



Notation: \mathbb{R} is the set of real numbers and \mathbb{C} is the set of complex numbers. If $F = \mathbb{R}$ or \mathbb{C} and n is a positive integer, we denote by $M_n(F)$ the set of $n \times n$ matrices with entries in F and by I_n the identity matrix in $M_n(F)$.

Problem 1 (15 pts). Let

$\mathbf{v}_1 = (1, 2, 0, 4)$, $\mathbf{v}_2 = (-1, 1, 3, -3)$, $\mathbf{v}_3 = (0, 1, -5, -2)$, $\mathbf{v}_4 = (-1, -9, -1, -4)$ be vectors in \mathbb{R}^4 . Let W_1 be the subspace spanned by \mathbf{v}_1 and \mathbf{v}_2 and let W_2 be the subspace spanned by \mathbf{v}_3 and \mathbf{v}_4 . Find the dimension and a basis of $W_1 \cap W_2$.

Problem 2. Let

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 3 & -4 & 1 \\ 3 & -8 & 5 \end{pmatrix}.$$

(1) (10pts) Find an invertible matrix $Q \in M_3(\mathbb{C})$ such that

$$Q^{-1}AQ = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 12 \\ 0 & 1 & 1 \end{pmatrix}.$$

(2) (15pts) Find an invertible matrix $P \in M_3(\mathbb{C})$ such that $P^{-1}AP$ is a diagonal matrix.

Problem 3. For any $A \in M_2(\mathbb{C})$, define

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.$$

(1) (5pts) Evaluate $\sin \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$.

(2) (15pts) Prove or disprove: There exists $A \in M_2(\mathbb{R})$ such that

$$\sin A = \begin{pmatrix} 1 & 2022 \\ 0 & 1 \end{pmatrix}.$$

Problem 4 (20pts). Let $A = (a_{ij}) \in M_n(\mathbb{C})$ and let $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$ be roots of characteristic polynomial of A (counted with multiplicity). Show that

$$AA^* = A^*A \text{ if and only if } \sum_{1 \leq i, j \leq n} |a_{ij}|^2 = \sum_{k=1}^n |\lambda_k|^2.$$

Problem 5 (20pts). Let $A, B \in M_n(\mathbb{C})$. Suppose that all of the eigenvalues of A and B are positive real numbers. If $A^4 = B^4$, prove that $A = B$.

Problem 2. Cyclic subspace, cyclic vector, companion matrix

(1)

$$\text{det}(A - xI) = \begin{vmatrix} -x & 0 & 0 \\ 1 & -x & 12 \\ 0 & 1 & 1-x \end{vmatrix} = -x^3 + x^2 + 12x = -x(x-4)(x+3)$$

eigenvalue of A is 0, 4, -3, resp. vector is

$$(0, \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{v_1}), (4, \underbrace{\begin{pmatrix} -7 \\ 1 \\ 29 \end{pmatrix}}_{v_2}), (-3, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}_{v_3}), \text{ let } v = v_1 + v_2 + v_3,$$

$$\text{Then } \{v, T v, T^2 v\} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 4 & 16 \\ 1 & -3 & 9 \end{pmatrix} \text{ (use } \{v_1, v_2, v_3\} \text{ basis), which det}$$

is not zero, Thus v is a cyclic vector, Moreover, resp. this basis,

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 12 \\ 0 & 1 & 1 \end{pmatrix}, Q = \begin{pmatrix} -6 & -28 & -112 \\ 3 & 1 & 25 \\ 31 & 113 & 473 \end{pmatrix}$$

(2) Diagonal

$$\text{Use previous result, } \lambda_A = 0, -3, 4, \text{ so } P = \begin{pmatrix} 1 & -7 & 0 \\ 1 & 1 & 1 \\ 1 & 29 & 1 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

Problem 1 $v \in W_1 \cap W_2 \Leftrightarrow \exists a, b, c, d \in \mathbb{F}$ such that

$$a \begin{pmatrix} 1 \\ 2 \\ 0 \\ 4 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \\ 3 \\ -3 \end{pmatrix} = c \begin{pmatrix} 0 \\ 1 \\ -5 \\ -2 \end{pmatrix} + d \begin{pmatrix} -1 \\ -9 \\ -1 \\ -4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 0 & -1 \\ 2 & 1 & 1 & -9 \\ 0 & 3 & -5 & -1 \\ 4 & -3 & -2 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \\ -c \\ -d \end{pmatrix} = \vec{0} \Rightarrow \begin{pmatrix} a \\ b \\ -c \\ -d \end{pmatrix} \in \ker \begin{pmatrix} 1 & -1 & 0 & -1 \\ 2 & 1 & 1 & -9 \\ 0 & 3 & -5 & -1 \\ 4 & -3 & -2 & -4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 3 & 1 & -7 \\ 0 & 3 & -5 & -1 \\ 0 & 1 & -2 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 3 & 1 & -7 \\ 0 & 0 & -6 & 6 \\ 0 & 0 & -7/3 & 7/3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a-b-d \\ 3b-c-7d \\ c-d \\ 0 \end{pmatrix} = \vec{0} \Rightarrow \begin{matrix} a = b+d \\ 3b = c+7d \\ c = d \end{matrix} \Rightarrow \begin{pmatrix} a \\ b \\ -c \\ -d \end{pmatrix} = \begin{pmatrix} b+c \\ 8/3c \\ -c \\ -c \end{pmatrix}$$

$$\Rightarrow b \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 8/3 \\ -1 \\ -1 \end{pmatrix} \quad \begin{pmatrix} a \\ b \\ -c \\ -d \end{pmatrix} \in \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 8 \\ -3 \\ -3 \end{pmatrix} \right\}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 3 \\ 8 \end{pmatrix} \quad a = x+3y \quad b = 8y$$

$$\Rightarrow W_1 \cap W_2 = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} -5 \\ 14 \\ 24 \\ -12 \end{pmatrix} \right\}, \dim W_1 \cap W_2 = 2$$

Prob. 3

(1) Note that $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \times 3 \\ 0 & 1 \end{pmatrix}$, so $\sin \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \begin{pmatrix} 1 & (2n+1) \times 3 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} & 3 \times \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \\ 0 & \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \end{pmatrix} = \begin{pmatrix} \sin 1 & 3 \cos 1 \\ 0 & \sin 1 \end{pmatrix} *$$

(2) Prove,

Note that when $\lambda_1 \neq 0$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\sin A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} \sin \lambda_1 & \lambda_1 \cos 1 \\ 0 & \sin \lambda_1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if $\sin A = \begin{pmatrix} 1 & 2022 \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} \sin \lambda_1 & \lambda_1 \cos 1 \\ 0 & \sin \lambda_1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 2022 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a \sin \lambda_1 + c \lambda_1 \cos 1, & b \sin \lambda_1 + d \lambda_1 \cos 1 \\ c \sin \lambda_1, & d \sin \lambda_1 \end{pmatrix} = \begin{pmatrix} a, & 2022a+b \\ c, & 2022c+d \end{pmatrix}$$

$$\text{if } \sin \lambda_1 = 1 \quad c = 0 \quad b = 0 \quad d = \frac{2022}{\lambda_1 \cos 1} \quad a = 1$$

then

$$\begin{pmatrix} 1 \times 1 + 0 & 2022 \\ 0 & \frac{2022 \times 1}{\lambda_1 \cos 1} \end{pmatrix} = \begin{pmatrix} 1 & 2022 \\ 0 & \frac{2022 \times 1}{\lambda_1 \cos 1} \end{pmatrix}$$

and

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{2022}{\lambda_1 \cos 1} \end{pmatrix} \text{ is invertible.}$$

$$\text{Thus, if } A = \begin{pmatrix} 1 & 0 \\ 0 & \frac{2022}{\lambda_1 \cos 1} \end{pmatrix}^{-1} \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{2022}{\lambda_1 \cos 1} \end{pmatrix}, \sin A = \begin{pmatrix} 1 & 2022 \\ 0 & 1 \end{pmatrix}$$

Prob. 4 Frobenius norm and trace of A^*A

Note that if $A = (a_1, \dots, a_n)$ $A^*A = \begin{pmatrix} \bar{a}_1 \\ \vdots \\ \bar{a}_n \end{pmatrix} (a_1, \dots, a_n)$, so we have

$$\text{trace}(A^*A) = \|A\|_2^2, \text{ where } \|A\|_2 = (\sum_{i,j} |a_{ij}|^2)^{1/2}.$$

Note that $\forall A \in M_n(\mathbb{C}) \quad \|A\|_F^2 = \sum_{A=U\Sigma U^*} \Sigma$, Σ is a upper-triangular,

$$\begin{aligned} \|A\|_2^2 &= \text{trace}(A^*A) = \text{trace}(U \Sigma^* U^* U \Sigma U^*) \\ &= \text{trace}(\Sigma^* \Sigma) = \|\Sigma\|_2^2 \geq \sum_{k=1}^n |\lambda_k|^2 \end{aligned}$$

Thus $\|A\|_2^2 = \sum_{k=1}^n |\lambda_k|^2$ iff Σ is a diagonal matrix, That is,
A is normal.

Prob. 5 Appendix : Higham "Func. of Matrix"

\Rightarrow From left Def. 1.1 and 1.4, $f: z \rightarrow z^{1/4}$

can be defined on spectral of A , so

$\exists p \in \text{Fix}$ such that $f(A) = p(A)$

Let $C = p(A)$, A and C , $A^{1/4}$, has positive spectral
commute,
 \hookrightarrow unique.

use "simultaneous triangularization", $A^{-1}C$ has

positive spectrum and $(A^{-1}C)^4 = I$, so $A^{-1}C$
 \hookrightarrow only positive root is 1

is diagonalizable, having 4th roots of unity $\Rightarrow A = C$

Definition 1.1. The function f is said to be defined on the spectrum of A if the values

$$f^{(j)}(\lambda_i), \quad j = 0: n_i - 1, \quad i = 1: s$$

exist. These are called the values of the function f on the spectrum of A .

In most cases of practical interest f is given by a formula, such as $f(t) = e^t$. However, the following definition of $f(A)$ requires only the values of f on the spectrum of A ; it does not require any other information about f . Indeed, any $\sum_{i=1}^s n_i$ arbitrary numbers can be chosen and assigned as the values of f on the spectrum of A . It is only when we need to make statements about global properties such as continuity that we will need to assume more about f .

Definition 1.2 (matrix function via Jordan canonical form). Let f be defined on the spectrum of $A \in \mathbb{C}^{n \times n}$ and let A have the Jordan canonical form (1.2). Then

$$f(A) := Z f(J) Z^{-1} = Z \text{diag}(f(J_k)) Z^{-1}, \quad (1.3)$$

where

$$f(J_k) := \begin{bmatrix} f(\lambda_k) & f'(\lambda_k) & \cdots & \frac{f^{(m_k-1)}(\lambda_k)}{(m_k-1)!} \\ & f(\lambda_k) & \ddots & \vdots \\ & & \ddots & f'(\lambda_k) \\ & & & f(\lambda_k) \end{bmatrix}. \quad (1.4)$$

Definition 1.4 (matrix function via Hermite interpolation). Let f be defined on the spectrum of $A \in \mathbb{C}^{n \times n}$ and let ψ be the minimal polynomial of A . Then $f(A) := p(A)$, where p is the polynomial of degree less than

$$\sum_{i=1}^s n_i = \deg \psi$$

that satisfies the interpolation conditions

$$p^{(j)}(\lambda_i) = f^{(j)}(\lambda_i), \quad j = 0: n_i - 1, \quad i = 1: s. \quad (1.7)$$

There is a unique such p and it is known as the Hermite interpolating polynomial.

An example is useful for clarification. Consider $f(t) = \sqrt{t}$ and

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}.$$

The eigenvalues are 1 and 4, so $s = 2$ and $n_1 = n_2 = 1$. We take $f(t)$ as the principal branch $t^{1/2}$ of the square root function and find that the required interpolant satisfying $p(1) = f(1) = 1$ and $p(4) = f(4) = 2$ is

$$p(t) = f(1) \frac{t-4}{1-4} + f(4) \frac{t-1}{4-1} = \frac{1}{3}(t+2).$$

Hence

$$f(A) = p(A) = \frac{1}{3}(A+2I) = \begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix}.$$

It is easily checked that $f(A)^2 = A$. Note that the formula $A^{1/2} = (A+2I)/3$ holds more generally for any diagonalizable $n \times n$ matrix A having eigenvalues 1 and/or 4.

