Calculus HW5

Date: Mar 13 Made by Eric

1.

Proof. Let x, y be the length of two sides of the rectangle with perimeter p. The area of the rectangle A is a function of x, y given by $A = xy$, and we know x, y must satisfy the constraint $x + y = \frac{p}{2}$

Let $g = x + y$. We have

$$\nabla A = (y, x) \tag{1}$$

$$\nabla g = (1, 1) \tag{2}$$

The method require us to solve

$$\begin{cases} \nabla A = \lambda \nabla g \\ g = \frac{p}{2} \end{cases} \tag{3}$$

By equation (3), we see $y = x$, so the rectangle is a square.

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2.

2.(a)

Let $g = x^4 - x^3 + y^2$, so the constraint is given by $g = 0$

Proof.

$$\nabla f = (1, 0) \tag{4}$$

$$\nabla g = (4x^3 - 3x^2, 2y) \tag{5}$$

Solving

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases} \tag{6}$$

We have $y = 0, x = 1, \lambda = 1$ ■

2.(b)

Define $h(x) := x^4 - x^3$

Because we are under the constraint $g = 0$, we see $h(x) = -y^2 \leq 0$

Because $h = x^3(x-1)$, we deduce $h \leq 0 \iff 0 \leq x \leq 1$. We see the minimum value for x is 0, for which we can take $y = 0$ to satisfy the constraint.

Notice $f(x,y) = x$, so it immediately follows that the minimum value for f is 0, and it happens at $f(0,0) = 0$.

Observe $\nabla g(0,0) = (0,0)$ and $\nabla f(0,0) = (1,0)$, we see no real number λ satisfy $(1,0) = \lambda(0,0)$.

2.(c)

Proof. The Lagrange multiplier theorem states that if $f(x_0, y_0)$ is a maximum or minimum of $f(x,y)$ under the constraint $g(x,y) = c$ and $\nabla g(x_0, y_0) \neq 0$, then there exists a Lagrange multiplier satisfy $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$. Although $f(0,0)$ is a minimum of f, we see $\nabla g(0,0) = (0,0)$, which does not satisfy the assumption of the Lagrange multiplier theorem. ■

3.

3.(a)

Proof. Let $g = \sum_{i=1}^{n} x_i$

$$\nabla f = \frac{1}{n}(x_1^{\frac{1-n}{n}}(x_2 \cdots x_n)^{\frac{1}{n}}, \ldots, (x_1 \cdots x_{n-1})^{\frac{1}{n}} x_n^{\frac{1-n}{n}}) \tag{7}$$

$$\nabla g = (1, \ldots, 1) \tag{8}$$

Solving $\nabla f = \lambda \nabla g$, we see

$$x_1^{\frac{1-n}{n}}(x_2 \cdots x_n)^{\frac{1}{n}} = (x_1 \cdots x_{n-1})^{\frac{1}{n}} x_n^{\frac{1-n}{n}} \tag{9}$$

Which give us

$$x_1^{-1} = x_n^{-1} \tag{10}$$

This tell us $x_1 = x_n$

Equation (9) is only an ordinary result of the symmetry of f. We see $x_1 = x_2 = \cdots = x_n$

Solving $g = c$, we see $\frac{c}{n} = x_i, \forall 1 \leq i \leq n$

So the maximum values for f is $f(\frac{c}{n}, \ldots, \frac{c}{n}) = \frac{c}{n}$

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3.(b)

Proof. Let $x_1 + \cdots + x_n = d$

RHS of the inequality is $\frac{d}{n}$. The desired result follows immediately from the fact that the LHS of the inequality have the maximum values $\frac{d}{n}$, by part (a).

The equality hold only when $x_1 = x_2 = \cdots = x_n$ also follows immediately from the discussion in part (a). ■

4.

