1.4 Script 2

Let A and B be two rings. Let M be an A-module, and let N be a (A,B)-**bimodule**. By N being a (A,B)-bimodule, we mean that N not only have both structure of A-module and B-module, but also satisfy $a(bx)=b(ax)$. Consider the tensor product $M\otimes_A N$. For any $b\in B$, we may define a A-bilinear map $M\times N\to M\otimes_A N$ by

$$(m,n)\mapsto m\otimes bn$$

Therefore, by universal property, there exists some unique A-linear map $\widetilde{b}:M\otimes_A N\to M\otimes_A N$. Doing this procedure for each $b\in B$, to claim $M\otimes_A N$ forms a (A,B)-bimodule, it remains to check that

(a) $b(x+y)=bx+by$.

(b) $(b_1+b_2)x=b_1x+b_2x$.

(c) $(b_1b_2)x=b_1(b_2x)$.

(d) $1_Bx=x$.

(e) $a(bx)=b(ax)$.

Question 1: Exercise 2.15

Let P be a B-module. Find an (A,B)-bimodule isomorphism between

$$(M\otimes_A N)\otimes_B P \text{ and } M\otimes_A(N\otimes_B P)$$

Proof. For each $p\in P$, the A-bilinear map from $M\times N$ to $M\otimes_A(N\otimes_B P)$ defined by $(m,n)\mapsto m\otimes(n\otimes p)$ induce a unique A-linear map $f_p:M\otimes_A N\to M\otimes_A(N\otimes_B P)$ that sends $m\otimes n$ to $m\otimes(n\otimes p)$. By expressing elements of $M\otimes_A N$ as finite sum of basic elements, one can see that f_p is also B-linear. Therefore, if we define $f:(M\otimes_A N)\times P\to M\otimes_A(N\otimes_B P)$ by

$$f(x,p)\triangleq f_p(x)$$

we see that f is B-linear in $M\otimes_A N$. Again, by expressing elements of $M\otimes_A N$ as finite sum of basic elements, one can see that f is also B-linear in P. Therefore, by universal property, there exists some B-linear mapping $\widetilde{f}:(M\otimes_A N)\otimes_B P\to M\otimes_A(N\otimes_B P)$ with action:

$$(m\otimes n)\otimes p\mapsto f_p(m\otimes n)=m\otimes(n\otimes p)$$

Tedious computation by expressing elements of $(M \otimes_A N) \otimes_B P$ into finite sum of basic elements shows that $\widetilde{f}$ is also A-linear. We have shown $\widetilde{f}$ is an (A, B)-bimodule homomorphism.

To finish the proof, one first use similar argument to construct some (A, B)-bimodule homomorphism $\widetilde{g} : M \otimes_A (N \otimes_B P) \to (M \otimes_A N) \otimes_B P$ with action:

$$m \otimes (n \otimes p) \mapsto (m \otimes n) \otimes p$$

And then, see that $\widetilde{g} \circ \widetilde{f} \in \text{End}_{(A,B)}[(M \otimes_A N) \otimes_B P]$ have the identity action on basic elements $x \otimes p$[6] to conclude by universal property that $\widetilde{g} \circ \widetilde{f}$ is the identity function. ■

Let $f : A \to B$ be a ring homomorphism. If N is a B-module, then the A-module structure on N defined by $an \triangleq f(a)n$ is called **restriction of scalars**. If M is an A-module, then the B-module structure on $B \otimes_A M$[a] defined by

$$b(b' \otimes m) \triangleq bb' \otimes m$$

is called **extension of scalars**.

[a] B is given an A-module structure by restriction of scalar.

Question 2: Proposition 2.16

Let A, B be two rings, and let B be an A-module, so we have a ring homomorphism $f : A \to B$ defined by $f(a) \triangleq a1_B$. Let N be a B-module, and give N an A-module structure using restriction of scalars with respect to f.

Show that if N is finitely generated as a B-module and if B is finitely generated as an A-module, then N is finitely generated as an A-module.

Proof. Suppose $n_1, \ldots, n_k$ generate N over B, and suppose $b_1, \ldots, b_m$ generate B over A. We claim $\{b_j n_i\}$ generates N over A. Let

$$b'_i = \sum_{j=1}^{m} a_{i,j} b_j$$

[6] Again, by expressing x as basic element $x = \sum m_i \otimes n_i$.

Compute

$$\begin{aligned}\sum_{i=1}^{k} b'_i n_i &= \sum_{i=1}^{k}\Big(\sum_{j=1}^{m} a_{i,j}b_j\Big)n_i\\ &= \sum_{i=1}^{k}\sum_{j=1}^{m}(a_{i,j}b_j)n_i\\ &= \sum_{i,j}(a_{i,j}b_j)n_i\\ &= \sum_{i,j}a_{i,j}(b_jn_i)\end{aligned}$$

For justification of last equality, compute

$$a(bn) = f(a)(bn) = (f(a)b)n = (ab)n$$

Remark: similar routine computation shows that N is in fact an (A, B)-bimodule. ■

Question 3: Proposition 2.17

Let $f : A \to B$ be a ring homomorphism, and let M be a finitely generated A-module, show that its extension of scalar $B \otimes_A M$ is finitely generated as a B-module.

