5.7 HW7

Question 73

1. a. Show that if a curve $C \subset S$ is both a line of curvature and a geodesic, then C is a plane curve.

b. Show that if a (nonrectilinear) geodesic is a plane curve, then it is a line of curvature.

c. Give an example of a line of curvature which is a plane curve and not a geodesic.

Proof. **(a)** We are required to show

$$\tau \text{ is 0 everywhere on } C$$

Frenet Equations give us

$$T' = \kappa_C N_C \text{ where } \kappa_C, N_C \text{ are the curvature and the normal of } C$$

Because C is a geodesic, we know N_C is parallel with N, where N is the normal of S. WOLG, we can let $N_C = N$. Now, because C is a line of curvature, we know

$$N'_C = N' = dN(\alpha') = \lambda T$$

where λ is the principal curvature and α is some arc-length parametrization of C.

Now by Frenet Equations, we have

$$\lambda T = N'_C = -\kappa_C T - \tau B$$

It then follows that $\tau = 0$. (done)

(b) Let α be an arc-length parametrization of C. Because C is a geodesic, again WOLG, we can let $N_C = N$. Then by Frenet equations, we have

$$\begin{aligned} dN(T) = N' = (N_C)' &= \kappa_C T + \tau B \\ &= \kappa_C T \quad (\tau \text{ is 0, since } C \text{ is plane curve }) \end{aligned}$$

This implies that $\alpha' = T$ is an eigenvalue of dN, which implies C is a line of curvature.

(c) Consider

$$C \triangleq S^2 \cap \{(x,y,\sqrt{2}) \in \mathbb{R}^3 : (x,y) \in \mathbb{R}^2\}$$

C is a line of curvature, since every direction is principal direction on S^2.

C is not a geodesic, since the only geodesic on S^2 is the great circles, while C isn't.

To see that only great circles on S^2 are geodesics, one note that given $p \in S$ and $w \in T_pS$, there exists only one geodesic passing through p with direction $\frac{w}{|w|}$. ■

Question 74

7. Intersect the cylinder $x^2 + y^2 = 1$ with a plane passing through the x axis and making an angle θ, $0 < \theta < \pi/2$, with the xy plane.

a. Show that the intersecting curve is an ellipse C.

b. Compute the absolute value of the geodesic curvature of C in the cylinder at the points where C meets their principal axes.

Proof. **(a)** The plane can be characterized by $z = \tan\theta y$. Then C can be characterized by

$$\begin{cases} x^2 + y^2 = 1 \\ z = (\tan\theta)y \end{cases}$$

Clearly, we can parametrized C by

$$\alpha(t) = \Big(\cos t, \sin t, \tan\theta \sin t\Big)$$

Observe

$$\alpha(t) = (\cos t)v + (\sin t)w \text{ where } v = (1,0,0) \text{ and } w = (0,1,\tan\theta)$$

This conclude that C is an ellipse.

(b) WOLG, we only have to compute κ_g for $\alpha(0)$ and $\alpha(\frac{\pi}{2})$.

Compute

$$\begin{cases} \alpha'(t) = \Big(-\sin t, \cos t, \tan\theta \cos t \Big) \\ \alpha''(t) = \Big(-\cos t, -\sin t, -\tan\theta \sin t \Big) \\ |\alpha' \times \alpha''| = \sec\theta \\ \kappa_C = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} \\ \kappa_C(0) = \cos^2\theta \text{ and } \kappa_C(\frac{\pi}{2}) = \sec\theta \\ \alpha'(0) = (0, 1, \tan\theta) = \sec\theta(0, \cos\theta, \sin\theta) \\ \alpha'(\frac{\pi}{2}) = (-1, 0, 0) \end{cases}$$

It is easily checked that the principal curvatures and directions at $\alpha(0) = (1, 0, 0)$ are

$$1 \text{ relative to } (0, 1, 0) \text{ and } 0 \text{ relative to } (0, 0, 1)$$

This implies κ_n at $\alpha(0)$ is $\cos^2\theta$, which implies $|\kappa_g| = \sqrt{\kappa_C^2 - \kappa_n^2} = 0$ at $\alpha(0)$.