4.(a)

Proof. Let $f(x_1,\dots,x_n,y_1,\dots,y_n) = \sum_{i=1}^n x_i y_i$ and $g(x_1,\dots,x_n,y_1,\dots,y_n) = \sum_{i=1}^n x_i^2$ and $h(x_1,\dots,x_n,y_1,\dots,y_n) = \sum_{i=1}^n y_i^2$

We are required to find the maximum of f subject to the constraint $g = h = 1$.

$$\nabla f = (y_1,\dots,y_n,x_1,\dots,x_n) \tag{11}$$

$$\nabla g = (2x_1,\dots,2x_n,0,\dots,0) \tag{12}$$

$$\nabla h = (0,\dots,0,2y_1,\dots,2y_n) \tag{13}$$

Solving $\nabla f = \lambda \nabla g + \mu \nabla h$, we have

$$(x_1,\dots,x_n) \parallel (y_1,\dots,y_n) \tag{14}$$

Further for $g = h = 1$, we have $(y_1,\dots,y_n) = \pm(x_1,\dots,x_n)$.

Then $f = \pm \sum_{i=1}^n x_i^2 = \pm 1$, since $g = 1$.

So the maximum of f is 1.

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4.(b)

Proof. Notice if $\sum a_j^2 = 0$, then the statement trivially hold true, so we only consider the case that $\sum a_j^2 \neq 0 \neq \sum b_j^2$

Notice $\sum x_i^2 = \frac{1}{\sum a_j^2} \sum a_i^2 = 1$ and $\sum y_i^2 = 1$, so by part (a), we see

$$\sum a_i b_i = \sqrt{\sum a_j^2 \sum b_j^2} \sum x_i y_i \leq \sqrt{\sum a_j^2 \sum b_j^2} \tag{15}$$

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5.

Given $a, b \in D$, by MVT, we can only guarantee that there exists a point c' between the line segment $\overline{ab}$ such that $\nabla f(c) \cdot (a-b) = f(a) - f(b)$.

We can choose $f = mx^3 - ny^3$ where $m, n > 0$ such that the hyperbola **(notice $\nabla f(c) \cdot (a-b) = 3m(a-b)_1 x^2 - 2n(a-b)_2 y^2$)** consisting of points c that satisfy $\nabla f(c) \cdot (a-b) = f(a) - f(b)$ does not intersect with D

6.

6.(a)

Let $D = \{(x,y)|0 \le x \le 4, |y| \le \sqrt{x}\}$

We are required to find $\int_D (1 + x^2y^2)dA$.

$$\int_{-2}^{2}\int_{y^2}^{4} 1 + x^2y^2 dxdy = \int_{-2}^{2} (x + \frac{1}{3}x^3y^2)|_{x=y^2}^{4} dy \tag{16}$$

Further compute above to the following

$$\int_{-2}^{2} (4 + \frac{64}{3}y^2) - (y^2 + \frac{1}{3}y^8)dy \tag{17}$$

Then we have answer

$$\frac{-1}{27}y^9 + \frac{61}{9}y^3 + 4y|_{y=-2}^{2} \tag{18}$$

6.(b)

We set up the triple integral

$$\int_0^2 \int_0^{\sqrt{4-z^2}} \int_0^{2y} 1dxdydz \tag{19}$$

And compute

$$\int_0^2 \int_0^{\sqrt{4-z^2}} \int_0^{2y} 1dxdydz = \int_0^2 \int_0^{\sqrt{4-z^2}} 2ydydz = \int_0^2 y^2|_0^{y=\sqrt{4-z^2}} dz \tag{20}$$

Further compute to have answer

$$\int_0^2 y^2|_0^{y=\sqrt{4-z^2}} dz = \int_0^2 4 - z^2 dz = \frac{-1}{3}z^3 + 4z|_0^{z=2} = \frac{16}{3} \tag{21}$$

7.

7.(a)

Let

$$I(a) = \int_{-a}^{a} e^{-x^2} dx$$

Notice

$$0 < \int_{\infty}^{\infty} e^{-x^2} < \int_{-\infty}^{-1} -xe^{-x^2}dx + \int_{-1}^{1} e^{-x^2}dx + \int_{1}^{\infty} xe^{-x^2}dx < \infty \tag{22}$$

So $\lim_{a\to\infty} I(a)$ exists.

