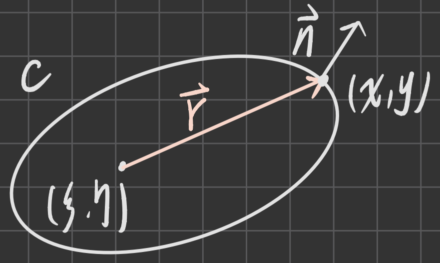
eg,,

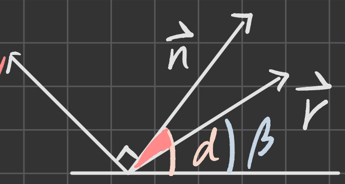


$$u(\xi,\eta)=\int_c \frac{\cos(\vec{r},\vec{n})}{r}ds,\ r=|\vec{r}|$$

$$\vec{r}=(x-\xi,y-\eta) \qquad \vec{r}\cdot\vec{n}=|\vec{r}||\vec{n}|\cos(\vec{r},\vec{n})=|\vec{r}|\cos(\vec{r},\vec{n})$$

$$\frac{\cos(\vec{r},\vec{n})}{|\vec{r}|}=\vec{r}^{\,-2}\cos(\vec{r},\vec{n})$$

Sol,,



$$x-\xi=r\cos\beta\ ;\ y-\eta=r\sin\beta$$

$$\cos(\vec{r},\vec{n})=\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

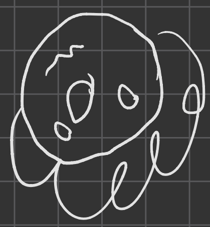
$$ds=(\cos(\alpha+\tfrac{\pi}{2}),\sin(\alpha+\tfrac{\pi}{2}))$$

$$=(\underset{dx}{-\sin\alpha},\underset{dy}{\cos\alpha})$$

$$=\frac{x-\xi}{r}\cos\alpha+\frac{y-\eta}{r}\sin\alpha$$

$$=\frac{x-\xi}{r}dy-\frac{y-\eta}{r}dx$$

$$u(\xi,\eta)=\int_c \overset{P}{\frac{-(y-\eta)}{r^2}}dx+\overset{Q}{\frac{(x-\xi)}{r}}dy$$



$$r=\sqrt{(x-\xi)^2+(y-\eta)^2},\ \frac{\partial r}{\partial x}=$$

$$\frac{\partial Q}{\partial x}=r^{-2}-2r^{-3}\frac{(x-\xi)^2}{r}$$

$$=\frac{r^2-2(x-\xi)^2}{r^4}=\frac{-(x-\xi)^2+(y-\eta)^2}{r^4}$$

$$-\frac{\partial P}{\partial y}=r^{-2}-2r^{-4}(y-\eta)^2=\frac{(x-\xi)^2-(y-\eta)^2}{r^4} \qquad \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0$$

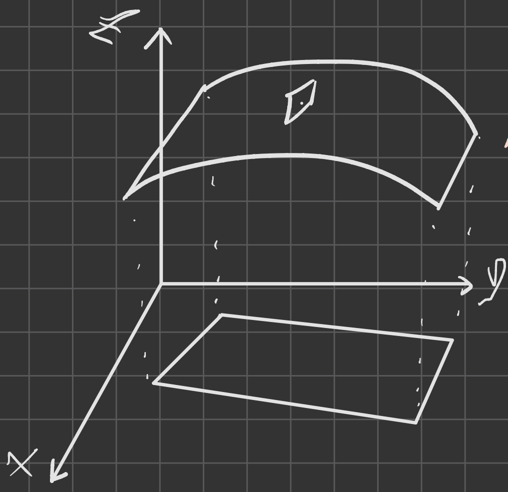
$$u(\xi,\eta)=\int_c \frac{-(y-\eta)}{r^2}dx+\frac{(x-\xi)}{r^2}dy=0,\ (\xi,\eta)\notin\Omega$$

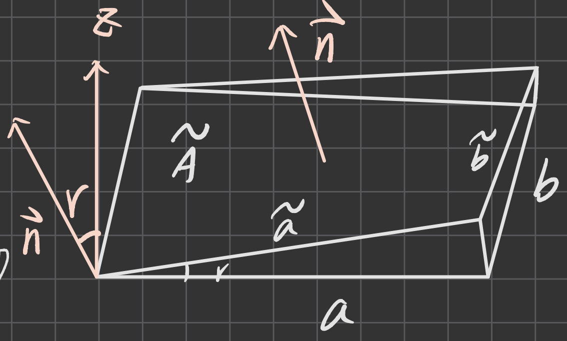
Sol,, $$u(\xi,\eta)=\int_c \frac{\cos(\vec{r},\vec{n})}{r}ds,\ r=|\vec{r}| \qquad \begin{cases}x=\xi+\varepsilon\cos t\\ y=\eta+\varepsilon\sin t\end{cases}$$

$$\vec{r}=(\varepsilon\cos t,\varepsilon\sin t)$$
$$\vec{n}=(\cos t,\sin t)$$

$$=\int_{D_\varepsilon(\xi,\eta)}\frac{\cos(\vec{r},\vec{n})}{r}ds=\int_0^{2\pi}ds=2\pi$$

$$\cos(\vec{r},\vec{n})=\varepsilon$$





$$\tilde{A} = \frac{1}{\cos r} A = \sqrt{1+f_x^2+f_y^2}\, A.$$

$$\vec{n} \cdot (0,0,1) = |\vec{n}|\,|(0,0,1)| \cos r = \cos r$$

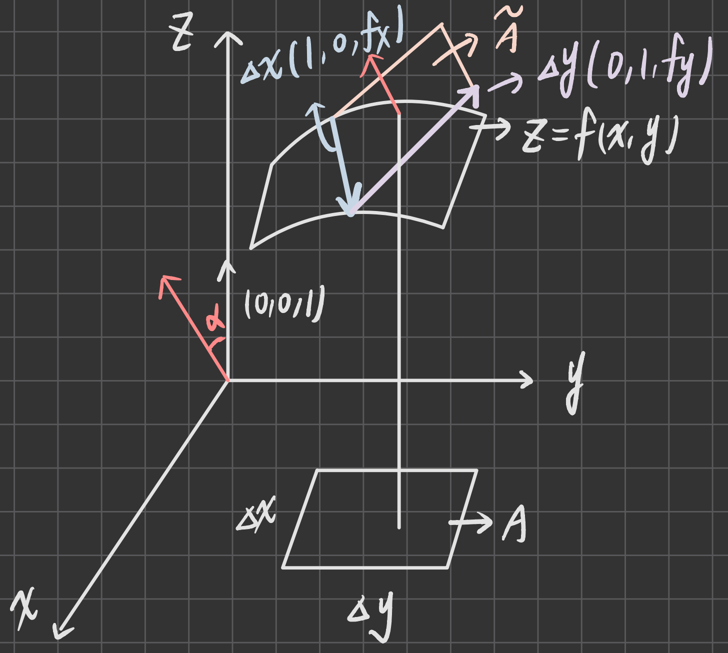
$$F(x,y,z) = z - f(x,y) = 0$$

$$\nabla F = (-f_x, -f_y, 1)$$

$$\vec{n} = \frac{(-f_x, -f_y, 1)}{\sqrt{1+f_x^2+f_y^2}}$$

$$\cos r = \frac{1}{\sqrt{1+f_x^2+f_y^2}}$$

$$\iint_S F(x,y,f(x,y))\, ds = \iint_\Omega F(x,y,f(x,y)) \sqrt{1+f_x^2+f_y^2}\, dA$$



