8

Maschke's Theorem

We now come to our first major result in representation theory, namely Maschke's Theorem. A consequence of this theorem is that every FG-module is a direct sum of irreducible FG-submodules, where as usual $F = \mathbb{R}$ or $\mathbb{C}$. (The assumption on F is important – see Example 8.2(2) below.) This essentially reduces representation theory to the study of irreducible FG-modules.

Maschke's Theorem

8.1 Maschke's Theorem
Let G be a finite group, let F be $\mathbb{R}$ or $\mathbb{C}$, and let V be an FG-module. If U is an FG-submodule of V, then there is an FG-submodule W of V such that

$$V = U \oplus W.$$

Before proving Maschke's Theorem, we illustrate it with some examples.

8.2 Examples
(1) Let $G = S_3$ and let $V = \text{sp}\,(v_1, v_2, v_3)$ be the permutation module for G over F (see Definition 4.10). Put

$$u = v_1 + v_2 + v_3 \text{ and } U = \text{sp}\,(u).$$

Then U is an FG-submodule of V, since $ug = u$ for all $g \in G$.

There are many *subspaces* W of V such that $V = U \oplus W$, for instance sp (v_2, v_3) and sp $(v_1, v_2 - 2v_3)$. But there is, in fact, only one *FG-submodule W* of V with $V = U \oplus W$. We shall find this W in an example after proving Maschke's Theorem (but you may like to look for it yourself now).

(2) The conclusion of Maschke's Theorem can fail if F is not $\mathbb{R}$ or $\mathbb{C}$. For example, let p be a prime number, let $G = C_p = \langle a: a^p = 1\rangle$, and take F to be the field of integers modulo p. Check that the function

$$a^j \to \begin{pmatrix} 1 & 0 \\ j & 1 \end{pmatrix} \quad (j = 0, 1, \ldots, p-1)$$

is a representation from G to GL $(2, F)$. The corresponding FG-module is $V = \mathrm{sp}\,(v_1, v_2)$, where, for $0 \leqslant j \leqslant p-1$,

$$\begin{aligned} v_1 a^j &= v_1, \\ v_2 a^j &= j v_1 + v_2. \end{aligned}$$

Clearly, $U = \mathrm{sp}(v_1)$ is an FG-submodule of V. But there is no FG-submodule W such that $V = U \oplus W$, since U is the only 1-dimensional FG-submodule of V, as can easily be seen.

Proof of Maschke's Theorem 8.1 We are given U, an FG-submodule of the FG-module V. Choose any *subspace* W_0 of V such that

$$V = U \oplus W_0.$$

(There are many choices for W_0 – simply take a basis $v_1, \ldots, v_m$ of U, extend it to a basis $v_1, \ldots, v_n$ of V, and let $W_0 = \mathrm{sp}\,(v_{m+1}, \ldots, v_n)$.)

For $v \in V$, we have $v = u + w$ for unique vectors $u \in U$ and $w \in W_0$, and we define $\phi: V \to V$ by setting $v\phi = u$. By Proposition 2.29, ϕ is a projection of V with kernel W_0 and image U.

We aim to modify the projection ϕ to create an FG-homomorphism from V to V with image U. To this end, define $\vartheta: V \to V$ by

(8.3) $$v\vartheta = \frac{1}{|G|} \sum_{g \in G} v g \phi g^{-1} \quad (v \in V).$$

It is clear that ϑ is an endomorphism of V and Im $\vartheta \subseteq U$.

We show first that ϑ is an FG-homomorphism. For $v \in V$ and $x \in G$,

$$(vx)\vartheta = \frac{1}{|G|}\sum_{g\in G}(vx)g\phi g^{-1}.$$

As g runs over the elements of G, so does $h = xg$. Hence

$$\begin{aligned}(vx)\vartheta &= \frac{1}{|G|}\sum_{h\in G} vh\phi h^{-1}x \\ &= \left(\frac{1}{|G|}\sum_{h\in G} vh\phi h^{-1}\right)x \\ &= (v\vartheta)x.\end{aligned}$$

Thus ϑ is an FG-homomorphism.

Next, we prove that $\vartheta^2 = \vartheta$. First note that for $u \in U$, $g \in G$, we have $ug \in U$, and so $(ug)\phi = ug$. Using this,

(8.4)

$$u\vartheta = \frac{1}{|G|}\sum_{g\in G} ug\phi g^{-1} = \frac{1}{|G|}\sum_{g\in G}(ug)g^{-1} = \frac{1}{|G|}\sum_{g\in G} u = u.$$

Now let $v \in V$. Then $v\vartheta \in U$, so by (8.4) we have $(v\vartheta)\vartheta = v\vartheta$. Consequently $\vartheta^2 = \vartheta$, as claimed.

