Calculus Done Taiwan

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Contents

Chapter 1

Selected topics

1.1 Compact

Let X be a topological space. We say X is **compact** if every of its open cover has a finite subcover. We say X is **sequentially compact** if every sequence in X has a **convergent** subsequence. We say X is **limit point compact** if every infinite subset has a limit point.

Let M be a metric space. We say M is **totally bounded** if for all ϵ, space M can be covered by finite number of open balls of radius ϵ. We say M is **complete**, if every Cauchy sequence in M converge.

Equivalent Definition 1.1.1. (Compactness of metric space) Let M be a metric space. The followings are equivalent:

(i) M is compact.

(ii) M is limit point compact.

(iii) M is sequentially compact.

(iv) M is totally bounded and complete.

Proof. (i) $\implies$ (ii): The argument by nature relies on contradiction. Assume for a contradiction that some sequence (x_n) of distinct elements in M whose image has no limit point[1]. For each $x \in M$, since x is not a limit point of $\{x_n\}$, there exists some open ball centered at x that contains at most one point of $\{x_n\}$, namely x itself if and only if it appears in the sequence. The collection of these open balls is then a open cover of M that has no finite subcover of M because $\{x_n\}$ is infinite, a contradiction.

[1]Note that if finite metric spaces are trivially limit point compact.

(ii) $\implies$ (iii) is clear.

(iii) $\implies$ (iv): It is easy to show Cauchy sequences converge in sequentially compact metric space. We now show M is totally bounded. Assume M is not for a contradiction. This implies the existence of some ϵ and some sequence of distinct elements in M whose every points are ϵ-away. Such sequence has no convergent subsequence, a contradiction.

(iv) $\implies$ (i): Again the argument by nature relies on contradiction. Assume for one that (U_α) is an open cover of M that has no finite subcover. Due to totally boundedness of M, we know M can be covered by some finite set of radius 1 open balls. Since (U_α) has no finite subcover, one of these radius 1 open balls can't be covered by finite numbers of (U_α). Denote the center of that radius 1 open ball by x_1. Clearly, $B_1(x_1)$ is again totally bounded, so it can be covered by some finite set of radius $\frac{1}{2}$ open balls centered inside. Again, because $B_1(x_1)$ can not be covered by finite numbers of (U_α), one of these radius $\frac{1}{2}$ open balls can't be covered by finite number of (U_α). Denote the center of that radius $\frac{1}{2}$ open ball by x_2. Repeating the same procedure, we have a Cauchy sequence $\{x_n\} \subseteq M$. Let x be the limit of this Cauchy sequence. Fix some α such that $x \in B_\epsilon(x) \subseteq U_\alpha$. Clearly, we may find N large enough, for example $N > 2/\epsilon$, so that $B_{\frac{1}{N}}(x_N) \subseteq B_\epsilon(x) \subseteq U_\alpha$, a contradiction to the construction of $B_{\frac{1}{N}}(x_N)$. $\blacksquare$

1.2 Arzelà–Ascoli theorem

Let X be a topological space, Y a metric space, and $\mathcal{F}$ a family of function from X to Y. We say $\mathcal{F}$ is **pointwise equicontinuous** if for all ϵ and $x \in X$, there exists a neighborhood $U \ni x$ such that $d(f(x), f(y)) \leq \epsilon$ for all $y \in U$ for all $f \in \mathcal{F}$. Give X a compatible metric. We say $\mathcal{F}$ is **uniform equicontinuous** if for all ϵ there exists δ such that $d(f(x), f(y)) \leq \epsilon$ for all δ-close $x, y \in X$ and $f \in \mathcal{F}$. Clearly uniform equicontinuity implies pointwise equicontinuity and that every $f \in \mathcal{F}$ is uniformly continuous. The converse isn't true.

Example 1.2.1. Let $\varphi(t) \triangleq \max\{0, 1 - |t|\}$. Define

$$f_n(x) \triangleq n \cdot \varphi(n(x-n)), \quad n \in \mathbb{N}$$

Every f_n is uniformly continuous on $\mathbb{R}$ and the collection $\mathcal{F} = \{f_n : \mathbb{R} \to \mathbb{R} | n \in \mathbb{N}\}$ is pointwise equicontinuous, but $\mathcal{F}$ is not uniform equicontinuous.

Theorem 1.2.2. (Pointwise equicontinuity implies uniform equicontinuity on compact domain) Let X, Y be two metric spaces and $\mathcal{F}$ a pointwise equicontinuous collection of functions from X to Y. If X is compact, then $\mathcal{F}$ is moreover uniform equicontinuous.

Proof. Fix ϵ. Because $\mathcal{F}$ is pointwise equicontinuous and X is compact, we know there exists a finite open cover $\mathcal{U} \triangleq \{B_{\delta_1 \diagup 2}(x_1), \dots, B_{\delta_n \diagup 2}(x_n)\}$ of X such that for all fixed i, we have:

$$d(f(x_i), f(y)) \leq \frac{\epsilon}{2}, \quad \text{for all } y \in B_{\delta_i}(x_i) \text{ and } f \in \mathcal{F}$$

Let $x, y \in X$ satisfies $d(x, y) < \min \delta_i \diagup 2$. Because $\mathcal{U}$ forms an open cover, we know $x \in B_{\delta_i \diagup 2}(x_i)$ for some i. Then clearly, $y \in B_{\delta_i}(x_i)$. This now give us:

$$d(f(x), f(y)) \leq d(f(x_i), f(x)) + d(f(x_i), f(y)) \leq \epsilon, \quad \text{for all } f \in \mathcal{F}$$

■

Theorem 1.2.3. (Arzelà–Ascoli theorem) Let X be a compact metric space, and let $\mathcal{F}$ be a collection of continuous function from X to $\mathbb{R}^n$. If $\mathcal{F}$ is pointwise equicontinuous and pointwise bounded, then:

(I) $\mathcal{F}$ is uniformly bounded.

(II) Every sequence $\{f_n\} \subseteq \mathcal{F}$ has a uniformly convergent subsequence.

Theorem 1.2.4. (Sufficient conditions for uniform equicontinuity) Let M be a metric space and $\mathcal{F}$ a collection of function from M to $\mathbb{R}$. If for some $K > 0$, every $f \in \mathcal{F}$

is K-Lipschitz, then $\mathcal{F}$ is uniform equicontinuous. In particular, if M is an open subset of $\mathbb{R}$ and $\mathcal{F}$ is a collection of differentiable functions whose derivatives are uniformly bounded by some K:

$$|f'(x)| \leq K, \quad \text{for all } x \in U \text{ and } f \in \mathcal{F} \tag{1.1}$$

then $f \in \mathcal{F}$ are all K-Lipschitz, thus making $\mathcal{F}$ uniform equicontinuous.

Proof. If the functions are all K-Lipschitz, then $\delta = \frac{\epsilon}{K}$ suffices to show uniform equicontinuity. If the functions are all differentiable and subject to condition 1.1, then **MVT** shows that they are all K-Lipschitz. ∎

Theorem 1.2.5. (Application scene of Arzelà–Ascoli theorem) Let X, Y be two metric spaces, and $\{f_n\}$ a sequence of functions from X to Y. If every subsequence contains a uniformly convergent subsequence, then every pointwise convergent subsequence uniformly converge.

Proof. Assume for a contradiction that $\{f_{n_k}\}$ is a pointwise convergent subsequence that doesn't converge uniformly. Let f be the limit of f_{n_k}. There exists some ϵ and some subsequence $\left\{f_{n_{k_l}}\right\}$ such that for all l there exists some $x_l \in X$ such that

$$d\left(f_{n_{k_l}}(x_l), f(x_l)\right) \geq \epsilon$$

Clearly, $\{f_{n_k}\}$ contain no uniformly convergent subsequence, a contradiction. ∎

1.3 Weierstrass approximation theorem

Theorem 1.3.1. (Weierstrass approximation theorem) Let $[a, b] \subseteq \mathbb{R}$ be compact, and $\mathcal{C}([a, b])$ the $\mathbb{C}$-vector space of continuous function $f : [a, b] \to \mathbb{R}$. Define the **uniform norm** on $\mathcal{C}([a, b])$ by $\|f\|_\infty \triangleq \max\{|f(x)| \in \mathbb{R} : x \in [a, b]\}$. Then the space of polynomial $\mathbb{C}[x]|_{[a,b]}$ is dense in $(\mathcal{C}([a, b]), \|\cdot\|_\infty)$.

Proof. ■

Let K be a compact metric space. Let A be a sub-$\mathbb{C}$-algebra of the $\mathbb{C}$-algebra of continuous functions $f : K \to \mathbb{C}$.

Theorem 1.3.2. (Stone-Weierstrass theorem)

1.4 Tests

Abstract

This section prove some basic result on sequence and series, which will be heavily used in next section on analytic functions and Chapter: Beauty. Although written in an almost glossary form, we present the Theorems in a structural order based on the necessity of notion of absolute convergence and limit superior. Note that in this section, z, v, w always represent complex numbers, and a, b, c always represent real numbers.

Theorem 1.4.1. **(Weierstrass M-test)** Given sequences $f_n : X \to \mathbb{C}$, and suppose

$$\forall n \in \mathbb{N}, \forall x \in X, |f_n(x)| \leq M_n$$

Then

$$\sum_{n=1}^{\infty} M_n \text{ converge } \implies \sum_{n=1}^{\infty} f_n \text{ uniformly converge}$$

Proof. The proof follows from noting

$$\forall x \in X, \left| \sum_{k=m}^{n} f_k(x) \right| \leq \sum_{k=m}^{n} |f_k(x)| \leq \sum_{k=m}^{n} M_k$$

∎

Note that in our proof of Weierstrass M-test, we reduce the proof for uniform convergence into uniform Cauchy, which is a technique we shall also use later in Abel's test for uniform convergence. We now prove summation by part, which is a result hold in all fields, and is the essence of the proof of Dirichlet's test and Abel's test for uniform convergence.

Theorem 1.4.2. **(Summation by Part)**

$$\begin{aligned} f_n g_n - f_m g_m &= \sum_{k=m}^{n-1} f_k \Delta g_k + g_k \Delta f_k + \Delta f_k \Delta g_k \\ &= \sum_{k=m}^{n-1} f_k \Delta g_k + g_{k+1} \Delta f_k \end{aligned}$$

Proof. The proof follows induction which is based on

$$f_m g_m + f_m \Delta g_m + g_m \Delta f_m + \Delta f_m \Delta g_m = f_{m+1} g_{m+1}$$

∎

Theorem 1.4.3. (Dirichlet's Test) Suppose

(a) $a_n \to 0$ monotonically.

(b) $\sum_{n=1}^{N} z_n$ is bounded.

We have

$$\sum a_n z_n \text{ converge}$$

Proof. Define $Z_n \triangleq \sum_{n=1}^{N} z_n$ and let M bound $|Z_n|$. Using summation by part by letting $f_k = a_k$ and $g_k = Z_{k-1}$, we have

$$\begin{aligned}
\left|\sum_{k=m}^{n} a_k z_k\right| &= \left|a_{n+1}Z_n - a_m Z_{m-1} - \sum_{k=m}^{n} Z_k(a_{k+1}-a_k)\right| \\
&\leq |a_{n+1}Z_n| + |a_m Z_{m-1}| + \left|\sum_{k=m}^{n} Z_k(a_{k+1}-a_k)\right| \\
(\because a_n \text{ is monotone }) \qquad &\leq M\Big(|a_{n+1}| + |a_m| + |a_{n+1}-a_m|\Big)
\end{aligned}$$

■

Theorem 1.4.4. (Abel's Test for Uniform Convergence) Suppose $g_n : X \to \mathbb{R}$ is a uniformly bounded pointwise monotone sequence. Then given a sequence $f_n : X \to \mathbb{R}$,

$$\sum f_n \text{ uniformly converge} \implies \sum f_n g_n \text{ uniformly converge}$$

Proof. Define $R_n \triangleq \sum_{k=n}^{\infty} f_k$. Let M uniformly bound g_n. Because $R_n \to 0$ uniformly, we can let N satisfy

$$\forall n \geq N, \forall x \in X, |R_n(x)| < \frac{\epsilon}{6M}$$

Then for all $n, m \geq N$, using summation by part, we have

$$\begin{aligned}
\left|\sum_{k=m}^{n} f_k g_k\right| &= \left|\sum_{k=m}^{n} g_k \Delta R_k\right| \\
&\leq |R_{n+1}g_{n+1}| + |R_{m+1}g_{m+1}| + \sum_{k=m}^{n} |R_{k+1}\Delta g_k| \\
(\because g_n \text{ is pointwise monotone }) \qquad &\leq |R_{n+1}g_{n+1}| + |R_{m+1}g_{m+1}| + \frac{\epsilon}{6M}|g_{n+1}-g_m| \leq \epsilon
\end{aligned}$$

■

Although the proofs of Dirichlet's test and Abel's test for uniform convergence are quite similar, one should note that the "ways" summation by part is applied are slightly different, as one use $R_n \triangleq \sum_{k=n}^{\infty} f_k$ instead of $\sum_{k=1}^{n} f_k$, like $Z_n \triangleq \sum_{j=1}^{n} z_j$. As corollaries of Dirichlet's test, one have the famous **alternating series test** and Abel's test for complex series.

Theorem 1.4.5. (Abel's Test for Complex Series) Suppose

(a) $\sum z_n$ converge.

(b) b_n is a bounded monotone sequence.

We have

$$\sum z_n b_n \text{ converge}$$

Proof. Denote $B \triangleq \lim_{n\to\infty} b_n$. By Dirichlet's Test, we know $\sum z_n(b_n - B)$ converge. The proof now follows form noting

$$\sum z_n b_n = \sum z_n(b_n - B) + B\sum z_n$$

■

We now introduce the idea of absolute convergence, which we shall use throughout the remaining of the section. By a **permutation** $\sigma : E \to E$ on some set E, we merely mean σ is a bijective function. We say $\sum z_n$ **absolutely converge** if $\sum |z_n|$ converge, and say $\sum z_n$ **unconditionally converge** if for all permutation $\sigma : \mathbb{N} \to \mathbb{N}$, the series $\sum z_{\sigma(n)}$ converge and converge to the same value.

Theorem 1.4.6. (Absolutely Convergent Series Unconditionally Converge)

$$\sum z_n \text{ absolutely converge } \implies \sum z_n \text{ unconditionally converge}$$

Proof. The fact $\sum z_n$ converge follows from noting

$$\left|\sum_{k=n}^{m} z_k\right| \leq \sum_{k=n}^{m} |z_k| \leq \sum_{k=n}^{\infty} |z_k|$$

Now, fix ϵ and permutation σ. Let N_1 and N_2 satisfy

$$\sum_{n=N_1}^{\infty} |z_n| < \frac{\epsilon}{2} \text{ and } \left|\sum_{n=N}^{\infty} z_n\right| < \frac{\epsilon}{2} \text{ for all } N > N_2$$

Let $M \triangleq \max\{N_1, N_2\}$. Observe that for all $N > \max_{1\le r\le M} \sigma^{-1}(r)$, we have

$$\left|\sum z_n - \sum_{n=1}^{N} z_{\sigma(n)}\right| \le \left|\sum_{n=M+1}^{\infty} z_n\right| + \sum_{n=M+1}^{\infty} |z_n| < \epsilon$$

■

Theorem 1.4.7. (Riemann Rearrangement Theorem) If $\sum a_n$ converge but not absolutely, then for each $L \in \overline{\mathbb{R}}$, there exists a permutation σ such that

$$\sum a_{\sigma(n)} = L$$

Proof. Define a_n^+ and a_n^- by

$$a_n^+ \triangleq \max\{a_n, 0\} \text{ and } a_n^- \triangleq \min\{a_n, 0\}$$

Because

$$\sum (a_n^+ + a_n^-) \text{ converge but } \sum (a_n^+ - a_n^-) = \infty$$

We know

$$\sum a_n^+ = \sum (-a_n^-) = \infty$$

WOLG, (why?), fix $L \in \mathbb{R}$ and suppose $a_n \neq 0$ for all n. Let $A = B = L$, and let two increasing sequence $\sigma^+, \sigma^- : \mathbb{N} \to \mathbb{N}$ satisfy

$$\sigma^+(k+1) = \min\left\{n \in \mathbb{N} : a_n > 0 \text{ and } n > \sigma^+(k)\right\}$$

and similar for σ^-. Now, recursively define p_k, q_k by

$$p_1 \text{ is the smallest number such that } \sum_{n=1}^{p_1} a_{\sigma^+(n)} \ge A \tag{1.2}$$

$$q_1 \text{ is the smallest number such that } \sum_{n=1}^{p_1} a_{\sigma^+(n)} + \sum_{n=1}^{q_1} a_{\sigma^-(n)} \le B \tag{1.3}$$

$$p_{k+1} \text{ is the smallest number such that } \sum_{n=1}^{p_{k+1}} a_{\sigma^+(n)} + \sum_{n=1}^{q_k} a_{\sigma^-(n)} \ge A \tag{1.4}$$

$$q_{k+1} \text{ is the smallest number such that } \sum_{n=1}^{p_{k+1}} a_{\sigma^+(n)} + \sum_{n=1}^{q_{k+1}} a_{\sigma^-(n)} \le B \tag{1.5}$$

We then define σ by

$$\sigma^+(1), \dots, \sigma^+(p_1), \sigma^-(1), \dots, \sigma^-(q_1), \sigma^+(p_1+1), \dots, \sigma^+(p_2), \sigma^-(q_1+1), \dots, \sigma^-(q_2), \dots$$

It then follows from

$$\left|\sum_{n=1}^{p} a_{\sigma^+}(n) + \sum_{n=1}^{q_k} a_{\sigma^-}(n) - L\right| \leq \min\left\{a_{\sigma^+(p_{k+1})}, \left|a_{\sigma^-(q_k)}\right|\right\} \text{ for all } p_k \leq p \leq p_{k+1}$$

and $a_n \to 0$ that $\sum a_{\sigma(n)} = L$. ■

Note that the method we deploy in the proof of Riemann rearrangement Theorem can be used to control the sequence to have arbitrary large set of subsequential limits by modifying the number of A, B in Equation (4.1), (4.2), (4.3) and (4.4).

Using Riemann rearrangement Theorem and equation

$$\max_{1\leq r\leq d} |x_n| \leq |\mathbf{x}| \leq \sum_{r=1}^{d} |x_r|$$

we can now generalize and strengthen Theorem 1.4.6 to

$$\begin{aligned}\sum \mathbf{x}_n \text{ absolutely converge } &\iff \sum_n x_{n,r} \text{ absolutely converge for all } r\\ &\iff \sum_n x_{n,r} \text{ unconditionally converge for all } r\\ &\iff \sum \mathbf{x}_n \text{ unconditionally converge}\end{aligned}$$

With this in mind, we can now well state the Fubini's Theorem for Double Series.

