Chapter 1

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Question 1

Let $S^1 \triangleq \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ be the unit circle in $\mathbb{R}^2$. Let $\mathbb{R}P^1$ be the real projective line. As usual, denote by $[x, y]$ the homogeneous coordinate on $\mathbb{R}P^1$. Show that the map $F : S^1 \to \mathbb{R}P^1$ defined by

$$F(x,y) \triangleq \begin{cases} [1-y, x] & \text{if } y \neq 1 \\ [x, 1+y] & \text{if } y \neq -1 \end{cases}$$

established a diffeomorphism.

Proof. Consider the atlas $\{(U_N, \varphi_N), (U_S, \varphi_S)\}$ for S^1 where $\varphi_N : U_N \triangleq S^1 \setminus \{(0,1)\} \to \mathbb{R}$ is defined by

$$\varphi_N(x,y) \triangleq \frac{x}{1-y}$$

and $\varphi_S : U_S \triangleq S^1 \setminus \{(0,-1)\}$ is defined by

$$\varphi_S(x,y) \triangleq \frac{x}{1+y}$$

Note that $\varphi_N(U_N) = \varphi_S(U_S) = \mathbb{R}$. Tedious algebra shows that

$$\varphi_N^{-1}(t) = \frac{(2t, t^2-1)}{t^2+1} \text{ and } \varphi_S^{-1}(t) = \frac{(2t, 1-t^2)}{t^2+1} \text{ for all } t \in \mathbb{R}$$

Consider the atlas $\{(V_1, \varphi_1), (V_2, \varphi_2)\}$ for $\mathbb{R}P^1$ where $\varphi_1 : V_1 \triangleq \{[x, y] \in \mathbb{R}P^1 : x \neq 0\} \to \mathbb{R}$ is defined by

$$\varphi_1([x,y]) \triangleq \frac{y}{x}$$

and $\varphi_2 : V_2 \triangleq \{[x,y] \in \mathbb{R}P^2 : y \neq 0\} \to \mathbb{R}$ is defined by

$$\varphi_2([x,y]) \triangleq \frac{x}{y}$$

Note that $\varphi_1(V_1) = \varphi_2(V_2) = \mathbb{R}$.

Compute

$$F \circ \varphi_N^{-1}(t) = F\Big(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\Big) = [2, 2t]$$

This shows that $V_1 = F(U_N)$ and

$$\varphi_1 \circ F \circ \varphi_N^{-1}(t) = t \tag{1.1}$$

which is clearly smooth. Compute

$$F \circ \varphi_S^{-1}(t) = F\Big(\frac{2t}{t^2+1}, \frac{1-t^2}{t^2+1}\Big) = [2t, 2]$$

This shows that $V_2 = F(U_S)$ and

$$\varphi_2 \circ F \circ \varphi_S^{-1}(t) = t \tag{1.2}$$

which is again smooth. We have shown $F : S^1 \to \mathbb{R}P^1$ is smooth. Note that Equation 1.1 and Equation 1.2 shows that not only F is one-to-one (because F is one-to-one on both U_N, U_S), but F is also onto, since $\mathbb{R}P^1 = V_1 \cup V_2 = F(U_N) \cup F(U_S) = F(U_N \cup U_S)$. We have shown F is a bijection; Thus, we can talk about its inverse $F^{-1} : \mathbb{R}P^1 \to S^1$. Again from Equation 1.1 and Equation 1.2, we have

$$\begin{cases} \varphi_N \circ F^{-1} \circ \varphi_1^{-1}(t) = (\varphi_1 \circ F \circ \varphi_N^{-1})^{-1}(t) = t \\ \varphi_S \circ F^{-1} \circ \varphi_2^{-1}(t) = (\varphi_2 \circ F \circ \varphi_S^{-1})^{-1}(t) = t \end{cases}$$

for all $t \in \mathbb{R}$. This shows that $F^{-1} : \mathbb{R}P^1 \to S^1$ is also smooth. We have shown F is indeed a diffeomorphism. ■

Question 2

Suppose M, N are both smooth manifold with M connected, and $F : M \to N$ is a smooth map such that $F_{*,p}$ is null for all $p \in M$. Show that F is a constant map.

