1.2 Equivalent Definition of Subspace and Product

Abstract

This section handles the definition of subspace and product topological space using the language of category theory.

Given some topological space X and some collection $(X_i)_{i\in I}$ of topological spaces, we say X **satisfy the universal property of product** and X **is a product space of** $(X_i)_{i\in I}$ if there exists some tuple of continuous maps

$$(\pi_i : X \to X_i)_{i\in I}$$

called **projection maps**, such that for each tuple of continuous maps $(F_i : Z \to X_i)_{i\in I}$, there exists exactly one continuous map $F : Z \to X$ such that for all i the following diagram commute

$$\begin{tikzcd} & X \arrow[d, "\pi_i"] \\ Z \arrow[ur, "F"] \arrow[r, "F_i"'] & X_i \end{tikzcd}$$

Immediately, one sees that if X' is another product space of $(X_i)_{i\in I}$, then X' is homeomorphic to X. In other words, the term "product space" is unique up to a homeomorphism.

Theorem 1.2.1. (A characteristic property of product topological space) Let X be the product of the collection $(X_i)_{i\in I}$ with projections $(\pi_i : X \to X_i)_{i\in I}$, and let G maps some topological space Z into X. Denote $G \circ \pi_i$ by G_i.

$$G \text{ is continuous} \iff (G_i)_{i\in I} \text{ are all continuous}$$

Proof. If G is continuous, then clearly $(G_i)_{i\in I}$ are all continuous. Suppose $(G_i)_{i\in I}$ are all continuous. By universal property, there exists a unique continuous map $G' : Z \to X$ such that $\pi_i \circ G' = G_i$ for all $i \in I$. To show G is continuous, we only have to show $G = G'$. Fix $z \in Z$, and let $i : \{z\} \to Z$ be the inclusion map. Because the source is singleton, $(\pi_i \circ G \circ i)_{i\in I}$ are all continuous. This implies there exists a unique continuous map $G'' : \{z\} \to X$ such that $\pi_i \circ G'' = \pi_i \circ G \circ i$ for all $i \in I$. Clearly, $G \circ i$ and $G' \circ i$ both are continuous maps that satisfy this condition. This implies $G(z) = G'(z)$. We have shown $G = G'$. ■

Albeit infamous, there also is **universal property of subspace** in study of topology. Let X, E be two topological spaces. We say E is a **subspace of** X if there exists a continuous map $i : E \to X$ such that for all topological spaces Z and maps $F : Z \to E$,

$$F \text{ is continuous} \iff i \circ F \text{ is continuous}$$

Immediately, we are required to show the consistency between the universal properties of product and subspace.

Theorem 1.2.2. (Consistency Between The Universal Properties of Product and Subspaces) Let $(E_i)_{i\in I}, (X_i)_{i\in I}$ be two collection of topological spaces. Let X be the product of $(X_i)_{i\in I}$ with projection maps $\pi_{i;X} : X \to X_i$, and let E be the product of $(E_i)_{i\in I}$ with projection maps $\pi_{i;E} : E \to E_i$. If for all $i \in I$, there exists some $i_i : E_i \to X_i$ that satisfy the universal property of subspace, then there exists some $i : E \to X$ that satisfy the universal property of subspace.

Proof. Because for each $i \in I$, the map $i_i \circ \pi_{i;E} : E \to X_i$ is continuous, by universal property, there exists a unique continuous map $i : E \to X$ such that the following diagram commute

$$\begin{tikzcd}
X_i & X \arrow[l, "\pi_{i;X}"'] \\
E_i \arrow[u, "i_i"] & E \arrow[l, "\pi_{i;E}"] \arrow[u, "i"']
\end{tikzcd}$$

It remains to show such $i : E \to X$ satisfy the universal property of subspace. Let $F : Z \to E$ be some map. If F is continuous, then $i \circ F$ is continuous because it is a composition of two continuous map. Suppose $i \circ F : Z \to X$ is continuous. It remains to show F is continuous. Because the diagram commute, we may deduce

$$i_i \circ \pi_{i;E} \circ F = \pi_{i;X} \circ i \circ F \text{ are continuous}$$

It then follows from universal property of subspace that $\pi_{i;E} \circ F$ are continuous. This by Theorem 1.2.1 implies F is continuous. ■

What's useful about the universal property is that, given two topological spaces X, Y, because both $X \times Y$ and $Y \times X$ when equipped with the product topology satisfy the universal property, we no longer have to distinguish them even though they are fundamentally "different" as sets. Similarly, we no longer need to distinguish between $(X \times Y) \times Z$ and $X \times (Y \times Z)$, because they are all homeomorphic to $X \times Y \times Z$.

