9 The Riemann Zeta Function

In order to make progress in number theory, it is sometimes necessary to use techniques from other areas of mathematics, such as algebra, analysis or geometry. In this chapter we give some number-theoretic applications of the theory of infinite series. These are based on the properties of the Riemann zeta function $\zeta(s)$, which provides a link between number theory and real and complex analysis. Some of the results we obtain have probabilistic interpretations in terms of random integers. For the background on convergence of infinite series, see Appendix C. For a detailed treatment of $\zeta(s)$, see Titchmarsh (1951).

9.1 Historical background

One of the most familiar examples of an infinite series is the *harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots.$$

Since number theory is mainly about the positive integers $n = 1, 2, 3 \ldots$, it is not surprising that this series is of interest to number theorists. Unfortunately, it diverges, but only just: the sum of the first n terms is about $\ln n$, and although this tends to $+\infty$ as $n \to \infty$, it does so rather slowly. To make the series converge, without losing its important number-theoretic properties, we replace its general term $1/n$ with the smaller term $1/n^s$, where $s > 1$. This gives rise to the *Riemann zeta function*, defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots. \tag{9.1}$$

Although this function is named after Riemann, who wrote a fundamental paper on its properties in 1859, it was in fact introduced about 120 years earlier by Euler, who showed that it can be expanded as a product

$$\zeta(s) = \prod_p \left(\frac{1}{1-p^{-s}}\right), \tag{9.2}$$

where p ranges over all the primes. This is a very powerful result, since it allows methods of analysis to be applied to the study of prime numbers. Euler regarded $\zeta(s)$ as a function of a real variable s, whereas Riemann's great contribution depended on allowing s to be a complex number, so that the even richer theory of complex functions could be used. One of the great unsolved problems in number theory is the Riemann Hypothesis (see Section 9), a conjecture concerning the complex zeros of $\zeta(s)$; a solution of this would resolve many important problems concerning the distribution of prime numbers.

Before dealing with questions of convergence, we will first outline a proof of (9.2), and then show a simple but effective application of this product formula. Each factor on the right-hand side of (9.2) can be expanded as a geometric series

$$\frac{1}{1-p^{-s}} = 1 + p^{-s} + p^{-2s} + \cdots = \sum_{e=0}^{\infty} p^{-es},$$

convergent since $|p^{-s}| = p^{-s} < 1$ for all $s > 0$. If we multiply these series together (and we will justify this later), then the general term in their product has the form

$$p_1^{-e_1 s} \ldots p_k^{-e_k s} = \frac{1}{n^s},$$

where $p_1, \ldots, p_k$ are distinct primes, each $e_i \geq 0$, and $n = p_1^{e_1} \ldots p_k^{e_k}$. By the Fundamental Theorem of Arithmetic (Theorem 2.3), every integer $n \geq 1$ has a unique factorisation of this form, so it contributes exactly one summand, equal to $1/n^s$, and hence (9.2) represents $\sum 1/n^s = \zeta(s)$. (We will prove this more rigorously later in the chapter, in Theorem 9.3.)

Exercise 9.1

Use a similar argument to outline a proof that

$$\prod_p (1-p^{-s}) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

where μ is the Möbius function, and hence show that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}.$$

Using (9.2), we can now sketch a quick proof of Theorem 2.6, that there are infinitely many primes: if there were only finitely many primes, then $\zeta(s)$ would approach a finite limit $\prod_p (1-p^{-1})^{-1}$ as $s \to 1$, whereas in fact $\zeta(s) \to +\infty$, as we shall shortly see.

9.2 Convergence

To justify the preceding arguments, we must first consider the convergence of the series (9.1). We will show that it converges for all real $s > 1$, and diverges for all real $s \leq 1$. Suppose first that $s > 1$. We group the terms together in blocks of length $1, 2, 4, 8, \ldots$, giving

$$\zeta(s) = 1 + \left(\frac{1}{2^s} + \frac{1}{3^s}\right) + \left(\frac{1}{4^s} + \cdots + \frac{1}{7^s}\right) + \left(\frac{1}{8^s} + \cdots + \frac{1}{15^s}\right) + \cdots.$$

Now

$$\begin{aligned}
\frac{1}{2^s} + \frac{1}{3^s} &\leq \frac{1}{2^s} + \frac{1}{2^s} = \frac{2}{2^s} = 2^{1-s}, \\
\frac{1}{4^s} + \cdots + \frac{1}{7^s} &\leq \frac{1}{4^s} + \cdots + \frac{1}{4^s} = \frac{4}{4^s} = (2^{1-s})^2, \\
\frac{1}{8^s} + \cdots + \frac{1}{15^s} &\leq \frac{1}{8^s} + \cdots + \frac{1}{8^s} = \frac{8}{8^s} = (2^{1-s})^3,
\end{aligned}$$

and so on, so we can compare (9.1) with the geometric series

$$1 + 2^{1-s} + (2^{1-s})^2 + (2^{1-s})^3 + \cdots.$$

This converges since $0 < 2^{1-s} < 1$, and hence so does (9.1) by the Comparison Test (Appendix C). In fact, this argument shows that $1 \leq \zeta(s) \leq f(s)$ for all $s > 1$, where

$$f(s) = \sum_{n=0}^{\infty} (2^{1-s})^n = \frac{1}{1 - 2^{1-s}}.$$

If $s \to +\infty$ then $2^{1-s} \to 0$ and so $f(s) \to 1$, giving

$$\lim_{s \to +\infty} \zeta(s) = 1.$$

We now show that (9.1) diverges for $s \leq 1$. This is obvious if $s \leq 0$, since then $1/n^s \not\to 0$ as $n \to \infty$, so let us assume that $s > 0$. By grouping the terms of (9.1) in blocks of length $1, 1, 2, 4, \ldots$, we have

$$\zeta(s) = 1 + \frac{1}{2^s} + \left(\frac{1}{3^s} + \frac{1}{4^s}\right) + \left(\frac{1}{5^s} + \cdots + \frac{1}{8^s}\right) + \cdots.$$

If $s \leq 1$ then

$$\begin{aligned} \frac{1}{2^s} &\geq \frac{1}{2}, \\ \frac{1}{3^s} + \frac{1}{4^s} &\geq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}, \\ \frac{1}{5^s} + \cdots + \frac{1}{8^s} &\geq \frac{1}{8} + \cdots + \frac{1}{8} = \frac{1}{2}, \end{aligned}$$

and so on, so (9.1) diverges by comparison with the divergent series $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$. In particular, by taking $s = 1$ we see that the harmonic series $\sum 1/n$ diverges.

