

# 臺灣大學數學系113學年度碩士班甄試筆試試題

## 科目：高等微積分

2023.11.02

1. (15%) If every closed and bounded set of a metric space  $(M, d)$  is compact, does it follow that  $(M, d)$  is complete? If your answer is “yes”, prove it; if your answer is “no”, give a counter-example.
2. (20%) Determine the values of  $h$  for which the following series converges uniformly on  $I_h = \{x \in \mathbb{R} : |x| \leq h\}$ :

$$\sum_{n=1}^{\infty} \frac{(n!)^2 x^n}{(2n)!} .$$

Show your work.

3. (10%+10%+5%) Consider

$$(*) \quad F(x) = \int_0^\infty \frac{e^{-xt} - e^{-t}}{t} dt$$

on  $I = \{x \in \mathbb{R} : \frac{1}{2} \leq x \leq 2\}$ .

- (a) Show that  $(*)$  converges on  $I$ , and  $F(x)$  is continuous on  $I$ .
  - (b) Show that
- $$F'(x) = \int_0^\infty -e^{-xt} dt .$$
- (c) Evaluate  $F(x)$ .
4. (5%+15%) Let  $f$  be a smooth function on  $\mathbb{R}^n$  with  $\det([\frac{\partial^2 f}{\partial x_i \partial x_j}]_{1 \leq i,j \leq n}) = 2$  everywhere.
    - (a) Show that there exist an open neighborhood  $U \subset \mathbb{R}^n$  of the origin and an open set  $V \subset \mathbb{R}^n$  such that  $\mathbf{x} = (x_1, \dots, x_n) \mapsto Df(\mathbf{x}) = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$  is a bijection map from  $U$  and  $V$ , and its inverse is also a smooth map.
    - (b) Denote the inverse map in part (a) by  $\xi(\mathbf{y}) = (\xi_1(\mathbf{y}), \dots, \xi_n(\mathbf{y}))$ . For any  $\mathbf{y} \in V$ , let

$$f^*(\mathbf{y}) = -f(\xi(\mathbf{y})) + \sum_{i=1}^n y_i \xi_i(\mathbf{y}) .$$

Compute  $\det([\frac{\partial^2 f^*}{\partial y_i \partial y_j}]_{1 \leq i,j \leq n})$ .

5. (20%) Let  $f(x)$  be a  $C^1$  function for  $x \in [0, \infty)$ . Suppose that  $f(x) \geq 0$  and  $f'(x) \leq 1$  for every  $x \geq 0$ , and  $\int_0^\infty f(x) dx$  converges. Does  $\lim_{x \rightarrow \infty} f(x)$  exist? If your answer is “yes”, determine the limit and prove it; if your answer is “no”, give a counter-example.

# 臺灣大學數學系 112 學年度碩士班甄試試題

## 科目：高等微積分

2022.10.20

### 1. (15 points)

Let  $M = \{f : [0, \infty) \rightarrow [0, \infty); \int_0^\infty f(x)^2 dx \leq 1\}$ . Evaluate the following:

$$\sup_{f \in M} \int_0^\infty f(x)e^{-x} dx.$$

### 2. (15 points)

Assume  $A \subset \mathbb{R}^n$  is compact and let  $a \in A$ . Suppose  $\{a_n\}$  is a sequence in  $A$  such that every convergent subsequence of  $\{a_n\}$  converges to  $a$ .

(1) Does the sequence  $\{a_n\}$  also converge to  $a$ ? Justify your result.

(2) Now assume  $A$  is not compact and suppose  $\{a_n\}$  is a sequence in  $A$  such that every convergent subsequence of  $\{a_n\}$  converges to  $a \in A$ . Does the sequence  $\{a_n\}$  also converge to  $a$ ? Justify your result.

### 3. (20 points)

Let  $M$  be a metric space and  $A \subset M$  a compact subset. Suppose  $f : A \rightarrow A$  is continuous and satisfies  $d(f(x), f(y)) \geq d(x, y)$  for all  $x, y$ . Prove that  $f$  is onto  $A$ , i.e.  $f(A) = A$ .

### 4. (20 points)

Define a sequence of functions  $\{f_n(x)\}$  on  $[0, 1]$  as:

$$f_n(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1 & \text{if } x \in (\frac{2k}{2^n}, \frac{2k+1}{2^n}], k = 0, 1, \dots, 2^{n-1}-1 \\ -1 & \text{if } x \in (\frac{2k+1}{2^n}, \frac{2k+2}{2^n}], k = 0, 1, \dots, 2^{n-1}-1 \end{cases}$$

Prove or disprove that we always have  $\lim_{n \rightarrow \infty} \int_0^1 f_n(x)g(x)dx = 0$  as long as  $g$  is a continuous function.

### 5. (20 points)

Denote  $P_2$  the set of all polynomials with real coefficients and degree  $\leq 2$ . Consider the function  $G : P_2 \rightarrow \mathbb{R}$  by

$$G(p) = \int_0^1 p(x)^2 dx.$$

Let  $S = \{p \in P_2; p(1) = 1\}$ . Does  $G$  attain any extremal value on  $S$ ? If yes, find  $p \in S$  such that  $G$  attains an extremal value at  $p$ .

### 6. (10 points)

Suppose  $f(x) : [a, b] \rightarrow \mathbb{R}$  is Riemann integrable, and  $g(x) : \mathbb{R} \rightarrow \mathbb{R}$  satisfies  $|g(x) - g(y)| \leq C|x - y|$  for all  $x, y$ . Prove that  $g(f(x))$  is Riemann integrable.

# 臺灣大學數學系 110 學年度碩士班甄試試題

## 科目：高等微積分

2020.10.23

1. (10 points)

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be continuously differentiable. Assume that its Jacobian matrix  $(\frac{\partial f_i}{\partial x_j})$  has rank  $n$  everywhere and  $f^{-1}(K)$  is compact whenever  $K$  is compact. Show that  $f(\mathbb{R}^n) = \mathbb{R}^n$

2. (15 points)

Let  $F(x, y, z) = (x, y, z)$ , and  $S$  be the boundary of the region  $x^2 + y^2 \leq z \leq \sqrt{2 - x^2 - y^2}$ , oriented so that the normal points out of the region. Compute  $\int_S F \cdot ndA$ .

3. (20 points)

Let  $M$  be a metric space with countable elements. Prove or disprove that  $M$  is disconnected.

4. (20 points)

Let  $f(x) = \frac{1}{4} + x - x^2$ . For a real number  $x$ , define  $x_{n+1} = f(x_n)$ , where  $x_0 = x$ . (1) Given  $x = 0$ , show that the sequence  $\{x_n\}$  converges and find its limit  $L$ . (2) Find all real numbers  $x$  such that their corresponding sequences all converge to  $L$ .

5. (20 points)

Let  $X$  consist of all real valued functions  $f$  on  $[0, 1]$  such that

(1)  $f(0) = 0$

(2)  $\|f\| = \sup\left\{\frac{|f(x)-f(y)|}{|x-y|^{1/3}}; x \neq y\right\}$  is finite.

Prove that  $\|\cdot\|$  is a norm for  $X$  and  $X$  is complete with respect to this norm.

6. (15 points)

Assume  $f : (a, b) \rightarrow \mathbb{R}$  is differentiable. Consider its derivative  $f'(x)$ , show that  $f'(x)$  never has a jump discontinuity.

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科目：高等微積分

2019.10.18

1. (15 points)

Consider a series  $\sum_{k=2}^{\infty} a_k \sin(kx)$ , where  $\{a_k\}$  is a sequence of real numbers. Is it possible to construct a sequence  $\{a_k\}$  such that  $\sum_{k=2}^{\infty} a_k \sin(kx)$  converges uniformly to  $\sin(x)$  on  $[0, \pi]$ ? Justify your result.

2. (15 points)

Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is uniformly continuous. Show that  $f$  must be bounded by a linear function, i.e., there exist constants  $A, B$  such that  $|f(x)| \leq A + B|x|$  for all  $x$ .

3. (20 points)

(1): Let  $f : M \rightarrow N$  be a map from metric space  $M$  to another metric space  $N$  with the property that if a sequence  $\{p_n\}$  in  $M$  converges, then the sequence  $\{f(p_n)\}$  in  $N$  also converges. Is  $f$  continuous? Justify your result.

(2): Let  $f : M \rightarrow \mathbb{R}$  where  $M$  is a metric space, and define  $G = \{(x, y) \in M \times \mathbb{R}; y = f(x)\}$ . If  $G$  is compact (in product space  $M \times \mathbb{R}$ ). Is  $f$  continuous? Justify your result.

4. (20 points)

Suppose  $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is an invertible linear map and  $g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  has continuous first order partial derivatives and satisfies  $\|g(x)\| \leq C\|x\|^{1+\epsilon}$  for some  $\epsilon > 0$  and all  $x \in \mathbb{R}^3$  ( $\|\cdot\|$  is the usual Euclidean norm). Show that  $f(x) = L(x) + g(x)$  is invertible near the origin 0.

5. (20 points)

(1): Find the volume of the ellipsoid  $(x+2y)^2 + (x-2y+z)^2 + 3z^2 \leq 1$ .

(2): Let  $C$  be a positively oriented simple closed curve. Find the curve  $C$  that maximizes the integral  $\int_C y^3 dx + (3x - x^3) dy$ .

6. (10 points)

Let  $f(x)$  be a real valued continuous function on  $[0, 1]$ . Find the limit

$$\lim_{n \rightarrow \infty} (n+1) \int_0^1 x^n f(x) dx.$$

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科目：高等微積分

2018.10.19

(1) [10 分] Does the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n \log(n+1)}{n}$$

converge? Does it converge absolutely? Justify your answer.

(2) [10+10 分] Consider the function

$$f(x, y) = \frac{1}{(1 - xy)^2}$$

defined on  $\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1, (x, y) \neq (1, 1)\}$ .

- (a) For any  $\kappa \in (0, 1)$ , let  $U_\kappa = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq \kappa, 0 \leq y \leq \kappa\}$ . Is  $f(x, y)$  uniformly continuous on  $U_\kappa$ ? Justify your answer.
- (b) Is  $f(x, y)$  uniformly continuous on  $\Omega$ ? Justify your answer.