$$(0, \Delta y, f_y \Delta y) = \Delta y (0,1,f_y) = \Delta y\, r_y$$

$$(\Delta x, 0, f_x \Delta x) = \Delta x (1,0,f_x) = \Delta x\, r_x$$

$$r(x,y) = (x,y,z)$$

$$r_x = (1,0,z_x)$$

$$r_y = (0,1,z_y)$$

$$\tilde{A} = |\Delta x\, r_x \times \Delta y\, r_y| = \Delta x \Delta y\, |r_x \times r_y| = A\, |r_x \times r_y|$$

$$r_x \times r_y = (-f_x, -f_y, 1), \quad \vec{n} = \frac{(-f_x, -f_y, 1)}{|r_x \times r_y|}$$

$$\tilde{A} = \frac{1}{\cos\alpha} A = |r_x \times r_y|\, A$$

$$\vec{n} \cdot (0,0,1) = |\vec{n}| \cdot |(0,0,1)| \cos\alpha = \cos\alpha$$

$$\cos\alpha = \frac{1}{|r_x \times r_y|}$$

$$\iint_S g(x,y,z(x,y))\, dr = \iint_\Omega \tilde{g}(x,y)\, |r_x \times r_y|\, dA$$

$$= \iint_\Omega \tilde{g}(x,y) \sqrt{1+f_x^2+f_y^2}\, dA$$

$$r(x,y) = (x,y,z(x,y)) = r(u,v) = (x(u,v), y(u,v), z(u,v))$$

$$\iint g(r(x,y))\, dr = \iint_S g(r(x,y))\, d\sigma = \underline{\iint_\Omega g(r(x,y))\, |r_x \times r_y|\, dx\,dy} \text{ 記}$$

$$= \iint_W g(r(u,v))\, |r_u \times r_v|\, du\,dv$$

$$= \iint_\Omega g(r(x,y))\, |r_u \times r_v| \left| \frac{\partial(u,v)}{\partial(x,y)} \right| dx\,dy$$

$$r_x = \left(\frac{\partial x}{\partial x}, \frac{\partial y}{\partial x}, \frac{\partial z}{\partial x} \right)$$

$$= \left(\frac{\partial x}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial x}, \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x}, \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \right)$$

$$= \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) + \frac{\partial v}{\partial x}\left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right)$$

$$= \frac{\partial u}{\partial x} r_u + \frac{\partial v}{\partial x} r_v$$

$$r_y = \left(\frac{\partial x}{\partial y}, \frac{\partial y}{\partial y}, \frac{\partial z}{\partial y} \right)$$

$$= \frac{\partial u}{\partial y} r_u + \frac{\partial v}{\partial y} r_v$$

$$|r_x \times r_y| = \left| \left(\frac{\partial u}{\partial x} r_u + \frac{\partial v}{\partial x} r_v \right) \times \left(\frac{\partial u}{\partial y} r_u + \frac{\partial v}{\partial y} r_v \right) \right|$$

$$= \left| \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\underbrace{(r_u \times r_v)}_{=0} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}(r_u \times r_v) \right.\right.$$

$$\left.+ \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}(r_v \times r_u) + \frac{\partial v}{\partial x}\frac{\partial v}{\partial x}\underbrace{(r_v \times r_u)}_{=0} \right|$$

$$= \left| \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \right)(r_u \times r_v) \right|$$

$$= \left| \frac{\partial(u,v)}{\partial(x,y)} \right| |r_u \times r_v|$$

Test! 20%

eg. $|\partial B_a(0)|$

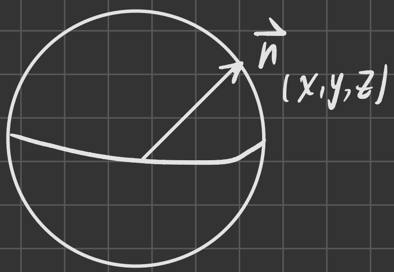
Sol. $x^2+y^2+z^2=a^2 \Rightarrow z=\sqrt{a^2-x^2-y^2}$

$$\Omega=\{(x,y),\ x^2+y^2\le a^2\}$$

$$z_x=\frac{-x}{\sqrt{a^2-x^2-y^2}},\quad z_y=\frac{-y}{\sqrt{a^2-x^2-y^2}},\quad 1+z_x^2+z_y^2=1+\frac{x^2+y^2}{a^2-x^2-y^2}=\frac{a^2}{a^2-x^2-y^2}$$

$$2\iint_S 1\,d\sigma = 2\iint_{B_a(0)}\sqrt{1+z_x^2+z_y^2}\,dxdy = 2\iint_{B_a(0)} a\frac{1}{\sqrt{a^2-x^2-y^2}}\,dxdy$$

$$=2a\int_0^r\int_0^{2\pi}\frac{1}{\sqrt{a^2-r^2}}\,r\,d\theta\,dr=4\pi a^2$$



$$\vec{n}=\frac{1}{a}(x,y,z)$$

$$\cos\alpha=\vec{n}\,(0,0,1)=\frac{z}{a}$$

$$\frac{1}{\cos\alpha}=\frac{a}{z}=\frac{a}{\sqrt{a^2-x^2-y^2}}$$

$$z = g(x,y), \quad F(x,y,z) = z - g(x,y)$$

$$\vec{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{(-g_x, -g_y, 1)}{\sqrt{1+g_x^2+g_y^2}} \qquad \vec{n}\cdot(0,0,1) = \frac{1}{\sqrt{1+g_x^2+g_y^2}}$$

$$r(x,y) = (x,y,z), \quad r_x = (1,0,z_x), \quad r_y = (0,1,z_y)$$

$$\vec{n} = \frac{r_x \times r_y}{|r_x \times r_y|} = \frac{(-z_x, -z_y, 1)}{|r_x \times r_y|} \longrightarrow \begin{vmatrix} i & j & k \\ 1 & 0 & z_x \\ 0 & 1 & z_y \end{vmatrix}$$

$$\iint_S f(x,y)\,d\sigma = \iint_\Omega (x,y) \frac{1}{\cos\alpha}\,dA = \iint_{\Omega x,y} f(x,y) \frac{1}{\vec{n}\cdot(0,0,1)}\,dA$$

$$= \iint_\Omega f\sqrt{1+g_x^2+g_y^2}\,dA$$

$$= \iint_{\Omega(x,y)} f(x,y,z)\,|r_x \times r_y|\,dA$$

$$= \iint_{\Omega u,v} f(u,v)\,|r_u \times r_v|\,du\,dv$$

eg., $S=\{(r,\theta)\mid x=r\cos\theta,\ y=r\sin\theta,\ z=\theta,\ 0\le\theta\le 2\pi,\ 0\le r\le 1\}$

Sol., $\iint_S \sqrt{1+x^2+y^2}\,d\sigma = \int_0^{2\pi}\int_0^1 \sqrt{1+\cos^2\theta+\sin^2\theta}\,|r_r\times r_\theta|\,dr\,d\theta$

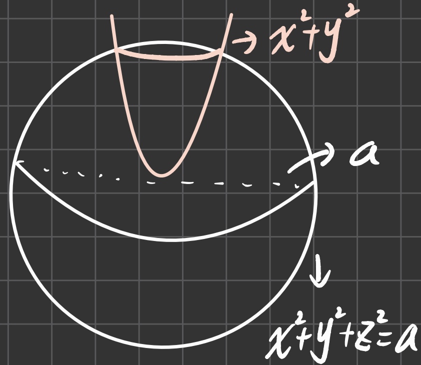
$r_r=(\cos\theta,\sin\theta,0)$, $r_\theta=(-r\sin\theta, r\cos\theta, 1)$