Similarly, it is easily checked that the principal curvatures and directions at $\alpha(\frac{\pi}{2}) = (0, 1, \tan\theta)$ are

$$1 \text{ relative to } (1, 0, 0) \text{ and } 0 \text{ relative to } (0, 0, 1)$$

This implies κ_n at $\alpha(\frac{\pi}{2})$ is 1, which implies $|\kappa_g| = \sqrt{\kappa_C^2 - \kappa_n^2} = \sqrt{\sec^2\theta - 1} = \tan\theta$. ■

Question 75

***9.** Consider two meridians of a sphere C_1 and C_2 which make an angle φ at the point p_1. Take the parallel transport of the tangent vector w_0 of C_1, along C_1 and C_2, from the initial point p_1 to the point p_2 where the two

meridians meet again, obtaining, respectively, w_1 and w_2. Compute the angle from w_1 to w_2.

Proof. Because meridians are geodesic, we can speak of the following parametrization.

Let $\alpha : [0, l] \to S$ be the geodesic parametrization from p_1 to p_2 along C_1, and let $\overline{\alpha} : [0, l] \to S$ be the geodesic parametrization from p_1 to p_2 along C_2. Note that the angle between $\alpha'(0)$ and $(\overline{\alpha})'(0)$ is given ϕ by premise.

WLOG, we now write

$$\begin{aligned} w_0 &= \cos\theta_0 e_1 + \sin\theta e_2 \\ \alpha'(0) &= \cos(\theta_0 + \psi_0)e_1 + \sin(\theta_0 + \psi_0)e_2 \\ (\overline{\alpha})'(0) &= \cos(\theta_0 + \psi_0 + \phi_0)e_1 + \sin(\theta_0 + \psi_0 + \phi_0)e_2 \end{aligned}$$

where $e_1 = (1,0,0), e_2 = (0,1,0)$ and p_1 is the north pole.

Because parallel transport along geodesics preserves the angle between the vector and the speed of the geodesic, we know

$$\text{the angle from } w_1 \text{ to } \alpha'(l) \text{ is still } \psi_0$$

and

$$\text{the angle from } w_2 \text{ to } (\overline{\alpha})'(l) \text{ is still } \psi_0 + \phi_0$$

It is clear that

$$\begin{aligned} \alpha'(l) &= -\alpha'(0) = \cos(\theta_0 + \psi_0 + \pi)e_1 + \sin(\theta_0 + \psi_0 + \pi)e_2 \\ (\overline{\alpha})'(l) &= -(\overline{\alpha})'(0) = \cos(\theta_0 + \psi_0 + \phi_0 + \pi)e_1 + \sin(\theta_0 + \psi_0 + \phi_0 + \pi)e_2 \end{aligned}$$

This give us

$$\begin{aligned} w_1 &= \cos(\theta_0 + 2\psi_0 + \pi)e_1 + \sin(\theta_0 + 2\psi_0 + \pi)e_2 \\ w_2 &= \cos(\theta_0 + 2\psi_0 + 2\phi_0 + \pi)e_1 + \sin(\theta_0 + 2\psi_0 + 2\phi_0 + \pi)e_2 \end{aligned}$$

Then the angle from w_1 to w_2 is $2\phi_0$, where ϕ_0 is the angle C_1, C_2 make at the north pole p_1. ■

Question 76

***10. Show that the geodesic curvature of an oriented curve $C \subset S$ at a point $p \in C$ is equal to the curvature of the plane curve obtained by projecting C onto the tangent plane $T_p(S)$ along the normal to the surface at p.**

