# Eric's note on Complex Geometry

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# Chapter 1

# Single Variable Complex Analysis

## 1.1 Quick Recap

Given a complex-valued function $u+iv = f(x+iy)$ defined on some neighborhood of $z \in \mathbb{C}$, we say $f$ is **complex-differentiable** at $z$ if there exists some complex number denoted by $f'(x)$ such that

$$\frac{f(z+h) - f(z) - f'(z)h}{h} \to 0 \text{ as } h \to 0; h \in \mathbb{C}$$

If $f$ is complex-differentiable at $z$, then obviously $f$ satisfies the **Cauchy-Riemann equation**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ at } z$$

The converse hold true under an extra condition. If $u, v$ satisfy the Cauchy-Riemann equation at $z$ and are real-differentiable at $z$, then from a direct estimation, $f$ is also complex-differentiable at $z$. Given open $U \subseteq \mathbb{C}$ and some complex-valued function $f : U \to \mathbb{C}$, we say $f$ is **holomorphic** if $f$ is complex-differentiable on all $z \in U$. In this note, if we say $\gamma : [a, b] \to \mathbb{C}$ is $C^1$, we mean there exists some $C^1$ map $\widetilde{\gamma} : (a - \epsilon, b + \epsilon) \to \mathbb{C}$ such that $\gamma = \widetilde{\gamma}|_{[a,b]}$. By a **contour**, precisely, we mean a map $\gamma : [a, b] \to \mathbb{C}$ such that for a finite set of points $\{a = x_0 < x_1 < \cdots < x_n < b = x_{n+1}\}$, the maps $\gamma|_{[x_i,x_{i+1}]}$ are $C^1$ and

$$\gamma'(t) \neq 0, \quad \text{for all } t \in [x_i, x_{i+1}]$$

Given some contour $\gamma : [a, b] \to \mathbb{C}$ and some $z$ that does not lie in the image of $\gamma$, we define the **winding number** of $z$ with respect to $\gamma$ to be

$$\text{Ind}_\gamma(z) \triangleq \frac{1}{2\pi i} \int_\gamma \frac{d\xi}{\xi - z}$$

If the contour $\gamma : [a,b] \to \mathbb{C}$ is **closed**[1], by setting

$$f(t) \triangleq \frac{1}{2\pi i}\int_a^{a+t} \frac{\gamma'(s)}{\gamma(s)-z} ds$$

and by noting that $\frac{d}{dt}[e^{-2\pi i f(t)}(\gamma(t)-z)]$ is zero everywhere, we see that the winding number $\mathrm{Ind}_\gamma(z)$ is indeed an integer as expected. Moreover, because $\mathrm{Ind}_\gamma$ is continuous on $\mathbb{C}\setminus\{\gamma(t) : t \in [a,b]\}$ [2], we see $\mathrm{Ind}_\gamma$ is constant on each connected component of $\mathbb{C}\setminus\{\gamma(t) : t \in [a,b]\}$. Now, by a **domain**, we mean an nonempty open connected subset of $\mathbb{C}$. Finally, we may state our version of **Cauchy's Integral Theorem**.

**Theorem 1.1.1. (Cauchy's Integral Theorem)** Given some domain $D$, some holomorphic function $f : D \to \mathbb{C}$, and some closed contour $\gamma : [a,b] \to D$ that does not wind around any point in $D \setminus \{\gamma(t) : t \in [a,b]\}$, we have

$$\int_\gamma f = 0$$

Cauchy's Integral Theorem is the cornerstone of complex analysis. Its proof fundamentally relies on triangulation and its special case for triangles. For brevity, the proof is presented here. Note that when integrating along the boundary of a disk, the orientation matters unless the integral equals 0. To simplify matters, we adopt the universal convention that integration is always performed counterclockwise. Now, by a geometric arguments using 'cuts', we have **Cauchy's Integral Formula**, stating that if $f$ is holomorphic on $|z-z_0| < r$, then

$$f(z) = \frac{1}{2\pi i}\int_{\partial B_\epsilon(z_0)} \frac{f(\xi)}{\xi - z} d\xi, \quad \text{for all } \epsilon < r$$

This with the decomposition

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0} + \frac{z-z_0}{(\xi-z_0)^2} + \cdots + \frac{(z-z_0)^n}{(\xi-z_0)^{n+1}} + \frac{(z-z_0)^{n+1}}{(\xi-z)(\xi-z_0)^{n+1}} \tag{1.1}$$

and an estimation using $|z-z_0| < |\xi - z_0|$ shows that holomorphic functions are locally power series

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi i}\Big(\int_{\partial B_\epsilon(z_0)} \frac{f(\xi)}{(\xi-z_0)^{n+1}} d\xi\Big)(z-z_0)^n, \quad \text{for all } \epsilon < r \text{ and } z \in B_\epsilon(z_0)$$

[1] $\gamma(a) = \gamma(b)$.

