10 ☐ PARAMETRIC EQUATIONS AND POLAR COORDINATES

10.1 Curves Defined by Parametric Equations

1. $x = t^2 + t, \quad y = 3^{t+1}, \quad t = -2, -1, 0, 1, 2$

t	-2	-1	0	1	2
x	2	0	0	2	6
y	$\frac{1}{3}$	1	3	9	27

Therefore, the coordinates are $\left(2, \frac{1}{3}\right)$, $(0, 1)$, $(0, 3)$, $(2, 9)$, and $(6, 27)$.

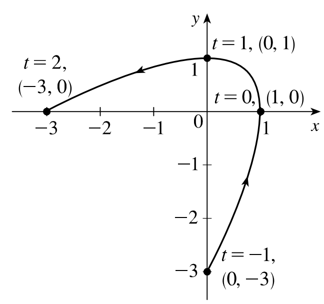
2. $x = \ln(t^2 + 1), \quad y = t/(t + 4), \quad t = -2, -1, 0, 1, 2$

t	-2	-1	0	1	2
x	$\ln 5$	$\ln 2$	0	$\ln 2$	$\ln 5$
y	-1	$-\frac{1}{3}$	0	$\frac{1}{5}$	$\frac{1}{3}$

Therefore, the coordinates are $(\ln 5, -1)$, $\left(\ln 2, -\frac{1}{3}\right)$, $(0, 0)$, $\left(\ln 2, \frac{1}{5}\right)$, and $\left(\ln 5, \frac{1}{3}\right)$.

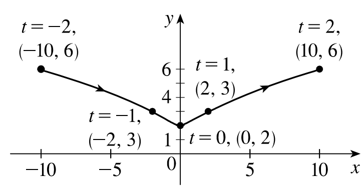
3. $x = 1 - t^2, \quad y = 2t - t^2, \quad -1 \le t \le 2$

t	-1	0	1	2
x	0	1	0	-3
y	-3	0	1	0



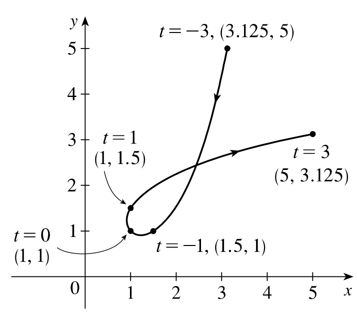
4. $x = t^3 + t, \quad y = t^2 + 2, \quad -2 \le t \le 2$

t	-2	-1	0	1	2
x	-10	-2	0	2	10
y	6	3	2	3	6



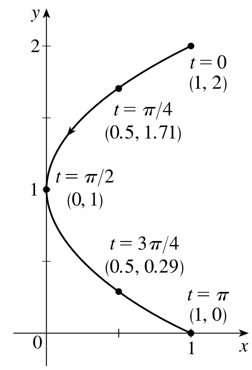
5. $x = 2^t - t, \quad y = 2^{-t} + t, \quad -3 \le t \le 3$

t	-3	-2	-1	0	1	2	3
x	3.125	2.25	1.5	1	1	2	5
y	5	2	1	1	1.5	2.25	3.125



6. $x = \cos^2 t, \quad y = 1 + \cos t, \quad 0 \le t \le \pi$

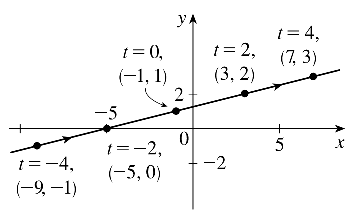
t	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
x	1	0.5	0	0.5	1
y	2	1.707	1	0.293	0



7. $x = 2t - 1, \quad y = \frac{1}{2}t + 1$

(a)

t	-4	-2	0	2	4
x	-9	-5	-1	3	7
y	-1	0	1	2	3



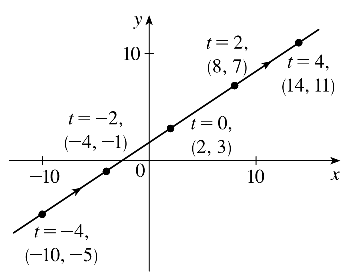
(b) $x = 2t - 1 \quad \Rightarrow \quad 2t = x + 1 \quad \Rightarrow \quad t = \frac{1}{2}x + \frac{1}{2}$, so

$y = \frac{1}{2}t + 1 = \frac{1}{2}\left(\frac{1}{2}x + \frac{1}{2}\right) + 1 = \frac{1}{4}x + \frac{1}{4} + 1 \quad \Rightarrow \quad y = \frac{1}{4}x + \frac{5}{4}$

8. $x = 3t + 2, \quad y = 2t + 3$

(a)

t	-4	-2	0	2	4
x	-10	-4	2	8	14
y	-5	-1	3	7	11



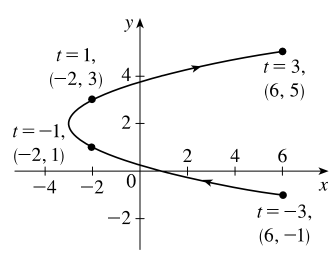
(b) $x = 3t + 2 \quad \Rightarrow \quad 3t = x - 2 \quad \Rightarrow \quad t = \frac{1}{3}x - \frac{2}{3}$, so

$y = 2t + 3 = 2\left(\frac{1}{3}x - \frac{2}{3}\right) + 3 = \frac{2}{3}x - \frac{4}{3} + 3 \quad \Rightarrow \quad y = \frac{2}{3}x + \frac{5}{3}$

9. $x = t^2 - 3, \quad y = t + 2, \quad -3 \le t \le 3$

(a)

t	-3	-1	1	3
x	6	-2	-2	6
y	-1	1	3	5



(b) $y = t + 2 \quad \Rightarrow \quad t = y - 2$, so

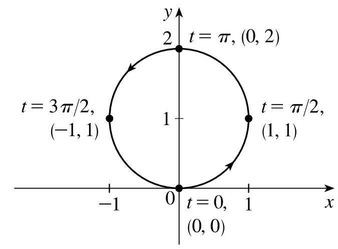
$x = t^2 - 3 = (y - 2)^2 - 3 = y^2 - 4y + 4 - 3 \quad \Rightarrow$

$x = y^2 - 4y + 1, -1 \le y \le 5$

10. $x = \sin t, \quad y = 1 - \cos t, \quad 0 \le t \le 2\pi$

(a)

t	0	$\pi/2$	π	$3\pi/2$	2π
x	0	1	0	-1	0
y	0	1	2	1	0



(b) $x = \sin t, y = 1 - \cos t$ [or $y - 1 = -\cos t$] $\Rightarrow$

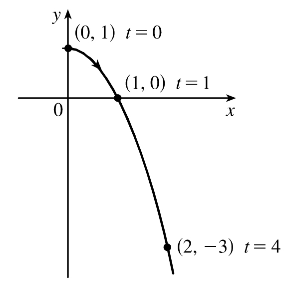
$x^2 + (y-1)^2 = (\sin t)^2 + (-\cos t)^2 \quad \Rightarrow \quad x^2 + (y-1)^2 = 1.$

As t varies from 0 to 2π, the circle with center $(0, 1)$ and radius 1 is traced out.

11. $x = \sqrt{t}, \ y = 1 - t$

(a)

t	0	1	2	3	4
x	0	1	1.414	1.732	2
y	1	0	-1	-2	-3



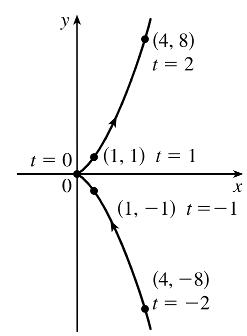
(b) $x = \sqrt{t} \quad \Rightarrow \quad t = x^2 \quad \Rightarrow \quad y = 1 - t = 1 - x^2$. Since $t \ge 0$, $x \ge 0$.

So the curve is the right half of the parabola $y = 1 - x^2$.

12. $x = t^2, \quad y = t^3$

(a)

t	-2	-1	0	1	2
x	4	1	0	1	4
y	-8	-1	0	1	8

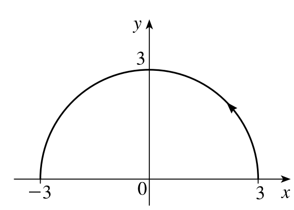


(b) $y = t^3 \quad \Rightarrow \quad t = \sqrt[3]{y} \quad \Rightarrow \quad x = t^2 = \left(\sqrt[3]{y}\right)^2 = y^{2/3}. \quad t \in \mathbb{R}, y \in \mathbb{R}, x \ge 0.$

13. (a) $x = 3\cos t, \quad y = 3\sin t, \quad 0 \le t \le \pi$

$x^2 + y^2 = 9\cos^2 t + 9\sin^2 t = 9(\cos^2 t + \sin^2 t) = 9$, which is the equation of a circle with radius 3. For $0 \le t \le \pi/2$, we have $3 \ge x \ge 0$ and $0 \le y \le 3$. For $\pi/2 < t \le \pi$, we have $0 > x \ge -3$ and $3 > y \ge 0$. Thus, the curve is the top half of the circle $x^2 + y^2 = 9$ traced counterclockwise.

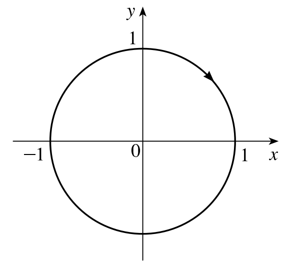
(b)



14. (a) $x = \sin 4\theta, \quad y = \cos 4\theta, \quad 0 \le \theta \le \pi/2$

$x^2 + y^2 = \sin^2 4\theta + \cos^2 4\theta = 1$, which is the equation of a circle with radius 1. When $\theta = 0$, we have $x = 0$ and $y = 1$. For $0 \le \theta \le \pi/4$, we have $x \ge 0$. For $\pi/4 < \theta \le \pi/2$, we have $x \le 0$. Thus, the curve is the circle $x^2 + y^2 = 1$ traced clockwise starting at $(0, 1)$.

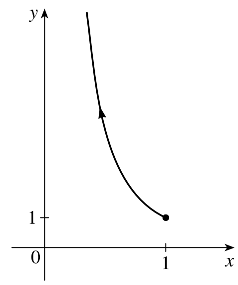
(b)



15. (a) $x = \cos\theta, \quad y = \sec^2\theta, \quad 0 \le \theta < \pi/2.$

$y = \sec^2\theta = \dfrac{1}{\cos^2\theta} = \dfrac{1}{x^2}$. For $0 \le \theta < \pi/2$, we have $1 \ge x > 0$ and $1 \le y$.

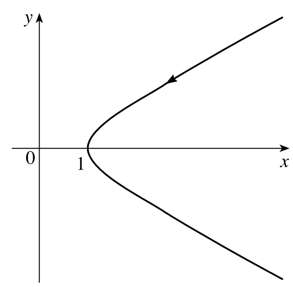
(b)



16. (a) $x = \csc t, y = \cot t, 0 < t < \pi$

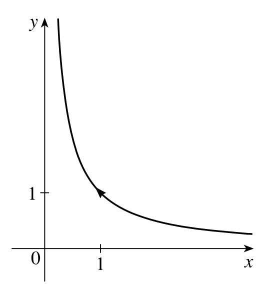
$y^2 - x^2 = \cot^2 t - \csc^2 t = 1$. For $0 < t < \pi$, we have $x > 1$. Thus, the curve is the right branch of the hyperbola $y^2 - x^2 = 1$.

(b)



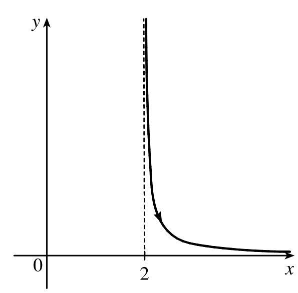
17. (a) $y = e^t = 1/e^{-t} = 1/x$ for $x > 0$ since $x = e^{-t}$. Thus, the curve is the portion of the hyperbola $y = 1/x$ with $x > 0$.

(b)



18. (a) $x = t + 2 \quad \Rightarrow \quad t = x - 2$. $y = 1/t = 1/(x-2)$. For $t > 0$, we have $x > 2$ and $y > 0$. Thus, the curve is the portion of the hyperbola $y = 1/(x-2)$ with $x > 2$.

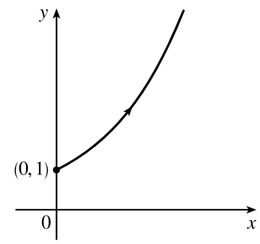
(b)



19. (a) $x = \ln t$, $y = \sqrt{t}$, $t \geq 1$.

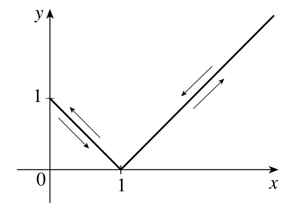
$x = \ln t \quad \Rightarrow \quad t = e^x \quad \Rightarrow \quad y = \sqrt{t} = e^{x/2}$, $x \geq 0$.

(b)



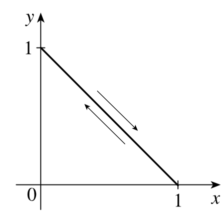
20. (a) $x = |t|$, $y = \Big|1 - |t|\Big| = |1 - x|$. For all t, we have $x \geq 0$ and $y \geq 0$. Thus, the curve is the portion of the absolute value function $y = |1 - x|$ with $x \geq 0$.

(b)



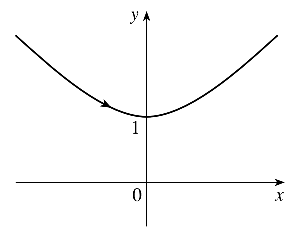
21. (a) $x = \sin^2 t$, $y = \cos^2 t$. $x + y = \sin^2 t + \cos^2 t = 1$. For all t, we have $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Thus, the curve is the portion of the line $x + y = 1$ or $y = -x + 1$ in the first quadrant.

(b)



22. (a) $x = \sinh t$, $y = \cosh t \quad \Rightarrow \quad y^2 - x^2 = \cosh^2 t - \sinh^2 t = 1$.
Since $y = \cosh t \geq 1$, we have the upper branch of the hyperbola $y^2 - x^2 = 1$.

(b)



23. The parametric equations $x = 5\cos t$ and $y = -5\sin t$ both have period 2π. When $t = 0$, we have $x = 5$ and $y = 0$. When $t = \pi/2$, we have $x = 0$ and $y = -5$. This is one-fourth of a circle. Thus, the object completes one revolution in $4 \cdot \frac{\pi}{2} = 2\pi$ seconds following a clockwise path.

24. The parametric equations $x = 3\sin\left(\frac{\pi}{4}t\right)$ and $y = 3\cos\left(\frac{\pi}{4}t\right)$ both have period $\dfrac{2\pi}{\pi/4} = 8$. When $t = 0$, we have $x = 0$ and $y = 3$. When $t = 2$, we have $x = 3$ and $y = 0$. This is one-fourth of a circle. Thus, the object completes one revolution in $4 \cdot 2 = 8$ seconds following a clockwise path.

25. $x = 5 + 2\cos \pi t$, $y = 3 + 2\sin \pi t \quad \Rightarrow \quad \cos \pi t = \dfrac{x-5}{2}$, $\sin \pi t = \dfrac{y-3}{2}$. $\cos^2(\pi t) + \sin^2(\pi t) = 1 \quad \Rightarrow$ $\left(\dfrac{x-5}{2}\right)^2 + \left(\dfrac{y-3}{2}\right)^2 = 1$. The motion of the particle takes place on a circle centered at $(5, 3)$ with a radius 2. As t goes from 1 to 2, the particle starts at the point $(3, 3)$ and moves counterclockwise along the circle $\left(\dfrac{x-5}{2}\right)^2 + \left(\dfrac{y-3}{2}\right)^2 = 1$ to $(7, 3)$ [one-half of a circle].

26. $x = 2 + \sin t$, $y = 1 + 3\cos t \quad \Rightarrow \quad \sin t = x - 2$, $\cos t = \dfrac{y-1}{3}$. $\sin^2 t + \cos^2 t = 1 \quad \Rightarrow \quad (x-2)^2 + \left(\dfrac{y-1}{3}\right)^2 = 1$.
The motion of the particle takes place on an ellipse centered at $(2, 1)$. As t goes from $\pi/2$ to 2π, the particle starts at the point $(3, 1)$ and moves counterclockwise three-fourths of the way around the ellipse to $(2, 4)$.

27. $x = 5\sin t,\, y = 2\cos t \quad\Rightarrow\quad \sin t = \dfrac{x}{5},\, \cos t = \dfrac{y}{2}.\quad \sin^2 t + \cos^2 t = 1 \quad\Rightarrow\quad \left(\dfrac{x}{5}\right)^2 + \left(\dfrac{y}{2}\right)^2 = 1$. The motion of the particle takes place on an ellipse centered at $(0, 0)$. As t goes from $-\pi$ to 5π, the particle starts at the point $(0, -2)$ and moves clockwise around the ellipse 3 times.

28. $y = \cos^2 t = 1 - \sin^2 t = 1 - x^2$. The motion of the particle takes place on the parabola $y = 1 - x^2$. As t goes from -2π to $-\pi$, the particle starts at the point $(0, 1)$, moves to $(1, 0)$, and goes back to $(0, 1)$. As t goes from $-\pi$ to 0, the particle moves to $(-1, 0)$ and goes back to $(0, 1)$. The particle repeats this motion as t goes from 0 to 2π.

29. We must have $1 \le x \le 4$ and $2 \le y \le 3$. So the graph of the curve must be contained in the rectangle $[1, 4]$ by $[2, 3]$.

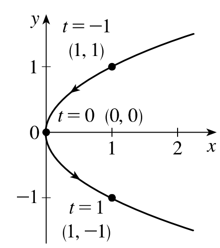
30. (a) From the first graph, we have $1 \le x \le 2$. From the second graph, we have $-1 \le y \le 1$. The only choice that satisfies either of those conditions is III.

(b) From the first graph, the values of x cycle through the values from -2 to 2 four times. From the second graph, the values of y cycle through the values from -2 to 2 six times. Choice I satisfies these conditions.

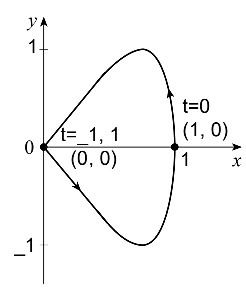
(c) From the first graph, the values of x cycle through the values from -2 to 2 three times. From the second graph, we have $0 \le y \le 2$. Choice IV satisfies these conditions.

(d) From the first graph, the values of x cycle through the values from -2 to 2 two times. From the second graph, the values of y do the same thing. Choice II satisfies these conditions.

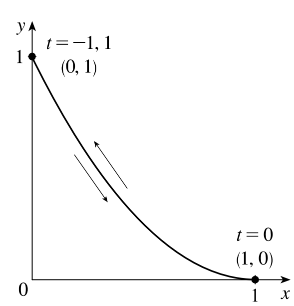
31. When $t = -1$, $(x, y) = (1, 1)$. As t increases to 0, x and y both decrease to 0. As t increases from 0 to 1, x increases from 0 to 1 and y decreases from 0 to -1. As t increases beyond 1, x continues to increase and y continues to decrease. For $t < -1$, x and y are both positive and decreasing. We could achieve greater accuracy by estimating x- and y-values for selected values of t from the given graphs and plotting the corresponding points.



32. When $t = -1$, $(x, y) = (0, 0)$. As t increases to 0, x increases from 0 to 1, while y first decreases to -1 and then increases to 0. As t increases from 0 to 1, x decreases from 1 to 0, while y first increases to 1 and then decreases to 0. We could achieve greater accuracy by estimating x- and y-values for selected values of t from the given graphs and plotting the corresponding points.



33. When $t = -1$, $(x, y) = (0, 1)$. As t increases to 0, x increases from 0 to 1 and y decreases from 1 to 0. As t increases from 0 to 1, the curve is retraced in the opposite direction with x decreasing from 1 to 0 and y increasing from 0 to 1. We could achieve greater accuracy by estimating x- and y-values for selected values of t from the given graphs and plotting the corresponding points.



34. (a) $x = t^4 - t + 1 = (t^4 + 1) - t > 0$ [think of the graphs of $y = t^4 + 1$ and $y = t$] and $y = t^2 \geq 0$, so these equations are matched with graph V.

(b) $y = \sqrt{t} \geq 0$. $x = t^2 - 2t = t(t-2)$ is negative for $0 < t < 2$, so these equations are matched with graph I.

(c) $x = t^3 - 2t = t(t^2 - 2) = t\left(t + \sqrt{2}\right)\left(t - \sqrt{2}\right)$, $y = t^2 - t = t(t-1)$. The equation $x = 0$ has three solutions and the equation $y = 0$ has two solutions. Thus, the curve has three y-intercepts and two x-intercepts, which matches graph II.

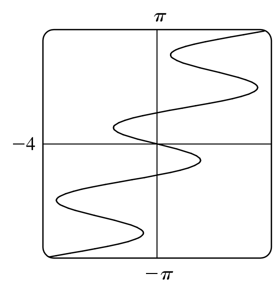
Alternate method: $x = t^3 - 2t$, $y = t^2 - t = \left(t^2 - t + \frac{1}{4}\right) - \frac{1}{4} = \left(t - \frac{1}{2}\right)^2 - \frac{1}{4}$ so $y \geq -\frac{1}{4}$ on this curve, whereas x is unbounded. These equations are matched with graph II.

(d) $x = \cos 5t$ has period $2\pi/5$ and $y = \sin 2t$ has period π, so x will take on the values -1 to 1, and then 1 to -1, before y takes on the values -1 to 1. Note that when $t = 0$, $(x, y) = (1, 0)$. These equations are matched with graph VI.

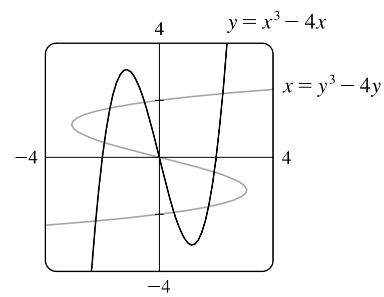
(e) $x = t + \sin 4t$, $y = t^2 + \cos 3t$. As t becomes large, t and t^2 become the dominant terms in the expressions for x and y, so the graph will look like the graph of $y = x^2$, but with oscillations. These equations are matched with graph IV.

(f) $x = t + \sin 2t$, $y = t + \sin 3t$. As t becomes large, t becomes the dominant term in the expressions for both x and y, so the graph will look like the graph of $y = x$, but with oscillations. These equations are matched with graph III.

35. Use $y = t$ and $x = t - 2\sin \pi t$ with a t-interval of $[-\pi, \pi]$.



36. Use $x_1 = t$, $y_1 = t^3 - 4t$ and $x_2 = t^3 - 4t$, $y_2 = t$ with a t-interval of $[-3, 3]$. There are 9 points of intersection; $(0, 0)$ is fairly obvious. The point in quadrant I is approximately $(2.2, 2.2)$, and by symmetry, the point in quadrant III is approximately $(-2.2, -2.2)$. The other six points are approximately $(\mp 1.9, \pm 0.5)$, $(\mp 1.7, \pm 1.7)$, and $(\mp 0.5, \pm 1.9)$.



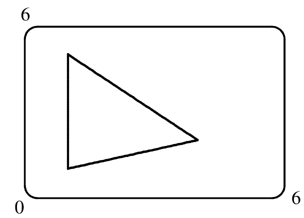
37. (a) $x = x_1 + (x_2 - x_1)t$, $y = y_1 + (y_2 - y_1)t$, $0 \leq t \leq 1$. Clearly the curve passes through $P_1(x_1, y_1)$ when $t = 0$ and through $P_2(x_2, y_2)$ when $t = 1$. For $0 < t < 1$, x is strictly between x_1 and x_2 and y is strictly between y_1 and y_2. For every value of t, x and y satisfy the relation $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, which is the equation of the line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

Finally, any point (x, y) on that line satisfies $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$; if we call that common value t, then the given parametric equations yield the point (x, y); and any (x, y) on the line between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ yields a value of t in $[0, 1]$. So the given parametric equations exactly specify the line segment from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$.

(b) $x = -2 + [3 - (-2)]t = -2 + 5t$ and $y = 7 + (-1 - 7)t = 7 - 8t$ for $0 \le t \le 1$.

38. For the side of the triangle from A to B, use $(x_1, y_1) = (1, 1)$ and $(x_2, y_2) = (4, 2)$. Hence, the equations are

$$x = x_1 + (x_2 - x_1)\,t = 1 + (4 - 1)\,t = 1 + 3t,$$
$$y = y_1 + (y_2 - y_1)\,t = 1 + (2 - 1)\,t = 1 + t.$$



Graphing $x = 1 + 3t$ and $y = 1 + t$ with $0 \le t \le 1$ gives us the side of the triangle from A to B. Similarly, for the side BC we use $x = 4 - 3t$ and $y = 2 + 3t$, and for the side AC we use $x = 1$ and $y = 1 + 4t$.

39. The result in Example 4 indicates the parametric equations have the form $x = h + r\sin bt$ and $y = k + r\cos bt$ where (h, k) is the center of the circle with radius r and $b = 2\pi/\text{period}$. (The use of positive sine in the x-equation and positive cosine in the y-equation results in a clockwise motion.) With $h = 0$, $k = 0$ and $b = 2\pi/4\pi = 1/2$, we have $x = 5\sin\left(\frac{1}{2}t\right)$, $y = 5\cos\left(\frac{1}{2}t\right)$.

40. As in Example 4, we use parametric equations of the form $x = h + r\cos bt$ and $y = k + r\sin bt$ where $(h, k) = (1, 3)$ is the center of the circle with radius $r = 1$ and $b = 2\pi/\text{period} = 2\pi/3$. (The use of positive cosine in the x-equation and positive sine in the y-equation results in a counterclockwise motion.) Thus, $x = 1 + \cos\left(\frac{2\pi}{3}t\right)$, $y = 3 + \sin\left(\frac{2\pi}{3}t\right)$.

41. The circle $x^2 + (y - 1)^2 = 4$ has center $(0, 1)$ and radius 2, so by Example 4 it can be represented by $x = 2\cos t$, $y = 1 + 2\sin t$, $0 \le t \le 2\pi$. This representation gives us the circle with a counterclockwise orientation starting at $(2, 1)$.

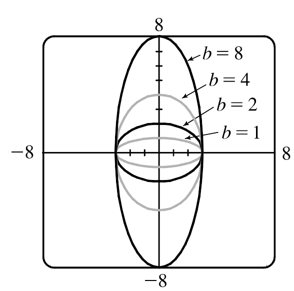
(a) To get a clockwise orientation, we could change the equations to $x = 2\cos t$, $y = 1 - 2\sin t$, $0 \le t \le 2\pi$.

(b) To get three times around in the counterclockwise direction, we use the original equations $x = 2\cos t$, $y = 1 + 2\sin t$ with the domain expanded to $0 \le t \le 6\pi$.

(c) To start at $(0, 3)$ using the original equations, we must have $x_1 = 0$; that is, $2\cos t = 0$. Hence, $t = \frac{\pi}{2}$. So we use $x = 2\cos t$, $y = 1 + 2\sin t$, $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$.

Alternatively, if we want t to start at 0, we could change the equations of the curve. For example, we could use $x = -2\sin t$, $y = 1 + 2\cos t$, $0 \le t \le \pi$.

42. (a) Let $x^2/a^2 = \sin^2 t$ and $y^2/b^2 = \cos^2 t$ to obtain $x = a\sin t$ and $y = b\cos t$ with $0 \le t \le 2\pi$ as possible parametric equations for the ellipse $x^2/a^2 + y^2/b^2 = 1$.



(b) The equations are $x = 3\sin t$ and $y = b\cos t$ for $b \in \{1, 2, 4, 8\}$.

(c) As b increases, the ellipse stretches vertically.

43. *Big circle:* It's centered at $(2, 2)$ with a radius of 2, so by Example 4, parametric equations are

$$x = 2 + 2\cos t, \qquad y = 2 + 2\sin t, \qquad 0 \leq t \leq 2\pi$$

Small circles: They are centered at $(1, 3)$ and $(3, 3)$ with a radius of 0.1. By Example 4, parametric equations are

(left) $\quad x = 1 + 0.1\cos t, \qquad y = 3 + 0.1\sin t, \qquad 0 \leq t \leq 2\pi$

and *(right)* $\quad x = 3 + 0.1\cos t, \qquad y = 3 + 0.1\sin t, \qquad 0 \leq t \leq 2\pi$

Semicircle: It's the lower half of a circle centered at $(2, 2)$ with radius 1. By Example 4, parametric equations are

$$x = 2 + 1\cos t, \qquad y = 2 + 1\sin t, \qquad \pi \leq t \leq 2\pi$$

To get all four graphs on the same screen with a typical graphing calculator, we need to change the last t-interval to$[0, 2\pi]$ in order to match the others. We can do this by changing t to $0.5t$. This change gives us the upper half. There are several ways to get the lower half—one is to change the "$+$" to a "$-$" in the y-assignment, giving us

$$x = 2 + 1\cos(0.5t), \qquad y = 2 - 1\sin(0.5t), \qquad 0 \leq t \leq 2\pi$$

44. If you are using a calculator or computer that can overlay graphs (using multiple t-intervals), the following is appropriate.

Left side: $x = 1$ and y goes from 1.5 to 4, so use

$$x = 1, \qquad y = t, \qquad 1.5 \leq t \leq 4$$

Right side: $x = 10$ and y goes from 1.5 to 4, so use

$$x = 10, \qquad y = t, \qquad 1.5 \leq t \leq 4$$

Bottom: x goes from 1 to 10 and $y = 1.5$, so use

$$x = t, \qquad y = 1.5, \qquad 1 \leq t \leq 10$$

Handle: It starts at $(10, 4)$ and ends at $(13, 7)$, so use

$$x = 10 + t, \qquad y = 4 + t, \qquad 0 \leq t \leq 3$$

Left wheel: It's centered at $(3, 1)$, has a radius of 1, and appears to go about 30° above the horizontal, so use

$$x = 3 + 1\cos t, \qquad y = 1 + 1\sin t, \qquad \tfrac{5\pi}{6} \leq t \leq \tfrac{13\pi}{6}$$

Right wheel: Similar to the left wheel with center $(8, 1)$, so use

$$x = 8 + 1\cos t, \qquad y = 1 + 1\sin t, \qquad \tfrac{5\pi}{6} \leq t \leq \tfrac{13\pi}{6}$$

If you are using a calculator or computer that cannot overlay graphs (using one t-interval), the following is appropriate. We'll start by picking the t-interval $[0, 2.5]$ since it easily matches the t-values for the two sides. We now need to find parametric equations for all graphs with $0 \leq t \leq 2.5$.

Left side: $x = 1$ and y goes from 1.5 to 4, so use

$$x = 1, \qquad y = 1.5 + t, \qquad 0 \leq t \leq 2.5$$

[continued]

Right side: $x = 10$ and y goes from 1.5 to 4, so use

$$x = 10, \qquad y = 1.5 + t, \qquad 0 \le t \le 2.5$$

Bottom: x goes from 1 to 10 and $y = 1.5$, so use

$$x = 1 + 3.6t, \qquad y = 1.5, \qquad 0 \le t \le 2.5$$

To get the x-assignment, think of creating a linear function such that when $t = 0$, $x = 1$ and when $t = 2.5$, $x = 10$. We can use the point-slope form of a line with $(t_1, x_1) = (0, 1)$ and $(t_2, x_2) = (2.5, 10)$.

$$x - 1 = \frac{10 - 1}{2.5 - 0}(t - 0) \quad \Rightarrow \quad x = 1 + 3.6t.$$

Handle: It starts at $(10, 4)$ and ends at $(13, 7)$, so use

$$x = 10 + 1.2t, \qquad y = 4 + 1.2t, \qquad 0 \le t \le 2.5$$

$(t_1, x_1) = (0, 10)$ and $(t_2, x_2) = (2.5, 13)$ gives us $x - 10 = \dfrac{13 - 10}{2.5 - 0}(t - 0) \quad \Rightarrow \quad x = 10 + 1.2t.$

$(t_1, y_1) = (0, 4)$ and $(t_2, y_2) = (2.5, 7)$ gives us $y - 4 = \dfrac{7 - 4}{2.5 - 0}(t - 0) \quad \Rightarrow \quad y = 4 + 1.2t.$

Left wheel: It's centered at $(3, 1)$, has a radius of 1, and appears to go about 30° above the horizontal, so use

$$x = 3 + 1\cos\left(\tfrac{8\pi}{15}t + \tfrac{5\pi}{6}\right), \qquad y = 1 + 1\sin\left(\tfrac{8\pi}{15}t + \tfrac{5\pi}{6}\right), \qquad 0 \le t \le 2.5$$

$(t_1, \theta_1) = \left(0, \frac{5\pi}{6}\right)$ and $(t_2, \theta_2) = \left(\frac{5}{2}, \frac{13\pi}{6}\right)$ gives us $\theta - \frac{5\pi}{6} = \dfrac{\frac{13\pi}{6} - \frac{5\pi}{6}}{\frac{5}{2} - 0}(t - 0) \quad \Rightarrow \quad \theta = \frac{5\pi}{6} + \frac{8\pi}{15}t.$

Right wheel: Similar to the left wheel with center $(8, 1)$, so use

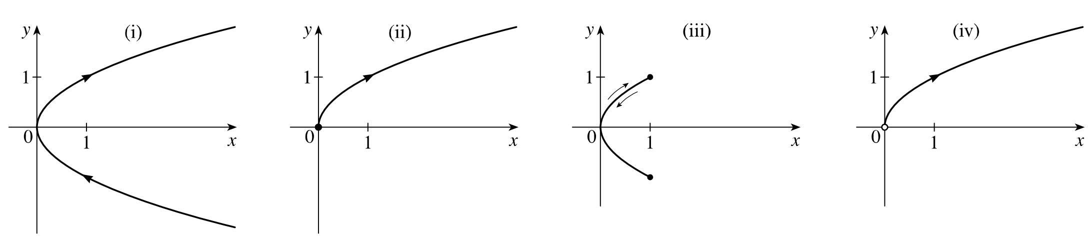
$$x = 8 + 1\cos\left(\tfrac{8\pi}{15}t + \tfrac{5\pi}{6}\right), \qquad y = 1 + 1\sin\left(\tfrac{8\pi}{15}t + \tfrac{5\pi}{6}\right), \qquad 0 \le t \le 2.5$$

45. (a) (i) $x = t^2, y = t \;\Rightarrow\; y^2 = t^2 = x$ (ii) $x = t, y = \sqrt{t} \;\Rightarrow\; y^2 = t = x$

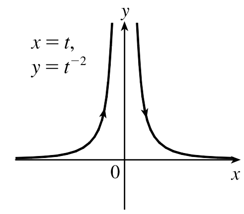
(iii) $x = \cos^2 t, y = \cos t \;\Rightarrow\; y^2 = \cos^2 t = x$ (iv) $x = 3^{2t}, y = 3^t \;\Rightarrow\; y^2 = (3^t)^2 = 3^{2t} = x.$

Thus, the points on all four of the given parametric curves satisfy the Cartesian equation $y^2 = x$.

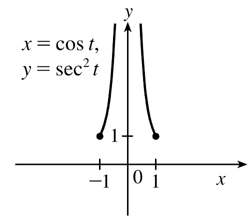
(b) The graph of $y^2 = x$ is a right-opening parabola with vertex at the origin. For curve (i), $x \ge 0$ and y is unbounded so the graph contains the entire parabola. For (ii), $y = \sqrt{t}$ requires that $t \ge 0$, so that both $x \ge 0$ and $y \ge 0$, which captures the upper half of the parabola, including the origin. For (iii), $-1 \le \cos t \le 1$ so the graph is the portion of the parabola contained in the intervals $0 \le x \le 1$ and $-1 \le y \le 1$. For (iv), $x > 0$ and $y > 0$, which captures the upper half of the parabola excluding the origin.



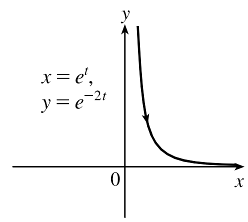
46. (a) $x = t$, so $y = t^{-2} = x^{-2}$. We get the entire curve $y = 1/x^2$ traversed in a left-to-right direction.



(b) $x = \cos t$, $y = \sec^2 t = \dfrac{1}{\cos^2 t} = \dfrac{1}{x^2}$. Since $\sec t \geq 1$, we only get the parts of the curve $y = 1/x^2$ with $y \geq 1$. We get the first quadrant portion of the curve when $x > 0$, that is, $\cos t > 0$, and we get the second quadrant portion of the curve when $x < 0$, that is, $\cos t < 0$.

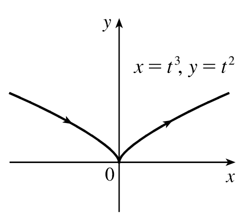


(c) $x = e^t$, $y = e^{-2t} = (e^t)^{-2} = x^{-2}$. Since e^t and e^{-2t} are both positive, we only get the first quadrant portion of the curve $y = 1/x^2$.