Proof. Fix $p_0 \in M$. Let $q_0 \triangleq F(p_0)$. Because N is Hausdorff, for each $q \in N \setminus \{q_0\}$, we may select some neighborhood V_q around q such that $q_0 \notin V_q$. This implies that $\{q_0\}$ is closed in N. Then because F is continuous (smooth), we know that $F^{-1}(q_0)$ is closed in M.

Let $p \in F^{-1}(q_0)$, (V, ψ) be a chart for N centering q_0, and (U, φ) be a chart for M centering p such that $F(U) \subseteq V$ and $\varphi(U) = B_r(\mathbf{0})$ where $B_r(\mathbf{0}) \subseteq \mathbb{R}^m$ is an open ball. Because F_* is null on U, we see that

$$d(\psi \circ F \circ \varphi^{-1})_{\mathbf{x}} = 0 \text{ for all } \mathbf{x} \in \varphi(U) \tag{1.3}$$

For all $\mathbf{x} \in \varphi(U)$, we may define $\gamma : [0,1] \to \varphi(U)$ by $\gamma(t) \triangleq t\mathbf{x}$ joining $\mathbf{0}, \mathbf{x}$. Using ordinary chain rule and Equation 1.3, we see that

$$(\psi \circ F \circ \varphi^{-1} \circ \gamma)'(t) = 0 \text{ for all } t \in (0,1)$$

This implies

$$(\psi^i \circ F \circ \varphi^{-1} \circ \gamma)'(t) = 0 \text{ for all } t \in (0,1) \text{ and } i \in \{1, \dots, n\}$$

We then can use MVT to deduce

$$\begin{aligned}
\psi^i \circ F(p) &= \psi^i \circ F \circ \varphi^{-1} \circ \gamma(0) \\
&= \psi^i \circ F \circ \varphi^{-1} \circ \gamma(1) \\
&= \psi^i \circ F \circ \varphi^{-1}(\mathbf{x}) \text{ for all } i \in \{1, \dots, n\}
\end{aligned}$$

We have shown for all $\mathbf{x} \in \varphi(U)$,

$$\psi \circ F(\mathbf{x}) = \psi \circ F(p)$$

In other words, F sends all points in U to $F(p) = q_0$. This implies $U \subseteq F^{-1}(q_0)$. It follows from U being a neighborhood of p and p is arbitrarily picked from $F^{-1}(q_0)$ that $F^{-1}(q_0)$ is open.

We have shown $F^{-1}(q_0)$ is clopen. It follows form M is connected that $F^{-1}(q_0)$ is either empty or M. Note that $F^{-1}(q_0)$ is non-empty because $F(p_0) = q_0$. It follows that $F^{-1}(q_0) = M$, i.e., F is a constant map. ■

Question 3

Consider the trace function $f : SL(2, \mathbb{R}) \to \mathbb{R}, A \mapsto \operatorname{tr}(A)$. What are the regular level sets of f?

Proof. Note that when we refer to $SL(2, \mathbb{R})$, we are using the unique topology and smooth structure that make $SL(2, \mathbb{R})$ an embedded submanifold of $M_2(\mathbb{R})$. Consider

$$U_a \triangleq \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{R}) : a \neq 0 \right\} \text{ and } U_b \triangleq \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{R}) : b \neq 0 \right\}$$

It is clear that $U_a \cup U_b = SL(2,\mathbb{R})$. Consider the chart $\varphi_a : U_a \to \mathbb{R}^3, \varphi_n : U_b \to \mathbb{R}^3$

$$\varphi_a\left(\begin{bmatrix} a & b \\ c & \frac{1+bc}{a} \end{bmatrix}\right) \triangleq \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ and } \varphi_b\left(\begin{bmatrix} a & b \\ \frac{ad-1}{b} & d \end{bmatrix}\right) \triangleq \begin{bmatrix} a \\ b \\ d \end{bmatrix}$$

