Method of Lagrange Multipliers

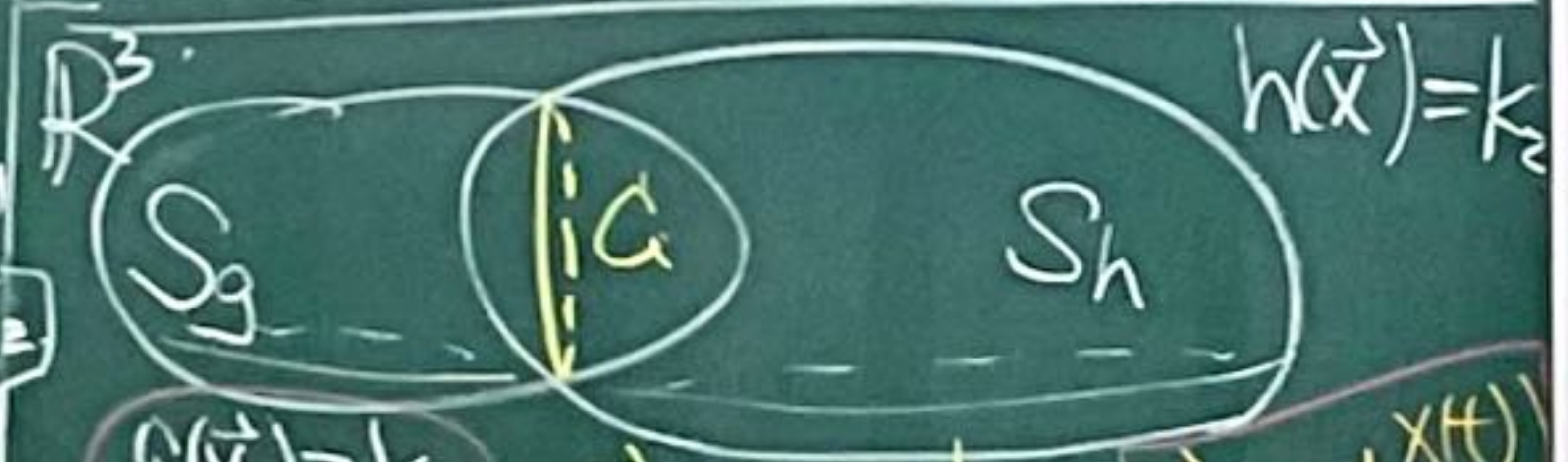
Find max & min values of $f(x,y,z)$ subject to the constraints: $g(x,y,z)=k_1$, $h(x,y,z)=k_2$

1° Solve $\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g = k_1,\ h = k_2 \end{cases}$ $\quad \vec{x} = (x,y,z)$

2° $\vec{x}_1, \vec{x}_2 \cdots \quad \lambda_1, \lambda_2 \cdots, \quad \mu_1, \mu_2 \cdots$

3° Compute $f(\vec{x}_1)\ f(\vec{x}_2) \cdots$

$\Rightarrow$ Largest one = max, smallest one = min



$C: \begin{cases} g = k_1 \\ h = k_2 \end{cases} : \vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$

$C \subset S_g \Rightarrow \frac{d}{dt} g(\vec{r}(t)) = \frac{d}{dt} g(x(t), y(t), z(t)) = \frac{d}{dt} k_1$

Chain Rule $\Rightarrow g_x \frac{dx}{dt} + g_y \frac{dy}{dt} + g_z \frac{dz}{dt} = 0$

$\Rightarrow \nabla g(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} = 0$

