Quiz 6, Advanced Calculus I, Yung Fu Fang

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Name: $\boxed{}$ Id: $\boxed{}$ $\boxed{\textit{Group}}$

1a. Consider the alternating series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\cdots := \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$.

Set $A_k = \sum_{n=1}^{k} a_n$. Give a direct proof that the series converges.

1b. Consider a rearrangement of the series in (1a), $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\cdots := \sum_{n=1}^{\infty} b_n$.

Find a formula for b_n. Set $B_k = \sum_{n=1}^{k} b_n$. Hint: sort it into two groups

$$\left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \cdots\right\} = \left\{\frac{1}{\boxed{}}\right\}_{n=1}^{\infty} = \left\{\frac{1}{\boxed{\text{4n-3}}} + \frac{1}{\boxed{}}\right\}_{n=1}^{\infty} \text{ and}$$

$$\left\{-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{6}, -\frac{1}{8}, -\frac{1}{10}, \cdots\right\} = \left\{-\frac{1}{\boxed{}}\right\}_{n=1}^{\infty} = \left\{-\frac{1}{\boxed{\text{4n-2}}} - \frac{1}{\boxed{}}\right\}_{n=1}^{\infty}.$$

Find a general formula for $B_{3k} = \sum_{n=1}^{3k}\left(\frac{1}{\boxed{}} + \frac{1}{\boxed{}} - \frac{1}{\boxed{}}\right)$.

1c. Prove that $B_{3k} - B_{3(k-1)}$ is positive for all $k \in \mathbb{N}$. Thus $\{B_{3k}\}_{k=1}^{\infty}$ is $\boxed{}$ sequence.

1d. Show that $\limsup B_k > \frac{5}{6}$.

1e. Show that $\sum_{n=1}^{\infty} a_n < \frac{5}{6}$

1f. If $\sum_{n=1}^{\infty} b_n$ converges, then $\boxed{} \neq \boxed{}$.

1g. Show that the series $\sum_{n=1}^{\infty} b_n$ converges

$\Longleftrightarrow$ Show the sequence $\{B_k\}_{k=1}^{\infty}$ converges

$\Longleftrightarrow$ Show that $\boxed{}$ is a $\boxed{}$ sequence.

$\Longleftrightarrow$ Given $\varepsilon > 0$, there is a large $N > 0$ such that $\left\| \boxed{} - \boxed{} \right\| < \varepsilon$ for all $k > j > N$

$\Longleftrightarrow$ Given $\varepsilon > 0$, $\exists$ a large $N > 0$ such that $\left\| \boxed{} + \cdots + \boxed{} \right\| < \varepsilon$ for all $k > j > N$