Quiz 9, Advanced Calculus I, Yung Fu Fang

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Name: Id: *Group*

Riemann-Stieltjes integral of bounded function f with respect to α over the interval $[a, b]$

Let α be a monotonically increasing function on $[a, b]$.

Let $\mathcal{P} =$ $\{x_j \mid a = x_0 \le x_1 \le \cdots \le x_n = b,\ \forall 0 \le j \le n\}$ be a partition of $[a, b]$.

$\Delta\alpha_j =$ $\alpha(x_j) - \alpha(x_{j-1})$

Upper Sum $U(\mathcal{P}, f, \alpha) =$ $\sum_{j=1}^{n} M_j \Delta\alpha_j$, where $M_j =$ $\sup f(x) \quad (x_{j-1} \le x \le x_j)$.

Lower Sum $\mathcal{L}(\mathcal{P}, f, \alpha) =$ $\sum_{j=1}^{n} m_j \Delta\alpha_j$, where $m_j =$ $\inf f(x) \quad (x_{j-1} \le x \le x_j)$.

Upper Integral $\overline{\int}_a^b f(x)d\alpha =$ $\inf U(P, f)$.

Lower Integral $\underline{\int}_a^b f(x)d\alpha =$ $\sup L(P, f)$.

f is called Riemann-Stieltjes integrable if $\overline{\int}_a^b f(x)d\alpha$ $=$ $\underline{\int}_a^b f(x)d\alpha$, which is called the Riemann-Stieltjes integral of f with respect to α over $[a, b]$ and denoted by $\int_a^b f(x)d\alpha(x)$.

The partition $\mathcal{P}^*$ is a refinement of the partition $\mathcal{P}_2$ if $P^* \supset P_2$.

The partition $\mathcal{P}^*$ is a common refinement of the partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ if $P^* = P_1 \cup P_2$.

If $\mathcal{P}^*$ is a refinement of $\mathcal{P}$, then the relations for Upper Sums, Lower Sums, Upper Integral, and Lower Integral of f are

$$L(P, f, \alpha) \le L(P^*, f, \alpha) \le \underline{\int} f\, d\alpha \le \overline{\int} f\, d\alpha \le U(P^*, f, \alpha) \le U(P, f, \alpha).$$

If f is Riemann-Stieltjes integrable with respect to α over $[a, b]$, then for every $\varepsilon > 0$, there is a partition $\mathcal{P}$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.