In the following, for a linear map $f : V \rightarrow V$, $\ker f$ and $\text{im } f$ denote the kernel and the image of f , respectively.

- ✓ 1. Let V be a finite-dimensional complex inner product space. Let $d : V \rightarrow V$ be a linear map satisfying $d^2 = 0$. Let $\delta : V \rightarrow V$ be the adjoint of d and $\Delta = d\delta + \delta d$. Prove the following.
 - ✓ [5%] $d\delta x = 0$ implies that $\delta x = 0$, and $\delta dx = 0$ implies that $dx = 0$, for all $x \in V$.
 - ✓ [10%] $\ker \Delta = \ker d \cap \ker \delta$.
 - ✓ [10%] There is the orthogonal decomposition $V = \ker \Delta \oplus \text{im } d \oplus \text{im } \delta$.
 - ✓ [5%] There is the orthogonal decomposition $\ker d = \ker \Delta \oplus \text{im } d$.
- ✓ 2. [10%] Let $V = \mathbb{R}^n$ be the space of column vectors, and M a positive definite symmetric $n \times n$ real matrix. Suppose the matrix $A \in M_n(\mathbb{R})$ satisfies $MAM^{-1} = A^t$. Show that there exists $P \in M_n(\mathbb{R})$ satisfying $P^t MP = I_n$ such that $P^{-1}AP$ is diagonal. (Here B^t denotes the transpose of the matrix B .)
- ✓ 3. (a) [10%] Let M be an invertible $n \times n$ complex matrix. Prove that there exists an invertible matrix A such that $A^2 = M$.
 - ✓ [10%] Let $n \geq 2$ and N be an $n \times n$ matrix over a field such that $N^n = 0$ but $N^{n-1} \neq 0$. Prove that there is no square matrix B such that $B^2 = N$.
- ✓ 4. [20%] Let V be a vector space over a field F and $u_1, \dots, u_n \in V$ are linearly independent. Show that, for any $v_1, \dots, v_n \in V$, $u_1 + \alpha v_1, \dots, u_n + \alpha v_n$ are linearly independent for all but finitely many values of $\alpha \in F$.
- ✓ 5. [20%] Let P be an $n \times n$ matrix with coefficients in a field. Suppose $\text{rank}(P) + \text{rank}(I_n - P) = n$. Prove that $P^2 = P$.

Prob. 1 direct sum

(a) Since $0 = \langle d\delta x, x \rangle = \langle \delta x, dx \rangle$, $\delta x = 0$. Similarly,

$$0 = \langle \delta dx, x \rangle = \langle dx, dx \rangle, \quad dx = 0.$$

(b) if $x \in \ker \Delta = \ker(d\delta + \delta d)$, $d\delta x + \delta dx = 0 \Rightarrow d^2 dx + d\delta dx = 0$
 $\Rightarrow dd dx = 0 \Rightarrow dx = 0 \Rightarrow x \in \ker d$.

Note that $d^2 = 0$ mean $\delta^2 = 0$ $\langle d^2 x, x \rangle = \langle x, \delta^2 x \rangle = 0$

Similar $d\delta x + \delta dx = 0 \Rightarrow \delta d\delta x + \delta^2 dx = 0 \Rightarrow \delta d\delta x = 0 \Rightarrow \delta x = 0 \quad x \in \ker d$
 $\Rightarrow \ker \Delta \subseteq \ker \delta \cap \ker d$, opposite, if $x \in \ker \delta \cap \ker d$, then

$$\Delta x = d\delta x + \delta dx = 0 \Rightarrow x \in \ker \Delta *$$

(c) First, we show that $\ker \Delta \perp \text{Im } d$ & $\ker \Delta \perp \text{Im } \delta$ & $\text{Im } d \perp \text{Im } \delta$

If $x \in \ker \Delta$, $y \in \text{Im } d \Rightarrow y = dv$ for some v ,

$$\langle x, y \rangle = \langle x, dv \rangle = \langle \delta x, v \rangle = 0 \quad (\text{previous result})$$

If $x \in \ker \Delta$, $y \in \text{Im } \delta \Rightarrow y = \delta v$ for some v ,

$$\langle x, y \rangle = \langle x, \delta v \rangle = \langle dx, v \rangle = 0 \quad (\text{previous result})$$

If $x \in \text{Im } \delta$, $y \in \text{Im } d$, $x = \delta v_1$, $y = dv_2$ for some v_1, v_2

$$\langle x, y \rangle = \langle \delta v_1, dv_2 \rangle = \langle v_1, d^2 v_2 \rangle = 0 \quad (d^2 = 0)$$

Next, we claim that $\forall v \in V$, $v = v_1 + v_2 + v_3$, $v_1 \in \ker \Delta$ $v_2 \in \text{Im } d$
 $v_3 \in \text{Im } \delta$

Note that Δ is self-adjoint, so $V = \ker \Delta \oplus \left(\bigoplus_{\lambda \neq 0} E_\lambda \right)$, then for any

$$x \in \bigoplus_{\lambda \neq 0} E_\lambda \quad x = \sum_{\lambda_i} x_{\lambda_i}, \quad x_{\lambda_i} \in E_{\lambda_i}, \quad x = \sum_{\lambda_i} \frac{1}{\lambda_i} \Delta x_{\lambda_i} \in \text{Im } \Delta$$

so $\bigoplus_{\lambda \neq 0} E_\lambda \subseteq \text{Im } \Delta$, By Rank-Kernel theorem, $\dim \bigoplus_{\lambda \neq 0} E_\lambda = \dim \text{Im } \Delta$,

so $\bigoplus_{\lambda \neq 0} E_\lambda = \text{Im } \Delta$. That mean $V = \ker \Delta \oplus \text{Im } \Delta$, However,

$$v = \frac{v_1}{\ker \Delta} + \frac{v_2}{\text{Im } \Delta} = \frac{v_1}{\ker \Delta} + \frac{\Delta v_3}{V_3 = \Delta v_3} = \frac{v_1}{\ker \Delta} + \frac{d\delta v_3}{\in \text{Im } d} + \frac{\delta d v_3}{\in \text{Im } \delta} \in \ker \Delta \oplus \text{Im } d \oplus \text{Im } \delta$$

(d) We have showed that $\ker \Delta \perp \text{Im } d$, since $\forall v \in \text{Im } d, v = dv_0$ for some v_0 , $dv = d^2 v_0 = 0$, so $\text{Im } d \subseteq \ker d$. Moreover, we have $\ker \Delta \oplus \text{Im } d \subseteq \ker d$ and since V is finite dimension, $V = \text{Im } \delta \oplus (\text{Im } \delta)^\perp = \text{Im } \delta \oplus \ker d = \text{Im } \delta \oplus \ker \Delta \oplus \text{Im } d$, so $\dim(\ker d) = \dim(\ker \Delta \oplus \text{Im } d)$

Thus $\ker d = \ker \Delta \oplus \text{Im } d$.

Problem 2 Self-adjoint, positive root

Note that $MA = A^t M$ mean $(MA)^t = A^t M^t = A^t M = MA$
 MA is self-adjoint, Moreover, $A = M^{-\frac{1}{2}}(MA) = M^{\frac{1}{2}}M^{\frac{1}{2}}S M^{\frac{1}{2}}M^{\frac{1}{2}}$
 Note that $M^{\frac{1}{2}}SM^{\frac{1}{2}}$ is self-adjoint, so A is diagonal and can be written as $A = M^{\frac{1}{2}}P_1 D P_1^{-1} M^{\frac{1}{2}}$, Let R be $M^{\frac{1}{2}}P_1$, then $R^{-1}AR = D$ and $R^t M R = P_1^t M^{\frac{1}{2}} M M^{-\frac{1}{2}} P_1 = P_1^t P_1 = I$ (since $M^{\frac{1}{2}}SM^{\frac{1}{2}}$ is self-adjoint $P^t = P^{-1}$)

Prob. 3 nilpotent

(a)

8.21 Description of operators on complex vector spaces

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of T . Then

- (a) $V = G(\lambda_1, T) \oplus \dots \oplus G(\lambda_m, T)$;
- (b) each $G(\lambda_j, T)$ is invariant under T ;
- (c) each $(T - \lambda_j I)|_{G(\lambda_j, T)}$ is nilpotent.

8.31 Identity plus nilpotent has a square root

Suppose $N \in \mathcal{L}(V)$ is nilpotent. Then $I + N$ has a square root.

Let λ_i be distinct eig-value of T , then by 8.21, each

$$T - \lambda_j I \mid_{G(\lambda_j, T)} = N_{\lambda_j} \Rightarrow T = I + \frac{1}{\lambda_j} N_{\lambda_j} \rightarrow \lambda_j \neq 0 \text{ since } T \text{ inv.}$$

By 8.31, \sqrt{T} exist on $G(\lambda_j, T)$, say \sqrt{T}_{λ_j} . By 8.21 again,

$V = \bigoplus_{i=1}^m G(\lambda_i, T)$, $v = \sum_{i=1}^m v_i$ $v_i \in G(\lambda_i, T)$, so we define

$$Rv = \sqrt{T}_{\lambda_1} v_1 + \dots + \sqrt{T}_{\lambda_j} v_j$$

Then

$$\vec{R} v = R(Rv) = R(\sqrt{T}_{\lambda_1} v_1 + \dots + \sqrt{T}_{\lambda_j} v_j)$$

$$= (\sqrt{T}_{\lambda_1})^2 v_1 + \dots + (\sqrt{T}_{\lambda_j})^2 v_j = T_{\lambda_1} v_1 + \dots + T_{\lambda_j} v_j = Tv$$

(b) Suppose not, $\exists B$ s.t. $B^2 = N$, Then $B^{2n} = N^n = 0$, Thus

$\min\text{-poly} \mid x^{2n}$, Moreover, $\min\text{-poly} \mid \text{char-poly}$ implies $B^n = 0$,
 $\because 2n-2 = n + n-2 \geq n$, $B^{2n-2} = 0$, but $N^{n-1} = B^{2n-2} \neq 0 \Rightarrow \leftarrow$

Problem 5.

Goal : $P(I-P)v = 0 \quad \forall v \in V$, note that $\text{rank } P + \ker P = n$,

so $\ker P = \text{rank } (I-P)$, but $\forall v \in \ker P$, $v = v - Pv = (I-P)v$, so

$\ker P \subseteq \text{rank } (I-P)$, That induces that $\ker P = \text{Im } (I-P)$, That is, $\forall v$

$$P(I-P)v = 0 \rightarrow (P - P^2)v = 0 \rightarrow Pv = P^2v \neq$$

Prob. 4

Let $\dim V = m$, $n \leq m$, extend u_1, \dots, u_n to a basis $\{u_i\} \cup \{z_j\}$.

To test $\{u_i + \alpha v_i\}_{i=1}^n$ linear indep. Let $x_i \in F$ such that

$$x_1(u_1 + \alpha v_1) + x_2(u_2 + \alpha v_2) + \dots + x_n(u_n + \alpha v_n) = 0 \quad (*)$$

Since $\{u_i\} \cup \{z_j\}$ is a basis, $v_i = \sum_{j=1}^n y_{ij} u_j + \sum_{j=n+1}^m y_{ij} z_j$

so $(*)$ become

$$(x_1 + \alpha y_{11} x_1 + \alpha y_{21} x_2 + \dots + \alpha y_{n1} x_n)u_1 + (\alpha y_{12} x_1 + x_2 + \alpha y_{22} x_2 + \dots + \alpha y_{n2} x_n)u_2 +$$

$$\dots + (\alpha y_{1n} x_1 + \alpha y_{2n} x_2 + \dots + x_n + \alpha y_{nn} x_n)u_n = 0$$

A

$$\Rightarrow \begin{pmatrix} 1 + \alpha y_{11}, \alpha y_{21}, \alpha y_{31}, \dots, \alpha y_{n1} \\ \alpha y_{12}, 1 + \alpha y_{22}, \alpha y_{32}, \dots, \alpha y_{n2} \\ \vdots & \vdots & & \vdots \\ \alpha y_{1n}, \alpha y_{2n}, \dots, 1 + \alpha y_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Then the problem become "How many α such that $\ker A \neq 0$, Note

that $A = I + \alpha \begin{pmatrix} y_1 & y_{21} & y_{n1} \\ \vdots & \vdots & \cdots \\ y_{1n} & y_{2n} & y_{nn} \end{pmatrix} = I + \alpha B$, Hence, when we discuss $\ker A \neq \{0\} \Leftrightarrow$ Does 0 is a eig-value of $I + \alpha B$, Let $\{\lambda_i\}_{i=1}^n$ be distinct eig-value of B , then eig-value of $A = \{1 + \alpha \lambda_i\}_{i=1}^n$, so $1 + \alpha \lambda_i = 0$ for some $i \Leftrightarrow \alpha = -\frac{1}{\lambda_i}$, $\lambda_i \neq 0$, That complete proof.

