7.1 give f. let $F(s) = \int_\alpha^\beta \overrightarrow{k(s,t)} f(t)\, dt$ kernel.

$\int_0^\infty e^{ct}\, dt = \frac{1}{c} e^{ct} \Big|_0^\infty = \frac{1}{c} e^{c\infty} - \frac{1}{c} \quad \begin{cases} -\frac{1}{c} & c<0 \\ \infty & c \geq 0 \end{cases}$

$\int_1^\infty t^{-p}\, dt = -\frac{1}{p} t^{-p} \Big|_1^\infty = -\frac{1}{p} \infty^{-p} + \frac{1}{p} \quad \begin{cases} \frac{1}{p-1} & p>1 \\ \infty & p \leq 1 \end{cases}$ $\quad k(s,t) = e^{-st}$

The Laplace transform of f is $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$

· Thm 7.1.2 Suppose that

$\langle i \rangle$ $f \in pc([0,A])$ for any $A>0$.

$\langle ii \rangle$ $\exists\, a,\ k>0,\ M>0$. s.t $|f(t)| \leq k e^{at}$, $t \geq M$

Then $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$ exists for $s>a$

$\int_0^\infty |e^{-st} f(t)|\, dt = \int_0^\infty e^{-st} |f(t)|\, dt = \int_0^M e^{-st} |f(t)|\, dt + \int_M^\infty |f(t)|\, dt$

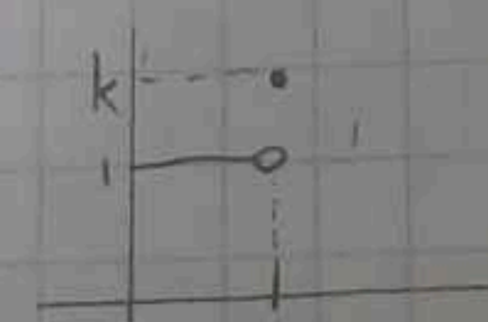
$\leq B \frac{-1}{s} e^{-st} \Big|_0^M + k \int_M^\infty e^{-st} e^{at}\, dt$

eg. $f(t) = 1$. $F(s) = \mathcal{L}\{f(t)\}$ 　在考試前要記起來.

$= \int_0^\infty e^{-st}\, dt = \frac{1}{s}$, $s>0$.

eg. $f(t) = e^{at}$ $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-(s-a)t}\, dt = \frac{1}{s-a}$, $s>0$

eg. $f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ k, & t=1 \\ 0, & t>1 \end{cases}$



eg. $f(t) = \cos(at)$, $t \geq 0$

$F(s) = \int_0^\infty e^{-st} \cos(at)\, dt$

$= -\frac{1}{s} e^{-st} \cos(at) \Big|_0^\infty - \int_0^\infty \frac{a}{s} \sin(at) e^{-st}\, dt$

$= \frac{1}{s} - \frac{a}{s} \int_0^\infty e^{-st} \sin(at)\, dt$

$F(s) = \frac{1}{s} + \frac{a}{s^2} e^{-st} \sin(at) \Big|_0^\infty - \frac{a^2}{s^2} \int_0^\infty e^{-st} \cos(at)\, dt$

$= \frac{1}{s} - \frac{a^2}{s^2} F(s) \Rightarrow \left(1 + \frac{a^2}{s^2}\right) F(s) = \frac{1}{s} \Rightarrow (s^2 + a^2) F(s) = s$

$\Rightarrow F(s) = \frac{s}{s^2 + a^2}$, $s>0$

eg. $f(t) = \sin(at)$, $t \geq 0$.

