(1) Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the map

$$(u, v, w) = F(x, y) = (x, y, xy).$$

Let $p = (x, y) \in \mathbb{R}^2$. Compute $F_*(\frac{\partial}{\partial x}|_p)$ as a linear combination of $\partial/\partial u$, $\partial/\partial v$ and $\partial/\partial w$ at $F(p)$.

(2) Let G be a Lie group with multiplication map $\mu : G \times G \to G$ and identity element e. Show that the differential at the identity of the multiplication map μ is addition:

$$\mu_{*,(e,e)} : T_eG \times T_eG \to T_eG, \qquad \mu_{*,(e,e)}(X_e, Y_e) = X_e + Y_e.$$

(Note that $T_{(p,q)}(M \times N)$ is isomorphic to $T_pM \times T_qN$ via the differentials of the two projections $\pi_1 : M \times N \to M$, $\pi_2 : M \times N \to N$.)

(3) Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ be the sphere in $\mathbb{R}^3$. Consider the function $h : S^2 \to \mathbb{R}$ given by $h(x, y, z) = z$. Find the critical points of h on S^2.

(4) A C^∞ map $f : M \to N$ is said to be transversal to a submanifold $S \subset N$ if for every $p \in f^{-1}(S)$,

$$f_*(T_pM) + T_{f(p)}S = T_{f(p)}N.$$

(If A and B are subspaces of a vector space, their sum $A+B$ is the subspace consisting of all $a+b$ with $a \in A$ and $b \in B$. The sum need not be a direct sum.) The goal of this exercise is to prove the *transversality theorem*: if a C^∞ map $f : M \to N$ is traversal to a regular submanifold S of codimension k in N, then $f^{-1}(S)$ is a regular submanifold of codimension k in M.

Let $p \in f^{-1}(S)$ and $(U, x^1, \dots, x^n)$ be an adapated chart centered at $f(p)$ for N relative to S such that $U \cap S = Z(x^{n-k+1}, \dots, x^n)$, the zero set of the functions $x^{n-k+1}, \dots, x^n$. Define $g : U \to \mathbb{R}^k$ to be the map

$$g = (x^{n-k+1}, \dots, x^n).$$

(a) Show that $f^{-1}(U) \cap f^{-1}(S) = (g \circ f)^{-1}(0)$.

(b) Show that $f^{-1}(U) \cap f^{-1}(S)$ is a regular level set of the function $g \circ f : f^{-1}(U) \to \mathbb{R}^k$.

(c) Prove the transversality theorem.