Éléments de géométrie algébrique

A. Grothendieck and J. Dieudonné
Publications mathématiques de l'I.H.É.S

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Contents

What this is

This whole chapter is written by the translators.

This is a community translation of Alexander Grothendieck's and Jean Dieudonné's *Éléments de géométrie algébrique* (EGA). As it is a work in progress by multiple people, there will probably be a few mistakes—if you spot any then please do let us know[1].
To contribute, please visit

`https://github.com/ryankeleti/ega`.

On est désolés, Grothendieck.

In defense of a translation

From Wikipedia[2]:

> In January 2010, Grothendieck wrote the letter "Déclaration d'intention de non-publication" to Luc Illusie, claiming that all materials published in his absence have been published without his permission. He asks that none of his work be reproduced in whole or in part and that copies of this work be removed from libraries.[3] [...] This order may have been reversed later in 2010.[4]

It is a matter of often heated contention as to whether or not any translation of Grothendieck's work should take place, given his extremely explicit views on the matter. By no means do we mean to argue that somehow Grothendieck's wishes should be invalidated or ignored, nor do we wish to somehow twist his earlier words around in order to justify what we have done: we fully accept that he himself would probably have branded this project "an abomination". With this in mind, it remains to explain why we have gone ahead anyway.

First, and possibly foremost, it does not make sense (to us) for an individual to own the rights to knowledge. Arguments can be made about how the EGA is the product of years and years of intense work by Grothendieck (and Dieudonné, amongst others), and so *this* is something that he 'owns' and has full control over. Indeed, it is true that there are almost innumerably many sentences in these works that only Grothendieck himself could have engineered, but, in translation, we have never improved anything, but only (regrettably, but almost certainly) worsened. The work in these pages is that of Grothendieck; we have been not much more than typesetters and eager readers. However, there is some important point to be made about the fact that Grothendieck collaborated and worked with many other incredibly proficient mathematicians during the writing of this treatise; although it is impossible to pinpoint which parts exactly others may have contributed (and by no means do we wish to imply that any of this work is derivative or fraudulent in any way whatsoever—EGA was written *by Grothendieck*) it seems fair that, in some amount, there are bits of the EGAs that 'belong' to a broader collection of minds.

It is a very good idea here to repeat the oft-quoted aphorism: "the work here is not ours, but any mistakes are"—it is very understandable for an author to not want their name on something that they have not themselves written, or, at the very least, read. This may be, in part, a reason for Grothendieck's wishes, but that is pure speculation. Even so, we include this above disclaimer.

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1 `https://github.com/ryankeleti/ega/issues`

2 `https://en.wikipedia.org/wiki/Alexander_Grothendieck#Retirement_into_reclusion_and_death`

3 Grothendieck's letter. *Secret Blogging Seminar.* 9 February 2010. Retrieved 3 September 2019.

4 Réédition des SGA. Archived from the original on 29 June 2016. Retrieved 12 November 2013.

Secondly, then, we note that the French version of EGA is still entirely readily accessible. Anybody reading these copies who is not a native French speaker, will probably be translating at least some part of EGA into English in their head, or into their notebooks, as they read. This document is just the product of a few people doing exactly that, but then passing on their efforts to make things just that little bit easier for anyone else who follows.

Lastly, to quote another adage, "the guilty person is often the loudest". If it seems like we are over-eager to defend ourselves because we know that we are somehow in the wrong, it is because we are, at least partially. Working on this translation has meant going against Grothendieck's explicit requests, and for that we are sorry. We only hope that the freedom of knowledge is an excusable defense.

Notes from the translators

Grothendieck's writing style in EGA is quite particular, most notably for its long sentence structure. As translators, we have tried to give the best possible approximation of this style in English, resisting the temptation to "streamline" things in places where the language is more dense than usual.

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Any translations about which we are not entirely sure will be marked with a (?).

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Whenever a note is made by the translators, it will be prefaced by "[Trans.]".

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Along the margins we have provided the page numbers corresponding to the original text (of the first edition), as published by *Publications mathématiques de l'I.H.É.S.*, where the EGA were published as the following volumes:[5]

- EGA I (*tome 4, 1960*)
- EGA II (*tome 8, 1961*)
- EGA III, part 1 (*tome 11, 1961*)
- EGA III, part 2 (*tome 17, 1963*)
- EGA IV, part 1 (*tome 20, 1964*)
- EGA IV, part 2 (*tome 24, 1965*)
- EGA IV, part 3 (*tome 28, 1966*)
- EGA IV, part 4 (*tome 32, 1967*).

Due to EGA being a collection of volumes (one non-preliminary chapter, or part of a chapter, per volume), the page numbers reset at every new chapter. In addition, the preliminary section is stretched out over multiple volumes. To combat this, we label the pages as

$$\mathbf{X} \mid p,$$

referring to Chapter X, page p. For EGA III and IV, which are split across multiple chapters, we label the pages as

$$\mathbf{X}\text{-}n \mid p,$$

referring to Chapter X, part n, page p. In the case of the preliminaries (which are often collectively referred to as EGA 0), the preliminaries from Chapter Y are denoted as $\mathbf{0_Y}$.

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Later volumes (EGA II, III, and IV) include errata for earlier chapters. Where possible, we have used these to 'update' our translation, and entirely replace whatever mistakes might have been in the original copies of EGA I and II. If the change is minor (e.g. 'intersection' replacing 'inter-section') then we will not mention it; if it is anything more fundamental (e.g. X' replacing X) then we will include some margin note on the relevant line detailing the location of the erratum (e.g. $\mathbf{Err}_{\mathrm{II}}$ to denote that the correction is listed in the Errata section of EGA II).

[5] PDFs of which can be found online, hosted by the *Grothendieck circle*.

EGA IV SECTIONS

In EGA IV-1, the summary included a tentative list of section that EGA IV would contain. As EGA IV was written, §5, §7, and §§11–21 would contain different sections than initially envisaged. We include the original listing here:

§5. Dimension and depth for preschemes.
§7. Application to the relations between a local Noetherian ring and its completion. Excellent rings.
§11. Topological properties of finitely presented flat morphisms. Local flatness criteria.
§12. Study of fibres of finitely presented flat morphisms.
§13. Equidimensional morphisms.
§14. Universally open morphisms.
§15. Study of fibres of a universally open morphism.
§16. Differential invariants. Differentially smooth morphisms.
§17. Smooth morphisms, unramified morphisms, and étale morphisms.
§18. Supplement on étale morphisms. Henselian local rings.
§19. Regular immersions and transversely regular immersions.
§20. Hyperplane sections; generic projections.
§21. Infinitesimal extensions.

MATHEMATICAL WARNINGS

EGA uses *prescheme* for what is now usually called a scheme, and *scheme* for what is now usually called a separated scheme — we have decided to translate "literally" or "historically", and thus continue to use the word *scheme* to mean separated scheme, and *prescheme* to mean scheme.

In some cases, we (the translators) have changed "$\to$" to "$\mapsto$" where appropriate.

Introduction

To Oscar Zariski and André Weil.

This memoir, and the many others will undoubtedly follow, are intended to form a treatise on the foundations of algebraic geometry. They do not, in principle, presume any particular knowledge of the subject, and it has even been recognised that such knowledge, despite its obvious advantages, could sometimes (because of the much-too-narrow interpretation—through the birational point of view—that it usually implies) be a hindrance to the one who wants to become familiar with the point of view and techniques presented here. However, we assume that the reader has a good knowledge of the following topics: **I** | 5

(a) *Commutative algebra*, as it is laid out, for example, in the volumes (in progress of being written) of the *Éléments* of N. Bourbaki (and, pending the publication of these volumes, in Samuel–Zariski [**SZ60**] and Samuel [**Sam53b**, **Sam53a**]).
(b) *Homological algebra*, for which we refer to Cartan–Eilenberg [**CE56**] (cited as (M)) and Godement [**God58**] (cited as (G)), as well as the recent article by A. Grothendieck [**Gro57**] (cited as (T)).
(c) *Sheaf theory*, where our main references will be (G) and (T); this theory provides the essential language for interpreting, in "geometric" terms, the essential notions of commutative algebra, and for "globalizing" them.
(d) Finally, it will be useful for the reader to have some familiarity with *functorial language*, which will be constantly used in this treatise, and for which the reader may consult (M), (G), and especially (T); the principles of this language and the main results of the general theory of functors will be described in more detail in a book currently in preparation by the authors of this treatise.

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It is not the place, in this introduction, to give a more or less summary description from the "schemes" point of view in algebraic geometry, nor the long list of reasons which made its adoption necessary, and in particular the systematic acceptance of nilpotent elements in the local rings of "manifolds" that we consider (which necessarily shifts the idea of rational maps into the background, in favor of those of regular maps or "morphisms"). To be precise, this treatise aims to systematically develop the language of schemes, and will demonstrate, we hope, its necessity. Although it would be easy to do so, we will not try to give here an "intuitive" introduction to the notions developed in **I** | 6
Chapter I. For the reader who would like to have a glimpse of the preliminary study of the subject matter of this treatise, we refer them to the conference by A. Grothendieck at the International Congress of Mathematicians in Edinburgh in 1958 [**Gro58**], and the exposé [**Gro**] of the same author. The work [**Ser55a**] (cited as (FAC)) of J.-P. Serre can also be considered as an intermediary exposition between the classical point of view and the schemes point of view in algebraic geometry, and, as such, its reading may be an excellent preparation for the reading of our *Éléments*.

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We give below the general outline planned for this treatise, subject to later modifications, especially concerning the later chapters.

Chapter I. — The language of schemes.
— II. — Elementary global study of some classes of morphisms.
— III. — Cohomology of algebraic coherent sheaves. Applications.
— IV. — Local study of morphisms.
— V. — Elementary procedures of constructing schemes.
— VI. — Descent. General method of constructing schemes.
— VII. — Group schemes, principal fibre bundles.
— VIII. — Differential study of fibre bundles.
— IX. — The fundamental group.
— X. — Residues and duality.
— XI. — Theories of intersection, Chern classes, Riemann–Roch theorem.
— XII. — Abelian schemes and Picard schemes.
— XIII. — Weil cohomology.

In principle, all chapters are considered open to changes, and supplementary sections could always be added later; such sections would appear in separate fascicles in order to minimize the inconveniences accompanying whatever mode of publication adopted. When the writing of such a section is foreseen or in progress during the publication of a chapter, it will be mentioned in the summary of the chapter in question, even if, owing to certain orders of urgency, its actual publication clearly ought to have been later. For the convenience of the reader, we give in "Chapter 0" the necessary tools in commutative algebra, homological algebra, and sheaf theory, that will be used throughout this treatise, that are more or less well known but for which it was not possible to give convenient references. It is recommended for the reader to not read Chapter 0 except whilst reading the actual treatise, when the results to which we refer seem unfamiliar. Besides, we think that in this **I** | 7
way, the reading of this treatise could be a good method for the beginner to familiarize themselves with commutative algebra and homological algebra, whose study, when not accompanied with tangible applications, is considered tedious, or even depressing, by many.

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It is outside of our capabilities to give a historic overview, or even a summary thereof, of the ideas and results described herein. The text will contain only those references considered particularly useful for comprehension, and we indicate the origin of only the most important results. Formally, at least, the subjects discussed in our work are reasonably new, which explains the scarcity of references made to the fathers of algebraic geometry from the 19th to the beginning of the 20th century, whose works we know only by hear-say. It is suitable, however, to say some words here about the works which have most directly influenced the authors and contributed to the development of scheme-theoretic point of view. We absolutely must mention the fundamental work (FAC) of J.-P. Serre first, which has served as an introduction to algebraic geometry for more that one young student (the author of this treatise being one), deterred by the dryness of the classic *Foundations* of A. Weil [**Wei46**]. It is there that it is shown, for the first time, that the "Zariski topology" of an "abstract" algebraic variety is perfectly suited to applying certain techniques from algebraic topology, and notably to be able to define a cohomology theory. Further, the definition of an algebraic variety given therein is that which translates most naturally to the idea that we develop here[6]. Serre himself had incidentally noted that the cohomology theory of affine algebraic varieties could be translated without difficulty by replacing the affine algebras over a field by arbitrary commutative rings. Chapters I and II of this treatise, and the first two paragraphs of Chapter III, can thus be considered, for the most part, as easy translations, to this bigger framework, of the principal results of (FAC) and a later article

[6]Just as J.-P. Serre informed us, it is right to note that the idea of defining the structure of a manifold by the data of a sheaf of rings is due to H. Cartan, who took this idea as the starting point of his theory of analytic spaces. Of course, just as in algebraic geometry, it would be important in "analytic geometry" to allow the use of nilpotent elements in local rings of analytic spaces. This extension of the definition of H. Cartan and J.-P. Serre has recently been broached by H. Grauert [**Gra60**], and there is room to hope that a systematic report of analytic geometry in this setting will soon see the light of day. It is also evident that the ideas and techniques developed in this treatise retain a sense of analytic geometry, even though one must expect more considerable technical difficulties in this latter theory. We can foresee that algebraic geometry, by the simplicity of its methods, will be able to serve as a sort of formal model for future developments in the theory of analytic spaces.

of the same author [Ser57]. We have also vastly profited from the *Séminaire de géométrie algébrique* de C. Chevalley [CC]; in particular, the systematic usage of "constructible sets" introduced by him has turned out to be extremely useful in the theory of schemes (cf. Chapter IV). We have also borrowed from him the study of morphisms from the point of view of dimension (Chapter IV), that translates with negligible change to the framework of schemes. It also merits noting that the idea of "schemes of local rings", introduced by Chevalley, naturally lends itself to being extended to algebraic geometry (not having, however, all the flexibility and generality that we intend to give it here); for the connections between this idea and our theory, see Chapter I, §8. One such extension has been developed by M. Nagata in a series of memoirs [Nag58a], which contain many special results concerning algebraic geometry over Dedekind rings[7].

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It goes without saying that a book on algebraic geometry, and especially a book dealing with the fundamentals, is of course influenced, if only by proxy, by mathematicians such as O. Zariski and A. Weil. In particular, the *Théorie des fonctions holomorphes* by Zariski [Zar51], reasonably flexible thanks to the cohomological methods and an existence theorem (Chapter III, §§4 and 5), is (along with the method of descent described in Chapter VI) one of the principal tools used in this treatise, and it seems to us one of the most powerful at our disposal in algebraic geometry.

The general technique in which it is employed can be sketched as follows (a typical example of which will be given in Chapter XI, in the study of the fundamental group). We have a proper morphism (Chapter II) $f : X \to Y$ of algebraic varieties (or, more generally, of schemes) that we wish to study on the neighbourhood of a point $y \in Y$, with the aim of resolving a problem P relative to a neighbourhood of y. We proceed step by step:

1st We can suppose that Y is affine, so that X becomes a scheme defined on the affine ring A of Y, and we can even replace A by the local ring of y. This reduction is always easy in practice (Chapter V) and brings us to the case where A is a *local* ring.

2nd We study the problem in question when A is a local *Artinian* ring. So that the problem P still makes sense when A is not assumed to be integral, we sometimes have to reformulate P, and it appears that we often obtain a better understanding of the problem in doing so, in an "infinitesimal" way.

3rd The theory of formal schemes (Chapter III, §§3, 4, and 5) lets us pass from the case of an Artinian ring to a *complete local ring*.

4th Finally, if A is an arbitrary local ring, considering "multiform (?) sections" of suitable schemes over X, approximating a given "formal" section (Chapter IV), will let us pass, by extension of scalars, to the completion of A, from a known result (about the scheme I | 9 induced by X by extension of scalars to the completion of A) to an analogous result for a finite simple (e.g. unramified) extension of A.

This sketch shows the importance of the systematic study of schemes defined over an Artinian ring A. The point of view of Serre in his formulation of the theory of local class fields, and the recent works of Greenberg, seem to suggest that such a study could be undertaken by functorially attaching, to some such scheme X, a scheme X' over the residue field k of A (assumed perfect) of dimension equal (in nice cases) to $n \dim X$, where n is the height of A.

As for the influence of A. Weil, it suffices to say that it is the need to develop the tools necessary to formulate, with full generality, the definition of "Weil cohomology", and to tackle the proof[8] of all the formal properties necessary to establish the famous conjectures in Diophantine geometry [Wei49], that has been one of the principal motivations for the writing of this treatise, as well as the desire to find the natural setting of the usual ideas and methods of algebraic geometry, and to give the authors the chance to understand said ideas and methods.

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[7] Among the works that come close to our point of view of algebraic geometry, we pick out the work of E. Kähler [Käh58] and a recent note of Chow and Igusa [CI58], which go back over certain results of (FAC) in the context of Nagata–Chevalley theory, as well as giving a Künneth formula.

[8] To avoid any misunderstanding, we point out that this task has barely been undertaken at the moment of writing this introduction, and still hasn't led to the proof of the Weil conjectures.

Finally, we believe it useful to warn the reader that they, as did all the authors themselves, will almost certainly have difficulty before becoming accustomed to the language of schemes, and to convince themselves that the usual constructions that suggest geometric intuition can be translated, in essentially only one sensible way, to this language. As in many parts of modern mathematics, the first intuition seems further and further away, in appearance, from the correct language needed to express the mathematics in question with complete precision and the desired level of generality. In practice, the psychological difficulty comes from the need to replicate some familiar set-theoretic constructions to a category that is already quite different from that of sets (the category of preschemes, or the category of preschemes over a given prescheme): Cartesian products, group laws, ring laws, module laws, fibre bundles, principal homogeneous fibre bundles, etc. It will most likely be difficult for the mathematician, in the future, to shy away from this new effort of abstraction (maybe rather negligible, on the whole, in comparison with that supplied by our fathers) to familiarize themselves with the theory of sets.

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The references are given following the numerical system; for example, in **III**, 4.9.3, the **III** indicates the volume, the 4 the chapter, the 9 the section, and the 3 the paragraph.[9] If we reference a volume from within itself then we omit the volume number.

[Trans] Page 10 in the original is left blank. **I** | 10

[9] *[Trans] This is not a direct translation of the original, but instead uses the language more familiar to modern book (and LaTeX document) layouts.*

CHAPTER 0

Preliminaries (EGA 0)

§1. Rings of fractions

1.0. Rings and Algebras $\mathbf{0_I}$ | 11

(1.0.1). All the rings considered in this treatise will have a *unit element*; all the modules over such a ring will be assumed to be *unitary*; the ring homomorphisms will always be assumed to *send the unit element to the unit element*; unless otherwise stated, a subring of a ring A will be assumed to *contain the unit element of A*. We will focus in particular on *commutative* rings, and when we speak of a ring without specifying any details, it will be implied that it is commutative. If A is a not-necessarily-commutative ring, by A-module we will we mean a *left* module unless stated otherwise.

(1.0.2). Let A and B be not-necessarily-commutative rings and $\phi : A \to B$ a homomorphism. Any left (resp. right) B-module M can be provided with a left (resp. right) A-module structure by $a \cdot m = \phi(a) \cdot m$ (resp. $m \cdot a = m \cdot \phi(a)$); when it will be necessary to distinguish M as an A-module or a B-module, we will denote by $M_{[\phi]}$ the left (resp. right) A-module defined as such. If L is an A-module, then a homomorphism $u : L \to M_{[\phi]}$ is a homomorphism of abelian groups such that $u(a \cdot x) = \phi(a) \cdot u(x)$ for $a \in A$, $x \in L$; we will also say that it is a *ϕ-homomorphism $L \to M$*, and that the pair (ϕ, u) (or, by abuse of language, u) is a *di-homomorphism* from (A, L) to (B, M). The pairs (A, L) consisting of a ring A and an A-module L thus form a *category* whose morphisms are di-homomorphisms.

(1.0.3). Under the hypotheses of (1.0.2), if $\mathfrak{J}$ is a left (resp. right) ideal of A, we denote by $B\mathfrak{J}$ (resp. $\mathfrak{J}B$) the left (resp. right) ideal $B\phi(\mathfrak{J})$ (resp. $\phi(\mathfrak{J})B$) of B generated by $\phi(\mathfrak{J})$; it is also the image of the canonical homomorphism $B \otimes_A \mathfrak{J} \to B$ (resp. $\mathfrak{J} \otimes_A B \to B$) of left (resp. right) B-modules.

(1.0.4). If A is a (commutative) ring, and B a not-necessarily-commutative ring, then the data of a structure of an *A-algebra* on B is equivalent to the data of a ring homomorphism $\phi : A \to B$ such that $\phi(A)$ is contained in the center of B. For all ideals $\mathfrak{J}$ of A, $\mathfrak{J}B = B\mathfrak{J}$ is then a two-sided ideal of B, and for every B-module M, $\mathfrak{J}M$ is then a B-module equal to $(B\mathfrak{J})M$.

(1.0.5). We will not dwell much on the notions of *modules of finite type* and (commutative) *algebras of finite type*; to say that an A-module M is of finite type means that there exists an exact sequence $\mathbf{0_I}$ | 12
$A^p \to M \to 0$. We say that an A-module M admits a *finite presentation* if it is isomorphic to the cokernel of a homomorphism $A^p \to A^q$, or, in other words, if there exists an exact sequence $A^p \to A^q \to M \to 0$. We note that for a *Noetherian* ring A, every A-module of finite type admits a finite presentation.

Let us recall that an A-algebra B is said to be *integral* over A if every element in B is a root in B of a monic polynomial with coefficients in A; equivalently, if every element of B is contained in a subalgebra of B which is an *A-module of finite type*. When this is so, and if B is commutative, the subalgebra of B generated by a finite subset of B is an A-module of finite type; for a commutative algebra B to be integral and of finite type over A, it is necessary and sufficient that B be an A-module of finite type; we also say that B is an *integral A-algebra of finite type* (or simply *finite*, if there is no chance of confusion). It should be noted that in these definitions, it is not assumed that the homomorphism $A \to B$ defining the A-algebra structure is injective.

(1.0.6). An *integral* ring (or an *integral domain*) is a ring in which the product of a finite family of elements $\neq 0$ is $\neq 0$; equivalently, in such a ring, we have $0 \neq 1$, and the product of two elements $\neq 0$ is $\neq 0$. A *prime* ideal of a ring A is an ideal $\mathfrak{p}$ such that $A/\mathfrak{p}$ is integral; this implies that $\mathfrak{p} \neq A$. For a ring A to have at least one prime ideal, it is necessary and sufficient that $A \neq \{0\}$.

(1.0.7). A *local* ring is a ring A in which there exists a unique maximal ideal, which is thus the complement of the invertible elements, and contains all the ideals $\neq A$. If A and B are local rings, and $\mathfrak{m}$ and $\mathfrak{n}$ their respective maximal ideals, then we say that a homomorphism $\phi : A \to B$ is *local* if $\phi(\mathfrak{m}) \subset \mathfrak{n}$ (or, equivalently, if $\phi^{-1}(\mathfrak{n}) = \mathfrak{m}$). By passing to quotients, such a homomorphism then defines a monomorphism from the residue field $A/\mathfrak{m}$ to the residue field $B/\mathfrak{n}$. The composition of any two local homomorphisms is a local homomorphism.

1.1. Radical of an ideal. Nilradical and radical of a ring

(1.1.1). Let $\mathfrak{a}$ be an ideal of a ring A; the *radical* of $\mathfrak{a}$, denoted by $\mathfrak{r}(\mathfrak{a})$, is the set of $x \in A$ such that $x^n \in \mathfrak{a}$ for an integer $n > 0$; it is an ideal containing $\mathfrak{a}$. We have $\mathfrak{r}(\mathfrak{r}(\mathfrak{a})) = \mathfrak{r}(\mathfrak{a})$; the relation $\mathfrak{a} \subset \mathfrak{b}$ implies $\mathfrak{r}(\mathfrak{a}) \subset \mathfrak{r}(\mathfrak{b})$; the radical of a finite intersection of ideals is the intersection of their radicals. If ϕ is a homomorphism from another ring A' to A, then we have $\mathfrak{r}(\phi^{-1}(\mathfrak{a})) = \phi^{-1}(\mathfrak{r}(\mathfrak{a}))$ for any ideal $\mathfrak{a} \subset A$. For an ideal to be the radical of an ideal, it is necessary and sufficient that it be an intersection of prime ideals. The radical of an ideal $\mathfrak{a}$ is the intersection of the *minimal* prime ideals which contain $\mathfrak{a}$; if A is Noetherian, there are finitely many of these minimal prime ideals.

The radical of the ideal (0) is also called the *nilradical* of A; it is the set $\mathfrak{N}$ of the nilpotent elements of A. We say that the ring A is *reduced* if $\mathfrak{N} = (0)$; for every ring A, the quotient $A/\mathfrak{N}$ of A by its nilradical is a reduced ring.

(1.1.2). Recall that the *radical* $\mathfrak{r}(A)$ of a (not-necessarily-commutative) ring A is the intersection of the maximal left ideals of A (and also the intersection of maximal right ideals). The radical of $A/\mathfrak{r}(A)$ is (0).

1.2. Modules and rings of fractions $\mathbf{0_I}$ | 13

(1.2.1). We say that a subset S of a ring A is *multiplicative* if $1 \in S$ and the product of two elements of S is in S. The examples which will be the most important in what follows are: 1st, the set S_f of powers f^n ($n \geqslant 0$) of an element $f \in A$; and 2nd, the complement $A - \mathfrak{p}$ of a *prime* ideal $\mathfrak{p}$ of A.

(1.2.2). Let S be a multiplicative subset of a ring A, and M an A-module; on the set $M \times S$, the relation between pairs (m_1, s_1) and (m_2, s_2):

"there exists an $s \in S$ such that $s(s_1 m_2 - s_2 m_1) = 0$"

is an equivalence relation. We denote by $S^{-1}M$ the quotient set of $M \times S$ by this relation, and by m/s the canonical image of the pair (m, s) in $S^{-1}M$; we call $i_M^S : m \mapsto m/1$ (also denoted i^S) the *canonical* map from M to $S^{-1}M$. This map is, in general, neither injective nor surjective; its kernel is the set of $m \in M$ such that there exists an $s \in S$ for which $sm = 0$.

On $S^{-1}M$ we define an additive group law by setting

$$(m_1/s_1) + (m_2/s_2) = (s_2 m_1 + s_1 m_2)/(s_1 s_2)$$

(one can check that it is independent of the choice of representative of the elements of $S^{-1}M$, which are equivalence classes). On $S^{-1}A$ we further define a multiplicative law by setting $(a_1/s_1)(a_2/s_2) = (a_1 a_2)/(s_1 s_2)$, and finally an exterior law on $S^{-1}M$, acted on by the set of elements of $S^{-1}A$, by setting $(a/s)(m/s') = (am)/(ss')$. It can then be shown that $S^{-1}A$ is endowed with a ring structure (called *the ring of fractions of A with denominators in S*) and $S^{-1}M$ with the structure of an $S^{-1}A$-module (called *the module of fractions of M with denominators in S*); for all $s \in S$, $s/1$ is invertible in $S^{-1}A$, its inverse being $1/s$. The canonical map i_A^S (resp. i_M^S) is a ring homomorphism (resp. a homomorphism of A-modules, $S^{-1}M$ being considered as an A-module by means of the homomorphism $i_A^S : A \to S^{-1}A$).

(1.2.3). If $S_f = \{f^n\}_{n \geqslant 0}$ for a $f \in A$, we write A_f and M_f instead of $S_f^{-1}A$ and $S_f^{-1}M$; when A_f is considered as algebra over A, we can write $A_f = A[1/f]$. A_f is isomorphic to the quotient algebra $A[T]/(fT-1)A[T]$. When $f = 1$, A_f and M_f are canonically identified with A and M; if f is nilpotent, then A_f and M_f are 0.

When $S = A - \mathfrak{p}$, with $\mathfrak{p}$ a prime ideal of A, we write $A_\mathfrak{p}$ and $M_\mathfrak{p}$ instead of $S^{-1}A$ and $S^{-1}M$; $A_\mathfrak{p}$ is a *local ring* whose maximal ideal $\mathfrak{q}$ is generated by $i_A^S(\mathfrak{p})$, and we have $(i_A^S)^{-1}(\mathfrak{q}) = \mathfrak{p}$; by passing to quotients, i_A^S gives a monomorphism from the integral ring $A/\mathfrak{p}$ to the field $A_\mathfrak{p}/\mathfrak{q}$, which can be identified with the field of fractions of $A/\mathfrak{p}$.

(1.2.4). The ring of fractions $S^{-1}A$ and the canonical homomorphism i_A^S are a solution to a *universal mapping problem*: any homomorphism u from A to a ring B such that $u(S)$ is composed of *invertible* elements in B factors uniquely as

$$u : A \xrightarrow{i_A^S} S^{-1}A \xrightarrow{u^*} B$$

where u^* is a ring homomorphism. Under the same hypotheses, let M be an A-module, N a B-module, and $v : M \to N$ a homomorphism of A-modules (for the B-module structure on N defined by $u : A \to B$); then v factors uniquely as $\mathbf{0_I}$ | 14

$$v : M \xrightarrow{i_M^S} S^{-1}M \xrightarrow{v^*} N$$

where v^* is a homomorphism of $S^{-1}A$-modules (for the $S^{-1}A$-module structure on N defined by u^*).

(1.2.5). We define a canonical isomorphism $S^{-1}A \otimes_A M \simeq S^{-1}M$ of $S^{-1}A$-modules, sending the element $(a/s) \otimes m$ to the element $(am)/s$, with the inverse isomorphism sending m/s to $(1/s) \otimes m$.

(1.2.6). For every ideal $\mathfrak{a}'$ of $S^{-1}A$, $\mathfrak{a} = (i_A^S)^{-1}(\mathfrak{a}')$ is an ideal of A, and $\mathfrak{a}'$ is the ideal of $S^{-1}A$ generated by $i_A^S(\mathfrak{a})$, which can be identified with $S^{-1}\mathfrak{a}$ (1.3.2). The map $\mathfrak{p}' \mapsto (i_A^S)^{-1}(\mathfrak{p}')$ is an isomorphism, for the structure given by ordering, from the set of *prime* ideals of $S^{-1}A$ to the set of prime ideals $\mathfrak{p}$ of A such that $\mathfrak{p} \cap S = \varnothing$. In addition, the local rings $A_\mathfrak{p}$ and $(S^{-1}A)_{S^{-1}\mathfrak{p}}$ are canonically isomorphic (1.5.1).

(1.2.7). When A is an *integral* ring, for which K denotes its field of fractions, the canonical map $i_A^S : A \to S^{-1}A$ is injective for any multiplicative subset S not containing 0, and $S^{-1}A$ is then canonically identified with a subring of K containing A. In particular, for every prime ideal $\mathfrak{p}$ of A, $A_\mathfrak{p}$ is a local ring containing A, with maximal ideal $\mathfrak{p}A_\mathfrak{p}$, and we have $\mathfrak{p}A_\mathfrak{p} \cap A = \mathfrak{p}$.

(1.2.8). If A is a *reduced* ring (1.1.1), so is $S^{-1}A$: indeed, if $(x/s)^n = 0$ for $x \in A$, $s \in S$, then this means that there exists an $s' \in S$ such that $s'x^n = 0$, hence $(s'x)^n = 0$, which, by hypothesis, implies $s'x = 0$, so $x/s = 0$.

1.3. Functorial properties

(1.3.1). Let M and N be A-modules, and u an A-homomorphism $M \to N$. If S is a multiplicative subset of A, we define a $S^{-1}A$-homomorphism $S^{-1}M \to S^{-1}N$, denoted by $S^{-1}u$, by setting $S^{-1}u(m/s) = u(m)/s$; if $S^{-1}M$ and $S^{-1}N$ are canonically identified with $S^{-1}A \otimes_A M$ and $S^{-1}A \otimes_A N$ (1.2.5), then $S^{-1}u$ is considered as $1 \otimes u$. If P is a third A-module, and v an A-homomorphism $N \to P$, we have $S^{-1}(v \circ u) = (S^{-1}v) \circ (S^{-1}u)$; in other words, $S^{-1}M$ is a *covariant functor in* M, from the category of A-modules to that of $S^{-1}A$-modules (A and S being fixed).

(1.3.2). The functor $S^{-1}M$ is *exact*; in other words, if the sequence

$$M \xrightarrow{u} N \xrightarrow{v} P$$

is exact, then so is the sequence

$$S^{-1}M \xrightarrow{S^{-1}u} S^{-1}N \xrightarrow{S^{-1}v} S^{-1}P.$$

In particular, if $u : M \to N$ is injective (resp. surjective), the same is true for $S^{-1}u$; if N and P are submodules of M, $S^{-1}N$ and $S^{-1}P$ are canonically identified with submodules of $S^{-1}M$, and we have $\mathbf{0_I}$ | 15

$$S^{-1}(N+P) = S^{-1}N + S^{-1}P \text{ and } S^{-1}(N \cap P) = (S^{-1}N) \cap (S^{-1}P).$$

(1.3.3). Let $(M_\alpha, \phi_{\beta\alpha})$ be an inductive system of A-modules; then $(S^{-1}M_\alpha, S^{-1}\phi_{\beta\alpha})$ is an inductive system of $S^{-1}A$-modules. Expressing the $S^{-1}M_\alpha$ and $S^{-1}\phi_{\beta\alpha}$ as tensor products ((1.2.5) and (1.3.1)), it follows from the permutability of the tensor product and inductive limit operations that we have a canonical isomorphism

$$S^{-1} \varinjlim M_\alpha \simeq \varinjlim S^{-1}M_\alpha$$

which is we can further express by saying that the functor $S^{-1}M$ (in M) *commutes with inductive limits*.

(1.3.4). Let M and N be A-modules; there is a canonical *functorial* (in M and N) isomorphism

$$(S^{-1}M) \otimes_{S^{-1}A} (S^{-1}N) \simeq S^{-1}(M \otimes_A N)$$

which sends $(m/s) \otimes (n/t)$ to $(m \otimes n)/st$.

(1.3.5). We also have a *functorial* (in M and N) homomorphism

$$S^{-1}\operatorname{Hom}_A(M,N) \longrightarrow \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$$

which sends u/s to the homomorphism $m/t \mapsto u(m)/st$. When M has a finite presentation, the above homomorphism is an *isomorphism*: it is immediate when M is of the form A^r, and we pass to the general case by starting with the exact sequence $A^p \to A^q \to M \to 0$ and using the exactness of the functor $S^{-1}M$ and the left-exactness of the functor $\operatorname{Hom}_A(M,N)$ in M. Note that this is always the case when A is *Noetherian* and the A-module M is *of finite type*.

1.4. Change of multiplicative subset

(1.4.1). Let S and T be multiplicative subsets of a ring A such that $S \subset T$; there exists a canonical homomorphism $\rho_A^{T,S}$ (or simply $\rho^{T,S}$) from $S^{-1}A$ to $T^{-1}A$, sending the element denoted a/s of $S^{-1}A$ to the element denoted a/s in $T^{-1}A$; we have $i_A^T = \rho_A^{T,S} \circ i_A^S$. For every A-module M, there exists, in the same way, an $S^{-1}A$-linear map from $S^{-1}M$ to $T^{-1}M$ (the latter considered as an $S^{-1}A$-module by the homomorphism $\rho_A^{T,S}$), which sends the element m/s of $S^{-1}M$ to the element m/s of $T^{-1}M$; we denote this map by $\rho_M^{T,S}$, or simply $\rho^{T,S}$, and we still have $i_M^T = \rho_M^{T,S} \circ i_M^S$; by the canonical identification (1.2.5), $\rho_M^{T,S}$ is identified with $\rho_A^{T,S} \otimes 1$. The homomorphism $\rho_M^{T,S}$ is a *functorial morphism* (or natural transformation) from the functor $S^{-1}M$ to the functor $T^{-1}M$, in other words, the diagram

$$\begin{tikzcd} S^{-1}M \arrow[r, "S^{-1}u"] \arrow[d, "\rho_M^{T,S}"'] & S^{-1}N \arrow[d, "\rho_N^{T,S}"] \\ T^{-1}M \arrow[r, "T^{-1}u"] & T^{-1}N \end{tikzcd}$$

is commutative, for every homomorphism $u : M \to N$; $T^{-1}u$ is entirely determined by $S^{-1}u$, since, $\mathbf{0_I}$ | 16
for $m \in M$ and $t \in T$, we have

$$(T^{-1}u)(m/t) = (t/1)^{-1}\rho^{T,S}((S^{-1}u)(m/1)).$$

(1.4.2). With the same notation, for A-modules M and N, the diagrams (cf. (1.3.4) and (1.3.5))

$$\begin{tikzcd} (S^{-1}M) \otimes_{S^{-1}A} (S^{-1}N) \arrow[r, "\sim"] \arrow[d] & S^{-1}(M \otimes_A N) \arrow[d] \\ (T^{-1}M) \otimes_{T^{-1}A} (T^{-1}N) \arrow[r, "\sim"] & T^{-1}(M \otimes_A N) \end{tikzcd}$$

$$\begin{tikzcd} S^{-1}\operatorname{Hom}_A(M,N) \arrow[r] \arrow[d] & \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \arrow[d] \\ T^{-1}\operatorname{Hom}_A(M,N) \arrow[r] & \operatorname{Hom}_{T^{-1}A}(T^{-1}M, T^{-1}N) \end{tikzcd}$$

are commutative.

(1.4.3). There is an important case, in which the homomorphism $\rho^{T,S}$ is *bijective*, when we then know that every element of T is a divisor of an element of S; we then identify the modules $S^{-1}M$ and $T^{-1}M$ via $\rho^{T,S}$. We say that S is *saturated* if every divisor in A of an element of S is in S; by replacing S with the set T of all the divisors of the elements of S (a set which is multiplicative and saturated), we see that we can always, if we wish, consider only modules of fractions $S^{-1}M$, where S is saturated.

(1.4.4). If S, T, and U are three multiplicative subsets of A such that $S \subset T \subset U$, then we have

$$\rho^{U,S} = \rho^{U,T} \circ \rho^{T,S}.$$

(1.4.5). Consider an *increasing filtered family* (S_α) of multiplicative subsets of A (we write $\alpha \leqslant \beta$ for $S_\alpha \subset S_\beta$), and let S be the multiplicative subset $\bigcup_\alpha S_\alpha$; let us put $\rho_{\beta\alpha} = \rho_A^{S_\beta,S_\alpha}$ for $\alpha \leqslant \beta$; according to (1.4.4), the homomorphisms $\rho_{\beta\alpha}$ define a ring A' as the *inductive limit* of the inductive system of rings $(S_\alpha^{-1}A, \rho_{\beta\alpha})$. Let ρ_α be the canonical map $S_\alpha^{-1}A \to A'$, and let $\phi_\alpha = \rho_A^{S,S_\alpha}$; as $\phi_\alpha = \phi_\beta \circ \rho_{\beta\alpha}$ for $\alpha \leqslant \beta$ according to (1.4.4), we can uniquely define a homomorphism $\phi : A' \to S^{-1}A$ such that the diagram

$$\begin{tikzcd}
& S_\alpha^{-1}A \arrow[dl, "\rho_\alpha"'] \arrow[d, "\rho_{\beta\alpha}"] \arrow[dr, "\phi_\alpha"] & \\
& S_\beta^{-1}A \arrow[dl, "\rho_\beta"] \arrow[dr, "\phi_\beta"'] & \\
A' \arrow[rr, "\phi"] & & S^{-1}A
\end{tikzcd} \qquad (\alpha \leqslant \beta)$$

is commutative. In fact, ϕ is an *isomorphism*; it is indeed immediate by construction that ϕ is surjective. On the other hand, if $\rho_\alpha(a/s_\alpha) \in A'$ is such that $\phi(\rho_\alpha(a/s_\alpha)) = 0$, then this means that $a/s_\alpha = 0$ in $S^{-1}A$, that is to say that there exists an $s \in S$ such that $sa = 0$; but there is a $\beta \geqslant \alpha$ such that $s \in S_\beta$, and consequently, as $\rho_\alpha(a/s_\alpha) = \rho_\beta(sa/ss_\alpha) = 0$, we find that ϕ is injective. The case for an A-module M is treated likewise, and we have thus defined canonical isomorphisms

$$\varinjlim S_\alpha^{-1}A \simeq (\varinjlim S_\alpha)^{-1}A, \ \varinjlim S_\alpha^{-1}M \simeq (\varinjlim S_\alpha)^{-1}M,$$

the second being *functorial* in M.

$\mathbf{0_I}$ | 17

(1.4.6). Let S_1 and S_2 be multiplicative subsets of A; then S_1S_2 is also a multiplicative subset of A. Let us denote by S_2' the canonical image of S_2 in the ring $S_1^{-1}A$, which is a multiplicative subset of this ring. For every A-module M there is then a functorial isomorphism

$$S_2'^{-1}(S_1^{-1}M) \simeq (S_1S_2)^{-1}M$$

which sends $(m/s_1)/(s_2/1)$ to the element $m/(s_1s_2)$.

1.5. Change of ring

(1.5.1). Let A and A' be rings, ϕ a homomorphism $A' \to A$, and S (resp. S') a multiplicative subset of A (resp. A'), such that $\phi(S') \subset S$; the composition homomorphism $A' \xrightarrow{\phi} A \to S^{-1}A$ factors as

$$A' \longrightarrow S'^{-1}A' \xrightarrow{\phi^{S'}} S^{-1}A,$$

by (1.2.4); where $\phi^{S'}(a'/s') = \phi(a')/\phi(s')$. If $A = \phi(A')$ and $S = \phi(S')$, then $\phi^{S'}$ is *surjective*. If $A' = A$ and ϕ is the identity, then $\phi^{S'}$ is exactly the homomorphism $\rho_A^{S,S'}$ defined in (1.4.1).

(1.5.2). Under the hypotheses of (1.5.1), let M be an A-module. There exists a canonical functorial morphism

$$\sigma : S'^{-1}(M_{[\phi]}) \longrightarrow (S^{-1}M)_{[\phi^{S'}]}$$

of $S'^{-1}A'$-modules, sending each element m/s' of $S'^{-1}(M_{[\phi]})$ to the element $m/\phi(s')$ of $(S^{-1}M)_{[\phi^{S'}]}$; indeed, we immediately see that this definition does not depend on the representative m/s' of the element in question. When $S = \phi(S')$, the homomorphism σ is *bijective*. When $A' = A$ and ϕ is the identity, σ is none other than the homomorphism $\rho_M^{S,S'}$ defined in (1.4.1).

When, in particular, we take $M = A$ the homomorphism ϕ defines an A'-algebra structure on A; $S'^{-1}(A_{[\phi]})$ is then endowed with a ring structure, with which it can be identified with $(\phi(S'))^{-1}A$, and the homomorphism $\sigma : S'^{-1}(A_{[\phi]}) \to S^{-1}A$ is a homomorphism of $S'^{-1}A'$-algebras.

(1.5.3). Let M and N be A-modules; by composing the homomorphisms defined in (1.3.4) and (1.5.2), we obtain a homomorphism

$$(S^{-1}M \otimes_{S^{-1}A} S^{-1}N)_{[\phi^{S'}]} \longleftarrow S'^{-1}((M \otimes N)_{[\phi]})$$

which is an isomorphism when $\phi(S') = S$. Similarly, by composing the homomorphisms in (1.3.5) and (1.5.2), we obtain a homomorphism

$$S'^{-1}((\mathrm{Hom}_A(M,N))_{[\phi]}) \longrightarrow (\mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N))_{[\phi^{S'}]}$$

which is an isomorphism when $\phi(S') = S$ and M admits a finite presentation.

(1.5.4). Let us now consider an A'-module N', and form the tensor product $N' \otimes_{A'} A_{[\phi]}$, which can be considered as an A-module by setting $a \cdot (n' \otimes b) = n' \otimes (ab)$. There is a functorial isomorphism of $S^{-1}A$-modules

$$\tau : (S'^{-1}N') \otimes_{S'^{-1}A'} (S^{-1}A)_{[\phi^{S'}]} \simeq S^{-1}(N' \otimes_{A'} A_{[\phi]})$$

which sends the element $(n'/s') \otimes (a/s)$ to the element $(n' \otimes a)/(\phi(s')s)$; indeed, we can show that when we replace n'/s' (resp. a/s) by another expression of the same element, $(n' \otimes a)/(\phi(s')s)$ does not change; on the other hand, we can define a homomorphism inverse to τ by sending $(n' \otimes a)/s$ to the element $(n'/1) \otimes (a/s)$: we use the fact that $S^{-1}(N' \otimes_{A'} A_{[\phi]})$ is canonically isomorphic to $(N' \otimes_{A'} A_{[\phi]}) \otimes_A S^{-1}A$ (1.2.5), so also to $N' \otimes_{A'} (S^{-1}A)_{[\psi]}$, where we denote by ψ the composite homomorphism $a' \mapsto \phi(a')/1$ from A' to $S^{-1}A$. $\mathbf{0_I}$ | 18

(1.5.5). If M' and N' are A'-modules, then by composing the isomorphisms (1.3.4) and (1.5.4), we obtain an isomorphism

$$S'^{-1}M' \otimes_{S'^{-1}A'} S'^{-1}N' \otimes_{S'^{-1}A'} S^{-1}A \simeq S^{-1}(M' \otimes_{A'} N' \otimes_{A'} A).$$

Likewise, if M' admits a finite presentation, we have by (1.3.5) and (1.5.4) an isomorphism

$$\mathrm{Hom}_{S'^{-1}A'}(S'^{-1}M', S'^{-1}N') \otimes_{S'^{-1}A'} S^{-1}A \simeq S^{-1}(\mathrm{Hom}_{A'}(M',N') \otimes_{A'} A).$$

(1.5.6). Under the hypotheses of (1.5.1), let T (resp. T') be another multiplicative subset of A (resp. A') such that $S \subset T$ (resp. $S' \subset T'$) and $\phi(T') \subset T$. Then the diagram

$$\begin{tikzcd} S'^{-1}A' \arrow[r, "\phi^{S'}"] \arrow[d, "\rho^{T',S'}"'] & S^{-1}A \arrow[d, "\rho^{T,S}"] \\ T'^{-1}A' \arrow[r, "\phi^{T'}"] & T^{-1}A \end{tikzcd}$$

is commutative. If M is an A-module, then the diagram

$$\begin{tikzcd} S'^{-1}(M_{[\phi]}) \arrow[r, "\sigma"] \arrow[d, "\rho^{T',S'}"'] & (S^{-1}M)_{[\phi^{S'}]} \arrow[d, "\rho^{T,S}"] \\ T'^{-1}(M_{[\phi]}) \arrow[r, "\sigma"] & (T^{-1}M)_{[\phi^{T'}]} \end{tikzcd}$$

is commutative. Finally, if N' is an A'-module, then the diagram

$$\begin{tikzcd} (S'^{-1}N') \otimes_{S'^{-1}A'} (S^{-1}A)_{[\phi^{S'}]} \arrow[r, "\sim", "\tau"'] \arrow[d] & S^{-1}(N' \otimes_{A'} A_{[\phi]}) \arrow[d, "\rho^{T,S}"] \\ (T'^{-1}N') \otimes_{T'^{-1}A'} (T^{-1}A)_{[\phi^{T'}]} \arrow[r, "\sim", "\tau"'] & T^{-1}(N' \otimes_{A'} A_{[\phi]}) \end{tikzcd}$$

is commutative, the left vertical arrow obtained by applying $\rho^{T',S'}_{N'}$ to $S'^{-1}N'$ and $\rho^{T,S}_A$ to $S^{-1}A$.

$\mathbf{0_I}$ | 19

(1.5.7). Let A'' be a third ring, $\phi' : A'' \to A'$ a ring homomorphism, and S'' a multiplicative subset of A'' such that $\phi'(S'') \subset S'$. Let $\phi'' = \phi \circ \phi'$; then we have

$$\phi''^{S''} = \phi^{S'} \circ \phi'^{S''}.$$

Let M be an A-module; evidently we have $M_{[\phi'']} = (M_{[\phi]})_{[\phi']}$; if σ' and σ'' are the homomorphisms defined by ϕ' and ϕ'' in the same way as how σ is defined in (1.5.2) by ϕ, then we have the transitivity formula

$$\sigma'' = \sigma \circ \sigma'.$$

Finally, let N'' be an A''-module; the A-module $N'' \otimes_{A''} A_{[\phi'']}$ is canonically identified with $(N'' \otimes_{A''} A'_{[\phi']}) \otimes_{A'} A_{[\phi]}$, and likewise the $S^{-1}A$-module $(S''^{-1}N'') \otimes_{S''^{-1}A''} (S^{-1}A)_{[\phi''^{S''}]}$ is canonically identified with $((S''^{-1}N'') \otimes_{S''^{-1}A''} (S'^{-1}A')_{[\phi'^{S''}]}) \otimes_{S'^{-1}A'} (S^{-1}A)_{[\phi^{S'}]}$. With these identifications, if τ' and τ'' are the isomorphisms defined by ϕ' and ϕ'' in the same way as how τ is defined in (1.5.4) by ϕ, then we have the transitivity formula

$$\tau'' = \tau \circ (\tau' \otimes 1).$$

(1.5.8). Let A be a subring of a ring B; for every *minimal* prime ideal $\mathfrak{p}$ of A, there exists a minimal prime ideal $\mathfrak{q}$ of B such that $\mathfrak{p} = A \cap \mathfrak{q}$. Indeed, $A_{\mathfrak{p}}$ is a subring of $B_{\mathfrak{p}}$ (1.3.2) and has *a single prime ideal* $\mathfrak{p}'$ (1.2.6); since $B_{\mathfrak{p}}$ is not 0, it has at least one prime ideal $\mathfrak{q}'$ and we necessarily have $\mathfrak{q}' \cap A_{\mathfrak{p}} = \mathfrak{p}'$; the prime ideal $\mathfrak{q}_1$ of B, the inverse image of $\mathfrak{q}'$, is thus such that $\mathfrak{q}_1 \cap A = \mathfrak{p}$, and *a fortiori* we have $\mathfrak{q} \cap A = \mathfrak{p}$ for every minimal prime ideal $\mathfrak{q}$ of B contained in $\mathfrak{q}_1$.

1.6. Identification of the module M_f as an inductive limit

(1.6.1). Let M be an A-module and f an element of A. Consider a sequence (M_n) of A-modules, all identical to M, and for each pair of integers $m \leqslant n$, let ϕ_{nm} be the homomorphism $z \mapsto f^{n-m}z$ from M_m to M_n; it is immediate that $((M_n), (\phi_{nm}))$ is an *inductive system* of A-modules; let $N = \varinjlim M_n$ be the inductive limit of this system. We define a canonical *functorial* A-isomorphism from N to M_f. For this, let us note that, for all n, $\theta_n : z \mapsto z/f^n$ is an A-homomorphism from $M = M_n$ to M_f, and it follows from the definitions that we have $\theta_n \circ \phi_{nm} = \theta_m$ for $m \leqslant n$. As a result, there exists an A-homomorphism $\theta : N \to M_f$ such that, if ϕ_n denotes the canonical homomorphism $M_n \to N$, then we have $\theta_n = \theta \circ \phi_n$ for all n. Since, by hypothesis, every element of M_f is of the form z/f^n for at least one n, it is clear that θ is surjective. On the other hand, if $\theta(\phi_n(z)) = 0$, or, in other words, if $z/f^n = 0$, then there exists an integer $k > 0$ such that $f^k z = 0$, so $\phi_{n+k,n}(z) = 0$, which gives $\phi_n(z) = 0$. We can therefore identify M_f with $\varinjlim M_n$ via θ.

(1.6.2). Now write $M_{f,n}$, ϕ^f_{nm}, and ϕ^f_n instead of M_n, ϕ_{nm}, and ϕ_n. Let g be another element of A. Since f^n divides $f^n g^n$, we have a functorial homomorphism

$$\rho_{fg,f} : M_f \longrightarrow M_{fg} \quad ((1.4.1) \text{ and } (1.4.3));$$

if we identify M_f and M_{fg} with $\varinjlim M_{f,n}$ and $\varinjlim M_{fg,n}$ respectively, then $\rho_{fg,f}$ identifies with the **0_{I} | 20**
inductive limit of the maps $\rho^n_{fg,f} : M_{f,n} \to M_{fg,n}$, defined by $\rho^n_{fg,f}(z) = g^n z$. Indeed, this follows immediately from the commutativity of the diagram

$$\begin{tikzcd} M_{f,n} \arrow[r, "\rho^n_{fg,f}"] \arrow[d, "\phi^f_n"'] & M_{fg,n} \arrow[d, "\phi^{fg}_n"] \\ M_f \arrow[r, "\rho_{fg,f}"] & M_{fg}. \end{tikzcd}$$

1.7. Support of a module

(1.7.1). Given an A-module M, we define the *support* of M, denoted by $\operatorname{Supp}(M)$, to be the set of prime ideals $\mathfrak{p}$ of A such that $M_{\mathfrak{p}} \neq 0$. For it to be the case that $M = 0$, it is necessary and sufficient that $\operatorname{Supp}(M) = \varnothing$, because if $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p}$, then the annihilator of an element $x \in M$ cannot be contained in any prime ideal of A, and so is the whole of A.

(1.7.2). If $0 \to N \to M \to P \to 0$ is an exact sequence of A-modules, then we have

$$\operatorname{Supp}(M) = \operatorname{Supp}(N) \cup \operatorname{Supp}(P)$$

because, for every prime ideal $\mathfrak{p}$ of A, the sequence $0 \to N_{\mathfrak{p}} \to M_{\mathfrak{p}} \to P_{\mathfrak{p}} \to 0$ is exact (1.3.2) and in order that $M_{\mathfrak{p}} = 0$, it is necessary and sufficient that $N_{\mathfrak{p}} = P_{\mathfrak{p}} = 0$.

(1.7.3). If M is the sum of a family (M_λ) of submodules, then $M_{\mathfrak{p}}$ is the sum of the $(M_\lambda)_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ of A ((1.3.3) and (1.3.2)), so $\operatorname{Supp}(M) = \bigcup_\lambda \operatorname{Supp}(M_\lambda)$.

(1.7.4). If M is an A-module *of finite type*, then $\operatorname{Supp}(M)$ is the set of prime ideals *containing the annihilator of* M. Indeed, if M is cyclic and generated by x, then to say that $M_{\mathfrak{p}} = 0$ is to say that that there exists an $s \notin \mathfrak{p}$ such that $s \cdot x = 0$, and thus that $\mathfrak{p}$ does not contain the annihilator of x. Now if M admits a finite system $(x_i)_{1 \leqslant i \leqslant n}$ of generators, and if $\mathfrak{a}_i$ is the annihilator of x_i, then it follows from (1.7.3) that $\operatorname{Supp}(M)$ is the set of the $\mathfrak{p}$ containing one of the $\mathfrak{a}_i$, or equivalently, the set of the $\mathfrak{p}$ containing $\mathfrak{a} = \bigcap_i \mathfrak{a}_i$, which is the annihilator of M.

(1.7.5). If M and N are two A-modules *of finite type*, then we have

$$\operatorname{Supp}(M \otimes_A N) = \operatorname{Supp}(M) \cap \operatorname{Supp}(N).$$

It is a question of seeing that, if $\mathfrak{p}$ is a prime ideal of A, then the condition $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} \neq 0$ is equivalent to "$M_{\mathfrak{p}} \neq 0$ and $N_{\mathfrak{p}} \neq 0$" (taking (1.3.4) into account). In other words, it is a question of seeing that, if P and Q are modules of finite type over a *local* ring $B \neq 0$, then $P \otimes_B Q \neq 0$. Let $\mathfrak{m}$ be the maximal ideal of B. By Nakayama's Lemma, the vector spaces $P/\mathfrak{m}P$ and $Q/\mathfrak{m}Q$ are not 0, and so it is the same for the tensor product $(P/\mathfrak{m}P) \otimes_{B/\mathfrak{m}} (Q/\mathfrak{m}Q) = (P \otimes_B Q) \otimes_B (B/\mathfrak{m})$, whence the conclusion.

In particular, if M is an A-module of finite type, and $\mathfrak{a}$ an ideal of A, then $\operatorname{Supp}(M/\mathfrak{a}M)$ is the set of prime ideals containing both $\mathfrak{a}$ and the annihilator $\mathfrak{n}$ of M (1.7.4), that is, the set of prime ideals containing $\mathfrak{a} + \mathfrak{n}$.

§2. Irreducible spaces. Noetherian spaces

2.1. Irreducible spaces $\mathbf{0}_{\mathbf{I}}$ | 21

(2.1.1). We say that a topological space X is *irreducible* if it is nonempty and if it is not a union of two distinct closed subspaces of X. It is equivalent to say that $X \neq \varnothing$ and the intersection of two nonempty open sets (and consequently of a finite number of open sets) of X is nonempty, or that every nonempty open set is everywhere dense, or that any closed set is *rare*[1], or, lastly, that all open sets of X are *connected*.

(2.1.2). For a subspace Y of a topological space X to be irreducible, it is necessary and sufficient that its closure $\overline{Y}$ be irreducible. In particular, any subspace which is the closure $\overline{\{x\}}$ of a singleton is irreducible; we will express the relation $y \in \overline{\{x\}}$ (equivalent to $\overline{\{y\}} \subset \overline{\{x\}}$) by saying that y is a *specialization of* x or that x is a *generalization of* y. When there exists, in an irreducible space X, a point x such that $X = \overline{\{x\}}$, we will say that x is a *generic point of* X. Any nonempty open subset of X then contains x, and any subspace containing x has x as a generic point.

(2.1.3). Recall that a *Kolmogoroff space* is a topological space X satisfying the axiom of separation:

(T_0) If $x \neq y$ are any two points of X, there is an open set containing one of the points x and y, but not the other.

[1][Trans] also known as *nowhere dense*.

If an irreducible Kolmogoroff space admits a generic point, it admits *exactly* one, since a nonempty open set contains any generic point.

Recall that a topological space X is said to be *quasi-compact* if, from any collection of open sets of X, one can extract a finite cover of X (or, equivalently, if any decreasing filtered family of nonempty closed sets has a nonempty intersection). If X is a quasi-compact space, then any nonempty closed subset A of X contains a *minimal* nonempty closed set M, because the set of nonempty closed subsets of A is inductive under the relation $\supset$; if, in addition, X is a Kolmogoroff space, M is necessarily a single point (or, as we say by abuse of language, is a *closed point*).

(2.1.4). In an irreducible space X, every nonempty open subspace U is irreducible, and if X admits a generic point x, x is also a generic point of U.

To prove this, let (U_α) be a cover (whose set of indices is nonempty) of a topological space X, consisting of nonempty open sets; if X is irreducible, it is necessary and sufficient that U_α is irreducible for all α, and that $U_\alpha \cap U_\beta \neq \varnothing$ for any α, β. The condition is clearly necessary; to see that it is sufficient, it suffices to prove that if V is a nonempty open subset of X, then $V \cap U_\alpha$ is nonempty for all α, since then $V \cap U_\alpha$ is dense in U_α for all α, and consequently V is dense in X. Now there is at least one index γ such that $V \cap U_\gamma \neq \varnothing$, so $V \cap U_\gamma$ is dense in U_γ, and as for all α, $U_\alpha \cap V_\alpha \neq \varnothing$, we also have $V \cap U_\alpha \cap U_\gamma \neq \varnothing$.

$\mathbf{0_I}$ | 22

(2.1.5). Let X be an irreducible space, and f a continuous map from X into a topological space Y. Then $f(X)$ is irreducible, and if x is a generic point of X, then $f(x)$ is a generic point of $f(X)$ and hence also of $\overline{f(X)}$. In particular, if, in addition, Y is irreducible and with a single generic point y, then for $f(X)$ to be everywhere dense, it is necessary and sufficient that $f(x) = y$.

(2.1.6). Any irreducible subspace of a topological space X is contained in a maximal irreducible subspace, which is necessarily closed. Maximal irreducible subspaces of X are called the *irreducible components* of X. If Z_1 and Z_2 are two irreducible components distinct from the space X, then $Z_1 \cap Z_2$ is a closed *rare* set in each of the subspaces Z_1, Z_2; in particular, if an irreducible component of X admits a generic point (2.1.2), such a point cannot belong to any other irreducible component. If X has only a *finite* number of irreducible components Z_i ($1 \leqslant i \leqslant n$), and if, for each i, we put $U_i = \bigcup_{j \neq i} Z_j$, then the U_i are open, irreducible, disjoint, and their union is dense in X. Let U be an **Err$_{\mathrm{II}}$** open subset of a topological space X. If Z is an irreducible subset of X that intersects U, then $Z \cap U$ is open and dense in Z, thus irreducible; conversely, for any irreducible closed subset Y of U, the closure $\overline{Y}$ of Y in X is irreducible and $\overline{Y} \cap U = Y$. We conclude that there is a *bijective correspondence* between the irreducible components of U and the irreducible components of X which intersect U.

(2.1.7). If a topological space X is a union of a *finite* number of irreducible closed subspaces Y_i, then the irreducible components of X are the maximal elements of the set of the Y_i, because if Z is an irreducible closed subset of X, then Z is the union of the $Z \cap Y_i$, from which one sees that Z must be contained in one of the Y_i. Let Y be a subspace of a topological space X, and suppose that Y has only a finite number of irreducible components Y_i, ($1 \leqslant i \leqslant n$); then the closures $\overline{Y_i}$ in X are the irreducible components of Y.

(2.1.8). Let Y be an irreducible space admitting a single generic point y. Let X be a topological space, and f a continuous map from X to Y. Then, for any irreducible component Z of X intersecting $f^{-1}(y)$, $f(Z)$ is dense in Y. The converse is not necessarily true; however, if Z has a generic point z, and if $f(Z)$ is dense in Y, then we must have $f(z) = y$ (2.1.5); in addition, $Z \cap f^{-1}(y)$ is then the closure of $\{z\}$ in $f^{-1}(y)$ and is therefore irreducible, and as an irreducible subset of $f^{-1}(y)$ containing z is necessarily contained in Z (2.1.6), z is a generic point of $Z \cap f^{-1}(y)$. As any irreducible component of $f^{-1}(y)$ is contained in an irreducible component of X, we see that, if any irreducible component Z of X intersecting $f^{-1}(y)$ admits a generic point, then there is a *bijective correspondence* between all these components and all the irreducible components $Z \cap f^{-1}(y)$ of $f^{-1}(y)$, the generic points of Z being identical to those of $Z \cap f^{-1}(y)$.

2.2. Noetherian spaces $\mathbf{0_I}$ | 23

(2.2.1). We say that a topological space X is *Noetherian* if the set of open subsets of X satisfies the *maximal* condition, or, equivalently, if the set of closed subsets of X satisfies the *minimal* condition. We say that X is *locally Noetherian* if each $x \in X$ admits a neighbourhood which is a Noetherian subspace.

(2.2.2). Let E be an ordered set satisfying the *minimal* condition, and let $\mathbf{P}$ be a property of the elements of E subject to the following condition: if $a \in E$ is such that for any $x < a$, $\mathbf{P}(x)$ is true, then $\mathbf{P}(a)$ is true. Under these conditions, $\mathbf{P}(x)$ *is true for all* $x \in E$ ("principle of Noetherian induction"). Indeed, let F be the set of $x \in E$ for which $\mathbf{P}(x)$ is false; if F were not empty, it would have a minimal element a, and as then $\mathbf{P}(x)$ is true for all $x < a$, $\mathbf{P}(a)$ would be true, which is a contradiction.

We will apply this principle in particular when E is a *set of closed subsets of a Noetherian space*.

(2.2.3). Any subspace of a Noetherian space is Noetherian. Conversely, any topological space that is a finite union of Noetherian subspaces is Noetherian.

(2.2.4). Any Noetherian space is quasi-compact; conversely, any topological space in which all open sets are quasi-compact is Noetherian.

(2.2.5). A Noetherian space has only a *finite* number of irreducible components, as we see by Noetherian induction.

§3. Supplement on sheaves

3.1. Sheaves with values in a category

(3.1.1). Let $\mathtt{C}$ be a category, $(A_\alpha)_{\alpha\in I}$, $(A_{\alpha\beta})_{(\alpha,\beta)\in I\times I}$ two families of objects of $\mathtt{C}$ such that $A_{\beta\alpha} = A_{\alpha\beta}$, and $(\rho_{\alpha\beta})_{(\alpha,\beta)\in I\times I}$ a family of morphisms $\rho_{\alpha\beta} : A_\alpha \to A_{\alpha\beta}$. We say that a pair consisting of an object A of $\mathtt{C}$ and a family of morphisms $\rho_\alpha : A \to A_\alpha$ is a *solution to the universal problem* defined by the data of the families (A_α), $(A_{\alpha\beta})$, and $(\rho_{\alpha\beta})$ if, for every object B of $\mathtt{C}$, the map which sends $f \in \operatorname{Hom}(B, A)$ to the family $(\rho_\alpha \circ f) \in \Pi_\alpha \operatorname{Hom}(B, A_\alpha)$ is a *bijection* of $\operatorname{Hom}(B, A)$ to the set of all (f_α) such that $\rho_{\alpha\beta} \circ f_\alpha = \rho_{\beta\alpha} \circ f_\beta$ for any pair of indices (α, β). If such a solution exists, it is unique up to an isomorphism.

(3.1.2). We will not recall the definition of a *presheaf* $U \mapsto \mathscr{F}(U)$ on a topological space X with values in a category $\mathtt{C}$ (G, I, 1.9); we say that such a presheaf is a *sheaf with values in* $\mathtt{C}$ if it satisfies the following axiom:

(F) *For any covering* (U_α) *of an open set* U *of* X *by open sets* U_α *contained in* U, *if we denote by* ρ_α (resp. $\rho_{\alpha\beta}$) *the restriction morphism*

$$\mathscr{F}(U) \longrightarrow \mathscr{F}(U_\alpha) \quad (\text{resp. } \mathscr{F}(U_\alpha) \longrightarrow \mathscr{F}(U_\alpha \cap U_\beta)),$$

the pair formed by $\mathscr{F}(U)$ *and the family* (ρ_α) *are a solution to the universal problem for* $(\mathscr{F}(U_\alpha))$, $\mathbf{0_I}$ | 24
$(\mathscr{F}(U_\alpha \cap U_\beta))$, *and* $(\rho_{\alpha\beta})$ *in* (3.1.1)[2].

Equivalently, we can say that, for each object T of $\mathtt{C}$, that the family $U \mapsto \operatorname{Hom}(T, \mathscr{F}(U))$ is a *sheaf of sets*.

(3.1.3). Assume that $\mathtt{C}$ is the category defined by a "type of structure with morphisms" Σ, the objects of $\mathtt{C}$ being the sets with structures of type Σ and morphisms those of Σ. Suppose that the category $\mathtt{C}$ also satisfies the following condition:

(E) If $(A, (\rho_\alpha))$ is a solution of a universal mapping problem *in the category* $\mathtt{C}$ for families (A_α), $(A_{\alpha\beta})$, $(\rho_{\alpha\beta})$, then it is also a solution of the universal mapping problem for the same families *in the category of sets* (that is, when we consider A, A_α, and $A_{\alpha\beta}$ as sets, ρ_α and $\rho_{\alpha\beta}$ as functions)[3].

[2] This is a special case of the more general notion of a (non-filtered) *projective limit* (*see* (T, I, 1.8) and the book in preparation announced in the introduction).

[3] It can be proved that it also means that the canonical functor $\mathtt{C} \to \mathtt{Set}$ *commutes with projective limits* (not necessarily filtered).

Under these conditions, the condition (F) gives that, when considered as a presheaf *of sets*, $U \mapsto \mathscr{F}(U)$ is a *sheaf*. In addition, for a map $u : T \to \mathscr{F}(U)$ to be a morphism of $\mathtt{C}$, it is necessary and sufficient, according to (F), that each map $\rho_\alpha \circ u$ is a morphism $T \to \mathscr{F}(U_\alpha)$, which means that the structure of type Σ on $\mathscr{F}(U)$ is the *initial structure* for the morphisms ρ_α. Conversely, suppose a presheaf $U \mapsto \mathscr{F}(U)$ on X, with values in $\mathtt{C}$, is a *sheaf of sets* and satisfies the previous condition; it is then clear that it satisfies (F), so it is a *sheaf with values in* $\mathtt{C}$.

(3.1.4). When Σ is a type of a group or ring structure, the fact that the presheaf $U \mapsto \mathscr{F}(U)$ with values in $\mathtt{C}$ is a sheaf of *sets* implies *ipso facto* that it is a sheaf with values in $\mathtt{C}$ (in other words, a sheaf of groups or rings within the meaning of (G))[4]. But it is not the same when, for example, $\mathtt{C}$ is the category of *topological rings* (with morphisms as continuous homomorphisms): a sheaf with values in $\mathtt{C}$ is a sheaf of rings $U \mapsto \mathscr{F}(U)$ such that for any open U and any covering of U by open sets $U_\alpha \subset U$, the topology of the ring $\mathscr{F}(U)$ is to be *the least fine* making the homomorphisms $\mathscr{F}(U) \to \mathscr{F}(U_\alpha)$ continuous. We will say in this case that $U \mapsto \mathscr{F}(U)$, considered as a sheaf of rings (without a topology), is *underlying* the sheaf of topological rings $U \mapsto \mathscr{F}(U)$. Morphisms $u_V : \mathscr{F}(V) \to \mathscr{G}(V)$ (V an arbitrary open subset of X) of sheaves of topological rings are therefore homomorphisms of the underlying sheaves of rings, such that u_V is *continuous* for all open $V \subset X$; to distinguish them from any homomorphisms of the sheaves of the underlying rings, we will call them continuous homomorphisms of sheaves of topological rings. We have similar definitions and conventions for sheaves of topological spaces or topological groups.

$\mathbf{0_I}$ | 25

(3.1.5). It is clear that for any category $\mathtt{C}$, if there is a presheaf (respectively a sheaf) $\mathscr{F}$ on X with values in $\mathtt{C}$ and U is an open set of X, the $\mathscr{F}(V)$ for open $V \subset U$ constitute a presheaf (or a sheaf) with values in $\mathtt{C}$, which we call the presheaf (or sheaf) *induced* by $\mathscr{F}$ on U and denote it by $\mathscr{F}|U$.

For any morphism $u : \mathscr{F} \to \mathscr{G}$ of presheaves on X with values in $\mathtt{C}$, we denote by $u|U$ the morphism $\mathscr{F}|U \to \mathscr{G}|U$ consisting of the u_V for $V \subset U$.

(3.1.6). Suppose now that the category $\mathtt{C}$ admits *inductive limits* (T, 1.8); then, for any presheaf (and in particular any sheaf) $\mathscr{F}$ on X with values in $\mathtt{C}$ and each $x \in X$, we can define the *stalk* $\mathscr{F}_x$ as the object of $\mathtt{C}$ defined by the inductive limit of the $\mathscr{F}(U)$ with respect to the filtered set (for $\supset$) of the open neighbourhoods U of x in X, and the morphisms $\rho_U^V : \mathscr{F}(V) \to \mathscr{F}(U)$. If $u : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves with values in $\mathtt{C}$, we define for each $x \in X$ the morphism $u_x : \mathscr{F}_x \to \mathscr{G}_x$ as the inductive limit of $u_U : \mathscr{F}(U) \to \mathscr{G}(U)$ with respect to all open neighbourhoods of x; we thus define $\mathscr{F}_x$ as a covariant functor in $\mathscr{F}$, with values in $\mathtt{C}$, for all $x \in X$.

When $\mathtt{C}$ is further defined by a kind of structure with morphisms Σ, we call *sections over* U of a *sheaf* $\mathscr{F}$ with values in $\mathtt{C}$ the elements of $\mathscr{F}(U)$, and we write $\Gamma(U, \mathscr{F})$ instead of $\mathscr{F}(U)$; for $s \in \Gamma(U, \mathscr{F})$, V an open set contained in U, we write $s|V$ instead of $\rho_V^U(s)$; for all $x \in U$, the canonical image of s in $\mathscr{F}_x$ is the *germ* of s at the point x, denoted by s_x (*we will never replace the notation* $s(x)$ *in this sense*, this notation being reserved for another notion relating to sheaves which will be considered in this treatise (5.5.1)).

If then $u : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves with values in $\mathtt{C}$, we will write $u(s)$ instead of $u_V(s)$ for all $s \in \Gamma(V, \mathscr{F})$. $\mathbf{Err_{II}}$

If $\mathscr{F}$ is a sheaf of commutative groups, or rings, or modules, we say that the set of $x \in X$ such that $\mathscr{F}_x \neq \{0\}$ is the *support* of $\mathscr{F}$, denoted $\operatorname{Supp}(\mathscr{F})$; this set is not necessarily closed in X.

When $\mathtt{C}$ is defined by a type of structure with morphisms, *we systematically refrain from using the point of view of "étalé spaces"* in terms of relating to sheaves with values in $\mathtt{C}$; in other words, we will never consider a sheaf as a topological space (nor even as the whole union of its stalks), and we will not consider also a morphism $u : \mathscr{F} \to \mathscr{G}$ of such sheaves on X as a continuous map of topological spaces.

[4] This is because in the category $\mathtt{C}$, any morphism that is a *bijection* (as a map of sets) is an *isomorphism*. This is no longer true when $\mathtt{C}$ is the category of topological spaces, for example.

3.2. Presheaves on an open basis

(3.2.1). We will restrict to the following categories $\mathtt{C}$ admitting *projective limits* (generalized, that is, corresponding to not necessarily filtered preordered sets, cf. (T, 1.8)). Let X be a topological space, $\mathfrak{B}$ an open basis for the topology of X. We will call a *presheaf on* $\mathfrak{B}$, *with values in* $\mathtt{C}$, a family of objects $\mathscr{F}(U) \in \mathtt{C}$, corresponding to each $U \in \mathfrak{B}$, and a family of morphisms $\rho_U^V : \mathscr{F}(V) \to \mathscr{F}(U)$ defined for any pair (U, V) of elements of $\mathfrak{B}$ such that $U \subset V$, with the conditions $\rho_U^U =$ identity and $\mathbf{0_I}$ | 26
$\rho_U^W = \rho_U^V \circ \rho_V^W$ if U, V, W in $\mathfrak{B}$ are such that $U \subset V \subset W$. We can associate a *presheaf with values in* $\mathtt{C}$: $U \mapsto \mathscr{F}(U)$ in the ordinary sense, taking for all open U, $\mathscr{F}'(U) = \varprojlim \mathscr{F}(V)$, where V runs through the ordered set (for $\subset$, *not filtered* in general) of $V \in \mathfrak{B}$ sets such that $V \subset U$, since the (V) form a projective system for the ρ_V^W ($V \subset W \subset U$, $V \in \mathfrak{B}$, $W \in \mathfrak{B}$). Indeed, if U, U' are two open sets of X such that $U \subset U'$, we define $\rho'^{U'}_U$ as the projective limit (for $V \subset U$) of the canonical morphisms $\mathscr{F}'(U') \to \mathscr{F}(V)$, in other words the unique morphism $\mathscr{F}'(U') \to \mathscr{F}'(U)$, which, when composed with the canonical morphisms $\mathscr{F}'(U) \to \mathscr{F}(V)$, gives the canonical morphisms $\mathscr{F}'(U') \to \mathscr{F}(V)$; the verification of the transitivity of $\rho'^{U'}_U$ is then immediate. Moreover, if $U \in \mathfrak{B}$, the canonical morphism $\mathscr{F}'(U) \to \mathscr{F}(U)$ is an isomorphism, allowing us to identify these two objects[5].

(3.2.2). For the presheaf $\mathscr{F}'$ thus defined to be a *sheaf*, it is necessary and sufficient that the presheaf $\mathscr{F}$ on $\mathfrak{B}$ satisfies the condition:

(F_0) *For any covering* (U_α) *of* $U \in \mathfrak{B}$ *by sets* $U_\alpha \in \mathfrak{B}$ *contained in* U, *and for any object* $T \in \mathtt{C}$, *the map which sends* $f \in \operatorname{Hom}(T, \mathscr{F}(U))$ *to the family* $(\rho^U_{U_\alpha} \circ f) \in \prod_\alpha \operatorname{Hom}(T, \mathscr{F}(U_\alpha))$ *is a bijection from* $\operatorname{Hom}(T, \mathscr{F}(U))$ *to the set of all* (f_α) *such that* $\rho_V^{U_\alpha} \circ f_\alpha = \rho_V^{U_\beta} \circ f_\beta$ *for any pair of indices* (α, β) *and any* $V \in \mathfrak{B}$ *such that* $V \subset U_\alpha \cap U_\beta$[6].

The condition is obviously necessary. To show that it is sufficient, consider first a second basis $\mathfrak{B}'$ of the topology of X, *contained in* $\mathfrak{B}$, and show that if $\mathscr{F}''$ denotes the presheaf induced by the subfamily $(\mathscr{F}(V))_{V \in \mathfrak{B}'}$, $\mathscr{F}''$ is *canonically isomorphic* to $\mathscr{F}'$. Indeed, first the projective limit (for $V \in \mathfrak{B}'$, $V \subset U$) of the canonical morphisms $\mathscr{F}'(U) \to \mathscr{F}(V)$ is a morphism $\mathscr{F}'(U) \to \mathscr{F}''(U)$ for all open U. If $U \in \mathfrak{B}$, this morphism is an isomorphism, because by hypothesis the canonical morphisms $\mathscr{F}''(U) \to \mathscr{F}(V)$ for $V \in \mathfrak{B}'$, $V \subset U$, factorize as $\mathscr{F}''(U) \to \mathscr{F}(U) \to \mathscr{F}(V)$, and it is immediate to see that the composition of morphisms $\mathscr{F}(U) \to \mathscr{F}''(U)$ and $\mathscr{F}''(U) \to \mathscr{F}(U)$ thus defined are the identities. This being so, for all open U, the morphisms $\mathscr{F}''(U) \to \mathscr{F}''(W) = \mathscr{F}(W)$ for $W \in \mathfrak{B}$ and $W \subset U$ satisfy the conditions characterizing the projective limit of $\mathscr{F}(W)$ ($W \in \mathfrak{B}$, $W \subset U$), which proves our assertion given the uniqueness of a projective limit up to isomorphism.

This being so, let U be any open set of X, (U_α) a covering of U by the open sets contained in U, and $\mathfrak{B}'$ the subfamily of $\mathfrak{B}$ formed by the sets of $\mathfrak{B}$ contained in at least one U_α; it is clear that $\mathbf{0_I}$ | 27
$\mathfrak{B}'$ is still a basis of the topology of U, so $\mathscr{F}'(U)$ (resp. $\mathscr{F}''(U_\alpha)$) is the projective limit of $\mathscr{F}(V)$ for $V \in \mathfrak{B}'$ and $V \subset U$ (resp., $V \subset U_\alpha$), the axiom (F) is then immediately verified by virtue of the definition of the projective limit.

When (F_0) is satisfied, we will say by abuse of language that the presheaf $\mathscr{F}$ on the basis $\mathfrak{B}$ is a sheaf.

(3.2.3). Let $\mathscr{F}, \mathscr{G}$ be two presheaves on a basis $\mathfrak{B}$, with values in $\mathtt{C}$; we define a *morphism* $u : \mathscr{F} \to \mathscr{G}$ as a family $(u_V)_{V \in \mathfrak{B}}$ of morphisms $u_V : \mathscr{F}(V) \to \mathscr{G}(V)$ satisfying the usual compatibility conditions with the restriction morphisms ρ_V^W. With the notation of (3.2.1), we have a morphism $u' : \mathscr{F}' \to \mathscr{G}'$ of (ordinary) presheaves by taking for u'_U the projective limit of the u_V for $V \in \mathfrak{B}$ and $V \subset U$; the

[5] If X is a *Noetherian* space, we can still define $\mathscr{F}'(U)$ and show that it is a presheaf (in the ordinary sense) when one supposes only that $\mathtt{C}$ admits projective limits for *finite* projective systems. Indeed, if U is any open set of X, there is a *finite* covering (V_i) of U consisting of sets of $\mathfrak{B}$; for every couple (i, j) of indices, let (V_{ijk}) be a finite covering of $V_i \cap V_j$ formed by sets of $\mathfrak{B}$. Let I be the set of i and triples (i, j, k), ordered only by the relations $i > (i, j, k)$, $j > (i, j, k)$; we then take $\mathscr{F}'(U)$ to be the projective limit of the system of $\mathscr{F}(V_i)$ and $\mathscr{F}(V_{ijk})$; it is easy to verify that this does not depend on the coverings (V_i) and (V_{ijk}) and that $U \mapsto \mathscr{F}'(U)$ is a presheaf.

[6] It also means that the pair formed by $\mathscr{F}(U)$ and the $\rho_\alpha = \rho^U_{U_\alpha}$ is a *solution to the universal problem* defined in (3.1.1) by the data of $A_\alpha = \mathscr{F}(U_\alpha)$, $A_{\alpha\beta} = \prod \mathscr{F}(V)$ (for $V \in \mathfrak{B}$ such that $V \subset U_\alpha \cap U_\beta$) and $\rho_{\alpha\beta} = (\rho''_V) : \mathscr{F}(U_\alpha) \to \prod \mathscr{F}(V)$ defined by the condition that for $V \in \mathfrak{B}$, $V' \in \mathfrak{B}$, $W \in \mathfrak{B}$, $V \cup V' \subset U_\alpha \cap U_\beta$, $W \subset V \cap V'$, $\rho^V_W \circ \rho''_V = \rho^{V'}_W \circ \rho''_{V'}$.

verification of the compatibility conditions with the $\rho'^{U'}_U$ follows from the functorial properties of the projective limit.

(3.2.4). If the category $\mathtt{C}$ admits inductive limits, and if $\mathscr{F}$ is a presheaf on the basis $\mathfrak{B}$, with values in $\mathtt{C}$, for each $x \in X$ the neighbourhoods of x belonging to $\mathfrak{B}$ form a cofinal set (for $\supset$) in the set of neighbourhoods of x, therefore, if $\mathscr{F}'$ is the (ordinary) presheaf corresponding to $\mathscr{F}$, the stalk $\mathscr{F}'_x$ is equal to $\varinjlim_{\mathfrak{B}} \mathscr{F}(V)$ over the set of $V \in \mathfrak{B}$ containing x. If $u : \mathscr{F} \to \mathscr{G}$ is morphism of presheaves on $\mathfrak{B}$ with values in $\mathtt{C}$, $u' : \mathscr{F}' \to \mathscr{G}'$ the corresponding morphism of ordinary presheaves, u'_x is likewise the inductive limit of the morphisms $u_V : \mathscr{F}(V) \to \mathscr{G}(V)$ for $V \in \mathfrak{B}$, $x \in V$.

(3.2.5). We return to the general conditions of (3.2.1). If $\mathscr{F}$ is an ordinary *sheaf* with values in $\mathtt{C}$, $\mathscr{F}_1$ the sheaf *on* $\mathfrak{B}$ obtained by the restriction of $\mathscr{F}$ to $\mathfrak{B}$, then the ordinary sheaf $\mathscr{F}'_1$ obtained from $\mathscr{F}_1$ by the procedure of (3.2.1) is canonically isomorphic to $\mathscr{F}$, by virtue of the condition (F) and the uniqueness properties of the projective limit. We identify the ordinary sheaf $\mathscr{F}$ with $\mathscr{F}'_1$.

If $\mathscr{G}$ is a second (ordinary) sheaf on X with values in $\mathtt{C}$, and $u : \mathscr{F} \to \mathscr{G}$ a morphism, the preceding remark shows that the data of the $u_V : \mathscr{F}(V) \to \mathscr{G}(V)$ *for only the* $V \in \mathfrak{B}$ completely determines u; conversely, it is sufficient, the u_V being given for $V \in \mathfrak{B}$, to verify the commutative diagram with the restriction morphisms ρ^W_V for $V \in \mathfrak{B}$, $W \in \mathfrak{B}$, and $V \subset W$, for there to exist a morphism u' and a unique $\mathscr{F}$ in $\mathscr{G}$ such that $u'_V = u_V$ for each $V \in \mathfrak{B}$ (3.2.3).

(3.2.6). Suppose that $\mathtt{C}$ admits projective limits. Then the category of *sheaves on X with values in* $\mathtt{C}$ admits *projective limits*; if $(\mathscr{F}_\lambda)$ is a projective system of sheaves on X with values in $\mathtt{C}$, the $\mathscr{F}(U) = \varprojlim_\lambda \mathscr{F}_\lambda(U)$ indeed define a presheaf with values in $\mathtt{C}$, and the verification of the axiom (F) follows from the transitivity of projective limits; the fact that $\mathscr{F}$ is then the projective limit of the $\mathscr{F}_\lambda$ is immediate.

When $\mathtt{C}$ is the category of sets, for each projective system $(\mathscr{H}_\lambda)$ such that $\mathscr{H}_\lambda$ is a *subsheaf* of $\mathscr{F}_\lambda$ $\mathbf{0_I}$ | 28
for each λ, $\varprojlim_\lambda \mathscr{H}_\lambda$ canonically identifies with a *subsheaf* of $\varprojlim_\lambda \mathscr{F}_\lambda$. If $\mathtt{C}$ is the category of abelian groups, the covariant functor $\varprojlim_\lambda \mathscr{F}_\lambda$ is *additive* and *left exact*.

3.3. Gluing sheaves

(3.3.1). Suppose still that the category $\mathtt{C}$ admits (generalized) projective limits. Let X be a topological space, $\mathfrak{U} = (U_\lambda)_{\lambda \in L}$ an open cover of X, and for each $\lambda \in L$, let $\mathscr{F}_\lambda$ be a sheaf on U_λ, with values in $\mathtt{C}$; for each pair of indices (λ, μ), suppose that we are given an *isomorphism* $\theta_{\lambda\mu} : \mathscr{F}_\mu|(U_\lambda \cap U_\mu) \simeq \mathscr{F}|(U_\lambda \cap U_\mu)$; in addition, suppose that for each triple (λ, μ, ν), if we denote by $\theta'_{\lambda\mu}, \theta'_{\mu\nu}, \theta'_{\lambda\nu}$ the restrictions of $\theta_{\lambda\mu}, \theta_{\mu\nu}, \theta_{\lambda\nu}$ to $U_\lambda \cap U_\mu \cap U_\nu$, then we have $\theta'_{\lambda\nu} = \theta'_{\lambda\mu} \circ \theta'_{\mu\nu}$ (*gluing condition* for the $\theta_{\lambda\mu}$). Then there exists a sheaf $\mathscr{F}$ on X, with values in $\mathtt{C}$, and for each λ an isomorphism $\eta_\lambda : \mathscr{F}|U_\lambda \simeq \mathscr{F}_\lambda$ such that, for each pair (λ, μ), if we denote by η'_λ and η'_μ the restrictions of η_λ and η_μ to $U_\lambda \cap U_\mu$, then we have $\theta_{\lambda\mu} = \eta'_\lambda \circ \eta'^{-1}_\mu$; in addition, $\mathscr{F}$ and the η_λ are determined up to unique isomorphism by these conditions. The uniqueness indeed follows immediately from (3.2.5). To establish the existence of $\mathscr{F}$, denote by $\mathfrak{B}$ the open basis consisting of the open sets contained in at least one U_λ, and for each $U \in \mathfrak{B}$, choose (by the Hilbert function τ) one of the $\mathscr{F}_\lambda(U)$ for one of the λ such that $U \subset U_\lambda$; if we denote this object by $\mathscr{F}(U)$, the ρ^V_U for $U \subset V$, $U \in \mathfrak{B}$, $V \in \mathfrak{B}$ are defined in an evident way (by means of the $\theta_{\lambda\mu}$), and the transitivity conditions is a consequence of the gluing condition; in addition, the verification of (F_0) is immediate, so the presheaf on $\mathfrak{B}$ thus clearly defines a sheaf, and we deduce by the general procedure (3.2.1) an (ordinary) sheaf still denoted $\mathscr{F}$ and which answers the question. We say that $\mathscr{F}$ is obtained by *gluing the $\mathscr{F}_\lambda$ by means of the $\theta_{\lambda\mu}$* and we usually identify the $\mathscr{F}_\lambda$ and $\mathscr{F}|U_\lambda$ by means of the η_λ.

It is clear that each sheaf $\mathscr{F}$ on X with values in $\mathtt{C}$ can be considered as being obtained by the gluing of the sheaves $\mathscr{F}_\lambda = \mathscr{F}|U_\lambda$ (where (U_λ) is an arbitrary open cover of X), by means of the isomorphisms $\theta_{\lambda\mu}$ reduced to the identity.

(3.3.2). With the same notation, let $\mathscr{G}_\lambda$ be a second sheaf on U_λ (for each $\lambda \in L$) with values in $\mathtt{C}$, and for each pair (λ, μ) let us be given an isomorphism $\omega_{\lambda\mu} : \mathscr{G}_\mu|(U_\lambda \cap U_\mu) \simeq \mathscr{G}_\lambda|(U_\lambda \cap U_\mu)$, these isomorphisms satisfying the gluing condition. Finally, suppose that we are given for each λ a morphism $u_\lambda : \mathscr{F}_\lambda \to \mathscr{G}_\lambda$, and that the diagrams

(3.3.2.1)
$$\begin{tikzcd} \mathscr{F}_\mu|(U_\lambda \cap U_\mu) \arrow[r, "u_\mu"] \arrow[d] & \mathscr{G}_\mu|(U_\lambda \cap U_\mu) \arrow[d] \\ \mathscr{F}_\lambda|(U_\lambda \cap U_\mu) \arrow[r, "u_\lambda"] & \mathscr{G}_\lambda|(U_\lambda \cap U_\mu) \end{tikzcd}$$

are commutative. Then, if $\mathscr{G}$ is obtained by gluing the $\mathscr{G}_\lambda$ by means of the $\omega_{\lambda\mu}$, there exists a unique morphism $u : \mathscr{F} \to \mathscr{G}$ such that the diagrams **0_I** | 29

$$\begin{tikzcd} \mathscr{F}|U_\lambda \arrow[r, "u|U_\lambda"] \arrow[d] & \mathscr{G}|U_\lambda \arrow[d] \\ \mathscr{F}_\lambda \arrow[r, "u_\lambda"] & \mathscr{G}_\lambda \end{tikzcd}$$

are commutative; this follows immediately from (3.2.3). The correspondence between the family (u_λ) and u is in a functorial bijection with the subset of $\prod_\lambda \mathrm{Hom}(\mathscr{F}_\lambda, \mathscr{G}_\lambda)$ satisfying the conditions (3.3.2.1) on $\mathrm{Hom}(\mathscr{F}, \mathscr{G})$.

(3.3.3). With the notation of (3.3.1), let V be an open set of X; it is immediate that the restrictions to $V \cap U_\lambda \cap U_\mu$ of the $\theta_{\lambda\mu}$ satisfy the gluing condition for the induced sheaves $\mathscr{F}_\lambda|(V \cap U_\lambda)$ and that the sheaves on V obtained by gluing the latter identifies canonically with $\mathscr{F}|V$.

3.4. Direct images of presheaves

(3.4.1). Let X, Y be two topological spaces, $\psi : X \to Y$ a continuous map. Let $\mathscr{F}$ be a presheaf on X with values in a category $\mathtt{C}$; for each open $U \subset Y$, let $\mathscr{G}(U) = \mathscr{F}(\psi^{-1}(U))$, and if U, V are two open subsets of Y such that $U \subset V$, let ρ_U^V be the morphism $\mathscr{F}(\psi^{-1}(V)) \to \mathscr{F}(\psi^{-1}(U))$; it is immediate that the $\mathscr{G}(U)$ and the ρ_U^V define a *presheaf* on Y with values in $\mathtt{C}$, that we call the *direct image of $\mathscr{F}$ by ψ* and we denote it by $\psi_*(\mathscr{F})$. If $\mathscr{F}$ is a sheaf, we immediately verify the axiom (F) for the presheaf $\mathscr{G}$, so $\psi_*(\mathscr{F})$ is a sheaf.

(3.4.2). Let $\mathscr{F}_1$, $\mathscr{F}_2$ be two presheaves of X with values in $\mathtt{C}$, and let $u : \mathscr{F}_1 \to \mathscr{F}_2$ be a morphism. When U varies over the set of open subsets of Y, the family of morphisms $u_{\psi^{-1}(U)} : \mathscr{F}_1(\psi^{-1}(U)) \to \mathscr{F}_2(\psi^{-1}(U))$ satisfies the compatibility conditions with the restriction morphisms, and as a result defines a morphism $\psi_*(u) : \psi_*(\mathscr{F}_1) \to \psi_*(\mathscr{F}_2)$. If $v : \mathscr{F}_2 \to \mathscr{F}_3$ is a morphism from $\mathscr{F}_2$ to a third presheaf on X with values in $\mathtt{C}$, we have $\psi_*(v \circ u) = \psi_*(v) \circ \psi_*(u)$; in other words, $\psi_*(\mathscr{F})$ is a *covariant functor* in $\mathscr{F}$, from the category of presheaves (resp. sheaves) on X with values in $\mathtt{C}$, to that of presheaves (resp. sheaves) on Y with values in $\mathtt{C}$.

(3.4.3). Let Z be a third topological space, $\psi' : Y \to Z$ a continuous map, and let $\psi'' = \psi' \circ \psi$. It is clear that we have $\psi''_*(\mathscr{F}) = \psi'_*(\psi_*(\mathscr{F}))$ for each presheaf $\mathscr{F}$ on X with values in $\mathtt{C}$; in addition, for each morphism $u : \mathscr{F} \to \mathscr{G}$ of such presheaves, we have $\psi''_*(u) = \psi'_*(\psi_*(u))$. In other words, ψ''_* is the *composition* of the functors ψ'_* and ψ_*, and this can be written as

$$(\psi' \circ \psi)_* = \psi'_* \circ \psi_*.$$

In addition, for each open set U of Y, the image under the restriction $\psi|\psi^{-1}(U)$ of the induced presheaf $\mathscr{F}|\psi^{-1}(U)$ is none other than the induced presheaf $\psi_*(\mathscr{F})|U$.

(3.4.4). Suppose that the category $\mathtt{C}$ admits inductive limits, and let $\mathscr{F}$ be a presheaf on X with values in $\mathtt{C}$; for all $x \in X$, the morphisms $\Gamma(\psi^{-1}(U), \mathscr{F}) \to \mathscr{F}_x$ (U an open neighbourhood of $\psi(x)$ in Y) form an inductive limit, which gives by passing to the limit a morphism $\psi_x : (\psi_*(\mathscr{F}))_{\psi(x)} \to \mathscr{F}_x$ of the stalks; in general, these morphisms are *neither injective or surjective*. It is functorial; indeed, if **0_I** | 30

$u : \mathscr{F}_1 \to \mathscr{F}_2$ is a morphism of presheaves on X with values in $\mathtt{C}$, the diagram

$$\begin{tikzcd}
(\psi_*(\mathscr{F}_1))_{\psi(x)} \arrow[r, "\psi_x"] \arrow[d, "(\psi_*(u))_{\psi(x)}"'] & (\mathscr{F}_1)_x \arrow[d, "u_x"] \\
(\psi_*(\mathscr{F}_2))_{\psi(x)} \arrow[r, "\psi_x"] & (\mathscr{F}_2)_x
\end{tikzcd}$$

is commutative. If Z is a third topological space, $\psi' : Y \to Z$ a continuous map, and $\psi'' = \psi' \circ \psi$, then we have $\psi''_x = \psi_x \circ \psi'_{\psi(x)}$ for $x \in X$.

(3.4.5). Under the hypotheses of (3.4.4), suppose in addition that ψ is a *homeomorphism* from X to the subspace $\psi(X)$ of Y. Then, for each $x \in X$, ψ_x is an *isomorphism*. This applies in particular to the canonical injection j of a subset X of Y into Y.

(3.4.6). Suppose that $\mathtt{C}$ be the category of groups, or of rings, etc. If $\mathscr{F}$ is a sheaf on X with values in $\mathtt{C}$, of support S, and if $y \notin \overline{\psi(S)}$, then it follows from the definition of $\psi_*(\mathscr{F})$ that $(\psi_*(\mathscr{F}))_y = \{0\}$, or in other words, that the support of $\psi_*(\mathscr{F})$ is contained in $\overline{\psi(S)}$; but it is not necessarily contained in $\psi(S)$. Under the same hypotheses, if j is the canonical injection of a subset X of Y into Y, the sheaf $j_*(\mathscr{F})$ induces $\mathscr{F}$ on X; if moreover X is *closed* in Y, $j_*(\mathscr{F})$ is the sheaf on Y which induces $\mathscr{F}$ on X and 0 on $Y - X$ (G, II, 2.9.2), but it is in general distinct from the latter when we suppose that X is locally closed but not closed.

3.5. Inverse images of presheaves

(3.5.1). Under the hypotheses of (3.4.1), if $\mathscr{F}$ (resp. $\mathscr{G}$) is a presheaf on X (resp. Y) with values in $\mathtt{C}$, then each morphism $u : \mathscr{G} \to \psi_*(\mathscr{F})$ of presheaves on Y is called a ϕ-*morphism* from $\mathscr{G}$ to $\mathscr{F}$, and we denote it also by $\mathscr{G} \to \mathscr{F}$. We denote also by $\mathrm{Hom}_\phi(\mathscr{G}, \mathscr{F})$ the set of $\mathrm{Hom}_Y(\mathscr{G}, \psi_*(\mathscr{F}))$ the ψ-morphisms from $\mathscr{G}$ to $\mathscr{F}$. For each pair (U, V), where U is an open set of X, V an open set of Y such that $\psi(U) \subset V$, we have a morphism $u_{U,V} : \mathscr{G}(V) \to \mathscr{F}(U)$ by composing the restriction morphism $\mathscr{F}(\psi^{-1}(V)) \to \mathscr{F}(U)$ and the morphism $u_V : \mathscr{G}(V) \to \psi_*(\mathscr{F})(V) = \mathscr{F}(\psi^{-1}(V))$; it is immediate that these morphisms render commutative the diagrams

(3.5.1.1)
$$\begin{tikzcd}
\mathscr{G}(V) \arrow[r, "u_{U,V}"] \arrow[d] & \mathscr{F}(U) \arrow[d] \\
\mathscr{G}(V') \arrow[r, "u_{U',V'}"] & \mathscr{F}(U')
\end{tikzcd}$$

for $U' \subset U$, $V' \subset V$, $\psi(U') \subset V'$. Conversely, the data of a family $(u_{U,V})$ of morphisms rendering commutative the diagrams (3.5.1.1) define a ψ-morphism u, since it suffices to take $u_V = u_{\psi^{-1}(V),V}$.

If the category $\mathtt{C}$ admits (generalized) projective limits, and if $\mathfrak{B}$, $\mathfrak{B}'$ are bases for the topologies $\mathbf{0_I}$ | 31
of X and Y respectively, to define a ψ-morphism u of *sheaves*, we can restrict to giving the $u_{U,V}$ for $U \in \mathfrak{B}$, $V \in \mathfrak{B}'$, and $\psi(U) \subset V$, satisfying the compatibility conditions of (3.5.1.1) for U, U' in $\mathfrak{B}$ and V, V' in $\mathfrak{B}'$; it indeed suffices to define u_W, for each open $W \subset Y$, as the projective limit of the $u_{U,V}$ for $V \in \mathfrak{B}'$ and $V \subset W$, $U \in \mathfrak{B}$ and $\psi(U) \subset V$.

When the category $\mathtt{C}$ admits inductive limits, we have, for each $x \in X$, a morphism $\mathscr{G}(V) \to \mathscr{F}(\psi^{-1}(V)) \to \mathscr{F}_x$, for each open neighbourhood V of $\psi(x)$ in Y, and these morphisms form an inductive system which gives by passing to the limit a morphism $\mathscr{G}_{\psi(x)} \to \mathscr{F}_x$.

(3.5.2). Under the hypotheses of (3.4.3), let $\mathscr{F}$, $\mathscr{G}$, $\mathscr{H}$ be presheaves with values in $\mathtt{C}$ on X, Y, Z respectively, and let $u : \mathscr{G} \to \psi_*(\mathscr{F})$, $v : \mathscr{H} \to \psi'_*(\mathscr{G})$ be a ψ-morphism and a ψ'-morphism respectively. We obtain a ψ''-morphism $w : \mathscr{H} \xrightarrow{v} \psi'_*(\mathscr{G}) \xrightarrow{\psi'_*(u)} \psi'_*(\psi_*(\mathscr{F})) = \psi''_*(\mathscr{F})$, that we call, by definition, the *composition* of u and v. We can therefore consider the pairs $(X, \mathscr{F})$ consisting of a topological space X and a presheaf $\mathscr{F}$ on X (with values in $\mathtt{C}$) as forming a *category*, the morphisms being the pairs $(\psi, \theta) : (X, \mathscr{F}) \to (Y, \mathscr{G})$ consisting of a continuous map $\psi : X \to Y$ and of a ψ-morphism $\theta : \mathscr{G} \to \mathscr{F}$.

(3.5.3). Let $\psi : X \to Y$ be a continuous map, $\mathscr{G}$ a *presheaf* on Y with values in $\mathtt{C}$. We call the *inverse image of* $\mathscr{G}$ *under* ψ the pair $(\mathscr{G}', \rho)$, where $\mathscr{G}'$ is a *sheaf* on X with values in $\mathtt{C}$, and $\rho : \mathscr{G} \to \mathscr{G}'$ a ψ-morphism (in other words a homomorphism $\mathscr{G} \to \psi_*(\mathscr{G}')$) such that, for each *sheaf* $\mathscr{F}$ on X with values in $\mathtt{C}$, the map

$$\mathrm{Hom}_X(\mathscr{G}', \mathscr{F}) \longrightarrow \mathrm{Hom}_\psi(\mathscr{G}, \mathscr{F}) \longrightarrow \mathrm{Hom}_Y(\mathscr{G}, \psi_*(\mathscr{F})) \tag{3.5.3.1}$$

sending v to $\psi_*(v) \circ \rho$, is a *bijection*; this map, being functorial in $\mathscr{F}$, then defines an isomorphism of functors in $\mathscr{F}$. The pair $(\mathscr{G}', \rho)$ is the solution of a universal problem, and we say it is *determined up to unique isomorphism* when it exists. We then write $\mathscr{G}' = \psi^*(\mathscr{G})$, $\rho = \rho_{\mathscr{G}}$, and by abuse of language, we say that $\psi^*(\mathscr{G})$ is *the inverse image sheaf* of $\mathscr{G}$ under ψ, and we agree that $\psi^*(\mathscr{G})$ is considered as equipped with a *canonical* ψ*-morphism* $\rho_{\mathscr{G}} : \mathscr{G} \to \psi^*(\mathscr{G})$, that is to say the *canonical homomorphism* of presheaves on Y:

$$\rho_{\mathscr{G}} : \mathscr{G} \longrightarrow \psi_*(\psi^*(\mathscr{G})). \tag{3.5.3.2}$$

For each homomorphism $v : \psi^*(\mathscr{G}) \to \mathscr{F}$ (where $\mathscr{F}$ is a sheaf on X with values in $\mathtt{C}$), we put $v^\flat = \psi_*(v) \circ \rho_{\mathscr{G}} : \mathscr{G} \to \psi_*(\mathscr{F})$. By definition, *each* morphism of presheaves $u : \mathscr{G} \to \psi_*(\mathscr{F})$ is of the form $v^\flat$ for a unique v, which we will denote $u^\sharp$. In other words, each morphism $u : \mathscr{G} \to \psi_*(\mathscr{F})$ of presheaves factorizes in a unique way as

$$u : \mathscr{G} \xrightarrow{\rho_{\mathscr{G}}} \psi_*(\psi^*(\mathscr{G})) \xrightarrow{\psi_*(u^\sharp)} \psi_*(\mathscr{F}). \tag{3.5.3.3}$$

$\mathbf{0_I}$ | 32

(3.5.4). Suppose now that the category $\mathtt{C}$ be such[7] that *each* presheaf $\mathscr{F}$ on Y with values in $\mathtt{C}$ admits an inverse image under ψ, and we denote it by $\psi^*(\mathscr{G})$.

We will see that we can define $\psi^*(\mathscr{G})$ as a *covariant functor* in $\mathscr{G}$, from the category of presheaves on Y with values in $\mathtt{C}$, to that of sheaves on X with values in $\mathtt{C}$, in such a way that the isomorphism $v \mapsto v^\flat$ is an *isomorphism of bifunctors*

$$\mathrm{Hom}_X(\psi^*(\mathscr{G}), \mathscr{F}) \simeq \mathrm{Hom}_Y(\mathscr{G}, \psi_*(\mathscr{F})) \tag{3.5.4.1}$$

in $\mathscr{G}$ and $\mathscr{F}$.

Indeed, for each morphism $w : \mathscr{G}_1 \to \mathscr{G}_2$ of presheaves on Y with values in $\mathtt{C}$, consider the composite morphism $\mathscr{G}_1 \xrightarrow{w} \mathscr{G}_2 \xrightarrow{\rho_{\mathscr{G}_2}} \psi_*(\psi^*(\mathscr{G}_2))$; to it corresponds a morphism $(\rho_{\mathscr{G}_2} \circ w)^\sharp : \psi^*(\mathscr{G}_1) \to \psi^*(\mathscr{G}_2)$, that we denote by $\psi^*(w)$. We therefore have, according to (3.5.3.3),

$$\psi_*(\psi^*(w)) \circ \rho_{\mathscr{G}_1} = \rho_{\mathscr{G}_2} \circ w. \tag{3.5.4.2}$$

For each morphism $u : \mathscr{G}_2 \to \psi_*(\mathscr{F})$, where $\mathscr{F}$ is a sheaf on X with values in $\mathtt{C}$, we have, according to (3.5.3.3), (3.5.4.2), and the definition of $u^\flat$, that

$$(u^\sharp \circ \psi^*(w))^\flat = \psi_*(u^\sharp) \circ \psi_*(\psi^*(w)) \circ \rho_{\mathscr{G}_1} = \psi_*(u^\sharp) \circ \rho_{\mathscr{G}_2} \circ w = u \circ w$$

where again

$$(u \circ w)^\sharp = u^\sharp \circ \psi^*(w). \tag{3.5.4.3}$$

If we take in particular for u a morphism $\mathscr{G}_2 \xrightarrow{w'} \mathscr{G}_3 \xrightarrow{\rho_{\mathscr{G}_3}} \psi_*(\psi^*(\mathscr{G}_3))$, it becomes $\psi^*(w' \circ w) = (\rho_{\mathscr{G}_3} \circ w' \circ w)^\sharp = (\rho_{\mathscr{G}_3} \circ w')^\sharp \circ \psi^*(w) = \psi^*(w') \circ \psi^*(w)$, hence our assertion.

Finally, for each sheaf $\mathscr{F}$ on X with values in $\mathtt{C}$, let $i_{\mathscr{F}}$ be the identity morphism of $\psi_*(\mathscr{F})$ and denote by

$$\sigma_{\mathscr{F}} : \psi^*(\psi_*(\mathscr{F})) \longrightarrow \mathscr{F}$$

the morphism $(i_{\mathscr{F}})^\sharp$; the formula (3.5.4.3) gives in particular the factorization

$$u^\sharp : \psi^*(\mathscr{G}) \xrightarrow{\psi^*(u)} \psi^*(\psi_*(\mathscr{F})) \xrightarrow{\sigma_{\mathscr{F}}} \mathscr{F} \tag{3.5.4.4}$$

for each morphism $u : \mathscr{G} \to \psi_*(\mathscr{F})$. We say that the morphism $\sigma_{\mathscr{F}}$ is *canonical*.

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[7] In the book mentioned in the introduction, we will give very general conditions on the category $\mathtt{C}$ ensuring the existence of inverse images of presheaves with values in $\mathtt{C}$.

(3.5.5). Let $\psi' : Y \to Z$ be a continuous map, and suppose that each presheaf $\mathscr{H}$ on Z with values in $\mathtt{C}$ admits an inverse image $\psi'^*(\mathscr{H})$ under ψ'. Then (with the hypotheses of (3.5.4)) each presheaf $\mathscr{H}$ on Z with values in $\mathtt{C}$ admits an inverse image under $\psi'' = \psi' \circ \psi$ and we have a canonical functorial isomorphism

$$\psi''^*(\mathscr{H}) \simeq \psi^*(\psi'^*(\mathscr{H})). \tag{3.5.5.1}$$

This indeed follows immediately from the definitions, taking into account that $\psi''_* = \psi'_* \circ \psi_*$. In addition, if $u : \mathscr{G} \to \psi_*(\mathscr{F})$ is a ψ-morphism, $v : \mathscr{H} \to \psi'_*(\mathscr{G})$ a ψ'-morphism, and $w = \psi'_*(u) \circ v$ their composition (3.5.2), then we have immediately that $w^\sharp$ is the composite morphism **0_{I} | 33**

$$w^\sharp : \psi^*(\psi'^*(\mathscr{H})) \xrightarrow{\psi^*(v^\sharp)} \psi^*(\mathscr{G}) \xrightarrow{u^\sharp} \mathscr{F}.$$

(3.5.6). We take in particular for ψ the identity map $1_X : X \to X$. Then if the inverse image under ψ of a presheaf $\mathscr{F}$ on X with values in $\mathtt{C}$ exists, we say that this inverse image is the *sheaf associated to the presheaf* $\mathscr{F}$. Each morphism $u : \mathscr{F} \to \mathscr{F}'$ from $\mathscr{F}$ to a *sheaf* $\mathscr{F}'$ with values in $\mathtt{C}$ factorizes in a unique way as $\mathscr{F} \xrightarrow{\rho_{\mathscr{F}}} 1_X^*(\mathscr{F}) \xrightarrow{u^\sharp} \mathscr{F}'$.

3.6. Simple and locally simple sheaves

(3.6.1). We say that a *presheaf* $\mathscr{F}$ on X, with values in $\mathtt{C}$, is *constant* if the canonical morphisms $\mathscr{F}(X) \to \mathscr{F}(U)$ are *isomorphisms* for each nonempty open $U \subset X$; we note that $\mathscr{F}$ is not necessarily a sheaf. We say that a *sheaf* is *simple* if it is the associated sheaf (3.5.6) of a constant presheaf. We say that a sheaf $\mathscr{F}$ is *locally simple* if each $x \in X$ admits an open neighbourhood U such that $\mathscr{F}|U$ is simple.

(3.6.2). Suppose that X is *irreducible* (2.1.1); then the following properties are equivalent:

(a) *$\mathscr{F}$ is a constant presheaf on X;*
(b) *$\mathscr{F}$ is a simple sheaf on X;*
(c) *$\mathscr{F}$ is a locally simple sheaf on X.*

Indeed, let $\mathscr{F}$ be a constant presheaf on X; if U, V are two nonempty open sets in X, then $U \cap V$ is nonempty, so $\mathscr{F}(X) \to \mathscr{F}(U) \to \mathscr{F}(U \cap V)$ and $\mathscr{F}(X) \to \mathscr{F}(U)$ are isomorphisms, and similarly both $\mathscr{F}(U) \to \mathscr{F}(U \cap V)$ and $\mathscr{F}(V) \to \mathscr{F}(U \cap V)$ are isomorphisms. We therefore conclude immediately that the axiom (F) of (3.1.2) is clearly satisfied, $\mathscr{F}$ is isomorphic to its associated sheaf, and as a result (a) implies (b).

Now let (U_α) be an open cover of X by nonempty open sets and let $\mathscr{F}$ be a sheaf on X such that $\mathscr{F}|U_\alpha$ is simple for each α; as U_α is irreducible, $\mathscr{F}|U_\alpha$ is a constant presheaf according to the above. As $U_\alpha \cap U_\beta$ is not empty, $\mathscr{F}(U_\alpha) \to \mathscr{F}(U_\alpha \cap U_\beta)$ and $\mathscr{F}(U_\beta) \to \mathscr{F}(U_\alpha \cap U_\beta)$ are isomorphisms, hence we have a canonical isomorphism $\theta_{\alpha\beta} : \mathscr{F}(U_\alpha) \to \mathscr{F}(U_\beta)$ for each pair of indices. But then if we apply the condition (F) for $U = X$, we see that for each index α_0, $\mathscr{F}(U_{\alpha_0})$ and the $\theta_{\alpha_0\alpha}$ are solutions to the universal problem, which (according to the uniqueness) implies that $\mathscr{F}(X) \to \mathscr{F}(U_{\alpha_0})$ is an isomorphism, and hence proves that (c) implies (a).

3.7. Inverse images of presheaves of groups or rings **0_{I} | 34**

(3.7.1). We will show that when we take $\mathtt{C}$ to be the category of sets, the inverse image under ψ for each presheaf $\mathscr{G}$ with values in $\mathtt{C}$ *always exists* (the notation and hypotheses on X, Y, ψ being that of (3.5.3)). Indeed, for each open $U \subset X$, define $\mathscr{G}'(U)$ as follows: an element s' of $\mathscr{G}'(U)$ is a family $(s'_x)_{x \in U}$, where $s'_x \in \mathscr{G}_{\psi(x)}$ for each $x \in U$, and where, for each $x \in U$, the following condition is satisfied: there exists an open neighbourhood V of $\psi(x)$ in Y, a neighbourhood $W \subset \psi^{-1}(V) \cap U$ of x, and an element $s \in \mathscr{G}(V)$ such that $s'_z = s_{\psi(x)}$ for all $z \in W$. We verify immediately that $U \mapsto \mathscr{G}'(U)$ clearly satisfies the axioms of a *sheaf*.

Now let $\mathscr{F}$ be a sheaf of sets on X, and let $u : \mathscr{G} \to \psi_*(\mathscr{F})$, $v : \mathscr{G}' \to \mathscr{F}$ be morphisms. We define $u^\sharp$ and $v^\flat$ in the following manner: if s' is a section of $\mathscr{G}'$ over a neighbourhood U of $x \in X$ and if V is an open neighbourhood of $\psi(x)$ and $s \in \mathscr{G}(V)$ such that we have $s'_z = s_{\psi(x)}$ for z in a neighbourhood of x contained in $\psi^{-1}(V) \cap U$, we take $u^\sharp_x(s'_x) = u_{\psi(x)}(s_{\psi(x)})$. Similarly, if $s \in \mathscr{G}(V)$ (V open in Y), $v^\flat(s)$ is the section of $\mathscr{F}$ over $\psi^{-1}(V)$, the image under v of the section s' of $\mathscr{G}'$ such that $s'_x = s_{\psi(x)}$

for all $x \in \psi^{-1}(V)$. In addition, the canonical homomorphism (3.5.3) $\rho : \mathscr{G} \to \psi_*(\psi^*(\mathscr{G}))$ is defined in the following manner: for each open $V \subset Y$ and each section $s \in \Gamma(V, \mathscr{G})$, $\rho(s)$ is the section $(s_{\psi(x)})_{x\in\psi^{-1}(V)}$ of $\psi^*(\mathscr{G})$ over $\psi^{-1}(V)$. The verification of the relations $(u^\sharp)^\flat = u$, $(v^\flat)^\sharp = v$, and $v^\flat = \psi_*(v) \circ \rho$ is immediate, and proves our assertion.

We check that, if $w : \mathscr{G}_1 \to \mathscr{G}_2$ is a homomorphism of sheaves of sets on Y, $\psi^*(w)$ is expressed in the following manner: if $s' = (s'_x)_{x\in U}$ is a section of $\psi^*(\mathscr{G}_1)$ over an open set U of X, then $(\psi^*(w))(s')$ is the family $(w_{\psi(x)}(s'_x))_{x\in U}$. Finally, it is immediate that for each open set V of Y, the inverse image of $\mathscr{G}|V$ under the restriction of ψ to $\psi^{-1}(V)$ is identical to the induced sheaf $\psi^*(\mathscr{G})|\psi^{-1}(V)$.

When ψ is the identity 1_X, we recover the definition of a sheaf of sets associated to a presheaf (G, II, 1.2). The above considerations apply without change when $\mathtt{C}$ is the category of groups or of rings (not necessarily commutative).

When X is any subset of a topological space Y, and j the canonical injection $X \to Y$, for each sheaf $\mathscr{G}$ on Y with values in a category $\mathtt{C}$, we call the *induced* sheaf of X by $\mathscr{G}$ the inverse image $j^*(\mathscr{G})$ (whenever it exists); for the sheaves of sets (or of groups, or of rings) we recover the usual definition (G, II, 1.5).

(3.7.2). Keeping the notation and hypotheses of (3.5.3), suppose that $\mathscr{G}$ is a *sheaf* of groups (resp. of rings) on Y. The definition of sections of $\psi^*(\mathscr{G})$ (3.7.1) shows (taking into account (3.4.4)) that the homomorphism of stalks $\psi_x \circ \rho_{\psi(x)} : \mathscr{G}_{\psi(x)} \to (\psi^*(\mathscr{G}))_x$ is a *functorial isomorphism in* $\mathscr{G}$, that identifies the two stalks; with this identification, $u_x^\sharp$ is identical to the homomorphism defined in (3.5.1), and in particular, we have $\operatorname{Supp}(\psi^*(\mathscr{G})) = \psi^{-1}(\operatorname{Supp}(\mathscr{G}))$.

An immediate consequence of this result is that *the functor* $\psi^*(\mathscr{G})$ *is exact in* $\mathscr{G}$ on the abelian category of sheaves of abelian groups.

3.8. Sheaves on pseudo-discrete spaces $\mathbf{0_I}$ | 35

(3.8.1). Let X be a topological space whose topology admits a basis $\mathfrak{B}$ consisting of open *quasi-compact* subsets. Let $\mathscr{F}$ be a *sheaf of sets* on X; if we equip each of the $\mathscr{F}(U)$ with the *discrete* topology, $U \mapsto \mathscr{F}(U)$ is a *presheaf of topological spaces*. We will see that there exists a *sheaf of topological spaces* $\mathscr{F}'$ *associated to* $\mathscr{F}$ (3.5.6) such that $\Gamma(U, \mathscr{F}')$ is the discrete space $\mathscr{F}(U)$ for each open *quasi-compact* subsets U. It will suffice to show that the presheaf $U \mapsto \mathscr{F}(U)$ of discrete topological spaces *on* $\mathfrak{B}$ satisfy the condition (F_0) of (3.2.2), and more generally that if U is an open quasi-compact subset and if (U_α) is a cover of U by sets of $\mathfrak{B}$, then the least fine topology $\mathscr{T}$ on $\Gamma(U, \mathscr{F})$ renders continuous the maps $\Gamma(U, \mathscr{F}) \to \Gamma(U_\alpha, \mathscr{F})$ is the *discrete* topology. There exists a finite number of indices α_i such that $U = \bigcup_i U_{\alpha_i}$. Let $s \in \Gamma(U, \mathscr{F})$ and let s_i be its image in $\Gamma(U_{\alpha_i}, \mathscr{F})$; the intersection of the inverse images of the sets $\{s_i\}$ is by definition a neighbourhood of s for $\mathscr{T}$; but since $\mathscr{F}$ is a sheaf of sets and the U_{α_i} cover U, this intersection is reduced to s, hence our assertion.

We note that if U is an open non quasi-compact subset of X, the topological space $\Gamma(U, \mathscr{F}')$ still has $\Gamma(U, \mathscr{F})$ as the underlying set, but the topology is not discrete in general: it is the least fine rendering commutative the maps $\Gamma(U, \mathscr{F}) \to \Gamma(V, \mathscr{F})$, for $V \in \mathfrak{B}$ and $V \subset U$ (the $\Gamma(V, \mathscr{F})$ being discrete).

The above considerations apply without modification to sheaves of groups or of rings (not necessarily commutative), and associate to them sheaves of *topological groups* or *topological rings*, respectively. To summarize, we say that the sheaf $\mathscr{F}'$ is the *pseudo-discrete* sheaf of *spaces* (resp. *groups*, *rings*) associated to a sheaf of sets (resp. groups, rings) $\mathscr{F}$.

(3.8.2). Let $\mathscr{F}, \mathscr{G}$ be two sheaves of sets (resp. groups, rings) on X, $u : \mathscr{F} \to \mathscr{G}$ a homomorphism. Then u is thus a *continuous* homomorphism $\mathscr{F}' \to \mathscr{G}'$, if we denote by $\mathscr{F}'$ and $\mathscr{G}'$ the pseudo-discrete sheaves associated to $\mathscr{F}$ and $\mathscr{G}$; this follows in effect from (3.2.5).

(3.8.3). Let $\mathscr{F}$ be a sheaf of sets, $\mathscr{H}$ a subsheaf of $\mathscr{F}$, $\mathscr{F}'$ and $\mathscr{H}'$ the pseudo-discrete sheaves associated to $\mathscr{F}$ and $\mathscr{H}$ respectively. Then, for each open $U \subset X$, $\Gamma(U, \mathscr{H}')$ is *closed* in $\Gamma(U, \mathscr{F}')$: indeed, it is the intersection of the inverse images of the $\Gamma(V, \mathscr{H})$ (for $V \in \mathfrak{B}$, $V \subset U$) under the continuous maps $\Gamma(U, \mathscr{F}) \to \Gamma(V, \mathscr{F})$, and $\Gamma(V, \mathscr{H})$ is closed in the discrete space $\Gamma(V, \mathscr{F})$.

§4. RINGED SPACES

4.1. Ringed spaces, sheaves of $\mathscr{A}$-modules, $\mathscr{A}$-algebras

(4.1.1). A *ringed space* (resp. topologically ringed space) is a pair $(X,\mathscr{A})$ consisting of a topological space X and a sheaf of (not necessarily commutative) rings (resp. of a sheaf of topological rings) $\mathscr{A}$ on X; we say that X is the *underlying* topological space of the ringed space $(X,\mathscr{A})$, and $\mathscr{A}$ the *structure sheaf*. The latter is denoted $\mathscr{O}_X$, and its stalk at a point $x\in X$ is denoted $\mathscr{O}_{X,x}$ or simply $\mathscr{O}_x$ when there is no chance of confusion.

We denote by 1 or e the *unit section* of $\mathscr{O}_X$ over X (the unit element of $\Gamma(X,\mathscr{O}_X)$).

As in this treatise we will have to consider in particular sheaves of *commutative* rings, it will be understood, when we speak of a ringed space $(X,\mathscr{A})$ without specification, that $\mathscr{A}$ is a sheaf of commutative rings.

The ringed spaces with not-necessarily-commutative structure sheaves (resp. the topologically ringed spaces) form a *category*, where we define a *morphism* $(X,\mathscr{A})\to(Y,\mathscr{B})$ as a couple $(\psi,\theta)=\Psi$ consisting of a continuous map $\psi: X\to Y$ and a ψ-*morphism* $\theta:\mathscr{B}\to\mathscr{A}$ (3.5.1) of sheaves of rings (resp. of sheaves of topological rings); the *composition* of a second morphism $\Psi'=(\psi',\theta'):(Y,\mathscr{B})\to(Z,\mathscr{C})$ and of Ψ, denoted $\Psi''=\Psi'\circ\Psi$, is the morphism (ψ'',θ'') where $\psi''=\psi'\circ\psi$, and θ'' is the composition of θ and θ' (equal to $\psi'_*(\theta)\circ\theta'$, cf. (3.5.2)). For ringed spaces, remember that we then have $\theta''^\sharp=\theta^\sharp\circ\psi^*(\theta'^\sharp)$ (3.5.5); therefore if $\theta'^\sharp$ and $\theta^\sharp$ are *injective* (resp. *surjective*), then the same is true of $\theta''^\sharp$, taking into account that $\psi_x\circ\rho_{\psi(x)}$ is an isomorphism for all $x\in X$ (3.7.2). We verify immediately, thanks to the above, that when ψ is an *injective* continuous map and when $\theta^\sharp$ is a *surjective* homomophism of sheaves of rings, the morphism (ψ,θ) is a *monomorphism* (T, 1.1) in the category of ringed spaces. **Err**$_{\mathrm{II}}$

By abuse of language, we will often replace ψ by Ψ in notation, for example in writing $\Psi^{-1}(U)$ in place of $\psi^{-1}(U)$ for a subset U of Y, when the is no risk of confusion.

(4.1.2). For each subset M of X, the pair $(M,\mathscr{A}|M)$ is evidently a ringed space, said to be *induced* on M by the ringed space $(X,\mathscr{A})$ (and is still called the *restriction* of $(X,\mathscr{A})$ to M). If j is the canonical injection $M\to X$ and ω is the identity map of $\mathscr{A}|M$, $(j,\omega^\flat)$ is a monomorphism $(M,\mathscr{A}|M)\to(X,\mathscr{A})$ of ringed spaces, called the *canonical injection*. The composition of a morphism $\Psi:(X,\mathscr{A})\to(Y,\mathscr{B})$ and this injection is called the *restriction* of Ψ to M.

(4.1.3). We will not revisit the definitions of $\mathscr{A}$-*modules* or *algebraic sheaves* on a ringed space $(X,\mathscr{A})$ (G, II, 2.2); when $\mathscr{A}$ is a sheaf of not necessarily commutative rings, by $\mathscr{A}$-module we will always mean "left $\mathscr{A}$-module" unless expressly stated otherwise. The $\mathscr{A}$-submodules of $\mathscr{A}$ will be called *sheaves of ideals* (left, right, or two-sided) in $\mathscr{A}$ or $\mathscr{A}$-*ideals*.

When $\mathscr{A}$ is a sheaf of commutative rings, and in the definition of $\mathscr{A}$-modules, we replace everywhere the *module* structure by that of an *algebra*, we obtain the definition of an $\mathscr{A}$-*algebra* on X. It is the same to say that an $\mathscr{A}$-algebra (not necessarily commutative) is a $\mathscr{A}$-module $\mathscr{C}$, given with a homomorphism of $\mathscr{A}$-modules $\phi:\mathscr{C}\otimes_{\mathscr{A}}\mathscr{C}\to\mathscr{C}$ and a section e over X, such that: 1st the diagram

$$\begin{tikzcd}
\mathscr{C}\otimes_{\mathscr{A}}\mathscr{C}\otimes_{\mathscr{A}}\mathscr{C} \arrow[r, "\phi\otimes 1"] \arrow[d, "1\otimes\phi"'] & \mathscr{C}\otimes_{\mathscr{A}}\mathscr{C} \arrow[d, "\phi"] \\
\mathscr{C}\otimes_{\mathscr{A}}\mathscr{C} \arrow[r, "\phi"] & \mathscr{C}
\end{tikzcd}$$

is commutative; 2nd for each open $U\subset X$ and each section $s\in\Gamma(U,\mathscr{C})$, we have $\phi((e|U)\otimes s)=\phi(s\otimes(e|U))=s$. We say that $\mathscr{C}$ is a commutative $\mathscr{A}$-algebra if the diagram

$$\begin{tikzcd}
\mathscr{C}\otimes_{\mathscr{A}}\mathscr{C} \arrow[rr, "\sigma"] \arrow[dr, "\phi"'] & & \mathscr{C}\otimes_{\mathscr{A}}\mathscr{C} \arrow[dl, "\phi"] \\
& \mathscr{C} &
\end{tikzcd}$$

is commutative, σ denoting the canonical symmetry (twist) map of the tensor product $\mathscr{C}\otimes_{\mathscr{A}}\mathscr{C}$.

The homomorphisms of $\mathscr{A}$-algebras are also defined as the homomorphisms of $\mathscr{A}$-modules in (G, II, 2.2), but naturally no longer form an abelian group.

If $\mathscr{M}$ is an $\mathscr{A}$-submodule of an $\mathscr{A}$-algebra $\mathscr{C}$, the $\mathscr{A}$-*subalgebra of* $\mathscr{C}$ *generated by* $\mathscr{M}$ is the sum of the images of the homomorphisms $\bigotimes^n\mathscr{M}\to\mathscr{C}$ (for each $n\geqslant 0$). This is also the sheaf associated to the presheaf $U \mapsto \mathscr{B}(U)$ of algebras, $\mathscr{B}(U)$ being the subalgebra of $\Gamma(U, \mathscr{C})$ generated by the submodule $\Gamma(U, \mathscr{M})$.

(4.1.4). We say that a sheaf of rings $\mathscr{A}$ on a topological space X is *reduced at a point x in X* if the stalk $\mathscr{A}_x$ is a *reduced* ring (1.1.1); we say that $\mathscr{A}$ is *reduced* if it is reduced at all points of X. Recall that a ring A is called *regular* if each of the local rings $A_{\mathfrak{p}}$ (where $\mathfrak{p}$ varies over the set of prime ideals of A) is a regular local ring; we will say that a sheaf of rings $\mathscr{A}$ on X is *regular at a point x* (resp. *regular*) if the stalk $\mathscr{A}_x$ is a regular ring (resp. if $\mathscr{A}$ is regular at each point). Finally, we will say that a sheaf of rings $\mathscr{A}$ on X is *normal at a point x* (resp. *normal*) if the stalk $\mathscr{A}_x$ is an *integral and integrally closed* ring (resp. if $\mathscr{A}$ is normal at each point). We will say that a ringed space $(X, \mathscr{A})$ has any of these preceeding properties if the sheaf of rings $\mathscr{A}$ has that property.

A *graded* sheaf of rings $\mathscr{A}$ is by definition a sheaf of rings that is the direct sum (G, II, 2,7) of a family $(\mathscr{A}_n)_{n \in \mathbf{Z}}$ of sheaves of abelian groups with the conditions $\mathscr{A}_m \mathscr{A}_n \subset \mathscr{A}_{m+n}$; a *graded $\mathscr{A}$-module* is an $\mathscr{A}$-module $\mathscr{F}$ that is the direct sum of a family $(\mathscr{F}_n)_{n \in \mathbf{Z}}$ of sheaves of abelian groups, satisfying the conditions $\mathscr{A}_m \mathscr{F}_n \subset \mathscr{F}_{m+n}$. It is equivalent to say that $(\mathscr{A}_m)_x (\mathscr{A}_n)_x \subset (\mathscr{A}_{m+n})_x$ (resp. $(\mathscr{A}_m)_x (\mathscr{F}_n)_x \subset (\mathscr{F}_{m+n})_x$) for each point x.

(4.1.5). Given a ringed space $(X, \mathscr{A})$ (not necessarily commutative), we will not recall here the definitions of the bifunctors $\mathscr{F} \otimes_{\mathscr{A}} \mathscr{G}$, $\mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{F})$, and $\operatorname{Hom}_{\mathscr{A}}(\mathscr{F}, \mathscr{G})$ (G, II, 2.8 and 2.2) in the categories of left or right (depending on the case) $\mathscr{A}$-modules, with values in the category of sheaves of abelian groups (or more generally of $\mathscr{C}$-modules, if $\mathscr{C}$ is the center of $\mathscr{A}$). The stalk $(\mathscr{F} \otimes_{\mathscr{A}} \mathscr{G})_x$ for each point $x \in X$ canonically identifies with $\mathscr{F}_x \otimes_{\mathscr{A}_x} \mathscr{G}_x$ and we define a canonical and functorial homomorphism $(\mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{G}))_x \to \operatorname{Hom}_{\mathscr{A}_x}(\mathscr{F}_x, \mathscr{G}_x)$ which is in general neither injective nor surjective. The bifunctors considered above are additive and in particular, commute with finite direct limits; $\mathscr{F} \otimes_{\mathscr{A}} \mathscr{G}$ is right exact in $\mathscr{F}$ and in $\mathscr{G}$, commutes with inductive limits, and $\mathscr{A} \otimes_{\mathscr{A}} \mathscr{G}$ (resp. $\mathscr{F} \otimes_{\mathscr{A}} \mathscr{A}$) canonically identifies with $\mathscr{G}$ (resp. $\mathscr{F}$). The functors $\mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{G})$ and $\operatorname{Hom}_{\mathscr{A}}(\mathscr{F}, \mathscr{G})$ are *left exact* in $\mathscr{F}$ and $\mathscr{G}$; more precisely, if we have an exact sequence of the form $0 \to \mathscr{G}' \to \mathscr{G} \to \mathscr{G}''$, the sequence

$$0 \longrightarrow \mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{G}') \longrightarrow \mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{G}) \longrightarrow \mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{G}'')$$

is exact, and if we have an exact sequence of the form $\mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$, the sequence

$$0 \longrightarrow \mathscr{H}om_{\mathscr{A}}(\mathscr{F}'', \mathscr{G}) \longrightarrow \mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{G}) \longrightarrow \mathscr{H}om_{\mathscr{A}}(\mathscr{F}', \mathscr{G})$$

is exact, with the analogous properties for the functor Hom. In addiiton, $\mathscr{H}om_{\mathscr{A}}(\mathscr{A}, \mathscr{G})$ canonically identifies with $\mathscr{G}$; finally, for each open $U \subset X$, we have

$$\Gamma(U, \mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{G}) = \operatorname{Hom}_{\mathscr{A}|U}(\mathscr{F}|U, \mathscr{G}|U).$$

For each left (resp. right) $\mathscr{A}$-module, we define the *dual* of $\mathscr{F}$ and denote it by $\mathscr{F}^{\vee}$ the right (resp. left) $\mathscr{A}$-module $\mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{A})$.

Finally, if $\mathscr{A}$ is a sheaf of commutative rings, $\mathscr{F}$ an $\mathscr{A}$-module, $U \mapsto \wedge^p \Gamma(U, \mathscr{F})$ is a presheaf whose associated sheaf is an $\mathscr{A}$-module denoted $\wedge^p \mathscr{F}$ and is called the *p-th exterior power of $\mathscr{F}$*; we verify easily that the canonical map of the presheaf $U \mapsto \wedge^p \Gamma(U, \mathscr{F})$ to the associated sheaf $\wedge^p \mathscr{F}$ is *injective*, and for each $x \in X$, $(\wedge^p \mathscr{F})_x = \wedge^p(\mathscr{F}_x)$. It is clear that $\wedge^p \mathscr{F}$ is a covariant functor in $\mathscr{F}$.

(4.1.6). Suppose that $\mathscr{A}$ is a sheaf of not-necessarily-commutative rings, $\mathscr{J}$ a left sheaf of ideals of $\mathscr{A}$, $\mathscr{F}$ an left $\mathscr{A}$-module; we then denote by $\mathscr{J}\mathscr{F}$ the $\mathscr{A}$-submodule of $\mathscr{F}$, the image of $\mathscr{J} \otimes_{\mathbf{Z}} \mathscr{F}$ (where $\mathbf{Z}$ is the sheaf associated to the constant presheaf $U \mapsto \mathbf{Z}$) under the canonical map $\mathscr{J} \otimes_{\mathbf{Z}} \mathscr{F} \to \mathscr{F}$; it is clear that for each $x \in X$, we have $(\mathscr{J}\mathscr{F})_x = \mathscr{J}_x \mathscr{F}_x$. When $\mathscr{A}$ is commutative, $\mathscr{J}\mathscr{F}$ is also the canonical image of $\mathscr{J} \otimes_{\mathscr{A}} \mathscr{F} \to \mathscr{F}$. It is immediate that $\mathscr{J}\mathscr{F}$ is also the $\mathscr{A}$-module associated to the presheaf $U \mapsto \Gamma(U, \mathscr{J})\Gamma(U, \mathscr{F})$. If $\mathscr{J}_1$, $\mathscr{J}_2$ are two left sheaves of ideals of $\mathscr{A}$, we have $\mathscr{J}_1(\mathscr{J}_2\mathscr{F}) = (\mathscr{J}_1\mathscr{J}_2)\mathscr{F}$.

(4.1.7). Let $(X_\lambda, \mathscr{A}_\lambda)_{\lambda \in L}$ be a family of ringed spaces; for each couple (λ, μ), suppose we are given an open subset $V_{\lambda\mu}$ of X_λ, and an isomorphism of ringed spaces $\phi_{\lambda\mu} : (V_{\mu\lambda}, \mathscr{A}_\mu|V_{\lambda\mu}) \simeq (V_{\lambda\mu}, \mathscr{A}_\lambda|V_{\lambda\mu})$, with $V_{\lambda\lambda} = X_\lambda$, $\phi_{\lambda\lambda}$ being the identity. Furthermore, suppose that, for each triple (λ, μ, ν), if we denote by $\phi'_{\mu\lambda}$ the restriction of $\phi_{\mu\lambda}$ to $V_{\lambda\mu} \cap V_{\lambda\nu}$, $\phi'_{\mu\lambda}$ is an isomorphism from $(V_{\lambda\mu} \cap V_{\lambda\nu}, \mathscr{A}_\lambda|(V_{\lambda\mu} \cap V_{\lambda\nu}))$ to $(V_{\mu\nu} \cap V_{\mu\lambda}, \mathscr{A}_\mu|(V_{\mu\nu} \cap V_{\mu\lambda}))$ and that we have $\phi'_{\lambda\nu} = \phi'_{\lambda\mu} \circ \phi'_{\mu\nu}$ (*gluing condition* for the $\phi_{\lambda\mu}$).

We can first consider the topological space obtained by gluing (by means of the $\phi_{\lambda\mu}$) of the X_λ along the $V_{\lambda\mu}$; if we identify X_λ with the corresponding open subset X'_λ in X, the hypotheses imply that the three sets $V_{\lambda\mu} \cap V_{\lambda\nu}$, $V_{\mu\nu} \cap V_{\mu\lambda}$, $V_{\nu\lambda} \cap V_{\nu\mu}$ identify with $X'_\lambda \cap X'_\mu \cap X'_\nu$. We can also transport to X'_λ the ringed space structure of X_λ, and if $\mathscr{A}'_\lambda$ are the transported sheaves of rings corresponding to the $\mathscr{A}_\lambda$, the $\mathscr{A}'_\lambda$ satisfy the gluing condition (3.3.1) and therefore define a sheaf of rings $\mathscr{A}$ on X; we say that $(X, \mathscr{A})$ is the ringed space obtained by *gluing the* $(X_\lambda, \mathscr{A}_\lambda)$ *along the* $V_{\lambda\mu}$, by means of the $\phi_{\lambda\mu}$. $\mathbf{0_I}$ | 39

4.2. Direct image of an $\mathscr{A}$-module

(4.2.1). Let $(X, \mathscr{A})$, $(Y, \mathscr{B})$ be two ringed spaces, $\Psi = (\psi, \theta)$ a morphism $(X, \mathscr{A}) \to (Y, \mathscr{B})$; $\psi_*(\mathscr{A})$ is then a sheaf of rings on Y, and θ a homomorphism $\mathscr{B} \to \psi_*(\mathscr{A})$ of sheaves of rings. Then let $\mathscr{F}$ be an $\mathscr{A}$-module; the direct image $\psi_*(\mathscr{F})$ is a sheaf of abelian groups on Y. In addition, for each open $U \subset Y$,

$$\Gamma(U, \psi_*(\mathscr{F})) = \Gamma(\psi^{-1}(U), \mathscr{F})$$

is equipped with the structure of a module over the ring $\Gamma(U, \psi_*(\mathscr{A})) = \Gamma(\psi^{-1}(U), \mathscr{A})$; the bilinear maps which define these structures are compatible with the restriction operations, defining on $\psi_*(\mathscr{F})$ the structure of a $\psi_*(\mathscr{A})$-module. The homomorphism $\theta : \mathscr{B} \to \psi_*(\mathscr{A})$ then defines also on $\psi_*(\mathscr{F})$ a *$\mathscr{B}$-module* structure; we say that this $\mathscr{B}$-module is the *direct image of $\mathscr{F}$ under the morphism* Ψ, and we denote it by $\Psi_*(\mathscr{F})$. If $\mathscr{F}_1$, $\mathscr{F}_2$ are two $\mathscr{A}$-modules over X and u an $\mathscr{A}$-homomorphism $\mathscr{F}_1 \to \mathscr{F}_2$, it is immediate (by considering the sections over the open subsets of Y) that $\psi_*(u)$ is a $\psi_*(\mathscr{A})$-homomorphism $\psi_*(\mathscr{F}_1) \to \psi_*(\mathscr{F}_2)$, and *a fortiori* a $\mathscr{B}$-homomorphism $\Psi_*(\mathscr{F}_1) \to \Psi_*(\mathscr{F}_2)$; as a $\mathscr{B}$-homomorphism, we denote it by $\Psi_*(u)$. So we see that Ψ_* is a *covariant functor* from the category of $\mathscr{A}$-modules to that of $\mathscr{B}$-modules. In addition, it is immediate that this functor is *left exact* (G, II, 2.12).

On $\psi_*(\mathscr{A})$, the structure of a $\mathscr{B}$-module and the structure of a sheaf of rings define a $\mathscr{B}$-algebra structure; we denote by $\Psi_*(\mathscr{A})$ this $\mathscr{B}$-algebra.

(4.2.2). Let $\mathscr{M}$, $\mathscr{N}$ be two $\mathscr{A}$-modules. For each open set U of Y, we have a canonical map

$$\Gamma(\psi^{-1}(U), \mathscr{M}) \times \Gamma(\psi^{-1}(U), \mathscr{N}) \longrightarrow \Gamma(\psi^{-1}(U), \mathscr{M} \otimes_{\mathscr{A}} \mathscr{N})$$

which is bilinear over the ring $\Gamma(\psi^{-1}(U), \mathscr{A}) = \Gamma(U, \psi_*(\mathscr{A}))$, and *a fortiori* over $\Gamma(U, \mathscr{B})$; it therefore defines a homomorphism

$$\Gamma(U, \Psi_*(\mathscr{M})) \otimes_{\Gamma(U,\mathscr{B})} \Gamma(U, \Psi_*(\mathscr{N})) \longrightarrow \Gamma(U, \Psi_*(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N}))$$

and as we check immediately that these homomorphisms are compatible with the restriction operations, they give a canonical functorial homomorphism of $\mathscr{B}$-modules

$$\Psi_*(\mathscr{M}) \otimes_{\mathscr{B}} \Psi_*(\mathscr{N}) \longrightarrow \Psi_*(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N}) \tag{4.2.2.1}$$

which is in general neither injective nor surjective. If $\mathscr{P}$ is a third $\mathscr{A}$-module, we check immediately that the diagram $\mathbf{0_I}$ | 40

$$\begin{tikzcd} \Psi_*(\mathscr{M}) \otimes_{\mathscr{B}} \Psi_*(\mathscr{N}) \otimes_{\mathscr{B}} \Psi_*(\mathscr{P}) \arrow[r] \arrow[d] & \Psi_*(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N}) \otimes_{\mathscr{B}} \Psi_*(\mathscr{P}) \arrow[d] \\ \Psi_*(\mathscr{M}) \otimes_{\mathscr{B}} \Psi_*(\mathscr{N} \otimes_{\mathscr{A}} \mathscr{P}) \arrow[r] & \Psi_*(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N} \otimes_{\mathscr{A}} \mathscr{P}) \end{tikzcd} \tag{4.2.2.2}$$

is commutative.

(4.2.3). Let $\mathscr{M}$, $\mathscr{N}$ be two $\mathscr{A}$-modules. For each open $U \subset Y$, we have by definition that $\Gamma(\psi^{-1}(U), \mathscr{H}om_{\mathscr{A}}(\mathscr{M}, \mathscr{N})) = \mathrm{Hom}_{\mathscr{A}|V}(\mathscr{M}|V, \mathscr{N}|V)$, where we put $V = \psi^{-1}(U)$; the map $u \mapsto \Psi_*(u)$ is a homomorphism

$$\mathrm{Hom}_{\mathscr{A}|V}(\mathscr{M}|V, \mathscr{N}|V) \longrightarrow \mathrm{Hom}_{\mathscr{B}|U}(\Psi_*(\mathscr{M})|U, \Psi_*(\mathscr{N})|U)$$

on the $\Gamma(U, \mathscr{B})$-module structures; these homomorphisms are compatible with the restriction operations, hence they define a canonical functorial homomorphism of $\mathscr{B}$-modules

$$\Psi_*(\mathscr{H}om_{\mathscr{A}}(\mathscr{M}, \mathscr{N})) \longrightarrow \mathscr{H}om_{\mathscr{B}}(\Psi_*(\mathscr{M}), \Psi_*(\mathscr{N})). \tag{4.2.3.1}$$

(4.2.4). If $\mathscr{C}$ is an $\mathscr{A}$-algebra, the composite homomorphism

$$\Psi_*(\mathscr{C}) \otimes_{\mathscr{B}} \Psi_*(\mathscr{C}) \longrightarrow \Psi_*(\mathscr{C} \otimes_{\mathscr{A}} \mathscr{C}) \longrightarrow \Psi_*(\mathscr{C})$$

defines on $\Psi_*(\mathscr{C})$ the structure of a *$\mathscr{B}$-algebra*, as a result of (4.2.2.2). We see similarly that if $\mathscr{M}$ is a $\mathscr{C}$-module, $\Psi_*(\mathscr{M})$ is canonically equipped with the structure of a $\Psi_*(\mathscr{C})$-module.

(4.2.5). Consider in particular the case where X is a *closed* subspace of Y and where ψ is the canonical injection $j : X \to Y$. If $\mathscr{B}' = \mathscr{B}|X = j^*(\mathscr{B})$ is the restriction of the sheaf of rings $\mathscr{B}$ to X, an $\mathscr{A}$-module $\mathscr{M}$ can be considered as a $\mathscr{B}'$-module by means of the homomorphism $\theta^\sharp : \mathscr{B}' \to \mathscr{A}$; then $\Psi_*(\mathscr{M})$ is the $\mathscr{B}$-module which induces $\mathscr{M}$ on X and 0 elsewhere. If $\mathscr{N}$ is a second $\mathscr{A}$-module, $\Psi_*(\mathscr{M}) \otimes_{\mathscr{B}} \Psi_*(\mathscr{N})$ canonically identifies with $\Psi_*(\mathscr{M} \otimes_{\mathscr{B}'} \mathscr{N})$ and $\mathscr{H}om_{\mathscr{B}}(\Psi_*(\mathscr{M}), \Psi_*(\mathscr{N}))$ with $\Psi_*(\mathscr{H}om_{\mathscr{B}'}(\mathscr{M}, \mathscr{N}))$.

(4.2.6). Let $(Z, \mathscr{C})$ be a third ringed space, $\Psi' = (\psi', \theta')$ a morphism $(Y, \mathscr{B}) \to (Z, \mathscr{C})$; if Ψ'' is the composite morphism $\Psi' \circ \Psi$, it is clear that we have $\Psi''_* = \Psi'_* \circ \Psi_*$.

4.3. Inverse image of an $\mathscr{A}$-module

(4.3.1). The hypotheses and notation being the same as (4.2.1), let $\mathscr{G}$ be a $\mathscr{B}$-module and $\psi^*(\mathscr{G})$ the inverse image (3.7.1) which is therefore a sheaf of abelian groups on X. The definition of sections of $\psi^*(\mathscr{G})$ and of $\psi^*(\mathscr{B})$ (3.7.1) shows that $\psi^*(\mathscr{G})$ is canonically equipped with a $\psi^*(\mathscr{B})$-module structure. On the other hand, the homomorphism $\theta^\sharp : \psi^*(\mathscr{B}) \to \mathscr{A}$ endows $\mathscr{A}$ with the a $\psi^*(\mathscr{B})$-module structure, which we denote by $\mathscr{A}_{[\theta]}$ when necessary to avoid confusion; the tensor product $\psi^*(\mathscr{G}) \otimes_{\psi^*(\mathscr{B})} \mathscr{A}_{[\theta]}$ is then equipped with an $\mathscr{A}$-module structure. We say that this $\mathscr{A}$-module is *the inverse image of $\mathscr{G}$ under the morphism* Ψ and we denote it by $\Psi^*(\mathscr{G})$. If $\mathscr{G}_1, \mathscr{G}_2$ are two $\mathscr{B}$-modules over Y, v a $\mathscr{B}$-homomorphism $\mathscr{G}_1 \to \mathscr{G}_2$, then $\psi^*(v)$, as we check immediately, is a $\psi^*(\mathscr{B})$-homomorphism from $\psi^*(\mathscr{G}_1)$ to $\psi^*(\mathscr{G}_2)$; as a result $\psi^*(v) \otimes 1$ is an $\mathscr{A}$-homomorphism $\Psi^*(\mathscr{G}_1) \to \Psi^*(\mathscr{G}_2)$, which we denote by $\Psi^*(v)$. So we define Ψ^* as a *covariant functor* from the category of $\mathscr{B}$-modules to that of $\mathscr{A}$-modules. Here, this functor (contrary to ψ^*) is no longer exact in general, but only *right exact*, the tensorization by $\mathscr{A}$ being a right exact functor to the category of $\psi^*(\mathscr{B})$-modules. $\mathbf{0_I}$ | 41

For each $x \in X$, we have $(\Psi^*(\mathscr{G}))_x = \mathscr{G}_{\psi(x)} \otimes_{\mathscr{B}_{\psi(x)}} \mathscr{A}_x$, according to (3.7.2). The support of $\Psi^*(\mathscr{G})$ is thus contained in $\psi^{-1}(\mathrm{Supp}(\mathscr{G}))$.

(4.3.2). Let $(\mathscr{G}_\lambda)$ be an inductive system of $\mathscr{B}$-modules, and let $\mathscr{G} = \varinjlim \mathscr{G}_\lambda$ be its inductive limit. The canonical homomorphisms $\mathscr{G}_\lambda \to \mathscr{G}$ define the $\psi^*(\mathscr{B})$-homomorphisms $\psi^*(\mathscr{G}_\lambda) \to \psi^*(\mathscr{G})$, which give a canonical homomorphism $\varinjlim \psi^*(\mathscr{G}_\lambda) \to \psi^*(\mathscr{G})$. As the stalk at a point of an inductive limit of sheaves is the inductive limit of the stalks at the same point (G, II, 1.11), the preceding canonical homomorphism is *bijective* (3.7.2). In addition, the tensor product commutes with inductive limits of sheaves, and we thus have a *canonical functorial isomorphism* $\varinjlim \Psi^*(\mathscr{G}_\lambda) \simeq \Psi^*(\varinjlim \mathscr{G}_\lambda)$ of $\mathscr{A}$-modules.

On the other hand, for a finite direct sum $\bigoplus_i \mathscr{G}_i$ of $\mathscr{B}$-modules, it is clear that $\psi^*(\bigoplus_i \mathscr{G}_i) = \bigoplus_i \psi^*(\mathscr{G}_i)$, therefore, by tensoring with $\mathscr{A}_{[\theta]}$,

$$\Psi^*\Big(\bigoplus_i \mathscr{G}_i\Big) = \bigoplus_i \Psi^*(\mathscr{G}_i). \tag{4.3.2.1}$$

By passing to the inductive limit, we deduce, in light of the above, that the above equality is still true for *any* direct sum.

(4.3.3). Let $\mathscr{G}_1$, $\mathscr{G}_2$ be two $\mathscr{B}$-modules; from the definition of the inverse images of sheaves of abelian groups (3.7.1), we obtain immediately a canonical homomorphism $\psi^*(\mathscr{G}_1) \otimes_{\psi^*(\mathscr{B})} \psi^*(\mathscr{G}_2) \to \psi^*(\mathscr{G}_1 \otimes_{\mathscr{B}} \mathscr{G}_2)$ of $\psi^*(\mathscr{B})$-modules, and the stalk at a point of a tensor product of sheaves being the tensor product of the stalks at this point (G, II, 2.8), we deduce from (3.7.2) that the above homomorphism is in fact a *isomorphism*. By tensoring with $\mathscr{A}$, we obtain a *canonical functorial isomorphism*

$$\Psi^*(\mathscr{G}_1) \otimes_{\mathscr{A}} \Psi^*(\mathscr{G}_2) \simeq \Psi^*(\mathscr{G}_1 \otimes_{\mathscr{B}} \mathscr{G}_2). \tag{4.3.3.1}$$

(4.3.4). Let $\mathscr{C}$ be a $\mathscr{B}$-algebra; the data of the algebra structure on $\mathscr{C}$ is the same as the data of a $\mathscr{B}$-homomorphism $\mathscr{C} \otimes_{\mathscr{B}} \mathscr{C} \to \mathscr{C}$ satisfying the associativity and commutativity conditions (conditions which are checked stalk-wise); the above isomorphism allows us to consider this homomorphism as a homomorphism of $\mathscr{A}$-modules $\Psi^*(\mathscr{C}) \otimes_{\mathscr{A}} \Psi^*(\mathscr{C}) \to \Psi^*(\mathscr{C})$ satisfying the same conditions, so $\Psi^*(\mathscr{C})$ is thus equipped with an $\mathscr{A}$-algebra structure. In particular, it follows immediately from the definitions that the $\mathscr{A}$-algebra $\Psi^*(\mathscr{B})$ is *equal to* $\mathscr{A}$ (up to a canonical isomorphism).

Similarly, if $\mathscr{M}$ is a $\mathscr{C}$-module, the data of this module structure is the same as that of a $\mathscr{B}$-homomorphism $\mathscr{C} \otimes_{\mathscr{B}} \mathscr{M} \to \mathscr{M}$ satisfying the associativity condition; hence we give a $\Psi^*(\mathscr{C})$-module structure on $\Psi^*(\mathscr{M})$. $\mathbf{0_I}$ | 42

(4.3.5). Let $\mathscr{J}$ be a sheaf of ideals of $\mathscr{B}$; as the functor ψ^* is exact, the $\psi^*(\mathscr{B})$-module $\psi^*(\mathscr{J})$ canonically identifies with a sheaf of ideals of $\psi^*(\mathscr{B})$; the canonical injection $\psi^*(\mathscr{J}) \to \psi^*(\mathscr{B})$ then gives a homomorphism of $\mathscr{A}$-modules $\Psi^*(\mathscr{J}) = \psi^*(\mathscr{J}) \otimes_{\psi^*(\mathscr{B})} \mathscr{A}_{[\theta]} \to \mathscr{A}$; we denote by $\Psi^*(\mathscr{J})\mathscr{A}$, or $\mathscr{J}\mathscr{A}$ if there is no fear of confusion, the image of $\Psi^*(\mathscr{J})$ under this homomorphism. So we have by definition $\mathscr{J}\mathscr{A} = \theta^\sharp(\psi^*(\mathscr{J}))\mathscr{A}$ and in particular, for each $x \in X$, $(\mathscr{J}\mathscr{A})_x = \theta^\sharp_x(\mathscr{J}_{\psi(x)})\mathscr{A}_x$, taking into account the canonical identification between the stalks of $\psi^*(\mathscr{J})$ and those of $\mathscr{J}$ (3.7.2). If $\mathscr{J}_1$, $\mathscr{J}_2$ are two sheaves of ideals of $\mathscr{B}$, then we have $(\mathscr{J}_1\mathscr{J}_2)\mathscr{A} = \mathscr{J}_1(\mathscr{J}_2\mathscr{A}) = (\mathscr{J}_1\mathscr{A})(\mathscr{J}_2\mathscr{A})$.

If $\mathscr{F}$ is an $\mathscr{A}$-module, we set $\mathscr{J}\mathscr{F} = (\mathscr{J}\mathscr{A})\mathscr{F}$.

(4.3.6). Let $(Z, \mathscr{C})$ be a third ringed space, $\Psi' = (\psi', \theta')$ a morphism $(Y, \mathscr{B}) \to (Z, \mathscr{C})$; if Ψ'' is the composite morphism $\Psi' \circ \Psi$, it follows from the definition (4.3.1) and from (4.3.3.1) that we have $\Psi''^* = \Psi^* \circ \Psi'^*$.

4.4. Relation between direct and inverse images

(4.4.1). The hypotheses and notation being the same as in (4.2.1), let $\mathscr{G}$ be a $\mathscr{B}$-module. By definition, a homomorphism $u : \mathscr{G} \to \Psi_*(\mathscr{F})$ of $\mathscr{B}$-modules is still called a Ψ*-morphisms from* $\mathscr{G}$ *to* $\mathscr{F}$, or simply a *homomorphism from* $\mathscr{G}$ *to* $\mathscr{F}$ and we write it as $u : \mathscr{G} \to \mathscr{F}$ when no confusion will occur. To give such a homomorphism is the same as giving, for each pair (U, V) where U is an open set of X, V an open set of Y such that $\psi(U) \subset V$, a *homomorphism* $u_{U,V} : \Gamma(V, \mathscr{G}) \to \Gamma(U, \mathscr{F})$ *of* $\Gamma(V, \mathscr{B})$*-modules*, $\Gamma(U, \mathscr{F})$ being considered as a $\Gamma(V, \mathscr{B})$-module by means of the ring homomorphism $\theta_{U,V} : \Gamma(V, \mathscr{B}) \to \Gamma(U, \mathscr{A})$; the $u_{U,V}$ must in addition render commutative the diagrams (3.5.1.1). It suffices, moreover, to define u by the data of the $u_{U,V}$ when U (resp. V) varies over a basis $\mathfrak{B}$ (resp. $\mathfrak{B}'$) for the topology of X (resp. Y) and to check the commutativity of (3.5.1.1) for these restrictions.

(4.4.2). Under the hypotheses of (4.2.1) and (4.2.6), let $\mathscr{H}$ be a $\mathscr{C}$-module, $v : \mathscr{H} \to \Psi'_*(\mathscr{G})$ a Ψ'-morphism; then $w : \mathscr{H} \xrightarrow{v} \Psi'_*(\mathscr{G}) \xrightarrow{\Psi'_*(u)} \Psi'_*(\Psi_*(\mathscr{F}))$ is a Ψ''-morphism which we call the *composition* of u and v.

(4.4.3). We will now see that we can define a canonical *isomorphism* of *bifunctors* in $\mathscr{F}$ and $\mathscr{G}$

$$\mathrm{Hom}_{\mathscr{A}}(\Psi^*(\mathscr{G}), \mathscr{F}) \simeq \mathrm{Hom}_{\mathscr{B}}(\mathscr{G}, \Psi_*(\mathscr{F})) \tag{4.4.3.1}$$

which we denote by $v \mapsto v^\flat_\theta$ (or simply $v \mapsto v^\flat$ if there is no chance of confusion); we denote by $u \mapsto u^\sharp_\theta$, or $u \mapsto u^\sharp$, the inverse isomorphism. This definition is the following: by composing $v : \Psi^*(\mathscr{G}) \to \mathscr{F}$ with the canonical map $\psi^*(\mathscr{G}) \to \Psi^*(\mathscr{G})$, we obtain a homomorphism of sheaves of groups $v' : \psi^*(\mathscr{G}) \to \mathscr{F}$, which is also a homomorphism of $\psi^*(\mathscr{B})$-modules. We obtain (3.7.1) a homomorphism $v'^\flat : \mathscr{G} \to \psi_*(\mathscr{F}) = \Psi_*(\mathscr{F})$, which is also a homomorphism of $\mathscr{B}$-modules as we check easily; we take $v^\flat_\theta = v'^\flat$. Similarly, for $u : \mathscr{G} \to \Psi_*(\mathscr{F})$, which is a homomorphism of $\mathscr{B}$-modules, we obtain (3.7.1) a homomorphism $u^\sharp : \psi^*(\mathscr{G}) \to \mathscr{F}$ of $\psi^*(\mathscr{B})$-modules, hence by tensoring with $\mathscr{A}$ we have a homomorphism of $\mathscr{A}$-modules $\Psi^*(\mathscr{G}) \to \mathscr{F}$, which we denote by $u^\sharp_\theta$. It is immediate to check that $(u^\sharp_\theta)^\flat_\theta = u$ and $(v^\flat_\theta)^\sharp_\theta = v$, so we have established the functorial nature in $\mathscr{F}$ of the isomorphism $v \mapsto v^\flat_\theta$. The functorial nature in $\mathscr{G}$ of $u \mapsto u^\sharp_\theta$ is then formally shown as in (3.5.4) (reasoning that would also prove the functorial nature of Ψ^* established in (4.3.1) directly). $\mathbf{0_I}$ | 43

If we take for v the identity homomorphism of $\Psi^*(\mathscr{B})$, $v^\flat_\theta$ is a homomorphism

$$\rho_{\mathscr{G}} : \mathscr{G} \longrightarrow \Psi_*(\Psi^*(\mathscr{G})); \tag{4.4.3.2}$$

if we take for u the identity homomorphism of $\Psi_*(\mathscr{F})$, $u^\sharp_\theta$ is a homomorphism

$$\sigma_{\mathscr{F}} : \Psi^*(\Psi_*(\mathscr{F})) \longrightarrow \mathscr{F}; \tag{4.4.3.3}$$

these homomorphisms will be called *canonical*. They are in general neither injective or surjective. We have canonical factorizations analogous to (3.5.3.3) and (3.5.4.4).

We note that if s is a section of $\mathscr{G}$ over an open set V of Y, $\rho_{\mathscr{G}}(s)$ is the section $s' \otimes 1$ of $\Psi^*)(\mathscr{G})$ over $\psi^{-1}(V)$, s' being such that $s'_x = s_{\psi(x)}$ for all $x \in \psi^{-1}(V)$. We also note that if $u : \mathscr{G} \to \psi_*(\mathscr{F})$ is a homomorphism, it defines for all $x \in X$ a homomorphism $u_x : \mathscr{G}_{\psi(x)} \to \mathscr{F}_x$ on the stalks, obtained by composing $(u^\sharp)_x : (\Psi^*(\mathscr{G}))_x \to \mathscr{F}_x$ and the canonical homomorphism $s_x \mapsto s_x \otimes 1$ from $\mathscr{G}_{\psi(x)}$ to $(\Psi^*(\mathscr{G}))_x = \mathscr{G}_{\psi(x)} \otimes_{\mathscr{B}_{\psi(x)}} \mathscr{A}_x$. The homomorphism u_x is obtained also by passing to the inductive limit relative to the homomorphisms $\Gamma(V, \mathscr{G}) \xrightarrow{u} \Gamma(\psi^{-1}(V), \mathscr{F}) \to \mathscr{F}_x$, where V varies over the neighbourhoods of $\psi(x)$.

(4.4.4). Let $\mathscr{F}_1, \mathscr{F}_2$ be $\mathscr{A}$-modules, $\mathscr{G}_1, \mathscr{G}_2$ be $\mathscr{B}$-modules, u_i $(i = 1, 2)$ a homomorphism from $\mathscr{G}_i$ to $\mathscr{F}_i$. We denote by $u_1 \otimes u_2$ the homomorphism $u : \mathscr{G}_1 \otimes_{\mathscr{B}} \mathscr{G}_2 \to \mathscr{F}_1 \otimes_{\mathscr{A}} \mathscr{F}_2$ such that $u^\sharp = (u_1)^\sharp \otimes (u_2)^\sharp$ (taking into account (4.3.3.1)); we check that u is also the composition $\mathscr{G}_1 \otimes_{\mathscr{B}} \mathscr{G}_2 \to \Psi_*(\mathscr{F}_1) \otimes_{\mathscr{B}} \Psi_*(\mathscr{F}_2) \to \Psi_*(\mathscr{F}_1 \otimes_{\mathscr{A}} \mathscr{F}_2)$, where the first arrow is the ordinary tensor product $u_1 \otimes_{\mathscr{B}} u_2$ and the second is the canonical homomorphism (4.2.2.1).

(4.4.5). Let $(\mathscr{G}_\lambda)_{\lambda \in L}$ be an inductive system of $\mathscr{B}$-modules, and, for each $\lambda \in L$, let u_λ be a homomorphism $\mathscr{G}_\lambda \to \Psi_*(\mathscr{F})$, form an inductive limit; we put $\mathscr{G} = \varinjlim \mathscr{G}_\lambda$ and $u = \varinjlim u_\lambda$; then the $(u_\lambda)^\sharp$ form an inductive system of homomorphisms $\Psi^*(\mathscr{G}_\lambda) \to \mathscr{F}$, and the inductive limit of this system is none other than $u^\sharp$.

(4.4.6). Let $\mathscr{M}$, $\mathscr{N}$ be two $\mathscr{B}$-modules, V an open set of Y, $U = \psi^{-1}(V)$; the map $v \mapsto \Psi^*(v)$ is a homomorphism

$$\mathrm{Hom}_{\mathscr{B}|V}(\mathscr{M}|V, \mathscr{N}|V) \longrightarrow \mathrm{Hom}_{\mathscr{A}|U}(\Psi^*(\mathscr{M})|U, \Psi^*(\mathscr{N})|U)$$

for the $\Gamma(V, \mathscr{B})$-module structures ($\mathrm{Hom}_{\mathscr{A}|U}(\Psi^*(\mathscr{M})|U, \Psi^*(\mathscr{N})|U)$ is normaly equipped with the a $\Gamma(U, \psi^*(\mathscr{B}))$-module structure, and thanks to the canonical homomorphism (3.7.2) $\Gamma(V, \mathscr{B}) \to$ **0_I** | 44
$\Gamma(U, \psi^*(\mathscr{B}))$, it is also a $\Gamma(V, \mathscr{B})$-module). We see immediately that these homomorphisms are compatible with the restriction morphisms, and as a result define a canonical functorial homomorphism

$$\gamma : \mathscr{H}\!om_{\mathscr{B}}(\mathscr{M}, \mathscr{N}) \longrightarrow \Psi_*(\mathscr{H}\!om_{\mathscr{A}}(\Psi^*(\mathscr{M}), \Psi^*(\mathscr{N}));$$

it also corresponds to this homomorphism the homomorphism

$$\gamma^\sharp : \Psi^*(\mathscr{H}\!om_{\mathscr{B}}(\mathscr{M}, \mathscr{N})) \longrightarrow \mathscr{H}\!om_{\mathscr{A}}(\Psi^*(\mathscr{M}), \Psi^*(\mathscr{N}))$$

and these canonical morphisms are functorial in $\mathscr{M}$ and $\mathscr{N}$.

(4.4.7). Suppose that $\mathscr{F}$ (resp. $\mathscr{G}$) is an $\mathscr{A}$-algebra (resp. a $\mathscr{B}$-algebra). If $u : \mathscr{G} \to \Psi_*(\mathscr{F})$ is a homomorphism of $\mathscr{B}$-algebras, $u^\sharp$ is a homomorphism $\Psi^*(\mathscr{G}) \to \mathscr{F}$ of $\mathscr{A}$-algebras; this follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathscr{G} \otimes_{\mathscr{B}} \mathscr{G} \arrow[r] \arrow[d] & \mathscr{G} \arrow[d, "u"] \\
\Psi_*(\mathscr{F} \otimes_{\mathscr{A}} \mathscr{F}) \arrow[r] & \Psi_*(\mathscr{F})
\end{tikzcd}$$

and from (4.4.4). Similarly, if $v : \Psi^*(\mathscr{G}) \to \mathscr{F}$ is a homomorphism of $\mathscr{A}$-algebras, $v^\flat : \mathscr{G} \to \Psi_*(\mathscr{F})$ is a homomorphism of $\mathscr{B}$-algebras.

(4.4.8). Let $(Z, \mathscr{C})$ be a third ringed space, $\Psi' = (\psi', \theta')$ a morphism $(Y, \mathscr{B}) \to (Z, \mathscr{C})$, and $\Psi'' : (X, \mathscr{A}) \to (Z, \mathscr{C})$ the composite morphism $\Psi' \circ \Psi$. Let $\mathscr{H}$ be a $\mathscr{C}$-module, u' a homomorphism from $\mathscr{H}$ to $\mathscr{G}$; the composition $v'' = v \circ v'$ is by definition the homomorphism from $\mathscr{H}$ to $\mathscr{F}$ defined by $\mathscr{H} \xrightarrow{v'} \Psi'_*(\mathscr{G}) \xrightarrow{\Psi'_*(v)} \Psi'_*(\Psi_*(\mathscr{F}))$; we check that $v''^\sharp$ is the homomorphism

$$\Psi^*(\Psi'^*(\mathscr{H})) \xrightarrow{\Psi^*(v'^\sharp)} \Psi^*(\mathscr{G}) \xrightarrow{v^\sharp} \mathscr{F}.$$

§5. Quasi-coherent and coherent sheaves

5.1. Quasi-coherent sheaves

(5.1.1). Let $(X, \mathscr{O}_X)$ be a ringed space, $\mathscr{F}$ an $\mathscr{O}_X$-module. The data of a homomorphism $u : \mathscr{O}_X \to \mathscr{F}$ of $\mathscr{O}_X$-modules is equivalent to that of the section $s = u(1) \in \Gamma(X, \mathscr{F})$. Indeed, when s is given, for each section $t \in \Gamma(U, \mathscr{O}_X)$, we necessarily have $u(t) = t \cdot (s|U)$; we say that u is *defined by the section* s. If now I is any set of indices, consider the direct sum sheaf $\mathscr{O}_X^{(I)}$, and for each $i \in I$, let h_i be the canonical injection of the i-th factor into $\mathscr{O}_X^{(I)}$; we know that $u \mapsto (u \circ h_i)$ is an isomorphism from $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X^{(I)}, \mathscr{F})$ to the product $(\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{F}))^I$. So there is a canonical one-to-one correspondence between the homomorphisms $u : \mathscr{O}_X^{(I)} \to \mathscr{F}$ and the *families of sections* $(s_i)_{i\in I}$ *of* $\mathscr{F}$ *over* X. The homomorphism u corresponding to (s_i) sends an element $(a_i) \in (\Gamma(U, \mathscr{O}_X))^{(I)}$ to $\sum_{i\in I} a_i \cdot (s_i|U)$.

We say that $\mathscr{F}$ is *generated by the family* (s_i) if the homomorphism $\mathscr{O}_X^{(I)} \to \mathscr{F}$ defined for each family is *surjective* (in other words, if, for each $x \in X$, $\mathscr{F}_x$ is an $\mathscr{O}_x$-module generated by the $(s_i)_x$). **0_{I}** | 45
We say that $\mathscr{F}$ is *generated by its sections over* X if it is generated by the family of all these sections (or by a subfamily), in other words, if there exists a surjective homomorphism $\mathscr{O}_X^{(I)} \to \mathscr{F}$ for a suitable I.

We note that an $\mathscr{O}_X$-module $\mathscr{F}$ can be such that there exists a point $x_0 \in X$ for which $\mathscr{F}|U$ is not generated by its sections over U, *regardless of the choice of neighbourhood U of* x_0: it suffices to take $X = \mathbf{R}$, for $\mathscr{O}_X$ the simple sheaf $\mathbf{Z}$, for $\mathscr{F}$ the algebraic subsheaf of $\mathscr{O}_X$ such that $\mathscr{F}_0 = \{0\}$, $\mathscr{F}_x = \mathbf{Z}$ for $x \neq 0$, and finally $x_0 = 0$: the only section of $\mathscr{F}|U$ over U is 0 for a neighbourhood U of 0.

(5.1.2). Let $f : X \to Y$ be a morphism of ringed spaces. If $\mathscr{F}$ is an $\mathscr{O}_X$-module generated by its sections over X, then the canonical homomorphism $f^*(f_*(\mathscr{F})) \to \mathscr{F}$ (4.4.3.3) is *surjective*; indeed, with the notation of (5.1.1), $s_i \otimes 1$ is a section of $f^*(f_*(\mathscr{F}))$ over X, and its image in $\mathscr{F}$ is s_i. The example in (5.1.1) where f is the identity shows that the inverse of this proposition is false in general.

If $\mathscr{G}$ is an $\mathscr{O}_Y$-module generated by its sections over Y, then $f^*(\mathscr{G})$ is generated by its sections over X, since f^* is a right exact functor. **$\mathrm{Err}_{\mathrm{II}}$**

(5.1.3). We say that an $\mathscr{O}_X$-module $\mathscr{F}$ is *quasi-coherent* if for each $x \in X$ there is an open neighbourhood U of x such that $\mathscr{F}|U$ is isomorphic to the *cokernel* of a homomorphism of the form $\mathscr{O}_X^{(I)}|U \to \mathscr{O}_X^{(J)}|U$, where I and J are sets of arbitrary indices. It is clear that $\mathscr{O}_X$ is itself a quasi-coherent $\mathscr{O}_X$-module, and that any direct sum of quasi-coherent $\mathscr{O}_X$-modules is again a quasi-coherent $\mathscr{O}_X$-module. We say that an *$\mathscr{O}_X$-algebra $\mathscr{A}$* is *quasi-coherent* if it is quasi-coherent as an $\mathscr{O}_X$-module.

(5.1.4). Let $f : X \to Y$ be a morphism of ringed spaces. If $\mathscr{G}$ is a quasi-coherent $\mathscr{O}_Y$-module, then $f^*(\mathscr{G})$ is a quasi-coherent $\mathscr{O}_X$-module. Indeed, for each $x \in X$, there is an open neighbourhood V of $f(x)$ in Y such that $\mathscr{G}|V$ is the cokernel of a homomorphism $\mathscr{O}_Y^{(I)}|V \to \mathscr{O}_Y^{(J)}|V$. If $U = f^{-1}(V)$, and if f_U is the restriction of f to U, then we have $f^*(\mathscr{G})|U = f_U^*(\mathscr{G}|V)$; as f_U^* is right exact and commutes with direct sums, $f_U^*(\mathscr{G}|V)$ is the cokernel of a homomorphism $\mathscr{O}_X^{(I)}|U \to \mathscr{O}_X^{(J)}|U$.

5.2. Sheaves of finite type

(5.2.1). We say that an $\mathscr{O}_X$-module $\mathscr{F}$ is *of finite type* if for each $x \in X$ there exists an open neighbourhood U of x such that $\mathscr{F}|U$ is generated by a *finite* family of sections over U, or if it is isomorphic to a sheaf quotient of a sheaf of the form $(\mathscr{O}_X|U)^p$ where p is finite. Each sheaf quotient of a sheaf of finite type is again a sheaf of finite type, as well as each finite direct sum and each finite tensor product of sheaves of finite type. An $\mathscr{O}_X$-module of finite type is not necessarily quasi-coherent, as we can see for the $\mathscr{O}_X$-module $\mathscr{O}_X/\mathscr{F}$, where $\mathscr{F}$ is the example in (5.1.1). If $\mathscr{F}$ is of finite type, then $\mathscr{F}_x$ is an $\mathscr{O}_x$-module of finite type for each $x \in X$, but the example in (5.1.1) shows that this condition is necessary but not sufficient in general.

(5.2.2). Let $\mathscr{F}$ be an $\mathscr{O}_X$-module *of finite type*. If s_i $(1 \leqslant i \leqslant n)$ are the sections of $\mathscr{F}$ over an open neighbourhood U of a point $x \in X$ and the $(s_i)_x$ generate $\mathscr{F}_x$, then there exists an open neighbourhood $V \subset U$ of x such that the $(s_i)_y$ generate $\mathscr{F}_y$ for all $y \in Y$ (FAC, I, 2, 12, prop. 1). In particular, we conclude that the support of $\mathscr{F}$ is *closed*.

Similarly, if $u : \mathscr{F} \to \mathscr{G}$ is a homomorphism such that $u_x = 0$, then there exists a neighbourhood U of x such that $u_y = 0$ for all $y \in U$. $\mathbf{0_I}$ | 46

(5.2.3). Suppose that X is *quasi-compact*, and let $\mathscr{F}$ and $\mathscr{G}$ be two $\mathscr{O}_X$-modules such that $\mathscr{G}$ is *of finite type*, $u : \mathscr{F} \to \mathscr{G}$ a *surjective* homomorphism. In addition, suppose that $\mathscr{F}$ is the inductive limit of an inductive system $(\mathscr{F}_\lambda)$ of $\mathscr{O}_X$-modules. Then there exists an index μ such that the homomorphism $\mathscr{F}_\mu \to \mathscr{G}$ is *surjective*. Indeed, for each $x \in X$, there exists a finite system of sections s_i of $\mathscr{G}$ over an open neighbourhood $U(x)$ of x such that the $(s_i)_y$ generate $\mathscr{G}_y$ for all $y \in U(x)$; there is then an open neighbourhood $V(x) \subset U(x)$ of x and n sections t_i of $\mathscr{F}$ over $V(x)$ such that $s_i|V(s) = u(t_i)$ for all i; we can also suppose that the t_i are the canonical images of sections of a similar sheaf $\mathscr{F}_{\lambda(x)}$ over $V(x)$. We then cover X with a finite number of neighbourhoods $V(x_k)$, and let μ be the maximal index of the $\lambda(x_k)$; it is clear that this index gives the answer.

Suppose still that X is quasi-compact, and let $\mathscr{F}$ be an $\mathscr{O}_X$-module of finite type generated by its sections over X (5.1.1); then $\mathscr{F}$ is generated by a *finite* subfamily of these sections: indeed, it suffices to cover X by a finite number of open neighbourhoods U_k such that, for each k, there is a finite number of sections s_{ik} of $\mathscr{F}$ over X whose restrictions to U_k generate $\mathscr{F}|U_k$; it is clear that the s_{ik} then generate $\mathscr{F}$.

(5.2.4). Let $f : X \to Y$ be a morphism of ringed spaces. If $\mathscr{G}$ is an $\mathscr{O}_Y$-module of finite type, then $f^*(\mathscr{G})$ is an $\mathscr{O}_X$-module of finite type. Indeed, for each $x \in X$, there is an open neighbourhood V of $f(x)$ in Y and a surjective homomorphism $v : \mathscr{O}_Y^p|V \to \mathscr{G}|V$. If $U = f^{-1}(V)$ and if f_U is the restriction of f to U, then we have $f^*(\mathscr{G})|U = f_U^*(\mathscr{G}|V)$; since f_U^* is right exact (4.3.1) and commutes with direct sums (4.3.2), $f_U^*(v)$ is a surjective homomorphism $\mathscr{O}_X^p|U \to f^*(\mathscr{G})|U$. $\mathbf{Err_{II}}$

(5.2.5). We say that an $\mathscr{O}_X$-module $\mathscr{F}$ *admits a finite presentation* if for each $x \in X$ there exists an open neighbourhood U of x such that $\mathscr{F}|U$ is isomorphic to a *cokernel of a* $(\mathscr{O}_X|U)$*-homomorphism* $\mathscr{O}_X^p|U \to \mathscr{O}_X^q|U$, p and q being two integers > 0. Such an $\mathscr{O}_X$-module is therefore of finite type and quasi-coherent. If $f : X \to Y$ is a morphism of ringed spaces, and if $\mathscr{G}$ is an $\mathscr{O}_Y$-module admitting a finite presentation, then $f^*(\mathscr{G})$ admits a finite presentation, as shown in the argument of (5.1.4).

(5.2.6). Let $\mathscr{F}$ be an $\mathscr{O}_X$-module admitting a finite presentation (5.2.5); then, for each $\mathscr{O}_X$-module $\mathscr{H}$, the canonical functorial homomorphism

$$(\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{H}))_x \longrightarrow \mathrm{Hom}_{\mathscr{O}_x}(\mathscr{F}_x,\mathscr{H}_x)$$

is *bijective* (T, 4.1.1).

(5.2.7). Let $\mathscr{F}$ and $\mathscr{G}$ be two $\mathscr{O}_X$-modules admitting a finite presentation. If for some $x \in X$, $\mathscr{F}_x$ and $\mathscr{G}_x$ are *isomorphic* as $\mathscr{O}_x$-modules, then there exists an open neighbourhood U of x such that $\mathscr{F}|U$ and $\mathscr{G}|U$ are *isomorphic*. Indeed, if $\phi : \mathscr{F}_x \to \mathscr{G}_x$ and $\psi : \mathscr{G}_x \to \mathscr{F}_x$ are an isomorphism and its inverse isomorphism, then there exists, according to (5.2.6), an open neighbourhood V of x and a section u (resp. v) of $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$ (resp. $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{G},\mathscr{F})$) over V such that $u_x = \phi$ (resp. $v_x = \psi$). As $(u \circ v)_x$ and $(v \circ u)_x$ are the identity automorphisms, there exists an open neighbourhood $U \subset V$ of x such that $(u \circ v)|U$ and $(v \circ u)|U$ are the identity automorphisms, hence the proposition. $\mathbf{0_I}$ | 47

5.3. Coherent sheaves

(5.3.1). We say that an $\mathscr{O}_X$-module $\mathscr{F}$ is *coherent* if it satisfies the two following conditions:

(a) $\mathscr{F}$ is of finite type.
(b) for each open $U \subset X$, integer $n > 0$, and homomorphism $u : \mathscr{O}_X^n|U \to \mathscr{F}|U$, the kernel of u is of finite type.

We note that these two conditions are of a *local* nature.

For most of the proofs of the properties of coherent sheaves in what follows, cf. (FAC, I, 2).

(5.3.2). Each coherent $\mathscr{O}_X$-module admits a finite presentation (5.2.5); the inverse is not necessarily true, since $\mathscr{O}_X$ itself is not necessarily a coherent $\mathscr{O}_X$-module.

Each $\mathscr{O}_X$-submodule *of finite type* of a coherent $\mathscr{O}_X$-module is coherent; each *finite* direct sum of coherent $\mathscr{O}_X$-modules is a coherent $\mathscr{O}_X$-module.

(5.3.3). If $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ is an exact sequence of $\mathscr{O}_X$-modules and if two of these $\mathscr{O}_X$-modules are coherent, then so is the third.

(5.3.4). If $\mathscr{F}$ and $\mathscr{G}$ are two coherent $\mathscr{O}_X$-modules, $u : \mathscr{F} \to \mathscr{G}$ a homomorphism, then $\mathrm{Im}(u)$, $\mathrm{Ker}(u)$, and $\mathrm{Coker}(u)$ are coherent $\mathscr{O}_X$-modules. In particular, if $\mathscr{F}$ and $\mathscr{G}$ are $\mathscr{O}_X$-submodules of a coherent $\mathscr{O}_X$-module, then $\mathscr{F} + \mathscr{G}$ and $\mathscr{F} \cap \mathscr{G}$ are coherent.

If $\mathscr{A} \to \mathscr{B} \to \mathscr{C} \to \mathscr{D} \to \mathscr{E}$ is an exact sequence of $\mathscr{O}_X$-modules, and if $\mathscr{A}$, $\mathscr{B}$, $\mathscr{D}$, $\mathscr{E}$ are coherent, then $\mathscr{C}$ is coherent. **Err$_{\mathrm{II}}$**

(5.3.5). If $\mathscr{F}$ and $\mathscr{G}$ are two coherent $\mathscr{O}_X$-modules, then so are $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ are $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$.

(5.3.6). Let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module, $\mathscr{J}$ a coherent sheaf of ideals of $\mathscr{O}_X$. Then the $\mathscr{O}_X$-module $\mathscr{J}\mathscr{F}$ is coherent, as the image of $\mathscr{J} \otimes_{\mathscr{O}_X} \mathscr{F}$ under the canonical homomorphism $\mathscr{J} \otimes_{\mathscr{O}_X} \mathscr{F} \to \mathscr{F}$ ((5.3.4) and (5.3.5)).

(5.3.7). We say that an $\mathscr{O}_X$-algebra $\mathscr{A}$ is *coherent* if it is coherent as an $\mathscr{O}_X$-module. In particular, $\mathscr{O}_X$ is a *coherent sheaf of rings* if and only if for each open $U \subset X$ and each homomorphism of the form $u : \mathscr{O}_X^p|U \to \mathscr{O}_X|U$, the kernel of u is an $(\mathscr{O}_X|U)$-module of finite type.

If $\mathscr{O}_X$ is a coherent sheaf of rings, then each $\mathscr{O}_X$-module $\mathscr{F}$ admitting a finite presentation (5.2.5) is coherent, according to (5.3.4).

The *annihilator* of an $\mathscr{O}_X$-module $\mathscr{F}$ is the kernel $\mathscr{J}$ of the canonical homomorphism $\mathscr{O}_X \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{F})$ which sends each section $s \in \Gamma(U, \mathscr{O}_X)$ to the multiplication by s map in $\mathrm{Hom}(\mathscr{F}|U, \mathscr{F}|U)$; if $\mathscr{O}_X$ is coherent and if $\mathscr{F}$ is a coherent $\mathscr{O}_X$-module, then $\mathscr{J}$ is coherent ((5.3.4) and (5.3.5)) and for each $x \in X$, $\mathscr{J}_x$ is the annihilator of $\mathscr{F}_x$ (5.2.6).

(5.3.8). Suppose that $\mathscr{O}_X$ is coherent; let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module, x a point of X, M a submodule of finite type of $\mathscr{F}_x$; then there exists an open neighbourhood U of x and a coherent $(\mathscr{O}_X|U)$-submodule $\mathscr{G}$ of $\mathscr{F}|U$ such that $\mathscr{G}_x = M$ (T, 4.1, Lemma 1). **0$_{\mathrm{I}}$ | 48**

This result, along with the properties of the $\mathscr{O}_X$-submodules of a coherent $\mathscr{O}_X$-module, impose the necessary conditions on the rings $\mathscr{O}_x$ such that $\mathscr{O}_X$ is coherent. For example (5.3.4), the intersection of two ideals of finite type of $\mathscr{O}_x$ must still be an ideal of finite type.

(5.3.9). Suppose that $\mathscr{O}_X$ is coherent, and let M be an $\mathscr{O}_x$-module admitting a finite presentation, therefore isomorphic to a cokernel of a homomorphism $\phi : \mathscr{O}_x^p \to \mathscr{O}_x^q$; then there exists an open neighbourhood U of X and a coherent $(\mathscr{O}_X|U)$-module $\mathscr{F}$ such that $\mathscr{F}_x$ is isomorphic to M. Indeed, according to (5.2.6), there exists a section u of $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X^p, \mathscr{O}_X^q)$ over an open neighbourhood U of x such that $u_x = \phi$; the cokernel $\mathscr{F}$ of the homomorphism $u : \mathscr{O}_X^p|U \to \mathscr{O}_X^q|U$ gives the answer (5.3.4).

(5.3.10). Suppose that $\mathscr{O}_X$ is coherent, and let $\mathscr{J}$ be a coherent sheaf of ideals of $\mathscr{O}_X$. For a $(\mathscr{O}_X/\mathscr{J})$-module $\mathscr{F}$ to be coherent, it is necessary and sufficient for it to be coherent as an $\mathscr{O}_X$-module. In particular, $\mathscr{O}_X/\mathscr{J}$ is a coherent sheaf of rings.

(5.3.11). Let $f : X \to Y$ be a morphism of ringed spaces, and suppose that $\mathscr{O}_X$ is coherent; then, for each coherent $\mathscr{O}_Y$-module $\mathscr{G}$, $f^*(\mathscr{G})$ is a coherent $\mathscr{O}_X$-module. Indeed, with the notation of (5.2.4), we can assume that $\mathscr{G}|V$ is the cokernel of a homomorphism $v : \mathscr{O}_Y^q|V \to \mathscr{O}_Y^p|V$; as f_U^* is right exact, $f^*(\mathscr{G})|U = f_U^*(\mathscr{G}|V)$ is the cokernel of the homomorphism $f_U^*(v) : \mathscr{O}_X^q|U \to \mathscr{O}_X^p|U$, hence our assertion.

(5.3.12). Let Y be a closed subset of X, $j : Y \to X$ the canonical injection, $\mathscr{O}_Y$ a sheaf of rings on Y, and set $\mathscr{O}_X = j_*(\mathscr{O}_Y)$. For an $\mathscr{O}_Y$-module $\mathscr{G}$ to be of finite type (resp. quasi-coherent, coherent), it is necessary and sufficient for $j_*(\mathscr{G})$ to be an $\mathscr{O}_X$-module of finite type (resp. quasi-coherent, coherent).

5.4. Locally free sheaves

(5.4.1). Let X be a ringed space. We say that an $\mathscr{O}_X$-module $\mathscr{F}$ is *locally free* if for each $x \in X$ there exists an open neighbourhood U of x such that $\mathscr{F}|U$ is isomorphic to a $(\mathscr{O}_X|U)$-module of the form $\mathscr{O}_X^{(I)}|U$, where I can depend on U. If for each U, I is finite, then we say that $\mathscr{F}$ is *of finite rank*; if for each U, I has the same finite number of elements n, we say that $\mathscr{F}$ is *of rank n*. A locally free $\mathscr{O}_X$-module of rank 1 is called *invertible* (cf. (5.4.3)). If $\mathscr{F}$ is a locally free $\mathscr{O}_X$-module of finite rank, then for each $x \in X$, $\mathscr{F}_x$ is a free $\mathscr{O}_x$-module of finite rank $n(x)$, and there exists a neighbourhood U of x such that $\mathscr{F}|U$, is of rank $n(x)$; if X is connected, then $n(x)$ is *constant*.

It is clear that each locally free sheaf is quasi-coherent, and if $\mathscr{O}_X$ is a coherent sheaf of rings, then each locally free $\mathscr{O}_X$-module of finite rank is coherent.

If $\mathscr{L}$ is locally free, then $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{F}$ is an *exact* functor in $\mathscr{F}$ to the category of $\mathscr{O}_X$-modules.

We will mostly consider locally free $\mathscr{O}_X$-modules of finite rank, and when we speak of locally $\mathbf{0_I}$ | 49
free sheaves without specifying, it will be understood that they are of *finite rank*.

Suppose that $\mathscr{O}_X$ is *coherent*, and let $\mathscr{F}$ be a *coherent* $\mathscr{O}_X$-module. Then, if at a point $x \in X$, $\mathscr{F}_x$ is an $\mathscr{O}_x$-module *free of rank n*, there exists a neighbourhood U of x such that $\mathscr{F}|U$ is *locally free of rank n*; in fact, $\mathscr{F}_x$ is then isomorphic to $\mathscr{O}_x^n$, and the proposition follows from (5.2.7). $\mathbf{Err_{II}}$

(5.4.2). If $\mathscr{L}$, $\mathscr{F}$ are two $\mathscr{O}_X$-modules, we have a canonical functorial homomorphism

$$\mathscr{L}^\vee \otimes_{\mathscr{O}_X} \mathscr{F} = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X) \otimes_{\mathscr{O}_X} \mathscr{F} \longrightarrow \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{F}) \tag{5.4.2.1}$$

defined in the following way: for each open set U, send any pair (u, t), where $u \in \Gamma(U, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X)) = \operatorname{Hom}(\mathscr{L}|U, \mathscr{O}_X|U)$ and $t \in \Gamma(U, \mathscr{F})$, to the element of $\operatorname{Hom}(\mathscr{L}|U, \mathscr{F}|U)$ which, for each $x \in U$, sends $s_x \in \mathscr{L}_x$ to the element $u_x(s_x)t_x$ of $\mathscr{F}_x$. If $\mathscr{L}$ is *locally free of finite rank*, then this homomorphism is *bijective*; the property being local, we can in fact reduce to the case where $\mathscr{L} = \mathscr{O}_X^n$; as for each $\mathscr{O}_X$-module $\mathscr{G}$, $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X^n, \mathscr{G})$ is canonically isomorphic to $\mathscr{G}^n$, we have reduced to the case $\mathscr{L} = \mathscr{O}_X$, which is immediate.

(5.4.3). If $\mathscr{L}$ is invertible, then so is its dual $\mathscr{L}^\vee = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X)$, since we can immediately reduce (as the question is local) to the case $\mathscr{L} = \mathscr{O}_X$. In addition, we have a canonical isomorphism

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X) \otimes_{\mathscr{O}_X} \mathscr{L} \simeq \mathscr{O}_X \tag{5.4.3.1}$$

as, according to (5.3.2), it suffices to define a canonical isomorphism $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{L}) \simeq \mathscr{O}_X$. For *each* $\mathscr{O}_X$-module $\mathscr{F}$, we have a canonical homomorphism $\mathscr{O}_X \simeq \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{F})$ (5.3.7). It remains to prove that if $\mathscr{F} = \mathscr{L}$ is invertible, then this homomorphism is bijective, and as the question is local, it reduces to the case $\mathscr{L} = \mathscr{O}_X$, which is immediate.

Due to the above, we put $\mathscr{L}^{-1} = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X)$, and we say that $\mathscr{L}^{-1}$ is the *inverse* of $\mathscr{L}$. The terminology "invertible sheaf" can be justified in the following way when X is a point and $\mathscr{O}_X$ is a *local* ring A with maximal ideal $\mathfrak{m}$; if M and M' are two A-modules (M being of finite type) such that $M \otimes_A M'$ is isomorphic to A, then as $(A/\mathfrak{m}) \otimes_A (M \otimes_A M')$ identifies with $(M/\mathfrak{m}M) \otimes_{A/\mathfrak{m}} (M'/\mathfrak{m}M')$, this latter tensor product of vector spaces over the field $A/\mathfrak{m}$ is isomorphic to $A/\mathfrak{m}$, which requires $M/\mathfrak{m}M$ and $M'/\mathfrak{m}M'$ to be of dimension 1. For each element $z \in M$ not in $\mathfrak{m}M$, we have $M = Az + \mathfrak{m}M$, which implies that $M = Az$ according to Nakayama's Lemma, M being of finite type. Moreover, as the annihilator of z kills $M \otimes_A M'$, which is isomorphic to A, this annihilator is $\{0\}$, and as a result M is *isomorphic to A*. In the general case, this shows that $\mathscr{L}$ is an $\mathscr{O}_X$-module of finite type, such that there exists an $\mathscr{O}_X$-module $\mathscr{F}$ for which $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{F}$ is isomorphic to $\mathscr{O}_X$, and if in addition the rings $\mathscr{O}_x$ are local rings, then $\mathscr{L}_x$ is an $\mathscr{O}_x$-module isomorphic to $\mathscr{O}_x$ for each $x \in X$. If $\mathscr{O}_X$ and $\mathscr{L}$ are assumed to be *coherent*, then we conclude that $\mathscr{L}$ is invertible according to (5.2.7).

(5.4.4). If $\mathscr{L}$ and $\mathscr{L}'$ are two invertible $\mathscr{O}_X$-modules, then so is $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{L}'$, since the question is local, we can assume that $\mathscr{L} = \mathscr{O}_X$, and the result is then trivial. For each integer $n \geqslant 1$, we denote by $\mathscr{L}^{\otimes n}$ the tensor product of n copies of the sheaf $\mathscr{L}$; we set by convention $\mathscr{L}^{\otimes 0} = \mathscr{O}_X$, and for $n \geqslant 1$, $\mathbf{0_I}$ | 50
$\mathscr{L}^{\otimes(-n)} = (\mathscr{L}^{-1})^{\otimes n}$. With these notation, there is then a *canonical functorial isomorphism*

$$\mathscr{L}^{\otimes m} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n} \simeq \mathscr{L}^{\otimes(n+m)} \tag{5.4.4.1}$$

for any rational integers m and n: indeed, by definition, we immediately reduce to the case where $m = -1$, $n = 1$, and the isomorphism in question is then that defined in (5.4.3).

(5.4.5). Let $f : Y \to X$ be a morphism of ringed spaces. If $\mathscr{L}$ is a locally free (resp. invertible) $\mathscr{O}_X$-module, then $f^*(\mathscr{L})$ is a locally free (resp. invertible) $\mathscr{O}_Y$-module: this follows immediately from that the inverse images of two locally isomorphic $\mathscr{O}_X$-modules are locally isomorphic, that f^* commutes with finite direct sums, and that $f^*(\mathscr{O}_X) = \mathscr{O}_Y$ (4.3.4). In addition, we know that we have a canonical functorial homomorphism $f^*(\mathscr{L}^\vee) \to (f^*(\mathscr{L}))^\vee$ (4.4.6), and when $\mathscr{L}$ is locally free, this homomorphism is *bijective*: indeed, we again reduce to the case where $\mathscr{L} = \mathscr{O}_X$ which is trivial. We conclude that if $\mathscr{L}$ is invertible, then $f^*(\mathscr{L}^{\otimes n})$ canonically identifies with $(f^*(\mathscr{L}))^{\otimes n}$ for each rational integer n.

(5.4.6). Let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module; we denote by $\Gamma_\bullet(X, \mathscr{L})$ or simply $\Gamma_\bullet(\mathscr{L})$ the abelian group direct sum $\bigoplus_{n\in\mathbf{Z}} \Gamma(X, \mathscr{L}^{\otimes n})$; we equip it with the structure of a *graded ring*, by corresponding to a pair (s_n, s_m), where $s_n \in \Gamma(X, \mathscr{L}^{\otimes n})$, $s_m \in \Gamma(X, \mathscr{L}^{\otimes m})$, the section of $\mathscr{L}^{\otimes(n+m)}$ over X which corresponds canonically (5.4.4.1) to the section $s_n \otimes s_m$ of $\mathscr{L}^{\otimes n} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes m}$; the associativity of this multiplication is verified in an immediate way. It is clear that $\Gamma_\bullet(X, \mathscr{L})$ is a covariant functor in $\mathscr{L}$, with values in the category of graded rings.

If now $\mathscr{F}$ is any $\mathscr{O}_X$-module, then we set

$$\Gamma_\bullet(\mathscr{L}, \mathscr{F}) = \bigoplus_{n\in\mathbf{Z}} \Gamma(X, \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}).$$

We equip this abelian group with the structure of a *graded module* over the graded ring $\Gamma_\bullet(\mathscr{L})$ in the following way: to a pair (s_n, u_m), where $s_n \in \Gamma(X, \mathscr{L}^{\otimes n})$ and $u_m \in \Gamma(X, \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes m})$, we associate the section of $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes(m+n)}$ which canonically corresponds (5.4.4.1) to $s_n \otimes u_m$; the verification of the module axioms are immediate. For X and $\mathscr{L}$ fixed, $\Gamma_\bullet(\mathscr{L}, \mathscr{F})$ is a covariant functor in $\mathscr{F}$ with values in the category of graded $\Gamma_\bullet(\mathscr{L})$-modules; for X and $\mathscr{F}$ fixed, it is a covariant functor in $\mathscr{L}$ with values in the category of abelian groups.

If $f : Y \to X$ is a morphism of ringed spaces, the canonical homomorphism (4.4.3.2) $\rho : \mathscr{L}^{\otimes n} \to f_*(f^*(\mathscr{L}^{\otimes n}))$ defines a homomorphism of abelian groups $\Gamma(X, \mathscr{L}^{\otimes n}) \to \Gamma(Y, f^*(\mathscr{L}^{\otimes n}))$, and as $f^*(\mathscr{L}^{\otimes n}) = (f^*(\mathscr{L}))^{\otimes n})$, it follows from the definitions of the canonical homomorphisms (4.4.3.2) and (5.4.4.1) that the above homomorphisms define a *functorial homomorphism of graded rings* $\Gamma_\bullet(\mathscr{L}) \to \Gamma_\bullet(f^*(\mathscr{L}))$. The same canonical homomorphism (4.4.3) similarly defines a homomorphism of abelian groups $\Gamma(X, \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}) \to \Gamma(Y, f^*(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}))$, and as

$$f^*(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}) = f^*(\mathscr{F}) \otimes_{\mathscr{O}_Y} (f^*(\mathscr{L}))^{\otimes n} \quad (4.3.3.1),$$

these homomorphism (for n variable) define a *di-homomorphism of graded modules* $\Gamma_\bullet(\mathscr{L}, \mathscr{F}) \to$ $\mathbf{0_I}$ | 51
$\Gamma_\bullet(f^*(\mathscr{L}), f^*(\mathscr{F}))$.

(5.4.7). One can show that there exists a *set* $\mathfrak{M}$ (also denoted $\mathfrak{M}(X)$) of invertible $\mathscr{O}_X$-modules such that each invertible $\mathscr{O}_X$-module is isomorphic to a unique element of $\mathfrak{M}$ [8]; we define on $\mathfrak{M}$ a composition law by sending two elements $\mathscr{L}$ and $\mathscr{L}'$ of $\mathfrak{M}$ to the unique element of $\mathfrak{M}$ isomorphic to $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{L}'$. With this composition law, $\mathfrak{M}$ *is a group isomorphic to the cohomology group* $\mathrm{H}^1(X, \mathscr{O}_X^*)$, where $\mathscr{O}_X^*$ is the subsheaf of $\mathscr{O}_X$ such that $\Gamma(U, \mathscr{O}_X^*)$ is the group of invertible elements of the ring $\Gamma(U, \mathscr{O}_X)$ for each open $U \subset X$ ($\mathscr{O}_X^*$ is therefore a sheaf of *multiplicative* abelian groups).

We will note that for all open $U \subset X$, the group of sections $\Gamma(U, \mathscr{O}_X^*)$ canonically identifies with the *automorphism group* of the $(\mathscr{O}_X|U)$-module $\mathscr{O}_X|U$, the identification sending a section ε of $\mathscr{O}_X^*$ over U to the automorphism u of $\mathscr{O}_X|U$ such that $u_x(s_x) = \varepsilon_x s_x$ for all $x \in X$ and all $s_x \in \mathscr{O}_x$. Then let $\mathfrak{U} = (U_\lambda)$ be an open cover of X; the data, for each pair of indices (λ, μ), of an automorphism $\theta_{\lambda\mu}$ of $\mathscr{O}_X|(U_\lambda \cap U_\mu)$ is the same as giving a 1-*cochain* of the cover $\mathfrak{U}$, with values in $\mathscr{O}_X^*$, and say that the $\theta_{\lambda\mu}$ satisfy the gluing condition (3.3.1), meaning that the corresponding cochain is a *cocycle*. Similarly, the data, for each λ, of an automorphism ω_λ of $\mathscr{O}_X|U_\lambda$ is the same as the data of a 0-cochain of the cover $\mathfrak{U}$, with values in $\mathscr{O}_X^*$, and its *coboundary* corresponds to the family of automorphisms $(\omega_\lambda|U_\lambda \cap U_\mu) \circ (\omega_\mu|U_\lambda \cap U_\mu)^{-1}$. We can send each 1-cocycle of $\mathfrak{U}$ with values in $\mathscr{O}_X^*$ to the element of $\mathfrak{M}$ isomorphic to an invertible $\mathscr{O}_X$-module obtained by gluing with respect to the family of

[8] See the book in preparation cited in the introduction.

automorphisms $(\theta_{\lambda\mu})$ corresponding to this cocycle, and to two cohomologous coycles correspond two equal elements of $\mathfrak{M}$ (3.3.2); in other words, we thus define a map $\phi_{\mathfrak{U}} : \mathrm{H}^1(\mathfrak{U}, \mathscr{O}_X^*) \to \mathfrak{M}$. In addition, if $\mathfrak{V}$ is a second open cover of X, finer than $\mathfrak{U}$, then the diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathfrak{U}, \mathscr{O}_X^*) \arrow[dd] \arrow[dr, "\phi_{\mathfrak{U}}"] & \\
& \mathfrak{M} \\
\mathrm{H}^1(\mathfrak{V}, \mathscr{O}_X^*) \arrow[ur, "\phi_{\mathfrak{V}}"'] &
\end{tikzcd}$$

where the vertical arrow is the canonical homomorphism (G, II, 5.7), is commutative, as a result of (3.3.3). By passing to the inductive limit, we therefore obtain a map $\mathrm{H}^1(X, \mathscr{O}_X^*) \to \mathfrak{M}$, the Čech cohomology group $\check{\mathrm{H}}^1(X, \mathscr{O}_X^*)$ identifying as we know with the first cohomology group $\mathrm{H}^1(X, \mathscr{O}_X^*)$ (G, II, 5.9, Cor. of Thm. 5.9.1). This map is *surjective*: indeed, by definition, for each invertible $\mathscr{O}_X$-module $\mathscr{L}$, there is an open cover (U_λ) of X such that $\mathscr{L}$ is obtained by gluing the sheaves $\mathscr{O}_X|U_\lambda$ (3.3.1). It is also *injective*, since it suffices to prove for the maps $\mathrm{H}^1(\mathfrak{U}, \mathscr{O}_X) \to \mathfrak{M}$, and this follows from (3.3.2). It remains to show that the bijection thus defined is a group homomorphism. $\mathbf{0_I}$ | 52
Given two invertible $\mathscr{O}_X$-modules $\mathscr{L}$ and $\mathscr{L}'$, there is an open cover (U_λ) such that $\mathscr{L}|U_\lambda$ and $\mathscr{L}'|U_\lambda$ are isomorphic to $\mathscr{O}_X|U_\lambda$ for each λ; so there is for each index λ an element a_λ (resp. a'_λ) of $\Gamma(U_\lambda, \mathscr{L})$ (resp. $\Gamma(U_\lambda, \mathscr{L}')$) such that the elements of $\Gamma(U_\lambda, \mathscr{L})$ (resp. $\Gamma(U_\lambda, \mathscr{L}')$) are the $s_\lambda \cdot a_\lambda$ (resp. $s_\lambda \cdot a'_\lambda$), where s_λ varies over $\Gamma(U_\lambda, \mathscr{O}_X)$. The corresponding cocycles $(\varepsilon_{\lambda\mu})$, $(\varepsilon'_{\lambda\mu})$ are such that $s_\lambda \cdot a_\lambda = s_\mu \cdot a_\mu$ (resp. $s_\lambda \cdot a'_\lambda = s_\mu \cdot a'_\mu$) over $U_\lambda \cap U_\mu$ is equivalent to $s_\lambda = \varepsilon_{\lambda\mu} s_\mu$ (resp. $s_\lambda = \varepsilon'_{\lambda\mu} s_\mu$) over $U_\lambda \cap U_\mu$. As the sections of $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{L}'$ over U_λ are the finite sums of the $s_\lambda s'_\lambda \cdot (a_\lambda \otimes a'_\lambda)$ where s_λ and s'_λ vary over $\Gamma(U_\lambda, \mathscr{O}_X)$, it is clear that the cocycle $(\varepsilon_{\lambda\mu}, \varepsilon'_{\lambda\mu})$ corresponds to $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{L}'$, which finishes the proof. [9]

(5.4.8). Let $f = (\psi, \omega)$ be a morphism $Y \to X$ of ringed spaces. The functor $f^*(\mathscr{L})$ to the category of free $\mathscr{O}_X$-modules defines a map (which we still denote f^* by abuse of language) from the set $\mathfrak{M}(X)$ to the set $\mathfrak{M}(Y)$. Second, we have a canonical homomorphism (T, 3.2.2)

$$\mathrm{H}^1(X, \mathscr{O}_X^*) \longrightarrow \mathrm{H}^1(Y, \mathscr{O}_Y^*). \tag{5.4.8.1}$$

When we canonically identify (5.4.7) $\mathfrak{M}(X)$ and $\mathrm{H}^1(X, \mathscr{O}_X^*)$ (resp. $\mathfrak{M}(Y)$ and $\mathrm{H}^1(Y, \mathscr{O}_Y^*)$), the homomorphism (5.4.8.1) *identifies with the map* f^*. Indeed, if $\mathscr{L}$ comes from a cocycle $(\varepsilon_{\lambda\mu})$ corresponding to an open cover (U_λ) of X, then it suffices to show that $f^*(\mathscr{L})$ comes from a cocycle whose cohomology class is the image under (5.4.8.1) of $(\varepsilon_{\lambda\mu})$. If $\theta_{\lambda\mu}$ is the automorphism of $\mathscr{O}_X|(U_\lambda \cap U_\mu)$ which corresponds to $\varepsilon_{\lambda\mu}$, then it is clear that $f^*(\mathscr{L})$ is obtained by gluing the $\mathscr{O}_Y|\psi^{-1}(U_\lambda)$ by means of the automorphisms $f^*(\theta_{\lambda\mu})$, and it then suffices to check that these latter automorphisms corresponds to the cocycle $(\omega^\sharp(\varepsilon_{\lambda\mu}))$, which follows immediately from the definitions (we can identify $\varepsilon_{\lambda\mu}$ with its canonical image under ρ (3.7.2), a section of $\psi^*(\mathscr{O}_X^*)$ over $\psi^{-1}(U_\lambda \cap U_\mu)$).

(5.4.9). Let $\mathscr{E}$ and $\mathscr{F}$ be two $\mathscr{O}_X$-modules, $\mathscr{F}$ assumed to be *locally free*, and let $\mathscr{G}$ be an *$\mathscr{O}_X$-module extension of $\mathscr{F}$ by $\mathscr{E}$*, in other words there exists an exact sequence $0 \to \mathscr{E} \xrightarrow{i} \mathscr{G} \xrightarrow{p} \mathscr{F} \to 0$. Then, for each $x \in X$, there exists an open neighbourhood U of x such that $\mathscr{G}|U$ is isomorphic to the *direct sum* $\mathscr{E}|U \oplus \mathscr{F}|U$. We can reduce to the case where $\mathscr{F} = \mathscr{O}_X^n$; let e_i $(1 \leqslant i \leqslant n)$ be the canonical sections (5.5.5) of $\mathscr{O}_X^n$; there then exists an open neighbourhood U of x and n sections s_i of $\mathscr{G}$ over U such that $p(s_i|U) = e_i|U$ for $1 \leqslant i \leqslant n$. That being so, let f be the homomorphism $\mathscr{F}|U \to \mathscr{G}|U$ defined by the sections $s_i|U$ (5.1.1). It is immediate that for each open $V \subset U$, and each section $s \in \Gamma(V, \mathscr{G})$ we have $s - f(p(s)) \in \Gamma(V, \mathscr{E})$, hence our assertion.

(5.4.10). Let $f : X \to Y$ be a morphism of ringed spaces, $\mathscr{F}$ an $\mathscr{O}_X$-module, and $\mathscr{L}$ a locally free $\mathscr{O}_Y$-module of finite rank. Then there exists a canonical isomorphism

$$f_*(\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{L} \simeq f_*(\mathscr{F} \otimes_{\mathscr{O}_X} f^*(\mathscr{L})). \tag{5.4.10.1}$$

[9] For a general form of this result, see the book cited in the note on p. 51.

Indeed, for each $\mathscr{O}_Y$-module $\mathscr{L}$, we have a canonical homomorphism $\mathbf{0_I}$ | 53

$$f_*(\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{L} \xrightarrow{1\otimes\rho} f_*(\mathscr{F}) \otimes_{\mathscr{O}_Y} f_*(f^*(\mathscr{L})) \xrightarrow{\alpha} f_*(\mathscr{F} \otimes_{\mathscr{O}_X} f^*(\mathscr{L})),$$

ρ the homomorphism (4.4.3.2) and α the homomorphism (4.2.2.1). To show that when $\mathscr{L}$ is locally free, this homomorphism is bijective, it suffices, since the questions is local, to consider the case where $\mathscr{L} = \mathscr{O}_X^n$; in addition, f_* and f^* commute with finite direct sums, so we can assume $n = 1$, and in this case the proposition follows immediately from the definitions and from the relation $f^*(\mathscr{O}_Y) = \mathscr{O}_X$.

5.5. Sheaves on a locally ringed space

(5.5.1). We say that a ringed space $(X, \mathscr{O}_X)$ is a *locally ringed space* if for each $x \in X$, $\mathscr{O}_x$ is a local ring; these ringed spaces will be by far the most frequent ringed spaces that we will consider in this work. We then denote by $\mathfrak{m}_x$ the *maximal ideal* of $\mathscr{O}_x$, by $k(x)$ the *residue field* $\mathscr{O}_x/\mathfrak{m}_x$; for each $\mathscr{O}_X$-module $\mathscr{F}$, each open set U of X, each point $x \in U$, and each section $f \in \Gamma(U, \mathscr{F})$, we denote by $f(x)$ the *class* of the germ $f_x \in \mathscr{F}_x$ mod. $\mathfrak{m}_x\mathscr{F}_x$, and we say that this is the *value* of f at the point x. The relation $f(x) = 0$ then means that $f_x \in \mathfrak{m}_x\mathscr{F}_x$; when this is so, we say (by abuse of language) that f *is zero at* x. We will take care not to confuse this relation with $f_x = 0$.

(5.5.2). Let X be a locally ringed space, $\mathscr{L}$ an invertible $\mathscr{O}_X$-module, and f a section of $\mathscr{L}$ over X. There is then an *equivalence* between the three following properties for a point $x \in X$:

(a) *f_x is a generator of $\mathscr{L}_x$;*
(b) $f_x \notin \mathfrak{m}_x\mathscr{L}_x$ (in other words, $f(x) \neq 0$);
(c) *there exists a section g of $\mathscr{L}^{-1}$ over an open neighbourhood V of x such that the canonical image of $f \otimes g$ in $\Gamma(V, \mathscr{O}_X)$ (5.4.3) is the unit section.*

Indeed, since the questions is local, we can reduce to the case where $\mathscr{L} = \mathscr{O}_X$; the equivalence of (a) and (b) are then evident, and it is clear that (c) implies (b). Conversely, if $f_x \notin \mathfrak{m}_x$, then f_x is invertible in $\mathscr{O}_x$, say $f_x g_x = 1_x$. By definition of germs of sections, this means that there exists a neighbourhood V of x and a section g of $\mathscr{O}_X$ over V such that $fg = 1$ in V, hence (c).

It follows immediately from the condition (c) that the set X_f of x satisfying the equivalent conditions (a), (b), (c) is *open* in X; following the terminology introduced in (5.5.1), this is the set of the x for which f *does not vanish*.

(5.5.3). Under the hypotheses of (5.5.2), let $\mathscr{L}'$ be a second invertible $\mathscr{O}_X$-module; then, if $f \in \Gamma(X, \mathscr{L})$, $g \in \Gamma(X, \mathscr{L}')$, we have

$$X_f \cap X_g = X_{f\otimes g}.$$

We can in fact reduce immediately to the case where $\mathscr{L} = \mathscr{L}' = \mathscr{O}_X$ (since the questions is local); as $f \otimes g$ then canonically identifies with the product fg, the proposition is evident.

$\mathbf{0_I}$ | 54

(5.5.4). Let $\mathscr{F}$ be a locally free $\mathscr{O}_X$ of rank n; it is immediate that $\wedge^p\mathscr{F}$ is a locally free $\mathscr{O}_X$-module of rank $\binom{n}{p}$ if $p \leqslant n$ and 0 if $p > n$, since the question is local and we can reduce to the case where $\mathscr{F} = \mathscr{O}_X^n$; in addition, for each $x \in X$, $(\wedge^p\mathscr{F})_x/\mathfrak{m}_x(\wedge^p\mathscr{F})_x$ is a vector space of dimension $\binom{n}{p}$ over $k(x)$, which canonically identifies with $\wedge^p(\mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x)$. Let $s_1, \ldots, s_p$ be the sections of $\mathscr{F}$ over an open subset U of X, and let $s = s_1 \wedge \cdots \wedge s_p$, which is a section of $\wedge^p\mathscr{F}$ over U (4.1.5); we have $s(x) = s_1(x) \wedge \cdots \wedge s_p(x)$, and as a result, we say that the $s_1(x), \ldots, s_p(x)$ are *linearly dependent* means $s(x) = 0$. We conclude that the *set of the $x \in X$ such that $s_1(x), \ldots, s_p(x)$ are linearly independent is open in* X: it suffices in fact, by reducing to the case where $\mathscr{F} = \mathscr{O}_X^n$, to apply (5.5.2) to the section image of s under one of the projections of $\wedge^p\mathscr{F} = \mathscr{O}_X^{\binom{n}{p}}$ to the $\binom{n}{p}$ factors.

In particular, if $s_1, \ldots, s_n$ are n sections of $\mathscr{F}$ over U such that $s_1(x), \ldots, s_n(x)$ are linearly independent for each point $x \in U$, then the homomorphism $u : \mathscr{O}_X^n|U \to \mathscr{F}|U$ defined by the s_i (5.1.1) is an *isomorphism*: indeed, we can restrict to the case where $\mathscr{F} = \mathscr{O}_X^n$ and where we canonically identify $\wedge^n\mathscr{F}$ and $\mathscr{O}_X$; $s = s_1 \wedge \cdots \wedge s_n$ is then an *invertible* section of $\mathscr{O}_X$ over U, and we define an inverse homomorphism for u by means of the Cramer formulas.

(5.5.5). Let $\mathscr{E}$ and $\mathscr{F}$ be two locally free $\mathscr{O}_X$-modules (of finite rank), and let $u : \mathscr{E} \to \mathscr{F}$ be a homomorphism. For there to exist a neighbourhood U of $x \in X$ such that $u|U$ is *injective* and that $\mathscr{F}|U$ is *the direct sum of the* $u(\mathscr{E})|U$ *and of a locally free* $(\mathscr{O}_X|U)$*-submodule* $\mathscr{G}$, it is necessary and sufficient that $u_x : \mathscr{E}_x \to \mathscr{F}_x$ gives, by passing to quotients, an *injective* homomorphism of vector spaces $\mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x \to \mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x$. The condition is indeed *necessary*, since $\mathscr{F}_x$ is then the direct sum of the free $\mathscr{O}_x$-modules $u_x(\mathscr{E}_x)$ and $\mathscr{G}_x$, so $\mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x$ is the direct sum of $u_x(\mathscr{E}_x)/\mathfrak{m}_x u_x(\mathscr{E}_x)$ and of $\mathscr{G}_x/\mathfrak{m}_x\mathscr{G}_x$. The condition is *sufficient*, since we can reduce to the case where $\mathscr{E} = \mathscr{O}_X^m$; let $s_1, \ldots, s_m$ be the images under u of the sections e_i of $\mathscr{O}_X^m$ such that $(e_i)_y$ is equal to the i-th element of the canonical basis of $\mathscr{O}_y^m$ for each $y \in Y$ (*canonical sections* of $\mathscr{O}_X^m$); by hypothesis, the $s_1(x), \ldots, s_m(x)$ are linearly independent, so if $\mathscr{F}$ is of rank n, then there exist $n-m$ sections $s_{m+1}, \ldots, s_n$ of $\mathscr{F}$ over a neighbourhood V of x such that the $s_i(x)$ $(1 \leqslant i \leqslant n)$ form a basis for $\mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x$. There then exists (5.5.4) a neighbourhood $U \subset V$ of x such that the $s_i(y)$ $(1 \leqslant i \leqslant n)$ form a basis for $\mathscr{F}_y/\mathfrak{m}_y\mathscr{F}_y$ for each $y \in V$, and we conclude (5.5.4) that there is an isomorphism from $\mathscr{F}|U$ to $\mathscr{O}_X^n|U$, sending the $s_i|U$ $(1 \leqslant i \leqslant m)$ to the $e_i|U$, which finishes the proof.

§6. Flatness

(6.0). The notion of flatness is due to J.-P. Serre [Ser56]; in the following, we omit the proofs of the results which are presented in the *Algèbre commutative* of N. Bourbaki, to which we refer the reader. We assume that all rings are commutative.[10]

If M, N are two A-modules, M' (resp. N') a submodule of M (resp. N), we denote by $\operatorname{Im}(M' \otimes_A N')$ the submodule of $M \otimes_A N$, the image under the canonical map $M' \otimes_A N' \to M \otimes_A N$. $\mathbf{0}_{\mathbf{I}}$ | 55

6.1. Flat modules

(6.1.1). Let M be an A-module. The following conditions are equivalent:

(a) The functor $M \otimes_A N$ is exact in N on the category of A-modules;
(b) $\operatorname{Tor}_i^A(M, N) = 0$ for each $i > 0$ and for each A-module N;
(c) $\operatorname{Tor}_1^A(M, N) = 0$ for each A-module N.

When M satisfies these conditions, we say that M is a *flat A-module*. It is clear that each free A-module is flat.

For M to be a flat A-module, it suffices that for each ideal $\mathfrak{J}$ of A, *of finite type*, the canonical map $M \otimes_A \mathfrak{J} \to M \otimes_A A = M$ is *injective*.

(6.1.2). Each inductive limit of flat A-modules is a flat A-module. For a direct sum $\bigoplus_{\lambda \in L} M_\lambda$ of A-modules to be a flat A-modules, it is necessary and sufficient that each of the A-modules M_λ is flat. In particular, every projective A-module is flat.

Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules, such that M'' is *flat*. Then, for each A-module N, the sequence

$$0 \longrightarrow M' \otimes_A N \longrightarrow M \otimes_A N \longrightarrow M'' \otimes_A N \longrightarrow 0$$

is exact. In addition, for M to be flat, is it necessary and sufficient that M' is (but it can be that M and M' are flat without $M'' = M/M'$ being so).

(6.1.3). Let M be a flat A-module, N any A-module; for two submodules N' N'' of N, we then have

$$\operatorname{Im}(M \otimes_A (N' + N'')) = \operatorname{Im}(M \otimes_A N') + \operatorname{Im}(M \otimes_A N''),$$
$$\operatorname{Im}(M \otimes_A (N' \cap N'')) = \operatorname{Im}(M \otimes_A N') \cap \operatorname{Im}(M \otimes_A N'')$$

(images taken in $M \otimes_A N$).

(6.1.4). Let M and N be two A-modules, M' (resp. N') a submodule of M (resp. N), and suppose that one of the modules M/M', N/N' is flat. Then we have $\operatorname{Im}(M' \otimes_A N') = \operatorname{Im}(M' \otimes_A N) \cap (M \otimes_A N')$ (images in $M \otimes_A N$). In particular, if $\mathfrak{J}$ is an ideal of A and if M/M' is flat, then we have $\mathfrak{J}M' = M' \cap \mathfrak{J}M$.

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[10] See the exposé cited of N. Bourbaki for the generalization from most of the results to the noncommutative case.

6.2. Change of ring When an additive group M is equipped with multiple modules structures relative to the rings A, B, ..., we say that M is flat as an A-module, B-module, ..., we sometimes also say that M is *A-flat*, *B-flat*, ...

(6.2.1). Let A and B be two rings, M an A-module, N an (A, B)-bimodule. If M is flat and if N is B-flat, then $M \otimes_A N$ is B-flat. In particular, if M and N are two flat A-modules, then $M \otimes_A N$ is a flat A-module. If B is an A-algebra and if M is a flat A-module, then the B-module $M_{(B)} = M \otimes_A B$ is flat. Finally, if B is an A-algebra which is flat as an A-module, and if N is a flat B-module, then N is also A-flat. $\mathbf{0_I}$ | 56

(6.2.2). Let A be a ring, B an A-algebra which is flat as an A-module. Let M, N be two A-modules, such that M admits a finite presentation; then the canonical homomorphism

$$\operatorname{Hom}_A(M, N) \otimes_A B \longrightarrow \operatorname{Hom}_B(M \otimes_A B, N \otimes_A B) \tag{6.2.2.1}$$

(sending $u \otimes b$ to the homomorphism $m \otimes b' \mapsto u(m) \otimes b'b$) is an isomorphism.

(6.2.3). Let $(A_\lambda, \phi_{\mu\lambda})$ be a filtered inductive system of rings; let $A = \varinjlim A_\lambda$. On the other hand, for each λ, let M_λ be an A_λ-module, and for $\lambda \leqslant \mu$ let $\theta_{\mu\lambda} : M_\lambda \to M_\mu$ be a $\phi_{\mu\lambda}$-homomorphism, such that $(M_\lambda, \theta_{\mu\lambda})$ is an inductive system; $M = \varinjlim M_\lambda$ is then an A-module. This being so, if for each λ, M_λ is a *flat* A_λ-module, then M is a *flat* A-module. Indeed, let $\mathfrak{J}$ be an ideal *of finite type* of A; by definition of the inductive limit, there exists an index λ and an ideal $\mathfrak{J}_\lambda$ of A_λ such that $\mathfrak{J} = \mathfrak{J}_\lambda A$. If we put $\mathfrak{J}'_\mu = \mathfrak{J}_\lambda A_\mu$ for $\mu \geqslant \lambda$, we also have $\mathfrak{J} = \varinjlim \mathfrak{J}'_\mu$ (where μ varies over the indices $\geqslant \lambda$), hence (the functor $\varinjlim$ being exact and commuting with tensor products)

$$M \otimes_A \mathfrak{J} = \varinjlim (M_\mu \otimes_{A_\mu} \mathfrak{J}'_\mu) = \varinjlim \mathfrak{J}'_\mu M_\mu = \mathfrak{J} M.$$

6.3. Local nature of flatness

(6.3.1). If A is a ring, S a multiplicative subset of A, $S^{-1}A$ is a *flat A-module*. Indeed, for each A-module N, $N \otimes_A S^{-1}A$ identifies with $S^{-1}N$ (1.2.5) and we know (1.3.2) that $S^{-1}N$ is an exact functor in N.

If now M is a flat A-module, $S^{-1}M = M \otimes_A S^{-1}A$ is a flat $S^{-1}A$-module (6.2.1), so it is also A-flat according to the above and from (6.2.1). In particular, if P is an $S^{-1}A$-module, we can consider it as an A-module isomorphic to $S^{-1}P$; for P to be A-flat, it is necessary and sufficient that it is $S^{-1}A$-flat.

(6.3.2). Let A be a ring, B an A-algebra, and T a multiplicative subset of B. If P is a B-module which is *A-flat*, $T^{-1}P$ is *A-flat*. Indeed, for each A-module N, we have $(T^{-1}P) \otimes_A N = (T^{-1}B \otimes_B P) \otimes_A N$ $= T^{-1}B \otimes_B (P \otimes_A N) = T^{-1}(P \otimes_A N)$; $T^{-1}(P \otimes_A N)$ is an exact functor in N, being the composition of the two exact functors $P \otimes_A N$ (in N) and $T^{-1}Q$ (in Q). If S is a multiplicative subset of A such that its image in B is *contained in* T, then $T^{-1}P$ is equal to $S^{-1}(T^{-1}P)$, so it is also $S^{-1}A$-flat according to (6.3.1).

(6.3.3). Let $\phi : A \to B$ be a ring homomorphism, M a B-module. The following properties are equivalent:

(a) M is a flat A-module.
(b) For each maximal ideal $\mathfrak{n}$ of B, $M_\mathfrak{n}$ is a flat A-module.
(c) For each maximal ideal $\mathfrak{n}$ of B, by setting $\mathfrak{m} = \phi^{-1}(\mathfrak{n})$, $M_\mathfrak{n}$ is a flat $A_\mathfrak{m}$-module.

Indeed, as $M_\mathfrak{n} = (M_\mathfrak{n})_\mathfrak{m}$, the equivalence of (b) and (c) follows from (6.3.1), and the fact that (a) implies (b) is a particular case of (6.3.2). It remains to see that (b) implies (a), that is to say, that for each injective homomorhism $u : N' \to N$ of A-modules, the homomorphism $v = 1 \otimes u : M \otimes_A N' \to M \otimes_A N$ is injective. We have that v is also a homomorphism of B-modules, and we know that for it to be injective, it suffices that for each maximal ideal $\mathfrak{n}$ of B, $v_\mathfrak{n} : (M \otimes_A N')_\mathfrak{n} \to (M \otimes_A N)_\mathfrak{n}$ is injective. But as $\mathbf{0_I}$ | 57

$$(M \otimes_A N)_\mathfrak{n} = B_\mathfrak{n} \otimes_B (M \otimes_A N) = M_\mathfrak{n} \otimes_A N,$$

$v_\mathfrak{n}$ is none other that the homomorphism $1 \otimes u : M_\mathfrak{n} \otimes_A N' \to M_\mathfrak{n} \otimes_A N$, which is injective since $M_\mathfrak{n}$ is A-flat.

In particular (by taking $B = A$), for an A-module M to be flat, it is necessary and sufficient that $M_{\mathfrak{m}}$ is $A_{\mathfrak{m}}$-flat for each maximal ideal $\mathfrak{m}$ of A.

(6.3.4). Let M be an A-module; if M is flat, and if $f \in A$ does not divide 0 in A, f does not kill any element $\neq 0$ in M, since the homomorphism $m \mapsto f \cdot m$ is expressed as $1 \otimes u$, where u is the multiplication $a \mapsto f \cdot a$ on A and M is identified with $M \otimes_A A$; if u is injective, it is the same for $1 \otimes u$ since M is flat. In particular, if A is *integral*, M is *torsion-free*.

Conversely, suppose that A is integral, M is torsion-free, and suppose that for each maximal ideal $\mathfrak{m}$ of A, $A_{\mathfrak{m}}$ is a *discrete valuation ring*; then M is *A-flat*. Indeed, it suffices (6.3.3) to prove that $M_{\mathfrak{m}}$ is $A_{\mathfrak{m}}$-flat, and we can therefore suppose that A is already a discrete valuation ring. But as M is the inductive limit of its submodules of finite type, and these latter submodules are torsion-free, we can in addition reduce to the case where M is of finite type (6.1.2). The proposition follows in this case from that M is a free A-module.

In particular, if A is an *integral* ring, $\phi : A \to B$ a ring homomorphism making B a *flat* A-module and $\neq \{0\}$, ϕ is necessarily *injective*. Conversely, if B is integral, A a subring of B, and if for each maximal ideal $\mathfrak{m}$ of A, $A_{\mathfrak{m}}$ is a discrete valuation ring, then B is *A-flat*.

6.4. Faithfully flat modules

(6.4.1). For an A-module M, the following four properties are equivalent:

(a) For a sequence $N' \to N \to N''$ of A-modules to be exact, it is necessary and sufficient that the sequence $M \otimes_A N' \to M \otimes_A N \to M \otimes_A N''$ is exact;
(b) M is flat for each A-module N, the relation $M \otimes_A N = 0$ implies $N = 0$;
(c) M is flat for each homomorphism $v : N \to N'$ of A-modules, the relation $1_M \otimes v = 0$, 1_M being the identity automorphism of M;
(d) M is flat for each maximal ideal $\mathfrak{m}$ of A, $\mathfrak{m}M \neq M$.

When M satisfies these conditions, we say that M is a *faithfully flat* A-module; M is then necessarily a *faithful* module. In addition, if $u : N \to N'$ is a homomorphism of A-modules, then for u to be injective (resp. surjective, bijective), it is necessary and sufficient that $1 \otimes u : M \otimes_A N \to M \otimes_A N'$ is so.

$\mathbf{0_I}$ | 58

(6.4.2). A free module $\neq \{0\}$ is faithfully flat; it is the same for the direct sum of a flat module and a faithfully flat module. If S is a multiplicative subset of A, then $S^{-1}A$ is a faithfully flat A-module if S consists of invertible elements (so $S^{-1}A = A$).

(6.4.3). Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules; if M' and M'' are flat, and if one of the two is faithfully flat, then M is also faithfully flat.

(6.4.4). Let A and B be two rings, M an A-module, N an (A, B)-bimodule. If M is faithfully flat and if N is a faithfully flat B-module, then $M \otimes_A N$ is a faithfully flat B-module. In particular, if M and N are two faithfully flat A-modules, then so is $M \otimes_A N$. If B is an A-algebra and if M is a faithfully flat A-module, the B-module $M_{(B)}$ is faithfully flat.

(6.4.5). If M is a faithfully flat A-modules and if S is a multiplicative subset of A, $S^{-1}M$ is a faithfully flat $S^{-1}A$-module, since $S^{-1}M = M \otimes_A (S^{-1}A)$ (6.4.4). Conversely, if for each maximal ideal $\mathfrak{m}$ of A, $M_{\mathfrak{m}}$ is a faithfully flat $A_{\mathfrak{m}}$-module, then M is a faithfully flat A-module, since M is A-flat (6.3.3), and we have

$$M_{\mathfrak{m}}/\mathfrak{m}M_{\mathfrak{m}} = (M \otimes_A A_{\mathfrak{m}}) \otimes_{A_{\mathfrak{m}}} (A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}}) = M \otimes_A (A/\mathfrak{m}) = M/\mathfrak{m}M,$$

so the hypotheses imply that $M/\mathfrak{m}M \neq 0$ for each maximal ideal $\mathfrak{m}$ of A, which proves our assertion (6.4.1).

6.5. Restriction of scalars

(6.5.1). Let A be a ring, $\phi : A \to B$ a ring homomorphism making B an A-algebra. Suppose that there exists a B-module N which is a *faithfully flat* A-module. Then, for each A-module M, the homomorphism $x \mapsto 1 \otimes x$ from M to $B \otimes_A M = M_{(B)}$ is *injective*. In particular, ϕ is injective; for each ideal $\mathfrak{a}$ of A, we have $\phi^{-1}(\mathfrak{a}B) = \mathfrak{a}$; for each maximal (resp. prime) ideal $\mathfrak{m}$ of A, there exists a maximal (resp. prime) ideal $\mathfrak{n}$ of B such that $\phi^{-1}(\mathfrak{n}) = \mathfrak{m}$.

(6.5.2). When the conditions of (6.5.1) are satisfied, we identify A with the subring of B by ϕ and more generally, for each A-module M, we identify M with an A-submodule of $M_{(B)}$. We note that if B is also *Noetherian*, then so is A, since the map $\mathfrak{a} \mapsto \mathfrak{a}B$ is an increasing injection from the set of ideals of A to the set of ideals of B; the existence of an infinite strictly increasing sequence of ideals of A thus implies the existence of an analogous sequence of ideals of B.

6.6. Faithfully flat rings

(6.6.1). Let $\phi : A \to B$ be a ring homomorphism making B an A-algebra. The following five properties are equivalent:

(a) B is a faithfully flat A-module (in other words, $M_{(B)}$ is an *exact* and *faithful* functor in M).
(b) The homomorphism ϕ is injective and the A-module $B/\phi(A)$ is flat. $\mathbf{0_I}$ | 59
(c) The A-module B is flat (in other words, the functor $M_{(B)}$ is *exact*), and for each A-module M. the homomorphism $x \mapsto 1 \otimes x$ from M to $M_{(B)}$ is injective.
(d) The A-module B is flat and for each ideal $\mathfrak{a}$ of A, we have $\phi^{-1}(\mathfrak{a}B) = \mathfrak{a}$.
(e) The A-module B is flat and for each maximal ideal $\mathfrak{m}$ of A, there exists a maximal ideal $\mathfrak{n}$ of B such that $\phi^{-1}(\mathfrak{n}) = \mathfrak{m}$.

When these conditions are satisfied, we identify A with a subring of B.

(6.6.2). Let A be a *local* ring, $\mathfrak{m}$ its maximal ideal, and B an A-algebra such that $\mathfrak{m}B \neq B$ (which is so when for example B is a local ring and $A \to B$ is a *local* homomorphism). If B is a *flat* A-module, B is a *faithfully flat* A-module. Indeed, this follows from (6.4.1, (d)). Under the indicated conditions, we $\mathbf{Err_{II}}$ thus see that if B is Noetherian, then so too is A (6.5.2).

(6.6.3). Let B be an A-algebra which is a faithfully flat A-module. For each A-module M and each A-submodule M' of M, we have (by identifying M with an A-submodule of $M_{(B)}$) $M' = M \cap M'_{(B)}$. For M to be a flat (resp. faithfully flat) A-module, it is necessary and sufficient that $M_{(B)}$ is a flat (resp. faithfully flat) B-module.

(6.6.4). Let B be an A-algebra, N a faithfully flat B-module. For B to be a flat (resp. faithfully flat) A-module, it is necessary and sufficient that N is.

In particular, let C be a B-algebra; if the ring C is faithfully flat over B and B is faithfully flat over A, then C is faithfully flat over A; if C is faithfully flat over B and over A, then B is faithfully flat over A.

6.7. Flat morphisms of ringed spaces

(6.7.1). Let $f : X \to Y$ be a morphism of ringed spaces, and let $\mathscr{F}$ be an $\mathscr{O}_X$-module. We say that $\mathscr{F}$ is *f-flat* (or *Y-flat* when there is no chance of confusion with f) *at a point* $x \in X$ if $\mathscr{F}_x$ is a flat $\mathscr{O}_{f(x)}$-module; we say that $\mathscr{F}$ is *f-flat over* $y \in Y$ if $\mathscr{F}$ is f-flat for all the points $x \in f^{-1}(y)$; we say that $\mathscr{F}$ is *f-flat* if $\mathscr{F}$ is f-flat at all the points of X. We say that the morphism f is *flat at* $x \in X$ (resp. *flat over* $y \in Y$, resp. *flat*) if $\mathscr{O}_X$ is f-flat at x (resp. f-flat over y, resp. f-flat). If f is a flat morphism, we then say that X is *flat over* Y, or *Y-flat*. $\mathbf{Err_{II}}$

(6.7.2). With the notation of (6.7.1), if $\mathscr{F}$ is f-flat at x, for each open neighbourhood U of $y = f(x)$, the functor $(f^*(\mathscr{G}) \otimes_{\mathscr{O}_X} \mathscr{F})_x$ in $\mathscr{G}$ is *exact* on the category of $(\mathscr{O}_Y|U)$-modules; indeed, this stalk canonically identifies with $\mathscr{G}_y \otimes_{\mathscr{O}_y} \mathscr{F}_x$, and our assertion follows from the definition. In particular, if f is a *flat* morphism, the functor f^* is *exact* on the category of $\mathscr{O}_Y$-modules.

(6.7.3). Conversely, suppose the sheaf of rings $\mathscr{O}_Y$ is *coherent*, and suppose that for *each* open neighbourhood U of y, the functor $(f^*(\mathscr{G}) \otimes_{\mathscr{O}_X} \mathscr{F})_x$ is exact in $\mathscr{G}$ on the category of *coherent* $(\mathscr{O}_Y|U)$-modules. Then $\mathscr{F}$ is *f-flat at x*. In fact, it suffices to prove that for each ideal of finite type $\mathfrak{J}$ of $\mathscr{O}_y$,

the canonical homomorphism $\mathfrak{J} \otimes_{\mathscr{O}_y} \mathscr{F}_x \to \mathscr{F}_x$ is injective (6.1.1). We know (5.3.8) that there then exists an open neighbourhood U of y and a coherent sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_Y|U$ such that $\mathscr{J}_y = \mathfrak{J}$, **0_I** | 60
hence the conclusion.

(6.7.4). The results of (6.1) for flat modules are immediately translated into propositions for sheaves with are f-flat at a point:

If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of $\mathscr{O}_X$-modules and if $\mathscr{F}''$ is f-flat at a point $x \in X$, then, for each open neighbourhood U of $y = f(x)$ and each $(\mathscr{O}_Y|U)$-module $\mathscr{G}$, the sequence

$$0 \longrightarrow (f^*(\mathscr{G}) \otimes_{\mathscr{O}_X} \mathscr{F}')_x \longrightarrow (f^*(\mathscr{G}) \otimes_{\mathscr{O}_X} \mathscr{F})_x \longrightarrow (f^*(\mathscr{G}) \otimes_{\mathscr{O}_X} \mathscr{F}'')_x \longrightarrow 0$$

is exact. For $\mathscr{F}$ to be f-flat at x, it is necessary and sufficient that $\mathscr{F}'$ is. We have similar statements for the corresponding notions of a f-flat $\mathscr{O}_X$-modules over $y \in Y$, or of a f-flat $\mathscr{O}_X$-module.

(6.7.5). Let $f : X \to Y$, $g : Y \to Z$ be two morphisms of ringed spaces; let $x \in X$, $y = f(x)$, and $\mathscr{F}$ be an $\mathscr{O}_X$-module. If $\mathscr{F}$ is f-flat at the point x and if the morphism g is flat at the point y, then $\mathscr{F}$ is $(g \circ f)$-flat at x (6.2.1). In particular, if f and g are flat morphisms, then $g \circ f$ is flat.

(6.7.6). Let X, Y be two ringed spaces, $f : X \to Y$ a *flat* morphism. Then the canonical homomorphism of bifunctors (4.4.6)

(6.7.6.1) $$f^*(\mathscr{H}om_{\mathscr{O}_Y}(\mathscr{F}, \mathscr{G})) \longrightarrow \mathscr{H}om_{\mathscr{O}_X}(f^*(\mathscr{F}), f^*(\mathscr{G}))$$

is an *isomorphism* when $\mathscr{F}$ admits a *finite presentation* (5.2.5).

Indeed, since the questions is local, we can assume that there exists an exact sequence $\mathscr{O}_Y^m \to \mathscr{O}_Y^n \to \mathscr{F} \to 0$. The two sides of (6.7.6.1) are right exact functors in $\mathscr{F}$ according to the hypotheses on f; we then have reduced to proving the proposition in the case where $\mathscr{F} = \mathscr{O}_Y$, in which the result is trivial.

(6.7.8). We say that a morphism $f : X \to Y$ of ringed spaces is *faithfully flat* if f is *surjective* and if, for each $x \in X$, $\mathscr{O}_x$ is a *faithfully flat* $\mathscr{O}_{f(x)}$-module. When X and Y are locally ringed spaces (5.5.1), it is equivalent to say that the morphism f is *surjective* and *flat* (6.6.2). When f is faithfully flat, f^* is an *exact* and *faithful* functor on the category of $\mathscr{O}_Y$-modules (6.6.1, a), and for an $\mathscr{O}_Y$-module $\mathscr{G}$ to be Y-flat, it is necessary and sufficient that $f^*(\mathscr{G})$ is (6.6.3).

§7. ADIC RINGS

7.1. Admissible rings

(7.1.1). Recall that in a topological ring A (not necessarily separated), we say that an element x is *topologically nilpotent* if 0 is *a* limit of the sequence $(x^n)_{n \geqslant 0}$. We say that a topological ring A is *linearly topologized* if there exists a fundamental system of neighbourhoods of 0 in A of (necessarily *open*) *ideals*.

Definition (7.1.2). — In a linearly topologized ring A, we say that an ideal $\mathfrak{J}$ is an *ideal of definition* if $\mathfrak{J}$ is open and if, for each neighbourhood V of 0, there exists a integer $n > 0$ such that $\mathfrak{J}^n \subset V$ **0_I** | 61
(which we express, by abuse of language, by saying that the sequence $(\mathfrak{J}^n)$ *tends to* 0*).* We say that a linearly topologized ring A is *preadmissible* if there exists in A an ideal of definition; we say that A is *admissible* if it is preadmissible and if in addition it is separated and complete.

It is clear that if $\mathfrak{J}$ is an ideal of definition, $\mathfrak{L}$ an open ideal of A, then $\mathfrak{J} \cap \mathfrak{L}$ is also an ideal of definition; the ideals of definition of a preadmissible ring A thus form a *fundamental system of neighbourhoods of* 0.

Lemma (7.1.3). — *Let A be a linearly topologized ring.*

(i) *For $x \in A$ to be topologically nilpotent, it is necessary and sufficient that for each open ideal $\mathfrak{J}$ of A, the canonical image of x in $A/\mathfrak{J}$ is nilpotent. The set $\mathfrak{T}$ of topologically nilpotent elements of A is an ideal.*

(ii) *Suppose that in addition A is preadmissible, and let $\mathfrak{J}$ be an ideal of definition for A. For $x \in A$ to be topologically nilpotent, it is necessary and sufficient that its canonical image in $A/\mathfrak{J}$ is nilpotent; the ideal $\mathfrak{T}$ is the inverse image of the nilradical of $A/\mathfrak{J}$ and is thus open.*

PROOF. (i) follows immediately from the definitions. To prove (ii), it suffices to note that for each neighbourhood V of 0 in A, there exists an $n > 0$ such that $\mathfrak{J}^n \subset V$; if $x \in A$ is such that $x^m \in \mathfrak{J}$, we have $x^{mq} \in V$ for $q \geqslant n$, so x is topologically nilpotent. □

Proposition (7.1.4). — *Let A be a preadmissible ring, $\mathfrak{J}$ an ideal of definition for A.*

(i) *For an ideal $\mathfrak{J}'$ of A to be contained in an ideal of definition, it is necessary and sufficient that there exists an integer $n > 0$ such that $\mathfrak{J}'^n \subset \mathfrak{J}$.*
(ii) *For an $x \in A$ to be contained in an ideal of definition, it is necessary and sufficient that it is topologically nilpotent.*

PROOF.

(i) If $\mathfrak{J}'^n \subset \mathfrak{J}$, then for each open neighbourhood V of 0 in A, there exists an m such that $\mathfrak{J}^m \subset V$, thus $\mathfrak{J}'^{mn} \subset V$.
(ii) The condition is evidently necessary; it is sufficient, since if it satisfied, then there exists an n such that $x^n \in \mathfrak{J}$, so $\mathfrak{J}' = \mathfrak{J} + Ax$ is an ideal of definition, because it is open, and $\mathfrak{J}'^n \subset \mathfrak{J}$. □

Corollary (7.1.5). — *In a preadmissible ring A, an open prime ideal contains all the ideals of definition.*

Corollary (7.1.6). — *The notation and hypotheses being that of (7.1.4), the following properties of an ideal $\mathfrak{J}_0$ of A are equivalent:*

(a) *$\mathfrak{J}_0$ is the largest ideal of definition of A;*
(b) *$\mathfrak{J}_0$ is a maximal ideal of definition;*
(c) *$\mathfrak{J}_0$ is an ideal of definition such that the ring $A/\mathfrak{J}_0$ is reduced.*

For there to exist an ideal $\mathfrak{J}_0$ to have these properties, it is necessary and sufficient that the nilradical of $A/\mathfrak{J}$ to be nilpotent; $\mathfrak{J}_0$ is then equal to the ideal $\mathfrak{T}$ of topologically nilpotent elements of A.

PROOF. It is clear that (a) implies (b), and (b) implies (c) according to (7.1.4, ii), and (7.1.3, ii); for the same reason, (c) implies (a). The latter assertion follows from (7.1.4, i) and (7.1.3, ii). □

When $\mathfrak{T}/\mathfrak{J}$, the nilradical of $A/\mathfrak{J}$, is nilpotent, and we denote by A_{red} the (reduced) quotient ring $A/\mathfrak{T}$.

$\mathbf{0_I}$ | 62

Corollary (7.1.7). — *A preadmissible Noetherian ring admits a largest ideal of definition.*

Corollary (7.1.8). — *If a preadmissible ring A is such that, for an ideal of definition $\mathfrak{J}$, the powers $\mathfrak{J}^n$ ($n > 0$) form a fundamental system of neighbourhoods of 0, it is the same for the powers $\mathfrak{J}'^n$ for each ideal of definition $\mathfrak{J}'$ of A.*

Definition (7.1.9). — We say that a preadmissible ring A is *preadic* if there exists an ideal of definition $\mathfrak{J}$ for A such that the $\mathfrak{J}^n$ form a fundamental system of neighbourhoods of 0 in A *(or equivalently, such that the $\mathfrak{J}^n$ are* open*)*. We call a ring *adic* if it is a separated and complete preadic ring.

If $\mathfrak{J}$ is an ideal of definition for a preadic (resp. adic) ring A, we say that A is a $\mathfrak{J}$-*preadic* (resp. $\mathfrak{J}$-*adic*) ring, and that its topology is the $\mathfrak{J}$-*preadic* (resp. $\mathfrak{J}$-*adic*) topology. More generally, if M is an A-module, the topology on M having for a fundamental system of neighbourhoods of 0 the submodules $\mathfrak{J}^n M$ is called the $\mathfrak{J}$-*preadic* (resp. $\mathfrak{J}$-*adic*) topology. According to (7.1.8), these topologies are independent of the choice of the ideal of definition $\mathfrak{J}$.

Proposition (7.1.10). — *Let A be an admissible ring, $\mathfrak{J}$ an ideal of definition for A. Then $\mathfrak{J}$ is contained in the radical of A.*

This statement is equivalent to any of the following corollaries:

Corollary (7.1.11). — *For each $x \in \mathfrak{J}$, $1 + x$ is invertible in A.*

Corollary (7.1.12). — *For $f \in A$ to be invertible in A, it is necessary and sufficient that its canonical image in $A/\mathfrak{J}$ is invertible in $A/\mathfrak{J}$.*

Corollary (7.1.13). — *For each A-module M of finite type, the relation $M = \mathfrak{J}M$ (equivalent to $M \otimes_A (A/\mathfrak{J}) = 0$) implies that $M = 0$.*

Corollary (7.1.14). — *Let $u : M \to N$ be a homomorphism of A-modules, N being of finite type; for u to be surjective, it is necessary and sufficient that $u \otimes 1 : M \otimes_A (A/\mathfrak{J}) \to N \otimes_A (A/\mathfrak{J})$ is.*

PROOF. The equivalence of (7.1.10) and (7.1.11) follows from Bourbaki, *Alg.*, chap. VIII, §6, no. 3, th. 1, and the equivalence of (7.1.10) and (7.1.10) and (7.1.13) follows from *loc. cit.*, th. 2; the fact that (7.1.10) implies (7.1.14) follows from *loc. cit.*, cor. 4 of the prop. 6; on the other hand, (7.1.14) implies (7.1.13) by applying the zero homomorphism. Finally, (7.1.10) implies that if f is invertible in $A/\mathfrak{J}$, then f is not contained in any maximal ideal of A, thus f is invertible in A, in other words, (7.1.10) implies (7.1.12); conversely, (7.1.12) implies (7.1.11).

It therefore remains to prove (7.1.11). Now as A is separated and complete, and the sequence $(\mathfrak{J}^n)$ tends to 0, it is immediate that the series $\sum_{n=0}^{\infty}(-1)^n x^n$ is convergent in A, and that if y is its sum, then we have $y(1+x) = 1$. □

7.2. Adic rings and projective limits

(7.2.1). Each projective limit of *discrete* rings is evidently a linearly topologized ring, separated and compact. Conversely, let A be a linearly topologized ring, and let $(\mathfrak{J}_\lambda)$ be a fundamental system of open neighbourhoods of 0 in A consisting of ideals. The canonical maps $\phi_\lambda : A \to A/\mathfrak{J}_\lambda$ form a **0** projective system of continuous representations and therefore define a continuous representation $\phi : A \to \varprojlim A/\mathfrak{J}_\lambda$; if A is *separated*, then ϕ is a topological isomorphism from A to an everywhere-dense subring of $\varprojlim A/\mathfrak{J}_\lambda$; if in addition A is *complete*, then ϕ is a topological isomorphism from A to $\varprojlim A/\mathfrak{J}_\lambda$.

Lemma (7.2.2). — *For a linearly topologized ring to be admissible, it is necessary and sufficient that it is isomorphic to a projective limit $A = \varprojlim A_\lambda$, where $(A_\lambda, u_{\lambda\mu})$ is a projective limit of discrete rings having for the set of indices a filtered ordered (by $\leqslant$) L which admits a smallest element denoted 0 and satisfies the following conditions: 1st. the $u_\lambda : A \to A_\lambda$ are sujective; 2nd. the kernel $\mathfrak{J}_\lambda$ of $u_{0\lambda} : A_\lambda \to A_0$ is nilpotent. When this is so, the kernel $\mathfrak{J}$ of $u_0 : A \to A_0$ is equal to $\varprojlim \mathfrak{J}_\lambda$.*

PROOF. The necessity of the condition follows from (7.2.1), by choosing $(\mathfrak{J}_\lambda)$ to be a fundamental system of neighbourhoods of 0 consisting of ideals of definitions contained in an ideal of definition $\mathfrak{J}_0$ and by applying (7.1.4, i). The converse follows from the definition of the projective limit and from (7.2.1), and the latter assertion is immediate. □

(7.2.3). Let A be an *admissible* topological ring, $\mathfrak{J}$ an ideal of A contained in an ideal of definition (in other words (7.1.4) such that $(\mathfrak{J}^n)$ tends to 0); we can consider on A the ring topology having for a fundamental system of neighbourhoods of 0 the powers $\mathfrak{J}^n$ ($n > 0$); we call again this the $\mathfrak{J}$-*preadic* topology. The hypothesis that A is admissible implies that $\bigcup_n \mathfrak{J}^n = 0$, therefore the $\mathfrak{J}$-preadic topology on A is *separated*; let $\widehat{A} = \varprojlim A/\mathfrak{J}^n$ be the completion of A for this topology (where the $A/\mathfrak{J}^n$ are equipped with the discrete topology), and denote by u the (not necessarily continuous) ring homomorphism $A \to \widehat{A}$, the projective limit of the sequence of homomorphisms $u_n : A \to A/\mathfrak{J}^n$. On the other hand, the $\mathfrak{J}$-preadic topology on A is finer than the given topology $\mathscr{T}$ on A; as A is separated and complete for $\mathscr{T}$, we can extend by continuity the identity map of A (equipped with the $\mathfrak{J}$-preadic topology) to A equipped with $\mathscr{T}$; this gives a continuous representation $v : \widehat{A} \to A$.

Proposition (7.2.4). — *If A is an admissible ring and $\mathfrak{J}$ is contained in an ideal of definition of A, then A is separated and complete for the $\mathfrak{J}$-preadic topology.*

PROOF. With the notation of (7.2.3), it is immediate that $v \circ u$ is the identity map of A. On the other hand, $u_n \circ v : \widehat{A} \to A/\mathfrak{J}^n$ is the extension by continuity (for the $\mathfrak{J}$-preadic topology on A and the discrete topology on $A/\mathfrak{J}^n$) of the canonical map u_n; in other words, it is the canonical map from $\widehat{A} = \varprojlim_k A/\mathfrak{J}^k$ to $A/\mathfrak{J}^n$; $u \circ v$ is therefore the projective limit of this sequence of maps, which is by definition the identity map of $\widehat{A}$; this proves the proposition. □

Corollary (7.2.5). — *Under the hypotheses of (7.2.3), the following conditions are equivalent:*

(a) *the homomorphism u is continuous;* **0**
(b) *the homomorphism v is bicontinous;*
(c) *A is a $\mathfrak{J}$-adic ring.*

Corollary (7.2.6). — *Let A be an admissible ring A, $\mathfrak{J}$ an ideal of definition for A. For A to be Noetherian, it is necessary and sufficient for $A/\mathfrak{J}$ to be Noetherian and for $\mathfrak{J}/\mathfrak{J}^2$ to be an $A/\mathfrak{J}$-module of finite type.*

These conditions are evidently necessary. Conversely, suppose the conditions are satisfied; as according to (7.2.4) A is complete for the $\mathfrak{J}$-preadic topology, for it to be Noetherian, it is necessary and sufficient that the associated graded ring $\mathrm{grad}(A)$ (for the filtration on the $\mathfrak{J}^n$) is ([**CC**, p .18–07, th. 4]). Now, let $a_1, \ldots, a_n$ be the elements of $\mathfrak{J}$ whose classes mod. $\mathfrak{J}^2$ are the generators of $\mathfrak{J}/\mathfrak{J}^2$ as a $A/\mathfrak{J}$-module. It is immediate by induction that the classes mod. $\mathfrak{J}^{m+1}$ of the monomials of total degree m in the a_i ($1 \leqslant i \leqslant n$) form a system of generators for the $A/\mathfrak{J}$-module $\mathfrak{J}^m/\mathfrak{J}^{m+1}$. We conclude that $\mathrm{grad}(A)$ is a ring isomorphic to a quotient of $(A/\mathfrak{J})[T_1, \ldots, T_n]$ (T_i indeterminates), which finishes the proof.

Proposition (7.2.7). — *Let (A_i, u_{ij}) be a projective system ($i \in \mathbf{N}$) of discrete rings, and for each integer i, let $\mathfrak{J}_i$ be the kernel in A_i of the homomorphism $u_{0i} : A_i \to A_0$. We suppose that:*

(a) *For $i \leqslant j$, u_{ij} is surjective and its kernel is $\mathfrak{J}_j^{i+1}$* (therefore A_i is isomorphic to $A_j/\mathfrak{J}_j^{i+1}$).

(b) *$\mathfrak{J}_1/\mathfrak{J}_1^2$ ($= \mathfrak{J}_1$) is a module of finite type over $A_0 = A_1/\mathfrak{J}_1$.*

Let $A = \varprojlim_i A_i$, and for each integer $n \geqslant 0$, let u_n be the canonical homomorphism $A \to A_n$, and let $\mathfrak{J}^{(n+1)} \subset A$ be its kernel. Then we have these conditions:

(i) *A is an adic ring, having $\mathfrak{J} = \mathfrak{J}^{(1)}$ for an ideal of definition.*

(ii) *We have $\mathfrak{J}^{(n)} = \mathfrak{J}^n$ for each $n \geqslant 1$.*

(iii) *$\mathfrak{J}/\mathfrak{J}^2$ is isomorphic to $\mathfrak{J}_1 = \mathfrak{J}_1/\mathfrak{J}_1^2$, and as a result is a module of finite type over $A_0 = A/\mathfrak{J}$.*

PROOF. The hypothesis of surjectivity on the u_{ij} implies that u_n is surjective; in addition, the hypothesis (a) implies that $\mathfrak{J}_j^{j+1} = 0$, therefore A is an admissible ring (7.2.2); by definition, the $\mathfrak{J}^{(n)}$ form a fundamental system of neighbourhoods of 0 in A, so (ii) implies (i). In addition, we have $\mathfrak{J} = \varprojlim_i \mathfrak{J}_i$ and the maps $\mathfrak{J} \to \mathfrak{J}_i$ are surjective, so (ii) implies (iii), and it remains to prove (ii). By definition, $\mathfrak{J}^{(n)}$ consists of the elements $(x_k)_{k\geqslant 0}$ of A such that $x_k = 0$ for $k < n$, therefore $\mathfrak{J}^{(n)}\mathfrak{J}^{(m)} \subset \mathfrak{J}^{(n+m)}$, in other words the $\mathfrak{J}^{(n)}$ form a *filtration* of A. On the other hand, $\mathfrak{J}^{(n)}/\mathfrak{J}^{(n+1)}$ is isomorphic to the projection from $\mathfrak{J}^{(n)}$ to A_n; as $\mathfrak{J}^{(n)} = \varprojlim_{i>n} \mathfrak{J}_i^n$, this projection is none other than $\mathfrak{J}_n^n$, which is a module over $A_0 = A_n/\mathfrak{J}_n$. Now let $a_j = (a_{jk})_{k\geqslant 0}$ be r elements of $\mathfrak{J} = \mathfrak{J}^{(1)}$ such that $a_{11}, \ldots, a_{r1}$ form a system of generators for $\mathfrak{J}_1$ over A_0; we will see that the set S_n of monomials of total degree n and the a_j generate the ideal $\mathfrak{J}^{(n)}$ of A. As $\mathfrak{J}_i^{i+1} = 0$, it is clear first of all that $S_n \subset \mathfrak{J}^{(n)}$; since A is complete for the filtration $(\mathfrak{J}^{(m)})$, it suffices to prove that the set $\overline{S}_n$ of classes mod. $\mathfrak{J}^{(n+1)}$ of elements of S_n generate the graded module $\mathrm{grad}(\mathfrak{J}^{(n)})$ over the graded ring $\mathrm{grad}(A)$ for the above filtration ([**CC**, p. 18–06, lemme]); according to the definition of the multiplication on $\mathrm{grad}(A)$, it suffices to prove that for each m, $\overline{S}_m$ is a system of generators for the A_0-module $\mathfrak{J}^{(m)}/\mathfrak{J}^{(m+1)}$, or that $\mathfrak{J}_m^m$ is generated by the monomials of degree m in the a_{jm} ($1 \leqslant j \leqslant r$). For this, it remains to show that $\mathfrak{J}_m$ is generated (as an A_m-module) by the monomials of degree $\leqslant m$ with respect to to a_{jm}; the proposition being evident by definition for $m = 1$, we argue by induction on m, and let $\mathfrak{J}'_m$ be the A_m-submodule of $\mathfrak{J}_m$ generated by these monomials. The relation $\mathfrak{J}_{m-1} = \mathfrak{J}_m/\mathfrak{J}_m^m$ and the induction hypothesis prove that $\mathfrak{J}_m = \mathfrak{J}'_m + \mathfrak{J}_m^m$, hence, since $\mathfrak{J}_m^{m+1} = 0$, we have $\mathfrak{J}_m^m = {\mathfrak{J}'_m}^m$, and finally $\mathfrak{J}_m = \mathfrak{J}'_m$. □

Corollary (7.2.8). — *Under the conditions of Proposition (7.2.7), for A to be Noetherian, it is necessary and sufficient that A_0 is.*

PROOF. This follows immediately from Corollary (7.2.6). □

Proposition (7.2.9). — *Suppose the hypotheses of Proposition (7.2.7): for each integer i, let M_i be an A_i-module, and for $i \leqslant j$, let $v_{ij} : M_j \to M_i$ be a u_{ij}-homomorphism, such that (M_i, v_{ij}) is a projective system. In addition, suppose that M_0 is an A_0-module of finite type and that the v_{ij} are surjective with kernel $\mathfrak{J}_j^{i+1} M_j$. Then $M = \varprojlim M_i$ is an A-module of finite type, and the kernel of the surjective u_n-homomorphism $v_n : M \to M_n$ is $\mathfrak{J}^{n+1}M$* (such that M_n identifies with $M/\mathfrak{J}^{n+1}M = M \otimes_A (A/\mathfrak{J}^{n+1})$).

PROOF. Let $z_h = (z_{hk})_{k\geqslant 0}$ be a system of s elements of M such that the z_{h0} $(1 \leqslant h \leqslant s)$ forms a system of generators for M_0; we will show that the z_h generate the A-module M. The A-module M is separated and complete for the filtration by the $M^{(n)}$, where $M^{(n)}$ is the set of $y = (y_k)_{k\geqslant 0}$ in M such that $y_k = 0$ for $k < n$; it is clear that we have $\mathfrak{J}^{(n)}M \subset M^{(n)}$ and that $M^{(n)}/M^{(n+1)} = \mathfrak{J}_n^n M_n$. We therefore have reduced to showing that the classes of the z_h modulo $M^{(0)}$ generate the graded module $\operatorname{grad}(M)$ (by the above filtration) over the graded ring $\operatorname{grad}(A)$ [CC, p. 18–06, lemme]; for this, we observe that it suffices to prove that the z_{hn} $(1 \leqslant h \leqslant s)$ generate the A_n-module M_n. We argue by induction on n, the proposition being evident by definition for $n = 0$; the relation $M_{n-1} = M_n/\mathfrak{J}_n^n M_n$ and the induction hypothesis show that if M'_n is the submodule of M_n generated by the z_{hn}, we have that $M_n = M'_n + \mathfrak{J}_n^n M_n$, and as $\mathfrak{J}_n$ is nilpotent, this implies that $M_n = M'_n$. Similarly, passing to the associated graded modules shows that the canonical map from $\mathfrak{J}^{(n)}$ to $M^{(n)}$ is surjective (thus bijection), in other words that $\mathfrak{J}^{(n)}M = \mathfrak{J}^n M$ is the kernel of $M \to M_{n-1}$. □ **Err$_{\mathrm{II}}$**

Corollary (7.2.10). — *Let (N_i, w_{ij}) be a second projective system of A_i-modules satisfying the conditions of Proposition (7.2.9), and let $N = \varprojlim N_i$. There is a bijective correspondence between the projective systems (h_i) of A_i-homomorphisms $h_i : M_i \to N_i$ and the homomorphisms of A-modules $h : M \to N$ (which is necessarily continuous for the $\mathfrak{J}$-adic topologies).*

PROOF. It is clear that if $h : M \to N$ is an A-homomorphism, then we have $h(\mathfrak{J}^n M) \subset \mathfrak{J}^n N$, hence the continuity of h; by passing to quotients, there corresponds to h a projective system of A_i-homomorphisms $h_i : M_i \to N_i$, whose projective limit is h, hence the corollary. □

Remark (7.2.11). — Let A be an adic ring with an ideal of definition $\mathfrak{J}$ such that $\mathfrak{J}/\mathfrak{J}^2$ is an $A/\mathfrak{J}$-module of finite type; it is clear that the $A_i = A/\mathfrak{J}^{i+1}$ satisfy the conditions of Proposition (7.2.7); as **0$_{\mathrm{I}}$ | 66** A is the projective limit of the A_i, we see that Proposition (7.2.7) gives the description of *all* the adic rings of the type considered (and in particular of all the *adic Noetherian* rings).

Example (7.2.12). — Let B be a ring, $\mathfrak{J}$ an ideal of B such that $\mathfrak{J}/\mathfrak{J}^2$ is a module of finite type over $B/\mathfrak{J}$ (or over B, equivalently); set $A = \varprojlim_n B/\mathfrak{J}^{n+1}$; A is the separated completion of B equipped with the $\mathfrak{J}$-preadic topology. If $A_n = B/\mathfrak{J}^{n+1}$, then it is immediate that the A_n satisfy the conditions of Proposition (7.2.7); therefore A is an adic ring and if $\overline{\mathfrak{J}}$ is the closure in A of the canonical image of $\mathfrak{J}$, then $\overline{\mathfrak{J}}$ is an ideal of definition for A, $\overline{\mathfrak{J}^n}$ is the closure of the canonical image of $\mathfrak{J}^n$, $A/\overline{\mathfrak{J}^n}$ identifies with $B/\mathfrak{J}^n$ and $\overline{\mathfrak{J}}/\overline{\mathfrak{J}}^2$ is isomorphic to $\mathfrak{J}/\mathfrak{J}^2$ as an $A/\overline{\mathfrak{J}}$-module. Similarly, if N is such that $N/\mathfrak{J}N$ is a B-module of finite type, and if we set $M_i = N/\mathfrak{J}^{i+1}N$, then $M = \varprojlim M_i$ is an A-module of finite type, isomorphic to the separated completion of N for the $\mathfrak{J}$-preadic topology, and $\overline{\mathfrak{J}^n}M$ identifies with the closure of the canonical image of $\mathfrak{J}^n N$, and $M/\overline{\mathfrak{J}^n}M$ identifies with $N/\mathfrak{J}^n N$.

7.3. Preadic Noetherian rings

(7.3.1). Let A be a ring, $\mathfrak{J}$ an ideal of A, and M an A-module; we denote by $\widehat{A} = \varprojlim_n A/\mathfrak{J}^n$ (resp. $\widehat{M} = \varprojlim_n M/\mathfrak{J}^n M$) the separated completion of A (resp. M) for the $\mathfrak{J}$-preadic topology. Let $M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$ be an exact sequence of A-modules; as $M/\mathfrak{J}^n M = M \otimes_A (A/\mathfrak{J}^n)$, the sequence

$$M'/\mathfrak{J}^n M' \xrightarrow{u_n} M/\mathfrak{J}^n M \xrightarrow{v_n} M''/\mathfrak{J}^n M'' \longrightarrow 0$$

is exact for each n. In addition, as $v(\mathfrak{J}^n M) = \mathfrak{J}^n v(M) = \mathfrak{J}^n M''$, $\widehat{v} = \varprojlim v_n$ is surjective (Bourbaki, *Top. gén.*, Chap. IX, 2nd ed., p. 60, Cor. 2). On the other hand, if $z = (z_k)$ is an element of the kernel of $\widehat{v}$, then for each integer k, there exists a $z'_k \in M'/\mathfrak{J}^k M'$ such that $u_k(z'_k) = z_k$; we conclude that there exists a $z' = (z'_n) \in \widehat{M'}$ such that the first k components of $\widehat{u}(z')$ coincide with the z; in other words, the image of $\widehat{M'}$ under $\widehat{u}$ is *dense* in the kernel of $\widehat{v}$.

If we suppose that A is *Noetherian*, then so is $\widehat{A}$, according to (7.2.12), $\mathfrak{J}/\mathfrak{J}^2$ is then an A-module of finite type. In addition, we have the following theorem.

Theorem (7.3.2). — (Krull's Theorem). *Let A be a* Noetherian *ring, $\mathfrak{J}$ an ideal of A, M an A-module* of finite type, *and M' a submodule of M; then the induced topology on M' by the $\mathfrak{J}$-preadic topology of M is identical to the $\mathfrak{J}$-preadic topology of M'.*

This follows immediately from

Lemma (7.3.2.1). — (Artin–Rees Lemma). *Under the hypotheses of (7.3.2), there exists an integer p such that, for $n \geqslant p$, we have*

$$M' \cap \mathfrak{J}^n M = \mathfrak{J}^{n-p}(M' \cap \mathfrak{J}^p M).$$

For the proof, see [**CC**, p. 2–04].

$\mathbf{0_I}$ | 67

Corollary (7.3.3). — *Under the hypotheses of (7.3.2), the canonical map $M \otimes_A \widehat{A} \to \widehat{M}$ is bijective, and the functor $M \otimes_A \widehat{A}$ is exact in M on the category of A-modules of finite type; as a result, the separated $\mathfrak{J}$-adic completion $\widehat{A}$ is a flat A-module (6.1.1).*

Proof. We first note that $\widehat{M}$ is an *exact* functor in M on the category of A-modules of finite type. Indeed, let $0 \to M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$ be an exact sequence; we have seen that $\widehat{v} : \widehat{M} \to \widehat{M''}$ is surjective (7.3.1); on the other hand, if i is the canonical homomorphism $M \to \widehat{M}$, it follows from Krull's Theorem (7.3.2) that the closure of $i(u(M'))$ in $\widehat{M}$ identifies with the separated completion of M' for the $\mathfrak{J}$-preadic topology; thus $\widehat{u}$ is injective, and according to (7.3.1), the image of $\widehat{u}$ is equal to the kernel of $\widehat{v}$.

This being so, the canonical map $M \otimes_A \widehat{A} \to \widehat{M}$ is obtained by passing to the projective limit of the maps $M \otimes_A \widehat{A} \to M \otimes_A (A/\mathfrak{J}^n) = M/\mathfrak{J}^n M$. It is clear that this map is bijective when M is of the form A^p. If M is an A-module of finite type, then we have an exact sequence $A^p \to A^q \to M \to 0$, hence, by virtue of the *right* exactness of the functors $M \otimes_A \widehat{A}$ and $\widehat{M}$ (in M) on the category of A-modules of finite type, we have the commutative diagram

$$\begin{tikzcd}
A^p \otimes_A \widehat{A} \arrow[r] \arrow[d] & A^q \otimes_A \widehat{A} \arrow[r] \arrow[d] & M \otimes_A \widehat{A} \arrow[r] \arrow[d] & 0 \\
\widehat{A^p} \arrow[r] & \widehat{A^q} \arrow[r] & \widehat{M} \arrow[r] & 0,
\end{tikzcd}$$

where the two rows are exact and the first two vertical arrows are isomorphisms; this immediately finishes the proof. □

Corollary (7.3.4). — *Let A be a Noetherian ring, $\mathfrak{J}$ an ideal of A, M and N two A-modules of finite type; we have the canonical functorial isomorphisms*

$$(M \otimes_A N)^\wedge \simeq \widehat{M} \otimes_{\widehat{A}} \widehat{N}, \ (\mathrm{Hom}_A(M, N))^\wedge \simeq \mathrm{Hom}_{\widehat{A}}(\widehat{M}, \widehat{N}).$$

Proof. This follows from Corollary (7.3.3), (6.2.1), and (6.2.2). □

Corollary (7.3.5). — *Let A be a Noetherian ring, $\mathfrak{J}$ an ideal of A. The following conditions are equivalent:*

(a) *$\mathfrak{J}$ is contained in the radical of A.*
(b) *$\widehat{A}$ is a faithfully flat A-module (6.4.1).*
(c) *Each A-module of finite type is separated for the $\mathfrak{J}$-preadic topology.*
(d) *Each submodule of an A-module of finite type is closed for the $\mathfrak{J}$-preadic topology.*

Proof. As $\widehat{A}$ is a flat A-module, the conditions (b) and (c) are equivalent, since (b) is equivalent to saying that if M is an A-module of finite type, then the canonical map $M \to \widehat{M} = M \otimes_A \widehat{A}$ is injective (6.6.1, c). It is immediate that (c) implies (d), since if N is a submodule of an A-module M of finite type, then M/N is separated for the $\mathfrak{J}$-preadic topology, so N is closed in M. We show that (d) implies (a): if $\mathfrak{m}$ is a maximal ideal of A, then $\mathfrak{m}$ is closed in A for the $\mathfrak{J}$-preadic topology, so $\mathfrak{m} = \bigcap_{p \geqslant 0}(\mathfrak{m} + \mathfrak{J}^p)$, and as $\mathfrak{m} + \mathfrak{J}^p$ is necessarily equal to A or to $\mathfrak{m}$, we have that $\mathfrak{m} + \mathfrak{J}^p = \mathfrak{m}$ for large enough p, hence $\mathfrak{J}^p \subset \mathfrak{m}$, and $\mathfrak{J} \subset \mathfrak{m}$ when $\mathfrak{m}$ is prime. Finally, (a) implies (b): indeed, let $\mathbf{0_I}$ | 68
P be the closure of $\{0\}$ in an A-module M of finite type, for the $\mathfrak{J}$-preadic topology; according to Krull's Theorem (7.3.2), the topology induced on P by the $\mathfrak{J}$-preadic topology of M is the $\mathfrak{J}$-preadic topology of P, so $\mathfrak{J}P = P$; as P is of finite type, it follows from Nakayama's Lemma that $P = 0$ ($\mathfrak{J}$ being contained in the radical of A). □

We note that the conditions of Corollary (7.3.5) are satisfied when A is a *local Noetherian ring* and $\mathfrak{J} \neq A$ is any ideal of A.

Corollary (7.3.6). — *If A is a $\mathfrak{J}$-preadic Noetherian ring, then each A-module of finite type is separated and complete for the $\mathfrak{J}$-preadic topology.*

PROOF. As we then have $\widehat{A} = A$, this follows immediately from Corollary (7.3.3). □

We conclude that Proposition (7.2.9) gives the description of *all* the modules of finite type over an adic Noetherian ring.

Corollary (7.3.7). — *Under the hypotheses of (7.3.2), the kernel of the canonical map $M \to \widehat{M} = M \otimes_A \widehat{A}$ is the set of the $x \in M$ killed by an element of $1 + \mathfrak{J}$.*

PROOF. For each $x \in M$ in this kernel, it is necessary and sufficient that the separated completion of the submodule Ax is 0 (by Krull's Theorem (7.3.2)), in other words, that $x \in \mathfrak{J}x$. □

7.4. Quasi-finite modules over local rings

Definition (7.4.1). — Given a local ring A, with maximal ideal $\mathfrak{m}$, we say that an A-module M is quasi-finite (over A) if $M/\mathfrak{m}M$ is of finite rank over the residue field $k = A/\mathfrak{m}$.

When A is *Noetherian*, the separated completion $\widehat{M}$ of M for the $\mathfrak{m}$-preadic topology is then an *$\widehat{A}$-module of finite type*; indeed, as $\mathfrak{m}/\mathfrak{m}^2$ is then an A-module of finite type, this follows from Example (7.2.12) and from the hypothesis on $M/\mathfrak{m}M$.

In particular, if we suppose that in addition A is *complete* and M is *separated* for the $\mathfrak{m}$-preadic topology (in other words, $\bigcap_n \mathfrak{m}^n M = 0$), then M is also an *A-module of finite type*: indeed, $\widehat{M}$ is then an A-module of finite type, and as M identifies with a submodule of $\widehat{M}$, M is also of finite type (and is indeed identical to its completion according to Corollary (7.3.6)).

Proposition (7.4.2). — *Let A, B be two local rings, $\mathfrak{m}$, $\mathfrak{n}$ their maximal ideals, and suppose that B is Noetherian. Let $\phi : A \to B$ be a local homomorphism, M a B-module of finite type. If M is a quasi-finite A-module, then the $\mathfrak{m}$-preadic and $\mathfrak{n}$-preadic topologies on M are identical, thus separated.*

PROOF. We note that by hypothesis $M/\mathfrak{m}M$ is of *finite length* as an A-module, thus also *a fortiori* as a B-module. We conclude that $\mathfrak{n}$ is the *unique prime ideal* of B containing the annihilator of $M/\mathfrak{m}M$: indeed, we immediately reduce (according to (1.7.4) and (1.7.2)) to the case where $M/\mathfrak{m}M$ is *simple*, thus necessarily isomorphic to $B/\mathfrak{n}$, and our assertion is evident in this case. On the other hand, as M is a B-module of finite type, the prime ideals which contain the annihilator of $M/\mathfrak{m}M$ are those which contain $\mathfrak{m}B + \mathfrak{b}$, where we denote by $\mathfrak{b}$ the annihilator of the B-module M (1.7.5). As B is Noetherian, we conclude ([**Sam53b**, p. 127, Cor. 4]) that $\mathfrak{m}B + \mathfrak{b}$ is an ideal of definition for B, in $\mathbf{0_I}$ | 69
other words there exists a $k > 0$ such that $\mathfrak{n}^k \subset \mathfrak{m}B + \mathfrak{b} \subset \mathfrak{n}$; as a result, for each $h > 0$, we have

$$\mathfrak{n}^{hk} \subset (\mathfrak{m}B + \mathfrak{b})^h M = \mathfrak{m}^h M \subset \mathfrak{n}^h M,$$

which proves that the $\mathfrak{m}$-preadic and $\mathfrak{n}$-preadic topologies on M are the same; the second is separated according to Corollary (7.3.5). □

Corollary (7.4.3). — *Under the hypotheses of Proposition (7.2.4), if in addition A is Noetherian and complete for the $\mathfrak{m}$-preadic topology, then M is an A-module of finite type.*

PROOF. Indeed, M is then separated for the $\mathfrak{m}$-preadic topology, and our assertion follows from the remark after Definition (7.4.1). □

(7.4.4). The most important case of Proposition (7.4.2) is when B is a quasi-finite A-module, i.e., $B/\mathfrak{m}B$ is an *algebra of finite rank over $k = A/\mathfrak{m}$*; furthermore, this condition can be broken down into the combination of the following:

(i) *$\mathfrak{m}B$ is an ideal of definition for B;*
(ii) *$B/\mathfrak{n}$ is an extension of finite rank of the field $A/\mathfrak{m}$.*

When this is so, every B-module of finite type is evidently a quasi-finite A-module.

Corollary (7.4.5). — *Under the hypotheses of Proposition (7.4.2), if $\mathfrak{b}$ is the annihilator of the B-module M, then $B/\mathfrak{b}$ is a quasi-finite A-module.*

PROOF. Suppose $M \neq 0$ (otherwise the corollary is evident). We can consider M as a module over the local Noetherian ring $B/\mathfrak{b}$; its annihilator then being 0, the proof of Proposition (7.4.2) shows that $\mathfrak{m}(B/\mathfrak{b})$ is an ideal of definition for $B/\mathfrak{b}$. On the other hand, $M/\mathfrak{n}M$ is a vector space of finite rank over $A/\mathfrak{m}$, being a quotient of $M/\mathfrak{m}M$, which is by hypothesis of finite rank over $A/\mathfrak{m}$; as $M \neq 0$, we have $M \neq \mathfrak{n}M$ by virtue of Nakayama's Lemma; as $M/\mathfrak{n}M$ is a vector space $\neq 0$ over $B/\mathfrak{n}$, the fact that it is of finite rank over $A/\mathfrak{m}$ implies that $B/\mathfrak{n}$ is also of finite rank over $A/\mathfrak{m}$; the conclusion follows from (7.4.4) applied to the ring $B/\mathfrak{b}$. □

7.5. Rings of restricted formal series

(7.5.1). Let A be a topological ring, linearly topologized, separated and complete; let $(\mathfrak{J}_\lambda)$ be a fundamental system of neighbourhoods of 0 in A consisting of (open) ideals, such that A canonically identifies with $\varprojlim A/\mathfrak{J}_\lambda$ (7.2.1). For each λ, let $B_\lambda = (A/\mathfrak{J}_\lambda)[T_1, \ldots, T_r]$, where the T_i are indeterminates; it is clear that the B_λ form a projective system of discrete rings. We set $\varprojlim B_\lambda = A\{T_1, \ldots, T_r\}$, and we will see that this topological ring is independent of the fundamental system of ideals $(\mathfrak{J}_\lambda)$ considered. More precisely, let A' be the subring of the ring of formal series $A[\![T_1, \ldots, T_r]\!]$ consisting of formal series $\sum_\alpha c_\alpha T^\alpha$ (with $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbf{N}^r$) such that $\lim c_\alpha = 0$ (according to the filter by compliments of finite subsets of $\mathbf{N}^r$); we say that these series are the *restricted* formal series in the T_i, with coefficients in A. For each neighbourhood V of 0 in A, let V' be the set of $x = \sum_\alpha c_\alpha T^\alpha \in A'$ $\mathbf{0_I}$ | 70
such that $c_\alpha \in V$ for all α. We verify immediately that the V' form a fundamental system of neighbourhoods of 0 defining on A' a *separated* ring topology; we will canonically define a *topological isomorphism* from the ring $A\{T_1, \ldots, T_r\}$ to A'. For each $\alpha \in \mathbf{N}^r$ and each λ, let $\phi_{\lambda,\alpha}$ be the map from $(A/\mathfrak{J}_\lambda)[T_1, \ldots, T_r]$ to $A/\mathfrak{J}\lambda$ which sends each polynomial in the first ring to coefficient of T^α in that polynomial. It is clear that the $\phi_{\lambda,\alpha}$ form a projective system of homomorphisms of $A/\mathfrak{J}_\lambda$-modules, so their projective limit is a continuous homomorphism $\phi_\alpha : A\{T_1, \ldots, T_r\} \to A$; we will see that, for each $y \in A\{T_1, \ldots, T_r\}$, the formal series $\sum_\alpha \phi_\alpha(y)T^\alpha$ is *restricted*. Indeed, if y_λ is the component of y in B_λ, and if we denote by H_λ the finite set of the $\alpha \in \mathbf{N}^r$ for which the coefficients of the polynomial y_λ are nonzero, then we have $\phi_{\lambda,\alpha}(y_\mu) \in \mathfrak{J}_\lambda$ for $\mathfrak{J}_\mu \subset \mathfrak{J}_\lambda$ and $\alpha \notin H_\lambda$, and by passing to the limit, $\phi_\alpha(y) \in \mathfrak{J}_\lambda$ for $\alpha \notin H_\lambda$. We thus define a ring homomorphism $\phi : A\{T_1, \ldots, T_r\} \to A'$ by setting $\phi(y) = \sum_\alpha \phi_\alpha(y)T^\alpha$, and it is immediate that ϕ is continuous. Conversely, if θ_λ is the canonical homomorphism $A \to A/\mathfrak{J}_\lambda$, then for each element $z = \sum_\alpha c_\alpha T^\alpha \in A'$ and each λ, there are only a finite number of indices α such that $\theta_\lambda(c_\alpha) \neq 0$, and as a result $\psi_\lambda(z) = \sum_\alpha \theta_\lambda(c_\alpha)T^\alpha$ is in B_λ; the ψ_λ are continuous and form a projective system of homomorphisms whose projective limit is a continuous homomorphism $\psi : A' \to A\{T_1, \ldots, T_r\}$; is remains to verify that $\phi \circ \psi$ and $\psi \circ \phi$ are the identity automorphisms, which is immediate.

(7.5.2). We identify $A\{T_1, \ldots, T_r\}$ with the ring A' of restricted formal series by means of the isomorphisms defined in (7.5.1). The canonical isomorphisms

$$((A/\mathfrak{J}_\lambda)[T_1, \ldots, T_r])[T_{r+1}, \ldots, T_s] \simeq (A/\mathfrak{J}_\lambda)[T_1, \ldots, T_s]$$

define, by passing to the projective limit, a canonical isomorphism

$$(A\{T_1, \ldots, T_r\})\{T_{r+1}, \ldots, T_s\} \simeq A\{T_1, \ldots, T_s\}.$$

(7.5.3). For every continuous homomorphism $u : A \to B$ from A to a linearly topologized ring B, separated and complete, and each system $(b_1, \ldots, b_r)$ of r elements of B, there exists *a unique continuous homomorphism* $\overline{u} : A\{T_1, \ldots, T_r\} \to B$, such that $\overline{u}(a) = u(a)$ for all $a \in A$ and $\overline{u}(T_j) = b_j$ for $1 \leqslant j \leqslant r$. It suffices to set

$$\overline{u}\Big(\sum_\alpha c_\alpha T^\alpha\Big) = \sum_\alpha u(c_\alpha) b_1^{\alpha_1} \cdots b_r^{\alpha_r};$$

the verification of the fact that the family $(u(c_\alpha)b_1^{\alpha_1} \cdots b_r^{\alpha_r})$ is summable in B and that $\overline{u}$ is continuous are immediate and left to the reader. We note that this property (for arbitrary B and b_j) *characterize* the topological ring $A\{T_1, \ldots, T_r\}$ up to unique isomorphism.

Proposition (7.5.4). —

(i) *If A is an admissible ring, then so is $A' = A\{T_1, \ldots, T_r\}$.*

(ii) *Let A be an adic ring, $\mathfrak{J}$ an ideal of definition for A such that $\mathfrak{J}/\mathfrak{J}^2$ is of finite type over $A/\mathfrak{J}$. If we set $\mathfrak{J}' = \mathfrak{J}A'$, then A' is also a $\mathfrak{J}'$-adic ring, and $\mathfrak{J}'/\mathfrak{J}'^2$ is of finite type over $A'/\mathfrak{J}'$. If in addition A is Noetherian, then so is A'.* $\mathbf{0_I}$ | 71

PROOF.

(i) If $\mathfrak{J}$ is an ideal of A, $\mathfrak{J}'$ the ideal of A' consisting of the $\sum_\alpha c_\alpha T^\alpha$ such that $c_\alpha \in \mathfrak{J}$ for all α, then $(\mathfrak{J}')^n \subset (\mathfrak{J}^n)'$; if $\mathfrak{J}$ is an ideal of definition for A, then $\mathfrak{J}'$ is also an ideal of definition for A'.