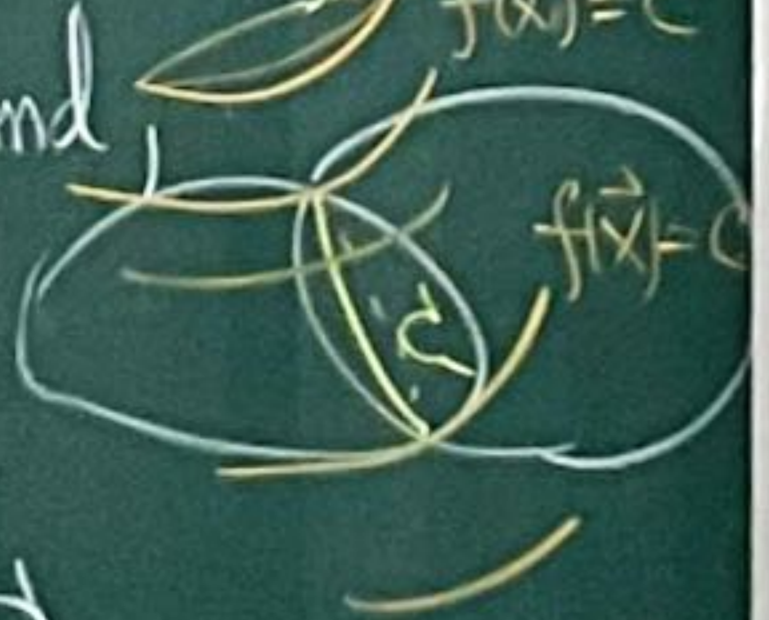
$\Rightarrow \nabla g|_C \perp \vec{r}'(t)$ $\boxed{\text{tangent vector of } C}$

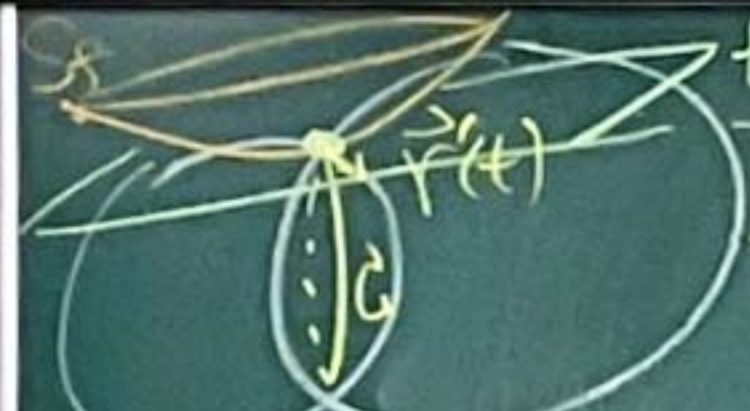
Similarly $\nabla h|_C \perp \vec{r}'(t)$

Thus $\{\nabla g(\vec{r}(t)), \nabla h(\vec{r}(t))\} \perp \vec{r}'(t)$

On the other hand

$S_f \equiv$ Level surface of f $= \{\vec{x} \mid f(\vec{x}) = c\}$ is tangent to C





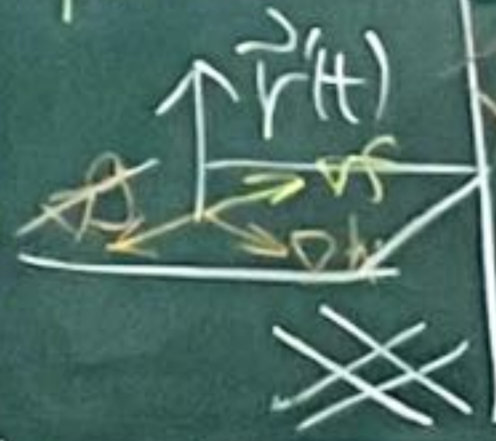
tangent plane of S_f $\in$ tangent vector $\vec{r}'(t)$

