14

Inner products of characters

We establish some significant properties of characters in this chapter, and in particular we prove the striking result (Theorem 14.21) that if two $\mathbb{C}G$-modules have the same character then they are isomorphic. Also, we describe a method for decomposing a given $\mathbb{C}G$-module as a direct sum of $\mathbb{C}G$-submodules, using characters.

The proofs rely on an inner product involving the characters of a group, and we describe this first.

Inner products

The characters of a finite group G are certain functions from G to $\mathbb{C}$. The set of all functions from G to $\mathbb{C}$ forms a vector space over $\mathbb{C}$, if we adopt the natural rules for adding functions and multiplying functions by complex numbers. That is, if ϑ, ϕ are functions from G to $\mathbb{C}$, and $\lambda \in \mathbb{C}$, then we define $\vartheta + \phi$: $G \to \mathbb{C}$ by

$$(\vartheta + \phi)(g) = \vartheta(g) + \phi(g) \quad (g \in G)$$

and we define $\lambda\vartheta$: $G \to \mathbb{C}$ by

$$\lambda\vartheta(g) = \lambda(\vartheta(g)) \quad (g \in G).$$

(We write these functions on the left to agree with our notation for characters.)

14.1 Example

Let $G = C_3 = \langle a \colon a^3 = 1\rangle$, and suppose that ϑ: $G \to \mathbb{C}$ and ϕ: $G \to \mathbb{C}$ are given by

	1	a	a^2
ϑ	2	i	-1
ϕ	1	1	1

This means that $\vartheta(1) = 2$, $\vartheta(a) = \mathrm{i}$, $\vartheta(a^2) = -1$ and $\phi(1) = \phi(a) = \phi(a^2) = 1$. Then $\vartheta + \phi$ and 3ϑ are given by

	1	a	a^2
$\vartheta + \phi$	3	$1 + \mathrm{i}$	0
3ϑ	6	$3\mathrm{i}$	-3

We shall often think of functions from G to $\mathbb{C}$ as row vectors, as in this example.

The vector space of all functions from G to $\mathbb{C}$ can be equipped with an inner product in a way which we shall describe shortly. The definition of an inner product on a vector space over $\mathbb{C}$ runs as follows. With every ordered pair of vectors ϑ, ϕ in the vector space, there is associated a complex number $\langle \vartheta, \phi \rangle$ which satisfies the following conditions:

(14.2) (a) $\langle \vartheta, \phi \rangle = \overline{\langle \phi, \vartheta \rangle}$ for all ϑ, ϕ;

(b) $\langle \lambda_1 \vartheta_1 + \lambda_2 \vartheta_2, \phi \rangle = \lambda_1 \langle \vartheta_1, \phi \rangle + \lambda_2 \langle \vartheta_2, \phi \rangle$ for all $\lambda_1, \lambda_2 \in \mathbb{C}$ and all vectors $\vartheta_1, \vartheta_2, \phi$;

(c) $\langle \vartheta, \vartheta \rangle > 0$ if $\vartheta \neq 0$.

Notice that condition (a) implies that $\langle \vartheta, \vartheta \rangle$ is always real, and that conditions (a) and (b) give

$$\langle \phi, \lambda_1 \theta_1 + \lambda_2 \vartheta_2 \rangle = \overline{\lambda}_1 \langle \phi, \vartheta_1 \rangle + \overline{\lambda}_2 \langle \phi, \vartheta_2 \rangle$$

for all $\lambda_1, \lambda_2 \in \mathbb{C}$ and all vectors $\vartheta_1, \vartheta_2, \phi$.

We now introduce an inner product on the vector space of all functions from G to $\mathbb{C}$. This will be of basic importance in our study of characters.

14.3 Definition
Suppose that ϑ and ϕ are functions from G to $\mathbb{C}$. Define

$$\langle \vartheta, \phi \rangle = \frac{1}{|G|} \sum_{g \in G} \vartheta(g)\overline{\phi(g)}.$$

It is transparent that the conditions of (14.2) hold, so $\langle\ ,\ \rangle$ is an inner product on the vector space of functions from G to $\mathbb{C}$.

