Recall $y=f(x)$ diff. at a if $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists $\Leftrightarrow$ $\Delta y = f'(a)\Delta x + \boxed{\varepsilon}\Delta x$

$z=f(x,y)$ diff. at (a,b) if $\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \boxed{\varepsilon_1}\Delta x + \boxed{\varepsilon_2}\Delta y$

Thm: f_x f_y exist and conti. $\Rightarrow f=$ diff.

$f: \mathbb{R}\to\mathbb{R} \Rightarrow f' \in \mathbb{R}$

$f: \mathbb{R}^2\to\mathbb{R} \Rightarrow f' \in \mathbb{R}^2$

In 1D, $\Delta y = f'(a)\Delta x + \varepsilon\Delta x$ derivatives of f

$f: \mathbb{R}^2\to\mathbb{R}^2 \Rightarrow f' \in M_{2\times 2}$ matrix

In 2D, $\Delta z = \langle f_x(a,b), f_y(a,b)\rangle \cdot \langle \Delta x, \Delta y\rangle + \langle \varepsilon_1, \varepsilon_2\rangle \cdot \langle \Delta x, \Delta y\rangle$

Def: Directional derivative of f at (x_0, y_0, z_0) in the direction of a unit vector $\underline{\underline{u}} = \langle a, b, c\rangle$

is $D_{\underline{\underline{\vec{u}}}} f(x_0, y_0, z_0) \equiv \lim_{t\to 0} \frac{f(x_0+ta,\, y_0+tb,\, z_0+tc) - f(x_0, y_0, z_0)}{t}$ (*)

if the limit exists.

Let $\vec{X}_0 = (x_0, y_0, z_0) \Longrightarrow$ rewrite (*)

$$D_{\vec{u}} f(\vec{X}_0) = \lim_{t\to 0} \frac{f(\vec{X}_0 + t\vec{u}) - f(\vec{X}_0)}{t}$$

Def: Gradient vector of f

$$\text{Grad}\, f(x, y, z) = \nabla f(x, y, z) = \langle f_x, f_y, f_z\rangle$$

$$\Rightarrow D_{\vec{u}} f(\vec{X}_0) = \nabla f(\vec{X}) \cdot \vec{u}$$

Thm: Suppose f = diff.

$\Rightarrow \max_{\vec{u}} D_{\vec{u}} f(\vec{x}) = |\nabla f(\vec{x})|$, it occurs when $\vec{u} \uparrow\uparrow \nabla f(\vec{x})$ #

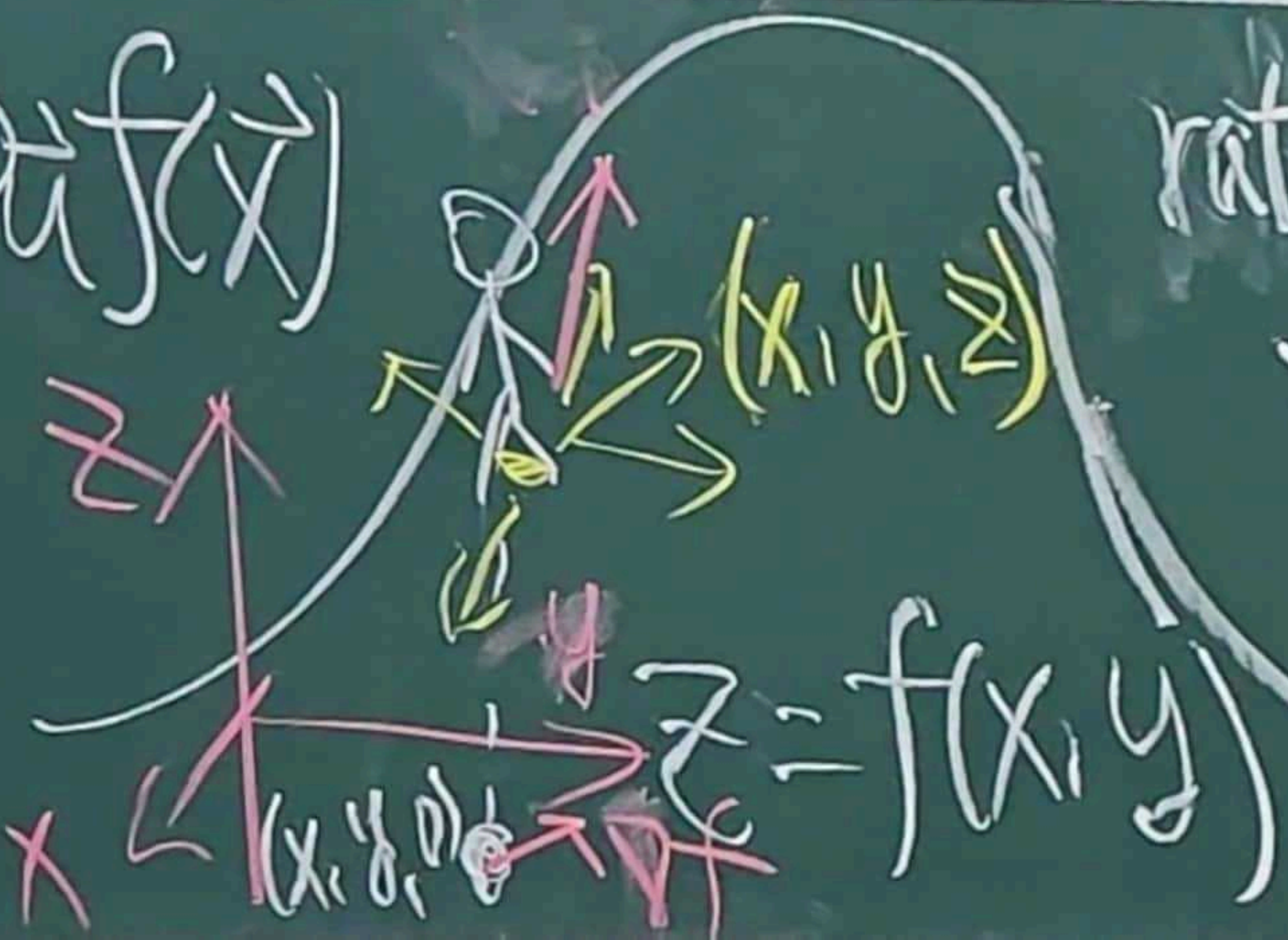
pf : Since $D_{\vec{u}} f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{u}$

