10

Irreducible modules and the group algebra

Let G be a finite group and $\mathbb{C}G$ be the group algebra of G over $\mathbb{C}$. Consider $\mathbb{C}G$ as the regular $\mathbb{C}G$-module. By Theorem 8.7, we can write

$$\mathbb{C}G = U_1 \oplus \ldots \oplus U_r$$

where each U_i is an irreducible $\mathbb{C}G$-module. We shall show in this chapter that *every* irreducible $\mathbb{C}G$-module is isomorphic to one of the $\mathbb{C}G$-modules $U_1, \ldots, U_r$. As a consequence, there are only finitely many non-isomorphic irreducible $\mathbb{C}G$-modules (a result which has already been established for abelian groups in Theorem 9.8). Also, in theory, to find all irreducible $\mathbb{C}G$-modules, it is sufficient to decompose $\mathbb{C}G$ as a direct sum of irreducible $\mathbb{C}G$-submodules. However, this is not really a practical way of finding the irreducible $\mathbb{C}G$-modules, unless G is a small group.

Irreducible submodules of $\mathbb{C}$G

We begin with another consequence of Maschke's Theorem.

10.1 Proposition
Let V and W be $\mathbb{C}G$-modules and let ϑ: $V \to W$ be a $\mathbb{C}G$-homomorphism. Then there is a $\mathbb{C}G$-submodule U of V such that $V =$ Ker $\vartheta \oplus U$ and $U \cong$ Im ϑ.

Proof Since Ker ϑ is a $\mathbb{C}G$-submodule of V by Proposition 7.2, there is by Maschke's Theorem a $\mathbb{C}G$-submodule U of V such that $V =$ Ker $\vartheta \oplus U$. Define a function $\bar{\vartheta}$: $U \to$ Im ϑ by

$$u\bar{\vartheta} = u\vartheta \quad (u \in U).$$

We show that $\overline{\vartheta}$ is a $\mathbb{C}G$-isomorphism from U to Im ϑ. Clearly $\overline{\vartheta}$ is a $\mathbb{C}G$-homomorphism, since ϑ is a $\mathbb{C}G$-homomorphism. If $u \in$ Ker $\overline{\vartheta}$ then $u \in$ Ker $\vartheta \cap U = \{0\}$; hence Ker $\overline{\vartheta} = \{0\}$. Now let $w \in$ Im ϑ; so $w = v\vartheta$ for some $v \in V$. Write $v = k + u$ with $k \in$ Ker ϑ, $u \in U$. Then

$$w = v\vartheta = k\vartheta + u\vartheta = u\vartheta = u\overline{\vartheta}.$$

Therefore Im $\overline{\vartheta}$ = Im ϑ. We have now established that $\overline{\vartheta}$: $U \to$ Im ϑ is an invertible $\mathbb{C}G$-homomorphism. Thus $U \cong$ Im ϑ, as required.

10.2 Proposition
Let V be a $\mathbb{C}G$-module, and write

$$V = U_1 \oplus \ldots \oplus U_s,$$

a direct sum of irreducible $\mathbb{C}G$-submodules U_i. If U is any irreducible $\mathbb{C}G$-submodule of V, then $U \cong U_i$ for some i.

Proof For $u \in U$, we have $u = u_1 + \ldots + u_s$ for unique vectors $u_i \in U_i$ $(1 \leqslant i \leqslant s)$. Define π_i: $U \to U_i$ by setting $u\pi_i = u_i$. Choosing i such that $u_i \neq 0$ for some $u \in U$, we have $\pi_i \neq 0$.

Now π_i is a $\mathbb{C}G$-homomorphism (see Proposition 7.11). As U and U_i are irreducible, and $\pi_i \neq 0$, Schur's Lemma 9.1(1) implies that π_i is a $\mathbb{C}G$-isomorphism. Therefore $U \cong U_i$.

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Of course it can happen that U is an irreducible $\mathbb{C}G$-submodule of $U_1 \oplus \ldots \oplus U_s$ (each U_i irreducible) without U being equal to any U_i, as the following example shows.

10.3 Example
Let G be any group and let V be a 2-dimensional $\mathbb{C}G$-module, with basis v_1, v_2, such that $vg = v$ for all $v \in V$ and $g \in G$. Then

$$V = U_1 \oplus U_2,$$

where U_1 = sp (v_1) and U_2 = sp (v_2) are irreducible $\mathbb{C}G$-submodules. However, U = sp ($v_1 + v_2$) is an irreducible $\mathbb{C}G$-submodule which is not equal to U_1 or U_2.

