5.6 HW3

Question 17

Definition:

(i) The Fourier transform of f on $\mathbb{R}$ is defined by $\widehat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx$.

(ii) The Fourier inverse transform of f on $\mathbb{R}$ is defined by $f(x) = \mathcal{F}^{-1}[\widehat{f}] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{ix\xi}\,d\xi$.

1. Show that $\widehat{f'} = i\xi\widehat{f}$ and $\widehat{xf} = i\frac{d}{d\xi}\widehat{f}$. (You may assume $f \to 0$ as $x \to \pm\infty$)

Proof. Compute

$$\begin{aligned}\hat{f'} - i\xi\hat{f} &= \int_{-\infty}^{\infty} \left(f'(x)e^{-i\xi x} - i\xi f(x)e^{-i\xi x}\right)dx \\ &= f(x)e^{-i\xi x}\Big|_{x=-\infty}^{\infty}\end{aligned}$$

Note that

$$\left|f(x)e^{-i\xi x}\right| = |f(x)|$$

Compute

$$\left|f(M)e^{-i\xi M} - f(-M)e^{i\xi M}\right| \leq |f(M)| + |f(-M)| \to 0 \text{ as } M \to \infty$$

This now implies

$$\hat{f'} - i\xi\hat{f} = \lim_{M\to\infty} f(x)e^{-i\xi x}\Big|_{x=-M}^{M} = 0$$

Define

$$\phi(x,\xi) \triangleq f(x)e^{-i\xi x}$$

It is clear that

$$\partial_\xi\phi(x,\xi) = -ixf(x)e^{-i\xi x} \text{ is continuous every where}$$

Then, we can apply Feynman's Trick to compute

$$\begin{aligned}
i\frac{d}{d\xi}\hat{f} &= i\frac{d}{d\xi}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx \\
&= i\int_{-\infty}^{\infty} -ixf(x)e^{-i\xi x}dx \\
&= \int_{-\infty}^{\infty} xf(x)e^{-i\xi x}dx = \widehat{xf}
\end{aligned}$$

$\blacksquare$

Theorem 5.6.1. (Gaussian Integral)

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

Proof. Fix $I = \int_{-\infty}^{\infty} e^{-x^2}dx$. Compute using Fubini's Theorem

$$\begin{aligned}
I^2 &= \int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy \\
&= \int_{0}^{\infty}\int_{0}^{2\pi} re^{-r^2}d\theta dr \\
&= 2\pi\int_{0}^{\infty} re^{-r^2}dr \\
&= -\pi e^{-r^2}\Big|_{r=0}^{\infty} = \pi
\end{aligned}$$

Because e^{-x^2} is a positive function, we now have

$$\int_{-\infty}^{\infty} e^{-x^2}dx = I = \sqrt{I^2} = \sqrt{\pi}$$

$\blacksquare$

Theorem 5.6.2. (Gaussian Integral)

$$\int_{-\infty}^{\infty} e^{-\frac{(x-a)^2}{b}}dx$$

Proof. Fix

$$y \triangleq \frac{x-a}{\sqrt{b}} \text{ and } \frac{dy}{dx} = \frac{1}{\sqrt{b}}$$

Compute using Theorem 5.6.1

$$\int_{-\infty}^{\infty} e^{-\frac{(x-a)^2}{b}} dx = \int_{-\infty}^{\infty} e^{-y^2} \sqrt{b} dy$$
$$= \sqrt{b\pi}$$

■

Question 18

2. Let $g(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$, $\sigma \neq 0$. $g(x)$ is called the normalized Gaussian function in $\mathbb{R}$. Find the Fourier transform of g on $\mathbb{R}$.