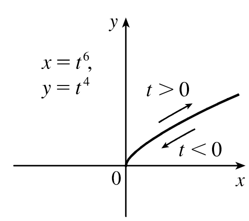
47. (a) $x = t^3 \quad \Rightarrow \quad t = x^{1/3}$, so $y = t^2 = x^{2/3}$.

We get the entire curve $y = x^{2/3}$ traversed in a left to right direction.



(b) $x = t^6 \quad \Rightarrow \quad t = x^{1/6}$, so $y = t^4 = x^{4/6} = x^{2/3}$.

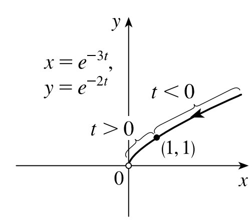
Since $x = t^6 \geq 0$, we only get the right half of the curve $y = x^{2/3}$.



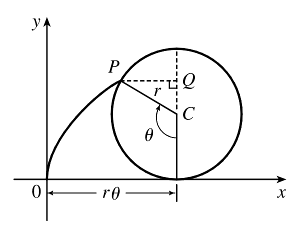
(c) $x = e^{-3t} = (e^{-t})^3 \quad [\text{so } e^{-t} = x^{1/3}]$,

$y = e^{-2t} = (e^{-t})^2 = (x^{1/3})^2 = x^{2/3}$.

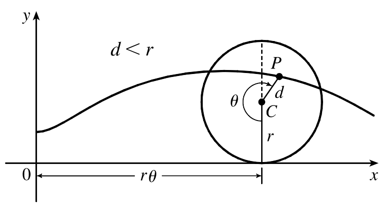
If $t < 0$, then x and y are both larger than 1. If $t > 0$, then x and y are between 0 and 1. Since $x > 0$ and $y > 0$, the curve never quite reaches the origin.

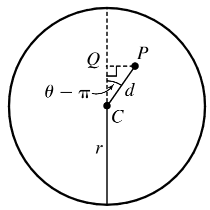


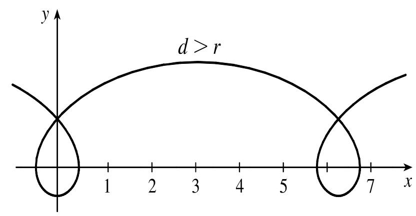
48. The case $\frac{\pi}{2} < \theta < \pi$ is illustrated. C has coordinates $(r\theta, r)$ as in Example 7, and Q has coordinates $(r\theta, r + r\cos(\pi - \theta)) = (r\theta, r(1 - \cos\theta))$ [since $\cos(\pi - \alpha) = \cos\pi\cos\alpha + \sin\pi\sin\alpha = -\cos\alpha$], so P has coordinates $(r\theta - r\sin(\pi - \theta), r(1 - \cos\theta)) = (r(\theta - \sin\theta), r(1 - \cos\,\theta))$ [since $\sin(\pi - \alpha) = \sin\pi\cos\alpha - \cos\pi\sin\alpha = \sin\alpha$]. Again we have the parametric equations $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.



49. The first two diagrams depict the case $\pi < \theta < \frac{3\pi}{2}$, $d < r$. As in Example 7, C has coordinates $(r\theta, r)$. Now Q (in the second diagram) has coordinates $(r\theta, r + d\cos(\theta - \pi)) = (r\theta, r - d\cos\theta)$, so a typical point P of the trochoid has coordinates $(r\theta + d\sin(\theta - \pi), r - d\cos\theta)$. That is, P has coordinates (x, y), where $x = r\theta - d\sin\theta$ and $y = r - d\cos\theta$. When $d = r$, these equations agree with those of the cycloid.

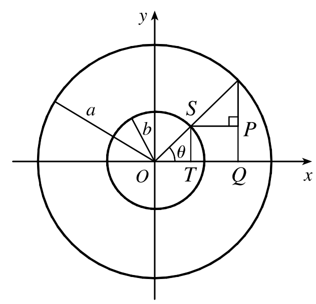






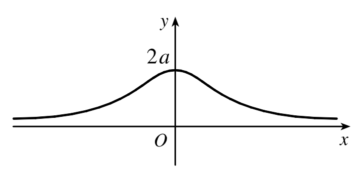
50. In polar coordinates, an equation for the circle is $r = 2a\sin\theta$. Thus, the coordinates of Q are $x = r\cos\theta = 2a\sin\theta\,\cos\theta$ and $y = r\sin\theta = 2a\sin^2\theta$. The coordinates of R are $x = 2a\cot\theta$ and $y = 2a$. Since P is the midpoint of QR, we use the midpoint formula to get $x = a(\sin\theta\,\cos\theta + \cot\theta)$ and $y = a(1 + \sin^2\theta)$.

51. It is apparent that $x = |OQ|$ and $y = |QP| = |ST|$. From the diagram, $x = |OQ| = a\cos\theta$ and $y = |ST| = b\sin\theta$. Thus, the parametric equations are $x = a\cos\theta$ and $y = b\sin\theta$. To eliminate θ we rearrange: $\sin\theta = y/b \quad\Rightarrow$ $\sin^2\theta = (y/b)^2$ and $\cos\theta = x/a \quad\Rightarrow\quad \cos^2\theta = (x/a)^2$. Adding the two equations: $\sin^2\theta + \cos^2\theta = 1 = x^2/a^2 + y^2/b^2$. Thus, we have an ellipse.



52. A has coordinates $(a\cos\theta, a\sin\theta)$. Since OA is perpendicular to AB, ΔOAB is a right triangle and B has coordinates $(a\sec\theta, 0)$. It follows that P has coordinates $(a\sec\theta, b\sin\,\theta)$. Thus, the parametric equations are $x = a\sec\theta$, $y = b\sin\theta$.

53. $C = (2a\cot\theta, 2a)$, so the x-coordinate of P is $x = 2a\cot\theta$. Let $B = (0, 2a)$. Then $\angle OAB$ is a right angle and $\angle OBA = \theta$, so $|OA| = 2a\sin\theta$ and $A = ((2a\sin\theta)\cos\theta, (2a\sin\theta)\sin\theta)$. Thus, the y-coordinate of P is $y = 2a\sin^2\theta$.



54. (a) Let θ be the angle of inclination of segment OP. Then $|OB| = \dfrac{2a}{\cos\theta}$.

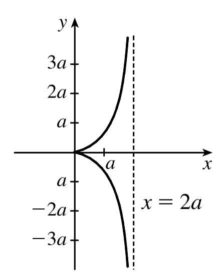
Let $C = (2a, 0)$. Then by use of right triangle OAC we see that $|OA| = 2a\cos\theta$.

Now

$$\begin{aligned} |OP| &= |AB| = |OB| - |OA| \\ &= 2a\left(\frac{1}{\cos\theta} - \cos\theta\right) = 2a\,\frac{1 - \cos^2\theta}{\cos\theta} = 2a\,\frac{\sin^2\theta}{\cos\theta} = 2a\sin\theta\,\tan\theta \end{aligned}$$

So P has coordinates $x = 2a\sin\theta\,\tan\theta\cdot\cos\theta = 2a\sin^2\theta$ and $y = 2a\sin\theta\,\tan\theta\cdot\sin\theta = 2a\sin^2\theta\,\tan\theta$.

(b)



55. (a) *Red particle*: $x = t + 5$, $y = t^2 + 4t + 6$

Blue particle: $x = 2t + 1$, $y = 2t + 6$

Substituting $x = 1$ and $y = 6$ into the parametric equations for the red particle gives $1 = t + 5$ and $6 = t^2 + 4t + 6$, which are both satisfied when $t = -4$. Making the same substitution for the blue particle gives $1 = 2t + 1$ and $6 = 2t + 6$, which are both satisfied when $t = 0$. Repeating the process for $x = 6$ and $y = 11$, the red particle's equations become $6 = t + 5$ and $11 = t^2 + 4t + 6$, which are both satisfied when $t = 1$. Similarly, the blue particle's equations become $6 = 2t + 1$ and $11 = 2t + 6$, which are both satisfied when $t = 2.5$. Thus, $(1, 6)$ and $(6, 11)$ are both intersection points, but they are not collision points, since the particles reach each of these points at different times.

(b) *Blue particle*: $x = 2t + 1 \quad \Rightarrow \quad t = \frac{1}{2}(x - 1)$.

Substituting into the equation for y gives $y = 2t + 6 = 2\left[\frac{1}{2}(x - 1)\right] + 6 = x + 5$.

Green particle: $x = 2t + 4 \quad \Rightarrow \quad t = \frac{1}{2}(x - 4)$.

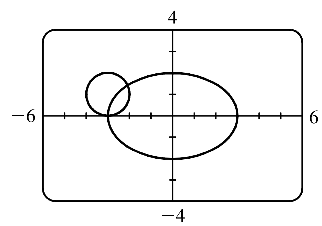
Substituting into the equation for y gives $y = 2t + 9 = 2\left[\frac{1}{2}(x - 4)\right] + 9 = x + 5$.

Thus, the green and blue particles both move along the line $y = x + 5$.

Now, the red and green particles will collide if there is a time t when both particles are at the same point. Equating the x parametric equations, we find $t + 5 = 2t + 4$, which is satisfied when $t = 1$, and gives $x = 1 + 5 = 6$. Substituting $t = 1$ into the red and green particles' y equations gives $y = (1)^2 + 4(1) + 6 = 11$ and $y = 2(1) + 9 = 11$, respectively. Thus, the red and green particles collide at the point $(6, 11)$ when $t = 1$.

56. (a) $x = 3\sin t,\ y = 2\cos t,\ 0 \le t \le 2\pi$;

$x = -3 + \cos t,\ y = 1 + \sin t,\ 0 \le t \le 2\pi$



There are 2 points of intersection:

$(-3, 0)$ and approximately $(-2.1, 1.4)$.

(b) A collision point occurs when $x_1 = x_2$ and $y_1 = y_2$ for the same t. So solve the equations:

$$3\sin t = -3 + \cos t \quad \textbf{(1)}$$
$$2\cos t = 1 + \sin t \quad \textbf{(2)}$$

From **(2)**, $\sin t = 2\cos t - 1$. Substituting into **(1)**, we get $3(2\cos t - 1) = -3 + \cos t \quad \Rightarrow \quad 5\cos t = 0 \quad (\star) \quad \Rightarrow$ $\cos t = 0 \quad \Rightarrow \quad t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. We check that $t = \frac{3\pi}{2}$ satisfies **(1)** and **(2)** but $t = \frac{\pi}{2}$ does not. So the only collision point occurs when $t = \frac{3\pi}{2}$, and this gives the point $(-3, 0)$. [We could check our work by graphing x_1 and x_2 together as functions of t and, on another plot, y_1 and y_2 as functions of t. If we do so, we see that the only value of t for which *both* pairs of graphs intersect is $t = \frac{3\pi}{2}$.]

(c) The circle is centered at $(3, 1)$ instead of $(-3, 1)$. There are still 2 intersection points: $(3, 0)$ and $(2.1, 1.4)$, but there are no collision points, since $(\star)$ in part (b) becomes $5\cos t = 6 \quad \Rightarrow \quad \cos t = \frac{6}{5} > 1$.

57. (a) $x = 1 - t^2$, $y = t - t^3$. The curve intersects itself if there are two distinct times $t = a$ and $t = b$ (with $a < b$) such that $x(a) = x(b)$ and $y(a) = y(b)$. The equation $x(a) = x(b)$ gives $1 - a^2 = 1 - b^2$ so that $a^2 = b^2$. Since $a \neq b$ by assumption, we must have $a = -b$. Substituting into the equation for y gives $y(-b) = y(b) \Rightarrow$ $-b - (-b)^3 = b - b^3 \Rightarrow 2b^3 - 2b = 0 \Rightarrow 2b(b-1)(b+1) = 0 \Rightarrow b = -1, 0, 1$. Since $a < b$, the only valid solution is $b = 1$, which corresponds to $a = -1$ and results in the coordinates $x = 0$ and $y = 0$. Thus, the curve intersects itself at $(0, 0)$ when $t = -1$ and $t = 1$.

(b) $x = 2t - t^3$, $y = t - t^2$. Similar to part (a), we try to find the times $t = a$ and $t = b$ with $a < b$ such that $x(a) = x(b)$ and $y(a) = y(b)$. The equation $y(a) = y(b)$ gives $a - a^2 = b - b^2 \Rightarrow 0 = a^2 - a + (b - b^2)$. Using the quadratic formula to solve for a, we get

$$a = \frac{1 \pm \sqrt{1 - 4(b - b^2)}}{2} = \frac{1 \pm \sqrt{4b^2 - 4b + 1}}{2} = \frac{1 \pm \sqrt{(2b-1)^2}}{2} = \frac{1 \pm (2b-1)}{2} \quad \Rightarrow \quad a = b \text{ or } a = 1 - b.$$

Since $a < b$ by assumption, we reject the first solution and substitute $a = 1 - b$ into $x(a) = x(b) \Rightarrow x(1-b) = x(b) \Rightarrow$ $2(1-b) - (1-b)^3 = 2b - b^3$. Expanding and simplifying gives $2b^3 - 3b^2 - b + 1 = 0$. By graphing the equation, we see that $b = \frac{1}{2}$ is a zero, so $2b - 1$ is a factor, and by long division $b^2 - b - 1$ is another factor. Hence, the solutions are $b = \frac{1}{2}$ and $b = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$ (found using the quadratic formula). Since $a = 1 - b$ and we require $a < b$, the only valid solution is $b = \frac{1}{2} + \frac{1}{2}\sqrt{5}$, which corresponds to $a = \frac{1}{2} - \frac{1}{2}\sqrt{5}$ and results in the coordinates $x = 2\left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right) - \left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^3 = -1$ and $y = \frac{1}{2} - \frac{1}{2}\sqrt{5} - \left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^2 = -1$. Thus, the curve intersects itself at $(-1, -1)$ when $t = \frac{1}{2} - \frac{1}{2}\sqrt{5}$ and $t = \frac{1}{2} + \frac{1}{2}\sqrt{5}$.

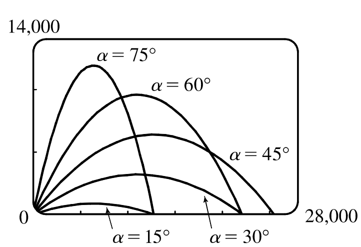
58. (a) If $\alpha = 30^\circ$ and $v_0 = 500$ m/s, then the equations become $x = (500\cos 30^\circ)t = 250\sqrt{3}t$ and $y = (500\sin 30^\circ)t - \frac{1}{2}(9.8)t^2 = 250t - 4.9t^2$. $y = 0$ when $t = 0$ (when the gun is fired) and again when $t = \frac{250}{4.9} \approx 51$ s. Then $x = \left(250\sqrt{3}\right)\left(\frac{250}{4.9}\right) \approx 22{,}092$ m, so the bullet hits the ground about 22 km from the gun.

The formula for y is quadratic in t. To find the maximum y-value, we will complete the square:

$$y = -4.9\left(t^2 - \tfrac{250}{4.9}t\right) = -4.9\left[t^2 - \tfrac{250}{4.9}t + \left(\tfrac{125}{4.9}\right)^2\right] + \tfrac{125^2}{4.9} = -4.9\left(t - \tfrac{125}{4.9}\right)^2 + \tfrac{125^2}{4.9} \le \tfrac{125^2}{4.9}$$

with equality when $t = \frac{125}{4.9}$ s, so the maximum height attained is $\frac{125^2}{4.9} \approx 3189$ m.

(b)



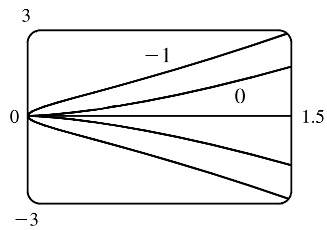
As α $(0^\circ < \alpha < 90^\circ)$ increases up to 45°, the projectile attains a greater height and a greater range. As α increases past 45°, the projectile attains a greater height, but its range decreases.

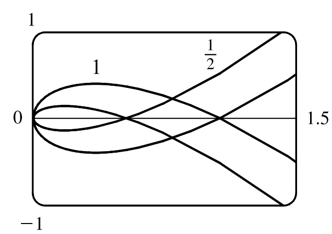
(c) $x = (v_0 \cos\alpha)t \;\Rightarrow\; t = \dfrac{x}{v_0\cos\alpha}$.

$$y = (v_0 \sin\alpha)t - \tfrac{1}{2}gt^2 \;\Rightarrow\; y = (v_0\sin\alpha)\,\frac{x}{v_0\cos\alpha} - \frac{g}{2}\left(\frac{x}{v_0\cos\alpha}\right)^2 = (\tan\alpha)x - \left(\frac{g}{2v_0^2\cos^2\alpha}\right)x^2,$$

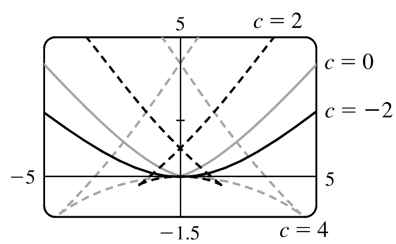
which is the equation of a parabola (quadratic in x).

59. $x = t^2, y = t^3 - ct$. We use a graphing device to produce the graphs for various values of c with $-\pi \le t \le \pi$. Note that all the members of the family are symmetric about the x-axis. For $c < 0$, the graph does not cross itself, but for $c = 0$ it has a cusp at $(0, 0)$ and for $c > 0$ the graph crosses itself at $x = c$, so the loop grows larger as c increases.

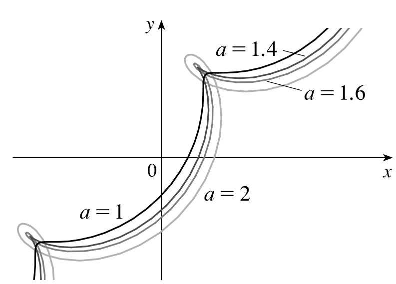




60. $x = 2ct - 4t^3, y = -ct^2 + 3t^4$. We use a graphing device to produce the graphs for various values of c with $-\pi \le t \le \pi$. Note that all the members of the family are symmetric about the y-axis. When $c < 0$, the graph resembles that of a polynomial of even degree, but when $c = 0$ there is a corner at the origin, and when $c > 0$, the graph crosses itself at the origin, and has two cusps below the x-axis. The size of the "swallowtail" increases as c increases.

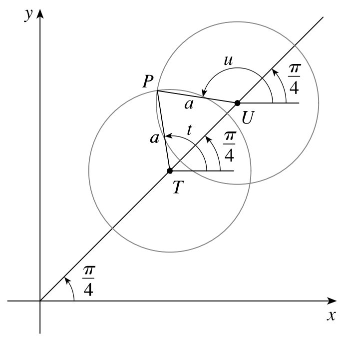


61. $x = t + a\cos t, y = t + a\sin t, a > 0$. From the first figure, we see that curves roughly follow the line $y = x$, and they start having loops when a is between 1.4 and 1.6. The loops increase in size as a increases.

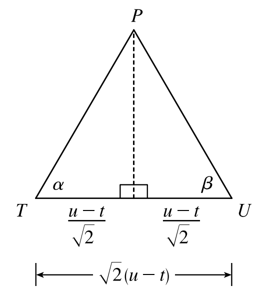


While not required, the following is a solution to determine the *exact* values for which the curve has a loop, that is, we seek the values of a for which there exist parameter values t and u such that $t < u$ and $(t + a\cos t, t + a\sin t) = (u + a\cos u, u + a\sin u)$.

[continued]



In the diagram at the left, T denotes the point (t,t), U the point (u,u), and P the point $(t+a\cos t, t+a\sin t) = (u+a\cos u, u+a\sin u)$. Since $\overline{PT} = \overline{PU} = a$, the triangle PTU is isosceles. Therefore its base angles, $\alpha = \angle PTU$ and $\beta = \angle PUT$ are equal. Since $\alpha = t - \frac{\pi}{4}$ and $\beta = 2\pi - \frac{3\pi}{4} - u = \frac{5\pi}{4} - u$, the relation $\alpha = \beta$ implies that $u + t = \frac{3\pi}{2}$ **(1)**.



Since $\overline{TU} = \text{distance}((t,t),(u,u)) = \sqrt{2(u-t)^2} = \sqrt{2}\,(u-t)$, we see that

$\cos\alpha = \dfrac{\frac{1}{2}\overline{TU}}{\overline{PT}} = \dfrac{(u-t)/\sqrt{2}}{a}$, so $u - t = \sqrt{2}\,a\cos\alpha$, that is,

$u - t = \sqrt{2}\,a\cos\left(t - \frac{\pi}{4}\right)$ **(2)**. Now $\cos\left(t - \frac{\pi}{4}\right) = \sin\left[\frac{\pi}{2} - \left(t - \frac{\pi}{4}\right)\right] = \sin\left(\frac{3\pi}{4} - t\right)$, so we can rewrite **(2)** as $u - t = \sqrt{2}\,a\sin\left(\frac{3\pi}{4} - t\right)$ **(2′)**. Subtracting **(2′)** from **(1)** and dividing by 2, we obtain $t = \frac{3\pi}{4} - \frac{\sqrt{2}}{2}a\sin\left(\frac{3\pi}{4} - t\right)$, or $\frac{3\pi}{4} - t = \frac{a}{\sqrt{2}}\sin\left(\frac{3\pi}{4} - t\right)$ **(3)**.

Since $a > 0$ and $t < u$, it follows from **(2′)** that $\sin\left(\frac{3\pi}{4} - t\right) > 0$. Thus from **(3)** we see that $t < \frac{3\pi}{4}$. [We have implicitly assumed that $0 < t < \pi$ by the way we drew our diagram, but we lost no generality by doing so since replacing t by $t + 2\pi$ merely increases x and y by 2π. The curve's basic shape repeats every time we change t by 2π.] Solving for a in **(3)**, we get $a = \dfrac{\sqrt{2}\left(\frac{3\pi}{4} - t\right)}{\sin\left(\frac{3\pi}{4} - t\right)}$. Write $z = \frac{3\pi}{4} - t$. Then $a = \dfrac{\sqrt{2}\,z}{\sin z}$, where $z > 0$. Now $\sin z < z$ for $z > 0$, so $a > \sqrt{2}$.

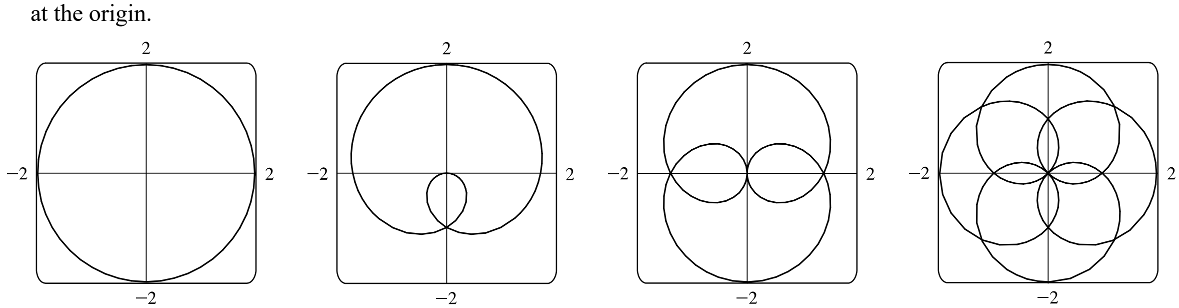
$\left[\text{As } z \to 0^+, \text{ that is, as } t \to \left(\frac{3\pi}{4}\right)^-, a \to \sqrt{2}\right]$.

62. Consider the curves $x = \sin t + \sin nt$, $y = \cos t + \cos nt$, where n is a positive integer. For $n = 1$, we get a circle of radius 2 centered at the origin. For $n > 1$, we get a curve lying on or inside that circle that traces out $n - 1$ loops as t ranges from 0 to 2π.

Note:

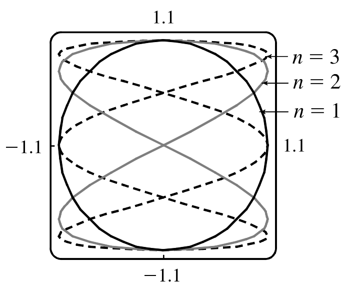
$$\begin{aligned} x^2 + y^2 &= (\sin t + \sin nt)^2 + (\cos t + \cos nt)^2 \\ &= \sin^2 t + 2\sin t\,\sin nt + \sin^2 nt + \cos^2 t + 2\cos t\,\cos nt + \cos^2 nt \\ &= (\sin^2 t + \cos^2 t) + (\sin^2 nt + \cos^2 nt) + 2(\cos t\,\cos nt + \sin t\,\sin nt) \\ &= 1 + 1 + 2\cos(t - nt) = 2 + 2\cos((1-n)t) \le 4 = 2^2, \end{aligned}$$

with equality for $n = 1$. This shows that each curve lies on or inside the curve for $n = 1$, which is a circle of radius 2 centered at the origin.

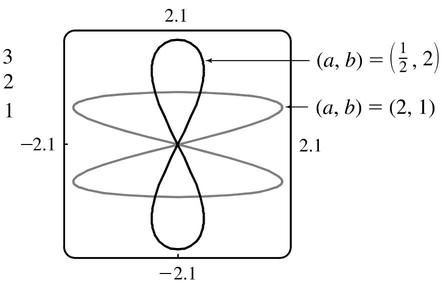


$n = 1$ $n = 2$ $n = 3$ $n = 5$

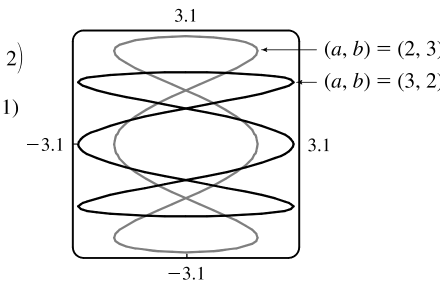
63. Note that all the Lissajous figures are symmetric about the x-axis. The parameters a and b simply stretch the graph in the x- and y-directions respectively. For $a = b = n = 1$ the graph is simply a circle with radius 1. For $n = 2$ the graph crosses itself at the origin and there are loops above and below the x-axis. In general, the figures have $n - 1$ points of intersection, all of which are on the y-axis, and a total of n closed loops.



$a = b = 1$

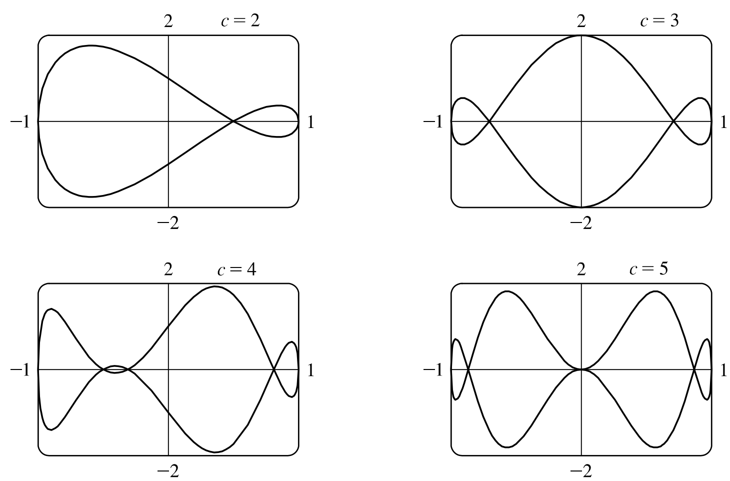


$n = 2$

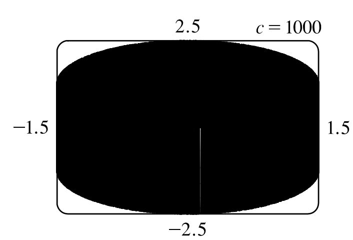


$n = 3$

64. $x = \cos t,\ y = \sin t - \sin ct$. If $c = 1$, then $y = 0$, and the curve is simply the line segment from $(-1, 0)$ to $(1, 0)$. The graphs are shown for $c = 2, 3, 4$ and 5.

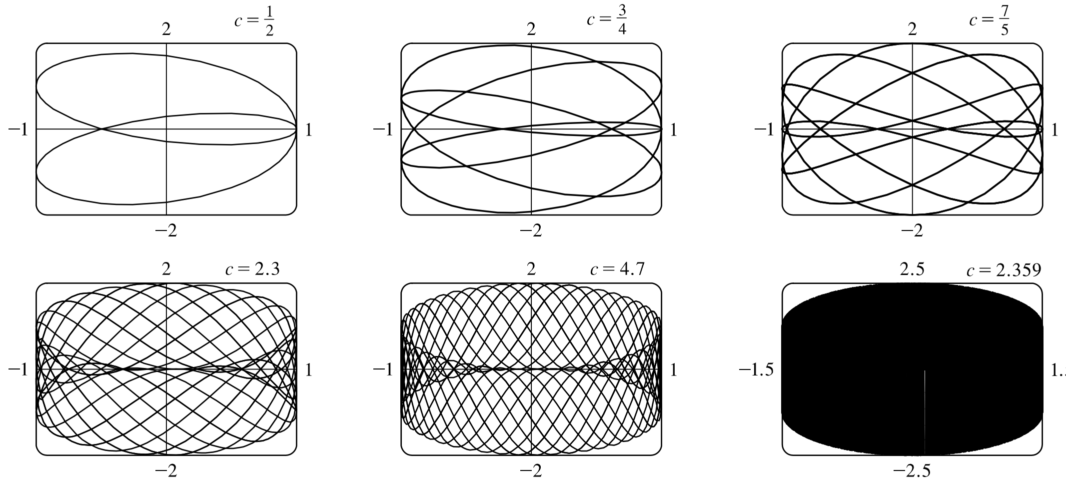


It is easy to see that all the curves lie in the rectangle $[-1, 1]$ by $[-2, 2]$. When c is an integer, $x(t + 2\pi) = x(t)$ and $y(t + 2\pi) = y(t)$, so the curve is closed. When c is a positive integer greater than 1, the curve intersects the x-axis $c + 1$ times and has c loops (one of which degenerates to a tangency at the origin when c is an odd integer of the form $4k + 1$).
As c increases, the curve's loops become thinner, but stay in the region bounded by the semicircles $y = \pm\left(1 + \sqrt{1 - x^2}\,\right)$ and the line segments from $(-1, -1)$ to $(-1, 1)$ and from $(1, -1)$ to $(1, 1)$. This is true because
$|y| = |\sin t - \sin ct| \le |\sin t| + |\sin ct| \le \sqrt{1 - x^2} + 1$. This curve appears to fill the entire region when c is very large, as shown in the figure for $c = 1000$.

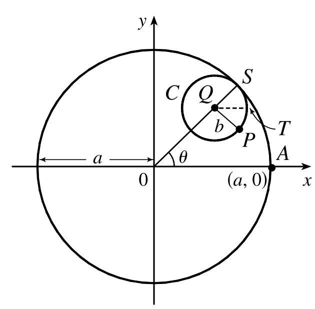


[continued]

When c is a fraction, we get a variety of shapes with multiple loops, but always within the same region. For some fractional values, such as $c = 2.359$, the curve again appears to fill the region.



DISCOVERY PROJECT Running Circles Around Circles

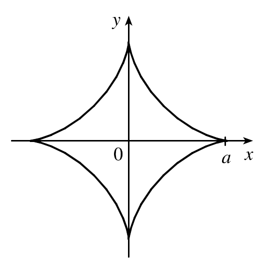


1. The center Q of the smaller circle has coordinates $((a-b)\cos\theta, (a-b)\sin\theta)$. Arc PS on circle C has length $a\theta$ since it is equal in length to arc AS (the smaller circle rolls without slipping against the larger.)

Thus, $\angle PQS = \dfrac{a}{b}\theta$ and $\angle PQT = \dfrac{a}{b}\theta - \theta$, so P has coordinates

$$x = (a-b)\cos\theta + b\cos(\angle PQT) = (a-b)\cos\theta + b\cos\left(\frac{a-b}{b}\theta\right)$$

and $$y = (a-b)\sin\theta - b\sin(\angle PQT) = (a-b)\sin\theta - b\sin\left(\frac{a-b}{b}\theta\right).$$

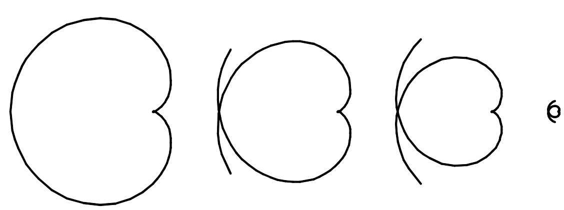


2. With $b = 1$ and a a positive integer greater than 2, we obtain a hypocycloid of a cusps. Shown in the figure is the graph for $a = 4$. Let $a = 4$ and $b = 1$. Using the sum identities to expand $\cos 3\theta$ and $\sin 3\theta$, we obtain

$$x = 3\cos\theta + \cos 3\theta = 3\cos\theta + \left(4\cos^3\theta - 3\cos\theta\right) = 4\cos^3\theta$$

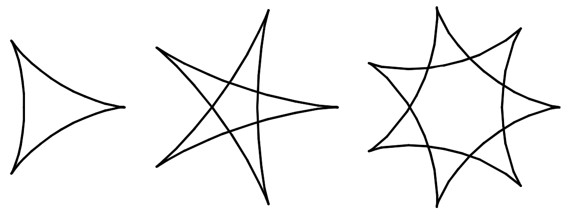
and $$y = 3\sin\theta - \sin 3\theta = 3\sin\theta - \left(3\sin\theta - 4\sin^3\theta\right) = 4\sin^3\theta.$$

3. The graphs at the right are obtained with $b = 1$ and $a = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, and $\frac{1}{10}$ with $-2\pi \le \theta \le 2\pi$. We conclude that as the denominator d increases, the graph gets smaller, but maintains the basic shape shown.

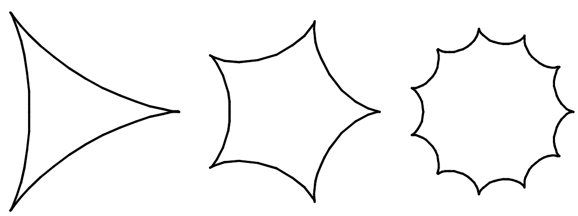


[continued]

Letting $d = 2$ and $n = 3$, 5, and 7 with $-2\pi \le \theta \le 2\pi$ gives us the following:



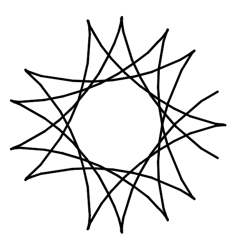
So if d is held constant and n varies, we get a graph with n cusps (assuming n/d is in lowest form). When $n = d + 1$, we obtain a hypocycloid of n cusps. As n increases, we must expand the range of θ in order to get a closed curve. The following graphs have $a = \frac{3}{2}$, $\frac{5}{4}$, and $\frac{11}{10}$.



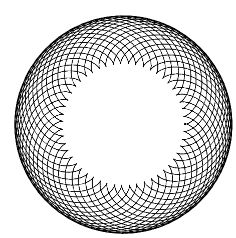
4. If $b = 1$, the equations for the hypocycloid are

$$x = (a-1)\cos\theta + \cos\left((a-1)\theta\right) \qquad y = (a-1)\sin\theta - \sin\left((a-1)\theta\right)$$

which is a hypocycloid of a cusps (from Problem 2). In general, if $a > 1$, we get a figure with cusps on the "outside ring" and if $a < 1$, the cusps are on the "inside ring". In any case, as the values of θ get larger, we get a figure that looks more and more like a washer. If we were to graph the hypocycloid for all values of θ, every point on the washer would eventually be arbitrarily close to a point on the curve.

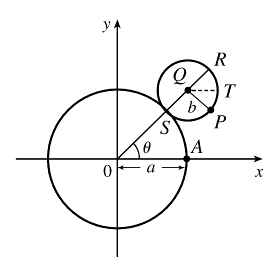


$a = \sqrt{2}, \quad -10\pi \le \theta \le 10\pi$



$a = e - 2, \quad 0 \le \theta \le 446$

5. The center Q of the smaller circle has coordinates $((a+b)\cos\theta, (a+b)\sin\theta)$.



Arc PS has length $a\theta$ (as in Problem 1), so that $\angle PQS = \dfrac{a\theta}{b}$, $\angle PQR = \pi - \dfrac{a\theta}{b}$,

and $\angle PQT = \pi - \dfrac{a\theta}{b} - \theta = \pi - \left(\dfrac{a+b}{b}\right)\theta$ since $\angle RQT = \theta$.

