9

Schur's Lemma

Schur's Lemma is a basic result concerning irreducible modules. Though simple in both statement and proof, Schur's Lemma is fundamental to representation theory, and we give an immediate application by determining all the irreducible representations of finite abelian groups.

Schur's Lemma concerns $\mathbb{C}G$-modules rather than $\mathbb{R}G$-modules, and since much of the ensuing theory depends on it, we shall deal with $\mathbb{C}G$-modules for the remainder of the book (except in Chapter 23).

Throughout, G denotes a finite group.

Schur's Lemma

9.1 Schur's Lemma

Let V and W be irreducible $\mathbb{C}G$-modules.

(1) *If ϑ: $V \to W$ is a $\mathbb{C}G$-homomorphism, then either ϑ is a $\mathbb{C}G$-isomorphism, or $v\vartheta = 0$ for all $v \in V$.*

(2) *If ϑ: $V \to V$ is a $\mathbb{C}G$-isomorphism, then ϑ is a scalar multiple of the identity endomorphism 1_V.*

Proof (1) Suppose that $v\vartheta \neq 0$ for some $v \in V$. Then Im $\vartheta \neq \{0\}$. As Im ϑ is a $\mathbb{C}G$-submodule of W by Proposition 7.2, and W is irreducible, we have Im $\vartheta = W$. Also by Proposition 7.2, Ker ϑ is a $\mathbb{C}G$-submodule of V; as Ker $\vartheta \neq V$ and V is irreducible, Ker $\vartheta = \{0\}$. Thus ϑ is invertible, and hence is a $\mathbb{C}G$-isomorphism.

(2) By (2.26), the endomorphism ϑ has an eigenvalue $\lambda \in \mathbb{C}$, and so Ker $(\vartheta - \lambda 1_V) \neq \{0\}$. Thus Ker $(\vartheta - \lambda 1_V)$ is a non-zero $\mathbb{C}G$-submodule of V. Since V is irreducible, Ker $(\vartheta - \lambda 1_V) = V$. Therefore

$$v(\vartheta - \lambda 1_V) = 0 \quad \text{for all } v \in V.$$

That is, $\vartheta = \lambda 1_V$, as required.

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Part (2) of Schur's Lemma has the following converse.

9.2 Proposition
Let V be a non-zero $\mathbb{C}G$-module, and suppose that every $\mathbb{C}G$-homomorphism from V to V is a scalar multiple of 1_V. Then V is irreducible.

Proof Suppose that V is reducible, so that V has a $\mathbb{C}G$-submodule U not equal to $\{0\}$ or V. By Maschke's Theorem, there is a $\mathbb{C}G$-submodule W of V such that

$$V = U \oplus W.$$

Then the projection π: $V \to V$ defined by $(u + w)\pi = u$ for all $u \in U$, $w \in W$ is a $\mathbb{C}G$-homomorphism (see Proposition 7.11), and is not a scalar multiple of 1_V, which is a contradiction. Hence V is irreducible.

■

We next interpret Schur's Lemma and its converse in terms of representations.

9.3 Corollary
Let ρ: $G \to \mathrm{GL}(n, \mathbb{C})$ be a representation of G. Then ρ is irreducible if and only if every $n \times n$ matrix A which satisfies

$$(g\rho)A = A(g\rho) \quad \textit{for all } g \in G$$

has the form $A = \lambda I_n$ with $\lambda \in \mathbb{C}$.

Proof As in Theorem 4.4(1), regard $\mathbb{C}^n$ as a $\mathbb{C}G$-module by defining $vg = v(g\rho)$ for all $v \in \mathbb{C}^n$, $g \in G$.

Let A be an $n \times n$ matrix over $\mathbb{C}$. The endomorphism $v \to vA$ of $\mathbb{C}^n$ is a $\mathbb{C}G$-homomorphism if and only if

$$(vg)A = (vA)g \quad \text{for all } v \in \mathbb{C}^n,\ g \in G;$$

that is, if and only if

$$(g\rho)A = A(g\rho) \quad \text{for all } g \in G.$$

The result now follows from Schur's Lemma 9.1 and Proposition 9.2.