To summarise, we have proved:

Theorem 9.1

The series (9.1) converges for all real $s > 1$, and diverges for all real $s \leq 1$.

There is an alternative proof based on the Integral Test (Appendix C), using the fact that $\int_1^{+\infty} x^{-s}\,dx$ converges if and only if $s > 1$.

Exercise 9.2

Show that if $s > 1$ then $\zeta(s) \geq (1 + f(s))/2$, where $f(s)$ is as defined above, and deduce that $\zeta(s) \to +\infty$ as $s \to 1$.

9.3 Applications to prime numbers

We can now give a more rigorous analytic proof of Theorem 2.6, that there are infinitely many primes. Suppose that there are only finitely many primes, say $p_1, \ldots, p_k$. For each prime $p = p_i$ we have $|1/p| < 1$, so there is a convergent geometric series

$$1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots = \frac{1}{1 - p^{-1}}.$$

It follows that if we multiply these k different series together, their product

$$\prod_{i=1}^{k}\left(1+\frac{1}{p_i}+\frac{1}{p_i^2}+\frac{1}{p_i^3}+\cdots\right)=\prod_{i=1}^{k}\left(\frac{1}{1-p_i^{-1}}\right) \tag{9.3}$$

is finite. Now these convergent series all consist of positive terms, so they are absolutely convergent. It follows (see Appendix C) that we can multiply out the series in (9.3) and rearrange the terms, without changing the product. If we take a typical term $1/p_1^{e_1}$ from the first series, $1/p_2^{e_2}$ from the second series, and so on, where each $e_i \geq 0$, then their product

$$\frac{1}{p_1^{e_1}}\cdot\frac{1}{p_2^{e_2}}\cdot\dots\cdot\frac{1}{p_k^{e_k}}=\frac{1}{p_1^{e_1}p_2^{e_2}\dots p_k^{e_k}}$$

will represent a typical term in the expansion of (9.3). By the Fundamental Theorem of Arithmetic, every integer $n \geq 1$ has a unique expression

$$n=p_1^{e_1}p_2^{e_2}\dots p_k^{e_k} \qquad (e_i \geq 0)$$

as a product of powers of the primes p_i, since we are assuming that these are the *only* primes; notice that we allow $e_i = 0$, in case n is not divisible by a particular prime p_i. This uniqueness implies that each n contributes exactly one term $1/n$ to (9.3), so the expansion takes the form

$$\prod_{i=1}^{k}\left(1+\frac{1}{p_i}+\frac{1}{p_i^2}+\frac{1}{p_i^3}+\cdots\right)=\sum_{n=1}^{\infty}\frac{1}{n}.$$

The right-hand side is the *harmonic series*, which is divergent. However, we have seen that the left-hand side is finite, so this contradiction proves that there must be infinitely many primes.

The next result, also due to Euler, develops this method a little further:

Theorem 9.2

If p_n denotes the n-th prime (in increasing order), then the series

$$\sum_{n=1}^{\infty}\frac{1}{p_n}=\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\cdots$$

diverges.

Proof

If $\sum 1/p_n$ converges to a finite sum l, then its partial sums must satisfy

$$\left|\sum_{n=1}^{N}\frac{1}{p_n}-l\right|\leq\frac{1}{2}$$

for all sufficiently large N, so that

$$\sum_{n>N} \frac{1}{p_n} \leq \frac{1}{2}$$

for any such N. This implies that the series

$$\sum_{k=1}^{\infty} \left(\sum_{n>N} \frac{1}{p_n} \right)^k \tag{9.4}$$

converges by comparison with the geometric series $\sum_{k=1}^{\infty} 1/2^k$. If q denotes the product $p_1 \ldots p_N$ then no integer of the form $qr + 1$ $(r \geq 1)$ can be divisible by any of $p_1, \ldots, p_N$, so it must be a product of primes p_n for $n > N$ (possibly with repetitions), say

$$qr + 1 = p_{n_1} \cdots p_{n_k}$$

where each $n_i > N$. Then the reciprocal $1/(qr + 1)$ of each such an integer appears as a summand $1/p_{n_1} \ldots p_{n_k}$ in the expansion of

$$\left(\sum_{n>N} \frac{1}{p_n} \right)^k,$$

and hence it appears (just once) as a summand in the expansion of (9.4). Since (9.4) converges, it follows that the series

$$\sum_{r=1}^{\infty} \frac{1}{qr+1},$$

which is contained within (9.4), also converges. However this series diverges, since its terms exceed those of the divergent series

$$\sum_{r=1}^{\infty} \frac{1}{qr+q} = \frac{1}{q} \sum_{r=1}^{\infty} \frac{1}{r+1} = \frac{1}{q} \sum_{r=2}^{\infty} \frac{1}{r}.$$

This contradiction shows that $\sum 1/p_n$ must diverge. □

Comments

1 It can be shown that

$$\frac{1}{p_1} + \cdots + \frac{1}{p_n} \to +\infty$$

about as fast as $\ln \ln n$, so this series diverges very slowly indeed.

2 Theorem 9.2 gives yet another proof that there are infinitely many primes, since a finite series must converge. It also shows that the primes are more densely distributed than the perfect squares: the series $\sum 1/n^2$ converges (by the Integral Test), so $1/n^2 \to 0$ faster than $1/p_n \to 0$ as $n \to \infty$, that is, primes occur more frequently than squares.

We can now use these ideas to give a rigorous proof of the product expansion (9.2):

Theorem 9.3

If $s > 1$ then

$$\zeta(s) = \prod_p \left(\frac{1}{1-p^{-s}}\right),$$

where the product is over all primes p.

Proof

The method is to consider the product $P_k(s)$ of the factors corresponding to the first k primes, and to show that $P_k(s) \to \zeta(s)$ as $k \to \infty$. Let $p_1, \ldots, p_k$ be the first k primes. Arguing as before with (9.3), we see that if $s > 0$ (so that the geometric series all converge) then

$$P_k(s) = \prod_{i=1}^{k} \left(\frac{1}{1-p_i^{-s}}\right) = \prod_{i=1}^{k} \left(1 + \frac{1}{p_i^s} + \frac{1}{p_i^{2s}} + \frac{1}{p_i^{3s}} + \cdots\right).$$