Proof. Let α be a geodesic parametrization of C around $\alpha(0) = p$. The orthogonal projection β of α onto T_pS is

$$\beta(s) = \alpha(s) + \langle p - \alpha(s), N(p)\rangle N(p)$$

where $N(p)$ is the normal of S at p. Compute

$$\beta'(s) = \alpha'(s) - \langle \alpha'(s), N(p)\rangle N(p)$$

Compute

$$\beta''(s) = \alpha''(s) - \langle \alpha''(s), N(p) \rangle N(p)$$

This give us

$$\beta'(0) = \alpha'(0) \text{ and } \beta''(0) = \frac{D\alpha'}{ds}(0)$$

Then because $|\beta'(0)| = |\alpha'(0)| = 1$, the curvature κ of β at p is then

$$\begin{aligned}
\kappa = \frac{|\beta'(0) \times \beta''(0)|}{|\beta'(0)|^3} = \left| \alpha'(0) \times \frac{D\alpha'}{ds}(0) \right| &= |\alpha'(0)| \cdot \left| \frac{D\alpha'}{ds}(0) \right| \sin\theta \\
&= \left| \frac{D\alpha'}{ds}(0) \right| \sin\theta
\end{aligned}$$

where θ is the angle between $\alpha'(0)$ and $\frac{D\alpha'}{ds}(0)$. On the other hand, we know

$$\begin{aligned}
\kappa_g(p) &= \langle \frac{D\alpha'}{ds}(0), N(p) \times \alpha'(0) \rangle \\
&= \left| \frac{D\alpha'}{ds}(0) \right| \cdot |N(p) \times \alpha'(0)| \cos\phi \\
&= \left| \frac{D\alpha'}{ds}(0) \right| \cos\phi
\end{aligned}$$

where ϕ is the angle between $\frac{D\alpha'}{ds}(0)$ and $N(p) \times \alpha'(0)$.

Note that $N \times \alpha'(0), \alpha'(0), \frac{D\alpha'}{ds}(0)$ are all in T_pS, and the angle from $\alpha'(0)$ to $N \times \alpha'(0)$ is $\frac{\pi}{2}$. This implies $\theta = \phi + \frac{\pi}{2}$, and conclude the result. ■

Question 77

14. Let S be an oriented regular surface and let $\alpha \colon I \to S$ be a curve parametrized by arc length. At the point $p = \alpha(s)$ consider the three unit vectors (the *Darboux trihedron*) $T(s) = \alpha'(s)$, $N(s) =$ the normal vector to S at p, $V(s) = N(s) \wedge T(s)$. Show tbat

$$\frac{dT}{ds} = 0 + aV + bN,$$
$$\frac{dV}{ds} = -aT + 0 + cN,$$
$$\frac{dN}{ds} = -bT - cV + 0,$$

where $a = a(s)$, $b = b(s)$, $c = c(s)$, $s \in I$. The above formulas are the analogues of Frenet's formulas for the trihedron T, V, N. To establish the geometrical meaning of the coefficients, prove that

a. $c = -\langle dN/ds, V \rangle$; conclude from this that $\alpha(I) \subset S$ is a line of curvature if and only if $c \equiv 0$ ($-c$ is called the *geodesic torsion* of α; cf. Exercise 19, Sec. 3-2).

b. b is the normal curvature of $\alpha(I) \subset S$ at p.

c. a is the geodesic curvature of $\alpha(I) \subset S$ at p.

Proof. It is clear that $T, N, V = N \times T$ form an orthonormal basis. Write

$$\begin{bmatrix} T' \\ V' \\ N' \end{bmatrix} = M \begin{bmatrix} T \\ V \\ N \end{bmatrix}$$

where M is a 3×3-matrix, and we are required to prove

(a) $M_{k,k} = 0$ for all k

(b) $M_{i,j} = -M_{j,i}$ for all i, j

Note that $M_{1,1} = T' \times T$, $M_{2,2} = V' \times V$, $M_{3,3} = N' \times N$. (a) follows from the fact T, V, M

are all unit.

Because T, V, M are orthogonal, we know

$$M_{1,2} = T' \cdot V \text{ and } M_{2,1} = V' \cdot T$$
$$M_{1,3} = T' \cdot V \text{ and } M_{3,1} = N' \cdot T$$
$$M_{2,3} = V' \cdot N \text{ and } M_{3,2} = N' \cdot V$$

(b) then follows from T, V, M are orthogonal, and the fact $(w_1 \cdot w_2)' = w_1' \cdot w_2 + w_1 \cdot w_2'$.

