Taylor series

$$f(x) = \underbrace{f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots}\ \cdots$$

$$= \underbrace{T_n(x)}_{\text{Taylor polynomial}} + \underbrace{R_n(x)}_{\text{error}}$$

(*)

$$T_n(x) \xrightarrow[n\to\infty]{} f(x) \iff R_n(x) \xrightarrow[n\to\infty]{} 0$$

(**)

Prop $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad \forall x \in \mathbb{R}$

pf $\frac{x^n}{n!} = \frac{x}{1}\frac{x}{2}\frac{x}{3}\cdots\frac{x}{[x]}\cdots\frac{x}{n} \xrightarrow[n\to\infty]{} 0$

$>1 \qquad <1<1$

※

<Ex2> Prove $e^x \overset{?}{=} \sum_{n=0}^{\infty} \frac{x^n}{n!}$ Taylor series of e^x

Recall Taylor's Inequality

necessary condition

If $\boxed{|f^{(n+1)}(x)| \le M}$ for $|x-a| < d$ (***)

$\Rightarrow$ Remainder $|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \xrightarrow[n\to\infty]{} 0$

Sol 2: $f(x) = e^x \Rightarrow f^{(n+1)}(x) = e^x \ \forall n \xrightarrow{(*)} T_n(x) \to f(x) = e^x \quad n\to\infty$

For $|x| \le d \in \mathbb{R}^+ \Rightarrow \boxed{|f^{(n+1)}(x)| = e^x \le e^d \equiv M} \iff e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$f(x) = e^x \ \forall n$

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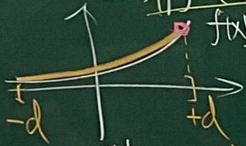

$$|R_n(x)| \overset{(***)}{\le} M\frac{|x-a|^{n+1}}{(n+1)!} \xrightarrow[n\to\infty]{(**)} 0$$

Rmk: $x = 1, \Rightarrow e = \sum_{n=0}^{\infty} \frac{1}{n!}$

$\iff \underbrace{S_n = \sum_{k=0}^{n} \frac{1}{k!}}_{\text{approximation of } e} \longrightarrow e \iff \underbrace{R_n(x)}_{\text{error}} \to 0$ ※

Rmk: $f(x)=e^x=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$

$=\sum_{n=0}^{\infty}\frac{e^a}{n!}(x-a)^n$

$=e^a\sum_{n=0}^{\infty}\frac{1}{n!}(x-a)^n=e^x$

$\underbrace{\qquad}_{=e^{x-\cancel{a}}}$

Maclaurin Series of $\sin x$

$f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots$

$=0+1\cdot x+0x^2+\frac{(-1)}{3!}x^3+0x^4+\frac{1}{5!}x^5+0\cdot x^6$

$=(-1)^0x+\frac{(-1)^{\frac{3-1}{2}}}{3!}x^3+\frac{(-1)^{\frac{5-1}{2}}}{5!}x^5+\frac{(-1)^{\frac{7-1}{2}}}{7!}x^7+\cdots \qquad \frac{-1}{7!}x^7+\cdots$

$=\sum_{k=1}^{\infty}\frac{(-1)^{\frac{(2k-1)-1}{2}}}{(2k-1)!}x^{(2k-1)}=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)!}x^{2k-1}$

$=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1}\overset{?}{=}\sin x$

⟨Ex⟩ Find Maclaurin series of $\sin x$

Sol $f(x)\big|_{x=0}=\sin x\big|_{x=0}=0$ $\quad f''(x)\big|_{x=0}=-\sin x\big|_{x=0}=0$

$f'(x)\big|_{x=0}=\cos x\big|_{x=0}=1$ $\quad f'''(x)\big|_{x=0}=-\cos x\big|_{x=0}=-1$

$\because \left|f^{(n+1)}(x)\right|=\left|\pm\sin x \text{ or } \pm\cos x\right|\le 1$

$\forall x,\ \forall n.$

$\xrightarrow[(**)'(*)]{(**)}$ $f(x)=\sin x=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1}$

$(**)'$ $\lim_{k\to\infty}\frac{|x|^{2k+1}}{(2k+1)!}=0$

※

$\tan^{-1}x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$, $\ln(1+x)=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^n}{n}$,

Important Taylor's Series

$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+\cdots$

$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=\cdots$

$\sin x=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}=\cdots$

$\cos x=\sum_{k=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}=\cdots$

$$\boxed{(1+x)^k=\sum_{n=0}^{\infty}\binom{k}{n}x^n}$$

Q: Find R

⟨Ex⟩. Find the sum of $\frac{1}{1\cdot 2}-\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}-\frac{1}{4\cdot 2^4}+\cdots$

$$=\frac{(-1)^{1-1}}{1\cdot 2^1}+\frac{(-1)^{2-1}}{2\cdot 2^2}+\frac{(-1)^{3-1}}{3\cdot 2^3}+\frac{(-1)^{4-1}}{4\cdot 2^4}+\cdots$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{1}{2}\right)^n \quad x=\frac{1}{2}$$

Recall $\ln(1+x)=\sum(-1)^{n+1}\frac{x^n}{n}$ $(*)$

$$\overset{(*)}{=}\ln\left(1+\frac{1}{2}\right)=\ln\frac{3}{2} \quad \#$$

⟨Ex⟩ (a) Evaluate $\int e^{-x^2}dx \neq$ elementary fn

(b) Evaluate $\int_0^1 e^{-x^2}dx$ with error <0.001

Sol: (a) $e^{x}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$ $\forall x$ Find R

$$\Rightarrow \int e^{-x^2}dx=\int 1+(-x^2)^1+\frac{(-x^2)^2}{2!}+\frac{(-x^2)^3}{3!}+\cdots dx \quad \text{Find R}$$

$$= x+(-1)^1\frac{x^3}{1!\,3}+\frac{(-1)^2x^5}{2!\,5}+\frac{(-1)^3x^7}{3!\,7}+\cdots+C$$

$$\int e^{-x^2}dx=\sum_{k=0}^{\infty}(-1)^k\boxed{\frac{x^{2k+1}}{k!(2k+1)}}+C \quad (\text{Find R})$$

(b). By Alternating Series Test, $\int_0^1 e^{-x^2}dx$

$$|S-S_k|\le b_{k+1}\le \max\left.\frac{x^{2k+1}}{k!(2k+1)}\right|_{x\in[0,1]}$$

$$\le\frac{1}{k!(2k+1)}\le 0.001 \quad \text{Find } k=5 \quad \#$$

⟨Ex⟩ Evaluate $\lim_{x\to 0}\frac{e^x-1-x}{x^2}$ $(*)$

Sol

(A) L'Hopital's Rule

(B) Use Maclaurin Series.

$$(*)=\lim\frac{\left(\cancel{1}+\cancel{x}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)-\cancel{1}-\cancel{x}}{x^2}$$

$$=\lim_{x\to 0}\left(\frac{1}{2}+\frac{1}{3!}x+\cdots\right)=\frac{1}{2} \quad \#$$