Chapter 6

Date: Mar 27 Made by Eric

Definition

Definition 1. *An **arithmetic** function $f(n)$ is a function that map all natural numbers to complex numbers*

Definition 2. *An arithmetic function is called **multiplicative** if*

$$f(mn) = f(m)f(n) \tag{1}$$

whenever $gcd(m, n) = 1$

Lemma 1. *A function f is multiplicative if and only if for all $n = p_1^{c_1} \cdots p_k^{c_k}$ we have $f(n) = f(p_1^{c_1}) \cdots f(p_k^{c_k})$*

Proof. From left to right it hold true because prime are co-prime to each other.

From right to left it hold true by simple computation. ■

Definition 3.

$$\tau(n) := \sum_{d|n} 1 \tag{2}$$

$$\sigma(n) := \sum_{d|n} d \tag{3}$$

$$\sigma_k(n) := \sum_{d|n} d^k \tag{4}$$

$$N(n) := n \tag{5}$$

$$u(n) := 1 \tag{6}$$

Lemma 2. *τ and σ are both multiplicative function.*

Proof.

$$\tau(p_1^{c_1} \cdots p_k^{c_k}) = \Pi_{i=1}^k (c_i + 1) = \Pi_{i=1}^k \tau(p_i^{c_i}) \tag{7}$$

$$\sigma(p_1^{c_1} \cdots p_n^{c_n}) = \sum_{d_1=1}^{c_1} \cdots \sum_{d_n=1}^{c_n} \Pi_{i=1}^n p_i^{d_i} = \Pi_{i=1}^n \sum_{d_i=1}^{c_i} p_i^{d_i} = \Pi_{i=1}^n \sigma(p_i^{c_i}) \tag{8}$$

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Definition 4. *the **identity function** I is defined as*

$$I(n) := \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases} \tag{9}$$

Definition 5. *the **Möubius function** is inductively defined as*

$$I(n) = \sum_{d|n} \mu(d) \tag{10}$$

Definition 6. *Let f, g be two arithmetic function. The **Dirichlet product**, or **convolution**, is the arithmetic function $f * g$ given by*

$$f * g(n) := \sum_{de=n} f(d)g(e) \tag{11}$$

Lemma 3.

$$f * g = g * f \tag{12}$$
$$(f * g) * h = f * (g * h) \tag{13}$$
$$f * I = f = I * f \tag{14}$$

Proof.

$$f * g(n) = \sum_{de=n} f(d)g(e) = \sum_{ed=n} g(e)f(d) = g * f(n) \tag{15}$$
$$(f * g) * h(n) = \sum_{de=n} f * g(d)h(e) = \sum_{de=n} \sum_{qr=d} f(q)g(r)h(e) \tag{16}$$
$$= \sum_{qre=n} f(q)g(r)h(e) = \sum_{qm=n} f(q) \sum_{re=m} g(r)h(e) = \sum_{qm=n} f(q)g * h(m) \tag{17}$$
$$= f * (g * h)(n) \tag{18}$$
$$f * I(n) = \sum_{de=n} f(d)I(e) = f(n) = \sum_{ed=n} I(e)f(d) = I * f(n) \tag{19}$$

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Definition 7. *G denote the set of all arithmetic function f that satisfy $f(1) \neq 0$*

Lemma 4. *$\langle G, * \rangle$ constitute an abelian group*

Proof. Arbitrarily pick f, g from G, we see

$$f * g(1) = f(1)g(1) \neq 0 \tag{20}$$
$$I(1) = 1 \neq 0 \implies I \in G \tag{21}$$

Pick $h(n) = \begin{cases} \frac{1}{f(1)} & n = 1 \\ -\frac{1}{f(1)} \sum_{d|n, d<n} h(d)f(\frac{n}{d}) & n > 1 \end{cases}$ **(This function h is defined by induction)**, and we see

$$f * h(1) = f(1)h(1) = 1 = I(1) \tag{22}$$
$$f*h(n) = \sum_{de=n} h(d)f(e) = h(n)f(1) + \sum_{d|n, d<n} h(d)f(\frac{n}{d}) = h(n)f(1) - h(n)f(1) = 0 \tag{23}$$

So $f^{-1} = h \in G$ ■

Definition 8. *Let f be an arithmetic function ans suppose $f(1) \neq 0$. The **Dirichlet inverse** f^{-1} of f is defined implicitly by $f^{-1} * f = I$*

Theorems

Theorem 5. *Suppose $f(n)$ are an arithmetic function that can be express in the form of*

$$f(n) = \sum_{d|n} g(d) \tag{24}$$

where g is an arithmetic function, then we have

$$g(n) = \sum_{d|n} \mu(d) f(\frac{n}{d}) \tag{25}$$

Proof.

$$\sum_{d|n} \mu(d) f(\frac{n}{d}) = \sum_{d|n} \mu(d) \sum_{e|\frac{n}{d}} g(e) = \sum_{d|n} \sum_{e|\frac{n}{d}} \mu(d) g(e) \tag{26}$$

$$= \sum_{ed|n} \mu(d) g(e) = \sum_{e|n} \sum_{d|\frac{n}{e}} \mu(d) g(e) = \sum_{e|n} I(\frac{n}{e}) g(e) = g(n) \tag{27}$$