(ii) Set $A_i = A/\mathfrak{J}^{i+1}$, and for $i \leqslant j$, let u_{ij} be the canonical homomorphism $A/\mathfrak{J}^{j+1} \to A/\mathfrak{J}^{i+1}$; set $A'_i = A_i[T_1, \ldots, T_r]$, and let u'_{ij} be the homomorphism $A'_j \to A'_i$ ($i \leqslant j$) obtained by applying u_{ij} to the coefficients of the polynomials in A'_j. We will show that the projective system (A'_i, u'_{ij}) satisfies the conditions of Proposition (7.2.7); as $\mathfrak{J}'$ is the kernel of $A' \to A'_0$, this proves the first assertion of (ii). It is clear that the u'_{ij} are surjective; the kernel $\mathfrak{J}'_i$ of u_{0i} is the set of polynomials in $A_i[T_1, \ldots, T_r]$ whose coefficients are in $\mathfrak{J}/\mathfrak{J}^{i+1}$; in particular, $\mathfrak{J}'_1$ is the set of polynomials in $A_1[T_1, \ldots, T_r]$ whose coefficients are in $\mathfrak{J}/\mathfrak{J}^2$. As $\mathfrak{J}/\mathfrak{J}^2$ is of finite type over $A_1 = A/\mathfrak{J}^2$, we see that $\mathfrak{J}'_1/\mathfrak{J}'^2_1$ is a module of finite type over A'_1 (or equivalently, over $A'_0 = A'_1/\mathfrak{J}'_1$). We will now show that the kernel of u_{ij} is $\mathfrak{J}'^{i+1}_j$. It is evident that $\mathfrak{J}'^{i+1}_j$ is contained in this kernel. On the other hand, let $a_1, \ldots, a_m$ be the elements of $\mathfrak{J}$ whose classes mod $\mathfrak{J}^2$ generate $\mathfrak{J}/\mathfrak{J}^2$; we verify immediately that the classes mod $\mathfrak{J}^{j+1}$ of monomials of degree $\leqslant j$ in the a_k ($1 \leqslant k \leqslant m$) generate $\mathfrak{J}/\mathfrak{J}^{j+1}$, and the classes of monomials of degree $> i$ and $\leqslant j$ generate $\mathfrak{J}^{i+1}/\mathfrak{J}^{j+1}$; a monomial in the T_k having such an element for a coefficient is thus a product of $i+1$ elements of $\mathfrak{J}'_i$, which establishes our assertion. Finally, if A is Noetherian, then so is $A'/\mathfrak{J}' = (A/\mathfrak{J})[T_1, \ldots, T_r]$, hence A' is Notherian (7.2.8).

□

Proposition (7.5.5). — *Let A be a Notherian $\mathfrak{J}$-adic ring, B an admissible topological ring, $\phi : A \to B$ a continuous homomorphism, making B and A-algebra. The following conditions are equivalent:*

(a) *B is Noetherian and $\mathfrak{J}B$-adic, and $B/\mathfrak{J}B$ is an algebra of finite type over $A/\mathfrak{J}$.*
(b) *B is topologically A-isomorphic to $\varprojlim B_n$, where $B_n = B_m/\mathfrak{J}^{n+1}B_m$ for $m \geqslant n$, and B_1 is an algebra of finite type over $A_1 = A/\mathfrak{J}^2$.*
(c) *B is topologically A-isomorphic to a quotient of an algebra of the form $A\{T_1, \ldots, T_r\}$ by an ideal (necessarily closed according to Corollary (7.3.6) and Proposition (7.5.4, ii)).*

PROOF. As A is Noetherian, so is $A' = A\{T_1, \ldots, T_r\}$ (7.5.4), so (c) implies that B is Noetherian; as $\mathfrak{J}' = \mathfrak{J}A'$ is an open neighbourhood of 0 in A' such that the $\mathfrak{J}'^n$ form a fundamental system of neighbourhoods of 0, the images $\mathfrak{J}^nB$ of the $\mathfrak{J}'^n$ form a fundamental system of neighbourhoods of 0 in B, and as B is separated and complete, B is a $\mathfrak{J}B$-adic ring. Finally, $B/\mathfrak{J}B$ is an algebra (over $(A/\mathfrak{J})$ quotient of $A'/\mathfrak{J}A' = (A/\mathfrak{J})[T_1, \ldots, T_r]$, so it is of finite type, which proves that (c) implies (a).

If B is $\mathfrak{J}B$-adic and Noetherian, then B is isomorphic to $\varprojlim B_n$, where $B_n = B/\mathfrak{J}^{n+1}B$ (7.2.11), and $\mathfrak{J}B/\mathfrak{J}^2B$ is a module of finite type over $B/\mathfrak{J}B$. Let $(a_j)_{1 \leqslant j \leqslant s}$ be a system of generators for the $B/\mathfrak{J}B$-module $\mathfrak{J}B/\mathfrak{J}^2B$, and let $(c_i)_{1 \leqslant i \leqslant r}$ be a system of elements of $B/\mathfrak{J}^2B$ such that the classes mod $\mathfrak{J}B/\mathfrak{J}^2B$ form a system of generators for the $A/\mathfrak{J}$-algebra $B/\mathfrak{J}B$; we see immediately that the c_ia_j form a system of generators for the $A/\mathfrak{J}^2$-algebra $B/\mathfrak{J}^2B$, hence (a) implies (b). $\mathbf{0_I}$ | 72

It remains to prove that (b) implies (c). The hypotheses imply that B_1 is a Noetherian ring, and as $B_1 = B_2/\mathfrak{J}^2B_2$, we have $\mathfrak{J}^2B_1 = 0$, hence $\mathfrak{J}B_1 = \mathfrak{J}B_1/\mathfrak{J}^2B_1$ is a B_0-module of finite type. The conditions of Proposition (7.2.7) are thus satisfied by the projective system (B_n) and B is a $\mathfrak{J}B$-adic ring. Let $(c_i)_{1 \leqslant i \leqslant r}$ be a finite system of elements of B whose classes mod $\mathfrak{J}B$ generate the $A/\mathfrak{J}$-algebra $B/\mathfrak{J}B$, and whose linear combinations with coefficients in $\mathfrak{J}$ are such that their classes mod $\mathfrak{J}^2B$ generate the B_0-module $\mathfrak{J}B/\mathfrak{J}^2B$. There exists a continuous A-homomorphism u from $A' = A\{T_1, \ldots, T_r\}$ to B which reduces to ϕ on A and is such that $u(T_i) = c_i$ for $1 \leqslant i \leqslant r$ (7.5.3); if we prove that u is *surjective*, then (c) will be established, since from $u(A') = B$ we deduce that $u(\mathfrak{J}^nA') = \mathfrak{J}^nB$, in other words that u is a strict morphism of topological rings and B is this isomorphic to a quotient of A' by a closed ideal. As B is complete for the $\mathfrak{J}B$-adic topology, it suffices ([**CC**, p. 18–07]) to show that the homomorphism $\operatorname{grad}(A') \to \operatorname{grad}(B)$, induced canonically by u for the $\mathfrak{J}$-adic filtrations on A' and B, is surjective. But by definition, the homomorphisms $A'/\mathfrak{J}A' \to B/\mathfrak{J}B$ and $\mathfrak{J}A'/\mathfrak{J}^2A' \to \mathfrak{J}B/\mathfrak{J}^2B$ induced by u are surjective; by induction on n, we immediately deduce that so is $\mathfrak{J}A'/\mathfrak{J}^nA' \to \mathfrak{J}B/\mathfrak{J}^nB$, and *a fortiori* so is $\mathfrak{J}^nA'/\mathfrak{J}^{n+1}A' \to \mathfrak{J}^nB/\mathfrak{J}^{n+1}B$, which finishes the proof. ☐

7.6. Completed rings of fractions

(7.6.1). Let A be a linearly topologized ring, $(\mathfrak{J}_\lambda)$ a fundamental system of neighbourhoods of 0 in A consisting of ideals, S a multiplicative subset of A. Let u_λ be the canonical homomorphism $A \to A_\lambda = A/\mathfrak{J}_\lambda$, and for $\mathfrak{J}_\mu \subset \mathfrak{J}_\lambda$, let $u_{\lambda\mu}$ be the canonical homomorphism $A_\mu \to A_\lambda$. Set $S_\lambda = u_\lambda(S)$, so that $u_{\lambda\mu}(S_\mu) = S_\lambda$. The $u_{\lambda\mu}$ canonically induce surjective homomorphisms $S_\mu^{-1}A_\mu \to S_\lambda^{-1}A_\lambda$, for which these rings form a projective system; we denote by $A\{S^{-1}\}$ the projective limit of this system. This definition does not depend on the fundamental system of neighbourhoods $(\mathfrak{J}_\lambda)$ chosen; indeed:

Proposition (7.6.2). — *The ring $A\{S^{-1}\}$ is topologically isomorphic to the separated completion of the ring $S^{-1}A$ for the topology which has a fundamental system of neighbourhoods of* 0 *consisting of the $S^{-1}\mathfrak{J}_\lambda$.*

Proof. If v_λ is the canonical homomorphism $S^{-1}A \to S_\lambda^{-1}A_\lambda$ induced by u_λ, then the kernel of v_λ is surjective, hence the proposition (7.2.1). ☐

Corollary (7.6.3). — *If S' is the canonical image of S in the separated completion $\widehat{A}$ of A, then $A\{S^{-1}\}$ canonically identifies with $\widehat{A}\{S'^{-1}\}$.*

We note that if A is separated and complete, then it is not necessarily the same for $S^{-1}A$ with the topology defined by the $S^{-1}\mathfrak{J}_\lambda$, as we see for example by taking S to be the set of the f^n $(n \geqslant 0)$, where f is topologically nilpotent but not nilpotent: indeed, $S^{-1}A$ is not 0 and on the other hand, for each λ there exists an n such that $f^n \in \mathfrak{J}_\lambda$, so $1 = f^n/f^n \in S^{-1}\mathfrak{J}_\lambda$ and $S^{-1}\mathfrak{J}_\lambda = S^{-1}A$.

$\mathbf{0_I}$ | 73

Corollary (7.6.4). — *If, in A,* 0 *does not belong to S, then the ring $A\{S^{-1}\}$ is not* 0*.*

Proof. Indeed, 0 does not belong to $\{1\}$ in the ring $S^{-1}A$; otherwise, we would have that $1 \in S^{-1}\mathfrak{J}_\lambda$ for each open ideal $\mathfrak{J}_\lambda$ of A, and it follows that $\mathfrak{J}_\lambda \cap S \neq \varnothing$ for all λ, contradicting the hypothesis. ☐

(7.6.5). We say that $A\{S^{-1}\}$ is the *completed ring of fractions* of A with denominators in S. With the above notation, it is clear that the inverse image of $S^{-1}\mathfrak{J}_\lambda$ in A contains $\mathfrak{J}_\lambda$, hence the canonical map $A \to S^{-1}A$ is continuous, and if we compose it with the canonical map $S^{-1}A \to A\{S^{-1}\}$, we obtain a canonical continuous homomorphism $A \to A\{S^{-1}\}$, the projective limit of the homomorphisms $A \to S_\lambda^{-1}A_\lambda$.

(7.6.6). The couple consisting of $A\{S^{-1}\}$ and the canonical homomorphism $A \to A\{S^{-1}\}$ are characterized by the following *universal property*: every continuous homomorphism u from A to a linearly topologized ring B, separated and complete, such that $u(S)$ consists of the invertible elements of B, uniquely factorizes as $A \to A\{S^{-1}\} \xrightarrow{u'} B$, where u' is continuous. Indeed, u uniquely factorizes as $A \to S^{-1}A \xrightarrow{v'} B$; as for each open ideal $\mathfrak{K}$ of B we have that $u^{-1}(\mathfrak{K})$ contains a $\mathfrak{J}_\lambda$, $v'^{-1}(\mathfrak{K})$ necessarily contains $S^{-1}\mathfrak{J}_\lambda$, so v' is continuous; since B is separated and complete, v' uniquely factorizes as $S^{-1}A \to A\{S^{-1}\} \xrightarrow{u'} B$, where u' is continuous; hence our assertion.

(7.6.7). Let B be a second linearly topologized ring, T a multiplicative subset of B, $\phi : A \to B$ a continuous homomorphism such that $\phi(S) \subset T$. According to the above, the continuous homomorphism $A \xrightarrow{\phi} B \to B\{T^{-1}\}$ uniquely factorizes as $A \to A\{S^{-1}\} \xrightarrow{\phi'} B\{T^{-1}\}$, where ϕ' is continuous. In particular, if $B = A$ and if ϕ is the identity, we see that for $S \subset T$ we have a continuous homomorphism $\rho^{T,S} : A\{S^{-1}\} \to A\{T^{-1}\}$ obtained by passing to the separated completion from $S^{-1}A \to T^{-1}A$; if U is a third multiplicative subset of A such that $S \subset T \subset U$, then we have $\rho^{U,S} = \rho^{U,T} \circ \rho^{T,S}$.

(7.6.8). Let S_1, S_2 be two multiplicative subsets of A, and let S_2' be the canonical image of S_2 in $A\{S_1^{-1}\}$; we then have a canonical topological isomorphism $A\{(S_1S_2)^{-1}\} \simeq A\{S_1^{-1}\}\{S_2'^{-1}\}$, as we see from the canonical isomorphism $(S_1S_2)^{-1}A \simeq S_2''^{-1}(S_1^{-1}A)$ (where S_2'' is the canonical image of S_2 in $S_1^{-1}A$), which is bicontinuous.

(7.6.9). Let $\mathfrak{a}$ be an *open* ideal of A; we can assume that $\mathfrak{J}_\lambda \subset \mathfrak{a}$ for all λ, and as a result $S^{-1}\mathfrak{J}_\lambda \subset S^{-1}\mathfrak{a}$ in the ring $S^{-1}A$, in other words, $S^{-1}\mathfrak{a}$ is an *open* ideal of $S^{-1}A$; we denote by $\mathfrak{a}\{S^{-1}\}$ its separated completion, equal to $\varprojlim(S^{-1}\mathfrak{a}/S^{-1}\mathfrak{J}_\lambda)$, which is an *open* ideal of $A\{S^{-1}\}$, isomorphic to the closure of the canonical image of $S^{-1}\mathfrak{a}$. In addition, *the discrete ring $A\{S^{-1}\}/\mathfrak{a}\{S^{-1}\}$ is canonically isomorphic to $S^{-1}A/S^{-1}\mathfrak{a} = S^{-1}(A/\mathfrak{a})$*. Conversely, if $\mathfrak{a}'$ is an open ideal of $A\{S^{-1}\}$, then $\mathfrak{a}'$ contains an ideal of the form $\mathfrak{J}_\lambda\{S^{-1}\}$ which is the inverse image of an ideal of $S^{-1}A/S^{-1}\mathfrak{J}_\lambda$, which is necessarily (1.2.6) of the form $S^{-1}\mathfrak{a}$, where $\mathfrak{a} \supset \mathfrak{J}_\lambda$. We conclude that $\mathfrak{a}' = \mathfrak{a}\{S^{-1}\}$. In particular (1.2.6):

Proposition (7.6.10). — *The map $\mathfrak{p} \mapsto \mathfrak{p}\{S^{-1}\}$ is an increasing bijection from the set of* open prime *ideals $\mathfrak{p}$ of A such that $\mathfrak{p} \cap S = \varnothing$ to the set of* open prime *ideals of $A\{S^{-1}\}$; in addition, the field of fractions of $A\{S^{-1}\}/\mathfrak{p}\{S^{-1}\}$ is canonically isomorphic to that of $A/\mathfrak{p}$.* $\mathbf{0_I}$ | 74

Proposition (7.6.11). —

(i) *If A is an admissible ring, then so is $A' = A\{S^{-1}\}$, and for every ideal of definition $\mathfrak{J}$ for A, $\mathfrak{J}' = \mathfrak{J}\{S^{-1}\}$ is an ideal of definition for A'.*

(ii) *Let A be an adic ring, $\mathfrak{J}$ an ideal of definition for A such that $\mathfrak{J}/\mathfrak{J}^2$ is of finite type over $A/\mathfrak{J}$; then A' is a $\mathfrak{J}'$-adic ring and $\mathfrak{J}'/\mathfrak{J}'^2$ is of finite type over $A'/\mathfrak{J}'$. If in addition A is Noetherian, then so is A'.*

PROOF.

(i) If $\mathfrak{J}$ is an ideal of definition for A, then it is clear that $S^{-1}\mathfrak{J}$ is an ideal of definition for the topological ring $S^{-1}A$, since we have $(S^{-1}\mathfrak{J})^n = S^{-1}\mathfrak{J}^n$. Let A'' be the separated ring associated to $S^{-1}A$, $\mathfrak{J}''$ the image of $S^{-1}\mathfrak{J}$ in A''; the image of $S^{-1}\mathfrak{J}^n$ is $\mathfrak{J}''^n$, so $\mathfrak{J}''^n$ tends to 0 in A''; as $\mathfrak{J}'$ is the closure of $\mathfrak{J}''$ in A', $\mathfrak{J}'^n$ is contained in the closure of $\mathfrak{J}''^n$, hence tends to 0 in A'.

(ii) Set $A_i = A/\mathfrak{J}^{i+1}$, and for $i \leqslant j$, let u_{ij} be the canonical homomorphism $A/\mathfrak{J}^{j+1} \to A/\mathfrak{J}^{i+1}$; let S_i be the canonical image of S in A_i, and set $A_i' = S_i^{-1}A_i$; finally, let $u_{ij}' : A_j' \to A_i'$ be the homomorphism canonically induced by u_{ij}. We show that the projective system (A_i', u_{ij}') satisfies the conditions of Proposition (7.2.7): it is clear that the u_{ij}' are surjective; on the other hand, the kernel of u_{ij}' is $S_j^{-1}(\mathfrak{J}^{i+1}/\mathfrak{J}^{j+1})$ (1.3.2), equal to $\mathfrak{J}_j'^{i+1}$, where $\mathfrak{J}_j' = S_j^{-1}(\mathfrak{J}/\mathfrak{J}^{j+1})$; finally, $\mathfrak{J}_1'/\mathfrak{J}_1'^2 = S_1^{-1}(\mathfrak{J}/\mathfrak{J}^2)$, and as $\mathfrak{J}/\mathfrak{J}^2$ is of finite type over $A/\mathfrak{J}^2$, $\mathfrak{J}_1'/\mathfrak{J}_1'^2$ is of finite type over A_1'. Finally, if A is Noetherian, then so is $A_0' = S_0^{-1}(A/\mathfrak{J})$, which finishes the proof (7.2.8).

□

Corollary (7.6.12). — *Under the hypotheses of Proposition (7.6.11, ii), we have $(\mathfrak{J}\{S^{-1}\})^n = \mathfrak{J}^n\{S^{-1}\}$.*

PROOF. This follows from Proposition (7.2.7) and the proof of Proposition (7.6.11). □

Proposition (7.6.13). — *Let A be an adic Noetherian ring, S a multiplicative subset of A; then $A\{S^{-1}\}$ is a flat A-module.*

PROOF. If $\mathfrak{J}$ is an ideal of definition for A, then $A\{S^{-1}\}$ is the separated completion of the Noetherian ring $S^{-1}A$ equipped with the $S{-}1\mathfrak{J}$-preadic topology; as a result (7.3.3) $A\{S^{-1}\}$ is a flat $S^{-1}A$-module; as $S^{-1}A$ is a flat A-module (6.3.1), the proposition follows from the transitivity of flatness (6.2.1). □

Corollary (7.6.14). — *Under the hypotheses of Proposition (7.6.13), let $S' \subset S$ be a second multiplicative subset of A; then $A\{S^{-1}\}$ is a flat $A\{S'^{-1}\}$-module.*

PROOF. By (7.6.8), $A\{S^{-1}\}$ canonically identifies with $A\{S'^{-1}\}\{S_0^{-1}\}$, where S_0 is the canonical image of S in $A\{S'^{-1}\}$, and $A\{S'^{-1}\}$ is Noetherian (7.6.11). □

(7.6.15). For each element f of a linearly topologized ring A, we denote by $A_{\{f\}}$ the completed ring of fractions $A\{S_f^{-1}\}$, where S_f is the multiplicative set of the f^n ($n \geqslant 0$); for each open ideal $\mathfrak{a}$ of A, we write $\mathfrak{a}_{\{f\}}$ for $\mathfrak{a}\{S_f^{-1}\}$. If g is a second element of A, then we have a canonical continuous homomorphism $A_{\{f\}} \to A_{\{fg\}}$ (7.6.7). When f varies over a multiplicative subset S of A, the $A_{\{f\}}$ form a filtered inductive system with the above homomorphisms; we set $A_{\{S\}} = \varinjlim_{f \in S} A_{\{f\}}$. For every $f \in S$, we have a homomorphism $A_{\{f\}} \to A\{S^{-1}\}$ (7.6.7), and these homomorphisms form an inductive system; by passing to the inductive limit, they thus define a canonical homomorphism $A_{\{S\}} \to A\{S^{-1}\}$. $\mathbf{0_I}$ | 75

Proposition (7.6.16). — *If A is a Noetherian ring, then $A\{S^{-1}\}$ is a flat module over $A_{\{S\}}$.*

PROOF. By (7.6.14), $A\{S^{-1}\}$ is flat for each of the rings $A_{\{f\}}$ for $f \in S$, and the conclusion follows from (6.2.3). □

Proposition (7.6.17). — *Let $\mathfrak{p}$ be an open prime ideal in an admissible ring A, and let $S = A - \mathfrak{p}$. Then the rings $A\{S^{-1}\}$ and $A_{\{S\}}$ are local rings, the canonical homomorphism $A_{\{S\}} \to A\{S^{-1}\}$ is local, and the residue fields of $A_{\{S\}}$ and $A\{S^{-1}\}$ are canonically isomorphic to the field of fractions of $A/\mathfrak{p}$.*

PROOF. Let $\mathfrak{J} \subset \mathfrak{p}$ be an ideal of definition for A; we have $S^{-1}\mathfrak{J} \subset S^{-1}\mathfrak{p} = \mathfrak{p}A_{\mathfrak{p}}$, so $A_{\mathfrak{p}}/S^{-1}\mathfrak{J}$ is a local ring; we conclude from Corollary (7.1.12), (7.6.9), and Proposition (7.6.11, i) that $A\{S^{-1}\}$ is a local ring. Set $\mathfrak{m} = \varinjlim_{f \in S} \mathfrak{p}_{\{f\}}$, which is an ideal in $A_{\{S\}}$; we will see that each element in $A_{\{S\}}$ not in $\mathfrak{m}$ is invertible. Indeed, such an element is the image in $A_{\{S\}}$ of an element $z \in A_{\{f\}}$ not in $\mathfrak{p}_{\{f\}}$, for an $f \in S$; its canonical image z_0 in $A_{\{f\}}/\mathfrak{J}_{\{f\}} = S_f^{-1}(A/\mathfrak{J})$ therefore is not in $S_f^{-1}(\mathfrak{p}/\mathfrak{J})$ (7.6.9), which means that $z_0 = \overline{x}/\overline{f}^k$, where $x \notin \mathfrak{p}$ and $\overline{x}$, $\overline{f}$ are the classes of x, f mod $\mathfrak{J}$. As $x \in S$, we have $g = xf \in S$, and in $S_g^{-1}A$, the canonical image $y_0 = x^{k+1}/g^k$ of $x/f^k \in S_f^{-1}A$ admits an inverse $x^{k-1}f^{2k}/g^k$. This implies *a fortiori* that the image of y_0 in $S_g^{-1}A/S_g^{-1}\mathfrak{J}$ is invertible, so ((7.6.9) and Corollary (7.1.12)) the canonical image y of z in $A_{\{g\}}$ is invertible; the image of z in $A_{\{S\}}$ (equal to that of y) is as a result invertible. We thus see that $A_{\{S\}}$ a local ring with maximal ideal $\mathfrak{m}$; in addition, the image of $\mathfrak{p}_{\{f\}}$ in $A\{S^{-1}\}$ is contained in the maximal ideal $\mathfrak{p}\{S^{-1}\}$ of this ring; *a fortiori*, the image of $\mathfrak{m}$ in $A\{S^{-1}\}$ is contained in $\mathfrak{p}\{S^{-1}\}$, so the canonical homomorphism $A_{\{S\}} \to A\{S^{-1}\}$ is local. Finally, as each element of $A\{S^{-1}\}/\mathfrak{p}\{S^{-1}\}$ is the image of an element in the ring $S_f^{-1}A$ for a suitable $f \in S$, the homomorphism $A_{\{S\}} \to A\{S^{-1}\}/\mathfrak{p}\{S^{-1}\}$ is surjective, and gives an isomorphism of the residue fields by passing to quotients. □

Corollary (7.6.18). — *Under the hypotheses of Proposition (7.6.17), if we suppose also that A is an adic Noetherian ring, then the local rings $A\{S^{-1}\}$ and $A_{\{S\}}$ are Noetherian, and $A\{S^{-1}\}$ is a faithfully flat $A_{\{S\}}$-module.*

PROOF. We know from before (7.6.11, ii) that $A\{S^{-1}\}$ is Noetherian and $A_{\{S\}}$-flat (7.6.16); as the homomorphism $A_{\{S\}} \to A\{S^{-1}\}$ is local, we conclude that $A\{S^{-1}\}$ is a faithfully flat $A_{\{S\}}$-module (6.6.2), and as a result that $A_{\{S\}}$ is Noetherian (6.5.2). □

7.7. Completed tensor products

(7.7.1). Let A be a linearly topologized ring, M, N two linearly topologized A-modules. Let $\mathfrak{J}$, V, W be open neighbourhoods of 0 in A, M, N respectively, which are A-modules, and such that $\mathfrak{J}M \subset V$, $\mathfrak{J}N \subset W$, so that M/V and N/W can be considered as $A/\mathfrak{J}$-modules. When $\mathfrak{J}$, V, W vary over the systems of open neighbourhoods satisfying these properties, it is immediate that the modules $(M/V) \otimes_{A/\mathfrak{J}} (N/W)$ form a projective system of modules over the projective system of rings $A/\mathfrak{J}$; $\mathbf{0_I}$ | 76
by passing to the projective limit, we thus obtain a module over the separated completion $\widehat{A}$ of A, which we call the *completed tensor product* of M and N and denote by $(M \otimes_A N)^\wedge$. If we have that M/V is canonically isomorphic to $\widehat{M}/\overline{V}$, where $\widehat{M}$ is the separated completion of M and $\overline{V}$ the closure in $\widehat{M}$ of the image of V, then we see that the completed tensor product $(M \otimes_A N)^\wedge$ canonically identifies with $(\widehat{M} \otimes_{\widehat{A}} \widehat{N})^\wedge$, which we denote by $\widehat{M} \widehat{\otimes}_{\widehat{A}} \widehat{N}$.

(7.7.2). With the above notation, the tensor products $(M/V) \otimes_A (N/W)$ and $(M/V) \otimes_{A/\mathfrak{J}} (N/W)$ identify canonically; they identify with $(M \otimes_A N)/(\mathrm{Im}(V \otimes_A N) + \mathrm{Im}(M \otimes_A W))$. We conclude that $(M \otimes_A N)^\wedge$ is *the separated completion of the A-module $M \otimes_A N$, equipped with the topology for which the submodules*

$$\mathrm{Im}(V \otimes_A N) + \mathrm{Im}(M \otimes_A W)$$

form a fundamental system of neighbourhoods of 0 (V and W varying over the set of open submodules of M and N respectively); we say for brevity that this topology is the *tensor product* of the given topologies on M and N.

(7.7.3). Let M', N' be two linearly topologized A-modules, $u : M \to M'$, $v : N \to N'$ two continuous homomorphisms; it is immediate that $u \otimes v$ is continuous for the tensor product topologies on $M \otimes_A N$ and $M' \otimes_A N'$ respectively; by passing to the separated completions, we obtain a continuous homomorphism $(M \otimes_A N)^\wedge \to (M' \otimes_A N')^\wedge$, which we denote by $u \widehat{\otimes} v$; $(M \otimes_A N)^\wedge$ is thus a *bifunctor* in M and N on the category of linearly topologized A-modules.

(7.7.4). We similarly define the completed tensor product of any finite number of linearly topologized A-modules; it is immediate that this product has the usual properties of associativity and commutativity.

(7.7.5). If B, C are two linearly topologized A-algebras, then the tensor product topology on $B \otimes_A C$ has for a fundamental system of neighbourhoods of 0 the *ideals* $\mathrm{Im}(\mathfrak{K} \otimes_A C) + \mathrm{Im}(B \otimes_A \mathfrak{L})$ of the algebra $B \otimes_A C$, $\mathfrak{K}$ (resp. $\mathfrak{L}$) varying over the set of open ideals of B (resp. C). As a result, $(B \otimes_A C)^\wedge$ is equipped with the structure of a *topological $\widehat{A}$-algebra*, the projective limit of the projective system of $A/\mathfrak{J}$-algebras $(B/\mathfrak{K}) \otimes_{A/\mathfrak{J}} (C/\mathfrak{L})$ ($\mathfrak{J}$ the open ideal of A such that $\mathfrak{J}B \subset \mathfrak{K}$, $\mathfrak{J}C \subset \mathfrak{L}$; it always exists). We say that this algebra is the *completed tensor product* of the algebras B and C.

(7.7.6). The A-algebra homomorphisms $b \mapsto b \otimes 1$, $c \mapsto 1 \otimes c$ from B and C to $B \otimes_A C$ are continuous when we equip the latter algebra with the tensor product topology; by composing with the canonical homomorphism from $B \otimes_A C$ to its separated completion, they give *canonical homomorphisms* $\rho : B \to (B \otimes_A C)^\wedge$, $\sigma : C \to (B \otimes_A C)^\wedge$. The algebra $(B \otimes_A C)^\wedge$ and the homomorphisms ρ and σ have in addition the following *universal property*: for every linearly topologized A-algebra D, separated and complete, and each pair of continuous A-homomorphisms $u : B \to D$, $v : C \to D$, there exists a unique continuous A-homomorphism $w : (B \otimes_A C)^\wedge \to D$ such that $u = w \circ \rho$ and $v = w \circ \sigma$. Indeed, there already exists a unique A-homomorphism $w_0 : B \otimes_A C \to D$ such that $u(b) = w_0(b \otimes 1)$ and $v(c) = w_0(1 \otimes c)$, and it remains to prove that w_0 is *continuous*, since it then gives a continuous homomorphism w by passing to the separated completion. If $\mathfrak{M}$ is an open ideal $\mathbf{0_I}$ | 77
of D, then there exists by hypothesis open ideals $\mathfrak{K} \subset B$, $\mathfrak{L} \subset C$ such that $u(\mathfrak{K}) \subset \mathfrak{M}$, $v(\mathfrak{L}) \subset \mathfrak{M}$; the image under w_0 of $\mathrm{Im}(\mathfrak{K} \otimes_A C) + \mathrm{Im}(B \otimes_A \mathfrak{L})$ is again contained in $\mathfrak{M}$, hence our assertion.

Proposition (7.7.7). — *If B and C are two preadmissible A-algebras, then $(B \otimes_A C)^\wedge$ is admissible, and if $\mathfrak{K}$ (resp. $\mathfrak{L}$) is an ideal of definition for B (resp. C), then the closure in $(B \otimes_A C)^\wedge$ of the canonical image of $\mathfrak{H} = \mathrm{Im}(\mathfrak{K} \otimes_A C) + \mathrm{Im}(B \otimes_A \mathfrak{L})$ is an ideal of definition.*

PROOF. It suffices to show that $\mathfrak{H}^n$ tends to 0 for the tensor product topology, which follows immediately from the inclusion

$$\mathfrak{H}^{2n} \subset \mathrm{Im}(\mathfrak{K}^n \otimes_A C) + \mathrm{Im}(B \otimes_A \mathfrak{L}^n).$$

□

Proposition (7.7.8). — *Let A be a preadic ring, $\mathfrak{J}$ an ideal of defintion for A, M an A-module* of finite type*, equipped with the $\mathfrak{J}$-preadic topology. For every topological adic Noetherian A-algebra B, $B \otimes_A M$ identifies with the completed tensor product $(B \otimes_A M)^\wedge$.*

Proof. If $\mathfrak{K}$ is an ideal of definition for B, there exists by hypothesis an integer m such that $\mathfrak{J}^m B \subset \mathfrak{K}$, so $\operatorname{Im}(B \otimes_A \mathfrak{J}^{nm} M) = \operatorname{Im}(\mathfrak{J}^{nm} B \otimes_A M) \subset \operatorname{Im}(\mathfrak{K}^n B \otimes_A M) = \mathfrak{K}^n (B \otimes_A M)$; we conclude that over $B \otimes_A M$, the tensor products of the topologies of B and M is the $\mathfrak{K}$-preadic topology. As $B \otimes_A M$ is a B-module of finite type, the proposition follows from Corollary (7.3.6). □

7.8. Topologies on modules of homomorphisms

(7.8.1). Let A be a *Noetherian $\mathfrak{J}$-adic* ring, M and N two A-modules of finite type, equipped with the $\mathfrak{J}$-preadic topology; we know (7.3.6) that they are separated and complete; in addition, every A-homomorphism $M \to N$ is automatically continuous, and the A-module $\operatorname{Hom}_A(M, N)$ is of finite type. For every integer $i \geqslant 0$, set $A_i = A/\mathfrak{J}^{i+1}$, $M_i = M/\mathfrak{J}^{i+1}M$, $N_i = N/\mathfrak{J}^{i+1}N$; for $i \leqslant j$, every homomorphism $u_j : M_j \to N_j$ maps $\mathfrak{J}^{i+1}M_j$ to $\mathfrak{J}^{i+1}N_j$, thus giving by passage to quotients a homomorphism $u_i : M_i \to N_i$, which defines a canonical homomorphism $\operatorname{Hom}_{A_j}(M_j, N_j) \to \operatorname{Hom}_{A_i}(M_i, N_i)$; in addition, the $\operatorname{Hom}_{A_i}(M_i, N_i)$ form a *projective system* for these homomorphisms, and it follows from Corollary (7.2.10) that there is a canonical isomorphism $\phi : \operatorname{Hom}_A(M, N) \to \varprojlim_i \operatorname{Hom}_{A_i}(M_i, N_i)$. In addition:

Proposition (7.8.2). — *If M and N are modules of finite type over a $\mathfrak{J}$-adic Noetherian ring A, then the submodules $\operatorname{Hom}_A(M, \mathfrak{J}^{i+1}N)$ form a fundamental system of neighbourhoods of 0 in $\operatorname{Hom}_A(M, N)$ for the $\mathfrak{J}$-adic topology, and the canonical isomorphism $\phi : \operatorname{Hom}_A(M, N) \to \varprojlim_i \operatorname{Hom}_{A_i}(M_i, N_i)$ is a topological isomorphism.*

Proof. We can consider M as the quotient of a free A-module L of finite type, and as a result identify $\operatorname{Hom}_A(M, N)$ as a submodule of $\operatorname{Hom}_A(L, N)$; in this identification, $\operatorname{Hom}_A(M, \mathfrak{J}^{i+1}N)$ is the intersection of $\operatorname{Hom}_A(M, N)$ and $\operatorname{Hom}_A(L, \mathfrak{J}^{i+1}N)$; as the induced topology on $\operatorname{Hom}_A(M, N)$ by the $\mathfrak{J}$-adic topology of $\operatorname{Hom}_A(L, N)$ is the $\mathfrak{J}$-adic (7.3.2), we have reduced to proving the first assertion for $M = L = A^m$; but then $\operatorname{Hom}_A(L, N) = N^m$, $\operatorname{Hom}_A(L, \mathfrak{J}^{i+1}N) = (\mathfrak{J}^{i+1}N)^m = \mathfrak{J}^{i+1}N^m$ and the **0**$_{\mathbf{I}}$ | 78
result is evident. To establish the second assertion, we note that the image of $\operatorname{Hom}_A(M, \mathfrak{J}^{i+1}N)$ in $\operatorname{Hom}_{A_j}(M_j, N_j)$ is zero for $j \leqslant i$, hence ϕ is continuous; conversely, the inverse image in $\operatorname{Hom}_A(M, N)$ of 0 of $\operatorname{Hom}_{A_i}(M_i, N_i)$ is $\operatorname{Hom}_A(M, \mathfrak{J}^{i+1}N)$, so ϕ is bicontinuous. □

If we only suppose that A is a *Noetherian $\mathfrak{J}$-preadic* ring, M and N two A-modules of finite type, separated for the $\mathfrak{J}$-preadic topology, then the following proof shows that the first assertion of Proposition (7.8.2) remains valid, and that ϕ is a topological isomorphism from $\operatorname{Hom}_A(M, N)$ to a submodule of $\varprojlim_i \operatorname{Hom}_{A_i}(M_i, N_i)$.

Proposition (7.8.3). — *Under the hypotheses of Proposition (7.8.2), the set of injective (resp. surjective, bijective) homomorphisms from M to N is an open subset of $\operatorname{Hom}_A(M, N)$.*

Proof. According to Corollaries (7.3.5) and (7.1.14), for u to be injective, it is necessary and sufficient that the corresponding homomorphism $u_0 : M/\mathfrak{J}M \to N/\mathfrak{J}N$ is, and the set of surjective homomorphisms from M to N is thus the inverse image under the continuous map $\operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{A_0}(M_0, N_0)$ of a subset of a discrete space. We now show that the set of injective homomorphisms is open; let v be such a homomorphism and set $M' = v(M)$; by the Artin–Rees Lemma (7.3.2.1), there exists an integer $k \geqslant 0$ such that $M' \cap \mathfrak{J}^{m+k}N \subset \mathfrak{J}^m M'$ for all $m > 0$; we will see that for all $w \in \mathfrak{J}^{k+1} \operatorname{Hom}_A(M, N)$, $u = v + w$ is injective, which will finish the proof. Indeed, let $x \in M$ be such that $u(x) = 0$; we prove that for every $i \geqslant 0$ the relation $x \in \mathfrak{J}^i M$ implies that $x \in \mathfrak{J}^{i+1}M$; this follows from $x \in \bigcap_{i \geqslant 0} \mathfrak{J}^i M = (0)$. Indeed, we then have $w(x) \in \mathfrak{J}^{i+k+1}N$, and as a result $w(x) = -v(x) \in M'$, so $v(x) \in M' \cap \mathfrak{J}^{i+k+1}N \subset \mathfrak{J}^{i+1}M'$, and as v is an isomorphism from M to M', $x \in \mathfrak{J}^{i+1}M$; q.e.d. □

§8. Representable functors

8.1. Representable functors $\mathbf{0}_{\mathbf{III}}$ | 5

(8.1.1). We denote by $\mathtt{Set}$ the category of sets. Let $\mathtt{C}$ be a category; for two objects X, Y of $\mathtt{C}$, we set $h_X(Y) = \mathrm{Hom}(Y, X)$; for each morphism $u : Y \to Y'$ in $\mathtt{C}$, we denote by $h_X(u)$ the map $v \mapsto vu$ from $\mathrm{Hom}(Y', X)$ to $\mathrm{Hom}(Y, X)$. It is immediate that with these definitions, $h_X : \mathtt{C} \to \mathtt{Set}$ is a *contravariant functor*, i.e., an object of the category $\mathtt{Hom}(\mathtt{C}^{\mathrm{op}}, \mathtt{Set})$, of covariant functors from the category $\mathtt{C}^{\mathrm{op}}$ (the dual of the category $\mathtt{C}$) to the category $\mathtt{Set}$ (T, 1.7, (d) and [Car]).

(8.1.2). Now let $w : X \to X'$ be a morphism in $\mathtt{C}$; for each $Y \in \mathtt{C}$ and each $v \in \mathrm{Hom}(Y, X) = h_X(Y)$, we have $wv \in \mathrm{Hom}(Y, X') = h_{X'}(Y)$; we denote by $h_w(Y)$ the map $v \mapsto wv$ from $h_X(Y)$ to $h_{X'}(Y)$. It is immediate that for each morphism $u : Y \to Y'$ in $\mathtt{C}$, the diagram

$$\begin{tikzcd} h_X(Y') \arrow[r, "h_X(u)"] \arrow[d, "h_w(Y')"'] & h_X(Y) \arrow[d, "h_w(Y)"] \\ h_{X'}(Y') \arrow[r, "h_{X'}(u)"] & h_{X'}(Y) \end{tikzcd}$$

is commutative; in other words, h_w is a *natural transformation (or functorial morphism)* $h_X \to h_{X'}$ (T, 1.2), also a morphism in the category $\mathtt{Hom}(\mathtt{C}^{\mathrm{op}}, \mathtt{Set})$ (T, 1.7, (d)). The definitions of h_X and of h_w therefore constitute the definition of a *canonical covariant functor*

(8.1.2.1) $$h : \mathtt{C} \longrightarrow \mathtt{Hom}(\mathtt{C}^{\mathrm{op}}, \mathtt{Set}), \quad X \longmapsto h_X.$$

(8.1.3). Let X be an object in $\mathtt{C}$, F a contravariant functor from $\mathtt{C}$ to $\mathtt{Set}$ (an object of $\mathtt{Hom}(\mathtt{C}^{\mathrm{op}}, \mathtt{Set})$). Let $g : h_X \to F$ be a *natural transformation*: for all $Y \in \mathtt{C}$, $g(Y)$ is thus a map $h_X(Y) \to F(Y)$ such that $\mathbf{0}_{\mathbf{III}}$ | 6
for each morphism $u : Y \to Y'$ in $\mathtt{C}$, the diagram

(8.1.3.1) $$\begin{tikzcd} h_X(Y') \arrow[r, "h_X(u)"] \arrow[d, "g(Y')"'] & h_X(Y) \arrow[d, "g(Y)"] \\ F(Y') \arrow[r, "F(u)"] & F(Y) \end{tikzcd}$$

is commutative. In particular, we have a map $g(X) : h_X(X) = \mathrm{Hom}(X, X) \to F(X)$, hence an element

(8.1.3.2) $$\alpha(g) = (g(X))(1_X) \in F(X)$$

and as a result a canonical map

(8.1.3.3) $$\alpha : \mathrm{Hom}(h_X, F) \longrightarrow F(X).$$

Conversely, consider an element $\xi \in F(X)$; for each morphism $v : Y \to X$ in $\mathtt{C}$, $F(v)$ is a map $F(X) \to F(Y)$; consider the map

(8.1.3.4) $$v \longmapsto (F(v))(\xi)$$

from $h_X(Y)$ to $F(Y)$; if we denote by $(\beta(\xi))(Y)$ this map,

(8.1.3.5) $$\beta(\xi) : h_X \longrightarrow F$$

is a *natural transformation*, since for each morphism $u : Y \to Y'$ in $\mathtt{C}$ we have $(F(vu))(\xi) = (F(v) \circ F(u))(\xi)$, which makes (8.1.3.1) commutative for $g = \beta(\xi)$. We have thus defined a canonical map

(8.1.3.6) $$\beta : F(X) \longrightarrow \mathrm{Hom}(h_X, F).$$

Proposition (8.1.4). — *The maps α and β are the inverse bijections of each other.*

Proof. We calculate $\alpha(\beta(\xi))$ for $\xi \in F(X)$; for each $Y \in \mathtt{C}$, $(\beta(\xi))(Y)$ is a map $g_1(Y) : v \mapsto (F(v))(\xi)$ from $h_X(Y)$ to $F(Y)$. We thus have

$$\alpha(\beta(\xi)) = (g_1(X))(1_X) = (F(1_X))(\xi) = 1_{F(X)}(\xi) = \xi.$$

We now calculate $\beta(\alpha(g))$ for $g \in \mathrm{Hom}(h_X, F)$; for each $Y \in \mathtt{C}$, $(\beta(\alpha(g)))(Y)$ is the map $v \mapsto (F(v))((g(X))(1_X))$; according to the commutativity of (8.1.3.1), this map is none other than $v \mapsto$

$(g(Y))((h_X(v))(1_X)) = (g(Y))(v)$ by definition of $h_X(v)$, in other words, it is equal to $g(Y)$, which finishes the proof. □

(8.1.5). Recall that a *subcategory* $\mathtt{C}'$ of a category $\mathtt{C}$ is defined by the condition that its objects are objects of $\mathtt{C}$, and that if X', Y' are two objects of $\mathtt{C}'$, then the set $\mathrm{Hom}_{\mathtt{C}'}(X', Y')$ of morphisms $X' \to Y'$ *in* $\mathtt{C}'$ is a subset of the set $\mathrm{Hom}_{\mathtt{C}}(X', Y')$ of morphisms $X' \to Y'$ *in* $\mathtt{C}$, the canonical map of "composition of morphisms"

$$\mathrm{Hom}_{\mathtt{C}'}(X', Y') \times \mathrm{Hom}_{\mathtt{C}'}(Y', Z') \longrightarrow \mathrm{Hom}_{\mathtt{C}'}(X', Z')$$

being the restriction of the canonical map $\mathbf{0_{III}}$ | 7

$$\mathrm{Hom}_{\mathtt{C}}(X', Y') \times \mathrm{Hom}_{\mathtt{C}}(Y', Z') \longrightarrow \mathrm{Hom}_{\mathtt{C}}(X', Z').$$

We say that $\mathtt{C}'$ is a *full* subcategory of $\mathtt{C}$ if $\mathrm{Hom}_{\mathtt{C}'}(X', Y') = \mathrm{Hom}_{\mathtt{C}}(X', Y')$ for every pair of objects in $\mathtt{C}'$. The subcategory $\mathtt{C}''$ of $\mathtt{C}$ consisting of the objects of $\mathtt{C}$ isomorphic to objects of $\mathtt{C}'$ is then again a full subcategory of $\mathtt{C}$, *equivalent* (T, 1.2) to $\mathtt{C}'$ as we verify easily.

A covariant functor $F : \mathtt{C}_1 \to \mathtt{C}_2$ is called *fully faithful* if for every pair of objects X_1, Y_1 of $\mathtt{C}_1$, the map $u \mapsto F(u)$ from $\mathrm{Hom}(X_1, Y_1)$ to $\mathrm{Hom}(F(X_1), F(Y_1))$ is *bijective*; this implies that the subcategory $F(\mathtt{C}_1)$ of $\mathtt{C}_2$ is *full*. In addition, if two objects X_1, X_1' have the same image X_2, then there exists a unique isomorphism $u : X_1 \to X_1'$ such that $F(u) = 1_{X_1}$. For each object X_2 of $F(\mathtt{C}_1)$, let $G(X_2)$ be one of the objects X_1 of $\mathtt{C}_1$ such that $F(X_1) = X_2$ (G is defined by means of the axiom of choice); for each morphism $v : X_2 \to Y_2$ in $F(\mathtt{C}_1)$, $G(v)$ will be the unique morphism $u : G(X_2) \to G(Y_2)$ such that $F(u) = v$; G is then a *functor* from $F(\mathtt{C}_1)$ to $\mathtt{C}_1$; FG is the identity functor on $F(\mathtt{C}_1)$, and the above shows that there exists an isomorphism of functors $\phi : 1_{\mathtt{C}_1} \to GF$ such that F, G, ϕ, and the identity $1_{F(\mathtt{C}_1)} \to FG$ defines an *equivalence* between the category $\mathtt{C}_1$ and the full subcategory $F(\mathtt{C}_1)$ of $\mathtt{C}_2$ (T, 1.2).

(8.1.6). We apply Proposition (8.1.4) to the case where F is $h_{X'}$, X' being any object of $\mathtt{C}$; the map $\beta : \mathrm{Hom}(X, X') \to \mathrm{Hom}(h_X, h_{X'})$ is none other than the map $w \mapsto h_w$ defined in (8.1.2); this map being *bijective*, we see with the terminology of (8.1.5) that:

Proposition (8.1.7). — *The canonical functor* $h : \mathtt{C} \to \mathtt{Hom}(\mathtt{C}^{\mathtt{op}}, \mathtt{Set})$ *is fully faithful.*

(8.1.8). Let F be a contravariant functor from $\mathtt{C}$ to $\mathtt{Set}$; we say that F is *representable* if there exists an object $X \in \mathtt{C}$ such that F is *isomorphic* to h_X; it follows from Proposition (8.1.7) that the data of an $X \in \mathtt{C}$ and an isomorphism of functors $g : h_X \to F$ determines X up to unique isomorphism. Proposition (8.1.7) then implies that h defines an *equivalence* between $\mathtt{C}$ and the full subcategory of $\mathtt{Hom}(\mathtt{C}^{\mathtt{op}}, \mathtt{Set})$ consisting of the *contravariant representable functors*. It follows from Proposition (8.1.4) that the data of a natural transformation $g : h_X \to F$ is equivalent to that of an element $\xi \in F(X)$; to say that g is an *isomorphism* is equivalent to the following condition on ξ: *for every object Y of $\mathtt{C}$ the map $v \mapsto (F(v))(\xi)$ from* $\mathrm{Hom}(Y, X)$ *to* $F(Y)$ *is bijective*. When ξ satisfies this condition, we say that the pair (X, ξ) *represents* the representable functor F, By abuse of language, we also say that the object $X \in \mathtt{C}$ represents F if there exists a $\xi \in F(X)$ such that (X, ξ) represents F, in other words if h_X is isomorphic to F.

Let F, F' be two contravariant representable functors from $\mathtt{C}$ to $\mathtt{Set}$, $h_X \to F$ and $h_{X'} \to F'$ two isomorphisms of functors. Then it follows from (8.1.6) that there is a canonical bijective correspondence between $\mathrm{Hom}(X, X')$ and the set $\mathrm{Hom}(F, F')$ of natural transformations $F \to F'$.

(8.1.9). *Example I. Projective limits*. The notion of a contravariant representable functor covers in particular the "dual" notion of the usual notion of a "solution to a universal problem". More $\mathbf{0_{III}}$ | 8
generally, we will see that the notion of the *projective limit* is a special case of the notion of a representable functor. Recall that in a category $\mathtt{C}$, we define a *projective system* by the data of a preordered set I, a family $(A_\alpha)_{\alpha \in I}$ of objects of $\mathtt{C}$, and for every pair of indices (α, β) such that $\alpha \leqslant \beta$, a morphism $u_{\alpha\beta} : A_\beta \to A_\alpha$. A *projective limit* of this system in $\mathtt{C}$ consists of an object B of $\mathtt{C}$ (denoted $\varprojlim A_\alpha$), and for each $\alpha \in I$, a morphism $u_\alpha : B \to A_\alpha$ such that: 1$^{\text{st}}$. $u_\alpha = u_{\alpha\beta} u_\beta$ for $\alpha \leqslant \beta$; 2$^{\text{nd}}$. for every object X of $\mathtt{C}$ and every family $(v_\alpha)_{\alpha \in I}$ of morphisms $v_\alpha : X \to A_\alpha$ such that $v_\alpha = u_{\alpha\beta} v_\beta$ for $\alpha \leqslant \beta$, there exists a unique morphism $v : X \to B$ (denoted $\varprojlim v_\alpha$) such that $v_\alpha = u_\alpha v$ for all $\alpha \in I$ (T, 1.8). This can be interpreted in the following way: the $u_{\alpha\beta}$ canonically define maps

$$\overline{u}_{\alpha\beta} : \mathrm{Hom}(X, A_\beta) \longrightarrow \mathrm{Hom}(X, A_\alpha)$$

which define a *projective system* of sets $(\mathrm{Hom}(X, A_\alpha), \overline{u}_{\alpha\beta})$, and (v_α) is by definition an element of the set $\varprojlim \mathrm{Hom}(X, A_\alpha)$; it is clear that $X \mapsto \varprojlim \mathrm{Hom}(X, A_\alpha)$ is a *contravariant functor* from $\mathtt{C}$ to $\mathtt{Set}$, and the existence of the projective limit B is equivalent to saying that $(v_\alpha) \mapsto \varprojlim v_\alpha$ is an *isomorphism* of functors in X

(8.1.9.1) $$\varprojlim \mathrm{Hom}(X, A_\alpha) \simeq \mathrm{Hom}(X, B),$$

in other words, that the functor $X \mapsto \varprojlim \mathrm{Hom}(X, A_\alpha)$ is *representable*.

(8.1.10). *Example II. Final objects*. Let $\mathtt{C}$ be a category, $\{a\}$ a singleton set. Consider the contravariant functor $F : \mathtt{C} \to \mathtt{Set}$ which sends every object X of $\mathtt{C}$ to the set $\{a\}$, and every morphism $X \to X'$ in $\mathtt{C}$ to the unique map $\{a\} \to \{a\}$. To say that this functor is *representable* means that there exists an object $e \in \mathtt{C}$ such that for every $Y \in \mathtt{C}$, $\mathrm{Hom}(Y, e) = h_e(Y)$ is a *singleton set*; we say that e is an *final object* of $\mathtt{C}$, and it is clear that two final objects of $\mathtt{C}$ are isomorphic (which allows us to define, in general with the axiom of choice, *one* final object of $\mathtt{C}$ which we then denote $e_{\mathtt{C}}$). For example, in the category $\mathtt{Set}$, the final objects are the singleton sets; in the category of *augmented algebras* over a field K (where the morphisms are the algebra homomorphisms compatible with the augmentation), K is a final object; in the category of *S-preschemes* (**I**, 2.5.1), S is a final object.

(8.1.11). For two objects X and Y of a category $\mathtt{C}$, set $h'_X(Y) = \mathrm{Hom}(X, Y)$, and for every morphism $u : Y \to Y'$, let $h_X(u)$ be the map $v \mapsto vu$ from $\mathrm{Hom}(X, Y)$ to $\mathrm{Hom}(X, Y')$; h'_X is then a *covariant functor* $\mathtt{C} \to \mathtt{Set}$, so we deduce as in (8.1.2) the definition of a canonical covariant functor $h' : \mathtt{C}^{\mathtt{op}} \to \mathtt{Hom}(\mathtt{C}, \mathtt{Set})$; a *covariant* functor F from $\mathtt{C}$ to $\mathtt{Set}$, in other words an object of $\mathtt{Hom}(\mathtt{C}, \mathtt{Set})$, is then *representable* if there exists an object $X \in \mathtt{C}$ (necessarily unique up to unique isomorphism) such that F is *isomorphic to h'_X*; we leave it to the reader to develop the "dual" notions of the above, which this time cover the notion of an *inductive limit*, and in particular the usual notion of a "solution to a universal problem".

8.2. Algebraic structures in categories

(8.2.1). Given two contravariant functors F and F' from $\mathtt{C}$ to $\mathtt{Set}$, recall that for every object $Y \in \mathtt{C}$, $\mathbf{0}_{\mathbf{III}}$ | 9
we set $(F \times F')(Y) = F(Y) \times F'(Y)$, and for every morphism $u : Y \to Y'$ in $\mathtt{C}$, we set $(F \times F')(u) = F(u) \times F'(u)$, which is the map $(t, t') \mapsto (F(u)(t), F'(u)(t'))$ from $F(Y') \times F'(Y')$ to $F(Y) \times F'(Y)$; $F \times F' : \mathtt{C} \to \mathtt{Set}$ is thus a *contravariant functor* (which is none other than the *product* of the objects F and F' in the category $\mathtt{Hom}(\mathtt{C}^{\mathtt{op}}, \mathtt{Set})$). Given an object $X \in \mathtt{C}$, we call an *internal composition law* on X a *natural transformation*

(8.2.1.1) $$\gamma_X : h_X \times h_X \longrightarrow h_X.$$

In other words (T, 1.2), for every object $Y \in \mathtt{C}$, $\gamma_X(Y)$ is a map $h_X(Y) \times h_X(Y) \to h_X(Y)$ (thus by definition an *internal composition law* on the set $h_X(Y)$) with the condition that for every morphism $u : Y \to Y'$ in $\mathtt{C}$, the diagram

$$\begin{tikzcd}
h_X(Y') \times h_X(Y') \arrow[r, "h_X(u) \times h_X(u)"] \arrow[d, "\gamma_X(Y')"'] & h_X(Y) \times h_X(Y) \arrow[d, "\gamma_X(Y)"] \\
h_X(Y') \arrow[r, "h_X(u)"] & h_X(Y)
\end{tikzcd}$$

is commutative; this implies that for the composition laws $\gamma_X(Y)$ and $\gamma_X(Y')$, $h_X(u)$ is a *homomorphism* from $h_X(Y')$ to $h_X(Y)$.

In a similar way, given two objects Z and X of $\mathtt{C}$, we call an *external composition law on X, with Z as its domain of operators* a natural transformation

(8.2.1.2) $$\omega_{X,Z} : h_Z \times h_X \longrightarrow h_X.$$

We see as above that for every $Y \in \mathtt{C}$, $\omega_{X,Z}(Y)$ is an external composition law on $h_X(Y)$, with $h_Z(Y)$ as its domain of operators and such that for every morphism $u : Y \to Y'$, $h_X(u)$ and $h_Z(u)$ form a *di-homomorphism* from $(h_Z(Y'), h_X(Y'))$ to $(h_Z(Y), h_X(Y))$.

(8.2.2). Let X' be a second object of $\mathtt{C}$, and suppose we are given an internal composition law $\gamma_{X'}$ on X'; we say that a morphism $w : X \to X'$ in $\mathtt{C}$ is a *homomorphism* for the composition laws if for every $Y \in \mathtt{C}$, $h_w(Y) : h_X(Y) \to h_{X'}(Y)$ is a *homomorphism* for the composition laws $\gamma_X(Y)$ and $\gamma_{X'}(Y)$. If X'' is a third object of $\mathtt{C}$ equipped with an internal composition law $\gamma_{X''}$ and $w' : X' \to X''$ is a morphism $\mathbf{0_{III}}$ | 10
in $\mathtt{C}$ which is a homomorphism for $\gamma_{X'}$ and $\gamma_{X''}$, then it is clear that the morphism $w'w : X \to X''$ is a homomorphism for the composition laws γ_X and $\gamma_{X''}$. An isomorphism $w : X \simeq X'$ in $\mathtt{C}$ is called an *isomorphism for the composition laws γ_X and $\gamma_{X'}$* if w is a homomorphism for these composition laws, and if its inverse morphism w^{-1} is a homomorphism for the composition laws $\gamma_{X'}$ and γ_X.

We define in a similar way the *di-homomorphisms* for pairs of objects of $\mathtt{C}$ equipped with external composition laws.

(8.2.3). When an internal composition law γ_X on an object $X \in \mathtt{C}$ is such that $\gamma_X(Y)$ is a *group law* on $h_X(Y)$ for *every* $Y \in \mathtt{C}$, we say that X, equipped with this law, is a $\mathtt{C}$-*group* or a *group object in* $\mathtt{C}$. We similarly define $\mathtt{C}$-*rings*, $\mathtt{C}$-*modules*, etc.

(8.2.4). Suppose that the *product* $X \times X$ of an object $X \in \mathtt{C}$ by itself exists in $\mathtt{C}$; by definition, we then have $h_{X\times X} = h_X \times h_X$ up to canonical isomorphism, since it is a particular case of the projective limit (8.1.9); an internal composition law on X can thus be considered as a functorial morphism $\gamma_X : h_{X\times X} \to h_X$, and thus canonically determine (8.1.6) an element $c_X \in \operatorname{Hom}(X \times X, X)$ such that $h_{c_X} = \gamma_X$; in this case, the data of an internal composition law on X is equivalent to the data of a morphism $X \times X \to X$; when $\mathtt{C}$ is the category $\mathtt{Set}$, we recover the classical notion of an internal composition law on a set. We have an analogous result for an external composition law when the product $Z \times X$ exists in $\mathtt{C}$.

(8.2.5). With the above notation, suppose that in addition $X \times X \times X$ exists in $\mathtt{C}$; the characterization of the product as an object representing a functor (8.1.9) implies the existence of canonical isomorphisms

$$(X \times X) \times X \simeq X \times X \times X \simeq X \times (X \times X);$$

if we canonically identity $X \times X \times X$ with $(X \times X) \times X$, then the map $\gamma_X(Y) \times 1_{h_X(Y)}$ identifies with $h_{c_X \times 1_X}(Y)$ for all $Y \in \mathtt{C}$. As a result, it is equivalent to say that for every $Y \in \mathtt{C}$, the internal law $\gamma_X(Y)$ is associative, or that the diagram of maps

$$\begin{tikzcd}
h_X(Y) \times h_X(Y) \times h_X(Y) \arrow[r, "\gamma_X(Y)\times 1"] \arrow[d, "1\times\gamma_X(Y)"'] & h_X(Y) \times h_X(Y) \arrow[d, "\gamma_X(Y)"] \\
h_X(Y) \times h_X(Y) \arrow[r, "\gamma_X(Y)"] & h_X(Y)
\end{tikzcd}$$

is commutative, or that the diagram of morphisms $\mathbf{0_{III}}$ | 11

$$\begin{tikzcd}
X \times X \times X \arrow[r, "c_X\times 1_X"] \arrow[d, "1_X\times c_X"'] & X \times X \arrow[d, "c_X"] \\
X \times X \arrow[r, "c_X"] & X
\end{tikzcd}$$

is commutative.

(8.2.6). Under the hypotheses of (8.2.5), if we want to express, for every $Y \in \mathtt{C}$, the internal law $\gamma_X(Y)$ as a *group law*, then it is first necessary that it is associative, and second that there exists a map $\alpha_X(Y) : h_X(Y) \to h_X(Y)$ having the properties of the *inverse* operation of a group; as for every morphism $u : Y \to Y'$ in $\mathtt{C}$, we have seen that $h_X(u)$ must be a group homomorphism $h_X(Y') \to h_X(Y)$, we first see that $\alpha_X : h_X \to h_X$ must be a *natural transformation*. On the other hand, one can express the characteristic properties of the inverse $s \mapsto s^{-1}$ of a group G without involving the identity element: it suffices to check that the two composite maps

$$(s,t) \longmapsto (s,s^{-1},t) \longmapsto (s,s^{-1}t) \longmapsto s(s^{-1}t),$$

$$(s,t) \longmapsto (s,s^{-1},t) \longmapsto (s,ts^{-1}) \longmapsto (ts^{-1})s$$

are equal to the second projection $(s,t) \mapsto t$ from $G \times G$ to G. By (8.1.3), we have $\alpha_X = h_{a_X}$, where $a_X \in \mathrm{Hom}(X,X)$; the first condition above then expresses that the composite morphism

$$X \times X \xrightarrow{(1_X,a_X)\times 1_X} X \times X \times X \xrightarrow{1_X \times c_X} X \times X \xrightarrow{c_X} X$$

is the second projection $X \times X \to X$ in $\mathtt{C}$, and the second condition is similar.

(8.2.7). Now suppose that there exists a *final object* e (8.1.10) in $\mathtt{C}$. Let us always assume that $\gamma_X(Y)$ is a group law on $h_X(Y)$ for every $Y \in \mathtt{C}$, and denote by $\eta_X(Y)$ the identity element of $\gamma_X(Y)$. As, for every morphism $u : Y \to Y'$ in $\mathtt{C}$, $h_X(u)$ is a group homomorphism, we have $\eta_X(Y) = (h_X(u))(\eta_X(Y'))$; taking in particular $Y' = e$, in which case u is the unique element ε of $\mathrm{Hom}(Y,e)$, we see that the element $\eta_X(e)$ completely determines $\eta_X(Y)$ for every $Y \in \mathtt{C}$. Set $e_X = \eta_X(X)$, the identity element of the group $h_X(X) = \mathrm{Hom}(X,X)$; the commutativity of the diagram

$$\begin{tikzcd} h_X(e) \arrow[r, "h_X(\varepsilon)"] \arrow[d, "h_{e_X}(e)"'] & h_X(Y) \arrow[d, "h_{e_X}(Y)"] \\ h_X(e) \arrow[r, "h_X(\varepsilon)"] & h_X(Y) \end{tikzcd}$$

(cf. (8.1.2)) shows that, on the set $h_X(Y)$, the map $h_{e_X}(Y)$ is none other than $s \mapsto \eta_X(Y)$ sending every element to the identity element. We then verify that the fact that $\eta_X(Y)$ is the identity element of $\gamma_X(Y)$ for every $Y \in \mathtt{C}$ is equivalent to saying that the composite morphism $\mathbf{0_{III}}$ | 12

$$X \xrightarrow{(1_X,1_X)} X \times X \xrightarrow{1_X \times e_X} X \times X \xrightarrow{c_X} X,$$

and the analog in which we swap 1_X and e_X, are both *equal* to 1_X.

(8.2.8). One could of course easily extend the examples of algebraic structures in categories. The example of groups was treated with enough detail, but latter on we will usually leave it to the reader to develop analogous notions for the examples of algebraic structures we will encounter.

§9. CONSTRUCTIBLE SETS

9.1. Constructible sets

Definition (9.1.1). — We say that a continuous map $f : X \to Y$ is *quasi-compact* if for every quasi-compact open subset U of Y, $f^{-1}(U)$ is quasi-compact. We say that a subset Z of a topological space X is *retrocompact* in X if the canonical injection $Z \to X$ is quasi-compact, in other words, if for every quasi-compact open subset U of X, $U \cap Z$ is quasi-compact.

A *closed* subset of X is retrocompact in X, but a quasi-compact subset of X is not necessarily retrocompact in X. If X is quasi-compact, every retrocompact open subset of X is quasi-compact. It is clear that every *finite* union of retrocompact sets in X is retrocompact in X, as every finite union of quasi-compact sets is quasi-compact. Every finite intersection of retrocompact open sets in X is a retrocompact open set in X. In a *locally Noetherian* space X, every quasi-compact set is a Noetherian subspace, and as a result *every subset* of X is retrocompact in X.

Definition (9.1.2). — Given a topological space X, we say that a subset of X is *constructible* if it belongs to the smallest set of subsets $\mathfrak{F}$ of X containing all the retrocompact open subsets of $\mathfrak{F}$ and is stable under finite intersections and complements (which implies that $\mathfrak{F}$ is also stable under finite unions).

Proposition (9.1.3). — *For a subset of X to be constructible, it is necessary and sufficient for it to be a finite union of sets of the form $U \cap \complement V$, where U and V are retrocompact open sets in X.*

PROOF. It is clear that the condition is sufficient. To see that it is necessary, consider the set $\mathfrak{G}$ of finite unions of sets of the form $U \cap \complement V$, where U and V are retrocompact open sets in X; it suffices to see that every complement of a set in $\mathfrak{G}$ is in $\mathfrak{G}$. Let $Z = \bigcup_{i \in I}(U_i \cap \complement V_i)$, where I is finite, U_i and V_i retrocompact open sets in X; we have $\complement Z = \bigcap_{i \in I}(V_i \cup \complement U_i)$, so Z is a finite union of sets which are intersections of a certain number of the V_i and of a certain number of the $\complement U_i$, thus of the form $V \cap \complement U$, where U is the union of a certain number of the U_i and V is the intersection of a certain $\mathbf{0}_{\mathbf{III}}$ | 13
number of the V_i; but we have noted above that finite unions and intersections of retrocompact open sets in X are retrocompact open sets in X, hence the conclusion. □

Corollary (9.1.4). — *Every constructible subset of X is retrocompact in X.*

PROOF. It suffices to show that if U and V are retrocompact open sets in X, then $U \cap \complement V$ is retrocompact in X; if W is a quasi-compact open set in X, then $W \cap U \cap \complement V$ is closed in the quasi-compact space $W \cap U$, hence it is quasi-compact. □

In particular:

Corollary (9.1.5). — *For an open subset U of X to be constructible, it is necessary and sufficient for it to be retrocompact in X. For a closed subset F of X to be constructible, it is necessary and sufficient for the open set $\complement F$ to be retrocompact.*

(9.1.6). An important case is when every quasi-compact open subset of X is retrocompact, in other words, when the intersection of two quasi-compact open subsets of X is quasi-compact (cf. (**I**, 5.5.6)). When X is also quasi-compact, this implies that the retrocompact open subsets of X are identical to the quasi-compact open subsets of X, and the constructible subsets of X are finite unions of sets of the form $U \cap \complement V$, where U and V are quasi-compact open sets.

Corollary (9.1.7). — *For a subset of a Noetherian space to be constructible, it is necessary and sufficient for it to be a finite union of locally closed subsets of X.*

Proposition (9.1.8). — *Let X be a topological space, U an open subset of X.*

(i) *If T is a constructible subset of X, then $T \cap U$ is a constructible subset of U.*
(ii) *In addition, suppose that U is retrocompact in X. For a subset Z of U to be constructible in X, it is necessary and sufficient for it to be constructible in U.*

PROOF.

(i) Using Proposition (9.1.3), we reduce to showing that if T is a retrocompact open set in X, then $T \cap U$ is a retrocompact open set in U, in other words, for every quasi-compact open $W \subset U$, $T \cap U \cap W = T \cap W$ is quasi-compact, which immediately follows from the hypothesis.
(ii) The condition is necessary by (i), so it remains to show that it is sufficient. By Proposition (9.1.3), it suffices to consider the case where Z is a retrocompact open set *in U*, because it will then follow that $U - Z$ is constructible in X, and if Z and Z' are two retrocompact opens *in U*, then $Z \cap (U - Z')$ will be constructible in X. If W is a quasi-compact open set in X and Z a retrocompact open set in U, then we have $Z \cap W = Z \cap (W \cap U)$, and by hypothesis $W \cap U$ is a quasi-compact open set in U; so $W \cap Z$ is quasi-compact, and as a result Z is a retrocompact open set in X, and *a fortiori* constructible in X.

□

Corollary (9.1.9). — *Let X be a topological space, $(U_i)_{i \in I}$ a finite cover of X by retrocompact open sets in X. For a subset Z of X to be constructible in X, it is necessary and sufficient for $Z \cap U_i$ to be constructible in U_i for all $i \in I$.*

(9.1.10). In particular, suppose that X is quasi-compact and every point of X admits a fundamental $\mathbf{0}_{\mathbf{III}}$ | 14
system of retrocompact open neighbourhoods in X (and *a fortiori* quasi-compact); then the condition for a subset Z of X to be constructible in X is of a *local* nature, in other words, it is necessary and sufficient that for every $x \in X$, there exists an open neighbourhood V of x such that $V \cap Z$ is constructible in V. Indeed, if this condition is satisfied, then there exists for every $x \in X$ an open neighbourhood V of x which is *retrocompact in X* and such that $V \cap Z$ is constructible in V, by the

hypotheses on X and by Proposition (9.1.8, i); it then suffices to cover X by a finite number of these neighbourhoods and to apply Corollary (9.1.9).

Definition (9.1.11). — Let X be a topological space. We say that a subset T of X is *locally constructible* in X if for every $x \in X$ there exists an open neighbourhood V of x such that $T \cap V$ is constructible in V.

It follows from Proposition (9.1.8, i) that if V is such that $V \cap T$ is constructible in V, then for every open $W \subset V$, $W \cap T$ is constructible in W. If T is locally constructible in X, then for every open set U in X, $T \cap U$ is locally constructible in U, as a result of the above remark. The same remark shows that the set of locally constructible subsets of X is stable under finite unions and finite intersections; on the other hand, it is clear that it is also stable under taking complements.

Proposition (9.1.12). — *Let X be a topological space. Every constructible set in X is locally constructible in X. The converse is true if X is quasi-compact and if its topology admits a basis formed by the retrocompact sets in X.*

PROOF. The first assertion follows from Definition (9.1.11) and the second from (9.1.10). □

Corollary (9.1.13). — *Let X be a topological space whose topology admits a basis formed by the retrocompact sets in X. Then every locally constructible subset T of X is retrocompact in X.*

PROOF. Let U be a quasi-compact open set in X; $T \cap U$ is locally constructible in U, hence constructible in U by Proposition (9.1.12), and as a result quasi-compact by Corollary (9.1.4). □

9.2. Constructible subsets of Noetherian spaces

(9.2.1). We have seen (9.1.7) that in a Noetherian space X, the constructible subsets of X are the *finite unions of locally closed subsets of X.*

The inverse image of a constructible set in X by a continuous map from a Noetherian space X' to X is constructible in X'. If Y is a constructible subset of a Noetherian space X, then the subsets of Y are constructible as subspaces of Y and are identical to those which are constructible as subspaces of X.

Proposition (9.2.2). — *Let X be an irreducible Noetherian space, E a constructible subset of X. For E to be everywhere dense in X, it is necessary and sufficient for E to contain a nonempty open subset of X.*

PROOF. The condition is evidently sufficient, as every nonempty open set is dense in X. Conversely, let $E = \bigcup_{i=1}^{n}(U_i \cap F_i)$ be a constructible subset of X, the U_i being nonempty open sets and the F_i closed in X; we then have $\overline{E} \subset \bigcup_i F_i$. As a result, if $\overline{E} = X$, then X is equal to one of the F_i, hence $E \supset U_i$, which finishes the proof. □

When X admits a generic point x (**0**, 2.1.2), the condition of Proposition (9.2.2) is equivalent to the relation $x \in E$.

Proposition (9.2.3). — *Let X be a Noetherian space. For a subset E of X to be constructible, it is necessary and sufficient that, for every irreducible closed subset Y of X, $E \cap Y$ is rare in Y or contains a nonempty open subset of Y.*

PROOF. The necessity of the condition follows from the fact that $E \cap Y$ must be a constructible subset of Y and from Proposition (9.2.2), since a nondense subset of Y is necessarily rare in the irreducible space Y (**0**, 2.1.1). To prove that the condition is sufficient, apply the principle of Noetherian induction (**0**, 2.2.2) to the set $\mathfrak{F}$ of closed subsets Y of X such that $Y \cap E$ is constructible (with respect to Y or with respect to X, which are equivalent): we can thus assume that for every closed subset $Y \neq X$ of X, $E \cap Y$ is constructible. First suppose that X is not irreducible, and let X_i $(1 \leqslant i \leqslant m)$ are its irreducible components, necessarily of finite number (**0**, 2.2.5); by hypothesis the $E \cap X_i$ are constructible, hence their union E is as well. Suppose now that X is irreducible; then by hypothesis, if E is rare, then $\overline{E} \neq X$ and $E = E \cap \overline{E}$ is constructible; if E contains a nonempty open subset U of X, then it is the union of U and $E \cap (X - U)$; but $X - U$ is a closed set distinct from X, so $E \cap (X - U)$ is constructible; as a result, E is itself constructible, which finishes the proof. □

Corollary (9.2.4). — *Let X be a Noetherian space, (E_α) an increasing filtered family of constructible subsets of X, such that*

(1st) *X is the union of the family (E_α).*

(2nd) *Every irreducible closed subset of X is contained in the closure of one of the E_α.*

Then there exists an index α such that $X = E_\alpha$.

When every irreducible closed subset of X admits a generic point, the hypothesis (1st) *can be omitted.*

Proof. We apply the principle of Noetherian induction (**0**, 2.2.2) to the set $\mathfrak{M}$ of closed subsets of X contained in at least one of the E_α; we can thus suppose that every closed subset $Y \neq X$ of X is contained in one of the E_α. The proposition is evident if X is not irreducible, because each of the irreducible components X_i of X ($1 \leqslant i \leqslant m$) is contained in an E_{α_i}, and there exists an E_α containing all of the E_{α_i}. Now suppose that X is irreducible. By hypothesis, there exists a β such that $X = \overline{E_\beta}$, so (9.2.2) E_β contains a nonempty open subset U of X. But then the closed set $X - U$ is contained in an E_γ, and it suffices to take an E_α containing E_β and E_γ. When every irreducible closed subset Y of X admits a generic point y, there exists α such that $y \in E_\alpha$, so $Y = \overline{\{y\}} \subset \overline{E_\alpha}$, and condition (2nd) is $\mathbf{0}_{\mathbf{III}}$ | 16
therefore a consequence of (1st). □

Proposition (9.2.5). — *Let X be a Noetherian space, x a point of X, and E a constructible subset of X. For E to be a neighbourhood of x, it is necessary and sufficient that for every irreducible closed subset Y of X containing x, $E \cap Y$ is dense in Y (if there exists a generic point y of Y, this also implies (9.2.2) that $y \in E$).*

Proof. The condition is evidently necessary; we will prove that it is sufficient. Applying the principle of Noetherian induction to the set $\mathfrak{M}$ of closed subsets Y of X containing x and such that $E \cap Y$ is a neighbourhood of x in Y, we can assume that every closed subset $Y \neq X$ of X containing x belongs to $\mathfrak{M}$. If X is not irreducible, then each of the irreducible components X_i of X containing x are distinct from X, hence $E \cap X_i$ is a neighbourhood of x with respect to X_i; as a result, E is a neighbourhood of x in the union of the irreducible components of X containing x, and as this union is a neighbourhood of x in X, so is E. If X is irreducible, then E is dense in X by hypothesis, so it contains a nonempty open subset U of X (9.2.2); the proposition is then evident if $x \in U$; otherwise, x is by hypothesis inside $E \cap (X - U)$ with respect to $X - U$, so the closure of $X - E$ in X does not contain x, and the complement of this closure is a neighbourhood of x contained in E, which finishes the proof. □

Corollary (9.2.6). — *Let X be a Noetherian space, E a subset of X. For E to be an open set in X, it is necessary and sufficient that for every irreducible closed subset Y of X intersecting E, $E \cap Y$ contains a nonempty open subset of Y.*

Proof. The condition is evidently necessary; conversely, if it is satisfied, then it implies that E is constructible by Proposition (9.2.3). In addition, Proposition (9.2.5) shows that E is then a neighbourhood of each of its points, hence the conclusion. □

9.3. Constructible functions

Definition (9.3.1). — Let h be a map from a topological space X to a set T. We say that h is *constructible* if $h^{-1}(t)$ is constructible for every $t \in T$, and empty except for finitely many values of t; then for every subset S of T, $h^{-1}(S)$ is constructible. We say that h is *locally constructible* if every $x \in X$ has an open neighbourhood V such that $h|V$ is constructible.

Every constructible function is locally constructible; the converse is true when X is quasi-compact and admits a basis formed by the retrocompact open sets in X (in particular, when X is Noetherian).

Proposition (9.3.2). — *Let h be a map from a Noetherian space X to a set T. For h to be constructible, it is necessary and sufficient that for every irreducible closed subset Y of X, there exists a nonempty subset U of Y, open with respect to Y, in which h is constant.*

Proof. The condition is necessary: indeed, by hypothesis, h does not take finitely many values t_i on Y, and each of the sets $h^{-1}(t_i) \cap Y$ is constructible in Y (9.2.1); as they can not all be rare subsets of the space Y, at least one of them contains a nonempty open set (9.2.3).

To see that the condition is sufficient, we apply the principle of Noetherian induction on the set $\mathfrak{M}$ of closed subsets Y of X such that the restriction $h|Y$ is constructible; we can thus assume that for every closed subset $Y \neq X$ of X, $h|Y$ is constructible. If X is not irreducible, then the restriction of h to each of the (finitely many) irreducible components X_i of X is constructible, and it then follows immediately from Definition (9.3.1) that h is constructible. If X is irreducible, then there exists by hypothesis a nonempty open subset U of X on which h is constant; on the other hand, the restriction of h to $X - U$ is constructible by hypothesis, and it follows immediately that h is constructible. □ $\mathbf{0_{III}}$ | 17

Corollary (9.3.3). — *Let X be a Noetherian space in which every irreducible closed subset admits a generic point. If h is a map from X to a set T such that, for every $t \in T$, $h^{-1}(t)$ is constructible, then h is constructible.*

Proof. If Y is an irreducible closed subset of X and y its generic point, then $Y \cap h^{-1}(h(y))$ is constructible and contains y, hence (9.2.2) this set contains a nonempty open subset of Y, and it suffices to apply Proposition (9.3.2). □

Proposition (9.3.4). — *Let X be a Noetherian space in which every irreducible closed subset admits a generic point, h a constructible map from X to an ordered set. For h to be upper semi-continuous on X, it is necessary and sufficient that for every $x \in X$ and every specialization* (**0**, 2.1.2) *x' of x, we have $h(x') \leqslant h(x)$.*

Proof. The function h does not take a finite number of values; therefore, to say that it is upper semi-continuous means that for every $x \in X$, the set E of the $y \in X$ such that $h(y) \leqslant h(x)$ is a neighbourhood of x. By hypothesis, E is a constructible subset of X; on the other hand,to say that an irreducible closed subset Y of X contains x means that its generic point y is a specialization of x; the conclusion then follows from Proposition (9.2.5). □

§10. Supplement on flat modules

For any proofs missing in (10.1) and (10.2), we refer the reader to Bourbaki, *Alg. comm.*, chap. II and III.

10.1. Relations between flat modules and free modules

(10.1.1). Let A be a ring, $\mathfrak{J}$ an ideal of A, and M an A-module; for every integer $p \geqslant 0$, we have a canonical homomorphism of $(A/\mathfrak{J})$-modules

(10.1.1.1) $$\phi_p : (M/\mathfrak{J}M) \otimes_{A/\mathfrak{J}} (\mathfrak{J}^p/\mathfrak{J}^{p+1}) \longrightarrow \mathfrak{J}^p M/\mathfrak{J}^{p+1}M,$$

which is evidently *surjective*. We denote by $\operatorname{gr}(A) = \oplus_{p \geqslant 0}\mathfrak{J}^p/\mathfrak{J}^{p+1}$ the graded ring associated to A filtered by the $\mathfrak{J}^p$, and by $\operatorname{gr}(M) = \oplus_{p \geqslant 0}\mathfrak{J}^p M/\mathfrak{J}^{p+1}M$ the graded $\operatorname{gr}(A)$-module associated to M filtered by the $\mathfrak{J}^p M$; we then have $\operatorname{gr}_p(A) = \mathfrak{J}^p/\mathfrak{J}^{p+1}$, and $\operatorname{gr}_p(M) = \mathfrak{J}^p M/\mathfrak{J}^{p+1}M$; the ϕ define a *surjective* homomorphism of graded $\operatorname{gr}(A)$-modules

(10.1.1.2) $$\phi : \operatorname{gr}_0(M) \otimes_{\operatorname{gr}_0(A)} \operatorname{gr}(A) \longrightarrow \operatorname{gr}(M).$$

$\mathbf{0_{III\text{-}1}}$ | 18

(10.1.2). Suppose that *one* of the following hypotheses is satisfied:

(i) $\mathfrak{J}$ is nilpotent;
(ii) A is Noetherian, $\mathfrak{J}$ is contained in the radical of A, and M is of finite type.

Then the following properties are equivalent.

(a) M is a free A-module.
(b) $M/\mathfrak{J}M = N \otimes_A (A/\mathfrak{J})$ is a free $(A/\mathfrak{J})$-module, and $\operatorname{Tor}_1^A(M, A/\mathfrak{J}) = 0$.
(c) $M/\mathfrak{J}M$ is a free $(A/\mathfrak{J})$-module, and the canonical homomorphism (10.1.1.2) is injective (and thus bijective).

(10.1.3). Suppose that $A/\mathfrak{J}$ is a *field* (in other words, that $\mathfrak{J}$ is maximal), and that one of the hypotheses, (i) and (ii), of (10.1.2) is satisfied. Then the following properties are equivalent.

(a) M is a free A-module.
(b) M is a projective A-module.
(c) M is a flat A-module.

(d) $\operatorname{Tor}_1^A(M, A/\mathfrak{J}) = 0$.
(e) The canonical homomorphism (10.1.1.2) is bijective.

This result can be applied, in particular, to the following two cases:

(i) M is an *arbitrary* module, over a local ring A whose maximal ideal $\mathfrak{J}$ is *nilpotent* (for example, a local Artinian ring);
(ii) M is a module *of finite type* over a *local Noetherian* ring.

10.2. Local flatness criteria

(10.2.1). With the hypotheses and notation of (10.1.1), consider the following conditions.

(a) M is a flat A-module.
(b) $M/\mathfrak{J}M$ is a flat $(A/\mathfrak{J})$-module, and $\operatorname{Tor}_1^A(M, A/\mathfrak{J}) = 0$.
(c) $M/\mathfrak{J}M$ is a flat $(A/\mathfrak{J})$-module, and the canonical homomorphism (10.1.1.2) is bijective.
(d) For all $n \geqslant 1$, $M/\mathfrak{J}^n M$ is a flat $(A/\mathfrak{J}^n)$-module.

Then we have the implications

$$(a) \implies (b) \implies (c) \implies (d),$$

and, if $\mathfrak{J}$ is *nilpotent*, then the four conditions are *equivalent*. This is also the case if A is Noetherian and M is *ideally separated*, that is to say, for every ideal $\mathfrak{a}$ of A, the A-module $\mathfrak{a} \otimes_A M$ is *separated* for the $\mathfrak{J}$-preadic topology.

(10.2.2). Let A be a Noetherian ring, B a commutative Noetherian A-algebra, $\mathfrak{J}$ an ideal of A such that $\mathfrak{J}B$ is contained in the radical of B, and M a B-*module* of finite type. Then, when M is considered as an A-*module*, the four conditions of (10.2.1) are *equivalent*. This result applies first and foremost in the case where A and B are *local* Noetherian rings, with the homomorphism $A \to B$ being a *local* homomorphism. More specifically, if $\mathfrak{J}$ is then the *maximal* ideal of A, we can, in conditions *(b)* and *(c)*, remove the hypothesis that $M/\mathfrak{J}M$ is flat, since it is automatically satisfied, and condition *(d)* implies that the modules $M/\mathfrak{J}^n M$ are *free* over the $A/\mathfrak{J}^n$. $\mathbf{0}_{\mathbf{III\text{-}1}}$ | 10

(10.2.3). With the hypotheses on A, B, $\mathfrak{J}$, and M from the start of (10.2.2), let $\widehat{A}$ be the separated completion of A for the $\mathfrak{J}$-preadic topology, and $\widehat{M}$ the separated completion of M for the $\mathfrak{J}B$-preadic topology. Then, for M to be a flat A-module, it is necessary and sufficient for $\widehat{M}$ to be a flat $\widehat{A}$-module.

(10.2.4). Let $\rho : A \to B$ be a local homomorphism of local Noetherian rings, k the residue field of A, and M and N both B-modules of finite type, with N assumed to be A-*flat*. Let $u : M \to N$ be a B-homomorphism. Then the following conditions are equivalent.

(a) u is injective, and $\operatorname{Coker}(u)$ is a flat A-module.
(b) $u \otimes 1 : M \otimes_A k \to N \otimes_A k$ is injective.

(10.2.5). Let $\rho : A \to B$ and $\sigma : B \to C$ be local homomorphisms of local Noetherian rings, k the residue field of A, and M a C-module of finite type. Suppose that B is a *flat* A-module. Then the following conditions are equivalent.

(a) M is a flat B-module.
(b) M is a flat A-module, and $M \otimes_A k$ is a flat $(B \otimes_A k)$-module.

Proposition (10.2.6). — *Let A and B be local Noetherian rings, $\rho : A \to B$ a local homomorphism, $\mathfrak{J}$ an ideal of B contained in the maximal ideal, and M a B-module of finite type. Suppose that, for all $n \geqslant 0$, $M_n = M/\mathfrak{J}^{n+1}M$ is a flat A-module. Then M is a flat A-module.*

Proof. We have to prove that, for every injective homomorphism $u : N' \to N$ of A-modules of finite type, $v = 1 \otimes u : M \otimes_A N' \to M \otimes_A N$ is injective. But $M \otimes_A N'$ and $M \otimes_A N$ are B-modules of finite type, and thus separated for the $\mathfrak{J}$-preadic topology ($\mathbf{0}_{\mathbf{I}}$, 7.3.5); it thus suffices to prove that the homomorphism $\widehat{v} : \widehat{M \otimes_A N'} \to \widehat{M \otimes_A N}$ of the separated completions is injective. But $\widehat{v} = \varprojlim v_n$, where v_n is the homomorphism $1 \otimes u : M_n \otimes_A N' \to M_n \otimes_A N$; since, by hypothesis, M_n is A-flat, v_n is injective for all n, and thus so too is v, because the functor $\varprojlim$ is left exact. □

Corollary (10.2.7). — *Let A be a Noetherian ring, B a local Noetherian ring, $\rho : A \to B$ a homomorphism, f an element of the maximal ideal of B, and M a B-module of finite type. Suppose that the homothety $f_M : x \to fx$ on M is injective, and that M/fM is a flat A-module. Then M is a flat A-module.*

PROOF. Let $M_i = f^i M$ for $i \geqslant 0$; since f_M is injective, M_i/M_{i+1} is isomorphic to M/fM, and thus A-flat for all $i \geqslant 0$; the exact sequence

$$0 \longrightarrow M_i/M_{i+1} \longrightarrow M/M_{i+1} \longrightarrow M/M_i \longrightarrow 0$$

gives us, by induction on i, that M/M_i is *A-flat* for all $i \geqslant 0$ (**$0_{\mathbf{I}}$**, 6.1.2); we can thus apply (10.2.6). We can also argue directly as follows: for every A-module N of finite type, $M \otimes_A N$ is a B-module of finite type; since f belongs to the radical $\mathfrak{n}$ of B, the (f)-adic topology on $M \otimes_A N$ is finer than the $\mathfrak{n}$-adic topology, and we know that the latter is *separated* (**$0_{\mathbf{I}}$,0.7.3.5**,) Now, since M/M_i is A-flat, we have that **$0_{\mathbf{III\text{-}1}}$** | 20

$$f^i(M \otimes_A N) = \operatorname{Im}(M_i \otimes_A N \longrightarrow M \otimes_A N) = \operatorname{Ker}(M \otimes_A N \longrightarrow (M/M_i) \otimes_A N)$$

by (**$0_{\mathbf{I}}$**, 6.1.2). So let N be an A-module of finite type, and N' a submodule of N, with canonical injection $j : N' \to N$; in the commutative diagram

$$\begin{tikzcd} M \otimes_A N' \arrow[r] \arrow[d, "1_M \otimes j"'] & (M/M_i) \otimes_A N' \arrow[d, "1_{M/M_i} \otimes j"] \\ M \otimes_A N \arrow[r] & (M/M_i) \otimes_A N \end{tikzcd}$$

$1_{M/M_i} \otimes j$ is injective, because M/M_i is A-flat; we thus conclude that

$$\operatorname{Ker}(M \otimes_A N' \longrightarrow M \otimes_A N) \subset \operatorname{Ker}(M \otimes_A N' \longrightarrow (M/M_i) \otimes_A N')$$

for any i; since the intersection (over i) of the latter kernel is 0, as we saw above, so too is the intersection (over i) of the former, and so M is A-flat. □

Proposition (10.2.8). — *Let A be a reduced Noetherian ring, and M an A-module of finite type. Suppose that, for every A-algebra B (which is then a discrete valuation ring), $M \otimes_A B$ is a flat B-module (and thus free* (10.1.3)*). Then M is a flat A-module.*

PROOF. We know that, for M to be flat, it is necessary and suffices for $M_{\mathfrak{m}}$ to be a flat $A_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$ of A (**$0_{\mathbf{I}}$**, 6.3.3); we can thus restrict to the case where A is *local* (**$0_{\mathbf{I}}$**, 1.2.8). So let $\mathfrak{m}$ be the maximal ideal of A, $\mathfrak{p}_i$ $(1 \leqslant i \leqslant r)$ the minimal prime ideals of A, and k the residue field $A/\mathfrak{m}$. We know (**II**, 7.1.7) that there exists, for each i, a discrete valuation ring B_i that has the same field of fractions K_i as the integral ring $A/\mathfrak{p}_i$, and that, further, dominates $A/\mathfrak{p}_i$. Let $M_i = M \otimes_A B_i$. By hypothesis, M_i is free over B_i, and so, denoting by k_i the residue field of B_i, we have

(10.2.8.1) $$\operatorname{rg}_{k_i}(M_i \otimes_{B_i} k_i) = \operatorname{rg}_{K_i}(M_i \otimes_{B_i} K_i).$$

But it is clear that the composite homomorphism $A \to A/\mathfrak{p}_i \to B_i$ is local, and so k is an extension of k_i, and that we have $M_i \otimes_{B_i} k_i = M \otimes_A k_i = (M \otimes_A k) \otimes_k k_i$, and also that $M_i \otimes_{B_i} K_i = M \otimes_A K_i$. Equation (10.2.8.1) thus implies that

$$\operatorname{rg}_k(M \otimes_A k) = \operatorname{rg}_{K_i}(M \otimes_A K_i) \quad \text{for } 1 \leqslant i \leqslant r$$

and since A is reduced, we know that this condition implies that M is a *free* A-module (Bourbaki, *Alg. comm.*, chap. II, § 3, n° 2, prop. 7). □

10.3. Existence of flat extensions of local rings

Proposition (10.3.1). — *Let A be a local Noetherian ring, with maximal ideal $\mathfrak{J}$, and residue field $k = A/\mathfrak{J}$. Let K be a field extension of k. Then there exists a local homomorphism from A to a local Noetherian ring B, such that $B/\mathfrak{J}B$ is k-isomorphic to K, and such that B is a flat A-module.*

The rest of this section is devoted to proving this proposition, step-by-step.

(10.3.1.1). First suppose that $K = k(T)$, where T is an indeterminate. In the ring of polynomials $A' = A[T]$, consider the prime ideal $\mathfrak{J}' = \mathfrak{J}A$, consisting of the polynomials that have coefficients in the ideal $\mathfrak{J}$; it is clear that $A'/\mathfrak{J}'$ is canonically isomorphic to $k[T]$. We will show that the ring of fractions $B = A'_{\mathfrak{J}}$ is that for which we are searching (that is, a ring which satisfies the conditions of the conclusion of the proposition); it is clearly a local Noetherian ring, with maximal ideal $\mathfrak{L} = \mathfrak{J}B$. Further, $B/\mathfrak{L} = (A'/\mathfrak{J}')_{\mathfrak{J}'} = (k[T])_{\mathfrak{J}'}$ is exactly the field of fractions K of $k[T]$. Finally, B is a flat A'-module, and A' is a free A-module, so B is a flat A-module ($\mathbf{0_I}$, 6.2.1). $\mathbf{0_{III\text{-}1}}$ | 21

(10.3.1.2). Now suppose that $K = k(t) = k[t]$, where t is algebraic over k; let $f \in k[T]$ be the minimal polynomial of t; there exists a monic polynomial $F \in A[T]$ whose canonical image in $k[T]$ is f. So let $A' = A[T]$, and let $\mathfrak{J}'$ be the ideal $\mathfrak{J}A' + (F)$ in A'. We will see that the quotient ring $B = A'/(F)$ is that for which we are searching. First of all, it is clear that B is a *free* A-module, and thus flat. The ring $A'/\mathfrak{J}'$ is isomorphic to

$$(A'/\mathfrak{J}A')/\big((\mathfrak{J}A' + (F))/\mathfrak{J}A'\big) = k[T]/(f) = K;$$

the image $\mathfrak{L}$ of $\mathfrak{J}'$ in B is thus maximal, and we evidently have that $\mathfrak{L} = \mathfrak{J}B$. Finally, B is a semi-local ring, because it is an A-module of finite type (Bourbaki, *Alg. comm.*, chap. IV, § 2, n° 5, cor. 3 of prop. 9), and its maximal ideals are in bijective correspondence with those of $B/\mathfrak{J}B$ ([**SZ60**, vol. I, p. 259]); the previous arguments then prove that B is a local ring.

Lemma (10.3.1.3). — *Let $(A_\lambda, f_{\mu\lambda})$ be a filtered inductive system of local rings, such that the $f_{\mu\lambda}$ are local homomorphisms; let $\mathfrak{m}_\lambda$ be the maximal ideal of A_λ, and let $K_\lambda = A_\lambda/\mathfrak{m}_\lambda$. Then $A' = \varinjlim A_\lambda$ is a local ring, with maximal ideal $\mathfrak{m} = \varinjlim \mathfrak{m}_\lambda$, and residue field $K = \varinjlim K_\lambda$. Further, if $\mathfrak{m}_\mu = \mathfrak{m}_\lambda A_\mu$ with $\lambda < \mu$, then we have $\mathfrak{m}' = \mathfrak{m}_\lambda A'$ for all λ. If, further, for $\lambda < \mu$, A_μ is a flat A_λ-module, and if all the A_λ are Noetherian, then A' is a flat Noetherian A_λ-modules for all λ.*

Proof. Since, by hypothesis, $(f_\mu\lambda)(\mathfrak{m}_\lambda) \subset \mathfrak{m}_\mu$ for $\lambda < \mu$, the $\mathfrak{m}_\lambda$ form an inductive system, and its limit $\mathfrak{m}'$ is evidently an ideal of A'. Further, if $x' \notin \mathfrak{m}'$, there exists a λ such that $x' = f_\lambda(x_\lambda)$ for some $x_\lambda \in A_\lambda$ (where $f_\lambda : A_\lambda \to A'$ denotes the canonical homomorphism); because $x' \notin \mathfrak{m}'$, we necessarily have that $x_\lambda \notin \mathfrak{m}_\lambda$, and so x_λ admits an inverse y_λ in A_λ, and $y' = f_\lambda(y_\lambda)$ is the inverse of x' in A', which proves that A' is a local ring with maximal ideal $\mathfrak{m}'$; the claim about K follows immediately from the fact that $\varinjlim$ is an exact functor. The hypothesis that $\mathfrak{m}_\mu = \mathfrak{m}_\lambda A_\mu$ implies that the canonical map $\mathfrak{m}_\lambda \otimes_{A_\lambda} A_\mu \to \mathfrak{m}_\mu$ is surjective; the equality $\mathfrak{m}' = \mathfrak{m}_\lambda A'$ then follows from, again, the fact that the functor $\varinjlim$ is exact and commutes with the tensor product.

Now suppose that, for $\lambda < \mu$, we have $\mathfrak{m}_\mu = \mathfrak{m}_\lambda A_\mu$, and that A_μ is a flat A_λ-module. Then A' is a flat A_λ-module for all λ, by ($\mathbf{0_I}$, 6.2.3); since A' and A_λ are local rings, and since $\mathfrak{m}' = \mathfrak{m}_\lambda A'$, A' is even a *faithfully flat* A_λ-module ($\mathbf{0_I}$, 6.6.2). Finally, suppose further that the A_λ are *Noetherian*; the $\mathfrak{m}_\lambda$-adic topologies are then separated ($\mathbf{0_I}$, 7.3.5); we now show that, from this, it follows that the $\mathfrak{m}'$-adic topology on A' is *separated*. Indeed, if $x' \in A'$ belongs to all the $\mathfrak{m}'^n$ ($n \geqslant 0$), then it is the image of some $x_\mu \in A_\mu$ for a specific index μ, and since the inverse image in A_μ of $\mathfrak{m}'^n = \mathfrak{m}_\mu^n A'$ is $\mathfrak{m}_\mu^n$ ($\mathbf{0_I}$, 6.6.1), x_μ belongs to all the $\mathfrak{m}_\mu^n$, so $x_\mu = 0$, by hypothesis, and so $x' = 0$. Denote by $\widehat{A}'$ the completion of A' for the $\mathfrak{m}'$-adic topology; the above shows that we have $A' \subset \widehat{A}'$. We will now show that $\widehat{A}'$ is *Noetherian* and A_λ*-flat* for all λ; from this, it will follow that $\widehat{A}'$ is A'-flat ($\mathbf{0_I}$, 6.2.3), $\mathbf{0_{III\text{-}1}}$ | 22 and since $\mathfrak{m}'\widehat{A}' \neq \widehat{A}'$, that $\widehat{A}'$ is a faithfully flat A'-module ($\mathbf{0_I}$, 6.6.2), whence the final conclusion that A' is *Noetherian* ($\mathbf{0_I}$, 6.5.2), which will finish the proof of the lemma.

We have $\widehat{A}' = \varprojlim_n A'/\mathfrak{m}'^n$; by the fact that A' is A_λ-flat, we have that

$$\mathfrak{m}'^n/\mathfrak{m}'^{n+1} = (\mathfrak{m}_\lambda^n/\mathfrak{m}_\lambda^{n+1}) \otimes_{A_\lambda} A' = (\mathfrak{m}_\lambda^n/\mathfrak{m}_\lambda^{n+1}) \otimes_{K_\lambda} (K_\lambda \otimes_{A_\lambda} A') = (\mathfrak{m}_\lambda^n/\mathfrak{m}_\lambda^{n+1}) \otimes_{K_\lambda} K;$$

since $\mathfrak{m}_\lambda^n/\mathfrak{m}_\lambda^{n+1}$ is a K_λ-vector space of finite dimension, $\mathfrak{m}_\lambda'^n/\mathfrak{m}_\lambda'^{n+1}$ is a K-vector space of finite dimension for all $n \geqslant 0$. It thus follows from ($\mathbf{0}_\mathbf{I}$, 7.2.12) and ($\mathbf{0}_\mathbf{I}$, 7.2.8) that $\widehat{A}'$ is *Noetherian*. We further know that the maximal ideal of $\widehat{A}'$ is $\mathfrak{m}'A'$, and that $\widehat{A}'/\mathfrak{m}'^n\widehat{A}'$ is isomorphic to $A'/\mathfrak{m}'^n$; since $A'/\mathfrak{m}'^n = (A_\lambda/\mathfrak{m}_\lambda^n) \otimes_{A_\lambda} A'$, we see that $A'/\mathfrak{m}'^n$ is a flat $(A_\lambda/\mathfrak{m}_\lambda^n)$-module ($\mathbf{0}_\mathbf{I}$, 6.2.1); criterion (10.2.2) is thus applicable to the Noetherian A_λ-algebra $\widehat{A}'$, and shows that $\widehat{A}'$ is A_λ-*flat*. □

(10.3.1.4). We now treat the general case. There exists an ordinal γ and, for every ordinal $\lambda \leqslant \gamma$, a subfield k_λ of K that contains k, such that (i) for all $\lambda < \gamma$, $k_{\lambda+1}$ is an extension of k_λ generated by a single element; (ii) for every limit ordinal μ, $k_\mu = \bigcup_{\lambda<\mu} k_\lambda$; and (iii) $K = k_\gamma$. In fact, it suffices to consider a bijection $\xi \mapsto t_\xi$ from the set of ordinals $\xi \leqslant \beta$ (for some suitable β) to K, and to define k_λ by transfinite induction (for $\lambda \leqslant \beta$) as the union of the k_μ for $\mu < \lambda$ if λ is a limit ordinal, and as $k_\nu(t_\xi)$ if $\lambda = \nu + 1$, where ξ is the smallest ordinal such that $t_\xi \notin k_\nu$; γ is then, by definition, the smallest ordinal $\leqslant \beta$ such that $k_\gamma = K$.

With this in mind, we will define, by transfinite induction, a family of local Noetherian rings A_λ for $\lambda \leqslant \gamma$, and local homomorphisms $f_{\mu\lambda} : A_\lambda \to A_\mu$ for $\lambda \leqslant \mu$, satisfying the following conditions:

(i) $(A_\lambda, f_{\mu\lambda})$ is an inductive system, and $A_0 = A$;
(ii) for all λ, we have a k-isomorphism $A_\lambda/\mathfrak{J}A_\lambda \simeq k_\lambda$;
(iii) for $\lambda \leqslant \mu$, A_μ is a flat A_λ-module.

So suppose that the A_λ and the $f_{\mu\lambda}$ are defined for $\lambda < \mu < \xi$, and suppose, first of all, that $\xi = \zeta + 1$, so that $k_\xi = k_\zeta(t)$. If t is transcendental over k_ζ, we define A_ζ, following the procedure of (10.3.1.1), to be equal to $(A_\zeta[t])_{\mathfrak{J}A_{\zeta[t]}}$; the canonical map is $f_{\zeta\xi}$, and, for $\lambda < \zeta$, we take $f_{\xi\lambda} = f_{\xi\zeta} \circ f_{\zeta\lambda}$; the verification of conditions (i) to (iii) is then immediate, given that what we have shown in (10.3.1.1). So now suppose that t is algebraic, and let h be its minimal polynomial in $k_\zeta[T]$, and H a monic polynomial in $A_\zeta[T]$ whose image in $k_\zeta[T]$ is h; we then take A_ξ to be equal to $A_\zeta[T](H)$, with the $f_{\xi\lambda}$ being defined as before; the verification of conditions (i) to (iii) then follows from what we have shown in (10.3.1.2).

Now suppose that ξ has no predecessor; we then take A_ξ to be the inductive limit of the inductive system of local rings $(A_\lambda, f_{\mu\lambda})$ for $\lambda < \xi$; we define $f_{\xi\lambda}$ as the canonical map for $\lambda < \xi$. The fact that A_ξ is local and Noetherian, that the $f_{\xi\lambda}$ are local homomorphism, and that conditions (i) to (iii) are satisfied for $\lambda \leqslant \xi$ then follows from the induction hypothesis, and from Lemma (10.3.1.3). $\mathbf{0}_{\mathbf{III}\text{-}\mathbf{1}}$ | 23
With this construction, it is clear that the ring $B = A_\gamma$ satisfies the conditions of (10.3.1). □

We note that, by (10.2.1, c), we have a canonical isomorphism

$$\mathrm{gr}(A) \otimes_k K \xrightarrow{\sim} \mathrm{gr}(B). \tag{10.3.1.5}$$

We can also replace B by its $\mathfrak{J}B$-adic completion $\widehat{B}$ without changing the conclusions of (10.3.1), because $\widehat{B}$ is a flat B-module ($\mathbf{0}_\mathbf{I}$, 7.3.3), and thus a flat A-module ($\mathbf{0}_\mathbf{I}$, 6.2.1).

We have also shown the following:

Corollary (10.3.2). — *If K is an extension of finite degree, then we can assume that B is a finite A-algebra.*

§11. Supplement on homological algebra

11.1. Review of spectral sequences

(11.1.1). In the following, we use a more general notion of a spectral sequence than that defined in (T, 2.4); keeping the notation of (T, 2.4), we call a *spectral sequence* in an abelian category $\mathtt{C}$ a system E consisting of the following parts:

(a) A family (E_r^{pq}) of objects of $\mathtt{C}$ defined for $p, q \in \mathbf{Z}$ and $r \geqslant 2$.
(b) A family of morphisms $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$ such that $d_r^{p+r,q-r+1} d_r^{pq} = 0$. We set $Z_{r+1}(E_r^{pq}) = \mathrm{Ker}(d_r^{pq})$ and $B_{r+1}(E_r^{pq}) = \mathrm{Im}(d_r^{p+r,q-r+1})$, so that
$$B_{r+1}(E_r^{pq}) \subset Z_{r+1}(E_r^{pq}) \subset E_r^{pq}.$$
(c) A family of isomorphisms $\alpha_r^{pq} : Z_{r+1}(E_r^{pq})/B_{r+1}(E_r^{pq}) \simeq E_{r+1}^{pq}$.

We then define for $k \geqslant r+1$, by induction on k, the subobjects $B_k(E_r^{pq})$ and $Z_k(E_r^{pq})$ as the inverse images, under the canonical morphism $E_r^{pq} \to E_r^{pq}/B_{r+1}(E_r^{pq})$ of the subobjects of this quotient identified via α_r^{pq} with the subobjects $B_k(E_{r+1}^{pq})$ and $Z_k(E_{r+1}^{pq})$ respectively. It is clear that we then have, up to isomorphism,

(11.1.1.1) $$Z_k(E_r^{pq})/B_k(E_r^{pq}) = E_k^{pq} \text{ for } k \geqslant r+1,$$

and, if we set $B_r(E_r^{pq}) = 0$ and $Z_r(E_r^{pq}) = E_r^{pq}$, then we have the inclusion relations

(11.1.1.1.2)
$$0 = B_r(E_r^{pq}) \subset B_{r+1}(E_r^{pq}) \subset B_{r+2}(E_r^{pq}) \subset \cdots \subset Z_{r+2}(E_r^{pq}) \subset Z_{r+1}(E_r^{pq}) \subset Z_r(E_r^{pq}) = E_r^{pq}.$$

The other parts of the data of E are then:

(d) Two subobjects $B_\infty(E_2^{pq})$ and $Z_\infty(E_2^{pq})$ of E_2^{pq} such that we have $B_\infty(E_2^{pq}) \subset Z_\infty(E_2^{pq})$ and, for every $k \geqslant 2$,
$$B_k(E_2^{pq}) \subset B_\infty(E_2^{pq}) \text{ and } Z_\infty(E_2^{pq}) \subset Z_k(E_2^{pq}).$$
We set

(11.1.1.1.3) $$E_\infty^{pq} = Z_\infty(E_2^{pq})/B_\infty(E_2^{pq}).$$

(e) A family (E^n) of objects of $\mathtt{C}$, each equipped with a *decreasing filtration* $(F^p(E^n))_{p\in\mathbf{Z}}$. As usual, we denote by $\mathrm{gr}(E^n)$ the graded object associated to the filtered object E^n, the direct sum of the $\mathrm{gr}_p(E^n) = F^p(E^n)/F^{p+1}(E^n)$. **0**$_{\mathbf{III}}$ | 24

(f) For every pair $(p,q) \in \mathbf{Z}\times\mathbf{Z}$, an isomorphism $\beta^{pq} : E_\infty^{pq} \simeq \mathrm{gr}_p(E^{p+q})$.

The family (E^n), without the filtrations, is called the *abutment* (or *limit*) of the spectral sequence E.

Suppose that the category $\mathtt{C}$ admits infinite direct sums, or that for every $r \geqslant 2$ and every $n \in \mathbf{Z}$, there are finitely many pairs (p,q) such that $p+q = n$ and $E_r^{pq} \neq 0$ (it suffices for it to hold for $r = 2$). Then the $E_r^{(n)} = \sum_{p+q=n} E_r^{pq}$ are defined, and we if denote by $d_r^{(n)}$ the morphism $E_r^{(n)} \to E_r^{(n+1)}$ whose restriction to E_r^{pq} is d_r^{pq} for every pair (p,q) such that $p+q = n$, then $d_r^{(n+1)} \circ d_r^{(n)} = 0$, in other words, $(E_r^{(n)})_{n\in\mathbf{Z}}$ is a *complex* $E_r^{(\bullet)}$ in $\mathtt{C}$, with differentials of degree $+1$, and it follows from (c) that $\mathrm{H}^n(E_n^{(\bullet)})$ is *isomorphic to* $E_{r+1}^{(n)}$ for every $r \geqslant 2$.

(11.1.2). A *morphism* $u : E \to E'$ from a spectral sequence E to a spectral sequence $E' = (E_r'^{pq}, E'^n)$ consists of systems of morphisms $u_r^{pq} : E_r^{pq} \to E_r'^{pq}$ and $u^n : E^n \to E'^n$, the u^n compatible with the filtrations on E^n and E'^n, and the diagrams

$$\begin{tikzcd}
E_r^{pq} \arrow[r, "d_r^{pq}"] \arrow[d, "u_r^{pq}"'] & E_r^{p+r,q-r+1} \arrow[d, "u_r^{p+r,q-r+1}"] \\
E_r'^{pq} \arrow[r, "d_r'^{pq}"] & E_r''^{p+r,q-r+1}
\end{tikzcd}$$

being commutative; in addition, by passing to quotients, u_r^{pq} gives a morphism $\overline{u}_r^{pq} : Z_{r+1}(Z_r^{pq})/B_{r+1}(E_r^{pq}) \to Z_{r+1}(E_r'^{pq})/B_{r+1}(E_r'^{pq})$ and we must have $\alpha_r'^{pq} \circ \overline{u}_r^{pq} = u_{r+1}^{pq} \circ \alpha_r^{pq}$; finally, we must have $u_2^{pq}(B_\infty(E_2^{pq})) \subset B_\infty(E_2'^{pq})$ and $u_2^{pq}(Z_\infty(E_2^{pq})) \subset Z_\infty(E_2'^{pq})$; by passing to quotients, u_2^{pq} then gives a morphism $u_\infty'^{pq} : E_\infty^{pq} \to E_\infty'^{pq}$, and the diagram

$$\begin{tikzcd}
E_\infty^{pq} \arrow[r, "u_\infty'^{pq}"] \arrow[d, "\beta^{pq}"'] & E_\infty'^{pq} \arrow[d, "\beta'^{pq}"] \\
\mathrm{gr}_p(E^{p+q}) \arrow[r, "\mathrm{gr}_p(u^{p+q})"] & \mathrm{gr}_p(E'^{p+q})
\end{tikzcd}$$

must be commutative.

The above definitions show, by induction on r, that if the u_2^{pq} are *isomorphisms*, then so are the u_r^{pq} for $r \geqslant 2$; *if in addition we know that $u_2^{pq}(B_\infty(E_2^{pq})) = B_\infty(E_2'^{pq})$ and $u_2^{pq}(Z_\infty(E_2^{pq})) = Z_\infty(E_2'^{pq})$ and the u^n are isomorphisms*, then we can conclude that u is an *isomorphism*.

(11.1.3). Recall that if $(F^p(X))_{p\in\mathbf{Z}}$ is a (decreasing) *filtration* of an object $X \in \mathtt{C}$, then we say that this filtration is *separated* if $\inf(F^p(X)) = 0$, *discrete* if there exists a p such that $F^p(X) = 0$, *exhaustive* (or *coseparated*) if $\sup(F^p(X)) = X$, *codiscrete* if there exists a p such that $F^p(X) = X$.

We say that a spectral sequence $E = (E_r^{pq}, E^n)$ is *weakly convergent* if we have $B_\infty(E_2^{pq}) = \sup_k(B_k(E_2^{pq}))$ and $Z_\infty(E_2^{pq}) = \inf_k(Z_k(E_2^{pq}))$ (in other words, the objects of $B_\infty(E_2^{pq})$ and $Z_\infty(E_2^{pq})$ are determined from the data of (a) and (c) of the spectral sequence E). We say that the spectral sequence E is *regular* if it is weakly convergent and if, in addition:

(1st) For every pair (p,q), the decreasing sequence $(Z_k(E_2^{pq}))_{k\geqslant 2}$ is *stable*; the hypothesis that E is weakly convergent then implies that $Z_\infty(E_2^{pq}) = Z_k(E_2^{pq})$ for k large enough (depending on p and q).

(2nd) For every n, the filtration $(F^p(E^n))_{p\in\mathbf{Z}}$ of E^n is *discrete* and *exhaustive*.

We say that the spectral sequence E is *coregular* if it is weakly convergent and if, in addition:

(3rd) For every pair (p,q), the increasing sequence $(B_k(E_2^{pq}))_{k\geqslant 2}$ is *stable*, which implies that $B_\infty(E_2^{pq}) = B_k(E_2^{pq})$, and as a result, $E_\infty^{pq} = \inf E_k^{pq}$.

(4th) For every n, the filtration of E^n is *codiscrete*.

Finally, we say that E is *biregular* if it is both regular and coregular, in other words if we have the following conditions:

(a) For every pair (p,q), the sequences $(B_k(E_2^{pq}))_{k\geqslant 2}$ and $(Z_k(E_2^{pq}))_{k\geqslant 2}$ are *stable* and we have $B_\infty(E_2^{pq}) = B_k(E_2^{pq})$ and $Z_\infty(E_2^{pq}) = Z_k(E_2^{pq})$ for k large enough (which implies that $E_\infty^{pq} = E_k^{pq}$).

(b) For every n, the filatration $(F^p(E^n))_{p\in\mathbf{Z}}$ is *discrete* and *codiscrete* (which we also call *finite*).

The spectral sequences defined in (T, 2.4) are thus biregular spectral sequences.

(11.1.4). Suppose that in the category $\mathtt{C}$, filtered inductive limits exist and the functor $\varinjlim$ is *exact* (which is equivalent to saying that the axiom (AB 5) of (T, 1.5) is satisfied (cf. T, 1.8)). The condition that the filtration $(F^p(X))_{p\in\mathbf{Z}}$ of an object $X \in \mathtt{C}$ is exhaustive is then expressed as $\varinjlim_{p\to-\infty} F^p(X) = X$. If a spectral sequence E is weakly convergent, then we have $B_\infty(E_2^{pq}) = \varinjlim_{k\to\infty} B_k(E_2^{pq})$; if in addition $u : E \to E'$ is a morphism from E to a weakly convergent spectral sequence E' in $\mathtt{C}$, then we have $u_2^{pq}(B_\infty(E_2^{pq})) = B_\infty(E_2'^{pq})$, by the exactness of $\varinjlim$. In addition:

Proposition (11.1.5). — *Let $\mathtt{C}$ be an abelian category in which filtered inductive limits are exact, E and E' two regular spectral sequences in $\mathtt{C}$, $u : E \to E'$ a morphism of spectral sequences. If the u_2^{pq} are isomorphisms, then so is u.*

PROOF. We already know (11.1.2) that the u_r^{pq} are isomorphisms and that

$$u_2^{pq}(B_\infty(E_2^{pq})) = B_\infty(E_2'^{pq});$$

the hypothesis that E and E' are regular also implies that $u_2^{pq}(Z_\infty(E_2^{pq})) = Z_\infty(E_2'^{pq})$, and as u_2^{pq} $\mathbf{0_{III}}$ | 26
is an isomorphism, so is $u_\infty'^{pq}$; we thus conclude that $\mathrm{gr}_p(u^{p+q})$ is also an isomorphism. But as the filtrations of the E^n and the E'^n are discrete and exhaustive, this implies that the u^n are also isomorphisms (Bourbaki, *Alg. comm.*, chap. III, §2, n°8, th. 1). □

(11.1.6). It follows from (11.1.1.2) and the definition (11.1.1.3) that if, for a spectral sequence E, we have $E_r^{pq} = 0$, then we have $E_k^{pq} = 0$ for $k \geqslant r$ and $E_\infty^{pq} = 0$. We say that a spectral sequence *degenerates* if there exists an integer $r \geqslant 2$ and, for every integer $n \in \mathbf{Z}$, an integer $q(n)$ such that $E_r^{n-q,q} = 0$ *for every* $q \neq q(n)$. We first deduce from the previous remark that we also have $E_k^{n-q,q} = 0$ for $k \geqslant r$ (including $k = \infty$) and $q \neq q(n)$. In addition, the definition of E_{r+1}^{pq} shows that we have $E_{r+1}^{n-q(n),q(n)} = E_r^{n-q(n),q(n)}$; if E is *weakly convergent*, then we also have $E_\infty^{n-q(n),q(n)} = E_r^{n-q(n),q(n)}$; in other words, for every $n \in \mathbf{Z}$, $\mathrm{gr}_p(E^n) = 0$ for $p \neq q(n)$ and $\mathrm{gr}_{q(n)}(E^n) = E_r^{n-q(n),q(n)}$. If in addition the filtration of E^n is *discrete* and *exhaustive*, then the spectral sequence E is *regular*, and we have $E^n = E_r^{n-q(n),q(n)}$ up to unique isomorphism.

(11.1.7). Suppose that filtered inductive limits exist and are exact in the category $\mathtt{C}$, and let $(E_\lambda, u_{\mu\lambda})$ be an inductive system (over a filtered set of indices) of spectral sequeneces in $\mathtt{C}$. Then the *inductive limit* of this inductive system exists in the additive category of spectral sequences of objects of $\mathtt{C}$: to see this, it suffices to define E_r^{pq}, d_r^{pq}, α_r^{pq}, $B_\infty(E_2^{pq})$, $Z_\infty(E_2^{pq})$, E^n, $F^p(E^n)$, and β^{pq} as the respective inductive limits of the $E_{r,\lambda}^{pq}$, $d_{r,\lambda}^{pq}$, $\alpha_{r,\lambda}^{pq}$, $B_\infty(E_{2,\lambda}^{pq})$, $Z_\infty(E_{2,\lambda}^{pq})$, E_λ^n, $F^p(E_\lambda^n)$, and β_λ^{pq}; the verification of the conditions of (11.1.1) follows from the exactness of the functor $\varinjlim$ on $\mathtt{C}$.

Remark (11.1.8). — Suppose that the category $\mathtt{C}$ is the category of A-modules over a *Noetherian* ring A (resp. a ring A). Then the definitions of (11.1.1) show that if, for a given r, the E_r^{pq} are A-modules *of finite type* (resp. *of finite length*), then so are each of the modules E_s^{pq} for $s \geqslant r$, hence so is E_∞^{pq}. If in addition the filtration of the abutment/limit (E^n) is *discrete* or *codiscrete* for all n, then we conclude that each of the E^n is also an A-module *of finite type* (resp. *of finite length*).

(11.1.9). We will have to consider conditions which ensure that a spectral sequence E is biregular is a "uniform" way in $p+q=n$. We will then use the following lemma:

Lemma (11.1.10). — *Let (E_r^{pq}) be a family of objects of $\mathtt{C}$ related by the data of* (a), (b), *and* (c) *of* (11.1.1). *For a fixed integer n, the following properties are equivalent:*

(a) *There exists an integer $r(n)$ such that for $r \geqslant r(n)$, $p+q=n$ or $p+q=n-1$, the morphisms d_r^{pq} are zero.*
(b) *There exists an integer $r(n)$ such that for $p+q=n$ or $p+q=n+1$, we have $B_r(E_2^{pq}) = B_s(E_2^{pq})$ for $s \geqslant r \geqslant r(n)$.*
(c) *There exists an integer $r(n)$ such that for $p+q=n$ or $p+q=n-1$, we have $Z_r(E_2^{pq}) = Z_s(E_2^{pq})$ for $s \geqslant r \geqslant r(n)$.*
(d) *There exists an integer $r(n)$ such that for $p+q=n$, we have $B_r(E_2^{pq}) = B_s(E_2^{pq})$ and $Z_r(E_2^{pq}) = Z_s(E_2^{pq})$ for $s \geqslant r \geqslant r(n)$.*

PROOF. According to the conditions (a), (b), and (c) of (11.1.1), we have that saying $Z_{r+1}(E_2^{pq}) = Z_r(E_2^{pq})$ is equivalent to saying that $d_r^{pq} = 0$ and that saying $B_r(E_2^{p+r,q-r+1}) = B_{r+1}(E_2^{p+r,q-r+1})$ is equivalent to saying that $d_r^{pq} = 0$; the lemma immediately follows from this remark. □

11.2. The spectral sequence of a filtered complex

(11.2.1). Given an abelian category $\mathtt{C}$, we will agree to denote by notation such as $K^\bullet$ the *complexes* $(K^i)_{i\in\mathbf{Z}}$ of objects of $\mathtt{C}$ whose differential is of degree $+1$, and by the notation such as $K_\bullet$ the complexes $(K_i)_{i\in\mathbf{Z}}$ of objects of $\mathtt{C}$ whose differential is of degree -1. To each complex $K^\bullet = (K^i)$ whose differential d is of degree $+1$, we can associate a complex $K'_\bullet = (K'_i)$ by setting $K'_i = K^{-i}$, the differential $K'_i \to K'_{i-1}$ being the operator $d : K^{-i} \to K^{-i-1}$; and *vice versa*, which, depending on the circumstances, will allow one to consider either one of the types of complexes and translate any result from one type into results for the other. We similarly denote by notation such as $K^{\bullet\bullet} = (K^{ij})$ (resp. $K_{\bullet\bullet} = (K_{ij})$) the *bicomplexes* (or *double complexes*) of objects of $\mathtt{C}$ in which the *two* differentials are of degree $+1$ (resp. -1); we can still pass from one type to the other by changing the signs of the indices, and we have similar notation and remarks for any multicomplexes. The notation $K^\bullet$ and $K_\bullet$ will also be used for $\mathbf{Z}$-*graded objects* of $\mathtt{C}$, which are not necessarily complexes (they can be considered as such for the *zero* differentials); for example, we write $\mathrm{H}^\bullet(K^\bullet) = (\mathrm{H}^i(K^\bullet))_{i\in\mathbf{Z}}$ for the *cohomology* of a complex $K^\bullet$ whose differential is of degree $+1$, and $H_\bullet(K_\bullet) = (\mathrm{H}_i(K_\bullet))_{i\in\mathbf{Z}}$ for the *homology* of a complex $K_\bullet$ whose differential is of degree -1; when we pass from $K^\bullet$ to $K'_\bullet$ by the method described above, we have $\mathrm{H}_i(K'_\bullet) = \mathrm{H}^{-i}(K^\bullet)$.

Recall in this case that for a complex $K^\bullet$ (resp. $K_\bullet$), we will write in general $Z^i(K^\bullet) = \mathrm{Ker}(K^i \to K^{i+1})$ ("object of cocycles") and $B^i(K^\bullet) = \mathrm{Im}(K^{i-1} \to K^i)$ ("object of coboundaries") (resp. $Z_i(K_\bullet) = \mathrm{Ker}(K_i \to K_{i-1})$ ("object of cycles") and $B_i(K_\bullet) = \mathrm{Im}(K_{i+1} \to K_i)$ ("object of boundaries")) so that $\mathrm{H}^i(K^\bullet) = Z^i(K^\bullet)/B^i(K^\bullet)$ (resp. $\mathrm{H}_i(K_\bullet) = Z_i(K_\bullet)/B_i(K_\bullet)$).

If $K^\bullet = (K^i)$ (resp. $K_\bullet = (K_i)$) is a complex in $\mathtt{C}$ and $T : \mathtt{C} \to \mathtt{C}'$ a functor from $\mathtt{C}$ to an abelian category $\mathtt{C}'$, then we denote by $T(K^\bullet)$ (resp. $T(K_\bullet)$) the complex $(T(K^i))$ (resp. $(T(K_i))$) in $\mathtt{C}'$.

We will not review the definition of the ∂-*functors* (T, 2.1), except to note that we *also* say ∂-functor in place of ∂^*-functor when the morphism ∂ decreases the degree of a unit, the context clarifying this point if there is cause for confusion.

Finally, we say that a *graded object* $(A_i)_{i\in\mathbf{Z}}$ of $\mathtt{C}$ is *bounded below* (resp. *above*) if there exists an i_0 such that $A_i = 0$ for $i < i_0$ (resp. $i > i_0$).

(11.2.2). Let $K^\bullet$ be a complex in $\mathtt{C}$ whose differential d is of degree $+1$, and suppose it is equipped with a *filtration* $F(K^\bullet) = (F^p(K^\bullet))_{p\in\mathbf{Z}}$ consisting of *graded* subobjects of $K^\bullet$, in other words, $F^p(K^\bullet) =$ $\mathbf{0_{III}}$ | 28
$(K^i \cap F^p(K^\bullet))_{i\in\mathbf{Z}}$; in addition, we assume that $d(F^p(K^\bullet)) \subset F^p(K^\bullet)$ for every $p \in \mathbf{Z}$. Let us quickly recall how one *functorially* defines a spectral sequence $E(K^\bullet)$ from $K^\bullet$ (M, XV, 4 and G, I, 4.3). For $r \geqslant 2$, the canonical morphism $F^p(K^\bullet)/F^{p+r}(K^\bullet) \to F^p(K^\bullet)/F^{p+1}(K^\bullet)$ defines a morphism in cohomology

$$\mathrm{H}^{p+q}(F^p(K^\bullet)/F^{p+r}(K^\bullet)) \longrightarrow \mathrm{H}^{p+q}(F^p(K^\bullet)/F^{p+1}(K^\bullet)).$$

We denote by $Z_r^{pq}(K^\bullet)$ the image of this morphism. Similarly, from the exact sequence

$$0 \longrightarrow F^p(K^\bullet)/F^{p+1}(K^\bullet) \longrightarrow F^{p-r+1}(K^\bullet)/F^{p+1}(K^\bullet) \longrightarrow F^{p-r+1}(K^\bullet)/F^p(K^\bullet) \longrightarrow 0,$$

we deduce from the exact sequence in cohomology a morphism

$$\mathrm{H}^{p+q-1}(F^{p-r+1}(K^\bullet)/F^p(K^\bullet)) \longrightarrow \mathrm{H}^{p+q}(F^p(K^\bullet)/F^{p+1}(K^\bullet)),$$

and we denote by $B_r^{pq}(K^\bullet)$ the image of this morphism; we show that $B_r^{pq}(K^\bullet) \subset Z_r^{pq}(K^\bullet)$ and we take $E_r^{pq}(K^\bullet) = Z_r^{pq}(K^\bullet)/B_r^{pq}(K^\bullet)$; we will not specify the definition of the d_r^{pq} or the α_r^{pq}.

We note here that all the $Z_r^{pq}(K^\bullet)$ and $B_r^{pq}(K^\bullet)$, for p and q fixed, are subobjects of the same object $\mathrm{H}^{p+q}(F^p(K^\bullet)/F^{p+1}(K^\bullet))$, which we denote by $Z_1^{pq}(K^\bullet)$; we set $B_1^{pq}(K^\bullet) = 0$, so that the above definitions of $Z_r^{pq}(K^\bullet)$ and $B_r^{pq}(K^\bullet)$ also work for $r = 1$; we still set $E_1^{pq}(K^\bullet) = Z_1^{pq}(K^\bullet)$. We define d_1^{pq} and α_1^{pq} such that the conditions of (11.1.1) are satisfied for $r = 1$. On the other hand, we define the subobjects $Z_\infty^{pq}(K^\bullet)$ as the image of the morphism

$$\mathrm{H}^{p+q}(F^p(K^\bullet)) \longrightarrow \mathrm{H}^{p+q}(F^p(K^\bullet)/F^{p+1}(K^\bullet)) = E_1^{pq}(K^\bullet),$$

and $B_\infty^{pq}(K^\bullet)$ as the image of the morphism

$$\mathrm{H}^{p+q-1}(K^\bullet/F^p(K^\bullet)) \longrightarrow \mathrm{H}^{p+q}(F^p(K^\bullet)/F^{p+1}(K^\bullet)) = E_1^{pq}(K^\bullet),$$

induced as above from the exact sequence in cohomology. We set $Z_\infty(E_2^{pq}(K^\bullet))$ and $B_\infty(E_2^{pq}(K^\bullet))$ to be the canonical images of $E_2^{pq}(K^\bullet)$ in $Z_\infty^{pq}(K^\bullet)$ and $B_\infty^{pq}(K^\bullet)$.

Finally, we denote by $F^p(\mathrm{H}^n(K^\bullet))$ the image in $\mathrm{H}^n(K^\bullet)$ of the morphism $\mathrm{H}^n(F^p(K^\bullet)) \to \mathrm{H}^n(K^\bullet)$ induced from the canonical injection $F^p(K^\bullet) \to K^\bullet$; by the exact sequence in cohomology, this is also the kernel of the morphism $\mathrm{H}^n(K^\bullet) \to \mathrm{H}^n(K^\bullet/F^p(K^\bullet))$. This defines a filtration on $E^n(K^\bullet) = \mathrm{H}^n(K^\bullet)$; we will not give here the definition of the isomorphisms β^{pq}.

(11.2.3). The *functorial* nature of $E(K^\bullet)$ is understood in the following way: given two *filtered* complexes $K^\bullet$ and $K'^\bullet$ in $\mathtt{C}$ and a morphism of complexes $u : K^\bullet \to K'^\bullet$ that is *compatible with the filtrations*, we induce in an evident way the morphisms u_r^{pq} (for $r \geqslant 1$) and u^n, and we show that these morphisms are compatible with the d_r^{pq}, α_r^{pq}, and β^{pq} in the sense of (11.1.2), and thus given a well-defined morphism $E(u) : E(K^\bullet) \to E(K'^\bullet)$ of spectral sequences. In addition, we show that if u and v are morphisms $K^\bullet \to K'^\bullet$ of the above type, *homotopic in degree* $\leqslant k$, then $u_r^{pq} = v_r^{pq}$ for $r > k$ and $u^n = v^n$ for all n (M, XV, 3.1).

(11.2.4). Suppose that filtered inductive limits in $\mathtt{C}$ are exact. Then if the filtration $(F^p(K^\bullet))$ of $\mathbf{0_{III}}$ | 29
$K^\bullet$ is *exhaustive*, then so is the filatration $(F^p(\mathrm{H}^n(K^\bullet)))$ for all n, since by hypothesis we have $K^\bullet = \varinjlim_{p\to-\infty} F^p(K^\bullet)$ and since the hypothesis on $\mathtt{C}$ implies that cohomology commutes with inductive limits. In addition, for the same reason, we have $B_\infty(E_2^{pq}(K^\bullet)) = \sup_k B_k(E_2^{pq}(K^\bullet))$. We say that the filtration $(F^p(K^\bullet))$ of $K^\bullet$ is *regular* if for every n there exists an integer $u(n)$ such that $\mathrm{H}^n(F^p(K^\bullet)) = 0$ for $p > u(n)$. This is particularly the case when the filtration of $K^\bullet$ is *discrete*. When the filtration of $K^\bullet$ is regular and exhaustive, and filtered inductive limits are exact in $\mathtt{C}$, we have (M, XV, 4) that the spectral sequence $E(K^\bullet)$ is *regular*.

11.3. The spectral sequences of a bicomplex

(11.3.1). With regard the conventions for bicomplexes, we follow those of (T, 2.4) rather than those of (M), the two differentials d' and d'' (of degree $+1$) of such a bicomplex $K^{\bullet\bullet} = (K^{ij})$ thus being assumed to be *permutable*. Suppose that *one* of the following two conditions is satisfied:

(1) *Infinite direct sums* exist in $\mathtt{C}$;

(2) For all $n \in \mathbf{Z}$, there is only a *finite* number of pairs (p, q) such that $p + q = n$ and $K^{pq} \neq 0$.

Then the bicomplex $K^{\bullet\bullet}$ defines a (simple) *complex* $(K'^n)_{n\in\mathbf{Z}}$ with $K'^n = \sum_{i+j=n} K^{ij}$, the differential d (of degree $+1$) of this complex being given by $dx = d'x + (-1)^i d''x$ for $x \in K^{ij}$. When we later speak of the spectral sequence of a (simple) *complex* that is *defined by a bicomplex* $K^{\bullet\bullet}$, it will always be understood that one of the above conditions is satisfied. We adopt the analogous conventions for multicomplexes.

We denote by $K^{i,\bullet}$ (resp. $K^{\bullet,j}$) the simple complex $(K^{ij})_{j\in\mathbf{Z}}$ (resp. $(K^{ij})_{i\in\mathbf{Z}}$), by $\mathrm{Z}^p_{\mathrm{II}}(K^{i,\bullet})$, $B^p_{\mathrm{II}}(K^{i,\bullet})$, $\mathrm{H}^p_{\mathrm{II}}(K^{i,\bullet})$ (resp. $\mathrm{Z}^p_{\mathrm{I}}(K^{\bullet,j})$, $B^p_{\mathrm{I}}(K^{\bullet,j})$, $\mathrm{H}^p_{\mathrm{I}}(K^{\bullet,j})$) its p objects of cocycles, of coboundaries, and of cohomology, respectively; the differential $d' : K^{i,\bullet} \to K^{i+1,\bullet}$ is a morphism of complexes, which thus gives an operator on the cocycles, coboundaries, and cohomology,

$$
\begin{aligned}
d' &: \mathrm{Z}^p_{\mathrm{II}}(K^{i,\bullet}) \longrightarrow \mathrm{Z}^p_{\mathrm{II}}(K^{i+1,\bullet}) \\
d' &: B^p_{\mathrm{II}}(K^{i,\bullet}) \longrightarrow B^p_{\mathrm{II}}(K^{i+1,\bullet}) \\
d' &: \mathrm{H}^p_{\mathrm{II}}(K^{i,\bullet}) \longrightarrow \mathrm{H}^p_{\mathrm{II}}(K^{i+1,\bullet})
\end{aligned}
$$

and it is clear that for these operators, $(\mathrm{Z}^p_{\mathrm{II}}(K^{i,\bullet}))_{i\in\mathbf{Z}}$, $(B^p_{\mathrm{II}}(K^{i,\bullet}))_{i\in\mathbf{Z}}$, and $(\mathrm{H}^p_{\mathrm{II}}(K^{i,\bullet}))_{i\in\mathbf{Z}}$ are complexes; we denote the complex $(\mathrm{H}^p_{\mathrm{II}}(K^{i,\bullet}))_{i\in\mathbf{Z}}$ by $\mathrm{H}^p_{\mathrm{II}}(K^{\bullet\bullet})$, and its q objects of cocycles, coboundaries, and cohomology by $\mathrm{Z}^q_{\mathrm{I}}(\mathrm{H}^p_{\mathrm{II}}(K^{\bullet\bullet}))$, $B^q_{\mathrm{I}}(\mathrm{H}^p_{\mathrm{II}}(K^{\bullet\bullet}))$, and $\mathrm{H}^q_{\mathrm{I}}(\mathrm{H}^p_{\mathrm{II}}(K^{\bullet\bullet}))$. We similarly define the complexes $\mathrm{H}^p_{\mathrm{I}}(K^{\bullet\bullet})$ and their cohomology objects $\mathrm{H}^q_{\mathrm{II}}(\mathrm{H}^p_{\mathrm{I}}(K^{\bullet\bullet}))$. Recall, however, that $\mathrm{H}^n(K^{\bullet\bullet})$ denotes the n object of the cohomology of the *(simple)* complex defined by $K^{\bullet\bullet}$.

(11.3.2). On the complex defined by a bicomplex $K^{\bullet\bullet}$, we can consider two canonical filtrations, $(F^p_{\mathrm{I}}(K^{\bullet\bullet}))$ and $(F^p_{\mathrm{II}}(K^{\bullet\bullet}))$, given by

$$
F^p_{\mathrm{I}}(K^{\bullet\bullet}) = \left(\sum_{i+j=n, i\geqslant p} K^{ij}\right)_{n\in\mathbf{Z}} \quad\text{and}\quad F^p_{\mathrm{II}}(K^{\bullet\bullet}) = \left(\sum_{i+j=n, j\geqslant p} K^{ij}\right)_{n\in\mathbf{Z}} \tag{11.3.2.1}
$$

which, by definition, are graded subobjects of the (simple) complex defined by $K^{\bullet\bullet}$, and thus make $\mathbf{0}_{\mathbf{III}}$ | 30
this complex a filtered complex; moreover, is is clear that these filtrations are *exhaustive* and *separated*.

There corresponds to each of these filtrations a spectral sequence (11.2.2); we denote by ${}'E(K^{\bullet\bullet})$ and ${}''E(K^{\bullet\bullet})$ the spectral sequences corresponding to $(F^p_{\mathrm{I}}(K^{\bullet\bullet}))$ and $(F^p_{\mathrm{II}}(K^{\bullet\bullet}))$ respectively, called the *spectral sequence of the bicomplex* $K^{\bullet\bullet}$, and both having as their abutment the cohomology $(\mathrm{H}^n(K^{\bullet\bullet}))$. We can further show (M, XV, 6) that we have

$$
{}'E^{pq}_2(K^{\bullet\bullet}) = \mathrm{H}^p_{\mathrm{I}}(\mathrm{H}^p_{\mathrm{II}}(K^{\bullet\bullet})) \quad\text{and}\quad {}''E^{pq}_2(K^{\bullet\bullet})) = \mathrm{H}^p_{\mathrm{II}}(\mathrm{H}^p_{\mathrm{I}}(K^{\bullet\bullet})). \tag{11.3.2.2}
$$

Every morphism $u : K^{\bullet\bullet} \to K'^{\bullet\bullet}$ of bicomplexes is *ipso facto* compatible with the filtrations of the same type of $K^{\bullet\bullet}$ and $K'^{\bullet\bullet}$, thus defines a morphism for each of the two spectral sequences; in addition, two *homotopic* morphisms define a homotopy *of order* $\leqslant 1$ of the corresponding filtered (simple) complexes, thus the *same* morphism for each of the two spectral sequences (M, XV, 6.1).

Proposition (11.3.3). — *Let $K^{\bullet\bullet} = (K^{ij})$ be a bicomplex in an abelian category $\mathtt{C}$.*

(i) *If there exist i_0 and j_0 such that $K^{ij} = 0$ for $i < i_0$ or $j < j_0$ (resp. $i > i_0$ or $j > j_0$), then the two spectral sequences ${}'E(K^{\bullet\bullet})$ and ${}''E(K^{\bullet\bullet})$ are biregular.*

(ii) *If there exist i_0 and i_1 such that $K^{ij} = 0$ for $i < i_0$ or $i > i_1$ (resp. if there exist j_0 and j_1 such that $K^{ij} = 0$ for $j < j_0$ or $j > j_1$), then the two spectral sequences ${}'E(K^{\bullet\bullet})$ and ${}''E(K^{\bullet\bullet})$ are biregular.*

(iii) *If there exists i_0 such that $K^{ij} = 0$ for $i > i_0$ (resp. if there exists j_0 such that $K^{ij} = 0$ for $j < j_0$), then the spectral sequence ${}'E(K^{\bullet\bullet})$ is regular.*

(iv) *If there exists i_0 such that $K^{ij} = 0$ for $i < i_0$ (resp. if there exists j_0 such that $K^{ij} = 0$ for $j > j_0$), then the spectral sequence ${}''E(K^{\bullet\bullet})$ is regular.*

PROOF. The proposition follows immediately from the definitions (11.1.3) and from (11.2.4), as well as from the following observations relating to the filtration F_{I} (and similar observations that we can deduce for F_{II} by exchange the roles of the two indices in $K^{\bullet\bullet}$):

1° If there exists i_0 such that $K^{ij} = 0$ for $i > i_0$, then the filtration $F_{\mathrm{I}}(K^{\bullet\bullet})$ is *discrete*.

2° If there exists i_0 such that $K^{ij} = 0$ for $i < i_0$, then the filtration $F_{\mathrm{I}}(K^{\bullet\bullet})$ is *co-discrete*. We can then immediate deduce that it is the same for the corresponding filtration $F_{\mathrm{I}}(\mathrm{H}^n(K^{\bullet\bullet}))$ for all n. Furthermore, the definition of the B_r^{pq} corresponding to the filtration $F_{\mathrm{I}}(K^{\bullet\bullet})$ (11.2.2) shows that for any pair (p,q), the sequence $(B_r^{pq})_{r\leqslant 2}$ is stable.

3° If there exists j_0 such that $K^{ij} = 0$ for $j < j_0$, then we have

$$F_{\mathrm{I}}^{p+r}(K^{\bullet\bullet}) \cap (\sum_{i+j=n} K^{ij}) = 0$$

as soon as $p + r + j_0 > n$, so $Z_r^{pq} = Z_\infty(E_2^{pq})$ for $r > q - j_0 + 1$. On the other hand, $\mathrm{H}^n(F_{\mathrm{I}}^p(K^{\bullet\bullet})) = 0$ for $p > n - j_0 + 1$.

4° If there exists j_0 such that $K^{ij} = 0$ for $j > j_0$, then we have

$$F_{\mathrm{I}}^{p-r+1}(K^{\bullet\bullet}) \cap (\sum_{i+j=n} K^{ij}) = \sum_{i+j=n} K^{ij}$$

as soon as $p - r + 1 + j_0 < n$, so $B_r^{pq} = B_\infty(E_2^{pq})$ for $r < j_0 - q + 1$. On the other hand, $\mathrm{H}^n(F_{\mathrm{I}}^p(K^{\bullet\bullet})) = \mathrm{H}^n(K^{\bullet\bullet})$ for $p + j_0 < n - 1$.

□

(11.3.4). Suppose that the bicomplex $K^{\bullet\bullet} = (K^{ij})$ is such that $K^{ij} = 0$ for $i < 0$ or $j < 0$. We know that we can define, for all $p \in \mathbf{Z}$, a canonical "edge-homomorphism". **0_{III}** | 31

$$'E_2^{p0}(K^{\bullet\bullet}) \to \mathrm{H}^p(K^{\bullet\bullet}) \tag{11.3.4.1}$$

(M, XV, 6). Recall that this is due to, on the one hand, the spectral sequence $'E(K^{\bullet\bullet})$, since $Z_r^{p0} = Z_{\mathrm{I}}^p(Z_{\mathrm{II}}^0(K^{\bullet\bullet}))$ for $2 \leqslant r \leqslant +\infty$, and on the other hand, that $\mathrm{H}^p(F_{\mathrm{I}}^{p+1}(K^{\bullet\bullet})) = 0$, so that the isomorphism $\beta^{p0} : {}'E_\infty^{p0} \xrightarrow{\sim} \mathrm{H}^p(F_1^p)/\mathrm{H}^p(F_1^{p+1})$ gives a homomorphism $'E_\infty^{p0} \to \mathrm{H}^p(F_1^p(K^{\bullet\bullet})) \to \mathrm{H}^p(K^{\bullet\bullet})$. The equality of all the Z_r^{p0} allows us to define a canonical homomorphism $'E_r^{p0} \to' E_s^{p0}$ for $r \leqslant s$, and, in particular, a homomorphism $'E_2^{p0} \to' E_\infty^{p0}$, from the composition of the edge-homomorphism $'E_2^{p0}(K^{\bullet\bullet}) \to \mathrm{H}^p(K^{\bullet\bullet})$. Furthermore, we can immediately verify that, in the class mod B_2^{p0} of an element $z \in \mathbf{Z}_{\mathrm{II}}^0(K^{\bullet\bullet}) \subset K^{p0}$ such that $d'z = 0$, the edge-homomorphism thus defined associates, to the class of mod B_∞^{p0} in $'E_\infty^{p0}$, the cohomology class of z in $H^p(K^{\bullet\bullet})$. We therefore finally see that the *edge-homomorphism* (11.3.4.1) *comes from, by passing to cohomology, the canonical injection* $Z_{\mathrm{II}}^0(K^{\bullet\bullet}) \to K^{\bullet\bullet}$ (where $K^{\bullet\bullet}$ is considered as simple complex). We naturally interpret the edge-homomorphism in the same way

$$''E_2^{p0}(K^{\bullet\bullet}) \to \mathrm{H}^p(K^{\bullet\bullet}) \tag{11.3.4.2}$$

as coming from the canonical injection $Z_{\mathrm{I}}^0(K^{\bullet\bullet}) \to K^{\bullet\bullet}$.

(11.3.5). Now let $K_{\bullet\bullet} = (K_{ij})$ be a bicomplex in C whose two differential operators are of degrees -1. We then write $K_{i,\bullet}$ (resp. $K_{\bullet,j}$) to mean the simple complex $(K_{ij})_{j\in\mathbf{Z}}$ (resp. $(K_{ij})_{i\in\mathbf{Z}}$), $\mathrm{H}_p^{\mathrm{II}}(K_{i,\bullet})$ (resp. $\mathrm{H}_p^{\mathrm{I}}(K_{\bullet,j})$) to mean the p-th homology object, $\mathrm{H}_p^{\mathrm{II}}(K_{\bullet\bullet})$ (resp. $\mathrm{H}_p^{\mathrm{I}}(K_{\bullet\bullet})$) to mean the complex $(\mathrm{H}_p^{\mathrm{II}}(K_{i,\bullet}))_{i\in\mathbf{Z}}$ (resp. $(\mathrm{H}_p^{\mathrm{I}}(K_{\bullet,j}))_{j\in\mathbf{Z}}$), and $\mathrm{H}_q^{\mathrm{I}}(\mathrm{H}_p^{\mathrm{II}}(K_{\bullet\bullet}))$ (resp. $\mathrm{H}_q^{\mathrm{II}}(\mathrm{H}_p^{\mathrm{I}}(K_{\bullet\bullet}))$) to mean the q-th homology object; we use analogous notation for the objects of cycles and objects of boundaries; finally, we will denote by $\mathrm{H}_n(K_{\bullet\bullet})$ the n-th homology object (when it exists) of the simple complex (with a differential operator of degree -1) defined by $K_{\bullet\bullet}$. Let $K'^{\bullet\bullet} = (K'^{ij})$ with $K'^{\bullet\bullet} = (K_{-i,-j})$ be the bicomplex with differential operators of degrees $+1$ associated to $K_{\bullet\bullet}$. By definition, the *spectral sequences* of $K_{\bullet\bullet}$ are those of $K'^{\bullet\bullet}$, that we write as $'E(K_{\bullet\bullet})$ and $''E(K_{\bullet\bullet})$, where, however, we change the notation, by putting

$$'E_{pq}^r(K_{\bullet\bullet}) =' E_r^{-p,-q}(K'^{\bullet\bullet}) \quad \text{and} \quad ''E_{pq}^r(K_{\bullet\bullet}) ='' E_r^{-p,-q}(K'^{\bullet\bullet}),$$

for $2 \leqslant r \leqslant \infty$. With this notation, we have

$$'E^2_{pq}(K_{\bullet\bullet}) = \mathrm{H}^{\mathrm{I}}_p(\mathrm{H}^{\mathrm{II}}_q(K_{\bullet\bullet})) \quad \text{and} \quad 'E^2_{pq}(K_{\bullet\bullet}) = \mathrm{H}^{\mathrm{II}}_q(\mathrm{H}^{\mathrm{I}}_p(K_{\bullet\bullet})).$$

To avoid sign errors, it will be preferable in general to return to the complex $K'^{\bullet\bullet}$ for the relations between these spectral sequences and their abutments. We note, however, the criteria corresponding to (11.3.3):

(11.3.6). The spectral sequences $'E(K_{\bullet\bullet})$ and $''E(K_{\bullet\bullet})$ are *biregular* in the following cases: $\mathbf{0}_{\mathrm{III}}$ | 32

(a) There exist i_0 and j_0 such that $K_{ij} = 0$ for $i > i_0$ and $j > j_0$ (resp. for $i < i_0$ and $j < j_0$);
(b) There exist i_0 and i_1 such that $K_{ij} = 0$ for $i < i_0$ and $i > i_1$;
(c) There exist j_0 and j_1 such that $K_{ij} = 0$ for $j < j_0$ and $j > j_1$.

The sequence $'E(K_{\bullet\bullet})$ is *regular* if there exists i_0 such that $K_{ij} = 0$ for $i < i_0$ and $K_{ij} = 0$ for $j > j_0$.

The sequence $''E(K_{\bullet\bullet})$ is *regular* if there exists i_0 such that $K_{ij} = 0$ for $i > i_0$ and $K_{ij} = 0$ for $j < j_0$.

11.4. Hypercohomology of a functor with respect to a complex $K^\bullet$

(11.4.1). Let C be an abelian category. Recall that we defined the *right resolution* (or *cohomological resolution*) of a object A in C to be a complex of objects in C, whose differential operator is of degree $+1$,

$$0 \to L^0 \to L^1 \to L^2 \ldots$$

endowed with a morphism $\varepsilon : A \to L^0$, called the *augmentation* of the resolution, which we can consider as a morphism of complexes

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\varepsilon"'] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ldots \\ 0 \arrow[r] & L^0 \arrow[r] & L^1 \arrow[r] & L^2 \arrow[r] & \ldots \end{tikzcd}$$

such that the sequence

$$0 \to A \xrightarrow{\varepsilon} L^0 \to L^1 \ldots$$

is exact. Similarly, a *left resolution* (or *homological resolution*) of A is a complex $0 \leftarrow L_0 \leftarrow L_1 \leftarrow L_2 \ldots$ of objects in C, whose differential operator is of degree -1, endowed with an *augmentation* $L_0 \xrightarrow{\varepsilon} A$ such that the sequence

$$0 \leftarrow A \xleftarrow{\varepsilon} L_0 \leftarrow L_1 \ldots$$

is exact.

When the right resolution $(L_i)_i \geqslant 0$ of a object A is such that $L_i = 0$ for $i \geqslant n+1$, we say that this resolution is *of length* $\leqslant n$. We define a left resolution of length $\leqslant n$ similarly. A resolution which is of of length $\leqslant n$ for some integer n is said to be *finite*.

A resolution of A is called *projective* (resp. *injective*) if the objects of C, apart from A, of which it is composed are *projective* (resp. *injective*). When C is the category of (left, say) modules over a ring, we say that a resolution of A is *flat* (resp. *free*) when the modules, apart from A, of which it is composed are *flat* (resp. *free*).

(11.4.2). Let $K^\bullet = (K^i)_{i\in\mathbf{Z}}$ be a complex of objects of C, whose differential operator is of degree $+1$. We define the *right Cartan–Eilenberg resolution* of $K^\bullet$ to be a pair consisting of a bicomplex $L^{\bullet\bullet} = (L^{ij})$ with differential operator of degree $+1$ and such that $L^{ij} = 0$ for $j \leqslant 0$, and a morphism of simple complexes $\varepsilon : K^\bullet \to L^{\bullet,0}$, such that the following condition are fulfilled: $\mathbf{0}_{\mathrm{III}}$ | 33

(i) For each index i, the sequences

$$\begin{aligned} 0 \to & K^i \xrightarrow{\varepsilon} L^{i0} \to L^{i1} \to \ldots \\ 0 \to & B^i(K^\bullet) \xrightarrow{\varepsilon} B^i_{\mathrm{I}}(L^{\bullet,0}) \to B^i_{\mathrm{I}}(L^{\bullet,1}) \to \ldots \\ 0 \to & Z^i(K^\bullet) \xrightarrow{\varepsilon} Z^i_{\mathrm{I}}(L^{\bullet,0}) \to Z^i_{\mathrm{I}}(L^{\bullet,1}) \to \ldots \\ 0 \to & \mathrm{H}^i(K^\bullet) \xrightarrow{\varepsilon} \mathrm{H}^i_{\mathrm{I}}(L^{\bullet,0}) \to \mathrm{H}^i_{\mathrm{I}}(L^{\bullet,1}) \to \ldots \end{aligned}$$

are exact, or, in other words, $(L^{i,\bullet}), (B^i_{\mathrm{I}}(L^{\bullet\bullet})), (Z^i_{\mathrm{I}}(L^{\bullet\bullet}))$ and $(\mathrm{H}^i_{\mathrm{I}}(L^{\bullet\bullet}))$ are *resolutions* of $K^i, B^i(K^\bullet), Z^i(K^\bullet)$ and $\mathrm{H}^i(K^\bullet)$ (respectively).

(ii) For each j, the simple complex $L^{\bullet,j}$ is *split*, or, in other words, the exact sequences

(11.4.2.1) $$0 \to B^i_{\mathrm{I}}(L^{\bullet,j}) \to Z^i_{\mathrm{I}}(L^{\bullet,j}) \to \mathrm{H}^i_{\mathrm{I}}(L^{\bullet,j}) \to 0$$

(11.4.2.2) $$0 \to Z^i_{\mathrm{I}}(L^{\bullet,j}) \to L^{ij} \to B^{i+1}_{\mathrm{I}}(L^{\bullet,j}) \to 0$$

are *split*.

We can prove (M, XVII, 1.2) that if every object of $\mathtt{C}$ is subobjects of an injective object, then every complex $K^\bullet$ of $\mathtt{C}$ admits an *injective* Cartan–Eilenberg resolution, i.e. consisting of injective objects of L^{ij} (the condition (ii) above then implying that the $(B^i_{\mathrm{I}}(L^{\bullet,j})), (Z^i_{\mathrm{I}}(L^{\bullet,j}))$ and $(\mathrm{H}^i_{\mathrm{I}}(L^{\bullet,j}))$ also consist of injective objects). Furthermore, for any morphsim $f : K^\bullet \to K'^\bullet$ of complexes of $\mathtt{C}$, any Cartan–Eilenberg resolution $L^{\bullet\bullet}$ of $K^\bullet$, and any *injective* Cartan–Eilenberg resolution $L'^{\bullet\bullet}$ of $K'^\bullet$, there exists a morphism of bicomplexes $F : L^{\bullet\bullet} \to L'^{\bullet\bullet}$ that is compatible with f and its augmentation, and if f and g are homotopic morphisms from $K^\bullet$ to $K'^\bullet$, then the corresponding morphisms from $L^{\bullet\bullet}$ to $L'^{\bullet\bullet}$ are also homotopic *(loc. cit.)*.

When $K^\bullet$ is *bounded below* (resp. *bounded above*), we can take $L^{\bullet\bullet}$ to be such that $L^{ij} = 0$ for $i < i_0$ (resp. $i > i_0$) if $K^i = 0$ for $i < i_0$ (resp. $i > i_0$) (M, XVII, 1.3).

Suppose, on the other hand, that there exists a integer n such that every object of $\mathtt{C}$ admits a *injective resolution of length* $\leqslant n$; then we can assume that we have $L^{ij} = 0$ for $j > n$ (M, XVII, 1.4).

(11.4.3). Now let T be an *additive covariant functor* from $\mathtt{C}$ to an abelian category $\mathtt{C}'$. Given a complex $K^\bullet$ of $\mathtt{C}$ and an *injective* Cartan–Eilenberg resolution of $L^{\bullet\bullet}$ of $K^\bullet$, suppose that the (simple) complex defined by the bicomplex $T(L^{\bullet\bullet})$ exists (11.3.1); then the two spectral sequences $'E(T(L^{\bullet\bullet}))$ and $''E(T(L^{\bullet\bullet}))$ of this bicomplex are called the *hypercohomology spectral sequences* of T with respect to $K^\bullet$; by (11.4.2) and (11.4.3), they effectively only depend on the complex $K^\bullet$ effectively, not on the choice of the injective Cartan–Eilenberg resolution $L^{\bullet\bullet}$; furthermore, they depend *functorially* on $K^\bullet$. They have the same abutment $\mathrm{H}^\bullet(T(L^{\bullet\bullet}))$, also called the *hypercohomology* of T with respect to $K^\bullet$, and denoted by $\mathrm{R}^\bullet T(K^\bullet)$. We can show that the E_2 terms of the two spectral sequences above are given by

(11.4.3.1) $$'E_2^{pq} = H^p(\mathrm{R}^q T(K^\bullet))$$

(11.4.3.2) $$''E_2^{pq} = \mathrm{R}^p T(H^q(K^\bullet))$$

where $\mathrm{R}^p T$ denotes, as usual, the p-th *derived functor* of T for $p \in \mathbf{Z}$; $\mathrm{R}^q T(K^\bullet)$ denotes the complexes $(\mathrm{R}^q T(K^i))_{i\in\mathbf{Z}}$. Unless explicitly otherwise mentioned, we will henceforth assume that every object of $\mathtt{C}$ is a subobject of an injective object of $\mathtt{C}$, so that injective Cartan–Eilenberg resolutions exist for any complex of $\mathtt{C}$. Since $L^{ij} = 0$ for $j < 0$, the criteria of (11.3.3) show that the *two* hypercohomology spectral sequences of T with respect to $K^\bullet$ exist and are *biregular* in the each of the following two cases: $\mathbf{0_{III}}$ | 34

(1) $K^\bullet$ is bounded below;
(2) Every object of $\mathtt{C}$ admits an injective resolution of length at most n, for some integer n independent of the object in question.

Indeed, we can suppose, in the first case, that (11.4.2) there exists i_0 such that $L^{ij} = 0$ for $i < i_0$, and, in the second case, that there exists j_1 such that $L^{ij} = 0$ for $j > j_1$; in each of these two cases, it is furthermore clear that, for any given n, there exist only a finite number of pairs (i,j) such that $L^{ij} \neq 0$ and $i + j = n$, which proves our claims.

If we assume that filtrant inductive limits exist in $\mathtt{C}'$ and are exact (which implies, in particular, the existence of infinite direct sums in $\mathtt{C}'$), then the complex defined by the bicomplex $T(L^{\bullet\bullet})$ exists, and criterion (11.3.3) shows that the sequence $'E(T(L^{\bullet\bullet}))$ is always *regular*.

If $K^\bullet$ is a complex such that all the K^i are zero except for a single K^{i_0}, then $\mathrm{R}^n T(K^\bullet)$ is isomorphic to $\mathrm{R}^{n-i_0} T(K^\bullet)$, as follows immediately from the definitions by taking a Cartan–Eilenberg resolution $L^{\bullet\bullet}$ such that $L^{ij} = 0$ for $i \neq i_0$.

If $K^\bullet$ and $K'^\bullet$ are two complexes of $\mathtt{C}$, and f and g are homotopic morphism from $K^\bullet$ to $K'^\bullet$, then the morphisms $\mathrm{R}^\bullet T(K^\bullet) \to \mathrm{R}^\bullet T(K'^\bullet)$ induced by f and g are identical, and the same is true for the morphisms of the cohomology spectral sequences.

Proposition (11.4.5). — *Suppose that filtrant inductive limits exist in $\mathtt{C}'$ and are exact. If $\mathrm{R}^n T(K^i) = 0$ for all $n > 0$ and all $i \in \mathbf{Z}$, then we have functorial isomorphisms*

$$\mathrm{R}^i T(K^\bullet) \xrightarrow{\sim} \mathrm{H}^i(T(K^\bullet)) \tag{11.4.5.1}$$

for all $i \in \mathbf{Z}$.

Proof. Indeed, the only non-zero E_2 terms of the first spectral sequence (11.4.3.1) are then ${}'E_2^{p0} = \mathrm{H}^p(T(K^\bullet))$; in other words, this sequence is *degenerate*; since it is regular (11.4.4), the conclusion follows from (11.1.6). □

(11.4.6). Now consider, for example, a covariant bifunctor $(M, N) \to T(M, N)$ from $\mathtt{C} \times \mathtt{C}$ to $\mathtt{C}''$, where $\mathtt{C}$, $\mathtt{C}'$, and $\mathtt{C}''$ are three abelian categories; we assume, for simplicity, that T is additive in each of its arguments, and furthermore that every object of $\mathtt{C}$ and every object of $\mathtt{C}'$ are subobjects of an injective object, and that filtrant inductive limits exist in $\mathtt{C}$ and are exact. We then define the *hypercohomology* of T with respect to the complexes $K^\bullet$ and $K'^\bullet$ of $\mathtt{C}$ and $\mathtt{C}'$ (respectively), with differential operators of degree $+1$, by taking $K^\bullet$ (resp. $K'^\bullet$) to be an injective Cartan–Eilenberg resolution $L^{\bullet\bullet}$ (resp. $L'^{\bullet\bullet}$); then $T(L^{\bullet\bullet}, L'^{\bullet\bullet})$ is a quadricomplex of $\mathtt{C}''$, which we consider as a *bicomplex* of $\mathtt{C}'$, with the degree of $T(L^{ij}, L'^{hk})$ being the integers $i + h$ and $j + k$. The *hypercohomology* of T with respect to $K^\bullet$ and $K'^\bullet$ is by definition the cohomology $\mathrm{H}^\bullet(T(L^{\bullet\bullet}, L'^{\bullet\bullet}))$ of this bicomplex (in other words, that of the associated simple complex) and is denoted by $\mathrm{R}^\bullet T(K^\bullet, K'^\bullet)$; it is the abutment of two spectral sequences whose E_2 terms are given by **0**$_{\mathbf{III}}$ | 35

$${}'E_2^{pq} = \mathrm{H}^p(\mathrm{R}^q T(K^\bullet, K'^\bullet)) \tag{11.4.6.1}$$

$${}''E_2^{pq} = \sum_{q'+q''=q} \mathrm{R}^p T(\mathrm{H}^{q'} K^\bullet, \mathrm{H}^{q''} K'^\bullet) \tag{11.4.6.2}$$

(cf. M, XVII, 2).

Here $\mathrm{R}^q T(K^\bullet, K'^\bullet)$ is the bicomplex $(\mathrm{R}^q T(K^i, K'^j))_{(i,j) \in \mathbf{Z} \times \mathbf{Z}}$, and the right-hand side of (11.4.6.1) is its cohomology when we consider it as a simple complex.

Furthermore, the first spectral sequence is always regular, and the two spectral sequences are biregular when there exists n such that every object of $\mathtt{C}$ and every object of $\mathtt{C}'$ admit an injective resolution of length $\leqslant n$, or when $K^\bullet$ and $K'^\bullet$ are bounded below; in the latter case, we can furthermore omit the hypothesis that inductive limits exist in $\mathtt{C}$ and $\mathtt{C}'$.

If $K^\bullet$ and $K'^\bullet$ are two other complexes of $\mathtt{C}$ and $\mathtt{C}'$ (respectively), f and g *homotopic* morphisms from $K^\bullet$ to $K_1^\bullet$, and f' and g' *homotopic* morphisms from $K'^\bullet$ to $K_1'^\bullet$, then the morphisms $\mathrm{R}^\bullet T(K^\bullet, K'^\bullet) \to \mathrm{R}^\bullet T(K_1^\bullet, K_1'^\bullet)$ induced by f and f' on the one hand, and by g and g' on the other hand, are identical, and the same is true for the morphisms of the hypercohomology spectral sequences.

We can generalize easily to any additive covariant multifunctor.

Proposition (11.4.7). — *Suppose that for any injective object $\mathtt{I}$ of $\mathtt{C}$ (resp. $\mathtt{I}'$ of $\mathtt{C}'$), $\mathtt{A}' \mapsto T(\mathtt{I}, \mathtt{A}')$ (resp. $\mathtt{A} \mapsto T(\mathtt{A}, \mathtt{I}')$) is an exact functor. Then, with the notation of (11.4.6), we have a canonical isomorphism*

$$\mathrm{R}^\bullet T(K^\bullet, K'^\bullet) \xrightarrow{\sim} \mathrm{H}^\bullet(T(L^{\bullet\bullet}, K'^\bullet)) \xrightarrow{\sim} \mathrm{H}^\bullet(T(K^\bullet, L'^{\bullet\bullet}))$$

where the last two terms are the cohomology of simple complexes defined by the tricomplexes $T(L^{\bullet\bullet}, K'^\bullet)$ and $T(K^\bullet, L^{\bullet\bullet})$ (respectively).

Proof. Let us define, for example, the first of these isomorphisms. The quadricomplex $T(L^{\bullet\bullet}, L'^{\bullet\bullet})$ can be considered as a bicomplex, where the degrees of $T(L^{ij}, L'^{hk})$ are the integers $i + j$ and $h + k$. Since, for each h, $L'^{h,\bullet}$ is a *resolution* of K'^h, we have, for this bicomplex, by virtue of the hypotheses on T, $\mathrm{H}^q_{\mathrm{II}}(T(L^{\bullet\bullet}, L'^{\bullet\bullet})) = 0$ for $q \neq 0$, and $\mathrm{H}^0_{\mathrm{II}}(T(L^{\bullet\bullet}, L'^{\bullet\bullet})) = T(L^{\bullet\bullet}, K'^\bullet)$; the first spectral sequence of this bicomplex is thus *degenerate*; since $L'^{hk} = 0$ for $k < 0$, this sequence is also regular (11.3.3), and the conclusion then follows from (11.1.6). □

We have similar results for a covariant multifunctor of any number n of arguments: in the calculation of the hypercohomology, it is not necessary to replace *all* the complexes by a Cartan–Eilenberg resolution, but only $n-1$ of them, provided that, when we fix $n-1$ arbitrary arguments by taking them to be *injective* objects, the covariant functor in the remaining argument is *exact*.

11.5. Passage to the inductive limit in the hypercohomology $\mathbf{0_{III}}$ | 36

11.6. Hypercohomology of a functor with respect to a complex $K_\bullet$

11.7. Hypercohomology of a functor with respect to a bicomplex $K_{\bullet\bullet}$

11.8. Supplement on the cohomology of simplicial complexes

11.9. A lemma on the complexes of finite type

11.10. Euler-Poincare characteristic of a complex of finite length modules

§12. Supplement on sheaf cohomology

12.1. Cohomology of sheaves of modules on ringed spaces

§13. Projective limits in homological algebra

§14. Combinatorial dimension of a topological space

Summary

§14. Combinatorial dimension of a topological space. **IV-1** | 5
§15. M-regular sequences and $\mathscr{F}$-regular sequences.
§16. Dimension and depth of Noetherian local rings.
§17. Regular rings.
§18. Supplement on extensions of algebras.
§19. Formally smooth algebras and Cohen rings.
§20. Derivations and differentials.
§21. Differentials in rings of characteristic p.
§22. Differential criteria for smoothness and regularity.
§23. Japanese rings.

Almost all of the preceding sections have been focused on the exposition of ideas of commutative algebra that will be used throughout Chapter IV. Even though a large amount of these ideas already appear in multiple works ([**CC**], [**Sam53a**], [**SZ60**], [**Ser55b**], [**Nag62**]), we thought that it would be more practical for the reader to have a coherent, vaguely independent exposition. Together with §§5, 6, and 7 of Chapter IV (where we use the language of schemes), these sections constitute, in the middle of our treatise, a miniature special treatise, somewhat independent of Chapters I to III, and one that aims to present, in a coherent manner, the properties of rings that "behave well" with respect to operations such as completion, or integral closure, by systematically associating these properties to more general ideas.[11]

[11] The majority of properties which we discuss were discovered by Chevalley, Zariski, Nagata, and Serre. The method used here was first developed in the autumn of 1961, in a course taught at Harvard University by A. Grothendieck.

14.1. Combinatorial dimension of a topological space $\mathbf{0}_{\mathbf{IV}\text{-}\mathbf{1}}$ | 6

(14.1.1). Let I be an ordered set; a *chain* of elements of I is, by definition, a strictly-increasing finite sequence $i_0 < i_1 < \cdots < i_n$ of elements of I ($n \geqslant 0$); by definition, the *length* of this chain is n. If X is a topological space, the set of its *irreducible closed* subsets is ordered by inclusion, and so we have the notion of a *chain* of irreducible closed subsets of X.

Definition (14.1.2). — Let X be a topological space. We define the combinatorial dimension of X (or simply the dimension of X, if there is no risk of confusion), denoted by $\operatorname{dimc}(X)$ (or simply $\dim(X)$), to be the upper bound of lengths of chains of irreducible closed subsets of X. For all $x \in X$, we define the combinatorial dimension of X at x (or simply the dimension of X at x), denoted by $\dim_x(X)$, to be the number $\inf_U(\dim(U))$, where U varies over the open neighbourhoods of x in X.

It follows from this definition that we have

$$\dim(\varnothing) = -\infty$$

(the upper bound in $\overline{\mathbf{R}}$ of the empty set being $-\infty$). If (X_α) is the family of irreducible components of X, then we have

$$\dim(X) = \sup_\alpha \dim(X_\alpha), \tag{14.1.2.1}$$

because every chain of irreducible closed subsets of X is, by definition, contained in some irreducible component of X, and, conversely, the irreducible components are closed in X, so every irreducible closed subset of an X_α is a irreducible closed subset of X.

Definition (14.1.3). — We say that a topological space X is equidimensional if all its irreducible components have the same dimension (which is thus equal to $\dim(X)$, by (14.1.2.1)).

Proposition (14.1.4). —

(i) *For every closed subset Y of a topological space X, we have* $\dim(Y) \leqslant \dim(X)$.
(ii) *If a topological space X is a finite union of closed subsets X_i, then we have* $\dim(X) = \sup_i \dim(X_i)$.

Proof. For every irreducible closed subset Z of Y, the closure $\overline{Z}$ of Z in X is irreducible ($\mathbf{0}_\mathbf{I}$, 2.1.2), and $\overline{Z} \cap Y = Z$, whence (i). Now, if $X = \bigcup_{i=1}^n X_i$, where the X_i are closed, then every irreducible closed subset of X is contained in one of the X_i ($\mathbf{0}_\mathbf{I}$, 2.1.1), and so every chain of irreducible closed subsets of X is contained in one of the X_i, whence (ii). □

From (14.1.4, i), we see that, for all $x \in X$, we can also write

$$\dim_x(X) = \lim_U \dim(U), \tag{14.1.4.1}$$

where the limit is taken over the downward-directed set of open neighbourhoods of x in X.

$\mathbf{0}_{\mathbf{IV}\text{-}\mathbf{1}}$ | 7

Corollary (14.1.5). — *Let X be a topological space, x a point of X, U a neighbourhood of x, and Y_i ($1 \leqslant i \leqslant n$) closed subsets of U such that, for all i, $x \in Y_i$, and such that U is the union of the Y_i. Then we have*

$$\dim_x(X) = \sup_i(\dim_x(Y_i)). \tag{14.1.5.1}$$

Proof. It follows from (14.1.4, ii) that we have $\dim_x(X) = \inf_V(\sup_i(\dim(Y_i \cap V)))$, where V ranges over the set of open neighbourhoods of x that are contained in U; similarly, we have $\dim_x(Y_i) = \inf_V(\dim(Y_i \cap V))$ for all i. The corollary is thus evident if

$$\sup_i(\dim_x(Y_i)) = +\infty;$$

if this were not the case, then there would be an open neighbourhood $V_0 \subset U$ of x such that $\dim(Y_i \cap V) = \dim_x(Y_i)$ for $1 \leqslant i \leqslant n$ and for all $V \subset V_0$, whence the conclusion. □

Proposition (14.1.6). — *For every topological space X, we have* $\dim(X) = \sup_{x \in X} \dim_x(X)$.

PROOF. It follows from Definition (14.1.2) and Proposition (14.1.4) that $\dim_x(X) \leqslant \dim(X)$ for all $x \in X$. Now, let $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ be a chain of irreducible closed subsets of X, and let $x \in Z_0$; for every open subset $U \subset X$ that contains x, $U \cap Z_i$ is irreducible ($\mathbf{0}_\mathbf{I}$, 2.1.6) and closed in U, and since we have $\overline{U \cap Z_i} = Z_i$ in X, the $U \cap Z_i$ are pairwise distinct; thus $\dim(U) \geqslant n$, which finishes the proof. □

Corollary (14.1.7). — *If (X_α) is an open, or closed and locally finite, cover of X, then* $\dim(X) = \sup_\alpha(\dim(X_\alpha))$.

PROOF. If X_α is a neighbourhood of $x \in X$, then $\dim_x(X) \leqslant \dim(X_\alpha)$, whence the claim for open covers. On the other hand, if the X_α are closed, and U is a neighbourhood of $x \in X$ which meets only finitely many of the X_α, then

$$\dim_x(X) \leqslant \dim(U) = \sup_\alpha(\dim(U \cap X_\alpha)) \leqslant \sup_\alpha(\dim(X_\alpha))$$

by (14.1.4), whence the other claim. □

Corollary (14.1.8). — *Let X be a Noetherian Kolmogoroff space ($\mathbf{0}_\mathbf{I}$, 2.1.3), and F the set of closed points of X. Then* $\dim(X) = \sup_{x \in F} \dim_x(X)$.

PROOF. With the notation from the proof of (14.1.6), it suffices to note that there exists a closed point in Z_0 ($\mathbf{0}_\mathbf{I}$, 2.1.3). □

Proposition (14.1.9). — *Let X be a nonempty Noetherian Kolmogoroff space. To have* $\dim(X) = 0$, *it is necessary and sufficient for X to be finite and discrete.*

PROOF. If a space X is separated (and *a fortiori* if X is a discrete space), then all the irreducible closed subsets of X are single points, and so $\dim(X) = 0$. Conversely, suppose that X is a Noetherian Kolmogoroff space such that $\dim(X) = 0$; since every irreducible component of X contains a closed point ($\mathbf{0}_\mathbf{I}$, 2.1.3), it must be exactly this single point. Since X has only a finite number of irreducible components, it is thus finite and discrete. □

Corollary (14.1.10). — *Let X be a Noetherian Kolmogoroff space. For a point $x \in X$ to be isolated, it is necessary and sufficient to have* $\dim_x(X) = 0$.

PROOF. The condition is clearly necessary (without any hypotheses on X). It is also sufficient, because it implies that $\dim(U) = 0$ for any open neighbourhood U of x, and since U is a Noetherian Kolmogoroff space, U is finite and discrete. □ $\mathbf{0}_{\mathbf{IV}\text{-}\mathbf{1}}$ | 8

Proposition (14.1.11). — *The function $x \mapsto \dim_x(X)$ is upper semi-continuous on X.*

PROOF. It is clear that this function is upper semi-continuous at every point where its value is $+\infty$. So suppose that $\dim_x(X) = n < +\infty$; then Equation (14.1.4.1) shows that there exists an open neighbourhood U_0 of x such that $\dim(U) = n$ for every open neighbourhood $U \subset U_0$ of x. So, for all $y \in U_0$ and every open neighbourhood $V \subset U_0$ of y, we have $\dim(V) \leqslant \dim(U_0) = n$ (14.1.4); we thus deduce from (14.1.4.1) that $\dim_y(X) \leqslant n$. □

Remark (14.1.12). — If X and Y are topological spaces, and $f : X \to Y$ a continuous map, then it can be the case that $\dim(f(X)) > \dim(X)$; we obtain such an example by taking X to be a discrete space with 2 points, a and b, and Y to be the set $\{a, b\}$ endowed with the topology for which[12] the closed sets are $\varnothing$, $\{a\}$, and $\{a, b\}$; if $f : X \to Y$ is the identity map, then $\dim(Y) = 1$ and $\dim(X) = 0$. We note that Y is the spectrum of a discrete valuation ring A, of which a is the unique closed point, and b the generic point; if K and k are the field of fractions and the residue field of A (respectively), then X is the spectrum of the ring $k \times K$, and f is the continuous map corresponding to the homomorphism $(\phi, \psi) : A \to k \times K$, where $\phi : A \to k$ and $\psi : A \to K$ are the canonical homomorphism (cf. (**IV**, 5.4.3)).

[12] *[Trans.] This is now often referred to as the* Sierpiński space, *or the* connected two-point set.

14.2. Codimension of a closed subset

Definition (14.2.1). — Given an irreducible closed subset Y of a topological space X, we define the combinatorial codimension (or simply codimension) of Y in X, denoted by $\operatorname{codim}(Y,X)$, to be the upper bound of the lengths of chains of irreducible closed subsets of X of which Y is the smallest element. If Y is an arbitrary closed subset of X, then we define the codimension of Y in X, again denoted by $\operatorname{codim}(Y,X)$, to be the lower bound of the codimensions in X of the irreducible components of Y. We say that X is equicodimensional if all the minimal irreducible closed subsets of X has the same codimension in X.

It follows from this definition that $\operatorname{codim}(\varnothing,X)=+\infty$, since the lower bound of the empty set of $\overline{\mathbf{R}}$ is $+\infty$. If Y is closed in X, and if (X_α) (resp. (Y_α)) is the family of irreducible components of X (resp. Y), then every Y_β is contained in some X_α, and, more generally, every chain of irreducible closed subsets of X of which Y_β is the smallest element is formed of subsets of some X_α; we thus have

$$\operatorname{codim}(Y,X)=\inf_\beta(\sup_\alpha(\operatorname{codim}(Y_\beta,X_\alpha))), \tag{14.2.1.1}$$

where, for every β, α ranges over the set of indices such that $Y_\beta\subset X_\alpha$.

Proposition (14.2.2). — *Let X be a topological space.*

(i) *If Φ is the set of irreducible closed subsets of X, then*

$$\dim(X)=\sup_{Y\in\Phi}(\operatorname{codim}(Y,X)). \tag{14.2.2.1}$$

$\mathbf{0}_{\mathbf{IV\text{-}1}}$ | 9

(ii) *For every nonempty closed subset Y of X, we have*

$$\dim(Y)+\operatorname{codim}(Y,X)\leqslant\dim(X). \tag{14.2.2.2}$$

(iii) *If Y, Z, and T are closed subsets of X such that $Y\subset Z\subset T$, then*

$$\operatorname{codim}(Y,Z)+\operatorname{codim}(Z,T)\leqslant\operatorname{codim}(Y,T). \tag{14.2.2.3}$$

(iv) *For a closed subset Y of X to be such that* $\operatorname{codim}(Y,X)=0$, *it is necessary and sufficient for Y to contain an irreducible component of X.*

PROOF. Claims (i) and (iv) are immediate consequences of Definition (14.2.1). To show (ii), we can restrict to the case where Y is irreducible, and then the equation follows from Definitions (14.1.1) and (14.2.1). Finally, to show (iii), we can, by Definition (14.2.1), first restrict to the case where Y is irreducible; then $\operatorname{codim}(Y,Z)=\sup_\alpha(\operatorname{codim}(Y,Z_\alpha))$ for the irreducible components Z_α of Z that contain Y; it is clear that $\operatorname{codim}(Y,T)\geqslant\operatorname{codim}(Y,Z)$, so the inequality is true if $\operatorname{codim}(Y,Z)=+\infty$; but if this were not the case, then there would exist some α such that $\operatorname{codim}(Y,Z)=\operatorname{codim}(Y,Z_\alpha)$, and by (14.2.1), we can restrict to the case where Z itself is irreducible; but then the inequality in (14.2.2.3) is an evident consequence of Definition (14.2.1). □

Proposition (14.2.3). — *Let X be a topological space, and Y a closed subset of X. For every open subset U of X, we have*

$$\operatorname{codim}(Y\cap U,U)\geqslant\operatorname{codim}(Y,X). \tag{14.2.3.1}$$

Furthermore, for this inequality (14.2.3.1) to be an equality, it is necessary and sufficient to have $\operatorname{codim}(Y,X)=\inf_\alpha(\operatorname{codim}(Y_\alpha,X))$, *where (Y_α) is the family of irreducible components of Y that meet U.*

PROOF. We know ($\mathbf{0}_\mathbf{I}$, 2.1.6) that $Z\mapsto\overline{Z}$ is a bijection from the set of irreducible closed subsets of U to the set of irreducible closed subsets of X that meet U, and, in particular, induces a correspondence between the irreducible components of $Y\cap U$ and the irreducible components of Y that meet U; if Y_α is one of the latter such components, then we have $\operatorname{codim}(Y_\alpha,X)=\operatorname{codim}(Y_\alpha\cap U,U)$, and the proposition then follows from Definition (14.2.1). □

Definition (14.2.4). — Let X be a topological space, Y a closed subset of X, and x a point of X. We define the codimension of Y in X at the point x, denoted by $\operatorname{codim}_x(Y,X)$, to be the number $\sup_U(\operatorname{codim}(Y\cap U,U))$, where U ranges over the set of open neighbourhoods of x in X.

By (14.2.3), we can also write

$$\operatorname{codim}_x(Y,X) = \lim_U(\operatorname{codim}(Y\cap U, U)), \tag{14.2.4.1}$$

where the limit is taken over the downward-directed set of open neighbourhoods of x in X. We note that we have

$$\operatorname{codim}_x(Y,X) = +\infty \text{ if } x \in X - Y.$$

$\mathbf{0_{IV\text{-}1}}$ | 10

Proposition (14.2.5). — *If $(Y_i)_{1\leqslant i\leqslant n}$ is a finite family of closed subsets of a topological space X, and Y is the union of this family, then*

$$\operatorname{codim}(Y,X) = \inf_i(\operatorname{codim}(Y_i,X)). \tag{14.2.5.1}$$

PROOF. Every irreducible component of one of the Y_i is contained in an irreducible component of Y, and, conversely, every irreducible component of Y is also an irreducible component of ome of the Y_i ($\mathbf{0_I}$, 2.1.1); the conclusion then follows from Definition (14.2.1) and the inequality in (14.2.2.3). □

Corollary. — *Let X be a topological space, and Y a locally-Noetherian closed subspace of X.*

(i) *For all $x\in X$, there exists only a finite number of irreducible components Y_i ($1\leqslant i\leqslant n$) of Y that contain x, and we have* $\operatorname{codim}_x(Y,X) = \inf_i(\operatorname{codim}(Y_i,X))$.
(ii) *The function $x\mapsto \operatorname{codim}_x(Y,X)$ is lower semi-continuous on X.*

PROOF. By hypothesis, there exists an open neighbourhood U_0 of x in X such that $Y\cap U_0$ is Noetherian, and so U_0 has only a finite number of irreducible components, which are the intersections of U_0 with the irreducible components of Y; *a fortiori* there are only a finite number of irreducible components Y_i ($1\leqslant i\leqslant n$) of Y that contain x, and we can, by replacing U_0 with an open neighbourhood $U\subset U_0$ of x that doesn't meet any of the Y_j that do not contain x, assume that the $Y_i\cap U$ are the irreducible components of $Y\cap U$; for every open neighbourhood $V\subset U$ of x in X, the $Y_i\cap V$ are thus the irreducible components of $Y\cap V$, and (14.2.3) then shows that $\operatorname{codim}(Y_i,X) = \operatorname{codim}(Y_i\cap V, V)$, which proves (i). Further, for every $x'\in U$, the irreducible components of Y that contain x' are certain Y_i, and so $\operatorname{codim}_{x'}(Y,X)\geqslant \operatorname{codim}_x(Y,X)$, which proves (ii). □

14.3. The chain condition

(14.3.1). In a topological space X, we say that a chain $Z_0\subset Z_1\subset\cdots\subset Z_n$ of irreducible closed subsets if *saturated* if there does not exist an irreducible closed subset Z', distinct from each of the Z_i, such that $Z_k\subset Z'\subset Z_{k+1}$ for any k.

Proposition (14.3.2). — *Let X be a topological space such that, for any two irreducible closed subsets Y and Z of X with $Y\subset Z$, we have* $\operatorname{codim}(Y,Z) < +\infty$. *The following two conditions are equivalent.*

(a) *Any two saturated chains of closed irreducible subsets of X that have the same first and last elements as one another have the same length.*
(b) *If Y, Z, and T are irreducible closed subsets of X such that $Y\subset Z\subset T$, then*

$$\operatorname{codim}(Y,T) = \operatorname{codim}(Y,Z) + \operatorname{codim}(Z,T). \tag{14.3.2.1}$$

PROOF. It is immediate that (a) implies (b). Conversely, suppose that (b) is satisfied, and we will show that if we have two saturated chains with the same first and last elements as one another, of lengths m and $n\leqslant m$ (respectively), then $m=n$. We proceed by induction on n, with the proposition being clear for $n=1$. So suppose that $1<n<m$, and let $Z_0\subset Z_1\subset\cdots\subset Z_n$ be a saturated chain such that there exists another saturated chain, with first element Z_0 and last element Z_n, of length m. Since $\operatorname{codim}(Z_0,Z_n)\geqslant m>n$, and $\operatorname{codim}(Z_0,Z_1)=1$, it follows from *(b)* that $\operatorname{codim}(Z_1,Z_n) = \operatorname{codim}(Z_0,Z_n)-1 > n-1$, which contradicts our induction hypothesis. □ $\mathbf{0_{IV\text{-}1}}$ | 11

When the conditions of (14.3.2) are satisfied, we say that X satisfies the *chain condition*, or that it is a *catenary space*. It is clear that every closed subspace of a catenary space is catenary.

Proposition (14.3.3). — *Let X be a Noetherian Kolmogoroff space of finite dimension. The following conditions are equivalent.*

(a) *Any two maximal chains of irreducible closed subsets of X have the same length.*
(b) *X is equidimensional, equicodimensional, and catenary.*
(c) *X is equidimensional, and, for any irreducible closed subsets Y and Z of X with $Y \subset Z$, we have*

$$\dim(Z) = \dim(Y) + \operatorname{codim}(Y, Z). \tag{14.3.3.1}$$

(d) *X is equicodimensional, and, for any irreducible closed subsets Y and Z of X with $Y \subset Z$, we have*

$$\operatorname{codim}(Y, Z) = \operatorname{codim}(Y, Z) + \operatorname{codim}(Z, X). \tag{14.3.3.2}$$

PROOF. The hypotheses on X imply that the first and last elements of a maximal chain of irreducible closed subsets of X are necessarily a closed point and an irreducible component of X (respectively) ($\mathbf{0_I}$, 2.1.3); further, every saturated chain with first element Y and last element Z (thus $Y \subset Z$) is contained in a maximal chain whose elements differ from those of the given chain, or are contained in Y, or contain Z. These remarks immediately establish the equivalence between (a) and (b), and also show that if (a) is satisfied, then we have, for every irreducible closed subset Y to X,

$$\dim(Y) + \operatorname{codim}(Y, X) = \dim(X); \tag{14.3.3.3}$$

from (14.3.2.1), we immediately deduce (14.3.3.1) and (14.3.3.2) from (14.3.3.3). Conversely, (14.3.3.1) implies (14.3.2.1), and so (14.3.3.1) implies the chain condition, by (14.3.2); further, by applying (14.3.3.1) to the case where Y is a single closed point x of X, and Z is an irreducible component of X, we get that $\operatorname{codim}(\{x\}, X) = \dim(Z)$; we thus conclude that (c) implies (b). Similarly, (14.3.3.2) implies (14.3.2.1), and thus the chain condition; further, with the same choice of Y and Z as above, (14.3.3.2) again implies that $\operatorname{codim}(\{x\}, X) = \dim(Z)$, and so (since every irreducible component of X contains a closed point, by ($\mathbf{0_I}$, 2.1.3)), (d) implies (b). □

We say that a Noetherian Kolmogoroff space is *biequidimensional* if it is of *finite dimension* and if it verifies any of the (equivalent) conditions of (14.3.3).

Corollary (14.3.4). — *Let X be a biequidimensional Noetherian Kolmogoroff space; then, for every closed point x of X, and every irreducible component Z of X, we have*

$$\dim(X) = \dim(Z) = \operatorname{codim}(\{x\}, X) = \dim_x(X). \tag{14.3.4.1}$$

$\mathbf{0_{IV\text{-}1}}$ | 12

PROOF. The last equality follows from the fact that, if $Y_0 = \{x\} \subset Y_1 \subset \cdots \subset Y_m$ is a maximal chain of irreducible closed subsets of X, and U an open neighbourhood of x, then the $U \cap Y_i$ are pairwise disjoint irreducible closed subsets of U (because $\overline{U \cap Y_i} = Y_i$), whence $\dim(U) = \dim(X)$, by (14.1.4). □

Corollary (14.3.5). — *Let X be a Noetherian Kolmogoroff space; if X is biequidimensional, then so is every union of irreducible components of X, and every irreducible closed subset of X. In addition, for every closed subset Y of X, we have*

$$\dim(Y) + \operatorname{codim}(Y, X) = \dim(X). \tag{14.3.5.1}$$

PROOF. Every chain of irreducible closed subsets of X is contained in an irreducible component of X, and so the first claim follows immediately from (14.3.3). Further, if X' is an irreducible closed subset of X, then X' trivially satisfies the conditions of (14.3.3, c), whence the second claim.

Finally, to show (14.3.5.1), note that we have seen, in the proof of (14.3.3), that this equation is true whenever Y is irreducible; if Y_i ($1 \leqslant i \leqslant m$) are the irreducible components of Y, then the Y_i for which $\dim(Y_i)$ is the largest are also those for which $\operatorname{codim}(Y_i, X)$ is the smallest; so (14.3.5.1) follows from the definitions of $\dim(Y)$ and $\operatorname{codim}(Y, X)$. □

Remark (14.3.6). — The reader will note that the proof of (14.3.2) applies to any ordered set, and the fact that we are working with the example of a set of irreducible closed subsets of a topological space is not used at all in the proof. It is the same in the proof of (14.3.3), which holds, more generally, for any ordered set E such that, for all $x \in E$, there exists some $z \leqslant x$ which is *minimal* in E, and such that the length of chains of elements of E is bounded.

CHAPTER I

The language of schemes (EGA I)

Summary

In §§1–8 we do little more than develop a language to be used in what follows. It should be noted, however, that, in accordance with the general spirit of this treatise, §§7–8 will be used less than the others, and in a less essential way; we speak of Chevalley schemes only for the purpose of relating to the language of Chevalley [**CC**] and Nagata [**Nag58a**]. Then, in §9, we give definitions and results concerning quasi-coherent sheaves, some of which are no longer simply a translation of known notions of commutative algebra into a "geometric" language, but are instead already of a global nature; they will be indispensable, in the following chapters, when it comes to the global study of morphisms. Finally, in §10, we introduce a generalization of the notion of a scheme, which will be used as an intermediary in Chapter III to conveniently formulate and prove the fundamental results of the cohomological study of proper morphisms; moreover, it should be noted that the notion of formal schemes seems indispensable in expressing certain facts about the "theory of modules" (classification problems of algebraic varieties). The results of §10 will not be used before §3 of Chapter III, and it is recommended to skip their reading until then. **I** | 79

§1. Affine schemes

1.1. The prime spectrum of a ring

(1.1.1). *Notation*. Let A be a (commutative) ring, and M an A-module. In this chapter and the following, we will constantly use the following notation: **I** | 80

- $\operatorname{Spec}(A)$ = *set of prime ideals* of A, also called the *prime spectrum* of A; for $x \in X = \operatorname{Spec}(A)$, it will often be convenient to write $\mathfrak{j}_x$ instead of x. For $\operatorname{Spec}(A)$ to be *empty*, it is necessary and sufficient for the ring A to be 0.
- $A_x = A_{\mathfrak{j}_x}$ = *(local) ring of fractions* $S^{-1}A$, where $S = A - \mathfrak{j}_x$.
- $\mathfrak{m}_x = \mathfrak{j}_x A_{\mathfrak{j}_x}$ = *maximal ideal of* A_x.
- $k(x) = A_x/\mathfrak{m}_x$ = *residue field of* A_x, canonically isomorphic to the field of fractions of the integral ring $A/\mathfrak{j}_x$, with which we identify it.
- $f(x)$ = *class of* f *mod.* $\mathfrak{j}_x$ in $A/\mathfrak{j}_x \subset k(x)$, for $f \in A$ and $x \in X$. We also say that $f(x)$ is the *value* of f at a point $x \in \operatorname{Spec}(A)$; the equations $f(x) = 0$ and $f \in \mathfrak{j}_x$ are *equivalent*.
- $M_x = M \otimes_A A_x$ = *module of fractions with denominators in* $A - \mathfrak{j}_x$.
- $\mathfrak{r}(E)$ = *radical of the ideal of* A *generated by a subset* E *of* A.

- $V(E)$ = *set of* $x \in X$ *such that* $E \subset \mathfrak{j}_x$ (or the set of $x \in X$ such that $f(x) = 0$ for all $f \in E$), for $E \subset A$. So we have

$$\mathfrak{r}(E) = \bigcap_{x \in V(E)} \mathfrak{j}_x. \tag{1.1.1.1}$$

- $V(f) = V(\{f\})$ for $f \in A$.
- $D(f) = X - V(f)$ = *set of* $x \in X$ *where* $f(x) \neq 0$.

Proposition (1.1.2). — *We have the following properties:*

(i) $V(0) = X$, $V(1) = \varnothing$.
(ii) *The relation* $E \subset E'$ *implies* $V(E) \supset V(E')$.
(iii) *For each family* (E_λ) *of subsets of* A, $V(\bigcup_\lambda E_\lambda) = V(\sum_\lambda E_\lambda) = \bigcap_\lambda V(E_\lambda)$.
(iv) $V(EE') = V(E) \cup V(E')$.
(v) $V(E) = V(\mathfrak{r}(E))$.

PROOF. The properties (i), (ii), (iii) are trivial, and (v) follows from (ii) and from equation (1.1.1.1). It is evident that $V(EE') \supset V(E) \cap V(E')$; conversely, if $x \notin V(E)$ and $x \notin V(E')$, then there exists $f \in E$ and $f' \in E'$ such that $f(x) \neq 0$ and $f'(x) \neq 0$ in $k(x)$, hence $f(x)f'(x) \neq 0$, i.e., $x \notin V(EE')$, which proves (iv). □

Proposition (1.1.2) shows, among other things, that sets of the form $V(E)$ (where E varies over the subsets of A) are the *closed sets* of a topology on X, which we will call the *spectral topology* [1]; unless expressly stated otherwise, we always assume that $X = \mathrm{Spec}(A)$ is equipped with the spectral topology.

(1.1.3). For each subset Y of X, we denote by $\mathfrak{j}(Y)$ the set of $f \in A$ such that $f(y) = 0$ for all $y \in Y$; I | 81
equivalently, $\mathfrak{j}(Y)$ is the intersection of the prime ideals $\mathfrak{j}_y$ for $y \in Y$. It is clear that the relation $Y \subset Y'$ implies that $\mathfrak{j}(Y) \supset \mathfrak{j}(Y')$ and that we have

$$\mathfrak{j}\left(\bigcup_\lambda Y_\lambda\right) = \bigcap_\lambda \mathfrak{j}(Y_\lambda) \tag{1.1.3.1}$$

for each family (Y_λ) of subsets of X. Finally we have

$$\mathfrak{j}(\{x\}) = \mathfrak{j}_x. \tag{1.1.3.2}$$

Proposition (1.1.4). —

(i) *For each subset* E *of* A, *we have* $\mathfrak{j}(V(E)) = \mathfrak{r}(E)$.
(ii) *For each subset* Y *of* X, $V(\mathfrak{j}(Y)) = \overline{Y}$, *the closure of* Y *in* X.

PROOF. (i) is an immediate consequence of the definitions and (1.1.1.1); on the other hand, $V(\mathfrak{j}(Y))$ is closed and contains Y; conversely, if $Y \subset V(E)$, we have $f(y) = 0$ for $f \in E$ and all $y \in Y$, so $E \subset \mathfrak{j}(Y)$, $V(E) \supset V(\mathfrak{j}(Y))$, which proves (ii). □

Corollary (1.1.5). — *The closed subsets of* $X = \mathrm{Spec}(A)$ *and the ideals of* A *equal to their radicals (in other words, those that are the intersection of prime ideals) correspond bijectively by the inclusion-reversing maps* $Y \mapsto \mathfrak{j}(Y)$, $\mathfrak{a} \mapsto V(\mathfrak{a})$; *the union* $Y_1 \cup Y_2$ *of two closed subsets corresponds to* $\mathfrak{j}(Y_1) \cap \mathfrak{j}(Y_2)$, *and the intersection of any family* (Y_λ) *of closed subsets corresponds to the radical of the sum of the* $\mathfrak{j}(Y_\lambda)$.

Corollary (1.1.6). — *If* A *is a Noetherian ring,* $X = \mathrm{Spec}(A)$ *is a Noetherian space.*

Note that the converse of this corollary is false, as shown by any non-Noetherian integral ring with a single prime ideal $\neq \{0\}$ (for example a nondiscrete valuation ring of rank 1).

As an example of ring A whose spectrum is not a Noetherian space, one can consider the ring $\mathscr{C}(Y)$ of continuous real functions on an infinite compact space Y; we know that, as a set, Y corresponds to the set of maximal ideals of A, and it is easy to see that the topology induced on Y by that of $X = \mathrm{Spec}(A)$ is the original topology of Y. Since Y is not a Noetherian space, the same is true for X.

[1] The introduction of this topology in algebraic geometry is due to Zariski. So this topology is usually called the "Zariski topology" on X.

Corollary (1.1.7). — *For each $x \in X$, the closure of $\{x\}$ is the set of $y \in X$ such that $\mathfrak{j}_x \subset \mathfrak{j}_y$. For $\{x\}$ to be closed, it is necessary and sufficient that $\mathfrak{j}_x$ is maximal.*

Corollary (1.1.8). — *The space $X = \operatorname{Spec}(A)$ is a Kolmogoroff space.*

PROOF. If x and y are two distinct points of X, we have either $\mathfrak{j}_x \not\subset \mathfrak{j}_y$ or $\mathfrak{j}_y \not\subset \mathfrak{j}_x$, so one of the points x, y does not belong to the closure of the other. □

(1.1.9). According to Proposition (1.1.2, iv), for two elements f, g of A, we have

$$D(fg) = D(f) \cap D(g). \tag{1.1.9.1}$$

Note also that the equality $D(f) = D(g)$ means, according to Proposition (1.1.4, i) and Proposition (1.1.2, v), that $\mathfrak{r}(f) = \mathfrak{r}(g)$, or that the minimal prime ideals containing (f) and (g) are the same; in particular, it is also the case when $f = ug$, where u is invertible.

Proposition (1.1.10). — **I | 82**

(i) *When f ranges over A, the sets $D(f)$ forms a basis for the topology of X.*
(ii) *For every $f \in A$, $D(f)$ is quasi-compact. In particular, $X = D(1)$ is quasi-compact.*

PROOF.

(i) Let U be an open set in X; by definition, we have $U = X - V(E)$ where E is a subset of A, and $V(E) = \bigcap_{f \in E} V(f)$, hence $U = \bigcup_{f \in E} D(f)$.
(ii) By (i), it suffices to prove that, if $(f_\lambda)_{\lambda \in L}$ is a family of elements of A such that $D(f) \subset \bigcup_{\lambda \in L} D(f_\lambda)$, then there exists a finite subset J of L such that $D(f) \subset \bigcup_{\lambda \in J} D(f_\lambda)$. Let $\mathfrak{a}$ be the ideal of A generated by the f_λ; we have, by hypothesis, that $V(f) \supset V(\mathfrak{a})$, so $\mathfrak{r}(f) \subset \mathfrak{r}(\mathfrak{a})$; since $f \in \mathfrak{r}(f)$, there exists an integer $n \geqslant 0$ such that $f^n \in \mathfrak{a}$. But then f^n belongs to the ideal $\mathfrak{b}$ generated by the finite subfamily $(f_\lambda)_{\lambda \in J}$, and we have $V(f) = V(f^n) \supset V(\mathfrak{b}) = \bigcap_{\lambda \in J} V(f_\lambda)$, that is to say, $D(f) \supset \bigcup_{\lambda \in J} D(f_\lambda)$.

□

Proposition (1.1.11). — *For each ideal $\mathfrak{a}$ of A, $\operatorname{Spec}(A/\mathfrak{a})$ is canonically identified with the closed subspace $V(\mathfrak{a})$ of $\operatorname{Spec}(A)$.*

PROOF. We know there is a canonical bijective correspondence (respecting the inclusion order structure) between ideals (resp. prime ideals) of $A/\mathfrak{a}$ and ideals (resp. prime ideals) of A containing $\mathfrak{a}$. □

Recall that the set $\mathfrak{N}$ of nilpotent elements of A (the *nilradical* of A) is an ideal equal to $\mathfrak{r}(0)$, the intersection of all the prime ideals of A (**0**, 1.1.1).

Corollary (1.1.12). — *The topological spaces* $\operatorname{Spec}(A)$ *and* $\operatorname{Spec}(A/\mathfrak{N})$ *are canonically homeomorphic.*

Proposition (1.1.13). — *For $X = \operatorname{Spec}(A)$ to be irreducible (**0**, 2.1.1), it is necessary and sufficient that the ring $A/\mathfrak{N}$ is integral (or, equivalently, that the ideal $\mathfrak{N}$ is prime).*

PROOF. By virtue of Corollary (1.1.12), we can restrict to the case where $\mathfrak{N} = 0$. If X is reducible, then there exist two distinct closed subsets Y_1 and Y_2 of X such that $X = Y_1 \cup Y_2$, so $\mathfrak{j}(X) = \mathfrak{j}(Y_1) \cap \mathfrak{j}(Y_2) = 0$, since the ideals $\mathfrak{j}(Y_1)$ and $\mathfrak{j}(Y_2)$ are distinct from (0) (1.1.5); so A is not integral. Conversely, if there are elements $f \neq 0$, $g \neq 0$ of A such that $fg = 0$, we have $V(f) \neq X$, $V(g) \neq X$ (since the intersection of all the prime ideals of A is (0)), and $X = V(fg) = V(f) \cup V(g)$. □

Corollary (1.1.14). —

(i) *In the bijective correspondence between closed subsets of $X = \operatorname{Spec}(A)$ and ideals of A equal to their radicals, the irreducible closed subsets of X correspond to the prime ideals of A. In particular, the irreducible components of X correspond to the minimal prime ideals of A.*
(ii) *The map $x \mapsto \overline{\{x\}}$ establishes a bijective correspondence between X and the set of closed irreducible subsets of X* (in other words, *all closed irreducible subsets of X admit exactly one generic point).*

PROOF. (i) follows immediately from (1.1.13) and (1.1.11); and for proving (ii), we can, by (1.1.11), restrict to the case where X is irreducible; then, according to Proposition (1.1.13), there exists a smaller prime ideal $\mathfrak{N}$ in A, which corresponds to the generic point of X; in addition, X admits at most one generic point since it is a Kolmogoroff space ((1.1.8) and (**0**, 2.1.3)). □ I | 83

Proposition (1.1.15). — *If $\mathfrak{J}$ is an ideal in A containing the radical $\mathfrak{N}(A)$, the only neighbourhood of $V(\mathfrak{J})$ in $X = \operatorname{Spec}(A)$ is the whole space X.*

PROOF. Each maximal ideal of A belongs, by definition, to $V(\mathfrak{J})$. As each ideal $\mathfrak{a} \neq A$ of A is contained in a maximal ideal, we have $V(\mathfrak{a}) \cap V(\mathfrak{J}) \neq 0$, whence the proposition. □

1.2. Functorial properties of prime spectra of rings

(1.2.1). Let A, A' be two rings, and

$$\phi : A' \longrightarrow A$$

a homomorphism of rings. For each prime ideal $x = \mathfrak{j}_x \in \operatorname{Spec}(A) = X$, the ring $A'/\phi^{-1}(\mathfrak{j}_x)$ is canonically isomorphic to a subring of $A/\mathfrak{j}_x$, and so it is integral, or, in other words, $\phi^{-1}(\mathfrak{j}_x)$ is a prime ideal of A'; we denote it by ${}^a\phi(x)$, and we have thus defined a map

$${}^a\phi : X = \operatorname{Spec}(A) \longrightarrow X' = \operatorname{Spec}(A')$$

(also denoted $\operatorname{Spec}(\phi)$), that we call the map *associated* to the homomorphism ϕ. We denote by ϕ^x the injective homomorphism from $A'/\phi^{-1}(\mathfrak{j}_x)$ to $A/\mathfrak{j}_x$ induced by ϕ by passing to quotients, as well as its canonical extension to a monomorphism of fields

$$\phi^x : k({}^a\phi(x)) \longrightarrow k(x);$$

for each $f' \in A'$, we therefore have, by definition, **Err**$_{\mathrm{II}}$

(1.2.1.1) $$\phi^x(f'({}^a\phi(x))) = (\phi(f'))(x) \qquad (x \in X).$$

Proposition (1.2.2). —

(i) *For each subset E' of A', we have*

(1.2.2.1) $$ {}^a\phi^{-1}(V(E')) = V(\phi(E')),$$

and in particular, for each $f' \in A'$,

(1.2.2.2) $$ {}^a\phi^{-1}(D(f')) = D(\phi(f')).$$

(ii) *For each ideal $\mathfrak{a}$ of A, we have*

(1.2.2.3) $$\overline{{}^a\phi(V(\mathfrak{a}))} = V(\phi^{-1}(\mathfrak{a})).$$

PROOF. The relation ${}^a\phi(x) \in V(E')$ is, by definition, equivalent to $E' \subset \phi^{-1}(\mathfrak{j}_x)$, so $\phi(E') \subset \mathfrak{j}_x$, and finally $x \in V(\phi(E'))$, hence (i). To prove (ii), we can suppose that $\mathfrak{a}$ is equal to its radical, since $V(\mathfrak{r}(\mathfrak{a})) = V(\mathfrak{a})$ (1.1.2, (v)) and $\phi^{-1}(\mathfrak{r}(\mathfrak{a})) = \mathfrak{r}(\phi^{-1}(\mathfrak{a}))$; if we set $Y = V(\mathfrak{a})$, and $\mathfrak{a}' = \mathfrak{j}({}^a\phi(Y))$, then we have $\overline{{}^a(Y)} = V(\mathfrak{a}')$ ((1.1.4, (ii))) the relation $f' \in \mathfrak{a}'$ is, by definition, equivalent to $f'(x') = 0$ for each $x \in {}^a\phi(Y)$, so, by Equation (1.2.1.1), it is also equivalent to $\phi(f')(x) = 0$ for each $x \in Y$, or to $\phi(f') \in \mathfrak{j}(Y) = \mathfrak{a}$, since $\mathfrak{a}$ is equal to its radical; hence (ii). □

Corollary (1.2.3). — *The map ${}^a\phi$ is continuous.*

We remark that, if A'' is a third ring, and ϕ' a homomorphism $A'' \to A'$, then we have ${}^a(\phi' \circ \phi) = {}^a\phi \circ {}^a\phi'$; this result, with Corollary (1.2.3), says that $\operatorname{Spec}(A)$ is a *contravariant functor* in A, from the category of rings to that of topological spaces.

Corollary (1.2.4). — *Suppose that ϕ is such that every $f \in A$ can be written as $f = h\phi(f')$, where h is invertible in A* (which, in particular, is the case when ϕ is *surjective*). *Then ${}^a\phi$ is a homeomorphism from X to ${}^a\phi(X)$.* I | 84

PROOF. We show that for each subset $E \subset A$, there exists a subset E' of A' such that $V(E) = V(\phi(E'))$; according to the (T_0) axiom (1.1.8) and the formula (1.2.2.1), this implies first of all that ${}^a\phi$ is injective, and then, by (1.2.2.1), that ${}^a\phi$ is a homeomorphism. But it suffices, for each $f \in E$, to take $f' \in A'$ such that $h\phi(f') = f$ with h invertible in A; the set E' of these elements f' is exactly what we are searching for. □

(1.2.5). In particular, when ϕ is the canonical homomorphism from A to a ring quotient $A/\mathfrak{a}$, we again get (1.1.12), and ${}^a\phi$ is the *canonical injection* of $V(\mathfrak{a})$, identified with $\operatorname{Spec}(A/\mathfrak{a})$, into $X = \operatorname{Spec}(A)$.

Another particular case of (1.2.4):

Corollary (1.2.6). — *If S is a multiplicative subset of A, the spectrum $\operatorname{Spec}(S^{-1}A)$ is canonically identified (with its topology) with the subspace of $X = \operatorname{Spec}(A)$ consisting of the x such that $\mathfrak{j}_x \cap S = \varnothing$.*

PROOF. We know by (0, 1.2.6) that the prime ideals of $S^{-1}A$ are the ideals $S^{-1}\mathfrak{j}_x$ such that $\mathfrak{j}_x \cap S = \varnothing$, and that we have $\mathfrak{j}_x = (i_A^S)^{-1}(S^{-1}\mathfrak{j}_x)$. It then suffices to apply Corollary (1.2.4) to the i_A^S. □

Corollary (1.2.7). — *For ${}^a\phi(X)$ to be dense in X', it is necessary and sufficient for each element of the kernel $\operatorname{Ker}\phi$ to be nilpotent.*

PROOF. Applying Equation (1.2.2.3) to the ideal $\mathfrak{a} = (0)$, we have $\widetilde{{}^a\phi(X)} = V(\operatorname{Ker}\phi)$, and for $V(\operatorname{Ker}\phi) = X'$ to hold, it is necessary and sufficient for $\operatorname{Ker}\phi$ to be contained in all the prime ideals of A', or, equivalently, in the nilradical $\mathfrak{r}'$ of A'. □ **Err$_{\mathrm{II}}$** **Err$_{\mathrm{II}}$**

1.3. Sheaf associated to a module

(1.3.1). Let A be a commutative ring, M an A-module, f an element of A, and S_f the multiplicative set consisting of the f^n, where $n \geqslant 0$. Recall that we set $A_f = S_f^{-1}A$, $M_f = S_f^{-1}M$. If S'_f is the saturated multiplicative subset of A consisting of the $g \in A$ which divide an element of S_f, we know that A_f and M_f are canonically identified with ${S'_f}^{-1}A$ and ${S'_f}^{-1}M$ (0, 1.4.3).

Lemma (1.3.2). — *The following conditions are equivalent:*

(a) $g \in S'_f$;
(b) $S'_g \subset S'_f$;
(c) $f \in \mathfrak{r}(g)$;
(d) $\mathfrak{r}(f) \subset \mathfrak{r}(g)$;
(e) $V(g) \subset V(f)$;
(f) $D(f) \subset D(g)$.

PROOF. This follows immediately from the definitions and (1.1.5). □

(1.3.3). If $D(f) = D(g)$, then Lemma (1.3.2, b) shows that $M_f = M_g$. More generally, if $D(f) \supset D(g)$, then $S'_f \subset S'_g$, and we know (0, 1.4.1) that there exists a canonical functorial homomorphism

$$\rho_{g,f} : M_f \longrightarrow M_g,$$

and if $D(f) \supset D(g) \supset D(h)$, we have (0, 1.4.4)

$$\rho_{h,g} \circ \rho_{g,f} = \rho_{h,f}. \tag{1.3.3.1}$$

When f ranges over the elements of $A - \mathfrak{j}_x$ (for a given x in $X = \operatorname{Spec}(A)$), the sets S'_f constitute **I | 85** an increasing filtered set of subsets of $A - \mathfrak{j}_x$, since for elements f and g of $A - \mathfrak{j}_x$, S'_f and S'_g are contained in S'_{fg}; since the union of the S'_f over $f \in A - \mathfrak{j}_x$ is $A - \mathfrak{j}_x$, we conclude (0, 1.4.5) that the A_x-module M_x is canonically identified with the *inductive limit* $\varinjlim M_f$, relative to the family of homomorphisms $(\rho_{g,f})$. We denote by

$$\rho_x^f : M_f \longrightarrow M_x$$

the canonical homomorphism for $f \in A - \mathfrak{j}_x$ (or, equivalently, $x \in D(f)$).

Definition (1.3.4). — We define the structure sheaf of the prime spectrum $X = \operatorname{Spec}(A)$ (resp. the sheaf associated to the A-module M), denoted by $\widetilde{A}$ or $\mathscr{O}_X$ (resp. $\widetilde{M}$) as the sheaf of rings (resp. the $\widetilde{A}$-module) associated to the presheaf $D(f) \mapsto A_f$ (resp. $D(f) \mapsto M_f$), defined on the basis $\mathfrak{B}$ of X consisting of the $D(f)$ for $f \in A$ ((1.1.10), (0, 3.2.1), and (0, 3.5.6)).

We saw (**0**, 3.2.4) that the stalk $\widetilde{A}_x$ (resp. $\widetilde{M}_x$) *can be identified with the ring* A_x (resp. *the* A_x*-module* M_x); we denote by

$$\theta_f : A_f \longrightarrow \Gamma(D(f), \widetilde{A})$$

$$(\text{resp. } \theta_f : M_f \longrightarrow \Gamma(D(f), \widetilde{M})),$$

the canonical map, so that, for all $x \in D(f)$ and all $\xi \in M_f$, we have

$$(\theta_f(\xi))_x = \rho^f_x(\xi). \tag{1.3.4.1}$$

Proposition (1.3.5). — *$\widetilde{M}$ is an exact functor, covariant in M, from the category of A-modules to the category of $\widetilde{A}$-modules.*

Proof. Indeed, let M, N be two A-modules, and u a homomorphism $M \to N$; for each $f \in A$, u corresponds canonically to a homomorphism u_f from the A_f-module M_f to the A_f-module N_f, and the diagram (for $D(g) \subset D(f)$)

$$\begin{tikzcd} M_f \arrow[r, "u_f"] \arrow[d, "\rho_{g,f}"'] & N_f \arrow[d, "\rho_{g,f}"] \\ M_g \arrow[r, "u_g"] & N_g \end{tikzcd}$$

is commutative (1.4.1); these homomorphisms then define a homomorphism of $\widetilde{A}$-modules $\widetilde{u} : \widetilde{M} \to \widetilde{N}$ (**0**, 3.2.3). In addition, for each $x \in X$, $\widetilde{u}_x$ is the inductive limit of the u_f for $x \in D(f)$ ($f \in A$), and as a result (**0**, 1.4.5), if we canonically identify $\widetilde{M}_x$ and $\widetilde{N}_x$ with M_x and N_x respectively, then $\widetilde{u}_x$ is identified with the homomorphism u_x canonically induced by u. If P is a third A-module, v a homomorphism $N \to P$, and $w = v \circ u$, it is immediate that $w_x = v_x \circ u_x$, so $\widetilde{w} = \widetilde{v} \circ \widetilde{u}$. We have therefore clearly defined a *covariant (in M) functor* $\widetilde{M}$, from the category of A-modules to that of $\widetilde{A}$-modules. *This functor is exact*, since, for each $x \in X$, M_x is an exact functor in M (**0**, 1.3.2); in addition, we have $\operatorname{Supp}(M) = \operatorname{Supp}(\widetilde{M})$, by definition ((**0**, 1.7.1) and (**0**, 3.1.6)). □

I | 86

Proposition (1.3.6). — *For each $f \in A$, the open subset $D(f) \subset X$ is canonically identified with the prime spectrum* $\operatorname{Spec}(A_f)$, *and the sheaf $\widetilde{M_f}$ associated to the A_f-module M_f is canonically identified with the restriction $\widetilde{M}|D(f)$.*

Proof. The first assertion is a particular case of (1.2.6). In addition, if $g \in A$ is such that $D(g) \subset D(f)$, then M_g is canonically identified with the module of fractions of M_f whose denominators are the powers of the canonical image of g in A_f (**0**, 1.4.6). The canonical identification of $\widetilde{M_f}$ with $\widetilde{M}|D(f)$ then follows from the definitions. □

Theorem (1.3.7). — *For each A-module M and each $f \in A$, the homomorphism*

$$\theta_f : M_f \longrightarrow \Gamma(D(f), \widetilde{M})$$

is bijective (in other words, the presheaf $D(f) \mapsto M_f$ is a *sheaf*). *In particular, M can be identified with $\Gamma(X, \widetilde{M})$ via θ_1.*

Proof. We note that, if $M = A$, then θ_f is a homomorphism of structure rings; Theorem (1.3.7) then implies that, if we identify the rings A_f and $\Gamma(D(f), \widetilde{A})$ via θ_f, the homomorphism $\theta_f : M_f \to \Gamma(D(f), \widetilde{M})$ is an isomorphism of *modules*.

We show first that θ_f is *injective*. Indeed, if $\xi \in M_f$ is such that $\theta_f(\xi) = 0$, then, for each prime ideal $\mathfrak{p}$ of A_f, there exists $h \notin \mathfrak{p}$ such that $h\xi = 0$; as the annihilator of ξ is not contained in any prime ideal of A_f, each A_f integral, and so $\xi = 0$.

It remains to show that θ_f is *surjective*; we can restrict to the case where $f = 1$ (the general case then following by "localizing", using (1.3.6)). Now let s be a section of $\widetilde{M}$ over X; according to (1.3.4) and (1.1.10, ii), there exists a *finite* cover $(D(f_i))_{i \in I}$ of X ($f_i \in A$) such that, for each $i \in I$, the restriction $s_i = s|D(f_i)$ is of the form $\theta_{f_i}(\xi_i)$, where $\xi_i \in M_{f_i}$. If i, j are indices of I, and if the restrictions of s_i and s_j to $D(f_i) \cap D(f_j) = D(f_i f_j)$ are equal, then it follows, by definition of M, that

(1.3.7.1) $$\rho_{f_i f_j, f_i}(\xi_i) = \rho_{f_i f_j, f_j}(\xi_j).$$

By definition, we can write, for each $i \in I$, $\xi_i = z_i / f_i^{n_i}$, where $z_i \in M$, and. since I is finite, by multiplying each z_i by a power of f_i, we can assume that all the n_i are equal to one single n. Then, by definition, (1.3.7.1) implies that there exists an integer $m_{ij} \geqslant 0$ such that $(f_i f_j)^{m_{ij}}(f_j^n z_i - f_i^n z_j) = 0$, and we can moreover suppose that the m_{ij} are equal to the one single m; then replacing the z_i by $f_i^m z_i$, it remains to prove the case where $m = 0$, in other words, the case where we have

(1.3.7.2) $$f_j^n z_i = f_i^n z_j$$

for any i, j. We have $D(f_i^n) = D(f_i)$, and since the $D(f_i)$ form a cover of X, the ideal generated by the f_i^n is A; in other words, there exist elements $g_i \in A$ such that $\sum_i g_i f_i^n = 1$. Then consider the element $z = \sum_i g_i z_i$ of M; in (1.3.7.2), we have $f_i^n z = \sum_j g_j f_i^n z_j = (\sum_j g_j f_j^n) z_i = z_i$, where, by definition, $\xi_i = z/1$ in M_{f_i}. We conclude that the s_i are the restrictions to $D(f_i)$ of $\theta_1(z)$, which **I** | 87
proves that $s = \theta_1(z)$ and finishes the proof. □

Corollary (1.3.8). — *Let M and N be A-modules; the canonical homomorphism $u \mapsto \widetilde{u}$ from* $\mathrm{Hom}_A(M,N)$ *to* $\mathrm{Hom}_{\widetilde{A}}(\widetilde{M},\widetilde{N})$ *is bijective. In particular, the equations $M = 0$ and $\widetilde{M} = 0$ are equivalent.*

Proof. Consider the canonical homomorphism $v \mapsto \Gamma(v)$ from $\mathrm{Hom}_{\widetilde{A}}(\widetilde{M},\widetilde{N})$ to $\mathrm{Hom}_{\Gamma(\widetilde{A})}(\Gamma(\widetilde{M}),\Gamma(\widetilde{N}))$; the latter module is canonically identified with $\mathrm{Hom}_A(M,N)$, by Theorem (1.3.7). It remains to show that $u \mapsto \widetilde{u}$ and $v \mapsto \Gamma(v)$ are inverses of each other; it is evident that $\Gamma(\widetilde{u}) = u$ by definition of $\widetilde{u}$; on the other hand, if we let $u = \Gamma(v)$ for $v \in \mathrm{Hom}_{\widetilde{A}}(\widetilde{M},\widetilde{N})$, then the map $w : \Gamma(D(f),\widetilde{M}) \to \Gamma(D(f),\widetilde{N})$ canonically induced from v is such that the diagram

$$\begin{tikzcd} M \arrow[r, "u"] \arrow[d, "\rho_{f,1}"'] & N \arrow[d, "\rho_{f,1}"] \\ M_f \arrow[r, "w"] & N_f \end{tikzcd}$$

is commutative; so we necessarily have that $w = u_f$ for all $f \in A$ (**0**, 1.2.4), which shows that $\widetilde{\Gamma(v)} = v$. □

Corollary (1.3.9). —

(i) *Let u be a homomorphism from an A-module M to an A-module N; then the sheaves associated to* $\mathrm{Ker}\, u$, $\mathrm{Im}\, u$, *and* $\mathrm{Coker}\, u$, *are* $\mathrm{Ker}\, \widetilde{u}$, $\mathrm{Im}\, \widetilde{u}$, *and* $\mathrm{Coker}\, \widetilde{u}$ *(respectively). In particular, for $\widetilde{u}$ to be injective (resp. surjective, bijective), it is necessary and sufficient for u to be so too.*

(ii) *If M is an inductive limit (resp. direct sum) of a family of A-modules (M_λ), then $\widetilde{M}$ is the inductive limit (resp. direct sum) of the family $(\widetilde{M_\lambda})$, via a canonical isomorphism.*

Proof.

(i) It suffices to apply the fact that $\widetilde{M}$ is an exact functor in M (1.3.5) to the two exact sequences of A-modules
$$0 \longrightarrow \mathrm{Ker}\, u \longrightarrow M \longrightarrow \mathrm{Im}\, u \longrightarrow 0,$$
$$0 \longrightarrow \mathrm{Im}\, u \longrightarrow N \longrightarrow \mathrm{Coker}\, u \longrightarrow 0.$$
The second claim then follows from Theorem (1.3.7).

(ii) Let $(M_\lambda, g_{\mu\lambda})$ be an inductive system of A-modules, with inductive limit M, and let g_λ be the canonical homomorphism $M_\lambda \to M$. Since we have $\widetilde{g_{\nu\mu}} \circ \widetilde{g_{\mu\lambda}} = \widetilde{g_{\nu\lambda}}$ and $\widetilde{g_\lambda} = \widetilde{g_\mu} \circ \widetilde{g_{\mu\lambda}}$ for $\lambda \leqslant \mu \leqslant \nu$, it follows that $(\widetilde{M_\lambda}, \widetilde{g_{\mu\lambda}})$ is an inductive system of sheaves on X, and if we denote by h_λ the canonical homomorphism $\widetilde{M_\lambda} \to \varinjlim \widetilde{M_\lambda}$, then there is a unique homomorphism $v : \varinjlim \widetilde{M_\lambda} \to \widetilde{M}$ such that $v \circ h_\lambda = \widetilde{g_\lambda}$. To see that v is bijective, it suffices to check, for each $x \in X$, that v_x is a bijection from $(\varinjlim \widetilde{M_\lambda})_x$ to $\widetilde{M}_x$; but $\widetilde{M}_x = M_x$, and
$$(\varinjlim \widetilde{M_\lambda})_x = \varinjlim (\widetilde{M_\lambda})_x = \varinjlim (M_\lambda)_x = M_x \quad (\mathbf{0},\ 1.3.3).$$

Conversely, it follows from the definitions that $(\widetilde{g_\lambda})_x$ and (h_λ) are both equal to the canonical map from $(M_\lambda)_x$ to M_x; since $(\widetilde{g_\lambda})_x = v_x \circ (h_\lambda)_x$, v_x is the identity. Finally, if M is the direct sum of two A-modules N and P, it is immediate that $\widetilde{M} = \widetilde{N} \oplus \widetilde{P}$; each direct sum being the inductive limit of finite direct sums, the claims of (ii) are thus proved. I | 88

□

We note that Corollary (1.3.8) proves that the sheaves isomorphic to the associated sheaves of A-modules form an *abelian category* (T, I, 1.4).

We also note that Corollary (1.3.9) implies that, if M is an A-module *of finite type* (that is to say, there exists a surjective homomorphism $A^n \to M$) then there exists a surjective homomorphism $\widetilde{A^n} \to \widetilde{M}$, or, in other words, the $\widetilde{A}$-module $\widetilde{M}$ is *generated by a finite family of sections over* X (**0**, 5.1.1), and vice versa.

(1.3.10). If N is a submodule of an A-module M, the canonical injection $j : N \to M$ gives, by (1.3.9), an injective homomorphism $\widetilde{N} \to \widetilde{M}$, which allows us to canonically identify $\widetilde{N}$ with an $\widetilde{A}$*-submodule* of $\widetilde{M}$; we will always assume that we have made this identification. If N and P are submodules of M, then we have

(1.3.10.1) $$(N+P)^\sim = \widetilde{N} + \widetilde{P},$$

(1.3.10.2) $$(N \cap P)^\sim = \widetilde{N} \cap \widetilde{P},$$

since $N + P$ and $N \cap P$ are the image of the canonical homomorphism $N \oplus P \to M$ and the kernel of the canonical homomorphism $M \to (M/N) \oplus (M/P)$ (respectively), and it suffices to apply (1.3.9).

We conclude from (1.3.10.1) and (1.3.10.2) that, if $\widetilde{N} = \widetilde{P}$, then we have $N = P$.

Corollary (1.3.11). — *On the category of sheaves isomorphic to the associated sheaves of A-modules, the functor* Γ *is exact.*

Proof. Let $\widetilde{M} \xrightarrow{\widetilde{u}} \widetilde{N} \xrightarrow{\widetilde{v}} \widetilde{P}$ be an exact sequence corresponding to two homomorphisms $u : M \to N$ and $v : N \to P$ of A-modules. If $Q = \operatorname{Im} u$ and $R = \operatorname{Ker} v$, we have $\widetilde{Q} = \operatorname{Im} \widetilde{u} = \operatorname{Ker} \widetilde{v} = \widetilde{R}$ (Corollary (1.3.9)), hence $Q = R$. □

Corollary (1.3.12). — *Let M and N be A-modules.*

(i) *The sheaf associated to* $M \otimes_A N$ *is canonically identified with* $\widetilde{M} \otimes_{\widetilde{A}} \widetilde{N}$.

(ii) *If, in addition, M admits a finite presentation, then the sheaf associated to* $\operatorname{Hom}_A(M, N)$ *is canonically identified with* $\mathscr{H}om_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$.

Proof.

(i) The sheaf $\mathscr{F} = \widetilde{M} \otimes_{\widetilde{A}} \widetilde{N}$ is associated to the presheaf
$$U \longmapsto \mathscr{F}(U) = \Gamma(U, \widetilde{M}) \otimes_{\Gamma(U,\widetilde{A})} \Gamma(U, \widetilde{N}),$$
with U varying over the basis (1.1.10, i) of X consisting of the $D(f)$, where $f \in A$. We know that $\mathscr{F}(D(f))$ is canonically identified with $M_f \otimes_{A_f} N_f$, by (1.3.7) and (1.3.6). Moreover, we know that the A_f-module $M_f \otimes_{A_f} N_f$ is canonically isomorphic to $(M \otimes_A N)_f$ (**0**, 1.3.4), which is itself canonically isomorphic to $\Gamma(D(f), (M \otimes_A N)^\sim)$ (Theorem (1.3.7) and Proposition (1.3.6)). In addition, we see immediately that the canonical isomorphisms
$$\mathscr{F}(D(f)) \simeq \Gamma(D(f), (M \otimes_A N)^\sim)$$
thus obtained satisfy the compatibility conditions with respect to the restriction operations (**0**, 1.4.2), so they define a canonical functorial isomorphism I | 89
$$\widetilde{M} \otimes_{\widetilde{A}} \widetilde{N} \simeq (M \otimes_A N)^\sim.$$

(ii) The sheaf $\mathscr{G} = \mathscr{H}om_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$ is associated to the presheaf
$$U \longmapsto \mathscr{G}(U) = \operatorname{Hom}_{\widetilde{A}|U}(\widetilde{M}|U, \widetilde{N}|U),$$
with U varying over the basis of X consisting of the $D(f)$. We know that $\mathscr{G}(D(f))$ is canonically identified with $\operatorname{Hom}_{A_f}(M_f, N_f)$ (Proposition (1.3.6) and Corollary (1.3.8)),

which, according to the hypotheses on M, is canonically identified with $(\mathrm{Hom}_A(M,N))_f$ (**0**, 1.3.5). Finally, $(\mathrm{Hom}_A(M,N))_f$ is canonically identified with $\Gamma(D(f),(\mathrm{Hom}_A(M,N))^{\sim})$ (Proposition (1.3.6) and Theorem (1.3.7)), and the canonical isomorphisms $\mathscr{G}(D(f)) \simeq \Gamma(D(f),(\mathrm{Hom}_A(M,N))^{\sim})$ thus obtained are compatible with the restriction operations (**0**, 1.4.2); they thus define a canonical isomorphism $\mathscr{H}om_{\widetilde{A}}(\widetilde{M},\widetilde{N}) \simeq (\mathrm{Hom}_A(M,N))^{\sim}$.

□

(1.3.13). Now let B be a (commutative) A-algebra; this can be understood by saying that B is an A-module such that we have some given element $e \in B$ and an A-homomorphism $\phi : B \otimes_A B \to B$, so that *(a)* the diagrams

$$\begin{tikzcd} B \otimes_A B \otimes_A B \arrow[r, "\phi\otimes 1"] \arrow[d, "1\otimes\phi"'] & B \otimes_A B \arrow[d, "\phi"] \\ B \otimes_A B \arrow[r, "\phi"] & B \end{tikzcd} \qquad \begin{tikzcd} B \otimes_A B \arrow[rr, "\sigma"] \arrow[dr, "\phi"'] & & B \otimes_A B \arrow[dl, "\phi"] \\ & B & \end{tikzcd}$$

(σ being the canonical symmetry map) are commutative; and *(b)* $\phi(e \otimes x) = \phi(x \otimes e) = x$. By Corollary (1.3.12), the homomorphism $\widetilde{\phi} : \widetilde{B} \otimes_{\widetilde{A}} \widetilde{B} \to \widetilde{B}$ of $\widetilde{A}$-modules satisfies the analogous conditions, and so it defines an $\widetilde{A}$-*algebra* structure on $\widetilde{B}$. In a similar way, the data of a B-module N is the same as the data of an A-module N and an A-homomorphism $\psi : B \otimes_A N \to N$ such that the diagram

$$\begin{tikzcd} B \otimes_A B \otimes_A N \arrow[r, "\phi\otimes 1"] \arrow[d, "1\otimes\psi"'] & B \otimes_A N \arrow[d, "\psi"] \\ B \otimes_A N \arrow[r, "\psi"] & N \end{tikzcd}$$

is commutative and $\psi(e \otimes n) = n$; the homomorphism $\widetilde{\psi} : \widetilde{B} \otimes_{\widetilde{A}} \widetilde{N} \to \widetilde{N}$ satisfies the analogous condition, and so defines a $\widetilde{B}$-*module* structure on $\widetilde{N}$.

In a similar way, we see that if $u : B \to B'$ (resp. $v : N \to N'$) is a homomorphism of A-algebras (resp. of B-modules), then $\widetilde{u}$ (resp. $\widetilde{v}$) is a homomorphism of $\widetilde{A}$-algebras (resp. of $\widetilde{B}$-modules), and $\operatorname{Ker}\widetilde{u}$ is a $\widetilde{B}$-ideal (resp. $\operatorname{Ker}\widetilde{v}$, $\operatorname{Coker}\widetilde{v}$, and $\operatorname{Im}\widetilde{v}$ are $\widetilde{B}$-modules). If N is a B-module, then $\widetilde{N}$ is a $\widetilde{B}$-module of finite type if and only if N is a B-module of finite type (**0**, 5.2.3).

If M, N are B-modules, then the $\widetilde{B}$-module $\widetilde{M} \otimes_{\widetilde{B}} \widetilde{N}$ is canonically identified with $(M \otimes_B N)^{\sim}$; **I** | 90
similarly $\mathscr{H}om_{\widetilde{B}}(\widetilde{M},\widetilde{N})$ is canonically identified with $(\mathrm{Hom}_B(M,N))^{\sim}$ whenever M admits a finite presentation; the proofs are similar to those for Corollary (1.3.12).

If $\mathfrak{J}$ is an ideal of B, and N is a B-module, then we have $(\mathfrak{J}N)^{\sim} = \widetilde{\mathfrak{J}} \cdot \widetilde{N}$.

Finally, if B is an A-algebra *graded* by the A-submodules B_n ($n \in \mathbf{Z}$), then the $\widetilde{A}$-algebra $\widetilde{B}$, the direct sum of the $\widetilde{A}$-modules $\widetilde{B}_n$ (1.3.9), is graded by these $\widetilde{A}$-submodules, the axiom of graded algebras saying that the image of the homomorphism $B_m \otimes B_n \to B$ is contained in B_{m+n}. Similarly, if M is a B-module graded by the submodules M_n, then $\widetilde{M}$ is a $\widetilde{B}$-module graded by the $\widetilde{M_n}$.

(1.3.14). If B is an A-algebra, and M a submodule of B, then the $\widetilde{A}$-subalgebra of $\widetilde{B}$ generated by $\widetilde{M}$ (**0**, 4.1.3) is the $\widetilde{A}$-subalgebra $\widetilde{C}$, where we denote by C the subalgebra of B generated by M. Indeed, C is the direct sum of the submodules of B which are the images of the homomorphisms $\bigotimes^n M \to B$ ($n \geqslant 0$), so it suffices to apply (1.3.9) and (1.3.12).

1.4. Quasi-coherent sheaves over a prime spectrum

Theorem (1.4.1). — *Let X be the prime spectrum of a ring A, V a quasi-compact open subset of X, and $\mathscr{F}$ an $(\mathscr{O}_X|V)$-module. The following four conditions are equivalent.*

(a) *There exists an A-module M such that $\mathscr{F}$ is isomorphic to $\widetilde{M}|V$.*
(b) *There exists a finite open cover (V_i) of V by sets of the form $D(f_i)$ $(f_i \in A)$ contained in V, such that, for each i, $\mathscr{F}|V_i$ is isomorphic to a sheaf of the form $\widetilde{M}_i$, where M_i is an A_{f_i}-module.*
(c) *The sheaf $\mathscr{F}$ is quasi-coherent* (**0**, 5.1.3).
(d) *The two following properties are satisfied:*
  (d1) *For each $f \in A$ such that $D(f) \subset V$, and for each section $s \in \Gamma(D(f), \mathscr{F})$, there exists an integer $n \geqslant 0$ such that $f^n s$ extends to a section of $\mathscr{F}$ over V.*
  (d2) *For each $f \in A$ such that $D(f) \subset V$ and for each section $t \in \Gamma(V, \mathscr{F})$ such that the restriction of t to $D(f)$ is 0, there exists an integer $n \geqslant 0$ such that $f^n t = 0$.*

(In the statement of the conditions (d1) and (d2), we have tacitly identified A and $\Gamma(\widetilde{A})$ using Theorem (1.3.7)).

Proof. The fact that (a) implies (b) is an immediate consequence of Proposition (1.3.6) and the fact that the $D(f_i)$ form a basis for the topology of X (1.1.10). As each A-module is isomorphic to the cokernel of a homomorphism of the form $A^{(I)} \to A^{(J)}$, (1.3.6) implies that each sheaf associated to an A-module is quasi-coherent; so (b) implies (c). Conversely, if $\mathscr{F}$ is quasi-coherent, each $x \in V$ has a neighbourhood of the form $D(f) \subset V$ such that $\mathscr{F}|D(f)$ is isomorphic to the cokernel of a homomorphism $\widetilde{A_f}^{(I)} \to \widetilde{A_f}^{(J)}$, so also to the sheaf $\widetilde{N}$ associated to the module N, the cokernel of the corresponding homomorphism $A_f^{(I)} \to A_f^{(J)}$ (Corollaries (1.3.8) and (1.3.9)); since V is quasi-compact, it is clear that (c) implies (b).

To prove that (b) implies (d1) and (d2), we first assume that $V = D(g)$ for some $g \in A$, and that I | 91 $\mathscr{F}$ is isomorphic to the sheaf $\widetilde{N}$ associated to an A_g-module N; by replacing X with V and A with A_g (1.3.6), we can reduce to the case where $g = 1$. Then $\Gamma(D(f), \widetilde{N})$ and N_f are canonically identified with one another (Proposition (1.3.6) and Theorem (1.3.7)), so a section $s \in \Gamma(D(f), \widetilde{N})$ is identified with an element of the form z/f^n, where $z \in N$; the section $f^n s$ is identified with the element $z/1$ of N_f and, as a result, is the restriction to $D(f)$ of the section of $\widetilde{N}$ over X that is identified with the element $z \in N$; hence (d1) in this case. Similarly, $t \in \Gamma(X, \widetilde{N})$ is identified with an element $z' \in N$, the restriction of t to $D(f)$ is identified with the image $z'/1$ of z' in N_f, and to say that this image is zero means that there exists some $n \geqslant 0$ such that $f^n z' = 0$ in N, or, equivalently, $f^n t = 0$.

To finish the proof, that (b) implies (d1) and (d2), it suffices to establish the following lemma.

Lemma (1.4.1.1). — *Suppose that V is the finite union of sets of the form $D(g_i)$, and that all of the sheaves $\mathscr{F}|D(g_i)$ and $\mathscr{F}|(D(g_i) \cap D(g_j)) = \mathscr{F}|D(g_i g_j)$ satisfy* (d1) *and* (d2)*; then $\mathscr{F}$ has the following two properties:*

(d′1) *For each $f \in A$ and for each section $s \in \Gamma(D(f) \cap V, \mathscr{F})$, there exists an integer $n \geqslant 0$ such that $f^n s$ extends to a section of $\mathscr{F}$ over V.*
(d′2) *For each $f \in A$ and for each section $t \in \Gamma(V, \mathscr{F})$ such that the restriction of t to $D(f) \cap V$ is 0, there exists an integer $n \geqslant 0$ such that $f^n t = 0$.*

We first prove (d′2): since $D(f) \cap D(g_i) = D(f g_i)$, there exists, for each i, an integer n_i such that the restriction of $(f g_i)^{n_i} t$ to $D(g_i)$ is zero: since the image of g_i in A_{g_i} is invertible, the restriction of $f^{n_i} t$ to $D(g_i)$ is also zero; taking n to be the largest of the n_i, we have proved (d′2).

To show (d′1), we apply (d1) to the sheaf $\mathscr{F}|D(g_i)$: there exists an integer $n_i \geqslant 0$ and a section s'_i of $\mathscr{F}$ over $D(g_i)$ extending the restriction of $(f g_i)^{n_i} s$ to $D(f g_i)$; since the image of g_i in A_{g_i} is invertible, there is a section s_i of $\mathscr{F}$ over $D(g_i)$ such that $s'_i = g_i^{n_i} s_i$, and s_i extends the restriction of $f^{n_i} s$ to $D(f g_i)$; in addition we can suppose that all the n_i are equal to a single integer n. By construction, the restriction of $s_i - s_j$ to $D(f) \cap D(g_i) \cap D(g_j) = D(f g_i g_j)$ is zero; by (d2) applied to the sheaf $\mathscr{F}|D(g_i g_j)$, there exists an integer $m_{ij} \geqslant 0$ such that the restriction to $D(g_i g_j)$ of $(f g_i g_j)^{m_{ij}}(s_i - s_j)$ is zero; since the image of $g_i g_j$ in $A_{g_i g_j}$ is invertible, the restriction of $f^{m_{ij}}(s_i - s_j)$ to $D(g_i g_j)$ is zero. We can then assume that all the m_{ij} are equal to a single integer m, and so there exists a section $s' \in \Gamma(V, \mathscr{F})$ extending the $f^m s_i$; as a result, this section extends $f^{n+m}s$, hence we have proved (d′1).

It remains to show that (d1) and (d2) imply (a). We first show that (d1) and (d2) imply that these conditions are satisfied for each sheaf $\mathscr{F}|D(g)$, where $g \in A$ is such that $D(g) \subset V$. It is evident for (d1); on the other hand, if $t \in \Gamma(D(g), \mathscr{F})$ is such that its restriction to $D(f) \subset D(g)$ is zero, there exists, by (d1), an integer $m \geqslant 0$ such that $g^m t$ extends to a section s of $\mathscr{F}$ over V; applying (d2), we see that there exists an integer $n \geqslant 0$ such that $f^n g^m t = 0$, and as the image of g in A_g is invertible, $f^n t = 0$. **I | 92**

That being so, since V is quasi-compact, Lemma (1.4.1.1) proves that the conditions (d′1) and (d′2) are satisfied. Consider then the A-module $M = \Gamma(V, \mathscr{F})$, and define a homomorphism of $\widetilde{A}$-modules $u : \widetilde{M} \to j_*(\mathscr{F})$, where j is the canonical injection $V \to X$. Since the $D(f)$ form a basis for the topology of X, it suffices, for each $f \in A$, to define a homomorphism $u_f : M_f \to \Gamma(D(f), j_*(\mathscr{F})) = \Gamma(D(f) \cap V, \mathscr{F})$, with the usual compatibility conditions (**0**, 3.2.5). Since the canonical image of f in A_f is invertible, the restriction homomorphism $M = \Gamma(V, \mathscr{F}) \to \Gamma(D(f) \cap V, \mathscr{F})$ factors as $M \to M_f \xrightarrow{u_f} \Gamma(D(f) \cap V, \mathscr{F})$ (**0**, 1.2.4), and the verification of these compatibility conditions for $D(g) \subset D(f)$ is immediate. This being so, we show that the condition (d′1) (resp. (d′2)) implies that each of the u_f are surjective (resp. injective), which proves that u is *bijective*, and as a result that $\mathscr{F}$ is the restriction to V of an $\widetilde{A}$-module isomorphic to $\widetilde{M}$. If $s \in \Gamma(D(f) \cap V, \mathscr{F})$, there exists, by (d′1), an integer $n \geqslant 0$ such that $f^n s$ extends to a section $z \in M$; we then have $u_f(z/f^n) = s$, so u_f is surjective. Similarly, if $z \in M$ is such that $u_f(z/1) = 0$, this means that the restriction to $D(f) \cap V$ of the section z is zero; according to (d′2), there exists an integer $n \geqslant 0$ such that $f^n z = 0$, hence $z/1 = 0$ in M_f, and so u_f is injective. □

Corollary (1.4.2). — *Each quasi-coherent sheaf over a quasi-compact open subset of X is induced by a quasi-coherent sheaf on X.*

Corollary (1.4.3). — *Every quasi-coherent $\mathscr{O}_X$-algebra over $X = \operatorname{Spec}(A)$ is isomorphic to an $\mathscr{O}_X$-algebra of the form $\widetilde{B}$, where B is an algebra over A; every quasi-coherent $\widetilde{B}$-module is isomorphic to a $\widetilde{B}$-module of the form $\widetilde{N}$, where N is a B-module.*

PROOF. Indeed, a quasi-coherent $\mathscr{O}_X$-algebra is a quasi-coherent $\mathscr{O}_X$-module, and therefore of the form $\widetilde{B}$, where B is an A-module; the fact that B is an A-algebra follows from the characterization of the structure of an $\mathscr{O}_X$-algebra using the homomorphism $\widetilde{B} \otimes_{\widetilde{A}} \widetilde{B} \to \widetilde{B}$ of $\widetilde{A}$-modules, as well as Corollary (1.3.12). If $\mathscr{G}$ is a quasi-coherent $\widetilde{B}$-module, it suffices to show, in a similar way, that it is also a quasi-coherent $\widetilde{A}$-module to conclude the proof; since the question is local, we can, by restricting to an open subset of X of the form $D(f)$, assume that $\mathscr{G}$ is the cokernel of a homomorphism $\widetilde{B}^{(I)} \to \widetilde{B}^{(J)}$ of $\widetilde{B}$-modules (and *a fortiori* of $\widetilde{A}$-modules); the proposition then follows from Corollaries (1.3.8) and (1.3.9). □

1.5. Coherent sheaves over a prime spectrum

Theorem (1.5.1). — *Let A be a* Noetherian *ring, $X = \operatorname{Spec}(A)$ its prime spectrum, V an open subset of X, and $\mathscr{F}$ an $(\mathscr{O}_X|V)$-module. The following conditions are equivalent.*

(a) *$\mathscr{F}$ is coherent.*
(b) *$\mathscr{F}$ is of finite type and quasi-coherent.*
(c) *There exists an A-module M of finite type such that $\mathscr{F}$ is isomorphic to the sheaf $\widetilde{M}|V$.*

PROOF. (a) trivially implies (b). To see that (b) implies (c), note that, since V is quasi-compact (**0**, 2.2.3), we have previously seen that $\mathscr{F}$ is isomorphic to a sheaf $\widetilde{N}|V$, where N is an A-module (1.4.1). We have $N = \varinjlim M_\lambda$, where M_λ run over the set of A-submodules of N of finite type, hence (1.3.9) $\mathscr{F} = \widetilde{N}|V = \varinjlim \widetilde{M_\lambda}|V$; but since $\mathscr{F}$ is of finite type, and V is quasi-compact, there exists an index λ such that $\mathscr{F} = \widetilde{M_\lambda}|V$ (**0**, 5.2.3). **I | 93**

Finally, we show that (c) implies (a). It is clear that $\mathscr{F}$ is then of finite type ((1.3.6) and (1.3.9)); in addition, since the questions is local, we can restrict to the case where $V = D(f)$, $f \in A$. Since A_f is Noetherian, we see that it suffices to prove that the kernel of a homomorphism $\widetilde{A^n} \to \widetilde{M}$, where M is an A-module, is of finite type. But such a homomorphism is of the form $\widetilde{u}$, where u is a homomorphism $A^n \to M$ (1.3.8), and if $P = \operatorname{Ker} u$ then we have $\widetilde{P} = \operatorname{Ker} \widetilde{u}$ (1.3.9). Since A is Noetherian, P is of finite type, which finishes the proof. □

Corollary (1.5.2). — *Under the hypotheses of (1.5.1), the sheaf $\mathscr{O}_X$ is a quasi-coherent sheaf of rings.*

Corollary (1.5.3). — *Under the hypotheses of (1.5.1), every coherent sheaf over an open subset of X is induced by a coherent sheaf on X.*

Corollary (1.5.4). — *Under the hypotheses of (1.5.1), every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ is the inductive limit of the coherent $\mathscr{O}_X$-submodules of $\mathscr{F}$.*

PROOF. Indeed, $\mathscr{F} = \widetilde{M}$, where M is an A-module, and M is the inductive limit of its submodules of finite type; we conclude the proof by appealing to (1.3.9) and (1.5.1). □

1.6. Functorial properties of quasi-coherent sheaves over a prime spectrum

(1.6.1). Let A, A' be rings,

$$\phi : A' \longrightarrow A$$

a homomorphism, and

$$^a\phi : X = \operatorname{Spec}(A) \longrightarrow X' = \operatorname{Spec}(A')$$

the continuous map associated to ϕ (1.2.1). We will define a *canonical homomorphism*

$$\widetilde{\phi} : \mathscr{O}_{X'} \longrightarrow {}^a\phi_*(\mathscr{O}_X)$$

of sheaves of rings. For each $f' \in A'$, we put $f = \phi(f')$; we have $^a\phi^{-1}(D(f')) = D(f)$ (1.2.2.2). The rings $\Gamma(D(f'), \widetilde{A'})$ and $\Gamma(D(f), \widetilde{A})$ are identified with $A'_{f'}$ and A_f (respectively) ((1.3.6) and (1.3.7)). The homomorphism ϕ canonically defines a homomorphism $\phi_{f'} : A'_{f'} \to A_f$ (**0**, 1.5.1), in other words, we have a homomorphism of rings

$$\Gamma(D(f), \widetilde{A'}) \longrightarrow \Gamma({}^a\phi^{-1}(D(f')), \widetilde{A}) = \Gamma(D(f'), {}^a\phi_*(\widetilde{A})).$$

In addition, these homomorphisms satisfy the usual compatibility conditions: for $D(f') \supset D(g')$, the diagram **I | 94**

$$\begin{tikzcd} \Gamma(D(f'), \widetilde{A'}) \arrow[r] \arrow[d] & \Gamma(D(f'), {}^a\phi_*(\widetilde{A})) \arrow[d] \\ \Gamma(D(g'), \widetilde{A'}) \arrow[r] & \Gamma(D(g'), {}^a\phi_*(\widetilde{A}) \end{tikzcd}$$

is commutative (**0**, 1.5.1); we have thus defined a homomorphism of $\mathscr{O}_{X'}$-algebras, as the $D(f')$ form a basis for the topology of X' (**0**, 3.2.3). The pair $\Phi = ({}^a\phi, \widetilde{\phi})$ is thus a *morphism* of ringed spaces

$$\Phi : (X, \mathscr{O}_X) \longrightarrow (X', \mathscr{O}_{X'}),$$

(**0**, 4.1.1).

We also note that, if we put $x' = {}^a\phi(x)$, then the homomorphism $\widetilde{\phi}_x^\sharp$ (**0**, 3.7.1) is exactly the homomorphism

$$\phi_x : A'_{x'} \longrightarrow A_x$$

canonically induced by $\phi : A' \to A$ (**0**, 1.5.1). Indeed, each $z' \in A'_{x'}$ can be written as g'/f', where f', g' are in A' and $f' \notin \mathfrak{j}_{x'}$; $D(f')$ is then a neighbourhood of x' in X', and the homomorphism $\Gamma(D(f'), \widetilde{A'}) \to \Gamma({}^a\phi^{-1}(D(f')), \widetilde{A})$ induced by $\widetilde{\phi}$ is exactly $\phi_{f'}$; by considering the section $s' \in \Gamma(D(f'), \widetilde{A'})$ corresponding to $g'/f' \in A'_{f'}$, we obtain $\widetilde{\phi}_x^\sharp(z') = \phi(g')/\phi(f')$ in A_x.

Example (1.6.2). — Let S be a multiplicative subset of A, and ϕ the canonical homomorphism $A \to S^{-1}A$; we have already seen (1.2.6) that $^a\phi$ is a *homeomorphism* from $Y = \operatorname{Spec}(S^{-1}A)$ to the subspace of $X = \operatorname{Spec}(A)$ consisting of the x such that $\mathfrak{j}_x \cap S = \varnothing$. In addition, for each x in this subspace, which is thus of the form $^a\phi(y)$ with $y \in Y$, the homomorphism $\widetilde{\phi}_y^\sharp : \mathscr{O}_x \to \mathscr{O}_y$ is *bijective* (**0**, 1.2.6); in other words, $\mathscr{O}_Y$ is identified with the sheaf on Y induced by $\mathscr{O}_X$.

Proposition (1.6.3). — *For every A-module M, there exists a canonical functorial isomorphism from the $\mathscr{O}_{X'}$-module $(M_{[\phi]})^\sim$ to the direct image $\Phi_*(\widetilde{M})$.*

PROOF. For purposes of abbreviation, we write $M' = M_{[\phi]}$, and for each $f' \in A'$, we put $f = \phi(f')$. The modules of sections $\Gamma(D(f'), \widetilde{M'})$ and $\Gamma(D(f), \widetilde{M})$ are identified, respectively, with the modules $M'_{f'}$ and M_f (over $A'_{f'}$ and A_f, respectively); in addition, the $A'_{f'}$-module $(M_f)_{[\phi_{f'}]}$ is canonically isomorphic to $M'_{f'}$ (0, 1.5.2). We thus have a functorial isomorphism of $\Gamma(D(f'), \widetilde{A'})$-modules: $\Gamma(D(f'), \widetilde{M'}) \simeq \Gamma({}^a\phi^{-1}(D(f')), \widetilde{M})_{[\phi_{f'}]}$ and these isomorphisms satisfy the usual compatibility conditions with the restrictions (0, 1.5.6), thus defining the desired functorial isomorphism. We note that, in a precise way, if $u : M_1 \to M_2$ is a homomorphism of A-modules, it can be considered as a homomorphism $(M_1)_{[\phi]} \to (M_2)_{[\phi]}$ of A'-modules; if we denote this homomorphism by $u_{[\phi]}$, then $\Phi_*(\widetilde{u})$ is identified with $(u_{[\phi]})^\sim$. □

This proof also shows that, for each A-*algebra* B, the canonical functorial isomorphism $(B_{[\phi]})^\sim \simeq$ I | 95
$\Phi_*(\widetilde{B})$ is an isomorphism of $\mathscr{O}_{X'}$-*algebras*; if M is a B-module, the canonical functorial isomorphism $(M_{[\phi]})^\sim \simeq \Phi_*(\widetilde{M})$ is an isomorphism of $\Phi_*(\widetilde{B})$-modules.

Corollary (1.6.4). — *The direct image functor Φ_* is exact on the category of quasi-coherent $\mathscr{O}_X$-modules.*

PROOF. Indeed, it is clear that $M_{[\phi]}$ is an exact functor in M and $\widetilde{M'}$ is an exact functor in M' (1.3.5). □

Proposition (1.6.5). — *Let N' be an A'-module, and N the A-module $N' \otimes_{A'} A_{[\phi]}$; then there exists a canonical functorial isomorphism from the $\mathscr{O}_X$-module $\Phi^*(\widetilde{N'})$ to $\widetilde{N}$.*

PROOF. We first remark that $j : z' \mapsto z' \otimes 1$ is an A'-homomorphism from N' to $N_{[\phi]}$: indeed, by definition, for $f' \in A'$, we have $(f'z') \otimes 1 = z' \otimes \phi(f') = \phi(f')(z' \otimes 1)$. We have (1.3.8) a homomorphism $\widetilde{j} : \widetilde{N'} \to (N_{[\phi]})^\sim$ of $\mathscr{O}_{X'}$-modules, and, thanks to (1.6.3), we can consider $\widetilde{j}$ as mapping $\widetilde{N'}$ to $\Phi_*(\widetilde{N})$. There canonically corresponds to this homomorphism $\widetilde{j}$ a homomorphism $h = \widetilde{j}^\sharp$ from $\Phi^*(\widetilde{N'})$ to $\widetilde{N}$ (0, 4.4.3); we will see that, for each stalk, h_x is *bijective*. Put $x' = {}^a\phi(x)$ and let $f' \in A'$ be such that $x' \in D(f')$; let $f = \phi(f')$. The ring $\Gamma(D(f), \widetilde{A})$ is identified with A_f, the modules $\Gamma(D(f), \widetilde{N})$ and $\Gamma(D(f'), \widetilde{N'})$ with N_f and $N'_{f'}$ (respectively); let $s \in \Gamma(D(f'), \widetilde{N'})$, identified with n'/f'^p ($n' \in N'$), and s be its image under $\widetilde{j}$ in $\Gamma(D(f), \widetilde{N})$; s is identified with $(n' \otimes 1)/f^p$. On the other hand, let $t \in \Gamma(D(f), \widetilde{A})$, identified with g/f^q ($g \in A$); then, by definition, we have $h_x(s'_x \otimes t_x) = t_x \cdot s_x$ (0, 4.4.3). But we can canonically identify N_f with $N'_{f'} \otimes_{A'_{f'}} (A_f)_{[\phi_{f'}]}$ (0, 1.5.4); s then corresponds to the element $(n'/f'^p) \otimes 1$, and the section $y \mapsto t_y \cdot s_y$ with $(n'/f'^p) \otimes (g/f^q)$. The compatibility diagram of (0, 1.5.6) show that h_x is exactly the canonical isomorphism

$$N'_{x'} \otimes_{A'_{x'}} (A_x)_{[\phi_{x'}]} \simeq N_x = (N' \otimes_{A'} A_{[\phi]})_x. \tag{1.6.5.1}$$

In addition, let $v : N'_1 \to N'_2$ be a homomorphism of A'-modules; since $\widetilde{v}_{x'} = v_{x'}$ for each $x' \in X'$, it follows immediately from the above that $\Phi^*(\widetilde{v})$ is canonically identified with $(v \otimes 1)^\sim$, which finishes the proof of (1.6.5). □

If B' is an A'-algebra, the canonical isomorphism from $\Phi^*(\widetilde{B'})$ to $(B' \otimes_{A'} A_{[\phi]})^\sim$ is an isomorphism of $\mathscr{O}_X$-algebras; if, in addition, N' is a B'-module, then the canonical isomorphism from $\Phi^*(\widetilde{N'})$ to $(N' \otimes_{A'} A_{[\phi]})^\sim$ is an isomorphism of $\Phi^*(\widetilde{B'})$-modules.

Corollary (1.6.6). — *The sections of $\Phi^*(\widetilde{N'})$, the canonical images of the sections s', where s' varies over the A'-module $\Gamma(\widetilde{N'})$, generate the A-module $\Gamma(\Phi^*(N'))$.*

PROOF. Indeed, these images are identified with the elements $z' \otimes 1$ of N, when we identify N' and N with $\Gamma(\widetilde{N'})$ and $\Gamma(\widetilde{N})$ (respectively) (1.3.7), and z' varies over N'. □

(1.6.7). In the proof of (1.6.5), we had proved in passing that the canonical map (**0**, 4.4.3.2) $\rho : \widetilde{N'} \to \Phi_*(\Phi^*(\widetilde{N'}))$ is exactly the homomorphism $\widetilde{j}$, where $j : N' \to N' \otimes_{A'} A_{[\phi]}$ is the homomorphism I | 96
$z' \mapsto z' \otimes 1$. Similarly, the canonical map (**0**, 4.4.3.3) $\sigma : \Phi^*(\Phi_*(\widetilde{M})) \to \widetilde{M}$ is exactly $\widetilde{p}$, where $p : M_{[\phi]} \otimes_{A'} A_{[\phi]} \to M$ is the canonical homomorphism, which sends each tensor product $z \otimes a$ ($z \in M$, $a \in A$) to $a \cdot z$; this follows immediately from the definitions ((0, 3.7.1), (0, 4.4.3), and (1.3.7)).

We conclude ((**0**, 4.4.3) and (**0**, 3.5.4.4)) that if $v : N' \to M_{[\phi]}$ is an A'-homomorphism, then $\widetilde{v}^\sharp = (v \otimes 1)^\sim$.

(1.6.8). Let N'_1 and N'_2 be A'-modules, and assume N'_1 admits a *finite presentation*; it then follows from (1.6.7) and (1.3.12, ii) that the canonical homomorphism (**0**, 4.4.6)

$$\Phi^*(\mathscr{H}om_{\widetilde{A'}}(\widetilde{N'_1}, \widetilde{N'_2})) \longrightarrow \mathscr{H}om_{\widetilde{A}}(\Phi^*(\widetilde{N'_1}), \Phi^*(\widetilde{N'_2}))$$

is exactly $\widetilde{\gamma}$, where γ denotes the canonical homomorphism of A-modules $\mathrm{Hom}_{A'}(N'_1, N'_2) \otimes_{A'} A \to \mathrm{Hom}_A(N'_1 \otimes_{A'} A, N'_2 \otimes_{A'} A)$.

(1.6.9). Let $\mathfrak{J}'$ be an ideal of A', and M an A-module; since, by definition, $\widetilde{\mathfrak{J}}'\widetilde{M}$ is the image of the canonical homomorphism $\Phi^*(\widetilde{\mathfrak{J}'}) \otimes_{\widetilde{A}} \widetilde{M} \to \widetilde{M}$, it follows from Proposition (1.6.5) and Corollary (1.3.12, i) that $\widetilde{\mathfrak{J}}'\widetilde{M}$ canonically identifies with $(\mathfrak{J}'M)^\sim$; in particular, $\Phi^*(\widetilde{\mathfrak{J}'})\widetilde{A}$ is identified with $(\mathfrak{J}'A)^\sim$, and, taking the right exactness of the functor Φ^* into account, the $\widetilde{A}$-algebra $\Phi^*((A'/\mathfrak{J}')^\sim)$ is identified with $(A/\mathfrak{J}'A)^\sim$.

(1.6.10). Let A'' be a third ring, ϕ' a homomorphism $A'' \to A'$, and write $\phi'' = \phi \circ \phi'$. It follows immediately from the definitions that ${}^a\phi'' = ({}^a\phi') \circ ({}^a\phi)$, and $\widetilde{\phi''} = \widetilde{\phi} \circ \widetilde{\phi'}$ (**0**, 1.5.7). We conclude that $\Phi'' = \Phi' \circ \Phi$; in other words, $(\mathrm{Spec}(A), \widetilde{A})$ is a *functor* from the category of rings to that of ringed spaces.

1.7. Characterization of morphisms of affine schemes

Definition (1.7.1). — We say that a ringed space $(X, \mathscr{O}_X)$ is an *affine scheme* if it is isomorphic to a ringed space of the form $(\mathrm{Spec}(A), \widetilde{A})$, where A is a ring; we then say that $\Gamma(X, \mathscr{O}_X)$, which is canonically identified with the ring A (1.3.7), is the ring of the affine scheme $(X, \mathscr{O}_X)$, and we denote it by $A(X)$ when there is no chance of confusion.

By abuse of language, when we speak of an *affine scheme* $\mathrm{Spec}(A)$; it will always be the ringed space $(\mathrm{Spec}(A), \widetilde{A})$.

(1.7.2). Let A and B be rings, and $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ the affine schemes corresponding to the prime spectra $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$. We have seen (1.6.1) that each ring homomorphism $\phi : B \to A$ corresponds to a morphism $\Phi = ({}^a\phi, \widetilde{\phi}) = \mathrm{Spec}(\phi) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$. We note that ϕ is entirely determined by Φ, since we have, by definition, $\phi = \Gamma(\widetilde{\phi}) : \Gamma(\widetilde{B}) \to \Gamma({}^a\phi_*(\widetilde{A}) = \Gamma(\widetilde{A})$.

Theorem (1.7.3). — [2] *Let $(X, \mathscr{O}_X)$ $(Y, \mathscr{O}_Y)$ be affine schemes. For a morphism of ringed spaces $(\psi, \theta) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ to be of the form $({}^a\phi, \widetilde{\phi})$, where ϕ is a homomorphism of rings $A(Y) \to A(X)$, it is necessary and sufficient that, for each $x \in X$, $\theta_x^\sharp$ is a local homomorphism: $\mathscr{O}_{\psi(x)} \to \mathscr{O}_x$.*

Proof. Let $A = A(X)$, $B = A(Y)$. The condition is necessary, since we saw (1.6.1) that $\widetilde{\phi}_x^\sharp$ is the homomorphism from $B_{{}^a\phi(x)}$ to A_x canonically induced by ϕ, and, by definition, of ${}^a\phi(x) = \phi^{-1}(\mathfrak{j}_x)$, this homomorphism is local. **I** | 97

We now prove that the condition is sufficient. By definition, θ is a homomorphism $\mathscr{O}_Y \to \psi_*(\mathscr{O}_X)$, and we canonically obtain a ring homomorphism

$$\phi = \Gamma(\theta) : B = \Gamma(Y, \mathscr{O}_Y) \longrightarrow \Gamma(Y, \psi_*(\mathscr{O}_X)) = \Gamma(X, \mathscr{O}_X) = A.$$

The hypotheses on $\theta_x^\sharp$ mean that this homomorphism induces, by passing to quotients, a monomorphism θ^x from the residue field $k(\psi(x))$ to the residue field $k(x)$, such that, for each section $f \in \Gamma(Y, \mathscr{O}_Y) = B$, we have $\theta^x(f(\psi(x))) = \phi(f)(x)$. The relation $f(\psi(x)) = 0$ is therefore equivalent to $\phi(f)(x) = 0$, which means that $\mathfrak{j}_{\psi(x)} = \mathfrak{j}_{{}^a\phi(x)}$, and we now write $\psi(x) = {}^a\phi(x)$ for each $x \in X$, or $\psi = {}^a\phi$. We also know that the diagram

$$\begin{tikzcd}
B = \Gamma(Y, \mathscr{O}_Y) \arrow[r, "\phi"] \arrow[d] & \Gamma(X, \mathscr{O}_X) = A \arrow[d] \\
B_{\psi(x)} \arrow[r, "\theta_x^\sharp"] & A_x
\end{tikzcd}$$

[2] *[Trans.] See (1.8) and the footnote there.*

is commutative (**0**, 3.7.2), which means that $\theta_x^\sharp$ is equal to the homomorphism $\phi_x : B_{\psi(x)} \to A_x$ canonically induced by ϕ (**0**, 1.5.1). As the data of the $\theta_x^\sharp$ completely characterize $\theta^\sharp$, and as a result θ (**0**, 3.7.1), we conclude that we have $\theta = \widetilde{\phi}$, by the definition of $\widetilde{\phi}$ (1.6.1). □

We say that a morphism (ψ, θ) of ringed spaces satisfying the condition of (1.7.3) is a *morphism of affine schemes*.

Corollary (1.7.4). — *If $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ are affine schemes, there exists a canonical isomorphism from the set of morphisms of affine schemes $\operatorname{Hom}((X, \mathscr{O}_X), (Y, \mathscr{O}_Y))$ to the set of ring homomorphisms from B to A, where $A = \Gamma(\mathscr{O}_X)$ and $B = \Gamma(\mathscr{O}_Y)$.*

Furthermore, we can say that the functors $(\operatorname{Spec}(A), \widetilde{A})$ in A and $\Gamma(X, \mathscr{O}_X)$ in $(X, \mathscr{O}_X)$ define an *equivalence* between the category of commutative rings and the opposite category of affine schemes (T, I, 1.2).

Corollary (1.7.5). — *If $\phi : B \to A$ is surjective, then the corresponding morphism $({}^a\phi, \widetilde{\phi})$ is a monomorphism of ringed spaces* (cf. (4.1.7)).

Proof. Indeed, we know that ${}^a\phi$ is injective (1.2.5), and, since ϕ is surjective, for each $x \in X$, $\phi_x^\sharp : B_{{}^a\phi(x)} \to A_x$, which is induced by ϕ by passing to rings of fractions, is also surjective (**0**, 1.5.1); hence the conclusion (**0**, 4.1.1). □

1.8. Morphisms from locally ringed spaces to affine schemes Due to a remark by J. Tate, the statements of Theorem (1.7.3) and Proposition (2.2.4) can be generalized as follows:[3] **II** | 217

Proposition (1.8.1). — *Let $(S, \mathscr{O}_S)$ be an affine scheme, and $(X, \mathscr{O}_X)$ a locally ringed space. Then there is a canonical bijection from the set of ring homomorphisms $\Gamma(S, \mathscr{O}_S) \to \Gamma(X, \mathscr{O}_X)$ to the set of morphisms* **II** | 218 *of ringed spaces $(\psi, \theta) : (X, \mathscr{O}_X) \to (S, \mathscr{O}_S)$ such that, for each $x \in X$, $\theta_x^\sharp$ is a local homomorphism $\mathscr{O}_{\psi(x)} \to \mathscr{O}_x$.*

Proof. We note first that if $(X, \mathscr{O}_X)$ and $(S, \mathscr{O}_S)$ are any two ringed spaces, then a morphism (ψ, θ) from $(X, \mathscr{O}_X)$ to $(S, \mathscr{O}_S)$ canonically defines a ring homomorphism $\Gamma(\theta) : \Gamma(S, \mathscr{O}_S) \to \Gamma(X, \mathscr{O}_X)$, hence a first map

$$\rho : \operatorname{Hom}((X, \mathscr{O}_X), (S, \mathscr{O}_S)) \longrightarrow \operatorname{Hom}(\Gamma(S, \mathscr{O}_S), \Gamma(X, \mathscr{O}_X)). \tag{1.8.1.1}$$

Conversely, under the stated hypotheses, we set $A = \Gamma(S, \mathscr{O}_S)$, and consider a ring homomorphism $\phi : A \to \Gamma(X, \mathscr{O}_X)$. For each $x \in X$, it is clear that the set of the $f \in A$ such that $\phi(f)(x) = 0$ is a *prime ideal* of A, since $\mathscr{O}_x/\mathfrak{m}_x = k(x)$ is a field; it is therefore an element of $S = \operatorname{Spec}(A)$, which we denote by ${}^a\phi(x)$. In addition, for each $f \in A$, we have, by definition (**0**, 5.5.2), that ${}^a\phi^{-1}(D(f)) = X_f$, which proves that ${}^a\phi$ is a *continuous map* $X \to S$. We then define a homomorphism

$$\widetilde{\phi} : \mathscr{O}_S \longrightarrow {}^a\phi_*(\mathscr{O}_X)$$

of $\mathscr{O}_S$-modules; for each $f \in A$, we have $\Gamma(D(f), \mathscr{O}_S) = A_f$ (1.3.6); for each $s \in A$, we associate to $s/f \in A_f$ the element $(\phi(s)|X_f)(\phi(f)|X_f)^{-1}$ of $\Gamma(X_f, \mathscr{O}_X) = \Gamma(D(f), {}^a\phi_*(\mathscr{O}_X))$, and we immediately see (by passing from $D(f)$ to $D(fg)$) that this is a well-defined homomorphism of $\mathscr{O}_S$-modules, hence a morphism $({}^a\phi, \widetilde{\phi})$ of ringed spaces. In addition, with the same notation, and setting $y = {}^a\phi(x)$ for brevity, we immediately see (**0**, 3.7.1) that we have $\widetilde{\phi}_x^\sharp(s_y/f_y) = (\phi(s)_x)(\phi(f)_x)^{-1}$; since the relation $s_y \in \mathfrak{m}_y$ is, by definition, equivalent to $\phi(s)_x \in \mathfrak{m}_x$, we see that $\widetilde{\phi}_x^\sharp$ is a *local* homomorphism $\mathscr{O}_y \to \mathscr{O}_x$, and we have thus defined a second map $\sigma : \operatorname{Hom}(\Gamma(S, \mathscr{O}_S), \Gamma(X, \mathscr{O}_X)) \to \mathfrak{L}$, where $\mathfrak{L}$ is the set of the morphisms $(\psi, \theta) : (X, \mathscr{O}_X) \to (S, \mathscr{O}_S)$ such that $\theta_x^\sharp$ is local for each $x \in X$. It remains to prove that σ and ρ (restricted to $\mathfrak{L}$) are inverses of each other; the definition of $\widetilde{\phi}$ immediately shows that $\Gamma(\widetilde{\phi}) = \phi$, and, as a result, that $\rho \circ \sigma$ is the identity. To see that $\sigma \circ \rho$ is the identity, start with a morphism $(\psi, \theta) \in \mathfrak{L}$ and let $\phi = \Gamma(\theta)$; the hypotheses on $\theta_x^\sharp$ mean that this morphism induces, by passing to quotients, a monomorphism $\theta^x : k(\psi(x)) \to k(x)$ such that for each section

[3] *[Trans.] The following section (I.1.8) was added in the errata of EGA II, hence the temporary change in page numbers, which refer to EGA II.*

$f \in A = \Gamma(S, \mathscr{O}_S)$, we have $\theta^x(f(\psi(x))) = \phi(f)(x)$; the equation $f(\phi(x)) = 0$ is therefore equivalent to $\phi(f)(x) = 0$, which proves that ${}^a\phi = \psi$. On the other hand, the definitions imply that the diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & \Gamma(X, \mathscr{O}_X) \arrow[d] \\ A_{\psi(x)} \arrow[r, "\theta_x^\sharp"] & \mathscr{O}_x \end{tikzcd}$$

is commutative, and it is the same for the analogous diagram where $\theta_x^\sharp$ is replaced by $\widetilde{\phi}_x^\sharp$, hence $\widetilde{\phi}_x^\sharp = \theta_x^\sharp$ (**0**, 1.2.4), and, as a result, $\widetilde{\phi} = \theta$. □

(1.8.2). When $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ are *locally* ringed spaces, we will consider the morphisms $(\psi, \theta) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ such that, for each $x \in X$, $\theta_x^\sharp$ is a *local* homomorphism $\mathscr{O}_{\psi(x)} \to \mathscr{O}_x$. Henceforth when we speak of a *morphism of locally ringed spaces*, it will always be a morphism like the above; **II** | 219
with this definition of morphisms, it is clear that the locally ringed spaces form a *category*; for any two objects X and Y of this category, $\mathrm{Hom}(X, Y)$ thus denotes the set of morphisms of locally ringed spaces from X to Y (the set denoted $\mathfrak{L}$ in (1.8.1)); when we consider the set of *morphisms of ringed spaces* from X to Y, we will denote it by $\mathrm{Hom}_{\mathrm{rs}}(X, Y)$ to avoid any confusion. The map (1.8.1.1) is then written as

(1.8.2.1) $$\rho : \mathrm{Hom}_{\mathrm{rs}}(X, Y) \longrightarrow \mathrm{Hom}(\Gamma(Y, \mathscr{O}_Y), \Gamma(X, \mathscr{O}_X))$$

and its restriction

(1.8.2.2) $$\rho' : \mathrm{Hom}(X, Y) \longrightarrow \mathrm{Hom}(\Gamma(Y, \mathscr{O}_Y), \Gamma(X, \mathscr{O}_X))$$

is a *functorial* map in X and Y on the category of locally ringed spaces.

Corollary (1.8.3). — *Let $(Y, \mathscr{O}_Y)$ be a locally ringed space. For Y to be an affine scheme, it is necessary and sufficient that, for each locally ringed space $(X, \mathscr{O}_X)$, the map* (1.8.2.2) *be bijective.*

Proof. Proposition (1.8.1) shows that the condition is necessary. Conversely, if we suppose that the condition is satisfied, and if we put $A = \Gamma(Y, \mathscr{O}_Y)$, then it follows from the hypotheses and from (1.8.1) that the functors $X \mapsto \mathrm{Hom}(X, Y)$ and $X \mapsto \mathrm{Hom}(X, \mathrm{Spec}(A))$, from the category of locally ringed spaces to that of sets, are *isomorphic*. We know that this implies the existence of a canonical isomorphism $X \to \mathrm{Spec}(A)$ (cf. **0**, 8). □

(1.8.4). Let $S = \mathrm{Spec}(A)$ be an affine scheme; denote by (S', A') the ringed space whose underlying space is *a point* and the structure sheaf A' is the (necessarily simple) sheaf on S' defined by the ring A. Let $\pi : S \to S'$ be the unique map from S to S'; on the other hand, we note that, for each open subset U of S, we have a canonical map $\Gamma(S', A') = \Gamma(S, \mathscr{O}_S) \to \Gamma(U, \mathscr{O}_S)$ which thus defines a *π-morphism* $\iota : A' \to \mathscr{O}_S$ of sheaves of rings. We have thus canonically defined a *morphism of ringed spaces* $i = (\pi, \iota) : (S, \mathscr{O}_S) \to (S', A')$. For each A-module M, we denote by M' the simple sheaf on S' defined by M, which is evidently an A'-module. It is clear that $i_*(\widetilde{M}) = M'$ (1.3.7).

Lemma (1.8.5). — *With the notation of* (1.8.4)*, for each A-module M, the canonical functorial $\mathscr{O}_S$-homomorphism* (**0**, 4.4.3.3)

(1.8.5.1) $$i^*(i_*(\widetilde{M})) \longrightarrow \widetilde{M}$$

is an isomorphism.

Proof. Indeed, the two parts of (1.8.5.1) are right exact (the functor $M \mapsto i_*(\widetilde{M})$ evidently being exact) and commute with direct sums; by considering M as the cokernel of a homomorphism $A^{(I)} \to A^{(J)}$, we can reduce to proving the lemma for the case where $M = A$, and it is evident in this case. □

Corollary (1.8.6). — *Let $(X, \mathscr{O}_X)$ be a ringed space, and $u : X \to S$ a morphism of ringed spaces. For each* **II** | 220
A-module M, we have (with the notation of (1.8.4)*) a canonical functorial isomorphism of $\mathscr{O}_X$-modules*

(1.8.6.1) $$u^*(\widetilde{M}) \simeq u^*(i^*(M')).$$

Corollary (1.8.7). — *Under the hypotheses of (1.8.6), we have, for each A-module M and each $\mathscr{O}_X$-module $\mathscr{F}$, a canonical isomorphism, functorial in M and $\mathscr{F}$,*

$$\mathrm{Hom}_{\mathscr{O}_S}(\widetilde{M}, u_*(\mathscr{F})) \simeq \mathrm{Hom}_A(M, \Gamma(X, \mathscr{F})). \tag{1.8.7.1}$$

PROOF. We have, according to (**0**, 4.4.3) and Lemma (1.8.5), a canonical isomorphism of bifunctors

$$\mathrm{Hom}_{\mathscr{O}_S}(\widetilde{M}, u_*(\mathscr{F})) \simeq \mathrm{Hom}_{A'}(M', i_*(u_*(\mathscr{F})))$$

and it is clear that the right hand side is exactly $\mathrm{Hom}_A(M, \Gamma(X, \mathscr{F}))$. We note that the canonical homomorphism (1.8.7.1) sends each $\mathscr{O}_S$-homomorphism $h : \widetilde{M} \to u_*(\mathscr{F})$ (in other words, each u-morphism $\widetilde{M} \to \mathscr{F}$) to the A-homomorphism $\Gamma(h) : M \to \Gamma(S, u_*(\mathscr{F})) = \Gamma(X, \mathscr{F})$. □

(1.8.8). With the notation of (1.8.4), it is clear (**0**, 4.1.1) that each morphism of ringed spaces $(\psi, \theta) : X \to S'$ is equivalent to the data of a ring homomorphism $A \to \Gamma(X, \mathscr{O}_X)$. We can thus interpret Proposition (1.8.1) as defining a canonical bijection $\mathrm{Hom}(X, S) \simeq \mathrm{Hom}(X, S')$ (where we understand that the right-hand side is the collection of morphisms of ringed spaces, since in general A is not a local ring). More generally, if X and Y are locally ringed spaces, and if (Y', A') is the ringed space whose underlying space is a point and whose sheaf of rings A' is the simple sheaf defined by the ring $\Gamma(Y, \mathscr{O}_Y)$, we can interpret (1.8.2.1) as a map

$$\rho : \mathrm{Hom}_{\mathrm{rs}}(X, Y) \longrightarrow \mathrm{Hom}(X, Y'). \tag{1.8.8.1}$$

The result of Corollary (1.8.3) is interpreted by saying that affine schemes are characterized among locally ringed spaces as those for which the restriction of ρ to $\mathrm{Hom}(X, Y)$:

$$\rho' : \mathrm{Hom}(X, Y) \longrightarrow \mathrm{Hom}(X, Y') \tag{1.8.8.2}$$

is *bijective* for *every* locally ringed space X. In the following chapter, we generalize this definition, which allows us to associate to *any* ringed space Z (and not only to a ringed space whose underlying space is a point) a locally ringed space which we will call $\mathrm{Spec}(Z)$; this will be the starting point for a "relative" theory of preschemes over any ringed space, extending the results of Chapter I.

(1.8.9). We can consider the pairs $(X, \mathscr{F})$ consisting of a locally ringed space X and an $\mathscr{O}_X$-module $\mathscr{F}$ as forming a category, a *morphism* in this category being a pair (u, h) consisting of a morphism of locally ringed spaces $u : X \to Y$ and a u-morphism $h : \mathscr{G} \to \mathscr{F}$ of modules; these morphisms (for **II** | 221
$(X, \mathscr{F})$ and $(Y, \mathscr{G})$ fixed) form a set which we denote by $\mathrm{Hom}((X, \mathscr{F}), (Y, \mathscr{G}))$; the map $(u, h) \mapsto (\rho'(u), \Gamma(h))$ is a canonical map

$$\mathrm{Hom}((X, \mathscr{F}), (Y, \mathscr{G})) \longrightarrow \mathrm{Hom}((\Gamma(Y, \mathscr{O}_Y), \Gamma(Y, \mathscr{G})), (\Gamma(X, \mathscr{O}_X), \Gamma(X, \mathscr{F}))) \tag{1.8.9.1}$$

functorial in $(X, \mathscr{F})$ and $(Y, \mathscr{G})$, the right-hand side being the set of di-homomorphisms corresponding to the rings and modules considered (**0**, 1.0.2).

Corollary (1.8.10). — *Let Y be a locally ringed space, and $\mathscr{G}$ an $\mathscr{O}_Y$-module. For Y to be an affine scheme and $\mathscr{G}$ to be a quasi-coherent $\mathscr{O}_Y$-module, it is necessary and sufficient that, for each pair $(X, \mathscr{F})$ consisting of a locally ringed space X and an $\mathscr{O}_X$-module $\mathscr{F}$, the canonical map (1.8.9.1) be bijective.*

We leave the proof, which is modelled on that of (1.8.3), using (1.8.1) and (1.8.7), to the reader.

Remark (1.8.11). — The statements (1.7.3), (1.7.4), and (2.2.4) are particular cases of (1.8.1), as well as the definition in (1.6.1); similarly, (2.2.5) follows from (1.8.7). Corollary (1.8.7) also implies (1.6.3) (and, as a result, (1.6.4)) as a particular case, since if X is an affine scheme and $\Gamma(X, \mathscr{F}) = N$, then the functors $M \mapsto \mathrm{Hom}_{\mathscr{O}_S}(\widetilde{M}, u_*(\widetilde{N}))$ and $M \mapsto \mathrm{Hom}_{\mathscr{O}_S}(\widetilde{M}, (N_{[\phi]})^{\sim})$ (where $\phi : A \to \Gamma(X, \mathscr{O}_X)$ corresponds to u) are isomorphic, by Corollaries (1.8.7) and (1.3.8). Finally, (1.6.5) (and, as a result, (1.6.6)) follow from (1.8.6), and the fact that, for each $f \in A$, the A_f-modules $N' \otimes_{A'} A_f$ and $(N' \otimes_{A'} A)_f$ (with the notation of (1.6.5)) are canonically isomorphic.

§2. Preschemes and morphisms of preschemes

2.1. Definition of preschemes

(2.1.1). Given a ringed space $(X, \mathscr{O}_X)$, we say that an open subset V of X is an *affine open* subset if the ringed space $(V, \mathscr{O}_X|V)$ is an affine scheme (1.7.1).

Definition (2.1.2). — We define a prescheme to be a ringed space $(X, \mathscr{O}_X)$ such that every point of X admits an affine open neighbourhood.

Proposition (2.1.3). — *If $(X, \mathscr{O}_X)$ is a prescheme, then its affine open subsets form a basis for the topology of X.* **I** | 98

Proof. If V is an arbitrary open neighbourhood of $x \in X$, then there exists by hypothesis an open neighbourhood W of x such that $(W, \mathscr{O}_X|W)$ is an affine scheme; we write A to mean its ring. In the space W, $V \cap W$ is an open neighbourhood of x; so there exists some $f \in A$ such that $D(f)$ is an open neighbourhood of x contained inside $V \cap W$ (1.1.10, i). The ringed space $(D(f), \mathscr{O}_X|D(f))$ is thus an affine scheme, isomorphic to A_f (1.3.6), whence the proposition. □

Proposition (2.1.4). — *The underlying space of a prescheme is a Kolmogoroff space.*

Proof. If x and y are two distinct points of a prescheme X, then it is clear that there exists an open neighbourhood of one of these points that does not contain the other if x and y are not in the same affine open subset; and if they are in the same affine open subset, this is a result of (1.1.8). □

Proposition (2.1.5). — *If $(X, \mathscr{O}_X)$ is a prescheme, then every closed irreducible subset of X admits exactly one generic point, and the map $x \mapsto \overline{\{x\}}$ is thus a bijection of X onto its set of closed irreducible subsets.*

Proof. If Y is a closed irreducible subset of X and $y \in Y$, and if U is an affine open neighbourhood of y in X, then $U \cap Y$ is dense in Y, and also irreducible ((**0**, 2.1.1) and (**0**, 2.1.4)); thus, by Corollary (1.1.14), $U \cap Y$ is the closure in U of a point x, and so $Y \subset \overline{U}$ is the closure of x in X. The uniqueness of the generic point of X is a result of Proposition (2.1.4) and of (**0**, 2.1.3). □

(2.1.6). If Y is a closed irreducible subset of X, and y its generic point, then the local ring $\mathscr{O}_y$ (also written $\mathscr{O}_{X/Y}$) is called the *local ring of X along Y*, or the *local ring of Y in X*.