The spirit of category theory is to not directly refer to the object. Indeed, given a collection of topological spaces $(X_i)_{i\in I}$, the universal property of product does not guarantee the existence of some X that can be the product of X_i. This existence is true. One can give a concrete construction of the product by setting

> $X \triangleq \prod_{i\in I} X_i$, the projection map $\pi_i((x_j)_{j\in I}) \triangleq x_i$, and the topology on X to be the smallest one that makes each π_i continuous.

It is clear that our concrete construction indeed satisfy the universal property. With this concrete construction, one see that X has the subbase

$$\bigcup_{i\in I}\{\pi_i^{-1}(U) : U \in \mathscr{T}_i\} \tag{1.2}$$

Thus having the basis

$$\Big\{\prod U_i : U_i \neq X_i \text{ for finitely many } \alpha \text{ and } U_i \in \mathcal{B}_i\Big\}$$

The subbase in Equation 1.2 gives another equivalent definition for product topology in the language of set theory.

Let E be a subset of X, we may equip E with the topology

$$\mathscr{T}_E \triangleq \{O \cap E : O \in \mathscr{T}\}$$

And see that E and the inclusion map $i : E \to X$ defined by

$$i(x) \triangleq x$$

indeed satisfy the universal property of subspace.

Theorem 1.2.3. (Equivalent Definition of Subspace Topology) Given a basis $\mathcal{B}$ and a subbase $\mathcal{B}'$ of $(X, \mathscr{T})$, the following topologies are identical:

(a) $\mathscr{T}_E$.

(b) The topology on E generated by the basis $\mathcal{B}_E \triangleq \{B \cap E : B \in \mathcal{B}\}$.

(c) The topology on E generated by the subbase $\mathcal{B}'_E \triangleq \{B' \cap E : B' \in \mathcal{B}'\}$.

(d) The smallest topology on E such that the inclusion map $i : E \to X$ is continuous.

Proof. Check straightforward. ∎

1.3 Connected and Compact

Given a topological space $(X, \mathscr{T})$, we say nonempty $E \subseteq X$ is

(a) **connected** if E can not be written as $E = A \cup B$ so that $\overline{A} \cap B = A \cap \overline{B} = \varnothing$ and $A \neq \varnothing \neq B$.

(b) **path-connected** if for each $p, q \in E$, there exists a continuous function $f : [0,1] \to E$ such that $f(0) = p, f(1) = q$.

(c) **compact** if every open cover has a finite subcover.

These three properties are often called **topological properties**, since they are invariant under continuous function, the "morphism" between topological space. Put more precisely, If $E \subseteq X$ satisfy a topological property and $f : X \to Y$ is continuous, then $f(E)$ also satisfy the topological property.

Immediately, one should again check the "natural" behaviors of subspace topology: Whether a set E is connected, path-connected, or compact is independent of the choices of ambient space. In other words, given $E \subseteq X$, E is connected, path-connected or compact in $(X, \mathscr{T})$ if and only if E is connected, path-connected or compact in $(E, \mathscr{T}_E)$.

Theorem 1.3.1. (Equivalent Definitions of Connected) Given a subset $E \subseteq X$, the following statements are equivalent

(a) E is connected in $(X, \mathscr{T})$.

(b) E is connected in $(E, \mathscr{T}_E)$.

(c) The only clopen sets in $(E, \mathscr{T}_E)$ are E and $\varnothing$.

(d) In $(E, \mathscr{T}_E)$, the only set that has empty boundary are E and $\varnothing$.

(e) All continuous function from $(E, \mathscr{T}_E)$ to $\{0, 1\}$ with discrete topology is constant.