$r_r\times r_\theta=(\sin\theta,-\cos\theta,r)$, $|r_r\times r_\theta|=\sqrt{\sin^2\theta+\cos^2\theta+r^2}=\sqrt{1+r^2}$

$\Rightarrow \int_0^{2\pi}\int_0^1\sqrt{1+r^2}\sqrt{1+r^2}\,dr\,d\theta = \int_0^{2\pi}\int_0^1(1+r^2)\,dr\,d\theta = 2\pi\cdot\frac{4}{3}=\frac{8}{3}\pi$

eg., $S_a: x^2+y^2+z^2=a^2,\ a>0$

$$f(x,y,z)=\begin{cases} x^2+y^2, & z\ge\sqrt{x^2+y^2} \\ 0, & z<\sqrt{x^2+y^2}\end{cases}$$



Sol., $\iint_{S_a} f(x,y,z)\,d\sigma = \iint_{\Omega_{u,v}} f(u,v)\,|\vec{r}_u\times\vec{r}_v|\,du\,dv$

$x^2+y^2+z^2=a^2$, $z^2=x^2+y^2$, $x^2+y^2=\frac{a^2}{2}$

$\Omega_{x,y}=\{(x,y)\mid x^2+y^2\le(\frac{a}{\sqrt{2}})^2\}$

$z=\sqrt{a^2-x^2-y^2}$, $z_x=\frac{-x}{\sqrt{a^2-x^2-y^2}}=\frac{-x}{z}$, $z_y=\frac{-y}{\sqrt{a^2-x^2-y^2}}=\frac{-y}{z}$

$r_x=(1,0,z_x)=(1,0,\frac{-x}{z})$, $r_y=(0,1,z_y)=(0,1,\frac{-y}{z})$

$r_x\times r_y=(\frac{-x}{z},\frac{-y}{z},1)$, $|\vec{r}_x\times\vec{r}_z|=\sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}$

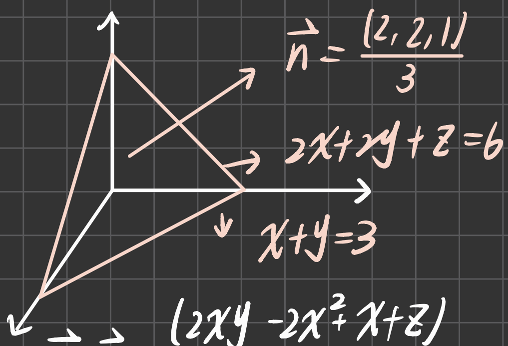
$\Rightarrow \iint_{\Omega_{x,y}}(x^2+y^2)\frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy$

$=a\int_0^{2\pi}\int_0^{\frac{a}{\sqrt{2}}} r^2\frac{r}{\sqrt{a^2-r^2}}\,dr\,d\theta = 2\pi a\int_0^{\frac{a}{\sqrt{2}}}\frac{r^3}{\sqrt{a^2-r^2}}\,dr$ $\left(\begin{array}{l}\text{let } r=a\sin\theta \\ dr=a\cos\theta\end{array}\right)$

$=\frac{8-5\sqrt{2}}{6}\pi a^4$

eg. $\vec{A} = (xy, -x^2, x+z)$

$S: 2x+2y+z=6,\ x,y,z \geq 0$



$\vec{A}\cdot\vec{n} = \frac{(2xy-2x^2+x+z)}{3}$

Sol. $\vec{r}_x = (1,0,-2),\ \vec{r}_y = (0,1,-2)$

$\vec{r}_x \times \vec{r}_y = (2,2,1),\ |\vec{r}_x \times \vec{r}_y| = 3$

$\Rightarrow \iint_S (\vec{A}\cdot\vec{n}_s)\,d\sigma = \iint_{\Omega x,y} (2xy - 2x^2 + 6 - 2x - 2y)\,dx\,dy$

$= \int_0^3 \int_0^{3-y} 2xy - 2x^2 - x - 2y + 6\,dx\,dy = \frac{27}{4}$

$\iint_S \vec{A}\cdot\vec{n}\,d\sigma,\ A = (P,Q,R)$

$= \iint_{\substack{\Omega x,y \\ \Omega u,v}} (\vec{A}\cdot\vec{n})\,|\vec{r}_{\underset{u}{x}} \times \vec{r}_{\underset{v}{y}}|\,dA \longrightarrow \vec{n} = \frac{\vec{r}_{\underset{u}{x}} \times \vec{r}_{\underset{v}{y}}}{|\vec{r}_{\underset{u}{x}} \times \vec{r}_{\underset{v}{y}}|}$

$= \iint_{\Omega x,y} \vec{A}\cdot(\vec{r}_{\underset{u}{x}} \times \vec{r}_{\underset{v}{y}})\,dx\,dy \longrightarrow \vec{r} = (x,y,z)$

$= \iint_{\Omega x,y} \begin{vmatrix} P & Q & R \\ 1 & 0 & z_x \\ 0 & 1 & z_y \end{vmatrix} dx\,dy \quad = \iint_{u,v} \begin{vmatrix} P & Q & R \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} du\,dv$

$= \iint_S P\,dy\,dz + Q\,dy\,dz + R\,dx\,dy \qquad \begin{pmatrix} +, & n_3 > 0 \\ -, & n_3 < 0 \end{pmatrix}$

$= \iint_{\Omega x,y} f(x,y)\,|r_x \times r_y|\,dx\,dy$

eg. $V=\{(x,y,z) \mid x^2+y^2=1,\ 0\le z\le 1\},\ S=\partial V$

$$\iint_S \underbrace{xy\,dydz}_{S_1}+\underbrace{x\,dzdx}_{S_2}+\underbrace{xyz\,dxdy}_{S_3}$$

Sol. $=\iint_S (xy,x,xyz)\cdot\vec{n}\,d\sigma$

$x=\cos\theta,\ y=\sin\theta,\ z=z,\ 0\le\theta\le 2\pi,\ 0\le z\le 1$

$$\iint_{S_1} d\sigma=\int_0^1\int_0^{2\pi}\begin{vmatrix}\cos\theta\sin\theta & \cos\theta & z\cos\theta\sin\theta\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{vmatrix}d\theta dz$$

$$=\int_0^1\int_0^{2\pi}\cos^2\theta\sin\theta+\sin\theta\cos\theta\,d\theta\,dz=0$$

$$\iint_{S_2} d\sigma=\iint_{\Omega xy} xyz\,d\sigma=\iint_{B_1} xy\,dxdy$$

$$\iint_{S_3} d\sigma=\iint_{\Omega x,y} -xy\underset{\uparrow\, z=0}{z}\,d\sigma=0$$

eg 11) $\vec{A} = (x, y, z)$, $S = x^2+y^2+z^2=4$, Find $\iint_S \vec{A}\cdot\vec{n}\, d\sigma$ $\Rightarrow 32\pi$

