Representation Theory of finite groups

Assignment Set 3

Due Day: Mar 29 (or Mar 31 if you are using latex)

You may use the results obtained in your previous homework problems when you think they are helpful.

Problem A. (6pts) Let G be a finite group and let $\tau : G \to GL_2(\mathbb{C})$ be a representation of degree 2. Suppose that there exist elements $g, h \in G$ so that the matrices $\tau(g)$ and $\tau(h)$ do NOT commute. Prove that the representation τ is irreducible.

Problem B. Consider S_3 acting on $V = \mathbb{C}^3 = \text{span}\{e_1, e_2, e_3\}$, the permutation module. We know that $\text{span}\{e_1 + e_2 + e_3\}$ is a 1-dim submodule of V, and we extend it to obtain a basis $\alpha = \{e_1 + e_2 + e_3, e_2, e_3\}$ for V.

(a) (2pts) Write down the matrices $[g]_\alpha$ for all $g \in S_3$. Note that your matrices should all looks like

$$[g]_\beta = \left[\begin{array}{c|cc} * & * & * \\ \hline 0 & * & * \\ 0 & * & * \end{array}\right]$$

(b) (5pts) Find another basis β for V such that for every $g \in S_3$, the matrix $[g]_\beta$ looks like

$$[g]_\beta = \left[\begin{array}{c|cc} * & 0 & 0 \\ \hline 0 & * & * \\ 0 & * & * \end{array}\right] = \left[\begin{array}{c|cc} * & 0 & 0 \\ \hline 0 & * & * \\ 0 & * & * \end{array}\right]$$

You need to write down all of $[g]_\beta$.

(c) (3pts) Show that the map $\Psi : S_3 \to GL_2(\mathbb{C})$ defined by $g \mapsto Z_g$, where Z_g is the submatrix consisting of the second and the third rows and columns of $[g]_\beta$ you found in (b), gives an irreducible representation of S_3.

Problem C. In this problem we provide a counterexample of Maschke's Theorem when the group is infinite. Consider the function $\phi : \mathbb{Z} \to GL_2(\mathbb{C})$ given by

$$\phi(k) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, \qquad \forall k \in \mathbb{Z}.$$

(a) (2pts) Show that ϕ is a representation for the infinite group $\mathbb{Z}$ (Equivalently, you can prove that $\mathbb{C}^2$, viewed as column vectors, is a left $\mathbb{Z}$-module by the action induced from the representation)

(b) (3pts) Show that ϕ is reducible (there exists a sub-representation or submodule).

(c) (4pts) Show that ϕ is indecomposable.