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科目：線性代數

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1. Let A be a 4×4 real symmetric matrix. Suppose that 1 and 2 are eigenvalues of A and the eigenspace for the eigenvalue 2 is 3-dimensional. Assume that $(1, -1, -1, 1)^t$ is an eigenvector for the eigenvalue 1. (Here v^t denotes the transpose of v .)

(a) Find an orthonormal basis for the eigenspace for the eigenvalue 2 of A . (10 points.)

(b) Find Av , where $v = (1, 0, 0, 0)^t$. (10 points.)

2. Let A be a real $n \times n$ matrix. Prove that

$$\text{rank}(A^2) - \text{rank}(A^3) \leq \text{rank}(A) - \text{rank}(A^2).$$

(10 points.)

3. Let V be a vector space of finite dimension over \mathbb{R} and S, T , and U be subspaces of V . Prove or disprove (by giving counterexamples) the following statements:

(a) $\dim(S + T) = \dim S + \dim T - \dim(S \cap T)$. (10 points.)

(b) $\dim(S + T + U) = \dim S + \dim T + \dim U - \dim(S \cap T) - \dim(T \cap U) - \dim(U \cap S) + \dim(S \cap T \cap U)$. (10 points.)

4. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$. Compute $\exp A$. (10 points.)

(b) Prove that $\det(\exp A) = \exp(\text{tr } A)$ for $A \in M(n, \mathbb{C})$. (10 points.)

(c) Prove or disprove (by giving counterexamples) that if A is nilpotent, then so is $\exp A - I_n$. Here a matrix M is said to be nilpotent if $M^k = 0$ for some positive integer k and I_n is the identity matrix of size n . (10 points.)

5. Let U and V be finite-dimensional vector spaces, and U^* and V^* be their dual spaces, respectively. For a linear transformation $T : U \rightarrow V$, define $T^* : V^* \rightarrow U^*$ by $(T^*f)(u) = f(Tu)$ for $f \in V^*$ and $u \in U$.

(a) Prove that T is injective if and only if T^* is surjective. (10 points.)

(b) Prove that T is surjective if and only if T^* is injective. (10 points.)

Prob. 1 Symmetric matrix

(1) Note that $\exists \theta$ s.t. $\theta\theta^t = I_4$ and $\theta A \theta^t = D$, so if $(1, [1, -1, -1, 1])$, then (2, $\text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$) $\xrightarrow{\text{obviously}}$

$$\begin{aligned} \Rightarrow A \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} &= A \left(\frac{1}{4} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right) \\ &= 1 \times \frac{1}{4} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} + 2 \times \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} + 2 \times \frac{1}{4} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{7}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \end{pmatrix} \end{aligned}$$

Prob. 2 Frobenius inequality rank

We show more general case

$$\text{rank}(A^{k+1}) - \text{rank}(A^{k+2}) \leq \text{rank}(A^k) - \text{rank}(A^{k+1})$$

Note that $\text{Im}(A^{k+1}) \subseteq \text{Im}(A^k)$, so there exist G (vec. space) such that

$$\Rightarrow \text{Im}(A^{k+1}) \oplus G = \text{Im} A^k, \text{ More, } A|_{\text{Im} A^k} = \text{Im}(A^{k+1}), \text{ so}$$

$$\text{Im}(A^{k+1}) = A(\text{Im}(A^{k+1})) + AG = \text{Im}(A^{k+2}) + AG$$

$$\begin{aligned} \text{so } \text{Rank}(A^{k+1}) &= \text{Rank}(A^{k+2}) + \text{Rank}(AG) - \text{Rank}(A^{k+2} \cap AG) \\ &\leq \text{Rank}(A^{k+2}) + \text{Rank}(AG) \\ &\leq \text{Rank}(A^{k+2}) + \text{Rank } G \leq \text{Rank}(A^{k+2}) + \text{Rank}(A^k) - \text{Rank}(A^k) \end{aligned}$$

$$\Rightarrow \text{Rank}(A^{k+1}) - \text{Rank}(A^{k+2}) \leq \text{Rank}(A^k) - \text{Rank}(A^{k+1}) \#$$

Prob. 3 Prove

(1) Let B be a basis of $S \cap T$, Then extend B to be $A \cup B$, a basis of S and $B \cup C$, a basis of T . We claim that $A \cup B \cup C$ is a basis of $S + T$. $\text{Span}(A \cup B \cup C) = S + T$ is obviously, to see it

$$A \cap B = \emptyset$$

$$B \cap C = \emptyset$$

is independent, let $\alpha_i \mid 1 \leq i \leq |A \cup B \cup C|$, $\alpha_1 V_1 + \alpha_2 V_2 + \dots + \alpha_n V_n = 0$
if $\exists \alpha_j \neq 0$ for some j such that $(*) = 0$, Then $\exists \alpha_j, V_j \in A$
and $\alpha_j \in C$ (since $A \cup B \cup C$ are linear indep.), so there is
 $x \in \langle A \rangle \cap \langle B \cup C \rangle \Rightarrow x \in S \cap T$ ($S = \text{Span}(A \cup B)$ $T = \text{Span}(B \cup C)$)
 $x \in \langle A \rangle \cap \langle B \rangle \Rightarrow x = 0$, a contradiction $\Rightarrow A \cup B \cup C$ linear ind.
 $\Rightarrow \dim S + \dim T = |A \cup B| + |B \cup C| = |A| + |B| + |C|$
 $= |A \cup B \cup C| + |B| = \dim(S+T) + \dim(S \cap T)$

(b) disprove, Let $S = \langle s_1 \rangle$ $T = \langle t_1 \rangle$ $L = \langle s_1 + t_1 \rangle$

$$\dim(S+T+L) = \dim(S+T) = 2$$

$$\begin{aligned} \dim S + \dim T + \dim L - \dim(S \cap T) - \dim(T \cap L) - \dim(S \cap L) \\ + \dim(S \cap T \cap L) = 1 + 1 + 1 - 0 - 0 - 0 - 0 = 3 \end{aligned}$$

Prob 4. $\exp A$

$$(a) A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},$$

$$\exp A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e & e \\ 0 & e \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -e & 2e \\ -e & e \end{pmatrix} = \begin{pmatrix} 0 & e \\ -e & 2e \end{pmatrix}$$

(b) Let $A = U \Sigma U^{-1}$, Where Σ is upper-triangular matrix

$$\exp A = \exp U \Sigma U^{-1} = U \exp \Sigma U^{-1} \Rightarrow \det(\exp A) = \det(\exp \Sigma)$$

Note that Σ^n is also a upper-tri matrix and entries on diagonal is

eig-value, entries of Σ^n is λ_i^n , Then

$$\det(\exp A) = \det(\exp \Sigma) = \exp(\sum_i \lambda_i) = \exp(\text{tr} A)$$

(c) Since $M^k = 0$ for some $k \in \mathbb{N}$, so $\exp(M) = \sum_{i=0}^k \frac{1}{i!} M^i$
 Then $\exp(M) - I = M \left(I_n + \frac{1}{1!} M + \dots + \frac{1}{k!} M^{k-1} \right)$

$$(\exp(M) - I)^k = M^k \left(I_n + \frac{1}{1!} M + \dots + \frac{1}{k!} M^{k-1} \right)^k = 0$$

So $\exp(M) - I$ is also a nilpotent.

Prob. 5 Relation between Range T, $\ker T^*$, M°

Let M be a set of vec. space V, The annihilator of M is defined by $M^\circ = \{ f \in V^* : f(M) = 0 \}$

Note that

$$\begin{aligned} \dim M^\circ &= \dim V^* - \dim T^* \\ &= \dim V - \dim M \quad \text{For } \dim V < \infty \end{aligned}$$

Note that

$$\begin{aligned} \ker(T^*) &= \{ f \in V^* : f(Tu) = 0 \} \\ &= \{ f \in V^* : f(\operatorname{Im} T) = 0 \} \\ &= (\operatorname{Im} T)^\circ \end{aligned}$$

$$\operatorname{Im}(T^*) = \{ g \in L^* : g = f|_{\operatorname{Im} T} \text{ for some } f \in V^* \}$$

If $g \in \operatorname{Im} T^*$, $\forall v \in \ker T$, $g(v) = f(Tv) = f(0) = 0$

so $g \in (\ker T)^\circ$, For the reverse inclusion, Let $g \in (\ker T)^\circ$.

Goal: $\exists f \in V^*$ such that $g = f \circ T$. Let $L = \ker T \oplus S$,

$S = \langle \beta_1, \dots, \beta_n \rangle$, define $f(T\beta_i) = g(\beta_i)$ (Note that $\{T\beta_i\}_{i=1}^n$ linear ind.)

Then $g = f \circ T$ on L, $g \in \operatorname{Im} T^*$.

Thus, we have ⁽¹⁾ $\text{Im } T^* = (\ker T)^\circ$ ⁽²⁾ $\ker T^* = (\text{Im } T)^\circ$

$$\text{(3)} \quad \dim U - \dim S = \dim S^\circ$$

So

$$\dim (\text{Im } T^*) = \dim (\ker T)^\circ = \dim U - \dim \ker T$$

$$\Rightarrow \dim (\text{Im } T^*) + \dim \ker T = \dim U = \dim U^*$$

and

$$\dim \ker T^* = \dim (\text{Im } T)^\circ = \dim V - \dim \text{Im } T$$

$$\Rightarrow \dim \ker T^* + \dim \text{Im } T = \dim V$$

T is injective $\Leftrightarrow \dim \ker T = 0 \Leftrightarrow \dim \text{Im } T^* = \dim U^*$

$\Leftrightarrow T^*$ is surjective

T is surjective $\Leftrightarrow \dim \text{Im } T = \dim V \Leftrightarrow \dim \ker T^* = 0$

$\Leftrightarrow T^*$ is injective.

臺灣大學數學系 108 學年度碩士班甄試試題

科目：線性代數

2018.10.19

- ✓ 1. Find all possible Jordan forms for 8×8 real matrices having $x^2(x - 2)^3$ as minimal polynomial. (20 points.)
- ✓ 2. Let V be a vector space over a field \mathbb{F} of infinite elements, and let v_1, \dots, v_n be vectors in V , where n is a positive integer. Suppose that $v_0 + zv_1 + \dots + z^n v_n = 0$ for infinitely many z in \mathbb{F} . Prove that all v_i 's are zero. (20 points.)
- ✓ 3. Let $V = M(n, \mathbb{R})$ be the vector space of all $n \times n$ matrices and $f : V \rightarrow \mathbb{R}$ be a linear transformation. Assume that $f(AB) = f(BA)$ for all $A, B \in V$ and $f(I_n) = n$, where I_n is the identity matrix in V . Prove that f is the trace function. (20 points.)
Hint: Consider the cases $A = E_{ij}$ and $B = E_{kl}$ for various E_{ij} and E_{kl} . Here E_{ij} denotes the matrix whose (i, j) -entry is 1 and whose other entries are 0.)
- ✓ 4. Let V be an n -dimensional vector space over \mathbb{R} and $B : V \times V \rightarrow \mathbb{R}$ be a symmetric bilinear form on V . (*Symmetric* means $B(u, v) = B(v, u)$ for all $u, v \in V$. *Bilinear* means that B is linear in each of the two variables.)
- ✓ (a) Let W be a vector subspace of V and let

$$W^\perp = \{u \in V : B(u, v) = 0 \text{ for all } v \in W\}.$$

Prove that if $\dim W = m$, then $\dim W^\perp \geq n - m$. (10 points.) *Hint:* Choose a basis $\{v_1, \dots, v_m\}$ for W and consider the map

$$u \mapsto (B(u, v_1), \dots, B(u, v_m))$$

- ✓ (b) from V into \mathbb{R}^m .
- ✓ (c) Prove that $V = W \oplus W^\perp$ if and only if the restriction of B to W is non-degenerate. (*Nondegenerate* means that $v = 0$ is the only vector of W such that $B(u, v) = 0$ for all $u \in W$.) (15 points.)
- ✓ (d) Prove that if B is nondegenerate on V , then there is a nonnegative integer p with $p \leq n$ and a basis $\{v_1, \dots, v_n\}$ such that

$$B(v_i, v_j) = \begin{cases} 1, & \text{if } 1 \leq i = j \leq p, \\ -1, & \text{if } p+1 \leq i = j \leq n, \\ 0, & \text{if } i \neq j. \end{cases}$$

(15 points.)