Thus, the coordinates of P are

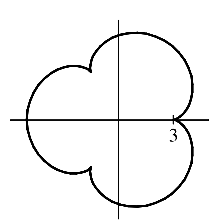
$$x = (a+b)\cos\theta + b\cos\left(\pi - \frac{a+b}{b}\theta\right) = (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\theta\right)$$

and $$y = (a+b)\sin\theta - b\sin\left(\pi - \frac{a+b}{b}\theta\right) = (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\theta\right).$$

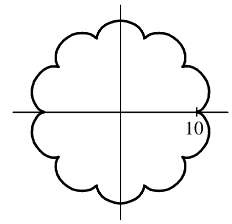
6. Let $b = 1$ and the equations become

$$x = (a+1)\cos\theta - \cos((a+1)\theta) \qquad\qquad y = (a+1)\sin\theta - \sin((a+1)\theta)$$

If $a = 1$, we have a cardioid. If a is a positive integer greater than 1, we get the graph of an "a-leafed clover", with cusps that are a units from the origin. (Some of the pairs of figures are not to scale.)

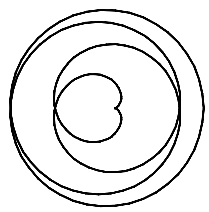


$a = 3$, $-2\pi \le \theta \le 2\pi$

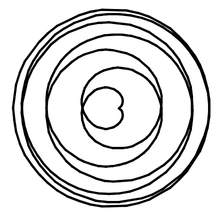


$a = 10$, $-2\pi \le \theta \le 2\pi$

If $a = n/d$ with $n = 1$, we obtain a figure that does not increase in size and requires $-d\pi \le \theta \le d\pi$ to be a closed curve traced exactly once.

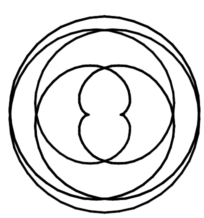


$a = \frac{1}{4}$, $-4\pi \le \theta \le 4\pi$

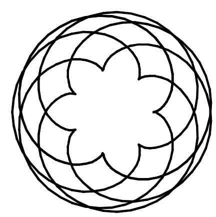


$a = \frac{1}{7}$, $-7\pi \le \theta \le 7\pi$

Next, we keep d constant and let n vary. As n increases, so does the size of the figure. There is an n-pointed star in the middle.

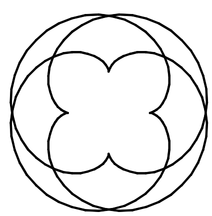


$a = \frac{2}{5}$, $-5\pi \le \theta \le 5\pi$

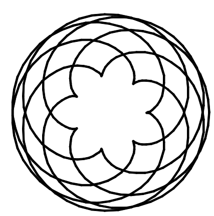


$a = \frac{7}{5}$, $-5\pi \le \theta \le 5\pi$

Now if $n = d + 1$ we obtain figures similar to the previous ones, but the size of the figure does not increase.

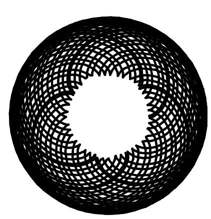


$a = \frac{4}{3}$, $-3\pi \le \theta \le 3\pi$

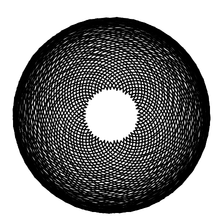


$a = \frac{7}{6}$, $-6\pi \le \theta \le 6\pi$

If a is irrational, we get washers that increase in size as a increases.



$a = \sqrt{2}$, $0 \le \theta \le 200$



$a = e - 2$, $0 \le \theta \le 446$

10.2 Calculus with Parametric Curves

1. $x = 2t^3 + 3t,\ y = 4t - 5t^2 \quad \Rightarrow \quad \dfrac{dx}{dt} = 6t^2 + 3,\ \dfrac{dy}{dt} = 4 - 10t$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4 - 10t}{6t^2 + 3}$.

2. $x = t - \ln t,\ y = t^2 - t^{-2} \quad \Rightarrow \quad \dfrac{dx}{dt} = 1 - t^{-1},\ \dfrac{dy}{dt} = 2t + 2t^{-3}$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t + 2t^{-3}}{1 - t^{-1}} \cdot \dfrac{t^3}{t^3} = \dfrac{2t^4 + 2}{t^3 - t^2}$.

3. $x = te^t,\ y = t + \sin t \quad \Rightarrow \quad \dfrac{dx}{dt} = te^t + e^t = e^t(t+1),\ \dfrac{dy}{dt} = 1 + \cos t$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1 + \cos t}{e^t(t+1)}$.

4. $x = t + \sin(t^2 + 2),\ y = \tan(t^2 + 2) \quad \Rightarrow \quad \dfrac{dx}{dt} = 1 + 2t\cos(t^2 + 2),\ \dfrac{dy}{dt} = 2t\sec^2(t^2 + 2)$, and

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t\sec^2(t^2 + 2)}{1 + 2t\cos(t^2 + 2)}$.

5. $x = t^2 + 2t,\ y = 2^t - 2t;\ (15, 2)$. $\quad \dfrac{dy}{dt} = 2^t \ln 2 - 2,\ \dfrac{dx}{dt} = 2t + 2$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2^t \ln 2 - 2}{2t + 2}$.

At $(15, 2)$, $x = t^2 + 2t = 15 \quad \Rightarrow \quad t^2 + 2t - 15 = 0 \quad \Rightarrow \quad (t+5)(t-3) = 0 \quad \Rightarrow \quad t = -5$ or $t = 3$. Only $t = 3$ gives $y = 2$. With $t = 3$, $\dfrac{dy}{dx} = \dfrac{2^3 \ln 2 - 2}{2(3) + 2} = \dfrac{4\ln 2 - 1}{4} = \ln 2 - \dfrac{1}{4} \approx 0.44$.

6. $x = t + \cos \pi t,\ y = -t + \sin \pi t;\ (3, -2)$. $\quad \dfrac{dy}{dt} = -1 + \pi \cos \pi t,\ \dfrac{dx}{dt} = 1 - \pi \sin \pi t$, and

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-1 + \pi\cos \pi t}{1 - \pi \sin \pi t}$. When $x = 3$, we have $t + \cos \pi t = 3 \quad \Rightarrow \quad \cos^2 \pi t = (3 - t)^2$ **(1)**. When $y = -2$, we have $-t + \sin \pi t = -2 \quad \Rightarrow \quad \sin^2 \pi t = (t - 2)^2$ **(2)**. Adding **(1)** and **(2)** gives $\sin^2 \pi t + \cos^2 \pi t = (t-2)^2 + (3-t)^2 \quad \Rightarrow$ $1 = t^2 - 4t + 4 + 9 - 6t + t^2 \quad \Rightarrow \quad 0 = 2t^2 - 10t + 12 \quad \Rightarrow \quad 0 = 2(t-2)(t-3) \quad \Rightarrow \quad t = 2$ or $t = 3$. Only $t = 2$ gives $y = -2$. With $t = 2$, $\dfrac{dy}{dx} = \dfrac{-1 + \pi \cos 2\pi}{1 - \pi \sin 2\pi} = \dfrac{-1 + \pi}{1 - 0} = \pi - 1 \approx 2.14$.

7. $x = t^3 + 1,\ y = t^4 + t;\quad t = -1.\ \dfrac{dy}{dt} = 4t^3 + 1,\ \dfrac{dx}{dt} = 3t^2$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4t^3 + 1}{3t^2}$. When $t = -1$, $(x, y) = (0, 0)$ and $dy/dx = -3/3 = -1$, so an equation of the tangent to the curve at the point corresponding to $t = -1$ is $y - 0 = -1(x - 0)$, or $y = -x$.

8. $x = \sqrt{t},\ y = t^2 - 2t;\quad t = 4.\ \dfrac{dy}{dt} = 2t - 2,\ \dfrac{dx}{dt} = \dfrac{1}{2\sqrt{t}}$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = (2t - 2)2\sqrt{t} = 4(t-1)\sqrt{t}$. When $t = 4$, $(x, y) = (2, 8)$ and $dy/dx = 4(3)(2) = 24$, so an equation of the tangent to the curve at the point corresponding to $t = 4$ is $y - 8 = 24(x - 2)$, or $y = 24x - 40$.

9. $x = \sin 2t + \cos t,\ y = \cos 2t - \sin t;\ t = \pi.\quad \dfrac{dy}{dt} = -2\sin 2t - \cos t,\ \dfrac{dx}{dt} = 2\cos 2t - \sin t$, and

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-2\sin 2t - \cos t}{2\cos 2t - \sin t}$. When $t = \pi$, $(x, y) = (-1, 1)$, and $\dfrac{dy}{dx} = \dfrac{1}{2}$, so an equation of the tangent to the curve at the point corresponding to $t = \pi$ is $y - 1 = \frac{1}{2}[x - (-1)]$, or $y = \frac{1}{2}x + \frac{3}{2}$.

10. $x = e^t \sin \pi t,\ y = e^{2t};\quad t = 0.\ \dfrac{dy}{dt} = 2e^{2t},\ \dfrac{dx}{dt} = e^t(\pi \cos \pi t) + (\sin \pi t)e^t = e^t(\pi \cos \pi t + \sin \pi t)$, and

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2e^{2t}}{e^t(\pi \cos \pi t + \sin \pi t)} = \dfrac{2e^t}{\pi \cos \pi t + \sin \pi t}$. When $t = 0$, $(x, y) = (0, 1)$ and $\dfrac{dy}{dx} = \dfrac{2}{\pi}$, so an equation of the tangent to the curve at the point corresponding to $t = 0$ is $y - 1 = \frac{2}{\pi}(x - 0)$, or $y = \frac{2}{\pi}x + 1$.

11. (a) $x = \sin t,\ y = \cos^2 t;\ \left(\frac{1}{2}, \frac{3}{4}\right).\quad \dfrac{dy}{dt} = 2\cos t(-\sin t),\ \dfrac{dx}{dt} = \cos t$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-2\sin t \cos t}{\cos t} = -2\sin t$.

At $\left(\frac{1}{2}, \frac{3}{4}\right)$, $x = \sin t = \frac{1}{2} \ \Rightarrow\ t = \frac{\pi}{6}$, so $dy/dx = -2\sin\frac{\pi}{6} = -2\left(\frac{1}{2}\right) = -1$, and an equation of the tangent is $y - \frac{3}{4} = -1\left(x - \frac{1}{2}\right)$, or $y = -x + \frac{5}{4}$.

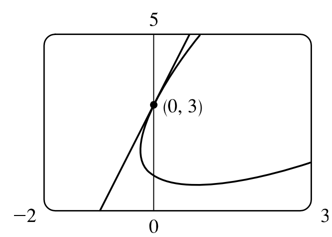
(b) $x = \sin t \ \Rightarrow\ x^2 = \sin^2 t = 1 - \cos^2 t = 1 - y$, so $y = 1 - x^2$, and $y' = -2x$. At $\left(\frac{1}{2}, \frac{3}{4}\right)$, $y' = -2 \cdot \frac{1}{2} = -1$, so an equation of the tangent is $y - \frac{3}{4} = -1\left(x - \frac{1}{2}\right)$, or $y = -x + \frac{5}{4}$.

12. (a) $x = \sqrt{t+4},\ y = 1/(t+4);\ \left(2, \frac{1}{4}\right).\quad \dfrac{dy}{dt} = -\dfrac{1}{(t+4)^2},\ \dfrac{dx}{dt} = \dfrac{1}{2\sqrt{t+4}}$, and

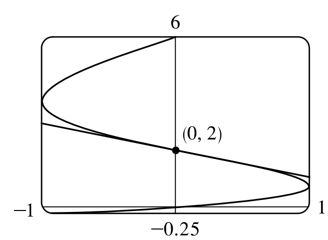
$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-1/(t+4)^2}{1/\left(2\sqrt{t+4}\right)} = -2(t+4)^{-3/2}$. At $\left(2, \frac{1}{4}\right)$, $x = \sqrt{t+4} = 2 \ \Rightarrow\ t + 4 = 4 \ \Rightarrow\ t = 0$ and $dy/dx = -2(4)^{-3/2} = -\frac{1}{4}$, so an equation of the tangent is $y - \frac{1}{4} = -\frac{1}{4}(x - 2)$, or $y = -\frac{1}{4}x + \frac{3}{4}$.

(b) $x = \sqrt{t+4} \ \Rightarrow\ x^2 = t + 4 \ \Rightarrow\ t = x^2 - 4$, so $y = \dfrac{1}{t+4} = \dfrac{1}{x^2 - 4 + 4} = \dfrac{1}{x^2}$, and $y' = -\dfrac{2}{x^3}$. At $\left(2, \frac{1}{4}\right)$, $y' = -2/2^3 = -1/4$, so an equation of the tangent is $y - \frac{1}{4} = -\frac{1}{4}(x - 2)$, or $y = -\frac{1}{4}x + \frac{3}{4}$.

13. $x = t^2 - t,\ y = t^2 + t + 1;\ (0, 3).\quad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t+1}{2t-1}$. To find the value of t corresponding to the point $(0, 3)$, solve $x = 0 \ \Rightarrow\ t^2 - t = 0 \ \Rightarrow\ t(t-1) = 0 \ \Rightarrow\ t = 0$ or $t = 1$. Only $t = 1$ gives $y = 3$. With $t = 1$, $dy/dx = 3$, and an equation of the tangent is $y - 3 = 3(x - 0)$, or $y = 3x + 3$.



14. $x = \sin \pi t,\ y = t^2 + t;\ (0, 2).\quad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t+1}{\pi \cos \pi t}$. To find the value of t corresponding to the point $(0, 2)$, solve $y = 2 \ \Rightarrow\ t^2 + t - 2 = 0 \ \Rightarrow\ (t+2)(t-1) = 0 \ \Rightarrow\ t = -2$ or $t = 1$. Either value gives $dy/dx = -3/\pi$, so an equation of the tangent is $y - 2 = -\frac{3}{\pi}(x - 0)$, or $y = -\frac{3}{\pi}x + 2$.



15. $x = t^2 + 1,\ y = t^2 + t \ \Rightarrow\ \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t+1}{2t} = 1 + \dfrac{1}{2t} \ \Rightarrow\ \dfrac{d^2y}{dx^2} = \dfrac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{dx/dt} = \dfrac{-1/(2t^2)}{2t} = -\dfrac{1}{4t^3}$.

The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $t < 0$.

16. $x = t^3 + 1,\ y = t^2 - t \ \Rightarrow\ \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t-1}{3t^2} = \dfrac{2}{3t} - \dfrac{1}{3t^2} \ \Rightarrow$

$\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \dfrac{-\frac{2}{3t^2} + \frac{2}{3t^3}}{3t^2} = \dfrac{\frac{2-2t}{3t^3}}{3t^2} = \dfrac{2(1-t)}{9t^5}$. The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $0 < t < 1$.

17. $x = e^t,\ y = te^{-t} \ \Rightarrow\ \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-te^{-t} + e^{-t}}{e^t} = \dfrac{e^{-t}(1-t)}{e^t} = e^{-2t}(1-t) \ \Rightarrow$

$\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \dfrac{e^{-2t}(-1) + (1-t)(-2e^{-2t})}{e^t} = \dfrac{e^{-2t}(-1-2+2t)}{e^t} = e^{-3t}(2t-3)$. The curve is CU when

$\dfrac{d^2y}{dx^2} > 0$, that is, when $t > \frac{3}{2}$.

18. $x = t^2 + 1,\ y = e^t - 1 \ \Rightarrow\ \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{e^t}{2t} \ \Rightarrow\ \dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \dfrac{\frac{2te^t - e^t \cdot 2}{(2t)^2}}{2t} = \dfrac{2e^t(t-1)}{(2t)^3} = \dfrac{e^t(t-1)}{4t^3}$.

The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $t < 0$ or $t > 1$.

19. $x = t - \ln t,\ y = t + \ln t$ [note that $t > 0$] $\Rightarrow\ \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1 + 1/t}{1 - 1/t} = \dfrac{t+1}{t-1} \ \Rightarrow$

$\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \dfrac{\frac{(t-1)(1) - (t+1)(1)}{(t-1)^2}}{(t-1)/t} = \dfrac{-2t}{(t-1)^3}$. The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $0 < t < 1$.

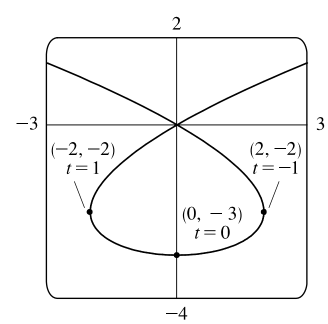
20. $x = \cos t,\ y = \sin 2t,\ 0 < t < \pi \ \Rightarrow\ \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2\cos 2t}{-\sin t} \ \Rightarrow$

$\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \dfrac{\frac{(-\sin t)(-4\sin 2t) - (2\cos 2t)(-\cos t)}{(-\sin t)^2}}{-\sin t} = \dfrac{(\sin t)(8\sin t\ \cos t) + [2(1 - 2\sin^2 t)](\cos t)}{(-\sin t)\sin^2 t}$

$= \dfrac{(\cos t)(8\sin^2 t + 2 - 4\sin^2 t)}{(-\sin t)\sin^2 t} = -\dfrac{\cos t}{\sin t} \cdot \dfrac{4\sin^2 t + 2}{\sin^2 t}$ $\quad$ [$(-\cot t) \cdot$ positive expression]

The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $-\cot t > 0 \ \Leftrightarrow\ \cot t < 0 \ \Leftrightarrow\ \frac{\pi}{2} < t < \pi$.

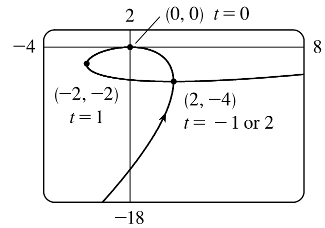
21. $x = t^3 - 3t,\ y = t^2 - 3.\ \ \dfrac{dy}{dt} = 2t$, so $\dfrac{dy}{dt} = 0 \ \Leftrightarrow\ t = 0 \ \Leftrightarrow$ $(x, y) = (0, -3)$. $\ \dfrac{dx}{dt} = 3t^2 - 3 = 3(t+1)(t-1)$, so $\dfrac{dx}{dt} = 0 \ \Leftrightarrow$ $t = -1$ or $1 \ \Leftrightarrow\ (x, y) = (2, -2)$ or $(-2, -2)$. The curve has a horizontal tangent at $(0, -3)$ and vertical tangents at $(2, -2)$ and $(-2, -2)$.



22. $x = t^3 - 3t,\ y = t^3 - 3t^2$. $\dfrac{dy}{dt} = 3t^2 - 6t = 3t(t-2)$, so $\dfrac{dy}{dt} = 0 \quad \Leftrightarrow$

$t = 0$ or $2 \quad \Leftrightarrow \quad (x,y) = (0,0)$ or $(2,-4)$. $\dfrac{dx}{dt} = 3t^2 - 3 = 3(t+1)(t-1)$,

so $\dfrac{dx}{dt} = 0 \quad \Leftrightarrow \quad t = -1$ or $1 \quad \Leftrightarrow \quad (x,y) = (2,-4)$ or $(-2,-2)$. The curve has horizontal tangents at $(0,0)$ and $(2,-4)$, and vertical tangents at $(2,-4)$ and $(-2,-2)$.



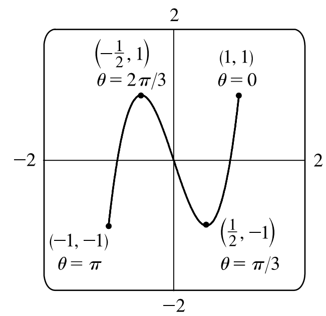
23. $x = \cos\theta,\ y = \cos 3\theta$. The whole curve is traced out for $0 \le \theta \le \pi$.

$\dfrac{dy}{d\theta} = -3\sin 3\theta$, so $\dfrac{dy}{d\theta} = 0 \quad \Leftrightarrow \quad \sin 3\theta = 0 \quad \Leftrightarrow \quad 3\theta = 0, \pi, 2\pi$, or $3\pi \quad \Leftrightarrow$

$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}$, or $\pi \quad \Leftrightarrow \quad (x,y) = (1,1), \left(\frac{1}{2},-1\right), \left(-\frac{1}{2},1\right)$, or $(-1,-1)$.

$\dfrac{dx}{d\theta} = -\sin\theta$, so $\dfrac{dx}{d\theta} = 0 \quad \Leftrightarrow \quad \sin\theta = 0 \quad \Leftrightarrow \quad \theta = 0$ or $\pi \quad \Leftrightarrow$

$(x,y) = (1,1)$ or $(-1,-1)$. Both $\dfrac{dy}{d\theta}$ and $\dfrac{dx}{d\theta}$ equal 0 when $\theta = 0$ and π.



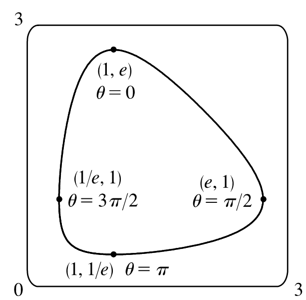
To find the slope when $\theta = 0$, we find $\displaystyle\lim_{\theta\to 0} \frac{dy}{dx} = \lim_{\theta\to 0} \frac{-3\sin 3\theta}{-\sin\theta} \overset{\text{H}}{=} \lim_{\theta\to 0} \frac{-9\cos 3\theta}{-\cos\theta} = 9$, which is the same slope when $\theta = \pi$. Thus, the curve has horizontal tangents at $\left(\frac{1}{2},-1\right)$ and $\left(-\frac{1}{2},1\right)$, and there are no vertical tangents.

24. $x = e^{\sin\theta}, y = e^{\cos\theta}$. The whole curve is traced out for $0 \le \theta < 2\pi$.

$\dfrac{dy}{d\theta} = -\sin\theta\, e^{\cos\theta}$, so $\dfrac{dy}{d\theta} = \theta \quad \Leftrightarrow \quad \sin\theta = 0 \quad \Leftrightarrow \quad \theta = 0$ or $\pi \quad \Leftrightarrow$

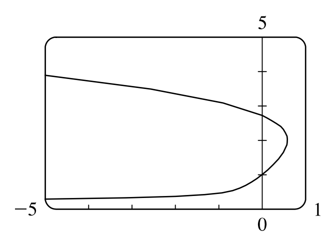
$(x,y) = (1,e)$ or $(1,1/e)$. $\dfrac{dx}{d\theta} = \cos\theta\, e^{\sin\theta}$, so $\dfrac{dx}{d\theta} = 0 \quad \Leftrightarrow \quad \cos\theta = 0 \quad \Leftrightarrow$

$\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2} \quad \Leftrightarrow \quad (x,y) = (e,1)$ or $(1/e,1)$. The curve has horizontal tangents at $(1,e)$ and $(1,1/e)$, and vertical tangents at $(e,1)$ and $(1/e,1)$.

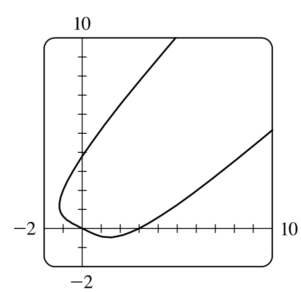


25. From the graph, it appears that the rightmost point on the curve $x = t - t^6, y = e^t$ is about $(0.6, 2)$. To find the exact coordinates, we find the value of t for which the graph has a vertical tangent, that is, $0 = dx/dt = 1 - 6t^5 \quad \Leftrightarrow \quad t = 1/\sqrt[5]{6}$. Hence, the rightmost point is

$$\left(1/\sqrt[5]{6} - 1/\left(6\sqrt[5]{6}\right), e^{1/\sqrt[5]{6}}\right) = \left(5\cdot 6^{-6/5}, e^{6^{-1/5}}\right) \approx (0.58, 2.01).$$



26. From the graph, it appears that the lowest point and the leftmost point on the curve $x = t^4 - 2t, y = t + t^4$ are $(1.5, -0.5)$ and $(-1.2, 1.2)$, respectively. To find the exact coordinates, we solve $dy/dt = 0$ (horizontal tangents) and $dx/dt = 0$ (vertical tangents).



[continued]

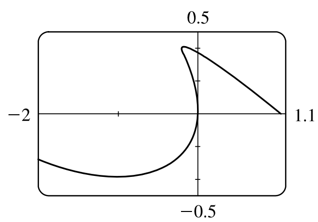
$\dfrac{dy}{dt} = 0 \quad\Leftrightarrow\quad 1 + 4t^3 = 0 \quad\Leftrightarrow\quad t = -\dfrac{1}{\sqrt[3]{4}}$, so the lowest point is

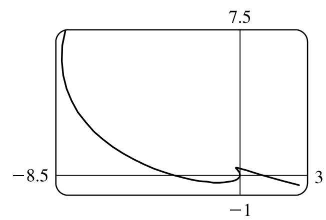
$$\left(\frac{1}{\sqrt[3]{256}} + \frac{2}{\sqrt[3]{4}}, -\frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{256}}\right) = \left(\frac{9}{\sqrt[3]{256}}, -\frac{3}{\sqrt[3]{256}}\right) \approx (1.42, -0.47)$$

$\dfrac{dx}{dt} = 0 \quad\Leftrightarrow\quad 4t^3 - 2 = 0 \quad\Leftrightarrow\quad t = \dfrac{1}{\sqrt[3]{2}}$, so the leftmost point is

$$\left(\frac{1}{\sqrt[3]{16}} - \frac{2}{\sqrt[3]{2}}, \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{16}}\right) = \left(-\frac{3}{\sqrt[3]{16}}, \frac{3}{\sqrt[3]{16}}\right) \approx (-1.19, 1.19)$$

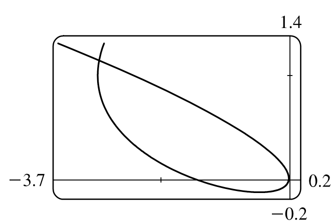
27. We graph the curve $x = t^4 - 2t^3 - 2t^2$, $y = t^3 - t$ in the viewing rectangle $[-2, 1.1]$ by $[-0.5, 0.5]$. This rectangle corresponds approximately to $t \in [-1, 0.8]$.

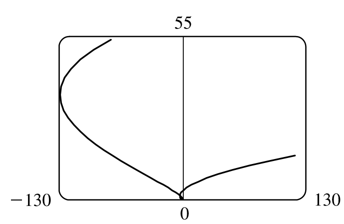




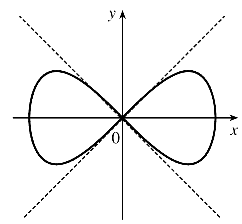
We estimate that the curve has horizontal tangents at about $(-1, -0.4)$ and $(-0.17, 0.39)$ and vertical tangents at about $(0, 0)$ and $(-0.19, 0.37)$. We calculate $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2 - 1}{4t^3 - 6t^2 - 4t}$. The horizontal tangents occur when $dy/dt = 3t^2 - 1 = 0 \quad\Leftrightarrow\quad t = \pm\frac{1}{\sqrt{3}}$, so both horizontal tangents are shown in our graph. The vertical tangents occur when $dx/dt = 2t(2t^2 - 3t - 2) = 0 \quad\Leftrightarrow\quad 2t(2t + 1)(t - 2) = 0 \quad\Leftrightarrow\quad t = 0, -\frac{1}{2}$ or 2. It seems that we have missed one vertical tangent, and indeed if we plot the curve on the t-interval $[-1.2, 2.2]$ we see that there is another vertical tangent at $(-8, 6)$.

28. We graph the curve $x = t^4 + 4t^3 - 8t^2$, $y = 2t^2 - t$ in the viewing rectangle $[-3.7, 0.2]$ by $[-0.2, 1.4]$. It appears that there is a horizontal tangent at about $(-0.4, -0.1)$, and vertical tangents at about $(-3, 1)$ and $(0, 0)$.



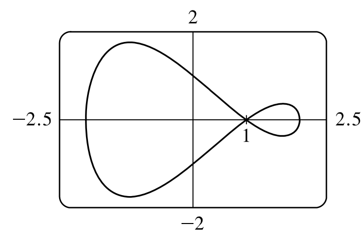


We calculate $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4t - 1}{4t^3 + 12t^2 - 16t}$, so there is a horizontal tangent where $dy/dt = 4t - 1 = 0 \quad\Leftrightarrow\quad t = \frac{1}{4}$. This point (the lowest point) is shown in the first graph. There are vertical tangents where $dx/dt = 4t^3 + 12t^2 - 16t = 0 \quad\Leftrightarrow\quad 4t(t^2 + 3t - 4) = 0 \quad\Leftrightarrow\quad 4t(t + 4)(t - 1) = 0$. We have missed one vertical tangent corresponding to $t = -4$, and if we plot the graph for $t \in [-5, 3]$, we see that the curve has another vertical tangent line at approximately $(-128, 36)$.



29. $x = \cos t,\ y = \sin t \cos t.\quad dx/dt = -\sin t,$

$dy/dt = -\sin^2 t + \cos^2 t = \cos 2t.$ $(x, y) = (0, 0) \quad\Leftrightarrow\quad \cos t = 0 \quad\Leftrightarrow\quad t$ is an odd multiple of $\frac{\pi}{2}$. When $t = \frac{\pi}{2}$, $dx/dt = -1$ and $dy/dt = -1$, so $dy/dx = 1$. When $t = \frac{3\pi}{2}$, $dx/dt = 1$ and $dy/dt = -1$. So $dy/dx = -1$. Thus, $y = x$ and $y = -x$ are both tangent to the curve at $(0, 0)$.



30. $x = -2\cos t,\ y = \sin t + \sin 2t$. From the graph, it appears that the curve crosses itself at the point $(1, 0)$. If this is true, then $x = 1 \quad\Leftrightarrow$ $-2\cos t = 1 \quad\Leftrightarrow\quad \cos t = -\frac{1}{2} \quad\Leftrightarrow\quad t = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$ for $0 \le t \le 2\pi$. Substituting either value of t into y gives $y = 0$, confirming that $(1, 0)$ is the point where the curve crosses itself. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\cos t + 2\cos 2t}{2\sin t}$.

When $t = \dfrac{2\pi}{3}$, $\dfrac{dy}{dx} = \dfrac{-1/2 + 2(-1/2)}{2(\sqrt{3}/2)} = \dfrac{-3/2}{\sqrt{3}} = -\dfrac{\sqrt{3}}{2}$, so an equation of the tangent line is $y - 0 = -\dfrac{\sqrt{3}}{2}(x - 1)$, or $y = -\dfrac{\sqrt{3}}{2}x + \dfrac{\sqrt{3}}{2}$. Similarly, when $t = \dfrac{4\pi}{3}$, an equation of the tangent line is $y = \dfrac{\sqrt{3}}{2}x - \dfrac{\sqrt{3}}{2}$.

31. $x = r\theta - d\sin\theta,\ y = r - d\cos\theta.$

(a) $\dfrac{dx}{d\theta} = r - d\cos\theta,\ \dfrac{dy}{d\theta} = d\sin\theta$, so $\dfrac{dy}{dx} = \dfrac{d\sin\theta}{r - d\cos\theta}$.

(b) If $0 < d < r$, then $|d\cos\theta| \le d < r$, so $r - d\cos\theta \ge r - d > 0$. This shows that $dx/d\theta$ never vanishes, so the trochoid can have no vertical tangent if $d < r$.

32. $x = a\cos^3\theta,\ y = a\sin^3\theta.$

(a) $\dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta,\ \dfrac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$, so $\dfrac{dy}{dx} = -\dfrac{\sin\theta}{\cos\theta} = -\tan\theta$.

(b) The tangent is horizontal $\quad\Leftrightarrow\quad dy/dx = 0 \quad\Leftrightarrow\quad \tan\theta = 0 \quad\Leftrightarrow\quad \theta = n\pi \quad\Leftrightarrow\quad (x, y) = (\pm a, 0)$.

The tangent is vertical $\quad\Leftrightarrow\quad \cos\theta = 0 \quad\Leftrightarrow\quad \theta$ is an odd multiple of $\frac{\pi}{2} \quad\Leftrightarrow\quad (x, y) = (0, \pm a)$.

(c) $dy/dx = \pm 1 \quad\Leftrightarrow\quad \tan\theta = \pm 1 \quad\Leftrightarrow\quad \theta$ is an odd multiple of $\frac{\pi}{4} \quad\Leftrightarrow\quad (x, y) = \left(\pm\frac{\sqrt{2}}{4}a, \pm\frac{\sqrt{2}}{4}a\right)$

[All sign choices are valid.]

33. $x = 3t^2 + 1,\ y = t^3 - 1 \quad\Rightarrow\quad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2}{6t} = \dfrac{t}{2}$. The tangent line has slope $\dfrac{1}{2}$ when $\dfrac{t}{2} = \dfrac{1}{2} \quad\Leftrightarrow\quad t = 1$, so the point is $(4, 0)$.

34. $x = 3t^2 + 1, y = 2t^3 + 1, \dfrac{dx}{dt} = 6t, \dfrac{dy}{dt} = 6t^2$, so $\dfrac{dy}{dx} = \dfrac{6t^2}{6t} = t \quad$ [even where $t = 0$].

So at the point corresponding to parameter value t, an equation of the tangent line is $y - (2t^3 + 1) = t[x - (3t^2 + 1)]$.

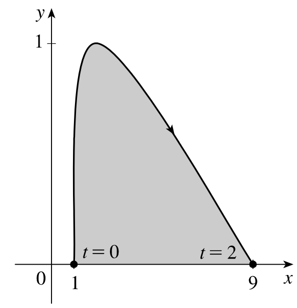
If this line is to pass through $(4, 3)$, we must have $3 - (2t^3 + 1) = t[4 - (3t^2 + 1)] \quad\Leftrightarrow\quad 2t^3 - 2 = 3t^3 - 3t \quad\Leftrightarrow$

$t^3 - 3t + 2 = 0 \;\Leftrightarrow\; (t-1)^2(t+2) = 0 \;\Leftrightarrow\; t = 1$ or -2. Hence, the desired equations are $y - 3 = x - 4$, or $y = x - 1$, tangent to the curve at $(4, 3)$, and $y - (-15) = -2(x - 13)$, or $y = -2x + 11$, tangent to the curve at $(13, -15)$.