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9.4 Examples

(1) Let $G = C_3 = \langle a\colon a^3 = 1\rangle$, and let $\rho\colon G \to \mathrm{GL}\,(2, \mathbb{C})$ be the representation for which

$$a\rho = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

(see Exercise 3.2). Since the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

commutes with all $g\rho$ ($g \in G$), Corollary 9.3 implies that ρ is reducible.

(2) Let $G = D_{10} = \langle a, b\colon a^5 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$, and let $\omega = e^{2\pi i/5}$. Check that there is a representation $\rho\colon G \to \mathrm{GL}\,(2, \mathbb{C})$ for which

$$a\rho = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \quad b\rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Assume that the matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

commutes with both $a\rho$ and $b\rho$. The fact that $(a\rho)A = A(a\rho)$ forces $\beta = \gamma = 0$; and then $(b\rho)A = A(b\rho)$ gives $\alpha = \delta$. Hence

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} = \alpha I.$$

Consequently ρ is irreducible, by Corollary 9.3.

Representation theory of finite abelian groups

Let G be a finite abelian group, and let V be an irreducible $\mathbb{C}G$-module. Pick $x \in G$. Since G is abelian,

$$vgx = vxg \quad \text{for all } g \in G,$$

and hence the endomorphism $v \to vx$ of V is a $\mathbb{C}G$-homomorphism. By Schur's Lemma 9.1(2), this endomorphism is a scalar multiple of the identity 1_V, say $\lambda_x 1_V$. Thus

$$vx = \lambda_x v \quad \text{for all } v \in V.$$

This implies that every subspace of V is a $\mathbb{C}G$-submodule. As V is irreducible, we deduce that $\dim V = 1$. Thus we have proved

9.5 Proposition
If G is a finite abelian group, then every irreducible $\mathbb{C}G$-module has dimension 1.

The next result is a major structure theorem for finite abelian groups. We shall not prove it here, but refer you to Chapter 9 of the book of J. B. Fraleigh listed in the Bibliography.

9.6 Theorem
Every finite abelian group is isomorphic to a direct product of cyclic groups.

We shall determine the irreducible representations of all direct products

$$C_{n_1} \times C_{n_2} \times \ldots \times C_{n_r}$$

where $n_1, \ldots, n_r$ are positive integers. By Theorem 9.6, this covers the irreducible representations of all finite abelian groups.

Let $G = C_{n_1} \times \ldots \times C_{n_r}$, and for $1 \leqslant i \leqslant r$, let c_i be a generator for C_{n_i}. Write

$$g_i = (1, \ldots, c_i, \ldots, 1) \quad (c_i \text{ in } i\text{th position}).$$

Then

$$G = \langle g_1, \ldots, g_r \rangle, \text{ with } g_i^{n_i} = 1 \text{ and } g_i g_j = g_j g_i \text{ for all } i, j.$$

Now let $\rho: G \to \mathrm{GL}(n, \mathbb{C})$ be an irreducible representation of G over $\mathbb{C}$. Then $n = 1$ by Proposition 9.5, so for $1 \leqslant i \leqslant r$, there exists $\lambda_i \in \mathbb{C}$ such that

$$g_i \rho = (\lambda_i)$$

(where of course (λ_i) is a 1×1 matrix). As g_i has order n_i, we have $\lambda_i^{n_i} = 1$; that is, λ_i is an n_ith root of unity. Also, the values $\lambda_1, \ldots, \lambda_r$ determine ρ, since for $g \in G$, we have $g = g_1^{i_1} \ldots g_r^{i_r}$ for some integers $i_1, \ldots, i_r$, and then

$$g\rho = (g_1^{i_1} \ldots g_r^{i_r})\rho = (\lambda_1^{i_1} \ldots \lambda_r^{i_r}). \tag{9.7}$$

For a representation ρ of G satisfying (9.7) for all $i_1, \ldots, i_r$, write

$$\rho = \rho_{\lambda_1, \ldots, \lambda_r}.$$

Conversely, given any n_ith roots of unity λ_i $(1 \leqslant i \leqslant r)$, the function

$$g_1^{i_1} \ldots g_r^{i_r} \to (\lambda_1^{i_1} \ldots \lambda_r^{i_r})$$

is a representation of G. There are $n_1 n_2 \ldots n_r$ such representations, and no two of them are equivalent.

