HWs

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CONTENTS

Chapter 1

HW for General Analysis

1.1 HW1

Theorem 1.1.1. $\mathbb{R}^n$ is complete.

Proof. Let $\mathbf{x}_k$ be an arbitrary Cauchy sequence in $\mathbb{R}^n$. We are required to show $\mathbf{x}_k$ converge in $\mathbb{R}^n$. For each k, denote $\mathbf{x}_k$ by $(x_{(1,k)}, \ldots, x_{(n,k)})$. We claim that for each $i \in \{1, \ldots, n\}$

$$x_{(i,k)} \text{ is a Cauchy sequence}$$

Fix i and $\epsilon > 0$. To show $x_{(i,k)}$ is a Cauchy sequence, we are required to find $N \in \mathbb{N}$ such that for all $r, m \geq N$ we have

$$\left| x_{(i,r)} - x_{(i,m)} \right| \leq \epsilon$$

Because $\mathbf{x}_k$ is a Cauchy sequence in $\mathbb{R}^n$, we know there exists $N \in \mathbb{N}$ such that for all $r, m \geq N$, we have

$$\left| \mathbf{x}_r - \mathbf{x}_m \right| < \epsilon$$

Fix such N and arbitrary $r, m \geq M$. Observe

$$\left| x_{(i,r)} - x_{(i,m)} \right| \leq \sqrt{\sum_{j=1}^{n} \left| x_{(j,r)} - x_{(j,m)} \right|^2} = \left| \mathbf{x}_r - \mathbf{x}_m \right| < \epsilon$$

We have proved that for each $i \in \{1, \ldots, n\}$, the real sequence $x_{(i,k)}$ is Cauchy. We now claim that for each $i \in \{1, \ldots, n\}$, we have

$$\limsup_{r \to \infty} x_{(i,r)} \in \mathbb{R} \text{ and } \lim_{k \to \infty} x_{(i,k)} = \limsup_{r \to \infty} x_{(i,r)}$$

Again fix i. Because $x_{(i,k)}$ is a Cauchy sequence, we know there exists some N such that for all $r, m \geq N$, we have

$$\left| x_{(i,r)} - x_{(i,m)} \right| < 1$$

This implies that for all $r \geq N$, we have

$$x_{(i,r)} < x_{(i,N)} + 1 \tag{1.1}$$

Equation 1.1 then tell us

$$x_{(i,N)} + 1 \text{ is an upper bound of } \{x_{(i,r)} : r \geq N\}$$

Then by definition of sup, we have

$$\sup\{x_{(i,r)} : r \geq N\} \leq x_{(i,N)} + 1 \in \mathbb{R}$$

This then implies $\limsup_{r \to \infty} x_{(i,r)} \in \mathbb{R}$. We now prove

$$\lim_{k \to \infty} x_{(i,k)} = \limsup_{r \to \infty} x_{(i,r)} \tag{1.2}$$

Fix $\epsilon > 0$. We are required to find N such that

$$\forall k \geq N, \left| x_{(i,k)} - \limsup_{r \to \infty} x_{(i,r)} \right| \leq \epsilon$$

Because $\{x_{(i,k)}\}_{k \in \mathbb{N}}$ is a Cauchy sequence, we can let N_0 satisfy

$$\forall k, m \geq N_0, \left| x_{(i,k)} - x_{(i,m)} \right| < \frac{\epsilon}{2}$$

Because $\sup\{x_{(i,k)} : k \geq N'\} \searrow \limsup_{r \to \infty} x_{(i,r)}$ as $N' \to \infty$, we know there exists $N_1 > N_0$ such that

$$\limsup_{r \to \infty} x_{(i,r)} - \frac{\epsilon}{2} < \sup\{x_{(i,k)} : k \geq N_0\} \leq \limsup_{r \to \infty} x_{(i,r)} + \frac{\epsilon}{2}$$