We have now established that ϑ: $V \to V$ is a projection and an FG-homomorphism. Moreover, (8.4) shows that Im $\vartheta = U$. Let W = Ker ϑ. Then W is an FG-submodule of V by Proposition 7.2, and $V = U \oplus W$ by Proposition 2.32.

This completes the proof of Maschke's Theorem.

■

8.5 Example

Let $G = S_3$ and let $V = \text{sp}(v_1, v_2, v_3)$ be the permutation module, with submodule $U = \text{sp}(v_1 + v_2 + v_3)$, as in Example 8.2(1). We use the proof of Maschke's Theorem to find an FG-submodule W of V such that $V = U \oplus W$.

First, let $W_0 = \text{sp}(v_1, v_2)$. Then $V = U \oplus W_0$ (but of course W_0 is not an FG-submodule). The projection ϕ onto U is given by

$$\phi: v_1 \to 0,\ v_2 \to 0,\ v_3 \to v_1 + v_2 + v_3.$$

Check now that the FG-homomorphism ϑ given by (8.3) is

$$\vartheta: v_i \to \tfrac{1}{3}(v_1 + v_2 + v_3) \quad (i = 1, 2, 3).$$

The required FG-submodule W is then Ker ϑ, so

$$W = \text{sp}(v_1 - v_2, v_2 - v_3).$$

(In fact, $W = \{\sum \lambda_i v_i : \sum \lambda_i = 0\}$, the FG-submodule constructed in Example 7.3(3).)

Note that if $\mathscr{B}$ is the basis $v_1 + v_2 + v_3, v_1, v_2$ of V, then for all $g \in G$, the matrix $[g]_{\mathscr{B}}$ has the form

$$[g]_{\mathscr{B}} = \begin{pmatrix} \blacksquare & 0 & 0 \\ \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{pmatrix}.$$

The zeros reflect the fact that U is an FG-submodule of V (see (5.4)). If instead we use $v_1 + v_2 + v_3, v_1 - v_2, v_2 - v_3$ as a basis $\mathscr{B}'$, then we get

$$[g]_{\mathscr{B}'} = \begin{pmatrix} \blacksquare & 0 & 0 \\ 0 & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare \end{pmatrix},$$

because $\text{sp}(v_1 - v_2, v_2 - v_3)$ is also an FG-submodule of V.

This example illustrates the matrix version of Maschke's Theorem: for an arbitrary finite group G, if we can choose a basis $\mathscr{B}$ of an FG-module V such that $[g]_{\mathscr{B}}$ has the form

$$\left(\begin{array}{c|c} * & 0 \\ \hline * & * \end{array}\right)$$

for all $g \in G$ (see (5.4)), then we can find a basis $\mathscr{B}'$ such that $[g]_{\mathscr{B}'}$ has the form

$$\left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$$

for all $g \in G$.

To put this another way, suppose that ρ is a reducible representation of a finite group G over F of degree n. Then we know that ρ is equivalent to a representation of the form

$$g \to \left(\begin{array}{c|c} X_g & 0 \\ \hline Y_g & Z_g \end{array}\right) \quad (g \in G),$$

for some matrices X_g, Y_g, Z_g, where X_g is $k \times k$ with $0 < k < n$.

Maschke's Theorem asserts further that ρ is equivalent to a representation of the form

$$g \to \left(\begin{array}{c|c} A_g & 0 \\ \hline 0 & B_g \end{array}\right),$$

where A_g is also a $k \times k$ matrix.

Consequences of Maschke's Theorem

We now use Maschke's Theorem to show that every non-zero FG-module is a direct sum of irreducible FG-submodules. (By an irreducible FG-submodule, we simply mean an FG-submodule which is an irreducible FG-module.)

8.6 Definition
An FG-module V is said to be *completely reducible* if $V = U_1 \oplus \cdots \oplus U_r$, where each U_i is an irreducible FG-submodule of V.

8.7 Theorem
If G is a finite group and $F = \mathbb{R}$ or $\mathbb{C}$, then every non-zero FG-module is completely reducible.

Proof Let V be a non-zero FG-module. The proof goes by induction on dim V. The result is true if dim $V = 1$, since V is irreducible in this case.

If V is irreducible then the result holds, so suppose that V is reducible. Then V has an FG-submodule U not equal to $\{0\}$ or V. By Maschke's Theorem, there is an FG-submodule W such that $V = U \oplus W$. Since dim U < dim V and dim W < dim V, we have, by induction,

$$U = U_1 \oplus \ldots \oplus U_r,\ W = W_1 \oplus \ldots \oplus W_s,$$

where each U_i and W_j is an irreducible FG-module. Then by (2.10),

$$V = U_1 \oplus \ldots \oplus U_r \oplus W_1 \oplus \ldots \oplus W_s,$$

a direct sum of irreducible FG-modules.