If X itself is irreducible and x its generic point then we say that $\mathscr{O}_x$ is the *ring of rational functions on X* (cf. §7).

Proposition (2.1.7). — *If $(X, \mathscr{O}_X)$ is a prescheme, then the ringed space $(U, \mathscr{O}_X|U)$ is a prescheme for every open subset U.*

Proof. This follows directly from Definition (2.1.2) and Proposition (2.1.3). □

We say that $(U, \mathscr{O}_X|U)$ is the prescheme *induced* on U by $(X, \mathscr{O}_X)$, or the *restriction* of $(X, \mathscr{O}_X)$ to U.

(2.1.8). We say that a prescheme $(X, \mathscr{O}_X)$ is *irreducible* (resp. *connected*) if the underlying space X is irreducible (resp. connected). We say that a prescheme is *integral* if it is *irreducible and reduced* (cf. (5.1.4)). We say that a prescheme $(X, \mathscr{O}_X)$ is *locally integral* if every $x \in X$ admits an open neighbourhood U such that the prescheme induced on U by $(X, \mathscr{O}_X)$ is integral.

2.2. Morphisms of preschemes

Definition (2.2.1). — Given two preschemes, $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$, we define a morphism (of preschemes) from $(X, \mathscr{O}_X)$ to $(Y, \mathscr{O}_Y)$ to be a morphism of ringed spaces (ψ, θ) such that, for all $x \in X$, $\theta_x^\sharp$ is a local homomorphism $\mathscr{O}_{\psi(x)} \to \mathscr{O}_x$.

By passing to quotients, the map $\mathscr{O}_{\psi(x)} \to \mathscr{O}_x$ gives us a monomorphism $\theta^x : k(\psi(x)) \to k(x)$, which lets us consider $k(x)$ as an *extension* of the field $k(\psi(x))$. I | 99

(2.2.2). The composition (ψ'', θ'') of two morphisms (ψ, θ), (ψ', θ') of preschemes is also a morphism of preschemes, since it is given by the formula $\theta''^\sharp = \theta^\sharp \circ \psi^*(\theta'^\sharp)$ (**0**, 3.5.5). From this we conclude that preschemes form a *category*; using the usual notation, we will write $\mathrm{Hom}(X, Y)$ to mean the set of morphisms from a prescheme X to a prescheme Y.

Example (2.2.3). — If U is an open subset of X, then the canonical injection (**0**, 4.1.2) of the induced prescheme $(U, \mathscr{O}_X|U)$ into $(X, \mathscr{O}_X)$ is a morphism of preschemes; it is further a *monomorphism* of ringed spaces (and *a fortiori* a monomorphism of preschemes), which follows rapidly from (**0**, 4.1.1).

Proposition (2.2.4). — [4] *Let $(X, \mathscr{O}_X)$ be a prescheme, and $(S, \mathscr{O}_S)$ an affine scheme associated to a ring A. Then there exists a canonical bijective correspondence between morphisms of preschemes from $(X, \mathscr{O}_X)$ to $(S, \mathscr{O}_S)$ and ring homomorphisms from A to $\Gamma(X, \mathscr{O}_X)$.*

PROOF. First note that, if $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ are two arbitrary ringed spaces, a morphism (ψ, θ) from $(X, \mathscr{O}_X)$ to $(Y, \mathscr{O}_Y)$ canonically defines a ring homomorphism $\Gamma(\theta) : \Gamma(Y, \mathscr{O}_Y) \to \Gamma(Y, \psi_*(\mathscr{O}_X)) = \Gamma(X, \mathscr{O}_X)$. In the case that we consider, everything boils down to showing that any homomorphism $\phi : A \to \Gamma(X, \mathscr{O}_X)$ is of the form $\Gamma(\theta)$ for exactly one θ. Now, by hypothesis, there is a covering (V_α) of X by affine open subsets; by composing ϕ with the restriction homomorphism $\Gamma(X, \mathscr{O}_X) \to \Gamma(V_\alpha, \mathscr{O}_X|V_\alpha)$, we obtain a homomorphism $\phi_\alpha : A \to \Gamma(V_\alpha, \mathscr{O}_X|V_\alpha)$ that corresponds to a unique morphism $(\psi_\alpha, \theta_\alpha)$ from the prescheme $(V_\alpha, \mathscr{O}_X|V_\alpha)$ to $(S, \mathscr{O}_S)$, by Theorem (1.7.3). Furthermore, for each pair of indices (α, β), each point of $V_\alpha \cap V_\beta$ admits an affine open neighbourhood W contained inside $V_\alpha \cap V_\beta$ (2.1.3); it is clear that, by composing ϕ_α and ϕ_β with the restriction homomorphisms to W, we obtain the same homomorphism $\Gamma(S, \mathscr{O}_S) \to \Gamma(W, \mathscr{O}_X|W)$, so, with the equation $(\theta_\alpha^\sharp)_x = (\phi_\alpha)_x$ for all $x \in V_\alpha$ and all α (1.6.1), the restriction to W of the morphisms $(\psi_\alpha, \theta_\alpha)$ and $(\psi_\beta, \theta_\beta)$ coincide. From this we conclude that there is a morphism $(\psi, \theta) : (X, \mathscr{O}_X) \to (S, \mathscr{O}_S)$ of ringed spaces, and only one such that its restriction to each V_α is $(\psi_\alpha, \theta_\alpha)$, and it is clear that this morphism is a morphism of preschemes and such that $\Gamma(\theta) = \phi$.

Let $u : A \to \Gamma(X, \mathscr{O}_X)$ be a ring homomorphism, and $v = (\psi, \theta)$ the corresponding morphism $(X, \mathscr{O}_X) \to (S, \mathscr{O}_S)$. For each $f \in A$, we have that

$$\psi^{-1}(D(f)) = X_{u(f)} \tag{2.2.4.1}$$

with the notation of (**0**, 5.5.2) relative to the locally free sheaf $\mathscr{O}_X$. In fact, it suffices to verify this formula when X itself is affine, and then this is nothing but (1.2.2.2). □

Proposition (2.2.5). — *Under the hypotheses of Proposition (2.2.4), let $\phi : A \to \Gamma(X, \mathscr{O}_X)$ be a ring homomorphism, $f : (X, \mathscr{O}_X) \to (S, \mathscr{O}_S)$ the corresponding morphism of preschemes, $\mathscr{G}$ (resp. $\mathscr{F}$) an $\mathscr{O}_X$-module (resp. $\mathscr{O}_S$-module), and $M = \Gamma(S, \mathscr{F})$. Then there exists a canonical bijective correspondence between f-morphisms $\mathscr{F} \to \mathscr{G}$ (**0**, 4.4.1) and A-homomorphisms $M \to (\Gamma(X, \mathscr{G}))_{[\phi]}$.* I | 100

PROOF. Reasoning as in Proposition (2.2.4), we reduce to the case where X is affine, and the proposition then follows from Proposition (1.6.3) and from Corollary (1.3.8). □

(2.2.6). We say that a morphism of preschemes $(\psi, \theta) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is *open* (resp. *closed*) if, for all open subsets U of X (resp. all closed subsets F of X), $\psi(U)$ is open (resp. $\psi(F)$ is closed) in Y. We say that (ψ, θ) is *dominant* if $\psi(X)$ is dense in Y, and *surjective* if ψ is surjective. We note that these conditions rely only on the continuous map ψ.

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[4] *[Trans.] See (1.8) and the footnote there.*

Proposition (2.2.7). — *Let*

$$f = (\psi, \theta) : (X, \mathscr{O}_X) \longrightarrow (Y, \mathscr{O}_Y)$$

and

$$g = (\psi', \theta') : (Y, \mathscr{O}_Y) \longrightarrow (Z, \mathscr{O}_Z)$$

be morphisms of preschemes.

(i) *If f and g are both open (resp. closed, dominant, surjective), then so is $g \circ f$.*
(ii) *If f is surjective and $g \circ f$ closed, then g is closed.*
(iii) *If $g \circ f$ is surjective, then g is surjective.*

PROOF. Claims (i) and (iii) are evident. Write $g \circ f = (\psi'', \theta'')$. If F is closed in Y then $\psi^{-1}(F)$ is closed in X, so $\psi''(\psi^{-1}(F))$ is closed in Z; but since ψ is surjective, $\psi(\psi^{-1}(F)) = F$, so $\psi''(\psi^{-1}(F)) = \psi'(F)$, which proves (ii). □

Proposition (2.2.8). — *Let $f = (\psi, \theta)$ be a morphism $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$, and (U_α) an open cover of Y. For f to be open (resp. closed, surjective, dominant), it is necessary and sufficient for its restriction to each induced prescheme $(\psi^{-1}(U_\alpha), \mathscr{O}_X|\psi^{-1}(U_\alpha))$, considered as a morphism of preschemes from this induced prescheme to the induced prescheme $(U_\alpha, \mathscr{O}_Y|U_\alpha)$ to be open (resp. closed, surjective, dominant).*

PROOF. The proposition follows immediately from the definitions, taking into account the fact that a subset F of Y is closed (resp. open, dense) in Y if and only if each of the $F \cap U_\alpha$ are closed (resp. open, dense) in U_α. □

(2.2.9). Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be two preschemes; suppose that X (resp. Y) has a finite number of irreducible components X_i (resp. Y_i) $(1 \leqslant i \leqslant n)$; let ξ_i (resp. η_i) be the generic point of X_i (resp. Y_i) (2.1.5). We say that a morphism

$$f = (\psi, \theta) : (X, \mathscr{O}_X) \longrightarrow (Y, \mathscr{O}_Y)$$

is *birational* if, for all i, $\psi^{-1}(\eta_i) = \{\xi_i\}$ and $\theta^\sharp_{\xi_i} : \mathscr{O}_{\eta_i} \to \mathscr{O}_{\xi_i}$ is an *isomorphism*. It is clear that a birational morphism is dominant (**0**, 2.1.8), and thus it is surjective if it is also closed.

Notation (2.2.10). — In all that follows, when there is no risk of confusion, we *suppress* the structure sheaf (resp. the morphism of structure sheaves) from the notation of a prescheme (resp. morphism of preschemes). If U is an open subset of the underlying space X of a prescheme, then whenever we speak of U as a prescheme we always mean the induced prescheme on U.

2.3. Gluing preschemes

(2.3.1). It follows from Definition (2.1.2) that every ringed space obtained by *gluing* preschemes (**0**, 4.1.7) is again a prescheme. In particular, although every prescheme admits, by definition, a cover by affine open sets, we see that every prescheme can actually be obtained by *gluing affine schemes*. **I** | 101

Example (2.3.2). — Let K be a field, $B = K[s]$ and $C = K[t]$ polynomial rings in one indeterminate over K, and define $X_1 = \operatorname{Spec}(B)$ and $X_2 = \operatorname{Spec}(C)$, which are isomorphic affine schemes. In X_1 (resp. X_2), let U_{12} (resp. U_{21}) be the affine open $D(s)$ (resp. $D(t)$) where the ring B_s (resp. C_t) is formed of rational fractions of the form $f(s)/s^m$ (resp. $g(t)/t^n$) with $f \in B$ (resp. $g \in C$). Let u_{12} be the isomorphism of preschemes $U_{21} \to U_{12}$ corresponding (2.2.4) to the isomorphism from B_s to C_t that, to $f(s)/s^m$, associates the rational fraction $f(1/t)/(1/t^m)$. We can glue X_1 and X_2 along U_{12} and U_{21} by using u_{12}, because there is clearly no gluing condition. We later show that the prescheme X obtained in this manner is a particular case of a general method of construction (**II**, 2.4.3). Here we show only that X *is not an affine scheme*; this will follow from the fact that the ring $\Gamma(X, \mathscr{O}_X)$ is *isomorphic* to K, and so its spectrum reduces to a point. Indeed, a section of $\mathscr{O}_X$ over X has a restriction over X_1 (resp. X_2), identified with an affine open of X, that is a polynomial $f(s)$ (resp. $g(t)$), and it follows from the definitions that we should have $g(t) = f(1/t)$, which is only possible if $f = g \in K$.

2.4. Local schemes

(2.4.1). We say that an affine scheme is a *local scheme* if it is the affine scheme associated to a local ring A; there then exists, in $X = \operatorname{Spec}(A)$, a single *closed point* $a \in X$, and for all other $b \in X$ we have that $a \in \overline{\{b\}}$ (1.1.7).

For all preschemes Y and points $y \in Y$, the local scheme $\operatorname{Spec}(\mathscr{O}_y)$ is called the *local scheme of Y at the point y*. Let V be an affine open subset of Y containing y, and B the ring of the affine scheme V; then $\mathscr{O}_y$ is canonically identified with B_y (1.3.4), and the canonical homomorphism $B \to B_y$ thus corresponds (1.6.1) to a morphism of preschemes $\operatorname{Spec}(\mathscr{O}_y) \to V$. If we compose this morphism with the canonical injection $V \to Y$, then we obtain a morphism $\operatorname{Spec}(\mathscr{O}_y) \to Y$ which is *independent* of the affine open subset V (containing y) that we chose: indeed, if V' is some other affine open subset containing y, then there exists a third affine open subset W that contains y and is such that $W \subset V \cap V'$ (2.1.3); we can thus assume that $V \subset V'$, and then if B' is the ring of V', so everything relies on remarking that the diagram

$$\begin{tikzcd} B' \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & \mathscr{O}_y & \end{tikzcd}$$

is commutative (**0**, 1.5.1). The morphism

$$\operatorname{Spec}(\mathscr{O}_y) \longrightarrow Y$$

thus defined is said to be *canonical*.

Proposition (2.4.2). — *Let $(Y, \mathscr{O}_Y)$ be a prescheme; for all $y \in Y$, let (ψ, θ) be the canonical morphism* $(\operatorname{Spec}(\mathscr{O}_y), \widetilde{\mathscr{O}}_y) \to (Y, \mathscr{O}_Y)$*. Then ψ is a homeomorphism from* $\operatorname{Spec}(\mathscr{O}_y)$ *to the subspace S_y of Y given by the z such that $y \in \overline{\{z\}}$* (or, equivalently, the *generalizations* of y (**0**, 2.1.2))*; furthermore, if $z = \psi(\mathfrak{p})$, then $\theta_z^\sharp : \mathscr{O}_z \to (\mathscr{O}_y)_{\mathfrak{p}}$ is an isomorphism; (ψ, θ) is thus a monomorphism of ringed spaces.* I | 102

Proof. Since the unique closed point a of $\operatorname{Spec}(\mathscr{O}_y)$ is contained in the closure of any point of this space, and since $\psi(a) = y$, the image of $\operatorname{Spec}(\mathscr{O}_y)$ under the continuous map ψ is contained in S_y. Since S_y is contained in every affine open containing y, one can consider just the case where Y is an affine scheme; but then this proposition follows from (1.6.2). □

We see (2.1.5) *that there is a bijective correspondence between* $\operatorname{Spec}(\mathscr{O}_y)$ *and the set of closed irreducible subsets of Y containing y.*

Corollary (2.4.3). — *For $y \in Y$ to be the generic point of an irreducible component of Y, it is necessary and sufficient for the only prime ideal of the local ring $\mathscr{O}_y$ to be its maximal ideal* (in other words, for $\mathscr{O}_y$ to be of *dimension zero*).

Proposition (2.4.4). — *Let $(X, \mathscr{O}_X)$ be a local scheme of some ring A, a its unique closed point, and $(Y, \mathscr{O}_Y)$ a prescheme. Every morphism $u = (\psi, \theta) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ then factors uniquely as $X \to \operatorname{Spec}(\mathscr{O}_{\psi(a)}) \to Y$, where the second arrow denotes the canonical morphism, and the first corresponds to a local homomorphism $\mathscr{O}_{\psi(a)} \to A$. This establishes a canonical bijective correspondence between the set of morphisms $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ and the set of local homomorphisms $\mathscr{O}_y \to A$ for $(y \in Y)$.*

Indeed, for all $x \in X$, we have that $a \in \overline{\{x\}}$, so $\psi(a) \in \overline{\{\psi(x)\}}$, which shows that $\psi(X)$ is contained in every affine open subset that contains $\psi(a)$. So it suffices to consider the case where $(Y, \mathscr{O}_Y)$ is an affine scheme of ring B, and then we have that $u = ({}^a\phi, \widetilde{\phi})$, where $\phi \in \operatorname{Hom}(B, A)$ (1.7.3). Further, we have that $\phi^{-1}(\mathfrak{j}_a) = \mathfrak{j}_{\psi(a)}$, and hence that the image under ϕ of any element of $B - \mathfrak{j}_{\psi(a)}$ is invertible in the local ring A; the factorization in the result follows from the universal property of the ring of fractions (**0**, 1.2.4). Conversely, to each local homomorphism $\mathscr{O}_y \to A$ there is a unique corresponding morphism $(\psi, \theta) : X \to \operatorname{Spec}(\mathscr{O}_y)$ such that $\psi(a) = y$ (1.7.3), and, by composing with the canonical morphism $\operatorname{Spec}(\mathscr{O}_y) \to Y$, we obtain a morphism $X \to Y$, which proves the proposition.

(2.4.5). The affine schemes whose ring is a field K have an underlying space that is just a point. If A is a local ring with maximal ideal $\mathfrak{m}$, then each local homomorphism $A \to K$ has kernel equal to $\mathfrak{m}$, and so factors as $A \to A/\mathfrak{m} \to K$, where the second arrow is a monomorphism. The morphisms $\operatorname{Spec}(K) \to \operatorname{Spec}(A)$ thus correspond bijectively to monomorphisms of fields $A/\mathfrak{m} \to K$.

Let $(Y, \mathscr{O}_Y)$ be a prescheme; for each $y \in Y$ and each ideal $\mathfrak{a}_y$ of $\mathscr{O}_y$, the canonical homomorphism $\mathscr{O}_y \to \mathscr{O}_y/\mathfrak{a}_y$ defines a morphism $\operatorname{Spec}(\mathscr{O}_y/\mathfrak{a}_y) \to \operatorname{Spec}(\mathscr{O}_y)$; if we compose this with the canonical morphism $\operatorname{Spec}(\mathscr{O}_y) \to Y$, then we obtain a morphism $\operatorname{Spec}(\mathscr{O}_y/\mathfrak{a}_y) \to Y$, again said to be *canonical*. For $\mathfrak{a}_y = \mathfrak{m}_y$, this says that $\mathscr{O}_y/\mathfrak{a}_y = k(y)$, and so Proposition (2.4.4) says that:

Corollary (2.4.6). — *Let $(X, \mathscr{O}_X)$ be a local scheme whose ring K is a field, ξ the unique point of* I | 103
X, and $(Y, \mathscr{O}_Y)$ a prescheme. Then each morphism $u : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ factors uniquely as $X \to \operatorname{Spec}(k(\psi(\xi))) \to Y$, where the second arrow denotes the canonical morphism, and the first corresponds to a monomorphism $k(\psi(\xi)) \to K$. This establishes a canonical bijective correspondence between the set of morphisms $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ and the set of monomorphisms $k(y) \to K$ (for $y \in Y$).

Corollary (2.4.7). — *For all $y \in Y$, every canonical morphism $\operatorname{Spec}(\mathscr{O}_y/\mathfrak{a}_y) \to Y$ is a monomorphism of ringed spaces.*

PROOF. We have already seen this when $\mathfrak{a}_y = 0$ (2.4.2), and it suffices to apply Corollary (1.7.5). □

Remark. — 2.4.8 Let X be a local scheme, and a its unique closed point. Since every affine open subset containing a is necessarily equal to the whole of X, every *invertible* $\mathscr{O}_X$-module (**0**, 5.4.1) is necessarily *isomorphic to* $\mathscr{O}_X$ (or, as we say, again, *trivial*). This property does not hold in general for an arbitrary affine scheme $\operatorname{Spec}(A)$; we will see in Chapter V that if A is a normal ring then this is true when A is a unique factorisation domain.

2.5. Preschemes over a prescheme

Definition (2.5.1). — Given a prescheme S, we say that the data of a prescheme X and a morphism of preschemes $\phi : X \to S$ defines a prescheme X *over the prescheme* S, or an *S-prescheme*; we say that S is the *base prescheme* of the S-prescheme X. The morphism ϕ is called the *structure morphism* of the S-prescheme X. When S is an affine scheme of ring A, we also say that X endowed with ϕ is a prescheme *over the ring* A (or an *A-prescheme*).

It follows from (2.2.4) that the data of a prescheme over a ring A is equivalent to the data of a prescheme $(X, \mathscr{O}_X)$ whose structure sheaf $\mathscr{O}_X$ is a sheaf of A-*algebras*. *An arbitrary prescheme can always be considered as a* **Z***-prescheme in a unique way*.

If $\phi : X \to S$ is the structure morphism of an S-prescheme X, we say that a point $x \in X$ is *over a point* $s \in S$ if $\phi(x) = s$. We say that X *dominates* S if ϕ is a dominant morphism (2.2.6).

(2.5.2). Let X and Y be S-preschemes; we say that a morphism of preschemes $u : X \to Y$ is a *morphism of preschemes over* S (or an *S-morphism*) if the diagram

$$\begin{tikzcd} X \arrow[rr, "u"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

(where the diagonal arrows are the structure morphisms) is commutative: this ensures that, for all $s \in S$ and $x \in X$ over s, $u(x)$ also lies over s.

It follows immediately from this definition that the composition of any two S-morphisms is an S-morphism; S-preschemes thus form a *category*.

We denote by $\operatorname{Hom}_S(X, Y)$ the set of S-morphisms from an S-prescheme X to an S-prescheme Y; the identity morphism of an S-prescheme X is denoted by 1_X.

When S is an affine scheme of ring A, we will also say *A-morphism* instead of S-morphism.

(2.5.3). If X is an S-prescheme, and $v : X' \to X$ a morphism of preschemes, then the composition I | 104
$X' \to X \to S$ endows X' with the structure of an S-prescheme; in particular, every prescheme induced by an open set U of X can be considered as an S-prescheme by the canonical injection.

If $u : X \to Y$ is an S-morphism of S-preschemes, then the restriction of u to any prescheme induced by an open subset U of X is also an S-morphism $U \to Y$. Conversely, let (U_α) be an open cover of X, and for each α let $u_\alpha : U_\alpha \to Y$ be an S-morphism; if, for all pairs of indices (α, β), the restrictions of u_α and u_β to $U_\alpha \cap U_\beta$ agree, then there exists an S-morphism $X \to Y$, and exactly one such that the restriction to each U_α is u_α.

If $u : X \to Y$ is an S-morphism such that $u(X) \subset V$, where V is an open subset of Y, then u, considered as a morphism from X to V, is also an S-morphism.

(2.5.4). Let $S' \to S$ be a morphism of preschemes; for all S'-preschemes, the composition $X \to S' \to S$ endows X with the structure of an S-prescheme. Conversely, suppose that S' is the induced prescheme of an open subset of S; let X be an S-prescheme and suppose that the structure morphism $f : X \to S$ is such that $f(X) \subset S'$; then we can consider X as an S'-prescheme. In this latter case, if Y is another S-prescheme whose structure morphism sends the underlying space to S', then every S-morphism from X to Y is also an S'-morphism.

(2.5.5). If X is an S-prescheme, with structure morphism $\phi : X \to S$, we define an *S-section of X* to be an S-morphism from S to X, that is to say a morphism of preschemes $\psi : S \to X$ such that $\phi \circ \psi$ is the identity on S. We denote by $\Gamma(X/S)$ the set of S-sections of X.

§3. Products of preschemes

3.1. Sums of preschemes Let (X_α) be any family of preschemes; let X be a topological space which is the *sum* of the underlying spaces X_α; X is then the union of pairwise disjoint open subspaces X'_α, and for each α there is a homomorphism ϕ_α from X_α to X'_α. If we equip each of the X'_α with the sheaf $(\phi_\alpha)_*(\mathscr{O}_{X_\alpha})$, it is clear that X becomes a prescheme, which we call the *sum* of the family of preschemes (X_α) and which we denote $\bigsqcup_\alpha X_\alpha$. If Y is a prescheme, then the map $f \mapsto (f \circ \phi_\alpha)$ is a *bijection* from the set $\operatorname{Hom}(X, Y)$ to the product set $\prod_\alpha \operatorname{Hom}(X_\alpha, Y)$. In particular, if the X_α are S-preschemes, with structure morphisms ψ_α, then X is an S-prescheme by the unique morphism $\psi : X \to S$ such that $\psi \circ \phi_\alpha = \psi_\alpha$ for each α. The sum of two preschemes X and Y is denoted by $X \sqcup Y$. It is immediate that, if $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$, then $X \sqcup Y$ is canonically identified with $\operatorname{Spec}(A \times B)$.

3.2. Products of preschemes

Definition (3.2.1). — Given S-preschemes X and Y, we say that a triple (Z, p_1, p_2), consisting of an S-prescheme Z, and S-morphisms $p_1 : Z \to X$ and $p_2 : Z \to Y$, is a product of the S-preschemes **I** | 105
X and Y, if, for each S-prescheme T, the map $f \mapsto (p_1 \circ f, p_2 \circ f)$ is a bijection from the set of S-morphisms from T to Z, to the set of pairs consisting of an S-morphism $T \to X$ and an S-morphism $T \to Y$ (in other words, a bijection

$$\operatorname{Hom}_S(T, Z) \simeq \operatorname{Hom}_S(T, X) \times \operatorname{Hom}_S(T, Y)).$$

This is the general notion of a *product* of two objects in a category, applied to the category of S-preschemes (T, I, 1.1); in particular, a product of two S-preschemes is *unique* up to a unique S-isomorphism. Because of this uniqueness, most of the time we will denote a product of two S-preschemes X and Y by $X \times_S Y$ (or simply $X \times Y$, when there is no chance of confusion), with the morphisms p_1 and p_2 (the *canonical projections* of $X \times_S Y$ to X and to Y, respectively) being suppressed in the notation. If $g : T \to X$ and $h : T \to Y$ are S-morphisms, we denote by $(g, h)_S$, or simply (g, h), the S-morphism $f : T \to X \times_S Y$ such that $p_1 \circ f = g$, $p_2 \circ f = h$. If X' and Y' are two S-preschemes, p'_1 and p'_2 the canonical projections of $X' \times_S Y'$ (assumed to exist), and $u : X' \to X$ and $v : Y' \to Y$ S-morphisms, then we write $u \times_S v$ (or simply $u \times v$) for the S-morphism $(u \circ p'_1, v \circ p'_2)_S$ from $X' \times_S Y'$ to $X \times_S Y$.

When S is an affine scheme given by some ring A, we often replace S by A is the above notation.

Proposition (3.2.2). — *Let X, Y, and S be affine schemes, given by rings B, C, and A (respectively). Let $Z = \operatorname{Spec}(B \otimes_A C)$, and let p_1 and p_2 be the S-morphisms corresponding (2.2.4) to the canonical A-homomorphisms $u : b \mapsto b \otimes 1$ and $v : c \mapsto 1 \otimes c$ (respectively) from B and C to $B \otimes_A C$; then (Z, p_1, p_2) is a product of X and Y.*

PROOF. According to (2.2.4), it suffices to check that, if, to each A-homomorphism $f : B \otimes_A C \to L$ (where L is an A-algebra), we associate the pair $(f \circ u, f \circ v)$, then this defines a bijection $\mathrm{Hom}_A(B \otimes_A C, L) \simeq \mathrm{Hom}_A(B, L) \times \mathrm{Hom}_A(C, L)$,[5] which follows immediately from the definitions and the fact that $b \otimes c = (b \otimes 1)(1 \otimes c)$. □

Corollary (3.2.3). — *Let T be an affine scheme given by some ring D, and $\alpha = ({}^a\rho, \widetilde{\rho})$ (resp. $\beta = ({}^a\sigma, \widetilde{\sigma})$) an S-morphism $T \to X$ (resp. $T \to Y$), where ρ (resp. σ) is an A-homomorphism from B (resp. C) to D; then $(\alpha, \beta)_S = ({}^a\tau, \widetilde{\tau})$, where τ is the homomorphism $B \otimes_A C \to D$ such that $\tau(b \otimes c) = \rho(b)\sigma(c)$.*

Proposition (3.2.4). — *Let $f : S' \to S$ be a* monomorphism *of preschemes* (T, I, 1.1), *and let X and Y be S'-preschemes, also considered as S-preschemes via f. Every product of the S-preschemes X and Y is then a product of the S'-preschemes X and Y, and vice versa.*

PROOF. Let $\phi : X \to S'$ and $\psi : Y \to S'$ be the structure morphisms. If T is an S-prescheme, and $u : T \to X$ and $v : T \to Y$ are S-morphisms, then we have, by definition, that $f \circ \phi \circ u = f \circ \psi \circ v = \theta$, the structure morphism of T; the hypotheses on f imply that $\phi \circ u = \psi \circ v = \theta'$, and so we can consider T as an S'-prescheme with structure morphism θ', and u and v as S'-morphisms. The conclusion of the proposition follows immediately, taking (3.2.1) into account. □

Corollary (3.2.5). — *Let X and Y be S-preschemes, with structure morphisms $\phi : X \to S$ and $\psi : Y \to S$, and let S' be an open subset of S such that $\phi(X) \subset S'$ and $\psi(Y) \subset S'$. Every product of the S-preschemes X and Y is then also a product of the S'-preschemes X and Y, and conversely.*

It suffices to apply (3.2.4) to the canonical injection $S' \to S$. I | 106

Theorem (3.2.6). — *Given S-preschemes X and Y, there exists a product $X \times_S Y$.*

The proof proceeds in several steps.

Lemma (3.2.6.1). — *Let (Z, p, q) be a product of X and Y, and U and V open subsets of X and Y, respectively. If we let $W = p^{-1}(U) \cap q^{-1}(V)$, then the triple consisting of W and the restrictions of p and q to W* (considered as the morphisms $W \to U$ and $W \to V$, respectively) *is a product of U and V.*

Indeed, if T is an S-prescheme, then we can identify the S-morphisms $T \to W$ with the S-morphisms $T \to Z$ mapping T to W. Then, if $g : T \to U$ and $h : T \to V$ are any two S-morphisms, we can consider them as S-morphisms from T to X and to Y, respectively, and, by hypothesis, there is then a unique S-morphism $f : T \to Z$ such that $g = p \circ f$ and $h = q \circ f$. Since $p(f(Y)) \subset U$, $q(f(T)) \subset V$, we have

$$f(T) \subset p^{-1}(U) \cap q^{-1}(V) = W,$$

hence our claim.

Lemma (3.2.6.2). — *Let Z be an S-prescheme, $p : Z \to X$ and $q : Z \to Y$ both S-morphisms, (U_α) an open cover of X, and (V_λ) an open cover of Y. Suppose that, for each pair (α, λ), the S-prescheme $W_{\alpha\lambda} = p^{-1}(U_\alpha) \cap q^{-1}(V_\lambda)$ and the restrictions of p and q to $W_{\alpha\lambda}$ form a product of U_α and V_λ. Then (Z, p, q) is a product of X and Y.*

We first show that, if f_1 and f_2 are S-morphisms $T \to Z$, then the equations $p \circ f_1 = p \circ f_2$ and $q \circ f_1 = q \circ f_2$ imply that $f_1 = f_2$. Indeed, Z is the union of the $W_{\alpha\lambda}$, so the $f_1^{-1}(W_{\alpha\lambda})$ form an open cover of T, and similarly for $f_2^{-1}(W_{\alpha\lambda})$. In addition, we have

$$f_1^{-1}(W_{\alpha\lambda}) = f_1^{-1}(p^{-1}(U_\alpha)) \cap f_1^{-1}(q^{-1}(V_\lambda)) = f_2^{-1}(p^{-1}(U_\alpha)) \cap f_2^{-1}(q^{-1}(V_\lambda)) = f_2^{-1}(W_{\alpha\lambda})$$

by hypothesis, and it thus reduces to noting that the the restrictions of f_1 and f_2 to $f_1^{-1}(W_{\alpha\lambda}) = f_2^{-1}(W_{\alpha\lambda})$ are identical for each pair of indices. But since these restrictions can be considered as S-morphisms from $f_1^{-1}(W_{\alpha\lambda})$ to $W_{\alpha\lambda}$, our claim follows from the hypotheses and Definition (3.2.1).

Suppose now that we are given S-morphisms $g : T \to X$ and $h : T \to Y$. Let $T_{\alpha\lambda} = g^{-1}(U_\alpha) \cap h^{-1}(V_\lambda)$; then the $T_{\alpha\lambda}$ form an open cover of T. By hypothesis, there exists an S-morphism $f_{\alpha\lambda}$ such that $p \circ f_{\alpha\lambda}$ and $q \circ f_{\alpha\lambda}$ are the restrictions of g and h to $T_{\alpha\lambda}$ (respectively). Now, we will show that

[5] The notation Hom_A denotes here the set of homomorphisms of *A-algebras*.

the restrictions of $f_{\alpha\lambda}$ and $f_{\beta\mu}$ to $T_{\alpha\lambda} \cap T_{\beta\mu}$ coincide, which will finish the proof of Lemma (3.2.6.2). The images of $T_{\alpha\lambda} \cap T_{\beta\mu}$ under $f_{\alpha\lambda}$ and $f_{\beta\mu}$ are contained in $W_{\alpha\lambda} \cap W_{\beta\mu}$ by definition. Since

$$W_{\alpha\lambda} \cap W_{\beta\mu} = p^{-1}(U_\alpha \cap U_\beta) \cap q^{-1}(V_\lambda \cap V_\mu),$$

it follows from Lemma (3.2.6.1) that $W_{\alpha\lambda} \cap W_{\beta\mu}$ and the restrictions to this prescheme of p and q form a *product* of $U_\alpha \cap U_\beta$ and $V_\lambda \cap V_\mu$. Since $p \circ f_{\alpha\lambda}$ and $p \circ f_{\beta\mu}$ coincide on $T_{\alpha\lambda} \cap T_{\beta\mu}$ and similarly for $q \circ f_{\alpha\lambda}$ and $q \circ f_{\beta\mu}$, we see that $f_{\alpha\lambda}$ and $f_{\beta\mu}$ coincide on $T_{\alpha\lambda} \cap T_{\beta\mu}$.

Lemma (3.2.6.3). — *Let (U_α) be an open cover of X, (V_λ) an open cover of Y, and suppose that, for each pair (α, λ), there exists a product of U_α and V_λ; then there exists a product of X and Y.* I | 107

Applying Lemma (3.2.6.1) to the open sets $U_\alpha \cap U_\beta$ and $V_\lambda \cap V_\mu$, we see that there exists a product of S-preschemes induced, respectively, by X and Y on these open sets; in addition, the uniqueness of the product shows that, if we set $i = (\alpha, \lambda)$ and $j = (\beta, \mu)$, then there is a canonical isomorphism h_{ij} (resp. h_{ji}) from this product to an S-prescheme W_{ij} (resp. W_{ji}) induced by $U_\alpha \times_S V_\lambda$ (resp. $U_\beta \times_S V_\mu$) on an open set; then $f_{ij} = h_{ij} \circ h_{ji}^{-1}$ is an isomorphism from W_{ji} to W_{ij}. In addition, for a third pair $k = (\gamma, \nu)$, we have $f_{ik} = f_{ij} \circ f_{jk}$ on $W_{ki} \cap W_{kj}$, by applying Lemma (3.2.6.1) to the open sets $U_\alpha \cap U_\beta \cap U_\gamma$ and $V_\lambda \cap V_\mu \cap V_\nu$ in U_β and V_μ, respectively. It follows that we have a prescheme Z, an open cover (Z_i) of the underlying space of Z, and, for each i, an isomorphism g_i from the induced prescheme Z_i to the prescheme $U_\alpha \times_S V_\lambda$, so that, for each pair (i, j), we have $f_{ij} = g_i \circ g_j^{-1}$ (2.3.1); in addition, we have $g_i(Z_i \cap Z_j) = W_{ij}$. If p_i, q_i, and θ_i are the projections and the structure morphism of the S-prescheme $U_\alpha \times_S V_\lambda$ (respectively), we immediately see that $p_i \circ g_i = p_j \circ g_j$ on $Z_i \cap Z_j$, and similarly for the two other morphisms. We can thus define the morphisms of preschemes $p : Z \to X$ (resp. $q : Z \to Y$, $\theta : Z \to S$) by the condition that p (resp. q, θ) coincide with $p_i \circ g_i$ (resp. $q_i \circ g_i$, $\theta_i \circ g_i$) on each of the Z_i; Z, equipped with θ, is then an S-prescheme. We now show that $Z'_i = p^{-1}(U_\alpha) \cap q^{-1}(V_\lambda)$ is equal to Z_i. For each index $j = (\beta, \mu)$, we have $Z_j \cap Z'_i = g_j^{-1}(p_j^{-1}(U_\alpha) \cap q_j^{-1}(V_\lambda))$. We have, by Lemma (3.2.6.1),

$$p_j^{-1}(U_\alpha) \cap q_j^{-1}(V_\lambda) = p_j^{-1}(U_\alpha \cap U_\beta) \cap q_j^{-1}(V_\lambda \cap V_\mu);$$

with the restrictions of p_j and q_j to $p_j^{-1}(U_\alpha) \cap q_j^{-1}(V_\lambda)$ defining, on this S-prescheme, the structure of a product of $U_\alpha \cap U_\beta$ and $V_\lambda \cap V_\mu$; but the uniqueness of the product then implies that $p_j^{-1}(U_\alpha) \cap q_j^{-1}(V_\lambda) = W_{ji}$. As a result, we have $Z_j \cap Z'_i = Z_j \cap Z_i$ for each j, hence $Z'_i = Z_i$. We then deduce from Lemma (3.2.6.2) that (Z, p, q) is a product of X and Y.

Lemma (3.2.6.4). — *Let $\phi : X \to S$ and $\psi : Y \to S$ be the structure morphisms of X and Y, (S_i) an open cover of S, and let $X_i = \phi^{-1}(S_i)$, $Y_i = \psi^{-1}(S_i)$. If each of the products $X_i \times_S Y_i$ exists, then $X \times_S Y$ exists.*

According to Lemma (3.2.6.3), everything follows from proving that the products $X_i \times_S Y_i$ exists for any i and j. Set $X_{ij} = X_i \cap X_j = \phi^{-1}(S_i \cap S_j)$, $Y_{ij} = Y_i \cap Y_j = \psi^{-1}(S_i \cap S_j)$; by Lemma (3.2.6.1), the product $Z_{ij} = X_{ij} \times_S Y_{ij}$ exists. We now note that, if T is an S-prescheme, and if $g : T \to X_i$ and $h : T \to Y_j$ are S-morphisms, then we necessarily have that $\phi(g(T)) = \psi(h(T)) \subset S_i \cap S_j$ by the definition of an S-morphism, and thus that $g(T) \subset X_{ij}$ and $h(T) \subset Y_{ij}$; it is then immediate that Z_{ij} is the product of X_i and Y_j.

(3.2.6.5). We can now complete the proof of Theorem (3.2.6). If S is an *affine scheme*, then there are covers (U_α) and (V_λ) of X and Y (respectively) consisting of affine open subsets; since $U_\alpha \times_S V_\lambda$ exists, by (3.2.2), $X \times_S Y$ exists similarly, by Lemma (3.2.6.3). If S is any prescheme, then there is a cover (S_i) of S consisting of affine open subsets. If $\phi : X \to S$ and $\psi : Y \to S$ are the structure morphisms, and if we set $X_i = \phi^{-1}(S_i)$ and $Y_i = \psi^{-1}(S_i)$, then the products $X_i \times_{S_i} Y_i$ exist, by the above; but then the products $X_i \times_S Y_i$ also exist (3.2.5), therefore $X \times_S Y$ exists similarly, by I | 108
Lemma (3.2.6.4). □

Corollary (3.2.7). — *Let $Z = X \times_S Y$ be the product of two S-preschemes, p and q the projections from Z to X and to Y (respectively),and ϕ (resp. ψ) the structure morphism of X (resp. Y). Let S' be an open subset of S, and U (resp. V) an open subset of X (resp. Y) contained in $\phi^{-1}(S')$ (resp. $\psi^{-1}(S')$). Then the product*

$U \times_{S'} V$ is canonically identified with the prescheme induced on Z by $p^{-1}(U) \cap q^{-1}(V)$ (considered as an S'-prescheme). In addition, if $f : T \to X$ and $g : T \to Y$ are S-morphisms such that $f(T) \subset U$ and $g(T) \subset V$, then the S'-morphism $(f,g)_{S'}$ can be identified with the restriction of $(f,g)_S$ to $p^{-1}(U) \cap q^{-1}(V)$.

PROOF. This follows from Corollary (3.2.5) and Lemma (3.2.6.1). ☐

(3.2.8). Let (X_α) and (Y_λ) be families of S-preschemes, and X (resp. Y) the sum of the family (X_α) (resp. (Y_λ)) (3.1). Then $X \times_S Y$ can be identified with the *sum* of the family $(X_\alpha \times_S Y_\lambda)$; this follows immediately from Lemma (3.2.6.3).

(3.2.9). [6] It follows from (1.8.1) that we can state (3.2.2) in the following manner: $Z = \operatorname{Spec}(B \otimes_A C)$ is not only a product of $X = \operatorname{Spec}(B)$ and $Y = \operatorname{Spec}(C)$ in the category of *S-preschemes*, but also in the category of *locally ringed spaces over S* (with a definition of S-morphisms modelled on that of (2.5.2)). The proof of (3.2.6) also proves that, for any two S-preschemes X and Y, the prescheme $X \times_S Y$ is not only the product of X and Y in the category of S-preschemes, but also in the category of locally ringed spaces over the prescheme S. **II** | 221

3.3. Formal properties of the product; change of the base prescheme

(3.3.1). The reader will notice that all the properties stated in this section, except (3.3.13) and (3.3.15), are true without modification in any category, whenever the products involved in the statements exist (since it is clear that the notions of an S-object and of an S-morphism can be defined exactly as in (2.5) for any object S of the category).

(3.3.2). First of all, $X \times_S Y$ is a *covariant bifunctor* in X and Y on the category of S-preschemes: it suffices in fact to note that the diagram

$$\begin{tikzcd} X \times Y \arrow[r, "f\times 1"] \arrow[d] & X' \times Y \arrow[r, "f'\times 1"] \arrow[d] & X'' \times Y \arrow[d] \\ X \arrow[r, "f"] & X' \arrow[r, "f'"] & X'' \end{tikzcd}$$

is commutative.

Proposition (3.3.3). — *For each S-prescheme X, the first (resp. second) projection from $X \times_S S$ (resp. $S \times_S X$) is a functorial isomorphism from $X \times_S S$ (resp. $S \times_S X$) to X, whose inverse isomorphism is $(1_X, \phi)_S$ (resp. $(\phi, 1_X)_S$), where we denote by ϕ the structure morphism $X \to S$; we can therefore write, up to a canonical isomorphism,*

$$X \times_S S = S \times_S X = X.$$

PROOF. It suffices to prove that the triple $(X, 1_X, \phi)$ is a product of X and S. If T is an S-prescheme, then the only S-morphism from T to S is necessarily the structure morphism $\psi : T \to S$. If f is an S-morphism from T to X, we necessarily have $\psi = \phi \circ f$, hence our claim. ☐

Corollary (3.3.4). — *Let X and Y be S-preschemes, with structure morphisms $\phi : X \to S$ and $\psi : Y \to S$. If we canonically identify X with $X \times_S S$, and Y with $S \times_S Y$, then the projections $X \times_S Y \to X$ and $X \times_S Y \to Y$ are identified with $1_X \times \psi$ and $\phi \times 1_Y$ (respectively).*

The proof is immediate and is left to the reader.

(3.3.5). We can define, in a manner similar to (3.2), the product of a finite number n of S-preschemes, and the existence of these products follows from (3.2.6) by induction on n, and by noting that $(X_1 \times_S X_2 \times_S \cdots \times_S X_{n-1}) \times_S X_n$ satisfies the definition of a product. The uniqueness of the product implies, as in any category, its *commutativity* and *associativity* properties. If, for example, p_1, p_2, and p_3 denote the projections from $X_1 \times_S X_2 \times_S X_3$, and if we identify this prescheme with $(X_1 \times_S X_2) \times_S X_3$, then the projection to $X_1 \times_S X_2$ is identified with $(p_1, p_2)_S$. **I** | 109

[6] *[Trans.] (3.2.9) is from the errata of EGA II, on page 221, whence the change in page numbering.*

(3.3.6). Let S and S' be preschemes, and $\phi : S' \to S$ a morphism, which lets us consider S' as an S-prescheme. For each S-prescheme X, consider the product $X \times_S S'$, and let p and π' be the projections to X and to S' (respectively). Equipped with π', this product is an S'-prescheme; when we consider it as such, we denote it by $X_{(S')}$ or $X_{(\phi)}$, and we say that this is the prescheme obtained by *base change* (or *a change of base*) from S to S' by means of the morphism ϕ, or the *inverse image* of X by ϕ. We note that, if π is the structure morphism of X, and θ the structure morphism of $X \times_S S'$, considered as an S-prescheme, then the diagram

$$\begin{tikzcd} X \arrow[d, "\pi"'] & X_{(S')} \arrow[l, "p"'] \arrow[d, "\pi'"] \arrow[dl, "\theta"'] \\ S & S' \arrow[l, "\phi"] \end{tikzcd}$$

is commutative.

(3.3.7). With the notation of (3.3.6), for each S-morphism $f : X \to Y$, we denote by $f_{(S')}$ the S'-morphism $f \times_S 1 : X_{(S')} \to Y_{(S')}$, and we say that $f_{(S')}$ is the *base change* (or *inverse image*) of f by ϕ. Therefore $X_{(S')}$ is a *covariant functor* in X, from the category of S-preschemes to that of S'-preschemes.

(3.3.8). The prescheme $X_{(S')}$ can be considered as a solution to a *universal mapping problem*: each S'-prescheme T is also an S-prescheme via ϕ; each S-morphism $g : T \to X$ is then uniquely written as $g = p \circ f$, where f is an S'-morphism $T \to X_{(S')}$, as follows from the definition of the product applied to the S-morphisms f and $\psi : T \to S'$ (the structure morphism of T).

Proposition (3.3.9). — ("Transitivity of base change"). *Let S'' be a prescheme, and $\phi' : S'' \to S$ a morphism. For each S-prescheme X, there exists a canonical functorial isomorphism from the S''-prescheme $(X_{(\phi)})_{(\phi')}$ to the S''-prescheme $X_{(\phi\circ\phi')}$.*

PROOF. Let T be a S''-prescheme, ψ its structure morphism, and g an S-morphism from T to X (T being considered as an S-prescheme with structure morphism $\phi \circ \phi' \circ \psi$). Since T is also an S'-prescheme with structure morphism $\phi' \circ \psi$, we can write $g = p \circ g'$, where g' is an S'-morphism $T \to X_{(\phi)}$, and then $g' = p' \circ g''$, where g'' is an S''-morphism $T \to (X_{(\phi)})_{(\phi')}$:

$$\begin{tikzcd} X \arrow[d, "\pi"'] & X_{(\phi)} \arrow[l, "p"'] \arrow[d, "\pi'"'] & (X_{(\phi)})_{(\phi')} \arrow[l, "p'"'] \arrow[d, "\pi''"] \\ S & S \arrow[l, "\phi"] & S''. \arrow[l, "\phi'"] \end{tikzcd}$$

So the result follows by the uniqueness of the solution to a universal mapping problem. □ **I** | 110

This result can be written as the equality (up to a canonical isomorphism) $(X_{(S')})_{(S'')} = X_{(S'')}$ (if there is no chance of confusion), or also as

(3.3.9.1) $$(X \times_S S') \times_{S'} S'' = X \times_S S'';$$

the functorial nature of the isomorphism defined in (3.3.9) can similarly be expressed by the transitivity formula for base change morphisms

(3.3.9.2) $$(f_{(S')})_{(S'')} = f_{(S'')}$$

for each S-morphism $f : X \to Y$.

Corollary (3.3.10). — *If X and Y are S-preschemes, then there exists a canonical functorial isomorphism from the S'-prescheme $X_{(S')} \times_{S'} Y_{(S')}$ to the S'-prescheme $(X \times_S Y)_{(S')}$.*

PROOF. We have, up to canonical isomorphism,

$$(X \times_S S') \times_{S'} (Y \times_S S') = X \times_S (Y \times_S S') = (X \times_S Y) \times_S S'$$

according to (3.3.9.1) and the associativity of products of S-preschemes. □

The functorial nature of the isomorphism defined in Corollary (3.3.10) can be expressed by the formula

$$(u_{(S')}, v_{(S')})_{S'} = ((u, v)_S)_{(S')} \tag{3.3.10.1}$$

for each pair of S-morphisms $u : T \to X$, $v : T \to Y$.

In other words, the base change functor $X_{(S')}$ *commutes with products*; it also commutes with sums (3.2.8).

Corollary (3.3.11). — *Let Y be an S-prescheme, and $f : X \to Y$ a morphism which makes X a Y-prescheme (and, as a result, also an S-prescheme). The prescheme $X_{(S')}$ is then identified with the product $X \times_Y Y_{(S')}$, the projection $X \times_Y Y_{(S')} \to Y_{(S')}$ being identified with $f_{(S')}$.*

Proof. Let $\psi : Y \to S$ be the structure morphism of Y; we have the commutative diagram

$$\begin{tikzcd} S' \arrow[d] & Y_{(S')} \arrow[l] \arrow[d] & X_{(S')} \arrow[l, "f_{(S')}"'] \arrow[d] \\ S & Y \arrow[l, "\psi"'] & X. \arrow[l, "f"'] \end{tikzcd}$$

We have that $Y_{(S')}$ is identified with $S'_{(\psi)}$, and $X_{(S')}$ with $S'_{(\psi \circ f)}$; taking (3.3.9) and (3.3.4) into account, we thus deduce the corollary. □

(3.3.12). Let $f : X \to X'$ and $g : Y \to Y'$ be S-morphisms which are *monomorphisms* of preschemes (T, I, 1.1); then $f \times_S g$ is a *monomorphism*. Indeed, if p and q are the projections of $X \times_S Y$, p' and q' the projections of $X' \times_S Y'$, and u and v both S-morphisms $T \to X \times_S Y$, then the equation $(f \times_S g) \circ u = (f \times_S g) \circ v$ implies that $p' \circ (f \times_S g) \circ u = p' \circ (f \times_S g) \circ v$, or, in other words, that $f \circ p \circ u = f \circ p \circ v$, and since f is a monomorphism, $p \circ u = p \circ v$; using the fact that g is a monomorphism, we similarly obtain $q \circ u = q \circ v$, hence $u = v$.

It follows that, for each base change $S' \to S$, **I** | 111

$$f_{(S')} : X_{(S')} \longrightarrow Y_{(S')}$$

is a monomorphism.

(3.3.13). Let S and S' be affine schemes of rings A and A' (respectively); a morphism $S' \to S$ then corresponds to a ring homomorphism $A \to A'$. If X is an S-prescheme, we denote by $X_{(A')}$ or $X \otimes_A A'$ the S'-prescheme $X_{(S')}$; when X is also affine of ring B, $X_{(A')}$ is affine of ring $B_{(A')} = B \otimes_A A'$ obtained by extension of scalars from the A-algebra B to A'.

(3.3.14). With the notation of (3.3.6), for each *S-morphism* $f : S' \to X$, we have that $f' = (f, 1_{S'})_S$ is an S'-morphism $S' \to X' = X_{(S')}$ such that $p \circ f' = f$, $\pi' \circ f' = 1_{S'}$, or, in other words, an *S'-section of of X'*; conversely, if f' is such an S'-section, then $f = p \circ f'$ is an S-morphism $S' \to X$. We thus define a canonical *bijective correspondence*

$$\mathrm{Hom}_S(S', X) \simeq \mathrm{Hom}_{S'}(S', X').$$

We say that f' is the *graph morphism* of f, and we denote it by Γ_f.

(3.3.15). Given a prescheme X, which we can always consider as a $\mathbf{Z}$-prescheme, it follows, in particular, from (3.3.14) that the *X-sections* of $X \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ (where T is an indeterminate) correspond bijectively with *morphisms* $\mathbf{Z}[T] \to X$. We will show that these X-sections also correspond bijectively with *sections of the structure sheaf $\mathscr{O}_X$ over X*. Indeed, let (U_α) be a cover of X by affine open subsets; let $u : X \to X \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ be an X-morphism, and let u_α be its restriction to U_α; if A_α is the ring of the affine scheme U_α, then $U_\alpha \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ is an affine scheme of ring $A_\alpha[T]$ (3.2.2), and u_α canonically corresponds to an A_α-homomorphism $A_\alpha[T] \to A_\alpha$ (1.7.3). Now, since such a homomorphism is completely determined by the data of the image of T in A_α, let $s_\alpha \in A_\alpha = \Gamma(U_\alpha, \mathscr{O}_X)$, and if we suppose that the restrictions of u_α and u_β to an affine open subset $V \subset U_\alpha \cap U_\beta$ coincide, then we see immediately that s_α and s_β coincide on V; thus the family (s_α) consists of the restrictions to U_α of a section s of $\mathscr{O}_X$ over X; conversely, it is clear that such a section defines a family (u_α) of morphisms which are the restrictions to U_α of an X-morphism $X \to X \otimes_{\mathbf{Z}} \mathbf{Z}[T]$. This result is generalized in (**II**, 1.7.12).

3.4. Points of a prescheme with values in a prescheme; geometric points

(3.4.1). Let X be a prescheme; for each prescheme T, we then denote by $X(T)$ the set $\mathrm{Hom}(T, X)$ of morphisms $T \to X$, and the elements of this set are called *the points of X with values in T*. If we associate to each morphism $f : T \to T'$ the map $u' \mapsto u' \circ f$ from $X(T')$ to $X(T)$, we see, for fixed X, that $X(T)$ is a *contravariant functor in T*, from the category of preschemes to that of sets. In addition, each morphism of preschemes $g : X \to Y$ defines a functorial homomorphism $X(T) \to Y(T)$, which sends $v \in X(T)$ to $g \circ v$.

(3.4.2). Given sets P, Q, and R, and maps $\phi : P \to R$ and $\psi : Q \to R$, we define the *fibre product of P and Q over R* (with respect to ϕ and ψ) as the subset of the product set $P \times Q$ consisting of the pairs (p, q) such that $\phi(p) = \psi(q)$; we denote it by $P \times_R Q$. Definition (3.2.1) of the product of S-preschemes can be interpreted, with the notation of (3.4.1), via the formula I | 112

$$(X \times_S Y)(T) = X(T) \times_{S(T)} Y(T). \tag{3.4.2.1}$$

the maps $X(T) \to S(T)$ and $Y(T) \to S(T)$ corresponding to the structure morphisms $X \to S$ and $Y \to S$.

(3.4.3). If we are given a prescheme S and we consider only the S-preschemes and S-morphisms, then we will denote by $X(T)_S$ the set $\mathrm{Hom}_S(T, X)$ of S-morphisms $T \to X$, and suppress the subscript S when there is no chance of confusion; we say that the elements of $X(T)_S$ are the *points* (or *S-points*, when there is a possibility of confusion) *of the S-prescheme X with values in the S-prescheme T*. In particular, an *S-section* of X is none other than a *point of X with values in S*. The formula (3.4.2.1) can then be written as

$$(X \times_S Y)(T)_S = X(T)_S \times Y(T)_S; \tag{3.4.3.1}$$

more generally, if Z is an S-prescheme, and X, Y, and T are Z-preschemes (thus *ipso facto* S-preschemes), then we have

$$(X \times_Z Y)(T)_S = X(T)_S \times_{Z(T)_S} Y(T)_S. \tag{3.4.3.2}$$

We note that, to show that a triple (W, r, s) consisting of an S-prescheme W and S-morphisms $r : W \to X$ and $s : W \to Y$ is a product of X and Y (over Z), it suffices, by definition, to check that, for *each S-prescheme T*, the diagram

$$\begin{tikzcd} W(T)_S \arrow[r, "r'"] \arrow[d, "s'"'] & X(T)_S \arrow[d, "\phi'"] \\ Y(T)_S \arrow[r, "\psi'"] & Z(T)_S \end{tikzcd}$$

makes $W(T)_S$ the fibre product of $X(T)_S$ and $Y(T)_S$ over $Z(T)_S$, where r' and s' correspond to r and s, and ϕ' and ψ' to the structure morphisms $\phi : X \to Z$ and $\psi : Y \to Z$.

(3.4.4). When T (resp. S) in the above is an affine scheme of ring B (resp. A), we replace T (resp. S) by B (resp. A) in the above notation, and we then call the elements of $X(B)$ the *points of X with values in the ring B*, and the elements of $X(B)_A$ the *points of the A-prescheme X with values in the A-algebra B*. We note that $X(B)$ and $X(B)_A$ are *covariant* functors in B. We similarly write $X(T)_A$ for the set of points of the A-prescheme X with values in the A-prescheme T.

(3.4.5). Consider, in particular, the case where T is of the form $\mathrm{Spec}(A)$, where A is a *local* ring; the elements of $X(A)$ then correspond bijectively to *local* homomorphisms $\mathscr{O}_x \to A$ for $x \in X$ (2.2.4); we say that the point x of the underlying space of X is the *location*[7] of the point of X with values in A to which it corresponds.

More specifically, we define the *geometric points* of a prescheme S to be the *points of X with values in a field K*: the data of such a point is equivalent to the data of its location x in the underlying I | 113
subspace of X, and of an *extension* K of $k(x)$; K will be called the *field of values* of the corresponding geometric point, and we say that this geometric point is *located at x*. We also define a map $X(K) \to X$, sending a geometric point with values in K to its location.

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[7] *[Trans.] We also say that* the geometric point lies over this x.

If $S' = \operatorname{Spec}(K)$ is an S-prescheme (in other words, if K is considered as an extension of the residue field $k(s)$, where $s \in S$), and if X is an S-prescheme, then an element of $X(K)_S$, or, as we say, a *geometric point of X lying over s with values in K*, consists of the data of a $k(s)$-monomorphism from the residue field $k(x)$ to K, where x is a point of X *lying over* s (therefore $k(x)$ is an extension of $k(s)$).

In particular, if $S = \operatorname{Spec}(K) = \{\xi\}$, *then the geometric points of X with values in K can be identified with the points $x \in X$ such that $k(x) = K$*; we say that these latter points are the *K-rational points of the K-prescheme X*; if K' is an extension of K, then the geometric points of X with values in K' bijectively correspond to the K'-rational points of $X' = X_{(K')}$ (3.3.14).

Lemma (3.4.6). — *Let X_i $(1 \leqslant i \leqslant n)$ be S-preschemes, s a point of S, and x_i $(1 \leqslant i \leqslant n)$ points of X_i lying over s. Then there exists an extension K of $k(s)$ and a geometric point of the product $Y = X_1 \times_S X_2 \times_S \cdots \times_S X_n$, with values in K, whose projections to the X_i are localized at the x_i.*

PROOF. There exist $k(s)$-monomorphisms $k(x_i) \to K$, all in the same extension K of $k(s)$ (Bourbaki, *Alg.*, chap. V, §4, prop. 2). The compositions $k(s) \to k(x_i) \to K$ are all identical, and so the morphisms $\operatorname{Spec}(K) \to X_i$ corresponding to the $k(x_i) \to K$ are all S-morphisms, and we thus conclude that they define a unique morphism $\operatorname{Spec}(K) \to Y$. If y is the corresponding point of Y, it is clear that its projection in each of the X_i is x_i. □

Proposition (3.4.7). — *Let X_i $(1 \leqslant i \leqslant n)$ be S-preschemes, and, for each index i, let x_i be a point of X_i. For there to exist a point y of $Y = X_1 \times_S X_2 \times \ldots \times_S X_n$ whose image is x_i under the ith projection for each $1 \leqslant i \leqslant n$, it is necessary and sufficient that the x_i all lie above the same point s of S.*

PROOF. The condition is evidently necessary; Lemma (3.4.6) proves that it is sufficient. □

In other words, if we denote by (X) the underlying set of X, we see that we have a canonical *surjective* function $(X \times_S Y) \to (X) \times_{(S)} (Y)$; we must point out that this function *is not injective* in general; in other words, *there can exist multiple distinct points z in $X \times_S Y$ that have the same projections $x \in X$ and $y \in Y$*; we have already seen this when S, X, and Y are prime spectra of fields k, K, and K' (respectively), since the tensor product $K \otimes_k K'$ has, in general, multiple distinct prime ideals (cf. (3.4.9)).

Corollary (3.4.8). — *Let $f : X \to Y$ be an S-morphism, and $f_{(S')} : X_{(S')} \to Y_{(S')}$ the S'-morphism induced by f by an extension $S' \to S$ of the base prescheme. Let p (resp. q) be the projection $X_{(S')} \to X$ (resp. $Y_{(S')} \to Y$); for every subset M of X, we have*

$$q^{-1}(f(M)) = f_{(S')}(p^{-1}(M)).$$

I | 114

PROOF. Indeed (3.3.11), $X_{(S')}$ can be identified with the product $X \times_Y Y_{(S')}$ thanks to the commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & X_{(S')} \arrow[l, "p"'] \arrow[d, "f_{(S')}"] \\ Y & Y_{(S')} \arrow[l, "q"'] \end{tikzcd}$$

By (3.4.7), the equation $q(y') = f(x)$ for $x \in M$ and $y' \in Y_{(S)}$ is equivalent to the existence of some $x' \in X_{(S')}$ such that $p(x') = x$ and $f_{(S')}(x') = y'$, whence the corollary. □

Lemma (3.4.6) can be made clearer in the following manner:

Proposition (3.4.9). — *Let X and Y be S-preschemes, x a point of X, and y a point of Y, with both x and y lying above the same point $s \in S$. The set of points of $X \times_S Y$ with projections x and y is in bijective correspondence with the set of types of extensions (?) composed of $k(x)$ and $k(y)$ considered as extensions of $k(s)$* (Bourbaki, *Alg.*, chap. VIII, §8, prop. 2).

PROOF. Let p (resp. q) be the projection from $X \times_S Y$ to X (resp. Y), and E the subspace $p^{-1}(x) \cap q^{-1}(y)$ of the underlying space of $X \times_S Y$. First, note that the morphisms $\operatorname{Spec}(k(x)) \to S$

and $\operatorname{Spec}(k(y)) \to S$ factor as $\operatorname{Spec}(k(x)) \to \operatorname{Spec}(k(s)) \to S$ and $\operatorname{Spec}(k(y)) \to \operatorname{Spec}(k(s)) \to S$; since $\operatorname{Spec}(k(s)) \to S$ is a monomorphism (2.4.7), it follows from (3.2.4) that we have

$$P = \operatorname{Spec}(k(x)) \times_S \operatorname{Spec}(k(y)) = \operatorname{Spec}(k(x)) \times_{\operatorname{Spec}(k(s))} \operatorname{Spec}(k(y)) = \operatorname{Spec}(k(x) \otimes_{k(s)} k(y)).$$

We will define two maps, $\alpha : P_0 \to E$ and $\beta : E \to P_0$, inverse to one another (where P_0 denotes the underlying set of the prescheme P). If $i : \operatorname{Spec}(k(x)) \to X$ and $j : \operatorname{Spec}(k(y)) \to Y$ are the canonical morphisms (2.4.5), we take α to be the map of underlying spaces corresponding to the morphism $i \times_S j$. On the other hand, every $z \in E$ defines, by hypothesis, two $k(s)$-monomorphisms, $k(x) \to k(z)$ and $k(y) \to k(z)$, and thus a $k(s)$-monomorphism $k(x) \otimes_{k(s)} k(y) \to k(z)$, and thus a morphism $\operatorname{Spec}(k(z)) \to P$; $\beta(z)$ will be the image of z in P_0 under this morphism. The verification of the fact that $\alpha \circ \beta$ and $\beta \circ \alpha$ are the identity maps follows from (2.4.5) and the definition of the product (3.2.1). Finally, we know that P_0 is in bijective correspondence with the set of types of extensions (?) composed of $k(x)$ and $k(y)$ (Bourbaki, *Alg.*, chap. VIII, §8, prop. 1). □

3.5. Surjections and injections

(3.5.1). In a general sense, consider a property **P** of morphisms of preschemes, and the following two propositions:

(i) If $f : X \to X'$ and $g : Y \to Y'$ are S-morphisms that have property **P**, then $f \times_S g$ also has property **P**.

(ii) If $f : X \to Y$ is an S-morphism that has property **P**, then every S'-morphism $f_{(S')} : X_{(S')} \to Y_{(S')}$, induced by f by an extension of the base prescheme, also has property **P**.

Since $f_{(S')} = f \times_S 1_{S'}$, we see that, if, for every prescheme X, the *identity* 1_X has property **P**, then (i) implies (ii); on the other hand, since $f \times_S g$ is the composite morphism

$$X \times_S Y \xrightarrow{f \times 1_Y} X' \times_S Y \xrightarrow{1_{X'} \times g} X' \times_S Y',$$

we see that, if the *composition* of two morphisms has property **P**, then so does the product $f \times_S g$, and so (ii) implies (i).

A first application of this remark is

Proposition (3.5.2). —

(i) *If $f : X \to X'$ and $g : Y \to Y'$ are surjective S-morphisms, then $f \times_S g$ is surjective.*

(ii) *If $f : X \to Y$ is a surjective S-morphism, then $f_{(S')}$ is surjective for every extension S' of the base prescheme.*

Proof. The composition of any two surjections being a surjection, it suffices to prove (ii); but this proposition follows immediately from (3.4.8) applied to $M = X$. □

Proposition (3.5.3). — *For a morphism $f : X \to Y$ to be surjective, it is necessary and sufficient that, for every field K and every morphism* $\operatorname{Spec}(K) \to Y$*, there exist an extension K' of K and a morphism* $\operatorname{Spec}(K') \to X$ *that make the following diagram commute:*

$$\begin{tikzcd} X & \operatorname{Spec}(K') \arrow[l] \arrow[d] \\ Y \arrow[from=u, "f"'] & \operatorname{Spec}(K). \arrow[l] \end{tikzcd}$$

Proof. The condition is sufficient because, for all $y \in Y$, it suffices to apply it to a morphism $\operatorname{Spec}(K) \to Y$ corresponding to a monomorphism $k(y) \to K$, with K being an extension of $k(y)$ (2.4.6). Conversely, suppose that f is surjective, and let $y \in Y$ be the image of the unique point of $\operatorname{Spec}(K)$; there exists some $x \in X$ such that $f(x) = y$; we will consider the corresponding monomorphism $k(y) \to k(x)$ (2.2.1); it then suffices to take K' to be the extension of $k(y)$ such that there exist $k(y)$-monomorphisms from $k(x)$ and K to K' (Bourbaki, *Alg.*, chap. V, §4, prop. 2); the morphism $\operatorname{Spec}(K') \to X$ corresponding to $k(x) \to K'$ is exactly that for which we are searching. □

With the language introduced in (3.4.5), we can say that *every geometric point of Y with values in K comes from a geometric point of X with values in an extension of K.*

Definition (3.5.4). — We say that a morphism $f : X \to Y$ of preschemes is *universally injective*, or a *radicial morphism*, if, for every field K, the corresponding map $X(K) \to Y(K)$ is injective.

It follows also from the definitions that every *monomorphism of preschemes* (T, 1.1) is radicial.

(3.5.5). For a morphism $f : X \to Y$ to be radicial, it suffices that the condition of Definition (3.5.4) hold for every *algebraically closed* field. In fact, if K is an arbitrary field, and K' an algebraically-closed extension of K, then the diagram

$$\begin{tikzcd} X(K) \arrow[r, "\alpha"] \arrow[d, "\phi"'] & Y(K) \arrow[d, "\phi'"] \\ X(K') \arrow[r, "\alpha'"] & Y(K') \end{tikzcd}$$

commutes, where ϕ and ϕ' come from the morphism $\mathrm{Spec}(K') \to \mathrm{Spec}(K)$, and α and α' corresponding to f. However, ϕ is injective, and so too is α', by hypothesis; hence α is necessarily injective.

Proposition (3.5.6). — *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of preschemes.*

(i) *If f and g are radicial, then so is $g \circ f$.*
(ii) *Conversely, if $g \circ f$ is radicial, then so is f.*

PROOF. Taking into account Definition (3.5.4), the proposition reduces to the corresponding claims for the maps $X(K) \to Y(K) \to Z(K)$, and these claims are evident. □

Proposition (3.5.7). —

(i) *If the S-morphisms $f : X \to X'$ and $g : X \to X'$ are radicial, then so is $f \times_S g$.*
(ii) *If the S-morphism $f : X \to Y$ is radicial, then so is $f_{(S')} : X_{(S')} \to Y_{(S')}$ for every extension $S' \to S$ of the base prescheme.*

PROOF. Given (3.5.1), it suffices to prove (i). We have seen (3.4.2.1) that

$$(X \times_S Y)(K) = X(K) \times_{S(K)} Y(K),$$
$$(X' \times_S Y')(K) = X'(K) \times_{S(K)} Y'(K),$$

with the map $(X \times_S Y)(K) \to (X' \times_S Y')(K)$ corresponding to $f \times_S g$ thus being identified with $(u, v) \to (f \circ u, g \circ v)$, and the proposition then follows. □

Proposition (3.5.8). — *For a morphism $f = (\psi, \theta) : X \to Y$ to be radicial, it is necessary and sufficient for ψ to be injective and for the monomorphism $\theta^x : k(\psi(x)) \to k(x)$ to make $k(x)$ a radicial extension of $k(\psi(x))$ for every $x \in X$.*

PROOF. We suppose that f is radicial and first show that the equation $\psi(x_1) = \psi(x_2) = y$ necessarily implies that $x_1 = x_2$. Indeed, there exists a field K, and an extension of $k(y)$, along with $k(y)$-monomorphisms $k(x_1) \to K$ and $k(x_2) \to K$ (Bourbaki, *Alg.*, chap. V, §4, prop. 2); the corresponding morphisms $u_1 : \mathrm{Spec}(K) \to X$ and $u_2 : \mathrm{Spec}(K) \to X$ are then such that $f \circ u_1 = f \circ u_2$, and so $u_1 = u_2$ by hypothesis, and this implies, in particular, that $x_1 = x_2$. We now consider $k(x)$ as the extension of $k(\psi(x))$ by means of θ^x: if $k(x)$ is not a radicial algebraically-closed extension, then there exist two distinct $k(\psi(x))$-monomorphisms from $k(x)$ to an algebraically-closed extension K of $k(\psi(x))$, and the two corresponding morphisms $\mathrm{Spec}(K) \to X$ would contradict the hypothesis. Conversely, taking (2.4.6) into account, it is immediate that the conditions stated are sufficient for f to be radicial. □

Corollary (3.5.9). — *If A is a ring, and S is a multiplicative set of A, then the canonical morphism* $\mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}(A)$ *is radicial.*

PROOF. Indeed, this morphism is a monomorphism (1.6.2). □

Corollary (3.5.10). — *Let $f : X \to Y$ be a radicial morphism, $g : Y' \to Y$ a morphism, and $X' = X_{(Y')} = X \times_Y Y'$. Then the radicial morphism $f_{(Y')}$ (3.5.7, ii) is a bijection from the underlying space of X to $g^{-1}(f(X))$; further, for every field K, the set $X'(K)$ can be identified with the subset of $Y'(K)$ given by the inverse image of the map $Y'(K) \to Y(K)$ (corresponding to g) from the subset $X(K)$ of $Y(K)$.*

PROOF. The first claim follows from (3.5.8) and (3.4.8); the second, from the commutativity of the following diagram: I | 117

$$\begin{tikzcd} X'(K) \arrow[r] \arrow[d] & Y'(K) \arrow[d] \\ X(K) \arrow[r] & Y(K) \end{tikzcd}$$

□

Remark (3.5.11). — We say that a morphism $f = (\psi, \theta)$ of preschemes is *injective* if the map ψ is injective. For a morphism $f = (\psi, \theta) : X \to Y$ to be radicial, it is necessary and sufficient that, for every morphism $Y' \to Y$, the morphism $f_{(Y')} : X_{(Y')} \to Y'$ be injective (which justifies the terminology of a *universally injective* morphism). In fact, the condition is necessary by (3.5.7, ii) and (3.5.8). Conversely, the condition implies that ψ is injective; if, for some $x \in X$, the monomorphism $\theta^x : k(\psi(x)) \to k(x)$ were not radicial, then there would be an extension K of $k(\psi(x))$, and two distinct morphisms $\operatorname{Spec}(K) \to X$ corresponding to the same morphism $\operatorname{Spec}(K) \to Y$ (3.5.8). But then, setting $Y' = \operatorname{Spec}(K)$, there would be two distinct Y'-sections of $X_{(Y')}$ (3.3.14), which contradicts the hypothesis that $f_{(Y')}$ is injective.

3.6. Fibres

Proposition (3.6.1). — *Let $f : X \to Y$ be a morphism, y a point of Y, and $\mathfrak{a}_y$ an ideal of definition for $\mathscr{O}_y$ for the $\mathfrak{m}_y$-preadic topology. Then the projection $p : X \times_Y \operatorname{Spec}(\mathscr{O}_y/\mathfrak{a}_y) \to X$ is a homeomorphism from the underlying space of the prescheme $X \times_Y \operatorname{Spec}(\mathscr{O}_y/\mathfrak{a}_y)$ to the fibre $f^{-1}(y)$ equipped with the topology induced from that of the underlying space of X.*

PROOF. Since $\operatorname{Spec}(\mathscr{O}_y/\mathfrak{a}_y) \to Y$ is radicial ((3.5.4) and (2.4.7)), since $\operatorname{Spec}(\mathscr{O}_y/\mathfrak{a}_y)$ is a single point, and since the ideal $\mathfrak{m}_y/\mathfrak{a}_y$ is nilpotent by hypothesis (1.1.12), we already know ((3.5.10) and (3.3.4)) that p identifies, as *sets*, the underlying space of $X \times_Y \operatorname{Spec}(\mathscr{O}_y/\mathfrak{a}_y)$ with $f^{-1}(y)$; everything reduces to proving that p is a homeomorphism. By (3.2.7), the question is local on X and Y, and so we can suppose that $X = \operatorname{Spec}(B)$ and $Y = \operatorname{Spec}(A)$, with B being an A-algebra. The morphism p then corresponds to the homomorphism $1 \otimes \phi : B \to B \otimes_A A'$, where $A' = A_y/\mathfrak{a}_y$ and ϕ is the canonical map from A to A'. Then every element of $B \otimes_A A'$ can be written as

$$\sum_i b_i \otimes \phi(a_i)/\phi(s) = \left(\sum_i (a_i b_i \otimes 1)\right)(1 \otimes \phi(s))^{-1},$$

where $s \notin \mathfrak{j}_y$, and Proposition (1.2.4) applies. □

(3.6.2). Throughout the rest of this treatise, whenever we consider a fibre $f^{-1}(y)$ of a morphism as having the structure of a $k(y)$-prescheme, *it will always be the prescheme obtained by transporting the structure of $X \times_Y \operatorname{Spec}(k(y))$ by the projection to X.* We will also write this (latter) product as $X \times_Y k(y)$, or $X \otimes_{\mathscr{O}_Y} k(y)$; more generally, if B is an $\mathscr{O}_y$-algebra, we will denote by $X \times_Y B$ or $X \otimes_{\mathscr{O}_Y} B$ the product $X \times_Y \operatorname{Spec}(B)$.

With the preceding convention, it follows from (3.5.10) that the points of X with values in an extension K of $k(y)$ are identified with the *points of $f^{-1}(y)$ with values in K.*

(3.6.3). Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms, and $h = g \circ f$ their composition; for all $z \in Z$, the fibre $h^{-1}(z)$ is a prescheme isomorphic to

$$X \times_Z \operatorname{Spec}(k(z)) = (X \times_Y Y) \times_Z \operatorname{Spec}(k(z)) = X \times_Y g^{-1}(z).$$

Inparticular, if U is an open subset of X, then the prescheme induced on $U \cap f^{-1}(y)$ by the prescheme $f^{-1}(y)$ is isomorphic to $f_U^{-1}(y)$ (f_U being the restriction of f to U), I | 118

Proposition (3.6.4). — (Transitivity of fibres) *Let $f : X \to Y$ and $g : Y' \to Y$ be morphisms; let $X' = X \times_Y Y' = X_{(Y')}$ and $f' = f_{(Y')} : X' \to Y'$. For every $y' \in Y'$, if we let $y = g(y')$, then the prescheme $f'^{-1}(y')$ is isomorphic to $f^{-1}(y) \otimes_{k(y)} k(y')$.*

PROOF. Indeed, it suffices to remark that the two preschemes $(X \otimes_Y k(y)) \otimes_{k(y)} k(y')$ and $(X \times_Y Y') \otimes_{Y'} k(y')$ are both canonically isomorphic to $X \times_Y \mathrm{Spec}(k(y'))$ by (3.3.9.1). □

In particular, if V is an open neighbourhood of y in Y, and we denote by f_V the restriction of f to the induced prescheme on $f^{-1}(V)$, then the preschemes $f^{-1}(y)$ and $f_V^{-1}(y)$ are canonically identified.

Proposition (3.6.5). — *Let $f : X \to Y$ be a morphism, y a point of Y, Z the local prescheme* $\mathrm{Spec}(\mathscr{O}_y)$, *and $p = (\psi, \theta)$ the projection $X \times_Y Z \to X$; then p is a homeomorphism from the underlying space of $X \times_Y Z$ to the subspace $f^{-1}(Z)$ of X (*when the underlying space of Z is identified with a subspace of Y, cf. (2.4.2)*), and, for all $t \in X \times_Y Z$, letting $z = \psi(t)$, we have that $\theta_t^\sharp$ is an isomorphism from $\mathscr{O}_x$ to $\mathscr{O}_t$.*

PROOF. Since Z (identified as a subspace of Y) is contained inside every affine open containing y (2.4.2), we can, as in (3.6.1), reduce to the case where $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ are affine schemes, with A being a B-algebra. Then $X \times_Y Z$ is the prime spectrum of $A \otimes_B B_y$, and this ring is canonically identified with $S^{-1}A$, where S is the image of $B - \mathfrak{j}_y$ in A (**0**, 1.5.2); since p then corresponds to the canonical homomorphism $A \to S^{-1}A$, the proposition follows from (1.6.2). □

3.7. Application: reduction of a prescheme mod. $\mathfrak{J}$ *This section, which makes use of notions and results from Chapter I and Chapter II, will not be used in what follows in this treatise, and is only intended for readers familiar with classical algebraic geometry.*

(3.7.1). Let A be a ring, X an A-prescheme, and $\mathfrak{J}$ an ideal of A; then $X_0 = X \otimes_A (A/\mathfrak{J})$ is an $(A/\mathfrak{J})$-prescheme, which we sometimes say is induced from X by *reduction* mod. $\mathfrak{J}$.

. This terminology is used foremost when A is a *local ring* and $\mathfrak{J}$ its maximal ideal, in such a way that X_0 is a prescheme over the residue field $k = A/\mathfrak{J}$ of A.

When A is also integral, with field of fractions K, we can consider the K-prescheme $X' = X \otimes_A K$. By an abuse of language which we will not use, it has been said, up until now, that X_0 is *induced by* X' by reduction mod. $\mathfrak{J}$. In the case where this language was used, A was a local ring of dimension 1 (most often a discrete valuation ring) and it was implied (be it more or less explicitly) that the given K-prescheme X' was a closed subprescheme of a K-prescheme P' (in fact, a projective space of the form (?) $\mathbf{P}_K^r$, cf. (**II**, 4.1.1)), itself of the form $P' = P \otimes_A K$, where P is a given A-prescheme (in fact, the A-scheme $\mathbf{P}_A^r$, with the notation of (**II**, 4.1.1)). In our language, the definition of X_0 in terms of X' is formulated as follows:

We consider the affine scheme $Y = \mathrm{Spec}(A)$, formed of two points, the unique closed point $y = \mathfrak{J}$ and the generic point (0), the singleton set U of the generic point being thus an open $U = \mathrm{Spec}(K)$ in Y. If X is an A-prescheme (or, in other words, a Y-prescheme), then $X \otimes_A K = X'$ is exactly the prescheme induced by X on $\psi^{-1}(U)$, denoting by ψ the structure morphism $X \to Y$. In particular, if ϕ is the structure morphism $P \to Y$, then a closed subprescheme X' of $P' = \phi^{-1}(U)$ is a (locally closed) subprescheme of P. If P is Noetherian (for example, if A is Noetherian and P is of finite type over A), then there exists a smaller closed subprescheme $X = \overline{X'}$ of G that through which X' factors (9.5.10), and X' is the prescheme induced by X on the open $\phi^{-1}(U) \cap X$, and so is isomorphic to $X \otimes_A K$ (9.5.10). *The immersion of X' into $P' = P \otimes_A K$ thus lets us canonically consider X' as being of the form $X' = X \otimes_A K$, where X is an A-prescheme.* We can then consider the reduced mod. $\mathfrak{J}$ prescheme $X_0 = X \otimes_A k$, which is exactly the fibre $\psi^{-1}(y)$ of the closed point y. Up until now, lacking the adequate terminology, we had avoided explicitly introducing the A-prescheme X. One ought to, however, note that all the claims normally made about the "reduced mod. $\mathfrak{J}$" prescheme X_0 should be seen as consequences of more complicated claims about X itself, and cannot be satisfactorily formulated or understood except by interpreting them as such. It seems also that any hypotheses made on X_0 always reduce to hypotheses on X itself (independent of the prior data of an immersion of X' in $\mathbf{P}_K^r$), which lets us give more intrinsic statements. **I | 119**

(3.7.3). Lastly, we draw attention to a very particular fact, which has undoubtedly contributed to slowing the conceptual clarification of the situation envisaged here: if A is a discrete valuation ring, and if X is *proper* over A (which is indeed the case if X is a closed subprescheme of some $\mathbf{P}_A^r$, cf. (**II**, 5.5.4)), then the points of X with values in A and the points of X' with values in k are in bijective correspondence (**II**, 7.3.8). This is why we often believe that facts about X' have been proved, when in reality we have proved facts about X, and these remain valuable (in this form) whenever we no longer suppose that the base local ring is of dimension 1.

§4. Subpreschemes and immersion morphisms

4.1. Subpreschemes

(4.1.1). As the notion of a quasi-coherent sheaf (**0**, 5.1.3) is local, a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ over a prescheme X can be defined by the following condition: for each affine open V of X, $\mathscr{F}|V$ is isomorphic to the sheaf associated to a $\Gamma(V, \mathscr{O}_X)$-module (1.4.1). It is clear that, over a prescheme X, the structure sheaf $\mathscr{O}_X$ is quasi-coherent, and that the kernels, cokernels, and images of homomorphisms of quasi-coherent $\mathscr{O}_X$-modules, as well as inductive limits and direct sums of quasi-coherent $\mathscr{O}_X$-modules, are also quasi-coherent (Theorem (1.3.7) and Corollary (1.3.9)).

Proposition (4.1.2). — *Let X be a prescheme, and $\mathscr{I}$ a quasi-coherent sheaf of ideals of $\mathscr{O}_X$. Then the support Y of the sheaf $\mathscr{O}_X/\mathscr{I}$ is closed, and if we denote by $\mathscr{O}_Y$ the restriction of $\mathscr{O}_X/\mathscr{I}$ to Y, then $(Y, \mathscr{O}_Y)$ is a prescheme.*

PROOF. It evidently suffices (2.1.3) to consider the case where X is an affine scheme, and to show that, in this case, Y is closed in X, and is an *affine scheme*. Indeed, if $X = \operatorname{Spec}(A)$, then we have $\mathscr{O}_X = \widetilde{A}$ and $\mathscr{I} = \widetilde{\mathfrak{I}}$, where $\mathfrak{I}$ is an ideal of A (1.4.1); Y is then equal to the closed subset $V(\mathfrak{I})$ of X and can be identified with the prime spectrum of the ring $B = A/\mathfrak{I}$ (1.1.11); in addition, if ϕ is the canonical homomorphism $A \to B = A/\mathfrak{I}$, then the direct image ${}^a\phi_*(\widetilde{B})$ is canonically identified with the sheaf $\widetilde{A}/\widetilde{\mathfrak{I}} = \mathscr{O}_X/\mathscr{I}$ (Proposition (1.6.3) and Corollary (1.3.9)), which finishes the proof. ☐ I | 120

We say that $(Y, \mathscr{O}_Y)$ is the *subprescheme* of $(X, \mathscr{O}_X)$ *defined by the sheaf of ideals* $\mathscr{I}$; this is a particular case of the more general notion of a *subprescheme*:

Definition (4.1.3). — We say that a ringed space $(Y, \mathscr{O}_Y)$ is a subprescheme of a prescheme $(X, \mathscr{O}_X)$ if:

1st. Y is a locally closed subspace of X;

2nd. if U denotes the largest open subset of X containing Y such that Y is closed in U (*equivalently*, the complement in X of the boundary of Y with respect to $\overline{Y}$), then $(Y, \mathscr{O}_Y)$ is a subprescheme of $(U, \mathscr{O}_X|U)$ defined by a quasi-coherent sheaf of ideals of $\mathscr{O}_X|U$.

We say that the subprescheme $(Y, \mathscr{O}_Y)$ of $(X, \mathscr{O}_X)$ is closed if Y is closed in X (in which case $U = X$).

It follows immediately from this definition and Proposition (4.1.2) that the closed subpreschemes of X are in canonical *bijective correspondence* with the quasi-coherent sheaves of ideals $\mathscr{J}$ of $\mathscr{O}_X$, since if two such sheaves $\mathscr{J}$ and $\mathscr{J}'$ have the same (closed) support Y, and if the restrictions of $\mathscr{O}_X/\mathscr{J}$ and $\mathscr{O}_X/\mathscr{J}'$ to Y are identical, then we have $\mathscr{J}' = \mathscr{J}$.

(4.1.4). Let $(Y, \mathscr{O}_Y)$ be a subprescheme of X, U the largest open subset of X such that Y is closed (and thus contained) in U, and V an open subset of X contained in U; then $V \cap Y$ is closed in V. In addition, if Y is defined by the quasi-coherent sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_X|U$, then $\mathscr{J}|V$ is a quasi-coherent sheaf of ideals of $\mathscr{O}_X|V$, and it is immediate that the prescheme induced by Y on $Y \cap V$ is the closed subprescheme of V defined by the sheaf of ideals $\mathscr{J}|V$. Conversely:

Proposition (4.1.5). — *Let $(Y, \mathscr{O}_Y)$ be a ringed space such that Y is a subspace of X, and there exists a cover (V_α) of Y by open subsets of X such that, for each α, $Y \cap V_\alpha$ is closed in V_α, and the ringed space $(Y \cap V_\alpha, \mathscr{O}_Y|(Y \cap V_\alpha))$ is a closed subprescheme of the prescheme induced on V_α by X. Then $(Y, \mathscr{O}_Y)$ is a subprescheme of X.*

PROOF. The hypotheses imply that Y is locally closed in X and that the largest open U in which Y is closed (and thus contained) contains all the V_α; we can thus reduce to the case where $U = X$ and Y is closed in X. We then define a quasi-coherent sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_X$ by taking $\mathscr{J}|V_\alpha$ to be the sheaf of ideals of $\mathscr{O}_X|V_\alpha$ which define the closed subprescheme $(Y \cap V_\alpha, \mathscr{O}_Y|(Y \cap V_\alpha))$, and, for each open subset W of X not intersecting Y, $\mathscr{J}|W = \mathscr{O}_X|W$. We see immediately, by Definition (4.1.3) and (4.1.4), that there exists a unique sheaf of ideals $\mathscr{J}$ satisfying these conditions, and that it defines the closed subprescheme $(Y, \mathscr{O}_Y)$. ☐

In particular, the *induced* (by X) prescheme on an *open subset* of X is a *subprescheme* of X.

Proposition (4.1.6). — *A subprescheme (resp. a closed subprescheme) of a subprescheme (resp. closed subprescheme) of X is canonically identified with a subprescheme (resp. closed subprescheme) of X.* **I | 121**

PROOF. Since a locally closed subset of a locally closed subspace of X is a locally closed subspace of X, it is clear (4.1.5) that the question is local and that we can thus suppose that X is affine; the proposition then follows from the canonical identification of $A/\mathfrak{J}'$ with $(A/\mathfrak{J})/(\mathfrak{J}'/\mathfrak{J})$, where A is some ring, and $\mathfrak{J}$ and $\mathfrak{J}'$ are ideals of A such that $\mathfrak{J} \subset \mathfrak{J}'$. $\square$

In what follows, we will always make use of the above identification.

(4.1.7). Let Y be a subprescheme of a prescheme X, and denote by ψ the canonical injection $Y \to X$ of the *underlying subspaces*; we know that the inverse image $\psi^*(\mathscr{O}_X)$ is the restriction $\mathscr{O}_X|Y$ (**0**, 3.7.1). If, for each $y \in Y$, we denote by ω_y the canonical homomorphism $(\mathscr{O}_X)_y \to (\mathscr{O}_Y)_y$, then these homomorphisms are the restrictions to stalks of a *surjective* homomorphism ω of sheaves of rings $\mathscr{O}_X|Y \to \mathscr{O}_Y$: indeed, it suffices to check locally on Y, that is to say, we can suppose that X is affine and that the subprescheme Y is closed; if in this case $\mathscr{I}$ is the sheaf of ideals of $\mathscr{O}_X$ which defines Y, then the ω_y are none other than the restriction to stalks of the homomorphism $\mathscr{O}_X|Y \to (\mathscr{O}_X/\mathscr{I})|Y$. We have thus defined a *monomorphism of ringed spaces* (**0**, 4.1.1) $j = (\psi, \omega^\flat)$ which is evidently a morphism $Y \to X$ of preschemes (2.2.1), and we call this the *canonical injection morphism*.

If $f : X \to Z$ is a morphism, we then say that the composite morphism $Y \xrightarrow{j} X \xrightarrow{f} Z$ is the *restriction* of f to the subprescheme Y.

(4.1.8). Conforming to the general definitions (T, I, 1.1), we will say that a morphism of preschemes $f : Z \to X$ is *majoré*[8] by the injection morphism $j : Y \to X$ of a subprescheme Y of X if f factors as $Z \xrightarrow{g} Y \xrightarrow{j} X$, where g is a morphism of preschemes; further, g is necessarily *unique* since j is a monomorphism.

Proposition (4.1.9). — *For a morphism $f : Z \to X$ to be factor through an injection morphism $j : Y \to X$, it is necessary and sufficient that $f(Z) \subset Y$ and, for all $z \in Z$, letting $y = f(z)$, that the homomorphism $(\mathscr{O}_X)_y \to \mathscr{O}_z$ corresponding to f factor as $(\mathscr{O}_Z)_y \to (\mathscr{O}_Y)_y \to \mathscr{O}_z$* (or equivalently, *for the kernel of $(\mathscr{O}_X)_y \to \mathscr{O}_z$ to contain the kernel of $(\mathscr{O}_X)_y \to (\mathscr{O}_Y)_y$).*

PROOF. The conditions are evidently necessary. To see that they are sufficient, we can reduce to the case where Y is a *closed* subprescheme of X, by replacing X by an open U such that Y is closed in U (4.1.3) if necessary; Y is then defined by a quasi-coherent sheaf of ideals $\mathscr{I}$ of $\mathscr{O}_X$. Let $f = (\psi, \theta)$, and let $\mathscr{J}$ be the sheaf of ideals of $\psi^*(\mathscr{O}_X)$, kernel of $\theta^\sharp : \psi^*(\mathscr{O}_X) \to \mathscr{O}_Z$; considering the properties of the functor ψ^* (**0**, 3.7.2), the hypotheses imply that, for each $z \in Z$, we have $(\psi^*(\mathscr{I}))_z \subset \mathscr{J}_z$, and, as a result, that $\psi^*(\mathscr{I}) \subset \mathscr{J}$. Thus $\theta^\sharp$ factors as

$$\psi^*(\mathscr{O}_X) \longrightarrow \psi^*(\mathscr{O}_X)/\psi^*(\mathscr{I}) = \psi^*(\mathscr{O}_X/\mathscr{I}) \xrightarrow{\omega} \mathscr{O}_Z,$$

the first arrow being the canonical homomorphism. Let ψ' be the continuous map $Z \to Y$ coinciding with ψ; it is clear that we have $\psi'^*(\mathscr{O}_Y) = \psi^*(\mathscr{O}_X/\mathscr{J})$; on the other hand, ω is evidently a local homomorphism, so $g = (\psi', \omega^\flat)$ is a morphism of preschemes $Z \to Y$ (2.2.1), which, according to **I | 122**
the above, is such that $f = j \circ g$, hence the proposition. $\square$

Corollary (4.1.10). — *For an injection morphism $Z \to X$ to be factor through the injection morphism $Y \to X$, it is necessary and sufficient for Z to be a subprescheme of Y.*

We then write $Z \leqslant Y$, and this condition is evidently an *ordering* on the set of subpreschemes of X.

[8] *[Trans.] There doesn't seem to be an English equivalent of this, except for 'bounded above', which doesn't make much sense in this context. We would normally just say that 'f factors through j', but to avoid having to entirely restructure the often-lengthy sentences in the original, we sometimes (but as little as we can) use 'majoré'.*

4.2. Immersion morphisms

Definition (4.2.1). — We say that a morphism $f : Y \to X$ is an immersion (resp. a closed immersion, an open immersion) if it factors as $Y \xrightarrow{g} Z \xrightarrow{j} X$, where g is an isomorphism, Z is a subprescheme of X (resp. a closed subprescheme, a subprescheme induced by an open set), and j is the injection morphism.

The subprescheme Z and the isomorphism g are then determined in a *unique* way, since if Z' is a second subprescheme of X, j' the injection $Z' \to X$, and g' an isomorphism $Y \to Z'$ such that $j \circ g = j' \circ g'$, then we have $j' = j \circ g \circ g'^{-1}$, hence $Z' \leqslant Z$ (4.1.10), and we can similarly show that $Z \leqslant Z'$, hence $Z' = Z$, and, since j is a monomorphism of preschemes, $g' = g$.

We say that $f = j \circ g$ is the *canonical factorization* of the immersion f, and the subprescheme Z and the isomorphism g are those *associated to* f.

It is clear that an immersion is a *monomorphism* of preschemes (4.1.7) and *a fortiori* a *radicial* morphism (3.5.4).

Proposition (4.2.2). —

(a) *For a morphism $f = (\psi, \theta) : Y \to X$ to be an open immersion, it is necessary and sufficient for ψ to be a homeomorphism from Y to some open subset of X, and, for all $y \in Y$, that the homomorphism $\theta^\sharp_y : \mathscr{O}_{\psi(y)} \to \mathscr{O}_y$ be bijective.*

(b) *For a morphism $f = (\psi, \theta) : Y \to X$ to be an immersion (resp. a closed immersion), it is necessary and sufficient for ψ to be a homeomorphism from Y to some locally closed (resp. closed) subset of X, and, for all $y \in Y$, that the homomorphism $\theta^\sharp_y : \mathscr{O}_{\psi(y)} \to \mathscr{O}_y$ be surjective.*

PROOF.

(a) The conditions are evidently necessary. Conversely, if they are satisfied, then it is clear that $\theta^\sharp$ is an isomorphism from $\mathscr{O}_Y$ to $\psi^*(\mathscr{O}_X)$, and $\psi^*(\mathscr{O}_X)$ is the sheaf induced by "transport of structure" via ψ^{-1} from $\mathscr{O}_X|\psi(Y)$; hence the conclusion.

(b) The conditions are evidently necessary—we prove that they are sufficient. Consider first the particular case where we suppose that X is an affine scheme, and that $Z = \psi(Y)$ is *closed* in X. We then know (**0**, 3.4.6) that $\psi_*(\mathscr{O}_Y)$ has support equal to Z, and that, denoting its restriction to Z by $\mathscr{O}'_Z$, the ringed space $(Z, \mathscr{O}'_Z)$ is induced from $(Y, \mathscr{O}_Y)$ by transport of structure via the homeomorphism ψ considered as a map from Y to Z. Let us now show that $f_*(\mathscr{O}_Y) = \psi_*(\mathscr{O}_Y)$ is a *quasi-coherent* $\mathscr{O}_X$-module. Indeed, for all $x \notin Z$, $\psi_*(\mathscr{O}_Y)$ restricted to a suitable neighbourhood of x is zero. On the contrary, if $z \in Z$, then we have $x = \psi(y)$ for some well-defined $y \in Y$; let V be an affine open neighbourhood of y in Y; $\psi(V)$ is then open in Z, and so equal to the intersection of Z with an open subset U of X, and the restriction of U to $\psi_*(\mathscr{O}_Y)$ is identical to the restriction of U to the direct image $(\psi_V)_*(\mathscr{O}_Y|V)$, where ψ_V is the restriction of ψ to V. The restriction of the morphism (ψ, θ) **I** | 123
to $(V, \mathscr{O}_Y|V)$ is a morphism from this aforementioned prescheme to $(X, \mathscr{O}_X)$, and, as a result, is of the form $({}^a\phi, \widetilde{\phi})$, where ϕ is the homomorphism from the ring $A = \Gamma(X, \mathscr{O}_X)$ to the ring $\Gamma(V, \mathscr{O}_Y)$ (1.7.3); we conclude that $(\psi_V)_*(\mathscr{O}_Y|V)$ is a quasi-coherent $\mathscr{O}_X$-module (1.6.3), which proves our assertion, due to the local nature of quasi-coherent sheaves. In addition, the hypothesis that ψ is a homeomorphism implies (**0**, 3.4.5) that, for all $y \in Y$, ψ_y is an isomorphism $(\psi_*(\mathscr{O}_Y))_{\psi(y)} \to \mathscr{O}_y$; since the diagram

$$\begin{tikzcd} \mathscr{O}_{\psi(y)} \arrow[r, "\theta_{\psi(y)}"] \arrow[d, "\psi_y \circ \rho_{\psi(y)}"'] & (\psi_*(\mathscr{O}_Y))_{\psi(y)} \arrow[d, "\psi_y"] \\ (\psi^*(\mathscr{O}_X))_y \arrow[r, "\theta^\sharp_y"] & \mathscr{O}_y \end{tikzcd}$$

is commutative, and the vertical arrows are the isomorphisms (**0**, 3.7.2), the hypothesis that $\theta^\sharp_y$ is surjective implies that $\theta_{\psi(y)}$ is surjective as well. Since the support of $\psi_*(\mathscr{O}_Y)$

is $Z = \psi(Y)$, θ is a *surjective* homomorphism from $\mathscr{O}_X = \widetilde{A}$ to the quasi-coherent $\mathscr{O}_X$-module $f_*(\mathscr{O}_Y)$. As a result, there exists a unique isomorphism ω from a sheaf quotient $\widetilde{A}/\widetilde{\mathfrak{J}}$ (with $\mathfrak{J}$ being an ideal of A) to $f_*(\mathscr{O}_Y)$ which, when composed with the canonical homomorphism $\widetilde{A} \to \widetilde{A}/\widetilde{\mathfrak{J}}$, gives θ (1.3.8); if $\mathscr{O}_Z$ denotes the restriction of $\widetilde{A}/\widetilde{\mathfrak{J}}$ to Z, then $(Z, \mathscr{O}_Z)$ is a subprescheme of $(X, \mathscr{O}_X)$, and f factors through the canonical injection of this subprescheme into X and the isomorphism (ψ_0, ω_0), where ψ_0 is ψ considered as a map from Y to Z, and ω_0 the restriction of ω to $\mathscr{O}_Z$.

We now pass to the general case. Let U be an affine open subset of X such that $U \cap \psi(Y)$ is closed in U and nonempty. By restricting f to the prescheme induced by Y on the open subset $\psi^{-1}(U)$, and by considering it as a morphism from this prescheme to the prescheme induced by X on U, we reduce to the first case; the restriction of f to $\psi^{-1}(U)$ is thus a closed immersion $\psi^{-1}(U) \to U$, canonically factoring as $j_U \circ g_U$, where g_U is an isomorphism from the prescheme $\psi^{-1}(U)$ to a subprescheme Z_U of U, and j_U is the canonical injection $Z_U \to U$. Let V be a second affine open subset of X such that $V \subset U$; since the restriction Z'_V of Z_U to V is a subprescheme of the prescheme V, the restriction of f to $\psi^{-1}(V)$ factors as $j'_V \circ g'_V$, where j'_V is the canonical injection $Z'_V \to V$ and g'_V is an isomorphism from $\psi^{-1}(V)$ to Z'_V. By the uniqueness of the canonical factorization of an immersion (4.2.1), we necessarily have that $Z'_V = Z_V$ and $g'_V = g_V$. We conclude (4.1.5) that there is a subprescheme Z of X whose underlying space is $\psi(Y)$, and whose restriction to each $U \cap \psi(Y)$ is Z_U; the g_U are then the restrictions to $\psi^{-1}(U)$ of an isomorphism $g : Y \to Z$ such that $f = j \circ g$, where j is the canonical injection $Z \to X$.

□

Corollary (4.2.3). — *Let X be an affine scheme. For a morphism $f = (\psi, \theta) : Y \to X$ to be a closed immersion, it is necessary and sufficient for Y to be an affine scheme, and the homomorphism $\Gamma(\psi) : \Gamma(\mathscr{O}_X) \to \Gamma(\mathscr{O}_Y)$ to be surjective.*

Corollary (4.2.4). —

(a) *Let f be a morphism $Y \to X$, and (V_λ) a cover of $f(Y)$ by open subsets of X. For f to be an immersion (resp. an open immersion), it is necessary and sufficient for its restriction to each of the induced preschemes $f^{-1}(V_\lambda)$ to be an immersion (resp. an open immersion) into V_λ.* **I** | 124

(b) *Let f be a morphism $Y \to X$, and (V_λ) an open cover of X. For f to be a closed immersion, it is necessary and sufficient for its restriction to each of the induced preschemes $f^{-1}(V_\lambda)$ to be a closed immersion into V_λ.*

PROOF. Let $f = (\psi, \theta)$; in the case (a), $\theta^\sharp_y$ is surjective (resp. bijective) for all $y \in Y$, and in the case (b), $\theta^\sharp_y$ is surjective for all $y \in Y$; it thus suffices to check, in case (a), that ψ is a homeomorphism from Y to a locally closed (resp. open) subset of X, and, in case (b), that ψ is a homeomorphism from Y to a closed subset of X. Now ψ is evidently injective, and sends each neighbourhood of y in Y to a neighbourhood of $\psi(y)$ is $\psi(Y)$ for all $y \in Y$, by virtue of the hypothesis; in case (a), $\psi(Y) \cap V_\lambda$ is locally closed (resp. open) in V_λ, so $\psi(Y)$ is locally closed (resp. open) in the union of the V_λ, and *a fortiori* in X; in case (b), $\psi(Y) \cap V_\lambda$ is closed in V_λ, so $\psi(Y)$ is closed in X since $X = \bigcup_\lambda V_\lambda$. □

Proposition (4.2.5). — *The composition of any two immersions (resp. of two open immersions, of two closed immersions) is an immersion (resp. an open immersion, a closed immersion).*

PROOF. This follows easily from (4.1.6). □

4.3. Products of immersions

Proposition (4.3.1). — *Let $\alpha : X' \to X$, $\beta : Y' \to Y$ be two S-morphisms; if α and β are immersions (resp. open immersions, closed immersions), then $\alpha \times_S \beta$ is an immersion (resp. an open immersion, a closed immersion). In addition, if α (resp. β) identifies X' (resp. Y') with a subprescheme X'' (resp. Y'') of X (resp. Y), then $\alpha \times_S \beta$ identifies the underlying space of $X' \times_S Y'$ with the subspace $p^{-1}(X'') \cap q^{-1}(Y'')$ of the underlying space of $X \times_S Y$, where p and q denote the projections from $X \times_S Y$ to X and Y respectively.*

PROOF. According to Definition (4.2.1), we can restrict to the case where X' and Y' are subpreschemes, and α and β the injection morphisms. The proposition has already been proven for the subpreschemes induced by open sets (3.2.7); since every subprescheme is a closed subprescheme of a prescheme induced by an open set (4.1.3), we can reduce to the case where X' and Y' are *closed* subpreschemes.

Let us first show that we can assume S to be *affine*. Let (S_λ) be a cover of S by affine open sets; if ϕ and ψ are the structure morphisms of X and Y, then let $X_\lambda = \phi^{-1}(S_\lambda)$ and $Y_\lambda = \psi^{-1}(S_\lambda)$. The restriction X'_λ (resp. Y'_λ) of X' (resp. Y') to $X_\lambda \cap X'$ (resp. $Y_\lambda \cap Y'$) is a closed subprescheme of X_λ (resp. Y_λ), the preschemes X_λ, Y_λ, X'_λ, and Y'_λ can be considered as S_λ-preschemes, and the products $X_\lambda \times_S Y_\lambda$ and $X_\lambda \times_{S_\lambda} Y_\lambda$ (resp. $X'_\lambda \times_S Y'_\lambda$ and $X'_\lambda \times_{S_\lambda} Y'_\lambda$) are identical (3.2.5). If the proposition is true when S is affine, then the restriction of $\alpha \times_S \beta$ to each of the $X'_\lambda \times_S Y'_\lambda$ is thus an immersion (3.2.7). Since the product $X'_\lambda \times_S Y'_\mu$ (resp. $X_\lambda \times_S Y_\mu$) can be identified with $(X'_\lambda \cap X'_\mu) \times_S (Y'_\lambda \cap Y'_\mu)$ (resp. $(X_\lambda \cap X_\mu) \times_S (Y_\lambda \cap Y_\mu)$) (3.2.6.4), the restriction of $\alpha \times_S \beta$ to each of the $X'_\lambda \times_S Y'_\mu$ is also an **I** | 125
immersion; the same is true for $\alpha \times_S \beta$ by (4.2.4).

Next, we show that we can assume X and Y to be *affine*. Indeed, let (U_i) (resp. (V_j)) be a cover of X (resp. Y) by affine open sets, and let X'_i (resp. Y'_j) be the restriction of X' (resp. Y') to $X' \cap U_i$ (resp. $Y' \cap V_j$), which is a closed subprescheme of U_i (resp. V_j); $U_i \times_S V_j$ can be identified with the restriction of $X \times_S Y$ to $p^{-1}(U_i) \cap q^{-1}(V_j)$ (3.2.7); similarly, if p' and q' are the projections from $X' \times_S Y'$, then $X'_i \times_S Y'_j$ can be identified with the restriction of $X' \times_S Y'$ to $p'^{-1}(X'_i) \cap q'^{-1}(Y'_j)$. Set $\gamma = \alpha \times_S \beta$; we have, by definition, $p \circ \gamma = \alpha \circ p'$ and $q \circ \gamma = \beta \circ q'$; since $X'_i = \alpha^{-1}(U_i)$ and $Y'_j = \beta^{-1}(V_j)$, we also have $p'^{-1}(X'_i) = \gamma^{-1}(p^{-1}(U_i))$ and $q'^{-1}(Y'_j) = \gamma^{-1}(q^{-1}(V_j))$, hence

$$p'^{-1}(X'_i) \cap q'^{-1}(Y'_j) = \gamma^{-1}(p^{-1}(U_i) \cap q^{-1}(V_j)) = \gamma^{-1}(U_i \times_S V_j),$$

and we then conclude as in the previous part of the proof.

So suppose X, Y, and S are affine, and let B, C, and A be their respective rings. Then B and C are A-algebras, and X' and Y' are affine schemes whose rings are quotient algebras B' and C' of B and C respectively. In addition, we have $\alpha = ({}^a\rho, \widetilde{\rho})$ and $\beta = ({}^a\sigma, \widetilde{\sigma})$, where ρ and σ are (respectively) the canonical homomorphisms $B \to B'$ and $C \to C'$ (1.7.3). With that in mind, we know that $X \times_S Y$ (resp. $X' \times_S Y'$) is an affine scheme with ring $B \otimes_A C$ (resp. $B' \otimes_A C'$), and $\alpha \times_S \beta = ({}^a\tau, \widetilde{\tau})$, where τ is the homomorphism $\rho \otimes \sigma$ from $B \otimes_A C$ to $B' \otimes_A C'$ (Proposition (3.2.2) and Corollary (3.2.3)); since this homomorphism is surjective, $\alpha \times_S \beta$ is an immersion. In addition, if $\mathfrak{b}$ (resp. $\mathfrak{c}$) is the kernel of ρ (resp. σ), then the kernel of τ is $u(\mathfrak{b}) + v(\mathfrak{c})$, where u (resp. v) is the homomorphism $b \mapsto b \otimes 1$ (resp. $c \mapsto 1 \otimes c$). Since $p = ({}^a u, \widetilde{u})$ and $q = ({}^a v, \widetilde{v})$, this kernel corresponds, in the prime spectrum of $B \otimes_A C$, to the closed set $p^{-1}(X') \cap q^{-1}(Y')$ ((1.2.2.1) and Proposition (1.1.2, iii)), which finishes the proof. □

Corollary (4.3.2). — *If $f : X \to Y$ is an immersion (resp. an open immersion, a closed immersion) and an S-morphism, then $f_{(S')}$ is an immersion (resp. an open immersion, a closed immersion) for every extension $S' \to S$ of the base prescheme.*

4.4. Inverse images of a subprescheme

Proposition (4.4.1). — *Let $f : X \to Y$ be a morphism, Y' a subprescheme (resp. a closed subprescheme, a prescheme induced by an open set) of Y, and $j : Y' \to Y$ the injection morphism. Then the projection $p : X \times_Y Y' \to X$ is an immersion (resp. a closed immersion, an open immersion); the underlying space of the subprescheme of X associated to p is $f^{-1}(Y')$; in addition, if j' is the injection morphism of this subprescheme into X, then for a morphism $h : Z \to X$ to be such that $f \circ h : Z \to Y$ factors through j, it is necessary and sufficient for h to factor through j'.*

Proof. Since $p = 1_X \times_Y j$ (3.3.4), the first claim follows from Proposition (4.3.1); the second is a particular case of Corollary (3.5.10) (after swapping the roles of X and Y'). Finally, if we have $f \circ h = j \circ h'$, where h' is a morphism $Z \to Y'$, then it follows from the definition of the product that we have $h = p \circ u$, where u is a morphism $Z \to X \times_Y Y'$, whence the last claim. □

We say that the subprescheme of X thus defined is the *inverse image* of the subprescheme Y' of Y under the morphism f, terminology which is consistent with that introduced more generally in (3.3.6). When we speak of $f^{-1}(Y')$ as a subprescheme of X, this will always be the subprescheme we mean. **I** | 126

When the preschemes $f^{-1}(Y')$ and X are identical, j' is the identity and each morphism $h : Z \to X$ thus factors through j', so *the morphism $f : X \to Y$ factors as $X \xrightarrow{g} Y' \xrightarrow{j} Y$.*

When y is a *closed* point of Y and $Y' = \operatorname{Spec}(k(y))$ is the smallest closed subprescheme of Y having $\{y\}$ as its underlying space (4.1.9), the closed subprescheme $f^{-1}(Y')$ is canonically isomorphic to the *fibre* $f^{-1}(y)$ defined in (3.6.2), and we will use this identification in all that follows.

Corollary (4.4.2). — *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms, and $h = g \circ f$ their composition. For each subprescheme Z' of Z, the subpreschemes $f^{-1}(g^{-1}(Z'))$ and $h^{-1}(Z')$ of X are identical.*

Proof. This follows from the existence of the canonical isomorphism $X \times_Y (Y \times_Z Z') \simeq X \times_Z Z'$ (3.3.9.1). □

Corollary (4.4.3). — *Let X' and X'' be subpreschemes of X, and $j' : X' \to X$, and $j'' : X'' \to X$ their injection morphisms; then ${j'}^{-1}(X'')$ and ${j''}^{-1}(X')$ are both equal to the greatest lower bound $\inf(X', X'')$ of X' and X'' for the ordering $\leqslant$ on subpreschemes, and this is canonically isomorphic to $X' \times_X X''$.*

Proof. This follows immediately from Proposition (4.4.1) and Corollary (4.1.10). □

Corollary (4.4.4). — *Let $f : X \to Y$ be a morphism, and Y' and Y'' subpreschemes of Y; then we have $f^{-1}(\inf(Y', Y'')) = \inf(f^{-1}(Y'), f^{-1}(Y''))$.*

Proof. This follows from the existence of the canonical isomorphism between $(X \times_Y Y') \times_X (X \times_Y Y'')$ and $X \times_Y (Y' \times_Y Y'')$ (3.3.9.1). □

Proposition (4.4.5). — *Let $f : X \to Y$ be a morphism, and Y' a closed subprescheme of Y defined by a quasi-coherent sheaf of ideals $\mathscr{K}$ of $\mathscr{O}_Y$ (4.1.3); the closed subprescheme $f^{-1}(Y')$ of X is then defined by the quasi-coherent sheaf of ideals $f^*(\mathscr{K})\mathscr{O}_X$ of $\mathscr{O}_X$.*

Proof. The statement is evidently local on X and Y; it thus suffices to note that if A is a B-algebra and $\mathfrak{K}$ an ideal of B, then we have $A \otimes_B (B/\mathfrak{K}) = A/\mathfrak{K}A$, and to then apply (1.6.9). □ **Err$_{\text{II}}$**

Corollary (4.4.6). — *Let X' be a closed subprescheme of X defined by a quasi-coherent sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_X$, and i the injection $X' \to X$; for the restriction $f \circ i$ of f to X' to factor through the injection $j : Y' \to Y$* (in other words, for it to factor as $j \circ g$, with g a morphism $X' \to Y'$), *it is necessary and sufficient that $f^*(\mathscr{K}) \subset \mathscr{J}$.*

Proof. It suffices to apply Proposition (4.4.1) to i, taking Proposition (4.4.5) into account. □

4.5. Local immersions and local isomorphisms

Definition (4.5.1). — Let $f : X \to Y$ be a morphism of preschemes. We say that f is a local immersion at a point $x \in X$ if there exists an open neighbourhood U of x in X and an open neighbourhood V of $f(x)$ in Y such that the restriction of f to the induced prescheme U is a closed immersion of U into the induced prescheme V. We say that f is a local immersion if f is a local immersion at each point of X.

Definition (4.5.2). — We say that a morphism $f : X \to Y$ is a local isomorphism at a point $x \in X$ if there exists an open neighbourhood U of x in X such that the restriction of f to the induced prescheme U is an open immersion of U into Y. We say that f is a local isomorphism if f is a local isomorphism at each point of X. **I** | 127

(4.5.3). An immersion (resp. a closed immersion) $f : X \to Y$ can be characterized as a local immersion such that f is a homeomorphism from the underlying space of X to a subset (resp. a closed subset) of Y. An open immersion f can be characterized as an *injective* local isomorphism.

Proposition (4.5.4). — *Let X be an irreducible prescheme, and $f : X \to Y$ a dominant injective morphism. If f is a local immersion, then f is an immersion, and $f(X)$ is open in Y.*

PROOF. Let $x \in X$, and let U be an open neighbourhood of x, and V an open neighbourhood of $f(x)$ in Y such that the restriction of f to U is a closed immersion into V; since U is dense in X, $f(U)$ is dense in Y by hypothesis, so $f(U) = V$, and f is a homeomorphism from U to V; the hypothesis that f is injective implies that $f^{-1}(V) = U$, hence the proposition. □

Proposition (4.5.5). —

(i) *The composition of any two local immersions (resp. of two local isomorphisms) is a local immersion (resp. a local isomorphism).*
(ii) *Let $f : X \to X'$ and $g : Y \to Y'$ be two S-morphisms. If f and g are local immersions (resp. local isomorphisms), then so too is $f \times_S g$.*
(iii) *If an S-morphism f is a local immersion (resp. a local isomorphism), then so too is $f_{(S')}$ for every extension $S' \to S$ of the base prescheme.*

PROOF. According to (3.5.1), it suffices to prove (i) and (ii).

(i) follows immediately from the transitivity of closed (resp. open) immersions (4.2.5) and from the fact that if f is a homeomorphism from X to a closed subset of Y, then for every open $U \subset X$, $f(U)$ is open in $f(X)$, so there exists an open subset V of Y such that $f(U) = V \cap f(X)$, and, as a result, $f(U)$ is closed in V.

To prove (ii), let p and q be the projections from $X \times_X Y$, and p' and q' the projections from $X' \times_S Y'$. There exists, by hypothesis, open neighbourhoods U, U', V, and V' of $x = p(z)$, $x' = p'(z')$, $y = q(z)$, and $y' = q'(z')$ (respectively), such that the restrictions of f and g to U and V (respectively) are closed (resp. open) immersions into U' and V' (respectively). Since the underlying spaces of $U \times_S V$ and $U' \times_S V'$ can be identified with the open neighbourhoods $p^{-1}(U) \cap q^{-1}(V)$ and $p'^{-1}(U') \cap q'^{-1}(V')$ of z and z' (respectively) (3.2.7), the proposition follows from Proposition (4.3.1). □

§5. Reduced preschemes; the separation condition

5.1. Reduced preschemes

Proposition (5.1.1). — *Let $(X, \mathscr{O}_X)$ be a prescheme, and $\mathscr{B}$ a quasi-coherent $\mathscr{O}_X$-algebra. Then there exists a unique quasi-coherent $\mathscr{O}_X$-module $\mathscr{N}$ whose stalk $\mathscr{N}_x$ at any $x \in X$ is the nilradical of the ring $\mathscr{B}_x$. When X is affine, and, consequently, $\mathscr{B} = \widetilde{B}$, where B is an algebra over $A(X)$, then we have $\mathscr{N} = \widetilde{\mathfrak{N}}$, where $\mathfrak{N}$ is the nilradical of B.*

PROOF. The statement is local, so it suffices to show the latter claim. We know that $\widetilde{\mathfrak{N}}$ is a quasi-coherent $\mathscr{O}_X$-module (1.4.1), and that its stalk at a point $x \in X$ is the ideal $\mathfrak{N}_x$ of the ring of fractions B_x; it remains to prove that the nilradical of B_x is contained in $\mathfrak{N}_x$, the converse inclusion being evident. Let z/s be an element of the nilradical of B_x, with $z \in B$, and $s \notin \mathfrak{j}_x$; by hypothesis, there exists an integer k such that $(z/s)^k = 0$, which implies that there exists some $t \notin \mathfrak{j}_x$ such that $tz^k = 0$. We conclude that $(tz)^k = 0$, and, as a result, that $z/s = (tz)/(ts) \in \mathfrak{N}_x$. □ **I** | 128

We say that the quasi-coherent $\mathscr{O}_X$-module $\mathscr{N}$ thus defined is the *nilradical* of the $\mathscr{O}_X$-algebra $\mathscr{B}$; in particular, we denote by $\mathscr{N}_X$ the nilradical of $\mathscr{O}_X$.

Corollary (5.1.2). — *Let X be a prescheme; the closed subprescheme of X defined by the sheaf of ideals $\mathscr{N}_X$ is the only reduced subprescheme (**0**, 4.1.4) of X that has X as its underlying space; it is also the smallest subprescheme of X that has X as its underlying space.*

PROOF. Since the structure sheaf of the closed subprescheme of Y defined by $\mathscr{N}_X$ is $\mathscr{O}_X/\mathscr{N}_X$, it is immediate that Y is reduced and has X as its underlying space, because $\mathscr{N}_x \neq \mathscr{O}_x$ for any $x \in X$. To show the other claims, note that a subprescheme Z of X that has X as its underlying space is defined by a sheaf of ideals $\mathscr{I}$ (4.1.3) such that $\mathscr{I}_x \neq \mathscr{O}_x$ for any $x \in X$. We can restrict to the case where X is affine, say $X = \operatorname{Spec}(A)$ and $\mathscr{I} = \widetilde{\mathfrak{I}}$, where $\mathfrak{I}$ is an ideal of A; then, for every $x \in X$, we have $\mathfrak{I}_x \subset \mathfrak{j}_x$, and so $\mathfrak{I}$ is contained in every prime ideal of A, and so also in their intersection $\mathfrak{N}$, the nilradical of A. This proves that Y is the small subprescheme of X that has X as its underlying space (4.1.9); furthermore, if Z is distinct from Y, we necessarily have $\mathscr{I}_x \neq \mathscr{N}_x$ for at least one $x \in X$, and so (5.1.1) Z is not reduced. ☐

Definition (5.1.3). — We define the reduced prescheme associated to a prescheme X, denoted by X_{red}, to be the unique reduced subprescheme of X that has X as its underlying space.

Saying that a prescheme X is reduced thus implies that $X = X_{\mathrm{red}}$.

Proposition (5.1.4). — *For the prime spectrum of a ring A to be a reduced (resp. integral) prescheme (2.1.7), it is necessary and sufficient for A to be a reduced (resp. integral) ring.*

PROOF. Indeed, it follows immediately from (5.1.1) that the condition $\mathscr{N} = (0)$ is necessary and sufficient for $X = \operatorname{Spec}(A)$ to be reduced; the claim corresponding to integral rings is then a consequence of (1.1.13). ☐

Since every ring of fractions $\neq \{0\}$ of an integral ring is integral, it follows from (5.1.4) that, for every *locally integral* prescheme X, $\mathscr{O}_x$ is an *integral* ring for every $x \in X$. The converse is true whenever the underlying space of X is *locally Noetherian*: indeed, X is then reduced, and if U is an affine open subset of X, which is a Noetherian space, then U has only a finite number of irreducible components, and so its ring A has only a finite number of minimal prime ideals (1.1.14). If two of the irreducible components of U had a common point x, then $\mathscr{O}_x$ would have at least two distinct minimal prime ideals, and would thus not be integral; the components of U are thus open subsets that are pairwise disjoint, and each of them is thus integral.

(5.1.5). Let $f = (\psi, \theta) : X \to Y$ be a morphism of preschemes; the homomorphism $\theta_x^\sharp : \mathscr{O}_{\psi(x)} \to \mathscr{O}_x$ **I** | 129
sends each nilpotent element of $\mathscr{O}_{\psi(x)}$ to a nilpotent element of $\mathscr{O}_x$; by passing to the quotients, $\theta^\sharp$ induces a homomorphism

$$\omega : \psi^*(\mathscr{O}_Y/\mathscr{N}_Y) \longrightarrow \mathscr{O}_X/\mathscr{N}_X.$$

It is clear that, for every $x \in X$, $\omega_x : \mathscr{O}_{\psi(x)}/\mathscr{N}_{\psi(x)} \to \mathscr{O}_x/\mathscr{N}_x$ is a local homomorphism, and so $(\psi, \omega^\flat)$ is a morphism of preschemes $X_{\mathrm{red}} \to Y_{\mathrm{red}}$, which we denote by f_{red}, and call the *reduced* morphism associated to f. It is immediate that, for morphisms $f : X \to Y$ and $g : Y \to Z$, we have $(g \circ f)_{\mathrm{red}} = g_{\mathrm{red}} \circ f_{\mathrm{red}}$, and so we have defined X_{red} as a *functor*, *covariant* in X.

The preceding definition shows that the diagram

$$\begin{tikzcd} X_{\mathrm{red}} \arrow[r, "f_{\mathrm{red}}"] \arrow[d] & Y_{\mathrm{red}} \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is commutative, where the vertical arrows are the injection morphisms; in other words, $X_{\mathrm{red}} \to X$ is a *functorial* morphism. We note in particular that, if X is reduced, then every morphism $f : X \to Y$ factors as $X \xrightarrow{f_{\mathrm{red}}} Y_{\mathrm{red}} \to Y$; in other words, f factors through the injection morphism $Y_{\mathrm{red}} \to Y$.

Proposition (5.1.6). — *Let $f : X \to Y$ be a morphism; if f is surjective (resp. radicial, an immersion, a closed immersion, an open immersion, a local immersion, a local isomorphism), then so too is f_{red}. Conversely, if f_{red} is surjective (resp. radicial), then so too is f.*

PROOF. The proposition is trivial if f is surjective; if f is radicial, then the proposition follows from the fact that, for every $x \in X$, the field $k(x)$ is the same for the preschemes X and X_{red} (3.5.8). Finally, if $f = (\psi, \theta)$ is an immersion, a closed immersion, or a local immersion (resp. an open immersion, or a local isomorphism), then the proposition follows from the fact that, if $\theta_x^\sharp$ is surjective (resp. bijective), then so too is the homomorphism obtained by passing to the quotients by the nilradicals $\mathscr{O}_{\psi(x)}$ and $\mathscr{O}_x$ ((5.1.2) and (4.2.2)) (cf. (5.5.12)). □

Proposition (5.1.7). — *If X and Y are S-preschemes, then the preschemes $X_{\mathrm{red}} \times_{S_{\mathrm{red}}} Y_{\mathrm{red}}$ and $X_{\mathrm{red}} \times_S Y_{\mathrm{red}}$ are identical, and canonically identified with a subprescheme of $X \times_S Y$ that has the same underlying subspace as the two aforementioned products.*

PROOF. The canonical identification of $X_{\mathrm{red}} \times_S Y_{\mathrm{red}}$ with a subprescheme of $X \times_S Y$ that has the same underlying space follows from (4.3.1). Furthermore, if ϕ and ψ are the structure morphisms $X_{\mathrm{red}} \to S$ and $Y_{\mathrm{red}} \to S$ (respectively), then they factor through S_{red} (5.1.5), and since $S_{\mathrm{red}} \to S$ is a monomorphism, the first claim of the proposition follows from (3.2.4). □

Corollary (5.1.8). — *The preschemes $(X \times_S Y)_{\mathrm{red}}$ and $(X_{\mathrm{red}} \times_{S_{\mathrm{red}}} Y_{\mathrm{red}})_{\mathrm{red}}$ are canonically identified with one another.*

PROOF. This follows from (5.1.2) and (5.1.7). □

We note that, even if X and Y are reduced preschemes, $X \times_S Y$ might not be reduced, because the tensor product of two reduced algebras can have nilpotent elements.

Proposition (5.1.9). — *Let X be a prescheme, and $\mathscr{I}$ a quasi-coherent sheaf of ideals of $\mathscr{O}_X$ such that $\mathscr{I}^n = 0$ for some integer $n > 0$. Let X_0 be the closed subprescheme $(X, \mathscr{O}_X/\mathscr{I})$ of X; for X to be an affine scheme, it is necessary and sufficient for X_0 to be an affine scheme.* **I** | 130

The condition is clearly necessary, so we will show that it is sufficient. If we set $X_k = (X, \mathscr{O}_X/\mathscr{I}^{k+1})$, it is enough to prove by induction on k that X_k is affine, and so we are led to consider the base case, where $\mathscr{I}^2 = 0$. We set

$$A = \Gamma(X, \mathscr{O}_X)$$
$$A_0 = \Gamma(X_0, \mathscr{O}_{X_0}) = \Gamma(X, \mathscr{O}_X/\mathscr{I}).$$

The canonical homomorphism $\mathscr{O}_X \to \mathscr{O}_X/\mathscr{I}$ induces a homomorphism of rings $\phi : A \to A_0$. We will see below that ϕ is *surjective*, which implies that

(5.1.9.1) $$0 \longrightarrow \Gamma(X, \mathscr{I}) \longrightarrow \Gamma(X, \mathscr{O}_X) \longrightarrow \Gamma(X, \mathscr{O}_X/\mathscr{I}) \longrightarrow 0$$

is an *exact* sequence. We now prove, assuming that this is true, the proposition. Note that $\mathfrak{K} = \Gamma(X, \mathscr{I})$ is an ideal whose square is zero in A, and thus a module over $A_0 = A/\mathfrak{K}$. By hypothesis, we have $X_0 = \operatorname{Spec}(A)$, and, since the underlying spaces of X_0 and X are identical, $\mathfrak{K} = \Gamma(X_0, \mathscr{I})$; Additionally, since $\mathscr{I}^2 = 0$, $\mathscr{I}$ is a quasi-coherent $(\mathscr{O}_X/\mathscr{I})$-module, so we have $\mathscr{I} \cong \widetilde{\mathfrak{K}}$ and $\mathfrak{K}_x = \mathscr{I}_x$ for all $x \in X_0$ (1.4.1). With this in mind, let $X' = \operatorname{Spec}(A)$, and consider the morphism $f = (\psi, \theta) : X \to X'$ of preschemes that corresponds to the identity map $A \to \Gamma(X, \mathscr{O}_X)$ (2.2.4). For every affine open subset V of X, the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \Gamma(V, \mathscr{O}_X|V) \arrow[d] \\ A_0 = A/\mathfrak{K} \arrow[r] & \Gamma(V, \mathscr{O}_{X_0}|V) \end{tikzcd}$$

commutes, whence the diagram

$$\begin{tikzcd} X' & X \arrow[l, "f"'] \\ X'_0 \arrow[u, "j'"] & X_0 \arrow[l, "f_0"] \arrow[u, "j"'] \end{tikzcd}$$

also commutes (X'_0 being the closed subprescheme of X' defined by the quasi-coherent sheaf of ideals $\widetilde{R}$, and j and j' being the canonical injection morphisms). But since X_0 is affine, f_0 is an isomorphism, and since the underlying continuous maps of j and j' are identity maps, we see straight away that $\psi : X \to X'$ is a homeomorphism. Furthermore, the equation $\mathfrak{K}_x = \mathscr{I}_x$ shows that the restriction of $\theta^\sharp : \psi^*(\mathscr{O}_{X'}) \to \mathscr{O}_X$ is an *isomorphism* from $\psi^*(\widetilde{\mathfrak{K}})$ to $\mathscr{I}$; additionally, by passing to the quotients, $\theta^\sharp$ gives an *isomorphism* $\psi^*(\mathscr{O}_X/\widetilde{\mathfrak{K}}) \to \mathscr{O}_X/\mathscr{I}$, because f_0 is an isomorphism; we thus immediately conclude, by the 5 lemma (M, I, 1.1), that $\theta^\sharp$ is itself an isomorphism, and thus that f is an *isomorphism*, and thus that X is affine. So everything reduces to proving the exactitude of (5.1.9.1), which will follow from showing that $\mathrm{H}^1(X, \mathscr{I}) = 0$. But $\mathrm{H}^1(X, \mathscr{I}) = \mathrm{H}^1(X_0, \mathscr{I})$, and we have seen **I** | 131
that $\mathscr{I}$ is a quasi coherent $\mathscr{O}_{X_0}$-module. Our proof will thus follow from

Lemma (5.1.9.2). — *If Y is an affine scheme, and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_Y$-module, then $\mathrm{H}^1(Y, \mathscr{F}) = 0$.*

Proof. This lemma will be proven in Chapter III, §1, as a consequence of the more general theorem that $\mathrm{H}^i(Y, \mathscr{F}) = 0$ for all $i > 0$. To give an independent proof, note that $\mathrm{H}^1(Y, \mathscr{F})$ can be identified with the module $\mathrm{Ext}^1_{\mathscr{O}_Y}(Y; \mathscr{O}_Y, \mathscr{F})$ of extensions classes of the $\mathscr{O}_Y$-module $\mathscr{O}_Y$ by the $\mathscr{O}_Y$-module $\mathscr{F}$ (T, 4.2.3); so everything reduces to proving that such an extension $\mathscr{G}$ is trivial. But, for all $y \in Y$, there is a neighbourhood V of y in Y such that $\mathscr{G}|V$ is isomorphic to $\mathscr{F}|Y \oplus \mathscr{O}_Y|V$ (**0**, 5.4.9); from this we conclude that $\mathscr{G}$ is a *quasi-coherent* $\mathscr{O}_Y$-module. If A is the ring of Y, then we have $\mathscr{F} = \widetilde{M}$ and $\mathscr{G} = \widetilde{N}$, where M and N are A-modules, and, by hypothesis, N is an extension of the A-module A by the A-module M (1.3.11). Since this extension is necessarily trivial, the lemma is proven, and thus so too is (5.1.9). □

Corollary (5.1.10). — *Let X be a prescheme such that $\mathscr{N}_X$ is nilpotent. For X to be an affine scheme, it is necessary and sufficient for X_{red} to be an affine scheme.*

5.2. Existence of a subprescheme with a given underlying space

Proposition (5.2.1). — *For every locally closed subspace Y of the underlying space of a prescheme X, there exists exactly one reduced subprescheme of X that has Y as its underlying space.*

Proof. The uniqueness follows from (5.1.2), so it remains only to show the existence of the prescheme in question.

If X is affine, given by some ring A, and Y closed in X, then the proposition is immediate: $\mathfrak{j}(Y)$ is the largest ideal $\mathfrak{a} \subset A$ such that $V(\mathfrak{a}) = Y$, and it is equal to its radical (1.1.4, i), so $A/\mathfrak{j}(Y)$ is a reduced ring.

In the general case, for every affine open $U \subset X$ such that $U \cap Y$ is closed in U, consider the closed subprescheme Y_U of U defined by the sheaf of ideals associated to the ideal $\mathfrak{j}(U \cap Y)$ of $A(U)$, which is reduced. We can show that, if V is an affine open subset of X contained in U, then Y_V is *induced* by Y_U on $V \cap Y$; but this induced prescheme is a closed subprescheme (of V) which is reduced and has $V \cap Y$ as its underlying space; the uniqueness of Y_V thus implies our claim. □

Proposition (5.2.2). — *Let X be reduced, $f : X \to Y$ a morphism, and Z a sub-prescheme closed over Y such that $f(X) \subset Z$. Then, f factors as $X \xrightarrow{g} Z \xrightarrow{j} Y$, where j is an injective morphism.*

Proof. It follows from the hypotheses that the closed subprescheme $f^{-1}(Z)$ of X has all of X as its underlying space (4.4.1); since X is reduced, this closed subprescheme agrees with X (5.1.2), and the proposition then follows from (4.4.1). □

Corollary (5.2.3). — *Let X be a reduced subprescheme of a prescheme Y; if Z is the closed reduced subprescheme of Y that has $\overline{X}$ as its underlying space, then X is a subprescheme induced on an open subset of Z.*

PROOF. There is indeed an open subset U of Y such that $X = U \cap \overline{X}$; since, by (5.2.2), X is a reduced subprescheme of Z, the subprescheme X is induced by Z on the open subspace X by uniqueness (5.2.1). □ I | 132

Corollary (5.2.4). — *Let $f : X \to Y$ be a morphism, and X' (resp. Y') a closed subprescheme of X (resp. Y) defined by a quasi-coherent sheaf of ideals $\mathscr{J}$ (resp. $\mathscr{K}$) of $\mathscr{O}_X$ (resp. $\mathscr{O}_Y$). Suppose that X' is reduced, and that $f(X') \subset Y'$. Then $f^*(\mathscr{K})\mathscr{O}_X \subset \mathscr{J}$.*

PROOF. Since, by (5.2.2), the restriction of f to X' factors as $X' \to Y' \to Y$, it suffices to apply (4.4.6). □

5.3. Diagonal; graph of a morphism

(5.3.1). Let X be an S-prescheme; we define the *diagonal morphism* of X in $X \times_S X$, denoted by $\Delta_{X|S}$, or Δ_X, or even Δ if no confusion may arise, to be the S-morphism $(1_X, 1_X)_S$, or, in other words, the unique S-morphism Δ_X such that

(5.3.1.1) $$p_1 \circ \Delta_X = p_2 \circ \Delta_X = 1_X,$$

where p_1 and p_2 are the projections of $X \times_S X$ (Definition (3.2.1)). If $f : T \to X$ and $g : T \to Y$ are S-morphisms, we immediately have that

(5.3.1.2) $$(f, g)_S = (f \times_S g) \circ \Delta_{T|S}.$$

The reader will note that the preceding definition and the results stated in (5.3.1) to (5.3.8) are true in any category, *provided that the products used within exist in the category*.

Proposition (5.3.2). — *Let X and Y be S-preschemes; if we make the canonical identification between $(X \times Y) \times (X \times Y)$ and $(X \times X) \times (Y \times Y)$, then the morphism $\Delta_{X \times Y}$ is identified with $\Delta_X \times \Delta_Y$.*

PROOF. Indeed, if $p_1 : X \times X \to X$ and $q_1 : Y \times Y \to Y$ are the projections onto the first component, then the projection onto the first component $(X \times Y) \times (X \times Y) \to X \times Y$ is identified with $p_1 \times q_1$, and we have

$$(p_1 \times q_1) \circ (\Delta_X \times \Delta_Y) = (p_1 \circ \Delta_X) \times (q_1 \circ \Delta_Y) = 1_{X \times Y}$$

and we can argue similarly for the projections onto the second component. □

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Corollary (5.3.4). — *For every extension $S' \to S$ of the base prescheme, $\Delta_{X_{S'}}$ is canonically identified with $(\Delta_X)_{(S')}$.*

PROOF. It suffices to remark that $(X \times_S X)_{(S')}$ is canonically identified with $X_{(S')} \times_{S'} X_{(S')}$ (3.3.10). □

Proposition (5.3.5). — *Let X and Y be S-preschemes, and $\phi : S \to T$ a morphism of preschemes, which lets us consider every S-prescheme as a T-prescheme. Let $f : X \to S$ and $g : Y \to S$ be the structure morphisms, p and q the projections of $X \times_S Y$, and $\pi = f \circ p = g \circ q$ the structure morphism $X \times_S Y \to S$. Then the diagram*

(5.3.5.1)
$$\begin{tikzcd} X \times_S Y \arrow[r, "(p,q)_T"] \arrow[d, "\pi"'] & X \times_T Y \arrow[d, "f \times_T g"] \\ S \arrow[r, "\Delta_{S|T}"] & S \times_T S \end{tikzcd}$$

commutes, and identifies $X \times_S Y$ with the product of the $(S \times_T S)$-preschemes S and $X \times_T Y$, and the projections with π and $(p, q)_T$. I | 133

PROOF. By (3.4.3), we are led to proving the corresponding proposition *in the category of sets*, replacing X, Y, and S by $X(Z)_T$, $Y(Z)_T$, and $S(Z)_T$ (respectively), with Z being an arbitrary T-prescheme. But, in the category of sets, the proof is immediate and left to the reader. □

Corollary (5.3.6). — *The morphism $(p, q)_T$ can be identified (letting $P = S \times_T S$) with $1_{X \times_T Y} \times_P \Delta_S$.*

PROOF. This follows from (5.3.5) and (3.3.4). □

Corollary (5.3.7). — *If $f : X \to Y$ is an S-morphism, then the diagram*

$$\begin{tikzcd} X \arrow[r, "(1_X,f)_S"] \arrow[d, "f"'] & X \times_S Y \arrow[d, "f \times_S 1_Y"] \\ Y \arrow[r, "\Delta_Y"] & Y \times_S Y \end{tikzcd}$$

commutes, and identifies X with the product of the $(Y \times_S Y)$-preschemes Y and $X \times_S Y$.

PROOF. It suffices to apply (5.3.5), replacing S by Y, and T by S, and noting that $X \times_Y Y = X$ (3.3.3). □

Proposition (5.3.8). — *For $f : X \to Y$ to be a monomorphism of preschemes, it is necessary and sufficient for $\Delta_{X|Y}$ to be an isomorphism from X to $X \times_Y X$.*

PROOF. Indeed, to say that f is a monomorphism implies that, for every Y-prescheme Z, the corresponding map $f' : X(Z)_Y \to Y(Z)_Y$ is an injection, and, since $Y(Z)_Y$ consists of a single element, this implies that $X(Z)_Y$ consists of a single element as well. But this can also be expressed by saying that $X(Z)_Y \times X(Z)_Y$ is canonically isomorphic to $X(Z)_Y$; the former is exactly the set $(X \times_Y X)(Z)_Y$ (3.4.3.1), which implies that $\Delta_{X|Y}$ is an isomorphism. □

Proposition (5.3.9). — *The diagonal morphism Δ_X is an immersion from X to $X \times_S X$.*

PROOF. Indeed, since the continuous maps p_1 and Δ_X from the underlying spaces are such that $p_1 \circ \Delta_X$ is the identity, Δ_X is a homeomorphism from X to $\Delta_X(X)$. Similarly, the composite homomorphism $\mathscr{O}_x \to \mathscr{O}_{\Delta_X(x)} \to \mathscr{O}_x$ (composed of the homomorphisms corresponding to p_1 and Δ_X) is the identity, which means that the homomorphism corresponding to Δ_X is surjective; the proposition thus follows from (4.2.2). □

We say that the subprescheme of $X \times_S X$ associated to the immersion Δ_X (4.2.1) is *the diagonal* of $X \times_S X$.

Corollary (5.3.10). — *Under the hypotheses of (5.3.5), $(p,q)_T$ is an immersion.*

PROOF. This follows from (5.3.6) and (4.3.1). □

We say (under the hypotheses of (5.3.5)) that $(p,q)_T$ is the *canonical immersion* of $X \times_S Y$ into $X \times_T Y$.

Corollary (5.3.11). — *Let X and Y be S-preschemes, and $f : X \to Y$ an S-morphism; then the graph morphism $\Gamma_f = (1_X, f)_S$ of f (3.3.14) is an immersion of X into $X \times_S Y$.*

PROOF. This is a particular case of Corollary (5.3.10), where we replace S by Y, and T by S (cf. (5.3.7)). □

The subprescheme of $X \times_S Y$ associated to the immersion Γ_f (4.2.1) is called *the graph* of the **I** | 134
morphism f; the subpreschemes of $X \times_S Y$ that are graphs of morphisms $X \to Y$ are characterised by the property that the restriction to such a subprescheme G of the projection $p_1 : X \times_S Y \to X$ is an *isomorphism* g from G to X: G is the the graph of the morphism $p_2 \circ g^{-1}$, where p_2 is the projection $X \times_S Y \to Y$.

When we take, in particular, $X = S$, then the S-morphisms $S \to Y$ (which are exactly the *S-sections* of Y (2.5.5)) are equal to their graph morphisms; the subpreschemes of Y that are the graphs of S-sections (in other words, those that are isomorphic to S by the restriction of the structure morphism $Y \to S$) are then also called the *images* of these sections, or, by an abuse of language, the *S-sections* of Y.

Corollary (5.3.12). — *With the hypotheses and notation of (5.3.11), for every morphism $g : S' \to S$, let f' be the inverse image of f under g (3.3.7); then $\Gamma_{f'}$ is the inverse image of Γ_f under g.*

PROOF. This is a particular case of (3.3.10.1). □

Corollary (5.3.13). — *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms; if $g \circ f$ is an immersion (resp. a local immersion), then so too is f.*

PROOF. Indeed, f factors as $X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$. Furthermore, p_2 can be identified with $(g \circ f) \times_Z 1_Y$ (3.3.4); if $g \circ f$ is an immersion (resp. a local immersion), then so too is p_2 ((4.3.1) and (4.5.5)), and since Γ_f is an immersion (5.3.11), we are done, by (4.2.4) (resp. (4.5.5)). □

Corollary (5.3.14). — *Let $j : X \to Y$ and $g : Z \to Z$ be S-morphisms. If j is an immersion (resp. a local immersion), then so too is $(j, g)_S$.*

PROOF. Indeed, if $p : Y \times_S Z \to Y$ is the projection onto the first component, then we have $j = p \circ (j, g)_S$, and it suffices to apply (5.3.13). □

Proposition (5.3.15). — *If $f : X \to Y$ is an S-morphism, then the diagram*

(5.3.15.1)
$$\begin{tikzcd} X \arrow[r, "\Delta_X"] \arrow[d, "f"'] & X \times_S X \arrow[d, "f \times_S f"] \\ Y \arrow[r, "\Delta_Y"] & Y \times_S Y \end{tikzcd}$$

commutes (in other words, Δ_X is a functorial morphism in the category of preschemes).

PROOF. The proof is immediate and left to the reader. □

Corollary (5.3.16). — *If X is a subprescheme of Y, then the diagonal $\Delta_X(X)$ can be identified with a subprescheme of $\Delta_Y(Y)$, and the underlying space can be identified with*
$$\Delta_Y(Y) \cap p_1^{-1}(X) = \Delta_Y \cap p_2^{-1}(X)$$
(p_1 and p_2 being the projections of $Y \times_S Y$).

PROOF. Applying (5.3.15) to the injection morphism $f : X \to Y$, we see that $f \times_S f$ is an immersion that identifies the underlying space of $X \times_S X$ with the subspace $p_1^{-1}(X) \cap p_2^{-1}(X)$ of $Y \times_S Y$ (4.3.1); further, if $z \in \Delta_Y(Y) \cap p_1^{-1}(X)$, then we have $z = \Delta_Y(y)$ and $y = p_1(z) \in X$, so **I** | 135
$y = f(y)$, and $z = \Delta_Y(f(y))$ belongs to $\Delta_X(X)$ by the commutativity of (5.3.15.1). □

Corollary (5.3.17). — *Let $f_1 : Y \to X$ and $f_2 : Y \to X$ be S-morphisms, and y a point of Y such that $f_1(y) = f_2(y) = x$, and such that the homomorphisms $k(x) \to k(y)$ corresponding to f_1 and f_2 are identical. Then, if $f = (f_1, f_2)_S$, the point $f(y)$ belongs to the diagonal $\Delta_{X|S}(X)$.*

PROOF. The two homomorphisms $k(x) \to k(y)$ corresponding to f_i $(i = 1, 2)$ define two S-morphisms $g_i : \operatorname{Spec}(k(y)) \to \operatorname{Spec}(k(x))$ such that the diagrams

$$\begin{tikzcd} \operatorname{Spec}(k(y)) \arrow[r, "g_i"] \arrow[d] & \operatorname{Spec}(k(x)) \arrow[d] \\ Y \arrow[r, "f_i"] & X \end{tikzcd}$$

commute. The diagram

$$\begin{tikzcd} \operatorname{Spec}(k(y)) \arrow[r, "(g_1,g_2)_S"] \arrow[d] & \operatorname{Spec}(k(x)) \times_S \operatorname{Spec}(k(x)) \arrow[d] \\ Y \arrow[r, "(f_1,f_2)_S"] & X \times_S X \end{tikzcd}$$

thus also commutes. But it follows from the equality $g_1 = g_2$ that the image under $(g_1, g_2)_S$ of the unique point of $\operatorname{Spec}(k(y))$ belongs to the diagonal of $\operatorname{Spec}(k(x)) \times_S \operatorname{Spec}(k(x))$; the conclusion then follows from (5.3.15). □

5.4. Separated morphisms and separated preschemes

Definition (5.4.1). — We say that a morphism of preschemes $f : X \to Y$ is *separated* if the diagonal morphism $X \to X \times_Y X$ is a *closed* immersion; we then also say that X is a *separated prescheme over* Y, or a Y*-scheme*. We say that a prescheme X is separated if it is separated over $\operatorname{Spec}(\mathbf{Z})$; we then also say that X is a *scheme*[9] *(cf. (5.5.7))*.

By (5.3.9), for X to be separated over Y, it is necessary and sufficient for $\Delta_X(X)$ to be a *closed subspace* of the underlying space of $X \times_Y X$.

Proposition (5.4.2). — *Let $S \to T$ be a separated morphism. If X and Y are S-preschemes, then the canonical immersion $X \times_S Y \to X \times_T Y$ (5.3.10) is closed.*

PROOF. Indeed, if we refer to the diagram in (5.3.5.1), we see that $(p, q)_T$ can be considered as being obtained from $\Delta_{S|T}$ by the extension $f \times_T g : X \times_T Y \to S \times_T S$ of the base prescheme $S \times_T S$; the proposition then follows from (4.3.2). ☐

Corollary (5.4.3). — *Let Y be an S-scheme, and $f : X \to Y$ an S-morphism. Then the graph morphism $\Gamma_f : X \to X \times_S Y$ (5.3.11) is a closed immersion.*

PROOF. This is a particular case of (5.4.2), where we replace S by Y, and T by S. ☐

Corollary (5.4.4). — *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms, with g separated. If $g \circ f$ is a closed immersion, then so too is f.*

PROOF. The proof using (5.4.3) is the same as that of (5.3.13) using (5.3.11). ☐

I | 136

Corollary (5.4.5). — *Let Z be an S-scheme, and $j : X \to Y$ and $g : X \to Z$ S-morphisms. If j is a closed immersion, then so too is $(j, g)_S : X \to Y \times_S Z$.*

PROOF. The proof using (5.4.4) is the same as that of (5.3.14) using (5.3.13). ☐

Corollary (5.4.6). — *If X is an S-scheme, then every S-section of X (2.5.5) is a closed immersion.*

PROOF. If $\phi : X \to S$ is the structure morphism, and $\psi : S \to X$ an S-section of X, it suffices to apply (5.4.5) to $\phi \circ \psi = 1_S$. ☐

Corollary (5.4.7). — *Let X be an integral prescheme with generic point s, and X an S-scheme. If two S-sections f and g are such that $f(s) = g(s)$, then $f = g$.*

PROOF. Indeed, if $x = f(s) = g(s)$, then the homomorphisms $k(x) \to k(s)$ corresponding to f and g are necessarily identical. If $h = (f, g)_S$, we thus deduce (5.3.17) that $h(s)$ belongs to the diagonal $Z = \Delta_X(X)$; but since $S = \overline{\{s\}}$, and since Z is closed by hypothesis, we have $h(S) \subset Z$. It then follows from (5.2.2) that h factors as $S \to Z \to X \times_S X$, and we thus conclude that $f = g$, by definition of the diagonal. ☐

Remark (5.4.8). — If we suppose, conversely, that the conclusion of (5.4.3) is true when $f = 1_Y$, then we can conclude that Y is separated over S; similarly, if we suppose that the conclusion of (5.4.5) applies to the two morphisms $Y \xrightarrow{\Delta_Y} Y \times_Z Y \xrightarrow{p_1} Y$, then we can conclude that Δ_Y is a closed immersion, and thus that Y is separated over Z; finally, if we assume that the conclusion of (5.4.6) is true for the Y section Δ_Y of the Y-prescheme $Y \times_S Y \to Y$, then this implies that Y is separated over S.

[9] *[Trans.] We repeat here the warning given at the very start of this translation: the early versions of the EGA use* prescheme *to mean is now usually called a scheme, and* scheme *for what is now usually called a separated scheme.*. Grothendieck himself later said that the more modern terminology was preferable, but we have decided to keep this translation 'historically accurate' by using the older nomenclature.

5.5. Separation criteria

Proposition (5.5.1). —

(i) *Every monomorphism of preschemes (and, in particular, every immersion) is a separated morphism.*
(ii) *The composition of any two separated morphisms is separated.*
(iii) *If $f : X \to X'$ and $g : Y \to Y'$ are separated S-morphisms, then $f \times_S g$ is separated.*
(iv) *If $f : X \to Y$ is a separated S-morphism, then the S'-morphism $f_{(S')}$ is separated for every extension $S' \to S$ of the base prescheme.*
(v) *If the composition $g \circ f$ is separated, then f is separated.*
(vi) *For a morphism f to be separated, it is necessary and sufficient for f_{red} (5.1.5) to be separated.*

PROOF. Note that (i) is an immediate consequence of (5.3.8). If $f : X \to Y$ and $g : Y \to Z$ are morphisms, then the diagram

(5.5.1.1)
$$\begin{tikzcd} X \arrow[rr, "\Delta_{X|Z}"] \arrow[dr, "\Delta_{X|Y}"'] & & X \times_Z X \\ & X \times_Y X \arrow[ur, "j"'] & \end{tikzcd}$$

commutes (where j denotes the canonical immersion (5.3.10)), as can be immediately verified. If f and g are separated, then $\Delta_{X|Y}$ is a closed immersion by definition, and j is a closed immersion by (5.4.2), whence $\Delta_{X|Z}$ is a closed immersion by (4.2.4), which proves (ii). Given (i) and (ii), (iii) and (iv) I | 137
are equivalent (3.5.1), so it suffices to prove (iv). But $X_{(S')} \times_{Y_{(S')}} X_{(S')}$ is canonically identified with $(X \times_Y X) \times_Y Y_{(S')}$ by (3.3.11) and (3.3.9.1), and we immediately see that the diagonal morphism $\Delta_{X_{(S')}}$ can then be identified with $\Delta_X \times_Y 1_{Y_{(S')}}$; the proposition then follows from (4.3.1).

To prove (v), consider, as in (5.3.13), the factorisation $X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$ of f, noting that $p_2 = (g \circ f) \times_Z 1_Y$; the hypothesis (that $g \circ f$ is separated) implies that g_2 is separated, by (iii) and (i), and, since Γ_f is an immersion, Γ_f is separated, by (i), whence f is separated, by (ii). Finally, to prove (vi), recall that the preschemes $X_{\mathrm{red}} \times_{Y_{\mathrm{red}}} X_{\mathrm{red}}$ and $X_{\mathrm{red}} \times_Y X_{\mathrm{red}}$ are canonically identified with one another (5.1.7); if we denote by j the injection $X_{\mathrm{red}} \to X$, then the diagram

$$\begin{tikzcd} X_{\mathrm{red}} \arrow[r, "\Delta_{X_{\mathrm{red}}}"] \arrow[d, "j"'] & X_{\mathrm{red}} \times_Y X_{\mathrm{red}} \arrow[d, "j \times_Y j"] \\ X \arrow[r, "\Delta_X"] & X \times_Y X \end{tikzcd}$$

commutes (5.3.15), and the proposition follows from the fact that the vertical arrows are homeomorphisms of the underlying spaces (4.3.1). □

Corollary (5.5.2). — *If $f : X \to Y$ is separated, then the restriction of f to any subprescheme of X is separated.*

PROOF. This follows from (5.5.1, i and ii). □

Corollary (5.5.3). — *If X and Y are S-preschemes such that Y is separated over S, then $X \times_S Y$ is separated over X.*

PROOF. This is a particular case of (5.5.1, iv). □

Proposition (5.5.4). — *Let X be a prescheme, and assume that its underlying space is a* finite *union of closed subsets X_k ($1 \leqslant k \leqslant n$); for each k, consider the reduced subprescheme of X that has X_k as its underlying space (5.2.1), and denote this again by X_k. Let $f : X \to Y$ be a morphism, and, for each k, let Y_k be a closed subset of Y such that $f(X_k) \subset Y_k$; we again denote by Y_k the reduced subprescheme of Y that has Y_k as its underlying space, so that the restriction $X_k \to Y$ of f to X_k factors as $X_k \xrightarrow{f_k} Y_k \to Y$ (5.2.2). For f to be separated, it is necessary and sufficient for all the f_k to be separated.*

PROOF. The necessity follows from (5.5.1, i, ii, and v). Conversely, if the condition of the statement is satisfied, then each of the restrictions $X_k \to Y$ of f is separated (5.5.1, (i) and (ii)); if p_1 and p_2 are the projections of $X \times_Y X$, then the subspace $\Delta_{X_k}(X_k)$ can be identified with the subspace $\Delta_X(X) \cap p_1^{-1}(X_k)$ of the underlying space of $X \times_Y X$ (5.3.16); these subspaces are closed in $X \times_Y X$, and thus so too is their union $\Delta_X(X)$. □

Suppose, in particular, that the X_k are the *irreducible components* of X; then we can suppose that the Y_k are the irreducible components of Y (**0**, 2.1.5); Proposition (5.5.4) then, in this case, leads to the idea of separation in the case of *integral* preschemes (2.1.7).

I | 138

Proposition (5.5.5). — *Let (Y_λ) be an open cover of a prescheme Y; for a morphism $f : X \to Y$ to be separated, it is necessary and sufficient for each of its restrictions $f^{-1}(Y_\lambda) \to Y_\lambda$ to be separated.*

PROOF. If we set $X_\lambda = f^{-1}(Y_\lambda)$, everything reduces, by taking (4.2.4, b) and the identification of the products $X_\lambda \times_Y X_\lambda$ and $X_\lambda \times_{Y_\lambda} X_\lambda$ into account, to proving that the $X_\lambda \times_Y X_\lambda$ form a cover of $X \times_Y X$. But if we set $Y_{\lambda\mu} = Y_\lambda \cap Y_\mu$ and $X_{\lambda\mu} = X_\lambda \cap X_\mu = f^{-1}(Y_{\lambda\mu})$, then $X_\lambda \times_Y X_\mu$ can be identified with the product $X_{\lambda\mu} \times_{Y_{\lambda\mu}} X_{\lambda\mu}$ (3.2.6.4), and so also with $X_{\lambda\mu} \times_Y X_{\lambda\mu}$ (3.2.5), and finally with an open subset of $X_\lambda \times_Y X_\lambda$, which proves our claim (3.2.7). □

Proposition (5.5.4) allows us, by taking an affine open cover of Y, to restrict our study of separated morphisms to just those that take values in affine schemes.

Proposition (5.5.6). — *Let Y be an* affine *scheme, X a prescheme, and (U_α) a cover of X by affine open subsets. For a morphism $f : X \to Y$ to be separated, it is necessary and sufficient for $U_\alpha \cap U_\beta$ to be, for every pair of indices (α, β), an affine open subset,* and *for the ring $\Gamma(U_\alpha \cap U_\beta, \mathscr{O}_X)$ to be generated by the union of the canonical images of the rings $\Gamma(U_\alpha, \mathscr{O}_X)$ and $\Gamma(U_\beta, \mathscr{O}_X)$.*

PROOF. The $U_\alpha \times_Y U_\beta$ form an open cover of $X \times_Y X$ (3.2.7); denoting the projections of $X \times_Y X$ by p and q, we have

$$\begin{aligned}\Delta_X^{-1}(U_\alpha \times_Y U_\beta) &= \Delta_X^{-1}(p^{-1}(U_\alpha) \cap q^{-1}(U_\beta))\\ &= \Delta_X^{-1}(p^{-1}(U_\alpha)) \cap \Delta_X^{-1}(q^{-1}(U_\beta)) = U_\alpha \cap U_\beta;\end{aligned}$$

so everything reduces to proving that the restriction of Δ_X to $U_\alpha \cap U_\beta$ is a closed immersion into $U_\alpha \times_Y U_\beta$. But this restriction is exactly $(j_\alpha, j_\beta)_Y$, where j_α (resp. j_β) denotes the injection morphism from $U_\alpha \cap U_\beta$ to U_α (resp. U_β), as follows from the definitions. Since $U_\alpha \times_Y U_\beta$ is an affine scheme whose ring is canonically isomorphic to $\Gamma(U_\alpha, \mathscr{O}_X) \otimes_{\Gamma(Y, \mathscr{O}_Y)} \Gamma(U_\beta, \mathscr{O}_X)$ (3.2.2), we see that $U_\alpha \cap U_\beta$ must be an affine scheme, and that the map $h_\alpha \otimes h_\beta \mapsto h_\alpha h_\beta$ from the ring $A(U_\alpha \times_Y U_\beta)$ to $\Gamma(U_\alpha \cap U_\beta, \mathscr{O}_X)$ must be surjective (4.2.3), which finishes the proof. □

Corollary (5.5.7). — *An affine scheme is separated* (and is thus a *scheme*, which justifies the terminology of (5.4.1)).

Corollary (5.5.8). — *Let Y be an affine scheme; for $f : X \to Y$ to be a separated morphism, it is necessary and sufficient for X to be separated* (in other words, for X to be a *scheme*).

PROOF. Indeed, we see that the criteria of (5.5.6) do not depend on f. □

Corollary (5.5.9). — *For a morphism $f : X \to Y$ to be separated, it is necessary and sufficient for the induced prescheme $f^{-1}(U)$ to be separated, for every open subset of U on which Y induces a separated prescheme, and it is sufficient for it to be the case for every affine open subset $U \subset Y$.*

PROOF. The necessity of the condition follows from (5.5.4) and (5.5.1, ii); the sufficiency follows from (5.5.4) and (5.5.8), taking into account the existence of affine open covers of Y. □

In particular, if X and Y are affine schemes, then *every* morphism $X \to Y$ is separated.

Proposition (5.5.10). — *Let Y be a* scheme, *and $f : X \to Y$ a morphism. For every affine open subset U of X, and every affine open subset V of Y, $U \cap f^{-1}(V)$ is affine.*

PROOF. Let p_1 and p_2 be the projections of $X \times_{\mathbf{Z}} Y$; the subspace $U \cap f^{-1}(V)$ is the image of $\Gamma_f(X) \cap p_1^{-1}(U) \cap p_2^{-1}(V)$ under p_1. But $p_1^{-1}(U) \cap p_2^{-1}(V)$ can be identified with the underlying space of the prescheme $U \times_{\mathbf{Z}} V$ (3.2.7), and is thus an affine scheme (3.2.2); since $\Gamma_f(X)$ is closed in $X \times_{\mathbf{Z}} Y$ (5.4.3), $\Gamma_f(X) \cap p_1^{-1}(U) \cap p_2^{-1}(V)$ is closed in $U \times_{\mathbf{Z}} V$, and so the prescheme induced by the subprescheme of $X \times_{\mathbf{Z}} Y$ associated to Γ_f (4.2.1) on the open subset $\Gamma_f(X) \cap p_1^{-1}(U) \cap p_2^{-1}(V)$ of its underlying space is a closed subprescheme of an affine scheme, and thus an affine scheme (4.2.3). The proposition then follows from the fact that Γ_f is an immersion. □ **I** | 139

Examples (5.5.11). — The prescheme from Example (2.3.2) ("the projective line over a field K") is *separated*, because, for the cover (X_1, X_2) of X by affine open subsets, $X_1 \cap X_2 = U_{12}$ is affine, and $\Gamma(U_{12}, \mathscr{O}_X)$, the ring of rational fractions of the form $f(s)/s^m$ with $f \in K[s]$, is generated by $K[s]$ and $1/s$, so the conditions of (5.5.6) are satisfied.

With the same choice of X_1, X_2, U_{12}, and U_{21} as in Example (2.3.2), now take u_{12} to be the isomorphism which sends $f(s)$ to $f(t)$; we now obtain, by gluing, a *non-separated integral* prescheme X, because the first condition of (5.5.6) is satisfied, but not the second. It is immediate here that $\Gamma(X, \mathscr{O}_X) \to \Gamma(X_1, \mathscr{O}_X) = K[s]$ is an isomorphism; the inverse isomorphism defines a morphism $f : X \to \operatorname{Spec}(K[s])$ that is surjective, and for every $y \in \operatorname{Spec}(K[s])$ such that $\mathfrak{j}_y \neq (0)$, $f^{-1}(y)$ consists of a single point, but for $\mathfrak{j}_y = (0)$, $f^{-1}(y)$ consists of *two* distinct points (we say that X is the "affine line over K with the point 0 doubled").

We can also give examples where *neither* of the two conditions of (5.5.6) are satisfied. First note that, in the prime spectrum Y of the ring $A = K[s,t]$ of polynomials in two indeterminates over a field K, the open subset U given by the union of $D(s)$ and $D(t)$ is *not an affine open subset*. Indeed, if z is a section of $\mathscr{O}_Y$ over U, there exist two integers $m, n \geqslant 0$ such that $s^m z$ and $t^n z$ are the restrictions of polynomials in s and t to U (1.4.1), which is clearly possible only if the section z extends to a section over the whole of Y, identified with a polynomial in s and t. If U were an affine open subset, then the injection morphism $U \to Y$ would be an isomorphism (1.7.3), which is a contradiction, since $U \neq Y$.

With the above in mind, take two affine schemes Y_1 and Y_2, prime spectra of the rings $A_1 = K[s_1, t_2]$ and $A_2 = K[s_2, t_2]$ (respectively); take $U_{12} = D(s_1) \cup D(t_1)$ and $U_{21} = D(s_2) \cup D(t_2)$, and take u_{12} to be the restriction of an isomorphism $Y_2 \to Y_1$ to U_{21} corresponding to the isomorphism of rings that sends $f(s_1, t_1)$ to $f(s_2, t_2)$; we then have an example where the conditions of (5.5.6) are not satisfied (the integral prescheme thus obtained is called "the affine plane over K with the point 0 doubled").

Remark (5.5.12). — Given some property $\mathbf{P}$ of morphisms of preschemes, consider the following propositions.

(i) *Every closed immersion has property $\mathbf{P}$.*
(ii) *The composition of any two morphisms that both have property $\mathbf{P}$ also has property $\mathbf{P}$.*
(iii) *If $f : X \to X'$ and $g : Y \to Y'$ are S-morphisms that have property $\mathbf{P}$, then $f \times_S g$ has property $\mathbf{P}$.* **I** | 140
(iv) *If $f : X \to Y$ is an S-morphism that has property $\mathbf{P}$, then every S'-morphism $f_{(S')}$ obtained by an extension $S' \to S$ of the base prescheme also has property $\mathbf{P}$.*
(v) *If the composition $g \circ f$ of two morphisms $f : X \to Y$ and $g : Y \to Z$ has property $\mathbf{P}$, and g is separated, then f has property $\mathbf{P}$.*
(vi) *If a morphism $f : X \to Y$ has property $\mathbf{P}$, then so too does f_{red} (5.1.5).*

If we suppose that (i) and (ii) are both true, then (iii) and (iv) are *equivalent*, and (v) and (vi) are *consequences* of (i), (ii), and (iii).

The first claim has already been shown (3.5.1). Consider the factorisation (5.3.13) $X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$ of f; the equation $p_2 = (g \circ f) \times_Z 1_Y$ shows that, if $g \circ f$ has property $\mathbf{P}$, then so too does p_2, by (iii); if g is separated, then Γ_f is a closed immersion (5.4.3), and so also has property $\mathbf{P}$, by (i); finally, by (ii), f has property $\mathbf{P}$.

Finally, consider the commutative diagram

$$\begin{tikzcd} X_{\mathrm{red}} \arrow[r, "f_{\mathrm{red}}"] \arrow[d] & Y_{\mathrm{red}} \arrow[d] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

where the vertical arrows are the closed immersions (5.1.5), and thus have property **P**, by (i). The hypothesis that f has property **P** implies, by (ii), that $X_{\mathrm{red}} \xrightarrow{f_{\mathrm{red}}} Y_{\mathrm{red}} \to Y$ has property **P**; finally, since a closed immersion is separated (5.5.1, i), f_{red} has property **P**, by (v).

Note that, if we consider the propositions

(i′) *Every immersion has property **P**;*

(v′) *If $g \circ f$ has property **P**, then so too does f;*

then the above arguments show that (v′) is a consequence of (i′), (ii), and (iii).

(5.5.13). Note that (v) and (vi) are again consequences of (i), (iii), and

(ii′) *If $j : X \to Y$ is a closed immersion, and $g : Y \to Z$ is a morphism that has property **P**, then $g \circ j$ has property **P**.*

Similarly, (v′) is a consequence of (i′), (iii), and

(ii″) *If $j : X \to Y$ is an immersion, and $g : Y \to Z$ is a morphism that has property **P**, then $g \circ j$ has property **P**.*

This follows immediately from the arguments of (5.5.12).

§6. FINITENESS CONDITIONS

6.1. Noetherian and locally Noetherian preschemes

Definition (6.1.1). — We say that a prescheme X is Noetherian (resp. locally Noetherian) if it is a finite union (resp. union) of affine open V_α in such a way that the ring of the of the induced scheme on each of the V_α is Noetherian.

It follows immediately from (1.5.2) that, if X is locally Noetherian, then the structure sheaf $\mathscr{O}_X$ is a *coherent sheaf of rings*, since the questions is a local one. Every *quasi-coherent $\mathscr{O}_X$-submodule* (resp. **I** | 141
quasi-coherent quotient $\mathscr{O}_X$-module) of a *coherent* $\mathscr{O}_X$-module $\mathscr{F}$ is *coherent*, as the question is once again a local one, and it suffices to apply (1.5.1), (1.4.1), and (1.3.10), combined with the fact that a submodule (resp. quotient module) of a module of finite type over a Noetherian ring is of finite type. In particular, every *quasi-coherent sheaf of ideals* of $\mathscr{O}_X$ is *coherent*.

If a prescheme X is a finite union (resp. union) of open subsets W_λ in such a way that the preschemes induced on the W_λ are Noetherian (resp. locally Noetherian), it is clear that X is Noetherian (resp. locally Noetherian).

Proposition (6.1.2). — *For a prescheme X to be Noetherian, it is necessary and sufficient for it to be locally Noetherian and have a quasi-compact underlying space. The underlying space itself is then also Noetherian.*

PROOF. The first claim follows immediately from the definitions and (1.1.10, ii). The second follows from (1.1.6) and the fact that every space that is a finite union of Noetherian subspaces is itself Noetherian (**0**, 2.2.3). □

Proposition (6.1.3). — *Let X be an affine scheme given by a ring A. The following conditions are equivalent:* (a) *X is Noetherian;* (b) *X is locally Noetherian;* (c) *A is Noetherian.*

PROOF. The equivalence between (a) and (b) follows from (6.1.2) ant the fact that the underlying space of every affine scheme is quasi-compact (1.1.10); it is furthermore clear that (c) implies (a). To see that (a) implies (c), we remark that there is a finite cover (V_i) of X by affine open subsets such that the ring A_i of the prescheme induced on V_i is Noetherian. So let $(\mathfrak{a}_n)$ be an increasing sequence of ideals of A; by a canonical bijective correspondence, there is a corresponding sequence $(\widetilde{\mathfrak{a}}_n)$ of sheaves of ideals in $\widetilde{A} = \mathscr{O}_X$; to see that the sequence $(\mathfrak{a}_n)$ is stable (?), it suffices to prove that the sequence $(\widetilde{\mathfrak{a}}_n)$ is. But the restriction $\widetilde{\mathfrak{a}}_n|V_i$ is a quasi-coherent sheaf of ideals of $\mathscr{O}_X|V_i$, being the inverse image of $\widetilde{\mathfrak{a}}_n$ under the canonical injection $V_i \to X$ (**0**, 5.1.4); $\widetilde{\mathfrak{a}}_n | V_i$ is thus of the form $\widetilde{\mathfrak{a}}_{ni}$, where $\mathfrak{a}_{ni}$ is an ideal of A_i (1.3.7). Since A_i is Noetherian, the sequence $(\mathfrak{a}_{ni})$ is stable for all i, whence the proposition. □

We note that the above argument proves also that *if X is a Noetherian prescheme, then every increasing sequence of coherent sheaves of ideals of $\mathscr{O}_X$ is stable (?)*.

Proposition (6.1.4). — *Every subprescheme of a Noetherian (resp. locally Noetherian) prescheme is Noetherian (resp. locally Noetherian).*

PROOF. If suffices to give a proof for a Noetherian prescheme X; further, by definition (6.1.1), we can also restrict to the case where X is an affine scheme. Since every subprescheme of X is a closed subprescheme of a prescheme induced on an open subset (4.1.3), we can restrict to the case of a subprescheme Y, either closed or induced on an open subset of X. The proof in the case where Y is closed is immediate, since if A is the ring of X, we know that Y is an affine scheme given by the ring $A/\mathfrak{J}$, where $\mathfrak{J}$ is an ideal of A (4.2.3); since A is Noetherian (6.1.3), so too is $A/\mathfrak{J}$.

Now suppose that Y is open in X; the underlying space of Y is Noetherian (6.1.2), hence quasi-compact, and thus a finite union of open subsets $D(f_i)$ $(f_i \in A)$; everything reduces to showing the proposition in the case where $Y = D(f)$ with $f \in A$. But then Y is an affine scheme whose ring is isomorphic to A_f (1.3.6); since A is Noetherian (6.1.3), so too is A_f. □

(6.1.5). We note that the *product* of two Noetherian S-preschemes is not necessarily Noetherian, even if the preschemes are affine, since the tensor product of two Noetherian algebras in not necessarily a Noetherian ring (cf. (6.3.8)).

Proposition (6.1.6). — *If X is a Noetherian prescheme, the nilradical $\mathscr{N}_X$ of $\mathscr{O}_X$ is nilpotent.*

PROOF. We can in fact cover X with a finite number of affine open subsets U_i, and it suffices to prove that there exists whole numbers n_i such that $(\mathscr{N}_X | U_i)^{n_i} = 0$; if n is the largest of the n_i, then we will have $\mathscr{N}_X^n = 0$. We can thus restrict to the case where $X = \operatorname{Spec}(A)$ is affine, with A a Noetherian ring; by (5.1.1) and (1.3.13), it suffices to observe that the nilradical of A is nilpotent ([**Sam53b**, p. 127, cor. 4]). □

Corollary (6.1.7). — *Let X be a Noetherian prescheme; for X to be an affine scheme, it is necessary and sufficient that X_{red} be affine.*

PROOF. This follows from (6.1.6) and (5.1.10). □

Lemma (6.1.8). — *Let X be a topological space, x a point of X, and U an open neighbourhood of x having only a finite number of irreducible components. Then there exists a neighbourhood V of x such that every open neighbourhood of x contained in V is connected.*

PROOF. Let U_i $(1 \leqslant i \leqslant m)$ be the irreducible components of U not containing x; the complement (in U) of the union of the U_i is an open neighbourhood V of X inside U, and thus so too in X; it is also, incidentally, the complement (in X) of the union of the irreducible components of X that do not contain x (**0**, 2.1.6). So let W be an open neighbourhood of X contained in V. The irreducible components of W are the intersections of W with the irreducible components of U (**0**, 2.1.6), so these components contain x; since they are connected, so too is W. □

Corollary (6.1.9). — *A locally Noetherian topological space is locally connected (which implies, amongst other things, that its connected components are open).*

Proposition (6.1.10). — *Let X be a locally Noetherian topological space. The following conditions are equivalent.*

(a) *The irreducible components of X are open.*
(b) *The irreducible components of X are exactly its connected components.*
(c) *The connected components of X are irreducible.*
(d) *Two distinct irreducible components of X have an empty intersection.*

Finally, if X is a prescheme, then these conditions are also equivalent to

(e) *For every $x \in X$,* $\operatorname{Spec}(\mathscr{O}_x)$ *is irreducible (or, in other words, the nilradical of $\mathscr{O}_x$ is prime).*

Proof. It is immediate that (a) implies (b), because an irreducible space is connected, and (a) implies that the irreducible components of X are the sets that are both open and closed. It is trivial that (b) implies (c); conversely, a closed set F containing a connected component C of X, with C distinct from F, cannot be irreducible, because not being connected means that F is the union of two disjoint nonempty sets that are both open and closed in F, and thus closed in X; as a result, (c) implies (b). We immediately conclude from this that (c) implies (d), since two distinct connected components have no points in common. **I** | 143

We have not yet used the fact that X is locally Noetherian. Suppose now that this is indeed the case, and we will show that (d) implies (a): by (**0**, 2.1.6), we can restrict ourselves to the case where the space X is Noetherian, and so has only a finite number of irreducible components. Since they are closed and pairwise disjoint, they are open.

Finally, the equivalence between (d) and (e) holds true even without the assumption that the underlying space of the prescheme X is locally Noetherian. We can in fact restrict ourselves to the case where $X = \operatorname{Spec}(A)$ is affine, by (**0**, 2.1.6); to say that x is contained in only one single irreducible component of X is to say that $\mathfrak{j}_x$ contains only one single minimal ideal of A (1.1.14), which is equivalent to saying that $\mathfrak{j}_x\mathcal{O}_x$ contains only one single minimal ideal of $\mathcal{O}_x$, whence the conclusion. □

Corollary (6.1.11). — *Let X be a locally Noetherian space. For X to be irreducible, it is necessary and sufficient that X be connected and nonempty, and that any two distinct irreducible components of X have an empty intersection. If X is a prescheme, this latter condition is equivalent to asking that $\operatorname{Spec}(\mathcal{O}_x)$ be irreducible for all $x \in X$.*

Proof. The second claim has already been shown in (6.1.10); the only thing thus remaining to show is that the conditions in the first claim are sufficient. But by (6.1.10), these conditions imply that the irreducible components of X are exactly its connected components, and since X is connected and nonempty, it is irreducible. □

Corollary (6.1.12). — *Let X be a locally Noetherian prescheme. For X to be integral, it is necessary and sufficient that X be connected and that $\mathcal{O}_x$ be integral for all $x \in X$.*

Proposition (6.1.13). — *Let X be a locally Noetherian prescheme, and let $x \in X$ be a point such that the nilradical $\mathcal{N}_x$ of $\mathcal{O}_x$ is prime (resp. such that $\mathcal{O}_x$ is reduced, resp. integral); then there exists an open neighbourhood U of x that is irreducible (resp. reduced, resp. integral).*

Proof. It suffices to consider two cases: where $\mathcal{N}_x$ is prime, and where $\mathcal{N}_x = 0$; the third hypotheses is a combination of the first two. If $\mathcal{N}_x$ is prime, then x belongs to only one single irreducible component Y of X (6.1.10); the union of the irreducible components of X that do not contain x is closed (the set of these components being locally finite), and the complement U of this union is thus open and contained in Y, and thus irreducible (**0**, 2.1.6) If $\mathcal{N}_x = 0$, we also have $\mathcal{N}_y = 0$ for any y in a neighbourhood of x, because $\mathcal{N}$ is quasi-coherent (5.1.1), and thus coherent, since X is locally Noetherian, and the conclusion then follows from (**0**, 5.2.2). □

6.2. Artinian preschemes

Definition (6.2.1). — We say that a prescheme is *Artinian* if it is affine, and given by an Artinian ring.

Proposition (6.2.2). — *Given a prescheme X, the following conditions are equivalent:* **I** | 144

(a) *X is an Artinian scheme;*
(b) *X is Noetherian and its underlying space is discrete;*
(c) *X is Noetherian and the points of its underlying space are closed* (the T_1 condition).

When any of the above hold, the underlying space of X is finite, and the ring A of X is the direct sum of local (Artinian) rings of points of X.

Proof. We know that (a) implies the last claim ([**SZ60**, p. 205, th. 3]), so every prime ideal of A is thus maximal and is the inverse image of a maximal ideal of one of the local components of A, and so the space X is finite and discrete; (a) thus implies (b), and (b) clearly implies (c). To see that (c) implies (a), we first show that X is then finite; we can indeed restrict to the case where X is affine, and we know that a Noetherian ring whose prime ideals are all maximal is Artinian ([SZ60, p. 203]), whence our claim. The underlying space X is then discrete, the topological sum of a finite number of points x_i, and the local rings $\mathscr{O}_{x_i} = A_i$ are Artinian; it is clear that X is isomorphic to the prime spectrum affine scheme of the ring A (the direct sum of the A_i) (1.7.3). □

6.3. Morphisms of finite type

Definition (6.3.1). — We say that a morphism $f : X \to Y$ is *of finite type* if Y is the union of a family (V_α) of affine open subsets having the following property:

(P) $f^{-1}(V_\alpha)$ is a finite union of affine open subsets $U_{\alpha i}$ that are such that each ring $A(U_{\alpha i})$ is an algebra of finite type over $A(V_\alpha)$.

We then say that X is a prescheme of finite type over Y, or a Y-prescheme of finite type.

Proposition (6.3.2). — *If $f : X \to Y$ is a morphism of finite type, then every affine open subset W of Y satisfies property* (P) *of* (6.3.1).

We first show

Lemma (6.3.2.1). — *If $T \subset Y$ is an affine open subset, satisfying property* (P)*, then, for every $g \in A(T)$, $D(g)$ also satisfies property* (P).

Proof. By hypothesis, $f^{-1}(T)$ is a finite union of affine open subsets Z_j, that are such that $A(Z_j)$ is an algebra of finite type over $A(T)$; let $\phi_j : A(T) \to A(Z_j)$ be the homomorphism of rings corresponding to the restriction of f to Z_j (2.2.4), and set $g_j = \phi_j(g)$; we then have $f^{-1}(D(g)) \cap Z_j = D(g_j)$ (1.2.2.2). But $A(D(g_j)) = A(Z_j)_{g_j} = A(Z_j)[1/g_j]$ is of finite type over $A(Z_j)$, and *a fortiori* over $A(T)$ by the hypothesis, and so also over $A(D(g)) = A(T)[1/g]$, which proves the lemma. □

Proof. With the above lemma, since W is quasi-compact (1.1.10), there exists a finite covering of W by sets of the form $D(g_i) \subset W$, where each g_i belongs to a ring $A(V_{\alpha(i)})$. Each $D(g_i)$, being quasi-compact, is a finite union of sets $D(h_{ik})$, where $h_{ik} \in A(W)$; if $\phi_i : A(W) \to A(D(g_i))$ is the canonical map, then we have $D(h_{ik}) = D(\phi_i(h_{ik}))$ by (1.2.2.2). By (6.3.2.1), each of the $f^{-1}(D(h_{ik}))$ admits a finite covering by affine open subsets U_{ijk}, that are such that the $A(U_{ijk})$ are algebras of finite type over $A(D(h_{ik})) = A(W)[1/h_{ik}]$, whence the proposition. □ **Err**$_{\text{II}}$

We can thus say that the notion of a prescheme of finite type over Y is *local on Y*. **I** | 145

Proposition (6.3.3). — *Let X and Y be affine schemes; for X to be of finite type over Y, it is necessary and sufficient that $A(X)$ be an algebra of finite type over $A(Y)$.*

Proof. Since the condition clearly suffices, we show that it is necessary. Set $A = A(Y)$ and $B = A(X)$; by (6.3.2), there exists a finite affine open cover (V_i) of X such that each of the rings $A(V_i)$ is an A-algebra of finite type. Further, since the V_i are quasi-compact, we can cover each of them with a finite number of open subsets of the form $D(g_{ij}) \subset V_i$, where $g_{ij} \in B$; if ϕ_i is a homomorphism $B \to A(V_i)$ that corresponds to the canonical injection $V_i \to X$, then we have $B_{g_{ij}} = (A(V_i))_{\phi_i(g_{ij})} = A(V_i)[1/\phi_i(g_{ij})]$, so $B_{g_{ij}}$ is an A-algebra of finite type. We can thus restrict to the case where $V_i = D(g_i)$ with $g_i \in B$. By hypothesis, there exists a finite subset F_i of B and an integer $n_i \geqslant 0$ such that B_{g_i} is the algebra generated over A by the elements $b_i/g_i^{n_i}$, where the b_i run over all of F_i. Since there are only finitely many of the g_i, we can assume that all the n_i are equal to the same integer n. Further, since the $D(g_i)$ form a cover of X, the ideal generated in B by the g_i is equal to B, or, in other words, there exist $h_i \in B$ such that $\sum_i h_i g_i = 1$. So let F be the finite subset of B given by the union of the F_i, the set of the g_i, and the set of the h_i; we will show that the subring $B' = A[F]$ of B is equal to B. By hypothesis, for every $b \in B$ and every i, the canonical image of b in B_{g_i} is of the form $b'_i/g_i^{m_i}$, where $b'_i \in B'$; by multiplying the b'_i by suitable powers of the g_i, we can again assume that all the m_i are equal to the same integer m. By the definition of the ring of fractions, there is thus an integer N (dependant on b) such that $N \geqslant m$ and $g_i^N b \in B'$ for all i; but, *in the ring B'*, the g_i^N generate the ideal B', because the g_i do (and the h_i belong to B'); there are thus $c_i \in B'$ such that $\sum_i c_i g_i^N = 1$, whence $b = \sum_i c_i g_i^N b \in B'$, Q.E.D. □

Proposition (6.3.4). —

(i) *Every closed immersion is of finite type.*
(ii) *The composition of any two morphisms of finite type is of finite type.*
(iii) *If $f : X \to X'$ and $g : Y \to Y'$ are S-morphisms of finite type, then $f \times_S g$ is of finite type.*
(iv) *If $f : X \to Y$ is an S-morphism of finite type, then $f_{(S')}$ is of finite type for any extension $g : S' \to S$ of the base prescheme.*
(v) *If the composition $g \circ f$ of two morphisms is of finite type, with g separated, then f is of finite type.*
(vi) *If a morphism f is of finite type, then f_{red} is of finite type.*

Proof. By (5.5.12), it suffices to prove (i), (ii), and (iv).

To show (i), we can restrict to the case of a canonical injection $X \to Y$, with X being a closed subprescheme of Y; further (6.3.2), we can assume that Y is affine, in which case X is also affine (4.2.3) and its ring is isomorphic to a quotient ring $A/\mathfrak{J}$, where A is the ring of Y and $\mathfrak{J}$ is an ideal of A; since $A/\mathfrak{J}$ is of finite type over A, the conclusion follows.

Now we show (ii). Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of finite type, and let U be an affine open subset of Z; g^{-1} admits a finite covering by affine open subsets V_i that are such that each $A(V_i)$ is an algebra of finite type over $A(U)$ (6.3.2); similarly. each of the f^{-1} admits a finite cover by affine open subsets W_{ij} that are such that each $A(W_{ij})$ is an algebra of finite type over $A(V_i)$, and so also an algebra of finite type over $A(U)$, whence the conclusion. I | 146

Finally, to show (iv), we can restrict to the case where $S = Y$; then $f_{(S')}$ is also equal to $f_{Y_{(S')}}$, where we consider f as a Y-morphism, and the base extension is $Y_{(S')} \to Y$ (3.3.9). So let p and q be the projections $X_{(S')} \to X$ and $X_{(S')} \to S'$. Let V be an affine open subset of S; $f^{-1}(V)$ is a finite union of affine open subsets W_i, each of which is such that $A(W_i)$ is an algebra of finite type over $A(V)$ (6.3.2). Let V' be an affine open subset of S' contained in $g^{-1}(V)$; since $f \circ p = g \circ q$, $q^{-1}(V')$ is contained in the union of the $p^{-1}(W_i)$; on the other hand, the intersection $p^{-1}(W_i) \cap q^{-1}(V')$ can be identified with the product $W_i \times VV'$ (3.2.7), which is an affine scheme whose ring is isomorphic to $A(W_i) \otimes_{A(V)} A(V')$ (3.2.2); this ring is, by hypothesis, an algebra of finite type over $A(V')$, which proves the proposition. □

Corollary (6.3.5). — *Let $f : X \to Y$ be an immersion morphism. If the underlying space of Y (resp. X) is locally Noetherian (resp. Noetherian), then f is of finite type.*

Proof. We can always assume that Y is affine (6.3.2); if the underlying space of Y is locally Noetherian, then we can further assume that it is Noetherian, and then the underlying space of X, which is a subspace, is also Noetherian. In other words, we can assume that Y is affine and that the underlying space of X is Noetherian; there then exists a covering of X by a finite number of affine open subsets $D(g_i) \subset Y$, where $g_i \in A(Y)$, that are such that the $X \cap D(g_i)$ are closed in $D(g_i)$ (and thus affine schemes (4.2.3)), because X is locally closed in Y (4.1.3). Then $A(X \cap D(g_i))$ is an algebra of finite type over $A(D(g_i))$, by (6.3.4, i) and (6.3.3), and $A(D(g_i)) = A(Y)_{g_i} = A(Y)[1/g_i]$ is of finite type over $A(Y)$, which finishes the proof. □

Corollary (6.3.6). — *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms. If $g \circ f$ if of finite type, with either X Noetherian or $X \times_Z Y$ locally Noetherian, then f is of finite type.*

Proof. This follows immediately from the proof of (5.5.12) and from (6.3.5) applied to the immersion morphism Γ_f. □

Proposition (6.3.7). — *Let $f : X \to Y$ be a morphism of finite type; if Y is Noetherian (resp. locally Noetherian), then X is Noetherian (resp. locally Noetherian).*

Proof. We can restrict to proving the proposition for when Y is Noetherian. Then Y is a finite union of affine open subsets V_i that are such that the $A(V_i)$ are Noetherian rings. By (6.3.2), each of the $f^{-1}(V_i)$ is a finite union of affine open subsets W_{ij} that are such that the $A(W_{ij})$ are algebras of finite type over $A(V_i)$, and thus Noetherian rings; this proves that X is Noetherian. □

Corollary (6.3.8). — *Let X be a prescheme of finite type over S. For every base extension $S' \to S$ with S' Noetherian (resp. locally Noetherian), $X_{(S')}$ is Noetherian (resp. locally Noetherian).*

Proof. This follows from (6.3.7), since $X_{(S')}$ is of finite type over S' by (6.3.4, iv). □

We can also says that, for a product $X \times_S Y$ of S-preschemes, if *one* of the factors X or Y is *of finite type* over S and *the other is Noetherian* (resp. *locally Noetherian*), then $X \times_S Y$ is *Noetherian* (resp. *locally Noetherian*). I | 147

Corollary (6.3.9). — *Let X be a prescheme of finite type over a locally Noetherian prescheme S. Then every S-morphism $f : X \to Y$ is of finite type.*

PROOF. In fact, we can assume that S is Noetherian; if $\phi : X \to S$ and $\psi : Y \to S$ are the structure morphisms, then we have $\phi = \psi \circ f$, and X is Noetherian by (6.3.7); f is thus of finite type by (6.3.6). □

Proposition (6.3.10). — *Let $f : X \to Y$ be a morphism of finite type. For f to be surjective, it is necessary and sufficient that, for every* algebraically closed *field Ω, the map $X(\Omega) \to Y(\Omega)$ that corresponds to f (3.4.1) be surjective.*

PROOF. The condition suffices, as we can see by considering, for all $y \in Y$, an algebraically closed extension Ω of $k(y)$, and the commutative diagram

$$\begin{tikzcd} X \arrow[dd, "f"'] & \\ & \mathrm{Spec}(\Omega) \arrow[ul] \arrow[dl] \\ Y & \end{tikzcd}$$

(cf. (3.5.3)). Conversely, suppose that f is surjective, and let $g : \{\xi\} = \mathrm{Spec}(\Omega) \to Y$ be a morphism, where Ω is an algebraically closed field. If we consider the diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & X_{(\Omega)} \arrow[l] \arrow[d, "f_{(\Omega)}"] \\ Y & \mathrm{Spec}(\Omega), \arrow[l] \end{tikzcd}$$

then it suffices to show that there exists a *rational point over* Ω in $X_{(\Omega)}$ ((3.3.14), (3.4.3), and (3.4.4)). Since f is surjective, $X_{(\Omega)}$ is nonempty (3.5.10), and since f is of finite type, so too is $f_{(\Omega)}$ (6.4.3, iv); thus $X_{(\Omega)}$ contains a nonempty affine open subset Z such that $A(Z)$ is an non-null algebra of finite type over Ω. By Hilbert's Nullstellensatz [**Zar47**], there exists an Ω-homomorphism $A(Z) \to \Omega$, and thus a section of $X_{(\Omega)}$ over $\mathrm{Spec}(\Omega)$, which proves the proposition. □

6.4. Algebraic preschemes

Definition (6.4.1). — Given a field K, we define an *algebraic K-prescheme* to be a prescheme X of finite type over K; K is called the base field of X. If in addition X is a scheme *(or if X is a* K-scheme, *which is equivalent* (5.5.8)*)*, we say that X is an *algebraic K-scheme*.

Every algebraic K-prescheme is *Noetherian* (6.3.7).

Proposition (6.4.2). — *Let X be an algebraic K-prescheme. For a point $x \in X$ to be closed, it is necessary and sufficient that $k(x)$ be an algebraic extension of K of finite degree.*

PROOF. We can assume that X is affine, with the ring A of X being a K-algebra of finite type. Indeed, the affine open subsets U of X such that $A(U)$ is a K-algebra of finite type form a finite cover of X (6.3.1). The closed points of X are thus the points such that $\mathfrak{j}_x$ is a maximal ideal of A, or, in I | 148
other words, such that $A/\mathfrak{j}_x$ is a field (necessarily equal to $k(x)$). Since $A/\mathfrak{j}_x$ is a K-algebra of finite type, we see that if x is closed, then $k(x)$ is a field that is an algebra of finite type over K, and so necessarily a K-algebra of *finite rank* [**Zar47**]. Conversely, if $k(x)$ is of finite rank over K, then so is $A/\mathfrak{j}_x \subset k(x)$, and since every integral ring that is also a K-algebra of finite rank is a field, we have that $A/\mathfrak{j}_x = k(x)$, and hence x is closed. □

Corollary (6.4.3). — *Let K be an algebraically-closed field, and X an algebraic K-prescheme; the closed points of X are then the rational points over K (3.4.4) and can be canonically identified with the points of X with values in K.*

Proposition (6.4.4). — *Let X be an algebraic prescheme over a field K. The following properties are equivalent.*

(a) *X is Artinian.*
(b) *The underlying space of X is discrete.*
(c) *The underlying space of X has only a finite number of closed points.*
(c′) *The underlying space of X is finite.*
(d) *The points of X are closed.*
(e) *X is isomorphic to $\operatorname{Spec}(A)$, where A is a K-algebra of finite rank.*

PROOF. Since X is Noetherian, it follows from (6.2.2) that the conditions (a), (b), and (d) are equivalent, and imply (c) and (c′); it is also clear that (e) implies (a). It remains to see that (c) implies (d) and (e); we can restrict to the case where X is affine. Then $A(X)$ is a K-algebra of finite type (6.3.3), and thus a Jacobson ring ([**CC**, p. 3-11 and 3-12]), in which there are, by hypothesis, only a finite number of maximal ideals. Since a finite intersection of prime ideals can only be a prime ideal if it is equal to one of the prime ideals being intersected, every prime ideal of $A(X)$ is thus maximal, whence (d). Further, we then know (6.2.2) that $A(X)$ is an Artinian K-algebra of finite type, and so necessarily of *finite rank* [**Zar47**]. □

(6.4.5). When the conditions of (6.4.4) are satisfied, we say that X is a scheme *finite over K* (cf. (**II**, 6.1.1)), or a *finite K-scheme*, of *rank* $[A : K]$, which we also denote by $\operatorname{rg}_K(X)$; if X and Y are finite K-schemes, we have

(6.4.5.1) $$\operatorname{rg}_K(X \sqcup Y) = \operatorname{rg}_K(X) + \operatorname{rg}_K(Y),$$

(6.4.5.2) $$\operatorname{rg}_K(X \times_K Y) = \operatorname{rg}_K(X)\operatorname{rg}_K(Y),$$

as a result of (3.2.2).

Corollary (6.4.6). — *Let X be a finite K-scheme. For every extension K' of K, $X \otimes_K K'$ as a finite K'-scheme, and its rank over K' is equal to the rank of X over K.*

PROOF. If $A = A(X)$, then we have $[A \otimes_K K' : K'] = [A : K]$. □

Corollary (6.4.7). — *Let X be a scheme finite over a field K; we let $n = \sum_{x \in X}[k(X) : K]_S$* (we recall that if K' is an extension of K, then $[K' : K]_S$ is the *separable rank* of K' over k, the rank of the largest algebraic separable extension of K contained in K'); *then for every algebraically closed extension Ω of K,* **I** | 149
the underlying space of $X \otimes_K \Omega$ has exactly n points, which can be identified with the points of X with values in Ω.

PROOF. We can clearly restrict to the case where the ring $A = A(X)$ is *local* (6.2.2); let $\mathfrak{m}$ be its maximal ideal, and $L = A/\mathfrak{m}$ its residue field, an algebraic extension of K. The points of X with values in Ω then correspond, bijectively, to the Ω-sections of $X \otimes_K \Omega$ ((3.4.1) and (3.3.14)), and also to the K-homomorphisms from L to Ω (1.7.3), whence the proposition (Bourbaki, *Alg.*, chap. V, §7, n° 5, prop. 8), taking (6.4.3) into account. □

(6.4.8). The number n defined in (6.4.7) is called the *separable rank* of A (or of X) over K, or also the *geometric number of points* of X; it is equal to the number of elements of $X(\Omega)_K$. It follows immediately from this definition that, for every extension K' of K, $X \otimes_K K'$ has the same geometric number of points as X. If we denote this number by $n(X)$, it is clear that, if X and Y are two schemes, finite over K, then

(6.4.8.1) $$n(X \sqcup Y) = n(X) + n(Y).$$

Under the same hypotheses, we also have

(6.4.8.2) $$n(X \times_K Y) = n(X)n(Y)$$

because of the interpretation of $n(X)$ as the number of elements of $X(\Omega)_K$ and Equation (3.4.3.1).

Proposition (6.4.9). — *Let K be a field, X and Y algebraic K-preschemes, $f : X \to Y$ a K-morphism, and Ω an algebraically closed extension of K of infinite transcendence degree over K. For f to be surjective, it is necessary and sufficient that the map $X(\Omega)_K \to Y(\Omega)_K$ that corresponds to f (3.4.1) be surjective.*

PROOF. The necessity follows from (6.3.10), noting that f is necessarily of finite type (6.3.9). To see that the condition is sufficient, we argue as in (6.3.10), noting that, for every $y \in Y$, $k(y)$ is an extension of K of finite type, and so is K-isomorphic to a subfield of Ω. □

Remark (6.4.10). — We will see in chapter IV that the conclusion of (6.4.9) still holds without the hypothesis on the transcendence degree of Ω over K.

Proposition (6.4.11). — *If $f : X \to Y$ is a morphism of finite type, then, for every $y \in Y$, the fibre $f^{-1}(y)$ is an algebraic prescheme over the residue field $k(y)$, and for every $x \in f^{-1}(y)$, $k(x)$ is an extension of $k(x)$ of finite type.*

PROOF. Since $f^{-1}(y) = X \otimes_Y k(y)$ (6.3.6), the proposition follows from (6.3.4, iv) and (6.3.3). □

Proposition (6.4.12). — *Let $f : X \to Y$ and $g : Y' \to Y$ be morphisms; set $X' = X \times_Y Y'$, and let $f' = f_{(Y')} : X' \to Y'$. Let $y' \in Y'$ and set $y = g(y')$; if the fibre $f^{-1}(y)$ is a finite algebraic scheme over $k(y)$, then the fibre $f'^{-1}(y)$ is a finite algebraic scheme over $k(y')$, and has the same rank and geometric number of points as $f^{-1}(y)$ does.*

PROOF. Taking into account the transitivity of fibres (3.6.5), this follows immediately from (6.4.6) and (6.4.8). □

(6.4.13). Proposition (6.4.11) shows that the morphisms of finite type that correspond, intuitively, to the "algebraic families of algebraic varieties", with the points of Y playing the role of "parameters", **I** | 150
which gives these morphisms a "geometric" meaning. The morphisms which are not of finite type will show up in the following mostly in questions of "changing the base prescheme", by localisation or completion, for example.

6.5. Local determination of a morphism

Proposition (6.5.1). — *Let X and Y be S-preschemes, with Y of finite type over S; let $x \in X$ and $y \in Y$ lie over the same point $s \in S$.*

(i) *If two S-morphisms $f = (\psi, \theta)$ and $f' = (\psi', \theta')$ from X to Y are such that $\psi(x) = \psi'(x) = y$, and the (local) $\mathscr{O}_s$-homomorphisms $\theta_x^\sharp$ and $\theta'^\sharp_x$ from $\mathscr{O}_y$ to $\mathscr{O}_x$ are identical, then f and f' agree on an open neighbourhood of x.*

(ii) *Suppose further that S is locally Noetherian. For every local $\mathscr{O}_s$-homomorphism $\phi : \mathscr{O}_y \to \mathscr{O}_x$, there exists an open neighbourhood U of x in X, and an S-morphism $f = (\psi, \theta)$ from U to Y such that $\psi(x) = y$ and $\theta_x^\sharp = \phi$.*

PROOF.

(i) Since the question is local on S, X, and Y, we can assume that S, X, and Y are affine, given by rings A, B, and C (respectively), and with f and f' of the form $({}^a\phi, \widetilde{\phi})$ and $({}^a\phi', \widetilde{\phi}')$ (respectively), where ϕ and ϕ' are A-homomorphisms from C to B such that $\phi^{-1}(\mathfrak{j}_x) = \phi'^{-1}(\mathfrak{j}_x) = \mathfrak{j}_y$, and the homomorphisms ϕ_x and ϕ'_x from C_y to B_x, induced by ϕ and ϕ', are identical; we can further suppose that C is an A-algebra *of finite type*. Let c_i ($1 \leqslant i \leqslant n$) be the generators of the A-algebra C, and set $b_i = \phi(c_i)$ and $b'_i = \phi'(c_i)$; by hypothesis, we have $b_i/1 = b'_i/1$ in the ring of fractions B_x ($1 \leqslant i \leqslant n$). This implies that there exist elements $s_i \in B - \mathfrak{j}_x$ such that $s_i(b_i - b'_i) = 0$ for $1 \leqslant i \leqslant n$, and we can clearly assume that all the s_i are equal to a single element $g \in B - \mathfrak{j}_x$. From this, we conclude that we have $b_i/1 = b'_i/1$ for $1 \leqslant i \leqslant n$ in the ring of fractions B_g; if i_g is the canonical homomorphism $B \to B_g$, we then have $i_g \circ \phi = i_g \circ \phi'$; so the restrictions of f and f' to $D(g)$ are identical.

(ii) We can restrict to the situation as in (i), and further assume that the ring A is Noetherian. Let c_i ($1 \leqslant i \leqslant n$) be the generators of the A-algebra C, and let $\alpha : A[X_1, \ldots, X_n] \to C$ be the homomorphism of polynomial algebras that sends X_i to c_i for $1 \leqslant i \leqslant n$. Also let i_y be the canonical homomorphism $C \to C_y$, and consider the composite homomorphism

$$\beta : A[X_1, \ldots, X_n] \xrightarrow{\alpha} C \xrightarrow{i_y} C_y \xrightarrow{\phi} B_x.$$

We denote by $\mathfrak{a}$ the kernel of β; since A is Noetherian, so too is $A[X_1, \ldots, X_n]$, and so $\mathfrak{a}$ admits a finite system of generators $Q_j(X_1, \ldots, X_n)$ $(1 \leqslant j \leqslant m)$. Furthermore, each of the elements $\phi(i_y(c_i))$ can be written in the form b_i/s_i, where $b_i \in B$ and $s_i \notin \mathfrak{j}_x$; we can further assume that all of the s_i are equal to a single element $g \in B - \mathfrak{j}_x$. With this, by hypothesis, we have $Q_j(b_1/g, \ldots, b_n/g) = 0$ in B_x; set I | 151

$$Q_j(X_1/T, \ldots, X_n/T) = P_j(X_1, \ldots, X_n, T)/T^{k_j}$$

where P_j is homogeneous of degree k_j. Then let $d_j = P_j(b_1, \ldots, b_n, g) \in B$. By hypothesis, we have $t_j d_j = 0$ for some $t_j \in B - \mathfrak{j}_x$ $(1 \leqslant j \leqslant m)$, and we can clearly assume that all the t_j are equal to a single element $h \in B - \mathfrak{j}_x$; from this we conclude that $P_j(hb_1, \ldots, hb_n, hg) = 0$ for $1 \leqslant j \leqslant m$. With this, consider the homomorphism ρ from $A[X_1, \ldots, X_n]$ to the ring of fractions B_{hg} which sends X_i to hb_i/hg $(1 \leqslant i \leqslant n)$; the image of $\mathfrak{a}$ under this homomorphism is 0, and is *a fortiori* the same as the image of the kernel $\alpha^{-1}(0)$ under ρ. So ρ factors as $A[X_1, \ldots, X_n] \xrightarrow{\alpha} C \xrightarrow{\gamma} B_{hg}$, with $\gamma(c_i) = hb_i/hg$, and it is clear that, if i_x is the canonical homomorphism $B_{hg} \to B_x$, then the diagram

(6.5.1.1)
$$\begin{tikzcd} C \arrow[r, "\gamma"] \arrow[d, "i_y"'] & B_{hg} \arrow[d, "i_x"] \\ C_y \arrow[r, "\phi"] & B_x \end{tikzcd}$$

is commutative; we thus have $\phi = \gamma_x$, and since ϕ is a local homomorphism, ${}^a\gamma(x) = y$; $f = ({}^a\gamma, \widetilde{\gamma})$ is thus an S-morphism from the neighbourhood $D(hg)$ of x to Y as claimed in the proposition.

□

Corollary (6.5.2). — *Under the hypotheses of (6.5.1, ii), if, further, X is of finite type over S, then we can assume that the morphism f is of finite type.*

PROOF. This follows from Corollary (6.3.6). □

Corollary (6.5.3). — *Suppose that the hypotheses of Proposition (6.5.1, ii), and suppose further that Y is integral, and that ϕ is an injective homomorphism. Then we can assume that $f = ({}^a\gamma, \widetilde{\gamma})$, where γ is injective.*

PROOF. Indeed, we can assume C to be integral (5.1.4), hence i_y injective; it then follows from the diagram (6.5.1.1) that γ is injective. □

Proposition (6.5.4). — *Let $f = (\psi, \theta) : X \to Y$ be a morphism of finite type, x a point of X, and $y = \psi(x)$.*

(i) *For f to be a local immersion at the point x (4.5.1), it is necessary and sufficient that $\theta_x^\sharp : \mathscr{O}_y \to \mathscr{O}_x$ be surjective.*

(ii) *Assume further that Y is locally Noetherian. For f to be a local isomorphism at the point x (4.5.2), it is necessary and sufficient that $\theta_x^\sharp$ be an isomorphism.*

PROOF.

(ii) By (6.5.1), there exists an open neighbourhood V of Y and a morphism $g : V \to X$ such that $g \circ f$ (resp. $f \circ g$) is defined and agrees with the identity on a neighbourhood of x (resp. y), whence we can easily see that f is a local isomorphism.

(i) Since the question is local on X and Y, we can assume that X and Y are affine, given by rings A and B (respectively); we have $f = ({}^a\phi, \widetilde{\phi})$, where ϕ is a homomorphism of rings $B \to A$ that makes A a B-algebra of finite type; we have $\phi^{-1}(\mathfrak{j}_x) = \mathfrak{j}_y$, and the homomorphism $\phi_x : B_y \to A_x$ induced by ϕ is *surjective*. Let (t_i) $(1 \leqslant i \leqslant n)$ be a system of generators of the B-algebra A; the hypothesis on ϕ_x implies that there exist $b_i \in B$ and some $c \in B - \mathfrak{j}_x$ such that, in the ring of fractions A_x, we have $t_i/1 = \phi(b_i)/\phi(c)$ for $1 \leqslant i \leqslant n$. Then (1.3.3) there exists some $a \in A - \mathfrak{j}_x$ such that, if we let $g = a\phi(c)$, we also have $t_i/1 = a\phi(b_i)/g$ *in the ring of fractions* A_g. With this, there exists, by hypothesis, a polynomial $Q(X_1, \ldots, X_n)$, with coefficients in the ring $\phi(B)$, such that $a = Q(t_1, \ldots, t_n)$; let $Q(X_1/T, \ldots, X_n/T) = P(X_1, \ldots, X_n, T)/T^m$, where P is homogeneous of degree m. In the ring A_g, we have I | 152

$$a/1 = a^m P(\phi(b_1), \ldots, \phi(b_n), \phi(c))/g^m = a^m \phi(d)/g^m$$

where $d \in B$. Since, in A_g, $g/1 = (a/1)(\phi(c)/1)$ is invertible by definition, so too are $a/1$ and $\phi(c)/1$, and we can thus write $a/1 = (\phi(d)/1)(\phi(c)/1)^{-m}$. From this we conclude that $\phi(d)/1$ is also invertible in A_g. So let $h = cd$; since $\phi(h)/1$ is invertible in A_g, the composite homomorphism $B \xrightarrow{\phi} A \to A_g$ factors as $B \to B_h \xrightarrow{\gamma} A_g$ (**0**, 1.2.4). We will show that γ is *surjective*; it suffices to show that the image of B_h in A_g contains the $t_i/1$ and $(g/1)^{-1}$. But we have $(g/1)^{-1} = (\phi(c)/1)^{m-1}(\phi(d)/1)^{-1} = \gamma(c^m/h)$, and $a/1 = \gamma(d^{m+1}/h^m)$, so $(a\phi(b_i))/1 = \gamma(b_i d^{m+1}/h^m)$, and since $t_i/1 = (a\phi(b_i)/1)(g/1)^{-1}$, our claim is proved. The choice of h implies that $\psi(D(g)) \subset D(h)$, and we also know that the restriction of f to $D(g)$ is equal to $({}^a\gamma, \widetilde{\gamma})$; since γ is surjective, this restriction is a closed immersion of $D(g)$ into $D(h)$ (4.2.3).

□

Corollary (6.5.5). — *Let $f = (\psi, \theta) : X \to Y$ be a morphism of finite type. Assume that X is irreducible, and denote by x its generic point, and let $y = \psi(x)$.*

(i) *For f to be a local immersion at any point of X, it is necessary and sufficient that $\theta_x^\sharp : \mathscr{O}_y \to \mathscr{O}_x$ be surjective.*

(ii) *Assume further that Y is irreducible and locally Noetherian. For f to be a local isomorphism at any point of X, it is necessary and sufficient that y be the generic point of Y* (or, equivalently (**0**, 2.1.4), that f be a *dominant* morphism) *and that $\theta_x^\sharp$ be an isomorphism* (in other words, that f be *birational* (2.2.9)).

Proof. It is clear that (i) follows from (6.5.4, i), taking into account the fact that every nonempty open subset of X contains x; similarly, (ii) follows from (6.5.4, ii). □

6.6. Quasi-compact morphisms and morphisms locally of finite type

Definition (6.6.1). — We say that a morphism $f : X \to Y$ is *quasi-compact* if the inverse image of any quasi-compact open subset of Y under f is quasi-compact.

Let $\mathfrak{B}$ be a base for the topology of Y consisting of quasi-compact open subsets (for example, affine open subsets); for f to be quasi-compact, it is necessary and sufficient that the inverse image of every set of $\mathfrak{B}$ under f be quasi-compact (or, equivalently, a *finite* union of affine open subsets), because every quasi-compact open subset of Y is a finite union of sets of $\mathfrak{B}$. For example, if X is *quasi-compact* and Y *affine*, then *every* morphism $f : X \to Y$ is quasi-compact: indeed, X is a finite union of affine open subsets U_i, and for every affine open subset V of Y, $U_i \cap f^{-1}(V)$ is affine (5.5.10), and so quasi-compact.

If $f : X \to Y$ is a quasi-compact morphism, it is clear that, for every open subset V of Y, the restriction of f to $f^{-1}(V)$ is a quasi-compact morphism $f^{-1}(V) \to V$. Conversely, if (U_α) is an open cover of Y, and $f : X \to Y$ a morphism such that the restrictions $f^{-1}(U_\alpha) \to U_\alpha$ are quasi-compact, then f is quasi-compact.

Definition (6.6.2). — We say that a morphism $f : X \to Y$ is *locally of finite type* if, for every $x \in X$, there exists an open neighbourhood U of x and an open neighbourhood $V \supset f(U)$ of y such that the restriction of f to U is a morphism of finite type from U to V. We then also say that X is a prescheme locally of finite type over Y, or a Y-prescheme locally of finite type. I | 153

It follows immediately from (6.3.2) that, if f is locally of finite type, then, for every open subset W of Y, the restriction of f to $f^{-1}(W)$ is a morphism $f^{-1}(W) \to W$ that is locally of finite type.

If Y is locally Noetherian and X locally of finite type over Y, then X is locally Noetherian thanks to (6.3.7).

Proposition (6.6.3). — *For a morphism $f : X \to Y$ to be of finite type, it is necessary and sufficient that it be quasi-compact and locally of finite type.*

PROOF. The necessity of the conditions is immediate, given (6.3.1) and the remark following (6.6.1). Conversely, suppose that the conditions are satisfied, and let U be an affine open subset of Y, given by some ring A; for all $x \in f^{-1}(U)$, there is, by hypothesis, a neighbourhood $V(x) \subset f^{-1}(U)$ of x, and a neighbourhood $W(x) \subset U$ of $y = f(x)$ containing $f(V(x))$, and such that the restriction of f to $V(x)$ is a morphism $V(x) \to W(x)$ of finite type. Replacing $W(x)$ with a neighbourhood $W_1(x) \subset W(x)$ of x of the form $D(g)$ (with $g \in A$), and $V(x)$ with $V(x) \cap f^{-1}(W_1(x))$, we can assume that $W(x)$ is of the form $D(g)$, and thus of finite type over U (because its ring can be written as $A[1/g]$); so $V(x)$ is of finite type over U. Further, $f^{-1}(U)$ is quasi-compact by hypothesis, and so the finite union of open subsets $V(x_i)$, which finishes the proof. □

Proposition (6.6.4). —

(i) *An immersion $X \to Y$ is quasi-compact if it is closed, or if the underlying space of Y is locally Noetherian, or if the underlying space of X is Noetherian.*
(ii) *The composition of any two quasi-compact morphisms is quasi-compact.*
(iii) *If $f : X \to Y$ is a quasi-compact S-morphism, then so too is $f_{(S')} : X_{(S')} \to Y_{(S')}$ for any extension $g : S \to S'$ of the base prescheme.*
(iv) *If $f : X \to X'$ and $g : Y \to Y'$ are two quasi-compact S-morphisms, then $f \times_S g$ is quasi-compact.*
(v) *If the composition of any two morphisms $f : X \to Y$ and $g : Y \to Z$ is quasi-compact, and if either g is separated or the underlying space of X is locally Noetherian, then f is quasi-compact.*
(vi) *For a morphism f to be quasi-compact, it is necessary and sufficient that f_{red} be quasi-compact.*

PROOF. We note that (vi) is evident because the property of being quasi-compact, for a morphism, depends only on the corresponding continuous map of underlying spaces. We will similarly prove the part of (v) corresponding to the case where the underlying space of X is locally Noetherian. Set $h = g \circ f$, and let U be a quasi-compact open subset of Y; $g(U)$ is quasi-compact (but not necessarily open) in Z, and so contained in a finite union of quasi-compact open subsets V_j (2.1.3), and $f^{-1}(U)$ is thus contained in the union of the $h^{-1}(V_j)$, which are quasi-compact subspaces of X, and thus Noetherian subspaces. We thus conclude (**0**, 2.2.3) that $f^{-1}(U)$ is a Noetherian space, and *a fortiori* quasi-compact.

To prove the other claims, it suffices to prove (i), (ii), and (iii) (5.5.12). But (ii) is evident, and (i) follows from (6.3.5) whenever the space Y is locally Noetherian or the space X is Noetherian, and is evident for a closed immersion. To show (iii), we can restrict to the case where $S = Y$ (3.3.11); **I** | 154
let $f' = f_{(S')}$, and let U' be a quasi-compact open subset of S'. For every $s' \in U'$, let T be an affine open neighbourhood of $g(s')$ in S, and let W be an affine open neighbourhood of s' contained in $U' \cap g^{-1}(T)$; it will suffice to show that $f'^{-1}(W)$ is quasi-compact; in other words, we can restrict to showing that, when S and S' are *affine*, the underlying space of $X \times_S S'$ is quasi-compact. But since X is then, by hypothesis, a finite union of affine open subsets V_j, $X \times_S S'$ is a union of the underlying spaces of the affine schemes $V_j \times_S S'$ ((3.2.2) and (3.2.7)), which proves the proposition. □

We note also that, if $X = X' \sqcup X''$ is the sum of two preschemes, a morphism $f : X \to Y$ is quasi-compact if and only if its restrictions to both X' and X'' are quasi-compact.

Proposition (6.6.5). — *Let $f : X \to Y$ be a quasi-compact morphism. For f to be dominant, it is necessary and sufficient that, for every generic point y of an irreducible component of Y, $f^{-1}(y)$ contain the generic point of an irreducible component of X.*

PROOF. It is immediate that the condition is sufficient (even without assuming that f is quasi-compact). To see that it is necessary, consider an affine open neighbourhood U of y; $f^{-1}(U)$ is quasi-compact, and so a *finite* union of affine open subsets V_i, and the hypothesis that f be dominant implies that y belongs to the closure *in* U of one of the $f(V_i)$. We can clearly assume X and Y to be reduced; since the closure in X of an irreducible component of V_i is an irreducible component on X

(**0**, 2.1.6), we can replace X by V_i, and Y by the closed reduced subprescheme of U that has $\overline{f(V_i)} \cap U$ as its underlying space (5.2.1), and we are thus led to proving the proposition when $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$ are affine and reduced. Since f is dominant, B is a subring of A (1.2.7), and the proposition then follows from the fact that every minimal prime ideal of B is the intersection of B with a minimal prime ideal of A (**0**, 1.5.8). □

Proposition (6.6.6). —

(i) *Every local immersion is locally of finite type.*
(ii) *If two morphisms $f : X \to Y$ and $g : Y \to Z$ are locally of finite type, then so too is $g \circ f$.*
(iii) *If $f : X \to Y$ is an S-morphism locally of finite type, then $f_{(S')} : X_{(S')} \to Y_{(S')}$ is locally of finite type for any extension $S' \to S$ of the base prescheme.*
(iv) *If $f : X \to X'$ and $g : Y \to Y'$ are S-morphisms locally of finite type, then $f \times_S g$ is locally of finite type.*
(v) *If the composition $g \circ f$ of two morphisms is locally of finite type, then f is locally of finite type.*
(vi) *If a morphism f is locally of finite type, then so too is f_{red}.*

Proof. By (5.5.12), it suffices to prove (i), (ii), and (iii). If $j : X \to Y$ is a local immersion then, for every $x \in X$, there is an open neighbourhood V of $j(x)$ in Y and an open neighbourhood U of x in X such that the restriction of j to U is a closed immersion $U \to V$ (4.5.1), and so this restriction is of finite type. To prove (ii), consider a point $x \in X$; by hypothesis, there is an open neighbourhood W of $g(f(x))$ and an open neighbourhood V of $f(x)$ such that $g(V) \subset W$ and such that V is of of finite type over W; furthermore, $f^{-1}(V)$ is locally of finite type over V (6.6.2), so there is an open neighbourhood U of x that is contained in $f^{-1}(V)$ and of is finite type over V; thus we have **I** | 155
$g(f(U)) \subset W$, and that U is of finite type over W (6.3.4, ii). Finally, to prove (iii), we can restrict to the case where $Y = S$ (3.3.11); for every $x' \in X' = X_{(S')}$, let x be the image of x' in X, s the image of x in S, T an open neighbourhood of s, T' the inverse image of T in S', and U an open neighbourhood of x that is of finite type over T and whose image is contained in T; then $U \times_S T' = U \times_T T'$ is an open neighbourhood of x' (3.2.7) that is of finite type over T' (6.3.4, iv). □

Corollary (6.6.7). — *Let X and Y be S-preschemes that are locally of finite type over S. If S is locally Noetherian, then $X \times_S Y$ is locally Noetherian.*

Proof. Indeed, X being locally of finite type over S means that it is locally Noetherian, and that $X \times_S Y$ is locally of finite type over X, and so $X \times_S Y$ is also locally Noetherian. □

Remark (6.6.8). — Proposition (6.3.10) and its proof extend immediately to the case where we suppose only that the morphism f is locally of finite type. Similarly, propositions (6.4.2) and (6.4.9) hold true when we suppose only that the preschemes X and Y in the claim are locally of finite type over the field K.

§7. Rational maps

7.1. Rational maps and rational functions

(7.1.1). Let X and Y be preschemes, U and V dense open subsets of X, and f (resp. g) a morphism from U (resp. V) to Y; we say that f and g are *equivalent* if they agree on a dense open subset of $U \cap V$. Since a finite intersection of dense open subsets of X is a dense open subset of X, it is clear that this relation is an *equivalence relation*.

Definition (7.1.2). — Given preschemes X and Y, we define a rational map from X to Y to be an equivalence class of morphisms from a dense open subset of X to a dense open subset of Y, under the equivalence relation defined in (7.1.1). If X and Y are S-preschemes, we say that such a class is a rational S-map if there exists a representative of the class that is also an S-morphism. We define a rational S-section of X to be any rational S-map from S to X. We define a rational function on a prescheme X to be any rational X-section on the X-prescheme $X \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ (where T is an indeterminate).

By an abuse of language, whenever we are discussing only S-preschemes, we will say "rational map" instead of "rational S-map" if no confusion may arise.

Let f be a rational map from X to Y, and U an open subset of X; if f_1 and f_2 are two morphisms belonging to the class of f, defined (respectively) on dense open subsets V and W of X, then the restrictions $f_1|(U \cap V)$ and $f_2|(U \cap W)$ agree on $U \cap V \cap W$, which is dense in U; the class f of morphisms thus defines a rational map from U to Y, called the *restriction of f to U*, and denoted by $f|U$.

If, to every S-morphism $f : X \to Y$, we take the corresponding rational S-map to which f belongs, we obtain a canonical map from $\mathrm{Hom}_S(X, Y)$ to the set of rational S-maps from X to Y. We denote by $\Gamma_{\mathrm{rat}}(X/Y)$ the set of rational Y-sections on X, and we thus have a canonical map $\Gamma(X/Y) \to \Gamma_{\mathrm{rat}}(X/Y)$. It is also clear that, if X and Y are S-preschemes, then the set of rational S-maps from X to Y is canonically identified with $\Gamma_{\mathrm{rat}}((X \times_S Y)/X)$ (3.3.14).

(7.1.3). It also follows from (7.1.2) and (3.3.14) that the rational functions on X are canonically identified with *equivalence classes of sections of the structure sheaf $\mathcal{O}_X$* over dense open subsets of X, where two such sections are equivalent if the agree on some dense open subset of X contained inside the intersection of the subsets on which they are defined. In particular, it follows that the rational functions on X form a *ring* $R(X)$.

(7.1.4). When X is an *irreducible* prescheme, every nonempty open subset of X is dense in X; so we can say that the nonempty open subsets of X are the *open neighbourhoods of the generic point x* of X. To say that two morphisms from nonempty open subsets of X to Y are equivalent thus means, in this case, that they have the *same germ* at the point x. In other words, the rational maps (resp. rational S-maps) $X \to Y$ are identified with the *germs of morphisms* (resp. *S-morphisms*) from nonempty open subsets of X to Y at the generic point x of X. In particular:

Proposition (7.1.5). — *If X is an irreducible prescheme, then the ring $R(X)$ of rational maps on X is canonically identified with the local ring $\mathcal{O}_x$ of the generic point x of X. It is a local ring of dimension 0, and thus a local Artinian ring when X is Noetherian; it is a field when X is integral, and, when X is further an affine scheme, it is identified with the field of fractions of $A(X)$.*

PROOF. Given the above, and the identification of rational functions with sections of $\mathcal{O}_X$ over a dense open subset of X, the first claim is nothing but the definition of the fibre of a sheaf above a point. For the other claims, we can reduce to the case where X is affine, given by some ring A; then $\mathfrak{j}_x$ is the nilradical of A, and $\mathcal{O}_x$ is thus of dimension 0; if A is integral, then $\mathfrak{j}_x = (0)$, and $\mathcal{O}_x$ is thus the field of fractions of A. Finally, if A is Noetherian, we know ([**Sam53b**, p. 127, cor. 4]) that $\mathfrak{j}_x$ is nilpotent, and $\mathcal{O}_x = A_x$ Artinian. □

If X is *integral*, then the ring $\mathcal{O}_z$ is integral for all $z \in X$; every affine open subset U containing z also contains x, and $R(U)$, being equal to the field of fractions of $A(U)$, is identified with $R(X)$; we thus conclude that $R(X)$ can also be identified with the *field of fractions of $\mathcal{O}_z$*: the canonical identification of $\mathcal{O}_z$ to a subring of $R(X)$ consists of associating, to every germ of a section $s \in \mathcal{O}_z$, the unique rational function on X, class of a section of $\mathcal{O}_X$, (necessarily defined on a dense open subset of X) having s as its germ at the point z.

(7.1.6). Now suppose that X has a *finite* number of irreducible components X_i ($1 \leqslant i \leqslant n$) (which will be the case whenever the underlying space of X is *Noetherian*); let X'_i be the open subset of X given by the complement of the $X_j \cap X_i$ for $j \neq i$ inside X_i; X'_i is irreducible, its generic point x_i is the generic point of X_i, and the X'_i are pairwise disjoint, with their union being dense in X (**0**, 2.1.6). For every dense open subset U of X, $U_i = U \cap X'_i$ is a nonempty dense open subset of X'_i, with the U_i being pairwise disjoint, and so $U' = \bigcup_i U'_i$ is dense in X. Giving a morphism from U' to Y consists of giving (arbitrarily) a morphism from each of the U_i to Y.

Thus:

Proposition (7.1.7). — *Let X and Y be two preschemes (resp. S-preschemes) such that X has a finite number of irreducible components X_i, with generic points x_i ($1 \leqslant i \leqslant n$). If R_i is the set of germs of morphisms (resp. S-morphisms) from open subsets of X to Y at the point x_i, then the set of rational maps (resp. rational S-maps) from X to Y can be identified with the product of the R_i ($1 \leqslant i \leqslant n$).*

Corollary (7.1.8). — *Let X be a Noetherian prescheme. The ring of rational functions on X is an Artinian ring, whose local components are the rings $\mathcal{O}_{x_i}$ of the generic points x_i of the irreducible components of X.*

Corollary (7.1.9). — *Let A be a Noetherian ring, and $X = \operatorname{Spec}(A)$. If Q is the complement of the union of the minimal prime ideals of A, then the ring of rational functions on X can be canonically identified with the ring of fractions $Q^{-1}A$.*

This will follow from the following lemma:

Lemma (7.1.9.1). — *For an element $f \in A$ to be such that $D(f)$ is dense in X, it is necessary and sufficient that $f \in Q$; every dense open subset of X contains an open subset of the form $D(f)$, where $f \in Q$.*

PROOF. To show (7.1.9.1), we again denote by X_i $(1 \leqslant i \leqslant n)$ the irreducible components of X; if $D(f)$ is dense in X then $D(f) \cap X_i \neq \varnothing$ for $1 \leqslant i \leqslant n$, and vice-versa; but this means that $f \notin \mathfrak{p}_i$ for $1 \leqslant i \leqslant n$, where we set $\mathfrak{p}_i = \mathfrak{j}(X_i)$, and since the $\mathfrak{p}_i$ are the minimal prime ideals of A (1.1.14), the conditions $f \notin \mathfrak{p}_i$ $(1 \leqslant i \leqslant n)$ are equivalent to $f \in Q$, whence the first claim of the lemma. For the other claim, if U is a dense open subset of X, the complement of U is a set of the form $V(\mathfrak{a})$, where $\mathfrak{a}$ is an ideal which is not contained in any of the $\mathfrak{p}_i$; it is thus not contained in their union ([**Nor53**, p. 13]), and there thus exists some $f \in \mathfrak{a}$ belonging to Q; whence $D(f) \subset U$, which finishes the proof. □

(7.1.10). Suppose again that X is irreducible, with generic point x. Since every nonempty open subset U of X contains x, and thus also contains every $z \in X$ such that $x \in \overline{\{z\}}$, every morphism $U \to Y$ can be composed with the canonical morphism $\operatorname{Spec}(\mathscr{O}_x) \to X$ (2.4.1); and any two morphisms into Y from two nonempty open subsets of X which agree on a nonempty open subset of X give, by composition, the same morphism $\operatorname{Spec}(\mathscr{O}_x) \to Y$. In other words, to every rational map from X to Y there is a corresponding well-defined morphism $\operatorname{Spec}(\mathscr{O}_x) \to Y$.

Proposition (7.1.11). — *Let X and Y be two S-preschemes; suppose that X is irreducible with generic point x, and that Y is of finite type over S. Any two rational S-maps from X to Y that correspond to the same S-morphism $\operatorname{Spec}(\mathscr{O}_x) \to Y$ are then identical. If we further suppose S to be locally Noetherian, then every S-morphism from $\operatorname{Spec}(\mathscr{O}_x)$ to Y corresponds to exactly one rational S-map from X to Y.*

PROOF. Taking into account that every nonempty subset of X is dense in X, this follows from (6.5.1). □

Corollary (7.1.12). — *Suppose that S is locally Noetherian, and that the other hypotheses of (7.1.11) are satisfied. The rational S-maps from X to Y can then be identified with points of the S-prescheme Y, with values in the S-prescheme $\operatorname{Spec}(\mathscr{O}_x)$.*

PROOF. This is nothing but (7.1.11), with the terminology introduced in (3.4.1). □

Corollary (7.1.13). — *Suppose that the conditions of (7.1.12) are satisfied. Let s be the image of x in S. The data of a rational S-map from X to Y is equivalent to the data of a point y of Y over s along with a local $\mathscr{O}_s$-homomorphism $\mathscr{O}_y \to \mathscr{O}_x = R(X)$.*

PROOF. This follows from (7.1.11) and (2.4.4). □

In particular:

Corollary (7.1.14). — *Under the conditions of (7.1.12), rational S-maps from X to Y depend only (for any given Y) on the S-prescheme $\operatorname{Spec}(\mathscr{O}_x)$, and, in particular, remain the same whenever X is replaced by $\operatorname{Spec}(\mathscr{O}_z)$, for any $z \in X$.*

PROOF. Since $z \in \overline{\{x\}}$, x is the generic point of $Z = \operatorname{Spec}(\mathscr{O}_z)$, and $\mathscr{O}_{X,x} = \mathscr{O}_{Z,z}$. □

When X is integral, $R(X) = \mathscr{O}_x = k(x)$ is a field (7.1.5); the preceding corollaries then specialize to the following:

Corollary (7.1.15). — *Suppose that the conditions of (7.1.12) are satisfied, and further that X is integral. Let s be the image of x in S. Then rational S-maps from X to Y can be identified with the geometric points of $Y \otimes_S k(s)$ with values in the extension $R(X)$ of $k(s)$, or, in other words, every such map is equivalent to the data of a point $y \in Y$ above s along with a $k(s)$-monomorphism from $k(y)$ to $k(x) = R(X)$.*

PROOF. The points of Y above s are identified with the points of $Y \otimes_S k(s)$ (3.6.3), and the local $\mathscr{O}_s$-homomorphisms $\mathscr{O}_y \to R(X)$ with the $k(s)$-monomorphisms $k(y) \to R(X)$. □

More precisely:

Corollary (7.1.16). — *Let k be a field, and X and Y two algebraic preschemes over k (6.4.1); suppose further that X is integral. Then the rational k-maps from X to Y can be identified with the geometric points of Y with values in the extension $R(X)$ of k (3.4.4).*

7.2. Domain of definition of a rational map

(7.2.1). Let X and Y be preschemes, and f rational map from X to Y. We say that f is *defined at a point* $x \in X$ if there exists a dense open subset U of X that contains x, and a morphism $U \to Y$ belonging to the equivalence class of f. The set of points $x \in X$ where f is defined is called the *domain of definition* of f; it is clear that it is an open dense subset of X.

Proposition (7.2.2). — *Let X and Y be S-preschemes such that X is reduced and Y is separated over S. Let f be a rational S-map from X to Y, with domain of definition U_0. Then there exists exactly one S-morphism $U_0 \to Y$ belonging to the class of f.* **I** | 159

Since, for every morphism $U \to Y$ belonging to the class of f, we necessarily have $U \subset U_0$, it is clear that the proposition will be a consequence of the following:

Lemma (7.2.2.1). — *Under the hypotheses of (7.2.2), let U_1 and U_2 be two dense open subsets of X, and $f_i : U_i \to Y$ $(i = 1, 2)$ two S-morphisms such that there exists an open subset $V \subset U_1 \cap U_2$, dense in X, and on which f_1 and f_2 agree. Then f_1 and f_2 agree on $U_1 \cap U_2$.*

Proof. We can clearly restrict to the case where $X = U_1 = U_2$. Since X (and thus V) is reduced, X is the smallest closed subprescheme of X containing V (5.2.2). Let $g = (f_1, f_2)_S : X \to Y \times_S Y$; since, by hypothesis, the diagonal $T = \Delta_Y(Y)$ is a closed subprescheme of $Y \times_S Y$, $Z = g^{-1}(T)$ is a closed subprescheme of X (4.4.1). If $h : V \to Y$ is the common restriction of f_1 and f_2 to V, then the restriction of g to V is $g' = (h, h)_S$, which factors as $g' = \Delta_Y \circ h$; since $\Delta_Y^{-1}(T) = Y$, we have that $g'^{-1}(T) = V$, and so Z is a closed subprescheme of X inducing V, thus containing V, which implies that $Z = X$. From the equation $g^{-1}(T) = X$, we deduce (4.4.1) that g factors as $\Delta_Y \circ f$, where f is a morphism $X \to Y$, which implies, by the definition of the diagonal morphism, that $f_1 = f_2 = f$. □

It is clear that the morphism $U_0 \to Y$ defined in (7.2.2) is the unique morphism of the class f that *cannot be extended* to a morphism from an open subset of X that strictly contains U_0. *Under the hypotheses of* (7.2.2), we can thus *identify* the rational maps from X to Y with the *non-extendible* (to strictly larger open subsets) morphisms from dense open subsets of X to Y. With this identification, Proposition (7.2.2) implies:

Corollary (7.2.3). — *With the hypotheses from (7.2.2) on X and Y, let U be a dense open subset of X. Then there exists a canonical bijective correspondence between S-morphisms from U to Y and rational S-maps from X to Y that are defined at all points of U.*

Proof. By (7.2.2), for every S-morphism f from U to Y, there exists exactly one rational S-map $\overline{f}$ from X to Y which extends f. □

Corollary (7.2.4). — *Let S be a scheme, X a reduced S-prescheme, Y an S-scheme, and $f : U \to Y$ an S-morphism from a dense open subset U of X to Y. If $\overline{f}$ is the rational $\mathbf{Z}$-map from X to Y that extends f, then $\overline{f}$ is an S-morphism (and thus the rational S-map from X to Y that extends f).*

Proof. Indeed, if $\phi : X \to S$ and $\psi : Y \to S$ are the structure morphisms, U_0 the domain of definition of $\overline{f}$, and j the injection $U_0 \to X$, then it suffices to show that $\psi \circ \overline{f} = \phi \circ j$, but this follows from (7.2.2.1), since f is an S-morphism. □

Corollary (7.2.5). — *Let X and Y be two S-preschemes; suppose that X is reduced, and that X and Y are separated over S. Let $p : Y \to X$ be an S-morphism (making Y an X-prescheme), U a dense open subset of X, and f a U-section of Y; then the rational map $\overline{f}$ from X to Y extending f is a rational X-section of Y.*

Proof. We have to show that $p \circ \overline{f}$ is the identity on the domain of definition of $\overline{f}$; since X is separated over S, this again follows from (7.2.2.1). □ **I** | 160

Corollary (7.2.6). — *Let X be a reduced prescheme, and U a dense open subset of X. Then there is a canonical bijective correspondence between sections of $\mathscr{O}_X$ over U and rational functions on X defined at every point of U.*

PROOF. Taking (7.2.3), (7.1.2), and (7.1.3) into account, it suffices to note that the X-prescheme $X \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ is separated over X (5.5.1, iv). □

Corollary (7.2.7). — *Let Y be a reduced prescheme, $f : X \to Y$ a separated morphism, U a dense open subset of Y, $g : U \to f^{-1}(U)$ a U-section of $f^{-1}(U)$, and Z the reduced subprescheme of X that has $\overline{g(U)}$ as its underlying space* (5.2.1)*. For g to be the restriction of a Y-section of X* (in other words (7.2.5), for the rational map from Y to X extending g to be defined everywhere)*, it is necessary and sufficient for the restriction of f to Z to be an isomorphism from Z to Y.*

PROOF. The restriction of f to $f^{-1}(U)$ is a separated morphism (5.5.1, i), so g is a closed immersion (5.4.6), and so $g(U) = Z \cap f^{-1}(U)$, and the subprescheme induced by Z on the open subset $g(U)$ of Z is identical to the closed subprescheme of $f^{-1}(U)$ associated to g (5.2.1). It is then clear that the stated condition is sufficient, because, if satisfied, and if $f_Z : Z \to Y$ is the restriction of f to Z, and $\overline{g} : Y \to Z$ is the inverse isomorphism, then $\overline{g}$ extends g. Conversely, if g is the restriction to U of a Y-section h of X, then h is a closed immersion (5.4.6), and so $h(Y)$ is closed, and, since it is contained in Z, is equal to Z, and it follows from (5.2.1) that h is necessarily an isomorphism from Y to the closed subprescheme Z of X. □

(7.2.8). Let X and Y be two S-preschemes, with X reduced, and Y separated over S. Let f be a rational S-map from X to Y, and let x be a point of X; we can compose f with the canonical S-morphism $\operatorname{Spec}(\mathscr{O}_x) \to X$ (2.4.1) provided that the intersection of $\operatorname{Spec}(\mathscr{O}_x)$ with the domain of definition of f is dense in $\operatorname{Spec}(\mathscr{O}_x)$ (identified with the set of $z \in X$ such that $x \in \overline{\{z\}}$ (2.4.2)). This will happen in the follow cases:

1st. X is *irreducible* (and thus *integral*), because then the generic point ξ of X is the generic point of $\operatorname{Spec}(\mathscr{O}_x)$; since the domain of definition U of f contains ξ, $U \cap \operatorname{Spec}(\mathscr{O}_x)$ contains ξ, and so is dense in $\operatorname{Spec}(\mathscr{O}_x)$.

2nd. X is *locally Noetherian*; our claim then follows from:

Lemma (7.2.8.1). — *Let X be a prescheme whose underlying space is locally Noetherian, and x a point of X. The irreducible components of* $\operatorname{Spec}(\mathscr{O}_x)$ *are the intersections of* $\operatorname{Spec}(\mathscr{O}_x)$ *with the irreducible components of X containing x. For an open subset $U \subset X$ to be such that $U \cap \operatorname{Spec}(\mathscr{O}_x)$ is dense* $\operatorname{Spec}(\mathscr{O}_x)$*, it is necessary and sufficient for it to have a nonempty intersection with the irreducible components of X that contain x* (which will be the case whenever U is *dense* in X).

PROOF. It suffices to show just the first claim, since the second then follows. Since $\operatorname{Spec}(\mathscr{O}_x)$ is contained in every affine open subset U that contains x, and since the irreducible components of U that contain x are the intersections of U with the irreducible components of X containing x (**0**, 2.1.6), we can suppose that X is affine, given by some ring A. Since the prime ideals of A_x correspond bijectively to the prime ideals of A that are contained in $\mathfrak{j}_x$ (2.1.6), the minimal prime ideals of A_x correspond to the minimal prime ideals of A that are contained in $\mathfrak{j}_x$, hence the lemma. □ **I** | 161

With this in mind, suppose that we are in one of the two cases mentioned in (7.2.8). If U is the domain of definition of the rational S-map f, then we denote by f' the rational map from $\operatorname{Spec}(\mathscr{O}_x)$ to Y which agrees (taking (2.4.2) into account) with f on $U \cap \operatorname{Spec}(\mathscr{O}_x)$; we say that this rational map is *induced by f*.

Proposition (7.2.9). — *Let S be a locally Noetherian prescheme, X a reduced S-prescheme, and Y an S-scheme of finite type. Suppose further that X is either irreducible or locally Noetherian. Then let f be a rational S-map from X to Y, and x a point of X. For f to be defined at a point x, it is necessary and sufficient for the rational map f' from* $\operatorname{Spec}(\mathscr{O}_x)$ *to Y, induced by f* (7.2.8)*, to be a morphism.*

PROOF. The condition clearly being necessary (since $\operatorname{Spec}(\mathscr{O}_x)$ is contained in every open subset containing x), we show that it is sufficient. By (6.5.1), there exists an open neighbourhood U of x in X, and an S-morphism g from U to Y that induces f' on $\operatorname{Spec}(\mathscr{O}_x)$. If X is irreducible, then U is dense in X, and, by (7.2.3), we can suppose that g is a rational S-map. Further, the generic point of

X belongs to both $\operatorname{Spec}(\mathscr{O}_x)$ and the domain of definition of f, and so s and g agree at this point, and thus on a nonempty open subset of X (6.5.1). But since f and g are rational S-maps, they are identical (7.2.3), and so f is defined at x.

If we now suppose that X is locally Noetherian, then we can suppose that U is Noetherian; then there are only a finite number of irreducible components X_i of X that contain x (7.2.8.1), and we can suppose that they are the only ones that have a nonempty intersection with U, by replacing, if needed, U with a smaller open subset (since there are only a finite number of irreducible components of X that have a nonempty intersection with U, because U is Noetherian). We then have, as above, that f and g agree on a nonempty open subset of each of the X_i. Taking into account the fact that each of the X_i is contained in $\overline{U}$, we consider the morphism f_1, defined on a dense open subset of $U \cup (X - \overline{U})$, equal to g on U, and to f on the intersection of $X - \overline{U}$ with the domain of definition of f. Since $U \cup (X - \overline{U})$ is dense in X, f_1 and f agree on a dense open subset of X, and since f is a rational map, f is an extension of f_1 (7.2.3), and is thus defined at the point x. ☐

7.3. Sheaf of rational functions

(7.3.1). Let X be a prescheme. For every open subset $U \subset X$, we denote by $R(U)$ the ring of rational functions on U (7.1.3); this is a $\Gamma(U, \mathscr{O}_X)$-algebra. Further, if $V \subset U$ is a second open subset of X, then every section of $\mathscr{O}_X$ over a dense (in X) open subset of V gives, by restriction to V, a section over a dense (in X) open subset of V, and if two sections agree on a dense (in X) open subset of U, then their restrictions to V agree on a dense (in X) open subset of V. We can thus define a di-homomorphism of algebras $\rho_{V,U} : R(U) \to R(V)$, and it is clear that, if $U \supset V \supset W$ are open subsets of X, then we have $\rho_{W,U} = \rho_{W,V} \circ \rho_{V,U}$; the $R(U)$ thus define a *presheaf* of algebras on X.

Definition (7.3.2). — We define the sheaf of rational functions on a prescheme X, denoted by $\mathscr{R}(X)$, to be the $\mathscr{O}_X$-algebra associated to the presheaf defined by the $R(U)$. **I** | 162

For every prescheme X and open subset $U \subset X$, it is clear that the induced sheaf $\mathscr{R}(X)|U$ is exactly $\mathscr{R}(U)$.

Proposition (7.3.3). — *Let X be a prescheme such that the family (X_λ) of its irreducible components is locally finite (which is the case whenever the underlying space of X is locally Noetherian). Then the $\mathscr{O}_X$-module $\mathscr{R}(X)$ is quasi-coherent, and for every open subset U of X that has a nonempty intersection with only finitely many of the components X_λ, $R(U)$ is equal to $\Gamma(U, \mathscr{R}(X))$, and can be canonically identified with the direct sum of the local rings of the generic points of the X_λ such that $U \cap X_\lambda \neq \varnothing$.*

Proof. We can evidently restrict to the case where X has only a finite number of irreducible components X_i, with generic points x_i ($1 \leqslant i \leqslant n$). The fact that $R(U)$ can be canonically identified with the direct sum of the $\mathscr{O}_{x_i} = R(X_i)$ such that $U \cap X_i \neq \varnothing$ then follows from (7.1.7). We will show that the presheaf $U \to R(U)$ satisfies the sheaf axioms, which will prove that $R(U) = \Gamma(U, \mathscr{R}(X))$. Indeed, it satisfies (F1) by what has already been discussed. To see that it satisfies (F2), consider a cover of an open subset U of X by open subsets $V_\alpha \subset U$; if the $s_\alpha \in R(V_\alpha)$ are such that the restrictions of s_α and s_β to $V_\alpha \cap V_\beta$ agree for every pair of indices, then we can conclude that, for every index i such that $U \cap X_i \neq \varnothing$, the components in $R(X_i)$ of all the s_α such that $V_\alpha \cap X_i \neq \varnothing$ are all the same; denoting this component by t_i, it is clear that the element of $R(U)$ that has the t_i as its components has s_α as its restriction to each V_α. Finally, to see that $\mathscr{R}(X)$ is quasi-coherent, we can restrict to the case where $X = \operatorname{Spec}(A)$ is affine; by taking U to be an affine open subset of the form $D(f)$, where $f \in A$, it follows from the above and from Definition (1.3.4) that we have $\mathscr{R}(X) = \widetilde{M}$, where M is the direct sum of the A-modules A_{x_i}. ☐

Corollary (7.3.4). — *Let X be a reduced prescheme that has only a finite number of irreducible components, and let X_i ($1 \leqslant i \leqslant n$) be the closed reduced preschemes of X that have the irreducible components of X as their underlying spaces (5.2.1). If h_i is the canonical injection $X_i \to X$, then $\mathscr{R}(X)$ is the direct sum of the $\mathscr{O}_X$-algebras $(h_i)_*(\mathscr{R}(X_i))$.*

Corollary (7.3.5). — *If X is irreducible, then every quasi-coherent $\mathscr{R}(X)$-module $\mathscr{F}$ is a simple sheaf.*

Proof. It suffices to show that every $x \in X$ admits a neighbourhood U such that $\mathscr{F}|U$ is a simple sheaf (**0**, 3.6.2); in other words, we are led to considering the case where X is affine; we can further suppose that $\mathscr{F}$ is the cokernel of a homomorphism $(\mathscr{R}(X))^I \to (\mathscr{R}(X))^J$ (**0**, 5.1.3), and everything then follows from showing that $\mathscr{R}(X)$ is a simple sheaf; but this is evident, because $\Gamma(U, \mathscr{R}(X)) = R(X)$ for every nonempty open subset U, where U contains the generic point of X. □

Corollary (7.3.6). — *If X is irreducible, then, for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{R}(X)$ is a simple sheaf; if, further, X is reduced (and thus integral), then $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{R}(X)$ is isomorphic to a sheaf of the form $(\mathscr{R}(X))^{(I)}$.*

PROOF. The second claim follows from the fact that $R(X)$ is a field. □

Proposition (7.3.7). — *Suppose that the prescheme X is locally integral or locally Noetherian. Then $\mathscr{R}(X)$ is a quasi-coherent $\mathscr{O}_X$-algebra; if, further, X is reduced (which will be the case whenever X is locally integral), then the canonical homomorphism $\mathscr{O}_X \to \mathscr{R}(X)$ is injective.* **I** | 163

PROOF. Since the questions is local, the first claim follows from (7.3.3); the second follows from (7.2.3). □

(7.3.8). [10] Let X and Y be two *integral* preschemes, which implies that $\mathscr{R}(X)$ (resp. $\mathscr{R}(Y)$) is a quasi-coherent $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module) (7.3.3). Let $f : X \to Y$ be a *dominant* morphism; then there exists a canonical homomorphism of $\mathscr{O}_X$-modules

$$\tau : f^*(\mathscr{R}(Y)) \longrightarrow \mathscr{R}(X). \tag{7.3.8.1}$$

PROOF. Suppose first that $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$ are affine, given by integral rings A and B, with f thus corresponding to an injective homomorphism $B \to A$ which extends to a monomorphism $L \to K$ from the field of fractions L of B to the field of fractions K of A. The homomorphism (7.3.8.1) then corresponds to the canonical homomorphism $L \otimes_B A \to K$ (1.6.5). In the general case, for each pair of nonempty affine open sets $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$, we define, as above, a homomorphism $\tau_{U,V}$ and we immediately have that, if $U' \subset U$, $V' \subset V$, $f(U') \subset V'$, then $\tau_{U,V}$ extends $\tau_{U',V'}$, and hence our assertion. If x and y are the generic points of X and Y respectively, then we have $f(x) = y$,

$$(f^*(\mathscr{R}(Y)))_x = \mathscr{O}_y \otimes_{\mathscr{O}_y} \mathscr{O}_x = \mathscr{O}_x$$

(**0**, 4.3.1) and τ_x is thus an *isomorphism*. □

7.4. Torsion sheaves and torsion-free sheaves

(7.4.1). Let X be an *integral* scheme. For every $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $\mathscr{O}_X \to \mathscr{R}(X)$ defines, by tensoring, a homomorphism (again said to be *canonical*) $\mathscr{F} \to \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{R}(X)$, which, on each fibre, is exactly the homomorphism $z \to z \otimes 1$ from $\mathscr{F}_x$ to $\mathscr{F}_x \otimes_{\mathscr{O}_X} R(X)$. The *kernel* $\mathscr{T}$ of this homomorphism is an $\mathscr{O}_X$-submodule of $\mathscr{F}$, called the *torsion sheaf* of $\mathscr{F}$; it is quasi-coherent if $\mathscr{F}$ is quasi-coherent ((4.1.1) and (7.3.6)). We say that $\mathscr{F}$ is *torsion free* if $\mathscr{T} = 0$, and that $\mathscr{F}$ is a *torsion sheaf* if $\mathscr{T} = \mathscr{F}$. For every $\mathscr{O}_X$-module $\mathscr{F}$, $\mathscr{F} / \mathscr{T}$ is torsion free. We deduce from (7.3.5) that:

Proposition (7.4.2). — *If X is an integral prescheme, then every torsion-free quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ is isomorphic to a subsheaf $\mathscr{G}$ of a simple sheaf of the form $(\mathscr{R}(X))^{(I)}$, generated (as a $\mathscr{R}(X)$-module) by $\mathscr{G}$.*

The cardinality of I is called the *rank* of $\mathscr{F}$; for every nonempty affine open subset U of X, the rank of $\mathscr{F}$ is equal to the rank of $\Gamma(U, \mathscr{F})$ as a $\Gamma(U, \mathscr{O}_X)$-module, as we see by considering the generic point of X, contained in U. In particular:

Corollary (7.4.3). — *On an integral prescheme X, every torsion-free quasi-coherent $\mathscr{O}_X$-module of rank* 1 *(in particular, every invertible $\mathscr{O}_X$-module) is isomorphic to an $\mathscr{O}_X$-submodule of $\mathscr{R}(X)$, and vice versa.*

Corollary (7.4.4). — *Let X be an integral prescheme, $\mathscr{L}$ and $\mathscr{L}'$ torsion-free $\mathscr{O}_X$-modules, and f (resp. f') a section of $\mathscr{L}$ (resp. $\mathscr{L}'$) over X. In order to have $f \otimes f' = 0$, it is necessary and sufficient for one of the sections f and f' to be zero.*

[10] *[Trans.] This paragraph was changed entirely in the Errata of EGA II.*

PROOF. Let x be the generic point of X; we have, by hypothesis, that $(f \otimes f')_x = f_x \otimes f'_x = 0$. Since $\mathscr{L}_x$ and $\mathscr{L}'_x$ can be identified with $\mathscr{O}_X$-submodules of the field $\mathscr{O}_x$, the above equation leads to $f_x = 0$ or $f'_x = 0$, and thus $f = 0$ or $f' = 0$, since $\mathscr{L}$ and $\mathscr{L}'$ are torsion free (7.3.5). □

Proposition (7.4.5). — *Let X and Y be integral preschemes, and $f : X \to Y$ a dominant morphism. For every torsion-free quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $f_*(\mathscr{F})$ is a torsion-free $\mathscr{O}_Y$-module.*

PROOF. Since f_* is left exact (**0**, 4.2.1), it suffices, by (7.4.2), to prove the proposition in the case where $\mathscr{F} = (\mathscr{R}(X))^{(I)}$. But every nonempty open subset U of Y contains the generic point of Y, so $f^{-1}(U)$ contains the generic point of X (**0**, 2.1.5), so we have that $\Gamma(U, f_*(\mathscr{F})) = \Gamma(f^{-1}(U), \mathscr{F}) = (R(X))^{(I)}$; in other words, $f_*(\mathscr{F})$ is the simple sheaf with fibre $(R(X))^{(I)}$, considered as a $\mathscr{R}(Y)$-module, and it is clearly torsion free. □ I | 164

Proposition (7.4.6). — *Let X be an integral prescheme, and x its generic point. For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type, the following conditions are equivalent:* (a) *$\mathscr{F}$ is a torsion sheaf;* (b) *$\mathscr{F}_x = 0$;* (c) $\operatorname{Supp}(\mathscr{F}) \neq X$.

PROOF. By (7.3.5) and (7.4.1), the equations $\mathscr{F}_x = 0$ and $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{R}(X) = 0$ are equivalent, so (a) and (b) are equivalent; then $\operatorname{Supp}(\mathscr{F})$ is closed in X (**0**, 5.2.2), and since every nonempty open subset of X contains x, (b) and (c) are equivalent. □

(7.4.7). We generalise (by an abuse of language) the definitions of (7.4.1) to the case where X is a *reduced* prescheme having only a *finite* number of irreducible components; it then follows from (7.3.4) that the equivalence between *a)* and *c)* in (7.4.6) still holds true for such a prescheme.

§8. CHEVALLEY SCHEMES

8.1. Allied local rings For each local ring A, we denote by $\mathfrak{m}(A)$ the maximal ideal of A.

Lemma (8.1.1). — *Let A and B be two local rings such that $A \subset B$; Then the following conditions are equivalent.*

(i) $\mathfrak{m}(B) \cap A = \mathfrak{m}(A)$.
(ii) $\mathfrak{m}(A) \subset \mathfrak{m}(B)$.
(iii) 1 *is not an element of the ideal of B generated by $\mathfrak{m}(A)$.*

PROOF. It is evident that (i) implies (ii), and (ii) implies (iii); lastly, if (iii) is true, then $\mathfrak{m}(B) \cap A$ contains $\mathfrak{m}(A)$, and does not contain 1, and is thus equal to $\mathfrak{m}(A)$.

When the equivalent conditions of (8.1.1) are satisfied, we say that B *dominates* A; this is equivalent to saying that the injection $A \to B$ is a *local* homomorphism. It is clear that, in the set of local subrings of a ring R, the relation given by domination is an order. □

(8.1.2). Now consider a *field* R. For all subrings A of R, we denote by $L(A)$ the set of local rings $A_{\mathfrak{p}}$, where $\mathfrak{p}$ ranges over the prime spectrum of A; such local rings are identified with the subrings of R containing A. Since $\mathfrak{p} = (\mathfrak{p}A_{\mathfrak{p}}) \cap A$, the map $\mathfrak{p} \mapsto A_{\mathfrak{p}}$ from $\operatorname{Spec}(A)$ to $L(A)$ is bijective.

Lemma (8.1.3). — *Let R be a field, and A a subring of R. For a local subring M of R to dominate a ring $A_{\mathfrak{p}} \in L(A)$, it is necessary and sufficient that $A \subset M$; the local ring $A_{\mathfrak{p}}$ dominated by M is then unique, and corresponds to $\mathfrak{p} = \mathfrak{m}(M) \cap A$.*

PROOF. If M dominates $A_{\mathfrak{p}}$, then $\mathfrak{m}(M) \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$, by (8.1.1), whence the uniqueness of $\mathfrak{p}$; on the other hand, if $A \subset M$, then $\mathfrak{m}M \cap A = \mathfrak{p}$ is prime in A, and since $A - \mathfrak{p} \subset M$, we have that $A_{\mathfrak{p}} \subset M$ and $\mathfrak{p}A_{\mathfrak{p}} \subset \mathfrak{m}(M)$, so M dominates $A_{\mathfrak{p}}$. □

Lemma (8.1.4). — *Let R be a field, M and N local subrings of R, and P the subring of R generated by $M \cup N$. Then the following conditions are equivalent.* I | 165

(i) *There exists a prime ideal $\mathfrak{p}$ of P such that $\mathfrak{m}(M) = \mathfrak{p} \cap M$ and $\mathfrak{m}(N) = \mathfrak{p} \cap N$.*
(ii) *The ideal $\mathfrak{a}$ generated in P by $\mathfrak{m}(M) \cup \mathfrak{m}(N)$ is distinct from P.*
(iii) *There exists a local subring Q of R simultaneously dominating both M and N.*

PROOF. It is clear that (i) implies (ii); conversely, if $\mathfrak{a} \neq P$, then $\mathfrak{a}$ is contained in a maximal ideal $\mathfrak{n}$ of P, and since $1 \notin \mathfrak{n}$, $\mathfrak{n} \cap M$ contains $\mathfrak{m}(M)$ and is distinct from M, so $\mathfrak{n} \cap M = \mathfrak{m}(M)$, and similarly $\mathfrak{n} \cap N = \mathfrak{m}(N)$. It is clear that, if Q dominates both M and N, then $P \subset Q$ and $\mathfrak{m}(M) = \mathfrak{m}(Q) \cap M = (\mathfrak{m}(Q) \cap P) \cap M$, and $\mathfrak{m}(N) = (\mathfrak{m}(Q) \cap P) \cap N$, so (iii) implies (i); the converse is evident when we take $Q = P_{\mathfrak{p}}$. □

When the conditions of (8.1.4) are satisfied, we say, with C. Chevalley, that the local rings M and N are *allied*.

Proposition (8.1.5). — *Let A and B be subrings of a field R, and C the subring of R generated by $A \cup B$. Then the following conditions are equivalent.*

(i) *For every local ring Q containing A and B, we have that $A_{\mathfrak{p}} = B_{\mathfrak{q}}$, where $\mathfrak{p} = \mathfrak{m}(Q) \cap A$ and $\mathfrak{q} = \mathfrak{m}(Q) \cap B$.*

(ii) *For all prime ideals $\mathfrak{r}$ of C, we have that $A_{\mathfrak{p}} = B_{\mathfrak{q}}$, where $\mathfrak{p} = \mathfrak{r} \cap A$ and $\mathfrak{q} = \mathfrak{r} \cap B$.*

(iii) *If $M \in L(A)$ and $N \in L(B)$ are allied, then they are identical.*

(iv) *$L(A) \cap L(B) = L(C)$.*

PROOF. Lemmas (8.1.3) and (8.1.4) prove that (i) and (iii) are equivalent; it is clear that (i) implies (ii) by taking $Q = C_{\mathfrak{r}}$; conversely, (ii) implies (i), because if Q contains $A \cup B$ then it contains C, and if $\mathfrak{r} = \mathfrak{m}(Q) \cap C$, then $\mathfrak{p} = \mathfrak{r} \cap A$ and $\mathfrak{q} = \mathfrak{r} \cap B$, by (8.1.3). It is immediate that (iv) implies (i), because if Q contains $A \cup B$ then it dominates a local ring $C_{\mathfrak{r}} \in L(C)$ by (8.1.3); by hypothesis we have that $C_{\mathfrak{r}} \in L(A) \cap L(B)$, and (8.1.1) and (8.1.3) prove that $C_{\mathfrak{r}} = A_{\mathfrak{p}} = B_{\mathfrak{q}}$. We prove finally that (iii) implies (iv). Let $Q \in L(C)$; Q dominates some $M \in L(A)$ and some $N \in L(B)$ (8.1.3), so M and N, being allied, are identical by hypothesis. As we then have that $C \subset M$, we know that M dominates some $Q' \in L(C)$ (8.1.3), so Q dominates Q', whence necessarily (8.1.3) $Q = Q' = M$, so $Q \in L(A) \cap L(B)$. Conversely, if $Q \in L(A) \cap L(B)$, then $C \subset Q$, so (8.1.3) Q dominates some $Q'' \in L(C) \subset L(A) \cap L(B)$; Q and Q'', being allied, are identical, so $Q'' = Q \in L(C)$, which completes the proof. □

8.2. Local rings of an integral scheme

(8.2.1). Let X be an *integral* prescheme, and R its field of rational functions, identical to the local ring of the generic point a of X; for all $x \in X$, we know that $\mathscr{O}_x$ can be canonically identified with a subring of R (7.1.5), and for every rational function $f \in R$, the domain of definition $\delta(f)$ of f is the open set of $x \in X$ such that $f \in \mathscr{O}_x$. It thus follows, from (7.2.6), that, for every open $U \subset X$, we have

$$\Gamma(U, \mathscr{O}_X) = \bigcap_{x \in U} \mathscr{O}_x. \tag{8.2.1.1}$$

Proposition (8.2.2). — *Let X be an integral prescheme, and R its field of rational fractions. For X to be a scheme, it is necessary and sufficient for the relation "$\mathscr{O}_x$ and $\mathscr{O}_y$ are allied" (8.1.4), for points x and y of X, to imply that $x = y$.* **I | 166**

PROOF. We suppose that this condition is satisfied, and aim to show that X is separated. Let U and V be two distinct affine open subsets of X, given by rings A and B (respectively), identified with subrings of R; U (resp. V) is thus identified (8.1.2) with $L(A)$ (resp. $L(B)$), and the hypotheses tell us (8.1.5) that C is the subring of R generated by $A \cup B$, and $W = U \cap V$ is identified with $L(A) \cap L(B) = L(C)$. Furthermore, we know ([CC], p. 5-03, 4 *bis*) that every subring E of R is equal to the intersection of the local rings belonging to $L(E)$; C is thus identified with the intersection of the rings $\mathscr{O}_z$ for $z \in W$, or, equivalently (8.2.1.1), with $\Gamma(W, \mathscr{O}_X)$. So consider the subprescheme induced by X on W; to the identity morphism $\phi : C \to \Gamma(W, \mathscr{O}_X)$ there corresponds (2.2.4) a morphism $\Phi = (\psi, \theta) : W \to \operatorname{Spec}(C)$; we will see that Φ is an *isomorphism* of preschemes, whence W is an *affine* open subset. The identification of W with $L(C) = \operatorname{Spec}(C)$ shows that ψ is *bijective*. On the other hand, for all $x \in W$, $\theta_x^\sharp$ is the injection $C_{\mathfrak{r}} \to \mathscr{O}_x$, where $\mathfrak{r} = \mathfrak{m}_x \cap C$, and, by definition, $C_{\mathfrak{r}}$ is identified with $\mathscr{O}_x$, so $\theta_x^\sharp$ is bijective. It thus remains to show that ψ is a *homeomorphism*, or, in other words, that for every closed subset $F \subset W$, $\psi(F)$ is closed in $\operatorname{Spec}(C)$. But F is the intersection of W with a closed subspace of U of the form $V(\mathfrak{a})$, where $\mathfrak{a}$ is an ideal of A; we will show that $\psi(F) = V(\mathfrak{a}C)$, which proves our claim. In fact, the prime ideals of C containing $\mathfrak{a}C$ are the prime ideals of C containing $\mathfrak{a}$, and so are the ideals of the form $\psi(x) = \mathfrak{m}_x \cap C$, where $\mathfrak{a} \subset \mathfrak{m}_x$ and $x \in W$; since $\mathfrak{a} \subset \mathfrak{m}_x$ is equivalent to $x \in V(\mathfrak{a}) = W \cap F$ for $x \in U$, we do indeed have that $\psi(F) = V(\mathfrak{a}C)$.

It follows that X is separated, because $U \cap V$ is affine and its ring C is generated by the union $A \cup B$ of the rings of U and V (5.5.6).

Conversely, suppose that X is separated, and let x and y be points of X such that $\mathscr{O}_x$ and $\mathscr{O}_y$ are allied. Let U (resp. V) be an affine open subset containing x (resp. y), of ring A (resp. B); we then know that $U \cap V$ is affine and that its ring C is generated by $A \cup B$ (5.5.6). If $\mathfrak{p} = \mathfrak{m}_x \cap A$ and $\mathfrak{q} = \mathfrak{m}_y \cap B$, then $A_\mathfrak{p} = \mathscr{O}_x$ and $B_\mathfrak{q} = \mathscr{O}_y$, and since $A_\mathfrak{p}$ and $B_\mathfrak{q}$ are allied, there exists a prime ideal $\mathfrak{r}$ of C such that $\mathfrak{p} = \mathfrak{r} \cap A$ and $\mathfrak{q} = \mathfrak{r} \cap B$ (8.1.4). But then there exists a point $z \in U \cap V$ such that $\mathfrak{r} = \mathfrak{m}_z \cap C$, since $U \cap V$ is affine, and so evidently $x = z$ and $y = z$, whence $x = y$. □

Corollary (8.2.3). — *Let X be an integral scheme, and x and y points of X. In order for $x \in \overline{\{y\}}$, it is necessary and sufficient for $\mathscr{O}_x \subset \mathscr{O}_y$, or, equivalently, for every rational function defined at x to also be defined at y.*

Proof. The condition is evidently necessary because the domain of definition $\delta(f)$ of a rational function $f \in R$ is open; we now show that it is sufficient. If $\mathscr{O}_x \subset \mathscr{O}_y$, then there exists a prime ideal $\mathfrak{p}$ of $\mathscr{O}_x$ such that $\mathscr{O}_y$ dominates $(\mathscr{O}_x)_\mathfrak{p}$ (8.1.3); but (2.4.2) there exists some $z \in X$ such that $x \in \overline{\{z\}}$ and $\mathscr{O}_z = (\mathscr{O}_x)_\mathfrak{p}$; since $\mathscr{O}_z$ and $\mathscr{O}_y$ are allied, we have that $z = y$ by (8.2.2), whence the corollary. □

Corollary (8.2.4). — *If X is an integral scheme then the map $x \to \mathscr{O}_x$ is injective; equivalently, if x and y are two distinct points of X, then there exists a rational function defined at one of these points but not the other.*

Proof. This follows from (8.2.3) and the axiom (T_0) (2.1.4). □ **I** | 167

Corollary (8.2.5). — *Let X be an integral scheme whose underlying space is Noetherian; letting f range over the field R of rational functions on X, the sets $\delta(f)$ generate the topology of X.*

In fact, every closed subset of X is thus a finite union of irreducible closed subsets, or, in other words, of the form $\overline{\{y\}}$ (2.1.5). But, if $x \notin \overline{\{y\}}$, then there exists a rational function f defined at x but not at y (8.2.3), or, equivalently, we have that $x \in \delta(f)$ and that $\delta(f)$ is not contained in $\overline{\{y\}}$. The complement of $\overline{\{y\}}$ is thus a union of sets of the form $\delta(f)$, and, by virtue of the first remark, every open subset of X is the union of finite intersections of open sets of the form $\delta(f)$.

(8.2.6). Corollary (8.2.5) shows that the topology of X is entirely characterised by the data of the local rings $(\mathscr{O}_x)_{x \in X}$ that have R as their field of fractions. It is equivalent to say that the closed subsets of X are defined in the following manner: given a finite subset $\{x_1, \ldots, x_n\}$ of X, consider the set of $y \in X$ such that $\mathscr{O}_y \subset \mathscr{O}_{x_i}$ for at least one index i, and these sets (over all choices of $\{x_1, \ldots, x_n\}$) are the closed subsets of X. Further, once the topology on X is known, the structure sheaf $\mathscr{O}_X$ is also determined by the family of the $\mathscr{O}_x$, since $\Gamma(U, \mathscr{O}_X) = \bigcap_{x \in U} \mathscr{O}_x$, by (8.2.1.1). The family $(\mathscr{O}_X)_{x \in X}$ thus completely determines the prescheme X when X is an integral scheme whose underlying space is Noetherian.

Proposition (8.2.7). — *Let X and Y be integral schemes, $f : X \to Y$ a dominant morphism (2.2.6), and K (resp.L) the field of rational functions on X (resp.Y). Then L can be identified with a subfield of K, and, for all $x \in X$, $\mathscr{O}_{f(x)}$ is the unique local ring of Y dominated by $\mathscr{O}_x$.*

Proof. If $f = (\psi, \theta)$ and a is the generic point of X, then $\psi(a)$ is the generic point of Y (**0**, 2.1.5); $\theta_a^\sharp$ is then a monomorphism of fields, from $L = \mathscr{O}_{\psi(a)}$ to $K = \mathscr{O}_a$. Since every nonempty affine open subset U of Y contains $\psi(a)$, it follows from (2.2.4) that the homomorphism $\Gamma(U, \mathscr{O}_Y) \to \Gamma(\psi^{-1}(U), \mathscr{O}_X)$ corresponding to f is the restriction of $\theta_a^\sharp$ to $\Gamma(U, \mathscr{O}_Y)$. So, for every $x \in X$, $\theta_x^\sharp$ is the restriction to $\mathscr{O}_{\psi(a)}$ of $\theta_a^\sharp$, and is thus a monomorphism. We also know that $\theta_x^\sharp$ is a local homomorphism, so, if we identify L with a subfield of K by $\theta_a^\sharp$, $\mathscr{O}_{\psi(x)}$ is dominated by $\mathscr{O}_x$ (8.1.1); it is also the only local ring of Y dominated by $\mathscr{O}_x$, since two local rings of Y that are allied are identical (8.2.2). □

Proposition (8.2.8). — *Let X be an* irreducible *prescheme, $f : X \to Y$ a local immersion (resp. local isomorphism), and suppose further that f is separated. Then f is an immersion (resp. an open immersion).*

PROOF. Let $f = (\psi, \theta)$; it suffices, in both cases, to prove that ψ is a *homeomorphism* from X to $\psi(X)$ (4.5.3). Replacing f by f_{red} ((5.1.6) and (5.5.1, vi)), we can assume that X and Y are *reduced*. If Y' is the closed reduced subprescheme of Y that has $\overline{\psi(X)}$ as its underlying space, then f factors as $X \xrightarrow{f'} Y' \xrightarrow{j} Y$, where j is the canonical injection (5.2.2). It follows from (5.5.1, v) that f' is again a separated morphism; further, f' is again a local immersion (resp. a local isomorphism), because, I | 168
since the condition is local on X and Y, we can restrict to the case where f is a closed immersion (resp. open immersion), and our claim then follows immediately from (4.2.2).

We can thus suppose that f is a *dominant* morphism, which leads to the fact that Y is, itself, irreducible (**0**, 2.1.5), and so X and Y are both *integral*. Further, since the condition is local on Y, we can suppose that Y is an affine scheme; since f is separated, X is a scheme (5.5.1, ii), and we are finally at the hypotheses of Proposition (8.2.7). Then, for all $x \in X$, $\theta_x^\sharp$ is injective; but the hypothesis that f is a local immersion implies that $\theta_x^\sharp$ is surjective (4.2.2), so $\theta_x^\sharp$ is bijective, or, equivalently (with the identification of Proposition (8.2.7)) we have that $\mathscr{O}_{\psi(x)} = \mathscr{O}_x$. This implies, by Corollary (8.2.4), that ψ is an *injective* map, which already proves the proposition when f is a local isomorphism (4.5.3). When we suppose that f is only a local immersion, for all $x \in X$ there exists an open neighbourhood U of x in X and an open neighbourhood V of $\psi(x)$ in Y such that the restriction of ψ to U is a homeomorphism from U to a *closed* subset of V. But U is dense in X, so $\psi(U)$ is dense in Y and *a fortiori* in V, which proves that $\psi(U) = V$; since ψ is injective, $\psi^{-1}(V) = U$ and this proves that ψ is a homeomorphism from X to $\psi(X)$. ☐

8.3. Chevalley schemes

(8.3.1). Let X be a *Noetherian* integral scheme, and R its field of rational functions; we denote by X' the set of local subrings $\mathscr{O}_x \subset R$, where x ranges over all points of X. The set X' satisfies the following three conditions.

- (Sch. 1) For all $M \in X'$, R is the field of fractions of M.
- (Sch. 2) There exists a finite set of Noetherian subrings A_i of R such that $X' = \bigcup_i L(A_i)$, and, for all pairs of indices i, j, the subring A_{ij} of R generated by $A_i \cup A_j$ is an algebra of finite type over A_i.
- (Sch. 3) Any two elements M and N of X' that are allied are identical.

We have seen in (8.2.1) that (Sch. 1) is satisfied, and (Sch. 3) follows from (8.2.2). To show (Sch. 2), it suffices to cover X by a finite number of affine open subsets U_i whose rings are Noetherian, and to take $A_i = \Gamma(U_i, \mathscr{O}_X)$; the hypothesis that X is a scheme implies that $U_i \cap U_j$ is affine, and also that $\Gamma(U_i \cap U_j, \mathscr{O}_X) = A_{ij}$ (5.5.6); further, since the space U_i is Noetherian, the immersion $U_i \cap U_j \to U_i$ is of finite type (6.3.5), so A_{ij} is an A_i-algebra of finite type (6.3.3).

(8.3.2). The structures whose axioms are (Sch. 1), (Sch. 2), and (Sch. 3) generalise "schemes", in the sense of C. Chevalley, who additionally supposes that R is an extension of finite type of a field K, and that the A_i are K-algebras of finite type (which renders a part of (Sch. 2) useless) [**CC**]. Conversely, if we have such a structure on a set X', then we can associate to it an integral scheme X by using the remarks from (8.2.6): the underlying space of X is equal to X' endowed with the topology defined in (8.2.6), and with the sheaf $\mathscr{O}_X$ such that $\Gamma(U, \mathscr{O}_X) = \bigcap_{x \in U} \mathscr{O}_x$ for all open $U \subset X$, with the evident definition of restriction homomorphisms. We leave to the reader the task of verifying that we thus obtain an integral scheme, whose local rings are the elements of X'; we will not use this result in what follows.

§9. SUPPLEMENT ON QUASI-COHERENT SHEAVES

9.1. Tensor product of quasi-coherent sheaves

Proposition (9.1.1). — *Let X be a prescheme (resp. a locally Noetherian prescheme). Let $\mathscr{F}$ and $\mathscr{G}$ be* I | 169
quasi-coherent (resp. coherent) $\mathscr{O}_X$-modules; then $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is quasi-coherent (resp. coherent); it is further of finite type if both $\mathscr{F}$ and $\mathscr{G}$ are of finite type. If $\mathscr{F}$ admits a finite presentation and if $\mathscr{G}$ is quasi-coherent (resp. coherent), then $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ is quasi-coherent (resp. coherent).

PROOF. Being a local proposition, we can suppose that X is affine (resp. Noetherian affine); further, if $\mathscr{F}$ is coherent, then we can assume that it is the cokernel of a homomorphism $\mathscr{O}_X^m \to \mathscr{O}_X^n$. The claims pertaining to quasi-coherent sheaves then follow from Corollaries (1.3.12) and (1.3.9); the claims pertaining to coherent sheaves follow from Theorem (1.5.1) and from the fact that if M and N are modules of finite type over a Noetherian ring A then $M \otimes_A N$ and $\operatorname{Hom}_A(M, N)$ are both A-modules of finite type. □

Definition (9.1.2). — Let X and Y be S-preschemes, p and q the projections of $X \times_S Y$, and $\mathscr{F}$ (resp.$\mathscr{G}$) a quasi-coherent $\mathscr{O}_X$-module (resp. quasi-coherent $\mathscr{O}_Y$-module). We define the tensor product of $\mathscr{F}$ and $\mathscr{G}$ over $\mathscr{O}_S$ (*or* over S), denoted by $\mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{G}$ (*or* $\mathscr{F} \otimes_S \mathscr{G}$) to be the tensor product $p^*(\mathscr{F}) \otimes_{\mathscr{O}_{X\times_S Y}} q^*(\mathscr{G})$ over the prescheme $X \times_S Y$.

If X_i $(1 \leqslant i \leqslant n)$ are S-preschemes, and $\mathscr{F}_i$ $(1 \leqslant i \leqslant n)$ are quasi-coherent $\mathscr{O}_{X_i}$-modules, then we similarly define the tensor product $\mathscr{F}_1 \otimes_S \mathscr{F}_2 \otimes_S \cdots \otimes_S \mathscr{F}_n$ over the prescheme $Z = X_1 \times_S X_2 \times_S \cdots \times_S X_n$; it is a *quasi-coherent* $\mathscr{O}_Z$-module by virtue of (9.1.1) and (**0**, 5.1.4); it is further *coherent* if all the $\mathscr{F}_i$ are coherent *and* Z is *locally Noetherian*, by virtue of (9.1.1), (**0**, 5.3.11), and (6.1.1).

Note that, if we take $X = Y = S$, then Definition (9.1.2) gives us back the tensor product of $\mathscr{O}_S$-modules. Furthermore, since $q^*(\mathscr{O}_Y) = \mathscr{O}_{X\times_S Y}$ (**0**, 4.3.4), the product $\mathscr{F} \otimes_S \mathscr{O}_Y$ is canonically identified with $p^*(\mathscr{F})$, and, in the same way, $\mathscr{O}_X \otimes_S \mathscr{G}$ is canonically identified with $q^*(\mathscr{G})$. In particular, if we take $Y = S$ and denote by f the structure morphism $X \to Y$, then we have that $\mathscr{O}_X \otimes_Y \mathscr{G} = f^*(\mathscr{G})$: the ordinary tensor product and the inverse image thus appear as particular cases of the general tensor product.

Definition (9.1.2) leads immediately to the fact that, for fixed X and Y, $\mathscr{F} \otimes_S \mathscr{G}$ is a *right-exact additive covariant bifunctor* in $\mathscr{F}$ and $\mathscr{G}$.

Proposition (9.1.3). — *Let S, X, and Y be affine schemes of rings A, B, and C (respectively), with B and C being A-algebras. Let M (resp. N) be a B-module (resp. C-module), and $\mathscr{F} = \widetilde{M}$ (resp. $\mathscr{G} = \widetilde{N}$) the associated quasi-coherent sheaf; then $\mathscr{F} \otimes_S \mathscr{G}$ is canonically isomorphic to the sheaf associated to the $(B \otimes_A C)$-module $M \otimes_A N$.*

PROOF. According to Proposition (1.6.5), $\mathscr{F} \otimes_S \mathscr{G}$ is canonically isomorphic to the sheaf associated to the $(B \otimes_A C)$-module I | 170

$$\big(M \otimes_B (B \otimes_A C)\big) \otimes_{B\otimes_A C} \big((B \otimes_A C) \otimes_C N\big)$$

and, by the canonical isomorphisms between tensor products, this latter module is isomorphic to

$$M \otimes_B (B \otimes_A C) \otimes_C N = (M \otimes_B B) \otimes_A (C \otimes_C N) = M \otimes_A N.$$

□

Proposition (9.1.4). — *Let $f : T \to X$ and $g : T \to Y$ be S-morphisms, and $\mathscr{F}$ (resp. $\mathscr{G}$) a quasi-coherent $\mathscr{O}_X$-module (resp. quasi-coherent $\mathscr{O}_Y$-module). Then*

$$(f,g)_S^*(\mathscr{F} \otimes_S \mathscr{G}) = f^*(\mathscr{F}) \otimes_{\mathscr{O}_T} g^*(\mathscr{G}).$$

PROOF. If p, q are the projections of $X \times_S Y$, then the formula follows from the equalities $(f,g)_S^* \circ p^* = f^*$ and $(f,g)_S^* \circ q^* = g^*$ (**0**, 3.5.5), and the fact that the inverse image of a tensor product of algebraic sheaves is the tensor product of their inverse images (**0**, 4.3.3). □

Corollary (9.1.5). — *Let $f : X \to X'$ and $g : Y \to Y'$ be S-morphisms, and $\mathscr{F}'$ (resp. $\mathscr{G}'$) a quasi-coherent $\mathscr{O}_{X'}$-module (resp. quasi-coherent $\mathscr{O}_{Y'}$-module). Then*

$$(f,g)_S^*(\mathscr{F}' \otimes_S \mathscr{G}') = f^*(\mathscr{F}') \otimes_S g^*(\mathscr{G}')$$

PROOF. This follows from (9.1.4) and the fact that $f \times_S g = (f \circ p, g \circ q)_S$, where p and q are the projections of $X \times_S Y$. □

Corollary (9.1.6). — *Let X, Y, and Z be S-preschemes, and $\mathscr{F}$ (resp. $\mathscr{G}$, $\mathscr{H}$) a quasi-coherent $\mathscr{O}_X$-module (resp. quasi-coherent $\mathscr{O}_Y$-module, quasi-coherent $\mathscr{O}_Z$-module); then the sheaf $\mathscr{F} \otimes_S \mathscr{G} \otimes_S \mathscr{H}$ is the inverse image of $(\mathscr{F} \otimes_S \mathscr{G}) \otimes_S \mathscr{H}$ by the canonical isomorphism from $X \times_S Y \times_S Z$ to $(X \times_S Y) \times_S Z$.*

PROOF. This isomorphism is given by $(p_1, p_2)_S \times_S p_3$, where p_1, p_2, and p_3 are the projections of $X \times_S Y \times_S Z$.

Similarly, the inverse image of $\mathscr{G} \otimes_S \mathscr{F}$ under the canonical isomorphism from $X \times_S Y$ to $Y \times_S X$ is $\mathscr{F} \otimes_S \mathscr{G}$. $\square$

Corollary (9.1.7). — *If X is an S-prescheme, then every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ is the inverse image of $\mathscr{F} \otimes_S \mathscr{O}_S$ by the canonical isomorphism from X to $X \times_S S$ (3.3.3).*

PROOF. This isomorphism is $(1_X, \phi)_S$, where ϕ is the structure morphism $X \to S$, and the corollary follows from (9.1.4) and the fact that $\phi^*(\mathscr{O}_S) = \mathscr{O}_X$. $\square$

(9.1.8). Let X be an S-prescheme, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module, and $\phi : S' \to S$ a morphism; we denote by $\mathscr{F}_{(\phi)}$ or $\mathscr{F}_{(S')}$ the quasi-coherent sheaf $\mathscr{F} \otimes_S \mathscr{O}_{S'}$ over $X \times_S S' = X_{(\phi)} = X_{(S')}$; so $\mathscr{F}_{(S')} = p^*(\mathscr{F})$, where p is the projection $X_{(S')} \to X$.

Proposition (9.1.9). — *Let $\phi'' : S'' \to S'$ be a morphism. For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ on the S-prescheme X, $(\mathscr{F}_{(\phi)})_{(\phi')}$ is the inverse image of $\mathscr{F}_{(\phi\circ\phi')}$ by the canonical isomorphism $(X_{(\phi)})_{(\phi')} \simeq X_{(\phi\circ\phi')}$ (3.3.9).*

PROOF. This follows immediately from the definitions and from (3.3.9), and is written

$$(\mathscr{F} \otimes_S \mathscr{O}_{S'}) \otimes_{S'} \mathscr{O}_{S''} = \mathscr{F} \otimes_S \mathscr{O}_{S''}. \tag{9.1.9.1}$$

$\square$

Proposition (9.1.10). — *Let Y be an S-prescheme, and $f : X \to Y$ an S-morphism. For every quasi-coherent $\mathscr{O}_Y$-module $\mathscr{G}$ and every morphism $S' \to S$, we have that $(f_{(S')})^*(\mathscr{G}_{(S')}) = (f^*(\mathscr{G}))_{(S')}$.*

PROOF. This follows immediately from the commutativity of the diagram I | 171

$$\begin{tikzcd} X_{(S')} \arrow[r, "f_{(S')}"] \arrow[d] & Y_{(S')} \arrow[d] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

$\square$

Corollary (9.1.11). — *Let X and Y be S-preschemes, and $\mathscr{F}$ (resp. $\mathscr{G}$) a quasi-coherent $\mathscr{O}_X$-module (resp. quasi-coherent $\mathscr{O}_Y$-module). Then the inverse image of the sheaf $(\mathscr{F}_{(S')}) \otimes_{(S')} (\mathscr{G}_{(S')})$ by the canonical isomorphism $(X \times_S Y)_{(S')} \simeq (X_{(S')}) \times_{S'} (Y_{(S')})$ (3.3.10) is equal to $(\mathscr{F} \otimes_S \mathscr{G})_{(S')}$.*

PROOF. If p and q are the projections of $X \times_S Y$, then the isomorphism in question is nothing but $(p_{(S')}, q_{(S')})_{S'}$; the corollary then follows from Propositions (9.1.4) and (9.1.10). $\square$

Proposition (9.1.12). — *With the notation from Definition (9.1.2), let z be a point of $X \times_S Y$, and let $x = p(z)$, and $y = q(z)$; the stalk $(\mathscr{F} \otimes_S \mathscr{G})_z$ is isomorphic to $(\mathscr{F}_x \otimes_{\mathscr{O}_x} \mathscr{O}_z) \otimes_{\mathscr{O}_z} (\mathscr{G}_y \otimes_{\mathscr{O}_y} \mathscr{O}_z) = \mathscr{F}_x \otimes_{\mathscr{O}_x} \mathscr{O}_z \otimes_{\mathscr{O}_y} \otimes \mathscr{G}_y$.*

PROOF. Since we can reduce to the affine case, the proposition follows from Equation (1.6.5.1). $\square$

Corollary (9.1.13). — *If $\mathscr{F}$ and $\mathscr{G}$ are of finite type, then*

$$\operatorname{Supp}(\mathscr{F} \otimes_S \mathscr{G}) = p^{-1}(\operatorname{Supp}(\mathscr{F})) \cap q^{-1}(\operatorname{Supp}(\mathscr{G})).$$

PROOF. Since $p^*(\mathscr{F})$ and $q^*(\mathscr{G})$ are both of finite type over $\mathscr{O}_{X \times_S Y}$, we reduce, by Proposition (9.1.12) and by (**0**, 1.7.5), to the case where $\mathscr{G} = \mathscr{O}_Y$, that is, it remains to prove the following equation:

$$\operatorname{Supp}(p^*(\mathscr{F})) = p^{-1}(\operatorname{Supp}(\mathscr{F})). \tag{9.1.13.1}$$

The same reasoning as in (**0**, 1.7.5) leads us to prove that, for all $z \in X \times_S Y$, we have $\mathscr{O}_z / \mathfrak{m}_x \mathscr{O}_z \neq 0$ (with $x = p(z)$), which follows from the fact that the homomorphism $\mathscr{O}_x \to \mathscr{O}_z$ is *local*, by hypothesis. $\square$

We leave it to the reader to extend the results in this section to the more general case of an arbitrary (but finite) number of factors, instead of just two.

