NCKU 112.1
Apostol

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Contents

Chapter 1

Introduction

1.1 Historical Introduction

1.1.1 Archimedes' method of exhaustion for the area of a parabolic segment

Before we proceed to a systematic treatment of integral calculus, we first introduce Archimedes' method of exhaustion.

Given the curve $f(x) = x^2$, we want to find the "area" bounded by the x-axis, the curve, the line $x = 0$ ant the line $x = b$.

We divide the base of the bounded segment into n equal part, each of length $\frac{b}{n}$. The points of subdivision correspond to the following values of x :

$$0, \frac{b}{n}, \frac{2b}{n}, \cdots, \frac{(n-1)b}{n}, \frac{nb}{n} = b \tag{1.1}$$

Now, we can draw the inner and outer rectangles. Let us denote by S_n the sum of the areas of the outer rectangles and s_n the sum of those of inner rectangles.

We compute

$$\begin{aligned} S_n &= \sum_{k=1}^{n} (\frac{b}{n})(\frac{kb}{n})^2 \quad (\text{Notice } (\frac{kb}{n})^2 = f(\frac{kb}{n})) && (1.2) \\ &= \frac{b^3}{n^3} \sum_{k=1}^{n} k^2 && (1.3) \\ &= \frac{b^3}{n^3}(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}) && (1.4) \\ & && (1.5) \end{aligned}$$

and compute

$$s_n = \frac{b^3}{n^3}\sum_{k=1}^{n-1} k^2 \tag{1.6}$$

$$= \frac{b^3}{n^3}(\frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}) \tag{1.7}$$

Notice that we haven't rigorously define the concept of "inner and outer" rectangle. So we can only define s_n and S_n as the value we just compute, and fill the gap between geometry truth and our definition with intuition.

Denote by A the area of the segment. We then by our intuition say for all $n \in \mathbb{N}$, $s_n \leq A \leq S_n$. Although we can't compute A the way we compute the area of a rectangle a triangle, we actually have a way to not just approximate A, but to get the exact value of A.

Theorem 1.1.1. Given the sequences $S_n = \sum_{k=1}^{n}(\frac{b}{n})(\frac{kb}{n})^2$ and $s_n = \sum_{k=1}^{n-1}(\frac{b}{n})(\frac{kb}{n})^2$, we have

$$\forall n \in \mathbb{N}, s_n \leq A \leq S_n \iff A = \frac{b^3}{3} \tag{1.8}$$

Proof. First we observe that for all $n \in \mathbb{N}$

$$s_n \leq A \leq S_n \tag{1.9}$$

$$\iff \frac{b^3}{n^3}(\frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}) \leq A \leq \frac{b^3}{n^3}(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}) \tag{1.10}$$

$$\iff \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6} \leq (\frac{n^3}{b^3})A \leq \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \tag{1.11}$$

$(\longleftarrow)$

Observe

$$\frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6} \leq \frac{n^3}{3} \leq \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \tag{1.12}$$

$$\iff -\frac{n^2}{2} + \frac{n}{6} \leq 0 \leq \frac{n^2}{2} + \frac{n}{6} \tag{1.13}$$

This is true and can be verified by induction.

$(\longrightarrow)$

We first show $S_n < \frac{b^3}{3} + \frac{b^3}{n}$.

$$\frac{1}{3n} < 1 \tag{1.14}$$
$$\iff \frac{1}{6n^2} < \frac{1}{2n} \tag{1.15}$$
$$\iff \frac{1}{2n} + \frac{1}{6n^2} < \frac{1}{n} \tag{1.16}$$
$$\iff \frac{b^3}{n^3}(\frac{n^2}{2} + \frac{n}{6}) < \frac{b^3}{n} \tag{1.17}$$
$$\iff S_n = \frac{b^3}{n^3}(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}) < \frac{b^3}{3} + \frac{b^3}{n} \text{ (done)} \tag{1.18}$$

