$\text{erf}(x) \triangleq \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2} dp$$

**Theorem 3.8.1. (Solution of Dirac Initial Condition)** If  $\phi$  is defined to be

$$\phi(x) \triangleq \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

then a solution is

$$Q(x, t) = \frac{1}{2} + \frac{\text{erf}\left(\frac{x}{\sqrt{4kt}}\right)}{2} \quad (3.6)$$

*Proof.* Note that our version of diffusion equation admits dilated solutions. This inspire us to guess

$$Q(x, t) \triangleq g\left(\frac{x}{\sqrt{4kt}}\right)$$

Direct computation yields

$$Q_t = \frac{-x}{2\sqrt{4kt}^{\frac{3}{2}}} g'\left(\frac{x}{\sqrt{4kt}}\right) \text{ and } Q_{xx} = g''\left(\frac{x}{\sqrt{4kt}}\right) \frac{1}{4kt}$$

If we let  $p = \frac{x}{\sqrt{4kt}}$ , we now have

$$Q_t = \frac{-pg'(p)}{2t} \text{ and } Q_{xx} = \frac{g''(p)}{4kt}$$

Plugging this back to diffusion equation and canceling the common terms, we have

$$\frac{g''(p)}{2} + pg'(p) = 0$$

The general solution to this ODE is

$$g(p) = c_1 \text{erf}(p) + c_2$$

In other words,

$$Q(x, t) = c_1 \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right) + c_2$$

Plugging this back to the initial condition, we see

$$Q(x, t) = \frac{1 + \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)}{2}$$

■

Differentiating [Equation 3.6](#) with respect to  $x$ , we have another solution

$$S(x, t) \triangleq \frac{1}{2\sqrt{\pi kt}} e^{-\frac{x^2}{4kt}}$$

Solution  $S$  is often called the **fundamental solution**, since for all initial condition  $\phi$  that have compact support, we gain a solution to the initial value problem by

$$u(x, t) \triangleq (S * \phi)(x, t)$$

where

$$(S * \phi)(x, t) = \int_{\mathbb{R}} S(x - y, t) \phi(y) dy$$

This is true because if we define  $F(x, y, t) = Q(x - y, t)$ , we have

$$\begin{aligned} u(x, t) &= \int_{\mathbb{R}} F_x(x, y, t) \phi(y) dy \\ &= \int_{\mathbb{R}} -F_y(x, y, t) \phi(y) dy \\ &= -F(x, y, t) \phi(y) \Big|_{y=-\infty}^{\infty} + \int_{\mathbb{R}} F(x, y, t) \phi'(y) dy \\ &= \int_{\mathbb{R}} Q(x - y, t) \phi'(y) dy \end{aligned}$$

and thus

$$\begin{aligned} \text{For all } x, \lim_{t \rightarrow 0} u(x, t) &= \int_{\mathbb{R}} \lim_{t \rightarrow 0} Q(x - y, t) \phi'(y) dy \\ &= \int_{-\infty}^x \phi'(y) dy = \phi(x) \end{aligned}$$



### Question 70

Solve the diffusion equation with the initial condition

$$\phi(x) \triangleq \begin{cases} 1 & \text{if } |x| < l \\ 0 & \text{if } |x| > l \end{cases}$$

*Proof.* Define

$$F(x, y, t) \triangleq Q(x - y, t)$$

The solution is exactly

$$\begin{aligned}
u(x, t) &= (S * \phi)(x, t) \\
&= \int_{\mathbb{R}} S(x - y, t) \phi(y) dy \\
&= \int_{-l}^l S(x - y, t) dy \\
&= \int_{-l}^l F_x(x, y, t) dy \\
&= \int_{-l}^l -F_y(x, y, t) dy = F(x, y, t) \Big|_{y=l}^{-l} = Q(x + l, t) - Q(x - l, t) = \frac{\operatorname{erf}(\frac{x+l}{\sqrt{4kt}}) - \operatorname{erf}(\frac{x-l}{\sqrt{4kt}})}{2}
\end{aligned}$$

■

### Question 71

Solve the diffusion equation with the initial condition

$$\phi(x) \triangleq \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$$

*Proof.* Define

$$F(x, y, t) \triangleq Q(x - y, t)$$

The solution is exactly

$$\begin{aligned}
u(x, t) &= (S * \phi)(x, t) \\
&= \int_{\mathbb{R}} S(x - y, t) \phi(y) dy \\
&= \int_0^\infty e^{-y} S(x - y, t) dy \\
&= \frac{1}{2\sqrt{t\pi k}} \int_0^\infty e^{\frac{-(x-y)^2}{4kt} - y} dy \\
&= \frac{1}{2\sqrt{t\pi k}} \int_0^\infty e^{\frac{-(y-(x-2kt))^2 + (x-2kt)^2 - x^2}{4kt}} dy \\
&= \frac{e^{kt-x}}{2\sqrt{t\pi k}} \int_0^\infty e^{\frac{-(y-(x-2kt))^2}{4kt}} dy \\
&= \frac{e^{kt-x}}{\sqrt{\pi}} \int_{\frac{2kt-x}{2\sqrt{kt}}}^\infty e^{-s^2} ds \quad (\because s = \frac{y - (x - 2kt)}{2\sqrt{kt}}) \\
&= \frac{e^{kt-x}}{\sqrt{\pi}} \left( \frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{2kt - x}{2\sqrt{kt}}\right) \right) \\
&= \frac{e^{kt-x}}{2} \left[ 1 - \operatorname{erf}\left(\frac{2kt - x}{2\sqrt{kt}}\right) \right]
\end{aligned}$$

■

### Question 72

Show that for any fixed  $\delta > 0$

$$\max_{\delta \leq |x| < \infty} S(x, t) \rightarrow 0 \text{ as } t \rightarrow 0$$

*Proof.*

■

### Question 73

Let  $\phi(x)$  be a continuous function such that  $|\phi(x)| \leq Ce^{ax^2}$ . Show that formula

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^\infty e^{\frac{-(x-y)^2}{4kt}} \phi(y) dy$$

for diffusion equation make sense for  $0 < t < \frac{1}{4ak}$  but not necessarily for larger  $t$ .

