13

Characters

Suppose that $\rho: G \to GL(n, \mathbb{C})$ is a representation of the finite group G. With each $n \times n$ matrix $g\rho$ ($g \in G$) we associate the complex number given by adding all the diagonal entries of the matrix, and call this number $\chi(g)$. The function $\chi: G \to \mathbb{C}$ is called the character of the representation ρ. Characters of representations have many remarkable properties, and they are the fundamental tools for performing calculations in representation theory. For example, we shall show later that two representations have the same character if and only if they are equivalent. Moreover, basic problems, such as deciding whether or not a given representation is irreducible, can be resolved by doing some easy arithmetic with the character of the representation. These facts are surprising, since from the definition of a representation $\rho: G \to GL(n, \mathbb{C})$, it appears that we must keep track of all the n^2 entries in each matrix $g\rho$, whereas the character records just one number for each matrix.

The theory of characters will occupy a considerable portion of the rest of the book. In this chapter we present some basic properties and examples.

The trace of a matrix

13.1 Definition

If $A = (a_{ij})$ is an $n \times n$ matrix, then the *trace* of A, written tr A, is given by

$$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}.$$

That is, the trace of A is the sum of the diagonal entries of A.

13.2 Proposition

Let $A = (a_{ij})$ *and* $B = (b_{ij})$ *be* $n \times n$ *matrices. Then*

$$\operatorname{tr}(A+B) = \operatorname{tr} A + \operatorname{tr} B, \textit{ and}$$

$$\operatorname{tr}(AB) = \operatorname{tr}(BA).$$

Moreover, if T is an invertible n × n matrix, then

$$\operatorname{tr}(T^{-1}AT) = \operatorname{tr} A.$$

Proof The ii-entry of $A + B$ is $a_{ii} + b_{ii}$, and the ii-entry of AB is $\sum_{j=1}^{n} a_{ij}b_{ji}$. Therefore

$$\operatorname{tr}(A+B) = \sum_{i=1}^{n}(a_{ii}+b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \operatorname{tr} A + \operatorname{tr} B,$$

and

$$\operatorname{tr}(AB) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ji}a_{ij} = \operatorname{tr}(BA).$$

For the last part,

$$\begin{aligned}\operatorname{tr}(T^{-1}AT) &= \operatorname{tr}((T^{-1}A)T)\\ &= \operatorname{tr}(T(T^{-1}A)) \quad \text{(by the second part)}\\ &= \operatorname{tr} A.\end{aligned}$$

■

Notice that, unlike the determinant function, the trace function is not multiplicative; that is, tr (AB) need not equal (tr A)(tr B).

Characters

13.3 Definition

Suppose that V is a $\mathbb{C}G$-module with a basis $\mathscr{B}$. Then the *character* of V is the function χ: $G \to \mathbb{C}$ defined by

$$\chi(g) = \mathrm{tr}\,[g]_{\mathscr{B}} \quad (g \in G).$$

The character of V does not depend on the basis $\mathscr{B}$, since if $\mathscr{B}$ and $\mathscr{B}'$ are bases of V, then

$$[g]_{\mathscr{B}'} = T^{-1}[g]_{\mathscr{B}}T$$

for some invertible matrix T (see (2.24)), and so by Proposition 13.2,

$$\mathrm{tr}\,[g]_{\mathscr{B}'} = \mathrm{tr}\,[g]_{\mathscr{B}} \quad \text{for all } g \in G.$$

Naturally enough, we define the character of a representation ρ: $G \to \mathrm{GL}(n, \mathbb{C})$ to be the character χ of the corresponding $\mathbb{C}G$-module $\mathbb{C}^n$, namely

$$\chi(g) = \mathrm{tr}\,(g\rho) \quad (g \in G).$$

13.4 Definition

We say that χ is a *character* of G if χ is the character of some $\mathbb{C}G$-module. Further, χ is an *irreducible* character of G if χ is the character of an irreducible $\mathbb{C}G$-module; and χ is *reducible* if it is the character of a reducible $\mathbb{C}G$-module.

You will have noticed that we are writing characters as functions on the left. That is, we write $\chi(g)$ and not $g\chi$.

13.5 Proposition

(1) *Isomorphic $\mathbb{C}G$-modules have the same character.*

(2) *If x and y are conjugate elements of the group G, then*

$$\chi(x) = \chi(y)$$

for all characters χ of G.