Choose $\vec{u} \uparrow\uparrow \nabla f(\vec{x})$ $= |\nabla f(\vec{x})| \underbrace{|\vec{u}|}_{= 1} \underbrace{\cos\theta}_{= 1}$

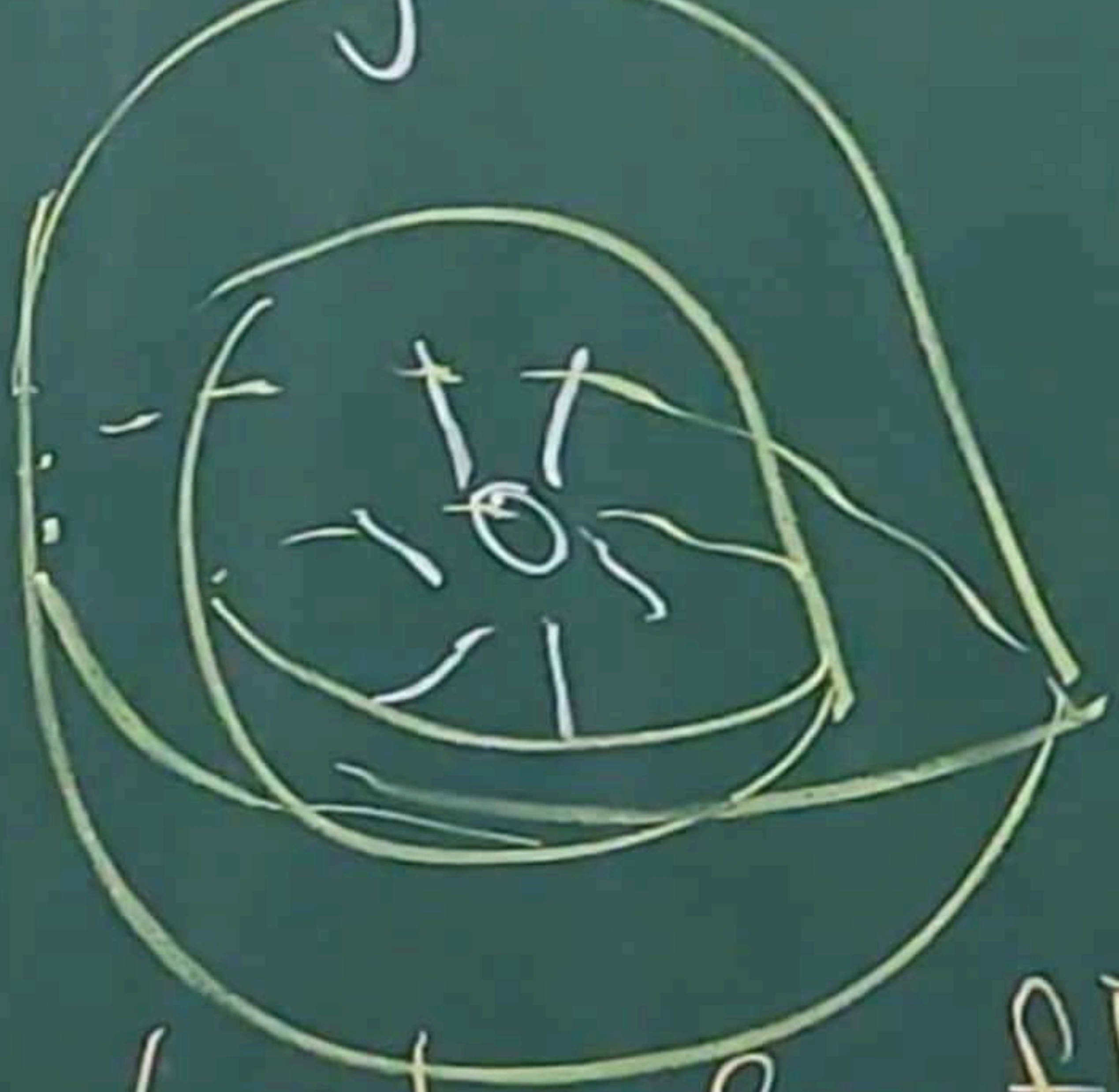
$\theta = 0$ $\leq |\nabla f(\vec{x})|$ #

Rmk $\underline{\nabla f(\vec{x})}$ = in the direction for which $z = f(x, y)$ ascends most rapid