10.4 Definitions

(1) If V is a $\mathbb{C}G$-module and U is an irreducible $\mathbb{C}G$-module, then we say that U is a *composition factor* of V if V has a $\mathbb{C}G$-submodule which is isomorphic to U.

(2) Two $\mathbb{C}G$-modules V and W are said to have a *common composition factor* if there is an irreducible $\mathbb{C}G$-module which is a composition factor of both V and W.

We now come to the main result of the chapter, which shows that every irreducible $\mathbb{C}G$-module is a composition factor of the regular $\mathbb{C}G$-module.

10.5 Theorem

Regard $\mathbb{C}G$ as the regular $\mathbb{C}G$-module, and write

$$\mathbb{C}G = U_1 \oplus \ldots \oplus U_r,$$

a direct sum of irreducible $\mathbb{C}G$-submodules. Then every irreducible $\mathbb{C}G$-module is isomorphic to one of the $\mathbb{C}G$-modules U_i.

Proof Let W be an irreducible $\mathbb{C}G$-module, and choose a non-zero vector $w \in W$. Observe that $\{wr: r \in \mathbb{C}G\}$ is a $\mathbb{C}G$-submodule of W; since W is irreducible, it follows that

$$W = \{wr\colon\ r \in \mathbb{C}G\}. \tag{10.6}$$

Now define ϑ: $\mathbb{C}G \to W$ by

$$r\vartheta = wr \quad (r \in \mathbb{C}G).$$

Clearly ϑ is a linear transformation, and Im $\vartheta = W$ by (10.6). Moreover, ϑ is a $\mathbb{C}G$-homomorphism, since for $r, s \in \mathbb{C}G$,

$$(rs)\vartheta = w(rs) = (wr)s = (r\vartheta)s.$$

By Proposition 10.1, there is a $\mathbb{C}G$-submodule U of $\mathbb{C}G$ such that

$$\mathbb{C}G = U \oplus \operatorname{Ker} \vartheta \text{ and } U \cong \operatorname{Im} \vartheta = W.$$

As W is irreducible, so is U. By Proposition 10.2 we have $U \cong U_i$ for some i; then $W \cong U_i$, and the result is proved.

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Theorem 10.5 shows that there is a finite set of irreducible $\mathbb{C}G$-modules such that every irreducible $\mathbb{C}G$-module is isomorphic to one of them. We record this fact in the following corollary.

10.7 Corollary
If G is a finite group, then there are only finitely many non-isomorphic irreducible $\mathbb{C}G$-modules.

According to Theorem 10.5, to find all the irreducible $\mathbb{C}G$-modules we need only decompose the regular $\mathbb{C}G$-module as a direct sum of irreducible $\mathbb{C}G$-submodules. We now do this for a couple of examples; however, this is not a practical method for studying $\mathbb{C}G$-modules in general.

10.8 Examples
(1) Let $G = C_3 = \langle a : a^3 = 1 \rangle$, and write $\omega = e^{2\pi i/3}$. Define $v_0, v_1, v_2 \in \mathbb{C}G$ by

$$\begin{aligned} v_0 &= 1 + a + a^2, \\ v_1 &= 1 + \omega^2 a + \omega a^2, \\ v_2 &= 1 + \omega a + \omega^2 a^2, \end{aligned}$$

and let $U_i = \text{sp}\,(v_i)$ for $i = 0, 1, 2$. Then $v_1 a = a + \omega^2 a^2 + \omega 1 = \omega v_1$, and similarly

$$v_i a = \omega^i v_i \quad \text{for } i = 0, 1, 2.$$

Hence U_i is a $\mathbb{C}G$-submodule of $\mathbb{C}G$ for $i = 0, 1, 2$.

It is easy to check that v_0, v_1, v_2 is a basis of $\mathbb{C}G$, and hence

$$\mathbb{C}G = U_0 \oplus U_1 \oplus U_2,$$

a direct sum of irreducible $\mathbb{C}G$-submodules U_i. By Theorem 10.5, every irreducible $\mathbb{C}G$-module is isomorphic to U_0, U_1 or U_2. The irreducible representation of G corresponding to U_i is the representation ρ_{ω^i} of Example 9.9(1).