Since $\nabla f \perp S_f \Leftrightarrow \nabla f \perp$ tangent plane

$\Rightarrow \nabla f \perp \vec{r}'(t)$

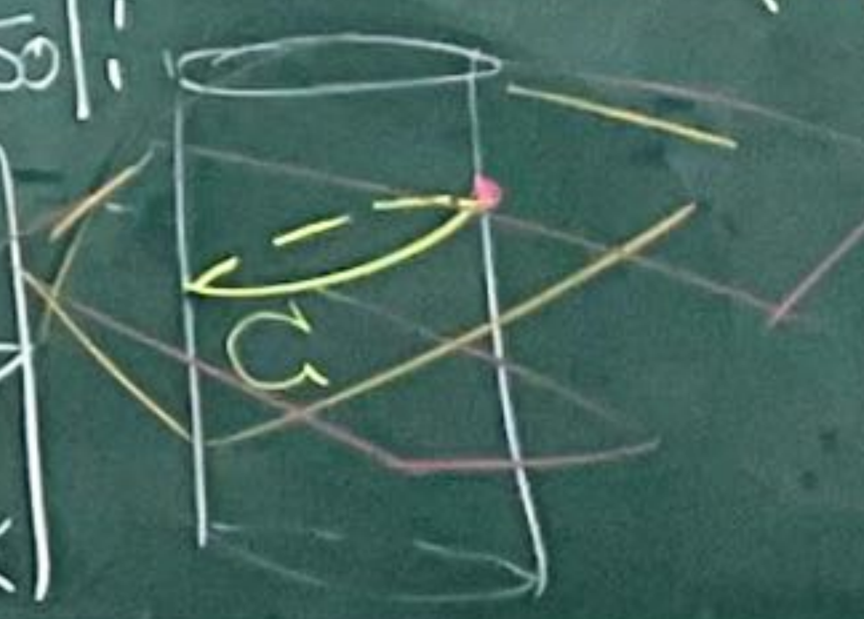
Also $\{\nabla g, \nabla h\} \perp \vec{r}'(t)$

Hence $\nabla f = \lambda \nabla g + \mu \nabla h$



(Ex) Find max of $f(\vec{x}) = x + 2y + 3z$ on $\begin{cases} x - y + z = 1 = g(\vec{x}) \\ x^2 + y^2 = 1 = h(\vec{x}) \end{cases} = C$.

Sol:



Solve $\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g = 1,\ h = 1 \end{cases}$

$\begin{cases} f_x = \lambda g_x + \mu h_x \\ f_y = \lambda g_y + \mu h_y \\ f_z = \lambda g_z + \mu h_z \\ g = 1 \\ h = 1 \end{cases} \Rightarrow$

$\begin{pmatrix} f = x + 2y + 3z \\ g = x - y + z \\ h = x^2 + y^2 \end{pmatrix}$

$\begin{cases} 1 = \lambda + \mu 2x \\ 2 = -\lambda + \mu 2y \\ 3 = \lambda + \mu \cdot 0 \\ x - y + z = 1 \\ x^2 + y^2 = 1 \end{cases} \Rightarrow \begin{cases} 1 = \lambda + 2\mu x \\ 2 = -\lambda + 2\mu y \\ 3 = \lambda \\ x - y + z = 1 \\ x^2 + y^2 = 1 \end{cases} \Rightarrow \begin{cases} \cancel{1} = \cancel{3} + 2\mu x \\ 5\,\cancel{2} = \cancel{-3} + 2\mu y \\ x - y + z = 1 \\ x^2 + y^2 = 1 \end{cases}$

$\Rightarrow \begin{cases} \mu x = -1 \\ \mu y = \frac{5}{2} \\ x - y + z = 1 \\ x^2 + y^2 = 1 \end{cases} \Rightarrow$

Ch 15 Multiple Integrals.

§15.1. Double integral over rectangles.