$D_{\vec{u}} f(\vec{x})$ rate of change of f in the direction $\vec{u}$



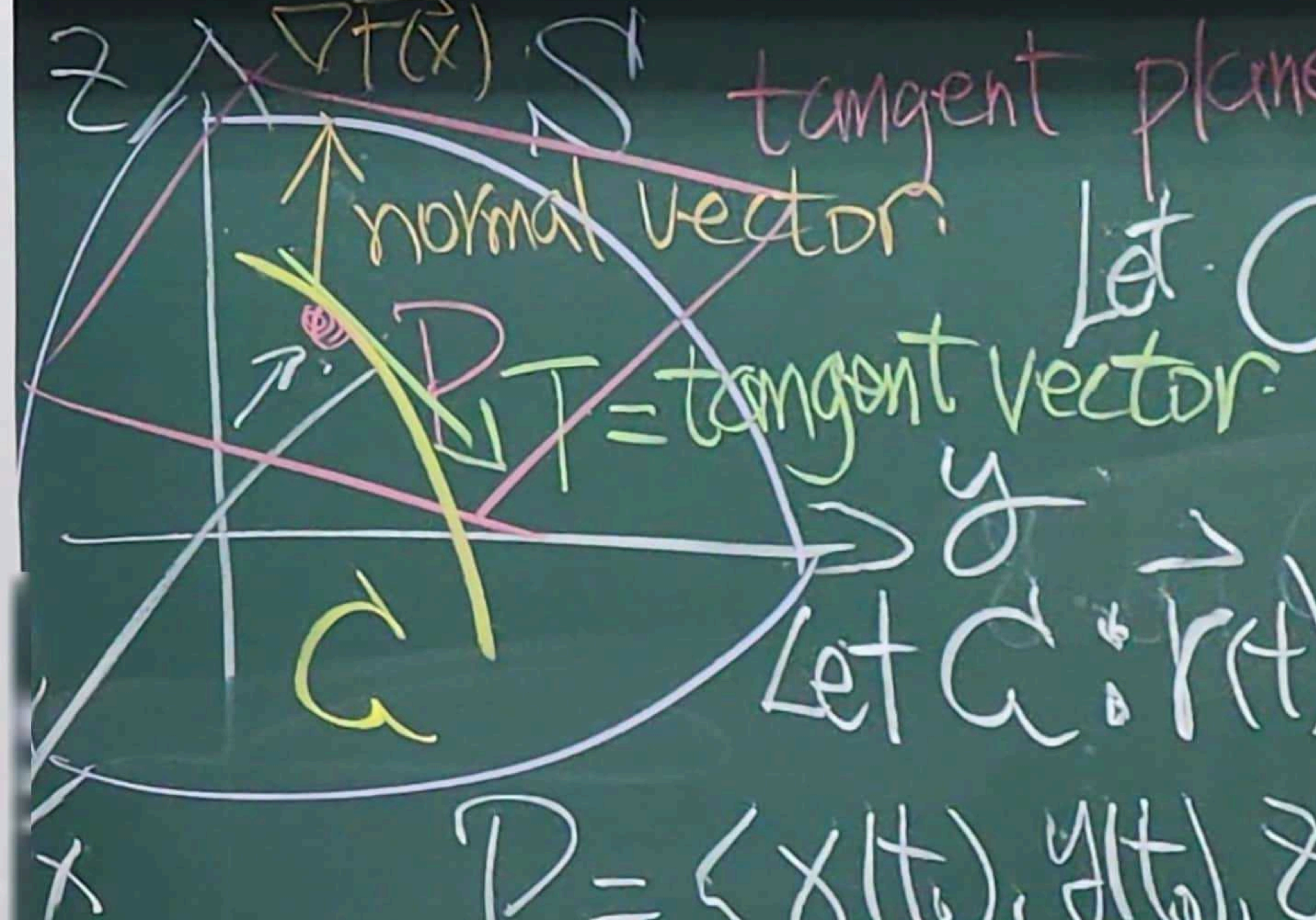
Tangent Planes to level surfaces



$w = f(x, y, z) = \text{const}$.

Suppose a surface S:

Level surface of F $F(x, y, z) = k$



Let C = curve $\subset S$

Let $C : \vec{r}(t) = \langle x(t), y(t), z(t) \rangle \subset S$

$P = \langle x(t_0), y(t_0), z(t_0) \rangle = \vec{r}(t_0)$

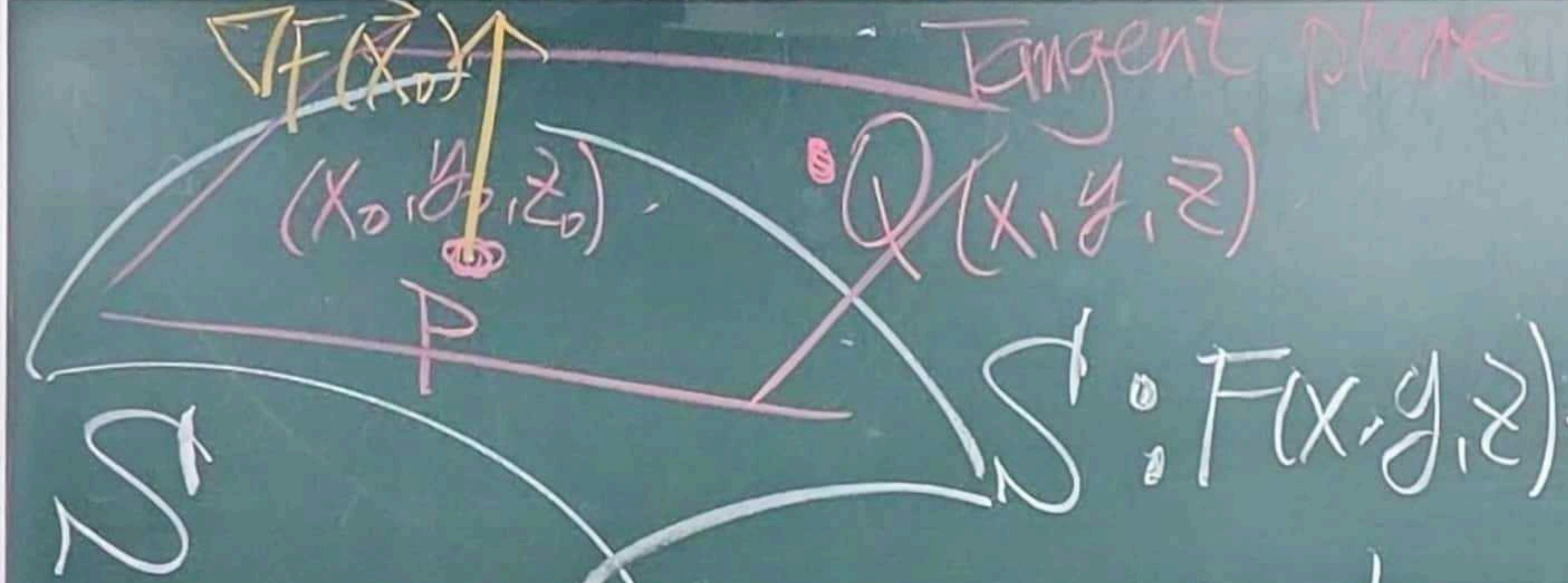
$\Rightarrow \frac{d}{dt} F(x(t), y(t), z(t)) = \frac{d}{dt} k$ chain rule

