

HWs

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Chapter 1

General Analysis HW

1.1 HW1

Question 1

Show \mathbb{R}^n is complete.

Proof. Let \mathbf{x}_k be an arbitrary Cauchy sequence in \mathbb{R}^n . We are required to show \mathbf{x}_k converge in \mathbb{R}^n . For each k , denote \mathbf{x}_k by $(x_{(1,k)}, \dots, x_{(n,k)})$. We claim that for each $i \in \{1, \dots, n\}$

$x_{(i,k)}$ is a Cauchy sequence

Fix i and $\epsilon > 0$. To show $x_{(i,k)}$ is a Cauchy sequence, we are required to find $N \in \mathbb{N}$ such that for all $r, m \geq N$ we have

$$|x_{(i,r)} - x_{(i,m)}| \leq \epsilon$$

Because \mathbf{x}_k is a Cauchy sequence in \mathbb{R}^n , we know there exists $N \in \mathbb{N}$ such that for all $r, m \geq N$, we have

$$|\mathbf{x}_r - \mathbf{x}_m| < \epsilon$$

Fix such N and arbitrary $r, m \geq M$. Observe

$$|x_{(i,r)} - x_{(i,m)}| \leq \sqrt{\sum_{j=1}^n |x_{(j,r)} - x_{(j,m)}|^2} = |\mathbf{x}_r - \mathbf{x}_m| < \epsilon$$

We have proved that for each $i \in \{1, \dots, n\}$, the real sequence $x_{(i,k)}$ is Cauchy. We now claim that for each $i \in \{1, \dots, n\}$, we have

$$\limsup_{r \rightarrow \infty} x_{(i,r)} \in \mathbb{R} \text{ and } \lim_{k \rightarrow \infty} x_{(i,k)} = \limsup_{r \rightarrow \infty} x_{(i,r)}$$

Again fix i . Because $x_{(i,k)}$ is a Cauchy sequence, we know there exists some N such that for all $r, m \geq N$, we have

$$|x_{(i,r)} - x_{(i,m)}| < 1$$

This implies that for all $r \geq N$, we have

$$x_{(i,r)} < x_{(i,N)} + 1 \quad (1.1)$$

Equation 1.1 then tell us

$$x_{(i,N)} + 1 \text{ is an upper bound of } \{x_{(i,r)} : r \geq N\}$$

Then by definition of sup, we have

$$\sup\{x_{(i,r)} : r \geq N\} \leq x_{(i,N)} + 1 \in \mathbb{R}$$

This then implies $\limsup_{r \rightarrow \infty} x_{(i,r)} \in \mathbb{R}$. We now prove

$$\lim_{k \rightarrow \infty} x_{(i,k)} = \limsup_{r \rightarrow \infty} x_{(i,r)} \quad (1.2)$$

Fix $\epsilon > 0$. We are required to find N such that

$$\forall k \geq N, \left| x_{(i,k)} - \limsup_{r \rightarrow \infty} x_{(i,r)} \right| \leq \epsilon$$

Because $\{x_{(i,k)}\}_{k \in \mathbb{N}}$ is a Cauchy sequence, we can let N_0 satisfy

$$\forall k, m \geq N_0, |x_{(i,k)} - x_{(i,m)}| < \frac{\epsilon}{2}$$

Because $\sup\{x_{(i,k)} : k \geq N'\} \searrow \limsup_{r \rightarrow \infty} x_{(i,r)}$ as $N' \rightarrow \infty$, we know there exists $N_1 > N_0$ such that

$$\limsup_{r \rightarrow \infty} x_{(i,r)} - \frac{\epsilon}{2} < \sup\{x_{(i,k)} : k \geq N_0\} \leq \limsup_{r \rightarrow \infty} x_{(i,r)} + \frac{\epsilon}{2}$$

Then because $\limsup_{r \rightarrow \infty} x_{(i,r)} - \frac{\epsilon}{2}$ is strictly smaller than the smallest upper bound of $\{x_{(i,k)} : k \geq N_1\}$, we see $\limsup_{n \rightarrow \infty} x_{(i,r)} - \frac{\epsilon}{2}$ is not an upper bound of $\{x_{(i,k)} : k \geq N_1\}$. This implies the existence of some N such that $N \geq N_1$ and

$$\limsup_{r \rightarrow \infty} x_{(i,r)} - \frac{\epsilon}{2} < x_{(i,N)} \leq \limsup_{r \rightarrow \infty} x_{(i,r)} + \frac{\epsilon}{2}$$

Now observe that for all $k \geq N$, because $N \geq N_1 \geq N_0$

$$\limsup_{r \rightarrow \infty} x_{(i,r)} - \epsilon < x_{(i,N)} - \frac{\epsilon}{2} < x_{(i,k)} < x_{(i,N)} + \frac{\epsilon}{2} \leq \limsup_{r \rightarrow \infty} x_{(i,r)} + \epsilon$$

This implies for all $k \geq N$, we have

$$\left| x_{(i,k)} - \limsup_{r \rightarrow \infty} x_{(i,r)} \right| \leq \epsilon$$

We have just proved [Equation 1.2](#). Lastly, to close out the proof, we show

$$\lim_{k \rightarrow \infty} \mathbf{x}_k = \left(\lim_{k \rightarrow \infty} x_{(1,k)}, \dots, \lim_{k \rightarrow \infty} x_{(n,k)} \right) \quad (1.3)$$

Fix $\epsilon > 0$. For each $i \in \{1, \dots, n\}$, let N_i satisfy

$$\forall r \geq N_i, \left| x_{(i,r)} - \lim_{k \rightarrow \infty} x_{(i,k)} \right| \leq \frac{\epsilon}{\sqrt{n}}$$

Observe that for all $r \geq \max_{i \in \{1, \dots, n\}} N_i$, we have

$$\begin{aligned} \left| \mathbf{x}_r - \left(\lim_{k \rightarrow \infty} x_{(1,k)}, \dots, \lim_{k \rightarrow \infty} x_{(n,k)} \right) \right| &= \sqrt{\sum_{i=1}^n \left| x_{(i,r)} - \lim_{k \rightarrow \infty} x_{(i,k)} \right|^2} \\ &\leq \sqrt{\sum_{i=1}^n \frac{\epsilon^2}{n}} = \epsilon \end{aligned}$$

We have proved [Equation 1.3](#). ■

Question 2

Show \mathbb{Q} is dense in \mathbb{R} .

Proof. Fix $x \in \mathbb{R}$ and $\epsilon > 0$. To show \mathbb{Q} is dense in \mathbb{R} , we have to find $q \in \mathbb{Q}$ such that $|x - q| < \epsilon$.

Let $m \in \mathbb{N}$ satisfy $\frac{1}{m} < \epsilon$. Let n be the largest integer such that $n \leq mx$. Because n is the largest integer such that $n \leq mx$, we know $mx - n < 1$, otherwise we can deduce $n + 1 \leq mx$, which is impossible, since $n + 1$ is an integer and n is the largest integer such that $n \leq mx$. We now see that

$$\frac{n}{m} \in \mathbb{Q} \text{ and } \left| x - \frac{n}{m} \right| = \frac{mx - n}{m} < \frac{1}{m} < \epsilon$$
■

Theorem 1.1.1. (Distance Formula) Given two subsets A, B of a metric space, we have

$$d(A, B) = \inf_{b \in B} d(A, b)$$

Proof. Fix arbitrary $b \in B$. It is clear that

$$d(A, B) \leq d(A, b)$$

It then follows $d(A, B) \leq \inf_{b \in B} d(A, b)$. Fix arbitrary $a \in A$ and $b_0 \in B$. Observe that

$$d(a, b_0) \geq d(A, b_0) \geq \inf_{b \in B} d(A, b)$$

It then follows $\inf_{b \in B} d(A, b) \leq d(A, B)$. ■

Question 3

Let E_1, E_2 be non-empty sets in \mathbb{R}^n with E_1 closed and E_2 compact. Show that there are points $x_1 \in E_1$ and $x_2 \in E_2$ such that

$$d(E_1, E_2) = |x_1 - x_2|$$

Deduce that $d(E_1, E_2)$ is positive if such E_1, E_2 are disjoint.

Proof. Because

(a) $f(x) \triangleq d(E_1, x)$ is a continuous function on \mathbb{R}^n .

(b) E_2 is compact.

It now follows by EVT there exists some $x_2 \in E_2$ such that

$$d(E_1, x_2) = \min_{x \in E_2} d(E_1, x) = \inf_{x \in E_2} d(E_1, x) = d(E_1, E_2)$$

where the last equality is proved above. We can now reduce the problem into finding x_1 in E_1 such that

$$d(x_1, x_2) = d(E_1, x_2)$$

For each $n \in \mathbb{N}$, let t_n satisfy

$$t_n \in E_1 \text{ and } d(t_n, x_2) < d(E_1, x_2) + \frac{1}{n}$$

Clearly, t_n is a bounded sequence. Then by Bolzano-Weierstrass Theorem, there exists a convergence subsequence t_{n_k} . Now, because E_1 is closed, we know

$$x_1 \triangleq \lim_{k \rightarrow \infty} t_{n_k} \in E_1$$

It then follows from the function $f(x) \triangleq d(x, x_2)$ being continuous on \mathbb{R}^n such that

$$d(x_1, x_2) = \lim_{k \rightarrow \infty} d(t_{n_k}, x_2) = d(E_1, x_2)$$

■

Question 4

Prove that the distance between two nonempty, compact, disjoint sets in \mathbb{R}^n is positive.

Proof. The proof follows from the result in last question while acknowledging compact is closed. ■

Question 5

Prove that if f is continuous on $[a, b]$, then f is Riemann-integrable on $[a, b]$.