(3) [10+15 分] For any  $n \in \mathbb{N}$ , consider  $f_n(x) = n x^n (1 - x)$  on  $I = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ .

- (a) Determine  $\lim_{n \rightarrow \infty} f_n(x)$  for every  $x \in I$ .
- (b) Is the convergence uniform on  $I$ ? Give your reason.

(4) [15+10 分] Let

$$F(x) = \int_0^\infty \frac{1 - \cos(xt)}{t^2 e^t} dt .$$

- (a) Can you switch the order of integration and differentiation to obtain the formulae for  $F'(x)$  and  $F''(x)$ ? Explain the reason.
- (b) Find the explicit<sup>1</sup> formula for  $F'(x)$  and  $F(x)$ .

(5) [10+10 分] Consider

$$\begin{aligned} F : \quad \mathbb{R}^4 &\rightarrow \mathbb{R}^2 \\ (x, y, u, v) &\mapsto \left( \int_{x-y^2}^{x^2+y} (e^{t^2} + u) dt, x^3 + v \right) . \end{aligned}$$

- (a) Prove that near  $(1, 1, 0, 0)$ , the two equations  $F(x, y, u, v) = (\int_0^2 e^{t^2} dt, 1)$  can be solved for  $u, v$  as continuously differentiable functions of  $x, y$ .
- (b) For the functions  $u(x, y)$  and  $v(x, y)$  in part (a), find all their first order partial derivatives at  $(x, y) = (1, 1)$ .

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<sup>1</sup>Not an improper integral.

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科目：高等微積分

2017. 10. 21

1. (15 points. No partial credit will be given if the answer is wrong.) Evaluate the integral

$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \frac{1}{1+3\sin^2 x} dx. \quad (\text{Hint. Let } u = \tan x.)$$

2. (15 points) State and prove Leibniz's criterion for convergence of alternating series.

3. (15 points) Let  $F$  be an  $\mathbf{R}$ -valued  $C^\infty$  function on  $\mathbf{R}^2$  such that at a point  $p = (a, b)$  we have

$$\frac{\partial F}{\partial x}(p) = \frac{\partial F}{\partial y}(p) = 0, \quad \frac{\partial^2 F}{\partial x^2}(p) > 0, \quad \text{and} \quad \frac{\partial^2 F}{\partial x^2}(p) \frac{\partial^2 F}{\partial y^2}(p) - \left( \frac{\partial^2 F}{\partial x \partial y}(p) \right)^2 > 0.$$

Show that there exists  $R > 0$  such that  $F(p) < F(q)$  for  $q \in \{(x, y) \in \mathbf{R}^2 \mid (x-a)^2 + (y-b)^2 < R^2\}$ .

4. We adopt the following definitions.

Let  $(X, d)$  be a metric space. (i) A family  $\mathcal{F}$  of  $\mathbf{R}$ -valued functions on  $X$  is *equicontinuous at a point  $x_0 \in X$*  (with respect to  $d$ ) if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall f \in \mathcal{F} \quad \forall x \in X \quad d(x, x_0) < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

(ii) A sequence  $f_n$  of  $\mathbf{R}$ -valued functions on  $X$  *converges compactly* to some function  $f$  on  $X$  if for every compact subset  $K$  of the metric space  $(X, d)$  the sequence  $f_n|_K$  converges to  $f|_K$  uniformly.

(15 points) Let  $f_n$  be a sequence of  $\mathbf{R}$ -valued functions on a metric space  $(X, d)$  which converges *pointwise* to a *continuous* function  $f$  on  $(X, d)$ . Suppose that  $\{f_n \mid n \in \mathbf{N}\}$  is equicontinuous at every point of  $X$ . Show that  $f_n$  converges compactly to  $f$  on  $X$ .

5. (15 points.) Compute the outward flux of the vector field  $(x + ye^z, e^x \sin(yz), ye^{zx})$  through the boundary of the region  $D = \left\{ (x, y, z) \in \mathbf{R}^3 \mid \left( \sqrt{x^2 + y^2} - 3 \right)^2 + z^2 < 1 \right\}$ .

6. (25 points) Show that the function  $f(x) = \sum_{n=0}^{\infty} \frac{\cos(ne)}{n^x}$  (where  $e = \sum_{m=0}^{\infty} \frac{1}{m!}$ ) is well-defined (i. e., the series converges) on  $(0, \infty)$  and is continuous.

7. (30 points. In your argument if any theorems are used you have to clearly verify that their conditions are fulfilled.) Let  $f$  and  $g$  be  $\mathbf{R}$ -valued  $C^\infty$  functions on  $\mathbf{R}^2$  and let  $S = \{(x, y) \in \mathbf{R}^2 \mid f(x, y) = 0\}$ . Suppose that at some point  $p = (a, b) \in S$  we have  $\frac{\partial f}{\partial x}(p) = -1, \frac{\partial f}{\partial y}(p) = 2, \frac{\partial g}{\partial x}(p) = 3, \frac{\partial g}{\partial y}(p) = -6,$

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(p) & \frac{\partial^2 f}{\partial x \partial y}(p) \\ \frac{\partial^2 f}{\partial y \partial x}(p) & \frac{\partial^2 f}{\partial y^2}(p) \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} \frac{\partial^2 g}{\partial x^2}(p) & \frac{\partial^2 g}{\partial x \partial y}(p) \\ \frac{\partial^2 g}{\partial y \partial x}(p) & \frac{\partial^2 g}{\partial y^2}(p) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}.$$

Show that there exists  $R > 0$  such that  $g(p) < g(q)$  for  $q \in S \cap \{(x, y) \in \mathbf{R}^2 \mid (x-a)^2 + (y-b)^2 < R^2\}$ .

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科目：高等微積分

2016.10.21

1. (15 points) Let  $A$  be the unit ball  $B_1(0)$  in  $\mathbf{R}^3$ . Compute

$$\int_A \cos(x+y+z) dx dy dz.$$

2. Let  $f$  be a real-valued function on  $\mathbf{R}$  which has period  $2\pi$  and is Riemann integrable on  $[-\pi, \pi]$ . We define its Fourier coefficients

$$a_n := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \quad (n = 0, 1, 2, \dots) \quad \text{and} \quad b_n := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \quad (n = 1, 2, \dots).$$

- (1) (10 points) Show that  $f(x)^2$  is Riemann integrable on  $[-\pi, \pi]$ .

- (2) (15 points) Show that the series  $\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$  converges.

3. (1) (15 points) Let  $\{a_n\}_{n=1}^{\infty}$  and  $\{b_n\}_{n=1}^{\infty}$  be sequences of real numbers and let  $B_n = b_1 + \cdots + b_n$  ( $n \in \mathbf{N}$ ). Suppose that  $a_n \searrow 0$  as  $n \rightarrow \infty$  and that there exists  $M > 0$  such that  $|B_n| \leq M$  for every  $n \in \mathbf{N}$ . Show that the series  $\sum_{n=1}^{\infty} a_n b_n$  converges.

- (2) (10 points) Show that the function series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin(nx)$$

converges uniformly on  $[-K, K]$  if  $|K| < \pi$ .

4. **Definition.** Let  $\mathcal{F}$  be a set of real-valued functions on a set  $X$ .  $\mathcal{F}$  is *uniformly bounded* if there exists  $M > 0$  such that  $|f(x)| \leq M$  for every  $x \in X$  and every  $f \in \mathcal{F}$ .

- (20 points) Let  $\{F_n\}_{n=1}^{\infty}$  be a sequence of convex functions on  $[-2, 2]$  and let  $f_n = F_n|_{[-1, 1]}$  ( $n \in \mathbf{N}$ ). Suppose that  $\{F_n | n \in \mathbf{N}\}$  is uniformly bounded. Show that there exists a subsequence of  $\{f_n\}_{n=1}^{\infty}$  which is uniformly convergent on  $[-1, 1]$ .

5. (15 points) Let  $f = (f_1, \dots, f_n) : U \rightarrow \mathbf{R}^n$  be a  $C^1$  map from an open set  $U$  in  $\mathbf{R}^n$ , and let  $g : V \rightarrow U$  be a continuous map from an open set  $V$  in  $\mathbf{R}^n$ . Suppose that

$$\det \left( \frac{\partial f_j}{\partial x_k}(x) \right) \neq 0 \quad \text{for every } x \in U,$$

and that  $f(g(x)) = x$  for every  $x \in V$ . Show that  $g$  is  $C^1$ .

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科目：高等微積分

2015.10.22

1. 30% (each 10%)

(i) Calculate  $\sup_{x \neq 0} \frac{x^2 - \ln(1+x^2)}{x^4} = ?$

(ii) Calculate  $\iint_A xy \sin(x^2 - y^2) dx dy$ , where

$$A = \{(x, y) : 0 < y < 1, x > y \text{ and } x^2 - y^2 < 1\}$$

(iii) Prove that  $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$  (Hint: use Fourier series)

2. 30% (each 10%)

Let  $C = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{3^n} : a_n \in \{0, 2\}, n \in \mathbb{N} \right\}.$

- (i) Prove that  $C \subset [0, 1]$ .
- (ii) Does  $C$  have interior point? Justify your answer.
- (iii) Is  $C$  a compact subset of  $\mathbb{R}$ ? Justify your answer.

3. 20% (each 10%)

Let  $A = \{f \in C^0[0, 1] : \|f\|_{C^0} \leq 1\}$  and  $B = \{f \in C^1[0, 1] : \|f\|_{C^1} \leq 1\}$ ,

where  $C^0[0, 1] = \{f : [0, 1] \rightarrow \mathbb{R} \mid f \text{ is continuous on } [0, 1]\}$ ,

$C^1[0, 1] = \{f : [0, 1] \rightarrow \mathbb{R} \mid f \text{ is of } C^1 \text{ on } [0, 1]\}$ ,  $\|\cdot\|_{C^0}$  is the

standard sup-norm, and  $\|\cdot\|_{C^1}$  is the standard  $C^1$ -norm.

- (i) Is  $A$  (sequentially) compact? Justify your answer.
- (ii) Prove that any sequence of  $B$  has a convergent subsequence in  $A$ .