Compute

$$\frac{\partial f \circ \varphi_a^{-1}}{\partial a} = 1 - \frac{1+bc}{a^2}, \frac{\partial f \circ \varphi_a^{-1}}{\partial b} = \frac{c}{a}, \frac{\partial f \circ \varphi_a^{-1}}{\partial c} = \frac{b}{a}$$

This implies there are only two critical points in U_a

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Compute

$$\frac{\partial f \circ \varphi_b^{-1}}{\partial a} = 1$$

This implies that U_b contain no critical points. The regular level sets then are $f^{-1}(r)$ where $r \neq \pm 2$. ■

Question 4

Consider the map $F : \mathbb{R}P^2 \to \mathbb{R}^5$ given by

$$F\Big([x,y,z]\Big) \triangleq \left(\frac{yz}{\sqrt{3}}, \frac{zx}{\sqrt{3}}, \frac{xy}{\sqrt{3}}, \frac{x^2-y^2}{2\sqrt{3}}, \frac{x^2+y^2-2z^2}{6}\right) \text{ for } (x,y,z) \in S^2$$

Shows that F is an immersion. Is F an embedding?

Proof. Consider the canonical atlas (U_i, Φ_i) for $\mathbb{R}P^2$

$$U_i \triangleq \{[\mathbf{x}] \in \mathbb{R}P^2 : \mathbf{x}^i \neq 0\}$$
$$\Phi_1([\mathbf{x}]) \triangleq \left(\frac{\mathbf{x}^2}{\mathbf{x}^1}, \frac{\mathbf{x}^3}{\mathbf{x}^1}\right), \Phi_2([\mathbf{x}]) \triangleq \left(\frac{\mathbf{x}^1}{\mathbf{x}^2}, \frac{\mathbf{x}^3}{\mathbf{x}^2}\right), \Phi_3([\mathbf{x}]) \triangleq \left(\frac{\mathbf{x}^1}{\mathbf{x}^3}, \frac{\mathbf{x}^2}{\mathbf{x}^3}\right)$$

Tedious algebra shows that $\Phi_i(U_i) = \mathbb{R}^2$ and

$$\begin{cases} \Phi_1^{-1}(a,b) = [1,a,b] \\ \Phi_2^{-1}(a,b) = [a,1,b] \\ \Phi_3^{-1}(a,b) = [a,b,1] \end{cases}$$

In the first chart

$$F \circ \Phi_1^{-1}(a,b) = \frac{(\frac{ab}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{1-a^2}{2\sqrt{3}}, \frac{1+a^2-2b^2}{6})}{a^2+b^2+1}$$

$$\begin{aligned}
\frac{\partial F}{\partial a} &= \frac{(a^2+b^2+1)(\frac{b}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}, \frac{-2a}{2\sqrt{3}}, \frac{2a}{6}) - 2a(\frac{ab}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{1-a^2}{2\sqrt{3}}, \frac{1+a^2-2b^2}{6})}{(a^2+b^2+1)^2} \\
&= \frac{(\frac{b^3-a^2b+b}{\sqrt{3}}, \frac{-2ab}{\sqrt{3}}, \frac{b^2-a^2+1}{\sqrt{3}}, \frac{-2ab^2-4a}{2\sqrt{3}}, \frac{6ab^2}{6})}{(a^2+b^2+1)^2} \\
\frac{\partial F}{\partial b} &= \frac{(a^2+b^2+1)(\frac{a}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0, 0, \frac{-4b}{6}) - 2b(\frac{ab}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{1-a^2}{2\sqrt{3}}, \frac{1+a^2-2b^2}{6})}{(a^2+b^2+1)^2} \\
&= \frac{(\frac{a^3-ab^2+a}{\sqrt{3}}, \frac{a^2-b^2+1}{\sqrt{3}}, \frac{-2ab}{\sqrt{3}}, \frac{2a^2b-2b}{2\sqrt{3}}, \frac{-6b-6a^2b}{6})}{(a^2+b^2+1)^2}
\end{aligned}$$