Proof. For (a) $\iff$ (b), use the identity $\forall A \subseteq E$, $\text{cl}_X(A) \cap E = \text{cl}_E(A)$. Check straightforward for (b) $\iff$ (c) and (d) $\iff$ (c) $\iff$ (e). ■

Three things to note here

(a) If $E \subseteq X$ is connected, E can not be covered by any two disjoint open sets intersecting with E in $(X, \mathscr{T}_X)$. The converse is not true. Consider finite subset of an infinite set with cofinite topology.

(b) Union of collection $(A_\alpha)_{\alpha \in J}$ of connected sets with non-empty intersection is connected. Prove this by a proof of contradiction. Path-connectedness has the same property, and the proof is much easier.

(c) Path-connectedness is strictly stronger than connectedness. This can be proved using a proof of contradiction and supremum. One famous examples of the "strict" part is Topologist's Sine Curve.

Although path-connectedness is strictly stronger than connectedness, there exists a quite general condition that implies the converse. We say a topological space X is **locally path-connected** if for each p and open set U containing p, there exists a path-connected open set V containing p and contained by U. Following our definition, we see that if X is both connected and locally path-connected, then X must be path-connected, since any path-connected component of X (due to X being locally path-connected) must be clopen in X.

Theorem 1.3.2. (Equivalent Definitions of Compactness) The following statements are equivalent

(a) E is compact in $(X, \mathscr{T})$.

(b) E is compact in $(E, \mathscr{T}_E)$.

(c) Given subbase $\mathcal{B}$ of $(X, \mathscr{T})$, every cover of E consisting of the elements of $\mathcal{B}$ has a finite subcover.

(d) Every infinite subset M of E has a complete limit point in E, that is, a point $x \in E$ such that all open set O containing x satisfy $|O \cap M| = |M|$.

(e) Every collection of closed sets of $(E, \mathscr{T}_E)$ that has finite intersection property has non-empty intersection.

(f) For all topological space Y, the projection $\pi_Y : E \times Y \to Y$ is a closed mapping.

Proof. For (b) $\iff$ (e), use proofs by contradiction. (b) $\iff$ (a) $\implies$ (c) are clear. We now prove

$$(c) \implies (b)$$

Fix $\mathcal{B}$. Assume E is not compact. Then the collection $\mathbb{S}$ of all open covers that have no finite subcover is non-empty. Let $\mathcal{C}$ be a maximal element of $\mathbb{S}$. It is clear that $\mathcal{C} \cap \mathcal{B}$ is not a cover of E by premise. Let $x \in E \setminus \bigcup(\mathcal{C} \cap \mathcal{B})$. Let U be an element of $\mathcal{C} \setminus \mathcal{B}$ containing x. Because $\mathcal{B}$ is a subbase, there exists finite $B_1, \dots, B_n \in \mathcal{B}$ such that $x \in B_1 \cap \dots \cap B_n \subseteq U$. Because $\mathcal{C}$ is a maximal element of $\mathbb{S}$, for all j, the collection $\mathcal{C} \cup \{B_j\}$ does not belong to $\mathbb{S}$. This implies that for each $j \in \{1, \dots, n\}$, there exists a finite sub-collection $\mathcal{C}_j \subseteq \mathcal{C}$

such that $\mathcal{C}_j \cup \{B_j\}$ covers E. Let $\mathcal{C}_F \triangleq \bigcup_{j=1}^{n} \mathcal{C}_j$. Because $\mathcal{C}_j \cup \{B_j\}$ are covers of E, $\mathcal{C}_F \cup \{B_1, \dots, B_n\}$ is a cover of E. This implies $\mathcal{C}_F \cup \{U\} \subseteq \mathcal{C}$ is a finite subcover. CaC (done)

We now prove

(a) $\implies$ (d)

Assume there exists infinite $M \subseteq E$ that has no complete limit point. Because of our assumption, for each $x \in E$, there exists an open set O_x containing x such that $|M \cap O_x| < |M|$. Because $(O_x)_{x \in E}$ is an open cover of E, there exists a finite sub-cover $(O_x)_{x \in I}$. Note that M is infinite, so we can deduce

$$|M| = \left| \bigcup_{x \in I} M \cap O_x \right| \leq \sum_{x \in I} |M \cap O_x| < |M| \text{ CaC (done)}$$

We now prove

(d) $\implies$ (a)

