Deadline : 2023/06/14, 17:00.

1. Determine whether or not the given set is open, connected, and simply-connected.

   (a) $\{(x, y) \mid 0 < y < 3\}$.

   (b) $\{(x, y) \mid 1 < |x| < 2\}$.

   (c) $\{(x, y) \mid 1 \le x^2 + y^x \le 4,\ y \ge 0\}$.

   (d) $\{(x, y) \mid (x, y) \ne (2, 3)\}$.

2. Let $\mathbf{F}(x, y) = \dfrac{\langle -y, x\rangle}{x^2 + y^2}$.

   (a) Show that $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

   (b) Show that $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ is not independent of path. Does this contradict to Theorem 6 in the textbook P.1186?

3. (a) If C is the line segment connecting the point (x_1, y_1) to the point (x_2, y_2), show that

   $$\int_C x\,dy - y\,dx = x_1y_2 - x_2y_1.$$

   (b) If the vertices of a polygon, in counterclockwise order, are (x_1, y_1), (x_2, y_2), $\cdots$, (x_n, y_n), show that the area of the polygon is

   $$A = \frac{1}{2}\left[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)\right].$$

   (c) Find the area of the pentagon with vertices $(0, 0)$, $(2, 1)$, $(1, 3)$, $(0, 2)$, and $(-1, 1)$.

4. (a) Let D be a region bounded by a simple closed path C in the xy-plane. Use Green's Theorem to prove that the coordinates of the centroid $(\bar{x}, \bar{y})$ of D are

   $$\bar{x} = \frac{1}{2A}\oint_C x^2\,dy \qquad \bar{y} = \frac{1}{2A}\oint_C y^2\,dx$$

   where A is the area of D.

   (b) Find the centroid of a quarter-circular region of radius a.

   (c) Find the centroid of the triangle with vertices $(0, 0)$, $(a, 0)$, and (a, b), where $a > 0$ and $b > 0$.

5. Let $f(x,y,z)$, $g(x,y,z)$ be real-valued functions and $\mathbf{F}(x,y,z)$, $\mathbf{G}(x,y,z)$ be vector-valued functions. Prove the identity, assuming that the appropriate partial derivatives exists and are continuous.

   (a) $\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$

   (b) $\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$

   (c) $\nabla \cdot (f\mathbf{F}) = f(\nabla \cdot \mathbf{F}) + \mathbf{F} \cdot \nabla f$

   (d) $\nabla \times (f\mathbf{F}) = f(\nabla \times \mathbf{F}) + (\nabla f) \times \mathbf{F}$

   (e) $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$

   (f) $\nabla \cdot (\nabla f \times \nabla g) = 0$

   (g) $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \Delta \mathbf{F}$

6. (a) Use Green's Theorem to prove **Green's first identity:**

$$\iint_D f \Delta g \, dA = \oint_C f(\nabla g) \cdot \mathbf{n} \, ds - \iint_D \nabla f \cdot \nabla g \, dA$$

   where D and C satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of f and g exist and are continuous. (The quantity $\nabla g \cdot \mathbf{n} = D_{\mathbf{n}} g$ occurs in the line integral; it is the directional derivative in the direction of the normal vector $\mathbf{n}$ and is called **normal derivative** of g.)

   (b) Use Green's first identity to prove **Green's second identity:**

$$\iint_D (f \Delta g - g \Delta f) \, dA = \oint_C (f \nabla g - g \nabla f) \cdot \mathbf{n} \, ds$$

   where D and C satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of f and g exist and are continuous.

7. Find the area of the part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies inside the cylinder $x^2 + y^2 = ax$.

8. Use Gauss's Law to find the charge enclosed by the cube with vertices $(\pm 1, \pm 1, \pm 1)$ if the electric field is

$$\mathbf{E}(x,y,z) = \langle x, y, z \rangle.$$

9. Evaluate

$$\int_C (y + \sin x) \, dx + (z^2 + \cos y) \, dy + x^3 \, dz$$

where C is the curve $\mathbf{r}(t) = \langle \sin t, \cos t, \sin 2t \rangle$, $0 \le t \le 2\pi$.

10. Use the Divergence Theorem to evaluate

$$\iint_S (2x + 2y + z^2)\, dS$$

where S is the sphere $x^2 + y^2 + z^2 = 1$.

11. Find the positively oriented simple closed curve C for which the value of the line integral

$$\int_C (y^3 - y)\, dx - 2x^3\, dy$$

is a maximum.

12. Let C be a simple closed piecewise-smooth space curve that lies in a plane with unit normal vector $\mathbf{n} = \langle a, b, c \rangle$ and has positive orientation with respect to $\mathbf{n}$. Show that the plane area enclosed by C is

$$\frac{1}{2} \int_C (bz - cy)\, dx + (cx - az)\, dy + (ay - bx)\, dz.$$