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \sin(at)\, dt = \frac{-1}{s} e^{-st} \sin(at) \Big|_0^\infty + \int_0^\infty \frac{a}{s} \cos(at)\, e^{-st}\, dt$$

$$= \frac{a}{s} \int_0^\infty e^{-st} \cos(at)\, dt = -\frac{a}{s^2} \cos(at) \Big|_0^\infty - \frac{a^2}{s^2} \int_0^\infty e^{-st} \sin(at)\, dt$$

$$= \frac{a}{s^2} + \frac{a^2}{s^2} F(s).$$

$$s^2 F(s) = a + a^2 F(s) \Rightarrow F(s) = \frac{a}{s^2+a^2}$$

eg. $\mathcal{L}\{C_1 f(t) + C_2 g(t)\}$

$$\mathcal{L}\{C_1 f(t) + C_2 g(t)\} = \int_0^\infty e^{-st} (C_1 f(t) + C_2 g(t))\, dt$$

$$= C_1 \int_0^\infty e^{-st} f(t)\, dt + C_2 \int_0^\infty e^{-st} g(t)\, dt$$

$$= C_1 \mathcal{L}\{f(t)\} + C_2 \mathcal{L}\{g(t)\}.$$

eg. $f(t) = 2e^{-4t} + 5 \sin(2t) - 7 \cos(3t)$, $t > 0$.

$$F(s) = 2 \frac{1}{s+4} + 5 \frac{2}{s^2+4} - 7 \frac{s}{s^2+9}, \quad s > 0.$$

eg. $\begin{cases} y' + 2y = 12 e^{3t} \\ y(0) = 3 \end{cases}$

$\sin : \frac{a}{s^2+a^2}$. $\cos \cdot \frac{s}{s^2+a^2}$

$$\mathcal{L}\{y' + 2y\} = 12 \mathcal{L}\{e^{3t}\}.$$

$$sY(s) - y(0) + 2Y(s) = 12 \frac{1}{s-3}, \quad (s+2) Y(s) = 3 + \frac{12}{s-3}$$

$$Y(s) = \frac{3}{s+2} + \frac{12}{(s+2)(s-3)} = \frac{3}{s+2} + \frac{A}{s+2} + \frac{B}{s-3} = \frac{3}{5} \frac{1}{s+2} + \frac{12}{5} \frac{1}{s-3}$$

$$y(t) = \frac{3}{5} e^{-2t} + \frac{12}{5} e^{3t}$$

eg. $\begin{cases} y^{(4)} - 4y = 0 \\ y(0)=0 \quad y'(0)=1 \\ y''(0)=0. \quad y'''(0)=0 \end{cases}$

$\mathcal{L}\{y^{(4)} - 4y\} = 0. \quad s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - 4Y(s) = 0$

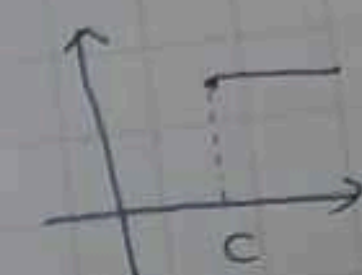
$\Rightarrow (s^4 - 4) Y(s) = s^2$

$\Rightarrow Y(s) = \frac{s^2}{(s^2+2)(s^2-2)} = \frac{1}{2(s^2+2)} + \frac{1}{2(s^2-2)} = \frac{\sqrt{2}}{\sqrt{2}\cdot 2(s^2+2)} - \frac{1}{4\sqrt{2}(s+\sqrt{2})} + \frac{1}{4\sqrt{2}(s-\sqrt{2})}$

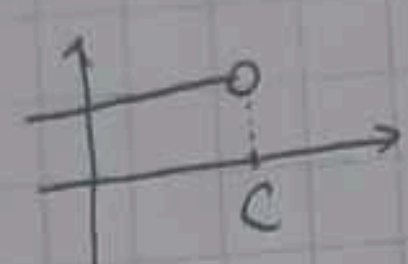
$\Rightarrow y(t) = \frac{1}{2\sqrt{2}} \sin(\sqrt{2}t) - \frac{1}{4\sqrt{2}} e^{-\sqrt{2}t} + \frac{1}{4\sqrt{2}} e^{\sqrt{2}t}$