If we expand this product, the general term in the resulting series is $1/n^s$ where $n = p_1^{e_1} \ldots p_k^{e_k}$ and each $e_i \geq 0$. The Fundamental Theorem of Arithmetic implies that each such n contributes just one term to $P_k(s)$, so

$$P_k(s) = \sum_{n \in A_k} \frac{1}{n^s},$$

where $A_k = \{n \mid n = p_1^{e_1} \ldots p_k^{e_k},\ e_i \geq 0\}$ is the set of integers n whose prime factors are among $p_1, \ldots, p_k$. Each $n \notin A_k$ is divisible by some prime $p > p_k$, and so $n > p_k$. It follows that if $s > 1$ then

$$|P_k(s) - \zeta(s)| = \sum_{n \notin A_k} \frac{1}{n^s} \leq \sum_{n > p_k} \frac{1}{n^s} = \zeta(s) - \sum_{n \leq p_k} \frac{1}{n^s}.$$

Since $s > 1$, the partial sums of the series $\sum 1/n^s$ converge to $\zeta(s)$, so in particular

$$\sum_{n \leq p_k} \frac{1}{n^s} \to \zeta(s)$$

as $k \to \infty$. Thus $|P_k(s) - \zeta(s)| \to 0$ as $k \to \infty$, so $P_k(s) \to \zeta(s)$ as required. □

Exercise 9.3

Show that $P_k(1) \to +\infty$ as $k \to \infty$, and deduce that for each $\varepsilon > 0$ there exists n such that $\phi(n)/n < \varepsilon$. (See Exercise 5.11 in Chapter 5 for a similar result, and for a probabilistic interpretation of this.)

A similar method gives a rigorous prooof of the following result. We will also prove this as part of a more general result later in this chapter (see Example 9.4).

Theorem 9.4

If $s > 1$ then

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p (1 - p^{-s}) = \frac{1}{\zeta(s)}.$$

Exercise 9.4

Prove Theorem 9.4.

9.4 Random integers

As an application of the Riemann zeta function, we will calculate the probability P that a pair of randomly-chosen integers are coprime. Since we do not wish to spend too much time on some of the finer details of probability theory, we will simply outline the main points. We will, in fact, use three different methods, leading to the formulae

$$\frac{1}{\zeta(2)}, \qquad \prod_p (1 - p^{-2}), \qquad \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}$$

for P, and the fact that they must all be equal gives an alternative proof of Theorem 9.4 in the case $s = 2$. (In fact, one can extend this to all integers $s \geq 2$ by calculating the probability that s randomly-chosen integers have greatest common divisor 1.) We will then show that $\zeta(2) = \pi^2/6$, so that $P = 6/\pi^2$.

There is an interesting geometric application of this result. An *integer point* in the plane $\mathbb{R}^2$ is a point with integer coordinates. Such a point A is *visible from the origin* if the line-segment AO joining A to the origin $O = (0, 0)$ contains no integer points other than A and O. It is easy to show that an integer point $A \neq O$ is visible from O if and only if its coordinates are coprime, and it follows from this that P represents the probability that a randomly-chosen integer point is visible from O. (Restricting to positive coordinates does not alter the probability.) To put this another way, P is the proportion of the integer lattice $\mathbb{Z}^2$ which can be seen from any given integer point.

Exercise 9.5

Show that an integer point $(x, y) \neq (0,0)$ is visible from O if and only if x and y are coprime.

There is an immediate problem in discussing randomly-chosen integers $x \in \mathbb{N}$ (or indeed randomly-chosen elements of any infinite set). If p_n denotes the probability $\Pr(x = n)$ that a particular integer n is chosen, then clearly

$$\sum_{n=1}^{\infty} p_n = 1 .$$

However, if we want all integers to have the same status, then p_n must be a constant, independent of n, so that $\sum p_n$ is either 0 or divergent.

One way of avoiding this difficulty is to assign probabilities to certain *sets* of integers, rather than to individual integers. For any integers c and n we will assign the probability

$$\Pr\bigl(x \equiv c \bmod (n)\bigr) = \frac{1}{n},$$

so that x has the same probability $1/n$ of lying in any of the n classes $[c] \in \mathbb{Z}_n$.

Now the Chinese Remainder Theorem (Theorem 3.10) implies that if m and n are coprime, then the solutions x of any pair of simultaneous congruences

$$x \equiv b \bmod (m),$$

$$x \equiv c \bmod (n)$$

form a single congruence class mod (mn); thus

$$\begin{aligned}\Pr\bigl(x \equiv b \bmod (m) \text{ and } x \equiv c \bmod (n)\bigr) \\ = \quad \frac{1}{mn} = \Pr\bigl(x \equiv b \bmod (m)\bigr). \Pr\bigl(x \equiv c \bmod (n)\bigr),\end{aligned}$$

so the pair of congruences are statistically independent. (Two events are *statistically independent* if the probability of them both happening is the product of their individual probabilities.) The same argument applies to any finite set of linear congruences with mutually coprime moduli.

Suppose now that x and y are chosen randomly from $\mathbb{N}$, as above, and that they are also chosen independently, that is, that

$$\Pr(x, y \in S) = \Pr(x \in S). \Pr(y \in S)$$

for any subset S of $\mathbb{N}$ for which these probabilites are defined. Let

$$P = \Pr\big(\gcd(x,y) = 1\big)$$

denote the probability that x and y are coprime. We will calculate P in three different ways.

Method A For each $n \in \mathbb{N}$, we have

$$\gcd(x,y) = n \iff \begin{cases} x \equiv 0 \bmod (n) & \text{and} \\ y \equiv 0 \bmod (n) & \text{and} \\ \gcd(x/n, y/n) = 1. \end{cases}$$

Now the conditions $x \equiv 0$ mod (n) and $y \equiv 0$ mod (n) are each satisfied with probability $1/n$; since x and y are chosen independently, these two conditions are simultaneously satisfied with probability $1/n^2$. When they are both satisfied, we can regard x/n and y/n as randomly-chosen integers, so they will be coprime (the third condition) with conditional probability P. It follows that the three conditions are simultaneously satisfied with probability P/n^2, so

$$\Pr\big(\gcd(x,y) = n\big) = \frac{P}{n^2}\,.$$

Now $\gcd(x,y)$ must take a unique value $n \in \mathbb{N}$ for each pair x, y, so the sum of all these probabilities must be equal to 1. Thus

$$1 = \sum_{n=1}^{\infty} \Pr\big(\gcd(x,y) = n\big) = \sum_{n=1}^{\infty} \frac{P}{n^2} = P \sum_{n=1}^{\infty} \frac{1}{n^2} = P\zeta(2)\,,$$

and hence

$$P = \frac{1}{\zeta(2)}\,.$$

Method B We have

$$\gcd(x,y) = 1 \iff \begin{cases} x \not\equiv 0 \bmod (p) \\ \quad\text{or} \\ y \not\equiv 0 \bmod (p) \end{cases}$$

for every prime p. For each p, the congruences $x \equiv 0$ and $y \equiv 0$ mod (p) each have probability p^{-1}, so $x \equiv 0 \equiv y$ mod (p) with probability p^{-2}, and hence $x \not\equiv 0$ or $y \not\equiv 0$ mod (p) with probability $1 - p^{-2}$. Now congruences modulo distinct primes are statistically independent, so we multiply these probabilities for all primes p to get

$$P = \prod_p \left(1 - p^{-2}\right).$$

(Strictly speaking, we need to justify the use of an infinite product here, since we have discussed statistical independence of finitely many congruences, but not infinitely many; for simplicity of exposition, we will omit the details of this.)