Theorem 1.4.8. (Fubini's Theorem for Double Series) If

$$\sum_n \sum_k |z_{n,k}| \text{ converge}$$

Then

$$\sum_{n,k} |z_{n,k}| \text{ converge and } \sum_{n,k} z_{n,k} = \sum_n \sum_k z_{n,k} = \sum_k \sum_n z_{n,k}$$

Proof. The fact $\sum z_{n,k}$ absolutely converge follow from

$$\sum_{n=1}^{N} \sum_{k=1}^{N} |z_{n,k}| \leq \sum_n \sum_k |z_{n,k}| \text{ for all } N$$

WOLG, it remains to prove

$$\sum_{n,k} z_{n,k} = \sum_n \sum_k z_{n,k}$$

Because $\sum_n \sum_k |z_{n,k}|$ converge, we can reduce the problem into proving the same statement for nonnegative series $a_{n,k}$. (why?)

$$\sum_n \sum_k |a_{n,k}| \text{ converge } \implies \sum_{n,k} a_{n,k} = \sum_n \sum_k a_{n,k}$$

Because

$$\sum_{n=1}^N \sum_{k=1}^N a_{n,k} \leq \sum_{n=1}^N \sum_k a_{n,k} \leq \sum_n \sum_k a_{n,k} \text{ for all } N$$

we see

$$\sum_{n,k} a_{n,k} \leq \sum_n \sum_k a_{n,k}$$

It remains to prove

$$\sum_{n,k} a_{n,k} \geq \sum_n \sum_k a_{n,k}$$

Fix N and ϵ. We reduce the problem into proving

$$\sum_{n,k} a_{n,k} \geq \sum_{n=1}^N \sum_k a_{n,k} - \epsilon$$

Let K satisfy

$$\text{For all } 1 \leq n \leq N, \sum_{k=K+1}^{\infty} a_{n,k} < \frac{\epsilon}{N}$$

It then follows

$$\sum_{n,k} a_{n,k} \geq \sum_{n=1}^N \sum_{k=1}^K a_{n,k} \geq \sum_{n=1}^N \sum_k a_{n,k} - \epsilon \text{ (done)}$$

$\blacksquare$

Example 1 (Counter-Example for Fubini's Theorem for Double Series)

$$a_{n,k} \triangleq \begin{cases} 1 & \text{if } n = k \\ -1 & \text{if } n = k+1 \\ 0 & \text{if otherwise} \end{cases}$$

$$\sum |a_{n,k}| = \infty \text{ and } \sum_n \sum_k a_{n,k} = 1 \text{ and } \sum_k \sum_n a_{n,k} = 0$$

Theorem 1.4.9. (Merten's Theorem for Cauchy Product) Suppose

(a) $\sum_{n=0}^{\infty} z_n$ converge absolutely

(b) $\sum_{n=0}^{\infty} z_n = Z$

(c) $\sum_{n=0}^{\infty} v_n = V$

(d) $w_n = \sum_{k=0}^{n} z_k v_{n-k}$

Then we have

$$\sum_{n=0}^{\infty} w_n = ZV$$

Proof. We prove

$$\left| V \sum_{n=0}^{N} z_n - \sum_{n=0}^{N} w_n \right| \to 0 \text{ as } N \to \infty$$

Compute

$$\begin{aligned} V \sum_{n=0}^{N} z_n - \sum_{n=0}^{N} w_n &= \sum_{n=0}^{N} z_n (V - \sum_{k=0}^{N-n} v_k) \\ &= \sum_{n=0}^{N} z_n \sum_{k=N-n+1}^{\infty} v_k \end{aligned}$$

Because $\sum_{k=n}^{\infty} v_k \to 0$ as $n \to \infty$, we know there exists M such that

$$\left| \sum_{k=n}^{\infty} v_k \right| < M \text{ for all } n$$

Let N_0 satisfy

$$\sum_{n=N_0+1}^{\infty} |z_n| < \frac{\epsilon}{2M}$$

Let $N_1 > N_0$ satisfy

$$\left| \sum_{k=N-N_0+1}^{\infty} v_k \right| < \frac{\epsilon}{2(N_0+1)\sum_n |z_n|} \text{ for all } N > N_1$$

Now observe that for all $N > N_1$

$$\left| \sum_{n=0}^{N} z_n \Big(\sum_{k=N-n+1}^{\infty} v_k \Big) \right| \leq \sum_{n=0}^{N_0} |z_n| \left| \sum_{k=N-n+1}^{\infty} v_k \right| + \sum_{n=N_0+1}^{N} |z_n| \left| \sum_{k=N-n+1}^{\infty} v_k \right| < \epsilon \text{ (done)}$$

■

We first define the **limit superior** by

$$\limsup_{n\to\infty} a_n \triangleq \lim_{n\to\infty} (\sup_{k\geq n} a_k)$$

Note that $\limsup_{n\to\infty} a_n$ must exists because $(\sup_{k\geq n} a_k)_n$ is a decreasing sequence.

Theorem 1.4.10. (Equivalent Definition for Limit Superior) If we let E be the set of subsequential limits of a_n

$$E \triangleq \left\{ L \in \overline{\mathbb{R}} : L = \lim_{k\to\infty} a_{n_k} \text{ for some } n_k \right\}$$

The set E is non-empty and

$$\max E = \limsup_{n\to\infty} a_n$$

Proof. Let $n_1 \triangleq 1$. Recursively, because

$$\sup_{j\geq n_k} a_k \geq \limsup_{n\to\infty} a_n > \limsup_{n\to\infty} a_n - \frac{1}{k} \text{ for each } k$$

We can let n_{k+1} be the smallest number such that

$$a_{n_{k+1}} > \limsup_{n\to\infty} a_n - \frac{1}{k}$$

It is straightforward to check $a_{n_k} \to \limsup_{n\to\infty} a_n$ as $k \to \infty$. Note that no subsequence can converge to $\limsup_{n\to\infty} a_n + \epsilon$ because there exists N such that $\sup_{k\geq N} a_k < \limsup_{n\to\infty} a_n + \epsilon$. ■

We can now state the **limit comparison test** as follows. Given a positive sequence b_n,

$$\limsup_{n\to\infty} \frac{|z_n|}{b_n} \in \mathbb{R} \text{ and } \sum b_n \text{ converge } \implies \sum z_n \text{ absolutely converge}$$
$$\liminf_{n\to\infty} \frac{b_n}{|z_n|} > 0 \text{ and } \sum z_n \text{ diverge } \implies \sum b_n \text{ diverge}$$

Theorem 1.4.11. (Geometric Series)

$$|z| < 1 \implies \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

Proof. The proof follows from noting

$$(1-z)\sum_{n=0}^{N} z^n = 1 - z^{N+1} \to 1 \text{ as } N \to \infty$$

■

Theorem 1.4.12. (Ratio and Root Test)

$$\limsup_{n\to\infty} \sqrt[n]{|z_n|} < 1 \text{ or } \limsup_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| < 1 \implies \sum z_n \text{ absolutely converge}$$
$$\liminf_{n\to\infty} \sqrt[n]{|z_n|} > 1 \text{ or } \liminf_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| > 1 \implies \sum z_n \text{ diverge}$$

Proof. The convergent part follows from comparison to an appropriate geometric series and the diverge part follows from noting $|z_n|$ does not converge to 0. ■

Theorem 1.4.13. (Root Test is Stronger Than Ratio Test)

$$\liminf_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| \leq \liminf_{n\to\infty} \sqrt[n]{|z_n|} \leq \limsup_{n\to\infty} \sqrt[n]{|z_n|} \leq \limsup_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right|$$

Proof. Fix ϵ and WLOG suppose $\liminf_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| > 0$. We prove

$$\liminf_{n\to\infty} \sqrt[n]{|z_n|} \geq \liminf_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| - \epsilon$$

Let $\alpha \in \mathbb{R}$ satisfy

$$\liminf_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| - \epsilon < \alpha < \liminf_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right|$$

Let N satisfy

$$\text{For all } n \geq N, \left|\frac{z_{n+1}}{z_n}\right| > \alpha$$

We then see

$$\sqrt[N+n]{|z_{N+n}|} \geq \sqrt[N+n]{|z_N|\,\alpha^n} = \alpha\Big(\frac{|z_N|^{\frac{1}{N+n}}}{\alpha^{\frac{N}{N+n}}}\Big) \to \alpha \text{ as } n \to \infty \text{ (done)}$$

The proof for the other side is similar. ∎

Theorem 1.4.14. (Root Test Trick) For all $k \in \mathbb{N}$

$$\limsup_{n\to\infty} |z_{n+k}|^{\frac{1}{n}} = \limsup_{n\to\infty} |z_n|^{\frac{1}{n}}$$

Proof. This is a direct corollary of equivalent definition for limit superior. ∎

Lastly, we prove Cauchy's condensation Test, whose existence is almost solely for investigating p-Series.

Theorem 1.4.15. (Cauchy's Condensation Test) Suppose $a_n \searrow 0$. We have

$$\sum_{n=0}^{\infty} 2^n a_{2^n} \text{ converge} \iff \sum_{n=1}^{\infty} a_n \text{ converge}$$

Proof. Observe that for all $N \in \mathbb{N}$

$$\sum_{n=0}^{N} 2^n a_{2^n} \geq \sum_{n=0}^{N}\sum_{k=1}^{2^n} a_{2^n+k-1} = \sum_{n=1}^{2^{N+1}-1} a_n$$

and

$$2\sum_{n=1}^{2^N-1} a_n = 2\sum_{n=1}^{N}\sum_{k=0}^{2^{n-1}-1} a_{2^{n-1}+k} \geq 2\sum_{n=1}^{N} 2^{n-1}a_{2^n} = \sum_{n=1}^{N} 2^n a_{2^n}$$

∎

Theorem 1.4.16. (p-Series)

$$\sum_{n=1}^{\infty}\frac{1}{n^p} \text{ converge} \iff p > 1$$

Proof. Observe that

$$\sum_{n=0}^{\infty} 2^n \frac{1}{(2^n)^p} = \sum_{n=0}^{\infty} (2^{1-p})^n$$

The result then follows from Cauchy's Condensation Test and geometric series. ∎

1.5 Analytic Functions

Abstract

This section introduces the concept of analytic functions and proves some of their basic properties, including the Identity Theorem. We will rely on the tools developed in the previous section on sequences and series. Note that throughout this section, z will always denote a complex number.

In this section, by a **power series**, we mean a pair (z_0, c_n) where $z_0 \in \mathbb{C}$ is called the **center** of power series, and $c_n \in \mathbb{C}$ are the coefficients sequence. By **radius of convergence**, we mean a unique $R \in \mathbb{R}_0^+ \cup \infty$ such that

$$\sum_{n=0}^{\infty} c_n(z-z_0)^n \begin{cases} \text{converge absolutely} & \text{if } |z-z_0|<R \\ \text{diverge} & \text{if } |z-z_0|>R \end{cases}$$

Such R always exist (and is unique, the uniqueness can be checked without computing the actual value of R) and is exactly

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{c_n}} \tag{1.6}$$

This result is called **Cauchy-Hadamard Theorem** and is proved by applying Root Test to $\sum c_n(z-z_0)^n$. Note that Cauchy-Hadamard Theorem does not tell us whether a power series converges at points of boundary of disk of convergence. It require extra works to determine if the power series converge at boundary.

Theorem 1.5.1. (Abel's Test for Power Series) Suppose $a_n \to 0$ monotonically and $\sum a_n z^n$ has radius of convergence R .

$$\text{The power series } \sum a_n z^n \text{ at least converge on } \overline{D_R(0)} \setminus \{R\}$$

Proof. Note that

$$\sum \frac{a_n}{R^n} z^n \text{ has radius of convergence } R$$

Fix $z \in \overline{D_R(0)} \setminus \{R\}$. Note that

$$\left| \sum_{n=0} \frac{z^n}{R^n} \right| = \left| \frac{1-(\frac{z}{R})^{N+1}}{1-\frac{z}{R}} \right| \leq \frac{2}{|1-\frac{z}{R}|} \text{ for all } N$$

It then follows from Dirichlet's Test that $\sum a_n (\frac{z}{R})^n$ converge. ∎

Example 2 (Discussion of Convergence on Boundary)

$$f_q(z) = \sum_{n=0}^{\infty} n^q z^n \text{ provided } q \in \mathbb{R}$$

It is clear that f_q has convergence radius 1 for all $q \in \mathbb{R}$. For boundary, we have

$$\begin{cases} q < -1 \implies f_q \text{ converge on } S^1 \\ -1 \leq q < 0 \implies f_q \text{ converge on } S^1 \setminus \{1\} \\ 0 \leq q \implies f_q \text{ diverge on } S^1 \end{cases}$$

Note that

(a) At $z = 1$, the discussion is just p-series.

(b) $n^q \searrow 0$ if and only if $q < 0$; and if $n^q \searrow 0$, then the series converge by Abel's test for power series.

(c) If $q \geq 0$, $n^q z^n$ does not converge to 0 on $S^1 \setminus \{1\}$

Notice that the fact $\sum c_n(z-z_0)^n$ absolutely converge in $D_R(z_0)$ implies the convergence is uniform on all $\overline{D_{R-\epsilon}(z_0)}$ by M-Test. However, on $D_R(z_0)$, the convergence is not always uniform.

Example 3 (Failure of Uniform Convergence on $D_R(z_0)$)

$$f(z) = \sum_{n=0}^{\infty} z^n$$

Note $R = 1$. Use Geometric series formula to show $f(z) = \frac{1}{1-z}$ on $D_1(0)$. It is then clear that f is unbounded on $D_1(0)$ while all partial sums $\sum_{k=0}^{n} z^k$ is bounded on $D_1(0)$.

We now introduce some terminologies. We say a complex function f is **analytic at** $z_0 \in \mathbb{C}$ if f there exists a power series (z_0, c_n) whose convergence radius is greater than 0 and f agrees with $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ on $D_R(z_0)$ for some R (of course, such R must not be strictly greater than the radius of convergence of (a, c_n)). It shall be quite clear that if f, g are both analytic at $z \in \mathbb{C}$ with radius $R_f \leq R_g$, then by Merten's Theorem for Cauchy product, $f + g$ and fg are analytic at z with radius at least R_f. We now investigate deeper into analytic functions.

Theorem 1.5.2. (Term by Term Differentiation) Given a power series (z_0, c_n) of convergence radius $R > 0$, if we define $f : D_R(z_0) \to \mathbb{C}$ by

$$f(z) \triangleq \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

Then f is holomorphic on $D_R(z_0)$ and its derivative at z_0 is also a power series with radius of convergence R

$$f'(z) = \sum_{n=0}^{\infty} (n+1) c_{n+1} (z - z_0)^n$$

Proof. Because $(n+1)^{\frac{1}{n}} \to 1$, we can use Theorem 1.4.14 to deduce

$$\limsup_{n\to\infty} \left((n+1) \left| c_{n+1} \right| \right)^{\frac{1}{n}} = \limsup_{n\to\infty} \left| c_{n+1} \right|^{\frac{1}{n}} = \limsup_{n\to\infty} \left| c_n \right|^{\frac{1}{n}}$$

which implies that the power series $\sum_{n=0}^{\infty} (n+1) c_{n+1} (z - z_0)^n$ is of radius of convergence R. We now prove

$$f'(z) = \sum_{n=0}^{\infty} (n+1) c_{n+1} (z - z_0)^n \text{ on } D_R(z_0)$$

Define $f_m : D_R(z_0) \to \mathbb{C}$ by

$$f_m(z) \triangleq \sum_{n=0}^{m} c_n (z - z_0)^n$$

Observe

(a) $f_m \to f$ pointwise on $D_R(a)$

(b) $f_m'(z) = \sum_{n=0}^{m-1} (n+1) c_{n+1} (z - z_0)^n$ for all m

Fix $z \in D_R(z_0)$. Proposition (b) allow us to reduce the problem into proving

$$f'(z) = \lim_{m\to\infty} f_m'(z) \text{ on } D_R(a) \tag{1.7}$$

Let $z \in D_r(z_0)$ where $r < R$. With proposition (a) in mind, to show Equation 1.7, by Theorem **??**, we only have to prove f_m' uniformly converge on $D_r(z_0)$, which follows from M-Test and the fact that $\sum_{n=0}^{\infty} (n+1) c_{n+1} (z - z_0)^n$ absolutely converge on $D_R(z_0)$. (done) ∎

Suppose

$$f(z) \triangleq \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

Now by repeatedly applying Theorem 1.5.2, we see

$$f^{(k)}(z) = \sum_{n=0}^{\infty} (n+k)\cdots(n+1)c_{n+k}(z-z_0)^n \text{ for all } k \in \mathbb{Z}_0^+ \tag{1.8}$$

This then give us

$$c_k = \frac{f^{(k)}(z_0)}{k!} \text{ for all } k \in \mathbb{Z}_0^+ \tag{1.9}$$

and

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots \text{ on } D_R(z_0) \tag{1.10}$$

Equation 1.10 is often called the **Taylor expansion of f at z_0**. Notably, Equation 1.9 tell us that if f is constant 0, then $c_n = 0$ for all n.

Example 4 (Smooth but not Analytic Function)

$$f(x) = \begin{cases} e^{\frac{-1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Use induction to show that

$$f^{(k)}(x) = P_k(\frac{1}{x})e^{-(\frac{1}{x})^2} \quad \exists P_k \in \mathbb{R}[x^{-1}], \forall k \in \mathbb{Z}_0^+, \forall x \in \mathbb{R}^*$$

and again use induction to show that

$$f^{(k)}(0) = 0 \quad \forall k \in \mathbb{Z}_0^+$$

The trick to show $f^{(k)}(0) = 0$ is let $u = \frac{1}{x}$.

Now, with Theorem 1.5.2, we see that f is not analytic at 0.

Example 5 (Bump Function)

$$f(x) = \begin{cases} e^{\frac{-1}{1-x^2}} & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

Use the same trick (but more advanced) to show f is smooth, and note that f is not analytic at ± 1.

Now, it comes an interesting question. Given a complex-valued function f analytic at z_0 with radius R, and suppose $z_1 \in D_R(z_0)$.

(a) Is f also analytic at z_1?

(b) What do we know about the radius of convergence of f at z_1?

(c) Suppose f is indeed analytic at z_1. It is trivial to see that the power series $(z_0, c_{0;n})$ and $(z_1, c_{1;n})$ must agree in the intersection of their convergence disks, and because f is given, we by Theorem 1.5.2 and Equation 1.9, have already known the value of $c_{1;n}$. Can we verify that the power series $(z_0, c_{0;n})$ and $(z_1, c_{1;n})$ do indeed agree with each other on the common convergence interval?

Taylor's Theorem for power series give satisfying answers to these problems.