$\Rightarrow \langle F_x, F_y, F_z \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle = 0$

$\Rightarrow \underbrace{\nabla F(\vec{x})}_{\text{Gradient of } F} \cdot \underbrace{\vec{r}'(t)}_{\text{tangent vector of } C} = 0 \quad \forall C$

$\Rightarrow \nabla F(\vec{x}) \perp$ Tangent plane at P

$\Rightarrow \nabla F(\vec{x}) \perp S$ at P.



$S : F(x, y, z) = k$

$Q(x, y, z) \in$ Tangent plane

eq'n of Tangent plane

$\Rightarrow \nabla F(\vec{X}_0) \cdot \overrightarrow{PQ} = 0$

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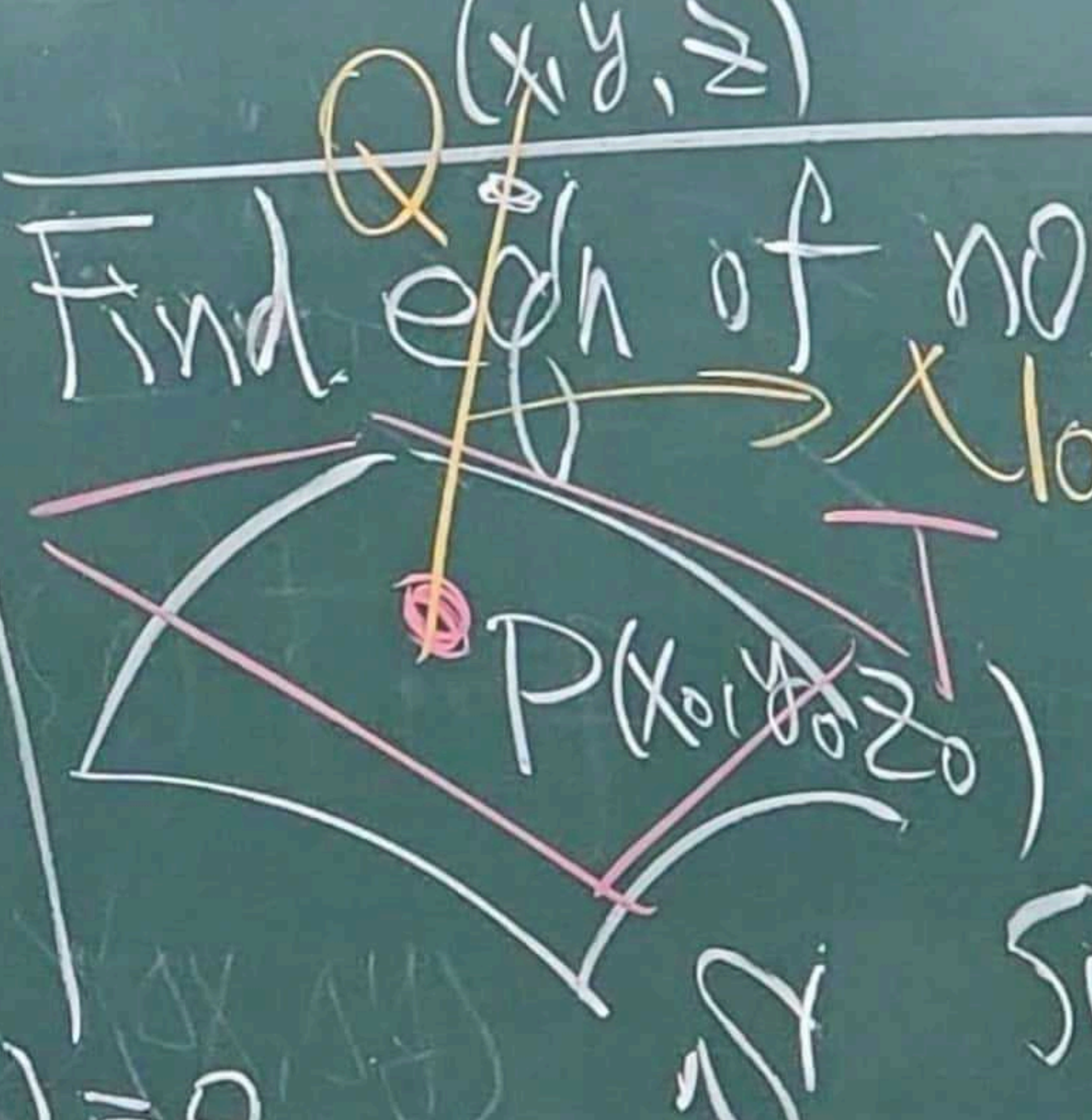
$\Rightarrow \langle f_x, f_y, f_z \rangle_{(\vec{x}_0)} \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

$\Rightarrow f_x(\vec{x}_0)(x - x_0) + f_y(\vec{x}_0)(y - y_0) + f_z(\vec{x}_0)(z - z_0) = 0.$

eq'n of plane $\underline{a}x + \underline{b}y + \underline{c}z = \underline{d}$.