Assume E is not compact. Let O be an open cover of E that has no finite subcover with smallest cardinality c. Well-order O by $O \triangleq \{O_\alpha\}_{\alpha < c}$. Use transfinite recursion to build $M \triangleq \{x_\alpha : \alpha < c\}$ where $x_\alpha \in E \setminus \bigcup_{\beta < \alpha} O_\beta$. Such x_α always exists; otherwise, there exists an open cover of E that has no finite subcover with cardinality smaller than c. To cause a contradiction, it remains to show

M has no complete limit point in E

Because O is an open cover of E, for all x, there exists some O_α containing x. Observe using the definition of M

$$|O_\alpha \cap M| \leq |\{x_\gamma : \gamma \leq \alpha\}| \leq |\alpha| < c = |M| \text{ CaC (done)}$$

Before we prove (a) $\implies$ (f), we first prove the Generalized Tube Lemma. That is,

> Given a product space $X \times Y$, compact $A \subseteq X$, compact $B \subseteq Y$, and $N \subseteq X \times Y$ open containing $A \times B$, there exists $U \subseteq X$ open, $V \subseteq Y$ open such that $A \times B \subseteq U \times V \subseteq N$.

First note that for all $(a, b) \in A \times B$, there exists $U_{(a,b)} \subseteq X$ open and $V_{(a,b)} \subseteq Y$ open such that $(a, b) \in U_{(a,b)} \times V_{(a,b)} \subseteq N$. Because A is compact and for all b, the collection $(U_{(a,b)})_{a \in A}$ is an open cover of A, there exists a finite subset $A_b \subseteq A$ for all b such that $A \subseteq \bigcup_{a \in A_b} U_{(a,b)}$. Now, let $U_b \triangleq \bigcup_{a \in A_b} U_{(a,b)}$ and $V_b \triangleq \bigcap_{a \in A_b} V_{(a,b)}$. It is clear that U_b, V_b are open, and it is straightforward to check $A \times \{b\} \subseteq U_b \times V_b \subseteq N$. Again, because B is compact and $(V_b)_{b\in B}$ is an open cover of B, there exists a finite subset $B_0 \subseteq B$ such that $B \subseteq \bigcup_{b\in B_0} V_b$. Let $V \triangleq \bigcup_{b\in B_0} V_b$ and $U \triangleq \bigcap_{b\in B_0} U_b$. It is straightforward to check U, V suffice. (done)

We now prove

$$\text{(a)} \implies \text{(f)}$$

Given $A \subseteq X \times Y$ closed, we are required to prove $\pi_Y(A)$ is closed. WOLG, assume $\pi_Y(A) \neq Y$. Fix $y \in Y \setminus \pi_Y(A)$. Because X and $\{y\}$ are compact and $X \times \{y\}$ is a subset of the open set A^c, by the Generalized Tube Lemma, there exists open $V \subseteq Y$ such that $X \times \{y\} \subseteq X \times V \subseteq A^c$. It is straightforward to check $V \cap \pi_Y(A) = \varnothing$. (done)

Lastly, we prove

$$\text{(f)} \implies \text{(a)}$$

Assume X is not compact. Let $(O_\alpha)_{\alpha\in J}$ be an open cover of X with no finite subcover. Consider the following construction:

(a) $\mathcal{U} \triangleq \{\bigcup_{\alpha\in I} O_\alpha : I \text{ is a finite subset of } J\}$ is an open cover of X with no finite subcover,

(b) $\mathcal{U}$ is closed under finite union,

(c) $\mathcal{F} \triangleq \{U^c : U \in \mathcal{U}\}$ is a collection of non-empty closed sets that has the finite intersection property.

(d) If we let $Y \triangleq X \cup \{p\}$ where $p \notin X$, then $\mathscr{T}_Y \triangleq \mathcal{P}(X) \cup \{\{p\} \cup A : \exists F \in \mathcal{F}, F \subseteq A \subseteq X\}$ is a topology on Y, where $\mathcal{P}(X)$ is the collection of all subsets of X.

(e) Let $C \triangleq \text{cl}_{X\times Y}\{(x,x) \in X \times Y : x \in X\}$.

(f) Fix $x \in X$. Because $\mathcal{U}$ is an open cover of X, there exists $U \in \mathcal{U}$ containing x. Note that $\{p\} \cup U^c$ is open in Y. This implies $U \times (\{p\} \cup U^c)$ is an open subset of $X \times Y$ containing (x, p). We have proved $C \subseteq X \times X$.