Theorem 1.5.3. (Taylor's Theorem for Power Series) Given a function f analytic at z_0 with radius R, and suppose $z_1 \in D_R(z_0)$. Then

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_1)}{k!}(z - z_1)^k \text{ on } D_{R-|z_1-z_0|}(z_1)$$

Proof. WOLG, let $z_0 = 0$. Suppose z satisfy $|z - z_1| < R - |z_1|$. By Equation 1.9, we can compute

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(z - z_1 + z_1)^k \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \sum_{n=0}^{k} \binom{k}{n} (z - z_1)^n z_1^{k-n} \\ &= \sum_{k=0}^{\infty} \sum_{n=0}^{k} \frac{f^{(k)}(0)}{k!} \binom{k}{n} (z - z_1)^n z_1^{k-n} \end{aligned}$$

Note that

$$\begin{aligned}
\sum_{k=0}^{\infty}\left|\sum_{n=0}^{\infty}\frac{f^{(k)}(0)}{k!}\binom{k}{n}(z-z_1)^n z_1^{k-n}\right| &\leq \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\left|\frac{f^{(k)}(0)}{k!}\right|\binom{k}{n}|z-z_1|^n\cdot|z_1|^{k-n}\\
&= \sum_{k=0}^{\infty}\left|\frac{f^{(k)}(0)}{k!}\right|\sum_{n=0}^{\infty}\binom{k}{n}|z-z_1|^n\cdot|z_1|^{k-n}\\
&= \sum_{k=0}^{\infty}\left|\frac{f^{(k)}(0)}{k!}\right|\big(|z-z_1|+|z_1|\big)^k
\end{aligned}$$

is a convergent series, by Cauchy-Hadamard Theorem and $|z-z_1|+|z_1|<R$; thus, we can use Fubini's Theorem for double series to deduce

$$\begin{aligned}
\sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{f^{(k)}(0)}{k!}\binom{k}{n}(z-z_1)^n z_1^{k-n} &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{f^{(k)}(0)}{k!}\binom{k}{n}(z-z_1)^n z_1^{k-n}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}\binom{k}{n}(z-z_1)^n z_1^{k-n}\\
&= \sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{f^{(k)}(0)}{k!}\binom{k}{n}(z-z_1)^n z_1^{k-n}\\
&= \sum_{n=0}^{\infty}\Big[\sum_{k=n}^{\infty}\frac{f^{(k)}(0)}{k!}\binom{k}{n}z_1^{k-n}\Big](z-z_1)^n
\end{aligned}$$

We have reduced the problem into proving

$$\sum_{k=n}^{\infty}\frac{f^{(k)}(0)}{k!}\binom{k}{n}z_1^{k-n}=\frac{f^{(n)}(z_1)}{n!}$$

Because z_1 is in $D_R(0)$, by Equation 1.8 and Equation 1.9, we can compute

$$\begin{aligned}
f^{(n)}(z_1) &= \sum_{k=0}^{\infty}(k+n)\cdots(k+1)\cdot\frac{f^{(n+k)}(0)}{(n+k)!}z_1^k\\
&= \sum_{k=n}^{\infty}(k)\cdots(k-n+1)\cdot\frac{f^{(k)}(0)}{k!}\cdot z_1^{k-n}\\
&= \sum_{k=n}^{\infty}\frac{f^{(k)}(0)}{(k-n)!}z_1^{k-n}
\end{aligned}$$

We now have

$$\frac{f^{(n)}(z_1)}{n!}=\sum_{k=n}^{\infty}\frac{f^{(k)}(0)}{n!(k-n)!}z_1^{k-n}=\sum_{k=n}^{\infty}\frac{f^{(k)}(0)}{k!}\binom{k}{n}z_1^{k-n}\text{ (done)}$$

$\blacksquare$

Lastly, to close this section, we prove the Identity Theorem, which is extremely useful in complex analysis.

Theorem 1.5.4. (Identity Theorem) Given two analytic complex-valued function $f,g:D\to\mathbb{C}$ defined on some open connected $D\subseteq\mathbb{C}$, if f,g agree on some subset $S\subseteq D$ such that S has a limit point in D, then f,g agree on the whole region D.

Proof. Define

$$T\triangleq\Big\{z\in D:f^{(k)}(z)=g^{(k)}(z)\text{ for all }k\geq 0\Big\}$$

Since D is connected, we can reduce the problem into proving T is non-empty, open and closed in D. Let c be a limit point of S in D. We first show

$$c\in T$$

Assume $c\notin T$. Let m be the smallest integer such that $f^{(m)}(c)\neq g^{(m)}(c)$. We can write the Taylor expansion of $f-g$ at c by

$$\begin{aligned}(f-g)(z)&=(z-c)^m\Big[\frac{(f-g)^{(m)}(c)}{m!}+\frac{(f-g)^{(m+1)}(c)}{(m+1)!}(z-c)+\cdots\Big]\\&\triangleq(z-c)^m h(z)\end{aligned}$$

Clearly, $h(c)\neq 0$. Now, because h is continuous at c (h is a well-defined power series at c with radius greater than 0), we see h is non-zero on some $B_\epsilon(c)$, which is impossible, since $(f-g)\equiv 0$ on $S\setminus\{c\}$ implies $h=0$ on $S\setminus\{c\}$. CaC (done)

Fix $z\in T$. Because f,g are analytic at z and $f^{(k)}(z)=g^{(k)}(z)$ for all k, we see $f-g$ is constant 0 on some open disk $B_\epsilon(z)$. We have proved that T is open. To see T is closed in D, one simply observe that

$$T=\bigcap_{k\geq 0}\Big\{z\in D:(f-g)^{(k)}(z)=0\Big\}$$

and $(f-g)^{(k)}$ is continuous on D. (done) $\blacksquare$

1.6 L'Hospital Rule

Abstract

This section state and prove the L'Hospital Rule, and provide examples to show the necessity of each hypotheses of L'Hospital Rule. Note that although L'Hospital Rule is not really directly used in most results in Theory of Calculus, it is used in the proof of Taylor's Theorem.

Theorem 1.6.1. (L'Hospital Rule) Let $I \subseteq \mathbb{R}$ be an open interval containing c and let $f, g : I \to \mathbb{R}$ be two function continuous on I and differentiable on I everywhere except possibly at c, where

$$g'(x) \neq 0 \text{ for all } x \in I \setminus \{c\}$$

If $\frac{f}{g}$ is indeterminate form, i.e.

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = L \text{ where } L \in \{0, \infty, -\infty\}$$

and

$$\lim_{x\to c} \frac{f'(x)}{g'(x)} \in \mathbb{R}$$

Then we have

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} \tag{1.11}$$

Proof. Suppose $I = (a, b)$. Note that since $g'(x) \neq 0$ on (c, b), by MVT, we know there exists at most one $x \in (c, b)$ such that $g(x) = 0$. With similar argument for (a, c), we see that

$$g(x) \neq 0 \text{ on } (c - \epsilon, c + \epsilon) \setminus \{c\} \text{ for some } \epsilon$$

We now see that the expression $\lim_{x\to c} \frac{f(x)}{g(x)}$ is at least well-defined, and WOLG, we can suppose $g(x) \neq 0$ on $I \setminus \{c\}$. Define $m, M : I \setminus \{c\} \to \mathbb{R}$ by

$$m(x) \triangleq \inf \frac{f'(t)}{g'(t)} \text{ and } M(x) \triangleq \sup \frac{f'(t)}{g'(t)} \text{ where } t \text{ ranges over values between } x \text{ and } c$$

Because the value $\frac{f'(t)}{g'(t)}$ converge at c, we can deduce

$$\lim_{x\to c} m(x) = \lim_{x\to c} M(x) = \lim_{x\to c} \frac{f'(x)}{g'(x)} \tag{1.12}$$

We now prove

$$\text{the case when } \lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$$

Fix $x \in I \setminus \{c\}$, and WOLG suppose $x < c$. By Cauchy's MVT, we know that

$$m(x) \leq \frac{f(x) - f(y)}{g(x) - g(y)} \leq M(x) \text{ for all } y \in (x, c)$$

Note that $g(x) \neq g(y)$ by MVT, since $g' \neq 0$ on $I \setminus \{c\}$. It now follows from

$$\lim_{y\to c^-} \frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f(x)}{g(x)}$$

that

$$m(x) \leq \frac{f(x)}{g(x)} \leq M(x)$$

The proof of Equation 1.11 then follows from Equation 1.12. (done)

We now prove

$$\text{the case when } \lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \infty$$

Again, fix $x \in I \setminus \{c\}$, and WOLG suppose $x < c$. By Cauchy's MVT, we know that

$$m(x) \leq \frac{f(y) - f(x)}{g(y) - g(x)} \leq M(x) \text{ for all } y \in (x, c)$$

Note that $g(x) \neq g(y)$ by MVT, since $g' \neq 0$ on $I \setminus \{c\}$. It now follows

$$\frac{f(y) - f(x)}{g(y) - g(x)} = \frac{\frac{f(y)}{g(y)} - \frac{f(x)}{g(y)}}{1 - \frac{g(x)}{g(y)}} \to \lim_{x\to c} \frac{f(x)}{g(x)} \text{ as } y \to c^-$$

The proof of Equation 1.11 then follows form Equation 1.12. (done) ■

1.7 Lebesgue's criteria for proper Riemann integrability

Let $f : [a, b] \to \mathbb{R}$ be bounded. We say f is **proper Riemann integrable** on $[a, b]$ if for all ϵ, there exists a **partition** $P = \{a = x_0 < \cdots < x_n = b\}$ such that

$$U(P) - L(P) \leq \epsilon$$

where

$$U(P) \triangleq \sum_{k=1}^{n} \left(\sup_{x \in I_k} f(x) \right) (x_k - x_{k-1}), \quad \text{where } I_k = [x_{k-1}, x_k]$$
$$\text{and } L(P) \triangleq \sum_{k=1}^{n} \left(\inf_{x \in I_k} f(x) \right) (x_k - x_{k-1}), \quad \text{where } I_k = [x_{k-1}, x_k]$$

Given some subset $I \subseteq [a, b]$, the **oscillation** of f on I is:

$$\text{osc}(f, I) \triangleq \sup_{x,y \in I} |f(x) - f(y)|$$

Because we clearly have:

$$\sup_{x \in I} f(x) - \inf_{x \in I} f(x) = \text{osc}(f, I)$$

we actually have:

$$U(P) - L(P) = \sum_{k=1}^{n} \text{osc}(f, I_k)(x_k - x_{k-1}), \quad \text{where } I_k = [x_{k-1}, x_k] \tag{1.13}$$

For all $x \in [a, b]$, we define the **oscillation** of f at x to be

$$\text{osc}(f, x) \triangleq \inf_{\delta > 0} \text{osc}\left(f, [x - \delta, x + \delta] \cap [a, b]\right)$$

Clearly, $f : [a, b] \to \mathbb{R}$ is continuous at x if and only if $\text{osc}(f, x) = 0$. This observation together with characterization 1.13 is the moral reason Lebesgue may link continuity together with proper Riemann integrability. We write:

$$D(\epsilon) \triangleq \{x \in [a, b] : \text{osc}(f, x) \geq \epsilon\}$$

Theorem 1.7.1. (Lebesgue's criteria for proper Riemann integrability) Let $f : [a, b] \to \mathbb{R}$ be bounded, and define

$$D_f \triangleq \{x \in [a, b] | f : [a, b] \to \mathbb{R} \text{ is discontinuous at } x\}$$

We have

$$f \text{ is proper Riemann integrable} \iff D_f \text{ have zero measure}$$

Proof. We first prove:

$$f \text{ is proper Riemann integrable } \implies |D_f| = 0$$

Because $D(\epsilon) \nearrow D_f$ as $\epsilon \to 0$, we must prove $|D(\epsilon)| = 0$ for all ϵ. Fix ϵ. Because f is proper Riemann integrable, we know for all n there exists a partition P_n of $[a,b]$ such that

$$U(P_n) - L(P_n) \leq \frac{1}{n}$$

Define $\mathcal{G}_{n,\epsilon} \triangleq$ to be the union of closed intervals of P_n that have oscillation $\geq \epsilon$. By definition of pointwise oscillation, clearly we have

$$D(\epsilon) - P_n \subseteq \mathcal{G}_{n,\epsilon}$$

The proof now follows from the inequality:

$$\epsilon\,|D(\epsilon)| = \epsilon\,|\mathcal{G}_{n,\epsilon}| \leq \sum_{\operatorname{osc}(f,I)\geq\epsilon} \operatorname{osc}(f,I)\,|I| \leq U(P_n) - L(P_n)$$

which holds for all n. (done)

We now prove:

$$|D_f| = 0 \implies f \text{ is proper Riemann integrable}$$

Fix ϵ. Let $M \triangleq \sup|f|$. Because D_f is measurable and null, there exists some $G \supseteq D_f$ open in $\mathbb{R}$ with $|G| \leq \epsilon \big/ 4M$.

Set $K \triangleq [a,b] - G$. Because for all $x \in K$, $f : [a,b] \to \mathbb{R}$ is continuous at x, we may select some closed interval $I_x \subseteq [a,b]$ such that $\operatorname{osc}(f, I_x) \leq \epsilon \big/ 2(b-a)$. Moreover, because K is compact, we may write

$$K \subseteq I_{x_1} \cup \cdots \cup I_{x_n}$$

Consider the partition $P \triangleq \{a,b\} \cup \partial I_{x_1} \cup \cdots \cup \partial I_{x_n}$. Let J be a closed interval of P. Taking the fact that $I_{x_1} \cup \cdots \cup I_{x_n}$ by nature only have finite number of connected components into account, we know J° either completely lies in the union $I_{x_1} \cup \cdots \cup I_{x_n}$ or completely lies outside of it, i.e., exactly one of the two statement:

$$J \subseteq I_{x_1} \cup \cdots \cup I_{x_n} \text{ or } J^\circ \subseteq [a,b] - I_{x_1} \cup \cdots \cup I_{x_n}$$

occurs. Moreover, if the former holds, we know $J \subseteq I_{x_i}$ for some i, since if $x \in J$ lies in I_{x_j} for some j, then $J \subseteq I_{x_j}$ by construction of P. Because of such, we may finally give the desired estimate:

$$\begin{aligned} U(P) - L(P) &= \sum_{J \subseteq \widetilde{I}} \operatorname{osc}(f, J)\,|J| + \sum_{J^\circ \subseteq [a,b] - \widetilde{I}} \operatorname{osc}(f, J)\,|J| \\ &\leq \sum_{J \subseteq \widetilde{I}} \frac{\epsilon}{2(b-a)}\,|J| + 2M \cdot \frac{\epsilon}{4M} \leq \epsilon \end{aligned}$$

where $\widetilde{I} \triangleq I_{x_1} \cup \cdots \cup I_{x_n}$. (done) ■

1.8 Lebesgue calculus

Theorem 1.8.1. (Dominated convergence theorem) Let $E \subseteq X$ be measurable and f_n be a sequence of complex-valued function measurable on E that converges almost everywhere to some f on E. If there exists some $\varphi \in L(E)$ such that $|f_n| \leq \varphi$ almost everywhere for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} \int_E |f - f_n| \, d\mathbf{x} = 0$$

Theorem 1.8.2. (Change of variables formula) Let $X \subseteq \mathbb{R}^n$ be open, $T : X \to \mathbb{R}^n$ be differentiable and injective. For all measurable function $f : \mathbb{R}^n \to [0, \infty]$, we have:

$$\int_{T(X)} f d\mathbf{m} = \int_X (f \circ T) \, |J_T| \, d\mathbf{m}$$

Proof. Rudin RAC Theorem 7.26 ■

Theorem 1.8.3. (Fubini's and Tonelli's theorem for double integral) Let $f : \mathbb{R}^{n+m} \to \mathbb{R}$ be measurable. If f satisfies one of the followings conditions:

(I) f is integrable.

(II) f is nonnegative.

then we have the formula:

$$\int_{\mathbb{R}^{n+m}} f(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^n} g(\mathbf{y}) d\mathbf{y}, \quad \text{where } g(\mathbf{y}) \triangleq \int_{\mathbb{R}^m} f(\mathbf{y}, \mathbf{z}) d\mathbf{z}$$

even if the integral may be $+\infty$.

Theorem 1.8.4. (Measure-theoretic Feymann's trick) Let X be an open subset of $\mathbb{R}$ and Ω a measure space. If $f : X \times \Omega \to \mathbb{R}$ satisfies:

(I) $f(x, \omega)$ is an integrable function of ω for each $x \in X$.

(II) For almost all $\omega \in \Omega$, the partial derivative $\partial f \diagup \partial x$ exists for all $x \in X$.

(III) There is an integrable function $\theta : \Omega \to \mathbb{R}$ such that for almost every $\omega \in \Omega$ we have:

$$\left| \frac{\partial f}{\partial x}(x, \omega) \right| \leq \theta(\omega), \quad \text{for all } x \in X$$

Then,

$$\frac{d}{dx} \int_\Omega f(x, \omega) d\omega = \int_\Omega \frac{\partial f}{\partial x} d\omega, \quad \text{for all } x \in X$$

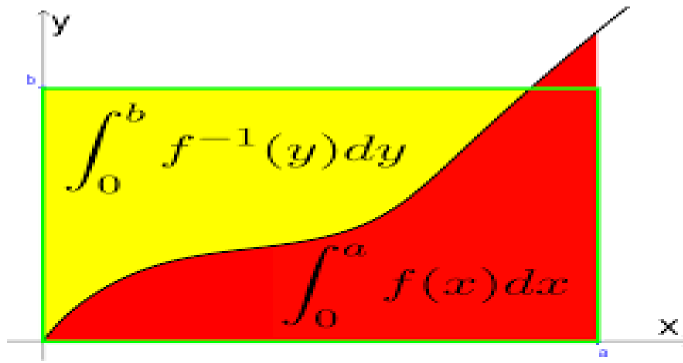
1.9 Young, Hölder, and Minkowski inequalities

Theorem 1.9.1. (A geometric estimation) If $\varphi : [0,\infty) \to \mathbb{R}$ is continuous and strictly increasing, then for all $a, b \in [0,\infty]$, we have:

$$ab \leq \int_0^a \varphi(x)dx + \int_0^b \varphi^{-1}(y)dy$$

where $\varphi^{-1} : [0,\infty) \to \mathbb{R}$ is the inverse function of φ. Moreover, equality holds if and only if $b = \varphi(a)$.

Proof. The proof is straightforward to come up after an observation of the graph:



■

We say a pair of numbers $p, q \in [1,\infty]$ are **exponential conjugated** if

$$\frac{1}{p} + \frac{1}{q} = 1$$

Theorem 1.9.2. (Young's inequality for product) If $a, b \in [0,\infty)$ and $p, q \in (1,\infty)$ is an exponential conjugated pair, then:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Moreover, equality holds if and only if $a^p = b^q$.