Prob. 1 Jordan Form

$$\left(\begin{array}{ccccc} 2 & 0 & 0 & & \\ 0 & 2 & 0 & & \\ 0 & 0 & 2 & & \\ & & & 2 & 0 \\ & & & 0 & 2 \\ & & & 0 & 0 \\ X & & & & \\ & & & & 0 \\ & & & & 0 \end{array} \right) \text{, } \left(\begin{array}{ccccc} 2 & 1 & 0 & & \\ 0 & 2 & 1 & & \\ 0 & 0 & 2 & & \\ & & & 2 & 0 \\ & & & 0 & 2 \\ & & & 0 & 0 \\ X & & & & \\ & & & & 0 \\ & & & & 0 \end{array} \right) \text{, } \left(\begin{array}{ccccc} 2 & 1 & 0 & & \\ 0 & 2 & 1 & & \\ 0 & 0 & 2 & & \\ & & & 0 & 1 \\ & & & 0 & 0 \\ X & & & & \\ & & & & 0 \\ & & & & 0 \end{array} \right)$$

$$\left(\begin{array}{ccccc} 0 & & & & \\ & 2 & 1 & 0 & \\ & 0 & 2 & 1 & \\ & 0 & 0 & 2 & \\ & & & 0 & 1 \\ & & & 0 & 0 \\ X & & & & \\ & & & & 0 \\ & & & & 0 \end{array} \right) \text{, } \left(\begin{array}{ccccc} 2 & 0 & 0 & & \\ 0 & 2 & 0 & & \\ 0 & 0 & 2 & & \\ & & & 2 & 0 \\ & & & 0 & 2 \\ & & & 0 & 0 \\ X & & & & \\ & & & & 0 \\ & & & & 0 \end{array} \right) \text{, } \left(\begin{array}{ccccc} 2 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ & & & 2 & 0 \\ & & & 0 & 2 \\ & & & 0 & 0 \\ X & & & & \\ & & & & 0 \\ & & & & 0 \end{array} \right)$$

Prob. 2 Fundamental theorem of Algebra

If some $v_i \neq 0 \Rightarrow$ some entry of $v_i \neq 0$, by FT of Algebra,
 $a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z a_0 = 0$ only have most n roots, That
 is contradiction.

Prob. 3 Trace func.

Since f is linear, if we show that $f(E_{ij}) = 0$ for any $i \neq j$,
 and $f(E_{ii}) = 1$, then f must be the trace func.

\Rightarrow Consider $f(E_{ij} E_{kl})$, we have following case.

$i = l$, $j = k$, but $i \neq j$. then

$$f(E_{ij} E_{kl}) = f(E_{il}) \Rightarrow f(E_{il}) = f(E_{kj}), \text{ since } f(I_n) = n, \\ f(E_{kl} E_{ij}) = f(E_{kj}) \Rightarrow f(E_{ii}) = 1$$

$i = l$, $j \neq k$

$$f(E_{ij} E_{kl}) = f(0) = 0 \quad \text{↑ } f \text{ is linear} \\ f(E_{kl} E_{ij}) = f(E_{kj}) \Rightarrow f(E_{kj}) = 0$$

Thus, f is the trace func.

Prob. 4 Bilinear symmetric func.

(a) Choose a basis of W , say $\{v_1, \dots, v_m\}$. Consider the map $\widehat{B}(u)$
 $V \rightarrow \mathbb{R}^m$ by $v \mapsto (B(v, v_1), \dots, B(v, v_m))$. Then by Rank -

$$\text{Null Theorem } \dim V = \dim \ker \widehat{B} + \dim \text{Im } \widehat{B}, \text{ so}$$

$$\dim W^\perp = \dim \ker \widehat{B} = \dim V - \dim \text{Im } \widehat{B}$$

$$\geq \dim V - \dim \mathbb{R}^m$$

$$= n - m.$$

(b) If $V = W \oplus W^\perp$, $v \in V$ such that $B|_W(u, v) = 0$ for all $u \in W$

mean $v \in W^\perp$, but $W \cap W^\perp = \{0\}$, so $v = 0$.

If $B|_W$ is non-degenerate, then $W \cap W^\perp = \{0\}$ obviously,

Note that $W + W^\perp \leq V$, but $\dim W + W^\perp = \dim W + \dim W^\perp$

large than n (by (a)), so $\dim V = \dim W + W^\perp = n$. Thus

$$V = W + W^\perp.$$

Hoffman, Linear Algebra : P. 369.

(c)

First, we show that \exists a basis of V , say $\{\alpha_i\}_{i=1}^n$, such that

$$f(\alpha_i, \alpha_j) = 0 \text{ for } i \neq j$$

We use the induction on n , if $B(u, v) = 0$ or $\dim V = 1$, then

the result is obvious true. Suppose that is true for $n < k$

When $n = k$, First, we find a vector α s.t. $f(\alpha, \alpha) \neq 0$, if no

this vector exist, note that $f(\alpha, \beta) = \frac{1}{4} [f(\alpha + \beta, \alpha + \beta) - f(\alpha - \beta, \alpha - \beta)]$.

Thus, if $f(\alpha, \alpha) = 0 \forall \alpha \in V \Rightarrow f(\alpha, \beta) = 0$ for any $\alpha, \beta \in V$, \Rightarrow

Hence, $\exists \alpha$ st. $f(\alpha, \alpha) \neq 0$. Let $W = \langle \alpha \rangle$ and $W^\perp = \langle \alpha \rangle^\perp$

We claim that $V = W \oplus W^\perp$, First, if $\beta \in W \cap W^\perp$, $\beta = c\alpha$ and $f(\alpha, \beta) = 0$, $f(\alpha, c\alpha) = c f(\alpha, \alpha) = 0 \Rightarrow c=0 \Rightarrow \beta=0$.

Next, if $\gamma \in V$, let $\beta = \gamma - \frac{f(\alpha, \gamma)}{f(\alpha, \alpha)} \alpha$

$$f(\alpha, \beta) = f(\alpha, \gamma) - \frac{f(\alpha, \gamma)}{f(\alpha, \alpha)} f(\alpha, \alpha) = 0$$

$\beta \in W^\perp$, Then $\gamma = \frac{f(\alpha, \gamma)}{f(\alpha, \alpha)} \alpha + \beta \in W + W^\perp$, so $V = W \oplus W^\perp$

By induction, $\exists \{x_2, \dots, x_n\}$ such that $f(x_i, x_j) = 0$, $W^\perp = \langle \{x_i\}_{i=2}^n \rangle$

Take $\{\alpha_1, x_2, \dots, x_n\}$, Then $V = \langle \alpha_1, \dots, x_n \rangle \neq$

Use this basis, Let $\beta_i = \alpha_i / |f(\alpha_i, \alpha_i)|^{1/2}$. Since $B(u, v)$ is non-degenerate, $f(x_i, \alpha_i) \neq 0$, otherwise, $f(\beta_i, x_i) = 0 \Rightarrow x_i = 0 \Rightarrow \Leftarrow$.

Then $\{\beta_1, \dots, \beta_n\}$ is the basis we want.

$$\left(f(\beta_i, \beta_i) = \frac{1}{|f(\alpha_i, \alpha_i)|} f(\alpha_i, \alpha_i) = \pm 1 \right)$$

By rearrange $\{\beta_1, \dots, \beta_n\}$ to $\{\hat{\beta}_1, \dots, \hat{\beta}_n\}$, we can find $p \leq n$ such that $f(\beta_i, \beta_i) = 1$ if $1 \leq i \leq p$; $f(\beta_i, \beta_i) = -1$ if $p+1 \leq i \leq n$.

(1) (20 points) Let $A = \begin{pmatrix} -1 & 3 & -2 \\ 2 & 3 & 0 \\ 11 & -6 & 7 \end{pmatrix}$. Find the lower triangular Jordan canonical form of A .

A. Please compute $\exp(tA)$ and derive the general solution to $x'(t) = A x(t)$, where $x(t)$ is a 3-dimensional column vector.

(2) (20 points) Let V be an n -dimensional complex vector space, and $T : V \rightarrow V$ be an invertible linear map such that $T^2 = 1$. (a) Show that T is diagonalizable, (b) Let S be the vector space of linear transformations from V to V that commute with T . Please express $\dim_{\mathbb{C}} S$ in terms of n and the trace of T .

(3) (20 points) Let $A = (A_{ij})$ be a real invertible skew-symmetric $2n \times 2n$ matrix.

(a) Show that all eigenvalues of A are pure imaginary.

(b) Define the Pfaffian $Pf(A)$ of A by

$$Pf(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) A_{\sigma(1), \sigma(2)} A_{\sigma(3), \sigma(4)} \cdots A_{\sigma(2n-1), \sigma(2n)}$$

Let B be any real $2n \times 2n$ matrix. Show that $Pf(BAB^T) = Pf(A) \det(B)$.

(c) Assuming the fact that there exists a real orthogonal $2n \times 2n$ matrix O such that

$$OAO^T = \text{diag} \left\{ \begin{pmatrix} 0 & m_1 \\ -m_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & m_2 \\ -m_2 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & m_n \\ -m_n & 0 \end{pmatrix} \right\},$$

where $m_i \in \mathbb{R}$ for $i = 1, \dots, n$. Show that $\det(A) = Pf(A)^2$.

(4) (20 points) Let $A, B \in M_n(\mathbb{C})$ be $n \times n$ complex matrices. Show that A and B are simultaneously triangularizable (i.e. there exists an invertible matrix $P \in GL_n(\mathbb{C})$ such that PAP^{-1} and PBP^{-1} are both upper triangular) if A and B commute.

Hint: Let λ be one of the eigenvalues of A . Try to show $B(\ker(A - \lambda I)) \subset \ker(A - \lambda I)$.

(5) (20 points) Show that

$$\begin{vmatrix} X_0 & X_1 & X_2 & \dots & X_{n-1} \\ X_{n-1} & X_0 & X_1 & \dots & X_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_1 & X_2 & X_3 & \dots & X_0 \end{vmatrix} = \prod_{j=0}^{n-1} \left(\sum_{k=0}^{n-1} \zeta^{jk} X_k \right)$$

where ζ is a primitive n -th root of unity.