(a) We know $\alpha(I)$ is a line of curvature if and only if N' is parallel with T everywhere. It follows from $N' = -bT - cV$ that $c \equiv 0$ if and only if $\alpha(I)$ is a line of curvature.

(b) We know

$$\kappa_\alpha N_\alpha = \frac{dT}{ds} = aV + bN$$

Then the normal curvature κ_n is

$$\kappa_n = \kappa_\alpha \langle N_\alpha, N \rangle = b$$

(c) Note that

$$\frac{d\alpha'}{ds} = bN + aN \times T$$

This give us

$$\frac{D\alpha'}{ds} = aN \times T = aN \times \alpha'$$

which implies a is the geodesic curvature.

Question 78

17. Let $\alpha: I \to R^3$ be a curve parametrized by arc length s, with nonzero curvature and torsion. Consider the parametrized surface (Sec. 2-3)

$$\mathbf{x}(s, v) = \alpha(s) + vb(s), \quad s \in I, -\epsilon < v < \epsilon, \epsilon > 0,$$

where b is the binormal vector of α. Prove that if ϵ is small, $\mathbf{x}(I \times (-\epsilon, \epsilon)) = S$ is a regular surface over which $\alpha(I)$ is a geodesic (*thus, every curve is a geodesic on the surface generated by its binormals*).

Proof. Compute, using Frenet Equations

$$\mathbf{x}_s = T + v\tau N_\alpha$$
$$\mathbf{x}_v = B$$

This give us

$$\mathbf{x}_s \times \mathbf{x}_v = -N_\alpha - v\tau T$$

which give us

$$N(s, 0) = N_\alpha$$

This implies $\alpha(I)$ has the same normal as S, which implies $\alpha(I)$ is a geodesic. ■

Question 79

1. Let $S \subset R^3$ be a regular, compact, connected, orientable surface which is not homeomorphic to a sphere. Prove that there are points on S where the Gaussian curvature is positive, negative, and zero.

Proof. Because S is not homeomorphic to a sphere, we know $\iint_S K d\sigma = 2\pi\chi(S) \leq 0$. Then the proof reduce to proving

$$S \text{ is elliptic at some } p \in S$$

Let q be an arbitrary point in $\mathbb{R}^3$. Note that the function $f : \mathbb{R}^3 \to \mathbb{R}_0^+$ defined by

$$f(p) \triangleq |p - q|$$

is continuous. Then because S is compact, we know f attain a maximum r_0 at some $p \in S$. Denote the sphere centering q with radius $r \in \mathbb{R}^+$ by $S^2(r)$. Note that $S^2(r_0) = f^{-1}(r_0)$. This implies that S is "contained" in $S^2(r_0)$ and

$$p \in S \cap S^2(r_0)$$

Now, given an arbitrary normal section C of S at p. By Section 1.7, Question 4, we see C must have normal curvature greater than $r_0 > 0$ at p. (Note that if $r_0 = 0$, then $S = \{q\}$, which is not a regular surface.) Because C is arbitrary, we now see that the two principal curvature must be greater than $r_0 > 0$. This implies $K > 0$. (done)

■

Question 80

2. Let T be a torus of revolution. Describe the image of the Gauss map of T and show, without using the Gauss-Bonnet theorem, that

$$\iint_T K\, d\sigma = 0.$$

Compute the Euler-Poincaré characteristic of T and check the above result with the Gauss-Bonnet theorem.

Proof. We are given the standard chart

$$\mathbf{x}(u, v) = \Big((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u\Big)$$

Some messy computation give us

$$N(u, v) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|} = (\cos u\cos v, \cos u\sin v, \sin u)$$

For each $\mathbf{x}(u_0, v_0) \in T$, there exists a circle

$$C = \{\mathbf{x}(u, v_0) : u \in [0, 2\pi]\}$$

containing $\mathbf{x}(u_0, v_0)$, and one can check that T is just the normal of such C.