35. The curve $x = t^3 + 1$, $y = 2t - t^2 = t(2 - t)$ intersects the x-axis when $y = 0$, that is, when $t = 0$ and $t = 2$. The corresponding values of x are 1 and 9. The shaded area is given by

$$\int_{x=1}^{x=9} (y_T - y_B)\,dx = \int_{t=0}^{t=2} [y(t) - 0]\,x'(t)\,dt = \int_0^2 (2t - t^2)(3t^2)\,dt$$

$$= 3\int_0^2 (2t^3 - t^4)\,dt = 3\Big[\tfrac{1}{2}t^4 - \tfrac{1}{5}t^5\Big]_0^2 = 3\big(8 - \tfrac{32}{5}\big) = \tfrac{24}{5}$$



36. The curve $x = \sin t$, $y = \sin t\cos t$, $0 \le t \le \pi/2$ intersects the x-axis when $y = 0$, that is, when $t = 0$ and $t = \pi/2$. The corresponding values of x are 0 and 1, so the area enclosed by the curve and the x-axis is given by

$$\int_{x=0}^{x=1} y\,dx = \int_{t=0}^{t=\pi/2} y(t)\,x'(t)\,dt = \int_0^{\pi/2} \sin t\cos t\,(\cos t)\,dt \overset{c}{=} -\int_1^0 u^2\,du = \big[\tfrac{1}{3}u^3\big]_0^1 = \tfrac{1}{3}$$

37. The curve $x = \sin^2 t$, $y = \cos t$ intersects the y-axis when $x = 0$, that is, when $t = 0$ and $t = \pi$. (Any integer multiple of π will result in $x = 0$, though we choose two values of t over which the curve is traced out once.) The corresponding values of y are 1 and -1, so the area enclosed by the curve and the y-axis is given by

$$\int_{y=-1}^{y=1} x\,dy = \int_{t=\pi}^{t=0} x(t)\,y'(t)\,dt = \int_\pi^0 \sin^2 t\,(-\sin t)\,dt = \int_0^\pi \sin^3 t\,dt \overset{67}{=} \left[-\frac{1}{3}(2 + \sin^2 t)\cos t\right]_0^\pi$$

$$= \tfrac{2}{3} - \big(-\tfrac{2}{3}\big) = \tfrac{4}{3}$$

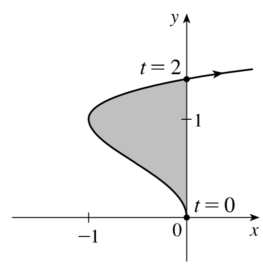
38. The curve $x = t^2 - 2t = t(t - 2)$, $y = \sqrt{t}$ intersects the y-axis when $x = 0$, that is, when $t = 0$ and $t = 2$. The corresponding values of y are 0 and $\sqrt{2}$. The shaded area is given by

$$\int_{y=0}^{y=\sqrt{2}} (x_R - x_L)\,dy = \int_{t=0}^{t=2} [0 - x(t)]\,y'(t)\,dt = -\int_0^2 (t^2 - 2t)\left(\frac{1}{2\sqrt{t}}\,dt\right)$$

$$= -\int_0^2 \left(\tfrac{1}{2}t^{3/2} - t^{1/2}\right)dt = -\left[\tfrac{1}{5}t^{5/2} - \tfrac{2}{3}t^{3/2}\right]_0^2$$

$$= -\left(\tfrac{1}{5}\cdot 2^{5/2} - \tfrac{2}{3}\cdot 2^{3/2}\right) = -2^{1/2}\big(\tfrac{4}{5} - \tfrac{4}{3}\big)$$

$$= -\sqrt{2}\,\big(-\tfrac{8}{15}\big) = \tfrac{8}{15}\sqrt{2}$$



39. $x = a\cos\theta$, $y = b\sin\theta$, $0 \le \theta \le 2\pi$. By symmetry of the ellipse about the x- and y-axes,

$$A = 4\int_{x=0}^{x=a} y\,dx = 4\int_{\theta=\pi/2}^{\theta=0} b\sin\theta\,(-a\sin\theta)\,d\theta = 4ab\int_0^{\pi/2} \sin^2\theta\,d\theta = 4ab\int_0^{\pi/2} \tfrac{1}{2}(1 - \cos 2\theta)\,d\theta$$

$$= 2ab\big[\theta - \tfrac{1}{2}\sin 2\theta\big]_0^{\pi/2} = 2ab\big(\tfrac{\pi}{2}\big) = \pi ab$$

40. By symmetry, the area of the shaded region is twice the area of the shaded portion above the x-axis. The top half of the loop is described by $x = 1 - t^2$, $y = t - t^3 = t(1 - t)(1 + t)$, $0 \le t \le 1$ with x-intercepts 0 and 1 corresponding to $t = 1$ and $t = 0$, respectively. Thus, the area of the shaded region is

$$2\int_0^1 y\,dx = 2\int_1^0 y(t)\,x'(t)\,dt = 2\int_1^0 (t - t^3)(-2t)\,dt = 4\int_0^1 (t^2 - t^4)\,dt = 4\big[\tfrac{1}{3}t^3 - \tfrac{1}{5}t^5\big]_0^1 = 4\big(\tfrac{1}{3} - \tfrac{1}{5}\big) = \tfrac{8}{15}.$$

41. $x = r\theta - d\sin\theta,\ \ y = r - d\cos\theta.$

$A = \int_0^{2\pi r} y\,dx = \int_0^{2\pi}(r - d\cos\theta)(r - d\cos\theta)\,d\theta = \int_0^{2\pi}(r^2 - 2dr\cos\theta + d^2\cos^2\theta)\,d\theta$

$\quad = \left[r^2\theta - 2dr\sin\theta + \frac{1}{2}d^2\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} = 2\pi r^2 + \pi d^2$

42. (a) By symmetry, the area of $\mathcal{R}$ is twice the area inside $\mathcal{R}$ above the x-axis. The top half of the loop is described by $x = t^2$, $y = t^3 - 3t$, $-\sqrt{3} \le t \le 0$, so, using the Substitution Rule with $y = t^3 - 3t$ and $dx = 2t\,dt$, we find that

$$\text{area} = 2\int_0^3 y\,dx = 2\int_0^{-\sqrt{3}}(t^3 - 3t)2t\,dt = 2\int_0^{-\sqrt{3}}(2t^4 - 6t^2)\,dt = 2\left[\tfrac{2}{5}t^5 - 2t^3\right]_0^{-\sqrt{3}}$$
$$= 2\left[\tfrac{2}{5}(-3^{1/2})^5 - 2(-3^{1/2})^3\right] = 2\left[\tfrac{2}{5}\left(-9\sqrt{3}\right) - 2\left(-3\sqrt{3}\right)\right] = \tfrac{24}{5}\sqrt{3}$$

(b) Here we use the formula for disks and use the Substitution Rule as in part (a):

$$\text{volume} = \pi\int_0^3 y^2\,dx = \pi\int_0^{-\sqrt{3}}(t^3 - 3t)^2 2t\,dt = 2\pi\int_0^{-\sqrt{3}}(t^6 - 6t^4 + 9t^2)t\,dt = 2\pi\left[\tfrac{1}{8}t^8 - t^6 + \tfrac{9}{4}t^4\right]_0^{-\sqrt{3}}$$
$$= 2\pi\left[\tfrac{1}{8}(-3^{1/2})^8 - (-3^{1/2})^6 + \tfrac{9}{4}(-3^{1/2})^4\right] = 2\pi\left[\tfrac{81}{8} - 27 + \tfrac{81}{4}\right] = \tfrac{27}{4}\pi$$

(c) By symmetry, the y-coordinate of the centroid is 0. To find the x-coordinate, we note that it is the same as the x-coordinate of the centroid of the top half of $\mathcal{R}$, the area of which is $\frac{1}{2}\cdot\frac{24}{5}\sqrt{3} = \frac{12}{5}\sqrt{3}$. So, using Formula 8.3.8 with $A = \frac{12}{5}\sqrt{3}$, we get

$$\overline{x} = \tfrac{5}{12\sqrt{3}}\int_0^3 xy\,dx = \tfrac{5}{12\sqrt{3}}\int_0^{-\sqrt{3}} t^2(t^3 - 3t)2t\,dt = \tfrac{5}{6\sqrt{3}}\left[\tfrac{1}{7}t^7 - \tfrac{3}{5}t^5\right]_0^{-\sqrt{3}}$$
$$= \tfrac{5}{6\sqrt{3}}\left[\tfrac{1}{7}(-3^{1/2})^7 - \tfrac{3}{5}(-3^{1/2})^5\right] = \tfrac{5}{6\sqrt{3}}\left[-\tfrac{27}{7}\sqrt{3} + \tfrac{27}{5}\sqrt{3}\right] = \tfrac{9}{7}$$

So the coordinates of the centroid of $\mathcal{R}$ are $(x, y) = \left(\frac{9}{7}, 0\right)$.

43. $x = 3t^2 - t^3$, $y = t^2 - 2t$. $\ dx/dt = 6t - 3t^2$ and $dy/dt = 2t - 2$, so $(dx/dt)^2 + (dy/dt)^2 = (6t - 3t^2)^2 + (2t - 2)^2 = 36t^2 - 36t^3 + 9t^4 + 4t^2 - 8t + 4 = 9t^4 - 36t^3 + 40t^2 - 8t + 4$. The endpoints of the curve both have $y = 3$, so the value of t at these points must satisfy $t^2 - 2t = 3 \ \Rightarrow\ t^2 - 2t - 3 = 0 \ \Rightarrow\ (t + 1)(t - 3) = 0 \ \Rightarrow\ t = -1$ or $t = 3$. Thus,

$$L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_{-1}^3 \sqrt{9t^4 - 36t^3 + 40t^2 - 8t + 4}\,dt \approx 15.2092$$

44. $x = t + e^{-t}$, $y = t^2 + t$. $\ dx/dt = 1 - e^{-t}$ and $dy/dt = 2t + 1$, so $(dx/dt)^2 + (dy/dt)^2 = (1 - e^{-t})^2 + (2t + 1)^2 = 1 - 2e^{-t} + e^{-2t} + 4t^2 + 4t + 1$. One endpoint of the curve has $y = 2$, so the value of t must satisfy $t^2 + t = 2 \ \Rightarrow\ t^2 + t - 2 = 0 \ \Rightarrow\ (t + 2)(t - 1) = 0 \ \Rightarrow\ t = -2$. (The solution $t = 1$ corresponds to $x \approx 1.37$, which is not an endpoint.) The other endoint has $y = 6 \ \Rightarrow\ t^2 + t = 6 \ \Rightarrow\ t^2 + t - 6 = 0 \ \Rightarrow\ (t + 3)(t - 2) = 0 \ \Rightarrow\ t = 2$. (The solution $t = -3$ corresponds to $x \approx 17.1$, which is not a point on the graph.) Thus,

$$L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_{-2}^2 \sqrt{2 - 2e^{-t} + e^{-2t} + 4t^2 + 4t}\,dt \approx 11.2485$$

45. $x = t - 2\sin t,\ y = 1 - 2\cos t,\ 0 \le t \le 4\pi$. $dx/dt = 1 - 2\cos t$ and $dy/dt = 2\sin t$, so

$(dx/dt)^2 + (dy/dt)^2 = (1 - 2\cos t)^2 + (2\sin t)^2 = 1 - 4\cos t + 4\cos^2 t + 4\sin^2 t = 5 - 4\cos t$. Thus,

$L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_0^{4\pi} \sqrt{5 - 4\cos t}\, dt \approx 26.7298$.

46. $x = t\cos t,\ y = t - 5\sin t$. $dx/dt = \cos t - t\sin t$ and $dy/dt = 1 - 5\cos t$, so

$(dx/dt)^2 + (dy/dt)^2 = (\cos t - t\sin t)^2 + (1 - 5\cos t)^2$. Observe that when $t = -\pi$, $(x, y) = (\pi, -\pi)$ and when $t = \pi$,

$(x, y) = (-\pi, \pi)$. Thus, $L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_{-\pi}^{\pi} \sqrt{(\cos t - t\sin t)^2 + (1 - 5\cos t)^2}\, dt \approx 22.8546$.

47. $x = \frac{2}{3}t^3,\ y = t^2 - 2,\ 0 \le t \le 3$. $dx/dt = 2t^2$ and $dy/dt = 2t$, so $(dx/dt)^2 + (dy/dt)^2 = 4t^4 + 4t^2 = 4t^2(t^2 + 1)$.

Thus,

$$L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_0^3 \sqrt{4t^2(t^2+1)}\, dt = \int_0^3 2t\sqrt{t^2+1}\, dt$$
$$= \int_1^{10} \sqrt{u}\, du \quad [u = t^2 + 1,\, du = 2t\, dt] \quad = \left[\tfrac{2}{3}u^{3/2}\right]_1^{10} = \tfrac{2}{3}(10^{3/2} - 1) = \tfrac{2}{3}\left(10\sqrt{10} - 1\right)$$

48. $x = e^t - t,\ y = 4e^{t/2},\ 0 \le t \le 2$. $dx/dt = e^t - 1$ and $dy/dt = 2e^{t/2}$, so

$(dx/dt)^2 + (dy/dt)^2 = (e^t - 1)^2 + (2e^{t/2})^2 = e^{2t} - 2e^t + 1 + 4e^t = e^{2t} + 2e^t + 1 = (e^t + 1)^2$. Thus,

$$L = \int_0^2 \sqrt{(e^t + 1)^2}\, dt = \int_0^2 \left|e^t + 1\right|\, dt = \int_0^2 (e^t + 1)\, dt = \left[e^t + t\right]_0^2 = (e^2 + 2) - (1 + 0) = e^2 + 1.$$

49. $x = t\sin t,\ y = t\cos t,\ 0 \le t \le 1$. $\dfrac{dx}{dt} = t\cos t + \sin t$ and $\dfrac{dy}{dt} = -t\sin t + \cos t$, so

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = t^2\cos^2 t + 2t\sin t\, \cos t + \sin^2 t + t^2\sin^2 t - 2t\sin t\, \cos t + \cos^2 t$$
$$= t^2(\cos^2 t + \sin^2 t) + \sin^2 t + \cos^2 t = t^2 + 1.$$

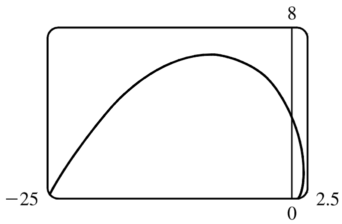
Thus, $L = \int_0^1 \sqrt{t^2 + 1}\, dt \overset{21}{=} \left[\frac{1}{2}t\sqrt{t^2+1} + \frac{1}{2}\ln\left(t + \sqrt{t^2+1}\right)\right]_0^1 = \frac{1}{2}\sqrt{2} + \frac{1}{2}\ln\left(1 + \sqrt{2}\right)$.

50. $x = 3\cos t - \cos 3t,\ y = 3\sin t - \sin 3t,\ 0 \le t \le \pi$. $\dfrac{dx}{dt} = -3\sin t + 3\sin 3t$ and $\dfrac{dy}{dt} = 3\cos t - 3\cos 3t$, so

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9\sin^2 t - 18\sin t\, \sin 3t + 9\sin^2 3t + 9\cos^2 t - 18\cos t\, \cos 3t + 9\cos^2 3t$$
$$= 9(\cos^2 t + \sin^2 t) - 18(\cos t\, \cos 3t + \sin t\, \sin 3t) + 9(\cos^2 3t + \sin^2 3t)$$
$$= 9(1) - 18\cos(t - 3t) + 9(1) = 18 - 18\cos(-2t) = 18(1 - \cos 2t)$$
$$= 18[1 - (1 - 2\sin^2 t)] = 36\sin^2 t.$$

Thus, $L = \int_0^\pi \sqrt{36\sin^2 t}\, dt = 6\int_0^\pi |\sin t|\, dt = 6\int_0^\pi \sin t\, dt = -6\left[\cos t\right]_0^\pi = -6\,(-1 - 1) = 12$.

51.



$x = e^t \cos t,\ y = e^t \sin t,\ 0 \le t \le \pi.$

$$\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 = [e^t(\cos t - \sin t)]^2 + [e^t(\sin t + \cos t)]^2$$
$$= (e^t)^2(\cos^2 t - 2\cos t\,\sin t + \sin^2 t)$$
$$+ (e^t)^2(\sin^2 t + 2\sin t\,\cos t + \cos^2 t$$
$$= e^{2t}(2\cos^2 t + 2\sin^2 t) = 2e^{2t}$$

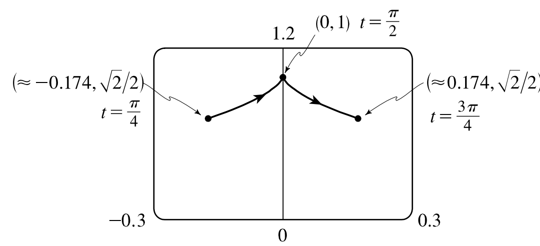
Thus, $L = \int_0^\pi \sqrt{2e^{2t}}\,dt = \int_0^\pi \sqrt{2}\,e^t\,dt = \sqrt{2}\,\left[e^t\right]_0^\pi = \sqrt{2}\,(e^\pi - 1).$

52. $x = \cos t + \ln(\tan \frac{1}{2}t), \quad y = \sin t, \quad \pi/4 \le t \le 3\pi/4.$

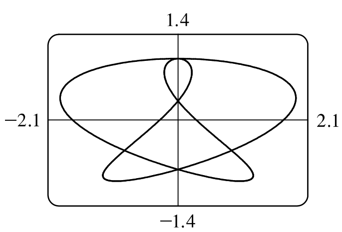
$$\frac{dx}{dt} = -\sin t + \frac{\frac{1}{2}\sec^2(t/2)}{\tan(t/2)} = -\sin t + \frac{1}{2\sin(t/2)\,\cos(t/2)} = -\sin t + \frac{1}{\sin t} \text{ and } \frac{dy}{dt} = \cos t, \text{ so}$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \sin^2 t - 2 + \frac{1}{\sin^2 t} + \cos^2 t = 1 - 2 + \csc^2 t = \cot^2 t. \text{ Thus,}$$

$$L = \int_{\pi/4}^{3\pi/4} |\cot t|\,dt = 2\int_{\pi/4}^{\pi/2} \cot t\,dt$$
$$= 2\Big[\ln|\sin t|\Big]_{\pi/4}^{\pi/2} = 2\left(\ln 1 - \ln \frac{1}{\sqrt{2}}\right)$$
$$= 2\left(0 + \ln\sqrt{2}\right) = 2\left(\tfrac{1}{2}\ln 2\right) = \ln 2.$$



53.



The figure shows the curve $x = \sin t + \sin 1.5t$, $y = \cos t$ for $0 \le t \le 4\pi$.

$dx/dt = \cos t + 1.5\cos 1.5t$ and $dy/dt = -\sin t$, so

$(dx/dt)^2 + (dy/dt)^2 = \cos^2 t + 3\cos t\,\cos 1.5t + 2.25\cos^2 1.5t + \sin^2 t.$

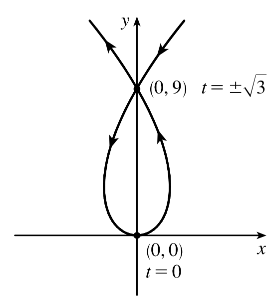
Thus, $L = \int_0^{4\pi} \sqrt{1 + 3\cos t\,\cos 1.5t + 2.25\cos^2 1.5t}\,dt \approx 16.7102.$

54. $x = 3t - t^3$, $y = 3t^2$. $\quad dx/dt = 3 - 3t^2$ and $dy/dt = 6t$, so

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (3 - 3t^2)^2 + (6t)^2 = (3 + 3t^2)^2$$

and the length of the loop is given by

$$L = \int_{-\sqrt{3}}^{\sqrt{3}} (3 + 3t^2)\,dt = 2\int_0^{\sqrt{3}} (3 + 3t^2)\,dt = 2\left[3t + t^3\right]_0^{\sqrt{3}}$$
$$= 2\left(3\sqrt{3} + 3\sqrt{3}\right) = 12\sqrt{3}$$



55. $x = \sin^2 t,\ y = \cos^2 t,\ 0 \le t \le 3\pi.$

$(dx/dt)^2 + (dy/dt)^2 = (2\sin t\,\cos t)^2 + (-2\cos t\sin t)^2 = 8\sin^2 t\cos^2 t = 2\sin^2 2t \quad \Rightarrow$

Distance $= \int_0^{3\pi} \sqrt{2}\,|\sin 2t|\,dt = 6\sqrt{2}\int_0^{\pi/2} \sin 2t\,dt$ [by symmetry] $= -3\sqrt{2}\left[\cos 2t\right]_0^{\pi/2} = -3\sqrt{2}\,(-1 - 1) = 6\sqrt{2}.$

[continued]

The full curve is traversed as t goes from 0 to $\frac{\pi}{2}$, because the curve is the segment of $x+y=1$ that lies in the first quadrant (since $x, y \geq 0$), and this segment is completely traversed as t goes from 0 to $\frac{\pi}{2}$. Thus, $L = \int_0^{\pi/2} \sin 2t\, dt = \sqrt{2}$, as above.

56. $x = \cos^2 t,\ y = \cos t,\ 0 \leq t \leq 4\pi.\quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (-2\cos t \sin t)^2 + (-\sin t)^2 = \sin^2 t\,(4\cos^2 t + 1)$

$$\begin{aligned}
\text{Distance } &= \int_0^{4\pi} |\sin t|\sqrt{4\cos^2 t + 1}\, dt = 4\int_0^{\pi} \sin t\sqrt{4\cos^2 t + 1}\, dt \\
&= -4\int_1^{-1} \sqrt{4u^2+1}\, du \quad [u = \cos t,\, du = -\sin t\, dt] \quad = 4\int_{-1}^{1} \sqrt{4u^2+1}\, du \\
&= 8\int_0^1 \sqrt{4u^2+1}\, du = 8\int_0^{\tan^{-1}2} \sec\theta \cdot \tfrac{1}{2}\sec^2\theta\, d\theta \qquad [2u = \tan\theta,\ 2\,du = \sec^2\theta\, d\theta] \\
&= 4\int_0^{\tan^{-1}2} \sec^3\theta\, d\theta \overset{71}{=} \Big[2\sec\theta\tan\theta + 2\ln|\sec\theta + \tan\theta|\Big]_0^{\tan^{-1}2} = 4\sqrt{5} + 2\ln\left(\sqrt{5}+2\right)
\end{aligned}$$

Thus, $L = \int_0^{\pi} |\sin t|\sqrt{4\cos^2 t + 1}\, dt = \sqrt{5} + \frac{1}{2}\ln\left(\sqrt{5}+2\right)$.

57. $x = 2t - 3,\ y = 2t^2 - 3t + 6.\quad dx/dt = 2$ and $dy/dt = 4t - 3$, so $v(t) = s'(t) = \sqrt{2^2 + (4t-3)^2}$. Thus, the speed of the particle at $t = 5$ is $v(5) = \sqrt{4 + (4\cdot 5 - 3)^2} = \sqrt{293} \approx 17.12$ m/s.

58. $x = 2 + 5\cos\left(\frac{\pi}{3}t\right),\ y = -2 + 7\sin\left(\frac{\pi}{3}t\right).\quad dx/dt = -\frac{5\pi}{3}\sin\left(\frac{\pi}{3}t\right)$ and $dy/dt = \frac{7\pi}{3}\cos\left(\frac{\pi}{3}t\right)$, so

$v(t) = s'(t) = \sqrt{\left[-\frac{5\pi}{3}\sin\left(\frac{\pi}{3}t\right)\right]^2 + \left[\frac{7\pi}{3}\cos\left(\frac{\pi}{3}t\right)\right]^2}$. Thus, the speed of the particle at $t = 3$ is

$$v(3) = \sqrt{\frac{25\pi^2}{9}\sin^2\pi + \frac{49\pi^2}{9}\cos^2\pi} = \sqrt{\frac{49\pi^2}{9}} = \frac{7\pi}{3} \approx 7.33 \text{ m/s}.$$

59. $x = e^t,\ y = te^t.\quad dx/dt = e^t$ and $dy/dt = te^t + e^t$, so $v(t) = s'(t) = \sqrt{(e^t)^2 + (te^t + e^t)^2}$. At (e, e), $x = e^t = e \Rightarrow t = 1$. Thus, the speed of the particle at (e, e) is $v(1) = \sqrt{e^2 + (e+e)^2} = \sqrt{5e^2} = \sqrt{5}\,e \approx 6.08$ m/s.

60. $x = t^2 + 1,\ y = t^4 + 2t^2 + 1.\quad dx/dt = 2t$ and $dy/dt = 4t^3 + 4t$, so $v(t) = s'(t) = \sqrt{4t^2 + (4t^3 + 4t)^2}$. At $(2, 4)$, $x = t^2 + 1 = 2 \Rightarrow t^2 = 1 \Rightarrow t = -1$ or $t = 1$. Both values result in the same speed since the function v is even. Thus, the speed of the particle at $(2, 4)$ is $v(1) = \sqrt{4\cdot 1^2 + (4\cdot 1^3 + 4\cdot 1)^2} = \sqrt{68} = 2\sqrt{17} \approx 8.25$ m/s.

61. $x = (v_0\cos\alpha)t,\ y = (v_0\sin\alpha)t - \frac{1}{2}gt^2.\quad dx/dt = v_0\cos\alpha$ and $dy/dt = v_0\sin\alpha - gt$, so speed $= v(t) = s'(t) = \sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{v_0^2\cos^2\alpha + (v_0\sin\alpha - gt)^2}$.

(a) The projectile hits the ground when $y = 0 \Rightarrow (v_0\sin\alpha)t - \frac{1}{2}gt^2 = 0 \Rightarrow t\left(v_0\sin\alpha - \frac{1}{2}gt\right) = 0 \Rightarrow t = 0$ or $t = \dfrac{2v_0\sin\alpha}{g}$. The second solution gives the time at which the projectile hits the ground, and at this time it will have a speed of

$$\begin{aligned}
v\left(\frac{2v_0\sin\alpha}{g}\right) &= \sqrt{v_0^2\cos^2\alpha + \left[v_0\sin\alpha - g\left(\frac{2v_0\sin\alpha}{g}\right)\right]^2} = \sqrt{v_0^2\cos^2\alpha + (-v_0\sin\alpha)^2} \\
&= \sqrt{v_0^2\cos^2\alpha + v_0^2\sin^2\alpha} = \sqrt{v_0^2(\cos^2\alpha + \sin^2\alpha)} = \sqrt{v_0^2} = v_0 \text{ m/s}.
\end{aligned}$$

Thus, the projectile hits the ground with the same speed at which it was fired.

(b) The projectile is at its highest point (maximum height) when $dy/dt = 0$. Thus, the speed of the projectile at this time is

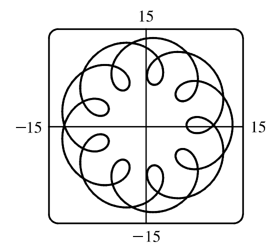
$v(t) = \sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{v_0^2 \cos^2\alpha + (0)^2} = v_0 \cos\alpha \text{ m/s}.$

62. $x = a\sin\theta,\ y = b\cos\theta,\ 0 \le \theta \le 2\pi.$

$$\begin{aligned}\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 &= (a\cos\theta)^2 + (-b\sin\theta)^2 = a^2\cos^2\theta + b^2\sin^2\theta = a^2(1-\sin^2\theta) + b^2\sin^2\theta \\ &= a^2 - (a^2-b^2)\sin^2\theta = a^2 - c^2\sin^2\theta = a^2\left(1 - \frac{c^2}{a^2}\sin^2\theta\right) = a^2(1 - e^2\sin^2\theta)\end{aligned}$$

So $L = 4\int_0^{\pi/2} \sqrt{a^2\left(1 - e^2\sin^2\theta\right)}\, d\theta$ [by symmetry] $= 4a\int_0^{\pi/2} \sqrt{1 - e^2\sin^2\theta}\, d\theta$.

63. (a) $x = 11\cos t - 4\cos(11t/2),\ y = 11\sin t - 4\sin(11t/2).$

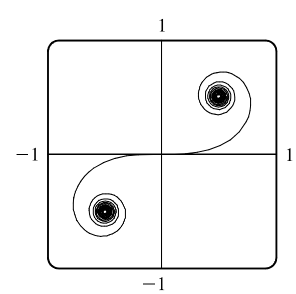


Notice that $0 \le t \le 2\pi$ does not give the complete curve because $x(0) \ne x(2\pi)$. In fact, we must take $t \in [0, 4\pi]$ in order to obtain the complete curve, since the first term in each of the parametric equations has period 2π and the second has period $\frac{2\pi}{11/2} = \frac{4\pi}{11}$, and the least common integer multiple of these two numbers is 4π.

(b) We use the CAS to find the derivatives dx/dt and dy/dt, and then use Theorem 5 to find the arc length. Recent versions of Maple express the integral $\int_0^{4\pi} \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$ as $88E\left(2\sqrt{2}\, i\right)$, where $E(x)$ is the elliptic integral $\displaystyle\int_0^1 \frac{\sqrt{1-x^2t^2}}{\sqrt{1-t^2}}\, dt$ and i is the imaginary number $\sqrt{-1}$.

Some earlier versions of Maple (as well as Mathematica) cannot do the integral exactly, so we use the command `evalf(Int(sqrt(diff(x,t)^2+diff(y,t)^2),t=0..4*Pi));` to estimate the length, and find that the arc length is approximately 294.03.

64. (a) It appears that as $t \to \infty$, $(x, y) \to \left(\frac{1}{2}, \frac{1}{2}\right)$, and as $t \to -\infty$, $(x, y) \to \left(-\frac{1}{2}, -\frac{1}{2}\right)$.



(b) By the Fundamental Theorem of Calculus, $x = C(t) = \int_0^t \cos(\pi u^2/2)\, du \Rightarrow$ $dx/dt = \cos\left(\frac{\pi}{2}t^2\right)$ and $y = S(t) = \int_0^t \sin(\pi u^2/2)\, du \Rightarrow dy/dt = \sin\left(\frac{\pi}{2}t^2\right)$, so by Theorem 5, the length of the curve from the origin to the point with parameter value t is

$$\begin{aligned}L &= \int_0^t \sqrt{\left(\tfrac{dx}{du}\right)^2 + \left(\tfrac{dy}{du}\right)^2}\, du = \int_0^t \sqrt{\cos^2\left(\tfrac{\pi}{2}u^2\right) + \sin^2\left(\tfrac{\pi}{2}u^2\right)}\, du \\ &= \int_0^t 1\, du = t \qquad [\text{or } -t \text{ if } t < 0]\end{aligned}$$

We have used u as the dummy variable so as not to confuse it with the upper limit of integration.

65. $x = a\cos^3\theta,\ y = a\sin^3\theta$. By symmetry,

$A = 4\int_0^a y\,dx = 4\int_{\pi/2}^0 a\sin^3\theta(-3a\cos^2\theta\,\sin\theta)\,d\theta = 12a^2\int_0^{\pi/2}\sin^4\theta\,\cos^2\theta\,d\theta$. Now

$$\begin{aligned}\int \sin^4\theta\,\cos^2\theta\,d\theta &= \int \sin^2\theta\left(\tfrac{1}{4}\sin^2 2\theta\right)d\theta = \tfrac{1}{8}\int(1-\cos 2\theta)\sin^2 2\theta\,d\theta \\ &= \tfrac{1}{8}\int\left[\tfrac{1}{2}(1-\cos 4\theta) - \sin^2 2\theta\,\cos 2\theta\right]d\theta = \tfrac{1}{16}\theta - \tfrac{1}{64}\sin 4\theta - \tfrac{1}{48}\sin^3 2\theta + C\end{aligned}$$

so $\int_0^{\pi/2}\sin^4\theta\,\cos^2\theta\,d\theta = \left[\frac{1}{16}\theta - \frac{1}{64}\sin 4\theta - \frac{1}{48}\sin^3 2\theta\right]_0^{\pi/2} = \frac{\pi}{32}$. Thus, $A = 12a^2\left(\frac{\pi}{32}\right) = \frac{3}{8}\pi a^2$.

66. By symmetry, the perimeter P of the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$ is given by

$$\begin{aligned}P &= 4\int_0^{\pi/2}\sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\,d\theta = 4\int_0^{\pi/2}\sqrt{[3a\cos^2\theta(-\sin\theta)]^2 + [3a\sin^2\theta(\cos\theta)]^2}\,d\theta \\ &= 4\int_0^{\pi/2}\sqrt{9a^2\sin^2\theta\cos^4\theta + 9a^2\sin^4\theta\cos^2\theta}\,d\theta = 4\int_0^{\pi/2} 3a\,|\sin\theta\cos\theta|\,\sqrt{\cos^2\theta + \sin^2\theta}\,d\theta \\ &= 6a\int_0^{\pi/2} 2\sin\theta\cos\theta\sqrt{1}\,d\theta = 6a\int_0^{\pi/2}\sin 2\theta\,d\theta = 6a\left[-\tfrac{1}{2}\cos 2\theta\right]_0^{\pi/2} = -3a(-1-1) = 6a\end{aligned}$$

67. $x = t\sin t,\ y = t\cos t,\ 0 \le t \le \pi/2$. $dx/dt = t\cos t + \sin t$ and $dy/dt = -t\sin t + \cos t$, so

$$\begin{aligned}(dx/dt)^2 + (dy/dt)^2 &= t^2\cos^2 t + 2t\sin t\,\cos t + \sin^2 t + t^2\sin^2 t - 2t\sin t\,\cos t + \cos^2 t \\ &= t^2(\cos^2 t + \sin^2 t) + \sin^2 t + \cos^2 t = t^2 + 1\end{aligned}$$

$S = \int 2\pi y\,ds = \int_0^{\pi/2} 2\pi t\cos t\sqrt{t^2+1}\,dt \approx 4.7394$.

68. $x = \sin t,\ y = \sin 2t,\ 0 \le t \le \pi/2$. $dx/dt = \cos t$ and $dy/dt = 2\cos 2t$, so $(dx/dt)^2 + (dy/dt)^2 = \cos^2 t + 4\cos^2 2t$.

$S = \int 2\pi y\,ds = \int_0^{\pi/2} 2\pi\sin 2t\sqrt{\cos^2 t + 4\cos^2 2t}\,dt \approx 8.0285$.

69. $x = t + e^t,\ y = e^{-t},\ 0 \le t \le 1$.

$dx/dt = 1 + e^t$ and $dy/dt = -e^{-t}$, so $(dx/dt)^2 + (dy/dt)^2 = (1+e^t)^2 + (-e^{-t})^2 = 1 + 2e^t + e^{2t} + e^{-2t}$.

$S = \int 2\pi y\,ds = \int_0^1 2\pi e^{-t}\sqrt{1 + 2e^t + e^{2t} + e^{-2t}}\,dt \approx 10.6705$.

70. $x = t^2 - t^3,\ y = t + t^4,\ 0 \le t \le 1$.

$(dx/dt)^2 + (dy/dt)^2 = (2t - 3t^2)^2 + (1 + 4t^3)^2 = 4t^2 - 12t^3 + 9t^4 + 1 + 8t^3 + 16t^6$, so

$S = \int 2\pi y\,ds = \int_0^1 2\pi(t + t^4)\sqrt{16t^6 + 9t^4 - 4t^3 + 4t^2 + 1}\,dt \approx 12.7176$.

71. $x = t^3,\ y = t^2,\ 0 \le t \le 1$. $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(3t^2\right)^2 + (2t)^2 = 9t^4 + 4t^2$.

$$\begin{aligned}S &= \int_0^1 2\pi y\sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\,dt = \int_0^1 2\pi t^2\sqrt{9t^4 + 4t^2}\,dt = 2\pi\int_0^1 t^2\sqrt{t^2(9t^2+4)}\,dt \\ &= 2\pi\int_4^{13}\left(\frac{u-4}{9}\right)\sqrt{u}\left(\tfrac{1}{18}\,du\right) \quad \left[\begin{array}{l} u = 9t^2 + 4,\ t^2 = (u-4)/9, \\ du = 18t\,dt,\ \text{so}\ t\,dt = \frac{1}{18}\,du \end{array}\right] \quad = \frac{2\pi}{9\cdot 18}\int_4^{13}\left(u^{3/2} - 4u^{1/2}\right)du \\ &= \tfrac{\pi}{81}\left[\tfrac{2}{5}u^{5/2} - \tfrac{8}{3}u^{3/2}\right]_4^{13} = \tfrac{\pi}{81}\cdot\tfrac{2}{15}\left[3u^{5/2} - 20u^{3/2}\right]_4^{13} \\ &= \tfrac{2\pi}{1215}\left[\left(3\cdot 13^2\sqrt{13} - 20\cdot 13\sqrt{13}\right) - (3\cdot 32 - 20\cdot 8)\right] = \tfrac{2\pi}{1215}\left(247\sqrt{13} + 64\right)\end{aligned}$$

72. $x = 2t^2 + 1/t,\ y = 8\sqrt{t},\ 1 \le t \le 3.$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(4t - \frac{1}{t^2}\right)^2 + \left(\frac{4}{\sqrt{t}}\right)^2 = 16t^2 - \frac{8}{t} + \frac{1}{t^4} + \frac{16}{t} = 16t^2 + \frac{8}{t} + \frac{1}{t^4} = \left(4t + \frac{1}{t^2}\right)^2.$$

$$\begin{aligned} S &= \int_1^3 2\pi y\sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\,dt = \int_1^3 2\pi\left(8\sqrt{t}\right)\sqrt{\left(4t + \frac{1}{t^2}\right)^2}\,dt = 16\pi\int_1^3 t^{1/2}(4t + t^{-2})\,dt \\ &= 16\pi\int_1^3 (4t^{3/2} + t^{-3/2})\,dt = 16\pi\left[\tfrac{8}{5}t^{5/2} - 2t^{-1/2}\right]_1^3 = 16\pi\left[\left(\tfrac{72}{5}\sqrt{3} - \tfrac{2}{3}\sqrt{3}\right) - \left(\tfrac{8}{5} - 2\right)\right] \\ &= 16\pi\left(\tfrac{206}{15}\sqrt{3} + \tfrac{6}{15}\right) = \tfrac{32\pi}{15}\left(103\sqrt{3} + 3\right) \end{aligned}$$

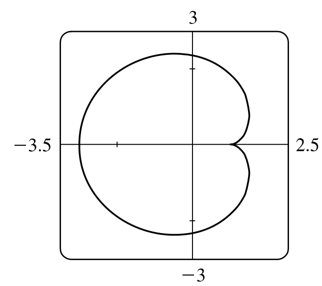
73. $x = a\cos^3\theta,\ y = a\sin^3\theta,\ 0 \le \theta \le \frac{\pi}{2}.$ $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (-3a\cos^2\theta\,\sin\theta)^2 + (3a\sin^2\theta\,\cos\theta)^2 = 9a^2\sin^2\theta\,\cos^2\theta.$

$S = \int_0^{\pi/2} 2\pi\cdot a\sin^3\theta\cdot 3a\sin\theta\,\cos\theta\,d\theta = 6\pi a^2\int_0^{\pi/2}\sin^4\theta\,\cos\theta\,d\theta = \frac{6}{5}\pi a^2\left[\sin^5\theta\right]_0^{\pi/2} = \frac{6}{5}\pi a^2$

74. $x = 2\cos\theta - \cos 2\theta,\ y = 2\sin\theta - \sin 2\theta \quad\Rightarrow$

$$\begin{aligned} \left(\tfrac{dx}{d\theta}\right)^2 + \left(\tfrac{dy}{d\theta}\right)^2 &= (-2\sin\theta + 2\sin 2\theta)^2 + (2\cos\theta - 2\cos 2\theta)^2 \\ &= 4[(\sin^2\theta - 2\sin\theta\,\sin 2\theta + \sin^2 2\theta) + (\cos^2\theta - 2\cos\theta\,\cos 2\theta + \cos^2 2\theta)] \\ &= 4[1 + 1 - 2(\cos 2\theta\cos\theta + \sin 2\theta\sin\theta)] = 8[1 - \cos(2\theta - \theta)] = 8(1 - \cos\theta) \end{aligned}$$

We plot the graph with parameter interval $[0, 2\pi]$, and see that we should only integrate between 0 and π. (If the interval $[0, 2\pi]$ were taken, the surface of revolution would be generated twice.) Also note that $y = 2\sin\theta - \sin 2\theta = 2\sin\theta(1 - \cos\theta)$. So



$$\begin{aligned} S &= \int_0^{\pi} 2\pi\cdot 2\sin\theta(1 - \cos\theta)\,2\sqrt{2}\sqrt{1 - \cos\theta}\,d\theta \\ &= 8\sqrt{2}\pi\int_0^{\pi}(1 - \cos\theta)^{3/2}\sin\theta\,d\theta = 8\sqrt{2}\pi\int_0^2\sqrt{u^3}\,du \qquad \begin{bmatrix} u = 1 - \cos\theta, \\ du = \sin\theta\,d\theta \end{bmatrix} \\ &= 8\sqrt{2}\pi\left[\left(\tfrac{2}{5}\right)u^{5/2}\right]_0^2 = \tfrac{16}{5}\sqrt{2}\pi(2^{5/2}) = \tfrac{128}{5}\pi \end{aligned}$$

75. $x = 3t^2,\ y = 2t^3,\ 0 \le t \le 5 \quad\Rightarrow\quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (6t)^2 + (6t^2)^2 = 36t^2(1 + t^2) \quad\Rightarrow$

$$\begin{aligned} S &= \int_0^5 2\pi x\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^5 2\pi(3t^2)6t\sqrt{1 + t^2}\,dt = 18\pi\int_0^5 t^2\sqrt{1 + t^2}\,2t\,dt \\ &= 18\pi\int_1^{26}(u - 1)\sqrt{u}\,du \quad \begin{bmatrix} u = 1 + t^2, \\ du = 2t\,dt \end{bmatrix} \quad = 18\pi\int_1^{26}(u^{3/2} - u^{1/2})\,du = 18\pi\left[\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right]_1^{26} \\ &= 18\pi\left[\left(\tfrac{2}{5}\cdot 676\sqrt{26} - \tfrac{2}{3}\cdot 26\sqrt{26}\right) - \left(\tfrac{2}{5} - \tfrac{2}{3}\right)\right] = \tfrac{24}{5}\pi\left(949\sqrt{26} + 1\right) \end{aligned}$$

76. $x = e^t - t, y = 4e^{t/2}, 0 \le t \le 1.$ $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (e^t - 1)^2 + (2e^{t/2})^2 = e^{2t} + 2e^t + 1 = (e^t + 1)^2.$

$$\begin{aligned} S &= \int_0^1 2\pi(e^t - t)\sqrt{(e^t - 1)^2 + (2e^{t/2})^2}\,dt = \int_0^1 2\pi(e^t - t)(e^t + 1)d \\ &= 2\pi\left[\tfrac{1}{2}e^{2t} + e^t - (t - 1)e^t - \tfrac{1}{2}t^2\right]_0^1 = \pi(e^2 + 2e - 6) \end{aligned}$$

77. If f' is continuous and $f'(t) \ne 0$ for $a \le t \le b$, then either $f'(t) > 0$ for all t in $[a, b]$ or $f'(t) < 0$ for all t in $[a, b]$. Thus, f is monotonic (in fact, strictly increasing or strictly decreasing) on $[a, b]$. It follows that f has an inverse. Set $F = g \circ f^{-1}$, that is, define F by $F(x) = g(f^{-1}(x))$. Then $x = f(t) \quad\Rightarrow\quad f^{-1}(x) = t$, so $y = g(t) = g(f^{-1}(x)) = F(x)$.