$\underline{f_x(\vec{x}_0)}\, x + \underline{f_y(\vec{x}_0)}\, y + \underline{f_z(\vec{x}_0)}\, z = \underline{f_x(\vec{x}_0)x_0 + f_y(\vec{x}_0)y_0 + f_z(\vec{x}_0)z_0}$

Find eq'n of normal line



$\overrightarrow{PQ} \perp S$

Since $\nabla F(\vec{x}_0) \perp S$

$$\Rightarrow \nabla F(\vec{x}_0) \,/\!/\, \overrightarrow{PQ} \qquad \vec{x}_0 = (x_0, y_0, z_0)$$

$$\Rightarrow (f_x, f_y, f_z)^{(\vec{x}_0)} \,/\!/\, (x - x_0, y - y_0, z - z_0).$$

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$$(f_x, f_y, f_z) = t(x - x_0, y - y_0, z - z_0)$$

$$\Rightarrow (f_x, f_y, f_z) = (tx - tx_0, ty - ty_0, tz - tz_0)$$

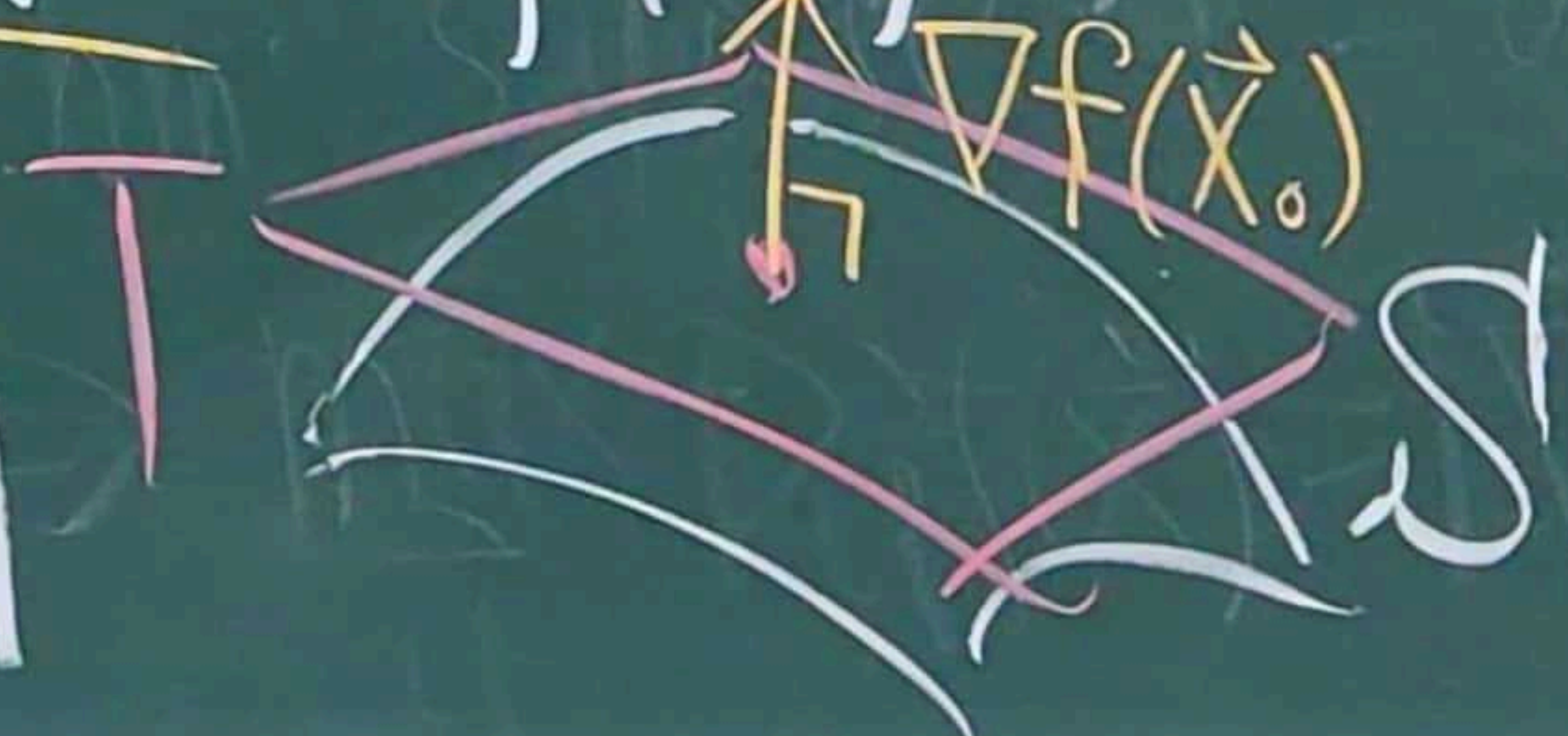
$$\Rightarrow \begin{cases} f_x = tx - tx_0 \\ f_y = ty - ty_0 \\ f_z = tz - tz_0 \end{cases} \Rightarrow \frac{f_x}{x - x_0} = \frac{f_y}{y - y_0} = \frac{f_z}{z - z_0} = t$$