Proof (1) Suppose that V and W are isomorphic $\mathbb{C}G$-modules. Then by (7.7), there are a basis $\mathscr{B}_1$ of V and a basis $\mathscr{B}_2$ of W such that

$$[g]_{\mathscr{B}_1} = [g]_{\mathscr{B}_2} \quad \text{for all } g \in G.$$

Consequently tr $[g]_{\mathscr{B}_1}$ = tr $[g]_{\mathscr{B}_2}$ for all $g \in G$, and so V and W have the same character.

(2) Assume that x and y are conjugate elements of G, so that $x = g^{-1}yg$ for some $g \in G$. Let V be a $\mathbb{C}G$-module, and let $\mathscr{B}$ be a basis of V. Then

$$[x]_{\mathscr{B}} = [g^{-1}yg]_{\mathscr{B}} = [g]_{\mathscr{B}}^{-1}[y]_{\mathscr{B}}[g]_{\mathscr{B}}.$$

Hence by Proposition 13.2, we have tr $[x]_{\mathscr{B}}$ = tr $[y]_{\mathscr{B}}$. Therefore $\chi(x) = \chi(y)$, where χ is the character of V.

■

The result corresponding to Proposition 13.5(1) for representations is that equivalent representations have the same character.

Later, we shall prove an astonishing converse of Proposition 13.5(1): if two $\mathbb{C}G$-modules have the same character, then they are isomorphic.

13.6 Examples

(1) Let $G = D_8 = \langle a, b\colon a^4 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$, and let $\rho\colon G \to \mathrm{GL}(2, \mathbb{C})$ be the representation for which

$$a\rho = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ b\rho = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(see Example 3.2(1)). Let χ be the character of this representation. The following table records g, $g\rho$ and $\chi(g)$ as g runs through G. (We obtain $\chi(g)$ by adding the two entries on the diagonal of $g\rho$.)

g	1	a	a^2	a^3
$g\rho$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
$\chi(g)$	2	0	-2	0

g	b	ab	a^2b	a^3b
$g\rho$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\chi(g)$	0	0	0	0

(2) Let $G = S_3$, and take V to be the 3-dimensional permutation module for G over $\mathbb{C}$ (see Definition 4.10). Let $\mathscr{B}$ be the natural basis of V; thus $\mathscr{B}$ is the basis v_1, v_2, v_3, where $v_i g = v_{ig}$ for $1 \leqslant i \leqslant 3$ and all $g \in G$. The matrices $[g]_{\mathscr{B}}$ ($g \in G$) are given by Exercise 4.1. We record these matrices, together with the character χ of V.

g	1	$(1\,2)$	$(1\,3)$
$[g]_{\mathscr{B}}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
$\chi(g)$	3	1	1

g	$(2\,3)$	$(1\,2\,3)$	$(1\,3\,2)$
$[g]_{\mathscr{B}}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\chi(g)$	1	0	0

(3) Let $G = C_3 = \langle a\colon a^3 = 1\rangle$. By Theorem 9.8, G has just three irreducible characters χ_1, χ_2, χ_3, with values

g	1	a	a^2
$\chi_1(g)$	1	1	1
$\chi_2(g)$	1	ω	ω^2
$\chi_3(g)$	1	ω^2	ω

where $\omega = e^{2\pi i/3}$.

(4) Let $G = D_6 = \langle a, b\colon a^3 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$ (so $G \cong S_3$). In Example 10.8(2), we found a complete set of non-isomorphic irreducible $\mathbb{C}G$-modules U_1, U_2, U_3. Thus if χ_i is the character of U_i for $1 \leqslant i \leqslant 3$, then the irreducible characters of G are χ_1, χ_2 and χ_3. The values of these characters on the elements of G can be calculated from the corresponding representations ρ_1, ρ_2, ρ_3 given in Example 10.8(2), and they are as follows:

g	1	a	a^2	b	ab	a^2b
$\chi_1(g)$	1	1	1	1	1	1
$\chi_2(g)$	1	1	1	-1	-1	-1
$\chi_3(g)$	2	-1	-1	0	0	0

Notice that in all the above examples, the characters given take few distinct values. This reflects the fact that by Proposition 13.5(2), every character is constant on conjugacy classes of G. Moreover, it is much quicker to write down the single complex number $\chi(g)$ for the group element g than to record the matrix which corresponds to g. Nevertheless, the character encapsulates a great deal of information about the representation. This will become clear as the theory of characters develops.