9.3 Step fn.

Let $c>0$. The unit step (or Heaviside fn) is $u_c(t) = \begin{cases} 0 & t<c \\ 1 & t\geq c \end{cases}$



eg. $y(t) = 1 - u_c(t) = \begin{cases} 1 & t\leq c \\ 0 & t>c \end{cases}$



eg. $\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\, dt$

Thm. $\mathcal{L}\{u_c(t) f(t-c)\} = \int_0^\infty e^{-st} u_c(t) f(t-c)\, dt = \int_c^\infty e^{-st} f(t-s)\, dt.$

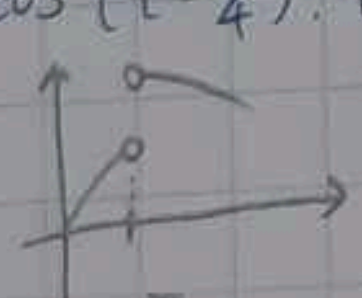
$x = t-c.$
$dx = dt$

$= \int_0^\infty e^{-s(x+c)} f(x)\, dx = e^{-sc} \int_0^\infty e^{-sx} f(x)\, dx$

$= e^{-cs} F(s).$

eg. $f(t) \begin{cases} \sin t, & 0\leq t<\frac{\pi}{4} \\ \sin t + \cos(t-\frac{\pi}{4}). & t\geq \frac{\pi}{4} \end{cases}$ find $\mathcal{L}\{f(t)\}$

$\int_0^{\frac{\pi}{4}} + \int_{\frac{\pi}{4}}^\infty$



, $f(t) = \sin t + h(t).$

$h(t) = \begin{cases} 0, & 0\leq t<\frac{\pi}{4}. \\ \cos(t-\frac{\pi}{4}), & t\geq \frac{\pi}{4} \end{cases} = u_{\frac{\pi}{4}}(t) \cos(t - \frac{\pi}{4}).$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} + \mathcal{L}\{u_{\frac{\pi}{4}}(t) \cos(t-\frac{\pi}{4})\}$

$= \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s} \mathcal{L}\{\cos t\} = \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s} \frac{s}{s^2+1}$

$\mathcal{L}^{-1}\{\frac{1}{s^2+1}\} = \sin t \qquad \mathcal{L}^{-1}\{\frac{s}{s^2+1}\} = \cos t$

$u_{\frac{\pi}{4}}$

eg. $\begin{cases} y'' + 4y = g(t) \\ y(0) = 0. \ y'(0) = 0 \end{cases}$

$g(t) = U_5(t)\,\frac{t-5}{5} - U_{10}(t)\,\frac{t-10}{5} = \begin{cases} 0, & 0 \le t < 5 \\ \frac{1}{5}(t-5), & 5 \le t < 10 \\ 1, & t \ge 10 \end{cases}$

$s^2 Y(s) + 4Y(s) = \mathcal{L}\{g(t)\} = \frac{1}{5}e^{-5s}\frac{1}{s^2} - \frac{1}{5}e^{-10s}\frac{1}{s^2}$

$\Rightarrow Y(s) = \frac{1}{5}(e^{-5s} - e^{-10s})\,\frac{1}{s^2(s^2+4)}$

Let $H(s) = \mathcal{L}\{h(t)\} = \frac{1}{s^2(s^2+4)}$, $y(t) = \frac{1}{5}U_5(t)\,h(t-5) - \frac{1}{5}U_{10}(t)\,h(t-10)$

$H(s) = \frac{1}{4s^2} - \frac{2}{8(s^2+4)}$

$h(t) = \frac{1}{4}t - \frac{1}{8}\sin 2t$

eg. $F(s)=\frac{1-e^{-3s}}{s^2}$, Find f

$\mathcal{L}^{-1}\{\frac{1}{s^2}\} - \mathcal{L}^{-1}\{e^{-3s}\frac{1}{s^2}\} = t - u_3(t)\,h(t-3)$ $\left(\begin{array}{l}\mathcal{L}\{h(t)\}=\frac{1}{s^2}\\ h(t)=t\end{array}\right)$