9.2. Direct image of a quasi-coherent sheaf

Proposition (9.2.1). — *Let $f : X \to Y$ be a morphism of preschemes. We suppose that there exists a cover (Y_α) of Y by affine opens having the following property: every $f^{-1}(Y_\alpha)$ admits a* finite *cover $(X_{\alpha i})$ by affine opens that are contained in $f^{-1}(Y_\alpha)$ and that are such that every intersection $X_{\alpha i} \cap X_{\alpha j}$ is itself a* finite *union of affine opens. With these hypotheses, for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $f_*(\mathscr{F})$ is a quasi-coherent $\mathscr{O}_Y$-module.*

Proof. Since this is a local condition on Y, we can assume that Y is equal to one of the Y_α, and thus omit the indices α.

(a) First, suppose that the $X_i \cap X_j$ are themselves *affine* opens. We set $\mathscr{F}_i = \mathscr{F}|X_i$ and $\mathscr{F}_{ij} = \mathscr{F}|(X_i \cap X_j)$, and let $\mathscr{F}'_i$ and $\mathscr{F}'_{ij}$ be the images of $\mathscr{F}_i$ and $\mathscr{F}_{ij}$ (respectively) by the restriction of f to X_i and to $X_i \cap X_j$ (respectively); we know that the $\mathscr{F}'_i$ and $\mathscr{F}'_{ij}$ are quasi-coherent (1.6.3). Set $\mathscr{G} = \bigoplus_i \mathscr{F}'_i$ and $\mathscr{H} = \bigoplus_{i,j} \mathscr{F}'_{ij}$; $\mathscr{G}$ and $\mathscr{H}$ are quasi-coherent $\mathscr{O}_Y$-modules; we will define a homomorphism $u : \mathscr{G} \to \mathscr{H}$ such that $f_*(\mathscr{F})$ is the *kernel* of u; it will follow from this that $f_*(\mathscr{F})$ is quasi-coherent (1.3.9). It suffices to define u as a homomorphism of presheaves; taking into account the definitions of $\mathscr{G}$ and $\mathscr{H}$, it thus suffices, for every open subset $W \subset Y$, to define a homomorphism **I | 172**
$$u_W : \bigoplus_i \Gamma(f^{-1}(W) \cap X_i, \mathscr{F}) \longrightarrow \bigoplus_{i,j} \Gamma(f^{-1}(W) \cap X_i \cap X_j, \mathscr{F})$$
that satisfies the usual compatibility conditions when we let W vary. If, for every section $s_i \in \Gamma(f^{-1}(W) \cap X_i, \mathscr{F})$, we denote by $s_{i|j}$ its restriction to $f^{-1}(W) \cap X_i \cap X_j$, then we set
$$u_W\big((s_i)\big) = (s_{i|j} - s_{j|i})$$
and the compatibility conditions are clearly satisfied. To prove that the kernel $\mathscr{R}$ of u is $f_*(\mathscr{F})$, we define a homomorphism from $f_*(\mathscr{F})$ to $\mathscr{R}$ by sending each section $s \in \Gamma(f^{-1}(W), \mathscr{F})$ to the family (s_i), where s_i is the restriction of s to $f^{-1}(W) \cap X_i$; axioms (F1) and (F2) of sheaves (G, II, 1.1) tell us that this homomorphism is *bijective*, which finishes the proof in this case.

(b) In the general case, the same reasoning applies once we have established that the $\mathscr{F}_{ij}$ are quasi-coherent. But, by hypothesis, $X_i \cap X_j$ is a finite union of affine opens X_{ijk}; and since the X_{ijk} are affine opens *in a scheme*, the intersection of any two of them is again an affine open (5.5.6). We are thus led to the first case, and so we have proved Proposition (9.2.1).

□

Corollary (9.2.2). — *The conclusion of Proposition (9.2.1) holds true in each of the following cases:*

(a) *f is separated and quasi-compact;*
(b) *f is separated and of finite type;*
(c) *f is quasi-compact, and the underlying space of X is locally Noetherian.*

Proof. In case (a), the $X_{\alpha i} \cap X_{\alpha j}$ are affine (5.5.6). Case (b) is a particular example of case (a) (6.6.3). Finally, in case (c), we can reduce to the case where Y is affine and the underlying space of X is Noetherian; then X admits a finite cover of affine opens (X_i), and the $X_i \cap X_j$, being quasi-compact, are finite unions of affine opens (2.1.3). □

9.3. Extension of sections of quasi-coherent sheaves

Theorem (9.3.1). — *Let X be a prescheme whose underlying space is Noetherian, or a scheme whose underlying space is quasi-compact. Let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module (0, 5.4.1), f a section of $\mathscr{L}$ over X, X_f the open set of $x \in X$ such that $f(x) \neq 0$ (0, 5.5.1), and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module.*

(i) *If $s \in \Gamma(X, \mathscr{F})$ is such that $s|X_f = 0$, then there exists an integer $n > 0$ such that $s \otimes f^{\otimes n} = 0$.*
(ii) *For every section $s \in \Gamma(X_f, \mathscr{F})$, there exists an integer $n > 0$ such that $s \otimes f^{\otimes n}$ extends to a section of $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ over X.*

PROOF.

(i) Since the underlying space of X is quasi-compact, and thus the union of finitely-many affine opens U_i with $\mathscr{L}|U_i$ isomorphic to $\mathscr{O}_X|U_i$, we can reduce to the case where X is affine and $\mathscr{L} = \mathscr{O}_X$. In this case, f can be identified with an element of $A(X)$, and we have that $X_f = D(f)$; s can be identified with an element of an $A(X)$-module M, and $s|X_f$ to the corresponding element of M_f, and the result is then trivial, recalling the definition of a module of fractions. I | 173
(ii) Again, X is a finite union of affine opens U_i $(1 \leqslant i \leqslant r)$ such that $\mathscr{L}|U_i \cong \mathscr{O}_X|U_i$, and, for every i, $(s \otimes f^{\otimes n})|(U_i \cap X_f)$ can be identified (by the aforementioned isomorphism) with $(f|(U_i \cap X_f))^n(s|(U_i \cap X_f))$. We then know (1.4.1) that there exists an integer $n > 0$ such that, for all i, $(s \otimes f^{\otimes n})|(U_i \cap X_f)$ extends to a section s_i of $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ over U_i. Let $s_{i|j}$ be the restriction of s_i to $U_i \cap U_j$; by definition we have that $s_{i|j} - s_{j|i} = 0$ on $X_f \cap U_i \cap U_j$. But, if X is a Noetherian space, then $U_i \cap U_j$ is quasi-compact; if X is a scheme, then $U_i \cap U_j$ is an affine open (5.5.6), and so again quasi-compact. By virtue of (i), there thus exists an integer m (independent of i and j) such that $(s_{i|j} - s_{j|i}) \otimes f^{\otimes m} = 0$. It immediately follows that there exists a section s' of $\mathscr{F} \otimes \mathscr{L}^{\otimes(n+m)}$ over X that restricts to $s_i \otimes f^{\otimes m}$ over each U_i, and restricts to $s \otimes f^{\otimes(n+m)}$ over X_f.

□

The following corollaries give an interpretation of Theorem (9.3.1) in a more algebraic language:

Corollary (9.3.2). — *With the hypotheses of (9.3.1), consider the graded ring $A_\bullet = \Gamma_\bullet(\mathscr{L})$ and the graded $A_\bullet$-module $M_* = \Gamma_\bullet(\mathscr{L}, \mathscr{F})$ (0, 5.4.6). If $f \in A_n$, where $n \in \mathbf{Z}$, then there is a canonical isomorphism $\Gamma(X_f, \mathscr{F}) \simeq ((M_*)_f)_0$* (the subgroup of the module of fractions $(M_*)_f$ consisting of elements of degree 0)*.*

Corollary (9.3.3). — *Suppose that the hypotheses of (9.3.1) are satisfied, and suppose further that $\mathscr{L} = \mathscr{O}_X$. Then, setting $A = \Gamma(X, \mathscr{O}_X)$ and $M = \Gamma(X, \mathscr{F})$, the A_f-module $\Gamma(X_f, \mathscr{F})$ is canonically isomorphic to M_f.*

Proposition (9.3.4). — *Let X be a Noetherian prescheme, $\mathscr{F}$ a coherent $\mathscr{O}_X$-module, and $\mathscr{J}$ a coherent sheaf of ideals of $\mathscr{O}_X$, such that the support of $\mathscr{F}$ is contained in that of $\mathscr{O}_X|\mathscr{J}$. Then there exists an integer $n > 0$ such that $\mathscr{J}^n\mathscr{F} = 0$.*

PROOF. Since X is a union of finitely-many affine opens whose rings are Noetherian, we can suppose that X is affine, given by some Noetherian ring A; then $\mathscr{F} = \widetilde{M}$, where $M = \Gamma(X, \mathscr{F})$ is an A-module of finite type, and $\mathscr{J} = \widetilde{\mathfrak{J}}$, where $\mathfrak{J} = \Gamma(X, \mathscr{J})$ is an ideal of A ((1.4.1) and (1.5.1)). Since A is Noetherian, $\mathfrak{J}$ admits a finite system of generators f_i $(1 \leqslant i \leqslant m)$. By hypothesis, every section of $\mathscr{F}$ over X is zero on each of the $D(f_i)$; if s_j $(1 \leqslant j \leqslant q)$ are sections of $\mathscr{F}$ that generate M, then there exists an integer h, independent of i and j, such that $f_i^h s_j = 0$ (1.4.1), whence $f_i^h s = 0$ for all $s \in M$. We thus conclude that, if $n = mh$, then $\mathfrak{J}^n M = 0$, and so the corresponding $\mathscr{O}_X$-module $\mathscr{J}^n\mathscr{F} = \widetilde{\mathfrak{J}^n M}$ (1.3.13) is zero. □

Corollary (9.3.5). — *With the hypotheses of (9.3.4), there exists a closed subprescheme Y of X, whose underlying space is the support of $\mathscr{O}_X/\mathscr{J}$, such that, if $j : Y \to X$ is the canonical injection, then $\mathscr{F} = j_*(j^*(\mathscr{F}))$.*

PROOF. First, note that the supports of $\mathscr{O}_X/\mathscr{J}$ and $\mathscr{O}_X/\mathscr{J}^n$ are the same, since, if $\mathscr{J}_x = \mathscr{O}_x$, then $\mathscr{J}_x^n = \mathscr{O}_x$, and we also have that $\mathscr{J}_x^n \subset \mathscr{J}_x$ for all $x \in X$. We can, thanks to (9.3.4), thus suppose that $\mathscr{J}\mathscr{F} = 0$; we can then take Y to be the closed subprescheme of X defined by $\mathscr{J}$, and since $\mathscr{F}$ is then an $(\mathscr{O}_X/\mathscr{J})$-module, the conclusion follows immediately. □

9.4. Extension of quasi-coherent sheaves

(9.4.1). Let X be a topological space, $\mathscr{F}$ a sheaf of sets (resp. of groups, of rings) on X, U an open subset of X, $\psi : U \to X$ the canonical injection, and $\mathscr{G}$ a subsheaf of $\mathscr{F}|U = \psi^*(\mathscr{F})$. Since ψ_* is left exact, $\psi_*(\mathscr{G})$ is a subsheaf of $\psi_*(\psi^*(\mathscr{F}))$; we denote by ρ the canonical homomorphism $\mathscr{F} \to \psi_*(\psi^*(\mathscr{F}))$ (**0**, 3.5.3), and we denote by $\overline{\mathscr{G}}$ the subsheaf $\rho^{-1}(\psi_*(\mathscr{G}))$ of $\mathscr{F}$. It follows immediately from the definitions that, for every open subset V of X, $\Gamma(V, \overline{\mathscr{G}})$ consists of sections $s \in \Gamma(V, \mathscr{F})$ whose restriction to $V \cap U$ is a section of $\mathscr{G}$ over $V \cap U$. We thus have that $\overline{\mathscr{G}}|U = \psi^*(\overline{\mathscr{G}}) = \mathscr{G}$, and that $\overline{\mathscr{G}}$ is the *largest* subsheaf of $\mathscr{F}$ that restricts to $\mathscr{G}$ over U; we say that $\overline{\mathscr{G}}$ is the *canonical extension* of the subsheaf $\mathscr{G}$ of $\mathscr{F}|U$ to a subsheaf of $\mathscr{F}$. I | 174

Proposition (9.4.2). — *Let X be a prescheme, and U an open subset of X such that the canonical injection $j : U \to X$ is a quasi-compact morphism* (which will be the case for *all* U if the underlying space of X is *locally Noetherian* (6.6.4, i)). *Then:*

(i) *for every quasi-coherent $(\mathscr{O}_X|U)$-module $\mathscr{G}$, $j_*(\mathscr{G})$ is a quasi-coherent $\mathscr{O}_X$-module, and $j_*(\mathscr{G})|U = j^*(j_*(\mathscr{G})) = \mathscr{G}$;*

(ii) *for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and every quasi-coherent $(\mathscr{O}_X|U)$-submodule $\mathscr{G}$, the canonical extension $\overline{\mathscr{G}}$ of $\mathscr{G}$ (9.4.1) is a quasi-coherent $\mathscr{O}_X$-submodule of $\mathscr{F}$.*

PROOF. If $j = (\psi, \theta)$ (ψ being the injection $U \to X$ of underlying spaces), then, by definition, we have that $j_*(\mathscr{G}) = \psi_*(\mathscr{G})$ for every $(\mathscr{O}_X|U)$-module $\mathscr{G}$, and, further, that $j^*(\mathscr{H}) = \psi^*(\mathscr{H}) = \mathscr{H}|U$ for every $\mathscr{O}_X$-module $\mathscr{H}$, by definition of the prescheme induced over an open subset. So (i) is thus a particular case of ((9.2.2, a)); for the same reason, $j_*(j^*(\mathscr{F}))$ is quasi-coherent, and since $\overline{\mathscr{G}}$ is the inverse image of $j_*(\mathscr{G})$ by the homomorphism $\rho : \mathscr{F} \to j_*(j^*(\mathscr{F}))$, (ii) follows from (4.1.1). □

Note that the hypothesis that the morphism $j : U \to X$ is quasi-compact holds whenever the open subset U is *quasi-compact* and X is a *scheme*: indeed, U is then a union of finitely-many affine opens U_i, and, for every affine open V of X, $V \cap U_i$ is an affine open (5.5.6), and thus quasi-compact.

Corollary (9.4.3). — *Let X be a prescheme, and U a quasi-compact open subset of X such that the injection morphism $j : U \to X$ is quasi-compact. Suppose as well that every quasi-coherent $\mathscr{O}_X$-module is the inductive limit of its quasi-coherent $\mathscr{O}_X$-submodules of finite type* (which will be the case if X is an *affine scheme*). *Then let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module, and $\mathscr{G}$ a quasi-coherent $(\mathscr{O}_X|U)$-submodule of $\mathscr{F}|U$* of finite type. *Then there exists a quasi-coherent $\mathscr{O}_X$-submodule $\mathscr{G}'$ of $\mathscr{F}$* of finite type *such that $\mathscr{G}'|U = \mathscr{G}$.*

PROOF. We have $\mathscr{G} = \overline{\mathscr{G}}|U$, and $\overline{\mathscr{G}}$ is quasi-coherent, from (9.4.2), so the inductive limit of its quasi-coherent $\mathscr{O}_X$-submodules $\mathscr{H}_\lambda$ of finite type. It follows that $\mathscr{G}$ is the inductive limit of the $\mathscr{H}_\lambda|U$, and thus equal to one of the $\mathscr{H}_\lambda|U$, since it is of finite type (**0**, 5.2.3). □

Remark (9.4.4). — Suppose that for *every* affine open $U \subset X$, the injection morphism $U \to X$ is quasi-compact. Then, if the conclusion of (9.4.3) holds for every affine open U and for every quasi-coherent $(\mathscr{O}_X|U)$-submodule $\mathscr{G}$ of $\mathscr{F}|U$ of finite type, it follows that $\mathscr{F}$ is the inductive limit I | 175
of its quasi-coherent $\mathscr{O}_X$-submodules of finite type. Indeed, for every affine open $U \subset X$, we have that $\mathscr{F}|U = \widetilde{M}$, where M is an $A(U)$-module, and since the latter is the inductive limit of its quasi-coherent submodules of finite type, $\mathscr{F}|U$ is the inductive limit of its $(\mathscr{O}_X|U)$-submodules of finite type (1.3.9). But, by hypothesis, each of these submodules is induced on U by a quasi-coherent $\mathscr{O}_X$-submodule $\mathscr{G}_{\lambda,U}$ of $\mathscr{F}$ of finite type. The finite sums of the $\mathscr{G}_{\lambda,U}$ are again quasi-coherent $\mathscr{O}_X$-modules of finite type, because this is a local property, and the case where X is affine was covered in (1.3.10); it is clear then that $\mathscr{F}$ is the inductive limit of these finite sums, whence our claim.

Corollary (9.4.5). — *Under the hypotheses of Corollary (9.4.3), for every quasi-coherent $(\mathscr{O}_X|U)$-module $\mathscr{G}$ of finite type, there exists a quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}'$ of finite type such that $\mathscr{G}'|U = \mathscr{G}$.*

PROOF. Since $\mathscr{F} = j_*(\mathscr{G})$ is quasi-coherent (9.4.2) and $\mathscr{F}|U = \mathscr{G}$, it suffices to apply Corollary (9.4.3) to $\mathscr{F}$. □

Lemma (9.4.6). — *Let X be a prescheme, L a well-ordered set, $(V_\lambda)_{\lambda\in L}$ a cover of X by affine opens, and U an open subset of X; for all $\lambda \in L$, we set $W_\lambda = \bigcup_{\mu<\lambda} V_\mu$. Suppose that: (1) for every $\lambda \in L$, $V_\lambda \cap W_\lambda$ is quasi-compact; and (2) the immersion morphism $U \to X$ is quasi-compact. Then, for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and every quasi-coherent $(\mathscr{O}_X|U)$-submodule $\mathscr{G}$ of $\mathscr{F}|U$* of finite type, *there exists a quasi-coherent $\mathscr{O}_X$-submodule $\mathscr{G}'$ of $\mathscr{F}$* of finite type *such that $\mathscr{G}'|U = \mathscr{G}$.*

PROOF. Let $U_\lambda = U \cup W_\lambda$; we will define a family $(\mathscr{G}'_\lambda)$ by induction, where $\mathscr{G}'_\lambda$ is a quasi-coherent $(\mathscr{O}_X|U_\lambda)$-submodule of $\mathscr{F}|U_\lambda$ of finite type, such that $\mathscr{G}'_\lambda|U_\mu = \mathscr{G}'_\mu$ for $\mu < \lambda$ and $\mathscr{G}'_\lambda|U = \mathscr{G}$. The unique $\mathscr{O}_X$-submodule $\mathscr{G}'$ of $\mathscr{F}$ such that $\mathscr{G}'|U_\lambda = \mathscr{G}'$ for all $\lambda \in L$ (**0**, 3.3.1) will then give us what we want. So suppose that the $\mathscr{G}'_\mu$ are defined and have the preceding properties for $\mu < \lambda$; if λ does not have a predecessor then we take $\mathscr{G}'_\lambda$ to be the unique $(\mathscr{O}_X|U_\lambda)$-submodule of $\mathscr{F}|U_\lambda$ such that $\mathscr{G}'_\lambda|U_\mu = \mathscr{G}'_\mu$ for all $\mu < \lambda$, which is allowed since the U_μ with $\mu < \lambda$ then form a cover of U_λ. If, conversely, $\lambda = \mu + 1$, then $U_\lambda = U_\mu \cup V_\mu$, and it suffices to define a quasi-coherent $(\mathscr{O}_X|V_\mu)$-submodule $\mathscr{G}''_\mu$ of $\mathscr{F}|V_\mu$ of finite type such that

$$\mathscr{G}''_\mu|(U_\mu \cap V_\mu) = \mathscr{G}'_\mu|(U_\mu \cap V_\mu);$$

and then to take $\mathscr{G}'_\lambda$ to be the $(\mathscr{O}_X|U_\lambda)$-submodule of $\mathscr{F}|U_\lambda$ such that $\mathscr{G}'_\lambda|U_\mu = \mathscr{G}'_\mu$ and $\mathscr{G}'_\lambda|V_\mu = \mathscr{G}''_\mu$ (**0**, 3.3.1). But, since V_μ is affine, the existence of $\mathscr{G}''_\mu$ is guaranteed by (9.4.3) as soon as we show that $U_\mu \cap V_\mu$ is quasi-compact; but $U_\mu \cap V_\mu$ is the union of $U \cap V_\mu$ and $W_\mu \cap V_\mu$, which are both quasi-compact by virtue of the hypotheses. □

Theorem (9.4.7). — *Let X be a prescheme, and U an open subset of X. Suppose that one of the following conditions is verified:*

(a) *the underlying space of X is locally Noetherian;*
(b) *X is a quasi-compact scheme and U is a quasi-compact open.*

Then, for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and every quasi-coherent $(\mathscr{O}_X|U)$-submodule $\mathscr{G}$ of $\mathscr{F}|U$ of finite type, *there exists a quasi-coherent $\mathscr{O}_X$-submodule $\mathscr{G}'$ of $\mathscr{F}$* of finite type *such that $\mathscr{G}'|U = \mathscr{G}$.*

PROOF. Let $(V_\lambda)_{\lambda\in L}$ be a cover of X by affine opens, with L assumed to be finite in case (b); **I** | 176
since L is equipped with the structure of a well-ordered set, it suffices to check that the conditions of (9.4.6) are satisfied. It is clear in the case of (a), since the spaces V_λ are Noetherian. For case (b), the $V_\lambda \cap \lambda_\mu$ are affine (5.5.6), and thus quasi-compact, and, since L is finite, $V_\lambda \cap W_\lambda$ is quasi-compact. Whence the theorem. □

Corollary (9.4.8). — *Under the hypotheses of* (9.4.7), *for every quasi-coherent $(\mathscr{O}_X|U)$-module $\mathscr{G}$ of finite type, there exists a quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}'$ of finite type such that $\mathscr{G}'|U = \mathscr{G}$.*

PROOF. It suffices to apply (9.4.7) to $\mathscr{F} = j_*(\mathscr{G})$, which is quasi-coherent (9.4.2) and such that $\mathscr{F}|U = \mathscr{G}$. □

Corollary (9.4.9). — *Let X be a prescheme whose underlying space is locally Noetherian, or a quasi-compact scheme. Then every quasi-coherent $\mathscr{O}_X$-module is the inductive limit of its quasi-coherent $\mathscr{O}_X$-submodules of finite type.*

PROOF. This follows from Theorem (9.4.7) and Remark (9.4.4). □

Corollary (9.4.10). — *Under the hypotheses of* (9.4.9), *if a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ is such that every quasi-coherent $\mathscr{O}_X$-submodule of finite type of $\mathscr{F}$ is generated by its sections over X, then $\mathscr{F}$ is generated by its sections over X.*

PROOF. Let U be an affine open neighbourhood of a point $x \in X$, and let s be a section of $\mathscr{F}$ over U; the $\mathscr{O}_X$-submodule $\mathscr{G}$ of $\mathscr{F}|U$ generated by s is quasi-coherent and of finite type, so there exists a quasi-coherent $\mathscr{O}_X$-submodule $\mathscr{G}'$ of $\mathscr{F}$ of finite type such that $\mathscr{G}'|U = \mathscr{G}$ (9.4.7). By hypothesis, there thus exists a finite number of sections t_i of $\mathscr{G}'$ over X and of sections a_i of $\mathscr{O}_X$ over a neighbourhood $V \subset U$ of x such that $s|V = \sum_i a_i(t_i|V)$, which proves the corollary. □

9.5. Closed image of a prescheme; closure of a subprescheme

Proposition (9.5.1). — *Let $f : X \to Y$ be a morphism of preschemes such that $f_*(\mathscr{O}_X)$ is a quasi-coherent $\mathscr{O}_Y$-module (which will be the case if f is quasi-compact* and*, in addition, either f is separated or X is locally Noetherian (9.2.2)). Then there exists a smaller subprescheme Y' of Y such that f factors through the canonical injection $j : Y' \to Y$ (*or, equivalently (4.4.1), such that the subprescheme $f^{-1}(Y')$ of X is *identical* to *X).*

More precisely:

Corollary (9.5.2). — *Under the conditions of (9.5.1), let $f = (\psi, \theta)$, and let $\mathscr{J}$ be the (quasi-coherent) kernel of the homomorphism $\theta : \mathscr{O}_Y \to f_*(\mathscr{O}_X)$. Then the closed subprescheme Y' of Y defined by $\mathscr{J}$ satisfies the conditions of (9.5.1).*

Proof. Since the functor ψ^* is exact, the canonical factorization $\theta : \mathscr{O}_Y \to \mathscr{O}_Y/\mathscr{J} \xrightarrow{\theta'} \psi_*(\mathscr{O}_X)$ gives (**0**, 3.5.4.3) a factorization $\theta^\sharp : \psi^*(\mathscr{O}_Y) \to \psi^*(\mathscr{O}_Y)/\psi^*(\mathscr{J}) \xrightarrow{\theta'^\sharp} \mathscr{O}_X$; since $\theta_x^\sharp$ is a local homomorphism for every $x \in X$, the same is true of ${\theta'}_x^\sharp$; if we denote by ψ_0 the continuous map ψ considered as a map from X to Y', and by θ_0 the restriction $\theta'|X' : (\mathscr{O}_Y/\mathscr{J})|Y' \to \psi_*(\mathscr{O}_X)|Y' = (\psi_0)_*(\mathscr{O}_X)$, **Err$_{\text{II}}$** then we see that $f_0 = (\psi_0, \theta_0)$ is a morphism of preschemes $X \to Y'$ (2.2.1) such that $f = j \circ f_0$. Now, if Y'' is a second closed subprescheme of Y, defined by a quasi-coherent sheaf of ideals $\mathscr{J}'$ **I | 177** of $\mathscr{O}_Y$, such that f factors through the injection $j' : Y'' \to Y$, then we should immediately have that $\psi(X) \subset Y''$, and thus that $Y' \subset Y''$, since Y'' is closed. Furthermore, for all $y \in Y''$, θ should factor as $\mathscr{O}_y \to \mathscr{O}_y/\mathscr{J}'_y \to (\psi_*(\mathscr{O}_X))_y$, which, by definition, leads to $\mathscr{J}'_y \subset \mathscr{J}_y$, and thus X' is a closed subprescheme of Y'' (4.1.10). □

Definition (9.5.3). — Whenever there exists a smaller subprescheme Y' of Y such that f factors through the canonical injection $j : Y' \to Y$, we say that Y' is the *closed image* prescheme of X under the morphism f.

Proposition (9.5.4). — *If $f_*(\mathscr{O}_X)$ is a quasi-coherent $\mathscr{O}_Y$-module, then the underlying space of the closed image of X under f is the closure $\overline{f(X)}$ in Y.*

Proof. As the support of $f_*(\mathscr{O}_X)$ is contained in $\overline{f(X)}$, we have (with the notation of (9.5.2)) $\mathscr{J}_y = \mathscr{O}_y$ for $y \notin \overline{f(X)}$, thus the support of $\mathscr{O}_Y/\mathscr{J}$ is contained in $\overline{f(X)}$. In addition, this support is closed and contains $f(X)$: indeed, if $y \in f(X)$, the unit element of the ring $(\psi_*(\mathscr{O}_X))_y$ is not zero, being the germ at y of the section

$$1 \in \Gamma(X, \mathscr{O}_X) = \Gamma(Y, \psi_*(\mathscr{O}_X));$$

since it is the image under θ of the unit element of $\mathscr{O}_y$, the latter does not belong to $\mathscr{J}_y$, hence $\mathscr{O}_y/\mathscr{J}_y \neq 0$; this finishes the proof. □

Proposition (9.5.5). — (Transitivity of closed images). *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of preschemes; we suppose that the closed image Y' of X under f exists, and that, if g' is the restriction of g to Y', then the closed image Z' of Y' under g' exists. Then the closed image of X under $g \circ f$ exists and is equal to Z'.*

Proof. It suffices (9.5.1) to show that Z' is the smallest closed subprescheme Z_1 of Z such that the closed subprescheme $(g \circ f)^{-1}(Z_1)$ of X (equal to $f^{-1}(g^{-1}(Z_1))$ by Corollary (4.4.2)) is equal to X; it is equivalent to say that Z' is the smallest closed subprescheme of Z such that f factors through the injection $g^{-1}(Z_1) \to Y$ (4.4.1). By virtue of the existence of the closed image Y', every Z_1 with this property is such that $g^{-1}(Z_1)$ factors through Y', which is equivalent to saying that $j^{-1}(g^{-1}(Z_1)) = g'^{-1}(Z_1) = Y'$, denoting by j the injection $Y' \to Y$. By the definition of Z', we indeed conclude that Z' is the smallest closed subprescheme of Z satisfying the preceding condition. □

Corollary (9.5.6). — *Let $f : X \to Y$ be an S-morphism such that Y is the closed image of X under f. Let Z be an S-scheme; if two S-morphisms g_1, g_2 from Y to Z are such that $g_1 \circ f = g_2 \circ f$, then $g_1 = g_2$.*

Proof. Let $h = (g_1, g_2)_S : Y \to Z \times_S Z$; since the diagonal $T = \Delta_Z(Z)$ is a closed subprescheme of $Z \times_S Z$, $Y' = h^{-1}(T)$ is a closed subprescheme of Y (4.4.1). Let $u = g_1 \circ f = g_2 \circ f$; we then have, by definition of the product, $h' = h \circ f = (u, u)_S$, so $h \circ f = \Delta_Z \circ u$; since $\Delta_Z^{-1}(T) = Z$, we have $h'^{-1}(T) = u^{-1}(Z) = X$, so $f^{-1}(Y') = X$. From this, we conclude (4.4.1) that the f factors through the canonical injection $Y' \to Y$, so $Y' = Y$ by hypothesis; it then follows (4.4.1) that h factors as $\Delta_Z \circ v$, where v is a morphism $Y \to Z$, which implies that $g_1 = g_2 = v$. □

Remark (9.5.7). — If X and Y are S-schemes, Proposition (9.5.6) implies that, when Y is the closed image of X under f, f is an *epimorphism* in the category of *S-schemes* (T, 1.1). We will show in Chapter V that, conversely, if the closed image Y' of X under f exists and if f is an epimorphism of S-schemes, then we necessarily have that $Y' = Y$. I | 178

Proposition (9.5.8). — *Suppose that the hypotheses of (9.5.1) are satisfied, and let Y' be the closed image of X under f. For every open V of Y, let $f_V : f^{-1}(V) \to V$ be the restriction of f; then the closed image of $f^{-1}(V)$ under f_V in V exists and is equal to the prescheme induced by Y' on the open $V \cap Y'$ of Y'* (in other words, to the subprescheme $\inf(V, Y)$ of Y (4.4.3)).

Proof. Let $X' = f^{-1}(V)$; since the direct image of $\mathscr{O}_{X'}$ by f_V is exactly the restriction of $f_*(\mathscr{O}_X)$ to V, it is clear that the kernel $\mathscr{J}'$ of the homomorphism $\mathscr{O}_V \to (f_V)_*(\mathscr{O}_{X'})$ is the restriction of $\mathscr{J}$ to V, from which the proposition quickly follows. □

We will see that this result can be understood as saying that taking the closed image commutes with an extension $Y_1 \to Y$ of the base prescheme, which is an *open immersion*. We will see in Chapter IV that it is the same for an extension $Y_1 \to Y$ which is a *flat* morphism, provided that f is separated and quasi-compact.

Proposition (9.5.9). — *Let $f : X \to Y$ be a morphism such that the closed image Y' of X under f exists.*

(i) *If X is reduced, then so too is Y'.*
(ii) *If the hypotheses of Proposition (9.5.1) are satisfied and X is irreducible (resp. integral), then so too is Y'.*

Proof. By hypothesis, the morphism f factors as $X \xrightarrow{g} Y' \xrightarrow{j} Y$, where j is the canonical injection. Since X is reduced, g factors as $X \xrightarrow{h} Y'_{\mathrm{red}} \xrightarrow{j'} Y'$, where j' is the canonical injection (5.2.2), and it then follows from the definition of Y' that $Y'_{\mathrm{red}} = Y'$. If, moreover, the conditions of Proposition (9.5.1) are satisfied, then it follows from (9.5.4) that $f(X)$ is dense in Y'; if X is irreducible, then so is Y' (**0**, 2.1.5). The claim about integral preschemes follows from the conjunction of the two others. □

Proposition (9.5.10). — *Let Y be a subprescheme of a prescheme X, such that the canonical injection $i : Y \to X$ is a quasi-compact morphism. Then there exists a smaller closed subprescheme $\overline{Y}$ of X containing Y; its underlying space is the closure of that of Y; the latter is open in its closure, and the prescheme Y is induced on this open by $\overline{Y}$.*

Proof. It suffices to apply Proposition (9.5.1) to the injection j, which is separated (5.5.1) and quasi-compact by hypothesis; (9.5.1) thus proves the existence of $\overline{Y}$, and (9.5.4) shows that its underlying space is the closure of Y in X; since Y is locally closed in X, it is open in $\overline{Y}$, and the last claim comes from (9.5.8) applied to an open subset V of X such that Y is closed in V. □

With the above notation, if the injection $V \to X$ is quasi-compact, and if $\mathscr{J}$ is the quasi-coherent sheaf of ideals of $\mathscr{O}_X|V$ defining the closed subprescheme Y of V, it follows from Proposition (9.5.1) that the quasi-coherent sheaf of ideals of $\mathscr{O}_X$ defining $\overline{Y}$ is the canonical extension (9.4.1) $\overline{\mathscr{J}}$ of $\mathscr{J}$, because it is clearly the largest quasi-coherent subsheaf of ideals of $\mathscr{O}_X$ inducing $\mathscr{J}$ on V.

Corollary (9.5.11). — *Under the hypotheses of Proposition (9.5.10), every section of $\mathscr{O}_{\overline{Y}}$ over an open V of $\overline{Y}$ that is zero on $V \cap Y$ is zero.* I | 179

Proof. By Proposition (9.5.8), we can reduce to the case where $V = \overline{Y}$. If we take into account that the sections of $\mathscr{O}_{\overline{Y}}$ over $\overline{Y}$ canonically correspond to the $\overline{Y}$-sections of $\overline{Y} \otimes_Z Z[T]$ (3.3.15) and that the latter is separated over $\overline{Y}$, then the corollary appears as a specific case of (9.5.6). □

When there exists a smaller closed subprescheme Y' of X containing a subprescheme Y of X, we say that Y' is the *closure* of Y in X, when there is little cause for confusion.

9.6. Quasi-coherent sheaves of algebras; change of structure sheaf

Proposition (9.6.1). — *Let X be a prescheme, and $\mathscr{B}$ a quasi-coherent $\mathscr{O}_X$-algebra (**0**, 5.1.3). For a $\mathscr{B}$-module $\mathscr{F}$ to be quasi-coherent (on the ringed space $(X,\mathscr{B})$) it is necessary and sufficient that $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module.*

PROOF. Since the question is local, we can assume X to be affine, given by some ring A, and thus $\mathscr{B} = \widetilde{B}$, where B is an A-algebra (1.4.3). If $\mathscr{F}$ is quasi-coherent on the ringed space $(X,\mathscr{B})$ then we can also assume that $\mathscr{F}$ is the cokernel of a $\mathscr{B}$-homomorphism $\mathscr{B}^{(I)} \to \mathscr{B}^{(J)}$; since this homomorphism is also an $\mathscr{O}_X$-homomorphism of $\mathscr{O}_X$-modules, and $\mathscr{B}^{(I)}$ and $\mathscr{B}^{(J)}$ are quasi-coherent $\mathscr{O}_X$-modules (1.3.9, ii), $\mathscr{F}$ is also a quasi-coherent $\mathscr{O}_X$-module (1.3.9, i).

Conversely, if $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module, then $\mathscr{F} = \widetilde{M}$, where M is a B-module (1.4.3); M is isomorphic to the cokernel of a B-homomorphism $B^{(I)} \to B^{(J)}$, so $\mathscr{F}$ is a $\mathscr{B}$-module isomorphic to the cokernel of the corresponding homomorphism $\mathscr{B}^{(I)} \to \mathscr{B}^{(J)}$ (1.3.13), which finishes the proof. □

In particular, if $\mathscr{F}$ and $\mathscr{G}$ are two quasi-coherent $\mathscr{B}$-modules, then $\mathscr{F} \otimes_{\mathscr{B}} \mathscr{G}$ is a quasi-coherent $\mathscr{B}$-module; similarly for $\mathscr{H}om(\mathscr{F},\mathscr{G})$ whenever we further suppose that $\mathscr{F}$ admits a finite presentation (1.3.13).

(9.6.2). Given a prescheme X, we say that a quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{B}$ is of *finite type* if, for all $x \in X$, there exists an open *affine* neighbourhood U of x such that $\Gamma(U,\mathscr{B}) = B$ is an algebra of finite type over $\Gamma(U,\mathscr{O}_X) = A$. We then have that $\mathscr{B}|U = \widetilde{B}$ and, for all $f \in A$, the induced $(\mathscr{O}_X|D(f))$-algebra $\mathscr{B}|D(f)$ is of finite type, because it is isomorphic to $(B_f)^\sim$, and $B_f = B \otimes_A A_f$ is clearly an algebra of finite type over A_f. Since the $D(f)$ form a basis for the topology of U, we thus conclude that if $\mathscr{B}$ is a quasi-coherent $\mathscr{O}_X$-algebra of finite type then, for every open V of X, $\mathscr{B}|V$ is a quasi-coherent $(\mathscr{O}_X|V)$-algebra of finite type.

Proposition (9.6.3). — *Let X be a locally Noetherian prescheme. Then every quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{B}$ of finite type is a coherent sheaf of rings (**0**, 5.3.7).*

PROOF. We can once again restrict to the case where X is an affine scheme given by a Noetherian ring A, and where $\mathscr{B} = \widetilde{B}$, with B being an A-algebra of finite type; B is then a Noetherian ring. With this, it remains to prove that the kernel $\mathscr{N}$ of a $\mathscr{B}$-homomorphism $\mathscr{B}^m \to \mathscr{B}$ is a $\mathscr{B}$-module of **I** | 180
finite type; but it is isomorphic (as a $\mathscr{B}$-module) to $\widetilde{N}$, where N is the kernel of the corresponding homomorphism of B-modules $B^m \to B$ (1.3.13). Since B is Noetherian, the submodule N of B^m is a B-module of finite type, so there exists a homomorphism $B^p \to B^m$ with image N; since the sequence $B^p \to B^m \to B$ is exact, so too is the corresponding sequence $\mathscr{B}^p \to \mathscr{B}^m \to \mathscr{B}$ (1.3.5), and since $\mathscr{N}$ is the image of $\mathscr{B}^p \to \mathscr{B}^m$ (1.3.9, i), this proves the proposition. □

Corollary (9.6.4). — *Under the hypotheses of (9.6.3), for a $\mathscr{B}$-module $\mathscr{F}$ to be coherent, it is necessary and sufficient that it be a quasi-coherent $\mathscr{O}_X$-module and a $\mathscr{B}$-module of finite type. If this is the case, and if $\mathscr{G}$ is a $\mathscr{B}$-submodule or a quotient module of $\mathscr{F}$, then in order for $\mathscr{G}$ to be a coherent $\mathscr{B}$-module, it is necessary and sufficient that it be a quasi-coherent $\mathscr{O}_X$-module.*

PROOF. Taking (9.6.1) into account, the conditions on $\mathscr{F}$ are clearly necessary; we will show that they are sufficient. We can restrict to the case where X is affine, given by some Noetherian ring A, where $\mathscr{B} = \widetilde{B}$, with B an A-algebra of finite type, where $\mathscr{F} = \widetilde{M}$, with M a B-module, and where there exists a surjective $\mathscr{B}$-homomorphism $\mathscr{B}^m \to \mathscr{F} \to 0$. We then have the corresponding exact sequence $B^m \to M \to 0$, so M is a B-module of finite type; further, the kernel P of the homomorphism $B^m \to M$ is then a B-module of finite type, since B is Noetherian. We thus conclude (1.3.13) that $\mathscr{F}$ is the cokernel of a $\mathscr{B}$-homomorphism $\mathscr{B}^m \to \mathscr{B}^n$, and is thus coherent, since $\mathscr{B}$ is a coherent sheaf of rings (**0**, 5.3.4). The same reasoning shows that a quasi-coherent $\mathscr{B}$-submodule (resp. a quotient $\mathscr{B}$-module) of $\mathscr{F}$ is of finite type, from whence the second part of the corollary. □

Proposition (9.6.5). — *Let X be a quasi-compact scheme, or a prescheme whose underlying space is Noetherian. For all quasi-compact $\mathscr{O}_X$-algebras $\mathscr{B}$ of finite type, there exists a quasi-coherent $\mathscr{O}_X$-submodule $\mathscr{E}$ of $\mathscr{B}$ of finite type such that $\mathscr{E}$ generates* (**0**, 4.1.3) *the $\mathscr{O}_X$-algebra $\mathscr{B}$.*

Proof. In fact, by hypothesis, there exists a finite cover (U_i) of X consisting of affine opens such that $\Gamma(U_i, \mathscr{B}) = B_i$ is an algebra of finite type over $\Gamma(U_i, \mathscr{O}_X) = A_i$; let E_i be a A_i-submodule of B_i of finite type that generates the A_i-algebra B_i; thanks to (9.4.7), there exists an $\mathscr{O}_X$-submodule $\mathscr{E}_i$ of $\mathscr{B}$, quasi-coherent and of finite type, such that $\mathscr{E}_i|U_i = \widetilde{E}_i$. It is clear that the sum $\mathscr{E}$ of the $\mathscr{E}_i$ is the desired object. □

Proposition (9.6.6). — *Let X be a prescheme whose underlying space is locally Noetherian, or a quasi-compact scheme. Then every quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{B}$ is the inductive limit of its quasi-coherent $\mathscr{O}_X$-subalgebras of finite type.*

Proof. In fact, it follows from (9.4.9) that $\mathscr{B}$ is the inductive limit (as an $\mathscr{O}_X$-module) of its quasi-coherent $\mathscr{O}_X$-submodules of finite type; the latter generating quasi-coherent $\mathscr{O}_X$-*subalgebras* of $\mathscr{B}$ of finite type (1.3.14), and so $\mathscr{B}$ is *a fortiori* their inductive limit. □

§10. Formal schemes

10.1. Formal affine schemes

(10.1.1). Let A be an *admissible* topological ring (**0**, 7.1.2); for each ideal of definition $\mathfrak{J}$ of A, $\operatorname{Spec}(A/\mathfrak{J})$ can be identified with the closed subspace $V(\mathfrak{J})$ of $\operatorname{Spec}(A)$ (1.1.11), the set of *open* prime ideals of A; this topological space does not depend on the ideal of definition $\mathfrak{J}$ considered; we denote this topological space by $\mathfrak{X}$. Let $(\mathfrak{J}_\lambda)$ be a fundamental system of neighbourhoods of 0 in A, consisting of ideals of definition, and for each λ, let $\mathscr{O}_\lambda$ be the structure sheaf of $\operatorname{Spec}(A/\mathfrak{J}_\lambda)$; this sheaf is induced on $\mathfrak{X}$ by $\widetilde{A}/\widetilde{\mathfrak{J}_\lambda}$ (which is zero outside of $\mathfrak{X}$). For $\mathfrak{J}_\mu \subset \mathfrak{J}_\lambda$, the canonical homomorphism $A/\mathfrak{J}_\mu \to A/\mathfrak{J}_\lambda$ thus defines a homomorphism $u_{\lambda\mu} : \mathscr{O}_\mu \to \mathscr{O}_\lambda$ of sheaves of rings (1.6.1), and $(\mathscr{O}_\lambda)$ is a *projective system of sheaves of rings* for these homomorphisms. Since the topology of $\mathfrak{X}$ admits a basis consisting of quasi-compact open subsets, we can associate to each $\mathscr{O}_\lambda$ a *pseudo-discrete sheaf of topological rings* (**0**, 3.8.1) which have $\mathscr{O}_\lambda$ as the underlying sheaf of rings (without topologies), and that we denote also by $\mathscr{O}_\lambda$; and the $\mathscr{O}_\lambda$ give again a *projective system of sheaves of topological rings* (**0**, 3.8.2). We denote by $\mathscr{O}_{\mathfrak{X}}$ the *sheaf of topological rings* on $\mathfrak{X}$, the projective limit of the system $(\mathscr{O}_\lambda)$; for each *quasi-compact* open subset U of $\mathfrak{X}$, $\Gamma(U, \mathscr{O}_{\mathfrak{X}})$ is a topological ring, the projective limit of the system of *discrete* rings $\Gamma(U, \mathscr{O}_\lambda)$ (**0**, 3.2.6). I | 181

Definition (10.1.2). — Given an admissible topological ring A, we define the formal spectrum of A, denoted by $\operatorname{Spf}(A)$, to be the closed subspace $\mathfrak{X}$ of $\operatorname{Spec}(A)$ consisting of the open prime ideals of A. We say that a topologically ringed space is a formal affine scheme if it is isomorphic to a formal spectrum $\operatorname{Spf}(A) = \mathfrak{X}$ equipped with a sheaf of topological rings $\mathscr{O}_{\mathfrak{X}}$ which is the projective limit of sheaves of pseudo-discrete topological rings $(\widetilde{A}/\widetilde{\mathfrak{J}_\lambda})|\mathfrak{X}$, where $\mathfrak{J}_\lambda$ varies over the filtered set of ideals of definition of A.

When we speak of a *formal spectrum* $\mathfrak{X} = \operatorname{Spf}(A)$ as a formal affine scheme, it will always be as the topologically ringed space $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ where $\mathscr{O}_{\mathfrak{X}}$ is defined as above.

We note that every *affine scheme* $X = \operatorname{Spec}(A)$ can be considered as a formal affine scheme in only one way, by considering A as a discrete topological ring: the topological rings $\Gamma(U, \mathscr{O}_X)$ are then discrete whenever U is quasi-compact (but not, in general, when U is an arbitrary open subset of X).

Proposition (10.1.3). — *If $\mathfrak{X} = \operatorname{Spf}(A)$, where A is an admissible ring, then $\Gamma(\mathfrak{X}, \mathscr{O}_X)$ is topologically isomorphic to A.*

Proof. Indeed, since $\mathfrak{X}$ is closed in $\operatorname{Spec}(A)$, it is quasi-compact, and so $\Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ is topologically isomorphic to the projective limit of the discrete rings $\Gamma(\mathfrak{X}, \mathscr{O}_\lambda)$; but $\Gamma(\mathfrak{X}, \mathscr{O}_\lambda)$ is isomorphic to $A/\mathfrak{J}_\lambda$ (1.3.7); since A is separated and complete, it is topologically isomorphic to $\varprojlim A/\mathfrak{J}_\lambda$ (**0**, 7.2.1), whence the proposition. □

Proposition (10.1.4). — *Let A be an admissible ring, $\mathfrak{X} = \operatorname{Spf}(A)$, and, for every $f \in A$, let $\mathfrak{D}(f) = D(f) \cap \mathfrak{X}$; then the topologically ringed space $(\mathfrak{D}(f), \mathscr{O}_{\mathfrak{X}}|\mathfrak{D}(f))$ is isomorphic to the formal affine spectrum* $\operatorname{Spf}(A_{\{f\}})$ (**0**, 7.6.15).

Proof. For every ideal of definition $\mathfrak{J}$ of A, the discrete ring $S_f^{-1}A/S_f^{-1}\mathfrak{J}$ is canonically identified with $A_{\{f\}}/\mathfrak{J}_{\{f\}}$ (**0**, 7.6.9), so, by (1.2.5) and (1.2.6), the topological space $\operatorname{Spf}(A_{\{f\}})$ is canonically identified with $\mathfrak{D}(f)$. Further, for every quasi-compact open subset U of $\mathfrak{X}$ contained in $\mathfrak{D}(f)$, $\Gamma(U, \mathscr{O}_\lambda)$ can be identified with the module of sections of the structure sheaf of $\operatorname{Spec}(S_f^{-1}A/S_f^{-1}\mathfrak{J}_\lambda)$ over U (1.3.6), so, setting $\mathfrak{Y} = \operatorname{Spf}(A_{\{f\}})$, $\Gamma(U, \mathscr{O}_{\mathfrak{X}})$ can be identified with the module of sections $\Gamma(U, \mathscr{O}_{\mathfrak{Y}})$, which proves the proposition. □

(10.1.5). As a sheaf of rings *without topology*, the structure sheaf $\mathscr{O}_{\mathfrak{X}}$ of $\operatorname{Spf}(A)$ admits, for every $x \in \mathfrak{X}$, a fibre which, by (10.1.4), can be identified with the inductive limit $\varinjlim A_{\{f\}}$ for the $f \notin \mathfrak{j}_x$. Then, by (**0**, 7.6.17) and (**0**, 7.6.18):

Proposition (10.1.6). — *For every $x \in \mathfrak{X} = \operatorname{Spf}(A)$, the fibre $\mathscr{O}_x$ is a local ring whose residue field is isomorphic to $k(x) = A_x/\mathfrak{j}_x A_x$. If, further, A is adic and Noetherian, then $\mathscr{O}_x$ is a Noetherian ring.*

Since $k(x)$ is not reduced at 0, we conclude from this that the *support* of the ring of sheaves $\mathscr{O}_{\mathfrak{X}}$ is *equal to* $\mathfrak{X}$.

10.2. Morphisms of formal affine schemes

(10.2.1). Let A, B be two admissible rings, and let $\phi : B \to A$ be a *continuous* morphism. The continuous map ${}^a\phi : \operatorname{Spec}(A) \to \operatorname{Spec}(B)$ (1.2.1) then maps $\mathfrak{X} = \operatorname{Spf}(A)$ to $\mathfrak{Y} = \operatorname{Spf}(B)$, since the inverse image under ϕ of an open prime ideal of A is an open prime ideal of B. Conversely, for all $g \in B$, ϕ defines a continuous homomorphism $\Gamma(\mathfrak{D}(g), \mathscr{O}_{\mathfrak{Y}}) \to \Gamma(\mathfrak{D}(\phi(g)), \mathscr{O}_{\mathfrak{X}})$ according to (10.1.4), (10.1.3), and (**0**, 7.6.7); since these homomorphisms satisfy the compatibility conditions for the restrictions corresponding to the change from g to a multiple of g, and since $\mathfrak{D}(\phi(g)) = {}^a\phi^{-1}(\mathfrak{D}(g))$, they define a *continuous* homomorphism of sheaves of topological rings $\mathscr{O}_{\mathfrak{Y}} \to {}^a\phi_*(\mathscr{O}_{\mathfrak{X}})$ (**0**, 3.2.5) that we denote by $\widetilde{\phi}$; we have thus defined a morphism $\Phi = ({}^a\phi, \widetilde{\phi})$ of topologically ringed spaces $\mathfrak{X} \to \mathfrak{Y}$. We note that, as a homomorphism of sheaves without topology, $\widetilde{\phi}$ defines a homomorphism $\widetilde{\phi}_x^\sharp : \mathscr{O}_{{}^a\phi(x)} \to \mathscr{O}_x$ on the stalks, for all $x \in \mathfrak{X}$.

Proposition (10.2.2). — *Let A and B be admissible topological rings, and let $\mathfrak{X} = \operatorname{Spf}(A)$ and $\mathfrak{Y} = \operatorname{Spf}(B)$. For a morphism $u = (\psi, \theta) : \mathfrak{X} \to \mathfrak{Y}$ of topologically ringed spaces to be of the form $({}^a\phi, \widetilde{\phi})$, where ϕ is a continuous ring homomorphism $B \to A$, it is necessary and sufficient that $\theta_x^\sharp$ be a local homomorphism $\mathscr{O}_{\phi(x)} \to \mathscr{O}_x$ for all $x \in \mathfrak{X}$.*

Proof. The condition is necessary: let $\mathfrak{p} = \mathfrak{j}_x \in \operatorname{Spf}(A)$, and let $\mathfrak{q} = \phi^{-1}(\mathfrak{j}_x)$; if $g \notin \mathfrak{q}$, then we have $\phi(g) \notin \mathfrak{p}$, and it is immediate that the homomorphism $B_{\{g\}} \to A_{\{\phi(g)\}}$ induced by ϕ (**0**, 7.6.7) sends $\mathfrak{q}_{\{g\}}$ to a subset of $\mathfrak{p}_{\{\phi(g)\}}$; by passing to the inductive limit, we see (taking (10.1.5) and (**0**, 7.6.17) into account) that $\widetilde{\phi}_x^\sharp$ is a local homomorphism.

Conversely, let (ψ, θ) be a morphism satisfying the condition of the proposition; by (10.1.3), θ defines a continuous ring homomorphism

$$\phi = \Gamma(\theta) : B = \Gamma(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}) \longrightarrow \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) = A.$$

By virtue of the hypothesis on θ, for the section $\phi(g)$ of $\mathscr{O}_{\mathfrak{X}}$ over $\mathfrak{X}$ to be an invertible germ at the point x, it is necessary and sufficient that g be an invertible germ at the point $\psi(x)$. But, by (**0**, 7.6.17), the sections of $\mathscr{O}_{\mathfrak{X}}$ (resp. $\mathscr{O}_{\mathfrak{Y}}$) over $\mathfrak{X}$ (resp. $\mathfrak{Y}$) that have a non-invertible germ at the point x (resp. $\psi(x)$) are exactly the elements of $\mathfrak{j}_x$ (resp. $\mathfrak{j}_{\psi(x)}$); the above remark thus shows that ${}^a\phi = \psi$. Finally, **I** | 183
for all $g \in B$ the diagram

$$\begin{tikzcd} B = \Gamma(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}) \arrow[r, "\phi"] \arrow[d] & \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) = A \arrow[d] \\ B_{\{g\}} = \Gamma(\mathfrak{D}(g), \mathscr{O}_{\mathfrak{Y}}) \arrow[r, "\Gamma(\theta_{\mathfrak{D}(g)})"] & \Gamma(\mathfrak{D}(\phi(g)), \mathscr{O}_{\mathfrak{X}}) = A_{\{\phi(g)\}} \end{tikzcd}$$

is commutative; by the universal property of completed rings of fractions (**0**, 7.6.6), $\theta_{\mathfrak{D}(g)}$ is equal to $\widetilde{\phi}_{\mathfrak{D}(g)}$ for all $g \in B$, and so (**0**, 3.2.5) we have $\theta = \widetilde{\phi}$. □

We say that a morphism (ψ, θ) of topologically ringed spaces satisfying the condition of Proposition (10.2.2) is a *morphism of formal affine schemes*. We can say that the functors $\operatorname{Spf}(A)$ in A and $\Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ in $\mathfrak{X}$ define an *equivalence* between the category of admissible rings and the opposite category of formal affine schemes (T, I, 1.2).

(10.2.3). As a particular case of (10.2.2), note that, for $f \in A$, the canonical injection of the formal affine scheme induced by $\mathfrak{X}$ on $\mathfrak{D}(f)$ corresponds to the continuous canonical homomorphism $A \to A_{\{f\}}$. Under the hypotheses of Proposition (10.2.2), let h be an element of B, and g an element of A that is a multiple of $\phi(h)$; we then have $\psi(\mathfrak{D}(g)) \subset \mathfrak{D}(h)$; the restriction of u to $\mathfrak{D}(g)$, considered as a morphism from $\mathfrak{D}(g)$ to $\mathfrak{D}(h)$, is the unique morphism v making the diagram

$$\begin{tikzcd} \mathfrak{D}(g) \arrow[r, "v"] \arrow[d] & \mathfrak{D}(h) \arrow[d] \\ \mathfrak{X} \arrow[r, "u"] & \mathfrak{Y} \end{tikzcd}$$

commutate.

This morphism corresponds to the unique continuous homomorphism $\phi' : B_{\{h\}} \to A_{\{g\}}$ (**0**, 7.6.7) making the diagram

$$\begin{tikzcd} A \arrow[d] & B \arrow[l, "\phi"'] \arrow[d] \\ A_{\{g\}} & B_{\{h\}} \arrow[l, "\phi'"'] \end{tikzcd}$$

commutate.

10.3. Ideals of definition for a formal affine scheme

(10.3.1). Let A be an admissible ring, $\mathfrak{J}$ an open ideal of A, and $\mathfrak{X}$ the formal affine scheme $\operatorname{Spf}(A)$. Let $(\mathfrak{J}_\lambda)$ be the set of those ideals of definition for A that are contained in $\mathfrak{J}$; then $\widetilde{\mathfrak{J}}/\widetilde{\mathfrak{J}}_\lambda$ is a sheaf of ideals of $\widetilde{A}/\widetilde{\mathfrak{J}}_\lambda$. Denote by $\mathfrak{J}^\Delta$ the projective limit of the sheaves on $\mathfrak{X}$ induced by $\widetilde{\mathfrak{J}}/\widetilde{\mathfrak{J}}_\lambda$, which is identified with a *sheaf of ideals* of $\mathscr{O}_{\mathfrak{X}}$ (**0**, 3.2.6). For every $f \in A$, $\Gamma(\mathfrak{D}(f), \mathfrak{J}^\Delta)$ is the projective limit of the $S_f^{-1}\mathfrak{J}/S_f^{-1}\mathfrak{J}_\lambda$, or, in other words, can be identified with the open ideal $\mathfrak{J}_{\{f\}}$ of the ring $A_{\{f\}}$ (**0**, 7.6.9), and, in particular, $\Gamma(\mathfrak{X}, \mathfrak{J}^\Delta) = \mathfrak{J}$; we conclude (the $\mathfrak{D}(f)$ forming a basis for the topology of $\mathfrak{X}$) that

$$\mathfrak{J}^\Delta|\mathfrak{D}(f) = (\mathfrak{J}_{\{f\}})^\Delta. \tag{10.3.1.1}$$

(10.3.2). With the notation of (10.3.1), for all $f \in A$, the canonical map from $A_{\{f\}} = \Gamma(\mathfrak{D}(f), \mathscr{O}_{\mathfrak{X}})$ **I** | 184
to $\Gamma(\mathfrak{D}(f), (\widetilde{A}/\widetilde{\mathfrak{J}})|\mathfrak{X}) = S_f^{-1}A/S_f^{-1}\mathfrak{J}$ is *surjective* and has $\Gamma(\mathfrak{D}(f), \mathfrak{J}^\Delta) = \mathfrak{J}_{\{f\}}$ as its kernel (**0**, 7.6.9); these maps thus define a *surjective* continuous homomorphism, said to be *canonical*, from the sheaf of topological rings $\mathscr{O}_{\mathfrak{X}}$ to the sheaf of discrete rings $(\widetilde{A}/\widetilde{\mathfrak{J}})|\mathfrak{X}$, whose kernel is $\mathfrak{J}^\Delta$; this homomorphism is none other than $\widetilde{\phi}$ (10.2.1), where ϕ is the continuous homomorphism $A \to A/\mathfrak{J}$; the morphism $({}^a\phi, \widetilde{\phi}) : \operatorname{Spec}(A/\mathfrak{J}) \to \mathfrak{X}$ of formal affine schemes (where ${}^a\phi$ is the identity homeomorphism from $\mathfrak{X}$ to itself) is also said to be *canonical*. We thus have, according to the above, a *canonical isomorphism*

$$\mathscr{O}_{\mathfrak{X}}/\mathfrak{J}^\Delta \simeq (\widetilde{A}/\widetilde{\mathfrak{J}})|\mathfrak{X}. \tag{10.3.2.1}$$

It is clear (since $\Gamma(\mathfrak{X}, \mathfrak{J}^\Delta) = \mathfrak{J}$) that the map $\mathfrak{J} \to \mathfrak{J}^\Delta$ is *strictly increasing*; according to the above, for $\mathfrak{J} \subset \mathfrak{J}'$, the sheaf $\mathfrak{J}'^\Delta/\mathfrak{J}^\Delta$ is canonically isomorphic to $\widetilde{\mathfrak{J}}'/\widetilde{\mathfrak{J}} = (\mathfrak{J}'/\mathfrak{J})^\sim$.

(10.3.3). The hypotheses and notation being the same as in (10.3.1), we say that a sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_{\mathfrak{X}}$ is a *sheaf of ideals of definition* for $\mathfrak{X}$ (or an *ideal sheaf of definition* for $\mathfrak{X}$) if, for all $x \in \mathfrak{X}$, there exists an open neighbourhood of x of the form $\mathfrak{D}(f)$, where $f \in A$, such that $\mathscr{J}|\mathfrak{D}(f)$ is of the form $\mathfrak{H}^\Delta$, where $\mathfrak{H}$ is an ideal of definition for $A_{\{f\}}$.

Proposition (10.3.4). — *For all $f \in A$, each sheaf of ideals of definition for $\mathfrak{X}$ induces a sheaf of ideals of definition for $\mathfrak{D}(f)$.*

PROOF. This follows from (10.3.1.1). □

Proposition (10.3.5). — *If A is an admissible ring, then every sheaf of ideals of definition for $\mathfrak{X} = \operatorname{Spf}(A)$ is of the form $\mathfrak{J}^\Delta$, where $\mathfrak{J}$ is a uniquely determined ideal of definition for A.*

PROOF. Let $\mathscr{J}$ be a sheaf of ideals of definition of $\mathfrak{X}$; by hypothesis, and since $\mathfrak{X}$ is quasi-compact, there are finitely-many elements $f_i \in A$ such that the $\mathfrak{D}(f_i)$ cover $\mathfrak{X}$ and such that $\mathscr{J}|\mathfrak{D}(f_i) = \mathfrak{H}_i^\Delta$, where $\mathfrak{H}_i$ is an ideal of definition for $A_{\{f_i\}}$. For each i, there exists an open ideal $\mathfrak{K}_i$ of A such that $(\mathfrak{K}_i)_{\{f_i\}} = \mathfrak{H}_i$ (**0**, 7.6.9); let $\mathfrak{K}$ be an ideal of definition for A containing all the $\mathfrak{K}_i$. The canonical image of $\mathscr{J}/\mathfrak{K}^\Delta$ in the structure sheaf $(A/\mathfrak{K})^\sim$ of $\operatorname{Spec}(A/\mathfrak{K})$ (10.3.2) is thus such that its restriction to $\mathfrak{D}(f_i)$ is equal to its restriction to $(\mathfrak{K}_i/\mathfrak{K})^\sim$; we conclude that this canonical image is a *quasi-coherent* sheaf on $\operatorname{Spec}(A/\mathfrak{K})$, and so is of the form $(\mathfrak{J}/\mathfrak{K})^\sim$, where $\mathfrak{J}$ is an ideal of definition for A containing $\mathfrak{K}$ (1.4.1), whence $\mathscr{J} = \mathfrak{J}^\Delta$ (10.3.2); in addition, since for each i there exists an integer n_i such that $\mathfrak{H}_i^{n_i} \subset \mathfrak{K}_{\{f_i\}}$, we will have, by setting n to be the largest of the n_i, that $(\mathscr{J}/\mathfrak{K}^\Delta)^n = 0$, and, as a result (10.3.2), that $((\mathfrak{J}/\mathfrak{K})^\sim)^n = 0$, whence finally that $(\mathfrak{J}/\mathfrak{K})^n = 0$ (1.3.13), which proves that $\mathfrak{J}$ is an ideal of definition for A (**0**, 7.1.4). □

Proposition (10.3.6). — *Let A be an adic ring, and $\mathfrak{J}$ an ideal of definition for A such that $\mathfrak{J}/\mathfrak{J}^2$ is an $(A/\mathfrak{J})$-module of finite type. For any integer $n > 0$, we then have $(\mathfrak{J}^\Delta)^n = (\mathfrak{J}^n)^\Delta$.*

PROOF. For all $f \in A$, we have (since $\mathfrak{J}^n$ is an open ideal)

$$(\Gamma(\mathfrak{D}(f), \mathfrak{J}^\Delta))^n = (\mathfrak{J}_{\{f\}})^n = (\mathfrak{J}^n)_{\{f\}} = \Gamma(\mathfrak{D}(f^n), (\mathfrak{J}^n)^\Delta)$$

by (10.3.1.1) and (**0**, 7.6.12). The result then follows from the fact that $(\mathfrak{J}^\Delta)^n$ is associated to the presheaf $U \mapsto (\Gamma(U, \mathfrak{J}^\Delta))^n$ (**0**, 4.1.6), since the $\mathfrak{D}(f)$ form a basis for the topology of $\mathfrak{X}$. □ **I** | 185

(10.3.7). We say that a family $(\mathscr{J}_\lambda)$ of sheaves of ideals of definition for $\mathfrak{X}$ is a *fundamental system of sheaves of ideals of definition* if each sheaf of ideals of definition for $\mathfrak{X}$ contains one of the $\mathscr{J}_\lambda$; since $\mathscr{J}_\lambda = \mathfrak{J}_\lambda^\Delta$, it is equivalent to say that the $\mathfrak{J}_\lambda$ form a *fundamental system of neighbourhoods of* 0 in A. Let (f_α) be a family of elements of A such that the $\mathfrak{D}(f_\alpha)$ cover $\mathfrak{X}$. If $(\mathscr{J}_\lambda)$ is a filtered decreasing family of sheaves of ideals of $\mathscr{O}_\mathfrak{X}$ such that, for each α, the family $(\mathscr{J}_\lambda|\mathfrak{D}(f_\alpha))$ is a fundamental system of sheaves of ideals of definition for $\mathfrak{D}(f_\alpha)$, then $(\mathscr{J}_\lambda)$ is a fundamental system of sheaves of ideals of definition for $\mathfrak{X}$. Indeed, for each sheaf of ideals of definition $\mathscr{J}$ for $\mathfrak{X}$, there is a finite cover of $\mathfrak{X}$ by $\mathfrak{D}(f_i)$ such that, for each i, $\mathscr{J}_{\lambda_i}|\mathfrak{D}(f_i)$ is a sheaf of ideals of definition for $\mathfrak{D}(f_i)$ that is contained in $\mathscr{J}|\mathfrak{D}(f_i)$. If μ is an index such that $\mathscr{J}_\mu \subset \mathscr{J}_{\lambda_i}$ for all i, then it follows from (10.3.3) that $\mathscr{J}_\mu$ is a sheaf of ideals of definition for $\mathfrak{X}$, evidently contained in $\mathscr{J}$, whence our claim.

10.4. Formal preschemes and morphisms of formal preschemes

(10.4.1). Given a topologically ringed space $\mathfrak{X}$, we say that an open $U \subset \mathfrak{X}$ is a *formal affine open* (resp. a *formal adic affine open*, resp. a *formal Noetherian affine open*) if the topologically ringed space induced on U by $\mathfrak{X}$ is a formal affine scheme (resp. a scheme whose ring is adic, resp. adic and Noetherian).

Definition (10.4.2). — A *formal prescheme* is a topologically ringed space $\mathfrak{X}$ which admits a formal affine open neighbourhood for each point. We say that the formal prescheme $\mathfrak{X}$ is adic (resp. locally Noetherian) if each point of $\mathfrak{X}$ admits a formal adic (resp. Noetherian) open neighbourhood. We say that $\mathfrak{X}$ is Noetherian if it is locally Noetherian and its underlying space is quasi-compact (and hence Noetherian).

Proposition (10.4.3). — *If $\mathfrak{X}$ is a formal prescheme (resp. a locally Noetherian formal prescheme), then the formal affine (resp. Noetherian affine) open sets form a basis for the topology of $\mathfrak{X}$.*

PROOF. This follows from Definition (10.4.2) and Proposition (10.1.4) by taking into account that, if A is an adic Noetherian ring, then so too is $A_{\{f\}}$ for all $f \in A$ (**0**, 7.6.11). □

Corollary (10.4.4). — *If $\mathfrak{X}$ is a formal prescheme (resp. a locally Noetherian formal prescheme, resp. a Noetherian formal prescheme), then the topologically ringed space induced on each open set of $\mathfrak{X}$ is also a formal prescheme (resp. a locally Noetherian formal prescheme, resp. a Noetherian formal prescheme).*

Definition (10.4.5). — Given two formal preschemes $\mathfrak{X}$ and $\mathfrak{Y}$, a morphism (of formal preschemes) from $\mathfrak{X}$ to $\mathfrak{Y}$ is a morphism (ψ, θ) of topologically ringed spaces such that, for all $x \in \mathfrak{X}$, $\theta_x^\sharp$ is a local homomorphism $\mathscr{O}_{\psi(x)} \to \mathscr{O}_x$.

It is immediate that the composition of any two morphisms of formal preschemes is again a morphism of formal preschemes; the formal preschemes thus form a *category*, and we denote by $\operatorname{Hom}(\mathfrak{X}, \mathfrak{Y})$ the set of morphisms from a formal prescheme $\mathfrak{X}$ to a formal prescheme $\mathfrak{Y}$.

If U is an open subset of $\mathfrak{X}$, then the canonical injection into $\mathfrak{X}$ of the formal prescheme induced I | 186
on U by $\mathfrak{X}$ is a morphism of formal preschemes (and in fact a *monomorphism* of topologically ringed spaces (**0**, 4.1.1)).

Proposition (10.4.6). — *Let $\mathfrak{X}$ be a formal prescheme, and $\mathfrak{S} = \operatorname{Spf}(A)$ a formal affine scheme. There exists a canonical bijective equivalence between the morphisms from the formal prescheme $\mathfrak{X}$ to the formal prescheme $\mathfrak{S}$ and the continuous homomorphisms from the ring A to the topological ring $\Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$.*

Proof. The proof is similar to that of (2.2.4), by replacing "homomorphism" with "continuous homomorphism", "affine open" with "formal affine open", and by using Proposition (10.2.2) instead of Theorem (1.7.3); we leave the details to the reader. ☐

(10.4.7). Given a formal prescheme $\mathfrak{S}$, we say that the data of a formal prescheme $\mathfrak{X}$ and a morphism $\phi : \mathfrak{X} \to \mathfrak{S}$ defines a formal prescheme $\mathfrak{X}$ *over* $\mathfrak{S}$ or an *formal $\mathfrak{S}$-prescheme*, ϕ being called the *structure morphism* of the $\mathfrak{S}$-prescheme $\mathfrak{X}$. If $\mathfrak{S} = \operatorname{Spf}(A)$, where A is an admissible ring, then we also say that the formal $\mathfrak{S}$-prescheme $\mathfrak{X}$ is a *formal A-prescheme* or a formal prescheme *over A*. An arbitrary formal prescheme can be considered as a formal prescheme over $\mathbf{Z}$ (equipped with the discrete topology).

If $\mathfrak{X}$ and $\mathfrak{Y}$ are formal $\mathfrak{S}$-preschemes, then we say that a morphism $u : \mathfrak{X} \to \mathfrak{Y}$ is a $\mathfrak{S}$-*morphism* if the diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[rr, "u"] \arrow[dr] & & \mathfrak{Y} \arrow[dl] \\ & \mathfrak{S} & \end{tikzcd}$$

(where the downwards arrows are the structure morphisms) is commutative. With this definition, the formal $\mathfrak{S}$-preschemes (for some fixed $\mathfrak{S}$) form a *category*. We denote by $\operatorname{Hom}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$ the set of $\mathfrak{S}$-morphisms from a formal $\mathfrak{S}$-prescheme $\mathfrak{X}$ to a formal $\mathfrak{S}$-prescheme $\mathfrak{Y}$. When $\mathfrak{S} = \operatorname{Spf}(A)$, we sometimes say *$A$-morphism* instead of *$\mathfrak{S}$-morphism*.

(10.4.8). Since every affine scheme can be considered as a formal affine scheme (10.1.2), every (usual) prescheme can be considered as a formal prescheme. In addition, it follows from Definition (10.4.5) that, for the *usual* preschemes, the morphisms (resp. S-morphisms) of *formal* preschemes coincide with the morphisms (resp. S-morphisms) defined in §2.

10.5. Sheaves of ideals of definition for formal preschemes

(10.5.1). Let $\mathfrak{X}$ be a formal prescheme; we say that an $\mathscr{O}_{\mathfrak{X}}$-ideal $\mathscr{J}$ is a *sheaf of ideals of definition* (or an *ideal sheaf of definition*) for $\mathfrak{X}$ if every $x \in \mathfrak{X}$ has a formal affine open neighbourhood U such that $\mathscr{J}|U$ is a sheaf of ideals of definition for the formal affine scheme induced on U by $\mathfrak{X}$ (10.3.3); by (10.3.1.1) and Proposition (10.4.3), for each open $V \subset \mathfrak{X}$, $\mathscr{J}|V$ is then a sheaf of ideals of definition for the formal prescheme induced on V by $\mathfrak{X}$.

We say that a family $(\mathscr{J}_\lambda)$ of sheaves of ideals of definition for $\mathfrak{X}$ is a *fundamental system of* I | 187
sheaves of ideals of definition if there exists a cover (U_α) of $\mathfrak{X}$ by formal affine open sets such that, for each α, the family of the $\mathscr{J}_\lambda|U_\alpha$ is a fundamental system of sheaves of ideals of definition (10.3.6) for the formal affine scheme induced on U_α by $\mathfrak{X}$. It follows from the last remark of (10.3.7) that, when $\mathfrak{X}$ is a formal affine scheme, this definition coincides with the definition given in (10.3.7). For an open subset V of $\mathfrak{X}$, the restrictions $\mathscr{J}_\lambda|V$ then form a fundamental system of sheaves of ideals of definition for the formal prescheme induced on V, according to (10.3.1.1). If $\mathfrak{X}$ is a *locally Noetherian*

formal prescheme, and $\mathscr{J}$ is a sheaf of ideals of definition for $\mathfrak{X}$, then it follows from Proposition (10.3.6) that the powers $\mathscr{J}^n$ form a fundamental system of sheaves of ideals of definition for $\mathfrak{X}$.

(10.5.2). Let $\mathfrak{X}$ be a formal prescheme, and $\mathscr{J}$ a sheaf of ideals of definition for $\mathfrak{X}$. Then the ringed space $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{J})$ is a (usual) *prescheme*, which is affine (resp. locally Noetherian, resp. Noetherian) when $\mathfrak{X}$ is a formal affine scheme (resp. a locally Noetherian formal scheme, resp. a Noetherian formal scheme); we can reduce to the affine case, and then the proposition has already been proved in (10.3.2). In addition, if $\theta : \mathscr{O}_{\mathfrak{X}} \to \mathscr{O}_{\mathfrak{X}}/\mathscr{J}$ is the canonical homomorphism, then $u = (1_{\mathfrak{X}}, \theta)$ is a *morphism* (said to be *canonical*) of formal preschemes $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{J}) \to (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$, because, again, this was proved in the affine case (10.3.2), to which it is immediately reduced.

Proposition (10.5.3). — *Let $\mathfrak{X}$ be a formal prescheme, and $(\mathscr{J}_\lambda)$ a fundamental system of sheaves of ideals of definition for $\mathfrak{X}$. Then the sheaf of topological rings $\mathscr{O}_{\mathfrak{X}}$ is the projective limit of the pseudo-discrete sheaves of rings* (**0**, 3.8.1) *$\mathscr{O}_{\mathfrak{X}}/\mathscr{J}_\lambda$.*

PROOF. Since the topology of $\mathfrak{X}$ admits a basis of formal quasi-compact affine open sets (10.4.3), we reduce to the affine case, where the proposition is a consequence of Proposition (10.3.5), (10.3.2), and Definition (10.1.1). □

It is not true that any formal prescheme admits a sheaf of ideals of definition. However:

Proposition (10.5.4). — *Let $\mathfrak{X}$ be a locally Noetherian formal prescheme. There exists a largest sheaf of ideals of definition $\mathscr{T}$ for $\mathfrak{X}$; this is the unique sheaf of ideals of definition $\mathscr{J}$ such that the prescheme $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{J})$ is reduced. If $\mathscr{J}$ is a sheaf of ideals of definition for $\mathfrak{X}$, then $\mathscr{T}$ is the inverse image under $\mathscr{O}_{\mathfrak{X}} \to \mathscr{O}_{\mathfrak{X}}/\mathscr{J}$ of the nilradical of $\mathscr{O}_{\mathfrak{X}}/\mathscr{J}$.*

PROOF. Suppose first that $\mathfrak{X} = \operatorname{Spf}(A)$, where A is an adic Noetherian ring. The existence and the properties of $\mathscr{T}$ follow immediately from Propositions (10.3.5) and (5.1.1), taking into account **Err$_{\text{II}}$** the existence and the properties of the largest ideal of definition for A ((**0**, 7.1.6) and (**0**, 7.1.7)).

To prove the existence and the properties of $\mathscr{T}$ in the general case, it suffices to show that, if $U \supset V$ are two Noetherian formal affine open subsets of X, then the largest sheaf of ideals of definition $\mathscr{T}_U$ for U induces the largest sheaf of ideals of definition $\mathscr{T}_V$ for V; but as $(V, (\mathscr{O}_{\mathfrak{X}}|V)/(\mathscr{T}_U|V))$ is reduced, this follows from the above. □

We denote by $\mathfrak{X}_{\text{red}}$ the (usual) reduced prescheme $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{T})$.

Corollary (10.5.5). — *Let $\mathfrak{X}$ be a locally Noetherian formal prescheme, and $\mathscr{T}$ the largest sheaf of ideals of definition for $\mathfrak{X}$; for each open subset V of $\mathfrak{X}$, $\mathscr{T}|V$ is the largest sheaf of ideals of definition for the formal prescheme induced on V by $\mathfrak{X}$.*

Proposition (10.5.6). — *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be formal preschemes, $\mathscr{J}$ (resp. $\mathscr{K}$) be a sheaf of ideals of definition for $\mathfrak{X}$ (resp. $\mathfrak{Y}$), and $f : \mathfrak{X} \to \mathfrak{Y}$ a morphism of formal preschemes.*

(i) *If $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subset \mathscr{J}$, then there exists a unique morphism $f' : (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{J}) \to (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{K})$ of usual preschemes making the diagram*

(10.5.6.1)
$$\begin{tikzcd}
(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) \arrow[r, "f"] & (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}) \\
(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{J}) \arrow[u] \arrow[r, "f'"] & (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{K}) \arrow[u]
\end{tikzcd}$$

commutate, where the vertical arrows are the canonical morphisms.

(ii) *Suppose that $\mathfrak{X} = \operatorname{Spf}(A)$ and $\mathfrak{Y} = \operatorname{Spf}(B)$ are formal affine schemes, $\mathscr{J} = \mathfrak{J}^\Delta$ and $\mathscr{K} = \mathfrak{K}^\Delta$, where $\mathfrak{J}$ (resp. $\mathfrak{K}$) is an ideal of definition for A (resp. B), and $f = ({}^a\phi, \widetilde{\phi})$, where $\phi : B \to A$ is a continuous homomorphism; for $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subset \mathscr{J}$ to hold, it is necessary and sufficient that $\phi(\mathfrak{K}) \subset \mathfrak{J}$, and, in this case, f' is then the morphism $({}^a\phi', \widetilde{\phi'})$, where $\phi' : B/\mathfrak{K} \to A/\mathfrak{J}$ is the homomorphism induced from ϕ by passing to quotients.*

PROOF.

(i) If $f = (\psi, \theta)$, then the hypotheses imply that the image under $\theta^\sharp : \psi^*(\mathscr{O}_{\mathfrak{Y}}) \to \mathscr{O}_{\mathfrak{X}}$ of the sheaf of ideals $\psi^*(\mathscr{K})$ of $\psi^*(\mathscr{O}_{\mathfrak{Y}})$ is contained in $\mathscr{J}$ (**0**, 4.3.5). By passing to quotients, we thus obtain from $\theta^\sharp$ a homomorphism of sheaves of rings

$$\omega : \psi^*(\mathscr{O}_{\mathfrak{Y}}/\mathscr{K}) = \psi^*(\mathscr{O}_{\mathfrak{Y}})/\psi^*(\mathscr{K}) \longrightarrow \mathscr{O}_{\mathfrak{X}}/\mathscr{J};$$

furthermore, since, for all $x \in \mathfrak{X}$, $\theta^\sharp_x$ is a *local* homomorphism, so too is ω_x. The morphism of ringed spaces $(\psi, \omega^\flat)$ is thus (2.2.1) the unique morphism f' of ringed spaces whose existence was claimed.

(ii) The canonical functorial correspondence between morphisms of formal affine schemes and continuous homomorphisms of rings (10.2.2) shows that, in the case considered, the relation $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subset \mathfrak{J}$ implies that we have $f' = ({}^a\phi', \widetilde{\phi'})$, where $\phi' : B/\mathfrak{K} \to A/\mathfrak{J}$ is the unique homomorphism making the diagram

(10.5.6.2)
$$\begin{tikzcd} B \arrow[r, "\phi"] \arrow[d] & A \arrow[d] \\ B/\mathfrak{K} \arrow[r, "\phi'"] & A/\mathfrak{J} \end{tikzcd}$$

commutate. The existence of ϕ' thus implies that $\phi(\mathfrak{K}) \subset \mathfrak{J}$. Conversely, if this condition is satisfied, then, denoting by ϕ' the unique homomorphism making the diagram (10.5.6.2) commutate and setting $f' = ({}^a\phi', \widetilde{\phi'})$, it is clear that the diagram (10.5.6.1) is commutative; considering the homomorphisms ${}^a\phi^*(\mathscr{O}_{\mathfrak{Y}}) \to \mathscr{O}_{\mathfrak{X}}$ and ${}^a\phi'^*(\mathscr{O}_{\mathfrak{Y}}/\mathscr{K}) \to \mathscr{O}_{\mathfrak{X}}/\mathscr{J}$ corresponding to f and f' respectively then leads to the fact that this implies the relation $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subset \mathscr{J}$.

□

It is clear that the correspondence $f \mapsto f'$ defined above is *functorial*.

10.6. Formal preschemes as inductive limits of preschemes

(10.6.1). Let $\mathfrak{X}$ be a formal prescheme, and $(\mathscr{J}_\lambda)$ a fundamental system of sheaves of ideals of definition for $\mathfrak{X}$; for each λ, let f_λ be the canonical morphism $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{J}_\lambda) \to \mathfrak{X}$ (10.5.2); for $\mathscr{J}_\mu \subset \mathscr{J}_\lambda$, the canonical morphism $\mathscr{O}_{\mathfrak{X}}/\mathscr{J}_\mu \to \mathscr{O}_{\mathfrak{X}}/\mathscr{J}_\lambda$ defines a canonical morphism $f_{\mu\lambda}$: **I** | 189
$(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{J}_\lambda) \to (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{J}_\mu)$ of (usual) preschemes such that $f_\lambda = f_\mu \circ f_{\mu\lambda}$. The preschemes $X_\lambda = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{J}_\lambda)$ and the morphisms $f_{\mu\lambda}$ thus form (by (10.4.8)) an *inductive system* in the category of formal preschemes.

Proposition (10.6.2). — *With the notation of (10.6.1), the formal prescheme $\mathfrak{X}$ and the morphisms f_λ form an inductive limit (T, I, 1.8) of the system $(X_\lambda, f_{\mu\lambda})$ in the category of formal preschemes.*

PROOF. Let $\mathfrak{Y}$ be a formal prescheme, and, for each index λ, let

$$g_\lambda = (\psi_\lambda, \theta_\lambda) : X_\lambda \longrightarrow \mathfrak{Y}$$

be a morphism such that $g_\lambda = g_\mu \circ f_{\mu\lambda}$ for $\mathscr{J}_\mu \subset \mathscr{J}_\lambda$. This latter condition and the definition of the X_λ imply first of all that the ψ_λ are identical to a single continuous map $\psi : \mathfrak{X} \to \mathfrak{Y}$ of the underlying spaces; in addition, the homomorphisms $\theta^\sharp_\lambda : \psi^*(\mathscr{O}_{\mathfrak{Y}}) \to \mathscr{O}_{X_i} = \mathscr{O}_{\mathfrak{X}}/\mathscr{J}_\lambda$ form a *projective system* of homomorphisms of sheaves of rings. By passing to the projective limit, there is an induced homomorphism $\omega : \psi^*(\mathscr{O}_{\mathfrak{Y}}) \to \varprojlim \mathscr{O}_{\mathfrak{X}}/\mathscr{J}_\lambda = \mathscr{O}_{\mathfrak{X}}$, and it is clear that the morphism $g = (\psi, \omega^\flat)$ of *ringed spaces* is the *unique* morphism making the diagrams

(10.6.2.1)
$$\begin{tikzcd} X_\lambda \arrow[rr, "g_\lambda"] \arrow[dr, "f_\lambda"'] & & \mathfrak{Y} \\ & \mathfrak{X} \arrow[ur, "g"'] & \end{tikzcd}$$

commutative. It remains to prove that g is a morphism of *formal preschemes*; the question is local on $\mathfrak{X}$ and $\mathfrak{Y}$, so we can assume that $\mathfrak{X} = \mathrm{Spf}(A)$ and $\mathfrak{Y} = \mathrm{Spf}(B)$, with A and B admissible rings, and with $\mathscr{J}_\lambda = \mathfrak{J}^\Delta_\lambda$, where $(\mathfrak{J}_\lambda)$ is a fundamental system of ideals of definition for A (10.3.5);

since $A = \varprojlim A/\mathfrak{J}_\lambda$, the existence of a morphism g of formal affine schemes making the diagrams (10.6.2.1) commutate then follows from the bijective correspondence (10.2.2) between morphisms of formal affine schemes and continuous ring homomorphisms, and from the definition of the projective limit. But the uniqueness of g as a morphism of ringed spaces shows that it coincides with the morphism in the beginning of the proof. □

The following proposition establishes, under certain additional conditions, the existence of the inductive limit of a given inductive system of (usual) preschemes in the category of formal preschemes:

Proposition (10.6.3). — *Let $\mathfrak{X}$ be a topological space, and $(\mathscr{O}_i, u_{ji})$ a projective system of sheaves of rings on $\mathfrak{X}$, with $\mathbf{N}$ for its set of indices. Let $\mathscr{J}_i$ be the kernel of $u_{0i} : \mathscr{O}_i \to \mathscr{O}_0$. Suppose that:*

(a) *the ringed space $(\mathfrak{X}, \mathscr{O}_i)$ is a prescheme X_i;*
(b) *for all $x \in \mathfrak{X}$ and all i, there exists an open neighbourhood U_i of x in $\mathfrak{X}$ such that the restriction $\mathscr{J}_i|U_i$ is nilpotent; and* **Err$_{\mathrm{II}}$**
(c) *the homomorphisms u_{ji} are surjective.*

Let $\mathscr{O}_\mathfrak{X}$ be the sheaf of topological rings given by the projective limit of the pseudo-discrete sheaves of rings $\mathscr{O}_i$, and let $u_i : \mathscr{O}_\mathfrak{X} \to \mathscr{O}_i$ be the canonical homomorphism. Then the topologically ringed space $(\mathfrak{X}, \mathscr{O}_\mathfrak{X})$ is a formal prescheme; the homomorphisms u_i are surjective; their kernels $\mathscr{J}^{(i)}$ form a fundamental system of sheaves of ideals of definition for $\mathfrak{X}$, and $\mathscr{J}^{(0)}$ is the projective limit of the sheaves of ideals $\mathscr{J}_i$. **I | 190**

Proof. We first note that, on each stalk, u_{ji} is a surjective homomorphism and *a fortiori* a local homomorphism; thus $v_{ij} = (1_\mathfrak{X}, u_{ji})$ is a morphism of preschemes $X_j \to X_i$ ($i \geqslant j$) (2.2.1). Suppose first that each X_i is an affine scheme of ring A_i. There exists a *ring* homomorphism $\phi_{ji} : A_i \to A_j$ such that $u_{ji} = \widetilde{\phi_{ji}}$ (1.7.3); as a result (1.6.3), the sheaf $\mathscr{O}_j$ is a quasi-coherent $\mathscr{O}_i$-module over X_i (for the external law defined by u_{ji}), associated to A_j considered as an A_i-module by means of ϕ_{ji}. For all $f \in A_i$, let $f' = \phi_{ji}(f)$; by hypothesis, the open sets $D(f)$ and $D(f')$ are identical in $\mathfrak{X}$, and the homomorphism from $\Gamma(D(f), \mathscr{O}_i) = (A_i)_f$ to $\Gamma(D(f), \mathscr{O}_j) = (A_j)_{f'}$ corresponding to u_{ji} is exactly $(\phi_{ji})_f$ (1.6.1). But when we consider A_j as an A_i-module, $(A_j)_{f'}$ is the $(A_i)_f$-module $(A_j)_f$, so we also have $u_{ji} = \widetilde{\phi_{ji}}$, where ϕ_{ji} is now considered as a homomorphism of A_i-*modules*. Then, since u_{ji} is surjective, we conclude that ϕ_{ji} is also surjective (1.3.9), and if $\mathfrak{J}_{ji}$ is the kernel of ϕ_{ji}, then the kernel of u_{ji} is a quasi-coherent $\mathscr{O}_i$-module equal to $\widetilde{\mathfrak{J}_{ji}}$. In particular, we have $\mathscr{J}_i = \widetilde{\mathfrak{J}_i}$, where $\mathfrak{J}_i$ is the kernel of $\phi_{0i} : A_i \to A_0$. Hypothesis (b) implies that $\mathscr{J}_i$ is *nilpotent*: indeed, since $\mathfrak{X}$ is quasi-compact, we can cover $\mathfrak{X}$ by a finite number of open sets U_k such that $(\mathscr{J}_i|U_k)^{n_k} = 0$, and, by setting n to be the largest of the n_k, we have $\mathscr{J}_i^n = 0$. We thus conclude that $\mathfrak{J}_i$ is nilpotent (1.3.13). Then the ring $A = \varprojlim A_i$ is admissible (**0**, 7.2.2), the canonical homomorphism $\phi_i : A \to A_i$ is surjective, and its kernel $\mathfrak{J}^{(i)}$ is equal to the projective limit of the $\mathfrak{J}_{ik}$ for $k \geqslant i$; the $\mathfrak{J}^{(i)}$ form a fundamental system of neighbourhoods of 0 in A. The claims of Proposition (10.6.3) follow in this case from (10.1.1) and (10.3.2), with $(\mathfrak{X}, \mathscr{O}_\mathfrak{X})$ being $\operatorname{Spf}(A)$.

Again, in this particular case, we note that, if $f = (f_i)$ is an element of the projective limit $A = \varprojlim A_i$, then all the open sets $D(f_i)$ (which are affine open sets in X_i) can be identified with the open subset $\mathfrak{D}(f)$ of $\mathfrak{X}$, and the prescheme induced on $\mathfrak{D}(f)$ by X_i thus being identified with the affine scheme $\operatorname{Spec}((A_i)_{f_i})$.

In the general case, we remark first that, for every quasi-compact open subset U of $\mathfrak{X}$, each of the $\mathscr{J}_i|U$ is nilpotent, as shown by the above reasoning. We will show that, for every $x \in \mathfrak{X}$, there exists an open neighbourhood U of x in $\mathfrak{X}$ which is an *affine open set* for *all* the X_i. Indeed, we take U to be an affine open set for X_0, and observe that $\mathscr{O}_{X_0} = \mathscr{O}_{X_i}/\mathscr{J}_i$. Since, by the above, $\mathscr{J}_i|U$ is nilpotent, U is also an affine open set for each X_i, by Proposition (5.1.9). This being so, for each U satisfying the preceding conditions, the study of the affine case as above shows that $(U, \mathscr{O}_X|U)$ is a formal prescheme whose $\mathscr{J}^{(i)}|U$ form a fundamental system of sheaves of ideals of definition, and $\mathscr{J}^{(0)}|U$ is the projective limit of the $\mathscr{J}_i|U$; whence the conclusion. □

Corollary (10.6.4). — *Suppose that, for $i \geqslant j$, the kernel of u_{ji} is $\mathscr{J}_i^{j+1}$ and that $\mathscr{J}_1/\mathscr{J}_1^2$ is of finite type over $\mathscr{O}_0 = \mathscr{O}_1/\mathscr{J}_1$. Then $\mathfrak{X}$ is an adic formal prescheme, and if $\mathscr{J}^{(n)}$ is the kernel of $\mathscr{O}_\mathfrak{X} \to \mathscr{O}_n$,* **I | 191**

then $\mathscr{J}^{(n)} = \mathscr{J}^{n+1}$ and $\mathscr{J}/\mathscr{J}^2$ is isomorphic to $\mathscr{J}_1$. If, in addition, X_0 is locally Noetherian (resp. Noetherian), then $\mathfrak{X}$ is locally Noetherian (resp. Noetherian).

PROOF. Since the underlying spaces of $\mathfrak{X}$ and X_0 are the same, the question is local, and we can suppose that all the X_i are affine; taking into account the fact that $\mathscr{J}_{ij} = \widetilde{\mathfrak{J}_{ji}}$ (with the notation of Proposition (10.3.6)), we can immediately reduce the problem to the corresponding claims of Proposition (**0**, 7.2.7) and Corollary (**0**, 7.2.8), by noting that $\mathfrak{J}_1/\mathfrak{J}_1^2$ is then an A_0-module of finite type (1.3.9). □

In particular, *every locally Noetherian formal prescheme* $\mathfrak{X}$ is the inductive limit of a sequence (X_n) of locally Noetherian (usual) preschemes satisfying the conditions of Proposition (10.3.6) and Corollary (10.6.4): it suffices to consider a sheaf of ideals of definition $\mathscr{J}$ for $\mathfrak{X}$ (10.5.4) and to take $X_n = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{J}^{n+1})$ ((10.5.1) and Proposition (10.6.2)).

Corollary (10.6.5). — *Let A be an admissible ring. For the formal affine scheme $\mathfrak{X} = \operatorname{Spf}(A)$ to be Noetherian, it is necessary and sufficient for A to be adic and Noetherian.*

PROOF. The condition is evidently sufficient. Conversely, suppose that $\mathfrak{X}$ is Noetherian, and let $\mathfrak{J}$ be an ideal of definition for A, and $\mathscr{J} = \mathfrak{J}^{\Delta}$ the corresponding sheaf of ideals of definition for $\mathfrak{X}$. The (usual) preschemes $X_n = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{J}^{n+1})$ are then affine and Noetherian, so the rings $A_n = A/\mathfrak{J}^{n+1}$ are Noetherian (6.1.3), whence we conclude that $\mathfrak{J}/\mathfrak{J}^2$ is an $A/\mathfrak{J}$-module of finite type. Since the $\mathscr{J}^n$ form a fundamental system of sheaves of ideals of definition for $\mathfrak{X}$ (10.5.1), we have $\mathscr{O}_{\mathfrak{X}} = \varprojlim \mathscr{O}_{\mathfrak{X}}/\mathscr{J}^n$ (10.5.3); we thus conclude (10.1.3) that A is topologically isomorphic to $\varprojlim A/\mathfrak{J}^n$, which is adic and Noetherian (**0**, 7.2.8). □

Remark (10.6.6). — With the notation of Proposition (10.6.3), let $\mathscr{F}_i$ be an $\mathscr{O}_i$-module, and suppose we are given, for $i \geqslant i$, a v_{ij}-morphism $\theta_{ji} : \mathscr{F}_i \to \mathscr{F}_j$ such that $\theta_{kj} \circ \theta_{ji} = \theta_{ki}$ for $k \leqslant j \leqslant i$. Since the underlying continuous map of v_{ij} is the identity, θ_{ji} is a homomorphism of sheaves of abelian groups to the space $\mathfrak{X}$; in addition, if $\mathscr{F}$ is the projective limit of the projective system $(\mathscr{F}_i)$ of sheaves of abelian groups, then the fact that the θ_{ji} are v_{ij}-morphisms lets us define an $\mathscr{O}_{\mathfrak{X}}$-module structure on $\mathscr{F}$ by passing to the projective limit; when equipped with this structure, we say that $\mathscr{F}$ is the *projective limit* (with respect to the θ_{ji}) of the system of $\mathscr{O}_i$-modules $(\mathscr{F}_i)$. In the particular case where $v_{ij}^*(\mathscr{F}_i) = \mathscr{F}_j$ and θ_{ji} is the *identity*, we say that $\mathscr{F}$ is the projective limit of a system $(\mathscr{F}_i)$ such that $v_{ij}^*(\mathscr{F}_i) = \mathscr{F}_j$ for $j \leqslant i$ (without mentioning the θ_{ji}).

(10.6.7). Let $\mathfrak{X}$ and $\mathfrak{Y}$ be formal preschemes, $\mathscr{J}$ (resp. $\mathscr{K}$) a sheaf of ideals of definition for $\mathfrak{X}$ (resp. $\mathfrak{Y}$), and $f : \mathfrak{X} \to \mathfrak{Y}$ a morphism such that $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subset \mathscr{J}$. We then have, for every integer $n > 0$, that $f^*(\mathscr{K}^n)\mathscr{O}_{\mathfrak{X}} = (f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}})^n \subset \mathscr{J}^n$; f thus induces (10.5.6) a morphism of (usual) preschemes $f_n : X_n \to Y_n$ by setting $X_n = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{J}^{n+1})$ and $Y_n = (\mathscr{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{K}^{n+1})$, and it immediately follows from the definitions that the diagrams

(10.6.7.1)
$$\begin{tikzcd} X_m \arrow[r, "f_m"] \arrow[d] & Y_m \arrow[d] \\ X_n \arrow[r, "f_n"] & Y_n \end{tikzcd}$$

commute for $m \leqslant n$; in other words, (f_n) is an *inductive system* of morphisms. **I** | 192

(10.6.8). Conversely, let (X_n) (resp. (Y_n)) be an inductive system of (usual) preschemes satisfying conditions (b) and (c) of Proposition (10.6.3), and let $\mathfrak{X}$ (resp. $\mathfrak{Y}$) be its inductive limit. By definition of the inductive limit, each sequence (f_n) of morphisms $X_n \to Y_n$ forms an inductive system that admits an *inductive limit* $f : \mathfrak{X} \to \mathfrak{Y}$, which is the unique morphism of formal preschemes that makes the diagrams

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d] & Y_n \arrow[d] \\ \mathfrak{X} \arrow[r, "f"] & \mathfrak{Y} \end{tikzcd}$$

commute.

Proposition (10.6.9). — *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be locally Noetherian formal preschemes, and $\mathscr{J}$ (resp. $\mathscr{K}$) be a sheaf of ideals of definition for $\mathfrak{X}$ (resp. $\mathfrak{Y}$); the map $f \mapsto (f_n)$ defined in (10.6.7) is a bijection from the set of morphisms $f : \mathfrak{X} \to \mathfrak{Y}$ such that $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subset \mathscr{J}$ to the set of sequences (f_n) of morphisms that make the diagrams (10.6.7.1) commutate.*

PROOF. If f is the inductive limit of this sequence, then it remains to show that $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subset \mathscr{J}$. The statement, being local on $\mathfrak{X}$ and $\mathfrak{Y}$, can be reduced to the case where $\mathfrak{X} = \operatorname{Spf}(A)$ and $\mathfrak{Y} = \operatorname{Spf}(B)$ are affine, with A and B adic Noetherian rings, and with $\mathscr{J} = \mathfrak{J}^\Delta$ and $\mathscr{K} = \mathfrak{K}^\Delta$, where $\mathfrak{J}$ (resp. $\mathfrak{K}$) is an ideal of definition for A (resp. B). We then have that $X_n = \operatorname{Spec}(A_n)$ and $Y_n = \operatorname{Spec}(B_n)$, with $A_n = A/\mathfrak{J}^{n+1}$ and $B_n = B/\mathfrak{K}^{n+1}$, by Proposition (10.3.6) and (10.3.2); $f_n = ({}^a\phi_n, \widetilde{\phi_n})$, where the homomorphisms $\phi_n : B_n \to A_n$ forms a projective system, thus $f = ({}^a\phi, \widetilde{\phi})$, and so $f = ({}^a\phi, \widetilde{\phi})$, where $\phi = \varprojlim \phi_n$. The commutativity of the diagram (10.6.7.1) for $m = 0$ then gives the condition $\phi_n(\mathfrak{K}/\mathfrak{K}^{n+1}) \subset \mathfrak{J}/\mathfrak{J}^{n+1}$ for all n, so, by passing to the projective limit, we have $\phi(\mathfrak{K}) \subset \mathfrak{J}$, which implies that $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subset \mathscr{J}$ (10.5.6, ii). □

Corollary (10.6.10). — *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be locally Noetherian formal preschemes, and $\mathscr{T}$ the largest sheaf of ideals of definition for $\mathfrak{X}$ (10.5.4).*

(i) *For every sheaf of ideals of definition $\mathscr{K}$ for $\mathfrak{Y}$ and every morphism $f : \mathfrak{X} \to \mathfrak{Y}$, we have $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subset \mathscr{T}$.*

(ii) *There is a canonical bijective correspondence between* $\operatorname{Hom}(\mathfrak{X}, \mathfrak{Y})$ *and the set of sequences (f_n) of morphisms making the diagrams (10.6.7.1) commute, where $X_n = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{T}^{n+1})$ and $Y_n = (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{K}^{n+1})$.*

PROOF. (ii) follows immediately from (i) and Proposition (10.6.9). To prove (i), we can reduce to the case where $\mathfrak{X} = \operatorname{Spf}(A)$ and $\mathfrak{Y} = \operatorname{Spf}(B)$, with A and B Noetherian, and with $\mathscr{T} = \mathfrak{T}^\Delta$ and $\mathscr{K} = \mathfrak{K}^\Delta$, where $\mathfrak{T}$ is the largest ideal of definition for A and $\mathfrak{K}$ is an ideal of definition for B. Let $f = ({}^a\phi, \widetilde{\phi})$, where $\phi : B \to A$ is a continuous homomorphism; since the elements of $\mathfrak{K}$ are topologically nilpotent (**0**, 7.1.4, ii), so too are those of $\phi(\mathfrak{K})$, and so $\phi(\mathfrak{K}) \subset \mathfrak{T}$, since $\mathfrak{T}$ is the set of topologically nilpotent elements of A (**0**, 7.1.6); hence, by Proposition (10.5.6, ii), we are done. □

Corollary (10.6.11). — *Let $\mathfrak{S}$, $\mathfrak{X}$, $\mathfrak{Y}$ be locally Noetherian formal preschemes, and $f : \mathfrak{X} \to \mathfrak{S}$ and $g : \mathfrak{Y} \to \mathfrak{S}$ the morphisms that make $\mathfrak{X}$ and $\mathfrak{Y}$ formal $\mathfrak{S}$-preschemes. Let $\mathscr{J}$ (resp. $\mathscr{K}$, $\mathscr{L}$) be a sheaf of ideals of definition for $\mathfrak{S}$ (resp. $\mathfrak{X}$, $\mathfrak{Y}$), and suppose that $f^*(\mathscr{J})\mathscr{O}_{\mathfrak{X}} \subset \mathscr{K}$ and $g^*(\mathscr{J})\mathscr{O}_{\mathfrak{Y}} = \mathscr{L}$; set $S_n = (\mathfrak{S}, \mathscr{O}_{\mathfrak{S}}/\mathscr{J}^{n+1})$, $X_n = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{K}^{n+1})$, and $Y_n = (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{L}^{n+1})$. Then there exists a canonical bijective correspondence between* $\operatorname{Hom}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$ *and the set of sequences (u_n) of S_n-morphisms $u_n : X_n \to Y_n$* **I** | 193
making the diagrams (10.6.7.1) commute.

PROOF. For each $\mathfrak{S}$-morphism $u : \mathfrak{X} \to \mathfrak{Y}$, we have by definition that $f = g \circ u$, whence

$$u^*(\mathscr{L})\mathscr{O}_{\mathfrak{X}} = u^*(g^*(\mathscr{J}\mathscr{O}_{\mathfrak{Y}})\mathscr{O}_{\mathfrak{X}} = f^*(\mathscr{J})\mathscr{O}_{\mathfrak{X}} \subset \mathscr{K},$$

and the corollary then follows from Proposition (10.6.9). □

We note that, for $m \leqslant n$, the data of a morphism $f_n : X_n \to Y_n$ determines a unique morphism $f_m : X_m \to Y_m$ making the diagram (10.6.7.1) commutate, since we immediately see that we can reduce to the affine case; we have thus defined a map $\phi_{mn} : \operatorname{Hom}_{S_n}(X_n, Y_n) \to \operatorname{Hom}_{S_m}(X_m, Y_m)$, and the $\operatorname{Hom}_{S_n}(X_n, Y_n)$ form, with the ϕ_{mn}, a *projective system of sets*; Corollary (10.6.11) then says that there is a canonical bijection

$$\operatorname{Hom}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y}) \simeq \varprojlim_n \operatorname{Hom}_{S_n}(X_n, Y_n).$$

10.7. Products of formal preschemes

(10.7.1). Let $\mathfrak{S}$ be a formal prescheme; formal $\mathfrak{S}$-preschemes form a category, and we can define a notion of a *product* of formal $\mathfrak{S}$-preschemes.

Proposition (10.7.2). — *Let $\mathfrak{X} = \operatorname{Spf}(B)$ and $\mathfrak{Y} = \operatorname{Spf}(C)$ be formal affine schemes over a formal affine scheme $\mathfrak{S} = \operatorname{Spf}(A)$. Let $\mathfrak{Z} = \operatorname{Spf}(B\widehat{\otimes}_A C)$, and let p_1 and p_2 be the $\mathfrak{S}$-morphisms corresponding (10.2.2) to the canonical (continuous) A-homomorphisms $\rho : B \to B\widehat{\otimes}_A C$ and $\sigma : C \to B\widehat{\otimes}_A C$; then $(\mathfrak{Z}, p_1, p_2)$ is a product of the formal affine $\mathfrak{S}$-schemes $\mathfrak{X}$ and $\mathfrak{Y}$.*

PROOF. By Proposition (10.4.6), it suffices to check that, if we associate, to each continuous A-homomorphism $\phi : B\widehat{\otimes}_A C \to D$ (where D is an admissible ring which is a topological A-algebra), the pair $(\phi \circ \rho, \phi \circ \sigma)$, then this defines a bijection

$$\operatorname{Hom}_A(B\widehat{\otimes}_A C, D) \simeq \operatorname{Hom}_A(B, D) \times \operatorname{Hom}_A(C, D),$$

which is exactly the universal property of the completed tensor product (**0**, 7.7.6). □

Proposition (10.7.3). — *Given formal $\mathfrak{S}$-preschemes $\mathfrak{X}$ and $\mathfrak{Y}$, their product $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ exists.*

PROOF. The proof is similar to that of Theorem (3.2.6), replacing affine schemes (resp. affine open sets) by formal affine schemes (resp. formal affine open sets), and replacing Proposition (3.2.2) by Proposition (10.7.2). □

All the formal properties of the product of preschemes ((3.2.7) and (3.2.8), (3.3.1) and (3.3.12)) hold true without modification for the product of formal preschemes.

(10.7.4). Let $\mathfrak{S}$, $\mathfrak{X}$, and $\mathfrak{Y}$ be formal preschemes, and let $f : \mathfrak{X} \to \mathfrak{S}$ and $g : \mathfrak{Y} \to \mathfrak{S}$ be morphisms. Suppose that there exist, in $\mathfrak{S}$, $\mathfrak{X}$, and $\mathfrak{Y}$ respectively, three fundamental systems of sheaves of ideals of definitions $(\mathscr{J}_\lambda)$, $(\mathscr{K}_\lambda)$, and $(\mathscr{L}_\lambda)$, all having the same set I of indices, and such that $f^*(\mathscr{J}_\lambda)\mathscr{O}_{\mathfrak{X}} \subset \mathscr{K}_\lambda$ and $g^*(\mathscr{J}_\lambda)\mathscr{O}_{\mathfrak{Y}} \subset \mathscr{L}_\lambda$ for all λ. Set $S_\lambda = (\mathfrak{S}, \mathscr{O}_{\mathfrak{S}}/\mathscr{J}_\lambda)$, $X_\lambda = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{K}_\lambda)$, and $Y_\lambda = (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{L}_\lambda)$; for $\mathscr{J}_\mu \subset \mathscr{J}_\lambda$, $\mathscr{K}_\mu \subset \mathscr{K}_\lambda$, and $\mathscr{L}_\mu \subset \mathscr{L}_\lambda$, note that S_λ (resp. X_λ, Y_λ) is a closed subprescheme of S_μ (resp. X_μ, Y_μ) that has the *same* underlying space (10.6.1). Since $S_\lambda \to S_\mu$ is **I** | 194
a monomorphism of preschemes, we see that the products $X_\lambda \times_{S_\lambda} Y_\lambda$ and $X_\lambda \times_{S_\mu} Y_\lambda$ are identical (3.2.4), since $X_\lambda \times_{S_\mu} Y_\lambda$ can be identified with a closed subprescheme of $X_\mu \times_{S_\mu} Y_\mu$ that has the *same* underlying space (4.3.1). With this in mind, the product $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ is the *inductive limit* of the usual preschemes $X_\lambda \times_{S_\lambda} Y_\lambda$: indeed, as we see in Proposition (10.6.2), we can reduce to the case where $\mathfrak{S}$, $\mathfrak{X}$, and $\mathfrak{Y}$ are formal affine schemes. Taking into account both Proposition (10.5.6, ii) and the hypotheses on the fundamental systems of sheaves of ideals of definition for $\mathfrak{S}$, $\mathfrak{X}$, and $\mathfrak{Y}$, we immediately see that our claim follows from the definition of the completed tensor product of two algebras (**0**, 7.7.1).

Furthermore, let $\mathfrak{Z}$ be a formal $\mathfrak{S}$-prescheme, $(\mathscr{M}_\lambda)$ a fundamental system of ideals of definition for $\mathfrak{Z}$ having I for its set of indices, and let $u : \mathfrak{Z} \to \mathfrak{X}$ and $v : \mathfrak{Z} \to \mathfrak{Y}$ be $\mathfrak{S}$-morphisms such that $u^*(\mathscr{K}_\lambda)\mathscr{O}_{\mathfrak{Z}} \subset \mathscr{M}_\lambda$ and $v^*(\mathscr{L}_\lambda)\mathscr{O}_{\mathfrak{Z}} \subset \mathscr{M}_\lambda$ for all λ. If we set $Z_\lambda = (\mathfrak{Z}, \mathscr{O}_{\mathfrak{Z}}/\mathscr{M}_\lambda)$, and if $u_\lambda : Z_\lambda \to X_\lambda$ and $v_\lambda : Z_\lambda \to Y_\lambda$ are the S_λ-morphisms corresponding to u and v (10.5.6), then we immediately have that $(u, v)_{\mathfrak{S}}$ is the inductive limit of the S_λ-morphisms $(u_\lambda, v_\lambda)_{S_\lambda}$.

The ideas of this section apply, in particular, to the case where $\mathfrak{S}$, $\mathfrak{X}$, and $\mathfrak{Y}$ are locally Noetherian, taking the systems consisting of the powers of a sheaf of ideals of definition (10.5.1) as the fundamental systems of sheaves of ideals of definition . However, we note that $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ is not necessarily locally Noetherian (see however (10.13.5)).

10.8. Formal completion of a prescheme along a closed subset

(10.8.1). Let X be a *locally Noetherian* (usual) prescheme, and X' a closed subset of the underlying space of X; we denote by Φ the set of *coherent* sheaves of ideals $\mathscr{J}$ of $\mathscr{O}_X$ such that the support of $\mathscr{O}_X/\mathscr{J}$ is X'. The set Φ is nonempty ((5.2.1), (4.1.4), (6.1.1)); we order it by the relation $\supset$.

Lemma (10.8.2). — *The ordered set Φ is filtered; if X is Noetherian, then, for all $\mathscr{J}_0 \in \Phi$, the set of powers $\mathscr{J}_0^n$ ($n > 0$) is cofinal in Φ.*

Proof. If $\mathscr{J}_1$ and $\mathscr{J}_2$ are in Φ, and if we set $\mathscr{J} = \mathscr{J}_1 \cap \mathscr{J}_2$, then $\mathscr{J}$ is coherent since $\mathscr{O}_X$ is coherent ((6.1.1) and (**0**, 5.3.4)), and we have $\mathscr{J}_x = (\mathscr{J}_1)_x \cap (\mathscr{J}_2)_x$ for all $x \in X$, whence $\mathscr{J}_x = \mathscr{O}_x$ for $x \notin X'$, and $\mathscr{J}_x \neq \mathscr{O}_x$ for $x \in X'$, which proves that $\mathscr{J} \in \Phi$. On the other hand, if X is Noetherian and if $\mathscr{J}_0$ and $\mathscr{J}$ are in Φ, then there exists an integer $n > 0$ such that $\mathscr{J}_0^n(\mathscr{O}_X/\mathscr{J}) = 0$ (9.3.4), which implies that $\mathscr{J}_0^n \subset \mathscr{J}$. □

(10.8.3). Now let $\mathscr{F}$ be a *coherent* $\mathscr{O}_X$-module; for all $\mathscr{J} \in \Phi$, we have that $\mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{J})$ is a coherent $\mathscr{O}_X$-module (9.1.1) with support contained in X', and we will usually identify it with its restriction to X'. When $\mathscr{J}$ varies over Φ, these sheaves form a *projective system* of sheaves of abelian groups.

Definition (10.8.4). — Given a closed subset X' of a locally Noetherian prescheme X and a coherent $\mathscr{O}_X$-module $\mathscr{F}$, we define the *completion of $\mathscr{F}$ along X'*, denoted by $\mathscr{F}_{/X'}$ (or $\widehat{\mathscr{F}}$ when there is little chance of confusion), to be the restriction to X' of the sheaf $\varprojlim_\Phi(\mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{J}))$; we say that its sections over X' are the *formal sections of $\mathscr{F}$ along X'*. I | 195

It is immediate that, for every open $U \subset X$, we have $(\mathscr{F}|U)_{/(U\cap X')} = (\mathscr{F}_{/X'})|(U \cap X')$.

By passing to the projective limit, it is clear that $(\mathscr{O}_X)_{/X'}$ is a sheaf of rings, and that $\mathscr{F}_{/X'}$ can be considered as an $(\mathscr{O}_X)_{/X'}$-module. In addition, since there exists a basis for the topology of X' consisting of quasi-compact open sets, we can consider $(\mathscr{O}_X)_{/X'}$ (resp. $\mathscr{F}_{/X'}$) as a *sheaf of topological rings* (resp. of *topological groups*), the projective limit of the *pseudo-discrete* sheaves of rings (resp. groups) $\mathscr{O}_X/\mathscr{F}$ (resp. $\mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{F}) = \mathscr{F}/\mathscr{J}\mathscr{F}$), and, by passing to the projective limit, $\mathscr{F}_{/X'}$ then becomes a *topological $(\mathscr{O}_X)_{/X'}$-module* ((**0**, 3.8.1) and (**0**, 3.8.2)); recall that, for every *quasi-compact* open $U \subset X$, $\Gamma(U \cap X', (\mathscr{O}_X)_{/X'})$ (resp. $\Gamma(U \cap X', \mathscr{F}_{/X'})$) is then the projective limit of the discrete rings (resp. groups) $\Gamma(U, \mathscr{O}_X/\mathscr{J})$ (resp. $\Gamma(U, \mathscr{F}/\mathscr{J}\mathscr{F})$).

Now, if $u : \mathscr{F} \to \mathscr{G}$ is a homomorphism of $\mathscr{O}_X$-modules, then there are canonically induced homomorphisms $u_{\mathscr{J}} : \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{J}) \to \mathscr{G} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{J})$ for all $\mathscr{J} \in \Phi$, and these homomorphisms form a projective system. By passing to the projective limit and restricting to X', these give a continuous $(\mathscr{O}_X)_{/X'}$-homomorphism $\mathscr{F}_{/X'} \to \mathscr{G}_{/X'}$, denoted $u_{/X'}$ or $\widehat{u}$, and called the *completion* of the homomorphism u along X'. It is clear that, if $v : \mathscr{G} \to \mathscr{H}$ is a second homomorphism of $\mathscr{O}_X$-modules, then we have $(v \circ u)_{/X'} = (v_{/X'}) \circ (u_{/X'})$, hence $\mathscr{F}_{/X'}$ is a *covariant additive functor* in $\mathscr{F}$ from the category of coherent $\mathscr{O}_X$-modules to the category of topological $(\mathscr{O}_X)_{/X'}$-modules.

Proposition (10.8.5). — *The support of $(\mathscr{O}_X)_{/X'}$ is X'; the topologically ringed space $(X', (\mathscr{O}_X)_{/X'})$ is a locally Noetherian formal prescheme, and, if $\mathscr{J} \in \Phi$, then $\mathscr{J}_{/X'}$ is a sheaf of ideals of definition for this formal prescheme. If $X = \operatorname{Spec}(A)$ is an affine scheme with Noetherian ring A, $\mathscr{J} = \widetilde{\mathfrak{J}}$ for some ideal $\mathfrak{J}$ of A, and $X' = V(\mathfrak{J})$, then $(X', (\mathscr{O}_X)_{/X'})$ is canonically identified with $\operatorname{Spf}(\widehat{A})$, where $\widehat{A}$ is the separated completion of A with respect to the $\mathfrak{J}$-preadic topology.*

Proof. We can evidently reduce to proving the latter claim. We know (**0**, 7.3.3) that the separated completion $\widehat{\mathfrak{J}}$ of $\mathfrak{J}$ with respect to the $\mathfrak{J}$-preadic topology can be identified with the ideal $\mathfrak{J}\widehat{A}$ of $\widehat{A}$, where $\widehat{A}$ is the Noetherian $\widehat{\mathfrak{J}}$-adic ring such that $\widehat{A}/\widehat{\mathfrak{J}}^n = A/\mathfrak{J}^n$ (**0**, 7.2.6). This latter equation shows that the open prime ideals of $\widehat{A}$ are the ideals $\widehat{\mathfrak{p}} = \mathfrak{p}\widehat{\mathfrak{J}}$, where $\mathfrak{p}$ is a prime ideal of A containing $\mathfrak{J}$, and that we have $\widehat{\mathfrak{p}} \cap A = \mathfrak{p}$, and hence $\operatorname{Spf}(\widehat{A}) = X'$. Since $\mathscr{O}_X/\mathscr{J}^n = (A/\mathfrak{J}^n)^\sim$, the proposition follows immediately from the definitions. □

We say that the formal prescheme defined above is the *completion of X along X'*, and we denote it by $X_{/X'}$ or $\widehat{X}$ when there is little chance of confusion. When we take $X' = X$, we can set $\mathscr{J} = 0$, and we thus have $X_{/X} = X$.

It is clear that, if U is a subprescheme induced on an open subset of X, then $U_{/(U\cap X')}$ is canonically identified with the formal subprescheme induced on $X_{/X'}$ by the open subset $U \cap X'$ of X'.

Corollary (10.8.6). — *The (usual) prescheme $\widehat{X}_{\text{red}}$ is the unique reduced subprescheme of X having X' as its underlying space (5.2.1). For $\widehat{X}$ to be Noetherian, it is necessary and sufficient for $\widehat{X}_{\text{red}}$ to be Noetherian, and it suffices that X be Noetherian.*

Proof. Since $\widehat{X}_{\text{red}}$ is determined locally (10.5.4), we can assume that X is an affine scheme of some Noetherian ring A; with the notation of Proposition (10.8.5), the ideal $\mathfrak{T}$ of topologically nilpotent elements of $\widehat{A}$ is the inverse image under the canonical map $\widehat{A} \to \widehat{A}/\widehat{\mathfrak{J}} = A/\mathfrak{J}$ of the nilradical of $A/\mathfrak{J}$ (**0**, 7.1.3), so $\widehat{A}/\mathfrak{T}$ is isomorphic to the quotient of $A/\mathfrak{J}$ by its nilradical. The first claim then follows from Propositions (10.5.4) and (5.1.1). If $\widehat{X}_{\text{red}}$ is Noetherian, then so too is its underlying space X', and so the $X'_n = \operatorname{Spec}(\mathscr{O}_X/\mathscr{J}^n)$ are Noetherian (6.1.2), and thus so too is $\widehat{X}$ (10.6.4); the converse is immediate, by Proposition (6.1.2). □

(10.8.7). The canonical homomorphisms $\mathscr{O}_X \to \mathscr{O}_X/\mathscr{J}$ (for $\mathscr{J} \in \Phi$) form a projective system, and give, by passing to the projective limit, a homomorphism of sheaves of rings $\theta : \mathscr{O}_X \to \psi_*((\mathscr{O}_X)_{/X'}) = \varprojlim_\Phi(\mathscr{O}_X/\mathscr{J})$, where ψ denotes the canonical injection $X' \to X$ of the underlying spaces. We denote by i (or i_X) the morphism (said to be *canonical*)

$$(\psi, \theta) : X_{/X'} \longrightarrow X$$

of ringed spaces.

By taking tensor products, for every coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphisms $\mathscr{O}_X \to \mathscr{O}_X/\mathscr{J}$ give homomorphisms $\mathscr{F} \to \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{J})$ of $\mathscr{O}_X$-modules which form a projective system, and thus give, by passing to the projective limit, a canonical functorial homomorphism $\gamma : \mathscr{F} \to \psi_*(\mathscr{F}_{/X'})$ of $\mathscr{O}_X$-modules.

Proposition (10.8.8). —

(i) *The functor $\mathscr{F}_{/X'}$ (in $\mathscr{F}$) is exact.*
(ii) *The functorial homomorphism $\gamma^\sharp : i^*(\mathscr{F}) \to \mathscr{F}_{/X'}$ of $(\mathscr{O}_X)_{/X'}$-modules is an isomorphism.*

Proof.

(i) It suffices to prove that, if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of coherent $\mathscr{O}_X$-modules, and if U is an affine open subset of X with Noetherian ring A, then the sequence

$$0 \longrightarrow \Gamma(U \cap X', \mathscr{F}'_{/X'}) \longrightarrow \Gamma(U \cap X', \mathscr{F}_{/X'}) \longrightarrow \Gamma(U \cap X', \mathscr{F}''_{/X'}) \longrightarrow 0$$

is exact. We have that $\mathscr{F}|U = \widetilde{M}$, $\mathscr{F}'|U = \widetilde{M'}$, and $\mathscr{F}''|U = \widetilde{M''}$, where M, M', and M'' are three A-modules of finite type such that the sequence $0 \to M' \to M \to M'' \to 0$ is exact ((1.5.1) and (1.3.11)); let $\mathscr{J} \in \Phi$, and let $\mathfrak{J}$ be an ideal of A such that $\mathscr{J}|U = \widetilde{\mathfrak{J}}$. We then have

$$\Gamma(U \cap X', \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X/\mathscr{J}^n) = M \otimes_A (A/\mathfrak{J}^n)$$

(1.3.12); so, by definition of the projective limit, we have

$$\Gamma(U \cap X', \mathscr{F}_{/X'}) = \varprojlim_n (M \otimes_A (A/\mathfrak{J}^n)) = \widehat{M},$$

the separated completion of M with respect to the $\mathfrak{J}$-preadic topology, and similarly

$$\Gamma(U \cap X', \mathscr{F}'_{/X'}) = \widehat{M'},\ \Gamma(U \cap X', \mathscr{F}''_{/X'}) = \widehat{M''};$$

our claim then follows, since A is Noetherian, and since the functor $\widehat{M}$ in M is exact on the category of A-modules of finite type (**0**, 7.3.3). **I** | 197

(ii) The question is local, so we can assume that we have an exact sequence $\mathscr{O}_X^m \to \mathscr{O}_X^n \to \mathscr{F} \to 0$ (**0**, 5.3.2); since $\gamma^\sharp$ is functorial, and the functors $i^*(\mathscr{F})$ and $\mathscr{F}_{/X'}$ are right exact (by (i) and (**0**, 4.3.1)), we have the commutative diagram

(10.8.8.1)
$$\begin{tikzcd} i^*(\mathscr{O}_X^m) \arrow[r] \arrow[d, "\gamma^\sharp"'] & i^*(\mathscr{O}_X^n) \arrow[r] \arrow[d, "\gamma^\sharp"'] & i^*(\mathscr{F}) \arrow[r] \arrow[d, "\gamma^\sharp"'] & 0 \\ (\mathscr{O}_X^m)_{/X'} \arrow[r] & (\mathscr{O}_X^n)_{/X'} \arrow[r] & \mathscr{F}_{/X'} \arrow[r] & 0 \end{tikzcd}$$

whose rows are exact. Furthermore, the functors $i^*(\mathscr{F})$ and $\mathscr{F}_{/X'}$ commute with finite direct sums ((**0**, 3.2.6) and (**0**, 4.3.2)), and we thus reduce to proving our claim for $\mathscr{F} = \mathscr{O}_X$. We have $i^*(\mathscr{O}_X) = (\mathscr{O}_X)_{/X'} = \mathscr{O}_{\widehat{X}}$ (**0**, 4.3.4), and that $\gamma^\sharp$ is a homomorphism of $\mathscr{O}_{\widehat{X}}$*-modules*; so it suffices to check that $\gamma^\sharp$ sends the unit section of $\mathscr{O}_{\widehat{X}}$ over an open subset of X' to itself, which is immediate, and shows, in this case, that $\gamma^\sharp$ is the identity. □

Corollary (10.8.9). — *The morphism $i : X_{/X'} \to X$ of ringed spaces is flat.*

Proof. This follows from (**0**, 6.7.3) and Proposition (10.8.8, i). □

Corollary (10.8.10). — *If $\mathscr{F}$ and $\mathscr{G}$ are coherent $\mathscr{O}_X$-modules, then there exist canonical functorial (in $\mathscr{F}$ and $\mathscr{G}$) isomorphisms*

(10.8.10.1)
$$(\mathscr{F}_{/X'}) \otimes_{(\mathscr{O}_X)_{/X'}} (\mathscr{G}_{/X'}) \simeq (\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_{/X'},$$

(10.8.10.2)
$$(\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}))_{/X'} \simeq \mathscr{H}om_{(\mathscr{O}_X)_{/X'}}(\mathscr{F}_{/X'},\mathscr{G}_{/X'}).$$

Proof. This follows from the canonical identification of $i^*(\mathscr{F})$ with $\mathscr{F}_{/X'}$; the existence of the first isomorphism is then a result which holds for all morphisms of ringed spaces (**0**, 4.3.3.1), and the second is a result which holds for all flat morphisms (**0**, 6.7.6), by Corollary (10.8.9). □

Proposition (10.8.11). — *For every coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $\Gamma(X,\mathscr{F}) \to \Gamma(X',\mathscr{F}_{/X'})$ induced by $\mathscr{F} \to \mathscr{F}_{/X'}$ has kernel consisting of the zero sections in some neighbourhood of X'.*

Proof. It follows from the definition of $\mathscr{F}_{/X'}$ that the canonical image of such a section is zero. Conversely, if $s \in \Gamma(X,\mathscr{F})$ has a zero image in $\Gamma(X',\mathscr{F}_{/X'})$, then it suffices to see that every $x \in X'$ admits a neighbourhood in X in which s is zero, and we can thus reduce to the case where $X = \operatorname{Spec}(A)$ is affine, A Noetherian, $X' = V(\mathfrak{J})$ for some ideal $\mathfrak{J}$ of A, and $\mathscr{F} = \widetilde{M}$ for some A-module M of finite type. Then $\Gamma(X',\mathscr{F}_{/X'})$ is the separated completion $\widehat{M}$ of M for the $\mathfrak{J}$-preadic topology, and the homomorphism $\Gamma(X,\mathscr{F}) \to \Gamma(X',\mathscr{F}_{/X'})$ is the canonical homomorphism $M \to \widehat{M}$. We know (**0**, 7.3.7) that the kernel of this homomorphism is the set of the $z \in M$ killed by an element of $1+\mathfrak{J}$. So we have $(1+f)s = 0$ for some $f \in \mathfrak{J}$; for every $x \in X'$ we have $(1_x + f_x)s_x = 0$, and, since $1_x + f_x$ is invertible in $\mathscr{O}_x$ ($\mathfrak{J}_x\mathscr{O}_x$ being contained in the maximal ideal of $\mathscr{O}_x$), we have $s_x = 0$, which proves the proposition. □

Corollary (10.8.12). — *The support of $\mathscr{F}_{/X'}$ is equal to* $\operatorname{Supp}(\mathscr{F}) \cap X'$.

Proof. It is clear that $\mathscr{F}_{/X'}$ is an $(\mathscr{O}_X)_{/X'}$-module of finite type ((10.8.8, ii) and (**0**, 5.2.4)), so **I** | 198
its support is closed (**0**, 5.2.2) and evidently contained in $\operatorname{Supp}(\mathscr{F}) \cap X'$. To show that it is equal to the latter set, we immediately reduce to proving that the equation $\Gamma(X',\mathscr{F}_{/X'}) = 0$ implies that $\operatorname{Supp}(\mathscr{F}) \cap X' = \varnothing$; this follows from Proposition (10.8.11) and Theorem (1.4.1). □

Corollary (10.8.13). — *Let $u : \mathscr{F} \to \mathscr{G}$ be a homomorphism of coherent $\mathscr{O}_X$-modules. For $u_{/X'} : \mathscr{F}_{/X'} \to \mathscr{G}_{/X'}$ to be zero, it is necessary and sufficient for u to be zero on a neighbourhood of X'.*

Proof. By Proposition (10.8.8, ii), $u_{/X'}$ can be identified with $i^*(u)$, so, if we consider u as a section over X of the sheaf $\mathscr{H} = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$, then $u_{/X'}$ is the section over X' of $i^*(\mathscr{H}) = \mathscr{H}_{/X'}$ to which it canonically corresponds ((10.8.10.2) and (**0**, 4.4.6)). It thus suffices to apply Proposition (10.8.11) to the coherent $\mathscr{O}_X$-module $\mathscr{H}$. □

Corollary (10.8.14). — *Let $u : \mathscr{F} \to \mathscr{G}$ be a homomorphism of coherent $\mathscr{O}_X$-modules. For $u_{/X'}$ to be a monomorphism (resp. an epimorphism), it is necessary and sufficient for u to be a monomorphism (resp. an epimorphism) on a neighbourhood of X'.*

PROOF. Let $\mathscr{P}$ and $\mathscr{N}$ be the cokernel and kernel (respectively) of u, so that we have the exact sequence $0 \to \mathscr{N} \xrightarrow{v} \mathscr{F} \xrightarrow{u} \mathscr{G} \xrightarrow{w} \mathscr{P} \to 0$, hence (10.8.8, i) the exact sequence

$$0 \longrightarrow \mathscr{N}_{/X'} \xrightarrow{v_{/X'}} \mathscr{F}_{/X'} \xrightarrow{u_{/X'}} \mathscr{G}_{/X'} \xrightarrow{w_{/X'}} \mathscr{P}_{/X'} \longrightarrow 0.$$

If $u_{/X'}$ is a monomorphism (resp. an epimorphism), then we have $v_{/X'} = 0$ (resp. $w_{/X'} = 0$), so there exists a neighbourhood of X' on which $v = 0$ (resp. $w = 0$) by Corollary (10.8.13). □

10.9. Extension of morphisms to completions

(10.9.1). Let X and Y be locally Noetherian (usual) preschemes, $f : X \to Y$ a morphism, and X' (resp. Y') a closed subset of the underlying space X (resp. Y) such that $f(X') \subset Y'$. Let $\mathscr{J}$ (resp. $\mathscr{K}$) be a sheaf of ideals of $\mathscr{O}_X$ (resp. $\mathscr{O}_Y$) such that the support of $\mathscr{O}_X/\mathscr{J}$ (resp. $\mathscr{O}_Y/\mathscr{K}$) is X' (resp. Y') and $f^*(\mathscr{K})\mathscr{O}_X \subset \mathscr{J}$; we note that there always exist such sheaves of ideals, since, for example, we can take $\mathscr{J}$ to be the largest sheaf of ideals of $\mathscr{O}_X$ defining a subprescheme of X with underlying space X' (5.2.1), and the hypothesis $f(X') \subset Y'$ implies that $f^*(\mathscr{K})\mathscr{O}_X \subset \mathscr{J}$ (5.2.4). For every integer $n > 0$ we have $f^*(\mathscr{K}^n)\mathscr{O}_X \subset \mathscr{J}^n$ (**0**, 4.3.5); as a result (4.4.6), if we set $X'_n = (X', \mathscr{O}_X/\mathscr{J}^{n+1})$ and $Y'_n = (Y', \mathscr{O}_Y/\mathscr{K}^{n+1})$, then f induces a morphism $f_n : X'_n \to Y'_n$, and it is immediate that the f_n form an inductive system. We denote its inductive limit (10.6.8) by $\widehat{f} : X_{/X'} \to Y_{/Y'}$, and we say (by abuse of language) that $\widehat{f}$ is the *extension of f to the completions of X and Y along X' and Y'*. It can be checked immediately that this morphism does not depend on the choice of sheaves of ideals $\mathscr{J}$ and $\mathscr{K}$ satisfying the above conditions. It suffices to consider the case where X and Y are Noetherian affine schemes with rings A and B (respectively); then $\mathscr{J} = \widetilde{\mathfrak{J}}$ and $\mathscr{K} = \widetilde{\mathfrak{K}}$, where $\mathfrak{J}$ (resp. $\mathfrak{K}$) is an ideal of A (resp. B), f corresponds to a ring homomorphism $\phi : B \to A$ such that $\phi(\mathfrak{K}) \subset \mathfrak{J}$ ((4.4.6) and (1.7.4)); *$\widehat{f}$ is then the morphism corresponding* (10.2.2) *to the continuous homomorphism $\widehat{\phi} : \widehat{B} \to \widehat{A}$*, where $\widehat{A}$ (resp. $\widehat{B}$) is the separated completion of A (resp. B) with respect to the $\mathfrak{J}$-preadic (resp. $\mathfrak{K}$-preadic) topology (10.6.8); we know that, if we replace $\mathscr{J}$ by another sheaf of ideals $\mathscr{J}' = \widetilde{\mathfrak{J}'}$ **I** | 199 such that the support of $\mathscr{O}_X/\mathscr{J}'$ is X', then the $\mathfrak{J}$-preadic and $\mathfrak{J}'$-preadic topologies on A are the same (10.8.2).

We note that, by definition, the continuous map $X' \to Y'$ of the underlying spaces of $X_{/X'}$ and $Y_{/Y'}$ corresponding to $\widehat{f}$ is exactly the restriction to X' of f.

(10.9.2). It follows immediately from the above definition that the diagram of morphisms of ringed spaces

$$\begin{tikzcd}
\widehat{X} \arrow[r, "\widehat{f}"] \arrow[d, "i_X"'] & \widehat{Y} \arrow[d, "i_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

commutes, with the vertical arrows being the canonical morphisms (10.8.7).

(10.9.3). Let Z be a third prescheme, $g : Y \to Z$ a morphism, and Z' a closed subset of Z such that $g(Y') \subset Z'$. If $\widehat{g}$ denotes the completion of the morphism g along Y' and Z', then it immediately follows from (10.9.1) that we have $(g \circ f)^\wedge = \widehat{g} \circ \widehat{f}$.

Proposition (10.9.4). — *Let X and Y be locally Noetherian S-preschemes, with Y of finite type over S. Let f and g be S-morphisms from X to Y such that $f(X') \subset Y'$ and $g(X') \subset Y'$. For $\widehat{f} = \widehat{g}$ to hold, it is necessary and sufficient for f and g to coincide on a neighbourhood of X'.*

PROOF. The condition is evidently sufficient (even without the finiteness hypothesis on Y). To see that it is necessary, we remark first that the hypothesis $\widehat{f} = \widehat{g}$ implies that $f(x) = g(x)$ for all $x \in X'$. Also, since the questions is local, we can assume that X and Y are affine open neighbourhoods of x and $y = f(x) = g(x)$ respectively (with Noetherian rings), that S is affine, and that $\Gamma(Y, \mathscr{O}_Y)$ is a $\Gamma(S, \mathscr{O}_S)$-algebra of finite type (6.3.3). Then f and g correspond to $\Gamma(S, \mathscr{O}_S)$-homomorphisms ρ and σ (respectively) from $\Gamma(Y, \mathscr{O}_Y)$ to $\Gamma(X, \mathscr{O}_X)$ (1.7.3), and, by hypothesis, the extensions by continuity of these homomorphisms to the separated completion of $\Gamma(Y, \mathscr{O}_Y)$ are the same. We conclude from Proposition (10.8.11) that, for every section $s \in \Gamma(Y, \mathscr{O}_Y)$, the sections $\rho(s)$ and $\sigma(s)$ coincide on a neighbourhood (depending on s) of X'; since $\Gamma(Y, \mathscr{O}_Y)$ is an algebra of finite type over $\Gamma(S, \mathscr{O}_S)$, we have that there exists a neighbourhood V of X' such that $\rho(s)$ and $\sigma(s)$ coincide on V for *every* section $s \in \Gamma(Y, \mathscr{O}_Y)$. If $h \in \Gamma(X, \mathscr{O}_X)$ is such that $D(h)$ is a neighbourhood of x contained in V, then we conclude from the above and from Theorem (1.4.1, d) that f and g coincide on $D(h)$. □

Proposition (10.9.5). — *Under the hypotheses of (10.9.1), for every coherent $\mathscr{O}_Y$-module $\mathscr{G}$, there exists a canonical functorial isomorphism of $(\mathscr{O}_X)_{/X'}$-modules*

$$(f^*(\mathscr{G}))_{/X'} \simeq \widehat{f}^*(\mathscr{G}_{/Y'}).$$

PROOF. If we canonically identify $(f^*(\mathscr{G}))_{/X'}$ with $i_X^*(f^*(\mathscr{G}))$, and $\widehat{f}^*(\mathscr{G}_{/Y'})$ with $\widehat{f}^*(i_Y^*(\mathscr{G}))$ (10.8.8), then the proposition follows immediately from the commutativity of the diagram in (10.9.2). □

(10.9.6). Now let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module, and let $\mathscr{G}$ be a coherent $\mathscr{O}_Y$-module. If $u : \mathscr{G} \to \mathscr{F}$ is an f-morphism, then it corresponds to an $\mathscr{O}_X$-homomorphism $u^\sharp : f^*(\mathscr{G}) \to \mathscr{F}$, thus, by completion, to a continuous $(\mathscr{O}_X)_{/X'}$-homomorphism $(u^\sharp)_{/X'} : (f^*(\mathscr{G}))_{/X'} \to \mathscr{F}_{/X'}$, and, by Proposition (10.9.5), there exists a unique $\widehat{f}$-morphism $v : \mathscr{G}_{/Y'} \to \mathscr{F}_{/X'}$ such that $v^\sharp = (u^\sharp)_{/X'}$. If we consider the triples $(\mathscr{F}, X, X')$ ($\mathscr{F}$ being a coherent $\mathscr{O}_X$-module, and X' a closed subset of X) as a *category*, with the morphisms $(\mathscr{F}, X, X') \to (\mathscr{G}, Y, Y')$ consisting of a morphism of preschemes $f : X \to Y$ such that $f(X') \subset Y'$ and an f-morphism $u : \mathscr{G} \to \mathscr{F}$, then we can say that $(X_{/X'}, \mathscr{F}_{/X'})$ is a *functor* in $(\mathscr{F}, X, X')$ with values in the category of pairs $(\mathfrak{Z}, \mathscr{H})$ consisting of a locally Noetherian formal prescheme $\mathfrak{Z}$ and an $\mathscr{O}_\mathfrak{Z}$-module $\mathscr{H}$, with the morphisms of the latter category being the pairs consisting of a morphism g of formal preschemes and a g-morphism. I | 200

Proposition (10.9.7). — *Let S, X, and Y be locally Noetherian preschemes, $g : X \to S$ and $h : Y \to S$ morphisms, S' a closed subset of S, and X' (resp. Y') a closed subset of X (resp. Y) such that $g(X') \subset S'$ (resp. $h(Y') \subset S'$); let $Z = X \times_S Y$; suppose that Z is locally Noetherian, and let $Z' = p^{-1}(X') \cap q^{-1}(Y')$, where p and q are the projections of $X \times_S Y$. With these conditions, the completion $Z_{/Z'}$ can be identified with the product $(X_{/X'}) \times_{S_{/S'}} (Y_{/Y'})$ of formal $S_{/S'}$-preschemes, where the structure morphisms are identified with $\widehat{g}$ and $\widehat{h}$, and the projections with $\widehat{p}$ and $\widehat{q}$.*

PROOF. It is immediate that the question is local for S, X, and Y, and we thus reduce to the case where $S = \operatorname{Spec}(A)$, $X = \operatorname{Spec}(B)$, $Y = \operatorname{Spec}(C)$, $S' = V(\mathfrak{J})$, $X' = V(\mathfrak{K})$, and $Y' = V(\mathfrak{L})$, with $\mathfrak{J}$, $\mathfrak{K}$, and $\mathfrak{L}$ ideals such that $\phi(\mathfrak{J}) \subset \mathfrak{K}$ and $\psi(\mathfrak{J}) \subset \mathfrak{L}$, where we denote by ϕ and ψ the homomorphisms $A \to B$ and $A \to C$ which correspond to g and h (respectively). We know that $Z = \operatorname{Spec}(B \otimes_A C)$ and that $Z' = V(\mathfrak{M})$, where $\mathfrak{M}$ is the ideal $\operatorname{Im}(\mathfrak{K} \otimes_A C) + \operatorname{Im}(B \otimes_A \mathfrak{L})$. The conclusion follows (10.7.2) from the fact that the completed tensor product $(\widehat{B} \otimes_{\widehat{A}} \widehat{C})^\wedge$ (where $\widehat{A}$, $\widehat{B}$, and $\widehat{C}$ are, respectively, the separated completions of A, B, and C with respect to the $\mathfrak{J}$-, $\mathfrak{K}$-, and $\mathfrak{L}$-preadic topologies) is the separated completion of the tensor product $B \otimes_A C$ with respect to the $\mathfrak{M}$-preadic topology (**0**, 7.7.2). □

In addition, we note that, if T is a locally Noetherian S-prescheme, $u : T \to X$ and $v : T \to Y$ both S-morphisms, and T' a closed subset of T such that $u(T') \subset X'$ and $v(T') \subset Y'$, then the extension to the completion $((u, v)_S)^\wedge$ can be identified with $(\widehat{u}, \widehat{v})_{S_{/S'}}$.

Corollary (10.9.8). — *Let X and Y be locally Noetherian S-preschemes such that $X \times_S Y$ is locally Noetherian; let S' be a closed subset of S, and X' (resp. Y') a closed subset of X (resp. Y) whose image in S is contained in S'. For every S-morphism $f : X \to Y$ such that $f(X') \subset Y'$, the graph morphism $\Gamma_{\widehat{f}}$ can be identified with the extension $(\Gamma_f)^\wedge$ of the graph morphism of f.*

Corollary (10.9.9). — *Let X and Y be locally Noetherian preschemes, $f : X \to Y$ a morphism, Y' a closed subset of Y, and $X' = f^{-1}(Y')$. Then the prescheme $X_{/X'}$ can be identified, by the commutative diagram*

$$\begin{tikzcd} X & X_{/X'} \arrow[l] \arrow[d, "\widehat{f}"] \\ Y \arrow[from=u, "f"'] & Y_{/Y'} \arrow[l] \end{tikzcd}$$

with the product $X \times_Y (Y_{/Y'})$ of formal preschemes.

PROOF. It suffices to apply Proposition (10.9.7), replacing S and S' by Y, and X and X' by X. □

Remark (10.9.10). — If S is the sum $X_1 \sqcup X_2$ (3.1), X' the union $X'_1 \cup X'_2$, where X'_i is a closed subset of X_i ($i = 1, 2$), then we have $X_{/X'} = X_{1/X'_1} \sqcup X_{2/X'_2}$. I | 201

10.10. Application to coherent sheaves on formal affine schemes

(10.10.1). In this paragraph, A denotes an *adic Noetherian ring*, and $\mathfrak{J}$ an ideal of definition for A. Let $X = \operatorname{Spec}(A)$, and $\mathfrak{X} = \operatorname{Spf}(A)$, which can be identified with the closed subset $V(\mathfrak{J})$ of X (10.1.2). In addition, Definitions (10.1.2) and (10.8.4) show that the *formal affine scheme* $\mathfrak{X}$ is identical the completion $X_{/\mathfrak{X}}$ of the affine scheme X along the closed subset $\mathfrak{X}$ of its underling space. To every coherent $\mathscr{O}_X$-module $\mathscr{F}$, corresponds an $\mathscr{O}_{\mathfrak{X}}$-module of finite type $\mathscr{F}_{/\mathfrak{X}}$, which is a sheaf of topological modules over the sheaf of topological rings $\mathscr{O}_{\mathfrak{X}}$. Every coherent $\mathscr{O}_X$-module $\mathscr{F}$ is of the form $\widetilde{M}$, where M is an A-module of finite type (1.5.1); we set $(\widetilde{M})_{/X} = M^\Delta$. In addition, if $u : M \to N$ is an A-homomorphism of A-modules of finite type, then it corresponds to a homomorphism $\widetilde{u} : \widetilde{M} \to \widetilde{N}$, and, as a result, to a continuous homomorphism $\widetilde{u}_{/X'} : (\widetilde{M})_{/X'} \to (\widetilde{N})_{/X'}$, which we denote by u^Δ. It is immediate that $(v \circ u)^\Delta = v^\Delta \circ u^\Delta$; we have thus defined a *covariant additive functor* M^Δ from the category of A-modules of finite type to the category of $\mathscr{O}_{\mathfrak{X}}$-modules of finite type. When A is a *discrete* ring, we have $M^\Delta = \widetilde{M}$.

Proposition (10.10.2). —

(i) *M^Δ is an exact functor in M, and there exists a canonical functorial isomorphism of A-modules $\Gamma(\mathfrak{X}, M^\Delta) \simeq M$.*

(ii) *If M and N are A-modules of finite type, then there exist canonical functorial isomorphisms*

$$(M \otimes_A N)^\Delta \simeq M^\Delta \otimes_{\mathscr{O}_{\mathfrak{X}}} N^\Delta, \tag{10.10.2.1}$$

$$(\operatorname{Hom}_A(M, N))^\Delta \simeq \mathscr{H}om_{\mathscr{O}_{\mathfrak{X}}}(M^\Delta, N^\Delta). \tag{10.10.2.2}$$

(iii) *The map $u \mapsto u^\Delta$ is a functorial isomorphism*

$$\operatorname{Hom}_A(M, N) \simeq \operatorname{Hom}_{\mathscr{O}_X}(M^\Delta, N^\Delta). \tag{10.10.2.3}$$

PROOF. The exactness of M^Δ follows from the exactness of the functors $\widetilde{M}$ (1.3.5) and $\mathscr{F}_{/X'}$ (10.8.8). By definition, $\Gamma(X, M^\Delta)$ is the separated completion of the A-module $\Gamma(X, \widetilde{M}) = M$ with respect to the $\mathfrak{J}$-preadic topology; but, since A is complete and M is of finite type, we know (**0**, 7.3.6) that M is separated and complete, which proves (i). The isomorphism (10.10.2.1) (resp. (10.10.2.2)) comes from the composition of the isomorphisms (1.3.12, i) and (10.8.10.1) (resp. (1.3.12, ii) and (10.8.10.2)). Finally, since $\operatorname{Hom}_A(M, N)$ is an A-module of finite type, we can apply (i), which identifies $\Gamma(\mathfrak{X}, (\operatorname{Hom}_A(M, N))^\Delta)$ with $\operatorname{Hom}_A(M, N)$, and we can use (10.10.2.2), which proves that the homomorphism (10.10.2.3) is an isomorphism. □

We deduce from Proposition (10.10.2) a series of results analogous to those of Theorem (1.3.7) and Corollary (1.3.12), whose formulation we leave to the reader.

We note that the exactness property of M^Δ, applied to the exact sequence $0 \to \mathfrak{J} \to A \to A/\mathfrak{J} \to$ I | 202
0, shows that the sheaf of ideals of $\mathscr{O}_{\mathfrak{X}}$ denoted here by $\mathfrak{J}^\Delta$ coincides with the one denoted also by $\mathfrak{J}^\Delta$ in (10.3.1), by (10.3.2).

Proposition (10.10.3). — *Under the hypotheses of (10.10.1), $\mathscr{O}_{\mathfrak{X}}$ is a coherent sheaf of rings.*

Proof. If $f \in A$, then we know that $A_{\{f\}}$ is an adic Noetherian ring (**0**, 7.6.11), and, since the question is local, we reduce (10.1.4) to proving that the kernel of the homomorphism $v : \mathscr{O}_{\mathfrak{X}}^n \to \mathscr{O}_{\mathfrak{X}}$ is an $\mathscr{O}_{\mathfrak{X}}$-module of finite type. We then have $v = u^\Delta$, where u is an A-homomorphism $A^n \to A$ (10.10.2); since A is Noetherian, the kernel of u is of finite type, or, equivalently, we have a homomorphism $A^m \xrightarrow{w} A^n$ such that the sequence $A^m \xrightarrow{w} A^n \xrightarrow{u} A$ is exact. We conclude (10.10.2) that the sequence $\mathscr{O}_{\mathfrak{X}}^m \xrightarrow{w^\Delta} \mathscr{O}_{\mathfrak{X}}^n \xrightarrow{v} \mathscr{O}_{\mathfrak{X}}$ is exact, which proves that the kernel of v is of finite type. □

(10.10.4). With the above notation, set $A_n = A/\mathfrak{J}^{n+1}$, and let S_n be the affine scheme $\operatorname{Spec}(A_n) = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{J}^{n+1})$, with $\mathscr{J} = \mathfrak{J}^\Delta$ the sheaf of ideals of definition for $\mathscr{O}_{\mathfrak{X}}$ corresponding to the ideal $\mathfrak{J}$. Let u_{mn} be the morphism of preschemes $X_m \to X_n$ corresponding to the canonical homomorphism $A_n \to A_m$ for $m \leqslant n$; the formal scheme $\mathfrak{X}$ is the inductive limit of the X_n with respect to the u_{mn} (10.6.3).

Proposition (10.10.5). — *Under the hypothesis of (10.10.1), let $\mathscr{F}$ be an $\mathscr{O}_{\mathfrak{X}}$-module. The following conditions are equivalent:*

(a) *$\mathscr{F}$ is a coherent $\mathscr{O}_{\mathfrak{X}}$-module;*
(b) *$\mathscr{F}$ is isomorphic to the projective limit (10.6.6) of a sequence $(\mathscr{F}_n)$ of coherent $\mathscr{O}_{X_n}$-modules such that $u_{mn}^*(\mathscr{F}_n) = \mathscr{F}_m$; and*
(c) *there exists an A-module M of finite type (determined up to canonical isomorphism by Proposition (10.10.2, i)) such that $\mathscr{F}$ is isomorphic to M^Δ.*

Proof. We first show that (b) implies (c). We have $\mathscr{F}_n = \widetilde{M_n}$, where M_n is an A_n-module of finite type, and the hypotheses imply that $M_m = M_n \otimes_{A_n} A_m$ for $m \leqslant n$ (1.6.5); the M_n thus form a projective system for the canonical di-homomorphisms $M_n \to M_m$ ($m \leqslant n$), and it follows immediately from the definition of the A_n that this projective system satisfies the conditions of (**0**, 7.2.9); as a result, its projective limit M is an A-module of finite type such that $M_n = M \otimes_A A_n$ for all n. We deduce that $\mathscr{F}_n$ is induced over X_n by $\widetilde{M} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\widetilde{\mathfrak{J}}^{n+1})$, and so $\mathscr{F} = M^\Delta$ by Definition (10.8.4).

Conversely, (c) implies (b); indeed, if u_n is the immersion morphism $X_n \to X$, then $u_n^*(\widetilde{M}) = (M \otimes_A A_n)^\sim$ is induced over X_n by $\widetilde{M} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\widetilde{\mathfrak{J}}^{n+1})$, and $M^\Delta = \varprojlim u_n^*(\widetilde{M})$ by Definition (10.8.4); since $u_m = u_n \circ u_{mn}$ for $m \leqslant n$, the $\mathscr{F}_n = u_n^*(\widetilde{M})$ satisfy the conditions of (b), whence our claim.

We now show that (c) implies (a): indeed, we have, by definition, that $\mathscr{O}_{\mathfrak{X}} = A^\Delta$; since M is the cokernel of a homomorphism $A^m \to A^n$, it follows from Proposition (10.10.2) that M^Δ is the cokernel of a homomorphism $\mathscr{O}_{\mathfrak{X}}^m \to \mathscr{O}_{\mathfrak{X}}^n$, and, since the sheaf of rings $\mathscr{O}_{\mathfrak{X}}$ is coherent (10.10.3), so too is M^Δ (**0**, 5.3.4).

Finally, (a) implies (b). Considered as an $\mathscr{O}_{\mathfrak{X}}$-module, we have that $\mathscr{O}_{X_n} = \mathscr{O}_{\mathfrak{X}}/\mathscr{J}^{n+1} = A_n^\Delta$; **I** | 203
but $\mathscr{F}_n = \mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_{X_n}$ is a coherent $\mathscr{O}_{\mathfrak{X}}$-module (**0**, 5.3.5), and, since it is also an $\mathscr{O}_{X_n}$-module, and $\mathscr{J}^{n+1}$ is coherent, we conclude that $\mathscr{F}_n$ is a coherent $\mathscr{O}_{X_n}$-module (**0**, 5.3.10), and it is immediate that $u_{mn}^*(\mathscr{F}_n) = \mathscr{F}_m$ for $m \leqslant n$ (recalling that the continuous map $X_m \to X_n$ of the underlying spaces is the identity on $\mathfrak{X}$). The sheaf $\mathscr{G} = \varprojlim \mathscr{F}_n$ is thus a coherent $\mathscr{O}_{\mathfrak{X}}$-module, since we have seen that (b) implies (a). The canonical homomorphisms $\mathscr{F} \to \mathscr{F}_n$ form a projective system, which, by passing to the limit, gives a canonical homomorphism $w : \mathscr{F} \to \mathscr{G}$, and it remains only to prove that w is bijective. The question is now *local*, so we can reduce to the case where $\mathscr{F}$ is the cokernel of a homomorphism $\mathscr{O}_{\mathfrak{X}}^p \to \mathscr{O}_{\mathfrak{X}}^q$; since this homomorphism is of the form v^Δ, where v is a homomorphism $A^m \to A^n$ (10.10.2), $\mathscr{F}$ is isomorphic to M^Δ, where $M = \operatorname{Coker} v$ (10.10.2). We then have, by Proposition (10.10.2), that $\mathscr{F}_n = M^\Delta \otimes_{\mathscr{O}_{\mathfrak{X}}} A_n^\Delta = (M \otimes_A A_n)^\Delta$, and, since the $\mathfrak{J}$-adic topology on $M \otimes_A A_n$ is discrete, we have $(M \otimes_A A_n)^\Delta = (M \otimes_A A_n)^\sim$ (as an $\mathscr{O}_{X_n}$-module); we have seen above that $M^\Delta = \varprojlim \mathscr{F}_n$, and w is thus the identity in this case. □

Corollary (10.10.6). — *If $\mathscr{F}$ satisfies condition* (b) *of Proposition (10.10.5), then the projective system $(\mathscr{F}_n)$ is isomorphic to the system of the $\mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_{X_n}$.*

(10.10.7). Now let A and B be adic Noetherian rings, and $\phi : B \to A$ a continuous homomorphism; we denote by $\mathfrak{J}$ (resp. $\mathfrak{K}$) an ideal of definition for A (resp. B) such that $\phi(\mathfrak{K}) \subset \mathfrak{J}$, and we set $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$, $\mathfrak{X} = \operatorname{Spf}(A)$, and $\mathfrak{Y} = \operatorname{Spf}(B)$. Let $f : X \to Y$ be the morphism of preschemes corresponding to ϕ (1.6.1), and $\widehat{f} : \mathfrak{X} \to \mathfrak{Y}$ its extension to the completions (10.9.1), which is also a morphism of formal preschemes that corresponds to ϕ (10.2.2).

Proposition (10.10.8). — *For every B-module N of finite type, there exists a canonical functorial isomorphism of $\mathscr{O}_{\mathfrak{X}}$-modules*

$$\widehat{f}^*(N^\Delta) \simeq (N \otimes_B A)^\Delta.$$

PROOF. Denoting by $i_X : \mathfrak{X} \to X$ and $i_Y : \mathfrak{Y} \to Y$ the canonical morphisms, we have (10.8.8), up to canonical functorial isomorphisms, $N^\Delta = i_Y^*(\widetilde{N})$ and

$$(N \otimes_B A)^\Delta = i_X^*((N \otimes_B A)^\sim) = i_X^*(f^*(\widetilde{N}))$$

(1.6.5); the proposition then follows from the commutativity of the diagram in (10.9.2). □

Corollary. — *For every ideal $\mathfrak{b}$ of B, we have $\widehat{f}^*(\mathfrak{b}^\Delta)\mathscr{O}_{\mathfrak{X}} = (\mathfrak{b}A)^\Delta$.*

PROOF. Let j be the canonical injection $\mathfrak{b} \to B$, to which corresponds the canonical injection $j^\Delta : \mathfrak{b}^\Delta \to \mathscr{O}_{\mathfrak{Y}}$ of sheaves of $\mathscr{O}_{\mathfrak{Y}}$-modules; by definition, $\widehat{f}^*(\mathfrak{b}^\Delta)\mathscr{O}_{\mathfrak{X}}$ is the image of the homomorphism $\widehat{f}^*(j^\Delta) : \widehat{f}^*(\mathfrak{b}^\Delta) \to \mathscr{O}_{\mathfrak{X}} = \widehat{f}^*(\mathscr{O}_{\mathfrak{Y}})$; but this homomorphism can be identified with $(j \otimes 1)^\Delta : (\mathfrak{b} \otimes_B A)^\Delta \to \mathscr{O}_{\mathfrak{X}} = (B \otimes_B A)^\Delta$ by Proposition (10.10.8). Since the image of $j \otimes 1$ is the ideal $\mathfrak{b}A$ of A, the image of $(j \otimes 1)^\Delta$ is thus $(\mathfrak{b}A)^\Delta$, by Proposition (10.10.2), whence the conclusion. □

10.11. Coherent sheaves on formal preschemes

Proposition (10.11.1). — *If $\mathfrak{X}$ is a locally Noetherian formal prescheme, then the sheaf of rings $\mathscr{O}_{\mathfrak{X}}$ is coherent, and every sheaf of ideals of definition for $\mathfrak{X}$ is coherent.* I | 204

PROOF. The question is local, so we can reduce to the case of a Noetherian affine formal scheme, and the proposition then follows from Propositions (10.10.3) and (10.10.5). □

(10.11.2). Let $\mathfrak{X}$ be a locally Noetherian formal prescheme, $\mathscr{J}$ a sheaf of ideals of definition for $\mathfrak{X}$, and X_n the locally Noetherian (usual) prescheme $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{J}^{n+1})$, so that $\mathfrak{X}$ is the *inductive limit* of the sequence (X_n) with respect to the canonical morphisms $u_{mn} : X_m \to X_n$ (10.6.3). With this notation:

Theorem (10.11.3). — *For an $\mathscr{O}_{\mathfrak{X}}$-module $\mathscr{F}$ to be coherent, it is necessary and sufficient for it to be isomorphic to a projective limit of a sequence $(\mathscr{F}_n)$, where the $\mathscr{F}_n$ are coherent $\mathscr{O}_{X_n}$-modules such that $u_{mn}^*(\mathscr{F}_n) = \mathscr{F}_m$ for $m \leqslant n$ (10.6.6). The projective system $(\mathscr{F}_n)$ is then isomorphic to the system of the $u_n^*(\mathscr{F}) = \mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_{X_n}$, where u_n is the canonical morphism $X_n \to \mathfrak{X}$.*

PROOF. The question is local, so we can reduce to the case where $\mathfrak{X}$ is a Noetherian affine formal scheme, and the theorem then is a consequence of Proposition (10.10.5) and Corollary (10.10.6). □

We can thus say that *the data of a coherent $\mathscr{O}_{\mathfrak{X}}$-module is equivalent to the data of a projective system $(\mathscr{F}_n)$ of coherent $\mathscr{O}_{X_n}$-modules such that $u_{mn}(\mathscr{F}_n) = \mathscr{F}_m$ for $m \leqslant n$.*

Corollary (10.11.4). — *If $\mathscr{F}$ and $\mathscr{G}$ are coherent $\mathscr{O}_{\mathfrak{X}}$-modules, then we can (with the notation of Theorem (10.11.3)) define a canonical functorial isomorphism*

$$\operatorname{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F}, \mathscr{G}) \simeq \varprojlim_n \operatorname{Hom}_{\mathscr{O}_{X_n}}(\mathscr{F}_n, \mathscr{G}_n). \tag{10.11.4.1}$$

PROOF. The projective limit on the right-hand side is understood to be taken with respect to the maps $\theta_n \mapsto u_{mn}^*(\theta_n)$ ($m \leqslant n$) from $\operatorname{Hom}_{\mathscr{O}_{X_n}}(\mathscr{F}_n, \mathscr{G}_n)$ to $\operatorname{Hom}_{\mathscr{O}_{X_m}}(\mathscr{F}_m, \mathscr{F}_m)$. The homomorphism (10.11.4.1) sends an element $\theta \in \operatorname{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F}, \mathscr{G})$ to the sequence $(u_n^*(\theta))$; we see that we can define an inverse homomorphism of the above by sending a projective system $(\theta_n) \in \varprojlim_n \operatorname{Hom}_{\mathscr{O}_{X_n}}(\mathscr{F}_n, \mathscr{G}_n)$ to its projective limit in $\operatorname{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F}, \mathscr{G})$, taking into account Theorem (10.11.3). □

Corollary (10.11.5). — *For a homomorphism $\theta : \mathscr{F} \to \mathscr{G}$ to be surjective, it is necessary and sufficient for the corresponding homomorphism $\theta_0 = u_0^*(\theta) : \mathscr{F}_0 \to \mathscr{G}_0$ to be surjective.*

PROOF. The question is local, so we reduce to the case where $\mathfrak{X} = \operatorname{Spf}(A)$ with A an adic Noetherian ring, $\mathscr{F} = M^{\Delta}$, $\mathscr{G} = N^{\Delta}$, and $\theta = u^{\Delta}$, where M and N are A-modules of finite type, and u is a homomorphism $M \to N$; we then have that $\theta_0 = \widetilde{u_0}$, where u_0 is the homomorphism $u \otimes 1 : M \otimes_A A/\mathfrak{J} \to N \otimes_A A/\mathfrak{J}$; the conclusion follows from the fact θ and u (resp. θ_0 and u_0) are simultaneously surjective ((1.3.9) and (10.10.2)) and that u and u_0 are simultaneously surjective (**0**, 7.1.14). □

(10.11.6). Theorem (10.11.3) shows that we can consider every coherent $\mathscr{O}_{\mathfrak{X}}$-module $\mathscr{F}$ as a *topological $\mathscr{O}_{\mathfrak{X}}$-module*, considering it as a projective limit of *pseudo-discrete* sheaves of groups $\mathscr{F}_n$ (**0**, 3.8.1). It then follows from Corollary (10.11.4) that every homomorphism $u : \mathscr{F} \to \mathscr{G}$ of coherent $\mathscr{O}_{\mathfrak{X}}$-modules is automatically *continuous* (**0**, 3.8.2). Furthermore, if $\mathscr{H}$ is a coherent $\mathscr{O}_{\mathfrak{X}}$-submodule of a coherent $\mathscr{O}_{\mathfrak{X}}$-module $\mathscr{F}$, then, for every open $U \subset \mathfrak{X}$, $\Gamma(U, \mathscr{H})$ is a *closed* subgroup of the topological group $\Gamma(U, \mathscr{F})$, since the functor Γ is left exact, and $\Gamma(U, \mathscr{H})$ is the kernel of the homomorphism $\Gamma(U, \mathscr{F}) \to \Gamma(U, \mathscr{F}/\mathscr{H})$, which is *continuous* by the above, since $\mathscr{F}/\mathscr{G}$ is coherent (**0**, 5.3.4); our claim follows from the fact that $\Gamma(U, \mathscr{F}/\mathscr{H})$ is a separated topological group. I | 205

Proposition (10.11.7). — *Let $\mathscr{F}$ and $\mathscr{G}$ be coherent $\mathscr{O}_{\mathfrak{X}}$-modules. We can define (with the notation of Theorem* (10.11.3)*) canonical functorial isomorphisms of topological $\mathscr{O}_{\mathfrak{X}}$-modules* (10.11.6)

$$\mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{G} \simeq \varprojlim_n (\mathscr{F}_n \otimes_{\mathscr{O}_{X_n}} \mathscr{G}_n), \tag{10.11.7.1}$$

$$\mathscr{H}om_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F}, \mathscr{G}) \simeq \varprojlim_n \mathscr{H}om_{\mathscr{O}_{X_n}}(\mathscr{F}_n, \mathscr{G}_n). \tag{10.11.7.2}$$

PROOF. The existence of the isomorphism (10.11.7.1) follows from the formula

$$\mathscr{F}_n \otimes_{\mathscr{O}_{X_n}} \mathscr{G}_n = (\mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_{X_n}) \otimes_{\mathscr{O}_{X_n}} (\mathscr{G} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_{X_n}) = (\mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{G}) \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_{X_n}$$

and from Theorem (10.11.3). The isomorphism (10.11.7.2), where both sides are considered as sheaves of modules without topology, follows from the definition of the sections of $\mathscr{H}om_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F}, \mathscr{G})$ and $\mathscr{H}om_{\mathscr{O}_{X_n}}(\mathscr{F}_n, \mathscr{G}_n)$, and from the existence of the isomorphism (10.11.4.1), mapping a prescheme induced on an arbitrary Noetherian formal affine open set to $\mathfrak{X}$. It remains to prove that the isomorphism (10.11.7.2) is bicontinuous over a quasi-compact set, and we can thus reduce to the case where $\mathfrak{X} = \operatorname{Spf}(A)$ with A an adic Noetherian ring, and hence (10.10.5) to the case where $\mathscr{F} = M^{\Delta}$ and $\mathscr{G} = N^{\Delta}$, with M and N both A-modules of finite type; taking (10.10.2.1), (10.10.2.3), and Corollary (1.3.12, ii) into account, we reduce to showing that the canonical isomorphism $\operatorname{Hom}_A(M, N) \simeq \varprojlim_n \operatorname{Hom}_{A_n}(M_n, N_n)$ (with $M_n = M \otimes_A A_n$ and $N_n = N \otimes_A A_n$) is continuous, which has already been proved in (**0**, 7.8.2). □

(10.11.8). Since $\operatorname{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F}, \mathscr{G})$ is the group of sections of the sheaf of topological groups $\mathscr{H}om_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F}, \mathscr{G})$, it is equipped with a group topology. If $\mathfrak{X}$ is *Noetherian*, then it follows from (10.11.7.2) that the subgroups $\operatorname{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F}, \mathscr{J}^n\mathscr{G})$ (for arbitrary n) form a fundamental system of neighbourhoods of 0 in this group.

Proposition (10.11.9). — *Let $\mathfrak{X}$ be a Noetherian formal prescheme, and $\mathscr{F}$ and $\mathscr{G}$ coherent $\mathscr{O}_{\mathfrak{X}}$-modules. In the topological group* $\operatorname{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F}, \mathscr{G})$*, the surjective (resp. injective, bijective) homomorphisms form an open set.*

PROOF. By Corollary (10.11.5), the set of surjective homomorphisms in $\operatorname{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F}, \mathscr{G})$ is the inverse image under the continuous map $\operatorname{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F}, \mathscr{G}) \to \operatorname{Hom}_{\mathscr{O}_{X_0}}(\mathscr{F}_0, \mathscr{G}_0)$ of a subset of the discrete group $\operatorname{Hom}_{\mathscr{O}_{X_0}}(\mathscr{F}_0, \mathscr{G}_0)$, whence the first claim. To show the second, we cover $\mathfrak{X}$ by a finite number of Noetherian formal affine subsets U_i. For $\theta \in \operatorname{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F}, \mathscr{G})$ to be injective, it is necessary and sufficient for all of the images under the (continuous) restriction maps $\operatorname{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F}, \mathscr{G}) \to \operatorname{Hom}_{\mathscr{O}_{\mathfrak{X}}|U_i)}(\mathscr{F}|U_i, \mathscr{G}|U_i)$ to be injective; we can thus reduce to the affine case, and then this has already been proved in (**0**, 7.8.3). □

10.12. Adic morphisms of formal preschemes

(10.12.1). Let $\mathfrak{X}$ and $\mathfrak{S}$ be *locally Noetherian* formal preschemes; we say that a morphism $f : \mathfrak{X} \to \mathfrak{S}$ is *adic* if there exists an ideal of definition $\mathscr{J}$ of $\mathfrak{S}$ such that $\mathscr{K} = f^*(\mathscr{J})\mathscr{O}_{\mathfrak{X}}$ is an ideal of definition of $\mathfrak{X}$; we then also say that $\mathfrak{X}$ is an *adic* $\mathfrak{S}$*-prescheme* (for f). Whenever this is the case, for *every* ideal of definition $\mathscr{J}_1$ of $\mathfrak{S}$, $\mathscr{K}_1 = f^*(\mathscr{J}_1)\mathscr{O}_{\mathfrak{X}}$ is an ideal of definition of $\mathfrak{X}$. Indeed, since the questions is local, we can assume that $\mathfrak{X}$ and $\mathfrak{S}$ are Noetherian and affine; there then exists a whole number n such that $\mathscr{J}^n \subset \mathscr{J}_1$ and $\mathscr{J}_1^n \subset \mathscr{J}$ ((10.3.6) and (**0**, 7.1.4)), whence $\mathscr{K}^n \subset \mathscr{K}_1$ and $\mathscr{K}_1^n \subset \mathscr{K}$. The first of these relations shows that $\mathscr{K}_1 = \mathfrak{K}_1^\Delta$, where $\mathfrak{K}_1$ is an open ideal of $A = \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$, and the second shows that $\mathfrak{K}_1$ is an ideal of definition of A (**0**, 7.1.4), whence our claim. I | 206

It follows immediately from the above that, if $\mathfrak{X}$ and $\mathfrak{Y}$ are adic $\mathfrak{S}$-preschemes, then *every* $\mathfrak{S}$*-morphism* $u : \mathfrak{X} \to \mathfrak{Y}$ *is adic*: indeed, if $f : \mathfrak{X} \to \mathfrak{S}$ and $g : \mathfrak{Y} \to \mathfrak{S}$ are the structure morphisms, and $\mathscr{J}$ is an ideal of definition of $\mathfrak{S}$, then we have $f = g \circ u$, and so $u^*(g^*(\mathscr{J})\mathscr{O}_{\mathfrak{Y}})\mathscr{O}_{\mathfrak{X}} = f^*(\mathscr{J})\mathscr{O}_{\mathfrak{X}}$ is an ideal of definition of $\mathfrak{X}$, and, by hypothesis, $g^*(\mathscr{J})\mathscr{O}_{\mathfrak{Y}}$ is an ideal of definition of $\mathfrak{Y}$.

(10.12.2). In what follows, we suppose that we have some fixed locally Noetherian formal prescheme $\mathfrak{S}$, and some ideal of definition $\mathscr{J}$ of $\mathfrak{S}$; we set $S_n = (\mathfrak{S}, \mathscr{O}_{\mathfrak{S}}/\mathscr{J}^{n+1})$. The (locally Noetherian) adic $\mathfrak{S}$-preschemes clearly form a *category*. We say that an inductive system (X_n) of locally Noetherian (usual) S_n-preschemes is an *adic inductive* (S_n)*-system* if the structure morphisms $f_n : X_n \to S_n$ are such that, for $m \leqslant n$, the diagrams

(10.12.2.1)
$$\begin{tikzcd} X_n \arrow[d, "f_n"'] & X_m \arrow[l] \arrow[d, "f_m"] \\ S_n & S_m \arrow[l] \end{tikzcd}$$

commute and *identify* X_m *with the product* $X_n \times_{S_n} S_m = (X_n)_{(S_m)}$. The adic inductive systems form a *category*: it suffices in fact to define a morphism $(X_n) \to (Y_n)$ of such systems to be an *inductive system of* S_n*-morphisms* $u_n : X_n \to Y_n$ such that u_m is identified with $(u_n)_{(S_m)}$ for $m \leqslant n$. With this in mind:

Theorem (10.12.3). — *There is a canonical equivalence between the category of adic* $\mathfrak{S}$*-preschemes and the category of adic inductive* (S_n)*-systems.*

The equivalence in question is obtained in the following way: if $\mathfrak{X}$ is an adic $\mathfrak{S}$-prescheme, and $f : \mathfrak{X} \to \mathfrak{S}$ is the structure morphism, then $\mathscr{K} = f^*(\mathscr{J})\mathscr{O}_{\mathfrak{X}}$ is an ideal of definition of $\mathfrak{X}$, and we associate to $\mathfrak{X}$ the inductive system of the $X_n = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{K}^{n+1})$, with the structure morphism $f_n : X_n \to S_n$ corresponding to f (10.5.6). We first show that (X_n) is an *adic inductive system*: if $f = (\psi, \theta)$, then $\psi^*(\mathscr{J})\mathscr{O}_{\mathfrak{X}} = \mathscr{K}$, so $\psi^*(\mathscr{J}^n)\mathscr{O}_{\mathfrak{X}} = \mathscr{K}^n$ for all n, and (by exactness of the functor ψ^*) $\mathscr{K}^{m+1}/\mathscr{K}^{n+1} = \psi^*(\mathscr{J}^{m+1}/\mathscr{J}^{n+1})(\mathscr{O}_{\mathfrak{X}}/\mathscr{K}^{n+1})$ for $m \leqslant n$; our conclusion thus follows from (4.4.5). Furthermore, it can be immediately verified that a $\mathfrak{S}$-morphism $u : \mathfrak{X} \to \mathfrak{Y}$ of adic $\mathfrak{S}$-preschemes corresponds (with the obvious notation) to an inductive system of S_n-morphisms I | 207
$u_n : X_n \to Y_n$ such that u_m is identified with $(u_n)_{(S_m)}$ for $m \leqslant n$.

The fact that this equivalence is well defined will follow from the more-precise following proposition.

Proposition (10.12.3.1). — *Let* (X_n) *be an inductive system of* S_n*-preschemes; suppose that the structure morphisms* $f_n : X_n \to S_n$ *are such that the diagrams in* (10.12.2.1) *commute and identify* X_m *with* $X_n \times_{S_n} S_m$ *for* $m \leqslant n$*. Then the inductive system* (X_n) *satisfies conditions* (b) *and* (c) *of* (10.6.3)*; let* $\mathfrak{X}$ *be the inductive limit, and* $f : \mathfrak{X} \to \mathfrak{S}$ *the morphism given by the inductive limit of the inductive system* (f_n)*. Then, if* X_0 *is locally Noetherian,* $\mathfrak{X}$ *is locally Noetherian, and* f *is an adic morphism.*

Proof. Since the sheaf of ideals of $\mathscr{O}_{S_n}$ that defines the subprescheme S_m of S_n is nilpotent, by (4.4.5), so too is the sheaf of ideals of $\mathscr{O}_{X_n}$ that defines the subprescheme X_m of X_n, and so the conditions of (10.6.3) are satisfied. Since the questions is local on $\mathfrak{X}$ and $\mathfrak{S}$, we can assume that $\mathfrak{S} = \operatorname{Spf}(A)$, $\mathscr{J} = \mathfrak{J}^\Delta$ (with A a Noetherian $\mathfrak{J}$-adic ring), and $X_n = \operatorname{Spec}(B_n)$; if $A_n = A/\mathfrak{J}^{n+1}$, then the hypothesis implies that B_0 is Noetherian, and if we set $\mathfrak{J}_n = \mathfrak{J}/\mathfrak{J}^{n+1}$, then $B_m = B_n/\mathfrak{J}_n^{m+1}B_n$. The kernel of $B_n \to B_0$ is thus $\mathfrak{K}_n = \mathfrak{J}_n B_n$, and the kernel of $B_n \to B_m$ is $\mathfrak{K}_n^{m+1}$ for $m \leqslant n$; further, since A_1 is Noetherian, $\mathfrak{J}_1$ is of finite type over A_1, and so $\mathfrak{K}_1 = \mathfrak{K}_1/\mathfrak{K}_1^2$ is of finite type over B_1, and *a fortiori* of finite type over $B_0 = B_1/\mathfrak{K}_1$; the fact that $\mathfrak{X}$ is Noetherian then follows from (10.6.4); if $B = \varprojlim B_n$, then we have $\mathfrak{X} = \operatorname{Spf}(B)$, and, if $\mathfrak{K}$ is the kernel of $B \to B_0$, then $B_n = B/\mathfrak{K}^{n+1}$. If $\rho_n : A/\mathfrak{J}^{n+1} \to B/\mathfrak{K}^{n+1}$ is the homomorphism corresponding to f_n, then we have that

$$\mathfrak{K}/\mathfrak{K}^{n+1} = (B/\mathfrak{K}^{n+1})\rho_n(\mathfrak{J}/\mathfrak{J}^{n+1})$$

since the homomorphism $\rho : A \to B$ corresponding to f is equal to $\varprojlim \rho_n$, and that the ideal $\mathfrak{J}B$ of B is dense in $\mathfrak{K}$, and, since every ideal of B is closed (**0**, 7.3.5), we also have that $\mathfrak{K} = \mathfrak{J}B$. If $\mathscr{K} = \mathfrak{K}^\Delta$, the equality $f^*(\mathscr{J})\mathscr{O}_\mathfrak{X} = \mathscr{K}$ then follows from (10.10.9), and finishes the proof. □

(10.12.3.2). The above equivalence gives, for adic $\mathfrak{S}$-preschemes $\mathfrak{X}$ and $\mathfrak{Y}$, a *canonical bijection*

$$\operatorname{Hom}_\mathfrak{S}(\mathfrak{X}, \mathfrak{Y}) \simeq \varprojlim_n \operatorname{Hom}_{S_n}(X_n, Y_n)$$

where the projective limit is relative to the maps $u_n \to (u_n)_{(S_m)}$ for $m \leqslant n$.

10.13. Morphisms of finite type

Proposition (10.13.1). — *Let $\mathfrak{Y}$ be a locally Noetherian formal prescheme, $\mathscr{K}$ an ideal of definition of $\mathfrak{Y}$, and $f : \mathfrak{X} \to \mathfrak{Y}$ a morphism of formal preschemes. Then the following conditions are equivalent.*

(a) *X is locally Noetherian, f is an adic morphism* (10.12.1), *and, if we set $\mathscr{J} = f^*(\mathscr{K})\mathscr{O}_\mathfrak{X}$, then the morphism $f_0 : (\mathfrak{X}, \mathscr{O}_\mathfrak{X}/\mathscr{J}) \to (\mathfrak{Y}, \mathscr{O}_\mathfrak{Y}/\mathscr{K})$ induced by f is of finite type.*

(b) *$\mathfrak{X}$ is locally Noetherian, and is the inductive limit of an adic inductive (Y_n)-system (X_n) such that the morphism $X_0 \to Y_0$ is of finite type.* I | 208

(1) *Every point of $\mathfrak{Y}$ has a Noetherian formal affine open neighbourhood V which satisfies the following property:*

 (**Q**) *$f^{-1}(V)$ is a finite union of Noetherian formal affine open subsets U_i such that the Noetherian adic ring $\Gamma(U_i, \mathscr{O}_\mathfrak{X})$ is topologically isomorphic to the quotient of a formal series algebra, restricted* (**0**, 7.5.1) *to $\Gamma(V, \mathscr{O}_\mathfrak{Y})$, by an ideal (which is necessarily closed).*

PROOF. It is immediate that (a) implies (b), by (10.12.3). To show that (b) implies (c), we can, since the question is local on $\mathfrak{Y}$, assume that $\mathfrak{Y} = \operatorname{Spf}(B)$, where B is Noetherian and adic; let $\mathscr{K} = \mathfrak{K}^\Delta$, with $\mathfrak{K}$ an ideal of definition of B. Since, by hypothesis, X_0 is of finite type over Y_0, X_0 is a finite union of affine open subsets U_i such that the ring A_{i0} of the affine scheme induced by X_0 on U_i is an algebra of finite type over the ring $B/\mathfrak{K}$ of Y_0 (6.3.2). By (5.1.9), U_i is also an affine open subset in each of the Noetherian preschemes X_n, and, if A_{in} is the ring of the affine scheme induced by X_n on U_i, then hypothesis (b) implies, for $m \leqslant n$, that A_{im} is isomorphic to $A_{in}/\mathfrak{K}^{m+1}A_{in}$. Consequently, the formal prescheme induced by $\mathfrak{X}$ on U_i is isomorphic to $\operatorname{Spf}(A_i)$, where $A_i = \varprojlim_n A_{in}$ (10.6.4); A_i is a $\mathfrak{K}A_i$-adic ring, and $A_i/\mathfrak{K}A_i$, being isomorphic to A_{i0}, is an algebra of finite type over $B/\mathfrak{K}$. We thus conclude (**0**, 7.5.5) that A_i is topologically isomorphic to a quotient of a formal series algebra restricted to B (by a necessarily closed ideal, because such an algebra is Noetherian (**0**, 7.5.4)).

To show that (c) implies (a), we can restrict to the case where $\mathfrak{X} = \operatorname{Spf}(A)$ is also affine, with A a Noetherian adic ring isomorphic to a quotient of a formal series algebra, restricted to B, by a closed ideal. Then (**0**, 7.5.5) $A/\mathfrak{K}A$ is an algebra of finite type over $B/\mathfrak{K}$, and $\mathfrak{K}A = \mathfrak{J}$ is an ideal of definition of A, and so, by (10.10.9), the conditions of (a) are satisfied. □

We note that, if the conditions of Proposition (10.13.1) are satisfied, then property (a) holds true for *any* ideal of definition $\mathscr{K}$ of $\mathfrak{Y}$ (by *(c)*), and so, in property (b), *all* the f_n are morphisms of finite type.

Corollary (10.13.2). — *If the conditions of* (10.13.1) *are satisfied, then every Noetherian formal affine open subset V of $\mathfrak{Y}$ has property (**Q**), and, if $\mathfrak{Y}$ is Noetherian, then so too is $\mathfrak{X}$.*

PROOF. This follows immediately from (10.13.1) and (6.3.2). □

Definition (10.13.3). — When the equivalent properties (a), (b), and (c) of (10.13.1) are satisfied, we say that the morphism f is of finite type, or that $\mathfrak{X}$ is a formal $\mathfrak{Y}$-prescheme of finite type, or a formal prescheme of finite type over $\mathfrak{Y}$.

Corollary (10.13.4). — *Let $\mathfrak{X} = \operatorname{Spf}(A)$ and $\mathfrak{Y} = \operatorname{Spf}(B)$ be Noetherian formal affine schemes; for $\mathfrak{X}$ to be of finite type over $\mathfrak{Y}$, it is necessary and sufficient for the Noetherian adic ring A to be isomorphic to the quotient of a formal series algebra, restricted to B, by some closed ideal.*

PROOF. With the notation of (10.13.1), if $\mathfrak{X}$ is of finite type over $\mathfrak{Y}$, then $A/\mathfrak{K}A$ is a $(B/\mathfrak{K})$-algebra of finite type by (6.3.3), and $\mathfrak{K}A$ is an ideal of definition of A (10.10.9). We are then done, by (**0**, 7.5.5). □

Proposition (10.13.5). —

(i) *The composition of any two morphisms (of formal preschemes) of finite type is again of finite type.* I | 209

(ii) *Let $\mathfrak{X}$, $\mathfrak{S}$, and $\mathfrak{S}'$ be locally Noetherian (resp. Noetherian) formal preschemes, and $f : \mathfrak{X} \to \mathfrak{S}$ and $\mathfrak{X} \to \mathfrak{S}'$ morphisms. If f is of finite type, then $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{S}'$ is locally Noetherian (resp. Noetherian) and of finite type over $\mathfrak{S}'$.*

(iii) *Let $\mathfrak{S}$ be a locally Noetherian formal prescheme, and $\mathfrak{X}'$ and $\mathfrak{Y}'$ formal $\mathfrak{S}$-preschemes such that $\mathfrak{X}' \times_{\mathfrak{S}} \mathfrak{Y}'$ is locally Noetherian. If $\mathfrak{X}$ and $\mathfrak{Y}$ are locally Noetherian formal $\mathfrak{S}$-preschemes, and $f : \mathfrak{X} \to \mathfrak{X}'$ and $g : \mathfrak{Y} \to \mathfrak{Y}'$ are $\mathfrak{S}$-morphisms of finite type, then $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ is locally Noetherian, and $f \times_{\mathfrak{S}} g$ is a $\mathfrak{S}$-morphism of finite type.*

PROOF. By the formal argument of (3.5.1), (iii) follows from (i) and (ii), so it suffices to prove (i) and (ii).

Let $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ be locally Noetherian formal preschemes, and $f : \mathfrak{X} \to \mathfrak{Y}$ and $g : \mathfrak{Y} \to \mathfrak{Z}$ morphisms of finite type. If $\mathscr{L}$ is an ideal of definition of $\mathfrak{Z}$, then $\mathscr{K} = g^*(\mathscr{L})\mathscr{O}_{\mathfrak{Y}}$ is an ideal of definition of $\mathfrak{Y}$, and $\mathscr{J} = f^*(g^*(\mathscr{L}))\mathscr{O}_{\mathfrak{X}}$ is an ideal of definition for $\mathfrak{X}$. Let $X_0 = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{J})$, $Y_0 = (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{K})$, and $Z_0 = (\mathfrak{Z}, \mathscr{O}_{\mathfrak{Z}}/\mathscr{L})$, and let $f_0 : X_0 \to Y_0$ and $g_0 : Y_0 \to Z_0$ be the morphisms corresponding to f and g (respectively). Since, by hypothesis, f_0 and g_0 are of finite type, so too is $g_0 \circ f_0$ (6.3.4), which corresponds to $g \circ f$; thus $g \circ f$ is of finite type, by (10.13.1).

Under the conditions of (ii), $\mathfrak{S}$ (resp. $\mathfrak{X}$, $\mathfrak{S}'$) is the inductive limit of a sequence (S_n) (resp. (X_n), (S'_n)) of locally Noetherian preschemes, and we can assume (10.13.1) that $X_m = X_n \times_{S_n} S_m$ for $m \leqslant n$. The formal prescheme $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{S}'$ is then the inductive limit of the preschemes $X_n \times_{S_n} S'_n$ (10.7.4), and we have that

$$X_m \times_{S_m} S'_m = (X_n \times_{S_n} S_m) \times_{S_m} S'_m = (X_n \times_{S_n} S'_n) \times_{S'_n} S'_m.$$

Furthermore, $X_0 \times_{S_0} S'_0$ is locally Noetherian, since X_0 is of finite type over S_0 (6.3.8). We thus conclude (10.12.3.1), first of all, that $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{S}'$ is locally Noetherian; then, since $X_0 \times_{S_0} S'_0$ is of finite type over S'_0 (6.3.8), it follows from (10.12.3.1) and (10.13.1) that $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{S}'$ is of finite type over $\mathfrak{S}'$, which proves (ii) (the claim about Noetherian preschemes being an immediate consequence of (6.3.8)). □

Corollary (10.13.6). — *Under the hypotheses of (10.9.9), if f is a morphism of finite type, then so too is its extension $\widehat{f}$ to the completions.*

10.14. Closed subpreschemes of formal preschemes

Proposition (10.14.1). — *Let $\mathfrak{X}$ be a locally Noetherian formal prescheme, and $\mathscr{A}$ a coherent sheaf of ideals of $\mathscr{O}_{\mathfrak{X}}$. If $\mathfrak{Y}$ if the (closed) support of $\mathscr{O}_{\mathfrak{X}}/\mathscr{A}$, then the topologically ringed space $(\mathfrak{Y}, (\mathscr{O}_{\mathfrak{X}}/\mathscr{A})|\mathfrak{Y})$ is a locally Noetherian formal prescheme that is Noetherian if $\mathfrak{X}$ is.*

PROOF. Note that $\mathscr{O}_{\mathfrak{X}}/\mathfrak{A}$ is coherent by (10.10.3) and (**0**, 5.3.4), so its support $\mathfrak{Y}$ is closed (**0**, 5.2.2). Let $\mathscr{J}$ be an ideal of definition of $\mathfrak{X}$, and let $X_n = (\mathfrak{X}/\mathscr{O}_{\mathfrak{X}}/\mathscr{J}^{n+1})$; the sheaf of rings $\mathscr{O}_{\mathfrak{X}}/\mathscr{A}$ is the projective limit of the sheaves $\mathscr{O}_{\mathfrak{X}}/(\mathscr{A} + \mathscr{J}^{n+1}) = (\mathscr{O}_{\mathfrak{X}}/\mathscr{A}) \otimes_{\mathscr{O}_{\mathfrak{X}}} (\mathscr{O}_{\mathfrak{X}}/\mathscr{J}^{n+1})$ (10.11.3), all of which have support $\mathfrak{Y}$. The sheaf $(\mathscr{A} + \mathscr{J}^{n+1})/\mathscr{J}^{n+1}$ is a coherent $\mathscr{O}_{\mathfrak{X}}$-module, since $\mathscr{J}^{n+1}$ is coherent, and so $(\mathscr{A} + \mathscr{J}^{n+1})/\mathscr{J}^{n+1}$ is also a coherent $(\mathscr{O}_{\mathfrak{X}}/\mathscr{J}^{n+1})$-module (**0**, 5.3.10); if Y_n is the closed subprescheme of X_n defined by this sheaf of ideals, it is immediate that $(\mathfrak{Y}, (\mathscr{O}_{\mathfrak{X}}/\mathscr{A})|\mathfrak{Y})$ is I | 210
the formal prescheme given by the inductive limit of the Y_n, and, since the conditions of (10.6.4) are satisfied, this proves that this formal prescheme is locally Noetherian, and further Noetherian if $\mathfrak{X}$ is (since then Y_0 is, by (6.1.4)). □

Definition (10.14.2). — We define a closed subprescheme of a formal prescheme $\mathfrak{X}$ to be any formal prescheme of the form $(\mathfrak{Y}, (\mathscr{O}_{\mathfrak{X}}/\mathscr{A})|\mathfrak{Y})$ with $\mathscr{A}$ a coherent ideal of $\mathscr{O}_{\mathfrak{X}}$; we say that this prescheme is the subprescheme defined by $\mathscr{A}$. **Err$_{\text{II}}$**

It is clear that the correspondence thus defined between coherent ideals of $\mathscr{O}_{\mathfrak{X}}$ and closed subpreschemes of $\mathfrak{X}$ is bijective. **Err$_{\text{II}}$**

The morphism of topologically ringed spaces $j = (\psi, \theta) : \mathfrak{Y} \to \mathfrak{X}$, where ψ is the injection $\mathfrak{Y} \to \mathfrak{X}$ and $\theta^\sharp$ the canonical homomorphism $\mathscr{O}_{\mathfrak{X}} \to \mathscr{O}_{\mathfrak{X}}/\mathscr{A}$, is evidently (10.4.5) a morphism of formal preschemes, and we call it the *canonical injection* from $\mathfrak{Y}$ to $\mathfrak{X}$. Note that, if $\mathfrak{X} = \mathrm{Spf}(A)$, or if A is Noetherian and adic, then $\mathscr{A} = \mathfrak{a}^\Delta$, where $\mathfrak{a}$ is an ideal of A (10.10.5), and it then follows immediately from the above that $\mathfrak{Y} = \mathrm{Spf}(A/\mathfrak{a})$, up to isomorphism, and that j corresponds (10.2.2) to the canonical homomorphism $A \to A/\mathfrak{a}$.

We say that a morphism $f : \mathfrak{Z} \to \mathfrak{X}$ of locally Noetherian formal preschemes is a *closed immersion* if it factors as $\mathfrak{Z} \xrightarrow{g} \mathfrak{Y} \xrightarrow{j} \mathfrak{X}$, where g is an isomorphism from $\mathfrak{Z}$ to a closed subprescheme $\mathfrak{Y}$ of $\mathfrak{X}$, and j is the canonical injection. Since j is a monomorphism of ringed spaces, g and $\mathfrak{Y}$ are necessarily *unique*.

Proposition (10.14.3). — *A closed immersion is a morphism of finite type.*

Proof. We can immediately restrict to the case where $\mathfrak{X}$ is a formal affine scheme $\mathrm{Spf}(A)$, and $\mathfrak{Y} = \mathrm{Spf}(A/\mathfrak{a})$; the proposition then follows from Proposition (10.13.1, *c*). □

Lemma (10.14.4). — *Let $f : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of locally Noetherian formal preschemes, and let (U_α) be a cover of $f(\mathfrak{Y})$ by Noetherian formal affine open subsets of $\mathfrak{X}$ such that the $f^{-1}(U_\alpha)$ are Noetherian formal affine open subsets of $\mathfrak{Y}$. For f to be a closed immersion, it is necessary and sufficient for $f(\mathfrak{Y})$ to be a closed subset of $\mathfrak{X}$ and, for all α, for the restriction of f to $f^{-1}(U_\alpha)$ to correspond* (10.4.6) *to a surjective homomorphism $\Gamma(U_\alpha, \mathscr{O}_{\mathfrak{X}}) \to \Gamma(f^{-1}(U_\alpha), \mathscr{O}_{\mathfrak{Y}})$.*

Proof. The conditions are clearly necessary. Conversely, if the conditions are satisfied, and if we denote by $\mathfrak{a}_\alpha$ the kernel of $\Gamma(U_\alpha, \mathscr{O}_{\mathfrak{X}}) \to \Gamma(f^{-1}(U_\alpha), \mathscr{O}_{\mathfrak{Y}})$, then we can define a coherent sheaf of ideals $\mathscr{A}$ of $\mathscr{O}_{\mathfrak{X}}$ by setting $\mathscr{A}|U_\alpha = \mathfrak{a}_\alpha^\Delta$ and taking $\mathscr{A}$ to be zero on the complement of the union of the U_α. Since $f(\mathfrak{Y})$ is closed, and since the support of $\mathfrak{a}_\alpha^\Delta$ is $U_\alpha \cap f(\mathfrak{Y})$, everything reduces to proving that $\mathfrak{a}_\alpha^\Delta$ and $\mathfrak{a}_\beta^\Delta$ induce the same sheaf on any Noetherian formal affine open subset $V \subset U_\alpha \cap U_\beta$. But the restriction to $f^{-1}(U_\alpha)$ of f is a closed immersion of this formal prescheme into U_α, $f^{-1}(V)$ is a Noetherian formal affine open subsets of $f^{-1}(U_\alpha)$, and the restriction of f to $f^{-1}(V)$ is a closed immersion; if $\mathfrak{b}$ is the kernel of the surjective homomorphism $\Gamma(V, \mathscr{O}_{\mathfrak{X}}) \to \Gamma(f^{-1}(V), \mathscr{O}_{\mathfrak{Y}})$ corresponding to this restriction, then it is immediate (10.10.2) that $\mathfrak{a}_\alpha^\Delta$ induces $\mathfrak{b}^\Delta$ on V. The sheaf of ideals $\mathscr{A}$ being thus defined, it is then clear that $f = g \circ j$, where $j : \mathfrak{Z} \to \mathfrak{X}$ is the canonical injection of the closed subprescheme $\mathfrak{Z}$ of $\mathfrak{X}$ defined by $\mathscr{A}$, and that g is an isomorphism from $\mathfrak{Y}$ to $\mathfrak{Z}$. □

Proposition (10.14.5). —

(i) *If $f : \mathfrak{Z} \to \mathfrak{Y}$ and $g : \mathfrak{Y} \to \mathfrak{X}$ are closed immersions of locally Noetherian formal preschemes, then $g \circ f$ is a closed immersion.* **I | 211**

(ii) *Let $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{S}$ be locally Noetherian formal preschemes, $f : \mathfrak{X} \to \mathfrak{S}$ a closed immersion, and $g : \mathfrak{Y} \to \mathfrak{S}$ a morphism. Then the morphism $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y} \to \mathfrak{Y}$ is a closed immersion.*

(iii) *Let $\mathfrak{S}$ be a locally Noetherian formal prescheme, and $\mathfrak{X}'$ and $\mathfrak{Y}'$ locally Noetherian formal $\mathfrak{S}$-preschemes such that $\mathfrak{X}' \times_{\mathfrak{S}} \mathfrak{Y}'$ is locally Noetherian. If $\mathfrak{X}$ and $\mathfrak{Y}$ are locally Noetherian $\mathfrak{S}$-preschemes, and $f : \mathfrak{X} \to \mathfrak{X}'$ and $g : \mathfrak{Y} \to \mathfrak{Y}'$ are $\mathfrak{S}$-morphisms that are closed immersions, then $f \times_{\mathfrak{S}} g$ is a closed immersion.*

Proof. By (3.5.1), it again suffices to prove (i) and (ii).

To prove (i), we can assume that $\mathfrak{Y}$ (resp. $\mathfrak{Z}$) is a closed subprescheme of $\mathfrak{X}$ (resp. $\mathfrak{Y}$) defined by a coherent sheaf $\mathscr{J}$ (resp. $\mathscr{K}$) of ideals of $\mathscr{O}_{\mathfrak{X}}$ (resp. $\mathscr{O}_{\mathfrak{Y}}$); if ψ is the injection $\mathfrak{Y} \to \mathfrak{X}$ of underlying spaces, then $\psi_*(\mathscr{K})$ is a coherent sheaf of ideals of $\psi_*(\mathscr{O}_{\mathfrak{Y}}) = \mathscr{O}_{\mathfrak{X}}/\mathscr{J}$ (**0**, 5.3.12), and thus also a coherent $\mathscr{O}_{\mathfrak{X}}$-module (**0**, 5.3.10); the kernel $\mathscr{K}_1$ of $\mathscr{O}_{\mathfrak{X}} \to (\mathscr{O}_{\mathfrak{X}}/\mathscr{J})/\psi_*(\mathscr{K})$ is thus a coherent sheaf of ideals of $\mathscr{O}_{\mathfrak{X}}$ (**0**, 5.3.4), and $\mathscr{O}_{\mathfrak{X}}/\mathscr{K}_1$ is isomorphic to $\psi_*(\mathscr{O}_{\mathfrak{Y}}/\mathscr{K})$, which proves that $\mathfrak{Z}$ is an isomorphism to a closed subprescheme of $\mathfrak{X}$.

To prove (ii), we can immediately restrict to the case where $\mathfrak{S} = \operatorname{Spf}(A)$, $\mathfrak{X} = \operatorname{Spf}(B)$, and $\mathfrak{Y} = \operatorname{Spf}(C)$, with A a Noetherian $\mathfrak{J}$-adic ring, $B = A/\mathfrak{a}$ (where $\mathfrak{a}$ is an ideal of A), and C a Noetherian topological adic A-algebra. Everything then reduces to proving that the homomorphism $C \to C\widehat{\otimes}_A (A/\mathfrak{a})$ is *surjective*: but $A/\mathfrak{a}$ is an A-module of finite type, and its topology is the $\mathfrak{J}$-adic topology; it then follows from (**0**, 7.7.8) that $C\widehat{\otimes}_A (A/\mathfrak{a})$ can be identified with $C \otimes_A (A/\mathfrak{a}) = C/\mathfrak{a}C$, whence our claim. $\square$

Corollary (10.14.6). — *Under the hypotheses of (10.14.5, ii), let $p : \mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y} \to \mathfrak{X}$ and $q : \mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y} \to \mathfrak{Y}$ be the projections, so that the diagram*

$$\begin{tikzcd} \mathfrak{X} \arrow[d, "f"'] & \mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y} \arrow[l, "p"'] \arrow[d, "q"] \\ \mathfrak{S} & \mathfrak{Y} \arrow[l, "g"] \end{tikzcd}$$

commutes. For every coherent $\mathscr{O}_{\mathfrak{X}}$ module $\mathscr{F}$, we then have a canonical isomorphism of $\mathscr{O}_{\mathfrak{Y}}$-modules

$$u : g^* f_*(\mathscr{F}) \simeq q_* p^*(\mathscr{F}). \tag{10.14.6.1}$$

Proof. We know that defining a homomorphism $g^* f_*(\mathscr{F}) \to q_* p^*(\mathscr{F})$ is equivalent to defining a homomorphism $f_*(\mathscr{F}) \to g_* q_* p^*(\mathscr{F}) = f_* p_* p^*(\mathscr{F})$ (**0**, 4.4.3): we take $u = f_*(\rho)$, where ρ is the canonical homomorphism $\mathscr{F} \to p_* p^*(\mathscr{F})$ (**0**, 4.4.3). To see that u is an isomorphism, we can immediately restrict to the case where $\mathfrak{S}$, $\mathfrak{X}$, and $\mathfrak{Y}$ are formal spectra of Noetherian adic rings A, B, and C (respectively), satisfying the conditions in (10.14.5, ii) above; we then have $\mathscr{F} = M^\Delta$, where M is an $(A/\mathfrak{a})$-module of finite type (10.10.5), and the two sides of (10.14.6.1) can then be identified, respectively, by (10.10.8), with $(C \otimes_A M)^\Delta$ and $((C/\mathfrak{a}C) \otimes_{A/\mathfrak{a}} M)^\Delta$, whence the corollary, since $(C/\mathfrak{a}C) \otimes_{A/\mathfrak{a}} M = (C \otimes_A (A/\mathfrak{a})) \otimes_{A/\mathfrak{a}} M$ is canonically identified with $C \otimes_A M$. $\square$

I | 212

Corollary (10.14.7). — *Let X be a locally Noetherian usual prescheme, Y a closed subprescheme of X, j the canonical injection $Y \to X$, X' a closed subset of X, and $Y' = Y \cap X'$; then $\widehat{j} : Y_{/Y'} \to X_{/X'}$ is a closed immersion, and, for every coherent $\mathscr{O}_Y$-module $\mathscr{F}$, we have*

$$\widehat{j}_*(\mathscr{F}_{/Y'}) = (j_*(\mathscr{F}))_{/X'}.$$

Proof. Since $Y' = j^{-1}(X')$, it suffices to use (10.9.9) and to apply (10.14.5) and (10.14.6). $\square$

10.15. Separated formal preschemes

Definition (10.15.1). — Let $\mathfrak{S}$ be a formal prescheme, $\mathfrak{X}$ a formal $\mathfrak{S}$-prescheme, and $f : \mathfrak{X} \to \mathfrak{S}$ the structure morphism. We define the diagonal morphism $\Delta_{\mathfrak{X}|\mathfrak{S}} : \mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{S}} \mathfrak{X}$ (also denoted by $\Delta_{\mathfrak{X}}$) to be the morphism $(1_{\mathfrak{X}}, 1_{\mathfrak{X}})_{\mathfrak{S}}$. We say that $\mathfrak{X}$ is separated over $\mathfrak{S}$, or is a formal $\mathfrak{S}$-scheme, or that f is a separated morphism, if the image of the underlying space of $\mathfrak{X}$ under $\Delta_{\mathfrak{X}}$ is a closed subset of the underlying space of $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{X}$. We say that a formal prescheme $\mathfrak{X}$ is separated, or is a formal scheme, if it is separated over $\mathbf{Z}$.

Proposition (10.15.2). — *Suppose that the formal preschemes $\mathfrak{S}$ and $\mathfrak{X}$ are inductive limits of sequences (S_n) and (X_n) (respectively) of usual preschemes, and that the morphism $f : \mathfrak{X} \to \mathfrak{S}$ is the inductive limit of a sequence of morphisms $f_n : X_n \to S_n$. For f to be separated, it is necessary and sufficient for the morphism $f_0 : X_0 \to S_0$ to be separated.*

Proof. Indeed, $\Delta_{\mathfrak{X}|\mathfrak{S}}$ is then the inductive limit of the sequence of morphisms $\Delta_{X_n|S_n}$ (10.7.4), and the image of the underlying space of $\mathfrak{X}$ (resp. of $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{X}$ under $\Delta_{\mathfrak{X}|\mathfrak{S}}$) is identical to the image of the underlying space of X_0 (resp. of $X_0 \times_{S_0} X_0$) under $\Delta_{X_0|S_0}$; whence the conclusion. $\square$

Proposition (10.15.3). — *Suppose that all the formal preschemes (resp. morphisms of formal preschemes) in what follows are inductive limits of sequences of usual preschemes (resp. of morphisms of usual preschemes).*

(i) *The composition of any two separated morphisms is separated.*
(ii) *If $f : \mathfrak{X} \to \mathfrak{X}'$ and $g : \mathfrak{Y} \to \mathfrak{Y}'$ are separated $\mathfrak{S}$-morphisms, then $f \times_{\mathfrak{S}} g$ is separated.*
(iii) *If $f : \mathfrak{X} \to \mathfrak{Y}$ is a separated $\mathfrak{S}$-morphism, then the $\mathfrak{S}'$-morphism $f_{(\mathfrak{S}')}$ is separated for every extension $\mathfrak{S}' \to \mathfrak{S}$ of the base formal prescheme.*

(iv) *If the composition $g \circ f$ of two morphisms is separated, then f is separated.*

(In the above, it is implicit that if the same formal prescheme $\mathfrak{Z}$ is mentioned more than once in the same proposition, we consider it as the inductive limit of the *same* sequence (Z_n) of usual preschemes wherever it is mentioned, and the morphisms from $\mathfrak{Z}$ to another formal prescheme (resp. from a formal prescheme to $\mathfrak{Z}$) as inductive limits of morphisms from Z_n to some usual preschemes (resp. from some usual preschemes to Z_n)).

PROOF. With the notation of (10.15.2), we have, in fact, that $(g \circ f)_0 = g_0 \circ f_0$ and $(f \times_{\mathfrak{S}} g)_0 = f_0 \times_{S_0} g_0$; the claims of (10.15.3) are then immediate consequences of (10.15.2) and the corresponding claims in (5.5.1) for usual preschemes. ☐

We leave it to the reader to state, for the same type of formal preschemes and morphisms as in (10.15.3), the propositions corresponding to (5.5.5), (5.5.9), and (5.5.10) (by replacing "affine open subset" by "formal affine open subset satisfying condition (b) of (10.6.3)"). I | 213

A similar argument also shows that every *Noetherian* formal affine scheme is separated, which justifies the terminology.

Proposition (10.15.4). — *Let $\mathfrak{S}$ be a locally Noetherian formal prescheme, and $\mathfrak{X}$ and $\mathfrak{Y}$ locally Noetherian formal $\mathfrak{S}$-preschemes such that $\mathfrak{X}$ or $\mathfrak{Y}$ is of finite type over $\mathfrak{S}$ (so that $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ is locally Noetherian) and such that $\mathfrak{Y}$ is separated over $\mathfrak{S}$. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be an $\mathfrak{S}$-morphism; then the graph morphism $\Gamma_f(1_{\mathfrak{X}}, f)_{\mathfrak{S}} : \mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ is a closed immersion.*

PROOF. We can assume that $\mathfrak{S}$ is the inductive limit of a sequence (S_n) of locally Noetherian preschemes, $\mathfrak{X}$ (resp. $\mathfrak{Y}$) the inductive limit of a sequence (X_n) (resp. (Y_n)) of S_n-preschemes, and f the inductive limit of a sequence $(f_n : X_n \to Y_n)$ of S_n-morphisms; then $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ is the inductive limit of the sequence $(X_n \times_{S_n} Y_n)$, and Γ_f the inductive limit of the sequence (Γ_{f_n}) (10.7.4); by hypothesis, Y_0 is separated over S_0 (10.15.2), so the space $\Gamma_{f_0}(X_0)$ is a closed subspace of $X_0 \times_{S_0} Y_0$; since the underlying spaces of $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ (resp. $\Gamma_f(\mathfrak{X})$) and $X_0 \times_{S_0} Y_0$ (resp. $\Gamma_{f_0}(X_0)$) are identical, we already see that $\Gamma_f(\mathfrak{X})$ is a *closed* subspace of $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$. Now note that, when (U, V) runs over the set of pairs consisting of a Noetherian formal affine open subset U (resp. V) of $\mathfrak{X}$ (resp $\mathfrak{Y}$) such that $f(U) \subset V$, the open subsets $U \times_S V$ form a cover of $\Gamma_f(\mathfrak{X})$ in $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$, and, if $f_U : U \to V$ is the restriction of f to U, then $\Gamma_{f_U} : U \to U \times_{\mathfrak{S}} V$ is the restriction of Γ_f to U. If we show that Γ_{f_U} is a closed immersion, then Γ_f will be a closed immersion (10.14.4), or, in other words, we are led to consider the case where $\mathfrak{S} = \operatorname{Spf}(A)$, $\mathfrak{X} = \operatorname{Spf}(B)$, and $\mathfrak{Y} = \operatorname{Spf}(C)$ are affine (with A, B, and C Noetherian adics), with f corresponding to a continuous A-homomorphism $\phi : C \to B$; then Γ_f corresponds to the unique continuous homomorphism $\omega : B \widehat{\otimes}_A C \to B$ which, when composed with the canonical homomorphisms $B \to B \widehat{\otimes}_A C$ and $C \to B \widehat{\otimes}_A C$, gives the identity and ϕ (respectively). But it is clear that ω is *surjective*, whence our claim. ☐

Corollary (10.15.5). — *Let $\mathfrak{S}$ be a locally Noetherian formal prescheme, and $\mathfrak{X}$ an $\mathfrak{S}$-prescheme of finite type; for $\mathfrak{X}$ to be separated over $\mathfrak{S}$, it is necessary and sufficient for the diagonal morphism $\mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{S}} \mathfrak{X}$ to be a closed immersion.*

Proposition (10.15.6). — *A closed immersion $j : \mathfrak{Y} \to \mathfrak{X}$ of locally Noetherian formal preschemes is a separated morphism.*

PROOF. With the notation of (10.14.2), $j_0 : Y_0 \to X_0$ is a closed immersion, thus a separated morphism, and so it suffices to apply (10.15.2). ☐

Proposition (10.15.7). — *Let X be a locally Noetherian (usual) prescheme, X' a closed subset of X, and $\widehat{X} = X_{/X'}$. For $\widehat{X}$ to be separated, it is necessary and sufficient that $\widehat{X}_{\mathrm{red}}$ be separated, and it is sufficient that X be separated.*

PROOF. With the notation of (10.8.5), for $\widehat{X}$ to be separated, it is necessary and sufficient for X'_0 to be separated (10.15.2), and since $\widehat{X}_{\mathrm{red}} = (X'_0)_{\mathrm{red}}$, it is equivalent to ask for $\widehat{X}_{\mathrm{red}}$ to be separated (5.5.1, vi). ☐

CHAPTER II

Elementary global study of some classes of morphisms (EGA II)

SUMMARY

The various classes of morphisms studied in this chapter are used extensively in cohomological **II** | 5
methods; further study using these methods will be done in Chapter III, where we make particular use of §§2, 4, and 5 of Chapter II. On a first reading, §8 can be omitted: it supplements the formalism developed in §§1 and 3, reducing to easy applications of this formalism, and we will use it less consistently than the other results of this chapter.

§1. AFFINE MORPHISMS

1.1. S-preschemes and $\mathscr{O}_S$-algebras

(1.1.1). Let S be a prescheme, X an S-prescheme, and $f : X \to S$ its structure morphism. We know (**0**, 4.2.4) that the direct image $f_*(\mathscr{O}_X)$ is an $\mathscr{O}_S$-algebra, which we denote $\mathscr{A}(X)$ when there is little **II** | 6
chance of confusion; if U is an open subset of S, then we have

$$\mathscr{A}(f^{-1}(U)) = \mathscr{A}(X)|U.$$

Similarly, for every $\mathscr{O}_X$-module $\mathscr{F}$ (resp. every $\mathscr{O}_X$-algebra $\mathscr{B}$), we write $\mathscr{A}(\mathscr{F})$ (resp. $\mathscr{A}(\mathscr{B})$) for the direct image $f_*(\mathscr{F})$ (resp. $f_*(\mathscr{B})$) which is an $\mathscr{A}(X)$-module (resp. an $\mathscr{A}(X)$-algebra) and not only an $\mathscr{O}_S$-module (resp. an $\mathscr{O}_S$-algebra).

(1.1.2). Let Y be a second S-prescheme, $g : Y \to S$ its structure morphism, and $h : X \to Y$ an S-morphism; we then have the commutative diagram

(1.1.2.1)
$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S. & \end{tikzcd}$$

We have by definition $h = (\psi, \theta)$, where $\theta : \mathscr{O}_Y \to h_*(\mathscr{O}_X) = \psi_*(\mathscr{O}_X)$ is a homomorphism of sheaves of rings; we thus obtain (**0**, 4.2.2) a homomorphism of $\mathscr{O}_S$-algebras $g_*(\theta) : g_*(\mathscr{O}_Y) \to g_*(h_*(\mathscr{O}_X)) = f_*(\mathscr{O}_X)$, in other words, a homomorphism of $\mathscr{O}_S$-algebras $\mathscr{A}(Y) \to \mathscr{A}(X)$, which we denote by $\mathscr{A}(h)$. If $h' : Y \to Z$ is a second S-morphism, then it is immediate that $\mathscr{A}(h' \circ h) = \mathscr{A}(h) \circ \mathscr{A}(h')$. We have thus defined a *contravariant functor* $\mathscr{A}(X)$ from the category of S-preschemes to the category of $\mathscr{O}_S$-algebras.

Now let $\mathscr{F}$ be an $\mathscr{O}_X$-module, $\mathscr{G}$ an $\mathscr{O}_Y$-module, and $u : \mathscr{G} \to \mathscr{F}$ an h-morphism, that is (**0**, 4.4.1) a homomorphism of $\mathscr{O}_Y$-modules $\mathscr{G} \to h_*(\mathscr{F})$. Then $g_*(u) : g_*(\mathscr{G}) \to g_*(h_*(\mathscr{F})) = f_*(\mathscr{F})$ is a homomorphism $\mathscr{A}(\mathscr{G}) \to \mathscr{A}(\mathscr{F})$ of $\mathscr{O}_S$-modules, which we denote by $\mathscr{A}(u)$; in addition, the pair $(\mathscr{A}(h), \mathscr{A}(u))$ form a *di-homomorphism* from the $\mathscr{A}(Y)$-module $\mathscr{A}(\mathscr{G})$ to the $\mathscr{A}(X)$-module $\mathscr{A}(\mathscr{F})$.

(1.1.3). If we fix the prescheme S, then we can consider the pairs $(X, \mathscr{F})$, where X is an S-prescheme and $\mathscr{F}$ is an $\mathscr{O}_X$-module, as forming a *category*, by defining a *morphism* $(X, \mathscr{F}) \to (Y, \mathscr{G})$ as a pair (h, u), where $h : X \to Y$ is an S-morphism and $u : \mathscr{G} \to \mathscr{F}$ is an h-morphism. We can then say that $(\mathscr{A}(X), \mathscr{A}(\mathscr{F}))$ is a *contravariant functor* with values in the category whose objects are pairs consisting of an $\mathscr{O}_S$-algebra and a module over that algebra, and the morphisms are the di-homomorphisms.

1.2. Affine preschemes over a prescheme

Definition (1.2.1). — Let X be an S-prescheme, and $f : X \to S$ its structure morphism. We say that X is *affine over* S if there exists a cover (S_α) of S by affine open sets such that for all α, the prescheme induced by X on the open set $f^{-1}(S_\alpha)$ is affine.

Example (1.2.2). — Every closed subprescheme of S is an affine S-prescheme over S ((**I**, 4.2.3) and (**I**, 4.2.4)).

Remark (1.2.3). — An affine prescheme X over S is not necessarily an affine scheme, as the example $X = S$ shows (1.2.2). On the other hand, if an affine scheme X is an S-prescheme, then X is not necessarily affine over S (see Example (1.3.3)). However, remember that if S is a *scheme*, then every **II | 7**
S-prescheme which is an affine scheme is affine over S (**I**, 5.5.10).

Proposition (1.2.4). — *Every S-prescheme which is affine over S is separated over S (in other words, it is an S-scheme).*

PROOF. This follows immediately from (**I**, 5.5.5) and (**I**, 5.5.8). □

Proposition (1.2.5). — *Let X be an S-scheme affine over S, and $f : X \to S$ its structure morphism. For every open $U \subset S$, $f^{-1}(U)$ is affine over U.*

PROOF. By Definition (1.2.1), we can reduce to the case where $S = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ are affine; then $f = ({}^a\phi, \widetilde{\phi})$, where $\phi : A \to B$ is a homomorphism. As the $D(g)$ for $g \in A$ form a basis for S, we reduce to the case where $U = D(g)$; but we then know that $f^{-1}(U) = D(\phi(g))$ (**I**, 1.2.2.2), hence the proposition. □

Proposition (1.2.6). — *Let X be an S-scheme affine over S, and $f : X \to S$ its structure morphism. For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $f_*(\mathscr{F})$ is a quasi-coherent $\mathscr{O}_S$-module.*

PROOF. Taking into account Proposition (1.2.4), this follows from (**I**, 9.2.2, a). □

In particular, the $\mathscr{O}_S$-algebra $\mathscr{A}(X) = f_*(\mathscr{O}_X)$ is *quasi-coherent*.

Proposition (1.2.7). — *Let X be an S-scheme affine over S. For every S-prescheme Y, the map $h \mapsto \mathscr{A}(h)$ from the set* $\operatorname{Hom}_S(Y, X)$ *to the set* $\operatorname{Hom}(\mathscr{A}(X), \mathscr{A}(Y))$ (1.1.2) *is bijective.*

PROOF. Let $f : X \to S$ and $g : Y \to S$ be the structure morphisms. First, suppose that $S = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ are affine; we must prove that for every homomorphism $\omega : f_*(\mathscr{O}_X) \to g_*(\mathscr{O}_Y)$ of $\mathscr{O}_S$-algebras, there exists a unique S-morphism $h : Y \to X$ such that $\mathscr{A}(h) = \omega$. By definition, for every open $U \subset S$, ω defines a homomorphism $\omega_U = \Gamma(U, \omega) : \Gamma(f^{-1}(U), \mathscr{O}_X) \to \Gamma(g^{-1}(U), \mathscr{O}_Y)$ of $\Gamma(U, \mathscr{O}_S)$-algebras. In particular, for $U = S$, this gives a homomorphism $\phi : \Gamma(X, \mathscr{O}_X) \to \Gamma(Y, \mathscr{O}_Y)$ of $\Gamma(S, \mathscr{O}_S)$-algebras, to which corresponds a well-defined S-morphism $h : Y \to X$, since X is affine (**I**, 2.2.4). It remains to prove that $\mathscr{A}(h) = \omega$, or, in other words, that, for every open set U of a basis for S, ω_U coincides with the homomorphism of algebras ϕ_U corresponding to the S-morphism $g^{-1}(U) \to f^{-1}(U)$, a restriction of h. We can reduce to the case where $U = D(\lambda)$, with $\lambda \in S$; then, if $f = ({}^a\rho, \widetilde{\rho})$, where $\rho : A \to B$ is a ring homomorphism, we have $f^{-1}(U) = D(\mu)$, where $\mu = \rho(\lambda)$, and $\Gamma(f^{-1}(U), \mathscr{O}_X)$ is the ring of fractions B_μ; the diagram

$$\begin{tikzcd} B \arrow[r, "\phi"] \arrow[d] & \Gamma(Y, \mathscr{O}_Y) \arrow[d] \\ B_\mu \arrow[r, "\phi_U"] & \Gamma(g^{-1}(U), \mathscr{O}_Y) \end{tikzcd}$$

is commutative, and so is the analogous diagram where ϕ_U is replaced by ω_U; the equality $\phi_U = \omega_U$ then follows from the universal property of rings of fractions (**0**, 1.2.4).

We now pass to the general case; let (S_α) be a cover of S by affine open sets such that the $f^{-1}(S_\alpha)$ are affine. Then every homomorphism $\omega : \mathscr{A}(X) \to \mathscr{A}(Y)$ of $\mathscr{O}_S$-algebras gives by restriction a family of homomorphisms **II | 8**

$$\omega_\alpha : \mathscr{A}(f^{-1}(S_\alpha)) \longrightarrow \mathscr{A}(g^{-1}(S_\alpha))$$

of $\mathscr{O}_{S_\alpha}$-algebras, hence a family of S_α-morphisms $h_\alpha : g^{-1}(S_\alpha) \to f^{-1}(S_\alpha)$ by the above. It remains to see that for every affine open set of a basis for $S_\alpha \cap S_\beta$, the restriction of h_α and h_β to $g^{-1}(U)$ coincide, which is evident since by the above, these restrictions both correspond to the homomorphism $\mathscr{A}(X)|U \to \mathscr{A}(Y)|U$, a restriction of ω. □

Corollary (1.2.8). — *Let X and Y be two S-schemes which are affine over S. For an S-morphism $h : Y \to X$ to be an isomorphism, it is necessary and sufficient for $\mathscr{A}(h) : \mathscr{A}(X) \to \mathscr{A}(Y)$ to be an isomorphism.*

PROOF. This follows immediately from Proposition (1.2.7) and from the functorial nature of $\mathscr{A}(X)$. □

1.3. Affine preschemes over S associated to an $\mathscr{O}_S$-algebra

Proposition (1.3.1). — *Let S be a prescheme. For every quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{B}$, there exists a prescheme X affine over S, defined up to unique S-isomorphism, such that $\mathscr{A}(X) = \mathscr{B}$.*

PROOF. The uniqueness follows from Corollary (1.2.8); we prove the existence of X. For every affine open $U \subset S$, let X_U be the prescheme $\operatorname{Spec}(\Gamma(U, \mathscr{B}))$; as $\Gamma(U, \mathscr{B})$ is a $\Gamma(U, \mathscr{O}_S)$-algebras, X_U is an S-prescheme (**I**, 1.6.1). In addition, as $\mathscr{B}$ is quasi-coherent, the $\mathscr{O}_S$-algebra $\mathscr{A}(X_U)$ canonically identifies with $\mathscr{B}|U$ ((**I**, 1.3.7), (**I**, 1.3.13), (**I**, 1.6.3)). Let V be a second affine open subset of S, and let $X_{U,V}$ be the prescheme induced by X_U on $f_U^{-1}(U \cap V)$, where f_U denotes the structure morphism $X_U \to S$; $X_{U,V}$ and $X_{V,U}$ are affine over $U \cap V$ (1.2.5), and by definition $\mathscr{A}(X_{U,V})$ and $\mathscr{A}(X_{V,U})$ canonically identify with $\mathscr{B}|(U \cap V)$. Hence there is (1.2.8) a canonical S-isomorphism $\theta_{U,V} : X_{U,V} \to X_{U,V}$; in addition, if W is a third affine open subset of S, and if $\theta'_{U,V}$, $\theta'_{V,W}$, and $\theta'_{U,W}$ are the restrictions of $\theta_{U,V}$, $\theta_{V,W}$, and $\theta_{U,W}$ to the inverse images of $U \cap V \cap W$ in X_V, X_W, and X_W respectively under the structure morphisms, then we have $\theta'_{U,V} \circ \theta'_{V,W} = \theta'_{U,W}$. As a result, there exists a prescheme X, a cover (T_U) of X by affine open sets, and for every U an isomorphism $\phi_U : X_U \to T_U$, such that ϕ_U maps $f_U^{-1}(U \cap V)$ to $T_U \cap T_V$, and we have $\theta_{U,V} = \phi_U^{-1} \circ \phi_V$ (**I**, 2.3.1). The morphism $g_U = f_U \circ \phi_U^{-1}$ makes T_U an S-prescheme, and the morphisms g_U and g_V coincide on $T_U \cap T_V$, hence X is an S-prescheme. In addition, it is clear by definition that X is affine over S and that $\mathscr{A}(T_U) = \mathscr{B}|U$, hence $\mathscr{A}(X) = \mathscr{B}$. □

We say that the S-scheme X defined in this way is *associated to the $\mathscr{O}_S$-algebra $\mathscr{B}$*, or is the *spectrum of $\mathscr{B}$*, and we denote it by $\operatorname{Spec}(\mathscr{B})$.

Corollary (1.3.2). — *Let X be a prescheme affine over S, $f : X \to S$ the structure morphism. For every affine open $U \subset S$, the induced prescheme on $f^{-1}(U)$ is the affine scheme with ring $\Gamma(U, \mathscr{A}(X))$.*

PROOF. As we can suppose that X is associated to an $\mathscr{O}_S$-algebra by Propositions (1.2.6) and (1.3.1), the corollary follows from the construction of X described in Proposition (1.3.1). □ **II | 9**

Example (1.3.3). — Let S be the affine plane over a field K, where the point 0 has been doubled (**I**, 5.5.11); with the notation of (**I**, 5.5.11), S is the union of two affine open sets Y_1 and Y_2; if f is the open immersion $Y_1 \to S$, then $f^{-1}(Y_2) = Y_1 \cap Y_2$ is not an affine open set in Y_1 (*loc. cit.*), hence we have an example of an affine scheme which is not affine over S.

Corollary (1.3.4). — *Let S be an affine scheme; for an S-prescheme X to be affine over S, it is necessary and sufficient for X to be an affine scheme.*

Corollary (1.3.5). — *Let X be a prescheme affine over a prescheme S, and let Y be an X-prescheme. For Y to be affine over S, it is necessary and sufficient for Y to be affine over X.*

PROOF. We immediately reduce to the case where S is an affine scheme, and then we can reduce to the case where X is an affine scheme (1.3.4); the two conditions of the statement then give that Y is an affine scheme (1.3.4). □

(1.3.6). Let X be a prescheme affine over S. To define a prescheme Y affine *over* X, it is equivalent, by Corollary (1.3.5), to give a prescheme Y affine *over* S, and an S-morphism $g : Y \to X$; in other words (Proposition (1.3.1) and (1.2.7)), it is equivalent to give a quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{B}$ and a homomorphism $\mathscr{A}(X) \to \mathscr{B}$ of $\mathscr{O}_S$-algebras (which can be considered as defining on $\mathscr{B}$ an $\mathscr{A}(X)$-algebra structure). If $f : X \to S$ is the structure morphism, then we have $\mathscr{B} = f_*(g_*(\mathscr{O}_Y))$.

Corollary (1.3.7). — *Let X be a prescheme affine over S; for X to be of finite type over S, it is necessary and sufficient for the quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{A}(X)$ to be of finite type* (**I**, 9.6.2).

PROOF. By definition (**I**, 9.6.2), we can reduce to the case where S is affine; then X is an affine scheme (1.3.4), and if $S = \operatorname{Spec}(A)$, $X = \operatorname{Spec}(B)$, then $\mathscr{A}(X)$ is the $\mathscr{O}_S$-algebra $\widetilde{B}$; as $\Gamma(U, \widetilde{B}) = B$, the corollary follows from (**I**, 9.6.2) and (**I**, 6.3.3). □

Corollary (1.3.8). — *Let X be a prescheme affine over S; for X to be reduced, it is necessary and sufficient for the quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{A}(X)$ to be reduced* (**0**, 4.1.4).

PROOF. The question is local on S; by Corollary (1.3.2), the corollary follows from (**I**, 5.1.1) and (**I**, 5.1.4). □

1.4. Quasi-coherent sheaves over an affine prescheme over S

Proposition (1.4.1). — *Let X be a prescheme affine over S, Y an S-prescheme, and $\mathscr{F}$ (resp. $\mathscr{G}$) a quasi-coherent $\mathscr{O}_X$-module (resp. an $\mathscr{O}_Y$-module). Then the map $(h, u) \mapsto (\mathscr{A}(h), \mathscr{A}(u))$ from the set of morphism $(Y, \mathscr{G}) \to (X, \mathscr{F})$ to the set of di-homomorphisms $(\mathscr{A}(X), \mathscr{A}(\mathscr{F})) \to (\mathscr{A}(Y), \mathscr{A}(\mathscr{G}))$ ((1.1.2) and (1.1.3)) is bijective.*

PROOF. The proof follows exactly as that of Proposition (1.2.7) by using (**I**, 2.2.5) and (**I**, 2.2.4), and the details are left to the reader. □

Corollary (1.4.2). — *If, in addition to the hypotheses of Proposition (1.4.1), we suppose that Y is affine over S, then for (h, u) to be an isomorphism, it is necessary and sufficient for $(\mathscr{A}(h), \mathscr{A}(u))$ to be a di-isomorphism.*

Proposition (1.4.3). — *For every pair $(\mathscr{B}, \mathscr{M})$ consisting of a quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{B}$ and a quasi-coherent $\mathscr{B}$-module $\mathscr{M}$* (considered as an $\mathscr{O}_S$-module or as a $\mathscr{B}$-module, which are equivalent (**I**, 9.6.1)), *there exists a pair $(X, \mathscr{F})$ consisting of a prescheme X affine over S and of a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, such that $\mathscr{A}(X) = \mathscr{B}$ and $\mathscr{A}(\mathscr{F}) = \mathscr{M}$; in addition, this couple is determined up to unique isomorphism.* **II** | 10

PROOF. The uniqueness follows from Proposition (1.4.1) and Corollary (1.4.2); the existence is proved as in Proposition (1.3.1), and we leave the details to the reader. □

We denote by $\widetilde{\mathscr{M}}$ the $\mathscr{O}_X$-module $\mathscr{F}$, and we say that it is *associated* to the quasi-coherent $\mathscr{B}$-module $\mathscr{M}$; for every affine open $U \subset S$, $\mathscr{M}|p^{-1}(U)$ (where p is the structure morphism $X \to S$) is canonically isomorphic to $(\Gamma(U, \mathscr{M}))^{\sim}$.

Corollary (1.4.4). — *On the category of quasi-coherent $\mathscr{B}$-modules, $\widetilde{\mathscr{M}}$ is an additive covariant exact functor in $\mathscr{M}$, which commutes with inductive limits and direct sums.*

PROOF. We immediately reduce to the case where S is affine, and the corollary then follows from (**I**, 1.3.5), (**I**, 1.3.9), and (**I**, 1.3.11). □

Corollary (1.4.5). — *Under the hypotheses of Proposition (1.4.3), for $\widetilde{\mathscr{M}}$ to be an $\mathscr{O}_X$-module of finite type, it is necessary and sufficient for $\mathscr{M}$ to be a $\mathscr{B}$-module of finite type.*

PROOF. We immediately reduce to the case where $S = \operatorname{Spec}(A)$ is an affine scheme. Then $\mathscr{B} = \widetilde{B}$, where B is an A-algebra of finite type (**I**, 9.6.2), and $\mathscr{M} = \widetilde{M}$, where M is a B-module (**I**, 1.3.13); *over the prescheme* X, $\mathscr{O}_X$ is associated to the ring B and $\widetilde{\mathscr{M}}$ to the B-module M; for $\widetilde{\mathscr{M}}$ to be of finite type, it is therefore necessary and sufficient for M to be of finite type (**I**, 1.3.13), hence our assertion. □

Proposition (1.4.6). — *Let Y be a prescheme affine over S, X and X' two preschemes affine over Y* (hence also over S (1.3.5)). *Let $\mathscr{B} = \mathscr{A}(Y)$, $\mathscr{A} = \mathscr{A}(X)$, and $\mathscr{A}' = \mathscr{A}(X')$. Then $X \times_Y X'$ is affine over Y* (thus also over S), *and $\mathscr{A}(X \times_Y X')$ canonically identifies with $\mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$.*

Proof. By (I, 9.6.1), $\mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$ is a quasi-coherent $\mathscr{B}$-algebra, thus also a quasi-coherent $\mathscr{O}_S$-algebra (I, 9.6.1); let Z be the spectrum of $\mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$ (1.3.1). The canonical $\mathscr{B}$-homomorphisms $\mathscr{A} \to \mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$ and $\mathscr{A}' \to \mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$ correspond (1.2.7) to Y-morphisms $Z \to X$ and $p' : Z \to X'$. To see that the triple (Z, p, p') is a product $X \times_Y X'$, we can reduce to the case where S is an affine scheme with ring C (I, 3.2.6.4). But then Y, X, and X' are affine schemes (1.3.4) whose rings B, A, and A' are C-algebras such that $\mathscr{B} = \widetilde{B}$, $\mathscr{A} = \widetilde{A}$, and $\mathscr{A}' = \widetilde{A'}$. We then know (I, 1.3.13) that $\mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$ canonically identifies with the $\mathscr{O}_S$-algebra $(A \otimes_B A')^\sim$, hence the ring $A(Z)$ identifies with $A \otimes_B A'$ and the morphisms p and p' correspond to the canonical homomorphisms $A \to A \otimes_B A'$ and $A' \to A \otimes_B A'$. The proposition then follows from (I, 3.2.2). □

Corollary (1.4.7). — *Let $\mathscr{F}$ (resp. $\mathscr{F}'$) be a quasi-coherent $\mathscr{O}_X$-module (resp. $\mathscr{O}_{X'}$-module); then $\mathscr{A}(\mathscr{F} \otimes_Y \mathscr{F}')$ canonically identifies with $\mathscr{A}(\mathscr{F}) \otimes_{\mathscr{A}(Y)} \mathscr{A}(\mathscr{F}')$.*

Proof. We know that $\mathscr{F} \otimes_Y \mathscr{F}'$ is quasi-coherent over $X \times_Y X'$ (I, 9.1.2). Let $g : Y \to S$, $f : X \to Y$, and $f' : X' \to Y$ be the structure morphisms, such that the structure morphism $h : Z \to S$ **II | 11**
is equal to $g \circ f \circ p$ and to $g \circ f' \circ p'$. We define a canonical homomorphism

$$\mathscr{A}(\mathscr{F}) \otimes_{\mathscr{A}(Y)} \mathscr{A}(\mathscr{F}') \longrightarrow \mathscr{A}(\mathscr{F} \otimes_Y \mathscr{F}')$$

in the following way: for every open $U \subset S$, we have canonical homomorphisms $\Gamma(f^{-1}(g^{-1}(U)), \mathscr{F}) \to \Gamma(h^{-1}(U), p^*(\mathscr{F}))$ and $\Gamma(f'^{-1}(g^{-1}(U)), \mathscr{F}') \to \Gamma(h^{-1}(U), p'^*(\mathscr{F}'))$ (0, 4.4.3), thus we obtain a canonical homomorphism

$$\Gamma(f^{-1}(g^{-1}(U)), \mathscr{F}) \otimes_{\Gamma(g^{-1}(U), \mathscr{O}_Y)} \Gamma(f'^{-1}(g^{-1}(U)), \mathscr{F}') \longrightarrow \Gamma(h^{-1}(U), p^*(\mathscr{F})) \otimes_{\Gamma(h^{-1}(U), \mathscr{O}_Z)} \Gamma(h^{-1}(U), p'^*(\mathscr{F}')).$$

To see that we have defined an isomorphism of $\mathscr{A}(Z)$-modules, we can reduce to the case where S (and as a result X, X', Y, and $X \times_Y X'$) are affine scheme, and (with the notation of Proposition (1.4.6)), $\mathscr{F} = \widetilde{M}$, $\mathscr{F}' = \widetilde{M'}$, where M (resp. M') is an A-module (resp. an A'-module). Then $\mathscr{F} \otimes_Y \mathscr{F}'$ identifies with the sheaf on $X \times_Y X'$ associated to the $(A \otimes_B A')$-module $M \otimes_B M'$ (I, 9.1.3), and the corollary follows from the canonical identification of the $\mathscr{O}_S$-modules $(M \otimes_B M')^\sim$ and $\widetilde{M} \otimes_{\widetilde{B}} \widetilde{M'}$ (where M, M', and B are considered as C-modules) ((I, 1.3.12) and (I, 1.6.3)). □

If we apply Corollary (1.4.7) in particular to the case where $X = Y$ and $\mathscr{F}' = \mathscr{O}_{X'}$, then we see that the $\mathscr{A}'$-module $\mathscr{A}(f'^*(\mathscr{F}))$ identifies with $\mathscr{A}(\mathscr{F}) \otimes_{\mathscr{B}} \mathscr{A}'$.

(1.4.8). In particular, when $X = X' = Y$ (X being affine over S), we see that if $\mathscr{F}$ and $\mathscr{G}$ are two quasi-coherent $\mathscr{O}_X$-modules, then we have

$$\mathscr{A}(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}) = \mathscr{A}(\mathscr{F}) \otimes_{\mathscr{A}(X)} \mathscr{A}(\mathscr{G}) \tag{1.4.8.1}$$

up to canonical functorial isomorphism. If in addition $\mathscr{F}$ admits a finite presentation, then it follows from (I, 1.6.3) and (I, 1.3.12) that

$$\mathscr{A}(\mathscr{H}om_X(\mathscr{F}, \mathscr{G})) = \mathscr{H}om_{\mathscr{A}(X)}(\mathscr{A}(\mathscr{F}), \mathscr{A}(\mathscr{G})) \tag{1.4.8.2}$$

up to canonical isomorphism.

Remark (1.4.9). — If X and X' are two preschemes affine over S, then the sum $X \sqcup X'$ is also affine over S, as the sum of two affine schemes is an affine scheme.

Proposition (1.4.10). — *Let S be a prescheme, $\mathscr{B}$ a quasi-coherent $\mathscr{O}_S$-algebra, and $X = \operatorname{Spec}(\mathscr{B})$. For a quasi-coherent sheaf of ideals $\mathscr{J}$ of $\mathscr{B}$, $\widetilde{\mathscr{J}}$ is quasi-coherent sheaf of ideals of $\mathscr{O}_X$, and the closed subprescheme Y of X defined by $\widetilde{\mathscr{J}}$ is canonically isomorphic to* $\operatorname{Spec}(\mathscr{B}/\mathscr{J})$.

Proof. It follows immediately from (I, 4.2.3) that Y is affine over S; by Proposition (1.3.1), we reduce to the case where S is affine, and the proposition then follows immediately from (I, 4.1.2). □

We can also express the result of Proposition (1.4.10) by saying that if $h : \mathscr{B} \to \mathscr{B}'$ is a *surjective* homomorphism of quasi-coherent $\mathscr{O}_S$-algebras, $\mathscr{A}(h) : \operatorname{Spec}(\mathscr{B}') \to \operatorname{Spec}(\mathscr{B})$ is a *closed immersion*.

Proposition (1.4.11). — *Let S be a prescheme, $\mathscr{B}$ a quasi-coherent $\mathscr{O}_S$-algebra, and $X = \operatorname{Spec}(\mathscr{B})$. For* **II | 12**
every quasi-coherent sheaf of ideals $\mathscr{K}$ of $\mathscr{O}_S$, we have (denoting by f the structure morphism $X \to S$) $f^*(\mathscr{K})\mathscr{O}_X = (\mathscr{K}\mathscr{B})^\sim$ up to canonical isomorphism.

Proof. Since the questions is local on S, we can reduce to the case where $S = \operatorname{Spec}(A)$ is affine, and in this case the proposition is none other than (**I**, 1.6.9). □

1.5. Change of base prescheme

Proposition (1.5.1). — *Let X be a prescheme affine over S. For every extension $g : S' \to S$ of the base prescheme, $X' = X_{(S')} = X \times_S S'$ is affine over S'.*

Proof. If f' is the projection $X' \to S'$, then it suffices to prove that $f'^{-1}(U')$ is an affine open set for every affine open subset U' of S' such that $g(U')$ is contained in an affine open subset U of S (1.2.1); we can thus reduce to the case where S and S' are affine, and it suffices to prove that X' is then an affine scheme (1.3.4). But then (1.3.4) X is an affine scheme, and if A, A', and B are the rings of S, S', and X respectively, then we know that X' is the affine scheme with ring $A' \otimes_A B$ (**I**, 3.2.2). □

Corollary (1.5.2). — *Under the hypotheses of Proposition (1.5.1), let $f : X \to S$ be the structure morphism, $f' : X' \to S'$ and $g' : X' \to X$ the projections, such that the diagram*

$$\begin{tikzcd} X & X' \arrow[l, "g'"'] \arrow[d, "f'"] \\ S \arrow[from=u, "f"'] & S' \arrow[l, "g"] \end{tikzcd}$$

is commutative. For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, there exists a canonical isomorphism of $\mathscr{O}_{S'}$-modules

(1.5.2.1) $$u : g^*(f_*(\mathscr{F})) \simeq f'_*(g'^*(\mathscr{F})).$$

In particular, there exists a canonical isomorphism from $\mathscr{A}(X')$ to $g^*(\mathscr{A}(X))$.

Proof. To define u, it suffices to define a homomorphism

$$v : f_*(\mathscr{F}) \longrightarrow g_*(f'_*(g'^*(\mathscr{F}))) = f_*(g'_*(g'^*(\mathscr{F})))$$

and to set $u = v^\sharp$ (**0**, 4.4.3). We take $v = f_*(\rho)$, where ρ is the canonical homomorphism $\mathscr{F} \to g'_*(g'^*(\mathscr{F}))$ (**0**, 4.4.3). To prove that u is an isomorphism, we can reduce to the case where S and S', hence X and X', are affine; with the notation of Proposition (1.5.1), we then have $\mathscr{F} = \widetilde{M}$, where M is a B-module. We then note immediately that $g^*(f_*(\mathscr{F}))$ and $f'_*(g'^*(\mathscr{F}))$ are both equal to the $\mathscr{O}_{S'}$-module associated to the A'-module $A' \otimes_A M$ (where M is considered as an A-module), and that u is the homomorphism associated to the identity ((**I**, 1.6.3), (**I**, 1.6.5), (**I**, 1.6.7)). □

Remark (1.5.3). — We do not have that Corollary (1.5.2) remains true when X is not assumed affine over S, even when $S' = \operatorname{Spec}(k(s))$ ($s \in S$) and $S' \to S$ is the canonical morphism (**I**, 2.4.5)—in which case X' is none other than the *fibre* $f^{-1}(s)$ (**I**, 3.6.2). In other words, when X is not affine over S, the operation "direct image of quasi-coherent sheaves" does not commute with the operation **II** | 13
of "passing to fibres". However, we will see in Chapter III (**III**, 4.2.4) a result in this sense, of an "asymptotic" nature, valid for *coherent* sheaves on X when f is proper (5.4) and S is Noetherian.

Corollary (1.5.4). — *For every prescheme X affine over S and every $s \in S$, the fibre $f^{-1}(s)$ (where f denoted the structure morphism $X \to S$) is an affine scheme.*

Proof. It suffices to apply Proposition (1.5.1) with $S' = \operatorname{Spec}(k(s))$ and to use Corollary (1.3.4). □

Corollary (1.5.5). — *Let X be an S-prescheme, S' a prescheme affine over S; then $X' = X_{(S')}$ is a prescheme affine over X. In addition, if $f : X \to S$ is the structure morphism, then there is a canonical isomorphism of $\mathscr{O}_X$-algebras $\mathscr{A}(X') \simeq f^*(\mathscr{A}(S'))$, and for every quasi-coherent $\mathscr{A}(S')$-module $\mathscr{M}$, a canonical di-isomorphism $f^*(\mathscr{M}) \simeq \mathscr{A}(\widetilde{f'^*(\mathscr{M})})$, denoting by $f' = f_{(S')}$ the structure morphism $X' \to S'$.*

Proof. It suffices to swap the roles of X and S' in (1.5.1) and (1.5.2). □

(1.5.6). Now let S, S' be two preschemes, $q : S' \to S$ a morphism, $\mathscr{B}$ (resp. $\mathscr{B}'$) a quasi-coherent $\mathscr{O}_S$-algebra (resp. $\mathscr{O}_{S'}$-algebra), $u : \mathscr{B} \to \mathscr{B}'$ a q-morphism (that is, a homomorphism $\mathscr{B} \to q_*(\mathscr{B}')$ of $\mathscr{O}_S$-algebras). If $X = \mathrm{Spec}(\mathscr{B})$, $X' = \mathrm{Spec}(\mathscr{B}')$, then we canonically obtain a morphism

$$v = \mathrm{Spec}(u) : X' \longrightarrow X$$

such that the diagram

(1.5.6.1)
$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d] & X \arrow[d] \\ S' \arrow[r, "q"] & S \end{tikzcd}$$

is commutative (the vertical arrows being the structure morphisms). Indeed, the data of u is equivalent to that of a homomorphism of quasi-coherent $\mathscr{O}_{S'}$-algebras $u^\sharp : q^*(\mathscr{B}) \to \mathscr{B}'$ (**0**, 4.4.3); this thus canonically defines an S'-morphism

$$w : \mathrm{Spec}(\mathscr{B}') \longrightarrow \mathrm{Spec}(q^*(\mathscr{B}))$$

such that $\mathscr{A}(w) = u^\sharp$ (1.2.7). On the other hand, it follows from (1.5.2) that $\mathrm{Spec}(q^*(\mathscr{B}))$ canonically identifies with $X \times_S S'$; the morphism v is the composition $X' \xrightarrow{w} X \times_S S' \xrightarrow{p_1} X$ of w with the first projection, and the commutativity of (1.5.6.1) follows from the definitions. Let U (resp. U') be an affine open of S (resp. S') such that $q(U') \subset U$, $A = \Gamma(U, \mathscr{O}_S)$, $A' = \Gamma(U', \mathscr{O}_{S'})$ their rings, $B = \Gamma(U, \mathscr{B})$, $B' = \Gamma(U', \mathscr{B}')$; the restriction of u to a $(q|U')$-morphism: $\mathscr{B}|U \to \mathscr{B}'|U'$ corresponds to a di-homomorphism of algebras $B \to B'$; if V, V' are the inverse images of U, U' in X, X' respectively, under the structure morphisms, then the morphism $V' \to V$, the restriction of v, corresponds (**I**, 1.7.3) to the above di-homomorphism.

(1.5.7). Under the same hypotheses as in (1.5.6), let $\mathscr{M}$ be a quasi-coherent $\mathscr{B}$-module; there is then a canonical isomorphism of $\mathscr{O}_{X'}$-modules

(1.5.7.1)
$$v^*(\widetilde{\mathscr{M}}) \simeq (q^*(\mathscr{M}) \otimes_{q^*(\mathscr{B})} \mathscr{B}')^\sim.$$

Indeed, the canonical isomorphism (1.5.2.1) gives a canonical isomorphism from $p_1^*(\widetilde{\mathscr{M}})$ to the **II** | 14
sheaf on $\mathrm{Spec}(q^*(\mathscr{B}))$ associated to the $q^*(\mathscr{B})$-module $q^*(\mathscr{M})$, and it then suffices to apply (1.4.7).

1.6. Affine morphisms

(1.6.1). We say that a morphism $f : X \to Y$ of preschemes is *affine* if it defines X as a prescheme affine over Y. The properties of preschemes affine over another translates as follows in this language:

Proposition (1.6.2). —

(i) *A closed immersion is affine.*
(ii) *The composition of two affine morphisms is affine.*
(iii) *If $f : X \to Y$ is an affine S-morphism, then $f_{(S')} : X_{(S')} \to Y_{(Y')}$ is affine for every base change $S' \to S$.*
(iv) *If $f : X \to Y$ and $f' : X' \to Y'$ are two affine S-morphisms, then*
$$f \times_S f' : X \times_S X' \longrightarrow Y \times_S Y'$$
is affine.
(v) *If $f : X \to Y$ and $g : Y \to Z$ are two morphisms such that $g \circ f$ is affine and g is separated, then f is affine.*
(vi) *If f is affine, then so if f_{red}.*

Proof. By (**I**, 5.5.12), it suffices to prove (i), (ii), and (iii). But (i) is none other than Example (1.2.2), and (ii) is none other than Corollary (1.3.5); finally, (iii) follows from Proposition (1.5.1), since $X_{(S')}$ identifies with the product $X \times_Y Y_{(S')}$ (**I**, 3.3.11). □

Corollary (1.6.3). — *If X is an affine scheme and Y is a scheme, then every morphism $f : X \to Y$ is affine.*

Proposition (1.6.4). — *Let Y be a locally Noetherian prescheme, $f : X \to Y$ a morphism of finite type. For f to be affine, it is necessary and sufficient for f_{red} to be.*

PROOF. By (1.6.2, vi), we see only need to prove the sufficiency of the condition. It suffices to prove that if Y is affine and Noetherian, then X is affine; but Y_{red} is then affine, so the same is true for X_{red} by hypothesis. Now X is Noetherian, so the conclusion follows from (**I**, 6.1.7). □

1.7. Vector bundle associated to a sheaf of modules

(1.7.1). Let A be a ring, E an A-module. Recall that we call the *symmetric algebra* on E and denote by $\mathbf{S}(E)$ (or $\mathbf{S}_A(E)$) the quotient algebra of the tensor algebra $\mathbf{T}(E)$ by the two-sided ideal generated by the elements $x \otimes y - y \otimes x$, where x and y vary over E. The algebra $\mathbf{S}(E)$ is characterized by the following universal property: if σ is the canonical map $E \to \mathbf{S}(E)$ (obtained by composing $E \to \mathbf{T}(E)$ with the canonical map $\mathbf{T}(E) \to \mathbf{S}(E)$), then every A-linear map $E \to B$, where B is a *commutative* A-algebra, factors uniquely as $E \xrightarrow{\sigma} \mathbf{S}(E) \xrightarrow{g} B$, where g is an A-homomorphism *of algebras*. We immediately deduce from this characterization that for two A-modules E and F, we have

$$\mathbf{S}(E \oplus F) = \mathbf{S}(E) \otimes \mathbf{S}(F)$$

up to canonical isomorphism; in addition, $\mathbf{S}(E)$ is a covariant functor in E, from the category of II | 15
A-modules to that of commutative A-algebras; finally, the above characterization also shows that if $E = \varinjlim E_\lambda$, then we have $\mathbf{S}(E) = \varinjlim \mathbf{S}(E_\lambda)$ up to canonical isomorphism. By abuse of language, a product $\sigma(x_1)\sigma(x_2)\cdots\sigma(x_n)$, where $x_i \in E$, is often denoted by $x_1x_2\cdots x_n$ if no confusion follows. The algebra $\mathbf{S}(E)$ is *graded*, $\mathbf{S}_n(E)$ being the set of linear combinations of n elements of E ($n \geqslant 0$); the algebra $\mathbf{S}(A)$ is canonically isomorphic to the polynomial algebra $A[T]$ is an indeterminate, and the algebra $\mathbf{S}(A^n)$ with the polynomial algebra in n indeterminates $A[T_1, \ldots, T_n]$.

(1.7.2). Let ϕ be a ring homomorphism $A \to B$. If F is a B-module, then the canonical map $F \to \mathbf{S}(F)$ gives a canonical map $F_{[\phi]} \to \mathbf{S}(F)_{[\phi]}$, which thus factors as $F_{[\phi]} \to \mathbf{S}(F_{[\phi]}) \to \mathbf{S}(F)_{[\phi]}$; the canonical homomorphism $\mathbf{S}(F_{[\phi]}) \to \mathbf{S}(F)_{[\phi]}$ is surjective, but not necessarily bijective. If E is an A-module, then every di-homomorphism $E \to F$ (that is to say, every A-homomorphism $E \to F_{[\phi]}$) thus canonically gives an A-homomorphism of algebras $\mathbf{S}(E) \to \mathbf{S}(F_{[\phi]}) \to \mathbf{S}(F)_{[\phi]}$, that is to say a di-homomorphism of algebras $\mathbf{S}(E) \to \mathbf{S}(F)$.

With the same notations, for every A-module E, $\mathbf{S}(E \otimes_A B)$ canonically identifies with the algebra $\mathbf{S}(E) \otimes_A B$; this follows immediately from the universal property of $\mathbf{S}(E)$ (1.7.1).

(1.7.3). Let R be a multiplicative subset of the ring A; apply (1.7.2) to the ring $B = R^{-1}A$, and remembering that $R^{-1}E = E \otimes_A R^{-1}A$, we see that we have $\mathbf{S}(R^{-1}E) = R^{-1}\mathbf{S}(E)$ up to canonical isomorphism. In addition, if $R' \supset R$ is a second multiplicative subset of A, then the diagram

$$\begin{tikzcd} R^{-1}E \arrow[r] \arrow[d] & R'^{-1}E \arrow[d] \\ \mathbf{S}(R^{-1}E) \arrow[r] & \mathbf{S}(R'^{-1}E) \end{tikzcd}$$

is commutative.

(1.7.4). Now let $(S, \mathscr{A})$ be a ringed space, and let $\mathscr{E}$ be a $\mathscr{A}$-module over S. If to any open $U \subset S$ we associate the $\Gamma(U, \mathscr{A})$-module $\mathbf{S}(\Gamma(U, \mathscr{E}))$, then we define (see the functorial nature of $\mathbf{S}(E)$ (1.7.2)) a presheaf of algebras; we say that the associated sheaf, which we denote by $\mathbf{S}(\mathscr{E})$ or $\mathbf{S}_{\mathscr{A}}(\mathscr{E})$ is the *symmetric $\mathscr{A}$-algebra* on the $\mathscr{A}$-module $\mathscr{E}$. It follows immediately from (1.7.1) that $\mathbf{S}(\mathscr{E})$ is a solution to a universal problem: every homomorphism of $\mathscr{A}$-modules $\mathscr{E} \to \mathscr{B}$, where $\mathscr{B}$ is an $\mathscr{A}$-algebra, factors uniquely as $\mathscr{E} \to \mathbf{S}(\mathscr{E}) \to \mathscr{B}$, the second arrow being a homomorphism of $\mathscr{A}$-algebras. There is thus a bijective correspondence between homomorphisms $\mathscr{E} \to \mathscr{B}$ of $\mathscr{A}$-modules and homomorphisms $\mathbf{S}(\mathscr{E}) \to \mathscr{B}$ of $\mathscr{A}$-algebras. In particular, every homomorphism $u : \mathscr{E} \to \mathscr{F}$ of $\mathscr{A}$-modules defines a homomorphism $\mathbf{S}(u) : \mathbf{S}(\mathscr{E}) \to \mathbf{S}(\mathscr{F})$ of $\mathscr{A}$-algebras, and $\mathbf{S}(\mathscr{E})$ is thus a covariant functor in $\mathscr{E}$.

By (1.7.2) and the commutativity of $\mathbf{S}$ with inductive limits, we have $(\mathbf{S}(\mathscr{E}))_x = \mathbf{S}(\mathscr{E}_x)$ for every II | 16
point $x \in S$. If $\mathscr{E}, \mathscr{F}$ are two $\mathscr{A}$-modules, then $\mathbf{S}(\mathscr{E} \oplus \mathscr{F})$ canonically identifies with $\mathbf{S}(\mathscr{E}) \otimes_{\mathscr{A}} \mathbf{S}(\mathscr{F})$, as we see for the corresponding presheaves.

We also note that $\mathbf{S}(\mathscr{E})$ is a graded $\mathscr{A}$-algebra, the infinite direct sum of the $\mathbf{S}_n(\mathscr{E})$, where the $\mathscr{A}$-module $\mathbf{S}_n(\mathscr{E})$ is the sheaf associated to the presheaf $U \mapsto \mathbf{S}_n(\Gamma(U, \mathscr{E}))$. If we take in particular

$\mathscr{E} = \mathscr{A}$, then we see that $\mathbf{S}_{\mathscr{A}}(\mathscr{A})$ identifies with $\mathscr{A}[T] = \mathscr{A} \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ (T indeterminate, $\mathbf{Z}$ being considered as a simple sheaf).

(1.7.5). Let $(T, \mathscr{B})$ be a second ringed space, f a morphism $(S, \mathscr{A}) \to (T, \mathscr{B})$. If $\mathscr{F}$ is a $\mathscr{B}$-module, then $\mathbf{S}(f^*(\mathscr{F}))$ canonically identifies with $f^*(\mathbf{S}(\mathscr{F}))$; indeed, if $f = (\psi, \theta)$, then by definition (**0**, 4.3.1),

$$\mathbf{S}(f^*(\mathscr{F})) = \mathbf{S}(\psi^*(\mathscr{F}) \otimes_{\psi^*(\mathscr{B})} \mathscr{A}) = \mathbf{S}(\psi^*(\mathscr{F})) \otimes_{\psi^*(\mathscr{B})} \mathscr{A}$$

(1.7.2); for every open U of S and every section h of $\mathbf{S}(\psi^*(\mathscr{F}))$ over U, h coincides, in a neighbourhood V of every point $s \in U$, with an element of $\mathbf{S}(\Gamma(V, \psi^*(\mathscr{F})))$; if we refer to the definition of $\psi^*(\mathscr{F})$ (**0**, 3.7.1) and take into account that every element of $\mathbf{S}(E)$ for a module E is a linear combination of a finite number of products of elements of E, then we see that there is a neighbourhood W of $\psi(s)$ in T, a section h' of $\mathbf{S}(\mathscr{F})$ over W, and a neighbourhood $V' \subset V \cap \psi^{-1}(W)$ of s such that h coincides with $t \mapsto h'(\psi(t))$ over V'; hence out assertion.

Proposition (1.7.6). — *Let A be a ring, $S = \operatorname{Spec}(A)$ its prime spectrum, $\mathscr{E} = \widetilde{M}$ the $\mathscr{O}_S$-module associated to an A-module M; then the $\mathscr{O}_S$-algebra $\mathbf{S}(\mathscr{E})$ is associated to the A-algebra $\mathbf{S}(M)$.*

Proof. For every $f \in A$, $\mathbf{S}(M_f) = (\mathbf{S}(M))_f$ (1.7.3), and the proposition thus follows from Definition (**I**, 1.3.4). □

Corollary (1.7.7). — *If S is a prescheme, $\mathscr{E}$ a quasi-coherent $\mathscr{O}_S$-module, then the $\mathscr{O}_S$-algebra $\mathbf{S}(\mathscr{E})$ is quasi-coherent. If in addition $\mathscr{E}$ is of finite type, then each of the $\mathscr{O}_S$-modules $\mathbf{S}_n(\mathscr{E})$ is of finite type.*

Proof. The first assertion is an immediate consequence of (1.7.6) and of (**I**, 1.4.1); the second follows from the fact that if E is an A-module of finite type, then $\mathbf{S}_n(E)$ is an A-module of finite type; we then apply (**I**, 1.3.13). □

Definition (1.7.8). — Let $\mathscr{E}$ be a quasi-coherent $\mathscr{O}_S$-module. We call the *vector bundle over S defined by $\mathscr{E}$* and denote by $\mathbf{V}(\mathscr{E})$ the spectrum (1.3.1) of the quasi-coherent $\mathscr{O}_S$-algebra $\mathbf{S}(\mathscr{E})$.

By (1.2.7), for every S-prescheme X, there is a canonical bijective correspondence between the S-morphisms $X \to \mathbf{V}(\mathscr{E})$ and the homomorphisms of *$\mathscr{O}_S$-algebras* $\mathbf{S}(\mathscr{E}) \to \mathscr{A}(X)$, thus also between these S-morphisms and the homomorphisms of *$\mathscr{O}_S$-modules* $\mathscr{E} \to \mathscr{A}(X) = f_*(\mathscr{O}_X)$ (where f is the structure morphism $X \to S$). In particular:

(1.7.9). Take for X a subprescheme induced by S on an *open* $U \subset S$. Then the S-morphisms $U \to \mathbf{V}(\mathscr{E})$ are none other than the U-sections (**I**, 2.5.5) of the U-prescheme induced by $\mathbf{V}(\mathscr{E})$ on the open $p^{-1}(U)$ (where p is the structure morphism $\mathbf{V}(\mathscr{E}) \to S$). From what we have just seen, these U-sections bijectively correspond to homomorphisms of $\mathscr{O}_S$-modules $\mathscr{E} \to j_*(\mathscr{O}_S|U)$ (where j is the canonical injection $U \to S$), or equivalently (**0**, 4.4.3) with the $(\mathscr{O}_S|U)$-homomorphisms $j^*(\mathscr{E}) = \mathscr{E}|U \to \mathscr{O}_S|U$. **II** | 17
In addition, it is immediate that the restriction to an open $U' \subset U$ of an S-morphism $U \to \mathbf{V}(\mathscr{E})$ corresponds to the restriction to U' of the corresponding homomorphism $\mathscr{E}|U \to \mathscr{O}_S|U$. We conclude that *the sheaf of germs of S-sections* of $\mathbf{V}(\mathscr{E})$ is canonically identified with the *dual* $\mathscr{E}^\vee$ of $\mathscr{E}$.

In particular, if we set $X = U = S$, then the *zero* homomorphism $\mathscr{E} \to \mathscr{O}_S$ corresponds to a canonical S-section of $\mathbf{V}(\mathscr{E})$, called the *zero S-section* (cf. (8.3.3)).

(1.7.10). Now take X to be the spectrum $\{\xi\}$ of a field K; the structure morphism $f : X \to S$ then corresponds to a monomorphism $k(s) \to K$, where $s = f(\xi)$ (**I**, 2.4.6); the S-morphisms $\{\xi\} \to \mathbf{V}(\mathscr{E})$ are none other than the *geometric points of* $\mathbf{V}(\mathscr{E})$ *with values in the extension K of* $k(s)$ (**I**, 3.4.5), points which are localized at the points of $p^{-1}(s)$. The set of these points, which we can call *the rational geometric fibre over K* of $\mathbf{V}(\mathscr{E})$ *over the point s*, is identified by (1.7.8) with the set of homomorphisms of $\mathscr{O}_S$-modules $\mathscr{E} \to f_*(\mathscr{O}_X)$, or, equivalently (**0**, 4.4.3) with the set of homomorphisms of $\mathscr{O}_X$-modules $f^*(\mathscr{E}) \to \mathscr{O}_X = K$. But we have by definition (**0**, 4.3.1) $f^*(\mathscr{E}) = \mathscr{E}_s \otimes_{\mathscr{O}_s} K = \mathscr{E}^s \otimes_{k(s)} K$, setting $\mathscr{E}^s = \mathscr{E}_s/\mathfrak{m}_s\mathscr{E}_s$; the geometric fibre of $\mathbf{V}(\mathscr{E})$ rational over K over s thus identifies with the *dual* of the *K-vector space $\mathscr{E}^s \otimes_{k(s)} K$*; if $\mathscr{E}^s$ or K is of finite dimension over $k(s)$, then this dual also identifies with $(\mathscr{E}^s)^\vee \otimes_{k(s)} K$, denoting by $(\mathscr{E}^s)^\vee$ the dual of the $k(s)$-vector space $\mathscr{E}^s$.

Proposition (1.7.11). —

(i) $\mathbf{V}(\mathscr{E})$ *is a contravariant functor in* $\mathscr{E}$ *from the category of quasi-coherent* $\mathscr{O}_S$*-modules to the category of affine* S*-schemes.*
(ii) *If* $\mathscr{E}$ *is an* $\mathscr{O}_S$*-module of finite type, then* $\mathbf{V}(\mathscr{E})$ *is of finite type over* S*.*
(iii) *If* $\mathscr{E}$ *and* $\mathscr{F}$ *are two quasi-coherent* $\mathscr{O}_S$*-modules, then* $\mathbf{V}(\mathscr{E} \oplus \mathscr{F})$ *canonically identifies with* $\mathbf{V}(\mathscr{E}) \times_S \mathbf{V}(\mathscr{F})$*.*
(iv) *Let* $g : S' \to S$ *be a morphism; for every quasi-coherent* $\mathscr{O}_S$*-module* $\mathscr{E}$*,* $\mathbf{V}(g^*(\mathscr{E}))$ *canonically identifies with* $\mathbf{V}(\mathscr{E})_{(S')} = \mathbf{V}(\mathscr{E}) \times_S S'$*.*
(v) *A surjective homomorphism* $\mathscr{E} \to \mathscr{F}$ *of quasi-coherent* $\mathscr{O}_S$*-modules corresponds to a closed immersion* $\mathbf{V}(\mathscr{F}) \to \mathbf{V}(\mathscr{E})$*.*

PROOF. (i) is an immediate consequence of (1.2.7), taking into account that every homomorphism of $\mathscr{O}_S$-modules $\mathscr{E} \to \mathscr{F}$ canonically defines a homomorphism of $\mathscr{O}_S$-algebras $\mathbf{S}(\mathscr{E}) \to \mathbf{S}(\mathscr{F})$. (ii) follows immediately from the definition (**I**, 6.3.1) and the fact that if E is an A-module of finite type, then $\mathbf{S}(E)$ is an A-algebra of finite type. To prove (iii), it suffices to start with the canonical isomorphism $\mathbf{S}(\mathscr{E} \oplus \mathscr{F}) \simeq \mathbf{S}(\mathscr{E}) \otimes_{\mathscr{O}_S} \mathbf{S}(\mathscr{F})$ (1.7.4) and to apply (1.4.6). Similarly, to prove (iv), it suffices to start with the canonical isomorphism $\mathbf{S}(g^*(\mathscr{E})) \simeq g^*(\mathbf{S}(\mathscr{E}))$ (1.7.5) and to apply (1.5.2). Finally, to establish (v), it suffices to remark that if the homomorphism $\mathscr{E} \to \mathscr{F}$ is surjective, then so is the corresponding homomorphism $\mathbf{S}(\mathscr{E}) \to \mathbf{S}(\mathscr{F})$ of $\mathscr{O}_S$-algebras, and the conclusion follows from (1.4.10). ☐

(1.7.12). Take in particular $\mathscr{E} = \mathscr{O}_S$; the prescheme $\mathbf{V}(\mathscr{O}_S)$ is the affine S-scheme, spectrum of the $\mathscr{O}_S$-algebra $\mathbf{S}(\mathscr{O}_S)$ which identifies with the $\mathscr{O}_S$-algebra $\mathscr{O}_S[T] = \mathscr{O}_S \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ (T indeterminate); this **II** | 18
is evident when $S = \operatorname{Spec}(\mathbf{Z})$, by virtue of (1.7.6), and we pass from there to the general case by considering the structure morphism $S \to \operatorname{Spec}(\mathbf{Z})$ and using (1.7.11, iv). Because of this result, we set $\mathbf{V}(\mathscr{O}_S) = S[T]$, and we thus have the formula

(1.7.12.1) $$S[T] = S \otimes_{\mathbf{Z}} \mathbf{Z}[T].$$

The identification of the sheaf of germs of S-sections of $S[T]$ with $\mathscr{O}_S$, already seen in (**I**, 3.3.15), here in a more general context, as a special case of (1.7.9).

(1.7.13). For every S-prescheme X, we have seen (1.7.8) that $\operatorname{Hom}_S(X, S[T])$ canonically identifies with $\operatorname{Hom}_{\mathscr{O}_S}(\mathscr{O}_S, \mathscr{A}(X))$, which is canonically isomorphic to $\Gamma(S, \mathscr{A}(X))$, and as a result is equipped with the structure of a ring; in addition, to every S-morphism $h : X \to Y$ there corresponds a morphism $\Gamma(\mathscr{A}(h)) : \Gamma(S, \mathscr{A}(Y)) \to \Gamma(S, \mathscr{A}(X))$ for the ring structures (1.1.2). When we equip $\operatorname{Hom}_S(X, S[T])$ with a ring structure as defined, then we can see that $\operatorname{Hom}(X, S[T])$ can be considered as a *contravariant functor* in X, from the category of S-preschemes to that of rings. On the other hand, $\operatorname{Hom}_S(X, \mathbf{V}(\mathscr{E}))$ is likewise identified with $\operatorname{Hom}_{\mathscr{O}_S}(\mathscr{E}, \mathscr{A}(X))$ (where $\mathscr{A}(X)$ is considered as an $\mathscr{O}_S$*-module*); as a result, we can canonically equip it with a *module* structure over the ring $\operatorname{Hom}_S(X, S[T])$, and we see as above that the pair

$$(\operatorname{Hom}_S(X, S[T]), \operatorname{Hom}(X, \mathbf{V}(\mathscr{E})))$$

is a contravariant functor in X, with values in the category whose elements are the pairs (A, M) consisting of a ring A and an A-module M, the morphisms being di-homomorphisms.

We will interpret these facts by saying that $S[T]$ is an *S-scheme of rings* and that $\mathbf{V}(\mathscr{E})$ is an *S-scheme of modules* on the S-scheme of rings $S[T]$ (cf. Chapter 0, §8).

(1.7.14). We will see that the structure of an S-scheme of modules defined on the S-scheme $\mathbf{V}(\mathscr{E})$ allows us to reconstruct the $\mathscr{O}_S$-module $\mathscr{E}$ up to unique isomorphism: for this, we will show that $\mathscr{E}$ is canonically isomorphic to an $\mathscr{O}_S$-submodule of $\mathbf{S}(\mathscr{E}) = \mathscr{A}(\mathbf{V}(\mathscr{E}))$, defined by means of this structure. Indeed (1.7.4) the set $\operatorname{Hom}_{\mathscr{O}_S}(\mathbf{S}(\mathscr{E}), \mathscr{A}(X))$ of homomorphisms of $\mathscr{O}_S$*-algebras* is canonically identified with $\operatorname{Hom}_{\mathscr{O}_S}(\mathscr{E}, \mathscr{A}(X))$, the set of homomorphisms of $\mathscr{O}_S$*-modules*: if h and h' are two elements of this latter set, s_i ($1 \leqslant i \leqslant n$) sections of $\mathscr{E}$ over an open $U \subset S$, t a section of $\mathscr{A}(X)$ over U, then we have by definition

$$(h + h')(s_1 s_2 \cdots s_n) = \prod_{i=1}^{n} (h(s_i) + h'(s_i))$$

and

$$(t \cdot h)(s_1 s_2 \cdots s_n) = t^n \prod_{i=1}^{n} h(s_i).$$

This being so, if z is a section of $\mathbf{S}(\mathscr{E})$ over U, then $h \mapsto h(z)$ is a map from $\mathrm{Hom}_S(X, \mathbf{V}(\mathscr{E})) = \mathrm{Hom}_{\mathscr{O}_S}(\mathbf{S}(\mathscr{E}), \mathscr{A}(X))$ to $\Gamma(U, \mathscr{A}(X))$. We will show that $\mathscr{E}$ is identified with a submodule of $\mathbf{S}(\mathscr{E})$ **II** | 19 such that, *for every open $U \subset S$, every section z of this $\mathscr{O}_S$-submodule of U, and every S-prescheme X, the map $h \mapsto h(z)$ from $\mathrm{Hom}_{\mathscr{O}_S}(\mathbf{S}(\mathscr{E})|U, \mathscr{A}(X)|U)$ to $\Gamma(U, \mathscr{A}(X))$ is a homomorphism of $\Gamma(U, \mathscr{A}(X))$-modules*.

It is immediate that $\mathscr{E}$ has this property; to show the converse, we can reduce to proving that when $S = \mathrm{Spec}(A)$, $\mathscr{E} = \widetilde{M}$, a section z of $\mathbf{S}(\mathscr{E})$ over S that (for $U = S$) has the property stated above is necessarily a section of $\mathscr{E}$; we then have $z = \sum_{n=0}^{\infty} z_n$, where $z_n \in \mathbf{S}_n(M)$, and it is a question of proving that $z_n = 0$ for $n \neq 1$. Set $B = \mathbf{S}(M)$ and take for X the prescheme $\mathrm{Spec}(B[T])$, where T is an indeterminate. The set $\mathrm{Hom}_{\mathscr{O}_S}(\mathbf{S}(\mathscr{E}), \mathscr{A}(X))$ identifies with the set of ring homomorphisms $h : B \to B[T]$ (**I**, 1.3.13), and from what we saw above, we have $(T \cdot h)(z) = \sum_{n=0}^{\infty} T^n h(z_n)$: the hypothesis on z implies that we have $\sum_{n=0}^{\infty} T^n h(z_n) = T \cdot \sum_{n=0}^{\infty} h(z_n)$ for every homomorphism h. In in particular we take for h the canonical injection, then $\sum_{n=0}^{\infty} T^n z_n = T \cdot \sum_{n=0}^{\infty} z_n$, which implies the conclusion $z_n = 0$ for $n \neq 1$.

Proposition (1.7.15). — *Let Y be a prescheme whose underlying space is Noetherian, or a quasi-compact scheme. Every affine Y-scheme X of finite type over Y is Y-isomorphic to a closed Y-subscheme of a Y-scheme of the form $\mathbf{V}(\mathscr{E})$, where $\mathscr{E}$ is a quasi-coherent $\mathscr{O}_Y$-module of finite type.*

PROOF. The quasi-coherent $\mathscr{O}_Y$-algebra $\mathscr{A}(X)$ is of finite type (1.3.7). The hypotheses imply that $\mathscr{A}(X)$ is generated by a quasi-coherent $\mathscr{O}_Y$-submodule of finite type $\mathscr{E}$ (**I**, 9.6.5); by definition, this implies that the canonical homomorphism $\mathbf{S}(\mathscr{E}) \to \mathscr{A}(X)$ canonically extending the injection $\mathscr{E} \to \mathscr{A}(X)$ is *surjective*; the conclusion then follows from (1.4.10). □

§2. HOMOGENEOUS PRIME SPECTRA

2.1. Generalities on graded rings and modules

Notation (2.1.1). — Given a *positively graded* ring S, we denote by S_n the subset of S consisting of homogeneous elements of degree n ($n \geqslant 0$), by S_+ the (direct) sum of the S_n for $n > 0$; we have $1 \in S_0$, S_0 is a subring of S, S_+ is a graded ideal of S, and S is the direct sum of S_0 and S_+. If M is a *graded* module over S (with positive or negative degrees), we similarly denote by M_n the S_0-module consisting of homogeneous elements of M of degree n (with $n \in \mathbf{Z}$).

For every integer $d > 0$, we denote by $S^{(d)}$ the direct sum of the S_{nd}; by considering the elements of S_{nd} as homogeneous of degree n, the S_{nd} define on $S^{(d)}$ a graded ring structure.

For every integer k such that $0 \leqslant k \leqslant d-1$, we denote by $M^{(d,k)}$ the direct sum of the M_{nd+k} **II** | 20 ($n \in \mathbf{Z}$); this is a graded $S^{(d)}$-module when we consider the elements of M_{nd+k} as homogeneous of degree n. We write $M^{(d)}$ in place of $M^{(d,0)}$.

With the above notation, for every integer n (positive or negative), we denote by $M(n)$ the graded S-module defined by $(M(n))_k = M_{n+k}$ for every $k \in \mathbf{Z}$. In particular, $S(n)$ will be a graded S-module such that $(S(n))_k = S_{n+k}$, by agreeing to set $S_n = 0$ for $n < 0$. We say that a graded S-module M is *free* if it is isomorphic, considered as a *graded* module, to a direct sum of modules of the form $S(n)$; as $S(n)$ is a monogeneous S-module, generated by the element 1 of S considered as an element of degree $-n$, it is equivalent to say that M admits a *basis* over S consisting of *homogeneous* elements.

We say that a graded S-module M *admits a finite presentation* if there exists an exact sequence $P \to Q \to M \to 0$, where P and Q are finite direct sums of modules of the form $S(n)$ and the homomorphisms are of degree 0 (cf. (2.1.2)).

(2.1.2). Let M and N be two graded S-modules; we define on $M \otimes_S N$ a *graded* S-module structure in the following way. On the tensor product $M \otimes_{\mathbf{Z}} N$, we can define a graded $\mathbf{Z}$-module structure (where $\mathbf{Z}$ is graded by $\mathbf{Z}_0 = \mathbf{Z}$, $\mathbf{Z}_n = 0$ for $n \neq 0$) by setting $(M \otimes_{\mathbf{Z}} N)_q = \bigoplus_{m+n=q} M_m \otimes_{\mathbf{Z}} N_n$ (as M and N are respectively direct sums of the M_m and the N_n, we know that we can canonically identify $M \otimes_{\mathbf{Z}} N$ with the direct sum of all the $M_m \otimes_{\mathbf{Z}} N_n$). This being so, we have $M \otimes_S N = (M \otimes_{\mathbf{Z}} N)/P$, where P is the $\mathbf{Z}$-submodule of $M \otimes_{\mathbf{Z}} N$ generated by the elements $(xs) \otimes y - x \otimes (sy)$ for $x \in M$,

$y \in N$, $s \in S$; it is clear that P is a *graded* $\mathbf{Z}$-submodule of $M \otimes_{\mathbf{Z}} N$, and we see immediately that we obtain a graded S-module structure on $M \otimes_S N$ by passing to the quotient.

For two graded S-modules M and N, recall that a homomorphism $u : M \to N$ of S-modules is said to be *of degree* k if $u(M_j) \subset N_{j+k}$ for all $j \in \mathbf{Z}$. If H_n denotes the set of all the homomorphisms of degree n from M to N, then we denote by $\operatorname{Hom}_S(M, N)$ the (direct) *sum* of the H_n ($n \in \mathbf{Z}$) in the S-module H of all the homomorphisms (of S-modules) from M to N; in general, $\operatorname{Hom}_S(M, N)$ is not equal to the later. However, we have $H = \operatorname{Hom}_S(M, N)$ when M is *of finite type*; indeed, we can then suppose that M is generated by a finite number of homogeneous elements x_i ($1 \leqslant i \leqslant n$), and every homomorphism $u \in H$ can be written in a unique way as $\sum_{k \in \mathbf{Z}} u_k$, where for each k, $u_k(x_i)$ is equal to the homogeneous component of degree $k + \deg(x_i)$ of $u(x_i)$ ($1 \leqslant i \leqslant n$), which implies that $u_k = 0$ except for a finite number of indices; we have by definition that $u_k \in H_k$, hence the conclusion.

We say that the elements of degree 0 of $\operatorname{Hom}_S(M, N)$ are the *homomorphisms of graded S-modules*. It is clear that $S_m H_n \subset H_{m+n}$, so the H_n define on $\operatorname{Hom}_S(M, N)$ a graded S-module structure.

It follows immediately from these definitions that we have

(2.1.2.1) $$M(m) \otimes_S N(n) = (M \otimes_S N)(m + n),$$

(2.1.2.2) $$\operatorname{Hom}_S(M(m), N(n)) = (\operatorname{Hom}_S(M, N))(n - m),$$

for two graded S-modules M and N.

Let S and S' be two graded rings; a homomorphism of *graded rings* $\phi : S \to S'$ is a homomor- **II** | 21
phism of rings such that $\phi(S_n) \subset S'_n$ for all $n \in \mathbf{Z}$ (in other words, ϕ must be a homomorphism *of degree* 0 of graded $\mathbf{Z}$-modules). The data of such a homomorphism defines on S' a *graded S'*-module structure; equipped with this structure and its graded ring structure, we say that S' is a *graded S'-algebra*.

If M is also a graded S-module, then the tensor product $M \otimes_S S'$ of *graded* S-modules is equipped in a natural way with a *graded S'*-module structure, the grading being defined as above.

Lemma (2.1.3). — *Let S be a ring graded in positive degrees. For a subset E of S_+ consisting of homogeneous elements to generate S_+ as an S-module, it is necessary and sufficient for E to generate S an an S_0-algebra.*

PROOF. The condition is evidently sufficient; we show that it is necessary. Let E_n (resp. E^n) be the set of elements of E equal to n (resp. $\leqslant n$); it suffices to show, by induction on $n > 0$, that S_n is the S_0-module generated by the elements of degree n which are products of elements of E^n. This is evident for $n = 1$ by virtue of the hypothesis; the latter also shows that $S_n = \sum_{p=0}^{n-1} S_p E_{n-p}$, and the induction argument is then immediate. □

Corollary (2.1.4). — *For S_+ to be an ideal of finite type, it is necessary and sufficient for S to be an S_0-algebra of finite type.*

PROOF. We can always assume that a finite system of generators of the S_0-algebra S (resp. of the S-ideal S_+) consists of homogeneous elements, by replacing each of the generators considered by its homogeneous components. □

Corollary (2.1.5). — *For S to be Noetherian, it is necessary and sufficient for S_0 to be Noetherian and for S to be an S_0-algebra of finite type.*

PROOF. The condition is evidently sufficient; it is necessary, since S_0 is isomorphic to S/S_+ and S_+ must be an ideal of finite type (2.1.4). □

Lemma (2.1.6). — *Let S be a ring graded in positive degrees, which is an S_0-algebra of finite type. Let M be a graded S-module of finite type. Then:*

(i) *The M_n are S_0-modules of finite type, and there exists an integer n_0 such that $M_n = 0$ for $n \leqslant n_0$.*
(ii) *There exists an integer n_1 and an integer $h > 0$ such that, for every integer $n \geqslant n_1$, we have $M_{n+h} = S_h M_n$.*
(iii) *For every pair of integers (d, k) such that $d > 0$, $0 \leqslant k \leqslant d - 1$, $M^{(d,k)}$ is an $S^{(d)}$-module of finite type.*
(iv) *For every integer $d > 0$, $S^{(d)}$ is an S_0-algebra of finite type.*

(v) *There exists an integer $h > 0$ such that $S_{mh} = (S_h)^m$ for all $m > 0$.*
(vi) *For every integer $n > 0$, there exists an integer m_0 such that $S_m \subset S_+^n$ for all $m \geqslant m_0$.*

PROOF. We can assume that S is generated (as an S_0-algebra) by homogeneous elements f_i, of degrees h_i ($1 \leqslant i \leqslant r$), and M is generated (as an S-module) by homogeneous elements x_j of degrees k_j ($1 \leqslant j \leqslant s$). It is clear that M_n is formed by linear combinations, with coefficients in S_0, of elements $f_1^{\alpha_1} \cdots f_r^{\alpha_r} x_j$ such that the α_i are integers $\geqslant 0$ satisfying $k_j + \sum_i \alpha_i h_i = n$; for each j, there are only finitely many systems (α_i) satisfying this equation, since the h_i are > 0, hence the first assertion of (i); the second is evident. On the other hand, let h be the l.c.m. of the h_i and set $g_i = f_i^{h/h_i}$ ($1 \leqslant i \leqslant r$) such that all the g_i are of degree h; let z_μ be the elements of M of the form $f_1^{\alpha_1} \cdots f_r^{\alpha_r} x_j$ with $0 \leqslant \alpha_i < h/h_i$ for $1 \leqslant i \leqslant r$; there are finitely many of these elements, so let n_1 be the largest of their degrees. It is clear that for $n \geqslant n_1$, every element of M_{n+h} is a linear combination of the z_μ whose coefficients are monomials of degree > 0 with respect to the g_i, so we have $M_{n+h} = S_h M_n$, which establishes (ii). In a similar way, we see (for all $d > 0$) that an element of $M^{(d,k)}$ is a linear combinations, with coefficients in S_0, of elements of the form $g^d f_1^{\alpha_1} \cdots f_r^{\alpha_r} x_j$ with $0 \leqslant \alpha_i < d$, g being a homogeneous element of S; hence (iii); (iv) then follows from (iii) and from Lemma (2.1.3), by taking $M = S_+$, since $(S_+)^{(d)} = (S^{(d)})_+$. The assertion of (v) is deduced from (ii) by taking $M = S$. Finally, for a given n, there are finitely many systems (α_i) such that $\alpha_i \geqslant 0$ and $\sum_i \alpha_i < n$, so if m_0 is the largest value of the sum $\sum_i \alpha_i h_i$ of these systems, then we have $S_m \subset S_+^n$ for $m > m_0$, which proves (vi). ☐ II | 22

Corollary (2.1.7). — *If S is Noetherian, then so is $S^{(d)}$ for every integer $d > 0$.*

PROOF. This follows from (2.1.5) and (2.1.6, iv). ☐

(2.1.8). Let $\mathfrak{p}$ be a *graded* prime ideal of the graded ring S; $\mathfrak{p}$ is thus a direct sum of the subgroups $\mathfrak{p}_n = \mathfrak{p} \cap S_n$. Suppose that $\mathfrak{p}$ *does not contain* S_+. Then if $f \in S_+$ is not in $\mathfrak{p}$, the relation $f^n x \in \mathfrak{p}$ is equivalent to $x \in \mathfrak{p}$; in particular, if $f \in S_d$ ($d > 0$), for all $x \in S_{m-nd}$, then the relation $f^n x \in \mathfrak{p}_m$ is equivalent to $x \in \mathfrak{p}_{m-nd}$.