Hint: You may first compute, for example,

$$\begin{pmatrix} X_0 & X_1 & X_2 & X_3 \\ X_3 & X_0 & X_1 & X_2 \\ X_2 & X_3 & X_0 & X_1 \\ X_1 & X_2 & X_3 & X_0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 & \zeta^3 \\ 1 & \zeta^2 & \zeta^4 & \zeta^6 \\ 1 & \zeta^3 & \zeta^6 & \zeta^9 \end{pmatrix}.$$

Prob. 1 Jordan Form

$$\det(A - \lambda I) = \left| \begin{pmatrix} -1-\lambda & 3 & -2 \\ 2 & 3-\lambda & 0 \\ 11 & -6 & 7-\lambda \end{pmatrix} \right| = (7-\lambda)(\lambda^2 - 2\lambda - 9) - 2(11\lambda - 45)$$

$$\begin{aligned} &= -22\lambda + 90 - \lambda^3 + 2\lambda^2 + 9\lambda + 7\lambda^2 - 14\lambda - 63 \\ &= -\lambda^3 + 9\lambda^2 - 27\lambda + 27 \\ &= -(\lambda - 3)^3 \end{aligned}$$

$$\Rightarrow A - 3I = \begin{pmatrix} -4 & 3 & -2 \\ 2 & 0 & 0 \\ 11 & -6 & 4 \end{pmatrix} \Rightarrow \ker(A - 3I) = \left\{ \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \right\}$$

$$(A - 3I) V_2 = V_1 \Rightarrow V_2 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

$$(A - 3I) V_3 = V_2 \Rightarrow V_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow A = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & -2 & 3 \end{pmatrix}}_{Q^{-1}} \underbrace{\begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \end{pmatrix}}_{\Lambda} \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & -2 & 3 \end{pmatrix}}_{Q}$$

$$\Rightarrow \exp A = Q^{-1} \exp \begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \end{pmatrix} Q$$

$$= Q^{-1} \sum_{k=1}^{\infty} \frac{1}{k!} \begin{pmatrix} 3^k & 0 & 0 \\ 1^k 3^{k-1} & 3^k & 0 \\ 1^k 3^{k-2} 1^k 3^{k-1} & 1^k 3^k & 0 \end{pmatrix} Q = Q^{-1} \begin{pmatrix} e^3 & 0 & 0 \\ e^3 & e^3 & 0 \\ \frac{e^3}{2} & e^3 & e^3 \end{pmatrix} Q$$

Prob. 2 Min-poly

$$(1) \text{ Since } T^2 = I \Rightarrow T^2 - I = 0 \Rightarrow \text{min-p of } T \mid x^2 - 1$$

\Rightarrow min-p of T is a product of linear form $(x-1)(x+1) \cdot (x+1) \cdot x-1$

$\Rightarrow T$ is diagonalizable (since T is invertible $\lambda_T = 1 \text{ or } -1 \text{ only}$)

(2) Find a basis of V such that $[T]_B$ is diagonal, Then

$ST = TS$ mean if $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, A, B, C, D are submatrix

Let $x = \dim(\ker(T - I))$, $y = \dim(\ker(T + I)) \Rightarrow x - y = \text{trace } T$

$$ST = TS \Leftrightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} A & -B \\ C & -D \end{pmatrix} = \begin{pmatrix} A & B \\ -C & -D \end{pmatrix} \Rightarrow \begin{cases} B = C = 0 \\ A, D \text{ arbitrary.} \end{cases}$$

Thus, $\dim S = x^2 + y^2$, Note that $x + y = n$ $x - y = \text{trace } T$

$$\Rightarrow \dim S = \frac{n^2 + (\text{trace } T)^2}{2}$$

Prob. 3 Skew symmetric matrix

(a) Let λ, v be a eigen-value / vector respectively, Then

$$\begin{aligned} \lambda \langle v, v \rangle &= \langle \lambda v, v \rangle = \langle Av, v \rangle = \langle v, -Av \rangle = \langle v, -\lambda v \rangle \\ &= -\bar{\lambda} \langle v, v \rangle \end{aligned}$$

$$\Rightarrow \lambda = -\bar{\lambda} \Rightarrow \lambda \text{ is pure imaginary.}$$

(b) $Pf(A) \equiv \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) A_{\sigma(1)\sigma(2)} A_{\sigma(3)\sigma(4)} \cdots A_{\sigma(2n-1)\sigma(2n)}$

We use exterior algebra to show !!

Exterior algebra $\{x^1, \dots, x^{2n}\}$, $x^i \wedge x^j = -x^j \wedge x^i$, $x^i \wedge x^i = 0$

$$\text{Def } w_i = \sum_{j=1}^n a_{ij} x^j \Rightarrow A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

$$\Rightarrow w_1 \wedge w_2 \wedge \cdots \wedge w_n = \det A \underbrace{x^1 \wedge x^2 \wedge \cdots \wedge x^n}_n$$

$$\text{Def } \omega = \sum_{i,j} a_{ij} e_i \wedge e_j \Rightarrow \frac{1}{n!} (\omega \wedge \omega \wedge \cdots \wedge \omega) = Pf(A) e_1 \wedge \cdots \wedge e_n$$

$$\text{or } \omega = \sum a_{ij} e_i \wedge e_j \Rightarrow (\omega)^n = 2^n n! Pf(A) e_1 \wedge \cdots \wedge e_n$$

Thus,

$$A \rightarrow \omega_A = \sum a_{ij} e_i \wedge e_j \xrightarrow{\wedge^n} 2^n n! Pf(A) e_1 \wedge \cdots \wedge e_n$$

$$BAB^T \mapsto \sum_{i,j} (BAB^T)_{ij} e_i \wedge e_j \rightarrow \sum_{i,j} \left(\sum_{k,l} b_{ik} b_{jl} a_{kl} \right) e_i \wedge e_j$$

$$\text{Let } f_k = \sum_i b_{ik} e_i, f_l = \sum_j b_{jl} e_j$$

$$\Rightarrow \sum_{i,j} \left(\sum_{k,l} b_{ik} b_{jl} a_{kl} \right) e_i \wedge e_j = \sum_{k,l} a_{kl} \left(\sum_{i,j} b_{ik} b_{jl} \right) e_i \wedge e_j$$

$$= \sum_{k,l} A_{kl} f_{k,n} f_{l,n} \xrightarrow{\wedge^n} \sum n! \text{Pf}(A) f_{1,n} \cdots f_{n,n} = \sum n! \text{Pf}(BAB^T) e_1 e_1 \cdots e_n e_n$$

But $f_{1,n} \cdots f_{n,n} = \det(B) e_1 e_1 \cdots e_n e_n$, so $\text{Pf}(A) \det B = \text{Pf}(BAB^T)$ *

(c) Skew-Symmetric Matrix \rightarrow spectral theorem

$\Rightarrow \forall M_{2n}(\mathbb{R})$, $A^T = -A$ matrix, $A = Q \Sigma Q^T$, Q is orthogonal

and

$$\Sigma = \begin{bmatrix} 0 & \lambda_1 & & \\ -\lambda_1 & 0 & X & \\ & \ddots & 0 & \lambda_n \\ & X & -\lambda_n & 0 \end{bmatrix}, \text{ where } \lambda_i \text{ is spectral of } A$$

\Rightarrow Note that $\det A = \det(Q \Sigma Q^T) = \det \Sigma = \lambda_1^2 \times \cdots \times \lambda_n^2$, where λ_i is positive part of eigenvalue

$$\Rightarrow \text{pf}(A) = \text{pf}(Q \Sigma Q^T) = \det(Q) \text{pf}(\Sigma) = \lambda_1 \cdots \lambda_n \Rightarrow (\text{pf}(A))^2 = \det A *$$

Prob. 4 Simultaneously triangular Hoffman P. 207

First, we claim $B(\ker(A-\lambda I)) \subset \ker(A-\lambda I)$, if $v \in B(\ker(A-\lambda I))$,

$$v = Bu, u \in \ker(A-\lambda I) \Rightarrow Av = ABu = BAu = B\lambda u = \lambda Bu = \lambda v \Rightarrow v \in \ker(A-\lambda I),$$

Second, Consider $A|_{\ker(A-\lambda I)}$ and $B|_{\ker(A-\lambda I)}$,

$\dim(\ker(A-\lambda I)) \neq 0$, if $\dim(\ker(A-\lambda I)) = \dim V$, since any matrix can be

written in $Q \Sigma Q^{-1}$, Σ is upper triangular, choose Q , each column vec.

$$\text{of } Q \in \ker(A-\lambda I) \Rightarrow [A]_Q = \lambda I, B = L \Rightarrow P = Q$$

if $\dim(\ker(A-\lambda I)) < \dim V$, Use induction, $A|_{\ker(A-\lambda I)}$ and $B|_{\ker(A-\lambda I)}$ can

be simultaneously triangularizable, More general, $B(\frac{\ker(A-\lambda I)}{G(\lambda)}) \subseteq \ker(A-\lambda I)^n$

Note that $V = G(\lambda_1) \oplus \cdots \oplus G(\lambda_n)$, each $\dim G(\lambda_i) < \dim V$, so it can find

a basis p_{λ_i} such that $A|_{G(\lambda_i)}$ and $B|_{G(\lambda_i)}$ are simu-tri, Take $p = \{p_i\}$,

we done.

Prob. 5 V -determinant

$$\left(\begin{array}{cccc} x_0 & x_1 & \cdots & x_{n-1} \\ x_{n-1} & x_0 & \cdots & x_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \cdots & x_0 \end{array} \right) \left(\begin{array}{cccc} 1 & 1 & \cdots & 1 \\ 1 & \xi & \xi^2 & \cdots & \xi^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \xi^{n-1} & \xi^{2n-2} & \cdots & \xi^{(n-1)^2} \end{array} \right) = \left(\begin{array}{cccc} A_1 & A_2 & \cdots & A_n \\ A_1 & \xi A_2 & & \xi^{n-1} A_n \\ \vdots & \vdots & & \vdots \\ A_1 & \xi^{n-1} A_2 & \cdots & \xi^{(n-1)^2} A_n \end{array} \right)$$

where $A_i = (x_0, \dots, x_{n-1}) \cdot (1, \xi^{\frac{i-1}{n}}, \dots, \xi^{\frac{(i-1)(n-1)}{n}}) = \sum_{k=0}^{n-1} (\xi^{(i-1)k} x_k)$

$$\begin{aligned}
 & \text{So } \det \begin{pmatrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_{n-1} & x_0 & \cdots & x_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \cdots & x_0 \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \xi & \xi^2 & \cdots & \xi^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \xi^{n-1} & \xi^{2n-2} & \cdots & \xi^{(n-1)^2} \end{pmatrix} = \det \begin{pmatrix} A_1 & A_2 & \cdots & A_n \\ A_1 & \xi A_2 & \cdots & \xi^{n-1} A_n \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & \xi^{n-1} A_2 & \cdots & \xi^{(n-1)^2} A_n \end{pmatrix} \\
 & = \prod_{i=1}^n A_i \times \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \xi & \xi^2 & \cdots & \xi^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \xi^{n-1} & \xi^{2n-2} & \cdots & \xi^{(n-1)^2} \end{pmatrix} = \left(\prod_{i=1}^n \sum_{k=0}^{n-1} [\xi^{(i-1)k} x_k] \right) \times \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \xi & \xi^2 & \cdots & \xi^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \xi^{n-1} & \xi^{2n-2} & \cdots & \xi^{(n-1)^2} \end{pmatrix} \\
 \Rightarrow & \det \begin{pmatrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_{n-1} & x_0 & \cdots & x_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \cdots & x_0 \end{pmatrix} = \prod_{j=0}^{n-1} \sum_{k=0}^{n-1} \left(\xi^{jk} x_k \right) *
 \end{aligned}$$

- ✓. (20%) Let $A \in M_{n \times n}(F)$ where F is a field.
- (a) Show that if k is the largest integer such that some $k \times k$ submatrix of A has a nonzero determinant, then $\text{rank}(A) = k$.
- (b) If A is nilpotent of index m (that is, $A^m = 0$ but $A^{m-1} \neq 0$), and if, for each vector v in F^n , W_v is defined to be the subspace generated by $v, Av, \dots, A^{m-1}v$, how large can the dimension of W_v be? (Justify your answer.)

- ✓. (30%)

✓) Let $A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$. Find the general solution to the system of differential equations

$$\frac{dX}{dt} = AX, \text{ where } X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

where for each i , $x_i(t)$ is a differentiable real-valued function of the real variable t .

- ✓) Let V be the space of all real polynomials having degree less than 4 with the inner product $\langle f(x), g(x) \rangle = \int_{-1}^1 f(t)g(t)dt$. Let T be a linear operator on V defined by $T(f(x)) = f'(x) + 3f(x)$.

Use the Gram-Schmidt process to replace $\beta = \{1, 1+x, x+x^2, x^2+x^3\}$ by an orthonormal basis for V and find the matrix representation of the adjoint T^* of T in this orthonormal basis.

- ✓. (30%)

- (a) Let $A \in M_{n \times n}(\mathbb{R})$. Show that there exists an orthogonal matrix Q and a positive semi-definite symmetric matrix P such that $A = QP$.
- (b) Let V be a finite-dimensional vector space over \mathbb{C} and T be a linear operator on V . Show that T is normal if and only if its adjoint $T^* = g(T)$ for some polynomial $g(x) \in \mathbb{C}[x]$.

- ✓. (20 %) Let $T \in \text{End}_{\mathbb{C}}(V)$ for a finite-dimensional \mathbb{C} -vector space V .

