§1. Holomorphic functions

Let

$z_i := x_i + \sqrt{-1}\, y_i$, $i = 1, \cdots, n$ complex coordinates on $\mathbb{C}^n$

$\bar{z}_i := x_i - \sqrt{-1}\, y_i$, $i = 1, \cdots, n$ conjugate coordinates on $\mathbb{C}^n$

Define

$$\frac{\partial}{\partial z_i} := \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right), \quad \frac{\partial}{\partial \bar{z}_i} := \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right)$$

Let $U \subset \mathbb{C}^n$ be an open subset.

Def: A differentiable function $f : U \to \mathbb{C}$ is said to be *holomorphic* on U if

$$\frac{\partial f}{\partial \bar{z}_i} = 0$$

$\forall\, i = 1, \cdots, n$.

Rmk: If $f(x,y) = u(x,y) + \sqrt{-1}\, v(x,y)$,

$$\frac{\partial f}{\partial \bar{z}_i} = 0 \iff \frac{\partial u}{\partial x_i} = -\frac{\partial v}{\partial y_i} \quad \text{(Cauchy-Riemann equations)}$$

$$\iff f(z_1, \cdots, z_n) \text{ is complex differentiable in } z_i$$

Let $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n) \in \mathbb{R}^n_{>0}$ and

$$B_\varepsilon(p) = \{ z \in \mathbb{C}^n : |z_i - p_i| < \varepsilon_i \} \quad \text{(polydisk)}$$

Thm : [Cauchy's integral formula]

Let $f : \overline{B_\varepsilon(p)} \longrightarrow \mathbb{C}$ be continuous and holomorphic on $B_\varepsilon(p)$.

Then for any $z \in B_\varepsilon(p)$,

$$f(z) = \frac{1}{(2\pi\sqrt{-1})^n} \int_{|\zeta_1 - p_1| = \varepsilon_1} \cdots \int_{|\zeta_n - p_n| = \varepsilon_n} \frac{f(\zeta_1, \cdots, \zeta_n)}{(\zeta_1 - z_1)\cdots(\zeta_n - z_n)} d\zeta_1 \cdots d\zeta_n$$

Pf : Apply Cauchy's integral formula in one variable to each z_i. □

Cor : If $f : U \to \mathbb{C}$ is holomorphic, then $\forall p \in U$, $\exists B_\varepsilon(p) \subset U$ such that

$$f(z) = \sum_{i_1, \cdots, i_n \geq 0} a_{i_1 \cdots i_n} (z_1 - p_1)^{i_1} \cdots (z_n - p_n)^{i_n}$$

$\forall z \in B_\varepsilon(p)$, where

$$a_{i_1 \cdots i_n} = \frac{1}{i_1! \cdots i_n!} \frac{\partial^{i_1 + \cdots + i_n} f}{\partial z_{i_1} \cdots \partial z_{i_n}}(p)$$

Cor : [Maximal principle]

If $f : U \longrightarrow \mathbb{C}$ is holomorphic and non-constant, then $|f|$ has no maximum on U.

Cor : [Identity mapping theorem]

If $f : U \to \mathbb{C}$ is holomorphic and $f|_V = 0$ for some open subset $V \subset U$, then $f = 0$.

<u>Cor</u>: [Liouville theorem]

Every bounded holomorphic $f: \mathbb{C}^n \longrightarrow \mathbb{C}$ is constant.

Let $f: U \to \mathbb{C}$ be holomorphic. Define

$$Z(f) := \{ z \in U : f(z) = 0 \}.$$

<u>Thm</u>: [Riemann extension theorem]

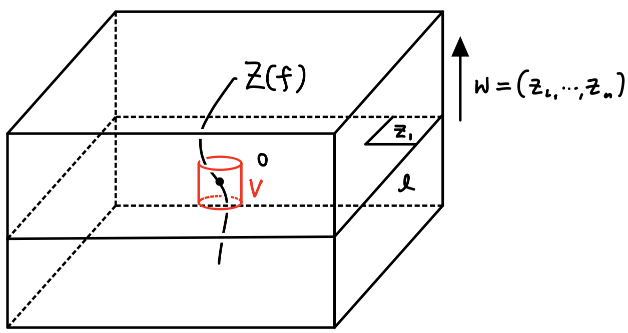
Let $f: U \longrightarrow \mathbb{C}$ be a non-constant holomorphic function.
If $g: U \setminus Z(f) \longrightarrow \mathbb{C}$ is holomorphic and locally bounded near $Z(f)$, then g can be extended to a holomorphic function $\tilde{g}: U \to \mathbb{C}$.