Proof. Let $\overline{\int_a^b} f dx$ and $\underline{\int_a^b} f dx$ respectively denote the upper and lower Darboux sums. We prove that

$$\overline{\int_a^b} f dx = \underline{\int_a^b} f dx$$

Fix ϵ . We reduce the problem into proving the existence of some partition $\{a = x_0, x_1, \dots, x_n = b\}$ such that

$$\sum_{i=1}^n [M_i - m_i] (x_i - x_{i-1}) \leq \epsilon$$

where

$$M_i \triangleq \sup_{t \in [x_{i-1}, x_i]} f(t) \text{ and } m_i \triangleq \inf_{t \in [x_{i-1}, x_i]} f(t)$$

Because f is continuous on the compact interval $[a, b]$, we know f is uniformly continuous on $[a, b]$. Let δ satisfy

$$|x - y| < \delta \text{ and } x, y \in [a, b] \implies |f(x) - f(y)| < \frac{\epsilon}{b - a}$$

Let n satisfy $\frac{b-a}{n} < \delta$. We claim the partition

$$\{a = x_0, x_1, \dots, x_n = b\} \text{ where } x_i \triangleq a + \frac{i(b-a)}{n} \text{ suffices}$$

Now, by EVT, we know that for each i , there exists some $t_{i,M}, t_{i,m} \in [x_{i-1}, x_i]$ such that

$$f(t_{i,m}) = m_i \text{ and } f(t_{i,M}) = M_i$$

Then because

$$|t_{i,m} - t_{i,M}| \leq x_i - x_{i-1} \leq \frac{b-a}{n} < \delta$$

We know $M_i - m_i < \frac{\epsilon}{b-a}$. This now give us

$$\begin{aligned} \sum_{i=1}^n [M_i - m_i] (x_i - x_{i-1}) &< \sum_{i=1}^n \frac{\epsilon}{(b-a)} (x_i - x_{i-1}) \\ &= \frac{\epsilon}{b-a} \sum_{i=1}^n (x_i - x_{i-1}) \\ &= \frac{\epsilon}{b-a} (b-a) = \epsilon \end{aligned}$$

■

Question 6

Find $\limsup_{n \rightarrow \infty} E_n$ and $\liminf_{n \rightarrow \infty} E_n$ where

$$E_n \triangleq \begin{cases} [\frac{-1}{n}, 1] & \text{if } n \text{ is odd} \\ [-1, \frac{1}{n}] & \text{if } n \text{ is even} \end{cases}$$

Proof. Fix arbitrary $n \in \mathbb{N}$. Let $p, q \geq n$ respectively be odd and even. We see

$$[0, 1] \subseteq E_p \text{ and } [-1, 0] \subseteq E_q$$

This now implies

$$[-1, 1] \subseteq \bigcup_{k \geq n} E_k$$

Then because n is arbitrary, it follows

$$\limsup_{n \rightarrow \infty} E_n = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k = [-1, 1]$$

Again, fix arbitrary $n \in \mathbb{N}$ and $\epsilon > 0$. Let p, q respectively be even and odd integers greater than $\max\{n, \frac{1}{\epsilon}\}$. We now see

$$\epsilon \notin [-1, \frac{1}{p}] = E_p \text{ and } -\epsilon \notin [\frac{-1}{q}, 1] = E_q$$

Because ϵ is arbitrary and clearly $0 \in E_k$ for all k , we now see

$$\bigcap_{k \geq n} E_k = \{0\}$$

Then because n is arbitrary, we see

$$\liminf_{n \rightarrow \infty} E_n = \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} E_k = \{0\}$$

■

Question 7

Show that

$$(\limsup_{n \rightarrow \infty} E_n)^c = \liminf_{n \rightarrow \infty} (E_n)^c$$

and

$$E_n \searrow E \text{ or } E_n \nearrow E \implies \limsup_{n \rightarrow \infty} E_n = \liminf_{n \rightarrow \infty} E_n = E$$

Proof. Fix arbitrary $x \in (\limsup_{n \rightarrow \infty} E_n)^c$. We can deduce

$$\exists n, x \notin \bigcup_{k \geq n} E_k$$

This implies

$$\exists n, x \in \bigcap_{k \geq n} E_k^c$$

Then we see

$$x \in \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} E_k^c = \liminf_{n \rightarrow \infty} E_n^c$$

We have proved $(\limsup_{n \rightarrow \infty} E_n)^c \subseteq \liminf_{n \rightarrow \infty} E_n^c$. We now prove the converse. Fix arbitrary $x \in \liminf_{n \rightarrow \infty} E_n^c$. We can deduce

$$\exists n, x \in \bigcap_{k \geq n} E_k^c$$

This implies

$$\exists n, x \notin \bigcup_{k \geq n} E_k$$

Then we see

$$x \notin \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k = \limsup_{n \rightarrow \infty} E_n$$

■

Theorem 1.1.2. (Equivalent Definition for Limit Superior) If we let E be the set of subsequential limits of a_n

$$E \triangleq \{L \in \mathbb{R} : L = \lim_{k \rightarrow \infty} a_{n_k} \text{ for some } n_k\}$$

The set E is non-empty and

$$\max E = \limsup_{n \rightarrow \infty} a_n$$

Proof. Let $n_1 \triangleq 1$. Recursively, because

$$\sup_{j \geq n_k} a_j \geq \limsup_{n \rightarrow \infty} a_n > \limsup_{n \rightarrow \infty} a_n - \frac{1}{k} \text{ for each } k$$

We can let n_{k+1} be the smallest number such that

$$a_{n_{k+1}} > \limsup_{n \rightarrow \infty} a_n - \frac{1}{k}$$

It is straightforward to check $a_{n_k} \rightarrow \limsup_{n \rightarrow \infty} a_n$ as $k \rightarrow \infty$. Note that no subsequence can converge to $\limsup_{n \rightarrow \infty} a_n + \epsilon$ because there exists N such that $\sup_{k \geq N} a_k < \limsup_{n \rightarrow \infty} a_n + \epsilon$. ■

Question 8

Show that

$$\limsup_{n \rightarrow \infty} (-a_n) = -\liminf_{n \rightarrow \infty} a_n$$

Proof. Note that $-a_{n_k}$ converge if and only if a_{n_k} converge. Then if we respectively define E and E^- to be the set of subsequential limits of a_n and $-a_n$, we see

$$E^- = \{-L \in \mathbb{R} : L \in E\}$$

We now see

$$\limsup_{n \rightarrow \infty} (-a_n) = \max E^- = -\min E = -\liminf_{n \rightarrow \infty} a_n$$

■

Question 9

Show that

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n \quad (1.4)$$

Proof. Fix arbitrary ϵ . Let N_a, N_b respectively satisfy

$$\sup_{n \geq N_a} a_n \leq \limsup_{n \rightarrow \infty} a_n + \frac{\epsilon}{2} \text{ and } \sup_{n \geq N_b} b_n \leq \limsup_{n \rightarrow \infty} b_n + \frac{\epsilon}{2}$$

Let $N \triangleq \max\{N_a, N_b\}$. We now see that

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \sup_{n \geq N} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n + \epsilon$$

The result then follows from ϵ being arbitrary. ■

Question 10

$$a_n, b_n \text{ is bounded non-negative} \implies \limsup_{n \rightarrow \infty} (a_n b_n) \leq (\limsup_{n \rightarrow \infty} a_n)(\limsup_{n \rightarrow \infty} b_n) \quad (1.5)$$

Proof. There are three cases we should consider

- (a) Both $\limsup_{n \rightarrow \infty} a_n$ and $\limsup_{n \rightarrow \infty} b_n$ equal 0.
- (b) Between $\limsup_{n \rightarrow \infty} a_n$ and $\limsup_{n \rightarrow \infty} b_n$, only one of them equals 0.
- (c) Neither $\limsup_{n \rightarrow \infty} a_n$ nor $\limsup_{n \rightarrow \infty} b_n$ equals to 0.

In the first case, because a_n, b_n are both non-negative, we can deduce

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = 0$$

which implies

$$\limsup_{n \rightarrow \infty} (a_n b_n) = \lim_{n \rightarrow \infty} a_n b_n = 0 = \lim_{n \rightarrow \infty} a_n \lim_{n \rightarrow \infty} b_n$$

For second case, WOLG, suppose $\limsup_{n \rightarrow \infty} a_n = 0$. Fix arbitrary ϵ . We can let N satisfy

$$\sup_{n \geq N} a_n < \frac{\epsilon}{\sup_{n \in \mathbb{N}} b_n}$$

Since for all $n \geq N$, we have

$$a_n b_n \leq \frac{b_n \epsilon}{\sup_{k \in \mathbb{N}} b_k} \leq \epsilon$$

We now see

$$\limsup_{n \rightarrow \infty} (a_n b_n) \leq \sup_{n \geq N} a_n b_n \leq \epsilon$$

The result

$$\limsup_{n \rightarrow \infty} a_n b_n = 0 = \limsup_{n \rightarrow \infty} a_n \limsup_{n \rightarrow \infty} b_n$$

then follows from ϵ being arbitrary.

Lastly, for the last case, let N_a, N_b respectively satisfy

$$\sup_{n \geq N_a} a_n \leq \limsup_{n \rightarrow \infty} a_n \sqrt{1 + \epsilon} \text{ and } \sup_{n \geq N_b} b_n \leq \limsup_{n \rightarrow \infty} b_n \sqrt{1 + \epsilon}$$

Let $N \triangleq \max\{N_a, N_b\}$, because for each $n \geq N$, we have

$$a_n b_n \leq \left(\sup_{k \geq N_a} a_k \right) \left(\sup_{k \geq N_b} b_k \right) \leq (1 + \epsilon) \left(\limsup_{n \rightarrow \infty} a_n \right) \left(\limsup_{n \rightarrow \infty} b_n \right)$$

It then follows that

$$\limsup_{n \rightarrow \infty} (a_n b_n) \leq \sup_{n \geq N} (a_n b_n) \leq (1 + \epsilon) \left(\limsup_{n \rightarrow \infty} a_n \right) \left(\limsup_{n \rightarrow \infty} b_n \right)$$

The result then follows from ϵ being arbitrary. ■

Question 11

Show that if either a_n or b_n converge, the equalities in [Equation 1.4](#) and [Equation 1.5](#) both hold true.

Proof. WOLG, suppose $\lim_{n \rightarrow \infty} a_n = L \in \mathbb{R}$. We then see

$$(a_{n_k} + b_{n_k}) \text{ converge} \iff b_{n,k} \text{ converge}$$

Let $E_{a,b}$ and E_b respectively be the set of subsequential limits of $(a_n + b_n)$ and b_n . We now have

$$E_{a,b} = \{L + L_b \in \mathbb{R} : L_b \in E_b\}$$

This give us

$$\limsup_{n \rightarrow \infty} (a_n + b_n) = \max E_{a,b} = L + \max E_b = \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$$

Now, additionally, suppose a_n, b_n are both bounded and nonnegative. Again because

$$a_{n_k} b_{n,k} \text{ converge} \iff b_{n,k} \text{ converge}$$

We see

$$E_{a,b} = \{L(L_b) \in \mathbb{R} : L_b \in E_b\}$$

This give us

$$\limsup_{n \rightarrow \infty} (a_n b_n) = \max E_{a,b} = L \max E_b = (\limsup_{n \rightarrow \infty} a_n)(\limsup_{n \rightarrow \infty} b_n)$$

■

Question 12

Give example for which inequality in [Equation 1.4](#) and [Equation 1.5](#) are not equalities.

Proof. If

$$a_n \triangleq \begin{cases} 1 & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ is even} \end{cases} \quad \text{and} \quad b_n \triangleq \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

we have

$$\limsup_{n \rightarrow \infty} (a_n + b_n) = 0 < 2 = \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$$

Let $L > 1$ and

$$a_n \triangleq \begin{cases} L - \frac{1}{k} & \text{if } n = 2k - 1 \\ (L - \frac{1}{k})^{-1} & \text{if } n = 2k \end{cases} \quad \text{and} \quad b_n \triangleq \begin{cases} (L - \frac{1}{k})^{-1} & \text{if } n = 2k - 1 \\ (L - \frac{1}{k}) & \text{if } n = 2k \end{cases}$$

We have

$$\limsup_{n \rightarrow \infty} a_n b_n = 1 < L^2 = \limsup_{n \rightarrow \infty} a_n \limsup_{n \rightarrow \infty} b_n$$

■

Question 13

Give an example of a decreasing sequence of nonempty closed sets in \mathbb{R}^n whose intersection is empty.