Proof. The proof follows from setting $\varphi(x) \triangleq x^{p-1}$ in the above geometric estimation. ■

Fix $1 \leq p < \infty$. Let $E \subseteq \mathbb{R}$ be measurable, and define for all $\mathbb{R}$-valued function f measurable on E that

$$\|f\|_p \triangleq \left(\int_E |f|^p\right)^{\frac{1}{p}}$$

and that

$$\|f\|_\infty \triangleq \inf\left\{M \in \mathbb{R}_0^+ : f \leq M \text{ a.e. on } E\right\}$$

Clearly, for all $1 \leq p \leq \infty$, the space $L^p(E)$ of $\mathbb{R}$-valued functions f measurable on E that satisfies $\|f\|_p < \infty$ forms a $\mathbb{R}$-vector space. Clearly, the **p-norms** $\|\cdot\|_p$ form semi-norms on $L^p(E)$.

Theorem 1.9.3. (L^p implies $L^{p+\epsilon}$ on finite domain) Let $1 \leq p < \infty$ and $E \subseteq \mathbb{R}$. Then:

$$|E| < \infty \text{ and } 0 \leq N \leq \infty \implies L^{p+N}(E) \subseteq L^p(E)$$

Proof. The $N = \infty$ case is clear. Let $f \in L^{p+N}(E)$. The proof then follows from direct estimation based on splitting:

$$E = \{x \in E : |f(x)| \leq 1\} \cup \{x \in E : |f(x)| \geq 1\}$$

■

Theorem 1.9.4. (Hölder's inequality) Let $p, q \in [1, \infty]$ be an exponential conjugated pair, $E \subseteq \mathbb{R}$ be measurable, and f, g be two $\mathbb{R}$-valued functions measurable on E. Then:

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

If moreover $p > 1$, then the equality hold if and only if there exists $c \in \mathbb{R}$ such that $|f|^p = c\,|g|^q$ a.e. on E.

Proof. The proof relies on *normalization*. Before such, we first have to handle the boundary case $p = 1$ and $\|f\|_p = 0$. They are all easy, with the latter an observation that $\|f\|_p = 0 \implies fg = 0$ a.e., which follows from definition of Lebesgue integral.

We may from now on suppose $1 < p < \infty$ and $\|f\|_p \neq 0 \neq \|g\|_q$. Define $\widehat{f}, \widehat{g}$ by

$$\widehat{f} \triangleq \frac{f}{\|f\|_p} \quad \text{and} \quad \widehat{g} \triangleq \frac{g}{\|g\|_q}$$

The proof then follows from Young's inequality:

$$\left\|\widehat{f} \cdot \widehat{g}\right\|_1 \leq \int_E \frac{\left|\widehat{f}\right|^p}{p} + \frac{|\widehat{g}|^q}{q} = \frac{1}{p} + \frac{1}{q} = 1$$

■

Let $E \subseteq \mathbb{R}$ be measurable, one may check that $\|\cdot\|_2$ satisfies **parallelogram law** and is thus induced by the obvious inner product:

$$\langle f, g\rangle \triangleq \int_E f\overline{g}$$

Therefore, Hölder's inequality become **Cauchy-Schwarz inequality** of L^2-space.

1.10 Fourier series

Let $L^2(0, P)$ be the space of complex-valued functions L^2 on $(0, P)$. By Hölder's inequality, we may well define a semi-$\mathbb{C}$-inner product on $L^2(0, P)$ by:

$$\langle f, g \rangle \triangleq \int_0^P f(x)\overline{g(x)}dx \in \mathbb{C}$$

which induces the L^2-norm $\|\cdot\|_2$. We then see that $\left\{ e^{-i2\pi n\frac{x}{P}} \big/ \sqrt{P} : n \in \mathbb{Z} \right\}$ forms an **orthonormal system**.
define **Fourier coefficients** by:

$$c_n \triangleq \langle f, \frac{e^{-i2\pi n\frac{x}{P}}}{P} \rangle$$

$$\frac{1}{P}\int_0^P f(x)e^{-i2\pi n\frac{x}{P}}dx, \quad \text{for all } n \in \mathbb{Z}$$

$$a_n \triangleq \frac{1}{P}\int_0^P f(x)\cos\left(2\pi n\frac{x}{P}\right)dx, \quad \text{for all } n \in \mathbb{N} \cup \{0\}$$

$$b_n \triangleq \frac{1}{P}\int_0^P f(x)\sin\left(2\pi n\frac{x}{P}\right)dx, \quad \text{for all } n \in \mathbb{N} \cup \{0\}$$

1.11 Operator norm

We define the **operator norm** of linear $T:\mathbb{R}^n\to\mathbb{R}^m$ by

$$\|T\|_{\text{op}} \triangleq \sup_{\mathbf{x}\neq 0}\frac{|T\mathbf{x}|}{|\mathbf{x}|}=\max_{\mathbf{x}=1}|T\mathbf{x}|$$

One may check that this forms a norm on the real vector space of linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$. **Singular Value Decomposition Theorem** said that for each linear $T:\mathbb{R}^n\to\mathbb{R}^m$, there exists some orthonormal basis $\{\mathbf{v}_1,\dots,\mathbf{v}_n\}$ such that $\{T(\mathbf{v}_1),\dots,T(\mathbf{v}_m)\}-\{\mathbf{0}\}$ is orthogonal. We call $|T(\mathbf{v}_1)|,\dots,|T(\mathbf{v}_n)|$ the **singular values** of T.

Theorem 1.11.1. (Operator norm is the largest singular value) The operator norm of linear $T:\mathbb{R}^n\to\mathbb{R}^m$ is exactly the largest singular value of T.

Proof. ■

Theorem 1.11.2. (Equivalence of norms on $\mathbb{R}^n$)

Now use the fact that determinant is continuous to conclude that collection of invertible function in $\text{End}_{\text{linear}}(\mathbb{R}^n)$ is open.

Theorem 1.11.3. (Inversion is Continuous) Let Ω be the space of invertible linear functions from $\mathbb{R}^n$ to itself. The mapping $\Omega\to\Omega, A\mapsto A^{-1}$ is continuous.

Proof. Stick to the one in General Analysis. ■

Theorem 1.11.4. (Differentiability Theorem) Suppose $\alpha=\{e_1,\dots,e_n\}$ is an orthonormal basis of $\mathbb{R}^n$, and $\beta=\{q_1,\dots,q_m\}$ is an orthonormal basis of $\mathbb{R}^m$. Suppose f maps an open set $E\subseteq\mathbb{R}^n$ to $\mathbb{R}^m$. Then

$$f\text{ is continuously differentiable on }E\iff\partial_i f_j\text{ exists and is continuous on }E\text{ for all }i,j$$

Proof. $(\longrightarrow)$

Fix i,j. Because f is differentiable on E, we know $\partial_i f_j$ exists on E by Theorem **??**. Fix $x\in E$. We only have to show

$$\partial_i f_j\text{ is continuous at }x$$

Fix ϵ. We wish

$$\text{to find }\delta\text{ such that }|\partial_i f_j(y)-\partial_i f_j(x)|\leq\epsilon\text{ for all }|y-x|<\delta$$

Because f is continuously differentaible at x, we know there exists δ such that

$$\|df_y-df_x\|_{\text{op}}<\epsilon\text{ for all }|y-x|\leq\delta$$

We claim

$$\text{such } \delta \text{ suffices}$$

By the the matrix representation, we know

$$\partial_i f_j(y) - \partial_i f_j(x) = (df_y - df_x)e_i \cdot q_j$$

Then by Cauchy-Inequality, we have

$$\begin{aligned} |\partial_i f_j(y) - \partial_i f_j(x)| &\leq |(df_y - df_x)e_i| \\ &\leq \|df_y - df_x\|_{\text{op}} < \epsilon \text{ (done)} \end{aligned}$$

$(\longleftarrow)$

We first show

$$f \text{ is differentiable on } E$$

We first prove

$$\forall j \in \{1, \dots, m\}, f_j : \mathbb{R}^n \to \mathbb{R} \text{ is differentiable on } E \implies f \text{ is differentiable on } E$$

Fix $x \in E$. We wish to prove

$$f \text{ is differentiable at } x$$

Define $A : E \to \mathbb{R}^m$ by

$$A(h) \triangleq \sum_{j=1}^{m} (df_j)_x(h) q_j$$

We claim

$$A \text{ suffices to be the } df_x$$

Using the fact q_j are orthonormal, we have

$$f(x+h) - f(x) - A(h) = \sum_{j=1}^{m} \big(f_j(x+h) - f_j(x) - (df_j)_x(h)\big) q_j$$

This give us

$$\begin{aligned} \lim_{h \to 0} \frac{|f(x+h) - f(x) - A(h)|}{|h|} &= \lim_{h \to 0} \frac{\left|\sum_{j=1}^{m} \big(f_j(x+h) - f_j(x) - (df_j)_x(h)\big) q_j\right|}{|h|} \\ &\leq \lim_{h \to 0} \frac{\sum_{j=1}^{m} |f_j(x+h) - f_j(x) - (df_j)_x(h)|}{|h|} = 0 \text{ (done)} \end{aligned}$$

Fix $j \in \{1, \dots, m\}$. We can now reduce the problem into

$$f_j : \mathbb{R}^n \to \mathbb{R} \text{ is differentiable on } E$$

Fix $x \in E$. We wish to prove

$$f_j \text{ is differentiable at } x$$

Express $h = \sum_{i=1}^n h_i e_i$. Define $B : E \to \mathbb{R}$ by

$$B(h) = \sum_{i=1}^n \partial_i f_j(x) h_i$$

We claim

$$B \text{ suffices to be } (df_j)_x$$

By continuity of each $\partial_i f_j$ on E, we can let δ satisfy

$$|\partial_i f_j(y) - \partial_i f_j(x)| < \frac{\epsilon}{n} \text{ for all } y \in B_\delta(x)$$

We claim

$$\frac{|f_j(y) - f_j(x) - B(y-x)|}{|y-x|} \leq \epsilon \text{ for all } y \in B_\delta(x)$$

Express $y - x = \sum_{k=1}^n h_k e_k$. Define $v_0, \dots, v_n \in \mathbb{R}^n$ by

$$v_0 \triangleq 0 \text{ and } v_k \triangleq \sum_{i=1}^k h_i e_i \text{ for all } k \in \{1, \dots, n\}$$

Now observe

$$\begin{aligned}
\frac{|f_j(y) - f_j(x) - B(y-x)|}{|y-x|} &= \frac{|f_j(x+v_n) - f_j(x) - B(\sum_{k=1}^n h_k e_k)|}{|y-x|} \\
&= \frac{\left|\left(\sum_{k=1}^n f_j(x+v_k) - f_j(x+v_{k-1})\right) - \sum_{k=1}^n \partial_k f_j(x) h_k\right|}{|y-x|} \\
&= \frac{|\sum_{k=1}^n f_j(x+v_k) - f_j(x+v_{k-1}) - \partial_k f_j(x) h_k|}{|y-x|} \\
&= \frac{|\sum_{k=1}^n f_j(x+v_{k-1}+h_k e_k) - f_j(x+v_{k-1}) - \partial_k f_j(x) h_k|}{|y-x|} \\
(\text{ For some } e_k \in (0,1) \text{ by MVT}) \quad &= \frac{|\sum_{k=1}^n \partial_k f_j(x+v_{k-1}+t_k e_k) h_k - \partial_k f_j(x) h_k|}{|y-x|} \\
&\leq \frac{\sum_{k=1}^n \left|\left(\partial_k f_j(x+v_{k-1}+t_k e_k) - \partial_k f_j(x)\right) h_k\right|}{|y-x|} \\
&< \frac{\sum_{k=1}^n \frac{\epsilon}{n} |h_k|}{|y-x|} \leq \epsilon \text{ (done)}
\end{aligned}$$

We now prove

$$f \text{ is continuously differentiable on } E$$

Fix ϵ and $x \in E$. We are required

$$\text{to find } \delta \text{ such that } \|df_y - df_x\|_{\text{op}} \leq \epsilon \text{ for all } y \in B_\delta(x)$$

Note that one can define a norm $\|\cdot\|_F$ called "Forbenius Norm" on $BL(\mathbb{R}^n, \mathbb{R}^n)$ by

$$\|A\|_F \triangleq \sqrt{\sum_{i=1}^n \sum_{j=1}^n a_{i,j}^2} \text{ where } [A]_\alpha^\beta = \begin{bmatrix} a_{1,1} & \cdots & a_{n,1} \\ \vdots & \ddots & \vdots \\ a_{1,n} & \cdots & a_{n,n} \end{bmatrix}$$

Because all norms on finite-dimensional real vector spaces are equivalent, we know there exists M such that for all $x \in E$, we have

$$\|df_x\|_{\text{op}} \leq M \|df_x\|_F$$

Because the partial derivatives are all continuous by definition, we can let δ satisfy

$$\big(\partial_i f_j(x+h)\big)^2 - \big(\partial_i f_j(x)\big)^2 < \frac{\epsilon^2}{M^2 n^2} \text{ for all } h \in B_\delta(0)$$

We claim

$$\text{such } \delta \text{ suffices}$$

Let $|y - x| < \delta$. We see

$$\|df_y - df_x\|_{\text{op}} \leq M \|df_y - df_x\|_F < M \sqrt{\sum_{i=1}^n \sum_{j=1}^n \frac{\epsilon^2}{M^2 n^2}} = \epsilon \text{ (done)}$$

$\blacksquare$

1.12 Inverse function theorem

Let X be a metric space. A **contraction** $f : X \to X$ is a K-Lipschitz function such that $K < 1$.

Example 1.12.1. Let $X \triangleq \mathbb{R}$. $f(x) \triangleq \frac{x}{2}$ is a $\frac{1}{2}$-contraction, and $g(x) \triangleq x$ for no $r \in [0, 1)$ is a r-contraction.

Theorem 1.12.2. (Banach's fixed point theorem) Let X be a metric space. Let $f : X \to X$ be a contraction. Then f admits at most one fixed point. Let $f^n : X \to X$ denote:

$$\overbrace{f \circ \cdots \circ f}^{n \text{ copies}} : X \to X$$

If X is moreover complete, then for all $x \in X$, the sequence $\{f^n(x)\} \subseteq X$ converges, with limit being a (the unique) fixed point of f.

Proof. It is easy to see from the definition that a contraction can only have at most one fixed point from the definition. Let X be complete and $x \in X$. Because f as a Lipschitz function is continuous, we know that if $\{f^n(x)\} \subseteq X$ converges, then its limit is a (the unique) fixed point of f:

$$f\left(\lim_{n\to\infty} f^n(x)\right) = \lim_{n\to\infty} f^{n+1}(x) = \lim_{n\to\infty} f^n(x)$$

It remains to prove $\{f^n(x)\} \subseteq X$ converges. Because X is complete, we only have to prove $\{f^n(x)\} \subseteq X$ is Cauchy. Fix ϵ. Because $0 \le K < 1$, we may find N large enough so that $\frac{K^N}{1-K} d(x, f(x)) \le \epsilon$. We claim that for all $n \ge N$ and $k \in \mathbb{N}$, we have:

$$d(f^n(x), f^{n+k}(x)) \le \epsilon, \quad \text{as desired.} \tag{1.14}$$

We now prove inequality 1.14. We first note that, using induction on k and triangle inequality, we have:

$$d(x, f^k(x)) = \sum_{j=0}^{k-1} K^j d(x, f^k(x)) \le \frac{1}{1-K} d(x, f^k(x))$$

This then implies

$$d(f^n(x), f^{n+k}(x)) \le K^n d(x, f^k(x)) \le \frac{K^n}{1-K} d(x, f(x)) \le \epsilon$$

as desired. ∎

Theorem 1.12.3. (Pointwise inverse function theorem) Let U be some neighborhood of $\mathbf{a} \in \mathbb{R}^n$. If differentiable $\mathbf{f} : U \to \mathbb{R}^n$ is C^1 at $\mathbf{a}$ and has full rank Jacobian at $\mathbf{a}$, then:

(i) $\mathbf{f}$ maps some $B_\delta(\mathbf{a}) \subseteq U$ injectively to some open subset of $\mathbb{R}^n$.

(ii) Moreover, the local inverse of $\mathbf{f}$ is differentiable at $\mathbf{f}(\mathbf{a})$.

Proof. ■

Corollary 1.12.4. (Inverse function theorem) Let U be some neighborhood of $\mathbf{a}$. If C^1 function $\mathbf{f} : U \to \mathbb{R}^n$ has full rank Jacobian at $\mathbf{a}$, then $\mathbf{f}$ forms a C^1-diffeomorphism between some neighborhood $W \ni \mathbf{a}$ contained by U, and $\mathbf{f}(W)$.

Proof. Because $\mathrm{GL}_n(\mathbb{R}) \subseteq \mathrm{M}_n(\mathbb{R})$ is open and $\mathbf{f}$ is C^1, we know there exists some neighborhood $U' \ni \mathbf{a}$ contained by U such that $\mathbf{f}$ has full rank Jacobian everywhere in U'. Applying pointwise inverse function theorem for every point in U', we get an injective C^1-function $\mathbf{f} : W \to \mathbb{R}^n$ with differentiable inverse. To see $(\mathbf{f})^{-1} : \mathbf{f}(W) \to W$ is also C^1, just observe:

$$\mathbf{y} \mapsto d\mathbf{f}^{-1}|_{\mathbf{y}} = \big(\mathbf{x} \mapsto (d\mathbf{f}|_{\mathbf{x}})^{-1}\big) \circ \big(\mathbf{y} \mapsto \mathbf{f}^{-1}(\mathbf{y})\big)$$

and recall that $GL_n(\mathbb{R}) \to GL_n(\mathbb{R}), A \mapsto A^{-1}$ is continuous. ■

1.13 Implicit function theorem

Theorem 1.13.1. (Implicit function theorem) Let $W \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be open and $(\mathbf{a}, \mathbf{b}) \in W$. Let $\mathbf{f} : W \to \mathbb{R}^n$ be a C^1 function such that the matrix:

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}\Big|_{(\mathbf{a},\mathbf{b})} & \cdots & \frac{\partial f_1}{\partial x_n}\Big|_{(\mathbf{a},\mathbf{b})} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}\Big|_{(\mathbf{a},\mathbf{b})} & \cdots & \frac{\partial f_n}{\partial x_n}\Big|_{(\mathbf{a},\mathbf{b})} \end{pmatrix}$$

is invertible. Then, there exists a pair of neighborhoods $U \ni (\mathbf{a}, \mathbf{b})$ and $V \ni \mathbf{b}$ such that: For all $\mathbf{y} \in V$ there exists a unique $\mathbf{x} \in \mathbb{R}^n$ that satisfies both $(\mathbf{x}, \mathbf{y}) \in U$ and $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{f}(\mathbf{a}, \mathbf{b})$. Moreover, if we define $\mathbf{g} : V \to \mathbb{R}^n$ by setting

$$\mathbf{f}(\mathbf{g}(\mathbf{y}), \mathbf{y}) = \mathbf{f}(\mathbf{a}, \mathbf{b}), \quad \text{for all } \mathbf{y} \in V$$

then $\mathbf{g}$ will be C^1 and clearly make $\mathbf{g}(\mathbf{b}) = \mathbf{a}$.