*Proof.*

■

### Question 74

Prove the uniqueness of the diffusion problem with Neumann boundary conditions:

$$u_t - ku_{xx} = f(x, t) \text{ for } 0 < x < l, t > 0$$

$$u(x, 0) = \phi(x)$$

$$u_x(0, t) = g(t) \text{ and } u_x(l, t) = h(t)$$

*Proof.* ■

### Question 75

Solve the diffusion equation with constant dissipation:

$$u_t - ku_{xx} + bu = 0 \text{ for } -\infty < x < \infty$$

$$u(x, 0) = \phi(x)$$

where  $b > 0$  is a constant. (Hint: Make the change of variables  $u(x, t) = e^{-bt}v(x, t)$ )

*Proof.* ■

### Question 76

Solve the diffusion equation with variable dissipation :

$$u_t - ku_{xx} + bt^2u = 0 \text{ for } -\infty < x < \infty$$

$$u(x, 0) = \phi(x)$$

where  $b > 0$  is a constant. (Hint: Make the change of variables  $u(x, t) = e^{\frac{-bt^3}{3}}v(x, t)$ )

*Proof.* ■

### Question 77

Solve the heat equation with convection:

$$u_t - ku_{xx} + Vu_x = 0 \text{ for } -\infty < x < \infty$$

$$u(x, 0) = \phi(x)$$

(Hint: Go to a moving frame of reference by substituting  $y = x - Vt$  )

*Proof.* ■

### Question 78

Show that  $S_x(x, y, t) \triangleq S(x, t)S(y, t)$  satisfy the diffusion equation  $S_t = k(S_{xx} + S_{yy})$ .

Deduce that  $S_2(x, y, t)$  is the source function for two-dimensional diffusion.

*Proof.*



### 3.9 3.1 Diffusion on the half line

Consider the following **Dirichlet problem**

$$\begin{cases} v_t - kv_{xx} = 0 & \text{for } 0 < x < \infty, 0 < t < \infty \\ \lim_{t \rightarrow 0} v(x, t) = \phi(x) & \text{for } 0 < x < \infty \\ v(0, t) = 0 & \text{for } 0 < t < \infty \end{cases}$$

where  $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}$ . Define  $\phi_{\text{odd}} : \mathbb{R} \rightarrow \mathbb{R}$  by

$$\phi_{\text{odd}}(x) \triangleq \begin{cases} \phi(x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\phi(x) & \text{if } x < 0 \end{cases}$$

It then follows that  $\phi_{\text{odd}}$  is an odd function, and we can solve the Cauchy problem with respect to this initial condition  $\phi_{\text{odd}}$  and have the solution

$$u(x, t) \triangleq \int_{-\infty}^{\infty} S(x - y, t) \phi_{\text{odd}}(y) dy$$

Now, because

$$S(x, t) = \frac{1}{2\sqrt{\pi kt}} e^{\frac{-x^2}{4kt}} \text{ is clearly even in } x$$

We can deduce

$$\begin{aligned} u(-x, t) &= \int_{-\infty}^{\infty} S(-x - y, t) \phi_{\text{odd}}(y) dy \\ &= - \int_{-\infty}^{\infty} S(-x - y, t) \phi_{\text{odd}}(-y) dy \\ &= - \int_{-\infty}^{\infty} S(x + y, t) \phi_{\text{odd}}(-y) dy \\ &= \int_{\infty}^{-\infty} S(x - z) \phi_{\text{odd}}(z) dz = -u(x, t) \quad (\because z = -y) \end{aligned}$$

In other words, we have deduced that  $u$  is an odd function in  $x$ . It then follows that  $u(0, t) = -u(-0, t) = 0$ . Then we see that the restriction  $v \triangleq u|_{(\mathbb{R}^+)^2}$  form a solution of the Dirichlet problem. In particular, we can express  $v$  in a form without usage of  $\phi_{\text{odd}}$

if we consider

$$\begin{aligned}
u(x, t) &= \int_{-\infty}^{\infty} S(x - y, t) \phi_{\text{odd}}(y) dy \\
&= \int_0^{\infty} S(x - y, t) \phi(y) dy + \int_{-\infty}^0 S(x - y, t) (-\phi(-y)) dy \\
&= \int_0^{\infty} [S(x - y, t) - S(x + y, t)] \phi(y) dy \\
&= \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left[ e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}} \right] \phi(y) dy
\end{aligned}$$

Now, consider the following **Neumann problem**

$$\begin{cases} w_t - kw_{xx} = 0 \text{ for } 0 < x < \infty, 0 < t < \infty \\ \lim_{t \rightarrow 0} w(x, t) = \phi(x) \text{ for } 0 \leq x < \infty \\ w_x(0, t) = 0 \text{ for } 0 < t < \infty \end{cases}$$

where  $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}$ . Define  $\phi_{\text{even}} : \mathbb{R} \rightarrow \mathbb{R}$  by

$$\phi_{\text{even}}(x) \triangleq \begin{cases} \phi(x) & \text{if } x \geq 0 \\ \phi(-x) & \text{if } x \leq 0 \end{cases}$$

It then follows that  $\phi_{\text{even}}$  is an even function, and we can solve the Cauchy problem with respect to this initial condition  $\phi_{\text{even}}$  and have the solution

$$u(x, t) \triangleq \int_{-\infty}^{\infty} S(x - y, t) \phi_{\text{even}}(y) dy$$

Again because  $S$  is even in  $x$ , we can deduce

$$\begin{aligned}
u(-x, t) &= \int_{-\infty}^{\infty} S(-x - y, t) \phi_{\text{even}}(y) dy \\
&= \int_{-\infty}^{\infty} S(-x - y, t) \phi_{\text{even}}(-y) dy \\
(\because z = -y) \quad &= - \int_{\infty}^{-\infty} S(-x + z, t) \phi_{\text{even}}(z) dz = u(x, t)
\end{aligned}$$

Now, we have proved that  $u$  is even in  $x$ . This then give  $u_x(0, t) = 0$ , and solve the **Neumann problem** by letting  $w \triangleq u|_{(\mathbb{R}^+)^2}$

### Question 79

Solve

$$\begin{aligned}u_t &= ku_{xx} \\ u(x, 0) &= e^{-x} \\ u(0, t) &= 0\end{aligned}$$

on the half line  $0 < x < \infty$

*Proof.* ■

### Question 80

Solve

$$\begin{aligned}u_t &= ku_{xx} \\ u(x, 0) &= 0 \\ u(0, t) &= 1\end{aligned}$$

on the half line  $0 < x < \infty$ .