Then because $\limsup_{r \to \infty} x_{(i,r)} - \frac{\epsilon}{2}$ is strictly smaller than the smallest upper bound of $\{x_{(i,k)} : k \geq N_1\}$, we see $\limsup_{n \to \infty} x_{(i,r)} - \frac{\epsilon}{2}$ is not an upper bound of $\{x_{(i,k)} : k \geq N_1\}$. This implies the existence of some N such that $N \geq N_1$ and

$$\limsup_{r \to \infty} x_{(i,r)} - \frac{\epsilon}{2} < x_{(i,N)} \leq \limsup_{r \to \infty} x_{(i,r)} + \frac{\epsilon}{2}$$

Now observe that for all $k \geq N$, because $N \geq N_1 \geq N_0$

$$\limsup_{r \to \infty} x_{(i,r)} - \epsilon < x_{(i,N)} - \frac{\epsilon}{2} < x_{(i,k)} < x_{(i,N)} + \frac{\epsilon}{2} \leq \limsup_{r \to \infty} x_{(i,r)} + \epsilon$$

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This implies for all $k \geq N$, we have

$$\left| x_{(i,k)} - \limsup_{r \to \infty} x_{(i,r)} \right| \leq \epsilon$$

We have just proved Equation 1.2. Lastly, to close out the proof, we show

$$\lim_{k \to \infty} \mathbf{x}_k = \left(\lim_{k \to \infty} x_{(1,k)}, \ldots, \lim_{k \to \infty} x_{(n,k)} \right) \tag{1.3}$$

Fix $\epsilon > 0$. For each $i \in \{1, \ldots, n\}$, let N_i satisfy

$$\forall r \geq N_i, \left| x_{(i,r)} - \lim_{k \to \infty} x_{(i,k)} \right| \leq \frac{\epsilon}{\sqrt{n}}$$

Observe that for all $r \geq \max_{i \in \{1,\ldots,n\}} N_i$, we have

$$\left| \mathbf{x}_r - \left(\lim_{k \to \infty} x_{(1,k)}, \ldots, \lim_{k \to \infty} x_{(n,k)} \right) \right| = \sqrt{\sum_{i=1}^n \left| x_{(i,r)} - \lim_{k \to \infty} x_{(i,k)} \right|^2}$$

$$\leq \sqrt{\sum_{i=1}^n \frac{\epsilon^2}{n}} = \epsilon$$

We have proved Equation 1.2.

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Theorem 1.1.2. ($\mathbb{Q}$ is dense in $\mathbb{R}$)

Chapter 2

Complex Analysis HW

2.1 HW1

Theorem 2.1.1.

$$(1+i)^n, \frac{(1+i)^n}{n}, \frac{n!}{(1+i)^n} \text{ all diverge as } n \to \infty$$

Proof. Note that

$$|(1+i)^n| = 2^{\frac{n}{2}} \to \infty \text{ as } n \to \infty$$

This implies $(1+i)$ is unbounded, thus diverge.

Note that

$$\left| \frac{(1+i)^n}{n} \right| = \frac{(\sqrt{2})^n}{n}$$

Observe

$$\frac{(\sqrt{2})^n}{n} = \frac{[(\sqrt{2}-1)+1]^n}{n} = \frac{\sum_{k=0}^{n} \binom{n}{k}(\sqrt{2}-1)^k}{n}$$

$$\geq \frac{\binom{n}{2}(\sqrt{2}-1)^2}{n} = (n-1)[\frac{(\sqrt{2}-1)^2}{2}] \to \infty \text{ as } n \to \infty$$

This implies $\frac{(1+i)^n}{n}$ is unbounded, thus diverge.