Compute

$$\det\left(\begin{bmatrix} \frac{\partial F^2}{\partial a} & \frac{\partial F^2}{\partial b} \\ \frac{\partial F^3}{\partial a} & \frac{\partial F^3}{\partial b} \end{bmatrix}\right) = \frac{4a^2b^2 - (1+(a^2-b^2))(1-(a^2-b^2))}{3(a^2+b^2+1)^4} = \frac{(a^2+b^2)^2-1}{3(a^2+b^2+1)^4} \tag{1.4}$$

$$\det\left(\begin{bmatrix} \frac{\partial F^3}{\partial a} & \frac{\partial F^3}{\partial b} \\ \frac{\partial F^4}{\partial a} & \frac{\partial F^4}{\partial b} \end{bmatrix}\right) = \frac{(b^2-a^2+1)2b(a^2-1) - 2ab(2ab^2+4a)}{6(a^2+b^2+1)^4} = \frac{2b(a^2+1)(-a^2-b^2-1)}{6(a^2+b^2+1)^4} \tag{1.5}$$

Equation 1.4 shows that F is immersion on $\mathbb{R}^2 \setminus S^1$ and Equation 1.5 shows that F is immersion on $\mathbb{R}^2 \setminus \{b=0\}$. Trivial computation shows that F is also an immersion on $(1,0)$ and $(-1,0)$. It follows that F is an immersion on U_1.

In the second chart,

$$F \circ \Phi_2^{-1}(a,b) = \frac{(\frac{b}{\sqrt{3}}, \frac{ab}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a^2-1}{2\sqrt{3}}, \frac{a^2+1-2b^2}{6})}{a^2+b^2+1}$$

$$
\begin{aligned}
\frac{\partial F}{\partial a} &= \frac{(a^2+b^2+1)(0, \frac{b}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{2a}{2\sqrt{3}}, \frac{2a}{6}) - 2a(\frac{b}{\sqrt{3}}, \frac{ab}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a^2-1}{2\sqrt{3}}, \frac{a^2+1-2b^2}{6})}{(a^2+b^2+1)^2} \\
&= \frac{(\frac{-2ab}{\sqrt{3}}, \frac{b^3-a^2b+b}{\sqrt{3}}, \frac{b^2-a^2+1}{\sqrt{3}}, \frac{2ab^2+4a}{2\sqrt{3}}, \frac{6ab^2}{6})}{(a^2+b^2+1)^2} \\
\frac{\partial F}{\partial b} &= \frac{(a^2+b^2+1)(\frac{1}{\sqrt{3}}, \frac{a}{\sqrt{3}}, 0, 0, \frac{-4b}{6}) - 2b(\frac{b}{\sqrt{3}}, \frac{ab}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a^2-1}{2\sqrt{3}}, \frac{a^2+1-2b^2}{6})}{(a^2+b^2+1)^2} \\
&= \frac{(\frac{a^2-b^2+1}{\sqrt{3}}, \frac{a^3-ab^2+a}{\sqrt{3}}, \frac{-2ab}{\sqrt{3}}, \frac{2b-2a^2b}{2\sqrt{3}}, \frac{-6b-6a^2b}{6})}{(a^2+b^2+1)^2}
\end{aligned}
$$

Compute

$$
\det\left(\begin{bmatrix} \frac{\partial F^1}{\partial a} & \frac{\partial F^1}{\partial b} \\ \frac{\partial F^3}{\partial a} & \frac{\partial F^3}{\partial b} \end{bmatrix}\right) = \frac{4a^2b^2 - (1+(a^2-b^2))(1-(a^2-b^2))}{3(a^2+b^2+1)^4} = \frac{(a^2+b^2)^2-1}{3(a^2+b^2+1)^4} \tag{1.6}
$$

$$
\det\left(\begin{bmatrix} \frac{\partial F^3}{\partial a} & \frac{\partial F^3}{\partial b} \\ \frac{\partial F^4}{\partial a} & \frac{\partial F^4}{\partial b} \end{bmatrix}\right) = \frac{(b^2-a^2+1)2b(1-a^2) + 2ab(2ab^2+4a)}{6(a^2+b^2+1)^4} = \frac{2b(a^2+1)(a^2+b^2+1)}{6(a^2+b^2+1)^4} \tag{1.7}
$$