Sol: $\iint_S \vec{A}\cdot\vec{n}\, d\sigma = \iint_{S^+} \vec{A}\cdot\vec{n}\, d\sigma + \iint_{S^-} \vec{A}\cdot\vec{n}\, d\sigma$

$$\iint_{S^+} \vec{A}\cdot\vec{n}\, d\sigma = \iint_{B_2(0)} \begin{vmatrix} x & y & z \\ 1 & 0 & \frac{-x}{z} \\ 0 & 1 & \frac{-y}{z} \end{vmatrix} \qquad z=\sqrt{4-x^2-y^2} \quad \frac{\partial z}{\partial x} = \frac{-x}{z} \quad \frac{\partial z}{\partial y} = \frac{-y}{z}$$

$$= \iint_{B_2(0)} \frac{x^2}{z} + \frac{y^2}{z} + z\, dxdy = \iint_{B_2(0)} \frac{x^2+y^2+z^2}{z}\, dxdy$$

$$= \iint_{B_2(0)} \frac{4}{\sqrt{4-x^2-y^2}}\, dxdy = 4\int_0^{2\pi}\int_0^2 \frac{r}{\sqrt{4-r^2}}\, drd\theta$$

$$= 8\pi \int_0^{2\pi} \frac{r}{\sqrt{4-r^2}}\, dr = 16\pi$$

$$\iint_{z=-\sqrt{4-x^2-y^2}} \vec{A}\cdot\vec{n}\, d\sigma = \iint_{B_2(0)} \begin{vmatrix} x & y & z \\ 0 & 1 & \frac{y}{\sqrt{4-x^2-y^2}} \\ 1 & 0 & \frac{x}{\sqrt{4-x^2-y^2}} \end{vmatrix} \qquad \frac{\partial z}{\partial x} = \frac{x}{\sqrt{4-x^2-y^2}} \quad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{4-x^2-y^2}}$$

$$= \iint_{B_2(0)} \frac{x^2}{\sqrt{4-x^2-y^2}} + \frac{y^2}{\sqrt{4-x^2-y^2}} + \sqrt{4-x^2-y^2}\, dxdy$$

$$= \iint_{B_2(0)} \frac{x^2+y^2+(4-x^2-y^2)}{\sqrt{4-x^2-y^2}}\, dxdy = \iint_{B_2(0)} \frac{4}{\sqrt{4-x^2-y^2}}\, dxdy = 16\pi$$

$$\iint_{x^2+y^2+z^2=4} \vec{A}\cdot\vec{n}\, d\sigma = \int_0^{2\pi}\int_0^{2\pi} \begin{vmatrix} 2\sin\varphi\cos\theta & 2\sin\varphi\sin\theta & 2\cos\varphi \\ 2\cos\varphi\cos\theta & 2\cos\varphi\sin\theta & -2\sin\varphi \\ -2\sin\varphi\sin\theta & 2\sin\varphi\cos\theta & 0 \end{vmatrix} d\varphi d\theta$$

$$= \int_0^{2\pi}\int_0^{2\pi} -8\sin^3\varphi + 8\sin\varphi\cos^2\varphi + 8\sin\varphi\cos^2\varphi + 8\sin^3\varphi\, d\varphi d\theta$$

$$= 16\pi \int_0^{\pi} \sin\varphi\, d\varphi = 32\pi$$

$$\text{flux} = \lim_{\Delta t \to 0} \frac{\left(\iint_S \vec{A}\cdot\vec{n}\, d\sigma\right)\Delta t}{\Delta t}$$

$\boxed{eg.}$ $\vec{F} = (x, y, 0)$. $S: x^2+y^2+z^2=a^2$, $a>0$

Find $\iint_S \vec{F}\cdot\vec{n}\, d\sigma$

Sol. $\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iint_{S^+} \vec{F}\cdot\vec{n}\, d\sigma = \iint_{S^-} \vec{F}\cdot\vec{n}\, d\sigma$

$$= \iint_{B_a(0)} \begin{vmatrix} x & y & 0 \\ 1 & 0 & \frac{-x}{\sqrt{a^2-x^2-y^2}} \\ 0 & 1 & \frac{-y}{\sqrt{a^2-x^2-y^2}} \end{vmatrix} dxdy + (-)\iint_{B_a(0)} \begin{vmatrix} x & y & 0 \\ 1 & 0 & \frac{-x}{\sqrt{a^2-x^2-y^2}} \\ 0 & 1 & \frac{-y}{\sqrt{a^2-x^2-y^2}} \end{vmatrix} dxdy$$

$$= 2\iint_{B_a(0)} \frac{x^2+y^2}{\sqrt{a^2-x^2-y^2}} dxdy = 2\int_0^a \int_0^{2\pi} \frac{r^2}{\sqrt{a^2-r^2}}\, r\, d\theta dr$$

$$= 4\pi \int_0^a \frac{r^3}{\sqrt{a^2-r^2}} dr = \frac{8}{3}\pi a^3$$

⟨ii⟩ $\vec{n} = \frac{(x,y,z)}{a}$, $\vec{F}\cdot\vec{n} = (x,y,0)\cdot\frac{(x,y,z)}{a} = \frac{x^2+y^2}{a}$

$$\begin{vmatrix} i & j & k \\ 1 & 0 & \frac{-x}{z} \\ 0 & 1 & \frac{-y}{z} \end{vmatrix} = \sqrt{\frac{a^2}{z^2}} = \frac{a}{|z|}$$

$$\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iint_{S^+} \frac{x^2+y^2}{a} d\sigma + \iint_{S^-} \frac{x^2+y^2}{a} d\sigma$$

eg. $\vec{F}=(x,y,z)$, $S: z=1-x^2-y^2,\ z\geq 0$

Find $\iint_S \vec{F}\cdot\vec{n}\,d\sigma$

Sol. $\iint_{B_1(0)} \begin{vmatrix} x & y & z \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} dxdy = \iint_{B_1(0)} 2x^2+2y^2+(1-x^2-y^2)dxdy$

$=\iint_{B_1(0)} 1+x^2+y^2\,dxdy = \frac{3}{2}\pi$

eg. $\vec{F}=\frac{q}{|\vec{r}|^3}\vec{r}$, $\vec{r}=(x,y,z)$, $S: x^2+y^2+z^2=a^2$

Find $\iint_S \vec{F}\cdot\vec{n}\,d\sigma$

Sol. $\iint_S \vec{F}\cdot\vec{n}\,d\sigma = 2\iint_{S^+}\vec{F}\cdot\vec{n}\,d\sigma = 2\iint_{S^+}\frac{q}{a^2}d\sigma$

$= \frac{2q}{a^2}\underbrace{\iint_{S^+} 1\,d\sigma}_{}\overset{2\pi a^2}{} = 4\pi q$

$\vec{n}=\frac{(x,y,z)}{a}=\frac{\vec{r}}{a}$

$\vec{F}\cdot\vec{n} = \frac{q}{|\vec{r}|^3}\cdot\frac{|\vec{r}|^2}{a} = \frac{q}{a|\vec{r}|}$

$\frac{q}{|\vec{r}|^3}\begin{vmatrix} x & y & z \\ 1 & 0 & \frac{-x}{z} \\ 0 & 1 & \frac{-y}{z} \end{vmatrix} = \frac{q}{|\vec{r}|^3}\frac{x^2+y^2+z^2}{z} = \frac{q}{|\vec{r}|^2}\frac{a^2}{z}$

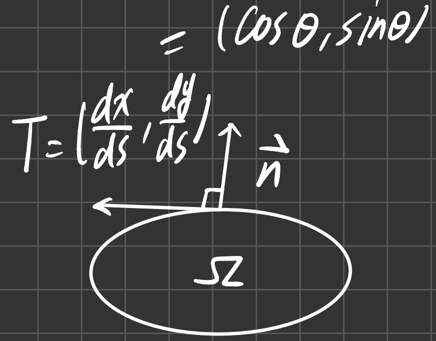
EX. $\vec{F} = (1, xy, 0)$, $S = \{x = u+v,\ y = u-v,\ z = u^2,\ 0 \le u, v \le 1\}$