4. 20%

Let  $f : [0,1] \rightarrow \mathbb{R}$  be a continuous function. Let  $F(x) = \int_0^x f(t) dt$  for

$x \in [0,1]$ . Suppose  $F(1) = 0$ . Determine if the value

$$\sup_{g \in C^0[0,1]} \frac{\int_0^1 F(x) g(x) dx}{\left( \int_0^1 G^2(x) dx \right)^{1/2}} \text{ exists, where } G(x) = \int_0^x g(t) dt \text{ for}$$

$x \in [0,1]$ , and  $C^0[0,1] = \{g : [0,1] \rightarrow \mathbb{R} \mid g \text{ is continuous on } [0,1]\}$ .

Justify your answer.

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科目：高等微積分

2014.10.24

1. (10%) Determine whether  $\int_0^\infty x^{-\frac{1}{3}} e^{-x} dx$  converges or not. Justify your answer.
2. (20% = 10% + 10%) For the following functions, determine whether they are *uniformly continuous* or not. Justify your answer.
  - (a)  $f(x) = \log(2014 + x^{10})$  for  $x \in \mathbb{R}$ , where  $\log$  is the natural logarithm function.
  - (b) For  $(x, y) \in [-1, 1] \times [-1, 1]$ ,
$$g(x, y) = \begin{cases} x \sin\left(\frac{y}{x}\right) & \text{when } x \neq 0, \\ 0 & \text{when } x = 0. \end{cases}$$
3. (20% = 10% + 10%) For the following sequences of continuous functions on  $\mathbb{R}^1$ , determine whether they have a *uniformly convergent* subsequence or not. Justify your answer.
  - (a)  $f_n(x) = \exp(-(x - n)^2)$ ,  $n \in \mathbb{N}$ .
  - (b)  $g_n(x) = \sin(x - n)$ ,  $n \in \mathbb{N}$ .
4. (20% = 10% + 10%) Consider the system of equations:

$$\begin{aligned} x + yu + z^2 + x^3 &= 0, \\ z + x^2 + z^2u - y^3 &= 0. \end{aligned}$$

It is clear that  $x = y = z = u = 0$  is a solution.

- (a) Near  $(0, 0, 0, 0)$ , can the system be solved for  $x, z$  as continuously differentiable functions of  $y, u \in (-\varepsilon, \varepsilon)$  for some  $\varepsilon > 0$ ? Justify your answer.
- (b) Near  $(0, 0, 0, 0)$ , can the system be solved for  $y, u$  as continuously differentiable functions of  $x, z \in (-\varepsilon, \varepsilon)$  for some  $\varepsilon > 0$ ? Justify your answer.
5. (20% = 10% + 10%) Suppose that  $\{a_n\}_{n=1}^\infty$  is a sequence such that  $\sum_{n=1}^\infty n|a_n|$  converges.
  - (a) Show that both  $\sum_{n=1}^\infty a_n \sin(nx)$  and  $\sum_{n=1}^\infty na_n \cos(nx)$  converges uniformly for all  $x \in \mathbb{R}$ .
  - (b) Let  $f(x) = \sum_{n=1}^\infty a_n \sin(nx)$ , and  $g(x) = \sum_{n=1}^\infty na_n \cos(nx)$ . Prove that the derivative of  $f(x)$  is  $g(x)$ .
6. (10%) Let  $f(x, y, z)$  be a smooth function on  $\mathbb{R}^3$ , and  $\kappa$  be a constant within  $(0, 3)$ . Prove that

$$(3 - \kappa) \iiint_{|\mathbf{x}| \leq r} f^2 d\mathbf{x} dy dz \leq r \iint_{|\mathbf{x}|=r} f^2 dS + \frac{1}{\kappa} \iiint_{|\mathbf{x}| \leq r} |\mathbf{x}|^2 |\nabla f|^2 d\mathbf{x} dy dz$$

for any  $r > 0$ . Here,  $\mathbf{x} = (x, y, z)$  and  $\nabla f$  is the gradient of  $f$ .

Hint: Apply the divergence theorem and the Cauchy–Schwarz inequality.

# 臺灣大學數學系 103 學年度碩士班甄試試題

## 科目：高等微積分

2013.10.18

請為每一題預留充分書寫空間，依題號次序 1.(a)(b)(c), 2.(a)(b), … 作答。若未依題目次序，其跳題作答之部份不予批閱、計分。

Answer the following questions in order.

All functions and sequences are real-valued.

1. (25% = 5+5+15)

- (a) State the definition that a sequence  $(a_n)$  is a *Cauchy sequence*.
- (b) State the definition that a function  $F : D \subset \mathbb{R} \rightarrow \mathbb{R}$  is *uniformly continuous* on  $D$ .
- (c) Let  $f(x)$  be a continuous function on  $\mathbb{R}$  and  $(x_n)$  be a Cauchy sequence. Use an  $\epsilon - \delta/\epsilon - N$  type argument to show that  $(f(x_n))$  is also a Cauchy sequence.

2. (25% = 5+20)

- (a) State the definition that a sequence of functions  $(h_n), h_n : D \subset \mathbb{R} \rightarrow \mathbb{R}$ , converges to  $H$  uniformly on  $D$ .
- (b) Let  $(g_n)$  be a sequence of differentiable functions on  $(a, b)$  such that  $\lim_{n \rightarrow \infty} g_n(x) = G(x)$  exists for all  $x \in (a, b)$ . Suppose that there exists a constant  $M > 0$  such that  $\sup_{x \in (a, b)} |g'_n(x)| < M$  for all  $n$ . Show that  $(g_n)$  converges to  $G$  uniformly on  $(a, b)$ .

3. (25% = 7+8+10) Let  $\lambda > 0$ ,  $J(\lambda) = \int_0^\infty \frac{dx}{(1+x)x^{2\lambda}}$  and  $\Gamma(\lambda) = \int_0^\infty x^{\lambda-1}e^{-x} dx$ .

- (a) Show that for  $\lambda > 0$  the improper integral  $\Gamma(\lambda)$  converges.
- (b) Find the range of  $\lambda > 0$  on which the improper integral  $J(\lambda)$  converges.
- (c) Show that  $J(\lambda) = \Gamma(2\lambda)\Gamma(1-2\lambda)$  when both sides are meaningful. Hint. Express  $(1+x)^{-1}$  as an integral.

4. (25% = 10+15) Let  $u_k, p_k, k = 1, \dots, n$ , be positive numbers and  $p_1 + \dots + p_n = 1$ .

- (a) Evaluate the limit

$$\lim_{t \rightarrow \infty} \left( \sum_{k=1}^n p_k u_k^t \right)^{\frac{1}{t}}.$$

- (b) Evaluate the limit

$$\lim_{t \rightarrow 0} \left( \sum_{k=1}^n p_k u_k^t \right)^{\frac{1}{t}}.$$

Hint. Use  $u^t = \exp(t \ln u)$  and Taylor expansions.

臺灣大學數學系 102 學年度碩士班甄試試題

科目：高等微積分

2012.10.19

Advanced Calculus

1. (40 points) True or False. Prove or disprove each of the following statements.

- a) If the real series  $\sum_{j=1}^{\infty} b_j = 1$  is conditional convergent (which is not absolutely convergent), then there exist a rearrangement of  $\{b_j\}$  such that  $\sum_{j=1}^{\infty} b_{\sigma(j)} = 2012$ , where  $\{\sigma(j)\}_{j=1}^{\infty}$  is a permutation of  $\{j\}_{j=1}^{\infty}$ .
- b) If, for some positive number  $M > 0$ , the partial sum  $|\sum_{j=1}^n b_j| \leq M$  for all  $n \in \mathbb{N}$  and  $\{a_j\}_{j=1}^{\infty}$  is a positive decreasing sequence that tends to 0, then  $\sum_{j=1}^{\infty} a_j b_j$  converges.
- c) If  $f(x)$  is a continuous function defined on the closed interval  $[-1, 1]$  such that  $f'(0) = 1$ , then there exists a  $\delta > 0$  such that  $f(x)$  is an increasing function for all  $x \in [-\delta, \delta]$ .
- d) Suppose  $g(x)$  is a nonnegative continuous function defined on the closed interval  $[0, 1]$  and  $f(x)$  is a positive monotone continuous function, then there exists a number  $\xi \in [0, 1]$  such that

$$\int_0^1 f(t)g(t)dt = f(\xi) \int_0^1 g(t)dt.$$

2. (50 points) Evaluate

$$\begin{aligned} &\int_0^1 x^9(10 \ln x + 1)dx, && \int_2^4 \int_{4/x}^{(20-4x)/(8-x)} (y-4)dydx, \\ &\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 dydx, && \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n+i}, \\ &\int_A x dx dy dz, \end{aligned}$$

where  $A$  is the region in  $\mathbb{R}^3$  bounded by the planes  $x = 0, y = 0, z = 2$  and the surface  $z = x^2 + y^2$ .

3. (10 points) Find the maximum of  $f(x, y, z) = x + y + 2z$  under the constraints

$$x^2 - xy + y^2 + z^2 = 2.$$

At which point does  $f(x, y, z)$  achieve its maximum ?

臺灣大學數學系  
101 學年度碩士班甄試試題  
科目：高等微積分

2011.10.21

1. (30 points) Calculate

(1)

$$\int \int_{\mathbb{R}^2} e^{-(13x^2+16xy+5y^2)} dx dy$$

(2)

$$\int_0^\infty \frac{\sin x}{x} dx.$$

2. (30 points)

(1) Suppose  $n$  is a positive integer,  $a_i \geq 0$ ,  $b_i \geq 0$  for  $i = 1, 2, \dots, n$ , and  $p$  and  $q$  are two positive numbers such that  $1/p + 1/q = 1$ . Prove the following Holder's inequality.

$$\sum_{i=1}^n a_i b_i \leq \left( \sum_{i=1}^n a_i^p \right)^{1/p} \left( \sum_{i=1}^n b_i^q \right)^{1/q}.$$

(2) Suppose  $a_1 \geq a_2 \geq a_3 \geq a_4$  and  $b_1 \geq b_2 \geq b_3 \geq b_4$ . Prove the following inequality.

$$a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4 \geq a_1 b_2 + a_2 b_4 + a_3 b_1 + a_4 b_3.$$

3. (30 points) Prove or disprove the following statements.