Notice

$$I(a)^2 = (\int_{-a}^{a} e^{-x^2}dx)(\int_{-a}^{a} e^{-y^2}dy) = \int_{-a}^{a}\int_{-a}^{a} e^{-(x^2+y^2)}dxdy \tag{23}$$

We now switch to polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, and compute the Jacobian matrix and its determinant.

$$\mathbf{J}(r,\theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \; |\mathbf{J}(r,\theta)| = r \tag{24}$$

Notice that $I(a)^2$ is an integral over a square and that a square is bigger than its incircle and smaller than its circumcircle, and notice that $e^{-(x^2+y^2)} > 0$ for all x, y. We have the following inequality

$$\int_0^a \int_0^{2\pi} re^{-r^2} d\theta dr < I(a)^2 < \int_0^{\sqrt{2}a} \int_0^{2\pi} re^{-r^2} d\theta dr \tag{25}$$

Notice that re^{-r^2} is easy to integrate with substitution $u = r^2, du = 2rdr$.

So we compute the inegral in the above inequality

$$\pi(1 - e^{-a^2}) < I(a)^2 < \pi(1 - e^{-2a^2}) \tag{26}$$

Then by squeeze theroem we have

$$I(a)^2 = \pi, \text{ as } a \to \infty \tag{27}$$

That is

$$\int_{\mathbb{R}} e^{-x^2}dx = \sqrt{I(a)^2} = \sqrt{\pi} \tag{28}$$

7.(b)

Rewrite the orginial integral as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-(x_1^2+\cdots+x_n^2)}dx_1\cdots dx_n \tag{29}$$

Compute the above to the below

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-(x_2^2+\cdots+x_n^2)}(\int_{-\infty}^{\infty} e^{-x_1^2}dx_1)dx_2\cdots dx_n \tag{30}$$

From part a we know $(\int_{-\infty}^{\infty} e^{-x_1^2}dx_1) = \sqrt{\pi}$. So we can write further compute the above to below

$$\sqrt{\pi}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-(x_2^2+\cdots+x_n^2)}dx_2\cdots dx_n \tag{31}$$

Using the same method of simply separating the variable, we have the answer $\pi^{\frac{n}{2}}$

7.(c)

Rewrite the integral as

$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} e^{-(x_{i+1}^2+\cdots+x_n^2)} (\int_{\mathbb{R}} x_i \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} e^{-(x_1^2+\cdots+x_i^2)} dx_1 \cdots dx_{i-1} dx_i) dx_{i+1} \cdots dx_n \tag{32}$$

Compute the above to below

$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} e^{-(x_{i+1}^2+\cdots+x_n^2)} \int_{\mathbb{R}} \pi^{\frac{i-1}{2}} x_i e^{-x_i^2} dx_i dx_{i+1} \cdots dx_n \tag{33}$$

Notice $x_i e^{-x_i^2}$ is an odd function and we immediately have answer 0

8.

Notice the surface consists of four surfaces congruent to each other, so we only have to compute the area of one of them and multiply 4 with it to obtain the answer.

We compute the one that have positive y-axis through its center.

Let $f(x,z) = y = \sqrt{1-z^2}$

$$A = \int_{x^2+z^2 \leq 1} \sqrt{f_x(x,z)^2 + f_z(x,z)^2 + 1} d(x,z) \tag{34}$$

Substituting $f_x = 0$, $f_z = \frac{-z}{\sqrt{1-z^2}}$, and notice this integral is over a circle where

$$A = \int_{-1}^{1} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \frac{1}{\sqrt{1-z^2}} dx dz \tag{35}$$

Notice that this is in fact an improper integral with respect to z where $\frac{1}{\sqrt{1-z^2}}$ approaches to infinity as z approaches to 0.

notice that $\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \frac{1}{\sqrt{1-z^2}} dz$ is an even function of z, so we know

$$A = 2 \lim_{t \to 1} \int_0^t \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \frac{1}{\sqrt{1-z^2}} dx dz = 2 \lim_{t \to 1} \int_0^t 2 dz = 4 \tag{36}$$

So the answer is $16 = 4 * 4$

9.