[2] One may prove this continuity by direct estimation.

Because all power series converge uniformly on disk with radius strictly smaller than its convergence radius, we may differentiate term by term and have **Taylor's Theorem for power series**

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

If $D \subseteq \mathbb{C}$ is a domain and $f : D \to \mathbb{C}$ is holomorphic, Taylor's Theorem for power series tell us that $\{z \in D : f^{(n)}(z) = 0 \text{ for all } n \geq 0\}$ is not only closed in $D$ but also open, and thus equals to $D$ if proved nonempty. One particularly weak condition for $T$ to be nonempty is that $f \equiv 0$ on some $S \subseteq D$, and $S$ has a limit point in $D$. This result is commonly referred to as **Identity Theorem**. By an **entire** function, we mean a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ defined on the whole complex plane. Obviously, for all $r > 0$ and $z \in B_r(0)$, we have

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \quad \text{where } c_n = \frac{1}{2\pi i}\int_{\partial B_r(0)} \frac{f(\xi)}{\xi^{n+1}} d\xi$$

If $f$ is bounded, then direct estimations show that $c_n = 0$ for all $n > 0$. This result is commonly referred to as **Liouville's Theorem**. Suppose $f$ is holomorphic on some annulus $r < |z - z_0| < R$. Cauchy's integral theorem, Cauchy's integral formula and a geometric argument using 'cuts' give us

$$f(z) = \frac{1}{2\pi i}\Big(\int_{\partial B_{R-\epsilon}(z_0)} \frac{f(\xi)}{\xi - z} d\xi - \int_{\partial B_{r+\epsilon}(z_0)} \frac{f(\xi)}{\xi - z} d\xi\Big)$$

This with decomposition 1.1 and the following decomposition:

$$\frac{1}{z-\xi} = \frac{1}{z-z_0} + \frac{\xi - z_0}{(z-z_0)^2} + \cdots + \frac{(\xi - z_0)^{n-1}}{(z-z_0)^n} - \frac{(\xi-z_0)^n}{(z-z_0)^n(\xi - z)}$$

and two estimations that use

$$|z - z_0| < |\xi - z_0| \text{ for } \xi \in \partial B_{R-\epsilon}(z_0) \text{ and } |\xi - z_0| < |z - z_0| \text{ for } \xi \in \partial B_{r+\epsilon}(z_0)$$

shows that

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \frac{1}{2\pi i}\Big(\int_{\partial B_{R-\epsilon}(z_0)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi\Big)(z-z_0)^n \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{2\pi i}\Big(\int_{\partial B_{r+\epsilon}(z_0)} (\xi - z_0)^{n-1} f(\xi) d\xi\Big)(z-z_0)^{-n} \end{aligned}$$

Because the integrands $(\xi - z_0)^k f(\xi)$ have no singularities on the annulus, again, we may apply a geometric argument using 'cuts' to simplify the expression into its **Laurent series**:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

where

$$c_n = \frac{1}{2\pi i} \int_{\partial B_\epsilon(z_0)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \text{ for all } r < \epsilon < R \text{ and } n \in \mathbb{Z} \tag{1.2}$$

If $f$ is defined and holomorphic on some deleted neighborhood $B_\epsilon(z_0) \setminus \{z_0\}$ but not defined on $\{z_0\}$, we say $z_0$ is an **isolated singularity** of $f$, and we write

$$\text{Res}(f, z_0) \triangleq c_{-1}$$

to denote the **residue** of $f$ at $z_0$. Let $D \subseteq \mathbb{C}$ be a simply connected domain, and suppose holomorphic $f$ is defined on $D$ except at some finite numbers of singularities. Let $\gamma : [a, b] \to D$ be some **simple**[3] closed contour, so that the image of $\gamma$ is a Jordan curve. Under this condition, we may apply the **Jordan Curve Theorem** to distinguish between the interior and the exterior of $\gamma$. A simple closed contour $\gamma$ is **positively oriented** if the winding number is positive in the region enclosed by $\gamma$. We may now state our version of **Cauchy's Residue Theorem**.

**Theorem 1.1.2. (Cauchy's Residue Theorem)** Let $D \subseteq \mathbb{C}$ be a simply connected domain, and let $\gamma : [a, b] \to D$ be some positively oriented simple closed contour. If $f$ is is defined and holomorphic on $D$ except at a finite set of points $\{z_1, \dots, z_n\}$ that are all enclosed by $\gamma$, then

$$\int_\gamma f(\xi) d\xi = 2\pi i \sum_{j=1}^{n} \text{Res}(f, z_j)$$

There are many distinct rigorous proofs for Cauchy's residue Theorem. None of them are trivial by some geometric argument. Again, for brevity, we present a proof here using Green's Theorem. Suppose $z_0$ is an isolated singularity of $f$, and $c_n$ are the coefficients of the Laurent series of $f$ about $z_0$. There are three types of singularities depending on $c_n$. If $c_n = 0$ for all $n < 0$, then $z_0$ is said to be a **removable singularity**. By direct estimation[4], we see that if $f$ is bounded on some deleted neighborhood of $z_0$, then $z_0$ is removable. This recognition is called **Riemann's removable singularity Theorem**. If there exists some sequence $n_k$ of integers that converges to $-\infty$ such that $c_{n_k} \neq 0$ for all $k$, then we say $z_0$