Proof. Compute

$$g'(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot \frac{-2x}{2\sigma^2} e^{\frac{-x^2}{2\sigma^2}} = -\frac{1}{\sigma^2} x g(x)$$

Using the statement of the first question, which we have proved, we now have

$$i\xi\widehat{g} = \widehat{g'} = -\frac{1}{\sigma^2}\widehat{xg} = \frac{-i}{\sigma^2}\frac{d\widehat{g}}{d\xi}$$

This give us the first order homogenoeous ODE

$$\frac{d}{d\xi}\widehat{g} + \sigma^2\xi\widehat{g} = 0$$

Compute the general solution

$$\widehat{g}(\xi) = Ce^{\frac{-\sigma^2\xi^2}{2}}$$

Compute using Theorem 5.6.2

$$C = \widehat{g}(0) = \int_{-\infty}^{\infty} g(x) dx = 1$$

We now have the answer

$$\widehat{g}(\xi) = e^{\frac{-\sigma^2\xi^2}{2}}$$

■

Question 19

3. The convolution of two functions f and g is defined by $f*g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy$. Show that $\widehat{f*g} = \widehat{f}\widehat{g}$. (You may assume the Fubini's Theorem always holds.)

Proof. Compute using Fubini's Theorem

$$\begin{aligned}\widehat{f*g}(\xi) &= \int_{-\infty}^{\infty} (f*g)(u)e^{-i\xi u}du \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u-y)g(y)e^{-i\xi u}dudy\end{aligned}$$

Compute using Fubini's Theorem

$$\begin{aligned}\widehat{f}\cdot\widehat{g}(\xi) &= \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx\int_{-\infty}^{\infty} g(y)e^{-\xi y}dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)e^{-i\xi(x+y)}dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u-y)g(y)e^{-i\xi u}dudy \quad \text{where } u = x+y \text{ and } \frac{du}{dx} = 1 \\ &= \widehat{f*g}(\xi)\end{aligned}$$

■

Question 20

4. For $0 < \alpha < 1$, define $C_\alpha := \Gamma(\frac{\alpha}{2})$, where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$ is the gamma function. Show that

$$C_\alpha \mathcal{F}^{-1}[\pi^{\frac{1}{2}}2^\alpha|\xi|^{-\alpha}\widehat{f}](x) = C_{1-\alpha}\int_{\mathbb{R}} \frac{f(y)}{|x-y|^{1-\alpha}}\,dy.$$

(You may assume the Fubini's Theorem always holds.)

Proof. Define

$$g(x) \triangleq \frac{1}{|x|^{1-\alpha}}$$

We see

$$\begin{aligned}\int_{\mathbb{R}} \frac{f(y)}{|x-y|^{1-\alpha}} dy &= \int_{\mathbb{R}} \frac{f(x-u)}{|u|^{1-\alpha}} du \quad (\because u = x - y) \\ &= \int_{\mathbb{R}} f(x-u) g(u) du = f * g(x)\end{aligned}$$

Compute

$$C_\alpha \mathcal{F}^{-1}[\pi^{\frac{1}{2}} 2^\alpha |\xi|^{-\alpha} \widehat{f}](x) = C_\alpha \pi^{\frac{1}{2}} 2^\alpha \mathcal{F}^{-1}[|\xi|^{-\alpha} \widehat{f}](x)$$

We now can reduce the problem into proving

$$C_\alpha \pi^{\frac{1}{2}} 2^\alpha \mathcal{F}^{-1}[|\xi|^{-\alpha} \widehat{f}](x) = C_{1-\alpha} f * g(x)$$

Using Fourier Inversion Theorem, and Convolution Theorem, we then can reduce the problem into proving

$$C_\alpha \pi^{\frac{1}{2}} 2^\alpha \frac{1}{|\xi|^\alpha} \widehat{f}(\xi) = C_{1-\alpha} \widehat{g}(\xi) \cdot \widehat{f}(\xi)$$

Then, we reduce the problem into

$$C_\alpha \pi^{\frac{1}{2}} 2^\alpha \frac{1}{|\xi|^\alpha} = C_{1-\alpha} \widehat{g}(\xi)$$