<u>Pf</u>: Since f is non-constant, for each point $p \in Z(f)$, there exist a line $\ell \subset \mathbb{C}^n$ such that $Z(f) \cap \ell = \{p\}$.

By a coordinate change, we assume

$$\ell = \{ (z_1, 0, \cdots, 0) \in B_\varepsilon(0) : z_1 \in \mathbb{C} \}$$

and $Z(f) \cap \ell = \{0\}$.



Thus there exist $\varepsilon_1, \cdots, \varepsilon_n > 0$ such that $f(z) \neq 0$ on

$$V = \{ z \in U : |z_1| = \varepsilon_1 \text{ and } |z_i| < \varepsilon_i \quad , i = 2, \cdots, n \}$$

By assumption, $g_w(z_1) := g(z_1, w)$ is bounded on $B_{\varepsilon_1}(0) \setminus Z(f)$

By the Riemann extension theorem in one variable, g_w can be extended to a holomorphic function $\tilde{g}_w : B_{\frac{\varepsilon_1}{2}}(0) \longrightarrow \mathbb{C}$ which is given by

$$\tilde{g}_w(z_1) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial B_{\varepsilon_1}(0)} \frac{g_w(\zeta)}{\zeta - z_1} \, d\zeta$$

Since the integrand is smooth on $\partial B_{\varepsilon_1}(0)$, we can differentiate under the integral sign to see that $\tilde{g}(z_1, w) := \tilde{g}_w(z_1)$ is holomorphic in (z_1, w). This gives an extension of g on the region enclosed by V. Doing this at every point on $Z(f)$ and applying the identity mapping theorem, we obtain a unique extension of g. $\square$

The following proposition is special for $n \geq 2$.

<u>Prop</u> [Hartog's theorem]

Suppose $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)$ and $\varepsilon' = (\varepsilon_1', \cdots, \varepsilon_n')$ are given such that $\varepsilon_i < \varepsilon_i' \ \forall \ i = 1, \cdots, n$. If $n > 1$, then any holomorphic function $f : B_\varepsilon(0) \setminus \overline{B_{\varepsilon'}(0)} \longrightarrow \mathbb{C}$ can be uniquely extended to $B_\varepsilon(0)$.

Pf: Let's assume $\varepsilon = (1, \cdots, 1)$. Note that

$$\mathbb{C}^n \setminus \overline{B_{\varepsilon'}(0)} = \{ z \mid |z_i| > \varepsilon'_i \text{ for some } i = 1, \cdots, n \}$$

We can then choose a small $\delta > 0$ s.t.

$B_\varepsilon(0) \setminus \overline{B_{\varepsilon'}(0)}$ contains the open set

$$V := \{ z \mid 1-\delta < |z_1| < 1,\ |z_{i\neq 1}| < 1 \} \cup \{ z \mid 1-\delta < |z_2| < 1,\ |z_{i \neq 2}| < 1 \}$$

Here we use $n > 1$

For fixed $w = (z_2, \cdots, z_n)$, $f_w(z_1) := f(z_1, w)$ is holomorphic on $1-\delta < |z_1| < 1$

$$\rightsquigarrow \quad f_w(z_1) = \sum_{k=-\infty}^{\infty} a_k(w) z^k \quad \text{(Laurent series)}$$

Cauchy integral formula

$$\Rightarrow \quad a_k(w) = \frac{1}{(2\pi\sqrt{-1})^k} \int_{|z_1| = 1-\frac{\delta}{2}} \frac{f_w(z_1)}{z_1^{k+1}} \, dz_1$$

Diffentiate under the integral sign w.r.t. w_i's, we see that $a_k(w)$ is holomorphic in on $\{ |z_{i\neq 1}| < 1 \}$.

On the other hand, when $1-\delta < |z_2| < 1$, $f_w(z_1)$ is holomorphic on $\{ |z_1| < 1 \}$

$$\Rightarrow \quad a_k(w) = 0 \quad \forall\, k < 0 \quad \text{when } 1-\delta < |z_2| < 1$$

Since $a_k(w)$ is holomorphic and vanishes on an open subset, identity principle implies a_k is identically zero.

We can then define the extension $\tilde{f} : B_\varepsilon(0) \to \mathbb{C}$ by

$$\tilde{f}(z_1, w) := \sum_{k=0}^{\infty} a_k(w) z_1^k$$

$\square$

Def: Let $U \subset \mathbb{C}^m$ be open. A map $F: U \longrightarrow \mathbb{C}^n$ is said to be holomorphic if all components $F_1, \cdots, F_n$ of F are holomorphic.