13.7 Definition

If χ is the character of the $\mathbb{C}G$-module V, then the dimension of V is called the *degree* of χ.

13.8 Examples

(1) In Example 13.6(1) we gave a character of D_8 of degree 2; in 13.6(2) we gave a character of S_3 of degree 3; and in 13.6(4) we saw that the irreducible characters of D_6 have degrees 1, 1 and 2.

(2) If V is any 1-dimensional $\mathbb{C}G$-module, then for each $g \in G$ there is a complex number λ_g such that

$$vg = \lambda_g v \quad \text{for all } v \in V.$$

The character χ of V is given by

$$\chi(g) = \lambda_g \quad (g \in G)$$

and χ has degree 1. Characters of degree 1 are called *linear* characters; they are, of course, irreducible characters.

Observe that Theorem 9.8 gives all the irreducible characters of finite abelian groups; in particular, they are all linear characters.

Every linear character of G is a homomorphism from G to the multiplicative group of non-zero complex numbers. In fact, these are the only non-zero characters of G which are homomorphisms (see Exercise 13.4).

(3) The character of the trivial $\mathbb{C}G$-module (see Definition 4.8(1)) is a linear character, called the *trivial* character of G. We denote it by 1_G. Thus

$$1_G\colon g \to 1 \quad \text{for all } g \in G.$$

Given any group G, we therefore know at least one of the irreducible characters of G, namely the trivial character. Finding all the irreducible characters is usually difficult.

The values of a character

The next result gives information about the complex numbers $\chi(g)$, where χ is a character of G and $g \in G$.

13.9 Proposition
Let χ be the character of a $\mathbb{C}G$-module V. Suppose that $g \in G$ and g has order m. Then

(1) $\chi(1) = \dim V$;
(2) $\chi(g)$ *is a sum of mth roots of unity*;
(3) $\chi(g^{-1}) = \overline{\chi(g)}$;
(4) $\chi(g)$ *is a real number if g is conjugate to g^{-1}*.

Proof (1) Let $n = \dim V$, and let $\mathscr{B}$ be a basis of V. Then the matrix $[1]_{\mathscr{B}}$ of the identity element 1 relative to $\mathscr{B}$ is equal to I_n, the $n \times n$ identity matrix. Consequently

$$\chi(1) = \operatorname{tr}[1]_{\mathscr{B}} = \operatorname{tr} I_n = n,$$

and so $\chi(1) = \dim V$.

(2) By Proposition 9.11 there is a basis $\mathscr{B}$ of V such that

$$[g]_{\mathscr{B}} = \begin{pmatrix} \omega_1 & & 0 \\ & \ddots & \\ 0 & & \omega_n \end{pmatrix}$$

where each ω_i is an mth root of unity. Therefore

$$\chi(g) = \omega_1 + \ldots + \omega_n,$$

a sum of mth roots of unity.

(3) We have

$$[g^{-1}]_{\mathscr{B}} = \begin{pmatrix} \omega_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \omega_n^{-1} \end{pmatrix}$$

and so $\chi(g^{-1}) = \omega_1^{-1} + \ldots + \omega_n^{-1}$. Every complex mth root of unity ω satisfies $\omega^{-1} = \overline{\omega}$, since for all real 3,

$$(e^{i\vartheta})^{-1} = e^{-i\vartheta},$$

which is the complex conjugate of $e^{i\vartheta}$. Therefore

$$\chi(g^{-1}) = \overline{\omega}_1 + \ldots + \overline{\omega}_n = \overline{\chi(g)}.$$

(4) If g is conjugate to g^{-1} then $\chi(g) = \chi(g^{-1})$ by Proposition 13.5(2). Also $\chi(g^{-1}) = \overline{\chi(g)}$ by (3), and so $\chi(g) = \overline{\chi(g)}$; that is, $\chi(g)$ is real.

■

When the element g of G has order 2, we can be much more specific about the possibilities for $\chi(g)$:

13.10 Corollary
Let χ be a character of G, and let g be an element of order 2 in G. Then $\chi(g)$ is an integer, and

$$\chi(g) \equiv \chi(1) \bmod 2.$$

Proof By Proposition 13.9, we have

$$\chi(g) = \omega_1 + \ldots + \omega_n,$$

where $n = \chi(1)$ and each ω_i is a square root of unity. Then each ω_i is $+1$ or -1. Suppose r of them are $+1$, and s are -1, so that

$$\chi(g) = r - s, \text{ and } \chi(1) = r + s.$$

Certainly then, $\chi(g) \in \mathbb{Z}$, and since $r - s = r + s - 2s \equiv r + s \bmod 2$, we have $\chi(g) \equiv \chi(1) \bmod 2$.