(1) Let  $f(x, y)$  be a real-valued function on  $\mathbb{R}^2$  such that both  $\frac{\partial^2 f(x_0, y_0)}{\partial x \partial y}$  and  $\frac{\partial^2 f(x_0, y_0)}{\partial y \partial x}$  exist. Then  $\frac{\partial^2 f(x_0, y_0)}{\partial x \partial y} = \frac{\partial^2 f(x_0, y_0)}{\partial y \partial x}$ .

(2) Let  $f_n(x)$  be a sequence of continuous functions defined on  $[0, 1]$ . If  $f_n(x)$  converges uniformly on  $[0, 1]$  as  $n \rightarrow \infty$ , then  $\lim_{n \rightarrow \infty} f_n(x)$  is a continuous function on  $[0, 1]$ .

4. (10 points) Suppose  $f(x)$  is a continuous function defined on  $[0, 1]$ . Prove that

$$\lim_{n \rightarrow \infty} \int_0^1 x^n f(x) dx = 0.$$

臺灣大學數學系  
100 學年度碩士班甄試試題  
科目：高等微積分

2010.10.22

- (1) (25 pts) Suppose the series  $f(x) = \sum_{n=0}^{\infty} a_n x^n$  converges for  $|x| < R$ .  
Show that  $f$  is continuous and differentiable on  $(-R, R)$  and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \text{ for } |x| < R.$$

- (2) (25 pts) Let  $\{f_n\}$  and  $f$  be defined on  $[0, 2]$ . Suppose that  $\{f_n\}$  are continuous and

$$\lim_{n \rightarrow \infty} f_n(x_n) = f(x)$$

for every sequence  $\{x_n\} \subset [0, 2]$  such that  $\lim_{n \rightarrow \infty} x_n = x$  and  $x \in [0, 2]$ .

- (a) Is it true that  $\{f_n\}$  converges uniformly to  $f$  on  $[0, 2]$ ?  
(b) Is it true that  $f$  is continuous on  $[0, 2]$ ?

- (3) (25 pts)

- (a) Prove that

$$\left| \int_0^1 f(x)g(x) dx \right| \leq \left( \int_0^1 f^2(x) dx \right)^{\frac{1}{2}} \left( \int_0^1 g^2(x) dx \right)^{\frac{1}{2}}.$$

- (b) Let  $h(x)$  be a continuous function on  $[0, 1]$ . Show that

$$\lim_{n \rightarrow \infty} \int_0^1 h(x) \sin(nx) dx = 0.$$

- (4) (25 pts) Let  $\mathbb{N} = \{1, 2, 3, \dots\}$  denote the natural numbers and  $E$  be defined as follows:  $A \in E$  if and only if  $A$  is a subset of  $\mathbb{N}$ . Show that there is a one-to-one and onto mapping from  $E$  to the open interval  $(0, 1)$ .

臺灣大學數學系  
九十九學年度碩士班甄試試題  
科目：高等微積分

2009.10.30

(1) (25 pts) Define  $f(0, 0) = 0$  and  $f(x, y) = \frac{x^3}{3x^2 + y^2}$ ,  $(x, y) \neq (0, 0)$ .

(a) Show that  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$  exist at  $(0, 0)$  and  $f$  is continuous at  $(0, 0)$ .

(b) Prove that  $f$  is not differentiable at  $(0, 0)$ .

(c) Is  $f^2$  differential at  $(0, 0)$ ?

(2) (25 pts) Assume  $a > 0$  and  $b > 0$ . Find all  $(a, b)$  such that  $\frac{\sin(x^a)}{1 + x^b}$  is uniformly continuous on  $\{x : x > 0\}$ .

(3) (25 pts)

(a) Let  $\{b_n\}$  be defined by  $b_1 = 1$ ;  $b_{2m} = \frac{b_{2m-1}}{4}$ ;  $b_{2m+1} = 1 + b_{2m}$ . Find  $\limsup_{n \rightarrow \infty} b_n$  and  $\liminf_{n \rightarrow \infty} b_n$ .

(b) Let  $\{c_n\}$  and  $\{d_n\}$  be two strictly increasing sequences of positive integers satisfying  $c_n + d_n < 1.5n$ . Define  $p(m) = 1$  if  $m \in \{c_n\}$ ,  $p(m) = 0$  if  $m \notin \{c_n\}$ ;  $q(m) = 1$  if  $m \in \{d_n\}$ ,  $q(m) = 0$  if  $m \notin \{d_n\}$ , where  $m$  is a positive integer. Find  $\limsup_{m \rightarrow \infty} (p(m) + q(m))$ .

(4) (25 pts)

(a) Let  $f(x)$  be an increasing function on  $[0, 1]$ . Prove that  $f$  is Riemann integrable.

(b) Assume  $g(x, y) < g(x, z)$  if  $y < z$  and  $g(x, y) < g(t, y)$  if  $x < t$ . Prove or disprove that the assumptions above imply  $g(x, y)$  is Riemann integrable on  $[0, 1] \times [0, 1]$ .

注意：考試開始鈴響前，不得翻閱試題，  
並不得書寫、畫記、作答。

國立清華大學 113 學年度碩士班考試入學試題

系所班組別：數學系

科目代碼：0101

考試科目：高等微積分

## 一作答注意事項一

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
2. 考試開始後，請於作答前先翻閱整份試題，是否有污損或試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。
3. 考生限在答案卷上標記「 由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。
4. 答案卷用盡不得要求加頁。
5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「國立清華大學試場規則及違規處理辦法」，無法因本試題封面作答注意事項中未列明而稱未知悉。

# 國立清華大學 113 學年度碩士班考試入學試題

系所班組別：數學系碩士班

考試科目（代碼）：高等微積分（0101）

共 2 頁，第 1 頁 \*請在【答案卷】作答

You need to explain how you get your answers in each problem. Please mark your answers clearly.

Problem 1: (10 points) Let  $P = x - y^3 - y$  and  $Q = x^3 + y^2$ . Find the line integral of  $\oint_C Pdx + Qdy$  where  $C$  is the simple closed curve  $x^2 + y^2 = 1$ . The line integral is taken counter-clockwisely.

Problem 2: (10 points) Find  $\int_2^{24} \frac{d[\sqrt{x}]}{\sqrt{x}}$  where  $[\cdot]$  is the greatest integer function.

Problem 3: Let  $f_n(x) = nx^n(1-x)$ ,  $n = 1, 2, 3, \dots$ .

(a) (5 points) Determine if the sequence  $\{f_n\}_{n=1}^{\infty}$  converges pointwisely on  $[0, 1]$ .

(b) (10 points) Determine if the family of functions  $\{f_n\}_{n=1}^{\infty}$  is equicontinuous on  $[0, 1]$ .

Problem 4: (10 points) Let  $f(x, y) = \frac{x^5 - y^3}{x^4 + y^2}$  when  $(x, y) \neq (0, 0)$  and  $f(0, 0) = 0$ . Determine if  $f$  is differentiable at  $(0, 0)$ .

Problem 5: (15 points) Let  $S := \{(x, \sin(\frac{1}{x})) \mid 0 < x < \pi\} \cup \{(0, y) \mid 0 \leq y \leq 1, y \in \mathbb{Q}\}$ . Determine if  $S$  is connected. You can use the fact that  $\{(x, \sin(\frac{1}{x})) \mid 0 < x < \pi\}$  is connected.

Problem 6: (15 points) Let  $f$  be an Riemann integrable function on  $[0, 1]$ . Suppose that for any  $0 < a < b < 1$ , there is at least one number  $c \in (a, b)$  such that  $f(c) = 1$  or  $f(c) = 0$ . If  $\int_0^1 f(x)dx = 1/2$ , prove that there is an uncountable subset  $S \subseteq [0, 1]$  such that  $f(s) = 0$  for any  $s \in S$ . Note that  $f$  may not be continuous.

Problem 7: (10 points) Prove that  $\sqrt[3]{x}$  is uniformly continuous on  $\mathbb{R}$ .

國立清華大學 113 學年度碩士班考試入學試題

系所班組別：數學系碩士班

考試科目（代碼）：高等微積分（0101）

共 2 頁，第 2 頁 \*請在【答案卷】作答

Problem 8: (15points) Let  $s_n(x) = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots + \frac{1}{n}\sin nx$ . Prove that the sequence  $\{s_n\}_{n=1}^{\infty}$  converges uniformly on  $[1, 2]$ .

注意：考試開始鈴響前，不得翻閱試題，  
並不得書寫、畫記、作答。

國立清華大學 112 學年度碩士班考試入學試題

系所班組別：數學系

科目代碼：0101

考試科目：高等微積分

## 一作答注意事項一

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國立清華大學 112 學年度碩士班考試入學試題

系所班組別：數學系碩士班

考試科目（代碼）：高等微積分 (0101)

共 2 頁第 1 頁 \* 請在 [答案卷] 作答

1. (10 pts) Let  $A$  be a nonempty set of real numbers which is bounded below. Let  $-A = \{-x | x \in A\}$ . Prove that  $\inf A = -\sup(-A)$ .

2. (15 pts) Let  $E$  be a nonempty subset of  $\mathbb{R}^n$ . Let  $E'$  be the set of all limit points (accumulation point). (i) Is  $E'$  a closed set? (ii) Does  $E$  and  $E'$  always have the same limit points? Prove the statements or give counterexamples for (i) and (ii). (iii) If  $E = \{(x, \sin \frac{1}{x}) | x \in (0, 1)\}$ . What is  $E'$ ?

3. (10 pts) Let  $a_1$  and  $a_{n+1} = 1 + \sqrt{a_n}$ ,  $n \in \mathbb{N}$ . Find

$$\limsup_{n \rightarrow \infty} a_n.$$

4. (15 pts) (i) Suppose  $a_n \geq 0$ ,  $n \in \mathbb{N}$  and  $\sum a_n$  diverges. Prove  $\sum a_n/(1 + a_n)$  also diverges. (ii) Suppose  $\sum a_n$  is a series of real numbers which converges absolutely. Prove that every rearrangement of  $\sum a_n$  converges to the same sum.

5. (10 pts) Use the finite open covering property of the compactness to show the following. If  $f$  is a continuous mapping of a compact metric space  $X$  into a metric space  $Y$ . Then  $f$  is uniformly continuous on  $X$ .

