(1) Let V be a finite dimensional vector space over $\mathbb{R}$. Show that, if $\dim(V) < 4$, then every non-zero element of $\wedge^2(V)$ can be written as a wedge product of two vectors in V. Give an example to show that this is not true in dimension 4.

(2) Let α be the 1-form $dz + xdy$ on $\mathbb{R}^3$.
  (a) Find a basis for $\ker \alpha$.
  (b) Compute $\alpha \wedge d\alpha$.
  (c) Find the vector field R that satisfies $\alpha(R) = 1$ and $\iota_R d\alpha = 0$, where ι denotes the interior multiplication.
  (d) Let R be the same vector field in (c), and let $\varphi_t : \mathbb{R}^3 \to \mathbb{R}^3$ denote its flow. Compute $\mathcal{L}_R\alpha$, where $\mathcal{L}$ denotes the Lie derivative, and compute $\varphi_t^*\alpha$ for any fixed t.

(3) Orient the unit sphere S^n in $\mathbb{R}^{n+1}$ as the boundary of the closed unit ball.
  (a) Show that an orientation form on S^n is

$$\omega = \sum_{i=1}^{n+1} (-1)^{i-1} x^i \, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n+1},$$

  where the caret $\widehat{}$ over dx^i indicates that dx^i is to be omitted.
  (b) Show that on S^2, ω is equal to

$$\begin{cases} \frac{dy\wedge dz}{x} & \text{for } x \neq 0, \\ \frac{dz\wedge dx}{y} & \text{for } y \neq 0, \\ \frac{dx\wedge dy}{z} & \text{for } z \neq 0. \end{cases}$$

  (c) Calculate $\int_{S^2} \omega$.

(4) Let M be a manifold of dimension n, and $\{U_i\}_{i\in I}$ be a countable open cover of M. Suppose that each U_i is diffeomorphic to $\mathbb{R}^n$ and all $U_{ij} := U_i \cap U_j$ and $U_{ijk} := U_i \cap U_j \cap U_k$ are either diffeomorphic to $\mathbb{R}^n$ or empty. (You may assume that every manifold has such a good cover.)
Choose a total order $<$ on I, and consider the following sequence of real vector spaces:

$$\mathcal{W}_1 = \prod_{i\in I} \mathbb{R} \xrightarrow{\lambda} \mathcal{W}_2 = \prod_{\substack{i<j\in I \\ U_{ij}\neq\emptyset}} \mathbb{R} \xrightarrow{\mu} \mathcal{W}_3 = \prod_{\substack{i<j<k\in I \\ U_{ijk}\neq\emptyset}} \mathbb{R}$$

where the linear maps are defined by

$$\lambda : (c_i)_{i\in I} \mapsto (c_i - c_j)_{i<j\in I: U_{ij}\neq\emptyset},$$

$$\mu : (c_{ij})_{i<j\in I: U_{ij}\neq\emptyset} \mapsto (c_{ij} + c_{jk} - c_{ik})_{i<j<k\in I: U_{ijk}\neq\emptyset},$$

which satisfy $\mu \circ \lambda = 0$.

  (a) Suppose α is a closed 1-form. Show that for each $i \in I$, we have $\alpha|_{U_i} = df_i$ for smooth function $f_i : U_i \to \mathbb{R}$. Show that there exists a unique element (c_{ij}) in $\mathcal{W}_2$ with $f_i|_{U_{ij}} - f_j|_{U_{ij}} = c_{ij}$ for all $i < j$, $U_{ij} \neq \emptyset$. Show that $\mu((c_{ij})) = 0$.

(b) Show that in Part (a), the element $(c_{ij}) + \operatorname{Im}\lambda \in \ker\mu/\operatorname{Im}\lambda$ is independent of the choice of f_i, and depends only on the cohomology class $[\alpha] \in H^1(M)$.

$\star$ From Parts (a) and (b), one defines a map $\Phi : H^1(M) \to \dfrac{\ker\mu}{\operatorname{Im}\lambda}$.

(c) Show that Φ is injective.

(d) Suppose $(c_{ij})_{i<j\in I:U_{ij}\neq\emptyset}$ lies in $\ker\mu \subset \mathcal{W}_2$. Choose a partition of unity $\{\rho_i\}_{i\in I}$ subordinate to $\{U_i\}_{i\in I}$. Define functions $f_i : U_i \to \mathbb{R}$ by

$$f_i = \sum_{\substack{j\in I, i<j\\ U_{ij}\neq\emptyset}} c_{ij}\rho_j|_{U_i} - \sum_{\substack{j\in I, j<i\\ U_{ij}\neq\emptyset}} c_{ji}\rho_j|_{U_i}.$$

Show that there exists a closed 1-form α such that these f_i and c_{ij} are possible choices in Part (a). Deduce that Φ is surjective.

$\star$ So Φ is an isomorphism.

(e) If M is a compact manifold, show that its first de Rham cohomology group $H^1(M)$ is finite-dimensional.