14.4 Example
As in Example 14.1, suppose that $G = C_3 = \langle a \colon a^3 = 1 \rangle$ and that ϑ and ϕ are given by

	1	a	a^2
ϑ	2	i	-1
ϕ	1	1	1

Then

$$\langle \theta, \phi \rangle = \tfrac{1}{3}(2 \cdot 1 + \mathrm{i} \cdot 1 - 1 \cdot 1) = \tfrac{1}{3}(1 + \mathrm{i}),$$
$$\langle \theta, \theta \rangle = \tfrac{1}{3}(2 \cdot 2 + \mathrm{i} \cdot \bar{\mathrm{i}} + (-1) \cdot (-1)) = 2,$$
$$\langle \phi, \phi \rangle = \tfrac{1}{3}(1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1) = 1.$$

Inner products of characters

We can exploit the fact that characters are constant on conjugacy classes to simplify slightly the calculation of the inner product of two characters.

14.5 Proposition

Assume that G has exactly l conjugacy classes, with representatives $g_1, \ldots, g_l$. Let χ and ψ be characters of G.

(1) $\langle \chi, \psi \rangle = \langle \psi, \chi \rangle = \dfrac{1}{|G|}\sum_{g\in G}\chi(g)\psi(g^{-1})$, *and this is a real number.*

(2) $\langle \chi, \psi \rangle = \displaystyle\sum_{i=1}^{l}\frac{\chi(g_i)\overline{\psi(g_i)}}{|C_G(g_i)|}$.

Proof (1) We have $\overline{\psi(g)} = \psi(g^{-1})$ for all $g \in G$, by Proposition 13.9(3). Therefore

$$\langle \chi, \psi \rangle = \frac{1}{|G|}\sum_{g\in G}\chi(g)\psi(g^{-1}).$$

Since $\{g^{-1}: g \in G\} = G$, we also have

$$\langle \chi, \psi \rangle = \frac{1}{|G|}\sum_{g\in G}\chi(g^{-1})\psi(g) = \langle \psi, \chi \rangle.$$

Since $\langle \psi, \chi \rangle = \overline{\langle \chi, \psi \rangle}$, it follows that $\langle \chi, \psi \rangle$ is real. (We shall prove later that $\langle \chi, \psi \rangle$ is, in fact, an integer.)

(2) Recall that g_i^G denotes the conjugacy class of G which contains g_i. Since characters are constant on conjugacy classes,

$$\sum_{g\in g_i^G}\chi(g)\overline{\psi(g)} = |g_i^G|\chi(g_i)\overline{\psi(g_i)}.$$

Now

$$G = \bigcup_{i=1}^{l} g_i^G \text{ and } |g_i^G| = |G|/|C_G(g_i)|,$$

by Corollary 12.3 and Theorem 12.8. Hence

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)} = \frac{1}{|G|} \sum_{i=1}^{l} \sum_{g \in g_i^G} \chi(g)\overline{\psi(g)}$$

$$= \sum_{i=1}^{l} \frac{|g_i^G|}{|G|} \chi(g_i)\overline{\psi(g_i)}$$

$$= \sum_{i=1}^{l} \frac{1}{|C_G(g_i)|} \chi(g_i)\overline{\psi(g_i)}.$$

■

14.6 Example

The alternating group A_4 has four conjugacy classes, with representatives

$$g_1 = 1, \quad g_2 = (1\,2)(3\,4), \quad g_3 = (1\,2\,3), \quad g_4 = (1\,3\,2)$$