■

Our next result gives the first inkling of the importance of characters, showing that we can determine the kernel of a representation just from knowledge of its character.

13.11 Theorem
Let ρ: $G \to \mathrm{GL}(n, \mathbb{C})$ be a representation of G, and let χ be the character of ρ.

(1) *For $g \in G$,*

$$|\chi(g)| = \chi(1) \Leftrightarrow g\rho = \lambda I_n \quad \text{for some } \lambda \in \mathbb{C}.$$

(2) Ker $\rho = \{g \in G: \chi(g) = \chi(1)\}$.

Proof (1) Let $g \in G$, and suppose that g has order m. If $g\rho = \lambda I_n$ with $\lambda \in \mathbb{C}$, then λ is an mth root of unity, and $\chi(g) = n\lambda$, so $|\chi(g)| = n = \chi(1)$.

Conversely, suppose that $|\chi(g)| = \chi(1)$. By Proposition 9.11, there is a basis $\mathscr{B}$ of $\mathbb{C}^n$ such that

$$[g]_{\mathscr{B}} = \begin{pmatrix} \omega_1 & & 0 \\ & \ddots & \\ 0 & & \omega_n \end{pmatrix}$$

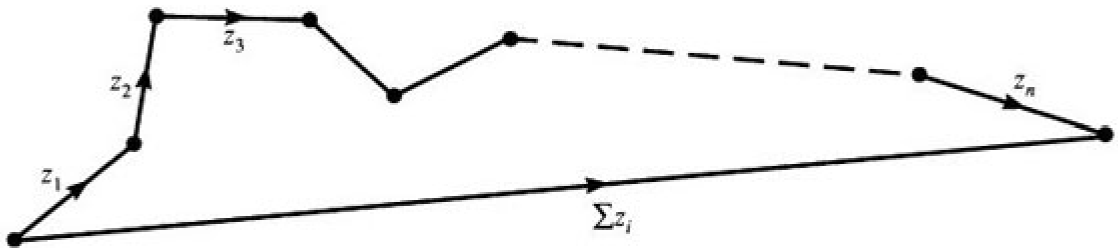
where each ω_i is an mth root of unity. Then

$$|\chi(g)| = |\omega_1 + \ldots + \omega_n| = \chi(1) = n. \tag{13.12}$$

Note now that for any complex numbers $z_1, \ldots, z_n$, we have

$$|z_1 + \ldots + z_n| \leqslant |z_1| + \ldots + |z_n|,$$

with equality if and only if the arguments of $z_1, \ldots, z_n$ are all equal. (To see this, consider the picture



in the Argand diagram.) Since $|\omega_i| = 1$ for all i, we deduce from (13.12) that $\omega_i = \omega_j$ for all i, j. Thus

$$[g]_{\mathscr{B}} = \begin{pmatrix} \omega_1 & & 0 \\ & \ddots & \\ 0 & & \omega_1 \end{pmatrix} = \omega_1 I_n.$$

Hence for all bases $\mathscr{B}'$ of $\mathbb{C}^n$ we have $[g]_{\mathscr{B}'} = \omega_1 I_n$, and so $g\rho = \omega_1 I_n$. This completes the proof of (1).

(2) If $g \in \operatorname{Ker} \rho$ then $g\rho = I_n$, and so $\chi(g) = n = \chi(1)$.

Conversely, suppose that $\chi(g) = \chi(1)$. Then by (1), we have $g\rho = \lambda I_n$ for some $\lambda \in \mathbb{C}$. This implies that $\chi(g) = \lambda\chi(1)$, whence $\lambda = 1$. Therefore $g\rho = I_n$, and so $g \in \operatorname{Ker} \rho$. Part (2) follows.

■

Motivated by Theorem 13.11(2), we define the kernel of a character as follows.