國立清華大學 112 學年度碩士班考試入學試題

系所班組別：數學系碩士班

考試科目（代碼）：高等微積分 (0101)

共 2 頁第 2 頁 \* 請在 [答案券] 作答

6. (15 pts) Give the reasons in your computation. (i) Let  $a > 0$ . Find

$$\lim_{n \rightarrow \infty} \int_a^{\infty} \frac{\sin(nx)}{nx} dx.$$

(ii) What is the answer for (i) if  $a = 0$  ?

(iii) Let  $[x]$  be the greatest integer function. That is  $[x] = \sup\{n \mid n \leq x, n \in \mathbb{Z}\}$ . Find

$$\int_0^2 x d[x].$$

7. (15 pts) Let  $f : \mathbb{R}^5 \rightarrow \mathbb{R}^2$  defined by  $f = (f_1, f_2)$  with

$$f_1(x_1, x_2, y_1, y_2, y_3) = 2e^{x_1} + x_2 y_1 - 4y_2 + 3$$

$$f_2(x_1, x_2, y_1, y_2, y_3) = x_2 \cos x_1 - 6x_1 + 2y_1 - y_3.$$

Note that  $f(0, 1, 3, 2, 7) = (0, 0)$ . (i) Prove that there exists a function  $g : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  defined on a neighborhood of  $(3, 2, 7)$  so that  $f(g(y_1, y_2, y_3), y_1, y_2, y_3) = 0$ . (ii) Find the derivative of  $g$  at  $(3, 2, 7)$ .

8. (10 pts) (i) Let  $(x, y) \in \mathbb{R}^2 \setminus (0, 0)$ ,  $P(x, y) = \frac{-y}{x^2+y^2}$  and  $Q(x, y) = \frac{x}{x^2+y^2}$ . Let  $C$  be the curve goes from  $(a, 0)$ ,  $a > 0$  to itself once along  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ,  $b > 0$ . Find

$$\oint_C P(x, y) dx + Q(x, y) dy.$$

(ii) Can we find a function  $f : \mathbb{R}^2 \setminus (0, 0) \rightarrow \mathbb{R}$  such that

$$\frac{\partial f}{\partial x} = P(x, y) \text{ and } \frac{\partial f}{\partial y} = Q(x, y)?$$

Give the reason for your answer.

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國立清華大學 111 學年度碩士班考試入學試題

系所班組別：數學系

科目代碼：0101

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國立清華大學 111 學年度碩士班考試入學試題

系所班組別：數學系碩士班

考試科目（代碼）：高等微積分（0101）

共 1 頁，第 1 頁

\*請在【答案卷、卡】作答

1. (10%) Find

$$\lim_{n \rightarrow \infty} \int_3^n \left(1 - \frac{x}{n}\right)^n dx$$

2. (10%) Let  $f : \mathbb{R} - \{0\} \rightarrow \mathbb{R}$  be defined by

$$f(x) = \frac{1}{x}$$

Is  $f$  a continuous function? Prove your claim.

3. (15%) Consider  $\mathbb{Q}$  as a topological subspace of  $\mathbb{R}$  where  $\mathbb{R}$  is given the Euclidean topology. Let  $f : \mathbb{Q} \rightarrow \mathbb{R}$  be a continuous function and  $A \subset \mathbb{Q}$  be a compact subset. Must  $f(A)$  be compact in  $\mathbb{R}$ ? Prove or give a counterexample.

4. (15%) Construct a sequence of Riemann integrable functions  $f_n : [0, 1] \rightarrow \mathbb{R}$  which converges pointwise to a function  $f : [0, 1] \rightarrow \mathbb{R}$  but

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx \neq \int_0^1 f(x) dx$$

5. (15%) Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be defined by

$$f(x, y) := \begin{cases} \left(\frac{\sin(x^2+y^2-1)}{x^2+y^2-1}, \cos(x^2+y^2-1)\right), & \text{if } x^2 + y^2 \neq 1 \\ (1, 1), & \text{if } x^2 + y^2 = 1 \end{cases}$$

Is  $f$  differentiable at  $(1, 0)$ ? Prove your claim.

6. Let

$$E = \left\{ \sum_{k=0}^n a_k x^{2k+1} \mid n \in \mathbb{N} \cup \{0\} \right\}$$

be the set of polynomials of odd degree in each term defined on  $[1, 2]$ .

- (a) (10%) Show that  $E$  is not closed in  $C^0([1, 2])$  where  $C^0([1, 2])$  is the space of continuous functions from  $[1, 2]$  to  $\mathbb{R}$  with the sup norm.

- (b) (10%) Is  $E$  dense in  $C^0([1, 2])$ ? Prove your claim.

7. (15%) Given a continuous function  $h : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$  such that

$$\sup_{x \in [0, 1]} \left\{ \int_0^1 |h(x, y)| dy \right\} < 1$$

Suppose that  $g : [0, 1] \rightarrow \mathbb{R}$  is a continuous function. Show that there is a unique continuous function  $f : [0, 1] \rightarrow \mathbb{R}$  such that

$$f(x) - \int_0^1 h(x, y) f(y) dy = g(x)$$

注意：考試開始鈴響前，不得翻閱試題，  
並不得書寫、畫記、作答。

國立清華大學 110 學年度碩士班考試入學試題

系所班組別：數學系

科目代碼：0101

考試科目：高等微積分

## 一作答注意事項一

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
2. 考試開始後，請於作答前先翻閱整份試題，是否有污損或試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。
3. 考生限在答案卷上標記「由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。
4. 答案卷用盡不得要求加頁。
5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「國立清華大學試場規則及違規處理辦法」，無法因本試題封面作答注意事項中未列明而稱未知悉。

國立清華大學 110 學年度碩士班考試入學試題

系所班組別：數學系碩士班

考試科目（代碼）：高等微積分（0101）

共 2 頁，第 1 頁 \*請在【答案卷、卡】作答

1. (10%) Calculate

$$\int_0^2 x \sin(3x+1) \sin(4x+1) dx$$

2. (10%) Calculate the line integral

$$\int_C \frac{xdy - ydx}{x^2 + y^2}$$

where  $C$  is the positive oriented curve in  $\mathbb{R}^2 - \{(0,0)\}$  parametrized by

$$\varphi(\theta) = \begin{cases} (\cos 4\pi\theta, \sin 4\pi\theta), & \text{if } \theta \in [0, \frac{1}{2}]; \\ (2 - \cos 4\pi\theta, -\sin 4\pi\theta), & \text{if } \theta \in [\frac{1}{2}, 1] \end{cases}$$

3. (10%) Determine if

$$\sum_{n=1}^{\infty} n \cos \frac{x}{n}$$

converges uniformly on  $[-\pi, \pi]$ .

4. (10%) Show that the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by

$$f(x, y) = \frac{1}{10} (\sin^2(3x+y), \cos(4x+2y))$$

has a fixed point.

5. (10%) Show that the set

$$X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + 2x_2^4 + 3(e^{x_1} + x_3^2)^6 = 2021\}$$

is compact.

6. (10%) Let  $X$  be a topological space and  $A, B \subset X$  be compact subsets.

Is  $A \cup B$  always a compact set? Prove or give a counterexample.

國立清華大學 110 學年度碩士班考試入學試題

系所班組別：數學系碩士班

考試科目（代碼）：高等微積分（0101）

共 2 頁，第 2 頁

\*請在【答案卷、卡】作答

7. (10%) Let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  be a function defined by

$$f(x, y, z) = \frac{x^2 \sin x}{x^2 + y^2 + z^2}$$

for  $(x, y, z) \in \mathbb{R}^3 - \{0\}$ . Can  $f$  be extended to a differentiable function on  $\mathbb{R}^3$ ?

8. Let  $\mathcal{C}^0([0, \pi]) := \{f : [0, \pi] \rightarrow \mathbb{R} | f \text{ is continuous}\}$  with the sup norm.

(a) (10%) Is the family  $\{\cos(nx)\}_{n=1}^{\infty}$  equicontinuous on  $[0, \pi]$ ?

(b) (10%) Does the sequence  $\{\cos(nx)\}_{n=1}^{\infty}$  in  $\mathcal{C}^0([0, \pi])$  have a convergent subsequence?

(c) (10%) Is the family  $\{\frac{1}{n} \sin(nx)\}_{n=1}^{\infty}$  in  $\mathcal{C}^0([0, \pi])$  compact?

注意：考試開始鈴響前，不得翻閱試題，  
並不得書寫、畫記、作答。

國立清華大學 109 學年度碩士班考試入學試題

系所班組別：數學系

科目代碼：0101

考試科目：高等微積分

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6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「國立清華大學試場規則及違規處理辦法」，無法因本試題封面作答注意事項中未列明而稱未知悉。

# 國立清華大學 109 學年度碩士班考試入學試題

系所班組別：數學系碩士班

考試科目（代碼）：高等微積分（0101）

共 1 頁，第 1 頁 \*請在【答案卷、卡】作答

1. (10 pts) Let  $f : (0, \infty) \rightarrow \mathbb{R}$  be differentiable. If  $f(x) \rightarrow 5$  and  $f'(x) \rightarrow \lambda$  as  $x \rightarrow \infty$ , prove that  $\lambda = 0$ .

2. (10 pts) Find the limit

$$\lim_{n \rightarrow \infty} \frac{n}{\log n} \left( n^{1/n} - 1 \right).$$

3. (12 pts) Prove that the function

$$f(x) = \sum_{n=1}^{\infty} \frac{x^2}{x^2 + n^2}$$

is continuous on  $\mathbb{R}$ .

4. (12 pts) Suppose that  $\{p_n\}$  is a sequence of polynomials, and that  $p_n \rightarrow f$  uniformly on the interval  $[0, 1]$ . Must  $f$  be differentiable?

5. (12 pts) Is there a simple closed curve  $C$  in the  $xy$ -plane which maximizes the value of

$$\oint_C y^3 dx + (3x - x^3) dy ?$$

If so, find the maximum value.

6. (12 pts) Let  $B = \{x \in \mathbb{R}^n / \|x\| \leq r\}$ , and suppose  $f : B \rightarrow \mathbb{R}^n$  satisfying  $\|f(0)\| \leq \frac{2}{3}r$  and

$$\|f(x) - f(y)\| \leq \frac{1}{3} \|x - y\| \quad \text{for all } x, y \in B.$$

Prove that there exists a unique  $x \in B$  such that  $f(x) = x$ .