(2) Let $G = D_6 = \langle a, b\colon a^3 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$. We decompose $\mathbb{C}G$ as a direct sum of irreducible $\mathbb{C}G$-submodules. Let $\omega = e^{2\pi i/3}$ and define

$$\begin{aligned} v_0 &= 1 + a + a^2, & w_0 &= bv_0 \quad (= b + ba + ba^2), \\ v_1 &= 1 + \omega^2 a + \omega a^2, & w_1 &= bv_1, \\ v_2 &= 1 + \omega a + \omega^2 a^2, & w_2 &= bv_2. \end{aligned}$$

As in (1) above, $v_i a = \omega^i v_i$ for $i = 0, 1, 2$, and so $\text{sp}\,(v_i)$ and $\text{sp}\,(w_i)$ are $\mathbb{C}\langle a\rangle$-modules. Next, note that

$$\begin{aligned} v_0 b &= w_0, & w_0 b &= v_0, \\ v_1 b &= w_2, & w_1 b &= v_2, \\ v_2 b &= w_1, & w_2 b &= v_1. \end{aligned}$$

Therefore, $\text{sp}\,(v_0, w_0)$, $\text{sp}\,(v_1, w_2)$ and $\text{sp}\,(v_2, w_1)$ are $\mathbb{C}\langle b\rangle$-modules, and hence are $\mathbb{C}G$-submodules of $\mathbb{C}G$. By the argument in Example 5.5(2), the $\mathbb{C}G$-submodules $U_3 = \text{sp}\,(v_1, w_2)$ and $U_4 = \text{sp}\,(v_2, w_1)$ are irreducible.

However, $sp(v_0, w_0)$ is reducible, as $U_1 = sp(v_0 + w_0)$ and $U_2 = sp(v_0 - w_0)$ are $\mathbb{C}G$-submodules.

Now $v_0, v_1, v_2, w_0, w_1, w_2$ is a basis of $\mathbb{C}G$, and hence

$$\mathbb{C}G = U_1 \oplus U_2 \oplus U_3 \oplus U_4,$$

a direct sum of irreducible $\mathbb{C}G$-submodules. Note that U_1 is the trivial $\mathbb{C}G$-module, and U_1 is not isomorphic to U_2, the other 1-dimensional U_i. But $U_3 \cong U_4$ (there is a $\mathbb{C}G$-isomorphism sending $v_1 \to w_1, w_2 \to v_2$).

We conclude from Theorem 10.5 that there are exactly three non-isomorphic irreducible $\mathbb{C}G$-modules, namely U_1, U_2 and U_3. Correspondingly, every irreducible representation of D_6 over $\mathbb{C}$ is equivalent to precisely one of the following:

$$\rho_1\colon a \to (1),\ b \to (1);$$

$$\rho_2\colon a \to (1),\ b \to (-1);$$

$$\rho_3\colon a \to \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix},\ b \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Summary of Chapter 10

1. Every irreducible $\mathbb{C}G$-module occurs as a composition factor of the regular $\mathbb{C}G$-module.

2. There are only finitely many non-isomorphic irreducible $\mathbb{C}G$-modules.

Exercises for Chapter 10

1. Let G be a finite group. Find a $\mathbb{C}G$-submodule of $\mathbb{C}G$ which is isomorphic to the trivial $\mathbb{C}G$-module. Is there only one such $\mathbb{C}G$-submodule?

2. Let $G = C_4$. Express $\mathbb{C}G$ as a direct sum of irreducible $\mathbb{C}G$-submodules. (Hint: copy the method of Example 10.8(1).)

3. Let $G = D_8 = \langle a, b : a^4 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$. Find a 1-dimensional $\mathbb{C}G$-submodule, sp (u_1) say, of $\mathbb{C}G$ such that

$$u_1 a = u_1, \; u_1 b = -u_1.$$

Find also 1-dimensional $\mathbb{C}G$-submodules, sp (u_2) and sp (u_3), such that

$$u_2 a = -u_2, \quad u_2 b = u_2, \text{ and}$$

$$u_3 a = -u_3, \quad u_3 b = -u_3.$$

4. Use the method of Example 10.8(2) to find all the irreducible representations of D_8 over $\mathbb{C}$.

5. Suppose that V is a non-zero $\mathbb{C}G$-module such that $V = U_1 \oplus U_2$, where U_1 and U_2 are isomorphic $\mathbb{C}G$-modules. Show that there is a $\mathbb{C}G$-submodule U of V which is not equal to U_1 or U_2, but is isomorphic to both of them.

6. Let $G = Q_8 = \langle a, b : a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1}\rangle$, and let V be the $\mathbb{C}G$-module given in Example 4.5(2). Thus V has basis v_1, v_2 and

$$v_1 a = \mathrm{i} v_1, \qquad v_1 b = v_2,$$

$$v_2 a = -\mathrm{i} v_2, \quad v_2 b = -v_1.$$

Show that V is irreducible, and find a $\mathbb{C}G$-submodule of $\mathbb{C}G$ which is isomorphic to V.