*Proof.* ■

### Question 81

Consider the following problem with a Robin boundary condition:

$$\begin{aligned}u_t &= ku_{xx} \text{ on the half line } 0 < x < \infty \\ u(x, 0) &= x \\ u_x(0, t) - 2u(0, t) &= 0\end{aligned}$$

Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  by

$$f(x) \triangleq \begin{cases} x & \text{if } x \geq 0 \\ x + 1 - e^{2x} & \text{if } x < 0 \end{cases}$$

and let

$$v(x, t) \triangleq \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt}} f(y) dy$$

(a) What PDE and initial condition does  $v(x, t)$  satisfy for  $-\infty < x < \infty$ ?

- (b) Let  $w = v_x - 2v$ . What PDE and initial condition does  $w(x, t)$  satisfy  $-\infty < x < \infty$ ?
- (c) Show that  $f'(x) - 2f(x)$  is an odd function.
- (d) Show that  $w$  is an odd function of  $x$ .
- (e) Deduce that  $u$  satisfy the Robin condition.

*Proof.*



### Question 82

Generalize the method of the last exercises to the case of general initial data  $\phi(x)$  and arbitrary constant coefficient for  $u(0, t)$  in the boundary condition.

*Proof.*





## 3.10 3.2 Reflection of waves

We now consider the **Dirichlet's problem for wave on the half line**  $(0, \infty)$

$$\text{DE: } v_{tt} - c^2 v_{xx} = 0 \text{ for } 0 < x < \infty, -\infty < t < \infty$$

$$\text{IC: } v(x, 0) = \phi(x), v_t(x, 0) = \psi(x) \text{ for } 0 < x < \infty$$

$$\text{BC: } v(0, t) = 0 \text{ for } -\infty < t < \infty$$

One can check that if we again extend  $\phi, \psi : \mathbb{R}^+ \rightarrow \mathbb{R}$  to odd function  $\phi_{\text{odd}}, \psi_{\text{odd}} : \mathbb{R} \rightarrow \mathbb{R}$

$$\phi_{\text{odd}}(x) \triangleq \begin{cases} \phi(x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\phi(-x) & \text{if } x < 0 \end{cases} \quad \text{and} \quad \psi_{\text{odd}}(x) \triangleq \begin{cases} \psi(x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\psi(-x) & \text{if } x < 0 \end{cases}$$

and solve the **Cauchy's problem for wave on the whole line** with respect to them

$$u(x, t) \triangleq \frac{\phi_{\text{odd}}(x + ct) + \phi_{\text{odd}}(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{\text{odd}}(y) dy$$

then its restriction  $v \triangleq u|_{[0, \infty) \times \mathbb{R}}$  is again a solution to the Dirichlet's problem for wave on the half line, where the boundary condition follows from  $u$  being odd in  $x$  as easily checked.

Consider also the Neumann problem

$$\text{DE: } v_{tt} - c^2 v_{xx} = 0 \text{ for } 0 < x < \infty, -\infty < t < \infty$$

$$\text{IC: } v(x, 0) = \phi(x), v_t(x, 0) = \psi(x) \text{ for } 0 < x < \infty$$

$$\text{BC: } v_x(0, t) = 0 \text{ for } -\infty < t < \infty$$

# Chapter 4

## PDE HW

### 4.1 PDE HW 1

**Theorem 4.1.1.**

Show  $u \mapsto u_x + uu_y$  is non-linear

*Proof.* See that

$$2u \mapsto 2u_x + 4uu_y \neq 2(u_x + uu_y) \quad (4.1)$$

■

**Theorem 4.1.2.**

Solve  $(1 + x^2)u_x + u_y = 0$

*Proof.* The characteristic curve has the derivative

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

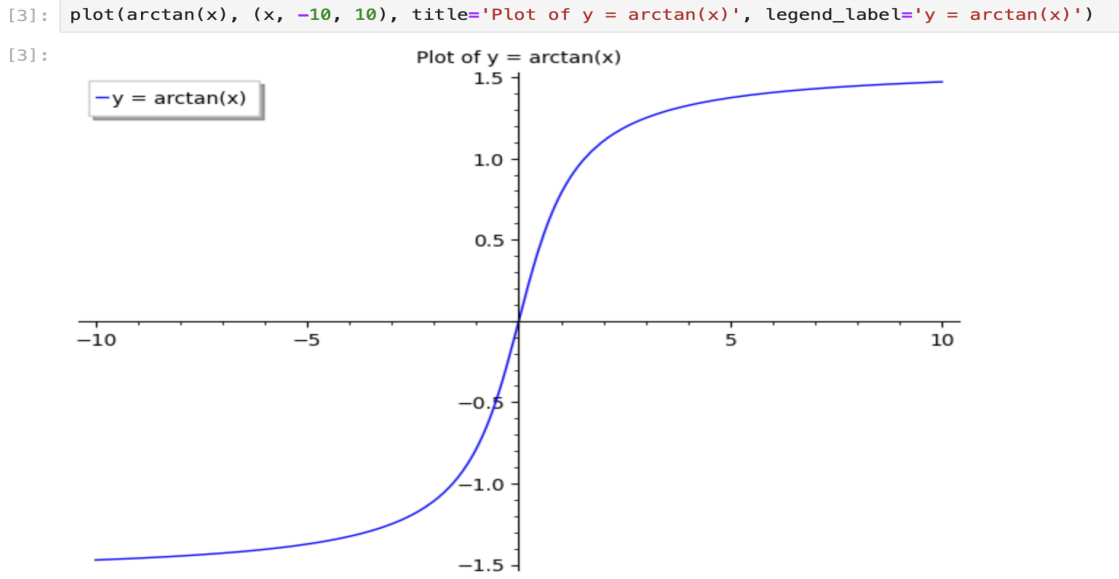
The solution to this ODE is

$$y = \arctan x + C$$

We now see that the solution to the PDE in [Equation 4.1](#) is

$u = f((\arctan x) - y)$  where  $f : \mathbb{R} \rightarrow \mathbb{R}$  is an arbitrary smooth function

A characteristic curve is as followed.