7. (16 pts) Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be defined by  $F(x) = L(x) + G(x)$ , where  $L$  is a linear isomorphism and  $G$  is a  $C^1$ -function. Suppose that there are positive constants  $M$  and  $\varepsilon$  such that  $\|G(x)\| \leq M \|x\|^{1+\varepsilon}$  for all  $x$  in a neighborhood of the origin. Prove that  $F$  is locally invertible near the origin.

8. (16 pts) Let  $f : (a, b) \rightarrow \mathbb{R}$  be a  $C^n$ -function, and suppose for some  $c \in (a, b)$ ,

$$f'(c) = f''(c) = \cdots = f^{(n-1)}(c) = 0, \quad \text{but} \quad f^{(n)}(c) \neq 0.$$

Prove that

- For  $n$  even,  $f$  has a local minimum at  $c$  if  $f^{(n)}(c) > 0$ , and a local maximum at  $c$  if  $f^{(n)}(c) < 0$ .
- For  $n$  odd, there is neither a local maximum nor a local minimum at  $c$ .

**國立成功大學 114 學年度「碩士班」甄試入學考試**  
**高等微積分**

1. (15%) Determine the following limit by  $\delta$ - $\epsilon$  definition.

$$\lim_{x \rightarrow 1} (2x^3 + 7x^2 + 5x + 8)$$

2. (15%) Show by definition that the following set is open:

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 < x + y\}$$

3. (15%) Assume function  $f$  is increasing on interval  $[20, 24]$ .

- (a) Show that the set of discontinuity of  $f$  is countable.  
(b) Show that  $f$  is Riemann integrable.

4. (15%) Denote  $x_n$  the unique positive root of the polynomial

$$f_n(x) = x^n + x^{n-1} + \cdots + x - 114, \quad n = 1, 2, 3, \dots$$

Show that the sequence  $\{x_n\}$  is convergent. Also find the limit.

5. (20%) Yes or No. Justify your answer.

- (a) Determine if  $f(x) = \sqrt{1+x^2}$  is uniformly continuous on  $(-\infty, \infty)$ .  
(b) Determine if  $f(x) = \sqrt{1+x^4}$  is uniformly continuous on  $(-\infty, \infty)$ .

6. (20%) Find the value of the limit. Justify your answer.

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{nx^n}{1+e^x} dx$$

# 國立成功大學 113 學年度「碩士班」甄試入學考試

## 高等微積分

### Advanced Calculus

October 19, 2023

1. (15 points) Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a function defined by

$$f(x) := \begin{cases} \frac{1}{k} & \text{when } x \in \mathbb{Q} \text{ and } x = \frac{q}{2^k} \text{ for some } q, k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that  $f$  is continuous at every irrational point.

2. (a). (10 points) Determine whether the following series is convergent or not:

$$\sum_{n=1}^{\infty} 5^{-n+(-1)^n}.$$

(b). (10 points) Prove that the power series  $\sum_{n=0}^{\infty} a_n x^n$  converges on  $(-R, R)$  when  $\sum_{n=0}^{\infty} R^{2n} a_n^2$  converges.

3. (15 points) Let  $X \subset \mathbb{R}^2$  be an open set and  $f : X \rightarrow \mathbb{R}$  be a continuous function with  $f(x) \neq 0$  for all  $x \in X$ . Prove that either  $f(x) > 0$  for all  $x \in X$ ,  $f(x) < 0$  for all  $x \in X$  or  $X$  is disconnected<sup>1</sup>.

4. (20 points) Let  $\{g_n\}_{n \in \mathbb{N}}$  be a sequence of positive, Riemann integrable functions defined on  $[0, 1]$  and  $g_n(x) \leq 1$  for all  $x \in [0, 1]$ ,  $n \in \mathbb{N}$ . Prove that the sequence

$$\left\{ f_n(x) := \int_0^x g_n(s) ds \right\}_{n \in \mathbb{N}}$$

has a subsequence which is uniformly convergent on  $[0, 1]$ .

5. (15 points) Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a Riemann integrable function. Prove that  $f^3$  is also Riemann integrable.

6. (15 points) Let  $S := \{(x, y, z) \in \mathbb{R}^3 | z = 7 - x^2 - y^2, z \geq 3\}$ ,  $\vec{n}$  be the outer normal of  $S$  and

$$V := (z^2, -3xy, x^3y^3)$$

be a vector field defined on  $\mathbb{R}^3$ . Find

$$\int_S \operatorname{curl}(V) \cdot \vec{n} ds.$$

<sup>1</sup>A set  $X$  is disconnected iff there exist two disjoint nonempty open sets  $A, B$  cover  $X$  and  $A \cap X \neq \emptyset, B \cap X \neq \emptyset$ .

# 國立成功大學 112 學年度「碩士班」甄試入學考試

## 高等微積分

1. (a) Determine the limit of  $f(x) = \frac{3x^2 - 5x + 4}{x + 1}$  at  $x = 2$  by  $\delta$ - $\epsilon$  definition. (10%)  
(b) Determine whether  $f(x)$  is uniformly continuous on  $[0, \infty)$ . (10%)
2. Show by definition that the interior of triangular region with vertices  $(1, 1), (5, 2), (3, 4)$  is an open set. (10%)
3. Denote  $x_n$  the positive root of the polynomial  $f_n(x) = x^n + \dots + x^2 + x - 1$ . Show that the sequence  $\{x_n\}$  is convergent and find the limit. (15%)
4. Given the fact that the following integral is convergent for all  $p > 0$ .
$$\int_1^\infty \frac{\sin x}{x^p} dx$$
For what values of  $p > 0$  is the integral convergent absolutely/conditionally? (15%)

5. (a) By observing the graph of  $y = \frac{n}{1+n^2x^2}$  as  $n$  increases and evaluating its integral on  $(-\infty, \infty)$ , find the value the following limit. (10%)
$$\lim_{n \rightarrow \infty} \int_{-\infty}^\infty \frac{n e^{\cos x}}{1 + n^2 x^2} dx$$
(b) Justify the convergence by  $N$ - $\epsilon$  definition. (10%)

6. (a) Evaluate the limit. (10%)
$$\lim_{n \rightarrow \infty} \left( \frac{3^n + 4^n + 5^n}{3} \right)^{\frac{1}{n}} + \left( \frac{3^{\frac{1}{n}} + 4^{\frac{1}{n}} + 5^{\frac{1}{n}}}{3} \right)^n$$
(b) Evaluate the limit. (10%)
$$\lim_{n \rightarrow \infty} \left( \int_0^1 e^{nx(1-x)} dx \right)^{\frac{1}{n}} + \left( \int_0^1 e^{\frac{x(1-x)}{n}} dx \right)^n$$

# 國立成功大學 111 學年度「碩士班」研究生甄試入學考試

## 高等微積分

Throughout the exam, the Euclidean spaces  $\mathbb{R}^n$  are all equipped with usual Euclidean metric  $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ .

1. (10 points) Evaluate the line integral

$$\int_C \mathbf{V} \cdot d\mathbf{r},$$

where the vector field is given by

$$\mathbf{V} = \langle 2y + xe^{x^2}, 4x + e^{y^2} \cos y \rangle,$$

and  $C$  is the path starting from  $(1, 0)$  to  $(0, -1)$  by a *counterclockwise* circular path, and then from  $(0, -1)$  to  $(0, 0)$  by a straight line.

2. For a continuous function  $f : [a, b] \rightarrow \mathbb{R}$  that is differentiable on  $(a, b)$ ,
  - (8 points) State and prove the *Rolle's Theorem* for  $f$ .
  - (7 points) State and prove the *Mean Value Theorem* for  $f$ .
3. (10 points) Given the fact that  $\mathbb{R}$  is complete, prove the *Monotone Convergence Theorem*:

*Any bounded monotonic sequence in  $\mathbb{R}$  is convergent.*

4. Consider the usual Euclidean space  $\mathbb{R}^n$ ,
  - (8 points) If  $E \subset K \subset \mathbb{R}^n$ , where  $E$  is closed and  $K$  is compact, then  $E$  is compact.
  - (8 points) Prove that if  $\{K_\alpha\}_{\alpha \in A}$  is any family of compact subsets of  $\mathbb{R}^n$ , then

$$\bigcap_{\alpha \in A} K_\alpha$$

is also compact.

5. (15 points) Prove that the sequence of functions  $\{f_n\}$  on  $[0, 1]$  given by

$$f_n(x) = nx(1 - x^2)^n$$

pointwise converges to a function  $f(x)$  but does not converge uniformly.

6. Given two metric spaces  $(E, d_E)$  and  $(F, d_F)$ ,
- (8 points) If  $f : E \rightarrow F$  is continuous and  $E$  is compact, then  $f$  is uniformly continuous.
  - (8 points) Use part (a) to prove that

$$f(x) = e^{\cos^2 x \sin x}$$

is uniformly continuous on  $\mathbb{R}$ .

7. (a) (4 points) State the *Inverse Function Theorem* on  $\mathbb{R}^n$ .
- (b) (4 points) State the *Implicit Function Theorem* on  $\mathbb{R}^{n+m}$ .
- (c) (10 points) These two theorems are in fact equivalent. Prove either one of the implications. (i.e. prove either (a)  $\Rightarrow$  (b) or (b)  $\Rightarrow$  (a)).

# 國立成功大學 110 學年度「碩士班」研究生甄試入學考試

## 高等微積分

1. (10%) Show that  $f(x) = \sqrt[3]{x}$  is uniformly continuous on  $\mathbb{R}$ .

Hint: find a constant  $c > 0$  such that  $|a - b|^3 \leq c|a^3 - b^3|$  for all  $a, b \in \mathbb{R}$ .

2. (15%) Denote  $\mathbf{x} = (x, y) \in \mathbb{R}^2$  and  $\|\mathbf{x}\| = \sqrt{x^2 + y^2}$ .

Let  $G \subseteq \mathbb{R}^2$  be an open set such that the closed disk  $\{\|\mathbf{x}\| \leq 1\} \subseteq G$ .