(see Example 12.18(1)). We shall see in Chapter 18 that there are characters χ and ψ of A_4 which take the following values on the representatives g_i:

g_i	g_1	g_2	g_3	g_4
$\lvert C_G(g_i)\rvert$	12	4	3	3
χ	1	1	ω	ω^2
ψ	4	0	ω^2	ω

(where $\omega = e^{2\pi i/3}$). Using part (2) of Proposition 14.5, we have

$$\langle \chi, \psi \rangle = \frac{1 \cdot 4}{12} + \frac{1 \cdot 0}{4} + \frac{\omega \cdot \overline{\omega}^2}{3} + \frac{\omega^2 \cdot \overline{\omega}}{3} = 0,$$

$$\langle \psi, \psi \rangle = \frac{4 \cdot 4}{12} + \frac{0 \cdot 0}{4} + \frac{\omega^2 \cdot \overline{\omega}^2}{3} + \frac{\omega \cdot \overline{\omega}}{3} = 2.$$

We advise you to check also that $\langle \chi, \chi \rangle = 1$, and to find the inner products of χ and ψ with the trivial character (which takes the value 1 on all elements of A_4).

We are now going to pave the way to proving the key fact (Theorem 14.12) that the irreducible characters of G form an orthonormal set of vectors in the vector space of functions from G to $\mathbb{C}$; that is, for distinct irreducible characters χ and ψ of G, we have $\langle \chi, \chi \rangle = 1$ and $\langle \chi, \psi \rangle = 0$.

Recall from Chapter 10 that the regular $\mathbb{C}G$-module is a direct sum of irreducible $\mathbb{C}G$-submodules, say

$$\mathbb{C}G = U_1 \oplus \ldots \oplus U_r,$$

and that every irreducible $\mathbb{C}G$-module is isomorphic to one of the $\mathbb{C}G$-modules $U_1, \ldots, U_r$. There are several ways of choosing $\mathbb{C}G$-submodules W_1 and W_2 of $\mathbb{C}G$ such that $\mathbb{C}G = W_1 \oplus W_2$ and W_1 and W_2 have no common composition factor (see Definition 10.4). For example, we may take W_1 to be the sum of those irreducible $\mathbb{C}G$-submodules U_i which are isomorphic to a given irreducible $\mathbb{C}G$-module, and then let W_2 be the sum of the remaining $\mathbb{C}G$-modules U_i. We shall investigate some consequences of writing $\mathbb{C}G$ like this; therefore, we temporarily adopt the following Hypothesis:

14.7 Hypothesis

Let $\mathbb{C}G = W_1 \oplus W_2$, where W_1 and W_2 are $\mathbb{C}G$-submodules which have no common composition factor. Write $1 = e_1 + e_2$ where $e_1 \in W_1$ and $e_2 \in W_2$.

Among other results, we shall derive a formula for e_1 in terms of the character of W_1.

We first look at the effect of applying the elements e_1 and e_2 of $\mathbb{C}G$ to W_1 and W_2.

14.8 Proposition

For all $w_1 \in W_1$ and $w_2 \in W_2$, we have

$$w_1e_1 = w_1, \quad w_2e_1 = 0,$$

$$w_1e_2 = 0, \quad w_2e_2 = w_2.$$

Proof If $w_1 \in W_1$ then the function $w_2 \to w_1w_2$ ($w_2 \in W_2$) is clearly a $\mathbb{C}G$-homomorphism from W_2 to W_1. But W_2 and W_1 have no common composition factor, so every $\mathbb{C}G$-homomorphism from W_2 to W_1 is zero, by Proposition 11.3. Therefore $w_1w_2 = 0$ for all $w_1 \in W_1$, $w_2 \in W_2$. Similarly $w_2w_1 = 0$. In particular, $w_1e_2 = w_2e_1 = 0$. Now

$$w_1 = w_1 1 = w_1(e_1 + e_2) = w_1e_1, \text{ and}$$

$$w_2 = w_2 1 = w_2(e_1 + e_2) = w_2e_2,$$

and this completes the proof.

