HWs

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Contents

Chapter 1

HW for General Analysis

1.1 HW1

Theorem 1.1.1. $\mathbb{R}^n$ is complete.

Proof. Let $\mathbf{x}_k$ be an arbitrary Cauchy sequence in $\mathbb{R}^n$. We are required to show $\mathbf{x}_k$ converge in $\mathbb{R}^n$. For each k, denote $\mathbf{x}_k$ by $(x_{(1,k)}, \dots, x_{(n,k)})$. We claim that for each $i \in \{1, \dots, n\}$

$$x_{(i,k)} \text{ is a Cauchy sequence}$$

Fix i and $\epsilon > 0$. To show $x_{(i,k)}$ is a Cauchy sequence, we are required to find $N \in \mathbb{N}$ such that for all $r, m \geq N$ we have

$$\left| x_{(i,r)} - x_{(i,m)} \right| \leq \epsilon$$

Because $\mathbf{x}_k$ is a Cauchy sequence in $\mathbb{R}^n$, we know there exists $N \in \mathbb{N}$ such that for all $r, m \geq N$, we have

$$|\mathbf{x}_r - \mathbf{x}_m| < \epsilon$$

Fix such N and arbitrary $r, m \geq M$. Observe

$$\left| x_{(i,r)} - x_{(i,m)} \right| \leq \sqrt{\sum_{j=1}^{n} \left| x_{(j,r)} - x_{(j,m)} \right|^2} = |\mathbf{x}_r - \mathbf{x}_m| < \epsilon$$

We have proved that for each $i \in \{1, \dots, n\}$, the real sequence $x_{(i,k)}$ is Cauchy. We now claim that for each $i \in \{1, \dots, n\}$, we have

$$\limsup_{r \to \infty} x_{(i,r)} \in \mathbb{R} \text{ and } \lim_{k \to \infty} x_{(i,k)} = \limsup_{r \to \infty} x_{(i,r)}$$

Again fix i. Because $x_{(i,k)}$ is a Cauchy sequence, we know there exists some N such that for all $r,m \geq N$, we have

$$\left|x_{(i,r)} - x_{(i,m)}\right| < 1$$

This implies that for all $r \geq N$, we have

$$x_{(i,r)} < x_{(i,N)} + 1 \tag{1.1}$$

Equation 1.1 then tell us

$$x_{(i,N)} + 1 \text{ is an upper bound of } \{x_{(i,r)} : r \geq N\}$$

Then by definition of sup, we have

$$\sup\{x_{(i,r)} : r \geq N\} \leq x_{(i,N)} + 1 \in \mathbb{R}$$

This then implies $\limsup_{r\to\infty} x_{(i,r)} \in \mathbb{R}$. We now prove

$$\lim_{k\to\infty} x_{(i,k)} = \limsup_{r\to\infty} x_{(i,r)} \tag{1.2}$$

Fix $\epsilon > 0$. We are required to find N such that

$$\forall k \geq N, \left|x_{(i,k)} - \limsup_{r\to\infty} x_{(i,r)}\right| \leq \epsilon$$

Because $\{x_{(i,k)}\}_{k\in\mathbb{N}}$ is a Cauchy sequence, we can let N_0 satisfy

$$\forall k,m \geq N_0, \left|x_{(i,k)} - x_{(i,m)}\right| < \frac{\epsilon}{2}$$

Because $\sup\{x_{(i,k)} : k \geq N'\} \searrow \limsup_{r\to\infty} x_{(i,r)}$ as $N' \to \infty$, we know there exists $N_1 > N_0$ such that

$$\limsup_{r\to\infty} x_{(i,r)} - \frac{\epsilon}{2} < \sup\{x_{(i,k)} : k \geq N_0\} \leq \limsup_{r\to\infty} x_{(i,r)} + \frac{\epsilon}{2}$$

Then because $\limsup_{r\to\infty} x_{(i,r)} - \frac{\epsilon}{2}$ is strictly smaller than the smallest upper bound of $\{x_{(i,k)} : k \geq N_1\}$, we see $\limsup_{n\to\infty} x_{(i,r)} - \frac{\epsilon}{2}$ is not an upper bound of $\{x_{(i,k)} : k \geq N_1\}$. This implies the existence of some N such that $N \geq N_1$ and

$$\limsup_{r\to\infty} x_{(i,r)} - \frac{\epsilon}{2} < x_{(i,N)} \leq \limsup_{r\to\infty} x_{(i,r)} + \frac{\epsilon}{2}$$

Now observe that for all $k \geq N$, because $N \geq N_1 \geq N_0$

$$\limsup_{r\to\infty} x_{(i,r)} - \epsilon < x_{(i,N)} - \frac{\epsilon}{2} < x_{(i,k)} < x_{(i,N)} + \frac{\epsilon}{2} \leq \limsup_{r\to\infty} x_{(i,r)} + \epsilon$$

This implies for all $k \geq N$, we have

$$\left| x_{(i,k)} - \limsup_{r\to\infty} x_{(i,r)} \right| \leq \epsilon$$

We have just proved Equation 1.2. Lastly, to close out the proof, we show

$$\lim_{k\to\infty} \mathbf{x}_k = \left(\lim_{k\to\infty} x_{(1,k)}, \dots, \lim_{k\to\infty} x_{(n,k)} \right) \tag{1.3}$$

Fix $\epsilon > 0$. For each $i \in \{1, \dots, n\}$, let N_i satisfy

$$\forall r \geq N_i, \left| x_{(i,r)} - \lim_{k\to\infty} x_{(i,k)} \right| \leq \frac{\epsilon}{\sqrt{n}}$$

Observe that for all $r \geq \max_{i\in\{1,\dots,n\}} N_i$, we have

$$\begin{aligned} \left| \mathbf{x}_r - \left(\lim_{k\to\infty} x_{(1,k)}, \dots, \lim_{k\to\infty} x_{(n,k)} \right) \right| &= \sqrt{\sum_{i=1}^{n} \left| x_{(i,r)} - \lim_{k\to\infty} x_{(i,k)} \right|^2} \\ &\leq \sqrt{\sum_{i=1}^{n} \frac{\epsilon^2}{n}} = \epsilon \end{aligned}$$

We have proved Equation 1.2.

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Theorem 1.1.2. ($\mathbb{Q}$ is dense in $\mathbb{R}$)