Equation 1.6 shows that F is immersion on $\mathbb{R}^2 \setminus S^1$ and Equation 1.7 shows that F is immersion on $\mathbb{R}^2 \setminus \{b = 0\}$. Trivial computation shows that F is also an immersion on $(1, 0)$ and $(-1, 0)$. It follows that F is an immersion on U_2.

In the third chart,

$$
F \circ \Phi_3^{-1}(a, b) = \frac{(\frac{b}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{ab}{\sqrt{3}}, \frac{a^2-b^2}{2\sqrt{3}}, \frac{a^2+b^2-2}{6})}{a^2+b^2+1}
$$

$$
\begin{aligned}
\frac{\partial F}{\partial a} &= \frac{(a^2+b^2+1)(0, \frac{1}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{2a}{2\sqrt{3}}, \frac{2a}{6}) - 2a(\frac{b}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{ab}{\sqrt{3}}, \frac{a^2-b^2}{2\sqrt{3}}, \frac{a^2+b^2-2}{6})}{(a^2+b^2+1)^2} \\
&= \frac{(\frac{-2ab}{\sqrt{3}}, \frac{-a^2+b^2+1}{\sqrt{3}}, \frac{b^3-a^2b+b}{\sqrt{3}}, \frac{4ab^2+6a}{2\sqrt{3}}, \frac{6a}{6})}{(a^2+b^2+1)^2} \\
\frac{\partial F}{\partial b} &= \frac{(a^2+b^2+1)(\frac{1}{\sqrt{3}}, 0, \frac{a}{\sqrt{3}}, \frac{-2b}{2\sqrt{3}}, \frac{2b}{6}) - 2b(\frac{b}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{ab}{\sqrt{3}}, \frac{a^2-b^2}{2\sqrt{3}}, \frac{a^2+b^2-2}{6})}{(a^2+b^2+1)^2} \\
&= \frac{(\frac{a^2-b^2+1}{\sqrt{3}}, \frac{-2ab}{\sqrt{3}}, \frac{a^3-ab^2+a}{\sqrt{3}}, \frac{-4a^2b-2b}{2\sqrt{3}}, \frac{6b}{6})}{(a^2+b^2+1)^2}
\end{aligned}
$$

Compute

$$\det\left(\begin{bmatrix}\frac{\partial F^1}{\partial a} & \frac{\partial F^1}{\partial b} \\ \frac{\partial F^5}{\partial a} & \frac{\partial F^5}{\partial b}\end{bmatrix}\right) = \frac{-2ab^2 - a(a^2-b^2+1)}{\sqrt{3}(a^2+b^2+1)^4} = \frac{-a(a^2+b^2+1)}{\sqrt{3}(a^2+b^2+1)^4} \tag{1.8}$$

$$\det\left(\begin{bmatrix}\frac{\partial F^2}{\partial a} & \frac{\partial F^2}{\partial b} \\ \frac{\partial F^5}{\partial a} & \frac{\partial F^5}{\partial b}\end{bmatrix}\right) = \frac{b(-a^2+b^2+1)+2a^2b}{\sqrt{3}(a^2+b^2+1)^2} = \frac{b(a^2+b^2+1)}{\sqrt{3}(a^2+b^2+1)^4} \tag{1.9}$$

Equation 1.8 and Equation 1.9 shows that F is immersion on $\mathbb{R}^2$ except possibly at 0. Trivial computation shows that F is also an immersion on 0. It follows that F is an immersion on U_3. In summary, we have shown F is an immersion by showing F_* is injective on all of its charts U_1, U_2, U_3 that covers $\mathbb{R}P^2$.