Sol. $\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iint_{\Omega_{u,v}} \begin{vmatrix} 1 & (u+v)(u-v) & 0 \\ 1 & 1 & 2u \\ 1 & -1 & 0 \end{vmatrix} = \iint_{\Omega_{u,v}} 2u + 2u(u^2 - v^2)\, du dv$

$$= \int_0^1\int_0^1 2u + 2u^3 - 2uv^2\, du dv = \int_0^1 u^2 + \frac{1}{2}u^4 - v^2u^2 \Big|_0^1 dv$$

$$= \int_0^1 \frac{3}{2} - v^2\, dv = \frac{3}{2} - \frac{1}{3} = \frac{7}{6}$$

Recall Green's Thm



$$\iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx dy = \int_C P dx + Q dy$$

In vector's form, let $v = (Q, -P)$

$$\iint_\Omega \nabla\cdot v\, dx dy \overset{?}{=} \int_{C=\partial\Omega} \vec{v}\cdot\vec{n}\, ds$$

$$\int_C P dx + Q dy = \int_\Gamma P\frac{dx}{ds} + Q\frac{dy}{ds}\, ds$$

$$= \int_\Gamma (P, Q)\cdot\left(\frac{dx}{ds}, \frac{dy}{ds}\right) ds$$

$$= \int_\Gamma (Q, -P)\cdot\underbrace{\left(\frac{dy}{ds}, \frac{-dx}{ds}\right)}_{=\vec{n}} ds$$

$$= \int_\Gamma \vec{v}\cdot\vec{n}\, ds = \int_C (\vec{v}\cdot\vec{n})\, ds$$

$$\vec{n} = \left(\cos\left(\theta - \frac{\pi}{2}\right), \sin\left(\theta - \frac{\pi}{2}\right)\right)$$
$$= (\sin\theta, -\cos\theta)$$
$$= \left(\frac{dy}{ds}, \frac{-dx}{ds}\right)$$

In 3-D, Thm: The divergence Thm

let T be a solid bounded by a closed surface S (no boundary)

If $\vec{v} = \vec{v}(x,y,z)$, then

$$\iiint_T (\nabla\cdot\vec{v})\,dxdydz = \iint_{S=\partial T} (\vec{v}\cdot\vec{n})\,d\sigma$$

$$\| $$

$$\iiint_T \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) dxdydz$$

eg.1 $\vec{v} = (x,y,z)$, $S: x^2+y^2+z^2=a^2$, $a>0$

$$\iint_S \vec{v}\cdot\vec{n}\,d\sigma = \iiint_T (\nabla\cdot\vec{v})\,dxdydz = 3\iiint_T 1\,dxdydz$$

$$= 3\cdot\frac{4}{3}\pi a^3 = 4\pi a^3$$

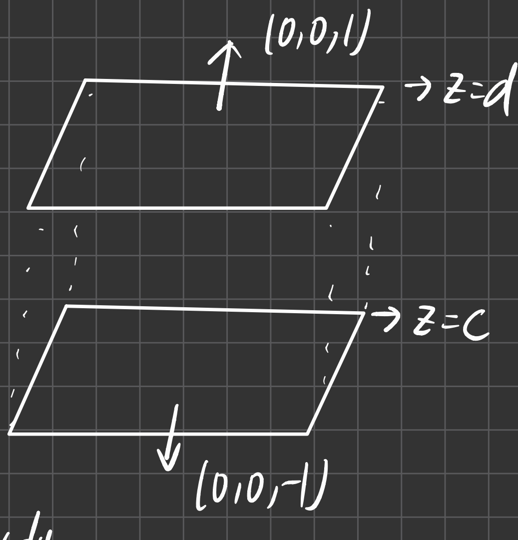
$$\iiint_T \nabla\cdot\vec{v}\,dxdydz = \iiint_T \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\,dxdydz$$

$$= \iint_{S=\partial T} \vec{v}\cdot\vec{n}\,d\sigma = \iint_S v_1n_1 + v_2n_2 + v_3n_3\,d\sigma$$

$$\iiint_T \frac{\partial v_1}{\partial x}\,dxdydz = \iint_S v_1n_1\,d\sigma$$

$$\iiint_T \frac{\partial v_2}{\partial y}\,dxdydz = \iint_S v_2n_2\,d\sigma$$

$$\iiint_T \frac{\partial v_3}{\partial z}\,dxdydz = \iint_S v_3n_3\,d\sigma$$

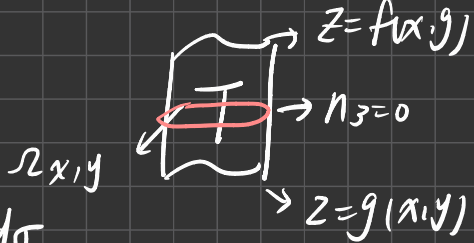


$$\iint_{\Omega_{x,y}}\int_c^d \frac{\partial V_3}{\partial z}\,dz\,dxdy$$

$$= \iint_{\Omega_{x,y}} V_3(x,y,d) - V_3(x,y,c)\,dxdy$$

$$= \iint_{\Omega_{x,y}} V_3(x,y,d)\,dxdy - \iint_{\Omega_{xy}} (x,y,c)\,dxdy$$

$$\iiint_T (\nabla\cdot\vec{v})\,dxdydz = \iint_{S=\partial T} \vec{v}\cdot\vec{n}\,d\sigma$$



$\vec{V} = (V_1, V_2, V_3)$, $\vec{n} = (n_1, n_2, n_3)$

To prove $\iiint \frac{\partial V_3}{\partial z}\,dxdydz = \iint_{S=\partial T} V_3 n_3\,d\sigma$

$$= \iint_{\Omega x,y} V_3(x,y,f(x,y)) - V_3(x,y,g(x,y))\,dxdy \qquad = \iint_{S_1} V_3 n_3\,d\sigma + \iint_{S_2} V_3 n_3\,d\sigma$$

$$n_3 = \vec{n}\,(0,0,-1) , \quad r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & z_x \\ 0 & 1 & z_y \end{vmatrix}$$
$$= -\vec{n}\,(0,0,1)$$

$$\iint_{S_1} V_3 n_3\,d\sigma + \iint_{S_2} V_3 n_3\,d\sigma = \iint_{S_1} V_3 \frac{1}{|r_x\times r_y|}\,d\sigma - \iint_{S_2} V_3 \frac{1}{|r_x\times r_y|}\,d\sigma$$

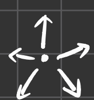
$$= \iint_{\Omega x,y} \frac{V_3(x,y,f(x,y))}{|r_x\times r_y|} |r_x\times r_y|\,dxdy - \iint_{\Omega x,y} \frac{V_3(x,y,g(x,y))}{|r_x\times r_y|} |r_x\times r_y|\,dxdy$$

$$= \iint_{\Omega x,y} V_3(x,y,f(x,y))\,|r_x\times r_y|\,dxdy - \iint_{\Omega x,y} V_3(x,y,g(x,y))\,dxdy$$

$$\iiint_T (\nabla\cdot\vec{v})\,dxdydz = \iint_{S=\partial T} \vec{v}\cdot\vec{n}\,d\sigma$$

$$T = B_\varepsilon(p) \Rightarrow \frac{\iiint_{B_\varepsilon(p)} (\nabla\cdot\vec{v})\,dxdydz}{Vol(B_\varepsilon(p))} = \frac{\iint_{\partial B_\varepsilon(p)} (\vec{V}\cdot\vec{n})\,d\sigma}{Vol(B_\varepsilon(p))}$$

let $\varepsilon \to 0$, $\nabla\cdot\vec{v}(p) =$

If $\nabla\cdot\vec{v}(p) > 0$, then  p sour Q

$\nabla\cdot\vec{v}(p) < 0$, then  p sink eg. $F=(x,y,z)$, $T=\{(x,y,z) \mid x^2+y^2+z^2=a^2\}$

