6.7 PDE HW7

Question 117

Solve

$$\begin{cases} u_{tt} = c^2 u_{xx} \text{ for } x \in (0,\infty) \textbf{ (Homogeneous DE)} \\ u(x,0) = x, u_t(x,0) = 0 \textbf{ (IC)} \\ u(0,t) = t^2 \textbf{ (BC)} \end{cases}$$

Proof. If we define

$$w = u - t^2$$

we see that w satisfy

$$\begin{cases} w_{tt} - c^2 w_{xx} = -2 \text{ for } x \in (0,\infty) \textbf{ (Non-homogeneous DE)} \\ w(x,0) = x, w_t(x,0) = 0 \textbf{ (IC)} \\ w(0,t) = 0 \textbf{ (Dirichlet BC)} \end{cases}$$

then if we define

$$\varphi_{\text{odd}}(x) \triangleq x \text{ and } f_{\text{odd}}(x,t) \triangleq \begin{cases} -2 & \text{if } x > 0 \\ 2 & \text{if } x < 0 \end{cases}$$

and

$$\begin{aligned} w(x,t) &\triangleq \frac{\varphi(x+ct) + \varphi(x-ct)}{2} + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f_{\text{odd}}(y,s)\,dyds \\ &= x + \begin{cases} -t^2 & \text{if } x - ct > 0 \\ \frac{x^2}{c^2} - \frac{2tx}{c} & \text{if } x - ct < 0 \end{cases} \end{aligned}$$

Note that we only consider when $x \geq 0$. This then give us

$$u(x,t) = \begin{cases} x & \text{if } x - ct > 0 \\ x + (t - \frac{x}{c})^2 & \text{if } x - ct < 0 \end{cases}$$

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