1. Harmonic Functions on discs

In this section, we aim to solve the following Dirichlet problem: Given a continuous function $g : C_R \to \mathbb{R}$, find a continuous function $u : \overline{D}(R) \to \mathbb{R}$ such that: u is harmonic on D_R, and $u|_{C_R} = g$.

Here,

$$D_R = \{(x, y) : x^2 + y^2 < R^2\}, \quad C_R = \{(x, y) : x^2 + y^2 = R^2\}.$$

Given a harmonic function u, we aim to find a holomorphic function $f : D_R \to \mathbb{C}$ such that $\operatorname{Re} f = u$. Denote $\operatorname{Im} f = v$. If f is holomorphic, then the Cauchy-Riemann equations hold:

$$v_x = -u_y, \quad v_y = u_x.$$

Since D_R is a $*$-domain, we may define:

$$v(P) = \int_{\overline{OP}} -u_y(x, y)\, dx + u_x(x, y)\, dy,$$

for any $P = (a, b) \in D_R$. Here, the line segment $\overline{OP}$ is defined by $\gamma(t) = (ta, tb)$ for $0 \le t \le 1$.

We can verify that $v_x = -u_y$ and $v_y = u_x$, and that v is also harmonic.

Definition 1.1. Let u, v be harmonic functions on a domain D of $\mathbb{R}^2$. We say that v is a harmonic conjugate to u if u and v satisfy the Cauchy-Riemann equations:

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

Given a harmonic function u on D_R, we construct a harmonic conjugate v on D_R. Define $f : D_R \to \mathbb{C}$ by:

$$f(z) = u(x, y) + iv(x, y).$$

Since u and v satisfy the Cauchy-Riemann equations with continuous partial derivatives on D_R, f is complex differentiable at every point of D_R. Thus, f is holomorphic on D_R.

Example 1.1. Let $u(x, y) = x^3 - 3xy^2$ on $\mathbb{R}^2$.

(1) Find a function v on $\mathbb{R}^2$ such that:

$$v_x = -u_y \quad \text{and} \quad v_y = u_x.$$

(2) Find a holomorphic function f on $\mathbb{C}$ such that:

$$f(z) = u(x, y) + iv(x, y),$$

where $z = x + iy$.

We define:

$$v(a, b) = C + \int_{\gamma} 6xy\,dx + 3(x^2 - y^2)\,dy.$$

By definition, with $dx = a\,dt$ and $dy = b\,dt$, we have:

$$v(P) = C + \int_0^1 \bigl(9a^2b - 3b^3\bigr)t^2\,dt = C + 3a^2b - b^3.$$

In other words:

$$v(x, y) = C + 3x^2y - y^3.$$

Thus, the corresponding complex function is:

$$f(z) = z^3 + C,$$

where $z = x + iy$.

Let $u : \overline{D}_R \to \mathbb{R}$ be a continuous function that is harmonic on D_R. Denote $g = u|_{C_R}$. Let $v : D_R \to \mathbb{R}$ be a harmonic conjugate to u. If $f = u + iv$, then f is holomorphic on D_R. By the Cauchy integral formula, $f(z)$ has the representation:

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \quad |z| < r,$$

for any $r < R$. Write $c_n = \alpha_n + i\beta_n$ and $z = re^{i\theta}$ with $-\pi \leq \theta \leq \pi$. It is known that:

$$u(r\cos\theta, r\sin\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta),$$

where:

$$a_0 = 2\alpha_0, \quad a_n = \alpha_n, \quad b_n = -\beta_n \quad \text{for } n \geq 1.$$

By the continuity of u, $g(e^{i\theta}) = \lim_{r \to R^-} u(r\cos\theta, r\sin\theta)$. Hence:

$$g(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} R^n(a_n \cos n\theta + b_n \sin n\theta).$$

Here, we denote $g(e^{i\theta})$ by $g(\theta)$ for simplicity. Then:

$$a_n = \frac{1}{R^n\pi} \int_{-\pi}^{\pi} g(\theta)\cos n\theta\,d\theta,$$

$$b_n = \frac{1}{R^n\pi} \int_{-\pi}^{\pi} g(\theta)\sin n\theta\,d\theta.$$

Substituting a_n and b_n into $u(r\cos\theta, r\sin\theta)$, we obtain:

$$u(r\cos\theta, r\sin\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\phi)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \cos n(\theta-\phi)\right] d\phi.$$

Lemma 1.1. For any $\xi \in \mathbb{R}$ and any $0 \le \rho < 1$,

$$\frac{1}{2} + \sum_{n=1}^{\infty} \rho^n \cos n\xi = \frac{1}{2}\cdot\frac{1-\rho^2}{1-2\rho\cos\xi+\rho^2}.$$

Proof. Let $w = \rho e^{i\xi}$. Since $\rho < 1$, we have $|w| < 1$. Then:

$$\begin{aligned}
\frac{1}{2} + \sum_{n=1}^{\infty} \rho^n \cos n\xi &= \frac{1}{2}\left(\sum_{n=0}^{\infty} w^n + \sum_{m=1}^{\infty} \overline{w}^m\right) \\
&= \frac{1}{2}\left(\frac{1}{1-w} + \frac{\overline{w}}{1-\overline{w}}\right) \\
&= \frac{1}{2}\cdot\frac{1-|w|^2}{|1-w|^2} \\
&= \frac{1}{2}\cdot\frac{1-\rho^2}{1-2\rho\cos\xi+\rho^2}.
\end{aligned}$$

$\square$

Using this lemma, we deduce that:

Theorem 1.1. The Dirichlet problem can be solved by the formula:

$$u(r\cos\theta, r\sin\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2} g(\phi)\, d\phi,$$

i.e., u is continuous on $\overline{D}_R$ and harmonic on D_R, with $u|_{C_R} = g$.

Example 1.2. Find a function u that is continuous on $\{x^2+y^2 \le 1\}$, harmonic on $\{x^2+y^2 < 1\}$, and satisfies:

$$u(\cos\theta, \sin\theta) = 1 + 2\sin\theta + \cos^2\theta + \sin^3\theta.$$