Show that there exists  $\epsilon > 0$  such that  $\{\|\mathbf{x}\| \leq 1 + \epsilon\} \subseteq G$ .

3. (15%) Let  $\{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$  be a sequence such that  $x_n \rightarrow x$ . Show that

$$\frac{(2n-1)x_1 + (2n-3)x_2 + \cdots + 3x_{n-1} + x_n}{n^2} \rightarrow x.$$

4. (15%) Let  $\{x_n\}_{n \in \mathbb{N}}$  be a sequence of positive real numbers. Show that

$$\limsup_{n \rightarrow \infty} \sqrt[n]{x_n} \leq \limsup_{n \rightarrow \infty} \frac{x_{n+1}}{x_n}.$$

Also provide an example such that  $\limsup_{n \rightarrow \infty} \sqrt[n]{x_n} < 1 < \limsup_{n \rightarrow \infty} \frac{x_{n+1}}{x_n}$ .

5. (15%) Let  $f(x)$  be a Riemann integrable function on  $[0, 1]$ . Show that

$$\lim_{n \rightarrow \infty} \int_0^1 f(x) \cos(nx) dx = 0.$$

Hint: first let  $f(x)$  be a piecewise constant function.

6. (15%) Determine the values of  $p \geq 0$  for which the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{(1+n^2)^p}$$

is divergent, conditionally convergent or absolutely convergent.

7. (15%) Evaluate the limit. Justify your calculation.

$$\lim_{n \rightarrow \infty} \int_0^1 n \cdot \sin\left(\frac{x^2}{n}\right) dx$$

# 國立成功大學 109 學年度「碩士班」研究生甄試入學考試

## 高等微積分

1. (15%) Suppose that  $E$  is a compact subset of  $\mathbb{R}$  and that  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Prove that if  $f$  is continuous on  $E$ , then  $f$  is uniformly continuous on  $E$ .
2. (a) (5%) State the Implicit Function Theorem.  
(b) (10%) Prove that there exist functions  $u(x, y)$ ,  $v(x, y)$ , and  $w(x, y)$ , and an  $r > 0$  such that  $u, v, w$  are continuously differentiable and satisfy the equations

$$\begin{aligned} u^5 + xv^2 - y + w &= 0 \\ v^5 + yu^2 - x + w &= 0 \\ w^4 + y^5 - x^4 &= 1 \end{aligned}$$

on  $B_r(1, 1)$ , and  $u(1, 1) = 1, v(1, 1) = 1, w(1, 1) = -1$ .

3. (a) (10%) Prove that if  $f : [0, 1] \rightarrow \mathbb{R}$  is continuous and  $\int_0^1 |f(x)| dx = 0$ , then  $f(x) = 0$  for all  $x \in [0, 1]$ .  
(b) (5%) State the Weierstrass Approximation Theorem.  
(c) (10%) Prove that if  $f : [0, 1] \rightarrow \mathbb{R}$  is continuous and
$$\int_0^1 f(x)x^k dx = 0 \quad \text{for } k = 0, 1, \dots,$$
then  $f(x) = 0$  for all  $x \in [0, 1]$ .
4. (a) (5%) State the Arzela-Ascoli Theorem.  
(b) (10%) Let  $f_n : [0, 1] \rightarrow \mathbb{R}$  be continuous and be such that  $f_n(0) = 0$  for every  $n \in \mathbb{N}$ . Suppose that the derivatives  $f'_n$  exist and are uniformly bounded on  $(0, 1)$ . Prove or disprove that  $f_n$  has a uniformly convergent subsequence.
5. (15%) Suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfies  $|f(x) - f(y)| \leq \frac{1}{2}|x - y|$  for all  $x, y \in \mathbb{R}$ . Prove that  $f$  has a unique fixed point.
6. (15%) Let  $f_n : [0, 1] \rightarrow \mathbb{R}$  be continuous for every  $n \in \mathbb{N}$ . Suppose that  $\{f_n\}$  converges uniformly to  $f$  on  $[0, 1]$  and that  $\{x_n\}$  is a sequence in  $[0, 1]$  converging to a point  $x \in [0, 1]$ . Prove or disprove that  $\lim_{n \rightarrow \infty} f_n(x_n) = f(x)$ .

# 國立成功大學 108 學年度「碩士班」研究生甄試入學考試

## 【基礎數學】：Part II. 高等微積分

1. (10 points) Evaluate

$$\int_0^1 \int_{\sqrt{x}}^1 e^{y^3} dy dx$$

by interchanging limits of integrations.

2. (10 points) A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is called *convex* if for every  $s < t$  and  $\lambda \in [0, 1]$ ,

$$f(\lambda s + (1 - \lambda)t) \leq \lambda f(s) + (1 - \lambda)f(t).$$

Prove that if a convex function is differentiable, its derivative  $f'$  is an increasing function:

$$s \leq t \Rightarrow f'(s) \leq f'(t).$$

Note: Here we do *NOT* assume the existence of  $f''$ . It is useful to consider the fact that for  $s < t$ , the graph of  $f$  over  $[s, t]$  always lies below the line through  $(s, f(s))$  and  $(t, f(t))$ .

3. For a subset  $E$  of a metric space  $(X, d)$ , let  $E^\circ$  be the set of interior points of  $E$ . Prove that

(a) (10 points)  $E^\circ$  is open.

(b) (10 points) If  $G \subset E$  and  $G$  is open, then  $G \subset E^\circ$

Note:  $x \in E$  is an *interior point* if there exists  $r > 0$  so that

$$B_r(x) = \{y \in X \mid d(x, y) < r\} \subset E.$$

4. (10 points) Given a sequence of Riemann integrable functions  $\{f_n\}$  on  $[a, b]$  and assume that  $f_n$  converges uniformly to a function  $f$ , it is known that  $f$  is also Riemann integrable on  $[a, b]$ . With this fact, prove that

$$\int_a^b f dx = \lim_{n \rightarrow \infty} \int_a^b f_n dx.$$

5. Let  $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map. Define the *operator norm* of  $A$  by

$$\|A\| := \sup_{\|\mathbf{x}\|_n \leq 1} \{|A\mathbf{x}|_m\}.$$

Here,  $|\cdot|_n$  and  $|\cdot|_m$  are the usual Euclidean lengths in  $\mathbb{R}^n$  and  $\mathbb{R}^m$ , respectively.

(a) (10 points) Prove that

$$\|A\| = \sup_{\|\mathbf{x}\|_n = 1} \{|A\mathbf{x}|_m\}.$$

(b) (10 points) Let  $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be defined by

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & -2 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

$$\|A\| = ?$$

6. (10 points) Prove that the function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  defined by

$$f(x, y, z) = (e^{x^3+2x}, e^{y-z} + \cos z, e^z)$$

is locally invertible. That is, for every  $P \in \mathbb{R}^3$ , there exists a neighborhood  $U$  of  $P$  so that  $f|_U : U \rightarrow f(U)$  is invertible.

7. Consider the sequence of functions  $\{f_n\}_{n=1}^\infty$  on  $[0, 1]$  given by

$$f_n(x) = \frac{\sin^4(nx)}{\sin^2(nx) + (1 - nx)^2}.$$

- (a) (10 points) Prove that  $\{f_n\}_{n=1}^\infty$  is uniformly bounded. That is, there exists  $M > 0$  so that  $|f_n(x)| \leq M$  for all  $n$ , and all  $x \in [0, 1]$ .
- (b) (10 points) Use Arzela-Ascoli Theorem to prove that  $\{f_n\}_{n=1}^\infty$  is *NOT* equicontinuous.

國立成功大學 107 學年度「碩士班」研究生甄試入學考試  
【基礎數學】：Part II. 高等微積分

1. (25 points) Given a set  $X$ ,
  - (a) (5 points) State the definition for a function  $d : X \times X \rightarrow \mathbb{R}$  to be a *metric*.
  - (b) (5 points) State the definition for a subset  $E \subset X$  to be *open* with respect to  $d$ .
  - (c) (5 points) State the definition for a subset  $E \subset X$  to be *closed* with respect to  $d$ .
  - (d) (5 points) Let  $X = \mathbb{R}$  and  $E = \{x\} \subset \mathbb{R}$  for some  $x \in \mathbb{R}$ . Prove that  $E$  is closed with respect to the metric  $d(x, y) = |x - y|$ .
  - (e) (5 points) Define a metric  $d$  on  $\mathbb{R}$  so that the subset  $E$  in part (d) is open. Explain your answer.
2. (15 points) Assume *Bolzano-Weierstrass Theorem* on  $\mathbb{R}$  with the usual Euclidean metric:

*Any bounded sequence has a convergent subsequence,*

prove the same theorem for  $\mathbb{R}^2$ , also with standard Euclidean metric

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

3. (15 points) Given a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,
  - (a) (5 points) State the definition for  $f$  to be *differentiable* at  $\mathbf{p} \in \mathbb{R}^n$ .
  - (b) (10 points) Show, *directly* (ie. without using big theorem) from the definition in part (a), that the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by

$$f(x, y) = (xy + x, y^2 + y)$$

is differentiable at  $(0, 0)$ .

4. (15 points) Consider the sequence of functions  $\{f_n\}$  defined on  $[0, 1]$  given by

$$f_n(x) = \log nx^2(1 - x^3)^n.$$

- (a) (5 points) Find  $f(x) := \lim_{n \rightarrow \infty} f_n(x)$ .
  - (b) (10 points) Prove that  $f_n$  are not uniformly convergent.
5. (15 points) If a function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  satisfies

$$|\phi(x) - \phi(y)| \leq \frac{1}{2}|x - y|,$$

prove that

- (a) (5 points)  $\phi$  is continuous (with respect to the usual metric  $d(x, y) = |x - y|$ ).
- (b) (10 points)  $\phi(x) = x$  for some  $x \in \mathbb{R}$ .

6. (15 points) Consider  $\mathcal{C}([0, 1])$ , the space of continuous complex valued functions on  $[0, 1]$ .

(a) (5 points) State the *Stone-Weirstrass Theorem* on this space.

(b) (10 points) Prove that if  $f \in \mathcal{C}([0, 1])$  satisfies the fact that

$$\int_0^1 x^n f(x) dx = 0 \quad \forall n \in \mathbb{N},$$

then  $f = 0$ . (Hint: first show that  $\int_0^1 (f(x))^2 dx = 0$  using part (a).)