(a) $\iint_{\partial T} \vec{F}\cdot\vec{n}\, ds$ (b) $\iiint_T \operatorname{div}\vec{F}\, dv$

Sol. (a) $\vec{n}=\frac{1}{a}(x,y,z)$,

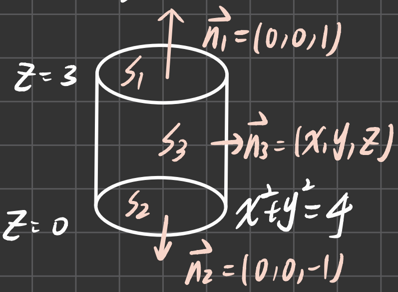
$$\vec{F}\cdot\vec{n}=(x,y,z)\frac{1}{a}(x,y,z)=\frac{1}{a}(x^2+y^2+z^2)=a$$

$$\iint_{\partial T}\vec{F}\cdot\vec{n}\,ds=a\iint_{\partial T}ds=a(4\pi a^2)=4\pi a^3$$

(b) $\iiint_T \operatorname{div}\vec{F}\,dv=3\iiint_T dv=3\cdot\frac{4}{3}\pi a^3=4\pi a^3$

eg. let $T=\begin{cases} x^2+y^2=4 \\ 0\le z\le 3\end{cases}$, $F=(x^3+\tan y z,\ y^3-e^{xz},\ 3z+x^3)$

Find $\iint_{\partial T}\vec{F}\cdot\vec{n}\,d\sigma$



Sol. $\iint_{S_1}\vec{F}\cdot\vec{n}\,d\sigma=\iint_{B_2(0)}\vec{F}\cdot(0,0,1)\,dxdy=\iint_{B_2(0)}(3z+x^3)\,dxdy$

$=\iint_{B_2(0)}(9+x^3)\,dxdy$ $(z=3)$

S_3: $x=2\cos\theta$
$y=2\sin\theta$
$z=z$

$\iint_{S_2}\vec{F}\cdot\vec{n}\,d\sigma=\iint_{B_2(0)}-x^3\,dxdy$ $(z=0)$

$$\iint_{S_3}\vec{F}\cdot\vec{n}\,d\sigma=\int_0^{2\pi}\int_0^3\begin{vmatrix} x^3+\tan yz & y^3-e^{xz} & 3z+x^3 \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1\end{vmatrix}dz\,d\theta$$

$$\iint_{\partial T}\vec{F}\cdot\vec{n}\,d\sigma=\iiint_T(\nabla\cdot\vec{F})\,dv=\iiint_T(3x^2+3y^2+3)\,dxdydz$$

$$=3\int_0^3\int_0^{2\pi}\int_0^2(r^2+1)r\,drd\theta dz=18\pi\int_0^2(r^3+r)\,dr=18\pi\left(\frac{r^4}{4}+\frac{r^2}{2}\Big|_0^2\right)$$

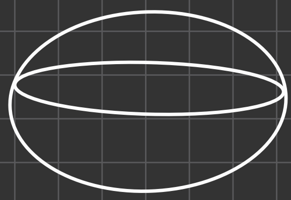
$$=108\pi$$

eg. T = a bounded domain

$\vec{F}(x,y,z) = \frac{\vec{r}}{|\vec{r}|^3}$, $\vec{r} = (x,y,z)$

Show that $\iint_{\partial T} \vec{F}\cdot\vec{n} = \begin{cases} 0, & (0,0,0) \in \overline{T} \\ 4\pi, & (0,0,0) \in T \end{cases}$

Sol. ⟨i⟩ $\iint_{\partial T} \vec{F}\cdot\vec{n}\, d\sigma = \iiint_T \operatorname{div}\vec{F}\, d\sigma = \iiint_T 0\, d\sigma = 0$

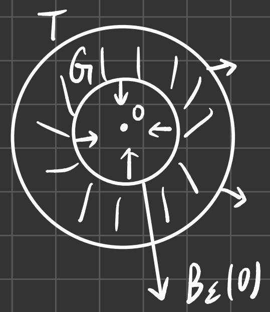


$$\Big[\ \operatorname{div}\vec{F} = \frac{\partial}{\partial x}\Big(\frac{x}{|\vec{r}|^3}\Big) + \frac{\partial}{\partial y}\Big(\frac{y}{|\vec{r}|^3}\Big) + \frac{\partial}{\partial z}\Big(\frac{z}{|\vec{r}|^3}\Big)$$

$$= |\vec{r}|^{-3} - 3|\vec{r}|^{-4}\frac{x^2}{|\vec{r}|} + |\vec{r}|^{-3} - 3|\vec{r}|^{-5}y^2 + |\vec{r}|^{-3} - 3|\vec{r}|^{-5}z^2$$

$$= 3|\vec{r}|^{-3} - 3|\vec{r}|^{-5}(x^2+y^2+z^2) = 3|\vec{r}|^{-3} - 3|\vec{r}|^{-3} = 0\ \Big]$$

⟨ii⟩ $\underline{\iint_{\partial T} \vec{F}\cdot\vec{n}\, d\sigma - \iint_{\partial B_\varepsilon(0)} \vec{F}\cdot\vec{n}\, d\sigma} + \iint_{\partial B_\varepsilon(0)} \vec{F}\cdot\vec{n}\, d\sigma$



$$= \iiint_G \operatorname{div}\vec{F}\, dv + \iint_{\partial B_\varepsilon(0)} \vec{F}\cdot\vec{n}\, d\sigma$$

$$= \iint_{\partial B_\varepsilon(0)} \vec{F}\cdot\vec{n}\, d\sigma = \iint_{\partial B_\varepsilon(0)} \frac{\vec{r}}{|\vec{r}|^3}\cdot\frac{\vec{r}}{\varepsilon}\, d\sigma$$

$$= \iint_{\partial B_\varepsilon(0)} \frac{1}{\varepsilon^2}\, d\sigma = 4\pi\varepsilon^2 \frac{1}{\varepsilon^2} = 4\pi$$

eg., $T=\begin{cases} x^2+y^2 \le 4 \\ 0 \le z \le 3 \end{cases}$, $\vec{F}=(4x, -2y^2, z^2)$

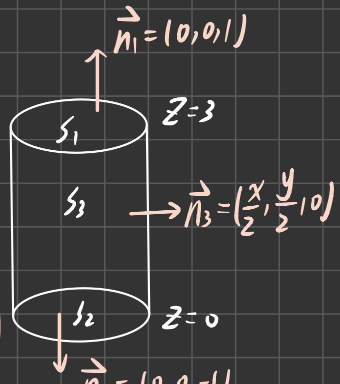
Find $\iint_{\partial T} \vec{F}\cdot\vec{n}\, d\sigma$

Sol., $\iint_{\partial T} \vec{F}\cdot\vec{n}\, d\sigma = \iiint_T \operatorname{div}\vec{F}\, dxdydz = \iiint_T (4-4y+2z)\, dxdydz$