13.13 Definition

If χ is a character of G, then the *kernel* of χ, written $\operatorname{Ker} \chi$, is defined by

$$\operatorname{Ker} \chi = \{g \in G : \chi(g) = \chi(1)\}.$$

By Theorem 13.11(2), if ρ is a representation of G with character χ, then $\operatorname{Ker} \rho = \operatorname{Ker} \chi$. In particular, $\operatorname{Ker} \chi \lhd G$. We call χ a *faithful* character if $\operatorname{Ker} \chi = \{1\}$.

13.14 Examples

(1) According to Example 13.6(4), the irreducible characters of the group $G = D_6 = \langle a, b\colon a^3 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$ are χ_1, χ_2, χ_3, with the following values:

g	1	a	a^2	b	ab	a^2b
$\chi_1(g)$	1	1	1	1	1	1
$\chi_2(g)$	1	1	1	−1	−1	−1
$\chi_3(g)$	2	−1	−1	0	0	0

Then $\operatorname{Ker} \chi_1 = G$, $\operatorname{Ker} \chi_2 = \langle a\rangle$ and $\operatorname{Ker} \chi_3 = \{1\}$. In particular, χ_3 is a faithful irreducible character of D_6.

(2) Let $G = D_8 = \langle a, b\colon a^4 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$, and let χ be the character of G given in Example 13.6(1):

g	1	a	a^2	a^3	b	ab	a^2b	a^3b
$\chi(g)$	2	0	−2	0	0	0	0	0

Then $\operatorname{Ker} \chi = \{1\}$, so χ is a faithful character. And since $|\chi(a^2)| = |-2| = \chi(1)$, Theorem 13.11(1) implies that if $\rho\colon G \to \mathrm{GL}(2, \mathbb{C})$ is a representation with character χ, then $a^2\rho = -I$.

We next prove a result which is sometimes useful for constructing a new character from a given one. For a character χ of G, define $\overline{\chi}\colon G \to \mathbb{C}$ by

$$\overline{\chi}(g) = \overline{\chi(g)} \quad (g \in G).$$

Thus the values of $\overline{\chi}$ are the complex conjugates of the values of χ.

13.15 Proposition

Let χ be a character of G. Then $\overline{\chi}$ is a character of G. If χ is irreducible, then so is $\overline{\chi}$.

Proof Suppose that χ is the character of a representation $\rho\colon G \to \mathrm{GL}(n, \mathbb{C})$. Thus

$$\chi(g) = \mathrm{tr}\,(g\rho) \quad (g \in G).$$

If $A = (a_{ij})$ is an $n \times n$ matrix over $\mathbb{C}$, then we define $\overline{A}$ to be the $n \times n$ matrix $(\overline{a}_{ij})$. Observe that if $A = (a_{ij})$ and $B = (b_{ij})$ are $n \times n$ matrices over $\mathbb{C}$, then

$$(\overline{AB}) = \overline{A}\,\overline{B}, \tag{13.16}$$

since the ij-entry of $\overline{A}\overline{B}$ is

$$\sum_{k=1}^{n} \overline{a}_{ik}\overline{b}_{kj},$$

which is equal to the complex conjugate of $\sum_{k=1}^{n} a_{ik}b_{kj}$, the ij-entry of AB.

It follows from (13.16) that the function $\overline{\rho}\colon G \to \mathrm{GL}(n, \mathbb{C})$ defined by

$$g\overline{\rho} = \overline{(g\rho)} \quad (g \in G)$$

is a representation of G. Since

$$\mathrm{tr}\,(g\overline{\rho}) = \mathrm{tr}\,\overline{(g\rho)} = \overline{\mathrm{tr}\,(g\rho)} = \overline{\chi(g)} \quad (g \in G),$$

the character of the representation $\overline{\rho}$ is $\overline{\chi}$.

It is clear that if ρ is reducible then $\overline{\rho}$ is reducible. Hence χ is irreducible if and only if $\overline{\chi}$ is irreducible.

■

The regular character

13.17 Definition

The *regular* character of G is the character of the regular $\mathbb{C}G$-module. We write the regular character as χ_{reg}.

In Theorem 13.19, we shall express the regular character in terms of the irreducible characters of G. First we need a preliminary result.

13.18 Proposition
Let V be a $\mathbb{C}G$-module, and suppose that

$$V = U_1 \oplus \ldots \oplus U_r,$$

a direct sum of irreducible $\mathbb{C}G$-modules U_i. Then the character of V is equal to the sum of the characters of the $\mathbb{C}G$-modules $U_1, \ldots, U_r$.

