2.2 Exercise 1

Let R be a complex algebra with 1_A and $a \in R$. Given a complex polynomial

$$f(Z) = a_0 + a_1 Z + \cdots + a_n Z^n,$$

we define the evaluation of f at a by

$$f(a) = a_0 1_A + a_1 a + \cdots + a_n a^n.$$

Question 26

Let $R = \mathbb{C}$ and $a = 1 + i$. Given $f(Z) = Z^3$. Evaluate $f(a)$.

Proof. $f(a) = (1+i)^3 = 2i(1+i) = -2 + 2i$ ∎

Question 27

Let $R = M_{2\times 2}(\mathbb{C})$ be the algebra of 2×2 complex matrices. Take

$$a = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

and $g(Z) = 3 + 2Z$. Evaluate $g(a)$.

Proof.

$$g(a) = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 2 & 5 \end{bmatrix}$$

∎

Question 28

Let R be the algebra of complex valued periodic functions of period 2π, i.e., $a \in R$ is a continuous function $a : \mathbb{R} \to \mathbb{C}$ so that $a(x + 2\pi) = a(x)$. Let $e(x) = \cos x + i \sin x$ and

$$h(Z) = 1 + Z + Z^2 + \cdots + Z^9.$$

Find $h(e)$.

Proof. Note that

$$\begin{aligned}(\cos x + i\sin x)(\cos y + i \sin y) &= (\cos x \cos y - \sin x \sin y) + i(\sin x \cos y + \cos x \sin y) \\ &= \cos(x+y) + i\sin(x+y)\end{aligned}$$

This give us

$$h(e) = \sum_{k=0}^{9} \cos(kx) + i \sin(kx)$$

$\blacksquare$