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Corollary 5.1.

$$f = g * u \implies g = f * \mu \tag{28}$$

Corollary 5.2.

$$u * \mu = I \tag{29}$$

Proof.

$$u * \mu(n) = \sum_{de=n} u(d)\mu(e) = \sum_{e|n} \mu(e) = I(n) \tag{30}$$

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Theorem 6. *Let g, h be multiplicative function*

$$g * h \textit{ are multiplicative}$$

Proof.

$$g * h(\Pi_{i=1}^r p_i^{c_i}) = \sum_{de=\Pi_{i=1}^r p_i^{c_i}} g(d)h(e) = \sum_{d_k \le c_k, \forall k} g(\Pi_{j=1}^r p_j^{d_j}) h(\Pi_{j=1}^r p_j^{c_j - d_j}) \tag{31}$$

$$= \sum_{d_k \le c_k, \forall k} \Pi_{j=1}^r g(p_j^{d_j}) h(p_j^{c_j - d_j}) = \Pi_{j=1}^r \sum_{d_i=1}^{c_i} g(p_i^{c_i}) h(p_i^{c_i - d_i}) = \Pi_{j=1}^r g * h(p_j^{c_j}) \tag{32}$$

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Exercises

8.3

Show that for each k, the function $\sigma_k(n) = \sum_{d|n} d^k$ is multiplicative

Proof.

$$\sigma_k(p_1^{c_1} \cdots p_r^{c_r}) = \sum_{d_1=1}^{c_1} \cdots \sum_{d_r=1}^{c_r} \Pi_{i=1}^r p_i^{kd_i} = \Pi_{i=1}^r \sum_{d_i=1}^{c_i} p_i^{kd_i} = \Pi_{i=1}^r \sigma(p_i^{c_i}) \tag{33}$$

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8.12

Prove that

$$\sum_{d|n} \tau(d)\mu(\frac{n}{d}) = \sum_{d|n} \mu(d)\tau(\frac{n}{d}) = 1 \tag{34}$$

and

$$\sum_{d|n} \sigma(d)\mu(\frac{n}{d}) = \sum_{d|n} \mu(d)\sigma(\frac{n}{d}) = n \tag{35}$$

for all $n \leq 1$. Verify these equations for $n = 12$

Proof. Notice that $\tau(n) = \sum_{d|n} g(d)$ where g is defined by $x \mapsto 1$

Then by Theorem 3, we see

$$\sum_{d|n} \mu(d)\tau(\frac{n}{d}) = g(n) = 1 \tag{36}$$

Notice that $\sigma(n) = \sum_{d|n} N(d)$

Then by Theorem 3, we see

$$\sum_{d|n} \mu(d)\sigma(\frac{n}{d}) = N(n) = n \tag{37}$$

Let $A = \sum_{d|12} \mu(d)\tau(\frac{12}{d})$ and $B = \sum_{d|12} \mu(d)\sigma(\frac{12}{d})$. Now we verify

$$A = \mu(1)\tau(12)+\mu(2)\tau(6)+\mu(3)\tau(4)+\mu(4)\tau(3)+\mu(6)\tau(2)+\mu(12)\tau(1) \tag{38}$$
$$= \tau(12) - \tau(6) - \tau(4) + 0\tau(3) + \tau(2) + 0\tau(1) \tag{39}$$
$$= 6 - 4 - 3 + 0 + 2 + 0 = 1 \tag{40}$$
$$B = \mu(1)\sigma(12)+\mu(2)\sigma(6)+\mu(3)\sigma(4)+\mu(4)\sigma(3)+\mu(6)\sigma(2)+\mu(12)\sigma(1) \tag{41}$$
$$= \sigma(12) - \sigma(6) - \sigma(4) + 0\sigma(3) + \sigma(2) + 0\sigma(1) \tag{42}$$
$$= 28 - 12 - 7 + 0 + 3 + 0 = 12 \tag{43}$$

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8.16

Express τ and σ as the convolution of of two simpler arithmetic function

Proof.

$$\tau(n) = \sum_{d|n} 1 = \sum_{de|n} u(d)u(e) = (u * u)(n) \tag{44}$$

$$\sigma(n) = \sum_{d|n} d = \sum_{de|n} N(d)u(e) = (N * u)(n) \tag{45}$$

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8.18

What arithmetic functions are represented by $\tau * \mu$ and by $\sigma * \mu$

Proof.

$$\tau * \mu = (u * u) * \mu = u * (u * \mu) = u * I = u \tag{46}$$

$$\sigma * \mu = (N * u) * \mu = N \tag{47}$$

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8.20

Show that if f is multiplicative and $f \neq 0$, then $f(1) \neq 0$ and f^{-1} is multiplicative

Proof. $f(n) = f(n)f(1) \implies f(1) = 1$

Notice $f^{-1}(1) = 1$

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