Proof. Define $\mathbf{F} : W \to \mathbb{R}^n \times \mathbb{R}^m$ by

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) \triangleq (\mathbf{f}(\mathbf{x}, \mathbf{y}), \mathbf{y})$$

By differentiability theorem, $\mathbf{F}$ is C^1. Clearly, the Jacobian of $\mathbf{F}$ at $(\mathbf{a}, \mathbf{b})$ is full rank. We may now apply inverse function theorem to find some smaller neighborhood $U \subseteq W$ of $(\mathbf{a}, \mathbf{b})$ such that:

(I) $\mathbf{F}$ is injective on U.

(II) $\mathbf{F}(U) \subseteq \mathbb{R}^n \times \mathbb{R}^m$ is open.

(III) $\mathbf{F}$ has full rank Jacobian everywhere in U.

(IV) The local inverse $\mathbf{F}^{-1} : \mathbf{F}(U) \to U$ is C^1.

Because $\mathbf{F}(U)$ is open, we know

$$V \triangleq \{\mathbf{y} \in \mathbb{R}^m : (\mathbf{f}(\mathbf{a}, \mathbf{b}), \mathbf{y}) \in \mathbf{F}(U)\} \text{ is open}$$

We claim such $U \ni (\mathbf{a}, \mathbf{b})$ and $V \ni \mathbf{b}$ suffices. Fix $\mathbf{y} \in V$. We are required to show the existence and uniqueness of $\mathbf{x} \in \mathbb{R}^n$ that satisfies both $(\mathbf{x}, \mathbf{y}) \in U$ and $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{f}(\mathbf{a}, \mathbf{b})$.

By definition of V, we know $(\mathbf{f}(\mathbf{a}, \mathbf{b}), \mathbf{y}) \in \mathbf{F}(U)$. Therefore, there exist $\mathbf{z} = (\mathbf{z}_1, \mathbf{z}_2) \in U$ such that $\mathbf{F}(\mathbf{z}) = (\mathbf{f}(\mathbf{a}, \mathbf{b}), \mathbf{y})$. By definition of $\mathbf{F}$, $\mathbf{z}_1$ is the $\mathbf{x}$ we are looking for. Moreover, because $\mathbf{F}$ is injective on U, we know $\mathbf{z}_1$ is the only option. We have shown the existence and uniqueness of $\mathbf{x}$. It remains to prove that $\mathbf{g}$ is C^1, which follows from differentiability theorem and that $(\mathbf{g}(\mathbf{y}), \mathbf{y}) = \mathbf{F}^{-1}(\mathbf{f}(\mathbf{a}, \mathbf{b}), \mathbf{y})$ for all $\mathbf{y} \in V$. ■

Note that $\mathbf{g}$ clearly satisfies:

$$\begin{pmatrix} \left.\frac{\partial g_1}{\partial y_1}\right|_{\mathbf{y}} & \cdots & \left.\frac{\partial g_1}{\partial y_m}\right|_{\mathbf{y}} \\ \vdots & \ddots & \vdots \\ \left.\frac{\partial g_n}{\partial y_1}\right|_{\mathbf{y}} & \cdots & \left.\frac{\partial g_n}{\partial y_m}\right|_{\mathbf{y}} \end{pmatrix} = - \begin{pmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_{(\mathbf{g}(\mathbf{y}),\mathbf{y})} & \cdots & \left.\frac{\partial f_1}{\partial x_n}\right|_{(\mathbf{g}(\mathbf{y}),\mathbf{y})} \\ \vdots & \ddots & \vdots \\ \left.\frac{\partial f_n}{\partial x_1}\right|_{(\mathbf{g}(\mathbf{y}),\mathbf{y})} & \cdots & \left.\frac{\partial f_n}{\partial x_n}\right|_{(\mathbf{g}(\mathbf{y}),\mathbf{y})} \end{pmatrix}^{-1} \cdot \begin{pmatrix} \left.\frac{\partial f_1}{\partial y_1}\right|_{(\mathbf{g}(\mathbf{y}),\mathbf{y})} & \cdots & \left.\frac{\partial f_1}{\partial y_m}\right|_{(\mathbf{g}(\mathbf{y}),\mathbf{y})} \\ \vdots & \ddots & \vdots \\ \left.\frac{\partial f_n}{\partial y_1}\right|_{(\mathbf{g}(\mathbf{y}),\mathbf{y})} & \cdots & \left.\frac{\partial f_n}{\partial y_m}\right|_{(\mathbf{g}(\mathbf{y}),\mathbf{y})} \end{pmatrix}$$

for all $\mathbf{y} \in V$, which in particular manifest the fact that $\mathbf{g}$ is C^1.

1.14 Legendre type formula

1.15 Miscellaneous

Theorem 1.15.1. (Stirling's approximation)

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \quad \text{as } n \to \infty$$

Theorem 1.15.2. (Lagrange multiplier theorem) Let $V \subseteq \mathbb{R}^n$ be open , $f, g_1, \dots, g_m \in C^1(V)$ with $m \leq n$.

Proof. See Wade Theorem 11.63 ∎

Theorem 1.15.3. (Divergence theorem) Let $M \subseteq \mathbb{R}^3$ be compact with piecewise smooth boundary and $\mathbf{F} : M \to \mathbb{R}^3$ be C^1. Then:

$$\int_{\partial M} \mathbf{F} \cdot \mathbf{n} dA = \int_M \nabla \cdot \mathbf{F} dV$$

Theorem 1.15.4. (Green's theorem) Let $C \subseteq \mathbb{R}^2$ be a positively oriented, piecewise smooth, and smooth curve. Let $D \subseteq \mathbb{R}^2$ be the region bounded by C. If $L, M \in C^1(D)$, then

$$\int_C L dx + M dy = \int_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dA$$

Chapter 2

NTU Math M.A. Program Entrance Exam

2.1 Year 114

Question 1: easy estimate

Let $\mathbf{x} = (x_1, x_2, x_3)$ be coordinate of $\mathbb{R}^3$. Let

$$f(\mathbf{x}) \triangleq \frac{x_1^3 + x_2^2 + i\ln(1 + x_2^2 + x_3^2)}{x_1^2 + x_2^2 + ix_3^2 \sin\left(\frac{x_1}{x_3}\right)} \qquad \text{if } x_1x_2x_3 \neq 0$$

$$f(\mathbf{x}) \triangleq 0 \qquad \text{if } x_1x_2x_3 = 0$$

Prove that

$$\left\{\mathbf{x} \in \mathbb{R}^3 : f \text{ is continuous at } \mathbf{x}\right\} = \left\{\mathbf{x} \in \mathbb{R}^3 : x_1x_2x_3 \neq 0\right\}$$

Proof. Because f is the quotient of two nonzero continuous function on $\mathbb{R}^3 - \left\{\mathbf{x} \in \mathbb{R}^3 : x_1x_2x_3 = 0\right\}$ we know f is continuous on it. Let

(i) $E_0 \triangleq \{\mathbf{0}\}$.

(ii) $E_1 \triangleq \left\{\mathbf{x} \in \mathbb{R}^3 : x_1 = x_2 = 0\right\} - E_0$.

(iii) $E_2 \triangleq \left\{\mathbf{x} \in \mathbb{R}^3 : x_1 = x_3 = 0\right\} - E_0$.

(iv) $E_3 \triangleq \left\{\mathbf{x} \in \mathbb{R}^3 : x_2 = x_3 = 0\right\} - E_0$.

(v) $E_4 \triangleq \left\{\mathbf{x} \in \mathbb{R}^3 : x_1 = 0\right\} - (E_0 \cup E_1 \cup E_2)$.

(vi) $E_5 \triangleq \{\mathbf{x} \in \mathbb{R}^3 : x_2 = 0\} - (E_0 \cup E_1 \cup E_3)$.

(vii) $E_6 \triangleq \{\mathbf{x} \in \mathbb{R}^3 : x_3 = 0\} - (E_0 \cup E_2 \cup E_3)$.

We shall prove in order that all of them is contained by $\{\mathbf{x} \in \mathbb{R}^3 : f \text{ is discontinuous at } \mathbf{x}\}$. To see f is discontinuous at $\mathbf{0}$, just observe

$$\lim_{t \to 0} \operatorname{Re} f(\pi t, t, t) = \frac{1}{\pi^2 + 1}$$

Fix $(0, 0, a) \in E_1$. To see f is discontinuous at $(0, 0, a)$, just observe

$$|f(t, t, a + t)| = \frac{\left|t^3 + t^2 + i \ln(1 + t^2 + (a + t)^2)\right|}{\left|2t^2 + i(a + t)^2 \sin\left(\frac{t}{a+t}\right)\right|} \to \infty, \qquad \text{as } t \to 0$$

Fix $(0, a, 0) \in E_2$. To see f is discontinuous at $(0, a, 0)$, just observe

$$\lim_{t \to 0} f(t, a + t, t) = \frac{a^2 + i \ln(1 + a^2)}{a^2}$$

Fix $(a, 0, 0) \in E_3$. To see f is discontinuous at $(a, 0, 0)$, just observe

$$\lim_{t \to 0} f(a + t, t, t) = a$$

Fix $(0, a, b) \in E_4$. To see f is discontinuous at $(0, a, b)$, just observe

$$\lim_{t \to 0} f(t, a + t, b + t) = \frac{a^2 + i \ln(1 + a^2 + b^2)}{a^2}$$

Fix $(a, 0, b) \in E_5$. To see f is discontinuous at $(a, 0, b)$, just observe

$$\lim_{t \to 0} f(a + t, b + t, 0) = \frac{a^3 + i \ln(1 + b^2)}{a^2 + ib^2 \sin\left(\frac{a}{b}\right)}$$

Fix $(a, b, 0) \in E_6$. To see f is discontinuous at $(a, b, 0)$, just observe

$$\lim_{t \to 0} f(a + t, t, b + t) = \frac{a^3 + b^2 + i \ln(1 + b^2)}{a^2 + b^2}$$

$\blacksquare$

Question 2: easy estimate

Let $\mathbf{x} = (x_1, x_2, x_3)$ be coordinate of $\mathbb{R}^3$. Let

$$f(\mathbf{x}) \triangleq \frac{x_1^3 + e^{-\frac{1}{x^2}} + i\sin^2\left(x_3^2\right)}{x_1^2 + x_2^2 + ix_3^3} \qquad \text{if } x_1x_2x_3 \neq 0$$
$$f(\mathbf{x}) \triangleq 0 \qquad \text{if } x_1x_2x_3 = 0$$

Is f differentiable at $\mathbf{0}$? (It isn't, as you may have guessed from the fact this is an exam question)

Proof. Because $\frac{\partial f}{\partial x_i}\big|_{\mathbf{0}} = 0$ for all i, if f is differentiable at origin, then we must have $df|_{\mathbf{0}} = (0, 0, 0)$. In other words, we only have to check whether

$$\frac{|f(\mathbf{x}) - f(\mathbf{0})|}{|\mathbf{x}|} = \frac{\left|x_1^3 + e^{-\frac{1}{x^2}} + i\sin^2\left(x_3^2\right)\right|}{|x_1^2 + x_2^2 + ix_3^3|\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

converge to 0 as $\mathbf{x} \to \mathbf{0}$. We claim it doesn't. Let $\mathbf{x} = (t, t, t)$. We have

$$\frac{|f(\mathbf{x}) - f(\mathbf{0})|}{|\mathbf{x}|} \geq \frac{|t^3|}{\left|2\sqrt{3}t^3 + \sqrt{3}it^4\right|} = \frac{1}{\left|2\sqrt{3} + \sqrt{3}it\right|} \to \frac{1}{2\sqrt{3}}, \qquad \text{as } t \searrow 0$$

∎

Question 3: Inverse and implicit function theorem. Legendre transformation.

Let $f(\mathbf{x}, \mathbf{y}) \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^m)$, where $\mathbf{x} = (x_1, \ldots, x_n)$ denote coordinates in $\mathbb{R}^n$ and $\mathbf{y} = (y_1, \ldots, y_m)$ denote coordinates in $\mathbb{R}^m$. Suppose $f(\mathbf{0}, \mathbf{0}) = 0$, $\frac{\partial f}{\partial x_j}\big|_{(\mathbf{0},\mathbf{0})} = 0$ for all $1 \leq j \leq n$, and that the n-by-n matrix $\left(\frac{\partial f}{\partial x_i \partial x_j}\right)\big|_{(\mathbf{0},\mathbf{0})}$ is invertible.

(I) Show that there exists open neighborhood V of $\mathbf{0} \in \mathbb{R}^m$ and smooth $g : V \to \mathbb{R}^n$ such that

$$\frac{\partial f}{\partial x_j}\bigg|_{(g(\mathbf{y}),\mathbf{y})} = 0, \qquad \text{for all } \mathbf{y} \in V \text{ and } j$$

(II) Show that there exists open neighborhoods $\Omega_1, \Omega \ni \mathbf{0} \in \mathbb{R}^n$ together with a smooth function $H : \Omega_1 \to \mathbb{R}$ that makes:

$$\left\{ \left(\mathbf{x}, \frac{\partial f}{\partial x_1}\bigg|_{(\mathbf{x},\mathbf{0})}, \dots, \frac{\partial f}{\partial x_n}\bigg|_{(\mathbf{x},\mathbf{0})} \right) \in \mathbb{R}^{2n} : \mathbf{x} \in \Omega \right\} \tag{2.1}$$

$$= \left\{ \left(\frac{\partial H}{\partial \xi_1}\bigg|_{\boldsymbol{\xi}}, \dots, \frac{\partial H}{\partial \xi_n}\bigg|_{\boldsymbol{\xi}}, \boldsymbol{\xi} \right) \in \mathbb{R}^{2n} : \boldsymbol{\xi} \in \Omega_1) \right\} \tag{2.2}$$

Proof. Define smooth $\mathbf{h} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ by

$$\mathbf{h}(\mathbf{x}, \mathbf{y}) \triangleq \left(\frac{\partial f}{\partial x_1}\bigg|_{(\mathbf{x},\mathbf{y})}, \dots, \frac{\partial f}{\partial x_n}\bigg|_{(\mathbf{x},\mathbf{y})} \right)$$

(I) now follows from applying implicit function theorem to $\mathbf{h}$. For (II), first note that $\mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\mathbf{G}(\mathbf{x}) \triangleq \left(\frac{\partial f}{\partial x_1}\bigg|_{(\mathbf{x},\mathbf{0})}, \dots, \frac{\partial f}{\partial x_n}\bigg|_{(\mathbf{x},\mathbf{0})} \right)$$

is smooth with full rank Jacobian at $\mathbf{0} \in \mathbb{R}^n$. This by inverse function theorem implies the existence of some neighborhood $\Omega \ni \mathbf{0} \in \mathbb{R}^n$ such that $\mathbf{G}$ forms a smooth diffeomorphism between Ω and the open set $\Omega_1 \triangleq \mathbf{G}(\Omega)$. Because set 2.1 is the graph of $\mathbf{G}$ on Ω, clearly we only have to show the existence of an $H : \Omega_1 \to \mathbb{R}$ whose gradient is exactly $\mathbf{G}^{-1} : \Omega_1 \to \Omega$.[1]

Denote $\mathbf{G}^{-1}$ by $\mathbf{T} : \Omega_1 \to \Omega$. We finish the proof by showing it suffices to define $H : \Omega_1 \to \mathbb{R}$ by

$$H(\boldsymbol{\xi}) \triangleq \left(\sum_{i=1}^{n} \xi_i \cdot T_i(\boldsymbol{\xi}) \right) - f\left(\mathbf{T}(\boldsymbol{\xi}), \mathbf{0}\right)$$

[1]Such is possible exactly because $\mathbf{G}$ itself is a gradient.

Fix i. Compute

$$\begin{aligned}
\left.\frac{\partial H}{\partial \xi_i}\right|_{\boldsymbol{\xi}} &= T_i(\boldsymbol{\xi}) + \left(\sum_{j=1}^{n} \xi_j \cdot \left.\frac{\partial T_j}{\partial \xi_i}\right|_{\boldsymbol{\xi}} \right) - \sum_{j=1}^{n} \left.\frac{\partial f}{\partial x_j}\right|_{(\mathbf{T}(\boldsymbol{\xi}),\mathbf{0})} \cdot \left.\frac{\partial T_j}{\partial \xi_i}\right|_{\boldsymbol{\xi}} \\
&= T_i(\boldsymbol{\xi}) + \left(\sum_{j=1}^{n} \xi_j \cdot \left.\frac{\partial T_j}{\partial \xi_i}\right|_{\boldsymbol{\xi}} \right) - \sum_{j=1}^{n} \mathbf{G}_j\left(\mathbf{T}(\boldsymbol{\xi})\right) \cdot \left.\frac{\partial T_j}{\partial \xi_i}\right|_{\boldsymbol{\xi}} \\
&= T_i(\boldsymbol{\xi}) + \left(\sum_{j=1}^{n} \xi_j \cdot \left.\frac{\partial T_j}{\partial \xi_i}\right|_{\boldsymbol{\xi}} \right) - \sum_{j=1}^{n} \xi_j \cdot \left.\frac{\partial T_j}{\partial \xi_i}\right|_{\boldsymbol{\xi}} = T_i(\boldsymbol{\xi})
\end{aligned}$$

■

Question 4: Convolution

Define for all $k \in \mathbb{N}$ the function $f_k : \mathbb{R}^n \to \mathbb{C}$ by:

$$f_k(\mathbf{x}) \triangleq k^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-k(|\mathbf{x}-\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^4) + i\sum_{j=1}^{n} x_j^2 y_j} d\mathbf{y}$$

Find its limit, and show that the convergence is always uniform on compact set.