Compute

$$\begin{aligned}
\widehat{g}(\xi) &= \int_{-\infty}^{\infty} |x|^{\alpha-1} e^{-i\xi x} dx \\
&= \int_{-\infty}^{\infty} |x|^{\alpha-1} \big(\cos(\xi x) - i\sin(\xi x)\big) dx \\
&= \int_{-\infty}^{\infty} |x|^{\alpha-1} \cos(\xi x) dx \quad (\because |x|^{\alpha-1}\sin(\xi x) \text{ is odd in } x\ ) \\
&= 2\int_{0}^{\infty} |x|^{\alpha-1} \cos(\xi x) dx \quad (\because |x|^{\alpha-1}\cos(\xi x) \text{ is even in } x\ ) \\
&= 2\int_{0}^{\infty} |x|^{\alpha-1} \operatorname{Re} e^{i\xi x} dx \\
&= \operatorname{Re} 2\int_{0}^{\infty} |x|^{\alpha-1} e^{i\xi x} dx \\
&= \operatorname{Re} 2\int_{0}^{\infty} \left|\frac{u}{\xi}\right|^{\alpha-1} e^{iu} \frac{du}{\xi} \quad (u \equiv \xi x) \\
&= \operatorname{Re} \frac{2}{|\xi|^{\alpha}} \int_{0}^{\infty} u^{\alpha-1} e^{iu} du \\
&= \operatorname{Re} \frac{2}{|\xi|^{\alpha}} e^{i\frac{\alpha\pi}{2}} \int_{0}^{\infty} x^{\alpha-1} e^{-x} dx \quad (\because \text{ Cauchy Integral Theorem }) \\
&= \operatorname{Re} \frac{2}{|\xi|^{\alpha}} e^{i\frac{\alpha\pi}{2}} \Gamma(\alpha) \\
&= \frac{2\cos\frac{\alpha\pi}{2}\Gamma(\alpha)}{|\xi|^{\alpha}}
\end{aligned}$$

We can reduce our problem into proving

$$\frac{\Gamma(\frac{\alpha}{2})\sqrt{\pi}2^{\alpha}}{|\xi|^{\alpha}} = \frac{2\cos\frac{\alpha\pi}{2}\Gamma(\alpha)\Gamma(\frac{1-\alpha}{2})}{|\xi|^{\alpha}}$$

Reduce to

$$\Gamma(\frac{\alpha}{2})\sqrt{\pi}2^{\alpha-1} = \cos\frac{\alpha\pi}{2}\Gamma(\alpha)\Gamma(\frac{1-\alpha}{2})$$

Note that the Legendre Duplication Formula give us

$$\Gamma(\frac{\alpha}{2})\Gamma(\frac{\alpha+1}{2}) = 2^{1-\alpha}\Gamma(\alpha)\sqrt{\pi}$$

This give us

$$\Gamma(\frac{\alpha}{2})\sqrt{\pi}2^{\alpha-1} = \frac{2^{1-\alpha}\Gamma(\alpha)\sqrt{\pi}}{\Gamma(\frac{\alpha+1}{2})}\sqrt{\pi}2^{\alpha-1}$$
$$= \frac{\Gamma(\alpha)\pi}{\Gamma(\frac{\alpha+1}{2})} \tag{5.27}$$

Note that Euler Reflection Formula give us

$$\Gamma(\frac{1-\alpha}{2})\Gamma(\frac{1+\alpha}{2}) = \frac{\pi}{\sin(\pi\frac{1+\alpha}{2})} = \frac{\pi}{\cos\frac{\alpha\pi}{2}}$$

This give us

$$\cos\frac{\alpha\pi}{2}\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha) = \cos\frac{\alpha\pi}{2}\Gamma(\alpha)\frac{\pi}{\cos\frac{\alpha\pi}{2}\Gamma(\frac{1+\alpha}{2})}$$
$$= \frac{\Gamma(\alpha)\pi}{\Gamma(\frac{\alpha+1}{2})} \tag{5.28}$$

Note that Equation 5.27 and Equation 5.28 are identical, and we are done. (done) ∎

Theorem 5.6.3. (Remainder of Taylor's Theorem in Mean Values Form) Given

$$f : I \subseteq \mathbb{R} \to \mathbb{R} \text{ is } n \text{ time continuously differentiable at } a \in I$$

Define

(a) $P_n(x) \triangleq \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!}$

(b) $R_n(x) \triangleq f(x) - P_n(x)$

If

(a) G is continuous on $[a, x]$

(b) G' exists and not equals to 0 on (a, x)

We have

$$\exists \xi \in (a, x), R_n(x) = \frac{f^{(n+1)}(\xi)(x-\xi)^n}{n!}\frac{G(x) - G(a)}{G'(\xi)}$$

