1.

$\because \|x\| = 5$

$c_1^2 + c_2^2 + c_3^2 = 25$

$\because \langle u_1, x \rangle = 4$

$\langle u_1, c_1 u_1 + c_2 u_2 + c_3 u_3 \rangle$

$= \langle u_1, c_1 u_1 \rangle$

$\therefore c_1 = 4$ #

$\because x \perp u_2$

$c_2 = 0$ #

$\therefore \langle u_2, c_1 u_1 + c_2 u_2 + c_3 u_3 \rangle = 0$

$\because c_1^2 + c_2^2 + c_3^2 = 25$

$c_3 = \pm 3$ #

2.

a)

2.

(a)

$$a_1 = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$$

$$a_2 = \begin{bmatrix} 6 \\ -6 \\ 3 \end{bmatrix}$$

$$q_1 = \frac{1}{\|a_1\|} a_1 = \begin{pmatrix} \frac{-2}{3} \\ \frac{1}{3} \\ \frac{-2}{3} \end{pmatrix} \quad \#$$

$$q_2 = \frac{a_2 - \frac{a_2 \cdot a_1}{a_1 \cdot a_1} a_1}{\left\| a_2 - \frac{a_2 \cdot a_1}{a_1 \cdot a_1} a_1 \right\|} = \frac{\begin{bmatrix} 6 \\ -6 \\ 3 \end{bmatrix} - \frac{-24}{9} \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}}{\sim}$$

$$= \frac{\begin{bmatrix} \frac{2}{3} \\ \frac{-10}{3} \\ \frac{-7}{3} \end{bmatrix}}{\sim}$$

$$= \frac{3}{\sqrt{151}} \begin{bmatrix} \frac{2}{3} \\ \frac{-10}{3} \\ \frac{-7}{3} \end{bmatrix} \quad \#$$

$$A = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$$

$$r_{11} = \|a_1\| = 3 \quad \#$$

$$r_{22} = \left\| a_2 - \frac{a_2 \cdot a_1}{a_1 \cdot a_1} a_1 \right\| = \frac{\sqrt{151}}{3} \quad \#$$

$$r_{12} = \|a_1\| \frac{a_2 \cdot a_1}{a_1 \cdot a_1} = -8 \quad \#$$

3.

Say A is the $n \times n$ triangular matrix

$$\begin{bmatrix} d_1 & a_{12} & a_{13} & \cdots \\ 0 & d_2 & & \ddots \\ 0 & 0 & \ddots & \\ \vdots & & & d_n \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\therefore \left| \begin{bmatrix} d_1-\lambda & a_{12} & & \\ 0 & d_2-\lambda & & \\ 0 & 0 & \ddots & \\ & & & d_n-\lambda \end{bmatrix} \right| = 0$$

using cofactor formula on the first column

$$(d_1-\lambda) \left| \begin{bmatrix} d_2-\lambda & a_{23} & \cdots & \\ 0 & d_3-\lambda & & \\ \vdots & & \ddots & \\ & & & d_n-\lambda \end{bmatrix} \right| = 0$$

repeating the cofactor formula again and again on the new first column, we got

$$(d_1-\lambda)(d_2-\lambda)\cdots(d_n-\lambda) = 0$$

#

4.

$$\left| \begin{bmatrix} 2-\lambda & -8 \\ 1 & -4-\lambda \end{bmatrix} \right| = 0$$

$$\lambda^2 + 2\lambda = 0$$

$$\lambda = -2 \text{ or } 0$$

$$D = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$N(A - \lambda I) = \text{span}\{(4,1)\} \text{ or span}\{(2,1)\}$$

$$X = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \#$$