We have proved the following theorem.

9.8 Theorem

Let G be the abelian group $C_{n_1} \times \ldots \times C_{n_r}$. The representations $\rho_{\lambda_1,\ldots,\lambda_r}$ of G constructed above are irreducible and have degree 1. There are $|G|$ of these representations, and every irreducible representation of G over $\mathbb{C}$ is equivalent to precisely one of them.

9.9 Examples

(1) Let $G = C_n = \langle a\colon a^n = 1\rangle$, and put $\omega = e^{2\pi i/n}$. The n irreducible representations of G over $\mathbb{C}$ are ρ_{ω^j} $(0 \leqslant j \leqslant n-1)$, where

$$a^k \rho_{\omega^j} = (\omega^{jk}) \quad (0 \leqslant k \leqslant n-1).$$

(2) The four irreducible $\mathbb{C}G$-modules for $G = C_2 \times C_2 = \langle g_1, g_2\rangle$ are V_1, V_2, V_3, V_4, where V_i is a 1-dimensional space with basis v_i $(i = 1, 2, 3, 4)$ and

$$\begin{aligned} v_1 g_1 &= v_1, & v_1 g_2 &= v_2; \\ v_2 g_1 &= v_2, & v_2 g_2 &= -v_2; \\ v_3 g_1 &= -v_3, & v_3 g_2 &= v_3; \\ v_4 g_1 &= -v_4, & v_4 g_2 &= -v_4. \end{aligned}$$

Diagonalization

Let $H = \langle g\rangle$ be a cyclic group of order n, and let V be a non-zero $\mathbb{C}H$-module. By Theorem 8.7,

$$V = U_1 \oplus \ldots \oplus U_r,$$

a direct sum of irreducible $\mathbb{C}H$-submodules U_i of V. Each U_i has dimension 1, by Proposition 9.5; let u_i be a vector spanning U_i. Put $\omega = e^{2\pi i/n}$. Then for each i, there exists an integer m_i such that

$$u_i g = \omega^{m_i} u_i.$$

Thus if $\mathscr{B}$ is the basis $u_1, \ldots, u_r$ of V, then

$$[g]_{\mathscr{B}} = \begin{pmatrix} \omega^{m_1} & & 0 \\ & \ddots & \\ 0 & & \omega^{m_r} \end{pmatrix}. \tag{9.10}$$

The following useful result is an immediate consequence of this.

9.11 Proposition
Let G be a finite group and V a $\mathbb{C}G$-module. If $g \in G$, then there is a basis $\mathscr{B}$ of V such that the matrix $[g]_{\mathscr{B}}$ is diagonal. If g has order n, then the entries on the diagonal of $[g]_{\mathscr{B}}$ are nth roots of unity.

Proof Let $H = \langle g \rangle$. As V is also a $\mathbb{C}H$-module, the result follows from (9.10).

■

Some further applications of Schur's Lemma

Our next application concerns an important subspace of the group algebra $\mathbb{C}G$.

9.12 Definition
Let G be a finite group. The *centre* of the group algebra $\mathbb{C}G$, written $Z(\mathbb{C}G)$, is defined by

$$Z(\mathbb{C}G) = \{z \in \mathbb{C}G\colon zr = rz \text{ for all } r \in \mathbb{C}G\}.$$

Using (2.5), it is easy to check that $Z(\mathbb{C}G)$ is a subspace of $\mathbb{C}G$.

For abelian groups G, the centre $Z(\mathbb{C}G)$ is the whole group algebra. For arbitrary groups G, we shall see that $Z(\mathbb{C}G)$ plays a crucial role in the study of representations of G (for example, its dimension is equal to the number of irreducible representations of G – see Chapter 15).