78. By Formula 8.2.5 with $y = F(x)$, $S = \int_a^b 2\pi F(x)\sqrt{1 + [F'(x)]^2}\,dx$. But by Formula 10.2.1,

$1 + [F'(x)]^2 = 1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \left(\dfrac{dy/dt}{dx/dt}\right)^2 = \dfrac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}$. Using the Substitution Rule with $x = x(t)$,

where $a = x(\alpha)$ and $b = x(\beta)$, we have $\left[\text{since } dx = \dfrac{dx}{dt}\,dt\right]$

$$S = \int_\alpha^\beta 2\pi\, F(x(t))\sqrt{\frac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}}\,\frac{dx}{dt}\,dt = \int_\alpha^\beta 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt,$$ which is Formula 10.2.9.

79. $\phi = \tan^{-1}\left(\dfrac{dy}{dx}\right) \;\Rightarrow\; \dfrac{d\phi}{dt} = \dfrac{d}{dt}\tan^{-1}\left(\dfrac{dy}{dx}\right) = \dfrac{1}{1 + (dy/dx)^2}\left[\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)\right]$. But $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\dot{y}}{\dot{x}} \;\Rightarrow$

$\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dt}\left(\dfrac{\dot{y}}{\dot{x}}\right) = \dfrac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2} \;\Rightarrow\; \dfrac{d\phi}{dt} = \dfrac{1}{1 + (\dot{y}/\dot{x})^2}\left(\dfrac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2}\right) = \dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2}$. Using the Chain Rule, and the fact

that $s = \displaystyle\int_0^t \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\,dt \;\Rightarrow\; \tfrac{ds}{dt} = \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2} = \left(\dot{x}^2 + \dot{y}^2\right)^{1/2}$, we have that

$\dfrac{d\phi}{ds} = \dfrac{d\phi/dt}{ds/dt} = \left(\dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2}\right)\dfrac{1}{(\dot{x}^2 + \dot{y}^2)^{1/2}} = \dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$. So $\kappa = \left|\dfrac{d\phi}{ds}\right| = \left|\dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}\right| = \dfrac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$.

80. $x = x$ and $y = f(x) \;\Rightarrow\; \dot{x} = 1$, $\ddot{x} = 0$ and $\dot{y} = \dfrac{dy}{dx}$, $\ddot{y} = \dfrac{d^2y}{dx^2}$.

So $\kappa = \dfrac{\left|1 \cdot (d^2y/dx^2) - 0 \cdot (dy/dx)\right|}{[1 + (dy/dx)^2]^{3/2}} = \dfrac{\left|d^2y/dx^2\right|}{[1 + (dy/dx)^2]^{3/2}}$.

81. $x = \theta - \sin\theta \;\Rightarrow\; \dot{x} = 1 - \cos\theta \;\Rightarrow\; \ddot{x} = \sin\theta$, and $y = 1 - \cos\theta \;\Rightarrow\; \dot{y} = \sin\theta \;\Rightarrow\; \ddot{y} = \cos\theta$. Therefore,

$\kappa = \dfrac{\left|\cos\theta - \cos^2\theta - \sin^2\theta\right|}{[(1 - \cos\theta)^2 + \sin^2\theta]^{3/2}} = \dfrac{\left|\cos\theta - (\cos^2\theta + \sin^2\theta)\right|}{(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)^{3/2}} = \dfrac{|\cos\theta - 1|}{(2 - 2\cos\theta)^{3/2}}$. The top of the arch is

characterized by a horizontal tangent, and from Example 2(b) in Section 10.2, the tangent is horizontal when $\theta = (2n - 1)\pi$,

so take $n = 1$ and substitute $\theta = \pi$ into the expression for κ: $\kappa = \dfrac{|\cos\pi - 1|}{(2 - 2\cos\pi)^{3/2}} = \dfrac{|-1 - 1|}{[2 - 2(-1)]^{3/2}} = \dfrac{1}{4}$.

82. (a) $y = x^2 \;\Rightarrow\; \dfrac{dy}{dx} = 2x \;\Rightarrow\; \dfrac{d^2y}{dx^2} = 2$. So $\kappa = \dfrac{\left|d^2y/dx^2\right|}{[1 + (dy/dx)^2]^{3/2}} = \dfrac{2}{(1 + 4x^2)^{3/2}}$, and at $(1, 1)$,

$\kappa = \dfrac{2}{5^{3/2}} = \dfrac{2}{5\sqrt{5}}$.

(b) $\kappa' = \dfrac{d\kappa}{dx} = -3(1 + 4x^2)^{-5/2}(8x) = 0 \;\Leftrightarrow\; x = 0 \;\Rightarrow\; y = 0$. This is a maximum since $\kappa' > 0$ for $x < 0$ and

$\kappa' < 0$ for $x > 0$. So the parabola $y = x^2$ has maximum curvature at the origin.

83. (a) Every straight line has parametrizations of the form $x = a + vt$, $y = b + wt$, where a, b are arbitrary and $v, w \neq 0$.

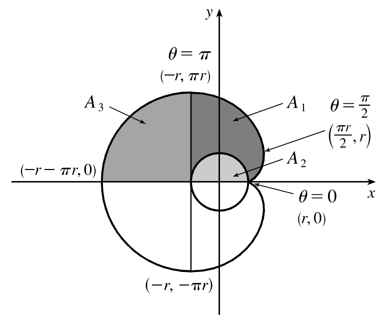
For example, a straight line passing through distinct points (a, b) and (c, d) can be described as the parametrized curve

$x = a + (c - a)t$, $y = b + (d - b)t$. Starting with $x = a + vt$, $y = b + wt$, we compute $\dot{x} = v$, $\dot{y} = w$, $\ddot{x} = \ddot{y} = 0$,

and $\kappa = \dfrac{|v \cdot 0 - w \cdot 0|}{(v^2 + w^2)^{3/2}} = 0$.

(b) Parametric equations for a circle of radius r are $x = r\cos\theta$ and $y = r\sin\theta$. We can take the center to be the origin. So $\dot{x} = -r\sin\theta \;\Rightarrow\; \ddot{x} = -r\cos\theta$ and $\dot{y} = r\cos\theta \;\Rightarrow\; \ddot{y} = -r\sin\theta$. Therefore,
$\kappa = \dfrac{\left|r^2\sin^2\theta + r^2\cos^2\theta\right|}{(r^2\sin^2\theta + r^2\cos^2\theta)^{3/2}} = \dfrac{r^2}{r^3} = \dfrac{1}{r}$. And so for any θ (and thus any point), $\kappa = \dfrac{1}{r}$.

84. If the cow walks with the rope taut, it traces out the portion of the involute in Exercise 85 corresponding to the range $0 \le \theta \le \pi$, arriving at the point $(-r, \pi r)$ when $\theta = \pi$. With the rope now fully extended, the cow walks in a semicircle of radius πr, arriving at $(-r, -\pi r)$. Finally, the cow traces out another portion of the involute, namely the reflection about the x-axis of the initial involute path. (This corresponds to the range $-\pi \le \theta \le 0$.) Referring to the figure, we see that the total grazing area is $2(A_1 + A_3)$. A_3 is one-quarter of the area of a circle of radius πr, so $A_3 = \frac{1}{4}\pi(\pi r)^2 = \frac{1}{4}\pi^3 r^2$. We will compute $A_1 + A_2$ and then subtract $A_2 = \frac{1}{2}\pi r^2$ to obtain A_1.



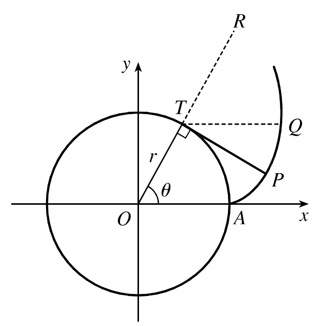
To find $A_1 + A_2$, first note that the rightmost point of the involute is $\left(\frac{\pi}{2}r, r\right)$. [To see this, note that $dx/d\theta = 0$ when $\theta = 0$ or $\frac{\pi}{2}$. $\theta = 0$ corresponds to the cusp at $(r, 0)$ and $\theta = \frac{\pi}{2}$ corresponds to $\left(\frac{\pi}{2}r, r\right)$.] The leftmost point of the involute is $(-r, \pi r)$. Thus, $A_1 + A_2 = \int_{\theta=\pi}^{\pi/2} y\,dx - \int_{\theta=0}^{\pi/2} y\,dx = \int_{\theta=\pi}^{0} y\,dx$.

Now $y\,dx = r(\sin\theta - \theta\cos\theta)\,r\theta\cos\theta\,d\theta = r^2(\theta\sin\theta\,\cos\theta - \theta^2\cos^2\theta)d\theta$. Integrate:
$(1/r^2)\int y\,dx = -\theta\cos^2\theta - \frac{1}{2}\left(\theta^2 - 1\right)\sin\theta\,\cos\theta - \frac{1}{6}\theta^3 + \frac{1}{2}\theta + C$. This enables us to compute

$$A_1 + A_2 = r^2\left[-\theta\cos^2\theta - \tfrac{1}{2}(\theta^2 - 1)\sin\theta\,\cos\theta - \tfrac{1}{6}\theta^3 + \tfrac{1}{2}\theta\right]_{\pi}^{0} = r^2\left[0 - \left(-\pi - \frac{\pi^3}{6} + \frac{\pi}{2}\right)\right] = r^2\left(\frac{\pi}{2} + \frac{\pi^3}{6}\right)$$

Therefore, $A_1 = (A_1 + A_2) - A_2 = \frac{1}{6}\pi^3 r^2$, so the grazing area is $2(A_1 + A_3) = 2\left(\frac{1}{6}\pi^3 r^2 + \frac{1}{4}\pi^3 r^2\right) = \frac{5}{6}\pi^3 r^2$.

85. The coordinates of T are $(r\cos\theta, r\sin\theta)$. Since TP was unwound from arc TA, TP has length $r\theta$. Also $\angle PTQ = \angle PTR - \angle QTR = \frac{1}{2}\pi - \theta$, so P has coordinates $x = r\cos\theta + r\theta\cos\left(\frac{1}{2}\pi - \theta\right) = r(\cos\theta + \theta\sin\theta)$, $y = r\sin\theta - r\theta\sin\left(\frac{1}{2}\pi - \theta\right) = r(\sin\theta - \theta\cos\theta)$.



DISCOVERY PROJECT Bézier Curves

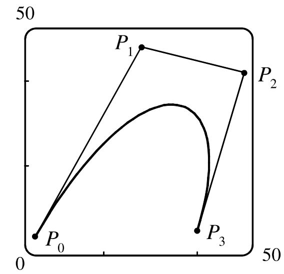
1. The parametric equations for a cubic Bézier curve are

$$x = x_0(1-t)^3 + 3x_1 t(1-t)^2 + 3x_2 t^2(1-t) + x_3 t^3$$
$$y = y_0(1-t)^3 + 3y_1 t(1-t)^2 + 3y_2 t^2(1-t) + y_3 t^3$$

where $0 \le t \le 1$. We are given the points $P_0(x_0, y_0) = (4, 1)$, $P_1(x_1, y_1) = (28, 48)$, $P_2(x_2, y_2) = (50, 42)$, and

$P_3(x_3, y_3) = (40, 5)$. The curve is then given by

$$x(t) = 4(1-t)^3 + 3 \cdot 28t(1-t)^2 + 3 \cdot 50t^2(1-t) + 40t^3$$
$$y(t) = 1(1-t)^3 + 3 \cdot 48t(1-t)^2 + 3 \cdot 42t^2(1-t) + 5t^3$$



where $0 \le t \le 1$. The line segments are of the form $x = x_0 + (x_1 - x_0)t$, $y = y_0 + (y_1 - y_0)t$:

P_0P_1	$x = 4 + 24t,$	$y = 1 + 47t$
P_1P_2	$x = 28 + 22t,$	$y = 48 - 6t$
P_2P_3	$x = 50 - 10t,$	$y = 42 - 37t$

2. It suffices to show that the slope of the tangent at P_0 is the same as that of line segment P_0P_1, namely $\dfrac{y_1 - y_0}{x_1 - x_0}$.

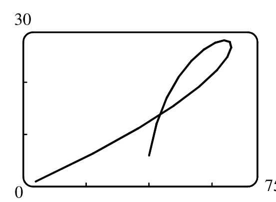
We calculate the slope of the tangent to the Bézier curve:

$$\frac{dy/dt}{dx/dt} = \frac{-3y_0(1-t)^2 + 3y_1\left[-2t(1-t) + (1-t)^2\right] + 3y_2\left[-t^2 + (2t)(1-t)\right] + 3y_3t^2}{-3x_0^2(1-t) + 3x_1[-2t(1-t) + (1-t)^2] + 3x_2[-t^2 + (2t)(1-t)] + 3x_3t^2}$$

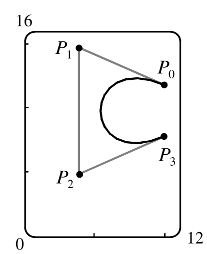
At point P_0, $t = 0$, so the slope of the tangent is $\dfrac{-3y_0 + 3y_1}{-3x_0 + 3x_1} = \dfrac{y_1 - y_0}{x_1 - x_0}$. So the tangent to the curve at P_0 passes through P_1. Similarly, the slope of the tangent at point P_3 [where $t = 1$] is $\dfrac{-3y_2 + 3y_3}{-3x_2 + 3x_3} = \dfrac{y_3 - y_2}{x_3 - x_2}$, which is also the slope of line P_2P_3.

3. It seems that if P_1 were to the right of P_2, a loop would appear.

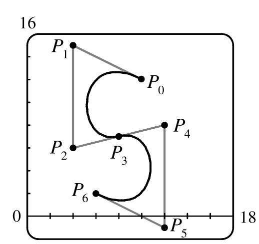
We try setting $P_1 = (110, 30)$, and the resulting curve does indeed have a loop.



4. Based on the behavior of the Bézier curve in Problems 1–3, we suspect that the four control points should be in an exaggerated C shape. We try $P_0(10, 12)$, $P_1(4, 15)$, $P_2(4, 5)$, and $P_3(10, 8)$, and these produce a decent C. If you are using a CAS, it may be necessary to instruct it to make the x- and y-scales the same so as not to distort the figure (this is called a “constrained projection” in Maple.)

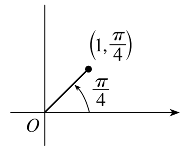


5. We use the same P_0 and P_1 as in Problem 4, and use part of our C as the top of an S. To prevent the center line from slanting up too much, we move P_2 up to $(4, 6)$ and P_3 down and to the left, to $(8, 7)$. In order to have a smooth joint between the top and bottom halves of the S (and a symmetric S), we determine points P_4, P_5, and P_6 by rotating points P_2, P_1, and P_0 about the center of the letter (point P_3). The points are therefore $P_4(12, 8)$, $P_5(12, -1)$, and $P_6(6, 2)$.



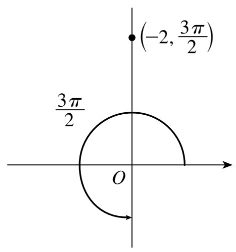
10.3 Polar Coordinates

1. (a) $\left(1, \frac{\pi}{4}\right)$



By adding 2π to $\frac{\pi}{4}$, we obtain the point $\left(1, \frac{9\pi}{4}\right)$, which satisfies the $r > 0$ requirement. The direction opposite $\frac{\pi}{4}$ is $\frac{5\pi}{4}$, so $\left(-1, \frac{5\pi}{4}\right)$ is a point that satisfies the $r < 0$ requirement.

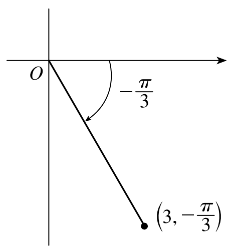
(b) $\left(-2, \frac{3\pi}{2}\right)$



$\boldsymbol{r} > \mathbf{0}$: $\left(-(-2), \frac{3\pi}{2} - \pi\right) = \left(2, \frac{\pi}{2}\right)$

$\boldsymbol{r} < \mathbf{0}$: $\left(-2, \frac{3\pi}{2} + 2\pi\right) = \left(-2, \frac{7\pi}{2}\right)$

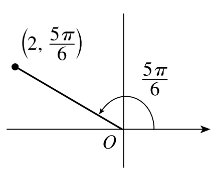
(c) $\left(3, -\frac{\pi}{3}\right)$



$\boldsymbol{r} > \mathbf{0}$: $\left(3, -\frac{\pi}{3} + 2\pi\right) = \left(3, \frac{5\pi}{3}\right)$

$\boldsymbol{r} < \mathbf{0}$: $\left(-3, -\frac{\pi}{3} + \pi\right) = \left(-3, \frac{2\pi}{3}\right)$

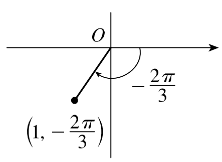
2. (a) $\left(2, \frac{5\pi}{6}\right)$



$\boldsymbol{r} > \mathbf{0}$: $\left(2, \frac{5\pi}{6} + 2\pi\right) = \left(2, \frac{17\pi}{6}\right)$

$\boldsymbol{r} < \mathbf{0}$: $\left(-2, \frac{5\pi}{6} - \pi\right) = \left(-2, -\frac{\pi}{6}\right)$

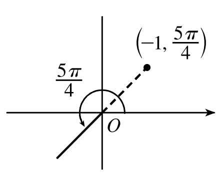
(b) $\left(1, -\frac{2\pi}{3}\right)$



$\boldsymbol{r} > \mathbf{0}$: $\left(1, -\frac{2\pi}{3} + 2\pi\right) = \left(1, \frac{4\pi}{3}\right)$

$\boldsymbol{r} < \mathbf{0}$: $\left(-1, -\frac{2\pi}{3} + \pi\right) = \left(-1, \frac{\pi}{3}\right)$

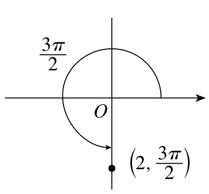
(c) $\left(-1, \frac{5\pi}{4}\right)$



$\boldsymbol{r} > \mathbf{0}$: $\left(-(-1), \frac{5\pi}{4} - \pi\right) = \left(1, \frac{\pi}{4}\right)$

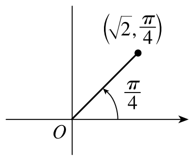
$\boldsymbol{r} < \mathbf{0}$: $\left(-1, \frac{5\pi}{4} - 2\pi\right) = \left(-1, -\frac{3\pi}{4}\right)$

3. (a)



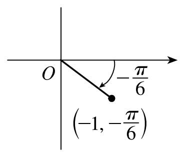
$x = 2\cos\frac{3\pi}{2} = 2(0) = 0$ and $y = 2\sin\frac{3\pi}{2} = 2(-1) = -2$ give us the Cartesian coordinates $(0, -2)$.

(b)



$x = \sqrt{2} \cos \dfrac{\pi}{4} = \sqrt{2}\left(\dfrac{1}{\sqrt{2}}\right) = 1$ and $y = \sqrt{2} \sin \dfrac{\pi}{4} = \sqrt{2}\left(\dfrac{1}{\sqrt{2}}\right) = 1$ give us the Cartesian coordinates $(1, 1)$.

(c)

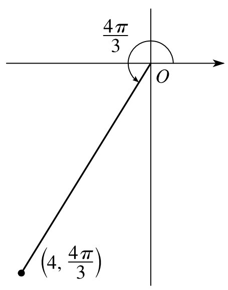


$x = -1 \cos\left(-\dfrac{\pi}{6}\right) = -1\left(\dfrac{\sqrt{3}}{2}\right) = -\dfrac{\sqrt{3}}{2}$ and

$y = -1 \sin\left(-\dfrac{\pi}{6}\right) = -1\left(-\dfrac{1}{2}\right) = \dfrac{1}{2}$ give us the Cartesian

coordinates $\left(-\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$.

4. (a)

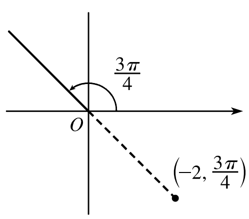


$x = 4 \cos \dfrac{4\pi}{3} = 4\left(-\dfrac{1}{2}\right) = -2$ and

$y = 4 \sin \dfrac{4\pi}{3} = 4\left(-\dfrac{\sqrt{3}}{2}\right) = -2\sqrt{3}$ give us the Cartesian

coordinates $\left(-2, -2\sqrt{3}\right)$.

(b)

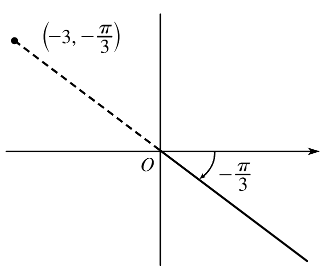


$x = -2 \cos \dfrac{3\pi}{4} = -2\left(-\dfrac{\sqrt{2}}{2}\right) = \sqrt{2}$ and

$y = -2 \sin \dfrac{3\pi}{4} = -2\left(\dfrac{\sqrt{2}}{2}\right) = -\sqrt{2}$ give us the Cartesian

coordinates $\left(\sqrt{2}, -\sqrt{2}\right)$.

(c)



$x = -3 \cos\left(-\dfrac{\pi}{3}\right) = -3\left(\dfrac{1}{2}\right) = -\dfrac{3}{2}$ and

$y = -3 \sin\left(-\dfrac{\pi}{3}\right) = -3\left(-\dfrac{\sqrt{3}}{2}\right) = \dfrac{3\sqrt{3}}{2}$ give us the Cartesian

coordinates $\left(-\dfrac{3}{2}, \dfrac{3\sqrt{3}}{2}\right)$.

5. (a) $x = -4$ and $y = 4 \;\;\Rightarrow\;\; r = \sqrt{(-4)^2 + 4^2} = 4\sqrt{2}$ and $\tan\theta = \frac{4}{-4} = -1 \;\;\left[\theta = -\frac{\pi}{4} + n\pi\right]$. Since $(-4, 4)$ is in the second quadrant, the polar coordinates are (i) $\left(4\sqrt{2}, \frac{3\pi}{4}\right)$ and (ii) $\left(-4\sqrt{2}, \frac{7\pi}{4}\right)$.

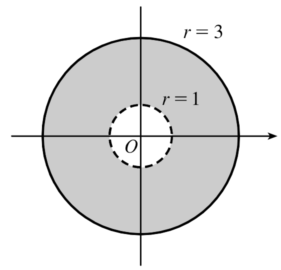
(b) $x = 3$ and $y = 3\sqrt{3} \;\;\Rightarrow\;\; r = \sqrt{3^2 + \left(3\sqrt{3}\right)^2} = \sqrt{9 + 27} = 6$ and $\tan\theta = \dfrac{3\sqrt{3}}{3} = \sqrt{3} \;\;\left[\theta = \frac{\pi}{3} + n\pi\right]$.

Since $\left(3, 3\sqrt{3}\right)$ is in the first quadrant, the polar coordinates are (i) $\left(6, \frac{\pi}{3}\right)$ and (ii) $\left(-6, \frac{4\pi}{3}\right)$.

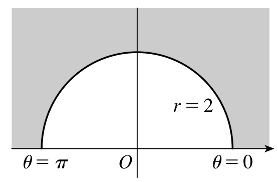
6. (a) $x = \sqrt{3}$ and $y = -1 \quad \Rightarrow \quad r = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2$ and $\tan\theta = \dfrac{-1}{\sqrt{3}} \quad [\theta = -\frac{\pi}{6} + n\pi]$. Since $\left(\sqrt{3}, -1\right)$ is in the fourth quadrant, the polar coordinates are (i) $\left(2, \frac{11\pi}{6}\right)$ and (ii) $\left(-2, \frac{5\pi}{6}\right)$.

(b) $x = -6$ and $y = 0 \quad \Rightarrow \quad r = \sqrt{(-6)^2 + 0^2} = 6$ and $\tan\theta = \frac{0}{-6} = 0 \quad [\theta = n\pi]$. Since $(-6, 0)$ is on the negative x-axis, the polar coordinates are (i) $(6, \pi)$ and (ii) $(-6, 0)$.

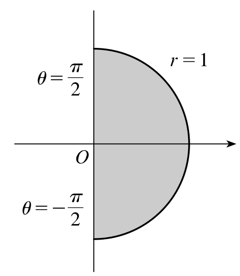
7. $1 < r \leq 3$. The curves $r = 1$ and $r = 3$ represent circles centered at O with radius 1 and 3, respectively. So $1 < r \leq 3$ represents the region outside the radius 1 circle and on or inside the radius 3 circle. Note that θ can take on any value.



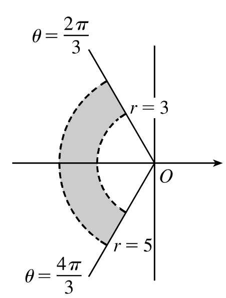
8. $r \geq 2,\ 0 \leq \theta \leq \pi$. This is the region on or outside the circle $r = 2$ in the first and second quadrants.



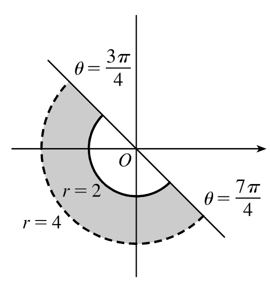
9. $0 \leq r \leq 1,\ -\pi/2 \leq \theta \leq \pi/2$. This is the region on or inside the circle $r = 1$ in the first and fourth quadrants.



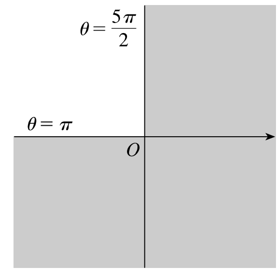
10. $3 < r < 5,\ 2\pi/3 \leq \theta \leq 4\pi/3$



11. $2 \leq r < 4, 3\pi/4 \leq \theta \leq 7\pi/4$



12. $r \geq 0,\ \pi \leq \theta \leq 5\pi/2$. This is the region in the third, fourth, and first quadrants including the origin and points on the negative x-axis and positive y-axis.



13. Converting the polar coordinates $\left(4, \frac{4\pi}{3}\right)$ and $\left(6, \frac{5\pi}{3}\right)$ to Cartesian coordinates gives us $\left(4\cos\frac{4\pi}{3}, 4\sin\frac{4\pi}{3}\right) = \left(-2, -2\sqrt{3}\right)$ and $\left(6\cos\frac{5\pi}{3}, 6\sin\frac{5\pi}{3}\right) = \left(3, -3\sqrt{3}\right)$. Now use the distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{[3 - (-2)]^2 + \left[-3\sqrt{3} - \left(-2\sqrt{3}\right)\right]^2}$$
$$= \sqrt{5^2 + \left(-\sqrt{3}\right)^2} = \sqrt{25 + 3} = \sqrt{28} = 2\sqrt{7}$$

14. The points (r_1, θ_1) and (r_2, θ_2) in Cartesian coordinates are $(r_1\cos\theta_1, r_1\sin\theta_1)$ and $(r_2\cos\theta_2, r_2\sin\theta_2)$, respectively. The *square* of the distance between them is

$$\begin{aligned}(r_2\cos\theta_2 - r_1\cos\theta_1)^2 &+ (r_2\sin\theta_2 - r_1\sin\theta_1)^2 \\ &= \left(r_2^2\cos^2\theta_2 - 2r_1r_2\cos\theta_1\cos\theta_2 + r_1^2\cos^2\theta_1\right) + \left(r_2^2\sin^2\theta_2 - 2r_1r_2\sin\theta_1\sin\theta_2 + r_1^2\sin^2\theta_1\right) \\ &= r_1^2\left(\sin^2\theta_1 + \cos^2\theta_1\right) + r_2^2\left(\sin^2\theta_2 + \cos^2\theta_2\right) - 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) \\ &= r_1^2 - 2r_1r_2\cos(\theta_1 - \theta_2) + r_2^2,\end{aligned}$$

so the distance between them is $\sqrt{r_1^2 - 2r_1r_2\cos(\theta_1 - \theta_2) + r_2^2}$.

15. $r^2 = 5 \quad\Leftrightarrow\quad x^2 + y^2 = 5$, a circle of radius $\sqrt{5}$ centered at the origin.

16. $r = 4\sec\theta \quad\Leftrightarrow\quad \dfrac{r}{\sec\theta} = 4 \quad\Leftrightarrow\quad r\cos\theta = 4 \quad\Leftrightarrow\quad x = 4$, a vertical line.

17. $r = 5\cos\theta \quad\Rightarrow\quad r^2 = 5r\cos\theta \quad\Leftrightarrow\quad x^2 + y^2 = 5x \quad\Leftrightarrow\quad x^2 - 5x + \frac{25}{4} + y^2 = \frac{25}{4} \quad\Leftrightarrow\quad \left(x - \frac{5}{2}\right)^2 + y^2 = \frac{25}{4}$, a circle of radius $\frac{5}{2}$ centered at $\left(\frac{5}{2}, 0\right)$. The first two equations are actually equivalent since $r^2 = 5r\cos\theta \quad\Rightarrow\quad r(r - 5\cos\theta) = 0 \quad\Rightarrow\quad r = 0$ or $r = 5\cos\theta$. But $r = 5\cos\theta$ gives the point $r = 0$ (the pole) when $\theta = 0$. Thus, the equation $r = 5\cos\theta$ is equivalent to the compound condition ($r = 0$ or $r = 5\cos\theta$).

18. $\theta = \dfrac{\pi}{3} \quad\Rightarrow\quad \tan\theta = \tan\dfrac{\pi}{3} \quad\Rightarrow\quad \dfrac{y}{x} = \sqrt{3} \quad\Leftrightarrow\quad y = \sqrt{3}\,x$, a line through the origin.

19. $r^2\cos 2\theta = 1 \quad\Leftrightarrow\quad r^2(\cos^2\theta - \sin^2\theta) = 1 \quad\Leftrightarrow\quad (r\cos\theta)^2 - (r\sin\theta)^2 = 1 \quad\Leftrightarrow\quad x^2 - y^2 = 1$, a hyperbola centered at the origin with foci on the x-axis.

20. $r^2\sin 2\theta = 1 \quad\Leftrightarrow\quad r^2(2\sin\theta\,\cos\theta) = 1 \quad\Leftrightarrow\quad 2(r\cos\theta)(r\sin\theta) = 1 \quad\Leftrightarrow\quad 2xy = 1 \quad\Leftrightarrow\quad xy = \frac{1}{2}$, a hyperbola centered at the origin with foci on the line $y = x$.

21. $x^2 + y^2 = 7 \quad \Rightarrow \quad (r\cos\theta)^2 + (r\sin\theta)^2 = 7 \quad \Rightarrow \quad r^2(\cos^2\theta + \sin^2\theta) = 7 \quad \Rightarrow \quad r^2 = 7 \quad \Rightarrow \quad r = \sqrt{7}$.
Note that $r = -\sqrt{7}$ produces the same curve as $r = \sqrt{7}$.

22. $x = -1 \quad \Rightarrow \quad r\cos\theta = -1 \quad \Rightarrow \quad r = -\dfrac{1}{\cos\theta} = -\sec\theta$

23. $y = \sqrt{3}\,x \quad \Rightarrow \quad \dfrac{y}{x} = \sqrt{3} \quad [x \neq 0] \quad \Rightarrow \quad \tan\theta = \sqrt{3} \quad \Rightarrow \quad \theta = \dfrac{\pi}{3}$ or $\dfrac{4\pi}{3}$ [either incudes the pole]

24. $y = -2x^2 \quad \Rightarrow \quad r\sin\theta = -2(r\cos\theta)^2 \quad \Rightarrow \quad r\sin\theta + 2r^2\cos^2\theta = 0 \quad \Rightarrow \quad r(\sin\theta + 2r\cos^2\theta) = 0 \quad \Rightarrow$
$r = 0$ or $r = -\dfrac{\sin\theta}{2\cos^2\theta} = -\dfrac{1}{2}\tan\theta\sec\theta$. $r = 0$ is included in $r = -\dfrac{1}{2}\tan\theta\sec\theta$ when $\theta = 0$, so the curve is represented by the single equation $r = -\frac{1}{2}\tan\theta\sec\theta$.

25. $x^2 + y^2 = 4y \quad \Rightarrow \quad r^2 = 4r\sin\theta \quad \Rightarrow \quad r^2 - 4r\sin\theta = 0 \quad \Rightarrow \quad r(r - 4\sin\theta) = 0 \quad \Rightarrow \quad r = 0$ or $r = 4\sin\theta$.
$r = 0$ is included in $r = 4\sin\theta$ when $\theta = 0$, so the curve is represented by the single equation $r = 4\sin\theta$.

26. $x^2 - y^2 = 4 \quad \Leftrightarrow \quad (r\cos\theta)^2 - (r\sin\theta)^2 = 4 \quad \Leftrightarrow \quad r^2\cos^2\theta - r^2\sin^2\theta = 4 \quad \Leftrightarrow \quad r^2(\cos^2\theta - \sin^2\theta) = 4 \quad \Leftrightarrow$
$r^2\cos 2\theta = 4$

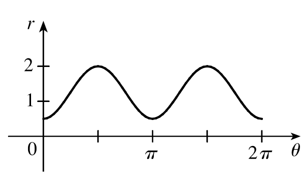
27. (a) The description leads immediately to the polar equation $\theta = \frac{\pi}{6}$, and the Cartesian equation $y = \tan\left(\frac{\pi}{6}\right)x = \frac{1}{\sqrt{3}}\,x$ is slightly more difficult to derive.

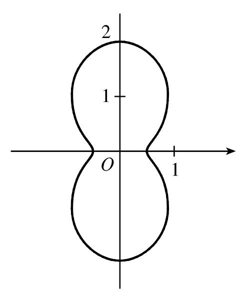
(b) The easier description here is the Cartesian equation $x = 3$.