■

14.9 Corollary
For the elements e_1 and e_2 of $\mathbb{C}G$ which appear in Hypothesis 14.7, we have

$$e_1^2 = e_1,\ e_2^2 = e_2 \textit{ and } e_1e_2 = e_2e_1 = 0.$$

Proof In Proposition 14.8, take $w_1 = e_1$ and $w_2 = e_2$.

■

Next, we evaluate e_1.

14.10. Proposition
Let χ be the character of the $\mathbb{C}G$-module W_1 which appears in Hypothesis 14.7. Then

$$e_1 = \frac{1}{|G|}\sum_{g\in G}\chi(g^{-1})g.$$

Proof Let $x \in G$. The function

$$\vartheta\colon w \to we_1x^{-1} \quad (w \in \mathbb{C}G)$$

is an endomorphism of $\mathbb{C}G$. We shall calculate the trace of ϑ in two ways.

First, for $w_1 \in W_1$ and $w_2 \in W_2$ we have, in view of Proposition 14.8,

$$w_1\vartheta = w_1 e_1 x^{-1} = w_1 x^{-1},$$
$$w_2\vartheta = w_2 e_1 x^{-1} = 0.$$

Thus ϑ acts on W_1 by $w_1 \to w_1x^{-1}$ and on W_2 by $w_2 \to 0$. By the definition of the character χ of W_1, the endomorphism $w_1 \to w_1x^{-1}$ of W_1 has trace equal to $\chi(x^{-1})$, and of course the endomorphism $w_2 \to 0$ of W_2 has trace 0. Therefore

$$\operatorname{tr}\vartheta = \chi(x^{-1}).$$

Secondly, $e_1 \in \mathbb{C}G$, so

$$e_1 = \sum_{g\in G} \lambda_g g$$

for some $\lambda_g \in \mathbb{C}$. By Proposition 13.20, the endomorphism $w \to wgx^{-1}$ ($w \in \mathbb{C}G$) of $\mathbb{C}G$ has trace 0 if $g \neq x$ and has trace $|G|$ if $g = x$. Hence, as $\vartheta\colon w \to w\sum_{g\in G}\lambda_g gx^{-1}$, we have

$$\operatorname{tr}\vartheta = \lambda_x|G|.$$

Comparing our two expressions for $\operatorname{tr}\vartheta$, we see that for all $x \in G$,

$$\lambda_x = \chi(x^{-1})/|G|.$$

Therefore

$$e_1 = \frac{1}{|G|}\sum_{g\in G}\chi(g^{-1})g.$$

■

14.11 Corollary

Let χ be the character of the $\mathbb{C}G$-module W_1 which appears in Hypothesis 14.7. Then

$$\langle \chi, \chi \rangle = \chi(1).$$

Proof Using the definition 6.3 of the multiplication in $\mathbb{C}G$, we deduce from Proposition 14.10 that the coefficient of 1 in e_1^2 is

$$\frac{1}{|G|^2}\sum_{g\in G}\chi(g^{-1})\chi(g) = \frac{1}{|G|}\langle \chi, \chi \rangle.$$

On the other hand, we know from Corollary 14.9 that $e_1^2 = e1$, and the coefficient of 1 in e_1 is $\chi(1)/|G|$. Hence $\langle \chi, \chi \rangle = \chi(1)$, as required.

■

We can now prove the main theorem concerning the inner product $\langle\ ,\ \rangle$.

14.12 Theorem

Let U and V be non-isomorphic irreducible $\mathbb{C}G$-modules, with characters χ and ψ, respectively. Then

$$\langle \chi, \chi \rangle = 1, \textit{ and}$$

$$\langle \chi, \psi \rangle = 0.$$

Proof Recall from Theorem 11.9 that $\mathbb{C}G$ is a direct sum of irreducible $\mathbb{C}G$-submodules, say

$$\mathbb{C}G = U_1 \oplus \ldots \oplus U_r,$$

where the number of $\mathbb{C}G$-submodules U_i which are isomorphic to U is dim U. Let m = dim U, and define W to be the sum of the m irreducible $\mathbb{C}G$-

submodules U_i which are isomorphic to U; let X be the sum of the remaining $\mathbb{C}G$-submodules U_i. Then

$$\mathbb{C}G = W \oplus X.$$

Moreover, every composition factor of W is isomorphic to U, and no composition factor of X is isomorphic to U. In particular, W and X have no common composition factor. The character of W is $m\chi$, since W is the direct sum of m $\mathbb{C}G$-submodules, each of which has character χ.