$= \int_0^3\int_0^{2\pi}\int_0^2 (4-4r\sin\theta+2z)\, r\, drd\theta dz$

$= 48\pi - \underbrace{12\int_0^{2\pi}\int_0^2 r^2\sin\theta\, drd\theta}_{=0} + 4\pi\int_0^3\int_0^2 zr\, drd\theta$

$= 48\pi + 36\pi = 84\pi$



$\iint_{\partial T} \vec{F}\cdot\vec{n}\, d\sigma = \iint_{S_1} \vec{F}\cdot\vec{n}\, d\sigma + \iint_{S_2} \vec{F}\cdot\vec{n}\, d\sigma + \iint_{S_3} \vec{F}\cdot\vec{n}\, d\sigma$

$= \iint_{S_1} 9\, d\sigma + \iint_{S_2} 0\, d\sigma + \underline{\iint_{3} 2x^2-y^3\, d\sigma}$ $\left(\begin{array}{l} x=2\cos\theta \\ y=2\sin\theta,\ 0\le\theta\le 2\pi \\ z=z,\ 0\le z\le 3 \end{array}\right)$

$= 36\pi + 2\int_0^3\int_0^{2\pi} 8\cos^2\theta - 8\sin^3\theta\, d\theta dz$

$= 36\pi + 48\int_0^{2\pi} \cos^2\theta - \sin^3\theta\, d\theta$

$= 36\pi + 48\pi = 84\pi$

$|\vec{r}_\theta \times \vec{r}_z| = \left|\begin{vmatrix} i & j & k \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}\right| = |(2\cos\theta, 2\sin\theta, 0)|$

$=2$

eg., $S=\begin{cases} z=1-x^2, & y=0 \\ z=0, & y=2 \end{cases}$, $\vec{F}=(x+\cos y, y+\sin z, z+e^x)$

Sol., $\iint_{\partial T} \vec{F}\cdot\vec{n}\, d\sigma = \iiint_T \operatorname{div}\vec{F}\, dV = 3\operatorname{vol}(T) = 3\left(\int_{-1}^1 (1-x^2)dx\right)\times 2 = 8$

eg., $\operatorname{vol}(T) = \frac{1}{3}\iint_{S=\partial T} \vec{F}\cdot\vec{n}\, d\sigma$

eg., $B_a(0) = \frac{1}{3}\iint_{\partial B_a(0)} (x,y,z)\cdot\frac{1}{a}(x,y,z)\, d\sigma$

$= \frac{a}{3}\iint_{\partial B_a(0)} 1\, d\sigma = \frac{a}{3}4\pi a^2 = \frac{4}{3}\pi a^3$

Thm: let V be a sinple connected domain, then $\operatorname{div}\vec{A}=0$ iff $\exists\, \vec{B}$ s.t. $\vec{A}=\nabla\times\vec{B}$

$$\nabla\times\vec{B}=\begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ Bx & By & Bz \end{vmatrix}$$

pf: $(\Leftarrow)$ $\operatorname{div}\vec{A}=\operatorname{div}(\nabla\times\vec{B})=\operatorname{div}(\partial_y B_3-\partial_z B_2,\partial_z B_1-\partial_x B_3,\partial_x B_2-\partial_y B_1)$

$$=\partial x(\partial_y B_3-\partial_z B_2)+\partial y(\partial_z B_1-\partial_x B_3)+\partial z(\partial_x B_2-\partial_y B_1)=0$$

$(\Rightarrow)$ $\partial_x A_1+\partial_y A_2+\partial_z A_3=0$

Find $\vec{B}=(B_1,B_2,B_3)$ s.t.

$\partial_y B_3-\partial_z B_2=A_1$ ①

$\partial_z B_1-\partial_x B_3=A_2$ ②

$\partial_x B_2-\partial_y B_1=A_3$ ③

let $B_3=$ costant

From ① : $-\partial_z B_2(x,y,z)=A_1(x,y,z)$

let $B_2(x,y,z)=\int_{z_0}^{z}A_1(x,y,t)\,dt$

From ② : $\partial_z B_1(x,y,z)=A_2(x,y,z)$

$\Rightarrow B_1(x,y,z)=B_1(x,y,z_0)+\int_{z_0}^{z}A_2(x,y,t)\,dt$ $=$ $-\int_{y_0}^{y}A_3(x,t,z_0)\,dt+\int_{z_0}^{z}A_2(x,y,t)\,dt$

From ③ $\int_{z_0}^{z}\frac{-\partial A_1(x,y,t)}{\partial x}dt-\int_{z_0}^{z}\frac{\partial A_2(x,y,t)}{\partial y}dt-\frac{\partial B_1(x,y,z_0)}{\partial y}=A_3(x,y,z)$

By $(*)$ $\int_{z_0}^{z}\frac{\partial A_3(x,y,t)}{\partial t}dt-\frac{\partial B_1(x,y,z_0)}{\partial y}=A_3(x,y,z)$

$$A_3(x,y,z)-A_3(x,y,z_0)-\frac{\partial B_1(x,y,z_0)}{\partial y}=A_3(x,y,z)$$

$$\frac{\partial B_1}{\partial y}(x,y,z_0)=-A_3(x,y,z_0)\Rightarrow B_1=-\int_{y_0}^{y}A_3(x,t,z_0)\,dt$$

Recall the Green's Thm

$$\int_C P\,dx + Q\,dy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\vec{F} = (P, Q, 0),\ (\nabla \times \vec{F}) \cdot (0,0,1) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

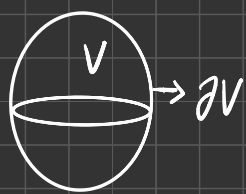
$$\Rightarrow \iint_\Omega (\nabla \times \vec{F}) \cdot \vec{n} = \int_{\partial\Omega} \vec{F} \cdot dr$$

In 3-D : Thm (Stoke's Thm)

Let S be a smooth surface with boundary curve $C = \partial S$

If $\vec{F} = \vec{F}(x, y, z)$,

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma = \int_{\partial S = C} \vec{F} \cdot dr$$



where $\vec{n}$ is the out normal of S

divergent Thm : $\iint_{S=\partial V} \vec{F} \cdot \vec{n}\, d\sigma = \iiint_V \operatorname{div} \vec{F}\, dx\,dy\,dz$

$\partial S = \phi$ (沒有邊界)

Cor., let $\vec{F} = (M, N, P)$ in a simple connected set Ω

Then $\vec{F}$ is conservative ($\vec{F} = \nabla f$) iff $\nabla \times \vec{F} = 0$

($\Rightarrow$) $\nabla \times \vec{F} = 0$

($\Leftarrow$) By Stoke's Thm

eg. $\vec{F} = (z^2, -2x, y^3)$, $S: x^2+y^2+z^2=1$, $z \geq 0$

Find $\iint_S (\nabla\times\vec{F})\cdot\vec{n}\, d\sigma$

Sol. $\nabla\times\vec{F} = (3y^2, 2z, -2) \longrightarrow \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ z^2 & -2x & y^3 \end{vmatrix}$

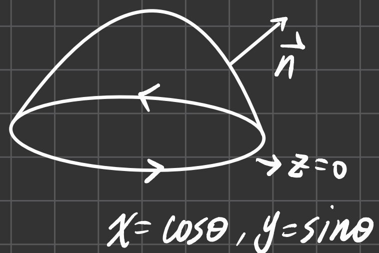
$\vec{n} = (x, y, z)$

$(\nabla\times\vec{F})\cdot\vec{n} = 3xy^2 + 2yz - 2z$

$$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\, d\sigma = \iint_{B(1,0)} (3xy^2+2yz-2z)\cdot\frac{1}{z}\, dxdy$$