Proof.

$$F_n \triangleq [n, \infty) \text{ suffices}$$

■

Question 14

Given an example of two disjoint, nonempty closed sets in E_1 and E_2 in \mathbb{R}^n for which $d(E_1, E_2) = 0$.

Proof. Let

$$E_1 \triangleq \left\{n - \frac{1}{n} \in \mathbb{R} : n \in \mathbb{N} \text{ and } n \geq 2\right\} \text{ and } E_2 \triangleq \left\{n - \frac{1}{2n} \in \mathbb{R} : n \in \mathbb{N} \text{ and } n \geq 2\right\}$$

To see $E_1 \cap E_2 = \emptyset$, suppose $n - \frac{1}{n} = k - \frac{1}{2k}$ where n, k are two natural numbers greater than 2. We then see $\frac{1}{n} - \frac{1}{2k} = n - k$, which is impossible, since

$$\left| \frac{1}{n} - \frac{1}{2k} \right| < \max\left\{\frac{1}{2k}, \frac{1}{n}\right\} < 1$$

The fact E_1, E_2 are closed follows from both of them being totally disconnected. Now observe that for all ϵ , there exists large enough n such that

$$\left(n + \frac{1}{n}\right) - \left(n + \frac{1}{2n}\right) < \frac{1}{n} < \epsilon$$

This implies $d(E_1, E_2) = 0$.

■

Question 15

If f is defined and uniformly continuous on E , show there is a function \bar{f} defined and continuous on \bar{E} such that $\bar{f} = f$ on E .

Proof. Define \bar{f} on E by $\bar{f} = f$. For each $x \in \bar{E} \setminus E$, associate x with a sequence $t_{n,x}$ in E converging to x . We now claim that for each $x \in \bar{E} \setminus E$ the limit

$$\lim_{n \rightarrow \infty} f(t_{n,x}) \text{ converge in } \mathbb{R}$$

Fix ϵ . Because f is uniformly continuous on E , we know there exists δ such that

$$a, b \in E \text{ and } |a - b| \leq \delta \implies |f(a) - f(b)| < \epsilon$$

Because $t_{n,x}$ converge, we know $t_{n,x}$ is Cauchy, then we know there exists N such that $|t_{n,x} - t_{m,x}| < \delta$ for all $n, m > N$, we then see that for all $n, m > N$, we have

$$|f(t_{n,x}) - f(t_{m,x})| < \epsilon$$

This implies $\{f(t_{n,x})\}_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathbb{R} , thus converge in \mathbb{R} .

Define

$$\bar{f}(x) \triangleq \lim_{n \rightarrow \infty} f(t_{n,x}) \text{ for all } x \in \bar{E} \setminus E$$

We are required to show \bar{f} is also continuous on $\bar{E} \setminus E$. Fix ϵ and $x \in \bar{E} \setminus E$. Let δ satisfy

$$a, b \in E \text{ and } |a - b| \leq \delta \implies |f(a) - f(b)| < \frac{\epsilon}{3}$$

We claim

$$\sup_{t \in B_{\frac{\delta}{2}}(x) \cap \bar{E}} |\bar{f}(t) - \bar{f}(x)| \leq \epsilon$$

Fix $t \in B_{\frac{\delta}{2}}(x) \cap \bar{E}$. There are two possibilities

(a) $t \in E$

(b) $t \in \bar{E} \setminus E$

If $t \in E$, let n satisfy

$$|f(t_{n,x}) - \bar{f}(x)| < \frac{\epsilon}{3} \text{ and } |t_{n,x} - x| < \frac{\delta}{2}$$

Because

$$|t_{n,x} - t| \leq |t_{n,x} - x| + |t - x| < \delta$$

we can deduce $|f(t_{n,x}) - f(t)| < \frac{\epsilon}{3}$. This now give us

$$|f(t) - \bar{f}(x)| \leq |f(t_{n,x}) - f(t)| + |f(t_{n,x}) - \bar{f}(x)| < \epsilon$$

If $t \in \bar{E} \setminus E$. Write $y = t$ and let $t_{n,y}$ be the associated sequence in E . Because $y \in B_{\frac{\delta}{2}}(x)$, we know there exists $t_{n,y}$ such that

$$t_{n,y} \in B_{\frac{\delta}{2}}(x) \text{ and } |f(t_{n,y}) - \bar{f}(y)| < \frac{\epsilon}{3}$$

Again, let m satisfy

$$t_{m,x} \in B_{\frac{\delta}{2}}(x) \text{ and } |f(t_{m,x}) - \bar{f}(x)| < \frac{\epsilon}{3}$$

We know $|t_{n,y} - t_{m,x}| \leq \delta$ because they both belong to $B_{\frac{\delta}{2}}(x)$. We can now deduce

$$|\bar{f}(y) - \bar{f}(x)| = |\bar{f}(y) - f(t_{n,y})| + |f(t_{n,y}) - f(t_{m,x})| + |f(t_{m,x}) - \bar{f}(x)| < \epsilon$$

which finish the proof. ■

Question 16

If f is defined and uniformly continuous on a bounded set E , show that f is bounded on E .

Proof. By last question, we can extend f to a continuous \bar{f} onto \bar{E} . Now because \bar{E} is compact and $|\bar{f}|$ is continuous on \bar{E} , by EVT, there exists $a \in \bar{E}$ such that

$$\sup_{x \in E} |f(x)| \leq \max_{x \in \bar{E}} |\bar{f}(x)| = \bar{f}(a)$$
■

1.2 HW2

Question 17

Construct a two-dimensional Cantor set in the unit square $[0, 1]^2$ as follows. Subdivide the square into nine equal parts and keep only the four closed corner squares, removing the remaining region (which form a cross). Then repeat this process in a suitably scaled version for the remaining squares, ad infinitum. Show that the resulting set is perfect, has plane measure zero, and equals $\mathcal{C} \times \mathcal{C}$.

Proof. Let $\mathcal{C}'_n \subseteq \mathbb{R}^2$ be the result after the n th stage of removal, and let $\mathcal{C}_n \subseteq \mathbb{R}$ be the result after the n th stage of removal in the construction of the classical ternary Cantor set. It is clear that

$$\mathcal{C}'_n = \mathcal{C}_n \times \mathcal{C}_n \text{ for all } n$$

It then follows

$$\bigcap_n \mathcal{C}'_n = \bigcap_n \mathcal{C}_n \times \mathcal{C}_n = \mathcal{C} \times \mathcal{C}$$

The fact that $\mathcal{C} \times \mathcal{C}$ has plane measure zero follows from [Lemma 1.2.1](#). Fix $(a, b) \in \mathcal{C} \times \mathcal{C}$. Because \mathcal{C} is perfect, there exists some $b' \in \mathcal{C}$ such that

$$0 < |b' - b| < \epsilon$$

To see that \mathcal{C}' is perfect, one see that

$$(a, b) \neq (a, b') \text{ and } (a, b') \in \mathcal{C}' \times \mathcal{C}' \text{ and } |(a, b) - (a, b')| = |b' - b| < \epsilon$$

■

Question 18

Construct a subset of $[0, 1]$ in the same manner as the Cantor set, except that at the k th stage, each interval removed has length $\delta 3^{-k}$, $0 < \delta < 1$. Show that the resulting set is perfect, has measure $1 - \delta$, and contains no intervals.

Proof. Let $\mathcal{C}'_n \subseteq \mathbb{R}$ be the result after the n th stage of removal according to the description. Clearly, each \mathcal{C}'_n has 2^n amount of connected component, we then can compute the length of $\mathcal{C}' \triangleq \bigcap \mathcal{C}'_n$ to be

$$1 - \sum_{k=1}^{\infty} 2^{k-1} \delta 3^{-k} = 1 - \frac{\delta \frac{3}{2}}{1 - \frac{2}{3}} = 1 - \delta$$

Since each \mathcal{C}'_n has 2^n amount of connected component of equal length and $\mathcal{C}'_n \subseteq [0, 1]$, we know the length of each connected component of \mathcal{C}'_n must not be greater than $\frac{1}{2^n}$. It then follows that no interval $[a, a + h]$ can be contained by all \mathcal{C}'_n because if $[a, a + h]$ is a subset of some connected component of \mathcal{C}'_k of some k , then the measure $h = |[a, a + h]|$ must be smaller than $\frac{1}{2^k}$, which is false when k is large enough. ■

Question 19

If E_k is a sequence of sets with $\sum |E_k|_e < \infty$, show that $\limsup_{n \rightarrow \infty} E_n$ has measure zero.

Proof. Note that

$$\sum_{k=N}^{\infty} |E_k|_e = \left(\sum_{k=1}^{\infty} |E_k|_e - \sum_{k=1}^{N-1} |E_k|_e \right) \rightarrow 0 \text{ as } N \rightarrow \infty$$

and note for all N we have

$$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \subseteq \bigcup_{k=N}^{\infty} E_k$$

Then for arbitrary ϵ , if we let N satisfy $\sum_{k=N}^{\infty} |E_k|_e < \epsilon$, we see that

$$\left| \limsup_{n \rightarrow \infty} E_n \right|_e = \left| \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \right|_e \leq \left| \bigcup_{k=N}^{\infty} E_k \right|_e \leq \sum_{k=N}^{\infty} |E_k|_e < \epsilon$$

■

Question 20

If E_1, E_2 are measurable, show that

$$|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|$$

Proof. Observe the following expression of each set in disjoint union

- (a) $E_1 = (E_1 \setminus E_2) \sqcup (E_1 \cap E_2)$
- (b) $E_2 = (E_2 \setminus E_1) \sqcup (E_1 \cap E_2)$
- (c) $E_1 \cup E_2 = (E_1 \setminus E_2) \sqcup (E_1 \cap E_2) \sqcup (E_2 \setminus E_1)$

It now follows

$$\begin{aligned} |E_1 \cup E_2| + |E_1 \cap E_2| &= |E_1 \setminus E_2| + |E_1 \cap E_2| + |E_1 \cap E_2| + |E_2 \setminus E_1| \\ &= |E_1| + |E_2| \end{aligned}$$

■

Lemma 1.2.1. Given two subsets Z_1, Z_2 of \mathbb{R} , if $|Z_1| = 0$, then $|Z_1 \times Z_2| = 0$.

Proof. Let $A_n \triangleq [n, n+1)$. Because

$$Z_1 \times Z_2 = \bigsqcup_{n \in \mathbb{Z}} Z_1 \times (A_n \cap Z_2)$$

To show $|Z_1 \times Z_2| = 0$, we only have to $|Z_1 \times (A_n \cap Z_2)| = 0$ for all $n \in \mathbb{Z}$. In other words, we can WOLG suppose Z_2 is bounded.