We now apply Corollary 14.11 to the character of W, and obtain

$$\langle m\chi, m\chi \rangle = m\chi(1).$$

As $\chi(1) = \dim U = m$, this yields $\langle \chi, \chi \rangle = 1$.

Next, let Y be the sum of those $\mathbb{C}G$-submodules U_i of $\mathbb{C}G$ which are isomorphic to either U or V, and let Z be the sum of the remaining $\mathbb{C}G$-submodules U_i. Then

$$\mathbb{C}G = Y \oplus Z,$$

and Y and Z have no common composition factor. The character of Y is $m\chi + n\psi$, where $n = \dim V$. By Corollary 14.11,

$$\begin{aligned} m\chi(1) + n\psi(1) &= \langle m\chi + n\psi, m\chi + n\psi \rangle \\ &= m^2\langle \chi, \chi \rangle + n^2\langle \psi, \psi \rangle + mn(\langle \chi, \psi \rangle + \langle \psi, \chi \rangle). \end{aligned}$$

Now $\langle \chi, \chi \rangle = \langle \psi, \psi \rangle = 1$, by the part of the theorem which we have already proved; and $\chi(1) = m$, $\psi(1) = n$. Therefore

$$\langle \chi, \psi \rangle + \langle \psi, \chi \rangle = 0.$$

By Proposition 14.5(1), $\langle \chi, \psi \rangle = \langle \psi, \chi \rangle$, and hence $\langle \chi, \psi \rangle = 0$.

■

Applications of Theorem 14.12

Let G be a finite group, and let $V_1, \ldots, V_k$ be a complete set of non-isomorphic irreducible $\mathbb{C}G$-modules (see Definition 11.11). If χ_i is the character of V_i ($1 \leqslant i \leqslant k$), then by Theorem 14.12, we have

$$\langle \chi_i, \chi_j \rangle = \delta_{ij} \quad \text{for all } i, j, \tag{14.13}$$

where δ_{ij} is the Kronecker delta function (that is, δ_{ij} is 1 if $i = j$ and is 0 if $i \neq j$). In particular, this implies that the irreducible characters $\chi_1, \ldots, \chi_k$ are all distinct.

Now let V be a $\mathbb{C}G$-module. By Theorem 8.7, V is equal to a direct sum of irreducible $\mathbb{C}G$-submodules. Each of these is isomorphic to some V_i, so there are non-negative integers $d_1, \ldots, d_k$ such that

$$\begin{aligned} V \cong (V_1 \oplus \ldots \oplus V_1) \oplus (V_2 \oplus \ldots \oplus V_2) \oplus \ldots \\ \oplus (V_k \oplus \ldots \oplus V_k), \end{aligned} \tag{14.14}$$

where for each i, there are d_i factors V_i.

Therefore the character ψ of V is given by

$$\psi = d_1\chi_1 + \ldots + d_k\chi_k. \tag{14.15}$$

Using (14.13), we obtain from this

$$\begin{aligned} \langle \psi, \chi_i \rangle = \langle \chi_i, \psi \rangle = d_i \text{ for } 1 \leqslant i \leqslant k, \text{ and} \\ \langle \psi, \psi \rangle = d_1^2 + \ldots + d_k^2. \end{aligned} \tag{14.16}$$

