RTFT HW4

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Problem A. Write $D_n = \langle a^{2m} = b^2 = 1, bab^{-1} = a^{-1} \rangle$

First consider a^i where $1 \leq i \leq 2m-1$ and $i \neq m$

Clearly $\langle a \rangle \subseteq C_G(a^i)$, so $(G : C_G(a^i)) \leq 2$

Notice $bab^{-1} = a^{-1} \implies a = ba^{-1}b^{-1} \implies ba^ib^{-1} = b(ba^{-1}b^{-1})^ib^{-1} = b^2a^{-i}b^{-2} = a^{-i}$

If $(G : C_G(a^i)) = 1$, then $G = C_G(a^i)$; yet $ba^ib^{-1} = a^{-i} = a^{2m-i} \neq a^i$ indicate $G \neq C_G(a^i)$, so we know $(G : C_G(a^i)) = 2$

Then $|(a^i)^G| = 2$

$a^{-i} = ba^ib^{-1} \implies a^{-1} \in (a^i)^G$

So we know $(a^i)^G = \{a^i, a^{-i}\}$

Now we consider a^m

Clearly $a \in C_G(a^m)$

Notice $ba = a^{-1}b$

So $ba^mb^{-1} = a^{-m}bb^{-1} = a^m$, that is $b \in C_G(a^m)$

Then $G = \langle a, b \rangle \subseteq C_G(a^m)$

So $|(a^m)^G| = (G : C_G(a^m)) = 1$

So $(a^m)^G = \{a^m\}$

So far we already find the following equivalence class $\{a^m\}, \{a, a^{-1}\}, \cdots, \{a^{m-1}, a^{m+1}\}$

And obviously we have equivalence class $\{e\}$

Notice all element in D_n can be uniquely expressed in the form a^kb^j, where $0 \leq k < n$ and $0 \leq j < 2$ **fact 1**

So we only have to find equivalence classes for the elements a^kb in G where $0 \leq k < 2m$

From **fact 1**, we deduce $b^G = \{a^kb^jb(a^kb^j))^{-1} | 0 \leq k < 2m, 0 \leq j < 2\} = \{a^kb^jbb^{-k}a^{-k} | 0 \leq k < 2m, 0 \leq j < 2\} = \{a^kba^{-k} | 0 \leq k < 2m\}$

Notice $bab^{-1} = a^{-1} \implies ba^{-1} = ab$

So $b^G = \{a^k b a^{-k} | 0 \le k < 2m, 0 \le j < 2\} = \{a^{2k}b | 0 \le k < 2m\}$

Again from **fact 1**, we deduce $(ab)^G = \{a^k b^j (ab)(a^k b^j)^{-1} | 0 \le k < 2m, 0 \le j < 2\}$

Notice $ba^{-1} = ab$ and $ba = a^{-1}b$

So we deduce, $(ab)^G = \{a^k b^j (ab)(a^k b^j)^{-1} | 0 \le k < 2m, 0 \le j < 2\} = \{a^{\pm 1} a^k b a^{-k} | 0 \le k < 2m\} = \{a^{2k \pm 1} b | 0 \le k < 2m\}$

Then from **fact 1**, our proof is completed by finding the remaining two equivalence classes as $\{a^{2k}b | 0 \le k < 2m\}$ and $\{a^{2k\pm1}b | 0 \le k < 2m\}$

Problem B.

Cycle type	(1,1,1,1,1)	(2,1,1,1)	(3,1,1)	(2,2,1)	(4,1)	(3,2)	(5)
Representative $x \in G$	e	(1,2)	(1,2,3)	(1,2)(3,4)	(1,2,3,4)	(1,2,3)(4,5)	(1,2,3,4,5
$\lvert x^G \rvert$	1	10	20	15	30	20	24
$\lvert C_G(x) \rvert$	120	12	6	8	4	6	3

Notice $(G : C_G(x)) = |x^G|$, so we only have to figure out $|x^G|$, then we can figure out $|C_G(x)| = \frac{|S_5|}{|x^G|} = \frac{120}{|x^G|}$

Now we give the reason of how we have the number for $|x^G|$ for each cycle type

Case: Cycle type $(1, 1, 1, 1, 1)$

Obviously e is the only choice, so $|x^G| = 1$

Case: Cycle type $(2, 1, 1, 1)$

$|x^G| = \binom{5}{2} = 10$

Case: Cycle type $(3, 1, 1)$

$|x^G| = \binom{5}{3} * 2 = 20$

2 stands for fixing one element and the two other elements can move freely

Case: Cycle type $(2, 2, 1)$

$|x^G| = \binom{5}{4} \frac{\binom{4}{2}}{2} = 15$

We pick 4 element for the two 2-cycle first, and pair them into two 2-cycle and have $\frac{\binom{4}{2}}{2}$ choices

Case: Cycle type $(4, 1)$

$|x^G| = \binom{5}{4} * 3! = 30$

$3!$ stands for fixing one element and the other 3 elements can move freely

Case: Cycle type $(3, 2)$

$|x^G| = \binom{5}{3} * 2 = 20$

2 stands for fixing one element in the 3-cycle and the other elements cam move freely

Case: Cycle type (5)

$|x^G| = 4! = 24$

$4!$ stands for fixing one element in the 4-cycle and the other elements can move freely