Proof. Let $\{m_1, \dots, m_n\}$ generates M over A. We claim $\{1_B \otimes m_i\}$ generate all the basic elements. Consider

$$\begin{aligned}b\otimes\sum a_im_i &= \sum b\otimes a_im_i\\ &= \sum b(1_B\otimes a_im_i)\\ &= \sum b(a_i1_B\otimes m_i) \quad (\because\ B \text{ is regarded as an } A\text{-module when we write } B\otimes_A M)\\ &= \sum b(f(a_i)\otimes m_i)\\ &= \sum bf(a_i)(1\otimes m_i)\end{aligned}$$

■

Let $M \xrightarrow{f} M'$ and $N \xrightarrow{g} N'$ be in the category of A-module. The function $h : M \times N \to M' \otimes N'$ defined by

$$h(x, y) \triangleq f(x) \otimes g(y)$$

is clearly A-bilinear. Therefore, we may induce some unique A-linear map $f \otimes g$:

$M \otimes N \to M' \otimes N'$ such that

$$(f \otimes g)(x \otimes y) = f(x) \otimes g(y)$$

Note that for each $M' \xrightarrow{f'} M''$ and $N' \xrightarrow{g'} N''$, we have

$$(f' \circ f) \otimes (g' \circ g) = (f' \otimes g') \circ (f \otimes g)$$

because they agree on the basic elements.

Question 4: Proposition 2.18 (Exaction of Tensor Product)

If

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0 \tag{1.1}$$

is an exact sequence of A-modules and homomorphism, then for any A-module N, the sequence

$$M' \otimes N \xrightarrow{f \otimes 1} M \otimes N \xrightarrow{g \otimes 1} M'' \otimes N \to 0$$

is also exact, where $1 \in \text{End}(N)$ is the identity mapping.

Proof. Because g is surjective, we may construct an **right inverse** $g^{-1} : M'' \to M$. That is, $g \circ g^{-1}(m'') = m''$ for all $m'' \in M''$. To see $g \otimes 1$ is surjective, just observe

$$\sum m_i'' \otimes n_i = (g \otimes 1)\Big(\sum g^{-1}(m_i'') \otimes n_i\Big)$$

After computing

$$(g \otimes 1) \circ (f \otimes 1) = (g \circ f) \otimes (1 \circ 1) = 0$$

we may reduce the problem into proving the factored map

$$\text{Coker}(f \otimes 1) \xrightarrow{\tilde{g}} M'' \otimes N$$

is injective. Consider the map $h : M'' \times N \to \text{Coker}(f \otimes 1)$ defined by

$$h(m'', n) \triangleq [g^{-1}(m'') \otimes n]$$

Clearly, h is linear in n. Using the fact $\text{Im}(f) = \text{Ker}(g)$ and computation

$$g(g^{-1}(am'') - ag^{-1}(m'')) = 0$$
$$g(g^{-1}(m_1'' + m_2'') - g^{-1}(m_1'') - g^{-1}(m_2'')) = 0$$

we may conclude that h is also linear in M''. Now, because h is bilinear, we may induce some linear $\widetilde{h} : M'' \otimes N \to \text{Coker}(f \otimes 1)$ with action

$$\widetilde{h}(m'' \otimes n) = [g^{-1}(m'') \otimes n]$$

Using universal property, it is east to check that $\widetilde{h} \circ \widetilde{g} \in \text{End}(\text{Coker}(f \otimes 1))$ is identity mapping. We have shown $\widetilde{g}$ is injective. ■

Note that the exaction of tensor product holds only for sequence of the form 1.1. One can't delete the zero space at the end and still reach the same conclusion. Consider

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f(x)=2x} \mathbb{Z}$$

where the underlying ring is $\mathbb{Z}$. The sequence

$$0 \longrightarrow \mathbb{Z} \otimes \text{Coker}(f) \xrightarrow{f \otimes 1} \mathbb{Z} \otimes \text{Coker}(f)$$

is not exact, because

$$(f \otimes 1)(x \otimes [y]) = 2x \otimes [y] = x \otimes [2y] = 0$$

implies $\text{Ker}(f \otimes 1) = \mathbb{Z} \otimes \text{Coker}(f)$, while

$$\mathbb{Z} \otimes \text{Coker}(f) \cong \text{Coker}(f) \neq 0$$

An A-module N is said to be **flat** if for any exact sequence

$$\cdots \to M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \to \cdots$$

in the category of A-modules, the sequence

$$\cdots \to M_{i-1} \otimes N \xrightarrow{f_{i-1} \otimes 1} M_i \otimes N \xrightarrow{f_i \otimes 1} M_{i+1} \otimes N \to \cdots$$

is also exact.