Method C. We have

$$\gcd(x,y)>1 \iff \begin{cases} x\equiv 0 \bmod (p) \\ \quad\text{and} \\ y\equiv 0 \bmod (p) \end{cases}$$

for some prime p. The event $\gcd(x,y)>1$ has probability $1-P$, so this must be the probability that $x\equiv 0\equiv y \bmod(p)$ for at least one prime p. We will now use the Inclusion–Exclusion Principle (see Exercise 5.10) to find an alternative expression for this probability. For each p, the event $x\equiv 0\equiv y \bmod (p)$ has probability p^{-2}, so adding these probabilities for all p we get a contribution

$$S_1=\sum_p p^{-2}$$

to $1-P$. From this we subtract a double sum

$$S_2=\sum_{p<q}(pq)^{-2}$$

to compensate for the double counting in S_1 of cases in which x and y are divisible by two primes $p<q$. We then add a triple sum

$$S_3=\sum_{p<q<r}(pqr)^{-2}$$

to allow for over-compensation in S_2 of integers divisible by three primes, and so on. Thus

$$1-P=S_1-S_2+S_3-\cdots,$$

where the general term S_k has the form

$$S_k=\sum(p_1\dots p_k)^{-2},$$

summing over all increasing k-tuples $p_1<\cdots<p_k$ of distinct primes. If we define $S_0=1$ then we can write

$$P=\sum_{k=0}^{\infty}(-1)^k S_k\,.$$

In this expression for P, every square-free integer $n=p_1\dots p_k\in\mathbb{N}$ contributes one summand $(-1)^k n^{-2}=\mu(n)n^{-2}$, where μ is the Möbius function, while all other integers $n \geq 1$ contribute nothing and satisfy $\mu(n) = 0$; using absolute convergence to justify rearrangement, we therefore have

$$P = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}.$$

Exercise 9.6

For each integer $s \geq 2$, let $P(s)$ denote the probability that s randomly- and independently-chosen integers have greatest common divisor 1 (so $P = P(2)$). Give three arguments to show that $P(s)$ is given by the formulae

$$\frac{1}{\zeta(s)}, \qquad \prod_p (1 - p^{-s}), \qquad \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

Exercise 9.7

Prove (in three different ways) that a single randomly-chosen integer x is square-free with probability $P = 1/\zeta(2)$. (Hint: consider $\mathrm{Sq}(x)$, the largest square factor of x.)

Exercise 9.8

For each integer $s \geq 2$, calculate (in three different ways) the probability $Q(s)$ that a randomly-chosen integer x should be s-th power-free, that is, divisible by no s-th power greater than 1.

9.5 Evaluating $\zeta(2)$

Having shown that $P = 1/\zeta(2)$, we now prove that

$$\zeta(2) = \frac{\pi^2}{6}. \tag{9.5}$$

Apostol (1983) gives an elementary proof of this, evaluating

$$\int_0^1 \int_0^1 (1 - xy)^{-1} \, dx \, dy$$

in two ways: first by writing $(1 - xy)^{-1} = \sum (xy)^n$ and integrating term by term, and second by using a change of variables to rotate the xy-plane through

$\pi/4$ and then using some straightforward trigonometric substitutions. A quicker but less elementary proof is simply to put $x = 1$ in the Fourier series expansion

$$x^2 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n \geq 1} \frac{(-1)^n}{n^2} \cos(n\pi x)$$

of the function x^2 on the interval $[-1, 1]$; we have $\cos(n\pi x) = (-1)^n$, so (9.5) follows immediately. (See Chapter IV of Churchill, 1963 for background on Fourier series.)

Instead, we will give a third proof, which has the advantage of extending to certain other values of $\zeta(s)$. We will use the infinite product expansion

$$\sin z = z \prod_{n \neq 0} \left(1 - \frac{z}{n\pi}\right) = z \prod_{n \geq 1} \left(1 - \frac{z^2}{n^2\pi^2}\right), \tag{9.6}$$

proofs of which can be found in books on complex variable theory, e.g. Jones and Singerman (1987, Chapter 3, Section 8). The first product in (9.6) is over all non-zero integers n, and the second product is obtained from the first by pairing the factors corresponding to $\pm n$. One can explain (if not rigorously prove) the first product by regarding $\sin z$ as behaving like a polynomial with infinitely many zeros at $z = n\pi$ $(n \in \mathbb{Z})$, so we have a 'factorisation'

$$\sin z = cz \prod_{n \neq 0} \left(1 - \frac{z}{n\pi}\right)$$

with

$$c = \lim_{z \to 0} \frac{\sin z}{z} = 1\,.$$

By expanding the second product in (9.6) and collecting powers of z, we obtain a power series for $\sin z$ which must coincide with its Taylor series expansion

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\,. \tag{9.7}$$

By comparing coefficients of z^3 in (9.6) and (9.7) we see that

$$-\sum_{n \geq 1} \frac{1}{n^2\pi^2} = -\frac{1}{3!}\,,$$

so multiplying through by $-\pi^2$ we obtain (9.5).

With the aid of the previous section and a pocket calculator, we immediately deduce:

Theorem 9.5

The probability that two randomly- and independently-chosen integers are coprime is given by

$$P = \frac{1}{\zeta(2)} = \frac{6}{\pi^2} = 0.607927101\ldots.$$

By Exercise 9.7, this is also the probability that a single randomly-chosen integer is square-free.

9.6 Evaluating $\zeta(2k)$

For many reasons, it would be useful to know the values of $\zeta(s)$ for *all* integers $s \geq 2$ (see Exercise 9.6, for instance). In 1978 Apéry proved a long-standing conjecture, that $\zeta(3)$ is irrational, but very little else is known about $\zeta(s)$ when s is odd. However, with a little extra work we can use (9.6) to evaluate $\zeta(s)$ for all even integers $s = 2k \geq 2$. Some of the techniques we use require rather careful analytic justification, using such concepts as uniform convergence, but for simplicity we will omit these details.

