NCKU 112.1
Sum of k-th Power

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Chapter 1

Sum of 1,2,3th power

1.1 k=1

Theorem 1.1.1. Given a natural number m, We have the identity

$$1 + \cdots + m = \frac{m^2 + m}{2} \tag{1.1}$$

Proof. Observe that given any natural number n

$$\begin{align} n^2 - (n-1)^2 &= n^2 - (n^2 - 2n + 1) \tag{1.2} \\ &= 2n - 1 \tag{1.3} \end{align}$$

Then we can deduce an identity

$$n = \frac{1}{2}[n^2 - (n-1)^2 + 1] \tag{1.4}$$

Then

$$\begin{align} \sum_{n=1}^{m} n &= \sum_{n=1}^{m} \frac{1}{2}[n^2 - (n-1)^2 + 1] \tag{1.5} \\ &= \frac{1}{2}[\sum_{n=1}^{m} n^2 - \sum_{n=1}^{m} (n-1)^2 + \sum_{n=1}^{m} 1] \tag{1.6} \\ &= \frac{1}{2}[\sum_{n=1}^{m} n^2 - \sum_{n=1}^{m-1} n^2 + m] \tag{1.7} \\ &= \frac{m^2 + m}{2} \tag{1.8} \end{align}$$

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1.2 k=2

Theorem 1.2.1. Given a natural number m, We have the identity

$$1^2 + \cdots + m^2 = \frac{m^3}{3} + \frac{m^2}{2} + \frac{m}{6} \tag{1.9}$$

Proof. Observe that given any natural number n

$$\begin{aligned} n^3 - (n-1)^3 &= n^3 - (n^3 - 3n^2 + 3n - 1) && (1.10)\\ &= 3n^2 - 3n + 1 && (1.11) \end{aligned}$$

Then we can deduce an identity

$$n^2 = \frac{1}{3}[n^3 - (n-1)^3 + 3n - 1] \tag{1.12}$$

Then

$$\begin{aligned} \sum_{n=1}^{m} n^2 &= \sum_{n=1}^{m} \frac{1}{3}[n^3 - (n-1)^3 + 3n - 1] && (1.13)\\ &= \frac{m^3}{3} + \sum_{n=1}^{m} n - \frac{m}{3} && (1.14)\\ &= \frac{m^3}{3} + \frac{m^2}{2} + \frac{m}{6} && (1.15) \end{aligned}$$

$\blacksquare$

1.3 k=3

Theorem 1.3.1. Given a natural number m, We have the identity

$$1^3 + \cdots + m^3 = \frac{m^4}{4} + \frac{m^3}{2} + \frac{m^2}{4} \tag{1.16}$$

Proof. Observe that given any natural number n

$$\begin{aligned} n^4 - (n-1)^4 &= n^4 - (n^4 - 4n^3 + 6n^2 - 4n + 1) && (1.17)\\ &= 4n^3 - 6n^2 + 4n - 1 && (1.18) \end{aligned}$$

Then we can deduce an identity

$$n^3 = \frac{1}{4}[n^4 - (n-1)^4 + 6n^2 - 4n + 1] \tag{1.19}$$

Then

$$\sum_{n=1}^{m} n^3 = \frac{m^4}{4} + \frac{3}{2}\sum_{n=1}^{m} n^2 - \sum_{n=1}^{m} n + \frac{m}{4} \tag{1.20}$$

$$= \frac{m^4}{4} + \frac{3}{2}(\frac{m^3}{3} + \frac{m^2}{2} + \frac{m}{6}) - (\frac{m^2+m}{2}) + \frac{m}{4} \tag{1.21}$$

$$= \frac{m^4}{4} + \frac{m^3}{2} + \frac{m^2}{4} \tag{1.22}$$

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Chapter 2

Generalization

2.1 Summary

So far, we have collected the following using the same method. It isn't difficult to show for all natural number k, the sum $\sum_{n=1}^{m} n^k$ can be expressed by a polynomial of m of $k+1$ degree.

$$\sum_{n=1}^{m} n^0 = m \tag{2.1}$$

$$\sum_{n=1}^{m} n^1 = \frac{m^2}{2} + \frac{m}{2} \tag{2.2}$$

$$\sum_{n=1}^{m} n^2 = \frac{m^3}{3} + \frac{m^2}{2} + \frac{m}{6} \tag{2.3}$$

$$\sum_{n=1}^{m} n^3 = \frac{m^4}{4} + \frac{m^3}{2} + \frac{m^2}{4} \tag{2.4}$$

Now, we use the same method to give an inductive formula of sum of k-th power. Notice that this formula is very inefficient if k is large.

