## 2.5 Exercise 3

**Question 45**

Provide a counterexample to the following statement: Suppose

$$f(z) = u(x,y) + iv(x,y)$$

is defined in a neighborhood of $z_0 = a + ib$. If the partial derivatives of $u$ and $v$ exist at $(a,b)$ and satisfy the Cauchy-Riemann equations $u_x(a,b) = v_y(a,b)$ and $u_y(a,b) = -v_x(a,b)$, then $f$ is holomorphic at $z_0$.

*Proof.* WOLG, let $a = b = 0$ and define

$$u(x,y) \triangleq \begin{cases} x & \text{if } y = 0 \\ y & \text{if } x = 0 \\ 0 & \text{if otherwise} \end{cases} \quad \text{and } v(x,y) \triangleq \begin{cases} -x & \text{if } y = 0 \\ y & \text{if } x = 0 \\ 0 & \text{if otherwise} \end{cases}$$

It is clear that

$$u_x = 1 = v_y \text{ and } u_y = 1 = -v_x \text{ at } (0,0)$$

but

$$\lim_{t \to 0; t \in \mathbb{R}} \frac{f(t) - f(0)}{t - 0} = \lim_{t \to 0; t \in \mathbb{R}} \frac{t - it}{t} = 1 - i$$

together with

$$\lim_{t \to 0; t \in \mathbb{R}} \frac{f(t + it) - f(0)}{t + it} = \lim_{t \to 0; t \in \mathbb{R}} \frac{0}{t + it} = 0$$

shows that $f$ is not holomorphic at $(0,0)$. ∎

**Question 46**

Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Suppose that $f$ is differentiable at $(a,b)$ and that $f'(x) = 0$ for all $x \in (a,b)$. Prove that $f$ is a constant function.

*Proof.* Assume $f(x) \neq f(y)$ for some $x \neq y \in [a,b]$. By MVT, we then see there exists some $t$ between $x, y$ (thus $t \in (a,b)$) such that $f'(t) = \frac{f(y)-f(x)}{y-x} \neq 0$, which is impossible. CaC ∎

## Question 47

Let $B = B_R(x_0)$ be the open ball in $\mathbb{R}^n$ centered at $x_0$ with radius $R > 0$. Prove that if $f : B \to \mathbb{R}$ is a differentiable function such that $\nabla f = 0$ on $B$, then $f$ is a constant function.

*Proof.* Let $\mathbf{x}, \mathbf{y}$ be two points in $B$. We are required to show $f(\mathbf{x}) = f(\mathbf{y})$. Define $g : [0, 1] \to B$ by

$$g(t) \triangleq f(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))$$

Note that $g$ is well-defined since $B$ is convex. Because $f$ is differentiable, we have

$$g'(t) = \nabla f(\mathbf{x} + y(\mathbf{y} - \mathbf{x})) \cdot (\mathbf{y} - \mathbf{x}) = 0$$

It then follows from the last question that

$$f(\mathbf{y}) = g(1) = g(0) = f(\mathbf{x})$$

■

## Question 48

Let $U$ be an open subset of $\mathbb{R}^n$. A function $f : U \to \mathbb{R}$ is called **locally constant** if, for each $x \in U$, there exists an open neighborhood $W$ of $x$ such that $W \subseteq U$ and $f : W \to \mathbb{R}$ is constant on $W$. Prove that $f$ is locally constant function if and only if $\nabla f = 0$ on $U$.

*Proof.* The if part follows from the last question by taking some small enough $r$ such that $B_r(x) \subseteq U$. We now prove the only if part. Fix arbitrary $x \in U$. Because $f$ is locally constant at $x$, we know there exists some $B_r(x)$ such that $f$ is constant on $B_r(x)$. Therefore, we can let $c \in \mathbb{R}$ satisfy

$$f(y) = c \text{ for all } y \in B_r(x)$$

To see $\nabla f(x) = 0$, just observe that for arbitrary axis $\mathbf{j}$

$$f_{\mathbf{j}}(x) = \lim_{t \to 0} \frac{f(x + t\mathbf{j}) - f(x)}{t} = 0$$

since $f(x + t\mathbf{j}) = c = f(x)$ as long as $|t| < r$. Because $\mathbf{j}$ is arbitrary, it then follows that $\nabla f(x) = 0$, and because $x$ is arbitrary selected from $U$, we have proved $\nabla f$ is 0 on $U$. ■

## Question 49

Let $D$ be an open, connected subset of $\mathbb{R}^n$. Prove that if $f : D \to \mathbb{R}$ is a locally constant function, then $f$ is a constant function.

*Proof.* Observe that for all $p \in D$, $f$ is constant on some neighborhood around $p$, thus continuous at $p$. We have shown $f : D \to \mathbb{R}$ is continuous. Fix $p \in D$, and let $c \triangleq f(p)$. Because $\{c\}$ is closed in $\mathbb{R}$ and $f : D \to \mathbb{R}$ is continuous, we know $f^{-1}(\{c\})$ is closed in $D$. We now show $f^{-1}(\{c\})$ is open in $D$. Fix arbitrary $q \in f^{-1}(\{c\})$. Because $f : D \to \mathbb{R}$ is locally constant, we know there exists some $r$ such that $B_r(q) \subseteq D$ and $f$ sends $B_r(q)$ to $f(q) = c$. It follows that $B_r(q) \subseteq f^{-1}(\{c\})$. Because $q$ is arbitrary selected from $f^{-1}(\{c\})$, we have shown $f^{-1}(\{c\})$ is open in $D$.

In conclusion, we have shown $f^{-1}(\{c\})$ is both open and closed in $D$. It then follows from $D$ being connected that $f^{-1}(\{c\}) = D$ or $\varnothing$. Because $p \in f^{-1}(\{c\})$, we can deduce $f^{-1}(\{c\}) = D$, i.e., $f$ send all points in $D$ to $c$, a constant function. ■