(1) Let $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be the unit circle in $\mathbb{R}^2$. Let $\mathbb{R}P^1$ be the real projective line, i.e., the quotient space of $\mathbb{R}^2 \setminus \{0\}$ by the equivalence relation:

$$(x, y) \sim (x', y') \quad \Leftrightarrow \quad (x', y') = t(x, y) \text{ for some nonzero real number } t.$$

As usual, denote by $[x, y]$ the homogeneous coordinates on $\mathbb{R}P^1$. Show that the map $F : S^1 \to \mathbb{R}P^1$ defined by

$$F(x, y) = \begin{cases} [1 - y, x], & \text{if } y \neq 1 \\ [x, 1 + y], & \text{if } y \neq -1 \end{cases}$$

establishes a diffeomorphism between S^1 and $\mathbb{R}P^1$. Hint: stereographic projection.

(2) Suppose M and N are smooth manifolds with M connected, and $F : M \to N$ is a smooth map such that $F_{*,p} : T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$. Show that F is a constant map.

(3) Consider the trace function on the special linear group $f\colon SL(2, \mathbb{R}) \to \mathbb{R}$ where $f(A) = tr(A)$. What are the regular level sets of f?

(4) Consider the map $F : \mathbb{R}P^2 \to \mathbb{R}^5$ given by

$$F : ([x, y, z]) \mapsto \left(\frac{yz}{\sqrt{3}}, \frac{zx}{\sqrt{3}}, \frac{xy}{\sqrt{3}}, \frac{x^2 - y^2}{2\sqrt{3}}, \frac{1}{6}(x^2 + y^2 - 2z^2) \right)$$

for $(x, y, z) \in S^2$. Show that F is an immersion. Is it an embedding?

(5) Consider the following vector fields on $\mathbb{R}^3$:

$$X = \frac{\partial}{\partial x}, \qquad Y = x\frac{\partial}{\partial z} + \frac{\partial}{\partial y}.$$

(a) Find $[X, Y]$.

(b) Assume that f is a smooth function on $\mathbb{R}^3$ such that

$$Xf = Yf = 0$$

at every point. Prove that f is a constant function. Hint: First show that $Zf = 0$ for any vector Z. Then use (and prove) that

$$f(c(1)) - f(c(0)) = \int_0^1 c'(t)f \, dt$$

for any smooth curve $c : [0, 1] \to \mathbb{R}^3$.