- (a) Show that we have an expression of T as $T = S + N$ with $S, N \in \text{End}_{\mathbb{C}}(V)$, such that S is diagonalisable, N is nilpotent and $SN = NS$.
- (b) Show that both S and N are uniquely defined by these conditions.
- (c) Show that there is a polynomial $p(x) \in \mathbb{C}[x]$ with $p(0) = 0$ such that $S = p(T)$.

Prob. 1 Rank

(a) We claim that column (A) has k linear independent vectors,

Let $A \in M_{n \times n}(F)$, B_k is submatrix of A with $\det B_k \neq 0$, Then

we focus on the corresp. column vec. of v_i from $B_k \Rightarrow \begin{pmatrix} ||B_k|| \\ \vdots \\ v_1 \dots v_k \text{ col}(A) \end{pmatrix}$

we note that $\{v_1, \dots, v_k\}$ are linear indep. Otherwise,

$\exists a_i \neq 0 \text{ for all } i$ s.t. $a_1 v_1 + \dots + a_k v_k = 0 \Rightarrow \det(B_k) = 0$ since its column

vec is linear dep. Thus $k \leq \text{rank}(A)$. For opposite, if $\text{rank } A = k_1 > k$,

then we can find v_1, \dots, v_k , linear indep vec of column (A)

such that $\begin{pmatrix} v_{11} & \dots & v_{k_11} \\ \vdots & \ddots & \vdots \\ v_{1n} & \dots & v_{k_1n} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_{k_1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad (\text{ } v_i = (v_{i1}, \dots, v_{in}))$

$$\Leftrightarrow x_1 = x_2 = \dots = x_{k_1} = 0$$

so we at least have k_1 effective eqn such that $| \begin{pmatrix} v_{1j} & \dots & v_{k_1j} \\ \vdots & \ddots & \vdots \\ v_{1j} & \dots & v_{k_1j} \end{pmatrix} | \neq 0$

That is a contradiction.

(b) Let $\beta = \{v, Av, \dots, A^{m-1}v\}$ and $a_i, 0 \leq i \leq m-1$ s.t. $\sum_{i=0}^{m-1} a_i A^i v = 0$

Apply A^{m-1} on $\sum_{i=0}^{m-1} a_i A^i v = 0 \Rightarrow a_0 A^{m-1} v = 0 \Rightarrow a_0$ if $A^{m-1}v \neq 0$.

But $A^{m-1} \neq 0$, we can find $v \in V$ s.t. $A^{m-1}v \neq 0$, Thus

$$\dim W_v = |\beta| = m \#$$

For a_1 , Apply A^{m-2} on $\sum_{i=1}^{m-1} a_i A^i v = 0 \Rightarrow a_1 A^{m-1} v = 0 \Rightarrow a_1 = 0$
 \Rightarrow For general, a_j , Apply A^{m-1-j} $\Rightarrow a_j A^{m-1} v = 0$

Prob. 2 Jordan Form

$$D_A = \underbrace{\begin{pmatrix} 1 & 2 & 2 \\ -\frac{1}{3} & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{A^{-1}} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} -3 & 6 & 6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix}}_{A}$$

\Rightarrow The solun. of $\frac{dx}{dt} = Ax \Rightarrow x = x_0 \exp(A) = x_0 A^{-1} \begin{pmatrix} e^1 & e^2 & e^3 \\ 0 & e^1 & e^2 \\ 0 & 0 & e^1 \end{pmatrix} A$

(b) Gram - Schmidt

$$V_1 = \frac{1}{\sqrt{2}}, V_2 = (1+x) - \langle 1+x, \frac{1}{\sqrt{2}} \rangle \frac{1}{\sqrt{2}} / \|V_2\| = \frac{3x}{\sqrt{6}}$$

$$V_3 = \frac{x^2 + x - \langle V_3, \frac{1}{\sqrt{2}} \rangle \frac{1}{\sqrt{2}} - \langle V_3, \frac{3}{\sqrt{6}} x \rangle \frac{3}{\sqrt{6}} x}{\|V_3\|} = \frac{3\sqrt{10}}{4} x^2 - \frac{\sqrt{10}}{4}$$

$$V_4 = \frac{x^3 + x^2 - [\frac{1}{3} + \frac{3}{5}x + x^2 - \frac{1}{3}]}{\|V_4\|} = -\frac{x^3 - \frac{3}{5}x}{\|V_4\|}$$

$$= \frac{5\sqrt{14}}{4} x^3 - \frac{3\sqrt{14}}{4} x \quad \rightarrow \frac{\sqrt{10}}{4} (3x^2 - 1) \quad \frac{\sqrt{14}}{4} (5x^3 - 3x)$$

$$\Rightarrow \beta = \{V_1, V_2, V_3, V_4\} = \left\{ \frac{1}{\sqrt{2}}, \frac{3x}{\sqrt{6}}, \frac{3\sqrt{10}}{4} x^2 - \frac{\sqrt{10}}{4}, \frac{5\sqrt{14}}{4} x^3 - \frac{3\sqrt{14}}{4} x \right\}$$

$$[T]_{\beta} = \begin{pmatrix} 3 & \sqrt{3} & 0 & \sqrt{7} \\ 0 & 3 & 4\sqrt{15} & 0 \\ 0 & 0 & 3 & 5\sqrt{14}/\sqrt{10} \\ 0 & 0 & 0 & 3 \end{pmatrix} \Rightarrow [T^*]_{\beta} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ \sqrt{3} & 3 & 0 & 0 \\ 0 & 4\sqrt{15} & 3 & 0 \\ \sqrt{7} & 0 & 5\sqrt{14}/\sqrt{10} & 3 \end{pmatrix}$$

Prob. 3 Singular value decomposition & Isometry decomposition

Note that $\forall A \in L(V)$, $A = U\Sigma V^*$, where $U^*U = VV^* = I$

and $A = S\sqrt{AA^*}$, S is isometric, let $A = IUV(V\Sigma V^*)$

$$\Rightarrow \begin{cases} Q = UV^* \\ P = V\Sigma V^* \end{cases} \quad \text{since } \begin{cases} UV^*(VU^*) = I \Rightarrow Q \text{ is orthonormal} \\ V\Sigma V^* = (V\Sigma V^*)^* \text{ and each eig-value of } \Sigma \geq 0 \Rightarrow P \text{ is positive.} \end{cases}$$

Moreover, if A is invertible, Q, V are unique

Note : if two positive semidefined matrix A, B such that $\tilde{A} = \tilde{B}$, then $A = B$

Let $A = QP = UZ$, we claim that $P = Z$, note that since A is invertible, P, Z are also invertible, $QP = UZ \Rightarrow U^*Q = ZP^{-1}$

Because Q, U are orthonormal, $ZP^{-1} = U^*Q$ is orthonormal, That is

$$I = (ZP^{-1})^* ZP^{-1} = (P^{-1})^* Z^* Z P^{-1} = P^{-1} Z^2 P^{-1} \Rightarrow Z^2 = P^2$$

By remark, $Z = P$.

Prob. 3. Normal, common vec and conjugate eig-value

(b) If $T^* = g(T)$ for some $g \in \mathbb{C}[x]$, Obviously $T^*T = TT^*$.

If $T^*T = TT^*$. By spectral theorem, $\exists \lambda_1, \dots, \lambda_n$ and v_1, \dots, v_n such that
 $v = \sum_{i=1}^n v_i$

$Tv = \sum_{i=1}^n \lambda_i v_i$, Remember that T and T^* has common eig-vector and
conjugate eig-value, So our goal is $T^*v = \sum_{i=1}^n \bar{\lambda}_i v_i = \sum_{i=1}^n g(T)v_i$

$$\Rightarrow g(T)v_i = \bar{\lambda}_i v_i \Rightarrow g(\lambda)v_i = \bar{\lambda}_i v_i \Rightarrow g(\lambda) = \bar{\lambda}$$

By Lagrange interpolation, we can find such $g(x)$ st. $g(\lambda) = \bar{\lambda}$

Then $T^* = g(T) \#$

Prob. 4 1a)

Theorem 12 (Primary Decomposition Theorem). Let T be a linear operator on the finite-dimensional vector space V over the field F . Let p be the minimal polynomial for T ,

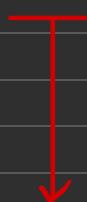
$$p = p_1^{r_1} \cdots p_k^{r_k}$$

where the p_i are distinct irreducible monic polynomials over F and the r_i are positive integers. Let W_i be the null space of $p_i(T)^{r_i}$, $i = 1, \dots, k$. Then

- (i) $V = W_1 \oplus \cdots \oplus W_k$;
- (ii) each W_i is invariant under T ;
- (iii) if T_i is the operator induced on W_i by T , then the minimal polynomial for T_i is $p_i^{r_i}$.

Theorem 11. Let T be a linear operator on a finite-dimensional space V . If T is diagonalizable and if c_1, \dots, c_k are the distinct characteristic values of T , then there exist linear operators E_1, \dots, E_k on V such that

- (i) $T = c_1 E_1 + \cdots + c_k E_k$;
- (ii) $I = E_1 + \cdots + E_k$;
- (iii) $E_i E_j = 0$, $i \neq j$;
- (iv) $E_i^2 = E_i$ (E_i is a projection);
- (v) the range of E_i is the characteristic space for T associated with c_i .



Theorem 13. Let T be a linear operator on the finite-dimensional vector space V over the field F . Suppose that the minimal polynomial for T decomposes over F into a product of linear polynomials. Then there is a diagonalizable operator D on V and a nilpotent operator N on V such that

- (i) $T = D + N$,
- (ii) $DN = ND$.

The diagonalizable operator D and the nilpotent operator N are uniquely determined by (i) and (ii) and each of them is a polynomial in T .

Note that in PDT \Rightarrow we find E_1, \dots, E_n such that $I = E_1 + \cdots + E_n$ and each $E_i E_j = 0$, $i \neq j$, E_i is a projection, Moreover, since \mathbb{C} is algebra closed, min-poly of T is a product of linear form and $E_i = (T - c_i)^{t_i}$ if min-poly = $\prod_{i=1}^n (x - c_i)^{t_i}$, let

$$S = C_1 E_1 + C_2 E_2 + \cdots + C_n E_n$$

By theorem 11 $\Rightarrow S$ is a diagonalizable operator

$$N = T - S = (T - C_1) E_1 + (T - C_2) E_2 + \cdots + (T - C_n) E_n$$

Note that range $E_i = \ker(T - C_i)^{t_i}$, so when n enough large
 $N^n = (T - C_1)^n E_1 + (T - C_2)^n E_2 + \dots + (T - C_n)^n E_n$
 $= 0$

when $n \geq \max \{t_i : i=1, \dots, n\}$. Thus N is a nilpotent.

Remark that each E_i is a polynomial form of T , so

S and N commute $\Rightarrow SN = NS$

(b) Note that S and N are poly form of T , we claim that for any D, M such that $T = S + N = D + M$ and $DM = MD$, D is diag and M is nilpotent, then $S = D$, $N = M$.