Proposition (2.1.9). — *Let n_0 be an integer > 0; for all $n \geqslant n_0$, let $\mathfrak{p}_n$ be a subgroup of S_n. For there to exist a graded prime ideal $\mathfrak{p}$ of S not containing S_+ and such that $\mathfrak{p} \cap S_n = \mathfrak{p}_n$ for all $n \geqslant n_0$, it is necessary and sufficient for the following conditions to be satisfied:*

(1st) *$S_m \mathfrak{p}_n \subset \mathfrak{p}_{m+n}$ for all $m \geqslant 0$ and all $n \geqslant n_0$.*
(2nd) *For $m \geqslant n_0$, $n \geqslant n_0$, $f \in S_m$, $g \in S_n$, the relation $fg \in \mathfrak{p}_{m+n}$ implies $f \in \mathfrak{p}_m$ or $g \in \mathfrak{p}_n$.*
(3rd) *$\mathfrak{p}_n \neq S_n$ for at least one $n \geqslant n_0$.*

In addition, the graded prime ideal $\mathfrak{p}$ is then unique.

PROOF. It is evident that the conditions (1st) and (2nd) are necessary. In addition, if $\mathfrak{p} \not\supset S_+$, then there exists at least one $k > 0$ such that $\mathfrak{p} \cap S_k \neq S_k$; if $f \in S_k$ is not in $\mathfrak{p}$, the relation $\mathfrak{p} \cap S_n = S_n$ implies $\mathfrak{p} \cap S_{n-mk} = S_{n-mk}$ according to (2.1.8); therefore, if $\mathfrak{p} \cap S_n = S_n$ for a certain value of n, we would have $\mathfrak{p} \supset S_+$ contrary to the hypothesis, which proves that (3rd) is necessary. Conversely, suppose that the conditions (1st), (2nd), and (3rd) are satisfied. Note that if for an integer $d \geqslant n_0$, $f \in S_d$ is not in $\mathfrak{p}_d$, then, if $\mathfrak{p}$ exists, $\mathfrak{p}_m$, for $m < n_0$, is necessarily equal to the set of the $x \in S_m$ such that $f^r x \in \mathfrak{p}_{m+rd}$, except for a finite number of values of r. This already proves that if $\mathfrak{p}$ exists, then it is unique. It remains to show that if we define the $\mathfrak{p}_m$ for $m < n_0$ by the previous condition, then $\mathfrak{p} = \sum_{n=0}^{\infty} \mathfrak{p}_n$ is a prime ideal. First, note that by virtue of (2nd), for $m \geqslant n_0$, $\mathfrak{p}_m$ is also defined as the set of the $x \in S_m$ such that $f^r x \in \mathfrak{p}_{m+rd}$ except for a finite number of values of r. This being II | 23
so, if $g \in S_m$, $x \in \mathfrak{p}_n$, then we have $f^r gx \in \mathfrak{p}_{m+n+rd}$ except for a finite number of values of r, so $gx \in \mathfrak{p}_{m+n}$, which proves that $\mathfrak{p}$ is an ideal of S. To establish that this ideal is prime, in other words that the ring $S/\mathfrak{p}$, graded by the subgroups $S_n/\mathfrak{p}_n$, is an integral domain, it suffices (by considering the components of higher degree of two elements of $S/\mathfrak{p}$) to prove that if $x \in S_m$ and $y \in S_n$ are such that $x \notin \mathfrak{p}_m$ and $y \notin \mathfrak{p}_n$, then $xy \notin \mathfrak{p}_{m+n}$. If not, for r large enough, we would have $f^{2r} xy \in \mathfrak{p}_{m+n+2rd}$; but we have $f^r y \notin \mathfrak{p}_{n+rd}$ for all $r > 0$; it then follows from (2nd) that, except for a finite number of values of r, we have $f^r x \in \mathfrak{p}_{m+rd}$, and we conclude that $x \in \mathfrak{p}_m$ contrary to the hypothesis. ☐

(2.1.10). We say that a subset $\mathfrak{J}$ of S_+ is an *ideal of* S_+ if it is an ideal of S, and $\mathfrak{J}$ is a *graded prime ideal of* S_+ if it is the intersection of S_+ and a graded prime ideal of S *not containing* S_+ (this prime ideal is also unique according to Proposition (2.1.9)). If $\mathfrak{J}$ is an ideal of S_+, the *radical of* $\mathfrak{J}$ *in* S_+ is the set of elements of S_+ which have a power in $\mathfrak{J}$, in other words the set $\mathfrak{r}_+(\mathfrak{J}) = \mathfrak{r}(\mathfrak{J}) \cap S_+$; in particular, the radical of 0 in S_+ is then called the *nilradical* of S_+ and denoted by $\mathfrak{N}_+$: this is the set of nilpotent elements of S_+. If $\mathfrak{J}$ is an *graded* ideal of S_+, then its radical $\mathfrak{r}_+(\mathfrak{J})$ is a *graded* ideal: by passing to the quotient ring $S/\mathfrak{J}$, we can reduce to the case $\mathfrak{J} = 0$, and it remains to see that if $x = x_h + x_{h+1} + \cdots + x_k$ is nilpotent, then so are the $x_i \in S_i$ $(1 \leqslant h \leqslant i \leqslant k)$; we can assume $x_k \neq 0$ and the component of highest degree of x^n is then x_k^n, hence x_k is nilpotent, and we then argue by induction on k. We say that the graded ring S is *essentially reduced* if $\mathfrak{N}_+ = 0$, in other words, if S_+ does not contain nilpotent elements $\neq 0$.

(2.1.11). We note that if, in the graded ring S, an element x is a zero-divisor, then so is its component of highest degree. We say that a ring S is *essentially integral* if the ring S_+ (*without the unit element*) does not contain a zero-divisor and is $\neq 0$; it suffices that a homogeneous element $\neq 0$ in S_+ is not a zero-divisor in this ring. It is clear that if $\mathfrak{p}$ is a graded prime ideal of S_+, then $S/\mathfrak{p}$ is essentially integral.

Let S be an essentially integral graded ring, and let $x_0 \in S_0$: if there then exists *a* homogeneous element $f \neq 0$ of S_+ such that $x_0 f = 0$, then we have $x_0 S_+ = 0$, since we have $(x_0 g)f = (x_0 f)g = 0$ for all $g \in S_+$, and the hypothesis thus implies $x_0 g = 0$. For S to be integral, it is necessary and sufficient for S_0 to be integral and the annihilator of S_+ in S_0 to be 0.

2.2. Rings of fractions of a graded ring

(2.2.1). Let S be a graded ring, in positive degrees, f a *homogeneous* element of S, of degree $d > 0$; then the ring of fractions $S' = S_f$ is graded, taking for S'_n the set of the x/f^k, where $x \in S_{n+kd}$ with $k \geqslant 0$ (we observe here that n can take arbitrary negative values); we denote the subring $S'_0 = (S_f)_0$ of S' consisting of elements *of degree* 0 by the notation $S_{(f)}$.

If $f \in S_d$, then the monomials $(f/1)^h$ in S_f (h a positive or negative integer) form a *free system* over the ring $S_{(f)}$, and the set of their linear combinations is none other than the ring $(S^{(d)})_f$, which is thus *isomorphic to* $S_{(f)}[T, T^{-1}] = S_{(f)} \otimes_{\mathbf{Z}} \mathbf{Z}[T, T^{-1}]$ (where T is an indeterminate). Indeed, if we have a relation $\sum_{h=-a}^{b} z_h (f/1)^h = 0$ with $z_h = x_h/f^m$, where the x_h are in S_{md}, then this relation is equivalent by definition to the existence of a $k > -a$ such that $\sum_{h=-a}^{b} f^{h+k} x_h = 0$, and as the degrees of the terms of this sum are distinct, we have $f^{h+k} x_h = 0$ for all h, hence $z_h = 0$ for all h. II | 24

If M is a graded S-module, then $M' = M_f$ is a graded S_f-module, M'_n being the set of the z/f^k with $z \in M_{n+kd}$ $(k \geqslant 0)$; we denote by $M_{(f)}$ the set of the homomogenous elements of degree 0 of M'; it is immediate that $M_{(f)}$ is an $S_{(f)}$-module and that we have $(M^{(d)})_f = M_{(f)} \otimes_{S_{(f)}} (S^{(d)})_f$.

Lemma (2.2.2). — *Let d and e be integers > 0, $f \in S_d$, $g \in S_e$. There exists a canonical ring isomorphism*

$$S_{(fg)} \simeq (S_{(f)})_{g^d/f^e};$$

if we canonically identify these two rings, then there exists a canonical module isomorphism

$$M_{(fg)} \simeq (M_{(f)})_{g^d/f^e}.$$

PROOF. Indeed, fg divides $f^e g^d$, and this latter element divides $(fg)^{de}$, so the graded rings S_{fg} and $S_{f^e g^d}$ are canonically identified; on the other hand, $S_{f^e g^d}$ also identifies with $(S_{f^e})_{g^d/1}$ (**0**, 1.4.6), and as $f^e/1$ is invertible in S_{f^e}, $S_{f^e g^d}$ also identifies with $(S_{f^e})_{g^d/f^e}$. The element g^d/f^e is of degree 0 in S_{f^e}; we immediately conclude that the subring of $(S_{f^e})_{g^d/f^e}$ consisting of elements of degree 0 is $(S_{(f^e)})_{g^d/f^e}$, and as we evidently have $S_{(f^e)} = S_{(f)}$, this proves the first part of the proposition; the second is established in a similar way. □

(2.2.3). Under the hypotheses of (2.2.2), it is clear that the canonical homomorphism $S_f \to S_{fg}$ (**0**, 1.4.1), which sends x/f^k to $g^k x/(fg)^k$, is of degree 0, thus gives by restriction a *canonical homomorphism* $S_{(f)} \to S_{(fg)}$, such that the diagram

$$\begin{tikzcd} & S_{(f)} \arrow[dl] \arrow[dr] & \\ S_{(fg)} \arrow[rr, "\sim"] & & (S_{(f)})_{g^d/f^e} \end{tikzcd}$$

is commutative. We similarly define a canonical homomorphism $M_{(f)} \to M_{(fg)}$.

Lemma (2.2.4). — *If f and g are two homogeneous elements of S_+, then the ring $S_{(fg)}$ is generated by the union of the canonical images of $S_{(f)}$ and $S_{(g)}$.*

PROOF. By virtue of Lemma (2.2.2), it suffices to see that $1/(g^d/f^e) = f^{d+e}/(fg)^d$ belongs to the canonical image of $S_{(g)}$ in $S_{(fg)}$, which is evident by definition. □

Proposition (2.2.5). — *Let d be an integer > 0 and let $f \in S_d$. Then there exists a canonical ring isomorphisms $S_{(f)} \simeq S^{(d)}/(f-1)S^{(d)}$; if we identify these two rings by this isomorphism, then there exists a canonical module isomorphism $M_{(f)} \simeq M^{(d)}/(f-1)M^{(d)}$.*

PROOF. The first of these isomorphisms is defined by sending x/f^n, where $x \in S_{nd}$, to the element $\overline{x}$, the class of x mod. $(f-1)S^{(d)}$; this map is well-defined, because we have the congruence $f^h x \equiv x \,(\text{mod. } (f-1)S^{(d)})$ for all $x \in S^{(d)}$, so if $f^h x = 0$ for an $h > 0$, then we have $\overline{x} = 0$. On the **II** | 25
other hand, if $x \in S_{nd}$ is such that $x = (f-1)y$ with $y = y_{hd} + y_{(h+1)d} + \cdots + y_{kd}$ with $y_{jd} \in S_{jd}$ and $y_{hd} \neq 0$, then we necessarily have $h = n$ and $x = -y_{hd}$, as well as the relations $y_{(j+1)d} = f y_{jd}$ for $h \leqslant j \leqslant k-1$, $f y_{kd} = 0$, which ultimately gives $f^{k-n}x = 0$; we send every class $\overline{x}$ mod. $(f-1)S^{(d)}$ of an element $x \in S_{nd}$ to the element x/f^n of $S_{(f)}$, since the preceding remark shows that this map is well-defined. It is immediate that these two maps thus defined are ring homomorphisms, each the reciprocal of the other. We proceed exactly the same way for M. □

Corollary (2.2.6). — *If S is Noetherian, then so is $S_{(f)}$ for f homogeneous of degree > 0.*

PROOF. This follows immediately from Corollary (2.1.7) and Proposition (2.2.5). □

(2.2.7). Let T be a multiplicative subset of S_+ consisting of *homogeneous* elements; $T_0 = T \cup \{1\}$ is then a multiplicative subset of S; as the elements of T_0 are homogeneous, the ring $T_0^{-1}S$ is still graded in the evident way; we denote by $S_{(T)}$ the subring of $T_0^{-1}S$ consisting of elements of order 0, that is to say, the elements of the form x/h, where $h \in T$ and x is homogeneous of degree equal to that of h. We know (**0**, 1.4.5) that $T_0^{-1}S$ is canonically identified with the inductive limit of the rings S_f, where f varies over T (with respect to the canonical homomorphisms $S_f \to S_{fg}$); as this identification respects the degrees, it identifies $S_{(T)}$ with the *inductive limit* of the $S_{(f)}$ for $f \in T$. For every graded S-module M, we similarly define the module $M_{(T)}$ (over the ring $S_{(T)}$) consisting of elements of degree 0 of $T_0^{-1}M$, and we see that this module is the inductive limit of the $M_{(f)}$ for $f \in T$.

If $\mathfrak{p}$ is a graded prime ideal of S_+, then we denote by $S_{(\mathfrak{p})}$ and $M_{(\mathfrak{p})}$ the ring $S_{(T)}$ and the module $M_{(T)}$ respectively, where T is the set of *homogeneous* elements of S_+ which do not belong to $\mathfrak{p}$.

2.3. Homogeneous prime spectrum of a graded ring

(2.3.1). Given a graded ring S, in positive degrees, we call the *homogeneous prime spectrum* of S and denote it by $\operatorname{Proj}(S)$ the set of graded prime ideals of S_+ (2.1.10), or equivalently the set of graded prime ideals of S *not containing* S_+; we will define a *scheme* structure having $\operatorname{Proj}(S)$ as the underlying set.

(2.3.2). For every subset E of S, let $V_+(E)$ be the set of graded prime ideals of S containing E and not containing S_+; this is thus the subset $V(E) \cap \operatorname{Proj}(S)$ of $\operatorname{Spec}(S)$. From (I, 1.1.2) we deduce:

(2.3.2.1) $$V_+(0) = \operatorname{Proj}(S),\ V_+(S) = V_+(S_+) = \varnothing,$$

(2.3.2.2) $$V_+\big(\textstyle\bigcup_\lambda E_\lambda\big) = \textstyle\bigcap_\lambda V_+(E_\lambda),$$

(2.3.2.3) $$V_+(EE') = V_+(E) \cup V_+(E').$$

We do not change $V_+(E)$ by replacing E with the graded ideal generated by E; in addition, if $\mathfrak{J}$ is a graded ideal of S, then we have

(2.3.2.4) $$V_+(\mathfrak{J}) = V_+\big(\textstyle\bigcup_{q \geqslant n}(\mathfrak{J} \cap S_q)\big)$$

for all $n > 0$: indeed, if $\mathfrak{p} \in \operatorname{Proj}(S)$ contains the homogeneous elements of $\mathfrak{J}$ of degree $\geqslant n$, then as II | 26
by hypothesis there exists a homogeneous element $f \in S_d$ not contained in $\mathfrak{p}$, for every $m \geqslant 0$ and every $x \in S_m \cap \mathfrak{J}$, we have $f^r x \in \mathfrak{J} \cap S_{m+rd}$ for all but finitely many values of r, so $f^r x \in \mathfrak{p} \cap S_{m+rd}$, which implies that $x \in \mathfrak{p} \cap S_m$ (2.1.9).

Finally, we have, for every graded ideal $\mathfrak{J}$ of S,

(2.3.2.5) $$V_+(\mathfrak{J}) = V_+(\mathfrak{r}_+(\mathfrak{J})).$$

(2.3.3). By definition, the $V_+(E)$ are the closed subsets of $X = \operatorname{Proj}(S)$ for the topology induced by the spectral topology of $\operatorname{Spec}(S)$, which we also call the *spectral topology* on X. For all $f \in S$, we set

(2.3.3.1) $$D_+(f) = D(f) \cap \operatorname{Proj}(S) = \operatorname{Proj}(S) - V_+(f)$$

and so, for any two elements f and g of S (I, 1.1.9.1),

(2.3.3.2) $$D_+(fg) = D_+(f) \cap D_+(g).$$

Proposition (2.3.4). — *The $D_+(f)$, as f runs over the set of homogeneous elements of S_+, form a base for the topology of $X = \operatorname{Proj}(S)$.*

PROOF. It follows from (2.3.2.2) and (2.3.2.4) that every closed subset of X is the intersection of sets of the form $V_+(f)$, where f is homogeneous of degree > 0. ☐

(2.3.5). Let f be a *homogeneous* element of S_+, of degree $d > 0$; for every graded prime ideal $\mathfrak{p}$ of S that does not contain f, we know that the set of the x/f^n, where $x \in \mathfrak{p}$ and $n \geqslant 0$, is a prime ideal of the ring of fractions S_f (**0**, 1.2.6); its intersection with $S_{(f)}$ is thus a prime ideal of $S_{(f)}$, which we denote by $\psi_f(\mathfrak{p})$: it is the set of the x/f^n for $n \geqslant 0$ and $x \in \mathfrak{p} \cap S_{nd}$. We have thus defined a map

$$\psi_f : D_+(f) \longrightarrow \operatorname{Spec}(S_{(f)});$$

furthermore, if $g \in S_e$ is another homogeneous element of S_+, then we have a commutative diagram

(2.3.5.1) $$\begin{tikzcd} D_+(f) \arrow[r, "\psi_f"] & \operatorname{Spec}(S_{(f)}) \\ D_+(fg) \arrow[u] \arrow[r, "\psi_{fg}"'] & \operatorname{Spec}(S_{(fg)}) \arrow[u] \end{tikzcd}$$

where the vertical arrow on the left is the inclusion, and the vertical arrow on the right is the map ${}^a\omega_{fg,f}$ induced by the canonical homomorphism $\omega = \omega_{fg,f} : S_{(f)} \to S_{(fg)}$ (I, 1.2.1). Indeed, if $x/f^n \in \omega^{-1}(\psi_{fg}(\mathfrak{p}))$, with $fg \notin \mathfrak{p}$, then, by definition, $g^n x/(fg)^n \in \psi_{fg}(\mathfrak{p})$, so $g^n x \in \mathfrak{p}$, and so $x \in \mathfrak{p}$; the converse is evident.

Proposition (2.3.6). — *The map ψ_f is a homeomorphism from $D_+(f)$ to $\operatorname{Spec}(S_{(f)})$.*

PROOF. Firstly, ψ_f is continuous; this is since, if $h \in S_{nd}$ is such that $h/f^n \in \psi_f(\mathfrak{p})$, then, by definition, $h \in \mathfrak{p}$, and conversely, and so $\psi_f^{-1}(D(h/f^n)) = D_+(hf)$, and our claim then follows from (2.3.3.2). Furthermore, the $D_+(hf)$, where h runs over the sets S_{nd}, form a topology of $D_+(f)$, by (2.3.4) and (2.3.3.2); the above thus proves, taking into account the (T_0) axiom, which holds in $D_+(f)$ **II | 27**
and in $\operatorname{Spec}(S_{(f)})$, that ψ_f is injective and that the inverse map $\psi_f(D_+(f)) \to D_+(f)$ is continuous. Finally, to see that ψ_f is surjective, we note that, if $\mathfrak{q}_0$ is a prime ideal of $S_{(f)}$, and if, for all $n > 0$, we denote by $\mathfrak{p}_n$ the set of $x \in S_n$ such that $x^d/f^n \in \mathfrak{q}_0$, then the $\mathfrak{p}_n$ satisfy the conditions of (2.1.9): if $x, y \in S_n$ are such that $x^d/f^n, y^d/f^n \in \mathfrak{q}_0$, then $(x+y)^{2d}/f^{2n} \in \mathfrak{q}_0$, whence $(x+y)^d/f^n \in \mathfrak{q}_0$, since $\mathfrak{q}_0$ is prime; this proves that the $\mathfrak{p}_n$ are subgroups of the S_n, and the verification of the other conditions of (2.1.9) is immediate, taking into account the fact that $\mathfrak{q}_0$ is prime. If $\mathfrak{p}$ is the graded prime ideal of S thus defined, then indeed $\psi_f(\mathfrak{p}) = \mathfrak{q}_0$, since, if $x \in S_{nd}$, then having $x/f^n \in \mathfrak{q}_0$ and $x^d/f^{nd} \in \mathfrak{q}_0$ is equivalent to $\mathfrak{q}_0$ being prime. □

Corollary (2.3.7). — *To have $D_+(f) = \varnothing$, it is necessary and sufficient for f to be nilpotent.*

PROOF. To have $\operatorname{Spec}(S_{(f)}) = \varnothing$, it is necessary and sufficient to have $S_{(f)} = 0$, or indeed to have $1 = 0$ in S_f, which means, by definition, that f is nilpotent. □

Corollary (2.3.8). — *Let E be a subset of S_+. Then the following conditions are equivalent:*

(a) $V_+(E) = X = \operatorname{Proj}(S)$.
(b) *Every element of E is nilpotent.*
(c) *The homogeneous components of every element of E are nilpotent.*

PROOF. It is clear that (c) implies (b), and that (b) implies (a). If $\mathfrak{J}$ is the graded ideal of S generated by E, then condition (a) is equivalent to requiring that $V_+(\mathfrak{J}) = X$; *a fortiori*, (a) implies that every homogeneous element $f \in \mathfrak{J}$ is such that $V_+(f) = X$, and so f is nilpotent by (2.3.7). □

Corollary (2.3.9). — *If $\mathfrak{J}$ is a graded ideal of S_+, then $\mathfrak{r}_+(\mathfrak{J})$ is the intersection of the graded prime ideals of S_+ that contain $\mathfrak{J}$.*

PROOF. By considering the graded ring $S/\mathfrak{J}$, we can reduce to the case where $\mathfrak{J} = 0$. We need to prove that, if $f \in S_+$ is not nilpotent, then there exists a graded prime ideal of S that does not contain f; but at least one of the homogeneous components of f is not nilpotent, and we can thus suppose f to be homogeneous; the claim then follows from (2.3.7). □

(2.3.10). For every subset Y of $X = \operatorname{Proj}(S)$, let $\mathfrak{j}_+(Y)$ be the set of $f \in S_+$ such that $Y \subset V_+(f)$; this is equivalent to saying that $\mathfrak{j}_+(Y) = \mathfrak{j}(Y) \cap S_+$; then $\mathfrak{j}_+(Y)$ is an ideal of S_+ that is equal to its radical in S_+.

Proposition (2.3.11). —
(i) *For every subset E of S_+, $\mathfrak{j}_+(V_+(E))$ is the radical in S_+ of the graded ideal of S_+ generated by E.*
(ii) *For every subset Y of X, $V_+(\mathfrak{j}_+(Y)) = \overline{Y}$, where $\overline{Y}$ is the closure of Y in X.*

PROOF.

(i) If $\mathfrak{J}$ is the graded ideal of S_+ generated by E, then $V_+(E) = V_+(\mathfrak{J})$, and the claim then follows from (2.3.9).
(ii) Since $V_+(\mathfrak{J}) = \bigcap_{f\in\mathfrak{J}} V_+(f)$, having $Y \subset V_+(\mathfrak{J})$ implies that $Y \subset V_+(f)$ for every $f \in \mathfrak{J}$, and thus $\mathfrak{j}_+(Y) \supset \mathfrak{J}$, whence $V_+(\mathfrak{j}_+(Y)) \subset V_+(\mathfrak{J})$, which proves (ii) by the definition of the closed subsets.

□

Corollary (2.3.12). — *The closed subsets Y of $X = \operatorname{Proj}(S)$ are in bijective correspondence with the graded ideals of S_+ that are equal to their radical in S_+, via the inclusion-reversing maps $Y \mapsto \mathfrak{j}_+(Y)$ and $\mathfrak{J} \mapsto V_+(\mathfrak{J})$; the union $Y_1 \cup Y_2$ of two closed subsets of X corresponds to $\mathfrak{j}_+(Y_1) \cap \mathfrak{j}_+(Y_2)$, and the* **II | 28**
intersection of an arbitrary family (Y_λ) of closed subsets corresponds to the radical in S_+ of the sum of the $\mathfrak{j}_+(Y_\lambda)$.

Corollary (2.3.13). — *Let $\mathfrak{J}$ be a graded ideal of S_+; to have $V_+(\mathfrak{J}) = \varnothing$, it is necessary and sufficient for every element f of S_+ to have a power f^n in $\mathfrak{J}$.*

This above corollary can also be expressed in one of the following equivalent forms:

Corollary (2.3.14). — *Let (f_α) be a family of homogeneous elements of S_+. For the $D_+(f_\alpha)$ to form a cover of $X = \operatorname{Proj}(S)$, it is necessary and sufficient for every element of S_+ to have a power in the ideal generated by the f_α.*

Corollary (2.3.15). — *Let (f_α) be a family of homogeneous elements of S_+, and f an element of S_+. Then the following are equivalent:*

(a) $D_+(f) \subset \bigcup_\alpha D_+(f_\alpha)$;
(b) $V_+(f) \supset \bigcap_\alpha V_+(f_\alpha)$;
(c) *f has a power in the ideal generated by the f_α.*

Corollary (2.3.16). — *For $X = \operatorname{Proj}(S)$ to be empty, it is necessary and sufficient for every element of S_+ to be nilpotent.*

Corollary (2.3.17). — *In the bijective correspondence described in (2.3.12), the* irreducible *closed subsets of X correspond to the graded* prime *ideals of S_+.*

PROOF. If $Y = Y_1 \cup Y_2$, where Y_1 and Y_2 are distinct closed subsets of Y, then

$$\mathrm{j}_+(Y) = \mathrm{j}_+(Y_1) \cap \mathrm{j}_+(Y_2)$$

with the ideals $\mathrm{j}_+(Y_1)$ and $\mathrm{j}_+(Y_2)$ being distinct from $\mathrm{j}_+(Y)$, and so $\mathrm{j}_+(Y)$ is not prime. Conversely, if $\mathfrak{J}$ is a graded ideal of S_+ that is not prime, then there exists elements $f, g \in S_+$ such that $f \notin \mathfrak{J}$ and $g \notin \mathfrak{J}$, but $fg \in \mathfrak{J}$; then $V_+(f) \supset V_+(\mathfrak{J})$ and $V_+(g) \supset V_+(\mathfrak{J})$, but $V_+(\mathfrak{J}) \subset V_+(f) \cup V_+(g)$, by (2.3.2.3); we thus conclude that $V_+(\mathfrak{J})$ is the union of the closed subsets $V_+(f) \cap V_+(\mathfrak{J})$ and $V_+(g) \cap V_+(\mathfrak{J})$, which are distinct from $V_+(\mathfrak{J})$. □

2.4. The scheme structure on $\operatorname{Proj}(S)$

(2.4.1). Let f and g be homogeneous elements of S_+; consider the affine schemes $Y_f = \operatorname{Spec}(S_{(f)})$, $Y_g = \operatorname{Spec}(S_{(g)})$, and $Y_{fg} = \operatorname{Spec}(S_{(fg)})$. By (2.2.2), the morphism $w_{fg,f} = ({}^a\omega_{fg,f}, \widetilde{\omega}_{fg,f})$ from Y_{fg} to Y_f, corresponding to the canonical homomorphism $\omega_{fg,f} : S_{(f)} \to S_{(fg)}$, is an *open immersion* (**I**, 1.3.6). Using the inverse homeomorphism of $\psi_f : D_+(f) \to Y_f$ (2.3.6), we can transport the affine scheme structure of Y_f to $D_+(f)$; by the commutativity of diagram (2.3.5.1), the affine scheme $D_+(fg)$ can thus be identified with the induced scheme on the open subset $D_+(fg)$ of the underlying space of the affine scheme $D_+(f)$. It is then clear (taking (2.3.4) into account) that $X = \operatorname{Proj}(S)$ is endowed with a unique *prescheme* structure, whose restriction to each $D_+(f)$ is the affine scheme that we have just defined. Furthermore:

Proposition (2.4.2). — *The prescheme* $\operatorname{Proj}(S)$ *is a scheme.*

PROOF. It suffices (**I**, 5.5.6) to show, for any homogeneous f and g in S_+, that $D_+(f) \cap D_+(g) = D_+(fg)$ is affine, and that its ring is generated by the canonical images of the rings of $D_+(f)$ and $D_+(g)$; the first point is evident by definition, and the second follows from (2.2.4). □

Whenever we speak of the homogeneous prime spectrum $\operatorname{Proj}(S)$ as a *scheme*, it will always mean with respect to the structure that we have just defined. **II** | 29

Example (2.4.3). — Let $S = K[T_1, T_2]$, where K is a field, T_1 and T_2 are indeterminates, and S is graded by total degree. It follows from (2.3.14) that $\operatorname{Proj}(S)$ is the union of $D_+(T_1)$ and $D_+(T_2)$; we immediately see that these affine schemes are isomorphic to $K[T]$, and that $\operatorname{Proj}(S)$ is obtained by the gluing of these two affine schemes as described in (**I**, 2.3.2) (cf. (7.4.14)).

Proposition (2.4.4). — *Let S be a positively graded ring, and let X be the scheme* $\operatorname{Proj}(S)$.

(i) *If $\mathfrak{N}_+$ is the nilradical of S_+ (2.1.10), then the scheme X_{red} is canonically isomorphic to* $\operatorname{Proj}(S/\mathfrak{N}_+)$; *in particular, if S is essentially reduced, then* $\operatorname{Proj}(S)$ *is reduced.*
(ii) *If S is essentially reduced, then, for X to be integral, it is necessary and sufficient for S to be essentially integral.*

PROOF.

(i) Let $\overline{S}$ be the graded ring $S/\mathfrak{N}_+$, and denote by $x \mapsto \overline{x}$ the canonical homomorphism $S \to \overline{S}$ of degree 0. For all $f \in S_d$ ($d > 0$), the canonical homomorphism $S_f \to \overline{S}$ (0, 1.5.1) is surjective and of degree 0, and thus gives, by restriction, a surjective homomorphism $S_{(f)} \to \overline{S}_{(\overline{f})}$; if we suppose that $f \notin \mathfrak{N}_+$, then we immediately see that $\overline{S}_{(\overline{f})}$ is reduced, and that the kernel of the above homomorphism is the nilradical of $S_{(f)}$, or, in other words, that $\overline{S}_{(\overline{f})} = (S_{(f)})_{\mathrm{red}}$. So to this homomorphism corresponds a closed immersion $D_+(\overline{f}) \to D_+(f)$ that identifies $D_+(\overline{f})$ with $(D_+(f))_{\mathrm{red}}$ (I, 5.1.2), and which is, in particular, a homeomorphism of the underlying spaces of these two affine schemes. Furthermore, if $g \notin \mathfrak{N}_+$ is another homogeneous element of S_+, then the diagram

$$\begin{tikzcd} S_{(f)} \arrow[r] \arrow[d] & \overline{S}_{(\overline{f})} \arrow[d] \\ S_{(fg)} \arrow[r] & \overline{S}_{(\overline{fg})} \end{tikzcd}$$

is commutative; since, further, the $D_+(f)$, for f homogeneous, of degree > 0, and $f \notin \mathfrak{N}_+$, form a cover of $X = \operatorname{Proj}(S)$ (2.3.7), we see that the morphisms $D_+(\overline{f}) \to D_+(f)$ are the restrictions of a closed immersion $\operatorname{Proj}(\overline{S}) \to \operatorname{Proj}(S)$ which is a homeomorphism of the underlying spaces; whence the conclusion (I, 5.1.2).

(ii) Suppose that S is essentially integral, or, in other words, that (0) is a graded prime ideal of S_+ that is distinct from S_+; then X is reduced, by (i), and irreducible, by (2.3.17). Conversely, suppose that S is essentially reduced and that X is integral; then, for homogeneous $f \neq 0$ in S_+, we have that $D_+(f) \neq \varnothing$ (2.3.7); the hypothesis that X is irreducible implies that $D_+(f) \cap D_+(g) \neq \varnothing$ for homogeneous $f, g \neq 0$ in S_+; thus $fg \neq 0$, by (2.3.3.2), and we thus conclude that S_+ has no zero divisors, whence the first claim.

□

(2.4.5). Given a commutative ring A, recall that we say that a graded ring S is a *graded A-algebra* if it is endowed with the structure of an A-algebra such that each of its subgroups S_n is an A-module; for this, it suffices for S_0 to be an A-algebra, or, in other words, we define the structure of a graded **II** | 30
A-algebra on S by defining the structure of an A-algebra on S_0 and setting $\alpha \cdot x = (\alpha \cdot 1)x$ for $\alpha \in A$ and $x \in S_n$.

Proposition (2.4.6). — *Suppose that S is a graded A-algebra. Then, on $X = \operatorname{Proj}(S)$, the structure sheaf $\mathscr{O}_X$ is an A-algebra (where A is considered as a simple sheaf on X); in other words, X is a scheme over* $\operatorname{Spec}(A)$.

PROOF. It suffices to note that, for every homogeneous f in S_+, $S_{(f)}$ is an algebra over A, and that the diagram

$$\begin{tikzcd} S_{(f)} \arrow[rr] & & S_{(fg)} \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

is commutative, for homogeneous f, g in S_+. □

Proposition (2.4.7). — *Let S be a positively graded ring.*

(i) *For every integer $d > 0$, there exists a canonical isomorphism from the scheme* $\operatorname{Proj}(S)$ *to the scheme* $\operatorname{Proj}(S^{(d)})$.
(ii) *Let S' be the graded ring such that $S_0 = \mathbf{Z}$ and $S'_n = S_n$ (considered as a $\mathbf{Z}$-module) for $n > 0$. Then there exists a canonical isomorphism from the scheme* $\operatorname{Proj}(S)$ *to the scheme* $\operatorname{Proj}(S')$.

PROOF.

(i) We first show that the map $\mathfrak{p} \mapsto \mathfrak{p} \cap S^{(d)}$ is a bijection from the set $\operatorname{Proj}(S)$ to the set $\operatorname{Proj}(S^{(d)})$. Indeed, suppose that we have a graded prime ideal $\mathfrak{p}' \in \operatorname{Proj}(S^{(d)})$, and let $\mathfrak{p}_{nd} = \mathfrak{p}' \cap S_{nd}$ ($n \geqslant 0$). For all $n > 0$ that are not multiples of d, define $\mathfrak{p}_n$ as the set of

$x \in S_n$ such that $x^d \in \mathfrak{p}_{nd}$; if $x, y \in \mathfrak{p}_n$, then $(x+y)^{2d} \in \mathfrak{p}_{2nd}$, and so $(x+y)^d \in \mathfrak{p}_{nd}$, since $\mathfrak{p}'$ is prime; it is immediate that the $\mathfrak{p}_n$ thus defined, for $n \geqslant 0$, satisfy the conditions of (2.1.9), and so there exists a unique prime ideal $\mathfrak{p} \in \operatorname{Proj}(S)$ such that $\mathfrak{p} \cap S^{(d)} = \mathfrak{p}'$. Since, for every homogeneous f in S_+, we have that $V_+(f) = V_+(f^d)$ (2.3.2.3), we see that the above bijection is a *homeomorphism* of topological spaces. Finally, with the same notation, $S_{(f)}$ and $S_{(f^d)}$ are canonically identified (2.2.2), and so $\operatorname{Proj}(S)$ and $\operatorname{Proj}(S^{(d)})$ are canonically identified as *schemes*.

(ii) If, to each $\mathfrak{p} \in \operatorname{Proj}(S)$, we associate the unique prime ideal $\mathfrak{p}' \in \operatorname{Proj}(S')$ such that $\mathfrak{p}' \cap S_n = \mathfrak{p} \cap S_n$ for every $n > 0$ (2.1.9), then it is clear that we have defined a canonical homeomorphism $\operatorname{Proj}(S) \xrightarrow{\sim} \operatorname{Proj}(S')$ of the underlying spaces, since $V_+(f)$ is the same set for S and S' when f is a homogeneous element of S_+. Since, further, $S_{(f)} = S'_{(f)}$, $\operatorname{Proj}(S)$ and $\operatorname{Proj}(S')$ can be identified as *schemes*.

$\square$

Corollary (2.4.8). — *If S is a graded A-algebra, and S'_A the graded A-algebra such that $(S'_A)_0 = A$ and $(S'_A)_n = S_n$ for $n > 0$, then there exists a canonical isomorphism from $\operatorname{Proj}(S)$ to $\operatorname{Proj}(S'_A)$.*

PROOF. In fact, these two schemes are both canonically isomorphic to $\operatorname{Proj}(S')$, using the notation of (2.4.7, (ii)). $\square$

2.5. The sheaf associated to a graded module

(2.5.1). Let M be a *graded* S-module. The, for every homogeneous f in S_+, $M_{(f)}$ is an $S_{(f)}$-module, and thus has a corresponding quasi-coherent associated sheaf $(M_{(f)})^\sim$ on the affine scheme $\operatorname{Spec}(S_{(f)})$, identified with $D_+(f)$ (**I**, 1.3.4).

II | 31

Proposition (2.5.2). — *There exists on $X = \operatorname{Proj}(S)$ exactly one quasi-coherent $\mathscr{O}_X$-module $\widetilde{M}$ such that, for f homogeneous in S_+, we have $\Gamma(D_+(f), \widetilde{M}) = M_{(f)}$, with the restriction homomorphism $\Gamma(D_+(f), \widetilde{M}) \to \Gamma(D_+(fg), \widetilde{M})$, for f and g homogeneous in S_+, being the canonical homomorphism $M_{(f)} \to M_{(fg)}$* (2.2.3).

PROOF. Suppose that $f \in S_d$ and $g \in S_e$. Since $D_+(fg)$ can be identified with the prime spectrum of $(S_{(f)})_{g^d/f^e}$ by (2.2.2), the restriction to $D_+(fg)$ of the sheaf $(M_{(f)})^\sim$ on $D_+(f)$ is canonically identified with the sheaf associated to the module $(M_{(f)})_{g^d/f^e}$ (**I**, 1.3.6), and thus also with $(M_{(fg)})^\sim$ (2.2.2); we thus conclude that there exists a canonical isomorphism

$$\theta_{g,f} : (M_{(f)})^\sim | D_+(fg) \xrightarrow{\sim} (M_{(g)})^\sim | D_+(fg)$$

such that, if h is a third homogeneous element of S_+, then $\theta_{f,h} = \theta_{f,g} \circ \theta_{g,h}$ in $D_+(fgh)$. Consequently (**0**, 3.3.1) there exists a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ on X, and, for every homogeneous f in S_+, an isomorphism η_f from $\mathscr{F}|D_+(f)$ to $(M_f)^\sim$ such that $\theta_{g,f} = \eta_g \circ \eta_f^{-1}$. If we then consider the sheaf $\mathscr{G}$ associated to the presheaf (on the base for the topology of X given by the $D_+(f)$) defined by $D_+(f) \mapsto M_{(f)}$, with the canonical homomorphisms $M_{(f)} \to M_{(fg)}$ as restriction homomorphisms, then the above proves that $\mathscr{F}$ and $\mathscr{G}$ are isomorphic (taking (**I**, 1.3.7) into account); the sheaf $\mathscr{G}$ is denoted by $\widetilde{M}$, and indeed satisfies the conditions of the statement. We have, in particular, $\widetilde{S} = \mathscr{O}_X$. $\square$

Definition (2.5.3). — We say that the quasi-coherent $\mathscr{O}_X$-module $\widetilde{M}$ defined in (2.5.2) is *associated* to the graded S-module M.

Recall that the graded S-modules form a category when we restrict from arbitrary homomorphisms of graded modules to homomorphisms *of degree* 0. With this convention:

Proposition (2.5.4). — *The functor $M \mapsto \widetilde{M}$ is an exact additive covariant functor from the category of graded S-modules to the category of quasi-coherent $\mathscr{O}_X$-modules, and it commutes with inductive limits and direct sums.*

PROOF. Indeed, since these properties are local, it suffices to show that they are satisfied for the sheaves of the form $\widetilde{M}|D_+(f) = (M_{(f)})^\sim$; but the functors $M \mapsto M_f$, $N \mapsto N_0$ (to the category of graded S_f-modules), and $P \mapsto \widetilde{P}$ (to the category of $S_{(f)}$-modules) all have the three properties of exactness and of commutativity with inductive limits and direct sums ((**I**, 1.3.5) and (**I**, 1.3.9)); whence the proposition. □

We denote by $\widetilde{u}$ the homomorphism $\widetilde{M} \to \widetilde{N}$ corresponding to a homomorphism $u : M \to N$ of degree 0. We immediately deduce from (2.5.4) that the results of (**I**, 1.3.9) and (**I**, 1.3.10) still hold for graded S-modules and homomorphisms of degree 0 (with the sense given here to $\widetilde{M}$), with the proofs being purely formal.

Proposition (2.5.5). — *For all $\mathfrak{p} \in X = \operatorname{Proj}(S)$, we have $\widetilde{M}_\mathfrak{p} = M_{(\mathfrak{p})}$.*

PROOF. By definition, $\widetilde{M}_\mathfrak{p} = \varinjlim \Gamma(D_+(f), \widetilde{M})$, where f runs over the set of homogeneous elements of S_+ such that $f \notin \mathfrak{p}$; since $\Gamma(D_+(f), \widetilde{M}) = M_{(f)}$, the proposition follows from the definition of $M_{(\mathfrak{p})}$ (2.2.7) □

In particular, the *local ring* $(\mathscr{O}_X)_\mathfrak{p}$ is exactly the ring $S_{(\mathfrak{p})}$, the set of fractions x/f with f homogeneous in S_+ and not belonging to $\mathfrak{p}$, and with x homogeneous of the same degree as f. **II** | 32

Even more particularly, if S is *essential integral*, so that $\operatorname{Proj}(S) = X$ is *integral* (2.4.4), and if $\xi = (0)$ is the generic point of X, then the *field of rational functions* $R(X) = \mathscr{O}_\xi$ is field consisting of elements fg^{-1} with f and g homogeneous of the same degree in S_+, and with $g \neq 0$.

Proposition (2.5.6). — *If, for all $z \in M$ and all homogeneous f in S_+, there exists a power of f that annihilates z, then $\widetilde{M} = 0$. This sufficient condition is also necessary if the S_0-algebra S is generated by the set S_1 of homogeneous elements of degree* 1.

PROOF. The condition $\widetilde{M} = 0$ is equivalent to $M_{(f)} = 0$ for all homogeneous f in S_+. On the other hand, if $f \in S_d$, to say that $M_{(f)} = 0$ implies that, for all homogeneous $z \in M$ whose degree is some multiple of d, there exists a power f^n such that $f^n z = 0$. If $M_{(f)} = 0$ for all $f \in S_1$, then the condition of the statement is thus satisfied for all $f \in S_1$; the condition is *a fortiori* satisfied for all homogeneous f in S_+ if S_1 generates S, since every homogeneous element of S_+ is then a linear combination of products of elements of S_1. □

Proposition (2.5.7). — *Let $d > 0$ be an integer, and let $f \in S_d$. Then, for all $n \in \mathbf{Z}$, the $(\mathscr{O}_X|D_+(f))$-module $(S(nd))^\sim|D_+(f)$ is canonically isomorphic to $\mathscr{O}_X|D_+(f)$.*

PROOF. Indeed, multiplication by the invertible element $(f/1)^n$ of S_f gives a bijection from $S_{(f)} = (S_f)_0$ to $(S_f)_{nd} = (S_f(nd))_0 = (S(nd)_f)_0 = S(nd)_{(f)}$; in other words, the $S_{(f)}$-modules $S_{(f)}$ and $S(nd)_{(f)}$ are canonically isomorphic, whence the proposition. □

Corollary (2.5.8). — *On the open subset $U = \bigcup_{f \in S_d} D_+(f)$, the restriction of the $\mathscr{O}_X$-module $(S(nd))^\sim$ is an invertible $(\mathscr{O}_X|U)$-module* (**0**, 5.4.1).

Corollary (2.5.9). — *If the ideal S_+ of S is generated by the set S_1 of homogeneous elements of degree* 1, *then the $\mathscr{O}_X$-module $(S(n))^\sim$ is invertible for all $n \in \mathbf{Z}$.*

PROOF. It suffices to remark that $X = \bigcup_{f \in S_1} D_+(f)$, by the hypothesis (2.3.14) and to apply (2.5.8) with $U = X$. □

(2.5.10). We set, for the rest of this section,

$$\mathscr{O}_X(n) = (S(n))^\sim \tag{2.5.10.1}$$

for all $n \in \mathbf{Z}$, and, for every open subset U of X, and every $(\mathscr{O}_X|U)$-module $\mathscr{F}$,

$$\mathscr{F}(n) = \mathscr{F} \otimes_{\mathscr{O}_X|U} (\mathscr{O}_X(n)|U) \tag{2.5.10.2}$$

for all $n \in \mathbf{Z}$. If the ideal S_+ is generated by S_1, then the functor $\mathscr{F}(n)$ is *exact* in $\mathscr{F}$ for all $n \in \mathbf{Z}$, since $\mathscr{O}_X(n)$ is then an *invertible* $\mathscr{O}_X$-module.

(2.5.11). Let M and N be graded S-modules. For all $f \in S_d$ ($d > 0$), we define a canonical functorial homomorphism of $S_{(f)}$-modules by

$$\lambda_f : M_{(f)} \otimes_{S_{(f)}} N_{(f)} \longrightarrow (M \otimes_S N)_{(f)} \tag{2.5.11.1}$$

by composing the homomorphism $M_{(f)} \otimes_{S_{(f)}} N_{(f)} \to M_f \otimes_{S_f} N_f$ (coming from the injections $M_{(f)} \to M_f$, $N_{(f)} \to N_f$, and $S_{(f)} \to S_f$) with the canonical isomorphism $M_f \otimes_{S_f} N_f \xrightarrow{\sim} (M \otimes_S N)_f$ (**0**, 1.3.4), and by noting that, by the definition of the tensor product of two graded modules, this latter isomorphism preserves degrees; for $x \in M_{md}$ and $y \in N_{nd}$ ($m, n \geqslant 0$), we thus have **II | 33**

$$\lambda_f((x/f^m) \otimes (y/f^n)) = (x \otimes y)/f^{m+n}.$$

It immediately follows from this definition that, if $g \in S_e$ ($e > 0$), then the diagram

$$\begin{tikzcd}
M_{(f)} \otimes_{S_{(f)}} N_{(f)} \arrow[r, "\lambda_f"] \arrow[d] & (M \otimes_S N)_{(f)} \arrow[d] \\
M_{(fg)} \otimes_{S_{(fg)}} N_{(fg)} \arrow[r, "\lambda_{fg}"'] & (M \otimes_S N)_{(fg)}
\end{tikzcd}$$

(where the vertical arrow on the right is the canonical homomorphism, and the one on the left comes from the canonical homomorphisms) commutes. Thus λ induces a canonical functorial homomorphism of $\mathscr{O}_X$-modules

$$\lambda : \widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N} \longrightarrow (M \otimes_S N)^{\sim}. \tag{2.5.11.2}$$

Consider, in particular, graded ideals $\mathfrak{J}$ and $\mathfrak{K}$ of S; since $\widetilde{\mathfrak{J}}$ and $\widetilde{\mathfrak{K}}$ are sheaves of ideals of $\mathscr{O}_X$, we have a canonical homomorphism $\widetilde{\mathfrak{J}} \otimes_{\mathscr{O}_X} \widetilde{\mathfrak{K}} \to \mathscr{O}_X$; the diagram

$$\begin{tikzcd}
\widetilde{\mathfrak{J}} \otimes_{\mathscr{O}_X} \widetilde{\mathfrak{K}} \arrow[rr, "\lambda"] \arrow[dr] & & (\widetilde{\mathfrak{J}} \otimes_S \widetilde{\mathfrak{K}})^{\sim} \arrow[dl] \\
& \mathscr{O}_X &
\end{tikzcd} \tag{2.5.11.3}$$

then commutes. Indeed, we can reduce to verifying this on each open subset $D_+(f)$ (for f homogeneous in S_+), and this follows immediately from the definition (2.5.11.1) of λ_f and from (**I**, 1.3.13).

Finally, note that, if M, N, and P are graded S-modules, then the diagram

$$\begin{tikzcd}
\widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N} \otimes_{\mathscr{O}_X} \widetilde{P} \arrow[r, "\lambda\otimes 1"] \arrow[d, "1\otimes\lambda"'] & (M \otimes_S N)^{\sim} \otimes_{\mathscr{O}_X} \widetilde{P} \arrow[d, "\lambda"] \\
\widetilde{M} \otimes_{\mathscr{O}_X} (N \otimes_S P)^{\sim} \arrow[r, "\lambda"'] & (M \otimes_S N \otimes_S P)^{\sim}
\end{tikzcd} \tag{2.5.11.4}$$

commutes. It again suffices to verify this on each open subset $D_+(f)$, and this follows immediately from the definitions and from (**I**, 1.3.13). **II | 34**

(2.5.12). Under the hypotheses of (2.5.11), we define a functorial canonical homomorphism of $S_{(f)}$-modules

$$\mu_f : (\mathrm{Hom}_S(M, N))_{(f)} \longrightarrow \mathrm{Hom}_{S_{(f)}}(M_{(f)}, N_{(f)}) \tag{2.5.12.1}$$

by sending u/f^n, where u is a homomorphism of degree nd, to the homomorphism $M_{(f)} \to N_{(f)}$ that sends x/f^m ($x \in M_{md}$) to $u(x)/f^{m+n}$. For $g \in S_e$ ($e > 0$), we again have a commutative diagram:

$$\begin{tikzcd}
(\mathrm{Hom}_S(M,N))_{(f)} \arrow[r, "\mu_f"] \arrow[d] & \mathrm{Hom}_{S_{(f)}}(M_{(f)}, N_{(f)}) \arrow[d] \\
(\mathrm{Hom}_S(M,N))_{(fg)} \arrow[r, "\mu_{fg}"'] & \mathrm{Hom}_{S_{(fg)}}(M_{(fg)}, N_{(fg)})
\end{tikzcd}$$

(where the vertical arrow on the left is the canonical homomorphism, and the one on the right comes from the canonical homomorphisms). We thus again conclude (taking (I, 1.3.8) into account) that the μ_f define a functorial canonical homomorphism of $\mathscr{O}_X$-modules

(2.5.12.2) $$\mu : (\mathrm{Hom}_S(M,N))^\sim \longrightarrow \mathscr{H}om_{\mathscr{O}_X}(\widetilde{M}, \widetilde{N})$$

Proposition (2.5.13). — *Suppose that the ideal S_+ is generated by S_1. Then the homomorphism λ (2.5.11.2) is an isomorphism; so too is the homomorphism μ (2.5.12.2) if the graded S-module M admits a finite presentation (2.1.1)*

Proof. Since X is the union of the $D_+(f)$ for $f \in S_1$ (2.3.14), we are led to proving that λ_f and μ_f are isomorphisms, under the given hypotheses, whenever f is homogeneous and *of degree* 1. But we can then define a $\mathbf{Z}$-bilinear map $M_m \times N_n \to M_{(f)} \otimes_{S_{(f)}} N_{(f)}$ by sending (x,y) to the element $(x/f^m) \otimes (y/f^n)$ (if $m < 0$, we write x/f^m to mean $f^{-m}x/1$); these maps define a $\mathbf{Z}$-linear map $M \otimes_{\mathbf{Z}} N \to M_{(f)} \otimes_{S_{(f)}} N_{(f)}$, and, if $s \in S_q$, this map sends $(sx) \otimes y$ to $(s/f^q)((x/f^m) \otimes (y/f^n))$ (for $x \in M_m$ and $y \in N_n$). We thus obtain a di-homomorphism of modules $\gamma_f : M \otimes_S N \to M_{(f)} \otimes_{S_{(f)}} N_{(f)}$, with respect to the canonical homomorphism $S \to S_{(f)}$ (sending $s \in S_q$ to s/f^q). Suppose furthermore that, for an element $\sum_i (x_i \otimes y_i)$ of $M \otimes_S N$ (with x_i and y_i homogeneous of degree m_i and n_i, respectively), we have that $f^r \sum_i (x_i \otimes y_i) = 0$, or, in other words, that $\sum_i (f^r x_i \otimes y_i) = 0$. We thus deduce, by (0, 1.3.4), that $\sum_i (f^r x_i / f^{m_i+r}) \otimes (y_i/f^{n_i}) = 0$, i.e. $\gamma_f(\sum_i (x_i \otimes y_i)) = 0$. Then γ_f factors as $M \otimes_S N \to (M \otimes_S N)_f \xrightarrow{\gamma'_f} M_{(f)} \otimes_{S_{(f)}} N_{(f)}$; if λ'_f is the restriction of γ' to $(M \otimes_S N)_{(f)}$, **II** | 35
then we can immediately show that λ_f and λ'_f are inverse $S_{(f)}$-homomorphisms, whence the first part of the proposition.

To prove the second part, suppose that M is the cokernel of a homomorphism $P \to Q$ of graded S-modules, with P and Q being direct sums of a finite number of modules of the form $S(n)$; using the left-exactness of $\mathrm{Hom}_S(L,N)$ in L, and the exactness of $M_{(f)}$ in M, we can immediately reduce to proving that μ_f is an isomorphism whenever $M = S(n)$. But, for any homogeneous z in N, let u_z be the homomorphism from $S(n)$ to N such that $u_z(1) = z$; we immediately see that $\eta : z \to u_z$ is an isomorphism of degree 0 from $N(-n)$ to $\mathrm{Hom}_S(S(n), N)$. There is a corresponding isomorphism

$$\eta_f : (N(-n))_{(f)} \longrightarrow (\mathrm{Hom}_S(S(n), N))_{(f)}.$$

Now let η'_f be the isomorphism $N_{(f)} \to \mathrm{Hom}_{S_{(f)}}(S(n)_{(f)}, N_{(f)})$ that, to any $z' \in N_{(f)}$, associates the homomorphism $v_{z'}$ that is such that $v_{z'}(s/f^k) = sz'/f^{n+k}$ (for $s \in S_{n+k} = (S(n))_k$). We easily note that the composed map

$$(N(-n))_{(f)} \xrightarrow{\eta_f} (\mathrm{Hom}_S(S(n), N))_{(f)} \xrightarrow{\mu_f} \mathrm{Hom}_{S_{(f)}}(S(n)_{(f)}, N_{(f)}) \xrightarrow{\eta'^{-1}_f} N_{(f)}$$

is the isomorphism $z/f^h \mapsto z/f^{h-n}$ from $(N(-n))_{(f)}$ to $N_{(f)}$, and thus μ_f is an isomorphism. □

If the ideal S_+ is generated by S_1, then we deduce from (2.5.13) that, for every graded ideal $\mathfrak{J}$ of S, and for every graded S-module M, we have

(2.5.13.1) $$\widetilde{\mathfrak{J}} \cdot \widetilde{M} = (\mathfrak{J} \cdot M)^\sim$$

up to canonical isomorphism; this follows from the commutativity of the diagram

$$\begin{tikzcd}
\widetilde{\mathfrak{J}} \otimes_{\mathscr{O}_X} \widetilde{M} \arrow[rr, "\lambda"] \arrow[dr] & & (\mathfrak{J} \otimes_S M)^\sim \arrow[dl] \\
& \widetilde{M} &
\end{tikzcd}$$

which we can verify as we did for (2.5.11.3).

Corollary (2.5.14). — *Suppose that S is generated by S_1. For any $m, n \in \mathbf{Z}$, we then have:*

(2.5.14.1) $$\mathscr{O}_X(m) \otimes_{\mathscr{O}_X} \mathscr{O}_X(n) = \mathscr{O}_X(m+n)$$

(2.5.14.2) $$\mathscr{O}_X(n) = (\mathscr{O}_X(1))^{\otimes n}$$

up to canonical isomorphism.

PROOF. The first equation follows from (2.5.13) and from the existence of the canonical isomorphism $S(m) \otimes_S S(n) \xrightarrow{\sim} S(m+n)$ of degree 0 that sends the element $1 \otimes 1$ (where the first 1 is in $(S(m))_{-m}$ and the second is in $(S(n))_{-n}$) to the element $1 \in (S(m+n))_{-(m+n)}$. It then suffices to prove the second equation for $n = -1$, and, by (2.5.13), this reduces to seeing that $\operatorname{Hom}_S(S(1), S)$ is canonically isomorphic to $S(-1)$, which can be immediately proven by going back to the definitions (2.1.2) and by remembering that $S(1)$ is a monogeneous S-module. □

II | 36

Corollary (2.5.15). — *Suppose that S is generated by S_1. Then, for every graded S-module M, and for every $n \in \mathbf{Z}$, we have*

(2.5.15.1) $$(M(n))^{\sim} = \widetilde{M}(n)$$

up to canonical isomorphism.

PROOF. This follows from definitions (2.5.10.2) and (2.5.10.1), from Proposition (2.5.13), and from the existence of a canonical isomorphism $M(n) \xrightarrow{\sim} M \otimes_S S(n)$ of degree 0 that, to every $z \in (M(n))_h = M_{n+h}$, associates $z \otimes 1 \in M_{n+h} \otimes (S(n))_{-n} \subset (M \otimes_S S(n))_h$. □

(2.5.16). We denote by S' the graded ring such that $S'_0 = \mathbf{Z}$, and $S'_n = S_n$ for $n > 0$. Then, if $f \in S_d$ ($d > 0$), we have that $(S(n))_{(f)} = (S'(n))_{(f)}$ for all $n \in \mathbf{Z}$, since an element of $(S'(n))_{(f)}$ is of the form x/f^k, with $x \in S'_{n+kd}$ ($k > 0$), and we can always take k to be such that $n + kd \neq 0$. Since $X = \operatorname{Proj}(S)$ and $X' = \operatorname{Proj}(S')$ are canonically identified (2.4.7, (ii)), we see that, for all $n \in \mathbf{Z}$, $\mathscr{O}_X(n)$ and $\mathscr{O}_{X'}(n)$ are the images of one another under the above identification.

Note also that, for all $d > 0$ and all $n \in \mathbf{Z}$, we have

$$(S^{(d)}(n))_h = S_{(n+h)d} = (S(nd))_{hd}$$

for $f \in S_d$, and thus $(S^{(d)}(n))_{(f)} = (S(nd))_{(f)}$. We know that the schemes $X = \operatorname{Proj}(S)$ and $X^{(d)} = \operatorname{Proj}(S^{(d)})$ are canonically identified (2.4.7, (ii)); the above shows that, if the S_0-algebra $S^{(d)}$ is generated by S_d, then $\mathscr{O}_X(nd)$ and $\mathscr{O}_{X^{(d)}}(n)$ are the images of one another under this identification, for all $n \in \mathbf{Z}$.

Proposition (2.5.17). — *Let $d > 0$ be an integer, and let $U = \bigcup_{f \in S_d} D_+(f)$. Then the restriction to U of the canonical homomorphism $\mathscr{O}_X(nd) \otimes_{\mathscr{O}_X} \mathscr{O}_X(-nd) \to \mathscr{O}_X$ is an isomorphism for every integer n.*

PROOF. By (2.5.16), we can restrict to the case where $d = 1$, and the conclusion then follows from the proof of (2.5.13). □

2.6. The graded S-module associated to a sheaf on $\mathrm{Proj}(S)$ *We suppose all throughout this section that the ideal S_+ of S is generated by the set S_1 of homogeneous elements of degree* 1*.*

(2.6.1). The $\mathscr{O}_X$-module $\mathscr{O}_X(1)$ is *invertible* (2.5.9); we thus define, for every $\mathscr{O}_X$-module $\mathscr{F}$ (**0**, 5.4.6),

$$\Gamma_\bullet(\mathscr{F}) = \Gamma_\bullet(\mathscr{O}_X(1), \mathscr{F}) = \bigoplus_{n\in\mathbf{Z}} \Gamma(X, \mathscr{F}(n)) \tag{2.6.1.1}$$

taking (2.5.14.2) into account. Recall (**0**, 5.4.6) that $\Gamma_\bullet(\mathscr{O}_X)$ is endowed with the structure of a *graded ring*, and $\Gamma_\bullet(\mathscr{F})$ with the structure of a *graded* $\Gamma_\bullet(\mathscr{O}_X)$*-module*.

Since $\mathscr{O}_X(n)$ is locally free, $\Gamma_\bullet(\mathscr{F})$ is a *left exact* additive covariant functor in $\mathscr{F}$; in particular, if $\mathscr{J}$ is a sheaf of ideals of $\mathscr{O}_X$, then $\Gamma_\bullet(\mathscr{J})$ is canonically identified with a *graded idea* of $\Gamma_\bullet(\mathscr{O}_X)$.

(2.6.2). Let M be a graded S-module; for every $f \in S_d$ $(d > 0)$, $x \mapsto x/1$ is a homomorphism of abelian groups $M_0 \to M_{(f)}$, and, since $M_{(f)}$ is canonically identified with $\Gamma(D_+(f), \widetilde{M})$, we thus **II | 37**
obtain a homomorphism of abelian groups $\alpha_0^f : M_0 \to \Gamma(D_+(f), \widetilde{M})$. It is clear that, for every $g \in S_e$ $(e > 0)$, the diagram

$$\begin{tikzcd}
 & \Gamma(D_+(f), \widetilde{M}) \arrow[dd] \\
M_0 \arrow[ur, "\alpha_0^f"] \arrow[dr, "\alpha_0^{fg}"'] & \\
 & \Gamma(D_+(fg), \widetilde{M})
\end{tikzcd}$$

commutes; this implies that, for all $x \in M_0$, the sections $\alpha_0^f(x)$ and $\alpha_0^g(x)$ of M agree on $D_+(f) \cap D_+(g)$, and thus there exists a unique section $\alpha_0(x) \in \Gamma(X, \widetilde{M})$ whose restriction to each $D_+(f)$ is $\alpha_0^f(x)$. We have thus defined (without using the hypothesis that S be generated by S_1) a homomorphism of abelian groups

$$\alpha_0 : M_0 \longrightarrow \Gamma(X, \widetilde{M}). \tag{2.6.2.1}$$

Applying this result to the graded S-module $M(n)$ (for each $n \in \mathbf{Z}$), we obtain, for each $n \in \mathbf{Z}$, a homomorphism of abelian groups

$$\alpha_n : M_n = (M(n))_0 \longrightarrow \Gamma(X, \widetilde{M}(n)) \tag{2.6.2.2}$$

(taking (2.5.15)); whence we obtain a functorial homomorphism (of degree 0) of graded abelian groups

$$\alpha : M \longrightarrow \Gamma_\bullet(\widetilde{M}) \tag{2.6.2.3}$$

(also denoted by α_M) which, on each M_n, agrees with α_n.

If we take, in particular, $M = S$, then we immediately see (taking into account the definition (**0**, 5.4.6) of multiplication in $\Gamma_\bullet(\mathscr{O}_X)$) that $\alpha : S \to \Gamma_\bullet(\mathscr{O}_X)$ is a homomorphism of graded rings, and that, for every graded S-module M, (2.6.2.3) is a di-homomorphism of graded modules.

Proposition (2.6.3). — *For every $f \in S_d$ $(d > 0)$, $D_+(f)$ is identical to the set of $\mathfrak{p} \in X$ on which the section $\alpha_d(f)$ of $\mathscr{O}_X(d)$ does not vanish* (**0**, 5.5.2)*.*

Proof. Since $X = \bigcup_{g\in S_1} D_+(g)$ by hypothesis, it suffices to show that, for all $g \in S_1$, the set of $\mathfrak{p} \in D_+(g)$ on which $\alpha_d(f)$ does not vanish is identical to $D_+(fg)$. But the restriction of $\alpha_d(f)$ to $D_+(g)$ is, by definition, the section corresponding to the element $f/1$ of $(S(d))_{(g)}$; under the canonical isomorphism $(S(d))_{(g)} \xrightarrow{\sim} S_{(g)}$ (2.5.7), this section of $\mathscr{O}_X(d)$ over $D_+(g)$ is identified with the section of $\mathscr{O}_X$ over $D_+(g)$ that corresponds to the element f/g^d of $S_{(g)}$; to say that this section vanishes at $\mathfrak{p} \in D_+(g)$ implies that $f/g^d \in \mathfrak{q}$, where $\mathfrak{q}$ is the prime ideal of $S_{(g)}$ corresponding to $\mathfrak{p}$ (2.3.6); by definition, this implies that $f \in \mathfrak{p}$, whence the proposition. ☐

(2.6.4). Now let $\mathscr{F}$ be an $\mathscr{O}_X$-modules, and set $M = \Gamma_\bullet(\mathscr{F})$; by the existence of the homomorphism of graded rings $\alpha : S \to \Gamma_\bullet(\mathscr{O}_X)$, we can consider M as a graded S-module. For every $f \in S_d$ $(d > 0)$, it follows from (2.6.3) that the restriction to $D_+(f)$ of the section $\alpha_d(f)$ of $\mathscr{O}_X(d)$ is invertible; thus so too is the restriction to $D_+(f)$ of the section $\alpha_d(f^n)$ of $\mathscr{O}_X(nd)$, for all $n > 0$. So let $z \in M_{nd} = \Gamma(X, \mathscr{F}(nd))$ $(n > 0)$; if there exists an integer $k \geqslant 0$ such that the restriction to $D_+(f)$ of $f^k z$, i.e. the section II | 38
$(z|D_+(f))(\alpha_d(f^k)|D_+(f))$ of $\mathscr{F}((n+k)d)$, is zero, then, by the above remark, we also have that $z|D_+(f) = 0$. This shows that we have defined an $S_{(f)}$-homomorphism $\beta_f : M_{(f)} \to \Gamma(D_+(f), \mathscr{F})$ by sending the element z/f^n to the section $(z|D_+(f))(\alpha_d(f^n)|D_+(f))^{-1}$ of $\mathscr{F}$ over $D_+(f)$. We can further immediately show that the diagram

(2.6.4.1)
$$\begin{tikzcd} M_{(f)} \arrow[r, "\beta_f"] \arrow[d] & \Gamma(D_+(f), \mathscr{F}) \arrow[d] \\ M_{(fg)} \arrow[r, "\beta_{fg}"'] & \Gamma(D_+(fg), \mathscr{F}) \end{tikzcd}$$

commutes for $g \in S_e$ $(e > 0)$. If we recall that $M_{(f)}$ is canonically identified with $\Gamma(D_+(f), \widetilde{M})$, and that the $D_+(f)$ form a base for the topology of X (2.3.4), then we see that the β_f come from a unique canonical homomorphism of $\mathscr{O}_X$-modules

(2.6.4.2)
$$\beta : (\Gamma_\bullet(\mathscr{F}))^\sim \longrightarrow \mathscr{F}$$

(also denoted by $\beta_{\mathscr{F}}$) which is evidently functorial.

Proposition (2.6.5). — *Let M be a graded S-module, and $\mathscr{F}$ an $\mathscr{O}_X$-module; then the composite homomorphisms*

(2.6.5.1)
$$\widetilde{M} \xrightarrow{\tilde{\alpha}} (\Gamma_\bullet(\widetilde{M}))^\sim \xrightarrow{\beta} \widetilde{M}$$

(2.6.5.2)
$$\Gamma_\bullet(\mathscr{F}) \xrightarrow{\alpha} \Gamma_\bullet((\Gamma_\bullet(\mathscr{F}))^\sim) \xrightarrow{\Gamma_\bullet(\beta)} \Gamma_\bullet(\mathscr{F})$$

are the identity isomorphisms.

Proof. The proof for (2.6.5.1) is local: in an open subset $D_+(f)$, it follows immediately from the definitions, along with the fact that β, applied to quasi-coherent sheaves, is determined by its action on the sections over $D_+(f)$ (I, 1.3.8). The proof for (2.6.5.2) is done for each degree separately: if we set $M = \Gamma_\bullet(\mathscr{F})$, then $M_n = \Gamma(X, \mathscr{F}(n))$, and $(\Gamma_\bullet(\widetilde{M}))_n = \Gamma(X, \widetilde{M}(n)) = \Gamma(X, (M(n))^\sim)$. But if $f \in S_1$ and $z \in M_n$, then $\alpha_n^f(z)$ is the element $z/1$ of $(M(n))_{(f)}$, equal to $(f/1)^n(z/f^n)$; it corresponds, via β_f, to the section

$$\Big((\alpha_1(f))^n|D_+(f)\Big)\Big((z|D_+(f))\,((\alpha_1(f))^n|D_+(f))^{-1}\Big)$$

over $D_+(f)$, i.e. the restriction of z to $D_+(f)$, which finishes the proof for (2.6.5.2). □

2.7. Finiteness conditions

Proposition (2.7.1). —
(i) *If S is a graded Noetherian ring, then $X = \operatorname{Proj}(S)$ is a Noetherian scheme.*
(ii) *If S is a graded A-algebra of finite type, then $X = \operatorname{Proj}(S)$ is a scheme of finite type over $Y = \operatorname{Spec}(A)$.*

II | 39

Proof.

(i) If S is Noetherian, then the ideal S_+ admits a finite system of homogeneous generators f_i $(1 \leqslant i \leqslant p)$, thus (2.3.14) the underlying space X is the union of the $D_+(f_i) = \operatorname{Spec}(S_{(f_i)})$, and everything then reduces to showing that each of the $S_{(f_i)}$ is Noetherian, which follows from (2.2.6).

(ii) The hypothesis implies that S_0 is an A-algebra of finite type, and that S is an S_0-algebra of finite type, and so S_+ is an ideal of finite type (2.1.4). We are thus reduced, as in (i), to showing that $S_{(f)}$ is an A-algebra of finite type for all $f \in S_d$. By (2.2.5), it suffices to show that $S^{(d)}$ is an A-algebra of finite type, which follows from (2.1.6).

□

(2.7.2). In what follows, we consider the following finiteness conditions for a graded S-module M:

(TF) There exists an integer n such that the submodule $\bigoplus_{k\geqslant n} M_k$ is an S-module of finite type.
(TN) There exists an integer n such that $M_k = 0$ for $k \geqslant n$.

If M satisfies (TN), then $M_{(f)} = 0$ for all homogeneous f in S_+, and thus $\widetilde{M} = 0$.

Let M and N be graded S-modules; we say that a homomorphism $u : M \to N$ of degree 0 is (TN)-*injective* (resp. (TN)-*surjective*, (TN)-*bijective*) if there exists an integer n such that $u_k : M_k \to N_k$ is injective (resp. surjective, bijective) for $k \geqslant n$. Saying that u is (TN)-injective (resp. (TN)-surjective) thus reduces to saying that $\operatorname{Ker} u$ (resp. $\operatorname{Coker} u$) satisfies (TN). By (2.5.4), if u is (TN)-injective (resp. (TN)-surjective, (TN)-bijective), then $\widetilde{u}$ is injective (resp. surjective, bijective); if u is (TN)-bijective, then we also say that u is a (TN)-*isomorphism*.

Proposition (2.7.3). — *Let S be a graded ring such that the ideal S_+ is of finite type, and let M be a graded S-module.*

(i) *If M satisfies condition (TF) then the $\mathscr{O}_X$-module $\widetilde{M}$ is of finite type.*
(ii) *Suppose that M satisfies (TF); for $\widetilde{M} = 0$, it is necessary and sufficient for M to satisfy (TN).*

PROOF. We have just seen that condition (TN) implies that $\widetilde{M} = 0$. If M satisfies (TF), then the graded submodule $M' = \bigoplus_{k\geqslant n} M_k$, which is, by hypothesis, of finite type, is such that M/M' satisfies (TN); thus $(M/M')^{\sim}$, and the exactness of the functor $\widetilde{M}$ (2.5.4) implies that $\widetilde{M} = \widetilde{M'}$; to prove that $\widetilde{M}$ is of finite type, we can thus reduce to the case where M is *of finite type* . Since the question is local, it suffices to prove that $M_{(f)}$ is an $S_{(f)}$-module of finite type (**I**, 1.3.9); but $M^{(d)}$ is an $S^{(d)}$-module of finite type (2.1.6, iii), and our claim then follows from (2.2.5).

Now suppose that M satisfies (TF) and that $\widetilde{M} = 0$; then, with the same notation as above, we have that $\widetilde{M'} = 0$, and condition (TN) for M' is equivalent to condition (TN) for M, so to prove that $\widetilde{M} = 0$ implies that M satisfies (TN), we can again restrict to the case where M is generated by a finite number of homogeneous elements x_i ($1 \leqslant i \leqslant p$); let $(f_j)_{1\leqslant j\leqslant q}$ be a system of homogeneous generators of the ideal S_+. By hypothesis, $M_{(f_j)} = 0$ for all j, and so there exists an integer n such that $f_j^n x_i = 0$ for any i and j. Let n_j be the degree of f_j, and let m be the largest value of $\sum_j r_j n_j$ for the system of finitely many integers (r_j) such that $\sum_j r_j \leqslant nq$; it is then clear that, if $k > m$, then $S_k x_i = 0$ **II** | 40
for all i; if h is the largest of the degrees of the x_i, then we conclude that $M_k = 0$ for $k > h + m$, which finishes the proof. □

Corollary (2.7.4). — *Let S be a graded ring such that the ideal S_+ is of finite type; for $X = \operatorname{Proj}(S) = \varnothing$, it is necessary and sufficient for there to exist n such that $S_k = 0$ for $k \geqslant n$.*

PROOF. The condition $X = \varnothing$ is equivalent to $\mathscr{O}_X = \widetilde{S} = 0$, and S is a monogeneous S-module. □

Theorem (2.7.5). — *Suppose that the ideal S_+ is generated by a finite number of homogeneous elements of degree 1; let $X = \operatorname{Proj}(S)$. Then, for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $\beta : (\Gamma_\bullet(\mathscr{F}))^{\sim} \to \mathscr{F}$ (2.6.4) is an isomorphism.*

PROOF. If S_+ is generated by a finite number of elements $f_i \in S_1$, then X is the union of the subspaces $\operatorname{Spec}(S_{(f_i)})$ (2.3.6), which are quasi-compact, and so X is quasi-compact; furthermore, X is a scheme (2.4.2); by (**I**, 9.3.2), (2.5.14.2), and (2.6.3), we have, for all $f \in S_d$ ($d > 0$), a canonical isomorphism $(\Gamma_\bullet(\mathscr{F}))_{(\alpha_d(f))} \xrightarrow{\sim} \Gamma(D_+(f), \mathscr{F})$; also, by definition, $(\Gamma_\bullet(\mathscr{F}))_{(\alpha_d(f))}$ (where $\Gamma_\bullet(\mathscr{F})$ is considered as a $\Gamma_\bullet(\mathscr{O}_X)$-module) is exactly $(\Gamma_\bullet(\mathscr{F}))_{(f)}$ (where $\Gamma_\bullet(\mathscr{F})$ is considered as an S-module); if we refer to the definition (**I**, 9.3.1) of the above canonical isomorphism, then we see that it agrees with the homomorphism β_f, whence the theorem. □

Remark (2.7.6). — If we suppose that the graded ring S is *Noetherian*, then the condition of (2.7.5) is satisfied *ipso facto* as soon as we suppose that the ideal S_+ is generated by the set S_1 of homogeneous elements of degree 1.

Corollary (2.7.7). — *Under the hypotheses of (2.7.5), every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ is isomorphic to an $\mathscr{O}_X$-module of the form $\widetilde{M}$, where M is a graded S-module.*

Corollary (2.7.8). — *Under the hypotheses of (2.7.5), every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type is isomorphic to an $\mathscr{O}_X$-module of the form $\widetilde{N}$, where N is a graded S-module of finite type.*

PROOF. We can suppose that $\mathscr{F} = \widetilde{M}$, where M is a graded S-module (2.7.7). Let $(f_\lambda)_{\lambda\in L}$ be a system of homogeneous generators of M; for every finite subset H of L, let M_H be the graded submodule of M generated by the f_λ such that $\lambda \in H$; it is clear that M is the inductive limit of its submodules M_H, and so $\mathscr{F}$ is the inductive limit of its sub-$\mathscr{O}_X$-modules $\widetilde{M_H}$ (2.5.4). But, since $\mathscr{F}$ is of finite type, and since the underlying space of X is quasi-compact, it follows from (**0**, 5.2.3) that $\mathscr{F} = \widetilde{M_H}$ for some finite subset H of L. □

Corollary (2.7.9). — *Under the hypotheses of (2.7.5), let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type. Then there exists an integer n_0 such that, for all $n \geqslant n_0$, $\mathscr{F}(n)$ is isomorphic to a quotient of an $\mathscr{O}_X$-module of the form $\mathscr{O}_X^k$ (for some $k > 0$ depending on n), and is thus generated by a finite number of its sections over X (**0**, 5.1.1).*

PROOF. By (2.7.8), we can suppose that $\mathscr{F} = \widetilde{M}$, where M is a quotient of a finite direct sum of S-modules of the form $S(m_i)$; by (2.5.4), we can thus restrict to the case where $M = S(m)$, and so $\mathscr{F}(n) = (S(m+n))^\sim = \mathscr{O}_X(m+n)$ (2.5.15). It thus suffices to prove

| 41

Lemma (2.7.9.1). — *Under the hypotheses of (2.7.5), for all $n \geqslant 0$, there exists an integer k (depending on n) and a surjective homomorphism $\mathscr{O}_X^k \to \mathscr{O}_X(n)$.*

It suffices (2.7.2) to show that, for suitable k, there is a (TN)-surjective homomorphism u *of degree* 0 from the graded product S-module S^k to $S(n)$. But $(S(n))_0 = S_n$, and, by hypothesis, $S_h = S_1^h$ for all $h > 0$, and so $SS_n = S_n + S_{n+1} + \ldots$. Since S_n is an S_0-module of finite type ((2.1.5) and (2.1.6, i)), consider a system of generators $(a_i)_{1\leqslant i\leqslant k}$ of this module; consider the homomorphism u that sends a_i to the i-th element of the canonical basis of S^k ($1 \leqslant i \leqslant k$); since Coker u can then be identified with $(S(n))_{-n} + \ldots + (S(n))_{-1}$, u is indeed the desired homomorphism. □

Corollary (2.7.10). — *Under the hypotheses of (2.7.5), let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type. Then there exists an integer n_0 such that, for all $n \geqslant n_0$, $\mathscr{F}$ is isomorphic to a quotient of an $\mathscr{O}_X$-module of the form $(\mathscr{O}_X(-n))^k$ (for some k depending on n).*

Proposition (2.7.11). — *Suppose that the hypotheses of (2.7.5) are satisfied, and let M be a graded S-module. Then:*

(i) *The canonical homomorphism $\widetilde{\alpha} : \widetilde{M} \to (\Gamma_\bullet(\widetilde{M}))^\sim$ is an isomorphism.*
(ii) *Let $\mathscr{G}$ be a quasi-coherent sub-$\mathscr{O}_X$-module of $\widetilde{M}$, and let N be the graded sub-S-module of M given by the inverse image of $\Gamma_\bullet(\mathscr{G})$ under α. Then $\widetilde{N} = \mathscr{G}$ (where $\widetilde{N}$ is identified, by (2.5.4), with a sub-$\mathscr{O}_X$-module of $\widetilde{M}$).*

PROOF. Since $\beta : (\Gamma_\bullet(\widetilde{M}))^\sim \to \widetilde{M}$ is an isomorphism (by (2.7.5)), $\widetilde{\alpha}$ is the inverse isomorphism (by (2.6.5.1)), whence (i). Let P be the graded submodule $\alpha(M)$ of $\Gamma_\bullet(\widetilde{M})$; since $\widetilde{M}$ is an exact functor (2.5.4), the image of $\widetilde{M}$ under $\widetilde{\alpha}$ is equal to $\widetilde{P}$, and so, by (i), $\widetilde{P} = (\Gamma_\bullet(\widetilde{M}))^\sim$. Set $Q = \Gamma_\bullet(\mathscr{G}) \cap P$; by the above, and by (2.5.4), we have that $\widetilde{Q} = (\Gamma_\bullet(\mathscr{G}))^\sim$, and so the restriction of β to $\widetilde{Q}$ is an *isomorphism* from this $\mathscr{O}_X$-module to G by (2.7.5). But, by the definition of N, and by (2.5.4), the restriction of the isomorphism $\widetilde{\alpha}$ to $\widetilde{N}$ is an isomorphism from $\widetilde{N}$ to $\widetilde{Q}$, whence the conclusion, by (2.6.5.1). □

2.8. Functorial behaviour

(2.8.1). Let S and S' be positively graded rings, and $\phi : S' \to S$ a homomorphism of graded rings. We denote by $G(\phi)$ the open subset of $X = \operatorname{Proj}(S)$ given by the complement of $V_+(\phi(S'_+))$, or, equivalently, the union of the $D_+(\phi(f'))$ where f' runs over the set of homogeneous elements of S'_+. The restriction to $G(\phi)$ of the continuous map ${}^a\phi$ from $\operatorname{Spec}(S)$ to $\operatorname{Spec}(S')$ (**I**, 1.2.1) is thus a continuous map from $G(\phi)$ to $\operatorname{Proj}(S')$, which we again denote, with an abuse of language, by ${}^a\phi$. If $f' \in S'_+$ is homogeneous, then

$$ {}^a\phi^{-1}(D_+(f')) = D_+(\phi(f')) \tag{2.8.1.1} $$

taking into account the fact that ${}^a\phi$ sends $G(\phi)$ to $\operatorname{Proj}(S')$, as well as (**I**, 1.2.2.2). The homomorphism ϕ also canonically defines (with the same notation) a homomorphism of graded rings $S'_{f'} \to S_f$, whence, by restriction to the degree 0 elements, a homomorphism $S'_{(f')} \to S_{(f)}$, which we denote II | 42
by $\phi_{(f)}$; there is a corresponding (**I**, 1.6.1) morphism of affine schemes $({}^a\phi_{(f)}, \widetilde{\phi}_{(f)}) : \operatorname{Spec}(S_{(f)}) \to \operatorname{Spec}(S'_{(f')})$. If we canonically identify $\operatorname{Spec}(S_{(f)})$ with the scheme induced by $\operatorname{Proj}(S)$ on $D_+(f)$ (2.3.6), then we have defined a morphism $\Phi_f : D_+(f) \to D_+(f')$, and ${}^a\phi_{(f)}$ is identified with the restriction of ${}^a\phi$ to $D_+(f)$. It is also immediate that, if g' is another homogeneous element of S'_+, and $g = \phi(g')$, then the diagram

$$\begin{tikzcd} D_+(f) \arrow[r, "\Phi_f"] & D_+(f') \\ D_+(fg) \arrow[u] \arrow[r, "\Phi_{fg}"'] & D_+(f'g') \arrow[u] \end{tikzcd}$$

commutes, by the fact that the diagram

$$\begin{tikzcd} S'_{(f')} \arrow[r, "\phi_{(f)}"] \arrow[d, "\omega_{f'g',f'}"'] & S_{(f)} \arrow[d, "\omega_{fg,f}"] \\ S'_{(f'g')} \arrow[r, "\phi_{(fg)}"'] & S_{(fg)} \end{tikzcd}$$

commutes. Taking the definition of $G(\phi)$, along with (2.3.3.2), we thus see that:

Proposition (2.8.2). — *Given a homomorphism of graded rings $\phi : S' \to S$, there exists exactly one morphism $({}^a\phi, \widetilde{\phi})$ from the induced prescheme $G(\phi)$ to $\operatorname{Proj}(S')$ (said to be* associated to ϕ*, and denoted by* $\operatorname{Proj}(\phi)$*) such that, for every homogeneous element $f' \in S'_+$, the restriction of this morphism to $D_+(\phi(f'))$ agrees with the morphism associated to the homomorphism $S'_{(f')} \to S_{(\phi(f'))}$ corresponding to ϕ.*

PROOF. With the above notation, if $f' \in S'_d$, then the diagram

$$\begin{tikzcd} S'_{(f')} \arrow[r, "\phi_{(f)}"] \arrow[d, "\sim"'] & S_{(f)} \arrow[d, "\sim"] \\ S'^{(d)}/(f'-1)S'^{(d)} \arrow[r] & S^{(d)}/(f-1)S^{(d)} \end{tikzcd} \tag{2.8.2.1}$$

commutes (the vertical arrows being the isomorphisms (2.2.5)). □

Corollary (2.8.3). — (i) *The morphism* $\operatorname{Proj}(\phi)$ *is affine.*
(ii) *If* $\operatorname{Ker}(\phi)$ *is nilpotent (and, in particular, if ϕ is injective), then the morphism* $\operatorname{Proj}(\phi)$ *is dominant.*

PROOF. Claim (i) is an immediate consequence of (2.8.2) and (2.8.1.1). Claim (ii) follows since, if $\operatorname{Ker}(\phi)$ is nilpotent, then, for every homogeneous f' in S'_+, we immediately see that $\operatorname{Ker}(\phi_f)$ (with $f = \phi(f')$) is nilpotent, and thus so too is $\operatorname{Ker}(\phi_{(f)})$; we then apply (2.8.2) and (**I**, 1.2.7) □

We note that there are, in general, morphisms from $\mathrm{Proj}(S)$ to $\mathrm{Proj}(S')$ that are not affine, and **II** | 43
that thus do not come from homomorphisms of graded rings $S' \to S$; an example is the structure morphism $\mathrm{Proj}(S) \to \mathrm{Spec}(A)$ when A is a field ($\mathrm{Spec}(A)$ thus being identified with $\mathrm{Proj}(A[T])$, cf. (3.1.7)); indeed, this follows from (**I**, 2.3.2).

(2.8.4). Let S'' be another positively graded ring, and $\phi' : S'' \to S'$ a homomorphism of graded rings, and set $\phi'' = \phi \circ \phi'$. By (2.8.1.1) and the formula ${}^a\phi'' = ({}^a\phi') \circ ({}^a\phi)$, we immediately see that $G(\phi'') \subset G(\phi)$, and that, if Φ, Φ', and Φ'' are the morphisms associated to ϕ, ϕ', and ϕ'' (respectively), then $\Phi'' = \Phi' \circ (\Phi|G(\phi''))$.

(2.8.5). Suppose that S (resp. S') is a graded A-algebra (resp. a graded A'-algebra), and let $\psi : A' \to A$ be a ring homomorphism such that the diagram

(2.8.5.1)
$$\begin{tikzcd} A' \arrow[r, "\psi"] \arrow[d] & A \arrow[d] \\ S' \arrow[r, "\phi"'] & S \end{tikzcd}$$

commutes. We can then consider $G(\phi)$ (resp. $\mathrm{Proj}(S')$) as a scheme over $\mathrm{Spec}(A)$ (*resp.* $\mathrm{Spec}(A')$); if Φ (resp. Ψ) is the morphism associated to ϕ (resp. ψ), then the diagram

$$\begin{tikzcd} G(\phi) \arrow[r, "\Phi"] \arrow[d] & \mathrm{Proj}(S') \arrow[d] \\ \mathrm{Spec}(A) \arrow[r, "\Psi"'] & \mathrm{Spec}(A') \end{tikzcd}$$

commutes: it suffices to prove this for the restriction of Φ to $D_+(f)$, where $f = \phi(f')$, with f' homogeneous in S'_+; this then follows from the fact that the diagram

$$\begin{tikzcd} A' \arrow[r, "\psi"] \arrow[d] & A \arrow[d] \\ S'_{(f')} \arrow[r, "\phi_{(f)}"'] & S_{(f)} \end{tikzcd}$$

commutes.

(2.8.6). Now let M be a graded S-module, and consider the S'-module $M_{[\phi]}$, which is evidently graded. Let f' be homogeneous in S'_+, and let $f = \phi(f')$; we know (**0**, 1.5.2) that there is a canonical isomorphism $(M_{[\phi]})_{f'} \xrightarrow{\sim} (M_f)_{[\phi_f]}$, and it is immediate that this isomorphism preserves degree, whence a canonical isomorphism $(M_{[\phi]})_{(f')} \xrightarrow{\sim} (M_{(f)})_{[\phi_{(f)}]}$. To this isomorphism, there canonically corresponds an isomorphism of sheaves $(M_{[\phi]})^\sim|D_+(f') \xrightarrow{\sim} (\Phi_f)_*(\widetilde{(M)}|D_+(f))$ ((2.5.2) and (**I**, 1.6.3)). Furthermore, if g' is another homogeneous element of S'_+, and $g = \phi(g')$, then the | 43
diagram

$$\begin{tikzcd} (M_{[\phi]})_{(f')} \arrow[r, "\sim"] \arrow[d] & (M_{(f)})_{[\phi_{(f)}]} \arrow[d] \\ (M_{[\phi]})_{(f'g')} \arrow[r, "\sim"] & (M_{(fg)})_{[\phi_{(fg)}]} \end{tikzcd}$$

commutes, whence we immediately conclude that the isomorphism

$$(M_{[\phi]})^\sim|D_+(f'g') \xrightarrow{\sim} (\Phi_{fg})_*(\widetilde{M}|D_+(fg))$$

is the restriction to $D_+(f'g')$ of the isomorphism $(M_{[\phi]})^\sim|D_+(f') \xrightarrow{\sim} (\Phi_f)_*(\widetilde{M}|D_+(f))$. Since Φ_f is the restriction to $D_+(f)$ of the morphism Φ, we see that, taking (2.8.1.1) into account, and setting $X' = \mathrm{Proj}(S)'$:

Proposition (2.8.7). — *There exists a canonical functorial isomorphism from the $\mathscr{O}_{X'}$-module $(M_{[\phi]})^{\sim}$ to the $\mathscr{O}_{X'}$-module $\Phi_*(\widetilde{M}|G(\phi))$.*

We thus immediately deduce a canonical functorial map from the set of ϕ-morphisms $M' \to M$ from a graded S'-module to the graded S-module M, to the set of Φ-morphisms $\widetilde{M'} \to \widetilde{M}|G(\phi)$. With the notation of (2.8.4), if M'' is a graded S''-module, then, to the composition of a ϕ-morphism $M' \to M$ and a ϕ'-morphism $M'' \to M'$, canonically corresponds the composition of $\widetilde{M'}G(\phi') \to \widetilde{M}|G(\phi'')$ and $\widetilde{M''} \to \widetilde{M'}|G(\phi')$.

Proposition (2.8.8). — *Under the hypotheses of (2.8.1), let M' be a graded S'-module. Then there exists a canonical functorial homomorphism ν from the $(\mathscr{O}_X|G(\phi))$-module $\Phi^*(\widetilde{M'})$ to the $(\mathscr{O}_X|G(\phi))$-module $(M' \otimes_{S'} S)^{\sim}|G(\phi)$. If the ideal S'_+ is generated by S'_1, then ν is an isomorphism.*

Proof. Indeed, for $f' \in S'_d$ $(d > 0)$, we define a canonical functorial homomorphism of $S_{(f)}$-modules (where $f = \phi(f')$)

(2.8.8.1) $$\nu_f : M'_{(f')} \otimes_{S'_{(f')}} S_{(f)} \longrightarrow (M' \otimes_{S'} S)_{(f)}$$

by composing the homomorphism $M'_{(f')} \otimes_{S'_{(f')}} S_{(f)} \to M'_{f'} \otimes_{S'_{f'}} S_f$ and the canonical isomorphism $M'_{f'} \otimes_{S'_{f'}} S_f \xrightarrow{\sim} (M' \otimes_{S'} S)_f$ (**0**, 1.5.4), and noting that the latter preserves degrees. We can immediately verify the compatibility of ν_f with the restriction operators from $D_+(f)$ to $D_+(fg)$ (for any $g' \in S'_+$ and $g = \phi(g')$), whence the definition of the homomorphism

$$\nu : \Phi^*(\widetilde{M'}) \longrightarrow (M' \otimes_{S'} S)^{\sim}|G(\phi)$$

taking (**I**, 1.6.5) into account. To prove the second claim, it suffices to show that ν_f is an isomorphism for all $f' \in S_1$, since $G(\phi)$ is then a union of the $D_+(\phi(f'))$. We first define a $\mathbf{Z}$-bilinear $M'_m \times S_n \to M'_{(f')} \otimes_{S'_{(f')}} S_{(f)}$ by sending (x', s) to the element $(x'/f'^m) \otimes (s/f^n)$ (with the convention that x'/f'^m is $f'^{-m}x'/1$ when $m < 0$). We claim that, in the proof of (2.5.13), this map gives rise to a **II** | 45
di-homomorphism of modules

$$\eta_f : M' \otimes_{S'} S \longrightarrow M'_{(f')} \otimes_{S_{(f')}} S_{(f)}.$$

Furthermore, if, for $r > 0$, we have $f^r \sum_i (x'_i \otimes s_i) = 0$, then this can also be written as $\sum_i (f'^r x'_i \otimes s_i) = 0$, whence, by (**0**, 1.5.4), $\sum_i (f'^r x_i / f'^{m_i + r}) \otimes (s_i / f^{n_i}) = 0$, i.e. $\eta_f(\sum_i x_i \otimes y_i)0 =$, which proves that η_f factors as $M' \otimes_{S'} S \to (M' \otimes_{S'} S)_f \xrightarrow{\eta'_f} M'_{(f')} \otimes_{S'_{(f')}} S_{(f)}$; we finally can prove that η'_f and ν_f are inverse isomorphisms to one another.

In particular, it follows from (2.1.2.1) that we have a canonical homomorphism

(2.8.8.2) $$\Phi^*(\mathscr{O}_{X'}(n)) \xrightarrow{\sim} \mathscr{O}_X(n)|G(\phi)$$

for all $n \in \mathbf{Z}$. □

(2.8.9). Let A and A' be rings, and $\psi : A' \to A$ a ring homomorphism, defining a morphism $\Psi : \operatorname{Spec}(A) \to \operatorname{Spec}(A')$. Let S' be a positively graded A'-algebra, and set $S = S' \otimes_{A'} A$, which is evidently an A-algebra graded by the $S'_n \otimes_{A'} A$; the map $\phi : s' \to s' \otimes 1$ is then a graded ring homomorphism that makes the diagram (2.8.5.1) commute. Since S_+ is here the A-module generated by $\phi(S'_+)$, we have $G(\phi) = \operatorname{Proj}(S) = X$; whence, setting $X' = \operatorname{Proj}(S')$, we have the commutative diagram

(2.8.9.1) $$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "p"'] & X' \arrow[d] \\ Y \arrow[r, "\Psi"'] & Y' \end{tikzcd}$$

Now let M' be a graded S'-module, and set $M = M' \otimes_{A'} A = M' \otimes_{S'} S$. Under these conditions:

Proposition (2.8.10). — *The diagram (2.8.9.1) identifies the scheme X with the product $X' \times_{Y'} Y$; furthermore, the canonical homomorphism $\nu : \Phi^*(\widetilde{M'}) \to \widetilde{M}$ (2.8.8) is an isomorphism.*

PROOF. The first claim will be proven if we show that, for every homogenous f' in S'_+, setting $f = \phi(f')$, the restrictions of Φ and p to $D_+(f)$ identify this scheme with the product $D_+(f') \times_{Y'} Y$ (**I**, 3.2.6.2); in other words, it suffices (**I**, 3.2.2) to prove that $S_{(f)}$ is canonically identified with $S_f \xrightarrow{\sim} S'_{f'} \otimes_{A'} A$, which is immediate by the existence of the canonical isomorphism $S_f \xrightarrow{\sim} S'_{f'} \otimes_{A'} A$ that preserves degrees (**0**, 1.5.4). The second claim then follows from the fact that $M'_{(f')} \otimes_{S'_{(f')}} S_{(f)}$ can be identified, by the above, with $M'_{(f')} \otimes_{A'} A$, and this can be identified with $M_{(f)}$, since M_f is canonically identified with $M'_{f'} \otimes_{A'} A$ by an isomorphism that preserves degrees. □

Corollary (2.8.11). — *For every integer $n \in \mathbf{Z}$, $\widetilde{M}(n)$ can be identified with $\Phi^*(\widetilde{M'}(n)) = \widetilde{M'}(n) \otimes_{Y'} \mathscr{O}_Y$; in particular, $\mathscr{O}_X(n) = \Phi^*(\mathscr{O}_{X'}(n)) = \mathscr{O}_{X'}(n) \otimes_{Y'} \mathscr{O}_Y$.*

PROOF. This follows from (2.8.10) and (2.5.15). □

II | 46

(2.8.12). Under the hypotheses of (2.8.9), for $f' \in S'_d$ ($d > 0$) and $f = \phi(f')$, the diagram

(2.8.12.1)
$$\begin{tikzcd} M'_{(f')} \arrow[r, "\sim"] \arrow[d] & M'^{(d)}/(f'-1)M'^{(d)} \arrow[d] \\ M_{(f)} \arrow[r, "\sim"] & M^{(d)}/(f-1)M^{(d)} \end{tikzcd}$$

(cf. (2.2.5)) commutes.

(2.8.13). Keep the notation and hypotheses of (2.8.9), and let $\mathscr{F}'$ be an $\mathscr{O}_{X'}$-module; if we set $\mathscr{F} = \Phi^*(\mathscr{F}')$, then, for all $n \in \mathbf{Z}$, we have $\mathscr{F}(n) = \Phi^*(\mathscr{F}'(n))$, by (2.8.11) and (**0**, 4.3.3). Then (**0**, 3.7.1) we have a canonical homomorphism

$$\Gamma(\rho) : \Gamma(X', \mathscr{F}'(n)) \longrightarrow \Gamma(X, \mathscr{F}(n))$$

which gives a canonical di-homomorphism of graded modules

$$\Gamma_\bullet(\mathscr{F}') \longrightarrow \Gamma_\bullet(\mathscr{F}).$$

Suppose that the ideal S_+ is generated by S_1, and that $\mathscr{F}' = \widetilde{M'}$, thus $\mathscr{F} = \widetilde{M}$ with $M = M' \otimes_{A'} A$. If f' is homogeneous in S'_+, and $f = \phi(f')$, then we have seen that $M_{(f)} = M'_{(f')} \otimes_{A'} A$, and the diagram

$$\begin{tikzcd} M'_0 \arrow[r] \arrow[d] & M'_{(f')} \arrow[d] & = \Gamma(D_+(f'), \widetilde{M'}) \\ M_0 \arrow[r] & M_{(f)} & = \Gamma(D_+(f), \widetilde{M}) \end{tikzcd}$$

thus commutes; we immediately conclude from this remark, and from the definition of the homomorphism α (2.6.2), that the diagram

(2.8.13.1)
$$\begin{tikzcd} M' \arrow[r, "\alpha_{M'}"] \arrow[d] & \Gamma_\bullet(\widetilde{M'}) \arrow[d] \\ M \arrow[r, "\alpha_M"'] & \Gamma_\bullet(\widetilde{M}) \end{tikzcd}$$

commutes. Similarly, the diagram

(2.8.13.2)
$$\begin{tikzcd} (\Gamma_\bullet(\mathscr{F}'))^\sim \arrow[r, "\beta_{\mathscr{F}'}"] \arrow[d] & \mathscr{F}' \arrow[d] \\ (\Gamma_\bullet(\mathscr{F}))^\sim \arrow[r, "\beta_{\mathscr{F}}"'] & \mathscr{F} \end{tikzcd}$$

commutes (the vertical arrow on the right being the canonical Φ-morphism $\mathscr{F}' \to \Phi^*(\mathscr{F}') = \mathscr{F}$).

II | 47

(2.8.14). Still keeping the notation and hypotheses of (2.8.9), let N' be another graded S'-module, and let $N = N' \otimes_{A'} A$. It is immediate that the canonical di-homomorphisms $M' \to M$ and $N' \to N$ give a di-homomorphism $M' \otimes_{S'} N' \to M \otimes_S N$ (with respect to the canonical ring homomorphism $S' \to S$), and thus also an S-homomorphism $(M' \otimes_{S'} N') \otimes_{A'} A \to M \otimes_S N$ of degree 0, to which corresponds (taking (2.8.10) into account) a homomorphism of $\mathscr{O}_X$-modules

$$\Phi^*((M' \otimes_{S'} N')^{\sim}) \longrightarrow (M \otimes_S N)^{\sim}. \tag{2.8.14.1}$$

Furthermore, we can immediately verify that the diagram

$$\begin{tikzcd} \Phi^*(\widetilde{M'} \otimes_{\mathscr{O}_{X'}} \widetilde{N'}) \arrow[r, "\sim"] \arrow[d, "\Phi^*(\lambda)"'] & \widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N} \arrow[d, "\lambda"] & = \Phi^*(\widetilde{M'}) \otimes_{\mathscr{O}_X} \Phi^*(\widetilde{N'}) \\ \Phi^*((M' \otimes_{S'} N')^{\sim}) \arrow[r] & (M \otimes_S N)^{\sim} & \end{tikzcd} \tag{2.8.14.2}$$

commutes, with the first line being the canonical isomorphism (**0**, 4.3.3). If the ideal S'_+ is generated by S'_1, then it is clear that S_+ is generated by S_1, and the two vertical arrows of (2.8.14.2) are then isomorphisms (2.5.13); it is thus also the case for (2.8.14.1).

We similarly have a canonical di-homomorphism $\operatorname{Hom}_{S'}(M', N') \to \operatorname{Hom}_S(M, N)$ by sending a homomorphism u' of degree k to the homomorphism $u' \otimes 1$, which is also of degree k; from this, we again deduce a S-homomorphism of degree 0

$$(\operatorname{Hom}_{S'}(M', N')) \otimes_{A'} A \longrightarrow \operatorname{Hom}_S(M, N)$$

whence a homomorphism of $\mathscr{O}_X$-modules

$$\Phi^*((\operatorname{Hom}_{S'}(M', N'))^{\sim}) \longrightarrow (\operatorname{Hom}_S(M, N))^{\sim}. \tag{2.8.14.3}$$

Furthermore, the diagram

$$\begin{tikzcd} \Phi^*((\operatorname{Hom}_{S'}(M', N'))^{\sim}) \arrow[r] \arrow[d, "\Phi^*(\mu)"'] & (\operatorname{Hom}_S(M, N))^{\sim} \arrow[d, "\mu"] \\ \Phi^*(\mathscr{H}om_{\mathscr{O}_{X'}}(\widetilde{M'}, \widetilde{N'})) \arrow[r] & \mathscr{H}om_{\mathscr{O}_X}(\widetilde{M}, \widetilde{N}) \end{tikzcd}$$

commutes (the second horizontal line being the canonical homomorphism (**0**, 4.4.6)).

II | 48

(2.8.15). With the notation and hypotheses of (2.8.1), it follows from (2.4.7) that we do not change the morphism Φ, up to isomorphism, when we replace S_0 and S'_0 by $\mathbf{Z}$, and ϕ_0 by the identity map, and thus when we replace S and S' by $S^{(d)}$ and ${S'}^{(d)}$ (respectively) ($d > 0$), and ϕ by its restriction $\phi^{(d)}$ to $S^{(d)}$.

2.9. Closed subschemes of a scheme $\operatorname{Proj}(S)$

(2.9.1). If $\phi : S \to S'$ is a homomorphism of graded rings, then we say that ϕ is (TN)-*surjective* (resp. (TN)-*injective*, (TN)-*bijective*) if there exists an integer n such that, for $k \geqslant n$, $\phi_k : S_k \to S'_k$ is *surjective* (resp. *injective*, *bijective*). Instead of saying that ϕ is (TN)-bijective, we sometimes say that it is a (TN)-*isomorphism*.

Proposition (2.9.2). — *Let S be a positively graded ring, and let $X = \operatorname{Proj}(S)$.*

(i) *If $\phi : S \to S'$ is a (TN)-surjective homomorphism of graded rings, then the corresponding morphism Φ (2.8.1) is defined on the whole of $\operatorname{Proj}(S')$, and is a closed immersion of $\operatorname{Proj}(S')$ into X. If $\mathfrak{I}$ is the kernel of ϕ, then the closed subscheme of X associated to Φ is defined by the quasi-coherent sheaf of ideals $\widetilde{\mathfrak{I}}$ of $\mathscr{O}_X$.*

(ii) *Suppose further that the ideal S_+ is generated by a finite number of homogeneous elements of degree* 1. *Let X' be a closed subscheme of X defined by a quasi-coherent sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_X$. Let $\mathfrak{J}$ be the graded ideal of S given by the inverse image of $\Gamma_\bullet(\mathscr{J})$ under the canonical homomorphism $\alpha : S \to \Gamma_\bullet(\mathscr{O}_X)$ (2.6.2), and set $S' = S/\mathfrak{J}$. Then X' is the subscheme associated to the closed immersion* $\operatorname{Proj}(S') \to X$ *corresponding to the canonical homomorphism of graded rings $S \to S'$.*

PROOF.

(i) We can suppose that ϕ is surjective (2.9.1). Since, by hypothesis, $\phi(S_+)$ generates S'_+, we have $G(\phi) = \operatorname{Proj}(S')$. Now, the second claim can be checked locally on X; so let f be a homogeneous element of S_+, and set $f' = \phi(f)$. Since ϕ is a surjective homomorphism of graded rings, we immediately see that $\phi_{(f')} : S_{(f)} \to S'_{(f')}$ is surjective, and that its kernel is $\mathfrak{J}_{(f)}$, which proves (i) (**I**, 4.2.3).

(ii) By (i), we are led to proving that the homomorphism $\widetilde{j} : \widetilde{\mathfrak{J}} \to \mathscr{O}_X$ induced by the canonical injection $j : \mathfrak{J} \to S$ is an isomorphism from $\widetilde{\mathfrak{J}}$ to $\mathscr{J}$, which follows from (2.7.11).

□

We note that $\mathfrak{J}$ is the *largest* of the graded ideals $\mathfrak{J}'$ of S such that $\widetilde{j}(\widetilde{\mathfrak{J}'}) = \mathscr{J}$, since we can immediately show, using the definitions (2.6.2), that this equation implies that $\alpha(\mathfrak{J}') \subset \Gamma_\bullet(\mathscr{J})$.

Corollary (2.9.3). — *Suppose that the hypotheses of (2.9.2, (i)) are satisfied, and further that the ideal S_+ is generated by S_1; then $\Phi^*((S(n))^\sim)$ is canonically isomorphic to $(S'(n))^\sim$ for all $n \in \mathbf{Z}$, and so $\Phi^*(\mathscr{F}(n))$ is canonically isomorphic to $\Phi^*(\mathscr{F})(n)$ for every $\mathscr{O}_X$-module $\mathscr{F}$.*

PROOF. This is a particular case of (2.8.8), taking (2.5.10.2) into account. □

Corollary (2.9.4). — *Suppose that the hypotheses of (2.9.2, (ii)) are satisfied. For the closed sub-prescheme X' of X to be integral, it is necessary and sufficient for the graded ideal $\mathfrak{J}$ to be prime in S.*

II | 49

PROOF. Since X' is isomorphic to $\operatorname{Proj}(S/\mathfrak{J})$, the condition is sufficient by (2.4.4). To see that it is necessary, consider the exact sequence $0 \to \mathscr{J} \to \mathscr{O}_X \to \mathscr{O}_X/\mathscr{J}$, which gives the exact sequence

$$0 \longrightarrow \Gamma_\bullet(\mathscr{J}) \longrightarrow \Gamma_\bullet(\mathscr{O}_X) \longrightarrow \Gamma_\bullet(\mathscr{O}_X/\mathscr{J}).$$

It suffices to prove that, if $f \in S_m$ and $g \in S_n$ are such that the image in $\Gamma_\bullet(\mathscr{O}_X/\mathscr{J})$ of $\alpha_{n+m}(fg)$ is zero, then the image of either $\alpha_m(f)$ or $\alpha_n(g)$ is zero. But, by definition, these images are sections of invertible $(\mathscr{O}_X/\mathscr{J})$-modules $\mathscr{L} = (\mathscr{O}_X/\mathscr{J})(m)$ and $\mathscr{L}' = (\mathscr{O}_X/\mathscr{J})(n)$ over the integral scheme X'; the hypothesis implies that the product of these two sections is zero in $\mathscr{L} \otimes \mathscr{L}'$ ((2.9.3) and (2.5.14.1)), and so one of them is zero by (**I**, 7.4.4). □

Corollary (2.9.5). — *Let A be a ring, M an A-module, S a graded A-algebra generated by the set S_1 of homogeneous elements of degree* 1, *$u : M \to S_1$ a surjective homomorphism of A-modules, and $\overline{u} : \mathbf{S}(M) \to S$ the homomorphism (of A-algebras) from the symmetric algebra $\mathbf{S}(M)$ of M to S that extends u. Then the morphism corresponding to $\overline{u}$ is a closed immersion of* $\operatorname{Proj}(S)$ *into* $\operatorname{Proj}(\mathbf{S}(M))$.

PROOF. Indeed, $\overline{u}$ is surjective by hypothesis, and so it suffices to apply (2.9.2) □

§3. HOMOGENEOUS SPECTRUM OF A SHEAF OF GRADED ALGEBRAS

3.1. Homogeneous spectrum of a quasi-coherent graded $\mathscr{O}_Y$-algebra

(3.1.1). Let Y be a prescheme, $\mathscr{S}$ a graded $\mathscr{O}_Y$-algebra, and $\mathscr{M}$ a graded $\mathscr{S}$-module. If $\mathscr{S}$ is *quasi-coherent*, then each of its homogenous components $\mathscr{S}_n$ is a *quasi-coherent* $\mathscr{O}_Y$-module, since they are the images of $\mathscr{S}$ under a homomorphism from $\mathscr{S}$ to itself ((**I**, 1.3.8) and (**I**, 1.3.9)); similarly, if $\mathscr{M}$ is quasi-coherent as an $\mathscr{O}_Y$-module, then its homogenous components $\mathscr{M}_n$ are also quasi-coherent, and the converse is also true. For an integer $d > 0$, we denote by $\mathscr{S}^{(d)}$ the direct sum of the $\mathscr{O}_Y$-modules $\mathscr{S}_{nd}$ (for $n \in \mathbf{Z}$), which is quasi-coherent if $\mathscr{S}$ is (**I**, 1.3.9); for every integer k such that $0 \leqslant k \leqslant d-1$, we denote by $\mathscr{M}^{(d,k)}$ (or $\mathscr{M}^{(d)}$, for $k = 0$) the direct sum of the $\mathscr{M}_{nd+k}$ (for $n \in \mathbf{Z}$), which is a graded $\mathscr{S}^{(d)}$-module, and quasi-coherent if both $\mathscr{S}$ and $\mathscr{M}$ are quasi-coherent (**I**, 9.6.1). We denote by $\mathscr{M}(n)$

the graded $\mathscr{S}$-module such that $(\mathscr{M}(n))_k = \mathscr{M}_{n+k}$ for all $k \in \mathbf{Z}$; if $\mathscr{S}$ and $\mathscr{M}$ are quasi-coherent, then $\mathscr{M}(n)$ is a quasi-coherent graded $\mathscr{S}$-module (**I**, 9.6.1).

We say that $\mathscr{M}$ is a graded $\mathscr{S}$-module *of finite type* (resp. admitting a *finite presentation*) if, for all $y \in Y$, there exists an open neighbourhood U of y, along with integers n_i (resp. integers m_i and n_j) such that there is a surjective degree 0 homomorphism $\bigoplus_{i=1}^r (\mathscr{S}(n_i)|U) \to \mathscr{M}|U$ (resp. such that $\mathscr{M}|U$ is isomorphic to the cokernel of a degree 0 homomorphism $\bigoplus_{i=1}^r (\mathscr{S}(m_i)|U) \to \bigoplus_{j1}^s (\mathscr{S}(n_J)|U)$).

Let U be an affine open of Y, with ring $A = \Gamma(U, \mathscr{O}_Y)$; by hypothesis, the graded $(\mathscr{O}_Y|U)$-algebra $\mathscr{S}|U$ is isomorphic to $\widetilde{S}$, where $S = \Gamma(U, \mathscr{S})$ is a graded A-algebra (**I**, 1.4.3); set $X_U = \operatorname{Proj}(\Gamma(U, \mathscr{S}))$. **II** | 50
Let $U' \subset U$ be another affine open of Y, with ring A', and let $j : U' \to U$ be the canonical injection, which corresponds to the restriction homomorphism $A \to A'$; we have that $\mathscr{S}|U' = j^*(\mathscr{S}|U)$, and so $S' = \Gamma(U', \mathscr{S})$ is canonically identified with $X_U \times_U U'$, and thus also with $f_U^{-1}(U')$, where we denote by f_U the structure morphism $X_U \to U$ (**I**, 4.4.1). We denote by $\sigma_{U',U}$ the canonical isomorphism $f_U^{-1}(U') \xrightarrow{\sim} X_{U'}$ thus defined, and by $\rho_{U',U}$ the open immersion $X_{U'} \to X_U$ obtained by composing $\sigma_{U',U}^{-1}$ with the canonical injection $f_U^{-1}(U') \to X_U$. It is immediate that, if $U'' \subset U'$ is another affine open of Y, then $\rho_{U'',U} = \rho_{U'',U'} \circ \rho_{U',U}$.

Proposition (3.1.2). — *Let Y be a prescheme. For every quasi-coherent positively graded $\mathscr{O}_Y$-algebra, there exists exactly one (up to Y-isomorphism) prescheme X over Y satisfying the following property: if $f : X \to Y$ is the structure morphism, then, for every affine open U of Y, there exists an* isomorphism *η_U from the induced prescheme $f^{-1}(U)$ to $X_U = \operatorname{Proj}(\Gamma(U, \mathscr{S}))$ such that, if V is another affine open of Y that is contained in U, then the diagram*

(3.1.2.1)
$$\begin{tikzcd} f^{-1}(V) \arrow[r, "\eta_V"] \arrow[d] & X_V \arrow[d, "\rho_{V,U}"] \\ f^{-1}(U) \arrow[r, "\eta_V"'] & X_U \end{tikzcd}$$

commutes.

Proof. Given affine opens U and V of Y, let $X_{U,V}$ be the prescheme induced on $f_U^{-1}(U \cap V)$ by X_U; we are going to define a Y-isomorphism $\theta_{U,V} : X_{V,U} \xrightarrow{\sim} X_{U,V}$. For this, we consider an affine open $W \subset U \cap V$: by composing the isomorphisms

$$f_U^{-1}(W) \xrightarrow{\sigma_{W,U}} X_W \xrightarrow{\sigma_{W,V}^{-1}} f_V^{-1}(W),$$

we obtain an isomorphism τ_W, and we immediately see that, if $W' \subset W$ is an affine open, then $\tau_{W'}$ is the restriction of τ_W to $f_U^{-1}(W')$; the τ_W are thus indeed the restrictions of a Y-isomorphism $\theta_{V,U}$. Further, if U, V, and W are affine open subsets of Y, and $\theta'_{U,V}$, $\theta'_{V,W}$, and $\theta'_{U,W}$ the restrictions of $\theta_{U,V}$, $\theta_{V,W}$, and $\theta_{U,W}$ (respectively) to the inverse images of $U \cap V \cap W$ in X_V, X_W, and X_W (respectively), then it follows from the above definitions that we have $\theta'_{U,V} \circ \theta'_{V,W} = \theta'_{U,W}$. The existence of some X satisfying the properties in the statement thus follows from (**I**, 2.3.1); its uniqueness up to Y-isomorphism is trivial, taking (3.1.2.1) into account. □

(3.1.3). We say that the prescheme X defined in (3.1.2) is the *homogeneous spectrum* of the quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S}$, and we denote it by $\operatorname{Proj}(\mathscr{S})$. It is immediate that $\operatorname{Proj}(\mathscr{S})$ is *separated over Y* ((2.4.2) and (**I**, 5.5.5)); if $\mathscr{S}$ is an $\mathscr{O}_Y$-algebra *of finite type* (**I**, 9.6.2), then $\operatorname{Proj}(\mathscr{S})$ is *of finite type* over Y ((2.7.1, (ii)) and (**I**, 6.3.1)).

If f is the structure morphism $X \to Y$, then it is immediate that, for every prescheme induced by Y on an open subset U of Y, $f^{-1}(U)$ can be identified with the homogeneous spectrum $\operatorname{Proj}(\mathscr{S}|U)$.

Proposition (3.1.4). — *Let $f \in \Gamma(Y, \mathscr{S}_d)$ for $d > 0$. Then there exists an open subset X_f of the underlying space of $X = \operatorname{Proj}(\mathscr{S})$ that satisfies the following property: for every affine open U of Y, we have $X_f \cap \phi^{-1}(U) = D_+(f|U)$ in $\phi^{-1}(U)$ identified with $X_U = \operatorname{Proj}(\Gamma(U, \mathscr{S}))$, where ϕ denotes the structure morphism $X \to Y$. Furthermore, the prescheme induced on X_f is affine over Y, and is canonically isomorphic* **II** | 51
to $\operatorname{Spec}(\mathscr{S}^{(d)}/(f-1)\mathscr{S}^{[d]})$ (1.3.1).

PROOF. We have $f|U \in \Gamma(U, \mathscr{S}_d) = (\Gamma(U, \mathscr{S}))_d$. If U and U' are affine opens of Y such that $U' \subset U$, then $f|U'$ is the image of $f|U$ under the restriction homomorphism

$$\Gamma(U, \mathscr{S}) \longrightarrow \Gamma(U', \mathscr{S})$$

and so $D_+(f|U')$ is equal (with the notation of (3.1.1)) to the prescheme induced on the inverse image $\rho^{-1}_{U',U}(D_+(f|U))$ in $X_{U'}$ (2.8.1); whence the first claim. Furthermore, the prescheme induced on $D_+(f|U)$ by X_U is canonically identified with $\operatorname{Spec}((\Gamma(U, \mathscr{S}))_{f|U})$, with these identifications being compatible with the restriction homomorphisms (2.8.1); the second claim then follows from (2.2.5) and from the commutativity of the diagram (2.8.2.1). ☐

We also say that X_f (as an open subset of the underlying space X) is the set of $x \in X$ where f *does not vanish*.

Corollary (3.1.5). — *If $f \in \Gamma(Y, \mathscr{S}_d)$ and $g \in \Gamma(Y, \mathscr{S}_e)$, then*

$$X_{fg} = X_f \cap X_g. \tag{3.1.5.1}$$

PROOF. It suffices to consider the intersection of the two sets with a set $\phi^{-1}(U)$, where U is an affine open in Y, and to then apply formula (2.3.3.2). ☐

Corollary (3.1.6). — *Let (f_α) be a family of sections of $\mathscr{S}$ over Y such that $f_\alpha \in \Gamma(Y, \mathscr{S}_{d_\alpha})$; if the sheaf of ideals of $\mathscr{S}$ generated by this family* (**0**, 5.1.1) *contains all the $\mathscr{S}_n$ starting from a certain rank, then the underlying space X is the union of the X_{f_α}.*

PROOF. For every affine open U of Y, $\phi^{-1}(U)$ is the union of the $X_{f_\alpha} \cap \phi^{-1}(U)$ (2.3.14). ☐

Corollary (3.1.7). — *Let $\mathscr{A}$ be a quasi-coherent $\mathscr{O}_Y$-algebra; set*

$$\mathscr{S} = \mathscr{A}[T] = \mathscr{A} \otimes_{\mathbf{Z}} \mathbf{Z}[T]$$

where T is an indeterminate (and $\mathbf{Z}$ and $\mathbf{Z}[T]$ are considered as simple sheaves over Y). Then $X = \operatorname{Proj}(\mathscr{S})$ is canonically identified with $\operatorname{Spec}(\mathscr{A})$. In particular, $\operatorname{Proj}(\mathscr{O}_Y[T])$ is identified with Y.

PROOF. By applying (3.1.6) to the unique section $f \in \Gamma(Y, \mathscr{S})$ that is equal to T at each point of Y< we see that $X_f = X$. Further, here we have $d = 1$, and $\mathscr{S}^{(1)}/(f-1)\mathscr{S}^{(1)} = \mathscr{S}/(f-1)\mathscr{S}$ is canonically isomorphic to $\mathscr{A}$, whence the corollary (1.2.2). ☐

Let $g \in \Gamma(Y, \mathscr{O}_Y)$; if we take $\mathscr{S} = \mathscr{O}_Y[T]$, then $g \in \Gamma(Y, \mathscr{S}_0)$; let

$$h = gT \in \Gamma(Y, \mathscr{S}_1).$$

If $X = \operatorname{Proj}(\mathscr{S})$, then the canonical identification defined in (3.1.7) identifies X_h with the open subset Y_g of Y (in the sense of (**0**, 5.5.2)): indeed, we can restrict to the case where $Y = \operatorname{Spec}(A)$ is affine, and everything then reduces (taking (2.2.5) into account) to the fact that the ring of fractions A_g is canonically identified with $A[T]/(gT-1)A[T]$ (**0**, 1.2.3).

Proposition (3.1.8). — *Let $\mathscr{S}$ be a quasi-coherent positively-graded $\mathscr{O}_Y$-algebra. Then*

(i) *For all $d > 0$, there exists a canonical Y-isomorphism from $\operatorname{Proj}(\mathscr{S})$ to $\operatorname{Proj}(\mathscr{S}^{(d)})$.* II | 52

(ii) *Let $\mathscr{S}'$ be the graded $\mathscr{O}_Y$-algebra given by the direct sum of $\mathscr{O}_Y$ with the $\mathscr{S}_n$ (for $n \geqslant 0$); then $\operatorname{Proj}(\mathscr{S}')$ and $\operatorname{Proj}(\mathscr{S})$ are canonically Y-isomorphic.*

(iii) *Let $\mathscr{L}$ be an invertible $\mathscr{O}_Y$-module* (**0**, 5.4.1), *and let $\mathscr{S}_{(\mathscr{L})}$ be the graded $\mathscr{O}_Y$-algebra given by the direct sum of the $\mathscr{S}_d \otimes \mathscr{L}^{\otimes d}$ (for $d \geqslant 0$); then $\operatorname{Proj}(\mathscr{S})$ and $\operatorname{Proj}(\mathscr{S}_{(\mathscr{L})})$ are canonically Y-isomorphic.*

PROOF. In each of the three cases, it suffices to define the isomorphism locally on Y, since the verification of compatibility with the restriction operations from one open subset to a smaller one is trivial. We can thus suppose that Y is affine, and then (i) follows from (2.4.7, (i)), and (ii) follows from (2.4.8). As for (iii), if we further suppose that $\mathscr{L}$ is isomorphic to $\mathscr{O}_Y$ (which we are allowed to do, since the question is local on Y), then the isomorphism between $\operatorname{Proj}(\mathscr{S})$ and $\operatorname{Proj}(\mathscr{S}_{(\mathscr{L})})$ is evident; to define a *canonical* isomorphism, let $Y = \operatorname{Spec}(A)$ and $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra, and let c be a generator of the free A-module L such that $\mathscr{L} = \widetilde{L}$; then, for all $n > 0$, $x_n \mapsto x_n \otimes c^{\otimes n}$ is an A-isomorphism from S_n to $S_n \otimes L^{\otimes n}$, and these A-isomorphisms define an A-isomorphism of graded algebras $\phi_c : S \to S_{(L)} = \bigoplus_{n\geqslant 0} S_n \otimes L^{\otimes n}$. So let $f \in S_+$ be homogeneous of degree d; for all $x \in S_{nd}$, we have that $(x \otimes c^{nd})/(f \otimes c^d)^n = (x \otimes (\varepsilon c)^{nd})/(f \otimes (\varepsilon c)^d)^n$ for every invertible element $\varepsilon \in A$, which shows that the isomorphism $S_{(f)} \to (S_{(L)})_{(f\otimes c^d)}$ induced from ϕ_c is *independent* of the generator c of L considered, and thus finishes the proof. □

(3.1.9). Recall ((**0**, 4.1.3) and (**I**, 1.3.14)) that, for the quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S}$ to be *generated by the $\mathscr{O}_Y$-module $\mathscr{S}_1$*, it is necessary and sufficient for there to exist an affine open cover (U_α) of Y such that the graded algebra $\Gamma(U_\alpha, \mathscr{S})$ over $\Gamma(U_\alpha, \mathscr{S}_0)$ is generated by the set $\Gamma(U_\alpha, \mathscr{S}_1)$ of its homogeneous elements of degree 1. For every open V of Y, $\mathscr{S}|V$ is then generated by the $(\mathscr{O}_Y|V)$-module $\mathscr{S}_1|V$.

Proposition (3.1.10). — *Suppose that there exists a finite affine open cover (U_i) of Y such that each graded algebra $\Gamma(U_i, \mathscr{S})$ is of finite type over $\Gamma(U_i\mathscr{O}_Y)$. Then there exists $d > 0$ such that $\mathscr{S}^{(d)}$ is generated by $\mathscr{S}_d$, with $\mathscr{S}_d$ an $\mathscr{O}_Y$-module of finite type.*

PROOF. Indeed, it follows from (2.1.6, (v)) that, for each i, there exists an integer m_i such that $\Gamma(U_i, \mathscr{S}_{nm_i}) = (\Gamma(U_i, \mathscr{S}_{m_i}))^n$ for all $n > 0$; it suffices to take d to be a common multiple of all the m_i, taking (2.1.6, (i)) into account. □

Corollary (3.1.11). — *Under the hypotheses of (3.1.10), $\mathrm{Proj}(\mathscr{S})$ is Y-isomorphic to a homogeneous spectrum $\mathrm{Proj}(\mathscr{S}')$, where $\mathscr{S}'$ is a graded $\mathscr{O}_Y$-algebra generated by $\mathscr{S}'_1$, with $\mathscr{S}'_1$ an $\mathscr{O}_Y$-module of finite type.*

PROOF. It suffices to take $\mathscr{S}' = \mathscr{S}^{(d)}$, where d satisfies the property of (3.1.10), and to then apply (3.1.8, (i)) □

(3.1.12). If $\mathscr{S}$ is a quasi-coherent positively-graded $\mathscr{O}_Y$-algebra, we know (**I**, 5.1.1) that its *nilradical* $\mathscr{N}$ is a quasi-coherent $\mathscr{O}_Y$-module; we say that $\mathscr{N}_+ = \mathscr{N} \cap \mathscr{S}_+$ is the *nilradical of $\mathscr{S}_+$*; this is a quasi-coherent graded $\mathscr{S}_0$-module, since we can immediately reduce to the case where Y is affine, and the proposition then follows from (2.1.10). For all $y \in Y$, $(\mathscr{N}_+)_y$ is then the nilradical of $(\mathscr{S}_+)_y = (\mathscr{S}_y)_+$ (**I**, 5.1.1). We say that the graded $\mathscr{O}_Y$-algebra $\mathscr{S}$ is *essentially reduced* if $\mathscr{N}_+ = 0$, which is equivalent to saying that $\mathscr{S}_y$ is an essentially reduced graded $\mathscr{O}_y$-algebra for all $y \in Y$. For every graded **II** | 53
$\mathscr{O}_Y$-algebra $\mathscr{S}$, $\mathscr{S}/\mathscr{N}_+$ is essentially reduced.

We say that $\mathscr{S}$ is *integral* if, for all $y \in Y$, $\mathscr{S}_y$ is an integral ring and, furthermore, $(\mathscr{S}_y)_+ = (\mathscr{S}_+)_y \neq 0$.

Proposition (3.1.13). — *Let $\mathscr{S}$ be a positively-graded $\mathscr{O}_Y$-algebra. If $X = \mathrm{Proj}(\mathscr{S})$, then the Y-scheme X_{red} is canonically isomorphic to $\mathrm{Proj}(\mathscr{S}/\mathscr{N}_+)$; in particular, if $\mathscr{S}$ is essentially reduced, then X is reduced.*

PROOF. The fact that $X' = \mathrm{Proj}(\mathscr{S}/\mathscr{N}_+)$ is reduced follows immediately from (2.4.4, (i)), since the property is local; further, for every affine open $U \subset Y$, ${\phi'}^{-1}(U)$ is equal to $(\phi^{-1}(U))_{\mathrm{red}}$ (where we denote by ϕ and ϕ' the structure morphisms $X \to Y$ and $X' \to Y$, respectively); we immediately see that the canonical U-morphisms ${\phi'}^{-1}(U) \to \phi^{-1}(U)$ are compatible with the restriction operations, and thus define a closed immersion $X' \to X$ that is a homeomorphism of the underlying spaces; whence the conclusion (**I**, 5.1.2). □

Proposition (3.1.14). — *Let Y be an integral prescheme, and $\mathscr{S}$ a quasi-coherent graded $\mathscr{O}_Y$-algebra such that $\mathscr{S}_0 = \mathscr{O}_Y$.*

(i) *If $\mathscr{S}$ is integral (3.1.12), then $X = \mathrm{Proj}(\mathscr{S})$ is integral, and the structure morphism $\phi : X \to Y$ is dominant.*
(ii) *Suppose furthermore that $\mathscr{S}$ is essentially reduced. Then, conversely, if X is integral and ϕ is dominant, then $\mathscr{S}$ is integral.*

PROOF.

(i) If (U_α) is a base of Y consisting of non-empty affine opens, then it suffices to prove the proposition in the case where Y is replaced by one of the U_α, and $\mathscr{S}$ by $\mathscr{S}|U_\alpha$: indeed, one one hand it will follow that the underlying space $\phi^{-1}(U_\alpha)$ are irreducible opens (and thus non-empty) of X such that $\phi^{-1}(U_\alpha) \cap \phi^{-1}(U_\beta) \neq \varnothing$ for any pair of indices (since $U_\alpha \cap U_\beta$ contains some U_γ), and so X is irreducible (**0**, 2.1.4); on the other hand, X will be reduced, since this is a local property, and so X will indeed be integral, with $\phi(X)$ dense in Y.

So suppose that $Y = \mathrm{Spec}(A)$, where A is integral (**I**, 5.1.4), and that $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra; the hypothesis is that, for every $y \in Y$, $\widetilde{S}_y = S_y$ is an integral graded ring such that $(S_y)_+ \neq 0$. It suffices to prove that S is an *integral* ring, since then we will have that $S_+ \neq 0$, and we can then apply (2.4.4, (ii)). But let $f, g \neq 0$ be elements of S, and suppose that $fg = 0$; for all $y \in Y$ we then have that $(f/1)(g/1) = 0$ in S_y, and so $f/1 = 0$ or $g/1 = 0$ by hypothesis. Suppose, for example, that $f/1 = 0$ in S_y; this implies that there exists $a \in A$ such that $a \notin \mathfrak{j}_y$ and $af = 0$; we then have, for *all* $z \in Y$, that $(a/1)(f/1) = 0$ in the *integral* ring S_z, and since $a/1 \neq 0$ (since A is integral), $f/1 = 0$, which implies that $f = 0$.

(ii) Since the question is local on Y, we can again suppose that $Y = \mathrm{Spec}(A)$, with A integral, and that $\mathscr{S} = \widetilde{S}$. By hypothesis, for all $y \in Y$, $(S_y)_+$ does not contain any non-zero nilpotent elements, and the same is true of $(S_0)_y = A_y$ by hypothesis; so S_y is a reduced ring for all $y \in Y$, and we thus conclude first of all that S itself is reduced (**I**, 5.1.1). The hypothesis that X is integral implies that S is essentially integral (2.4.4, (ii)), and everything then reduces to showing that the annihilator $\mathfrak{J}$ of S_+ in $A = S_0$ is just 0 (2.1.11). If this were not the case, we would have that $(S_h)_+ = 0$ for some $h \neq 0$ in $\mathfrak{J}$, and thus (3.1.1) that $\phi^{-1}(D(h)) = \varnothing$, and $\phi(X)$ would not be dense in Y, contradicting the hypothesis (since $D(h) \neq \varnothing$, since h is not nilpotent). **II** | 54

□

3.2. Sheaf on $\mathrm{Proj}(\mathscr{S})$ associated to a graded $\mathscr{S}$-module

(3.2.1). Let Y be a prescheme, $\mathscr{S}$ a quasi-coherent positively-graded $\mathscr{O}_Y$-algebra, and $\mathscr{M}$ a quasi-coherent graded $\mathscr{S}$-module (on $(Y, \mathscr{O}_Y)$, or, equivalently (**I**, 9.6.1), on the ringed space $(Y, \mathscr{S})$). With the notation of (3.1.1), we denote by $\widetilde{\mathscr{M}}_U$ the quasi-coherent $\mathscr{O}_{X_U}$-module $(\Gamma(U, \mathscr{M}))^\sim$; for $U' \subset U$, $\Gamma(U', \mathscr{M})$ is canonically identified with $\Gamma(U, \mathscr{M}) \otimes_A A'$ (**I**, 1.6.4); thus we have $\widetilde{\mathscr{M}}_{U'} = \rho^*_{U',U}(\widetilde{\mathscr{M}}_U)$ (2.8.11).

Proposition (3.2.2). — *There exists on* $\mathrm{Proj}(\mathscr{S}) = X$ *exactly one quasi-coherent* $\mathscr{O}_X$*-module* $\mathscr{M}$ *such that, for every affine open* U *of* Y*, we have* $\eta^*_U((\Gamma(U, \mathscr{M}))^\sim) = \widetilde{M}|f^{-1}(U)$ *(denoting by* η_U *the isomorphism* $f^{-1}(U) \xrightarrow{\sim} \mathrm{Proj}(\Gamma(U, \mathscr{S}))$*), where* f *is the structure morphism* $X \to Y$*.*

Proof. Since $\rho_{U',U}$ is identified with the injection morphism $f^{-1}(U') \to f^{-1}(U)$ (3.1.2.1), the proposition follows immediately from the relation $\widetilde{\mathscr{M}}_{U'} = \rho^*_{U',U}(\widetilde{\mathscr{M}}_U)$ and from the gluing principle for sheaves (**0**, 3.3.1). □

We say that $\widetilde{\mathscr{M}}$ is the $\mathscr{O}_X$-module *associated to* the quasi-coherent graded $\mathscr{S}$-module $\mathscr{M}$.

Proposition (3.2.3). — *Let* $\mathscr{M}$ *be a quasi-coherent graded* $\mathscr{S}$*-module, and let* $f \in \Gamma(Y, \mathscr{S}_d)$ *(for* $d > 0$*). If* ξ_f *is the canonical isomorphism from* X_f *to the* Y*-prescheme* $Z_f = \mathrm{Spec}(\mathscr{S}^{(d)}/(f-1)\mathscr{S}^{(d)})$ *(3.1.4), then* $(\xi_f)_*(\widetilde{\mathscr{M}}|X_f)$ *is the* $\mathscr{O}_{Z_f}$*-module* $(\mathscr{M}^{(l)}/(f-1)\mathscr{M}^{(d)})$ *(1.4.3).*

Proof. Since the question is local on Y, we can immediately reduce to (2.2.5), taking into account the commutativity of the diagram in (2.8.12.1). □

Proposition (3.2.4). — *The* $\mathscr{O}_X$*-module* $\widetilde{\mathscr{M}}$ *is an exact additive covariant functor in* $\mathscr{M}$*, from the category of quasi-coherent graded* $\mathscr{S}$*-modules to the category of quasi-coherent* $\mathscr{O}_X$*-modules, that commutes with inductive limits and direct sums.*

Proof. Since the question is local on Y, we can reduce to (**I**, 1.3.11), (**I**, 1.3.9), and (2.5.4). □

In particular, if $\mathscr{N}$ is a quasi-coherent graded sub-$\mathscr{S}$-module of $\mathscr{M}$, then $\widetilde{\mathscr{N}}$ is canonically identified with with a quasi-coherent sub-$\mathscr{O}_X$-module of $\widetilde{\mathscr{M}}$; in particular, for every quasi-coherent graded sheaf $\mathscr{J}$ of ideals of $\mathscr{S}$, $\widetilde{\mathscr{J}}$ is a quasi-coherent sheaf of ideals of $\mathscr{O}_X$.

If $\mathscr{M}$ is a quasi-coherent graded $\mathscr{S}$-module, and $\mathscr{I}$ a quasi-coherent sheaf of ideals of $\mathscr{O}_Y$, then $\mathscr{I}\mathscr{M}$ is a quasi-coherent graded sub-$\mathscr{S}$-module of $\mathscr{M}$, and we have

$$(\mathscr{I}\mathscr{M})^\sim = \mathscr{I} \cdot \widetilde{\mathscr{M}} \tag{3.2.4.1}$$

(where the right-hand side is defined as in (**0**, 4.3.5)). It suffices to verify this formula in the case where $Y = \operatorname{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$, with S a graded A-algebra, $\mathscr{M} = \widetilde{M}$, with M a graded S-module, and $\mathscr{I} = \mathfrak{I}$, with $\mathfrak{I}$ an ideal of A. For every homogeneous element f of S_+, the restriction to $D_+(f) = \operatorname{Spec}(S_{(f)})$ of the left-hand side of (3.2.4.1) can be associated with $(\mathfrak{I}M)_{(f)} = \mathfrak{I} \cdot M_{(f)}$, and the same is true of the restriction of the right-hand side, given (**I**, 1.3.13) and (**i**, 1.6.9). **II** | 55

Proposition (3.2.5). — *Let $f \in \Gamma(Y, \mathscr{S}_d)$ (for $d > 0$). On the open subset X_f, the $(\mathscr{O}_X|X_f)$-module $(\mathscr{S}(nd))^{\sim}X_f$ is canonically isomorphic to $\mathscr{O}_X|X_f$ for all $n \in \mathbf{Z}$. In particular, if the $\mathscr{O}_Y$-algebra $\mathscr{S}$ is generated by $\mathscr{S}_1$ (3.1.9), then the $\mathscr{O}_X$-modules $(\mathscr{S}(n))^{\sim}$ are invertible for all $n \in \mathbf{Z}$.*

Proof. Indeed, for every affine open U of Y, we defined in (2.5.7) a canonical isomorphism from $(\mathscr{S}(nd))^{\sim}|(X_f \cap \phi^{-1}(U))$ to $\mathscr{O}_X|(X_f \cap \phi^{-1}(U))$, taking (3.1.4) into account (where ϕ is the structure morphism $X \to Y$); it is immediate that these isomorphisms are compatible with the restriction from U to an affine open $U' \subset U$, whence the first claim. To prove the second, it suffices to note that, if $\mathscr{S}$ is generated by $\mathscr{S}_1$, then there is a cover (U_α) of Y by affine opens such that $\Gamma(U_\alpha, \mathscr{S})$ is generated by $\Gamma(U_\alpha, \mathscr{S})_1 = \Gamma(U_\alpha, \mathscr{S}_1)$; we can then apply the result of (2.5.9), since the property of being invertible is local. □

We again set, for all $n \in \mathbf{Z}$,

(3.2.5.1) $$\mathscr{O}_X(n) = (\mathscr{S}(n))^{\sim}$$

and, for all $\mathscr{O}_X$-modules $\mathscr{F}$,

(3.2.5.2) $$\mathscr{F}(n) = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(n).$$

It follows immediately from these definitions that, for every open subset U of Y, we have

$$((\mathscr{S}|U)(n))^{\sim} = \mathscr{O}_X(n)|f^{-1}(U)$$

where f is the structure morphism $X \to Y$.

Proposition (3.2.6). — *Let $\mathscr{M}$ and $\mathscr{N}$ be quasi-coherent graded $\mathscr{S}$-modules. Then there exists a canonical functorial (in $\mathscr{M}$ and $\mathscr{N}$) homomorphism*

(3.2.6.1) $$\lambda : \widetilde{\mathscr{M}} \otimes_{\mathscr{O}_X} \widetilde{\mathscr{N}} \longrightarrow (\mathscr{M} \otimes_{\mathscr{S}} \mathscr{N})^{\sim}$$

and a canonical functorial (in $\mathscr{M}$ and $\mathscr{N}$) homomorphism

(3.2.6.2) $$\mu : (\mathscr{H}om_{\mathscr{S}}(\mathscr{M}, \mathscr{N}))^{\sim} \longrightarrow \mathscr{H}om_{\mathscr{O}_X}(\widetilde{\mathscr{M}}, \widetilde{\mathscr{N}}).$$

Furthermore, if $\mathscr{S}$ is generated by $\mathscr{S}_1$ (3.1.9), then λ is an isomorphism; if, further, $\mathscr{M}$ admits a finite presentation (3.1.1), then μ is an isomorphism.

Proof. The isomorphisms λ and μ were defined in (2.5.11.2) and (2.5.12.2) in the case where Y is affine; since these definitions are local, they transfer immediately to the general case considered here, taking (2.8.14) into account. □

Corollary (3.2.7). — *If $\mathscr{S}$ is generated by $\mathscr{S}_1$, then, for any $m, n \in \mathbf{Z}$,*

(3.2.7.1) $$\mathscr{O}_X(m) \otimes_{\mathscr{O}_X} \mathscr{O}_X(n) = \mathscr{O}_X(m+n)$$

(3.2.7.2) $$\mathscr{O}_X(n) = (\mathscr{O}_X(1))^{\otimes n}$$

up to canonical isomorphism.

II | 56

Corollary (3.2.8). — *If $\mathscr{S}$ is generated by $\mathscr{S}_1$, then, for any graded $\mathscr{S}$-module $\mathscr{M}$ and any $n \in \mathbf{Z}$,*

(3.2.8.1) $$(\mathscr{M}(n))^{\sim} = \widetilde{\mathscr{M}}(n)$$

up to canonical isomorphism.

Proof. This follows from the corresponding properties in the case where Y is affine ((2.5.14) and (2.5.15)), along with (2.8.11). □

Remarks (3.2.9). —

(1) If $\mathscr{S} = \mathscr{A}[T]$, with $\mathscr{A}$ a quasi-coherent $\mathscr{O}_Y$-algebra (3.1.7), then we immediately see that all the invertible $\mathscr{O}$-modules $\mathscr{O}(n)$ are canonically isomorphic to $\mathscr{O}_X$.

Furthermore, let $\mathscr{N}$ be a quasi-coherent $\mathscr{A}$-module, and set $\mathscr{M} = \mathscr{N} \otimes_{\mathscr{A}} \mathscr{A}[T]$. It then follows from (3.2.3) and (3.1.7) that, under the canonical identification of $X = \operatorname{Proj}(\mathscr{A}[T])$ with $X' = \operatorname{Spec}(\mathscr{A})$, the $\mathscr{O}_X$-module $\widetilde{\mathscr{M}}$ is identified with the $\mathscr{O}_{X'}$-module $\widetilde{\mathscr{N}}$ associated to $\mathscr{N}$ (in the sense of (1.4.3)).

(2) Let $\mathscr{S}$ be an arbitrary graded $\mathscr{O}_Y$-algebra, and $\mathscr{S}'$ the graded $\mathscr{O}_Y$-algebra such that $\mathscr{S}'_0 = \mathscr{O}_Y$ and $\mathscr{S}'_n = \mathscr{S}_n$ for all $n > 0$; then the canonical isomorphism from $X = \operatorname{Proj}(\mathscr{S})$ to $X' = \operatorname{Proj}(\mathscr{S}')$ (3.1.8, (ii)) identifies $\mathscr{O}_X(n)$ with $\mathscr{O}_{X'}(n)$ for all $n \in \mathbf{Z}$. This follows from the same proposition for the affine case (2.5.16) and from the fact that the identifications, for the affine opens of Y, commute with the restriction operations. Similarly, let $X^{(d)} = \operatorname{Proj}(\mathscr{S}^{(d)})$; then the canonical isomorphism from X to $X^{(d)}$ (3.1.8, (i)) identifies $\mathscr{O}_X(nd)$ with $\mathscr{O}_{X^{(d)}}(n)$ for all $n \in \mathbf{Z}$.

Proposition (3.2.10). — *Let $\mathscr{L}$ be an invertible $\mathscr{O}_Y$-module, and g the canonical isomorphism from $X_{(\mathscr{L})} = \operatorname{Proj}(\mathscr{S}_{(\mathscr{L})})$ to $X = \operatorname{Proj}(\mathscr{S})$ (3.1.8, (iii)). Then, for any $n \in \mathbf{Z}$, $g_*(\mathscr{O}_{X_{(\mathscr{L})}}(n))$ is canonically isomorphic to $\mathscr{O}_X(n) \otimes_Y \mathscr{L}^{\otimes n}$.*

PROOF. Suppose first of all that Y is affine, of ring A, and that $\mathscr{L} = \widetilde{L}$, where L is a free monogenous A-module. With the notation from the proof of (3.1.8, (iii)), we define, for $f \in S_d$, an isomorphism from $(S(n))_{(f)} \otimes_A L^{\otimes n}$ to $(S_{(L)}(n))_{(f \otimes c^d)}$ by sending $(x/f^k) \otimes c^n$, where $x \in S_{kd+n}$, to the element $(x \otimes c^{n+kd})/(f \otimes c^d)^k$; it is immediate that this isomorphism is independent of the chosen generator c of L; further, the isomorphisms thus defined for each $f \in S_+$ are compatible with the restriction operators $D_+(f) \to D_+(fg)$. Finally, in the general case, we easily see, from the definitions (3.1.1), that the isomorphisms thus defined for each affine open U of Y are compatible with passing from U to an affine open $U' \subset U$. □

3.3. Graded $\mathscr{S}$-module associated to a sheaf on $\operatorname{Proj}(\mathscr{S})$ *Throughout this entire section we suppose that the graded $\mathscr{O}_Y$-algebra $\mathscr{S}$ is generated by $\mathscr{S}_1$ (3.1.9).* Recall that, by (3.1.8, (i)), this restrictive assumption is not essential, thanks to finiteness conditions (3.1.10).

(3.3.1). Let p be the structure morphism $X = \operatorname{Proj}(\mathscr{S}) \to Y$. For every $\mathscr{O}_X$-module $\mathscr{F}$, set

$$\boldsymbol{\Gamma}_*(\mathscr{F}) = \bigoplus_{n \in \mathbf{Z}} p_*(\mathscr{F}(n)) \tag{3.3.1}$$

and, in particular, II | 57

$$\boldsymbol{\Gamma}_*(\mathscr{O}_X) = \bigoplus_{n \in \mathbf{Z}} p_*(\mathscr{O}_X(n)). \tag{3.3.1.2}$$

We know (**0**, 4.2.2) that there exists a canonical homomorphism

$$p_*(\mathscr{F}) \otimes_{\mathscr{O}_Y} p_*(\mathscr{G}) \longrightarrow p_*(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})$$

for $\mathscr{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$; we thus deduce from (3.2.7.1) that $\boldsymbol{\Gamma}_*(\mathscr{O}_X)$ is endowed with the structure of a *graded $\mathscr{O}_Y$-algebra*, and (3.2.5.2) defines the structure of a *graded $\boldsymbol{\Gamma}_*(\mathscr{O}_X)$-module* on $\boldsymbol{\Gamma}_*(\mathscr{F})$.

By (3.2.5), and by left exactness of the functor p_* (**0**, 4.2.1), $\boldsymbol{\Gamma}_*(\mathscr{F})$ is an additive and *left exact* covariant functor in $\mathscr{F}$ from the category of $\mathscr{O}_X$-modules to the category of graded $\mathscr{O}_Y$-modules (where the morphisms are the homomorphisms of degree 0). In particular, if $\mathscr{J}$ is a sheaf of ideals of $\mathscr{O}_X$, then $\boldsymbol{\Gamma}_*(\mathscr{J})$ can be identified with a *graded sheaf of ideals* in $\boldsymbol{\Gamma}_*(\mathscr{O}_X)$.

(3.3.2). Let $\mathscr{M}$ be a quasi-coherent graded $\mathscr{S}$-module. For every affine open U of Y, we defined in (2.6.2) a homomorphism of abelian groups

$$\alpha_{0,U} : \Gamma(U, \mathscr{M}_0) \longrightarrow \Gamma(p^{-1}(U), \widetilde{\mathscr{M}}).$$

It is immediate that these homomorphisms commute with the restriction operations (2.8.13.1) and thus define (without using the hypothesis that $\mathscr{S}$ is generated by $\mathscr{S}_1$) a homomorphism of sheaves of abelian groups

$$\alpha_0 : \mathscr{M}_0 \longrightarrow p_*(\widetilde{\mathscr{M}}). \tag{3.3.2.1}$$

Applying this result to each of the $\mathscr{M}_n = (\mathscr{M}(n))_0$, and taking (3.2.8.1) into account, we can define a homomorphism of sheaves of abelian groups

$$\alpha_n : \mathscr{M}_n \longrightarrow p_*(\widetilde{\mathscr{M}}(n)) \tag{3.3.2.2}$$

for all $n \in \mathbf{Z}$, whence a functorial homomorphism (of degree 0) of graded sheaves of abelian groups

$$\alpha : \mathscr{M} \longrightarrow \mathbf{\Gamma}_*(\widetilde{\mathscr{M}}) \tag{3.3.2.3}$$

(also denoted $\alpha_{\mathscr{M}}$).

By taking $\mathscr{M} = \mathscr{S}$ in particular, we see that $\alpha : \mathscr{S} \to \mathbf{\Gamma}_*(\mathscr{O}_X)$ is a homomorphism of graded $\mathscr{O}_Y$-algebra, and that (3.3.2.3) is a di-homomorphism of graded modules, with respect to this homomorphism of graded algebras.

We again note that to each of the α_n there corresponds (**0**, 4.4.3) a canonical homomorphism of $\mathscr{O}_X$-modules

$$\alpha_n^\sharp : p^*(\mathscr{M}_n) \longrightarrow \widetilde{\mathscr{M}}(n). \tag{3.3.2.4}$$

We can easily verify that this homomorphism is exactly the one which corresponds functorially (3.2.4) to the canonical homomorphism (of degree 0) of *graded* $\mathscr{O}_Y$-modules

$$\mathscr{M}_n \otimes_{\mathscr{O}_Y} \mathscr{S} \longrightarrow \mathscr{M}(n) \tag{3.3.2.5}$$

where the grading of the right-hand side comes naturally from that of $\mathscr{S}$. We can restrict to the case **II** | 57
where $Y = \operatorname{Spec}(A)$ is affine, $\mathscr{M} = \widetilde{M}$, and $\mathscr{S} = \widetilde{S}$, with the graded A-algebra S being generated by S_1, so that, as f runs over S_1, the $D_+(f)$ form a cover of X. By the definitions (2.6.2), we see then see, taking (**I**, 1.6.7) into account, that the restriction to $D_+(f)$ of the homomorphism (3.3.2.4) corresponds (**I**, 1.3.8) to the homomorphism of $S_{(f)}$-modules $M_n \otimes_A S_{(f)} \to (S(n))_{(f)}$ that sends $x \otimes 1$ (where $x \in M_n$) to $x/1$; this proves the claim.

Proposition (3.3.3). — *For every section $f \in \Gamma(Y, \mathscr{S}_d)$ (where $d > 0$), X_f is identical to the set of points of X where $\alpha_d(f)$ (considered as a section of $\mathscr{O}_X(d)$) does not vanish* (**0**, 5.5.2).

Proof. (Note that $\alpha_d(f)$ is a section of $p_*(\mathscr{O}_X(d))$ over Y, but by definition such a section is also a section of $\mathscr{O}(d)$ over X (**0**, 4.2.1)). The definition of X_f (3.1.4) lets us reduce to the case where Y is affine, which has already been dealt with in (2.6.3). □

(3.3.4). From now on, we suppose, in addition to the hypothesis at the start of this section, that, for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the $p_*(\mathscr{F}(n))$ are *quasi-coherent* on Y, so that $\mathbf{\Gamma}_*(\mathscr{F}) = \bigoplus_{n\in\mathbf{Z}} p_*(\mathscr{F}(n))$ is also a quasi-coherent $\mathscr{O}_Y$-module ((**I**, 1.4.1) and (**I**, 1.3.9)); this will always be the case if X is *of finite type* over Y (**I**, 9.2.2). We thus conclude that $(\mathbf{\Gamma}_*(\mathscr{F}))^\sim$ is defined, and is a quasi-coherent $\mathscr{O}_X$-module. For every affine open U of Y< we have ((**I**, 1.3.9) and (**I**, 2.5.4))

$$\begin{aligned}\Big(\Gamma(U, \bigoplus_{n\in\mathbf{Z}} p_*(\mathscr{F}(n)))\Big)^\sim &= \bigoplus_{n\in\mathbf{Z}} \Big(\Gamma(U, p_*(\mathscr{F}(n)))\Big)^\sim \\ &= \bigoplus_{n\in\mathbf{Z}} \Big(\Gamma(p^{-1}(U), \mathscr{F}(n))\Big)^\sim \\ &= \Big(\bigoplus_{n\in\mathbf{Z}} \Gamma(p^{-1}(U), \mathscr{F}(n))\Big)^\sim \\ &= (\mathbf{\Gamma}_*(\mathscr{F}|p^{-1}(U)))^\sim\end{aligned}$$

and so (2.6.4) we have a canonical homomorphism

$$\beta_U : \Big(\Gamma(U, \bigoplus_{n\in\mathbf{Z}}) p_*(\mathscr{F}(n))\Big)^\sim \longrightarrow \mathscr{F}|p^{-1}(U).$$

Furthermore, the commutativity of (2.8.13.2) shows that these homomorphism commute with the restriction operations on Y; we thus obtain a canonical functorial homomorphism

$$\beta : (\mathbf{\Gamma}_*(\mathscr{F}))^\sim \longrightarrow \mathscr{F} \tag{3.3.4.1}$$

(also denoted $\beta_{\mathscr{F}}$) for quasi-coherent $\mathscr{O}_X$-modules.

Proposition (3.3.5). — *Let $\mathscr{M}$ be a quasi-coherent graded $\mathscr{S}$-module, and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module; then the composite homomorphisms*

$$\widetilde{\mathscr{M}} \xrightarrow{\widetilde{\alpha}} (\mathbf{\Gamma}_*(\widetilde{\mathscr{M}}))^{\sim} \xrightarrow{\beta} \widetilde{\mathscr{M}} \tag{3.3.5.1}$$

$$\mathbf{\Gamma}_*(\mathscr{F}) \xrightarrow{\alpha} \mathbf{\Gamma}_*((\mathbf{\Gamma}_*(\mathscr{F}))^{\sim}) \xrightarrow{\mathbf{\Gamma}_*(\beta)} \mathbf{\Gamma}_*(\mathscr{F}) \tag{3.3.5.2}$$

are the identity isomorphisms.

PROOF. The question is local on Y, so we can reduce to (2.6.5). □

3.4. Finiteness conditions

II | 59

Proposition (3.4.1). — *Let Y be a prescheme, and $\mathscr{S}$ a quasi-coherent $\mathscr{O}_Y$-algebra generated by $\mathscr{S}_1$ (3.1.9); suppose further that $\mathscr{S}_1$ is of finite type. Then $X = \operatorname{Proj}(\mathscr{S})$ is of finite type over Y.*

PROOF. Since the question is local on Y, we can suppose that Y is affine of ring A; then $\mathscr{S} = \widetilde{S}$, where $S = \Gamma(Y, \mathscr{S})$, and by hypothesis S is an A-algebra generated by $S_1 = \Gamma(Y, \mathscr{S}_1)$, where we can further suppose that S_1 is an A-module of finite type ((**I**, 1.3.9) and (**I**, 1.3.12)). Then S is a graded A-algebra of finite type, and we can reduce to (2.7.1, (ii)). □

(3.4.2). Let $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra; for a quasi-coherent graded $\mathscr{S}$-module $\mathscr{M}$, consider the following finiteness conditions:

(**TF**) There exists an integer n such that the $\mathscr{S}$-module $\bigoplus_{k \geqslant n} \mathscr{M}_k$ is of finite type.
(**TN**) There exists an integer n such that $\mathscr{M}_k = 0$ for $k \geqslant n$.

If $\mathscr{M}$ satisfies (**TN**), then $\widetilde{\mathscr{M}} = 0$, since this is a local property on Y (2.7.2).

Let $\mathscr{M}$ and $\mathscr{N}$ be quasi-coherent graded $\mathscr{S}$-modules; we say that a homomorphism $u : \mathscr{M} \to \mathscr{N}$ of degree 0 is *(**TN**)-injective* (resp. *(**TN**)-surjective*, *(**TN**)-bijective*) if there exists an integer n such that $u_k : \mathscr{M}_k \to \mathscr{N}_k$ is injective (resp. surjective, bijective) for $k \geqslant n$; then $\widetilde{u} : \widetilde{\mathscr{M}} \to \widetilde{\mathscr{N}}$ is injective (resp. surjective, bijective) by (2.7.2), since this is a local property on Y, and taking (**I**, 1.3.9) into account; if u is (**TN**)-bijective, then we also say that u is a *(**TN**)-isomorphism*.

Proposition (3.4.3). — *Let Y be a prescheme, and $\mathscr{S}$ a quasi-coherent graded $\mathscr{O}_Y$-algebra generated by $\mathscr{S}_1$, with S_1 assumed to be of finite type. Let $\mathscr{M}$ be a quasi-coherent graded $\mathscr{S}$-module.*

*(i) If $\mathscr{M}$ satisfies (**TF**), then $\widetilde{\mathscr{M}}$ is of finite type.*
*(ii) Suppose that $\mathscr{M}$ satisfies (**TF**); for $\widetilde{\mathscr{M}} = 0$, it is necessary and sufficient for $\mathscr{M}$ to satisfy (**TN**).*

PROOF. Since the questions are local on Y, we can reduce to the case where Y is affine of ring A, $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra such that the ideal S_+ is of finite type, and $\mathscr{M} = \widetilde{M}$, where M is a graded S-module; the proposition then follows from (2.7.3). □

Theorem (3.4.4). — *Let Y be a prescheme, and $\mathscr{S}$ a quasi-coherent graded $\mathscr{O}_Y$-algebra generated by $\mathscr{S}_1$, where $\mathscr{S}_1$ is assumed to be of finite type; let $X = \operatorname{Proj}(\mathscr{S})$. For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism β (3.3.4) is an isomorphism.*

PROOF. Note first of all that β is defined, by (3.4.1). To see that β is an isomorphism, we can reduce to the case where Y is affine of ring A, $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra generated by S_1, and S_1 is an A-module of finite type. It then suffices to apply (2.7.5). □

Corollary (3.4.5). — *Under the hypotheses of (3.4.4), every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ is isomorphic to an $\mathscr{O}_X$-module of the form $\widetilde{\mathscr{M}}$, where $\mathscr{M}$ is a quasi-coherent graded $\mathscr{S}$-module. If, further, $\mathscr{F}$ is of finite type, and if we assume that Y is a quasi-compact scheme, or that the underlying space of Y is Noetherian, then we can take $\mathscr{M}$ to be of finite type.*

II | 60

PROOF. The first claim follows immediately from (3.4.4) by taking $\mathscr{M} = \mathbf{\Gamma}_*(\mathscr{F})$. To prove the second, it suffices to show that $\mathscr{M}$ is the inductive limit of its *graded* sub-$\mathscr{S}$-modules of finite type $\mathscr{N}_\lambda$: indeed, it will follow from this that $\widetilde{\mathscr{M}}$ is the inductive limit of the $\widetilde{\mathscr{N}_\lambda}$ (3.2.4), and so $\mathscr{F}$ is the inductive limit of the $\beta(\widetilde{\mathcal{N}_\lambda})$; since X is quasi-compact ((3.4.1) and (**I**, 6.3.1)) and $\mathscr{F}$ is of finite type, $\mathscr{F}$ will necessarily be equal to one of the $\beta(\widetilde{\mathcal{N}_\lambda})$ (**0**, 5.2.3).

To define the $\mathcal{N}_\lambda$ having $\mathscr{M}$ as their inductive limit, it suffices to consider, for each $n \in \mathbf{Z}$, the quasi-coherent $\mathscr{O}_Y$-module $\mathscr{M}_n$, which is the inductive limit of its sub-$\mathscr{O}_Y$-modules $\mathscr{M}_n^{(\mu_n)}$ of finite type, by the hypotheses on Y (**I**, 9.4.9); it is immediate that $\mathscr{P}_{\mu_n} = \mathscr{S} \cdot \mathscr{M}_n^{(\mu_n)}$ is a graded $\mathscr{S}$-module of finite type, and we immediately see that taking $\mathcal{N}_\lambda$ to be the finite sums of the $\mathscr{S}$-modules of the form $\mathscr{P}_{\mu_n}$ gives the desired objects. $\square$

Corollary (3.4.6). — *Suppose that the hypotheses of* (3.4.4) *are satisfied, and further that the underlying space of Y is quasi-compact; let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type. Then there exists n_0 such that, for all $n \geqslant n_0$, the canonical homomorphism $\sigma : p^*(p_*(\mathscr{F}(n))) \to \mathscr{F}(n)$* (**0**, 4.4.3) *is surjective.*

Proof. For all $y \in Y$, let U be an affine open neighbourhood of y in Y. There exists an integer $n_0(U)$ such that, for all $n \geqslant n_0(U)$, $\mathscr{F}(n)|p^{-1}(U)$ is generated by a finite number of its sections over $p^{-1}(U)$ (2.7.9); but these sections are the canonical images of sections of $p^*(p_*(\mathscr{F}(n)))$ over $p^{-1}(U)$ ((**0**, 3.7.1) and (**0**, 4.4.3)), so $\mathscr{F}(n)|p^{-1}(U)$ is equal to the canonical image of $p^*(p_*(\mathscr{F}(n)))|p^{-1}(U)$. Finally, since Y is quasi-compact, there exists a finite cover of Y by affine opens U_i, and taking n_0 to be the largest of the $n_0(U_i)$ finishes the proof. $\square$

Remarks (3.4.7). — If $p = (\psi, \theta) : X \to Y$ is a morphism of ringed spaces, and $\mathscr{F}$ an $\mathscr{O}_X$-module, the fact that the canonical homomorphism $\sigma : p^*(p_*(\mathscr{F})) \to \mathscr{F}$ is surjective can be explained in the following way (**0**, 4.4.1): for all $x \in X$, and every section s of $\mathscr{F}$ over an open neighbourhood V of x, there exists an open neighbourhood U of $p(x)$ in Y, a finite number of sections t_i (for $1 \leqslant i \leqslant m$) of $\mathscr{F}$ over $p^{-1}(U)$, a neighbourhood $W \subset V \cap p^{-1}(U)$ of x, and sections a_i (for $1 \leqslant i \leqslant m$) of $\mathscr{O}_X$ over W such that

$$s|W = \sum_{i=1}^{m} a_i \cdot (t_i|W).$$

If Y is an *affine scheme* and $p_*(\mathscr{F})$ is *quasi-coherent*, this condition is equivalent to $\mathscr{F}$ being *generated by its sections over X* (**0**, 5.5.1): indeed, if $Y = \operatorname{Spec}(A)$, we can suppose that $U = D(f)$ with $f \in A$; then there exists an integer $n > 0$ and sections s_i of $\mathscr{F}$ over X such that t_i is the restriction to $p^{-1}(U)$ of $s_i g^n$, where $g = \theta(f)$ (by applying (**I**, 1.4.1) to $p_*(\mathscr{F})$); since g is invertible over $p^{-1}(U)$, we thus have

$$s|W = \sum_i b_i \cdot (s_i|W)$$

where $b_i = a_i(g|W)^{-n}$, whence the claim. If Y is affine, then the corollary (3.4.6) recovers (2.7.9).

We thus conclude that, when Y is an arbitrary prescheme, the following three conditions, for a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ such that $p_*(\mathscr{F})$ is *quasi-coherent*, are equivalent: **II** | 61

a) *The canonical homomorphism $\sigma : p^*(p_*(\mathscr{F})) \to \mathscr{F}$ is surjective.*
b) *There exists a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{G}$ and a surjective homomorphism $p^*(\mathscr{G}) \to \mathscr{F}$.*
c) *For every affine open U of Y, $\mathscr{F}|p^{-1}(U)$ is generated by its sections over $p^{-1}(U)$.*

Indeed, we have just proven the equivalence between *a)* and *c)*. It is also clear that *a)* implies *b)*, since $p_*(\mathscr{F})$ is quasi-coherent by hypothesis. Conversely, every homomorphism $u : p^*(\mathscr{G}) \to \mathscr{F}$ factors as $p^*(\mathscr{G}) \to p^*(p_*(\mathscr{F})) \xrightarrow{\sigma} \mathscr{F}$ (**0**, 3.5.4.4), so if u is surjective then so too is σ, which proves that *b)* implies *a)*.

Corollary (3.4.8). — *Suppose that the hypotheses of* (3.4.4) *are satisfied, and suppose further that Y is a quasi-compact scheme, or that the underlying space of Y is Noetherian. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type; then there exists an integer n_0 such that, for all $n \geqslant n_0$, $\mathscr{F}$ is isomorphic to a quotient of an $\mathscr{O}_X$-module of the form $(p^*(\mathscr{G}))(-n)$, where $\mathscr{G}$ is a quasi-coherent $\mathscr{O}_Y$-module of finite type (that depends on n).*

Proof. Since the structure morphism $X \to Y$ is separated and of finite type, $p_*(\mathscr{F}(n))$ is quasi-coherent (**I**, 9.2.2, b)), and thus the inductive limit of its quasi-coherent sub-$\mathscr{O}_Y$-modules of finite type, by the hypotheses on Y (**I**, 9.4.9). We thus deduce, by (3.4.6), (**0**, 4.3.2), and (**0**, 5.2.3), that $\mathscr{F}(n)$ is the canonical image of an $\mathscr{O}_X$-module of the form $p^*(\mathscr{G})$, where $\mathscr{G}$ is a quasi-coherent sub-$\mathscr{O}_Y$-module of $p_*(\mathscr{F}(n))$ of finite type; the corollary then follows from (3.2.5.2) and (3.2.7.1). $\square$

3.5. Functorial behaviour

(3.5.1). Let Y be a prescheme, and $\mathscr{S}$ and $\mathscr{S}'$ quasi-coherent positively-graded $\mathscr{O}_Y$-algebras; let $X = \operatorname{Proj}(\mathscr{S})$ and $X' = \operatorname{Proj}(\mathscr{S}')$, and let p and p' be the structure morphisms from X and X' to Y. Let $\phi : \mathscr{S}' \to \mathscr{S}$ be an $\mathscr{O}_Y$-homomorphism of graded algebras. For every affine open U of Y, let $S_U = \Gamma(U, \mathscr{S})$ and $S'_U = \Gamma(U, \mathscr{S}')$; the homomorphism ϕ defines a homomorphism $\phi_U : S'_U \to S_U$ of graded A_U-algebras, where $A_u = \Gamma(U, \mathscr{O}_Y)$. There is a corresponding open subset $G(\phi_U)$ in $p^{-1}(U)$ and morphism $\Phi_U : G(\phi_U) \to p'^{-1}(U)$ (2.8.1). Furthermore, if $V \subset U$ is an affine open, then the diagram

(3.5.1.1)
$$\begin{tikzcd} S'_U \arrow[r, "\phi_U"] \arrow[d] & S_U \arrow[d] \\ S'_V \arrow[r, "\phi_V"'] & S_V \end{tikzcd}$$

commutes, and we immediately see, by definition (2.8.1), that we have

$$G(\phi_V) = G(\phi_U) \cap p^{-1}(V)$$

and that Φ_V is the restriction to $G(\phi_V)$ of Φ_U. We have thus defined an open subset $G(\phi)$ of X such that $G(\phi) \cap p^{-1}(U) = G(\phi_U)$ for every affine open $U \subset Y$, and an *affine* Y-morphism $\Phi : G(\phi) \to X'$, which we say is *associated* to ϕ, and which we denote by $\operatorname{Proj}(\phi)$. If, for all $y \in Y$, there is an affine **II** | 62
neighbourhood U of y such that the $\Gamma(U, \mathscr{O}_Y)$-module $\Gamma(U, \mathscr{S}_+)$ is generated by $\phi(\Gamma(U, \mathscr{S}'_+))$, then $G(\phi_U) = p^{-1}(U)$, and so $G(\phi) = X$.

Proposition (3.5.2). —

(i) If $\mathscr{M}$ is a quasi-coherent graded $\mathscr{S}$-module, then there exists a canonical functorial isomorphism from the $\mathscr{O}_{X'}$-module $(\mathscr{M}_{[\phi]})^\sim$ to the $\mathscr{O}_{X'}$-module $\Phi_*(\widetilde{\mathscr{M}}|G(\phi))$.

(ii) If $\mathscr{M}'$ is a quasi-coherent graded $\mathscr{S}'$-module, then there exists a canonical functorial isomorphism ν from the $(\mathscr{O}_X|G(\phi))$-module $\Phi^*(\widetilde{\mathscr{M}'})$ to the $(\mathscr{O}_X|G(\phi))$-module $(\mathscr{M}' \otimes_{\mathscr{S}'} \mathscr{S})^\sim|G(\phi)$. If $\mathscr{S}'$ is generated by $\mathscr{S}'_1$, then ν is an isomorphism.

Proof. The homomorphisms in question are indeed already defined if Y is affine ((2.8.7) and (2.8.8)), and in the general case it suffices to check that they are compatible with the restriction of an affine open of Y to a smaller open, which follows immediately from the commutativity of (3.5.1.1). □

In particular, for all $n \in \mathbf{Z}$, we have a canonical homomorphism

(3.5.2.1)
$$\Phi^*(\mathscr{O}_{X'}(n)) \longrightarrow \mathscr{O}_X(n)|G(\phi).$$

Proposition (3.5.3). — *Let Y and Y' be preschemes, $\psi : Y' \to Y$ a morphism, and $\mathscr{S}$ a quasi-coherent graded $\mathscr{O}_Y$-algebra; set $\mathscr{S}' = \psi^*(\mathscr{S})$. Then the Y'-scheme $X' = \operatorname{Proj}(\mathscr{S}')$ is canonically identified with $\operatorname{Proj}(\mathscr{S}) \times_Y Y'$. Furthermore, if $\mathscr{M}$ is a quasi-coherent graded $\mathscr{S}$-module, then the $\mathscr{O}_{X'}$-module $(\psi^*(\mathscr{M}))^\sim$ can be identified with $\widetilde{\mathscr{M}} \otimes_Y \mathscr{O}_{Y'}$.*

Proof. Note first of all that $\psi^*(\mathscr{S})$ and $\psi^*(\mathscr{M})$ are quasi-coherent $\mathscr{O}_{Y'}$-modules, as are their homogenous components (**0**, 5.1.4). Let U be an affine open of Y, $U' \subset \psi^{-1}(U)$ an affine open of Y', and A and A' the rings of U and U', respectively; then $\mathscr{S}|U = \widetilde{S}$, where S is a graded A-algebra, and $\mathscr{S}'|U'$ can be identified with $(S \otimes_A A')^\sim$ (**I**, 1.6.5); the first claim then follows from (2.8.10) and (**I**, 3.2.6.2), since we immediately see that the projection $\operatorname{Proj}(\mathscr{S}'|U') \to \operatorname{Proj}(\mathscr{S}|U)$ defined by the above identification is compatible with the restriction operations on U and U', and thus indeed defines a morphism $\operatorname{Proj}(\mathscr{S}') \to \operatorname{Proj}(\mathscr{S})$. Now let

$$q : \operatorname{Proj}(\mathscr{S}) \longrightarrow Y$$
$$q' : \operatorname{Proj}(\mathscr{S}') \longrightarrow Y'$$

be the structure morphisms; $q'^{-1}(U')$ can then be identified with $q^{-1}(U) \times_U U'$, and the two sheaves $(\psi^*(\mathscr{M}))^\sim|q'^{-1}(U')$ and $(\widetilde{\mathscr{M}} \otimes_Y \mathscr{O}_{Y'})|q'^{-1}(U')$ are then both canonically identified with $(M \otimes_A A')^\sim$, where we set $M = \Gamma(U, \mathscr{M})$, by (2.8.10) and (I, 1.6.5); whence the second claim, since we can again immediately see the compatibility of the above identifications with the restriction operations. □

Corollary (3.5.4). — *With the notation of* (3.5.3), $\mathscr{O}_{X'}(n)$ *is canonically identified with* $\mathscr{O}_X(n) \otimes_Y \mathscr{O}_{Y'}$ *for all* $n \in \mathbf{Z}$ *(where* $X = \operatorname{Proj}(\mathscr{S})$*).*

PROOF. Indeed, with the notation of (3.5.3), it is clear that $\psi^*(\mathscr{S}(n)) = \mathscr{S}'(n)$ for all $n \in \mathbf{Z}$. □

(3.5.5). Keeping the above notation, denote by Ψ the canonical projection $X' \to X$, and set $\mathscr{M}' = \psi^*(\mathscr{M})$; we further suppose that $\mathscr{S}$ is generated by $\mathscr{S}_1$, and that X is of finite type over Y; it then **II** | 63
follows that $\mathscr{S}'$ is generated by $\mathscr{S}'_1$ (as can be seen by reducing to the case where Y and Y' are affine), and that X' is of finite type over Y' (I, 6.3.4). Let $\mathscr{F}$ be an $\mathscr{O}_X$-module, and set $\mathscr{F}' = \Psi^*(\mathscr{F})$; it then follows from (3.5.4) and (0, 4.3.3) that $\mathscr{F}'(n) = \Psi^*(\mathscr{F}(n))$ for all $n \in \mathbf{Z}$. We further define a canonical Ψ-homomorphism $\theta_n : q_*(\mathscr{F}(n)) \to q'_*(\mathscr{F}'(n))$ in the following way: given the commutativity of the diagram

$$\begin{tikzcd} X & X' \arrow[l, "\Psi"'] \arrow[d, "q'"] \\ Y & Y' \arrow[l, "\psi"] \arrow[from=1-1, to=2-1, "q"'] \end{tikzcd}$$

it is enough to define a homomorphism $q_*(\mathscr{F}(n)) \to \psi_*(q'_*(\Psi^*(\mathscr{F}(n)))) = q_*(\Psi_*(\Psi^*(\mathscr{F}(n))))$, and it suffices to take the homomorphism $\theta_n = q_*(\rho_n)$, where ρ_n is the canonical homomorphism $\mathscr{F}(n) \to \Psi_*(\Psi^*(\mathscr{F}(n)))$ (0, 4.4.3). It is immediate that, for every affine open U of Y and every affine open U' of Y' such that $U' \subset \psi^{-1}(U)$, the homomorphism θ_n gives, on sections, the canonical homomorphism (0, 3.7.2) $\Gamma(q^{-1}(U), \mathscr{F}(n)) \to \Gamma(q'^{-1}(U'), \mathscr{F}'(n))$. The commutativity of (2.8.13.2) then shows that, if $\mathscr{F}$ is quasi-coherent, the diagram

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\rho"] & \mathscr{F}' \\ (\mathbf{\Gamma}_*(\mathscr{F}))^\sim \arrow[u, "\beta_{\mathscr{F}}"] \arrow[r, "\widetilde{\theta}"'] & (\mathbf{\Gamma}_*(\mathscr{F}'))^\sim \arrow[u, "\beta_{\mathscr{F}'}"'] \end{tikzcd}$$

commutes (the top horizontal arrow being the canonical Ψ-morphism $\mathscr{F} \to \Psi^*(\mathscr{F})$).

Similarly, the commutativity of (2.8.13.1) shows that the diagram

$$\begin{tikzcd} \mathbf{\Gamma}_*(\widetilde{\mathscr{M}}) \arrow[r, "\theta"] & \mathbf{\Gamma}_*(\widetilde{\mathscr{M}'}) \\ \mathscr{M} \arrow[u, "\alpha_{\mathscr{M}}"] \arrow[r, "\rho"'] & \mathscr{M}' \arrow[u, "\alpha_{\mathscr{M}'}"'] \end{tikzcd}$$

commutes (the bottom horizontal arrow being the canonical ψ-morphism $\mathscr{M} \to \psi^*(\mathscr{M})$).

(3.5.6). Now consider preschemes Y and Y', a morphism $g : Y' \to Y$, a quasi-coherent graded $\mathscr{O}_Y$-algebra (resp. quasi-coherent graded $\mathscr{O}_{Y'}$-algebra) $\mathscr{S}$ (resp. $\mathscr{S}'$), and a g-morphism of graded algebras $u : \mathscr{S} \to \mathscr{S}'$, i.e. a $\mathscr{O}_Y$-homomorphism of graded algebras $\mathscr{S} \to g_*(\mathscr{S}')$; we already know that this is equivalent to giving a $\mathscr{O}_{Y'}$-homomorphism of graded algebras $u^\sharp : g^*(\mathscr{S}) \to \mathscr{S}'$. We thus canonically obtain from $u^\sharp$ a Y'-morphism $W = \operatorname{Proj}(u^\sharp) : G(u^\sharp) \to \operatorname{Proj}(g^*(\mathscr{S}))$, where $G(u^\sharp)$ is an open of $X' = \operatorname{Proj}(\mathscr{S}')$ (3.5.1). We also know that $X'' = \operatorname{Proj}(g^*(\mathscr{S}))$ is canonically identified with $X \times_Y Y'$, by taking $X = \operatorname{Proj}(\mathscr{S})$ (3.5.3); composing the first projection $p : X \times_Y Y' \to X$ with **II** | 64
$\operatorname{Proj}(u^\sharp)$, we thus obtain a morphism $v : G(u^\sharp) \to X$, which we denote by $\operatorname{Proj}(u)$, and which is such that the diagram

$$\begin{tikzcd} G(u^\sharp) \arrow[r, "v"] \arrow[d] & X \arrow[d] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

commutes.

Furthermore, for every quasi-coherent graded $\mathscr{O}_Y$-module $\mathscr{M}$, we have a canonical v-morphism

$$\nu : \widetilde{\mathscr{M}} \longrightarrow (g^*(\mathscr{M}) \otimes_{g^*(\mathscr{S})} \mathscr{S}')^{\sim}|G(u^\sharp). \tag{3.5.6.1}$$

Indeed, $\nu^\sharp$ is given by composing the homomorphisms

$$v^*(\widetilde{\mathscr{M}}) = w^*(p^*(\widetilde{\mathscr{M}})) \longrightarrow w^*((g^*(\mathscr{M}))^{\sim}) \longrightarrow (g^*(\mathscr{M}) \otimes_{g^*(\mathscr{S})} \mathscr{S}')^{\sim}|G(u^\sharp)$$

where the first arrow comes from the isomorphism (3.5.3) and the second is the homomorphism (3.5.2, (i)); if $\mathscr{S}$ is generated by $\mathscr{S}_1$, then it follows from (3.5.2) that $\nu^\sharp$ is an *isomorphism*.

As a particular case of (3.5.6.1), we have, for all $n \in \mathbf{Z}$, a canonical v-morphism

$$\nu : \mathscr{O}_X(n) \longrightarrow \mathscr{O}_{X'}(n)|G(u^\sharp). \tag{3.5.6.2}$$

3.6. Closed subpreschemes of a prescheme $\mathrm{Proj}(\mathscr{S})$

(3.6.1). Let Y be a prescheme, and $\phi : \mathscr{S} \to \mathscr{S}'$ a degree 0 homomorphism of quasi-coherent graded $\mathscr{O}_Y$-algebras. We say that ϕ is *(**TN**)-surjective* (resp. *(**TN**)-injective*, *(**TN**)-bijective*) if there exists n such that, for all $k \geqslant n$, $\phi_k : \mathscr{S}_k \to \mathscr{S}'_k$ is surjective (resp. injective, bijective). If this is the case, then we can reduce the study of the corresponding morphism $\Phi : \mathrm{Proj}(\mathscr{S}') \to \mathrm{Proj}(\mathscr{S})$ to the case where ϕ is *surjective* (resp. *injective*, *bijective*). We prove this as in (2.9.1) (which is the particular case where Y is affine) by using (3.1.8). Instead of saying that ϕ is (**TN**)-bijective, we also say that it is a *(**TN**)-isomorphism*.

Proposition (3.6.2). — *Let Y be a prescheme, and $\mathscr{S}$ a quasi-coherent graded $\mathscr{O}_Y$-algebra; set $X = \mathrm{Proj}(\mathscr{S})$.*

*(i) If $\phi : \mathscr{S} \to \mathscr{S}'$ is a (**TN**)-surjective homomorphism of graded $\mathscr{O}_Y$-algebras, then the corresponding morphism $\Phi = \mathrm{Proj}(\phi)$ (3.5.1) is defined on all of $\mathrm{Proj}(\mathscr{S}')$ and is a closed immersion of $\mathrm{Proj}(\mathscr{S}')$ into X. If $\mathscr{J}$ is the kernel of ϕ, then the closed subprescheme of X associated to Φ is defined by the quasi-coherent sheaf of ideals $\widetilde{\mathscr{J}}$ in $\mathscr{O}_X$.*

*(ii) Suppose further that $\mathscr{S}_0 = \mathscr{O}_Y$, that $\mathscr{S}$ is generated by $\mathscr{S}_1$, and that $\mathscr{S}_1$ is of finite type. Let X' be a closed subprescheme of $X = \mathrm{Proj}(\mathscr{S})$, defined by a quasi-coherent sheaf of ideals $\mathscr{I}$ in $\mathscr{O}_X$. Let $\mathscr{J}$ be the quasi-coherent graded sheaf of ideals of $\mathscr{S}$, given by the inverse image of $\mathbf{\Gamma}_*(\mathscr{I})$ by the canonical homomorphism $\alpha : \mathscr{S} \to \mathbf{\Gamma}_*(\mathscr{O}_X)$ (3.3.2), and set $\mathscr{S}' = \mathscr{S}|\mathscr{J}$. Then X' is the subprescheme associated (**I**, 4.2.1) to the closed immersion $\mathrm{Proj}(\mathscr{S}') \to X$ corresponding to the canonical homomorphism $\mathscr{S} \to \mathscr{S}'$ of graded $\mathscr{O}_Y$-algebras.* **II** | 65

PROOF.

(i) We can assume that ϕ is surjective (3.6.1). Then, for every affine open U of Y, $\Gamma(U, \mathscr{S}) \to \Gamma(U, \mathscr{S}')$ is surjective (**I**, 1.3.9), so (3.5.1) $G(\phi) = X$. We can then immediately reduce to proving the proposition in the case where Y is affine, and this follows from (2.9.2, (i)).

(ii) We can reduce to proving that the homomorphism $\widetilde{\mathscr{J}} \to \mathscr{O}_X$ induced by the canonical injection $\mathscr{J} \to \mathscr{S}$ is an isomorphism from $\widetilde{\mathscr{J}}$ to $\mathscr{I}$; since the question is local on Y, we can take Y to be affine of ring A, which implies that $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra generated by S_1, with S_1 of finite type over A. It then suffices to apply (2.9.2, (ii)).

□

Corollary (3.6.3). — *Under the conditions of (3.6.2, (i)), suppose further that $\mathscr{S}$ is generated by $\mathscr{S}_1$. Then $\Phi^*(\mathscr{O}_X(n))$ is canonically identified with $\mathscr{O}_{X'}(n)$ for all $n \in \mathbf{Z}$.*

PROOF. We have defined such a canonical isomorphism when Y is affine (2.9.3); in the general case, it suffices to show that the isomorphisms thus defined for each affine open U of Y are compatible with the passage from U to an affine open $U' \subset U$, which is immediate. □

Corollary (3.6.4). — *Let Y be a prescheme, $\mathscr{S}$ a quasi-coherent graded $\mathscr{O}_Y$-algebra generated by $\mathscr{S}_1$, $\mathscr{M}$ a quasi-coherent $\mathscr{O}_Y$-module, u a surjective $\mathscr{O}_Y$-homomorphism $\mathscr{M} \to \mathscr{S}_1$, and $\overline{u} : \mathbf{S}_{\mathscr{O}_Y}(\mathscr{M}) \to \mathscr{S}$ the homomorphism of graded $\mathscr{O}_Y$-algebras that extends u (1.7.4). Then the morphism corresponding to $\overline{u}$ is a closed immersion of $\mathrm{Proj}(\mathscr{S})$ into $\mathrm{Proj}(\mathbf{S}_{\mathscr{O}_Y}(\mathscr{M}))$.*

PROOF. Indeed, $\overline{u}$ is surjective by hypothesis, and we apply (3.6.1, (i)). □

3.7. Morphisms from a prescheme to a homogeneous spectrum

(3.7.1). Let $q : X \to Y$ be a morphism of preschemes, $\mathscr{L}$ an invertible $\mathscr{O}_X$-module, and $\mathscr{S}$ a quasi-coherent positively-graded $\mathscr{O}_Y$-algebra; then $q^*(\mathscr{S})$ is a quasi-coherent positively-graded $\mathscr{O}_X$-algebra. Consider the quasi-coherent graded $\mathscr{O}_X$-algebra $\mathscr{S}' = \bigoplus_{n\geqslant 0} \mathscr{L}^{\otimes n}$; suppose that we have some $\mathscr{O}_X$-homomorphism of graded algebras

$$\psi : q^*(\mathscr{S}) \longrightarrow \mathscr{S}' = \bigoplus_{n\geqslant 0} \mathscr{L}^{\otimes n}$$

which is equivalent to having a q-morphism of graded algebras

$$\psi^\flat : \mathscr{S} \longrightarrow q_*(\mathscr{S}').$$

We know that $\operatorname{Proj}(\mathscr{S}')$ is canonically identified with X ((3.1.7) and (3.1.8, (iii))); we thus canonically obtain from ψ an open subset $G(\psi)$ of X, and a Y-morphism

$$r_{\mathscr{L},\psi} : G(\psi) \longrightarrow \operatorname{Proj}(\mathscr{S}) = P \tag{3.7.1.1}$$

which we call the morphism *associated to $\mathscr{L}$ and ψ*; recall (3.5.6) that this morphism is obtained by composing the Y-morphism **II** | 66

$$\tau = \operatorname{Proj}(\psi) : G(\psi) \longrightarrow \operatorname{Proj}(q^*(\mathscr{S}))$$

with the first projection $\pi : \operatorname{Proj}(q^*(\mathscr{S})) = P \times_Y X \to P$.

(3.7.2). We will explicitly describe $r = r_{\mathscr{L},\psi}$ in the case where $Y = \operatorname{Spec}(A)$ is affine, so that $\mathscr{S} = \widetilde{S}$, where S is a positively-graded A-algebra. Suppose first of all that $X = \operatorname{Spec}(B)$ is also affine, and that we have $\mathscr{L} = \widetilde{L}$, where L is a free B-module of rank 1. Then $q^*(\mathscr{S}) = (S \otimes_A B)^\sim$ (I, 1.6.5); if c is a generator of L, then $\psi_n : q^*(\mathscr{S}_n) \to \mathscr{L}^{\otimes n}$ corresponds to a homomorphism $w_n : s \otimes b \mapsto bv_n(s)c^{\otimes n}$ from $S_n \otimes A_B$to $L^{\otimes n}$, where $v_n : S_n \to B$ is a homomorphism of A-modules, constituting a homomorphism of algebras $S \to B$. Let $f \in S_d$ (where $d > 0$), and set $g = v_d(f)$; we have $\pi^{-1}(D_+(f)) = D_+(f \otimes 1)$ by (2.8.10) and the identification of $D_+(f)$ with $\operatorname{Spec}(S_{(f)})$ (2.3.6); we also know, from (2.8.1.1) (taking into account the canonical identification of X with $\operatorname{Proj}(\mathscr{S}')$), that

$$\tau^{-1}(D_+(f \otimes 1)) = D(g)$$

whence

$$r^{-1}(D_+(f)) = D(g). \tag{3.7.2.1}$$

Furthermore, the morphism $\tau = \operatorname{Proj}(\psi)$, restricted to $D(g)$, corresponds to the homomorphism that sends $(s \otimes 1)/(f \otimes 1)^n$ (where $s \in S_{nd}$) to $v_{nd}(s)/g^n$ (2.8.1), and the projection π, restricted to $D_+(f \otimes 1)$, corresponds t the homomorphism $s/f^n \mapsto (s \otimes 1)/(f \otimes 1)^n$; we thus conclude that r, restricted to $D(g)$, corresponds to the homomorphism of A-algebras $\omega : S_{(f)} \to B_g$ such that $\omega(s/f^n) = v_{nd}(s)/g^n$ (where $s \in S_{nd}$ and $n > 0$). Passing to the case where X is arbitrary (but Y still affine), we thus obtain, taking (2.8.1) into account, the following:

Proposition (3.7.3). — *If $Y = \operatorname{Spec}(A)$ is affine and $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra, then, for all $f \in S_d = \Gamma(Y, \mathscr{S}_d)$, we have*

$$r_{\mathscr{L},\psi}^{-1}(D_+(f)) = X_{\psi^\flat(f)} \tag{3.7.3.1}$$

(where $\psi^\flat(f) \in \Gamma(X, \mathscr{L}^{\otimes d})$) and the restriction $X_{\psi^\flat(f)} \to D_+(f) = \operatorname{Spec}(S_{(f)})$ of $r_{\mathscr{L},\psi}$ corresponds (I, 2.2.4) to the homomorphism of algebras

$$\psi^\flat_{(f)} : S_{(f)} \longrightarrow \Gamma(X_{\psi^\flat(f)}, \mathscr{O}_X) \tag{3.7.3.2}$$

such that, for $s \in S_{nd} = \Gamma(Y, \mathscr{S}_{nd})$,

$$\psi^\flat_{(f)}(s/f^n) = (\psi^\flat(s)|X_{\psi^\flat(f)})(\psi^\flat(f)|X_{\psi^\flat(f)})^{-n}. \tag{3.7.3.3}$$

We say that $r_{\mathscr{L},\psi}$ is *everywhere defined* if $G(\psi) = X$. For this to be the case, it is evidently necessary and sufficient that $G(\psi) \cap q^{-1}(U) = q^{-1}(U)$ for every affine open $U \subset Y$; in other words, the question is *local* on Y. If Y is affine, then $G(\psi)$ is the union of the $r^{-1}(D_+(f))$ over f homogeneous in S_+ (2.8.1); by (3.7.3.1), the $X_{\psi^\flat(f)}$ must then form a *cover* of X; in other words:

Corollary (3.7.4). — *Under the hypotheses of (3.7.3), for $r_{\mathscr{L},\psi}$ to be everywhere defined, it is necessary and sufficient that, for every $x \in X$, there exist an integer $n > 0$ and a section s of $\mathscr{S}_n$ over Y such that, if we set $t(= \psi^\flat(s) \in \Gamma(X, \mathscr{L}^{\otimes n})$, then $t(x) \neq 0$.*

Note that this condition is always satisfied if ψ is *(**TN**)-surjective.* **II | 67**

Similarly, the question of if $r_{\mathscr{L},\psi}$ is *dominant* is local on Y, and we have:

Corollary (3.7.5). — *Under the hypotheses of (3.7.3), for $r_{\mathscr{L},\psi}$ to be dominant, it is necessary and sufficient that, for every integer $n > 0$, every section $s \in S_n$ such that $\psi^\flat(s) \in \Gamma(X, \mathscr{L}^{\otimes n})$ is locally nilpotent is itself nilpotent.*

PROOF. We have to show that $r^{-1}_{\mathscr{L},\psi}(D_+(s))$ is not empty if $D_+(s)$ is empty, and the corollary thus follows from (3.7.3.1) and (2.3.7). $\square$

Proposition (3.7.6). — *Let $q : X \to Y$ be a morphism, $\mathscr{L}$ an invertible $\mathscr{O}_X$-module, $\mathscr{S}$ and $\mathscr{S}'$ quasi-coherent graded $\mathscr{O}_Y$-algebras, $u : \mathscr{S}' \to \mathscr{S}$ a homomorphism of graded algebras, and $\psi : q^*(\mathscr{S}) \to \bigoplus_{n \geqslant 0} \mathscr{L}^{\otimes n}$ a homomorphism of graded algebras; let $\psi' = \psi \circ q^*(u)$ be the composite homomorphism. If $r_{\mathscr{L},\psi'}$ is everywhere defined, then so too is $r_{\mathscr{L},\psi}$; if u is (**TN**)-surjective, then if $r_{\mathscr{L},\psi'}$ is dominant, so too is $r_{\mathscr{L},\psi}$; conversely, if u is (**TN**)-injective, then if $r_{\mathscr{L},\psi}$ is dominant, so too is $r_{\mathscr{L},\psi'}$.*

PROOF. We know that $G(\psi') \subset G(\psi)$ (2.8.4), whence the first claim; if u is (**TN**)-surjective, then $\mathrm{Proj}(u) : \mathrm{Proj}(S) \to \mathrm{Proj}(S')$ is everywhere defined and a closed immersion; since $r_{\mathscr{L},\psi'}$ is the composition of $\mathrm{Proj}(u)$ and the restriction to $G(\psi')$ of $r_{\mathscr{L},\psi}$, we thus conclude that, if $r_{\mathscr{L},\psi'}$ is dominant, then so too is $r_{\mathscr{L},\psi}$. Finally, if u is (**TN**)-injective, then we know that $\mathrm{Proj}(u)$ is a dominant morphism from $G(u)$ to $\mathrm{Proj}(S')$ (2.8.3); since $G(\psi')$ is the inverse image of $G(u)$ under $r_{\mathscr{L},\psi}$, we see that, if $r_{\mathscr{L},\psi}$ is dominant, then so too is $r_{\mathscr{L},\psi'}$. $\square$

Proposition (3.7.7). — *Let Y be a quasi-compact prescheme, $q : X \to Y$ a quasi-compact morphism, $\mathscr{L}$ an invertible $\mathscr{O}_X$-module, and $\mathscr{S}$ a quasi-coherent graded $\mathscr{O}_Y$-algebra given by the filtered inductive limit of an inductive system $(\mathscr{S}^\lambda)$ of quasi-coherent $\mathscr{O}_Y$-algebras. Let $\phi_\lambda : \mathscr{S}^\lambda \to \mathscr{S}$ be the canonical homomorphism, $\psi : q^*(\mathscr{S}) \to \bigoplus_{n \geqslant 0} \mathscr{L}^{\otimes n}$ a homomorphism of graded rings, and set $\psi_\lambda = \psi \circ q^*(\phi_\lambda)$. For $r_{\mathscr{L},\psi}$ to be everywhere defined, it is necessary and sufficient that there exist some λ such that $r_{\mathscr{L},\psi_\lambda}$ be everywhere defined; $r_{\mathscr{L},\psi_\mu}$ is then everywhere defined for $\mu \geqslant \lambda$.*

PROOF. The condition is sufficient by (3.7.6). Conversely, suppose that $r_{\mathscr{L},\psi}$ is everywhere defined; we can reduce to the case where Y is affine, since, if, for every affine open $U \subset Y$ there exists $\lambda(U)$ such that the restriction of $r_{\mathscr{L},\psi_{\lambda(U)}}$ to $q^{-1}(U)$ is everywhere defined, then it will suffice (Y being quasi-compact) to cover Y by a finite number of affine opens U_i, and to take $\lambda \geqslant \lambda(U_i)$ for all indices i, by (3.7.6). If Y is affine, then the hypothesis implies that, for all $x \in X$, there exists a section $s^{(x)}$ of S_n such that, if we set $t^{(x)} = \psi^\flat(s^{(x)})$, then $t^{(x)}(x) \neq 0$ (where $t^{(x)}$ is considered as a section of $\mathscr{L}^{\otimes n}$ over X), which implies that $t^{(x)}(z) \neq 0$ for any z in a neighbourhood $V(x)$ of x. Cover X by a finite number of $V(x_i)$, and let $s^{(i)}$ be the corresponding sections of S; then there exists some λ such that the $s^{(i)}$ are all of the form $\phi_\lambda({s'_\lambda}^{(i)})$, with ${s'_\lambda}^{(i)} \in S^\lambda$ for all i; it then follows from (3.7.4) that $r_{\mathscr{L},\psi_\lambda}$ is everywhere defined. The final claim is a trivial consequence of (3.7.6). $\square$

Corollary (3.7.8). — *Under the hypotheses of (3.7.7), if the $r_{\mathscr{L},\psi_\lambda}$ are dominant, then so too is $r_{\mathscr{L},\psi}$; the converse is true if the ϕ_λ are all injective.*

II | 68

PROOF. The second claim is a particular case of (3.7.6); to show that $r_{\mathscr{L},\psi}$ is dominant, we can restrict to the case where Y is affine; if $s \in S$ is such that $\psi^\flat(s)$ is locally nilpotent, since we can write $s = \phi_\lambda(s_\lambda)$ for at least one λ, we conclude from the hypotheses and from (3.7.5) that s_λ is nilpotent, and thus so too is s, and the criteria of (3.7.5) then apply. $\square$

Remarks (3.7.9). —

(i) With the notation of (3.7.1), and taking (3.2.10) into account, we have, for all $n \in \mathbf{Z}$, a canonical homomorphism

$$\theta : r^*_{\mathscr{L},\psi}(\mathscr{O}_P(n)) \longrightarrow \mathscr{L}^{\otimes n} \tag{3.7.9.1}$$

defined in a general way in (3.5.6.2). We immediately see that, under the conditions of (3.7.3), the restriction of this homomorphism to $X_{\psi^\flat(f)}$ sends s/f^k (where $s \in S_{n+kd}$) to the element $(\psi^\flat(s)|X_{\psi^\flat(f)})(\psi^\flat(f)|X_{\psi^\flat(f)})^{-k}$.

(ii) Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module, and suppose that q is quasi-compact and separated, so that, for all $n \geqslant 0$, $q_*(\mathscr{F} \otimes \mathscr{L}^{\otimes n})$ is a quasi-coherent $\mathscr{O}_Y$-module (**I**, 9.2.2). Let $\mathscr{M}' = \bigoplus_{n\geqslant 0} \mathscr{F} \otimes \mathscr{L}^{\otimes n}$, which is a quasi-coherent graded $\mathscr{S}'$-module, and consider its image $\mathscr{M} = q_*(\mathscr{M}') = \bigoplus_{n\geqslant 0}(\mathscr{F} \otimes \mathscr{L}^{\otimes n})$ (which is a quasi-coherent $\mathscr{S}$-module, via the homomorphism $\psi^\flat$). We are going to show the existence of a canonical homomorphism of $\mathscr{O}_X$-modules

$$\xi : r^*_{\mathscr{L},\psi}(\widetilde{\mathscr{M}}) \longrightarrow \mathscr{F}|G(\psi). \tag{3.7.9.2}$$

Indeed, we have already defined (3.5.6.1) a canonical homomorphism

$$r^*_{\mathscr{L},\psi}(\widetilde{\mathscr{M}}) \longrightarrow (q^*(\mathscr{M}) \otimes_{q^*(\mathscr{S})} \mathscr{S}')^\sim|G(\psi) \tag{3.7.9.3}$$

where the right-hand side is considered as a quasi-coherent sheaf on $\operatorname{Proj}(\mathscr{S}')$. We also have a canonical homomorphism

$$q^*(q_*(\mathscr{M}')) \otimes_{q^*(\mathscr{S})} \mathscr{S}' \longrightarrow \mathscr{M}' \tag{3.7.9.4}$$

for any quasi-coherent graded $\mathscr{S}'$-module $\mathscr{M}'$: for any open U of X, any section t' of $q^*(q_*(\mathscr{M}'_h))$ over U, and any section b' of $\mathscr{S}'_k$ over U, we send $t' \otimes b'$ to the section $b'\sigma(t')$ of $\mathscr{M}'_{h+k}$, where $\sigma(t')$ is the section of $\mathscr{M}'_h$ over U that corresponds canonically (**0**, 4.4.3) to t'. We thus obtain a canonical homomorphism

$$(q^*(q_*(\mathscr{M}')) \otimes_{q^*(\mathscr{S})} \mathscr{S}')^\sim|G(\psi) \longrightarrow \widetilde{\mathscr{M}'}|G(\psi) \tag{3.7.9.5}$$

and, finally, since $\widetilde{\mathscr{M}'}$ is canonically identified with $\mathscr{F}$ (3.2.9, (i)), we indeed obtain the desired canonical homomorphism.

Under the conditions of (3.7.3), the restriction of this homomorphism to $X_{\psi^\flat(f)}$ acts as follows: given a section t_{nd} of $\mathscr{F} \otimes \mathscr{L}^{\otimes nd}$ over X, if t'_{nd} is the section t_{nd} considered as a section of $q_*(\mathscr{F} \otimes \mathscr{L}^{\otimes n})$ over Y, then it sends the element t'_{nd}/f^n to the section $(t_{nd}|X_{\psi^\flat(f)})(\psi^\flat(f)|X_{\psi^\flat(f)})^{-n}$ of $\mathscr{F}$ over $X_{\psi^\flat(f)}$.

3.8. Criteria for immersion into a homogeneous spectrum

II | 69

(3.8.1). With the notation of (3.7.1), the question of if $r_{\mathscr{L},\psi}$ is an immersion (resp. an open immersion, a closed immersion) is clearly *local on* Y.

Proposition (3.8.2). — *Under the hypotheses of* (3.7.3), *for $r_{\mathscr{L},\psi}$ to be everywhere defined and an immersion, it is necessary and sufficient that there exist a family of sections $s_\alpha \in S_{n_\alpha}$ (where $n_\alpha > 0$) such that, if we set $f_\alpha = \psi^\flat(s_\alpha)$, the following conditions are satisfied:*

(i) The X_{f_α} form a cover of X.

(ii) The X_{f_α} are affine opens.

(iii) For every α and every $t \in \Gamma(X_{f_\alpha}, \mathscr{O}_X)$, there exists an integer $m > 0$ and some $s \in S_{mn_\alpha}$ such that $t = (\psi^\flat(s)|X_{f_\alpha})(f_\alpha|X_{f_\alpha})^{-m}$.

For $r_{\mathscr{L},\psi}$ to be everywhere defined and an open immersion, it is necessary and sufficient that there exist a family (s_α) satisfying conditions (i), (ii), and (iii) above, as well as:

(iv) For all $n > 0$ and every $s \in S_{nn\alpha}$ such that $\psi^\flat(s)|X_{f_\alpha} = 0$, there exists an integer $k > 0$ such that $s^k_\alpha = 0$.

For $r_{\mathscr{L},\psi}$ to be everywhere defined and a closed immersion, it is necessary and sufficient that there exist a family (s_α) satisfying conditions (i), (ii), and (iii) above, as well as:

(v) The $D_+(s_\alpha)$ form a cover of $P = \operatorname{Proj}(S)$.

PROOF. For r to be an immersion (resp. a closed immersion), it is necessary and sufficient that there exist a cover of $r(G(\psi))$ (resp. of P) by the $D_+(s_\alpha)$, such that, if we set $V_\alpha = r^{-1}(D_+(s_\alpha))$, then the restriction of r to U_α is a *closed* immersion of V_α into $D_+(s_\alpha)$ (**I**, 4.2.4). Condition (i) says that r is everywhere defined and also that the $D_+(s_\alpha)$ cover $r(X)$, by (3.7.3.1); since $D_+(s_\alpha)$ is affine, conditions (ii) and (iii) say that the restriction of r to X_{f_α} is a closed immersion into $D_+(s_\alpha)$, by (**I**, 4.2.3); finally, since conditions (iii) and (iv) say that $\psi^\flat_{(s_\alpha)}$ is bijective (using the notation of (3.7.3.2)), conditions (ii), (iii), and (iv) say that the restriction of r to X_{f_α} is an isomorphism to $D_+(s_\alpha)$ for all α, and so conditions (i), (ii), (iii), and (iv) together say that r is an open immersion. □

Corollary (3.8.3). — *Under the hypotheses of (3.7.6), if $r_{\mathscr{L},\psi'}$ is everywhere defined and an immersion, then so too is $r_{\mathscr{L},\psi}$. If we further suppose that u is (**TN**)-surjective, then if $r_{\mathscr{L},\psi'}$ is an open (resp. closed) immersion, so too is $r_{\mathscr{L},\psi}$.*

PROOF. By (3.8.2), there exists a family $s'_\alpha \in S'_{n_\alpha}$ such that, if we set $f_\alpha = \psi'^\flat(s'_\alpha)$, conditions (i), (ii), and (iii) are satisfied. But if we set $s_\alpha = u(s'_\alpha)$, then we also have that $f_\alpha = \psi^\flat(s_\alpha)$, and if $t = (\psi'^\flat(s')|X_{f_\alpha})(f_\alpha|X_{f_\alpha})^{-m}$, then also $t = (\psi^\flat(s)|X_{f_\alpha})(f_\alpha|X_{f_\alpha})^{-m}$ by setting $s = u(s')$, whence the first claim. The second claim follows immediately from the fact that $\operatorname{Proj}(u)$ is a closed immersion. □

Proposition (3.8.4). — *Suppose that the hypotheses of (3.7.7) are satisfied, and further that $q : X \to Y$ is a morphism of finite type. Then, for $r_{\mathscr{L},\psi}$ to be everywhere defined and an immersion, it is necessary and sufficient that there exist some λ such that $r_{\mathscr{L},\psi_\lambda}$ is everywhere defined and an immersion; in this case, $r_{\mathscr{L},\psi_\mu}$ is also everywhere defined and an immersion for all $\mu \geqslant \lambda$.*

II | 70

PROOF. Taking (3.8.3) into account, it suffices to show that, if $r_{\mathscr{L},\psi}$ is everywhere defined and an immersion, then so too is $r_{\mathscr{L},\psi_\lambda}$ for at least one λ. By the same argument as in (3.7.7), using the fact that Y is quasi-compact, (3.8.2) shows the existence of a *finite* family $(s_i \in S_{n_i})$ of elements of S satisfying conditions (i), (ii), and (iii). The morphism $X_{f_i} \to Y$ (where $f_i = \psi^\flat(s_i)$) is of finite type: it is a morphism of affine schemes, and is thus quasi-compact (**I**, 6.6.1), and also locally of finite type, since q is of finite type (**I**, 6.3.2), and the conclusion then follows from (**I**, 6.6.3). The ring B_i of X_{f_i} is thus an A-algebra of finite type (**I**, 6.6.3); let (t_{ij}) be a family of generators of this algebra. By hypothesis, there exist elements $s'_{ij} \in S_{m_{ij}n_i}$ such that

$$t_{ij} = (\psi^\flat(s'_{ij})|X_{f_i})(\psi^\flat(s_i)|X_{f_i})^{-m_{ij}}.$$

Also by hypothesis, there exists some λ and elements $s_{i\lambda} \in S^\lambda_{n_i}$ and $s'_{ij\lambda} \in S^\lambda_{m_{ij}n_i}$ whose images under ϕ_λ are s_i and s'_{ij}, respectively; it is clear that $r_{\mathscr{L},\psi_\lambda}$ satisfies conditions (i), (ii), and (iii) of (3.8.2). □

Proposition (3.8.5). — *Let Y be a quasi-compact scheme, or a prescheme whose underlying space is Noetherian, $q : X \to Y$ a morphism* of finite type, *$\mathscr{L}$ an invertible $\mathscr{O}_X$-module, $\mathscr{S}$ a quasi-coherent graded $\mathscr{O}_Y$-algebra, and $\psi : q^*(\mathscr{S}) \to \bigoplus_{n\geqslant 0} \mathscr{L}^{\otimes n}$ a homomorphism of graded algebras. For $r_{\mathscr{L},\psi}$ to be everywhere defined and an immersion, it is necessary and sufficient that there exist an integer $n > 0$ and a quasi-coherent sub-$\mathscr{O}_Y$-module* of finite type *$\mathscr{E}$ of $\mathscr{S}_n$ such that*

a) the homomorphism $\psi_n \circ q^*(j_n) : q^*(\mathscr{E}) \to \mathscr{L}^{\otimes n}$ (where $j_n : \mathscr{E} \to \mathscr{S}$ is the canonical injection) is surjective; and

b) if we denote by $\mathscr{S}'$ the (graded) sub-$\mathscr{O}_Y$-algebra of $\mathscr{S}$ generated by $\mathscr{E}$, and by ψ' the homomorphism $\psi \circ q^*(j')$ (where j' is the injection of $\mathscr{S}'$ into $\mathscr{S}$), $r_{\mathscr{L},\psi'}$ is everywhere defined and an immersion.

If this is the case, then every quasi-coherent sub-$\mathscr{O}_Y$-module $\mathscr{E}'$ of $\mathscr{S}_n$ that contains $\mathscr{E}$ also possesses the same property, as does the image of $\otimes^k\mathscr{E}$ in $\mathscr{S}_{kn}$ for all $k > 0$.

PROOF. The fact that the condition is sufficient, and the last two claims, are particular cases of (3.8.3), taking into account the canonical isomorphism between $\operatorname{Proj}(\mathscr{S})$ and $\operatorname{Proj}(\mathscr{S}^{(k)})$ (3.1.8).

Let (U_i) be a finite cover of Y by affine opens, and set $A_i = A(U_i)$. Since $q^{-1}(U_i)$ is compact, the hypothesis that $r_{\mathscr{L},\psi}$ be everywhere defined and an immersion implies, along with (3.8.2), the existence of a finite family $(s_{ij} \in S^{(i)}_{n_{ij}})$ of elements of $S^{(i)} = \Gamma(U_i, \mathscr{S})$ satisfying conditions (i), (ii), and

(iii). We see, as in the proof of (3.8.4), that the morphism $X_{f_{ij}} \to U_i$ (where $f_{ij} = \psi^\flat(s_{ij})$) is of finite type, and that the ring B_{ij} of $X_{f_{ij}}$ is thus an A_i-algebra of finite type, having a system of generators of the form $(\psi^\flat(t_{ijk})X_{f_{ij}})(f_{ij}|X_{f_{ij}})^{-m_{ijk}}$, where $t_{ijk} \in S^{(i)}_{m_{ijk}n_{ij}}$. Let n be a common multiple of the $m_{ijk}n_{ij}$; replacing (for each (i,j,k)) s_{ij} by some power s_{ij}^ρ such that $\rho m_{ijk}n_{ij} = n$, and multiplying t_{ijk} by $s_{ij}^{\rho - m_{ijk}}$, we can assume that, for each i, the s_{ij} and t_{ijk} belong to $S^{(i)}_n$ and that $m_{ijk} = 1$. Let E_i be the sub-A_i-module of $S^{(i)}$ generated by these elements (for fixed i). Then there exists a coherent sub-$\mathscr{O}_Y$-module $\mathscr{E}_i$ of $\mathscr{S}_n$ of finite type such that $\mathscr{E}_i|U_i = (E_i)^\sim$ (I, 9.4.7). It is clear that the sub- II | 71
$\mathscr{O}_Y$-module $\mathscr{E}$ of $\mathscr{S}_n$ given by the sum of the $\mathscr{E}_i$ is the desired object (since each section f_{ij} is such that, for all $x \in X_{f_{ij}}$, there exists an affine neighbourhood $V \subset X_{f_{ij}}$ of x such that $f|V$ is a basis for $\Gamma(V, \mathscr{L}^{\otimes n})$). □

§4. Projective bundles; ample sheaves

4.1. Definition of projective bundles

Definition (4.1.1). — Let Y be a prescheme, $\mathscr{E}$ a quasi-coherent $\mathscr{O}_Y$-module, and $\mathbf{S}_{\mathscr{O}_Y}(\mathscr{E})$ the symmetric $\mathscr{O}_Y$-algebra of $\mathscr{E}$ (1.7.4), which is quasi-coherent (1.7.7). We define the *projective bundle on Y defined by $\mathscr{E}$*, denoted $\mathbf{P}(\mathscr{E})$, to be the Y-scheme $P = \operatorname{Proj}(\mathbf{S}_{\mathscr{O}_Y}(\mathscr{E}))$. The $\mathscr{O}_P$-module $\mathscr{O}_P(1)$ is called the *fundamental sheaf on P*.

When Y is affine of ring A, then we have $\mathscr{E} = \widetilde{E}$ for some A-module E, and we then write $\mathbf{P}(E)$ instead of $\mathbf{P}(\widetilde{E})$.

When we take $\mathscr{E} = \mathscr{O}_Y^n$, we write $\mathbf{P}_Y^{n-1}$ instead of $\mathbf{P}(\mathscr{E})$; if, further, Y is affine of ring A, then we also write $\mathbf{P}_A^{n-1}$ instead of $\mathbf{P}_Y^{n-1}$. Since $\mathbf{S}_{\mathscr{O}_Y}(\mathscr{O}_Y)$ is canonically identified with $\mathscr{O}_Y[T]$ (1.7.4), $\mathbf{P}_Y^0$ is canonically identified with Y (3.1.7); Example (2.4.3) is then exactly $\mathbf{P}_K^1$.

(4.1.2). Let $\mathscr{E}$ and $\mathscr{F}$ be quasi-coherent $\mathscr{O}_Y$-modules; let $u : \mathscr{E} \to \mathscr{F}$ be an $\mathscr{O}_Y$-homomorphism; there is a canonically corresponding homomorphism $\mathbf{S}(u) : \mathbf{S}_{\mathscr{O}_Y}(\mathscr{E}) \to \mathbf{S}_{\mathscr{O}_Y}(\mathscr{F})$ of graded $\mathscr{O}_Y$-algebras (1.7.4). If u is *surjective*, then so too is $\mathbf{S}(u)$, and thus (3.6.2, (i)) $\operatorname{Proj}(\mathbf{S}(u))$ is a *closed immersion* $\mathbf{P}(\mathscr{F}) \to \mathbf{P}(\mathscr{E})$, which we denote by $\mathbf{P}(u)$. We can thus say that $\mathbf{P}(\mathscr{E})$ is a *contravariant functor* in $\mathscr{E}$, with the condition that we only consider *surjective* morphisms of quasi-coherent $\mathscr{O}_Y$-modules.

Still supposing that u is surjective, and letting $P = \mathbf{P}(\mathscr{E})$, $Q = \mathbf{P}(\mathscr{F})$, and $j = \mathbf{P}(u)$, we have, up to isomorphism, that

(4.1.2.1) $$j^*(\mathscr{O}_P(n)) = \mathscr{O}_Q(n) \qquad \text{for all } n \in \mathbf{Z}$$

by (3.6.3).

(4.1.3). Now let $\psi : Y' \to Y$ be a morphism, and let $\mathscr{E}' = \psi^*(\mathscr{E})$; then $\mathbf{S}_{\mathscr{O}_{Y'}}(\mathscr{E}') = \psi^*(\mathbf{S}_{\mathscr{O}_Y}(\mathscr{E}))$ (1.7.5); thus (3.5.3)

(4.1.3.1) $$\mathbf{P}(\psi^*(\mathscr{E})) = \mathbf{P}(\mathscr{E}) \times_Y Y'$$

up to canonical isomorphism; furthermore, we clearly have that

$$\psi^*((\mathbf{S}_{\mathscr{O}_Y}(\mathscr{E}))(n)) = (\mathbf{S}_{\mathscr{O}_{Y'}}(\mathscr{E}'))(n)$$

for all $n \in \mathbf{Z}$, whence, letting $P = \mathbf{P}(\mathscr{E})$ and $P' = \mathbf{P}(\mathscr{E}')$, we have (3.5.4), up to isomorphism, that

(4.1.3.2) $$\mathscr{O}_{P'}(n) = \mathscr{O}_p(n) \otimes_Y \mathscr{O}_{Y'} \qquad \text{for all } n \in \mathbf{Z}.$$

II | 72

Proposition (4.1.4). — *Let $\mathscr{L}$ be an invertible $\mathscr{O}_Y$-module. For every quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$, there exists a canonical Y-isomorphism $i_{\mathscr{L}} : \mathbf{P}(\mathscr{E}) \xrightarrow{\sim} \mathbf{P}(\mathscr{E} \otimes \mathscr{L})$; furthermore, if we let $P = \mathbf{P}(\mathscr{E})$ and $Q = \mathbf{P}(\mathscr{E} \otimes \mathscr{L})$, then $i_{\mathscr{L}}^*(\mathscr{O}_Q(n))$ is canonically isomorphic to $\mathscr{O}_P(n) \otimes_Y \mathscr{L}^{\otimes n}$ for all $n \in \mathbf{Z}$.*

Proof. Note first of all that, if A is a ring, E an A-module, and L a *free monogenous* A-module, then we can canonically define a homomorphism of A-modules

$$\mathbf{S}_n(E \otimes L) \longrightarrow \mathbf{S}_n(E) \otimes L^{\otimes n}$$

by sending $(x_1 \otimes y_1) \ldots (x_n \otimes y_n)$ to the element

$$(x_1 x_2 \ldots x_n) \otimes (y_1 \otimes y_2 \otimes \ldots \otimes y_n) \qquad (x_i \in E, y_i \in L, \text{for } i \leqslant i \leqslant n);$$

we can immediately see (by restricting to the case where $L = A$) that this homomorphism is in fact an isomorphism. We thus obtain a canonical isomorphism of graded A-algebras $\mathbf{S}_A(E \otimes L) \xrightarrow{\sim} \bigoplus_{n \geqslant 0} \mathbf{S}_n(E) \otimes L^{\otimes n}$. By returning to the conditions of (4.1.4), the above remarks allow us to define a canonical isomorphism of graded $\mathscr{O}_Y$-algebras

$$\mathbf{S}_{\mathscr{O}_Y}(\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{L}) \xrightarrow{\sim} \bigoplus_{n \geqslant 0} \mathbf{S}_n(\mathscr{E}) \otimes_{\mathscr{O}_Y} \mathscr{L}^{\otimes n} \tag{4.1.4.1}$$

by defining this isomorphism as an isomorphism of presheaves, and taking into account (1.7.4), (**I**, 1.3.9), and (**I**, 1.3.12). The proposition then follows from (3.1.8, (iii)) and (3.2.10). □

(4.1.5). With the hypotheses of (4.1.1), let $P = \mathbf{P}(\mathscr{E})$, and denote by p the structure morphism $P \to Y$. Since, by definition, $\mathscr{E} = (\mathbf{S}_{\mathscr{O}_Y}(\mathscr{E}))_1$, we have a canonical homomorphism $\alpha_1 : \mathscr{E} \to p_*(\mathscr{O}_P(1))$ (3.3.2.2), and thus (**0**, 4.4.3) also a canonical homomorphism

$$\alpha_1^\sharp : p^*(\mathscr{E}) \longrightarrow \mathscr{O}_P(1). \tag{4.1.5.1}$$

Proposition (4.1.6). — *The canonical homomorphism* (4.1.5.1) *is surjective.*

PROOF. We have seen, in (3.3.2), that $\alpha_1^\sharp$ corresponds functorially to the canonical homomorphism $\mathscr{E} \otimes_{\mathscr{O}_Y} \mathbf{S}_{\mathscr{O}_Y}(\mathscr{E}) \to (\mathbf{S}_{\mathscr{O}_Y}(\mathscr{E}))(1)$; since, by definition, $\mathscr{E}$ generates $\mathbf{S}_{\mathscr{O}_Y}(\mathscr{E})$, this homomorphism is surjective, whence the conclusion, by (3.2.4) □

4.2. Morphisms from a prescheme to a projective bundle

(4.2.1). Keeping the notation of (4.1.5), let X be a Y-prescheme, $q : X \to Y$ the structure morphism, and let $r : X \to P$ be a Y-*morphism* such that the following diagram commutes:

$$\begin{tikzcd} P \arrow[d, "p"'] & X \arrow[l, "r"'] \arrow[dl, "q"] \\ Y & \end{tikzcd}$$

II | 73

Since the functor r^* is right exact (**0**, 4.3.1), we obtain, from the surjective homomorphism in (4.1.5.1), a surjective homomorphism

$$r^*(\alpha_1^\sharp) : r^*(p^*(\mathscr{E})) \longrightarrow r^*(\mathscr{O}_P(1)).$$

But $r^*(p^*(\mathscr{E})) = q^*(\mathscr{E})$, and $r^*(\mathscr{O}_P(1))$ is locally isomorphic to $r^*(\mathscr{O}_P) = \mathscr{O}_X$, or, in other words, the latter is an *invertible* sheaf $\mathscr{L}_r$ on $\mathscr{O}_X$, and so we have defined, given r, a canonical surjective $\mathscr{O}_X$-homomorphism

$$\phi_r : q^*(\mathscr{E}) \longrightarrow \mathscr{L}_r. \tag{4.2.1.1}$$

When $Y = \operatorname{Spec}(A)$ is affine, and $\mathscr{E} = \widetilde{E}$, we can further clarify this homomorphism in the following way: given $f \in E$, it follows from (2.6.3) that

$$r^{-1}(D_+(f)) = X_{\phi_r^\flat(f)}. \tag{4.2.1.2}$$

Now let V be an affine open subset of X that is contained inside $r^{-1}(D_+(f))$, and let B be its ring, which is an A-algebra; let $S = \mathbf{S}_A(E)$; the restriction of r to V corresponds to an A-homomorphism $\omega : \mathbf{S}_f \to B$, and we have that $q^*(\mathscr{E})|V = (E \otimes_A B)^\sim$ and $\mathscr{L}_r|V = \widetilde{L_r}$, whence $L_r = (S(1))_{(f)} \otimes_{S_{(f)}} B_{[\omega]}$ (**I**, 1.6.5). The restriction of ϕ_r to $q^*(\mathscr{E})|V$ corresponds to the B-homomorphism $u : E \otimes_A B \to L_r$, which sends $x \otimes 1$ to $(x/1) \otimes f = (f/1) \otimes \omega(x/f)$. The canonical extension of ϕ_r to a homomorphism of $\mathscr{O}_X$-algebras

$$\psi_r : q^*(\mathbf{S}(\mathscr{E})) = \mathbf{S}(q^*(\mathscr{E})) \longrightarrow \mathbf{S}(\mathscr{L}_r) = \bigoplus_{n \geqslant 0} \mathscr{L}_r^{\otimes n}$$

is thus such that the restriction of ψ_r to $q^*(\mathbf{S}_n(\mathscr{E}))|V$ corresponds to the homomorphism $\mathbf{S}_n(\mathscr{E} \otimes_A B) = \mathbf{S}_n(E) \otimes_A B \to L_r^{\otimes n}$ that sends $s \otimes 1$ to $(f/1)^{\otimes n} \otimes \omega(s/f^n)$.

(4.2.2). Conversely, suppose that we are given a morphism $q : X \to Y$, an invertible $\mathscr{O}_X$-module $\mathscr{L}$, and a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$; to each homomorphism $\phi : q^*(\mathscr{E}) \to \mathscr{L}$ there canonically corresponding homomorphism of quasi-coherent $\mathscr{O}_X$-algebras

$$\psi : \mathbf{S}(q^*(\mathscr{E})) = q^*(\mathbf{S}(\mathscr{E})) \longrightarrow \bigoplus_{n\geqslant 0} \mathscr{L}^{\otimes n}$$

and thus (3.7.1) a Y-morphism $r_{\mathscr{L},\psi} : G(\psi) \to \operatorname{Proj}(\mathbf{S}(\mathscr{E})) = \mathbf{P}(\mathscr{E})$, which we denote by $r_{\mathscr{L},\phi}$. If ϕ is *surjective*, then so too is ψ, and thus (3.7.4) $r_{\mathscr{L},\phi}$ is *everywhere defined*. Furthermore, with the notation of (4.2.1) and (4.2.2):

Proposition (4.2.3). — *Given a morphism $q : X \to Y$ and a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$, maps $r \to (\mathscr{L}_r, \phi_r)$ and $(\mathscr{L}, \phi) \to r_{\mathscr{L},\phi}$ give a bijective correspondence between the set of Y-morphisms $r : X \to \mathbf{P}(\mathscr{E})$ and the set of equivalence classes of pairs $(\mathscr{L}, \phi)$ of an invertible $\mathscr{O}_X$-module $\mathscr{L}$ and a surjective homomorphism $\phi : q^*(\mathscr{E}) \to \mathscr{L}$, where such pairs $(\mathscr{L}, \phi)$ and $(\mathscr{L}', \phi')$ are defined to be equivalent if there exists an $\mathscr{O}_X$-isomorphism $\tau : \mathscr{L} \xrightarrow{\sim} \mathscr{L}'$ such that $\phi' = \tau \circ \phi$.*

Proof. Start first with a Y-morphism $r : X \to \mathbf{P}(\mathscr{E})$, and construct $\mathscr{L}_r$ and ϕ_r (4.2.1), and let $r' = r_{\mathscr{L}_r,\phi_r}$; it follows immediately from (4.2.1) and (3.7.2) that the morphisms r and r' are identical (by taking the generator of $\mathscr{L}_r$ to be the element $(f/1) \otimes 1$ to define the homomorphisms v_n of (3.7.2)). Conversely, take a pair $(\mathscr{L}, \phi)$ and construct $r'' = r_{\mathscr{L},\phi}$, and then $\mathscr{L}_{r''}$ and $\phi_{r''}$; we will show **II | 74**
that there exists a canonical isomorphism $\tau : \mathscr{L}_{r''} \xrightarrow{\sim} \mathscr{L}$ such that $\phi = \tau \circ \phi_{r''}$; to define it, we can restrict to the case where $Y = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ are affine, and (with the notation of (4.2.1) and (3.7.2)) associate to each element $(x/1) \otimes 1$ of $L_{r''}$ (where $x \in E$) the element $v_1(x)c$ of L. We immediately see that τ does not depend on the chosen generator c of L; since v_1 is surjective by hypothesis, to prove that τ is an isomorphism it suffices to to show that, if $x/1 = 0$ in $(S(1))_{(f)}$, then $v_1(x)/1 = 0$ in B_g; but the first equality implies that $f^n x = 0$ in $\mathbf{S}_{n+1}(E)$ for some n, and this implies that $v_{n+1}(f^n x) = g^n v_1(x) = 0$ in B, whence the conclusion. Finally, it is immediate that, for any two equivalent pairs $(\mathscr{L}, \phi)$ and $(\mathscr{L}', \phi')$, we have $r_{\mathscr{L},\phi} = r_{\mathscr{L}',\phi'}$. ☐

In particular, for $X = Y$:

Theorem (4.2.4). — *The set of Y-sections of $\mathbf{P}(\mathscr{E})$ is in canonical bijective correspondence with the set of quasi-coherent sub-$\mathscr{O}_Y$-modules $\mathscr{F}$ of $\mathscr{E}$ such that $\mathscr{E}/\mathscr{F}$ is invertible.*

We note that this property corresponds to the classical definition of "projective space" as the set of hyperplanes of a vector space (the classical case corresponding to $Y = \operatorname{Spec}(K)$, where K is a field, and $\mathscr{E} = \widetilde{E}$, where E is a finite-dimensional K-vector space; the $\mathscr{F}$ having the property described in (4.2.4) then correspond to the hyperplanes of E, and we already know that the Y-sections of $\mathbf{P}(\mathscr{E})$ are then the *K-rational points of* $\mathbf{P}(\mathscr{E})$ (**I**, 3.4.5)).

Remark (4.2.5). — Since there is a canonical bijective correspondence between Y-morphisms from X to P and their graph morphisms, X-sections of $P \times_Y X$ (**I**, 3.3.14), we see that, conversely, (4.2.3) can be deduced from (4.2.4). Denote by $\operatorname{Hyp}_Y(X, \mathscr{E})$ the set of quasi-coherent sub-$\mathscr{O}_X$-modules $\mathscr{F}$ of $\mathscr{E} \otimes_Y \mathscr{O}_X = q^*(\mathscr{E})$ such that $q^*(\mathscr{E})/\mathscr{F}$ is an invertible $\mathscr{O}_X$-module. If $g : X' \to X$ is a Y-morphism, then it follows from the fact that g^* is right exact that $g^*q^*(\mathscr{E})/\mathscr{F}) = g^*q^*(\mathscr{E}))/g^*(\mathscr{F})$, and so the latter sheaf is invertible, and thus $\operatorname{Hyp}_Y(X, \mathscr{E})$ is a *contravariant functor* into the category of Y-preschemes. We can thus interpret the theorem (4.2.4) as defining a *canonical isomorphism* of functors $\operatorname{Hom}_Y(X, \mathbf{P}(\mathscr{E}))$ and $\operatorname{Hyp}_Y(X, \mathscr{E})$, where both functors are contravariant in the variable X and map into the category of Y-preschemes. This also gives a characterisation of the projective bundle $P = \mathbf{P}(\mathscr{E})$ by the following *universal property*, which is much closer to the geometric intuition than the constructions from §§2–3: for every morphism $q : X \to Y$ and every invertible $\mathscr{O}_X$-module $\mathscr{L}$ that is a quotient of $\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{O}_X$, there exists a unique Y-morphism $r : X \to P$ such that $\mathscr{L} = r^*(\mathscr{O}_P(1))$.

We will see later that we can similarly define, amongst other things, "Grassmannian" schemes.

Corollary (4.2.6). — *Suppose that every invertible $\mathscr{O}_Y$-module is trivial* (**I**, 2.4.8)*. Let V be the group $\operatorname{Hom}_{\mathscr{O}_Y}(\mathscr{E}, \mathscr{O}_Y)$, considered as a module over the ring $A = \Gamma(Y, \mathscr{O}_Y)$, and let V^* be the subset of V consisting of surjective homomorphisms. Then the set of Y-sections of $\mathbf{P}(\mathscr{E})$ is canonically identified with V^*/A^*, where A^* is the group of units of A.*

In particular: **II** | 75

(1) The corollary (4.2.6) applies whenever Y is a *local scheme* (**I**, 2.4.8). Let Y be an arbitrary prescheme, y a point of Y, and $Y' = \operatorname{Spec}(k(y))$; then the fibre $p^{-1}(y)$ of $\mathbf{P}(\mathscr{E})$ can, by (4.1.3.1), be identified with $\mathbf{P}(\mathscr{E}^y)$, where $\mathscr{E}^y = \mathscr{E}_y \otimes_{\mathscr{O}_y} k(y) = \mathscr{E}_y/\mathfrak{m}_y\mathscr{E}_y$ is considered as a vector space over $k(y)$. More generally, if K is an extension of $k(y)$, then $p^{-1}(y) \otimes_{k(y)} K$ can be identified with $\mathbf{P}(\mathscr{E}^y \otimes_{k(y)} K)$. The corollary (4.2.6) then shows that the set of *geometric points of* $\mathbf{P}(\mathscr{E})$ *with values in the extension* K *of* $k(y)$ (**I**, 3.4.5), which we can also call the *rational geometric fibre over* K *of* $\mathbf{P}(\mathscr{E})$ *over the point* y, can be identified with the *projective space* associated to the *dual* of the K-vector space $\mathscr{E}^y \otimes_{k(y)} K$.

(2) Suppose that Y is affine of ring A, and, further, that every invertible $\mathscr{O}_Y$-module is trivial; further, take $\mathscr{E} = \mathscr{O}_Y^n$; then, in (4.2.6), V can be identified with A^n (**I**, 1.3.8), and V^* with the sets of systems $(f_i)_{1\leqslant i\leqslant n}$ of elements of A that generate the ideal A; any two such systems define the same Y-section of $\mathbf{P}_Y^{n-1} = \mathbf{P}_A^{n-1}$, or, in other words, *the same point of* $\mathbf{P}_A^{n-1}$ *with values in* A, if and only if one of them can be obtained from the other by multiplication by an invertible element of A.

These properties justify the terminology "projective bundle" for $\mathbf{P}(\mathscr{E})$. We note that the definitions that we will similarly obtain for "projective space" is in fact *dual* to the classical definition; this is imposed upon us by the necessity of being able to define $\mathbf{P}(\mathscr{E})$ for *arbitrary* quasi-coherent $\mathscr{O}_Y$-modules $\mathscr{E}$, and not just locally free ones.

Remark (4.2.7). — We will see, in Chapter V, that, if Y is connected and locally Noetherian, and if $\mathscr{E}$ is locally free, then, letting $P = \mathbf{P}(\mathscr{E})$, every invertible $\mathscr{O}_P$-module is isomorphic to an $\mathscr{O}_P$-module of the form $\mathscr{L}' \otimes_{\mathscr{O}_Y} \mathscr{O}_P(m)$, with $\mathscr{L}'$ some invertible $\mathscr{O}_Y$-module, well defined up to isomorphism, and m some well defined integer. In other words, $\mathrm{H}^1(P, \mathscr{O}_P^*)$ is isomorphic to $\mathbf{Z} \times \mathrm{H}^1(Y, \mathscr{O}_Y^*)$ (**0**, 5.4.7). We will also see ((**III**, 2.1.14), taking (**0**, 5.4.10) into account) that $p_*(\mathscr{L}^{\otimes m}) = 0$ if $m < 0$, and $p_*(\mathscr{L}^{\otimes m})$ is isomorphic to $\mathscr{L}' \otimes_{\mathscr{O}_Y} (\mathbf{S}_{\mathscr{O}_Y}(\mathscr{E}))_m$ if $m \geqslant 0$. If $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_Y$-module, then every Y-morphism $\mathbf{P}(\mathscr{E}) \to \mathbf{P}(\mathscr{F})$ is determined by the data of an invertible $\mathscr{O}_Y$-module, an integer $m \geqslant 0$, and an $\mathscr{O}_Y$-homomorphism $\psi : \mathscr{F} \to \mathscr{L}' \otimes_{\mathscr{O}_Y} (\mathbf{S}_{\mathscr{O}_Y}(\mathscr{E}))_m$ such that the corresponding homomorphism $\psi^\sharp$ of $\mathscr{O}_{\mathbf{P}(\mathscr{F})}$-modules is surjective. We will also see that, if the Y-morphism in question is an isomorphism, then $m = 1$ and $\mathscr{F}$ is isomorphic to $\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{L}'$ (the converse of (4.1.4)). This will allow us to determine the sheaf of germs of automorphisms of $\mathbf{P}(\mathscr{E})$ as the quotient of the sheaf of groups $\mathscr{A}ut(\mathscr{E})$ (which is locally isomorphic to $\mathbf{GL}(n, \mathscr{O}_Y)$ is $\mathscr{E}$ is of rank n) by $\mathscr{O}_Y^*$.

(4.2.8). Keeping the notation of (4.2.1), let $u : X' \to X$ be a morphism; if the Y-morphism $r : X \to P$ corresponds to the homomorphism $\phi : q^*(\mathscr{E}) \to \mathscr{L}$, then the Y-morphism $r \circ u$ corresponds to $u^*(\phi) : u^*(q^*(\mathscr{E})) \to u^*(\mathscr{L})$, as follows immediately from the definitions.

(4.2.9). Let $\mathscr{E}$ and $\mathscr{F}$ be quasi-coherent $\mathscr{O}_Y$-modules, $v : \mathscr{E} \to \mathscr{F}$ a surjective homomorphism, and $j = \mathbf{P}(v)$ the corresponding closed immersion $\mathbf{P}(\mathscr{F}) \to \mathbf{P}(\mathscr{E})$ (4.1.2). If the Y-morphism $r : X \to \mathbf{P}(\mathscr{F})$ corresponds to the homomorphism $\phi : q^*(\mathscr{F}) \to \mathscr{L}$, then the Y-morphism $j \circ r$ **II** | 76
corresponds to $q^*(\mathscr{E}) \xrightarrow{q^*(v)} q^*(\mathscr{F}) \xrightarrow{\phi} \mathscr{L}$; this again follows from the definition given in (4.2.1).

(4.2.10). Let $\psi : Y' \to Y$ be a morphism, and let $\mathscr{E}' = \psi^*(\mathscr{E})$. If the Y-morphism $r : X \to P$ corresponds to the homomorphism $\phi : q^*(\mathscr{E}) \to \mathscr{L}$, then the Y'-morphism

$$r_{(Y')} : X_{(Y')} \longrightarrow P' = \mathbf{P}(\mathscr{E}')$$

corresponds to $\phi_{(Y')} : q_{(Y')}^*(\mathscr{E}') = q^*(\mathscr{E}) \otimes_Y \mathscr{O}_{Y'} \to \mathscr{L} \otimes_Y \mathscr{O}_{Y'}$. Indeed, by (4.1.3.1), we have the commutative diagram

$$\begin{tikzcd}
Y' \arrow[d] & P' = P_{(Y')} \arrow[l, "p_{(Y')}"'] \arrow[d, "u"] & X_{(Y')} \arrow[l, "r_{(Y')}"'] \arrow[d, "v"] \\
Y & P \arrow[l, "p"] & X \arrow[l, "r"]
\end{tikzcd}$$

From (4.1.3.1), we have

$$(r_{(Y')})^*(\mathscr{O}_{P'}(1)) = (r_{(Y')})^*(u^*(\mathscr{O}_P(1))) = v^*(r^*(\mathscr{O}_P(1))) = v^*(\mathscr{L}) = \mathscr{L} \otimes_Y \mathscr{O}_{Y'};$$

we also know that $u^*(\alpha_1^\sharp)$ is exactly the canonical homomorphism $\alpha_1^\sharp : (p_{(Y')})^*(\mathscr{E}') \to \mathscr{O}_{P'}(1)$; we can see this by explicitly calculating the canonical homomorphisms $\alpha_1^\sharp$ to P and P' as in (4.1.6). Whence our claim.

4.3. The Segre morphism

(4.3.1). Let Y be a prescheme, and $\mathscr{E}$ and $\mathscr{F}$ quasi-coherent $\mathscr{O}_Y$-modules; let $P_1 = \mathbf{P}(\mathscr{E})$ and $P_2 = \mathbf{P}(\mathscr{F})$, and denote the structure morphisms by $p_1 : P_1 \to Y$ and $p_2 : P_2 \to Y$. Let $Q = P_1 \times_Y P_2$, and let $q_1 : Q \to P_1$ and $q_2 : Q \to P_2$ be the canonical projections; then the $\mathscr{O}_Q$-module $\mathscr{L} = \mathscr{O}_{P_1}(1) \otimes_Y \mathscr{O}_{P_2}(1) = q_1^*(\mathscr{O}_{P_1}(1)) \otimes_{\mathscr{O}_Q} q_2^*(\mathscr{O}_{P_2}(1))$ is invertible, since it is the tensor product of of two invertible $\mathscr{O}_Q$-modules (**0**, 5.4.4). Also, if $r = p_1 \circ q_1 = p_2 \circ q_2$ is the structure morphism $Q \to Y$, then $r^*(\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{F}) = q_1^*(p_1^*(\mathscr{E})) \otimes_{\mathscr{O}_Q} q_2^*(p_2^*(\mathscr{F}))$ (**0**, 4.3.3); the canonical surjective homomorphisms (4.1.5.1) $p_1^*(\mathscr{E}) \to \mathscr{O}_{P_1}(1)$ and $p_2^*(\mathscr{F}) \to \mathscr{O}_{P_2}(1)$ thus give, by taking the tensor product, a canonical homomorphism

$$s : r^*(\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{F}) \longrightarrow \mathscr{L} \tag{4.3.1.1}$$

which is evidently surjective; from this we obtain (4.2.2) a canonical morphism, called the *Segre morphism*:

$$\varsigma : \mathbf{P}(\mathscr{E}) \times_Y \mathbf{P}(\mathscr{F}) \longrightarrow \mathbf{P}(\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{F}). \tag{4.3.1.2}$$

We can study the morphism ς more explicitly in the case where $Y = \operatorname{Spec}(A)$ is affine, and $\mathscr{E} = \widetilde{E}$ and $\mathscr{F} = \widetilde{F}$, where E and F are A-modules, whence $\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{F} = (E \otimes_A F)^\sim$ (**I**, 1.3.12); let $R = \mathbf{S}_A(E)$, $S = \mathbf{S}_A(F)$, and $T = \mathbf{S}_A(E \otimes_A F)$; let $f \in E$ and $g \in F$, and consider the affine open

$$D_+(f) \times_Y D_+(g) = \operatorname{Spec}(B)$$

of Q, where $B = R_{(f)} \otimes_A S_{(g)}$; the restriction of $\mathscr{L}$ to this affine open is $\widetilde{L}$, where II | 77

$$L = (R(1))_{(f)} \otimes_A (S(1))_{(g)}$$

and the element $c = (f/1) \otimes (g/1)$ is a generator of L considered as a free B-module (2.5.7). The homomorphism (4.3.1.1) corresponds to the homomorphism

$$(x \otimes y) \otimes b \longmapsto b((x/1) \otimes (y/1))$$

from $(E \otimes_A F) \otimes_A B$ to L. With the notation of (3.7.2), we thus have that $v_1(x \otimes y) = (x/f) \otimes (y/g)$; the restriction of ς to $D_+(f) \times_Y D_+(g)$ is a morphism from this affine scheme to $D_+(f \otimes g)$, corresponding to the ring homomorphism $\omega : T_{(f\otimes g)} \to R_{(f)} \otimes_A S_{(g)}$ defined by

$$\omega((x \otimes y)/(f \otimes g)) = (x/f) \otimes (y/g) \tag{4.3.1.3}$$

for $x \in E$ and $y \in F$.

(4.3.2). It follows from (4.2.3) that we have a canonical isomorphism

$$\tau : \varsigma^*(\mathscr{O}_P(1)) \xrightarrow{\sim} \mathscr{O}_{P_1}(1) \otimes_Y \mathscr{O}_{P_2}(1) \tag{4.3.2.1}$$

where we let $P = \mathbf{P}(\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{F})$. We will show that, for $x \in \Gamma(Y, \mathscr{E})$ and $y \in \Gamma(Y, \mathscr{F})$, we have

$$\tau(\alpha_1(x \otimes y)) = \alpha_1(x) \otimes \alpha_1(y). \tag{4.3.2.2}$$

Indeed, we can restrict to the case where Y is affine, and we then have, with the notation of (4.3.1) and (2.6.2), that $\alpha_1^{f\otimes g}(x \otimes y) = (x \otimes y)/1$ in $(T(1))_{(f\otimes g)}$, that $\alpha_1^f(x) = x/1$ in $(R(1))_{(f)}$, and that $\alpha_1^g(y) = y/1$ in $(S(1))_{(g)}$. The definition of τ given in (4.2.3) and the calculation of v_1 done in (4.3.1) then immediately prove the claim (4.3.2.2). From this we obtain the equation

$$\varsigma^{-1}(P_{x\otimes y}) = (P_1)_x \times_Y (P_2)_y \tag{4.3.2.3}$$

with the notation of (3.1.4). Indeed, taking (3.3.3) into account, the equation (4.3.2.2) (by restricting to the affine case, with the help of (**I**, 3.2.7) and (**I**, 3.2.3)) leaves us only to prove the following lemma:

Lemma (4.3.2.4). — *Let B and B' be A-algebras, and let $Y = \operatorname{Spec}(A)$, $Z = \operatorname{Spec}(B)$, and $Z' = \operatorname{Spec}(B')$; then $D(t \otimes t') = D(t) \times_Y D(t')$ for any $t \in B$, $t' \in B'$.*

PROOF. Indeed, if $p : Z \times_Y Z' \to Z$ and $p' : Z \times_Y Z' \to Z'$ are the canonical projections, then it follows from (I, 1.2.2.2) that $p^{-1}(D(t)) = D(t \otimes 1)$ and $p'^{-1}(D(t')) = D(1 \otimes t')$; the conclusion follows from (I, 3.2.7) and (I, 1.1.9.1), since $(t \otimes 1)(1 \otimes t') = t \otimes t'$. □

Proposition (4.3.3). — *The Segre morphism is a closed immersion.*

PROOF. Since the question is local on Y, we can restrict to the case where Y is affine. With the notation of (4.3.1) and (4.3.1), the $D_+(f \otimes g)$ then form a basis for the topology of P, since the $f \otimes g$ generate T when f runs over E and g runs over F. By (4.3.2.3), we also know that $\varsigma^{-1}(D_+(f \otimes g)) = D_+(f) \times_Y D_+(g)$. It thus suffices (I, 4.2.4) to prove that the restriction of ς to $D_+(f) \times_Y D_+(g)$ is a closed immersion into $D_+(f \otimes g)$. But, with the same notation, the equation (4.3.1.3) shows that ω is *surjective*, which completes the proof. □

(4.3.4). The Segre morphism is *functorial* in $\mathscr{E}$ and $\mathscr{F}$, if we consider only *surjective* homomorphisms of quasi-coherent $\mathscr{O}_Y$-modules. Indeed, we must then show that, if $\mathscr{E} \to \mathscr{E}'$ is a surjective $\mathscr{O}_Y$-homomorphism, then the diagram **II | 78**

$$\begin{tikzcd}
\mathbf{P}(\mathscr{E}') \times \mathbf{P}(\mathscr{F}) \arrow[r, "j\times 1"] \arrow[d, "\varsigma"'] & \mathbf{P}(\mathscr{E}) \times \mathbf{P}(\mathscr{F}) \arrow[d, "\varsigma"] \\
\mathbf{P}(\mathscr{E}' \otimes \mathscr{F}) \arrow[r] & \mathbf{P}(\mathscr{E} \otimes \mathscr{F})
\end{tikzcd}$$

commutes, where j denotes the canonical closed immersion $\mathbf{P}(\mathscr{E}') \to \mathbf{P}(\mathscr{E})$. Let $P_1' = \mathbf{P}(\mathscr{E}')$ and keep the notation from (4.3.1); then $j \times 1$ is a closed immersion (I, 4.3.1) and, up to isomorphism,

$$(j \times 1)^*(\mathscr{O}_{P_1}(1) \otimes \mathscr{O}_{P_2}(1)) = j^*(\mathscr{O}_{P_1}(1)) \otimes \mathscr{O}_{P_2}(1) = \mathscr{O}_{P_1'}(1) \otimes \mathscr{O}_{P_2}(1)$$

by (4.1.2.1) and (I, 9.1.5); our claim then follows from (4.2.8) and (4.2.9).

(4.3.5). With the notation of (4.3.1), let $\psi : Y' \to Y$ be a morphism, and let $\mathscr{E}' = \psi^*(\mathscr{E})$ and $\mathscr{F}' = \psi^*(\mathscr{F})$; then the Segre morphism $\mathbf{P}(\mathscr{E}') \times \mathbf{P}(\mathscr{F}') \to \mathbf{P}(\mathscr{E}' \otimes \mathscr{F}')$ can be identified with $\varsigma_{(Y')}$. Indeed, keeping the notation of (4.3.1), let $P_1' = \mathbf{P}(\mathscr{E}')$ and $P_2 = \mathbf{P}(\mathscr{F}')$; we know (4.1.3.1) that P_i' can be identified with $(P_i)_{(Y')}$ $(i = 1, 2)$, and so the structure morphism $P_1' \times P_2' \to Y'$ can be identified with $r_{(Y')}$. Also $\mathscr{E}' \otimes \mathscr{F}'$ can be identified with $\psi^*(\mathscr{E} \otimes \mathscr{F})$, and so $\mathbf{P}(\mathscr{E}' \otimes \mathscr{F}')$ can be identified with $(\mathbf{P}(\mathscr{E} \otimes \mathscr{F}))_{(Y')}$ (4.1.3.1). Finally, $\mathscr{O}_{P_1'}(1) \otimes_Y \mathscr{O}_{P_2'}(1) = \mathscr{L}'$ can be identified with $\mathscr{L} \otimes_Y \mathscr{O}_{Y'}$, by (4.1.3.1) and (I, 9.1.11). The canonical homomorphism $(r_{(Y)})^*(\mathscr{E}' \otimes \mathscr{F}') \to \mathscr{L}'$ can then be identified with $s_{(Y')}$, and our claim follows from (4.2.10).

Remark (4.3.6). — The prescheme given by the *sum* of $\mathbf{P}(\mathscr{E})$ and $\mathbf{P}(\mathscr{F})$ is even canonically isomorphic to a *closed subprescheme of* $\mathbf{P}(\mathscr{E} \oplus \mathscr{F})$. Indeed, the surjective homomorphisms $\mathscr{E} \oplus \mathscr{F} \to \mathscr{E}$ and $\mathscr{E} \oplus \mathscr{F} \to \mathscr{F}$ correspond to closed immersions $\mathbf{P}(\mathscr{E}) \to \mathbf{P}(\mathscr{E} \oplus \mathscr{F})$ and $\mathbf{P}(\mathscr{F}) \to \mathbf{P}(\mathscr{E} \oplus \mathscr{F})$; everything then reduces to showing that the underlying spaces of the closed subpreschemes of $\mathbf{P}(\mathscr{E} \oplus \mathscr{F})$ obtained in this way have empty intersection. Since the question is local on Y, we can adopt the notation of (4.3.1); but $\mathbf{S}_n(E)$ and $\mathbf{S}_n(F)$ can be identified with submodules of $\mathbf{S}_n(E \oplus F)$ with intersection consisting only of 0; if $\mathfrak{p}$ is a graded prime ideal of $\mathbf{S}(E)$ such that $\mathfrak{p} \cap \mathbf{S}_n(E) \neq \mathbf{S}_n(E)$ for any $n \geqslant 0$, then there exists a corresponding graded prime ideal of $\mathbf{S}(E \oplus F)$ whose intersection with $\mathbf{S}_n(E)$ is $\mathfrak{p} \cap \mathbf{S}_n(E)$, but who also *contains* $\mathbf{S}_+(F)$, as we immediately see; thus no point in $\operatorname{Proj}(\mathbf{S}(E))$ can have the same image in $\operatorname{Proj}(\mathbf{S}(E \oplus F))$ as any point in $\operatorname{Proj}(\mathbf{S}(F))$.

4.4. Immersions into projective bundles; very ample sheaves

Proposition (4.4.1). — *Let Y be a quasi-compact scheme, or a prescheme whose underlying space is Noetherian, $q : X \to Y$ a morphism* of finite type, *and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module.*

(i) *Let $\mathscr{S}$ be a quasi-coherent positively-graded $\mathscr{O}_Y$-algebra, and $\psi : q^*(\mathscr{S}) \to \bigoplus_{n\geqslant 0} \mathscr{L}^{\otimes n}$ a homomorphism of graded algebras. For $r_{\mathscr{L},\psi}$ to be everywhere defined and an immersion, it is necessary and sufficient for there to exist an integer $n \geqslant 0$ and a quasi-coherent sub-$\mathscr{O}_Y$-module* of finite type **II** | 79
$\mathscr{E}$ of $\mathscr{S}_n$ such that the homomorphism $\psi' = \psi_n \circ q^*(j) : q^*(\mathscr{E}) \to \mathscr{L}^{\otimes n} = \mathscr{L}'$ (where j is the injection $\mathscr{E} \to \mathscr{S}_n$) is surjective and such that the morphism $r_{\mathscr{L}',\psi'} : X \to \mathbf{P}(\mathscr{E})$ is an immersion.

(ii) *Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_Y$-module, and $\phi : q^*(\mathscr{F}) \to \mathscr{L}$ a surjective homomorphism. For the morphism $r_{\mathscr{L},\phi}$ to be an immersion $X \to \mathbf{P}(\mathscr{F})$, it is necessary and sufficient for there to exist a quasi-coherent sub-$\mathscr{O}_Y$-module* of finite type *$\mathscr{E}$ of $\mathscr{F}$ such that the homomorphism $\phi' = \phi \circ q(j) : q^*(\mathscr{E}) \to \mathscr{L}$ (where j is the canonical injection $\mathscr{E} \to \mathscr{F}$) is surjective and such that the morphism $r_{\mathscr{L},\phi'} : X \to \mathbf{P}(\mathscr{E})$ is an immersion.*

Proof.

(i) The fact that $r_{\mathscr{L},\phi}$ is everywhere defined and is an immersion is equivalent, by (3.8.5), to the existence of some $n \geqslant 0$ and $\mathscr{E}$ such that, if $\mathscr{S}'$ is the subalgebra of $\mathscr{S}$ generated by $\mathscr{E}$, the homomorphism $q^*(\mathscr{E}) \to \mathscr{L}^{\otimes n}$ is surjective and the morphism $r_{\mathscr{L},\psi'} : X \to \operatorname{Proj}(\mathscr{S}')$ is everywhere defined and is an immersion. We already have a canonical surjective homomorphism $\mathbf{S}(\mathscr{E}) \to \mathscr{S}'$ to which there exists a corresponding closed immersion $\operatorname{Proj}(\mathscr{S}') \to \mathbf{P}(\mathscr{E})$ (3.6.2); whence the conclusion.

(ii) Since $\mathscr{F}$ is the inductive limit of its quasi-coherent submodules of finite type $\mathscr{E}_\lambda$ (**I**, 9.4.9), $\mathbf{S}(\mathscr{F})$ is the inductive limit of the $\mathbf{S}(\mathscr{E}_\lambda)$; the conclusion then follows from (3.8.4), by observing that we can take all the n_i in the proof of (3.8.4) to be equal to 1: indeed, supposing that Y is affine, if $r = r_{\mathscr{L},\phi}$ is an immersion, then $r(X)$ is a quasi-compact subspace of $\mathbf{P}(\mathscr{F})$ that we can cover by finitely many open subsets of $\mathbf{P}(\mathscr{F})$ of the form $D_+(f)$, with $f \in F$, such that $D_+(f) \cap r(X)$ is closed.

□

Definition (4.4.2). — Let Y be a prescheme, and $q : X \to Y$ a morphism. We say that an invertible $\mathscr{O}_X$-module $\mathscr{L}$ is *very ample for q*, or *relative to q* (or *very ample for* (or *relative to*) Y, or simply *very ample*, if q is clear from the context) if there exists a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ and a Y-immersion i from X to $P = \mathbf{P}(\mathscr{E})$ such that $\mathscr{L}$ is isomorphic to $i^*(\mathscr{O}_P(1))$.

It is equivalent (4.2.3) to say that there exists a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ and a *surjective* homomorphism $\phi : q^*(\mathscr{E}) \to \mathscr{L}$ such that $r_{\mathscr{L},\phi} : X \to \mathbf{P}(\mathscr{E})$ is an *immersion*.

We note that the existence of a very ample (for Y) $\mathscr{O}_X$-module implies that q is *separated* ((3.1.3) and (**I**, 5.5.1, (i) and (ii))).

Corollary (4.4.3). — *Suppose that there exists a quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S}$, generated by $\mathscr{S}_1$, and a Y-immersion $i : X \to P = \operatorname{Proj}(\mathscr{S})$ such that $\mathscr{L}$ is isomorphic to $i^*(\mathscr{O}_P(1))$; then $\mathscr{L}$ is very ample relative to q.*

Proof. If $\mathscr{F} = \mathscr{S}_1$, then the canonical homomorphism $\mathbf{S}(\mathscr{F}) \to \mathscr{S}$ is surjective, and so, by compositing with the corresponding closed immersion $\operatorname{Proj}(\mathscr{S}) \to \mathbf{P}(\mathscr{F})$ (3.6.2) and the immersion i, we obtain an immersion $j : X \to \mathbf{P}(\mathscr{F}) = P'$ such that $\mathscr{L}$ is isomorphic to $j^*(\mathscr{O}_{P'}(1))$ (3.6.3). □

Proposition (4.4.4). — *Let $q : X \to Y$ be a quasi-compact morphism, and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module. Then the following properties are equivalent:*

(a) *$\mathscr{L}$ is very ample relative to q.*

(b) *$q_*(\mathscr{L})$ is quasi-coherent, the canonical homomorphism $\sigma : q^*(q_*(\mathscr{L})) \to \mathscr{L}$ is surjective, and the morphism $r_{\mathscr{L},\sigma} : X \to \mathbf{P}(q_*(\mathscr{L}))$ is an immersion.*

Proof. Since q is quasi-compact, we know that $q_*(\mathscr{L})$ is quasi-coherent if q is separated (**I**, 9.2.2).

II | 80

We know (3.4.7) that the existence of a surjective homomorphism $\phi : q^*(\mathscr{E}) \to \mathscr{L}$ (with $\mathscr{E}$ a quasi-coherent $\mathscr{O}_Y$-module) implies that σ is surjective; furthermore, given the factorisation $q^*(\mathscr{E}) \to q^*(q_*(\mathscr{L})) \xrightarrow{\sigma} \mathscr{L}$ of ϕ, there is a canonically corresponding factorisation

$$q^*(\mathbf{S}(\mathscr{E})) \longrightarrow q^*(\mathbf{S}(q_*(\mathscr{L}))) \longrightarrow \bigoplus_{n \geqslant 0} \mathscr{L}^{\otimes n}$$

and so (3.8.3) the hypothesis that $r_{\mathscr{L},\phi}$ is an immersion implies that so too is $j = r_{\mathscr{L},\sigma}$; furthermore (4.2.4), $\mathscr{L}$ is isomorphic to $j^*(\mathscr{O}_{P'}(1))$, where $P' = \mathbf{P}(q_*(\mathscr{L}))$. We thus see that (a) and (b) are equivalent. □

Corollary (4.4.5). — *Suppose that q is quasi-compact. For $\mathscr{L}$ to be very ample relative to Y, it is necessary and sufficient for there to exist an open cover (U_α) of Y such that $\mathscr{L}|q^{-1}(U_\alpha)$ is very ample relative to U_α for every α.*

PROOF. Indeed, condition (b) of (4.4.4) is local on Y. □

Proposition (4.4.6). — *Let Y be a quasi-compact scheme, or a prescheme whose underlying space is Noetherian, $q : X \to Y$ a morphism* of finite type*, and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module. Then conditions (a) and (b) of (4.4.4) are equivalent to the following:*

- (a′) *There exists a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$* of finite type *and a surjective homomorphism ϕ : $q^*(\mathscr{E}) \to \mathscr{L}$ such that $r_{\mathscr{L},\phi}$ is an immersion.*
- (b′) *There exists a coherent sub-$\mathscr{O}_Y$-module $\mathscr{E}$ of $q_*(\mathscr{L})$* of finite type *that has the properties stated in condition (a′).*

PROOF. It is clear that (a′) or (b′) imply (a); also (a) implies (a′), by (4.4.1), and similarly (b) implies (b′). □

Corollary (4.4.7). — *Suppose that Y is a quasi-compact scheme, or a Noetherian prescheme. If $\mathscr{L}$ is very ample for q, then there exists a quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S}$ such that $\mathscr{S}_1$ is of finite type and generates $\mathscr{S}$, and also a* dominant *open Y-immersion $i : X \to P = \operatorname{Proj}(\mathscr{S})$ such that $\mathscr{L}$ is isomorphic to $i^*(\mathscr{O}_P(1))$.*

PROOF. Indeed, condition (b) of (4.4.6) is satisfied; the structure morphism $p : \mathbf{P}(\mathscr{E}) = P' \to Y$ is then separated and of finite type (3.1.3), and so P' is a quasi-compact scheme (resp. a Noetherian prescheme) if Y is a quasi-compact scheme (resp. a Noetherian prescheme). Let Z be the closure (**I**, 9.5.11) of the subprescheme X' of P' associated to the immersion $j = r_{\mathscr{L},\phi}$ from X into P'; it is clear that j factors as a dominant open immersion $i : X \to Z$ followed by the canonical injection $Z \to P'$. But Z can be identified with a prescheme $\operatorname{Proj}(\mathscr{S})$, where $\mathscr{S}$ is a graded $\mathscr{O}_Y$-algebra equal to the quotient of $\mathbf{S}(\mathscr{E})$ by a quasi-coherent graded sheaf of ideals (3.6.2), and it is clear that $\mathscr{S}_1$ is of finite type and generates $\mathscr{S}$; furthermore, $\mathscr{O}_Z(1)$ is the inverse image of $\mathscr{O}_{P'}(1)$ by the canonical injection (3.6.3), and so $\mathscr{L} = i^*(\mathscr{O}_Z(1))$. □

Proposition (4.4.8). — *Let $q : X \to Y$ be a morphism, $\mathscr{L}$ a very ample (relative to q) $\mathscr{O}_X$-module, and $\mathscr{L}'$ an invertible $\mathscr{O}_X$-module, such that there exists a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}'$ and a surjective homomorphism $q^*(\mathscr{E}') \to \mathscr{L}'$. Then $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{L}'$ is very ample relative to q.*

PROOF. The hypothesis implies the existence of a Y-morphism $r' : X \to P' = \mathbf{P}(\mathscr{E}')$ such that $\mathscr{L}' = r'^*(\mathscr{O}_{P'}(1))$ (4.2.1). There is, by hypothesis, a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ and a Y-immersion **II** | 81
$r : X \to P = \mathbf{P}(\mathscr{E})$ such that $\mathscr{L} = r^*(\mathscr{O}_P(1))$. Let $Q = \mathbf{P}(\mathscr{E} \otimes \mathscr{E}')$, and consider the Segre morphism $\varsigma : P \times_Y P' \to Q$ (4.3.1). Since r is an immersion, so too is $(r, r')_Y : X \to P \times_Y P'$ (**I**, 5.3.14); but since ς is an immersion (4.3.3), so too is $r'' : X \xrightarrow{(r,r')} P \times_Y P' \xrightarrow{\varsigma} Q$. But also (4.3.2.1) $\varsigma(\mathscr{O}_Q(1))$ is isomorphic to $\mathscr{O}_P(1) \otimes_Y \mathscr{O}_{P'}(1)$, and so (**I**, 9.1.4) $r''^*(\mathscr{O}_Q(1))$ is isomorphic to $\mathscr{L} \otimes \mathscr{L}'$, which proves the proposition. □

Corollary (4.4.9). — *Let $q : X \to Y$ be a morphism.*

1. *Let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, and $\mathscr{K}$ an invertible $\mathscr{O}_Y$-module. For $\mathscr{L}$ to be very ample relative to q, it is necessary and sufficient for $\mathscr{L} \otimes q^*(\mathscr{K})$ to be so.*
2. *If $\mathscr{L}$ and $\mathscr{L}'$ are very ample (relative to q) $\mathscr{O}_X$-modules, then so too is $\mathscr{L} \otimes \mathscr{L}'$; in particular, $\mathscr{L}^{\otimes n}$ is very ample relative to q for all $n > 0$.*

Proof. Claim (ii) is an immediate consequence of (4.4.8), as well as the necessity of condition (i); conversely, if $\mathscr{L} \otimes q^*(\mathscr{K})$ is very ample, then so too is $(\mathscr{L} \otimes q^*(\mathscr{K})) \otimes q^*(\mathscr{K}^{-1})$, by the above, and the latter $\mathscr{O}_X$-module is isomorphic to $\mathscr{L}$ ((**0**, 5.4.3) and (**0**, 5.4.5)). □

Proposition (4.4.10). —

(i) *For every prescheme Y, every invertible $\mathscr{O}_Y$-module $\mathscr{L}$ is very ample relative to the identity morphism 1_Y.*

(i *bis*) *Let $f : X \to Y$ be a morphism, and $j : X' \to X$ an immersion. If $\mathscr{L}$ is a very ample (relative to f) $\mathscr{O}_X$-module, then $j^*(\mathscr{L})$ is very ample relative to $f \circ j$.*

(ii) *Let Z be a quasi-compact prescheme, $f : X \to Y$ a morphism of finite type, $g : Y \to Z$ a quasi-compact morphism, $\mathscr{L}$ a very ample (relative to f) $\mathscr{O}_X$-module, and $\mathscr{K}$ a very ample (relative to g) $\mathscr{O}_Y$-module. Then there exists some integer $n_0 > 0$ such that $\mathscr{L} \otimes f^*(\mathscr{K}^{\otimes n})$ is very ample relative to $g \circ f$ for all $n \geqslant n_0$.*

(iii) *Let $f : X \to Y$ and $g : Y' \to Y$ be morphisms, and let $X' = X_{(Y')}$. If $\mathscr{L}$ is a very ample (relative to f) $\mathscr{O}_X$-module, then $\mathscr{L}' = \mathscr{L} \otimes_Y \mathscr{O}_{Y'}$ is a very ample (relative to $f_{(Y')}$) $\mathscr{O}_X$-module.*

(iv) *Let $f_i : X_i \to Y_i$ ($i = 1, 2$) be S-morphism. If $\mathscr{L}_i$ is a very ample (relative to f_i) $\mathscr{O}_{X_i}$-module ($i = 1, 2$), then $\mathscr{L}_1 \otimes_S \mathscr{L}_2$ is very ample relative to $f_1 \times_S f_2$.*

(v) *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms. If an $\mathscr{O}_X$-module $\mathscr{L}$ is very ample relative to $g \circ f$, then it is also very ample relative to f.*

(vi) *Let $f : X \to Y$ be a morphism, and j the canonical injection $X_{\mathrm{red}} \to X$. If an $\mathscr{O}_X$-module $\mathscr{L}$ is very ample relative to f, then $j^*(\mathscr{L})$ is very ample relative to f_{red}.*

Proof. Property (i *bis*) follows immediately from the definition (4.4.2), and it is immediate that (vi) follows formally from (i *bis*) and (v), by an argument copied from the proof of (**I**, 5.5.12), which we leave to the reader. To prove (v), we consider, as in (**I**, 5.5.12), the factorisation $X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$, where $p_2 = (g \circ f) \times 1_Y$. It follows from the hypothesis and from (i) and (iv) that $\mathscr{L} \otimes_{\mathscr{O}_Z} \mathscr{O}_Y$ is very ample for p_2; but also $\mathscr{L} = \Gamma_f^*(\mathscr{L} \otimes_{\mathscr{O}_Z} \mathscr{O}_Y)$ (**I**, 9.1.4), and Γ_f is an immersion (**I**, 5.3.11); we can thus apply (i *bis*).

To prove (i), we apply the definition (4.4.2) with $\mathscr{E} = \mathscr{L}$, and note that then $\mathbf{P}(\mathscr{E})$ can be identified with Y (4.1.4). **II** | 82

Now we prove (iii). There exists a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ and a Y-immersion $i : X \to \mathbf{P}(\mathscr{E}) = P$ such that $\mathscr{L} = i^*(\mathscr{O}_P(1))$; if we let $\mathscr{E}' = g^*(\mathscr{E})$, then $\mathscr{E}'$ is a quasi-coherent $\mathscr{O}_{Y'}$-module, and we have that $P' = \mathbf{P}(\mathscr{E}') = P_{(Y')}$ (4.1.3.1), that $i_{(Y')}$ is an immersion from $X_{(Y')}$ into P' (**I**, 4.3.2), and that $\mathscr{L}'$ is isomorphic to $(i_{(Y')})^*(\mathscr{O}_{P'}(1))$ (4.2.10).

To prove (iv), note that there is, by hypothesis, a Y_i-immersion $r_i : X_i \to P_i = \mathbf{P}(\mathscr{E}_i)$, where $\mathscr{E}_i$ is a quasi-coherent $\mathscr{O}_{Y_i}$-module, and $\mathscr{L}_i = r_i^*(\mathscr{O}_{P_i}(1))$ ($i = 1, 2$); $r_1 \times_S r_2$ is an S-immersion of $X_1 \times_S X_2$ into $P_1 \times_S P_2$ (**I**, 4.3.1), and the inverse image of $\mathscr{O}_{P_1}(1) \otimes_S \mathscr{O}_{P_2}(1)$ under this immersion is $\mathscr{L}_1 \otimes_S \mathscr{L}_2$. Now let $T = Y_1 \times_S Y_2$, and let p_1 and p_2 be the projections from T to Y_1 and Y_2, respectively. If we let $P_i' = \mathbf{P}(p_i^*(\mathscr{E}_i))$ ($i = 1, 2$), then $P_i' = P_i \times_{Y_i} T$, by (4.1.3.1), and so

$$P_1' \times_T P_2' = (P_1 \times_{Y_1} T) \times_T (P_2 \times_{Y_2} T) = P_1 \times_{Y_1} (T \times_{Y_2} P_2) = P_1 \times_{Y_1} (Y_1 \times_S P_2) = P_1 \times_S P_2$$

up to canonical isomorphism. Similarly, $\mathscr{O}_{P_i'}(1) = \mathscr{O}_{P_i}(1) \otimes_{Y_i} \mathscr{O}_T$ (4.1.3.2), and an analogous calculation (based in particular on (**I**, 9.1.9.1) and (**I**, 9.1.2)) shows that, in the above identification, $\mathscr{O}_{P_1'}(1) \otimes_T \mathscr{O}_{P_2'}(1)$ can be identified with $\mathscr{O}_{P_1} \otimes_S \mathscr{O}_{P_2}(1)$. We can thus consider $r_1 \times_S r_2$ as a T-immersion from $X_1 \times_S X_2$ into $P_1' \times_T P_2'$, with the inverse image of $\mathscr{O}_{P_1'}(1) \otimes_T \mathscr{O}_{P_2'}(1)$ under this immersion being $\mathscr{L}_1 \otimes_S \mathscr{L}_2$. We then finish the argument as in (4.4.8) by using the Segre morphism.

It remains only to prove (ii). We can first of all restrict to the case where Z is an affine scheme, since, in general, there exists a finite cover (U_i) of Z by affine opens; if the proposition were proven for $\mathscr{K}|g^{-1}(U_i)$, $\mathscr{L}|f^{-1}(g^{-1}(U))$, and an integer n_i, then it would suffice to take n_0 to be the largest of the n_i to prove the proposition for $\mathscr{K}$ and $\mathscr{L}$ (4.4.5). The hypothesis implies that f and g are separated morphisms, and so X and Y are quasi-compact *schemes*.

There is an immersion $r : X \to P = \mathbf{P}(\mathscr{E})$, where $\mathscr{E}$ is a quasi-coherent $\mathscr{O}_Y$-module *of finite type*, and $\mathscr{L} = r^*(\mathscr{O}_P(1))$, by (4.4.6). We will see that there exists a very ample (relative to the composed morphism $P \xrightarrow{h} Y \xrightarrow{g} Z$) $\mathscr{O}_P$-module $\mathscr{M}$ such that $\mathscr{O}_P(1)$ is isomorphic to $\mathscr{M} \otimes_Y \mathscr{K}^{\otimes(-m)}$ for some integer m. For $n \geqslant m+1$, $\mathscr{O}_P(1) \otimes_Y \mathscr{K}^{\otimes n}$ will then be very ample for Z, by hypothesis and by (iv) applied to the morphisms $h : P \to Y$ and 1_Y; since r is an immersion and $\mathscr{L} \otimes f^*(\mathscr{K}^{\otimes n}) = r^*(\mathscr{O}_P(1) \otimes_Y \mathscr{K}^{\otimes n})$, the conclusion will then follow from (i *bis*). To prove our claim concerning $\mathscr{O}_P(1)$, we will use the following lemma:

Lemma (4.4.10.1). — *Let Z be a quasi-compact scheme, or a prescheme whose underlying space is Noetherian, and let $g : Y \to Z$ be a quasi-compact morphism, $\mathscr{K}$ a very ample (with respect to g) invertible $\mathscr{O}_Y$-module, and $\mathscr{E}$ a quasi-coherent $\mathscr{O}_Y$-module of finite type. Then there exists an integer m_0 such that, for all $m \geqslant m_0$, $\mathscr{E}$ is isomorphic to a quotient of an $\mathscr{O}_Y$-module of the form $g^*(\mathscr{F}) \otimes \mathscr{K}^{\otimes(-m)}$, where $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_Z$-module of finite type (depending on m).*

This lemma will be proven in (4.5.10.1); the reader can verify that (4.4.10) is not used anywhere in (4.5).

Assuming this lemma, there exists a closed immersion j_1 from P to **II | 83**

$$P_1 = \mathbf{P}(g^*(\mathscr{F}) \otimes \mathscr{K}^{\otimes(-m)})$$

such that $\mathscr{O}_P(1)$ is isomorphic to $j_1^*(\mathscr{O}_{P_1}(1))$ (4.1.2). Now, there exists an isomorphism from P_1 to $P_2 = \mathbf{P}(g^*(\mathscr{F}))$, sending $\mathscr{O}_{P_2}(1) \otimes_Y \mathscr{K}^{\otimes(-m)}$ to $\mathscr{O}_{P_1}(1)$ (4.1.4); we thus have a closed immersion $j_2 : P \to P_2$ such that $\mathscr{O}_P(1)$ is isomorphic to $j_2^*(\mathscr{O}_{P_2}(1)) \otimes_Y \mathscr{K}^{\otimes(-m)}$. Finally, P_2 can be identified with $P_3 \times_Z Y$, where $P_3 = \mathbf{P}(\mathscr{F})$, and $\mathscr{O}_{P_2}(1)$ with $\mathscr{O}_{P_3}(1) \otimes_Z \mathscr{O}_Y$ (4.1.3). By definition, $\mathscr{O}_{P_3}(1)$ is very ample for Z; since so too is $\mathscr{K}$, we conclude, from (iv), that $\mathscr{O}_{P_2}(1) \otimes_Y \mathscr{K}$ is very ample for Z; so too is $\mathscr{M} = j_2^*(\mathscr{O}_{P_2}(1) \otimes_Y \mathscr{K})$ by (i *bis*), and $\mathscr{O}_P(1)$ is isomorphic to $\mathscr{M} \otimes_Y \mathscr{K}^{\otimes(-m-1)}$, which finishes the proof. □

Proposition (4.4.11). — *Let $f : X \to Y$ and $f' : X' \to Y$ be morphisms, X'' the sum prescheme $X \sqcup X'$, and f'' the morphism $X'' \to Y$ that agrees with f (resp. f') on X (resp. X'). Let $\mathscr{L}$ (resp. $\mathscr{L}'$) be an invertible $\mathscr{O}_X$-module (resp. invertible $\mathscr{O}_{X'}$-module), and let $\mathscr{L}''$ be the invertible $\mathscr{O}_{X''}$-module that agrees with $\mathscr{L}$ (resp. $\mathscr{L}'$) on X (resp. X'). For $\mathscr{L}''$ to be very ample relative to f'', it is necessary and sufficient for $\mathscr{L}$ to be very ample relative to f and for $\mathscr{L}'$ to be very ample relative to f'.*

PROOF. We can immediately restrict to the case where Y is affine. If $\mathscr{L}''$ is very ample then so too are $\mathscr{L}$ and $\mathscr{L}'$, by (4.4.10, (i *bis*)). Conversely, if $\mathscr{L}$ and $\mathscr{L}'$ are very ample, then it follows immediately from the definition (4.4.2) and from (4.3.6) that $\mathscr{L}''$ is very ample. □

4.5. Ample sheaves

(4.5.1). Given a prescheme X and an invertible $\mathscr{O}_X$-module $\mathscr{L}$, we define, for every $\mathscr{O}_X$-module $\mathscr{F}$ (when there will be no confusion possible over $\mathscr{L}$) $\mathscr{F}(n) = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}$ ($n \in \mathbf{Z}$); we also define $S = \bigoplus_{n \geqslant 0} \Gamma(X, \mathscr{L}^{\otimes n})$ (a graded subring of the ring $\Gamma_\bullet(\mathscr{L})$ defined in (**0**, 5.4.6)). If we consider X as a $\mathbf{Z}$-prescheme, and we denote by p the structure morphism $X \to \operatorname{Spec}(\mathbf{Z})$, then there is a bijective correspondence between homomorphisms $p^*(\widetilde{S}) \to \bigoplus_{n \geqslant 0} \mathscr{L}^{\otimes n}$ of graded $\mathscr{O}_X$-algebras and endomorphisms of the graded ring S (**I**, 2.2.5); the homomorphism $\varepsilon : p^*(\widetilde{S}) \to \bigoplus_{n \geqslant 0} \mathscr{L}^{\otimes n}$ that then corresponds to the *identity* automorphism of S is said to be *canonical*. There is a corresponding (3.7.1) morphism $G(\varepsilon) \to \operatorname{Proj}(S)$ that is also said to be *canonical*.

Theorem (4.5.2). — *Let X be a quasi-compact scheme or a prescheme whose underlying space is Noetherian, $\mathscr{L}$ an invertible $\mathscr{O}_X$-module, and S the graded ring $\bigoplus_{n \geqslant 0} \Gamma(X, \mathscr{L}^{\otimes n})$. Then the following conditions are equivalent:*

- (a) *When f runs over the set of homogeneous elements of S_+, the X_f form a base for the topology of X.*
- (a′) *When f runs over the set of homogeneous elements of S_+, the X_f that are affine form a cover of X.*
- (b) *The canonical morphism $G(\varepsilon) \to \operatorname{Proj}(S)$ (4.5.1) is everywhere defined and is a dominant open immersion.* **II | 84**
- (b′) *The canonical morphism $G(\varepsilon) \to \operatorname{Proj}(S)$ is everywhere defined and is a homeomorphism from the underlying space of X to a subspace of $\operatorname{Proj}(S)$.*
- (c) *For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, if we denote by $\mathscr{F}_n$ the sub-$\mathscr{O}_X$-module of $\mathscr{F}(n)$ generated by the sections of $\mathscr{F}(n)$ over X, then $\mathscr{F}$ is the sum of the sub-$\mathscr{O}_X$-modules $\mathscr{F}_n(-n)$ over the integers $n > 0$.*
- (c′) *Property* (c) holds for every quasi-coherent sheaf of ideals of $\mathscr{O}_X$.

Furthermore, if (f_α) is a family of homogeneous elements of S_+ such that the X_{f_α} are affine, then the restriction to $\bigcup_\alpha X_{f_\alpha}$ of the canonical morphism $X \to \operatorname{Proj}(S)$ is an isomorphism from $\bigcup_\alpha X_{f_\alpha}$ to $\bigcup_\alpha(\operatorname{Proj}(S))_{f_\alpha}$.

PROOF. It is clear that (b) implies (b′), and (b′) implies (a) by (3.7.3.1) (taking into account the fact that $\varepsilon^\flat$ is the identity). Condition (a) implies (a′), since every $x \in X$ has an affine neighbourhood U such that $\mathscr{L}|U$ is isomorphic to $\mathscr{O}_X|U$; if $f \in \Gamma(X, \mathscr{L}^{\otimes n})$ is such that $x \in X_f \subset U$, then X_f is also the set of $x' \in U$ such that $(f|U)(x') \neq 0$, and it is thus an affine open subset (**I**, 1.3.6). To prove that (a′) implies (b), it suffices to prove the last claim of the theorem, and to further prove that, if $X = \bigcup_\alpha X_{f_\alpha}$, then condition (iv) of (3.8.2) is satisfied. This latter point follows immediately from (**I**, 9.3.1, (i)). As for the last claim of (4.5.2), since X_{f_α} is the inverse image of $(\operatorname{Proj}(S))_{f_\alpha}$ under $G(\varepsilon) \to \operatorname{Proj}(S)$, it suffices to apply (**I**, 9.3.2). Thus (a), (a′), (b), and (b′) are all equivalent.

To show that (a′) implies (c), note that, if X_f is affine (with $f \in S_k$), then $\mathscr{F}|X_f$ is generated by its sections over X_f (**I**, 1.3.9); on the other hand (**I**, 9.3.1, (ii)), such a section s is of the form $(t|X_f) \otimes (f|X_f)^{-m}$, where $t \in \Gamma(X, \mathscr{F}(km))$; by definition, t is also a section of $\mathscr{F}_{km}$, so s is indeed a section of $\mathscr{F}_{km}(-km)$ over X_f, which proves (c). It is clear that (c) implies (c′), so it remains only to show that (c′) implies (a). But let U be an open neighbourhood of $x \in X$, and let $\mathscr{J}$ be a quasi-coherent sheaf of ideals of $\mathscr{O}_X$ defining a closed subprescheme of X that has $X - U$ as its underlying space (**I**, 5.2.1). Hypothesis (c′) implies that there exists an integer $n > 0$ and a section f of $\mathscr{J}(n)$ over X such that $f(x) \neq 0$. But we clearly have $f \in S_n$, and $x \in X_f \subset U$, which proves (a). □

When X is a prescheme whose underlying space is Noetherian, the equivalent conditions of (4.5.2) imply that X is a *scheme*, since it is isomorphic to a subprescheme of the scheme $S = \operatorname{Proj}(A)$, by (4.5.2, (b)).

Definition (4.5.3). — We say that an invertible $\mathscr{O}_X$-module $\mathscr{L}$ is *ample* if X is a quasi-compact scheme and if the equivalent conditions of (4.5.2) are satisfied.

It evidently follows from criterion (a) of (4.5.2) that, if $\mathscr{L}$ is an ample $\mathscr{O}_X$-module, then, for every open subset U of X, $\mathscr{L}|U$ is an ample $(\mathscr{O}_X|U)$-module.

It follows from the proof of (4.5.2) that the *affine* X_f form a base for the topology of X. Furthermore:

Corollary (4.5.4). — *Let $\mathscr{L}$ be an ample $\mathscr{O}_X$-module. For every finite subspace Z of X and every neighbourhood U of Z, there exists an integer n and some $f \in \Gamma(X, \mathscr{L}^{\otimes n})$ such that X_f is an affine neighbourhood of Z contained in U.*

PROOF. By (4.5.2, (b)), it suffices to prove that, for every finite subset Z' of $\operatorname{Proj}(S)$ and every open neighbourhood U of Z', there exists a homogeneous element $f \in S_+$ such that $Z \subset (\operatorname{Proj}(S))_f \subset U$ (2.4.1). But, by definition, the closed set Y, complement of U in $\operatorname{Proj}(S)$, is of the form $V_+(\mathfrak{I})$, where $\mathfrak{I}$ is a graded ideal of S that does not contain S_+ (2.3.2); also, the points of Z' are, by definition, graded ideals $\mathfrak{p}_i$ of S_+ that do not contain $\mathscr{J}$ (2.3.1). There thus exists an element $f \in \mathfrak{I}$ that does not belong to any of the $\mathfrak{p}_i$ (Bourbaki, *Alg. comm.*, chap. II, §1, no. 1, prop. 2), and, since the $\mathfrak{p}_i$ are graded, the argument made *loc. cit.* shows that we can even take f to be homogeneous; this element then satisfies the claim. □ **II** | 85

Proposition (4.5.5). — *Suppose that X is a quasi-compact scheme or a prescheme whose underlying space is Noetherian. Then conditions (a) to (c′) of (4.5.2) are equivalent to the following:*

- (d) *For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type, there exists an integer n_0 such that, for all $n \geqslant n_0$, $\mathscr{F}(n)$ is generated by its sections over X.*
- (d′) *For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type, there exist integers $n > 0$ and $k > 0$ such that $\mathscr{F}$ is isomorphic to a quotient of the $\mathscr{O}_X$-module $\mathscr{L}^{\otimes(-n)} \otimes \mathscr{O}_X^k$.*
- (d″) *Property (d′) holds for every quasi-coherent sheaf of ideals of $\mathscr{O}_X$ of finite type.*

PROOF. Since X is quasi-compact, if a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type is such that $\mathscr{F}(n)$ (which is of finite type) is generated by its sections over X, then $\mathscr{F}(n)$ is generated by a *finite* number of these sections (**0**, 5.2.3), and so (d) implies (d′), and it is clear that (d′) implies (d″).