Now, fix ϵ . We are required to find a countable closed cube cover $Q_n \times C_n$ for $Z_1 \times Z_2$ such that $\sum_n |Q_n \times C_n| < \epsilon$. Let $C_n = C$ for all n where C is a compact interval containing Z_2 , and let Q_n be a countable compact interval cover for Z_1 such that $\sum |Q_n| < \frac{\epsilon}{|C|}$. It then follows $\sum_n |Q_n \times C_n| = \sum_n |Q_n| |C| < \epsilon$. ■

Theorem 1.2.2. (Product of Finite Measure Set) If E_1 and E_2 are measurable subset of \mathbb{R} and $|E_1|, |E_2| < \infty$, then $E_1 \times E_2$ is measurable in \mathbb{R}^2 and

$$|E_1 \times E_2| = |E_1| |E_2|$$

Proof. Write $E_1 \triangleq H_1 \sqcup Z_1$ and $E_2 \triangleq H_2 \sqcup Z_2$ where $H_1, H_2 \in F_\sigma$ and $|H_1| = |E_1|$ and $|H_2| = |E_2|$. Now observe

$$E_1 \times E_2 = (H_1 \times H_2) \cup (Z_1 \times E_2) \cup (E_1 \times Z_2)$$

Note that if we write $H_1 = \bigcap F_{1,n}$ and $H_2 = \bigcap F_{2,n}$, we see $H_1 \times H_2 = \bigcap F_{1,n} \times F_{2,n} \in F_\sigma$ in \mathbb{R}^2 , it now follows from **Lemma 1.2.1** that $E_1 \times E_2$ is measurable.

Now, let S_n be a decreasing sequence of open set containing E_1 such that $|S_n \setminus E_1| < \frac{1}{n}$, and let T_n be a decreasing sequence of open set containing E_2 such that $|T_n \setminus E_2| < \frac{1}{n}$. In other words,

$$E_1 = S \setminus Z_1 \text{ and } E_2 = T \setminus Z_2 \text{ where } \begin{cases} S \triangleq \bigcap S_n \\ T \triangleq \bigcap T_n \\ |Z_1| = |Z_2| = 0 \end{cases}$$

We now have

$$S \times T = (E_1 \times E_2) \cup (Z_1 \times E_2) \cup (E_1 \times Z_2)$$

This then implies $|S \times T| = |S \times T|_e \leq |E_1 \times E_2|_e + |Z_1 \times E_2|_e + |E_1 \times Z_2|_e = |E_1 \times E_2|$, where the last inequality follows from [Lemma 1.2.1](#). The reverse inequality is clear, since $E_1 \times E_2 \subseteq S \times T$. We have proved $|E_1 \times E_2| = |S \times T|$.

Now, for each n , write

$$S_n = \bigcup_{k \in \mathbb{N}} I_{k, S_n} \text{ and } T_n = \bigcup_{k \in \mathbb{N}} I_{k, T_n}$$

where $(I_{k, S_n})_k$ and $(I_{k, T_n})_k$ are non-overlapping compact interval. It then follows that

$$|S_n \times T_n| = \sum_{i, j} |I_{i, S_n} \times I_{j, T_n}| = \sum_{i, j} |I_{i, S_n}| \times |I_{j, T_n}| = \sum_i |I_{i, S_n}| \sum_j |I_{j, T_n}| = |S_n| |T_n|$$

Write $S \triangleq \bigcap_{n \in \mathbb{N}} S_n$ and $T \triangleq \bigcap_{n \in \mathbb{N}} T_n$. Because

- (a) Each $S_n \times T_n$ is open.
- (b) $|S_n \times T_n| = |S_n| |T_n|$ is bounded ($\because |S_n| \searrow |E_1| < \infty$).
- (c) $S_n \times T_n \searrow S \times T$

We can now deduce

$$\begin{aligned} |E_1 \times E_2| &= |S \times T| = \lim_{n \rightarrow \infty} |S_n \times T_n| \\ &= \lim_{n \rightarrow \infty} |S_n| |T_n| \\ &= |E_1| |E_2| \end{aligned}$$

■

Question 21

If E_1 and E_2 are measurable subset of \mathbb{R} , then $E_1 \times E_2$ is a measurable subset of \mathbb{R}^2 and $|E_1 \times E_2| = |E_1| |E_2|$

Proof. Define

$$A_n \triangleq \{x \in \mathbb{R} : n \leq x < n + 1\}$$

Because

$$E_1 = \bigcup_{n \in \mathbb{Z}} E_1 \cap A_n \text{ and } E_2 = \bigcup_{k \in \mathbb{Z}} E_2 \cap A_k$$

We can deduce

$$E_1 \times E_2 = \bigcup_{n,k \in \mathbb{Z}} (E_1 \cap A_n) \times (E_2 \cap A_k)$$

Note that **Theorem 1.2.2** tell us $(E_1 \cap A_n) \times (E_2 \cap A_k)$ is measurable, which implies $E_1 \times E_2$ is measurable. **Theorem 1.2.2** also tell us $|(E_1 \cap A_n) \times (E_2 \cap A_k)| = |E_1 \cap A_n| |E_2 \cap A_k|$, which allow us to deduce

$$\begin{aligned} |E_1 \times E_2| &= \sum_{n,k \in \mathbb{Z}} |(E_1 \cap A_n) \times (E_2 \cap A_k)| = \sum_{n,k \in \mathbb{Z}} |E_1 \cap A_n| |E_2 \cap A_k| \\ &= \sum_{n \in \mathbb{Z}} |E_1 \cap A_n| \sum_{k \in \mathbb{Z}} |E_2 \cap A_k| = |E_1| |E_2| \end{aligned}$$

■

Question 22

Give an example that shows that the image of a measurable set under a continuous transformation may not be measurable. See also Exercise 10 of Chapter 7.

Proof. Consider the Cantor-Lebesgue function denoted by $f : [0, 1] \rightarrow [0, 1]$ and denote the classical ternary Cantor set by \mathcal{C} . Let V be a Vitali set contained by $[0, 1]$. Because $f(\mathcal{C}) = [0, 1]$, we know there exists $E \subseteq \mathcal{C}$ such that $f(E) = V$. Such E is measurable since $|E|_e \leq |\mathcal{C}| = 0$, yet its continuous image $V = f(E)$ is by definition non-measurable. ■

Question 23

Show that there exists disjoint E_1, E_2, \dots such that $|\bigcup E_k|_e < \sum |E_k|_e$ with strict inequality.

Proof. Let V be a Vitali Set contained by $[0, 1]$. Enumerate $[0, 1] \cap \mathbb{Q}$ by x_n . Define

$$E_n \triangleq \{v + x_n \in \mathbb{R} : v \in V\} \text{ for all } n$$

The sequence E_n is disjoint, since if $p \in E_n \cap E_m$, then there exists some pair v_n, v_m belong to V such that

$$v_n + x_n = p = v_m + x_m \tag{1.6}$$

which is impossible, since Equation 1.6 implies that $v_n \neq v_m$ and v_n, v_m are of difference of a rational number.

Now, note that for arbitrary n and $v \in V$, because $v \in V \subseteq [0, 1]$ and $x_n \in [0, 1]$, we have $v + x_n \in [0, 2]$. This implies

$$\bigsqcup_n E_n \subseteq [0, 2] \text{ and } \left| \bigsqcup_n E_n \right|_e \leq 2$$

Because V is non-measurable by definition, we know $|V|_e > 0$, and since outer measure is translation invariant, we can now deduce

$$\sum_n |E_n|_e = \sum_n |V|_e = \infty > 2 \geq \left| \bigsqcup_n E_n \right|_e$$

■

Question 24

Show that there exists decreasing sequence E_k of sets such that

- (a) $E_k \searrow E$.
- (b) $|E_k|_e < \infty$.
- (c) $\lim_{k \rightarrow \infty} |E_k|_e > |E|_e$

Proof. Let V be a Vitali Set contained by $[0, 1]$. Enumerate $[0, 1] \cap \mathbb{Q}$ by x_n . Define

$$V + x_n \triangleq \{v + x_n \in \mathbb{R} : v \in V\}$$

In last question, we have proved that $V + x_n$ is pairwise disjoint. Define for all $n \in \mathbb{N}$

$$E_n \triangleq \bigsqcup_{k \geq n} V + x_k$$

Observe that

$$E_k \searrow \bigcap E_n = \emptyset$$

which implies $|\bigcap E_n|_e = 0$, but

$$\lim_{n \rightarrow \infty} |E_n|_e = \lim_{n \rightarrow \infty} \left| \bigsqcup_{k \geq n} V + x_k \right| \geq \lim_{n \rightarrow \infty} |V + x_n| = |V| > 0$$

■

Question 25

Let Z be a subset of \mathbb{R} with measure zero. Show that the set $\{x^2 : x \in Z\}$ also has measure zero.

Proof. Fix $Z_n \triangleq Z \cap [-n, n]$. Since

$$|\{x^2 : x \in Z\}| \leq \sum_{n=1}^{\infty} |\{x^2 : x \in Z_n\}|_e$$

We only have to prove

$$|\{x^2 : x \in Z_n\}|_e = 0 \text{ for all } n$$

Fix ϵ, n . Let I_k be a compact interval cover of Z_n such that $\sum |I_k| < \frac{\epsilon}{2n}$. We shall suppose $I_k \subseteq [-n, n]$, since if not, we can just let $I'_k \triangleq I_k \cap [-n, n]$.

Now, define

$$I_k^2 \triangleq \{x^2 : x \in I_k\}$$

Clearly, I_k^2 are all compact intervals, and if we write $I_k \triangleq [a_k, b_k]$, we have the following inequalities

$$\begin{cases} 0 \leq a_k \leq b_k \implies |I_k^2| = b_k^2 - a_k^2 = |I_k| (b_k + a_k) \leq 2n |I_k| \\ a_k \leq 0 \leq b_k \implies |I_k^2| = \max\{a_k^2, b_k^2\} \leq (b_k - a_k)^2 = |I_k| (b_k - a_k) \leq 2n |I_k| \\ a_k \leq b_k \leq 0 \implies |I_k^2| = a_k^2 - b_k^2 = |I_k| (-a_k - b_k) \leq 2n |I_k| \end{cases}$$

Note that $\{I_k^2\}_{k \in \mathbb{N}}$ is a compact interval cover of $\{x^2 : x \in Z_n\}$, we now see

$$|\{x^2 : x \in Z_n\}|_e \leq \sum_k |I_k^2| \leq 2n \sum |I_k| < \epsilon$$

■

1.3 Brunn-Minkowski Inequality

Abstract

This HW assignment require us to prove Brunn-Minkowski Inequality for rectangles, whose proof relies on the AM-GM inequality.