Significance of Gradient vector

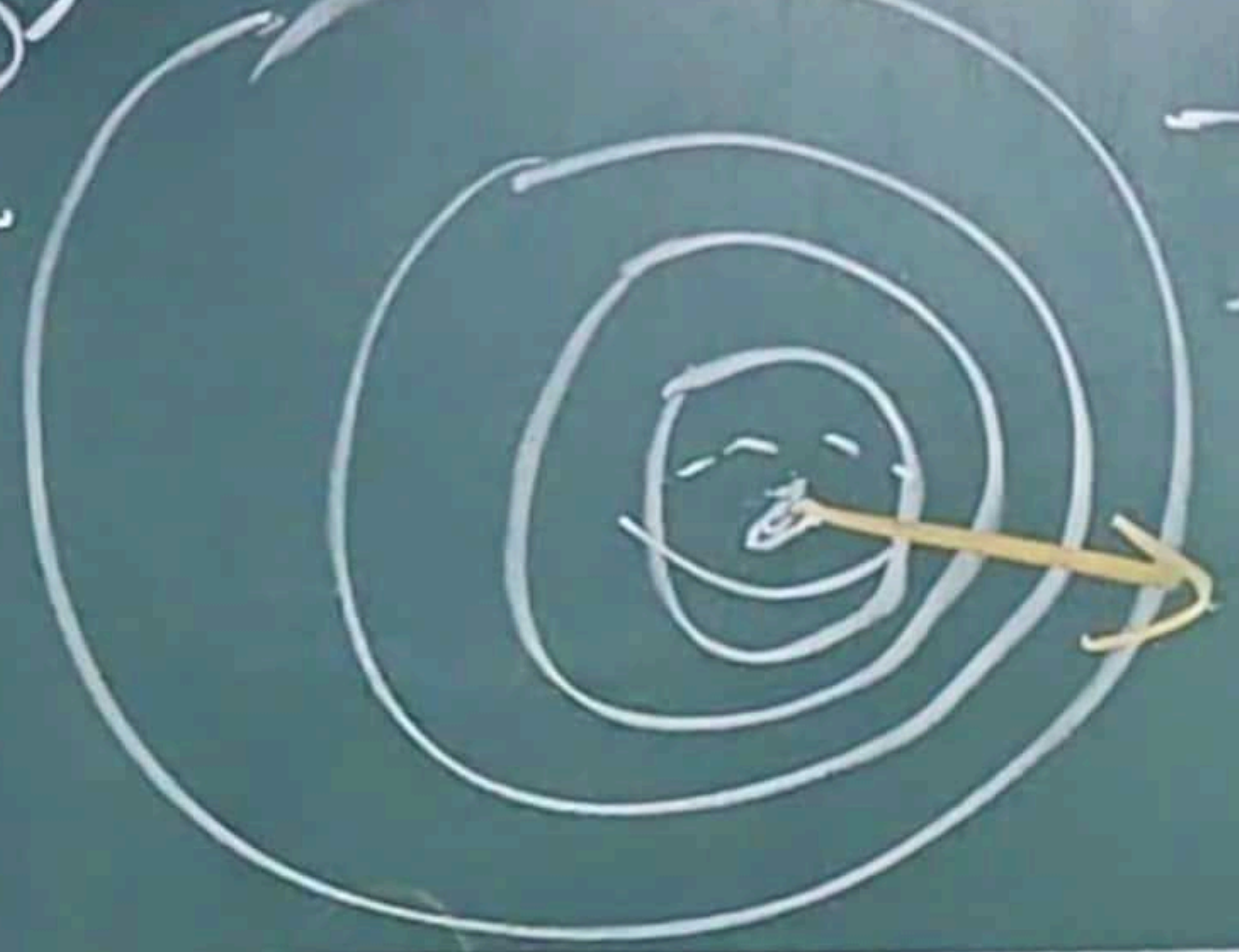
Summary: $w = f(x, y, z)$, $P(x_0, y_0, z_0) \in Dom(f)$

1° $\nabla f(\vec{x}_0) =$ direction of fastest ↑ of f

2° $\nabla f(\vec{x}_0) \perp$ Level surface S of f through P.