(g) It is clear that X is not closed in Y. Now observe that π_Y maps the closed set C to the open set $X \subseteq Y$. CaC (done)

■

Let's now compare the three topological properties we listed. Because the projection maps are continuous, we know that if $X \triangleq \prod_{i\in I} X_i$ satisfy any topological property, then all of its components satisfy the same topological property. What is difficult to prove is the converse.

Theorem 1.3.3. (Product of Connected Space)

$$X_i \text{ are all connected} \implies X \text{ is connected}$$

Proof. Fix $i \in I$. Let $F : X \to \{0,1\}$ be continuous. For all $(x_k)_{k\neq i}$, if we define $F' : X_i \to \{0,1\}$ by

$$F'(x_i) \triangleq F(x_j) \text{ where } x_j = x_i \text{ or } x_k$$

Then F' is continuous. It then follows from X_i is continuous that F' is constant. It then follows that all similarly defined F' are constant. This shows that F is constant. ■

Using universal property, it is easy to see that if X is path-connected, then all X_i are path-connected. To show the same for compactness, we first note that the subbase definition for compactness we proved in Theorem 1.3.2 is called **Alexander Subbase Theorem**.

Theorem 1.3.4. (Tychonoff's Theorem)

$$X_i \text{ are all compact} \implies X \text{ is compact}$$

Proof. We know

$$\bigcup_{i\in I}\{\pi_i^{-1}(U_i) \subseteq X : U_i \in \mathscr{T}_i\} \text{ is a subbase for } X$$

Let C be a cover of X consisting of element of this subbase. Partition C into $\bigcup_i C_i$ where C_i contains sets of the form $\pi_i^{-1}(U_i)$. There must exists some i such that

$$X_i = \bigcup_{j\in J} U_j \text{ where } C_i = \{\pi_i^{-1}(U_j) : j \in J\} \tag{1.3}$$

otherwise for each fixed i we may select some $x_i \in X_i$ that is not covered by U_j, and the point $(x_i)_{i\in I}$ will not be covered by C. Fix i such that Equation 1.3 holds true. Because X_i is compact, we know there exists a finite sub-cover of X_i consisting of (U_j), its inverse projection is a finite subcover of X consisting of elements of X. ■

Chapter 3

Pathological Example

3.1 Topologist's Sine Curve

Topologist's since curve is a famous pathological example of topological object being connected but not path-connected. One construction is setting

$$T \triangleq \Big\{(x, \sin(x^{-1})) \in \mathbb{R}^2 : x \in (0,1]\Big\} \cup \{\mathbf{0}\}$$

Theorem 3.1.1. (Topologist's Since Curve is Connected)

$$T \text{ is connected}$$

Proof. Let $f : T \to \{0,1\}$ be continuous. Define continuous $g : (0,1] \to T$ by

$$g(x) \triangleq (x, \sin(x^{-1}))$$

Equivalent Definitions of Connected says that $f \circ g$ is a constant. Note that g is a bijection between $(0,1]$ and $T \setminus \{\mathbf{0}\}$, so $g^{-1} : T \setminus \{\mathbf{0}\} \to (0,1]$ exists. We may now conclude f is constant on $T \setminus \{\mathbf{0}\}$, since

$$f(p) = f \circ g \circ g^{-1}(p) \text{ for all } p \in T \setminus \{\mathbf{0}\}$$

To see f is constant on the whole T, just observe

$$f(\mathbf{0}) = \lim_{n\to\infty} f(\frac{1}{n\pi}, \sin(n\pi))$$

■

Theorem 3.1.2. (Topologist's Since Curve is not path-connected)

$$T \text{ is not path-connected}$$

Proof. Assume for a contradiction that $f : [0,1] \to T$ is a continuous map such that

$$f(0) = \mathbf{0} \text{ and } f(1) = (1, \sin 1)$$

Note that for all $\alpha \in (0,1)$, f must pass through $(\alpha, \sin(\alpha^{-1}))$, otherwise the connected image of f is separated by the two open half plane defined by $x = \alpha$, which is impossible. In other words, f is surjective. We have shown $T = f([0,1])$ is compact. Therefore, if we define

$$L_n \triangleq T \cap \{(x,y) \in \mathbb{R}^2 : x \leq \frac{1}{n} \text{ and } y = 1\}$$