Theorem 1.3.1. (Brunn-Minkowski Inequality for Rectangles) Suppose $A, B \subseteq \mathbb{R}^n$ are two compact subset of \mathbb{R}^n . If we define their **Minkowski Sum** $A + B$ to be

$$A + B \triangleq \{a + b \in \mathbb{R}^n : a \in A \text{ and } b \in B\}$$

then we have the following inequality

$$|A|^{\frac{1}{n}} + |B|^{\frac{1}{n}} \leq |A + B|^{\frac{1}{n}}$$

Proof. Suppose

$$A = \prod_{j=1}^n [a_j, a_j + h_{a;j}] \text{ and } B = \prod_{j=1}^n [b_j, b_j + h_{b;j}]$$

We first claim

$$A + B = \prod_{j=1}^n [a_j + b_j, a_j + b_j + h_{a;j} + h_{b;j}] \quad (1.7)$$

Fix arbitrary $x \in A + B$. By definition, we know there exists some $p \in A$ and $q \in B$ such that $x = p + q$. Since $p \in A$ and $q \in B$, we have

$$a_j \leq p_j \leq a_j + h_{a;j} \text{ and } b_j \leq q_j \leq b_j + h_{b;j} \text{ for all } j \in \{1, \dots, n\}$$

It then follows that

$$a_j + b_j \leq p_j + q_j \leq a_j + b_j + h_{a;j} + h_{b;j} \text{ for all } j \in \{1, \dots, n\}$$

Then because $x = p + q$, we know

$$x_j = p_j + q_j \in [a_j + b_j, a_j + b_j + h_{a;j} + h_{b;j}] \text{ for all } j \in \{1, \dots, n\}$$

This implies

$$x \in \prod_{j=1}^n [a_j + b_j, a_j + b_j + h_{a;j} + h_{b;j}]$$

Because x is arbitrary selected from $A + B$, we have in fact proved

$$A + B \subseteq \prod_{j=1}^n [a_j + b_j, a_j + b_j + h_{a;j} + h_{b;j}]$$

We now fix arbitrary $x \in \prod_{j=1}^n [a_j + b_j, a_j + b_j + h_{a;j} + h_{b;j}]$. For each $j \in \{1, \dots, n\}$, define

$$y_j \triangleq \begin{cases} a_j + h_{a;j} & \text{if } x_j \geq a_j + b_j + h_{a;j} \\ x_j - b_j & \text{if } x_j < a_j + b_j + h_{a;j} \end{cases} \quad \text{and } z_j \triangleq x_j - y_j$$

Fix arbitrary $j \in \{1, \dots, n\}$. If $x_j < a_j + b_j + h_{a;j}$, by definition of y_j , we have

$$y_j = x_j - b_j < (a_j + b_j + h_{a;j}) - b_j = a_j + h_{a;j}$$

and since $x_j \geq a_j + b_j$, we also have

$$y_j = x_j - b_j \geq a_j$$

We have proved that

$$x_j < a_j + b_j + h_{a;j} \implies y_j \in [a_j, a_j + h_{a;j}]$$

Note that

$$x_j \geq a_j + b_j + h_{a;j} \implies y_j = a_j + h_{a;j} \in [a_j, a_j + h_{a;j}]$$

We now have

$$y_j \in [a_j, a_j + h_{a;j}]$$

Again, if $x_j \geq a_j + b_j + h_{a;j}$, by definition

$$b_j \leq x_j - a_j - h_{a;j} = x_j - y_j = z_j$$

and since $x_j \leq a_j + b_j + h_{a;j} + h_{b;j}$, we also have

$$z_j = x_j - a_j - h_{a;j} \leq b_j + h_{b;j}$$

We have proved that

$$x_j \geq a_j + b_j + h_{a;j} \implies z_j \in [b_j, b_j + h_{b;j}]$$

Note that

$$x_j < a_j + b_j + h_{a;j} \implies z_j = x_j - y_j = x_j - (x_j - b_j) = b_j \in [b_j, b_j + h_{b;j}]$$

In conclusion, we have proved $y_j \in [a_j, a_j + h_{a;j}]$ and $z_j \in [b_j, b_j + h_{b;j}]$ for all $j \in \{1, \dots, n\}$, since j is arbitrary selected from $\{1, \dots, n\}$. It now follows from $A = \prod_{j=1}^n [a_j, a_j + h_{a;j}]$, $B = \prod_{j=1}^n [b_j, b_j + h_{b;j}]$ if we define $y \triangleq (y_1, \dots, y_n)$, $z \triangleq (z_1, \dots, z_n)$, then we have

$$y \in A \text{ and } z \in B \text{ and } x = y + z \in A + B$$

We have proved

$$\prod_{j=1}^n [a_j + b_j, a_j + b_j + h_{a;j} + h_{b;j}] \subseteq A + B$$

Thus Equation 1.7 is proved.

Now, by direct computation, we know that

$$|A + B| = \prod_{j=1}^n (h_{a;j} + h_{b;j}) \text{ and } |A| = \prod_{j=1}^n h_{a;j} \text{ and } |B| = \prod_{j=1}^n h_{b;j}$$

This then give us

$$(a) \quad |A + B|^{\frac{1}{n}} = (\prod_{j=1}^n h_{a;j} + h_{b;j})^{\frac{1}{n}}.$$

$$(b) \quad |A|^{\frac{1}{n}} = (\prod_{j=1}^n h_{a;j})^{\frac{1}{n}}.$$

$$(c) \quad |B|^{\frac{1}{n}} = (\prod_{j=1}^n h_{b;j})^{\frac{1}{n}}.$$

It is clear that to prove $|A|^{\frac{1}{n}} + |B|^{\frac{1}{n}} \leq |A + B|^{\frac{1}{n}}$, we only have to prove

$$\frac{|A|^{\frac{1}{n}}}{|A + B|^{\frac{1}{n}}} + \frac{|B|^{\frac{1}{n}}}{|A + B|^{\frac{1}{n}}} \leq 1$$

Note that by AM-GM inequality, we have

$$\frac{|A|^{\frac{1}{n}}}{|A + B|^{\frac{1}{n}}} = \left(\prod_{j=1}^n \frac{h_{a;j}}{h_{a;j} + h_{b;j}} \right)^{\frac{1}{n}} \leq \frac{1}{n} \sum_{j=1}^n \frac{h_{a;j}}{h_{a;j} + h_{b;j}}$$

Similarly by AM-GM inequality, we have

$$\frac{|B|^{\frac{1}{n}}}{|A + B|^{\frac{1}{n}}} = \left(\prod_{j=1}^n \frac{h_{b;j}}{h_{a;j} + h_{b;j}} \right)^{\frac{1}{n}} \leq \frac{1}{n} \sum_{j=1}^n \frac{h_{b;j}}{h_{a;j} + h_{b;j}}$$

We then can deduce

$$\frac{|A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}}{|A + B|^{\frac{1}{n}}} \leq \frac{1}{n} \sum_{j=1}^n \frac{h_{a;j} + h_{b;j}}{h_{a;j} + h_{b;j}} = 1$$

Our proof is finished. ■

Chapter 2

Complex Analysis HW

2.1 HW1

Theorem 2.1.1.

$$(1+i)^n, \frac{(1+i)^n}{n}, \frac{n!}{(1+i)^n} \text{ all diverge as } n \rightarrow \infty$$

Proof. Note that

$$|(1+i)^n| = 2^{\frac{n}{2}} \rightarrow \infty \text{ as } n \rightarrow \infty$$

This implies $(1+i)$ is unbounded, thus diverge.

Note that

$$\left| \frac{(1+i)^n}{n} \right| = \frac{(\sqrt{2})^n}{n}$$

Observe

$$\begin{aligned} \frac{(\sqrt{2})^n}{n} &= \frac{[(\sqrt{2}-1)+1]^n}{n} = \frac{\sum_{k=0}^n \binom{n}{k} (\sqrt{2}-1)^k}{n} \\ &\geq \frac{\binom{n}{2} (\sqrt{2}-1)^2}{n} = (n-1) \left[\frac{(\sqrt{2}-1)^2}{2} \right] \rightarrow \infty \text{ as } n \rightarrow \infty \end{aligned}$$

This implies $\frac{(1+i)^n}{n}$ is unbounded, thus diverge.

Note that

$$\left| \frac{n!}{(1+i)^n} \right| = \frac{n!}{(\sqrt{2})^n}$$

Note that for all $k \geq 8$, we have

$$\frac{k}{\sqrt{2}} \geq \frac{\sqrt{8}}{\sqrt{2}} = 2$$

This implies

$$\frac{n!}{(\sqrt{2})^n} = \prod_{k=1}^n \frac{k}{\sqrt{2}} = \frac{7!}{(\sqrt{2})^7} \prod_{k=8}^n \frac{k}{\sqrt{2}} \geq \frac{7!}{(\sqrt{2})^7} \prod_{k=8}^n 2 \geq \frac{7!2^{n-8+1}}{(\sqrt{2})^7} \rightarrow \infty$$

which implies $\frac{n!}{(1+i)^n}$ is unbounded, thus diverge. ■

Theorem 2.1.2.

$$n!z^n \text{ converge} \iff z = 0$$

Proof. If $z = 0$, then $n!z^n = 0$ for all n , which implies $n!z^n \rightarrow 0$. Now, suppose $z \neq 0$. Let $M \in \mathbb{N}$ satisfy $|z| > \frac{1}{M}$. Observe

$$|n!z^n| = \left| \prod_{k=1}^n kz \right| = \left| \prod_{k=1}^{2M-1} kz \right| \left| \prod_{k=2M}^n kz \right| \geq \left| \prod_{k=1}^{2M-1} kz \right| \left| \prod_{k=2M}^n 2Mz \right| \geq \left| \prod_{k=1}^{2M-1} kz \right| 2^{n-2M+1} \rightarrow \infty$$

This implies $n!z^n$ is unbounded, thus diverge. ■

Theorem 2.1.3.

$$u_n \rightarrow u \implies v_n \triangleq \sum_{k=1}^n \frac{u_k}{n} \rightarrow u$$

Proof. Because

$$\sum_{k=1}^n \frac{u_k}{n} = \sum_{k \leq \sqrt{n}} \frac{u_k}{n} + \sum_{\sqrt{n} < k \leq n} \frac{u_k}{n}$$

It suffices to prove

$$\sum_{k \leq \sqrt{n}} \frac{u_k}{n} \rightarrow 0 \text{ and } \sum_{\sqrt{n} < k \leq n} \frac{u_k}{n} \rightarrow u \text{ as } n \rightarrow \infty$$

Because u_n converge, we can let M bound $|u_n|$. Observe

$$\left| \sum_{k \leq \sqrt{n}} \frac{u_k}{n} \right| \leq \sum_{k \leq \sqrt{n}} \left| \frac{u_k}{n} \right| \leq \sum_{k \leq \sqrt{n}} \frac{M}{n} \leq \frac{M\sqrt{n}}{n} = \frac{M}{\sqrt{n}} \rightarrow 0 \text{ as } n \rightarrow \infty \text{ (done)}$$

Because

$$\sum_{\sqrt{n} < k \leq n} \frac{u_k}{n} = \frac{n - \lceil \sqrt{n} \rceil + 1}{n} \sum_{\sqrt{n} < k \leq n} \frac{u_k}{n - \lceil \sqrt{n} \rceil + 1}$$

and

$$\lim_{n \rightarrow \infty} \frac{n - \lceil \sqrt{n} \rceil + 1}{n} = 1$$

We can reduce the problem into proving

$$\lim_{n \rightarrow \infty} \sum_{\sqrt{n} < k \leq n} \frac{u_k}{n - \lceil \sqrt{n} \rceil + 1} = u$$

Fix ϵ . Let N satisfy that for all $n \geq N$, we have $|u_n - u| < \epsilon$. Then for all $n \geq N^2$, we have

$$\begin{aligned} \left| \left(\sum_{\sqrt{n} < k \leq n} \frac{u_k}{n - \lceil \sqrt{n} \rceil + 1} \right) - u \right| &= \left| \sum_{\sqrt{n} < k \leq n} \frac{u_k - u}{n - \lceil \sqrt{n} \rceil + 1} \right| \\ &\leq \sum_{\sqrt{n} < k \leq n} \frac{|u_k - u|}{n - \lceil \sqrt{n} \rceil + 1} \\ &\leq \sum_{\sqrt{n} < k \leq n} \frac{\epsilon}{n - \lceil \sqrt{n} \rceil + 1} = \epsilon \text{ (done)} \end{aligned}$$

■

2.2 Exercise 1

Let R be a complex algebra with 1_A and $a \in R$. Given a complex polynomial

$$f(Z) = a_0 + a_1Z + \cdots + a_nZ^n,$$

we define the evaluation of f at a by

$$f(a) = a_01_A + a_1a + \cdots + a_na^n.$$

Question 26

Let $R = \mathbb{C}$ and $a = 1 + i$. Given $f(Z) = Z^3$. Evaluate $f(a)$.