$\left(z_x = \frac{x}{z},\ z_y = \frac{y}{z},\ \sqrt{1+z_x^2+z_y^2} = \frac{1}{z}\right)$

$$= \iint_{B(1,0)} \underbrace{\frac{3xy^2}{\sqrt{1-x^2-y^2}}}_{=0} dxdy + \iint_{B(1,0)} \underbrace{2y}_{=0} - 2\, dxdy = -2\pi$$



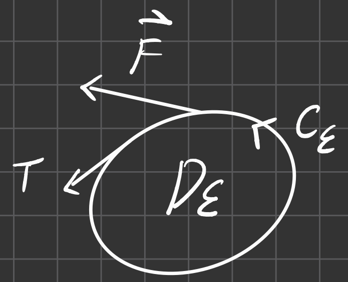
⟨ii⟩ $(x,y,z) = (\cos t, \sin t, 0)$

$$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\, d\sigma = \int_{\partial S} \vec{F}\cdot dr$$

$$= \int_C z^2dx - 2xdy + y^3dz = \int_{\partial B(1,0)} \underset{z=0}{\underline{z^2}}dx - 2xdy$$

$$= \int_0^{2\pi} -2\cos^2\theta\, d\theta = -2\pi$$

$$\frac{\iint_{D_\varepsilon} (\nabla\times\vec{F})\cdot\vec{n}\, d\sigma}{A(D_\varepsilon)} = \frac{\int_{C_\varepsilon} \vec{F}\cdot dr}{A(D_\varepsilon)}$$



let $\varepsilon \to 0$, $(\nabla\times\vec{F})\cdot(0,0,1) = \dfrac{\int_{C_\varepsilon}\vec{F}dr}{A(D_\varepsilon)}$

So, $\nabla\times\vec{F}$ measures the rotational tendency of $\vec{F}$

If $\nabla\times\vec{F} = \vec{0}$, identically, then the flow is called irrational

$$\underline{\iint_S (\nabla\times\vec{F})\cdot\vec{n}\, d\sigma} \overset{?}{=} \int_{\Gamma=\partial S} \vec{F}\cdot dr$$

$\vec{F} = (P, Q, R)$, $z = z(x,y)$, $\nabla\times\vec{F} = (R_y - Q_z, P_z - R_x, Q_x - P_y)$

$$\int_\Gamma \vec{F}\cdot dr = \int_\Gamma P\,dx + Q\,dy + R\,dz$$

$$= \int_\Gamma P\,dx + Q\,dy + R z_x\,dx + R z_y\,dy$$

$$= \int_{\partial\Omega} (P + R z_x)dx + (Q + R z_y)dy$$

$$= \iint_\Omega \frac{\partial}{\partial x}(Q + R z_y) - \frac{\partial}{\partial y}(P + R z_x)\,dxdy$$

$$\frac{\partial}{\partial x}(Q + R z_y) = Q_x + Q_z z_x + (R_x + \cancel{R_z z_x}) z_y + R\cancel{z_x z_y}$$

$$-)\ \frac{\partial}{\partial y}(P + R z_y) = P_y + P_z z_y + (R_y + \cancel{R_z z_y}) z_x + R\cancel{z_x z_y}$$

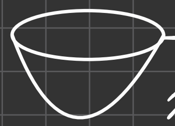
$$\iint_\Omega Q_x - P_y + Q_z z_x - P_z z_y + R_x z_y - R_y z_x\, dxdy$$

$$\|$$

$$= \iint_\Omega \begin{vmatrix} R_y - Q_z & P_z - R_x & Q_x - P_y \\ 1 & 0 & z_x \\ 0 & 1 & z_y \end{vmatrix} dxdy = \iint_\Omega -z_x(R_y - Q_z) - z_y(P_z - R_x) + Q_x - P_y\, dxdy$$

eg. $\vec{F} = (3y, -xz, yz^2)$, $S: 2z = x^2+y^2,\ 0 \le z \le 2$

Find $\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma$



Sol. ⟨i⟩ $\nabla \times \vec{F} = (z^2+x,\ 0,\ -z-3)$

$$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\,d\sigma = \iint_{B_2(0)} - \begin{vmatrix} z^2+x & 0 & -z-3 \\ 1 & 0 & x \\ 0 & 1 & y \end{vmatrix} dxdy$$

$$= \iint_{B_2(0)} xz^2 + x^2 + z + 3\, dxdy$$

$$= \iint_{B_2(0)} \frac{x}{4}(x^2+y^2)^2 + x^2 + \frac{x^2+y^2}{2} + 3\, dxdy = 20\pi$$

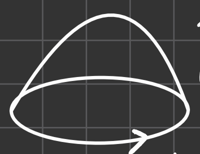
⟨ii⟩ $$\int_C \vec{F}\cdot dr = \int_C 3y\,dx - xz\,dy + yz^2 \underbrace{dz}_{=0}$$

$$= \int_C 3y\,dx - 2x\,dy = \int_{2\pi}^{0} -12\sin^2\theta - 8\cos^2\theta\, d\theta$$

$$= \int_0^{2\pi} 8 + 4\sin^2\theta\, d\theta = 20\pi$$

eg. $\vec{F} = (x^3 - x^2y + z^3,\ x^2y + y^3,\ xz + z^2)$

$$S: \begin{cases} x^2+y^2+z^2 = 4 \\ z \ge 0 \end{cases}$$



Sol. $$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\,d\sigma = \int_{\partial S} \vec{F}\cdot dr$$

$$= \int_{\partial S} (x^3 - xy + z^3)dx + (x^2y + y^3)dy + (xz + z^2)dz$$

$$= \int_{\partial B_2(0)} (x^3 - x^2y)dx + (x^2y + y^3)dy = 8\pi$$

eg. $\vec{F} = (e^{z^2}, 4z-y, 8x\sin y)$

$$S : \begin{cases} 4-x^2-y^2 = z \\ 0 \le z \le 4 \end{cases}$$

Sol. $\iint_S (\nabla\times\vec{F})\cdot\vec{n}\, d\sigma = \int_{\partial S} \vec{F}\cdot dr = \int_{\partial S} e^{z^2}dx + (4z-y)\,dy + 8x\sin y\, \underbrace{dz}_{=0}$

$= \int_{\partial B_2(0)} dx - y\,dy = \int_0^{2\pi} -2\sin\theta - 4\sin\theta\cos\theta\, d\theta = 0$

eg. $C : r(t) = (a\sin t, b\sin t, c\cos t)$, $0 \le t \le 2\pi$ $abc \ne 0$

Evaluate $I = \int_C y^3z^2dx + 3xy^2z^2dy + 2xy^3z\,dz$

$\vec{F} = (y^3z^2, 3xyz^2, 2xy^3z)$ $\nabla\times\vec{F} \ne 0$

Sol. $0 = \iint_S (\nabla\times\vec{F})\cdot\vec{n}\, d\sigma = \int_C \vec{F}\cdot dr$

$\iint_S f\, d\sigma = \iint_\Omega f \cdot |r_x \times r_y|\, dx\,dy$, $r = (x, y, z)$

$$\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iint \pm \begin{vmatrix} F_1 & F_2 & F_3 \\ & r_u & \\ & r_v & \end{vmatrix} du\,dv$$