Assume $A > \frac{b^3}{3}$. Find an n such that $n > \frac{b^3}{A - \frac{b^3}{3}}$

$$n > \frac{b^3}{A - \frac{b^3}{3}} \tag{1.19}$$
$$\iff A - \frac{b^3}{3} > \frac{b^3}{n} \tag{1.20}$$
$$\iff A > \frac{b^3}{3} + \frac{b^3}{n} > S_n \text{ CaC} \tag{1.21}$$

We now show $s_n > \frac{b^3}{3} - \frac{b^3}{n}$

$$-\frac{1}{2n} < \frac{1}{6n^2} \tag{1.22}$$
$$\iff -\frac{1}{n} < \frac{1}{-2n} + \frac{1}{6n^2} \tag{1.23}$$
$$\iff -\frac{b^3}{n^3} < \frac{b^3}{-2n} + \frac{b^3}{6n^2} \tag{1.24}$$
$$\iff \frac{b^3}{3} - \frac{b^3}{n} < \frac{b^3}{n^3}(\frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}) = s_n \text{ (done)} \tag{1.25}$$

Assume $A < \frac{b^3}{3}$. Find an n such that $n > \frac{b^3}{\frac{b^3}{3} - A}$

$$n > \frac{b^3}{\frac{b^3}{3} - A} \tag{1.26}$$
$$\iff \frac{b^3}{3} - A > \frac{b^3}{n} \tag{1.27}$$
$$\iff s_n > \frac{b^3}{3} - \frac{b^3}{n} > A \text{ CaC} \tag{1.28}$$

$\blacksquare$

1.1.2 Exercises

Question 1

Given two sequences $s_n = \frac{b^4}{n^4}[1^3 + \cdots + (n-1)^3]$, $S_n = \frac{b^4}{n^4}[1^3 + \cdots + n^3]$ and assuming inequality $1 + \cdots + (n-1)^3 < \frac{n^4}{4} < 1 + \cdots + n^3$.

Prove that

$$\forall n \in \mathbb{N}, s_n < A < S_n \iff A = \frac{b^4}{4} \tag{1.29}$$

Solution

$(\longleftarrow)$

Observe

$$1 + \cdots + (n-1)^3 < \frac{n^4}{4} < 1 + \cdots + n^3 \tag{1.30}$$

$$\iff \frac{n^4}{b^4}(s_n) < \frac{n^4}{4} < \frac{n^4}{b^4}(S_n) \tag{1.31}$$

$$\iff s_n < \frac{b^4}{4} < S_n \tag{1.32}$$

$(\longrightarrow)$

We first show $S_n < \frac{b^4}{4} + \frac{b^4}{n}$

$$1 + \cdots + (n-1)^3 < \frac{n^4}{4} \tag{1.33}$$

$$\iff 1 + \cdots + n^3 < \frac{n^4}{4} + n^3 \tag{1.34}$$

$$\iff \frac{n^4}{b^4}(S_n) < \frac{n^4}{4} + n^3 \tag{1.35}$$

$$\iff S_n < \frac{b^4}{4} + \frac{b^4}{n} \text{ (done)} \tag{1.36}$$

Assume $A > \frac{b^4}{4}$. Find an n such that $\frac{b^4}{A-\frac{b^4}{4}} < n$. Observe

$$\frac{b^4}{A-\frac{b^4}{4}} < n \tag{1.37}$$

$$\iff \frac{b^4}{n} < A - \frac{b^4}{4} \tag{1.38}$$

$$\iff S_n < \frac{b^4}{4} + \frac{b^4}{n} < A \text{ CaC} \tag{1.39}$$

We now show $s_n > \frac{b^4}{4} - \frac{b^4}{n}$

$$\frac{n^4}{4} < 1 + \cdots + n^3 \tag{1.40}$$

$$\iff \frac{n^4}{4} - n^3 < 1 + \cdots + (n-1)^3 \tag{1.41}$$

$$\iff \frac{n^4}{4} - n^3 < \frac{n^4}{b^4}(s_n) \tag{1.42}$$

$$\iff \frac{b^4}{4} - \frac{b^4}{n} < s_n \text{ (done)} \tag{1.43}$$

Assume $A < \frac{b^4}{4}$. Find an n such that $\frac{b^4}{\frac{b^4}{4}-A} < n$. Observe