■

Another useful consequence of Maschke's Theorem is the next proposition.

8.8 Proposition

Let V be an FG-module, where $F = \mathbb{R}$ or $\mathbb{C}$ and G is a finite group. Suppose that U is an FG-submodule of V. Then there exists a surjective FG-homomorphism from V onto U.

Proof By Maschke's Theorem, there is an FG-submodule W of V such that $V = U \oplus W$. Then the function $\pi: V \to U$ which is defined by

$$\pi: u + w \to u \quad (u \in U, w \in W)$$

is an FG-homomorphism onto U, by Proposition 7.11.

■

Theorem 8.7 tells us that every non-zero FG-module is a direct sum of irreducible FG-modules. Thus, in order to understand FG-modules, we may concentrate upon the irreducible FG-modules. We begin our study of these in the next chapter.

Summary of Chapter 8

Assume that G is a finite group and $F = \mathrm{R}$ or $\mathbb{C}$.

1. Maschke's Theorem says that for every FG-submodule U of an FG-module V, there is an FG-submodule W with

$$V = U \oplus W.$$

2. Every non-zero FG-module V is a direct sum of irreducible FG-modules:

$$V = U_1 \oplus \ldots \oplus U_r.$$

Exercises for Chapter 8

1. Let $G = \langle x: x^3 = 1 \rangle \cong C_3$, and let V be the 2-dimensional $\mathbb{C}G$-module with basis v_1, v_2, where

$$v_1 x = v_2, \; v_2 x = -v_1 - v_2.$$

(This *is* a $\mathbb{C}G$-module, by Exercise 3.2.)

Express V as a direct sum of irreducible $\mathbb{C}G$-submodules.

2. If $G = C_2 \times C_2$, express the group algebra $\mathbb{R}G$ as a direct sum of 1-dimensional $\mathbb{R}G$-submodules.
3. Find a group G, a $\mathbb{C}G$-module V and a $\mathbb{C}G$-homomorphism ϑ: $V \to V$ such that $V \neq \operatorname{Ker} \vartheta \oplus \operatorname{Im} \vartheta$.
4. Let G be a finite group and let ρ: $G \to \mathrm{GL}\,(2, \mathbb{C})$ be a representation of G. Suppose that there are elements g, h in G such that the matrices $g\rho$ and $h\rho$ do not commute. Prove that ρ is irreducible.

   (You may care to revisit Example 5.5(2) and Exercises 5.1, 5.3, 5.4, 6.6 in the light of this result.)

5. Suppose that G is the infinite group

$$\left\{ \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$$

   and let V be the $\mathbb{C}G$-module $\mathbb{C}^2$, with the natural multiplication by elements of G (so that for $v \in V$, $g \in G$, the vector vg is just the product of the row vector v with the matrix g).

   Show that V is not completely reducible.

   (This shows that Maschke's Theorem fails for infinite groups – compare Example 8.2(2).)

6. *An alternative proof of Maschke's Theorem for $\mathbb{C}G$-modules.*

   Let V be a $\mathbb{C}G$-module with basis $v_1, \ldots, v_n$ and suppose that U is a $\mathbb{C}G$-submodule of V. Define a complex inner product (,) on V as follows (see (14.2) for the definition of a complex inner product): for $\lambda_i, \mu_j \in \mathbb{C}$,

$$\left(\sum_{i=1}^{n} \lambda_i v_i, \sum_{j=1}^{n} \mu_j v_j \right) = \sum_{i=1}^{n} \lambda_i \overline{\mu}_i.$$

Define another complex inner product [,] on V by

$$[u, v] = \sum_{x \in G} (ux, vx) \quad (u, v \in V).$$

(1) Verify that [,] is a complex inner product, which satisfies

$$[ug, vg] = [u, v] \quad \text{for all } u, v \in V \text{ and } g \in G.$$

(2) Suppose that U is a $\mathbb{C}G$-submodule of V, and define

$$U^{\perp} = \{v \in V\colon [u, v] = 0 \text{ for all } u \in U\}.$$

Show that $U^{\perp}$ is a $\mathbb{C}G$-submodule of V.

(3) Deduce Maschke's Theorem. (Hint: it is a standard property of complex inner products that $V = U \oplus U^{\perp}$ for all subspaces U of V.)

7. Prove that for every finite simple group G, there exists a faithful irreducible $\mathbb{C}G$-module.