■

### Theorem 4.1.3.

$$\text{Solve } au_x + bu_y + cu = 0 \quad (4.2)$$

*Proof.* Fix

$$\begin{cases} x' \triangleq ax + by \\ y' \triangleq bx - ay \end{cases}$$

This map is clearly a diffeomorphism. Compute

$$\begin{cases} u_x = \frac{\partial u}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'} \frac{\partial y'}{\partial x} = au_{x'} + bu_{y'} \\ u_y = \frac{\partial u}{\partial x'} \frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'} \frac{\partial y'}{\partial y} = bu_{x'} - au_{y'} \end{cases}$$

Plugging it back into the PDE in [Equation 4.2](#), we have

$$cu + (a^2 + b^2)u_{x'} = 0 \quad (4.3)$$

If  $c = a^2 + b^2 = 0$ , then all smooth functions are solution. If  $a^2 + b^2 = 0$  but  $c \neq 0$ , then clearly the only solution is  $u = \tilde{0}$ . If  $a^2 + b^2 \neq 0$  but  $c = 0$ , then  $u_{x'} = \tilde{0}$ , which implies  $u = f(y')$  where  $y' = bx - ay$  and  $f$  can be arbitrary smooth function.

Now, suppose  $a^2 + b^2 \neq 0 \neq c$ , note that the PDE in [Equation 4.3](#) is just an ODE of the form

$$y + \frac{a^2 + b^2}{c}y' = 0$$

The general solution to this ODE is

$$y = Ce^{\frac{-ct}{a^2+b^2}}$$

In other words, the general solution of the PDE in [Equation 4.3](#) is

$$u = Ce^{\frac{-cx'}{a^2+b^2}} = Ce^{\frac{-c(ax+by)}{a^2+b^2}}$$



## 4.2 PDE HW 2

### Question 83

Consider heat flow in a long circular cylinder where the temperature depends only on  $t$  and on the distance  $r$  to the axis of the cylinder. Here  $r = \sqrt{x^2 + y^2}$  is the cylindrical coordinate. From the three dimensional heat equation derive the equation  $u_t = k(u_{rr} + \frac{u_r}{r})$

*Proof.* Write the three dimensional heat equation by

$$u_t = k\Delta u$$

Note that the Laplacian  $\Delta u$  when written in cylindrical coordinate is

$$\Delta u = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} + u_{zz}$$

Because the premise says that  $u$  is constant in  $z$  and  $\theta$ , we know  $u_{\theta\theta} = u_{zz} = 0$

$$\Delta u = u_{rr} + \frac{u_r}{r}$$

This give us

$$u_t = k(u_{rr} + \frac{u_r}{r})$$

■

## 4.3 PDE HW 3

### Question 84

Find a solution of

$$\begin{cases} u_t = u_{xx} \\ u(x, 0) = x^2 \end{cases}$$

*Proof.* Clearly  $u = x^2 + 2t$  suffices. ■

### Question 85

Consider the ODE

$$\begin{cases} u'' + u' = f \\ u'(0) = u(0) = \frac{1}{2}(u'(l) + u(l)) \end{cases}$$

where  $f$  is given.

- (a) Is the solution unique?
- (b) Does a solution necessarily exist, or is there a condition that  $f$  must satisfy for existence?

*Proof.* The solution space of linear homogeneous ODE  $u'' + u' = 0$  is spanned by  $e^{-x}$  and constant. If we add in the initial condition  $u'(0) = u(0)$ , then the solution space become the subspace spanned by  $e^{-x} - 2$ . One can check that if  $u \in \text{span}(e^{-x} - 2)$ , then

$$u(0) = \frac{1}{2}(u'(l) + u(l)) \text{ for all } l \in \mathbb{R}$$

We now know the solution of the original ODE is not unique, since any solution added by  $e^{-x} - 2$  is again a solution.

Integrating both side on  $[0, l]$ , we see that given the boundary conditions,  $f$  must satisfy

$$\begin{aligned} \int_0^l f(x)dx &= \int_0^l u'' + u'dx \\ &= u(l) + u'(l) - u(0) - u'(0) = 0 \end{aligned}$$
■

### Question 86

Find the regions in the  $xy$  plane where the equation

$$(1+x)u_{xx} + 2xyu_{xy} - y^2u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch them.