[3] By simple, we mean $\gamma(t) = \gamma(s)$ if and only if $|t - s| = |a - b|$

[4] For each $n < 0$, let $\epsilon \to 0$ in Equation 1.2.

is an **essential singularity**. The last type of singularities is perhaps the most interesting. If there exists some $m < 0$ such that $c_m \neq 0$ and $c_n = 0$ for all $n < m$, we say $z_0$ is a **pole** of $f$ with multiplicity $m$. In such case, obviously we may define some $g$ by

$$g(z) \triangleq (z - z_0)^m f(z) \text{ for all } z \neq z_0$$

so that $z_0$ is merely a removable singularity of $g$, and $g(z_0) \neq 0$ after the removal. Now, because $g$ is continuous at $z_0$, we see $f$ is nonzero on some neighborhood around $z_0$, and we may compute on that neighborhood:

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} - \frac{m}{z - z_0} \text{ for all } z \neq z_0$$

This implies

$$\text{Res}\left(\frac{f'}{f}, z_0\right) = -m$$

Similarly, if $z_0$ is a **zero**[5] of $f$ with multiplicity $k$, we may define $g$ by

$$g(z) \triangleq (z - z_0)^{-k} f(z)$$

and compute

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} + \frac{k}{z - z_0} \text{ for all } z \neq z_0 \text{ in some neighborhood of } z_0$$

to deduce

$$\text{Res}\left(\frac{f'}{f}, z_0\right) = k$$

These observations together with Cauchy's Residue Theorem now give us the **Argument Principle**. Given simply connected domain $D \subseteq \mathbb{C}$, positively oriented simple closed contour $\gamma : [a, b] \to D$ and some $f$ **meromorphic**[6] on $D$, if $f$ has no zeros and has no poles on the image of $\gamma$, then

$$\int_\gamma \frac{f'(\xi)}{f(\xi)} d\xi = 2\pi i(Z - P)$$

[5]By $z_0$ being a zero of $f$ with multiplicity $k$, we mean $f$ is holomorphically defined on some neighborhood of $z_0$, $f(z_0) = 0$, and $k$ is the smallest integer such that $c_k \neq 0$ where $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$. Note that zeros and poles are dual to each other.

[6]By $f$ being meromorphic on open $U$, we mean that $f$ is holomorphic on $U$ except on a finite set of poles.

where $Z$ and $P$ are respectively the numbers of zeros and poles enclosed by $\gamma$ counted with multiplicity.

Now, let $D$ be some simply connected domain, let $\gamma : [a, b] \to D$ be some positively oriented simple closed contour, and let $f, g : D \to \mathbb{C}$ be two holomorphic function. If we require that $|g| < |f|$ on the image of $\gamma$, then obviously neither $f$, $f + g$ nor $1 + \frac{g}{f}$ can have a zero on the image of $\gamma$, so after we compute

$$\frac{(1 + \frac{g}{f})'}{1 + \frac{g}{f}} = \frac{(f + g)'}{f + g} - \frac{f'}{f}$$

we may apply the argument principle to $f$ and $g$ to conclude

$$Z_{f+g} - Z_f = \int_\gamma \frac{(1 + \frac{g}{f})'}{1 + \frac{g}{f}} d\xi$$

where $Z_{f+g}$ and $Z_f$ are the numbers of zeros of $f + g$ and $f$ enclosed by $\gamma$ counted with multiplicity. Moreover, if we define $\widetilde{\gamma} : [a, b] \to \mathbb{C}$ by

$$\widetilde{\gamma}(t) \triangleq 1 + \frac{g(t)}{f(t)}$$

we see

$$\int_{\widetilde{\gamma}} \frac{d\xi}{\xi} = \int_\gamma \frac{(1 + \frac{g}{f})'}{1 + \frac{g}{f}} d\xi = Z_{f+g} - Z_f$$

Finally, noting that $\widetilde{\gamma}$ as a simple closed contour lies in $\mathrm{Re}(z) > 0$ [7], and that $\log(\xi)$ is well defined on $\mathrm{Re}(z) > 0$ with derivative $\frac{1}{\xi}$[8], we can finally conclude the **Rouché's theorem**:

$$Z_{f+g} - Z_f = 0$$

[7]This is because $\left|\frac{g}{f}\right| < 1$ on $\gamma$.

[8]By real inverse function theorem.

## 1.2 Riemann Mapping Theorem

## 1.3 Cauchy's Integral and Residue Theorem