$$\frac{b^4}{\frac{b^4}{4}-A} < n \tag{1.44}$$

$$\iff \frac{b^4}{n} < \frac{b^4}{4} - A \tag{1.45}$$

$$\iff A < \frac{b^4}{4} - \frac{b^4}{n} < s_n \text{ CaC} \tag{1.46}$$

Question 2

Use the same method to find the area when $f(x) = ax^3 + c$

Solution

We first show $s_n = bc + \frac{b^4}{n^4}a(1 + \cdots + (n-1)^3)$. Observe

$$
\begin{aligned}
s_n &= \sum_{k=0}^{n-1} \frac{b}{n} f(\frac{bk}{n}) && (1.47)\\
&= \sum_{k=0}^{n-1} \frac{b}{n}(a\frac{b^3k^3}{n^3} + c) && (1.48)\\
&= \sum_{k=0}^{n-1} \frac{b^4}{n^4}ak^3 + \frac{bc}{n} && (1.49)\\
&= bc + \frac{b^4}{n^4}a\sum_{k=0}^{n-1} k^3 && (1.50)\\
&= bc + \frac{b^4}{n^4}a(1 + \cdots + (n-1)^3) \text{ (done)} && (1.51)
\end{aligned}
$$

Also, we see

$$
\begin{aligned}
S_n &= \sum_{k=1}^{n} \frac{b}{n} f(\frac{bk}{n}) && (1.52)\\
&= bc + \frac{b^4}{n^4}a\sum_{k=1}^{n} k^3 && (1.53)\\
&= bc + \frac{b^4}{n^4}a(1 + \cdots + n^3) \text{ (done)} && (1.54)
\end{aligned}
$$

Now we prove

$$
\forall n \in \mathbb{N}, s_n < A < S_n \iff A = bc + \frac{ab^4}{4} \qquad (1.55)
$$

$(\longleftarrow)$

Observe

$$
\begin{aligned}
& 1 + \cdots + (n-1)^3 < \frac{n^4}{4} < 1 + \cdots n^3 && (1.56)\\
\iff & \frac{b^4}{n^4}a(1 + \cdots + (n-1)^3) < \frac{ab^4}{4} < \frac{b^4}{n^4}a(1 + \cdots + n^3) && (1.57)\\
\iff & s_n < bc + \frac{ab^4}{4} < S_n && (1.58)
\end{aligned}
$$

$(\longrightarrow)$

We first show $S_n < bc + \frac{ab^4}{4} + \frac{ab^4}{n}$ and $s_n > bc + \frac{ab^4}{4} - \frac{ab^4}{n}$. Observe

$$1 + \cdots + (n-1)^3 < \frac{n^4}{4} \tag{1.59}$$

$$\iff 1 + \cdots + n^3 < \frac{n^4}{4} + n^3 \tag{1.60}$$

$$\iff bc + \frac{b^4}{n^4}a(1 + \cdots + n^3) < bc + \frac{b^4}{n^4}a(\frac{n^4}{4} + n^3) \tag{1.61}$$

$$\iff S_n < bc + \frac{ab^4}{4} + \frac{ab^4}{n} \text{ (done)} \tag{1.62}$$

and observe

$$\frac{n^4}{4} < 1 + \cdots + n^4 \tag{1.63}$$

$$\iff \frac{n^4}{4} - n^3 < 1 + \cdots + (n-1)^3 \tag{1.64}$$

$$\iff bc + \frac{b^4}{n^4}a(\frac{n^4}{4} - n^3) < bc + \frac{b^4}{n^4}a(1 + \cdots + (n-1)^3) \tag{1.65}$$