Theorem 2.1.1. Given a natural number m, We have the identity

$$\sum_{n=1}^{m} n^k = \frac{1}{k+1}[m^{k+1} + \sum_{i=0}^{k-1} \binom{k+1}{i} (-1)^{k-i+1} \sum_{n=1}^{m} n^i] \tag{2.5}$$

Proof.

$$n^{k+1} - (n-1)^{k+1} = -\sum_{i=0}^{k} \binom{k+1}{i} n^i (-1)^{k+1-i} \tag{2.6}$$
$$= \sum_{i=0}^{k} \binom{k+1}{i} n^i (-1)^{k-i} \tag{2.7}$$
$$= \binom{k+1}{k} n^k (-1)^0 + \sum_{i-0}^{k-1} \binom{k+1}{i} n^i (-1)^{k-i} \tag{2.8}$$
$$= (k+1)n^k + \sum_{i=0}^{k-1} \binom{k+1}{i} n^i (-1)^{k-i} \tag{2.9}$$
$$(k+1)n^k = n^{k+1} - (n-1)^{k+1} + \sum_{i=0}^{k-1} \binom{k+1}{i} n^i (-1)^{k-i+1} \tag{2.10}$$
$$n^k = \frac{1}{k+1}[n^{k+1} - (n-1)^{k+1} + \sum_{i=0}^{k-1} \binom{k+1}{i} n^i (-1)^{k-i+1}] \tag{2.11}$$
$$\sum_{n=1}^{m} n^k = \frac{1}{k+1}[m^{k+1} + \sum_{n=1}^{m} \sum_{i=0}^{k-1} \binom{k+1}{i} n^i (-1)^{k-i+1}] \tag{2.12}$$
$$= \frac{1}{k+1}[m^{k+1} + \sum_{i=0}^{k-1} \binom{k+1}{i} (-1)^{k-i+1} \sum_{n=1}^{m} n^i] \tag{2.13}$$

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Now we show an intreseting property of the sum of k-th power.

Theorem 2.1.2. If m, k are natural numbers, then $\sum_{n=1}^{m} n^k$ is a polynomial of m, where the sum of coefficient is 1

Proof. Let f be a function that maps a polynomial of m to the sum of coefficients of the polynomial. We prove our theorem by induction.

We know that $\sum_{n=1}^{m} n^0 = m$, which finish the proof for base case. Given a natural number r, assume that $\forall u \leq r \in \mathbb{N}, f(\sum_{n=1}^{m} n^u) = 1$.

Notice

$$0 = (1-1)^{r+2} = \sum_{i=0}^{r+2} \binom{r+2}{i} (1)^i (-1)^{r+2-i} \tag{2.14}$$
$$= [\sum_{i=0}^{r} \binom{r+2}{i} (-1)^{r-i}] - (r+2) + 1 \tag{2.15}$$
$$r+1 = \sum_{i=0}^{r} \binom{r+2}{i} (-1)^{r-i} \tag{2.16}$$

Notice that f is a linear function. We use this fact to deduce

$$\begin{align}
\sum_{n=1}^{m} n^{r+1} &= \frac{1}{r+2}[m^{r+2} + \sum_{i=0}^{r} \binom{r+2}{i} (-1)^{r-i} \sum_{n=1}^{m} n^i] \tag{2.17}\\
f(\sum_{n=1}^{m} n^{r+1}) &= f(\frac{1}{r+2}[m^{r+2} + \sum_{i=0}^{r} \binom{r+2}{i} (-1)^{r-i} \sum_{n=1}^{m} n^i]) \tag{2.18}\\
&= \frac{1}{r+2}[f(m^{r+2}) + \sum_{i=0}^{r} \binom{r+2}{i} (-1)^{r-i} f(\sum_{n=1}^{m} n^i)] \tag{2.19}\\
&= \frac{1}{r+2}(1 + \sum_{i=0}^{r} \binom{r+2}{i} (-1)^{r-i}) \tag{2.20}\\
&= \frac{1}{r+2}(1 + r + 1) \tag{2.21}\\
&= 1 \tag{2.22}
\end{align}$$

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