*Proof.* The discriminant is exactly

$$\begin{aligned}(xy)^2 - (1+x)(-y^2) &= x^2y^2 + xy^2 + y^2 \\ &= y^2(x^2 + x + 1) \\ &= y^2\left[\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\right]\end{aligned}$$

It then follows that the equation is parabolic if and only if  $y = 0$ , and hyperbolic if and only if  $y \neq 0$ . ■

## 4.4 PDE HW 4

### Question 87

Solve

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x, 0) = e^x \\ u_t(x, 0) = \sin x \end{cases}$$

*Proof.* Let

$$u \triangleq f(x + ct) + g(x - ct)$$

Plugging the initial conditions, we know

$$\begin{cases} f(x) + g(x) = e^x \\ f'(x) - g'(x) = \frac{\sin x}{c} \end{cases}$$

This give us

$$f'(x) = \frac{e^x + \frac{\sin x}{c}}{2} \text{ and } g'(x) = \frac{e^x - \frac{\sin x}{c}}{2}$$

Then by FTC, we have

$$\begin{cases} f(x) = \frac{e^x - \frac{\cos x}{c}}{2} + f(0) - \frac{1}{2} + \frac{1}{2c} \\ g(x) = \frac{e^x + \frac{\cos x}{c}}{2} + g(0) - \frac{1}{2} - \frac{1}{2c} \end{cases}$$

Note that  $f(0) + g(0) = e^0 = 1$ , which cancel the constant terms in  $u$ , i.e.

$$\begin{aligned} u &= f(x + ct) + g(x - ct) \\ &= \frac{e^{x+ct} + e^{x-ct} + \frac{-\cos(x+ct) + \cos(x-ct)}{c}}{2} \end{aligned}$$

■



### Question 88

Solve

$$\begin{cases} u_{xx} + u_{xt} - 20u_{tt} = 0 \\ u(x, 0) = \phi(x) \\ u_t(x, 0) = \psi(x) \end{cases}$$

*Proof.* The PDE can be write in the form of

$$(\partial_x + 5\partial_t)(\partial_x - 4\partial_t)u = 0$$

which have the general solution

$$u(x, t) = f(5x - t) + g(4x + t)$$

We now see

$$f(5x) + g(4x) = \phi(x) \text{ and } -f'(5x) + g'(4x) = \psi(x)$$

This then give us

$$f'(5x) = \frac{(\phi' - 4\psi)(x)}{9} \text{ and } g'(4x) = \frac{(\phi' + 5\psi)(x)}{9}$$

and thus

$$\begin{aligned} f(x) &= f(0) + \int_0^x f'(s)ds \\ &= f(0) + \int_0^x \frac{(\phi' - 4\psi)(\frac{s}{5})}{9} ds \\ &= f(0) + \frac{5}{9} \left[ \phi\left(\frac{x}{5}\right) - \phi(0) \right] - \frac{4}{9} \int_0^x \psi\left(\frac{s}{5}\right) ds \end{aligned}$$

and similarly

$$\begin{aligned} g(x) &= g(0) + \int_0^x g'(s)ds \\ &= g(0) + \int_0^x \frac{(\phi' + 5\psi)(\frac{s}{4})}{9} ds \\ &= g(0) + \frac{4}{9} \left[ \phi\left(\frac{x}{4}\right) - \phi(0) \right] + \frac{5}{9} \int_0^x \psi\left(\frac{s}{4}\right) ds \end{aligned}$$

Noting that  $f(0) + g(0) = u(0, 0) = \psi(0)$ , we now have

$$\begin{aligned} u(x, t) &= f(5x - t) + g(4x + t) \\ &= \frac{5\phi(\frac{5x-t}{5}) + 4\phi(\frac{4x+t}{4})}{9} - \frac{4}{9} \int_0^{5x-t} \psi(\frac{s}{5}) ds + \frac{5}{9} \int_0^{4x+t} \psi(\frac{s}{4}) ds \end{aligned}$$

■

# Chapter 5

## Differential Geometry HW

### 5.1 HW1

---

#### Abstract

In this HW, we give precise definition to  $\mathbb{P}^n$  and  $\mathbb{R}P^n$ , and we rigorously show

- (a)  $\mathbb{R}P^n$  has a smooth structure.
- (b)  $\mathbb{P}^n$  is homeomorphic to  $\mathbb{R}P^n$
- (c)  $\mathbb{P}^n$  has a smooth structure.

We also solved [the other two questions](#). Note that in this PDF, brown text is always a clickable hyperlink reference.

---

Define an equivalence relation on  $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$  by

$$\mathbf{x} \sim \mathbf{y} \stackrel{\Delta}{\iff} \mathbf{y} = \lambda \mathbf{x} \text{ for some } \lambda \in \mathbb{R}^*$$

Let  $\mathbb{R}P^n \triangleq (\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}) / \sim$  be the quotient space and let

$$V_i \triangleq \{\mathbf{x} \in \mathbb{R}^{n+1} : x^i \neq 0\} \text{ for each } 1 \leq i \leq n+1$$

By definition, it is clear that

$$\text{either } \pi^{-1}(\pi(\mathbf{x})) \subseteq V_i \text{ or } \pi^{-1}(\pi(\mathbf{x})) \subseteq V_i^c$$

Then if we define  $\phi_i : V_i \rightarrow \mathbb{R}^n$  by

$$\phi_i(\mathbf{x}) \triangleq \left( \frac{\mathbf{x}^1}{\mathbf{x}^i}, \dots, \frac{\mathbf{x}^{i-1}}{\mathbf{x}^i}, \frac{\mathbf{x}^{i+1}}{\mathbf{x}^i}, \dots, \frac{\mathbf{x}^{n+1}}{\mathbf{x}^i} \right)$$

because  $\pi(\mathbf{x}) = \pi(\mathbf{y}) \implies \phi_i(\mathbf{x}) = \phi_i(\mathbf{y})$ , we can well induce a map

$$\Phi_i : U_i \triangleq \pi(V_i) \subseteq \mathbb{R}P^n \rightarrow \mathbb{R}^n; \pi(\mathbf{x}) \mapsto \phi_i(\mathbf{x})$$

Note that one has the equation

$$\Phi_i(\pi(\mathbf{x})) = \phi_i(\mathbf{x}) \text{ for all } \mathbf{x} \in V_i$$

**Theorem 5.1.1. (Real Projective Space with a differentiable atlas)** We have

$\mathbb{R}P^n$  with atlas  $\{(U_i, \Phi_i) : 1 \leq i \leq n+1\}$  is a differentiable manifold

*Proof.* We are required to prove

- (a)  $(U_i, \Phi_i)$  are all charts.
- (b)  $\{(U_i, \Phi_i) : 1 \leq i \leq n+1\}$  form a differentiable atlas.
- (c)  $\mathbb{R}P^n$  is Hausdorff.
- (d)  $\mathbb{R}P^n$  is second-countable.