By Theorem 1.1.1, we see that $\mathbb{R}P^2 \simeq \mathbb{P}^2$. This implies $\mathbb{R}P^2$ is compact because $\mathbb{P}^2$ is a quotient space of the compact S^3. It is clear from our above computation that $F : \mathbb{R}P^2 \to \mathbb{R}^5$ is continuous. We presented a proof in Theorem 1.1.2 that $F : \mathbb{R}P^2 \to \mathbb{R}^5$ is one-to-one. It follows from Theorem 1.1.3 that F is a topological embedding. Then because F is a smooth immersion, we see that F is also a smooth embedding. ■

Question 5

Consider the following vector field on $\mathbb{R}^3$

$$X = \frac{\partial}{\partial x} \text{ and } Y = x\frac{\partial}{\partial z} + \frac{\partial}{\partial y}$$

(a) Find $[X, Y]$.

(b) Suppose $f \in C^\infty(\mathbb{R}^3)$ satisfy $Xf = Yf = 0$ at all points. Prove that f is a constant function.

Proof. Compute

$$\begin{aligned}
[X, Y]f &= XYf - YXf \\
&= X\left(x\frac{\partial f}{\partial z} + \frac{\partial f}{\partial y}\right) - Y\left(\frac{\partial f}{\partial x}\right) \\
&= \frac{\partial f}{\partial z} + x\frac{\partial f}{\partial x\partial z} + \frac{\partial f}{\partial x\partial y} - x\frac{\partial f}{\partial x\partial z} - \frac{\partial f}{\partial x\partial y} \\
&= \frac{\partial f}{\partial z}
\end{aligned}$$

If $Xf = Yf = 0$, then

$$\frac{\partial f}{\partial z} = XYf - YXf = X0 - Y0 = 0 \text{ and } \frac{\partial f}{\partial x} = Xf = 0$$

Therefore,

$$\frac{\partial f}{\partial y} = Yf - x\frac{\partial f}{\partial z} = 0$$

We have shown $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$ at all points. Then for every $p, q \in \mathbb{R}^3$, if we let $\gamma : [0,1] \to \mathbb{R}^3$ be the line linearly joining p, q, we see that

$$\begin{aligned} f(p) - f(q) &= f \circ \gamma(1) - f \circ \gamma(0) \\ &= \int_0^1 (f \circ \gamma)'(t) dt \\ &= \int_0^1 df_{\gamma(t)}(\gamma'(t)) dt = 0 \end{aligned}$$

It follows from p, q being arbitrary that f is constant. ■

1.1 Appendix

Alternatively, we can characterize $\mathbb{R}P^n$ by identifying the antipodal pints on $S^n \triangleq \{\mathbf{x} \in \mathbb{R}^{n+1} : |\mathbf{x}| = 1\}$ as one point

$$\mathbf{x} \sim \mathbf{y} \overset{\triangle}{\iff} \mathbf{x} = \mathbf{y} \text{ or } \mathbf{x} = -\mathbf{y}$$

and let $\mathbb{P}^n \triangleq S^n \backslash \sim$ be the quotient space.

Theorem 1.1.1. (Equivalent Definitions of Real Projective Space)

$$\mathbb{R}P^n \text{ and } \mathbb{P}^n \text{ are homeomorphic}$$

Proof. Define $F : \mathbb{P}^n \to \mathbb{R}P^n$ by

$$\{\mathbf{x}, -\mathbf{x}\} \mapsto \{\lambda \mathbf{x} : \lambda \in \mathbb{R}^*\}$$

It is straightforward to check that F is well-defined and bijective. Define $f : S^n \to \mathbb{R}P^n$ by

$$f = \pi \circ \mathbf{id}$$

where $\mathbf{id} : S^n \to \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ and $\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{R}P^n$ are continuous. Check that

$$f = F \circ p$$

where $p : S^n \to \mathbb{P}^n$ is the quotient mapping. It now follows from the universal property that F is continuous, and since $\mathbb{P}^n$ is compact and $\mathbb{R}P^n$ is Hausdorff, it also follows that F is a homeomorphism between $\mathbb{R}P^n$ and $\mathbb{P}^n$. ■

Theorem 1.1.2. (One-to-one of the specified function) The map $F : \mathbb{R}P^2 \to \mathbb{R}^5$ in question 4 is one-to-one.