Proof. WLOG suppose $x > a$. Define $F : (a, x) \to \mathbb{R}$ by

$$F(t) = \sum_{k=0}^{n} \frac{f^{(k)}(t)(x-t)^k}{k!}$$

By Cauchy's MVT, we know

$$\exists \xi \in (a, x), \frac{F'(\xi)}{G'(\xi)} = \frac{F(x) - F(a)}{G(x) - G(a)}$$

Compute

$$F(x) = f(x)$$

Compute

$$F(a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!} = P_n(x)$$

Compute

$$\begin{aligned} F'(\xi) &= \sum_{k=0}^{n} \frac{f^{(k+1)}(\xi)(x-\xi)^k - kf^{(k)}(\xi)(x-\xi)^{k-1}}{k!} \\ &= \frac{f^{(n+1)}(\xi)(x-\xi)^n}{n!} \end{aligned}$$

We now have

$$\frac{\frac{f^{(n+1)}(\xi)(x-\xi)^n}{n!}}{G'(\xi)} = \frac{R_n(x)}{G(x) - G(a)}$$

Then we can deduce

$$R_n(x) = \frac{f^{(n+1)}(\xi)(x-\xi)^n}{n!} \frac{G(x) - G(a)}{G'(\xi)}$$

■

Corollary 5.6.4. (Lagarange Form of Remainders in Taylor's Theorem) Let

$$G(t) = (x-t)^{n+1}$$

We have

$$\exists \xi \in (a, x), R_n(x) = \frac{f^{(n+1)}(\xi)(x-a)^{n+1}}{(n+1)!}$$

Proof. Compute

$$\begin{aligned} G'(\xi) &= -(n+1)(x-\xi)^n \\ G(x) &= 0 \\ G(a) &= (x-a)^{n+1} \end{aligned}$$

The result now follows from Theorem 5.6.3. ■

Theorem 5.6.5. ($\sin x \le x$)

$$|\sin x| \le |x| \qquad (x \in [\frac{-\pi}{2}, \frac{\pi}{2}])$$

Proof. Because $|\sin x|$ and $|x|$ are both odd and positive, WOLG, we only have to prove when $x \in (0, \frac{\pi}{2}]$. Compute the Taylor polynomials to second degree and its remainder.

$$\sin x = x - \cos(\xi)\frac{x^3}{3!} \text{ for some } \xi \in (0, x)$$

Because $0 < \xi < x$, it is now clear that

$$0 < \sin x = x - \cos(\xi)\frac{x^3}{3!} \le x$$

This then implies

$$|\sin x| \le |x|$$

∎

Question 21

5. Determine whether the Dirichlet kernel $D_N(x) = \sum_{n=-N}^{N} e^{-inx} = \frac{\sin(N+\frac{1}{2})x}{\sin(\frac{x}{2})}$ is a good kernel?

Proof. No. Compute using Theorem 5.6.5

$$\begin{aligned} \int_{-\pi}^{\pi} |D_N(x)|\, dx &= \int_{-\pi}^{\pi} \left| \frac{\sin(N+\frac{1}{2})x}{\sin(\frac{x}{2})} \right| dx \\ &\ge 2\int_{-\pi}^{\pi} \frac{|\sin(N+\frac{1}{2})x|}{|x|} dx \end{aligned}$$

Using $u = (N+\frac{1}{2})x, dx = \frac{du}{N+\frac{1}{2}}$, we have the approximation