$$\iff bc + \frac{ab^4}{4} - \frac{ab^4}{n} < s_n \text{ (done)} \tag{1.66}$$

Assume $A > bc + \frac{ab^4}{4}$, and find an n such that $\frac{ab^4}{A - bc - \frac{ab^4}{4}} < n$. Observe

$$\frac{ab^4}{A - bc - \frac{ab^4}{4}} < n \tag{1.67}$$

$$\iff \frac{ab^4}{n} < A - bc - \frac{ab^4}{4} \tag{1.68}$$

$$\iff S_n < bc + \frac{ab^4}{4} + \frac{ab^4}{n} < A \text{ CaC} \tag{1.69}$$

Assume $A < bc + \frac{ab^4}{4}$, and find an n such that $\frac{ab^4}{bc + \frac{ab^4}{4} - A} < n$. Observe

$$\frac{ab^4}{bc + \frac{ab^4}{4} - A} < n \tag{1.70}$$

$$\iff \frac{ab^4}{n} < bc + \frac{ab^4}{4} - A \tag{1.71}$$

$$\iff A < bc + \frac{ab^4}{4} - \frac{ab^4}{n} < s_n \text{ CaC} \tag{1.72}$$

Question 3

given inequalities

$$\forall n, k \in \mathbb{N}, 1^k + \cdots + (n-1)^k < \frac{n^{k+1}}{k+1} < 1^k + \cdots + n^k \tag{1.73}$$

and sequences

$$s_n = \frac{b^{k+1}}{n^{k+1}}(1^k + \cdots + (n-1)^k) \text{ and } S_n = \frac{b^{k+1}}{n^{k+1}}(1^k + \cdots + n^k) \tag{1.74}$$

Show that for all $k \in \mathbb{N}$

$$\forall n \in \mathbb{N}, s_n < A < S_n \iff A = \frac{b^{k+1}}{k+1} \tag{1.75}$$

Solution

$(\longleftarrow)$

Observe that for all $n \in \mathbb{N}$

$$1^k + \cdots + (n-1)^k < \frac{n^{k+1}}{k+1} < 1^k + \cdots + n^k \tag{1.76}$$

$$\iff \frac{b^{k+1}}{n^{k+1}}(1^k + \cdots + (n-1)^k) < \frac{b^{k+1}}{n^{k+1}}\frac{n^{k+1}}{k+1} < \frac{b^{k+1}}{n^{k+1}}(1^k + \cdots + n^k) \tag{1.77}$$

$$\iff s_n < \frac{b^{k+1}}{k+1} < S_n \tag{1.78}$$

$(\longrightarrow)$

We first show $S_n < \frac{b^{k+1}}{k+1} + \frac{b^{k+1}}{n}$ and $s_n > \frac{b^{k+1}}{k+1} - \frac{b^{k+1}}{n}$. Observe that for all $n \in \mathbb{N}$

$$1^k + \cdots + (n-1)^k < \frac{n^{k+1}}{k+1} \tag{1.79}$$

$$\iff 1^k + \cdots + n^k < \frac{n^{k+1}}{k+1} + n^k \tag{1.80}$$

$$\iff \frac{b^{k+1}}{n^{k+1}}(1^k + \cdots + n^k) < \frac{b^{k+1}}{n^{k+1}}(\frac{n^{k+1}}{k+1} + n^k) \tag{1.81}$$

$$\iff S_n < \frac{b^{k+1}}{k+1} + \frac{b^{k+1}}{n} \text{ (done)} \tag{1.82}$$

and

$$\frac{n^{k+1}}{k+1} < 1 + \cdots + n^k \tag{1.83}$$

$$\iff \frac{n^{k+1}}{k+1} - n^k < 1 + \cdots + (n-1)^k \tag{1.84}$$

$$\iff \frac{b^{k+1}}{n^{k+1}}(\frac{n^{k+1}}{k+1} - n^k) < \frac{b^{k+1}}{n^{k+1}}(1 + \cdots + (n-1)^k) \tag{1.85}$$