Proof. Define $\mathbf{z} \triangleq \sqrt{k}(\mathbf{y}-\mathbf{x})$. Change of variables formula give us:

$$f_k(\mathbf{x}) = \int_{\mathbb{R}^n} e^{-|\mathbf{z}|^2 - \frac{|\mathbf{z}|^4}{k} + i\sum_{j=1}^{n} x_j^2 \left(\frac{z_j}{\sqrt{k}} + x_j\right)} d\mathbf{z}$$

Because the integrands are dominated by $e^{-|\mathbf{z}|^2} \in L(\mathbb{R}^n)$, we know by DCT that:

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} e^{-|\mathbf{z}|^2 - \frac{|\mathbf{z}|^4}{k} + i\sum_{j=1}^{n} x_j^2 \left(\frac{z_j}{\sqrt{k}} + x_j\right)} d\mathbf{z} = \int_{\mathbb{R}^n} e^{-|\mathbf{z}|^2 + i\sum_{j=1}^{n} x_j^3} d\mathbf{z}$$

Fix ϵ and M, and let $K' \triangleq \{\mathbf{x} \in \mathbb{R}^n : |x_j| \le M \text{ for all } j\}$. We are required to find N such that:

$$\int_{\mathbb{R}^n} e^{-|\mathbf{z}|^2} \left| 1 - e^{-\frac{|\mathbf{z}|^4}{k} + i\sum_{j=1}^{n} \frac{x_j^2 z_j}{\sqrt{k}}} \right| d\mathbf{z} \le \epsilon, \qquad \text{for all } k \ge N \text{ and } \mathbf{x} \in K'$$

Let $P > 0$ satisfies:

$$\int_{\mathbb{R}^n - K} e^{-|\mathbf{z}|^2} d\mathbf{z} \le \frac{\epsilon}{4}, \qquad \text{where } K \triangleq \{\mathbf{z} \in \mathbb{R}^n : |z_j| \le P \text{ for all } j\} \tag{2.3}$$

Because we have:

$$\left|1-e^{-\frac{|\mathbf{z}|^4}{k}+i\sum_{j=1}^{n}\frac{x_j^2 z_j}{\sqrt{k}}}\right|\le 2,\qquad \text{for all } \mathbf{x},\mathbf{z}\in\mathbb{R}^n$$

By bound 2.3, we have:

$$\int_{\mathbb{R}^n-K} e^{-|\mathbf{z}|^2}\left|1-e^{-\frac{|\mathbf{z}|^4}{k}+i\sum_{j=1}^{n}\frac{x_j^2 z_j}{\sqrt{k}}}\right|d\mathbf{z}\le\frac{\epsilon}{2},\qquad \text{for all } \mathbf{x}\in\mathbb{R}^n$$

Therefore, it only remains to find N so that

$$\int_{K} e^{-|\mathbf{z}|^2}\left|1-e^{-\frac{|\mathbf{z}|^4}{k}+i\sum_{j=1}^{n}\frac{x_j^2 z_j}{\sqrt{k}}}\right|d\mathbf{z}\le\frac{\epsilon}{2},\qquad \text{for all } k\ge N \tag{2.4}$$

Clearly, the function $g_k:K'\times K\to\mathbb{C}$ defined by:

$$g_k(\mathbf{x},\mathbf{z})\triangleq-\frac{|\mathbf{z}|^4}{k}+i\sum_{j=1}^{n}\frac{x_j^2 z_j}{\sqrt{k}}$$

converges uniformly to 0, which implies the function $h_k:K'\times K\to\mathbb{R}$ defined by

$$h_k(\mathbf{x},\mathbf{z})\triangleq\left|1-e^{-\frac{|\mathbf{z}|^4}{k}+i\sum_{j=1}^{n}\frac{x_j^2 z_j}{\sqrt{k}}}\right|\le\left|-\frac{|\mathbf{z}|^4}{k}+i\sum_{j=1}^{n}\frac{x_j^2 z_j}{\sqrt{k}}\right|e^{\left|-\frac{|\mathbf{z}|^4}{k}+i\sum_{j=1}^{n}\frac{x_j^2 z_j}{\sqrt{k}}\right|}$$

converge to 0 uniformly. This implies the existence of some N such that

$$h_k(\mathbf{x},\mathbf{z})\le\frac{\epsilon}{2}\cdot\pi^{-\frac{n}{2}},\qquad \text{for all } k\ge N \text{ and } (\mathbf{x},\mathbf{z})\in K'\times K$$

We have found the desired N for inequality 2.4 ■

Question 5: Convolution

Define for all $k\in\mathbb{N}$ the function $f_k:\mathbb{R}^n\to\mathbb{C}$ by:

$$f_k(\mathbf{x})\triangleq k^{\frac{n}{2}}\int_{|\mathbf{y}|<M} e^{-k(|\mathbf{x}-\mathbf{y}|^2+|\mathbf{x}-\mathbf{y}|^4)+i\sum_{j=1}^{n}x_j^2 y_j}d\mathbf{y}$$

Find its limit, and show that the convergence is always uniform on compact set.

Proof. Define $\mathbf{z}\triangleq\sqrt{k}(\mathbf{y}-\mathbf{x})$. Change of variables formula give us:

$$f_k(\mathbf{x})=\int_{\left|\frac{\mathbf{z}}{\sqrt{k}}+\mathbf{x}\right|<M} e^{-|\mathbf{z}|^2-\frac{|\mathbf{z}|^4}{k}+i\sum_{j=1}^{n}x_j^2\left(\frac{z_j}{\sqrt{k}}+x_j\right)}d\mathbf{z}$$

Let $A_k(\mathbf{x}) \triangleq \left\{ \mathbf{z} \in \mathbb{R}^n : \left| \frac{\mathbf{z}}{\sqrt{k}} + \mathbf{x} \right| < M \right\}$. It is clear that:

$$A_k(\mathbf{x})$$

Because the integrands are dominated by $e^{-|\mathbf{z}|^2} \in L(\mathbb{R}^n)$, we know by DCT that:

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} e^{-|\mathbf{z}|^2 - \frac{|\mathbf{z}|^4}{k} + i\sum_{j=1}^n x_j^2\left(\frac{z_j}{\sqrt{k}} + x_j\right)} d\mathbf{z} = \int_{\mathbb{R}^n} e^{-|\mathbf{z}|^2 + i\sum_{j=1}^n x_j^3} d\mathbf{z}$$

$\blacksquare$

Question 6: Arzelà–Ascoli theorem

Let $U \subseteq \mathbb{R}$ be open and $f_n : U \to \mathbb{R}$ be smooth. Suppose that for all compact $K \subseteq U$ and $m \in \mathbb{N} \cup \{0\}$, there exists a constant $C_{K,m} > 0$ such that

$$\sup\left\{ \left| f_n^{(m)}(x) \right| \in \mathbb{R} : x \in K, n \in \mathbb{N} \right\} \leq C_{K,m}$$

Suppose f_k has a pointwise limit $f : U \to \mathbb{R}$. Show that

(i) f is smooth.

(ii) For all $m \in \mathbb{N} \cup \{0\}$ and compact $K \subseteq U$, sequence $f_n^{(m)}$ converges to $f^{(m)}$ uniformly on K.

Proof. The proof is done via induction. The base case is to show that for all compact $K \subseteq U$, $f_n|_K \to f|_K$ uniformly. Because for all $x \in U$, we may find $[x-\epsilon, x+\epsilon] \subseteq U$, such suffices to show f is continuous at x, thus proving f is continuous on whole U.

Fix compact $K \subseteq U$, we now show $f_n \to f$ uniformly on K. Clearly $\{f_n|_K\}$ is uniformly bounded by $C_{K,0}$. Fix $x \in K$. Let $\delta \leq \epsilon / C_{K,1}$, and let $[x-\delta, x+\delta] \subseteq K$. By **MVT**, we have:

$$|f_n(y) - f_n(x)| \leq C_{K,1} |y - x| \leq \epsilon, \quad \text{for all } n \text{ and } y \in [x-\delta, x+\delta]$$

We have shown that $\{f_n|_K\}$ is pointwise equicontinuous. We may now apply Arzelà–Ascoli theorem to see that every subsequence of $\{f_n|_K\}$ has a uniformly convergent subsequence. Assume for a contradiction that $f_n|_K$ doesn't uniformly converge to $f|_K$. Then, there exists ϵ, some subsequence f_{n_k}, and some sequence $\{x_k \in K\}$ such that $|f_{n_k}(x_k) - f(x_k)| \geq \epsilon$ for all $k \in \mathbb{N}$. Such subsequence $\{f_{n_k}\}$ clearly have no uniform convergent subsequence, a contradiction. We have proved the base case.

Suppose $f \in C^{(m-1)}(U)$ and for all i and compact $K \subseteq U$, $f_n^{(m-i)}|_K \to f^{(m-i)}|_K$ uniformly. The inductive case it to show that for all compact $K \subseteq U$, $f_n^{(m)}|_K$ uniformly converge. Because for all $x \in U$, we may find $[x-\epsilon, x+\epsilon] \subseteq U$, such suffices to show $f \in C^m(U)$.

Fix K. Clearly $\left\{ f_n^{(m)}|_K \right\}$ is uniformly bounded by $C_{K,m}$. Fix $x \in K$. Let $\delta \leq \epsilon \diagup C_{K,m+1}$, and let $[x-\delta, x+\delta] \subseteq K$. By **MVT**, we have:

$$\left| f_n^{(m)}(y) - f_n^{(m)}(x) \right| \leq C_{k,m+1} |y-x| \leq \epsilon, \quad \text{for all } n \text{ and } y \in [x-\delta, x+\delta]$$

We have shown that $\left\{ f_n^{(m)}|_K \right\}$ is pointwise equicontinuous. We may now apply Arzelà–Ascoli theorem to see that every subsequence of $\left\{ f_n^{(m)}|_K \right\}$ has a uniformly convergent subsequence. This implies $f \in C^m(K^\circ)$, which moreover implies $C^m(K)$ if do the same thing on a slightly larger K. Assume for a contradiction that for some $x \in K$, $f_n^{(m)}(x) \not\to f^{(m)}(x)$. Then there exists some subsequence n_k such that $f_{n_k}^{(m)}|_K$ uniformly converge and $f_{n_k}^{(m)}(x)$ that converges to some number $[-\infty, \infty]$ that isn't $f^{(m)}(x)$. This cause a contradiction to the premise that $f_n^{(m-i)}|_K \to f_n^{(m)}|_K$ pointwise. We have shown that $f_n^{(m)}|_K \to f^{(m)}|_K$ pointwise. Assume for a contradiction that $f_n^{(m)}|_K$ doesn't uniformly converge to $f^{(m)}|_K$. Then there exists ϵ, some subsequence f_{n_k}, and some sequence $\{x_k \in K\}$ such that $\left| f_{n_k}^{(m)}(x_k) - f^{(m)}(x_k) \right| \geq \epsilon$ for all $k \in \mathbb{N}$. Such subsequence $\left\{ f_{n_k}^{(m)}|_K \right\}$ clearly have no uniform convergent subsequence, a contradiction. We have proved the inductive case.

∎

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Question 7: metric topology

If every closed and bounded set of a metric space M is compact, does it follows that (M,d) is complete?

Proof. Yes. Let $\{x_n\}$ be a Cauchy sequence in M. To prove $\{x_n\}$ converge in M, one let E be the closure of $\{x_n\}$, and prove that E is indeed bounded. This by premise implies E is compact, which implies there exists some convergent subsequence $x_{n_k} \to x \in E$, since compactness is equivalent to sequential compactness for metric space. The proof then follows from proving $x_n \to x$. ■

Question 8: convergent divergent, Stirling's formula

Determine the values of h for which the following series converges uniformly on $I_h = \{x \in \mathbb{R} : |x| \leq h\}$.

$$\sum_{n=1}^{\infty} \frac{(n!)^2 x^n}{(2n)!} \tag{2.5}$$

Proof. Defining $c_n \triangleq (n!)^2 \big/ (2n)!$, we may write series 2.5 as $\sum c_n x^n$. Because

$$\frac{c_{n+1}}{c_n} = \frac{(n+1)^2}{(2n+2)(2n+1)} \text{ converges to } \frac{1}{4}$$

and because root test is stronger than ratio test, i.e., we have

$$\liminf_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| \leq \liminf_{n\to\infty} \sqrt[n]{|c_n|} \leq \limsup_{n\to\infty} \sqrt[n]{|c_n|} \leq \limsup_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|$$

we know $\sqrt[n]{c_n} \to \frac{1}{4}$. This together with Weierstrass M-test and Cauchy-Hadamard theorem implies series 2.5 converges uniformly on I_h at least for any $h < 4$. We now show that series 2.5 diverge for $x = 4$ by showing $(n!)^2 4^n \big/ (2n)! \to \sqrt{\pi}$. Fix ϵ. By Stirling's formula, we know there exists N such that for all $n \geq N$, we have

$$1 - \epsilon \leq \frac{(n!)^2}{2\pi n \left(\frac{n}{e}\right)^{2n}} \leq 1 + \epsilon \quad \text{ and } \quad 1 - \epsilon \leq \frac{(2n)!}{\sqrt{2\pi n} \left(\frac{2n}{e}\right)^{2n}} \leq 1 + \epsilon$$

Because for all $n \in \mathbb{N}$, we have:

$$\frac{2\pi n \left(\frac{n}{e}\right)^{2n} 4^n}{\sqrt{4\pi n} \left(\frac{2n}{e}\right)^{2n}} = \sqrt{\pi}$$

We now see that for all $n \geq N$, we have:

$$(1-\epsilon)(1+\epsilon)^{-1} \leq \frac{(n!)^2 4^n}{(2n)!} \cdot \pi^{\frac{-1}{2}} \leq (1+\epsilon)(1-\epsilon)^{-1}$$

as desired. ■

Question 9: measure-theoretic Feymann's trick

Consider

$$F(x) \triangleq \int_0^\infty \frac{e^{-xt} - e^{-t}}{t} dt$$

on $I \triangleq \left\{x \in \mathbb{R} : \frac{1}{2} \leq x \leq 2\right\}$.

(i) Show that F is defined on I and F is continuous on I.

(ii) Show that

$$F'(x) = \int_0^\infty -e^{-xt} dt$$

(iii) Evaluate $F(x)$

Proof. Fix $x \in [\frac{1}{2}, 2]$, and define $f(t) \triangleq (e^{-xt} - e^{-t})/t$. We first show:

(i) f is integrable as $t \to 0$.

(ii) f is integrable as $t \to \infty$.

The former is easy, as applying L'Hospital rule, we see f moreover converges at 0^+. For the latter, we have to observe that because $x \geq \frac{1}{2}$, we have:

$$\left|e^{-xt} - e^{-t}\right| \leq \left|e^{-xt}\right| + \left|e^{-t}\right| \leq e^{\frac{-t}{2}} + e^{-t} \leq 2e^{\frac{-t}{2}} \tag{2.6}$$

for any large t. The latter now follows from **Comparison test** and the fact e^{-ct} integrable as $t \to \infty$ for positive c. We now prove (ii) using measure-theoretic Feymann's trick, and the continuity of F will follow. To perform Feymann's trick, we have to establish the existence of some L^1 function $\theta : (0, \infty) \to \mathbb{R}$ such that for almost every $t \in (0, \infty)$, we have:

$$\left|\frac{\partial f}{\partial x}(x,t)\right| \leq \theta(t), \quad \text{for all } x \in \left[\frac{1}{2}, 2\right]$$

Clearly, $\theta(t) \triangleq \exp\left(-\frac{t}{2}\right)$ suffices. Lastly, we compute $F(x)$ on $[1/2, 2]$ using **FTC**:

$$F(1) = 0 \quad \text{and} \quad F'(x) = \int_0^\infty -e^{-xt}dt = \frac{1}{-x} \implies F(x) = \ln\left(\frac{1}{x}\right)$$

∎

Question 10: inverse function theorem, Legendre transform

Consider smooth $f : \mathbb{R}^n \to \mathbb{R}$ whose **Hessian Determinant** is 2 everywhere. We denote its **gradient** $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ by F.

(i) Show that there exists neighborhood $U \subseteq \mathbb{R}^n$ of origin such that the restriction of F forms a smooth diffeomorphism from U to the image of U. Note that the image is guaranteed to be open by **invariance of domain** if we really prove that the action is injective on U.

(ii) Denote the inverse map in part (i) by $\boldsymbol{\xi}(\mathbf{y}) \triangleq (\xi_1(\mathbf{y}), \dots, \xi_n(\mathbf{y}))$. For any $\mathbf{y}$ in the image of U, define

$$f^*(\mathbf{y}) \triangleq -f(\boldsymbol{\xi}(\mathbf{y})) + \sum_{i=1}^n y_i \xi_i(\mathbf{y})$$

Compute the Hessian Determinant of f^*.

Proof. (i) is a direct consequence of the **Inverse function theorem** and the observation that $dF = \text{Hess} f$. Fix k, and compute:

$$\begin{aligned}
\frac{\partial f^*}{\partial y_k}\bigg|_{\mathbf{y}} &= -\left(\sum_{j=1}^n \frac{\partial f}{\partial x_j}\bigg|_{\boldsymbol{\xi}(\mathbf{y})} \frac{\partial \xi_j}{\partial y_k}\bigg|_{\mathbf{y}}\right) + \left(\sum_{i \neq k} y_i \frac{\partial \xi_i}{\partial y_k}\bigg|_{\mathbf{y}}\right) + \xi_k(\mathbf{y}) + y_k \frac{\partial \xi_k}{\partial y_k}\bigg|_{\mathbf{y}} \\
&= -\left(\sum_{j=1}^n y_j \frac{\partial \xi_j}{\partial y_k}\bigg|_{\mathbf{y}}\right) + \left(\sum_{i \neq k} y_i \frac{\partial \xi_i}{\partial y_k}\bigg|_{\mathbf{y}}\right) + \xi_k(\mathbf{y}) + y_k \frac{\partial \xi_k}{\partial y_k}\bigg|_{\mathbf{y}} \\
&= \xi_k(\mathbf{y})
\end{aligned}$$

Note that the second equality hold true by definition of $\boldsymbol{\xi}$. We now see that the matrix

$$\text{Hess} f^* = (d\boldsymbol{\xi})^t = d\boldsymbol{\xi} = (dF)^{-1} = (\text{Hess} f)^{-1}$$

always have determinant $\frac{1}{2}$. ∎

Question 11: estimate with proof of contradiction

Let C^1 function $f : [0, \infty) \to \mathbb{R}$ satisfies:

(i) $f(x) \geq 0$ and $f'(x) \leq 1$ for all $x \geq 0$.

(ii) $\int_0^\infty f(x)dx$ converges.

Does $f(x)$ converge as $x \to \infty$?