Proof. $f(a) = (1 + i)^3 = 2i(1 + i) = -2 + 2i$ ■

Question 27

Let $R = M_{2 \times 2}(\mathbb{C})$ be the algebra of 2×2 complex matrices. Take

$$a = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

and $g(Z) = 3 + 2Z$. Evaluate $g(a)$.

Proof.

$$g(a) = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 2 & 5 \end{bmatrix}$$
 ■

Question 28

Let R be the algebra of complex valued periodic functions of period 2π , i.e., $a \in R$ is a continuous function $a : \mathbb{R} \rightarrow \mathbb{C}$ so that $a(x + 2\pi) = a(x)$. Let $e(x) = \cos x + i \sin x$ and

$$h(Z) = 1 + Z + Z^2 + \cdots + Z^9.$$

Find $h(e)$.

Proof. Note that

$$\begin{aligned} (\cos x + i \sin x)(\cos y + i \sin y) &= (\cos x \cos y - \sin x \sin y) + i(\sin x \cos y + \cos x \sin y) \\ &= \cos(x + y) + i \sin(x + y) \end{aligned}$$

This give us

$$h(e) = \sum_{k=0}^9 \cos(kx) + i \sin(kx)$$

■

Chapter 3

PDE intro HW

3.1 HW1

Theorem 3.1.1.

Show $u \mapsto u_x + uu_y$ is non-linear

Proof. See that

$$2u \mapsto 2u_x + 4uu_y \neq 2(u_x + uu_y) \quad (3.1)$$

■

Theorem 3.1.2.

Solve $(1 + x^2)u_x + u_y = 0$

Proof. The characteristic curve has the derivative

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

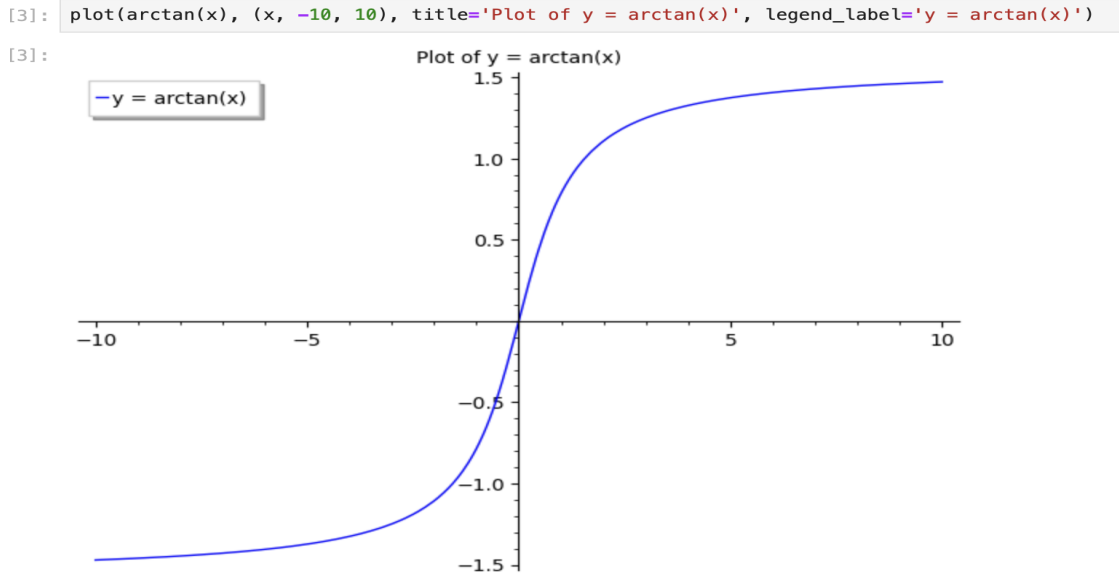
The solution to this ODE is

$$y = \arctan x + C$$

We now see that the solution to the PDE in [Equation 3.1](#) is

$u = f((\arctan x) - y)$ where $f : \mathbb{R} \rightarrow \mathbb{R}$ is an arbitrary smooth function

A characteristic curve is as followed.



■

Theorem 3.1.3.

$$\text{Solve } au_x + bu_y + cu = 0 \quad (3.2)$$

Proof. Fix

$$\begin{cases} x' \triangleq ax + by \\ y' \triangleq bx - ay \end{cases}$$

This map is clearly a diffeomorphism. Compute

$$\begin{cases} u_x = \frac{\partial u}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'} \frac{\partial y'}{\partial x} = au_{x'} + bu_{y'} \\ u_y = \frac{\partial u}{\partial x'} \frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'} \frac{\partial y'}{\partial y} = bu_{x'} - au_{y'} \end{cases}$$

Plugging it back into the PDE in [Equation 3.2](#), we have

$$cu + (a^2 + b^2)u_{x'} = 0 \quad (3.3)$$

If $c = a^2 + b^2 = 0$, then all smooth functions are solution. If $a^2 + b^2 = 0$ but $c \neq 0$, then clearly the only solution is $u = \tilde{0}$. If $a^2 + b^2 \neq 0$ but $c = 0$, then $u_{x'} = \tilde{0}$, which implies $u = f(y')$ where $y' = bx - ay$ and f can be arbitrary smooth function.

Now, suppose $a^2 + b^2 \neq 0 \neq c$, note that the PDE in [Equation 3.3](#) is just an ODE of the form

$$y + \frac{a^2 + b^2}{c}y' = 0$$

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The general solution to this ODE is

$$y = Ce^{\frac{-ct}{a^2+b^2}}$$

In other words, the general solution of the PDE in [Equation 3.3](#) is

$$u = Ce^{\frac{-cx'}{a^2+b^2}} = Ce^{\frac{-c(ax+by)}{a^2+b^2}}$$

■

3.2 HW2

Question 29

Consider heat flow in a long circular cylinder where the temperature depends only on t and on the distance r to the axis of the cylinder. Here $r = \sqrt{x^2 + y^2}$ is the cylindrical coordinate. From the three dimensional heat equation derive the equation $u_t = k(u_{rr} + \frac{u_r}{r})$

Proof. Write the three dimensional heat equation by

$$u_t = k\Delta u$$

Note that the Laplacian Δu when written in cylindrical coordinate is

$$\Delta u = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} + u_{zz}$$

Because the premise says that u is constant in z and θ , we know $u_{\theta\theta} = u_{zz} = 0$

$$\Delta u = u_{rr} + \frac{u_r}{r}$$

This give us

$$u_t = k(u_{rr} + \frac{u_r}{r})$$



3.3 1.2 First Order Linear Equations

(Principle of Geometric Method) Given a first order homogeneous linear PDE with the form

$$u_x + g(x, y)u_y = 0$$

We know if a curve $\gamma(x) = (x, y)$ satisfy

$$\gamma'(x) = c_x(1, g(x, y)) \text{ for some } c_x$$

Then

$$(u \circ \gamma)'(x) = 0 \text{ for all } x$$

Since

$$\gamma'(x) = (1, \frac{dy}{dx})$$

To find γ , we only wish to solve

$$\frac{dy}{dx} = g(x, y)$$

Question 30

Solve

$$(1 + x^2)u_x + u_y = 0$$

Proof. The ODE

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

has general solution $y = \arctan x + C$, so

$$u(x, y) = f(y - \arctan x)$$

■

Question 31

Solve

$$\begin{cases} yu_x + xu_y = 0 \\ u(0, y) = e^{-y^2} \end{cases}$$

In which region of the xy plane is the solution uniquely determined?

Proof. We are required to solve the ODE

$$\frac{dy}{dx} = \frac{x}{y}$$

Separating the variables and integrate

$$\int yy'dx = \int xdx \implies \frac{y^2}{2} = \frac{x^2}{2} + C$$

This now implies

$$u(x, y) = f(y^2 - x^2)$$

Initial Condition then give us

$$u = e^{x^2 - y^2}$$

■

Question 32

Solve the equation

$$u_x + u_y = 1$$

Proof. Clearly $u = \frac{x}{2} + \frac{y}{2}$ is a solution. We now solve the PDE

$$v_x + v_y = 0$$

Solving the ODE

$$\frac{dy}{dx} = 1$$

we have

$$y = x + C$$

Thus the general solution is

$$u = \frac{x}{2} + \frac{y}{2} + f(y - x)$$



Question 33

Solve

$$\begin{cases} u_x + u_y + u = e^{x+2y} \\ u(x, 0) = 0 \end{cases}$$

Proof. Let $\gamma(x) = x + C$, we have

$$(u \circ \gamma)' + (u \circ \gamma) = e^{3x+2C}$$

We now solve the ODE

$$y' + y = e^{3x+2C}$$

The particular solution is clearly

$$y = \frac{1}{4}e^{3x+2C}$$

Thus the general solution is

$$y = \frac{e^{3x+2C}}{4} + \tilde{C}e^{-x}$$

We then now know

$$(u \circ \gamma)(x) = \frac{e^{3x+2C}}{4} + \tilde{C}e^{-x} \tag{3.4}$$

In other words,

$$u(x, x + C) = \frac{e^{3x+2C}}{4} + \tilde{C}e^{-x}$$

Putting in the initial conditions, we have

$$0 = u(-C, 0) = \frac{e^{-C}}{4} + \tilde{C}e^C$$

This implies

$$\tilde{C} = \frac{-e^{-2C}}{4}$$

Putting the back into Equation 3.4, we have

$$u(x, x + C) = \frac{e^{3x+2C} - e^{-x-2C}}{4}$$

So

$$u(x, y) = \frac{e^{3x+2(y-x)} - e^{-x-2(y-x)}}{4}$$

■

Question 34

Use the coordinate method to solve the equation

$$u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2$$

Proof. Let

$$\begin{cases} \xi \triangleq 2x - y \\ \eta \triangleq x + 2y \end{cases}$$

We see

$$\begin{cases} u_\xi = \frac{2}{5}u_x - \frac{1}{5}u_y \\ u_\eta = \frac{1}{5}u_x + \frac{2}{5}u_y \end{cases}$$

We then can rewrite

$$5u_\eta + \xi u = \xi \eta \tag{3.5}$$

which clearly have particular solution

$$u = \eta - \frac{5}{\xi}$$

To solve the linear homogeneous PDE

$$5u_\eta + \xi u = 0$$

Observe that for all fixed ξ , the PDE is just an ODE whose solution is exactly $u = C_\xi e^{\frac{-\xi\eta}{5}}$. We now know the general solution for **PDE 3.5** is exactly

$$u = \eta - \frac{5}{\xi} + f(\xi)e^{\frac{-\xi\eta}{5}}$$

Then

$$u = x + 2y - \frac{5}{2x - y} + e^{\frac{-(2x-y)(x+2y)}{5}} f(2x - y)$$

■

3.4 1.4 Initial and Boundary Condition

Question 35

Find a solution of

$$\begin{cases} u_t = u_{xx} \\ u(x, 0) = x^2 \end{cases}$$

Proof. Clearly $u = x^2 + 2t$ suffices. ■

3.5 1.5 Well Posed Problems

Question 36

Consider the ODE

$$\begin{cases} u'' + u = 0 \\ u(0) = 0 \text{ and } u(L) = 0 \end{cases}$$

Is the solution unique? Does the answer depend on L ?