$$\iff \frac{b^{k+1}}{k+1} - \frac{b^{k+1}}{n} < s_n \text{ (done)} \tag{1.86}$$

Assume $A > \frac{b^{k+1}}{k+1}$. We find an n such that $\frac{b^{k+1}}{A - \frac{b^{k+1}}{k+1}} < n$. Observe

$$\frac{b^{k+1}}{A - \frac{b^{k+1}}{k+1}} < n \tag{1.87}$$

$$\iff \frac{b^{k+1}}{n} < A - \frac{b^{k+1}}{k+1} \tag{1.88}$$

$$\iff S_n < \frac{b^{k+1}}{n} + \frac{b^{k+1}}{k+1} < A \text{ CaC} \tag{1.89}$$

Assume $A < \frac{b^{k+1}}{k+1}$. We find an n such that $n > \frac{b^{k+1}}{\frac{b^{k+1}}{k+1} - A}$. Observe

$$\frac{b^{k+1}}{\frac{b^{k+1}}{k+1} - A} < n \tag{1.90}$$

$$\iff \frac{b^{k+1}}{n} < \frac{b^{k+1}}{k+1} - A \tag{1.91}$$

$$\iff A < \frac{b^{k+1}}{k+1} - \frac{b^{k+1}}{n} < s_n \text{ CaC} \tag{1.92}$$

1.2 Some Basic Concepts of the Theory of Sets

1.2.1 Exercises

Question 4

Use the roster notation to designate the following sets of real numbers.

(a) $A = \{x : x^2 - 1 = 0\}$

(b) $B = \{x : (x-1)^2 = 0\}$

(c) $C = \{x : x + 8 = 9\}$

(d) $D = \{x : x^3 - 2x^2 + x = 2\}$

(e) $E = \{x : (x+8)^2 = 9^2\}$

(f) $F = \{x : (x^2 + 16x)^2 = 17^2\}$

Question 5

For the sets in Exercise 1, note that $B \subseteq A$. List all the inclusion relation $\subseteq$ that hold among the sets A, B, C, D, E, F

Question 6

Let $A = \{1\}, B = \{1, 2\}$. Discuss the validity of the following statements

(a) $A \subset B$

(b) $1 \in A$

(c) $A \subseteq B$

(d) $1 \subseteq A$

(e) $A \in B$

(f) $1 \subset B$

Question 7

Solve Exercise 3 if $A = \{1\}$ and $B = \{\{1\}, 1\}$

Question 8

Given the set $S - \{1, 2, 3, 4\}$. Display all subsets of S.

1.3 A Set of Axioms for the Real-Number System

1.3.1 The order axioms

The below axiom is to define the positive real number set, and we will use the positive real number set to define order.

Axiom 1.3.1.

$$x \in \mathbb{R}^+ \text{ and } y \in \mathbb{R}^+ \implies x + y \in \mathbb{R}^+ \text{ and } xy \in \mathbb{R}^+ \tag{1.93}$$

Axiom 1.3.2. If $x \neq 0$, either $x \in \mathbb{R}^+$ or $-x \in \mathbb{R}^+$, but not both.

Axiom 1.3.3. $0 \notin \mathbb{R}^+$

Definition 1.3.1.

$$x \text{ is } \textbf{positive} \iff x \in \mathbb{R}^+ \tag{1.94}$$

Definition 1.3.2.

$$x \text{ is } \textbf{negative} \iff -x \in \mathbb{R}^+ \tag{1.95}$$

Theorem 1.3.1.

$$x \text{ is } \textbf{positive} \iff -x \text{ is } \textbf{negative} \tag{1.96}$$

Corollary 1.3.1.

$$x \text{ is } \textbf{negative} \iff -x \text{ is } \textbf{positive} \tag{1.97}$$

Theorem 1.3.2. If x is a real number, then either $x = 0$ or x is positive or x is negative. Only one of them hold true.