Proof. Yes. We will moreover show that $f(x)$ converge to 0 as $x \to \infty$. Assume for a contradiction that there exists some ϵ and $x_n \nearrow \infty$ such that $f(x_n) \geq \epsilon$ for all n and WLOG, $x_{n+1} - x_n \geq \epsilon$. Fix n. Because f is C^1 with derivative always smaller than 1, by **MVT**, every $x \in [x_n - \epsilon, x_n)$ must satisfies $[f(x_n) - f(x)] \diagup (x_n - x) \leq 1$. In other words, every $x \in [x_n - \epsilon, x_n)$ satisfies

$$\epsilon - (x_n - x) \leq f(x_n) - (x_n - x) \leq f(x)$$

We may now estimate

$$\int_{x_n - \epsilon}^{x_n} f(x)dx \geq \int_{x_n - \epsilon}^{x_n} \epsilon - (x_n - x)dx = \frac{\epsilon^2}{2}$$

This cause a contradiction:

$$\int_0^\infty f(x)dx \geq \sum_n \int_{x_n - \epsilon}^{x_n} f(x)dx = \infty$$

$\blacksquare$

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Question 12: Cauchy-Schwarz for integral

Let

$$M \triangleq \left\{ f : [0,\infty] \to [0,\infty) \,\middle|\, \int_0^\infty f^2(x)dx \le 1 \right\}$$

Evaluate

$$\sup_{f\in M} \int_0^\infty f(x)e^{-x}dx \tag{2.7}$$

Proof. This question asks for Cauchy-Schwarz for integral. Use it to show

$$\sup_{f\in M} \int_0^\infty f(x)e^{-x}dx \le \|f\|_2 \cdot \left\|e^{-x}\right\|_2 \le \frac{1}{\sqrt{2}}$$

And (again use it to) show the equality holds when $f = \sqrt{2}e^{-x}$. ∎

Question 13: metric topology

Let $a \in A \subseteq \mathbb{R}^n$, set A be compact, and all convergent subsequences of the sequence $(a_n) \subseteq A$ converge to a.

(i) Does (a_n) converge to a?

(ii) If we remove the hypothesis of A being compact, does (a_n) converge to a?

Proof. For (i), yes. Assume not for a contradiction. There exists some ϵ and subsequence (a_{n_k}) such that every a_{n_k} is ϵ-away from a. By definition of compact metric space, there exists some convergent subsequence of (a_{n_k}), while being a subsequence of (a_n), converges to a point other than a, a contradiction. For (ii), no. Let $A \triangleq [0,1)$ and a_n be $\frac{1}{n}$ if n is even and $1-\frac{1}{n}$ if n is odd. ∎

Question 14: metric topology, proof by contradiction

Let A be some compact metric space, and let continuous $f : A \to A$ never maps two points strictly closer to each other. Show that f is onto.

Proof. Assume $a_0 \in A - f(A)$ for a contradiction, and define $a_n \triangleq f^n(a_0)$. Because f is continuous and A is compact, the image $f(A)$ is compact. This implies the existence of some positive real r smaller than the distance between a_0 and p for any $p \in f(A)$. Because

$a_n \in f(A)$ for all positive n, we know $d(a_n, a_0) \geq r$ for every positive n. Note that by induction,

$$d(a_n, a_{n+1}) = d(f(a_{n-1}), f(a_n)) \geq d(a_{n-1}, a_n) \geq r, \quad \text{for all } n$$

which by induction implies

$$d(a_n, a_m) = d(f(a_{n-1}), f(a_{m-1})) \geq d(a_{n-1}, a_{m-1}) \geq r, \quad \text{for all } 0 \leq n < m. \tag{2.8}$$

The fact that $\{a_n\}$ is a sequence in $f(A)$ that satisfies inequality 2.8 contradicts to the sequential compactness of $f(A)$. ■

Question 15: estimates of integrals

Define a sequence of function $\{f_n(x)\}$ on $[0, 1]$ as:

$$f_n(x) \triangleq \begin{cases} 1 & \text{if } x = 0 \\ 1 & \text{if } x \in (\frac{2k}{2^n}, \frac{2k+1}{2^n}], k \in \{0, 1, \dots, 2^{n-1} - 1\} \\ -1 & \text{if } x \in (\frac{2k+1}{2^n}, \frac{2k+2}{2^n}], k \in \{0, 1, \dots, 2^{n-1} - 1\} \end{cases}$$

Let g be a continuous function. Prove or disprove that $\lim_{n\to\infty} \int_0^1 f_n g dx$ always converge to 0.

Remark. The original question, as stated, lacks precision. It doesn't specify the domain of the function g. If g is only defined on $(0, 1)$, then the product $f_n g$ may fail to be integrable on $(0, 1)$, making the expression $\int_0^1 f_n g\, dx$ ill-defined—for instance, if $g(x) = \frac{1}{x}$. At a minimum, the question should require $g \in L^1(0, 1)$, so that each $f_n g$ is integrable on $(0, 1)$. With this added hypothesis, one can show that the sequence of integrals $\int_0^1 f_n g\, dx$ always converges to zero, regardless of whether g has singularities at the endpoints.

I will just suppose g is continuous on $[0, 1]$.

Proof. Write

$$\int_0^1 f_n g dx = \sum_{k=0}^{2^{n-1}-1} \int_{\frac{2k}{2^n}}^{\frac{2k+1}{2^n}} g(x) - g(x + \frac{1}{2^n}) dx$$

Because **continuous function on a compact domain is uniformly continuous**, for all ϵ, there exists some N such that

$$\left| g(x) - g(x + \frac{1}{2^n}) \right| \leq \epsilon, \quad \text{for all } n \geq N \text{ and } x \in [0, 1 - \frac{1}{2^n}]$$

We may now estimate:

$$\left|\int_0^1 f_n g dx\right| \leq \sum_{k=0}^{2^{n-1}-1} \int_{\frac{2k}{2^n}}^{\frac{2k+1}{2^n}} \left|g(x) - g(x + \frac{1}{2^n})\right| dx \leq \sum_{k=0}^{2^{n-1}-1} \frac{\epsilon}{2^n} = \frac{\epsilon}{2}$$

for all $n \geq N$. ■

Question 16: Lagrange multiplier theorem

Let P_2 be the set of real polynomial with degree no greater than 2, and let $S \triangleq \{p \in P_2 : p(1) = 1\}$. Define $G : P_2 \to \mathbb{R}$ by

$$G(p) \triangleq \int_0^1 p^2(x)dx$$

Does $G|_S$ have any extreme value? If it does, find them.

Remark: The original phrasing of the question is misleading.

Proof. Because $F : \mathbb{R}^3 \to P_2, (a, b, c) \mapsto ax^2 + bx + c$ is surjective, the set on which $G|_S$ attains extreme values is exactly the image of the set on which $G \circ F|_{F^{-1}(S)}$ attains extreme values under F. We have transformed our **optimization problem** to the form in which the **objective function** is:

$$G \circ F(a, b, c) = \frac{a^2}{5} + \frac{ab}{2} + \frac{2ac + b^2}{3} + bc + c^2$$

and there is only one **constraint function**, being:

$$g(a, b, c) \triangleq a + b + c - 1$$

This is best solved using **Lagrange multiplier theorem for single constraint**. Note that there are lots of versions of Lagrange multiplier theorem for single constraint, and one of them requires:

(i) Both objective and constraint functions are C^1.

(ii) The dimension of codomain of the single constraint function is no greater than the dimension of its domain.

Moreover, Lagrange multiplier theorem can only detect local extremum on which the Jacobian of the constraint function is full rank. As harsh as these conditions seems, note that our case clearly satisfies both hypotheses and note that the Jacobian of our constraint function g is full rank globally, so we don't need to worry about any of these conditions.

Compute:

$$\nabla(G \circ F)(a,b,c) = \left(\frac{2a}{5} + \frac{b}{2} + \frac{2c}{3}, \frac{a}{2} + \frac{2b}{3} + c, \frac{2a}{3} + b + 2c\right)$$

Compute:

$$\nabla g(a,b,c) = (1,1,1)$$

We are now required to solve a system of linear equations with four unknown:

$$\begin{cases} \frac{2}{5}a + \frac{1}{2}b + \frac{2}{3}c & = \lambda \\ \frac{1}{2}a + \frac{2}{3}b + c & = \lambda \\ \frac{2}{3}a + b + 2c & = \lambda \\ a + b + c & = 1 \end{cases}$$

One may verify that there is only one solution, i.e.,

$$(a,b,c) = \left(\frac{10}{3}, -\frac{8}{3}, \frac{1}{3}\right) \text{ and } \lambda = \frac{2}{9}$$

It is easy to see that this is a local minimum. Because the Jacobian of our constraint function g is full rank globally, we know this local minimum is the only local extremum and also a global minimum. ■

Question 17: Lebesgue's criteria for proper Riemann integrability

Let $C \in \mathbb{R}^+$, function $f : [a,b] \to \mathbb{R}$ be proper Riemann integrable, and $g : \mathbb{R} \to \mathbb{R}$ be C-Lipschitz continuous. Prove that $g \circ f : [a,b] \to \mathbb{R}$ is proper Riemann integrable.

Remark: The original question write "Riemann integrable" instead of "proper Riemann integrable." The statement doesn't hold true if we include improper Riemann integrable f. For example, let $[a,b] \triangleq [0,1]$, $f(x) \triangleq (\sin(1/x))/x$, and $g(x) \triangleq |x|$.

Proof. There are essentially two way to "answer" this question. One is to construct an honest bound. Another is to cheat by quoting Lebesgue's criteria for proper Riemann integrability. The honest bound proof for this question and the proof for Lebesgue's criteria for proper Riemann integrability are morally the same. To quote Lebesgue's criteria for proper Riemann integrability, one only have to prove that $g \circ f$ is continuous at $x \in [a,b]$ if f is continuous at x. ■

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We say a topological space X is **locally compact** if for all $x \in X$, there exists some compact $K \subseteq X$ such that $x \in K^\circ$. We say a map $f : X \to Y$ is **proper** if f is continuous and for all compact $K \subseteq Y$, $f^{-1}(K)$ is compact.

Question 18: Inverse function theorem, Proper map

Show that C^1 proper map $f : \mathbb{R}^n \to \mathbb{R}^n$ (proper as preimage of compact subspace is compact) whose Jacobian is globally full rank is surjective.

Proof. Because $\mathbb{R}^n$ is connected, to prove f is surjective we only have to prove image of f is clopen in $\mathbb{R}^n$. The image of f is open in $\mathbb{R}^n$ by inverse funcion theorem. The image of f is closed in $\mathbb{R}^n$ is morally a topological result. In particular, we prove theorem 2.4.1, which finish the proof. ∎

Theorem 2.4.1. (Proper map is closed, if the target is locally compact Hausdorff) Let $f : X \to Y$ be proper with Y locally compact, then for all closed $A \subseteq X$, $f(A) \subseteq Y$ is closed.

Proof. Fix closed $A \subseteq X$. Let $y \in Y - f(A)$. We are required to find neighborhood of y disjoint from $f(A)$. Because Y is locally compact, there exists compact $K \subseteq Y$ such that $y \in K^\circ$. Therefore, $f^{-1}(K)$ is compact by properness of f. Now, because $E \triangleq A \cap f^{-1}(K)$ is closed in $f^{-1}(K)$, we know E is compact. (why?)

It now follows from continuity of E that $f(E)$ is compact. Because Y is Hausdorff, this implies $f(E) \subseteq Y$ is closed. (why?)

The rest of the proof then follows from checking that $K^\circ - f(E)$ is a neighborhood of y disjoint from $f(A)$. ∎

Question 19: divergence theorem

Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the identity map, and let $M \triangleq \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq z \leq \sqrt{2 - (x^2 + y^2)}\right\}$ be positively oriented, so the normal vector field point "outward. Compute:

$$\int_{\partial M} \mathbf{F} \cdot \mathbf{n} dA$$

Proof. By divergence theorem and Fubini's theorem, we have

$$\int_{\partial M} \mathbf{F} \cdot \mathbf{n} dA = \int_M \nabla \cdot \mathbf{F} dV = \int_M 3 dV$$

To perform change of variables, we set:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

We have Jacobian $|J_T| = |r|$, and because $x^2+y^2 = \sqrt{2-(x^2+y^2)} \implies r^2 = x^2+y^2 = 1$, we also have:

$$T^{-1}(M) = \left\{\theta \in [0, 2\pi] \quad \text{and} \quad r \in [0,1] \quad \text{and} \quad z \in [r^2, \sqrt{2-r^2}]\right\}$$

We may now compute

$$\int_M 3dV = \int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} 3r dz dr d\theta$$

Because using change of variables, we have:

$$\int_0^1 r\sqrt{2-r^2} dr = \int_1^2 \frac{\sqrt{u}}{2} du = \frac{2^{\frac{3}{2}}-1}{3}, \quad \text{where we set } u = 2-r^2$$

We then have:

$$6\pi \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} 3r dz dr d\theta = \pi \left(\frac{-3}{2} + 2\left(2^{\frac{3}{2}}-1\right)\right)$$

■

Question 20: metric topology

Let M be a metric space with countable elements. Prove or disprove that M is connected.

Proof. If M has only one element, then M is trivially connected. We now prove that if countable M has more than one element then M is disconnected. Let $x \in M$. Because $\{d(y,x) \in \mathbb{R}^+ : y \in M\}$ is countable and $(0,r) \subseteq \mathbb{R}^+$ isn't for any $r \in \mathbb{R}^+$, there exists some r such that $d(y,x) \neq r$ for all r. The proof then follows from showing $B_r(x)$ is closed. ■

Question 21: high school math

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) \triangleq \frac{1}{4} + x - x^2$. Let f^2 denote $f \circ f$, and f^n similarly.

(i) Show that $\{f^n(0)\}$ converges, and find its limit.

(ii) Find all $x \in \mathbb{R}$ that make $\{f^n(x)\}$ converge to $\lim f^n(0)$.

Proof. Completing the square:

$$f(x) = \frac{1}{4} + x - x^2 = -(x - \frac{1}{2})^2 + \frac{1}{2}$$

One can see:

$$f^n(x) = -(x - \frac{1}{2})^{2^n} + \frac{1}{2}$$

which implies:

$$f^n(0) \to \frac{1}{2}$$

and moreover implies:

$$f^n(x) \to \frac{1}{2} \iff x \in \left(\frac{-1}{2}, \frac{3}{2} \right)$$

■

Question 22: Holder space is Banach

Let X consist of all real-valued function f on $[0,1]$ such that $f(0) = 0$ and

$$\|f\| \triangleq \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\frac{1}{3}}} \quad \text{is finite.}$$

Show that X forms a vector space, $\|\cdot\|$ forms a norm on X, and X is complete with respect to this norm.

Remark: More precisely, we are required to show X forms a $\mathbb{R}$-vector space.

Proof. To prove X forms a normed $\mathbb{R}$-vector space, we are required to prove:

(i) $0 \in X$

(ii) $\|f\| = 0 \implies f = 0$

(iii) For all $r \in \mathbb{R}$ and $f \in X$, we have $\|rf\| = |r| \cdot \|f\|$

(iv) For all $f, g \in X$, we have $\|f+g\| \leq \|f\| + \|g\|$

Clearly (i) holds, with the observation $\|0\| = 0$. For (ii), simply use the fact that for $f \neq 0 \in X$, there exists some $x \in [0,1]$ such that $\|f\| \geq |f(x) - f(0)| \diagup |x|^{\frac{1}{3}} > 0$. (iii) and (iv) are easy.[2]

It remains to prove $(X, \|\cdot\|)$ is Banach. Let $\{f_n\} \subseteq (X, \|\cdot\|)$ be a Cauchy sequence. Proving $\{f_n\}$ converges in $(X, \|\cdot\|)$ consist of three steps:

(i) Showing that for all $x \in [0,1]$, $\{f_n(x)\} \subseteq \mathbb{R}$ is Cauchy. In particular, $f_n \to f$ pointwise for some $f : [0,1] \to \mathbb{R}$.

(ii) Showing that $f \in X$.

(iii) Showing that $\|f_n - f\| \to 0$.

Fix $x \in [0,1]$. Our first goal is to prove that $\{f_n(x)\} \subseteq \mathbb{R}$ is a Cauchy sequence. Fix ϵ. Because $\{f_n\} \subseteq X$ is Cauchy, we know there exists some N such that:

$$\|f_n - f_k\| \leq \epsilon x^{\frac{-1}{3}}, \quad \text{for all } n, k \geq N$$

It then follows from $f_n(0) = f_k(0) = 0$ that:

$$|f_n(x) - f_k(x)| \leq |(f_n - f_k)(x) - (f_n - f_k)(0)| \leq \|f_n - f_k\| \, x^{\frac{1}{3}} \leq \epsilon, \quad \text{for all } n, k \geq N$$

We have shown $\{f_n(x)\}$ is Cauchy. It then follows from completeness of $\mathbb{R}$ that $f_n \to f$ for some $f : [0,1] \to \mathbb{R}$ pointwise. We shall now show $f \in X$. Fix $x \neq y$. Because for all n, the inequality:

$$\begin{aligned}\frac{|f(x) - f(y)|}{|x-y|^{\frac{1}{3}}} &\leq \frac{|(f - f_n)(x) - (f - f_n)(y)|}{|x-y|^{\frac{1}{3}}} + \frac{|f_n(x) - f_n(y)|}{|x-y|^{\frac{1}{3}}} \\ &\leq \frac{|f(x) - f_n(x)|}{|x-y|^{\frac{1}{3}}} + \frac{|f(y) - f_n(y)|}{|x-y|^{\frac{1}{3}}} + \frac{|f_n(x) - f_n(y)|}{|x-y|^{\frac{1}{3}}}\end{aligned}$$

holds true, we know:

$$\frac{|f(x) - f(y)|}{|x-y|^{\frac{1}{3}}} \leq \sup \|f_n\| < \infty$$

In other words, $\|f\| \leq \sup \|f_n\| < \infty$, as desired. Lastly, it remains to prove $\|f_n - f\| \to 0$. Fix n and $x \neq y$. Clearly, we have:

$$\begin{aligned}\frac{|(f_n - f)(x) - (f_n - f)(y)|}{|x-y|^{\frac{1}{3}}} &= \lim_{k\to\infty} \frac{|(f_n - f_{n+k})(x) - (f_n - f_{n+k})(y)|}{|x-y|^{\frac{1}{3}}} \\ &\leq \sup_k \|f_n - f_{n+k}\|\end{aligned}$$

[2]Remember that the definition of sup is the smallest upper bound.

This implies (very importantly, helping us getting rid of the dependence on x, y) that:

$$\|f_n - f\| \leq \sup_k \|f_n - f_{n+k}\|, \qquad \text{for all } n$$

which implies

$$\lim_{n\to\infty} \|f_n - f\| \leq \lim_{n\to\infty} \sup_k \|f_n - f_{n+k}\| = 0$$

as desired. ■

Question 23: standard, MVT, proof by contradiction

Let $f : (a, b) \to \mathbb{R}$ be differentiable. Show that $f' : (a, b) \to \mathbb{R}$ has no jump discontinuity.