First, since $DM = MD$, $T = D + M \Rightarrow D, M$ commute with T

$\Rightarrow D, M$ commute with S, N . More, $S - D = M - N$

\Rightarrow Since D, S commute and diagonalizable $\Rightarrow D, S$ can be simultaneous

diagonal, Because M, N commute and nilpotent $\Rightarrow M - N$ is nilpotent

$\Rightarrow S - D = M - N$ is diagonal and nilpotent

$\Rightarrow (S - D)^n = 0$ for some $n \in \mathbb{N} \Rightarrow$ min-poly of $S - D \mid x^n$

\Rightarrow Since $S - D$ diagonal, min-poly of $S - D$ is $h(x) = x$, That mean

$S - D = 0 \Rightarrow S = D \Rightarrow M = N$

(c) $S = p(T)$ for some $p(x) \in \mathbb{C}[x]$ have proved in PDT, Let

h be a min-poly of T (i.e. $h(T) = 0$), if $h(0) \neq 0$, then

$$S = p(T) = p(T) - \frac{p(0)}{h(0)} h(T) \equiv \tilde{p}(T), \quad \tilde{p}(0) = p(0) - \frac{p(0)}{h(0)} h(0) = 0$$

if $h(0) = 0$, That mean $h(x) = x^{t_1} \times (x - c_2)^{t_2} \times \dots \times (x - c_n)^{t_n}$, Recall

that $S = p(T) = \sum_{i=1}^n C_i E_i = \sum_{i=1}^n C_i f_i(T) g_i(T)$, where

$$\sum_{i=1}^n f_i(x) g_i(x) = 1; \quad f_i(x) \equiv -\frac{h(x)}{(x - c_i)^{t_i}} \quad (C_i = 0)$$

Thus, if $h(0) = 0$ $S = P(T) = \sum_{i=1}^n c_i f_i(T) g_i(T)$, $c_i \neq 0$ $i \neq 1$. Then

$$P(x) = \sum_{i=1}^n c_i f_i(x) g_i(x), \text{ but } x^{t_1} \mid f_i(x) \text{ if } i \neq 1, \text{ so } P(0) = 0.$$

Note that PDT tell us that, if we know min-poly of T , say $h(x)$

$$h(x) = (x - c_1)^{t_1} \times (x - c_2)^{t_2} \times \cdots \times (x - c_n)^{t_n}.$$

$$\text{Then } V = \ker (T - c_1)^{t_1} \oplus \ker (T - c_2)^{t_2} \oplus \cdots \oplus \ker (T - c_n)^{t_n}$$

$$I = E_1 + E_2 + \cdots + E_n$$

$$E_i = f_i(T) g_i(T), \quad f_i(x) = \frac{h(x)}{(x - c_i)^{t_i}}, \quad \sum_{i=1}^n f_i(x) g_i(x) = 1$$

$$S = c_1 E_1 + \cdots + c_n E_n = \sum_{i=1}^n c_i f_i(T) g_i(T)$$

$$H = T - \sum_{i=1}^n c_i f_i(T) g_i(T) *$$

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線性代數

✓. (10 points) Let A, B be two $m \times n$ matrix. Show that $|\text{rank}(A) - \text{rank}(B)| \leq \text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$.

✓. (16 points) Let A be an $n \times n$ matrix and $r_k = \text{rank}(A^k)$.

(a). Show that $\lim_{k \rightarrow \infty} r_k$ exist.

(b). If $r_3 \neq r_4$, Is A diagonalizable? Show your answer.



✓. (8 points) Let $A = [a_{ij}]$ be an $n \times n$ matrix satisfying the condition that each a_{ij} is either equal to 1 or to -1. Show that $\det(A)$ is an integer multiple of 2^{n-1} .

✓. (16 points) Let S, T be linear operator on V such that $S^2 = S$. Show that the range of S is invariant under T if and only if $STS = TS$. Show that both the range and null space of S are invariant under T if and only if $ST = TS$.

✓. (20 points) Define a real vector space $V = \{f(x) \mid f(x) = ax^2 + bx + c, a, b, c \in \mathbb{R}\}$, with inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$.

(a). Find an orthonormal basis for V .

(b). Using (a), find $f \in V$ to maximize $f(\frac{1}{2})$ subject to the constraint $\langle f, f \rangle = 1$.

✓. (16 points) Let $A = \begin{bmatrix} -2 & 3 & 1 \\ 0 & a & 3 \\ 0 & -3 & 4-a \end{bmatrix}$. Find the condition of a such that A is diagonalizable over real number.

✓. (14 points) Let A be an $n \times n$ real symmetric matrix. Show that the matrix $A^2 + A + I$ is positive-definite.

Prob. 1. Rank / null space

(a) Note $v \in R(A+B)$ $v = (A+B)\varphi$ for some $\varphi \in V$, then $v \in R(A)+R(B)$

$$\begin{aligned} \text{so } \dim R(A+B) &\leq \dim R(A) + \dim R(B) - \dim(R(A) \cap R(B)) \\ &\leq \dim R(A) + \dim R(B) \end{aligned}$$

$$\Rightarrow \text{rank}(A+B) \leq \text{rank } A + \text{rank } B$$

$$\begin{aligned} \text{by (a)} \quad \text{rank}(A) &= \text{rank}(A+B-B) \leq \text{rank}(A+B) + \text{rank}(-B) \\ &= \text{rank}(A+B) + \text{rank}(B) \end{aligned}$$

$$\Rightarrow \text{rank}(A) - \text{rank}(B) \leq \text{rank}(A+B)$$

$$\text{similar } \text{rank}(B) - \text{rank}(A) \leq \text{rank}(A+B)$$

$$\Rightarrow |\text{rank}(A) - \text{rank}(B)| \leq \text{rank}(A+B)$$

Prob. Decreasing of image

Since $R(A^{k+1}) \subseteq R(A^k)$, so $\text{rank}(A^{k+1}) \leq \text{rank}(A^k)$

r_k is a positive decreasing seq., so it converge.

Note that $\text{rank}(A) = \text{Number of eig-value} \neq 0$, if A is diagonalizable, then nonzero eigenvalue of A also a nonzero of A^k and $\text{rank}(A^k) = \text{rank}(A)$, Thus, A is not diagonalizable if $r_3 \neq r_4$.

Prob. 3 Determine

Use elementary operator $A \rightarrow B$, entries of B are $\{0, 2, -2\}$

By def of det of B , $\det B = \sum_{\sigma \in S_{n-1}} a_{1\sigma(1)} \cdots a_{n-1\sigma(n-1)}$, then each term is 0 or $k 2^{n-1}$, $k \in \mathbb{Z}$. So $2^{n-1} \mid \det A = \det B$

Prob. 4 Projection

(a) \Rightarrow if $STS = TS$, let $v \in R(S)$, $v = S\varphi$

$$\Rightarrow Tv = TS\varphi = STS\varphi \in R(S) \Rightarrow T(R(S)) \subseteq R(S)$$

\Rightarrow if $T(R(S)) \subseteq R(S)$, then $TS\varphi = S\psi$ for some $\psi \in V$,

$$\text{then } STS\varphi = S^2\varphi = S\varphi = T\varphi \Rightarrow STS = TS$$

(b) If $TS = ST$, let $v \in R(S)$ and $u \in N(S)$

$$Tv = TSv = STv \in R(S) \Rightarrow T(R(S)) \subseteq R(S)$$

$$STu = TSu = Tu = 0 \Rightarrow Tu \in N(S) \Rightarrow T(N(S)) \subseteq N(S)$$

If $T(R(S)) \subseteq R(S)$ & $T(N(S)) \subseteq N(S)$, Note that $\forall v \in V$

$$v = (I-S)v + Sv \in N(S) + R(S)$$

$$T(I-S)v \in N(S) \text{ and } TSv \in R(S) \quad \xrightarrow{\text{Previous result}}$$

$$\Rightarrow STv = ST((I-S)v + Sv) = STSv = TSv \neq$$

Prob. 5 Gram-Schmidt

Note that $\{1, x, x^2\}$ is a basis of V $x^2 - x + \frac{1}{6}$

$$V_1 = 1 \quad V_2 = \frac{x - \frac{1}{2}}{\|V_2\|} = 2\sqrt{3}x - \sqrt{3} \quad \uparrow$$

$$V_3 = \frac{x^2 - (\langle x^2, 1 \rangle 1 + \langle x^2, V_2 \rangle V_2)}{\|V_3\|} = \frac{x^2 - \frac{1}{3} - \frac{\sqrt{3}}{6}(2\sqrt{3}x - \sqrt{3})}{\|V_3\|}$$

$$= 6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}$$

$$\text{Let } f(x) = aV_1 + bV_2 + cV_3 \quad \langle f, f \rangle = 1 \Rightarrow a^2 + b^2 + c^2 = 1$$

$$f(\frac{1}{2}) = a - \frac{\sqrt{5}}{2}c \Rightarrow (1, 0, \frac{\sqrt{5}}{2}) \parallel (2a, 2b, 2c)$$

$$\text{and } a^2 + b^2 + c^2 = 1 \quad 2a = 1 \quad 2b = 0 \quad 2c = \frac{\sqrt{5}}{2}$$

$$\text{if } \lambda \neq 0 \Rightarrow b = 0 \quad a = \frac{1}{2}\lambda \quad c = \frac{\sqrt{5}}{2\lambda}, \text{ let } k = \frac{1}{2}\lambda$$

$$k^2 + \frac{5}{4}k^2 = 1 \quad k^2 = \frac{4}{9} \quad k = \pm \frac{2}{3} \quad \lambda = \pm \frac{3}{2} \quad a = \pm \frac{3}{3}$$

$$c = \pm \frac{\sqrt{5}}{3} \quad (a, b, c) = \left(\frac{2}{3}, 0, \frac{\sqrt{5}}{3}\right) \text{ or } \left(-\frac{2}{3}, 0, -\frac{\sqrt{5}}{3}\right)$$

Then extremum of $f(\frac{1}{2}) = \frac{-1}{6} \& \frac{1}{6}$, since $a^2 + b^2 + c^2 = 1$ is compact

in \mathbb{R}^3 , $f(\frac{1}{2})$ has a max on it $\Rightarrow \max f(\frac{1}{2}) = \frac{1}{6}$

$$f = \frac{-2}{3} \times 1 + \frac{-\sqrt{5}}{3} \times (6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}) = -10x^2 + 10x - \frac{7}{3} \neq$$

Prob. 6 symmetric matrix

By spectrum theorem, $A = \theta^{-1} D \theta$ for some D : diagonal,

Then $\tilde{A}^T + A + I = \theta^{-1} (D^2 + D + I) \theta$ is symmetric and since

$\forall \lambda \in \mathbb{R}$, $\lambda^2 + \lambda + 1 > 0$, so $\tilde{A}^T + A + I$ has positive eig-value,

$\Rightarrow A$ is positive defined.

Prob. 5 diagonalizable.

$$A = \begin{pmatrix} -2 & 3 & 1 \\ 0 & a & 3 \\ 0 & -3 & 4-a \end{pmatrix}$$

A is diagonalizable \Leftrightarrow min-poly of A is linear product

Note, min-poly & char-poly has same root

$$\det(A - \chi I) = (-2 - \chi) \begin{vmatrix} a-\chi & 3 \\ -3 & 4-a-\chi \end{vmatrix} = 0$$

$$= -(\chi+2) [(a-\chi)(\chi+a-4)+9] = 0$$

$$= -(\chi+2) (\underline{\chi^2 - 4\chi - a(a-4) + 9}) = 0$$

\rightarrow must be reducible

$$\Rightarrow 16 - 4(-a(a-4) + 9) > 0 \quad \text{or} \quad 16 - 4(-a(a-4) + 9) = 0$$

$$\Rightarrow a^2 - 4a - 5 > 0 \quad \begin{matrix} -1 \\ \hline 5 \end{matrix} \quad a^2 - 4a - 5 = 0$$

$$a > 5 \text{ or } a = -1$$

$$(a-5)(a+1) = 0 \quad a = -1 \text{ or } 5$$

must be diagonalizable

$$A = \begin{pmatrix} -2 & 3 & 1 \\ 0 & 5 & 3 \\ 0 & -3 & -1 \end{pmatrix} \text{ or } \begin{pmatrix} -2 & 3 & 1 \\ 0 & -1 & 3 \\ 0 & -3 & 5 \end{pmatrix}$$

$$\lambda = -2, 2 \text{ (rep)} \quad \lambda = -2, 2$$

$$A - 2I = \begin{pmatrix} -4 & 3 & 1 \\ 0 & 3 & 3 \\ 0 & -3 & -3 \end{pmatrix} \Rightarrow \ker(A - 2I) = \{ \neq 2 \} \Rightarrow A \text{ is not diagonalizable,}$$

(for $a=5$)

$$A - 2I = \begin{pmatrix} -4 & 3 & 1 \\ 0 & -3 & 3 \\ 0 & -3 & 3 \end{pmatrix} \Rightarrow \ker(A - 2I) = \{ \neq 2 \} \Rightarrow A \text{ is not diagonalizable,}$$

(for $a=-1$)

$$\Rightarrow A \text{ is diagonalizable} \Leftrightarrow a > 5 \text{ or } a < -1$$

國立成功大學 111 學年度「碩士班」研究生甄試入學考試

線性代數

In this test, all vector spaces are finite dimensional over \mathbb{C} .

- ✓ 1. (15 points) Let T be a linear operator on a vector space V . Prove that T is diagonalizable if and only if its minimal polynomial is square-free.