Question 5

Show that for an A-module N, the following are equivalents

(a) N is flat.

(b) If $0 \to M' \longrightarrow M \longrightarrow M'' \to 0$ is exact, then $0 \to M' \otimes N \longrightarrow M \otimes N \longrightarrow M'' \otimes N \to 0$ is also exact.

(c) If $f : M' \to M$ is injective, then $f \otimes 1 : M' \otimes N \to M \otimes N$ is injective.

(d) If $f : M' \to N$ is injective and M, M' are finitely generated, then $f \otimes 1 : M' \otimes N \to M \otimes N$ is injective.

Proof. From (a) to (b) is definition. We now prove from (b) to (a). Consider the exact sequence

$$\cdots \to M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \to \cdots$$

We may split this into a short exact sequence

$$0 \longrightarrow \mathrm{Im}(f_{i-1}) \hookrightarrow M_i \xrightarrow{f_i} \mathrm{Im}(f_i) \longrightarrow 0$$

By (b), the short sequence

$$0 \longrightarrow \mathrm{Im}(f_{i-1}) \otimes N \hookrightarrow M_i \otimes N \xrightarrow{f_i \otimes 1} \mathrm{Im}(f_i) \otimes N \longrightarrow 0$$

is also exact. This implies

$$\mathrm{Ker}(f_i \otimes 1) = \mathrm{Im}(f_{i-1}) \otimes N = \mathrm{Im}(f_{i-1} \otimes 1)$$

We have shown

$$\cdots \to M_{i-1} \otimes N \xrightarrow{f_{i-1} \otimes 1} M_i \otimes N \xrightarrow{f_i \otimes 1} M_{i+1} \otimes N \to \cdots$$

is also exact, thus proving (a). From (b) to (c), we simply let $M'' \triangleq \mathrm{Coker}(f)$ and let $M \to M''$ be the quotient map. From (c) to (b) follows from right exaction and

$$\mathrm{Im}(f \otimes 1) = \mathrm{Im}(f) \otimes N = \mathrm{Ker}(g) \otimes N = \mathrm{Ker}(g \otimes 1)$$

From (c) to (d) is clear. It only remains to show from (d) to (c).

Fix

$$u = \sum_{i=1}^{n} x_i \otimes y_i \in \mathrm{Ker}(f \otimes 1)$$

Let M_0' be the submodule of M' generated by $\{x_1, \dots, x_n\}$, and let $u_0' \in M_0' \otimes N$ be the element

$$u_0' \triangleq \sum_{i=1}^{n} x_i \otimes y_i \in M_0' \otimes N$$

By Corollary 2.13, there exists some finitely generated submodule M_0 of M such that $u_0 \in M_0 \otimes N$ defined by

$$u_0 \triangleq \sum_{i=1}^{n} f(x_i) \otimes y_i \in M_0 \otimes N$$

equals to 0. Note that because $\{x_1, \dots, x_n\}$ generates M_0' and M_0 contains $\{f(x_1), \dots, f(x_n)\}$, so M_0 contains $f(M_0')$, and obviously

$$f|_{M_0'} : M_0' \to M_0 \text{ is injective.}$$

We now see from (d) that

$$f|_{M_0'} \otimes 1 : M_0' \otimes N \to M_0 \otimes N \text{ is injective.}$$

Compute

$$(f|_{M_0'} \otimes 1)(u_0') = \sum_{i=1}^{n} f(x_i) \otimes y_i = u_0 = 0$$

We see $u_0' = \sum_{i=1}^{n} x_i \otimes y_i \in M_0' \otimes N$ is zero. Now consider the universal property

$$\begin{tikzcd} M_0' \times N \arrow[r] \arrow[dr] & M_0' \otimes N \arrow[d, dashed, "\phi"] \\ & M' \otimes N \end{tikzcd}$$

We may see $u = \phi(u_0')$ is zero. Finishing the proof. ■

Question 6: Exercise 2.20

Let ring B be an (A, B)-bimodule, and let M be a flat A-module. Show that the extension of scalar $B \otimes_A M$ is a flat B-module.

Proof. Let $g : P' \to P$ be an injective B-module homomorphism. We are required to show

$$P' \otimes_B (B \otimes_A M) \xrightarrow{g \otimes 1} P \otimes_B (B \otimes_A M)$$

is also injective. We have the isomorphism

$$P' \otimes_B (B \otimes_A M) \cong (P' \otimes_B B) \otimes_A M \cong P' \otimes_A M$$

It now follows from M being flat that $g \otimes 1$ is injective.

■