重積分

Review

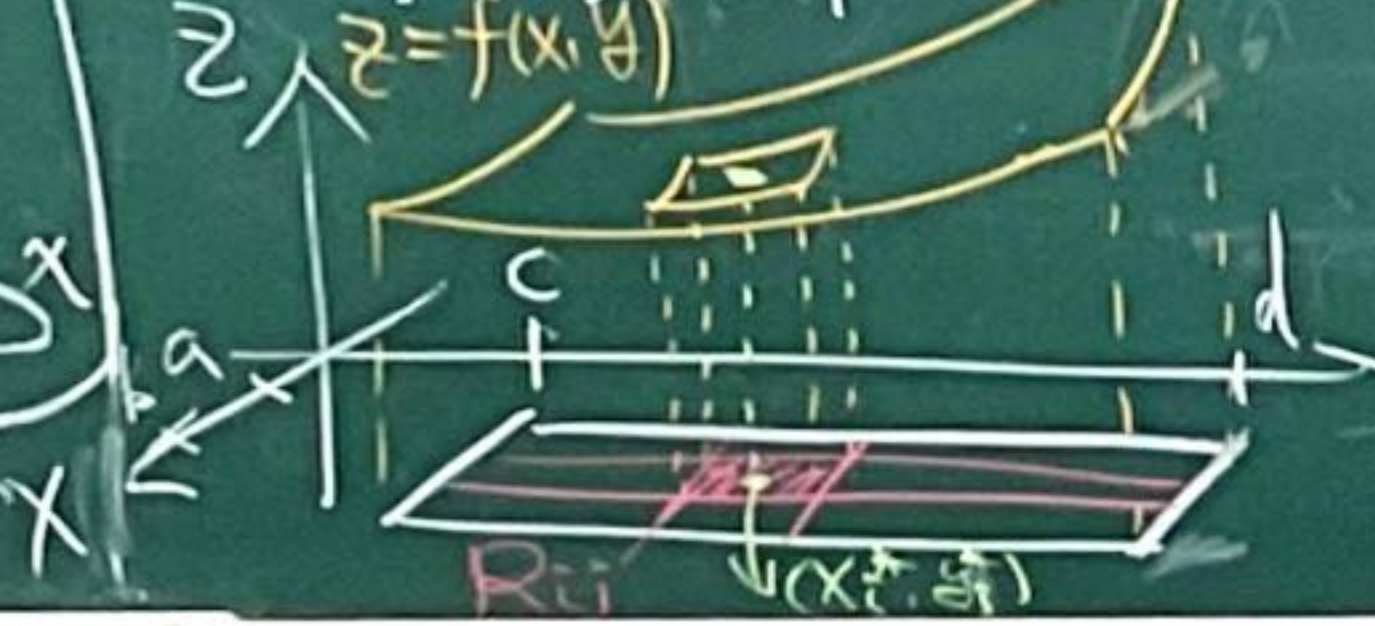
$\int_a^b f(x)\,dx \sim \boxed{\sum_{i=1}^{n} f(x_i^*)\Delta x_i}$

$\xleftarrow[n\to\infty]{}$ Riemann Sum

Consider $z=f(x,y)$ defined on R

rectangle $R=[a,b]\times[c,d]$.

$=\{(x,y)\in\mathbb{R}^2 \mid a\le x\le b,\ c\le y\le d\}$



$S=\{(x,y,z)\in\mathbb{R}^3 \mid 0\le z\le f(x,y),\ (x,y)\in R\}$

Let $f(x,y)\ge 0$

$[a,b]\to\{a=x_0, x_1, \cdots, x_n=b\}$

$[c,d]\to\{c=y_0, y_1, \cdots, y_m=d\}$

$R=\bigcup_{\substack{i=1,\cdots,n\\ j=1,\cdots,m}} R_{ij}=\bigcup_{i,j}\left\{(x,y)\in R \,\middle|\, \begin{matrix} x_{i-1}\le x\le x_i\\ y_{j-1}\le y\le y_j\end{matrix}\right\}$

Total Volume of $S\equiv V\approx \sum_{j=1}^{m}\sum_{i=1}^{n} \underbrace{f(x_i^*, y_j^*)}_{\text{height}}\underbrace{\Delta x_i\Delta y_j}_{\text{base}}$

Riemann Sum

$\iint_R f(x,y)\,\underbrace{dx\,dy}_{=dA} \xleftarrow[m,n\to\infty]{\lim}$

if the limit exists

$\Leftrightarrow$ For every $\varepsilon>0$, $\exists N$ s.t.

$\left|\iint_R f(x,y)\,dx\,dy - \sum_{j=1}^{m}\sum_{i=1}^{n} f(x_i^*, y_j^*)\Delta A\right|<\varepsilon$