Proof This is immediate from (7.10).

■

13.19 Theorem
Let $V_1, \cdots, V_k$ be a complete set of non-isomorphic irreducible $\mathbb{C}G$-modules (see Definition 11.11), and for $i = 1, \ldots, k$ let χ_i be the character of V_i and $d_i = \chi_i(1)$. Then

$$\chi_{\text{reg}} = d_1\chi_1 + \ldots + d_k\chi_k.$$

Proof By Theorem 11.9,

$$\mathbb{C}G \cong (V_1 \oplus \ldots \oplus V_1) \oplus (V_2 \oplus \ldots \oplus V_2) \oplus \ldots \\ \oplus (V_k \oplus \ldots \oplus V_k),$$

where for each i there are d_i factors V_i. Now the result follows from Proposition 13.18.

■

The values of χ_{reg} on the elements of G are easily described, and are given in the next result.

13.20 Proposition

If χ_{reg} is the regular character of G, then

$$\chi_{\text{reg}}(1) = |G|, \textit{ and}$$

$$\chi_{\text{reg}}(g) = 0 \textit{ if } g \neq 1.$$

Proof Let $g_1, \ldots, g_n$ be the elements of G, and let $\mathscr{B}$ be the basis $g_1, \ldots, g_n$ of $\mathbb{C}G$. By Proposition 13.9(1), $\chi_{\text{reg}}(1) = \dim \mathbb{C}G = |G|$.

Now let $g \in G$ with $g \neq 1$. Then for $1 \leqslant i \leqslant n$, we have $g_i g = g_j$ for some j with $j \neq i$. Therefore the ith row of the matrix $[g]_{\mathscr{B}}$ has zeros in every place except column j; in particular, the ii-entry is zero for all i. It follows that

$$\chi_{\text{reg}}(g) = \text{tr}\,[g]_{\mathscr{B}} = 0.$$

■

13.21 Example

We illustrate Theorem 13.19 and Proposition 13.20 for the group $G = D_6$. By Example 13.6(4), the irreducible characters of G are χ_1, χ_2, χ_3:

g	1	a	a^2	b	ab	a^2b
$\chi_1(g)$	1	1	1	1	1	1
$\chi_2(g)$	1	1	1	-1	-1	-1
$\chi_3(g)$	2	-1	-1	0	0	0

We calculate $\chi_1 + \chi_2 + 2\chi_3$:

$(\chi_1 + \chi_2 + 2\chi_3)(g)$	6	0	0	0	0	0

This is the regular character of G, by Theorem 13.19; and it takes the value $|G|$ on 1, and the value 0 on all non-identity elements of G, illustrating Proposition 13.20.

Permutation characters

In the case where G is a subgroup of the symmetric group S_n, there is an easy construction using the permutation module which produces a character of degree n, and we now describe this.

Suppose that G is a subgroup of S_n, so that G is a group of permutations of $\{1, \ldots, n\}$. The permutation module V for G over $\mathbb{C}$ has basis $v_1, \ldots, v_n$, where for all $g \in G$,

$$v_i g = v_{ig} \quad (1 \leqslant i \leqslant n)$$

(see Definition 4.10). Let $\mathscr{B}$ denote the basis $v_1, \ldots, v_n$. Then the ii-entry in the matrix $[g]_{\mathscr{B}}$ is 0 if $ig \neq i$, and is 1 if $ig = i$. Therefore the character π of the permutation module V is given by

$$\pi(g) = (\text{the number of } i \text{ such that } ig = i).$$

For $g \in G$, let

$$\text{fix}\,(g) = \{i\colon 1 \leqslant i \leqslant n \text{ and } ig = i\}.$$

Then

$$\pi(g) = |\text{fix}(g)| \quad (g \in G). \tag{13.22}$$

We call π the *permutation character* of G.

13.23 Example

Let $G = S_4$. Then by Example 12.16(3), G has five conjugacy classes, with representatives

$$1, (1\,2), (1\,2\,3), (1\,2)(3\,4), (1\,2\,3\,4).$$

The permutation character π takes the values

g_i	1	(12)	(123)	(12)(34)	(1234)
$\pi(g_i)$	4	2	1	0	0

13.24 Proposition
Let G be a subgroup of S_n. Then the function ν: $G \to \mathbb{C}$ defined by

$$\nu(g) = |\mathrm{fix}\,(g)| - 1 \quad (g \in G)$$

is a character of G.