28. (a) Because its center is not at the origin, it is more easily described by its Cartesian equation, $(x-2)^2 + (y-3)^2 = 5^2$.

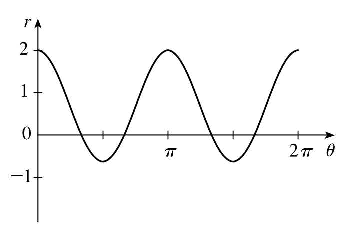
(b) This circle is more easily given in polar coordinates: $r = 4$. The Cartesian equation is also simple: $x^2 + y^2 = 16$.

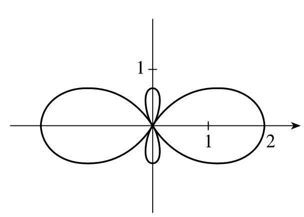
29. For $\theta = 0$, π, and 2π, r has its minimum value of about 0.5. For $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, r attains its maximum value of 2. We see that the graph has a similar shape for $0 \le \theta \le \pi$ and $\pi \le \theta \le 2\pi$.



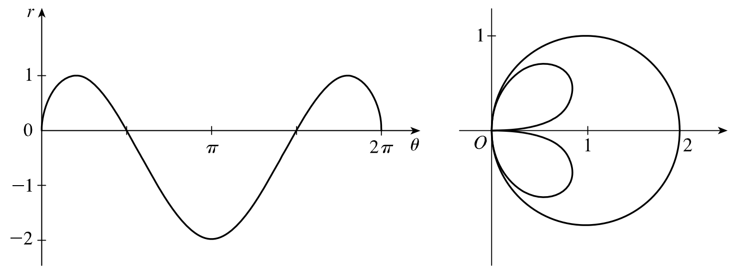


30. For $\theta = 0$, π, and 2π, r has its maximum value of 2. For $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, r has its minimum value of about -0.7.
Also, $r = 0$ for what appears to be $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, and $\frac{5\pi}{3}$, so the curve passes through the pole at these angles. r is negative in the intervals $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$ and $\left[\frac{4\pi}{3}, \frac{5\pi}{3}\right]$, so the curve will lie on the opposite side of the pole. The graph has a similar shape for $0 \le \theta \le \pi$ and $\pi \le \theta \le 2\pi$.

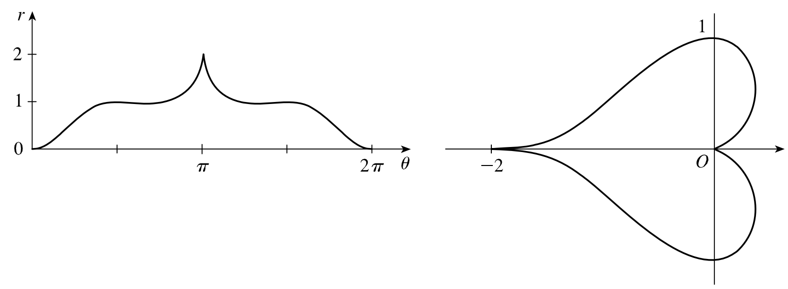




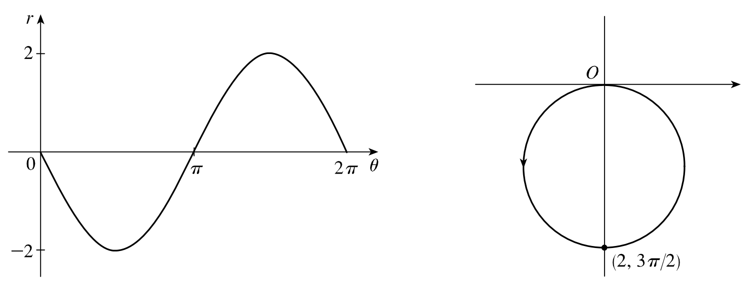
31. r has a maximum value of approximately 1 slightly before $\theta = \frac{\pi}{4}$ and slightly after $\theta = \frac{7\pi}{4}$. r has a minimum value of -2 when $\theta = \pi$. The graph touches the pole ($r = 0$) when $\theta = 0, \frac{\pi}{2}, \frac{3\pi}{2}$, and 2π. Since r is positive in the θ-intervals $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$, and negative in the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, the graph lies entirely in the first and fourth quadrants.



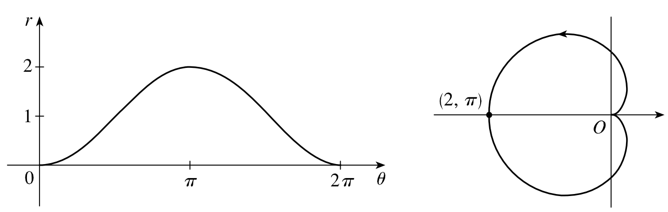
32. r increases from 0 to 1 (local max) in the interval $\left[0, \frac{\pi}{2}\right]$. It then decreases slightly, after which r increases to a maximum of 2 at $\theta = \pi$. The graph is symmetric about $\theta = \pi$, so the polar curve is symmetric about the polar axis.



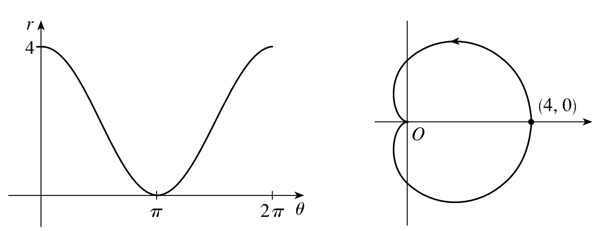
33. $r = -2\sin\theta$



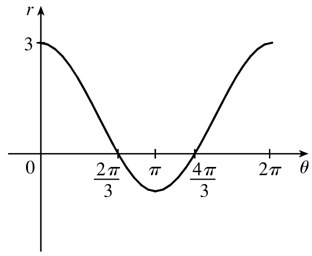
34. $r = 1 - \cos\theta$

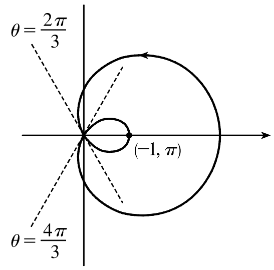


35. $r = 2(1 + \cos\theta)$

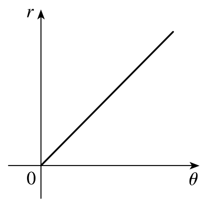


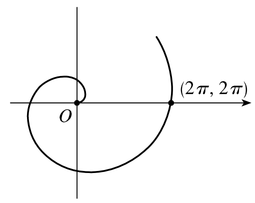
36. $r = 1 + 2\cos\theta$



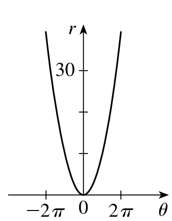


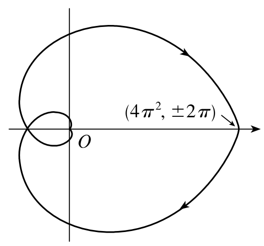
37. $r = \theta, \quad \theta \geq 0$



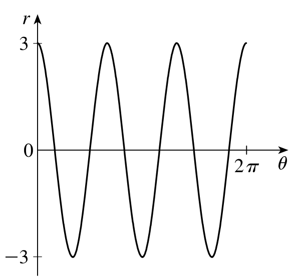


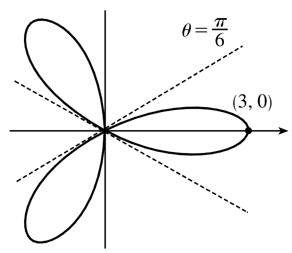
38. $r = \theta^2, \ -2\pi \leq \theta \leq 2\pi$



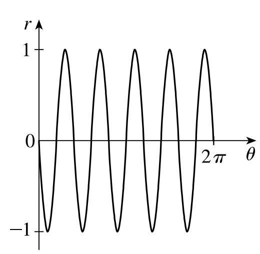


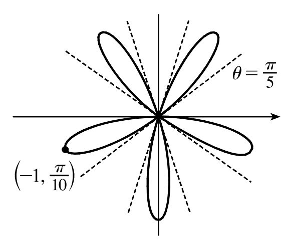
39. $r = 3\cos 3\theta$



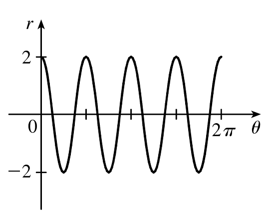


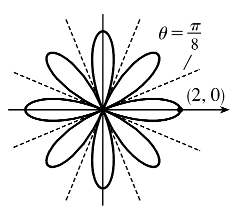
40. $r = -\sin 5\theta$



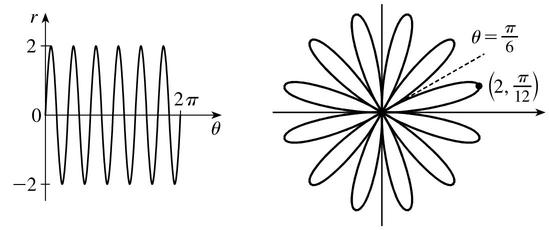


41. $r = 2\cos 4\theta$

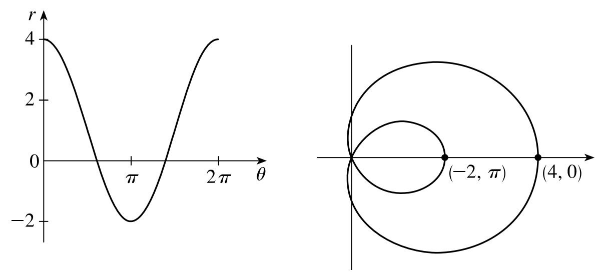




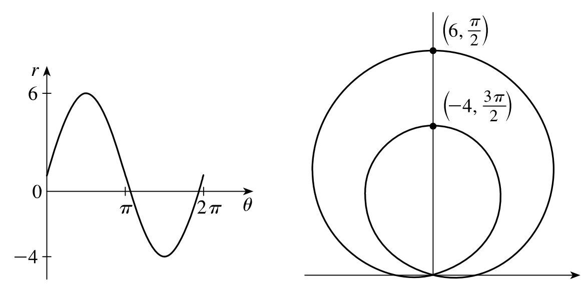
42. $r = 2\sin 6\theta$



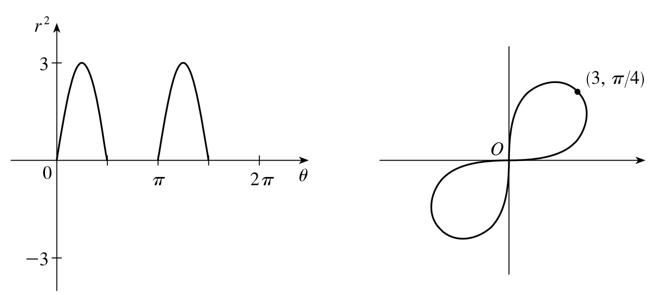
43. $r = 1 + 3\cos\theta$



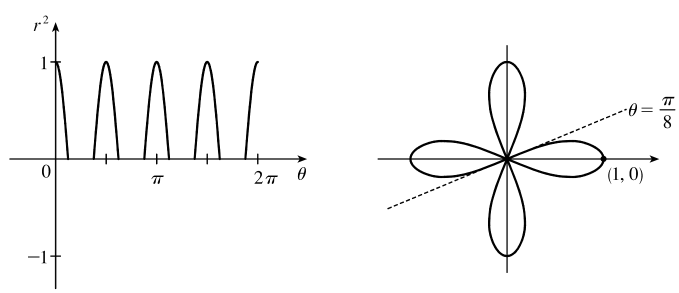
44. $r = 1 + 5\sin\theta$



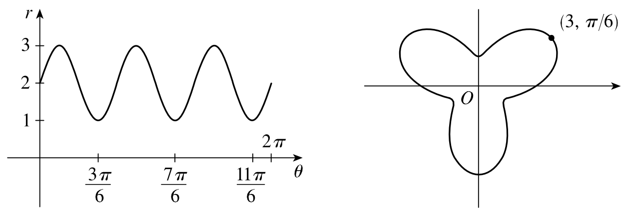
45. $r^2 = 9\sin 2\theta$



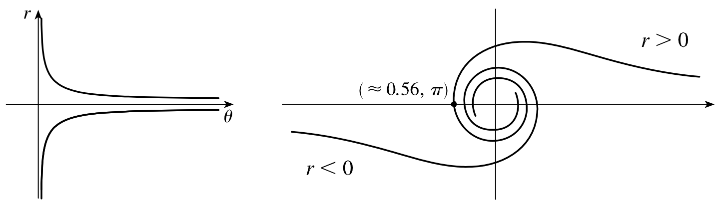
46. $r^2 = \cos 4\theta$



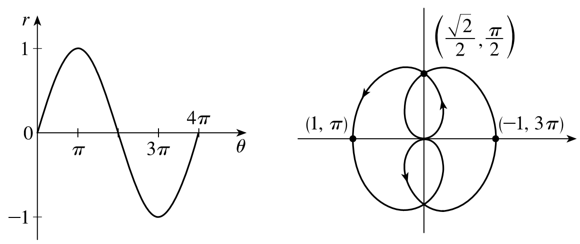
47. $r = 2 + \sin 3\theta$



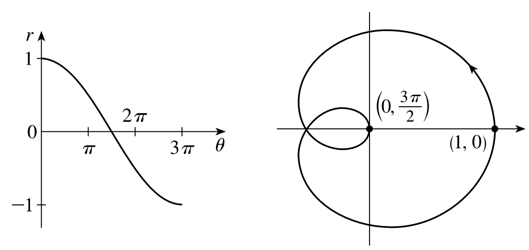
48. $r^2\theta = 1 \quad \Leftrightarrow \quad r = \pm 1/\sqrt{\theta}$ for $\theta > 0$



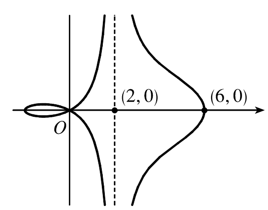
49. $r = \sin(\theta/2)$



50. $r = \cos(\theta/3)$

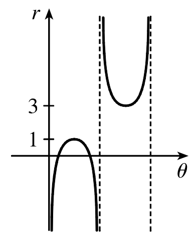


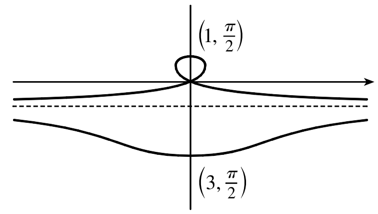
51. $x = r\cos\theta = (4 + 2\sec\theta)\cos\theta = 4\cos\theta + 2$. Now, $r \to \infty \quad \Rightarrow$ $(4 + 2\sec\theta) \to \infty \quad \Rightarrow \quad \theta \to \left(\frac{\pi}{2}\right)^-$ or $\theta \to \left(\frac{3\pi}{2}\right)^+$ [since we need only consider $0 \le \theta < 2\pi$], so $\lim\limits_{r\to\infty} x = \lim\limits_{\theta\to\pi/2^-} (4\cos\theta + 2) = 2$. Also, $r \to -\infty \quad \Rightarrow \quad (4 + 2\sec\theta) \to -\infty \quad \Rightarrow \quad \theta \to \left(\frac{\pi}{2}\right)^+$ or $\theta \to \left(\frac{3\pi}{2}\right)^-$, so $\lim\limits_{r\to-\infty} x = \lim\limits_{\theta\to\pi/2^+} (4\cos\theta + 2) = 2$. Therefore, $\lim\limits_{r\to\pm\infty} x = 2 \quad \Rightarrow \quad x = 2$ is a vertical asymptote.



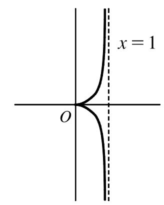
52. $y = r\sin\theta = (2 - \csc\theta)\cdot\sin\theta = 2\sin\theta - 1$. $r \to \infty \quad \Rightarrow \quad (2 - \csc\theta) \to \infty \quad \Rightarrow \quad \csc\theta \to -\infty \quad \Rightarrow \quad \theta \to \pi^+$ [since we need only consider $0 \le \theta < 2\pi$] and so $\lim\limits_{r\to\infty} y = \lim\limits_{\theta\to\pi^+} 2\sin\theta - 1 = -1$. Also $r \to -\infty \quad \Rightarrow$

$(2 - \csc\theta) \to -\infty \quad\Rightarrow\quad \csc\theta \to \infty \quad\Rightarrow\quad \theta \to \pi^-$ and so $\lim\limits_{r\to-\infty} x = \lim\limits_{\theta\to\pi^-} 2\sin\theta - 1 = -1$. Therefore $\lim\limits_{r\to\pm\infty} y = -1 \quad\Rightarrow\quad y = -1$ is a horizontal asymptote.



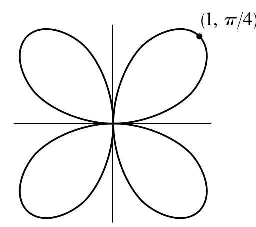


53. To show that $x = 1$ is an asymptote, we must prove $\lim\limits_{r\to\pm\infty} x = 1$.



$x = (r)\cos\theta = (\sin\theta\,\tan\theta)\cos\theta = \sin^2\theta$. Now, $r \to \infty \quad\Rightarrow\quad \sin\theta\,\tan\theta \to \infty \quad\Rightarrow$ $\theta \to \left(\frac{\pi}{2}\right)^-$, so $\lim\limits_{r\to\infty} x = \lim\limits_{\theta\to\pi/2^-} \sin^2\theta = 1$. Also, $r \to -\infty \quad\Rightarrow\quad \sin\theta\,\tan\theta \to -\infty \quad\Rightarrow$ $\theta \to \left(\frac{\pi}{2}\right)^+$, so $\lim\limits_{r\to-\infty} x = \lim\limits_{\theta\to\pi/2^+} \sin^2\theta = 1$. Therefore, $\lim\limits_{r\to\pm\infty} x = 1 \quad\Rightarrow\quad x = 1$ is a vertical asymptote. Also notice that $x = \sin^2\theta \ge 0$ for all θ, and $x = \sin^2\theta \le 1$ for all θ. And $x \ne 1$, since the curve is not defined at odd multiples of $\frac{\pi}{2}$. Therefore, the curve lies entirely within the vertical strip $0 \le x < 1$.

54. The equation is $(x^2 + y^2)^3 = 4x^2y^2$, but using polar coordinates we know that $x^2 + y^2 = r^2$ and $x = r\cos\theta$ and $y = r\sin\theta$. Substituting into the given equation: $r^6 = 4r^2\cos^2\theta\, r^2\sin^2\theta \quad\Rightarrow\quad r^2 = 4\cos^2\theta\,\sin^2\theta \quad\Rightarrow$ $r = \pm 2\cos\theta\,\sin\theta = \pm\sin 2\theta$. $r = \pm\sin 2\theta$ is sketched at right.



55. (a) We see that the curve $r = 1 + c\sin\theta$ crosses itself at the origin, where $r = 0$ (in fact the inner loop corresponds to negative r-values,) so we solve the equation of the limaçon for $r = 0 \quad\Leftrightarrow\quad c\sin\theta = -1 \quad\Leftrightarrow\quad \sin\theta = -1/c$. Now if $|c| < 1$, then this equation has no solution and hence there is no inner loop. But if $c < -1$, then on the interval $(0, 2\pi)$ the equation has the two solutions $\theta = \sin^{-1}(-1/c)$ and $\theta = \pi - \sin^{-1}(-1/c)$, and if $c > 1$, the solutions are $\theta = \pi + \sin^{-1}(1/c)$ and $\theta = 2\pi - \sin^{-1}(1/c)$. In each case, $r < 0$ for θ between the two solutions, indicating a loop.

(b) For $0 < c < 1$, the dimple (if it exists) is characterized by the fact that y has a local maximum at $\theta = \frac{3\pi}{2}$. So we determine for what c-values $\dfrac{d^2y}{d\theta^2}$ is negative at $\theta = \frac{3\pi}{2}$, since by the Second Derivative Test this indicates a maximum:

$$y = r\sin\theta = \sin\theta + c\sin^2\theta \quad\Rightarrow\quad \frac{dy}{d\theta} = \cos\theta + 2c\sin\theta\,\cos\theta = \cos\theta + c\sin 2\theta \quad\Rightarrow\quad \frac{d^2y}{d\theta^2} = -\sin\theta + 2c\cos 2\theta.$$

At $\theta = \frac{3\pi}{2}$, this is equal to $-(-1) + 2c(-1) = 1 - 2c$, which is negative only for $c > \frac{1}{2}$. A similar argument shows that for $-1 < c < 0$, y only has a local minimum at $\theta = \frac{\pi}{2}$ (indicating a dimple) for $c < -\frac{1}{2}$.

56. (a) The graph of $r = \cos 3\theta$ is a three-leaved rose, which is graph II.

(b) $r = \ln\theta,\ 1 \le \theta \le 6\pi$. r increases as θ increases and there are almost three full revolutions. The graph must be either III or VI. As θ increases, r grows slowly in VI and quickly in III. Since $r = \ln\theta$ grows slowly, its graph must be VI.

(c) $r = \cos(\theta/2)$. For $\theta = 0$, $r = 1$, and as θ increases to π, r decreases to 0. Only graph V satisfies those values.

(d) $r = \cos(\theta/3)$. For $\theta = 0$, $r = 1$ and as θ increases to $3\pi/2$, $\theta/3 \to \pi/2$, and r decreases to 0. Only graph IX satisfies those values.

(e) $r = \sec(\theta/3)$. The secant function is never equal to zero, so this graph never intersects the pole. As $\theta \to 3\pi/2$, $\theta/3 \to \pi/2$, and $r \to \infty$. Only graph VII has these properties.

(f) $r = \sec\theta \quad \Rightarrow \quad r\cos\theta = 1 \quad \Rightarrow \quad x = 1$. This is the graph of a vertical line, so it must be graph VIII.

(g) $r = \theta^2,\ 0 \le \theta \le 8\pi$. See part (b). This is graph III.

(h) Since $-1 \le \cos 3\theta \le 1$, $1 \le 2 + \cos 3\theta \le 3$, so $r = 2 + \cos 3\theta$ is never 0; that is, the curve never intersects the pole. The graph must be I or IV. For $0 \le \theta \le 2\pi$, the graph assumes its minimum r-value of 1 three times, at $\theta = \frac{\pi}{3}$, π, and $\frac{5\pi}{3}$, so it must be graph IV.

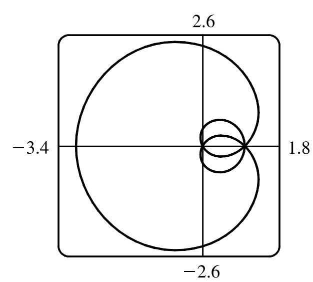
(i) $r = 2 + \cos(3\theta/2)$. As in part (h), this graph never intersects the pole, so it must be graph I.

57. $r = a\sin\theta + b\cos\theta \quad \Rightarrow \quad r^2 = ar\sin\theta + br\cos\theta \quad \Rightarrow \quad x^2 + y^2 = ay + bx \quad \Rightarrow$

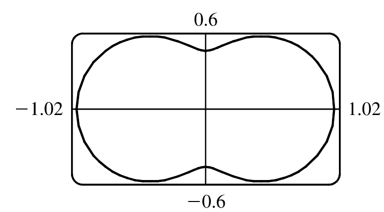
$x^2 - bx + \left(\frac{1}{2}b\right)^2 + y^2 - ay + \left(\frac{1}{2}a\right)^2 = \left(\frac{1}{2}b\right)^2 + \left(\frac{1}{2}a\right)^2 \quad \Rightarrow \quad \left(x - \frac{1}{2}b\right)^2 + \left(y - \frac{1}{2}a\right)^2 = \frac{1}{4}(a^2 + b^2)$, and this is a circle with center $\left(\frac{1}{2}b, \frac{1}{2}a\right)$ and radius $\frac{1}{2}\sqrt{a^2 + b^2}$.

58. These curves are circles which intersect at the origin and at $\left(\frac{1}{\sqrt{2}}a, \frac{\pi}{4}\right)$. At the origin, the first circle has a horizontal tangent and the second a vertical one, so the tangents are perpendicular here. For the first circle [$r = a\sin\theta$], $dy/d\theta = a\cos\theta\,\sin\theta + a\sin\theta\,\cos\theta = a\sin 2\theta = a$ at $\theta = \frac{\pi}{4}$ and $dx/d\theta = a\cos^2\theta - a\sin^2\theta = a\cos 2\theta = 0$ at $\theta = \frac{\pi}{4}$, so the tangent here is vertical. Similarly, for the second circle [$r = a\cos\theta$], $dy/d\theta = a\cos 2\theta = 0$ and $dx/d\theta = -a\sin 2\theta = -a$ at $\theta = \frac{\pi}{4}$, so the tangent is horizontal, and again the tangents are perpendicular.

59. $r = 1 + 2\sin(\theta/2)$. The parameter interval is $[0, 4\pi]$.

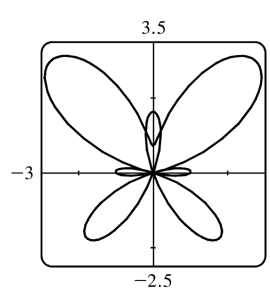


60. $r = \sqrt{1 - 0.8\sin^2\theta}$. The parameter interval is $[0, 2\pi]$.



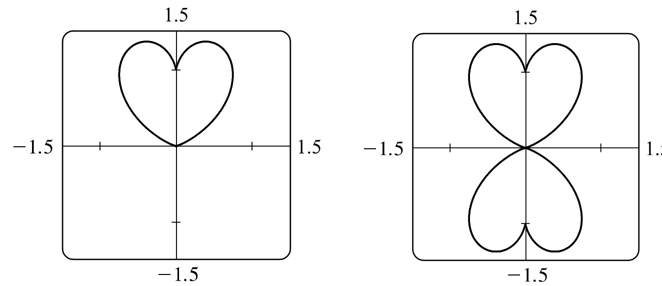
61. $r = e^{\sin\theta} - 2\cos(4\theta)$.

The parameter interval is $[0, 2\pi]$.

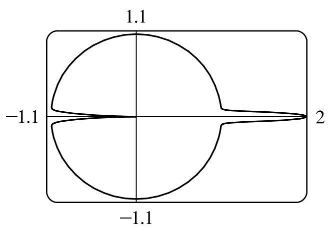


62. $r = |\tan\theta|^{|\cot\theta|}$. The parameter interval $[0, \pi]$ produces the heart-shaped valentine curve shown in the first window.

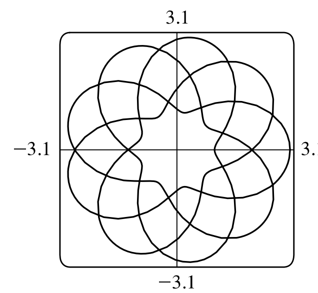
The complete curve, including the reflected heart, is produced by the parameter interval $[0, 2\pi]$, but perhaps you'll agree that the first curve is more appropriate.



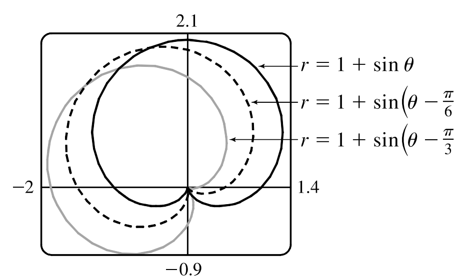
63. $r = 1 + \cos^{999}\theta$. The parameter interval is $[0, 2\pi]$.



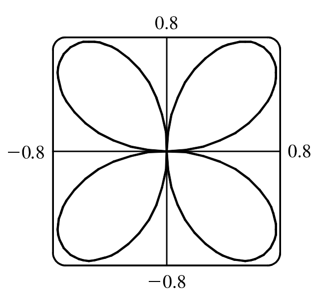
64. $r = 2 + \cos(9\theta/4)$. The parameter interval is $[0, 8\pi]$.



65. It appears that the graph of $r = 1 + \sin\left(\theta - \frac{\pi}{6}\right)$ is the same shape as the graph of $r = 1 + \sin\theta$, but rotated counterclockwise about the origin by $\frac{\pi}{6}$. Similarly, the graph of $r = 1 + \sin\left(\theta - \frac{\pi}{3}\right)$ is rotated by $\frac{\pi}{3}$. In general, the graph of $r = f(\theta - \alpha)$ is the same shape as that of $r = f(\theta)$, but rotated counterclockwise through α about the origin. That is, for any point (r_0, θ_0) on the curve $r = f(\theta)$, the point $(r_0, \theta_0 + \alpha)$ is on the curve $r = f(\theta - \alpha)$, since $r_0 = f(\theta_0) = f((\theta_0 + \alpha) - \alpha)$.



66.



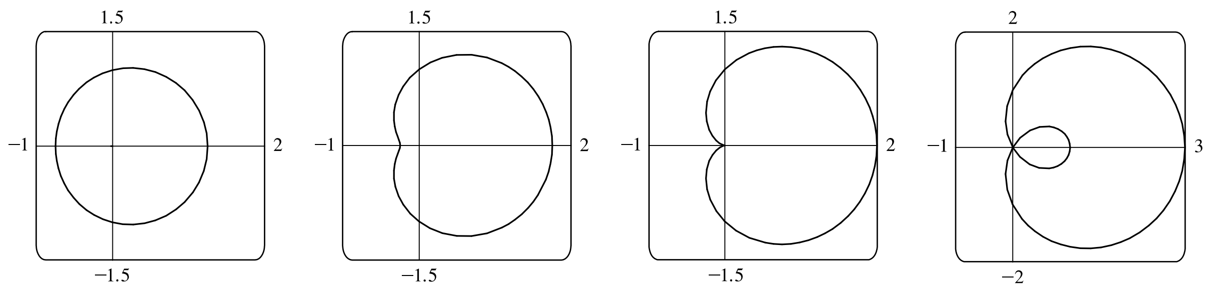
From the graph, the highest points seem to have $y \approx 0.77$. To find the exact value, we solve $dy/d\theta = 0$. $y = r\sin\theta = \sin\theta\,\sin 2\theta \quad \Rightarrow$

$$\begin{aligned} dy/d\theta &= 2\sin\theta\,\cos 2\theta + \cos\theta\,\sin 2\theta \\ &= 2\sin\theta\,(2\cos^2\theta - 1) + \cos\theta\,(2\sin\theta\,\cos\theta) \\ &= 2\sin\theta\,(3\cos^2\theta - 1) \end{aligned}$$

In the first quadrant, this is 0 when $\cos\theta = \frac{1}{\sqrt{3}} \quad \Leftrightarrow \quad \sin\theta = \sqrt{\frac{2}{3}} \quad \Leftrightarrow$

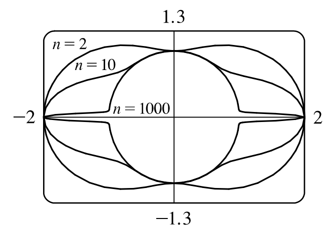
$y = 2\sin^2\theta\cos\theta = 2 \cdot \frac{2}{3} \cdot \frac{1}{\sqrt{3}} = \frac{4}{9}\sqrt{3} \approx 0.77.$

67. Consider curves with polar equation $r = 1 + c\cos\theta$, where c is a real number. If $c = 0$, we get a circle of radius 1 centered at the pole. For $0 < c \leq 0.5$, the curve gets slightly larger, moves right, and flattens out a bit on the left side. For $0.5 < c < 1$, the left side has a dimple shape. For $c = 1$, the dimple becomes a cusp. For $c > 1$, there is an internal loop. For $c \geq 0$, the rightmost point on the curve is $(1 + c, 0)$. For $c < 0$, the curves are reflections through the vertical axis of the curves with $c > 0$.

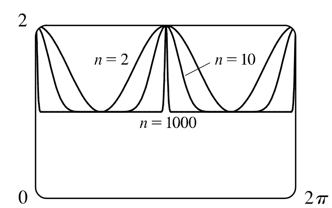


$c = 0.25$ $c = 0.75$ $c = 1$ $c = 2$

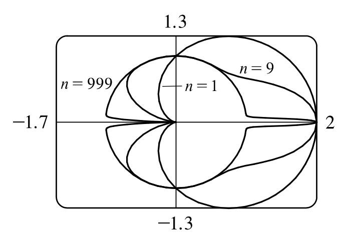
68. Consider the polar curves $r = 1 + \cos^n \theta$, where n is a positive integer. First, let $\boldsymbol{n}$ **be an even positive integer**. The first figure shows that the curve has a peanut shape for $n = 2$, but as n increases, the ends are squeezed. As n becomes large, the curves look more and more like the unit circle, but with spikes to the points $(2, 0)$ and $(2, \pi)$.



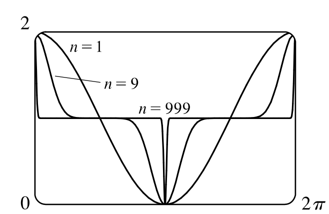
The second figure shows r as a function of θ in Cartesian coordinates for the same values of n. We can see that for large n, the graph is similar to the graph of $y = 1$, but with spikes to $y = 2$ for $x = 0$, π, and 2π. (Note that when $0 < \cos\theta < 1$, $\cos^{1000}\theta$ is very small.)



Next, let $\boldsymbol{n}$ **be an odd positive integer**. The third figure shows that the curve is a cardioid for $n = 1$, but as n increases, the heart shape becomes more pronounced. As n becomes large, the curves again look more like the unit circle, but with an outward spike to $(2, 0)$ and an inward spike to $(0, \pi)$.

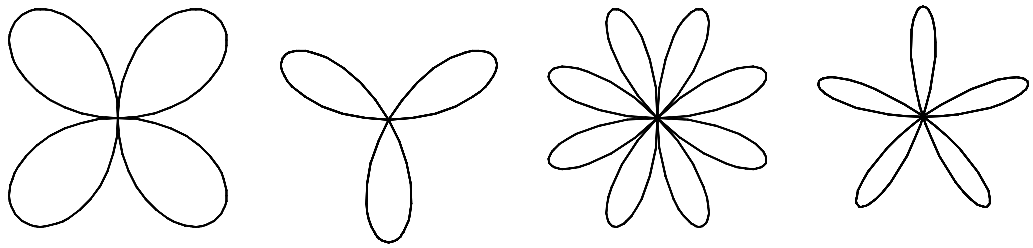


The fourth figure shows r as a function of θ in Cartesian coordinates for the same values of n. We can see that for large n, the graph is similar to the graph of $y = 1$, but spikes to $y = 2$ for $x = 0$ and π, and to $y = 0$ for $x = \pi$.



DISCOVERY PROJECT Families of Polar Curves

1. (a) $r = \sin n\theta$.

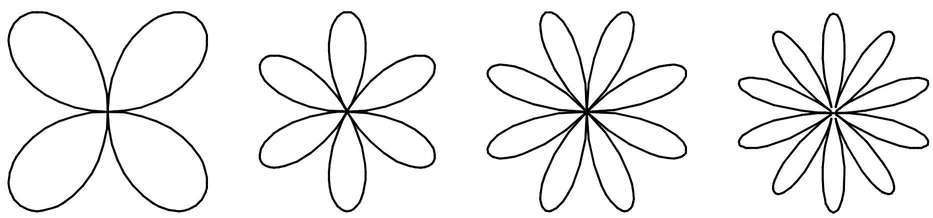


$n = 2$ $n = 3$ $n = 4$ $n = 5$

From the graphs, it seems that when n is even, the number of loops in the curve (called a rose) is $2n$, and when n is odd, the number of loops is simply n. This is because in the case of n odd, every point on the graph is traversed twice, due to the fact that

$$r(\theta + \pi) = \sin[n(\theta + \pi)] = \sin n\theta \, \cos n\pi + \cos n\theta \, \sin n\pi = \begin{cases} \sin n\theta & \text{if } n \text{ is even} \\ -\sin n\theta & \text{if } n \text{ is odd} \end{cases}$$

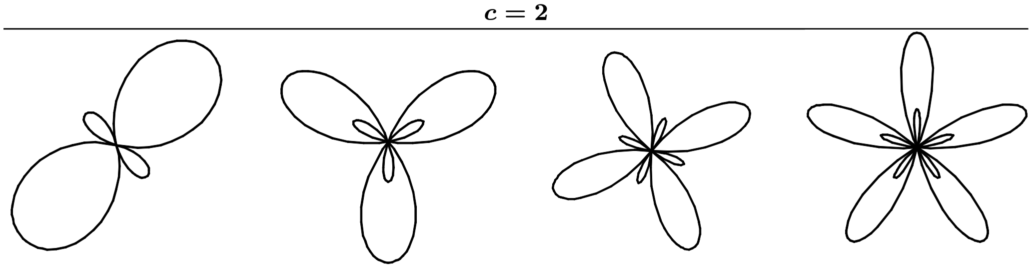
(b) The graph of $r = |\sin n\theta|$ has $2n$ loops whether n is odd or even, since $r(\theta + \pi) = r(\theta)$.