By taking logarithms in (9.6), we have

$$\ln \sin z = \ln z + \sum_{n\geq 1} \ln\left(1 - \frac{z^2}{n^2\pi^2}\right),$$

and differentiating term by term we have

$$\cot z = \frac{1}{z} - \sum_{n\geq 1} \frac{2z}{n^2\pi^2}\left(1 - \frac{z^2}{n^2\pi^2}\right)^{-1}.$$

If we use the geometric series to write

$$\frac{2z}{n^2\pi^2}\left(1 - \frac{z^2}{n^2\pi^2}\right)^{-1} = \frac{2z}{n^2\pi^2}\sum_{k\geq 0}\left(\frac{z^2}{n^2\pi^2}\right)^k = 2\sum_{k\geq 0}\frac{z^{2k+1}}{n^{2k+2}\pi^{2k+2}} = 2\sum_{k\geq 1}\frac{z^{2k-1}}{n^{2k}\pi^{2k}},$$

and then collect powers of z, we get

$$\cot z = \frac{1}{z} - 2\sum_{k\geq 1}\sum_{n\geq 1}\frac{z^{2k-1}}{n^{2k}\pi^{2k}} = \frac{1}{z} - 2\sum_{k\geq 1}\frac{\zeta(2k)z^{2k-1}}{\pi^{2k}}, \tag{9.8}$$

which is the Laurent series for $\cot z$.

We will now compare (9.8) with a second expansion for $\cot z$. The exponential series

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$$

implies that

$$\frac{e^t - 1}{t} = 1 + \frac{t}{2!} + \frac{t^2}{3!} + \cdots,$$

and the reciprocal of this has a Taylor series expansion which can be written in the form

$$\frac{t}{e^t - 1} = \left(1 + \frac{t}{2!} + \frac{t^2}{3!} + \cdots\right)^{-1} = \sum_{m \geq 0} \frac{B_m}{m!} t^m \tag{9.9}$$

for certain constants $B_0, B_1, \ldots$, known as the *Bernoulli numbers*. Now

$$\begin{aligned}
\frac{t}{e^t - 1} &= \frac{t}{2}\left(\frac{e^t + 1}{e^t - 1} - 1\right) \\
&= \frac{t}{2}\left(\frac{e^{t/2} + e^{-t/2}}{e^{t/2} - e^{-t/2}} - 1\right) \\
&= \frac{t}{2}\left(\coth\frac{t}{2} - 1\right) \\
&= \frac{t}{2}\left(i\cot\frac{it}{2} - 1\right)
\end{aligned}$$

where $i = \sqrt{-1}$. Putting $z = it/2$ we get

$$\frac{t}{e^t - 1} = z\cot z - \frac{z}{i} = z\cot z + iz,$$

so dividing by z and using (9.9) we have

$$\cot z = -i + \frac{1}{z}\sum_{m \geq 0} \frac{B_m}{m!} t^m = -i + \sum_{m \geq 0} \frac{B_m}{m!}\left(\frac{2}{i}\right)^m z^{m-1}.$$

By comparing coefficients with those in (9.8), we see that if $m = 2k \geq 2$ then

$$-2\frac{\zeta(2k)}{\pi^{2k}} = \frac{B_{2k}}{(2k)!}\left(\frac{2}{i}\right)^{2k},$$

so that

$$\zeta(2k) = \frac{(-1)^{k-1} 2^{2k-1} \pi^{2k} B_{2k}}{(2k)!}. \tag{9.10}$$

Thus

$$\zeta(2) = \pi^2 B_2, \qquad \zeta(4) = -\frac{\pi^4 B_4}{3}, \qquad \zeta(6) = \frac{2\pi^6 B_6}{45},$$

and so on.

To evaluate the Bernoulli numbers, we write (9.9) in the form

$$t = \sum_{m \geq 0} \frac{B_m}{m!} t^m . (e^t - 1) = \sum_{m \geq 0} \frac{B_m}{m!} t^m . \sum_{n \geq 1} \frac{1}{n!} t^n . \tag{9.11}$$

If we put $m+n=r$, we find that the coefficient of t^r in the right-hand side of (9.11) is

$$\sum_{m+n=r}\frac{B_m}{m!\,n!}=\sum_{m=0}^{r-1}\frac{B_m}{m!\,(m-r)!}=\frac{1}{r!}\sum_{m=0}^{r-1}\binom{r}{m}B_m\,.$$

Comparing this with the left-hand side of (9.11), we see that

$$\frac{1}{r!}\sum_{m=0}^{r-1}\binom{r}{m}B_m=\begin{cases}1 & \text{if } r=1,\\ 0 & \text{if } r>1.\end{cases} \tag{9.12}$$

For $r=1,2,\ldots$, this is an infinite sequence of linear equations

$$\begin{aligned} B_0 &= 1\,,\\ B_0+2B_1 &= 0\,,\\ B_0+3B_1+3B_2 &= 0\,,\end{aligned}$$

and so on, which we can solve in succession to find each B_m. (A more efficient but less elementary method for evaluating Bernoulli numbers is given in Graham *et al.*, 1989, Chapter 6, Section 5.) The first few values are

$$B_0=1,\quad B_1=-\frac{1}{2},\quad B_2=\frac{1}{6},\quad B_3=0,\quad B_4=-\frac{1}{30},\quad B_5=0,\quad B_6=\frac{1}{42},$$

and so on. (In particular, $B_m=0$ for all odd $m>1$, reflecting the fact that $\cot z$ is an odd function.) Substituting these values for even m in (9.10), we get

$$\zeta(2)=\frac{\pi^2}{6}=1.64493406\ldots,$$

$$\zeta(4)=\frac{\pi^4}{90}=1.08232323\ldots,$$

$$\zeta(6)=\frac{\pi^6}{945}=1.01734306\ldots,$$

so that in the notation of Exercises 9.6 and 9.8 we have

$$P(2)=Q(2)=\frac{6}{\pi^2}=0.607927101\ldots,$$

$$P(4)=Q(4)=\frac{90}{\pi^4}=0.923938402\ldots,$$

$$P(6)=Q(6)=\frac{945}{\pi^6}=0.982952592\ldots,$$

and so on.

The coefficients in the linear equations (9.12) are all rational numbers, so it follows by induction on r that the Bernoulli numbers are all rational. It then follows from (9.10) that $\zeta(2k)$ is a rational multiple of π^{2k}, so $P(2k)$ is a rational multiple of π^{-2k}. Now a complex number is said to be *algebraic* if it is a root of some non-trivial polynomial with integer coefficients (for instance, $\sqrt{2}$ is a root of $x^2 - 2$); all other complex numbers are called *transcendental*. In 1882 Lindemann proved that π is transcendental; it follows easily that π^{2k}, and hence $\zeta(2k)$ and $P(2k)$, are also transcendental, and are therefore irrational.