Since every quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}$ is the inductive limits of its sub-$\mathscr{O}_X$-modules of finite type (**I**, 9.4.9), to satisfy condition (c') of (4.5.2), it suffices to do so for a quasi-coherent sheaf of ideals of $\mathscr{O}_X$ that is of finite type, and (d'') thus implies (c'). It remains only to show that, if $\mathscr{L}$ is ample, then property (d) is satisfied. Consider a finite cover of X by X_{f_i} ($f_i \in S_{n_i}$), that we can assume to be affine; by replacing the f_i with suitable powers (which does not alter the X_{f_i}), we can assume that all the n_i are equal to one single integer m. The sheaf $\mathscr{F}|X_{f_i}$, being of finite type, by hypothesis, is generated by a finite number of its sections h_{ij} over X_{f_i} (**I**, 1.3.13); so there exists an integer k_0 such that the section $h_{ij} \otimes f_i^{\otimes k_0}$ extends to a section of $\mathscr{F}(k_0 m)$ over X for every pair (i,j) (**I**, 9.3.1). *A fortiori*, the $h_{ij} \otimes f_i^{\otimes k_0}$ extend to sections of $\mathscr{F}(km)$ over X for every $k \geqslant k_0$, and, for these values of k, $\mathscr{F}(km)$ is thus generated by its sections over X. For every p such that $0 < p < m$, $\mathscr{F}(p)$ is also of finite type, and so there exists an integer k_p such that $\mathscr{F}(p)(km) = \mathscr{F}(p+km)$ is generated by its sections over X for all $k \geqslant k_p$. Taking n_0 to be the largest of the $k_p m$, we thus conclude that $\mathscr{F}(n)$ is generated by its sections over X for all $n \geqslant n_0$, since such an n is of the form $n = km + p$, with $k \geqslant k_p$ and $0 \leqslant p < m$. □

Proposition (4.5.6). — *Let X be a quasi-compact scheme, and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module.*

(i) *Let $n > 0$ be an integer. For $\mathscr{L}$ to be ample, it is necessary and sufficient for $\mathscr{L}^{\otimes n}$ to be ample.*
(ii) *Let $\mathscr{L}'$ be an invertible $\mathscr{O}_X$-module such that, for all $x \in X$, there exists an integer $n > 0$ and a section s' of $\mathscr{L}'^{\otimes n}$ over X such that $s'(x) \neq 0$. Then, if $\mathscr{L}$ is ample, so too is $\mathscr{L} \otimes \mathscr{L}'$.* **II** | 86

PROOF. Property (i) is an evident consequence of criterion (a) of (4.5.2), since $X_{f^{\otimes n}} = X_f$. On the other hand, if $\mathscr{L}$ is ample, then, for every $x \in X$ and every neighbourhood U of x, there exists some $m > 0$ and $f \in \Gamma(X, \mathscr{L}^{\otimes m})$ such that $x \in X_f \subset U$ (4.5.2, (a)); if $f' \in \Gamma(X, \mathscr{L}'^{\otimes n})$ is such that $f'(x) \neq 0$, then $s(x) \neq 0$ for $s = f^{\otimes n} \otimes f'^{\otimes m} \in \Gamma(X, (\mathscr{L} \otimes \mathscr{L}')^{\otimes mn})$, and so $x \in X_s \subset X_f \subset U$, which proves that $\mathscr{L} \otimes \mathscr{L}'$ is ample (4.5.2, (a)). □

Corollary (4.5.7). — *The tensor product of two ample $\mathscr{O}_X$-modules is ample.*

Corollary (4.5.8). — *Let $\mathscr{L}$ be an ample $\mathscr{O}_X$-module, and $\mathscr{L}'$ an invertible $\mathscr{O}_X$-module; then there exists an integer $n_0 > 0$ such that $\mathscr{L}^{\otimes n} \otimes \mathscr{L}'$ is ample and generated by its sections over X for $n \geqslant n_0$.*

PROOF. It follows from (4.5.5) that there exists an integer m_0 such that $\mathscr{L}^{\otimes m} \otimes \mathscr{L}'$ is generated by its sections over X for all $m \geqslant m_0$; by (4.5.6), we can then take $n_0 = m_0 + 1$. □

Remark (4.5.9). — Let $P = \mathrm{H}^1(X, \mathscr{O}_X^\times)$ be the group of classes of invertible $\mathscr{O}_X$-modules (**0**, 5.4.7), and let P^+ be the subset of P consisting of classes of ample sheaves. Suppose that P^+ is *non-empty*. Then it follows from (4.5.7) and (4.5.8) that

$$P^+ + P^+ \subset P^+ \quad \text{and} \quad P^+ - P^+ = P$$

or, in other words, $P^+ \cup \{0\}$ is the set of *positive* elements in P for a *preorder* structure on P that is compatible with its group structure, and is even *archimedean*, by (4.5.8). This is why we sometimes say "positive sheaf" instead of ample sheaf, and "negative sheaf" for the inverse of an ample sheaf (but we will not use this terminology).

Proposition (4.5.10). — *Let Y be an affine scheme, $q : X \to Y$ a quasi-compact separated morphism, and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module.*

(i) *If $\mathscr{L}$ is very ample for q, then $\mathscr{L}$ is ample.*
(ii) *Suppose further that the morphism q is* of finite type. *Then, for $\mathscr{L}$ to be ample, it is necessary and sufficient for it to posses one of the following properties:*
  (e) *There exists $n_0 > 0$ such that, for every integer $n \geqslant n_0$, $\mathscr{L}^{\otimes n}$ is very ample for q.*
  (e') *There exists $n > 0$ such that $\mathscr{L}^{\otimes n}$ is very ample for q.*

PROOF. The first claim follows from the definition (4.4.2) of a very ample $\mathscr{O}_X$-module: if A is the ring of Y, then there exists an A-module E and a surjective homomorphism

$$\psi : q^*((\mathbf{S}(E))^\sim) \longrightarrow \bigoplus_{n \geqslant 0} \mathscr{L}^{\otimes n}$$

such that $i = r_{\mathscr{L},\psi}$ is an everywhere-defined immersion $X \to P = \mathbf{P}(\widetilde{E})$ and such that $\mathscr{L} = i^*(\mathscr{O}_P(1))$; since the $D_+(f)$ for f homogeneous in $(\mathbf{S}(E))_+$ form a base for the topology of P, and since $i^{-1}(D_+(f)) = X_{\psi^\flat(f)}$, by (3.7.3.1), we see that condition (a) of (4.5.2) is satisfied, and so $\mathscr{L}$ is ample.

Now to prove that, if q is of finite type and $\mathscr{L}$ is ample, then condition (e) is satisfied. Firstly, it follows from criterion (b) of (4.5.2) and from (4.4.1, (i)) that there exists an integer k_0 such that $\mathscr{L}^{\otimes k_0}$ **II** | 87
is very ample relative to q. Also, by (4.5.5), there exists an integer m_0 such that, for all $m \geqslant m_0$, $\mathscr{L}^{\otimes m}$ is generated by its sections over X. Let $n_0 = k_0 + m_0$; if $n \geqslant n_0$, then we can write $n = k_0 + m$ with $m \geqslant m_0$, whence $\mathscr{L}^{\otimes n} = \mathscr{L}^{\otimes k_0} \otimes \mathscr{L}^{\otimes m}$. Since $\mathscr{L}^{\otimes m}$ is generated by its sections over X, it follows from (4.4.8) and (3.4.7) that $\mathscr{L}^{\otimes n}$ is very ample relative to q. Finally, it is clear that (e) implies (e'), and (e') implies that $\mathscr{L}$ is ample by (i) and by (4.5.6, (i))

(4.5.10.1). *[Proof of Lemma (4.4.10.1)].* Let $\mathscr{E}(n) = \mathscr{E} \otimes \mathscr{K}^{\otimes n}$; since g is separated (4.4.2), the proof of (3.4.8) applies, and shows that the canonical homomorphism $g^*(g_*(\mathscr{E}(n))) \to \mathscr{E}(n)$ is surjective for n large enough. Furthermore, since Z is quasi-compact, the proof of (3.4.6) means that if suffices to prove the claim in the case where Z is affine. But $\mathscr{K}$ is then ample, by (4.5.10, (i)), and the conclusion follows from (4.5.5, (d')).

□

Corollary (4.5.11). — *Let Y be an affine scheme, $q : X \to Y$ a separated morphism of finite type, $\mathscr{L}$ an ample $\mathscr{O}_X$-module, and $\mathscr{L}'$ an invertible $\mathscr{O}_X$-module. Then there exists an integer n_0 such that, for all $n \geqslant n_0$, $\mathscr{L}^{\otimes n} \otimes \mathscr{L}'$ is very ample relative to q.*

Proof. There exists m_0 such that, for $m \geqslant m_0$, $\mathscr{L}^{\otimes m} \otimes \mathscr{L}'$ is generated by its sections over X (4.5.8); there also exists k_0 such that $\mathscr{L}^{\otimes k}$ is very ample relative to q for $k \geqslant k_0$. Then $\mathscr{L}^{\otimes (k+m_0)} \otimes \mathscr{L}'$ is very ample if $k \geqslant k_0$ ((4.4.8) and (3.4.7)). □

Remark (4.5.12). — We do not know if the hypothesis that an $\mathscr{O}_X$-module $\mathscr{L}$ is such that $\mathscr{L}^{\otimes n}$ is very ample (relative to q) implies the same conclusion for $\mathscr{L}^{\otimes (n+1)}$.

Proposition (4.5.13). — *Let X be a quasi-compact prescheme, Z a closed prescheme of X defined by a* nilpotent *quasi-coherent sheaf $\mathscr{J}$ of ideals of $\mathscr{O}_X$, and j the canonical injection $Z \to X$. For an invertible $\mathscr{O}_X$-module $\mathscr{L}$ to be ample, it is necessary and sufficient for $\mathscr{L}' = j^*(\mathscr{L})$ to be an ample $\mathscr{O}_Z$-module.*

Proof. The condition is *necessary*. Indeed, for every section f of $\mathscr{L}^{\otimes n}$ over X, let f' be its canonical image $f \otimes 1$, which is a section of $\mathscr{L}'^{\otimes n} = \mathscr{L}^{\otimes n} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{J})$ over the space Z (which is identical to X); it is clear that $X_f = Z_{f'}$, and so the criterion (a) of (4.5.2) shows that $\mathscr{L}'$ is ample.

To see that the condition is *sufficient*, note first of all that we can restrict to the case where $\mathscr{J}^2 = 0$, by considering the (finite) sequence of preschemes $X_k = (X, \mathscr{O}_X/\mathscr{J}^{k+1})$ with each prescheme being a closed subprescheme of the next, defined by a square-zero sheaf of ideals. But X is a scheme, since X_{red} is a scheme by hypothesis ((4.5.3) and (**I**, 5.5.1)). Criterion (a) of (4.5.2) shows that it suffices to prove

Lemma (4.5.13.1). — *Under the hypotheses of (4.5.13), suppose further that $\mathscr{J}$ is square-zero; with $\mathscr{L}$ being an invertible $\mathscr{O}_X$-module, let g be a section of $\mathscr{L}'^{\otimes n}$ over Z such that Z_g is affine. Then there exists an integer $m > 0$ such that $g^{\otimes m}$ is the canonical image of a section f of $\mathscr{L}^{\otimes nm}$ over X.*

Proof. We have the exact sequence of $\mathscr{O}_X$-modules

$$0 \longrightarrow \mathscr{J}(n) \longrightarrow \mathscr{O}_X(n) = \mathscr{L}^{\otimes n} \longrightarrow \mathscr{O}_Z(n) = \mathscr{L}'^{\otimes n} \longrightarrow 0$$

since $\mathscr{F}(n)$ is an exact functor in $\mathscr{F}$; from this, we have the exact sequence of cohomology **II** | 88

$$0 \longrightarrow \Gamma(X, \mathscr{J}(n)) \longrightarrow \Gamma(X, \mathscr{L}^{\otimes n}) \longrightarrow \Gamma(X, \mathscr{L}'^{\otimes n}) \xrightarrow{\partial} \mathrm{H}^1(X, \mathscr{J}(n))$$

that sends, in particular, g to an element $\partial g \in \mathrm{H}^1(X, \mathscr{J}(n))$. □

Note that, since $\mathscr{J}^2 = 0$, $\mathscr{J}$ can be considered as a quasi-coherent $\mathscr{O}_Z$-module, and, for all k, $\mathscr{L}'^{\otimes k} \otimes_{\mathscr{O}_Z} \mathscr{J}(n) = \mathscr{J}(n+k)$; for every section $s \in \Gamma(X, \mathscr{L}'^{\otimes k})$, tensor multiplication with s is thus a homomorphism $\mathscr{J}(n) \to \mathscr{J}(n+k)$ of $\mathscr{O}_Z$-modules, which then gives a homomorphism $\mathrm{H}^i(X, \mathscr{J}(n)) \xrightarrow{s} \mathrm{H}^i(X, \mathscr{J}(n+k))$ of cohomology groups.

With this, we will see that

$$g^{\otimes m} \otimes \partial g = 0 \tag{4.5.13.2}$$

for $m > 0$ large enough. In fact, Z_g is an *affine open* subset of Z, and so $\mathrm{H}^1(Z_g, \mathscr{J}(n)) = 0$ when $\mathscr{J}(n)$ is considered as an $\mathscr{O}_Z$*-module* (I, 5.1.9.2). In particular, if we set $g' = g|Z_g$, and if we consider its image under the map $\partial : \Gamma(Z_g, \mathscr{L}'^{\otimes n}) \to \mathrm{H}^1(Z_g, \mathscr{J}(n))$, then $\partial g' = 0$. To better explain this equation, we note that, in dimension 1, the cohomology of a sheaf of abelian groups is the same as its Čech cohomology (G, II, 5.9); to calculate ∂g, we must thus consider a fine-enough open cover (U_α) of X, that we can suppose to be *finite* and consisting of affine opens, and take, for each α, a section $g_\alpha \in \Gamma(U_\alpha, \mathscr{L}^{\otimes n})$ whose canonical image in $\Gamma(U_\alpha, \mathscr{L}'^{\otimes n})$ is $g|U_\alpha$, and to consider the cocycle class $(g_{\alpha|\beta} - g_{\beta|\alpha})$, with $g_{\alpha|\beta}$ being the restriction of g_α to $U_\alpha \cap U_\beta$ (with this cocycle taking values in $\mathscr{J}(n)$). We can further suppose that $\partial g'$ is calculated in the same way, by means of a cover given by the $U_\alpha \cap Z_g$, and restrictions $g_\alpha|(U_\alpha \cap Z_g)$ (by replacing, if necessary, (U_α) by a finer cover); the equation $\partial g' = 0$ then implies that there exists, for each α, a section $h_\alpha \in \Gamma(U_\alpha \cap Z_g, \mathscr{J}(n))$ such that $(g_{\alpha|\beta} - g_{\beta|\alpha})|(U_\alpha \cap U_\beta \cap Z_g) = h_{\alpha|\beta} - h_{\beta|\alpha}$, where $h_{\alpha|\beta}$ denotes the restriction of h_α to $U_\alpha \cap U_\beta \cap Z_g$ (G, II, 5.11). Then there exists an integer $m > 0$ such that $g^{\otimes m} \otimes h_\alpha$ is the restriction to $U_\alpha \cap Z_g$ of a section $t_\alpha \in \Gamma(U_\alpha, \mathscr{J}(n+nm))$ for all α (I, 9.3.1); thus $g^{\otimes m} \otimes (g_{\alpha|\beta} - g_{\beta|\alpha}) = t_{\alpha|\beta} - t_{\beta|\alpha}$ for every pair of indices, which proves (4.5.13.2).

Now note that, if $s \in \Gamma(X, \mathscr{O}_Z(p))$ and $t \in \Gamma(X, \mathscr{O}_Z(q))$, then, in the group $\mathrm{H}^1(X, \mathscr{J}(p+q))$,

$$\partial(s \otimes t) = (\partial s) \otimes t + s \otimes (\partial t). \tag{4.5.13.3}$$

Indeed, we can again calculate the two members by considering an open cover (U_α) of X, and, for each α, a section $s_\alpha \in \Gamma(U_\alpha, \mathscr{O}_X(p))$ (resp. $t_\alpha \in \Gamma(U_\alpha, \mathscr{O}_X(q))$) whose canonical image in $\Gamma(U_\alpha, \mathscr{O}_Z(p)$ (resp. $\Gamma(U_\alpha, \mathscr{O}_Z(q))$) is $s|U_\alpha$ (resp. $t|U_\alpha$); equation (4.5.13.3) then follows from the equations

$$(s_{\alpha|\beta} \otimes t_{\alpha|\beta}) - (s_{\beta|\alpha} \otimes t_{\beta|\alpha}) = (s_{\alpha|\beta} - s_{\beta|\alpha}) \otimes t_{\alpha|\beta} + s_{\beta|\alpha} \otimes (t_{\alpha|\beta} - t_{\beta|\alpha})$$

with the same notation as above. By induction on k, we thus have

$$\partial(g^{\otimes k}) = (k g^{\otimes(k-1)}) \otimes (\partial g) \tag{4.5.13.4}$$

and we thus conclude from (4.5.13.2) and (4.5.13.4) that $\partial(g^{\otimes(m+1)} = 0$; thus $g^{\otimes(m+1)}$ is the canonical II | 89
image of a section f of $\mathscr{L}^{\otimes n(m+1)}$ over X, which proves (4.5.13). □

Corollary (4.5.14). — *Let X be a Noetherian scheme, and j the canonical injection $X_{\mathrm{red}} \to X$. For an invertible $\mathscr{O}_X$-module $\mathscr{L}$ to be ample, it is necessary and sufficient for $j^*(\mathscr{L})$ to be an ample $\mathscr{O}_{X_{\mathrm{red}}}$-module.*

PROOF. This follows from (I, 6.1.6) □

4.6. Relatively ample sheaves

Definition (4.6.1). — Let $f : X \to Y$ be a quasi-compact morphism, and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module. We say that $\mathscr{L}$ is *ample relative to* f, or *relative to* Y, or f*-ample*, or Y*-ample* (or even simply *ample* if no confusion may arise with the notion defined in (4.5.3)) if there exists an affine open cover (U_α) of Y such that, if we set $X_\alpha = f^{-1}(U_\alpha)$, then $\mathscr{L}|X_\alpha$ is an ample $\mathscr{O}_{X_\alpha}$-module for all α.

The existence of an f-ample $\mathscr{O}_X$-module implies that f is necessarily *separated* ((4.5.3) and (I, 5.5.5)).

Proposition (4.6.2). — *Let $f : X \to Y$ be a quasi-compact morphism, and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module. If $\mathscr{L}$ is very ample relative to f, then it is ample relative to f.*

PROOF. This follows from the local (on Y) character of the notion of a very ample sheaf (4.4.5), from the definition (4.6.1), and from criterion (4.5.10, (i)). □

Proposition (4.6.3). — *Let $f : X \to Y$ be a quasi-compact morphism, and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module, and let $\mathscr{S}$ be the graded $\mathscr{O}_Y$-algebra $\bigoplus_{n \geqslant 0} f_*(\mathscr{L}^{\otimes n})$. Then the following conditions are equivalent:*

(a) *$\mathscr{L}$ is f-ample.*

(b) *$\mathscr{S}$ is quasi-coherent, and the canonical homomorphism $\sigma : f^*(\mathscr{S}) \to \bigoplus_{n\geqslant 0} \mathscr{L}^{\otimes n}$ (**0**, 4.4.3) is such that the Y-morphism $r_{\mathscr{L},\sigma} : G(\sigma) \to \mathrm{Proj}(\mathscr{S}) = P$ is everywhere defined and is an dominant open immersion.*

(b′) *The morphism f is separated, and the Y-morphism $r_{\mathscr{L},\sigma}$ is everywhere defined and is a homeomorphism from the underlying space of X to a subspace of $\mathrm{Proj}(\mathscr{S})$.*

Furthermore, if any of the above are satisfied, then, for all $n \in \mathbf{Z}$, the canonical homomorphism

$$r^*_{\mathscr{L},\sigma}(\mathscr{O}_P(n)) \longrightarrow \mathscr{L}^{\otimes n} \tag{4.6.3.1}$$

defined in (3.7.9.1) is an isomorphism.

Finally, for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, if we set $\mathscr{M} = \bigoplus_{n\geqslant 0} f_*(\mathscr{F} \otimes \mathscr{L}^{\otimes n})$, then the canonical homomorphism

$$r^*_{\mathscr{L},\sigma}(\widetilde{\mathscr{M}}) \longrightarrow \mathscr{F} \tag{4.6.3.2}$$

defined in (3.7.9.2) is an isomorphism.

PROOF. We note that (a) implies that f is separated, and thus that $\mathscr{S}$ is quasi-coherent (**I**, 9.2.2, (a)). Since $r_{\mathscr{L},\sigma}$ being an everywhere defined immersion is of a local (on Y) character, to prove that (a) implies (b) we can suppose that Y is affine and $\mathscr{L}$ ample; the claim then follows from (4.5.2, (b)). It is clear that (b) implies (b′); finally, to prove that (b′) implies (a), it suffices to consider an affine open cover (U_α) of Y and to apply the criterion (4.5.2, (b′)) to each sheaf $\mathscr{L}|f^{-1}(U_\alpha)$. **II** | 90

For the final two claims, we use the fact that $\sigma^\flat$ is here the identity, and the clarification of the homomorphisms (3.7.9.1) and (3.7.9.2); from this, it immediately follows that (4.6.3.1) is an isomorphism. As for (4.6.3.2), we can restrict to the case where Y is affine, and thus $\mathscr{L}$ ample; it is clear that the homomorphism (4.6.3.2) is injective, and criterion (4.5.2, (c)) shows that it is surjective, whence the conclusion. □

Corollary (4.6.4). — *Let (U_α) be an open cover of Y. For $\mathscr{L}$ to be ample relative to Y, it is necessary and sufficient for $\mathscr{L}|f^{-1}(U_\alpha)$ to be ample relative to U_α for all α.*

PROOF. Condition (b) is in fact local on Y. □

Corollary (4.6.5). — *Let $\mathscr{K}$ be an invertible $\mathscr{O}_Y$-module. For $\mathscr{L}$ to be Y-ample, it is necessary and sufficient for $\mathscr{L} \otimes f^*(\mathscr{K})$ to be Y-ample.*

PROOF. This is an evident consequence of (4.6.4), by taking the U_α to be such that $\mathscr{K}|U_\alpha$ is isomorphic to $\mathscr{O}_Y|U_\alpha$ for all α. □

Corollary (4.6.6). — *Suppose that Y is affine; for $\mathscr{L}$ to be Y-ample, it is necessary and sufficient for $\mathscr{L}$ to be ample.*

PROOF. This is an immediate consequence of the definition (4.6.1) and the criteria (4.6.3, (b)) and (4.5.2, (b)), since here $\mathrm{Proj}(\mathscr{S}) = \mathrm{Proj}(\Gamma(Y, \mathscr{S}))$ by definition. □

Corollary (4.6.7). — *Let $f : X \to Y$ be a quasi-compact morphism. Suppose that there exists a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$, and a Y-morphism $g : X \to P = \mathbf{P}(\mathscr{E})$ that is a homeomorphism from the underlying space of X to a subspace of $\mathbf{P}$; then $\mathscr{L} = g^*(\mathscr{O}_P(1))$ is Y-ample.*

PROOF. We can assume Y to be affine; the corollary then follows from the criterion (4.5.2, (a)), from equation (3.7.3.1), and from (4.2.3). □

Proposition (4.6.8). — *Let X be a quasi-compact scheme or a prescheme whose underlying space is Noetherian, and let $f : X \to Y$ be a quasi-compact separated morphism. For an invertible $\mathscr{O}_X$-module $\mathscr{L}$ to be f-ample, it is necessary and sufficient for one of the following equivalent conditions to be satisfied:*

(c) *For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type, there exists an integer $n_0 > 0$ such that, for all $n \geqslant n_0$, the canonical homomorphism $\sigma : f^*(f_*(\mathscr{F} \otimes \mathscr{L}^{\otimes n})) \to \mathscr{F} \otimes \mathscr{L}^{\otimes n}$ is surjective.*

(c′) *For every quasi-coherent sheaf $\mathscr{J}$ of ideals of $\mathscr{O}_X$ of finite type, there exists an integer $n > 0$ such that the canonical homomorphism $\sigma : f^*(f_*(\mathscr{J} \otimes \mathscr{L}^{\otimes n})) \to \mathscr{J} \otimes \mathscr{L}^{\otimes n}$ is surjective.*

PROOF. Since X is quasi-compact, so too is $f(X)$, and so there exists a finite cover (U_i) of $f(X)$ consisting of affine open subsets U_i of Y. To prove condition (c) when $\mathscr{L}$ is f-ample, we can replace Y by the U_i, and X by the $f^{-1}(U_i)$, since, if we obtain, for each i, an integer n_i such that (c) holds true (for U_i, $f^{-1}(U_i)$, and $\mathscr{L}|f^{-1}(U_i)$) for all $n \geqslant n_i$, then it suffices to take n_0 to be the largest of the n_i in order to obtain (c) for Y, X, and $\mathscr{L}$. But if Y is affine, condition (c) follows from (4.5.5, (d)), taking (4.6.6) into account. It is trivial that (c) implies (c′). Finally, to prove that (c′) implies that $\mathscr{L}$ is f-ample, we can again restrict to the case where Y is affine: in fact, every quasi-coherent sheaf $\mathscr{J}_i$ of ideals of $\mathscr{O}_X|f^{-1}(U_i)$ of finite type is the restriction of a coherent sheaf of ideals of $\mathscr{O}_X$ of finite type (**I**, 9.4.7), and hypothesis (c′) implies that $\mathscr{J}_i \otimes (\mathscr{L}^{\otimes n}|f^{-1}(U_i))$ is generated by its sections (taking **II** | 91
(**I**, 9.2.2) and (3.4.7) into account); it thus suffices to apply criterion (4.5.5, (d″)). □

Proposition (4.6.9). — *Let $f : X \to Y$ be a quasi-compact morphism, and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module.*

(i) *Let $n > 0$ be an integer. For $\mathscr{L}$ to be f-ample, it is necessary and sufficient for $\mathscr{L}^{\otimes n}$ to be f-ample.*
(ii) *Let $\mathscr{L}'$ be an invertible $\mathscr{O}_X$-module, and suppose that there exists an integer $n > 0$ such that the canonical homomorphism $\sigma : f^*(f_*(\mathscr{L}'^{\otimes n})) \to \mathscr{L}'^{\otimes n}$ is surjective. Then, if $\mathscr{L}$ if f-ample, so too is $\mathscr{L} \otimes \mathscr{L}'$.*

PROOF. We can in fact immediately restrict to the case where Y is affine, and the proposition is then an immediate consequence of (4.5.6). □

Corollary (4.6.10). — *The tensor product of two f-ample $\mathscr{O}_X$-modules is f-ample.*

Proposition (4.6.11). — *Let Y be a quasi-compact prescheme, $f : X \to Y$ a morphism of finite type, and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module. For $\mathscr{L}$ to be f-ample, it is necessary and sufficient for it to posses one of the following equivalent properties:*

(d) *There exists some $n_0 > 0$ such that, for every integer $n \geqslant n_0$, $\mathscr{L}^{\otimes n}$ is very ample relative to f.*
(d′) *There exists some $n > 0$ such that $\mathscr{L}^{\otimes n}$ is very ample relative to f.*

PROOF. If $\mathscr{L}$ is ample relative to f, then there exists a *finite* cover (U_i) of Y by affine open subsets such that the $\mathscr{L}|f^{-1}(U_i)$ are ample. We thus conclude (4.5.10) that there exists an integer n_0 such that $\mathscr{L}^{\otimes n}|f^{-1}(U_i)$ is very ample relative to $f^{-1}(U_i) \to U_i$ for all $n \geqslant n_0$ and every i, and so $\mathscr{L}^{\otimes n}$ is very ample relative to f (4.5.5). Conversely, (d′) already implies that $\mathscr{L}^{\otimes n}$ is f-ample (4.6.2), and thus so too is $\mathscr{L}$ (4.6.9, (i)). □

Corollary (4.6.12). — *Let Y be a quasi-compact prescheme, $f : X \to Y$ a morphism of finite type, and $\mathscr{L}$ and $\mathscr{L}'$ invertible $\mathscr{O}_X$-modules. If $\mathscr{L}$ is f ample, then there exists some n_0 such that $\mathscr{L}^{\otimes n} \otimes \mathscr{L}'$ is very ample relative to f for all $n \geqslant n_0$.*

PROOF. We argue as in (4.6.11), by using a finite affine open cover of Y and (4.5.11). □

Proposition (4.6.13). —

(i) *For every prescheme Y, every invertible $\mathscr{O}_Y$-module $\mathscr{L}$ is ample relative to the identity morphism 1_Y.*
(i *bis*) *Let $f : X \to Y$ be a quasi-compact morphism, and $j : X' \to X$ a quasi-compact morphism that is a homeomorphism from the underlying space of X' to a subspace of X. If $\mathscr{L}$ is an $\mathscr{O}_X$-module that is ample relative to f, then $j^*(\mathscr{L})$ is ample relative to $f \circ j$.*
(ii) *Let Z be a quasi-compact prescheme, $f : X \to Y$ and $g : Y \to Z$ quasi-compact morphisms, $\mathscr{L}$ an $\mathscr{O}_X$-module that is ample relative to f, and $\mathscr{K}$ an $\mathscr{O}_Y$-module that is ample relative to g. Then there exists an integer $n_0 > 0$ such that $\mathscr{L} \otimes f^*(\mathscr{K}^{\otimes n})$ is ample relative to $g \circ f$ for all $n \geqslant n_0$.*
(iii) *Let $f : X \to Y$ be a quasi-compact morphism, and $g : Y' \to Y$ a morphism, and let $X' = X_{(Y')}$. If $\mathscr{L}$ is an $\mathscr{O}_X$-module that is ample relative to f, then $\mathscr{L}' = \mathscr{L} \otimes_Y \mathscr{O}_{Y'}$ is an $\mathscr{O}_{X'}$-module that is ample relative to $f_{(Y')}$.*
(iv) *Let $f_i : X_i \to Y_i$ ($i = 1,2$) be quasi-compact S-morphisms. If $\mathscr{L}_i$ is an $\mathscr{O}_{X_i}$-modules that is ample relative to f_i ($i = 1,2$), then $\mathscr{L}_1 \otimes_S \mathscr{L}_2$ is ample relative to $f_1 \times_S f_2$.*
(v) *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms such that $g \circ f$ is quasi-compact. If an $\mathscr{O}_X$-module* **II** | 92
$\mathscr{L}$ is ample relative to $g \circ f$, and if g is separated or the underlying space of X is locally Noetherian, then $\mathscr{L}$ is ample relative to f.
(vi) *Let $f : X \to Y$ be a quasi-compact morphism, and j the canonical injection $X_{\mathrm{red}} \to X$. If $\mathscr{L}$ is an $\mathscr{O}_X$-module that is ample relative to f, then $j^*(\mathscr{L})$ is ample relative to f_{red}.*

PROOF. Note first of all that (v) and (vi) follow from (i), (i *bis*), and (iv) by the same argument as in (4.4.10), by using (4.6.4) instead of (4.4.5); we leave the details of the argument to the reader. Claim (i) is trivially a consequence of (4.4.10, (i)) and (4.6.2). To prove (i *bis*), (iii), and (iv), we will use the following lemma:

Lemma (4.6.13.1). — (i) *Let $u : Z \to S$ be a morphism, $\mathscr{L}$ an invertible $\mathscr{O}_S$-module, and s a section of $\mathscr{L}$ over S, and let s' be the canonically corresponding section $u^*(\mathscr{L}) = \mathscr{L}'$ over Z. Then $Z_{s'} = u^{-1}(S_s)$.*

(ii) *Let Z and Z' be S-preschemes, p and p' the projections of $T = Z \times_S Z'$, $\mathscr{L}$ (resp. $\mathscr{L}'$) an invertible $\mathscr{O}_Z$-module (resp. invertible $\mathscr{O}_{Z'}$-module), and let t (resp. t') be a section of $\mathscr{L}$ (resp. $\mathscr{L}'$) over Z (resp. Z'), and s (resp. s') the corresponding section of $p^*(\mathscr{L})$ (resp. $p'^*(\mathscr{L}')$) over $Z \times_S Z'$. Then $T_{s\otimes s'} = Z_t \times_S Z'_{t'}$.*

PROOF. It follows from the definitions that we can restrict to the case where all the preschemes in question are affine. Furthermore, in (i), we can suppose that $\mathscr{L} = \mathscr{O}_B$; claim (i) then follows immediately from (**I**, 1.2.2.2). Similarly, in (ii), we can restrict to the case where $\mathscr{L} = \mathscr{O}_Z$ and $\mathscr{L}' = \mathscr{O}_{Z'}$, and then the claim reduces to Lemma (4.3.2.4). □

We now prove (i *bis*). We can assume that Y is affine (4.6.4), and thus that $\mathscr{L}$ is ample (4.6.6); if s runs over the set of the union of the $\Gamma(X, \mathscr{L}^{\otimes n})$ $(n > 0)$, then the X_s form a base for the topology of X (4.5.2, (a)), and so, by hypothesis, the $j^{-1}(X_s)$ form a base for the topology of X'; we thus conclude, by Lemma (4.6.13.1, (i)) and (4.5.2, (a)), that $j^*(\mathscr{L})$ is ample.

Next we prove (iii). We can again suppose that Y and Y' are affine (4.6.4), whence it follows that the projection $h : X' \to X$ is affine (1.5.5). Since $\mathscr{L}$ is ample (4.6.6), if s runs over the set of sections of the $\mathscr{L}^{\otimes n}$ $(n > 0)$ over X such that X_s is affine, then the X_s cover X (4.5.2, (a')), and so the $h^{-1}(X_s)$ are affine (1.2.5) and cover X'; it thus follows again from Lemma (4.6.13.1, (i)) and from (4.5.2, (a')) that $\mathscr{L}'$ is ample, since the morphism $f_{(Y')}$ is quasi-compact (**I**, 6.6.4, (iii)).

To prove (iv), note first of all that $f_1 \times_S f_2$ is quasi-compact (**I**, 6.6.4, (iv)). We can further suppose that S, Y_1, and Y_2 are affine ((4.6.4) and (**I**, 3.2.7)), and thus that $\mathscr{L}_i$ is ample $(i = 1, 2)$ (4.6.6). The open subsets $(X_1)_{s_1} \times_S (X_2)_{s_2}$ form a cover of $X_1 \times_S X_2$, where s_i runs over the sections of $\mathscr{L}_i^{\otimes n_i}$ such that $(X_i)_{s_i}$ is affine (4.5.2, (a')). Then, replacing s_1 and s_2 with suitable powers, which does not change the $(X_i)_{s_i}$, we can assume that $n_1 = n_2$. We thus deduce, from (4.6.13.1, (ii)) and (4.5.2, (a')), that $\mathscr{L}_1 \otimes_S \mathscr{L}_2$ is ample, whence the claim, since $Y_1 \times_S Y_2$ is affine (4.6.6).

It remains only to prove (ii). By the same argument as in (4.4.10), but here using (4.6.4), we can restrict to the case where Z is affine. Since $\mathscr{K}$ is then ample, and Y quasi-compact, there exists a finite number of sections $s_i \in \Gamma(Y, \mathscr{K}^{\otimes k_i})$ such that the Y_{s_i} are *affine* and cover Y (4.5.2, (a')); **II** | 93 replacing the s_i with suitable powers, we can further suppose that all the k_i are equal to one single integer k. Let s'_i be the sections of $f^*(\mathscr{K}^{\otimes k})$ over X that canonically correspond to the s_i, so that the $X_{s'_i} = f^{-1}(Y_{s_i})$ (4.6.13.1, (i)) cover X. Since $\mathscr{L}|X_{s'_i}$ is ample ((4.6.4) and (4.6.6)), there exists, for each i, a finite number of sections $t_{ij} \in \Gamma(X, \mathscr{L}^{\otimes n_{ij}})$ such that the X_{ij} are affine, contained in the $X_{s'_i}$, and cover $X_{s'_i}$ (4.5.2, (a')); we can also suppose that all the n_{ij} are equal to one single integer n. With this in mind, X is separated and quasi-compact, and so there exists an integer $m > 0$, and, for every (i, j), a section

$$u_{ij} \in \Gamma(X, \mathscr{L}^{\otimes n} \otimes_X f^*(\mathscr{K}^{\otimes mk}))$$

such that $t_{ij} \otimes s_i'^{\otimes m}$ is the restriction to $X_{s'_i}$ of u_{ij} (**I**, 9.3.1); furthermore, $X_{u_{ij}} = X_{t_{ij}}$, and so the $X_{u_{ij}}$ are affine and cover X. We can also suppose that m is of the form nr; if we set $n_0 = rk$, then we see (4.5.2, (a')) that $\mathscr{L} \otimes_{\mathscr{O}_X} f^*(\mathscr{K}^{\otimes n_0})$ is ample. Furthermore, there exists $h_0 > 0$ such that $\mathscr{K}^{\otimes h}$ is generated by its sections over Y for all $h \geqslant h_0$ (4.5.5); *a fortiori*, $f^*(\mathscr{K}^{\otimes h})$ is generated by its sections over X for all $h \geqslant h_0$, by definition of the inverse images ((**0**, 3.7.1) and (4.4.1)). We thus conclude that $\mathscr{L} \otimes f^*(\mathscr{K}^{\otimes n_0 + h})$ is ample for all $h \geqslant h_0$ (4.5.6), which finishes the proof. □

Remark (4.6.14). — Under the conditions of (ii), we refrain from believing that $\mathscr{L} \otimes f^*(\mathscr{K})$ is ample for $g \circ f$; in fact, since $\mathscr{L} \otimes f^*(\mathscr{K}^{-1})$ is also ample for f (4.6.5), we would conclude that $\mathscr{L}$ is ample for $g \circ f$; taking, in particular, g to be the identity morphism, *every* invertible $\mathscr{O}_X$-module would be ample for f, which is not the case in general (see (5.1.6), (5.3.4, (i)), and (5.3.1)).

Proposition (4.6.15). — *Let $f : X \to Y$ be a quasi-compact morphism, $\mathscr{J}$ a quasi-coherent locally-nilpotent sheaf of ideals of $\mathscr{O}_X$, Z the closed subprescheme of X defined by $\mathscr{J}$, and $j : Z \to X$ the canonical injection. For an invertible $\mathscr{O}_X$-module $\mathscr{L}$ to be ample for f, it is necessary and sufficient for $j^*(\mathscr{L})$ to be ample for $f \circ j$.*

PROOF. Since the question is local on Y (4.6.4), we can suppose Y to be affine; since X is then quasi-compact, we can suppose $\mathscr{J}$ to be nilpotent. Taking (4.6.6) into account, the proposition is then exactly the same as (4.5.13). □

Corollary (4.6.16). — *Let X be a locally Noetherian prescheme, and $f : X \to Y$ a quasi-compact morphism. For an invertible $\mathscr{O}_X$-module $\mathscr{L}$ to be ample for f, it is necessary and sufficient for its inverse image $\mathscr{L}'$ under the canonical injection $X_{\mathrm{red}} \to X$ to be ample for f_{red}.*

PROOF. We have already seen that the condition is necessary (4.6.13, (vi)); conversely, if it is satisfied, then we can restrict, to prove that $\mathscr{L}$ is ample for f, to the case where Y is affine (4.6.4); then Y_{red} is also affine, and so $\mathscr{L}'$ is ample (4.6.6), and so too is $\mathscr{L}$ by (4.5.13), since X is then Noetherian and X_{red} a closed subprescheme of X defined by a quasi-coherent nilpotent sheaf of ideals (**I**, 6.1.6). □

Proposition (4.6.17). — *With the notation and hypotheses of (4.4.11), for $\mathscr{L}''$ to be ample relative to f'', it is necessary and sufficient for $\mathscr{L}$ to be ample relative to f and $\mathscr{L}'$ ample relative to f'.*

II | 94

PROOF. The necessity of the condition follows from (4.6.13, (i *bis*)),. To see that the condition is sufficient, we can restrict to the case where Y is affine, and then the fact that $\mathscr{L}''$ is ample follows from criterion (4.5.2, (a)) applied to $\mathscr{L}$, $\mathscr{L}'$, and $\mathscr{L}''$, noting that a section of $\mathscr{L}$ over X can be extended (by 0) to a section of $\mathscr{L}''$ over X''. □

Proposition (4.6.18). — *Let Y be a quasi-compact prescheme, $\mathscr{S}$ a quasi-coherent graded $\mathscr{O}_Y$-algebra of finite type, and $X = \mathrm{Proj}(\mathscr{S})$, and let $f : X \to Y$ the structure morphism. Then f is of finite type, and there exists an integer $d > 0$ such that $\mathscr{O}_X(d)$ is invertible and f-ample.*

PROOF. By (3.1.10), there exists an integer $d > 0$ such that $\mathscr{S}^{(d)}$ is generated by $\mathscr{S}_d$. We know that, under the canonical isomorphism between X and $X^{(d)} = \mathrm{Proj}(\mathscr{S}^{(d)})$, $\mathscr{O}_X(d)$ is identified with $\mathscr{O}_{X^{(d)}}(1)$ (3.2.9, (ii)). We thus see that we can restrict to the case where $\mathscr{S}$ is generated by $\mathscr{S}_1$; the proposition then follows from (4.4.3) and (4.6.2) (taking into account the fact that f is a morphism of finite type (3.4.1)). □

§5. QUASI-AFFINE MORPHISMS; QUASI-PROJECTIVE MORPHISMS; PROPER MORPHISMS; PROJECTIVE MORPHISMS

5.1. Quasi-affine morphisms

Definition (5.1.1). — We define a quasi-affine scheme to be a scheme isomorphic to some subscheme induced on some quasi-compact open subset of an affine scheme. We say that a morphism $f : X \to Y$ is quasi-affine, or that X (considered as a Y-prescheme via f) is a quasi-affine Y-scheme, if there exists a cover (U_α) of Y by affine open subsets such that the $f^{-1}(U_\alpha)$ are quasi-affine schemes.

It is clear that a quasi-affine morphism is *separated* ((**I**, 5.5.5) and (**I**, 5.5.8)) and *quasi-compact* (**I**, 6.6.1); every affine morphisms is evidently quasi-affine.

Recall that, for any prescheme X, setting $A = \Gamma(X, \mathscr{O}_X)$, the identity homomorphism $A \to A = \Gamma(X, \mathscr{O}_X)$ defines a morphism $X \to \mathrm{Spec}(A)$, said to be *canonical* (**I**, 2.2.4); this is nothing but the canonical morphism defined in (4.5.1) for the specific case where $\mathscr{L} = \mathscr{O}_X$, if we remember that $\mathrm{Proj}(A[T])$ is canonically identified with $\mathrm{Spec}(A)$ (3.1.7).

Proposition (5.1.2). — *Let X be a quasi-compact scheme or a prescheme whose underlying space is Noetherian, and A the ring $\Gamma(X, \mathscr{O}_X)$. The following conditions are equivalent.*

(a) *X is a quasi-affine scheme.*
(b) *The canonical morphism $u : X \to \mathrm{Spec}(A)$ is an open immersion.*

(b′) *The canonical morphism $u : X \to \mathrm{Spec}(A)$ is a homeomorphism from X to some subspace of the underlying space of* $\mathrm{Spec}(A)$.
(c) *The $\mathscr{O}_X$-module $\mathscr{O}_X$ is very ample relative to u* (4.4.2).
(c′) *The $\mathscr{O}_X$-module $\mathscr{O}_X$ is ample* (4.5.1).
(d) *When f ranges over A, the X_f form a basis for the topology of X.*
(d′) *When f varies over A, the X_f that are affine form a cover of X.* **II** | 95
(e) *Every quasi-coherent $\mathscr{O}_X$-module is generated by its sections over X.*
(e′) *Every quasi-coherent sheaf of ideals of $\mathscr{O}_X$ of finite type is generated by its sections over X.*

PROOF. It is clear that (b) implies (a), and (a) implies (c) by (4.4.4, b) applied to the identity morphism (taking into account the remark preceding this proposition); Furthermore, (c) implies (c′) (4.5.10, i), and (c′), (b), and (b′) are all equivalent by (4.5.2, b) and (4.5.2, b′). Finally, (c′) is the same as each of (d), (d′), (e), and (e′) by (4.5.2, a), (4.5.2, a′), (4.5.2, c), and (4.5.5, d″). □

We further observe that, with the previous notation, the X_f that are affine form a *basis* for the topology of X, and that the canonical morphism u is *dominant* (4.5.2).

Corollary (5.1.3). — *Let X be a quasi-compact prescheme. If there exists a morphism $v : X \to Y$ from X to some affine scheme Y (which would be a homeomorphism from X to some open subspace of Y), then X is quasi-affine.*

PROOF. There exists a family (g_α) of sections of $\mathscr{O}_Y$ over Y such that the $D(g_\alpha)$ form a basis for the topology of $v(X)$; if $v = (\psi, \theta)$ and we set $f_\alpha = \theta(g_\alpha)$, then we have $X_{f_\alpha} = \psi^{-1}(D(g_\alpha))$ (**I**, 2.2.4.1), so the X_{f_α} form a basis for the topology of X, and the criterion (5.1.2, d) is satisfied. □

Corollary (5.1.4). — *If X is a quasi-affine scheme, then* every *invertible $\mathscr{O}_X$-module is very ample (relative to the canonical morphism), and* a fortiori *ample.*

PROOF. Such a module $\mathscr{L}$ is generated by its sections over X (5.1.2, e), so $\mathscr{L} \otimes \mathscr{O}_X = \mathscr{L}$ is very ample (4.4.8). □

Corollary (5.1.5). — *Let X be a quasi-compact prescheme. If there exists an invertible $\mathscr{O}_X$-module $\mathscr{L}$ such that $\mathscr{L}$ and $\mathscr{L}^{-1}$ are ample, then X is a quasi-affine scheme.*

PROOF. Indeed, $\mathscr{O}_X = \mathscr{L} \otimes \mathscr{L}^{-1}$ is then ample (4.5.7). □

Proposition (5.1.6). — *Let $f : X \to Y$ be a quasi-compact morphism. Then the following conditions are equivalent.*
(a) *The morphism f is quasi-affine.*
(b) *The $\mathscr{O}_Y$-algebra $f_*(\mathscr{O}_X) = \mathscr{A}(X)$ is quasi-coherent, and the canonical morphism $X \to \mathrm{Spec}(\mathscr{A}(X))$ corresponding to the identity morphism $\mathscr{A}(X) \to \mathscr{A}(X)$* (1.2.7) *is an open immersion.*
(b′) *The $\mathscr{O}_Y$-algebra $\mathscr{A}(X)$ is quasi-coherent, and the canonical morphism $X \to \mathrm{Spec}(\mathscr{A}(X))$ is a homeomorphism from X to some subspace of* $\mathrm{Spec}(\mathscr{A}(X))$.
(c) *The $\mathscr{O}_X$-module $\mathscr{O}_X$ is very ample for f.*
(c′) *The $\mathscr{O}_X$-module $\mathscr{O}_X$ is ample for f.*
(d) *The morphism f is separated, and, for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $\sigma : f^*(f_*(\mathscr{F})) \to \mathscr{F}$* (**0**, 4.4.3) *is surjective.*

Furthermore, whenever f is quasi-affine, every invertible $\mathscr{O}_X$-module $\mathscr{L}$ is very ample relative to f.

PROOF. The equivalence between (a) and (c′) follows from the local (on Y) character of the f-ampleness (4.6.4), Definition (5.1.1), and the criterion (5.1.2, c′). The other properties are local on Y and thus follow immediately from (5.1.2) and (5.1.4), taking into account the fact that $f_*(\mathscr{F})$ is quasi-coherent whenever f is separated (**I**, 9.2.2, a). □ **II** | 96

Corollary (5.1.7). — *Let $f : X \to Y$ be a quasi-affine morphism. For every open subset U of Y, the restriction $f^{-1}(U) \to U$ of f is quasi-affine.*

Corollary (5.1.8). — *Let Y be an affine scheme, and $f : X \to Y$ a quasi-compact morphism. For f to be quasi-affine, it is necessary and sufficient for X to be a quasi-affine scheme.*

PROOF. This is an immediate consequence of (5.1.6) and (4.6.6). □

Corollary (5.1.9). — *Let Y be a quasi-compact scheme or a prescheme whose underlying space is Noetherian, and $f : X \to Y$ a morphism of* finite type*. If f is quasi-affine, then there exists a quasi-coherent $\mathscr{O}_Y$-subalgebra $\mathscr{B}$ of $\mathscr{A}(X) = f_*(\mathscr{O}_X)$ of* finite type (**I**, 9.6.2) *such that the morphism $X \to \operatorname{Spec}(\mathscr{B})$ corresponding to the canonical injection $\mathscr{B} \to \mathscr{A}(X)$ is an immersion. Further, every quasi-coherent $\mathscr{O}_Y$-subalgebra $\mathscr{B}'$ of finite type over $\mathscr{A}(X)$ containing $\mathscr{B}$ has the same property.*

PROOF. Indeed, $\mathscr{A}(X)$ is the inductive limit of its quasi-coherent $\mathscr{O}_Y$-subalgebras of finite type (**I**, 9.6.5); the result is then a particular case of (3.8.4), taking into account the identification of $\operatorname{Spec}(\mathscr{A}(X))$ with $\operatorname{Proj}(\mathscr{A}(X)[T])$ (3.1.7). □

Proposition (5.1.10). —

(i) *A quasi-compact morphism $X \to Y$ that is a homeomorphism from the underlying space of X to some subspace of the underlying space of Y (so, in particular, any closed immersion) is quasi-affine.*
(ii) *The composition of any two quasi-affine morphisms is quasi-affine.*
(iii) *If $f : X \to Y$ is a quasi-affine S-morphism, then $f_{(S')} : X_{(S')} \to Y_{(S')}$ is a quasi-affine morphism for any extension $S' \to S$ of the base prescheme.*
(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are quasi-affine S-morphisms, then $f \times_S g$ is quasi-affine.*
(v) *If $f : X \to Y$ and $g : Y \to Z$ are morphisms such that $g \circ f$ is quasi-affine, and if g is separated or the underlying space of X is locally Noetherian, then f is quasi-affine.*
(vi) *If f is a quasi-affine morphism, then so is f_{red}.*

PROOF. Taking into account the criterion (5.1.6, c′), all of (i), (iii), (iv), (v), and (vi) follow immediately from (4.6.13, i *bis*), (4.6.13, iii), (4.6.13, iv), (4.6.13, v), and (4.6.13, vi) (respectively). To prove (ii), we can restrict to the case where Z is affine, and then the claim follows directly from applying (4.6.13, ii) to $\mathscr{L} = \mathscr{O}_X$ and $\mathscr{K} = \mathscr{O}_Y$. □

Remark (5.1.11). — Let $f : X \to Y$ and $g : Y \to Z$ be morphisms such that $X \times_Z Y$ is locally Noetherian. Then the graph immersion $\Gamma_f : X \to X \times_Z Y$ is quasi-affine, since it is quasi-compact (**I**, 6.3.5), and since (**I**, 5.5.12) shows that, in (v), the conclusion still holds true if we remove the hypothesis that g is separated.

Proposition (5.1.12). — *Let $f : X \to Y$ be a quasi-compact morphism, and $g : X' \to X$ a quasi-affine morphism. If $\mathscr{L}$ is an ample (for f) $\mathscr{O}_X$-module, then $g^*(\mathscr{L})$ is an ample (for $f \circ g$) $\mathscr{O}_{X'}$-module.*

PROOF. Since $\mathscr{O}_{X'}$ is very ample for g, and the question is local on Y (4.6.4), it follows from (4.6.13, ii) that there exists (for Y affine) an integer n such that

$$g^*(\mathscr{L}^{\otimes n}) = (g^*(\mathscr{L}))^{\otimes n}$$

is ample for $f \circ g$, and so $g^*(\mathscr{L})$ is ample for $f \circ g$ (4.6.9) □

5.2. Serre's criterion

Theorem (5.2.1). — (Serre's criterion). *Let X be a quasi-compact scheme or a prescheme whose underlying space is Noetherian. The following conditions are equivalent.*

(a) *X is an affine scheme.*
(b) *There exists a family of elements $f_\alpha \in A = \Gamma(X, \mathscr{O}_X)$ such that the X_{f_α} are affine, and such that the ideal generated by the f_α in A is equal to A itself.*
(c) *The functor $\Gamma(X, \mathscr{F})$ is exact in $\mathscr{F}$ on the category of quasi-coherent $\mathscr{O}_X$-modules, or, in other words, if*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0 \tag{*}$$

is an exact sequence of quasi-coherent $\mathscr{O}_X$-modules, then the sequence

$$0 \longrightarrow \Gamma(X, \mathscr{F}') \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{F}'') \longrightarrow 0$$

is also exact.
(c′) *Condition* (c) *holds for every exact sequence $(*)$ of quasi-coherent $\mathscr{O}_X$-modules such that $\mathscr{F}$ is isomorphic to an $\mathscr{O}_X$-submodule of $\mathscr{O}_X^n$ for some finite n.*
(d) *$\mathrm{H}^1(X, \mathscr{F}) = 0$ for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$.*
(d′) *$\mathrm{H}^1(X, \mathscr{J}) = 0$ for every quasi-coherent sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_X$.*

PROOF. It is evident that (a) implies (b); furthermore, (b) implies that the X_{f_α} cover X, because, by hypothesis, the section 1 is a linear combination of the f_α, and the $D(f_\alpha)$ thus cover $\operatorname{Spec}(A)$. The final claim of (4.5.2) thus implies that $X \to \operatorname{Spec}(A)$ is an isomorphism.

We know that (a) implies (c) (**I**, 1.3.11), and (c) trivially implies (c′). We now prove that (c′) implies (b). First of all, (c′) implies that, for every *closed* point $x \in X$ and every open neighbourhood U of x, there exists some $f \in A$ such that $x \in X_f \subset X - U$. Let $\mathscr{J}$ (resp. $\mathscr{J}'$) be the quasi-coherent sheaf of ideals of $\mathscr{O}_X$ defining the closed reduced subprescheme of X that has $X - U$ (resp. $(X - U) \cup \{x\}$) as its underlying space (**I**, 5.2.1); it is clear that we have $\mathscr{J}' \subset \mathscr{J}$, and that $\mathscr{J}'' = \mathscr{J}/\mathscr{J}'$ is a quasi-coherent $\mathscr{O}_X$ module that has support equal to $\{x\}$, and such that $\mathscr{J}''_x = k(x)$. Hypothesis (c′) applied to the exact sequence $0 \to \mathscr{J}' \to \mathscr{J} \to \mathscr{J}'' \to 0$ shows that $\Gamma(X, \mathscr{J}) \to \Gamma(X, \mathscr{J}'')$ is surjective. The section of $\mathscr{J}''$ whose germ at x is 1_x is thus the image of some section $f \in \Gamma(X, \mathscr{J}) \subset \Gamma(X, \mathscr{O}_X)$, and we have, by definition, that $f(x) = 1_x$ and $f(y) = 0$ in $X - U$, which establishes our claim. Now, if U is affine, then so is X_f (**I**, 1.3.6), so the union of the X_f that are affine ($f \in A$) is an open set Z that contains *all the closed points* of X; since X is a quasi-compact Kolmogoroff space, we necessarily have $Z = X$ (**0**, 2.1.3). Because X is quasi-compact, there are a *finite* number of elements $f_i \in A$ ($1 \leqslant i \leqslant n$) such that the X_{f_i} are affine and cover X. So consider the homomorphism $\mathscr{O}_X^n \to \mathscr{O}_X$ defined by the sections f_i (**0**, 5.1.1); since, for all $x \in X$, at least one of the $(f_i)_x$ is invertible, this homomorphism is *surjective*, and we thus have an exact sequence $0 \to \mathscr{R} \to \mathscr{O}_X^n \to \mathscr{O}_X \to 0$, where $\mathscr{R}$ is a quasi-coherent $\mathscr{O}_X$-submodule of $\mathscr{O}_X$. It then follows from (c′) that the corresponding homomorphism $\Gamma(X, \mathscr{O}_X^n) \to \Gamma(X, \mathscr{O}_X)$ is surjective, which **II** | 98 proves (b).

Finally, (a) implies (d) (**I**, 5.1.9.2), and (d) trivially implies (d′). It remains to show that (d′) implies (c′). But if $\mathscr{F}'$ is a quasi-coherent $\mathscr{O}_X$-submodule of $\mathscr{O}_X^n$, then the filtration $0 \subset \mathscr{O}_X \subset \mathscr{O}_X^2 \subset \ldots \subset \mathscr{O}_X^n$ defines a filtration of $\mathscr{F}'$ given by the $\mathscr{F}'_k = \mathscr{F} \cap \mathscr{O}_X^k$ ($0 \leqslant k \leqslant n$), which are quasi-coherent $\mathscr{O}_X$-modules (**I**, 4.1.1), and $\mathscr{F}'_{k+1}/\mathscr{F}'_k$ is isomorphic to a quasi-coherent $\mathscr{O}_X$-submodule of $\mathscr{O}_X^{k+1}/\mathscr{O}_X^k = \mathscr{O}_X$, which is to say, a quasi-coherent sheaf of ideals of $\mathscr{O}_X$. Hypothesis (d′) thus implies that $\mathrm{H}^1(X, \mathscr{F}'_{k+1}/\mathscr{F}'_k) = 0$; the exact cohomology sequence $\mathrm{H}^1(X, \mathscr{F}'_k) \to \mathrm{H}^1(X, \mathscr{F}'_{k+1}) \to \mathrm{H}^1(X, \mathscr{F}'_{k+1}/\mathscr{F}'_k) = 0$ then lets us prove by induction on k that $H^1(X, \mathscr{F}'_k) = 0$ for all k. □

Remark (5.2.1.1). — When X is a *Noetherian* prescheme, we can replace "quasi-coherent" by "coherent" in the statements of (c′) and (d′). Indeed, in the proof of the fact that (c′) implies (b), $\mathscr{J}$ and $\mathscr{J}'$ are then *coherent* sheaves of ideals, and, furthermore, every quasi-coherent submodule of a coherent module is coherent (**I**, 6.1.1); whence the conclusion.

Corollary (5.2.2). — *Let $f : X \to Y$ be a separated quasi-compact morphism. The following conditions are equivalent.*

- (a) *The morphism f is an affine morphism.*
- (b) *The functor f_* is exact on the category of quasi-coherent $\mathscr{O}_X$-modules.*
- (c) *For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we have $\mathrm{R}^1 f_*(\mathscr{F}) = 0$.*
- (c′) *for every quasi-coherent sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_X$, we have $\mathrm{R}^1 f_*(\mathscr{J}) = 0$.*

PROOF. All these conditions are local on Y, by definition of the functor $\mathrm{R}^1 f_*$ (T, 3.7.3), and so we can assume that Y is affine. If f is affine, then X is affine, and property (b) is nothing more than (**I**, 1.6.4). Conversely, we now show that (b) implies (a): for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we have that $f_*(\mathscr{F})$ is a quasi-coherent $\mathscr{O}_Y$-module (**I**, 9.2.2, a). By hypothesis, the functor $f_*(\mathscr{F})$ is exact in $\mathscr{F}$, and the functor $\Gamma(Y, \mathscr{G})$ is exact in $\mathscr{G}$ (in the category of quasi-coherent $\mathscr{O}_Y$-modules) because Y is affine (**I**, 1.3.11); so $\Gamma(Y, f_*(\mathscr{F})) = \Gamma(X, \mathscr{F})$ is exact in $\mathscr{F}$, which proves our claim, by (5.2.1, c).

If f is affine, then $f^{-1}(U)$ is affine for every affine open subset U of Y (1.3.2), and so $\mathrm{H}^1(f^{-1}(U), \mathscr{F}) = 0$ (5.2.1, d), which, by definition, implies that $\mathrm{R}^1 f_*(\mathscr{F}) = 0$. Finally, suppose that condition (c′) is satisfied; the exact sequence of terms of low degree in the Leray spectral sequence (G, II, 4.17.1 and I, 4.5.1) give, in particular, the exact sequence

$$0 \longrightarrow \mathrm{H}^1(Y, f_*(\mathscr{J})) \longrightarrow \mathrm{H}^1(X, \mathscr{J}) \longrightarrow \mathrm{H}^0(Y, \mathrm{R}^1 f_*(\mathscr{J})).$$

Since Y is affine, and $f_*(\mathscr{J})$ quasi-coherent (I, 9.2.2, a), we have that $\mathrm{H}^1(Y, f_*(\mathscr{J})) = 0$ (5.2.1); hypothesis (c′) thus implies that $\mathrm{H}^1(X, \mathscr{J}) = 0$, and we conclude, by (5.2.1), that X is an affine scheme. □

Corollary (5.2.3). — *If $f : X \to Y$ is an affine morphism, then, for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $\mathrm{H}^1(Y, f_*(\mathscr{F})) \to \mathrm{H}^1(X, \mathscr{F})$ is bijective.*

II | 99

Proof. We have the exact sequence

$$0 \longrightarrow \mathrm{H}^1(Y, f_*(\mathscr{F})) \longrightarrow \mathrm{H}^1(X, \mathscr{F}) \longrightarrow \mathrm{H}^0(Y, \mathrm{R}^1 f_*(\mathscr{F}))$$

of terms of low degree in the Leray spectral sequence, and the conclusion follows from (5.2.2). □

Remark (5.2.4). — In Chapter III, §1, we prove that, if X is affine, then we have $\mathrm{H}^i(X, \mathscr{F}) = 0$ for all $i > 0$ and all quasi-coherent $\mathscr{O}_X$-modules $\mathscr{F}$.

5.3. Quasi-projective morphisms

Definition (5.3.1). — We say that a morphism $f : X \to Y$ is *quasi-projective*, or that X (considered as a Y-prescheme via f) is *quasi-projective over Y*, or that X is a *quasi-projective Y-scheme*, if f is of finite type and there exists an invertible f-ample $\mathscr{O}_X$-module.

We note that this notion *is not local on Y*: the counterexamples of Nagata [**Nag58b**] and Hironaka show that, even if X and Y are non-singular algebraic schemes over an algebraically closed field, every point of Y can have an affine neighbourhood U such that $f^{-1}(U)$ is quasi-projective over U, without f being quasi-projective.

We note that a quasi-projective morphism is necessarily *separated* (4.6.1). When Y is quasi-compact, it is equivalent to say either that f is quasi-projective, or that f is of finite type and there exists a *very ample* (relative to f) $\mathscr{O}_X$-module ((4.6.2) and (4.6.11)). Further:

Proposition (5.3.2). — *Let Y be a quasi-compact scheme or a prescheme whose underlying space is Noetherian, and let X be a Y-prescheme. The following conditions are equivalent.*

(a) *X is a quasi-projective Y-scheme.*
(b) *X is of finite type over Y, and there exists some quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ of finite type such that X is Y-isomorphic to a subprescheme of $\mathbf{P}(\mathscr{E})$.*
(c) *X is of finite type over Y, and there exists some quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S}$ such that $\mathscr{S}_1$ is of finite type and generates $\mathscr{S}$, and such that X is Y-isomorphic to a induced subprescheme on some everywhere-dense open subset of $\mathrm{Proj}(\mathscr{S})$.*

Proof. This follows immediately from the previous remark and from (4.4.3), (4.4.6), and (4.4.7). □

We note that, whenever Y is a *Noetherian* prescheme, we can, in conditions (b) and (c) of (5.3.2), remove the hypothesis that X is of finite type over Y, since this automatically satisfied (I, 6.3.5).

Corollary (5.3.3). — *Let Y be a quasi-compact scheme such that there exists an ample $\mathscr{O}_Y$-module $\mathscr{L}$ (4.5.3). For a Y-scheme X to be quasi-projective, it is necessary and sufficient for it to be of finite type over Y and also isomorphic to a Y-subscheme of a projective bundle of the form $\mathbf{P}^r_Y$.*

Proof. If $\mathscr{E}$ is a quasi-coherent $\mathscr{O}_Y$-module of finite type, then $\mathscr{E}$ is isomorphic to a quotient of an $\mathscr{O}_Y$-module $\mathscr{L}^{\otimes(-n)} \otimes_{\mathscr{O}_Y} \mathscr{O}_Y^k$ (4.5.5), and so $\mathbf{P}(\mathscr{E})$ is isomorphic to a closed subscheme of $\mathbf{P}^{k-1}_Y$ ((4.1.2) and (4.1.4)). □

Proposition (5.3.4). —

(i) *A quasi-affine morphism of finite type (and, in particular, a quasi-compact immersion, or an affine morphism of finite type) is quasi-projective.*
(ii) *If $f : X \to Y$ and $g : Y \to Z$ are quasi-projective, and if Z is quasi-compact, then $g \circ f$ is quasi-projective.* II | 100
(iii) *If $f : X \to Y$ is a quasi-projective S-morphism, then $f_{(S')} : X_{(S')} \to Y_{(S')}$ is quasi-projective for every extension $S' \to S$ of the base prescheme.*

(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are quasi-projective S-morphisms, then $f \times_S g$ is quasi-projective.*
(v) *If $f : X \to Y$ and $g : Y \to Z$ are morphisms such that $g \circ f$ is quasi-projective, and if g is separated or X locally Noetherian, then f is quasi-projective.*
(vi) *If f is a quasi-projective morphism, then so is f_{red}.*

PROOF. (i) follows from (5.1.6) and (5.1.10, i). The other claims are immediate consequences of Definition (5.3.1), of the properties of morphisms of finite type (**I**, 6.3.4), and of (4.6.13). □

Remark (5.3.5). — We note that we can have f_{red} being quasi-projective without f being quasi-projective, even if we assume that Y is the spectrum of an algebra of finite rank over $\mathbf{C}$ and that f is proper.

Corollary (5.3.6). — *If X and X' are quasi-projective Y-schemes, then $X \sqcup X'$ is a quasi-projective Y-scheme.*

PROOF. This follows from (4.6.18). □

5.4. Proper morphisms and universally closed morphisms

Definition (5.4.1). — We say that a morphism of preschemes $f : X \to Y$ is *proper* if it satisfies the following two conditions:

(a) f is separated and of finite type; and
(b) for every prescheme Y' and every morphism $Y' \to Y$, the projection $f_{(Y')} : X \times_Y Y' \to Y'$ is a closed morphism (**I**, 2.2.6).

When this is the case, we also say that X (considered as a Y-prescheme with structure morphism f) is proper over Y.

It is immediate that conditions (a) and (b) are *local* on Y. To show that the image of a closed subset Z of $X \times_Y Y'$ under the projection $q : X \times_Y Y' \to Y'$ is closed in Y, it suffices to see that $q(Z) \cap U'$ is closed in U' for every affine open subset U' of Y'; since $q(Z) \cap U' = q(Z \cap q^{-1}(U'))$, and since $q^{-1}(U')$ can be identified with $X \times_Y U'$ (**I**, 4.4.1), we see that to satisfy condition (b) of Definition (5.4.1), we can restrict to the case where Y is an *affine* scheme. We further see (5.3.6) that, if Y is locally Noetherian, then we can even restrict to proving (b) in the case where Y' is of finite type over Y.

It is clear that every proper morphism is *closed*.

Proposition (5.4.2). —

(i) *A closed immersion is a proper morphism.*
(ii) *The composition of two proper morphisms is proper.*
(iii) *If X and Y are S-preschemes, and $f : X \to Y$ a proper S-morphism, then $f_{(S')} : X_{(S')} \to Y_{(S')}$ is proper for every extension $S' \to S$ of the base prescheme.*
(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are proper S-morphisms, then $f \times_S g : X \times_S Y \to X' \times_S Y'$ is a proper S-morphism.*

II | 101

PROOF. It suffices to prove (i), (ii), and (iii) (**I**, 3.5.1). In each of the three cases, verifying condition (a) of (5.4.1) follows from previous results ((**I**, 5.5.1) and (6.4.3)); it remains to verify condition (b). It is immediate in case (i), because if $X \to Y$ is a closed immersion, then so is $X \times_Y Y' \to Y \times_Y Y' = Y'$ ((**I**, 4.3.2) and (3.3.3)). To prove (ii), consider two proper morphisms $X \to Y$ and $Y \to Z$, and a morphism $Z' \to Z$. We can write $X \times_Z Z' = X \times_Y (Y \times_Z Z')$ (**I**, 3.3.9.1), and so the projection $X \times_Z Z' \to Z'$ factors as $X \times_Y (Y \times_Z Z') \to Y \times_Z Z' \to Z'$. Taking the initial remark into account, (ii) follows from the fact that the composition of two closed morphisms is closed. Finally, for every morphism $S' \to S$, we can identify $X_{(S')}$ with $X \times_Y Y_{(S')}$ (**I**, 3.3.11); for every morphism $Z \to Y_{(S')}$, we can write

$$X_{(S')} \times_{Y_{(S')}} Z = (X \times_Y Y_{(S')}) \times_{Y_{(S')}} Z = X \times_Y Z;$$

since by hypothesis $X \times_Y Z \to Z$ is closed, this proves (iii). □

Corollary (5.4.3). — *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms such that $g \circ f$ is proper.*

(i) *If g is separated, then f is proper.*
(ii) *If g is separated and of finite type, and if f is surjective, then g is proper.*

PROOF. (i) follows from (5.4.2) by the general procedure (**I**, 5.5.12). To prove (ii), we need only verify that condition (b) of Definition (5.4.1) is satisfied. For every morphism $Z' \to Z$, the diagram

$$\begin{tikzcd} X \times_Z Z' \arrow[r, "f\times 1_{Z'}"] \arrow[dr, "p"'] & Y \times_Z Z' \arrow[d, "p'"] \\ & Z' \end{tikzcd}$$

(where p and p' are the projections) commutes (**I**, 3.2.1); furthermore, $f \times 1_{Z'}$ is surjective because f is surjective (**I**, 3.5.2), and p is a closed morphism by hypothesis. Every closed subset F of $Y \times_Z Z'$ is thus the image under $f \times 1_{Z'}$ of some closed subset E of $X \times_Z Z'$, so $p'(F) = p(E)$ is closed in Z' by hypothesis, whence the corollary. □

Corollary (5.4.4). — *If X is a proper prescheme over Y, and $\mathscr{S}$ a quasi-coherent $\mathscr{O}_Y$-algebra, then every Y-morphism $f : X \to \mathrm{Proj}(\mathscr{S})$ is proper (and* a fortiori *closed).*

PROOF. The structure morphism $p : \mathrm{Proj}(\mathscr{S}) \to Y$ is separated, and $p \circ f$ is proper by hypothesis. □

Corollary (5.4.5). — *Let $f : X \to Y$ be a separated morphism of finite type. Let $(X_i)_{1\leqslant i\leqslant n}$ (resp. $(Y_i)_{1\leqslant i\leqslant n}$) be a finite family of closed subpreschemes of X (resp. Y), and j_i (resp. h_i) the canonical injection $X_i \to X$ (resp. $Y_i \to Y$). Suppose that the underlying space of X is the union of the X_i, and that, for all i, there is a morphism $f_i : X_i \to Y_i$, such that the diagram*

$$\begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[d, "j_i"'] & Y_i \arrow[d, "h_i"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes. Then, for f to be proper, it is necessary and sufficient for all of the f_i to be proper.

II | 102

PROOF. If f is proper, then so is $f \circ j_i$, because j_i is a closed immersion (5.4.2); since h_i is a closed immersion, and thus a separated morphism, f_i is proper, by (5.4.3). Conversely, suppose that all of the f_i are proper, and consider the prescheme Z given by the *sum* of the X_i; let u be the morphism $Z \to X$ which reduces to j_i on each X_i. The restriction of $f \circ u$ to each X_i is equal to $f \circ j_i = h_i \circ f_i$, and is thus proper, because both the h_i and the f_i are (5.4.2); it then follows immediately from Definition (5.4.1) that u is proper. But since by hypothesis u is surjective, we conclude that f is proper by (5.4.3). □

Corollary (5.4.6). — *Let $f : X \to Y$ be a separated morphism of finite type; for f to be proper, it is necessary and sufficient for $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ to be proper.*

PROOF. This is a particular case of (5.4.5), with $n = 1$, $X_1 = X_{\mathrm{red}}$, and $Y_1 = Y_{\mathrm{red}}$ (**I**, 5.1.5). □

(5.4.7). If X and Y are Noetherian preschemes, and $f : X \to Y$ a separated morphism of finite type, then we can, to show that f is proper, restrict to the the case of *dominant* morphisms and *integral* preschemes. Indeed, let X_i $(1 \leqslant i \leqslant n)$ be the (finitely many) irreducible components of X, and consider, for each i, the unique closed reduced subprescheme of X that has X_i as its underlying space, which we again denote by X_i (**I**, 5.2.1). Let Y_i be the unique closed reduced subprescheme of Y that has $\overline{f(X_i)}$ as its underlying space. If g_i (resp. h_i) is the injection morphism $X_i \to X$ (resp. $Y_i \to Y$), then we conclude that $f \circ g_i = h_i \circ f_i$, where f_i is a dominant morphism $X_i \to Y_i$ (**I**, 5.2.2); we are then under the right conditions to apply (5.4.5), and for f to be proper, it is necessary and sufficient for all the f_i to be proper.

Corollary (5.4.8). — *Let X and Y be separated S-preschemes of finite type over S, and $f : X \to Y$ an S-morphism. For f to be proper, it is necessary and sufficient that, for every S-prescheme S', the morphism $f \times_S 1_{S'} : X \times_S S' \to Y \times_S S'$ be closed.*

PROOF. First note that, if $g : X \to S$ and $h : Y \to S$ are the structure morphisms, then we have, by definition, $g = h \circ f$, and so f is separated and of finite type ((**I**, 5.5.1) and (6.3.4)). If f is proper, then so is $f \times_S 1_{S'}$ (5.4.2); *a fortiori*, $f \times_S 1_{S'}$ is closed. Conversely, suppose that the conditions of the statement are satisfied, and let Y' be a Y-prescheme; Y' can also be considered as an S-prescheme, and since $Y \to S$ is separated, $X \times_Y Y'$ can be identified with a closed subprescheme of $X \times_S Y'$ (**I**, 5.4.2). In the commutative diagram

$$\begin{tikzcd} X \times_Y Y' \arrow[r, "f \times 1_{Y'}"] \arrow[d] & Y \times_Y Y' = Y' \arrow[d] \\ X \times_S Y' \arrow[r, "f \times 1_{S'}"] & Y \times_S Y', \end{tikzcd}$$

the vertical arrows are closed immersions; it thus immediately follows that if $f \times 1_{S'}$ is a closed morphism, then so is $f \times 1_{Y'}$ □

Remark (5.4.9). — We say that a morphism $f : X \to Y$ is *universally closed* if it satisfies condition (b) of Definition (5.4.1). The reader will observe that, in (5.4.2) to (5.4.8), we can replace every occurrence of "proper" with "universally closed" without changing the validity of the results (and in the hypotheses of (5.4.3), (5.4.5), (5.4.6), and (5.4.8), we can omit the finiteness conditions). **II** | 103

(5.4.10). Let $f : X \to Y$ be a morphism of finite type. We say that a closed subset Z of X is *proper on* Y (or Y-*proper*, or *proper for* f) if the restriction of f to a closed subprescheme of X, with underlying space Z (**I**, 5.2.1), is *proper*. Since this restriction is then separated, it follows from (5.4.6) and (**I**, 5.5.1, vi) that the preceding property *does not depend* on the closed subprescheme of X that has Z as its underlying space. If $g : X' \to X$ is a *proper* morphism, then $g^{-1}(Z)$ is a *proper* subset of X': if T is a subprescheme of X that has Z as its underlying space, it suffices to note that the restriction of g to the closed subprescheme $g^{-1}(T)$ of X' is a proper morphism $g^{-1}(T) \to T$, by (5.4.2, iii), and to then apply (5.4.2, ii). Further, if X'' is a Y-*scheme* of finite type, and $u : X \to X''$ a Y-morphism, then $u(Z)$ is a *proper* subset of X''; indeed, let us take T to be the closed reduced subprescheme of X having Z as its underlying space; then the restriction of f to T is proper, and thus so is the restriction of u to T (5.4.3, i), thus $u(Z)$ is closed in X''; let T'' be a closed subprescheme of X'' having $u(Z)$ as its underlying space (**I**, 5.2.1), such that $u|T$ factors as $T \xrightarrow{v} T'' \xrightarrow{j} X''$, where j is the canonical injection (**I**, 5.2.2), and v is thus proper and surjective (5.4.5); if g is the restriction to T'' of the structure morphism $X'' \to Y$, then g is separated and of finite type, and we have that $f|T = g \circ v$; it thus follows from (5.4.3, ii) that g is proper, whence our assertion.

It follows, in particular, from these remarks that, if Z is a Y-proper subset of X, then

(1) for every closed subprescheme X' of X, $Z \cap X'$ is a Y-proper subset of X'; and
(2) if X is a subprescheme of a Y-scheme of finite type X'', then Z is also a Y-proper subset of X'' (and so, in particular, is *closed in* X'').

5.5. Projective morphisms

Proposition (5.5.1). — *Let X be a Y-prescheme. The following conditions are equivalent.*

(a) *X is Y-isomorphic to a* closed *subprescheme of a projective bundle $\mathbf{P}(\mathscr{E})$, where $\mathscr{E}$ is a quasi-coherent $\mathscr{O}_Y$-module of finite type.*

(b) *There exists a quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S}$ such that $\mathscr{S}_1$ is of finite type and generates $\mathscr{S}$, and such that X is Y-isomorphic to $\operatorname{Proj}(\mathscr{S})$.*

PROOF. Condition (a) implies (b), by (3.6.2, ii): if $\mathscr{J}$ is a quasi-coherent graded sheaf of ideals of $\mathbf{S}(\mathscr{E})$, then the quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S} = \mathbf{S}(\mathscr{E})/\mathscr{J}$ is generated by $\mathscr{S}_1$, and $\mathscr{S}_1$, the canonical image of $\mathscr{E}$, is an $\mathscr{O}_Y$-module of finite type. Condition (b) implies (a) by (3.6.2) applied to the case where $\mathscr{M} \to \mathscr{S}_1$ is the identity map. □

II | 104

Definition (5.5.2). — We say that a Y-prescheme X is *projective* on Y, or is a projective Y-scheme, if it satisfies either of the (equivalent) conditions (a) and (b) of (5.5.1). We say that a morphism $f : X \to Y$ is projective if it makes X a projective Y-scheme.

It is clear that if $f : X \to Y$ is projective, then there exists a *very ample* (relative to f) $\mathscr{O}_X$-module (4.4.2).

Theorem (5.5.3). —

(i) *Every projective morphism is quasi-projective and proper.*

(ii) *Conversely, let Y be a quasi-compact scheme or a prescheme whose underlying space is Noetherian; then every morphism $f : X \to Y$ that is quasi-projective and proper is projective.*

PROOF.

(i) It is clear that if $f : X \to Y$ is projective, then it is of finite type and quasi-projective (thus, in particular, separated); furthermore, it follows immediately from (5.5.1, b) and (3.5.3) that if f is projective, then so is $f \times_Y 1_{Y'} : X \times_Y Y' \to Y'$ for every morphism $Y' \to Y$. To show that f is universally closed, it is thus enough to show that a projective morphism f is *closed*. Since the question is local on Y, we can suppose that $Y = \operatorname{Spec}(A)$, thus (5.5.1) $X = \operatorname{Proj}(S)$, where S is a graded A-algebra generated by a finite number of elements of S_1. For all $y \in Y$, the fibre $f^{-1}(y)$ can be identified with $\operatorname{Proj}(S) \times_Y \operatorname{Spec}(k(y))$ (**I**, 3.6.1), and so also with $\operatorname{Proj}(S \otimes_A k(y))$ (2.8.10); so $f^{-1}(y)$ is empty if and only if $S \otimes_A k(y)$ satisfies condition (TN) (2.7.4), or, in other words, if $S_n \otimes_A k(y) = 0$ for sufficiently large n. But since $(S_n)_y$ is an $\mathscr{O}_y$-module of finite type, the preceding condition implies that $(S_n)_y = 0$ for sufficiently large N, by Nakayama's lemma. If $\mathfrak{a}_n$ is the annihilator in A of the A-module S_n, then the preceding condition also implies that $\mathfrak{a}_n \subset \mathfrak{j}_n$ for sufficiently large n (**0**, 1.7.4). But since $S_n S_1 = S_{n+1}$, by hypothesis, we have that $\mathfrak{a}_n \subset \mathfrak{a}_{n+1}$, and if $\mathfrak{a}$ is the union of the $\mathfrak{a}_n$, then we see that $f(X) = V(\mathfrak{a})$, which proves that $f(X)$ is closed in Y. If now X' is an arbitrary closed subset of X, then there exists a closed subprescheme of X that has X' as its underlying space (**I**, 5.2.1), and it is clear (5.5.1, a) that the morphism $X' \to X \xrightarrow{f} Y$ is projective, and so $f(X')$ is closed in Y.

(ii) The hypothesis on Y and the fact that f is quasi-projective implies the existence of a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ of finite type, as well as a Y-immersion $j : X \to \mathbf{P}(\mathscr{E})$ (5.3.2). But since f is proper, j is *closed*, by (5.4.4), and so f is projective.

□

Remark (5.5.4). —

(i) Let $f : X \to Y$ be a morphism such that f is proper, such that there exists a *very ample* (relative to f) $\mathscr{O}_X$-module $\mathscr{L}$, and such that the quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E} = f_*(\mathscr{L})$ is *of finite type*. Then f is a *projective* morphism: indeed (4.4.4), there is then a Y-immersion $r : X \to \mathbf{P}(\mathscr{E})$, and, since f is proper, r is a *closed* immersion (5.4.4). We will see in Chapter III, §3, that when Y is *locally Noetherian*, the third condition above ($\mathscr{E}$ being of finite type) is a consequence of the first two, and so the first two conditions *characterise*, in this case, the projective morphisms, and if Y is quasi-compact, then we can replace the second condition (the existence of a very ample (relative to f) $\mathscr{O}_X$-module $\mathscr{L}$) by the hypothesis that there exists an *ample* (relative to f) $\mathscr{O}_X$-module (4.6.11).

(ii) Let Y be a quasi-compact scheme such that there exists an ample $\mathscr{O}_Y$-module. For a Y-scheme X to be *projective*, it is necessary and sufficient for it to be Y-isomorphic to a *closed* Y-subscheme of a projective bundle of the form $\mathbf{P}^r_Y$. The condition is clearly sufficient. Conversely, if X is projective over Y, then it is quasi-projective, and so there exists a II | 105
Y-immersion j of X into some $\mathbf{P}^r_Y$ (5.3.3) that is *closed*, by (5.4.4) and (5.5.3).

(iii) The argument of (5.5.3) shows that, for every prescheme Y and every integer $r \geqslant 0$, the structure morphism $\mathbf{P}^r_Y \to Y$ is *surjective*, because if we set $\mathscr{S} = \mathbf{S}_{\mathscr{O}_Y}(\mathscr{O}_Y^{r+1})$, then we evidently have $\mathscr{S}_y = \mathbf{S}_{k(y)}(k(y)^{r+1})$ (1.7.3), and so $(\mathscr{S}_n)_y \neq 0$ for any $y \in Y$ or any $n \geqslant 0$.

(iv) It follows from the examples of Nagata [**Nag58b**] that there exist proper morphisms that are not quasi-projective.

Proposition (5.5.5). —

(i) *A closed immersion is a projective morphism.*

(ii) *If $f : X \to Y$ and $g : Y \to Z$ are projective morphisms, and if Z is a quasi-compact scheme or a prescheme whose underlying space is Noetherian, then $g \circ f$ is projective.*

(iii) *If $f : X \to Y$ is a projective S-morphism, then $f_{(S')} : X_{(S')} \to Y_{(S')}$ is projective for every extension $S' \to S$ of the base prescheme.*

(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are projective S-morphisms, then so is $f \times_S g$.*

(v) *If $g \circ f$ is a projective morphism, and if g is separated, then f is projective.*

(vi) *If f is projective, then so is f_{red}.*

PROOF. (i) follows immediately from (3.1.7). We have to show (iii) and (iv) separately, because of the restriction introduced on Z in (ii) (cf. (**I**, 3.5.1)). To show (iii), we restrict to the case where $S = Y$ (**I**, 3.3.11), and the claim then immediately follows from (5.5.1, b) and (3.5.3). To show (iv), we are immediately led to the case where $X = \mathbf{P}(\mathscr{E})$ and $X = \mathbf{P}(\mathscr{E}')$, where $\mathscr{E}$ (resp. $\mathscr{E}'$) is a quasi-coherent $\mathscr{O}_Y$-module (resp. quasi-coherent $\mathscr{O}_{Y'}$-module) of finite type. Let p and p' be the canonical projections of $T = Y \times_S Y'$ to Y and Y' (respectively); by (4.1.3.1), we have $\mathbf{P}(p^*(\mathscr{E})) = \mathbf{P}(\mathscr{E}) \times_Y T$ and $\mathbf{P}(p'^*(\mathscr{E}')) = \mathbf{P}(\mathscr{E}') \times_{Y'} T$; whence

$$\begin{aligned}\mathbf{P}(p^*(\mathscr{E})) \times_T \mathbf{P}(p'^*(\mathscr{E}')) &= (\mathbf{P}(\mathscr{E}) \times_Y T) \times_T (T \times_{Y'} \mathbf{P}(\mathscr{E}')) \\ &= \mathbf{P}(\mathscr{E}) \times_Y (T \times_{Y'} \mathbf{P}(\mathscr{E}')) = \mathbf{P}(\mathscr{E}) \times_S \mathbf{P}(\mathscr{E}')\end{aligned}$$

by replacing T with $Y \times_S Y'$, and using (**I**, 3.3.9.1). But $p^*(\mathscr{E})$ and $p'^*(\mathscr{E}')$ are of finite type over T (**0**, 5.2.4), and thus so is $p^*(\mathscr{E}) \otimes_{\mathscr{O}_T} p'^*(\mathscr{E}')$; since $\mathbf{P}(p^*(\mathscr{E})) \times_T \mathbf{P}(p'^*(\mathscr{E}'))$ can be identified with a closed subprescheme of $p^*(\mathscr{E}) \otimes_{\mathscr{O}_T} p'^*(\mathscr{E}')$ (4.3.3), this proves (iv). To show (v) and (vi), we can apply (**I**, 5.5.13), because every closed subprescheme of a projective Y-scheme is a projective Y-scheme, by (5.5.1, a).

It remains to prove (ii); by the hypothesis on Z, this follows from (5.5.3), (5.3.4, ii), and (5.4.2, ii). □

Proposition (5.5.6). — *If X and X' are projective Y-schemes, then $X \sqcup X'$ is a projective Y-scheme.*

PROOF. This is an evident consequence of (5.5.2) and (4.3.6). □

Proposition (5.5.7). — *Let X be a projective Y-scheme, and $\mathscr{L}$ a Y-ample $\mathscr{O}_X$-module; then, for every section f of $\mathscr{L}$ over X, X_f is affine over Y.*

II | 106

PROOF. Since the question is local on Y, we can assume that $Y = \operatorname{Spec}(A)$; furthermore, $X_{f^{\otimes n}} = X_f$, so by replacing $\mathscr{L}$ with some suitable $\mathscr{L}^{\otimes n}$, we can assume that $\mathscr{L}$ is very ample relative to the structure morphism $q : X \to Y$ (4.6.11). The canonical homomorphism $\sigma : q^*(q_*(\mathscr{L})) \to \mathscr{L}$ is thus surjective, and the corresponding morphism

$$r = r_{\mathscr{L},\sigma} : X \longrightarrow P = \mathbf{P}(q_*(\mathscr{L}))$$

is an immersion such that $\mathscr{L} = r^*(\mathscr{O}_P(1))$ (4.4.4); furthermore, since X is proper over Y, the immersion r is closed (5.4.4). But by definition, $f \in \Gamma(Y, q_*(\mathscr{L}))$, and $\sigma^\flat$ is the identity of $q_*(\mathscr{L})$; it then follows from Equation (3.7.3.1) that we have $X_f = r^{-1}(D_+(f))$; so X_f is a closed subprescheme of the affine scheme $D_+(f)$, and is thus also an affine scheme. □

In the particular case where $Y = X$, we obtain (taking (4.6.13, i) into account) the following corollary, whose direct proof is immediate anyway:

Corollary (5.5.8). — *Let X be a prescheme, and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module. For every section f of $\mathscr{L}$ over X, X_f is affine over X (and thus also an affine scheme whenever X is an affine scheme).*

5.6. Chow's lemma

Theorem (5.6.1). — (Chow's lemma). *Let S be a prescheme, and X an S-scheme of finite type. Suppose that the following conditions are satisfied:*

(a) *S is Noetherian;*
(b) *S is a quasi-compact scheme, and X has a finite number of irreducible components.*

Under these hypotheses,

(i) *there exists a* quasi-projective *S-scheme X', and an S-morphism $f : X' \to X$ that is both*projective *and* surjective*;*
(ii) *we can take X' and f to be such that there exists an open subset $U \subset X$ for which $U' = f^{-1}(U)$ is dense in X', and for which the restriction of f to U' is an isomorphism $U' \simeq U$; and*
(iii) *if X is reduced (resp. irreducible, integral), then we can assume that X' is reduced (resp. irreducible, integral).*

Proof. The proof proceeds in multiple steps.

(A) We can first restrict to the case where X is *irreducible*. Indeed, in hypothesis (a), X is Noetherian, and so, in the two hypotheses, the irreducible components X_i of X are finite in number. If the theorem is shown to be true for each of the closed reduced preschemes of X having the X_i as their underlying spaces, and if X'_i and $f_i : X'_i \to X_i$ are the prescheme and the morphism corresponding to X_i (respectively), then the prescheme X' given by the *sum* of the X'_i, and the morphism $f : X' \to X$ whose restriction to each X'_i is $j_i \circ f_i$ (where j_i is the canonical injection $X_i \to X$) satisfy the conclusion of the theorem. It is immediate that X' is reduced if all of the X'_i are; furthermore, we can satisfy (ii) by taking U to be the union of the sets $U_i \cap \complement\left(\bigcup_{j \neq i} X_j\right)$. Finally, since the X'_i are quasi-projective over S, so is X' (5.3.6); similarly, the morphisms $X'_i \to X$ are projective by (5.5.5, i) and (5.5.5, ii), and so f is projective (5.5.6), and is clearly surjective, by definition. **II** | 107

(B) Now suppose that X is *irreducible*. Since the structure morphism $r : X \to S$ is of finite type, there exists a finite cover (S_i) of S by affine open subsets, and for each i there is a finite cover (T_{ij}) of $r^{-1}(S_i)$ by affine open subsets, and the morphisms $T_{ij} \to S_i$ are of finite type, and so quasi-projective (5.3.4, i); since in both hypotheses (a) and (b) the immersion $S_i \to S$ is quasi-compact, it is also quasi-projective (5.3.4, i), and so the restriction of r to T_{ij} is a quasi-projective morphism (5.3.4, ii). Denote the T_{ij} by U_k ($1 \leqslant k \leqslant n$). There exists, for each index k, an open immersion $\phi_k : U_k \to P_k$, where P_k is projective over S ((5.3.2) and (5.5.2)). Let $U = \bigcap_k U_k$; since X is irreducible, and the U_k nonempty, U is nonempty, and thus dense in X; the restrictions of the ϕ_k to U define a morphism

$$\phi : U \longrightarrow P = P_1 \times_S P_2 \times_S \cdots \times_S P_n$$

such that the diagrams

(5.6.1.1)
$$\begin{tikzcd} U \arrow[r, "\phi"] \arrow[d, "j_k"'] & P \arrow[d, "p_k"] \\ U_k \arrow[r, "\phi_k"] & P_k \end{tikzcd}$$

commute, where j_k is the canonical injection $U \to U_k$, and p_k the canonical projection $P \to P_k$. If j is the canonical injection $U \to X$, then the morphism $\psi = (j, \phi)_S : U \to X \times_S P$

is an *immersion* (I, 5.3.14). In hypothesis (a), $X \times_S P$ is locally Noetherian ((3.4.1), (I, 6.3.7), and (I, 6.3.8)); in hypothesis (b), $X \times_S P$ is a quasi-compact scheme ((I, 5.5.1) and (I, 6.6.4)); in both cases, the *closure* X' in $X \times_S P$ of the subprescheme Z associated to ψ (and so with underlying space $\psi(U)$) exists, and ψ factors as

(5.6.1.2) $$\psi : U \xrightarrow{\psi'} X' \xrightarrow{h} X \times_S P$$

where ψ' is an *open immersion* and h a *closed immersion* (I, 9.5.10). Let $q_1 : X \times_S P \to X$ and $q_2 : X \times_S P \to P$ be the canonical projections; we set

(5.6.1.3) $$f : X' \xrightarrow{h} X \times_S P \xrightarrow{q_1} X,$$

(5.6.1.4) $$g : X' \xrightarrow{h} X \times_S P \xrightarrow{q_2} P.$$

We will see that X' and f satisfy the conclusion of the theorem.

(C) First we show that f is *projective* and *surjective*, and that the restriction of f to $U' = f^{-1}(U)$ is an *isomorphism* from U' to U. Since the P_k are projective over S, so is P (5.5.5, iv), and so $X \times_S P$ is projective over X (5.5.5, iii), and thus so is X', which is a closed subprescheme of $X \times_S P$. Furthermore, we have $f \circ \psi' = q_1 \circ (h \circ \psi') = q_1 \circ \psi = j$, so $f(X')$ contains the open everywhere-dense subset U of X; but f is a *closed* morphism (5.5.3), so $f(X') = X$. Now note that $q_1^{-1}(U) = U \times_S P$ is induced on an open subset of $X \times_S P$, and, by definition, the prescheme $U' = h^{-1}(U \times_S P)$ is induced by X' on the open subset U'; it is thus the closure *relative to* $U \times_S P$ of the prescheme Z (I, 9.5.8). But the immersion ψ factors as **II** | 108
$U \xrightarrow{\Gamma_\phi} U \times_S P \xrightarrow{j \times 1} X \times_S P$, and since P is separated over S, the graph morphism Γ_ϕ is a closed immersion (I, 5.4.3), and so Z is a *closed* subprescheme of $U \times_S P$, whence $U' = Z$. Since ψ is an immersion, the restriction of f to U' is an isomorphism onto U, and the inverse of ψ'; finally, by the definition of X', U' is dense in X'.

(D) We now show that g is an *immersion*, which will imply that X' is *quasi-projective* over S, because P is projective over S. Set

$$\begin{aligned} V_k &= \phi_k(U_k) \quad \text{(open subset of } P_k) \\ W_k &= p_k^{-1}(V_k) \quad \text{(open subset of } P) \\ U'_k &= f^{-1}(U_k) \quad \text{(open subset of } X') \\ U''_k &= g^{-1}(W_k) \quad \text{(open subset of } X'). \end{aligned}$$

It is clear that the U'_k form an open cover of X'; we will first see that the U''_k also form an open cover of X', by showing that $U'_k \subset U''_k$. For this, it will suffice to show that the diagram

(5.6.1.5) $$\begin{tikzcd} U'_k \arrow[r, "g|U'_k"] \arrow[d, "f|U'_k"'] & P \arrow[d, "p_k"] \\ U_k \arrow[r, "\phi_k"] & P_k \end{tikzcd}$$

commutes. But the prescheme $U'_k = h^{-1}(U_k \times_S P)$ is induced by X' on the open subset U'_k, and is thus the closure of $Z = U' \subset U'_k$ relative to U'_k (I, 9.5.8). To show the commutativity of (5.6.1.5), it thus suffices (since P_k is an S-scheme) to show that composing the diagram with the canonical injection $U' \to U'_k$ (or, equivalently, thanks to the isomorphism from U' to U, with ψ) gives us a commutative diagram (I, 9.5.6). But, by definition, the diagram thus obtains is exactly (5.6.1.1), whence our claim.

The W_k thus form an open cover of $g(X')$; to show that g is an immersion, it suffices to show that each of the restrictions $g|U''_k$ is an immersion into W_k (I, 4.2.4). For this, consider the morphism $u_k : W_k \xrightarrow{p_k} V_k \xrightarrow{\phi_k^{-1}} U_k \to X$; since X is separated over S, the graph morphism $\Gamma_{u_k} : W_k \to X \times_S W_k$ is a closed immersion (I, 5.4.3), and so the graph $T_k = \Gamma_{u_k}(W_k)$ is a closed subprescheme of $X \times_S W$; if we show that $U' \to X \times_S W_k$ factors

through this subprescheme, then the map from the subprescheme induced by X' on the open subset X''_k of X' to $X \times_S W_k$ will also factor through this graph, by (I, 9.5.8). Since the restriction of q_2 to T_k is an isomorphism onto W_k, the restriction of g to X''_k will be an immersion into W_k, and our claim will be proven. Let v_k be the canonical injection $U' \to X \times_S W_k$; we have to show that there exists a morphism $w_k : U' \to W_k$ such that $v_k = \Gamma_{u_k} \circ w_k$. By the definition of the product, it suffices to prove that $q_1 \circ v_k = u_k \circ q_2 \circ v_k$ (I, 3.2.1), or, by composing on the right with the isomorphism $\psi' : U \to U'$, that $q_1 \circ \psi =$ **II** | 109
$u_k \circ q_2 \circ \psi$. But since $q_1 \circ \psi = j$ and $q_2 \circ \psi = \phi$, our claim follows from the commutativity of (5.6.1.1), taking into account the definition of u_k.

(E) It is clear that since U, and thus U', is irreducible, so is the X' from the preceding construction, and the morphism f is thus *birational* (I, 2.2.9). If in addition X is reduced, then so is U', and hence X' is also reduced (I, 9.5.9). This finishes the proof.

□

Corollary (5.6.2). — *Suppose that one of the hypotheses,* (a) *and* (b), *of* (5.6.1) *is satisfied. For X to be proper over S, it is necessary and sufficient for there to exist a projective scheme X' over S, and a surjective S−morphism $f : X' \to X$ (which is thus projective, by* (5.5.5, v)*). Whenever this is the case, we can further choose f to be such that there exists a dense open subset U of X for which the restriction of f to $f^{-1}(U)$ is an isomorphism $f^{-1}(U) \simeq U$, and for which $f^{-1}(U)$ is dense in X'. If in addition X is irreducible (resp. reduced), then we can assume that X' is also irreducible (resp. reduced); when X and X' are irreducible, f is a birational morphism.*

PROOF. The condition is sufficient, by (5.5.3) and (5.4.3, ii). It is necessary because, with the notation of (5.6.1), if X is proper over S, then X' is proper over S, because it is projective over X, and thus proper over X (5.5.3), and our claim follows from (5.4.2, ii); furthermore, since X' is quasi-projective over S, it is projective over S, by (5.5.3). □

Corollary (5.6.3). — *Let S be a locally Noetherian prescheme, and X an S-scheme of finite type over S, with structure morphism $f_0 : X \to S$. For X to be proper over S, it is necessary and sufficient that, for every morphism* of finite type *$S' \to S$, $(f_0)_{(S')} : X_{(S')} \to S'$ be a closed morphism. It even suffices for this condition to be verified only for every S-prescheme of the form $S' = S \otimes_{\mathbf{Z}} \mathbf{Z}[T_1, \ldots, T_n]$ (where the T_i are indeterminates).*

PROOF. The condition being clearly necessary, we now show that it is sufficient. Since the question is local on S and S' (5.4.1), we can suppose that S and S' are affine and Noetherian. By Chow's lemma, there exists a projective S-scheme P, an immersion $j : X' \to P$, and a surjective projective morphism $f : X' \to X$, such that the diagram

$$\begin{tikzcd} X \arrow[d, "f_0"'] & X' \arrow[l, "f"'] \arrow[d, "j"] \\ S & P \arrow[l, "r"] \end{tikzcd}$$

commutes. Since P is of finite type over S, the first hypothesis implies that the projection $q_2 : X \times_S P \to P$ is a *closed* morphism. But the immersion j is the composition of q_2 and the morphism $f \times 1$ from $X' \times_S P$ to $X \times_S P$; but f, being projective, is proper (5.5.3), and so $f \times 1$ is closed. We thus conclude that j is a closed immersion, and thus proper (5.4.2, i). Furthermore, the structure morphism $r : P \to S$ is projective, and thus proper (5.5.3), so $f_0 \circ f = r \circ j$ is proper (5.4.2, ii); finally, since f is surjective, f_0 is proper, by (5.4.3).

To prove the proposition using only the second, weaker hypothesis (where S' is of the form $S \otimes_{\mathbf{Z}} \mathbf{Z}[T_1, \ldots, T_n]$), it suffices to show that it implies the first. But, if S' is affine and of finite type over $S = \operatorname{Spec}(A)$, then we have $S' = \operatorname{Spec}(A[c_1, \ldots, c_n])$ (I, 6.3.3), and S' is thus isomorphic to a closed **II** | 110
subprescheme of $S'' = \operatorname{Spec}(A[T_1, \ldots, T_n])$ (where the T_i are indeterminates). In the commutative

diagram

$$\begin{array}{ccc} X \times_S S' & \xrightarrow{1_X \times j} & X \times_S S'' \\ {\scriptstyle (f_0)_{(S')}}\downarrow & & \downarrow{\scriptstyle (f_0)_{(S'')}} \\ S' & \xrightarrow{\quad j \quad} & S'' \end{array}$$

both j and $1_X \times j$ are closed immersions (**I**, 4.3.1), and $(f_0)_{(S')}$ is closed by hypothesis; thus $(f_0)_{(S'')}$ is also closed. □

§6. Integral morphisms and finite morphisms

6.1. Preschemes integral over another prescheme

Definition (6.1.1). — Let X be an S-prescheme, with structure morphism $f : X \to S$. We say that X is *integral over* S, or that f is an *integral morphism*, if there exists a cover (S_α) of S by affine opens such that, for all α, the induced prescheme $f^{-1}(S_\alpha)$ is an affine scheme whose ring B_α is an integral algebra over the ring A_α of S_α. We say that X is *finite over* S, or that f is a *finite morphism* if X is integral and of finite type over S.

If S is affine of ring A, then we also say "integral (resp. finite) over A" instead of "integral (resp. finite) over S".

(6.1.2). It is clear that, if X is integral over S, then it is *affine* over S. For an affine prescheme X over S to be integral (resp. finite) over S it is necessary and sufficient that the associated quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{A}(X)$ be such that there exist a cover (S_α) of S by affine opens having the property that, for all α, $\Gamma(S_\alpha, \mathscr{A}(X))$ is an integral (resp. integral and of finite type) algebra over $\Gamma(S_\alpha, \mathscr{O}_S)$. A quasi-coherent $\mathscr{O}_S$-algebra with this property is said to be *integral* (resp. *finite*) over $\mathscr{O}_S$. Giving an integral (resp. finite) prescheme over S is thus (1.3.1) the same as giving a quasi-coherent $\mathscr{O}_S$-algebra that is integral (resp. finite) over $\mathscr{O}_S$. Note that a quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{B}$ is finite if and only if it is an *$\mathscr{O}_S$-module of finite type* (**I**, 1.3.9); it is equivalent to say that $\mathscr{B}$ is an *integral* $\mathscr{O}_S$-algebra *of finite type*, since an algebra that is integral and of finite type over a ring A is an A-module of finite type.

Proposition (6.1.3). — *Let S be a locally Noetherian prescheme. For an affine prescheme X over S to be finite over S, it is necessary and sufficient that the $\mathscr{O}_S$-algebra $\mathscr{A}(X)$ be coherent.*

Proof. Taking the preceding remark into account, this reduces to noting that, if S is locally Noetherian, then the quasi-coherent $\mathscr{O}_S$-modules of finite type are exactly the coherent $\mathscr{O}_S$-modules (**I**, 1.5.1). □

Proposition (6.1.4). — *Let X be an integral (resp. finite) prescheme over S, with structure morphism $f : X \to S$. Then, for every affine open $U \subset S$ of ring A, $f^{-1}(U)$ is an affine scheme whose ring B is an integral (resp. finite) algebra over A.*

II | 111

Proof. We first prove the following lemma:

Lemma (6.1.4.1). — *Let A be a ring, M an A-module, and $(g_i)_{1 \leqslant i \leqslant m}$ a finite system of elements of A such that the $D(g_i)$ (for $1 \leqslant i \leqslant m$) cover* $\operatorname{Spec}(A)$. *If, for all i, M_{g_i} is an A_{g_i}-module of finite type, then M is an A-module of finite type.*

Proof. We can assume that M_{g_i} admits a finite system of generators (m_{ij}/g_i^n) with $m_{ij} \in M$, with n the same for all indices i. We will show that the m_{ij} for a system of generators of M. By hypothesis, for each i, there exist $a_{ij} \in A$ and some integer p (independent of i) such that, in M_{g_i}, $m/1 = (\sum_i a_{ij} m_{ij})/g_i^p$; this implies that there exists an integer $r \geqslant p$ such that, for all i, we have $g_i^r m \in M'$. But, since the $D(g_i^r) = D(g_i)$ cover $\operatorname{Spec}(A)$, the ideal of A generated by the g_i^r is equal to A, or, in other words, there exist elements $a_i \in A$ such that $\sum_i a_i g_i^r$; then $m = (\sum_i a_i g_i^r) m \in M'$, whence the lemma. □

Now we already know (1.3.2) that $f^{-1}(U)$ is affine. If ϕ is the homomorphism $A \to B$ corresponding to f, then there exists a finite cover of U by opens $D(g_i)$ (where $g_i \in A$) such that, if $h_i = \phi(g_i)$, B_{h_i} is an integral (resp. integral and finite) algebra over A_{g_i}. Indeed, there exists a cover of U by affine opens $V_\alpha \subset U$ such that, if $A_\alpha = A(V_\alpha)$, $B_\alpha = A(f^{-1}(V_\alpha))$ is an integral (resp. finite) algebra over A_α. Every $x \in U$ belongs to some V_α, so there exists $g \in A$ such that $x \in D(g) \subset V_\alpha$; if g_α is the image of g in A_α, then $A(D(g)) = A_g = (A_\alpha)_{g_\alpha}$; let $h = \phi(g)$, and let h_α be the image of g_α in B_α; we have

$$A(D(h)) = B_h = (B_\alpha)_{h_\alpha}$$

and, since B_α is integral (resp. finite) over A_α, $(B_\alpha)_{h_\alpha}$ is integral (resp. finite) over $(A_\alpha)_{g_\alpha}$. It now suffices to use the fact that U is quasi-compact to obtain the desired cover.

If we suppose first of all that the B_{h_i} are integral and finite over the A_{g_i}, then since B_{h_i} can also be written as B_{g_i} as an A_{g_i}-module, Lemma (6.1.4.1) shows that, in this case, B is an A-module of finite type.

Now suppose only that each B_{h_i} is integral over A_{g_i}; let $b \in B$, and let C be the sub-A-algebra of B generated by b. For all i, C_{h_i} is the algebra over A_{g_i} generated by $b/1$ in B_{h_i}; it follows from the hypothesis that each C_{h_i} is an A_{g_i}-module of finite type, and so (6.1.4.1) C is an A-module of finite type, which proves that B is integral over A. □

Proposition (6.1.5). —

(i) A closed immersion is finite (and a fortiori *integral).*
(ii) The composition of two finite (resp. integral) morphisms is finite (resp. integral).
(iii) If $f : X \to Y$ is a finite (resp. integral) S-morphism, then $f_{(S')} : X_{(S')} \to Y_{(S')}$ is finite (resp. integral) for any base extension $S' \to S$.
(iv) If $f : X \to Y$ and $g : X' \to Y'$ are finite (resp. integral) S-morphisms, then $f \times_S g : X \times_S Y \to X' \times_S Y'$ is finite (resp. integral).
(v) If $f : X \to Y$ and $g : Y \to Z$ are morphisms such that $g \circ f$ is finite (resp. integral), if g is separated, then f is finite (resp. integral).
(vi) If $f : X \to Y$ is a finite (resp. integral) morphism, then f_{red} is finite (resp. integral).

II | 112

PROOF. By (**I**, 5.5.12), it suffices to prove (i), (ii), and (iii). To prove that a closed immersion $X \to S$ is finite, we can restrict to the case where $S = \operatorname{Spec}(A)$, and everything then follows from noting that a quotient ring $A/\mathfrak{J}$ is a monogeneous A-module. To prove that the composition of two finite (resp. integral) morphism $X \to Y$, $Y \to Z$ is finite (resp. integral), we can again assume that Z (and thus X and Y (1.3.4)) is affine, and then the claim is equivalent to saying that, if B is a finite (resp. integral) A-algebra and C a finite (resp. integral) B-algebra, then C is a finite (resp. integral) A-algebra, which is immediate. Finally, to prove (iii), we can restrict to the case where $S = Y$, since $X_{(S')}$ can be identified with $X \times_Y Y_{(S')}$ (**I**, 3.3.11); we can further suppose that $S = \operatorname{Spec}(A)$ and $S' = \operatorname{Spec}(A')$; then X is affine of ring B (1.3.4), and $X_{(S')}$ affine of ring $A' \otimes_A B$, and it suffices to note that, if B is a finite (resp. integral) A-algebra, then $A' \otimes_A B$ is a finite (resp. integral) A'-algebra. □

We also note that, if X and Y are S-preschemes that are finite (resp. integral) over S, then their *sum* $X \sqcup Y$ is a finite (resp. integral) prescheme over S, since this reduces to showing that, if B and C are finite (resp. integral) A-algebras over A, then so too is $B \times C$.

Corollary (6.1.6). — *If X is an integral (resp. finite) prescheme over S, then, for every open $U \subset S$, $f^{-1}(U)$ is integral (resp. finite) over U.*

PROOF. This is a particular case of (6.1.5, (iii)). □

Corollary (6.1.7). — *Let $f : X \to Y$ be a finite morphism. Then, for all $y \in Y$, the fibre $f^{-1}(y)$ is a finite algebraic scheme over $k(y)$, and* a fortiori *its underlying space is discrete and finite.*

PROOF. Indeed, as a $k(y)$-prescheme, $f^{-1}(y)$ can be identified with $X \times_Y \operatorname{Spec}(k(y))$ (**I**, 3.6.1), which is finite over $\operatorname{Spec}(k(y))$ (6.1.5, (iii)); it is thus an affine scheme whose ring is an algebra of finite rank over $k(y)$ (6.1.4). The corollary then follows from (**I**, 6.4.4). □

Corollary (6.1.8). — *Let X and S be integral preschemes, and $f : X \to S$ a* dominant *morphism. If f is integral (resp. finite) then the field $R(X)$ of rational functions on X is algebraic (resp. algebraic of finite degree) over the field $R(S)$ of rational functions on S.*

PROOF. Let s be the generic point of S; the $k(s)$-prescheme $f^{-1}(s)$ is integral (resp. finite) over $\operatorname{Spec}(k(s))$ (6.1.5, (iii)) and contains, by hypothesis, the generic point x of X; since the local ring of x in $f^{-1}(s)$ is equal to $k(x)$ (**I**, 3.6.5), and is thus a local ring of an integral (resp. finite) algebra over $k(s)$ (6.1.4), whence the corollary. □

Remark (6.1.9). — The hypothesis that g be *separated* is essential for the validity of (6.1.5, (v)): if Y is not separated over S, then the identity 1_Y is the composite morphism $Y \xrightarrow{\Delta_Y} Y \times_Z Y \xrightarrow{p_1} Y$, but Δ_Y is not an integral morphism, as follows from (6.1.10):

Proposition (6.1.10). — *Every integral morphism is universally closed.*

PROOF. Let $f : X \to Y$ be an integral morphism; by (6.1.5, (iii)), it suffices to show that f is *closed*. Let Z be a closed subset of X; then there exists a subprescheme of X whose underlying space is Z (**I**, 5.4.1), and it thus follows from (6.1.5, (i) and (ii)) that it suffices to prove that $f(X)$ is *closed* in Y. By (6.1.5, (vi)), we can suppose that X and Y are *reduced*; further, if T is the closed reduced subprescheme of Y whose underlying space is $\overline{f(X)}$ (**I**, 5.2.1) then we know that f factors as $X \to T \xrightarrow{j} Y$, where j is the injection morphism (**I**, 5.2.2), and since j is separated (**I**, 5.5.1, (i)), it follows from (6.1.5, (v)) that g is an integral morphism. We can thus suppose that $f(X)$ is *dense* in Y. Finally, since the question is local on Y, we can restrict to the case where $Y = \operatorname{Spec}(A)$. Then $X = \operatorname{Spec}(B)$, where B is an A-algebra that is integral over A (6.1.4); furthermore, A is reduced (**I**, 5.1.4), and the hypothesis that $f(X)$ be dense in Y implies that the homomorphism $\phi : A \to B$ corresponding to f is *injective* (**I**, 1.2.7). Under these conditions, saying that $f(X) = Y$ implies that every prime ideal of A is the intersection with A of a prime ideal of B, which is exactly the first theorem of Cohen–Seidenberg ([**?**, t. I, p. 257, th. 3]). □ **II** | 113

Corollary (6.1.11). — *Every finite morphism $f : X \to Y$ is projective.*

PROOF. Since f is affine, $\mathscr{O}_X$ is a *very ample* $\mathscr{O}_X$-module with respect to f (5.1.2); furthermore, $f_*(\mathscr{O}_X)$ is a quasi-coherent $\mathscr{O}_Y$-module *of finite type* (6.1.2); finally, f is separated, of finite type, and universally closed (6.1.10), and thus satisfies the conditions of criterion (5.5.4, (i)). □

Proposition (6.1.12). — *Let $f : X' \to X$ be a finite morphism, and let $\mathscr{B} = f_*(\mathscr{O}_{X'})$ (which is a quasi-coherent $\mathscr{O}_X$-algebra, and a $\mathscr{O}_X$-module of finite type). Let $\mathscr{F}'$ be a quasi-coherent $\mathscr{O}_{X'}$-module; for $\mathscr{F}'$ to be locally free of rank r, it is necessary and sufficient that $f_*(\mathscr{F}')$ be a locally free $\mathscr{B}$-module of rank r.*

PROOF. It is clear that, if $f_*(\mathscr{F}')|U$ is isomorphic to $\mathscr{B}^r|U$ (where $U \subset X$ is open), then $\mathscr{F}'|f^{-1}(U)$ is isomorphic to $\mathscr{O}_{X'}^r|f^{-1}(U)$ (1.4.2). Conversely, suppose that $\mathscr{F}'$ is locally free of rank r; we will show that $f_*(\mathscr{F}')$ is locally isomorphic to $\mathscr{B}^r$ as a $\mathscr{B}$-module. Let x be a point of X; as U runs over a fundamental system of affine neighbourhoods of x, $f^{-1}(U)$ runs over a fundamental system of affine neighbourhoods (1.2.5) of the finite set $f^{-1}(x)$, since f is closed (6.1.10). The proposition then follows from the following lemma: □

Lemma (6.1.12.1). — *Let Y be a prescheme, $\mathscr{E}$ a locally free $\mathscr{O}_Y$-module of rank r, and Z a finite subset of Y contained inside some affine open V. Then there exists a neighbourhood $U \subset V$ of Z such that $\mathscr{E}|U$ is isomorphic to $\mathscr{O}_Y^r|U$.*

PROOF. We can evidently assume that Y is affine; for all $z_i \in Z$, there exists in the closure $\overline{\{z_i\}}$ at least one closed point z_i' (**0**, 2.1.3); if Z' is the set of the z_i' then every neighbourhood of Z' is a neighbourhood of Z, and we can thus assume that Z is discrete and closed in Y. Consider the closed reduced subprescheme of Y that has Z has its underlying space (**I**, 5.2.1) and let $j : Z \to Y$ be the canonical injection; $j^*(\mathscr{E}) = \mathscr{E} \otimes_Y \mathscr{O}_Z$ is locally free of rank r on the discrete scheme Z, and is thus isomorphic to $\mathscr{O}_Z^r$; in other words, there exist r sections s_i (for $1 \leqslant i \leqslant r$) of $\mathscr{E} \otimes_Y \mathscr{O}_Z$ over Z such that the homomorphism $\mathscr{O}_Z^r \to \mathscr{E} \otimes_Y \mathscr{O}_Z$ defined by these sections is bijective. But $Y = \operatorname{Spec}(A)$ is affine, Z is defined by an ideal $\mathfrak{J}$ of A, and we have $\mathscr{E} = \widetilde{M}$, where M is an A-module; the s_i are elements of $M \otimes_A (A/\mathfrak{J})$ and are thus images of r elements $t_i \in M = \Gamma(Y, \mathscr{E})$. For all $z_j \in Z$ there is thus a **II** | 114 neighbourhood V_j of z_j such that the restrictions of the t_i to V_j defined an isomorphism $\mathscr{O}_Y^r|V_j \to \mathscr{E}|V_j$ (**0**, 5.5.4); the neighbourhood U given by the union of the V_j is the desired neighbourhood. □

Proposition (6.1.13). — *Let $g : X' \to X$ be an integral morphism of preschemes, Y a normal locally integral prescheme, and f a rational map from Y to X' such that $g \circ f$ is an everywhere defined rational map* (**I**, 7.2.1). *Then f is everywhere defined.*

PROOF. If f_1 and f_2 are morphisms (from dense open subsets of Y to X') in the class f, then it is clear that $g \circ f_1$ and $g \circ f_2$ are equivalent morphisms, which justifies the notation $g \circ f$ for their equivalence class. We recall also that, if we further suppose Y to be *locally Noetherian*, then the hypothesis that Y is normal already implies that Y is locally integral (**I**, 6.1.13).

To prove the proposition, note first of all that the question is local on Y, and so we can suppose that there exists in the class $g \circ f$ a *morphism* $h : Y \to X$. Consider the inverse image $Y' = X'_{(h)} = X'_{(Y)}$, and note that the morphism $g' = g_{(Y)} : Y' \to Y$ is *integral* (6.1.5, (iii)). Given the correspondence between rational maps from Y to X' and rational Y-sections of Y' (**I**, 7.1.2), we see that it suffices to prove the specific case of (6.1.13) where $X = Y$; in other words, the following: □

Corollary (6.1.14). — *Let X be a normal locally integral prescheme, $g : X' \to X$ an* integral *morphism, and f a rational X-section of X'. Then f is everywhere defined.*

PROOF. Since the question is local on X, we can assume that X is integral, and then f is identified with a morphism from an open U of X to X' (**I**, 7.2.2) that is a U-section of $g^{-1}(U)$. Since g is separated, f is a closed immersion from U into $g^{-1}(U)$ (**I**, 5.4.6); let Z be the closed subprescheme of $g^{-1}(U)$ associated to f (**I**, 4.2.1), which is isomorphic to U, and thus integral; let X_1 be the reduced subprescheme of X' whose underlying space is the closure $\overline{Z}$ of Z in X' (**I**, 5.2.1); then Z is an induced subprescheme on an open of X_1 (**I**, 5.2.3), and, since it is irreducible, so too is X_1, which is thus integral. The morphism f can then be considered as a rational X-section of X_1; since the restriction of g to X_1 is an integral morphism (6.1.5, (i) and (ii)), we can finally reduce to proving (6.1.14) in the specific case where $X' = X_1$; in other words, the following: □

Corollary (6.1.15). — *Let X be a normal integral prescheme, X' an integral prescheme, and $g : X' \to X$ an* integral *morphism. If there exists a rational X-section f of X', then g is an isomorphism.*

PROOF. Since the question is local on X, we can assume that X is affine of integral ring A, and then X' is affine of ring A' with A' integral over A (6.1.4) and integral; furthermore, the argument of (6.1.14) shows that there exists a dense open of X that is isomorphic to a dense open of X', and so A and A' have the same field of fractions. Also, by (**I**, 8.2.1.1), and the hypothesis that the $\mathscr{O}_x$ are integrally closed, the ring A is integrally closed, and so $A' = A$, which finishes the proof of (6.1.13) □

6.2. Quasi-finite morphisms

Proposition (6.2.1). — *Let $f : X \to Y$ be morphism locally of finite type, and x a point of X. Then the following conditions are equivalent:* **II** | 115

a) The point x is isolated in its fibre $f^{-1}(f(x))$.
b) The ring $\mathscr{O}_x$ is a quasi-finite $\mathscr{O}_{f(x)}$-module (**0**, 7.4.1).

PROOF. Since the question is clearly local on X and on Y, we can assume that $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$ are affine, with A a B-algebra of finite type (**I**, 6.3.3). Furthermore, we can replace X by $X \times_Y \operatorname{Spec}(\mathscr{O}_{f(x)})$ without changing either the fibre $f^{-1}(f(x))$ or the local ring $\mathscr{O}_x$ (**I**, 3.6.5); we can thus assume that B is a local ring (equal to $\mathscr{O}_{f(x)}$); if $\mathfrak{n}$ is the maximal ideal of B, then $f^{-1}(f(x))$ is an affine scheme of ring $A/\mathfrak{n}A$, of finite type over $k(f(x)) = B/\mathfrak{n}$ (**I**, 6.4.11). With this, if (a) is satisfied then we can further suppose that $f^{-1}(f(x))$ consists of the single point x; thus $A/\mathfrak{n}A$ is of finite rank over $B/\mathfrak{n}$ (**I**, 6.4.4), or, in other words, A is a quasi-finite B-module. Conversely, if A is a quasi-finite B-module, then $f^{-1}(f(x))$ is an Artinian affine scheme, and thus discrete (**I**, 6.4.4); so x is isolated in its fibre, and this shows that (b) implies (a). □

Corollary (6.2.2). — *Let $f : X \to Y$ be a morphism of finite type. Then the following conditions are equivalent:*

a) *Every point $x \in X$ is isolated in its fibre $f^{-1}(f(x))$ (or, in other words, the subspace $f^{-1}(f(x))$ is* discrete*).*
b) *For all $x \in X$, the prescheme $f^{-1}(f(x))$ is a finite $k(f(x))$-prescheme.*
c) *For all $x \in X$, the ring $\mathscr{O}_x$ is a quasi-finite $\mathscr{O}_{f(x)}$-module.*

PROOF. The equivalence of (a) and (c) follows from (6.2.1). Since $f^{-1}(f(x))$ is an algebraic $k(f(x))$-prescheme (**I**, 6.4.11), the equivalence of (a) and (b) follows from (**I**, 6.4.4). □

Definition (6.2.3). — If $f : X \to Y$ is a morphism of finite type satisfying the equivalent conditions of (6.2.2), we say that f is *quasi-finite*, or that X is *quasi-finite over Y*.

It is clear that every *finite* morphism is quasi-finite (6.1.8).

Proposition (6.2.4). —

(i) *An immersion $X \to Y$ that is closed, or such that X is Noetherian, is a quasi-finite morphism.*
(ii) *If $f : X \to Y$ and $g : Y \to Z$ are quasi-finite morphisms, then $g \circ f$ is quasi-finite.*
(iii) *If X and Y are S-preschemes, and $f : X \to Y$ a quasi-finite S-morphism, then $f_{(S')} : X_{(S')} \to Y_{(S')}$ is quasi-finite for any base extension $g : S' \to S$.*
(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are quasi-finite S-morphisms, then*
$$f \times_S g : X \times_S Y \longrightarrow X' \times_S Y'$$
is quasi-finite.
(v) *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms such that $g \circ f$ is quasi-finite; if, further, g is separated, or X is Noetherian, or $X \times_Z Y$ is locally Noetherian, then f is quasi-finite.*
(vi) *If f is quasi-finite, then f_{red} is quasi-finite.*

PROOF. If $f : X \to Y$ is an immersion, then every fibre consists of a single point, and claim (i) then follows from ((**I**, 6.3.4, (i)) and (**I**, 6.3.5)). To prove (ii), note first of all that $h = g \circ f$ is of finite type (**I**, 6.3.4, (ii)); furthermore, if $z = h(x)$ and $y = f(x)$, then y is isolated in $g^{-1}(z)$, and so there exists an open neighbourhood V of y in Y that does not meet any point of $g^{-1}(z)$ apart from y; thus $f^{-1}(V)$ is an open neighbourhood of x that does not meet any of the $f^{-1}(y')$, where $y' \neq y$ is in $g^{-1}(z)$; since x is isolated in $f^{-1}(y)$, it is isolated in $h^{-1}(z) = f^{-1}(g^{-1}(z))$. To prove (iii), we can **II** | 116
reduce to the case where $Y = S$ (**I**, 3.3.11); we again note first of all that $f' = f_{(S')}$ is of finite type (**I**, 6.3.4, (iii)); also, if $x' \in X' = X_{(S')}$, and if we set $y' = f'(x')$ and $y = g(y')$, then $f'^{-1}(y')$ can be identified with $f^{-1}(y) \otimes_{k(y)} k(y')$ (**I**, 3.6.5); since $f^{-1}(y)$ is of finite rank over $k(y)$ by hypothesis, $f'^{-1}(y')$ is of finite rank over $k(y')$, and thus discrete. Claims (iv), (v), and (vi) follows from (i), (ii), and (iii) by the general method (**I**, 5.5.12), except for when the hypotheses in (v) are not "g is separated"; in these cases, we remark first of all that, if x is isolated in $f^{-1}(g^{-1}(g(f(x))))$, then it is *a fortiori* isolated in $f^{-1}(f(x))$; the fact that f is of finite type then follows from (**I**, 6.3.6). □

Proposition (6.2.5). — *Let A be a* complete *local Noetherian ring, X an A-scheme locally of finite type, and x a point of X over the closed point y of $Y = \mathrm{Spec}(A)$. Suppose that x is isolated in its fibre $f^{-1}(y)$ (where f is the structure morphism $X \to Y$). Then $\mathscr{O}_x$ is an A-module of finite type, and X is Y-isomorphic to the sum* (**I**, 3.1) *of $X' = \mathrm{Spec}(\mathscr{O}_x)$ (which is a finite Y-scheme) and an A-scheme X''.*

PROOF. It follows from (6.2.1) that $\mathscr{O}_x$ is a quasi-finite A-module. Since $\mathscr{O}_x$ is Noetherian (**I**, 6.3.7), and the homomorphism $A \to \mathscr{O}_x$ is local, the hypothesis that A is *complete* implies that $\mathscr{O}_x$ is an A-module *of finite type* (**0**, 7.4.3). Let $X' = \mathrm{Spec}(\mathscr{O}_x)$ be the local scheme of X at the point x (**I**, 2.4.1), and let $g : X' \to X$ be the canonical morphism. Since the composite $X' \xrightarrow{g} X \xrightarrow{f} Y$ is finite (6.1.1), and since f is separated, g is finite (6.1.5, (v)), and so $g(X')$ is *closed* in X (6.1.10); also, since g is of finite type, g is a local isomorphism at the closed point x' of X' by the definition of g and (**I**, 6.5.4); but since X' is the only open neighbourhood of x', this implies that g is an open immersion, and so $g(X')$ is also *open* in X, which finishes the proof. □

Corollary (6.2.6). — *Let A be a* complete *local Noetherian ring, $Y = \mathrm{Spec}(A)$, and $f : X \to Y$ a separated quasi-finite morphism. Then X is Y-isomorphic to a sum $X' \sqcup X''$, where X' is a finite Y-scheme, and X'' is a quasi-finite Y-scheme such that, if y is the closed point of Y, then $X'' \cap f^{-1}(y) = \varnothing$.*

PROOF. Indeed, the fibre $f^{-1}(y)$ is finite and discrete by hypothesis, and the corollary then follows, by induction on the number of points in this fibre, from (6.2.5). □

Remark (6.2.7). — In Chapter V, we will see that, if Y is *locally Noetherian*, then a separated quasi-finite morphism $X \to Y$ is necessarily *quasi-affine*.

6.3. Integral closure of a prescheme

Proposition (6.3.1). — *Let $(X, \mathscr{A})$ be a ringed space, $\mathscr{B}$ a (commutative) $\mathscr{A}$-algebra, and f a section of $\mathscr{B}$ over X. Then the following properties are equivalent:*

*a) The sub-$\mathscr{A}$-module of $\mathscr{B}$ generated by the f^n for $n \geqslant$ (**0**, 5.1.1) is of finite type.*
b) There exists a sub-$\mathscr{A}$-algebra $\mathscr{C}$ of $\mathscr{B}$ that is an $\mathscr{A}$-module of finite type and such that $f \in \Gamma(X, \mathscr{C})$.
c) For all $x \in X$, f_x is integral over the fibre $\mathscr{A}_x$.

II | 117

PROOF. Since the sub-$\mathscr{A}$-module of $\mathscr{B}$ generated by the f^n is an $\mathscr{A}$-algebra, it is clear that (a) implies (b). On the other hand, (b) implies that, for all $x \in X$, the $\mathscr{A}_x$-module $\mathscr{C}_x$ is of finite type, which implies that every element of the algebra $\mathscr{C}_x$, and, in particular, f_x, is integral over $\mathscr{A}_x$. Finally, if we have, for some $x \in X$, a relation of the form

$$f_x^n + (a_1)_x f_x^{n-1} + \ldots + (a_n)_x = 0$$

where the a_i (for $1 \leqslant i \leqslant n$) are sections of $\mathscr{A}$ over an open neighbourhood U of x, then the section $f^n|U + a_1 \cdot f^{n-1}|U + \ldots + a_n$ is zero over a neighbourhood $V \subset U$ of x, from which it immediately follows that the $f^k|V$ (for $k \geqslant 0$) are linear combinations of the $f^j|V$ (for $0 \leqslant j \leqslant n-1$) with coefficients in $\Gamma(V, \mathscr{A})$; we thus conclude that (c) implies (a). □

If the equivalent conditions of (6.3.1) are satisfied then we say that the section f is *integral over* $\mathscr{A}$.

Corollary (6.3.2). — *Under the hypotheses of (6.3.1), there exists a (unique) sub-$\mathscr{A}$-module $\mathscr{A}'$ of $\mathscr{B}$ such that, for all $x \in X$, $\mathscr{A}'_x$ is the set of germs $f_x \in \mathscr{B}_x$ that are integral over $\mathscr{A}_x$. For every open $U \subset X$, the sections of $\mathscr{A}'$ over U are the sections $f \in \Gamma(U, \mathscr{B})$ that are integral over $\mathscr{A}|U$.*

PROOF. The existence of $\mathscr{A}'$ is immediate, by taking $\Gamma(U, \mathscr{A}')$ to be the set of $f \in \Gamma(U, \mathscr{B})$ such that f_x is integral over $\mathscr{A}_x$ for all $x \in U$. The second claim follows immediately from (6.3.1). □

It is clear that $\mathscr{A}'$ is a sub-$\mathscr{A}$-algebra of $\mathscr{B}$; we say that it is the *integral closure of $\mathscr{A}$ in $\mathscr{B}$*.

(6.3.3). Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be ringed spaces, and let

$$g = (\psi, \theta) : X \longrightarrow Y$$

be a morphism. Let $\mathscr{C}$ (resp. $\mathscr{D}$) be an $\mathscr{A}$-algebra (resp. $\mathscr{B}$-algebra), and let

$$u : \mathscr{D} \longrightarrow \mathscr{C}$$

be a g-morphism (**0**, 4.4.1). Then, if $\mathscr{A}'$ (resp. $\mathscr{B}'$) is the integral closure of $\mathscr{A}$ (resp. $\mathscr{B}$) in $\mathscr{C}$ (resp. $\mathscr{D}$), the *restriction* of u to $\mathscr{B}'$ is a g-morphism

$$u' : \mathscr{B}' \longrightarrow \mathscr{A}'.$$

Indeed, if j is the canonical injection $\mathscr{B}' \to \mathscr{D}$, then it suffices to show that

$$v = u^\sharp \circ g^*(j) : j^*(\mathscr{B}') \longrightarrow \mathscr{C}'$$

sends $g^*(\mathscr{B}')$ to $\mathscr{A}'$. But an element of $g^*(\mathscr{B}')_x = \mathscr{B}'_{\psi(x)} \otimes_{\mathscr{B}_{\psi(x)}} \mathscr{A}_x$ is integral over $\mathscr{A}_x$ by the definition of $\mathscr{B}'$, and thus so too is its images under v_x, which proves the claim.

Proposition (6.3.4). — *Let X be a prescheme, and $\mathscr{A}$ a quasi-coherent $\mathscr{O}_X$-algebra. Then the integral closure $\mathscr{O}'_X$ of $\mathscr{O}_X$ in $\mathscr{A}$ is a quasi-coherent $\mathscr{O}_X$-algebra, and, for every affine open U of X, $\Gamma(U, \mathscr{O}'_X)$ is the integral closure of $\Gamma(U, \mathscr{O}_X)$ in $\Gamma(U, \mathscr{A})$.*

PROOF. We can restrict to the case where $X = \mathrm{Spec}(B)$ is affine, and $\mathscr{A} = \widetilde{A}$, where A is a B-algebra; let B' be the integral closure of B in A. Everything then reduces to proving that, for all $x \in X$, an element of A_x which is integral over B_x necessarily belongs to B'_x, which follows from the fact that, for a commutative ring C, the operations of taking the integral closure in a C-algebra and passing to a ring of fractions (with respect to a multiplicative subset of C) commute [SZ60, t. I, pp. 261 and 257]. □ **II | 118**

The X-scheme $X' = \mathrm{Spec}(\mathscr{O}'_X)$ is then called the *integral closure of X with respect to $\mathscr{A}$* (or *with respect to* $\mathrm{Spec}(\mathscr{A})$); it is clear that X' is *integral* over X (6.1.2).

We immediately deduce from (6.1.4) that, if $f : X' \to X$ is the structure morphism, then, for every open U of X, $f^{-1}(U)$ is the *integral closure of the prescheme induced by X on U, with respect to $\mathscr{A}|U$.*

(6.3.5). Let X and Y be preschemes, $f : X \to Y$ a morphism, $\mathscr{A}$ (resp. $\mathscr{B}$) a quasi-coherent $\mathscr{O}_X$-algebra (resp. quasi-coherent $\mathscr{O}_Y$-algebra), and let $u : \mathscr{B} \to \mathscr{A}$ be an f-morphism. We have seen (6.3.3) that we obtain an f-morphism $u' : \mathscr{O}'_Y \to \mathscr{O}'_X$, where $\mathscr{O}'_X$ (resp. $\mathscr{O}'_Y$) is the integral closure of $\mathscr{O}_X$ (resp. $\mathscr{O}_Y$) in $\mathscr{A}$ (resp. $\mathscr{B}$). Then, if X' (resp. Y') is the integral closure of X (resp. Y) with respect to $\mathscr{A}$ (resp. $\mathscr{B}$), we canonically obtain from u a morphism $f' = \mathrm{Spec}(u') : X' \to Y'$ (1.5.6) such that the diagram

(6.3.5.1)
$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes.

(6.3.6). Suppose that X has only a *finite* number of irreducible components X_i (for $1 \leqslant i \leqslant r$), with generic points ξ_i, and consider, in particular, the integral closure of X with respect to some quasi-coherent $\mathscr{R}(X)$*-algebra* $\mathscr{A}$ (quasi-coherent as either an $\mathscr{O}_X$-algebra or as an $\mathscr{R}(X)$-algebra, since the two are equivalent). We know (I, 7.3.5) that $\mathscr{A}$ is a direct sum of r quasi-coherent $\mathscr{O}_X$-algebras $\mathscr{A}_i$, where the support of $\mathscr{A}_i$ is contained inside X_i, and the sheaf induced by $\mathscr{A}_i$ on X_i is a simple sheaf whose fibre A_i is an algebra over $\mathscr{O}_{\xi_i}$. It is then clear (6.3.4) that the integral closure $\mathscr{O}'_X$ of $\mathscr{O}_X$ in $\mathscr{A}$ is the *direct sum* of the integral closures $\mathscr{O}_X^{(i)}$ of $\mathscr{O}_X$ in each of the $\mathscr{A}$, and thus the integral closure $X' = \mathrm{Spec}(\mathscr{O}'_X)$ of X with respect to $\mathscr{A}$ is the X-scheme given by the *sum* of the $\mathrm{Spec}(\mathscr{O}_X^{(i)}) = X'_i$ (for $1 \leqslant i \leqslant r$).

Suppose further that the $\mathscr{O}_X$-algebra $\mathscr{A}$ is *reduced*, or, equivalently, that each of the algebras A_i are reduced, and can thus be considered as an algebra over the *field* $k(\xi_i)$ (equal to the field of rational functions of the reduced subprescheme of X that has X_i as its underlying space); then (1.3.8) each of the X'_i is a *reduced* X-scheme, and X' is also the integral closure of X_{red}. Suppose further that each of the algebras A_i is a *direct sum of a finite number of fields* K_{ij} (for $1 \leqslant j \leqslant s_i$); if $\mathscr{K}_{ij}$ is the sub-algebra of $\mathscr{A}_i$ corresponding to K_{ij}, then it is clear that $\mathscr{O}_X^{(i)}$ is the *direct sum* of the integral closures $\mathscr{O}_X^{(ij)}$ of $\mathscr{O}_X$ in each of the $\mathscr{K}_{ij}$. Then X'_i is the *sum* of the X-schemes $X'_{ij} = \mathrm{Spec}(\mathscr{O}_X^{(ij)})$ (for $1 \leqslant j \leqslant s_i$). Furthermore, under these hypotheses, and with this notation, we have:

II | 119

Proposition (6.3.7). — *Each of the X'_{ij} is an integral normal X-prescheme, and its field of rational functions $R(X'_{ij})$ is canonically identified with the algebraic closure K'_{ij} of $k(\xi_i)$ in K_{ij}.*

PROOF. By the above, we can suppose that X is integral, and so $r = 1$ and $s_1 = 1$, so that the unique algebra A_1 is a field K; let ξ be the generic point of X, and let $f : X \to X'$ be the structure morphism. For every non-empty affine open U of X, $f^{-1}(U)$ can be identified with the spectrum of the integral closure B'_U in the field K of the integral ring $B_U = \Gamma(Y, \mathscr{O}_X)$ (6.3.4); since the ring B'_U is integral and integrally closed, so too are the local rings of the points of its spectrum, and thus $f^{-1}(U)$ is by definition an *integral* and *normal* scheme ((0, 4.1.4) and (I, 5.1.4)). Furthermore, since

(0) is the only prime ideal of B'_U over the prime ideal (0) of B_U [**SZ60**, t. 1, p. 259], $f^{-1}(\xi)$ consists of a single point ξ', and $k(\xi')$ is the field of fractions K' of B'_U, which is exactly the algebraic closure of $k(\xi)$ in K. Finally, X' is irreducible, since, if U runs over the set of non-empty affine opens of X, then the $f^{-1}(U)$ give an open cover of X' by irreducible opens; furthermore, the intersection $f^{-1}(U \cap V)$ of any two of these opens contains ξ', and is thus non-empty, and we conclude by (**0**, 2.1.4). □

Corollary (6.3.8). — *Let X be a reduced prescheme that has only a finite number of irreducible components X_i (where $1 \leqslant i \leqslant r$), and let ξ_i be the generic point of X_i. Then the integral closure X' of X with respect to $\mathscr{R}(X)$ is the sum of the r X-schemes X'_i, which are integral and normal. If $f : X' \to X$ is the structure morphism, then $f^{-1}(\xi_i)$ consists only of the generic point ξ'_i of X'_i, and we have $k(\xi'_i) = k(\xi_i)$, or, in other words, that f is birational.*

In this case, we say that X' is the *normalisation* of the reduced prescheme X; we note that f, since it is birational and integral, is *surjective* (6.1.10). Then in order to have $X' = X$, it is necessary and sufficient that X be *normal*. If X is an integral prescheme, then it follows from (6.3.8) that its normalisation X' is integral.

(6.3.9). Let X and Y be integral preschemes, $f : X \to Y$ a dominant morphism, and $L = R(X)$ (resp. $K = R(Y)$) the field of rational functions of X (resp. Y); there is a canonical injection $K \to L$ corresponding to f, and if we identify K (resp. L) with the simple sheaf $\mathscr{R}(Y)$ (resp. $\mathscr{R}(X)$), this injection is an f-morphism. Let K_1 (resp. L_1) be an extension of K (resp. L), and suppose we have a morphism $K_1 \to L_1$ such that the diagram

$$\begin{tikzcd} K_1 \arrow[r] & L_1 \\ K \arrow[u] \arrow[r] & L \arrow[u] \end{tikzcd}$$

commutes; if K_1 (resp. L_1) is considered as a simple sheaf on Y (resp. X), and thus as an $\mathscr{R}(Y)$-algebra (resp. $\mathscr{R}(X)$-algebra), this implies that $K_1 \to L_1$ is an f-morphism. With this, if X' (resp. Y') is the integral closure of X (resp. Y) with respect to L_1 (resp. K_1), then X' (resp. Y') is a normal integral prescheme (6.3.6) whose field of rational functions is canonically identified with the algebraic closure L' (resp. K') of L (resp. K) in L_1 (resp K_1), and there exists a (necessarily dominant) canonical morphism $f' : X' \to Y'$ that makes the diagram (6.3.5.1) commute. **II** | 120

The most important case is when we take $L_1 = L$, with K_1 then being an extension of K contained in L, and when we suppose X to be integral and *normal*, so that $X' = X$. The above then shows that, since X is normal, if Y' is the integral closure of Y with respect to a field $K_1 \subset L = R(X)$, then every dominant morphism $f : X \to Y$ factors as

$$f : X \xrightarrow{f'} Y' \longrightarrow Y$$

where f' is dominant; also, since the monomorphism $K_1 \to L$ is given, f' is necessarily unique, as we see by reducing to the case where X and Y are affine. We thus see that, given Y, L, and a K-monomorphism $K_1 \to L$, the integral closure Y' of Y with respect to K_1 is the solution of a *universal problem*.

Remark (6.3.10). — Consider the hypothesis of (6.3.6), and suppose further that each of the algebras A_i is *of finite rank* over $k(\xi_i)$ (which implies that A_i is a direct sum of a finite number of fields); we can then show, in certain cases, that the structure morphism $X' \to X$ is not just integral, but even *finite*. Restrict to the case where X is *reduced*; since the question is local on X, we can further suppose that X is affine of ring C, and that C has only a finite number of minimal ideals $\mathfrak{p}_i$ (for $1 \leqslant i \leqslant r$) such the that $C_i = C/\mathfrak{p}_i$ are integral; then X' is finite over X if the integral closure of each C_i in finite-dgree extension of its field of fractions is a *C-module of finite type* (6.3.4). We know that this condition is always satisfied if C is an *algebra of finite type over a field* [**SZ60**, t. I, p. 267, th. 9], or *over* **Z** [**Nag58a**, I,p. 93, th. 3], or *over a complete Noetherian local ring* [**Nag55**, p. 298]. We thus conclude that $X' \to X$ will be a finite morphism whenever X is a scheme *of finite type* over a field, or over **Z**, or over a complete Noetherian local ring.

6.4. Determinant of an endomorphism of $\mathscr{O}_X$-modules

(6.4.1). Let A be a (commutative) ring, E a free A-module of rank n, and u an endomorphism of E; recall that, to define the *characteristic polynomial* of u, we consider the endomorphism $u \otimes 1$ of the free $A[T]$-module of rank n, $E \otimes_A A[T]$ (where T is an indeterminate), and we set

(6.4.1.1) $$P(u,T) = \det(T \cdot I - (u \otimes 1))$$

(where I is the identity automorphism of $E \otimes_A A[T]$). We have

(6.4.1.2) $$P(u,T) = T^n - \sigma_1(u)T^{n-1} + \ldots + (-1)^n \sigma_n(u)$$

where $\sigma_i(u)$ is an element of A, equal to a homogeneous polynomial of degree i (with integer coefficients) with respect to the entries of the matrix of u in some arbitrary basis of E; we say that the $\sigma_i(u)$ are the *elementary symmetric functions* of u; in particular, we have that $\sigma_1(u) = \operatorname{Tr} u$ and $\sigma_n(u) = \det u$. Recall that, by the Hamilton–Cayley theorem, we have

(6.4.1.3) $$P(u,u) = u^n - \sigma_1(u)u^{n-1} + \ldots + (-1)^n \sigma_n(u) = 0$$

which can also be written as **II** | 121

(6.4.1.4) $$(\det u) \cdot I_E = uQ(u)$$

(where I_E is the identity automorphism of E), where

(6.4.1.5) $$Q(u) = (-1)^{n+1}(u^{n-1} - \sigma_1(u)u^{n-2} + \ldots + (-1)^{n-1}\sigma_{n-1}(u)).$$

Let $\phi : A \to B$ be a ring homomorphism, which makes B into an A-algebra; consider the B-module $E_{(B)} = E \otimes_A B$, which is free of rank n, and the extension $u \otimes 1$ of u to an endomorphism of $E_{(B)}$; it is immediate that we have $\sigma_i(u \otimes 1) = \phi(\sigma_i(u))$ for all indices i.

(6.4.2). Now suppose that A is an *integral* ring, with field of fractions K, and let E be an A-module *of finite type* (but not necessarily free). Let n be the *rank* of E, i.e. the rank of the free K-module $E \otimes_A K$; to every endomorphism u of E there canonically corresponds the endomorphism $u \otimes 1$ of $E \otimes_A K$; by an abuse of language, we also define the *characteristic polynomial* of u, denoted by $P(u,T)$, to be the polynomial $P(u \otimes 1, T)$, whose coefficients $\sigma_i(u \otimes 1)$ (which belong to K) are also denoted by $\sigma_i(u)$ and called the *elementary symmetric functions of* u; in particular, $\det u = \det(u \otimes 1)$ by definition. With this notation, Equations (6.4.1.3) to (6.4.1.5) make sense and still hold true, if we interpret u^j as the composite homomorphism $E \to E \otimes_A K$ of the endomorphism $u^j \otimes 1 = (u \otimes 1)^j$ of $E \otimes_A K$ and the canonical homomorphism $x \mapsto x \otimes 1$.

If F is the torsion submodule of E, and if $E_0 = E/F$, then $u(F) \subset F$, and so, by passing to quotients, u gives an endomorphism u_0 of E_0; furthermore, $E \otimes_A K$ can be identified with $E_0 \otimes_A K$, and $u \otimes 1$ with $u_0 \otimes 1$, and so $\sigma_i(u) = \sigma_i(u_0)$ for $1 \leqslant i \leqslant n$.

If E is *torsion free*, then E is canonically identified with a sub-A-module of $E \otimes_A K$, and the relation $u \otimes 1 = 0$ is equivalent to $u = 0$. If A is a free A-module, then the two definitions of $\sigma_i(u)$ given in (6.4.1) and (6.4.2) coincide, by the above, which justifies the notation used.

We also note that, if E is a *torsion module*, or, in other words, if $E_0 = \{0\}$, then the exterior algebra of E_0 consists of K, and the determinant of the unique endomorphism u_0 of E_0 is *equal to* 1.

Proposition (6.4.3). — *Let A be an integral ring, E an A-module of finite type, and u an endomorphism of E; then the elementary symmetric functions $\sigma_i(u)$ of u (and, in particular,* $\det u$*) are elements of K that are integral over A.*

Proof. Let A' be the integral closure of A; since $A'[T]$ is an integrally closed ring [**Jaf60**, p. 99], it is the integral closure of $A[T]$ in its field of fractions $K(T)$. Replacing u by $T \cdot I - u \otimes 1$, and A by $A[T]$, we see that we can reduce to proving that u is integral over A. If n is the rank of E, then $\det(u) = \det(\vee^n u)$ and $(\vee^n u \otimes 1) = \vee^n(u \otimes 1)$, so we can suppose that $n = 1$. But then the map $u \mapsto \det u$ is a homomorphism from the A-module $\operatorname{Hom}_A(E,E)$ to K; since E is of finite type, $\operatorname{Hom}_A(E,E)$ is isomorphic to a sub-A-module of the A-module of finite type E^n (if E admits a system of n generators), and so the elements $\det u$ belong to a sub-A-module of K of finite type, and are thus integral over A. $\square$

II | 122

Corollary (6.4.4). — *Under the hypotheses of (6.4.3), if we further suppose A to be normal, then the $\sigma_i(u)$ (and, in particular,* $\operatorname{Tr} u$ *and* $\det u$*) belong to A.*

Proposition (6.4.5). — *Let A be an integral ring, E an A-module of finite type of rank n, and u an endomorphism of E such that the $\sigma_i(u)$ belong to A. For u to be an automorphism of E, it is necessary that* $\det u$ *be invertible in A; this condition is sufficient if E is torsion free.*

Proof. The condition is *sufficient* when E is torsion free, since the hypothesis and equations (6.4.1.4) and (6.4.1.5) (which hold in E, and not just in $E \otimes_A K$, since E is torsion free) prove that $(\det u)^{-1}Q(u)$ is the inverse of u.

The condition is *necessary*, since, if u is invertible, then it follows from (6.4.3) that $\det(u^{-1})$ belongs to the integral closure A' of A in its field of fractions K, and is evidently the inverse of $\det u$ *in* A'. Our claim then follows from: □

Lemma (6.4.5.1). — *Let A be a subring of a ring A' such that A' is* integral *over A. If an element $x \in A$ is invertible in A', then it is also invertible in A.*

Proof. In the contrary case, x would belong to a maximal ideal $\mathfrak{m}$ of A, and it follows from the first theorem of Cohen–Seidenberg [**SZ60**, t. I, p. 257, th. 3] that there would be a maximal ideal $\mathfrak{m}'$ of A' such that $\mathfrak{m} = \mathfrak{m}' \cap A$; we would then have that $x \in \mathfrak{m}'$, which is a contradiction. □

Corollary (6.4.6). — *Let A be an integral and integrally closed ring, E a torsion-free A-module of finite type, and u an endomorphism of E> For u to be an automorphism of E, it is necessary and sufficient that* $\det u$ *be invertible in A.*

Proof. This follows from (6.4.4) and (6.4.5). □

Remark (6.4.7). — We will later need a generalisation of the above results. Consider a *reduced* Noetherian ring A; let $\mathfrak{p}_\alpha$ (for $1 \leqslant \alpha \leqslant r$) be its minimal ideals, K_α the field of fractions of the integral ring $A/\mathfrak{p}_\alpha$, and K the total ring of fractions of A, which is the *direct sum* of the fields K_α. Let E be an A-module of finite type, and *suppose* that $E \otimes_A K$ is a *free* K-module of rank n (which here is merely a consequence of the other hypotheses); equivalently, we can ask that all the K_α-vector spaces $E \otimes_A K_\alpha = E_\alpha$ have *the same dimension n*; if then u is an endomorphism of E, we again set $P(u,T) = P(u \otimes 1, T)$ and $\sigma_j(u) = \sigma_j(u \otimes 1)$, and, in particular, $\det u = \det(u \otimes 1)$; the $\sigma_j(u)$ are thus elements of K. It is immediate that $E \otimes_A K$ is the direct sum of the E_α, and that the latter are stable under $u \otimes 1$; the restriction of $u \otimes 1$ to E_α is exactly the extension u_α of u to this K_α-vector space; we thus conclude that $\sigma_j(u)$ is the element of K who components in the K_α are the $\sigma_j(u_\alpha)$. Since the integral closure of A in K is the *direct sum* of the integral closures of A in the K_α, the $\sigma_j(u)$ are *integral* over A.

Lemma (6.4.7.1). — *The sub-A-algebra of K generated by all the elements $\sigma_j(u)$ (for $1 \leqslant j \leqslant n$), where u runs over* $\operatorname{Hom}_A(E,E)$*, is an A-module of finite type.*

Proof. It suffices to prove that the sub-$A[T]$-algebra of $K[T]$ generated by the $P(u,T)$ is an $A[T]$-module of finite type, since if the $F_i(T)$ (for $1 \leqslant i \leqslant m$) form a system of generators for this $A[T]$-module, then the coefficients of the $F_i(T)$ are integral over A, and thus generate an A-algebra that is an A-module of finite type [**SZ60**, t. I, p. 255, th. 1]. We can thus replace A by $A[T]$ (which II | 123
is Noetherian), and E by $E \otimes_A A[T] = E'$, which is such that $E' \otimes_{A[T]} K[T] = E \otimes_A K[T]$ is a free $K[T]$-module of rank n. Using the initial notation, we thus see that it suffices to prove that the A-module generated by the elements $\det u$, where u runs over $\operatorname{Hom}_A(E,E)$, is of finite type; *a fortiori* (since every submodule of an A-module of finite type is of finite type) it suffices to prove that, as v runs over the set of endomorphisms of $\wedge^n E$, the A-module generated by the $\det v$ is of finite type; in other words, we can again reduce to the case where $n = 1$. But then the proposition follows from the fact that $\operatorname{Hom}_A(E,E)$ is an A-module of finite type, and that $v \mapsto \det v$ is a homomorphism from this A-module into K. □

Let F be the kernel of the canonical homomorphism $E \to E \otimes_A K$, and let $E_0 = E/F$; we see, as above, that $E \otimes_A K$ can be identified with $E_0 \otimes_A K$, that $u(F) \subset F$, and that, if u_0 is the endomorphism of E_0 induced from u by passing to the quotient, then $u \otimes 1$ can be identified with $u_0 \otimes 1$, and $\sigma_j(u) = \sigma_j(u_0)$ for all j. *If we have $F = 0$*, then equations (6.4.1.3) and (6.4.1.5) still make sense and hold true whenever E is identified with a submodule of $E \otimes_A K$, and the u^j with homomorphisms $E \to E \otimes_A K$; then Proposition (6.4.5) extends also to this case, with the same proof.

(6.4.8). Let $(X, \mathscr{A})$ be a ringed space, $\mathscr{E}$ a *locally free* $\mathscr{A}$-module (of finite rank), and u an endomorphism of $\mathscr{E}$. By hypothesis, there exists a base $\mathfrak{B}$ for the topology of X such that, for all $V \in \mathfrak{B}$, $\mathscr{E}|V$ is isomorphic to some $\mathscr{A}^n|V$ (where n may vary with V). Let u be an endomorphism of $\mathscr{E}$; for all $V \in \mathfrak{B}$, u_V is thus an endomorphism of the $\Gamma(V, \mathscr{A})$-module $\Gamma(V, \mathscr{E})$, which is free by hypothesis; the determinant $\det u_V$ is thus defined and belongs to $\Gamma(V, \mathscr{A})$. Furthermore, if $e_1, \ldots, e_n$ form a basis of $\Gamma(V, \mathscr{E})$, then their restrictions to every open $W \subset V$ form a basis of $\Gamma(W, \mathscr{E})$ over $\Gamma(W, \mathscr{A})$, and so $\det u_W$ is the restriction of $\det u_V$ to W. There thus exists exactly one section of $\mathscr{A}$ over X, which we denote by $\det u$, and call the *determinant* of u, such that the restriction of $\det u$ to each $V \in \mathfrak{B}$ is $\det u_V$. It is clear that, for all $x \in X$, we have $(\det u)_x = \det u_x$; for endomorphisms u and v of $\mathscr{E}$, we have

$$\det(u \circ v) = (\det u)(\det v) \tag{6.4.8.1}$$

as well as

$$\det(1_{\mathscr{E}}) = 1_{\mathscr{A}} \tag{6.4.8.2}$$

and, if the rank of $\mathscr{E}$ is constant (which will be the case (**0**, 5.4.1) if X is connected) and equal to n,

$$\det(s \cdot u) = s^n \det u \tag{6.4.8.3}$$

for all $s \in \Gamma(X, \mathscr{A})$ (note that $\det(0) = 0_{\mathscr{A}}$ if $n \geqslant 1$, but $\det(0) = 1_{\mathscr{A}}$ for $n = 0$). Furthermore, for u to be an *automorphism* of $\mathscr{E}$, it is necessary and sufficient that u be *invertible* in $\Gamma(X, \mathscr{A})$.

If the rank of $\mathscr{E}$ is constant, then we can similarly define the elementary symmetric functions $\sigma_i(u)$, which are elements of $\Gamma(X, \mathscr{A})$; we again have the relations in (6.4.1.3) and (6.4.1.5).

We have thus defined a homomorphism $\det : \operatorname{Hom}_{\mathscr{A}}(\mathscr{E}, \mathscr{E}) \to \Gamma(X, \mathscr{A})$ of multiplicative **II** | 124
monoids; if we note that $\operatorname{Hom}_{\mathscr{A}}(\mathscr{E}, \mathscr{E}) = \Gamma(X, \mathscr{H}om_{\mathscr{A}}(\mathscr{E}, \mathscr{E}))$ by definition, then we see that we can replace X in this definition by an arbitrary open U of X, which immediately defines a homomorphism $\det : \mathscr{H}om_{\mathscr{A}}(\mathscr{E}, \mathscr{E}) \to \mathscr{A}$ of sheaves of multiplicative monoids. If $\mathscr{E}$ has constant rank, then we similarly define homomorphisms $\sigma_i : \mathscr{H}om_{\mathscr{A}}(\mathscr{E}, \mathscr{E}) \to \mathscr{A}$ of sheaves of sets; for $i = 1$, the homomorphism $\sigma_1 = \operatorname{Tr}$ is a homomorphism of $\mathscr{A}$-modules.

Let $(Y, \mathscr{B})$ be another ringed space, and let $f : (X, \mathscr{A}) \to (Y, \mathscr{B})$ be a morphism of ringed spaces; if $\mathscr{F}$ is a locally free $\mathscr{B}$-module, then $f^*(\mathscr{F})$ is a locally free $\mathscr{A}$-module (which is of the same rank as $\mathscr{F}$ if the latter is of constant rank) (**0**, 5.4.5). For every endomorphism v of $\mathscr{F}$, $f^*(v)$ is an endomorphism of $f^*(\mathscr{F})$, and it follows immediately from the definitions that $\det f^*(v)$ is the section of $\mathscr{A} = f^*(\mathscr{B})$ over X that canonically corresponds to $\det v \in \Gamma(Y, \mathscr{B})$. We can further say that the homomorphism $f^*(\det) : f^*(\mathscr{H}om_{\mathscr{B}}(\mathscr{F}, \mathscr{F})) \to f^*(\mathscr{B}) = \mathscr{A}$ is the composition

$$f^*(\mathscr{H}om_{\mathscr{B}}(\mathscr{F}, \mathscr{F})) \xrightarrow{\gamma^\sharp} \mathscr{H}om_{\mathscr{A}}(f^*(\mathscr{F}), f^*(\mathscr{F})) \xrightarrow{\det} \mathscr{A}$$

(**0**, 4.4.6). We have analogous results for the σ_i.

(6.4.9). Now suppose that X is a *locally integral prescheme*, so that the sheaf $\mathscr{R}(X)$ of rational functions on X is a locally simple sheaf of fields (**I**, 7.4.3), and quasi-coherent as an $\mathscr{O}_X$-module. If $\mathscr{E}$ is a quasi-coherent $\mathscr{O}_X$-module *of finite type*, then $\mathscr{E}' = \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{R}(X)$ is a locally free $\mathscr{R}(X)$-module (**I**, 7.3.6); for every endomorphism u of $\mathscr{E}$, $u \otimes 1_{\mathscr{R}(X)}$ is then an endomorphism of $\mathscr{E}'$, and $\det(u \otimes 1)$ is a section of $\mathscr{R}(X)$ over X, which we also call the *determinant* of u, and also denote by $\det u$. It follows from (6.4.3) that $\det u$ is a section of the *integral closure* of $\mathscr{O}_X$ in $\mathscr{R}(X)$ (6.3.2); if, further, X is *normal*, then $\det u$ is a *section of* $\mathscr{O}_X$ over X, and if we further suppose that $\mathscr{E}$ is *torsion free*, then for u to be an automorphism of $\mathscr{E}$ it is necessary and sufficient that $\det u$ be *invertible*, by (6.4.6). Equations (6.4.8.1) to (6.4.8.3) still hold true; from the homomorphism $u \mapsto \det u$, applied to the modules of sections of $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E})$, we obtain a homomorphism of sheaves $\det : \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E}) \to \mathscr{R}(X)$, which takes values in $\mathscr{O}_X$ if X is normal. We have analogous definitions and results for the other elementary symmetric functions $\sigma_j(u)$, if $\mathscr{E}'$ is of *constant rank*; if, further, X is *normal*, then the $\sigma_j(u)$ are *sections of* $\mathscr{O}_X$ over X.

Finally, let X and Y be integral preschemes, and $f : X \to Y$ a *dominant* morphism. We know that there exists a canonical homomorphism $f^*(\mathscr{R}(Y)) \to \mathscr{R}(X)$ (**I**, 7.3.8), whence we obtain, for every quasi-coherent $\mathscr{O}_Y$-module $\mathscr{F}$ of finite type, a canonical homomorphism $\theta : f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{R}(Y)) \to f^*(\mathscr{F}) \otimes_{\mathscr{O}_X} \mathscr{R}(X)$. If v is an endomorphism of $\mathscr{F}$, then $f^*(v \otimes 1_{\mathscr{R}(Y)})$ is an endomorphism of **II** | 125
$f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{R}(Y))$, and we have a commutative diagram

$$\begin{tikzcd}
f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{R}(Y)) \arrow[r, "f^*(v\otimes 1)"] \arrow[d, "\theta"'] & f^*(\mathscr{F} \otimes \mathscr{O}_Y \mathscr{R}(Y)) \arrow[d, "\theta"] \\
f^*(\mathscr{F}) \otimes_{\mathscr{O}_X} \mathscr{R}(X) \arrow[r, "f^*(v)\otimes 1"'] & f^*(\mathscr{F}) \otimes_{\mathscr{O}_X} \mathscr{R}(X)
\end{tikzcd}$$

We thus easily conclude that $\det f^*(v)$ is the canonical image, under the homomorphism $f^*(\mathscr{R}(Y)) \to \mathscr{R}(X)$, of the section $\det v$ of $\mathscr{R}(Y)$; indeed, we can immediately reduce to the case where $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$ are affine, with A and B integral rings with fields of fractions K and L (resp.), with the homomorphism $B \to A$ an injection and thus extending to a monomorphism $L \to K$; if $\mathscr{F} = \widetilde{M}$, where M is a B-module of finite type, then the rank of $M \otimes_B L$ over L is *equal* to that of $(M \otimes_B A) \otimes_A K$ over K, and $\det((u \otimes 1) \otimes 1)$ is the image in K of $\det(u \otimes 1)$ for any endomorphism u of M, whence the conclusion.

(6.4.10). Finally, suppose that X is a *locally Noetherian reduced prescheme*, so that the sheaf $\mathscr{R}(X)$ of rational functions on X is again a quasi-coherent $\mathscr{O}_X$-module (**I**, 7.3.4); let $\mathscr{E}$ be a *coherent* $\mathscr{O}_X$-module such that $\mathscr{E}' = \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{R}(X)$ is *locally free* (of finite rank). By (6.4.7), if $\mathscr{E}'$ is of constant rank, then we can, for any endomorphism u of $\mathscr{E}$, define the $\sigma_j(u)$, which are sections of $\mathscr{R}(X)$ over X. If we do not suppose $\mathscr{E}'$ to be of constant rank, we can still define the homomorphism $\det : \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E}) \to \mathscr{R}(X)$.

6.5. Norm of an invertible sheaf

(6.5.1). Let $(X, \mathscr{A})$ be a ringed space and $\mathscr{B}$ a (commutative) $\mathscr{A}$-algebra. The $\mathscr{A}$-module $\mathscr{B}$ is canonically identified with a sub-$\mathscr{A}$-module of $\mathscr{H}om_{\mathscr{A}}(\mathscr{B}, \mathscr{B})$, and a section f of $\mathscr{B}$ over an open U of X is identified with multiplication by this section. If $(X, \mathscr{A})$ and $\mathscr{B}$ satisfy one of the conditions given in (6.4.8), (6.4.9), or (6.4.10), then we can define $\det(f)$ (and, in certain cases, the $\sigma_j(f)$) as sections of $\mathscr{A}$ or $\mathscr{R}(X)$ over U, that we call the *norm* of f (resp. the *elementary symmetric functions* of f), and denote by $N_{\mathscr{B}/\mathscr{A}}(f)$. We will suppose that *one* of the following conditions is satisfied:

(I) *$\mathscr{B}$ is a locally free $\mathscr{A}$-module (of finite rank).*

(II) *$(X, \mathscr{A})$ is a locally Noetherian reduced prescheme, $\mathscr{B}$ is a coherent $\mathscr{A}$-module such that $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{R}(X)$ is a locally free $\mathscr{R}(X)$-module, and, for every section $f \in \Gamma(U, \mathscr{B})$ over an open U of X, the norm $N_{\mathscr{B}/\mathscr{A}}(f)$ is a section of $\mathscr{A}$ over U.*

Note that the latter condition is automatically satisfied whenever the locally Noetherian prescheme **II** | 126
X is *normal* (6.4.9).

The hypothesis that $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{R}(X)$ is locally free can also be expressed in the following way: denote by X_α the closed reduced subpreschemes of X whose underlying spaces are the irreducible components of X (**I**, 5.2.1), which are thus locally Noetherian integral preschemes. Every $x \in X$ belongs to a finite number of the subspaces X_α; on the other hand, $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{R}(X_\alpha)$ is a locally free $\mathscr{R}(X_\alpha)$-module of constant rank k_α (**I**, 7.3.6); to say that $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{R}(X)$ is a locally free $\mathscr{R}(X)$-module implies that, for all $x \in X$, *the ranks k_α that correspond to the indices such that $x \in X_\alpha$ are equal*. This is a local statement, and we can reduce to the case $X = \operatorname{Spec}(C)$, where C is a reduced Noetherian ring, and $\mathscr{B} = \widetilde{D}$, where D is a C-algebra that is a C-module of finite type; if $\mathfrak{p}_i$ (for $1 \leqslant i \leqslant m$) are the minimal prime ideals of C, then the total ring of fractions L of C is the direct sum of the fields of fractions K_i of the integral rings $C/\mathfrak{p}_i$, and $D \otimes_C L$ is the direct sum of the $D \otimes_C K_i$, whence the conclusion.

It is clear that, under hypothesis (I) or (II), we have thus defined a *homomorphism of sheaves of multiplicative monoids* $N_{\mathscr{B}/\mathscr{A}} : \mathscr{B} \to \mathscr{A}$, also denoted by N if no confusion may arise, and we call this homomorphism the *norm*. For sections f and g of $\mathscr{B}$ over the same open U, we thus have

(6.5.1.1) $$N_{\mathscr{B}/\mathscr{A}}(fg) = N\mathscr{B}/\mathscr{A}(f)\mathscr{B}/\mathscr{A}(g)$$

for the corresponding sections of $\mathscr{A}$ over U;

(6.5.1.2) $$N\mathscr{B}/\mathscr{A}(1_{\mathscr{B}}) = 1_{\mathscr{A}};$$

finally, for any section s of $\mathscr{A}$ over U,

(6.5.1.3) $$N\mathscr{B}/\mathscr{A}(s \cdot 1_{\mathscr{B}}) = s^n$$

if the rank of $\mathscr{B}$ is constant and equal to n (for $s = 0_{\mathscr{A}}$, this formula gives $N(0_{\mathscr{B}}) = 0_{\mathscr{A}}$ if $n \geqslant 1$, and $N(0_{\mathscr{B}}) = N(1_{\mathscr{B}}) = 1_{\mathscr{A}}$ if $n = 0$).

In hypothesis (I), for $f \in \Gamma(U, \mathscr{B})$ to be invertible it is necessary and sufficient that $N(f) \in \Gamma(U, \mathscr{A})$ be invertible. In hypothesis (II), this condition is necessary; it is sufficient (by (6.4.7)) if we suppose that $\mathscr{B} \to \mathscr{B} \otimes_{\mathscr{A}} \mathscr{R}(X)$ is *injective* and that the following more restrictive hypothesis is satisfied:

(II *bis*) *$(X, \mathscr{A})$ is a locally Noetherian reduced prescheme, $\mathscr{B}$ is a coherent $\mathscr{A}$-module such that $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{R}(X)$ is a locally free $\mathscr{R}(X)$-module, and, for any section $f \in \Gamma(U, \mathscr{B})$ over an open U such that $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{R}(X)|U$ is of constant rank n on $\mathscr{R}(X)|U$, the $\sigma_j(f)$ (for $1 \leqslant j \leqslant n$) are sections of $\mathscr{A}$ over U.*

(We again note that this condition is satisfied if X is *normal*.)

(6.5.2). Suppose that one of the hypotheses (I) or (II) of (6.5.1) are satisfied, and let $\mathscr{L}'$ be an *invertible* $\mathscr{B}$-module. We will canonically associate (up to unique isomorphism) an *invertible* $\mathscr{A}$-module in the following way. Denote by $\mathscr{A}^*$ (resp. $\mathscr{B}^*$) the subsheaf of $\mathscr{A}$ (resp. of $\mathscr{B}$) such that $\Gamma(U, \mathscr{A}^*)$ (resp. $\Gamma(U, \mathscr{B}^*)$) is the set of invertible elements of $\Gamma(U, \mathscr{A})$ (resp. of $\Gamma(U, \mathscr{B})$) for every open $U \subset X$; these **II** | 127
are sheaves of multiplicative groups, and $N_{\mathscr{B}/\mathscr{A}}$ restricted to $\mathscr{B}^*$ is a *homomorphism* $\mathscr{B}^* \to \mathscr{A}^*$ of sheaves of groups (6.5.1). Let $\mathfrak{L}$ be the set of pairs (U_λ, ν_λ) that have the following property: U_λ is an open of X, and ν_λ is an isomorphism $\mathscr{L}'|U_\lambda \xrightarrow{\sim} \mathscr{B}|U_\lambda$ of $(B|U_\lambda)$-modules. By hypothesis, the U_λ form a cover of X; for arbitrary indices λ and μ, set $\omega_{\lambda\mu} = (\eta_\lambda|U_\lambda \cap U_\mu) \circ (\nu_\mu|U_\lambda \cap U_\mu)^{-1}$, which is an automorphism of $\mathscr{B}|U_\lambda \cap U_\mu$ which is canonically identified with a section of $\mathscr{B}^*$ over $U_\lambda \cap U_\mu$, and $(\omega_{\lambda\mu})$ is a 1-cocycle for the cover $\mathfrak{U} = (U_\lambda)$ with values in $\mathscr{B}^*$ (**0**, 5.4.7). The fact that $N_{\mathscr{B}/\mathscr{A}} : \mathscr{B}^* \to \mathscr{A}^*$ is a homomorphism implies that $(N_{\mathscr{B}/\mathscr{A}}\omega_{\lambda\mu})$ is a 1-cocycle of $\mathfrak{U}$ with values in $\mathscr{A}^*$, and thus corresponds (up to unique isomorphism) to an invertible $\mathscr{A}$-module $\mathscr{L}$ (**0**, 5.4.7) which we denote by $N_{\mathscr{B}/\mathscr{A}}(\mathscr{L}')$, and call the *norm* of the invertible $\mathscr{B}$-module $\mathscr{L}'$.

Let $\mathfrak{M}$ be a subset of $\mathfrak{L}$ such that the corresponding U_λ still form a cover of X, and let $\mathfrak{V}$ be the resulting cover; the restriction of the cocycle $(\omega_{\lambda\mu})$ to $\mathfrak{V}$ defines a 1-cocycle $(N_{\mathscr{B}/\mathscr{A}}\omega_{\lambda\mu})$ of $\mathfrak{V}$ with values in $\mathscr{A}^*$, the restriction of the 1-cocycle $(N_{\mathscr{B}/\mathscr{A}}\omega_{\lambda\mu})$ of $\mathfrak{U}$; it is clear that there is a canonical isomorphism from the invertible $\mathscr{A}$-module defined by this 1-cocycle of $\mathfrak{V}$ to $N_{\mathscr{B}/\mathscr{A}}(\mathscr{L}')$, which allows us to define the invertible $\mathscr{A}$-module by an arbitrary sub-cover of $\mathfrak{U}$. This possibility immediately shows that, if $\mathscr{L}'_1$ and $\mathscr{L}'_2$ are invertible $\mathscr{B}$-modules, then, by (6.5.1.1) and (6.5.1.2),

(6.5.2.1) $$N(\mathscr{L}'_1 \otimes_{\mathscr{B}} \mathscr{L}'_2) = N(\mathscr{L}'_1) \otimes_{\mathscr{A}} N(\mathscr{L}'_2)$$

and

(6.5.2.2) $$N_{\mathscr{B}/\mathscr{A}}(\mathscr{B}) = \mathscr{A}$$

as well as

(6.5.2.3) $$N(\mathscr{L}'^{-1}) = (N(\mathscr{L}'))^{-1}$$

up to canonical isomorphisms. Furthermore, it follows from (6.5.1.3) that, if $\mathscr{L}$ is an invertible $\mathscr{A}$-module = and if $\mathscr{B}$ is of constant rank n on $\mathscr{A}$ in case (I) (resp. $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{R}(X)$ is of constant rank n on $\mathscr{R}(X)$ in case (II)), we have, up to canonical isomorphism,

(6.5.2.4) $$N_{\mathscr{B}/\mathscr{A}}(\mathscr{L} \otimes_{\mathscr{A}} \mathscr{B}) = \mathscr{L}^{\otimes n}.$$

(6.5.3). We now show that $N_{\mathscr{B}/\mathscr{A}}$ is a covariant *functor* in the category of invertible $\mathscr{B}$-modules. Let $h' : \mathscr{L}'_1 \to \mathscr{L}'_2$ be a homomorphism of invertible $\mathscr{B}$-modules, and let $\mathfrak{V} = (U_\lambda)$ be an open cover of X such that, for all λ, we have $(\mathscr{B}|U_\lambda)$-isomorphisms $\eta_\lambda^{(1)} : \mathscr{L}'_1|U_\lambda \xrightarrow{\sim} \mathscr{B}|U_\lambda$ and $\eta_\lambda^{(2)} : \mathscr{L}'_2|U_\lambda \xrightarrow{\sim} \mathscr{B}|U_\lambda$; there is thus, for each λ, an endomorphism h'_λ of $\mathscr{B}|U_\lambda$ such that $h'_\lambda \circ \eta_\lambda^{(1)} = \eta_\lambda^{(2)} \circ (h'|U_\lambda)$, and we can evidently identify h'_λ with a section of $\mathscr{B}$ over U_λ (**0**, 5.1.1). So, for every pair (λ, μ)

of indices, the restrictions to $U_\lambda \cap U_\mu$ of $(\eta_\lambda^{(2)})^{-1} \circ h'_\lambda \circ \eta_\lambda^{(1)}$ and $(\eta_\mu^{(2)})^{-1} \circ h'_\mu \circ \eta_\mu^{(1)}$ agree. We thus obtain, for 1-cocycles $(\omega_{\lambda\mu}^{(1)})$ and $(\omega_{\lambda\mu}^{(2)})$ with values in $\mathscr{B}^*$ corresponding to $\mathscr{L}'_1$ and $\mathscr{L}'_2$, the relation

$$\omega_{\lambda\mu}^{(2)} h'_\mu = h'_\lambda \omega_{\lambda\mu}^{(1)}.$$

If we set $h_\lambda = N(h'_\lambda)$, we thus have the analogous relations **II** | 128

$$N(\omega_{\lambda\mu}^{(2)}) h_\mu = h_\lambda N(\omega_{\lambda\mu}^{(1)})$$

and so the h_λ define a homomorphism $N(\mathscr{L}'_1) \to N(\mathscr{L}'_2)$ which we denote by $N_{\mathscr{B}/\mathscr{A}}(h')$, or $N(h')$. In hypothesis (I), for h' to be an *isomorphism*, it is necessary and sufficient (since it is a local question) for $N_{\mathscr{B}/\mathscr{A}}(h')$ to be an isomorphism. In hypothesis (II), this condition is again necessary; it is sufficient if hypothesis (II *bis*) is satisfied and $\mathscr{B} \to \mathscr{B} \otimes_{\mathscr{A}} \mathscr{R}(X)$ is injective.

Take, in particular, $\mathscr{L}'_1 = \mathscr{B}$; the homomorphisms $\mathscr{B} \to \mathscr{L}'$ can then be identified (**0**, 5.1.1) with the sections of $\mathscr{L}'$ over X, whence a canonical map

$$N_{\mathscr{B}/\mathscr{A}} : \Gamma(X, \mathscr{L}') \longrightarrow \Gamma(X, N_{\mathscr{B}/\mathscr{A}}(\mathscr{L}')).$$

It again follows from (6.5.1.1) that, if $f'_1 \in \Gamma(X, \mathscr{L}'_1)$ and $f'_2 \in \Gamma(X, \mathscr{L}'_2)$, then

(6.5.3.1) $$N(f'_1 \otimes f'_2) = N(f'_1) \otimes N(f'_2).$$

For every invertible $\mathscr{A}$-module $\mathscr{L}$ and every section $f \in \Gamma(X, \mathscr{L})$, we have, taking (6.5.2.4) into account, that

(6.5.3.2) $$N_{\mathscr{B}/\mathscr{A}}(f \otimes 1_{\mathscr{B}}) = f^{\otimes n}$$

whenever $\mathscr{B}$ is of constant rank n in hypothesis (I) (resp. whenever $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{R}(X)$ is of constant rank n in hypothesis (II)). Finally, for the homomorphism $\mathscr{B} \to \mathscr{L}'$ corresponding to a section f' of $\mathscr{L}'$ over X to be an isomorphism, it is necessary and sufficient for f'_x to be a basis for $\mathscr{L}'_x$ for all $x \in X$; in hypothesis (I), this condition is thus equivalent to saying that $(N(f'))_x$ is a basis for $(N(\mathscr{L}'))_x$ for all x; in hypothesis (II), this condition is again necessary, and it is sufficient whenever $\mathscr{B}$ satisfies hypothesis (II *bis*) and $\mathscr{B} \to \mathscr{B} \otimes_{\mathscr{A}} \mathscr{R}(X)$ is injective.

(6.5.4). Let $(X, \mathscr{A})$ and $(X', \mathscr{A}')$ be ringed spaces, $f : X' \to X$ a morphism, $\mathscr{B}$ an $\mathscr{A}$-algebra, and $\mathscr{B}' = f^*(\mathscr{B})$ the inverse image $\mathscr{A}'$-algebra. Suppose that one of the following hypotheses is satisfied:

(1) $\mathscr{B}$ satisfies hypothesis (I) of (6.5.1).

(2) $(X, \mathscr{A})$ and $\mathscr{B}$ satisfy hypothesis (II) of (6.5.1), $(X', \mathscr{A}')$ is a locally Noetherian reduced prescheme, and, if we denote by X_α and X'_β the closed reduced subpreschemes of X and X' (respectively) that have the irreducible components of these spaces as their underlying spaces, then the restriction of f to each X'_β is a *dominant* morphism from X'_β to X_α.

Under these conditions, $\mathscr{B}'$ satisfies hypothesis (I) or hypothesis (II) (respectively) of (6.5.1). The first claim is immediate; to establish the second, it suffices to show that, for all $x' \in X'$, the ranks of the $\mathscr{B}' \otimes_{\mathscr{O}_{X'}} \mathscr{R}(X'_\beta)$ are *the same* for the β such that $x' \in X'_\beta$. But if the restriction of f to X'_β is a dominant morphism to X_α, then the rank of $\mathscr{B}' \otimes_{\mathscr{O}_{X'}} \mathscr{R}(X'_\beta)$ is equal to that of $\mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{R}(X_\alpha)$ (as we immediately see by reducing to the affine case, as in (6.4.9)), whence our claim, by hypothesis (II) and (6.5.1).

With this in mind, it follows from (6.4.8), (6.4.9), and (6.4.10) that, if s is a section of $\mathscr{B}$ over an open $U \subset X$, and s' the corresponding section of $\mathscr{B}'$ over $f^{-1}(U)$, then $N_{\mathscr{B}'/\mathscr{A}'}(s')$ is the section of $\mathscr{A}'$ over $f^{-1}(U)$ that corresponds to the section $N_{\mathscr{B}/\mathscr{A}}(s)$ of $\mathscr{A}$ over U. **II** | 129

If $\mathscr{M}$ is an invertible $\mathscr{B}$-module, then we deduce from the above that, if $\mathscr{M}' = f^*(\mathscr{M})$ (which is an invertible $\mathscr{B}'$-module), we have $N_{\mathscr{B}'/\mathscr{A}'}(\mathscr{M}') = f^*(N_{\mathscr{B}/\mathscr{A}}(\mathscr{M}))$ up to canonical isomorphism.

(6.5.5). Suppose from now on that $(X, \mathscr{A})$ is a *prescheme*. The data of a quasi-coherent $\mathscr{A}$-algebra $\mathscr{B}$, which is an *$\mathscr{A}$-module of finite type*, is then equivalent, as we know, to that of a *finite* morphism $g : X' \to X$ such that $g_*(\mathscr{O}_{X'}) = \mathscr{B}$, defined up to X-isomorphism ((6.1.2) and (1.3.1)). Furthermore, the data of a quasi-coherent $\mathscr{O}_{X'}$-module $\mathscr{F}'$ is equivalent to that of a quasi-coherent $\mathscr{B}$-module $\mathscr{F}$ such that $g_*(\mathscr{F}) = \mathscr{F}$ (1.4.3), and for $\mathscr{F}'$ to be invertible it is necessary and sufficient that $\mathscr{F}$ be invertible (6.1.12). To translate the above results in terms of finite morphisms g, it will be necessary to suppose either that $g_*(\mathscr{O}_{X'})$ is a *locally free* $\mathscr{O}_X$-module (of finite type) or that $(X, \mathscr{O}_X)$ and $g_*(\mathscr{O}_{X'})$ satisfy hypothesis (II). For every invertible $\mathscr{O}_{X'}$-module $\mathscr{L}'$, we thus set

$$N_{X'/X}(\mathscr{L}') = N_{g_*(\mathscr{O}_{X'})/\mathscr{O}_X}(g_*(\mathscr{L}')) \tag{6.5.5.1}$$

and we call this the *norm* (with respect to g) of $\mathscr{L}'$. Similarly, if $h' : \mathscr{L}'_1 \to \mathscr{L}'_2$ is a homomorphism of invertible $\mathscr{O}_{X'}$-modules, then we set

$$N_{X'/X}(h') = N_{g_*(\mathscr{O}_{X'})/\mathscr{O}_X}(g_*(h')) : N_{X'/X}(\mathscr{L}'_1) \longrightarrow N_{X'/X}(\mathscr{L}'_2). \tag{6.5.5.2}$$

In particular, for $\mathscr{L}'_1 = \mathscr{O}_{X'}$, we thus obtain a canonical map

$$N_{X'/X} : \Gamma(X', \mathscr{L}') \longrightarrow \Gamma(X, N_{X'/X}(\mathscr{L}')). \tag{6.5.5.3}$$

We leave to the reader the majority of these translations, and we restrict ourselves to spelling out the details of the following:

Proposition (6.5.6). — *Let $g : X' \to X$ be a finite morphism, and suppose that either $g_*(\mathscr{O}_{X'})$ is a locally free $\mathscr{O}_X$-module or that $(X, \mathscr{O}_X)$ and $g_*(\mathscr{O}_{X'})$ satisfy* (II bis) *(which will be the case, in particular, if X is locally Noetherian and normal). For a homomorphism $h' : \mathscr{L}'_1 \to \mathscr{L}'_2$ of invertible $\mathscr{O}_{X'}$-modules to be an isomorphism, it is necessary and sufficient, in the first hypothesis, that $N_{X'/X}(h')$ be an isomorphism; in the second hypothesis, this condition is again necessary, and is sufficient if the homomorphism $g_*(\mathscr{O}_{X'}) \to g_*(\mathscr{O}_{X'}) \otimes_{\mathscr{O}_X} \mathscr{R}(X)$ is injective.*

PROOF. Note that we use here the fact that, for $g_*(h')$ to be an isomorphism, it is necessary and sufficient for h' to be an isomorphism (1.4.2). □

Corollary (6.5.7). — *Let $g : X' \to X$ be a finite morphism, and suppose that either $g_*(\mathscr{O}_{X'})$ is a locally free $\mathscr{O}_X$-module or that $(X, \mathscr{O}_X)$ and $g_*(\mathscr{O}_{X'})$ satisfy* (II bis) *and $g_*(\mathscr{O}_{X'}) \to g_*(\mathscr{O}_{X'}) \otimes_{\mathscr{O}_X} \mathscr{R}(X)$ is injective. Let $\mathscr{L}'$ be an invertible $\mathscr{O}_{X'}$-module, f' a section of $\mathscr{L}'$ over X', and $f = N_{X'/X}(f')$ the corresponding section of $\mathscr{L} = N_{X'/X}(\mathscr{L}')$ over X (6.5.5.1). Then $g(X' \setminus X'_{f'}) = X \setminus X_f$, and X_f is the largest open subset* II | 130
U of X such that $g^{-1}(U) \subset X'_{f'}$.

PROOF. Indeed, $g(X' \setminus X'_{f'})$ is closed in X (6.1.10); it thus suffices to prove the last claim. But the relation $U \subset X_f$ is equivalent to the fact that the homomorphism $\mathscr{O}_X|U \to \mathscr{L}|U$ defined by $f|U$ is an isomorphism. By (6.5.6), this is equivalent to saying that the homomorphism $\mathscr{O}_{X'}|g^{-1}(U) \to \mathscr{L}'|g^{-1}(U)$ defined by $f'|g^{-1}(U)$ is an isomorphism, which is equivalent to the relation $g^{-1}(U) \subset X'_{f'}$. □

Proposition (6.5.8). — *Let $g : X' \to X$ be a finite morphism, and $f : Y \to X$ a morphism; let $Y' = X'_{(Y)}$, $g' = g_{(Y)}$, and $f' = f_{(X')}$, so that the diagram*

$$\begin{tikzcd} X' \arrow[d, "g"'] & Y' \arrow[l, "f'"'] \arrow[d, "g'"] \\ X & Y \arrow[l, "f"] \end{tikzcd}$$

commutes.

Suppose that either $g_*(\mathscr{O}_{X'})$ is a locally free $\mathscr{O}_X$-module or that $(X, \mathscr{O}_X)$ and $g_*(\mathscr{O}_{X'})$ satisfy (II). Suppose further that Y is a locally Noetherian reduced prescheme, and that the restriction of f to any irreducible component of Y is a dominant morphism to an irreducible component of X. Then, for every invertible $\mathscr{O}_{X'}$-module $\mathscr{L}'$, we have

$$N_{Y'/Y}(f'^*(\mathscr{L}')) = f^*(N_{X'/X}(\mathscr{L}'))$$

up to canonical isomorphism.

PROOF. Note that we have $f^*(g_*(\mathscr{L}')) = g'_*(f'^*(\mathscr{L}'))$, by (1.5.2), and, in particular, $g'_*(\mathscr{O}_{Y'}) = f^*(g_*(\mathscr{O}_{X'}))$; if $g_*(\mathscr{O}_{X'})$ is locally free, then so too is $g'_*(\mathscr{O}_{Y'})$. The conclusion then follows from the definitions and (6.5.4). □

Remark (6.5.9). — We later generalise the notion of norm developed above, by placing it relation to the notion of direct image of a divisor.

6.6. Application: criteria for ampleness

Proposition (6.6.1). — *Let Y be a prescheme, $f : X \to Y$ a quasi-compact morphism, and $g : X' \to X$ a finite and surjective morphism. Suppose that either $g_*(\mathscr{O}_{X'})$ is a locally free $\mathscr{O}_X$-module or that $(X, \mathscr{O}_X)$ and $g_*(\mathscr{O}_{X'})$ satisfy (II* bis*). Then, for every invertible $\mathscr{O}_{X'}$-module $\mathscr{L}'$ that is ample for $f \circ g$, $\mathscr{L} = N_{X'/X}(\mathscr{L}')$ is ample for f.*

PROOF. We can suppose Y to be affine (4.6.4), and then, by (4.6.6), the statement is equivalent to: □

Corollary (6.6.2). — *Let X be a quasi-compact prescheme, $g : X' \to X$ a finite surjective morphism, such that either $g_*(\mathscr{O}_{X'})$ is a locally free $\mathscr{O}_X$-module or that $(X, \mathscr{O}_X)$ and $g_*(\mathscr{O}_{X'})$ satisfy (II* bis*). Then, for every ample $\mathscr{O}_{X'}$-module $\mathscr{L}'$, $\mathscr{L} = N_{X'/X}(\mathscr{L}')$ is ample.*

PROOF. In the second hypothesis, we can further suppose that the canonical homomorphism $g_*(\mathscr{O}_{X'}) \to g_*(\mathscr{O}_{X'}) \otimes_{\mathscr{O}_X} \mathscr{R}(X)$ is *injective*. Indeed, if not, let $\mathscr{T}$ be the kernel of this homomorphism, **II** | 131
which is a coherent ideal of $\mathscr{B} = g_*(\mathscr{O}_{X'})$ (**I**, 6.1.1), and set $X'' = \operatorname{Spec}(\mathscr{B}/\mathscr{T})$; we thus have a commutative diagram

$$\begin{tikzcd} X'' \arrow[rr, "h"] \arrow[dr, "g'"'] & & X' \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

where h is a closed immersion (1.4.10). Furthermore, we know that the support of $\mathscr{T}$ is a closed set (**0**, 5.2.2) that is rare in X (**I**, 7.4.6), whence we conclude that, for the generic point x of an irreducible component of X, there is an affine open neighbourhood U of x such that $\mathscr{B}|U = (\mathscr{B}/\mathscr{T})|U$. Since g is, by hypothesis, surjective, we thus conclude that $x \in g'(X'')$; g' is thus dominant, and, since it is a finite morphism, it is *surjective* (6.1.10); by definition,

$$g'_*(\mathscr{O}_{X''}) \otimes_{\mathscr{O}_X} \mathscr{R}(X) = (\mathscr{B}/\mathscr{T}) \otimes_{\mathscr{O}_X} \mathscr{R}(X) = g_*(\mathscr{O}_{X'}) \otimes_{\mathscr{O}_X} \mathscr{R}(X)$$

thus $(X, \mathscr{O}_X)$ and $g'_*(\mathscr{O}_{X''})$ satisfy (II *bis*), and furthermore $g'_*(\mathscr{O}_{X''}) \to g'_*(\mathscr{O}_{X''}) \otimes_{\mathscr{O}_X} \mathscr{R}(X)$ is injective. Finally, $h^*(\mathscr{L}') = \mathscr{L}''$ is an ample $\mathscr{O}_{X''}$-module (4.6.13, (i *bis*)), and we have that $N_{X''/X}(\mathscr{L}'') = N_{X'/X}(\mathscr{L}')$. Indeed, to define these two invertible $\mathscr{O}_X$-modules, we can use the same affine open cover (U_λ) of X such that the restrictions of $g_*(\mathscr{L}')$ and $g'_*(\mathscr{L}'')$ to U_λ are isomorphic to $\mathscr{B}|U_\lambda$ and $(\mathscr{B}/\mathscr{T})|U_\lambda$ (respectively). We immediately see that, for every isomorphism $\eta_\lambda : g_*(\mathscr{L}')|U_\lambda \to \mathscr{B}|U_\lambda$, there is a canonically corresponding isomorphism

$$\eta'_\lambda : g'_*(\mathscr{L}'')|U_\lambda \longrightarrow (\mathscr{B}/\mathscr{T})|U_\lambda$$

so that, if $(\omega_{\lambda\mu})$ and $(\omega'_{\lambda\mu})$ are the 1-cocycles corresponding to the systems of isomorphisms (η_λ) and (η'_λ) (6.5.2), $\omega'_{\lambda\mu}$ is the canonical image in $\Gamma(U_\lambda \cap U_\mu, \mathscr{B}/\mathscr{T})$ of $\omega_{\lambda\mu} \in \Gamma(U_\lambda \cap U_\mu, \mathscr{B})$. By the definition of $\mathscr{T}$, we thus conclude that

$$N_{\mathscr{B}/\mathscr{A}}(\omega_{\lambda\mu}) = N_{(\mathscr{B}/\mathscr{T})/\mathscr{A}}(\omega'_{\lambda\mu})$$

(where $\mathscr{A} = \mathscr{O}_X$), whence the claimed equality.

So suppose that the homomorphism $g_*(\mathscr{O}_{X'}) \to g_*(\mathscr{O}_{X'}) \otimes_{\mathscr{O}_X} \mathscr{R}(X)$ is injective when we are in hypothesis (II *bis*). It suffices to prove that, as f runs over the sections of $\mathscr{L}^{\otimes n}$ (for $n > 0$) over X, the X_f form a base for the topology of X (4.5.2). But let $x \in X$, and let U be an arbitrary neighbourhood of x; since $g^{-1}(x)$ is finite (6.1.7) and $\mathscr{L}'$ is ample, there exists an integer $n > 0$ and a section f' of $\mathscr{L}'^{\otimes n}$ over X' such that $X'_{f'}$ is a neighbourhood of $g^{-1}(x)$ contained inside $g^{-1}(U)$ (4.5.4). Since

$$\mathscr{L}^{\otimes n} = N_{X'/X}(\mathscr{L}'^{\otimes n})$$

it suffices to take $f = N_{X'/X}(f')$; indeed, then $X \setminus X_f = g(X' \setminus X'_{f'})$ (6.5.7), and so $x \in X_f \subset U$. □

Corollary (6.6.3). — *Under the hypotheses of (6.6.1), for an invertible $\mathscr{O}_X$-module $\mathscr{L}$ to be ample for f, it is necessary and sufficient that $\mathscr{L}' = g^*(\mathscr{L})$ be ample for $f \circ g$.*

PROOF. The condition is necessary, since g is affine (5.1.12). To prove that the condition is sufficient, we can suppose that Y is affine (4.6.4), and so X and X' are quasi-compact and $\mathscr{L}'$ is ample (4.6.6), and we need to show that $\mathscr{L}$ is ample. But the set of points $x \in X$ that admit a neighbourhood where $g_*(\mathscr{O}_{X'})$ (resp. $g_*(\mathscr{O}_{X'}) \otimes_{\mathscr{O}_X} \mathscr{R}(X)$) has a given rank n in the first (resp. second) hypothesis is simultaneously open and closed in X, and so X is the prescheme given by the sum of a finite number of these opens, and we can thus suppose that it is equal to one of them (4.6.17). But then $N_{X'/X}(\mathscr{L}') = \mathscr{L}^{\otimes n}$, and so $\mathscr{L}^{\otimes n}$ is ample by (6.6.2), and thus so too is $\mathscr{L}$ (4.5.6). □ **II** | 132

Corollary (6.6.4). — *Suppose that the hypotheses of (6.6.1) are satisfied, and suppose further that $f : X \to Y$ is of finite type. Then, for f to be quasi-projective, it is necessary and sufficient that $f \circ g$ be quasi-projective. If we further suppose that Y is a quasi-compact scheme, or a prescheme whose underlying space is Noetherian, then for f to be projective, it is necessary and sufficient that $f \circ g$ be projective.*

PROOF. The hypothesis implies that $f \circ g$ is of finite type. Taking into account the definition of quasi-projective morphisms (5.3.1), the first claim follows from (6.6.1) and (6.6.3). Taking into account this result, along with (5.5.3, (ii)), it remains to show that, if f is quasi-projective, then for f to be proper, it is necessary and sufficient that $f \circ g$ be proper. But f is then separated (5.3.1) and of finite type; since g is surjective, our claim follows from (5.4.2, (ii)) and (5.4.3, (ii)). □

In particular:

Corollary (6.6.5). — *Let X be a prescheme of finite type over a field K, and K' a finite-degree extension of K. For X to be projective (resp. quasi-projective) over K, it is necessary and sufficient that $X' = X \otimes_K K'$ be projective (resp. quasi-projective) over K'.*

PROOF. The condition is necessary ((5.3.4, (iii)) and (5.5.5, (iii))). Conversely, suppose that it is satisfied, and let $g : X' \to X$ be the canonical projection. It is clear that g is a finite morphism (6.1.5, (iii)) and surjective (**I**, 3.5.2, (ii)). Furthermore, $g_*(\mathscr{O}_{X'})$ is a locally free $\mathscr{O}_X$-module, since it is isomorphic to $\mathscr{O}_X \otimes_K K'$ (1.5.2). It then follows from the hypothesis and from (6.1.11) and (5.5.5, (ii)) that X' is projective (resp. quasi-projective) *over* K; we then deduce from (6.6.4) that X is projective (resp. quasi-projective) over K. □

In Chapter V, we will show that the statement of (6.6.5) remains true when K' is an *arbitrary* extension of K.

The rest of this section is dedicated to the proof of the criterion in (6.6.11), which is a rather technical refinement of (6.6.1); it can be omitted on a first reading.

Lemma (6.6.6). — *Let X be a reduced Noetherian prescheme, and $\mathscr{E}$ a coherent $\mathscr{O}_X$-module such that $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{R}(X)$ is a locally free $\mathscr{R}(X)$-module of constant rank n. Then there exists a reduced Noetherian prescheme Z and a birational finite morphism $h : Z \to X$ that has the following property: the morphisms of sheaves of sets $\sigma_i : \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E}) \to \mathscr{R}(X)$ (for $1 \leqslant i \leqslant n$) (cf. (6.4.10)) send $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E})$ to the coherent $\mathscr{O}_X$-algebra $h_*(\mathscr{O}_Z)$.*

PROOF. Consider an affine open U of X, of ring $A(U) = A$; let $E = \Gamma(U, \mathscr{E})$, and let C_U be the subalgebra of $R(U)$ generated by the $\sigma_i(u)$ where u runs over $\operatorname{Hom}_A(E, E)$; we have seen (6.4.7.1) that this A-algebra is of *finite rank*. Furthermore, it is clear that forming the algebras C_U commutes with the restriction operations of an affine open U to an affine open $U' \subset U$. We have thus defined a *finite* sub-$\mathscr{O}_X$-algebra $\mathscr{C}$ of $\mathscr{R}(X)$ such that $\Gamma(U, \mathscr{C}) = C_U$ for every affine open U of X. We take $Z = \operatorname{Spec}(\mathscr{C})$, and take h to be the structure morphism, which is thus finite (6.1.2); since $\mathscr{C}$ is reduced, Z is a reduced Noetherian prescheme (1.3.8). Finally, the total ring of fractions of C_U is $R(U)$, by definition, and since C_U is contained inside the integral closure of $A(U)$ in $R(U)$, there is a bijective correspondence between minimal prime ideals of $A(U)$ and minimal prime ideals of C_U [**SZ60**, t. I, p. 259], which proves that h is birational and finishes the proof. □

Corollary (6.6.7). — *Under the hypotheses of (6.6.6), let W be an open of X such that, for all $x \in W$, either X is normal at the point x, or $\mathscr{E}_x$ is a free $\mathscr{O}_x$-module. Then we can suppose h to be defined such that the restriction of h to $h^{-1}(W)$ is an isomorphism from $h^{-1}(W)$ to W.*

PROOF. Either hypothesis implies that, if $U \subset W$ is an affine open, then, with the notation of (6.6.6), $(\sigma_i(u))_x \in A_x$ for all $x \in U$ (6.4.3), and so $\sigma_i(u) \in A$, and the conclusion follows from the definition of h given in (6.6.6). □

(6.6.8). Let X be a reduced Noetherian prescheme, and $g : X' \to X$ a finite surjective morphism, so that $\mathscr{B} = g_*(\mathscr{O}_{X'})$ is a *coherent* $\mathscr{O}_X$-algebra; suppose further that $\mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{R}(X)$ is a *locally free* $\mathscr{R}(X)$-module *of constant rank n*. We can then apply Lemma (6.6.6), taking $\mathscr{E} = \mathscr{B}$, whence, with the notation of (6.6.6), we obtain a homomorphism of sheaves of multiplicative monoids $\sigma_n : \mathscr{H}om_{\mathscr{O}_X}(\mathscr{B}, \mathscr{B}) \to h_*(\mathscr{O}_Z)$, and by composing this homomorphism with the canonical homomorphism $\mathscr{B} \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{B}, \mathscr{B})$ (6.5.1), we thus obtain a homomorphism of sheaves of multiplicative monoids:

$$N' : \mathscr{B} = g_*(\mathscr{O}_{X'}) \longrightarrow h_*(\mathscr{O}_Z) = \mathscr{C}. \tag{6.6.8.1}$$

With this in mind, for every invertible $\mathscr{O}_{X'}$-module $\mathscr{L}'$, $g_*(\mathscr{L}')$ is an invertible $\mathscr{B}$-module (6.1.12), and the method of (6.5.2) allows us to functorially associate to $\mathscr{L}'$ an invertible $\mathscr{C}$-module, which we denote by $N'(g_*(\mathscr{L}'))$.

Lemma (6.6.9). — *Let X be a reduced Noetherian prescheme, and $g : X' \to X$ a finite surjective morphism such that $g_*(\mathscr{O}_{X'}) \otimes_{\mathscr{O}_X} \mathscr{R}(X)$ is a locally free $\mathscr{R}(X)$-module of constant rank. Then there exists a reduced Noetherian prescheme Z and a finite birational morphism $h : Z \to X$ that has the following property: for every ample $\mathscr{O}_{X'}$-module $\mathscr{L}'$, the invertible $\mathscr{O}_Z$-module $\mathscr{M}$ such that $h_*(\mathscr{M}) = N'(g_*(\mathscr{L}'))$ (using the notation of* (6.6.8)*) is ample.*

PROOF. Suppose first of all that the homomorphism $\mathscr{B} \to \mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{R}(X)$ is *injective*. Define Z and h as in (6.6.6) (with $\mathscr{E} = g_*(\mathscr{O}_{X'})$). Let $z \in Z$; we have to show that there exists an integer $m > 0$ and a section t of $\mathscr{M}^{\otimes m}$ over Z such that Z_t is an affine open that contains z (4.5.2). Let $x = h(z)$, and let U be an affine open of X that contains x; then $h^{-1}(U)$ is an affine open neighbourhood of z, and it suffices to find t such that $z \in Z_t \subset h^{-1}(U)$, since Z_t will then necessarily be affine (5.5.8). There exists, by hypothesis, an integer $n > 0$ and a section s' of $\mathscr{L}'^{\otimes n}$ over X' such that

$$g^{-1}(x) \subset X'_{s'} \subset g^{-1}(U) \tag{6.6.9.1}$$

by (4.5.4). By definition, s' is also a section of $g_*(\mathscr{L}')$ over X, and it corresponds, as in (6.5.2), to a section $s = N'(s')$ of $N'(g_*(\mathscr{L}'))$ over X. We will show that, if t is the section s considered as a section of $\mathscr{M}$ over Z, then t is the desired section. Set **II** | 134

$$V = X \setminus g(X' \setminus X'_{s'}) \tag{6.6.9.2}$$

which is an open of X that contains x and is contained in U, by (6.6.9.1) and (6.1.10). We will show that

$$h^{-1}(V) \subset Z_t \subset h^{-1}(U) \tag{6.6.9.3}$$

which will finish the proof. It is equivalent to say that the set T of $y \in X$ such that s_y is invertible contains V and is contained in U. For this, consider first of all an affine open W contained in V; then $g^{-1}(W)$ is an affine open in X', and by (6.6.9.2) $s'_{y'}$ is invertible for all $y' \in g^{-1}(W)$; by the hypotheses on X and $\mathscr{B}$, we can apply the results of (6.4.7), and we see that, if $y = g(y')$, then s_y is invertible; in other words, $V \subset T$. On the other hand, it also follows from (6.4.7) that, conversely, if s_y is invertible, then so too is $s'_{y'}$, which implies that $y' \in g^{-1}(U)$ by (6.6.9.1), and so $y \in U$, whence $T \subset U$ in this case.

We pass from this to the general case by the same argument as in (6.6.2), replacing X' by X'' such that $g'_*(\mathscr{O}_{X''}) \to g'_*(\mathscr{O}_{X''}) \otimes_{\mathscr{O}_X} \mathscr{R}(X)$ is injective, and $\mathscr{L}'$ by an ample $\mathscr{O}_{X''}$-module $\mathscr{L}''$ such that $N'(g_*(\mathscr{L}')) = N'(g'_*(\mathscr{L}''))$. Lemma (6.6.9) is then proven in all cases (with a suitable choice of h). □

Corollary (6.6.10). — *Suppose that the hypotheses of* (6.6.9) *are satisfied; for every invertible $\mathscr{O}_X$-module $\mathscr{L}$ such that $g^*(\mathscr{L})$ is ample, $h^*(\mathscr{L})$ is ample.*

PROOF. If we set $\mathscr{L}' = g^*(\mathscr{L})$, then $g_*(\mathscr{L}') = \mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{B}$ (**0**, 5.4.10), so

$$N'(g_*(\mathscr{L}')) = (\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{C})^{\otimes n}$$

(by the same argument as for (6.5.2.4)). We thus conclude that $\mathscr{M} = (h^*(\mathscr{L}))^{\otimes n}$, and since $\mathscr{M}$ is ample, so too is $h^*(\mathscr{L})$ (4.5.6). □

Proposition (6.6.11). — *Let Y be an affine scheme, X an reduced Noetherian prescheme, $f : X \to Y$ a quasi-compact morphism, and $g : X' \to X$ a finite surjective morphism. Let W be an open subset of X such that, for all $x \in W$, either X is normal at the point x or there exists an open neighbourhood $T \subset W$ of x such that $(g_*(\mathscr{O}_{X'}))|T$ is a locally free $(\mathscr{O}_X|T)$-module. Then there exists a reduced Y-prescheme Z and a* finite birational *Y-morphism $h : Z \to X$ such that the restriction of h to $h^{-1}(W)$ is an isomorphism from $h^{-1}(W)$ to W and has the following property: for every invertible $\mathscr{O}_X$-module $\mathscr{L}$ such that $g^*(\mathscr{L})$ is ample for $f \circ g$, $h^*(\mathscr{L})$ is ample for $f \circ h$.*

Proof. Since Y is affine, $g^*(\mathscr{L})$ is ample, and the problem thus reduces to proving that, for a suitable choice of h, $h^*(\mathscr{L})$ is ample (4.6.6). We will show that we can replace g by a finite surjective morphism $g' : X'' \to X$ such that $g'^*(\mathscr{L})$ is ample and $g'_*(\mathscr{O}_{X''}) \otimes_{\mathscr{O}_X} \mathscr{R}(X)$ is a *locally free* $\mathscr{R}(X)$-module *of constant rank*; we will then have reduced to the conditions of (6.6.10), and the proposition will thus be proven.

For this, let $\mathscr{B} = g_*(\mathscr{O}_{X'})$; denote by X_i (for $1 \leqslant i \leqslant n$) the closed reduced subpreschemes of X that have the irreducible components of X as their underlying space (**I**, 5.2.1); they are integral by hypothesis. Let X'_i be the closed subprescheme $g^{-1}(X_i)$ of X', and $g_i : X'_i \to X_i$ the morphism **II | 135**
g restricted to X'_i, which is finite (6.1.5, (iii)) and surjective; let k_i be the *rank* of the $\mathscr{O}_{X_i}$-algebra $\mathscr{B}_i = (g_i)_*(\mathscr{O}_{X'_i})$. Since $\mathscr{B}_i \otimes_{\mathscr{O}_{X_i}} \mathscr{R}(X_i)$ is a constant presheaf (**I**, 7.3.5), the rank k_i is also the rank of the $(\mathscr{O}_X|U)$-algebra $\mathscr{B}|U$ for every open U of X that does not meet the *only* irreducible component X_i. If T is an open subset of X such that $\mathscr{B}|T$ is isomorphic to $\mathscr{O}_X^m|T$, then it follows from the above remark that the numbers k_i are equal to m for all the indices i such that $T \cap X_i \neq \varnothing$. So let U be the open subset of X given by the points that admit a neighbourhood on which $\mathscr{B}$ is a locally free $\mathscr{O}_X$-module, and let U_j (for $1 \leqslant j \leqslant s$) be its connected components, which are open in X and finitely many (since U is Noetherian); denote by V_j the closed subprescheme of X' given by the *closure* of the subprescheme induced on the open $g^{-1}(U_j)$ (**I**, 9.5.11). By the above, for all indices i such that $X_i \cap U_j \neq \varnothing$, the ranks k_i are all equal to a single integer m_j; note also that a single X_i cannot meet two U_j with different indices j. Let i_λ be the indices i such that $X_i \cap U = \varnothing$. Consider the product k of all the k_i, set $n_i = k/k_i$, and let X'' be the prescheme defined as follows. For each j (for $1 \leqslant j \leqslant s$), consider k/m_j preschemes isomorphic to V_j, and for each λ, k/k_{i_λ} preschemes isomorphic to X'_{i_λ}; then X'' is the *sum* of all these preschemes. We define a morphism $g'' : X'' \to X'$ that reduces to the canonical injection on each of the summands of X''; it is clear that g'' is a finite dominant morphism, and thus surjective (since a finite morphism is closed (6.1.10)); set $g' = g \circ g''$, which is a finite surjective morphism $X'' \to X$; we have $g'^*(\mathscr{L}) = g''^*(g^*(\mathscr{L}))$, and so $g'^*(\mathscr{L})$ is an ample $\mathscr{O}_{X''}$-module (5.1.12). It is then clear that, for this new prescheme X'', the ranks defined as the k_i for X' are all *equal to* k; taking (**I**, 7.3.3) into account, we immediately conclude that, for every affine open T of X, $(g'_*(\mathscr{O}_{X''}) \otimes_{\mathscr{O}_X} \mathscr{R}(X))|T$ is an $(\mathscr{R}(X)|T)$-module isomorphic to $(\mathscr{R}(X)|T)^k$. □

Corollary (6.6.12). — *If, in the statement of (6.6.11), we have $W = X$, then for an invertible $\mathscr{O}_X$-module $\mathscr{L}$ to be ample for f, it is necessary and sufficient that $g^*(\mathscr{L})$ be ample for $f \circ g$.*

Remark (6.6.13). — In Chapter III, we will see that, if Y is Noetherian, and f of finite type, and if the restriction of f to the closed reduced subprescheme of X that has $X \setminus W$ as its underlying space is *proper*, then the conclusion of (6.6.12) still holds true. But we will give, in Chapter V, examples of algebraic schemes X over a field K (with the structure morphism $X \to \operatorname{Spec}(K)$ not proper) whose normalisation X' is quasi-affine, but which are not quasi-affine (in that $\mathscr{O}_X$ is not ample, even though $\mathscr{O}_{X'}$ is (5.1.12) and the morphism $X' \to X$ is finite and surjective (6.3.10)). We will see in the next section that this circumstance cannot happen when we replace "quasi-affine" by "affine".

6.7. Chevalley's theorem We are going to prove (with the help of Serre's criterion (5.2.1)) the following theorem, which was proven by C. Chevalley by other methods, in the case of *algebraic* schemes.

II | 136

Theorem (6.7.1). — *Let X be an affine scheme, Y a Noetherian prescheme, and $f : X \to Y$ a finite surjective morphism. Then Y is an affine scheme.*

PROOF. It is clear that $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ is finite (6.1.5, (vi)); since X_{red} is an affine scheme, and since saying that Y is affine is equivalent to saying that Y_{red} is affine (since Y is Noetherian (**I**, 6.1.7)), we see that we can suppose Y to be *reduced*. For every closed subset Y' of Y, there exists exactly one reduced subprescheme of Y that has Y' as its underlying space (**I**, 5.1.2); its inverse image $f^{-1}(Y')$, which is canonically isomorphic to $X \times_Y Y'$ (**I**, 4.4.1), is affine as a closed subprescheme of X, and the restriction of f to $f^{-1}(Y')$, which can be identified with $f \times_Y 1_{Y'}$, is a finite surjective morphism (6.1.5, (iii)). By the principal of Noetherian induction (**0**, 2.2.2), we can thus (taking (**I**, 6.1.7) into account) reduce to proving the theorem under the hypothesis that for *every* closed subset $Y' \neq Y$, every closed subprescheme of Y that has Y' as its underlying space is affine. We thus conclude that, *for every coherent $\mathscr{O}_Y$-module $\mathscr{F}$ whose (closed) support Z is distinct from Y, we have* $\mathrm{H}^1(Y, \mathscr{F}) = 0$. Indeed, there exists a closed subprescheme Y' of Y that has Z as its underlying space and is such that, if $j : Y \to Y$ is the canonical injection, then $\mathscr{F} = j_*(j^*(\mathscr{F}))$ (**I**, 9.3.5); then (5.2.3) $\mathrm{H}^1(Y, \mathscr{F}) = \mathrm{H}^1(Y', j^*(\mathscr{F})) = 0$, by (**I**, 5.1.9.2).

Suppose first of all that Y is not irreducible, and let Y' be an irreducible component of Y, and $Y'' = Y \setminus Y'$; we again denote by Y' the closed reduced subprescheme of Y that has Y' as its underlying space, and by j the canonical injection $Y' \to Y$. Let $\mathscr{F}$ be a coherent $\mathscr{O}_Y$-module, and consider the canonical homomorphism

$$\rho : \mathscr{F} \longrightarrow \mathscr{F}' = j_*(j^*(\mathscr{F}))$$

(**0**, 4.4.3); $\mathscr{F}'$ is a coherent $\mathscr{O}_Y$-module by (**0**, 5.3.10) and (**0**, 5.3.12), since $j_*(\mathscr{O}_{Y'}) = \mathscr{O}_Y / \mathscr{J}$, where we denote by $\mathscr{J}$ the sheaf of ideals of $\mathscr{O}_Y$ that defines the subprescheme Y'. Then $\mathscr{G} = \operatorname{Ker} \rho$ and $\mathscr{K} = \operatorname{Im} \rho$ are also coherent $\mathscr{O}_Y$-modules (**0**, 5.3.4); but, by definition, the fibre $\mathscr{F}'_y$ of $\mathscr{F}'$ at the generic point y of Y' is equal to $\mathscr{F}_y$, since y is interior to Y' and thus $\mathscr{J}_y = 0$, since Y is reduced. We thus conclude that y is not contained in the (closed) support of $\mathscr{G}$; also, the support of $\mathscr{F}'$ (and *a fortiori* that of $\mathscr{K}$) is contained in Y'; in other words, the supports of $\mathscr{G}$ and $\mathscr{K}$ are *distinct from* Y. We thus deduce that $\mathrm{H}^1(Y, \mathscr{G}) = \mathrm{H}^1(Y, \mathscr{K}) = 0$, and the exact sequence of cohomology applied to the exact sequence $0 \to \mathscr{G} \to \mathscr{F} \to \mathscr{K} \to 0$ implies that $\mathrm{H}^1(Y, \mathscr{F}) = 0$. We thus conclude by Serre's criterion (5.2.1).

So suppose that Y is irreducible, and thus *integral*. We can also suppose that X is *integral*: if we denote by X_i the closed reduced subpreschemes of X that have the irreducible components of X as their underlying space (**I**, 5.2.1), and by g_i the restriction of g to X_i, then at least one of the g_i is dominant, and since it is a finite morphism (6.1.5), it is surjective (6.1.10); since X_i is also an affine scheme, we see that we can replace X by X_i in the statement.

Lemma (6.7.1.1). — *Let X and Y be integral Noetherian preschemes, x (resp. y) the generic point of X (resp. Y), and $f : X \to Y$ a finite surjective morphism. Let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module such that there exists an affine open neighbourhood U of y and a section $g \in \Gamma(X, \mathscr{L})$ such that $x \in X_g \subset f^{-1}(U)$. Then there exist* II | 137 *integers $m, n > 0$, a homomorphism $u : \mathscr{O}_Y^m \to f_*(\mathscr{L}^{\otimes n})$, and an open neighbourhood V of y such that the restriction $u|V$ is an isomorphism from $\mathscr{O}_Y^m|V$ to $f_*(\mathscr{L}^{\otimes n})|V$.*

PROOF. Let C be the (integral) ring of U, and $k = \mathscr{O}_y$ its field of fractions: since f is finite, $U' = f^{-1}(U)$ is affine (1.3.2); let D be its (integral) ring with field of fractions $K = \mathscr{O}_x$; by hypothesis, D is a C-module of finite type (6.1.4), and so K is an extension of finite rank of k. The fibre $f^{-1}(y) = X \times_Y \operatorname{Spec}(k(y)) = X \times_Y \operatorname{Spec}(\mathscr{O}_y)$ can be identified with $\operatorname{Spec}(K)$ (**I**, 3.6.5); let s_i (for $1 \leqslant i \leqslant m$) be elements of D that form a basis of K over k. There exists $n > 0$ such that the sections $(s_i|X_g)g^{\otimes n}$ of $\mathscr{L}^{\otimes n}$ over X_g extend to sections b_i (for $1 \leqslant i \leqslant m$) of $\mathscr{L}^{\otimes n}$ over X (**I**, 9.3.1). The b_i are also, by definition, sections of $f_*(\mathscr{L}^{\otimes n})$ over Y, and thus define a homomorphism $u : \mathscr{O}_Y^m \to f_*(\mathscr{L}^{\otimes n})$ (**0**, 5.1.1); we will show that u is the desired homomorphism. We have that

$\mathscr{L}^{\otimes n}|U' = \widetilde{M}$, where M is a D-module of finite type, so if ϕ is the injection $C \to D$ corresponding to the morphism $f^{-1}(U) \to U$ given by the restriction of f, then $M_{[\phi]}$ is a C-module of finite type; then

$$f_*(\mathscr{L}^{\otimes n})|U = (M_{[\phi]})^{\sim}$$

(**I**, 1.6.3) is coherent, and since U is an arbitrary affine open of Y, $f_*(\mathscr{L}^{\otimes n})$ is coherent; furthermore, $u|U = \widetilde{\theta}$, where θ is a C-homomorphism $C^m \to M_{[\phi]}$, and $u_y = \theta_y$ is the homomorphism $\theta \otimes 1 : K^m = C^m \otimes K \to M_{[\phi]} \otimes K$; but the latter is, by definition, an *isomorphism*, since the $(b_i)_x$ form a basis of $M_{[\phi]} \otimes K$ over k, with g_x being, by hypothesis, a generator of $(\mathscr{L}^{\otimes n})_x$. We thus conclude that the supports of the $\mathscr{O}_Y$-modules $\operatorname{Ker} u$ and $\operatorname{Coker} u$ do not contain y; since these $\mathscr{O}_Y$-modules are coherent (**0**, 5.3.3), their supports are closed (**0**, 5.2.2), whence the lemma. □

With this, the hypotheses of Lemma (6.7.1.1) are satisfied in the case that we are considering, by taking $\mathscr{L} = \mathscr{O}_X$, since X is affine (**I**, 1.1.10); set $\mathscr{A} = \mathscr{O}_Y$ and $\mathscr{B} = f_*(\mathscr{O}_X)$. By Serre's criterion (5.2.1.1), it suffices to prove that, for every coherent $\mathscr{O}_Y$-module $\mathscr{F}$, we have that $\mathrm{H}^1(Y, \mathscr{F}) = 0$; it even suffices to prove this in the case where $\mathscr{F} \subset \mathscr{O}_Y$, which implies that $\mathscr{F}$ is torsion free, since Y is integral; in fact, we will show that $\mathrm{H}^1(Y, \mathscr{O}_Y) = 0$ for *every* coherent *torsion-free* $\mathscr{O}_Y$-module $\mathscr{F}$. But the homomorphism $u : \mathscr{A}^m \to \mathscr{B}$ defines a homomorphism

$$v : \mathscr{G} = \mathscr{H}om_{\mathscr{A}}(\mathscr{B}, \mathscr{F}) \longrightarrow \mathscr{H}om_{\mathscr{A}}(\mathscr{A}^m, \mathscr{F}) = \mathscr{F}^m.$$

We will first show that v is *injective*: by hypothesis, $\mathscr{T} = \operatorname{Coker} u$ has a support that does not meet V, and is thus a *torsion* $\mathscr{O}_Y$-module (**I**, 7.4.6); the exact sequence

$$\mathscr{A}^m \longrightarrow \mathscr{B} \longrightarrow \mathscr{T} \longrightarrow 0$$

gives, by left exactness of the functor $\mathscr{H}om_{\mathscr{A}}$, the exact sequence

$$0 \longrightarrow \mathscr{H}om_{\mathscr{A}}(\mathscr{T}, \mathscr{F}) \longrightarrow \mathscr{G} \xrightarrow{v} \mathscr{F}^m.$$

But since $\mathscr{F}$ is torsion free, we have that $\mathscr{H}om_{\mathscr{A}}(\mathscr{T}, \mathscr{F}) = 0$ (**0**, 5.2.6), whence our claim. We thus have the exact sequence

$$0 \longrightarrow \mathscr{G} \longrightarrow \mathscr{F}^m \longrightarrow \operatorname{Coker} v \longrightarrow 0$$

where $\mathscr{G}$ and $\operatorname{Coker} v$ are coherent $\mathscr{O}_Y$-modules ((**0**, 5.3.4) and (**0**, 5.3.5)); by the exact sequence **II** | 138
of cohomology, it would suffice to show that $\mathrm{H}^1(Y, \mathscr{G}) = \mathrm{H}^1(Y, \operatorname{Coker} v) = 0$ in order to deduce that $\mathrm{H}^1(Y, \mathscr{F}^m) = (\mathrm{H}^1(Y, \mathscr{F}))^m = 0$, and thus that $\mathrm{H}^1(Y, \mathscr{F}) = 0$. But the restriction $v|V$ is an isomorphism, and so the support of $\operatorname{Coker} v$ is distinct from Y, whence $\mathrm{H}^1(Y, \operatorname{Coker} v) = 0$ by the hypothesis at the start. Now, $\mathscr{G}$ is a coherent $\mathscr{B}$-*module* (**I**, 9.6.4); since X is affine over Y, there exists a quasi-coherent $\mathscr{O}_X$-module $\mathscr{K}$ such that $\mathscr{G}$ is isomorphic to $f_*(\mathscr{K})$ (1.4.3); since X is affine, we have that $\mathrm{H}^1(X, \mathscr{K}) = 0$ (**I**, 5.1.9.2), and so $\mathrm{H}^1(Y, \mathscr{G}) = 0$ by (5.2.3), which finishes the proof of Theorem (6.7.1). □

Corollary (6.7.2). — *Let X be a Noetherian prescheme, $(X_i)_{1 \leqslant i \leqslant n}$ a finite cover of the space X consisting of closed subsets. For X to be affine, it is necessary and sufficient that, for each i, there exist a closed subprescheme of X that has X_i as its underlying space and is affine.*

Proof. Indeed, if this is the case, then let X' be the scheme given by the *sum* of the X_i; it is clear that X' is affine, and we define a surjective morphism $f : X' \to X$ by taking the restriction of f to X_i to be the canonical injection. Everything reduces to showing that f is *finite*, by (6.7.1), and this we have already seen in (6.1.5). □

Corollary (6.7.3). — *For a Noetherian prescheme X to be affine, it is necessary and sufficient that the closed reduced subpreschemes whose underlying spaces are the irreducible components of X be affine.*

§7. Valuative criteria

In this section, we give valuative criteria for separation and properness for a given morphism, that is, criteria which introduce a variable auxiliary scheme of the form $\operatorname{Spec}(A)$, where A is a valuation ring. Under certain suitable "Noetherian" hypotheses, we can refine our criteria and restrict to the case where A is a *discrete* valuation ring. This will be the only case that we need to concern ourselves with in all that follows, and we introduce arbitrary valuation rings, in the general case, only to discuss the links with the classical study of such objects.

7.1. Reminder on valuation rings

(7.1.1). Amongst the many diverse equivalent properties that characterise valuation rings, we will use the following: a ring A is said to be a *valuation ring* if it is an integral ring which is not a field, and A is *maximal* in the set of local rings strictly contained in the field of fractions K of A under the domination relation (**I**, 8.1.1). Recall that a valuation ring is *integrally closed*. If A is a valuation ring, then so too is $A_{\mathfrak{p}}$ for any prime ideal $\mathfrak{p} \neq 0$ of A.

(7.1.2). Let K be a field, and A a local subring of K that is not a field; then there exists a valuation ring that both dominates A and has K as its field of fractions ([**CC**, p. 1-07, lemma 2]). **II** | 139

Now let B be a valuation ring, k its residue field, K its field of fractions, and L an extension of k. Then there exists a *complete* valuation ring C that dominates B and whose residue field is L. Indeed, L is the algebraic extension of a pure transcendental extension $L' = k(T_\mu)_{\mu \in M}$; we know that we can extend the valuation of B corresponding to B to a valuation of $K' = K(T_\mu)_{\mu \in M}$ in such a way that L' is the residue field of this valuation ([**Jaf60**, p. 98]); replacing B by the completion of the ring of this extended valuation, we see that that we can restrict to the case where B is complete and L is an algebraic closure of k. If $\overline{K}$ is an algebraic closure of K, we can then extend the valuation that defines B to $\overline{K}$, and the corresponding residue field is an algebraic closure of k, as we can see by lifting to $\overline{K}$ the coefficients of a unitary polynomial of $k[T]$. We are thus finally led to the case where $L = k$, and it then suffices to take C to be the completion of B in order to satisfy our claim.

(7.1.3). Let K be a field, and A a subring of K; the integral closure A' of A in K is the intersection of the valuation rings that contain A and have K as their field of fractions ([**Sam53b**, p. 51, th. 2]). Proposition (7.1.2) can then be expressed geometrically in an equivalent form:

Proposition (7.1.4). — *Let Y be a prescheme, $p : X \to Y$ a morphism, x a point of X, $y = p(x)$, and $y' \neq y$ a specialisation* (**0**, 2.1.2) *of y. Then there exists a local scheme Y' which is the spectrum of some valuation ring, and a separated morphism $f : Y' \to Y$ such that, denoting the unique closed point of Y' by a and the generic point of Y' by b, we have $f(a) = y'$ and $f(b) = y$. We can furthermore suppose that one of the two additional following properties are satisfied:*

(i) *Y' is the spectrum of a complete valuation ring whose residue field is algebraically closed, and there exists a $k(y)$-homomorphism $k(x) \to k(b)$.*
(ii) *There exists a $k(y)$-isomorphism $k(x) \xrightarrow{\sim} k(b)$.*

Proof. Let Y_1 be the closed reduced subprescheme of Y that has $\overline{\{y\}}$ as its underlying space (**I**, 5.2.1), and let X_1 be the closed subprescheme given by the inverse image $p^{-1}(Y_1)$; since $y' \in \overline{\{y\}}$ by hypothesis, and since $k(x)$ is the same in X and in X_1, we can assume that Y is *integral*, with generic point y; $\mathcal{O}_{y'}$ is then an integral local ring that is not a field, and whose field of fractions is $\mathcal{O}_y = k(y)$, and $k(x)$ is then an extension of $k(y)$. To satisfy the conditions $f(a) = y'$ and $f(b) = y$ as well as the additional condition (i) (resp. (ii)), we take $Y' = \operatorname{Spec}(A')$, where A' is a valuation ring that dominates $\mathcal{O}_{y'}$ (resp. a valuation ring that dominates $\mathcal{O}_{y'}$ and whose field of fractions is $k(x)$); the existence such an of A' is guaranteed by (7.1.2). □

(7.1.5). Recall that a local ring A is said to be *of dimension 1* if there exists a prime ideal distinct from the maximal ideal $\mathfrak{m}$, and if every prime ideal of A distinct from $\mathfrak{m}$ is a *minimal* prime ideal; when A is *integral*, it is equivalent to ask that $\mathfrak{m}$ and (0) be the only prime ideals, with $\mathfrak{m} \neq (0)$; in other words, $Y = \operatorname{Spec}(A)$ consists of two points a and b: a is the unique *closed* point, we have $\mathfrak{j}_a = \mathfrak{m}$, and **II** | 140
$k(a) = k$ is the *residue field* $k = A/\mathfrak{m}$; b is the *generic point* of Y, $\mathfrak{j}_b = (0)$, with the set $\{b\}$ being the unique open subset of Y distinct from both $\varnothing$ and Y (an open subset which is thus *everywhere dense*), and $k(b) = K$ is the *field of fractions* of A.

(7.1.6). For a local ring A, Noetherian and of dimension 1, we know ([**CC**, pp. 2-08 and 17-01]) that the following conditions are equivalent:

(a) A is normal;
(b) A is regular;

(c) A is a valuation ring;

furthermore, A is then a *discrete valuation ring*. Propositions (7.1.2) and (7.1.3) then have the following analogues for discrete valuation rings:

Proposition (7.1.7). — *Let A be an integral local Noetherian ring that is not a field, K its field of fractions, and L an extension of finite type of K; then there exists a discrete valuation ring that dominates A and has L as its field of fractions.*

PROOF. Suppose first of all that $L = K$. Let $\mathfrak{m}$ be the maximal ideal of A, $(x_1, \ldots, x_n)$ a system of non-null generators of $\mathfrak{m}$, and B the subring $A[x_2/x_1, \ldots, x_n/x_1]$ of K, which is Noetherian. It is immediate that the ideal $\mathfrak{m}B$ of B is identical to the principal ideal $x_1 B$; if $\mathfrak{p}$ is a minimal prime ideal of $x_1 B$, then $\mathfrak{p}$ is of rank 1 ([SZ60, t. I, p. 277]); in other words, $B_\mathfrak{p}$ is a local Noetherian ring *of dimension 1*; it is clear that $\mathfrak{p}B_\mathfrak{p} \cap A$ is an ideal of A that contains $\mathfrak{m}$ and that does not contain 1, and is thus equal to $\mathfrak{m}$, and so $B_\mathfrak{p}$ *dominates* A (I, 8.1.1). It follows from the Krull-Akizuki Theorem ([Nag55, p. 293]) that the integral closure C of $B_\mathfrak{p}$ is a Noetherian ring (even though C is not necessarily a $B_\mathfrak{p}$-module of finite type); if $\mathfrak{n}$ is a maximal ideal of C, then $C_\mathfrak{n}$ is a normal local Noetherian ring of dimension 1 ([Nag55, p. 295]), and thus a discrete valuation ring that dominates $B_\mathfrak{p}$ and *a fortiori* A.

Now, if L is an extension of finite type of K, we can, by the above, restrict to the case where A is already a discrete valuation ring. Let w be a valuation of K associated to A; there exists a discrete valuation w' of L that *extends* w: we can restrict, by induction on the number of generators of L, to the case where $L = K(\alpha)$, and then the proposition is classical ([Jaf60, p. 106]). □

Corollary (7.1.8). — *Let A be a Noetherian integral ring, K its field of fractions, and L an extension of finite type of K. Then the integral closure of A in L is the intersection of the discrete valuation rings that have L as their field of fractions and that contain A.*

PROOF. Indeed, such a discrete valuation ring, being normal, contains *a fortiori* every element of L that is integral over A. It thus suffices to prove that, if $x \in L$ is not integral over A, then there exists a discrete valuation ring C that has L as its field of fractions, contains A, and does not contain x. The hypothesis on x implies that $x \notin B = A[1/x]$, or, in other words, that $1/x$ is not invertible in the Noetherian ring B. There is thus a prime ideal $\mathfrak{p}$ of B that contains $1/x$. The integral local ring $B_\mathfrak{p}$ is Noetherian and contained in L, which is an extension of finite type of the field of fractions of $B_\mathfrak{p}$ (with the latter containing K). By (7.1.7), there thus exists a discrete valuation ring C that dominates $B_\mathfrak{p}$ and has L as its field of fractions; since $1/x \in \mathfrak{p}B_\mathfrak{p}$ belongs to the maximal ideal of C, we have that $x \notin C$, which concludes the proof. □

The geometric form of (7.1.7) is the following:

II | 141

Proposition (7.1.9). — *Let Y be a locally Noetherian prescheme, $p : X \to Y$ a morphism locally of finite type, x a point of X, $y = p(x)$, and $y' \neq y$ a specialisation of y. Then there exists a local scheme Y', spectrum of a discrete valuation ring, a separated morphism $f : Y' \to Y$, and a rational Y-map g from Y' to X, such that, denoting the closed point of Y' by a, and the generic point of Y' by b, we have $f(a) = y'$, $f(b) = y$, $g(b) = x$, and such that, in the commutative diagram*

(7.1.9.1)
$$\begin{tikzcd} & k(x) \arrow[dl, "\gamma"'] \\ k(b) & k(y) \arrow[l, "\phi"] \arrow[u, "\pi"'] \end{tikzcd}$$

(where π, ϕ, and γ are the homomorphisms corresponding to p, f, and g, respectively) the morphism γ is a bijection.

PROOF. As in (7.1.4), we can restrict to the case where Y is integral with generic point y (taking (I, 6.4.3, iv) into account), and, since the question is local on X and Y, we can assume that p is of finite type; we are then in the situation of (7.1.4), with the additional property that $k(x)$ is an extension *of finite type* of $k(y)$ (I, 6.4.11) and that $\mathscr{O}_{y'}$ is Noetherian; this lets us apply (7.1.7) and take $Y' = \operatorname{Spec}(A')$, where A' is a discrete valuation ring that dominates $\mathscr{O}_{y'}$ and whose field of fractions is $k(x)$. We have thus defined a commutative diagram (7.1.9.1) where γ is a bijection, with π and ϕ corresponding to the morphisms p and f. Furthermore, since X and Y are locally Noetherian (**I**, 6.6.2) and since Y' is integral, there exists exactly one rational Y-map g from Y' to X to which corresponds the isomorphism γ (**I**, 7.1.15), which finishes the proof. □

7.2. Valuative criterion for separatedness

Proposition (7.2.1). — *Let X and Y be preschemes, and $f : X \to Y$ a quasi-compact morphism. The following two conditions are equivalent:*

(a) *The morphism f is closed.*
(b) *For all $x \in X$, and every specialisation $y' \neq y$ of $y = f(x)$, there exists a specialisation x' of x such that $f(x') = x$.*

PROOF. Condition (b) implies that $f(\overline{\{x\}}) = \overline{\{y\}}$, and is thus a consequence of (a). To show that (b) implies (a), consider a closed subset X' of the underlying space X; let $Y' = \overline{f(X')}$, and show that $Y' = f(X')$ as follows. Consider the closed reduced subpreschemes of X and Y whose underlying spaces are X' and Y' (respectively) (**I**, 5.2.1); there then exists a morphism $f' : X' \to Y'$ such that the diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes (**I**, 5.2.2), and, since f is quasi-compact, so too is f'. We are thus led to proving that, if f is a quasi-compact and *dominant* morphism, then condition (b) implies that $f(X) = Y$. But let y' be a point of Y, and let y be the generic point of an irreducible component of Y that contains y'; by (b), it suffices to show that $f^{-1}(y)$ is not empty. But we know that this property is a consequence of the fact that f is quasi-compact and dominant (**I**, 6.6.5). □ **II** | 142

Corollary (7.2.2). — *Let $f : X \to Y$ be a quasi-compact immersion. For the underlying space X to be closed in Y, it is necessary and sufficient for it to contain every specialisation (in Y) of all of its points.*

Proposition (7.2.3). — *Let Y be a prescheme (resp. a locally Noetherian prescheme), and $f : X \to Y$ a morphism (resp. a morphism locally of finite type). The following conditions are equivalent:*

(a) *f is separated.*
(b) *The diagonal morphism $X \to X \times_Y X$ is quasi-compact, and, for every Y-prescheme of the form $Y' = \operatorname{Spec}(A)$, with A some valuation ring (resp. some discrete valuation ring), any two Y-morphisms from Y' to X that agree on the generic point of Y' are equal.*
(c) *The diagonal morphism $X \to X \times_Y X$ is quasi-compact, and, for every Y-prescheme of the form $Y' = \operatorname{Spec}(A)$, with A some valuation ring (resp. some discrete valuation ring), any two Y'-sections of $X' = X_{(Y')}$ that agree on the generic point of Y' are equal.*

PROOF. The equivalence of (b) and (c) follows from the bijective correspondence between Y-morphisms from Y' to X and Y'-sections of X' (**I**, 3.3.14). If X is separated over Y, condition (b) is satisfied, by (**I**, 7.2.2.1), since Y' is integral. It remains to show that (b) implies that the diagonal morphism $\Delta : X \to X \times_Y X$ is closed, and it is equivalent to show that it satisfies the criteria of (7.2.2). But let z be a point of the diagonal $\Delta(X)$, and $z' \neq z$ a specialisation of z in $X \times_Y X$. There then exists (7.1.4) a valuation ring A and a morphism f from $Y' = \operatorname{Spec}(A)$ to $X \times_Y X$ such that f sends the closed point a of Y' to z', and the generic point b of Y' to z; this morphism makes Y' an $(X \times_Y X)$-prescheme, and *a fortiori* a Y-prescheme. If we compose the two projections of $X \times_Y X$ with f, then we obtain two Y-morphisms, g_1 and g_2, from Y' to X, which, by hypothesis, agree on the point b; they are thus equal to one single morphism g, which implies (**I**, 5.3.1) that f factors as $f = \Delta \circ g$, and thus $z' \in \Delta(X)$. If we suppose that Y is locally Noetherian and f is locally of finite type, then $X \times_Y X$ is locally Noetherian (**I**, 6.6.7); we can thus follow the same argument as before by supposing that A is a discrete valuation ring, by (7.1.9). □

Remark (7.2.4). — (i) The hypothesis that the morphism Δ is quasi-compact is always satisfied whenever Y is locally Noetherian and f is locally of finite type, because $X \times_Y X$ is then locally Noetherian (**I**, 6.6.4, i). In the general case, this also implies that, for every cover (U_α) of X by affine opens, the sets $U_\alpha \cap U_\beta$ are *quasi-compact*.

(ii) For f to be separated, it is *sufficient* for condition (b) or (c) to be satisfied for some valuation ring A that is *complete* and whose residue field is *algebraically closed*; this follows from the proofs of (7.2.3) and (7.2.4).

II | 143

7.3. Valuative criterion for properness

Proposition (7.3.1). — *Let A be a valuation ring, $Y = \operatorname{Spec}(A)$, b the generic point of Y, X an integral* scheme, *and $f : X \to Y$ a* closed *morphism such that $f^{-1}(b)$ consists of a single point x and such that the corresponding homomorphism $k(b) \to k(x)$ is bijective. Then f is an isomorphism.*

Proof. Since f if closed and dominant, we have that $f(X) = Y$; it suffices (**I**, 4.2.2) to prove that, for all $y' \neq b$ in Y, there exists *exactly one* point x' such that $f(x') = y'$, and that the corresponding homomorphism $\mathscr{O}_{y'} \to \mathscr{O}_{x'}$ is bijective, since then f will be a homeomorphism. But if $f(x') = y'$ then $\mathscr{O}_{x'}$ is a local ring contained in $K = k(x) = k(b)$ and dominating $\mathscr{O}_{y'}$; the latter is the local ring $A_{y'}$, and is thus a valuation ring (7.1.1) that has K as its field of fractions. Also, $\mathscr{O}_{x'} \neq K$, since x' is not the generic point of X (**0**, 2.1.3); we thus conclude that $\mathscr{O}_{x'} = \mathscr{O}_{y'}$. Since X is an integral scheme, the fact that $\mathscr{O}_{x'} = \mathscr{O}_{x''}$ implies that $x' = x''$ (**I**, 8.2.2), which finishes the proof. □

(7.3.2). Let A be a valuation ring, K its field of fractions, $Y = \operatorname{Spec}(A)$, and b the generic point of Y, such that $\mathscr{O}_b = k(b)$ is equal to K; let $f : X \to Y$ be a morphism. We know (**I**, 7.1.4) that the *rational Y-sections* of X are in bijective correspondence with the *germs* of Y-sections (defined in a neighbourhood of b) at the point b, whence we have a canonical map

$$\Gamma_{\mathrm{rat}}(X/Y) \longrightarrow \Gamma(f^{-1}(b)/\operatorname{Spec}(K)) \tag{7.3.2.1}$$

with the elements of $\Gamma(f^{-1}(b)/\operatorname{Spec}(K))$ being identified, by definition (**I**, 3.4.5), with the *points of $f^{-1}(b) = X \otimes_A K$ that are rational over K*. When f is *separated*, it follows from (**I**, 5.4.7) that the map (7.3.2.1) is *injective*, since Y is an integral scheme.

Composing (7.3.2.1) with the canonical map $\Gamma(X/Y) \to \Gamma_{\mathrm{rat}}(X/Y)$ (**I**, 7.1.2), we obtain a canonical map

$$\Gamma(X/Y) \longrightarrow \Gamma(f^{-1}(b)/\operatorname{Spec}(K)). \tag{7.3.2.2}$$

When f is *separated*, this map is again *injective* (**I**, 5.4.7).

Proposition (7.3.3). — *Let A be a valuation ring with field of fractions K, $Y = \operatorname{Spec}(A)$, b the generic point of Y, and $f : X \to Y$ a* separated *and* closed *morphism. Then the canonical map (7.3.2.2) is bijective (which is equivalent to saying that it is* surjective, *and implies that the rational Y-sections of X are* everywhere defined*).*

Proof. So let x be a point of $f^{-1}(b)$ that is *rational* over K. Since f is separated, so too is the morphism $f^{-1}(b) \to \operatorname{Spec}(K)$ corresponding to f (**I**, 5.5.1, iv), and, since every section of $f^{-1}(b)$ is a closed immersion (**I**, 5.4.6), $\{x\}$ is closed *in* $f^{-1}(b)$. Consider the closed reduced subprescheme X' of X that has the closure $\overline{\{x\}}$ of $\{x\}$ *in* X as its underlying space. It is clear that the restriction of f to X' satisfies the hypotheses of (7.3.1), and is thus an *isomorphism* from X' to Y, whose inverse isomorphism is the desired Y-section of X. □

(7.3.4). To state the two following results, we use a terminology that will be justified and discussed in chapter IV: if F is a subset of a prescheme Y, we define the *codimension* of F in Y, denoted $\operatorname{codim}_Y F$, **II** | 144
to be the lower bound of the integers $\dim(\mathscr{O}_z)$ over all z in F.

Corollary (7.3.5). — *Let Y be a locally Noetherian reduced prescheme, and N the set of points $y \in Y$ where Y is not regular (**0**, 4.1.4); suppose that* $\operatorname{codim}_Y N \geqslant 2$. *Let $f : X \to Y$ be a morphism of finite type, both* separated *and* closed, *and let g be a rational Y-section of X; if Y' is the set of points of Y where g is not defined (a set which is* closed *(**I**, 7.2.1)), then* $\operatorname{codim}_Y Y' \geqslant 2$.

PROOF. It suffices to prove that g is defined at every point $z \in Y$ such that $\dim \mathscr{O}_z \leqslant 1$. If $\dim \mathscr{O}_z = 0$, then z is the generic point of an irreducible component of Y (**I**, 1.1.14), and so belongs to every everywhere-dense open subset of Y, and, in particular, to the domain of definition of g. So suppose that $\dim \mathscr{O}_z = 1$; by hypothesis, $\mathscr{O}_z$ is then a regular Noetherian local ring, and thus (7.1.6) a *discrete valuation ring*. Let $Z = \operatorname{Spec}(\mathscr{O}_z)$; since $U = Y - Y'$ is everywhere dense, $U \cap Z$ is nonempty (**I**, 2.4.2); let g' be the rational map from Z to X induced by g (**I**, 7.2.8); it suffices to show that g' is a *morphism* (**I**, 7.2.9). But g' can be considered as a rational Z-section of the Z-prescheme $f^{-1}(Z) = X \times_Y Z$; it is clear that the morphism $f^{-1}(Z) \to Z$ corresponding to f is closed, and it follows from (**I**, 5.5.1, i) that it is separated; we thus conclude from (7.3.3) that g' is everywhere defined; since Z is reduced, and X is separated over Y, g' is a morphism (**I**, 7.2.2). □

Corollary (7.3.6). — *Let S be a locally Noetherian prescheme, and X and Y both S-preschemes; suppose that Y is reduced, and further that the set N of points $y \in Y$ where Y is not regular is such that* $\operatorname{codim}_Y N \geqslant 2$; *suppose finally that the structure morphism $X \to S$ is proper. Let f be a rational S-map from Y to X, and let Y' be the points of Y where f is not defined; then* $\operatorname{codim}_Y Y' \geqslant 2$.

PROOF. We know (**I**, 7.1.2) that we can identify the rational S-maps from Y to X with the rational Y-sections of $X \times_S Y$; since the structure morphism $X \times_S Y \to Y$ is closed (5.4.1), we can apply (7.3.5), whence the corollary. □

Remark (7.3.7). — The hypotheses on Y in (7.3.5) and (7.3.6) will be satisfied in particular when Y is *normal* (**0**, 4.1.4), by (7.1.6).

We can characterise the universally closed morphisms (resp. proper morphisms) by a converse of (7.3.3):

Theorem (7.3.8). — *Let Y be a prescheme (resp. a locally Noetherian prescheme), and $f : X \to Y$ a quasi-compact separated morphism (resp. a morphism of finite type). The following conditions are equivalent:*

(a) *f is universally closed (resp. proper).*
(b) *For every Y-scheme of the form $Y' = \operatorname{Spec}(A)$, where A is a valuation ring (resp. a discrete valuation ring) with field of fractions K, the canonical map*

$$\operatorname{Hom}_Y(Y', X) \longrightarrow \operatorname{Hom}_Y(\operatorname{Spec}(K), X)$$

corresponding to the canonical injection $A \to K$ is surjective (resp. bijective). **II** | 145

(c) *For every Y-scheme of the form $Y' = \operatorname{Spec}(A)$, where A is a valuation ring (resp. a discrete valuation ring), the canonical map (7.3.2.2) with respect to the Y'-prescheme $X_{(Y')}$ is surjective (resp. bijective).*

PROOF. The equivalence of (b) and (c) follows immediately from (**I**, 3.3.14); (a) implies (b), since (a) implies, in either case, that $f_(Y')$ is separated (**I**, 5.5.1, iv) and closed, and it suffices to apply (7.3.3). It remains to prove that (b) implies (a). We first consider the case where Y is arbitrary, and f is separated and quasi-compact. If condition (b) is satisfied by f, then it is also satisfied by $f_{(Y'')} : X_{(Y'')} \to Y''$, where Y'' is an arbitrary Y-prescheme, thanks to the equivalence between (b) and (c), and the fact that $X_{(Y'')} \times_{Y''} Y' = X \times_Y Y'$ for every morphism $Y' \to Y''$ (**I**, 3.3.9.1); since, further, $f_{(Y'')}$ is separated and quasi-compact whenever f is ((**I**, 5.5.1, iv) and (**I**, 6.6.4, iii)), we are led to proving that (b) implies that f is *closed*. For this, it suffices to verify condition (b) of (7.2.1). So let $x \in X$, and y' be a specialisation of $y = f(x)$, distinct from y; by (7.1.4), there exists a scheme Y', the spectrum of some valuation ring, and a separated morphism $g : Y' \to Y$ such that, letting a denote the closed point and b the generic point of Y', we have that $g(a) = y'$, $g(b) = y$, and that there exists a $k(y)$-homomorphism $k(x) \to k(b)$. The latter corresponds canonically to a Y-morphism $\operatorname{Spec}(k(b)) \to X$ (**I**, 2.4.6), and it thus follows from (b) that there exists a Y-morphism $h : Y' \to X$ to which the previous morphism corresponds. We then have that $h(b) = x$; if we set $h(a) = x'$, then x' is a specialisation of x, and we have that $f(x') = f(h(a)) = g(a) = y'$.

If now Y is locally Noetherian and f of finite type, then hypothesis (b) implies, first of all, that f is *separated*, by (7.2.3), with the diagonal morphism $X \to X \times_Y X$ being quasi-compact (7.2.4). Further, to show that f is proper, it suffices to show that $f_{(Y'')} : X_{(Y'')} \to Y''$ is *closed* for every Y-prescheme Y'' *of finite type*, taking (5.6.3) into account. Since Y'' is then locally Noetherian, we can follow the same reasoning as in the first case by taking Y' to be the spectrum of a discrete valuation ring, and applying (7.1.9) instead of (7.1.4). □

Remarks (7.3.9). — (i) Whenever Y is an arbitrary prescheme and f a separated morphism, for f to be universally closed, it *suffices* that condition (b) or (c) be satisfied for the *complete* valuation rings A whose residue field is *algebraically closed*; this follows from the above proof and from (7.1.4).

(ii) From criterion (c) of (7.3.8) we obtain a new proof of the fact that a projective morphism $X \to Y$ is closed (5.5.3), and it is closer to the classical approach. We can indeed assume that Y is affine, and thus that X can be identified with a closed subprescheme of a projective bundle $\mathbf{P}^n_Y$ (5.3.3); to prove that $X \to Y$ is closed, it suffices to verify that the structure morphism $\mathbf{P}^n_Y \to Y$ is closed, and criteria (c) of (7.3.8), combined with (4.1.3.1), tells us that we can reduce to proving the following fact: *if Y is the spectrum of a valuation ring A, with field of fractions K, then every point of $\mathbf{P}^n_Y$ with values in K comes from (by restriction to the generic point of Y) a point of $\mathbf{P}^n_Y$ with values in A.* But every invertible $\mathscr{O}_Y$-module is trivial (**I**, 2.4.8); so it follows from (4.2.6) that a point of $\mathbf{P}^n_Y$ with values in K can be identified with a class of elements $(\zeta c_0, \zeta c_1, \ldots, \zeta c_n)$ of K, where $\zeta \neq 0$ and the c are elements of K that are not all zero. However, by multiplying the c_i by an element of A of suitable valuation, we can **II** | 146
suppose that the c_i all belong to A, and that at least one of them is invertible. But then (4.2.6) the system $(c_0, \ldots, c_n)$ also defines a point of $\mathbf{P}^n_Y$ with values in A, which proves our claim.

(iii) Criteria (7.2.3) and (7.3.8) are particularly simple when we consider the data of a Y-prescheme X as being equivalent to the data of the functor

$$X(Y') = \operatorname{Hom}_Y(Y', X)$$

for Y-preschemes Y'; these criteria allow us, for example, to prove that, under certain conditions, the "Picard schemes" are proper.

Corollary (7.3.10). — *Let Y be an integral scheme (resp. a locally Noetherian integral scheme), X an integral scheme, and $f : X \to Y$ a dominant morphism.*

(i) *If f is quasi-compact and universally closed, then every valuation ring whose field of fractions is the field $R(X)$ of rational functions on X, and which is dominated by a local ring Y, also dominates by a local ring of X.*

(ii) *Conversely, suppose that f is of finite type, and that the property described in (i) is verified by every valuation ring (resp. every discrete valuation ring) that has $R(X)$ as its field of fractions. Then f is proper.*

Proof. Note first of all that the hypotheses imply, in any case, that f is separated (**I**, 5.5.9).

(i) Let $K = R(Y)$, $L = R(X)$, y a point of Y, and A a valuation ring that dominates $\mathscr{O}_y$ and has L as its field of fractions; the injection $\mathscr{O}_y \to A$ then defines a morphism h from $Y' = \operatorname{Spec}(A)$ to Y (**I**, 2.4.4) such that $h(a) = y$, where we write a to denote the closed point of Y'; furthermore, if η is the generic point of Y, which is also the generic point of $\operatorname{Spec}(\mathscr{O}_y)$, then we have $h(b) = \eta$, writing b to denote the generic point of Y' (since $K \subset L$ by hypothesis). If ξ is the generic point of X, then $k(\xi) = k(b) = L$ by hypothesis, whence we have a Y-morphism $g : \operatorname{Spec}(L) \to X$ such that $g(b) = \xi$; by (7.3.8), g comes from a Y-morphism $g' : Y' \to X$. If $x = g'(a)$, it is clear that A dominates $\mathscr{O}_x$.

(ii) Since the questions is local on Y, we can always suppose that Y is affine (resp. affine and Noetherian). Since f is of finite type, we can apply, in either case, Chow's lemma (5.6.1). There is thus a projective morphism $p : P \to Y$, an immersion morphism $j : X' \to P$, and a projective morphism $g : X' \to X$ that is both surjective and birational, with X integral, such that the diagram

$$\begin{tikzcd} P \arrow[d, "p"'] & X' \arrow[l, "j"'] \arrow[d, "g"] \\ Y & X \arrow[l, "f"] \end{tikzcd}$$

commutes. It suffices to prove that j is a *closed* immersion, since then $f \circ g = p \circ j$ will be a projective morphism, and thus closed, and, since g is surjective, f will also be proper (5.4.3). Let Z be the closed reduced subprescheme of P that has $\overline{j(X')}$ as its underlying space (**I**, 5.2.1); since X' is integral, j factors as $i \circ h$, where $i : Z \to P$ is the canonical injection, $h : X' \to Z$ a dominant open immersion (**I**, 5.2.3), and Z is integral; furthermore, **II** | 147
Z is projective over Y, and we see that we can restrict to the case where P is *integral* and j is *dominant* and *birational*, and everything then reduces to showing that j is *surjective*. But let $z \in P$; then $\mathscr{O}_z$ is an integral (resp. integral and Noetherian) local ring whose field of fractions is

$$L = R(P) = R(X') = R(X).$$

We can restrict to the case where z is not the generic point of P. There then exists ((7.1.2) and (7.1.7)) a valuation ring (resp. a discrete valuation ring) A which dominates $\mathscr{O}_z$ and has L as its field of fractions. *A fortiori*, A dominates $\mathscr{O}_y$, where $y = p(z)$, and, by hypothesis, there thus exists some $x \in X$ such that A dominates $\mathscr{O}_x$. Since g is proper, the first part of the proof shows that A also dominates $\mathscr{O}_{x'}$ for some $x' \in X'$; it then follows that $\mathscr{O}_z$ and $\mathscr{O}_{j(x')} = \mathscr{O}_{x'}$ are allied (**I**, 8.1.4), and, since P is a scheme, this implies that $z = j(x')$ (**I**, 8.2.2) and finishes the proof.

□

Corollary (7.3.11). — *Let X and Y be integral schemes, and $f : X \to Y$ a dominant, quasi-compact, and universally closed morphism. Suppose further that Y is affine of (integral) ring B. Then $\Gamma(X, \mathscr{O}_X)$ is canonically isomorphic to a subring of the integral closure of B in $R(X)$.*

PROOF. Indeed (**I**, 8.2.1.1), $\Gamma(X, \mathscr{O}_X)$ can be identified with the intersection of the $\mathscr{O}_x$ over $x \in X$; by (7.3.10), (7.1.2), and (7.1.3), $\Gamma(X, \mathscr{O}_X)$ is then contained in the intersection of the valuation rings that contain B and that have $R(X)$ as their field of fractions; the conclusion then follows from (7.1.3). □

Remarks (7.3.12). — Under the hypotheses of (7.3.11), and when we suppose that $R(X)$ is an extension of finite type of $R(Y)$, we can, in many cases, conclude that $\Gamma(X, \mathscr{O}_X)$ is a *module of finite type* over the ring $B = \Gamma(Y, \mathscr{O}_X)$. For example, this will be the case whenever B is an *algebra of finite type over a field*, since we then know that the integral closure of B in an extension of finite type of its field of fractions is a B-module of finite type ([**SZ60**, t. I, p. 267, th. 9]); the conclusion then follows from (7.3.11) and the fact that B is Noetherian.

In particular, *a proper affine scheme X over a field K is finite*. Indeed, by (1.6.4), (5.4.6), and (**I**, 6.4.4, (c)), we can restrict to the case where X is *reduced*. Furthermore, it suffices to prove that each of the closed subpreschemes of X that have an irreducible component of X as their underlying space (of which there are finitely many) is finite over K, which means (taking (5.4.5) into account) that we are finally reduced to the case where X is *integral*. But then the result follows from the above remarks.

In chapter III, we will again prove this above proposition by other methods, and as a consequence of more general results, by showing that, if $f : X \to Y$ is proper and Y is locally Noetherian, $f_*(\mathscr{F})$ is coherent for any coherent $\mathscr{O}_X$-module $\mathscr{F}$ (**III**, 4.4.2).

Finally, note that criterion (7.3.10) is taken as the *definition* of proper morphisms in classical algebraic geometry. We only mention this here as a remark, since criterion (7.3.8) seems more manageable in all the applications with which we are familiar.

II | 148

7.4. Algebraic curves and function fields of dimension 1 The aim of this section is to show how to formulate the classical notion of algebraic curves (as introduced, by example, in the book of C. Chevalley [**Che51**]) in the language of schemes. All throughout this section, *we write k to mean a field, all the schemes in question are k-schemes of finite type, and all the morphisms are k-morphisms*.

Proposition (7.4.1). — *Let X be a prescheme of finite type over k (and thus Noetherian); let x_i ($1 \leqslant i \leqslant n$) be the generic points of the irreducible components X_i of X, and let $K_i = k(x_i)$ ($1 \leqslant i \leqslant n$). Then the following conditions are equivalent:*

(a) *Each of the K_i is an extension of k with transcendence degree equal to* 1.
(b) *For every closed point x of X, the local ring $\mathscr{O}_x$ is of dimension* 1 (7.1.5).
(c) *The closed irreducible subsets of X that are distinct from the X_i are exactly the closed points of X.*

PROOF. Since X is quasi-compact, every closed irreducible subset F of X contains a closed point (**0**, 2.1.3). By (**I**, 2.4.2), there is a bijective correspondence between the prime ideals of $\mathscr{O}_x$ and the closed irreducible subsets of X that contain x (**I**, 1.1.14); the equivalence between (b) and (c) follows immediately from this. Now, if $\mathfrak{p}_\alpha$ ($1 \leqslant \alpha \leqslant r$) are the minimal prime ideals of the local Noetherian ring $\mathscr{O}_x$, then the local rings $\mathscr{O}_x/\mathfrak{p}_\alpha$ are integral, and have the K_i such that $x \in X_i$ as their fields of fractions. Furthermore, we know ([**CC**, p. 4-06, th. 2]) that the dimension of a local integral k-algebra of finite type is equal to the transcendence degree over k of its field of fractions. Finally, the dimension of $\mathscr{O}_x$ is bounded above by the dimensions of the $\mathscr{O}_x/\mathfrak{p}_\alpha$; but condition (a) implies that these dimensions are equal to 1, and so (a) implies (b); conversely, if $\mathscr{O}_x$ is of dimension 1, then none of the $\mathfrak{p}_\alpha$ can be equal to the maximal ideal of $\mathscr{O}_x$, otherwise $\mathscr{O}_x$ would be of dimension 0; thus each of the $\mathscr{O}_x/\mathfrak{p}_\alpha$ are of dimension 1, which shows that (b) implies (a). □

We note that, under the conditions of (7.4.1), the set X is either *empty* or *infinite*, as an immediate result of (**I**, 6.4.4).

Definition (7.4.2). — We define an *algebraic curve over k* to be a non-empty algebraic *scheme* over k that satisfies the conditions of (7.4.1).

In the language of dimensions, which will be introduced in Chapter IV, this can be expressed by saying that an algebraic curve over k is a non-empty algebraic k-scheme *whose irreducible components are all of dimension* 1.

We note that, if X is an algebraic curve over k, then the closed reduced subpreschemes X_i ($1 \leqslant i \leqslant n$) of X that have the irreducible components of X as their underlying space are also algebraic curves over k.

Corollary (7.4.3). — *Let X be an irreducible algebraic curve. The only non-closed point of X is its generic point. The closed subsets of X that are distinct from X are the finite sets of closed points; these are also the only subsets of X that are not everywhere dense.*

PROOF. If a point $x \in X$ is not closed, then its closure in X is an irreducible closed subset of X, and thus necessarily the whole of X, by (7.4.1), and thus x is the generic point of X. A closed subset F of X that is distinct from X cannot contain the generic point of X, and so all its points are closed **II** | 149
(in X, and *a fortiori* in F); by considering the closed reduced subpreschemes of X that have F as their underlying space (**I**, 5.2.1), it thus follows from (**I**, 6.2.2) that F is finite and discrete. The closure in X of any infinite subset of X is thus necessarily equal to X itself. □

If X is an arbitrary algebraic curve, by applying (7.4.3) to the irreducible components of X, we see that the only non-closed points of X are the generic points of these components.

Corollary (7.4.4). — *Let X and Y be irreducible algebraic curves over k, and $f : X \to Y$ a k-morphism. For f to be dominant, it is necessary and sufficient for $f^{-1}(y)$ to be finite for all $y \in Y$.*

PROOF. Indeed, if f is not dominant, then $f(X)$ is necessarily a *finite* subset of Y, by (7.4.3), and so it is not possible for $f^{-1}(y)$ to be finite for every point of Y, since otherwise X would be finite, which is a contradiction (7.4.1). Conversely, if f is dominant, then for any $y \in Y$ distinct from the generic point η of Y, we have that $f^{-1}(y)$ is closed in X, since $\{y\}$ is closed in Y (7.4.3); also, by hypothesis, $f^{-1}(y)$ does not contain the generic point ξ of X, and is thus finite, by (7.4.3). Finally, to see that, when f is dominant, $f^{-1}(\eta)$ is finite, we note that the fibre $f^{-1}(\eta)$ is an irreducible scheme of finite type over $k(\eta)$, and with generic point ξ ((**I**, 6.3.9) and (**I**, 6.4.11)). Since $k(\xi)$ and $k(\eta)$ are extensions of finite type of k, both of transcendence degree 1, we have that $k(\xi)$ is necessarily an extension of finite degree of $k(\eta)$, and so ξ is closed in $f^{-1}(\eta)$ (**I**, 6.4.2), and $f^{-1}(\eta)$ thus consists of a single point ξ. □

We will see, in Chapter III, that, if $f : X \to Y$ is a *proper* morphism of Noetherian preschemes such that $f^{-1}(y)$ is finite for all $y \in Y$, then f is necessarily *finite*; it will thus follow from (7.4.4) that a proper dominant morphism from an irreducible algebraic curve to an algebraic curve is *finite*.

Corollary (7.4.5). — *Let X be an algebraic curve over k. For X to be regular, it is necessary and sufficient for X to be normal, or for the local rings of its closed points to be discrete valuation rings.*

Proof. This follows immediately from (7.4.1, (b)) and (7.1.6). □

Corollary (7.4.6). — *Let X be a reduced algebraic curve, and $\mathscr{A}$ a reduced coherent $\mathscr{R}(X)$-algebra; then the integral closure X' of X with respect to $\mathscr{A}$ (6.3.4) is a normal algebraic curve, and the canonical morphism $X' \to X$ is finite.*

Proof. The fact that $X' \to X$ is finite follows from (6.3.10); X' is thus an algebraic k-scheme; furthermore, if x_i ($1 \leqslant i \leqslant n$) are the generic points of the irreducible components of X, and x'_j ($1 \leqslant j \leqslant m$) the generic points of the irreducible components of X', then each of the $k(x'_j)$ is a finite algebraic extension of one of the $k(x_i)$ (6.3.6), and thus of transcendence degree 1 over k. So X' is indeed an algebraic curve over k, and, furthermore, we know that X' is a finite sum of normal integral schemes ((6.3.6) and (6.3.7)). □

(7.4.7). We say that an algebraic curve X over k is *complete* if it is *proper* over k.

Corollary (7.4.8). — *For a reduced algebraic curve X over k to be complete, it is necessary and sufficient for its normalisation X' to be complete.*

II | 150

Proof. The canonical morphism $f : X' \to X$ is finite (7.4.6), and thus proper (6.1.11) and surjective (6.3.8); if $g : X \to \operatorname{Spec}(k)$ is the structure morphism, then g and $\circ f$ are both proper, by (5.4.2, (ii)) and (5.4.3, (ii)), since g is separated by hypothesis. □

Proposition (7.4.9). — *Let X be a* normal *algebraic curve over k, and Y a* proper *algebraic k-scheme over k. Then every rational k-map from X to Y is everywhere defined, or, in other words, is a morphism.*

Proof. It follows from (7.3.7) that, at the points $x \in X$ where such a map is not defined, the dimension of $\mathscr{O}_x$ must be $\geqslant 2$, and so the set of such points is empty; the final claim follows from (**I**, 7.2.3). □

Corollary (7.4.10). — *A normal algebraic curve over k is quasi-projective over k.*

Proof. Since X is a finite sum of normal integral algebraic curves (6.3.8), we can restrict to the case where X is integral (5.3.6). Since X is quasi-compact, it is covered by a finite number of affine open subsets U_i ($1 \leqslant i \leqslant n$), and, since each of these U_i is of finite type over k, for each i there exists some integer n_i along with a k-immersion $f_i : U_i \to \mathbf{P}_k^{n_i}$ ((5.3.3) and (5.3.4, (i))). Since U_i is dense in X, it follows from (7.4.9) that f_i can be extended to a k-morphism $g_i : X \to \mathbf{P}_k^{n_i}$, whence we obtain a k-morphism $g = (g_1, \ldots, g_n)_k$ from X to the product P of the $\mathbf{P}_k^{n_i}$ over k. Furthermore, for each i, since the restriction of g_i to U_i is an immersion, so too is the restriction of g to U_i (**I**, 5.3.14). Since the U_i cover X, and since g is separated (**I**, 5.5.1, (v)), g is an immersion from X into P (**I**, 8.2.8). Since the Segre morphism (4.3.3) gives an immersion of P into $\mathbf{P}_k^N$, this proves that X is quasi-projective. □

Corollary (7.4.11). — *Any normal algebraic curve X is isomorphic to the scheme induced by some complete normal algebraic curve $\widehat{X}$ on some everywhere dense open subset, and this $\widehat{X}$ is unique up to unique isomorphism.*

PROOF. If X_1 and X_2 are complete normal curves, then it follows from (7.4.9) that every isomorphism from any dense open U_1 in X_1 to any dense open U_2 in X_2 can be uniquely extended to an isomorphism from X_1 to X_2; whence the uniqueness claim. To prove the existence of $\widehat{X}$, it suffices to note that we can consider X as a subscheme of a projective bundle $\mathbf{P}_k^n$ (7.4.10). Let $\overline{X}$ be the closure of X in $\mathbf{P}_k^n$ (**I**, 9.5.11); since X is induced by $\overline{X}$ on a dense open subset of $\overline{X}$ (**I**, 9.5.10), the generic points x_i of the irreducible components of X are also the generic points of the irreducible components of $\overline{X}$, and the $k(x_i)$ are the same for both of these schemes, and so (7.4.1) $\overline{X}$ is an algebraic curve over k that is reduced (**I**, 9.5.9) and projective over k (5.5.1), whence complete (5.5.3). So we take for $\widehat{X}$ the *normalisation* of $\overline{X}$, which is again complete (7.4.8); furthermore, if $h : \widehat{X} \to \overline{X}$ is the canonical morphism, then the restriction of h to $h^{-1}(X)$ is an isomorphism to X, since X is normal (6.3.4), and since $h^{-1}(X)$ contains the generic points of the irreducible components of $\widehat{X}$ (6.3.8), it is dense in $\widehat{X}$, which finishes the proof. □

II | 151

Remark (7.4.12). — We will show, in Chapter V, that the conclusion of (7.4.10) still holds true without the assumption that the curve is normal (or even reduced); we will also show that, for an algebraic curve (reduced or not) to be affine, it is necessary and sufficient for its (reduced) irreducible components to *not* be complete.

Corollary (7.4.13). — *Let X be a* normal *irreducible curve over the field $K = R(X)$, and Y a* complete *integral curve over the field $L = R(Y)$. Then there is a canonical bijective correspondence between* dominant *k-morphisms $X \to Y$ and k-monomorphisms $L \to K$.*

PROOF. By (7.4.9), rational k-map from X to Y can be identified with k-morphisms $u : X \to Y$. Since the dominant morphisms $u : X \to Y$ are characterised by being those such that $u(x) = y$ (writing x and y to denote the generic points of X and Y, respectively), the corollary follows from these remarks and from (**I**, 7.1.13). □

(7.4.14). We can refine the result of (7.4.13) in the case where Y is the *projective line* $\mathbf{P}_k^1 = \mathrm{Proj}(k[T_0, T])$, where T_0 and T are indeterminates. Then Y is an integral scheme (2.4.4), and the scheme induced on the open subset $D_+(T_0)$ of Y is isomorphic to $\mathrm{Spec}(k[T])$ (2.3.6), and so the generic point of Y is the ideal (0) of $k[T]$, and the field of rational functions of Y is $k(T)$, which proves that Y is a complete algebraic curve over k. Furthermore, the only graded prime ideal of $S = k[T_0, T]$ that contains T_0 and is distinct from S_+ is the principal ideal (T_0), and so the complement of $D_+(T_0)$ in $Y = \mathbf{P}_k^1$ consists of *one closed point*, called the "point at infinity", which we denote by ∞ (for a general study of the links between vector bundles and projective bundles, see (8.4)). With these notations:

Corollary (7.4.15). — *Let X be a* normal *irreducible curve over the field $K = R(X)$. Then there exists a canonical bijective correspondence between the set K and the set of morphisms u from X to $\mathbf{P}_k^1$ that are distinct from the constant morphism with value ∞. For such a rational map to be dominant, it is necessary and sufficient for the corresponding element of K to be transcendent over k.*

PROOF. This claim follows immediately from (7.4.9) and the following:

Lemma. — *Let X be an integral prescheme over k, and let $K = R(X)$ be its field of rational functions. Then there exists a canonical bijective correspondence between the set K and the set of rational maps u from X to $\mathbf{P}_k^1$ that are distinct from the constant morphism with value ∞. For such a rational map to be dominant, it is necessary and sufficient for the corresponding element of K to be transcendent over k.*

First of all, rational maps from X to $\mathbf{P}_k^1$ correspond bijectively to points of $\mathbf{P}_k^1$ with values in the extension K of k (**I**, 7.1.12). If such a point is located (**I**, 3.4.5) at the generic point of $\mathbf{P}_k^1$, then the corresponding rational map is clearly dominant. In the converse case, since every point of $\mathbf{P}_k^1$ that is distinct from the generic point is closed (7.4.3), the image of the domain of definition U of u by the unique morphism $U \to \mathbf{P}_k^1$ of the class u (**I**, 7.2.2) consists of one closed point y of $\mathbf{P}_k^1$, and this morphism (which is not necessarily everywhere defined on X) is thus not dominant; as an abuse of language, we thus say that the rational map u is "constant, of value y". It remains to place in bijective correspondence the points of $\mathbf{P}_k^1$ with value in K that are located (**I**, 3.4.5) *not* at ∞, and the elements of K, and then to verify that the location of such a point is the generic point of $\mathbf{P}^1_k$ if and only if it corresponds to an element that is transcendental over k. But this is immediate (4.2.6, example 1). □ II | 152

Corollary (7.4.16). — *Let X and Y be algebraic curves over k that are* normal, complete, *and irreducible; let $K = R(X)$ and $L = R(Y)$ be their fields. Then there exists a canonical bijective correspondence between the set of k-isomorphisms $X \xrightarrow{\sim} Y$ and the set of k-isomorphisms $L \xrightarrow{\sim} K$.*

PROOF. This is an evident consequence of (7.4.13). □

(7.4.17). This corollary (7.4.16) shows that an algebraic curve over k that is normal, complete, and irreducible, is *determined by its field of rational functions K up to unique isomorphism*; by definition, K is an extension of finite type of k, of transcendence degree 1 (we classically call this a *field of algebraic functions of one variable*). Furthermore:

Proposition (7.4.18). — *For every extension K of k of finite type and of transcendence degree* 1*, there exists an algebraic curve X (determined up to unique isomorphism) that is normal, complete, and irreducible, and such that $R(X) = K$. The set of local rings of X can be identified* (**I**, 8.2.1) *with the set consisting of the elements of K and the elements of the valuation rings that contain k and have K as their field of fractions.*

PROOF. Indeed, K is an extension of finite degree of a pure transcendental extension $k(T)$ of k, which can be identified, as we have seen, with the field of rational functions of the projective line $Y = \mathbf{P}^1_k$. Let X be the *integral closure* of Y with respect to K (6.3.4); then X is a normal algebraic curve over the field K (6.3.7), and it is complete, since the morphism $X \to Y$ is finite (7.4.6). The local rings $\mathscr{O}_x$ of X are either the field K, when x is the generic point, or discrete valuation rings that contain k and have K as their field of fractions, when x is distinct from the generic point (7.4.5). Conversely, let A be such a ring; since the morphism $X \to \operatorname{Spec}(k)$ is proper, the fact that A dominates k implies that A also dominates a local ring $\mathscr{O}_x$ of X (7.3.10); since the latter is a valuation ring that has K as a field of fractions, it is necessarily equal to A. □

Remarks (7.4.19). — It follows from (7.4.16) and (7.4.18) that *the data of an algebraic curve over k that is normal, complete, and irreducible, is essentially equivalent to the data of an extension K of k that is of finite type and of transcendence degree* 1. We note that, if k' is an extension of the base field k, then $X \otimes_k k'$ will again be a complete algebraic curve over k' (5.4.2, (iii)), but, in general, it will be neither reduced nor irreducible. It will, however, be both reduced and irreducible if K is a separable extension of k, and k is algebraically closed in K (this can be expressed, in classical terminology, which we will not use, by saying that K is a "regular extension" of k). But even in this case, it is possible for $X \otimes_k k'$ to not be normal. The reader will find details on these questions in Chapter IV.

§8. BLOWUP SCHEMES; BASED CONES; PROJECTIVE CLOSURE

8.1. Blowup preschemes

(8.1.1). Let Y be a prescheme, and, for every integer $n \geqslant 0$, let $\mathscr{I}_n$ be a quasi-coherent sheaf of ideals of $\mathscr{O}_Y$; suppose that the following conditions are satisfied:

(8.1.1.1) $$\mathscr{I}_0 = \mathscr{O}_Y,\ \mathscr{I}_n \subset \mathscr{I}_m \text{ for } m \leqslant n,$$

(8.1.1.2) $$\mathscr{I}_m\mathscr{I}_n \subset \mathscr{I}_{m+n} \text{ for any } m, n.$$

We note that these hypotheses imply II | 153

(8.1.1.3) $$\mathscr{I}_1^n \subset \mathscr{I}_n.$$

Set

(8.1.1.4) $$\mathscr{S} = \bigoplus_{n\geqslant 0} \mathscr{I}_n.$$

It follows from (8.1.1.1) and (8.1.1.2) that $\mathscr{S}$ is a quasi-coherent graded $\mathscr{O}_Y$-algebra, and thus defines a Y-scheme $X = \operatorname{Proj}(\mathscr{S})$. If $\mathscr{J}$ is an *invertible* sheaf of ideals of $\mathscr{O}_Y$, then $\mathscr{I}_n \otimes_{\mathscr{O}_Y} \mathscr{J}^{\otimes n}$ is canonically identified with $\mathscr{I}_n\mathscr{J}^n$. If we then replace the $\mathscr{I}_n$ by the $\mathscr{I}_n\mathscr{J}^n$, and, in doing so, replace $\mathscr{S}$ by a quasi-coherent $\mathscr{O}_Y$-algebra $\mathscr{S}_{(\mathscr{J})}$, then $X_{(\mathscr{J})} = \operatorname{Proj}(\mathscr{S}_{(\mathscr{J})})$ is canonically isomorphic to X (3.1.8).

(8.1.2). Suppose that Y is *locally integral*, so that the sheaf $\mathscr{R}(Y)$ of rational functions is a quasi-coherent $\mathscr{O}_Y$-algebra (**I**, 7.3.7). We say that an $\mathscr{O}_Y$-submodule $\mathscr{I}$ of $\mathscr{R}(Y)$ is a *fractional ideal* of $\mathscr{R}(Y)$ if it is of *finite type* (**0**, 5.2.1). Suppose we have, for all $n \geqslant 0$, a quasi-coherent fractional ideal $\mathscr{I}_n$ of $\mathscr{R}(Y)$, such that $\mathscr{I}_0 = \mathscr{O}_Y$, and such that condition (8.1.1.2) (but not necessarily the second condition (8.1.1.1)) is satisfied; we can then again define a quasi-coherent graded $\mathscr{O}_Y$-algebra by Equation (8.1.1.4), and the corresponding Y-scheme $X = \operatorname{Proj}(\mathscr{S})$; we will again have a canonical isomorphism from X to $X_{\mathscr{J}}$ for every *invertible* fractional ideal $\mathscr{J}$ of $\mathscr{R}(Y)$.

Definition (8.1.3). — Let Y be a prescheme (resp. a locally integral prescheme), and $\mathscr{I}$ a quasi-coherent ideal of $\mathscr{O}_Y$ (resp. a quasi-coherent fractional ideal of $\mathscr{R}(Y)$). We say that the Y-scheme $X = \operatorname{Proj}(\bigoplus_{n\geqslant 0} \mathscr{I}^n)$ is obtained by blowing up the ideal $\mathscr{I}$, or is the blow-up prescheme of Y relative to $\mathscr{I}$. When $\mathscr{I}$ is a quasi-coherent ideal of $\mathscr{O}_Y$, and Y' is the closed subprescheme of Y defined by $\mathscr{I}$, we also say that X is the Y-scheme obtained by blowing up Y'.

By definition, $\mathscr{S} = \bigoplus_{n\geqslant 0} \mathscr{I}^n$ is then generated by $\mathscr{S}_1 = \mathscr{I}$; if $\mathscr{I}$ is an $\mathscr{O}_Y$-module of *finite type*, then X is *projective* over Y (5.5.2). Without any hypotheses on $\mathscr{I}$, the $\mathscr{O}_X$-module $\mathscr{O}_X(1)$ is *invertible* (3.2.5) and *very ample*, by (4.4.3) applied to the structure morphism $X \to Y$.

We note that, if $j : X \to Y$ is the structure morphism, then the restriction of f to $f^{-1}(Y - Y')$ is an *isomorphism* to $Y - Y'$ whenever $\mathscr{I}$ is an *ideal of* $\mathscr{O}_Y$ and Y' is the closed subprescheme that it defines: indeed, since the questions is local on Y, it suffices to assume that $\mathscr{I} = \mathscr{O}_Y$, and our claim then follows from (3.1.7).

If we replace $\mathscr{I}$ by $\mathscr{I}^d$ ($d > 0$), then the blow-up Y-scheme X is replaced by a canonically isomorphic Y-scheme X' (8.1.1); similarly, for every *invertible* ideal (resp. *invertible* fractional ideal) $\mathscr{J}$, the blow-up prescheme $X_{(\mathscr{J})}$ relative to the ideal $\mathscr{I}\mathscr{J}$ is canonically isomorphic to X (8.1.1).

In particular, whenever $\mathscr{I}$ is an *invertible* ideal (resp. *invertible* fractional ideal), the Y-scheme obtained by blowing up $\mathscr{I}$ is *isomorphic to* Y (3.1.7).

Proposition (8.1.3). — *Let Y be an integral prescheme.*

(i) *For every sequence $(\mathscr{I}_n)$ of quasi-coherent fractional ideals of $\mathscr{R}(Y)$ that satisfies (8.1.1.2) and* **II** | 154
such that $\mathscr{I}_0 = \mathscr{O}_Y$, the Y-scheme $X = \operatorname{Proj}(\bigoplus_{n\geqslant 0} \mathscr{I}^n)$ is integral, and the structure morphism $f : X \to Y$ is dominant.

(ii) *Let $\mathscr{I}$ be a quasi-coherent fractional ideal of $\mathscr{R}(Y)$, and let X be the Y-scheme given by the blow up of Y relative to $\mathscr{I}$. If $\mathscr{I} \neq 0$, then the structure morphism $f : X \to Y$ is then birational and surjective.*

PROOF.

(i) This follows from the fact that $\mathscr{S} = \bigoplus_{n\geqslant 0} \mathscr{I}_n$ is an *integral* $\mathscr{O}_Y$-algebra ((3.1.12) and (3.1.14)), since, for all $y \in Y$, $\mathscr{O}_y$ is an integral ring (**I**, 5.1.4).

(ii) By (i), X is integral; if, furthermore, x and y are the generic points of X and Y (respectively), then we have $f(x) = y$, and it remains to show that $k(x)$ is of rank 1 over $k(y)$. But x is also the generic point of the fibre $f^{-1}(y)$; if ψ is the canonical morphism $Z \to Y$, where $Z = \operatorname{Spec}(k(y))$, then the prescheme $f^{-1}(y)$ can be identified with $\operatorname{Proj}(\mathscr{S}')$, where $\mathscr{S}' = \psi^*(\mathscr{S})$ (3.5.3). But it is clear that $\mathscr{S}' = \bigoplus_{n\geqslant 0}(\mathscr{I}_y)^n$, and, since $\mathscr{I}$ is a quasi-coherent fractional ideal of $\mathscr{R}(Y)$ that is not zero, $\mathscr{I}_y \neq 0$ (**I**, 7.3.6), whence $\mathscr{I}_y = k(y)$; then $\operatorname{Proj}(\mathscr{S}')$ can be identified with $\operatorname{Spec}(k(y))$ (3.1.7), whence the conclusion.

□

We show a *converse* of (8.1.4) in (**III**, 2.3.8).

(8.1.5). We return to the setting and notation of (8.1.1). By definition, the injection homomorphisms $\mathscr{I}_{n+1} \to \mathscr{I}_n$ (8.1.1.1) define, for every $k \in \mathbf{Z}$, an injective homomorphism of degree zero of graded $\mathscr{S}$-modules

$$u_k : \mathscr{S}_+(k+1) \longrightarrow \mathscr{S}(k); \tag{8.1.5.1}$$

since $\mathscr{S}_+(k+1)$ and $\mathscr{S}(k+1)$ are canonically **(TN)**-isomorphic, they give a canonical correspondence between u_k and an injective homomorphism of $\mathscr{O}_X$-modules (3.4.2):

$$\widetilde{u}_k : \mathscr{O}_X(k+1) \longrightarrow \mathscr{O}_X(k). \tag{8.1.5.2}$$

Recall as well (3.2.6) that we have defined canonical homomorphisms

$$\lambda : \mathscr{O}_X(h) \otimes_{\mathscr{O}_X} \mathscr{O}_X(k) \longrightarrow \mathscr{O}_X(h+k) \tag{8.1.5.3}$$

and, since the diagram

$$\begin{tikzcd}
\mathscr{S}(h) \otimes_{\mathscr{S}} \mathscr{S}(k) \otimes_{\mathscr{S}} \mathscr{S}(l) \arrow[r] \arrow[d] & \mathscr{S}(h+k) \otimes_{\mathscr{S}} \mathscr{S}(l) \arrow[d] \\
\mathscr{S}(h) \otimes_{\mathscr{S}} \mathscr{S}(k+l) \arrow[r] & \mathscr{S}(h+k+l)
\end{tikzcd}$$

commutes, it follows from the functoriality of the λ (3.2.6) that the homomorphisms (8.1.5.3) define the structure of a *quasi-coherent graded $\mathscr{O}_X$-algebra* on

$$\mathscr{S}_X = \bigoplus_{n \in \mathbf{Z}} \mathscr{O}_X(n). \tag{8.1.5.4}$$

Furthermore, the diagram

$$\begin{tikzcd}
\mathscr{S}(h) \otimes_{\mathscr{S}} \mathscr{S}(k+1) \arrow[r] \arrow[d, "1\otimes u_k"'] & \mathscr{S}(h+k+1) \arrow[d, "u_{k+h}"] \\
\mathscr{S}(h) \otimes_{\mathscr{S}} \mathscr{S}(k) \arrow[r] & \mathscr{S}(h+k)
\end{tikzcd}$$

commutes; the functoriality of the λ then implies that we have a commutative diagram

$$\begin{tikzcd}
\mathscr{O}_X(h) \otimes_{\mathscr{O}_X} \mathscr{O}_X(k+1) \arrow[r, "\lambda"] \arrow[d, "1\otimes\widetilde{u}_k"'] & \mathscr{O}_X(h+k+1) \arrow[d, "\widetilde{u}_{k+h}"] \\
\mathscr{O}_X(h) \otimes_{\mathscr{O}_X} \mathscr{O}_X(k) \arrow[r, "\lambda"] & \mathscr{O}_X(h+k)
\end{tikzcd} \tag{8.1.5.5}$$

where the horizontal arrows are the canonical homomorphisms. We can thus say that the $\widetilde{u}_k$ define an *injective homomorphism* (of degree zero) *of graded $\mathscr{S}_X$-modules*

$$\widetilde{u} : \mathscr{S}_X(1) \longrightarrow \mathscr{S}_X. \tag{8.1.5.6}$$

(8.1.6). Keeping the notation from (8.1.5), we now note that, for $n \geqslant 0$, the composite homomorphism $\widetilde{v}_n = \widetilde{u}_{n-1} \circ \widetilde{u}_{n-2} \circ \ldots \circ \widetilde{u}_0$ is an *injective* homomorphism $\mathscr{O}_X(n) \to \mathscr{O}_X$; we denote by $\mathscr{I}_{n,X}$ its image, which is thus a quasi-coherent ideal of $\mathscr{O}_X$, *isomorphic* to $\mathscr{O}_X(n)$. Furthermore, the diagram

$$\begin{tikzcd}
\mathscr{O}_X(m) \otimes_{\mathscr{O}_X} \mathscr{O}_X(n) \arrow[r, "\lambda"] \arrow[d, "\widetilde{v}_m\otimes\widetilde{v}_n"'] & \mathscr{O}_X(m+n) \arrow[d, "\widetilde{v}_{m+n}"] \\
\mathscr{O}_X \arrow[r, "\mathrm{id}"'] & \mathscr{O}_X
\end{tikzcd}$$

commutes for $m \geqslant 0$, $n \geqslant 0$. We thus deduce the following inclusions:

$$\mathscr{I}_{0,X} = \mathscr{O}_X, \quad \mathscr{I}_{n,X} \subset \mathscr{I}_{m,X} \qquad \text{for } 0 \leqslant m \leqslant n; \tag{8.1.6.1}$$

$$\mathscr{I}_{m,X}\mathscr{I}_{n,X} \subset \mathscr{I}_{m+n,X} \qquad \text{for } m \geqslant 0,\ n \geqslant 0. \tag{8.1.6.2}$$

II | 156

Proposition (8.1.7). — *Let Y be a prescheme, $\mathscr{I}$ a quasi-coherent ideal of $\mathscr{O}_Y$, and $X = \operatorname{Proj}(\bigoplus_{n\geqslant 0} \mathscr{I}^n)$ the Y-scheme given by blowing up $\mathscr{I}$. We then have, for all $n > 0$, a canonical isomorphism*

$$\mathscr{O}_X(n) \xrightarrow{\sim} \mathscr{I}^n \mathscr{O}_X = \mathscr{I}_{n,X} \tag{8.1.7.1}$$

*(cf. (**0**, 4.3.5)), and thus that $\mathscr{I}^n\mathscr{O}_X$ is a very-ample invertible $\mathscr{O}_X$-module if $n > 0$.*

Proof. The last claim is immediate, since $\mathscr{O}_X(1)$ is invertible (3.2.5) and very ample for Y by definition ((4.4.3) and (4.4.9)). Also by definition, the image of v_n is exactly $\mathscr{I}^n\mathscr{S}$, and (8.1.7.1) then follows from the exactness of the functor $\widetilde{\mathscr{M}}$ (3.2.4) and from Equation (3.2.4.1). $\square$

Corollary (8.1.8). — *Under the hypotheses of (8.1.7), if $f : X \to Y$ is the structure morphism, and Y' the closed subprescheme of Y defined by $\mathscr{I}$, then the closed subprescheme $X' = f^{-1}(Y')$ of X is defined by $\mathscr{I}\mathscr{O}_X$ (which is canonically isomorphic to $\mathscr{O}_X(1)$), from which we obtain a canonical short exact sequence*

(8.1.8.1) $$0 \longrightarrow \mathscr{O}_X(1) \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_{X'} \longrightarrow 0.$$

PROOF. This follows from (8.1.7.1) and from (**I**, 4.4.5). □

(8.1.9). Under the hypotheses of (8.1.7), we can be more precise about the structure of the $\mathscr{I}_{n,X}$. Note that the homomorphism

$$\widetilde{u}_{-1} : \mathscr{O}_X \longrightarrow \mathscr{O}_X(-1)$$

canonically corresponds to a section s of $\mathscr{O}_X(-1)$ over X, which we call the *canonical section* (relative to $\mathscr{I}$) (**0**, 5.1.1). In the diagram in (8.1.5.5), the horizontal arrows are isomorphisms (3.2.7); by replacing h with k, and k with -1 in this diagram, we obtain that $\widetilde{u}_k = 1_k \otimes \widetilde{u}_{-1}$ (where 1_k denotes the identity on $\mathscr{O}_X(h)$), or, equivalently, that the homomorphism $\widetilde{u}_k$ is given exactly by *tensoring with the canonical section s* (for all $k \in \mathbf{Z}$). The homomorphism $\widetilde{u}$ (8.1.5.6) can then be understood in the same way.