- ✓ 2. (15 points) Let V be a vector space. A linear operator S on V is semisimple if for every S -invariant subspace W of V there exists an S -invariant subspace W' of V such that $V = W \oplus W'$. Prove that every diagonalizable operator on V is semisimple, and deduce that every linear operator T on V can be decomposed uniquely as $T = S + N$, where S is semisimple, N is nilpotent, and $SN = NS$.

- ✓ 3. (15 points) Let T and U be normal operators on an inner product space V such that $TU = UT$. Prove that $UT^* = T^*U$, where T^* is the adjoint of T .

- ✓ 4. (15 points) Let T and U be Hermitian operators on an inner product space $(V, \langle \cdot, \cdot \rangle)$ such that $\langle T(x), x \rangle > 0$ for all nonzero $x \in V$. Prove that UT is diagonalizable and has only real eigenvalues.

- ✓ 5. (15 points) Find the total number of distinct equivalence classes of congruent $n \times n$ real symmetric matrices and justify your answer.

- ✓ 6. (15 points) Let A be an $n \times n$ complex matrix, t be a variable, and I be the identity matrix. Prove that

$$\det(I - tA) = \exp \left(- \sum_{i \geq 1} \frac{\text{tr}(A^i)t^i}{i} \right).$$

- ✓ 7. (10 points) Let $A = (a_{i,j})$ be a $2n \times 2n$ matrix such that $A^T = -A$. The Phaffian of A is defined as

$$\text{pf}(A) := \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) \prod_{i=1}^n a_{\sigma(2i-1), \sigma(2i)},$$

where S_{2n} is the symmetric group of order $2n$ and $\text{sgn}(\sigma)$ is the signature of σ . Prove that for any $2n \times 2n$ matrix B ,

$$\text{pf}(BAB^T) = \det(B) \text{pf}(A).$$

Prob. 1

\Leftarrow if T is diagonalizable, Then $T = \theta^{-1}D\theta$, Note that $p(T)$ can be rewritten by $\theta^{-1}p(D)\theta$, Let $p(t) = \prod_{i=1}^n (t - c_i)$, c_i is an eigenvalue of T . $\forall v \in V$, $v = \sum_{i=1}^n v_i$

$$v_i \in \ker(T - c_i I) \quad p(T)v = \prod_{i=1}^n (T - c_i I)v = \sum_{i=1}^n (T - c_i I)v_i = 0, \text{ so } p(T)v = 0$$

\Rightarrow By definition of min-poly of $T \Rightarrow \text{min-poly} | p$. so min-poly of T is square free.

\Rightarrow Let W be the subspace spanned by all of the char-vector of T , and suppose $W \neq V$, We need follows lemma

Let $\dim V < \infty$ over \mathbb{F} , Let $p(x) = (x - c_1)^{r_1} \times \dots \times (x - c_n)^{r_n}$ $c_i \in \mathbb{F}$.

Let $W \subset V$, $T(W) \subseteq W$, there exist a vec $\alpha \in V$ s.t.

(a) $\alpha \notin W \quad (T - c_i)\alpha \in W$, for some c_i

Thus, $\exists \alpha \in W$ and c_i of T s.t. $\beta = (T - c_i I)\alpha \in W$.

$\Rightarrow \beta = \beta_1 + \dots + \beta_k$, $T\beta_i = c_i \beta_i \quad 1 \leq i \leq k$, Thus,

$\forall h(x) \in \mathbb{F}[x]$, $h(T)\beta = h(c_1)\beta_1 + \dots + h(c_k)\beta_k \in W$

Now $p = (x - c_i)g$, $g \in \mathbb{F}[x]$. Also

$$\underline{g(x) - g(c_i) = (x - c_i)h} \quad \text{for some } h \in \mathbb{F}[x]$$

$$\underline{\Rightarrow [g(T) - g(c_i)]\alpha = (T - c_i)h\alpha = h(T - c_i)\alpha \in W}$$

and $0 = p(T)\alpha = (T - c_i)g(T)\alpha$, $\underline{g(T)\alpha \text{ is eig-vec.}}$

$$g(T)\alpha \in W \Rightarrow g(c_i)\alpha = g(T)\alpha - h(T - c_i)\alpha \in W$$

Since $\alpha \notin W \Rightarrow \underline{g(c_i) = 0} \Rightarrow$ That is contradicts the

face that p has distinct roots

Prob. 2 PDT, semisimple operator,

Note if $T(W) \subseteq W$, then $V = W \oplus W'$ iff W is T -admissible

$$\begin{cases} \text{if } T(W) \subseteq W, \quad \forall p \in V, \quad p = w + w', \quad f(T)p = f(T)w + f(T)w' \\ \quad V = W \oplus W' \\ \Rightarrow f(T)p \in W \Rightarrow \frac{f(T)p}{\in W} = \frac{f(T)w}{\in W} \quad (f(T)w' = 0) \end{cases}$$

Note if min-poly of T is irr over F , T is semi-simple

Let $W \leq V$, $T(W) \subseteq W \Rightarrow$ Test $f(T)p \in W \rightarrow f(T)p = f(T)\alpha$
for some $\alpha \in W$.

if $f(T)p = 0$, that is easy to see (p : min-poly of f)

$$\begin{aligned} \text{if } f(T)p \neq 0, \quad p + f &\Rightarrow \gcd(p, f) = 1 \Rightarrow \exists g_1, g_2 \quad g_1 p + g_2 f = 1 \\ &\Rightarrow g_1(T)p(T) + g_2(T)f(T) = I \Rightarrow I = g_2(T)f(T) \\ &\Rightarrow p = \frac{g_2(T)}{g_2(T)(W) \subseteq W} \frac{f(T)p}{\in W} \Rightarrow p \in W \end{aligned}$$

Note T is semi-simple \Leftrightarrow min-poly of T is a product of irr poly
(distinct)

if T is semi-simple and $g^2 | p$ g is irr, let $W = \ker g(T)$

$T(W) \subseteq W$ obvious. $p = g^2 h \Rightarrow g(T)h(T) \neq 0$, $\exists p$ s.t. $g(T)h(T)p \neq 0$

\Rightarrow but $g(T)h(T) \in W$ and if $gh(T)p = gh(T)\alpha$, $\alpha \in W$

$gh\alpha = hg\alpha = h(g\alpha) = 0 \rightarrow W$ is not T -admissible $\rightarrow V \neq W \oplus W'$

if min-poly of T = product of irr poly, let $W \leq V$, $T(W) \subseteq W$
(distinct)

Use PDT, $V = W_1 \oplus \cdots \oplus W_n$ $W_i = \ker p_i(T)$, $T_i = T|_{W_i}$

$\Rightarrow W \cap W_i \leq W_i$ and $T_i(W \cap W_i) \subseteq W \cap W_i$, by previous note

$$W_i = V_i \oplus W \cap W_i, \quad V = (\bigoplus_{i=1}^n V_i) \oplus (\bigoplus_{i=1}^n W \cap W_i) = W' \oplus W$$

Thus, recall that \mathbb{C} is algebraically closed field, if min-poly of T is a product of distinct irr poly mean $\text{min-poly} = \prod_{i=1}^n (x - c_i)$

By theorem, T is diagonalizable

$$\text{By PDT, } V = W_1 \oplus \cdots \oplus W_n, \quad W_j = \ker(T - c_j)^{r_j} \quad p(t) = \prod_{j=1}^n (t - c_j)^{r_j}$$

$$\text{So } I = E_1 + \cdots + E_n, \quad E_j = (T - c_j)^{r_j} \quad T = TE_1 + TE_2 + \cdots + TE_n$$

and let $D = c_1 E_1 + \cdots + c_n E_n \Rightarrow$ By theorem D is diagonalizable.

therefore, by note, D is semi-simple.

Take $H = T - D = (T - c_1)E_1 + \cdots + (T - c_n)E_n$, Then

$$H^{\max r_j} = (T - c_1)^r E_1 + \cdots + (T - c_n)^r E_n = 0$$

Recall that E_j has a poly form of T , Hence D and H commute.

Prob. 4 Symmetric / positive

(a) diagonalizable, since T is positive, $T = R^*$ for some Hermitian operator

then $LT = L R^* = \bar{R}^* R L R$, where $R L R$ is a Hermitian

so LT is similar to Hermitian operator, so LT is diagonalizable

(b) Moreover, LT has only real eigenvalue since $R L R$ has only real eig-value.

Prob. 3 Simultaneous Diagonalizable

Since $LT = TL$, T and L can be diagonal use same P , i.e. $T = \bar{P}^* D_1 P$ and $L = \bar{P}^* D_2 P$

Note that T, T^* use same eig-vec but conjugate eig-value, $T^* = \bar{P}^* D_1^* P$

Therefore, $LT^* = \bar{P}^* D_2 P^* D_1^* P = \bar{P}^* D_1^* D_2 P = T^* L$

Prob. 6 ODE

Note that if we define $f(t) = \det(I - tA)$, $f(0) = 1$, use technique in ODE, We show that $f'(t) + p(t)f(t) = 0$

$$\Rightarrow f(t) = f(0) \exp \left(\int_0^t p(s) ds \right), \quad p(t) = \sum_{i=0}^{\infty} \text{tr}(A^{i+1}) t^i *$$

Note that $\det(I - tA) = (1 - \lambda_1 t)(1 - \lambda_2 t) \cdots (1 - \lambda_n t)$, where λ_j $1 \leq j \leq n = \dim V$ is eig-value of A (no need to be distinct).

$$\begin{aligned} \frac{df}{dt} &= (-\lambda_1) \times \prod_{j \neq 1} (1 - \lambda_j t) + (-\lambda_2) \prod_{j \neq 2} (1 - \lambda_j t) + \cdots + (-\lambda_n) \prod_{j \neq n} (1 - \lambda_j t) \\ &= \frac{-\lambda_1}{1 - \lambda_1 t} \prod_{j \neq 1} (1 - \lambda_j t) + \cdots + \frac{-\lambda_n}{1 - \lambda_n t} \prod_{j \neq n} (1 - \lambda_j t) \\ &= \left(\sum_{j=1}^n \frac{-\lambda_j}{1 - \lambda_j t} \right) f \quad \text{Note that } \text{tr}A = \lambda_1 + \cdots + \lambda_n ; \quad \text{tr}A^k = \lambda_1^k + \cdots + \lambda_n^k \\ &= \left(\sum_{j=1}^n -\lambda_j \sum_{k=0}^{\infty} (\lambda_j t)^k \right) f \quad \sum_{j=1}^n \left(\sum_{k=0}^{\infty} -\lambda_j^{k+1} t^k \right) = \sum_{k=0}^{\infty} -\text{tr}(A^{k+1}) t^k \\ &= \left(\sum_{k=0}^{\infty} -\text{tr}(A^{k+1}) t^k \right) f \quad \Rightarrow \frac{df}{dt} + \left(\sum_{k=0}^{\infty} \text{tr}(A^{k+1}) t^k \right) f = 0 * \end{aligned}$$

Prob. 5 Congruent $n \times n$ real symmetric matrix (Sylvester's law of inertia)

Note: A, B is congruent if $A = P^t B P$ for some $P \in GL(\mathbb{R}^n)$

Sylvester's law of inertia state that

$A, B \in M_{n \times n}(\mathbb{R})$, A, B are symmetric, Then

A and B is congruent \Leftrightarrow the number of negative / positive λ_j are same

If A is symmetric, $A = Q^T D Q$ for some Q is a orthonormal matrix,

Since D entry is real, can be rewritten as $W \Sigma W^T$, where Σ is a diagonal matrix

with entries $= +1, -1, 0$, W is a diagonal matrix with $w_{ii} = \sqrt{|D_{ii}|}$

So $A = Q^T W \Sigma W^T Q \rightarrow A$ is congruent with Σ
 \rightarrow if A is congruent with $B \Rightarrow A, B$ has same number
 or $+, -$ eig-value.

\Rightarrow Therefore, the different eig-class of congruent class is

$$x+y+z = n, \quad x = \# \text{ of } +1, \# \text{ of } -1, \# \text{ of } 0, \quad x,y,z \geq 0$$

$$\Rightarrow (x,y,z) \text{ has } \frac{(n+2)!}{n! 2!} = \frac{(n+2)(n+1)}{2} \#$$