Definition 1.3.3.

$$x \text{ is } \textbf{less than } y \iff x < y \iff y - x \text{ is positive} \tag{1.98}$$

Theorem 1.3.3.

$$x \text{ is } \textbf{positive} \iff 0 < x \tag{1.99}$$

Corollary 1.3.2.

$$x \text{ is } \textbf{negative} \iff x < 0 \tag{1.100}$$

Definition 1.3.4.

$$y \text{ is } \textbf{greater than } x \iff y > x \iff x < y \iff x \text{ is less than } y \tag{1.101}$$

Corollary 1.3.3.

$$x \text{ is } \textbf{positive} \iff x > 0 \tag{1.102}$$

Corollary 1.3.4.

$$x \text{ is } \textbf{negative} \iff 0 > x \tag{1.103}$$

Definition 1.3.5.

$$x \text{ is } \textbf{less than or equal to } y \iff x \leq y \iff x < y \text{ or } x = y \tag{1.104}$$

Definition 1.3.6.

$$y \text{ is } \textbf{greater than or equal to } x \iff y \geq x \iff y > x \text{ or } y = x \tag{1.105}$$

Theorem 1.3.4.

$$x \leq y \iff y \geq x \tag{1.106}$$

Definition 1.3.7. We say x is **nonnegative** if $0 \leq x$

Theorem 1.3.5. If a, b are two real numbers, then exactly one of the three relations $a < b$, $a = b$, $a > b$ holds.

Proof. Let $x = b - a$. Either $x = 0$ or $x \neq 0$, but not both. If $x = 0$, then $b - a = 0$, then $b = a$. If $x \neq 0$, then either $x > 0$ or $x < 0$, but not both. If $x > 0$, then $b - a > 0$, then $a < b$. If $x < 0$, then $b - a < 0$, then $a > b$. ∎

Theorem 1.3.6.

$$a < b \text{ and } b < c \implies a < c \tag{1.107}$$

Proof. $a < b$ and $b < c \implies b - a > 0$ and $c - b > 0 \implies (b-a)+(c-b) > 0 \implies c - a > 0 \implies a < c$ ∎

Theorem 1.3.7.

$$a < b \implies a + c < b + c \tag{1.108}$$

Proof. $a < b \implies b - a > 0 \implies (b+c) - (a+c) > 0 \implies a + c < b + c$ ∎

Theorem 1.3.8.

$$a < b \text{ and } c > 0 \implies ac < bc \tag{1.109}$$

Proof. $a < b \implies b - a > 0$ and $c > 0 \implies c(b-a) > 0 \implies bc - ac > 0 \implies ac < bc$ ∎

Theorem 1.3.9.

$$a \neq 0 \implies a^2 > 0 \tag{1.110}$$

Proof. $a \neq 0 \implies a > 0$ or $a < 0 \implies a^2 > 0$ or $-a > 0 \implies a^2 > 0$ or $(-a)^2 > 0 \implies a^2 > 0$ ∎

Theorem 1.3.10.

$$1 > 0 \tag{1.111}$$

Proof. Assume $1 < 0$, so $-1 > 0$. Arbitrarily pick a positive real number a. $a > 0$ and $-a = (-1)a > 0$ CaC ∎

Theorem 1.3.11.

$$a < b \text{ and } c < 0 \implies ac > bc \tag{1.112}$$

Proof. $a < b$ and $c < 0 \implies b - a > 0$ and $-c > 0 \implies -c(b-a) > 0 \implies ac - bc > 0 \implies ac > bc$ ∎

Theorem 1.3.12.

$$a < b \implies -a > -b \tag{1.113}$$

Proof. $a < b \implies b - a > 0 \implies -a - (-b) > 0 \implies -a > -b$ ∎

Corollary 1.3.5.

$$a < 0 \implies -a > 0 \tag{1.114}$$

Theorem 1.3.13.