We see L_n is a decreasing sequence of non-empty compact sets. This cause a contradiction since the intersection of L_n is clearly empty. ■

3.2 Long Line

Let X be some set and let $\leq$ be a relation on X. We say $\leq$ is a **total order** if

(i) $x \leq x$ for all $x \in X$ (Reflexive)

(ii) $x \leq y$ and $y \leq z \implies x \leq z$ (Transitive)

(iii) $x \leq y$ and $y \leq x \implies x = y$ (Antisymmetric)

(iv) For all $x, y \in X$, either $x \leq y$ or $y \leq x$.

Let $\leq$ be a total order of X. By the **order topology** of X, we mean the topology generated by the basis

$$\Big\{[\min X, x] \subseteq X : x \in X\Big\} \cup \Big\{(x, y) \subseteq X : x, y \in X\Big\} \cup \Big\{[x, \max X] : x \in X\Big\}$$

where the first collection is considered empty if $\min X$ does not exist, and the third collection is considered empty if $\max X$ does not exist. It is easy to check that the referred basis is closed under finite intersection, thus justifying our practice of calling it a basis. It is also easy to check that the definition of order topology is compatible with definition of subspace topology. Now, let ω_1 be the first uncountable ordinal, let $X \triangleq \omega_1 \times (0, 1)$, and define

$$(x, y) < (x', y') \overset{\triangle}{\iff} x < x' \text{ or } (x = x' \text{ and } y < y')$$

Equipping X with the order topology, we say X is the **long line**. To see that X is locally Euclidean, observe for each $(\alpha, t_0) \in X$, its neighborhood $\{(\alpha, t) \in X : t \in (0, 1)\}$ is homeomorphic to the unit open interval. This fact also shows that the long line is Hausdorff.

Theorem 3.2.1. (Long Line is not second countable)

$$X \text{ is not second countable}$$

Proof. Let $A = \{(\alpha_n, t_n) : n \in \mathbb{N}\}$ be a countable subset of X. Note that each ordinal $\alpha_n \in \omega_1$ as a set is countable because $\omega_1 > \alpha_n$ is the first uncountable ordinal. It then follows that ω_1 strictly contain the union of α_n. Therefore, we may find $\beta \in \omega_1$ such that $\alpha_n < \beta$ for all n. Observe that the element $(\beta, \frac{1}{2}) \in X$ has a neighborhood $\{(\beta, t) \in X : t \in (0, 1)\}$ that does not contain any element of A. This implies that A is not dense. We have shown no countable subset of X is dense, i.e., X is not separable. This shows X is not second countable. ■

In most practice, one requires topological manifolds to be

(i) Locally Euclidean.

(ii) Hausdorff.

(iii) Second Countable.

Because Euclidean space are both Hausdorff and second countable, one may question the necessity of these two requirement. Long line shows that second countable is indeed necessary. For a topological space being locally Euclidean and second countable but not Hausdorff, see "a line with two origins".

3.3 Weierstrass Function

Suppose $0 < a < 1, b$ is an odd integer, and

$$ab > 1 + \frac{3\pi}{2} \tag{3.1}$$

Define for each $n \in \mathbb{N}$, $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) \triangleq a^n \cos(b^n \pi x)$$

Note that

$$\sum_n (\sup |f_n|) = \sum_n a^n \in \mathbb{R}$$

So if we define a **Weierstrass function** $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) \triangleq \sum_{n=0}^{\infty} f_n(x)$$

the function is well-defined and continuous by Weierstrass M-Test. Fix $x_0 \in \mathbb{R}$. We wish to show f is not differentiable at x_0. For each n let j_n be the unique integer satisfying

$$\frac{-1}{2} < b^n x_0 - j_n \leq \frac{1}{2}$$

Define

$$x_n \triangleq \frac{j_n - 1}{b^n} \text{ and } \alpha_n \triangleq b^n x_0 - j_n$$