Note that

$$\left| \frac{n!}{(1+i)^n} \right| = \frac{n!}{(\sqrt{2})^n}$$

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Note that for all $k \geq 8$, we have

$$\frac{k}{\sqrt{2}} \geq \frac{\sqrt{8}}{\sqrt{2}} = 2$$

This implies

$$\frac{n!}{(\sqrt{2})^n} = \prod_{k=1}^{n} \frac{k}{\sqrt{2}} = \frac{7!}{(\sqrt{2})^7} \prod_{k=8}^{n} \frac{k}{\sqrt{2}} \geq \frac{7!}{(\sqrt{2})^7} \prod_{k=8}^{n} 2 \geq \frac{7! 2^{n-8+1}}{(\sqrt{2})^7} \to \infty$$

which implies $\frac{n!}{(1+i)^n}$ is unbounded, thus diverge. ∎

Theorem 2.1.2.

$$n! z^n \text{ converge } \iff z = 0$$

Proof. If $z = 0$, then $n! z^n = 0$ for all n, which implies $n! z^n \to 0$. Now, suppose $z \neq 0$. Let $M \in \mathbb{N}$ satisfy $|z| > \frac{1}{M}$. Observe

$$|n! z^n| = \left| \prod_{k=1}^{n} kz \right| = \left| \prod_{k=1}^{2M-1} kz \right| \left| \prod_{k=2M}^{n} kz \right| \geq \left| \prod_{k=1}^{2M-1} kz \right| \left| \prod_{k=2M}^{n} 2Mz \right| \geq \left| \prod_{k=1}^{2M-1} kz \right| 2^{n-2M+1} \to \infty$$

This implies $n! z^n$ is unbounded, thus diverge. ∎

Theorem 2.1.3.

$$u_n \to u \implies v_n \triangleq \sum_{k=1}^{n} \frac{u_k}{n} \to u$$

Proof. Because

$$\sum_{k=1}^{n} \frac{u_k}{n} = \sum_{k \leq \sqrt{n}} \frac{u_k}{n} + \sum_{\sqrt{n} < k \leq n} \frac{u_k}{n}$$

It suffices to prove

$$\sum_{k \leq \sqrt{n}} \frac{u_k}{n} \to 0 \text{ and } \sum_{\sqrt{n} < k \leq n} \frac{u_k}{n} \to u \text{ as } n \to \infty$$

Because u_n converge, we can let M bound $|u_n|$. Observe

$$\left| \sum_{k \leq \sqrt{n}} \frac{u_k}{n} \right| \leq \sum_{k \leq \sqrt{n}} \left| \frac{u_k}{n} \right| \leq \sum_{k \leq \sqrt{n}} \frac{M}{n} \leq \frac{M\sqrt{n}}{n} = \frac{M}{\sqrt{n}} \to 0 \text{ as } n \to 0 \text{ (done)}$$

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Because

$$\sum_{\sqrt{n}<k\leq n} \frac{u_k}{n} = \frac{n-\lceil\sqrt{n}\rceil+1}{n} \sum_{\sqrt{n}<k\leq n} \frac{u_k}{n-\lceil\sqrt{n}\rceil+1}$$

and

$$\lim_{n\to\infty} \frac{n-\lceil\sqrt{n}\rceil+1}{n} = 1$$

We can reduce the problem into proving

$$\lim_{n\to\infty} \sum_{\sqrt{n}<k\leq n} \frac{u_k}{n-\lceil\sqrt{n}\rceil+1} = u$$

Fix ϵ. Let N satisfy that for all $n \geq N$, we have $|u_n - u| < \epsilon$. Then for all $n \geq N^2$, we have

$$\left|\left(\sum_{\sqrt{n}<k\leq n} \frac{u_k}{n-\lceil\sqrt{n}\rceil+1}\right) - u\right| = \left|\sum_{\sqrt{n}<k\leq n} \frac{u_k-u}{n-\lceil\sqrt{n}\rceil+1}\right|$$

$$\leq \sum_{\sqrt{n}<k\leq n} \frac{|u_k-u|}{n-\lceil\sqrt{n}\rceil+1}$$

$$\leq \sum_{\sqrt{n}<k\leq n} \frac{\epsilon}{n-\lceil\sqrt{n}\rceil+1} = \epsilon \text{ (done)}$$

$\blacksquare$