$$
\begin{aligned}
2\int_{-(N+\frac{1}{2})\pi}^{(N+\frac{1}{2})\pi} \frac{|\sin u|}{\left|\frac{u}{N+\frac{1}{2}}\right|}\frac{1}{N+\frac{1}{2}}du &= 4\int_0^{(N+\frac{1}{2})\pi} \frac{|\sin u|}{|u|}du \\
&\geq 4\Big(\int_0^{\pi} \frac{\sin u}{u}du + \int_{\pi}^{N\pi} \frac{|\sin u|}{|u|}du + \int_{N\pi}^{(N+\frac{1}{2})\pi} \frac{|\sin u|}{|u|}du\Big) \\
&\geq 4\Big(\int_0^{\pi} \frac{\sin u}{\pi}du + \int_{\pi}^{N\pi} \frac{|\sin u|}{|u|}du + \int_{N\pi}^{(N+\frac{1}{2})\pi} \frac{|\sin u|}{(N+\frac{1}{2})\pi}du\Big) \\
&= 4\int_{\pi}^{N\pi} \frac{|\sin u|}{|u|}du + \frac{8}{\pi} + \frac{4}{(N+\frac{1}{2})\pi} \\
&= 4\sum_{k=1}^{N-1}\int_{k\pi}^{(k+1)\pi} \frac{|\sin u|}{|u|}du + \frac{8}{\pi} + \frac{4}{(N+\frac{1}{2})\pi} \\
&\geq 4\sum_{k=1}^{N-1}\frac{1}{(k+1)\pi}\int_{k\pi}^{(k+1)\pi} |\sin u|\, du + \frac{8}{\pi} + \frac{4}{(N+\frac{1}{2})\pi} \\
&= 4\sum_{k=1}^{N-1}\frac{2}{(k+1)\pi} + \frac{8}{\pi} + \frac{4}{(N+\frac{1}{2})\pi} \\
&= \frac{8}{\pi}\sum_{k=1}^{N-1}\frac{1}{k+1} + \frac{8}{\pi} + \frac{4}{(N+\frac{1}{2})\pi} \to \infty
\end{aligned}
$$

where the last expression tends to infinity because $\sum_{k=1}^{N}\frac{1}{k}$ tends to infinity and the other two terms stay bounded.

We have now seen

$$
\begin{aligned}
\int_{-\pi}^{\pi} |D_N(x)|\, dx &\geq 2\int_{-\pi}^{\pi} \frac{\left|\sin(N+\frac{1}{2})x\right|}{|x|}dx \\
&= 2\int_{-(N+\frac{1}{2})\pi}^{(N+\frac{1}{2})\pi} \frac{|\sin u|}{\left|\frac{u}{N+\frac{1}{2}}\right|}\frac{1}{N+\frac{1}{2}}du \to \infty \text{ as } N \to \infty
\end{aligned}
$$

This shows that the Dirichlet's Kernel $D_N(x)$ does NOT satisfy the second criterion. ■

Lemma 5.6.6.

$$D_N(x) \triangleq \sum_{n=-N}^{N} e^{-inx} = 1 + 2\sum_{n=1}^{N}\cos nx = \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}$$

Proof.

$$\sum_{n=-N}^{N} e^{-inx} = 1 + 2\sum_{n=1}^{N} \big(\cos nx + i\sin nx + \cos nx - i\sin nx\big)$$
$$= 1 + 2\sum_{n=1}^{N} \cos nx$$

■

Lemma 5.6.7.

$$|\sin x| \geq \frac{|x|}{2} \qquad (x \in [-\frac{\pi}{2}, \frac{\pi}{2}])$$

Proof. Because both $|\sin x|$ and $\frac{|x|}{2}$ are both odd and positive, WOLG, it suffices to just prove for $x \in (0, \frac{\pi}{2}]$.

Notice that $\sin x$ is concave on $[0, \frac{\pi}{2}]$ by computing second derivative.

Then, for all $x \in [0, \frac{\pi}{2}]$, we have

$$\sin x \geq \sin 0 + x\frac{\sin\frac{\pi}{2} - \sin 0}{\frac{\pi}{2} - 0}$$

This give us

$$\sin x \geq \frac{2x}{\pi} \geq \frac{x}{2} \quad (\because 2 \geq \frac{\pi}{2})$$

■

Question 22

6. Determine whether the Fejér kernel $F_N(x) = \dfrac{1}{N}\displaystyle\sum_{n=0}^{N-1} D_n(x) = \dfrac{1}{N}\dfrac{\sin^2(\frac{Nx}{2})}{\sin^2(\frac{x}{2})}$ is a good kernel?