Summarizing, we have

14.17 Theorem

Let $\chi_1, \ldots, \chi_k$ be the irreducible characters of G. If ψ is any character of G, then

$$\psi = d_1\chi_1 + \ldots + d_k\chi_k$$

for some non-negative integers $d_1, \ldots, d_k$. Moreover,

$$d_i = \langle \psi, \chi_i \rangle \quad \textit{for } 1 \leqslant i \leqslant k, \textit{ and}$$

$$\langle \psi, \psi \rangle = \sum_{i=1}^{k} d_i^2.$$

14.18 Example

Recall from Example 13.6(4) that the irreducible characters of $S_3 \cong D_6$ are χ_1, χ_2, χ_3, taking the following values on the conjugacy class representatives 1, (1 2), (1 2 3):

Now let ψ be the character of the 3-dimensional permutation module

g_i $\lvert C_{S_3}(g_i)\rvert$	1 6	(1 2) 2	(1 2 3) 3
χ_1	1	1	1
χ_2	1	−1	1
χ_3	2	0	−1

for S_3. By Example 13.6(2), we know that

$$\psi(1) = 3,\ \psi(1\,2) = 1,\ \psi(1\,2\,3) = 0.$$

Therefore, by Proposition 14.5(2),

$$\langle \psi, \chi_1 \rangle = \frac{3 \cdot 1}{6} + \frac{1 \cdot 1}{2} + 0 = 1.$$

Similarly, $\langle \psi, \chi_2 \rangle = 0$ and $\langle \psi, \chi_3 \rangle = 1$. Thus by Theorem 14.17,

$$\psi = \chi_1 + \chi_3.$$

(This can of course be checked immediately by comparing the values of ψ and $\chi_1 + \chi_3$ on each conjugacy class representative.)

A more substantial calculation along these lines is given in Example 15.7.

We shall see many more applications of the important Theorem 14.17.

14.19 Definition

Suppose that ψ is a character of G, and that χ is an irreducible character of G. We say that χ is a *constituent* of ψ if $\langle \psi, \chi \rangle \neq 0$. Thus, the constituents of ψ are the irreducible characters χ_i of G for which the integer d_i in the expression $\psi = d_1\chi_1 + \ldots + d_k\chi_k$ is non-zero.

The next result is another significant consequence of Theorem 14.12. It gives us a quick and effective method of determining whether or not a given $\mathbb{C}G$-module is irreducible.

14.20 Theorem

Let V be a $\mathbb{C}G$-module with character ψ. Then V is irreducible if and only if $\langle \psi, \psi \rangle = 1$.

Proof If V is irreducible then $\langle \psi, \psi \rangle = 1$ by Theorem 14.12.

Conversely, assume that $\langle \psi, \psi \rangle = 1$. We have

$$\psi = d_1\chi_1 + \ldots + d_k\chi_k$$

for some non-negative integers d_i, and by (14.16),

$$1 = \langle \psi, \psi \rangle = d_1^2 + \ldots + d_k^2.$$

It follows that one of the integers d_i is 1 and the rest are zero. Then by (14.14), $V \cong V_i$ for some i, and so V is irreducible.

■

We are now in a position to prove the remarkable result that 'a $\mathbb{C}G$-module is determined by its character'. It is this fact which motivates our study of characters in much of the rest of the book, for it means that many questions about $\mathbb{C}G$-modules can be answered using character theory.

14.21 Theorem
Suppose that V and W are $\mathbb{C}G$-modules, with characters χ and ψ, respectively. Then V and W are isomorphic if and only if $\chi = \psi$.

Proof In Proposition 13.5 we proved the elementary fact that if $V \cong W$ then $\chi = \psi$. It is the converse which is the substantial part of this theorem.