# 國立成功大學 106 學年度「碩士班」研究生甄試入學考試

## 【基礎數學】: Part I. 線性代數

Note:  $\mathbb{R}$  denotes the field of real numbers, and  $n$  denotes a positive integer.

1. (10%) Is there a linear transformation  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  such that  $T(1, -2, 0) = (1, 1)$ ,  $T(3, -5, 1) = (2, 3)$ , and  $T(-1, 3, 1) = (3, 0)$ ? Justify your answer.
2. (15%) Let  $V$  be the vector space of all polynomials of degree at most  $n$  with real coefficients. For  $i = 0, 1, \dots, n$ , let  $p_i(x) = x^i + x^{i+1} + \dots + x^n \in V$ . Show that  $\{p_0(x), p_1(x), \dots, p_n(x)\}$  is a basis for  $V$ .
3. (20%) Let  $A$  be an  $n \times n$  real matrix such that  $A^2 = A$ . Show that the trace of  $A$  is equal to the rank of  $A$ . Is  $A$  similar over  $\mathbb{R}$  to a diagonal matrix? Justify your answer.
4. (20%) Let  $T$  be a linear operator on a finite-dimensional vector space such that  $\text{rank}(T^2) = \text{rank}(T)$ . Show that  $N(T) \cap R(T) = \{0\}$ . (Here  $N(T)$  and  $R(T)$  are the null space and the range of  $T$  respectively.)
5. (15%) Let  $V$  be the vector space of all polynomials of degree at most 3 with real coefficients. Let  $D$  be the linear operator on  $V$  defined by  $D(p) = p'$  for  $p \in V$ . Find the Jordan form of  $D$ .
6. (20%) Let  $T$  and  $U$  be linear operators on an  $n$ -dimensional vector space  $V$ . Suppose that  $\{v, T(v), \dots, T^{n-1}(v)\}$  is a basis for  $V$  for some  $v \in V$ , and that  $TU = UT$ . Show that  $U = p(T)$  for some polynomial  $p$ .

# 國立成功大學 105 學年度「碩士班」研究生甄試入學考試

## 【基礎數學】：Part I. 高等微積分

### 1. ENTRANCE EXAM

- (1) (15 Points) Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a function defined by

$$f(x, y) = \begin{cases} \frac{x}{y} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

(a) (5 Points) Use definition to show that  $f_x(0, 0)$  and  $f_y(0, 0)$  exist.

(b) (5 Points) Prove that  $f$  is not continuous at  $(0, 0)$ .

(c) (5 Points) Is  $f$  differentiable at  $(0, 0)$ ? Explain.

- (2) (15 Points) Let  $a > 0$ . The following iterated integral can be rewritten as a double integral:

$$\int_{-a}^a \left( \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)} dy \right) dx = \iint_D e^{-(x^2+y^2)} dA.$$

Sketch the region  $D$  and evaluate the iterated integral by computing the double integral.

- (3) (15 Points)

(a) (7 Points) State Rolle's Theorem.

(b) (8 Points) Let  $f : (0, \infty) \rightarrow \mathbb{R}$  be a continuously differentiable function. Suppose that the equation

$$\frac{x^2}{2} + \ln x = f(x)$$

has at least two distinct solutions (on  $(0, \infty)$ ). Show that there exists a positive real number  $t$  such that  $f'(t) \geq 2$ .

- (4) (15 Points)

(a) (7 Points) State the Bolzano-Weierstrass Theorem (in  $\mathbb{R}^n$ .)

(b) (8 Points) Prove or disprove that the sequence of real numbers  $\{a_n\}$  defined by

$$a_n = e^{\sin n}, \quad n \in \mathbb{N}$$

has a convergent subsequence.

- (5) (15 Points) For each vector  $x \in \mathbb{R}^n$ , we define its Euclidean norm to be

$$\|x\|_{\mathbb{R}^n} = \sqrt{x_1^2 + \cdots + x_n^2},$$

where  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ . For any two vectors  $x, y \in \mathbb{R}^n$ , we define their inner product to be

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n = \sum_{i=1}^n x_i y_i.$$

Let  $a_1, \dots, a_m$  be unit vectors in  $\mathbb{R}^n$ , i.e.  $\|a_i\|_{\mathbb{R}^n} = 1$  for all  $1 \leq i \leq n$ . Define a function  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  by

$$T(x) = (a_1 \cdot x, \dots, a_m \cdot x), \quad x \in \mathbb{R}^n.$$

(a) (7 Points) Prove that

$$\|T(x)\|_{\mathbb{R}^m} \leq \sqrt{m} \|x\|_{\mathbb{R}^n} \text{ for any } x \in \mathbb{R}^n$$

(b) (8 Points) Prove that  $T$  is uniformly continuous on  $\mathbb{R}^n$  via  $\epsilon - \delta$  language.

- (6) (25 Points) Let  $X = C([0, 1])$  be the space of real valued continuous functions on  $[0, 1]$  equipped with the norm

$$\|f\|_\infty = \sup_{x \in [0, 1]} |f(x)|.$$

For each  $f, g \in X$ , we set  $d(f, g) = \|f - g\|_\infty$ . It is well known that  $(X, d)$  is a complete metric space, i.e every Cauchy sequence in  $X$  is convergent.

For each  $n \in \mathbb{N} \cup \{0\}$ , define  $f_n : [0, 1] \rightarrow \mathbb{R}$  inductively by  $f_0 = 0$  and

$$f_{n+1}(x) = 1 + \int_0^x t f_n(t) dt, \quad n \geq 0.$$

Then  $f_n \in X$  for all  $n \in \mathbb{N}$ .

(a) (7 Points) Prove that

$$\|f_{n+1} - f_n\|_\infty \leq \frac{1}{2} \|f_n - f_{n-1}\|_\infty, \quad \text{for all } n \in \mathbb{N}.$$

(b) (9 Points) Use completeness of  $X$  to show that  $\{f_n : n \in \mathbb{N}\}$  is uniformly convergent. (Hint: use (a) to show that  $\{f_n\}$  is a uniform Cauchy sequence.)

(c) (9 Points) Let  $f$  be the uniform limit of  $\{f_n : n \in \mathbb{N}\}$  in  $X$  i.e.

$$f = \lim_{n \rightarrow \infty} f_n \text{ in } X.$$

Show that  $f$  is continuously differentiable on  $[0, 1]$  and solve for  $f$ .

# 國立成功大學 104 學年度「碩士班」研究生甄試入學考試

## 【基礎數學】: Part I. 高等微積分

1. (10 %) Suppose  $f$  is a real, three times differentiable function on  $[-1, 1]$ , such that

$$f(-1) = 0, \quad f(0) = 0, \quad f(1) = 1, \quad f'(0) = 0.$$

Prove that  $f^{(3)}(x) \geq 3$  for some  $x \in (-1, 1)$ .

2. (12 %) Let

$$f_m(x) = \lim_{n \rightarrow \infty} (\cos m! \pi x)^{2^n} \quad (x \in \mathbb{R}, m \in \mathbb{N}).$$

Show that  $\{f_m\}$  converges to a function  $f$  in  $\mathbb{R}$ , but not uniformly.

3. (a) (10 %) Show that  $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ .

(b) (8 %) Show that  $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ , where  $\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1} dx$  is the gamma function,  $0 < \alpha < \infty$ .

4. (15 %) Let  $f : [0, 1] \rightarrow [0, 1]$  be continuous on  $[0, 1]$  and differentiable on  $(0, 1)$ . Assume that  $f(0) = 0$  and  $f(1) = 1$ . Show that for each  $n \in \mathbb{N}$ , there are  $n$  distinct points  $a_1, a_2, \dots, a_n \in [0, 1]$  such that

$$\frac{1}{f'(a_1)} + \frac{1}{f'(a_2)} + \dots + \frac{1}{f'(a_n)} = n.$$

5. (a) (5 %) State the Weierstrass approximation theorem.

(b) (10 %) Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a continuous function such that  $\int_0^1 f(x) x^n dx = 0$  for all nonnegative integers. Prove or disprove that  $f(x) = 0$  for all  $x \in [0, 1]$ .

6. (a) (7 %) State the Arzela-Ascoli theorem.

(b) (8 %) Let

$$f_n(x) = \frac{x^2}{x^2 + (1 - nx)^2} \quad (0 \leq x \leq 1, n = 1, 2, 3, \dots).$$

Show that  $\{f_n\}$  is not equicontinuous on  $[0, 1]$ .

7. Define

$$f(x, y) = \begin{cases} \sin\left(\frac{y^2}{x}\right) \cdot \sqrt{x^2 + y^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

(a) (7 %) Show that the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is continuous at  $(0, 0)$  and has directional derivatives in every direction at  $(0, 0)$ .

(b) (8 %) Show that there is no plane that is tangent to the graph of  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  at the point  $(0, 0, f(0, 0))$ .

# 國立成功大學 103 學年度「碩士班」研究生甄試入學考試

## 【基礎數學】: Part I. 高等微積分

### advanced calculus

1. Prove that  $\lim_{x \rightarrow \infty} (x^5 - 3x^4 - x^2 + 1)^{-1} = 0$ . (20 points)

2. Define a function  $f$  by

$$f(x) = \begin{cases} x \cdot \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Prove or disprove that  $f$  is uniformly continuous on  $\mathbb{R}$ . (20 points)

3. Let  $f$  be a smooth function on  $(-1, 1)$  and  $f(x) = \sum_{n=0}^{\infty} a_n x^n$  for  $x \in (-1, 1)$ . Let  $\{x_m\}$  be a sequence with  $x_m \neq 0$  for all  $m \in \mathbb{N}$ . Assume that  $\{x_m\}$  converges to zero with  $f(x_m) = 0$ . Show that  $f = 0$  on  $(-1, 1)$ . (20 points)

4. Let  $f$  be continuous on  $[0, 1]$  and suppose that  $f(0) = 0$ . Prove that

$$\lim_{n \rightarrow \infty} \int_0^1 f(x^n) = 0.$$

(20 points)

5. Evaluate the integral  $\int_0^\infty e^{-x^2} dx$ . (20 points)