9.13 Example

The elements 1 and $\sum_{g\in G} g$ lie in $Z(\mathbb{C}G)$. Indeed, if H is any normal subgroup of G, then

$$\sum_{h\in H} h \in Z(\mathbb{C}G).$$

To see this, write $z = \sum_{h\in H} h$. Then for all $g \in G$,

$$g^{-1}zg = \sum_{h\in H} g^{-1}hg = \sum_{h\in H} h = z,$$

and so $zg = gz$. Consequently $zr = rz$ for all $r \in \mathbb{C}G$.

For example, if $G = D_6 = \langle a, b: a^3 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$, then $\{1\}$, $\langle a\rangle$ and G are normal subgroups of G, so the elements

$$1,\ 1 + a + a^2 \text{ and } 1 + a + a^2 + b + ab + a^2b$$

lie in $Z(\mathbb{C}G)$. We shall see later that these elements in fact form a basis of $Z(\mathbb{C}G)$.

We use Schur's Lemma to prove the following important property of the elements of $Z(\mathbb{C}G)$.

9.14 Proposition

Let V be an irreducible $\mathbb{C}G$-module, and let $z \in Z(\mathbb{C}G)$. Then there exists $\lambda \in \mathbb{C}$ such that

$$vz = \lambda v \quad \textit{for all } v \in V.$$

Proof For all $r \in \mathbb{C}G$ and $v \in V$, we have

$$vrz = vzr,$$

and hence the function $v \to vz$ is a $\mathbb{C}G$-homomorphism from V to V. By Schur's Lemma 9.1(2), this $\mathbb{C}G$-homomorphism is equal to $\lambda 1_V$ for some $\lambda \in \mathbb{C}$, and the result follows.

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Some elements of the centre of $\mathbb{C}G$ are provided by the centre of G, which we now define.

9.15 Definition
The *centre* of G, written $Z(G)$, is defined by

$$Z(G) = \{z \in G\colon zg = gz \text{ for all } g \in G\}.$$

Clearly $Z(G)$ is a normal subgroup of G, and is a subset of $Z(\mathbb{C}G)$.

Although we have seen in Proposition 6.6 that for every finite group G there is a faithful $\mathbb{C}G$-module, it is not necessarily the case that there is a faithful *irreducible* $\mathbb{C}G$-module. Indeed, the following result shows that the existence of a faithful irreducible $\mathbb{C}G$-module imposes a strong restriction on the structure of G.

9.16 Proposition
If there exists a faithful irreducible $\mathbb{C}G$-module, then $Z(G)$ is cyclic.

Proof Let V be a faithful irreducible $\mathbb{C}G$-module. If $z \in Z(G)$ then z lies in $Z(\mathbb{C}G)$, and hence by Proposition 9.14, there exists $\lambda_z \in \mathbb{C}$ such that

$$vz = \lambda_z v \quad \text{for all } v \in V.$$

Since V is faithful, the function

$$z \to \lambda_z \quad (z \in Z(G))$$

is an injective homomorphism from $Z(G)$ into the multiplicative group $\mathbb{C}^*$ of non-zero complex numbers. Therefore $Z(G) \cong \{\lambda_z: z \in Z(G)\}$, which, being a finite subgroup of $\mathbb{C}^*$, is cyclic (see Exercise 1.7).

■

We remark that the converse of Proposition 9.16 is false, since in Exercise 25.6, we give an example of a group G such that $Z(G)$ is cyclic but there exists no faithful irreducible $\mathbb{C}G$-module.

9.17 Example

If G is an abelian group, then $G = Z(G)$, and so by Proposition 9.16, there is no faithful irreducible $\mathbb{C}G$-module unless G is cyclic. For example, $C_2 \times C_2$ has no faithful irreducible representation (compare Example 9.9(2)).

The irreducible representations of non-abelian groups are more difficult to construct than those of abelian groups. In particular, they do not all have degree 1, as is shown by the following converse to Proposition 9.5.

9.18 Proposition

Suppose that G is a finite group such that every irreducible $\mathbb{C}G$-module has dimension 1. Then G is abelian.