$= t - u_3(t)(t-3) = t - \begin{cases}0, & 0\le t<3\\ t-3, & t\ge 3\end{cases} = \begin{cases} t, & 0\le t<3 \\ 3, & t\ge 3\end{cases}$

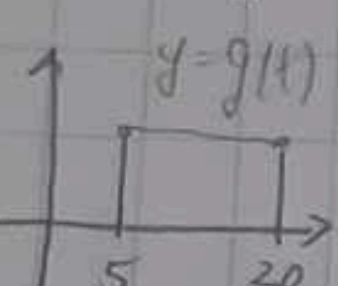
· Thm 2 沒抄到

eg. $F(s)=\frac{s-1}{s^2-2s+5}$, Find $\mathcal{L}^{-1}\{F(s)\}$

$F(s)=\frac{s-1}{s^2-2s+5} = \frac{s-1}{(s-1)^2+4} = G(s-1)$. $G(s)=\frac{s}{s^2+4}$

$\mathcal{L}^{-1}\{F(s)\} = e^t g(t) = e^t\cos 2t$, $\mathcal{L}\{g(t)\}=G(s)$, $\cos 2t = g(t)$

eg. $\begin{cases} 2y''+y'+2y=g(t) \\ y(0)=0. \quad y'(0)=0\end{cases}$ where $g(t)=u_5(t)-u_{20}(t) = \begin{cases}1, & 5\le t<20 \\ 0, & 0\le t<5 \text{ and } t\ge 20\end{cases}$ 考



$2\mathcal{L}\{y''\}+\mathcal{L}\{y'\}+2\mathcal{L}\{y\} = \mathcal{L}\{g(t)\} = \mathcal{L}\{u_5(t)\}-\mathcal{L}\{u_{20}(t)\}$

$2(s^2Y(s)-sy(0)-y'(0)) + Y(s) - y(0) + 2Y(s) = e^{-5s}\frac{1}{s} - e$

$\Rightarrow (2s^2+s+2)Y(s) = \frac{1}{s}(e^{-5s}-e^{-20s})$

$Y(s) = e^{-5s}\frac{1}{s(2s^2+s+2)} - e^{-20s}\frac{1}{s(2s^2+s+2)}$

Let $H(s)=\mathcal{L}\{h(t)\}=\frac{1}{s(2s^2+s+2)}$, $y(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20)$

$H(s)=\frac{1}{s(2s^2+s+2)} = \frac{a}{s}+\frac{bs+c}{2s^2+s+2} \Rightarrow a=\frac{1}{2}.\ b=-1.\ c=-\frac{1}{2}.$

$=\frac{1}{2}\frac{1}{s} - \frac{s+\frac{1}{2}}{2s^2+s+2} = \frac{1}{2s} - \frac{1}{2}\frac{(s+\frac{1}{4})+\frac{1}{4}}{(s+\frac{1}{4})^2+\frac{15}{16}} = \frac{1}{2s} - \frac{1}{2}\frac{s+\frac{1}{4}}{(s+\frac{1}{4})^2+(\frac{\sqrt{15}}{4})^2} - \frac{1}{2\sqrt{15}}\frac{\frac{\sqrt{15}}{4}}{(s+\frac{1}{4})^2+(\frac{\sqrt{15}}{4})^2}$

$h(t) = \frac{1}{2} - \frac{1}{2}e^{-\frac{1}{4}t}\cos(\frac{\sqrt{15}}{4}t) - \frac{1}{2\sqrt{15}}e^{-\frac{1}{4}t}\sin(\frac{\sqrt{15}}{4}t)$

7-1×3 (25分) Laplace transform $(1, t, e^{at}, \cos at, \sin at)$ C^3_3

$\frac{1}{s}$ $\frac{1}{s^2}$ $\frac{1}{s-a}$ $\frac{s}{s^2+a^2}$ $\frac{a}{s^2+a^2}$

7-2×1

7-3×2 (剛好一種一題)

7-4×1 (上面那題) 7-6×1