Proof. Let $(x_1, y_1, z_1), (x_2, y_2, z_2) \in S^2$. Suppose

$$F(x_1, y_1, z_1) = F(x_2, y_2, z_2)$$

Observe that

$$\begin{aligned} 6F^5(x_1, y_1, z_1) &= x_1^2 + y_1^2 - 2z_1^2 \\ &= x_1^2 + y_1^2 + z_1^2 - 3z_1^2 = 1 - 3z_1^2 \end{aligned}$$

Similarly we have

$$6F^5(x_2, y_2, z_2) = 1 - 3z_2^2$$

This give us

$$z_1^2 = \frac{1 - 6F^5(x_1, y_1, z_1)}{3} = \frac{1 - 6F^5(x_2, y_2, z_2)}{3} = z_2^2$$

Therefore,

$$|z_1| = |z_2|$$

If $z_1 = z_2 \neq 0$, we may deduce

$$\begin{aligned} x_1 &= \frac{\sqrt{3}F^2(x_1, y_1, z_1)}{z_1} = \frac{\sqrt{3}F^2(x_2, y_2, z_2)}{z_2} = x_2 \\ y_1 &= \frac{\sqrt{3}F^1(x_1, y_1, z_1)}{z_1} = \frac{\sqrt{3}F^1(x_2, y_2, z_2)}{z_2} = y_2 \end{aligned}$$

which implies $[x_1, y_1, z_1] = [x_2, y_2, z_2]$. If $z_1 = -z_2 \neq 0$, we may deduce

$$\begin{aligned} x_1 &= \frac{\sqrt{3}F^2(x_1, y_1, z_1)}{z_1} = \frac{\sqrt{3}F^2(x_2, y_2, z_2)}{-z_2} = -x_2 \\ y_1 &= \frac{\sqrt{3}F^1(x_1, y_1, z_1)}{z_1} = \frac{\sqrt{3}F^2(x_2, y_2, z_2)}{-z_2} = -y_2 \end{aligned}$$

which implies $[x_1, y_1, z_1] = [x_2, y_2, z_2]$. If $|z_1| = 0$, we may deduce

$$x_1^2 - y_1^2 = 2\sqrt{3}F^4(x_1, y_1, z_1) = 2\sqrt{3}F^4(x_2, y_2, z_2) = x_2^2 - y_2^2$$

This with the fact $x_1^2 + y_1^2 = 1 = x_2^2 + y_2^2 \quad (\because (x_1, y_1, z_1), (x_2, y_2, z_2) \in S^2$ and $z_1 = z_2 = 0)$ let us deduce

$$x_1^2 = x_2^2 \text{ and } y_1^2 = y_2^2$$

In other words, $|x_1| = |x_2|$ and $|y_1| = |y_2|$. Lastly, observe

$$x_1y_1 = \sqrt{3}F^3(x_1, y_1, z_1) = \sqrt{3}F^3(x_2, y_2, z_2) = x_2y_2$$

This shows that $(x_1, y_1, z_1) = (x_1, y_1, 0) = \pm(x_2, y_2, 0) = \pm(x_2, y_2, z_2)$, which implies $[x_1, y_1, z_1] = [x_2, y_2, z_2]$. ■

Theorem 1.1.3. (Homeomorphism between Compact Space and Hausdorff Space) Suppose

(a) X is compact.

(b) Y is Hausdorff.

(c) $f : X \to Y$ is a continuous bijective function.

Then

$$f \text{ is a homeomorphism between } X \text{ and } Y$$

Proof. Because closed subset of compact set is compact and continuous function send compact set to compact set, we see for each closed $E \subseteq X$, $f(E) \subseteq Y$ is compact. The result then follows from $f(E) \subseteq Y$ being closed since Y is Hausdorff. ■