$n = 2$ $n = 3$ $n = 4$ $n = 5$

2. $r = 1 + c \sin n\theta$. We vary n while keeping c constant at 2. As n changes, the curves change in the same way as those in Exercise 1: the number of loops increases. Note that if n is even, the smaller loops are outside the larger ones; if n is odd, they are inside.

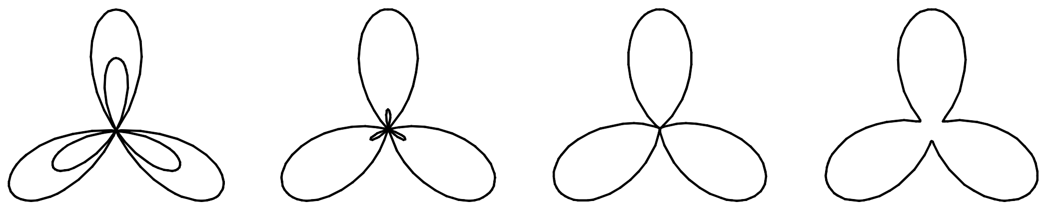
$\boldsymbol{c = 2}$



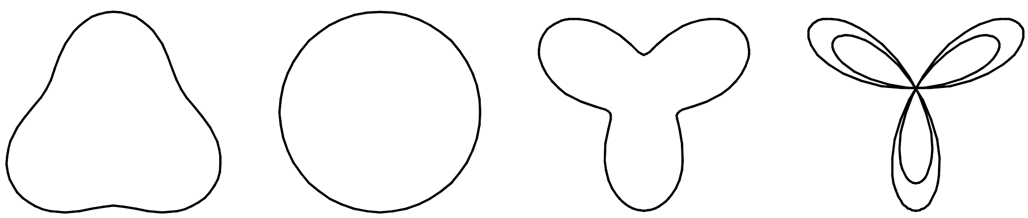
$n = 2$ $n = 3$ $n = 4$ $n = 5$

Now we vary c while keeping $n = 3$. As c increases toward 0, the entire graph gets smaller (the graphs below are not to scale) and the smaller loops shrink in relation to the large ones. At $c = -1$, the small loops disappear entirely, and for $-1 < c < 1$, the graph is a simple, closed curve (at $c = 0$ it is a circle). As c continues to increase, the same changes are seen, but in reverse order, since $1 + (-c) \sin n\theta = 1 + c \sin n(\theta + \pi)$, so the graph for $c = c_0$ is the same as that for $c = -c_0$, with a rotation through π. As $c \to \infty$, the smaller loops get relatively closer in size to the large ones. Note that the distance between the outermost points of corresponding inner and outer loops is always 2. Maple's `animate` command (or Mathematica's `Animate`) is very useful for seeing the changes that occur as c varies.

$\boldsymbol{n = 3}$



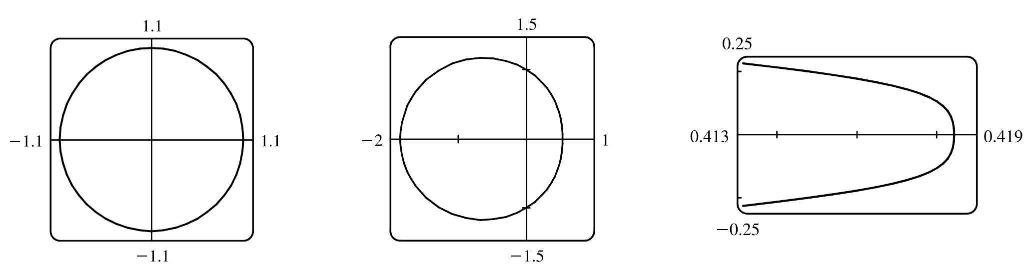
$c = -4$ $c = -1.4$ $c = -1$ $c = -0.8$



$c = -0.2$ $c = 0$ $c = 0.5$ $c = 8$

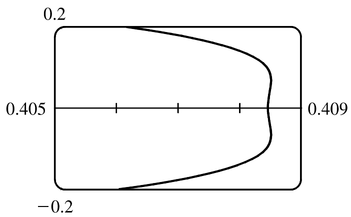
3. $r = \dfrac{1 - a\cos\theta}{1 + a\cos\theta}$. We start with $a = 0$, since in this case the curve is simply the circle $r = 1$.

As a increases, the graph moves to the left, and its right side becomes flattened. As a increases through about 0.4, the right side seems to grow a dimple, which upon closer investigation (with narrower θ-ranges) seems to appear at $a \approx 0.42$ [the actual value is $\sqrt{2} - 1$]. As $a \to 1$, this dimple becomes more pronounced, and the curve begins to stretch out horizontally, until at $a = 1$ the denominator vanishes at $\theta = \pi$, and the dimple becomes an actual cusp. For $a > 1$ we must choose our parameter interval carefully, since $r \to \infty$ as $1 + a\cos\theta \to 0 \quad \Leftrightarrow \quad \theta \to \pm\cos^{-1}(-1/a)$. As a increases from 1, the curve splits into two parts. The left part has a loop, which grows larger as a increases, and the right part grows broader vertically, and its left tip develops a dimple when $a \approx 2.42$ [actually, $\sqrt{2} + 1$]. As a increases, the dimple grows more and more pronounced. If $a < 0$, we get the same graph as we do for the corresponding positive a-value, but with a rotation through π about the pole, as happened when c was replaced with $-c$ in Exercise 2.

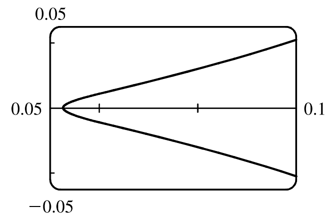


$a = 0$ $a = 0.3$ $a = 0.41, |\theta| \le 0.5$

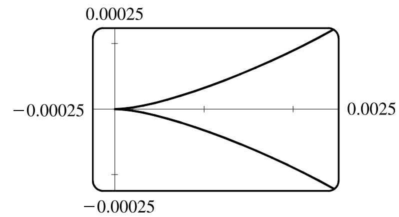
[continued]



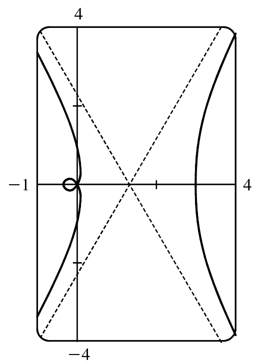
$a = 0.42, |\theta| \leq 0.5$



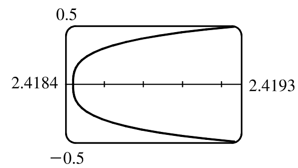
$a = 0.9, |\theta| \leq 0.5$



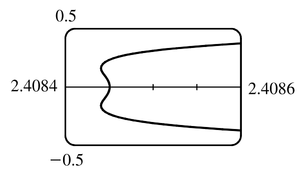
$a = 1, |\theta| \leq 0.1$



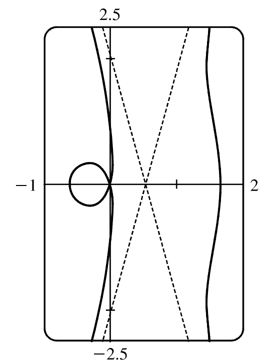
$a = 2$



$a = 2.41, |\theta - \pi| \leq 0.2$



$a = 2.42, |\theta - \pi| \leq 0.2$



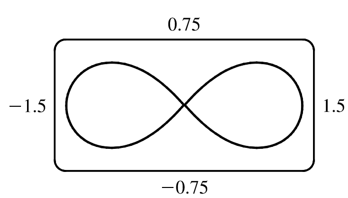
$a = 4$

4. Most graphing devices cannot plot implicit polar equations, so we must first find an explicit expression (or expressions) for r in terms of θ, a, and c. We note that the given equation, $r^4 - 2c^2r^2\cos 2\theta + c^4 - a^4 = 0$, is a quadratic in r^2, so we use the quadratic formula and find that

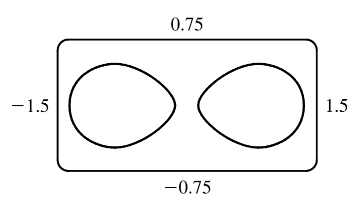
$$r^2 = \frac{2c^2\cos 2\theta \pm \sqrt{4c^4\cos^2 2\theta - 4(c^4 - a^4)}}{2} = c^2\cos 2\theta \pm \sqrt{a^4 - c^4\sin^2 2\theta}$$

so $r = \pm\sqrt{c^2\cos 2\theta \pm \sqrt{a^4 - c^4\sin^2 2\theta}}$. So for each graph, we must plot four curves to be sure of plotting all the points which satisfy the given equation. Note that all four functions have period π.

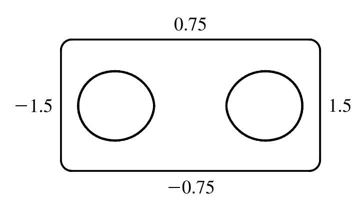
We start with the case $a = c = 1$, and the resulting curve resembles the symbol for infinity. If we let a decrease, the curve splits into two symmetric parts, and as a decreases further, the parts become smaller, further apart, and rounder. If instead we let a increase from 1, the two lobes of the curve join together, and as a increases further they continue to merge, until at $a \approx 1.4$, the graph no longer has dimples, and has an oval shape. As $a \to \infty$, the oval becomes larger and rounder, since the c^2 and c^4 terms lose their significance. Note that the shape of the graph seems to depend only on the ratio c/a, while the size of the graph varies as c and a jointly increase.



$(a, c) = (1, 1)$

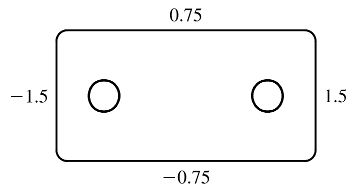


$(a, c) = (0.99, 1)$

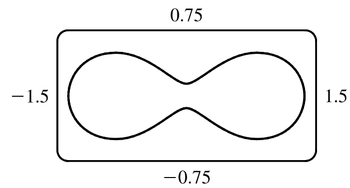


$(a, c) = (0.9, 1)$

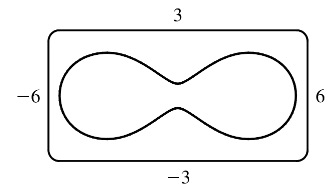
[continued]



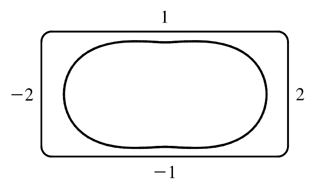
$(a, c) = (0.6, 1)$



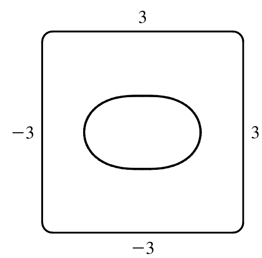
$(a, c) = (1.01, 1)$



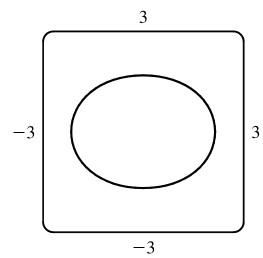
$(a, c) = (4.04, 4)$



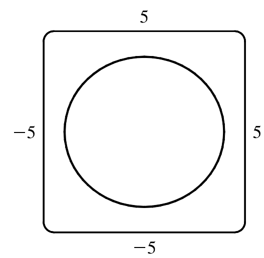
$(a, c) = (1.3, 1)$



$(a, c) = (1.5, 1)$



$(a, c) = (2, 1)$



$(a, c) = (4, 1)$

10.4 Calculus in Polar Coordinates

1. $r = \sqrt{2\theta},\ 0 \le \theta \le \pi/2.$

$$A = \int_0^{\pi/2} \frac{1}{2} r^2\, d\theta = \int_0^{\pi/2} \frac{1}{2}\left(\sqrt{2\theta}\right)^2 d\theta = \int_0^{\pi/2} \theta\, d\theta = \left[\frac{1}{2}\theta^2\right]_0^{\pi/2} = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{8}$$

2. $r = e^\theta,\ 3\pi/4 \le \theta \le 3\pi/2.$

$$A = \int_{3\pi/4}^{3\pi/2} \tfrac{1}{2} r^2\, d\theta = \int_{3\pi/4}^{3\pi/2} \tfrac{1}{2}(e^\theta)^2\, d\theta = \int_{3\pi/4}^{3\pi/2} \tfrac{1}{2} e^{2\theta}\, d\theta = \tfrac{1}{2}\left[\tfrac{1}{2}e^{2\theta}\right]_{3\pi/4}^{3\pi/2} = \tfrac{1}{4}(e^{3\pi} - e^{3\pi/2})$$

3. $r = \sin\theta + \cos\theta,\ 0 \le \theta \le \pi.$

$$A = \int_0^{\pi} \frac{1}{2} r^2\, d\theta = \int_0^{\pi} \frac{1}{2}(\sin\theta + \cos\theta)^2\, d\theta = \int_0^{\pi} \frac{1}{2}(\sin^2\theta + 2\sin\theta\,\cos\theta + \cos^2\theta)\, d\theta = \int_0^{\pi} \frac{1}{2}(1 + \sin 2\theta)\, d\theta$$
$$= \tfrac{1}{2}\left[\theta - \tfrac{1}{2}\cos 2\theta\right]_0^{\pi} = \tfrac{1}{2}\left[\left(\pi - \tfrac{1}{2}\right) - \left(0 - \tfrac{1}{2}\right)\right] = \tfrac{\pi}{2}$$

4. $r = 1/\theta,\ \pi/2 \le \theta \le 2\pi.$

$$A = \int_{\pi/2}^{2\pi} \frac{1}{2} r^2\, d\theta = \int_{\pi/2}^{2\pi} \frac{1}{2}\left(\frac{1}{\theta}\right)^2 d\theta = \int_{\pi/2}^{2\pi} \frac{1}{2}\theta^{-2}\, d\theta = \frac{1}{2}\left[-\frac{1}{\theta}\right]_{\pi/2}^{2\pi}$$
$$= \tfrac{1}{2}\left(-\tfrac{1}{2\pi} + \tfrac{2}{\pi}\right) = \tfrac{1}{2}\left(-\tfrac{1}{2\pi} + \tfrac{4}{2\pi}\right) = \tfrac{3}{4\pi}$$

5. $r^2 = \sin 2\theta,\ 0 \le \theta \le \pi/2.$

$$A = \int_0^{\pi/2} \tfrac{1}{2} r^2\, d\theta = \int_0^{\pi/2} \tfrac{1}{2}\sin 2\theta\, d\theta = \left[-\tfrac{1}{4}\cos 2\theta\right]_0^{\pi/2} = -\tfrac{1}{4}(\cos\pi - \cos 0) = -\tfrac{1}{4}(-1 - 1) = \tfrac{1}{2}$$

6. $r = 2 + \cos\theta,\ \pi/2 \le \theta \le \pi.$

$$A = \int_{\pi/2}^{\pi} \tfrac{1}{2} r^2\, d\theta = \int_{\pi/2}^{\pi} \tfrac{1}{2}(2 + \cos\theta)^2\, d\theta = \int_{\pi/2}^{\pi} \tfrac{1}{2}(4 + 4\cos\theta + \cos^2\theta)\, d\theta = \int_{\pi/2}^{\pi} \tfrac{1}{2}[4 + 4\cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)]\, d\theta$$
$$= \int_{\pi/2}^{\pi} \left(\tfrac{9}{4} + 2\cos\theta + \tfrac{1}{4}\cos 2\theta\right) d\theta = \left[\tfrac{9}{4}\theta + 2\sin\theta + \tfrac{1}{8}\sin 2\theta\right]_{\pi/2}^{\pi} = \left(\tfrac{9\pi}{4} + 0 + 0\right) - \left(\tfrac{9\pi}{8} + 2 + 0\right) = \tfrac{9\pi}{8} - 2$$

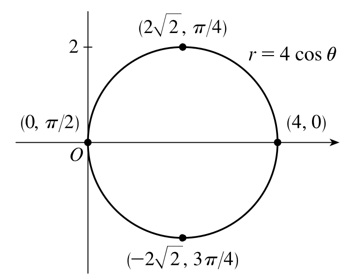
7. $r = 4 + 3\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

$$\begin{aligned} A &= \int_{-\pi/2}^{\pi/2} \tfrac{1}{2}(4+3\sin\theta)^2\,d\theta = \tfrac{1}{2}\int_{-\pi/2}^{\pi/2}(16+24\sin\theta+9\sin^2\theta)\,d\theta \\ &= \tfrac{1}{2}\int_{-\pi/2}^{\pi/2}(16+9\sin^2\theta)\,d\theta \qquad \text{[by Theorem 5.5.7]} \\ &= \tfrac{1}{2}\cdot 2\int_0^{\pi/2}\left[16+9\cdot\tfrac{1}{2}(1-\cos 2\theta)\right]d\theta \qquad \text{[by Theorem 5.5.7]} \\ &= \int_0^{\pi/2}\left(\tfrac{41}{2}-\tfrac{9}{2}\cos 2\theta\right)d\theta = \left[\tfrac{41}{2}\theta-\tfrac{9}{4}\sin 2\theta\right]_0^{\pi/2} = \left(\tfrac{41\pi}{4}-0\right)-(0-0) = \tfrac{41\pi}{4} \end{aligned}$$

8. $r = \sqrt{\ln\theta},\ 1 \le \theta \le 2\pi$.

$$\begin{aligned} A &= \int_1^{2\pi}\tfrac{1}{2}\left(\sqrt{\ln\theta}\right)^2 d\theta = \int_1^{2\pi}\tfrac{1}{2}\ln\theta\,d\theta = \left[\tfrac{1}{2}\theta\ln\theta\right]_1^{2\pi} - \int_1^{2\pi}\tfrac{1}{2}\,d\theta \qquad \begin{bmatrix} u=\ln\theta, & dv=\frac{1}{2}\,d\theta \\ du=(1/\theta)\,d\theta, & v=\frac{1}{2}\theta \end{bmatrix} \\ &= [\pi\ln(2\pi)-0] - \left[\tfrac{1}{2}\theta\right]_1^{2\pi} = \pi\ln(2\pi)-\pi+\tfrac{1}{2} \end{aligned}$$

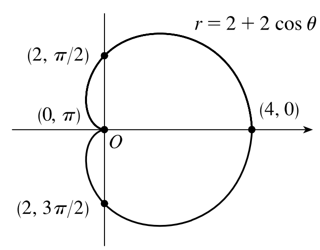
9. The area is bounded by $r = 4\cos\theta$ for $\theta = 0$ to $\theta = \pi$.



$$\begin{aligned} A &= \int_0^{\pi}\tfrac{1}{2}r^2\,d\theta = \int_0^{\pi}\tfrac{1}{2}(4\cos\theta)^2 d\theta = \int_0^{\pi} 8\cos^2\theta\,d\theta \\ &= 8\int_0^{\pi}\tfrac{1}{2}(1+\cos 2\theta)\,d\theta = 4\left[\theta+\tfrac{1}{2}\sin 2\theta\right]_0^{\pi} = 4\pi \end{aligned}$$

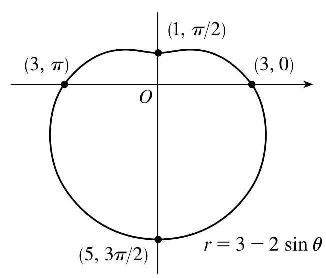
Also, note that this is a circle with radius 2, so its area is $\pi(2)^2 = 4\pi$.

10.



$$\begin{aligned} A &= \int_0^{2\pi}\tfrac{1}{2}r^2\,d\theta = \int_0^{2\pi}\tfrac{1}{2}(2+2\cos\theta)^2 d\theta \\ &= \int_0^{2\pi}\tfrac{1}{2}(4+8\cos\theta+4\cos^2\theta)\,d\theta \\ &= \int_0^{2\pi}\tfrac{1}{2}\left[4+8\cos\theta+4\cdot\tfrac{1}{2}(1+\cos 2\theta)\right]d\theta \\ &= \int_0^{2\pi}(3+4\cos\theta+\cos 2\theta)\,d\theta = \left[3\theta+4\sin\theta+\tfrac{1}{2}\sin 2\theta\right]_0^{2\pi} = 6\pi \end{aligned}$$

11.

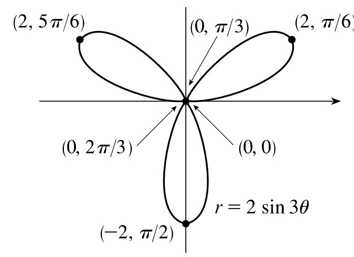


$$\begin{aligned} A &= \int_0^{2\pi}\tfrac{1}{2}r^2\,d\theta = \int_0^{2\pi}\tfrac{1}{2}(3-2\sin\theta)^2 d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi}(9-12\sin\theta+4\sin^2\theta)\,d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi}\left[9-12\sin\theta+4\cdot\tfrac{1}{2}(1-\cos 2\theta)\right]d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi}(11-12\sin\theta-2\cos 2\theta)\,d\theta = \tfrac{1}{2}\Big[11\theta+12\cos\theta-\sin 2\theta\Big]_0^{2\pi} \\ &= \tfrac{1}{2}[(22\pi+12)-12] = 11\pi \end{aligned}$$

12. The area is bounded by $r = 2\sin 3\theta$ for $\theta = 0$ to $\theta = \pi$.

$$A = \int_0^{\pi} \tfrac{1}{2} r^2\, d\theta = \int_0^{\pi} \tfrac{1}{2}(2\sin 3\theta)^2 d\theta = \int_0^{\pi} 2\sin^2 3\theta\, d\theta$$

$$= \int_0^{\pi} 2 \cdot \tfrac{1}{2}(1 - \cos 6\theta)\, d\theta = \left[\theta - \tfrac{1}{6}\sin 6\theta\right]_0^{\pi} = \pi$$

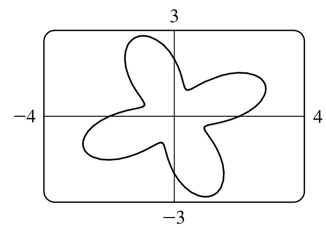


13. $A = \int_0^{2\pi} \tfrac{1}{2} r^2\, d\theta = \int_0^{2\pi} \tfrac{1}{2}(2 + \sin 4\theta)^2\, d\theta = \tfrac{1}{2}\int_0^{2\pi} (4 + 4\sin 4\theta + \sin^2 4\theta)\, d\theta$

$$= \tfrac{1}{2}\int_0^{2\pi} \left[4 + 4\sin 4\theta + \tfrac{1}{2}(1 - \cos 8\theta)\right] d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi} \left(\tfrac{9}{2} + 4\sin 4\theta - \tfrac{1}{2}\cos 8\theta\right) d\theta = \tfrac{1}{2}\left[\tfrac{9}{2}\theta - \cos 4\theta - \tfrac{1}{16}\sin 8\theta\right]_0^{2\pi}$$

$$= \tfrac{1}{2}[(9\pi - 1) - (-1)] = \tfrac{9}{2}\pi$$

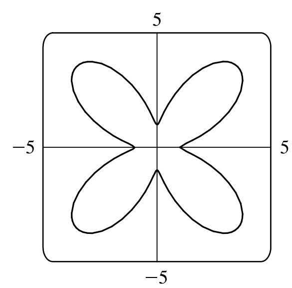


14. $A = \int_0^{2\pi} \tfrac{1}{2} r^2\, d\theta = \int_0^{2\pi} \tfrac{1}{2}(3 - 2\cos 4\theta)^2\, d\theta = \tfrac{1}{2}\int_0^{2\pi} (9 - 12\cos 4\theta + 4\cos^2 4\theta)\, d\theta$

$$= \tfrac{1}{2}\int_0^{2\pi} \left[9 - 12\cos 4\theta + 4 \cdot \tfrac{1}{2}(1 + \cos 8\theta)\right] d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi} (11 - 12\cos 4\theta + 2\cos 8\theta)\, d\theta = \tfrac{1}{2}\left[11\theta - 3\sin 4\theta + \tfrac{1}{4}\sin 8\theta\right]_0^{2\pi}$$

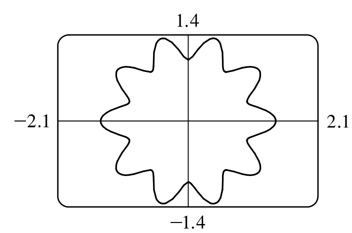
$$= \tfrac{1}{2}(22\pi) = 11\pi$$



15. $A = \int_0^{2\pi} \tfrac{1}{2} r^2\, d\theta = \int_0^{2\pi} \tfrac{1}{2}\left(\sqrt{1 + \cos^2 5\theta}\right)^2 d\theta$

$$= \tfrac{1}{2}\int_0^{2\pi} (1 + \cos^2 5\theta)\, d\theta = \tfrac{1}{2}\int_0^{2\pi} \left[1 + \tfrac{1}{2}(1 + \cos 10\theta)\right] d\theta$$

$$= \tfrac{1}{2}\left[\tfrac{3}{2}\theta + \tfrac{1}{20}\sin 10\theta\right]_0^{2\pi} = \tfrac{1}{2}(3\pi) = \tfrac{3}{2}\pi$$



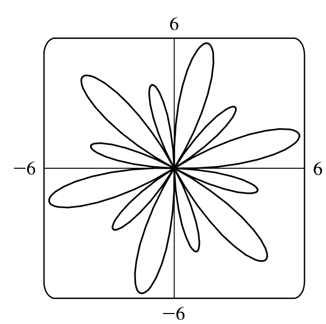
16. $A = \int_0^{2\pi} \tfrac{1}{2} r^2\, d\theta = \int_0^{2\pi} \tfrac{1}{2}(1 + 5\sin 6\theta)^2\, d\theta$

$$= \tfrac{1}{2}\int_0^{2\pi} (1 + 10\sin 6\theta + 25\sin^2 6\theta)\, d\theta$$

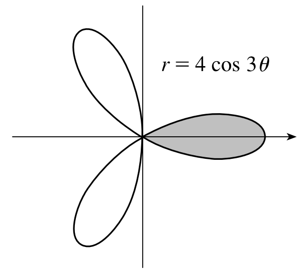
$$= \tfrac{1}{2}\int_0^{2\pi} \left[1 + 10\sin 6\theta + 25 \cdot \tfrac{1}{2}(1 - \cos 12\theta)\right] d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi} \left[\tfrac{27}{2} + 10\sin 6\theta - \tfrac{25}{2}\cos 12\theta\right] d\theta = \tfrac{1}{2}\left[\tfrac{27}{2}\theta - \tfrac{5}{3}\cos 6\theta - \tfrac{25}{24}\sin 12\theta\right]_0^{2\pi}$$

$$= \tfrac{1}{2}\left[\left(27\pi - \tfrac{5}{3}\right) - \left(-\tfrac{5}{3}\right)\right] = \tfrac{27}{2}\pi$$



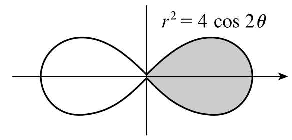
17. The curve passes through the pole when $r = 0 \quad \Rightarrow \quad 4\cos 3\theta = 0 \quad \Rightarrow \quad \cos 3\theta = 0 \quad \Rightarrow \quad 3\theta = \frac{\pi}{2} + \pi n \quad \Rightarrow$ $\theta = \frac{\pi}{6} + \frac{\pi}{3}n$. The part of the shaded loop above the polar axis is traced out for $\theta = 0$ to $\theta = \pi/6$, so we'll use $-\pi/6$ and $\pi/6$ as our limits of integration.



$$A = \int_{-\pi/6}^{\pi/6} \tfrac{1}{2}(4\cos 3\theta)^2\, d\theta = 2\int_0^{\pi/6} \tfrac{1}{2}(16\cos^2 3\theta)\, d\theta$$

$$= 16\int_0^{\pi/6} \tfrac{1}{2}(1 + \cos 6\theta)\, d\theta = 8\left[\theta + \tfrac{1}{6}\sin 6\theta\right]_0^{\pi/6} = 8\left(\tfrac{\pi}{6}\right) = \tfrac{4}{3}\pi$$

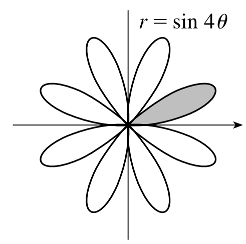
18. The curve given by $r^2 = 4\cos 2\theta$ passes through the pole when $r = 0 \quad \Rightarrow \quad 4\cos 2\theta = 0 \quad \Rightarrow \quad \cos 2\theta = 0 \quad \Rightarrow$ $2\theta = \frac{\pi}{2} + \pi n \quad \Rightarrow \quad \theta = \frac{\pi}{4} + \frac{\pi}{2}n$. The part of the shaded loop above the polar axis is traced out for $\theta = 0$ to $\theta = \pi/4$, so we'll use $-\pi/4$ to $\pi/4$ as our limits of integration.



$$A = \int_{-\pi/4}^{\pi/4} \tfrac{1}{2}(4\cos 2\theta)\, d\theta = 2\int_0^{\pi/4} 2\cos 2\theta\, d\theta = 2\Big[\sin 2\theta\Big]_0^{\pi/4}$$

$$= 2\sin\tfrac{\pi}{2} = 2(1) = 2$$

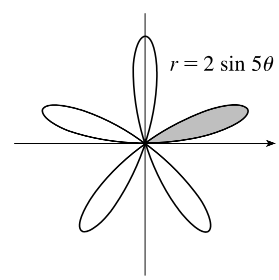
19. $r = 0 \quad \Rightarrow \quad \sin 4\theta = 0 \quad \Rightarrow \quad 4\theta = \pi n \quad \Rightarrow \quad \theta = \frac{\pi}{4}n$.



$$A = \int_0^{\pi/4} \tfrac{1}{2}(\sin 4\theta)^2\, d\theta = \tfrac{1}{2}\int_0^{\pi/4} \sin^2 4\theta\, d\theta = \tfrac{1}{2}\int_0^{\pi/4} \tfrac{1}{2}(1 - \cos 8\theta)\, d\theta$$

$$= \tfrac{1}{4}\left[\theta - \tfrac{1}{8}\sin 8\theta\right]_0^{\pi/4} = \tfrac{1}{4}\left(\tfrac{\pi}{4}\right) = \tfrac{1}{16}\pi$$

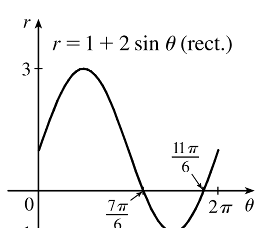
20. $r = 0 \quad \Rightarrow \quad 2\sin 5\theta = 0 \quad \Rightarrow \quad \sin 5\theta = 0 \quad \Rightarrow \quad 5\theta = \pi n \quad \Rightarrow \quad \theta = \frac{\pi}{5}n$.

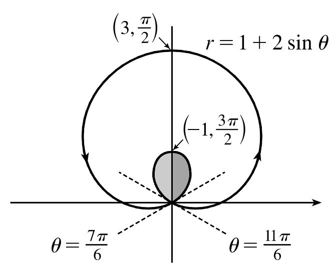


$$A = \int_0^{\pi/5} \tfrac{1}{2}(2\sin 5\theta)^2\, d\theta = \tfrac{1}{2}\int_0^{\pi/5} 4\sin^2 5\theta\, d\theta$$

$$= 2\int_0^{\pi/5} \tfrac{1}{2}(1 - \cos 10\theta)\, d\theta = \left[\theta - \tfrac{1}{10}\sin 10\theta\right]_0^{\pi/5} = \tfrac{\pi}{5}$$

21.





This is a limaçon, with inner loop traced out between $\theta = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$ [found by solving $r = 0$].