Exercise 9.9

Assuming Lindemann's result, prove the remarks in the last sentence.

9.7 Dirichlet series

We defined the Riemann zeta function as $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$, and then saw that $1/\zeta(s) = \sum_{n=1}^{\infty} \mu(n)/n^s$. These are just two examples of an important class of series of this general form.

Definition

If f is an arithmetic function, then its *Dirichlet series* is the series

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

For convenience, we will often abbreviate this to $F(s) = \sum f(n)/n^s$, with the convention that $\sum$ without limits denotes $\sum_{n=1}^{\infty}$. Just as generating functions $A(x) = \sum a_n x^n$ are useful for studying sequences (a_n) defined by recurrence relations, Dirichlet series $F(s)$ are useful for studying arithmetic functions f, especially those associated with primes and divisors. For instance, in 1837 Dirichlet used series of this type, called *L-series*, to prove Theorem 2.10, that if a and b are coprime then there are infinitely many primes $p \equiv b \bmod (a)$. The arithmetic functions u, N and μ have particularly simple Dirichlet series:

Example 9.1

If $f = u$ then $F(s) = \sum u(n)/n^s = \sum 1/n^s = \zeta(s)$.

Example 9.2

If $f = N$ then $F(s) = \sum N(n)/n^s = \sum n/n^s = \sum 1/n^{s-1} = \zeta(s-1)$.

Example 9.3

If $f = \mu$ then $F(s) = \sum \mu(n)/n^s = 1/\zeta(s)$ by Theorem 9.4.

The next result helps to explain the importance of Dirichlet series: multiplication of Dirichlet series corresponds to the Dirichlet product of arithmetic functions.

Theorem 9.6

Suppose that

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \qquad G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s} \qquad \text{and} \qquad H(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s},$$

where $h = f * g$. Then

$$H(s) = F(s)G(s)$$

for all s such that $F(s)$ and $G(s)$ both converge absolutely.

Proof

If $F(s)$ and $G(s)$ both converge absolutely, then we can multiply these series and rearrange their terms to give

$$\begin{aligned}
F(s)G(s) &= \sum_{n=1}^{\infty} \frac{f(n)}{n^s} . \sum_{n=1}^{\infty} \frac{g(n)}{n^s} \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{f(m)g(n)}{(mn)^s} \\
&= \sum_{k=1}^{\infty} \sum_{mn=k} \frac{f(m)g(n)}{k^s} \\
&= \sum_{k=1}^{\infty} \frac{(f * g)(k)}{k^s} \\
&= \sum_{k=1}^{\infty} \frac{h(k)}{k^s} \\
&= H(s) .
\end{aligned}$$

□

Example 9.4

If we take $f = \mu$ and $g = u$, then $h = f * g = \mu * u = I$ by our definition of μ (Chapter 8, Sections 3 and 6). Now $I(1) = 1$ and $I(n) = 0$ for all $n > 1$, so $H(s) = \sum I(n)/n^s = 1$ for all s. We have $F(s) = \sum \mu(n)/n^s$, and $G(s) = \sum u(n)/n^s = \sum 1/n^s = \zeta(s)$, both absolutely convergent for $s > 1$; hence Theorem 9.6 gives

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.\zeta(s) = 1\,,$$

so that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$$

for all $s > 1$, proving part of Theorem 9.4.

Example 9.5

Let $f = \phi$ and $g = u$. As before, $G(s) = \zeta(s)$ is absolutely convergent for $s > 1$. Now $1 \leq \phi(n) \leq n$ for all n, so $F(s) = \sum \phi(n)/n^s$ is absolutely convergent by comparison with $\sum n/n^s = \zeta(s-1)$ for $s - 1 > 1$, that is, for $s > 2$. Thus Theorem 9.6 is valid for $s > 2$. Now Theorem 5.8 gives $\phi * u = N$, so

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s}.\zeta(s) = \sum_{n=1}^{\infty} \frac{N(n)}{n^s} = \sum_{n=1}^{\infty} \frac{n}{n^s} = \zeta(s-1)\,,$$

and hence

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}$$

for all $s > 2$.

Exercise 9.10

Show that

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \zeta(s)^2$$

for all $s > 1$.

Exercise 9.11

Express $\sum_{n=1}^{\infty} \sigma_k(n)/n^s$ in terms of the Riemann zeta function, where $\sigma_k(n) = \sum_{d|n} d^k$.

Exercise 9.12

Liouville's function λ is defined by

$$\lambda(p_1^{e_1}\dots p_k^{e_k}) = (-1)^{e_1+\dots+e_k}$$

where $p_1, \dots, p_k$ are distinct primes. Show that

$$\sum_{d|n} \lambda(d) = \begin{cases} 1 & \text{if } n \text{ is a perfect square,} \\ 0 & \text{otherwise,} \end{cases}$$

and hence show that

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}$$

for all $s > 1$.

Exercise 9.13

Let $\nu(n)$ be the number of distinct primes dividing n (so that $\nu(60) = 3$, for instance). Show that

$$\sum_{n=1}^{\infty} \frac{\nu(n)}{n^s} = \zeta(s) \sum_{p} \frac{1}{p^s},$$

where p ranges over the set of primes. For which real s is this valid?

9.8 Euler products

Many Dirichlet series have product expansions analogous to that in Theorem 9.3, in which the factors are indexed by the primes. These are called *Euler products*. First we need to consider a stronger form of multiplicativity.

Definition

An arithmetic function f is *completely multiplicative* if $f(mn) = f(m)f(n)$ for all positive integers m and n.

Example 9.6

The functions N, u and I are completely multiplicative, whereas the multiplicative functions μ and ϕ are not.

Theorem 9.7

(a) If f is multiplicative, and $\sum_{n=1}^{\infty} f(n)$ is absolutely convergent, then

$$\sum_{n=1}^{\infty} f(n) = \prod_{p} \left(1 + f(p) + f(p^2) + \cdots\right).$$

(b) If f is completely multiplicative, and $\sum_{n=1}^{\infty} f(n)$ is absolutely convergent, then

$$\sum_{n=1}^{\infty} f(n) = \prod_{p} \left(\frac{1}{1 - f(p)}\right).$$

In each case, p ranges over all the primes.