9.(a)

$$\int_0^2 \int_0^{2-z} \int_0^{x^2} (x+y) dy dx dz = \frac{8}{3} \tag{37}$$

9.(b)

Notice f is an even function of y when x, z are fixed, so we can evaluate the triple integral as follows

$$I = \int\int\int_E f(x,y,z)dV = \frac{1}{2}(\int_0^1 \int_{x^2+y^2\leq 1} f(x,y,z)dAdz + \int_1^2 \int_{x^2+y^2\leq 2-z} f(x,y,z)dAdx) \tag{38}$$

We use polar coordinates $x = r\cos\theta, y = r\sin\theta$, and notice $dxdy = rdrd\theta$, so we have

$$I = \frac{1}{2}(\int_0^1 \int_0^{2\pi} \int_0^1 r^3 drd\theta dz + \int_1^2 \int_0^{2\pi} \int_0^{\sqrt{2-z}} r^3 drd\theta dz) \tag{39}$$

Further compute

$$I = \frac{1}{2}(\frac{\pi}{2} + \frac{\pi}{2}\int_1^2 (2-z)^2 dz) = \frac{\pi}{4}(1 + \frac{7}{3} - 6 + 4) \tag{40}$$

9.(c)

Notice f is an even function for x, y, z when the other variables are fixed, so we can evaluate the triple integral as follows

$$I = \frac{1}{4}\int_{\{4\leq x^2+y^2+z^2\leq 9\}} \sqrt{x^2+y^2+z^2}dV \tag{41}$$

Use spherical coordinates $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$, and we have

$$I = \frac{1}{4}\int_0^{\pi} \int_0^{2\pi} \int_2^3 \rho^3 \sin\phi d\rho d\theta d\phi = \frac{65}{4}\pi \tag{42}$$

10.

10.(a)

First we observe

$$\int_0^1 \int_0^1 \frac{1}{1-xy}dydx = \int_0^1 \frac{\ln 1-x}{-x}dx \tag{43}$$

Notice the Taylor series $\ln 1 - x = -\sum_{n=1}^{\infty} \frac{x^n}{n}$, so we have

$$\int_0^1 \frac{\ln 1-x}{-x}dx = \int_0^1 \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}dx = \sum_{n=1}^{\infty} \int_0^1 \frac{x^{n-1}}{n}dx = \sum_{n=1}^{\infty} \frac{1}{n^2} \tag{44}$$

10.(b)

$$\begin{cases} x = \frac{\sqrt{2}}{2}(u-v) \\ y = \frac{\sqrt{2}}{2}(u+v) \end{cases} \implies \mathbf{J} = 1 \text{ and } \frac{1}{1-xy} = \frac{2}{2-u^2+v^2} \tag{45}$$

Let $I = \int_0^1 \int_0^1 \frac{1}{1-xy} dxdy$

$$I = 2\int_0^{\frac{\sqrt{2}}{2}} \int_0^u \frac{2}{2-u^2+v^2} dvdu + 2\int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} \int_0^{\sqrt{2}-u} \frac{2}{2-u^2+v^2} dvdu \tag{46}$$

$$I = 4\int_0^{\frac{\sqrt{2}}{2}} \int_0^u \frac{1}{(2-u^2)+v^2} dvdu + 4\int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} \int_0^{\sqrt{2}-u} \frac{1}{(2-u^2)+v^2} dvdu \tag{47}$$

$$I = 4\int_0^{\frac{\sqrt{2}}{2}} \frac{1}{\sqrt{2-u^2}} \arctan \frac{u}{\sqrt{2-u^2}} du + 4\int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} \frac{1}{\sqrt{2-u^2}} \arctan \frac{\sqrt{2}-u}{\sqrt{2-u^2}} du \tag{48}$$

Do substitution $u = \sqrt{2}\sin t$ in first term and substitution $u = \sqrt{2}\cos x$ in second term, and we have

$$I = 4\int_0^{\frac{\pi}{6}} \arctan(\tan t) dt + 4\int_0^{\frac{\pi}{3}} \arctan(\frac{1-\cos x}{\sin x}) dx \tag{49}$$

Notice the identity $\tan\frac{x}{2} = \frac{1-\cos x}{\sin x}$, so we have

$$I = 4\int_0^{\frac{\pi}{6}} tdt + 4\int_0^{\frac{\pi}{3}} \frac{x}{2} dx = \frac{\pi^2}{18} + \frac{\pi^2}{9} = \frac{\pi^2}{6} \tag{50}$$