Def: A map $F: U \to V$ between two open subsets $U, V \subset \mathbb{C}^n$ is said to be biholomorphic if it is holomorphic and has a holomorphic inverse

Def: Let $F: U \subset \mathbb{C}^m \longrightarrow \mathbb{C}^n$ be holomorphic. The complex Jacobian is the matrix

$$J(F)(z) := \left(\frac{\partial F_i}{\partial z_j}(z) \right)_{\substack{i=1,\cdots,n \\ j=1,\cdots,m}}$$

Given a smooth $F: U \subset \mathbb{C}^m \to \mathbb{C}^n$. We have the differential $dF_z : T_z U \longrightarrow T_{F(z)} \mathbb{C}^n$. With respect to

$$\left\{ \frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_m}, \frac{\partial}{\partial \bar{z}_1}, \cdots, \frac{\partial}{\partial \bar{z}_m} \right\} \subset T_z U \otimes \mathbb{C}$$
$$\left\{ \frac{\partial}{\partial w_1}, \cdots, \frac{\partial}{\partial w_n}, \frac{\partial}{\partial \bar{w}_1}, \cdots, \frac{\partial}{\partial \bar{w}_n} \right\} \subset T_{F(z)} \mathbb{C}^n \otimes \mathbb{C}$$

The complexification $dF_{z,\mathbb{C}}$ of dF_z is given by the real Jacobian

$$J_{\mathbb{R}}(F) = \begin{pmatrix} \left(\frac{\partial F_i}{\partial z_j}\right)_{i,j=1,\cdots,n} & \left(\frac{\partial F_i}{\partial \bar{z}_j}\right)_{i,j=1,\cdots,n} \\ \left(\frac{\partial \bar{F}_i}{\partial z_j}\right)_{i,j=1,\cdots,n} & \left(\frac{\partial \bar{F}_i}{\partial \bar{z}_j}\right)_{i,j=1,\cdots,n} \end{pmatrix}$$

When F is holomorphic,

$$J_{\mathbb{R}}(F) = \begin{pmatrix} J(F) & \\ & \overline{J(F)} \end{pmatrix}$$

In particular, $\det(J_{\mathbb{R}}(F)) = |\det(J(F))| \geq 0$

$\Rightarrow F : U \longrightarrow \mathbb{C}^m$ is orientation preserving

The holomorphic version of inverse and implicit function theorem hold. However, in contrast to the smooth world, we have

<u>Prop</u>: If $F : U \to V$ is a holomorphic bijection between two open subsets $U, V \subset \mathbb{C}^n$, then $F^{-1} : V \longrightarrow U$ is holomorphic.

<u>Pf</u>: We prove by induction on $n = \dim_{\mathbb{C}}(U) = \dim_{\mathbb{C}}(V)$. □

For $n=1$. If $f'(z_0) = 0$ for some $z_0 \in U$, by a coordinate change, we may assume $f(0) = f'(0) = 0$.

Then $f(z) = z^d h(z)$ for some $d > 1$ and non-vanishing h.

This contradict injectivity of f.

Assume this has been proven for all $k < n$.

Let $z \in U$ be such that $\det(J(f)(z)) = 0$. We claim that $J(f)(z) = 0$. Suppose not. Then $\exists z \in U$ such that $rk(J(F)(z)) = k < n$. We may assume $\left(\frac{\partial F_i}{\partial z_j}(z)\right)_{i,j=1,\cdots,k}$ is invertible. By inverse function theorem,

$$\tilde{z}_i := \begin{cases} F_i(z) & i = 1, \cdots, k \\ z_i & i = k+1, \cdots, n \end{cases}$$

give local coordinates around $z \in U$. The map F maps $U' = \{\tilde{z} \mid \tilde{z}_i = 0, i = 1, \cdots, k\} \cap U$ bijectively onto $V' = \{w \mid w_i = 0, i = 1, \cdots, k\} \cap V$ which both have $\dim < n$.

However, $J(F|_{U'})(z)$ is singular and this contradicts the induction hypothesis. Hence $J(F)(z) = 0$.

Suppose now $\exists$ a regular point z of

$$g := \det(J(f)) : U \longrightarrow \mathbb{C}$$

on the fiber $g^{-1}(0)$. By implicit function theorem, there is a neighbourhood $W \subset g^{-1}(0)$ of z such that W is biholomorphic to an open subset of $\mathbb{C}^{n-1}$. But $f|_W : W \to \mathbb{C}^n$ has vanishing Jacobian, meaning $f|_W$ is constant, Contradicting injectivity of F. $\square$