If f is continuous and φ monotone increasing on $[a,b]$, then $\exists\ \xi \in [a,b]$ s.t.

$$\int_a^b f\,d\varphi = f(\xi)(\varphi(b)-\varphi(a))$$

proof: By Thm 6.8 of Rudin f conti on $[a,b] \Rightarrow f \in \mathcal{R}(\varphi)$

f conti on compact set, then by EVT, $\exists\ x_m, x_M \in [a,b]$ s.t. $M = \sup_{[a,b]} f = f(x_M)$

$$m = \inf_{[a,b]} f = f(x_m)$$

if $M = m$, then f = constant $\Rightarrow$ trivial

$$\int_a^b f\,d\varphi = \alpha(\varphi(b)-\varphi(a)) \quad \text{if } f = \alpha \text{ on } [a,b]$$

we may assume $M > m$ (then either $x_m < x_M$ or $x_M < x_m$)

$\forall\ P$ (partition) of $[a,b]$ we have

$$m(\varphi(b)-\varphi(a)) \le L(f,\varphi,P) \le U(f,\varphi,P) \le M(\varphi(b)-\varphi(a))$$

$$\Rightarrow \begin{cases} m(\varphi(b)-\varphi(a)) \le \inf_P U(f,\varphi,P) = \int_a^b f\,d\varphi \\ \int_a^b f\,d\varphi = \sup_P L(f,\varphi,P) \le M(\varphi(b)-\varphi(a)) \end{cases}$$

$$\Rightarrow m(\varphi(b)-\varphi(a)) \le \int_a^b f\,d\varphi \le M(\varphi(b)-\varphi(a))$$

we may assume $\varphi(a) < \varphi(b)$, otherwise $\int_a^b f\,d\varphi = 0$, trivial

$$\Rightarrow f(x_m) = m \le \underbrace{\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f\,d\varphi}_{I} \le M = f(x_M)$$

if $I = m$ or M, then $\xi = x_m, x_M \in [a,b]$

if $I \in (m,M) = (f(x_m), f(x_M))$, then by IVT $\exists\ \xi \in (x_m, x_M)$ or $(x_M, x_m) \subset [a,b]$ s.t. $f(\xi) = I \Rightarrow \int_a^b f\,d\varphi = f(\xi)(\varphi(b)-\varphi(a))$