Proof

(a) The proof follows that used for Theorem 9.3. Let $p_1, \ldots, p_k$ be the first k primes, and let

$$P_k = \prod_{i=1}^{k} \left(1 + f(p_i) + f(p_i^2) + \cdots\right).$$

The general term in the expansion of P_k is $f(p_1^{e_1}) \ldots f(p_k^{e_k}) = f(p_1^{e_1} \ldots p_k^{e_k})$, because f is multiplicative. Thus

$$P_k = \sum_{n \in A_k} f(n)$$

where $A_k = \{\, n \mid n = p_1^{e_1} \ldots p_k^{e_k},\ e_i \geq 0 \,\}$. We have

$$\left| P_k - \sum_{n=1}^{\infty} f(n) \right| = \left| \sum_{n \notin A_k} f(n) \right| \leq \sum_{n \notin A_k} |f(n)| \leq \sum_{n > p_k} |f(n)|,$$

since $n > p_k$ for each $n \notin A_k$. Now $\sum_{n=1}^{\infty} |f(n)|$ converges, so as $k \to \infty$ we have $\sum_{n > p_k} |f(n)| \to 0$ and hence $|P_k - \sum_{n=1}^{\infty} f(n)| \to 0$; thus $P_k \to \sum_{n=1}^{\infty} f(n)$ as $k \to \infty$, as required.

(b) If f is completely multiplicative, then $f(p^e) = f(p)^e$ for each prime-power p^e, so part (a) gives

$$\begin{aligned} \sum_{n=1}^{\infty} f(n) &= \prod_{p} \left(1 + f(p) + f(p^2) + \cdots\right) \\ &= \prod_{p} \left(1 + f(p) + f(p)^2 + \cdots\right) \\ &= \prod_{p} \left(\frac{1}{1 - f(p)}\right). \end{aligned}$$

□

We can apply this result to Dirichlet series:

Corollary 9.8

Suppose that $\sum_{n=1}^{\infty} f(n)/n^s$ converges absolutely. If f is multiplicative, then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \cdots\right),$$

and if f is completely multiplicative, then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(\frac{1}{1 - f(p)p^{-s}}\right).$$

Proof

In Theorem 9.7, we simply replace $f(n)$ with $f(n)n^{-s}$, which is multiplicative (or completely multiplicative) if and only if $f(n)$ is. □

Example 9.7

The function u is completely multiplicative, so as a special case we get Theorem 9.3, that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(\frac{1}{1 - p^{-s}}\right)$$

for all $s > 1$.

Example 9.8

The Möbius function $\mu(n)$ is multiplicative, with $\mu(p) = -1$ and $\mu(p^e) = 0$ for all $e \geq 2$, so

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \left(1 + \frac{\mu(p)}{p^s} + \frac{\mu(p^2)}{p^{2s}} + \cdots\right) = \prod_p \left(1 - p^{-s}\right)$$

for all $s > 1$. Inverting the factors in this product we obtain $1/\zeta(s)$ by the previous example, so this completes the proof of Theorem 9.4 which we promised earlier.

Exercise 9.14

Find the Euler product expansion for the Dirichlet series $\sum_{n=1}^{\infty}|\mu(n)|/n^s$, and hence show that $\sum_{n=1}^{\infty}|\mu(n)|/n^s = \zeta(s)/\zeta(2s)$ for $s > 1$. Deduce that if $n > 1$ then $\sum_d \lambda(d) = 0$, where d ranges over the divisors of n such that n/d is square-free. (Here λ is Liouville's function, defined in Exercise 9.12.)

Exercise 9.15

Show that $\prod_{p\leq x}(1-p^{-1}) \to 0$ as $x \to +\infty$.

9.9 Complex variables

In considering Dirichlet series $F(s) = \sum f(n)/n^s$, such as the Riemann zeta function $\sum 1/n^s$, we have assumed (often implicitly) that the variable s is real. For many purposes, this is adequate, but for some more advanced applications it is necessary to allow s to be complex. The advantage of this is that functions of a complex variable are often easier to deal with than those of a real variable: in particular, their domains of definition can often be extended by analytic continuation, and they can be integrated by the calculus of residues, techniques which are not available if we restrict to real variables.

Our earlier results on Dirichlet series and Euler products all extend to the case where s is a complex variable, provided we have absolute convergence. We therefore need to consider the subset of the complex plane $\mathbb{C}$ on which a Dirichlet series converges absolutely. We will see that, just as a power series converges absolutely on a disc (which may be the whole plane or a single point), a Dirichlet series has a half-plane of absolute convergence, which may be the whole plane or the empty set.

Following the traditional (if slightly bizarre) notation we put

$$s = \sigma + it \in \mathbb{C} \qquad \text{where} \qquad \sigma, t \in \mathbb{R}.$$

Then $n^s = n^{\sigma+it} = n^{\sigma}.n^{it} = n^{\sigma}.e^{it\ln(n)}$ with $n^{\sigma} > 0$ and $|e^{it\ln(n)}| = 1$, so $|n^s| = n^{\sigma}$. Now suppose that $F(s)$ converges absolutely (that is, $\sum|f(n)/n^s|$ converges) at some point $s = a + ib \in \mathbb{C}$; if $\sigma \geq a$ then

$$\left|\frac{f(n)}{n^{\sigma+it}}\right| = \left|\frac{f(n)}{n^{\sigma}}\right| \leq \left|\frac{f(n)}{n^{a}}\right| = \left|\frac{f(n)}{n^{a+ib}}\right|,$$

so $\sum f(n)/n^{\sigma+it}$ converges absolutely by the Comparison Test. This implies:

Theorem 9.9

Suppose that $\sum_{n=1}^{\infty} |f(n)/n^s|$ neither converges for all $s \in \mathbb{C}$, nor diverges for all $s \in \mathbb{C}$. Then there exists $\sigma_a \in \mathbb{R}$ such that $\sum_{n=1}^{\infty} |f(n)/n^s|$ converges for all $s = \sigma + it$ with $\sigma > \sigma_a$, and diverges for all $s = \sigma + it$ with $\sigma < \sigma_a$.

Proof

We take σ_a to be the least upper bound of all $a \in \mathbb{R}$ such that $\sum |f(n)/n^s|$ diverges at $s = a+ib$; by the preceding argument this coincides with the greatest lower bound of all $a \in \mathbb{R}$ such that $\sum |f(n)/n^s|$ converges at $s = a + ib$. □

Definition

We call σ_a the *abscissa of absolute convergence* of $F(s)$, and $\{s = \sigma + it \in \mathbb{C} \mid \sigma > \sigma_a\}$ its *half-plane of absolute convergence.*

Note that the theorem says nothing about the behaviour of $F(s)$ when $\sigma = \sigma_a$. Note also that there are two other extreme possibilities, not covered by the theorem: $F(s)$ may converge absolutely for all $s \in \mathbb{C}$, or for no $s \in \mathbb{C}$; we then write $\sigma_a = -\infty$ or $+\infty$ respectively. A similar but more complicated argument shows that there exists $\sigma_c \leq \sigma_a$, called the *abscissa of convergence*, such that $F(s)$ converges for $\sigma > \sigma_c$ and diverges for $\sigma < \sigma_c$; if $\sigma_c < \sigma_a$, then convergence is conditional for $\sigma_c < \sigma < \sigma_a$.

