Deadline : 2022/12/07, 17:00.

1. For what values of r is
$$\int_0^\infty x^r e^{-x}\,dx$$
convergent?

2. Use Comparison Theorem to determine whether the integral converges
$$(a)\int_1^\infty \frac{x}{\sqrt{1+x^5}}\,dx \quad (b)\int_1^\infty 2^{-x^2}\,dx \quad (c)\int_0^\infty (1+x^5)^{-\frac{1}{6}}\,dx$$
$$(d)\int_\pi^\infty \frac{\sin^2(2x)}{x^2}\,dx \quad (e)\int_1^\infty \frac{\ln x}{x^2}\,dx \quad (f)\int_e^\infty \frac{1}{\sqrt{x+1}\ln x}\,dx$$

3. (**87' Calculus Exam**) Determine that the improper integral
$$\int_{-\infty}^\infty e^{x-e^x}\,dx$$
is convergent or divergent.

4. If f is nonnegative, show that $\displaystyle\int_{-\infty}^\infty f(x)\,dx = L$ if and only if $\displaystyle\lim_{c\to\infty}\int_{-c}^c f(x)\,dx = L$.

5. (a) Show that
$$\lim_{b\to\infty}\int_{-b}^b \sin x\,dx = 0.$$
   (b) Determine whether the integral
$$\int_{-\infty}^\infty \sin x\,dx$$
   converges.

6. **(Cauchy Criterion)** Let $f(x)$ be integrable on every bounded intervals. Show that $\displaystyle\int_a^\infty f(x)\,dx$ converges if and only if for every $\varepsilon > 0$, there exists $A > a$ such that for any $s, t > A$, we have
$$\left|\int_s^t f(x)\,dx\right| < \varepsilon.$$

Note: In this problem, you should do the direction ($\Rightarrow$) without any additional assumption. For the direction ($\Leftarrow$), you can do it under an additional assumption that $f(x) \geq 0$ for every $x \geq a$.

7. Prove that if $\displaystyle\int_a^\infty |f(x)|\,dx$ is convergent, then $\displaystyle\int_a^\infty f(x)\,dx$ is convergent.

   Hint: Use Cauchy Criterion.

8. Find the exact length of the curve

   (a) $y = \dfrac{x^5}{6} + \dfrac{1}{10x^3}$, $1 \leq x \leq 2$.

   (b) $x = \dfrac{y^4}{8} + \dfrac{1}{4y^2}$, $1 \leq y \leq 2$.

   (c) $y = 3 + \dfrac{1}{2}\cosh(2x)$, $0 \leq x \leq 1$.

   (d) $y = \ln(1 - x^2)$, $0 \leq x \leq \frac{1}{2}$.

   (e) $y = \displaystyle\int_1^x \sqrt{\sqrt{t} - 1}\,dt$, $1 \leq x \leq 16$.

9. Find the arc length function for the curve $y = \sin^{-1} x + \sqrt{1 - x^2}$ with starting point $(0, 1)$.