Proof. We know the general solution space is exactly spanned by $\cos x$ and $\sin x$. Because

(a) $u(0) = 0$.

(b) $\sin 0 = 0$

(c) $\cos 0 = 1$

we know the solution of our original ODE must be of the form

$$u(x) = C \sin x$$

This implies that the solution is unique if and only if $2\pi \not\equiv L \pmod{2\pi}$ ■

Question 37

Consider the ODE

$$\begin{cases} u'' + u' = f \\ u'(0) = u(0) = \frac{1}{2}(u'(l) + u(l)) \end{cases}$$

where f is given.

(a) Is the solution unique?

(b) Does a solution necessarily exist, or is there a condition that f must satisfy for existence?

Proof. The solution space of linear homogeneous ODE $u'' + u' = 0$ is spanned by e^{-x} and constant. If we add in the initial condition $u'(0) = u(0)$, then the solution space become the subspace spanned by $e^{-x} - 2$. One can check that if $u \in \text{span}(e^{-x} - 2)$, then

$$u(0) = \frac{1}{2}(u'(l) + u(l)) \text{ for all } l \in \mathbb{R}$$

We now know the solution of the original ODE is not unique, since any solution added by $e^{-x} - 2$ is again a solution.

Integrating both side on $[0, l]$, we see that given the boundary conditions, f must satisfy

$$\int_0^l f(x)dx = 0$$

■

Question 38

Consider the Neumann problem

$$\Delta u = f(x, y, z) \text{ in } D \text{ and } \frac{\partial u}{\partial n} = 0 \text{ on bdy } D$$

- (a) What can we add to any solution to get another solution?
- (b) Use the divergence Theorem and the PDE to show that

$$\iiint_D f(x, y, z) dx dy dz = 0$$

is a necessary condition for the Neumann problem to have a solution.

Proof. Clearly, constants suffices, and observe

$$\iiint_D f dx dy dz = \iiint_D \Delta u dx dy dz = \iiint_D \nabla \cdot (\nabla u) dx dy dz = \iint_{\text{bdy } D} \nabla u \cdot \mathbf{n} dS = 0$$

■

Question 39

Consider the equation

$$u_x + y u_y = 0$$

with the boundary condition $u(x, 0) = \phi(x)$.

- (a) $\phi(x) = x \implies$ no solution exists
- (b) $\phi(x) = 1 \implies$ multiple solutions exist.

Proof. Using the geometric method, we see the characteristic curve is exactly $y = \tilde{C}e^x$. Thus the general solution is of the form

$$u(x, y) = f(e^{-x}y)$$

The boundary condition implies

$$\phi(x) = u(x, 0) = f(0)$$

The result then follows. ■

3.6 Types of Second-Order Equations

Consider the constant coefficient Linear PDE

$$u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0$$

Ignoring the term with order less than 2, we have

$$u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} = 0$$

Or equivalently

$$(\partial_x + a_{12}\partial_y)^2u + (a_{22} - a_{12}^2)\partial_{yy}u = 0$$

Now, if we set

$$x \triangleq \xi \text{ and } y \triangleq a_{12}\xi + (|a_{22} - a_{12}^2|)^{\frac{1}{2}}\eta$$

we see that

$$\begin{cases} a_{22} - a_{12}^2 > 0 \implies u_{\xi\xi} + u_{\eta\eta} = 0 & \text{(Elliptic)} \\ a_{22} - a_{12}^2 = 0 \implies u_{\xi\xi} = 0 & \text{(Parabolic)} \\ a_{22} - a_{12}^2 < 0 \implies u_{\xi\xi} - u_{\eta\eta} = 0 & \text{(Hyperbolic)} \end{cases}$$

More generally, if we are given

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} = 0$$

then the discriminant is exactly

$$a_{11}a_{22} - a_{12}^2$$

Question 40

What is the type of each of the following equations.

(a) $u_{xx} - u_{xy} + u_{yy} + \cdots + u = 0$.

(b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$

Question 41

Find the regions in the xy plane where the equation

$$(1+x)u_{xx} + 2xyu_{xy} - y^2u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch them.

Proof. The discriminant is exactly

$$\begin{aligned}(xy)^2 - (1+x)(-y^2) &= x^2y^2 + xy^2 + y^2 \\ &= y^2(x^2 + x + 1) \\ &= y^2\left[\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\right]\end{aligned}$$

It then follows that the equation is parabolic if and only if $y = 0$, and hyperbolic if and only if $y \neq 0$. ■

Question 42

Reduce the elliptic equation

$$u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u = 0$$

to the form

$$v_{xx} + v_{yy} + cv = 0$$

by a change of dependent variable

$$u \triangleq ve^{\alpha x + \beta y}$$

then a change of scale

$$y' = \gamma y$$

Question 43

Consider the equation $3u_y + u_{xy} = 0$.

(a) What is its type?

(b) Find the general solution. (Hint: Substitute $v = u_y$).

(c) With the auxiliary conditions $u(x, 0) = e^{-3x}$ and $u_y(x, 0) = 0$, does a solution

exist? Is it unique?

3.7 PDE HW 3

Question 44

Find a solution of

$$\begin{cases} u_t = u_{xx} \\ u(x, 0) = x^2 \end{cases}$$

Proof. Clearly $u = x^2 + 2t$ suffices. ■

Question 45

Consider the ODE

$$\begin{cases} u'' + u = 0 \\ u(0) = 0 \text{ and } u(L) = 0 \end{cases}$$

Is the solution unique? Does the answer depend on L ?

Proof. We know the general solution space is exactly spanned by $\cos x$ and $\sin x$. Because

(a) $u(0) = 0$.

(b) $\sin 0 = 0$

(c) $\cos 0 = 1$

we know the solution of our original ODE must be of the form

$$u(x) = C \sin x$$

This implies that the solution is unique if and only if $2\pi \not\equiv L \pmod{2\pi}$ ■

Question 46

Find the regions in the xy plane where the equation

$$(1+x)u_{xx} + 2xyu_{xy} - y^2u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch them.

Proof. The discriminant is exactly

$$\begin{aligned}(xy)^2 - (1+x)(-y^2) &= x^2y^2 + xy^2 + y^2 \\ &= y^2(x^2 + x + 1) \\ &= y^2[(x + \frac{1}{2})^2 + \frac{3}{4}]\end{aligned}$$

It then follows that the equation is parabolic if and only if $y = 0$, and hyperbolic if and only if $y \neq 0$. ■

Chapter 4

Differential Geometry HW

4.1 HW1

Abstract

In this HW, we give precise definition to \mathbb{P}^n and $\mathbb{R}P^n$, and we rigorously show

- (a) $\mathbb{R}P^n$ has a smooth structure.
- (b) \mathbb{P}^n is homeomorphic to $\mathbb{R}P^n$
- (c) \mathbb{P}^n has a smooth structure.

Note that in this PDF, brown text is always a clickable hyperlink reference.

Define an equivalence relation on $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ by

$$\mathbf{x} \sim \mathbf{y} \stackrel{\Delta}{\iff} \mathbf{y} = \lambda \mathbf{x} \text{ for some } \lambda \in \mathbb{R}^*$$

Let $\mathbb{R}P^n \triangleq (\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}) / \sim$ be the quotient space and let

$$V_i \triangleq \{\mathbf{x} \in \mathbb{R}^{n+1} : \mathbf{x}^i \neq 0\} \text{ for each } 1 \leq i \leq n+1$$

By definition, it is clear that

$$\text{either } \pi^{-1}(\pi(\mathbf{x})) \subseteq V_i \text{ or } \pi^{-1}(\pi(\mathbf{x})) \subseteq V_i^c$$

Then if we define $\phi_i : V_i \rightarrow \mathbb{R}^n$ by

$$\phi_i(\mathbf{x}) \triangleq \left(\frac{\mathbf{x}^1}{\mathbf{x}^i}, \dots, \frac{\mathbf{x}^{i-1}}{\mathbf{x}^i}, \frac{\mathbf{x}^{i+1}}{\mathbf{x}^i}, \dots, \frac{\mathbf{x}^{n+1}}{\mathbf{x}^i} \right)$$

because $\pi(\mathbf{x}) = \pi(\mathbf{y}) \implies \phi_i(\mathbf{x}) = \phi_i(\mathbf{y})$, we can well induce a map

$$\Phi_i : U_i \triangleq \pi(V_i) \subseteq \mathbb{R}P^n \rightarrow \mathbb{R}^n; \pi(\mathbf{x}) \mapsto \phi_i(\mathbf{x})$$

Note that one has the equation

$$\Phi_i(\pi(\mathbf{x})) = \phi_i(\mathbf{x}) \text{ for all } \mathbf{x} \in V_i$$

Theorem 4.1.1. (Real Projective Space with a differentiable atlas) We have

$\mathbb{R}P^n$ with atlas $\{(U_i, \Phi_i) : 1 \leq i \leq n+1\}$ is a differentiable manifold

Proof. We are required to prove

- (a) (U_i, Φ_i) are all charts.
- (b) $\{(U_i, \Phi_i) : 1 \leq i \leq n+1\}$ form a differentiable atlas.
- (c) $\mathbb{R}P^n$ is Hausdorff.
- (d) $\mathbb{R}P^n$ is second-countable.

Because $\pi^{-1}(U_i) = V_i$ and V_i is clearly open in $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$, we know $U_i \subseteq \mathbb{R}P^n$ is open. Note that clearly, $\Phi_i(U_i) = \mathbb{R}^n$. To show (U_i, Φ_i) is a chart, it remains to show that Φ_i is a homeomorphism between U_i and \mathbb{R}^n . It is straightforward to check Φ_i is one-to-one on U_i . This implies Φ_i is a bijective between U_i and \mathbb{R}^n .