Exercise 9.16

Find examples of Dirichlet series for which $\sigma_a = -\infty$ and $\sigma_a = +\infty$.

Example 9.9

Theorem 9.1 states that $\sum 1/n^s$ converges (absolutely) for all real $s > 1$ and diverges for $s \leq 1$. This series therefore has $\sigma_a = \sigma_c = 1$, so it converges absolutely for all $s = \sigma + it \in \mathbb{C}$ with $\sigma > 1$, and diverges for $\sigma < 1$. Similarly, $\sum(-1)^n/n^s$ has $\sigma_a = 1$, but in this case $\sigma_c = 0$ since the series converges for all $s > 0$ by the Alternating Test (Appendix C), but diverges for $s \leq 0$.

Example 9.10

If f is bounded, say $|f(n)| \leq M$ for all n, then $|f(n)/n^s| \leq M/n^\sigma$ where $s = \sigma + it$, so $\sum f(n)/n^s$ converges absolutely whenever $\sigma > 1$, by comparison with $\sum M/n^\sigma$. (It may converge absolutely for smaller σ, depending on the particular function f.) This applies to $f = \mu$ for example, with $M = 1$. More generally, if there are constants M and k such that $|f(n)| \le Mn^k$ for all n, then $\sum f(n)/n^s$ converges absolutely for $\sigma > 1+k$ by comparison with $\sum Mn^k/n^\sigma$. Now $|\phi(n)| \le n$ for all n, so taking $k = 1$ we see that $\sum \phi(n)/n^s$ converges absolutely for $\sigma > 2$.

A complex function $F(s)$ is said to be *analytic* if it is differentiable with respect to s.

Theorem 9.10

A Dirichlet series $\sum_{n=1}^{\infty} f(n)/n^s$ represents an analytic function $F(s)$ for $\sigma > \sigma_c$, with derivative $F'(s) = -\sum_{n=1}^{\infty} f(n)\ln(n)/n^s$.

Proof (Outline proof)

For each $n \ge 1$, the function $f(n)/n^s = f(n)e^{-s\ln(n)}$ is analytic for all s (since the exponential function e^s is analytic), with derivative $-f(n)\ln(n)/n^s$. One now shows that $\sum f(n)/n^s$ converges uniformly on all compact (closed, bounded) subsets of the half-plane $\sigma > \sigma_c$, and then quotes the theorem that a uniformly convergent series of analytic functions has an analytic sum, which may be differentiated term by term. For full details, see Apostol (1976, Chapter 11, Section 7). □

For example, the series $\sum 1/n^s$ defines an analytic function $\zeta(s)$ on the half-plane $\sigma > 1$. Riemann used analytic continuation to extend the domain of $\zeta(s)$: the resulting function, also denoted by $\zeta(s)$, is analytic on $\mathbb{C} \setminus \{1\}$, with a simple pole at $s = 1$ (this means that $(s-1)\zeta(s)$ is analytic at $s = 1$, so that $\zeta(s)$ diverges there like $1/(s-1)$). Note that we do *not* claim that the series $\sum 1/n^s$ converges outside the half-plane $\sigma > 1$: what Riemann showed is that there is a function $\zeta(s)$ which is analytic for all $s \neq 1$, and which agrees with $\sum 1/n^s$ for $\sigma > 1$. This is analogous to the situation with power series: for instance the geometric series $1 + z + z^2 + \cdots$ converges (absolutely) for all z with $|z| < 1$, and within this disc of convergence its sum is given by $(1-z)^{-1}$; however, this function $(1-z)^{-1}$ is analytic for *all* $z \neq 1$, even though the series diverges for $|z| \ge 1$.

Exercise 9.17

Show that $\zeta'(s) = -\sum \ln(n)/n^s$ and $-\zeta'(s)/\zeta(s) = \sum \Lambda(n)/n^s$ for all s with $\sigma > 1$, where Λ is Mangoldt's function (see Exercise 8.24).

Riemann showed that the extended function $\zeta(s)$ has zeros at $s = -2, -4, -6, \ldots$; these are called the *trivial zeros*, and he showed that the remaining *non-trivial zeros* all lie in the *critical strip* $0 \le \sigma \le 1$. The celebrated *Riemann Hypothesis* is the conjecture that these non-trivial zeros all lie on the line $\sigma = 1/2$. A great deal is now known about the location of the zeros of $\zeta(s)$: for instance, Hardy showed in 1914 that there are infinitely many on the line $\sigma = 1/2$. Despite strong evidence in its favour, the Riemann Hypothesis is still unproved; since many conjectures about the distribution of prime numbers depend on this result, the resolution of this problem remains one of the greatest challenges of number theory.

9.10 Supplementary exercises

Exercise 9.18

Let $\tau_k(n)$ be the number of k-tuples $(d_1, \ldots, d_k)$ of positive integers d_i such that $d_1 \ldots d_k = n$ (so that $\tau_2 = \tau$, for instance). Show that $\sum_{n=1}^{\infty} \tau_k(n)/n^s = \zeta(s)^k$ for all $s = \sigma + it$ with $\sigma > 1$.

Exercise 9.19

Show that $\sum_{n=1}^{\infty} \tau(n)^2/n^s = \zeta(s)^4/\zeta(2s)$ for $\sigma > 1$.

Exercise 9.20

Recall that $\pi(x)$ is the number of primes $p \le x$. Show that if $q(x)$ denotes the number of square-free integers $m \le x$, then

$$2^{\pi(x)} \ge q(x) \ge x\left(2 - \frac{\pi^2}{6}\right),$$

and hence

$$\begin{aligned}\pi(x) &\ge \log_2 x + \log_2\left(2 - \frac{\pi^2}{6}\right) \\ &= \frac{\ln x}{\ln 2} + \log_2\left(2 - \frac{\pi^2}{6}\right).\end{aligned}$$

(This estimate for $\pi(x)$ is very weak: for instance it gives $\pi(10^9) \ge 28$, whereas in fact $\pi(10^9) \approx 5 \times 10^7$.)

Exercise 9.21

Let $f_k(n)$ denote the number of subgroups of finite index n in the group $\mathbb{Z}^k$ (see Exercise 8.26). Express the Dirichlet series $F_k(s) = \sum_{n=1}^{\infty} f_k(n)/n^s$ of f_k in terms of the Riemann zeta function. For which $s \in \mathbb{C}$ is your expression valid?