Proof By Theorem 8.7, we can write

$$\mathbb{C}G = V_1 \oplus \ldots \oplus V_n,$$

where each V_i is an irreducible $\mathbb{C}G$-submodule of the regular $\mathbb{C}G$-module $\mathbb{C}G$. Then $\dim V_i = 1$ for all i, since we are assuming that all irreducible $\mathbb{C}G$-modules have dimension 1. For $1 \leqslant i \leqslant n$, let v_i be a vector spanning V_i. Then $v_1, \ldots, v_n$ is a basis of $\mathbb{C}G$; call it $\mathscr{B}$. For all $x, y \in G$, the matrices $[x]_{\mathscr{B}}$ and $[y]_{\mathscr{B}}$ are diagonal, and hence they commute. Since the representation

$$g \to [g]_{\mathscr{B}} \quad (g \in G)$$

of G is faithful (see Proposition 6.6), we deduce that x and y commute. Hence G is abelian, as required.

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Summary of Chapter 9

1. Schur's Lemma states that every $\mathbb{C}G$-homomorphism between irreducible $\mathbb{C}G$-modules is either zero or a $\mathbb{C}G$-isomorphism. Also, the only $\mathbb{C}G$-homomorphisms from an irreducible $\mathbb{C}G$-module to itself are scalar multiples of the identity.
2. The centre $Z(\mathbb{C}G)$ of the group algebra $\mathbb{C}G$ consists of those elements which commute with all elements of $\mathbb{C}G$. The elements of $Z(\mathbb{C}G)$ act as scalar multiples of the identity on all irreducible $\mathbb{C}G$-modules.
3. All irreducible $\mathbb{C}G$-modules for a finite abelian group G have dimension 1, and there are precisely $|G|$ of them.

Exercises for Chapter 9

1. Write down the irreducible representations over $\mathbb{C}$ of the groups C_2, C_3 and $C_2 \times C_2$.
2. Let $G = C_4 \times C_4$.
   (a) Find a non-trivial irreducible representation ρ of G such that $g^2\rho = (1)$ for all $g \in G$.
   (b) Prove that there is no irreducible representation σ of G such that $g\sigma = (-1)$ for all elements g of order 2 in G.
3. Let G be the finite abelian group $C_{n_1} \times \ldots \times C_{n_r}$. Prove that G has a faithful representation of degree r. Can G have a faithful representation of degree less than r?
4. Suppose that $G = D_8 = \langle a, b\colon a^4 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$. Check that there is a representation ρ of G over $\mathbb{C}$ such that

$$a\rho = \begin{pmatrix} -7 & 10 \\ -5 & 7 \end{pmatrix}, \quad b\rho = \begin{pmatrix} -5 & 6 \\ -4 & 5 \end{pmatrix}.$$

Find all 2×2 matrices M such that $M(g\rho) = (g\rho)M$ for all $g \in G$. Hence determine whether or not ρ is irreducible.

Do the same for the representation σ of G, where

$$a\sigma = \begin{pmatrix} 5 & -6 \\ 4 & -5 \end{pmatrix}, \quad b\sigma = \begin{pmatrix} -5 & 6 \\ -4 & 5 \end{pmatrix}.$$

5. Show that if V is an irreducible $\mathbb{C}G$-module, then there exists $\lambda \in \mathbb{C}$ such that

$$v\left(\sum_{g \in G} g\right) = \lambda v \quad \text{for all } v \in V.$$

6. Let $G = D_6 = \langle a, b\colon a^3 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$. Write $\omega = e^{2\pi i/3}$, and let W be the irreducible $\mathbb{C}G$-submodule of the regular $\mathbb{C}G$-module defined by

$$W = \mathrm{sp}\,(1 + \omega^2 a + \omega a^2,\ b + \omega^2 ab + \omega a^2 b)$$

(see Exercise 6.6).

(a) Show that $a + a^{-1} \in Z(\mathbb{C}G)$.

(b) Find $\lambda \in \mathbb{C}$ such that

$$w(a + a^{-1}) = \lambda w$$

for all $w \in W$. (Compare Proposition 9.14.)

7. Which of the following groups have a faithful irreducible representation?

(a) C_n (n a positive integer);

(b) D_8;

(c) $C_2 \times D_8$;

(d) $C_3 \times D_8$.