Fix open $E \subseteq \mathbb{R}^n$. We see

$$\pi^{-1}(\Phi_i^{-1}(E)) = \phi_i^{-1}(E)$$

Then because $\phi_i : V_i \rightarrow \mathbb{R}^n$ is clearly continuous, we see $\phi_i^{-1}(E)$ is open in $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$, and it follows from definition of quotient topology $\Phi_i^{-1}(E) \subseteq \mathbb{R}P^n$ is open. Then because U_i is open in $\mathbb{R}P^n$, we see $\Phi_i^{-1}(E)$ is open in U_i . We have proved $\Phi_i : U_i \rightarrow \mathbb{R}^n$ is continuous.

Define $\Psi_i : \mathbb{R}^n \rightarrow V_i$ by

$$\Psi(\mathbf{x}^1, \dots, \mathbf{x}^n) = (\mathbf{x}^1, \dots, \mathbf{x}^{i-1}, 1, \mathbf{x}^i, \dots, \mathbf{x}^n)$$

Observe that for all $\mathbf{x} \in \Phi_i(U_i)$, we have

$$\Phi_i^{-1}(\mathbf{x}) = \pi(\Psi_i(\mathbf{x}))$$

It then follows from $\Psi_i : \mathbb{R}^n \rightarrow V_i$ and $\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}P^n$ are continuous that $\Phi_i^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}P^n$ is continuous.

We have proved that (Ψ_i, U_i) are all charts. Now, because V_i clearly cover \mathbb{R}^{n+1} , we know U_i also cover $\mathbb{R}P^n$. We have proved $\{(U_i, \Phi) : 1 \leq i \leq n+1\}$ form an atlas. The fact $\mathbb{R}P^n$ is second-countable follows.

Fix $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \Phi_i(U_i \cap U_j)$. We compute

$$\begin{aligned} \Phi_j \circ \Phi_i^{-1}(\mathbf{x}^1, \dots, \mathbf{x}^n) &= \Phi_j\left([\mathbf{x}^1, \dots, \mathbf{x}^{i-1}, 1, \mathbf{x}^i, \mathbf{x}^{i+1}, \dots, \mathbf{x}^n]\right) \\ &= \begin{cases} \left(\frac{\mathbf{x}^1}{\mathbf{x}^j}, \dots, \frac{\mathbf{x}^{j-1}}{\mathbf{x}^j}, \frac{\mathbf{x}^{j+1}}{\mathbf{x}^j}, \dots, \frac{\mathbf{x}^{i-1}}{\mathbf{x}^j}, \frac{1}{\mathbf{x}^j}, \frac{\mathbf{x}^i}{\mathbf{x}^j}, \dots, \frac{\mathbf{x}^n}{\mathbf{x}^j}\right) & \text{if } j < i \\ \left(\frac{\mathbf{x}^1}{\mathbf{x}^{j-1}}, \dots, \frac{\mathbf{x}^{i-1}}{\mathbf{x}^{j-1}}, \frac{1}{\mathbf{x}^{j-1}}, \frac{\mathbf{x}^i}{\mathbf{x}^{j-1}}, \dots, \frac{\mathbf{x}^{j-2}}{\mathbf{x}^{j-1}}, \frac{\mathbf{x}^j}{\mathbf{x}^{j-1}}, \dots, \frac{\mathbf{x}^n}{\mathbf{x}^{j-1}}\right) & \text{if } j > i \end{cases} \end{aligned}$$

This implies our atlas is indeed differentiable.

Before we prove $\mathbb{R}P^n$ is Hausdorff, we first prove that $\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}P^n$ is an **open mapping**. Let $U \subseteq \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ be open. Observe that

$$\pi^{-1}(\pi(U)) = \{t\mathbf{x} \in \mathbb{R}^{n+1} : t \neq 0 \text{ and } \mathbf{x} \in U\}$$

Fix $t_0\mathbf{x} \in \pi^{-1}(\pi(U))$. Let $B_\epsilon(\mathbf{x}) \subseteq U$. Observe that

$$B_{|t_0|\epsilon}(t_0\mathbf{x}) \subseteq t_0B_\epsilon(\mathbf{x}) \subseteq t_0U \subseteq \pi^{-1}(\pi(U))$$

This implies $\pi^{-1}(\pi(U))$ is open. **(done)**

Now, **because π is open, to show $\mathbb{R}P^n$ is Hausdorff, we only have to show**

$$R_\pi \triangleq \{(\mathbf{x}, \mathbf{y}) \in (\mathbb{R}^{n+1} \setminus \{\mathbf{0}\})^2 : \pi(\mathbf{x}) = \pi(\mathbf{y})\} \text{ is closed}$$

Define $f : (\mathbb{R}^{n+1} \setminus \{\mathbf{0}\})^2 \rightarrow \mathbb{R}$ by

$$f(\mathbf{x}, \mathbf{y}) \triangleq \sum_{i \neq j} (\mathbf{x}^i \mathbf{y}^j - \mathbf{x}^j \mathbf{y}^i)^2$$

Note that f is clearly continuous and $f^{-1}(0) = R_\pi$, which finish the proof. ■

Alternatively, we can characterize $\mathbb{R}P^n$ by identifying the antipodal points on $S^n \triangleq \{\mathbf{x} \in \mathbb{R}^{n+1} : |\mathbf{x}| = 1\}$ as one point

$$\mathbf{x} \sim \mathbf{y} \iff \mathbf{x} = \mathbf{y} \text{ or } \mathbf{x} = -\mathbf{y}$$

and let $\mathbb{P}^n \triangleq S^n / \sim$ be the quotient space.

Theorem 4.1.2. (Equivalent Definitions of Real Projective Space)

$\mathbb{R}P^n$ and \mathbb{P}^n are homeomorphic

Proof. Define $F : \mathbb{P}^n \rightarrow \mathbb{R}P^n$ by

$$\{\mathbf{x}, -\mathbf{x}\} \mapsto \{\lambda \mathbf{x} : \lambda \in \mathbb{R}^*\}$$

It is straightforward to check that F is well-defined and bijective. Define $f : S^n \rightarrow \mathbb{R}P^n$ by

$$f = \pi \circ \mathbf{id}$$

where $\mathbf{id} : S^n \rightarrow \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ and $\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}P^n$ are continuous. Check that

$$f = F \circ p$$

where $p : S^n \rightarrow \mathbb{P}^n$ is the quotient mapping. It now follows from the universal property that F is continuous, and since \mathbb{P}^n is compact and $\mathbb{R}P^n$ is Hausdorff, it also follows that F is a homeomorphism between $\mathbb{R}P^n$ and \mathbb{P}^n . ■

Knowing that $F : \mathbb{P}^n \rightarrow \mathbb{R}P^n$ is a homeomorphism and $\mathbb{R}P^n$ is a smooth manifold, we see that \mathbb{P}^n is Hausdorff and second-countable, and if we define the atlas

$$\{(F^{-1}(U_i), \Phi_i \circ F) : 1 \leq i \leq n+1\}$$

We see this atlas is indeed smooth, since

$$(\Phi_i \circ F) \circ (\Phi_j \circ F)^{-1} = \Phi_i \circ \Phi_j^{-1}$$

4.2 Appendix

Theorem 4.2.1. (Homeomorphism between Compact Space and Hausdorff Space)
Suppose

- (a) X is compact.
- (b) Y is Hausdorff.
- (c) $f : X \rightarrow Y$ is a continuous bijective function.

Then

f is a homeomorphism between X and Y

Proof. Because closed subset of compact set is compact and continuous function send compact set to compact set, we see for each closed $E \subseteq X$, $f(E) \subseteq Y$ is compact. The result then follows from $f(E) \subseteq Y$ being closed since Y is Hausdorff. ■

Theorem 4.2.2. (Hausdorff and Quotient) If $\pi : X \rightarrow Y$ is an open mapping, and we define

$$R_\pi \triangleq \{(x, y) \in X^2 : \pi(x) = \pi(y)\}$$

Then

$$R_\pi \text{ is closed} \iff Y \text{ is Hausdorff}$$

Proof. Suppose R_π is closed. Fix some x, y such that $\pi(x) \neq \pi(y)$. Because R_π is closed, we know there exists open neighborhood U_x, U_y such that $U_x \times U_y \subseteq (R_\pi)^c$. It is clear that $\pi(U_x), \pi(U_y)$ are respectively open neighborhood of $\pi(x)$ and $\pi(y)$. To see $\pi(U_x)$ and $\pi(U_y)$ are disjoint, **assume** that $\pi(a) \in \pi(U_x) \cap \pi(U_y)$. Let $a_x \in U_x$ and $a_y \in U_y$ satisfy $\pi(a_x) = \pi(a) = \pi(a_y)$, which is impossible because $(a_x, a_y) \in (R_\pi)^c$. **CaC**

Suppose Y is Hausdorff. Fix some x, y such that $\pi(x) \neq \pi(y)$. Let U_x, U_y be open neighborhoods of $\pi(x), \pi(y)$ separating them. Observe that $(x, y) \in \pi^{-1}(U_x) \times \pi^{-1}(U_y) \subseteq (R_\pi)^c$ ■

4.3 Example: $S^1, \mathbb{R} \setminus \mathbb{Z}$ diffeomorphism

Equip $S^1 \triangleq \{(x, y) \in \mathbb{R}^2 : |(x, y)| = 1\}$ with the standard four projection chart smooth atlas as one of them being

$$V \triangleq \{(x, y) \in \mathbb{R}^2 : y > 0\} \text{ and } \phi_V : V \rightarrow \mathbb{R}; (x, y) \mapsto x$$

Let $p : \mathbb{R} \rightarrow \mathbb{R} \setminus \mathbb{Z}$ be the quotient map and let

$$U_0 \triangleq p\left((0, 1)\right) \text{ and } U_1 \triangleq p\left(\left(-\frac{1}{2}, \frac{1}{2}\right)\right)$$

which are both open as one can readily check. Define $\phi_0 : U_0 \rightarrow (0, 1)$ by

$$\phi_0(p(t)) \triangleq t_0 \text{ where } t_0 \in (0, 1) \text{ and } p(t_0) = p(t)$$

and $\phi_1 : U_1 \rightarrow (-\frac{1}{2}, \frac{1}{2})$ by

$$\phi_1(p(t)) \triangleq t_0 \text{ where } t_0 \in \left(-\frac{1}{2}, \frac{1}{2}\right) \text{ and } p(t_0) = p(t)$$

Clearly, the function $G : \mathbb{R} \setminus \mathbb{Z} \rightarrow S^1$ well-defined by $G(p(x)) \triangleq (\cos 2\pi x, \sin 2\pi x)$ is a homeomorphism, as one can check that

- (a) G is a continuous bijection. (Using Universal property of quotient map)
- (b) $\mathbb{R} \setminus \mathbb{Z}$ is compact. (by finite sub-cover definition)
- (c) S^1 is Hausdorff.

We now compute that $\phi_V \circ G \circ \phi_0^{-1}$ is defined on whole $(0, 1)$, and is exactly

$$\phi_V \circ G \circ \phi_0^{-1}(t) \triangleq \cos 2\pi t \text{ smooth}$$