Because  $\pi^{-1}(U_i) = V_i$  and  $V_i$  is clearly open in  $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ , we know  $U_i \subseteq \mathbb{R}P^n$  is open. Note that clearly,  $\Phi_i(U_i) = \mathbb{R}^n$ . To show  $(U_i, \Phi_i)$  is a chart, it remains to show that  $\Phi_i$  is a homeomorphism between  $U_i$  and  $\mathbb{R}^n$ . It is straightforward to check  $\Phi_i$  is one-to-one on  $U_i$ . This implies  $\Phi_i$  is a bijective between  $U_i$  and  $\mathbb{R}^n$ .

Fix open  $E \subseteq \mathbb{R}^n$ . We see

$$\pi^{-1}(\Phi_i^{-1}(E)) = \phi_i^{-1}(E)$$

Then because  $\phi_i : V_i \rightarrow \mathbb{R}^n$  is clearly continuous, we see  $\phi_i^{-1}(E)$  is open in  $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ , and it follows from definition of quotient topology  $\Phi_i^{-1}(E) \subseteq \mathbb{R}P^n$  is open. Then because  $U_i$  is open in  $\mathbb{R}P^n$ , we see  $\Phi_i^{-1}(E)$  is open in  $U_i$ . We have proved  $\Phi_i : U_i \rightarrow \mathbb{R}^n$  is continuous.

Define  $\Psi_i : \mathbb{R}^n \rightarrow V_i$  by

$$\Psi(\mathbf{x}^1, \dots, \mathbf{x}^n) = (\mathbf{x}^1, \dots, \mathbf{x}^{i-1}, 1, \mathbf{x}^i, \dots, \mathbf{x}^n)$$

Observe that for all  $\mathbf{x} \in \Phi_i(U_i)$ , we have

$$\Phi_i^{-1}(\mathbf{x}) = \pi(\Psi_i(\mathbf{x}))$$

It then follows from  $\Psi_i : \mathbb{R}^n \rightarrow V_i$  and  $\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}P^n$  are continuous that  $\Phi_i^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}P^n$  is continuous.

We have proved that  $(\Psi_i, U_i)$  are all charts. Now, because  $V_i$  clearly cover  $\mathbb{R}^{n+1}$ , we know  $U_i$  also cover  $\mathbb{R}P^n$ . We have proved  $\{(U_i, \Phi) : 1 \leq i \leq n+1\}$  form an atlas. The fact  $\mathbb{R}P^n$  is second-countable follows.

Fix  $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \Phi_i(U_i \cap U_j)$ . We compute

$$\begin{aligned} \Phi_j \circ \Phi_i^{-1}(\mathbf{x}^1, \dots, \mathbf{x}^n) &= \Phi_j\left([\mathbf{x}^1, \dots, \mathbf{x}^{i-1}, 1, \mathbf{x}^i, \mathbf{x}^{i+1}, \dots, \mathbf{x}^n]\right) \\ &= \begin{cases} \left(\frac{\mathbf{x}^1}{\mathbf{x}^j}, \dots, \frac{\mathbf{x}^{j-1}}{\mathbf{x}^j}, \frac{\mathbf{x}^{j+1}}{\mathbf{x}^j}, \dots, \frac{\mathbf{x}^{i-1}}{\mathbf{x}^j}, \frac{1}{\mathbf{x}^j}, \frac{\mathbf{x}^i}{\mathbf{x}^j}, \dots, \frac{\mathbf{x}^n}{\mathbf{x}^j}\right) & \text{if } j < i \\ \left(\frac{\mathbf{x}^1}{\mathbf{x}^{j-1}}, \dots, \frac{\mathbf{x}^{i-1}}{\mathbf{x}^{j-1}}, \frac{1}{\mathbf{x}^{j-1}}, \frac{\mathbf{x}^i}{\mathbf{x}^{j-1}}, \dots, \frac{\mathbf{x}^{j-2}}{\mathbf{x}^{j-1}}, \frac{\mathbf{x}^j}{\mathbf{x}^{j-1}}, \dots, \frac{\mathbf{x}^n}{\mathbf{x}^{j-1}}\right) & \text{if } j > i \end{cases} \end{aligned}$$

This implies our atlas is indeed differentiable.

Before we prove  $\mathbb{R}P^n$  is Hausdorff, we first prove that  $\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}P^n$  is an open mapping. Let  $U \subseteq \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$  be open. Observe that

$$\pi^{-1}(\pi(U)) = \{t\mathbf{x} \in \mathbb{R}^{n+1} : t \neq 0 \text{ and } \mathbf{x} \in U\}$$

Fix  $t_0\mathbf{x} \in \pi^{-1}(\pi(U))$ . Let  $B_\epsilon(\mathbf{x}) \subseteq U$ . Observe that

$$B_{|t_0|\epsilon}(t_0\mathbf{x}) \subseteq t_0B_\epsilon(\mathbf{x}) \subseteq t_0U \subseteq \pi^{-1}(\pi(U))$$

This implies  $\pi^{-1}(\pi(U))$  is open. (done)

Now, because  $\pi$  is open, to show  $\mathbb{R}P^n$  is Hausdorff, we only have to show

$$R_\pi \triangleq \{(\mathbf{x}, \mathbf{y}) \in (\mathbb{R}^{n+1} \setminus \{\mathbf{0}\})^2 : \pi(\mathbf{x}) = \pi(\mathbf{y})\} \text{ is closed}$$

Define  $f : (\mathbb{R}^{n+1} \setminus \{\mathbf{0}\})^2 \rightarrow \mathbb{R}$  by

$$f(\mathbf{x}, \mathbf{y}) \triangleq \sum_{i \neq j} (\mathbf{x}^i \mathbf{y}^j - \mathbf{x}^j \mathbf{y}^i)^2$$

Note that  $f$  is clearly continuous and  $f^{-1}(0) = R_\pi$ , which finish the proof. ■

Alternatively, we can characterize  $\mathbb{R}P^n$  by identifying the antipodal points on  $S^n \triangleq \{\mathbf{x} \in \mathbb{R}^{n+1} : |\mathbf{x}| = 1\}$  as one point

$$\mathbf{x} \sim \mathbf{y} \iff \mathbf{x} = \mathbf{y} \text{ or } \mathbf{x} = -\mathbf{y}$$

and let  $\mathbb{P}^n \triangleq S^n \setminus \sim$  be the quotient space.