for all $m,n\ge N$ and for any $(x_i^*, y_j^*)\in R_{ij}$

Def f is integrable if the limit of Riemann sum exists

Prop If f is $\boxed{\text{bdd and conti}}$ on R

$\Rightarrow$ f is integrable on R

Prop If $f(x,y)\ge 0 \Rightarrow$ Volume of $S\equiv V=\iint_R f(x,y)\,dA$

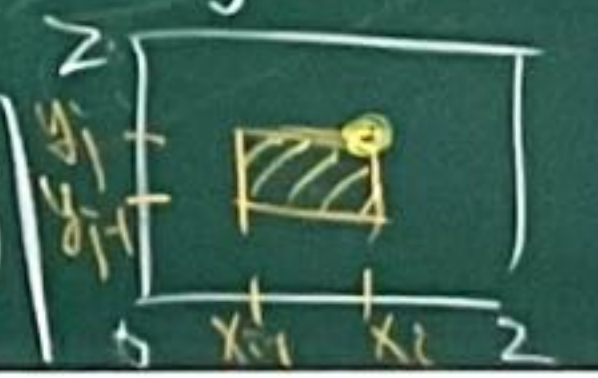
<Ex> Estimate the volume of the solid

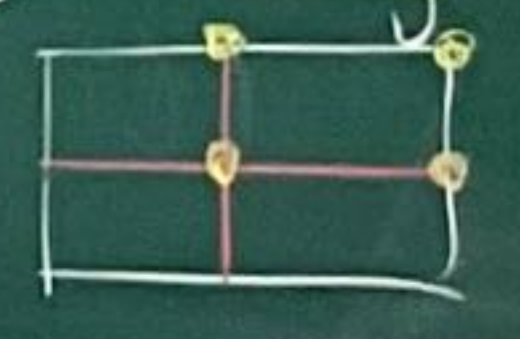
$R=[0,2]\times[0,2]$

$f(x,y)=16-x^2-2y^2$ (*)

Sample pt is the upper right corner

of $R_{ij}\equiv(x_i^*, y_j^*)=(x_i, y_j)$





For $m=2,\ n=2 \Rightarrow \Delta A=\Delta x_i\Delta y_j=1$

$V\approx\sum_{j=1}^{2}\sum_{i=1}^{2} f(\underbrace{x_i^*, y_j^*}_{=(x_i,\ y_j)})\Delta A$

$=f(1,1)\cdot 1+f(1,2)\cdot 1+f(2,1)\cdot 1+f(2,2)\cdot 1 \overset{(*)}{=} \boxed{34}$

$m=n=4 \Rightarrow \Delta A=\frac{1}{4},\ V\approx 41.5$

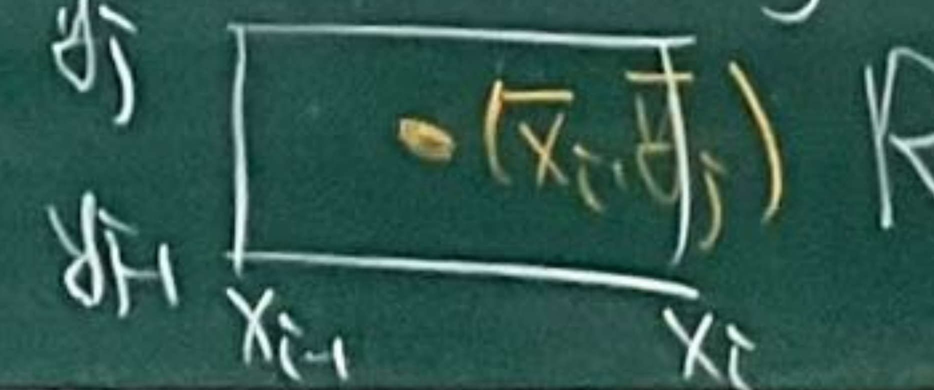
$m=n=8 \Rightarrow \Delta A=\frac{1}{16},\ V\approx 44.875$

$m=n=16 \Rightarrow \Delta A=\frac{1}{64},\ V\approx 46.468$

Midpt Rule for Double Integrals

$\iint_R f(x,y)\,dA\approx\sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\Delta A$

$\bar{x}_i=\frac{1}{2}(x_{i-1}+x_i),\ \bar{y}_j=\frac{1}{2}(y_{j-1}+y_j)$



Recall average of f on [a,b], $f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$

Def average of f on R, $f_{ave} = \frac{1}{\underbrace{|R|}_{\text{area of } R}}\iint_R f(x,y)\,dA$

If $f(x,y) \geq 0 \Rightarrow |R| \times f_{ave} = \iint_R f(x,y)\,dA$