Thus, for all $n \geqslant 0$, the homomorphism $\widetilde{v}_n : \mathscr{O}_X(n) \to \mathscr{O}_X$ is given exactly by tensoring with $s^{\otimes n}$; we thus deduce:

Corollary (8.1.10). — *With the notation of (8.1.8), the underlying space of X' is the set of $x \in X$ such that $s(x) = 0$, where s denotes the canonical section of $\mathscr{O}_X(-1)$.*

PROOF. Indeed, if c_x is a generator of the fibre $(\mathscr{O}_X(1))_x$ at a point x, then $s_x \otimes c_x$ is canonically identified with a generator of the fibre of $\mathscr{I}_{1,X}$ at the point x, and is thus invertible if and only if $s_x \notin \mathfrak{m}_x(\mathscr{O}_x(-1))_x$, or, equivalently, if and only if $s(x) \neq 0$. □

Proposition. — *Let Y be an integral prescheme, $\mathscr{I}$ a quasi-coherent fractional ideal of $\mathscr{R}(Y)$, and X the Y-scheme given by blowing up $\mathscr{I}$. Then $\mathscr{I}\mathscr{O}_X$ is an invertible $\mathscr{O}_X$-module that is very ample for Y.*

PROOF. Since the questions is local on Y (4.4.5), we can reduce to the case where $Y = \operatorname{Spec}(A)$, with A some integral ring of ring of fractions K, and $\mathscr{I} = \widetilde{\mathfrak{I}}$, with $\mathfrak{I}$ some fractional ideal of K; there then exists an element $a \neq 0$ of A such that $a\mathfrak{I} \subset A$. Let $S = \bigoplus_{n\geqslant 0} \mathfrak{I}^n$; the map $x \mapsto ax$ is an A-isomorphism from $\mathfrak{I}^{n+1} = (S(1))_n$ to $a\mathfrak{I}^{n+1} = a\mathfrak{I}S_n \subset \mathfrak{I}^n = S_n$, and thus defines a (TN)-isomorphism of degree zero of graded S-modules $S_+(1) \to a\mathfrak{I}S$. On the other hand, $x \mapsto a^{-1}x$ is an isomorphism of degree zero of graded S-modules $a\mathfrak{I}S \xrightarrow{\sim} \mathfrak{I}S$. We thus obtain, by composition (3.2.4), an isomorphism of $\mathscr{O}_X$-modules $\mathscr{O}_X(1) \xrightarrow{\sim} \mathscr{I}\mathscr{O}_X$, and, since S is generated by $S_1 = \mathfrak{I}$, $\mathscr{O}_X(1)$ is invertible (3.2.5) and very ample ((4.4.3) and (4.4.9)), whence our claim. □ **II | 157**

8.2. Preliminary results on the localisation of graded rings

(8.2.1). Let S be a graded ring, but not assumed (for the moment) to be only in positive degree. We define

(8.2.1.1) $$S^{\geqslant} = \bigoplus_{n\geqslant 0} S_n, \qquad S^{\leqslant} = \bigoplus_{n\leqslant 0} S_n$$

which are both graded subrings of S, in only positive and negative degrees (respectively). If f is a homogeneous elements of degree d (positive or negative) of S, then the ring of fraction $S_f = S'$ is again endowed with the structure of a graded ring, by taking S'_n ($n \in \mathbf{Z}$) to be the set of the x/f^k for $x \in S_{n+kd}$ ($k \geqslant 0$); we define $S_{(f)} = S'_0$, and will write $S_f^{\geqslant}$ and $S_f^{\leqslant}$ for $S'^{\geqslant}$ and $S'^{\leqslant}$ (respectively). If $d > 0$, then

(8.2.1.2) $$(S^{\geqslant})_f = S_f$$

since, if $x \in S_{n+kd}$ with $n + kd < 0$, then we can write $x/f^k = xf^h/f^{h+k}$, and we also have that $n + (h+k)d > 0$ for h sufficiently large and > 0. We thus conclude, by definition, that

(8.2.1.3) $$(S^{\geqslant})_{(f)} = (S_f^{\geqslant})_0 = S_{(f)}.$$

If M is a graded S-module, then we similarly define

$$M^{\geqslant} = \bigoplus_{n\geqslant 0} M_n, \qquad M^{\leqslant} = \bigoplus_{n\leqslant 0} M_n \tag{8.2.1.4}$$

which are (respectively) a graded $S^{\geqslant}$-module and a graded $S^{\leqslant}$-module, and their intersection is the S_0 module M_0. If $f \in S_d$, then we define M_f to be the graded S_f-module whose elements of degree n are the z/f^k for $z \in M_{n+kd}$ ($k \geqslant 0$); we denote by $M_{(f)}$ the set of elements of degree zero of M_f, and this is an $S_{(f)}$-module, and we will write $M_f^{\geqslant}$ and $M_f^{\leqslant}$ to mean $(M_f)^{\geqslant}$ and $(M_f)^{\leqslant}$ (respectively). If $d > 0$, then we see, as above, that

$$(M^{\geqslant})_f = M_f \tag{8.2.1.5}$$

and

$$(M^{\geqslant})_{(f)} = (M_f^{\geqslant})_0 = M_{(f)}. \tag{8.2.1.6}$$

(8.2.2). Let $\mathbf{z}$ be an indeterminate, we we will call the *homogenisation variable*. If S is a graded ring (in positive or negative degrees), then the polynomial algebra[1]

$$\widehat{S} = S[\mathbf{z}] \tag{8.2.2.1}$$

is a graded S-algebra, where we define the degree of $f\mathbf{z}^n$ ($n \geqslant 0$), with f homogeneous, as **II** | 158

$$\deg(f\mathbf{z}^n) = n + \deg f. \tag{8.2.2.2}$$

Lemma (8.2.3). — (i) *There are canonical isomorphisms of (non-graded) rings*

$$\widehat{S}_{(\mathbf{z})} \xrightarrow{\sim} \widehat{S}/(\mathbf{z}-1)\widehat{S} \xrightarrow{\sim} S. \tag{8.2.3.1}$$

(ii) *There is a canonical isomorphism of (non-graded) rings*

$$\widehat{S}_{(f)} \xrightarrow{\sim} S_f^{\leqslant} \tag{8.2.3.2}$$

for all $f \in S_d$ with $d > 0$.

Proof. The first of the isomorphisms in (8.2.3.1) was defined in (2.2.5), and the second is trivial; the isomorphism $\widehat{S}_{(\mathbf{z})} \xrightarrow{\sim} S$ thus defined thus gives a correspondence between $x\mathbf{z}^n/\mathbf{z}^{n+k}$ (where $\deg(x) = k$ for $k \geqslant -n$) and the element x. The homomorphism (8.2.3.2) gives a correspondence between $x\mathbf{z}^n/f^k$ (where $\deg(x) = kd - n$) and the element x/f^k of degree $-n$ in $S_f^{\leqslant}$, and it is again clear that this does indeed give an isomorphism. □

(8.2.4). Let M be a graded S-module. It is clear that the S-module

$$\widehat{M} = M \otimes_S \widehat{S} = M \otimes_S S[\mathbf{z}] \tag{8.2.4.1}$$

is the direct sum of the S-modules $M \otimes S\mathbf{z}^n$, and thus of the abelian groups $M_k \otimes S\mathbf{z}^n$ ($k \in \mathbf{Z}$, $n \geqslant 0$); we define on $\widehat{M}$ the structure of a graded $\widehat{S}$-module by setting

$$\deg(x \otimes \mathbf{z}^n) = n + \deg x \tag{8.2.4.2}$$

for all homogeneous x in M. We leave it to the reader to prove the analogue of (8.2.3):

Lemma (8.2.5). — (i) *There is a canonical di-isomorphism of (non-graded) modules*

$$\widehat{M}_{(\mathbf{z})} \xrightarrow{\sim} M. \tag{8.2.5.1}$$

(ii) *For all $f \in S_d$ ($d > 0$), there is a di-isomorphism of (non-graded) modules*

$$\widehat{M}_{(f)} \xrightarrow{\sim} M_f^{\leqslant}. \tag{8.2.5.2}$$

[1] This should not be confused with the use of the notation $\widehat{S}$ to denote the completed separation of a ring.

(8.2.6). Let S be a *positively*-graded ring, and consider the decreasing sequence of graded ideals of S

(8.2.6.1) $$S_{[n]} = \bigoplus_{m \geqslant n} S_m \qquad (n \geqslant 0)$$

(so, in particular, we have $S_{[0]} = S$ and $S_{[1]} = S_+$). Since it is evident that $S_{[m]}S_{[n]} \subset S_{[m+n]}$, we can define a *graded ring* $S^\natural$ by setting

(8.2.6.2) $$S^\natural = \bigoplus_{n \geqslant 0} S^\natural_n \quad \text{with} \quad S^\natural_n = S_{[n]}.$$

$S^\natural_0$ is then the ring S considered as a *non-graded* ring, and $S^\natural$ is thus an $S^\natural_0$-algebra. For every homogeneous element $f \in S_d$ ($d > 0$), we denote by $f^\natural$ the element f considered as belonging to $S_{[d]} = S^\natural_d$. With this notation:

II | 159

Lemma (8.2.7). — *Let S be a positively-graded ring, and f a homogeneous element of S_d ($d > 0$). There are canonical ring isomorphisms*

(8.2.7.1) $$S_f \xrightarrow{\sim} \bigoplus_{n \in \mathbf{Z}} S(n)_{(f)}$$

(8.2.7.2) $$(S_f^{\geqslant})_{f/1} \xrightarrow{\sim} S_f$$

(8.2.7.3) $$S^\natural_{(f^\natural)} \xrightarrow{\sim} S_f^{\geqslant}$$

where the first two are isomorphisms of graded rings.

PROOF. It is immediate, by definition, that we have $(S_f)_n = (S(n)_f)_0$, whence the isomorphism in (8.2.7.1), which is exactly the identity. Next, since $f/1$ is invertible in S_f, there is a canonical isomorphism $S_f \xrightarrow{\sim} (S_f^{\geqslant})_{f/1} = (S_f)_{f/1}$, by (8.2.1.2) applied to S_f; the inverse isomorphism is, by definition, the isomorphism in (8.2.7.2). Finally, if $x = \sum_{m \geqslant n} y_m$ is an element of $S_{[n]}$ with $n = kd$, then the element $x/(f^\natural)^k$ corresponds to the element $\sum_m y_m/f^k$ of $S_f^{\geqslant}$, and we can quickly verify that this defines an isomorphism (8.2.7.3). $\square$

(8.2.8). If M is a graded S-module, then we similarly define, for all $n \in \mathbf{Z}$,

(8.2.8.1) $$M_{[n]} = \bigoplus_{m \geqslant n} M_m$$

and, since $S_{[m]}M_{[n]} \subset M_{[m+n]}$ ($m \geqslant 0$), we can define a graded $S^\natural$-module $M^\natural$ by setting

(8.2.8.2) $$M^\natural = \bigoplus_{n \in \mathbf{Z}} \quad \text{with} \quad M^\natural_n = M_{[n]}.$$

We leave to the reader the proof of:

Lemma (8.2.9). — *With the notation of (8.2.7) and (8.2.8), there are canonical di-isomorphisms of modules*

(8.2.9.1) $$M_f \xrightarrow{\sim} \bigoplus_{n \in \mathbf{Z}} M(n)_{(f)}$$

(8.2.9.2) $$(M_f^{\geqslant})_{f/1} \xrightarrow{\sim} M_f$$

(8.2.9.3) $$M^\natural_{(f^\natural)} \xrightarrow{\sim} M_f^{\geqslant}$$

where the first two are di-isomorphisms of graded modules.

Lemma (8.2.10). — *Let S be a positively-graded ring.*

(i) *For $S^\natural$ to be an $S^\natural_0$-algebra of finite type (resp. a Noetherian $S^\natural_0$-algebra), it is necessary and sufficient for S to be an $S^\natural_0$-algebra of finite type (resp. a Noetherian $S^\natural_0$-algebra).*

(ii) *For $S^\natural_{n+1} = S^\natural_1 S^\natural_n$ ($n \geqslant n_0$), it is necessary and sufficient for $S_{n+1} = S_1 S_n$ ($n \geqslant n_0$).*

(iii) *For $S^\natural_n = S^\natural_1$ $(n \geqslant n_0)$, it is necessary and sufficient for $S_n = S^n_1$ $(n \geqslant n_0)$.*
(iv) *If (f_α) is a set of homogeneous elements of S_+ such that S_+ is the radical in S_+ of the ideal of S_+ generated by the f_α, then $S^\natural_+$ is the radical in $S^\natural_+$ of the ideal of $S^\natural_+$ generated by the $f^\natural_\alpha$.*

PROOF.

(i) If $S^\natural$ is an $S^\natural_0$-algebra of finite type, then $S_+ = S^\natural_1$ is a module of finite type over $S = S^\natural_0$, by (2.1.6, i), and so S is an S_0-algebra of finite type (2.1.4); if $S^\natural$ is a Noetherian ring, then so too is $S^\natural_0 = S$ (2.1.5). Conversely, if S is an S_0-algebra of finite type, then we know (2.1.6, ii) **II** | 160 that there exist $h > 0$ and $m_0 > 0$ such that $S_{n+h} = S_h S_n$ for $n \geqslant m_0$; we can clearly assume that $m_0 \geqslant h$. Furthermore, the S_m are S_0-modules of finite type (2.1.6, i). So, if $n \geqslant m_0 + h$, then $S^\natural_n = S_h S^\natural_{n-h} = S^\natural_h S^\natural_{n-h}$; and if $m < m_0 + h$ then, letting $E = S_{m_0} + \ldots + S_{m_0+h-1}$, we have that

$$S^\natural_m = S_m + \ldots + S_{m_0+h-1} + S_h E + S^2_h E + \ldots.$$

For $1 \leqslant m \leqslant m_0$, let G_m be the union of the finite systems of generators of the S_0-modules S_i for $m \leqslant i \leqslant m_0 + h - 1$, considered as a subset of $S_{[m]}$. For $m_0 + 1 \leqslant m \leqslant m_0 + h - 1$, let G_m be the union of the finite system of generators of the S_0-modules S_i for $m \leqslant i \leqslant m_0 + h - 1$ and of $S_h E$, considered as a subset of $S_{[m]}$. It is clear that $S^\natural_m = S^\natural_0 G_m$ for $1 \leqslant m \leqslant m_0 + h - 1$, and thus the union G of the G_m for $1 \leqslant m \leqslant m_0 + h - 1$ is a system of generators of the $S^\natural_0$-algebra $S^\natural$. We thus conclude that, if $S = S^\natural_0$ is a Noetherian ring, then so too is $S^\natural$.

(ii) It is clear that, if $S_{n+1} = S_1 S_n$ for $n \geqslant n_0$, then $S^\natural_{n+1} = S_1 S^\natural_n$, and *a fortiori* $S^\natural_{n+1} = S^\natural_1 S^\natural_n$ for $n \geqslant n_0$. Conversely, this last equality can be written as

$$S_{n+1} + S_{n+2} + \ldots = (S_1 + S_2 + \ldots)(S_n + S_{n+1} + \ldots)$$

and comparing terms of degree $n+1$ (in S) on both sides gives that $S_{n+1} = S_1 S_n$.

(iii) If $S_n = S^n_1$ for $n \geqslant n_0$, then $S^\natural_n = S^n_1 + S^{n+1}_1 + \ldots$; since $S^\natural_1$ contains $S_1 + S^2_1 + \ldots$, we have that $S^\natural_n \subset S^{\natural n}_1$, and thus $S^\natural_n = S^{\natural n}_1$ for $n \geqslant n_0$. Conversely, the only terms of $S^{\natural n}_1 = (S_1 + S_2 + \ldots)^n$ that are of degree n in S are those of S^n_1; the equality $S^\natural_n = S^{\natural n}_1$ thus implies that $S_n = S^n_1$.

(iv) It suffices to show that, if an element $g \in S_{k+h}$ is considered as an element of $S^\natural_k$ $(k > 0$, $h \geqslant 0)$, then there exists an integer $n > 0$ such that g^n is a linear combination (in $S^\natural_{kn}$) of the $f^\natural_\alpha$ with coefficients in $S^\natural$. By hypothesis, there exists an integer m_0 such that, for $m \geqslant m_0$, we have, *in* S, that $g^m = \sum_\alpha c_{\alpha m} f_\alpha$, where the indices α here are *independent of* m; furthermore, we can clearly assume that the $c_{\alpha m}$ are homogeneous, with

$$\deg(c_{\alpha m}) = m(k+h) - \deg f_\alpha$$

in S. So take m_0 sufficiently large enough to ensure that $km_0 > \deg f_\alpha$ for all the f_α that appear in g^{m_0}; for all α, let $c'_{\alpha m}$ be the element $c_{\alpha m}$ considered as having degree $km - \deg f_\alpha$ *in* $S^\natural$; we then have, in $S^\natural$, that $g^m = \sum_\alpha c'_{\alpha m} f^\natural_\alpha$, which finishes the proof.

□

(8.2.11). Consider the graded S_0-algebra

(8.2.11.1) $$S^\natural \otimes_S S_0 = S^\natural / S_+ S^\natural = \bigoplus_{n \geqslant 0} S_{[n]} / S_+ S_{[n]}.$$

Since S_n is a quotient S_0-module of $S_{[n]}/S_+ S_{[n]}$, there is a canonical homomorphism of graded S_0-algebras

(8.2.11.2) $$S^\natural \otimes_S S_0 \longrightarrow S$$

which is clearly *surjective*, and thus corresponds (2.9.2) to a canonical *closed immersion*

(8.2.11.3) $$\mathrm{Proj}(S) \longrightarrow \mathrm{Proj}(S^\natural \otimes_S S_0).$$

II | 161

Proposition (8.2.12). — *The canonical morphism (8.2.11.3) is bijective. For the homomorphism (8.2.11.2) to be (TN)-bijective, it is necessary and sufficient for there to exist some n_0 such that $S_{n+1} = S_1 S_n$ for $n \geqslant n_0$. If this latter condition is satisfied, then (8.2.11.3) is an isomorphism; the converse is true whenever S is Noetherian.*

PROOF. To prove the first claim, it suffices (2.8.3) to show that the kernel $\mathfrak{I}$ of the homomorphism (8.2.11.2) consists of *nilpotent* elements. But if $f \in S_{[n]}$ is an element whose class modulo $S_+ S_{[n]}$ belongs to this kernel, then this implies that $f \in S_{[n+1]}$; then f^{n+1}, considered as an element of $S_{[n(n+1)]}$, is also an element of $S_+ S_{[n(n+1)]}$, since it can be written as $f \cdot f^n$; so the class of f^{n+1} modulo $S_+ S_{[n(n+1)]}$ is zero, which proves our claim. Since the hypothesis that $S_{n+1} = S_1 S_n$ for $n \geqslant n_0$ is equivalent to $S^\natural_{n+1} = S^\natural_1 S^\natural_n$ for $n \geqslant n_0$ (8.2.10, ii), this hypothesis is equivalent, by definition, to the fact that (8.2.11.2) is (TN)-injective, and thus (TN)-bijective, and so (8.2.11.3) is an isomorphism, by (2.9.1). Conversely, if (8.2.11.3) is an isomorphism, then the sheaf $\widetilde{\mathfrak{I}}$ on $\operatorname{Proj}(S^\natural \otimes_S S_0)$ is zero (2.9.2, i); since $S^\natural \otimes_S S_0$ is Noetherian, as a quotient of $S^\natural$ (8.2.10, i), we conclude from (2.7.3) that $\mathfrak{I}$ satisfies condition (TN), and so $S^\natural_{n+1} = S^\natural_1 S^\natural_n$ for $n \geqslant n_0$, and this finishes the proof, by (8.2.10, ii). ☐

(8.2.13). Consider now the canonical injections $(S_+)^n \to S_{[n]}$, which define an injective homomorphism of degree zero of graded rings

$$\bigoplus_{n \geqslant 0} (S_+)^n \longrightarrow S^\natural. \tag{8.2.13.1}$$

Proposition (8.2.14). — *For the homomorphism (8.2.13.1) to be a (TN)-isomorphism, it is necessary and sufficient for there to exist some n_0 such that $S_n = S_1^n$ for all $n \geqslant n_0$. Whenever this is the case, the morphisms corresponding to (8.2.13.1) is everywhere defined and also an isomorphism*

$$\operatorname{Proj}(S^\natural) \xrightarrow{\sim} \operatorname{Proj}(\bigoplus_{n \geqslant 0} (S_+)^n);$$

the converse is true whenever S is Noetherian.

PROOF. The first two claims are evident, given (8.2.10, iii) and (2.9.1). The third will follow from (8.2.10, i and iii) and the following lemma:

Lemma (8.2.14.1). — *Let T be a positively-graded ring that is also a T_0-algebra of finite type. If the morphism corresponding to the injective homomorphism $\bigoplus_{n \geqslant 0} T_1^n \to T$ is everywhere defined and also an isomorphism $\operatorname{Proj}(T) \to \operatorname{Proj}(\bigoplus_{n \geqslant 0} T_1^n)$, then there exists some n_0 such that $T_n = T_1^n$ for $n \geqslant n_0$.*

Let g_i $(1 \leqslant i \leqslant r)$ be generators of the T_0-module T_1. The hypothesis implies first of all that the $D_+(g_i)$ cover $\operatorname{Proj}(T)$ (2.8.1). Let $(h_j)_{1 \leqslant j \leqslant s}$ be a system of homogeneous elements of T_+, with $\deg(h_j) = n_j$, that form, with the g_i, a system of generators of the ideal T_+, or, equivalently (2.1.3), a system of generators of T as a T_0-algebra; if we set $T' = \bigoplus_{n \geqslant 0} T_1^n$, then the element $h_j / g_i^{n_j}$ of the ring $T_{(g_i)}$ must, by hypothesis, belong to the subring $T'_{(g_i)}$, and so there exists some integer k such that $T_1^k h_j \subset T_1^{k+n_j}$ for all j. We thus conclude, by induction on r, that $T_1^k h_j^r \subset T'$ for all $r \geqslant 1$, and, by definition of the h_j, we thus have that $T_1^k T \subset T'$. Also, there exists, for all j, an integer m_j such that $h_j^{m_j}$ belongs to the ideal of T generated by the g_i (2.3.14), so $h_j^{m_j} \in T_1 T$, and $h_j^{m_j k} \in T_1^k T \subset T'$. **II** | 162
There is thus an integer $m_0 \geqslant k$ such that $h_j^m \in T_1^{mn}$ for $m \geqslant m_0$. So, if q is the largest of the integers n_j, then $n_0 = qsm_0 + k$ is the required number. Indeed, an element of S_n, for $n \geqslant n_0$, is the sum of monomials belonging to $T_1^\alpha u$, where u is a product of powers of the h_j; if $\alpha \geqslant k$, then it follows from the above that $T_1^\alpha u \subset T_1^n$; in the other case, one of the exponents of the h_j is $\geqslant m_0$, so $u \in T_1^\beta v$, where $\beta \geqslant k$ and v is again a product of powers of the h_j; we can then reduce to the previous case, and so we conclude that $T_1^\alpha u \subset T_1^n$ in all cases. ☐

Remark (8.2.15). — The condition $S_n = S_1^n$ for $n \geqslant n_0$ clearly implies that $S_{n+1} = S_1 S_n$ for $n \geqslant n_0$, but the converse is not necessarily true, even if we assume that S is Noetherian. For example, let K be a field, $A = K[\mathbf{x}]$, and $B = K[\mathbf{y}]/\mathbf{y}^2 K[\mathbf{y}]$, where $\mathbf{x}$ and $\mathbf{y}$ are indeterminates, with $\mathbf{x}$ taken to have degree 1 and $\mathbf{y}$ to have degree 2, and let $S = A \otimes_K B$, so that S is a graded algebra over K that has a basis given by the elements 1, $\mathbf{x}^n$ ($n \geqslant 1$), and $\mathbf{x}^n\mathbf{y}$ ($n \geqslant 0$). It is immediate that $S_{n+1} = S_1 S_n$ for $n \geqslant 2$, but $S_1^n = K\mathbf{x}^n$ while $S_n = K\mathbf{x}^n + K\mathbf{x}^n\mathbf{y}$ for $n \geqslant 2$.

8.3. Based cones

(8.3.1). Let Y be a prescheme; in all of this section, we will consider only *Y-preschemes* and *Y-morphisms*. Let $\mathscr{S}$ be a quasi-coherent *positively*-graded $\mathscr{O}_Y$-algebra; *we further assume that* $\mathscr{S}_0 = \mathscr{O}_Y$. Following the notation introduced in (8.2.2), we let

$$\widehat{\mathscr{S}} = \mathscr{S}[\mathbf{z}] = \mathscr{S} \otimes_{\mathscr{O}_Y} \mathscr{O}_Y[\mathbf{z}] \tag{8.3.1.1}$$

which we consider as a positively-graded $\mathscr{O}_Y$-algebra by defining the degrees as in (8.2.2.2), so that, for every affine open subset U of Y, we have

$$\Gamma(U, \widehat{\mathscr{S}}) = (\Gamma(U, \mathscr{S}))[\mathbf{z}].$$

In what follows, we write

$$X = \operatorname{Proj}(\mathscr{S}), \quad C = \operatorname{Spec}(\mathscr{S}), \quad \widehat{C} = \operatorname{Proj}(\widehat{\mathscr{S}}) \tag{8.3.1.2}$$

(where, in the definition of C, we consider $\mathscr{S}$ as a non-graded $\mathscr{O}_Y$-algebra), and we say that C (resp. $\widehat{C}$) is the *affine cone* (resp. *projective cone*) defined by $\mathscr{S}$; we will sometimes say "cone" instead of "affine cone". By an abuse of language, we also say that C (resp. $\widehat{C}$) is the *affine cone based at X (?)* (resp. the *projective cone based at X (?)*)[2], with the implicit understanding that the prescheme X is given in the form $\operatorname{Proj}(\mathscr{S})$; finally, we say that $\widehat{C}$ is the *projective closure* of C (with the data of $\mathscr{S}$ being implicit in the structure of C).

Proposition (8.3.2). — *There exist canonical Y-morphisms*

$$Y \xrightarrow{\varepsilon} C \xrightarrow{i} \widehat{C} \tag{8.3.2.1}$$

$$X \xrightarrow{j} \widehat{C} \tag{8.3.2.2}$$

such that ε and j are closed immersions, and i is an affine morphism, which is a dominant open immersion, for which

$$i(C) = \widehat{C} - j(X); \tag{8.3.2.3}$$

furthermore, $\widehat{C}$ is the smallest closed subprescheme of $\widehat{C}$ containing $i(C)$.

PROOF. To define i, consider the open subset of $\widehat{C}$ given by

$$\widehat{C}_{\mathbf{z}} = \operatorname{Spec}(\widehat{\mathscr{S}}/(\mathbf{z}-1)\widehat{\mathscr{S}}) \tag{8.3.2.4}$$

(3.1.4), where $\mathbf{z}$ is canonically identified with a section of $\mathscr{S}$ over Y. The isomorphism $i : C \xrightarrow{\sim} \widehat{C}_{\mathbf{z}}$ then corresponds to the canonical isomorphism (8.2.3.1)

$$\widehat{\mathscr{S}}/(\mathbf{z}-1)\widehat{\mathscr{S}} \xrightarrow{\sim} \mathscr{S}.$$

The morphism ε corresponds to the augmentation homomorphism $\mathscr{S} \to \mathscr{S}_0 = \mathscr{O}_Y$, which has kernel $\mathscr{S}_+$ (1.2.7), and, since the latter is surjective, ε is a closed immersion (1.4.10). Finally, j corresponds (3.5.1) to the surjective homomorphism of degree zero $\widehat{\mathscr{S}} \to \mathscr{S}$, which restricts to the identity on $\mathscr{S}$ and is zero on $\mathbf{z}\widehat{\mathscr{S}}$, which is its kernel; j is everywhere defined, and is a closed immersion, by (3.6.2).

To prove the other claims of (8.3.2), we can clearly restrict to the case where $Y = \operatorname{Spec}(A)$ is affine, and $\mathscr{S} = \widetilde{S}$, with S a graded A-algebra, whence $\widehat{\mathscr{S}} = (\widehat{S})^\sim$; the homogeneous elements f of S_+ can then be identified with sections of $\widehat{\mathscr{S}}$ over Y, and the open subset of $\widehat{C}$, denoted $D_+(f)$ in (2.3.3), can then be written as $\widehat{C}_f$ (3.1.4); similarly, the open subset of C denoted $D(f)$ in (**I**, 1.1.1) can be written as C_f (**0**, 5.5.2). With this in mind, it follows from (2.3.14) and from the definition of $\widehat{S}$

[2] *[Trans.] A more literal translation of the French* (cône projetant (affine/projectif)) *would be the* projecting (affine/projective) cone, *but it seems that this terminology already exists to mean something else.*

that, in this case, the open subsets $\widehat{C}_{\mathbf{z}} = i(C)$ and $\widehat{C}_f$ (with f homogeneous in S_+) form a *cover* of $\widehat{C}$. Furthermore, with this notation,

$$i^{-1}(\widehat{C}_f) = C_f; \tag{8.3.2.5}$$

indeed, $\widehat{C}_f \cap i(C) = \widehat{C}_f \cap \widehat{C}_{\mathbf{z}} = \widehat{C}_{f\mathbf{z}} = \operatorname{Spec}(\widehat{S}_{(f\mathbf{z})})$. But, if $d = \deg(f)$, then $\widehat{S}_{(f\mathbf{z})}$ is canonically isomorphic to $(\widehat{S}_{(\mathbf{z})})_{f/\mathbf{z}^d}$ (2.2.2), and it follows from the definition of the isomorphism in (8.2.3.1) that the image of $(\widehat{S}_{(\mathbf{z})})_{f/\mathbf{z}^d}$ under the corresponding isomorphism of rings of fractions is exactly S_f. Since $C_f = \operatorname{Spec}(S_f)$, this proves (8.3.2.5) and shows, at the same time, that the morphism i is affine; furthermore, the restriction of i to C_f, considered as a morphism to $\widehat{C}_f$, corresponds (I, 1.7.3) to the canonical homomorphism $\widehat{S}_{(f)} \to \widehat{S}_{(f\mathbf{z})}$, and, by the above and (8.2.3.2), we can claim the following result:

(8.3.2.6). If $Y = \operatorname{Spec}(A)$ is affine, and $\mathscr{S} = \widetilde{S}$, then, for every homogeneous f in S_+, $\widehat{C}_f$ is canonically identified with $\operatorname{Spec}(S_f^{\leqslant})$, and the morphism $C_f \to \widehat{C}_f$ given by restricting i then corresponds to the canonical injection $S_f^{\leqslant} \to S_f$.

Now note that (for Y affine) the complement of $\widehat{C}_{\mathbf{z}}$ in $\widehat{C} = \operatorname{Proj}(\widehat{S})$ is, by definition, the set of graded prime ideals of $\widehat{S}$ containing $\mathbf{z}$, which is exactly $j(X)$, by definition of j, which proves (8.3.2.3). **II** | 164

Finally, to prove the last claim of (8.3.2), we can assume that Y is affine. With the above notation, note that, in the ring $\widehat{S}$, $\mathbf{z}$ is not a zero divisor; since $i(C) = \widehat{C}$, it suffices to prove the following lemma:

Lemma (8.3.2.7). — *Let T be a positively-graded ring, $Z = \operatorname{Proj}(T)$, and g a homogeneous element of T of degree $d > 0$. If g is not a zero divisor in T, then Z is the smallest closed subprescheme of Z that contains $Z_g = D_+(g)$.*

By (I, 4.1.9), the question is local on Z; for every homogeneous element $h \in T_e$ ($e > 0$), it thus suffices to prove that Z_h is the smallest closed subprescheme of Z_h that contains Z_{gh}; it follows from the definitions and from (I, 4.3.2) that this condition is equivalent to asking for the canonical homomorphism $T_{(h)} \to T_{(gh)}$ to be *injective*. But this homomorphism can be identified with the canonical homomorphism $T_{(h)} \to (T_{(h)})_{g^e/h^d}$ (2.2.3). But since g^e is not a zero divisor in T, g^e/h^d is not a zero divisor in T_h (nor *a fortiori* in $T_{(h)}$), since the fact that $(g^e/h^d)(t/h^m) = 0$ (for $t \in T$ and $m > 0$) implies the existence of some $n > 0$ such that $h^n g^e t = 0$, whence $h^n t = 0$, and thus $t/h^m = 0$ in T_h. This thus finishes the proof (0, 1.2.2). □

(8.3.3). We will often identify the affine cone C with the subprescheme induced by the projective cone $\widehat{C}$ on the open subset $i(C)$ by means of the open immersion i. The closed subprescheme of C associated to the closed immersion ε is called the *vertex prescheme* (?) of C; we also say that ε, which is a Y-section of C, is the *vertex section* (?), or the *null section*, or C; we can identify Y with the vertex prescheme (?) of C by means of ε. Also, $i \circ \varepsilon$ is a Y-section of $\widehat{C}$, and thus also a closed immersion (I, 5.4.6), corresponding to the canonical surjective homomorphism of degree zero $\widehat{\mathscr{S}} = \mathscr{S}[\mathbf{z}] \to \mathscr{O}_Y[\mathbf{z}]$ (3.1.7), whose kernel is $\mathscr{S}_+[\mathbf{z}] = \mathscr{S}_+\widehat{\mathscr{S}}$; the subprescheme of $\widehat{C}$ associated to this closed immersion is also called the *vertex prescheme* (?) of $\widehat{C}$, and $i \circ \varepsilon$ the *vertex section* (?) of $\widehat{C}$; it can be identified with Y by means of $i \circ \varepsilon$. Finally, the closed subprescheme of $\widehat{C}$ associated to j is called the *part at infinity* of $\widehat{C}$, and can be identified with X by means of j.

(8.3.4). The subpreschemes of C (resp. $\widehat{C}$) induced on the *open* subsets

$$E = C - \varepsilon(Y), \qquad \widehat{E} = \widehat{C} - i(\varepsilon(Y)) \tag{8.3.4.1}$$

are called (by an abuse of language) the *pointed affine cone* and the *pointed projective cone* (respectively) defined by $\mathscr{S}$; we note that, despite this nomenclature, *E is not necessarily affine* over Y, nor $\widehat{E}$ projective over Y (8.4.3). When we identify C with $i(C)$, we thus have the underlying spaces

$$C \cup \widehat{E} = \widehat{C}, \qquad C \cap \widehat{E} = E \tag{8.3.4.2}$$

so that $\widehat{C}$ can be considered as being obtained by *gluing* the open subpreschemes C and $\widehat{E}$; furthermore, by (8.3.2.3),

$$E = \widehat{E} - j(X). \tag{8.3.4.3}$$

If $Y = \operatorname{Spec}(A)$ is affine, then, with the notation of (8.3.2),

$$E = \bigcup C_f, \qquad \widehat{E} = \bigcup \widehat{C}_f, \qquad C_f = C \cap \widehat{C}_f \tag{8.3.4.4}$$

where f runs over the set of homogeneous elements of S_+ (or only a subset M of this set, with M generating an ideal of S_+ whose radical in S_+ is S_+ itself, or, equivalently, such that the X_f for $f \in M$ cover X (2.3.14)). The gluing of C and $\widehat{C}_f$ along C_f is thus determined by the injection morphisms $C_f \to C$ and $C_f \to \widehat{C}_f$, which, as we have seen (8.3.2.6), correspond (respectively) to the canonical homomorphisms $S \to S_f$ and $S_f^{\leqslant} \to S_f$.

Proposition (8.3.5). — *With the notation of (8.3.1) and (8.3.4), the morphism associated (3.5.1) to the canonical injection $\phi : \mathscr{S} \to \widehat{\mathscr{S}} = \mathscr{S}[\mathbf{z}]$ is a surjective affine morphism (called the canonical retraction)*

$$p : \widehat{E} \longrightarrow X \tag{8.3.5.1}$$

such that

$$p \circ j = 1_X. \tag{8.3.5.2}$$

PROOF. To prove the proposition, we can restrict to the case where Y is affine. Taking into account the expression in (8.3.4.4) for $\widehat{E}$, the fact that the domain of definition $G(\phi)$ of p is equal to $\widehat{E}$ will follow from the first of the following claims:

(8.3.5.3). If $Y = \operatorname{Spec}(A)$ is affine, and $\mathscr{S} = \widetilde{S}$, then, for all homogeneous $f \in S_+$,

$$p^{-1}(X_f) = \widehat{C}_f \tag{8.3.5.4}$$

and the restriction of p to $\widehat{C}_f = \operatorname{Spec}(S_f^{\leqslant})$, considered as a morphism from $\widehat{C}_f$ to X_f, corresponds to the canonical injection $S_{(f)} \to S_f^{\leqslant}$. If, further, $f \in S_1$, then $\widehat{C}_f$ is isomorphic to $X_f \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ (where T is an indeterminate).

Indeed, Equation (8.3.5.4) is exactly a particular case of (2.8.1.1), and the second claim is exactly the definition of $\operatorname{Proj}(\phi)$ whenever Y is affine (2.8.1). Then Equation (8.3.5.2) and the fact that p is surjective show that the composition $\mathscr{S} \to \widehat{\mathscr{S}} \to \mathscr{S}$ of the canonical homomorphisms is the identity on $\mathscr{S}$. Finally, the last claim of (8.3.5.3) follows from the fact that $S_f^{\leqslant}$ is isomorphic to $S_{(f)}[T]$ whenever $f \in S_1$ (2.2.1). ☐

Corollary (8.3.6). — *The restriction*

$$\pi : E \longrightarrow X \tag{8.3.6.1}$$

of p to E is a surjective affine morphism. If Y is affine and f homogeneous in S_+, then

$$\pi^{-1}(X_f) = C_f \tag{8.3.6.2}$$

and the restriction of π to C_f corresponds to the canonical injection $S_{(f)} \to S_f$. If, further, $f \in S_1$, then C_f is isomorphic to $X_f \otimes_{\mathbf{Z}} \mathbf{Z}[T, T^{-1}]$ (where T is an indeterminate).

PROOF. Equation (8.3.6.2) follows immediately from (8.3.5.3) and (8.3.2.5), and shows the surjectivity of π; we have already seen that the immersion i, restricted to C_f, corresponds to the II | 166
injection $S_f^{\leqslant} \to S_f$ (8.3.2). Finally, the last claim is a consequence of the fact that, for $f \in S_1$, S_f is isomorphic to $S_{(f)}[T, T^{-1}]$ (2.2.1). ☐

Remark (8.3.7). — Whenever Y is affine, the elements of the underlying space of E are the (not-necessarily-graded) prime ideals $\mathfrak{p}$ of S not containing S_+, by definition of the immersion ε (8.3.2). For such an ideal $\mathfrak{p}$, the $\mathfrak{p} \cap S_n$ clearly satisfy the conditions of (2.1.9), and so there exists exactly one *graded* prime ideal $\mathfrak{q}$ of S such that $\mathfrak{q} \cap S_n = \mathfrak{p} \cap S_n$ for all n; the map $\pi : E \to X$ of underlying spaces can then be understood via the equation

$$\pi(\mathfrak{p}) = \mathfrak{q}. \tag{8.3.7.1}$$

Indeed, to prove this equation, it suffices to consider some homogeneous f in S_+ such that $\mathfrak{p} \in D(f)$, and to note that $\mathfrak{q}_{(f)}$ is the inverse image of $\mathfrak{p}_f$ under the injection $S_{(f)} \to S_f$.

Corollary (8.3.8). — *If $\mathscr{S}$ is generated by $\mathscr{S}_1$, then the morphisms p and π are of finite type; for all $x \in X$, the fibre $p^{-1}(x)$ is isomorphic to* $\operatorname{Spec}(k(x)[T])$*, and the fibre π^{-1} isomorphic to* $\operatorname{Spec}(k(x)[T, T^{-1}])$

PROOF. This follows immediately from (8.3.5) and (8.3.6) by noting that, whenever Y is affine and S is generated by S_1, the X_f, for $f \in S_1$, form a cover of X (2.3.14). □

Remark (8.3.9). — The pointed affine cone corresponding to the graded $\mathscr{O}_Y$-algebra $\mathscr{O}_Y[T]$ (where T is an indeterminate) can be identified with $G_m = \operatorname{Spec}(\mathscr{O}_Y[T, T^{-1}])$, since it is exactly C_T, as we have seen in (8.3.2) (see (8.4.4) for a more general result). This prescheme is canonical endowed with the structure of a "*Y-scheme in commutative groups*". This idea will be explained in detail later on, but, for now, can be quickly summarised as follows. A Y-scheme in groups is a Y-scheme G endowed with two Y-morphisms, $p : G \times_Y G \to G$ and $s : G \to G$, that satisfy conditions formally analogous to the axioms of the composition law and the symmetry law of a group: the diagram

$$\begin{tikzcd} G \times G \times G \arrow[r, "p\times 1"] \arrow[d, "1\times p"'] & G \times G \arrow[d, "p"] \\ G \times G \arrow[r, "p"'] & G \end{tikzcd}$$

should commute ("associativity"), and there should be a condition which corresponds to the fact that, for groups, the maps

$$(x, y) \longmapsto (x, x^{-1}, y) \longmapsto (x, x^{-1}y) \longmapsto x(x^{-1}y)$$

and

$$(x, y) \longmapsto (x, x^{-1}, y) \longmapsto (x, yx^{-1}) \longmapsto (yx^{-1})x$$

should both reduce to $(x, y) \mapsto y$; the sequence of morphisms corresponding, for example, to the first composite map is

$$G \times G \xrightarrow{(1,s)\times 1} G \times G \times G \xrightarrow{1\times p} G \times G \xrightarrow{p} G$$

and the reader should write down the second sequence.

It is immediate (**I**, 3.4.3) that the data of a structure of a Y-scheme in groups on a Y-scheme G is equivalent to the data, for *every* Y-prescheme Z, of a *group* structure on the set $\operatorname{Hom}_Y(Z, G)$, where these structures should be such that, for every Y-morphism $Z \to Z'$, the corresponding map $\operatorname{Hom}_Y(Z', G) \to \operatorname{Hom}_Y(Z, G)$ is a group homomorphism. In the particular case of G_m that we consider here, $\operatorname{Hom}_Y(Z, G)$ can be identified with the set of Z-sections of $Z \times_Y G_m$ (**I**, 3.3.14), and thus with the set of Z-sections of $\operatorname{Spec}(\mathscr{O}_Z[T, T^{-1}])$; finally, the same reasoning as in (**I**, 3.3.15) shows that this set is canonically identified with the set of *invertible* elements of the ring $\Gamma(Z, \mathscr{O}_Z)$, and the group structure on this set is the structure coming from the multiplication in the ring $\Gamma(Z, \mathscr{O}_Z)$. The reader can verify that the morphisms p and s from above are obtained in the following way: they correspond, by (1.2.7) and (1.4.6), to the homomorphisms of $\mathscr{O}_Y$-algebras **II | 167**

$$\pi : \mathscr{O}_Y[T, T^{-1}] \longrightarrow \mathscr{O}_Y[T, T^{-1}, T', T'^{-1}]$$
$$\sigma : \mathscr{O}_Y[T, T^{-1}] \longrightarrow \mathscr{O}_Y[T, T^{-1}]$$

and are entirely defined by the data of $\pi(T) = TT'$ and $\sigma(T) = T^{-1}$.

With this in mind, G_m can be considered as a "*universal domain of operators*" for every *affine cone* $C = \operatorname{Spec}(\mathscr{S})$, where $\mathscr{S}$ is a quasi-coherent positively-graded $\mathscr{O}_Y$-algebra. This means that we can canonically define a Y-morphism $G_m \times_Y C \to C$ which has the formal properties of an external law of a set endowed with a group of operators; or, again, as above for schemes in groups, we can give, for every Y-prescheme Z, an external law on $\operatorname{Hom}_Y(Z, C)$, having the group $\operatorname{Hom}_Y(Z, G_m)$ as its set of operators, with the usual axioms of sets endowed with a group of operators, and a compatibility condition with respect to the Y-morphisms $Z \to Z'$. In the current case, the morphism $G_m \times_Y C \to C$ is defined by the data of a homomorphism of $\mathscr{O}_Y$-algebras $\mathscr{S} \to \mathscr{S} \otimes_{\mathscr{O}_Y} \mathscr{O}_Y[T, T^{-1}] = \mathscr{S}[T, T^{-1}]$, which associates, to each section $s_n \in \Gamma(U, \mathscr{S}_n)$ (where U is an open subset of Y), the section $s_n T^n \in \Gamma(U, \mathscr{S} \otimes_{\mathscr{O}_Y} \mathscr{O}_Y[T, T^{-1}])$.

Conversely, suppose that we are given a quasi-coherent, *a priori non-graded*, $\mathscr{O}_Y$-algebra, and, on $C = \operatorname{Spec}(\mathscr{S})$, a structure of a "*$Y$-scheme in sets endowed with a group of operators*" that has the Y-scheme in groups G_m as its domain of operators; then we canonically obtain a *grading* of $\mathscr{O}_Y$-algebras on $\mathscr{S}$. Indeed, the data of a Y-morphism $G_m \times_Y C \to C$ is equivalent to that of a homomorphism of $\mathscr{O}_Y$-algebras $\psi : \mathscr{S} \to \mathscr{S}[T, T^{-1}]$, which can be written as $\psi = \sum_{n \in \mathbf{Z}} \psi_n T^n$, where the $\psi_n : \mathscr{S} \to \mathscr{S}$ are homomorphisms of $\mathscr{O}_Y$-modules (with $\psi_n(s) = 0$ except for finitely many n for every section $s \in \Gamma(U, \mathscr{S})$, for any open subset U of Y). We can then prove that the axioms of sets endowed with a group of operators imply that the $\psi_n(\mathscr{S}) = \mathscr{S}_n$ define a grading (in positive or negative degree) of $\mathscr{O}_Y$-algebras on $\mathscr{S}$, with the ψ_n being the corresponding projectors. We also have the notation of a structure of an "*affine cone*" on every affine Y-scheme, defined in a "geometric" way without any reference to any prior grading. We will not further develop this point of view here, and we leave the work of precisely formulating the definitions and results corresponding to the information given above to the reader. **II** | 168

8.4. Projective closure of a vector bundle

(8.4.1). Let Y be a prescheme, and $\mathscr{E}$ a quasi-coherent $\mathscr{O}_Y$-module. If we take $\mathscr{S}$ to be the graded $\mathscr{O}_Y$-algebra $\mathbf{S}_{\mathscr{O}_Y}(\mathscr{E})$, then Definition (8.3.1.1) shows that $\widehat{\mathscr{S}}$ can be identified with $\mathbf{S}_{\mathscr{O}_Y}(\mathscr{E} \oplus \mathscr{O}_Y)$. With the affine cone $\operatorname{Spec}(\mathscr{S})$ defined by $\mathscr{S}$ being, by definition, $\mathbf{V}(\mathscr{E})$, and $\operatorname{Proj}(\mathscr{S})$ being, by definition, $\mathbf{P}(\mathscr{E})$, we see that:

Proposition (8.4.2). — *The projective closure of a vector bundle $\mathbf{V}(\mathscr{E})$ on Y is canonically isomorphic to $\mathbf{P}(\mathscr{E} \oplus \mathscr{O}_Y)$, and the part at infinity of the latter is canonically isomorphic to $\mathbf{P}(\mathscr{E})$.*

Remark (8.4.3). — Take, for example, $\mathscr{E} = \mathscr{O}_Y^r$ with $r \geqslant 2$; then the pointed cones E and $\widehat{E}$ defined by $\mathscr{S}$ are nether affine nor projective on Y if $Y \neq \varnothing$. The second claim is immediate, because $\widehat{C} = \mathbf{P}(\mathscr{O}_Y^{r+1})$ is projective on Y, and the underlying spaces of E and $\widehat{E}$ are non-closed open subsets of $\widehat{C}$, and so the canonical immersions $E \to \widehat{C}$ and $\widehat{E} \to \widehat{C}$ are not projective (5.5.3), and we conclude by appealing to (5.5.5, v). Now, supposing, for example, that $Y = \operatorname{Spec}(A)$ is affine, and $r = 2$, then $C = \operatorname{Spec}(A[T_1, T_2])$, and E is then the prescheme induced by C on the open subset $D(T_1) \cup D(T_2)$; but we have already seen that the latter is not affine (**I**, 5.5.11); *a fortiori* $\widehat{E}$ cannot be affine, since E is the open subset where the section $\mathbf{z}$ over $\widehat{E}$ does not vanish (8.3.2).

However:

Proposition (8.4.4). — *If $\mathscr{L}$ is an invertible $\mathscr{O}_Y$-module, then there are canonical isomorphisms for both the pointed cones E and $\widehat{E}$ corresponding to $C = \mathbf{V}(\mathscr{L})$:*

$$\operatorname{Spec}\left(\bigoplus_{n \in \mathbf{Z}} \mathscr{L}^{\otimes n}\right) \xrightarrow{\sim} E \tag{8.4.4.1}$$

$$\mathbf{V}(\mathscr{L}^{-1}) \xrightarrow{\sim} \widehat{E}. \tag{8.4.4.2}$$

Furthermore, there exists a canonical isomorphism from the projective closure of $\mathbf{V}(\mathscr{L})$ to the projective closure of $\mathbf{V}(\mathscr{L}^{-1})$ that sends the null section (resp. the part at infinity) of the former to the part at infinity (resp. the null section) of the second.

PROOF. We have here that $\mathscr{S} = \bigoplus_{n \geqslant 0} \mathscr{L}^{\otimes n}$; the canonical injection

$$\mathscr{S} \longrightarrow \bigoplus_{n \in \mathbf{Z}} \mathscr{L}^{\otimes n}$$

defines a canonical dominant morphism

$$\operatorname{Spec}\left(\bigoplus_{n \in \mathbf{Z}} \mathscr{L}^{\otimes n}\right) \longrightarrow \mathbf{V}(\mathscr{L}) = \operatorname{Spec}\left(\bigoplus_{n \geqslant 0} \mathscr{L}^{\otimes n}\right) \tag{8.4.4.3}$$

and it suffices to prove that this morphism is an isomorphism from the scheme $\operatorname{Spec}(\bigoplus_{n \in \mathbf{Z}} \mathscr{L}^{\otimes n})$ to E. Since the questions is local on Y, we can assume that $Y = \operatorname{Spec}(A)$ is affine and that $\mathscr{L} = \mathscr{O}_Y$, **II** | 169 and so $\mathscr{S} = (A[T])^\sim$ and $\bigoplus_{n\in\mathbf{Z}} \mathscr{L}^{\times n} = (A[T,T^{-1}])^\sim$. But $A[T,T^{-1}]$ is the ring of fractions $A[T]_T$ of $A[T]$, and thus (8.4.4.3) identifies $\bigoplus_{n\in\mathbf{Z}} \mathscr{L}^{\otimes n}$ (?) with the prescheme induced by $C = \mathbf{V}(\mathscr{L})$ on the open subset $D(T)$; the complement $V(T)$ of this open subset in C is the underlying space of the closed subprescheme of C defined by the ideal $TA[T]$, which is exactly the null section of C, and so $E = D(T)$.

The isomorphism in (8.4.4.2) will be a consequence of the last claim, since $\mathbf{V}(\mathscr{L}^{-1})$ is the complement of the part at infinity of its projective closure, and $\widehat{E}$ is the complement of the null section of the projective closure $C = \mathbf{V}(\mathscr{L})$. But these projective closures are $\mathbf{P}(\mathscr{L}^{-1} \oplus \mathscr{O}_Y)$ and $\mathbf{P}(\mathscr{L} \oplus \mathscr{O}_Y)$ (respectively); but we can write $\mathscr{L} \oplus \mathscr{O}_Y = \mathscr{L} \otimes (\mathscr{L}^{-1} \oplus \mathscr{O}_Y)$. The existence of the desired canonical isomorphism then follows from (4.1.4), and everything reduces to showing that this isomorphism swaps the null sections and the parts at infinity. For this, we can reduce to the case where $Y = \operatorname{Spec}(A)$ is affine, $L = Ac$, and $L^{-1} = Ac'$, with the canonical isomorphism $L \otimes L^{-1} \to A$ sending $c \otimes c'$ to the element 1 of A. Then $\mathbf{S}(L \oplus A)$ is the tensor product of $A[\mathbf{z}]$ with $\bigoplus_{n\geqslant 0} Ac^{\otimes n}$, and $\mathbf{S}(L^{-1} \oplus A)$ is the tensor product of $A[\mathbf{z}]$ with $\bigoplus_{n\geqslant 0} Ac'^{\otimes n}$, and the isomorphism defined in (4.1.4) sends $\mathbf{z}^h \otimes c'^{\otimes(n-h)}$ to the element $\mathbf{z}^{n-h} \otimes c^{\otimes h}$. But, in $\mathbf{P}(\mathscr{L}^{-1} \oplus \mathscr{O}_Y)$, the part at infinity is the set of points where the section $\mathbf{z}$ vanishes, and the null section is the section of points where the section c' vanishes; since we have analogous definitions for $\mathbf{P}(\mathscr{L} \oplus \mathscr{O}_Y)$, the conclusion follows immediately from the above explanation. $\square$

8.5. Functorial behaviour

(8.5.1). Let Y and Y' be prescheme, $q : Y' \to Y$ a morphism, and $\mathscr{S}$ (resp. $\mathscr{S}'$) a quasi-coherent *positively*-graded $\mathscr{O}_Y$-algebra (resp. quasi-coherent *positively*-graded $\mathscr{O}_{Y'}$-algebra). Consider a q-morphism of graded algebras

(8.5.1.1) $$\phi : \mathscr{S} \longrightarrow \mathscr{S}'.$$

We know (1.5.6) that this corresponds, canonically, to a morphism

$$\Phi = \operatorname{Spec}(\phi) : \operatorname{Spec}(\mathscr{S}') \longrightarrow \operatorname{Spec}(\mathscr{S})$$

such that the diagram

(8.5.1.2) $$\begin{tikzcd} C' \arrow[r, "\Phi"] \arrow[d] & C \arrow[d] \\ Y' \arrow[r, "q"'] & Y \end{tikzcd}$$

commutes, where we write $C = \operatorname{Spec}(\mathscr{S})$ and $C' = \operatorname{Spec}(\mathscr{S}')$. *Suppose, further, that* $\mathscr{S}_0 = \mathscr{O}_Y$ *and* $\mathscr{S}'_0 = \mathscr{O}_{Y'}$; let $\varepsilon : Y \to C$ and $\varepsilon : Y' \to C'$ be the canonical immersions (8.3.2); we then have a commutative diagram

(8.5.1.3) $$\begin{tikzcd} Y' \arrow[r, "q"] \arrow[d, "\varepsilon'"'] & Y \arrow[d, "\varepsilon"] \\ C' \arrow[r, "\Phi"'] & C \end{tikzcd}$$

which corresponds to the diagram **II | 170**

$$\begin{tikzcd} \mathscr{S} \arrow[r, "\phi"] \arrow[d] & \mathscr{S}' \arrow[d] \\ \mathscr{O}_Y \arrow[r] & \mathscr{O}_{Y'} \end{tikzcd}$$

where the vertical arrows are the augmentation homomorphisms, and so the commutativity follows from the hypothesis that ϕ is assumed to be a homomorphism of *graded* algebras.

Proposition (8.5.2). — *If E (resp. E') is the pointed affine cone defined by $\mathscr{S}$ (resp. $\mathscr{S}'$), then $\Phi^{-1}(E) \subset E'$; if, further,* $\operatorname{Proj}(\phi) : G(\phi) \to \operatorname{Proj}(\mathscr{S})$ *is everywhere defined (or, equivalently, if $G(\phi) = \operatorname{Proj}(\mathscr{S}')$), then $\Phi^{-1}(E) = E'$, and conversely.*

PROOF. The first claim follows from the commutativity of (8.5.1.3). To prove the second, we can restrict to the case where $Y = \operatorname{Spec}(A)$ and $Y' = \operatorname{Spec}(A')$ are affine, and $\mathscr{S} = \widetilde{S}$ and $\mathscr{S}' = \widetilde{S}'$. For every homogeneous f in S_+, writing $f' = \phi(f)$, we have that $\Phi^{-1}(C_f) = C'_{f'}$ (I, 2.2.4.1); saying that $G(\phi) = \operatorname{Proj}(S')$ implies that the radical (in S'_+) of the ideal generated by the $f' = \phi(f)$ is S'_+ itself ((2.8.1) and (2.3.14)), and this is equivalent to saying that the $C'_{f'}$ cover E' (8.3.4.4). □

(8.5.3). The q-morphism ϕ canonically extends to a q-morphism of graded algebras

(8.5.3.1) $$\widehat{\phi} : \widehat{\mathscr{S}} \longrightarrow \widehat{\mathscr{S}'}$$

by letting $\widehat{\phi}(\mathbf{z}) = \mathbf{z}$. This induces a morphism

$$\widehat{\Phi} = \operatorname{Proj}(\widehat{\phi}) : G(\widehat{\phi}) \longrightarrow \widehat{C} = \operatorname{Proj}(\widehat{\mathscr{S}})$$

such that the diagram

$$\begin{tikzcd} G(\widehat{\phi}) \arrow[r, "\widehat{\Phi}"] \arrow[d] & \widehat{C} \arrow[d] \\ Y' \arrow[r, "q"'] & Y \end{tikzcd}$$

commutes (3.5.6). It follows immediately from the definitions that, if we write $i : C \to \widehat{C}$ and $i' : C' \to \widehat{C}'$ to mean the canonical open immersions (8.3.2), then $i'(C') \subset G(\widehat{\phi})$, and the diagram

(8.5.3.2) $$\begin{tikzcd} C' \arrow[r, "\Phi"] \arrow[d, "i"'] & C \arrow[d, "i'"] \\ G(\widehat{\phi}) \arrow[r, "\widehat{\Phi}"'] & \widehat{C} \end{tikzcd}$$

commutes. Finally, if we let $X = \operatorname{Proj}(\mathscr{S})$ and $X' = \operatorname{Proj}(\mathscr{S}')$, and if $j : X \to \widehat{C}$ and $j' : X' \to \widehat{C}'$ are the canonical closed immersions (8.3.2), then it follows from the definition of these immersions that $j'(G(\phi)) \subset G(\widehat{\phi})$, and that the diagram II | 171

(8.5.3.3) $$\begin{tikzcd} G(\phi) \arrow[r, "\operatorname{Proj}(\phi)"] \arrow[d, "j'"'] & X \arrow[d, "j"] \\ G(\widehat{\phi}) \arrow[r, "\widehat{\Phi}"'] & \widehat{C} \end{tikzcd}$$

commutes.

Proposition (8.5.4). — *If $\widehat{E}$ (resp. $\widehat{E}'$) is the pointed projective cone defined by $\mathscr{S}$ (resp. by $\mathscr{S}'$), then $\widehat{\Phi}^{-1}(\widehat{E}) \subset \widehat{E}'$; furthermore, if $p : \widehat{E} \to X$ and $p' : \widehat{E}' \to X'$ are the canonical retractions, then $p'(\widehat{\Phi}^{-1}(\widehat{E})) \subset G(\widehat{\phi})$, and the diagram*

(8.5.4.1) $$\begin{tikzcd} \widehat{\Phi}^{-1}(\widehat{E}) \arrow[r, "\widehat{\Phi}"] \arrow[d, "p'"'] & \widehat{E} \arrow[d, "p"] \\ G(\phi) \arrow[r, "\operatorname{Proj}(\phi)"'] & X \end{tikzcd}$$

commutes. If $\operatorname{Proj}(\phi)$ *is everywhere defined, then so too is $\widehat{\Phi}$, and we have that $\widehat{\Phi}^{-1}(\widehat{E}) = \widehat{E}'$*

PROOF. The first claim follows from the commutativity of Diagrams (8.5.1.3) and (8.5.3.2), and the two following claims from the definition of the canonical retractions (8.3.5) and the definition of $\widehat{\phi}$. To see that $\widehat{\Phi}$ is everywhere defined whenever $\operatorname{Proj}(\phi)$ is, we can restrict to the case where $Y = \operatorname{Spec}(A)$ and $Y' = \operatorname{Spec}(A')$ are affine, and where $\mathscr{S} = \widetilde{S}$ and $\mathscr{S}' = \widetilde{S}'$; the hypothesis is that, when f runs over the set of homogeneous elements of S_+, the radical in S'_+ of the ideal generated in S'_+ by the $\phi(f)$ is S'_+ itself; we thus immediately conclude that the radical in $(S'[\mathbf{z}])_+$ of the ideal generated by $\mathbf{z}$ and the $\phi(f)$ is $(S'[\mathbf{z}])_+$ itself, whence our claim; this also shows that $\widehat{E}'$ is the union of the $\widehat{C}'_{\phi(f)}$, and hence equal to $\widehat{\Phi}^{-1}(\widehat{E})$. □

Corollary (8.5.5). — *Whenever* $\operatorname{Proj}(\phi)$ *is everywhere defined, the inverse image under* $\widehat{\Phi}$ *of the underlying space of the part at infinity (resp. of the vertex prescheme) of* $\widehat{C}'$ *is the underlying space of the part at infinity (resp. of the vertex prescheme) of* $\widehat{C}$*.*

Proof. This follows immediately from (8.5.4) and (8.5.2), taking into account the equalities (8.3.4.1) and (8.3.4.2). □

8.6. A canonical isomorphism for pointed cones

(8.6.1). Let Y be a prescheme, $\mathscr{S}$ a quasi-coherent positively-graded $\mathscr{O}_Y$-algebra *such that* $\mathscr{S}_0 = \mathscr{O}_Y$, and let X be the Y-scheme $\operatorname{Proj}(\mathscr{S})$. We are going to apply the results of (8.5) to the case where $Y' = X$, and $q : X \to Y$ is the structure morphism; let

(8.6.1.1) $$\mathscr{S}_X = \bigoplus_{n\in\mathbf{Z}} \mathscr{O}_X(n)$$

which is a quasi-coherent graded $\mathscr{O}_X$-algebra, with multiplication defined by means of the canonical homomorphisms (3.2.6.1) **II | 172**

$$\mathscr{O}_X(m) \otimes_{\mathscr{O}_X} \mathscr{O}_X(n) \longrightarrow \mathscr{O}_X(m+n)$$

whose associativity is ensured by the commutative diagram in (2.5.11.4). Let $\mathscr{S}'$ be the quasi-coherent positively-graded $\mathscr{O}_X$-subalgebra $\mathscr{S}_X^{\geqslant} = \bigoplus_{n\geqslant 0} \mathscr{O}_X(n)$ of $\mathscr{S}_X$.

Finally, consider the canonical q-morphism

(8.6.1.2) $$\alpha : \mathscr{S} \longrightarrow \mathscr{S}_X^{\geqslant}$$

defined in (3.3.2.3) as a homomorphism $\mathscr{S} \to q_*(\mathscr{S}_X)$, but which clearly sends $\mathscr{S}$ to $q_*(\mathscr{S}_X^{\geqslant})$. Write

(8.6.1.3) $$C_X = \operatorname{Spec}(\mathscr{S}_X^{\geqslant}), \quad \widehat{C}_X = \operatorname{Proj}(\mathscr{S}_X^{\geqslant}[\mathbf{z}]), \quad X' = \operatorname{Proj}(\mathscr{S}_X^{\geqslant})$$

and denote by E_X and $\widehat{E}_X$ the corresponding pointed affine and pointed projective cones (respectively); denote the canonical morphisms defined in (8.3) by $\varepsilon_X : X \to C_X$, $i_X : C \to \widehat{C}_X$, $j_X : X' \to \widehat{C}_X$, $p_X : \widehat{E}_X \to X'$, and $\pi_X : E_X \to X'$.

Proposition (8.6.2). — *The structure morphism* $u : X' \to X$ *is an* isomorphism, *and the morphism* $\operatorname{Proj}(\alpha)$ *is everywhere defined and identical to* u*. The morphism* $\operatorname{Proj}(\widehat{\alpha}) : \widehat{C}_X \to \widehat{C}$ *is everywhere defined, and its restrictions to* $\widehat{E}_X$ *and* E_X *are* isomorphisms *to* $\widehat{E}$ *and* E *(respectively). Finally, if we identify* X' *with* X *via* u*, then the morphisms* p_X *and* π_X *are identified with the structure morphisms of the* X*-preschemes* $\widehat{E}_X$ *and* E_X*.*

Proof. We can clearly restrict to the case where $Y = \operatorname{Spec}(A)$ is affine, and $\mathscr{S} = \widetilde{S}$; then X is the union of affine open subsets X_f, where f runs over the set of homogeneous elements of S_+, with the ring of each X_f being $S_{(f)}$. It follows from (8.2.7.1) that

(8.6.2.1) $$\Gamma(X_f, \mathscr{S}_X^{\geqslant}) = S_f^{\geqslant}.$$

So $u^{-1}(X_f) = \operatorname{Proj}(S_f^{\geqslant})$. But if $f \in S_d$ ($d > 0$), then $\operatorname{Proj}(S_f^{\geqslant})$ is canonically isomorphic to $\operatorname{Proj}((S_f^{\geqslant})^{(d)})$ (2.4.7), and we also know that $(S_f^{\geqslant})^{(d)} = (S^{(d)})_f^{\geqslant}$ can be identified with $S_{(f)}[T]$ (2.2.1) by the map $T \mapsto f/1$; we thus conclude (3.1.7) that the structure morphism $u^{-1}(X_f) \to X_f$ is an isomorphism, whence the first claim. To prove the second, note that the restriction $u^{-1}(X_f) \cap G(\alpha) \to X = \operatorname{Proj}(S)$ of $\operatorname{Proj}(\alpha)$ corresponds to the canonical map $x \mapsto x/1$ from S to $S_f^{\geqslant}$ (2.6.2); we thus deduce, first of all, that $G(\alpha) = X'$, and then, taking into account the fact that $u^{-1}(X_f) = (u^{-1}(X_f))_{f/1}$, that it follows from (2.8.1.1) that the image of $u^{-1}(X_f)$ under $\operatorname{Proj}(\alpha)$ is contained in X_f, and the restriction of $\operatorname{Proj}(\alpha)$ to $u^{-1}(X_f)$, considered as a morphism to $X_f = \operatorname{Spec}(S_{(f)})$, is indeed identical to that of u. Finally, applying (8.3.5.4) to p_X instead of p, we see that $p_X^{-1}(u^{-1}(X_f)) = \operatorname{Spec}((S_f^{\geqslant})_{f/1}^{\leqslant})$, and this open subset is, by (8.5.4.1), the inverse image under $\operatorname{Proj}(\widehat{\alpha})$ of $p^{-1}(X_f) = \operatorname{Spec}(S_f^{\leqslant})$ (8.3.5.3). Taking (8.2.3.2) into account, the restriction of $\operatorname{Proj}(\widehat{\alpha})$ to

$p_X^{-1}(u^{-1}(X_f))$ corresponds to the isomorphism inverse to (8.2.7.2), restricted to $S_f^{\leqslant}$, whence the third claim; the last claim is evident by definition.

We note also that it follows from the commutative diagram in (8.5.3.2) that *the restriction to* C_X *of* $\mathrm{Proj}(\widehat{\alpha})$ *is exactly the morphism* $\mathrm{Spec}(\alpha)$. □ II | 173

Corollary (8.6.3). — *Considered as X-schemes, $\widehat{E}_X$ is canonically isomorphic to* $\mathrm{Spec}(\mathscr{S}_X^{\leqslant})$*, and E_X to* $\mathrm{Spec}(\mathscr{S}_X)$*.*

PROOF. Since we know that the morphisms p_X and π_X are affine ((8.3.5) and (8.3.6)), it suffices (given (1.3.1)) to prove the corollary in the case where $Y = \mathrm{Spec}(A)$ is affine and $\mathscr{S} = \widetilde{S}$. The first claim follows from the existence of the canonical isomorphisms (8.2.7.2) $(S_f^{\geqslant})_{f/1}^{\leqslant} \xrightarrow{\sim} S_f^{\leqslant}$ and from the fact that these isomorphisms are compatible with the map sending f to fg (where f and g are homogeneous in S_+). Similarly, applying (8.3.6.2) to π_X instead of π, we see that $\pi_X^{-1}(u^{-1}(X_f)) = \mathrm{Spec}((S_f^{\geqslant})_{f/1})$ for f homogeneous in S_+, and the second claim then follows from the existence of the canonical isomorphisms (8.2.7.2) $(S_f^{\geqslant})_{f/1} \xrightarrow{\sim} S_f$.

We can then say that $\widehat{C}_X$, considered as an X-scheme, is given by *gluing* the affine X-schemes $C_X = \mathrm{Spec}(\mathscr{S}_X^{\geqslant})$ and $\widehat{E}_X = \mathrm{Spec}(\mathscr{S}_X^{\leqslant})$ over X, where the intersection of the two affine X-schemes is the open subset $E_X = \mathrm{Spec}(\mathscr{S}_X)$. □

Corollary (8.6.4). — *Assume that $\mathscr{O}_X(1)$ is an invertible $\mathscr{O}_X$-module, and that $\mathscr{S}_X$ is isomorphic to $\bigoplus_{n\in\mathbf{Z}}(\mathscr{O}_X(1))^{\otimes n}$ (which will be the case, in particular, whenever $\mathscr{S}$ is generated by $\mathscr{S}_1$ ((3.2.5) and (3.2.7))). Then the pointed projective cone $\widehat{E}$ can be identified with the rank-1 vector bundle $\mathbf{V}(\mathscr{O}_X(-1))$ on X, and the pointed affine cone E with the subprescheme of this vector bundle induced on the complement of the null section. With this identification, the canonical retraction $\widehat{E} \to X$ is identified with the structure morphism of the X-scheme $\mathbf{V}(\mathscr{O}_X(-1))$. Finally, there exists a canonical Y-morphism $\mathbf{V}(\mathscr{O}_X(1)) \to C$, whose restriction to the complement of the null section of $\mathbf{V}(\mathscr{O}_X(1))$ is an isomorphism from this complement to the pointed affine cone E.*

PROOF. If we write $\mathscr{L} = \mathscr{O}_X(1)$, then $\mathscr{S}_X^{\geqslant}$ is identical to $\mathbf{S}_{\mathscr{O}_X}(\mathscr{L})$, and so $\widehat{E}_X$ is canonically identified with $\mathbf{V}(\mathscr{L}^{-1})$, by (8.6.3), and C_X with $\mathbf{V}(\mathscr{L})$. The morphism $\mathbf{V}(\mathscr{L}) \to C$ is the restriction of $\mathrm{Proj}(\widehat{\alpha})$, and the claims of the corollary are then particular cases of (8.6.2). □

We note that the inverse image under the morphism $\mathbf{V}(\mathscr{O}_X(1)) \to C$ of the underlying space of the vertex prescheme of C is the underlying space of the null section of $\mathbf{V}(\mathscr{O}_X(1))$ (8.5.5); but, in general, the corresponding subpreschemes of C and of $\mathbf{V}(\mathscr{O}_X(1))$ are not isomorphic. This problem will be studied below.

8.7. Blowing up based cones

(8.7.1). Under the conditions of (8.6.1), we have, writing $r = \mathrm{Proj}(\widehat{\alpha})$, a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i_X\circ\varepsilon_X"] \arrow[d, "q"'] & \widehat{C}_X \arrow[d, "r"] \\ Y \arrow[r, "i\circ\varepsilon"] & \widehat{C} \end{tikzcd} \tag{8.7.1.1}$$

by (8.5.1.3) and (8.5.3.2); furthermore, the restriction of r to the complement $\widehat{C}_X - i_X(\varepsilon_X(X))$ of the null section is an *isomorphism* to the complement $\widehat{C} - i(\varepsilon(Y))$ of the null section, by (8.6.2). If we suppose, to simplify things, that Y is affine, that $\mathscr{S}$ is of finite type and generated by $\mathscr{S}_1$, and that X is projective over Y and $\widehat{C}_X$ projective over X (5.5.1), then $\widehat{C}_X$ is projective over Y (5.5.5, ii), and *a fortiori* over $\widehat{C}$ (5.5.5, v). We then have a projective Y-morphism $r : \widehat{C}_X \to \widehat{C}$ (whose restriction to C_X is a projective Y-morphism $C_X \to C$) that *contracts X to Y* (?) and that induces an *isomorphism* when we restrict to the *complements of X and Y*. We thus have a connection between C_X and C, analogous to that which exists between a blow-up prescheme and the original prescheme (8.1.3). We will effectively show that C_X can be identified with the homogeneous spectrum of a graded $\mathscr{O}_C$-algebra. II | 174

(8.7.2). Keeping the notation of (8.6.1), consider, for all $n \geqslant 0$, the quasi-coherent ideal

$$\mathscr{S}_{[n]} = \bigoplus_{m \geqslant n} \mathscr{S}_m \tag{8.7.2.1}$$

of the graded $\mathscr{O}_Y$-algebra $\mathscr{S}$. It is clear that

$$\mathscr{S}_{[0]} = \mathscr{S}, \qquad \mathscr{S}_{[n]} \subset \mathscr{S}_{[m]} \qquad \text{for } m \leqslant n \tag{8.7.2.2}$$

$$\mathscr{S}_n \mathscr{S}_{[m]} \subset \mathscr{S}_{[m+n]}. \tag{8.7.2.3}$$

Consider the $\mathscr{O}_C$-module associated to $\mathscr{S}_{[n]}$, which is a quasi-coherent ideal of $\mathscr{O}_C = \widetilde{\mathscr{S}}$ (1.4.4)

$$\mathscr{I}_n = (\mathscr{S}_{[n]})^{\sim}. \tag{8.7.2.4}$$

We thus deduce, from (8.7.2.2) and (8.7.2.3), using (1.4.4) and (1.4.8.1), the analogous formulas

$$\mathscr{I}_{[0]} = \mathscr{O}_C, \qquad \mathscr{I}_{[n]} \subset \mathscr{I}_{[m]} \qquad \text{for } m \leqslant n \tag{8.7.2.5}$$

$$\mathscr{I}_n \mathscr{I}_{[m]} \subset \mathscr{I}_{[m+n]}. \tag{8.7.2.6}$$

We are thus in the setting of (8.1.1), which leads us to introduce the quasi-coherent graded $\mathscr{O}_C$-algebra

$$\mathscr{S}^{\natural} = \bigoplus_{n \geqslant 0} \mathscr{I}_n = \left(\bigoplus_{n \geqslant 0} \mathscr{S}_{[n]} \right)^{\sim}. \tag{8.7.2.7}$$

Proposition (8.7.3). — *There is a canonical C-isomorphism*

$$h : C_X \xrightarrow{\sim} \operatorname{Proj}(\mathscr{S}^{\natural}). \tag{8.7.3.1}$$

PROOF. Suppose first of all that $Y = \operatorname{Spec}(A)$ is affine, so that $\mathscr{S} = \widetilde{S}$, with S a positively-graded A-algebra, and $C = \operatorname{Spec}(S)$. Definition (8.7.2.4) then shows, with the notation of (8.2.6), that $\mathscr{S}^{\natural} = (S^{\natural})^{\sim}$. To define (8.7.3.1), consider a homogeneous element $f \in S_d$ ($d > 0$) and the corresponding element $f^{\natural} \in S^{\natural}$ (8.2.6); the S-isomorphism in (8.2.7.3) then defines a C-isomorphism

$$\operatorname{Spec}(S_f^{\geqslant}) \xrightarrow{\sim} \operatorname{Spec}(S^{\natural}_{(f^{\natural})}). \tag{8.7.3.2}$$

But with the notation of (8.6.2), if $v : C_X \to X$ is the structure morphism, then it follows from (8.6.2.1) that $v^{-1}(X_f) = \operatorname{Spec}(S_f^{\geqslant})$. We also have that $\operatorname{Spec}(S^{\natural}_{(f^{\natural})}) = D_+(f^{\natural})$, which means that (8.7.3.2) defines an isomorphism $v^{-1}(X_f) \to D_+(f^{\natural})$. Furthermore, if $g \in S_e$ ($e > 0$), then the diagram **II | 175**

$$\begin{tikzcd} v^{-1}(X_{fg}) \arrow[r, "\sim"] \arrow[d] & D_+(f^{\natural}g^{\natural}) \arrow[d] \\ v^{-1}(X_f) \arrow[r, "\sim"] & D_+(f^{\natural}) \end{tikzcd}$$

commutes, by definition of the isomorphism in (8.2.7.3). Finally, by definition, S_+ is generated by the homogeneous f, and so it follows from (8.2.10, iv) and from (2.3.14) that the $D_+(f^{\natural})$ form a cover of $\operatorname{Proj}(S^{\natural})$, and that the $v^{-1}(X_f)$ form a cover of C_X, since the X_f form a cover of X; in this case, we have thus defined the isomorphism (8.7.3.1).

To prove (8.7.3) in the general case, it suffices to show that, if U and U' are affine open subsets of Y, given by rings A and A' (respectively), and such that $U' \subset U$, then, setting $\mathscr{S}|U = \widetilde{S}$ and $\mathscr{S}|U' = \widetilde{S}'$, the diagram

$$\begin{tikzcd} C_{U'} \arrow[r] \arrow[d] & \operatorname{Proj}(S'^{\natural}) \arrow[d] \\ C_U \arrow[r] & \operatorname{Proj}(S^{\natural}) \end{tikzcd} \tag{8.7.3.3}$$

commutes. But S is canonically identified with $S \otimes_A A'$, and so $S'^{\natural}$ is canonically identified with

$$S^{\natural} \otimes_S S' = S^{\natural} \otimes_A A';$$

thus $\mathrm{Proj}(S'^{\natural}) = \mathrm{Proj}(S^{\natural}) \times_U U'$ (2.8.10); similarly, if $X = \mathrm{Proj}(S)$ and $X' = \mathrm{Proj}(S')$, then $X' = X \times_U U'$ and $\mathscr{S}_{X'} = \mathscr{S}_X \otimes_{\mathscr{O}_U} U'$ (3.5.4), or, equivalently, $\mathscr{S}_{X'} = j^*(\mathscr{S}_X)$, where j is the projection $X' \to X$. We then (1.5.2) have that $C_{U'} = C_U \times_X X' = C_U \times_U U'$, and the commutativity of (8.7.3.3) is then immediate. □

Remark (8.7.4). — (i) The end of the proof of (8.7.3) can be immediately generalised in the following way. Let $g : Y' \to Y$ be a morphism, $\mathscr{S}' = g^*(\mathscr{S})$, and $X' = \mathrm{Proj}(\mathscr{S}')$; then we have a commutative diagram

(8.7.4.1)
$$\begin{array}{ccc} C_{X'} & \longrightarrow & \mathrm{Proj}(\mathscr{S}'^{\natural}) \\ \downarrow & & \downarrow \\ C_X & \longrightarrow & \mathrm{Proj}(\mathscr{S}^{\natural}) \end{array}$$

Now let $\phi : \mathscr{S}'' \to \mathscr{S}$ be a homomorphism of graded $\mathscr{O}_Y$-algebras such that, if we write $X'' = \mathrm{Proj}(\mathscr{S}'')$, then $u = \mathrm{Proj}(\phi) : X \to X''$ is everywhere defined; we also have a Y-morphism $v : C \to C''$ (with $C'' = \mathrm{Spec}(\mathscr{S}'')$) such that $\mathscr{A}(v) = \phi$, and, since II | 176
ϕ is a homomorphism of graded algebras, ϕ induces a v-morphism of graded algebras $\psi : \mathscr{S}''^{\natural} \to \mathscr{S}^{\natural}$ (1.4.1). Furthermore, it follows from (8.2.10, iv) and from the hypothesis on ϕ that $\mathrm{Proj}(\psi)$ is everywhere defined. Finally, taking (3.5.6.1) into account, there is a canonical u-morphism $\mathscr{S}_{X''} \to \mathscr{S}_X$, whence (1.5.6) a morphism $w : C_{X''} \to C_X$. With this in mind, the diagram

(8.7.4.2)
$$\begin{array}{ccc} C_{X''} & \xrightarrow{\sim} & \mathrm{Proj}(\mathscr{S}''^{\natural}) \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle \mathrm{Proj}(\psi)} \\ C_X & \xrightarrow{\sim} & \mathrm{Proj}(\mathscr{S}^{\natural}) \end{array}$$

is commutative, as we can immediately verify by restricting to the case where Y is affine.

(ii) Note that, by (8.7.2.5) and (8.7.2.6), we have $\mathscr{I}_1^m \subset \mathscr{I}_m \subset \mathscr{I}_1$ for all $m > 0$. But, by definition, $\mathscr{I}_1 = (\mathscr{S}_+)^\sim$, and so $\mathscr{I}_1$ defines the closed subprescheme $\varepsilon(Y)$ in C ((1.4.10) and (8.3.2)); we thus conclude that, for all $m > 0$, *the support of $\mathscr{O}_C/\mathscr{I}_m$ is contained in the underlying space of the vertex prescheme $\varepsilon(Y)$*; on the inverse image of the pointed affine cone E, the structure morphism $\mathrm{Proj}(\mathscr{S}^{\natural}) \to C$ thus restricts to an *isomorphism* (by (8.7.3) and (8.7.1)). Furthermore, by canonically identifying C with an open subset of $\widehat{C}$ (8.3.3), we can clearly extend the ideals $\mathscr{I}_m$ of $\mathscr{O}_C$ to ideals $\mathscr{J}_m$ of $\mathscr{O}_{\widehat{C}}$, by asking for it to agree with $\mathscr{O}_{\widehat{C}}$ on the open subset $\widehat{E}$ of $\widehat{C}$. If we define $\mathscr{T} = \bigoplus_{n\geqslant 0} \mathscr{J}_m$, which is a quasi-coherent graded $\mathscr{O}_{\widehat{C}}$-algebra, we can extend the isomorphism (8.7.3.1) to a $\widehat{C}$-isomorphism

(8.7.4.3)
$$\widehat{C}_X \xrightarrow{\sim} \mathrm{Proj}(\mathscr{T}).$$

Indeed, over $\widehat{E}$, it follows from the above that $\mathrm{Proj}(\mathscr{T})$ is canonically identified with $\widehat{E}$, and we thus define the isomorphism (8.7.4.3) over $\widehat{E}$ by asking for it to agree with the canonical isomorphism $\widehat{E}_X \to \widehat{E}$ (8.6.2); it is clear that this isomorphism and (8.7.3.1) then agree over $\widehat{E}$.

Corollary (8.7.5). — *Suppose that there exists some $n_0 > 0$ such that*

(8.7.5.1)
$$\mathscr{S}_{n+1} = \mathscr{S}_1\mathscr{S}_n \qquad \textit{for } n \geqslant n_0.$$

Then the vertex subprescheme (?) of C_X (isomorphic to X) is the inverse image under the canonical morphism $r : C_X \to C$ of the vertex subprescheme of C (isomorphic to Y). Conversely, if this property is true, and if we further assume that Y is Noetherian and that $\mathscr{S}$ is of finite type, then there exists some $n_0 > 0$ such that (8.7.5.1) holds true.

Proof. Since the first claim is local on Y, we can assume that $Y = \mathrm{Spec}(A)$ is affine, so that $\mathscr{S} = \widetilde{S}$, with S a positively-graded A-algebra. The claim then follows from (8.2.12), since $\mathrm{Proj}(S^{\natural} \otimes_S S_0) = C_X \times_C \varepsilon(Y)$ (by the identification in (8.7.3.1)), or, in other words, since this prescheme is the inverse image of $\varepsilon(Y)$ in C_X (**I**, 4.4.1). The converse also follows from (8.2.12) whenever Y is

Noetherian affine and S is of finite type. If Y is Noetherian (but not necessarily affine) and $\mathscr{S}$ is of finite type, then there exists a finite cover of Y by Noetherian affine open subsets U_i, and we then deduce from the above that, for all i, there exists an integer n_i such that $\mathscr{S}_{n+1}|U_i = (\mathscr{S}_1|U_i)(\mathscr{S}_n|U_i)$ for $n \geqslant n_i$; the largest of the n_i then ensures that (8.7.5.1) holds true. □ II | 177

(8.7.6). Now consider the C-prescheme Z given by *blowing up* the *vertex subprescheme* $\varepsilon(Y)$ in the affine cone C; by Definition (8.1.3), it is exactly the prescheme $\operatorname{Proj}(\bigoplus_{n\geqslant 0}\mathscr{S}_+^n)$; the canonical injection

(8.7.6.1) $$\iota : \bigoplus_{n\geqslant 0}\mathscr{S}_+^n \longrightarrow \mathscr{S}^\natural$$

defines (by the identification in (8.7.3)) a canonical dominant C-morphism

(8.7.6.2) $$G(\iota) \longrightarrow Z$$

where $G(\iota)$ is an open subset of C_X (3.5.1); note that it could be the case that $G(\iota) \neq C_X$, as shown by the example where $Y = \operatorname{Spec}(K)$, with K a field, and $\mathscr{S} = \widetilde{S}$, with $S = K[\mathbf{y}]$, where $\mathbf{y}$ is an indeterminate *of degree* 2; if R_n denotes the set $(S_+)^n$, considered as a subset of $S_{[n]} = S_n^\natural$, then $S_+^\natural$ is not the radical in $S_+^\natural$ of the ideal generated by the union of the R_n (cf. (2.3.14)).

Corollary (8.7.7). — *Assume that there exists some $n_0 > 0$ such that*

(8.7.7.1) $$\mathscr{S}_n = \mathscr{S}_1^n \quad \textit{for } n \geqslant n_0.$$

Then the canonical morphism (8.7.6.2) is everywhere defined, and is an isomorphism $C_X \xrightarrow{\sim} Z$. Conversely, if this property is true, and if we further assume that Y is Noetherian and that $\mathscr{S}$ is of finite type, then there exists some n_0 such that (8.7.7.1) holds true.

PROOF. The first claim is local on Y, and thus follows from (8.2.14); the converse follows similarly, arguing as in (8.7.5). □

Remark (8.7.8). — Since condition (8.7.7.1) implies (8.7.5.1), we see that, whenever it holds true, not only can C_X be identified with the prescheme given by blowing up the vertex (identified with Y) of the affine cone C, but also the vertex (identified with X) of C_X can be identified with the closed subprescheme given by the inverse image of the vertex Y of C. Furthermore, hypothesis (8.7.7.1) implies that, on $X = \operatorname{Proj}(\mathscr{S})$, the $\mathscr{O}_X$-modules $\mathscr{O}_X(n)$ are invertible ((3.2.5) and (3.2.9)), and that $\mathscr{O}_X(n) = \mathscr{L}^{\otimes n}$ with $\mathscr{L} = \mathscr{O}_X(1)$ ((3.2.7) and (3.2.9)); by Definition (8.6.1.1), C_X is thus the *vector bundle* $\mathbf{V}(\mathscr{L})$ on X, and its vertex is the *null section* of this vector bundle.

8.8. Ample sheaves and contractions

(8.8.1). Let Y be a prescheme, $f : X \to Y$ a *separated* and *quasi-compact* morphism, and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module that is *ample relative to* f. Consider the positively-graded $\mathscr{O}_Y$-algebra

(8.8.1.1) $$\mathscr{S} = \mathscr{O}_Y \oplus \bigoplus_{n\geqslant 1} f_*(\mathscr{L}^{\otimes n})$$

which is quasi-coherent (**I**, 9.2.2, a). There is a canonical homomorphisms of graded $\mathscr{O}_X$-algebras II | 178

(8.8.1.2) $$\tau : f^*(\mathscr{S}) \longrightarrow \bigoplus_{n\geqslant 0}\mathscr{L}^{\otimes n}$$

which, in degrees $\geqslant 1$, agrees with the canonical homomorphism $\sigma : f^*(f_*(\mathscr{L}^{\otimes n})) \to \mathscr{L}^{\otimes n}$ (**0**, 4.4.3), and is the identity in degree 0. The hypothesis that $\mathscr{L}$ is f-ample then implies ((4.6.3) and (3.6.1)) that the corresponding Y-morphism

(8.8.1.3) $$r = r_{\mathscr{L},\tau} : X \longrightarrow P = \operatorname{Proj}(\mathscr{S})$$

is everywhere defined and is a *dominant open immersion*, and that

(8.8.1.4) $$r^*(\mathscr{O}_P(n)) = \mathscr{L}^{\otimes n} \qquad \text{for all } n \in \mathbf{Z}.$$

Proposition (8.8.2). — *Let $C = \operatorname{Spec}(\mathscr{S})$ be the affine cone defined by $\mathscr{S}$; if $\mathscr{L}$ is f-ample, then there exists a canonical Y-morphism*

(8.8.2.1)
$$g : V = \mathbf{V}(\mathscr{L}) \longrightarrow C$$

such that the diagram

(8.8.2.2)
$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "f"'] & \mathbf{V}(\mathscr{L}) \arrow[r, "\pi"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r, "\varepsilon"] & C \arrow[r, "\psi"] & Y \end{tikzcd}$$

commutes, where ψ and π are the structure morphisms, and j and ε the canonical immersions sending X and Y (respectively) to the null section of $\mathbf{V}(\mathscr{L})$ and the vertex prescheme of C (respectively). Furthermore, the restriction of g to $\mathbf{V}(\mathscr{L}) - j(X)$ is an open immersion

(8.8.2.3)
$$\mathbf{V}(\mathscr{L}) - j(X) \longrightarrow E = C - \varepsilon(Y)$$

into the pointed affine cone E corresponding to $\mathscr{S}$.

Proof. With the notation of (8.8.1), let $\mathscr{S}_P^{\geqslant} = \bigoplus_{n\geqslant 0} \mathscr{O}_P(n)$ and $C_p = \operatorname{Spec}(\mathscr{S}_P^{\geqslant})$. We know (8.6.2) that there is a canonical morphism $h = \operatorname{Spec}(\alpha) : C_p \to C$ such that the diagram

(8.8.2.4)
$$\begin{tikzcd} C_P \arrow[r] \arrow[d, "h"'] & P \arrow[d, "p"] \\ C \arrow[r, "\psi"] & Y \end{tikzcd}$$

commutes; furthermore, if $\varepsilon_P : P \to C_P$ is the canonical immersion, then the diagram

(8.8.2.5)
$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "\varepsilon_P"'] & C_P \arrow[d, "h"] \\ Y \arrow[r, "\varepsilon"] & C \end{tikzcd}$$

commutes (8.7.1.1), and, finally, the restriction of H to the pointed affine cone E_P is an *isomorphism* $E_P \xrightarrow{\sim} E$ (8.6.2). It follows from (8.8.1.4) that

$$r^*(\mathscr{S}_P^{\geqslant}) = \mathbf{S}_{\mathscr{O}_X}(\mathscr{L})$$

and so we have a canonical P-morphism $q : \mathbf{V}(\mathscr{L}) \to C_P$, with the commutative diagram **II** | 179

(8.8.2.6)
$$\begin{tikzcd} \mathbf{V}(\mathscr{L}) \arrow[r, "\pi"] \arrow[d, "q"'] & X \arrow[d, "r"] \\ C_P \arrow[r] & P \end{tikzcd}$$

identifying $\mathbf{V}(\mathscr{L})$ with the product $C_P \times_P X$ (1.5.2); since r is an open immersion, so too is q (I, 4.3.2). Furthermore, the restriction of q to $\mathbf{V}(\mathscr{L}) - j(X)$ sends this prescheme to E_p, by (8.5.2), and the diagram

(8.8.2.7)
$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "r"'] & \mathbf{V}(\mathscr{L}) \arrow[d, "q"] \\ P \arrow[r, "\varepsilon_P"] & C_P \end{tikzcd}$$

is commutative (since it is a particular case of (8.5.1.3)). The claims of (8.8.2) immediately follow from these facts, by taking g to be the composite morphism $h \circ q$. □

Remark (8.8.3). — Assume further that Y is a *Noetherian* prescheme, and that f is a *proper* morphism. Since r is then *proper*(5.4.4), and thus closed, and since it is also a dominant open immersion, r is necessarily an *isomorphism* $X \xrightarrow{\sim} P$. Furthermore, we will see, in Chapter III (**III**, 2.3.5.1), that $\mathscr{S}$ is then necessarily an $\mathscr{O}_Y$-algebra *of finite type*. It then follows that $\mathscr{S}^\natural$ is an $\mathscr{S}_0^\natural$-algebra *of finite type*

((8.2.10, i) and (8.7.2.7)); since C_P is C-isomorphic to $\mathrm{Proj}(\mathscr{S}^{\natural})$ (8.7.3), we see that the morphism $h : C_P \to C$ is *projective*; since the morphism r is an isomorphism, so too is $q : \mathbf{V}(\mathscr{L}) \to C_P$, and we thus conclude that the morphism $g : \mathbf{V}(\mathscr{L}) \to C$ is *projective*. Furthermore, since the restriction of h to E_P is an isomorphism to E, and since q is an isomorphism, the restriction (8.8.2.3) of g is an isomorphism $\mathbf{V}(\mathscr{L}) - j(X) \xrightarrow{\sim} E$.

If we further assume that L is *very ample* for f, then, as we will also see in Chapter III (**III**, 2.3.5.1), there exists some integer $n_0 > 0$ such that $\mathscr{S}_n = \mathscr{S}_1^n$ for $n \geqslant n_0$. We then conclude, by (8.7.7), that $\mathbf{V}(\mathscr{L})$ can be identified with the prescheme Z given by *blowing up the vertex prescheme* (identified with Y) *in the affine cone* C, and that the *null section* of $\mathbf{V}(\mathscr{L})$ (identified with Y) is the *inverse image* of the vertex subprescheme Y of C.

Some of the above results can in fact be proven even without the Noetherian hypothesis:

Corollary (8.8.4). — *Let Y be a prescheme (resp. a quasi-compact scheme), $f : X \to Y$ a proper morphism, and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module that is ample relative to f. Then the morphism in* (8.8.2.1) *is proper (resp. projective), and its restriction* (8.8.2.3) *is an isomorphism.*

Proof. To prove that g is proper, we can restrict to the case where Y is affine, and it then suffices to consider the case where Y is a quasi-compact scheme. The same arguments as in (8.8.3) first of all show that r is an *isomorphism* $X \xrightarrow{\sim} P$; then q is also an isomorphism, and, since the restriction of h to E_P is an isomorphism $E_P \xrightarrow{\sim} E$, we have already seen that (8.8.2.3) is an isomorphism. It remains only to prove that g is *projective*.

Since f is of finite type, by hypothesis, we can apply (3.8.5) to the homomorphism τ from **II** | 180
(8.8.1.2): there is an integer $d > 0$ and a quasi-coherent $\mathscr{O}_Y$-submodule $\mathscr{E}$ of finite type of $\mathscr{S}_d$ such that, if $\mathscr{S}'$ is the $\mathscr{O}_Y$-subalgebra of $\mathscr{S}$ generated by $\mathscr{E}$, and $\tau' = \tau \circ q^*(\phi)$ (where ϕ is the canonical injection $\mathscr{S}' \to \mathscr{S}$), then $r' = r_{\mathscr{L},\tau'}$ is an immersion

$$X \longrightarrow P' = \mathrm{Proj}(\mathscr{S}').$$

Furthermore, since ϕ is injective, r' is also a *dominant immersion* (3.7.6); the same argument as for r then shows that r' is a *surjective closed immersion*; since r' factors as $X \xrightarrow{r} \mathrm{Proj}(\mathscr{S}) \xrightarrow{\Phi} \mathrm{Proj}(\mathscr{S}')$, where $\Phi = \mathrm{Proj}(\phi)$, we thus conclude that Φ is also a *surjective closed immersion*. But this implies that Φ is an *isomorphism*; we can restrict to the case where $Y = \mathrm{Spec}(A)$ is affine, and $\mathscr{S} = \widetilde{S}$ and $\mathscr{S}' = \widetilde{S}'$, with S a graded A-algebra and S' a graded subalgebra of S. For every homogeneous element $t \in S'$, we have that $S'_{(t)}$ is a subring of $S_{(t)}$; if we return to the definition of $\mathrm{Proj}(\phi)$ (2.8.1), we see that it suffices to prove that, if B' is a subring of a ring B, and if the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(B')$ corresponding to the canonical injection $B' \to B$ is a closed immersion, then this morphism is necessarily an *isomorphism*; but this follows from (**I**, 4.2.3). Furthermore, $\Phi^*(\mathscr{O}_{P'}(n)) = \mathscr{O}_P(n)$ ((3.5.2, ii) and (3.5.4)), and so $r'^*(\mathscr{O}_{P'}(n))$ is isomorphic to $\mathscr{L}^{\otimes n}$ (4.6.3). Let $\mathscr{S}'' = \mathscr{S}'^{(d)}$, so that (3.1.8, i) X is canonically identified with $P'' = \mathrm{Proj}(\mathscr{S}'')$, and $\mathscr{L}'' = \mathscr{L}^{\otimes d}$ with $\mathscr{O}_{P''}(1)$ (3.2.9, ii).

Now, if $C'' = \mathrm{Spec}(\mathscr{S}'')$, then $\mathscr{S}_{P''}^{\geqslant} = \bigoplus_{n\geqslant 0} \mathscr{O}_{P''}(n)$ can be identified with $\bigoplus_{n\geqslant 0} \mathscr{L}''^{\otimes n}$, and thus $C_{P''} = \mathrm{Spec}(\mathscr{S}_{P''}^{\geqslant})$ with $\mathbf{V}(\mathscr{L}'')$; we also know (8.7.3) that $C_{P''}$ is C''-isomorphic to $\mathrm{Proj}(\mathscr{S}''^{\natural})$; by the definition of $\mathscr{S}''$, we know that $\mathscr{S}''^{\natural}$ is generated by $\mathscr{S}_1''^{\natural}$, and that $\mathscr{S}_1''^{\natural}$ is of finite type over $\mathscr{S}_0''^{\natural} = \mathscr{S}''$ ((8.2.10, i and iii)), and so $\mathrm{Proj}(\mathscr{S}''^{\natural})$ is *projective* over C'' (5.5.1). Consider the diagram

(8.8.4.1)
$$\begin{tikzcd}
\mathbf{V}(\mathscr{L}) \arrow[r, "g"] \arrow[d, "u"'] & \mathrm{Spec}(\mathscr{S}) = C \arrow[d, "v"] \\
\mathbf{V}(\mathscr{L}'') \arrow[r, "g''"] & \mathrm{Spec}(\mathscr{S}'') = C''
\end{tikzcd}$$

where g and g'' correspond, by (1.5.6), to the canonical j-morphisms

$$\mathscr{S} \longrightarrow \bigoplus_{n\geqslant 0} \mathscr{L}^{\otimes n} \quad \text{and} \quad \mathscr{S}'' \longrightarrow \bigoplus_{n\geqslant 0} \mathscr{L}''^{\otimes n}$$

(3.3.2.3) (see (8.8.5) below), and v and u to the inclusion morphisms $\mathscr{S}'' \to \mathscr{S}$ and $\bigoplus_{n\geqslant 0} \mathscr{L}^{\otimes nd} \to \bigoplus_{n\geqslant 0} \mathscr{L}^{\otimes n}$ (respectively); it is immediate (3.3.2) that this diagram is commutative. We have just seen that g'' is a projective morphism; we also know that u is a *finite* morphism. Since the question is local

on X, we can assume that X is affine of ring A, and that $\mathscr{L} = \mathscr{O}_X$; everything then reduces to noting that the ring $A[T]$ is a module of finite type over its subring $A[T^d]$ (with T an indeterminate). Since Y is a quasi-compact scheme, and since C'' is affine over Y, we know that C'' is also a quasi-compact scheme, and so $g'' \circ u$ is a projective morphism (5.5.5, ii); by commutativity of (8.8.4.1), $v \circ g$ is also **II** | 181
projective, and, since v is affine, thus separated, we finally conclude that g is projective (5.5.5, v). ☐

(8.8.5). Consider again the situation in (8.8.1). We will see that the morphism $g : \mathbf{V}(\mathscr{L}) \to C$ can be also be defined in a way that works for any invertible (but not necessarily ample) $\mathscr{O}_X$-module $\mathscr{L}$. For this, consider the f-morphism

$$\tau^\flat : \mathscr{S} \longrightarrow \bigoplus_{n\geqslant 0} \mathscr{L}^{\otimes n} \tag{8.8.5.1}$$

corresponding to the morphism τ from (8.8.1.2). This induces (1.5.6) a morphism $g' : V \to C$ such that, if $\pi : V \to X$ and $\psi : C \to Y$ are the structure morphisms, the diagrams

$$\begin{tikzcd} X \arrow[d, "f"'] & V \arrow[l, "\pi"'] \arrow[d, "g'"] \\ Y & C \arrow[l, "\psi"'] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "j"] \arrow[d, "f"'] & V \arrow[d, "g'"] \\ Y \arrow[r, "\varepsilon"] & C \end{tikzcd} \tag{8.8.5.2}$$

commute ((8.5.1.2) and (8.5.1.3)). We will show that (if we assume that $\mathscr{L}$ is f-ample) *the morphisms g and g' are identical*.

Since the questions is local on Y, we can assume that $Y = \operatorname{Spec}(A)$ is affine, and (by (8.8.1.3)) identify X with an open subset of $P = \operatorname{Proj}(S)$, where $S = A \oplus \bigoplus_{n\geqslant 0} \Gamma(X, \mathscr{L}^{\otimes n})$; we then deduce, by (8.8.1.4), that $\Gamma(X, \mathscr{O}_P(n)) = \Gamma(X, \mathscr{L}^{\otimes n})$ for all $n \in \mathbf{Z}$. Taking into account the definition of $h = \operatorname{Spec}(\alpha)$, where α is the canonical p-morphism $\widetilde{S} \to \mathscr{S}_P^{\geqslant}$ (8.6.1.2), we have to show that the restriction to X of $\alpha^\sharp : p^*(\widetilde{S}) \to \mathscr{S}_P^{\geqslant}$ is identical to τ. Taking (**0**, 4.4.3) into account, it suffices to show that, if we compose the canonical homomorphism $\alpha_n : S_n \to \Gamma(P, \mathscr{O}_P(n))$ with the restriction homomorphism $\Gamma(P, \mathscr{O}_P(n)) \to \Gamma(X, \mathscr{O}_P(n)) = \Gamma(X, \mathscr{L}^{\otimes n})$, then we obtain the identity, for all $n > 0$; but this follows immediately from the definition of the algebra S and of α_n (2.6.2).

Proposition (8.8.6). — *Assume (with the notation of (8.8.5)) that, if we write $f = (f_0, \lambda)$, then the homomorphism $\lambda : \mathscr{O}_Y \to j_*(\mathscr{O}_X)$ is bijective; then:*

(i) *if we write $g = (g_0, \mu)$, then $\mu : \mathscr{O}_C \to g_*(\mathscr{O}_V)$ is an isomorphism; and*
(ii) *if X is integral (resp. locally integral and normal), then C is integral (resp. normal).*

PROOF. Indeed, the f-morphism $\tau^\flat$ is then an *isomorphism*

$$\tau^\flat : \mathscr{S} = \psi_*(\mathscr{O}_C) \longrightarrow f_*(\pi_*(\mathscr{O}_V)) = \psi_*(g_*(\mathscr{O}_V))$$

and the Y-morphism g can be considered as that for which the homomorphism $\mathscr{A}(g)$ (1.1.2) is equal to $\tau^\flat$. To see that μ is an isomorphism of $\mathscr{O}_C$-modules, it suffices (1.4.2) to see that $\mathscr{A}(\mu) : \psi_*(\mathscr{O}_C) \to \psi_*(g_*(\mathscr{O}_V))$ is an isomorphism. But, by Definition (1.1.2), we have that $\mathscr{A}(\mu) = \mathscr{A}(g)$, whence the conclusion of (i).

To prove (ii), we can restrict to the case where Y is affine, and so $\mathscr{S} = \widetilde{S}$, with $S = \bigoplus_{n\geqslant 0} \Gamma(X, \mathscr{L}^{\otimes n})$; **II** | 182
the hypothesis that X is integral implies that the ring S is integral (**I**, 7.4.4), and thus so too is C (**I**, 5.1.4). To show that C is normal, we will use the following lemma:

Lemma (8.8.6.1). — *Let Z be a normal integral prescheme. Then the ring $\Gamma(Z, \mathscr{O}_Z)$ is integral and integrally closed.*

PROOF. It follows from (**I**, 8.2.1.1) that $\Gamma(Z, \mathscr{O}_Z)$ is the intersection, in the field of rational functions $R(Z)$, of the integrally closed rings $\mathscr{O}_z$ over all $z \in Z$. ☐

With this in mind, we first show that V is *locally integral* and *normal*; for this, we can restrict to the case where $X = \operatorname{Spec}(A)$ is affine, with ring A integral and integrally closed (6.3.8), and where $\mathscr{L} = \mathscr{O}_X$. Since then $V = \operatorname{Spec}(A[T])$, and $A[T]$ is integral and integrally closed [**Jaf60**, p. 99], this proves our claim. For every affine open subset U of C, $g^{-1}(U)$ is quasi-compact, since the morphism g is quasi-compact; since V is locally integral, the connected components of $g^{-1}(U)$ are open integral preschemes in $g^{-1}(U)$, and thus finite in number, and, since V is normal, these preschemes are also normal (6.3.8). Then $\Gamma(U, \mathscr{O}_C)$, which is equal to $\Gamma(g^{-1}(U), \mathscr{O}_V)$, by (i), is the direct sum (?) of finitely-many integral and integrally closed rings (8.8.6.1), which proves that C is normal (6.3.4). □

8.9. Grauert's ampleness criterion: statement We intend to show that the properties proven in (8.8.2) *characterise* f-ample $\mathscr{O}_X$-modules, and, more precisely, to prove the following criterion:

Theorem (8.9.1). — (Grauert's criterion). *Let Y be a prescheme, $p : X \to Y$ a separated and quasi-compact morphism, and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module. For $\mathscr{L}$ to be ample relative to p, it is necessary and sufficient for there to exist a Y-prescheme C, a Y-section $\varepsilon : Y \to C$ of C, and a Y-morphism $q : \mathbf{V}(\mathscr{L}) \to C$, satisfying the following properties:*

(i) *the diagram*

(8.9.1.1)
$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "p"'] & \mathbf{V}(\mathscr{L}) \arrow[d, "q"] \\ Y \arrow[r, "\varepsilon"] & C \end{tikzcd}$$

commutes, where j is the null section of the vector bundle $\mathbf{V}(\mathscr{L})$; and

(ii) *the restriction of q to $\mathbf{V}(\mathscr{L}) - j(X)$ is a quasi-compact open immersion*
$$\mathbf{V}(\mathscr{L}) - j(X) \longrightarrow X$$
whose image does not intersect $\varepsilon(Y)$.

Note that, if C is *separated* over Y, we can, in condition (ii), remove the hypothesis that the open immersion is quasi-compact; to see that this property (of quasi-compactness) is in fact a consequence of the other conditions, we can restrict to the case where Y is affine, and the claim then follows from (**I**, 5.5.1)i and (**I**, 5.5.10). We can also remove the same hypothesis if we assume that X is Noetherian, **II** | 183
since then V is also Noetherian, and the claim follows from (**I**, 6.3.5).

Corollary (8.9.2). — *If the morphism $p : X \to Y$ is proper, then we can, in the statement of Theorem (8.9.1), assume that q is proper, and replace "open immersion" by "isomorphism".*

In a more suggestive manner, we can say (whenever $p : X \to Y$ is proper) that *$\mathscr{L}$ is ample relative to p if and only if we can* "contract" *the null section of the vector bundle $\mathbf{V}(\mathscr{L})$ to the base prescheme Y.* An important particular case is that where Y is the spectrum of a field, and where the operation of "contraction" consists of contract the null section $\mathbf{V}(\mathscr{L})$ *to a single point*.

(8.9.3). The necessity of the conditions in Theorem (8.9.1) and Corollary (8.9.2) follow immediately from (8.8.2) and (8.8.4).

To show that the conditions of (8.9.1) suffices, consider a slightly more general situation. For this, let (with the notation of (8.8.2))
$$\mathscr{S}' = \bigoplus_{n \geqslant 0} \mathscr{L}^{\otimes n}$$
and
$$V = \mathbf{V}(\mathscr{L}) = \operatorname{Spec}(\mathscr{S}').$$

The closed subprescheme $j(X)$, null section of $\mathbf{V}(\mathscr{L})$, is defined by the quasi-coherent sheaf of ideals $\mathscr{J} = (\mathscr{S}'_+)^\sim$ of $\mathscr{O}_V$ (1.4.10). This $\mathscr{O}_V$-module is *invertible*, since this property is local on X, and this reduces to remarking that the ideal $TA[T]$ in a ring of polynomials $A[T]$ is a free cyclic $A[T]$-module. Furthermore, it is immediate (again, because the question is local on X) that
$$\mathscr{L} = j^*(\mathscr{J})$$
and
$$j_*(\mathscr{L}) = \mathscr{J} / \mathscr{J}^2.$$

Now, if
$$\pi : \mathbf{V}(\mathscr{L}) \longrightarrow X$$
is the structure morphism, then $\pi_*(\mathscr{J}) = \mathscr{S}'_+$ and $\pi_*(\mathscr{J}/\mathscr{J}^2) = \mathscr{L}$; there are thus canonical homomorphisms $\mathscr{L} \to \pi_*(\mathscr{J}) \to \mathscr{L}$, the first being the canonical injection $\mathscr{L} \to \mathscr{S}'_+$, and the second

the canonical projection from $\mathscr{S}'_+$ to $\mathscr{S}'_1 = \mathscr{L}$, and their composition being the identity. We can also canonically embed $\pi_*(\mathscr{J}) = \mathscr{S}'_+ = \bigoplus_{n\geqslant 1} \mathscr{L}^{\otimes n}$ into the *product* $\prod_{n\geqslant 1} \mathscr{L}^{\otimes n} = \varprojlim_n \pi_*(\mathscr{J}/\mathscr{J}^{n+1})$ (since $\pi_*(\mathscr{J}/\mathscr{J}^{n+1}) = \mathscr{L} \oplus \mathscr{L}^{\otimes 2} \oplus \ldots \oplus \mathscr{L}^{\otimes n}$), and we thus have canonical homomorphisms

$$\mathscr{L} \longrightarrow \varprojlim \pi_*(\mathscr{J}/\mathscr{J}^{n+1}) \longrightarrow \mathscr{L} \tag{8.9.3.1}$$

whose composition is the identity.

With this in mind, the generalisation of (8.9.1) that we are going to prove is the following:

Proposition (8.9.4). — *Let Y be a prescheme, V a Y-prescheme, and X a closed subprescheme of V defined by an ideal $\mathscr{J}$ of $\mathscr{O}_V$, which is an* invertible *$\mathscr{O}_V$-module; if $j : X \to V$ is the canonical injection, then let* II | 184 *$\mathscr{L} = j^*(\mathscr{J}) = \mathscr{J} \otimes_{\mathscr{O}_V} \mathscr{O}_X$, so that $j_*(\mathscr{L}) = \mathscr{J}/\mathscr{J}^2$. Assume that the structure morphism $p : X \to Y$ is separated and quasi-compact, and that the following conditions are satisfied:*

(i) *there exists a Y-morphism $\pi : V \to X$ of finite type such that $\pi \circ j = 1_X$, and so $\pi_*(\mathscr{J}/\mathscr{J}^2) = \mathscr{L}$;*

(ii) *there exists a homomorphism of $\mathscr{O}_X$-modules $\phi : \mathscr{L} \to \varprojlim \pi_*(\mathscr{J}/\mathscr{J}^{n+1})$ such that the composition*
$$\mathscr{L} \xrightarrow{\phi} \varprojlim \pi_*(\mathscr{J}/\mathscr{J}^{n+1}) \xrightarrow{\alpha} \pi_*(\mathscr{J}/\mathscr{J}^2) = \mathscr{L}$$
(where α is the canonical homomorphism) is the identity;

(iii) *there exists a Y-prescheme C, a Y-section ε of C, and a Y-morphism $q : V \to C$ such that the diagram*

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "p"'] & V \arrow[d, "q"] \\ Y \arrow[r, "\varepsilon"'] & C \end{tikzcd} \tag{8.9.4.1}$$

commutes; and

(iv) *the restriction of q to $W = V - j(X)$ is a quasi-compact open immersion into C, whose image does not intersect $\varepsilon(Y)$.*

Then $\mathscr{L}$ is ample relative to p.

8.10. Grauert's ampleness criterion: proof

Lemma (8.10.1). — *Let $\pi : V \to X$ be a morphism, $j : X \to V$ an X-section of V that is also a closed immersion, and $\mathscr{J}$ a quasi-coherent ideal of $\mathscr{O}_V$ that defines the closed subprescheme of V associated to j. Then the following all hold true.*

(i) *For all $n \geqslant 0$, $\pi_*(\mathscr{O}_V/\mathscr{J}^{n+1})$ and $\pi_*(\mathscr{J}/\mathscr{J}^{n+1})$ are quasi-coherent $\mathscr{O}_X$-modules, and $\pi_*(\mathscr{O}_V/\mathscr{J}) = \mathscr{O}_X$ and $\pi_*(\mathscr{J}/\mathscr{J}^2) = j^*(\mathscr{J})$.*

(ii) *If $X = \{\xi\} = \operatorname{Spec}(k)$, where k is a field, then $\varprojlim \pi_*(\mathscr{O}_V/\mathscr{J}^{n+1})$ is isomorphic to the separated completion of the local ring $\mathscr{O}_{j(\xi)}$ for the $\mathfrak{m}_{j(\xi)}$-preadic topology.*

(iii) *Assume that $\mathscr{J}$ is an invertible $\mathscr{O}_V$-module (which implies that*
$$\mathscr{L} = j^*(\mathscr{J}) = \pi_*(\mathscr{J}/\mathscr{J}^2)$$
is an invertible $\mathscr{O}_X$-module), and that there exists a homomorphism $\phi : \mathscr{L} \to \varprojlim \pi_*(\mathscr{J}/\mathscr{J}^{n+1})$ such that the composition $\mathscr{L} \xrightarrow{\phi} \varprojlim \pi_*(\mathscr{J}/\mathscr{J}^{n+1}) \xrightarrow{\alpha} \pi_*(\mathscr{J}/\mathscr{J}^2)$ (where α is the canonical homomorphism) is the identity. If we write $\mathscr{S} = \bigoplus_{n\geqslant 0} \mathscr{L}^{\otimes n}$, then ϕ canonically induces an isomorphism of $\mathscr{O}_X$-algebras from the completion $\widehat{\mathscr{S}}$ of $\mathscr{S}$ relative to its canonical filtration (the completion being isomorphic to the product $\prod_{n\geqslant 0} \mathscr{L}^{\otimes n}$) to $\varprojlim \pi_*(\mathscr{O}_V/\mathscr{J}^{n+1})$.

PROOF. Note first of all that the support of the $\mathscr{O}_V$-module $\mathscr{O}_V/\mathscr{J}^{n+1}$ is $j(X)$, and the support of $\mathscr{J}/\mathscr{J}^{n+1}$ is contained in $j(X)$. In the case of (ii), $j(X)$ is a closed point $j(\xi)$ of V, and, by definition, II | 185 $\pi_*(\mathscr{O}_V/\mathscr{J}^{n+1})$ is the fibre of $\mathscr{O}_V/\mathscr{J}^{n+1}$ at the point $j(\xi)$, or, equivalently, setting $C = \mathscr{O}_{j(\xi)}$, and denoting by $\mathfrak{m}$ the maximal ideal of C, the C-module $C/\mathfrak{m}^{n+1}$; claim (ii) is then evident.

To prove (i), note that the question is local on X; we can thus restrict to the case where X is affine. Let U be an affine open subset of V; then $j(X) \cap U$ is an affine open subset of $j(X)$, so $U_0 = \pi(j(X) \cap U)$, which is isomorphic to it, is an affine open subset of X; for every affine open subset $W_0 \subset U_0$ in X, $W = \pi^{-1}(W_0) \cap U$ is an affine open subset of V, since X is a scheme (**I**, 5.5.10); in particular, $U' = U \cap \pi^{-1}(U_0)$ is an affine open subset of V, and clearly $\pi(U') = U_0$ and $j(U_0) = j(X) \cap U$. Then, by definition, $\Gamma(W_0, \pi_*(\mathscr{O}_V/\mathscr{J}^{n+1})) = \Gamma(\pi^{-1}(W_0), \mathscr{O}_V/\mathscr{J}^{n+1})$; but since every point of $\pi^{-1}(W_0)$ not belonging to $j(W_0)$ has an open neighbourhood in $\pi^{-1}(W_0)$ not intersecting $j(X)$, and in which $\mathscr{O}_V/\mathscr{J}^{n+1}$ is thus zero, it is clear that the sections of $\mathscr{O}_V/\mathscr{J}^{n+1}$ over $\pi^{-1}(W_0)$ and over W are in bijective correspondence. In other words, if π' is the restriction of π to U', then the $(\mathscr{O}_X|U_0)$-modules $\pi_*(\mathscr{O}_V/\mathscr{J}^{n+1})|U_0$ and $\pi'_*((\mathscr{O}_V/\mathscr{J}^{n+1})|U')$ are identical. Since U' and U_0 are affine, and since the U_0 cover X, we thus conclude (**I**, 1.6.3) that $\pi_*(\mathscr{O}_V/\mathscr{J}^{n+1})$ is quasi-coherent, and the proof is identical for $\pi_*(\mathscr{J}/\mathscr{J}^{n+1})$.

Finally, to prove (iii), note that $\mathscr{S}$ is exactly $\mathbf{S}_{\mathscr{O}_X}(\mathscr{L})$; so ϕ canonically induces a homomorphism of $\mathscr{O}_X$-algebras $\psi : \mathscr{S} \to \varprojlim \pi_*(\mathscr{O}_V/\mathscr{J}^{n+1})$ (1.7.4); furthermore, this homomorphism sends $\mathscr{L}^{\otimes n}$ to $\varprojlim_m \pi_*(\mathscr{J}^n/\mathscr{J}^{n+1})$, and is thus continuous for the topologies considered, and indeed then extends to a homomorphism $\widehat{\psi} : \widehat{\mathscr{S}} \to \varprojlim \pi_*(\mathscr{O}_V/\mathscr{J}^{n+1})$. To see that this is indeed an isomorphism, we can, as in the proof of (i), restrict to the case where $X = \operatorname{Spec}(A)$ and $V = \operatorname{Spec}(B)$ are affine, with $\mathscr{J} = \widetilde{\mathfrak{J}}$, where $\mathfrak{J}$ is an ideal of B; there is an injection $A \to B$ corresponding to π that identifies A with a subring of B that is *complementary* to B, and $\mathscr{L}$ (resp. $\pi_*(\mathscr{O}_V/\mathscr{J}^{n+1})$) is the quasi-coherent $\mathscr{O}_X$-module associated to the A-module $L = \mathfrak{J}/\mathfrak{J}^2$ (resp. $B/\mathfrak{J}^{n+1}$). Since $\mathscr{J}$ is an *invertible* $\mathscr{O}_V$-module, we can further assume that $\mathfrak{J} = Bt$, where t is not a zero divisor in B. From the fact that $B = A \oplus Bt$, we deduce that, for all $n > 0$,

$$B = A \oplus At \oplus At^2 \oplus \ldots \oplus At^n \oplus Bt^{n+1}$$

and so there exists a canonical A-isomorphism from the ring of formal series $A[[T]]$ to $C = \varprojlim B/\mathfrak{J}^{n+1}$ that sends T to t. We also have that $L = A\bar{t}$, where $\bar{t}$ is the class of t modulo Bt^2, and the homomorphism ϕ sends, by hypothesis, $\bar{t}$ to an element $t' \in C$ that is congruent to t modulo Ct^2. We thus deduce, by induction on n, that

$$A \oplus At' \oplus \ldots \oplus At'^n \oplus Ct^{n+1} = A \oplus At \oplus \ldots \oplus At^n \oplus Ct^{n+1}$$

which proves that the homomorphism $\widehat{\psi}$ does indeed correspond to an isomorphism from $\prod_{n \geqslant 0} L^{\otimes n}$ to C. □

Lemma (8.10.2). — *Under the hypotheses of Lemma (8.10.1), let $g : X' \to X$ be a morphism, write $V' = V \times_X X'$, and let $\pi' : V' \to X'$ and $g : V' \to V$ be the canonical projections, so that we have the commutative diagram* **II** | 186

$$\begin{tikzcd} V & V' \arrow[l, "g'"'] \arrow[d, "\pi'"] \\ X \arrow[from=u, "\pi"'] & X' \arrow[l, "g"] \end{tikzcd}$$

Then $j' = j \times 1_{X'}$ is an X'-section of V' that is also a closed immersion, and $\mathscr{J}' = g'^*(\mathscr{J})\mathscr{O}_{V'}$ is the quasi-coherent ideal of $\mathscr{O}_{V'}$ that defines the closed subprescheme of V' associated to j'. Furthermore, $\pi'_*(\mathscr{O}_{V'}/\mathscr{J}'n+1) = g^*(\pi_*(\mathscr{O}_V/\mathscr{J}^{n+1}))$. Finally, $\mathscr{J}'$ is an $\mathscr{O}_{V'}$-module that is canonically isomorphic to $g'^*(\mathscr{J})$, and is, in particular, invertible if $\mathscr{J}$ is an invertible $\mathscr{O}_V$-module.

Proof. The fact that j' is a closed immersion follows from (**I**, 4.3.1), and it is an X'-section of V' by functoriality of extension of the base prescheme. Furthermore, if Z (resp. Z') is the closed subprescheme of V (resp. V') associated to j (resp. j'), then $Z' = g'^{-1}(Z)$ (**I**, 4.3.1), and the second claim then follows from (**I**, 4.4.5). To prove the other claims, we see, as in (8.10.1), that we can restrict to the case where X, V, and X' (and thus also V') are affine; we keep the notation from the proof of (8.10.1), and let $X' = \operatorname{Spec}(A')$. Then $V' = \operatorname{Spec}(B')$, where $B' = B \otimes_A A'$, and $\mathscr{J}' = \widetilde{\mathfrak{J}''}$, where $\mathfrak{J}' = \operatorname{Im}(\mathfrak{J} \otimes_A A')$. Then $B'/\mathfrak{J}'^{n+1} = (B/\mathfrak{J}^{n+1}) \otimes_A A'$; furthermore, since $\mathfrak{J}$ is a direct factor (as an A-module) of B, $\mathfrak{J} \otimes_A A'$ is a direct factor (as an A'-module) of B', and is thus canonically identified with $\mathfrak{J}'$. □

Corollary (8.10.3). — *Assume that the hypotheses of Lemma (8.10.1) are satisfied, and assume further that π is of finite type, and that $\mathscr{J}$ is an invertible $\mathscr{O}_V$-module. Then, for all $x \in X$, the local ring at the point $j(x)$ of the fibre $\pi^{-1}(x)$ is a regular (thus integral) ring of dimension 1, whose completion is isomorphic to the formal series ring $k(x)[[T]]$ (where T is an indeterminate); furthermore, there exists exactly one irreducible component of $\pi^{-1}(x)$ that contains $j(x)$.*

PROOF. Since $\pi^{-1}(x) = V \times_X \operatorname{Spec}(k(x))$, we are led, by (8.10.2), to the case where X is the spectrum of a field K. Since π is of finite type (**I**, 6.4.3, iv), $\mathscr{O}_{j(x)}$ is a Noetherian local ring, and thus separated for the $\mathfrak{m}_{j(x)}$-preadic topology (**0**, 7.3.5); it follows from (8.10.1, ii and iii) that the completion of this ring is isomorphic to $K[[T]]$, and so $\mathscr{O}_{j(x)}$ is regular and of dimension 1 ([**CC**, p. 17-01, th. 1]); finally, since $\mathscr{O}_{j(x)}$ is integral, $j(x)$ belongs to exactly one of the (finitely many) irreducible components of V (**I**, 5.1.4). □

Corollary (8.10.4). — *Suppose that the hypotheses of Lemma (8.10.1) are satisfied, and assume further that $\mathscr{J}$ is an invertible $\mathscr{O}_V$-module. Let $W = V - j(X)$; for every quasi-coherent ideal $\mathscr{K}$ of $\mathscr{O}_X$, let $\mathscr{K}_V = \pi^*(\mathscr{K})\mathscr{O}_V$ and $\mathscr{K}_W = \mathscr{K}_V|W$. Then $\mathscr{K}_V$ is the largest quasi-coherent ideal of $\mathscr{O}_V$ whose restriction to W is $\mathscr{K}_W$.*

PROOF. Indeed, we see as in (8.10.1) that the question is local on X and V; we can thus reuse the notation from the proof of (8.10.1), with $\mathfrak{J} = Bt$, where t is not a zero divisor in B. Furthermore, we have $W = \operatorname{Spec}(B_t)$ and $\mathscr{K} = \widetilde{\mathfrak{K}}$, where $\mathfrak{K}$ is an ideal of A; whence $\pi^*(\mathscr{K})\mathscr{O}_V = (\mathfrak{K}.B)^\sim$ (**I**, 1.6.9), $\mathscr{K}_W = (\mathfrak{K}.B_t)^\sim$, and the largest ideal of B whose canonical image in B_t is $\mathfrak{K}.B_t$ is the inverse image of $\mathfrak{K}.B_t$, that is, the set of $s \in B$ such that, for some integer $n > 0$, we have $t^n s \in \mathfrak{K}.B$. We have to show that this last relation implies that $s \in \mathfrak{K}.B$, or again that the canonical image of t is not a zero divisor in $B/\mathfrak{K}B = (A/\mathfrak{K}) \otimes_A B$, which follows from (8.10.2) applied to $X' = \operatorname{Spec}(A/\mathfrak{K})$. □ **II** | 187

Corollary (8.10.5). — *Suppose that the hypotheses of (8.10.3) are satisfied; let $W = V - j(X)$, x be a point of X, $\mathscr{K}$ a quasi-coherent ideal of $\mathscr{O}_X$, and z the generic point of the irreducible component of $\pi^{-1}(x)$ that contains $j(x)$ (8.10.3).*

(i) *Let g be a section of $\mathscr{O}_V$ over V such that $g|W$ is a section of $\mathscr{K}_W$ over W (using the notation from (8.10.4)). Then g is a section of $\mathscr{K}_V$; if further $g(z) \neq 0$, and if, for every integer $m > 0$, we denote by g_m^x the germ at the point x of the canonical image g_m of g in $\Gamma(X, \pi_*(\mathscr{O}_V/\mathscr{J}^{m+1}))$, then there exists an integer $m > 0$ such that the image of g_m^x in*

$$(\pi_*(\mathscr{O}_V/\mathscr{J}^{m+1}))_x \otimes_{\mathscr{O}_x} k(x)$$

is $\neq 0$.

(ii) *Suppose further that the conditions of (8.10.1, iii) are fulfilled. Then, if there exists a section g of $\mathscr{K}_V$ over V such that $g(z) \neq 0$, then there exists an integer $n \geqslant 0$ and a section f of $\mathscr{K}.\mathscr{L}^{\otimes n} = \mathscr{K} \otimes \mathscr{L}^{\otimes n} \subset \mathscr{L}^{\otimes n}$ such that $f(x) \neq 0$. If g is a section of $\mathscr{J}$, we can take $n > 0$.*

PROOF.

(i) Since the ideal of $\mathscr{O}_W$ generated by $g|W$ is contained in $\mathscr{K}_W$ by hypothesis, the ideal of $\mathscr{O}_V$ generated by g is contained in $\mathscr{K}_V$ by (8.10.4), or, in other words, g is a section of $\mathscr{K}_V$. To prove the second claim of (i), we can again assume that X and V are affine, and reuse the notation from (8.10.1); the fibre $\pi^{-1}(x)$ is then affine of ring $B' = B \otimes_A k(x)$, and there exists in B' an element t' which is not a zero divisor and is such that $B' = k(x) \oplus B't'$. Since $j(x)$ is a specialisation of z and since $g(z) \neq 0$, we necessarily have that $g_{(j)x} \neq 0$. But $\mathscr{O}_{j(x)}$ is a separated local ring (8.10.3), and thus embeds into its completion, and the image of g in this completion is thus not null. But this completion is isomorphic to $\varprojlim_n (B'/B't'^{n+1})$ (8.10.3); if $g' = g \otimes 1 \in B'$, there then exists an integer m such that $g' \notin B't'^{m+1}$, or, again, the image g'_m of g' in $B'/B't'^{m+1}$ is not null. But since g'_m is exactly the image of g_m^x, our claim is proved.

(ii) By (8.10.1, iii), $\pi_*(\mathscr{O}_V/\mathscr{J}^{m+1})$ is isomorphic to the direct sum of the $\mathscr{L}^{\otimes k}$ for $0 \leqslant k \leqslant m$; we denote by f_k the section of $\mathscr{L}^{\otimes k}$ over X that is the component of the element of $\bigoplus_{k=0}^m \Gamma(X, \mathscr{L}^{\otimes k})$ which corresponds to g_m by this isomorphism. Choosing m as in (i), there is thus an index k such that $f_k(x) \neq 0$, by (i). To see that f_k is a section of $\mathscr{K}\mathscr{L}^{\otimes k}$, it suffices to consider, as above, the case where X and V are affine, and this follows immediately from the fact that $g \in \mathfrak{K}.B$ (with the notation from (8.10.4)). The final claim follows from the fact that the hypothesis $g \in \Gamma(V, \mathscr{J})$ implies that $f_0 = 0$.

□

(8.10.6). *Proof of* (8.9.4). The question is local on Y (4.6.4); since ε is a Y-section, we can thus replace C by an affine open neighbourhood U of a point of $\varepsilon(Y)$ such that $\varepsilon(Y) \cap U$ is closed in U. In other words, we can assume that C is affine, and that Y is a closed subprescheme of C (and thus also affine) defined by a quasi-coherent sheaf $\mathscr{I}$ of ideals of $\mathscr{O}_C$. Since p is separated and quasi-compact, **II** | 188
X is thus a quasi-compact scheme, and we are reduced to proving that $\mathscr{L}$ is *ample* (4.6.4). By criterion (4.5.2, a)), we must thus prove the following: for every quasi-coherent ideal $\mathscr{K}$ of $\mathscr{O}_X$ and every point $x \in X$ not belonging to the support of $\mathscr{O}_X/\mathscr{K}$, there exists an integer $n > 0$ and a section f of $\mathscr{K} \otimes \mathscr{L}^{\otimes n}$ over X such that $f(x) \neq 0$.

For this, set

$$\mathscr{K}_V = \pi^*(\mathscr{K})\mathscr{O}_V$$
$$\mathscr{K}_W = \mathscr{K}_V|W$$

where $W = V - j(X)$; since the restriction of q to W is a quasi-compact immersion to C, it follows from (**I**, 9.4.2) that $\mathscr{K}_W$ is the restriction to W of a quasi-coherent ideal $\mathscr{K}'_V$ of $\mathscr{O}_V$ of the form

$$\mathscr{K}'_V = q^*(\mathscr{K}_C)\mathscr{O}_V$$

where $\mathscr{K}_C$ is a quasi-coherent ideal of $\mathscr{O}_C$. Furthermore, since, by hypotheses, $q^{-1}(Y) \subset j(X)$, and since Y is defined by the ideal $\mathscr{I}$, the restriction to W of $q^*(\mathscr{I})\mathscr{O}_V$ is identical to that of $\mathscr{O}_V$, and so $\mathscr{K}_W$ is also the restriction to W of $q^*(\mathscr{I}\mathscr{K}_C)\mathscr{O}_V$, and we can thus suppose that $\mathscr{K}_C \subset \mathscr{I}$, whence

$$\mathscr{K}'_V \subset q^*(\mathscr{I})\mathscr{O}_V \subset \mathscr{J} \tag{8.10.6.1}$$

taking into account (**I**, 4.4.6) and the commutativity of (8.9.4.1). Furthermore, we deduce from (8.10.4) that

$$\mathscr{K}'_V \subset \mathscr{K}_V. \tag{8.10.6.2}$$

With this in mind, it follows from (8.10.3) that $j(x)$ belongs to exactly one irreducible component of $\pi^{-1}(x)$; let z be the generic point of this component, and let $z' = q(z)$. By (8.10.5), the proof will be finished (taking (8.10.6.1) and (8.10.6.2) into account) if we show the existence of a section g of $\mathscr{K}'_V$ over V such that $g(z) \neq 0$. But, by hypothesis, $\mathscr{K}$ has a restriction equal to that of $\mathscr{O}_X$ in an open neighbourhood of x; also, it follows from (8.10.3) that $z \neq j(x)$, and so $z \in W$, and thus $(\mathscr{K}_W)_z = \mathscr{O}_{V,z}$, whence, by definition, $(\mathscr{K}_C)_{z'} = \mathscr{O}_{C,z}$. Since C is affine, there is thus a section g' of $\mathscr{K}_C$ over C such that $g'(z') \neq 0$, and by taking g to be the section of $\mathscr{K}'_V$ corresponding canonically to g', we indeed have $g(z) \neq 0$, which finishes the proof.

Remark (8.10.7). — We ignore the question of whether or not condition (ii) in (8.9.4) is superfluous or not. In any case, the conclusion does not hold if we do not assume the existence of a Y-morphism $\pi : V \to X$ such that $\pi \circ j = 1_X$; we briefly point out how we can indeed construct a counterexample, whose details will not be developed until later on. We take $Y = \operatorname{Spec}(k)$, where k is a field, and $C = \operatorname{Spec}(A)$, where $A = k[T_1, T_2]$, and the Y-section ε corresponding to the augmentation homomorphism $A \to k$. We denote by C' the scheme induced by C by blowing up the closed point $a = \varepsilon(Y)$ of C; if D is the inverse image of a in C', we consider in D a closed point b, and we denote **II** | 189
by V the scheme induced by C' by blowing up b; X is the closed subprescheme of V given by the inverse image of a by the structure morphism $q : V \to C$. We now show that X is the union of two irreducible components, X_1 and X_2, where X_1 is the inverse image of b in V. It is immediate that the ideal $\mathscr{J}$ of $\mathscr{O}_V$ that defines X is again invertible, we we can show that $j^*(\mathscr{J}) = \mathscr{L}$ (where j is the canonical injection $X \to V$) is not ample, by considering the "degree" of the inverse image of $\mathscr{L}$ in X_1, which would be > 0 if $\mathscr{L}$ were ample, but we can show (by an elementary intersection calculation) that it is in fact equal to 0.

8.11. Uniqueness of contractions

Lemma (8.11.1). — *Let U and V be preschemes, and $h = (h_0, \lambda) : U \to V$ a surjective morphism. Suppose that*

(1) $\lambda : \mathscr{O}_V \to h_*(\mathscr{O}_U) = (h_0)_*(\mathscr{O}_U)$ is an isomorphism;
*(2) the underlying space of V can be identified with the quotient of the underlying space of U by the relation $h_0(x) = h_0(y)$ (*a condition which always holds whenever the morphism h is *open* or *closed*, or, *a fortiori* when h is *proper*.)

Then, for every prescheme W, the map

$$\mathrm{Hom}(V, W) \longrightarrow \mathrm{Hom}(U, W) \tag{8.11.1.1}$$

that, to each morphism $v = (v_0, \nu)$ from V to W, associates the morphism $u = v \circ h = (u_0, \mu)$, is a bijection from $\mathrm{Hom}(V, W)$ *to the set of u such that u_0 is constant on every fibre $h_0^{-1}(x)$.*

Proof. It is clear that, if $u = v \circ h$, so that $u_0 = v_0 \circ h_0$, then u_0 is constant on every set $h_0^{-1}(x)$. Conversely, if u has this property, we will show that there exists exactly one $v \in \mathrm{Hom}(V, W)$ such that $u = v \circ h$. The existence and uniqueness of the continuous map $v_0 : V \to W$ such that $u_0 = v_0 \circ h_0$ follows from the hypotheses, since h_0 can be identified with the canonical map from U to U/R. We can also, replacing V by some isomorphic prescheme if necessary, suppose that λ is the identity; by hypothesis, μ is then a homomorphism $\mu : \mathscr{O}_W \to (u_0)_*(\mathscr{O}_U) = (v_0)_*((h_0)_*(\mathscr{O}_U))$ such that the corresponding homomorphism $\mu^\sharp : u_0^*(\mathscr{O}_W) \to \mathscr{O}_U$ is *local* on every fibre. Since $(v_0)_*((h_0)_*(\mathscr{O}_U)) = (v_0)_*(\mathscr{O}_V)$, we necessarily have that $\nu = u$, and everything then reduces to showing that the corresponding homomorphism $\nu^\sharp : v_0^*(\mathscr{O}_W) \to \mathscr{O}_V$ is local on every fibre. But every $y \in V$ is of the form $h_0(x)$ for some $x \in U$; let $z = v_0(y) = u_0(x)$. Then (**0**, 3.5.5) the homomorphism $\mu_x^\sharp$ factors as

$$\mu_x^\sharp : \mathscr{O}_z \xrightarrow{\nu_y^\sharp} \mathscr{O}_y \xrightarrow{\lambda_x^\sharp} \mathscr{O}_x.$$

By hypothesis, $\lambda_x^\sharp$ and $\mu_x^\sharp$ are local homomorphisms; thus $\lambda_x^\sharp$ sends every invertible element of $\mathscr{O}_y$ to an invertible element of $\mathscr{O}_x$; if $\nu_y^\sharp$ sent a non-invertible element of $\mathscr{O}_z$ to an invertible element of $\mathscr{O}_y$, then $\mu_x^\sharp$ would send this element of $\mathscr{O}_z$ to an invertible element of $\mathscr{O}_x$, contradicting the hypothesis, whence the lemma. □

Corollary (8.11.2). — *Let U be an integral prescheme, and V a normal prescheme; then every morphism $h : U \to V$ that is universally closed, birational, and radicial, is also an isomorphism.*

Proof. If $h = (h_0, \lambda)$, then it follows from the hypotheses that h_0 is injective and closed, and that $h_0(U)$ is dense in V, and so h_0 is a *homeomorphism* from U to V. To prove the corollary, it will suffice **II** | 190
to show that $\lambda : \mathscr{O}_V \to (h_0)_*(\mathscr{O}_U)$ is an isomorphism: we can then apply (8.11.1), which proves that the map (8.11.1.1) is bijective (the fibres $h_0^{-1}(x)$ each consisting of a single point); thus h will be an isomorphism. The question clearly being local on V, we can suppose that $V = \mathrm{Spec}(A)$ is affine, of an integral and integrally closed ring (8.8.6.1); h then corresponds (**I**, 2.2.4) to a homomorphism $\phi : A \to \Gamma(U, \mathscr{O}_U)$, and everything reduces to showing that ϕ is an isomorphism. But, if K is the field of fractions of A, then $\Gamma(U, \mathscr{O}_U)$ has, by hypothesis, K as its field of fractions, and A is a subring of $\Gamma(U, \mathscr{O}_U)$, with ϕ being the canonical injection (**I**, 8.2.7). Since the morphism h satisfies the hypotheses of (7.3.11), $\Gamma(U, \mathscr{O}_U)$ is a subring of the integral closure of A in K, and is thus identical to A by hypothesis. □

Remark (8.11.3). — We will see in chapter III (**III**, 4.4.11) that, whenever V is a *locally Noetherian* prescheme, every morphism $h : U \to V$ that is proper and quasi-finite (in particular, every morphism satisfying the hypotheses of (8.11.2)) is necessarily *finite*. The conclusion of (8.11.2) then follows in this case from (6.1.15).

(8.11.4). We will now see that, in Grauert's criterion, we can often prove that the prescheme C and the "contraction" q are determined in an *essentially unique* manner.

Lemma (8.11.5). — *Let Y be a prescheme, $p : X \to Y$ a proper morphism, $\mathscr{L}$ a p-ample invertible $\mathscr{O}_X$-module, C a Y-prescheme, $\varepsilon : Y \to C$ a Y-section, and $q : V = \mathbf{V}(\mathscr{L}) \to C$ a Y-morphism, all such* *that the diagram in (8.9.1.1) commutes. Suppose further that, if $p = (p_0, \theta)$, then $\theta : \mathscr{O}_Y \to p_*(\mathscr{O}_X)$ is an isomorphism. Let $\mathscr{S}' = \bigoplus_{n \geqslant 0} p_*(\mathscr{L}^{\otimes n})$ and $C' = \operatorname{Spec}(\mathscr{S}')$, and let $q' : \mathbf{V}(\mathscr{L}) \to C'$ be the canonical Y-morphism (8.8.5). Then there exists exactly one Y-morphism $u : C' \to C$ such that $q = u \circ q'$.*

PROOF. The hypothesis on θ implies, in particular, that p is surjective; since, by (8.8.4), the restriction of q' to $\mathbf{V}(\mathscr{L}) - j(X)$ is an *isomorphism* to $C' - \varepsilon'(Y)$ (where ε is the vertex section of C'), it follows from (8.8.4) that q' is *proper* and *surjective*; furthermore, by (8.8.6), if we let $q' = (q'_0, \tau)$, then $\tau : \mathscr{O}_{C'} \to q'_*(\mathscr{O}_V)$ is an isomorphism. We are thus in a situation where we can apply (8.11.1), and we will have proven the lemma if we show that q is constant on every fibre $q'^{-1}(z')$, where $z' \in C'$. But this condition is trivially satisfied for $z' \notin \varepsilon'(Y)$. If $z' \in \varepsilon'(Y)$, then there exists exactly one $y \in Y$ such that $z' = \varepsilon'(y)$, and, by commutativity of (8.8.5.2) and the fact that q' sends $\mathbf{V}(\mathscr{L}) - j(X)$ to $C' - \varepsilon'(Y)$, $q'^{-1}(z') = j(p^{-1}(y))$; the commutativity of the diagram in (8.9.1.1) then proves our claim. □

Corollary (8.11.6). — *Under the hypotheses of (8.11.5), suppose further that q is proper, and that the restriction of q to $\mathbf{V}(\mathscr{L}) - j(X)$ is an isomorphism to $C - \varepsilon(Y)$. Then the morphism u is universally closed, surjective, and radicial, and its restriction to $C' - \varepsilon'(Y)$ is an isomorphism to $C - \varepsilon(Y)$.*

PROOF. Since q' is an isomorphism from $\mathbf{V}(\mathscr{L}) - j(X)$ to $C' - \varepsilon'(Y)$ (8.8.4), the last claim follows immediately from the fact that $q = u \circ q'$. Furthermore, the commutativity of the diagrams in (8.8.5.2) **II** | 191
and (8.9.1.1) shows that the restriction of u to the closed subprescheme $\varepsilon'(Y)$ of C' is an isomorphism to the closed subprescheme $\varepsilon(Y)$ of C, from which we immediately deduce that, for all $z' \in \varepsilon'(Y)$, if $z = u(z')$, then u defines an isomorphism from $k(z)$ to $k(z')$. These remarks prove that u is bijective and radicial; furthermore, if $\psi : C \to Y$ and $\psi' : C' \to Y$ are the structure morphisms, then $\psi' = \psi \circ u$, and, since ψ' is separated (1.2.4), so too is u (**I**, 5.5.1, v). We have already seen, in the proof of (8.11.5), that q' is surjective; since $q = u \circ q'$ is proper, we finally conclude, from (5.4.3) and (5.4.9), that u is universally closed. □

Proposition (8.11.7). — *Let Y be a prescheme, X an* integral *prescheme, $p : X \to Y$ a proper morphism, $\mathscr{L}$ a p-ample invertible $\mathscr{O}_X$-module, C a* normal *Y-prescheme, $\varepsilon : Y \to C$ a Y-section, and $q : V = \mathbf{V}(\mathscr{L}) \to C$ a Y morphism, all such that the diagram in (8.9.1.1) commutes. Suppose further that, if $p = (p_0, \theta)$, then $\theta : \mathscr{O}_Y \to p_*(\mathscr{O}_X)$ is an isomorphism. Let $\mathscr{S}' = \bigoplus_{n \geqslant 0} p_*(\mathscr{L}^{\otimes n})$ and $C' = \operatorname{Spec}(\mathscr{S}')$, and let $q' : \mathbf{V}(\mathscr{L}) \to C'$ be the canonical Y-morphism (8.8.5). Then the unique Y-morphism $u : C' \to C$ such that $q = u \circ q'$ is an* isomorphism.

PROOF. It follows from (8.8.6) that C' is integral; since u is a homeomorphism of the underlying subspaces $C' \to C$ (u being bijective and closed, by (8.11.6)), C is irreducible, thus integral, and, since the restriction of u to a non-empty open subset of C' is an isomorphism to an open subset of C, u is birational. Since C is assumed to be normal, it suffices to apply (8.11.2) to obtain the conclusion. □

Remark (8.11.8). — (i) The hypothesis that C is normal implies that X is also normal. Indeed, $C' = \operatorname{Spec}(\mathscr{S}')$ is then normal, being isomorphic to C, and integral, by (8.8.6); we thus conclude that $\operatorname{Proj}(\mathscr{S}')$ is *normal*. Indeed, the question is local on Y; if Y is affine, with $\mathscr{S}' = \widetilde{S'}$, then the ring $S' = \Gamma(C', \mathscr{S}')$ is integral and integrally closed (8.8.6.1), and so, for every homogeneous element $f \in S'_+$, the graded ring S'_f is integral and integrally closed [**SZ60**, t. I, p. 257 and 261], and thus so too is the ring $S'_{(f)}$ of its degree-zero terms, because the intersection of S'_f with the field of fractions of $S'_{(f)}$ is equal to $S'_{(f)}$; this proves our claim (6.3.4). Finally, since X is isomorphic to an open subprescheme of $\operatorname{Proj}(\mathscr{S}')$ (8.8.1), X is indeed normal. We can thus express (8.11.7) in the following form: *If X is integral and normal, and $p = (p_0, \theta) : X \to Y$ is a proper morphism such that $\theta : \mathscr{O}_Y \to p_*(\mathscr{O}_X)$ is an isomorphism, then, for every p-ample $\mathscr{O}_X$-module $\mathscr{L}$, there exists exactly one way of contracting the null section of $V = \mathbf{V}(\mathscr{L})$ to obtain a normal Y-scheme C and a proper Y-morphism $q : V \to C$.*

(ii) When p is proper, the hypothesis $p_*(\mathscr{O}_X) = \mathscr{O}_Y$ can be considered as an auxiliary hypothesis, not really restricting the generality of the result. Indeed, if it is not satisfied, then it suffices to replace Y with the Y-scheme $Y' = \operatorname{Spec}(p_*(\mathscr{O}_X))$, and to consider X as a Y'-scheme. We will return to this general method in chapter III, § 4.

8.12. Quasi-coherent sheaves on based cones

(8.12.1). Let us use the hypotheses and notation of (8.3.1). Let $\mathscr{M}$ be a *quasi-coherent graded $\mathscr{S}$-module*; to avoid any confusion, we denote by $\widetilde{\mathscr{M}}$ the quasi-coherent $\mathscr{O}_C$-module associated to $\mathscr{M}$ (1.4.3) when $\mathscr{M}$ is considered as a *non-graded* $\mathscr{S}$-module, and by $\mathscr{P}roj_0(\mathscr{M})$ the quasi-coherent $\mathscr{O}_X$-module associated to $\mathscr{M}$, $\mathscr{M}$ being considered this time as a graded $\mathscr{S}$-module (in other words, the $\mathscr{O}_X$-module denoted by $\widetilde{\mathscr{M}}$ in (3.2.2)). In addition, we set **II | 192**

(8.12.1.1) $$\mathscr{M}_X = \mathscr{P}roj_0(\mathscr{M}) = \bigoplus_{n\in\mathbf{Z}} \mathscr{P}roj_0(\mathscr{M}(n));$$

the quasi-coherent graded $\mathscr{O}_X$-algebra $\mathscr{S}_X$ being defined by (8.6.1.1), $\mathscr{P}roj(\mathscr{M})$ is equipped with a structure of a *(quasi-coherent) graded $\mathscr{S}_X$-module*, by means of the canonical homomorphisms (3.2.6.1)

(8.12.1.2) $$\mathscr{O}_X(m)\otimes_{\mathscr{O}_X}\mathscr{P}roj_0(\mathscr{M}(n)) \longrightarrow \mathscr{P}roj_0(\mathscr{S}(m)\otimes_{\mathscr{S}}\mathscr{M}(n)) \longrightarrow \mathscr{P}roj_0(\mathscr{M}(m+n)),$$

the verification of the axioms of sheaves of modules being done using the commutative diagram in (2.5.11.4).

If $Y = \operatorname{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$, and $\mathscr{M} = \widetilde{M}$, where S is a graded A-algebra and M is a graded S-module, then, for every homogeneous element $f \in S_+$, we have

(8.12.1.3) $$\Gamma(X_f, \mathscr{P}roj(\widetilde{M})) = M_f$$

by the definitions and (8.2.9.1).

Now consider the quasi-coherent graded $\widehat{\mathscr{S}}$-module

(8.12.1.4) $$\widehat{\mathscr{M}} = \mathscr{M}\otimes_{\mathscr{S}}\widehat{\mathscr{S}}$$

($\widehat{\mathscr{S}}$ being defined by (8.3.1.1)); this induces a quasi-coherent graded $\mathscr{O}_{\widehat{C}}$-module $\mathscr{P}roj_0(\widehat{\mathscr{M}})$, which we will also denote by

(8.12.1.5) $$\mathscr{M}^{\square} = \mathscr{P}roj_0(\widehat{\mathscr{M}}).$$

It is clear (3.2.4) that $\mathscr{M}^{\square}$ is an additive functor which is *exact* in $\mathscr{M}$, commuting with direct sums and with inductive limits.

Proposition (8.12.2). — *With the notation of (8.3.2), we have canonical functorial isomorphisms*

(8.12.2.1) $$i^*(\mathscr{M}^{\square}) \xrightarrow{\sim} \widetilde{\mathscr{M}}, \quad j^*(\mathscr{M}^{\square}) \xrightarrow{\sim} \mathscr{P}roj_0(\mathscr{M}).$$

Indeed, $i^*(\mathscr{M}^{\square})$ is canonically identified with $(\widehat{\mathscr{M}}/(\mathbf{z}-1)\widehat{\mathscr{M}})^{\sim}$ on $\operatorname{Spec}(\widehat{\mathscr{S}}/(\mathbf{z}-1)\widehat{\mathscr{S}})$ by (3.2.3); the first of the canonical isomorphisms (8.12.2.1) is then immediately induced (1.4.1) by the canonical isomorphism $\widehat{\mathscr{M}}/(\mathbf{z}-1)\widehat{\mathscr{M}} \xrightarrow{\sim} \mathscr{M}$. The canonical immersion $j : X \to C$ corresponds to the canonical homomorphism $\widehat{\mathscr{S}} \to \mathscr{S}$ with kernel $\mathbf{z}\widehat{\mathscr{S}}$ (8.3.2); the second homomorphism (8.12.2.1) is the particular case of the canonical homomorphism (3.5.2, ii), since here we have $\widehat{\mathscr{M}}\otimes_{\widehat{\mathscr{S}}}\mathscr{S} = \mathscr{M}$; to verify that this is an isomorphism, we can restrict to the case where $Y = \operatorname{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$, and $\mathscr{M} = \widetilde{M}$; by appealing to (2.8.8), the proof that, for all homogeneous f in S_+, the preceding homomorphism, restricted to X_f, restricts to an isomorphism, is then immediate.

By an abuse of language, we again say, thanks to the existence of the first isomorphism (8.12.2.1), that $\mathscr{M}^{\square}$ is the *projective closure* of the $\mathscr{O}_X$-module $\widetilde{\mathscr{M}}$ (it being implicit that the data of the $\mathscr{O}_C$-module $\widetilde{\mathscr{M}}$ includes the grading of the $\mathscr{S}$-module $\mathscr{M}$). **II | 193**

(8.12.3). With the notation of (8.3.5), we have a canonical functorial homomorphism

(8.12.3.1) $$p^*(\mathscr{P}roj(\mathscr{M})) \longrightarrow \mathscr{M}^{\square}|\widehat{E}.$$

Indeed, this is a particular case of the homomorphism $\nu^\sharp$ defined more generally in (3.5.6). If $Y = \operatorname{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$, and $\mathscr{M} = \widetilde{M}$, then, by appealing to (2.8.8), the restriction of (8.12.3.1) to $p^{-1}(X_f) = \widehat{C}_f$ (for some homogeneous f in S_+) corresponds to the canonical homomorphism

(8.12.3.2) $$M_{(f)}\otimes_{S_{(f)}} S_f^{\leqslant} \longrightarrow M_f^{\leqslant}$$

taking into account (8.2.3.2) and (8.2.5.2).

(8.12.4). Let us place ourselves in the settings of (8.5.1), and assume its hypotheses and keep its notation. It follows from (1.5.6) that, for every quasi-coherent graded $\mathscr{S}$-module $\mathscr{S}$, we have, on one hand, a canonical isomorphism

$$\Phi^*(\widetilde{\mathscr{M}}) \xrightarrow{\sim} (q^*(\mathscr{M}) \otimes_{q^*(\mathscr{S})} \mathscr{S}')^\sim \tag{8.12.4.1}$$

of $\mathscr{O}_{C'}$-modules; on the other hand, (3.5.6) implies the existence of a canonical $\mathrm{Proj}(\phi)$-morphism

$$\mathscr{P}roj_0\mathscr{M} \longrightarrow (\mathscr{P}roj_0(q^*(\mathscr{M})) \otimes_{q^*(\mathscr{S})} \mathscr{S}')|G(\phi) \tag{8.12.4.2}$$

and also of a canonical $\widehat{\Phi}$-morphism

$$\mathscr{P}roj_0\widehat{\mathscr{M}} \longrightarrow (\mathscr{P}roj_0(q^*(\widehat{\mathscr{M}})) \otimes_{q^*(\widehat{\mathscr{S}})} \widehat{\mathscr{S}}')|G(\widehat{\phi}). \tag{8.12.4.3}$$

(8.12.5). Consider now the setting of (8.6.1), with the same notation; we thus take $Y' = X$, the morphism $q : X \to Y$ being the structure morphism, and ϕ the canonical q-morphism (8.6.1.2). We then have a canonical isomorphism

$$q^*(\mathscr{M}) \otimes_{q^*(\mathscr{S})} \mathscr{S}_X^{\geqslant} \xrightarrow{\sim} \mathscr{M}_X^{\geqslant} \tag{8.12.5.1}$$

by setting $\mathscr{M}_X^{\geqslant} = \bigoplus_{n\geqslant 0} \mathscr{P}roj_0(\mathscr{M}(n))$. We can indeed restrict to the case where $Y = \mathrm{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$, and $\mathscr{M} = \widetilde{M}$, and define the isomorphism (8.12.5.1) on each of the affine open subsets X_f (where f is homogeneous in S_+), by verifying the compatibility with taking a homogeneous multiple of f. But the restriction to X_f of the left-hand side of (8.12.5.1) is $\widetilde{M}' = ((M \otimes_A S_{(f)}) \otimes_{S\otimes_A S_{(f)}} S_f^{\geqslant})^\sim$ by (8.6.2.1); since we have a canonical isomorphism from $M \otimes_A S_{(f)}$ to $M \otimes_S (S \otimes_A S_{(f)})$, we have an induced isomorphism from $\widetilde{M}'$ to $(M \otimes_S S_f^{\geqslant})^\sim$, and the latter is canonically isomorphic, by (8.2.9.1), to the restriction to X_f of the right-hand side of (8.12.5.1), and satisfies the required compatibility conditions. **II** | 194

Replacing $\mathscr{M}$ by $\widehat{\mathscr{M}}$, $\mathscr{S}$ by $\widehat{\mathscr{S}}$, and $\mathscr{S}_X$ by $(\mathscr{S}_X^{\geqslant})^\wedge$ in the previous argument, we similarly have a canonical isomorphism

$$q^*(\widehat{\mathscr{M}}) \otimes_{q^*(\widehat{\mathscr{S}})} (\mathscr{S}_X^{\geqslant})^\wedge \xrightarrow{\sim} (\mathscr{M}_X^{\geqslant})^\wedge. \tag{8.12.5.2}$$

If we recall (8.6.2) that the structure morphism $u : \mathrm{Proj}(\mathscr{S}_X^{\geqslant}) \to X$ is an isomorphism, then we deduce, first of all, from the above, that we have a canonical u-isomorphism

$$\mathscr{P}roj_0\mathscr{M} \xrightarrow{\sim} \mathscr{P}roj_0(\mathscr{M}_X^{\geqslant}) \tag{8.12.5.3}$$

as a particular case of (8.12.4.2). We note that, with the notation from the proof of (8.6.2), this reduces to seeing that the canonical homomorphism $M_{(f)} \otimes_{S_{(f)}} (S_f^{\geqslant})^{(d)} \to (M_f^{\geqslant})^{(d)}$ is an isomorphism whenever $f \in S_d$, which is immediate.

Secondly, the isomorphism (8.12.5.2) gives us, by this time applying (8.12.4.3) to the canonical morphism $r = \mathrm{Proj}(\widehat{\alpha}) : \widehat{C}_X \to \widehat{C}$, a canonical r-morphism

$$\mathscr{M}^\square \longrightarrow (\mathscr{M}_X^{\geqslant})^\square. \tag{8.12.5.4}$$

Recall now (8.6.2) that the restrictions of r to the pointed cones $\widehat{E}_X$ and E_X are *isomorphisms* to $\widehat{E}$ and E (respectively). Furthermore:

Proposition (8.12.6). — *The restrictions to $\widehat{E}_X$ and E_X of the canonical r-morphism* (8.12.5.4) *are isomorphisms*

$$\mathscr{M}^\square|\widehat{E} \xrightarrow{\sim} (\mathscr{M}_X^{\geqslant})^\square|\widehat{E}_X \tag{8.12.6.1}$$

$$\mathscr{M}^\sim|\widehat{E} \xrightarrow{\sim} (\mathscr{M}_X^{\geqslant})^\sim|\widehat{E}_X. \tag{8.12.6.2}$$

PROOF. We restrict to the case where Y is affine, as in the proof of (8.6.2) (whose notation we adopt); by reducing to definitions (2.8.8), we have to show that the canonical homomorphism

$$\widehat{M}_{(f)} \otimes_{\widehat{S}_{(f)}} (S_f^{\geqslant})^\wedge_{(f/1)} \longrightarrow (M \otimes_S S_f^{\geqslant})^\wedge_{(f/1)}$$

is an isomorphism; but, by (8.2.3.2) and (8.2.5.2), the left-hand side is canonically identified with $M_f^{\leqslant} \otimes_{S_f^{\leqslant}} (S_f^{\geqslant})_{f/1}^{\leqslant}$, and thus with $M_f^{\leqslant}$, by (8.2.7.2), and the right-hand side with $(M_f^{\geqslant})_{f/1}^{\leqslant}$, and thus also with $M_f^{\leqslant}$, by (8.2.9.2), whence the conclusion concerning (8.12.6.1); (8.12.6.2) then follows from (8.12.6.1) and (8.12.2.1). □

Corollary (8.12.7). — *With the identifications of (8.6.3), the restriction of $(\mathscr{M}_X^{\geqslant})^{\square}$ to $\widehat{E}_X$ can be identified with $(\mathscr{M}_X^{\leqslant})^{\sim}$, and the restriction of $(\mathscr{M}_X^{\geqslant})^{\square}$ to E_x with $\widetilde{\mathscr{M}}_X$.*

PROOF. We can restrict to the affine case, and this follows from the identification of $(M_f^{\geqslant})_{f/1}^{\leqslant}$ with $M_f^{\leqslant}$, and of $(M_f^{\geqslant})_{f/1}$ with M_f (8.2.9.2). □

Proposition (8.12.8). — *Under the hypotheses of (8.6.4), the canonical homomorphism (8.12.3.1) is an isomorphism.*

PROOF. Taking into account the fact that $\operatorname{Proj}(\mathscr{S}_X^{\geqslant}) \to X$ is an isomorphism (8.6.2), and the isomorphisms (8.12.5.4) and (8.12.6.1), we are led to proving the corresponding proposition for the canonical homomorphism $p_X^*(\mathscr{P}roj_0(\mathscr{M}_X^{\geqslant})) \to (\mathscr{M}_X^{\geqslant})^{\square}|E_X$, or, in other words, we can restrict to the case where $\mathscr{S}_1$ is an invertible $\mathscr{O}_Y$-module, and where $\mathscr{S}$ is generated by $\mathscr{S}_1$. With the notation of (8.12.3), we then have, for some $f \in S_1$, that $S_f^{\leqslant} = S_{(f)}[1/f]$, and the canonical homomorphism $M_{(f)} \otimes_{S_{(f)}} S_f^{\leqslant} \to M_f^{\leqslant}$ is an isomorphism, by the definition of $M_f^{\leqslant}$. □ II | 195

(8.12.9). Consider now the quasi-coherent $\mathscr{S}$-modules

$$\mathscr{M}_{[n]} = \bigoplus_{m \geqslant n} \mathscr{M}_m$$

and (with the notation of (8.7.2)) the quasi-coherent graded $\mathscr{S}^{\natural}$-module

$$\mathscr{M}^{\natural} = \left(\bigoplus_{n \geqslant 0} \mathscr{M}_{[n]} \right)^{\sim}. \tag{8.12.9.1}$$

We have seen (8.7.3) that there exists a canonical C-isomorphism $h : C_X \xrightarrow{\sim} \operatorname{Proj}(\mathscr{S}^{\natural})$. Furthermore:

Proposition (8.12.10). — *There exists a canonical h-isomorphism*

$$\mathscr{P}roj_0(\mathscr{M}^{\natural}) \xrightarrow{\sim} \widetilde{\mathscr{M}}_X. \tag{8.12.10.1}$$

PROOF. We argue as in (8.7.3), this time using the existence of the di-isomorphism (8.2.9.3) instead of (8.2.7.3). We leave the details to the reader. □

8.13. Projective closures of subsheaves and closed subschemes

(8.13.1). With hypotheses and notation as in (8.12.1), consider a *not-necessarily graded* quasi-coherent sub-$\mathscr{S}$-module $\mathscr{N}$ of $\mathscr{M}$. We can then consider the quasi-coherent $\mathscr{O}_C$-module $\widetilde{\mathscr{N}}$ associated to $\mathscr{N}$, which is a sub-$\mathscr{O}_C$-module of $\widetilde{\mathscr{M}}$. We have seen elsewhere (8.12.2.1) that $\widetilde{\mathscr{M}}$ can be identified with the restriction of $\mathscr{M}^{\square}$ to C. Since the canonical injection $i : C \to \widehat{C}$ is an affine morphism (8.3.2), and *a fortiori* quasi-compact, the *canonical extension* $(\widetilde{\mathscr{N}})^{-}$, the largest sub-$\mathscr{O}_{\widehat{C}}$-module contained in $\mathscr{M}^{\square}$ and inducing $\widetilde{\mathscr{N}}$ on C, is a *quasi-coherent* $\mathscr{O}_{\widehat{C}}$-module (**I**, 9.4.2). We will give a more explicit description by using a graded $\widehat{\mathscr{S}}$-module.

(8.13.2). For this, consider, for every integer $n \geqslant 0$, the homomorphism $\bigoplus_{i \leqslant n} \mathscr{M}_i \to \mathscr{M}$ which, for every open U of Y, sends the family

$$(s_i) \in \bigoplus_{i \leqslant n} \Gamma(U, \mathscr{M}_i)$$

to the section $\sum_i s_i \in \Gamma(U, \mathscr{M})$. Denote by $\mathscr{N}_n'$ the inverse image of $\mathscr{N}$ by this homomorphism, which is a quasi-coherent sub-$\mathscr{S}$-module of $\bigoplus_{i \leqslant n} \mathscr{M}_i$. Now consider the homomorphism $\bigoplus_{i \leqslant n} \mathscr{M}_i \to \widehat{\mathscr{M}} = \mathscr{M}[\mathbf{z}]$ which sends (s_i) to the section $\sum_{i \leqslant n} s_i \mathbf{z}^{n-i} \in \Gamma(U, \widehat{\mathscr{M}}_n)$, and let $\mathscr{N}_n$ be the image of

$\mathscr{N}'_n$ under this homomorphism; we immediately have that $\overline{\mathscr{N}} = \bigoplus_{n\geqslant 0}\mathscr{N}_n$ is a (quasi-coherent) sub-$\widehat{\mathscr{S}}$-module of $\widehat{\mathscr{M}}$; we say that $\overline{\mathscr{N}}$ is induced from $\mathscr{N}$ by *homogenisation*, via the "homogenising variable" $\mathbf{z}$. We note that, if $\mathscr{N}$ is already a *graded* sub-$\mathscr{S}$-module of $\mathscr{M}$, then $\overline{\mathscr{N}}$ can be identified with the direct sum of the components $\widehat{\mathscr{N}_n}$ of degree $n \geqslant 0$ in $\widehat{\mathscr{N}} = \mathscr{N}[\mathbf{z}]$. **II** | 196

Proposition (8.13.3). — *The $\mathscr{O}_{\widehat{C}}$-module $\mathscr{P}roj_0(\overline{\mathscr{N}})$ is the canonical extension $(\widetilde{\mathscr{N}})^-$ of $\widetilde{\mathscr{N}}$ to $\widehat{C}$.*

Proof. The question is local on Y and $\widehat{C}$ by the definition of the canonical extension (**I**, 9.4.1). We can thus already suppose that $Y = \operatorname{Spec}(A)$ is affine, with $\mathscr{S} = \widetilde{S}$, $\mathscr{M} = \widetilde{M}$, and $\mathscr{N} = \widetilde{N}$, where N is a non-necessarily-graded sub-S-module of M. Furthermore (8.3.2.6), $\widehat{C}$ is a union of affine opens $\widehat{C}_z = C$ and $\widehat{C}_f = \operatorname{Spec}(S_f^{\leqslant})$ (with f homogeneous in S_+). It thus suffices to show that: (1) the restriction of $\mathscr{P}roj_0(\overline{\mathscr{N}})$ to C is $\widetilde{\mathscr{N}}$; (2) the restriction of $\mathscr{P}roj_0(\overline{\mathscr{N}})$ to each $\widehat{C}_f$ is the canonical extension of the restriction of $\mathscr{N}$ to $C \cap \widehat{C}_f = \operatorname{Spec}(S_f)$ (8.3.2.6). For the first point, note that $\mathscr{P}roj_0(\overline{\mathscr{N}})|C$ can be identified with $(\overline{N}_{(\mathbf{z})})^\sim$ (8.3.2.4); but $\overline{N}_{(\mathbf{z})}$ is canonically identified (2.2.5) with the image of $\overline{N}$ in $\widehat{M}/(\mathbf{z}-1)\widehat{M}$, and by the canonical isomorphism of the latter with M (8.2.5), this image can be identified with N, by the definition of $\overline{N}$ given in (8.13.2).

To prove the second point, note that the injection $i : C \cap \widehat{C}_f \to \widehat{C}$ corresponds to the canonical injection $S_f^{\leqslant} \to S_f$ (8.3.2.6); we also have that $\Gamma(\widehat{C}_f, \mathscr{M}^\square) = M_f^{\leqslant}$, that $\Gamma(\widehat{C}_f, i_*(\widetilde{\mathscr{N}})) = N$, and, by (8.12.2.1), that $\Gamma(\widehat{C}_f, i_*(i^*(\mathscr{M}^\square))) = M_f$. Taking (**I**, 9.4.2) into account, we are thus led to showing that $\overline{N}_{(f)} \subset \widehat{M}_{(f)} = M_f^{\leqslant}$ is canonically identified with the inverse image of N_f under the canonical injection $M_f^{\leqslant} \to M_f$. Indeed, let $d = \deg(f) > 0$, and suppose that an element $(\sum_{k\leqslant md} x_k)/f^m$ of M_f (with $x_k \in M_k$) is of the form y/f^m with $y \in N$. By multiplying y and the x_k by one single suitable f^h, we can already assume that $\sum_{k\leqslant md} x_k = y$. But in the identification of (8.2.5.2), $(\sum_{k\leqslant md} x_k)/f^m$ corresponds to $\sum_{k\leqslant md} x_k\mathbf{z}^{md-k}/f^m$, and this is indeed an element of $\overline{N}_{(f)}$, since $\sum_{k\leqslant md} x_k \in N$; the converse is evident. □

Remark (8.13.4). — (i) The most important case of application of (8.13.3) is that where $\mathscr{M} = \mathscr{S}$, with $\widetilde{\mathscr{N}}$ then being an *arbitrary* quasi-coherent sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_C$ (1.4.3), corresponding bijectively to a *closed subprescheme* Z of C. Then the canonical extension $\overline{\mathscr{J}}$ of $\mathscr{J}$ is the quasi-coherent sheaf of ideals of $\mathscr{O}_{\widehat{C}}$ that defines the *closure* $\overline{Z}$ of Z in $\widehat{C}$ (**I**, 9.5.10); Proposition (8.13.3) gives a canonical way of defining $\overline{Z}$ by using a graded ideal in $\widehat{\mathscr{S}} = \mathscr{S}[\mathbf{z}]$.

(ii) Suppose, to simplify things, that Y is affine, and adopt the notation from the proof of (8.13.3). For every non-zero $x \in N$, let $d(x)$ be the largest degree of the homogeneous components x_i of x in M; by definition, $\overline{N}$ is the submodule of $\widehat{M}$ consisting of 0 and elements of the form $h(x,k) = \mathbf{z}^k \sum_{i\leqslant d(x)} x_i\mathbf{z}^{d(x)-i}$ (for integral $k \geqslant 0$); it is thus generated, as a module over $\widehat{S} = S[\mathbf{z}]$, by the elements of the form

$$h(x,0) = \sum_{i\leqslant d(x)} x_i\mathbf{z}^{d(x)-i}.$$

We say that $h(x,0)$ is induced from x by *homogenisation* via the "homogenising variable" $\mathbf{z}$. **II** | 197
But since $h(x,0)$ does not depend additively on x (nor *a fortiori* S-linearly), *we will refrain from believing* (even when $M = S$) that the $h(x,0)$ form a *system of generators* of the graded $\widehat{S}$-module $\overline{N}$ when we let x run over a *system of generators* of the S-module N. This is, however, the case (considered only in elementary algebraic geometry) when N is a *free cyclic* S-module, since, if t is a basis of N, then $h(t,0)$ generates the $\widehat{S}$-module $\overline{N}$.

8.14. Supplement on sheaves associated to graded $\mathscr{S}$-modules

(8.14.1). Let Y be a prescheme, $\mathscr{S}$ a *positively-graded* quasi-coherent $\mathscr{O}_Y$-algebra, $X = \operatorname{Proj}(\mathscr{S})$, and $q : X \to Y$ the structure morphism (which is separated, by (3.1.3)). Using the notation of (8.12.1), we have defined a functor $\mathscr{M}_X = \mathscr{P}\!roj(\mathscr{M})$ in $\mathscr{M}$, from the category of quasi-coherent graded $\mathscr{S}$-modules to the category of quasi-coherent graded $\mathscr{S}_X$-modules; it is further clear (3.2.4) that this is an *additive* and *exact* functor, commuting with inductive limits.

Note, furthermore, that it follows immediately from the definition (8.12.1.1) that we have

$$\mathscr{P}\!roj(\mathscr{M}(n)) = (\mathscr{P}\!roj(\mathscr{M}))(n) \quad \text{for all } n \in \mathbf{Z}. \tag{8.14.1.1}$$

(8.14.2). We will first extend the canonical homomorphisms λ and μ, defined in (3.2.6), to $\mathscr{S}_X$-modules of the form $\mathscr{P}\!roj(\mathscr{M})$. For this, note that, for any $m \in \mathbf{Z}$ and $n \in \mathbf{Z}$, we have, by (2.1.2.1), a canonical homomorphism of $\mathscr{O}_X$-modules

$$\lambda_{mn} : \mathscr{P}\!roj_0((\mathscr{H}\!om_{\mathscr{S}}(\mathscr{M},\mathscr{N}))(n-m)) \longrightarrow \mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{P}\!roj_0(\mathscr{M}(m)), \mathscr{P}\!roj_0(\mathscr{N}(n))) \tag{8.14.2.1}$$

for any quasi-coherent graded $\mathscr{S}$-modules $\mathscr{M}$ and $\mathscr{N}$. This induces a homomorphism

$$\mu_k : \mathscr{P}\!roj_0((\mathscr{H}\!om_{\mathscr{S}}(\mathscr{M},\mathscr{N}))(k)) \longrightarrow (\mathscr{H}\!om_{\mathscr{S}_X}(\mathscr{P}\!roj(\mathscr{M}), \mathscr{P}\!roj(\mathscr{N})))_k \tag{8.14.2.2}$$

given by sending every $u \in \Gamma(U, \mathscr{P}\!roj_0((\mathscr{H}\!om_{\mathscr{S}}(\mathscr{M},\mathscr{N}))(k)))$ to the homomorphism $\mu_k(u)$, of degree k, of graded $\mathbf{Z}$-modules $\Gamma(U, \mathscr{P}\!roj(\mathscr{M})) \to \Gamma(U, \mathscr{P}\!roj(\mathscr{N}))$ (where U is open in X) which, in each $\Gamma(U, \mathscr{P}\!roj_0(\mathscr{M}(m)))$, agrees with $\mu_{m,m+k}(u)$; furthermore, by returning to the definition of the μ_{mn} (2.5.12.1), we immediately see that $\mu_k(u)$ is in fact a homomorphism of degree k of graded $\Gamma(U, \mathscr{S}_X)$-modules, and, furthermore, that the μ_k define a homomorphism of *graded $\mathscr{S}_X$-modules*

$$\mathscr{P}\!roj(\mathscr{H}\!om_{\mathscr{S}}(\mathscr{M},\mathscr{N})) \longrightarrow \mathscr{H}\!om_{\mathscr{S}_X}(\mathscr{P}\!roj(\mathscr{M}), \mathscr{P}\!roj(\mathscr{N})). \tag{8.14.2.3}$$

Similarly, taking the associativity diagram (2.5.11.4) into account, the homomorphisms (8.14.2.1) give a homomorphism of *graded $\mathscr{S}_X$-modules*

$$\lambda : \mathscr{P}\!roj(\mathscr{M}) \otimes_{\mathscr{S}_X} \mathscr{P}\!roj(\mathscr{N}) \longrightarrow \mathscr{P}\!roj(\mathscr{M} \otimes_{\mathscr{S}} \mathscr{N}). \tag{8.14.2.4}$$

II | 198

Proposition (8.14.3). — *The homomorphism (8.14.2.4) is bijective; so too is (8.14.2.3) whenever the graded $\mathscr{S}$-module $\mathscr{M}$ admits a finite presentation (3.1.1).*

Proof. The question is clearly local on X and Y; we can thus suppose that $Y = \operatorname{Spec}(A)$ is affine, with $\mathscr{S} = \widetilde{S}$, $\mathscr{M} = \widetilde{M}$, and $\mathscr{N} = \widetilde{N}$, where S is a positively-graded A-algebra, and M and N are graded S-modules. If f is a homogeneous element of S_+, then the homomorphisms (8.14.2.1) and (8.14.2.2), restricted to the affine open $D_+(f)$, correspond to the canonical homomorphisms (2.5.11.1) and (2.5.12.1):

$$M(m)_{(f)} \otimes_{S_{(f)}} N(n)_{(f)} \longrightarrow (M \otimes_S N)(m+n)_{(f)}$$
$$(\operatorname{Hom}_S(M,N))(n-m)_{(f)} \longrightarrow \operatorname{Hom}_{S_{(f)}}(M(m)_{(f)}, N(n)_{(f)}).$$

If we refer to the definitions of these homomorphisms, we thus see (taking (8.2.9.1) into account) that the restriction of (8.14.2.4) to $D_+(f)$ corresponds to the canonical homomorphism

$$M_f \otimes_{S_f} N_f \longrightarrow (M \otimes_S N)_f$$

defined in (**0**, 1.3.4), and we know that this latter homomorphism is an isomorphism. Similarly, the restriction of (8.14.2.3) to $D_+(f)$ corresponds to the canonical homomorphism (**0**, 1.3.5)

$$(\operatorname{Hom}_S(M,N))_f \longrightarrow \operatorname{Hom}_{S_f}(M_f, N_f)$$

taking into account the fact that, since M is of finite type, the module $\operatorname{Hom}_S(M,N)$, the direct sum of the subgroups consisting of *homogeneous* homomorphisms of S-modules (2.1.2), agrees with the set of *all* homomorphisms $M \to N$ of S-modules. The hypothesis that M admits a finite presentation then implies (**0**, 1.3.5) that the canonical homomorphism in question is indeed an isomorphism. □

Proposition (8.14.4). — *If U is a quasi-compact open of X, then there exists an integer d such that, for every integer n that is a multiple of d, $\mathscr{O}_X(n)|U$ is invertible, with its inverse being $\mathscr{O}_X(-n)|U$.*

PROOF. Since $q(U)$ is quasi-compact, it is covered by a finite number of affine opens V_i, and so every $x \in U$ is contained in some affine open of the form $D_+(f)$, where f is a homogeneous element of degree > 0 of one of the rings $\Gamma(V_i, \mathscr{S})$. Since U is quasi-compact, we can cover it by a finite number of such opens $D_+(f_j)$; let d be a common multiple of the degrees of the f_j. This d satisfies the desired property, by (2.5.17). □

(8.14.5). With the hypotheses and notation of (8.14.1), we defined, in (3.3.2), canonical homomorphisms of $\mathscr{O}_Y$-modules

$$\alpha_n : \mathscr{M}_n \longrightarrow q_*(\mathscr{P}roj_0(\mathscr{M}(n))) \qquad (n \in \mathbf{Z}). \tag{8.14.5.1}$$

Generalising the notation of (3.3.1), we set, for every *graded $\mathscr{S}_X$-module $\mathscr{F}$*,

$$\mathbf{\Gamma}_*(\mathscr{F}) = \bigoplus_{n\in\mathbf{Z}} q_*(\mathscr{F}_n). \tag{8.14.5.2}$$

In particular, $\mathbf{\Gamma}(\mathscr{S}_X) = \bigoplus_{n\in\mathbf{Z}} q_*(\mathscr{O}_X(n))$ is the graded $\mathscr{O}_Y$-algebra denoted by $\mathbf{\Gamma}_*(\mathscr{O}_X)$ in (3.3.1.2); it is clear that $\mathbf{\Gamma}(\mathscr{F})$ is a *graded $\mathbf{\Gamma}_*(\mathscr{S}_X)$-algebra* (**0**, 4.2.2). When we take $\mathscr{M} = \mathscr{S}$ in the homomorphisms (8.14.5.1), we obtain the homomorphism of graded $\mathscr{O}_Y$-algebras **II | 199**

$$\alpha : \mathscr{S} \longrightarrow \mathbf{\Gamma}(\mathscr{S}_X) \tag{8.14.5.3}$$

previously defined in (3.3.2), and which makes $\mathbf{\Gamma}_*(\mathscr{F})$ a *graded $\mathscr{S}$-module*; the homomorphisms (8.14.5.1) then define a homomorphism (of degree 0) of *graded $\mathscr{S}$-modules*

$$\alpha : \mathscr{M} \longrightarrow \mathbf{\Gamma}_*(\mathscr{P}roj(\mathscr{M})). \tag{8.14.5.4}$$

(8.14.6). In general, for a quasi-coherent graded $\mathscr{S}_X$-module $\mathscr{F}$, it is not certain that the graded $\mathscr{S}$-module $\mathbf{\Gamma}_*(\mathscr{F})$ will necessarily be quasi-coherent. Consider an open X' of X such that the restriction $q' : X' \to Y$ of q to X' is a *quasi-compact* morphism. Since q' is further separated, $q'_*(\mathscr{F}')$ is then a quasi-coherent $\mathscr{O}_Y$-module for every quasi-coherent $\mathscr{O}_{X'}$ module $\mathscr{F}'$ (**I**, 9.2.2, b). We set

$$\mathscr{S}_{X'} = \mathscr{S}_X|X' = \bigoplus_{n\in\mathbf{Z}} \mathscr{O}_X(n)|X' \tag{8.14.6.1}$$

and, for every graded $\mathscr{S}_{X'}$-module $\mathscr{F}'$,

$$\mathbf{\Gamma}'_*(\mathscr{F}') = \bigoplus_{n\in\mathbf{Z}} q'_*(\mathscr{F}'_n). \tag{8.14.6.2}$$

The previous remark then shows that, if $\mathscr{F}'$ is a quasi-coherent $\mathscr{S}_{X'}$-module, then $\mathbf{\Gamma}'_*(\mathscr{F}')$ is a graded *quasi-coherent* $\mathscr{S}$-module (**I**, 9.6.1).

We note also that the canonical injection $j : X' \to X$ is *quasi-compact*, because $q' = q \circ j$ is quasi-compact and q is separated (**I**, 6.6.4, v). Then $\mathscr{F} = j_*(\mathscr{F}')$ is a quasi-coherent graded $\mathscr{S}_X$-module for every quasi-coherent graded $\mathscr{S}_{X'}$-module $\mathscr{F}'$, and it follows from the previous definitions that

$$\mathbf{\Gamma}'_*(\mathscr{F}') = \mathbf{\Gamma}_*(\mathscr{F}). \tag{8.14.6.3}$$

With the same hypotheses on X', for every quasi-coherent graded $\mathscr{S}$-module $\mathscr{M}$, we set

$$\mathscr{P}roj'(\mathscr{M}) = \mathscr{P}roj(\mathscr{M})|X' \tag{8.14.6.4}$$

which is a quasi-coherent graded $\mathscr{S}_{X'}$-module. The canonical homomorphism

$$\mathscr{P}roj(\mathscr{M}) \longrightarrow j_*(\mathscr{P}roj'(\mathscr{M}))$$

(**0**, 4.4.3) thus gives a canonical homomorphism $\mathbf{\Gamma}_*(\mathscr{P}roj(\mathscr{M})) \to \mathbf{\Gamma}'_*(\mathscr{P}roj'(\mathscr{M}))$ of graded $\mathscr{S}$-modules, and, by composition with (8.14.5.4), we obtain a functorial canonical homomorphism (of degree 0) of quasi-coherent graded $\mathscr{S}$-modules

$$\alpha' : \mathscr{M} \longrightarrow \mathbf{\Gamma}'_*(\mathscr{P}roj'(\mathscr{M})). \tag{8.14.6.5}$$

(8.14.7). Keeping the hypotheses on X' from (8.14.6), let $\mathscr{F}'$ be a *quasi-coherent graded $\mathscr{S}_{X'}$-module* such that $\mathscr{P}roj'(\mathbf{\Gamma}'_*(\mathscr{F}'))$ is also a graded *quasi-coherent* $\mathscr{S}_{X'}$-module. We will define a functorial canonical homomorphism (of degree 0) of graded $\mathscr{S}_{X'}$-modules **II | 200**

$$\beta' : \mathscr{P}roj'(\mathbf{\Gamma}'_*(\mathscr{F}')) \longrightarrow \mathscr{F}'. \tag{8.14.7.1}$$

Suppose first of all that $Y = \mathrm{Spec}(A)$ is affine, and that $\mathscr{S} = \widetilde{S}$, where S is a positively-graded A-algebra; then $\mathbf{\Gamma}'_*(\mathscr{F}') = \widetilde{M}$, where $M = \bigoplus n \in \mathbf{Z}\Gamma(X', \mathscr{F}'_n)$ is a graded S-module. Let $f \in S_d$ be such that $D_+(f) \subset X'$; by definition (2.6.2), $\alpha_d(f)$ restricted to $D_+(f)$ is the section of $\mathscr{O}_X(d)$ over $D_+(f)$ corresponding to the element $f/1$ of $(S(d))_{(f)}$, and is thus invertible; thus so too is $\alpha_d(f^n)$ for every $n > 0$. From this, we immediately conclude that we have defined an S_f-homomorphism (of degree 0) of graded modules $\beta_f : M_f \to \Gamma(D_+(f), \mathscr{F}')$ by sending each element $z/f^n \in M_f$ (where $z \in M$) to the section $(z|D_+(f))(\alpha_d(f^n)|D_+(f))^{-1}$ of $\mathscr{F}'$ over $D_+(f)$. Furthermore, we have a commutative diagram corresponding to (2.6.4.1), whence the definition of β' in this case. To pass to the general case, we must consider an A-algebra A', the graded A'-algebra $S' = S \otimes_A A'$, and use the commutative diagram analogous to (2.8.13.2); we leave the details to the reader.

Proposition (8.14.8). — *If X' is an open of $X = \mathrm{Proj}(\mathscr{S})$ such that $q' : X' \to Y$ is quasi-compact, then the homomorphism β' defined in (8.14.7) is bijective.*

PROOF. We can clearly restrict to the case where Y is affine, and everything then reduces to proving (with the notation of (8.14.7)) that the homomorphism $\beta_f : M_f \to \Gamma(D_+(f), \mathscr{F}')$ is an isomorphism. But replacing f by one of its powers changes neither $D_+(f)$ nor β_f; since X' is *quasi-compact* by hypothesis, we can always assume, by (8.14.4), that the sheaf $\mathscr{O}_X(d)$ is *invertible*. Since X' is a scheme (because q' is separated), the proposition is then exactly (**I**, 9.3.1). □

Corollary (8.14.9). — *Under the hypotheses of (8.14.8), every quasi-coherent graded $\mathscr{S}_{X'}$-module is isomorphic to a graded $\mathscr{S}_{X'}$-module of the form $\mathscr{P}roj'(\mathscr{M})$, where $\mathscr{M}$ is a quasi-coherent graded $\mathscr{S}$-module. Further, if $\mathscr{F}'$ is of finite type, and if we assume that Y is a quasi-compact scheme, or a prescheme whose underlying space is Noetherian, then we can assume that $\mathscr{M}$ is of finite type.*

PROOF. The proof starting from (8.14.8) follows exactly the same route as the proof of (3.4.5) starting from (3.4.4), and we leave the details to the reader. □

Proposition (8.14.10). — *Under the hypotheses of (8.14.7), let $\mathscr{M}$ be a quasi-coherent graded $\mathscr{S}$-module, and $\mathscr{F}'$ a quasi-coherent graded $\mathscr{S}_{X'}$-module; the composite homomorphisms*

(8.14.10.1) $$\mathscr{P}roj'(\mathscr{M}) \xrightarrow{\mathscr{P}roj'(\alpha')} \mathscr{P}roj'(\mathbf{\Gamma}'_*(\mathscr{P}roj'(\mathscr{M}))) \xrightarrow{\beta'} \mathscr{P}roj'(\mathscr{M})$$

(8.14.10.2) $$\mathbf{\Gamma}'_*(\mathscr{F}') \xrightarrow{\alpha'} \mathbf{\Gamma}'_*(\mathscr{P}roj'(\mathbf{\Gamma}'_*(\mathscr{F}'))) \xrightarrow{\mathbf{\Gamma}'_*(\beta')} \mathbf{\Gamma}'_*(\mathscr{F}')$$

are the identity isomorphisms.

PROOF. The question is local on Y, and the proof follows as in (2.6.5); we leave the details to the reader. □

Remark (8.14.11). — In chapter III (**III**, 2.3.1), we will see that, when Y is *locally Noetherian*, and $\mathscr{S}$ is a quasi-coherent graded $\mathscr{O}_Y$-algebra *of finite type* (in which case we can take $X' = X$), then **II | 201**
the homomorphism α (8.14.5.4) is **(TN)**-*bijective* for every quasi-coherent graded $\mathscr{S}$-module $\mathscr{M}$ satisfying condition **(TF)**.

Remark (8.14.12). — The situation described in (8.14.4) is a particular case of the following. Let X be a ringed space, and $\mathscr{S}$ a (positively- and negatively-) graded $\mathscr{O}_X$-algebra; suppose that there exists an integer $d > 0$ such that $\mathscr{S}_d$ and $\mathscr{S}_{-d}$ are *invertible*, with the canonical homomorphism

(8.14.12.1) $$\mathscr{S}_d \otimes_{\mathscr{O}_X} \mathscr{S}_{-d} \longrightarrow \mathscr{O}_X$$

being an *isomorphism* (such that $\mathscr{S}_{-d}$ is identified with $\mathscr{S}_d^{-1}$). We then say that the graded $\mathscr{O}_X$-algebra $\mathscr{S}$ is *periodic*, *of period d*. This nomenclature stems from the following property: *under the preceding hypotheses, for every graded $\mathscr{S}$-module $\mathscr{F}$, the canonical homomorphism*

(8.14.12.1) $$\mathscr{S}_d \otimes \mathscr{F}_n \longrightarrow \mathscr{F}_{n+d}$$

is an isomorphism for all $n \in \mathbf{Z}$. Indeed, the question is local on X, and we can assume that $\mathscr{S}_d$ has an *invertible* section s over X, with its inverse s' being a section of $\mathscr{S}_{-d}$. The homomorphism $\mathscr{F}_{n+d} \to \mathscr{S}_d \otimes \mathscr{F}_n$, which sends each section $z \in \Gamma(U, \mathscr{F}_{n+d})$ to the section $(s|U) \otimes (s'|U)z$ of

$\mathscr{S}_d \otimes \mathscr{F}_n$ over U, is then the inverse of (8.14.12.2), whence our claim. This induces, for all $k \in \mathbf{Z}$, a canonical isomorphism

$$(\mathscr{S}_d)^{\otimes k} \otimes \mathscr{F}_n \xrightarrow{\sim} \mathscr{F}_{n+kd}.$$

Then *the data of a graded $\mathscr{S}$-module $\mathscr{F}$ is equivalent to the data of $\mathscr{S}_0$-modules $\mathscr{F}_i$ ($0 \leqslant i \leqslant d-1$) and canonical homomorphisms*

$$\mathscr{S}_i \otimes \mathscr{F}_j \longrightarrow \mathscr{F}_{i+j} \qquad \text{for } 0 \leqslant i,j \leqslant d-1$$

(setting $\mathscr{F}_{i+j} = \mathscr{S}_d \otimes_{\mathscr{S}_0} \mathscr{F}_{i+j-d}$ whenever $i+j \geqslant d$). Of course, for theses homomorphisms to give a well-defined $\mathscr{S}$-module structure on the direct sum of the $(\mathscr{S}_d)^{\otimes k} \otimes \mathscr{F}_i$ ($k \in \mathbf{Z}$, $0 \leqslant i \leqslant d-1$), they should satisfy some associativity conditions that we will not explain.

In the case where $d = 1$ (which is the one considered in (3.3)), we can thus say that the category of graded $\mathscr{S}$-modules (resp. quasi-coherent $\mathscr{S}$-modules if X is a prescheme and $\mathscr{S}$ is quasi-coherent) is *equivalent* to the category of arbitrary $\mathscr{S}_0$-modules (resp. quasi-coherent $\mathscr{S}_0$-modules); it is in this way that we can think of the results of this paragraph as generalising those of §3. Furthermore, we see that, under suitable finiteness conditions, the results of this paragraph (along with (8.14.11)) reduces, in some sense, the study of quasi-coherent graded algebras on a prescheme, and graded modules "modulo **(TN)**" on such algebras, to the study of the particular case where the algebras in question are *periodic* (and where condition **(TN)** for $\mathscr{M}$ (3.4.2) thus implies that $\mathscr{M} = 0$).

Remark (8.14.13). — Under the hypotheses of (8.14.1), let d be an integer > 0; we have defined a canonical Y-isomorphism h from X to $X^{(d)} = \operatorname{Proj}(\mathscr{S}^{(d)})$ (3.1.8). For every quasi-coherent graded $\mathscr{S}$-module $\mathscr{M}$ and every integer k such that $0 \leqslant k \leqslant d-1$, we also have (with the notation of (3.1.1)) a canonical h-isomorphism **II | 202**

(8.14.13.1)
$$(\mathscr{P}roj(\mathscr{M}))^{(d,k)} \xleftarrow{\sim} \mathscr{P}roj(\mathscr{M}^{(d,k)}).$$

Suppose, first of all, that $Y = \operatorname{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$, and $\mathscr{M} = \widetilde{M}$, where S is a positively-graded A-algebra, and M a graded S-module. We know, for every $f \in S_e$ ($e > 0$), that h sends $D_+(f)$ to $D_+(f^d)$, and corresponds to the canonical isomorphism $S_{(f^d)} \to S_{(f)}$ (2.2.2). The restriction of (8.14.13.1) to $D_+(f^d)$ then corresponds to the canonical di-isomorphism $M_{f^d} \to M_f$ restricted to the elements of M_{f^d} whose degree is congruent to k (modulo d). We leave to the reader the task of showing that these isomorphisms are compatible with passing from f to some homogeneous multiple fg, and then that there is an analogous compatibility with passing from S to a graded A'-algebra $S' = S \otimes_A A'$, where A' is some A-algebra. In particular, this gives us an h-isomorphism

(8.14.13.2)
$$(\mathscr{S}^{(d)})_{X^{(d)}} \xrightarrow{\sim} (\mathscr{S}_X)^{(d)}$$

that respects the multiplicative structures of both the source and the target, and that, thanks to (8.14.13.1), becomes an h-di-isomorphism from a graded $(\mathscr{S}^{(d)})_{X^{(d)}}$-module to a graded $(\mathscr{S}_X)^{(d)}$-module. Similarly, we have an h-isomorphism

(8.14.13.3)
$$\mathscr{P}roj_0(\mathscr{S}^{(d,k)}(n)) \xrightarrow{\sim} \mathscr{O}_X(nd+k),$$

which completes the result of (3.2.9, ii).

The isomorphism in (8.14.13.1) immediately induces an isomorphism of graded $\mathscr{S}^{(d)}$-modules

(8.14.13.4)
$$\mathbf{\Gamma}_*^{(d)}(\mathscr{P}roj(\mathscr{M}^{(d,k)})) \xrightarrow{\sim} \mathbf{\Gamma}_*((\mathscr{P}roj(\mathscr{M}))^{(d,k)})$$

where $\mathbf{\Gamma}_*^{(d)}$ corresponds to the structure morphism $q^{(d)} : X^{(d)} \to Y$; it can be immediately verified that the canonical homomorphism α (8.14.5.4), and the analogous homomorphism $\alpha^{(d)}$ for $X^{(d)}$, make the following diagram commute:

(8.14.13.5)
$$\begin{tikzcd} & \mathscr{M}^{(d,k)} \arrow[dl, "\alpha^{(d)}"'] \arrow[dr, "\alpha"] & \\ \mathbf{\Gamma}_*^{(d)}(\mathscr{P}roj(\mathscr{M}^{(d,k)})) \arrow[rr, "\sim"] & & \mathbf{\Gamma}_*((\mathscr{P}roj(\mathscr{M}))^{(d,k)}) \end{tikzcd}$$

where we proceed by supposing that Y is affine and then calculating the restrictions of the images under $\alpha^{(d)}$ and α of some single element of $M^{(d,k)}$ to the open subsets $D_+(f^d)$ and $D_+(f)$ (using the same notation as above).

Proposition (8.14.14). — *Let Y be a quasi-compact prescheme, $\mathscr{S}$ a quasi-coherent graded $\mathscr{O}_Y$-algebra of finite type, and $\mathscr{M}$ a quasi-coherent graded $\mathscr{S}$-module satisfying condition* **(TF)***; let $X = \operatorname{Proj}(\mathscr{S})$. Then $\mathscr{S}_X$ is a periodic graded $\mathscr{O}_X$-algebra (8.14.12), and there exists some period d of $\mathscr{S}_X$ such that the $(\mathscr{P}roj(\mathscr{M}))^{(d,k)}$* II | 203
$(0 \leqslant k \leqslant d-1)$ are $(\mathscr{S}_X)^{(d)}$-modules of finite type.

Proof. Indeed, (3.1.10) proves that there exists some d such that $\mathscr{S}^{(d)}$ is generated by $\mathscr{S}_d = (\mathscr{S}^{(d)})_1$, with the latter being an $\mathscr{S}_0$-module of finite type. To prove the first claim, we can thus, by (8.14.13.2), restrict to the case where $d = 1$, and the proposition then follows from (3.2.7). Furthermore, taking (8.14.13.1) into account, the second claim is a consequence of (2.1.6, iii) and (3.4.3). □

CHAPTER III

Cohomological study of coherent sheaves (EGA III)

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Summary

This chapter gives the fundamental theorems concerning the cohomology of coherent algebraic sheaves, with the exception of theorems explaining the theory of residues (duality theorems), which will be the subject of a later chapter. Amongst all those included here, there are essentially six fundamental theorems, and each one is the subject of one of the first six chapters. These results will prove to be essential tools in all that follows, even in questions which are not truly cohomological in their nature; the reader will see the first such examples starting from §4. §7 gives some more technical results, but ones which are constantly used in applications. Finally, in §§8–11, we will develop certain results, related to the duality of coherent sheaves, that are particularly important for applications, and which can be explained even before the introduction of the full general theory of residues.

The content of §§1 and 2 is due to J.-P. Serre, and the reader will observe that we have had only to follow (FAC). §8 and 9 are equally inspired by (FAC) (the changes necessitated by the different contexts, however, being less evident). Finally, as we said in the Introduction, §4 should be considered as the formalisation, in modern language, of the fundamental "invariance theorem" of Zariski's "theory of holomorphic functions".

We draw attention to the fact that the results of n°3.4 (and the preliminary propositions of (**0**, 13.4 to 13.7)) will not be used in what follows Chapter III, and can thus be skipped in a first reading.

§1. Cohomology of affine schemes

1.1. Review of the exterior algebra complex

(1.1.1). Let A be a ring, $\mathbf{f} = (f_i)_{1\leqslant i\leqslant r}$ a system of r elements of A. The *exterior algebra complex* $K_\bullet(\mathbf{f})$ corresponding to $\mathbf{f}$ is a chain complex (G, I, 2.2) defined in the following way: the graded A-module $K_\bullet(\mathbf{f})$ is equal to the *exterior algebra* $\wedge(A^r)$, graded in the usual way, and the boundary map is the *interior multiplication* $i_\mathbf{f}$ by $\mathbf{f}$ considered as an element of the dual $(A^r)^\vee$; we recall that $i_\mathbf{f}$ is an

[1] EGA IV does not depend on §§8–11, and will probably be published before these chapters. *[Trans.] These last four chapters were never published.*

antiderivation of degree -1 of $\wedge(A^r)$, and if $(\mathbf{e}_i)_{1\leqslant i\leqslant r}$ is the canonical basis of A^r, then we have $i_{\mathbf{f}}(\mathbf{e}_i) = f_i$; the verification of the condition $i_{\mathbf{f}} \circ i_{\mathbf{f}} = 0$ is immediate.

An equivalent definition is the following: for each i, we consider a chain complex $K_\bullet(f_i)$ defined as follows: $K_0(f_i) = K_1(f_i) = A$, $K_n(f_i) = 0$ for $n \neq 0, 1$: the boundary map is defined by the condition that $d_1 : A \to A$ is *multiplication by* f_i. We then take $K_\bullet(\mathbf{f})$ to be the *tensor product* $K_\bullet(f_1) \otimes K_\bullet(f_2) \otimes \cdots \otimes K_\bullet(f_r)$ (G, I, 2.7) with its total degree; the verification of the isomorphism from this complex to the complex defined above is immediate.

(1.1.2). For every A-module M, we define the *chain complex*

$$K_\bullet(\mathbf{f}, M) = K_\bullet(\mathbf{f}) \otimes_A M \tag{1.1.2.1}$$

and the *cochain complex* (G, I, 2.2)

$$K^\bullet(\mathbf{f}, M) = \mathrm{Hom}_A(K_\bullet(\mathbf{f}, M). \tag{1.1.2.2}$$

If g is a k-cochain of this latter complex, and if we set

$$g(i_1, \ldots, i_k) = g(\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}),$$

then g identifies with an *alternating* map from $[1, r]^k$ to M, and it follows from the above definitions that we have

$$d^k g(i_1, i_2, \ldots, i_{k+1}) = \sum_{h=1}^{k+1} (-1)^{h-1} f_{i_h} g(i_1, \ldots, \widehat{i_h}, \ldots, i_{k+1}). \tag{1.1.2.3}$$

(1.1.3). From the above complexes, we deduce as usual the *homology and cohomology A-modules* (G, I, 2.2) **III** | 83

$$\mathrm{H}_\bullet(\mathbf{f}, M) = \mathrm{H}_\bullet(K_\bullet(\mathbf{f}, M)), \tag{1.1.3.1}$$

$$\mathrm{H}^\bullet(\mathbf{f}, M) = \mathrm{H}^\bullet(K^\bullet(\mathbf{f}, M)). \tag{1.1.3.2}$$

We define an *A-isomorphism* $K_\bullet(\mathbf{f}, M) \simeq K^\bullet(\mathbf{f}, M)$ by sending each chain $z = \sum(\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}) \otimes z_{i_1,\ldots,i_k}$ to the cochain g_z such that $g_z(j_1, \ldots, j_{r-k}) = \varepsilon z_{i_1,\ldots,i_k}$, where $(j_h)_{1\leqslant h\leqslant r-k}$ is the strictly increasing sequence complementary to the strictly increasing sequence $(i_h)_{1\leqslant h\leqslant k}$ in $[1, r]$ and $\varepsilon = (-1)^\nu$, where ν is the number of inversions of the permutation $i_1, \ldots, i_k, j_1, \ldots, j_{r-k}$ of $[1, r]$. We verify that $g_{dz} = d(g_z)$, which gives an isomorphism

$$\mathrm{H}^i(\mathbf{f}, M) \simeq \mathrm{H}_{r-i}(\mathbf{f}, M) \text{ for } 0 \leqslant i \leqslant r. \tag{1.1.3.3}$$

In this chapter, we will especially consider the cohomology modules $\mathrm{H}^\bullet(\mathbf{f}, M)$.

For a given $\mathbf{f}$, it is immediate (G, I, 2.1) that $M \mapsto \mathrm{H}^\bullet(\mathbf{f}, M)$ is a *cohomological functor* (T, II, 2.1) from the category of A-modules to the category of graded A-modules, zero in degrees < 0 and $> r$. In addition, we have

$$\mathrm{H}^0(\mathbf{f}, M) = \mathrm{Hom}_A(A/(\mathbf{f}), M), \tag{1.1.3.4}$$

denoting by $(\mathbf{f})$ the ideal of A generated by $f_1, \ldots, f_r$; this follows immediately from (1.1.2.3), and it is clear that $\mathrm{H}^0(\mathbf{f}, M)$ identifies with the submodule of M *killed by* $(\mathbf{f})$. Similarly, we have by (1.1.2.3) that

$$\mathrm{H}^r(\mathbf{f}, M) = M / \left(\sum_{i=1}^r f_i M \right) = (A/(\mathbf{f})) \otimes_A M. \tag{1.1.3.5}$$

We will use the following known result, which we will recall a proof of to be complete:

Proposition (1.1.4). — *Let A be a ring, $\mathbf{f} = (f_i)_{1\leqslant i\leqslant r}$ a finite family of elements of A, and M an A-module. If, for $1 \leqslant i \leqslant r$, the scaling $z \mapsto f_i \cdot z$ on $M_{i-1} = M/(f_1 M + \cdots + f_{i-1} M)$ is injective, then we have $\mathrm{H}^i(\mathbf{f}, M) = 0$ for $i \neq r$.*

It suffices to prove that $\mathrm{H}_i(\mathbf{f}, M) = 0$ for all $i > 0$ according to (1.1.3.3). We argue by induction on r, the case $r = 0$ being trivial. Set $\mathbf{f}' = (f_i)_{1\leqslant i\leqslant r-1}$; this family satisfies the conditions in the statement, so if we set $L_\bullet = K_\bullet(\mathbf{f}', M)$, then we have $\mathrm{H}_i(L_\bullet) = 0$ for $i > 0$ by hypothesis, and $\mathrm{H}_0(L_\bullet) = M_{r-1}$ by virtue of (1.1.3.3) and (1.1.3.5). To abbreviate, set $K_\bullet = K_\bullet(f_r) = K_0 \oplus K_1$, with $K_0 = K_1 = A$, $d_1 : K_1 \to K_0$ multiplication by f_r; we have by definition (1.1.1) that $K_\bullet(\mathbf{f}, M) = K_\bullet \otimes_A L_\bullet$. We have the following lemma:

Lemma (1.1.4.1). — *Let $K_\bullet$ be a chain complex of free A-modules, zero except in dimensions* 0 *and* 1*. For every chain complex $L_\bullet$ of A-modules, we have an exact sequence*

$$0 \longrightarrow \mathrm{H}_0(K_\bullet \otimes \mathrm{H}_p(L_\bullet)) \longrightarrow \mathrm{H}_p(K_\bullet \otimes L_\bullet) \longrightarrow \mathrm{H}_1(K_\bullet \otimes \mathrm{H}_{p-1}(L_\bullet)) \longrightarrow 0$$

for every index p.

This is a particular case of an exact sequence of low-order terms of the Künneth spectral sequence (M, XVII, 5.2 (a) and G, I, 5.5.2); it can be proved directly as follows. Consider K_0 and K_1 as chain complexes (zero in dimensions $\neq 0$ and $\neq 1$ respectively); we then have an exact sequence of complexes **III | 84**

$$0 \longrightarrow K_0 \otimes L_\bullet \longrightarrow K_\bullet \otimes L_\bullet \longrightarrow K_1 \otimes L_\bullet \longrightarrow 0,$$

to which we can apply the exact sequence in homology

$$\cdots \longrightarrow \mathrm{H}_{p+1}(K_1 \otimes L_\bullet) \xrightarrow{\partial} \mathrm{H}_p(K_0 \otimes L_\bullet) \longrightarrow \mathrm{H}_p(K_\bullet \otimes L_\bullet) \longrightarrow \mathrm{H}_p(K_1 \otimes L_\bullet) \xrightarrow{\partial} \mathrm{H}_{p-1}(K_0 \otimes L_\bullet) \longrightarrow \cdots.$$

But it is evident that $\mathrm{H}_p(K_0 \otimes L_\bullet) = K_0 \otimes \mathrm{H}_p(L_\bullet)$ and $\mathrm{H}_p(K_1 \otimes L_\bullet) = K_1 \otimes \mathrm{H}_{p-1}(L_\bullet)$ for all p; in addition, we verify immediately that the operator $\partial : K_1 \otimes \mathrm{H}_p(L_\bullet) \to K_0 \otimes \mathrm{H}_p(L_\bullet)$ is none other than $d_1 \otimes 1$; the lemma thus follows from the above exact sequence and the definition of $\mathrm{H}_0(K_\bullet \otimes \mathrm{H}_p(L_\bullet))$ and $\mathrm{H}_1(K_\bullet \otimes \mathrm{H}_{p-1}(L_\bullet))$.

The lemma having been established, the end of the proof of Proposition (1.1.4) is immediate: the induction hypothesis of Lemma (1.1.4.1) gives $\mathrm{H}_p(K_\bullet \otimes L_\bullet) = 0$ for $p \geqslant 2$; in addition if we show that $\mathrm{H}_1(K_\bullet, \mathrm{H}_0(L_\bullet)) = 0$, then we also deduce from Lemma (1.1.4.1) that $\mathrm{H}_1(K_\bullet \otimes L_\bullet) = 0$; but by definition, $\mathrm{H}_1(K_\bullet, \mathrm{H}_0(L_\bullet))$ is none other than the kernel of the scaling $z \mapsto f_r \cdot z$ on M_{r-1}, and as by hypothesis this kernel is zero, this finishes the proof.

(1.1.5). Let $\mathbf{g} = (g_i)_{1 \leqslant i \leqslant r}$ be a second sequence of r elements of A, and set $\mathbf{fg} = (f_i g_i)_{1 \leqslant i \leqslant r}$. We can define a canonical homomorphism of complexes

$$\phi_{\mathbf{g}} : K_\bullet(\mathbf{fg}) \longrightarrow K_\bullet(\mathbf{f}) \tag{1.1.5.1}$$

as the canonical extension to the exterior algebra $\wedge(A^r)$ of the A-linear map $(x_1, \ldots, x_r) \mapsto (g_1 x_1, \ldots, g_r x_r)$ from A^r to itself. To see that we have a homomorphism of complexes, it suffices to note, in general, that if $u : E \to F$ is an A-linear map, and if $\mathbf{x} \in F^\vee$ and $\mathbf{y} = {}^t u(\mathbf{x}) \in E^\vee$, then we have the formula

$$(\wedge u) \circ i_{\mathbf{y}} = i_{\mathbf{x}} \circ (\wedge u); \tag{1.1.5.2}$$

indeed, the two elements are antiderivations of $\wedge F$, and it suffices to check that they coincide on F, which follows immediately from the definitions.

When we identify $K_\bullet(\mathbf{f})$ with the tensor product of the $K_\bullet(f_i)$ (1.1.1), $\phi_{\mathbf{g}}$ is the tensor product of the ϕ_{g_i}, where ϕ_{g_i} is the identity in degree 0 and multiplication by g_i in degree 1.

(1.1.6). In particular, for every pair of integers m and n such that $0 \leqslant n \leqslant m$, we have homomorphisms of complexes

$$\phi_{\mathbf{f}^{m-n}} : K_\bullet(\mathbf{f}^m) \longrightarrow K_\bullet(\mathbf{f}^n) \tag{1.1.6.1}$$

and as a result, homomorphisms

$$\phi_{\mathbf{f}^{m-n}} : K^\bullet(\mathbf{f}^n, M) \longrightarrow K^\bullet(\mathbf{f}^m, M), \tag{1.1.6.2}$$

$$\phi_{\mathbf{f}^{m-n}} : \mathrm{H}^\bullet(\mathbf{f}^n, M) \longrightarrow \mathrm{H}^\bullet(\mathbf{f}^m, M). \tag{1.1.6.3}$$

The latter homomorphisms evidently satisfy the transitivity condition $\phi_{\mathbf{f}^{m-p}} = \phi_{\mathbf{f}^{m-n}} \circ \phi_{\mathbf{f}^{n-p}}$ for $p \leqslant n \leqslant m$; they therefore define two *inductive systems* of A-modules; we set **III | 85**

$$\mathrm{C}^\bullet((\mathbf{f}), M) = \varinjlim_n K^\bullet(\mathbf{f}^n, M), \tag{1.1.6.4}$$

$$\mathrm{H}^\bullet((\mathbf{f}), M) = \mathrm{H}^\bullet(\mathrm{C}^\bullet((\mathbf{f}), M)) = \varinjlim_n \mathrm{H}^\bullet(\mathbf{f}^n, M), \tag{1.1.6.5}$$

the last equality following from the fact that passing to the inductive limit commutes with the functor $\mathrm{H}^\bullet$ (G, I, 2.1). We will later see (1.4.3) that $\mathrm{H}^\bullet((\mathbf{f}), M)$ does not depend on the *ideal* $(\mathbf{f})$ of A (and similarly on the $(\mathbf{f})$-pre-adic topology on A), which justifies the notations.

It is clear that $M \mapsto C^\bullet((\mathbf{f}), M)$ is an exact A-linear functor, and $M \mapsto \mathrm{H}^\bullet((\mathbf{f}), M)$ is a cohomological functor.

(1.1.7). Set $\mathbf{f} = (f_i) \in A^r$ and $\mathbf{g} = (g_i) \in A^r$; denote by $e_\mathbf{g}$ the left multiplication by the vector $\mathbf{g} \in A^r$ on the exterior algebra $\wedge(A^r)$; we know that we have the *homotopy formula*

$$i_\mathbf{f} e_\mathbf{g} + e_\mathbf{g} i_\mathbf{f} = \langle \mathbf{g}, \mathbf{f} \rangle 1 \tag{1.1.7.1}$$

in the A-module A^r (1 denotes the identity automorphism of A^r); this relation also implies that *in the complex $K_\bullet(\mathbf{f})$* we have

$$d e_\mathbf{g} + e_\mathbf{g} d = \langle \mathbf{g}, \mathbf{f} \rangle 1. \tag{1.1.7.2}$$

If the ideal $(\mathbf{f})$ is equal to A, then there exists a $\mathbf{g} \in A^r$ such that $\langle \mathbf{g}, \mathbf{f} \rangle = \sum_{i=1}^r g_i f_i = 1$. As a result (G, I, 2.4):

Proposition (1.1.8). — *Suppose that the ideal $(\mathbf{f})$ generated by the f_i is equal to A. Then the complex $K_\bullet(\mathbf{f})$ is homotopically trivial, and so are the complexes $K_\bullet(\mathbf{f}, M)$ and $K^\bullet(\mathbf{f}, M)$ for every A-module M.*

Corollary (1.1.9). — *If $(\mathbf{f}) = A$, then we have $\mathrm{H}^\bullet(\mathbf{f}, M) = 0$ and $\mathrm{H}^\bullet((\mathbf{f}), M) = 0$ for every A-module M.*

PROOF. Indeed, we then have $(\mathbf{f}^n) = A$ for all n. □

Remark (1.1.10). — With the same notations as above, set $X = \mathrm{Spec}(A)$ and Y the closed sub-prescheme of X defined by the ideal $(\mathbf{f})$. We will prove in §9 that $\mathrm{H}^\bullet((\mathbf{f}), M)$ is isomorphic to the cohomology $\mathrm{H}^\bullet_Y(X, \widetilde{M})$ corresponding to the antifilter Φ of closed subsets of Y (T, 3.2). We will also show that Proposition (1.2.3) applied to X and to $\mathscr{F} = \widetilde{M}$ is a particular case of an exact sequence in cohomology

$$\cdots \longrightarrow \mathrm{H}^p_Y(X, \mathscr{F}) \longrightarrow \mathrm{H}^p(X, \mathscr{F}) \longrightarrow \mathrm{H}^p(X - Y, \mathscr{F}) \longrightarrow \mathrm{H}^{p+1}_Y(X, \mathscr{F}) \longrightarrow \cdots.$$

1.2. Čech cohomology of an open cover

Notation (1.2.1). — In this section, we denote:

(1) X a prescheme;
(2) $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module; **III** | 86
(3) $A = \Gamma(X, \mathscr{O}_X)$, $M = \Gamma(X, \mathscr{F})$;
(4) $\mathbf{f} = (f_i)_{1 \leqslant i \leqslant r}$ a finite system of elements of A;
(5) $U_i = X_{f_i}$, the open set (**0**, 5.5.2) of the $x \in X$ such that $f_i(x) \neq 0$;
(6) $U = \bigcup_{i=1}^r U_i$;
(7) $\mathfrak{U}$ the cover $(U_i)_{1 \leqslant i \leqslant r}$ of U.

(1.2.2). Suppose that X is either a prescheme whose underlying space is *Noetherian* or a *scheme* whose underlying space is *quasi-compact*. We then know (**I**, 9.3.3) that we have $\Gamma(U_i, \mathscr{F}) = M_{f_i}$. We set

$$U_{i_0 i_1 \cdots i_p} = \bigcap_{k=0}^{p} U_{i_k} = X_{f_{i_0} f_{i_1} \cdots f_{i_p}}$$

(**0**, 5.5.3); so we also have

$$\Gamma(U_{i_0 i_1 \cdots i_p}, \mathscr{F}) = M_{f_{i_0} f_{i_1} \cdots f_{i_p}}. \tag{1.2.2.1}$$

We have (**0**, 1.6.1) that $M_{f_{i_0} f_{i_1} \cdots f_{i_p}}$ identifies with the inductive limit $\varinjlim_n M^{(n)}_{i_0 i_1 \cdots i_p}$, where the inductive system is formed by the $M^{(n)}_{i_0 i_1 \cdots i_p} = M$, the homomorphisms $\phi_{nm} : M^{(m)}_{i_0 i_1 \cdots i_p} \to M^{(n)}_{i_0 i_1 \cdots i_p}$ being multiplication by $(f_{i_0} f_{i_1} \cdots f_{i_p})^{n-m}$ for $m \leqslant n$. We denote by $C^p_n(M)$ the set of *alternating* maps from $[1, r]^{p+1}$ to M (for all n); these A-modules also form an inductive system with respect to the ϕ_{nm}. If $C^p(\mathfrak{U}, \mathscr{F})$ is the group of *alternating* Čech p-cochains relative to the cover $\mathfrak{U}$, with coefficients in $\mathscr{F}$ (G, II, 5.1), then it follows from the above that we can write

$$C^p(\mathfrak{U}, \mathscr{F}) = \varinjlim_n C^p_n(M). \tag{1.2.2.2}$$

With the notations of (1.1.2), $C_n^p(M)$ identifies with $K^{p+1}(\mathbf{f}^n, M)$, and the map ϕ_{nm} identifies with the map $\phi_{\mathbf{f}^{n-m}}$ defined in (1.1.6). We thus have, for every $p \geqslant 0$, a canonical functorial isomorphism

$$C^p(\mathfrak{U}, \mathscr{F}) \simeq C^{p+1}((\mathbf{f}), M). \tag{1.2.2.3}$$

In addition, the formula (1.1.2.3) and the definition of the cohomology of a cover (G, II, 5.1) shows that the isomorphisms (1.2.2.3) are compatible with the coboundary maps.

Proposition (1.2.3). — *If X is a prescheme whose underlying space in Noetherian or a scheme whose underlying space is quasi-compact, then there exists a canonical functorial isomorphism in $\mathscr{F}$*

$$\mathrm{H}^p(\mathfrak{U}, \mathscr{F}) \simeq \mathrm{H}^{p+1}((\mathbf{f}), M) \text{ for } p \geqslant 1. \tag{1.2.3.1}$$

In addition, we have a functorial exact sequence in $\mathscr{F}$

$$0 \longrightarrow \mathrm{H}^0((\mathbf{f}), M) \longrightarrow M \longrightarrow \mathrm{H}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathrm{H}^1((\mathbf{f}), M) \longrightarrow 0. \tag{1.2.3.2}$$

III | 87

Proof. The isomorphisms (1.2.3.1) are immediate consequences of what we saw in (1.2.2). On the other hand, we have $C^0(\mathfrak{U}, \mathscr{F}) = C^1((\mathbf{f}), M)$; as a result, $\mathrm{H}^0(\mathfrak{U}, \mathscr{F})$ identifies with the subgroup of 1-cocycles of $C^1((\mathbf{f}), M)$; as $M = C^0((\mathbf{f}), M)$, the exact sequence (1.2.3.2) is none other than the one given by the definition of the cohomology groups $\mathrm{H}^0((\mathbf{f}), M)$ and $\mathrm{H}^1((\mathbf{f}), M)$. □

Corollary (1.2.4). — *Suppose that the X_{f_i} are quasi-compact and that there exists $g_i \in \Gamma(U, \mathscr{F})$ such that $\sum_i g_i(f_i|U) = 1|U$. Then for every quasi-coherent $(\mathscr{O}_X|U)$-module $\mathscr{G}$, we have $\mathrm{H}^p(\mathfrak{U}, \mathscr{G}) = 0$ for $p > 0$; if in addition $U = X$, then the canonical homomorphism (1.2.3.2) $M \to \mathrm{H}^0(\mathfrak{U}, \mathscr{F})$ is bijective.*

Proof. As by hypothesis the $U_i = X_{f_i}$ are quasi-compact, so is U, and we can reduce to the case where $U = X$; the hypothesis then implies that $\mathrm{H}^p((\mathbf{f}), M) = 0$ for all $p \geqslant 0$ (1.1.9). The corollary then follows immediately from (1.2.3.1) and (1.2.3.2). □

We note that since $\mathrm{H}^0(\mathfrak{U}, \mathscr{F}) = \mathrm{H}^0(U, \mathscr{F})$ (G, II, 5.2.2), we have again proved (**I**, 1.3.7) as a special case.

Remark (1.2.5). — Suppose that X is an *affine scheme*; then the $U_i = X_{f_i} = D(f_i)$ are affine open sets, as well as the $U_{i_0 i_1 \cdots i_p}$ (but U is not necessarily affine). In this case, the functors $\Gamma(X, \mathscr{F})$ and $\Gamma(U_{i_0 i_1 \cdots i_p}, \mathscr{F})$ are exact in $\mathscr{F}$ (**I**, 1.3.11). If we have an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of quasi-coherent $\mathscr{O}_X$-modules, then the sequence of complexes

$$0 \longrightarrow C^\bullet(\mathfrak{U}, \mathscr{F}') \longrightarrow C^\bullet(\mathfrak{U}, \mathscr{F}) \longrightarrow C^\bullet(\mathfrak{U}, \mathscr{F}'') \longrightarrow 0$$

is exact, and thus gives an exact sequence in cohomology

$$\cdots \longrightarrow \mathrm{H}^p(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathrm{H}^p(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathrm{H}^p(\mathfrak{U}, \mathscr{F}'') \xrightarrow{\partial} \mathrm{H}^{p+1}(\mathfrak{U}, \mathscr{F}') \longrightarrow \cdots.$$

On the other hand, if we set $M' = \Gamma(X, \mathscr{F}')$ and $M'' = \Gamma(X, \mathscr{F}'')$, then the sequence $0 \to M' \to M \to M'' \to 0$ is exact; as $C^\bullet((\mathbf{f}), M)$ is an exact functor in M, we also have the exact sequence in cohomology

$$\cdots \longrightarrow \mathrm{H}^p((\mathbf{f}), M') \longrightarrow \mathrm{H}^p((\mathbf{f}), M) \longrightarrow \mathrm{H}^p((\mathbf{f}), M'') \xrightarrow{\partial} \mathrm{H}^{p+1}((\mathbf{f}), M') \longrightarrow \cdots.$$

This being so, as the diagram

$$\begin{tikzcd}
0 \arrow[r] & C^\bullet(\mathfrak{U}, \mathscr{F}') \arrow[r] \arrow[d] & C^\bullet(\mathfrak{U}, \mathscr{F}) \arrow[r] \arrow[d] & C^\bullet(\mathfrak{U}, \mathscr{F}'') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^\bullet((\mathbf{f}), M') \arrow[r] & C^\bullet((\mathbf{f}), M) \arrow[r] & C^\bullet((\mathbf{f}), M'') \arrow[r] & 0
\end{tikzcd}$$

is commutative, we conclude that the diagrams **III | 88**

$$\begin{tikzcd}
\mathrm{H}^p(\mathfrak{U}, \mathscr{F}'') \arrow[r, "\partial"] \arrow[d] & \mathrm{H}^{p+1}(\mathfrak{U}, \mathscr{F}') \arrow[d] \\
\mathrm{H}^{p+1}((\mathbf{f}), M'') \arrow[r, "\partial"] & \mathrm{H}^{p+2}((\mathbf{f}), M')
\end{tikzcd} \tag{1.2.5.1}$$

are commutative for all p (G, I, 2.1.1).

1.3. Cohomology of an affine scheme

Theorem (1.3.1). — *Let X be an affine scheme. For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we have $\mathrm{H}^p(X, \mathscr{F}) = 0$ for all $p > 0$.*

Proof. Let $\mathfrak{U}$ be a finite cover of X by the affine open sets $X_{f_i} = D(f_i)$ $(1 \leqslant i \leqslant r)$; we then know that the ideal of $A = \Gamma(X, \mathscr{O}_X)$ generated by the f_i is equal to A. We thus conclude from Corollary (1.2.4) that we have $\mathrm{H}^p(\mathfrak{U}, \mathscr{F}) = 0$ for $p > 0$. As there are finite covers of X by affine open sets which are arbitrarily fine (I, 1.1.10), the definition of Čech cohomology (G, II, 5.8) shows that we also have $\check{\mathrm{H}}^p(X, \mathscr{F}) = 0$ for $p > 0$. But this also applies to every prescheme X_f for $f \in A$ (I, 1.3.6), hence $\check{\mathrm{H}}^p(X_f, \mathscr{F}) = 0$ for $p > 0$. As we have $X_f \cap X_g = X_{fg}$, we deduce that we also have $\mathrm{H}^p(X, \mathscr{F}) = 0$ for all $p > 0$, by virtue of (G, II, 5.9.2). □

Corollary (1.3.2). — *Let Y be a prescheme, $f : X \to Y$ an affine morphism (II, 1.6.1). For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we have $\mathrm{R}^q f_*(\mathscr{F}) = 0$ for $q > 0$.*

Proof. By definition $\mathrm{R}^q f_*(\mathscr{F})$ is the $\mathscr{O}_Y$-module associated to the presheaf $U \mapsto \mathrm{H}^q(f^{-1}(U), \mathscr{F})$, where U varies over the open subsets of Y. But the affine open sets form a basis for Y, and for such an open set U, $f^{-1}(U)$ is affine (II, 1.3.2), hence $\mathrm{H}^q(f^{-1}(U), \mathscr{F}) = 0$ by Theorem (1.3.1), which proves the corollary. □

Corollary (1.3.3). — *Let Y be a prescheme, $f : X \to Y$ an affine morphism. For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $\mathrm{H}^p(Y, f_*(\mathscr{F})) \to \mathrm{H}^p(X, \mathscr{F})$ (**0**, 12.1.3.1) is bijective for all p.*

Proof. It suffices (by (**0**, 12.1.7)) to show that the edge homomorphisms $''E_2^{p0} = \mathrm{H}^p(Y, f_*(\mathscr{F})) \to \mathrm{H}^p(X, \mathscr{F})$ of the second spectral sequence of the composite functor Γf_* are bijective. But the E_2-term of this spectral sequence is given by $''E_2^{pq} = \mathrm{H}^p(Y, \mathrm{R}^q f_*(\mathscr{F}))$ (G, II, 4.17.1), so it follows from Corollary (1.3.2) that $''E_2^{pq} = 0$ for $q > 0$, and the spectral sequence degenerates; hence our assertion (**0**, 11.1.6). □

Corollary (1.3.4). — *Let $f : X \to Y$ be an affine morphism, $g : Y \to Z$ a morphism. For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $\mathrm{R}^p g_*(f_*(\mathscr{F})) \to \mathrm{R}^p(g \circ f)_*(\mathscr{F})$ (**0**, 12.2.5.1) is bijective for all p.* **III** | 89

Proof. It suffices to note that, according to Corollary (1.3.3), for every affine open subset W of Z, the canonical homomorphism $\mathrm{H}^p(g^{-1}(W), f_*(\mathscr{F})) \to \mathrm{H}^p(f^{-1}(g^{-1}(W)), \mathscr{F})$ is bijective; this proves that the homomorphism of presheaves defining the canonical homomorphism $\mathrm{R}^p g_*(f_*(\mathscr{F})) \to \mathrm{R}^p(g \circ f)_*(\mathscr{F})$ is bijective (**0**, 12.2.5). □

1.4. Application to the cohomology of arbitrary preschemes

Proposition (1.4.1). — *Let X be a scheme, $\mathfrak{U} = (U_\alpha)$ be a cover of X by affine open sets. For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the cohomology modules $\mathrm{H}^\bullet(X, \mathscr{F})$ and $\mathrm{H}^\bullet(\mathfrak{U}, \mathscr{F})$ (over $\Gamma(X, \mathscr{O}_X)$) are canonically isomorphic.*

Proof. As X is a scheme, every finite intersection V of open sets in the cover $\mathfrak{U}$ is affine (I, 5.5.6), so $\mathrm{H}^q(V, \mathscr{F}) = 0$ for $g \geqslant 1$ by Theorem (1.3.1). The proposition then follows from a theorem of Leray (G, II, 5.4.1). □

Remark (1.4.2). — We note that the result of Proposition (1.4.1) is still true when the finite intersections of the sets U_α are affine, even when we do not necessarily assume that X is a scheme.

Corollary (1.4.3). — *Let X be a scheme with quasi-compact underlying space, $A = \Gamma(X, \mathscr{O}_X)$, and $\mathbf{f} = (f_i)_{1 \leqslant i \leqslant r}$ a finite sequence of elements of A such that the X_{f_i} (notation of (1.2.1)) are affine. Then (with the notations of (1.2.1)), for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we have a canonical isomorphism which is functorial in $\mathscr{F}$*

(1.4.3.1) $$\mathrm{H}^q(U, \mathscr{F}) \simeq \mathrm{H}^{q+1}((\mathbf{f}), M) \text{ for } q \geqslant 1,$$

and an exact sequence which is functorial in $\mathscr{F}$

(1.4.3.2) $$0 \longrightarrow \mathrm{H}^0((\mathbf{f}), M) \longrightarrow M \longrightarrow \mathrm{H}^0(U, \mathscr{F}) \longrightarrow \mathrm{H}^1((\mathbf{f}), M) \longrightarrow 0.$$

PROOF. This follows immediately from Propositions (1.4.1) and (1.2.3). □

(1.4.4). If X is an *affine scheme*, then it follows from Remark (1.2.5) and Proposition (1.4.1) that for all $q \geqslant 0$, the diagrams

$$\begin{tikzcd} \mathrm{H}^q(U,\mathscr{F}'') \arrow[r, "\partial"] \arrow[d] & \mathrm{H}^{q+1}(U,\mathscr{F}') \arrow[d] \\ \mathrm{H}^{q+1}((\mathbf{f}),M'') \arrow[r, "\partial"] & \mathrm{H}^{q+2}((\mathbf{f}),M') \end{tikzcd} \tag{1.4.4.1}$$

corresponding to an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of quasi-coherent $\mathscr{O}_X$-modules (with the notations of Remark (1.2.5)) are commutative.

Proposition (1.4.5). — *Let X be a quasi-compact scheme, $\mathscr{L}$ an invertible $\mathscr{O}_X$-module, and consider the graded ring $A_\bullet = \Gamma_\bullet(\mathscr{L})$ (**0**, 5.4.6); then $\mathrm{H}^\bullet(\mathscr{F},\mathscr{L}) = \bigoplus_{n\in\mathbf{Z}} \mathrm{H}^\bullet(X,\mathscr{F}\otimes\mathscr{L}^{\otimes n})$ is a graded $A_\bullet$-module, and for all $f \in A_n$, we have a canonical isomorphism* **III | 90**

$$\mathrm{H}^\bullet(X_f,\mathscr{F}) \simeq (\mathrm{H}^\bullet(\mathscr{F},\mathscr{L}))_{(f)} \tag{1.4.5.1}$$

of $(A_\bullet)_{(f)}$-modules.

PROOF. As X is a quasi-compact scheme, we can calculate the cohomology of all the $\mathscr{O}_X$-modules $\mathscr{F}\otimes\mathscr{L}^{\otimes n}$ using the same finite cover $\mathfrak{U} = (U_i)$ consisting of the affine open sets such that the restriction $\mathscr{L}|U_i$ is isomorphic to $\mathscr{O}_X|U_i$ for each i (1.4.1). It is then immediate that the $U_i \cap X_f$ are affine open sets (**I**, 1.3.6), and we can thus calculate the cohomology $\mathrm{H}^\bullet(X_f,\mathscr{F}\otimes\mathscr{L}^{\otimes n})$ using the cover $\mathfrak{U}|X_f = (U_i \cap X_f)$ (1.4.1). It is immediate that for all $f \in A_n$, multiplication by f defines a homomorphism $\mathrm{C}^\bullet(\mathfrak{U},\mathscr{F}\otimes\mathscr{L}^m) \to \mathrm{C}^\bullet(\mathfrak{U},\mathscr{F}\otimes\mathscr{L}^{\otimes(m+n)})$, hence a homomorphism $\mathrm{H}^\bullet(\mathfrak{U},\mathscr{F}\otimes\mathscr{L}^{\otimes m}) \to \mathrm{H}^\bullet(\mathfrak{U},\mathscr{F}\otimes\mathscr{L}^{\otimes(m+n)})$, which establishes the first assertion. On the other hand, for a given $f \in A_n$, it follows from (**I**, 9.3.2) that we have an isomorphism of complexes of $(A_\bullet)_{(f)}$-modules

$$\mathrm{C}^\bullet(\mathfrak{U}|X_f,\mathscr{F}) \simeq \left(\mathrm{C}^\bullet\left(\mathfrak{U},\bigoplus_{n\in\mathbf{Z}}\mathscr{F}\otimes\mathscr{L}^{\otimes n}\right)\right)_{(f)},$$

taking into account (**I**, 1.3.9, ii). Passing to the cohomology of these complexes, we induce the isomorphism (1.4.5.1), recalling that the functor $M \mapsto M_{(f)}$ is exact on the category of graded $A_\bullet$-modules. □

Corollary (1.4.6). — *Suppose that the hypotheses of Proposition (1.4.5) are satisfied, and in addition suppose that $\mathscr{L} = \mathscr{O}_X$. If we set $A = \Gamma(X,\mathscr{O}_X)$, then for all $f \in A$, we have a canonical isomorphism $\mathrm{H}^\bullet(X_f,\mathscr{F}) \simeq (\mathrm{H}^\bullet(X,\mathscr{F}))_f$ of A_f-modules.*

Corollary (1.4.7). — *Let X be a quasi-compact scheme, f an element of $\Gamma(X,\mathscr{O}_X)$.*

(i) *Suppose that the open set X_f is affine. Then for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, every $i > 0$, and every $\xi \in \mathrm{H}^i(X,\mathscr{F})$, there exists an integer $n > 0$ such that $f^n\xi = 0$.*
(ii) *Conversely, suppose that X_f is quasi-compact and that for every quasi-coherent sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_X$ and every $\zeta \in \mathrm{H}^1(X,\mathscr{J})$, there exists an $n > 0$ such that $f^n\zeta = 0$. Then X_f is affine.*

PROOF.

(i) If X_f is affine, then we have $\mathrm{H}^i(X_f,\mathscr{F}) = 0$ for all $i > 0$ (1.3.1), so the assertion follows directly from Corollary (1.4.6).
(ii) By virtue of Serre's criterion (**II**, 5.2.1), it suffices to prove that for every quasi-coherent sheaf of ideals $\mathscr{K}$ of $\mathscr{O}_X|X_f$, we have $\mathrm{H}^1(X_f,\mathscr{K}) = 0$. As X_f is a quasi-compact open set in a quasi-compact scheme X, there exists a quasi-coherent sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_X$ such that $\mathscr{K} = \mathscr{J}|X_f$ (**I**, 9.4.2). According to Corollary (1.4.6), we have $\mathrm{H}^1(X_f,\mathscr{K}) = (\mathrm{H}^1(X,\mathscr{J}))_f$, and the hypothesis implies that the right hand side is zero, hence the assertion.

□

Remark (1.4.8). — We note that Corollary (1.4.7, i) gives a simpler proof of the relation (**II**, 4.5.13.2).

Lemma (1.4.9). — *Let X be a quasi-compact scheme, $\mathfrak{U} = (U_i)_{1 \leqslant i \leqslant n}$ a finite cover of X by affine open sets, and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module. The complex of sheaves $\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$ defined by the cover $\mathfrak{U}$ (G, II, 5.2) is then a quasi-coherent $\mathscr{O}_X$-module.* **III | 91**

PROOF. It follows from the definitions (G, II, 5.2) that $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ is the direct sum of the direct image sheaves of the $\mathscr{F}|U_{i_0 \cdots i_p}$ under the canonical injection $U_{i_0 \cdots i_p} \to X$. The hypothesis that X is a scheme implies that these injections are affine morphisms (**I**, 5.5.6), hence the $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ are quasi-coherent (**II**, 1.2.6). □

Proposition (1.4.10). — *Let $u : X \to Y$ be a separated and quasi-compact morphism. For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the $\mathrm{R}^q u_*(\mathscr{F})$ are quasi-coherent $\mathscr{O}_Y$-modules.*

PROOF. The question is local on Y, so we can suppose that Y is affine. Then X is a finite union of affine open sets U_i $(1 \leqslant i \leqslant n)$; let $\mathfrak{U}$ be the cover (U_i). In addition, as Y is a scheme, it follows from (**I**, 5.5.10) that for every affine open $V \subset Y$, the canonical injection $u^{-1}(V) \to X$ is an affine morphism; we conclude (Proposition (1.4.1) and (G, II, 5.2)) that we have a canonical isomorphism

(1.4.10.1) $$\mathrm{H}^\bullet(u^{-1}(V), \mathscr{F}) \simeq \mathrm{H}^\bullet(\Gamma(V, \mathscr{K}^\bullet)),$$

where we set $\mathscr{K}^\bullet = u_*(\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F}))$. According to Lemma (1.4.1) and (**I**, 9.2.2), $\mathscr{K}^\bullet$ is a quasi-coherent $\mathscr{O}_Y$-module; moreover, it constitutes a *complex of sheaves* since so is $\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$. It then follows from the definition of the cohomology $\mathscr{H}^\bullet(\mathscr{K}^\bullet)$ (G, II, 4.1) that the latter consists of quasi-coherent $\mathscr{O}_Y$-modules (**I**, 4.1.1). As (for V affine in Y) the functor $\Gamma(V, \mathscr{G})$ is exact in $\mathscr{G}$ on the category of quasi-coherent $\mathscr{O}_Y$-modules, we have (G, II, 4.1)

(1.4.10.2) $$\mathrm{H}^\bullet(\Gamma(V, \mathscr{K}^\bullet)) = \Gamma(V, \mathscr{H}^\bullet(\mathscr{K}^\bullet)).$$

Finally, we note that it follows from the definition of the canonical homomorphism

$$\mathrm{H}^\bullet(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathrm{H}^\bullet(X, \mathscr{F}),$$

given in (G, II, 5.2), that if $V' \subset V$ is a second affine open subset of Y, then the diagram

$$\begin{tikzcd} \mathrm{H}^\bullet(u^{-1}(V), \mathscr{F}) \arrow[r, "\sim"] \arrow[d] & \mathrm{H}^\bullet(\Gamma(V, \mathscr{K}^\bullet)) \arrow[d] \\ \mathrm{H}^\bullet(u^{-1}(V'), \mathscr{F}) \arrow[r, "\sim"] & \mathrm{H}^\bullet(\Gamma(V', \mathscr{K}^\bullet)) \end{tikzcd}$$

is commutative. We thus conclude from the above that the isomorphisms (1.4.10.1) define an isomorphism of $\mathscr{O}_Y$-modules

(1.4.10.3) $$\mathrm{R}^\bullet u_*(\mathscr{F}) \simeq \mathscr{H}^\bullet(\mathscr{K}^\bullet),$$

and as a result, $\mathrm{R}^\bullet u_*(\mathscr{F})$ is quasi-coherent. □ **III | 92**

In addition, it follows from (1.4.10.3), (1.4.10.2), and (1.4.10.1) that:

Corollary (1.4.11). — *Under the hypotheses of Proposition (1.4.10), for every affine open set V of Y, the canonical homomorphism*

(1.4.11.1) $$\mathrm{H}^q(u^{-1}(V), \mathscr{F}) \longrightarrow \Gamma(V, \mathrm{R}^1 u_*(\mathscr{F}))$$

is an isomorphism for all $q \geqslant 0$.

Corollary (1.4.12). — *Suppose that the hypotheses of Proposition (1.4.10) are satisfied, and in addition suppose that Y is quasi-compact. Then there exists an integer $r > 0$ such that for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and every integer $q > r$, we have $\mathrm{R}^q u_*(\mathscr{F}) = 0$. If Y is affine, then we can take for r an integer such that there exists a cover of X consisting of r affine open sets.*

PROOF. As we can cover Y by a finite number of affine open sets, we can reduce to proving the second assertion, by virtue of Corollary (1.4.11). If $\mathfrak{U}$ is a cover of X by r affine open sets, then we have $\mathrm{H}^q(\mathfrak{U}, \mathscr{F}) = 0$ for $q > r$, since the cochains of $C^q(\mathfrak{U}, \mathscr{F})$ are alternating; the assertion thus follows from Proposition (1.4.1). □

Corollary (1.4.13). — *Suppose that the hypotheses of Proposition (1.4.10) are satisfied, and in addition suppose that $Y = \mathrm{Spec}(A)$ is affine. Then for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and every $f \in A$, we have*

$$\Gamma(Y_f, \mathrm{R}^q u_*(\mathscr{F})) = (\Gamma(Y, \mathrm{R}^q u_*(\mathscr{F}))_f$$

up to canonical isomorphism.

PROOF. This follows from the fact that $\mathrm{R}^q u_*(\mathscr{F})$ is a quasi-coherent $\mathscr{O}_Y$-module (**I**, 1.3.7). □

Proposition (1.4.14). — *Let $f : X \to Y$ be a separated and quasi-compact morphism, $g : Y \to Z$ an affine morphism. For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $\mathrm{R}^p(g \circ f)_*(\mathscr{F}) \to g_*(\mathrm{R}^p f_*(\mathscr{F}))$ (**0**, 12.2.5.2) is bijective for all p.*

PROOF. For every affine open subset W of Z, $g^{-1}(W)$ is an affine open subset of Y. The homomorphism of presheaves defining the canonical homomorphism

$$\mathrm{R}^p(g \circ f)_*(\mathscr{F}) \longrightarrow g_*(\mathrm{R}^p f_*(\mathscr{F}))$$

(**0**, 12.2.5) is thus bijective by Corollary (1.4.11). □

Proposition (1.4.15). — *Let $u : X \to Y$ be a separated morphism of finite type, $v : Y' \to Y$ a flat morphism of preschemes (**0**, 6.7.1); let $u' = u_{(Y')}$, such that we have the commutative diagram*

(1.4.15.1)
$$\begin{tikzcd} X \arrow[d, "u"'] & X' = X_{(Y')} \arrow[l, "v'"'] \arrow[d, "u'"] \\ Y & Y'. \arrow[l, "v"'] \end{tikzcd}$$

Then for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $\mathrm{R}^q u'_*(\mathscr{F}')$ is canonically isomorphic to $\mathrm{R}^q u_*(\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y'} = v^*(\mathrm{R}^q u_*(\mathscr{F}))$ for all $q \geqslant 0$, where $\mathscr{F}' = v'^*(\mathscr{F}) = \mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y'}$.

III | 93

PROOF. The canonical homomorphism $\rho : \mathscr{F} \to v'_*(v'^*(\mathscr{F}))$ (**0**, 4.4.3.2) defines by functoriality a homomorphism

(1.4.15.2)
$$\mathrm{R}^q u_*(\mathscr{F}) \longrightarrow \mathrm{R}^q u_*(v'_*(\mathscr{F}')).$$

On the other hand, we have, by setting $w = u \circ v' = v \circ u'$, the canonical homomorphisms (**0** ,12.2.5.1 and 12.2.5.2)

(1.4.15.3)
$$\mathrm{R}^q u_*(v'_*(\mathscr{F}')) \longrightarrow \mathrm{R}^q w_*(\mathscr{F}') \longrightarrow v_*(\mathrm{R}^q u'_*(\mathscr{F}')).$$

Composing (1.4.15.3) and (1.4.15.2), we have a homomorphism

$$\psi : \mathrm{R}^q u_*(\mathscr{F}) \longrightarrow v_*(\mathrm{R}^q u'_*(\mathscr{F}')),$$

and finally we obtain a canonical homomorphism (whose definition does not make *any assumptions* on v)

(1.4.15.4)
$$\psi^\sharp : v^*(\mathrm{R}^q u_*(\mathscr{F})) \longrightarrow \mathrm{R}^q u'_*(\mathscr{F}'),$$

and it is necessary to prove that it is an isomorphism when v is *flat*. It is clear that the question is local on Y and Y', and we can therefore suppose that $Y = \mathrm{Spec}(A)$ and $Y' = \mathrm{Spec}(B)$; we will also use the following lemma:

Lemma (1.4.15.5). — *Let $\phi : A \to B$ be a ring homomorphism, $Y = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, $f : X \to Y$ the morphism corresponding to ϕ, and M a B-module. For the $\mathscr{O}_X$-module $\widetilde{M}$ to be f-flat (**0**, 6.7.1), it is necessary and sufficient for M to be a flat A-module. In particular, for the morphism f to be flat, it is necessary and sufficient for B to be a flat A-module.*

This follows from the definition (**0**, 6.7.1) and from (**0**, 6.3.3), taking into account (**I**, 1.3.4).

This being so, it follows from (1.4.11.1) and the definitions of the homomorphisms (1.4.15.3) (cf. (**0**, 12.2.5)) that ψ then corresponds to the composite morphism

$$\mathrm{H}^q(X, \mathscr{F}) \xrightarrow{\rho_q} \mathrm{H}^q(X, v'_*(v'^*(\mathscr{F}))) \xrightarrow{\theta_q} \mathrm{H}^q(X', v'^*(v'_*(v'^*(\mathscr{F})))) \xrightarrow{\sigma_q} \mathrm{H}^q(X', v'^*(\mathscr{F})),$$

where ρ_q and σ_q are the homomorphisms in cohomology corresponding to the canonical morphisms ρ and $\sigma : v'^*(v'_*(\mathscr{G}')) \to \mathscr{G}'$, and θ_q is the ϕ-morphism (**0**, 12.1.3.1) relative to the $\mathscr{O}_X$-module $v'_*(v'^*(\mathscr{F}))$. But by the functoriality of θ_q, we have the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^q(X,\mathscr{F}) \arrow[r, "\rho_q"] \arrow[d, "\theta_q"'] & \mathrm{H}^q(X, v'_*(v'^*(\mathscr{F}))) \arrow[d, "\theta_q"] \\
\mathrm{H}^q(X', v'^*(\mathscr{F})) \arrow[r, "v'^*(\rho_q)"] & \mathrm{H}^q(X', v'^*(v'_*(v'^*(\mathscr{F})))),
\end{tikzcd}$$

and as by definition (**0**, 4.4.3) $v'^*(\rho)$ is the inverse of σ, we see that the composite morphism considered above is finally none other than θ_q; as a result, $\psi^\sharp$ is the associated B-homomorphism $\mathrm{H}^q(X,\mathscr{F}) \otimes_A B \to \mathrm{H}^q(X',\mathscr{F}')$. As u is of finite type, X is a finite union of affine open sets U_i $(1 \leqslant i \leqslant r)$; let $\mathfrak{U}$ be the cover (U_i). As v is an affine morphism, so is v' (**II**, 1.6.2, iii), and as a result the $U'_i = v'^{-1}(U_i)$ form an affine open cover $\mathfrak{U}'$ of X'. We then know (**0**, 12.1.4.2) that the diagram **III** | 94

$$\begin{tikzcd}
\mathrm{H}^q(\mathfrak{U},\mathscr{F}) \arrow[r, "\theta_q"] \arrow[d] & \mathrm{H}^q(\mathfrak{U}',\mathscr{F}') \arrow[d] \\
\mathrm{H}^q(X,\mathscr{F}) \arrow[r, "\theta_q"] & \mathrm{H}^q(X',\mathscr{F}')
\end{tikzcd}$$

is commutative, and the vertical arrows are isomorphisms since X and X' are schemes (**I**, 1.4.1). As a result, it suffices to prove that the canonical ϕ-morphism $\theta_q : \mathrm{H}^q(\mathfrak{U},\mathscr{F}) \to \mathrm{H}^q(\mathfrak{U}',\mathscr{F}')$ is such that the associated B-homomorphism

$$\mathrm{H}^q(\mathfrak{U},\mathscr{F}) \otimes_A B \longrightarrow \mathrm{H}^q(\mathfrak{U}',\mathscr{F}')$$

is an isomorphism. For every sequence $\mathbf{s} = (i_k)_{0 \leqslant k \leqslant p}$ of $p+1$ indices of $[1,r]$, set $U_{\mathbf{s}} = \bigcap_{k=0}^p U_{i_k}$, $U'_{\mathbf{s}} = \bigcap_{k=0}^p U'_{i_k} = v'^{-1}(U_{\mathbf{s}})$, $M_{\mathbf{s}} = \Gamma(U_{\mathbf{s}},\mathscr{F})$, and $M'_{\mathbf{s}} = \Gamma(U'_{\mathbf{s}},\mathscr{F}')$. The canonical map $M_{\mathbf{s}} \otimes_A B \to M'_{\mathbf{s}}$ is an isomorphism (**I**, 1.6.5), hence the canonical map $C^p(\mathfrak{U},\mathscr{F}) \otimes_A B \to C^p(\mathfrak{U}',\mathscr{F}')$ is an isomorphism, by which $d \otimes 1$ identifies with the coboundary map $C^p(\mathfrak{U}',\mathscr{F}') \to C^{p+1}(\mathfrak{U}',\mathscr{F}')$. As B is a *flat* A-module, it follows from the definition of the cohomology modules that the canonical map $\mathrm{H}^q(\mathfrak{U},\mathscr{F}) \otimes_A B \to \mathrm{H}^q(\mathfrak{U}',\mathscr{F}')$ is an isomorphism (**0**, 6.1.1). This result will later be generalized in §6. □

Corollary (1.4.16). — *Let A be a ring, X an A-scheme of finite type, and B an A-algebra which is faithfully flat over A. For X to be affine, it is necessary and sufficient for $X \otimes_A B$ to be.*

Proof. The condition is evidently necessary (**I**, 3.2.2); we show that it is sufficient. As X is separated over A and the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is flat, it follows from Proposition (1.4.1) that we have

(1.4.16.1) $$\mathrm{H}^i(X \otimes_A B, \mathscr{F} \otimes_A B) = \mathrm{H}^i(X,\mathscr{F}) \otimes_A B$$

for every $i \geqslant 0$ and every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$. If $X \otimes_A B$ is affine, the left hand side of (1.4.16.1) is zero for $i = 1$, hence so is $\mathrm{H}^1(X,\mathscr{F})$ since B is a faithfully flat A-module. As X is a quasi-compact scheme, we finish the proof by Serre's criterion (**II**, 5.2.1). □

Proposition (1.4.17). — *Let X be a prescheme, $0 \to \mathscr{F} \xrightarrow{u} \mathscr{G} \xrightarrow{v} \mathscr{H} \to 0$ an exact sequence of $\mathscr{O}_X$-modules. If $\mathscr{F}$ and $\mathscr{H}$ are quasi-coherent, then so is $\mathscr{G}$.*

III | 95

Proof. The question is local on X, so we can suppose that $X = \mathrm{Spec}(A)$ is affine, and it then suffices to prove that $\mathscr{G}$ satisfies the conditions (d1) and (d2) of (**I**, 1.4.1) (with $V = X$). The verification of (d2) is immediate, because if $t \in \Gamma(X,\mathscr{G})$ is zero when restricted to $D(f)$, then so is its image $v(t) \in \Gamma(X,\mathscr{H})$; therefore there exists an $m > 0$ such that $f^m v(t) = v(f^m t) = 0$ (**I**, 1.4.1), and as Γ is left exact, $f^m t = u(s)$, where $s \in \Gamma(X,\mathscr{F})$; as u is injective, the restriction of s to $D(f)$

is zero, hence (**I**, 1.4.1) there exists an integer $n > 0$ such that $f^n s = 0$; we finally deduce that $f^{m+n} t = u(f^n s) = 0$.

We now check (d1); let $t' \in \Gamma(D(f), \mathscr{G})$; as $\mathscr{H}$ is quasi-coherent, there exists an integer m such that $f^m v(t') = v(f^m t')$ extends to a section $z \in \Gamma(X, \mathscr{H})$ (**I**, 1.4.1). But in virtue of Theorem (1.3.1) (or (**I**, 5.1.9.2)) applied to the quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the sequence $\Gamma(X, \mathscr{G}) \to \Gamma(X, \mathscr{H}) \to 0$ is exact, so there exists $t \in \Gamma(X, \mathscr{G})$ such that $z = v(t)$; we thus see that $v(f^m t' - t'') = 0$, denoting by t'' the restriction of t to $D(f)$; thus we have $f^m t' - t'' = u(s')$, where $s' \in \Gamma(D(f), \mathscr{F})$. But as $\mathscr{F}$ is quasi-coherent, there exists an integer $n > 0$ such that $f^n s'$ extends to a section $s \in \Gamma(X, \mathscr{F})$; as $f^{m+n} t' - f^n t'' = u(f^n s')$, we see that $f^{m+n} t'$ is the restriction to $D(f)$ of a section $f^n t + u(f^n s) \in \Gamma(X, \mathscr{G})$, which finishes the proof. □

§2. Cohomological study of projective morphisms

2.1. Explicit calculations of certain cohomology groups

§3. Finiteness theorem for proper morphisms

3.1. The dévissage lemma

Definition (3.1.1). — Let $\mathtt{C}$ be an abelian category. We say that a subset $\mathtt{C}'$ of the set of objects of $\mathtt{C}$ is *exact* if $0 \in \mathtt{C}'$ and if, for every exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathtt{C}$ such that two of the objects A, A', A'' are in $\mathtt{C}'$, then the third is also in $\mathtt{C}'$.

Theorem (3.1.2). — *Let X be a Noetherian prescheme; we denote by $\mathtt{C}$ the abelian category of coherent $\mathscr{O}_X$-modules. Let $\mathtt{C}'$ be an exact subset of $\mathtt{C}$, X' a closed subset of the underlying space of X. Suppose that for every closed irreducible subset Y of X', with generic point y, there exists an $\mathscr{O}_X$-module $\mathscr{G} \in \mathtt{C}'$ such that $\mathscr{G}_y$ is a $k(y)$-vector space of dimension 1. Then every coherent $\mathscr{O}_X$-module with support contained in X' is in $\mathtt{C}'$ (and in particular, if $X' = X$, then we have $\mathtt{C}' = \mathtt{C}$).*

Proof. Consider the following property $\mathbf{P}(Y)$ of a closed subset Y of X': every coherent $\mathscr{O}_X$-module with support contained in Y is in $\mathtt{C}'$. By virtue of the principle of Noetherian induction (**0**, 2.2.2), we see that we can reduce to showing that *if Y is a closed subset of X' such that the property $\mathbf{P}(Y')$ is true for every closed subset Y' of Y, distinct from Y, then $\mathbf{P}(Y)$ is true.*

Therefore, let $\mathscr{F} \in \mathtt{C}$ have support contained in Y, and we show that $\mathscr{F} \in \mathtt{C}'$. Denote also by Y the closed reduced subprescheme of X having Y for its underlying space (**I**, 5.2.1); it is defined by a coherent sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_X$. We know (**I**, 9.3.4) that there exists an integer $n > 0$ such that $\mathscr{J}^n \mathscr{F} = 0$; for $1 \leqslant k \leqslant n$, we thus have an exact sequence

$$0 \longrightarrow \mathscr{J}^{k-1}\mathscr{F} / \mathscr{J}^k \mathscr{F} \longrightarrow \mathscr{F} / \mathscr{J}^k \mathscr{F} \longrightarrow \mathscr{F} / \mathscr{J}^{k-1} \mathscr{F} \longrightarrow 0$$

of coherent $\mathscr{O}_X$-modules ((**I**, 5.3.6) and (**I**, 5.3.3)); as $\mathtt{C}'$ is exact, we see, by induction on k, that it suffices to show that each of the $\mathscr{F}_k = \mathscr{J}^{k-1}\mathscr{F} / \mathscr{J}^k \mathscr{F}$ is in $\mathtt{C}'$. We thus reduce to proving that $\mathscr{F} \in \mathtt{C}'$ under the additional hypothesis that $\mathscr{J}\mathscr{F} = 0$; it is equivalent to say that $\mathscr{F} = j_*(j^*(\mathscr{F}))$, where j is the canonical injection $Y \to X$. Let us now consider two cases:

(a) Y is *reducible*. Let $Y = Y' \cap Y''$, where Y' and Y'' are closed subsets of Y, distinct from Y; denote also by Y' and Y'' the closed reduced subpreschemes of X having Y and Y'' for their respective underlying spaces, which are defined respectively by sheaves of ideals $\mathscr{J}'$ and $\mathscr{J}''$ of $\mathscr{O}_X$. Set $\mathscr{F}' = \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X / \mathscr{J}')$ and $\mathscr{F}'' = \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X / \mathscr{J}'')$. The canonical homomorphisms $\mathscr{F} \to \mathscr{F}'$ and $\mathscr{F} \to \mathscr{F}''$ thus define a homomorphism $u : \mathscr{F} \to \mathscr{F}' \oplus \mathscr{F}''$. We show that for every $z \notin Y' \cap Y''$, the homomorphism $u_z : \mathscr{F}_z \to \mathscr{F}'_z \oplus \mathscr{F}''_z$ is *bijective*. Indeed, we have $\mathscr{J}' \cap \mathscr{J}'' = \mathscr{J}$, since the question is local and the above equality follows **III** | 116
from ((**I**, 5.2.1) and (**I**, 1.1.5)); if $z \notin Y''$, then we have $\mathscr{J}'_z = \mathscr{J}_z$, hence $\mathscr{F}'_z = \mathscr{F}_z$ and $\mathscr{F}''_z = 0$, which establishes our assertion in this case; we reason similarly for $z \notin Y'$. As a result, the kernel and cokernel of u, which are in $\mathtt{C}$ (**0**, 5.3.4), have their support in $Y' \cap Y''$, and thus is in $\mathtt{C}'$ by hypothesis; for the same reason, $\mathscr{F}'$ and $\mathscr{F}''$ are in $\mathtt{C}'$, hence also $\mathscr{F}' \oplus \mathscr{F}''$, as $\mathtt{C}'$ is exact. The conclusion then follows from the consideration of the two exact sequences

$$0 \longrightarrow \operatorname{Im} u \longrightarrow \mathscr{F}' \oplus \mathscr{F}'' \longrightarrow \operatorname{Coker} u \longrightarrow 0,$$

$$0 \longrightarrow \operatorname{Ker} u \longrightarrow \mathscr{F} \longrightarrow \operatorname{Im} u \longrightarrow 0,$$

and the hypothesis that $\mathtt{C}'$ is exact.

(b) Y is irreducible, and as a result, the subprescheme Y of X is *integral*. If y is its generic point, then we have $(\mathscr{O}_Y)_y = k(y)$, and as $j^*(\mathscr{F})$ is a coherent $\mathscr{O}_Y$-module, $\mathscr{F}_y = (j^*(\mathscr{F}))_y$ is a $k(y)$-vector space of finite dimension m. By hypothesis, there is a coherent $\mathscr{O}_X$-module $\mathscr{G} \in \mathtt{C}'$ (necessarily of support Y) such that $\mathscr{G}_y$ is a $k(y)$-vector space of dimension 1. As a result, there is a $k(y)$-isomorphism $(\mathscr{G}_y)^m \simeq \mathscr{F}_y$, which is also an $\mathscr{O}_Y$-isomorphism, and as $\mathscr{G}^m$ and $\mathscr{F}$ are coherent, there exists an open neighbourhood W of y in X and an isomorphism $\mathscr{G}^m|W \simeq \mathscr{F}|W$ (**0**, 5.2.7). Let $\mathscr{H}$ be the graph of this isomorphism, which is a coherent $(\mathscr{O}_X|W)$-submodule of $(\mathscr{G}^m \oplus \mathscr{F})|W$, canonically isomorphic to $\mathscr{G}^m|W$ and to $\mathscr{F}|W$; there thus exists a coherent $\mathscr{O}_X$-submodule $\mathscr{H}_0$ of $\mathscr{G}^m \oplus \mathscr{F}$, inducing $\mathscr{H}$ on W and 0 on $X - Y$, since $\mathscr{G}^m$ and $\mathscr{F}$ have Y for their support (**I**, 9.4.7). The restrictions $v : \mathscr{H}_0 \to \mathscr{G}^m$ and $w : \mathscr{H}_0 \to \mathscr{F}$ of the canonical projections of $\mathscr{G}^m \oplus \mathscr{F}$ are then homomorphisms of coherent $\mathscr{O}_X$-modules, which, on W and on $X - Y$, reduce to isomorphisms; in other words, the kernels and cokernels of v and w have their support in the closed set $Y - (Y \cap W)$, distinct from Y. They are in $\mathtt{C}'$; on the other hand, we have $\mathscr{G}^m \in \mathtt{C}'$ since $\mathscr{G} \in \mathtt{C}'$ and since $\mathtt{C}'$ is exact. We conclude successively, by the exactness of $\mathtt{C}'$, that $\mathscr{H}_0 \in \mathtt{C}'$, then $\mathscr{F} \in \mathtt{C}'$. Q.E.D.

□

Corollary (3.1.3). — *Suppose that the exact subset $\mathtt{C}'$ of $\mathtt{C}$ has in addition the property that any coherent direct factor of a coherent $\mathscr{O}_X$-module $\mathscr{M} \in \mathtt{C}'$ is also in $\mathtt{C}'$. In this case, the conclusion of Theorem (3.1.2) is still valid when the condition "$\mathscr{G}_y$ is a $k(y)$-vector space of dimension 1" is replaced by $\mathscr{G}_y \neq 0$ (this is equivalent to* $\operatorname{Supp}(\mathscr{G}) = Y$*).*

PROOF. The reasoning of Theorem (3.1.2) must be modified only in the case (b); now $\mathscr{G}_y$ is a $k(y)$-vector space of dimension $q > 0$, and as a result, we have an $\mathscr{O}_Y$-isomorphism $(\mathscr{G}_y)^m \simeq (\mathscr{F}_y)^q$; the end of the reasoning in Theorem (3.1.2) then proves that $\mathscr{F}^q \in \mathtt{C}'$, and the additional hypothesis on $\mathtt{C}'$ implies that $\mathscr{F} \in \mathtt{C}'$. □

3.2. The finiteness theorem: the case of usual schemes

Theorem (3.2.1). — *Let Y be a locally Noetherian prescheme, $f : X \to Y$ a proper morphism. For every coherent $\mathscr{O}_X$-module $\mathscr{F}$, the $\mathscr{O}_Y$-modules $\mathrm{R}^q f_*(\mathscr{F})$ are coherent for $q \geqslant 0$.*

PROOF. Since the questions is local on Y, we can suppose Y Noetherian, thus X Noetherian (**I**, 6.3.7). The coherent $\mathscr{O}_X$-modules $\mathscr{F}$ for which the conclusion of Theorem (3.2.1) is true forms an *exact* subset $\mathtt{C}'$ of the category $\mathtt{C}$ of coherent $\mathscr{O}_X$-modules. Indeed, let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of coherent $\mathscr{O}_X$-modules; suppose for example that $\mathscr{F}'$ and $\mathscr{F}''$ belong to $\mathtt{C}'$; we have the long exact sequence in cohomology **III | 117**

$$\mathrm{R}^{q-1} f_*(\mathscr{F}'') \xrightarrow{\partial} \mathrm{R}^q f_*(\mathscr{F}') \longrightarrow \mathrm{R}^q f_*(\mathscr{F}) \longrightarrow \mathrm{R}^q f_*(\mathscr{F}'') \xrightarrow{\partial} \mathrm{R}^{q+1} f_*(\mathscr{F}'),$$

in which by hypothesis the outer four terms are coherent; it is the same for the middle term $\mathrm{R}^q f_*(\mathscr{F})$ by ((**0**, 5.3.4) and (**0**, 5.3.3)). We show in the same way that when $\mathscr{F}$ and $\mathscr{F}'$ (resp. $\mathscr{F}$ and $\mathscr{F}''$) are in $\mathtt{C}'$, then so is $\mathscr{F}''$ (resp. $\mathscr{F}'$). In addition, every coherent *direct factor* $\mathscr{F}'$ of an $\mathscr{O}_X$-module $\mathscr{F} \in \mathtt{C}'$ belongs to $\mathtt{C}'$: indeed, $\mathrm{R}^q f_*(\mathscr{F}')$ is then a direct factor of $\mathrm{R}^q f_*(\mathscr{F})$ (G, II, 4.4.4), therefore it is of finite type, and as it is quasi-coherent (1.4.10), it is coherent, as Y is Noetherian. By virtue of Corollary (3.1.3), we reduce to proving that when X is *irreducible* with generic point x, there exists *one* coherent $\mathscr{O}_X$-module $\mathscr{F}$ belonging to $\mathtt{C}'$, such that $\mathscr{F}_x \neq 0$: indeed, if this point is established, then it can be applied to any irreducible closed subprescheme Y of X, since if $j : Y \to X$ is the canonical injection, then $f \circ j$ is proper (**II**, 5.4.2), and if $\mathscr{G}$ is a coherent $\mathscr{O}_Y$-module with support Y, then $j_*(\mathscr{G})$ is a coherent $\mathscr{O}_X$-module such that $\mathrm{R}^q (f \circ j)_*(\mathscr{G}) = \mathrm{R}^q f_*(j_*(\mathscr{G}))$ (G, II, 4.9.1), therefore we can apply Corollary (3.1.3).

By virtue of Chow's lemma (**II**, 5.6.2), there exists an irreducible prescheme X' an a *projective* and surjective morphism $g : X' \to X$ such that $f \circ g : X' \to Y$ is *projective*. There exists an ample $\mathscr{O}_X$-module $\mathscr{L}$ for g (**II**, 5.3.1); we apply the fundamental theorem of projective morphisms (2.2.1) to

$g : X' \to X$ and with $\mathscr{L}$: there thus exists an integer n such that $\mathscr{F} = g_*(\mathscr{O}_{X'}(n))$ is a coherent $\mathscr{O}_X$-module and $\mathrm{R}^q g_*(\mathscr{O}_{X'}(n)) = 0$ for all $q > 0$; in addition, as $g^*(g_*(\mathscr{O}_{X'}(n))) \to \mathscr{O}_{X'}(n)$ is surjective for n large enough (2.2.1), we see that we can suppose, at the generic point x of X, that we have $\mathscr{F}_x \neq 0$ (**II**, 3.4.7). On the other hand, as $f \circ g$ is projective as Y is Noetherian, the $\mathrm{R}^q(f \circ g)_*(\mathscr{O}_{X'}(n))$ are *coherent* (2.2.1). This being so, $\mathrm{R}^\bullet(f \circ g)_*(\mathscr{O}_{X'}(n))$ is the abutment of a Leray spectral sequence, whose E_2-term is given by $E_2^{pq} = \mathrm{R}^p f_*(\mathrm{R}^q g_*(\mathscr{O}_{X'}(n)))$; the above shows that this spectral sequence degenerates, and we then know (**0**, 11.1.6) that $E_2^{p0} = \mathrm{R}^p f_*(\mathscr{F})$ is isomorphic to $\mathrm{R}^p(f \circ g)_*(\mathscr{O}_{X'}(n))$, which finishes the proof. □

Corollary (3.2.2). — *Let Y be a locally Noetherian prescheme. For every proper morphism $f : X \to Y$, the direct image under f of any coherent $\mathscr{O}_X$-module is a coherent $\mathscr{O}_Y$-module.*

Corollary (3.2.3). — *Let A be a Noetherian ring, X a proper scheme over A; for every coherent $\mathscr{O}_X$-module $\mathscr{F}$, the $\mathrm{H}^p(X, \mathscr{F})$ are A-modules of finite type, and there exists an integer $r > 0$ such that for every coherent $\mathscr{O}_X$-module $\mathscr{F}$ and all $p > r$, $\mathrm{H}^p(X, \mathscr{F}) = 0$.*

Proof. The second assertion has already been proved (1.4.12); the first follows from the finiteness theorem (3.2.1), taking into account Corollary (1.4.11). □

In particular, if X is a *proper algebraic scheme* over a field k, then, for every coherent $\mathscr{O}_X$-module $\mathscr{F}$, the $\mathrm{H}^p(X, \mathscr{F})$ are *finite-dimensional* k-vector spaces.

Corollary (3.2.4). — *Let Y be a locally Noetherian prescheme, $f : X \to Y$ a morphism of finite type. For every coherent $\mathscr{O}_X$-module $\mathscr{F}$ whose support in proper over Y (**II**, 5.4.10), the $\mathscr{O}_Y$-modules $\mathrm{R}^q f_*(\mathscr{F})$ are coherent.*

III | 118

Proof. Since the questions is local on Y, we can suppose Y Noetherian, and it is the same for X (**I**, 6.3.7). By hypothesis, every closed subprescheme Z of X whose underlying space is $\mathrm{Supp}(\mathscr{F})$ is proper over Y, in other words, if $j : Z \to X$ is the canonical injection, then $f \circ j : Z \to Y$ is proper. We can suppose that Z is such that $\mathscr{F} = j_*(\mathscr{G})$, where $\mathscr{G} = j^*(\mathscr{F})$ is a coherent $\mathscr{O}_Z$-module (**I**, 9.3.5); as we have $\mathrm{R}^q f_*(\mathscr{F}) = \mathrm{R}^q(f \circ j)_*(\mathscr{G})$ by Corollary (1.3.4), the conclusion follows immediately from Theorem (3.2.1). □

3.3. Generalization of the finiteness theorem (usual schemes)

Proposition (3.3.1). — *Let Y be a Noetherian prescheme, $\mathscr{S}$ a quasi-coherent $\mathscr{O}_Y$-algebra of finite type, graded in positive degrees, $Y' = \mathrm{Proj}(\mathscr{S})$, and $g : Y' \to Y$ the structure morphism. Let $f : X \to Y$ be a proper morphism, $\mathscr{S}' = f^*(\mathscr{S})$, $\mathscr{M} = \bigoplus_{k \in \mathbf{Z}} \mathscr{M}_k$ a quasi-coherent graded $\mathscr{S}'$-module of finite type. Then the $\mathrm{R}^p f_*(\mathscr{M}) = \bigoplus_{k \in \mathbf{Z}} \mathrm{R}^p f_*(\mathscr{M}_k)$ are graded $\mathscr{S}$-modules of finite type for all p. Suppose in addition that the $\mathscr{S}$ are generated by $\mathscr{S}_1$; then, for every $p \in \mathbf{Z}$, there exists an integer k_p such that for all $k \geqslant k_p$ and all $r > 0$, we have*

$$\mathrm{R}^p f_*(\mathscr{M}_{k+r}) = \mathscr{S}_r \mathrm{R}^p f_*(\mathscr{M}_k). \tag{3.3.1.1}$$

Proof. The first assertion is identical to the statement of Theorem (2.4.1, i), where we have simply replaced "projective morphism" by "proper morphism". In the proof of Theorem (2.4.1, i), the hypothesis on f was only used to show (with the notation of this proof) that $\mathrm{R}^p f'_*(\widetilde{\mathscr{M}})$ is a coherent $\mathscr{O}_{Y'}$-module. With the hypothesis of Proposition (3.3.1), f' is proper (**II**, 5.4.2, iii), so we can resume without change in the proof of Theorem (2.4.1, i), thanks to the finiteness theorem (3.2.1).

As for the second assertion, it suffices to remark that there is a finite affine open cover (U_i) of Y such that the restrictions to the U_i of the two sides of (3.3.1.1) are equal for all $k \geqslant k_{p,i}$ (**II**, 2.1.6, ii); it suffices to take for k_p the largest of the $k_{p,i}$. □

Corollary (3.3.2). — *Let A be a Noetherian ring, $\mathfrak{m}$ an ideal of A, X a proper A-scheme, and $\mathscr{F}$ a coherent $\mathscr{O}_X$-module. Then, for all $p \geqslant 0$, the direct sum $\bigoplus_{k \geqslant 0} \mathrm{H}^p(X, \mathfrak{m}^k \mathscr{F})$ is a module of finite type over the ring $S = \bigoplus_{k \geqslant 0} \mathfrak{m}^k$; in particular, there exists an integer $k_p \geqslant 0$ such that for all $k \geqslant k_p$ and all $r > 0$, we have*

$$\mathrm{H}^p(X, \mathfrak{m}^{k+r} \mathscr{F}) = \mathfrak{m}^r \mathrm{H}^p(X, \mathfrak{m}^k \mathscr{F}). \tag{3.3.2.1}$$

PROOF. It suffices to apply Proposition (3.3.1) with $Y = \mathrm{Spec}(A)$, $\mathscr{S} = \widetilde{S}$, $\mathscr{M}_k = \mathfrak{m}^k\mathscr{F}$, taking into account Corollary (1.4.11). □

It should be remembered that the S-module structure on $\bigoplus_{k\geqslant 0}\mathrm{H}^p(X, \mathfrak{m}^k\mathscr{F})$ is obtained by considering, for every $a \in \mathfrak{m}^r$, the map $\mathrm{H}^p(X, \mathfrak{m}^k\mathscr{F}) \to \mathrm{H}^p(X, \mathfrak{m}^{k+r}\mathscr{F})$, which comes from the passage to cohomology of the multiplication map $\mathfrak{m}^r\mathscr{F} \to \mathfrak{m}^{k+r}\mathscr{F}$ defined by a (2.4.1).

3.4. Finiteness theorem: the case of formal schemes The results of this section (except the definition (3.4.1)) will not be used in the rest of this chapter. **III** | 119

(3.4.1). Let $\mathfrak{X}$ and $\mathfrak{S}$ be two locally Noetherian formal preschemes (I, 10.4.2), $f : \mathfrak{X} \to \mathfrak{S}$ a morphism of formal preschemes. We say that f is a *proper* morphism if it satisfies the following conditions:

1st. *f is a morphism of finite type* (I, 10.13.3).

2nd. *If $\mathscr{K}$ is a sheaf of ideals of definition for $\mathfrak{S}$ and if we set $\mathscr{J} = f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}}$, $X_0 = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{J})$, $S_0 = (\mathfrak{S}, \mathscr{O}_{\mathfrak{S}}/\mathscr{K})$, then the morphism $f_0 : X_0 \to S_0$ induced by f* (I, 10.5.6) *is proper*.

It is immediate that this definition does not depend on the sheaf of ideals of definition $\mathscr{K}$ for $\mathscr{S}$ considered; indeed, if $\mathscr{K}'$ is a second sheaf of ideals of definition such that $\mathscr{K}' \subset \mathscr{K}$, and if we set $\mathscr{J}' = f^*(\mathscr{K}')\mathscr{O}_{\mathfrak{X}}$, $X_0' = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{J}')$, $S_0' = (\mathscr{S}, \mathscr{O}_{\mathfrak{S}}/\mathscr{K}')$, then the morphism $f_0' : X_0' \to S_0'$ induced by f is such that the diagram

$$\begin{tikzcd} X_0 \arrow[r, "f_0"] \arrow[d, "i"'] & S_0 \arrow[d, "j"] \\ X_0' \arrow[r, "f_0'"] & S_0' \end{tikzcd}$$

is commutative, i and j being surjective immersions; it is equivalent to say that f_0 or f_0' is proper, by virtue of (II, 5.4.5).

We note that, for all $n \geqslant 0$, if we set $X_n = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{J}^{n+1})$, $S_n = (\mathfrak{S}, \mathscr{O}_{\mathfrak{S}}/\mathscr{K}^{n+1})$, then the morphism $f_n : X_n \to S_n$ induced by f (I, 10.5.6) is proper for all n whenever it is for $n = 0$ (II, 5.4.6).

If $g : Y \to Z$ is a proper morphism of locally Noetherian usual preschemes, Z' a closed subset of Z, Y' a closed subset of Y such that $g(Y') \subset Z'$, then the extension $\widehat{g} : Y_{/Y'} \to Z_{/Z'}$ of g to the completions (I, 10.9.1) is a proper morphism of formal preschemes, as it follows from the definition and from (II, 5.4.5).

Let $\mathfrak{X}$ and $\mathfrak{S}$ be two locally Noetherian formal preschemes, $f : \mathfrak{X} \to \mathfrak{S}$ a morphism *of finite type* (I, 10.13.3); the notation being the same as above, we say that a subset Z of the underlying space of $\mathfrak{X}$ is *proper* over $\mathfrak{S}$ (or proper for f) if, considered as a subset of X_0, Z is *proper over* S_0 (II, 5.4.10). All the properties of proper subsets of usual preschemes stated in (II, 5.4.10) are still true for the proper subsets of formal preschemes, as it follows immediately from the definitions.

Theorem (3.4.2). — *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be locally Noetherian formal preschemes, $f : \mathfrak{X} \to \mathfrak{Y}$ a proper morphism. For every coherent $\mathscr{O}_{\mathfrak{X}}$-module $\mathscr{F}$, the $\mathscr{O}_{\mathfrak{Y}}$-modules $\mathrm{R}^q f_*(\mathscr{F})$ are coherent for all $q \geqslant 0$.*

Let $\mathscr{J}$ be a sheaf of ideals of definition for $\mathfrak{Y}$, $\mathscr{K} = f^*(\mathscr{J})\mathscr{O}_{\mathfrak{X}}$, and consider the $\mathscr{O}_{\mathfrak{X}}$-modules

(3.4.2.1) $$\mathscr{F}_k = \mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{Y}}} (\mathscr{O}_{\mathfrak{Y}}/\mathscr{J}^{k+1}) = \mathscr{F}/\mathscr{K}^{k+1}\mathscr{F} \quad (k \geqslant 0)$$

which evidently form a *projective system* of topological $\mathscr{O}_{\mathfrak{X}}$-modules, such that $\mathscr{F} = \varprojlim_k \mathscr{F}_k$ (I, 10.11.3). **III** | 120
On the other hand, it follows from Theorem (3.4.2) that each of the $\mathrm{R}^q f_*(\mathscr{F})$, being coherent, is naturally equipped with a topological $\mathscr{O}_{\mathfrak{Y}}$-module structure (I, 10.11.6), and so are the $\mathrm{R}^q f_*(\mathscr{F}_k)$. The canonical homomorphisms $\mathscr{F} \to \mathscr{F}_k = \mathscr{F}/\mathscr{K}^{k+1}\mathscr{F}$ canonically correspond to homomorphisms

$$\mathrm{R}^q f_*(\mathscr{F}) \longrightarrow \mathrm{R}^q f_*(\mathscr{F}_k),$$

which are necessarily continuous for the topological $\mathscr{O}_{\mathfrak{Y}}$-module structures above (I, 10.11.6), and form a projective system, giving the limit a canonical functorial homomorphism

(3.4.2.2) $$\mathrm{R}^q f_*(\mathscr{F}) \longrightarrow \varprojlim_k \mathrm{R}^q f_*(\mathscr{F}_k),$$

which will be a continuous homomorphism of topological $\mathscr{O}_{\mathfrak{Y}}$-modules. We will prove along with Theorem (3.4.2) the

Corollary (3.4.3). — *Each of the homomorphisms (3.4.2.2) is a topological isomorphism. In addition, if $\mathfrak{Y}$ is Noetherian, then the projective system $(\mathrm{R}^q f_*(\mathscr{F}/\mathscr{K}^{k+1}\mathscr{F}))_{k\geqslant 0}$ satisfies the* (ML)*-condition* (**0**, 13.1.1).

We will begin by establishing Theorem (3.4.2) and Corollary (3.4.3) when Y is a Noetherian formal affine scheme (**I**, 10.4.1):

Corollary (3.4.4). — *Under the hypotheses of Theorem (3.4.2), suppose in addition that $\mathfrak{Y} = \mathrm{Spf}(A)$, where A is an adic Noetherian ring. Let $\mathfrak{J}$ be an ideal of definition for A, and set $\mathscr{F}_k = \mathscr{F}/\mathfrak{J}^{k+1}\mathscr{F}$ for $k \geqslant 0$. Then the $\mathrm{H}^n(\mathfrak{X},\mathscr{F})$ are A-modules of finite type; the projective system $(\mathrm{H}^n(\mathfrak{X},\mathscr{F}_k))_{k\geqslant 0}$ satisfies the* (ML)*-condition for all n; if we set*

$$N_{n,k} = \mathrm{Ker}\left(\mathrm{H}^n(\mathfrak{X},\mathscr{F}) \longrightarrow \mathrm{H}^n(\mathfrak{X},\mathscr{F}_k)\right) \tag{3.4.4.1}$$

(also equal to $\mathrm{Im}(\mathrm{H}^n(\mathfrak{X},\mathfrak{J}^{k+1}\mathscr{F}) \to \mathrm{H}^n(\mathfrak{X},\mathscr{F}))$ by the exact sequence in cohomology), then the $N_{n,k}$ define on $\mathrm{H}^n(\mathfrak{X},\mathscr{F})$ a $\mathfrak{J}$-good filtration (**0**, 13.7.7)*; finally, the canonical homomorphism*

$$\mathrm{H}^n(\mathfrak{X},\mathscr{F}) \longrightarrow \varprojlim_k \mathrm{H}^n(\mathfrak{X},\mathscr{F}_k) \tag{3.4.4.2}$$

is a topological isomorphism for all n (the left hand side being equipped with the $\mathfrak{J}$-adic topology, the $\mathrm{H}^n(\mathfrak{X},\mathscr{F}_k)$ with the discrete topology).