### Theorem 5.1.2. (Equivalent Definitions of Real Projective Space)

$\mathbb{R}P^n$  and  $\mathbb{P}^n$  are homeomorphic

*Proof.* Define  $F : \mathbb{P}^n \rightarrow \mathbb{R}P^n$  by

$$\{\mathbf{x}, -\mathbf{x}\} \mapsto \{\lambda \mathbf{x} : \lambda \in \mathbb{R}^*\}$$

It is straightforward to check that  $F$  is well-defined and bijective. Define  $f : S^n \rightarrow \mathbb{R}P^n$  by

$$f = \pi \circ \text{id}$$

where  $\text{id} : S^n \rightarrow \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$  and  $\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}P^n$  are continuous. Check that

$$f = F \circ p$$

where  $p : S^n \rightarrow \mathbb{P}^n$  is the quotient mapping. It now follows from the universal property that  $F$  is continuous, and since  $\mathbb{P}^n$  is compact and  $\mathbb{R}P^n$  is Hausdorff, it also follows that  $F$  is a homeomorphism between  $\mathbb{R}P^n$  and  $\mathbb{P}^n$ . ■

Knowing that  $F : \mathbb{P}^n \rightarrow \mathbb{R}P^n$  is a homeomorphism and  $\mathbb{R}P^n$  is a smooth manifold, we see that  $\mathbb{P}^n$  is Hausdorff and second-countable, and if we define the atlas

$$\{(F^{-1}(U_i), \Phi_i \circ F) : 1 \leq i \leq n+1\}$$

We see this atlas is indeed smooth, since

$$(\Phi_i \circ F) \circ (\Phi_j \circ F)^{-1} = \Phi_i \circ \Phi_j^{-1}$$

### Question 89

Let  $X$  be a set equipped with

- (a) a collection  $(U_\alpha)_{\alpha \in I}$  of subsets that covers  $X$ .
- (b) a collection of bijection  $\phi_\alpha : U_\alpha \rightarrow \phi_\alpha(U_\alpha)$  that maps  $U_\alpha$  to an open subset  $\phi_\alpha(U_\alpha)$  of  $\mathbb{R}^n$ .
- (c) For each  $\alpha, \beta \in I$ , the set  $\phi_\alpha(U_\alpha \cap U_\beta)$  is open.
- (d) For each  $\alpha, \beta \in I$ ,  $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)$  is smooth.

Give  $X$  a topology so that  $X$  is a smooth manifold.

*Proof.* If we define  $E \subseteq X$  is open if and only if

$$\phi_\alpha(U_\alpha \cap E) \text{ is open for all } \alpha$$

we see that given arbitrary collection of open sets  $(E_j)_{j \in J}$ , we have

$$\phi_\alpha(U_\alpha \cap \bigcup_{j \in J} E_j) = \bigcup_{j \in J} \phi_\alpha(U_\alpha \cap E_j) \text{ for all } \alpha \in I$$

thus open sets are closed under union, and given two open sets  $E_1, E_2$ , we have

$$\phi_\alpha(U_\alpha \cap E_1 \cap E_2) \subseteq \phi_\alpha(U_\alpha \cap E_1) \cap \phi_\alpha(U_\alpha \cap E_2) \text{ for all } \alpha \in I$$

Note that if  $\mathbf{x} \in \phi_\alpha(U_\alpha \cap E_1) \cap \phi_\alpha(U_\alpha \cap E_2)$ , then there exists  $p_1 \in U_\alpha \cap E_1$  and  $p_2 \in U_\alpha \cap E_2$  such that  $\phi_\alpha(p_1) = \phi_\alpha(p_2) = \mathbf{x}$ . Because  $\phi_\alpha$  is one-to-one, we can deduce  $p_1 = p_2 \in E_2$ , it then follows

$$\mathbf{x} = \phi(p_1) \in \phi_\alpha(U_\alpha \cap E_1 \cap E_2)$$

We now see

$$\phi_\alpha(U_\alpha \cap E_1) \cap \phi_\alpha(U_\alpha \cap E_2) \subseteq \phi_\alpha(U_\alpha \cap E_1 \cap E_2) \text{ for all } \alpha \in I$$

We have proved that our topology on  $X$  is well-defined.

Note that  $U_\alpha$  is open in  $X$  follows from premise (c). Thus, if some  $E \subseteq U_\alpha$  is open in  $U_\alpha$ , then  $E$  is open in  $X$  and  $\phi_\alpha(E) = \phi_\alpha(U_\alpha \cap E)$  is open. We have proved that  $\phi_\alpha : U_\alpha \rightarrow \phi_\alpha(U_\alpha)$  is an open mapping. The fact that  $\phi_\alpha : U_\alpha \rightarrow \phi_\alpha(U_\alpha)$  is continuous trivially follows from

- (a)  $U_\alpha$  is open in  $X$ .
- (b) our definition of topology on  $X$ .
- (c)  $\phi_\alpha : U_\alpha \rightarrow \phi_\alpha(U_\alpha)$  is a bijection.

We have proved that  $(U_\alpha, \phi_\alpha)$  are indeed charts. The fact that they form a smooth atlas follows from premise (a) and (d). ■

### Question 90

Let  $\mathbb{R}$  be the real line with the differentiable structure given by the maximal atlas of the chart  $(\mathbb{R}, \varphi = \mathbf{id} : \mathbb{R} \rightarrow \mathbb{R})$ , where  $\mathbf{id}$  is the identity map, and let  $\mathbb{R}'$  be the real line with the differentiable structure given by the maximal atlas of the chart  $(\mathbb{R}', \psi : \mathbb{R}' \rightarrow \mathbb{R})$ , where  $\psi(x) = x^{1/3}$ .