It is obvious that x_n converges to x_0. Our goal here is to show

$$\frac{f(x_n) - f(x_0)}{x_n - x_0} \text{ diverges.}$$

Compute

$$\begin{aligned}\frac{f(x_n)-f(x_0)}{x_n-x_0} &= \sum_{m=0}^{\infty} a^m \cdot \frac{\cos(b^m\pi x_n)-\cos(b^m\pi x_0)}{x_n-x_0} \\ &= \sum_{m=0}^{n-1}(ab)^m \frac{\cos(b^m\pi x_n)-\cos(b^m\pi x_0)}{b^m(x_n-x_0)} \\ &+ \sum_{m=0}^{\infty} a^{n+m} \cdot \frac{\cos(b^{n+m}\pi x_n)-\cos(b^{n+m}\pi x_0)}{x_n-x_0}\end{aligned}$$

Theorem 3.3.1. (An absolute upper bound for the first sum) For each $n \in \mathbb{N}$, we have

$$\left|\sum_{m=0}^{n-1}(ab)^n \cdot \frac{\cos(b^m\pi x_n)-\cos(b^m\pi x_0)}{b^m(x_n-x_0)}\right| \leq \frac{\pi(ab)^n}{ab-1}$$

Proof. Using trigonometric identity, we may give bound to each terms of the first sum

$$\left|\frac{\cos(b^m\pi x_n)-\cos(b^m\pi x_0)}{b^m(x_n-x_0)}\right| = \left|-\pi\sin(b^m\pi\frac{x_m+x_0}{2})\cdot\frac{\sin(b^m\pi\frac{x_n-x_0}{2})}{b^m\pi\frac{x_n-x_0}{2}}\right| \leq \pi$$

This give us a bound for the first sum

$$\left|\sum_{m=0}^{n-1}(ab)^n \cdot \frac{\cos(b^m\pi x_n)-\cos(b^m\pi x_0)}{b^m(x_n-x_0)}\right| \leq \pi\sum_{m=0}^{n-1}(ab)^m = \frac{\pi(ab)^n}{ab-1}$$

■

Theorem 3.3.2. (An absolute lower bound for the second sum) For each $n \in \mathbb{N}$, we have

$$\left|\sum_{m=0}^{\infty} a^{n+m} \cdot \frac{\cos(b^{n+m}\pi x_n)-\cos(b^{n+m}\pi x_0)}{x_n-x_0}\right| \geq (ab)^n\frac{2}{3}$$

Proof. Because b is odd, we have

$$\cos(b^{n+m}\pi x_n) = \cos(b^m(j_n-1)\pi) = (-1)^{j_n-1}$$

And have

$$\cos(b^{n+m}\pi x_0) = \cos(b^m(\alpha_n+j_n)\pi) = (-1)^{j_n}\cos(b^m\alpha_n\pi)$$

We may now simplify

$$\sum_{m=0}^{\infty} a^{n+m} \cdot \frac{\cos(b^{n+m}\pi x_n) - \cos(b^{n+m}\pi x_0)}{x_n - x_0} = (ab)^n(-1)^{j_n} \sum_{m=0}^{\infty} \frac{1+\cos(b^m\alpha_n\pi)}{1+\alpha_n} a^m$$

Because $\frac{-1}{2} < \alpha_n \leq \frac{1}{2}$, we know

$$\text{All terms of } \sum_{m=0}^{\infty} \frac{1+\cos(b^m\alpha_n\pi)}{1+\alpha_n} a^m \text{ are non-negative.}$$

This give us

$$\left| \sum_{m=0}^{\infty} a^{n+m} \cdot \frac{\cos(b^{n+m}\pi x_n) - \cos(b^{n+m}\pi x_0)}{x_n - x_0} \right| = (ab)^n \sum_{m=0}^{\infty} \frac{1+\cos(b^m\alpha_n\pi)}{1+\alpha_n} a^m$$

To close out the proof, just observe

$$m = 0 \implies \frac{1+\cos(b^m\alpha_n\pi)}{1+\alpha_n} a^m \geq \frac{2}{3}$$

■

Theorem 3.3.1 together with Theorem 3.3.2 give us the absolute lower bound

$$\left| \frac{f(x_n) - f(x_0)}{x_n - x_0} \right| \geq (ab)^n \left[\frac{2}{3} - \frac{\pi}{ab-1} \right]$$

Which by hypothesis 3.1 implies the sequence indeed diverges.