Thus, suppose that $\chi = \psi$. Again let $V_1, \ldots, V_k$ be a complete set of non-isomorphic irreducible $\mathbb{C}G$-modules with characters $\chi_1, \ldots, \chi_k$. We know by (14.14) that there are non-negative integers c_i, d_i $(1 \leqslant i \leqslant k)$ such that

$$V \cong (V_1 \oplus \ldots \oplus V_1) \oplus (V_2 \oplus \ldots \oplus V_2) \oplus \ldots \oplus (V_k \oplus \ldots \oplus V_k)$$

with c_i factors V_i for each i, and

$$W \cong (V_1 \oplus \ldots \oplus V_1) \oplus (V_2 \oplus \ldots \oplus V_2) \oplus \ldots \oplus (V_k \oplus \ldots \oplus V_k)$$

with d_i factors V_i for each i. By (14.16),

$$c_i = \langle \chi, \chi_i \rangle, \, d_i = \langle \psi, \chi_i \rangle \quad (1 \leqslant i \leqslant k).$$

Since $\chi = \psi$, it follows that $c_i = d_i$ for all i, and hence $V \cong W$.

■

14.22 Example
Let $G = C_3 = \langle a \colon a^3 = 1 \rangle$, and let $\rho_1, \rho_2, \rho_3, \rho_4$ be the representations of G over $\mathbb{C}$ for which

$$a\rho_1 = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}, a\rho_2 = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix},$$

$$a\rho_3 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, a\rho_4 = \begin{pmatrix} 1 & \omega^{-1} \\ 0 & \omega \end{pmatrix}$$

($\omega = e^{2\pi i/3}$). The characters ψ_i of the representations ρ_i ($i = 1, 2, 3, 4$) are

	1	a	a^2
ψ_1	2	2ω	$2\omega^2$
ψ_2	2	-1	-1
ψ_3	2	-1	-1
ψ_4	2	$1 + \omega$	$1 + \omega^2$

Hence by Theorem 14.21, the representations ρ_2 and ρ_3 are equivalent, but there are no other equivalences among ρ_1, ρ_2, ρ_3 and ρ_4.

The next theorem is another consequence of Theorem 14.12.

14.23 Theorem
Let $\chi_1, \ldots, \chi_k$ be the irreducible characters of G. Then $\chi_1, \ldots, \chi_k$ are linearly independent vectors in the vector space of all functions from G to $\mathbb{C}$
.

Proof Assume that

$$\lambda_1\chi_1 + \ldots + \lambda_k\chi_k = 0 \quad (\lambda_i \in \mathbb{C}).$$

Then for all i, using (14.13) we have

$$0 = \langle \lambda_1\chi_1 + \ldots + \lambda_k\chi_k, \chi_i \rangle = \lambda_i.$$

Therefore $\chi_1, \ldots, \chi_k$ are linearly independent.

■

We now relate inner products of characters to the spaces of $\mathbb{C}G$-homomorphisms which we constructed in Chapter 11.

14.24 Theorem

Let V and W be $\mathbb{C}G$-modules with characters χ and ψ, respectively. Then

$$\dim(\mathrm{Hom}_{\mathbb{C}G}(V, W)) = \langle \chi, \psi \rangle.$$

Proof We know from (14.14) that there are non-negative integers c_i, d_i ($1 \leqslant i \leqslant k$) such that

$$V \cong (V_1 \oplus \ldots \oplus V_1) \oplus (V_2 \oplus \ldots \oplus V_2) \oplus \ldots \oplus (V_k \oplus \ldots \oplus V_k)$$

with c_i factors V_i for each i, and

$$W \cong (V_1 \oplus \ldots \oplus V_1) \oplus (V_2 \oplus \ldots \oplus V_2) \oplus \ldots \oplus (V_k \oplus \ldots \oplus V_k)$$

with d_i factors V_i for each i. By Proposition 11.2, for any i, j we have

$$\dim(\mathrm{Hom}_{\mathbb{C}G}(V_i, V_j)) = \delta_{ij}.$$

Hence, using (11.5)(3) we see that

$$\dim(\mathrm{Hom}_{\mathbb{C}G}(V, W)) = \sum_{i=1}^{k} c_i d_i.$$

On the other hand,

$$\chi = \sum_{i=1}^{k} c_i \chi_i \text{ and } \psi = \sum_{i=1}^{k} d_i \chi_i$$