- (a) Show that these two differentiable structures are distinct.
- (b) Show that there is a diffeomorphism between  $\mathbb{R}$  and  $\mathbb{R}'$ . (Hint: The identity map  $\mathbb{R} \rightarrow \mathbb{R}$  is not the desired diffeomorphism.)

*Proof.* To see these two smooth structure are distinct. Observe

$$\mathbf{id}^{-1} \circ \psi : \mathbb{R} \rightarrow \mathbb{R}; x \mapsto x^{\frac{1}{3}} \text{ is not smooth at } 0$$

We claim  $\phi : \mathbb{R} \mapsto \mathbb{R}'$  defined by

$$\phi(x) \triangleq x^3 \text{ is a diffeomorphism}$$

It is clear that  $\phi$  is a homeomorphism. To see  $\phi$  is a smooth mapping from  $\mathbb{R}$  to  $\mathbb{R}'$ , observe that

$$\psi \circ \phi \circ \mathbf{id}^{-1}(x) = x$$

To see  $\phi^{-1}$  is a smooth mapping from  $\mathbb{R}'$  to  $\mathbb{R}$ , observe that

$$\mathbf{id} \circ \phi \circ \psi^{-1}(x) = x$$

We have proved that  $\phi$  is a diffeomorphism between  $\mathbb{R}$  and  $\mathbb{R}'$ . ■



## 5.2 Appendix

**Theorem 5.2.1. (Homeomorphism between Compact Space and Hausdorff Space)**  
Suppose

- (a)  $X$  is compact.
- (b)  $Y$  is Hausdorff.
- (c)  $f : X \rightarrow Y$  is a continuous bijective function.

Then

$f$  is a homeomorphism between  $X$  and  $Y$

*Proof.* Because closed subset of compact set is compact and continuous function send compact set to compact set, we see for each closed  $E \subseteq X$ ,  $f(E) \subseteq Y$  is compact. The result then follows from  $f(E) \subseteq Y$  being closed since  $Y$  is Hausdorff. ■

**Theorem 5.2.2. (Hausdorff and Quotient)** If  $\pi : X \rightarrow Y$  is an open mapping, and we define

$$R_\pi \triangleq \{(x, y) \in X^2 : \pi(x) = \pi(y)\}$$

Then

$$R_\pi \text{ is closed} \iff Y \text{ is Hausdorff}$$

*Proof.* Suppose  $R_\pi$  is closed. Fix some  $x, y$  such that  $\pi(x) \neq \pi(y)$ . Because  $R_\pi$  is closed, we know there exists open neighborhood  $U_x, U_y$  such that  $U_x \times U_y \subseteq (R_\pi)^c$ . It is clear that  $\pi(U_x), \pi(U_y)$  are respectively open neighborhood of  $\pi(x)$  and  $\pi(y)$ . To see  $\pi(U_x)$  and  $\pi(U_y)$  are disjoint, **assume** that  $\pi(a) \in \pi(U_x) \cap \pi(U_y)$ . Let  $a_x \in U_x$  and  $a_y \in U_y$  satisfy  $\pi(a_x) = \pi(a) = \pi(a_y)$ , which is impossible because  $(a_x, a_y) \in (R_\pi)^c$ . **CaC**

Suppose  $Y$  is Hausdorff. Fix some  $x, y$  such that  $\pi(x) \neq \pi(y)$ . Let  $U_x, U_y$  be open neighborhoods of  $\pi(x), \pi(y)$  separating them. Observe that  $(x, y) \in \pi^{-1}(U_x) \times \pi^{-1}(U_y) \subseteq (R_\pi)^c$  ■

### 5.3 Example: $S^1, \mathbb{R} \setminus \mathbb{Z}$ diffeomorphism

Equip  $S^1 \triangleq \{(x, y) \in \mathbb{R}^2 : |(x, y)| = 1\}$  with the standard four projection chart smooth atlas as one of them being

$$V \triangleq \{(x, y) \in \mathbb{R}^2 : y > 0\} \text{ and } \phi_V : V \rightarrow \mathbb{R}; (x, y) \mapsto x$$

Let  $p : \mathbb{R} \rightarrow \mathbb{R} \setminus \mathbb{Z}$  be the quotient map and let

$$U_0 \triangleq p\left((0, 1)\right) \text{ and } U_1 \triangleq p\left(\left(-\frac{1}{2}, \frac{1}{2}\right)\right)$$

which are both open as one can readily check. Define  $\phi_0 : U_0 \rightarrow (0, 1)$  by

$$\phi_0(p(t)) \triangleq t_0 \text{ where } t_0 \in (0, 1) \text{ and } p(t_0) = p(t)$$

and  $\phi_1 : U_1 \rightarrow (-\frac{1}{2}, \frac{1}{2})$  by

$$\phi_1(p(t)) \triangleq t_0 \text{ where } t_0 \in \left(-\frac{1}{2}, \frac{1}{2}\right) \text{ and } p(t_0) = p(t)$$

Clearly, the function  $G : \mathbb{R} \setminus \mathbb{Z} \rightarrow S^1$  well-defined by  $G(p(x)) \triangleq (\cos 2\pi x, \sin 2\pi x)$  is a homeomorphism, as one can check that

- (a)  $G$  is a continuous bijection. (Using Universal property of quotient map)
- (b)  $\mathbb{R} \setminus \mathbb{Z}$  is compact. (by finite sub-cover definition)
- (c)  $S^1$  is Hausdorff.

We now compute that  $\phi_V \circ G \circ \phi_0^{-1}$  is defined on whole  $(0, 1)$ , and is exactly

$$\phi_V \circ G \circ \phi_0^{-1}(t) \triangleq \cos 2\pi t \text{ smooth}$$