Proof Let $v_1, \ldots, v_n$ be the basis of the permutation module V as above, and let

$$u = v_1 + \ldots + v_n, \text{ and } U = \mathrm{sp}\,(u).$$

Observe that $ug = u$ for all $g \in G$, so U is a $\mathbb{C}G$-submodule of V. Indeed, U is isomorphic to the trivial $\mathbb{C}G$-module, so the character of U is the trivial character 1_G (see Example 13.8(3)). By Maschke's Theorem 8.1, there is a $\mathbb{C}G$-submodule W of V such that

$$V = U \oplus W.$$

Let ν be the character of W. Then

$$\pi = 1_G + \nu,$$

so $|\mathrm{fix}(g)| = 1 + \nu(g)$ for all $g \in G$, and therefore

$$\nu(g) = |\mathrm{fix}(g)| - 1 \quad (g \in G).$$

■

13.25 Example
Let $G = A_4$, a subgroup of S_4. By Example 12.18(1), the conjugacy classes of G are represented by

$$1,\ (1\,2)(3\,4),\ (1\,2\,3),\ (1\,3\,2).$$

The values of the character v of G are

g_i	1	$(1\,2)(3\,4)$	$(1\,2\,3)$	$(1\,3\,2)$
$\nu(g_i)$	3	-1	0	0

Summary of Chapter 13

1. A character is obtained from a representation by taking the trace of each matrix.
2. Characters are constant on conjugacy classes.
3. Isomorphic $\mathbb{C}G$-modules have the same character.
4. For all characters χ of G, and all $g \in G$, the complex number $\chi(g)$ is a sum of roots of unity, and $\chi(g^{-1}) = \overline{\chi(g)}$.
5. The character of a representation determines the kernel of the representation.
6. The regular character χ_{reg} of G takes the value $|G|$ on the identity and the value 0 on all other elements of G.
7. If G is a subgroup of S_n, then the function v which is given by

$$\nu(g) = |\text{fix}(g)| - 1 \quad (g \in G)$$

   is a character of G.

Exercises for Chapter 13

1. Let $G = D_{12} = \langle a, b\colon a^6 = b^2 = 1,\ b^{-1}ab = a^{-1}\rangle$, and let ρ_1, ρ_2 be the representations of G for which

$$a\rho_1 = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix},\ b\rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (\text{where } \omega = e^{2\pi i/3})$$

and

$$a\rho_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\ b\rho_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Find the characters of ρ_1 and ρ_2. Find also Ker ρ_1 and Ker ρ_2; check that your answers are consistent with Theorem 13.11.

2. Find all the irreducible characters of C_4. Write the regular character of C_4 as a linear combination of these.
3. Let χ be the character of the 7-dimensional permutation module for S_7. Find $\chi(x)$ for $x = (1\ 2)$ and for $x = (1\ 6)(2\ 3\ 5)$.
4. Prove that the only non-zero characters of G which are homomorphisms are the linear characters.
5. Assume that χ is an irreducible character of G. Suppose that $z \in Z(G)$ and that z has order m. Prove that there exists an mth root of unity $\lambda \in \mathbb{C}$ such that for all $g \in G$,

$$\chi(zg) = \lambda\chi(g).$$

6. Prove that if χ is a faithful irreducible character of the group G, then $Z(G) = \{g \in G\colon |\chi(g)| = \chi(1)\}$.
7. Let ρ be a representation of the group G over $\mathbb{C}$.

   (a) Show that $\delta\colon g \to \det(g\rho)$ $(g \in G)$ is a linear character of G.
   (b) Prove that $G/\text{Ker}\ \delta$ is abelian.
   (c) Assume that $\delta(g) = -1$ for some $g \in G$. Show that G has a normal subgroup of index 2.

8. Let g be a group of order $2k$, where k is an odd integer. By considering the regular representation of G, show that G has a normal subgroup of index 2.
9. Let χ be a character of a group G, and let g be an element of order 2 in G. Show that either

(1) $\chi(g) \equiv \chi(1) \bmod 4$, or
(2) G has a normal subgroup of index 2.

(Compare Corollary 13.10. Hint: use Exercise 7.)

0. Prove that if x is a non-identity element of the group G, then $\chi(x) \neq \chi(1)$ for some irreducible character χ of G.