Set

$$S = \mathrm{gr}(A) = \bigoplus_{k\geqslant 0} \mathfrak{J}^k/\mathfrak{J}^{k+1},\ \mathscr{M} = \mathrm{gr}(\mathscr{F}) = \bigoplus_{k\geqslant 0} \mathfrak{J}^k\mathscr{F}/\mathfrak{J}^{k+1}\mathscr{F}.$$

We know that $\mathfrak{J}^\Delta$ is a sheaf of ideals of definition for $\mathfrak{Y}$ (**I**, 10.3.1); let $\mathscr{K} = f^*(\mathfrak{J}^\Delta)\mathscr{O}_\mathfrak{X}$, $X_0 = (\mathfrak{X}, \mathscr{O}_\mathfrak{X}/\mathscr{K})$, $Y_0 = (\mathfrak{Y}, \mathscr{O}_\mathfrak{Y}/\mathfrak{J}^\Delta) = \mathrm{Spec}(A_0)$, with $A_0 = A/\mathfrak{J}$. It is clear that the $\mathscr{M}_k = \mathfrak{J}^k\mathscr{F}/\mathfrak{J}^{k+1}\mathscr{F}$ are coherent $\mathscr{O}_{X_0}$-modules (**I**, 10.11.3). Consider on the other hand the quasi-coherent graded $\mathscr{O}_{X_0}$-algebra

$$\mathscr{S} = \mathscr{O}_{X_0} \otimes_{A_0} S = \mathrm{gr}(\mathscr{O}_\mathfrak{X}) = \bigoplus_{k\geqslant 0} \mathscr{K}^k/\mathscr{K}^{k+1}. \tag{3.4.4.4}$$

The hypothesis that $\mathscr{F}$ is an $\mathscr{O}_\mathfrak{X}$-module of finite type implies first that $\mathscr{M}$ is a graded $\mathscr{S}$-module *of finite type*. Indeed, the question is local on $\mathfrak{X}$, and we can thus suppose that $\mathfrak{X} = \mathrm{Spf}(B)$, where B is an adic Noetherian ring, and $\mathscr{F} = N^\Delta$, where N is a B-module of finite type (**I**, 10.10.5); we have in addition $X_0 = \mathrm{Spec}(B_0)$, where $B_0 = B/\mathfrak{J}B$, and the quasi-coherent $\mathscr{O}_{X_0}$-modules $\mathscr{S}$ and $\mathscr{M}$ are respectively equal to $\widetilde{S'}$ and $\widetilde{M'}$, where $S' = \bigoplus_{k\geqslant 0}((\mathfrak{J}^k/\mathfrak{J}^{k+1}) \otimes_{A_0} B_0)$ and $M' = \bigoplus_{k\geqslant 0}((\mathfrak{J}^k/\mathfrak{J}^{k+1}) \otimes_{A_0} N_0)$, with $N_0 = N/\mathfrak{J}N$; we then evidently have $M' = S' \otimes_{B_0} N_0$, and as N_0 is a B_0-module of finite type, M' is a S'-module of finite type, hence our assertion (**I**, 1.3.13). **III** | 121

As the morphism $f_0 : X_0 \to Y_0$ is *proper* by hypothesis, we can apply Corollary (3.3.2) to $\mathscr{S}$, $\mathscr{M}$, and the morphism f_0: taking into account Corollary (1.4.11), we conclude that for *all* $n \geqslant 0$, $\bigoplus_{k\geqslant 0} \mathrm{H}^n(X_0, \mathscr{M}_k)$ is a graded S-module *of finite type*. This proves that the condition (F_n) of (**0**, 13.7.7) is satisfied for *all* $n \geqslant 0$, when we consider the strictly projective system $(\mathscr{F}/\mathfrak{J}^k\mathscr{F})_{k\geqslant 0}$ of sheaves of abelian groups on X_0, each equipped with its natural "filtered A-module" structure. We can thus apply (**0**, 13.7.7), which proves that:

1st. The projective system $(\mathrm{H}^n(\mathfrak{X},\mathscr{F}_k))_{k\geqslant 0}$ satisfies the (ML)-condition.
2nd. If $\mathrm{H}'^n = \varprojlim_k \mathrm{H}^n(\mathfrak{X},\mathscr{F}_k)$, then H'^n is an A-module of finite type.
3rd. The filtration defined on H'^n by the kernels of the canonical homomorphisms $\mathrm{H}'^n \to \mathrm{H}^n(\mathfrak{X},\mathscr{F}_k)$ is $\mathfrak{J}$-good.

Note that on the other hand, if we set $X_k = (\mathfrak{X}, \mathscr{O}_\mathfrak{X}/\mathscr{K}^{k+1})$, then $\mathscr{F}_k$ is a coherent $\mathscr{O}_{X_k}$-module (**I**, 10.11.3), and if U is an affine open set in X_0, then U is also an affine open set in each of the X_k (**I**, 5.1.9), so $\mathrm{H}^n(U,\mathscr{F}_k) = 0$ for all $n > 0$ and all k (1.3.1) and $\mathrm{H}^0(U,\mathscr{F}_k) \to \mathrm{H}^0(U,\mathscr{F}_h)$ is surjective for $h \leqslant k$ (**I**, 1.3.9). We are thus in the conditions of (**0**, 13.3.2) and applying (**0**, 13.3.1) proves that H'^n canonically identifies with $\mathrm{H}^n(\mathfrak{X}, \varprojlim_k \mathscr{F}_k) = \mathrm{H}^n(\mathfrak{X},\mathscr{F})$; this finishes the proof of Corollary (3.4.4).

(3.4.5). We return to the proof of (3.4.2) and (3.4.3). We first prove the propositions for the case $\mathfrak{Y} = \mathrm{Spf}(A)$ envisaged in (3.4.4); for this, for all $g \in A$, apply (3.4.4) to the Noetherian affine formal scheme induced on the open set $\mathfrak{Y}_g = \mathfrak{D}(g)$ of $\mathfrak{Y}$, which is equal to $\mathrm{Spf}(A_{\{g\}})$, and to the formal prescheme induced by $\mathfrak{X}$ on $f^{-1}(\mathfrak{Y}_g)$; note that $\mathfrak{Y}_g$ is also an affine open set in the prescheme $Y_k = (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/(\mathfrak{J}^{\Delta})^{k+1})$, and as $\mathscr{F}_k$ is a coherent $\mathscr{O}_{X_k}$-module, we have

$$\mathrm{H}^n(f^{-1}(\mathfrak{Y}_g), \mathscr{F}_k) = \Gamma(\mathfrak{Y}_g, \mathrm{R}^n f_*(\mathscr{F}_k))$$

for all $k \geqslant 0$ by virtue of Corollary (1.4.11). The canonical homomorphism

$$\mathrm{H}^n(f^{-1}(\mathfrak{Y}_g), \mathscr{F}) \longrightarrow \varprojlim_k \Gamma(\mathfrak{Y}_g, \mathrm{R}^n f_*(\mathscr{F}_k))$$

is an isomorphism; but we have (**0**, 3.2.6)

$$\varprojlim_k \Gamma(\mathfrak{Y}_g, \mathrm{R}^n f_*(\mathscr{F}_k)) = \Gamma(\mathfrak{Y}_g, \varprojlim_k \mathrm{R}^n f_*(\mathscr{F}_k)),$$

and as the sheaf $\mathrm{R}^n f_*(\mathscr{F})$ is the sheaf associated to the presheaf $\mathfrak{Y}_g \mapsto \mathrm{H}^n(f^{-1}(\mathfrak{Y}_g), \mathscr{F})$ on the $\mathfrak{Y}_g$ (**0**, 3.2.1), we have shown that the homomorphism (3.4.2.2) is *bijective*. Let us now prove that $\mathrm{R}^n f_*(\mathscr{F})$ is a coherent $\mathscr{O}_{\mathfrak{Y}}$-module, and more precisely that we have **III | 122**

(3.4.5.1) $$\mathrm{R}^n f_*(\mathscr{F}) = \big(\mathrm{H}^n(\mathfrak{X}, \mathscr{F})\big)^{\Delta}.$$

With the above notation, we have, since $\mathscr{F}_k$ is a coherent $\mathscr{O}_{X_k}$-module (1.4.13),

$$\Gamma(\mathfrak{Y}_g, \mathrm{R}^n f_*(\mathscr{F}_k)) = (\Gamma(\mathfrak{Y}, \mathrm{R}^n f_*(\mathscr{F}_k)))_g = (\mathrm{H}^n(\mathfrak{X}, \mathscr{F}_k))_g.$$

Now the $\mathrm{H}^n(\mathfrak{X}, \mathscr{F}_k)$ form a projective system satisfying (ML), and their projective limit $\mathrm{H}^n(\mathfrak{X}, \mathscr{F})$ is an A-module of finite type. We conclude (**0**, 13.7.8) that we have

$$\varprojlim_k \big((\mathrm{H}^n(\mathfrak{X}, \mathscr{F}_k))_g\big) = \mathrm{H}^n(\mathfrak{X}, \mathscr{F}) \otimes_A A_{\{g\}} = \Gamma(\mathfrak{Y}_g, (\mathrm{H}^n(\mathfrak{X}, \mathscr{F}))^{\Delta}),$$

taking into account (**I**, 10.10.8) applied to A and $A_{\{g\}}$; this proves (3.4.5.1) since $\Gamma(\mathfrak{Y}_g, \mathrm{R}^n f_*(\mathscr{F})) = \varprojlim_k \Gamma(\mathfrak{Y}_g, \mathrm{R}^n f_*(\mathscr{F}_k))$.

As (3.4.2.2) is then an isomorphism of coherent $\mathscr{O}_{\mathfrak{Y}}$-modules, it is necessarily a *topological* isomorphism (**I**, 10.11.6). Finally, it follows from the relations $\mathrm{R}^n f_*(\mathscr{F}_k) = (\mathrm{H}^n(\mathfrak{X}, \mathscr{F}_k))^{\Delta}$ that the projective system $(\mathrm{R}^n f_*(\mathscr{F}_k))_{k\geqslant 0}$ satisfies (ML) (**I**, 10.10.2).

Once (3.4.2) and (3.4.3) are proved in the case where the formal prescheme $\mathfrak{Y}$ is affine Noetherian, it is immediate to pass to the general case for (3.4.2) and the first assertion of (3.4.3), which are local on $\mathfrak{Y}$. As for the second assertion of (3.4.3), it suffices, $\mathfrak{Y}$ being Noetherian, to cover it by a finite number of Noetherian affine open sets U_i and to note that the restrictions of the projective system $(\mathrm{R}^q f_*(\mathscr{F}_k))$ to each of the U_i satisfies (ML).

Along the way, we have in addition proved:

Corollary (3.4.6). — *Under the hypotheses of Corollary (3.4.4), the canonical homomorphism*

(3.4.6.1) $$\mathrm{H}^q(\mathfrak{X}, \mathscr{F}) \longrightarrow \Gamma(\mathfrak{Y}, \mathrm{R}^q f_*(\mathscr{F}))$$

is bijective.

§4. The fundamental theorem of proper morphisms. Applications

4.1. The fundamental theorem

§5. An existence theorem for coherent algebraic sheaves

5.1. Statement of the theorem

§6. Local and global Tor functors; Künneth formula

6.1. Introduction

§7. Base change for homological functors of sheaves of modules

7.1. Functors of A-modules

CHAPTER IV

Local study of schemes and their morphisms (EGA IV)

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Summary

IV-1 | 222

The subjects discussed in the chapter call for the following remarks. **IV-1** | 223

(a) The common property of all the subjects discussed is that they all related to *local* properties of preschemes or morphisms, i.e. considered at a point, or the points of a fibre, or on a (non-specified) neighbourhood of a point or of a fibre. These properties are generally of a *topological*, *differential*, or *dimensional* nature (i.e. bringing the ideas of *dimension* and *depth* into play), and are linked to the properties of the *local rings* at the points considered. One type of problem is the relating, for a given morphisms $f : X \to Y$ and point $x \in X$, of the properties of X at x with those of Y at $y = f(x)$ and those of the fibre $X_y = f^{-1}(y)$ at x. Another is the determining of the topological nature (for example, the constructibility, or the fact of being open or closed) of the set of points $x \in X$ at which X has a certain property, or for which the fibre $X_{f(x)}$ passing through x has a certain property at x. Similarly, we are interested in the topological nature of the set of points $y \in Y$ such that X has a certain property at all the points of the fibre X_y, or those such that this fibre itself has a certain property.

[1]The order and content of §§11–21 are given only as an indication of what the titles will be, and will possibly be modified before their publication. *[Trans.] This was indeed the case: many of §§11–21 ended up having entirely different titles or content. See here.*

(b) The most important idea for the following chapters is that of *flat morphisms of finite presentation*, as well as the particular cases of *smooth morphisms* and *étale morphisms*. Their detailed study (as well as that of connected questions) really starts in §11.

(c) Sections §§1–10 can be considered as being preliminary in nature, and as developing three types of techniques, used, not only in the other sections of the chapter, but also, of course, in the follow chapters:

  (c1) Sections §§1–4 are envisaged as treating the diverse aspects of the idea of *change of base*, above all in relation with the conditions of *finiteness* or *flatness*; we there initiate the technique of *descent*, with its most elementary aspects (the questions of "effectiveness" linked to this technique will be studied in Chapter V).

  (c2) Sections §§5–7 are focused on what we may call *Noetherian* techniques, since the preschemes considered are always locally Noetherian, whereas, on the contrary, there is generally no finiteness condition imposed on the *morphisms*; this is essentially due to the fact that the ideas of dimension and depth are hardly manageable except in the case of Noetherian local rings. Recall that §7 constitutes a "delicate (?)" theory of Noetherian local rings, not much used in what follows in the chapter.

  (c3) Sections §§8–10 describe, amongst other things, the means of *eliminating the Noetherian hypotheses* on the preschemes considered, by substituting such hypotheses for suitable ones of *finiteness* ("finite presentation") on the *morphisms* considered: the advantage of this substitution is that the latter such hypotheses (those of finiteness on the morphisms) are *stable under base change*, which is not the case for the Noetherian hypotheses on the preschemes. The technique permitting this substitution relies, in some part, on the use of the idea of the *projective limit* of preschemes, thanks to which we can reduce a question to the same question with *Noetherian* hypotheses; on the other hand, it relies on the systematic use of *constructible sets*, which have the double interest of being preserved under taking inverse images (of arbitrary morphisms) and by direct images **IV-1** | 224 (of morphisms of finite presentation), and having manageable topological properties in locally Noetherian preschemes. The same techniques often even allow to restrict to the case of more specific Noetherian rings, for example the **Z**-*algebras of finite type*, and it is here that the properties of "excellent" rings (studied in §7) intervene in a decisive manner. Independently of the question of elimination of Noetherian hypotheses, the techniques of §§8–10, elementary in nature, find constant use in nearly all applications.

§1. Relative finiteness conditions. Constructible sets in preschemes

In this section, we will resume the exposé of "finiteness conditions" for a morphism of preschemes $f : X \to Y$ given in (**I**, 6.3 and 6.6). There are essentially two notions of "finiteness" of a *global* nature on X, that of *quasi-compact* morphism (defined in (**I**, 6.6.1)) and that of a *quasi-separated* morphism; on the other hand, there are two notions of "finiteness" of a *local* nature on X, that of a morphism *locally of finite type* (defined in (**I**, 6.6.2)) and that of a morphism *locally of finite presentation*. By combining these local notions with the preceding global notions, we obtain the notion of a morphism *of finite type* (defined in (**I**, 6.3.1)) and of a morphism *of finite presentation*. For the convenience of the reader, we will give again in this section the properties stated in (**I**, 6.3 and 6.6), referring to their labels in Chapter I for their proofs.

In n^{os}1.8 and 1.9, we complete, in the context of preschemes, and making use of the previous notions of finiteness, the results on constructible sets given in ($\mathbf{0}_{\text{III}}$, §9).

1.1. Quasi-compact morphisms

Definition (1.1.1). — We say that a morphism of preschemes $f : X \to Y$ is *quasi-compact* if the continuous map f from the topological space X to the topological space Y is quasi-compact (**0**, 9.1.1), in other words, if the inverse image $f^{-1}(U)$ of every quasi-compact open subset U of Y is quasi-compact (cf. (**I**, 6.6.1)).

If $\mathfrak{B}$ is a basis for the topology of Y consisting of affine open sets, then for f to be quasi-compact, it is necessary and sufficient that for all $V \in \mathfrak{B}$, $f^{-1}(V)$ is a *finite union of affine open sets*. For example, if Y is affine and X is quasi-compact, *every* morphism $f : X \to Y$ is quasi-compact (**I**, 6.6.1).

If $f : X \to Y$ is a quasi-compact morphism, then it is clear that for every open subset V of Y, the restriction of f to $f^{-1}(V)$ is a quasi-compact morphism $f^{-1}(V) \to V$. Conversely, if (U_α) is an open cover of Y and $f : X \to Y$ is a morphism such that the restrictions $f^{-1}(U_\alpha) \to U_\alpha$ are quasi-compact, then f is quasi-compact. As a result, if $f : X \to Y$ is an S-morphism of S-preschemes, and if there **IV-1** | 225 exists an open cover (S_λ) of S such that the restrictions $g^{-1}(S_\lambda) \to h^{-1}(S_\lambda)$ of f (where g and h are the structure morphisms) are quasi-compact, then f is quasi-compact.

§2. Base change and flatness

§3. Associated prime cycles and primary decomposition

§4. Change of base field for algebraic preschemes

§5. Dimension, depth, and regularity in locally Noetherian preschemes

§6. Flat morphisms of locally Noetherian preschemes

§7. Relations between a local Notherian ring and its completion. Excellent rings

§8. Projective limits of preschemes

§9. Constructible properties

§10. Jacobson preschemes

§11. Topological properties of finitely presented flat morphisms. Flatness criteria

§12. Study of fibres of finitely presented flat morphisms

§13. Equidimensional morphisms

§14. Universally open morphisms

§15. Study of fibres of a universally open morphism

§16. Differential invariants. Differentially smooth morphisms

In this paragraph we will present, in global form, some notions of differential calculus par- **IV-4** | 5 ticularly useful in algebraic geometry. We will ignore many classic developments in differential geometry (connections, infinitesimal transformations associated to vector fields, jets, etc.), although these notions are translated in a particularly natural way for schemes. We will similarly ignore phenomena exclusive to characteristic $p > 0$ (some of which are seen, in the affine case, in (**0**, 21). For certain complements to the differential formalism for preschemes the reader may consult Exposés II and VII of [**eAG64**] as well as subsequent chapters of this treatise.

16.1. Normal invariants of an immersion

(16.1.1). Let $(X, \mathscr{O}_X), (Y, \mathscr{O}_Y)$ be two ringed spaces and $f = (\psi, \theta) : Y \to X$ a morphism of ringed spaces (**0**, 4.1.1) such that the homomorphism

$$\theta^\# : \psi^*(\mathscr{O}_X) \longrightarrow \mathscr{O}_Y$$

is surjective, so that $\mathscr{O}_Y$ is identified with a sheaf of quotient rings $\psi^*(\mathscr{O}_X)/\mathscr{I}_f$. We can then endow $\psi^*(\mathscr{O}_X)$ with the $\mathscr{I}_f$-preadic filtration.

Definition (16.1.2). — The $\mathscr{O}_Y$-augmented sheaf of rings $\psi^*(\mathscr{O}_X)/\mathscr{I}_f^{n+1}$ is called the n'th *normal invariant* of f; the ringed space $(Y, \psi^*(\mathscr{O}_X)/\mathscr{I}_f^{n+1})$ is called the n'th *infinitesimal neighbourhood* of Y along f and is denoted by $Y_f^{(n)}$ or simply $Y^{(n)}$. The sheaf of graded rings associated to the sheaf of filtered rings $\psi^*(\mathscr{O}_X)$

(16.1.2.1) $$\mathscr{G}r_\bullet(f) = \bigoplus_{n \geqslant 0} (\mathscr{I}_f^n / \mathscr{I}_f^{n+1})$$

is called the sheaf of graded rings *associated to* f. The sheaf $\mathscr{G}r_1(f) = \mathscr{I}_f/\mathscr{I}_f^2$ is called the *conormal sheaf* of f (that will be denoted by $\mathscr{N}_{Y/X}$ when there is no risk of confusion).

It is clear that the $\mathscr{O}_{Y^{(n)}} = \psi^*(\mathscr{O}_X)/\mathscr{I}_f^{n+1}$ (that we also denote $\mathscr{O}_{Y_f^{(n)}}$) form a projective system of **IV-4** | 6
sheaves of rings on Y, the transition homomorphism $\phi_{nm} : \mathscr{O}_{Y^{(m)}} \to \mathscr{O}_{Y^{(n)}}$ for $n \leqslant m$ identifies $\mathscr{O}_{Y^{(n)}}$ with the quotient of $\mathscr{O}_{Y^{(m)}}$ by the power $(\mathscr{I}_f/\mathscr{I}_f^{n+1})^m$ of the *augmentation ideal* of $\mathscr{O}_{Y^{(n)}}$, kernel of $\phi_{0n} : \mathscr{O}_{Y^{(n)}} \to \mathscr{O}_Y$. The $Y^{(n)}$ therefore form a inductive system of ringed spaces, all having underlying space Y, and we have canonical morphisms of ringed spaces $h_n : Y^{(n)} \to X$ equal to (ψ, θ_n), where $\theta_n^\#$ is the canonical morphism $\psi^*(\mathscr{O}_X) \to \psi^*(\mathscr{O}_X)/\mathscr{I}_f^{n+1}$. It is clear that the sheaf $\mathscr{G}r_\bullet(f)$ is a sheaf of graded algebras over the sheaf of rings $\mathscr{O}_Y = \mathscr{G}r_0(f)$ and the $\mathscr{G}r_k(f)$ of $\mathscr{O}_Y$-modules.

As with every sheaf of filtered rings, we have a *canonical surjective homomorphism* of graded $\mathscr{O}_Y$-algebras

(16.1.2.2) $$\mathbf{S}^\bullet_{\mathscr{O}_Y}(\mathscr{G}r_1(f)) \longrightarrow \mathscr{G}r_\bullet(f)$$

which coincide in degrees 0 and 1 with the identities.

Examples (16.1.3). —

(i) Suppose that X is a locally ringed space, Y is reduced to a single point y (endowed with a ring $\mathscr{O}_y$) and that, if $x = \psi(y)$, $\theta^\# : \mathscr{O}_x \to \mathscr{O}_y$ is a *surjective* homomorphism of rings having as kernel the maximal ideal $\mathfrak{m}_x$ of $\mathscr{O}_x$. So the $\mathscr{O}_{Y^{(n)}}$ are identified with the rings $\mathscr{O}_x/\mathfrak{m}_x^{n+1}$ and $\mathscr{G}r_\bullet(f)$ with the graded ring associated with the local ring $\mathscr{O}_x$ endowed with the $\mathfrak{m}_x$-preadic filtration.

(ii) Suppose that Y is a closed subset of an open subspace U of X and that the $\mathscr{O}_Y$ is induced on Y by a quotient sheaf $\mathscr{O}_U/\mathscr{I}$, where $\mathscr{I}$ is an ideal of $\mathscr{O}_U$ such that $\mathscr{I}_x = \mathscr{O}_x$ for every $x \notin Y$; if X is a locally ringed space we also suppose that $\mathscr{I}_x \neq \mathscr{O}_x$ for $y \in Y$ so that $(Y, \mathscr{O}_Y)$ is a locally ringed space.

Let $\psi_0 : Y \to U$ be the canonical injection and denote by $\theta_0 : \mathscr{O}_U \to (\psi_0)_*(\mathscr{O}_Y)$ the homomorphism such that $\theta_0^\#$ is the canonical homomorphism $\psi_0^*(\mathscr{O}_U) = \mathscr{O}_U|Y \to (\mathscr{O}_U/\mathscr{I})|Y$, so that $j_0 = (\psi_0, \theta_0) : Y \to U$ is a morphism of ringed spaces (and of locally ringed spaces if X is a locally ringed space); if $i : U \to X$ is the canonical injection (morphism of ringed spaces), $j = i \circ j_0$ is the morphism (ψ, θ) of Y to X where $\psi : Y \to X$ is the canonical injection and $\theta : \mathscr{O}_X \to \psi_*(\mathscr{O}_Y)$ is the homomorphism such that $\theta^\# = \theta_0^\#$. Since $\theta^\#$ is surjective we can apply the previous definitions; $\mathscr{O}_{Y^{(n)}}$ is equal to $\psi_0^*(\mathscr{O}_U/\mathscr{I}^{n+1})$, and we have $(\psi_0)_*(\mathscr{O}_{Y^{(n)}}) = \mathscr{O}_U/\mathscr{I}^{n+1}$, and $\mathscr{G}r_n(j) = \mathscr{G}r_n(j_0) = \psi_0^*(\mathscr{I}^n/\mathscr{I}^{n+1}) = j_0^*(\mathscr{I}^n/\mathscr{I}^{n+1})$.

(16.1.4). The example (16.1.3, (ii)) shows that in general the $\mathscr{O}_{Y^{(n)}}$ are *not canonically endowed with a structure of an $\mathscr{O}_Y$-module*, or *a fortiori* with a structure of an $\mathscr{O}_Y$-algebra. The data of such structure is equivalent to the data of a homomorphism of sheaves of rings $\lambda_n : \mathscr{O}_Y \to \mathscr{O}_{Y^{(n)}}$, right inverse to the augmentation morphism ϕ_{0n}; it is also equivalent to the data of a morphism of ringed spaces $(1_Y, \lambda_n) : Y^{(n)} \to Y$ left inverse to the canonical morphism $(1_Y, \phi_{0n}) : Y \to Y^{(n)}$.

Proposition (16.1.5). — *Let $f = (\psi, \theta) : Y \to X$ be an immersion of preschemes. Then:*

(i) *$\mathscr{G}r_\bullet(f)$ is a quasi-coherent graded $\mathscr{O}_Y$-algebra.* **IV-4** | 7

(ii) *The $Y^{(n)}$ are preschemes, canonically isomorphic to subpreschemes of X.*

(iii) *Every homomorphism of sheaves of rings $\lambda_n : \mathscr{O}_Y \to \mathscr{O}_{Y^{(n)}}$, right inverse to the augmentation homomorphism ϕ_{0n}, makes the $\mathscr{O}_{Y^{(n)}}$ and $\mathscr{O}_{Y^{(k)}}$ for $k \leqslant n$ quasi-coherent $\mathscr{O}_Y$-algebras; the $\mathscr{O}_Y$-module structures induced from the above structures on the $\mathscr{G}r_k(f)$ for $k \leqslant n$ coincide with the ones defined in (16.1.2).*

PROOF. (i) Since the question is local on X and Y, we can reduce to the case where Y is a closed subpreschemes of X defined by an quasi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$; since $\mathscr{O}_Y$ is the restriction to Y of $\mathscr{O}_X/\mathscr{I}$ the assertion (i) is evident, and $Y^{(n)}$ is the closed subprescheme of X defined by the quasi-coherent ideal $\mathscr{I}^{n+1}$ of $\mathscr{O}_X$. Finally, to prove (iii) we notice that the data of λ_n makes the ideal $\mathscr{I}/\mathscr{I}^n$ of the augmentation ϕ_{0n} and their quotients $\mathscr{I}/\mathscr{I}^{k+1} (1 \leqslant k \leqslant n)$ $\mathscr{O}_Y$-modules, and it suffices to prove by induction on k that the $\mathscr{I}/\mathscr{I}^{k+1}$ are quasi-coherent $\mathscr{O}_Y$-modules and the structure of quotient $\mathscr{O}_Y$-module induced on $\mathscr{I}^k/\mathscr{I}^{k+1}$ is the same as defined on (16.1.2). The second assertion is immediate, $\mathscr{I}^k/\mathscr{I}^{k+1}$ being killed by $\mathscr{I}/\mathscr{I}^{n+1}$; the first result, by induction on k, is trivial for $k = 1$ and for $\mathscr{I}/\mathscr{I}^{k+1}$ being an extension of $\mathscr{I}/\mathscr{I}^k$ by $\mathscr{I}^k/\mathscr{I}^{k+1}$ (1.4.17). □

Corollary (16.1.6). — *Under the general hypotheses of (16.1.5), if the immersion f is locally of finite presentation then the $\mathscr{G}r_n(f)$ are quasi-coherent $\mathscr{O}_Y$-modules of finite type.*

PROOF. Indeed, with the notation from the proof of (16.1.5), $\mathscr{I}$ is an ideal of finite type of $\mathscr{O}_X$ (1.4.7), therefore the $\mathscr{I}^n/\mathscr{I}^{n+1}$ are $\mathscr{O}_Y$-modules of finite type, hence the conclusion. □

Corollary (16.1.7). — *Under the general hypotheses of (16.1.5), let $g : X \to Y$ be a morphism of preschemes, left inverse to f. Therefore, for every n, the composite morphism $(1, \lambda_n) : Y^{(n)} \xrightarrow{h_n} X \xrightarrow{g} Y$ defines a homomorphism of sheaves of rings $\lambda_n : \mathscr{O}_Y \to \mathscr{O}_{Y^{(n)}}$ right inverse to the augmentation ϕ_{0n}, making $\mathscr{O}_{Y^{(n)}}$ a quasi-coherent $\mathscr{O}_Y$-algebra; via these homomorphisms, the transition homomorphism $\phi_{nm} : \mathscr{O}_{Y^{(m)}} \to \mathscr{O}_{Y^{(n)}}$ ($n \leqslant m$) are homomorphisms of $\mathscr{O}_Y$-algebras. Also, if g is locally of finite type, then the $\mathscr{O}_{Y^{(n)}}$ are quasi-coherent $\mathscr{O}_Y$-modules of finite type.*

PROOF. The first assertion is an immediate result from the definitions and (16.1.5). On the other hand, if g is locally of finite type, then f is locally of finite presentation (1.4.3, (v)); the $\mathscr{G}r_n(f)$ being then quasi-coherent $\mathscr{O}_Y$-modules of finite type by (16.1.6), the same goes for the $\mathscr{O}_Y$-modules $\mathscr{I}/\mathscr{I}^{n+1}$, being extensions of a finite number of the $\mathscr{G}r_k(f)$ (**III**, 1.4.17). □

Proposition (16.1.8). — *Let X be a locally Noetherian prescheme, $j : Y \to X$ an immersion; Then the $Y^{(n)}$ are locally Noetherian preschemes, the $\mathscr{G}r_n(j)$ are coherent $\mathscr{O}_Y$-modules and the $\mathscr{G}r_\bullet(j)$ is a coherent sheaf of rings over the space Y.*

PROOF. Everything is local on X and Y, so we reduce to the case where X is affine and j is a closed immersion and therefore all the assertions are evident except for the last, which follows from the fact that if A is a Noetherian ring and $\mathfrak{J}$ is an ideal of A, then $\mathrm{gr}^\bullet_{\mathfrak{J}}(A)$ is a Noetherian ring, taking into account the exactness of the functor ψ^* and (**0**, 5.3.7). □

Proposition (16.1.9). — *Let X be a prescheme, $j : Y \to X$ an immersion locally of finite presentation, y a point of Y. The following conditions are equivalent:* **IV-4** | 8

(a) *There exists an open neighbourhood U of y in Y such that $j|U$ is a homeomorphism of U onto an open set of X.*

(b) *There is an integer $n > 0$ such that the canonical homomorphism*

$$(\phi_{n-1,n})_y : \mathscr{O}_{Y^{(n)},y} \longrightarrow \mathscr{O}_{Y^{(n-1)},y}$$

is bijective.

(c) *There is an integer $n > 0$ such that $(\mathscr{G}r_n(j))_y = 0$.*

In addition, if the integer n satisfies (b) *or* (c), *then there is a neighbourhood V of y in Y such that $\mathscr{G}r_m(j)|V = 0$ for $m \geqslant n$ and that $\phi_{nm}|V : \mathscr{O}_{Y^{(m)}}|V \to \mathscr{O}_{Y^{(n)}}|V$ is bijective for $m \geqslant n$.*

PROOF. Since the questions is local on Y, we can restrict ourselves to the case where j is a closed immersion, Y being defined by a quasi-coherent ideal *of finite type* $\mathfrak{I}$ of $\mathscr{O}_X$. The equivalence of (b) and (c), for a given n, is immediate; also, since $\mathscr{I}^n/\mathscr{I}^{n+1}$ is an $\mathscr{O}_X$-module of finite type, there is an open neighbourhood U of y in X such that $\mathscr{I}^n|U = \mathscr{I}^{n+1}|U$ (**0**, 5.2.2), so we also have $\mathscr{I}^n|U = \mathscr{I}^m|U$ for $m \geqslant n$ proving the last assertions. To prove that (a) implies (b), we can restrict ourselves to the cases where the underlying space of Y is equal to the underlying space of X and where $\mathscr{I}$ is generated by a finite number of sections over X: since $\mathscr{I}$ is contained in the nilradical $\mathscr{N}$ of $\mathscr{O}_X$ (**I**, 5.1.2), it is now nilpotent which proves b). Finally, to prove that (b) implies (a), we can restrict ourselves to the case where $\mathscr{I}^n = \mathscr{I}^m$; therefore, for every $y \in Y$, since $\mathscr{I}_y \subset \mathfrak{m}_y$, maximal ideal of $\mathscr{O}_{X,y}$, we must have $\mathscr{I}_y^n = 0$ because of Nakayama's lemma, since $\mathscr{I}_y$ is an ideal of finite type. The set of $x \in X$ such that $\mathscr{I}_x^n = 0$ is an open U of X contained in Y (**0**, 5.2.2); since on the other hand $\mathscr{I}_x \neq 0$ for $x \notin Y$, we must have $U = Y$. $\square$

Corollary (16.1.10). — *For a restriction of the immersion j to an open neighbourhood of y in Y to be an open immersion (in other words, for j to be a* local isomorphism *on the point y), it is necessary and sufficient that $(\mathscr{G}r_1(j))_y = (\mathscr{N}_{Y/X})_y = 0$.*

PROOF. The condition is clearly necessary, and the previous reasoning applied to $n = 1$ proves that it is sufficient. $\square$

Remark (16.1.11). —

(i) Under the conditions of the definition (16.1.1), the projective limit of the projective system $(\mathscr{O}_{Y_{(n)}}, \phi_{nm})$ of sheaves of rings over Y is called the *normal invariant of infinite order* of f, and sometimes denoted by $\mathscr{O}_{Y_{(\infty)}}$. When X is a locally noetherian prescheme, $j : Y \to X$ a closed immersion, Y then is a closed subprescheme of X defined by a coherent ideal $\mathscr{I}$ and $\mathscr{O}_{Y_{(\infty)}}$ is exactly the *formal completion* of $\mathscr{O}_X$ along Y (**I**, 10.8.4), and $Y^{(\infty)} = (Y, \mathscr{O}_{Y_{(\infty)}})$ is the formal prescheme that is the *completion* of X along Y (**I**, 10.8.5). In all cases, we could say that $Y^{(\infty)}$ is the *formal neighbourhood* of Y in X (via the morphism f). In the particular case we have just considered, it is the formal prescheme that is the inductive limit of the infinitesimal neighbourhoods of order n.

(ii) Note that for a morphism of preschemes $f = (\psi, \theta) : Y \to X$, it can happen that the homomorphism $\theta^\# : \psi^*(\mathscr{O}_X) \to \mathscr{O}_Y$ is surjective without f being a local immersion and **IV-4** | 9
without f being injective. We have an example by taking Y to be a sum of preschemes Y_λ all isomorphic to $\operatorname{Spec}(\mathscr{O}_x)$, where $x \in X$, ad taking f to be the morphism equal to the canonical morphism in each of the Y_λ.

16.2. Functorial properties of the normal invariants of an immersion

(16.2.1). Let $f = (\psi, \theta) : Y \to X$ and $f' = (\psi', \theta') : Y' \to X'$ by two morphisms of ringed spaces such that $\theta^\#$ and $\theta'^\#$ are surjective; consider a commutative diagram of morphisms of ringed spaces

(16.2.1.1)
$$\begin{tikzcd} Y \arrow[r, "f"] & X \\ Y' \arrow[u, "u"] \arrow[r, "f'"'] & X' \arrow[u, "v"'] \end{tikzcd}$$

Let $u = (\rho, \lambda), v = (\sigma, \mu)$. We have $\rho^*(\psi^*(\mathscr{O}_X)) = \psi'^*(\sigma^*(\mathscr{O}_X))$ and as a result a commutative diagram of homomorphisms of sheaves of rings over Y'

$$\begin{tikzcd} \rho^*(\psi^*(\mathscr{O}_X)) = \psi'^*(\sigma^*(\mathscr{O}_X)) \arrow[r, "\psi'^*(\mu^\#)"] \arrow[d, "\rho^*(\theta^\#)"'] & \psi'^*(\mathscr{O}_{X'}) \arrow[d, "\theta'^\#"] \\ \rho^*(\mathscr{O}_Y) \arrow[r, "\lambda^\#"'] & \mathscr{O}_{Y'} \end{tikzcd}$$

from which we conclude, if $\mathscr{I}$ and $\mathscr{I}'$ are the kernels of $\theta^\#$ and $\theta'^\#$, that we have $\psi'^*(\mu^\#)(\rho^*(\mathscr{I})) \subset \mathscr{I}'$, having in mind the exactness of the functor ρ^*. We deduce that, for every integer n, $\psi'^*(\mu^\#)(\rho^*(\mathscr{I}^n)) \subset$

$\mathscr{I}'^n$, which shows that $\psi'^*(\mu^\#)$ defines, passing to the quotients, a homomorphism of sheaves of rings

(16.2.1.2) $$\nu_n : \rho^*(\psi^*(\mathscr{O}_X)/\mathscr{I}^{n+1}) \longrightarrow \psi'^*(\mathscr{O}_{X'})/\mathscr{I}'^{n+1}$$

and therefore a morphism of ringed spaces $w_n = (\rho, \nu_n) : Y'^{(n)} \to Y^{(n)}$ (which, for $n = 0$, is none other than u). It follows immediately from this definition that the diagrams

$$\begin{tikzcd} Y^{(n)} \arrow[r, "h_{mn}"] & Y^{(m)} \arrow[r, "h_m"] & X \\ Y'^{(n)} \arrow[u, "w_n"] \arrow[r, "h'_{mn}"'] & Y'^{(m)} \arrow[u, "w_m"] \arrow[r, "h'_m"'] & X' \arrow[u, "v"'] \end{tikzcd} \qquad (n \leqslant m)$$

(where the horizontal arrows are the canonical morphisms (16.1.2)) are commutative.

By passage to the quotients via the morphisms (16.2.1.2), and taking into account the exactness **IV-4** | 10
of the functor ρ^*, we obtain a di-homomorphism of graded algebras (relative to the morphism $\lambda^\# : \rho^*(\mathscr{O}_Y) \to \mathscr{O}_{Y'}$)

(16.2.1.3) $$\mathrm{gr}(u) : \rho^*(\mathscr{G}r_\bullet(f)) \longrightarrow \mathscr{G}r_\bullet(f')$$

(or, if you like, a ρ-morphism (**0**, 3.5.1) $\mathscr{G}r_\bullet(f) \to \mathscr{G}r_\bullet(f')$), and in particular a di-homomorphism of conormal sheaves

$$\mathrm{gr}_1(u) : \rho^*(\mathscr{G}r_1(f)) \longrightarrow \mathscr{G}r_1(f').$$

It is also immediate that these homomorpisms give rise to a commutative diagram

(16.2.1.4) $$\begin{tikzcd} \rho^*(\mathbf{S}^\bullet_{\mathscr{O}_Y}(\mathscr{G}r_1(f))) \arrow[r] \arrow[d, "\mathbf{S}(\mathrm{gr}_1(u))"'] & \rho^*(\mathscr{G}r_\bullet(f)) \arrow[d, "\mathrm{gr}(u)"] \\ \mathbf{S}^\bullet_{\mathscr{O}_Y}(\mathscr{G}r_1(f')) \arrow[r] & \mathscr{G}r_\bullet(f') \end{tikzcd}$$

where the horizontal arrow are the canonical morphisms (16.1.2.2).

Finally, if we have a commutative diagram of morphisms of ringed spaces

$$\begin{tikzcd} Y \arrow[r, "f"] & X \\ Y' \arrow[u, "u"] \arrow[r, "f'"'] & X' \arrow[u, "v"'] \\ Y'' \arrow[u, "u'"] \arrow[r, "f''"'] & X'' \arrow[u, "v'"'] \end{tikzcd}$$

where $f'' = (\psi'', \theta'')$ is such that $\theta''^\#$ is surjective, and if w'_n and w''_n are defined from u', v' for one and $u'' = u \circ u'$, $v'' = v \circ v'$ for the other, we have $w''_n = w_n \circ w'_n$, which follows immediately from the definitions and from (**0**, 3.5.5); we have also $\mathrm{gr}(u'') = \mathrm{gr}(u') \circ \rho'^*(\mathrm{gr}(u))$ if $u' = (\rho', \lambda')$. Therefore we can say that $Y^{(n)}$ and $\mathscr{G}r_\bullet(f)$ *depend functorially* on f.

Proposition (16.2.2). — *With the notation and hypotheses of* (16.2.1), *suppose also that f, f', u, and v are morphisms of preschemes. We have:*

(i) *The morphisms $w_n : Y'^{(n)} \to Y^{(n)}$ are morphisms of preschemes.*

(ii) *If $Y' = Y \times_X X'$, u and f' the canonical projections, and if f is an immersion or if v is flat, we have $Y'^{(n)} = Y^{(n)} \times_X X'$.*

(iii) *If $Y' = Y \times_X X'$ and if v is flat (resp. if f is an immersion), the homomorphism*

$$\mathrm{Gr}(u) = \mathrm{gr}(u) \otimes I : \mathscr{G}r_\bullet(f) \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y'} \longrightarrow \mathscr{G}r_\bullet(f')$$

is bijective (resp. surjective).

PROOF.

(i) The hypotheses immediately imply that, for every $y' \in Y'$, $\rho^*_{y'}(\theta^{\#}_{\psi'(y')})$ is a *local* homomorphism (I, 1.6.2), so w_n is a morphism of preschemes (I, 2.2.1). **IV-4** | 11

(ii) and (iii) If f is an immersion, we can restrict ourselves to the case where f is a closed immersion, Y being defined by a quasi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$ and $Y^{(n)}$ by the ideal $\mathscr{I}^{n+1}$; the assertions follows from (I, 4.4.5).

Second, suppose that v is flat; we can restrict ourselves to the case where $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$, $X' = \operatorname{Spec}(A')$ are affines, A' being a flat A-module; so $Y' = \operatorname{Spec}(B')$ where $B' = B \otimes_A A'$; in addition, if $\mathfrak{I}$ is the kernel of the homomorphism $A \to B$, the kernel $\mathfrak{I}'$ of $A' \to B'$ is identified with $\mathfrak{I} \otimes_A A'$ by flatness, and $\mathscr{I}'^n/\mathscr{I}'^{n+1}$ is equal to

$$\psi'^*(\sigma^*((\mathfrak{I}^n/\mathfrak{I}^{n+1})^\sim) \otimes_{\sigma^*(\mathscr{O}_X)} \mathscr{O}_{X'}) =$$
$$\psi'^*(\sigma^*((\mathfrak{I}^n/\mathfrak{I}^{n+1}))^\sim) \otimes_{\psi'^*(\sigma^*(\mathscr{O}_X))} \psi'^*(\mathscr{O}_{X'}) = \rho^*(\mathscr{I}^n/\mathscr{I}^{n+1}) \otimes_{\rho^*(\psi^*(\mathscr{O}_X))} \psi'^*(\mathscr{O}_{X'})$$

and in particular for $n = 0$, we have

$$\mathscr{O}_{Y'} = \rho^*(\mathscr{O}_Y) \otimes_{\rho^*(\psi^*(\mathscr{O}_X))} \psi'^*(\mathscr{O}_{X'})$$

from which we have canonical isomorphism of $\mathscr{I}'^n/\mathscr{I}'^{n+1}$ with

$$\rho^*(\mathscr{I}^n/\mathscr{I}^{n+1}) \otimes_{\rho^*(\mathscr{O}_Y)} \mathscr{O}_{Y'} = (\mathscr{I}^n/\mathscr{I}^{n+1}) \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y'}$$

which proves (iii). Let now $C_n = \Gamma(Y, \mathscr{O}_{Y^{(n)}}), C'_n = \Gamma(Y', \mathscr{O}_{Y'^{(n)}})$. As $Y^{(n)}$ and $Y'^{(n)}$ are affine schemes (16.1.5), the kernel $\mathfrak{K}_n$ (resp. $\mathfrak{K}'_n$) of the homomorphism $C_n \to C_{n-1}$ (resp. $C'_n \to C'_{n-1}$) is $\Gamma(Y, \mathscr{I}^n/\mathscr{I}^{n+1})$ (resp. $\Gamma(Y, \mathscr{I}'^n/\mathscr{I}'^{n+1})$); therefore we can deduce from the above results that $\mathfrak{K}'_n = \mathfrak{K}_n \otimes_A A'$. Now, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{K}_n \otimes_A A' \arrow[r] \arrow[d, "r"] & C_n \otimes_A A' \arrow[r] \arrow[d, "s_n"] & C_{n-1} \otimes_A A' \arrow[r] \arrow[d, "s_{n-1}"] & 0 \\
0 \arrow[r] & \mathfrak{K}'_n \arrow[r] & C'_n \arrow[r] & C'_{n-1} \arrow[r] & 0
\end{tikzcd}$$

where the vertical arrow of the left is bijective and the two lines are exact (A' being a flat A-module). We deduce by induction that s_n is bijective for every n, because it is true by hypothesis for $n = 0$, and is deduced by application of the five lemma for all n. That proves the second assertion of (ii).

□

Corollary (16.2.3). — *Let $g : X \to Y$, $u : Y' \to Y$ be two morphisms of preschemes, $X' = X \times_Y Y'$, $g' : X' \to Y'$ and $v : X' \to X$ by the canonical projections. Let $f : Y \to X$ by a Y-section of X (and therefore an immersion), $f' = f_{(Y')} : Y' \to X'$ the Y'-section of X' deduced from f by the base change u. We have:*

(i) *The morphism $w_n : {Y'_{f'}}^{(n)} \to Y_f^{(n)}$ corresponding to f, f', u, v (16.2.1) and the canonical morphism $h'_n : {Y'_{f'}}^{(n)} \to X'$ identifies ${Y'_{f'}}^{(n)}$ with the product $Y_f^{(n)} \times_X X'$.*

(ii) *If we endow $\mathscr{O}_{Y_f^{(n)}}$ (resp. $\mathscr{O}_{{Y'_{f'}}^{(n)}}$) with the structure of an $\mathscr{O}_Y$-algebra defined by g (resp. with the structure of an $\mathscr{O}_{Y'}$-algebra defined by g') (16.1.5, (iii)), then the homomorphism of $\mathscr{O}_{Y'}$-algebras*

(16.2.3.1)
$$\rho^*(\mathscr{O}_{Y_f^{(n)}}) \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y'} \longrightarrow \mathscr{O}_{{Y'_{f'}}^{(n)}}$$

induced by the homomorphism v_n (16.2.1.2) is bijective. Also, the homomorphism of $\mathscr{O}_{Y'}$-modules **IV-4** | 12

(16.2.3.2)
$$\operatorname{Gr}_1(u) : \mathscr{G}r_1(f) \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y'} \longrightarrow \mathscr{G}r_1(f')$$

is bijective.

PROOF.

(i) Let us first note that $f' : Y' \to X'$ and $u : Y' \to Y$ identifies Y' with the product $Y \times_X X'$ (via the structure morphisms $f : Y \to X$ and $v : X' \to X$) (14.5.12.1). The conclusion of (i) now follows from (16.2.2, (ii)), the morphism g being an immersion.

(ii) The commutative diagram

$$\begin{tikzcd} Y_f^{(n)} \arrow[d, "h_n"] & Y'^{(n)}_{f'} \arrow[l, "w_n"'] \arrow[d, "h'_n"] \\ X \arrow[d, "g"] & X' \arrow[l, "v"'] \arrow[d, "g'"] \\ Y & Y' \arrow[l, "u"'] \end{tikzcd}$$

identifies $Y'^{(n)}_{f'}$ with the product $Y_f^{(n)} \times_X X'$, so (I, 3.3.9) it identifies (via the morphisms $g' \circ h'_n$ and w_n) $Y'^{(n)}_{f'}$ to the product $Y_f^{(n)} \times_Y Y'$. Since $Y_f^{(n)}$ (resp. $Y'^{(n)}_{f'}$) is the affine prescheme over Y (resp. over Y') associated with the $\mathscr{O}_Y$-algebran $\mathscr{O}_{Y_f^{(n)}}$ (resp. to the $\mathscr{O}_{Y'}$-algebran $\mathscr{O}_{Y'^{(n)}_{f'}}$), the fact that the canonical homomorphism (16.2.3.1) is bijective follows from (II, 1.5.2). Finally, the canonical homomorphism (16.2.3.1) is compatible with the augmentations $\mathscr{O}_{Y_f^{(n)}} \to \mathscr{O}_Y$ and $\mathscr{O}_{Y'^{(n)}_{f'}} \to \mathscr{O}_{Y'}$; since $\mathscr{O}_{Y_f^{(n)}}$ is a direct sum (as an $\mathscr{O}_Y$-module) of $\mathscr{O}_Y$ and the augmentation ideal $\mathscr{I}/\mathscr{I}^{n+1}$, we can therefore see that the canonical homomorphism (16.2.3.1), restricted to $\mathscr{I}/\mathscr{I}^{n+1} \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y'}$, is a bijection of the latter onto $\mathscr{I}'/\mathscr{I}'^{n+1}$. For $n = 1$ this shows that $\mathrm{Gr}_1(u)$ is bijective.

□

We note that, under the hypotheses of (16.2.3), the homomorphisms $\mathrm{Gr}_n(u)$ are *surjective* in view of the above, but are not bijective in general for $n \geqslant 2$. However:

Corollary (16.2.4). — *Under the hypotheses of* (16.2.3), *suppose that $u : Y' \to Y$ is a flat morphism (resp. that the $\mathscr{G}r_n(f)$ are flat $\mathscr{O}_Y$-modules for $n \leqslant m$). Then the homomorphism*

$$\mathrm{Gr}_n(u) : \mathscr{G}r_n(f) \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y'} \longrightarrow \mathscr{G}r_n(f')$$

is bijective for all n (resp. for $n \leqslant m$).

Proof. If u is flat, then we deduce by base change that the same is true for $v : X' \to X$, and we already know in this case that $\mathrm{Gr}(u)$ is bijective (16.2.2, (iii)). If the $\mathscr{G}r_n(f)$ are flat for $n \leqslant m$, then we first see by induction on n that the same holds for $\mathscr{I}/\mathscr{I}^{n+1}$ for $n \leqslant m$, because of the exact sequences

$$0 \longrightarrow \mathscr{I}^n/\mathscr{I}^{n+1} \longrightarrow \mathscr{I}/\mathscr{I}^{n+1} \longrightarrow \mathscr{I}/\mathscr{I}^n \longrightarrow 0$$

(**0**, 6.1.2); in addition, we have the commutative diagram **IV-4** | 13

$$\begin{tikzcd} 0 \arrow[r] & (\mathscr{I}^n/\mathscr{I}^{n+1}) \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y'} \arrow[r] \arrow[d] & (\mathscr{I}/\mathscr{I}^{n+1}) \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y'} \arrow[r] \arrow[d] & (\mathscr{I}/\mathscr{I}^n) \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y'} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{I}'^n/\mathscr{I}'^{n+1} \arrow[r] & \mathscr{I}'/\mathscr{I}'^{n+1} \arrow[r] & \mathscr{I}'/\mathscr{I}'^n \arrow[r] & 0 \end{tikzcd}$$

in which the lines are exact (the first by flatness (**0**, 6.1.2)) and the two last vertical arrows are bijective by virtue of (16.2.2, (ii)); hence the conclusion. □

Remarks (16.2.5). —

(i) The reasoning of (16.2.2, (i)) still applies to (16.2.1.1) when these are morphisms of *locally ringed spaces* (**I**, 1.8.2).

(ii) In (16.2.2, (ii)), the conclusion is no longer necessarily valid if we only suppose that v and f are morphisms of preschemes (f satisfying the condition of (16.1.1)). For example (with the notation of the proof of (16.2.2, (ii))), it can happen that $\mathfrak{I} = 0$ but the kernel $\mathfrak{I}'$ of $A' \to B' = B \otimes_A A'$ is not zero and that $B' \neq 0$, in which case we have $Y^{(n)} = Y$ for all n, but $Y'^{(n)} \neq Y'$. We have an example of this by taking $A = \mathbf{Z}$, $B = \mathbf{Q}$, $A' = \prod_{h=1}^{\infty}(\mathbf{Z}/m^h\mathbf{Z})$ where $m > 1$.

(16.2.6). Consider the particular case of the diagram (16.2.1.1) where $X' = X$, v is the identity, X a prescheme, Y a subprescheme of X, Y' a subprescheme of Y, f, u, and $f' = f \circ u$ the canonical injections; the di-homomorphism (16.2.1.3) gives us, by tensoring with $\mathscr{O}_{Y'}$ over $\rho^*(\mathscr{O}_Y)$, a homomorphism of graded $\mathscr{O}_{Y'}$-algebras

$$u^*(\mathscr{G}r_\bullet(f)) \longrightarrow \mathscr{G}r_\bullet(f'). \tag{16.2.6.1}$$

On the other hand, we identify $\mathscr{O}_Y$ to $\psi^*(\mathscr{O}_X)/\mathscr{I}_f$ and $\mathscr{O}_{Y'}$ to $\rho^*(\mathscr{O}_Y)/\mathscr{I}_u$; since ρ^* is an exact functor, we have $\rho^*(\mathscr{O}_Y) = \rho^*(\psi^*(\mathscr{O}_X))/\rho^*(\mathscr{I}_f) = \psi'^*(\mathscr{O}_X)/\rho^*(\mathscr{I}_f)$, and since $\mathscr{O}_{Y'}$ is moreover identified with $\psi'^*\mathscr{O}_X/\mathscr{I}_{f'}$, we see that $\mathscr{I}_u = \mathscr{I}_{f'}/\rho^*(\mathscr{I}_f)$. We deduce that for every integer n there is a canonical homomorphism $\mathscr{I}_{f'}^n/\mathscr{I}_{f'}^{n+1} \to \mathscr{I}_u^n/\mathscr{I}_u^{n+1}$, from which we have a canonical morphism of graded $\mathscr{O}_{Y'}$-algebras

$$\mathscr{G}r_\bullet(f') \longrightarrow \mathscr{G}r_\bullet(u). \tag{16.2.6.2}$$

Proposition (16.2.7). — *Let X be a prescheme, Y a subprescheme of X, Y' a subprescheme of Y, $j : Y' \to Y$ the canonical injection. We then have an exact sequence of conormal sheaves ($\mathscr{O}_{Y'}$-modules)*

$$j^*(\mathscr{N}_{Y/X}) \longrightarrow \mathscr{N}_{Y'/X} \longrightarrow \mathscr{N}_{Y'/Y} \longrightarrow 0 \tag{16.2.7.1}$$

where the arrows are the degree 1 *components of the canonical homomorphisms* (16.2.6.1) *and* (16.2.6.2).

Proof. The problem being local, we can restrict to the case where $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(A/\mathfrak{J})$ and $Y' = \operatorname{Spec}(A/\mathfrak{K})$, $\mathfrak{J}$ and $\mathfrak{K}$ being ideals of A such that $\mathfrak{J} \subset \mathfrak{K}$; everything reduces to seeing that the sequence of canonical morphisms $\mathfrak{J}/\mathfrak{K}\mathfrak{J} \to \mathfrak{K}/\mathfrak{K}^2 \to (\mathfrak{K}/\mathfrak{J})/(\mathfrak{K}/\mathfrak{J})^2 \to 0$ is exact, which is **IV-4** | 14
immediate given that the image of $\mathfrak{J}/\mathfrak{K}\mathfrak{J}$ in $\mathfrak{K}/\mathfrak{K}^2$ is $(\mathfrak{J}+\mathfrak{K}^2)/\mathfrak{K}^2$ and that $(\mathfrak{K}/\mathfrak{J})/(\mathfrak{K}/\mathfrak{J})^2$ is identified with $\mathfrak{K}/(\mathfrak{J}+\mathfrak{K}^2)$. □

It is easy to give examples where the sequence (16.2.7.1) extended on the left by 0 is not exact; with the above notation, it suffices to take $A = k[T]$, $\mathfrak{J} = AT^2$, $\mathfrak{K} = AT$, because then $(\mathfrak{J}+\mathfrak{K}^2)/\mathfrak{K}^2 = 0$ and $\mathfrak{J}/\mathfrak{K}\mathfrak{J} \neq 0$. See however (16.9.13) and (19.1.5) for some cases where the extended sequence is indeed exact.

16.3. Fundamental differential invariants of morphisms of preschemes

Definition (16.3.1). — Let $f : X \to S$ be a morphism of preschemes, $\Delta_f : X \to X \times_S X$ the corresponding diagonal morphism, which is an immersion (**I**, 5.3.9). We denote by $\mathscr{P}_f^n$ or $\mathscr{P}_{X/S}^n$, and call the *sheaf of principal parts of order n of the S-prescheme X*, the $\mathscr{O}_X$-augmented sheaf of rings, n-th normal invariant of Δ_f (16.1.2). We will also write $\mathscr{P}_f^\infty = \mathscr{P}_{X/S}^\infty = \varprojlim_n \mathscr{P}_{X/S}^n$, $\mathscr{G}r_n(\mathscr{P}_f) = \mathscr{G}r_n(\mathscr{P}_{X/S}) = \mathscr{G}r_n(\Delta_f)$ (16.1.2); the $\mathscr{O}_X$-module $\mathscr{G}r_1(\Delta_f)$, augmentation sheaf of ideals of $\mathscr{P}_{X/S}^1$, is denoted by Ω_f^1 or $\Omega_{X/S}^1$, and is called the $\mathscr{O}_X$-module of 1-*differentials* of f, or of X with respect to S, or of the S-prescheme X.

It follows from this definition that $\mathscr{P}_{X/S}^0$ is canonically identified with $\mathscr{O}_X$ (16.1.2).
We have (16.1.2.2) a canonical surjective morphism of graded $\mathscr{O}_X$-algebras

$$\mathbf{S}^\bullet_{\mathscr{O}_X}(\Omega_{X/S}^1) \longrightarrow \mathscr{G}r_\bullet(\mathscr{P}_{X/S}). \tag{16.3.1.1}$$

And it follows from Definition (16.3.1) that for every open U of X we have $\mathscr{P}_{f|U}^n = \mathscr{P}_f^n|U$, $\mathscr{P}_{f|U}^\infty = \mathscr{P}_f^\infty|U$, $\mathscr{G}r_n(\mathscr{P}_{f|U}) = \mathscr{G}r_n(\mathscr{P}_f)|U$, $\Omega_{f|U}^1 = \Omega_f^1|U$ (in other words, the notions introduced are *local* on X).

(16.3.2). Denote by p_1, p_2 the two canonical projections of the product $X \times_S X$; since Δ_f is an X-section of $X \times_S X$ for both p_1 and p_2, *each* of these morphisms define, for all n, a homomorphism of sheaves of rings $\mathscr{O}_X \to \mathscr{P}_{X/S}^n$, right inverse of the augmentation $\mathscr{P}_{X/S}^n \to \mathscr{O}_X$ (16.1.7); we can also say that we thus define on $\mathscr{P}_{X/S}^n$ *two quasi-coherent augmented $\mathscr{O}_X$-algebra* structures; the corresponding $\mathscr{O}_X$-module structures on on $\mathscr{G}r_n(\mathscr{P}_{X/S}^n)$ are the same. We also have, by passing to the limit, two $\mathscr{O}_X$-algebra structures on $\mathscr{P}_{X/S}^\infty$.

(16.3.3). The morphism $s = (p_2, p_1)_S : X \times_S X \to X \times_S X$ is an *involutive automorphism* of $X \times_S X$, called the *canonical symmetry*, such that

$$p_1 \circ s = p_2, \qquad p_2 \circ s = p_1, \qquad s \circ \Delta_f = \Delta_f. \tag{16.3.3.1}$$

If we put $s = (\rho, \lambda)$, $p_i = (\pi_i, \mu_i)$ $(i = 1, 2)$, $\Delta_f = (\delta, \nu)$, $\lambda^\#$ is then an isomorphism of $\rho^*(\pi_1^*(\mathscr{O}_X))$ onto $\pi_2^*(\mathscr{O}_X)$, and $\delta^*(\lambda^\#)$ fixes $\delta^*(\mathscr{O}_{X\times_S X})$ and the kernel $\mathscr{I}$ of the homomorphism $\nu^\# : \delta^*(\mathscr{O}_{X\times_S X}) \to \mathscr{O}_X$. Therefore:

Proposition (16.3.4). — *The homomorphism $\sigma = \delta^*(\lambda^\#)$ induced from s (and also called the* canonical symmetry*) is an involutive automorphism of the projective system $(\mathscr{P}^n_{X/S})$ of $\mathscr{O}_X$-augmented sheaves of rings, and as a result also of the projective limit $\mathscr{P}^\infty_{X/S}$. This automorphism permutes the $\mathscr{O}_X$-algebra structure on $\mathscr{P}^n_{X/S}$ and on $\mathscr{P}^\infty_{X/S}$.* **IV-4** | 15

(16.3.5). In what follows, the two $\mathscr{O}_X$-algebra structures defined on the $\mathscr{P}^n_{X/S}$ and on $\mathscr{P}^\infty_{X/S}$ will play very different roles: *we will now agree, unless said otherwise, that when $\mathscr{P}^n_{X/S}$ or $\mathscr{P}^\infty_{X/S}$ is considered as an $\mathscr{O}_X$-algebra, it is the algebra structure induced by p_1.*

For every open U of X and every section $t \in \Gamma(U, \mathscr{O}_X)$, we will simply denote by $t.1$ or even t the image of t under the structure morphism $\Gamma(U, \mathscr{O}_X) \to \Gamma(U, \mathscr{P}^n_{X/S})$ (resp. $\Gamma(U, \mathscr{O}_X) \to \Gamma(U, \mathscr{P}^\infty_{X/S})$) (that is to say, the homomorphism corresponding to p_1).

Definition (16.3.6). — We denote by d^n_f, or $d^n_{X/S}$ (resp. d^∞_f, or $d^\infty_{X/S}$), or simply d^n (resp. d^∞), the homomorphism of sheaves of rings $\mathscr{O}_X \to \mathscr{P}^n_f = \mathscr{P}^n_{X/S}$ (resp. $\mathscr{O}_X \to \mathscr{P}^\infty_f = \mathscr{P}^\infty_{X/S}$) induced by p_2. For every open U of X, and every $t \in \Gamma(U, \mathscr{O}_X)$, $d^n t$ (resp. $d^\infty t$) is called the *principal part of order n* (resp. *principal part of infinite order*) of t. We set $dt = d^1 t - t$, and we say that dt is the differential of t (an element of $\Gamma(U, \Omega^1_{X/S})$, also denoted $d_{X/S}(t)$).

It follows immediately [2] from this definition that we have

$$d(t_1 t_2) = t_1 dt_2 + t_2 dt_1 \tag{16.3.6.1}$$

for every t_1, t_2 in $\Gamma(U\mathscr{O}_X)$, that is, d is a *derivation* of the ring $\Gamma(U, \mathscr{O}_X)$ in the $\Gamma(U, \mathscr{O}_X)$-module $\Gamma(U, \Omega^1_{X/S})$.

In all notation introduced in (16.3.1) and (16.3.6), we will sometimes replace S by A when $S = \operatorname{Spec}(A)$.

(16.3.7). Suppose in particular that $S = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ are affine schemes, B then being an A-algebra. Then Δ_f corresponds to the canonical surjective homomorphism $\pi : B \otimes_A B \to B$ such that $\pi(b \otimes b') = bb'$, with kernel $\mathfrak{J} = \mathfrak{J}_{B/A}$ (**0**, 20.4.1); $\mathscr{P}^n_f$ is the structure sheaf of the prescheme $\operatorname{Spec}(P^n_{B/A})$, where

$$P^n_{B/A} = (B \otimes_A B)/\mathfrak{J}^{n+1};$$

$\mathscr{G}r_\bullet(\mathscr{P}_f)$ is the quasi-coherent $\mathscr{O}_X$-module corresponding to the graded B-module

$$\operatorname{gr}^\bullet_{\mathfrak{J}}(B \otimes_A B) = \bigoplus_{n \geqslant 0} (\mathfrak{J}^n / \mathfrak{J}^{n+1});$$

in particular $\Omega^1_f = \Omega^1_{X/S}$ is the quasi-coherent $\mathscr{O}_X$-module corresponding to the B-module of 1-differentials of B over A, $\Omega^1_{B/A}$ (**0**, 20.4.3). The projection morphisms $p_1 : X \times_S X \to X$, $p_2 : X \times_S X \to X$ corresponding to the two homomorphisms of rings $j_1 : B \to B \otimes_A B$, $j_2 : B \to B \otimes_A B$ such that $j_1(b) = b \otimes 1$, $j_2(b) = 1 \otimes b$, so that (by the convention of (16.3.5)), $P^n_{B/A}$ is always considered as a B-algebra via the composite homomorphism $B \xrightarrow{j_1} B \otimes_A B \to P^n_{B/A}$; the ring homomorphism $B \xrightarrow{j_2} B \otimes_A B \to P^n_{B/A}$ is denoted by $d^n_{B/A}$ and corresponds to $d^n_{X/S}$ acting on $\Gamma(X, \mathscr{O}_X)$; for every $t \in B$, dt is equal to $d_{B/A} t$, defined in (**0**, 20.4.6).

If $\pi_n : B \otimes_A B \to P^n_{B/A}$ is the canonical homomorphism, so we have, in light of the preceding definitions,

$$\pi_n(b \otimes b') = b \cdot \pi_n(1 \otimes b') = b \cdot d^n_{B/A}(b') \quad \text{for } b \in B, b' \in B. \tag{16.3.7.1}$$

[2] [Trans.] This is, locally we have (**0**, 20.1.1).

Proposition (16.3.8). — *The image of the canonical homomorphism $d^n_{X/S} : \mathscr{O}_X \to \mathscr{P}^n_{X/S}$ generates the $\mathscr{O}_X$-module $\mathscr{P}^n_{X/S}$.* IV-4 | 16

PROOF. We immediately reduce to the case where $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$ are affine and the proposition follows from (16.3.7.1) since π_n is surjective. We note that in general $d^n_{X/S}$ *is not surjective* (even for $n = 1$). □

Proposition (16.3.9). — *Suppose that $f : X \to S$ is a morphism locally of finite type. Then the $\mathscr{P}^n_f$ and the $\mathscr{G}r_n(\mathscr{P}_f)$ are quasi-coherent $\mathscr{O}_X$-modules of finite type.*

PROOF. This follows from (16.1.6) and from the fact that Δ_f is locally of finite presentation (**I**, 4.3.1). □

16.4. Functorial properties of differential invariants

(16.4.1). Consider a commutative diagram of morphisms of preschemes

(16.4.1.1)
$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l, "u"'] \arrow[d, "f'"] \\ S & S' \arrow[l, "w"] \end{tikzcd}$$

We deduce a commutative diagram

$$\begin{tikzcd} X \arrow[d, "\Delta_f"'] & & X' \arrow[ll, "u"'] \arrow[d, "\Delta_{f'}"] \\ X \times_S X & & X' \times_{S'} X' \arrow[ll, "v"] \end{tikzcd}$$

where v is the composite homomorphism (**I**, 5.3.5) and (**I**, 5.3.15).

(16.4.1.2)
$$X' \times_{S'} X' \xrightarrow{(p_1', p_2')_S} X' \times_S X' \xrightarrow{u \times_S u} X \times_S X.$$

So we induce from u and v, as explained in (16.2.1), homomorphisms of augmented sheaves of rings

(16.4.1.3)
$$\nu_n : \rho^*(\mathscr{P}^n_{X/S}) \longrightarrow \mathscr{P}^n_{X'/S'}$$

(where we put $u = (\rho, \lambda)$); these homomorphisms form a projective system, and therefore give at the limit a homomorphism of sheaves of graded rings

(16.4.1.4)
$$\nu_\infty : \rho^*(\mathscr{P}^\infty_{X/S}) \longrightarrow \mathscr{P}^\infty_{X'/S'};$$

on the other hand, by passing to the quotient, the homomorphisms ν_n give rise to a di-homomorphism of graded algebras (relative to $\lambda^\#$):

(16.4.1.5)
$$\mathrm{gr}(u) : \rho^*(\mathscr{G}r_\bullet(\mathscr{P}_{X/S})) \longrightarrow \mathscr{G}r_\bullet(\mathscr{P}_{X'/S'}).$$

(16.4.2). If we have a commutative diagram IV-4 | 17

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l, "u"'] \arrow[d, "f'"] & X'' \arrow[l, "u'"'] \arrow[d, "f''"] \\ S & S' \arrow[l, "w"] & S'' \arrow[l, "w'"] \end{tikzcd}$$

we deduce a commutative diagram

$$\begin{tikzcd} X \arrow[d, "\Delta_f"'] & & X' \arrow[ll, "u"'] \arrow[d, "\Delta_{f'}"] & & X'' \arrow[ll, "u'"'] \arrow[d, "\Delta_{f''}"] \\ X \times_S X & & X' \times_{S'} X' \arrow[ll, "v"] & & X'' \times_{S''} X'' \arrow[ll, "v'"] \end{tikzcd}$$

where v' is defined from u', w', f', f'' as v is from u, w, f, f'. We verify immediately that if $u'' = u \circ u'$, $w'' = w \circ w'$, then the composite homomorphism $v \circ v'$ is equal to the homomorphism v'' deduced from u'', v'', f, f'' as v is from u, w, f, f'. If we put $u' = (\rho', \lambda')$, $u'' = (\rho'', \lambda'')$ it follows (16.2.1) that the homomorphism $\nu''_n : \rho''^*(\mathscr{P}^n_{X/S}) \to \mathscr{P}^n_{X''/S''}$ is equal to the composite

$$\rho'^*(\rho^*(\mathscr{P}^n_{X/S})) \xrightarrow{\rho'^*(\nu^\#_n)} \rho'^*(\mathscr{P}^n_{X'/S'}) \xrightarrow{\nu'_n} \mathscr{P}^n_{X''/S''},$$

and we have analogous transitivity properties for the homomorphisms (16.4.1.4) and (16.4.1.5), which lets us say that the $\mathscr{P}^n_{X/S}$, $\mathscr{P}^\infty_{X/S}$ and $\mathscr{G}r_\bullet(\mathscr{P}_{X/S})$ *depend functorially on f*.

(16.4.3). We verify immediately (for example, by restricting ourselves to the affine case with help of (16.3.7)) that with the notation of (16.4.1), the diagram

(16.4.3.1)
$$\begin{tikzcd} \rho^*(\mathscr{O}_X) \arrow[r, "\lambda^\#"] \arrow[d] & \mathscr{O}_{X'} \arrow[d] \\ \rho^*(\mathscr{P}^n_{X/S}) \arrow[r, "\nu_n"'] & \mathscr{P}^n_{X'/S'} \end{tikzcd}$$

where the vertical arrows are the ones defining the algebra structure chosen in (16.3.5) (that is to say, the ones coming from the first projections) is commutative; the same goes for the diagram

(16.4.3.2)
$$\begin{tikzcd} \rho^*(\mathscr{O}_X) \arrow[r, "\lambda^\#"] \arrow[d, "\rho^*(d^n_{X/S})"'] & \mathscr{O}_{X'} \arrow[d, "d^n_{X'/S'}"] \\ \rho^*(\mathscr{P}^n_{X/S}) \arrow[r, "\nu_n"'] & \mathscr{P}^n_{X'/S'} \end{tikzcd}$$

the vertical arrows defining here the algebra structure from the second projection; besides, if σ and **IV-4** | 18
σ' are the canonical symmetries corresponding to f and f' (16.3.4), we have

$$\nu_n \circ \rho^*(\sigma) = \sigma' \circ \nu_n$$

which switches one diagram with the other. We deduce from (16.4.3.1) a canonical homomorphism of *augmented $\mathscr{O}_{X'}$-algebras*

(16.4.3.3)
$$P^n(u) : u^*(\mathscr{P}^n_{X/S}) = \mathscr{P}^n_{X/S} \otimes_{\mathscr{O}_X} \mathscr{O}_{X'} \longrightarrow \mathscr{P}^n_{X'/S'}$$

and it follows from (16.4.3.2) that the diagram

(16.4.3.4)
$$\begin{tikzcd} \mathscr{O}_{X'} \arrow[r, "\mathrm{id}"] \arrow[d, "u^*(d^n_{X/S})"'] & \mathscr{O}_{X'} \arrow[d, "d^n_{X'/S'}"] \\ u^*(\mathscr{P}^n_{X/S}) \arrow[r, "P^n(u)"'] & \mathscr{P}^n_{X'/S'} \end{tikzcd}$$

is commutative. We deduce a homomorphism of *graded $\mathscr{O}_{X'}$-algebras*

(16.4.3.5)
$$\mathrm{Gr}_\bullet(u) : u^*(\mathscr{G}r_\bullet(\mathscr{P}_{X/S})) \longrightarrow \mathscr{G}r_\bullet(\mathscr{P}_{X'/S'})$$

and in particular a homomorphism of $\mathscr{O}_{X'}$-modules

(16.4.3.6)
$$\mathrm{Gr}_1(u) : \Omega^1_{X/S} \otimes_{\mathscr{O}_X} \mathscr{O}_{X'} \longrightarrow \Omega^1_{X'/S'}$$

giving rise to a commutative diagram

(16.4.3.7)
$$\begin{tikzcd} \mathscr{O}_{X'} \arrow[r, "\mathrm{id}"] \arrow[d, "d_{X/S}\otimes 1"'] & \mathscr{O}_{X'} \arrow[d, "d_{X'/S'}"] \\ \Omega^1_{X/S} \otimes_{\mathscr{O}_X} \mathscr{O}_{X'} \arrow[r] & \Omega^1_{X'/S'} \end{tikzcd}$$

(16.4.4). When $S = \mathrm{Spec}(A)$, $S' = \mathrm{Spec}(A')$, $X = \mathrm{Spec}(B)$, $X' = \mathrm{Spec}(B')$ are affine, so that we have a commutative diagram of ring homomorphisms

$$\begin{tikzcd} B \arrow[r] & B' \\ A \arrow[r] \arrow[u] & A' \arrow[u] \end{tikzcd}$$

the image of $\mathfrak{I}_{B/A}$ in $B' \otimes_{A'} B'$ is contained in $\mathfrak{I}_{B'/A'}$, and the homomorphism ν_n corresponds to the homomorphism of rings $P^n_{B/A} \to P^n_{B'/A'}$ induced from the homomorphism $B \otimes_A B \to B' \otimes_{A'} B'$ by passing to quotients. The homomorphism (16.4.3.6) corresponds to the homomorphism defined in (**0**, 20.5.4.1), and the commutative diagram (16.4.3.7) to the diagram (**0**, 20.5.4.2).

Proposition (16.4.5). — *Suppose that $X' = X \times_S S'$, f' and u the canonical projections. Then the canonical homomorphisms $P^n(u)$ (16.4.3.3) and $\mathrm{Gr}_1(u)$ (16.4.3.6) are bijective.* **IV-4** | 19

PROOF. We have $X' \times_{S'} X' = (X \times_S X) \times_S S'$, and it suffices to apply (16.2.3, (ii)) replacing g by the first $p_1 : X \times_S X \to X$ and f by the diagonal Δ_f. □

We note that under the hypotheses of (16.4.5) the homomorphism $\mathrm{Gr}_\bullet(u)$ (16.4.3.5) is *surjective*, but not bijective in general. However (16.2.4):

Corollary (16.4.6). — *Under the hypotheses of (16.4.5), suppose in addition that $w : S \to S'$ is flat (resp. that $\mathscr{G}r_n(\mathscr{P}^n_{X/S})$ are flat $\mathscr{O}_X$-modules for $n \leqslant m$); then the homomorphism*

$$\mathrm{Gr}_n(u) : u^*(\mathscr{G}r(\mathscr{P}^n_{X/S})) \longrightarrow \mathscr{G}r(\mathscr{P}^n_{X'/S'})$$

is bijective for each n (resp. for $n \leqslant m$).

PROOF. Indeed, if w is flat, then so is $v : X' \times_{S'} X' \to X \times_S X$, so the conclusion follows from (16.2.4). □

(16.4.7). Let S be a prescheme, $\mathscr{E}$ a quasi-coherent $\mathscr{O}_S$-Module, and set $X = \mathbf{V}(\mathscr{E})$ (**II**, 1.7.8), the vector bundle associated to $\mathscr{E}$, equal to $\mathrm{Spec}(\mathbf{S}_{\mathscr{O}_S}(\mathscr{E}))$. Let $f : X \to S$ be the structure morphism. For every open U of S and every section $t \in \Gamma(U, \mathscr{E})$, t is identified with a section of $\mathbf{S}_{\mathscr{O}_S}(\mathscr{E})$ over U; let t' be its image in $\Gamma(f^{-1}(U), \mathscr{O}_X) = \Gamma(U, f_*(\mathscr{O}_X)) = \Gamma(U, \mathbf{S}_{\mathscr{O}_S}(\mathscr{E}))$, and set

(16.4.7.1) $$\delta(t) = d^n_{X/S}(t') - t' \in \Gamma(f^{-1}(U), \mathscr{P}^n_{X/S});$$

it is clear that δ is a di-homomorphism of modules (corresponding to the homomorphism of rings $\Gamma(U, \mathscr{O}_S) \to \Gamma(f^{-1}(U), \mathscr{O}_X)$) of $\Gamma(U, \mathscr{E})$ into $\Gamma(f^{-1}(U), \mathscr{P}^n_{X/S})$, and therefore the image belongs to the augmentation ideal of $\Gamma(f^{-1}(U), \mathscr{P}^n_{X/S})$. We deduce (by varying U) a canonical homomorphism of $\mathscr{O}_X$-algebras

(16.4.7.2) $$f^*(\mathbf{S}_{\mathscr{O}_S}(\mathscr{E})) \longrightarrow \mathscr{P}^n_{X/S}$$

and in view of the above remark, if $\mathscr{K}$ is the ideal kernel of augmentation $\mathbf{S}_{\mathscr{O}_S}(\mathscr{E}) \to \mathscr{O}_S$, the image of $\mathscr{K}^{n+1}$ by (16.4.7.2) is zero, so that by factoring by $\mathscr{K}^{n+1}$, we finally have a canonical homomorphism

(16.4.7.3) $$\delta_n : f^*(\mathbf{S}_{\mathscr{O}_S}(\mathscr{E})/\mathscr{K}^{n+1}) \longrightarrow \mathscr{P}^n_{X/S}.$$

Proposition (16.4.8). — *Under the conditions of (16.4.7), the homomorphisms δ_n are bijective and form a projective system of isomorphisms; we deduce an isomorphism of graded $\mathscr{O}_S$-algebras*

(16.4.8.1) $$f^*(\mathbf{S}^\bullet_{\mathscr{O}_S}(\mathscr{E})) \longrightarrow \mathscr{G}r_\bullet(\mathscr{P}_{X/S}).$$

PROOF. The fact that homomorphisms (16.4.7.3) form a projective system follows immediately from their definition. To prove they are isomorphisms, it suffices to prove that (16.4.8.1) is an isomorphism, since both filtrations involved in (16.4.7.3) are finite (Bourbaki, *Alg. comm.*, chap. III, §2, no 8, cor. 3 of th. 1). To do this, consider the split exact sequence of $\mathscr{O}_S$-modules **IV-4** | 20

(16.4.8.2) $$0 \longrightarrow \mathscr{E} \xrightarrow{u} \mathscr{E} \oplus \mathscr{E} \xrightarrow{v} \mathscr{E} \longrightarrow 0$$

where, for every pair of sections s, t of $\mathscr{E}$ over an open U of S, we take $u(s) = (-s, s)$ and $v(s,t) = s + t$. We have

$$X \times_S X = \operatorname{Spec}(\mathbf{S}_{\mathscr{O}_S}(\mathscr{E}) \otimes_{\mathscr{O}_S} \mathbf{S}_{\mathscr{O}_S}(\mathscr{E})) = \operatorname{Spec}(\mathbf{S}_{\mathscr{O}_S}(\mathscr{E} \oplus \mathscr{E}))$$

((**II**, 1.4.6) and (**II**, 1.7.11)), and the diagonal morphism $X \to X \times_S X$ corresponds (**II**, 1.2.7) to the homomorphism of $\mathscr{O}_X$-algebras $\mathbf{S}(v) : \mathbf{S}_{\mathscr{O}_S}(\mathscr{E} \oplus \mathscr{E}) \to \mathbf{S}_{\mathscr{O}_S}(\mathscr{E})$ (**II**, 1.7.4), such that if $\mathscr{I}$ is the kernel of this homomorphism, then we have

$$\mathscr{P}^n_{X/S} = f^*(\mathbf{S}_{\mathscr{O}_S}(\mathscr{E} \oplus \mathscr{E})/\mathscr{I}^{n+1}).$$

The proposition now will be a consequence of the following lemma: □

Lemma (16.4.8.3). — *Let Y be a ringed space, $0 \to \mathscr{F}' \xrightarrow{u} \mathscr{F} \xrightarrow{v} \mathscr{F}'' \to 0$ an exact sequence of $\mathscr{O}_Y$ modules such that each point $y \in Y$ has an open neighbourhood V such that the sequence $0 \to \mathscr{F}'|V \to \mathscr{F}|V \to \mathscr{F}''|V \to 0$ is split. Let $\mathscr{I}$ be the kernel ideal of $\mathbf{S}(v)$:*

$$\mathbf{S}_{\mathscr{O}_Y}(\mathscr{F}) \longrightarrow \mathbf{S}_{\mathscr{O}_Y}(\mathscr{F}''),$$

and let $\operatorname{gr}^\bullet_{\mathscr{I}}(\mathbf{S}_{\mathscr{O}_Y}(\mathscr{F}))$ be the graded $\mathscr{O}_Y$-algebra associated to the $\mathscr{O}_Y$-algebra $\mathbf{S}_{\mathscr{O}_Y}(\mathscr{F})$ endowed with the $\mathscr{I}$-preadic filtration. *Then the homomorphism of graded $\mathscr{O}_Y$-algebras*

$$\mathbf{S}^\bullet_{\mathscr{O}_Y}(\mathscr{F}') \otimes_{\mathscr{O}_Y} \mathbf{S}^\bullet_{\mathscr{O}_Y}(\mathscr{F}'') \longrightarrow \operatorname{gr}^\bullet_{\mathscr{I}}(\mathbf{S}_{\mathscr{O}_Y}(\mathscr{F})) \tag{16.4.8.4}$$

(where the first member is the graded tensor product of symmetric $\mathscr{O}_Y$-algebras endowed with the canonical gradation (**II**, 1.7.4) *and* (**II**, 2.1.2)*), induced by the canonical injection*

$$\mathscr{F}' \longrightarrow \mathscr{I} = \operatorname{gr}^1_{\mathscr{I}}(\mathbf{S}_{\mathscr{O}_Y}(\mathscr{F})),$$

is bijective.

Proof. The injection $\mathscr{F}' \to \mathscr{I}$ indeed canonically gives a homomorphism of graded $\mathscr{O}_Y$-algebras $\mathbf{S}^\bullet_{\mathscr{O}_Y}(\mathscr{F}') \to \operatorname{gr}^\bullet_{\mathscr{I}}(\mathbf{S}^\bullet_{\mathscr{O}_Y}(\mathscr{F}))$, and since the second member is by definition a graded $\mathbf{S}^\bullet_{\mathscr{O}_Y}(\mathscr{F}'')$-algebra, we induce the canonical homomorphism (16.4.8.4) by tensoring the above with $\mathbf{S}^\bullet_{\mathscr{O}_Y}(\mathscr{F}'')$. To prove the lemma we can, being a local problem, restrict to the case where $\mathscr{F} = \mathscr{F}' \oplus \mathscr{F}''$, u and v the canonical homomorphisms. Then the graded algebra $\mathbf{S}^\bullet_{\mathscr{O}_Y}(\mathscr{F})$ is canonically identified with the graded tensor product $\mathbf{S}^\bullet_{\mathscr{O}_Y}(\mathscr{F}') \otimes_{\mathscr{O}_Y} \mathbf{S}^\bullet_{\mathscr{O}_Y}(\mathscr{F}'')$ (**II**, 1.7.4), and it is immediate that $\mathscr{I}$ is therefore the ideal $\mathscr{I}' \otimes_{\mathscr{O}_Y} \mathbf{S}^\bullet_{\mathscr{O}_Y}(\mathscr{F}'')$, where $\mathscr{I}'$ is the augmentation ideal of $\mathbf{S}^\bullet_{\mathscr{O}_Y}(\mathscr{F}')$, that is to say the (direct) sum of the $\mathbf{S}^m_{\mathscr{O}_Y}(\mathscr{F}')$ for $m \geqslant 1$. We conclude that $\mathscr{I}^n = \mathscr{I}'^n \otimes_{\mathscr{O}_Y} \mathbf{S}^\bullet_{\mathscr{O}_Y}(\mathscr{F}'')$, where this time $\mathscr{I}'^n$ is the direct sum of the $\mathbf{S}^m_{\mathscr{O}_Y}(\mathscr{F}')$ for $m \geqslant n$; we have therefore $\mathscr{I}^n/\mathscr{I}^{n+1} = \mathbf{S}^n_{\mathscr{O}_Y}(\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathbf{S}^\bullet_{\mathscr{O}_Y}(\mathscr{F}'')$, which proves that (16.4.8.4) is bijective. □

Having proved the lemma, it remains to see that the homomorphism (16.4.8.1) is the image by f^* of the homomorphism (16.4.8.4) corresponding to the exact sequence (16.4.8.2); we can easily see that it follows from the definition of u (16.4.8.2) and of δ (16.4.7.1), given the definition of the $\mathscr{O}_X$-algebra structures of $\mathscr{P}^n_{X/S}$ and of the $d^n_{X/S}$ (16.3.5) and (16.4.3.6). **IV-4** | 21

In particular:

Corollary (16.4.9). — *Under the conditions of* (16.4.7), *we have a canonical isomorphism*

$$\operatorname{gr}_1(\delta) : f^*(\mathscr{E}) \simeq \Omega^1_{X/S}. \tag{16.4.9.1}$$

Corollary (16.4.10). — *If $S = \operatorname{Spec}(A)$, $\mathscr{E} = \mathscr{O}_S^m$, so that*

$$X = \operatorname{Spec}(A[T_1, \dots, T_m]),$$

then $\mathscr{P}^n_{X/S}$ is canonically identified with the $\mathscr{O}_X$-algebra corresponding to the quotient $A[T_1, \dots, T_m]$-algebra $A[T_1, \dots, T_m, U_1, \dots, U_m]/\mathfrak{K}^{n+1}$, where the U_i ($1 \leqslant i \leqslant m$) are m new indeterminates and $\mathfrak{K}$ is the ideal generated by $U_1, \dots, U_m$.

We thus recover in particular the structure of $\Omega^1_{X/S}$ in this case (**0**, 20.5.13).

In addition, note that the $d^n_{X/S}$ then corresponds to a polynomial $F(T_1, \dots, T_m)$, the class modulo $\mathfrak{K}^{n+1}$ of $F(T_1 + U_1, \dots, T_m + U_m)$, which follows from the definition (16.4.7.1).

Proposition (16.4.11). — *Let $f : X \to S$ be a morphism, $g : S \to X$ a S-section of X, $S^{(n)}$ the n-th infinitesimal neighbourhood of S by the immersion g (16.1.2). Then there exists a unique isomorphism of $\mathscr{O}_S$-algebras*

(16.4.11.1)
$$\varpi_n : g^*(\mathscr{P}^n_{X/S}) \longrightarrow \mathscr{O}_{S^{(n)}_g}$$

(via the $\mathscr{O}_S$-algebra structure on $\mathscr{O}_{S^{(n)}_g}$ defined by f (16.1.7)), making the diagram

(16.4.11.2)
$$\begin{tikzcd} \mathscr{O}_S = g^*(\mathscr{O}_X) \arrow[rr, "\lambda_n"] \arrow[dr, "g^*(d^n_{X/S})"'] & & \mathscr{O}_{S^{(n)}_g} \\ & g^*(\mathscr{P}^n_{X/S}) \arrow[ur, "\varpi_n"'] & \end{tikzcd}$$

commutative (where λ_n is the structure morphism).

PROOF. In light of (I, 5.3.7), where we replace X, Y, S by S, X, S respectively and f by g, the diagrams

(16.4.11.3)
$$\begin{tikzcd} S \arrow[r, "g"] \arrow[d, "g"'] & X \arrow[d, "\Delta_f"] \\ X \arrow[r, "(g\circ f, 1_X)_S"'] & X \times_S X \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r, "g"] \arrow[d, "g"'] & X \arrow[d, "\Delta_f"] \\ X \arrow[r, "(1_X, g\circ f)_S"'] & X \times_S X \end{tikzcd}$$

identifies S with the product of the $(X \times_S X)$-preschemes X and X by the morphisms Δ_f and $(g \circ f, 1_X)_S$ (resp. $(1_X, g \circ f)_S$). On the other hand, the diagrams **IV-4** | 22

(16.4.11.4)
$$\begin{tikzcd} X \arrow[r, "(g\circ f, 1_X)_S"] \arrow[d, "f"'] & X \arrow[d, "p_1"] \\ S \arrow[r, "g"'] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "(1_X, g\circ f)_S"] \arrow[d, "f"'] & X \arrow[d, "p_2"] \\ S \arrow[r, "g"'] & X \end{tikzcd}$$

identify X to the product of X-preschemes S and $X \times_S X$ via the morphisms g and p_1 (resp. p_2) (particular case of the associativity formula (I, 3.3.9.1)). We can say that Δ_f, considered as an X-section of $X \times_S X$ (relative to p_1 or p_2) plays the role of a *universal section* for the S-sections of X: each of these sections g in fact are deduced by *base change* $(g \circ f, 1_X)_S : X \to X \times_S X$. The definition of the homomorphism $\bar{\omega}_n$ and the fact that it is bijective follows from the remarks of (16.2.3, (ii)) applied to the first diagram (16.4.11.4). The commutativity of the first diagram (16.4.11.4) follows also from (16.2.3, (ii)) this time applied to the second diagram (16.11.4). To explain ϖ_n, we can restrict ourselves to the case where g is a closed immersion: Indeed, for every $s \in S$, there is an open neighbourhood W of s in S such that $g(W)$ is closed in an open set U of X, and it is clear that $g|W$ is a W-section of the morphism $U \cap f^{-1}(W)$. We can then suppose that S is a closed subprescheme of X defined by a quasi-coherent ideal $\mathscr{K}$. Then the preceding definitions show that if W is an open of S, t is a section of $\mathscr{O}_X$ over $f^{-1}(W)$, $\varpi_n(d^n t|W)$ is equal to the canonical image of t in $\Gamma(W, (\mathscr{O}_X/\mathscr{K}^{n+1})|W)$. The uniqueness of ϖ_n then follows since the image of $\mathscr{O}_X$ under $d^n_{X/S}$ generates the $\mathscr{O}_X$-module $\mathscr{P}^n_{X/S}$ (16.3.8). □

Corollary (16.4.12). — *Let k be a field, X a k-prescheme, x a point of X* rational over k. *Then $(\mathscr{P}^n_{X/S})_x \otimes_{\mathscr{O}_x} k(x)$ is canonically isomorphic (as an augmented $k(x)$-algebra) to $\mathscr{O}_x/\mathfrak{m}_x^{n+1}$.*

PROOF. It suffices to consider the unique k-section g of X such that $g(\operatorname{Spec}(k)) = \{x\}$. □

Corollary (16.4.13). — *Let $f : X \to S$ be a morphism, s a point of S, $X_s = X \times_S \operatorname{Spec}(k(s))$ the fibre of f in s. If $x \in X_s$ is* rational over $k(s)$, *$(\mathscr{P}^n_{X/S})_x \otimes_{\mathscr{O}_s} k(s)$ is canonically isomorphic to $\mathscr{O}_{X_s,x}/\mathfrak{m}'^{\,n+1}_x$, is the maximal ideal of $\mathscr{O}_{X_s,x}$; more precisely, this isomorphism sends $(d^n t)_x \otimes 1$* (where t is a section of $\mathscr{O}_X$ over an open neighbourhood of x in X) *to the class of $t_x \otimes 1$ modulo $\mathfrak{m}'^{\,n+1}_x$.*

PROOF. This follows from (16.4.5) and (16.4.12). □

The preceding corollaries justify the terminology "sheaf of principal parts of order n".

Proposition (16.4.14). — *Let $\rho : A \to B$ be a morphism of rings, S a multiplicative subset of B. Then the canonical homomorphisms*

(16.4.14.1) $$S^{-1}P^n_{B/A} \longrightarrow P^n_{S^{-1}B/A}$$

deduced from the canonical homomorphisms $P^n_{B/A} \to P^n_{S^{-1}B/A}$ (16.4.4), form a projective system and are bijective. **IV-4** | 23

PROOF. It suffices to remark that $S^{-1}((B \otimes_A B)/\mathfrak{I}^{n+1}) = S^{-1}(B \otimes_A B)/(S^{-1}\mathfrak{I})^{n+1}$ by flatness, and that $S^{-1}(B \otimes_A B) = (S^{-1}B) \otimes_A (S^{-1}B)$ (**I**, 1.3.4). □

Corollary (16.4.15). — *The notation being that of (16.4.14), let R be a multiplicative subset of A such that $\rho(R) \subset S$. Then we have canonical isomorphisms*

(16.4.15.1) $$S^{-1}P^n_{B/A} \simeq P^n_{S^{-1}B/R^{-1}A}$$

forming a projective system.

PROOF. It evidently suffices to define canonical isomorphisms

(16.4.15.2) $$P^n_{S^{-1}B/A} \simeq P^n_{S^{-1}B/R^{-1}A}$$

that is to say that we reduce to the case there $\rho(R)$ is consists of *invertible* elements of B. But then the isomorphism (16.4.15.2) is simply induced by the canonical isomorphism $B \otimes_A B \to B \otimes_{R^{-1}A} B$ by passing to quotients (**I**, 1.5.3). □

Corollary (16.4.16). — *Let $f : X \to S$ be a morphism of preschemes, x a point of X, $s = f(x)$. Then we have canonical isomorphisms*

(14.4.16.1) $$(\mathscr{P}^n_{X/S})_x \simeq P^n_{\mathscr{O}_x/\mathscr{O}_s}$$

forming a projective system.

We deduce from these isomorphisms of the associated graded rings, and in particular a canonical isomorphism

(16.4.16.2) $$(\Omega^1_{X/S})_x \simeq \Omega^1_{\mathscr{O}_x/\mathscr{O}_s}.$$

Corollary (16.4.17). — *Let k be a field, K the field of rational functions $k(T_1, \ldots, T_r)$. Then, for every integer n, the homomorphism of $K[U_1, \ldots, U_r]$ (U_i indeterminates) into $P^n_{K/k}$ which sends U_i to $d^n T_i - T_i.1$ is surjective and defines an isomorphism from the quotient $K[U_1, \ldots, U_n]/\mathfrak{m}^{n+1}$ (where $\mathfrak{m}$ is the ideal generated by the U_i) to $P^n_{K/k}$.*

PROOF. This follows from (16.4.8), (16.4.10) and (16.4.14), where we take $A = k$, $B = k[T_1, \ldots, T_r]$ and $S = B - \{0\}$. □

We thus recover the fact that the dT_i form a basis of the K-vector space $\Omega^1_{K/k}$ (**0**, 20.5.10).

(16.4.18). Let $f : X \to Y$, $g : Y \to Z$ be two morphisms of preschemes, and consider the canonical homomorphism of augmented $\mathscr{O}_X$-algebras (16.4.3.3)

(16.4.18.1) $$g_{X/Y/Z} : \mathscr{P}^n_{X/Z} \longrightarrow \mathscr{P}^n_{X/Y}$$

(16.4.18.2) $$f_{X/Y/Z} : f^*(\mathscr{P}^n_{Y/Z}) \longrightarrow \mathscr{P}^n_{X/Z}.$$

Then $g_{X/Y/Z}$ is surjective, and its kernel is the sheaf of ideals generated by the image under $f_{X/Y/Z}$ of the augmentation ideal of $f^*(\mathscr{P}^n_{X/Z})$.

PROOF. First note that $g_{X/Y/Z}$ corresponds to the case in (16.4.3.3) where $X' = X$, $S' = Y$ and **IV-4** | 24
$S = Z$, $u = 1_X$, and $f_{X/Y/Z}$ to the case where we replace X', X, S, S' by X, Y, Z, Z respectively and u, f by f, g respectively.

We have a commutative diagram (**I**, 3.5)

(16.4.18.3)
$$\begin{tikzcd} X \arrow[r, "\Delta_f"] \arrow[dr, "f"'] & X \times_Y X \arrow[r, "j"] \arrow[d, "p"] & X \times_Z X \arrow[d, "f \times_Z f"] \\ & Y \arrow[r, "\Delta_g"'] & Y \times_Z Y \end{tikzcd}$$

where $j = (1_X, 1_X)_Z$ is an immersion, $j \circ \Delta_f = \Delta_{g \circ f}$, and p is the structure morphism. Since we can restrict ourselves to the case where X, Y and Z are affine, we can suppose that the immersions Δ_f, Δ_g and j are closed, so that $\mathscr{O}_X$ and $\mathscr{O}_{X \times_Y X}$ are identified respectively with $\mathscr{O}_{X \times_Z X}/\mathscr{I}$ and $\mathscr{O}_{X \times_Z X}/\mathscr{L}$, where $\mathscr{L} \supset \mathscr{I}$ are two quasi-coherent ideals corresponding respectively to the immersions $\Delta_{g \circ f}$ and j. The $\mathscr{O}_X$-algebra $\mathscr{P}^n_{X/Z}$ is identified with $\mathscr{O}_{X \times_Z X}/\mathscr{I}^{n+1}$, and $\mathscr{P}^n_{X/Y}$ is identified with $\mathscr{O}_{X \times_Y X}/(\mathscr{I}/\mathscr{L})^{n+1}$, which is to say with $\mathscr{O}_{X \times_Z X}/(\mathscr{I}^{n+1} + \mathscr{L})$, and therefore with the quotient of $\mathscr{P}^n_{X/Z}$ by $(\mathscr{I}^{n+1} + \mathscr{L})/\mathscr{I}^{n+1}$. But we know (*loc. cit*) that if p and j make $X \times_Y X$ the product of the $(Y \times_Z Y)$-preschemes Y and $X \times_Z X$, so if $\mathscr{O}_Y$ is identified to $\mathscr{O}_{Y \times_Z Y}/\mathscr{K}$, where $\mathscr{K}$ is the ideal corresponding to Δ_g, $\mathscr{L}$ is equal to $(f \times_Z f)^*(\mathscr{K}).\mathscr{O}_{X \times_Z X}$ (I, 4.4.5). Since $(\mathscr{I}^{n+1} + \mathscr{L})/\mathscr{I}^{n+1}$ is the ideal of $\mathscr{P}^n_{X/Z}$ generated by the image of $\mathscr{L}$, we deduce the proposition. □

Corollary (16.4.19). — *With the notation of (16.4.18), we have an exact sequence of quasi-coherent $\mathscr{O}_X$-modules*

(16.4.19.1)
$$f^*(\Omega^1_{Y/Z}) \xrightarrow{f_{X/Y/Z}} \Omega^1_{X/Z} \xrightarrow{g_{X/Y/Z}} \Omega^1_{X/Y} \longrightarrow 0.$$

When X, Y, Z are affine, we recover the exact sequence (**0**, 5.7.1).

Proposition (16.4.20). — *Let $f : Y \to Z$ be a morphism, $j : X \to Y$ a closed immersion, $\mathscr{K}$ the quasi-coherent sheaf of ideals of $\mathscr{O}_Y$ corresponding to j. It follows that $\mathscr{P}^n_{X/Y} = \mathscr{O}_X = \mathscr{O}_Y/\mathscr{K}$, the canonical homomorphism $j_{X/Y/Z} : j^*(\mathscr{P}^n_{Y/Z}) \to \mathscr{P}^n_{X/Z}$ is surjective, and its kernel is the ideal of $j^*(\mathscr{P}^n_{Y/Z})$ generated by $j^*(\mathscr{O}_Y \cdot d^n_{Y/Z}(\mathscr{K}))$ (it should be noted that $d^n_{Y/Z}(\mathscr{K})$ is a subsheaf of abelian groups of $\mathscr{P}^n_{X/Z}$, but not an $\mathscr{O}_Y$-module in general).*

PROOF. We know (I, 5.3.8) that the diagonal $\Delta_j : X \to X \times_Y X$ is an isomorphism, from which the first assertion follows. If ϖ_1 and ϖ_2 are the two canonical homomorphisms of algebras $\mathscr{O}_Y \to \mathscr{P}^n_{Y/Z}$ corresponding respectively to the two canonical projections p_1, p_2 of $Y \times_Z Y \to Y$, recall that by definition ((16.3.5) and (16.3.6)) ϖ_1 is the structure homomorphism of the $\mathscr{O}_Y$-algebra $\mathscr{P}^n_{Y/Z}$ and $\varpi_2 = d^n_{Y/Z}$. The $\mathscr{O}_X$-algebra $j^*(\mathscr{P}^n_{Y/Z})$ is therefore identified with $\mathscr{P}^n_{Y/Z}/\varpi_1(\mathscr{K})\mathscr{P}^n_{Y/Z}$ and its quotient by the ideal generated by $j^*(d^n_{Y/Z}(\mathscr{K}))$ to $\mathscr{P}^n_{Y/Z}/(\varpi_1(\mathscr{K}) + \varpi_2(\mathscr{K}))\mathscr{P}^n_{Y/Z}$. Now note that we have a commutative diagram **IV-4** | 25

$$\begin{tikzcd} Y \arrow[d, "\Delta_f"'] & X \arrow[l, "j"'] \arrow[d, "\Delta_{f \circ j}"] \\ Y \times_Z Y & X \times_Z X \arrow[l, "j \times_Z j"] \end{tikzcd}$$

identifying X with the product of the $(Y \times_Z Y)$-preschemes Y and $X \times_Z X$ (I, 5.3.7). Since $j \times_Z j$ is an immersion, we therefore deduce from this remark and from (16.2.2) that if $\Delta^n_{Y/Z}$ and $\Delta^n_{X/Z}$ denote the infinitesimal neighbourhoods of order n of Y and X by the canonical immersions Δ_f and $\Delta_{f \circ j}$ respectively, then we have a diagram

$$\begin{tikzcd} \Delta^n_{Y/Z} \arrow[d] & \Delta^n_{X/Z} \arrow[l] \arrow[d] \\ Y \times_Z Y & X \times_Z X \arrow[l, "j \times_Z j"] \end{tikzcd}$$

making $\Delta^n_{X/Z}$ the product of the $(Y \times_Z Y)$-preschemes $\Delta^n_{Y/Z}$ and $X \times_Z X$. We can also say that $\mathscr{P}^n_{X/Z}$ is identified with the sheaf of rings $\mathscr{P}^n_{Y/Z} \otimes_{\mathscr{O}_{Y\times_Z Y}} \mathscr{O}_{X\times_Z X}$. But we see immediately that (for example, by restricting to the affine case) that $\mathscr{O}_{X\times_Z X} = \mathscr{O}_{Y\times_Z Y}/(p_1^*(\mathscr{K}) + p_2^*(\mathscr{K}))\mathscr{O}_{Y\times_Z Y}$. Therefore $\mathscr{P}^n_{X/Z}$ is identified with the quotient of $\mathscr{P}^n_{Y/Z}$ by the ideal generated by the image in $\mathscr{P}^n_{Y/Z}$ of $p_1^*(\mathscr{K}) + p_2^*(\mathscr{K})$. But by definition this ideal is generated by $\varpi_1(\mathscr{K}) + \varpi_2(\mathscr{K})$. ☐

Corollary (16.4.21). — *Let $f : Y \to Z$ be a morphism, $j : X \to Y$ an immersion. We have an exact sequence of quasi-coherent $\mathscr{O}_X$-modules*

(16.4.21.1) $$\mathscr{N}_{X/Y} \longrightarrow j^*(\Omega^1_{Y/Z}) \longrightarrow \Omega^1_{X/Z} \longrightarrow 0.$$

When X, Y, Z are affine, we recover the exact sequence (**0**, 20.5.12.1).

Corollary (16.4.22). — *If $f : X \to S$ is a morphism locally of finite presentation, $\mathscr{P}^n_{X/S}$ and $\Omega^1_{X/S}$ are quasi-coherent $\mathscr{O}_X$-modules of finite presentation.*

Proof. We immediately reduce to the case where $S = \operatorname{Spec}(A)$ is affine, $X = \operatorname{Spec}(B)$, where $B = A[T_1, \ldots, T_r]/\mathfrak{K}$, $\mathfrak{K}$ being an ideal of finite type of $C = A[T_1, \ldots, T_r]$. Applying (16.4.20) where $Z = S$, $Y = \operatorname{Spec}(C)$ and $\mathscr{K} = \widetilde{\mathfrak{K}}$. Then $j^*(\mathscr{P}^n_{Y/Z})$ is a free $\mathscr{O}_X$-module of finite rank (16.4.10) and the hypothesis on $\mathfrak{K}$ implies that $j^*(\mathscr{O}_Y.d^n_{Y/Z}(\mathscr{K}))$ generates a quasi-coherent $\mathscr{O}_X$-module of finite type; hence the conclusion. ☐

Proposition (16.4.23). — *Let X, Y be two S-preschemes, $Z = X \times_S Y$ their product, $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ the canonical projections. Then the canonical homomorphism*

(16.4.23.1) $$p_{Z/X/S} \oplus q_{Z/Y/S} : p^*(\Omega^1_{X/S}) \oplus q^*(\Omega^1_{Y/S}) \longrightarrow \Omega^1_{(X\times_S Y)/S}$$

is bijective.

Proof. The commutative diagram **IV-4** | 26

$$\begin{tikzcd} Y \arrow[d, "g"'] & X \times_S Y \arrow[l, "q"'] \arrow[d, "h"] & X \times_S Y \arrow[l, "\mathrm{id}"'] \arrow[d, "p"] \\ S & S \arrow[l, "\mathrm{id}"] & X \arrow[l, "f"] \end{tikzcd}$$

gives us a factorization of the canonical *isomorphism* $P^n(p)$ (16.4.5)

$$p^*(\mathscr{P}^n_{X/S}) \longrightarrow \mathscr{P}^n_{Z/S} \longrightarrow \mathscr{P}^n_{Z/Y}$$

and similarly, switching X with Y, we have a factorization of the *isomorphism* $P^n(q)$

$$q^*(\mathscr{P}^n_{Y/S}) \longrightarrow \mathscr{P}^n_{Z/S} \longrightarrow \mathscr{P}^n_{Z/Y}.$$

This proves that the canonical homomorphism (16.4.18.1)

$$p_{Z/X/S} : p^*(\mathscr{P}^n_{X/S}) \longrightarrow \mathscr{P}^n_{Z/S} \quad (\text{resp. } q_{Z/X/S} : q^*(\mathscr{P}^n_{Y/S}) \longrightarrow \mathscr{P}^n_{Z/S})$$

is *injective*, and that the kernel of the canonical surjective homomorphism (16.4.18.2)

$$\mathscr{P}^n_{Z/S} \longrightarrow \mathscr{P}^n_{Z/Y} \quad (\text{resp. } \mathscr{P}^n_{Z/S} \longrightarrow \mathscr{P}^n_{Z/X})$$

is direct summand of the image $p_{Z/X/S}$ (resp. $q_{Z/Y/S}$). On the other hand, this kernel is, by virtue of (16.4.18), generated by the image by $q_{Z/Y/S}$ (resp. $p_{Z/X/S}$) of the augmentation ideal of $q^*(\mathscr{P}^n_{Y/S})$ (resp. $p^*(\mathscr{P}^n_{X/S})$). We conclude the proposition by considering the case $n = 1$. ☐

We immediately generalize (16.4.23) to the case of a product of any finite number of S-preschemes.

Remarks (16.4.24). —

(i) We will see (17.2.3) that when the morphism $f : X \to Y$ in (16.4.18) is *smooth*, the homomorphism $f_{X/Y/Z}$ in (16.4.19.1) is locally *left invertible* and in particular injective. Similarly, when the morphism $f \circ j : X \to Z$ of (16.4.20) is *smooth*, the homomorphism on the left in (16.4.21.1) is locally *left invertible* and *a fortiori* injective (17.2.5). In Chapter V, we will also give a variant, in the case of modules over a prescheme, of the "imperfection modules" studied in (**0**, 20.6), and the exact sequences where they occur.

(ii) Let X be a topological space, $\mathscr{A}$ a sheaf of rings over X and $\mathscr{B}$ a $\mathscr{A}$-algebra over X. Then it is clear that
$$U \longmapsto P^n_{\Gamma(U,\mathscr{B})/\Gamma(U,\mathscr{A})} \quad (U \text{ open in } X)$$
is a presheaf of augmented $\Gamma(U,\mathscr{B})$-algebras, and therefore the associated sheaf $\mathscr{P}^n_{\mathscr{B}/\mathscr{A}}$ is an augmented $\mathscr{B}$-algebra. In the particular case where X is a prescheme, $f = (\psi,\theta) : X \to S$ a morphism of preschemes, it follows easily from (16.4.16) and from the exactness of the functor $\varinjlim$ that $\mathscr{P}^n_{X/S}$ is canonically isomorphic to $\mathscr{P}^n_{\mathscr{O}_X/\psi^*(\mathscr{O}_S)}$. It follows that the formalism developed in the present paragraph could be considered as a particular case **IV-4** | 27 of a differential formalism for ringed spaces endowed with a sheaf of algebras over the structure sheaf. However, we did not start with this point of view, which is less intuitive and less convenient for applications. It also seems that, for various kinds of "varieties", the "global" constructions of the $\mathscr{P}^n$ analogous to those we have used here are also better suited for applications.

16.5. Relative tangent sheaves and bundles; derivations.

(16.5.1). Let $f = (\psi,\theta) : X \to S$ be a morphism of ringed spaces. For every $\mathscr{O}_X$-module $\mathscr{F}$, we say *S-derivation* (or *(X/S)-derivation*, or *f-derivation*) *of $\mathscr{O}_X$ to $\mathscr{F}$* for every homomorphism of *sheaves of additive groups $D : \mathscr{O}_X \to \mathscr{F}$* satisfying the following conditions:

(a) for every open V of X, and all pair of sections (t_1,t_2) of $\mathscr{O}_X$ over V, we have

(16.5.1.1) $$D(t_1t_2) = t_1D(t_2) + D(t_1)t_2;$$

(b) for every open V of X, every section t of $\mathscr{O}_X$ over V, and every section s of $\mathscr{O}_S$ over an open U of S such that $V \subset f^{-1}(U)$, we have

(16.5.1.2) $$D((s|V)t) = (s|V)D(t).$$

It is clear that this amounts to saying that, for all $x \in X$, the homomorphism of additive groups $D_x : \mathscr{O}_x \to \mathscr{F}_x$ is an *$\mathscr{O}_{f(x)}$-derivation*.

Another interpretation consists of considering the $\mathscr{O}_X$-algebra $\mathscr{D}_{\mathscr{O}_X}(\mathscr{F})$ as equal to $\mathscr{O}_X \oplus \mathscr{F}$, the algebra structure being defined by the condition that for every open V of X, the product of two sections of $\mathscr{O}_X$ (resp. of a section of $\mathscr{O}_X$ and a section of $\mathscr{F}$) over V is defined by the ring structure of $\Gamma(V,\mathscr{O}_X)$ (resp. the $\Gamma(V,\mathscr{O}_X)$-module structure on $\Gamma(V,\mathscr{F})$), and the product of two sections of $\mathscr{F}$ over V is chosen to be zero; then $\mathscr{F}$ is an ideal of $\mathscr{D}_{\mathscr{O}_X}(\mathscr{F})$, the kernel of the canonical augmentation $\mathscr{D}_{\mathscr{O}_X}(\mathscr{F}) \to \mathscr{O}_X$, and to say that D is an S-derivation of $\mathscr{O}_X$ to $\mathscr{F}$ means that $1_{\mathscr{O}_X} + D$ is an *$\mathscr{O}_S$-homomorphism of algebras* from $\mathscr{O}_X$ to $\mathscr{D}_{\mathscr{O}_X}(\mathscr{F})$, which, composed with the augmentation, gives $1_{\mathscr{O}_X}$.

The S-derivations of $\mathscr{O}_X$ to $\mathscr{F}$ clearly form a *$\Gamma(X,\mathscr{O}_X)$-module* $\mathrm{Der}_{\mathscr{O}_S}(\mathscr{O}_X,\mathscr{F})$.

When $\mathscr{F} = \mathscr{O}_X$, an S-derivation of $\mathscr{O}_X$ to itself is simply called an *S-derivation of $\mathscr{O}_X$*.

Proposition (16.5.2). — *Let A be a ring, B an A-algebra, L a B-module; let $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, $\mathscr{F} = \widetilde{L}$. Then the map $D \mapsto \Gamma(D)$ which sends every S-derivation D of $\mathscr{O}_X$ to $\mathscr{F}$ to the map $\Gamma(D) : t \mapsto D(t)$ of B to L, is an isomorphism of B-modules from $\mathrm{Der}_S(\mathscr{O}_X,\mathscr{F})$ to $\mathrm{Der}_A(B,L)$ (cf.* (**0**, 20.1.2)*).*

Proof. This follows immediately from the given interpretation of S-derivations in terms of **IV-4** | 28 homomorphisms of algebras, analogous to the interpretation given in (**0**, 20.1.6), and from the canonical correspondence between homomorphisms of $\mathscr{O}_X$-algebras and homomorphisms of B-algebras ((**I**, 1.3.13) and (**I**, 1.3.8)). □

Proposition (16.5.3). — *Let $f = (\psi,\theta) : X \to S$ be a morphism of preschemes.*

(i) *The differential $d_{X/S} : \mathscr{O}_X \to \Omega^1_{X/S}$* (16.3.6) *is an S-derivation.*

(ii) *For every $\mathscr{O}_X$-module $\mathscr{F}$, the map $u \mapsto u \circ d_{X/S}$ is an isomorphism of $\Gamma(X,\mathscr{O}_X)$-modules*

(16.5.3.1) $$\mathrm{Hom}_{\mathscr{O}_X}(\Omega^1_{X/S},\mathscr{F}) \simeq \mathrm{Der}_S(\mathscr{O}_X,\mathscr{F}).$$

Proof. The assertion (i) has already been written (16.3.6). On the other hand, it is immediate (in light of (**0**, 20.4.8)) that $u \mapsto u \circ d_{X/S}$ is injective, considering the restrictions to a fibre $\mathscr{O}_x$ of the two sides and using (16.4.16.2). To see that the homomorphism (16.5.3.1) is surjective, consider an S-derivation $D : \mathscr{O}_X \to \mathscr{F}$; for every affine open $V = \mathrm{Spec}(B)$ of X, such that $f(V)$ is contained in an affine open $U = \operatorname{Spec}(A)$ of S, $D_V : B \to \Gamma(V, \mathscr{F})$ is an A-derivation, and therefore there exists a *unique* B-homomorphism $u_V : \Omega^1_{B/A} \to \Gamma(V, \mathscr{F})$ such that $D_V = u_V \circ d_{B/A}$ (**0**, 20.4.8); in addition, the uniqueness of u_V shows immediately that for an affine open $W \subset V$ we have $u_W = u_V|W$, and therefore the u_V define a homomorphism of $\mathscr{O}_X$-modules $u : \mathscr{O}_X \to \mathscr{F}$ answering the question. □

(16.5.4). With the notation of (16.5.1), for every open U of X, $\operatorname{Der}_S(\mathscr{O}_U, \mathscr{F}|U)$ is a $\Gamma(U, \mathscr{O}_X)$-module and it is clear that the map $U \mapsto \operatorname{Der}_S(\mathscr{O}_U, \mathscr{F}|U)$ is a presheaf; in fact, it is even a *sheaf* (and therefore an $\mathscr{O}_X$-module), in light of the pointwise characterization of S-derivations, seen in (16.5.1). This $\mathscr{O}_X$-module is denoted by $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{F})$ and is called the *sheaf of S-derivations of $\mathscr{O}_X$ in $\mathscr{F}$*, and what we have seen is further expressed in the following corollary:

Corollary (16.5.5). — *For every $\mathscr{O}_X$-module $\mathscr{F}$, the homomorphism of $\mathscr{O}_X$-modules induced by $u \mapsto u \circ d_{X/S}$*

(16.5.5.1) $$\mathscr{H}om_{\mathscr{O}_X}(\Omega^1_{X/S}, \mathscr{F}) \longrightarrow \mathscr{D}er_S(\mathscr{O}_X, \mathscr{F})$$

is bijective.

Corollary (16.5.6). —
(i) *If the morphism $f : X \to S$ is locally of finite presentation and if $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module, then $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{F})$ is a quasi-coherent $\mathscr{O}_X$-module.*
(ii) *If in addition S is locally Noetherian and if $\mathscr{F}$ is coherent, then $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{F})$ is a coherent $\mathscr{O}_X$-module.*

PROOF. The assertion (i) follows from the isomorphism (16.5.5.1), from (16.4.22), and (**I**, 1.3.12); the assertion (ii) follows from (**0**, 5.3.5). □

(16.5.7). We set

(16.5.7.1) $$\mathfrak{G}_{X/S} = \mathscr{H}om_{\mathscr{O}_X}(\Omega^1_{X/S}, \mathscr{O}_X) = \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X),$$

and say that it is the *sheaf of S-derivations of $\mathscr{O}_X$*, or even the *tangent sheaf of X relative to S*: it is therefore the *dual* of the $\mathscr{O}_X$-module $\Omega^1_{X/S}$. If f is locally of finite presentation, $\mathfrak{G}_{X/S}$ is a quasi-coherent $\mathscr{O}_X$-module; if in addition S is locally Noetherian, then $\mathfrak{G}_{X/S}$ is coherent (16.5.6). **IV-4** | 29

(16.5.8). Suppose in particular that $\Omega^1_{X/S}$ is a *locally free* $\mathscr{O}_X$-module (of finite rank) (which will be the case then f is *smooth* (17.2.3)); then $\mathfrak{G}_{X/S}$ is locally free $\mathscr{O}_X$-module of the same rank as $\Omega^1_{X/S}$ at each point. More specifically, suppose that $\Omega^1_{X/S}$ is of rank n at a point x; then there are n sections s_i ($1 \leqslant i \leqslant n$) of $\mathscr{O}_X$ over an affine neighbourhood U of x such that the canonical images of the ds_i in $\Omega^1_{X/S} \otimes_{\mathscr{O}_X} k(x)$ form a basis if this $k(x)$-vector space; by virtue of Nakayama's lemma, the germs $(ds_i)_x = d(s_i)_x$ of the ds_i at the point x form a basis of the $\mathscr{O}_x$-module $(\Omega^1_{X/S})_x$, and therefore, by restricting U, we can suppose that the ds_i form a *basis* of the $\Gamma(U, \mathscr{O}_X)$-module $\Gamma(U, \Omega^1_{X/S})$. So the $\Gamma(U, \mathscr{O}_X)$-module $\Gamma(U, \mathfrak{G}_{X/S})$ is dual to the above; we denote by $(D_i)_{1 \leqslant i \leqslant n}$ or $\left(\frac{\partial}{\partial s_i}\right)_{1 \leqslant i \leqslant n}$ the dual basis of $(ds_i)_{1 \leqslant i \leqslant n}$, so that, by (16.5.3), we have

(16.5.8.1) $$D_i s_j = \langle D_i, ds_j \rangle = \left\langle \frac{\partial}{\partial s_i}, ds_j \right\rangle = \delta_{ij} \quad \text{(Kronecker's symbol).}$$

Every $\Gamma(S, \mathscr{O}_S)$-derivation of the $\Gamma(S, \mathscr{O}_S)$-algebra $\Gamma(U, \mathscr{O}_X)$ is therefore written in an unique way as

$$D = \sum_{i=1}^{n} a_i D_i = \sum_{i=1}^{n} a_i \frac{\partial}{\partial s_i},$$

where the a_i ($1 \leqslant i \leqslant n$) are sections of $\mathscr{O}_X$ over U. For every section $g \in \Gamma(U, \mathscr{O}_X)$, if we put $dg = \sum_{i=1}^n c_i ds_i$, then we have $c_i = \langle D_i, dg \rangle = D_i g$ by virtue of (16.5.8.1), in other words,

(16.5.8.2) $$dg = \sum_{i=1}^{n} (D_i g) ds_i = \sum_{i=1}^{n} \frac{\partial g}{\partial s_i} ds_i.$$

(16.5.9). Let D_1, D_2 be two S-derivations of $\mathscr{O}_X$. For every open U of X, if D_1^U, D_2^U are the corresponding derivations of the ring $\Gamma(U, \mathscr{O}_X)$, the *bracket*

$$[D_1^U, D_2^U] = D_1^U \circ D_2^U - D_2^U \circ D_1^U$$

is also a derivation in this ring, and therefore the $\psi^*(\mathscr{O}_S)$-endomorphism of $\mathscr{O}_X$

(16.5.9.1) $$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$$

is also an S-derivation; as we immediately check that this bracket satisfies the Jacobi identity, we have thus defined on $\mathrm{Der}_S(\mathscr{O}_X, \mathscr{O}_X)$ a $\Gamma(S, \mathscr{O}_S)$-*Lie algebra* structure. Since the definition of this structure commutes with the restriction to an open of X, we thus see that $\mathfrak{G}_{X/S}$ is canonically equipped with a $\psi^*(\mathscr{O}_S)$-*Lie algebra* structure. Note that the mapping $(D_1, D_2) \mapsto [D_1, D_2]$ is *not* $\Gamma(X, \mathscr{O}_X)$-*bilinear*.

(16.5.10). For every base change $g : S' \to S$, if we set $X' = X \times_S S'$, then we see (16.4.5) that we have a canonical isomorphism

(16.5.10.1) $$\Omega^1_{X/S} \otimes_S S' \simeq \Omega^1_{X'/S'},$$

from which we deduce, by (16.5.10.1), a canonical homomorphism (Bourbaki, *Alg.*, chap. II, 3rd ed., §5, n°3) **IV-4** | 30

(16.5.10.2) $$\mathfrak{G}_{X/S} \otimes_{\mathscr{O}_S} \mathscr{O}_{S'} \longrightarrow \mathfrak{G}_{X'/S'},$$

which is neither injective nor surjective in general. However:

Proposition (16.5.11). — (i) *If $g : S' \to S$ is a flat morphism and if f is locally of finite type (resp. locally of finite presentation), then the homomorphism (16.5.10.2) is injective (resp. bijective).*

(ii) *If $\Omega^1_{X/S}$ is a locally free $\mathscr{O}_X$-module of finite type, then the homomorphism (16.5.10.2) is bijective.*

PROOF. The assertion (ii) follows from Bourbaki, *Alg.*, chap. II, 3rd ed., §5, n°3, prop. 7. The assertion (i) follows similarly from Bourbaki, *Alg. Comm.*, chap. I, §2, n°10, prop. 11 and from the fact that if f is locally of finite type (resp. locally of finite presentation), then $\Omega^1_{X/S}$ is an $\mathscr{O}_X$-module of finite type (resp. of finite presentation) ((16.3.9) (16.4.22)). ☐

(16.5.12). Since $\Omega^1_{X/S}$ is a quasi-coherent $\mathscr{O}_X$-module, we can consider the vector bundle over X defined by $\Omega^1_{X/S}$ (**II**, 1.7.8)

(16.5.12.1) $$T_{X/S} = \mathbf{V}(\Omega^1_{X/S})$$

which is called the *tangent bundle of X relative to S*. We have therefore a canonical bijection (**II**, 1.7.9)

$$\Gamma(T_{X/S}/S) \simeq \mathrm{Hom}_{\mathscr{O}_X}(\Omega^1_{X/S}, \mathscr{O}_X) = \Gamma(X, \mathfrak{G}_{X/S})$$

by definition of $\mathfrak{G}_{X/S}$, and we can replace X by an open set U of X in this isomorphism; so we can say that the *tangent sheaf* of X relative to S is isomorphic to the *sheaf of germs of S-sections* of the tangent bundle of X relative to S. If $f : X \to Y$ is an S-morphism, we saw (16.4.19) that we have a canonical homomorphism $f_{X/Y/S} : f^*(\Omega^1_{Y/S}) \to \Omega^1_{X/S}$, which, having in mind that

$$\mathbf{V}(f^*(\Omega^1_{Y/S})) = \mathbf{V}(\Omega^1_{Y/S}) \times_Y X \quad (\mathbf{II}, 1.7.11),$$

gives us an X-morphism $T_{X/S}(f) : T_{X/S} \to T_{Y/S} \times_Y X$. If $g : Y \to Z$ is a second S-morphism, we have $T_{X/S}(g \circ f) = (T_{Y/S}(g) \times 1_X) \circ T_{X/S}(f)$ (**0**, 20.5.4.1).

It follows from (16.5.10.1) and from (**II**, 1.7.11) that for every base change $g : S' \to S$ we have a canonical isomorphism

(16.5.12.2) $$T_{X'/S'} \simeq T_{X/S} \times_S S' = T_{X/S} \times_X X'.$$

(16.5.13). For every point $x \in X$, we define the *tangent space of X at the point x* (relative to S) to be the set of points in the fibre $T_{X/S} \times_X \mathrm{Spec}(k(x))$ that are *rational over* $k(x)$, that is, the set

(16.5.13.1) $$T_{X/S}(x) = \mathrm{Hom}_{k(x)}(\Omega^1_{X/S} \otimes_{\mathscr{O}_x} k(x), k(x)),$$

which is the *dual* of the $k(x)$-vector space $\Omega^1_{\mathscr{O}_x/\mathscr{O}_s}/\mathfrak{m}_x \cdot \Omega^1_{\mathscr{O}_x/\mathscr{O}_s}$. When $\Omega^1_{X/S}$ is an $\mathscr{O}_X$-module of *finite type*, then $T_{X/S}(x)$ is a vector space of finite rank over $k(x)$, and for every base change $g : S \to S'$, **IV-4** | 31
and every point $x' \in X' = X \times_S S'$ over x, we have a canonical isomorphism

(16.5.13.2) $$T_{X'/S'}(x') \simeq T_{X/S} \otimes_{k(x)} k(x').$$

If x is *rational over* $k(s)$, where $s = f(x)$ (so that $k(s) \to k(x)$ is an isomorphism), it follows from (16.4.13) that we have a canonical isomorphism

$$T_{X/S}(x) = T_{X_s/k(s)}(x) = \mathrm{Hom}_{k(s)}(\mathfrak{m}'_x/\mathfrak{m}'^2_x, k(x)), \tag{16.5.13.3}$$

where $\mathfrak{m}'_x$ is the maximal ideal of $\mathscr{O}_{X_s,x} = \mathscr{O}_{X,x}/\mathfrak{m}_s\mathscr{O}_{X,x}$. In the case where S is the spectrum of a field k, we recover the definition of the Zariski tangent space of a point $x \in X$ *rational over* k, as the dual of $\mathfrak{m}_x/\mathfrak{m}_x^2$.

Let Y be a second S-prescheme and let $g : Y \to X$ be an S-morphism; then we have a canonical homomorphism of $\mathscr{O}_Y$-modules (16.4.19)

$$g_{Y/X/S} : g^*(\Omega^1_{X/S}) \longrightarrow \Omega^1_{Y/S}. \tag{16.5.13.4}$$

Now note that if $y \in Y$ and $x = g(y)$, then we have

$$g^*(\Omega^1_{X/S}) \otimes_{\mathscr{O}_Y} k(y) = (\Omega^1_{X/S} \otimes_{\mathscr{O}_X} k(x)) \otimes_{k(x)} k(y)$$

and consequently, if $\Omega^1_{X/S}$ is an $\mathscr{O}_X$-module *of finite type*, then we can identify

$$\mathrm{Hom}_{k(y)}(g^*(\Omega^1_{X/S}) \otimes_{\mathscr{O}_Y} k(y), k(y))$$

with $T_{X/S}(x) \otimes_{k(x)} k(y)$. We therefore deduce from the homomorphism (16.5.13.4) a homomorphism of *$k(y)$-vector spaces*

$$T_y(g) : T_{Y/S}(y) \longrightarrow T_{X/S}(x) \otimes_{k(x)} k(y) \tag{16.5.13.5}$$

called the *linear map tangent to g at the point y*. When y is *rational over* $k(s)$, we can identify $k(s)$, $k(y)$, and $k(x)$, and $T_y(g)$ is then a homomorphism of $k(s)$-vector spaces $T_{Y/S}(y) \to T_{X/S}(x)$; also note that in this case, $g^*(\Omega^1_{X/S}) \otimes_{\mathscr{O}_Y} k(y)$ is identified with $\Omega^1_{X/S} \otimes_{\mathscr{O}_X} k(x)$, and the above homomorphism is therefore defined without any finiteness conditions on $\Omega^1_{X/S}$ and it is none other than the homomorphism $T_{Y/S}(g)$ (16.5.12) restricted to the fibre of $T_{Y/S}$ at the point y.

(16.5.14). The interpretation of derivations of an A-algebra B to a B-module L, given in (**0**, 20.1.1), translates to the language of preschemes in the following way.

Consider two morphisms of preschemes $f : X \to S$, $g : Y \to S$, and a closed subprescheme Y_0 of Y defined by a *zero-square* ideal $\mathscr{J}$ of $\mathscr{O}_Y$ (so that Y and Y_0 have the *same underlying subspace*). Suppose we are given an S-morphism $u_0 : Y_0 \to X$, so that we have a commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & Y_0 \arrow[l, "u_0"'] \arrow[d, "j"] \\ S & Y \arrow[l, "g"] \arrow[ul, dashed, "u"'] \end{tikzcd} \tag{16.5.14.1}$$

and we suggest looking for an *S-morphism* $u : Y \to X$ such that $u_0 = u \circ j$ (in other words, if it is possible to complete the diagram above by the dotted arrow u, keeping it *commutative*). **IV-4** | 32

For that, consider an affine open $U = \mathrm{Spec}(C)$ of Y; its inverse image $j^{-1}(U)$ is the affine open $U_0 = \mathrm{Spec}(C/\mathfrak{L})$, where $\mathfrak{L} = \Gamma(U, \mathscr{J})$, a *zero-square* ideal in C; suppose that U is small enough so that $u_0(U_0)$ is contained in an affine open $V = \mathrm{Spec}(B)$ of X and that $g(U) = f(u_0(U_0))$ is contained in an affine open $W = \mathrm{Spec}(A)$ of S, so that B and C are A-algebras and $u_0|U_0$ corresponds to an A-homomorphism ψ from B to $C/\mathfrak{L}$; Let $P(U_0)$ be the set of restrictions $u|U$ of the sought homomorphisms, which corresponds canonically to A-homomorphisms of algebras $\phi : B \to C$ such that the *composite* $B \xrightarrow{\phi} C \to C/\mathfrak{L}$ *is equal to* ψ. So we know (**0**, 20.1.1) that the set of such homomorphisms is either empty or of the form $\phi_1 + \mathrm{Der}_A(B, \mathfrak{L})$; when $P(U_0)$ is not empty, the additive group $\mathrm{Der}_A(B, \mathfrak{L})$ acts by addition on $P(U_0)$, which is therefore an *affine space* for the additive group $\mathrm{Der}_A(B, \mathfrak{L})$ (or even a *principal homogeneous space* (or *torsor*) *under* $\mathrm{Der}_A(B, \mathfrak{L})$).

Now notice that, since $\mathfrak{L}$ is equipped with a B-module structure via ψ, we have an *isomorphism* $v \mapsto v \circ d_{B/A}$ of $\mathrm{Hom}_B(\Omega^1_{B/A}, \mathfrak{L})$ onto $\mathrm{Der}_A(B, \mathfrak{L})$ (**0**, 20.4.8). Besides, as $\mathfrak{L}$ is square-zero, therefore a $(C/\mathfrak{L})$-module, every B-homomorphism $v : \Omega^1_{B/A} \to \mathfrak{L}$ can be considered as a $(C/\mathfrak{L})$-homomorphism $\Omega^1_{B/A} \otimes_B (C/\mathfrak{L}) \to \mathfrak{L}$. As $\mathscr{I}$ is square-zero, it can be considered as a quasi-coherent

$\mathscr{O}_{Y_0}$-module; let's introduce the $\mathscr{O}_{Y_0}$-module

$$\mathscr{G} = \mathscr{H}om(u_0^*(\Omega^1_{X/S}), \mathscr{I}); \tag{16.5.14.2}$$

it follows from the fact that $\Omega^1_{B/A} = \Gamma(V, \Omega^1_{X/S})$ (16.3.7) that we can write $\mathrm{Der}_A(B, \mathfrak{L}) = \Gamma(U_0, \mathscr{G})$.

As $P(U_0)$ is defined as a set of S-morphisms $U \to X$, it is clear that $U_0 \mapsto P(U_0)$ is a *sheaf of sets* $\mathscr{P}$ on Y_0. We can use this fact to prove that the map $h : \Gamma(U_0, \mathscr{G}) \times P(U_0) \to P(U_0)$ defining the torsor structure on $P(U_0)$ is independent of choice of V and W and also that, if $U' \subset U$ is a second affine open of Y, U'_0 its inverse image in Y_0, then the diagram

$$\begin{tikzcd} \Gamma(U_0, \mathscr{G}) \times P(U_0) \arrow[r, "h"] \arrow[d] & P(U_0) \arrow[d] \\ \Gamma(U'_0, \mathscr{G}) \times P(U'_0) \arrow[r, "h'"'] & P(U'_0) \end{tikzcd} \tag{16.5.14.3}$$

is commutative (the vertical arrows being the restrictions). In light of the above remark, we reduce to proving the commutativity of the above diagram when h is defined as such from affine opens V, W and h' from affine opens $V' \subset V$ and $W' \subset W$. But because of the preceding description of h, this follows from the commutativity of the diagram (**0**, 20.5.4.2). **IV-4** | 33

The mapping $\Gamma(U_0, \mathscr{G}) \times P(U_0) \to P(U_0)$ therefore define a homomorphism of *sheaf of sets* $m : \mathscr{G} \times \mathscr{P} \times \mathscr{P}$ such that, for all open sets U_0 for which $\Gamma(U_0, \mathscr{P}) \neq \varnothing$, $m_{U_0} : \Gamma(U_0, \mathscr{G}) \times \Gamma(U_0, \mathscr{P}) \to \Gamma(U_0, \mathscr{P})$ is an external law defining in $\Gamma(U_0, \mathscr{P})$ a torsor structure for the group $\Gamma(U_0, \mathscr{G})$.

(16.5.15). In general, when we are given a sheaf of sets $\mathscr{P}$ over a topological space Z, a sheaf of groups $\mathscr{G}$ (not necessarily commutative), and a homomorphism of sheaves of sets $m : \mathscr{G} \times \mathscr{P} \to \mathscr{P}$ such that, for every open $U \subset Z$ such that $\Gamma(U, \mathscr{P}) \neq \varnothing$, $m_U : \Gamma(U, \mathscr{G}) \times \Gamma(U, \mathscr{P}) \to \Gamma(U, \mathscr{P})$ makes $\Gamma(U, \mathscr{P})$ a *torsor* under the group $\Gamma(U, \mathscr{G})$, then we say that $\mathscr{P}$ is a *pseudo-torsor* (or *formally principal homogeneous sheaf*) under the sheaf of groups $\mathscr{G}$. We say that $\mathscr{P}$ is a *torsor* (or *principal homogeneous sheaf*) under $\mathscr{G}$ [3] if in addition $\Gamma(U, \mathscr{P}) \neq \varnothing$ for every open $U \neq \varnothing$ in a suitable basis for the topology of Z.

For the general theory of torsors, we refer to [**eAG64**]; we will limit ourselves to recalling the canonical correspondence between isomorphism classes of torsors (for a *given* $\mathscr{G}$) and elements from the cohomology set $\mathrm{H}^1(Z, \mathscr{G})$. Consider a torsor $\mathscr{P}$ under $\mathscr{G}$ and an open cover (U_λ) of Z such that $\Gamma(U_\lambda, \mathscr{P}) \neq \varnothing$ for every λ; denote by p_λ an element of $\Gamma(U_\lambda, \mathscr{P})$. For every pair of indices λ, μ such that $U_\lambda \cap U_\mu \neq \varnothing$, there then exists a unique element $\gamma_{\lambda\mu}$ of $\Gamma(U_\lambda \cap U_\mu, \mathscr{G})$ such that $\gamma_{\lambda\mu} \cdot (p_\mu | U_\lambda \cap U_\mu) = p_\lambda | U_\lambda \cap U_\mu$; in addition, if λ, μ, ν are three indices such that $U_\lambda \cap U_\mu \cap U_\nu \neq \varnothing$, then the restrictions $\gamma'_{\lambda\mu}$, $\gamma'_{\mu\nu}$, $\gamma'_{\lambda\nu}$ of $\gamma_{\lambda\mu}$, $\gamma_{\mu\nu}$, $\gamma_{\lambda\nu}$ to $U_\lambda \cap U_\mu \cap U_\nu$ satisfy the condition $\gamma'_{\lambda\nu} = \gamma'_{\lambda\mu}\gamma'_{\mu\nu}$; in other words, $(\lambda, \mu) \mapsto \gamma_{\lambda\mu}$ is a 1-*cocycle* of the cover (U_λ) with values in $\mathscr{G}$. If, for every λ, p'_λ is a second element of $\Gamma(U_\lambda, \mathscr{P})$, then there exists a unique element $\beta_\lambda \in \Gamma(U_\lambda, \mathscr{G})$ such that $p'_\lambda = \beta_\lambda \cdot p_\lambda$, and the 1-cocycle $(\gamma'_{\lambda\mu})$ corresponding to the family (p'_λ) is given by $\gamma'_{\lambda\mu} = \beta_\lambda \gamma_{\lambda\mu} \beta_\mu^{-1}$, that is, it is *cohomologous* to $\gamma_{\lambda\mu}$. Conversely, the data of a 1-cocycle $(\gamma_{\lambda\mu})$ defines, for every pair (λ, μ), an automorphism $\theta_{\lambda\mu}$ of the sheaf *of sets* $\mathscr{G} | U_\lambda \cap U_\mu$, namely the right translation by $\gamma_{\lambda\mu}$, and the fact that it is a cocycle shows that we can *glue* the sheaves of sets $\mathscr{G} | U_\lambda$ via the automorphisms $\theta_{\lambda\mu}$ (**0**, 3.3.1); we thus obtain a torsor under $\mathscr{G}$, denoted $\mathscr{P}$, and if we take for p_λ the unit section over U_λ, then the corresponding 1-cocycle is none other than the given 1-cocycle $(\gamma_{\lambda\mu})$; in addition, if we replace $(\gamma_{\lambda\mu})$ by a 1-cocycle $\gamma'_{\lambda\mu} = \beta_\lambda \gamma_{\lambda\mu} \beta_\mu^{-1}$ cohomologous to it, then we check immediately that the torsor obtained is isomorphic to $\mathscr{P}$.

In particular, if $(\gamma_{\lambda\mu})$ is a 1-*coboundary*, in other words of the form $\gamma_{\lambda\mu} = \beta_\lambda \beta_\mu^{-1}$, then the torsor $\mathscr{P}$ obtained is *isomorphic to* $\mathscr{G}$ (considered as a torsor under itself by left translations); we say in this case that $\mathscr{P}$ is *trivial*, and the converse is evident.

In particular, it follows from (**III**, 1.3.1) that we have:

Proposition (16.5.16). — *Let Z be an affine scheme, $\mathscr{G}$ a quasi-coherent $\mathscr{O}_Z$-module; then every torsor over $\mathscr{G}$ is trivial.*

[3] [Trans.] This is nowadays more commonly called a $\mathscr{G}$-torsor rather then a torsor *under* $\mathscr{G}$.

Returning to the problem considered in (16.5.13), we thus obtain: **IV-4 | 34**

Proposition (16.5.17). — *Let X, Y be two S-preschemes, Y_0 a closed subprescheme of Y defined by a quasi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_Y$ such that $\mathscr{I}^2 = 0$, $j : Y_0 \to Y$ the canonical injection. Let $u_0 : Y_0 \to X$ be an S-morphism, and $\mathscr{P}$ the sheaf of sets on Y such that, for every open U of Y, $\Gamma(U, \mathscr{P})$ is the set of S-morphisms $u : U \to X$ such that $u_0|U_0 = u \circ (j|U_0)$, where $U_0 = j^{-1}(U)$. Then there exists on $\mathscr{P}$ the structure of a pseudo-torsor over the $\mathscr{O}_{Y_0}$-module $\mathscr{G} = \mathscr{H}om_{\mathscr{O}_{Y_0}}(u_0^*(\Omega^1_{X/S}), \mathscr{I})$.*

In particular:

Corollary (16.5.18). — *With the notation of (16.5.16), suppose that Y is affine and $\Omega^1_{X/S}$ is of finite presentation; if there is a open cover (U_α) of Y, and, for every index α, an S-morphism $v_\alpha : U_\alpha \to X$ such that, if $U^0_\alpha = j^{-1}(U_\alpha)$, we have $v_\alpha \circ (j|U^0_\alpha) = u_0|U^0_\alpha$, then there is an S-morphism $u : Y \to X$ such that $u \circ j = u_0$.*

PROOF. Indeed, $\mathscr{G}$ is a quasi-coherent $\mathscr{O}_{Y_0}$-module (**I**, 1.3.12); by (16.5.16) and the fact that Y_0 is then affine, the sheaf $\mathscr{P}$, which is by hypothesis a torsor over $\mathscr{G}$, and not only a pseudo-torsor, is *trivial*; but if w is an isomorphism from $\mathscr{G}$ to $\mathscr{P}$ (as it is a torsor over $\mathscr{G}$), the image under w of the zero section of $\mathscr{G}$ is the S-morphism we want. □

16.6. Sheaf of p-differentials and exterior differentials.

(16.6.1). Let $f : X \to S$ be a morphism of preschemes. We define the *sheaf of p-differentials of X relative to S* (p integer) to be the *p'th exterior power* (**0**, 4.1.5) of the $\mathscr{O}_X$-module $\Omega^1_{X/S}$, denoted by

$$\Omega^p_{X/S} = \bigwedge^p(\Omega^1_{X/S}). \tag{16.6.1.1}$$

So we have $\Omega^0_{X/S} = \mathscr{O}_X$, and $\Omega^p_{X/S} = 0$ for $p < 0$; the $\Omega^p_{X/S}$ are the homogeneous components of the exterior algebra of $\Omega^1_{X/S}$

$$\Omega^\bullet_{X/S} = \bigwedge(\Omega^1_{X/S}) = \bigoplus_{p \in \mathbf{Z}} \bigwedge^p(\Omega^1_{X/S}), \tag{16.6.1.2}$$

which is therefore a quasi-coherent graded anti-commutative $\mathscr{O}_X$-algebra whose elements of degree 1 are square-zero. For every affine U of X, we have $\Gamma(U, \Omega^\bullet_{X/S}) = \bigwedge(\Gamma(U, \Omega^1_{X/S}))$, where $\Gamma(U, \Omega^1_{X/S})$ is considered as a $\Gamma(U, \mathscr{O}_X)$-module.

When $S = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ are affines, B being then an A-algebra, we have (**0**, 4.1.5) $\Omega^p_{X/S} = (\Omega^p_{B/A})^\sim$, by putting $\Omega^p_{B/A} = \bigwedge^p \Omega^1_{B/A}$.

Theorem (16.6.2). — *There is one and only one endomorphism d of the sheaf of additive groups $\Omega^\bullet_{X/S}$ with the following properties:*

(i) $d \circ d = 0$.
(ii) *For every open set U of X and every section $f \in \Gamma(U, \mathscr{O}_X)$ we have $df = d_{X/S}f$.* **IV-4 | 35**
(iii) *For every open set U of X, every pair of integers p, q and every pair of sections $\omega'_p \in \Gamma(U, \Omega^p_{X/S})$, $\omega''_q \in \Gamma(U, \Omega^q_{X/S})$, we have*

$$d(\omega'_p \wedge \omega''_q) = (d\omega'_p) \wedge \omega''_q + (-1)^p \omega'_p \wedge d\omega''_q. \tag{16.6.2.1}$$

Also, d is an endomorphism of graded $\psi^*(\mathscr{O}_X)$-modules of degree $+1$.

PROOF. Suppose that we have proved the existence of an endomorphism d. For every affine open U of X, every section of $\Omega^p_{X/S}$ over U is (because of (ii)) a linear combination if a finite number of elements of the form $g(df_1 \wedge df_2 \wedge \cdots \wedge df_p)$, where g and the f_i are sections of $\mathscr{O}_X$ over U (**0**, 20.4.7). The conditions (i) and (iii) then show, by induction on p, that we necessarily have

$$d(g(df_1 \wedge df_2 \wedge \cdots \wedge df_p)) = dg \wedge df_1 \wedge df_2 \wedge \cdots \wedge df_p. \tag{16.6.2.2}$$

This therefore proves the *uniqueness* of d and the last claim of the theorem. By virtue of this uniqueness property, to show the existence of d, we can restrict ourselves to the case where $S = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ are affines. Now (Bourbaki, *Alg.*, chap. III, 3rd ed., §10) to define an

A-derivation D of degree $+1$ of an exterior algebra $\bigwedge(M)$ (where M is a B-module and an A-algebra), such derivation taking its values in a *graded anti-commutative* A-algebra $C = \bigoplus_{n=0}^{\infty} C_n$, whose elements of degree 1 are square-zero, it suffices to give arbitrarily an A-derivation D_0 of B in C_1 and an A-homomorphism D_1 of M in C_2; then it exists one and only one A-anti-derivation D of $\Lambda(M)$ in C coinciding with D_0 in B and D_1 in M.

In the present case, D_0 is necessarily equal to $d_{B/A}$ by (ii); we reduce to seeing, having (16.6.2.2) in mind, that there is an A-homomorphism u of $\Omega^1_{B/A}$ in $\Omega^2_{B/A}$ such that

(16.6.2.3) $$u(g.df) = dg \wedge df$$

for whichever f, g in A; it suffices to show that there is an A-homomorphism $v : B \otimes_A \Omega^1_{B/A} \to \Omega^2_{B/A}$ such that

(16.6.2.4) $$v(g.\omega) = dg \wedge \omega$$

for $g \in B$ and $\omega \in \Omega^1_{B/A}$. Finally, since $\Omega^1_{B/A} = \mathfrak{I}/\mathfrak{I}^2$ (where $\mathfrak{I} = \mathfrak{I}_{B/A}$ is the kernel of the canonical homomorphism $B \times_A B \to B$) and that $\Omega^1_{B/A}$ is generated by elements of the form $g.df$, it is enough to define an A-homomorphism $w : B \otimes_A (B \otimes_A B) \to \Omega^2_{B/A}$ such that

(16.6.2.5) $$w(g' \otimes g \otimes f) = dg' \wedge (g.df)$$

and such that w is zero on the image of $B \otimes A\mathfrak{I}^2$. Or, since the second member of (16.6.2.5) is A-trilinear in g', g and f, the existence of w verifying (16.6.2.5) is immediate. Since, on the other hand, $\mathfrak{I}$ is generated by elements of the form $1 \otimes x - x \otimes 1$ ($x \in B$), we reduce to checking that when $z = (1 \otimes x - x \otimes 1)(1 \otimes y - y \otimes 1)$ we have $w(g' \otimes z) = 0$. Or, since $z = 1 \otimes (xy) + (xy) \otimes 1 - x \otimes y - y \otimes x$, the formula (16.6.2.4) shows that it is enough to see that we have $d(xy) - x.dy - y.dx = 0$, **IV-4** | 36
which is to say that d is a derivation.

It remains to be shown that d verifies the condition (i). Now, the square of an anti-derivation is a derivation (Bourbaki, *loc. cit.*), and since $\Omega^\bullet_{B/A}$ is generated by $\Omega^1_{B/A}$ as a B-algebra, it is enough to verify that $d(dz) = 0$ for $z \in B$ and $x \in \Omega^1_{B/A}$; in the first case, this follows from the formula (16.6.2.3) when $g = 1$; for the second, we can restrict ourselves to the case where $z = g.df$ with f, g in B, and then we have, because of (16.6.2.1) and (16.6.2.3),

$$d(d(g.df)) = d(dg \wedge df) = (d(dg)) \wedge (df) - (dg) \wedge (d(df)) = 0.$$

□

Definition (16.6.3). — The anti-derivation d defined in (16.6.2) (also denoted by $d_{X/S}$) is called the *exterior differential* on X (relative to S).

Proposition (16.6.4). — *For every base change $g : S' \to S$, if we put $X' = X \times_S S'$, the canonical morphism*

(16.6.4.1) $$\Omega^\bullet_{X/S} \otimes_S S' \longrightarrow \Omega^\bullet_{X'/S'}$$

deduced from the isomorphism (16.5.10.1) is bijective. Also, if s is a section of $\Omega^\bullet_{X/S}$ over an open set U of X, $s \otimes 1$ its inverse image, section of $\Omega^\bullet_{X'/S'}$ over the inverse image U' of U in X', we have $d_{X'/S'}(s \otimes 1) = d_{X/S}(s) \otimes 1$.

PROOF. The first claim is immediate, the formation of the exterior algebra of a module commutes with extending the scalar ring. To prove the second, we can, because of (16.6.2.2), restrict ourselves to the case where $s \in \Gamma(U, \mathscr{O}_X)$, and in this case the claim has already been proven (16.4.3.7). □

(16.6.5). Suppose that $\Omega^1_{X/S}$ is an locally free $\mathscr{O}_X$-module of rank n in a point x, so that we have n sections $s_i \in \Gamma(U, \mathscr{O}_X)$ such that the ds_i form a basis for the $\Gamma(U, \mathscr{O}_X)$-module $\Gamma(U, \Omega^1_{X/S})$ (16.5.8). Then, for every integer $p \geqslant 1$, the p-differentials $ds_{i_1} \wedge ds_{i_2} \wedge \cdots \wedge ds_{i_p}$ (for $i_1 \leqslant i_2 \leqslant \ldots \leqslant i_p$ elements of $[1, n]$) form a basis of $\binom{n}{p}$ elements of $\Gamma(U, \Omega^p_{X/S})$ over $\Gamma(U, \mathscr{O}_X)$. Also the formula (16.6.2.2) shows that for every section $g \in \Gamma(U, \mathscr{O}_X)$, we have

(16.6.5.1) $$d(g.ds_{i_1} \wedge ds_{i_2} \wedge \cdots \wedge ds_{i_p}) = \sum_k (-1)^r \frac{\partial g}{\partial s_k} ds_{i_1} \wedge \cdots \wedge ds_{i_r} \wedge ds_k \wedge ds_{i_{r+1}} \wedge \cdots \wedge ds_{i_p}$$

where, in the second member, k varies in the set of the $n - p$ indexes different from the i_h, i_r being the biggest index $< k$.

We note that the relation $d(dg) = 0$ for every section $g \in \Gamma(U, \mathscr{O}_X)$ expresses itself in the form

$$D_i(D_j g) = D_j(D_i g) \quad \text{for } i \neq j;$$

in other words, the derivations D_i defined in (16.5.7) commute with each other.

16.7. The $\mathscr{P}^n_{X/S}(\mathscr{F})$.

(16.7.1). Let $f : X \to S$ be a morphism of preschemes, $\mathscr{F}$ an $\mathscr{O}_X$-module. We denote by $X^{(n)}_{\Delta_f}$ the *n'th infinitesimal neighbourhood* of X via the diagonal morphism $\Delta_f : X \to X \times_S X$, by $h_n : X^{(n)}_{\Delta_f} \to X \times_S X$ IV-4 | 37
the canonical morphism (16.1.2), and consider the two composite morphisms

$$p^{(n)}_1 : X^{(n)}_{\Delta_f} \xrightarrow{h_n} X \times_S X \xrightarrow{p_1} X, \quad p^{(n)}_2 : X^{(n)}_{\Delta_f} \xrightarrow{h_n} X \times_S X \xrightarrow{p_2} X$$

so that, by definition, $p^{(n)}_1$ corresponds to the homomorphism of sheaves of rings $\mathscr{O}_X \to \mathscr{P}^n_{X/S}$ which we have chosen to define the $\mathscr{O}_X$-algebra structure on $\mathscr{P}^n_{X/S}$ (16.3.5), and $p^{(n)}_2$ to the homomorphism of sheaves of rings $d^n_{X/S} : \mathscr{O}_X \to \mathscr{P}^n_{X/S}$ (16.3.6). Since $X^{(n)}_{\Delta_f}$ and X have the same underlying subspace, we can write

(16.7.1.1) $$\mathscr{P}^n_{X/S} = (p^{(n)}_1)_*((p^{(n)}_2)^*(\mathscr{O}_X)).$$

More generally, we define

(16.7.1.2) $$\mathscr{P}^n_{X/S}(\mathscr{F}) = (p^{(n)}_1)_*((p^{(n)}_2)^*(\mathscr{F})).$$

so that $\mathscr{P}^n_{X/S} = \mathscr{P}^n_{X/S}(\mathscr{O}_X)$; by definition, $\mathscr{P}^n_{X/S}(\mathscr{F})$ is an $\mathscr{O}_X$-module.

(16.7.2). If we come back to the definition of the inverse image of modules on ringed spaces (**0**, 4.3.1) and having in mind that $X^{(n)}_{\Delta_f}$ and X have the same underlying space, we see that we can write the definition (16.7.1.2) in the form

(16.7.2.1) $$\mathscr{P}^n_{X/S}(\mathscr{F}) = \mathscr{P}^n_{X/S} \otimes_{\mathscr{O}_X} \mathscr{F},$$

but where you have to be careful that, in the interpretation of the symbol $\otimes$, $\mathscr{P}^n_{X/S}$ is endowed with the structure of $\mathscr{O}_X$-module defined by *the homomorphism of sheaves of rings* $d^n_{X/S} : \mathscr{O}_X \to \mathscr{P}^n_{X/S}$. It follows immediately from such formula (or directly from (16.7.1.2)) that $\mathscr{P}^n_{X/S}(\mathscr{F})$ is canonically equipped with a $\mathscr{P}^n_{X/S}$-module structure.

Proposition (16.7.3). — (i) *The functor $\mathscr{F} \mapsto \mathscr{P}^n_{X/S}(\mathscr{F})$ from the category of $\mathscr{O}_X$-modules to the category of $\mathscr{P}^n_{X/S}$-modules is right exact, and commutes with arbitrary inductive limits; it is exact when $\mathscr{P}^n_{X/S}$ is flat.*

(ii) *If $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module (resp. of finite type, resp. of finite presentation), then $\mathscr{P}^n_{X/S}(\mathscr{F})$ is quasi-coherent (resp. of finite type, resp. of finite presentation).*

PROOF. The claims from (i) follow from (16.7.2.1) and the consideration of the symmetry of $\mathscr{P}^n_{X/S}$ (16.3.4). The claims from (ii) follow from the right exactness of the functor $\mathscr{F} \mapsto \mathscr{P}^n_{X/S}(\mathscr{F})$. □

(16.7.4). The two structures of $\mathscr{O}_X$-module on $\mathscr{P}^n_{X/S}$ define in $\mathscr{P}^n_{X/S}(\mathscr{F})$ two structures of $\mathscr{O}_X$-modules, which happen to be permutable, and therefore an $\mathscr{O}_X$-bimodule structure. It is convenient to denote the structure coming from the structure homomorphism $\mathscr{O}_X \to \mathscr{P}^n_{X/S}$ (chosen in (16.3.5)) on the *left* and the one coming from the homomorphism $d^n_{X/S} : \mathscr{O}_X \to \mathscr{P}^n_{X/S}$ on the *right*. On other words, for every open U of X, and every triplet $a \in \Gamma(U, \mathscr{O}_X)$, $b \in \Gamma(U, \mathscr{P}^n_{X/S})$, $t \in \Gamma(U, \mathscr{F})$, we have by definition

(16.7.4.1) $$a(b \otimes t) = (ab) \otimes t, \quad (b \otimes t)a = (b.d^n a) \otimes t = b \otimes (at) = (d^n a).(b \otimes t).$$

The $\mathscr{O}_X$-module structure coming from the definition (16.7.1.2) is therefore, under these conventions, the *left* $\mathscr{O}_X$-module structure. If $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module, then the same is true for $\mathscr{P}^n_{X/S}(\mathscr{F})$ IV-4 | 38
for any one of its $\mathscr{O}_X$-module structures. If also $\mathscr{F}$ is of finite type (resp. of finite presentation) and $f : X \to S$ is locally of finite type (resp. locally of finite presentation), $\mathscr{P}^n_{X/S}(\mathscr{F})$ is (for any one of its

$\mathscr{O}_X$-module structures) of finite type (resp. of finite presentation), which is a consequence of (16.3.9) and (16.4.22).

(16.7.5). The definition (16.7.2.1) entails the existence of a homomorphism of sheaves of commutative groups

(16.7.5.1) $$d^n_{X/S,\mathscr{F}} : \mathscr{F} \longrightarrow \mathscr{P}^n_{X/S}(\mathscr{F}) \quad (\text{also denoted } d^n_{X/S})$$

such that, in the notations of (16.7.4), we have

(16.7.5.2) $$d^n_{X/S,\mathscr{F}}(t) = 1 \otimes t$$

and consequently, because of (16.7.4.1)

(16.7.5.3) $$d^n_{X/S,\mathscr{F}}(at) = (1 \otimes t)a = (d^n_{X/S,\mathscr{F}}(t)).a$$

(IV.16.7.5.4) $$d^n_{X/S,\mathscr{F}}(at) = (d^n_{X/S,\mathscr{F}}(a)).(1 \otimes t) = (d^n_{X/S,\mathscr{F}}(a))(d^n_{X/S,\mathscr{F}}(t)).$$

Therefore it is $\mathscr{O}_X$-linear for the structure of *right* $\mathscr{O}_X$-module on $\mathscr{P}^n_{X/S}(\mathscr{F})$, and *semilinear* (relative to the homomorphism σ (16.3.4)) for the *left* $\mathscr{O}_X$-module structure.

Proposition (16.7.6). — *The right $\mathscr{O}_X$-module $\mathscr{P}^n_{X/S}(\mathscr{F})$ is generated by the image of $\mathscr{F}$ by the homomorphism $d_{X/S,\mathscr{F}}$.*

PROOF. This is an immediate consequence of (16.7.5.3) and of the particular case $\mathscr{F} = \mathscr{O}_X$ (16.3.8). ☐

(16.7.7). The canonical homomorphisms of sheaves of rings

$$\phi_{nm} : \mathscr{P}^m_{X/S} \longrightarrow \mathscr{P}^n_{X/S}$$

for $n \leqslant m$ (16.1.2) define, because of (16.7.2.1), canonical homomorphisms

$$\mathscr{P}^m_{X/S}(\mathscr{F}) \longrightarrow \mathscr{P}^n_{X/S}(\mathscr{F}) \quad (n \leqslant m)$$

which are homomorphisms of $\mathscr{O}_X$-bimodules in light of (16.1.6) and (7.4.1); also we have commutative diagrams

$$\begin{tikzcd} \mathscr{P}^m_{X/S}(\mathscr{F}) \arrow[rr] & & \mathscr{P}^n_{X/S}(\mathscr{F}) \\ & \mathscr{F} \arrow[ul, "d^n_{X/S,\mathscr{F}}"] \arrow[ur, "d^n_{X/S,\mathscr{F}}"'] & \end{tikzcd}$$

We have therefore a projective system of $\mathscr{O}_X$-bimodules $(\mathscr{P}^n_{X/S}(\mathscr{F}))$, and we define

(16.7.7.1) $$\mathscr{P}^\infty_{X/S}(\mathscr{F}) = \varprojlim \mathscr{P}^n_{X/S}(\mathscr{F}).$$

Also, this shows that the homomorphisms (16.7.5.1) form a projective system of homomorphisms, and therefore define a canonical homomorphism

(16.7.7.2) $$d^\infty_{X/S,\mathscr{F}} : \mathscr{F} \to \mathscr{P}^\infty_{X/S}(\mathscr{F}).$$

IV-4 | 39

(16.7.8). Let $\mathscr{F}, \mathscr{G}$ be two $\mathscr{O}_X$-modules; it follows immediately from the definition (16.7.2.1) that we have a canonical isomorphism of $\mathscr{P}^n_{X/S}$-modules

(16.7.8.1) $$\mathscr{P}^n_{X/S}(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}) \simeq \mathscr{P}^n_{X/S}(\mathscr{F}) \otimes_{\mathscr{P}^n_{X/S}} \mathscr{P}^n_{X/S}(\mathscr{G})$$

(Bourbaki, *Alg.*, chap. II, 3$^{\text{rd}}$ed., §5, n.1, prop. 3).

We conclude in particular (or we see directly from the definition (16.7.2.1)) that if $\mathscr{F}$ has an $\mathscr{O}_X$-algebra structure (not necessarily associative), $\mathscr{P}^n_{X/S}(\mathscr{F})$ has a canonical $\mathscr{O}_X$-algebra structure; the latter is associative (resp. commutative, res. unital, resp. a Lie algebra) if $\mathscr{F}$ is so. Also the canonical homomorphisms $\mathscr{P}^m_{X/S}(\mathscr{F}) \to \mathscr{P}^n_{X/S}(\mathscr{F})$ for $n \leqslant m$ (16.7.7) are then algebra di-homomorphisms; similarly, (16.7.5.1) is then an $\mathscr{O}_X$-algebra homomorphisms when $\mathscr{P}^n_{X/S}(\mathscr{F})$ is equipped with the $\mathscr{O}_X$-algebra structure from its structure of *right* $\mathscr{O}_X$-module.

With the same notations, we equally have a canonical homomorphisms of $\mathscr{P}^n_{X/S}$-modules

(16.7.8.2) $$\mathscr{P}^n_{X/S}(\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})) \longrightarrow \mathscr{H}om_{\mathscr{P}^n_{X/S}}(\mathscr{P}^n_{X/S}(\mathscr{F}), \mathscr{P}^n_{X/S}(\mathscr{G}))$$

(Bourbaki, *Alg.*, chap. II, 3[rd]ed., §5, n.3), which is bijective when $\mathscr{P}^n_{X/S}$ is *locally free of finite type* (*loc. cit.*, prop. 7).

(16.7.9). Suppose we are in the situation described in (16.4.1); then from the canonical homomorphism $P^n(u)$ (16.4.3.3) we deduce immediately a canonical homomorphism of $\mathscr{O}_{X'}$-bimodules

$$u^*(\mathscr{P}^n_{X/S}(\mathscr{F})) \longrightarrow \mathscr{P}^n_{X'/S'}(u^*(\mathscr{F})). \tag{16.7.9.1}$$

We leave it to the reader to extend the properties seen in (16.4) in the case $\mathscr{F} = \mathscr{O}_X$.

Remark (16.7.10). — The definition of $\mathscr{P}^n_{X/S}(\mathscr{F})$ in the form (16.7.1.2) still makes sense when $\mathscr{F}$ is a sheaf of sets (the inverse image of a sheaf of sets by $p_2^{(n)}$ being defined in (**0**, 3.7.1)); a variant of this definition allows us to define the "jet schemes" (relatively to S) for any prescheme X.

16.8. Differential operators. [4]

Definition (16.8.1). — Let $f = (\psi, \theta) : X \to S$ be a morphism of preschemes, $\mathscr{F}, \mathscr{G}$ two $\mathscr{O}_X$-modules, n an integer $\geqslant 0$. We say that a morphism $D : \mathscr{F} \to \mathscr{G}$ of sheaves of additive groups is a differential operator of order $\leqslant n$ (relative to S) if there is a homomorphism of $\mathscr{O}_X$-modules $u : \mathscr{P}^n_{X/S}(\mathscr{F}) \to \mathscr{G}$ (where $\mathscr{P}^n_{X/S}(\mathscr{F})$ is equipped with the structure of left $\mathscr{O}_X$-modules (16.7.4)) such that we have $D = u \circ d^n_{X/S,\mathscr{F}}$.

It is clear, because of the existence of canonical morphisms

$$\mathscr{P}^m_{X/S}(\mathscr{F}) \longrightarrow \mathscr{P}^n_{X/S}(\mathscr{F})$$

for $n \leqslant m$ (16.7.7), that a differential operator of order $\leqslant n$ is a differential operator of order $\leqslant m$ **IV-4** | 40
for $n \leqslant m$. If $D : \mathscr{F} \to \mathscr{G}$ is a differential operator of order $\leqslant n$, then, for every open set U of X, $D|U : \mathscr{F}|U \to \mathscr{G}|U$ is a differential operator of order $\leqslant n$.

We say that a homomorphism $D : \mathscr{F} \to \mathscr{G}$ is a *differential operator* (relative to S) if, for every $x \in X$, there is an open neighbourhood U of x and an integer $n \geqslant 0$ such that $D|U : \mathscr{F}|U \to \mathscr{G}|U$ is a differential operator of order $\leqslant n$. The *order* of a differential operator is the least upper bound of all integers n so that D is a differential operator of order $\leqslant n$ (and therefore $+\infty$ if there is no such integer); such order is always finite if X is *quasi-compact*. The differential operator of order 0 are exactly the homomorphisms of $\mathscr{O}_X$-modules $\mathscr{F} \to \mathscr{G}$; the operators of order < 0 are *zero* by convention. For $n \geqslant 0$ a differential operator of order n is not in general a homomorphism of $\mathscr{O}_X$-modules but always a homomorphism of $\psi^*(\mathscr{O}_S)$-modules.

When $\mathscr{F} = \mathscr{O}_X$, a differential operator of order $\leqslant 1$ of $\mathscr{O}_X$ to $\mathscr{G}$ can be put in the form of $v + D$, where $v : \mathscr{O}_X \to \mathscr{G}$ is an $\mathscr{O}_X$-homomorphism, and D is an *S-derivation* (16.5.1) of $\mathscr{O}_X$ to $\mathscr{G}$: this results from the structure of $P_{B/A}$ (**0**, 20.4.8).

(16.8.2). To describe in a more precise manner a differential operator of order $\leqslant n$, $D : \mathscr{F} \to \mathscr{G}$, it suffices, for every open set U of X whose image in S is contained in an affine open set V, to characterize the homomorphism $D = D_U : \Gamma(U, \mathscr{F}) \to \Gamma(U, \mathscr{G})$. If we put $\Gamma(V, \mathscr{O}_S) = A$, $\Gamma(U, \mathscr{O}_X) = B$, so that B is an A-algebra, we have $\Gamma(U, \mathscr{P}^n_{X/S}(\mathscr{F})) = (B \otimes_A B)/\mathfrak{I}^{n+1}$, where we abbreviate $\mathfrak{I} = \mathfrak{I}_{B/A}$. Also put $M = \Gamma(U, \mathscr{F})$, $N = \Gamma(U, \mathscr{G})$; then the definition of D means that for every pair (U, V) satisfying the above, the A-homomorphism $D : M \to N$ factors through

$$M \longrightarrow ((B \otimes_A B)/\mathfrak{I}^{n+1}) \otimes_B M \xrightarrow{v} N$$

where the first arrow is the canonical morphism $t \mapsto 1 \otimes t$, and v is a B-homomorphism, the structure of B-module coming from the first factor (whereas we recall that in the formation of the tensor product over B, the structure of B-module on $(B \otimes_A B)/\mathfrak{I}^{n+1}$ comes from the second factor B). Note also that the B-module $((B \otimes_A B)/\mathfrak{I}^{n+1}) \otimes_B M$ is isomorphic to $(B \otimes_A M)/\mathfrak{I}^{n+1}(B \otimes_A M)$, where $(B \otimes_A M)$ is considered as a $(B \otimes_A B)$ module and its structure of B-module comes from $t \mapsto 1 \otimes t$ of B in $B \otimes_A B$. Let then D' be the B-homomorphism of $B \otimes_A M$ to N such that $D'(b \otimes t) = bD(t)$; then condition of factorization of D is to say that D' must be zero on the B-module $\mathfrak{I}^{n+1}(B \otimes_A M)$.

[4] For a more general formalism, see the exposé VII of [**eAG64**] (due to P. Gabriel).

(16.8.3). It is clear that the set of differential operators of order $\leqslant n$ from $\mathscr{F}$ to $\mathscr{G}$ forms an additive group, denoted by $\mathrm{Diff}^n_{X/S}(\mathscr{F},\mathscr{G})$; when $\mathscr{F} = \mathscr{G} = \mathscr{O}_X$, we also write $\mathrm{Diff}^n_{X/S}$ instead of $\mathrm{Diff}^n_{X/S}(\mathscr{O}_X,\mathscr{O}_X)$.

We have seen (16.8.1), that given two open sets $U \supset V$ of X, we have a restriction homomorphism

$$\mathrm{Diff}^n_{X/S}(\mathscr{F}|U,\mathscr{G}|U) \longrightarrow \mathrm{Diff}^n_{X/S}(\mathscr{F}|V,\mathscr{G}|V)$$

from which we deduce that $U \mapsto \mathrm{Diff}^n_{U/S}(\mathscr{F}|U,\mathscr{G}|U)$ is a presheaf of additive groups; in fact, it is **IV-4** | 41
actually a *sheaf*, since for an open set U varying in X, the homomorphisms $u \mapsto u \circ d^n_{U/S,\mathscr{F}|U}$ are *isomorphisms* of sheaves of additive groups

(16.8.3.1) $$\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{P}^n_{U/S}(\mathscr{F}|U),\mathscr{G}|U) \simeq \mathrm{Diff}^n_{U/S}(\mathscr{F}|U,\mathscr{G}|U),$$

because of the fact that the image of $\mathscr{F}$ by $d^n_{X/S,\mathscr{F}}$ generates $\mathscr{P}^n_{X/S}(\mathscr{F})$ (16.7.6). We denote this sheaf by $\mathscr{D}\!\mathit{iff}^n_{X/S}(\mathscr{F},\mathscr{G})$, and we have:

Proposition (16.8.4). — *The isomorphisms (16.8.3.1) define an isomorphism of sheaves of additive groups*

(18.4.1) $$\mathscr{H}\!\mathit{om}_{\mathscr{O}_X}(\mathscr{P}^n_{X/S}(\mathscr{F}),\mathscr{G}) \simeq \mathscr{D}\!\mathit{iff}^n_{X/S}(\mathscr{F},\mathscr{G}).$$

When $\mathscr{F} = \mathscr{G} = \mathscr{O}_X$, we also write $\mathscr{D}\!\mathit{iff}^n_{X/S}$ instead of $\mathscr{D}\!\mathit{iff}^n_{X/S}(\mathscr{O}_X,\mathscr{O}_X)$; it results from (16.6.4) that $\mathscr{D}\!\mathit{iff}^n_{X/S}$ is the *dual* of $\mathscr{P}^n_{X/S}$; so we also write $\langle t, D\rangle$ instead of $u(t)$ if t is a section of $\mathscr{P}^n_{X/S}$ over an open set and if u is a homomorphism from $\mathscr{P}^n_{X/S}$ to $\mathscr{O}_X$ corresponding to D.

(16.8.5). When $\mathscr{P}^n_{X/S}(\mathscr{F})$ has an $\mathscr{O}_X$-bimodule structure (16.7.4), we deduce canonicallly an $\mathscr{O}_X$-bimodule structure on $\mathscr{H}\!\mathit{om}_{\mathscr{O}_X}(\mathscr{P}^n_{X/S}(\mathscr{F}),\mathscr{G})$, and therefore on $\mathscr{D}\!\mathit{iff}^n_{X/S}(\mathscr{F},\mathscr{G})$ because of (16.8.4.1). More precisely, to the left $\mathscr{O}_X$-module structure on $\mathscr{P}^n_{X/S}(\mathscr{F})$ corresponds, because of (16.8.1), the left $\mathscr{O}_X$-module structure on $\mathscr{D}\!\mathit{iff}^n_{X/S}(\mathscr{F},\mathscr{G})$ explained as follows: for every open set U of X, every section $a \in \Gamma(U,\mathscr{O}_X)$ and every differential operator $D : \mathscr{F}|U \to \mathscr{G}|U$, aD is the differential operator which, for every section $t \in \Gamma(U,\mathscr{F})$, makes correspond the section

(16.8.5.1) $$(aD)(t) = a(D(t))$$

of $\Gamma(U,\mathscr{G})$. Similarly, the right $\mathscr{O}_X$-module structure on $\mathscr{D}\!\mathit{iff}^n_{X/S}(\mathscr{F},\mathscr{G})$ is made explicit as follows: under the same notations as above, Da is the operator which, to every $t \in \Gamma(U,\mathscr{F})$, makes correspond the section

(16.8.5.2) $$(Da)(t) = D(at).$$

Proposition (16.8.6). — *If $f : X \to S$ is a morphism locally of finite presentation, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module of finite presentation and $\mathscr{G}$ a quasi-coherent $\mathscr{O}_X$-module, then $\mathscr{D}\!\mathit{iff}^n_{X/S}(\mathscr{F},\mathscr{G})$ is a quasi-coherent $\mathscr{O}_X$-module for any of the structures defined in (16.8.5).*

PROOF. The proposition follows from the fact that, under these hypothesis, $\mathscr{P}^n_{X/S}$ is a quasi-coherent $\mathscr{O}_X$-module of finite presentation (16.7.4) and of (**I**, 1.3.12) ☐

(16.8.7). The set of differential operators (of unspecified order (16.8.1)) is denoted by $\mathrm{Diff}_{X/S}(\mathscr{F},\mathscr{G})$; we also see as in (16.8.3) that $U \mapsto \mathrm{Diff}_{U/S}(\mathscr{F}|U,\mathscr{G}|U)$ is a sheaf of additive groups, which we will denote by $\mathscr{D}\!\mathit{iff}_{X/S}(\mathscr{F},\mathscr{G})$. It is immediate that $\mathscr{D}\!\mathit{iff}_{X/S}(\mathscr{F},\mathscr{G})$ is the reunion of the increasing filtered family of its subsheaves $\mathscr{D}\!\mathit{iff}^n_{X/S}(\mathscr{F},\mathscr{G})$; if X is quasi-compact, $\mathrm{Diff}_{X/S}(\mathscr{F},\mathscr{G})$ is similarly **IV-4** | 42
the union of its subgroups $\mathrm{Diff}^n_{X/S}(\mathscr{F},\mathscr{G})$ (16.8.1). The $\mathscr{O}_X$-bimodule structure on the $\mathscr{D}\!\mathit{iff}^n_{X/S}(\mathscr{F},\mathscr{G})$ induce therefore an $\mathscr{O}_X$-bimodule structure on $\mathscr{D}\!\mathit{iff}_{X/S}(\mathscr{F},\mathscr{G})$, further explained in (16.8.5.1) and (16.8.5.2).

Note that, for $n \leqslant m$, we have a commutative diagram

(16.8.7.1) $$\begin{tikzcd} \mathscr{H}\!\mathit{om}_{\mathscr{O}_X}(\mathscr{P}^n_{X/S}(\mathscr{F}),\mathscr{G}) \arrow[r, "\sim"] \arrow[d] & \mathscr{D}\!\mathit{iff}^n_{X/S}(\mathscr{F},\mathscr{G}) \arrow[d] \\ \mathscr{H}\!\mathit{om}_{\mathscr{O}_X}(\mathscr{P}^m_{X/S}(\mathscr{F}),\mathscr{G}) \arrow[r, "\sim"] & \mathscr{D}\!\mathit{iff}^m_{X/S}(\mathscr{F},\mathscr{G}) \end{tikzcd}$$

where the horizontal arrows are the isomorphisms (16.8.4.1) and the horizontal arrow on the left comes from the canonical morphism $\mathscr{P}^m_{X/S}(\mathscr{F}) \to \mathscr{P}^n_{X/S}(\mathscr{F})$ (16.7.7). For every open set U of X, we then endow $\Gamma(U, \mathscr{P}^{\infty}_{X/S}(\mathscr{F})) = \varprojlim \Gamma(U, \mathscr{P}^{n}_{X/S}(\mathscr{F}))$ of the projective limit topology of the discrete topologies on $\Gamma(U, \mathscr{P}^{n}_{X/S}(\mathscr{F}))$, which defines on $\Gamma(U, \mathscr{P}^{\infty}_{X/S}(\mathscr{F}))$ a topological $\Gamma(U, \mathscr{O}_X)$-bimodule structure, so that $\mathscr{P}^{\infty}_{X/S}(\mathscr{F})$ shows itself as a sheaf valued in the category of topological commutative groups (**0**, 3.2.6). So [**God58**, II, 1.11], the limit of the inductive system of sheaves of commutative groups $(\mathscr{H}om_{\mathscr{O}_X}(\mathscr{P}^{n}_{X/S}(\mathscr{F}), \mathscr{G}))$ is precisely the sheaf of *continuous* germs of homomorphisms from $\mathscr{P}^{\infty}_{X/S}(\mathscr{F})$ to $\mathscr{G}$ (the latter equipped with the discrete topology): the continuous homomorphisms $\Gamma(U, \mathscr{P}^{\infty}_{X/S}(\mathscr{F}))$ into the discrete group $\mathscr{G}$ indeed correspond bijectively to the inductive systems of group homomorphisms $\Gamma(U, \mathscr{P}^{n}_{X/S}(\mathscr{F})) \to \Gamma(U, \mathscr{G})$. We can furthermore express (16.8.4) by saying there is a canonical isomorphism

$$\mathscr{H}om.\mathit{cont}_{\mathscr{O}_X}(\mathscr{P}^{\infty}_{X/S}(\mathscr{F}), \mathscr{G}) \simeq \mathscr{D}\!\mathit{iff}_{X/S}(\mathscr{F}, \mathscr{G})$$

where the first member denotes the sheaf of germs of continuous homomorphisms from $\mathscr{P}^{\infty}_{X/S}(\mathscr{F})$ to $\mathscr{G}$.

Proposition (16.8.8). — *Let $\mathscr{F}, \mathscr{G}$ be two $\mathscr{O}_X$-modules, $D : \mathscr{F} \to \mathscr{G}$ a homomorphism of $\psi^*(\mathscr{O}_S)$-modules, n an integer $\geqslant 0$. The following conditions are equivalent:*

(a) D is a differential operator of order $\leqslant n$.

(b) For all sections a of $\mathscr{O}_X$ over an open set U, the homomorphism $D_a : \mathscr{F}|U \to \mathscr{G}|U$ such that, for every section t of $\mathscr{F}$ over an open set $V \subset U$, we have

(16.8.8.1) $$D_a(t) = D(at) - aD(t)$$

is a differential operator of order $\leqslant n-1$.

(c) For every open set U of X, every family $(a_i)_{1 \leqslant i \leqslant n+1}$ of $n+1$ sections of $\mathscr{O}_X$ over U and every section t of $\mathscr{F}$ over U, we have the identity

(16.8.8.2) $$\sum_{H \subset I_{n+1}} (-1)^{\mathrm{Card}(H)} \Big(\prod_{i \in H} a_i\Big) D\Big(\Big(\prod_{i \notin H} a_i\Big) t\Big) = 0$$

(where I_{n+1} is the interval $1 \leqslant i \leqslant n+1$ of $\mathbb{N}$).

IV-4 | 43

Proof. Lets prove first the equivalence of (a) and (b). By definition, to prove that D is a differential operator of order $\leqslant n$, it suffices to prove so for the restriction $D|U : \mathscr{F}|U \to \mathscr{G}|U$ to any affine open set U of X, and on the other hand the property (c) is valid for every open set U if it is so on every affine open set. We can therefore restrict ourselves to the case where $S = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ are affines. Because of (16.8.2) (where we use the same notations), the condition (a) means that the A-homomorphism $D' : B \otimes_A M \to N$ such that $D'(b \otimes t) = bD(t)$ is zero on $\mathfrak{I}^{n+1}(B \otimes_A M)$, which is, because of (**0**, 20.4.4), equivalent to saying that D' is annihilates every element of the form

$$\Big(\prod_{i=1}^{n+1} (a_i \otimes 1 - 1 \otimes a_i)\Big).(1 \otimes t)$$

where $a_i \in B$ and $t \in M$. Now, this element can be written as $\sum_{H \subset I_{n+1}} (\prod_{i \in H} a_i) \otimes ((\prod_{i \notin H} a_i) t)$, and the image under D' of this element is the first member of (16.8.8.2), which proves the equivalence of (a) and (c).

Let us prove now the equivalence of (b) and (c). Let us reason by induction on n, the statement being trivial for $n = 0$. Writing a_{n+1} instead of a in the condition (b), we see, by the induction hypothesis, that condition (b) means that for every family $(a_i)_{1 \leqslant i \leqslant n}$ of n sections of $\mathscr{O}_X$ over U and every section t of $\mathscr{F}$ over U, that

$$\sum_{H' \subset I_{n+1}} (-1)^{\mathrm{Card}(H')} \Big(\prod_{i \in H'} a_i\Big) D_{a_{n+1}}\Big(\Big(\prod_{i \notin H'} a_i\Big) t\Big) = 0.$$

But if we replace on this relation $D_{a_{n+1}}$ by the definition (16.8.8.1), we check immediately that we have, up to sign, the first member of (16.8.8.2); from which we conclude. ∎

Proposition (16.8.9). — *If $D : \mathscr{F} \to \mathscr{G}$ is a differential operator of order $\leqslant n$, and $D' : \mathscr{G} \to \mathscr{H}$ a differential operator of order $\leqslant n'$, then $D' \circ D : \mathscr{F} \to \mathscr{H}$ is a differential operator of order $\leqslant n + n'$.*

PROOF. By hypothesis, we can write $D = u \circ d^n_{X/S,\mathscr{F}}$ and $D' = v \circ d^n_{X/S,\mathscr{G}}$, where $u : \mathscr{P}^n_{X/S} \otimes_{\mathscr{O}_X} \mathscr{F} \to \mathscr{G}$ and $v : \mathscr{P}^{n'}_{X/S} \otimes_{\mathscr{O}_X} \mathscr{G} \to \mathscr{H}$ are $\mathscr{O}_X$-homomorphisms. It all comes down to showing that the composite homomorphism of sheaves of additive groups

$$\mathscr{F} \xrightarrow{d^n_{X/S,\mathscr{F}}} \mathscr{P}^n_{X/S} \otimes_{\mathscr{O}_X} \mathscr{F} \xrightarrow{u} \mathscr{G} \xrightarrow{d^{n'}_{X/S,\mathscr{G}}} \mathscr{P}^{n'}_{X/S} \otimes_{\mathscr{O}_X} \mathscr{G}$$

factors as

$$\mathscr{F} \xrightarrow{d^{n+n'}_{X/S,\mathscr{F}}} \mathscr{P}^{n+n'}_{X/S} \otimes_{\mathscr{O}_X} \mathscr{F} \xrightarrow{w} \mathscr{P}^{n'}_{X/S} \otimes_{\mathscr{O}_X} \mathscr{G}$$

where w is an $\mathscr{O}_X$-homomorphism. It suffices to prove the

Lemma (16.8.9.1). — *There is one and only one $\mathscr{O}_X$-homomorphism*

(16.8.9.2) $$\delta : \mathscr{P}^{n+n'}_{X/S} \longrightarrow \mathscr{P}^{n'}_{X/S}(\mathscr{P}^n_{X/S}) = \mathscr{P}^{n'}_{X/S} \otimes_{\mathscr{O}_X} \mathscr{P}^n_{X/S}$$

making the following diagram commute **IV-4** | 44

(16.8.9.3)
$$\begin{tikzcd} \mathscr{O}_X \arrow[r, "d^{n+n'}_{X/S}"] \arrow[d, "d^n_{X/S}"'] & \mathscr{P}^{n+n'}_{X/S} \arrow[d, "\delta"] \\ \mathscr{P}^n_{X/S} \arrow[r, "d^{n'}_{X/S,\mathscr{P}^n_{X/S}}"'] & \mathscr{P}^{n'}_{X/S}(\mathscr{P}^n_{X/S}) \end{tikzcd}$$

We will then have, indeed, a commutative diagram deduced from (16.8.9.3) by tensorization with $\mathscr{F}$

$$\begin{tikzcd} \mathscr{F} \arrow[r, "d^{n+n'}_{X/S,\mathscr{F}}"] \arrow[d, "d^n_{X/S,\mathscr{F}}"'] & \mathscr{P}^{n+n'}_{X/S}(\mathscr{F}) \arrow[d, "\delta\otimes 1"] \\ \mathscr{P}^n_{X/S}(\mathscr{F}) \arrow[r, "d^{n'}_{X/S,\mathscr{P}^n_{X/S}(\mathscr{F})}"'] & \mathscr{P}^{n'}_{X/S}(\mathscr{P}^n_{X/S}((\mathscr{F}))) \end{tikzcd}$$

and on the other hand, we verify immediately the from definition (16.7.5) that the diagram

$$\begin{tikzcd} \mathscr{P}^n_{X/S}(\mathscr{F}) \arrow[r, "u"] \arrow[d, "d^{n'}_{X/S,\mathscr{P}^n_{X/S}(\mathscr{F})}"'] & \mathscr{G} \arrow[d, "d^{n'}_{X/S,\mathscr{G}}"] \\ \mathscr{P}^{n'}_{X/S}(\mathscr{P}^n_{X/S}((\mathscr{F}))) \arrow[r, "1\otimes u"'] & \mathscr{P}^{n'}_{X/S}(\mathscr{G}) \end{tikzcd}$$

is commutative. We finish the proof by taking w to be the composite $\mathscr{O}_X$-homomorphism

$$\mathscr{P}^{n+n'}_{X/S}(\mathscr{F}) \xrightarrow{\delta\otimes 1} \mathscr{P}^{n'}_{X/S}(\mathscr{P}^n_{X/S}(\mathscr{F})) \xrightarrow{1\otimes u} \mathscr{P}^{n'}_{X/S}(\mathscr{G}).$$

□

PROOF (16.8.9.3). It remains to prove the lemma (16.8.9.3). Considering (16.7.6), which proves the uniqueness of δ, we are brought back to the case where $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ are affines; letting $\mathfrak{I} = \mathfrak{I}_{B/A}$, it suffices to define a canonical homomorphism of B-modules

$$\phi : (B \otimes_A B)/\mathfrak{I}^{n+n'+1} \longrightarrow ((B \otimes_A B)/\mathfrak{I}^{n'+1}) \otimes_B ((B \otimes_A B)/\mathfrak{I}^{n+1})$$

the B-module structure of the two members coming from the first B factor; recall that on tensor product of the second member, $(B \otimes_A B)/\mathfrak{I}^{n'+1}$ must be considered as a right B-module by its second B factor , and $(B \otimes_A B)/\mathfrak{I}^{n+1}$ as a left B-module by its first B factor (16.7.2). It is the same to define a homomorphism of B-modules **IV-4** | 45

$$\phi_0 : B \otimes_A B \longrightarrow ((B \otimes_A B)/\mathfrak{I}^{n'+1}) \otimes_B ((B \otimes_A B)/\mathfrak{I}^{n+1})$$

and prove it is zero on $\mathfrak{I}^{n+n'+1}$. Now, we immediately define a homomorphism by the condition that

$$\phi_0(b \otimes b') = \pi_{n'}(b \otimes 1) \otimes \pi_n(1 \otimes b') \quad \text{for } b, b' \text{ in } B$$

under the notations of (16.3.7). Also, it is immediate that ϕ_0 is a homomorphism of *rings*. Now, we can write

$$\phi_0(b \otimes 1 - 1 \otimes b) = \pi_{n'}(b \otimes 1 - 1 \otimes b) \otimes \pi_n(1 \otimes 1) + \pi_{n'}(1 \otimes b) \otimes \pi_n(1 \otimes 1) - \pi_{n'}(1 \otimes 1) \otimes \pi_n(1 \otimes b)$$

and we have

$$\pi_{n'}(1 \otimes b) \otimes \pi_n(1 \otimes 1) = \pi_{n'}(1 \otimes 1)b \otimes \pi_n(1 \otimes 1) = \pi_{n'}(1 \otimes 1) \otimes b\pi_n(1 \otimes 1) = \pi_{n'}(1 \otimes 1) \otimes \pi_n(b \otimes 1)$$

from which, finally

$$\phi_0(b \otimes 1 - 1 \otimes b) = \pi_{n'}(b \otimes 1 - 1 \otimes b) \otimes \pi_n(1 \otimes 1) + \pi_{n'}(1 \otimes 1) \otimes \pi_n(b \otimes 1 - 1 \otimes b). \tag{16.8.9.4}$$

A product of $n + n' + 1$ of terms of the form (16.8.9.4) is therefore necessarily zero, because the same is true for the product of $n + 1$ terms of the form $\pi_n(b \otimes 1 - 1 \otimes b)$ and of $n' + 1$ terms of the form $\pi_{n'}(b \otimes 1 - 1 \otimes b)$. The conclusion therefore results from (**0**, 20.4.4). □

Corollary (16.8.10). — *The sheaf $\mathscr{D}\!\mathit{iff}_{X/S}(\mathscr{O}_X, \mathscr{O}_X)$ (also denoted $\mathscr{D}\!\mathit{iff}_{X/S}$) is canonically endowed with the structure of sheaf of rings, and the $\mathscr{D}\!\mathit{iff}^n_{X/S}$ form an increasing filtration compatible with such structure.*

In particular, $\mathscr{D}\!\mathit{iff}^0_{X/S}$ is a sheaf of subrings of $\mathscr{D}\!\mathit{iff}_{X/S}$, which is canonically identified with $\mathscr{O}_X$ (16.8.1). The formulas (16.8.5.1) and (16.8.5.2) show that the structure of $\mathscr{O}_X$-bimodule of $\mathscr{D}\!\mathit{iff}_{X/S}$ comes from the multiplication on the left and on the right by sections of $\mathscr{O}_X$ considered as a sheaf of subrings of $\mathscr{D}\!\mathit{iff}_{X/S}$.

Remarks (16.8.11). — (i) Suppose that $\mathscr{F} = \oplus_{\lambda \in L} \mathscr{F}_\lambda$; then it is clear (16.7.2.1) that $\mathscr{P}^n_{X/S}(\mathscr{F}) = \oplus_{\lambda \in L} \mathscr{P}^n_{X/S}(\mathscr{F}_\lambda)$; since the functor $U \mapsto \Gamma(U, \mathscr{F})$ commutes with the formation of arbitrary direct sums, $d^n_{X/S,\mathscr{F}}$ is the homomorphism whose restriction to each $\mathscr{F}_\lambda$ is $d^n_{X/S,\mathscr{F}_\lambda} : \mathscr{F}_\lambda \to \mathscr{P}^n_{X/S}(\mathscr{F}_\lambda)$; then we conclude immediately that we have

$$\mathrm{Diff}^n_{X/S}(\mathscr{F}, \mathscr{G}) = \prod_{\lambda \in L} \mathrm{Diff}^n_{X/S}(\mathscr{F}_\lambda, \mathscr{G}),$$

and therefore also (**0**, 3.2.6)

$$\mathscr{D}\!\mathit{iff}^n_{X/S}(\mathscr{F}, \mathscr{G}) = \prod_{\lambda \in L} \mathscr{D}\!\mathit{iff}^n_{X/S}(\mathscr{F}_\lambda, \mathscr{G}).$$

Moreover, if $\mathscr{G} = \prod_{\mu \in M} \mathscr{G}_\mu$ (**0**, 3.2.6), we have

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{P}^n_{X/S}(\mathscr{F}), \mathscr{G}) = \prod_{\mu \in M} \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{P}^n_{X/S}(\mathscr{F}), \mathscr{G}_\mu),$$

every homomorphism u from $\mathscr{P}^n_{X/S}(\mathscr{F})$ to $\mathscr{G}$ corresponds bijectively to the family of its **IV-4** | 46
composites $u_\mu : \mathscr{P}^n_{X/S}(\mathscr{F}) \to \mathscr{G} \to \mathscr{G}_\mu$. We have therefore

$$\mathrm{Diff}^n_{X/S}(\mathscr{F}, \mathscr{G}) = \prod_{\mu \in M} \mathrm{Diff}^n_{X/S}(\mathscr{F}, \mathscr{G}_\mu),$$

and consequently also

$$\mathscr{D}\!\mathit{iff}^n_{X/S}(\mathscr{F}, \mathscr{G}) = \prod_{\mu \in M} \mathscr{D}\!\mathit{iff}^n_{X/S}(\mathscr{F}, \mathscr{G}_\mu).$$

(ii) So far, we have hardly encountered differential operators $\mathscr{F} \to \mathscr{G}$ where $\mathscr{F}$ and $\mathscr{G}$ are not locally free of finite rank, in which case the structure is reduced locally, because of (i), to the case of the sheaf $\mathscr{D}\!\mathit{iff}_{X/S}$; the latter will be studied later (16.11) in a particular case.

16.9. Regular and quasi-regular immersions

Definition (16.9.1). — Let X be a ringed space. We say that an ideal $\mathscr{I}$ of $\mathscr{O}_X$ is regular (resp. quasi-regular) if, for every point $x \in \operatorname{Supp}(\mathscr{O}_X/\mathscr{I})$, there is an open neighbourhood of x in X and a regular sequence (**0**, 15.2.2) (resp. quasi-regular (**0**, 15.2.2)) of elements of $\Gamma(U, \mathscr{O}_X)$ which generates $\mathscr{I}|U$.

We say that a regular (resp. quasi-regular) sequence of sections of $\mathscr{O}_X$ over U which generates $\mathscr{I}|U$ is called a *regular system* (resp. *quasi-regular system*) of generators of $\mathscr{I}|U$.

Definition (16.9.2). — Let $j : Y \to X$ be an immersion of preschemes and let U be an open set such that $j(Y) \subset U$ and that j is a closed immersion of Y in U. We say that j is regular (resp. quasi-regular) if the closed subprescheme $j(Y)$ of U associated to j is defined by a regular (resp. quasi-regular) ideal of $\mathscr{O}_U$ (condition independent of the choice of U).

We say that a subprescheme Y of a prescheme X is *regularly immersed* (resp. *quasi-regularly immersed*) if the canonical injection $j : Y \to X$ is a regular immersion (resp. quasi-regular immersion). If Y is a closed subprescheme and $\mathscr{I}$ is the ideal of $\mathscr{O}_X$ that defines Y, it is the same as asking $\mathscr{I}$ to be *regular* (resp. *quasi-regular*).

For example, if A is an *integral* ring, f and element $\neq 0$ of A, the closed subprescheme $V(f)$ of $\operatorname{Spec}(A)$ (isomorphic to $\operatorname{Spec}(A/(f))$) is *regularly immersed* in $\operatorname{Spec}(A)$.

Every regular ideal is quasi-regular (**0**, 15.2.2); every regular immersion is quasi-regular (cf. (16.9.11) for a converse).

Proposition (16.9.3). — *Let X be a ringed space, $\mathscr{I}$ an ideal of $\mathscr{O}_X$, $(f_i)_{1 \leqslant i \leqslant m}$ a finite sequence of sections of $\mathscr{O}_X$ over X generating $\mathscr{I}$. For the (f_i) to be a quasi-regular sequence* (**0**, 15.2.2), *it is necessary and sufficient that the following conditions are verified:*

(i) *The canonical images of f_i in $\mathscr{I}/\mathscr{I}^2$ form a basis of this $\mathscr{O}_X/\mathscr{I}$-module.*
(ii) *The canonical surjective homomorphism* (16.1.2.2)

$$\mathbf{S}^{\bullet}_{\mathscr{O}_X/\mathscr{I}}(\mathscr{I}/\mathscr{I}^2) \longrightarrow \mathscr{G}r^{\bullet}_{\mathscr{I}}(\mathscr{O}_X)$$

is bijective.

Also, if this is true, then every sequence $(f'_i)_{1 \leqslant i \leqslant n}$ of n sections of $\mathscr{I}$ over X which generates $\mathscr{I}$ is quasi-regular.

IV-4 | 47

Proof. The two conditions stated above only translate the ones in (**0**, 15.2.2), given the definition of the canonical homomorphisms (16.2.1.1). The last claim follows from the fact that, if a module M over a commutative ring A admits a basis of n elements, every system of n generators of M is a basis (Bourbaki, *Alg. comm.*, chap. II, §3, cor. 5 of th. 1). □

Corollary (16.9.4). — *Let X be a locally ringed space, $\mathscr{I}$ an ideal of $\mathscr{O}_X$. For $\mathscr{I}$ to be quasi-regular, it is necessary and sufficient that it verifies the following conditions:*

(i) *$\mathscr{I}$ is of finite type.*
(ii) *$\mathscr{I}/\mathscr{I}^2$ is a locally free $\mathscr{O}_X/\mathscr{I}$-module.*
(iii) *The canonical homomorphism*

$$\mathbf{S}^{\bullet}_{\mathscr{O}_X/\mathscr{I}}(\mathscr{I}/\mathscr{I}^2) \longrightarrow \mathscr{G}r^{\bullet}_{\mathscr{I}}(\mathscr{O}_X) \tag{16.9.4.1}$$

is bijective.

Proof. The necessity of the conditions follows immediately from (16.9.3). To see that they are sufficient, it is enough to show, because of (16.9.3), that if, in a point $x \in \operatorname{Supp}(\mathscr{O}_X/\mathscr{I})$, there is a neighbourhood U of x in X and n-sections f_i $(1 \leqslant i \leqslant n)$ of $\mathscr{I}$ over U such that the image in $\mathscr{I}/\mathscr{I}^2$ form a basis of $(\mathscr{I}/\mathscr{I}^2)|U$ over $(\mathscr{O}_X/\mathscr{I})|U$, so there will be a neighbourhood $V \subset U$ of x such that the $f_i|V$ generate $\mathscr{I}|V$. Now, by hypothesis, we have $\mathscr{I}_x \neq \mathscr{O}_x$, so that $\mathscr{I}_x$ is contained in the maximal ideal of $\mathscr{O}_X$; since $\mathscr{I}_x$ is an $\mathscr{O}_x$ module of finite type and the classes of $(f_i)_x$ in $\mathscr{I}_x/\mathscr{I}_x^2$ generate this $(\mathscr{O}_x/\mathscr{I}_x)$-module, Nakayama's lemma shows that the $(f_i)_x$ generate $\mathscr{I}_x$. Since $\mathscr{I}$ is of finite type, we conclude by (**0**, 5.2.2). □

Corollary (16.9.5). — *Let X be a locally ringed space, $\mathscr{I}$ a quasi-regular ideal of $\mathscr{O}_X$, $(f_i)_{1\leqslant i\leqslant n}$ a sequence of sections of $\mathscr{I}$ over X, x a point of* $\operatorname{Supp}(\mathscr{O}_X/\mathscr{I})$. *The following conditions are equivalent:*

- *(a) There is a neighbourhood of x in X such that $f_i|U$ form a quasi-regular sequence of elements $\Gamma(U,\mathscr{O}_X)$ generating $\mathscr{I}|U$.*
- *(b) The $(f_i)_x$ form a system of generators of $\mathscr{I}_x$ whose size is as small as possible.*
- *(b′) The $(f_i)_x$ form a minimal set of generators of $\mathscr{I}_x$.*
- *(c) If $\bar{f}_i$ is the canonical image of f_i in $\Gamma(X,\mathscr{I}/\mathscr{I}^2)$, the $(\bar{f}_i)_x$ form a basis for the $(\mathscr{O}_x/\mathscr{I}_x)$-module $\mathscr{I}_x/\mathscr{I}_x^2$.*

PROOF. By hypothesis, $\mathscr{O}_x$ is a local ring, $\mathscr{I}_x$ an ideal of finite type of $\mathscr{O}_x$ contained in the maximal ideal of $\mathscr{O}_x$; the equivalence of (b), (b′) and (c) results from Nakayama's lemma (Bourbaki, *Alg. comm.*, chap. II, §3, n°2, prop. 5). It is clear that (a) implies (c) because of (16.9.3); on the other hand, from (**0**, 5.2.2) it follows that if condition (c) is verified (and therefore so is (b)), there is a neighbourhood U of x in X such that $(\mathscr{I}/\mathscr{I}^2)|U$ has constant rank equal to n, and that the $f_i|U$ generate $\mathscr{I}|U$; it suffices now to apply the last assertion of (16.9.3) to U. □

Remarks (16.9.6). — (i) Under the general hypothesis of (16.9.5), for the sequence (f_i) to generate $\mathscr{I}$, it is not enough that the $(\bar{f}_i)_y$ form a basis of the $(\mathscr{O}_y/\mathscr{I}_y)$-module $(\mathscr{I}_y/\mathscr{I}_y^2)$ for all $y\in X$. We have an example by taking $X=\operatorname{Spec}(A)$, where A is a Dedekind ring, **IV-4** | 48
and $\mathscr{I}=\widetilde{\mathfrak{I}}$, where $\mathfrak{I}$ is a *non-principal* ideal of A; then indeed $\mathscr{I}_y/\mathscr{I}_y^2=0$ in every point y different from $x\in X$ corresponding to $\mathfrak{I}$, and $\mathscr{I}_x/\mathscr{I}_x^2$ has rank 1 over the field $\mathscr{O}_x/\mathscr{I}_x$; also, $\mathscr{I}$ is evidently a regular ideal.

(ii) In (16.9.5), one cannot replace "quasi-regular" by "regular", even when X is a prescheme (cf. (16.9.12)). Indeed, denote by B the ring of germs of infinitely differentiable functions on the point 0 on $\mathbb{R}$; it has a maximal ideal $\mathfrak{m}$ generated by the germ of t of the identity mapping to 0, and the intersection $\mathfrak{n}$ of the $\mathfrak{m}^k$ for $k>0$ is not reduced to 0. Now let A be the quotient ring $B[T]/\mathfrak{n}TB[T]$, and let f_1, f_2 be the canonical images in A of the elements t and T of $B[T]$. The sequence (f_1,f_2) is *regular* in A: indeed, f_1 is not a zero divisor in A, because the relation $tP[T]\in\mathfrak{n}TB[T]$, for a polynomial $P\in B[T]$, implies that the products of t by the coefficients of P belongs to the ideal $\mathfrak{n}$, and it results immediately that the coefficients are the same in $\mathfrak{n}$, so $P[T]\in\mathfrak{n}TB[T]$. Since B/tB is isomorphic to $\mathbb{R}$, A/f_1A is isomorphic to the ring of polynomials $\mathbb{R}[T]$, therefore integral, and the image of f_2 in A/f_1A, being equal to T, is not a zero divisor, so that our claim is true. However, f_2 is a zero divisor in A, since for every non-zero element $x\in\mathfrak{n}$, the image of x in A is $\neq 0$, but the image of xT is zero. We conclude that the sequence (f_2,f_1) is *not regular* in A; on the other hand, the ideal $\mathfrak{I}=f_1A+f_2A$ is distinct from A, so the conditions (b), (b′) and (c) of (16.9.5) do not imply the condition (a) when we replace "quasi-regular" by "regular".

(16.9.7). If $X=\operatorname{Spec}(A)$ is an affine scheme, we'll say that the ideal $\mathfrak{I}$ of A is *regular* (resp. *quasi-regular*) if the ideal $\mathscr{J}=\widetilde{\mathfrak{I}}$ of $\mathscr{O}_X$ is regular (resp. quasi-regular); we note that this notion is *local* and does not imply the existence of a *system of generators* of $\mathfrak{I}$ forming in A a regular (resp. quasi-regular) sequence as the example (16.9.5) shows; however this is true if A is *local* (16.9.5).

The proposition (16.9.4) can be translated in terms of quasi-regular immersions in the following manner:

Proposition (16.9.8). — *Let $j:Y\to X$ be a morphism of preschemes; for j to be a quasi-regular immersion, it is necessary and sufficient that j satisfies the following conditions:*

- *(i) j is an immersion locally of finite presentation.*
- *(ii) The conormal sheaf $\mathscr{G}r^1(j)=\mathscr{N}_{Y/X}$* (16.1.2) *is a locally free $\mathscr{O}_Y$-module.*
- *(iii) The canonical homomorphism*

$$\mathbf{S}^\bullet_{\mathscr{O}_Y}(\mathscr{G}r^1(j))\longrightarrow\mathscr{G}r^\bullet(j)$$

(16.1.2.2) *is bijective.*

PROOF. The problem being local on Y, we can restrict ourselves to the case where j is the canonical injection of a closed subprescheme Y of X, so the translation of (16.9.4) into (16.9.8) results from the description of $\mathscr{G}r^1(j)$ and $\mathscr{G}r^\bullet(j)$ in terms of the ideal $\mathscr{I}$ of $\mathscr{O}_X$ defining the subprescheme Y (16.1.3, (ii)). □

Corollary (16.9.9). — *Let Y be a prescheme, X an Y-prescheme, $j : Y \to X$ a Y-section of X, so that the n'th normal invariant $\mathscr{A}^{(n)}$ of j (16.1.2) is an augmented $\mathscr{O}_Y$-algebra (16.1.7); take $\mathscr{A}^{(\infty)} = \varprojlim \mathscr{A}^{(n)}$. For j to be a quasi-regular immersion it is necessary and sufficient that j is locally of finite presentation, and that every $y \in Y$ admits an affine open neighbourhood U of ring C such that $\mathscr{A}^{(\infty)}|U$ is isomorphic as an augmented $\mathscr{O}_U$-algebra to $\mathscr{O}_U[[T_1, \ldots, T_n]]$.*

PROOF. We can reduce to the case where j is a closed immersion by restricting to a small neighbourhood of y (see the argument on (16.4.11)), and therefore $\mathscr{O}_Y$ is identified with the quotient algebran $\mathscr{O}_X/\mathscr{I}$ and the canonical surjective homomorphism $\mathscr{O}_X \to \mathscr{O}_Y$ admits a right inverse (16.1.7). We can therefore suppose $X = \mathrm{Spec}(B)$, $Y = \mathrm{Spec}(A)$ are affines, B being then an augmented A-algebra, and the augmentation ideal $\mathfrak{I}$ being of finite type. Since $\mathscr{A}^{(n)}$ is identified with $(B/\mathfrak{I}^{n+1})^\sim$, the corollary follows from the equivalence of (b) and (c) in (**0**, 19.5.4) when $B/\mathfrak{I} = A$. □ **IV-4** | 49

We note that, on the affine case considered, the fact that j is an quasi-regular immersion is further equivalent, because of (**0**, 19.5.4), to saying that B is a *formally smooth* A-algebra for the $\mathfrak{I}$-preadic topology.

We also note that the condition that j is locally of finite presentation is always satisfied when $X \to Y$ is locally of finite type (1.4.3, (v)).

Proposition (16.9.10). — *Let X be a locally Noetherian prescheme, Y a subprescheme of X, $j : Y \to X$ the canonical injection, y a point of Y.*

(i) *For there to exist an open neighbourhood U of y in X such that the restriction $Y \cap U \to U$ of j be a regular immersion, it is necessary and sufficient that the kernel $\mathscr{I}_y$ of the surjective homomorphism $\mathscr{O}_{X,y} \to \mathscr{O}_{Y,y}$ is generated by a regular sequence of elements of $\mathscr{O}_{X,y}$.*

(ii) *For the immersion j to be regular, it is necessary and sufficient that it is quasi-regular.*

PROOF. (i) We can reduce to the case where Y is a subprescheme defined by a *coherent* ideal $\mathscr{I}$ of $\mathscr{O}_X$. The condition is clearly necessary. Conversely, if $\mathscr{I}_y$ is generated by a regular sequence $(s_i)_y$, where the s_i are sections of $\mathscr{I}$ over an open neighbourhood U of y in X, we can suppose that the s_i generate $\mathscr{I}|U$ (**0**, 5.5.2) and form a regular sequence (**0**, 15.2.4), from which the claim follows.

(ii) The fact that a quasi-regular immersion is regular follows from (i) and the identification of quasi-regular and regular sequences of $\mathscr{O}_{X,y}$, formed from elements of the maximal ideal (**0**, 15.1.11)

□

Corollary (16.9.11). — *Let X be a locally Noetherian prescheme; then every quasi-regular ideal of $\mathscr{O}_X$ is regular.*

Remarks (16.9.12). — (i) We note that a regular immersion is not in general a flat morphism, and therefore *a fortiori* neither are quasi-regular morphisms in the sense of (6.8.1).

(ii) Let A be a local Noetherian ring; it follows immediately from (16.9.4) and from (**0**, 17.1.1) that for A to be *regular*, it is necessary and sufficient that its maximal ideal $\mathfrak{m}$ is *quasi-regular* (or regular, which amounts to the same thing given that A is Noetherian). For an affine Noetherian scheme X to be *regular*, it is necessary and sufficient that for every closed point $x \in X$, the canonical injection $\mathrm{Spec}(k(x)) \to X$ to be a *regular* immersion.

Proposition (16.9.13). — *Let X be a locally Noetherian prescheme, Y a subprescheme of X, Y' a subprescheme of Y, such that the canonical injection $j : Y' \to Y$ is regular. Then the sequence of $\mathscr{O}_{Y'}$-modules*

$$0 \longrightarrow j^*(\mathscr{N}_{Y/X}) \longrightarrow \mathscr{N}_{Y'/X} \longrightarrow \mathscr{N}_{Y'/Y} \longrightarrow 0 \tag{16.9.13.1}$$

is exact; furthermore, for every $x \in X$, there is an open neighbourhood U of x such that the restrictions to U of the homomorphisms of (16.9.13.1) form a split exact sequence. **IV-4** | 50

Let us first prove the following lemma:

Lemma (16.9.13.2). — *Let A be a ring, $\mathfrak{J}$ an ideal of A, $A' = A/\mathfrak{J}$, $(f_i)_{1\leqslant i\leqslant r}$ a sequence of elements of A which is A'-regular, $\mathfrak{K} = \sum_i f_i A$, $\mathfrak{L} = \mathfrak{J} + \mathfrak{K}$, $\mathfrak{K}' = \sum_i f_i A'$, so that $C = A/\mathfrak{L}$ is isomorphic to $A'/\mathfrak{K}'$. For every integer $n > 0$, and every integer $N \geqslant n$, we have the relation*

(16.9.13.3) $$\mathfrak{J} \cap \mathfrak{K}^n = \mathfrak{J}\mathfrak{K}^n + \mathfrak{J}\mathfrak{K}^N.$$

Proof. It is clearly sufficient to prove that every element of the first is contained in the second, and, by induction on n, we reduce to the case $N = n + 1$. An element of the first member of (16.9.13.3), being in $\mathfrak{K}^n$, is written as $P(f_1, \ldots, f_r)$, where $P \in A[T_1, \ldots, T_r]$ is homogeneous of degree n. If f'_i is the canonical image of f_i in A', the hypothesis $P(f_1, \ldots, f_n) \in \mathfrak{J}$ means that $P(f'_1, \ldots, f'_n) = 0$. But $P(f'_1, \ldots, f'_n) \in \mathfrak{K}'^n$, so the canonical image of $P(f'_1, \ldots, f'_n)$ in $\mathfrak{K}'^n/\mathfrak{K}'^{n+1}$ is zero. Now the hypothesis that (f_i) is A'-regular implies that the canonical homomorphism $\mathbf{S}^n_C(\mathfrak{K}'/\mathfrak{K}'^2) \to \mathfrak{K}'^n/\mathfrak{K}'^{n+1}$ is bijective (**0**, 15.1.9); we conclude that the coefficients of P are in $\mathfrak{L} = \mathfrak{J} + \mathfrak{K}$. It follows immediately that $P(f_1, \ldots, f_n) \in \mathfrak{J}\mathfrak{K}^n + \mathfrak{K}^{n+1}$, and since $P(f_1, \ldots, f_n) \in \mathfrak{J}$, we finally have $P(f_1, \ldots, f_n) \in \mathfrak{J}\mathfrak{K}^n + \mathfrak{J}\mathfrak{K}^{n+1}$ which proves the lemma. □

By taking the quotient of both side of (16.9.13.3) by $\mathfrak{J}\mathfrak{K}^n$, we see that the relations (16.9.13.3) for $N \geqslant n$ entail

(16.9.13.4) $$(\mathfrak{J} \cap \mathfrak{K}^n)/\mathfrak{J}\mathfrak{K}^n \subset \bigcap_{N\geqslant n} \mathfrak{K}^N.(A/(\mathfrak{J}\mathfrak{K}^n)).$$

We deduce the

Corollary (16.9.13.5). — *Suppose that the hypothesis of (16.9.13.2) are verified and also that the ring A is Noetherian and $\mathfrak{K}$ is contained in the radical of A. Then for every integer $n > 0$,*

(16.9.13.6) $$\mathfrak{J} \cap \mathfrak{K}^n = \mathfrak{J}\mathfrak{K}^n.$$

Proof. Indeed, the second member of (16.9.13.4) is then zero, given that $A/\mathfrak{J}\mathfrak{K}^n$ is an A-module of finite type (Bourbaki, *Alg. comm.*, chap. III, §3, n°3, prop. 6). □

Taking in particular $n = 2$ in (16.9.13.6), and remarking that we have $\mathfrak{L}^2 = \mathfrak{J}^2 + \mathfrak{J}\mathfrak{K} + \mathfrak{K}^2 = \mathfrak{J}\mathfrak{L} + \mathfrak{K}^2$; since $\mathfrak{J}\mathfrak{L} \subset \mathfrak{L}^2$, we deduce that

$$\mathfrak{J} \cap \mathfrak{L}^2 = \mathfrak{J}\mathfrak{L} + (\mathfrak{J} \cap \mathfrak{K}^2) = \mathfrak{J}\mathfrak{L} + \mathfrak{J}\mathfrak{K}^2 = \mathfrak{J}\mathfrak{L},$$

in other words,

(16.9.13.7) $$\mathfrak{J} \cap \mathfrak{L}^2 = \mathfrak{J}\mathfrak{L},$$

which we can also express in saying that the canonical homomorphism

$$\mathfrak{J}/\mathfrak{J}\mathfrak{L} \longrightarrow (\mathfrak{J} + \mathfrak{J}^2)/\mathfrak{J}^2$$

is bijective.

Proof (16.9.13). Having demonstrated the lemmas, let us prove the first claim of (16.9.13): It is clearly enough to prove that the sequence of stalks of the sheaves appearing in (16.9.13.1), in a point $x \in Y'$, is exact. Now, if we take $A = \mathscr{O}_{X,x}$, we can write $\mathscr{O}_{Y,x} = A' = A/\mathfrak{J}$, where $\mathfrak{J}$ is an ideal contained in the maximal ideal of A, then $\mathscr{O}_{Y',x} = A'/\mathfrak{K}'$, where $\mathfrak{K}'$ is generated by an A'-*regular* sequence of elements of A', which themselves are images of elements from a A'-regular sequence of elements of A belonging to the maximal ideal of A. If $\mathfrak{K}$ is the ideal generated by the latter and $\mathfrak{L} = \mathfrak{J} + \mathfrak{K}$, we have $\mathscr{O}_{Y',x} = A/\mathfrak{L}$, and since we are in the situation of (16.9.13.5), the canonical homomorphism $\mathfrak{J}/\mathfrak{J}\mathfrak{L} \to (\mathfrak{J} + \mathfrak{J}^2)/\mathfrak{J}^2$ is bijective. But this shows that the sequence **IV-4 | 51**

$$0 \longrightarrow \mathfrak{J}/\mathfrak{J}\mathfrak{L} \longrightarrow \mathfrak{L}/\mathfrak{L}^2 \longrightarrow (\mathfrak{J}/\mathfrak{L})/(\mathfrak{J}/\mathfrak{L})^2 \longrightarrow 0$$

is exact (see the demonstration of (16.2.7)), and the modules making up this sequence are precisely the stalks in x of the sheaves of (16.9.13.1). The second claim follows from the fact that $\mathscr{N}_{X/Y}$ is a locally free $\mathscr{O}_{Y'}$-module (16.9.8) and Bourbaki, *Alg.*, chap. II, 3rd ed., §1, n°11, prop. 21. □

16.10. Differentially smooth morphisms.

Definition (16.10.1). — We say that a morphism of preschemes $f : X \to S$ is differentially smooth (or that X is differentially smooth over S) if it satisfies the following conditions:

(i) $\Omega^1_{X/S}$ is a locally projective $\mathscr{O}_X$-module, which is to say that every point $x \in X$ admits an affine open neighbourhood U such that $\Gamma(U, \Omega^1_{X/S})$ is a projective $\Gamma(U, \mathscr{O}_X)$-module (not necessarily of finite type).

(ii) The canonical morphism (16.3.1.1)

$$\mathbf{S}^{\bullet}_{\mathscr{O}_X}(\Omega^1_{X/S}) \longrightarrow \mathscr{G}r_{\bullet}(\mathscr{P}_{X/S})$$

is bijective.

In particular, if $\Omega^1_{X/S}$ is locally free of finite rank, the $\mathscr{P}^n_{X/S}$ are locally free of finite rank $\mathscr{O}_X$-modules (being extensions of such modules).

We say that f is *differentially smooth at a point* $x \in X$ if there is an open neighbourhood U of x in X such that $f|U$ is differentially smooth.

We will see later (17.12.4) that a smooth morphism is differentially smooth, which justifies the terminology; but the converse is not true; indeed, a *monomorphism* $f : X \to S$ is differentially smooth, since $\Omega^1_{X/S} = 0$ because of (**I**, 5.3.8), and consequently the surjective homomorphism (16.3.1.1) is clearly bijective; however a monomorphism is not even necessarily flat, neither *a fortiori* smooth. Lets limit ourselves to proving the following proposition:

Proposition (16.10.2). — *Let A be a ring, B a formally smooth A-algebra for the discrete topology* (**0**, 19.3.1)*. Then* $\operatorname{Spec}(B)$ *is differentially smooth over* $\operatorname{Spec}(A)$*.*

Proof. Indeed, $B \otimes_A B$ is then (with the discrete topology) a formally smooth B-algebra (by on or the other canonical homomorphisms $b \mapsto b \otimes 1$, $b \mapsto 1 \otimes b$ of B to $B \otimes_A B$) (**0**, 19.3.5, (iii)); **IV-4** | 52
therefore $B \otimes_A B$ is a formally smooth A-algebra with the discrete topology (**0**, 19.3.5, (ii)). Taking $\mathfrak{I} = \mathfrak{I}_{B/A}$, it follows that $B \otimes_A B$ is a formally smooth A-algebra for the $\mathfrak{I}$-preadic topology (**0**, 19.3.8); since by hypothesis $B = (B \otimes_A B)/\mathfrak{I}$ is a formally smooth A-algebra for the discrete topology, the proposition follows from the equivalence of (a) and (b) in (**0**, 19.5.4). ☐

Proposition (16.10.3). — *For a morphism $f : X \to S$ to be differentially smooth, it is necessary and sufficient that for every $x \in X$, there is an affine open neighbourhood of x, of ring A, such that $\Gamma(U, \mathscr{P}^{\infty}_{X/S})$ is an augmented topological A-algebra isomorphic to the completion $\widehat{B}$, where $B = \mathbf{S}_A(V)$, V being a projective A-module and B being endowed with the B^+-preadic topology (where B^+ is the augmentation ideal). If $\Omega^1_{X/S}$ is locally free of finite rank, we can replace $\widehat{B}$ with the ring of formal series $A[[T_1, \dots, T_n]]$.*

Proof. The notion of a differentially smooth morphism is clearly local on X, so we reduce to the case where $S = \operatorname{Spec}(B)$, $X = \operatorname{Spec}(C)$. Consider $C \otimes_B C$ as a C-algebra (by the first factor); take $\mathfrak{I} = \mathfrak{I}_{C/B}$ and endow C with the $\mathfrak{I}$-preadic topology; we can apply to the topological C algebra $C \otimes_B C$ and to the ideal $\mathfrak{I}$ of $C \otimes_B C$ the equivalence of (b) and (c) of (**0**, 19.5.4), given that $(C \otimes_B C)/\mathfrak{I} = C$ is clearly a formally smooth C-algebra for the discrete topologies. The topology of $\Gamma(U, \mathscr{P}^{\infty}_{X/S})$ is clearly the projective limit topology of this ring (16.1.11). ☐

We note that the integer n of the proposition (16.10.3) is the *rank* of $\Omega^1_{X/S}$ in the point x. We shall see (17.13.5) that when f is differentially smooth and locally of finite type, n is equal to the dimension of the fiber $f^{-1}(f(x))$.

Proposition (16.10.4). — *Let $f : X \to S$, $g : S' \to S$ be two morphisms, and take $X' = X \times_S S'$, $f' = f_{(S')} : X' \to S'$.*

(i) *If f is differentially smooth, the same is true for f'.*

(ii) *Conversely, if g is faithfully flat and quasi-compact, and if f' is differentially smooth and $\Omega^1_{X'/S'}$ is a finite type $\mathscr{O}_{X'}$-module, f is differentially smooth and $\Omega^1_{X/S}$ is a finite type $\mathscr{O}_X$-module.*

Proof. Indeed, if f is differentially smooth, the $\mathscr{G}r_n(\mathscr{P}^n_{X/S})$ are *flat* $\mathscr{O}_X$-modules; therefore by (16.4.6), the homomorphism $\mathscr{G}r_n(\mathscr{P}^n_{X/S}) \otimes_{\mathscr{O}_X} \mathscr{O}_{X'} \to \mathscr{G}r_n(\mathscr{P}^n_{X'/S'})$ are bijective for every n, because of the commutative of the diagram (16.2.1.3), it follows from the definition (16.10.1) that f' is differentially smooth. On the other hand, if g is faithfully flat and quasi-compact, it follows also from (16.4.6) that $\mathscr{G}r_n(\mathscr{P}^n_{X/S}) \otimes_{\mathscr{O}_X} \mathscr{O}_{X'} \to \mathscr{G}r_n(\mathscr{P}^n_{X'/S'})$ is bijective for every n. Suppose also that f' is differentially smooth and $\Omega^1_{X'/S'}$ is of finite rank. Since the canonical projection $X' \to X$ is a faithfully flat and quasi-compact morphism, it results first from (2.5.2) that $\Omega^1_{X/S}$ is an $\mathscr{O}_X$-module locally free of finite rank, then from (2.2.7) that the canonical homomorphism (16.3.1.1) is bijective, and therefore f is differentially smooth. □

Proposition (16.10.5). — *For a morphism locally of finite type $f : X \to S$ to be differentially smooth, it is necessary and sufficient that the diagonal immersion $\Delta_f : X \to X \times_S X$ to be quasi-regular.*

IV-4 | 53

PROOF. Being a local problem, we can reduce to the case where S and X are affines, and therefore the diagonal subprescheme of $X \times_S X$ is closed. The hypothesis that f is locally of finite type implies that Δ_f is locally of finite presentation (**I**, 4.3.1), therefore the diagonal prescheme of $X \times_S X$ is defined by an ideal $\mathscr{I}$ of finite type, and $\Omega^1_{X/S} = \mathscr{I}/\mathscr{I}^2$ is an $\mathscr{O}_X$-module of finite type. The proposition is now immediate from the comparison of the conditions of (16.10.1) and (16.9.4). □

Remark (16.10.6). — Let $f : X \to S$ be a morphism such that the $\mathscr{O}_X$-module $\Omega^1_{X/S}$ is locally free of finite rank. It results from (**0**, 20.4.7) that every $x \in X$ has an open neighbourhood such that there is a finite family $(z_\lambda)_{\lambda \in L}$ of sections of $\mathscr{O}_X$ over U for which $(dz_\lambda)_{\lambda \in L}$ forms a *basis* of the $\Gamma(U, \mathscr{O}_X)$-module $\Gamma(U, \Omega^1_{X/S})$.

16.11. Differential operators on a differentially smooth S-prescheme

(16.11.1). Let $f : X \to S$ be a morphism, U an open set of X, $(z_\lambda)_{\lambda \in L}$ a family of sections of $\mathscr{O}_X$ over U such that the dz_λ form system of generators of $\Omega^1_{X/S}|U = \Omega^1_{U/S}$. Let m be an integer or the symbol ∞, and take, for every λ

(16.11.1.1) $$\zeta_\lambda = \delta z_\lambda = d^m z_\lambda - z_\lambda \in \Gamma(U, \mathscr{P}^m_{X/S}).$$

We will also use the usual notations from analysis; for every $\mathbf{p} = (p_\lambda) \in \mathbb{N}^{(L)}$ (so that $p_\lambda = 0$ except for a finite number of indices), we put

(16.11.1.2) $$|\mathbf{p}| = \sum_\lambda p_\lambda, \quad \mathbf{p} = \prod_\lambda (p_\lambda!)$$

(16.11.1.3) $$\binom{\mathbf{p}}{\mathbf{q}} = \mathbf{p}!/(\mathbf{q}!(\mathbf{p}-\mathbf{q})!), \quad \text{for } \mathbf{p}, \mathbf{q} \text{ in } \mathbb{N}^{(L)}, \mathbf{q} \leqslant \mathbf{p}$$

and we adopt the convention that $\binom{\mathbf{p}}{\mathbf{q}} = 0$ when $\mathbf{q} \not\leqslant \mathbf{p}$,

(16.11.1.4) $$\mathbf{z}^{\mathbf{p}} = \prod_\lambda (z_\lambda)^{p_\lambda}, \quad \boldsymbol{\zeta}^{\mathbf{p}} = \prod_\lambda (\zeta_\lambda)^{p_\lambda}.$$

We therefore have, with this notation

(16.11.1.5) $$d^m(\mathbf{z}^{\mathbf{p}}) = (d^m(\mathbf{z}))^{\mathbf{p}} = (\boldsymbol{\zeta} + \mathbf{z})^{\mathbf{p}} = \sum_{\mathbf{q} \leqslant \mathbf{p}} \binom{\mathbf{p}}{\mathbf{q}} \mathbf{z}^{\mathbf{p}-\mathbf{q}} \boldsymbol{\zeta}^{\mathbf{q}}$$

(16.11.1.6) $$\boldsymbol{\zeta}^{\mathbf{p}} = (d^m \mathbf{z} - \mathbf{z})^{\mathbf{p}} = \sum_{\mathbf{q} \leqslant \mathbf{p}} (-1)^{|\mathbf{p}-\mathbf{q}|} \binom{\mathbf{p}}{\mathbf{q}} \mathbf{z}^{\mathbf{p}-\mathbf{q}} d^m(\mathbf{z}^{\mathbf{q}}).$$

Since the dz_λ generate $\Omega^1_{X/S}$, and are the images of δz_λ, and as the canonical morphism (16.3.1.1) is surjective, we conclude that for finite m, the δz_λ generate the $\mathscr{O}_U$-algebra $\mathscr{P}^n_{U/S}$ (Bourbaki, *Alg. comm.*, chap. III, §2, n°8, cor. 2 du th. 1). Therefore the $\boldsymbol{\zeta}^{\mathbf{p}}$ (for $|\mathbf{p}| \leqslant m$) generate the $\mathscr{O}_U$-module $\mathscr{P}^n_{U/S}$. A differential operator $D \in \mathrm{Diff}^m_{U/S}$ is consequently entirely determined by the values of $\langle \boldsymbol{\zeta}^{\mathbf{p}}, D \rangle$for $|\mathbf{p}| \leqslant m$, or, which amounts to the same by (16.11.1.5) and (16.11.1.6), by the values of the **IV-4** | 54
$\langle d^m(\mathbf{z}), D \rangle = D(\mathbf{z}^{\mathbf{p}})$ for $|\mathbf{p}| \leqslant m$; more precisely, it follows from (16.11.5) that we have

(16.11.1.7) $$D(\mathbf{z}^{\mathbf{p}}) = \langle d^m(\mathbf{z}^{\mathbf{p}}), D \rangle = \sum_{\mathbf{q} \leqslant \mathbf{p}} \binom{\mathbf{p}}{\mathbf{q}} \langle \boldsymbol{\zeta}^{\mathbf{q}}, D \rangle \mathbf{z}^{\mathbf{p}-\mathbf{q}}.$$

Theorem (16.11.2). — *Let $f : X \to S$ be a morphism, U an open set of X, $(z_\lambda)_{\lambda\in L}$ a family of sections of $\mathscr{O}_X$ over U such that the family $(dz_\lambda)_{\lambda\in L}$ generates $\Omega^1_{X/S}|U = \Omega^1_{U/S}$. The following conditions are equivalent:*

(a) $f|U$ is differentially smooth and (dz_λ) is a basis of the $\mathscr{O}_U$-module $\Omega^1_{U/S}$.

(b) There is a family $(D_{\mathbf{p}})_{\mathbf{p}\in\mathbb{N}^{(L)}}$ of differential operators of $\mathscr{O}_U$ to itself verifying the conditions

(16.11.2.1) $$D_{\mathbf{p}}(\mathbf{z}^{\mathbf{q}}) = \binom{\mathbf{q}}{\mathbf{p}}\mathbf{z}^{\mathbf{q}-\mathbf{p}}, \quad (\mathbf{p}, \mathbf{q} \textit{ in } \mathbb{N}^{(L)}).$$

Also, when these conditions are verified, the family $(D_{\mathbf{p}})$ is uniquely determined by the conditions (16.11.2.1) and satisfies the relations

(16.11.2.2) $$D_{\mathbf{q}} \circ D_{\mathbf{p}} = D_{\mathbf{p}} \circ D_{\mathbf{q}} = \frac{(\mathbf{p}+\mathbf{q})!}{\mathbf{p}!\mathbf{q}!}D_{\mathbf{p}+\mathbf{q}} \quad (\mathbf{p}, \mathbf{q} \textit{ in } \mathbb{N}^{(L)}).$$

Finally, if L is finite, for every integer m, the $D_{\mathbf{p}}$ such that $|\mathbf{p}| \leqslant m$ form a basis of the $\mathscr{O}_U$-module $\mathscr{D}\!\mathit{iff}^m_{U/S}$, in other words, every differential operator of order $\leqslant m$ on U can be written uniquely as

$$D = \sum_{|\mathbf{p}|\leqslant m} a_{\mathbf{p}} D_{\mathbf{p}}$$

where the $a_{\mathbf{p}}$ are sections of $\mathscr{O}_X$ over U.

Proof. Note first that because of (16.11.1.6) and (16.11.1.5), we verify immediately that the conditions (16.11.2.1) are equivalent to

(16.11.2.3) $$\langle \boldsymbol{\zeta}^{\mathbf{p}}, D_{\mathbf{q}}\rangle = \delta_{\mathbf{pq}} \quad \text{(Kronecker's symbol)}.$$

The existence of the family $(D_{\mathbf{p}})$ verifying these conditions implies first (by taking $|\mathbf{p}| = 1$) that the dz_λ are linearly independent, and therefore form a basis of the $\mathscr{O}_U$-module $\Omega^1_{U/S}$. Then, for every integer $m \geqslant 1$, we deduce similarly from (16.11.2.3) that the $\boldsymbol{\zeta}^{\mathbf{p}}$ such that $|\mathbf{p}| \leqslant m$ are linearly independent; it follows that the canonical homomorphism (16.3.1.1) is injective, and therefore bijective, which proves that (b) implies (a). The converse follows immediately from the definition (16.10.1), the fact that $\boldsymbol{\zeta}^{\mathbf{p}}$ form a basis of $\mathscr{P}^n_{U/S}$ for $|\mathbf{p}| \leqslant m$ implies the existence and uniqueness of a family of homomorphisms $u_{\mathbf{q},m} : \mathscr{P}^n_{U/S} \to \mathscr{O}_U(|\mathbf{q}| \leqslant m)$ such that $\langle \zeta^{\mathbf{p}}, u_{\mathbf{q},m}\rangle = \delta\mathbf{p}, \mathbf{q}$ for $|\mathbf{p}| \leqslant m$, $|\mathbf{q}| \leqslant m$. For a given value of $\mathbf{q}$, the differential operators corresponding to $u_{\mathbf{q},m}$ for $m \geqslant |\mathbf{q}|$ are identified with the same operator $D_{\mathbf{q}}$. This proves that (a) implies (b), and also that the family $(D_{\mathbf{q}})$ is uniquely determined, and that, if L is finite, for $|\mathbf{p}| \leqslant m$, the $D_{\mathbf{p}}$ form a basis of the dual $\mathscr{D}\!\mathit{iff}^m_{U/S}$ of $\mathscr{P}^n_{U/S}$. Finally, the relations (16.11.2.2) follows immediately from the expression of the values of the three operators considered on the $\mathbf{z}^{\mathbf{r}}$, and of the fact that the $\boldsymbol{\zeta}^{\mathbf{r}}$ for $|\mathbf{r}| \leqslant m$ generate $\mathscr{P}^n_{U/S}$. □

IV-4 | 55

Remarks (16.11.3). — (i) The fact that, because of (16.11.2.2), the $D_{\mathbf{p}}$ are pairwise permutable naturally do not imply that the $\mathscr{O}_U$-algebra $\mathscr{D}\!\mathit{iff}_{U/S}$ is commutative, the $D_{\mathbf{p}}$ do not commute with the sections of $\mathscr{O}_U$ unless $n = 0$.

(ii) The indices $\mathbf{p}$ such that $|\mathbf{p}| = 1$ are the $\boldsymbol{\epsilon}_\lambda = (\epsilon_{\lambda\mu})_{\mu\in L}$ where $\epsilon_{\lambda\mu} = 0$ if $\mu \neq \lambda$ and $\epsilon_{\lambda\lambda} = 1$; when L is finite, the operators $D_{\boldsymbol{\epsilon}_\lambda}$ are exactly the S-derivations D_i introduced in (16.5.7). We note that in general (contrary to what happens in classical analysis), it is not true that a differential operator of any order can be written as a linear combination of powers of D_i (cf. (16.12)).

(iii) For every integer $r \geqslant 1$, we can define the notion of *differentially smooth up to order r* by replacing in (16.10.1) the condition (ii) by the condition that the homomorphisms

$$\mathbf{S}^m_{\mathscr{O}_X}(\Omega^1_{X/S}) \longrightarrow \mathscr{G}\!r_m(\mathscr{P}^n_{X/S})$$

are bijective *for every $m \leqslant r$*. The argument of (16.11.2) proves also that if, in the condition (a), we replace "differentially smooth" by "differentially smooth up to order r", this condition is equivalent to (b) by restricting ourselves to $\mathbf{p} \in \mathbb{N}^{(L)}$, $\mathbf{q} \in \mathbb{N}^{(L)}$ such that $|\mathbf{p}| \leqslant r$, $|\mathbf{q}| \leqslant r$.

16.12. Case of characteristic zero: Jacobian criterion for differentially smooth morphisms.

(16.12.1). We say that a prescheme X is *of characteristic* p (p equal to zero or a prime number) if, for every affine open set U of X, the ring $\Gamma(U, \mathcal{O}_X)$ is of characteristic p (**0**, 21.1.1). It follows from (**0**, 21.1.3) that for X to be of characteristic 0, it is necessary and sufficient that for every *closed* point x of X, the residue field $k(x)$ is of characteristic 0, or even that X can be given a structure of $\mathbb{Q}$-prescheme (necessarily unique).

Theorem (16.12.2). — *Let X be a scheme of characteristic 0, $f : X \to S$ a morphism. If $\Omega^1_{X/S}$ is a locally free $\mathcal{O}_X$-module (not necessarily of finite type), f is differentially smooth.*

PROOF. The problem being local on X, we can suppose there is a family (z_λ) of sections of $\mathcal{O}_X$ over X such that the (dz_λ) is a basis for the $\mathcal{O}_X$-module $\Omega^1_{X/S}$. Applying the criterion (16.11.2), it is enough for the operators

$$D_{\mathbf{p}} = (\mathbf{p}!)^{-1} \prod_\lambda D_\lambda^{p_\lambda}$$

(where the D_λ are the coordinate forms corresponding to the basis (dz_λ)) to verify the relations (16.11.2.1), which is a consequence of the fact that the D_λ are derivations. □

(16.12.3). The theorem above is not true if we discard the hypothesis that X is of characteristic 0. For example, if $S = \operatorname{Spec}(k)$, where k is a field of characteristic $p > 0$, $X = \operatorname{Spec}(K)$ where $K = k(\alpha)$ where $\alpha \notin k$, $\alpha^p \in k$, we verify immediately that $\Omega^1_{X/S}$ has rank 1, and that the morphism $X \to S$ **IV-4 | 56**
has rank 1, and that the morphism $X \to S$ is differentially smooth up to order $p-1$ (16.11.3, (iii)), but not of order p. However, the proof of (16.12.2) proves that if $\Omega^1_{X/S}$ is locally free, and if $n!1_{\mathcal{O}_X}$ is inversible in $\Gamma(X, \mathcal{O}_X)$, then X is differentially smooth over S up to order n.

§17. SMOOTH MORPHISMS, UNRAMIFIED (OR NET) MORPHISMS, AND ÉTALE MORPHISMS.

In this paragraph, we revisit the concepts studied in ($\mathbf{0}_{\text{III}}$, 9), expressed in the geometric language of schemes from a global point of view, for preschemes locally of finite presentation over a given base.

Most of the results (except 17.7, 17.8, 17.9, 17.13, and 17.16) are reduced to various properties already encountered in ($\mathbf{0}_{\text{III}}$, 9).

For more specific results on étale morphisms, the reader should consult §18.

17.1. Formally smooth morphisms, formally unramified morphisms, formally étale morphisms.

Definition (17.1.1). — Let $f : X \to Y$ be a morphism of preschemes. We say that f is *formally smooth* (resp. *formally unramified*, resp. *formally étale*) if, for all affine schemes Y', all closed subschemes Y'_0 of Y' defined by a nilpotent ideal $\mathcal{J}$ of $\mathcal{O}_{Y'}$, and every morphism $Y' \to Y$, the map

(17.1.1.1) $$\operatorname{Hom}_Y(Y', X) \longrightarrow \operatorname{Hom}_Y(Y'_0, X)$$

induced by the canonical map $Y'_0 \to Y'$, is *surjective* (resp. *injective*, resp. *bijective*).

One also says that X is *formally smooth* (resp. *formally unramified*, resp. *formally étale*) over Y.

It is clear that for f to be formally étale, it is necessary and sufficient for f to be formally smooth and formally unramified.

Remark (17.1.2). —

(i) Suppose that $Y = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ are affine, so that f comes from a homomorphism of rings $\phi : A \to B$. According to (**0**, 19.3.1) and (**0**, 19.10.1), saying that f is formally smooth (resp. formally unramified, resp. formally étale) means that, via ϕ, B is a *formally smooth* (resp. *formally unramified*, resp. *formally étale*) A-algebra, for the *discrete* topologies on A and B.

(ii) To verify that f is formally smooth (resp. formally unramified, resp. formally étale), we can, in Definition (17.1.1), restrict to the case where $\mathscr{J}^2 = 0$. To see this, if f satisfies the corresponding condition of Definition (17.1.1) in the particular case $\mathscr{J}^2 = 0$, and if we have $\mathscr{J}^n = 0$, then we consider the closed subscheme Y'_j of Y' defined by the sheaf of ideals $\mathscr{J}^{j+1}$ for $0 \leqslant j \leqslant n-1$, so that Y'_j is a closed subscheme of Y'_{j+1} defined by a square-zero sheaf of ideals; the hypotheses imply that each of the maps

$$\operatorname{Hom}_Y(Y'_{j+1}, X) \longrightarrow \operatorname{Hom}_Y(Y'_j, X) \quad (0 \leqslant j \leqslant n-1)$$

is surjective (resp. injective, resp. bijective); by composition, we conclude that the same holds for (17.1.1.1). **IV** | 57

(iii) Note that the properties of the morphism f defined in (17.1.1) are properties of the *representable functor* ($\mathbf{0}_{\mathbf{III}}$, 8.1.8)

$$Y' \longmapsto \operatorname{Hom}_Y(Y', X)$$

from the category of Y-preschemes to the category of sets; they keep a meaning for *any* contravariant functor with the same domain and codomain, representable or not.

(iv) Assume that the morphism f is formally unramified (resp. formally étale); consider an *arbitrary* Y-prescheme Z and a closed subprescheme Z_0 of Z defined by a *locally nilpotent* sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_Z$. Then the map

(17.1.2.1)
$$\operatorname{Hom}_Y(Z, X) \longrightarrow \operatorname{Hom}_Y(Z_0, X)$$

induced by the canonical injection $Z_0 \to Z$, is still injective (resp. bijective). To see this, let (U_α) be an affine open covering of Z such that the sheaves of ideals $\mathscr{J}|U_\alpha$ are nilpotent, and for each α, let U_α^0 be the inverse image of U_α in Z_0, which is the closed subprescheme of U_α defined by $\mathscr{J}|U_\alpha$. Let $f_0 : Z_0 \to X$ by a Y-morphism; by hypothesis, for each α, there is at most one (resp. one and only one) Y-morphism $f_\alpha : U_\alpha \to X$ whose restriction to Z_0 coincides with $f_0|U_\alpha$. We immediately conclude that if f_α and f_β are defined, then, for each affine open $V \subset U_\alpha \cap U_\beta$, we have $f_\alpha|V = f_\beta|V$, as the restrictions of these morphisms to the inverse image V_0 of V in Z_0 coincide. There is therefore at most one (resp. one and only one) Y-morphism $f : Z \to X$ whose restriction to Z_0 coincides with f_0.

Proposition (17.1.3). —

(i) *A monomorphism of preschemes is formally unramified; an open immersion is formally étale.*

(ii) *The composition of two formally smooth (resp. formally unramified, resp. formally étale) morphisms is formally smooth (resp. formally unramified, resp. formally étale).*

(iii) *If $f : X \to Y$ is a formally smooth (resp. formally unramified, resp. formally étale) S-morphism, then so is $f_{(S')} : X_{(S')} \to Y_{(S')}$ for any base extension $S' \to S$.*

(iv) *If $f : X \to X'$ and $g : Y \to Y'$ are two formally smooth (resp. formally unramified, resp. formally étale) S-morphisms, then so is $f \times_S g : X \times_S Y \to X' \times_S Y'$.*

(v) *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms; if $g \circ f$ is formally unramified, then so is f.*

(vi) *If $f : X \to Y$ is a formally unramified morphism, then so is $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$.*

Proof. According to (**I**, 5.5.12), it suffices to prove (i), (ii), and (iii). The assertions in (i) are both trivial. To prove (ii), consider two morphisms $f : X \to Y$, $g : Y \to Z$, an affine scheme Z', a closed subscheme Z'_0 of Z defined by a nilpotent ideal and a morphism $Z' \to Z$. Suppose that f and g formally smooth, and consider a Z-morphism $u_0 : Z'_0 \to X$; the hypothesis on g implies that there **IV** | 58
exists a Z-morphism $v : Z' \to Y$ such that $f \circ u_0 = v \circ j$ (where $j : Z'_0 \to Z$ is the canonical injection); the hypothesis on f then implies that there exists a morphism $u : Z' \to X$ such that $f \circ u = v$ and $u \circ j = u_0$, therefore $(g \circ f) \circ u$ is equal to the given morphism $Z' \to Z$ and $u \circ j = u_0$, which proves that $g \circ f$ is formally smooth; we argue the same way when we suppose that f and g are formally unramified.

Finally, to prove (iii), let $X' = X_{S'}$, $Y' = Y_{S'}$, $f' = f_{S'}$; consider an affine scheme Y'', a closed subscheme Y''_0 defined by a nilpotent sheaf of ideals, and a morphism $g : Y'' \to Y'$ making Y'' a Y'-prescheme; we then know by (**I**, 3.3.8) that $\operatorname{Hom}_{Y'}(Y'', X')$ is canonically identified with

$\mathrm{Hom}_Y(Y'', X)$, and $\mathrm{Hom}_{Y'}(Y_0'', X')$ with $\mathrm{Hom}_Y(Y_0'', X)$, and the conclusion follows immediately from Definition (17.1.1). □

We note that a *closed immersion* is not necessarily formally smooth.

Proposition (17.1.4). — *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms, and suppose that g is formally unramified. Then, if $g \circ f$ is formally smooth (resp. formally étale), so is f.*

PROOF. Let Y' be an affine scheme, Y_0' a closed subscheme of Y' defined by a nilpotent sheaf of ideals, $h : Y' \to Y$ a morphism, $j : Y_0' \to Y'$ the canonical injection, $u_0 : Y_0' \to Y$ a Y-morphism, such that $f \circ u_0 = h \circ j$. Suppose that $g \circ f$ is formally smooth; then there exists a morphism $u : Y' \to X$ such that $u \circ j = u_0$ and $(g \circ f) \circ u = g \circ h$. But these two relations imply that $f \circ u$ and h are Z-morphisms from Y' to Y such that $(f \circ u) \circ j = h \circ j$; by virtue of the hypothesis that g is formally unramified, we get that $f \circ u = h$, in other words that u is a Y-morphism; thus f is formally smooth. Taking into account (17.1.3, (v)), this proves the proposition. □

Corollary (17.1.5). — *Suppose that g is formally étale; then, for $g \circ f$ to be formally smooth (resp. formally unramified, resp. formally étale), it is necessary and sufficient that f is.*

PROOF. This follows from (17.1.4) and (17.1.3, (ii) and (iv)). □

Proposition (17.1.6). — *Let $f : X \to Y$ be a morphism of preschemes.*

(i) *Let (U_α) be an open covering of X and, for each α, let $i_\alpha : U_\alpha \to X$ be the canonical injection. For f to be formally smooth (resp. formally unramified, resp. formally étale), it is necessary and sufficient that each $f \circ i_\alpha$ is.*

(ii) *Let (V_λ) be an open covering of Y. For f to be formally smooth (resp. formally unramified, resp. formally étale), it is necessary and sufficient that each of the restrictions $f^{-1}(V_\lambda) \to V_\lambda$ of f is.*

PROOF. First note that (ii) is a consequence of (i): if $j_\lambda : V_\lambda \to Y$ and $i_\lambda : f^{-1}(V_\lambda) \to X$ are the canonical injections, then the restriction $f_\lambda : f^{-1}(V_\lambda) \to V_\lambda$ of f is such that $j_\lambda \circ f_\lambda = f \circ i_\lambda$; if f is formally smooth (resp. formally unramified), then so is $f \circ i_\lambda$ since i_λ is formally étale (17.1.3); but since j_λ is formally étale, this means that f_λ is formally smooth (resp. formally unramified), by virtue of (17.1.5). Conversely, if all the f_λ are formally smooth (resp. formally unramified), the same applies to $j_\lambda \circ f_\lambda$ (17.1.3), so also to f in virtue of (i).

If we take into account that the i_α are formally étale, everything comes down to proving that if the $f \circ i_\alpha$ are formally smooth (resp. formally unramified), then the same applies to f. **IV** | 59

Therefore let Y' be an affine scheme, Y_0' a closed subscheme of Y' defined by a nilpotent ideal $\mathscr{J}$, which we may assume to satisfy $\mathscr{J}^2 = 0$ (17.1.2, (ii)), and finally let $g : Y' \to Y$ be a morphism. Suppose we are given a Y-morphism $u_0 : Y_0' \to X$; denote by W_α (resp. W_α^0) the prescheme induced by Y' (resp. Y_0') on the open subset $u_0^{-1}(U_\alpha)$ (we recall that Y' and Y_0' share the *same underlying topological space*). Let us first suppose that the $f \circ i_\alpha$ are *formally unramified*, and show that, if u' and u'' are two Y-morphisms from Y' to X whose restrictions to Y_0' coincide, then we have $u' = u''$. Indeed, taking into account (17.1.2, (iv)), the hypothesis that the $f \circ i_\alpha$ are formally unramified implies that for all α, we have $u'|W_\alpha = u''|W_\alpha$, since the restrictions of both Y-morphisms to W_α^0 coincide. Hence the conclusion follows.

Now suppose that the $f \circ i_\alpha$ are *formally smooth* and prove the existence of a Y-morphism $u : Y' \to X$ whose restriction to Y_0' is u_0. Now, since Y' is an *affine scheme*, we can apply (16.5.17), the hypotheses of which are satisfied, and the conclusion of which precisely proves the existence of u. □

We can therefore say that the notions introduced in (17.1.1) are *local* on X and Y, which always allows, in virtue of (17.1.2, (i)), to be reduced to the study of formally smooth (resp. formally unramified, resp. formally étale) *algebras*.

17.2. General properties of differentials

Proposition (17.2.1). — *For a morphism $f : X \to Y$ to be formally unramified, it is necessary and sufficient that $\Omega^1_f = 0$ (what we still write $\Omega^1_{X/Y} = 0$ (16.3.1)).*

PROOF. Taking into account (17.1.6), we reduce to the case where $Y = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ are affine, and the conclusion then follows from (**0**, 20.7.4) and the interpretation of $\Omega^1_{X/Y}$ in this case (16.3.7). □

Corollary (17.2.2). — *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms. For f being formally unramified, it is necessary and sufficient that the canonical morphism (16.4.19)*

$$f^*(\Omega^1_{Y/Z}) \longrightarrow \Omega^1_{X/Z}$$

is surjective.

PROOF. This is an immediate consequence of (17.2.1) and the exact sequence (16.4.19.1). □

Proposition (17.2.3). — *Let $f : X \to Y$ be a formally smooth morphism.*

(i) *The $\mathscr{O}_X$-module $\Omega^1_{X/Y}$ is locally projective (16.10.1). If f is locally of finite type, then $\Omega^1_{X/Y}$ is locally free and of finite type.*

(ii) *For all morphisms $g : Y \to Z$, the sequence (16.4.19) of $\mathscr{O}_X$-modules*

(17.2.3.1) $$0 \longrightarrow f^*(\Omega^1_{Y/Z}) \longrightarrow \Omega^1_{X/Z} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$$

is exact; moreover, for each $x \in X$, there exists an open neighbourhood U of x such that the restrictions to U of the homomorphisms in (17.2.3.1) form a split *exact sequence.*

IV | 60

PROOF.

(i) We know (16.3.9) that if f is locally of finite type, then Ω^1_f is an $\mathscr{O}_X$-module of finite type. To prove that, in all cases, it is locally projective, we can reduce, by virtue of (17.1.6), to the case where $Y = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ are affine, and the result follows from the hypothesis on f and from (**0**, 20.4.9) and (**0**, 19.2.1).

(ii) Again, we can restrict to the case where X, Y, and Z are affine (17.1.6), and the conclusion in this case follows from the interpretation of the sheaves of modules in the sequence (17.2.3.1) and from (**0**, 20.5.7).

□

Corollary (17.2.4). — *If $f : X \to Y$ is formally étale, then, for all morphisms $g : Y \to Z$, the canonical homomorphism of $\mathscr{O}_X$-modules*

$$f^*(\Omega^1_{Y/Z}) \longrightarrow \Omega^1_{X/Z}$$

is bijective.

PROOF. This follows from the exactness of the sequence (17.2.3.1) and from the fact that we then have $\Omega^1_{X/Y} = 0$ (17.2.1). □

Proposition (17.2.5). — *Let $f : X \to Y$ be a morphism, X' a subprescheme of X such that the composite morphism $X' \xrightarrow{j} X \xrightarrow{f} Y$ (where j is the canonical injection) is formally smooth. Then the sequence of $\mathscr{O}_X$-modules (16.4.21)*

(17.2.5.1) $$0 \longrightarrow \mathscr{N}_{X'/X} \longrightarrow \Omega^1_{X/Y} \otimes_{\mathscr{O}_X} \mathscr{O}_{X'} \longrightarrow \Omega^1_{X'/Y} \longrightarrow 0$$

is exact; moreover, for each $x \in X$, there exists an open neighbourhood U of x such that the restrictions to U of the homomorphisms in (17.2.5.1) form a split *exact sequence.*

PROOF. By virtue of (17.1.6), we reduce to the case where $Y = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ are affine, and $X' = \operatorname{Spec}(B/\mathfrak{J})$, where $\mathfrak{J}$ is an ideal of B. The conormal sheaf $\mathscr{N}_{X'/X}$ then corresponds to the B-module $\mathfrak{J}/\mathfrak{J}^2$ (16.1.3), and the conclusion follows from (**0**, 20.5.14). □

Proposition (17.2.6). — *Let X and Y be two preschemes, $f : X \to Y$ a morphism locally of finite type. The following conditions are equivalent:*

(a) *f is a monomorphism.*
(b) *f is radicial and formally unramified.*
(c) *For each $y \in Y$, the fibre $f^{-1}(y)$ is empty or $k(y)$-isomorphic to* $\mathrm{Spec}(k(y))$ *(in other words, it is reduced to a single point z such that $k(y) \to \mathscr{O}_z/\mathfrak{m}_y\mathscr{O}_z$ is an isomorphism).*

PROOF. The fact that (a) implies (c) follows from (8.11.5.1). It is clear that (c) implies that f is radicial; let us prove that it also follows from (c) that $\Omega^1_{X/Y} = 0$, which will prove that (c) implies (b) (17.2.1). Note that the $\mathscr{O}_X$-module $\Omega^1_{X/Y}$ is quasi-coherent of finite type (16.3.9). It follows from (**I**, 9.1.13.1) that, for $(\Omega^1_{X/Y})_x = 0$, it is necessary and sufficient that if we set $Y_1 = \mathrm{Spec}(k(y))$, $X_1 = f^{-1}(y) = X \times_Y Y_1$, then we have $(\Omega^1_{X_1/Y_1})_x = 0$; but as the morphism $f_1 : X_1 \to Y_1$ induced by f is formally unramified by virtue of the hypothesis (c) (17.1.3), the conclusion follows from (17.2.1). Finally, let us prove that (b) implies (a); for this, consider the diagonal morphism $g = \Delta_f : X \to X \times_Y X$; since f is radicial, g is surjective (1.8.7.1); on the other hand, $\Omega^1_{X/Y}$ is by definition the conormal sheaf $\mathscr{G}r_1(g)$ of the immersion g (16.3.1), and to say that f is formally unramified therefore means that $\mathscr{G}r_1(g) = 0$ (17.2.1). In addition, g is locally of finite presentation **IV** | 61
(1.4.3.1); therefore the hypothesis $\mathscr{G}r_1(g) = 0$ implies that g is an open immersion (16.1.10); being surjective, this immersion is an isomorphism, hence f is a monomorphism (**I**, 5.3.8). □

17.3. Smooth morphisms, unramified morphisms, étale morphisms

Definition (17.3.1). — We say that a morphism $f : X \to Y$ is *smooth* (resp. *unramified*, or *net* [5] resp. *étale*) if it is locally of finite presentation and formally smooth (resp. formally unramified, resp. formally étale).

We then also say that X is *smooth* (resp. *unramified*, resp. *étale*) *over* Y.

We will see later (17.5.2) that this definition of a smooth morphism coincides with the definition already given in (6.8.1); until then, we will exclusively use definition (17.3.1).

It is clear that saying that f is étale means that it is *both* smooth and unramified.

Remark (17.3.2). —

(i) Note that definition (17.3.1) can be phrased using only the functor
$$Y' \longmapsto \mathrm{Hom}_Y(Y', X)$$
considered in (17.1.2, (iii)) because to say that f is locally of finite presentation is equivalent to saying that the preceding functor *commutes with projective limits of affine schemes* (8.14.2).

(ii) Let A be a ring and B an A-algebra. We say that B is a *smooth* (resp. *unramified*, resp. *étale*) A-algebra if the corresponding morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is smooth (resp. unramified, resp. étale). It is equivalent to say that B is an A-algebra *of finite presentation* (1.4.6) that is furthermore formally smooth (resp. formally unramified, resp. formally étale) for the discrete topologies.

(iii) It follows from (17.1.6) and the definition of a morphism locally of finite presentation (1.4.2) that the notion of a smooth (resp. unramified, resp. étale) morphism is *local on X and on Y*.

Proposition (17.3.3). —

(i) *An open immersion is étale. For an immersion to be unramified, it is necessary and sufficient to it be locally of finite presentation.*

(ii) *The composition of two smooth (resp. unramified, resp. étale) morphisms is smooth (resp. unramified, resp. étale).*

(iii) *If $f : X \to Y$ is a smooth (resp. unramified, resp. étale) S-morphism, then so is $f_{(S')} : X_{(S')} \to Y_{(S')}$ for any base extension $S' \to S$.*

(iv) *If $f : X \to X'$ and $g : Y \to Y'$ are smooth (resp. unramified, resp. étale) S-morphisms, then so is $f \times_S g : X \times_S Y \to X' \times_S Y'$.* **IV** | 62

[5] The words "net" and "formally net" seem more preferable to the terminology used in "unramified" (resp. formally unramified") and will be used almost exclusively in Chapter V. In this chapter, we have kept the old terminology so as not to conflict with **0**, 19.10.

(v) *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be two morphisms; if* g *is locally of finite type and if* $g \circ f$ *is unramified, then* f *is unramified.*

PROOF. This follows from (1.4.3) and (17.1.3). □

Proposition (17.3.4). — *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be two morphisms, and suppose that* g *is unramified. Then, if* $g \circ f$ *is smooth (resp. unramified, resp. étale), so is* f.

PROOF. As g and $g \circ f$ are locally of finite presentation, so is f (1.4.3, (v)); the conclusion thus follows from (17.1.4) and (17.1.3, (v)). □

Corollary (17.3.5). — *Suppose that* g *is étale; then, for* f *to be smooth (resp. unramified, resp. étale) it is necessary and sufficient that* $g \circ f$ *is.*

PROOF. This follows from (17.3.4) and (17.3.3, (ii)). □

Proposition (17.3.6). — *Let* $g : Y \to S$ *and* $h : X \to S$ *be two morphisms locally of finite presentation. For an* S*-morphism* $f : X \to Y$ *to be unramified, it is necessary and sufficient that the canonical homomorphism* (16.4.19)

$$f^*(\Omega^1_{Y/S}) \longrightarrow \Omega^1_{X/S}$$

is surjective.

PROOF. As f is locally of finite presentation (1.4.3, (v)), the proposition follows from (17.2.2). □

Definition (17.3.7). — Let $f : X \to Y$ be a morphism. We say that f is *smooth* (resp. *unramifed*, resp. *étale*) at a point $x \in X$, if there exists an open neighbourhood U of x in X such that the restriction $f|U$ is a smooth (resp. unramified, resp. étale) morphism from U to Y.

We then also say that X is *smooth* (resp. *unramified*, resp. *étale*) *over* Y *at the point* x.

Taking into account remark (17.3.2, (iii)), it is equivalent to say that f is smooth (resp. unramified, resp. étale) and to say that f is smooth (resp. unramified, resp. étale) at all points of X.

It is clear that the set of points of X at which the morphism $f : X \to Y$ is smooth (resp. unramified, resp. étale) is *open* in X.

Proposition (17.3.8). — *For all preschemes* Y *and all locally free* $\mathscr{O}_Y$*-modules* $\mathscr{E}$ *of finite type, the vector bundle prescheme* $\mathbf{V}(\mathscr{E})$ (**II**, 1.7.8) *associated to* $\mathscr{E}$ *is a smooth* Y*-prescheme.*

PROOF. Indeed (17.3.2, (iii)), we can restrict ourselves to the case where $Y = \operatorname{Spec}(A)$ is affine and $\mathbf{V}(\mathscr{E}) = \operatorname{Spec}(A[T_1, \ldots, T_r])$; as $A[T_1, \ldots, T_r]$ is a formally smooth A-algebra for the discrete topologies (**0**, 19.3.2), and of finite presentation, this proves the proposition (17.3.2, (ii)) □

Corollary (17.3.9). — *Under the hypotheses of* (17.3.8), *the projective prescheme* $\mathbf{P}(\mathscr{E})$ (**II**, 4.1.1) *is a smooth* Y*-prescheme.*

PROOF. We can still restrict to the case where $Y = \operatorname{Spec}(A)$ is affine and $\mathbf{P}(\mathscr{E}) = \mathbf{P}^r_Y$. We then know (**II**, 2.3.14) that we have a finite open cover of $\mathbf{P}^r_A$ by the $D_+(T_i)$ ($0 \leqslant i \leqslant r$) respectively equal to the spectrum of the ring $S_{(f)}$, where we wrote S for $A[T_1, \ldots, T_r]$ and f for T_i; but it follows immediately from the definition of $S_{(f)}$ (**II**, 2.2.1), that this ring, in this case, is isomorphic to $A[T_0, \ldots, T_{i-1}, T_{i+1}, \ldots, T_r]$; hence the corollary follows by (17.3.8). □

17.4. Characterizations of unramified morphisms. **IV** | 63

§18. Supplement on étale morphisms. Henselian local rings and strictly local rings

§19. Regular immersions and normal flatness

§20. Meromorphic functions and pseudo-morphisms

20.0. Introduction Most of the concepts and results of §§20 and 21 directly relate to Chapter I, and hardly depend on Chapters I and IV, except for the occasional usage of the notion of depth and of a local regular ring (in (20.6), (21.11), (21.13), and (21.15)), of Zariski's "Main theorem" in (20.4) and (21.12), and the properties of transversely regular immersions in (20.6) and (21.15).

In §20, we introduce several variants of the concept of a rational map, already studied in (I, 7) **IV-4** | 226 from a point of view still fairly close to the classical point of view, and for this reason quite ill-suited to the case of not necessarily reduced preschemes. The notions and results of §20 are used in §21 (n$^{\text{os}}$(21.1) and (21.7)) to develop the general notion of a divisor and its most elementary properties. This notion is especially convenient when the local rings of the preschemes considered are Noetherian and integrally closed, and especially when they are also *factorial* ((21.6) and (21.7)), because of their identification in the latter case with the notion of a 1-*codimensional cycle* (a linear combination of irreducible subpreschemes of codimension 1). In (21.9), we determine the divisors on a Noetherian prescheme of dimension 1 but not necessarily normal, which is useful for various applications. The (21.11) and (21.12) give two important theorems, due respectively to Auslander–Buchsbaum and Van der Waerden, and relate the notion of a factorial ring (the n$^{\text{os}}$(21.9), (21.11), and (21.12) are independent of each other). In the n$^{\text{os}}$(21.13) and (21.14), also independent of the previous three, we study a useful variant of the notion of a local factorial ring, that of a local *parafactorial* ring, which is introduce in particular **[?]** in the development of comparison theorem of the Picard group of a projective prescheme X over a field k and a "hyperplane section". We will see in (21.14.1) (Ramanujam–Samuel theorem) that the local parafactorial rings are much more numerous than one would have expected *a priori*.

In (20.5), (20.6), and (21.15), we review the previous notions but from a point of view "relative" to a fixed base prescheme. For the moment these notions are used only relatively rarely; in particular, the concept of a relative divisor is hardly used except when it comes to positive divisors, and in this case it is explained advantageously without the help of the notion of a relative meromorphic functions, using the notion of a transversely regular immersion of codimension 1. It will therefore be advantageous to omit these sections on a first reading.

20.1. Meromorphic functions

(20.1.1). Let $(X, \mathscr{O}_X)$ be a ringed space, and let $\mathscr{S}$ be a subsheaf *of sets* of $\mathscr{O}_X$. For every open U of X, consider the *ring of fractions* $\Gamma(U, \mathscr{O}_X)[\Gamma(U, \mathscr{S})^{-1}]$ (Bourbaki, *Alg. comm.*, chap. II, §2, n$^{\text{o}}$1). It is immediate that the map $U \mapsto \Gamma(U, \mathscr{O}_X)[\Gamma(U, \mathscr{S})^{-1}]$ is a *presheaf of rings* ((**0**, 1.5.1) and (**0**, 1.5.7)). We denote by $\mathscr{O}_X[\mathscr{S}^{-1}]$ the *sheaf of rings* associated to this presheaf and we say that this is the *sheaf of rings of fractions of $\mathscr{O}_X$ with denominators in $\mathscr{S}$*; this is a *flat* $\mathscr{O}_X$-module. It is immediate that for every $x \in X$, we have a canonical isomorphism

$$(\mathscr{O}_X[\mathscr{S}^{-1}])_x \simeq \mathscr{O}_x[\mathscr{S}_x^{-1}], \tag{20.1.1.1}$$

since the reasoning of (**0**, 1.4.5) generalizes immediately in the case where we have an inductive system $(A_\alpha, \phi_{\beta\alpha})$ of rings, and for each index α a subset S_α of A_α such that $\phi_{\beta\alpha}(S_\alpha) \subset S_\beta$ for $\alpha \leqslant \beta$; **IV-4** | 227 we then take for S the inductive limit in $A = \varinjlim A_\alpha$ of the inductive limit of the subsets (S_α).

(20.1.2). Now let $\mathscr{F}$ be an $\mathscr{O}_X$-module. We then set

$$\mathscr{F}[\mathscr{S}^{-1}] = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X[\mathscr{S}^{-1}], \tag{20.1.2.1}$$

and we say that this is the *sheaf of modules of fractions of $\mathscr{F}$ with denominators in $\mathscr{S}$*; it is immediate that it is associated to the presheaf of modules $U \mapsto \Gamma(U, \mathscr{F})[\Gamma(U, \mathscr{S})^{-1}]$, and that for every $x \in X$, we have a canonical isomorphism

(20.1.2.2) $$(\mathscr{F}[\mathscr{S}^{-1}])_x \simeq \mathscr{F}_x[\mathscr{S}_x^{-1}].$$

(20.1.3). We will focus here on the case where $\mathscr{S}$ is the subsheaf $\mathscr{S}(\mathscr{O}_X)$ of $\mathscr{O}_X$ such that for every open U, $\Gamma(U, \mathscr{S})$ is the *set of regular elements* of the ring $\Gamma(U, \mathscr{O}_X)$; it is immediate that it is a sheaf (and not only a presheaf), the regularity of a section of $\mathscr{O}_X$ over U being verified "fibre by fibre" (i.e. meaning that the germ of the section in x is regular in $\mathscr{O}_X$ for all $x \in U$), in other words $\mathscr{S}(\mathscr{O}_X)_x$ is none other then the set of regular elements of $\mathscr{O}_{X,x}$. The corresponding sheaf of rings

$$\mathscr{M}_X = \mathscr{O}_X[\mathscr{S}^{-1}]$$

is called the *sheaf of germs of meromorphic functions on* X, and the sections of $\mathscr{M}_X$ over X are called the *meromorphic functions on* X; they form a ring which we denote by $M(X)$. For every $\mathscr{O}_X$*-module* $\mathscr{F}$,

$$\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{M}_X = \mathscr{F}[\mathscr{S}^{-1}]$$

is also denoted $\mathscr{M}_X(\mathscr{F})$ and called the *sheaf of germs of meromorphic sections of* $\mathscr{F}$; its sections over X form an $M(X)$-module denoted by $M(X, \mathscr{F})$, whose elements are called *meromorphic sections of* $\mathscr{F}$ *over* X. These definitions imply that for every open U of X, we have a canonical isomorphism $\mathscr{M}_X(\mathscr{F})|U \simeq \mathscr{M}_U(\mathscr{F}|U)$, in particular $\mathscr{M}_X|U \simeq \mathscr{M}_U$.

(20.1.3.1). If X is a *reduced prescheme*, then we note that if an element $s \in \Gamma(U, \mathscr{O}_X)$ is such that $s_\xi \neq 0$ for every maximal point ξ of U, then s is *regular*. Indeed, if $st = 0$ for a $t \in \Gamma(U, \mathscr{O}_X)$, then we have $s_\xi t_\xi = 0$, so $t_\xi = 0$ since $\mathscr{O}_{X,\xi}$ is a field, and say that $t_\xi = 0$ for every maximal point ξ of X means that $t = 0$: we are immediately reduced to the case where U is affine, and an element of a reduced ring which belongs to all the minimal prime ideals is zero by definition. The converse is true if the set of irreducible components of X is *locally finite*. We immediately reduce to the case where $X = \operatorname{Spec}(A)$ is affine; if $\mathfrak{p}_i$ $(1 \leqslant i \leqslant n)$ are the minimal prime ideals of A and if $s \in \mathfrak{p}_i$ for an index i, then there exists $t \in A$ such that $t \in \mathfrak{p}_j$ for $j \neq i$ and $t \notin \mathfrak{p}_i$ (Bourbaki, *Alg. comm.*, chap. II, §1, n°1, Prop. 1); so we have $st \in \mathfrak{p}_i$ for all i, and as a result $st = 0$ since A is reduced; s is therefore nonregular.

(20.1.4). For every open U of X, the homomorphism $t \mapsto t/1$ from $\Gamma(U, \mathscr{O}_X)$ to $\Gamma(U, \mathscr{O}_X)[\Gamma(U, \mathscr{S})^{-1}]$ (which is none other than the *total ring of fractions* of $\Gamma(U, \mathscr{O}_X)$) is injective; these homomorphisms **IV-4** | 228
thus define a *canonical injective homomorphism*

(20.1.4.1) $$i : \mathscr{O}_X \longrightarrow \mathscr{M}_X$$

which allows us to identify $\mathscr{O}_X$ with a subsheaf of $\mathscr{M}_X$. Given a meromorphic function $\phi \in M(X)$, we say that ϕ is *defined* on an open U of X if $\phi|U$ is a *section of* $\mathscr{O}_U$ over U; the axioms of sheaves show that there is, for a given section ϕ, a *largest* open on which ϕ is defined; we call this the *domain of definition* of ϕ and denote it by $\operatorname{dom}(\phi)$.

(20.1.5). For every $\mathscr{O}_X$-module $\mathscr{F}$, we obtain from (20.1.4.1) a di-homomorphism consisting of i and the homomorphism of sheaves of additive groups

(20.1.5.1) $$1_{\mathscr{F}} \otimes i : \mathscr{F} \longrightarrow \mathscr{M}_X(\mathscr{F}) = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{M}_X.$$

We note that the latter is not injective in general; when it is injective, we say that $\mathscr{F}$ is *strictly torsion-free*: this means that for every open U of X and every section $s \in \Gamma(U, \mathscr{O}_X)$ which is a regular element in this ring the homothety $z \mapsto sz$ of $\Gamma(U, \mathscr{F})$ is injective; this condition is evidently satisfied if $\mathscr{F}$ is *locally free*.

Proposition (20.1.6). — *Let X be a locally Noetherian prescheme, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module. For $\mathscr{F}$ to be strictly torsion-free, it is necessary and sufficient that* $\operatorname{Ass}(\mathscr{F}) \subset \operatorname{Ass}(\mathscr{O}_X)$.

PROOF. We immediately reduce to the case where $X = \operatorname{Spec}(A)$ is affine, $\mathscr{F} = \widetilde{M}$, and we know that the elements s of A belonging to an ideal of $\operatorname{Ass}(M)$ are exactly those for which the homothety $z \mapsto sz$ is not injective (Bourbaki, *Alg. comm.*, chap. IV, §1, n°1, cor. 2 of prop. 2). □

(20.1.7). If u is a section of $\mathscr{M}_X(\mathscr{F})$ over X, then we say that u is *defined* at a point $x \in X$ if there exists an open neighbourhood V of x in X such that $u|V$ is the image of a section of $\mathscr{F}$ over V under the di-homomorphism (20.1.5.1). We say that u is *defined* on an open U of X if it is defined at every point in U; there is still a larger open in which u is defined, called the *domain of definition* of u and denoted

$\mathrm{dom}(u)$. When $\mathscr{F}$ is strictly torsion-free, such that $\mathscr{F}$ is identified by (20.1.5.1) with a subsheaf of $\mathscr{M}_X(\mathscr{F})$, then saying u is defined on U means that $u|V$ is a *section of* $\mathscr{F}$ *over* U.

(20.1.8). In accordance with the general notation of ($\mathbf{0_I}$, 5.4.7), we denote by $\mathscr{M}_X^*$ the sheaf of multiplicative groups such that $\Gamma(U, \mathscr{M}_X^*)$ is (for every open U of X) the group of *invertible elements* of $\Gamma(U, \mathscr{M}_X)$. This sheaf is none other than the sheaf $\mathscr{S}(\mathscr{M}_X)$ defined in (20.1.3): indeed, if $s \in \Gamma(U, \mathscr{S}(\mathscr{M}_X))$, then for every $x \in U$, there exists an open neighbourhood $V \subset U$ of x such that $s|V$ is a regular element in the *total ring of fractions* of $\Gamma(V, \mathscr{O}_X)$, and we know that such an element is necessarily invertible in this ring of fractions. We say that the sections of $\mathscr{M}_X^*$ over X are the *regular meromorphic functions* (note that we are deviating here from the terminology followed by certain authors, who call "regular" meromorphic functions those which are *sections of* $\mathscr{O}_X$, identified with a subsheaf of $\mathscr{M}_X$).

Let $\mathscr{L}$ be an *invertible* $\mathscr{O}_X$*-module* ($\mathbf{0_I}$, 5.4.1); then it is clear that $\mathscr{M}_X(\mathscr{L}) = \mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{M}_X$ is an *invertible* $\mathscr{M}_X$*-module*. Let U be an open such that $\mathscr{L}|U$ is isomorphic to $\mathscr{O}_U$; as every automorphism of $\mathscr{M}_U$ is multiplication by an invertible element of $\Gamma(U, \mathscr{M}_X)$ ($\mathbf{0_I}$, 5.4.7), it is equivalent to say that a section $s \in \Gamma(U, \mathscr{M}_X(\mathscr{L}))$ has an invertible image in $\Gamma(U, \mathscr{M}_X)$ under *an* isomorphism or by *any* isomorphism on $\Gamma(U, \mathscr{M}_X)$; we say in this case that s is a *regular meromorphic section of* $\mathscr{L}$ over U; a section s of $\mathscr{L}$ over X is called a *regular meromorphic section of* $\mathscr{L}$ if, for every open U such that $\mathscr{L}|U$ is isomorphic to $\mathscr{O}_U$, $s|U$ is a regular meromorphic section of $\mathscr{L}$ over U. We denote by $(\mathscr{M}_X(\mathscr{L}))^*$ the subsheaf of $\mathscr{M}_X(\mathscr{L})$ such that for every open U, $\Gamma(U, (\mathscr{M}_X(\mathscr{L}))^*)$ is the set of regular meromorphic sections of $\mathscr{L}$ over U. Let s be a meromorphic section of $\mathscr{L}$ over X (i.e. a section of $\mathscr{M}_X(\mathscr{L})$); it defines a homomorphism $h_s : \mathscr{M}_X \to \mathscr{M}_X(\mathscr{L})$ which sends every section t of $\mathscr{M}_X$ over an open U to $(s|U)t$. It follows immediately from the above that for s to be *regular*, it is necessary and sufficient for h_s to be *injective*, and in fact h_s is then a *bijective* homomorphism from $\mathscr{M}_X$ to $\mathscr{M}_X(\mathscr{L})$, and its restriction to $\mathscr{M}_X^*$ is a bijection to $(\mathscr{M}_X(\mathscr{L}))^*$. We conclude that the homothety $t \mapsto ts$ is an isomorphism from $M(X)$ to $M(X, \mathscr{L})$. IV-4 | 229

(20.1.9). Let s be a regular meromorphic section of an invertible $\mathscr{O}_X$-module $\mathscr{L}$ over X; then for every $\mathscr{O}_X$-module $\mathscr{F}$, s similarly defines a homomorphism $h_s \otimes 1_{\mathscr{F}} : \mathscr{M}_X(\mathscr{F}) \to \mathscr{M}_X(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L})$, which is again *bijective*.

(20.1.10). Let s be a meromorphic section of an invertible $\mathscr{O}_X$-module $\mathscr{L}$ over X; for s to be regular, it is necessary and sufficient for there to exists a meromorphic section s' of $\mathscr{L}^{-1}$ over X such that the canonical image of $s \otimes s'$ in $\mathscr{M}_X$ ($\mathbf{0_I}$, 5.4.3) is the unit section, and this section s' is then unique: indeed, the necessity of the local existence of such a section is evident, and its local uniqueness implies its global (and unique) existence; moreover, the existence of s' is trivially sufficient for s to be regular. We will take $s' = s^{-1}$.

Finally, if $\mathscr{L}'$ is a second invertible $\mathscr{O}_X$-module, s (resp. s') a regular meromorphic section of $\mathscr{L}$ (resp. $\mathscr{L}'$) over X, then $s \otimes s'$ is evidently a regular meromorphic section of $\mathscr{L} \otimes \mathscr{L}'$ over X.

(20.1.11). If $f : X' \to X$ is a morphism of ringed spaces, then there is in general no natural map sending a meromorphic function on X to a meromorphic function on X'. For example, if X is the spectrum of a local integral domain A, X' its residue field k, then there is no natural homomorphism from the field of fractions K of A to k, and we can only send an element of K to an element of k if it is already in A.

In general, if $f = (\psi, \theta)$, then for every open U of X, denote by $\mathscr{S}_f(U)$ the set of *regular* sections $s \in \Gamma(U, \mathscr{O}_X)$ such that the image of s under

$$\Gamma(\theta^\#) : \Gamma(U, \mathscr{O}_X) \longrightarrow \Gamma(f^{-1}(U), \mathscr{O}_{X'})$$

is a *regular* section. It is immediate that $U \mapsto \mathscr{S}_f(U)$ is a *subsheaf* of the sheaf of sets $\mathscr{S}(\mathscr{O}_X)$, which we denote by $\mathscr{S}_f$. We set $\mathscr{M}_f = \mathscr{O}_X[\mathscr{S}_f^{-1}]$; this is a subsheaf of rings of $\mathscr{M}_X$, and we canonically obtain from $\theta^\# : \psi^*(\mathscr{O}_X) \to \mathscr{O}_{X'}$ a homomorphism of sheaves of rings $\theta'^\# : \psi^*(\mathscr{M}_f) \to \mathscr{M}_{X'}$ extending $\theta^\#$ (Bourbaki, *Alg. comm.*, chap. II, §2, n°1, prop. 2); hence, recalling that $f^*(\mathscr{M}_f) = \psi^*(\mathscr{M}_f) \otimes_{\psi^*(\mathscr{O}_X)} \mathscr{O}_{X'}$, we get a canonical homomorphism of $\mathscr{O}_{X'}$-algebras IV-4 | 230

(20.1.11.1) $$f^*(\mathscr{M}_f) \longrightarrow \mathscr{M}_{X'}.$$

For every meromorphic function ϕ on X that *is a section of* $\mathscr{M}_f$, $\Gamma(\theta'^{\#})(\phi)$ is a meromorphic function on X', called the *inverse image of* ϕ *under* f, and denoted by $\phi \circ f$ is there is no cause for confusion.

Similarly, if $\mathscr{F}$ is an $\mathscr{O}_X$-module, then we set $\mathscr{M}_f(\mathscr{F}) = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{M}_f$, and we immediately obtain from $\theta'^{\#}$ a canonical homomorphism (which is also written as $u \mapsto u \circ f$)

$$\Gamma(X, \mathscr{M}_f(\mathscr{F})) \longrightarrow \Gamma(X', \mathscr{M}_{X'}(f^*(\mathscr{F}))).$$

In addition, if $u \in \Gamma(X, \mathscr{M}_f(\mathscr{F}))$ is defined (20.1.7) at a point x, then u coincides, on a neighbourhood U of x, with a section of the form $\sum_i h_i \otimes (t_i/s_i)$, where the h_i belong to $\Gamma(U, \mathscr{F})$, the t_i to $\Gamma(U, \mathscr{O}_X)$, and the s_i to $\Gamma(U, \mathscr{S}_f)$. As by hypothesis the images of the s_i in $\Gamma(f^{-1}(U), \mathscr{O}_{X'})$ are regular, we see that $u \circ f$ is defined at every point of $f^{-1}(U)$; in other words, we have

$$f^{-1}(\mathrm{dom}(u)) \subset \mathrm{dom}(u \circ f). \tag{20.1.11.2}$$

We will see later (20.6.5, (i)) examples (with $\mathscr{F} = \mathscr{O}_X$) where the two sides of (20.1.11.2) can be different.

Consider in particular the case where $\mathscr{M}_f = \mathscr{M}_X$; then, if $\mathscr{L}$ is an invertible $\mathscr{O}_X$-module, the image in $\mathscr{M}_{X'}(f^*(\mathscr{L}))$, under $\Gamma(\theta'^{\#})$, of a *regular* meromorphic section of $\mathscr{L}$ over X (20.1.8) is a *regular* meromorphic section of $f^*(\mathscr{L})$ over X', as it follows immediately from the definition of its sections, and from the fact that a homomorphism of rings sends an invertible element to an invertible element.

Let $f' : X'' \to X'$ be a second morphism of ringed spaces, and suppose that $\mathscr{M}_f = \mathscr{M}_X$ and $\mathscr{M}_{f'} = \mathscr{M}_{X'}$; then, if we set $f'' = f \circ f'$, we also have $\mathscr{M}_{f''} = \mathscr{M}_X$, and we immediately see that for every meromorphic section u of $\mathscr{F}$ over X, we have $u \circ f'' = (u \circ f) \circ f'$.

Proposition (20.1.12). — *If the morphism $f : X' \to X$ is flat* ($\mathbf{0_I}$, 6.7.1)*, then we have $\mathscr{M}_f = \mathscr{M}_X$, and the homomorphism $\phi \mapsto \phi \circ f$ is defined on all of $M(X)$. In addition, if f is a (flat) morphism of locally ringed spaces, then we have* $\mathrm{dom}(\phi \circ f) = f^{-1}(\mathrm{dom}(\phi))$*; if in addition f is surjective (thus faithfully flat), then the homomorphism $\phi \mapsto \phi \circ f$ is injective.*

PROOF. The first assertion follows from the fact that, if B is an A-algebra which is a flat A-module, then every element of A not a divisor of 0 in A is not a divisor of 0 in B ($\mathbf{0_I}$, 6.3.4). To prove the other assertions, note that, for every $x' \in X'$, if $x = f(x')$, then $\mathscr{O}_{X',x'}$ is a flat $\mathscr{O}_{X,x}$-module, and as the homorpism $\mathscr{O}_{X,x} \to \mathscr{O}_{X',x'}$ is *local* by hypothesis, it is injective (($\mathbf{0_I}$, 6.5.1) and ($\mathbf{0_I}$, 6.6.2)); if we set $A = \mathscr{O}_{X,x}$, $B = \mathscr{O}_{X',x'}$, such that A identifies with a subring of B, then $(f^*(\mathscr{M}_X))_{x'}$ is equal to $S^{-1}A \otimes_A B = S^{-1}B$, where S is the set of regular elements of A, $(\mathscr{M}_{X'})_{x'}$ is equal to $T^{-1}B$, where T is the set of regular elements of B, and as we have seen that $S \subset T$, the homomorphism $S^{-1}B \to T^{-1}B$ **IV-4** | 231
is injective; in other words, this proves that the homomorphism (20.1.11.1) $f^*(\mathscr{M}_X) \to \mathscr{M}_{X'}$ is *injective* (hence the last assertion of the statement). The quotient $f^*(\mathscr{M}_X)/\mathscr{O}_{X'}$ identifies with an $\mathscr{O}_{X'}$-submodule of $\mathscr{M}_{X'}/\mathscr{O}_{X'}$, and $(f^*(\mathscr{M}_X)/\mathscr{O}_{X'})_{x'}$ identifies with $(\mathscr{M}_X/\mathscr{O}_X)_x \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{X',x'}$. Then suppose that $x \notin \mathrm{dom}(\phi)$; the image of ϕ_x in $(\mathscr{M}_X/\mathscr{O}_X)_x$ is therefore $\neq 0$; by faithful flatness, we deduce that that it is the same for the image of $(\phi \circ f)_{x'}$, so $x' \notin \mathrm{dom}(\phi \circ f)$, which finishes the proof. □

Remark (20.1.13). — Let X be a *reduced* complex analytic space; then the notion of a meromorphic function on X defined above coincides with the usual notion. Consider on the other hand a prescheme Y, locally of finite type over the field $\mathbf{C}$; we then know that we can associate to Y an analytic space Y^{an} having the same underlying topological space, and the canonical morphism $f : Y^{\mathrm{an}} \to Y$ is *flat* [**?**]; by virtue of (20.1.12), the canonical homomorphism $u \mapsto u \circ f$ from $M(Y)$ to $M(Y^{\mathrm{an}})$ is therefore always defined and is injective; but it is not *surjective* in general. For example, when $Y = \mathbf{V}_0^r$ ($\mathbf{Err_{III}}$, 14) is the affine space of dimension r over $\mathbf{C}$, $M(Y)$ canonically identifies with the field $R(Y)$ of rational functions on Y (20.2.13, (i)), while $M(Y^{\mathrm{an}})$ is the field of usual meromorphic functions on $\mathbf{C}^r$. Because of this fact, it is often preferable, in algebraic geometry, to abstain from the terminology introduced in this section, and to use the equivalent terminology of "pseudo-function" which will be defined below.

20.2. Pseudo-morphisms and pseudo-functions

§21. DIVISORS

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