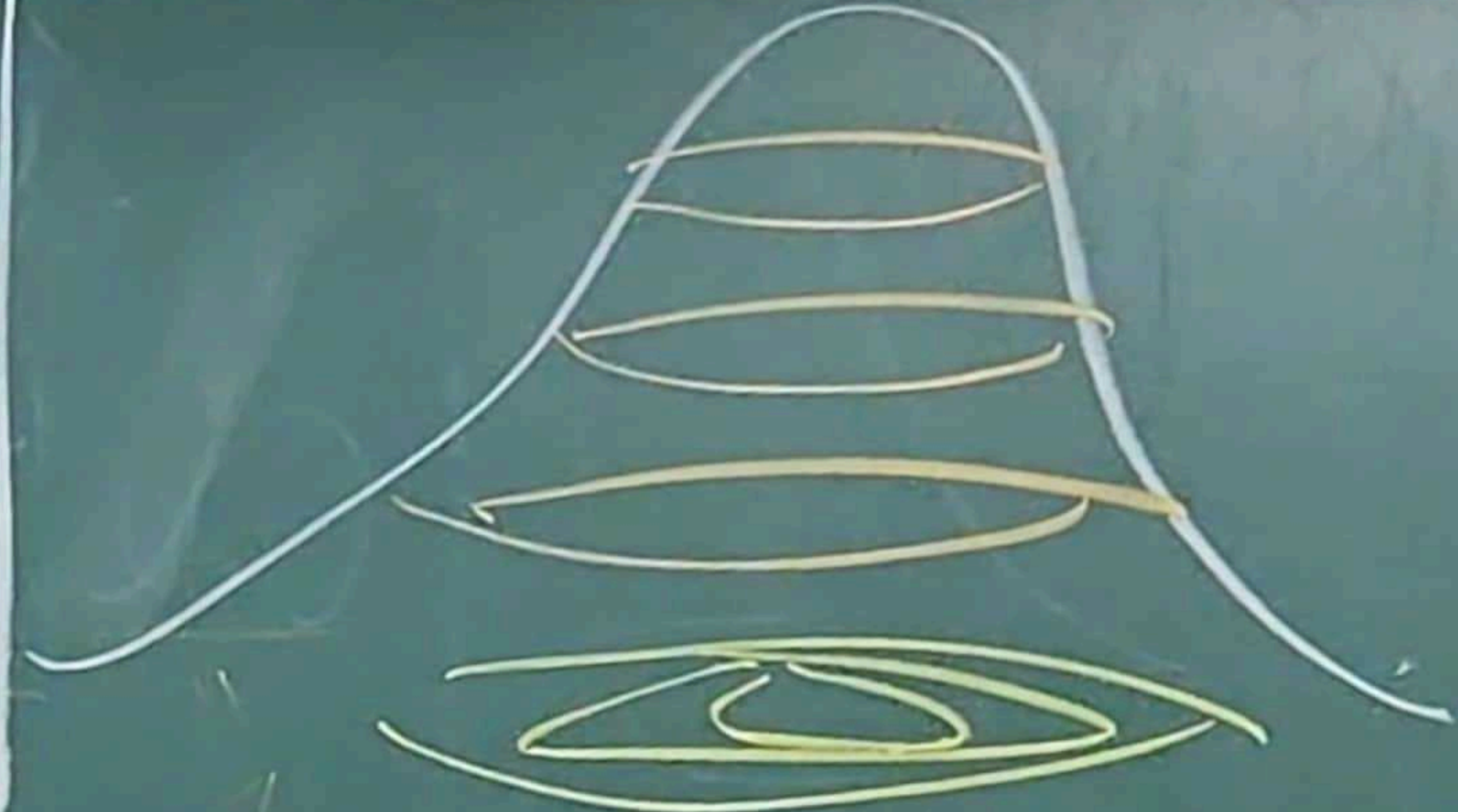


$\mathbb{R}^3$:

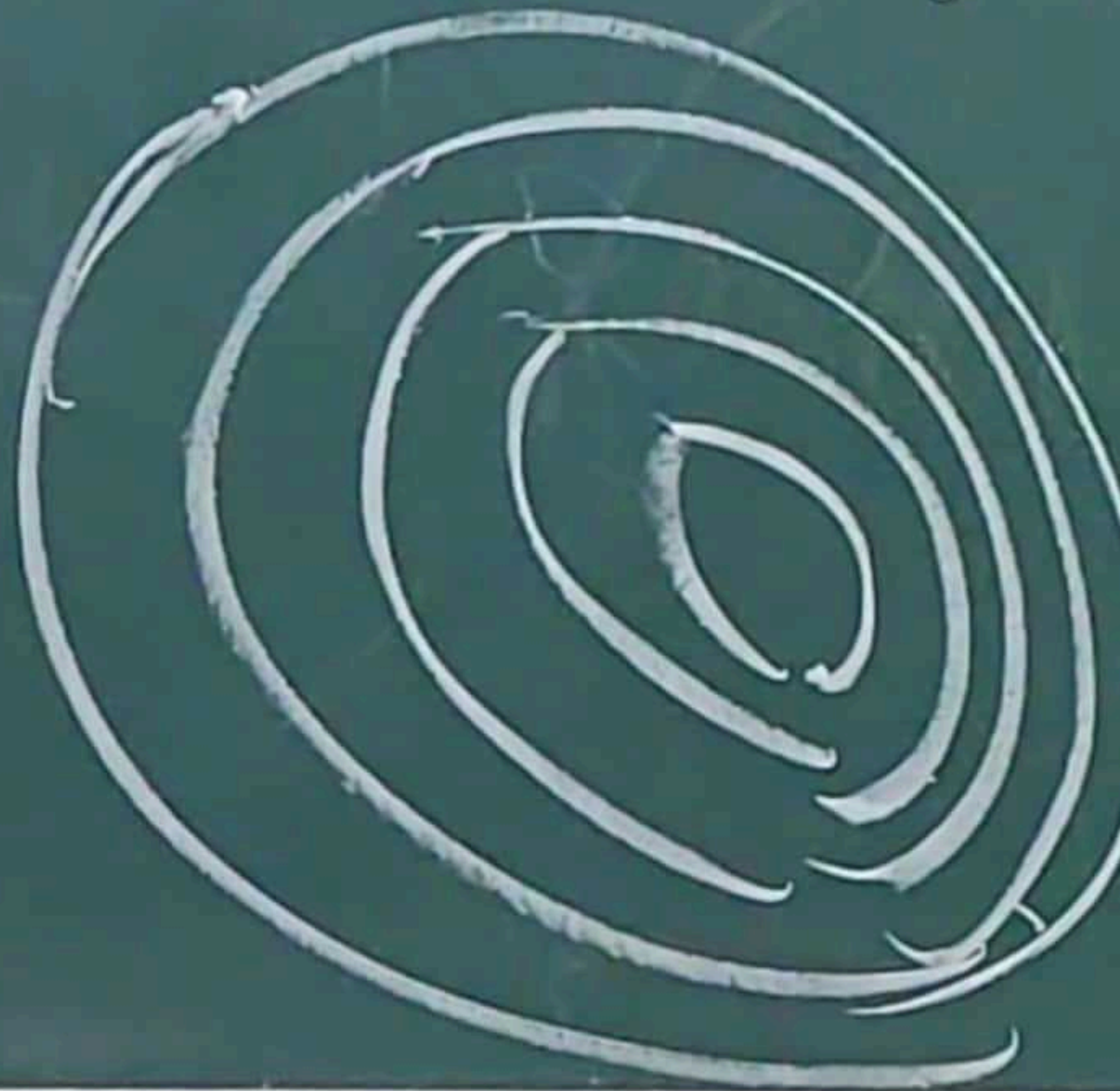


$f(x,y,z) = k$

∇f



$z = f(x,y) = k$ level curves



$\nabla f(\vec{x}) = (f_x(\vec{x}), f_y(\vec{x}))$

1° $\nabla f(\vec{x})$ = direction of fastest ↑ of f

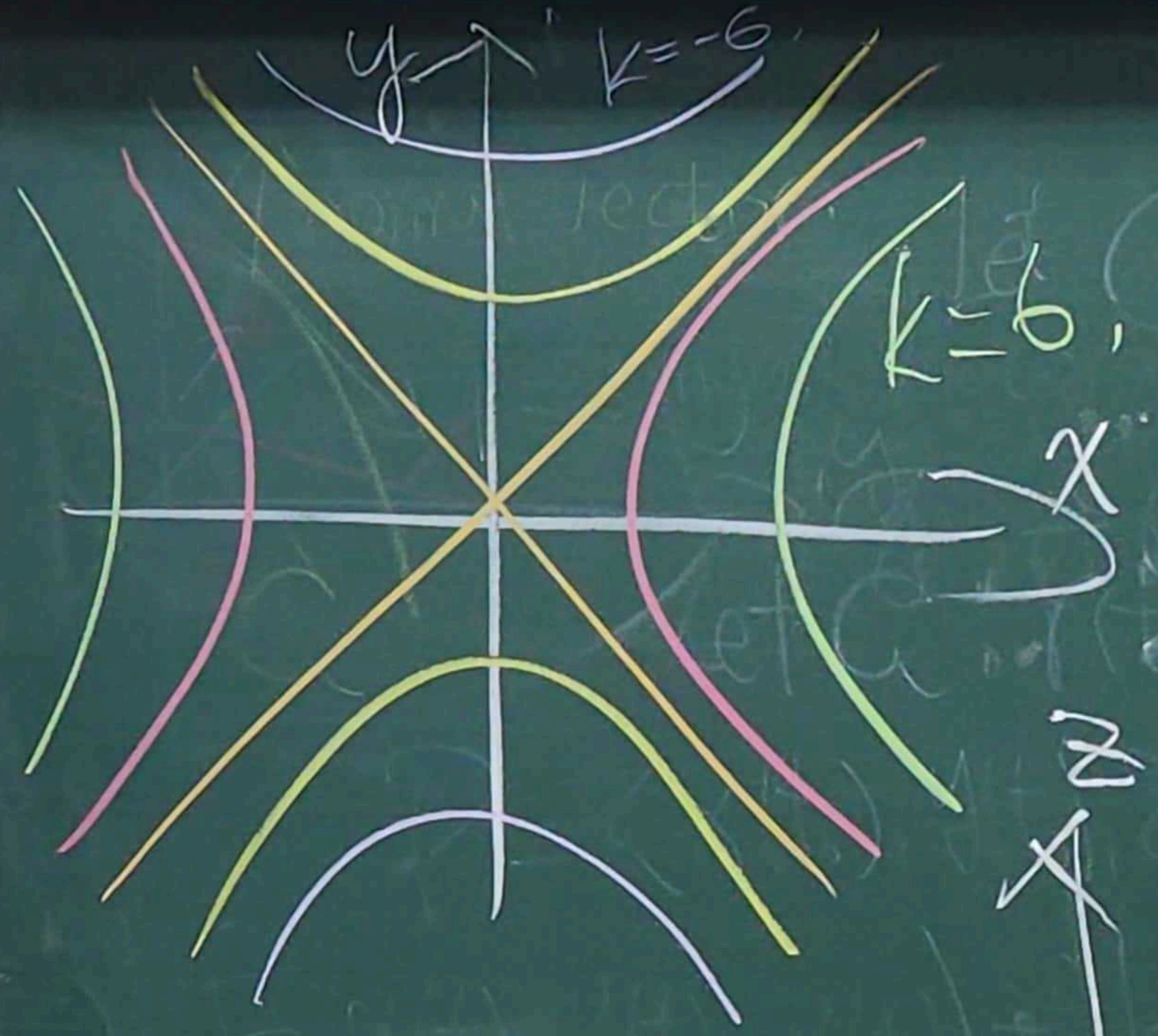
2° $\nabla f(\vec{x}_0) \perp$ level curve C of f through P

<Ex> $f(x,y) = x^2 - y^2$ height

level curves $f(x,y) = x^2 - y^2 = k$

Graph for $k = -9, -6, -3, 0,$
$3, 6, 9$.

Graph $\nabla f(\vec{x})$



$$x^2-y^2=0$$

$$x^2-y^2=-3$$

$$x^2-y^2=3$$

$$z=x^2-y^2$$

3-D