Proof. Yes. For first condition, compute

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)dx &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{N+1}\sum_{n=0}^{N}\frac{\sin(n+\frac{1}{2})x}{\sin\frac{x}{2}}dx \\
&= \frac{1}{2\pi(N+1)}\sum_{n=0}^{N}\int_{-\pi}^{\pi} D_n(x)dx \\
&= \frac{1}{2\pi(N+1)}\sum_{n=0}^{N}\int_{-\pi}^{\pi}(1+2\sum_{k=1}^{n}\cos kx)dx \quad (\text{Lemma 5.6.6}) \\
&= \frac{1}{2\pi(N+1)}\sum_{n=0}^{N} 2\pi = 1 \quad (\because \int_{-\pi}^{\pi}\cos kx = 0)
\end{aligned}$$

For second condition, just not that F_N is positive, so

$$\int_{-\pi}^{\pi} |F_n(x)|\, dx = \int_{-\pi}^{\pi} F_n(x) = 2\pi$$

For third condition, suppose $0 < \delta \leq |x| \leq \pi$.

Using Lemma 5.6.7 to compute

$$0 \leq F_n(x) = \frac{\sin^2\frac{nx}{2}}{n\sin^2\frac{x}{2}} \leq \frac{1}{n\sin^2\frac{x}{2}} \leq \frac{1}{n(\frac{x}{4})^2} \leq \frac{1}{n(\frac{\delta}{4})^2} \searrow 0 \text{ as } n\to\infty$$

Then

$$\int_{\delta\leq|x|\leq\pi} F_n(x)dx \leq \int_{\delta\leq|x|\leq\pi}\frac{16}{n\delta^2}dx = \frac{32(\pi-\delta)}{n\delta^2} \searrow 0 \text{ as } n\to\infty$$

■

Question 23

7. The **Poisson kernel** is given by $P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}$, $-\pi \leq \theta \leq \pi$. Show that if $0 \leq r < 1$, then $P_r(\theta) = \dfrac{1-r^2}{1-2r\cos\theta + r^2}$.

Proof. Compute

$$
\begin{aligned}
P_r(\theta) &= \sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta} \\
&= 1 + \sum_{n=1}^{\infty} r^n e^{in\theta} + \sum_{n=1}^{\infty} r^n e^{-in\theta} \\
&= 1 + \frac{re^{i\theta}}{1-re^{i\theta}} + \frac{re^{-i\theta}}{1-re^{-i\theta}} \\
&= 1 + \frac{re^{i\theta}(1-re^{-i\theta}) + re^{-i\theta}(1-re^{i\theta})}{(1-re^{i\theta})(1-re^{-i\theta})} \\
&= 1 + \frac{re^{i\theta}+re^{-i\theta}-2r^2}{1-re^{i\theta}-re^{-i\theta}+r^2} \\
&= 1 + \frac{2r\cos\theta - 2r^2}{1-2r\cos\theta+r^2} = \frac{1-r^2}{1-2r\cos\theta+r^2}
\end{aligned}
$$

∎

Question 24

8. If $0 \leq r < 1$, Determine whether the Poisson kernel kernel is a good kernel?

Proof. Yes. For first condition, compute

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}d\theta \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} r^{|n|}\int_{-\pi}^{\pi} e^{in\theta}d\theta \\
&= \frac{1}{2\pi}r^0 2\pi = 1
\end{aligned}
$$

For second condition, note that

$$1-2r\cos\theta+r^2 \geq 1-2r+r^2 = (1-r)^2 \in \mathbb{R}^+$$

Then because $1-r^2 \in \mathbb{R}^+$, we see

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2} \in \mathbb{R}^+$$

We now have

$$\int_{-\pi}^{\pi} |P_r(\theta)|\, d\theta = \int_{-\pi}^{\pi} P_r(\theta) d\theta = 1$$

Note that P_r is even, we then can reduce proving the third critirion into proving

$$P_r(\theta) \to 0 \text{ uniformly on } [\delta, \pi] \text{ as } r \nearrow 1$$

Compute

$$P_r'(\theta) = \frac{-2r \sin\theta(1 - r^2)}{(1 - 2r\cos\theta + r^2)^2} < 0 \text{ on } [\delta, \pi]$$

This then give us

$$P_r(\theta) \leq P_r(\delta) \text{ on } [\delta, \pi]$$

Compute

$$P_r(\delta) = \frac{1 - r^2}{(1 - r)^2 + 2r(1 - \cos\delta)} \to 0 \text{ as } r \nearrow 1$$

and we are done (done) ■