Observe

$$\begin{aligned}
\frac{d}{du}\mathbf{x}(u, v_0) &= \Big(- r \sin u \cos v_0, -r \sin u \sin v_0, r \cos u \Big) \\
\frac{d}{du}N(u, v_0) &= \Big(- \sin u \cos v_0, - \sin u \sin v_0, \cos u \Big)
\end{aligned}$$

This then implies for all $\mathbf{x}(u, v) \in S$, one of the principal curvature is $\frac{1}{r}$.

Observe

$$\begin{aligned}
\frac{d}{dv}\mathbf{x}(u_0, v) &= \Big(- (a + r \cos u_0) \sin v, (a + r \cos u_0) \cos v, 0 \Big) \\
\frac{d}{dv}N(u_0, v) &= \Big(- \cos u_0 \sin v, \cos u_0 \cos v, 0 \Big)
\end{aligned}$$

Then then implies for all $\mathbf{x}(u, v)$, another principal curvature is $\frac{\cos u}{a + r \cos u}$. We now have

$$K(u, v) = \frac{\cos u}{r(a + r \cos u)}$$

Compute

$$\begin{aligned}
\mathbf{x}_u &= (-r \sin u \cos v, -r \sin u \sin v, r \cos u) \\
\mathbf{x}_v &= (-(a + r \cos u) \sin v, (a + r \cos u) \cos v, 0) \\
E &= r^2 \text{ and } F = 0 \text{ and } G = (a + r \cos u)^2 \\
EG - F^2 &= r^2 (a + r \cos u)^2
\end{aligned}$$

Note the symmetry

$$\begin{aligned}
K(\frac{\pi}{2} - t, v) &= -K(\frac{\pi}{2} + t, v) \\
K(\frac{-\pi}{2} - t, v) &= -K(\frac{-\pi}{2} + t, v) \\
(EG - F^2)(\frac{\pi}{2} - t, v) &= (EG - F^2)(\frac{\pi}{2} + t, v) \\
(EG - F^2)(\frac{-\pi}{2} - t, v) &= (EG - F^2)(\frac{-\pi}{2} + t, v) \text{ for all } t \in [0, \frac{\pi}{2}], v \in [0, 2\pi]
\end{aligned}$$

This give

$$\iint_R K d\sigma = \iint_{[0,2\pi]^2} K(u,v)\sqrt{EG-F^2}dudv = 0$$

It now follows from Gauss-Bonnet that $\chi(T) = 0$. ■

Question 81

4. Compute the Euler-Poincaré characteristic of

a. An ellipsoid.

***b.** The surface $S = \{(x, y, z) \in R^3; x^2 + y^{10} + z^6 = 1\}$.

Proof. **(a)** Given ellipsoid S

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

We can map S to S^2 by

$$f(x,y,z) = (ax, by, cz)$$

It is clear that $f : S \to S^2$ is continuous, one-to-one and onto, and admits an continuous inverse

$$f^{-1}(x,y,z) = (\frac{x}{a}, \frac{y}{b}, \frac{c}{z})$$

It is now established that f is a homeomorphism between S and S^2. Then we see ellipsoid have the same Euler-Poincare characteristic as S^2, i.e. 2.

To see $\chi(S^2) = 2$, one can use the triangulation $\{T_1, \dots, T_8\}$, in which each T_k is the intersection between one of the octant and S^2. We then have $F = 8$, $E = 12$, $V = 6$, so $\chi(S^2) = 8 - 12 + 6 = 2$.

(b) Map S to S^2 by

$$f(x,y,z) = (x, y^5, z^3)$$

It is clear that f continuous, one-to-one and onto, and admits a continuous inverse

$$f^{-1}(x,y,z) = \left(x, \begin{cases} y^{\frac{1}{5}} & \text{if } y \geq 0 \\ -(-y)^{\frac{1}{5}} & \text{if } y < 0 \end{cases}, \begin{cases} z^{\frac{1}{3}} & \text{if } z \geq 0 \\ -(-z)^{\frac{1}{3}} & \text{if } z < 0 \end{cases} \right)$$

It is now established that f is a homeomorphism between S and S^2. Then we see $\chi(S) = \chi(S^2) = 2$ ■