Proof. Assume for a contradiction that there exists some $x \in (a, b)$ that makes $f'(x-) < f'(x) \leq f'(x+)$, without loss of generality. By **MVT**, for each $n \gg 0$ there exists some $t_n \in (x - \frac{1}{n}, x)$ such that:

$$\frac{f(x) - f(x - \frac{1}{n})}{\frac{1}{n}} = f'(t_n)$$

The contradiction follows from noting

$$\lim_{n\to\infty} f'(t_n) = f'(x-) < f'(x) = \lim_{n\to\infty} \frac{f(x) - f(x - \frac{1}{n})}{\frac{1}{n}}$$

■

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Question 24: Uniform implies L^2, Orthogonality

Does there exist real sequence $\{a_n\}$ that makes $\sum_{n=2}^{\infty} a_n \sin(nx)$ converges uniformly to $\sin(x)$ on $[0, \pi]$?

Proof. No. Using formula:

$$\sin(nx)\sin(mx) = \frac{\cos((n-m)x) - \cos((n+m)x)}{2}$$

It is easily computed that $\sin(x)$ has standard L^2-distance $\geq \sqrt{\pi/2}$ with any partial sum $a_2 \sin(2x) + \cdots + a_N \sin(nx)$. However, uniform convergence on domain of finite measure implies L^2-convergence, a contradiction. ■

Question 25

Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous. Show that f must be bounded by a linear function, i.e., there exists constant $A, B \in \mathbb{R}$ such that

$$|f(x)| \leq A + B|x|, \quad \text{for all } x \in \mathbb{R}$$

Proof. ■

Question 26

Let M, N be two metric spaces, and let $f : M \to N$ be a function.

(i) Given the hypothesis that for each convergent sequence $\{x_n\} \subseteq M$, the sequence $\{f(x_n)\} \subseteq N$ also converges, is f continuous?

(ii) Suppose the function $g : M \to \mathbb{R}$ makes $E \triangleq \{(x, g(x)) \in M \times \mathbb{R} : x \in M\}$ compact in the product topology. Is g continuous?

Proof. The answers for both questions are positive. Assume for a contradiction that f is discontinuous at some $x \in M$. This implies the existence of some ϵ such that for all δ, we have $f(B_\delta(x)) \not\subseteq B_\epsilon(f(x))$. For all $n \in \mathbb{N}$, fix $x_n \in B_{\frac{1}{n}}(x)$ that satisfies $f(x_n) \notin B_\epsilon(f(x))$. We now see that the image of the convergent sequence $(x_1, x, x_2, x, x_3, x, \dots)$ doesn't converge, a contradiction.

Before we prove the continuity of g, we need to first make three necessary topological remarks:

(a) $g(M)$ is compact because E is compact and the projection $M \times \mathbb{R} \to \mathbb{R}$ is continuous by definition of product topology.

(b) $M \times \mathbb{R}$ is Hausdorff because it is the product of two Hausdorff spaces. Therefore E as a compact subspace of $M \times \mathbb{R}$ is closed in $M \times \mathbb{R}$.

(c) Product topology have the property that if $y_n \to y$ in topological space Y and $z_n \to z$ in topological space Z, then $(y_n, z_n) \to (y, z)$ in $Y \times Z$.

We may now prove the continuity of g using the result from question (i). Suppose $x_n \to x$ in M. We are required to prove $g(x_n) \to g(x)$. Assume not for a contradiction: There exists some ϵ and subsequence such that $g(x_{n_k})$ are all ϵ-away from $g(x)$. Because $g(M)$ is sequentially compact, there exists some $y \in M$ and some convergent subsequence $g(x_{n_{k_l}}) \to g(y)$. Because $x_n \to x$, we now have $(x_{n_{k_l}}, g(x_{n_{k_l}})) \to (x, g(y))$. Moreover, this by closedness of E implies $(x, g(y)) \in E$, which implies $g(y) = g(x)$, a contradiction to the assumption that $g(x_{n_k})$ are all ϵ-away from $g(x)$.

■

Question 27

Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be an invertible linear map, and let function $g : \mathbb{R}^3 \to \mathbb{R}^3$ be C^1 while satisfying:

$$\text{For some } \epsilon, |g(x)| \leq C\,|x|^{1+\epsilon} \text{ for all } x \in \mathbb{R}^3$$

Show that $L + g : \mathbb{R}^3 \to \mathbb{R}^3$ is invertible near the origin.

Proof. ■

Question 28: change of variable, multiple integral

Find the volume of the ellipsoid:

$$(x + 2y)^2 + (x - 2y + z)^2 + 3z^2 \leq 1$$

Proof. To perform change of variables, we set:

$$t = \sqrt{3}z \quad \text{and} \quad u = x + 2y \quad \text{and} \quad v = x - 2y + z$$

which linearly and injectively maps the unit ball to

$$M \triangleq \left\{ (x, y, z) \in \mathbb{R}^3 : (x + 2y)^2 + (x - 2y + z)^2 + 3z^2 \leq 1 \right\}$$

Because the Jacobian $|J_T|$ is $\left|\frac{-\sqrt{3}}{12}\right|$, we see that

$$\int_M dV = \int_{B_1(\mathbf{0})} \frac{\sqrt{3}}{12} dV = \frac{\sqrt{3}}{12} \cdot \mathrm{Vol}(B_1(\mathbf{0})) = \frac{\sqrt{3}}{12} \cdot \frac{4}{3}\pi = \frac{\sqrt{3}\pi}{9}$$

■

Question 29: Green's theorem

Let $C \subseteq \mathbb{R}^2$ be a positively oriented simple closed curve. Find the curve C that maximize the integral:

$$\int_C y^3 dx + (3x - x^3) dy \tag{2.9}$$

Proof. Let D be the region bounded by C. By Green's theorem, we have

$$\int_C y^3 dx + (3x - x^3) dy = \int_D \left(\frac{d(3x - x^3)}{dx} - \frac{dy^3}{dy} \right) dA = \int_D \left(3 - 3x^2 - 3y^2\right) dA$$

We now see that the curve that maximize value 2.9 is the unit circle $x^2 + y^2 = 1$. ■

Question 30: Estimate with Weierstrass approximation theorem

Let $f : [0, 1] \to \mathbb{R}$ be continuous. Find the limit:

$$\lim_{n\to\infty} (n + 1) \int_0^1 x^n f(x) dx$$

Remark: Every kind of this question wish you to find a "key" estimate.

Proof. We claim

$$\lim_{n\to\infty} (n + 1) \int_0^1 x^n f(x) dx = f(1) \tag{2.10}$$

Let $P \in \mathbb{R}[x]$ be any polynomial. The key estimate of this question came from splitting:

$$\begin{aligned}(n + 1) \int_0^1 x^n f(x) dx - f(1) = (n + 1) &\int_0^1 x^n \left(f(x) - P(x)\right) dx \\ &+ (n + 1) \int_0^1 x^n P(x) dx - P(1) \\ &+ P(1) - f(1)\end{aligned}$$

which implies:

$$\left|(n+1)\int_0^1 x^n f(x)dx - f(1)\right| \leq (n+1)\left|\int_0^1 x^n \left(f(x)-P(x)\right)dx\right| \tag{2.11}$$
$$+\left|(n+1)\int_0^1 x^n P(x)dx - P(1)\right| \tag{2.12}$$
$$+\left|P(1)-f(1)\right| \tag{2.13}$$

Computing:

$$\begin{aligned}(n+1)\left|\int_0^1 x^n \left(f(x)-P(x)\right)dx\right| &\leq (n+1)\int_0^1 x^n \left|f(x)-P(x)\right| dx \\ &\leq \|f-P\|_\infty (n+1)\int_0^1 x^n dx = \|f-P\|_\infty\end{aligned}$$

We see that both term 2.11 and term 2.13 can be bounded by $\|f-P\|_\infty$ which according to Weierstrass approximation theorem can be made arbitrarily small. For the remaining term 2.12,write $P = c_k x^k + \cdots + c_0$ to see:

$$(n+1)\int_0^1 x^n P(x)dx = (n+1)\left(\frac{c_k}{k+n+1} + \cdots + \frac{c_0}{n+1}\right)$$

which clearly converges to $c_k + \cdots + c_0 = P(1)$ as $n \to \infty$. ■

2.6 Year 108

Question 31: Alternating series test and Comparison test

Does the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n+1)}{n}$$

converge? Does it absolutely converge?

Proof. The fact that it converges follows from alternating series test[3]. The that that it doesn't absolutely converge follows from comparing it with $\sum \frac{1}{n}$. ■

Question 32

Define $\Omega \triangleq \{(x,y) \in \mathbb{R}^2 : 0 \le x, y \le 1\} - \{(1,1)\}$, and define $f : \Omega \to \mathbb{R}$ by

$$f(x,y) \triangleq \frac{1}{(1-xy)^2}$$

(i) For any $\kappa \in (0,1)$, let $U_\kappa \triangleq \{(x,y) \in^2 : 0 \le x, y \le \kappa\}$. Is f uniformly continuous on U_κ ?

(ii) Is f uniformly continuous on Ω?

Proof. ■

Question 33

Define $f_n : [0,1] \to \mathbb{R}$ for all $n \in \mathbb{N}$ by $f_n(x) \triangleq nx^n(1-x)$.

(i) Determine $\lim_{n\to\infty} f_n(x)$ for all $x \in [0,1]$.

(ii) Is the convergence uniform on $[0,1]$?

Proof. Clearly $f_n(x) \to 0$ for all $x \in [0,1]$. Compute:

$$f_n\left(\frac{n}{n+1}\right) = \left(\frac{n}{n+1}\right)^{n+1} \to e^{-1}$$

Remark: The idea came from noting that f_n are polynomials of only two terms and that $f_n(0) = f_n(1) = 0$. This let us see that for all n, we have $\|f_n\|_\infty = f_n(x)$ for some x

[3]Alternating series test follows from noting the partial sum sequence of the suspected series is Cauchy

dependent on n that makes $f_n'(x) = 0$. Such x, if nonzero, can be easily computed to be $\frac{n}{n+1}$. ■

Question 34

Define

$$F(x) \triangleq \int_0^\infty \frac{1-\cos(xt)}{t^2 e^t} dt$$

(i) Can we switch the order of differentiation and integration to obtain formulas for $F'(x)$ and $F''(X)$?

(ii) Find explicit formulas (Not an improper integral) for $F'(x)$ and $F''(x)$.

Proof. ■

Question 35

Consider the function $F : \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$F(x,y,u,v) \triangleq \left(\int_{x-y^2}^{x^2+y} (e^{t^2} + u) dt, x^3 + v \right)$$

(i) Prove that near $(1,1,0,0)$, the two equations $F(x,y,u,v) = (\int_0^2 e^{t^2} dt, 1)$ can be solved for u, v as C^1 functions of x, y.

(ii) Find the first order partial derivatives of the functions $u(x,y)$ and $v(x,y)$ in part (i).

Proof. ■

2.7 Year 107

Question 36

Evaluate the integral

$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \frac{1}{1+3\sin^2(x)}dx$$

Hint: Let $u = \tan x$.

Proof. ∎

Question 37

State and prove Leibniz criterion for convergence of alternating series.

Proof. ∎

Question 38

Let $p \in \mathbb{R}^2$ and let smooth $F : \mathbb{R}^2 \to \mathbb{R}$ satisfies:

$$\left.\frac{\partial F}{\partial x}\right|_p = \left.\frac{\partial F}{\partial y}\right|_p = 0$$

$$\left.\frac{\partial^2 F}{\partial x^2}\right|_p > 0$$

$$\left.\left(\frac{\partial^2 F}{\partial x^2}\cdot\frac{\partial^2 F}{\partial y^2} - \frac{\partial^2 F}{\partial x \partial y}\right)\right|_p > 0$$

Show that p is a local minimum of F.

Proof. ∎

Question 39

Let X be a metric space. We say a sequence $(f_n : X \to \mathbb{R})$ of real-valued functions **converges compactly** if for every compact $K \subseteq X$, the sequence $(f_n|_K : K \to \mathbb{R})$ converges uniformly.

Show that if a pointwise equicontinuous sequence $(f_n : X \to \mathbb{R})$ of real-valued functions pointwise converge to some continuous function $f : X \to \mathbb{R}$, then the convergence is compact.

Proof. ■

Question 40

Compute the outward flux of the vector field $(x + ye^z, e^x \sin(yz), ye^{zx})$ through the boundary of the region $\left\{(x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2 + y^2} - 3)^2 + z^2 < 1\right\}$.

Proof. ■

Question 41

Show that the function

$$f(x) \triangleq \sum_{n=1}^{\infty} \frac{\cos(ne)}{n^x}$$

is well-defined and continuous on $(0, \infty)$.

Proof. ■

Question 42

Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be smooth, and define $S \triangleq \{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}$. Suppose $p \in S$ satisfies

$$\nabla f|_p = (-1, 2)$$
$$\nabla g|_p = (3, -6)$$
$$\left.\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}\right|_p = \begin{pmatrix} 1 & 3 \\ 3 & 0 \end{pmatrix}$$
$$\left.\begin{pmatrix} \frac{\partial^2 g}{\partial x^2} & \frac{\partial^2 g}{\partial x \partial y} \\ \frac{\partial^2 g}{\partial y \partial x} & \frac{\partial^2 g}{\partial y^2} \end{pmatrix}\right|_p = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$$

Show that p is a local minimum for $g|_S$.

Proof. ■

2.8 Year 106

Question 43: tricky multiple integral

Let M be the unit ball $B_1(0)$ in $\mathbb{R}^3$. Compute

$$\int_M \cos(x+y+z)dxdydz$$

Proof. Defining vectors $\mathbf{v},\mathbf{w},\mathbf{s}\in\mathbb{R}^3$ by

$$\mathbf{v}\triangleq(1,1,1) \quad \text{and} \quad \mathbf{w}\triangleq\left(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}},0\right) \quad \text{and} \quad \mathbf{s}\triangleq\left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{-2}{\sqrt{6}}\right)$$

We have an injective differentiable map:

$$\begin{aligned}(t,\theta,r)&\mapsto t\mathbf{v}+r(\cos\theta)\mathbf{w}+r\left(\sin\theta\right)\mathbf{s}\\ &=\left(t+\frac{r\cos\theta}{\sqrt{2}}+\frac{r\sin\theta}{\sqrt{6}},t+\frac{-r\cos\theta}{\sqrt{2}}+\frac{r\sin\theta}{\sqrt{6}},t+\frac{-2r\sin\theta}{\sqrt{6}}\right)\end{aligned}$$

Such that when we perform the change of variable with Fubini's theorem, we see:

$$\int_M \cos(x+y+z)dV=\int_0^{2\pi}\int_{\frac{-1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}}\int_0^{\sqrt{1-3t^2}}\cos(3t)\sqrt{3}r\,drdtd\theta$$

with the Jacobian being:

$$\left|\det\begin{pmatrix}\frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t}\\ \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r}\\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta}\end{pmatrix}\right|=\sqrt{3}r$$

Compute

$$\int_0^{2\pi}\int_{\frac{-1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}}\int_0^{\sqrt{1-3t^2}}\cos(3t)\sqrt{3}r\,drdtd\theta=\sqrt{3}\pi\int_{\frac{-1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}}\cos(3t)(1-3t^2)dt$$

Because

$$\cos(t)t^2=\frac{d}{dt}\left((\sin t)t^2+2(\cos t)t-2\sin t\right)$$

We may compute

$$\sqrt{3}\pi\int_{\frac{-1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}}\cos(3t)(1-3t^2)dt=\pi\left[\sin(\sqrt{3})\left(\frac{4\sqrt{3}}{9}\right)+\cos(\sqrt{3})\left(\frac{-4}{3}\right)\right]$$

∎

Question 44: Lebesgue's criteria, Bessel's inequality

Let $f : [-\pi, \pi] \to \mathbb{R}$ be proper Riemann integrable, and define its Fourier coefficients by

$$a_n \triangleq \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \qquad \text{for all } n \in \mathbb{N}\cup\{0\}$$

$$b_n \triangleq \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx, \qquad \text{for all } n \in \mathbb{N}$$

(i) Show that $f^2 : [-\pi, \pi]$ is proper Riemann integrable.

(ii) Show that the series $\sum a_n^2 + b_n^2$ converges.

Proof. (i) is a direct consequence of the fact that if $f : [-\pi, \pi] \to \mathbb{R}$ is continuous at $x \in [-\pi, \pi]$, then $f^2 : [-\pi, \pi] \to \mathbb{R}$ is continuous at x, and Lebesgue's criteria for proper Riemann integrability.

(ii) is just Bessel's inequality. Let $N \geq 0$. We have:

$$\begin{aligned}
&\frac{1}{\pi}\int_{-\pi}^{\pi}\left(\sum_{n=0}^{N} a_n\cos(nx) + b_n\sin(nx)\right)\left(f(x) - \sum_{n=0}^{N} a_n\cos(nx) + b_n\sin(nx)\right)dx \\
&= \left(\sum_{n=0}^{N} a_n^2 + b_n^2\right) - \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\sum_{n=0}^{N} a_n\cos(nx) + b_n\sin(nx)\right)^2 dx \\
&= \left(\sum_{n=0}^{N} a_n^2 + b_n^2\right) - \left(\sum_{n=0}^{N} a_n^2 + b_n^2\right) = 0
\end{aligned}$$

This give us:

$$\begin{aligned}
\int_{-\pi}^{\pi} f^2(x)dx &= \int_{-\pi}^{\pi}\left(\sum_{n=0}^{N} a_n\cos(nx) + b_n\sin(nx)\right)^2 dx \\
&\quad + \int_{-\pi}^{\pi}\left(f(x) - \sum_{n=0}^{N} a_n\cos(nx) + b_n\sin(nx)\right)^2 dx \\
&= \pi\left(\sum_{n=0}^{N} a_n^2 + b_n^2\right) + \int_{-\pi}^{\pi}\left(f(x) - \sum_{n=0}^{N} a_n\cos(nx) + b_n\sin(nx)\right)^2 dx
\end{aligned}$$

Which moreover give us:

$$\sum_{n=0}^{N} a_n^2 + b_n^2 \leq \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)$$

It then follows from monotone convergence theorem that $\sum a_n^2 + b_n^2$ converges. ■

Question 45

(i) Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers, and define $B_n \triangleq b_1 + \cdots + b_n$. Show that if $a_n \searrow 0$ and $\{B_n\}$ is bounded, then the series $\sum a_n b_n$ converges.

(ii) Show that the function series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\sin(nx)$$

converges uniformly on $[-\pi+\epsilon, \pi-\epsilon]$.

Proof. ■

Question 46

Let $(F_n : [-2,2] \to \mathbb{R})$ be a uniformly bounded sequence of convex functions. Show that it has a subsequence that converges uniformly on $[-1,1]$.

Proof. ■

Question 47

Let $U \subseteq \mathbb{R}^n$ be open, let C^1 function $f : U \to \mathbb{R}^n$ has globally full rank Jacobian, let $V \subseteq \mathbb{R}^n$ be open, and let continuous $g : V \to U$ satisfies $f \circ g(x) = x$ for all $x \in V$. Show that g is also C^1.

Proof. ■