$$ab > 0 \implies a, b > 0 \text{ or } a, b < 0 \tag{1.115}$$

Proof. If $a > 0$, Assume $b < 0$, then $-ab = a(-b) > 0$ CaC . If $a < 0$, Assume $b > 0$, then $-ab = (-a)b > 0$ CaC ∎

Theorem 1.3.14.

$$a < c \text{ and } b < d \implies a + b < c + d \tag{1.116}$$

Proof. $a < c$ and $b < d \implies c - a > 0$ and $d - b > 0 \implies c + d - (a + b) > 0 \implies a + b < c + d$ ∎

1.3.2 The Least-upper-bound Axiom

Definition 1.3.8. Let S be a set of real numbers and x be a real number.

$$x \text{ is an } \textbf{upper bound} \text{ of } S \iff S \text{ is } \textbf{bounded above} \text{ by } x \iff \forall y \in S, y \leq x \tag{1.117}$$

$$S \text{ is } \textbf{unbounded above} \iff \forall x \in \mathbb{R}, \exists y \in S, x \leq y \tag{1.118}$$

$$x \text{ is an } \textbf{lower bound} \text{ of } S \iff S \text{ is } \textbf{bounded below} \text{ by } x \iff \forall y \in S, y \geq x \tag{1.119}$$

$$S \text{ is } \textbf{unbounded below} \iff \forall x \in \mathbb{R}, \exists y \in S, x \geq y \tag{1.120}$$

$$s = \max S \iff s \text{ is the } \textbf{maximum element} \text{ of } S \iff s \in S \text{ and } \forall y \in S, y \leq s \tag{1.121}$$

$$s = \min S \iff s \text{ is the } \textbf{minimum element} \text{ of } S \iff s \in S \text{ and } \forall y \in S, y \geq s \tag{1.122}$$

Definition 1.3.9.

$$\begin{gathered} x = \sup S \iff x \text{ is an } \textbf{least upper bound} \text{ of } S \iff x \text{ is an } \textbf{supremum} \text{ of } S \iff \\ x \text{ is an upper bound of } S \text{ and no number less than } x \text{ is an upper bound of } S \\ \iff \forall y \in S, y \leq x \text{ and } \forall z < x, \exists y \in S, z < y \end{gathered} \tag{1.123}$$

Definition 1.3.10.

$$\begin{gathered} x = \inf S \iff x \text{ is an } \textbf{greatest lower bound} \text{ of } S \iff x \text{ is an } \textbf{infimum} \text{ of } S \iff \\ x \text{ is an lower bound of } S \text{ and no number greater than } x \text{ is an lower bound of } S \\ \iff \forall y \in S, y \geq x \text{ and } \forall z > x, \exists y \in S, z > y \end{gathered} \tag{1.124}$$

Theorem 1.3.15. An bounded above set S have exactly one least upper bound

Proof. Assume x and y are two different least upper bound of S. WOLG, let $x < y$. Because y is an least upper bound of S, we know $\exists s \in S, x < s$ CaC to that x is an upper bound of S ∎

Corollary 1.3.6. An bounded below set S have exactly one greatest lower bound.

Axiom 1.3.4. (Real Numbers Set is a completed order filed) Every nonempty set S of real numbers which is bounded above has a supremum; that is, there is a real number B such that $B = \sup S$.

Theorem 1.3.16. Every nonempty set S of real number which is bounded below has an infimum

Proof. Define $-S := \{-s : s \in S\}$. We know $-S$ is nonempty since S is nonempty. So by completeness axiom, there exists a real number B such that $\forall x \in -S, x \leq B$ and $\forall y < B, \exists x \in -S, y < x$. Then we know $\forall x \in S, -x \leq B$, which implies $\forall x \in S, x \geq -B$; that is, $-B$ is an lower bound of S. Also we know $\forall y > -B, \exists x \in S, x < y$, which implies that $-B$ is an infimum of $-S$ ■