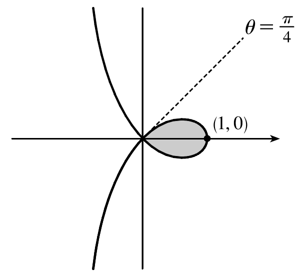
$$A = 2\int_{7\pi/6}^{3\pi/2} \tfrac{1}{2}(1 + 2\sin\theta)^2\, d\theta = \int_{7\pi/6}^{3\pi/2} \left(1 + 4\sin\theta + 4\sin^2\theta\right) d\theta = \int_{7\pi/6}^{3\pi/2} \left[1 + 4\sin\theta + 4\cdot\tfrac{1}{2}(1 - \cos 2\theta)\right] d\theta$$

$$= \Big[\theta - 4\cos\theta + 2\theta - \sin 2\theta\Big]_{7\pi/6}^{3\pi/2} = \left(\tfrac{9\pi}{2}\right) - \left(\tfrac{7\pi}{2} + 2\sqrt{3} - \tfrac{\sqrt{3}}{2}\right) = \pi - \tfrac{3\sqrt{3}}{2}$$

22. To determine when the strophoid $r = 2\cos\theta - \sec\theta$ passes through the pole, we solve

$r = 0 \;\Rightarrow\; 2\cos\theta - \dfrac{1}{\cos\theta} = 0 \;\Rightarrow\; 2\cos^2\theta - 1 = 0 \;\Rightarrow\; \cos^2\theta = \dfrac{1}{2} \;\Rightarrow$

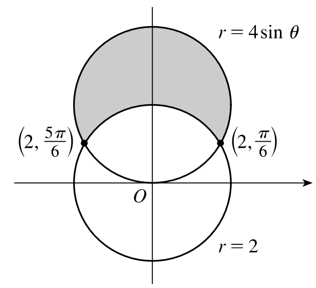
$\cos\theta = \pm\dfrac{1}{\sqrt{2}} \;\Rightarrow\; \theta = \frac{\pi}{4}$ or $\theta = \frac{3\pi}{4}$ for $0 \le \theta \le \pi$ with $\theta \ne \frac{\pi}{2}$.



$$A = 2\int_0^{\pi/4} \tfrac{1}{2}(2\cos\theta - \sec\theta)^2\,d\theta = \int_0^{\pi/4}(4\cos^2\theta - 4 + \sec^2\theta)\,d\theta$$
$$= \int_0^{\pi/4}\left[4\cdot\tfrac{1}{2}(1+\cos 2\theta) - 4 + \sec^2\theta\right]d\theta = \int_0^{\pi/4}(-2 + 2\cos 2\theta + \sec^2\theta)\,d\theta$$
$$= \left[-2\theta + \sin 2\theta + \tan\theta\right]_0^{\pi/4} = \left(-\tfrac{\pi}{2} + 1 + 1\right) - 0 = 2 - \tfrac{\pi}{2}$$

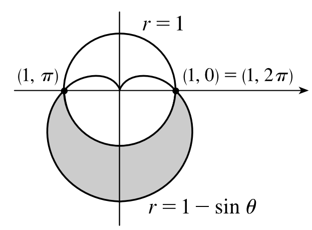
23. $4\sin\theta = 2 \;\Rightarrow\; \sin\theta = \frac{1}{2} \;\Rightarrow\; \theta = \frac{\pi}{6}$ or $\frac{5\pi}{6} \;\Rightarrow$

$$A = \int_{\pi/6}^{5\pi/6} \tfrac{1}{2}\left[(4\sin\theta)^2 - 2^2\right]d\theta = 2\int_{\pi/6}^{\pi/2}\tfrac{1}{2}(16\sin^2\theta - 4)\,d\theta$$
$$= \int_{\pi/6}^{\pi/2}\left[16\cdot\tfrac{1}{2}(1-\cos 2\theta) - 4\right]d\theta = \int_{\pi/6}^{\pi/2}(4 - 8\cos 2\theta)\,d\theta$$
$$= \Big[4\theta - 4\sin 2\theta\Big]_{\pi/6}^{\pi/2} = (2\pi - 0) - \left(\tfrac{2\pi}{3} - 2\sqrt{3}\right) = \tfrac{4\pi}{3} + 2\sqrt{3}$$

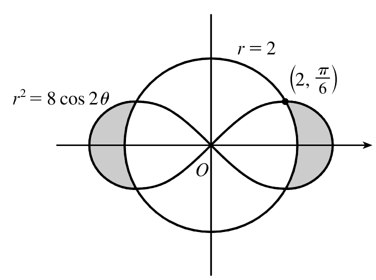


24. $1 - \sin\theta = 1 \;\Rightarrow\; \sin\theta = 0 \;\Rightarrow\; \theta = 0$ or $\pi \;\Rightarrow$

$$A = \int_\pi^{2\pi}\tfrac{1}{2}\left[(1-\sin\theta)^2 - 1\right]d\theta = \tfrac{1}{2}\int_\pi^{2\pi}(\sin^2\theta - 2\sin\theta)\,d\theta$$
$$= \tfrac{1}{4}\int_\pi^{2\pi}(1 - \cos 2\theta - 4\sin\theta)\,d\theta = \tfrac{1}{4}\left[\theta - \tfrac{1}{2}\sin 2\theta + 4\cos\theta\right]_\pi^{2\pi}$$
$$= \tfrac{1}{4}\pi + 2$$



25. To find the area inside the leminiscate $r^2 = 8\cos 2\theta$ and outside the circle $r = 2$, we first note that the two curves intersect when $r^2 = 8\cos 2\theta$ and $r = 2$, that is, when $\cos 2\theta = \frac{1}{2}$. For $-\pi < \theta \le \pi$, $\cos 2\theta = \frac{1}{2} \;\Leftrightarrow\; 2\theta = \pm\pi/3$ or $\pm 5\pi/3 \;\Leftrightarrow\; \theta = \pm\pi/6$ or $\pm 5\pi/6$. The figure shows that the desired area is 4 times the area between the curves from 0 to $\pi/6$. Thus,

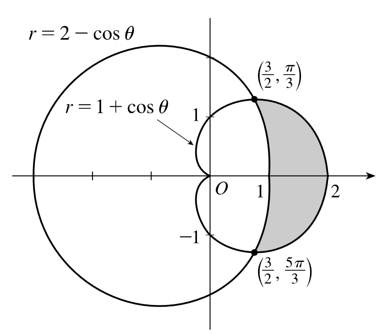


$$A = 4\int_0^{\pi/6}\left[\tfrac{1}{2}(8\cos 2\theta) - \tfrac{1}{2}(2)^2\right]d\theta = 8\int_0^{\pi/6}(2\cos 2\theta - 1)\,d\theta$$
$$= 8\Big[\sin 2\theta - \theta\Big]_0^{\pi/6} = 8\left(\sqrt{3}/2 - \pi/6\right) = 4\sqrt{3} - 4\pi/3$$

26. $1 + \cos\theta = 2 - \cos\theta \;\Rightarrow\; 2\cos\theta = 1 \;\Rightarrow\; \cos\theta = \frac{1}{2} \;\Rightarrow$

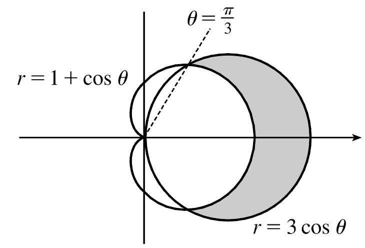
$\theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$.

$$A = 2\int_0^{\pi/3}\tfrac{1}{2}\left[(1+\cos\theta)^2 - (2-\cos\theta)^2\right]d\theta \qquad \text{[by symmetry]}$$
$$= \int_0^{\pi/3}(1 + 2\cos\theta + \cos^2\theta - 4 + 4\cos\theta - \cos^2\theta)\,d\theta$$
$$= \int_0^{\pi/3}(6\cos\theta - 3)\,d\theta = \Big[6\sin\theta - 3\theta\Big]_0^{\pi/3} = 6\left(\tfrac{\sqrt{3}}{2}\right) - 3\left(\tfrac{\pi}{3}\right)$$
$$= 3\sqrt{3} - \pi$$



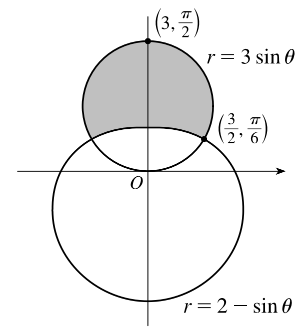
27. $3\cos\theta = 1+\cos\theta \quad\Leftrightarrow\quad \cos\theta = \frac{1}{2} \quad\Rightarrow\quad \theta = \frac{\pi}{3}$ or $-\frac{\pi}{3}$.

$$A = 2\int_0^{\pi/3} \tfrac{1}{2}\left[(3\cos\theta)^2 - (1+\cos\theta)^2\right] d\theta$$
$$= \int_0^{\pi/3} (8\cos^2\theta - 2\cos\theta - 1)\, d\theta = \int_0^{\pi/3} [4(1+\cos 2\theta) - 2\cos\theta - 1]\, d\theta$$
$$= \int_0^{\pi/3} (3 + 4\cos 2\theta - 2\cos\theta)\, d\theta = \left[3\theta + 2\sin 2\theta - 2\sin\theta\right]_0^{\pi/3}$$
$$= \pi + \sqrt{3} - \sqrt{3} = \pi$$



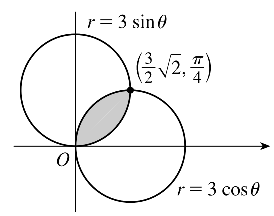
28. $3\sin\theta = 2 - \sin\theta \quad\Rightarrow\quad 4\sin\theta = 2 \quad\Rightarrow\quad \sin\theta = \frac{1}{2} \quad\Rightarrow\quad \theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.

$$A = 2\int_{\pi/6}^{\pi/2} \tfrac{1}{2}\left[(3\sin\theta)^2 - (2-\sin\theta)^2\right] d\theta$$
$$= \int_{\pi/6}^{\pi/2} (9\sin^2\theta - 4 + 4\sin\theta - \sin^2\theta]\, d\theta$$
$$= \int_{\pi/6}^{\pi/2} (8\sin^2\theta + 4\sin\theta - 4)\, d\theta$$
$$= 4\int_{\pi/6}^{\pi/2} \left[2\cdot\tfrac{1}{2}(1-\cos 2\theta) + \sin\theta - 1\right] d\theta$$
$$= 4\int_{\pi/6}^{\pi/2} (\sin\theta - \cos 2\theta)\, d\theta = 4\left[-\cos\theta - \tfrac{1}{2}\sin 2\theta\right]_{\pi/6}^{\pi/2}$$
$$= 4\left[(0-0) - \left(-\tfrac{\sqrt{3}}{2} - \tfrac{\sqrt{3}}{4}\right)\right] = 4\left(\tfrac{3\sqrt{3}}{4}\right) = 3\sqrt{3}$$



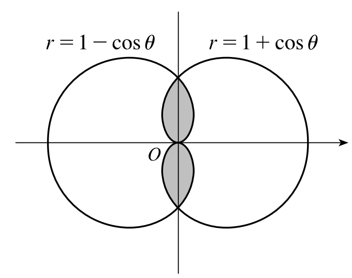
29. $3\sin\theta = 3\cos\theta \quad\Rightarrow\quad \dfrac{3\sin\theta}{3\cos\theta} = 1 \quad\Rightarrow\quad \tan\theta = 1 \quad\Rightarrow\quad \theta = \frac{\pi}{4} \quad\Rightarrow$

$$A = 2\int_0^{\pi/4} \tfrac{1}{2}(3\sin\theta)^2\, d\theta = \int_0^{\pi/4} 9\sin^2\theta\, d\theta = \int_0^{\pi/4} 9\cdot\tfrac{1}{2}(1-\cos 2\theta)\, d\theta$$
$$= \int_0^{\pi/4} \left(\tfrac{9}{2} - \tfrac{9}{2}\cos 2\theta\right) d\theta = \left[\tfrac{9}{2}\theta - \tfrac{9}{4}\sin 2\theta\right]_0^{\pi/4} = \left(\tfrac{9\pi}{8} - \tfrac{9}{4}\right) - (0-0)$$
$$= \tfrac{9\pi}{8} - \tfrac{9}{4}$$



30. $A = 4\int_0^{\pi/2} \frac{1}{2}(1-\cos\theta)^2\, d\theta = 2\int_0^{\pi/2} (1 - 2\cos\theta + \cos^2\theta)\, d\theta$

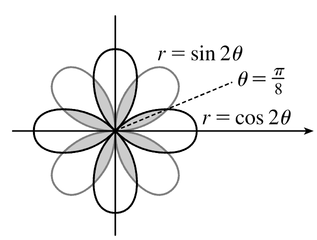
$$= 2\int_0^{\pi/2} \left[1 - 2\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\right] d\theta$$
$$= 2\int_0^{\pi/2} \left(\tfrac{3}{2} - 2\cos\theta + \tfrac{1}{2}\cos 2\theta\right) d\theta = \int_0^{\pi/2} (3 - 4\cos\theta + \cos 2\theta)\, d\theta$$
$$= \left[3\theta - 4\sin\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \tfrac{3\pi}{2} - 4$$



31. $\sin 2\theta = \cos 2\theta \quad\Rightarrow\quad \dfrac{\sin 2\theta}{\cos 2\theta} = 1 \quad\Rightarrow\quad \tan 2\theta = 1 \quad\Rightarrow\quad 2\theta = \frac{\pi}{4} \quad\Rightarrow$

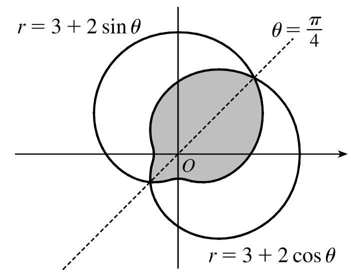
$\theta = \frac{\pi}{8} \quad\Rightarrow$

$$A = 8\cdot 2\int_0^{\pi/8} \tfrac{1}{2}\sin^2 2\theta\, d\theta = 8\int_0^{\pi/8} \tfrac{1}{2}(1-\cos 4\theta)\, d\theta$$
$$= 4\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/8} = 4\left(\tfrac{\pi}{8} - \tfrac{1}{4}\cdot 1\right) = \tfrac{\pi}{2} - 1$$



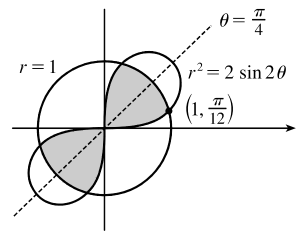
32. $3 + 2\cos\theta = 3 + 2\sin\theta \quad\Rightarrow\quad \cos\theta = \sin\theta \quad\Rightarrow\quad \theta = \frac{\pi}{4}$ or $\frac{5\pi}{4}$.

$$A = 2\int_{\pi/4}^{5\pi/4} \tfrac{1}{2}(3 + 2\cos\theta)^2\,d\theta = \int_{\pi/4}^{5\pi/4}(9 + 12\cos\theta + 4\cos^2\theta)\,d\theta$$
$$= \int_{\pi/4}^{5\pi/4}\left[9 + 12\cos\theta + 4\cdot\tfrac{1}{2}(1+\cos 2\theta)\right]d\theta$$
$$= \int_{\pi/4}^{5\pi/4}(11 + 12\cos\theta + 2\cos 2\theta)\,d\theta = \Big[11\theta + 12\sin\theta + \sin 2\theta\Big]_{\pi/4}^{5\pi/4}$$
$$= \left(\tfrac{55\pi}{4} - 6\sqrt{2} + 1\right) - \left(\tfrac{11\pi}{4} + 6\sqrt{2} + 1\right) = 11\pi - 12\sqrt{2}$$



33. From the figure, we see that the shaded region is 4 times the shaded region from $\theta = 0$ to $\theta = \pi/4$. $r^2 = 2\sin 2\theta$ and $r = 1 \quad\Rightarrow$ $2\sin 2\theta = 1^2 \quad\Rightarrow\quad \sin 2\theta = \frac{1}{2} \quad\Rightarrow\quad 2\theta = \frac{\pi}{6} \quad\Rightarrow\quad \theta = \frac{\pi}{12}$.

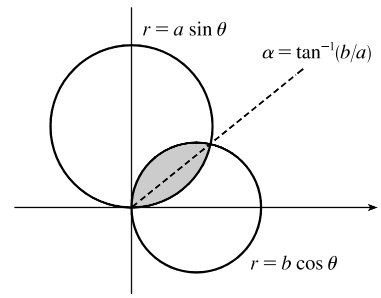
$$A = 4\int_0^{\pi/12} \tfrac{1}{2}(2\sin 2\theta)\,d\theta + 4\int_{\pi/12}^{\pi/4}\tfrac{1}{2}(1)^2\,d\theta$$
$$= \int_0^{\pi/12} 4\sin 2\theta\,d\theta + \int_{\pi/12}^{\pi/4} 2\,d\theta = \Big[-2\cos 2\theta\Big]_0^{\pi/12} + \Big[2\theta\Big]_{\pi/12}^{\pi/4}$$
$$= \left(-\sqrt{3} + 2\right) + \left(\tfrac{\pi}{2} - \tfrac{\pi}{6}\right) = -\sqrt{3} + 2 + \tfrac{\pi}{3}$$



34. $a\sin\theta = b\cos\theta \quad\Rightarrow\quad \dfrac{\sin\theta}{\cos\theta} = \dfrac{b}{a} \quad\Rightarrow\quad \tan\theta = \dfrac{b}{a}$. Let $\alpha = \tan^{-1}(b/a)$.

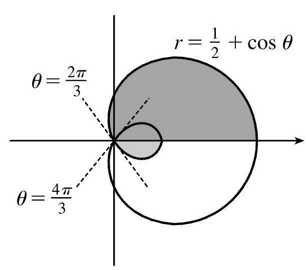
Then

$$A = \int_0^{\alpha} \tfrac{1}{2}(a\sin\theta)^2\,d\theta + \int_{\alpha}^{\pi/2}\tfrac{1}{2}(b\cos\theta)^2\,d\theta$$
$$= \tfrac{1}{4}a^2\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\alpha} + \tfrac{1}{4}b^2\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{\alpha}^{\pi/2}$$
$$= \tfrac{1}{4}\alpha(a^2 - b^2) + \tfrac{1}{8}\pi b^2 - \tfrac{1}{4}(a^2 + b^2)(\sin\alpha\,\cos\alpha)$$
$$= \tfrac{1}{4}(a^2 - b^2)\,\tan^{-1}(b/a) + \tfrac{1}{8}\pi b^2 - \tfrac{1}{4}ab$$

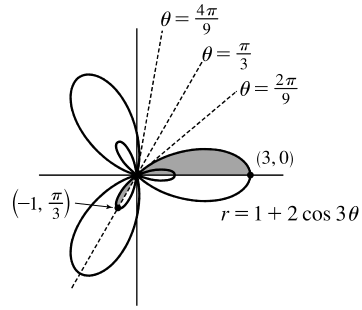


35. The darker shaded region (from $\theta = 0$ to $\theta = 2\pi/3$) represents $\frac{1}{2}$ of the desired area plus $\frac{1}{2}$ of the area of the inner loop. From this area, we'll subtract $\frac{1}{2}$ of the area of the inner loop (the lighter shaded region from $\theta = 2\pi/3$ to $\theta = \pi$), and then double that difference to obtain the desired area.

$$A = 2\left[\int_0^{2\pi/3}\tfrac{1}{2}\left(\tfrac{1}{2} + \cos\theta\right)^2 d\theta - \int_{2\pi/3}^{\pi}\tfrac{1}{2}\left(\tfrac{1}{2} + \cos\theta\right)^2 d\theta\right]$$
$$= \int_0^{2\pi/3}\left(\tfrac{1}{4} + \cos\theta + \cos^2\theta\right)d\theta - \int_{2\pi/3}^{\pi}\left(\tfrac{1}{4} + \cos\theta + \cos^2\theta\right)d\theta$$
$$= \int_0^{2\pi/3}\left[\tfrac{1}{4} + \cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\right]d\theta$$
$$- \int_{2\pi/3}^{\pi}\left[\tfrac{1}{4} + \cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\right]d\theta$$
$$= \left[\frac{\theta}{4} + \sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi/3} - \left[\frac{\theta}{4} + \sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{2\pi/3}^{\pi}$$
$$= \left(\tfrac{\pi}{6} + \tfrac{\sqrt{3}}{2} + \tfrac{\pi}{3} - \tfrac{\sqrt{3}}{8}\right) - \left(\tfrac{\pi}{4} + \tfrac{\pi}{2}\right) + \left(\tfrac{\pi}{6} + \tfrac{\sqrt{3}}{2} + \tfrac{\pi}{3} - \tfrac{\sqrt{3}}{8}\right)$$
$$= \tfrac{\pi}{4} + \tfrac{3}{4}\sqrt{3} = \tfrac{1}{4}\left(\pi + 3\sqrt{3}\right)$$



36. $r = 0 \Rightarrow 1 + 2\cos 3\theta = 0 \Rightarrow \cos 3\theta = -\frac{1}{2} \Rightarrow 3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ [for $0 \le 3\theta \le 2\pi$] $\Rightarrow \theta = \frac{2\pi}{9}, \frac{4\pi}{9}$. The darker shaded region (from $\theta = 0$ to $\theta = 2\pi/9$) represents $\frac{1}{2}$ of the desired area plus $\frac{1}{2}$ of the area of the inner loop. From this area, we'll subtract $\frac{1}{2}$ of the area of the inner loop (the lighter shaded region from $\theta = 2\pi/9$ to $\theta = \pi/3$), and then double that difference to obtain the desired area.



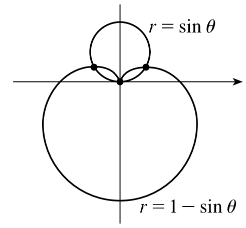
$$A = 2\left[\int_0^{2\pi/9} \tfrac{1}{2}(1 + 2\cos 3\theta)^2\, d\theta - \int_{2\pi/9}^{\pi/3} \tfrac{1}{2}(1 + 2\cos 3\theta)^2\, d\theta\right]$$

Now
$$\begin{aligned} r^2 &= (1 + 2\cos 3\theta)^2 = 1 + 4\cos 3\theta + 4\cos^2 3\theta = 1 + 4\cos 3\theta + 4 \cdot \tfrac{1}{2}(1 + \cos 6\theta) \\ &= 1 + 4\cos 3\theta + 2 + 2\cos 6\theta = 3 + 4\cos 3\theta + 2\cos 6\theta \end{aligned}$$

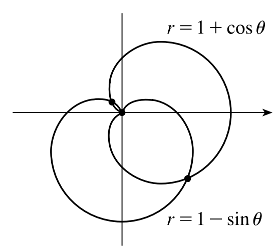
and $\int r^2 d\theta = 3\theta + \frac{4}{3}\sin 3\theta + \frac{1}{3}\sin 6\theta + C$, so

$$\begin{aligned} A &= \left[3\theta + \tfrac{4}{3}\sin 3\theta + \tfrac{1}{3}\sin 6\theta\right]_0^{2\pi/9} - \left[3\theta + \tfrac{4}{3}\sin 3\theta + \tfrac{1}{3}\sin 6\theta\right]_{2\pi/9}^{\pi/3} \\ &= \left[\left(\tfrac{2\pi}{3} + \tfrac{4}{3} \cdot \tfrac{\sqrt{3}}{2} + \tfrac{1}{3} \cdot \tfrac{-\sqrt{3}}{2}\right) - 0\right] - \left[(\pi + 0 + 0) - \left(\tfrac{2\pi}{3} + \tfrac{4}{3} \cdot \tfrac{\sqrt{3}}{2} + \tfrac{1}{3} \cdot \tfrac{-\sqrt{3}}{2}\right)\right] \\ &= \tfrac{4\pi}{3} + \tfrac{4}{3}\sqrt{3} - \tfrac{1}{3}\sqrt{3} - \pi = \tfrac{\pi}{3} + \sqrt{3} \end{aligned}$$

37. The pole is a point of intersection. $\sin\theta = 1 - \sin\theta \Rightarrow 2\sin\theta = 1 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. So the other points of intersection are $\left(\frac{1}{2}, \frac{\pi}{6}\right)$ and $\left(\frac{1}{2}, \frac{5\pi}{6}\right)$.



38. The pole is a point of intersection. $1 + \cos\theta = 1 - \sin\theta \Rightarrow \cos\theta = -\sin\theta \Rightarrow \dfrac{\cos\theta}{\sin\theta} = -1 \Rightarrow \cot\theta = -1 \Rightarrow \theta = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$. So the other points of intersection are $\left(1 - \frac{1}{2}\sqrt{2}, \frac{3\pi}{4}\right)$ and $\left(1 + \frac{1}{2}\sqrt{2}, \frac{7\pi}{4}\right)$.



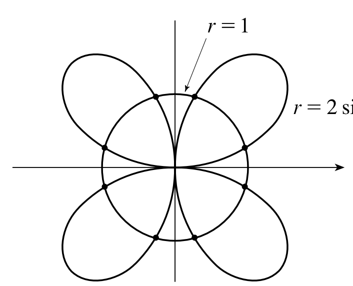
39. $2\sin 2\theta = 1 \Rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$, or $\frac{17\pi}{6}$.

By symmetry, the eight points of intersection are given by

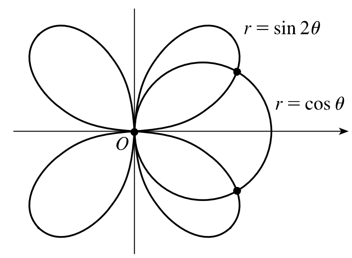
$(1, \theta)$, where $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}$, and $\frac{17\pi}{12}$, and

$(-1, \theta)$, where $\theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}$, and $\frac{23\pi}{12}$.

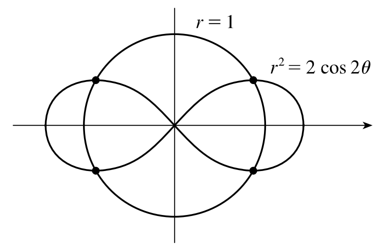
[There are many ways to describe these points.]



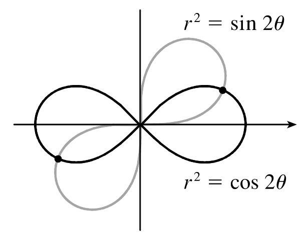
40. The pole is a point of intersection. $\cos\theta = \sin 2\theta = 2\sin\theta\cos\theta \Rightarrow$ $\cos\theta(1-2\sin\theta) = 0 \Rightarrow \cos\theta = 0$ or $\sin\theta = \frac{1}{2} \Rightarrow$ $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}$, or $\frac{5\pi}{6}$, so the two remaining intersection points are $\left(\frac{\sqrt{3}}{2}, \frac{\pi}{6}\right)$ and $\left(-\frac{\sqrt{3}}{2}, \frac{5\pi}{6}\right)$.



41. $r = 1$ and $r^2 = 2\cos 2\theta \Rightarrow 1^2 = 2\cos 2\theta \Rightarrow \cos 2\theta = \frac{1}{2} \Rightarrow$ $2\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$, or $\frac{11\pi}{3} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$, or $\frac{11\pi}{6}$. Thus, the four points of intersection are $\left(1, \frac{\pi}{6}\right)$, $\left(1, \frac{5\pi}{6}\right)$, $\left(1, \frac{7\pi}{6}\right)$, and $\left(1, \frac{11\pi}{6}\right)$.



42. Clearly the pole is a point of intersection. $\sin 2\theta = \cos 2\theta \Rightarrow$ $\tan 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{4} + 2n\pi$ [since $\sin 2\theta$ and $\cos 2\theta$ must be positive in the equations] $\Rightarrow \theta = \frac{\pi}{8} + n\pi \Rightarrow \theta = \frac{\pi}{8}$ or $\frac{9\pi}{8}$. So the curves also intersect at $\left(\frac{1}{\sqrt[4]{2}}, \frac{\pi}{8}\right)$ and $\left(\frac{1}{\sqrt[4]{2}}, \frac{9\pi}{8}\right)$.



43. The shaded region lies outside the rose $r = \sin 2\theta$ and inside the limaçon $r = 3 + 2\cos\theta$, so its area is given by

$$\begin{aligned} A &= \int_0^{2\pi} \tfrac{1}{2}\left[(3+2\cos\theta)^2 - (\sin 2\theta)^2\right] d\theta = \tfrac{1}{2}\int_0^{2\pi} \left(9 + 12\cos\theta + 4\cos^2\theta - \sin^2 2\theta\right) d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi} \left[9 + 12\cos\theta + 4\cdot\tfrac{1}{2}(1+\cos 2\theta) - \tfrac{1}{2}(1-\cos 4\theta)\right] d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi} \left[\tfrac{21}{2} + 12\cos\theta + 2\cos 2\theta + \tfrac{1}{2}\cos 4\theta\right] d\theta \\ &= \tfrac{1}{2}\left[\tfrac{21}{2}\theta + 12\sin\theta + \sin 2\theta + \tfrac{1}{8}\sin 4\theta\right]_0^{2\pi} = \tfrac{1}{2}(21\pi) = \tfrac{21\pi}{2} \end{aligned}$$

44. $r = \sqrt{2}\cos\theta$ and $r^2 = \sqrt{3}\sin 2\theta \Rightarrow \left(\sqrt{2}\cos\theta\right)^2 = \sqrt{3}\sin 2\theta \Rightarrow 2\cos^2\theta = 2\sqrt{3}\sin\theta\cos\theta \Rightarrow$ $2\cos^2\theta - 2\sqrt{3}\sin\theta\cos\theta = 0 \Rightarrow 2\cos\theta\left(\cos\theta - \sqrt{3}\sin\theta\right) = 0 \Rightarrow \cos\theta = 0$ or $\sqrt{3}\sin\theta = \cos\theta \Rightarrow$ $\tan\theta = \dfrac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{2}$ or $\frac{\pi}{6}$ for $0 \le \theta \le \frac{\pi}{2}$. The shaded region is comprised of the area swept out by the lemniscate $r^2 = \sqrt{3}\sin 2\theta$ in the interval $0 \le \theta \le \pi/6$ and the portion of the circle $r = \sqrt{2}\cos\theta$ in the interval $\pi/6 \le \theta \le \pi/2$. Thus,

$$\begin{aligned} A &= \int_0^{\pi/6} \frac{1}{2}\left(\sqrt{3}\sin 2\theta\right) d\theta + \int_{\pi/6}^{\pi/2} \frac{1}{2}\left(\sqrt{2}\cos\theta\right)^2 d\theta = \frac{\sqrt{3}}{2}\left[-\frac{1}{2}\cos 2\theta\right]_0^{\pi/6} + \int_{\pi/6}^{\pi/2} \cos^2\theta\, d\theta \\ &= -\frac{\sqrt{3}}{4}\left(\frac{1}{2} - 1\right) + \int_{\pi/6}^{\pi/2} \frac{1}{2}(1+\cos 2\theta)\, d\theta = \frac{\sqrt{3}}{8} + \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_{\pi/6}^{\pi/2} = \frac{\sqrt{3}}{8} + \left[\frac{\pi}{4} - \left(\frac{\pi}{12} + \frac{\sqrt{3}}{8}\right)\right] = \frac{\pi}{6} \end{aligned}$$

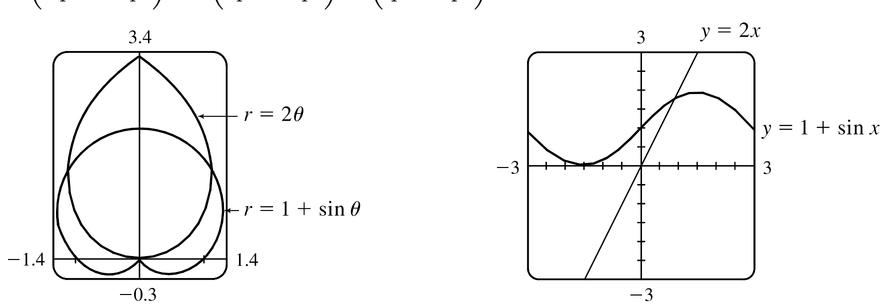
45. $1+\cos\theta = 3\cos\theta \quad\Rightarrow\quad 1 = 2\cos\theta \quad\Rightarrow\quad \cos\theta = \frac{1}{2} \quad\Rightarrow\quad \theta = \frac{\pi}{3}$. The area swept out by $r = 1+\cos\theta$, $\pi/3 \le \theta \le \pi$, contains the shaded region plus the portion of the circle $r = 3\cos\theta$, $\pi/3 \le \theta \le \pi/2$. Thus, the area of the shaded region is given by

$$\begin{aligned}
A &= \int_{\pi/3}^{\pi} \tfrac{1}{2}(1+\cos\theta)^2\,d\theta - \int_{\pi/3}^{\pi/2} \tfrac{1}{2}(3\cos\theta)^2\,d\theta = \tfrac{1}{2}\int_{\pi/3}^{\pi} (1+2\cos\theta+\cos^2\theta)\,d\theta - \tfrac{9}{2}\int_{\pi/3}^{\pi/2} \cos^2\theta\,d\theta \\
&= \tfrac{1}{2}\int_{\pi/3}^{\pi} \left[1+2\cos\theta+\tfrac{1}{2}(1+\cos 2\theta)\right] d\theta - \tfrac{9}{2}\int_{\pi/3}^{\pi/2} \tfrac{1}{2}(1+\cos 2\theta)\,d\theta \\
&= \tfrac{1}{2}\int_{\pi/3}^{\pi} \left(\tfrac{3}{2}+2\cos\theta+\tfrac{1}{2}\cos 2\theta\right) d\theta - \tfrac{9}{4}\int_{\pi/3}^{\pi/2} (1+\cos 2\theta)\,d\theta \\
&= \tfrac{1}{2}\left[\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_{\pi/3}^{\pi} - \tfrac{9}{4}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{\pi/3}^{\pi/2} \\
&= \tfrac{1}{2}\left[\tfrac{3\pi}{2} - \left(\tfrac{\pi}{2} + \sqrt{3} + \tfrac{\sqrt{3}}{8}\right)\right] - \tfrac{9}{4}\left[\tfrac{\pi}{2} - \left(\tfrac{\pi}{3} + \tfrac{\sqrt{3}}{4}\right)\right] = \tfrac{1}{2}\left(\pi - \tfrac{9\sqrt{3}}{8}\right) - \tfrac{9}{4}\left(\tfrac{\pi}{6} - \tfrac{\sqrt{3}}{4}\right) = \tfrac{\pi}{2} - \tfrac{3\pi}{8} = \tfrac{\pi}{8}
\end{aligned}$$

46. The pole is reached when $r = 1 - 2\sin\theta = 0 \quad\Rightarrow\quad \sin\theta = \frac{1}{2} \quad\Rightarrow\quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}$, or $\frac{13\pi}{6}$. The curve's inner loop is traced from $\theta = \pi/6$ to $\theta = 5\pi/6$ (corresponding to negative r-values), while the outer loop is traced from $\theta = 5\pi/6$ to $\theta = 13\pi/6$. From the figure, we see that the area of the outer loop minus the area of the inner loop gives the area of the shaded region. Thus,

$$\begin{aligned}
A &= \int_{5\pi/6}^{13\pi/6} \tfrac{1}{2}(1-2\sin\theta)^2\,d\theta - \int_{\pi/6}^{5\pi/6} \tfrac{1}{2}(1-2\sin\theta)^2\,d\theta \\
&= \int_{5\pi/6}^{13\pi/6} \tfrac{1}{2}(1-4\sin\theta+4\sin^2\theta)\,d\theta - \int_{\pi/6}^{5\pi/6} \tfrac{1}{2}(1-4\sin\theta+4\sin^2\theta)\,d\theta \\
&= \int_{5\pi/6}^{13\pi/6} \left[\tfrac{1}{2} - 2\sin\theta + 2\cdot\tfrac{1}{2}(1-\cos 2\theta)\right] d\theta - \int_{\pi/6}^{5\pi/6} \left[\tfrac{1}{2} - 2\sin\theta + 2\cdot\tfrac{1}{2}(1-\cos 2\theta)\right] d\theta \\
&= \left[\tfrac{1}{2}\theta + 2\cos\theta + \theta - \tfrac{1}{2}\sin 2\theta\right]_{5\pi/6}^{13\pi/6} - \left[\tfrac{1}{2}\theta + 2\cos\theta + \theta - \tfrac{1}{2}\sin 2\theta\right]_{\pi/6}^{5\pi/6} \\
&= \left(\tfrac{13\pi}{12} + \sqrt{3} + \tfrac{13\pi}{6} - \tfrac{\sqrt{3}}{4}\right) - \left(\tfrac{5\pi}{12} - \sqrt{3} + \tfrac{5\pi}{6} + \tfrac{\sqrt{3}}{4}\right) - \left(\tfrac{5\pi}{12} - \sqrt{3} + \tfrac{5\pi}{6} + \tfrac{\sqrt{3}}{4}\right) + \left(\tfrac{\pi}{12} + \sqrt{3} + \tfrac{\pi}{6} - \tfrac{\sqrt{3}}{4}\right) \\
&= \left(\tfrac{13\pi}{4} + \tfrac{3\sqrt{3}}{4}\right) - 2\left(\tfrac{5\pi}{4} - \tfrac{3\sqrt{3}}{4}\right) + \left(\tfrac{\pi}{4} + \tfrac{3\sqrt{3}}{4}\right) = \pi + 3\sqrt{3}
\end{aligned}$$

47.

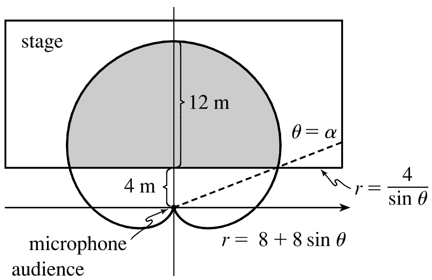


From the first graph, we see that the pole is one point of intersection. By zooming in or using the cursor, we find the θ-values of the intersection points to be $\alpha \approx 0.88786 \approx 0.89$ and $\pi - \alpha \approx 2.25$. (The first of these values may be more easily estimated by plotting $y = 1 + \sin x$ and $y = 2x$ in rectangular coordinates; see the second graph.) By symmetry, the total area contained is twice the area contained in the first quadrant, that is,

$$A = 2\int_0^{\alpha} \tfrac{1}{2}(2\theta)^2\, d\theta + 2\int_{\alpha}^{\pi/2} \tfrac{1}{2}(1+\sin\theta)^2\, d\theta = \int_0^{\alpha} 4\theta^2\, d\theta + \int_{\alpha}^{\pi/2} \left[1 + 2\sin\theta + \tfrac{1}{2}(1-\cos 2\theta)\right] d\theta$$

$$= \left[\tfrac{4}{3}\theta^3\right]_0^{\alpha} + \left[\theta - 2\cos\theta + \left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right)\right]_{\alpha}^{\pi/2} = \tfrac{4}{3}\alpha^3 + \left[\left(\tfrac{\pi}{2}+\tfrac{\pi}{4}\right) - \left(\alpha - 2\cos\alpha + \tfrac{1}{2}\alpha - \tfrac{1}{4}\sin 2\alpha\right)\right] \approx 3.4645$$

48.



We need to find the shaded area A in the figure. The horizontal line representing the front of the stage has equation $y = 4 \Leftrightarrow r\sin\theta = 4 \Rightarrow r = 4/\sin\theta$. This line intersects the curve $r = 8 + 8\sin\theta$ when $8 + 8\sin\theta = \dfrac{4}{\sin\theta} \Rightarrow 8\sin\theta + 8\sin^2\theta = 4 \Rightarrow 2\sin^2\theta + 2\sin\theta - 1 = 0 \Rightarrow$

$$\sin\theta = \frac{-2 \pm \sqrt{4+8}}{4} = \frac{-2 \pm 2\sqrt{3}}{4} = \frac{-1+\sqrt{3}}{2} \quad \text{[the other value is less than } -1\text{]} \quad \Rightarrow \quad \theta = \sin^{-1}\!\left(\frac{\sqrt{3}-1}{2}\right).$$

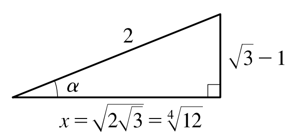
This angle is about 21.5° and is denoted by α in the figure.

$$A = 2\int_{\alpha}^{\pi/2} \tfrac{1}{2}(8+8\sin\theta)^2\, d\theta - 2\int_{\alpha}^{\pi/2} \tfrac{1}{2}(4\csc\theta)^2\, d\theta = 64\int_{\alpha}^{\pi/2}(1 + 2\sin\theta + \sin^2\theta)\, d\theta - 16\int_{\alpha}^{\pi/2}\csc^2\theta\, d\theta$$

$$= 64\int_{\alpha}^{\pi/2}\left(1 + 2\sin\theta + \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right) d\theta + 16\int_{\alpha}^{\pi/2}(-\csc^2\theta)\, d\theta = 64\left[\tfrac{3}{2}\theta - 2\cos\theta - \tfrac{1}{4}\sin 2\theta\right]_{\alpha}^{\pi/2} + 16\left[\cot\theta\right]_{\alpha}^{\pi/2}$$

$$= 16\left[6\theta - 8\cos\theta - \sin 2\theta + \cot\theta\right]_{\alpha}^{\pi/a} = 16[(3\pi - 0 - 0 + 0) - (6\alpha - 8\cos\alpha - \sin 2\alpha + \cot\alpha)]$$

$$= 48\pi - 96\alpha + 128\cos\alpha + 16\sin 2\alpha - 16\cot\alpha$$



From the figure, $x^2 + \left(\sqrt{3}-1\right)^2 = 2^2 \Rightarrow x^2 = 4 - \left(3 - 2\sqrt{3} + 1\right) \Rightarrow x^2 = 2\sqrt{3} = \sqrt{12}$, so $x = \sqrt{2\sqrt{3}} = \sqrt[4]{12}$. Using the trigonometric relationships for a right triangle and the identity $\sin 2\alpha = 2\sin\alpha\,\cos\alpha$, we continue:

$$A = 48\pi - 96\alpha + 128\cdot\frac{\sqrt[4]{12}}{2} + 16\cdot 2\cdot\frac{\sqrt{3}-1}{2}\cdot\frac{\sqrt[4]{12}}{2} - 16\cdot\frac{\sqrt[4]{12}}{\sqrt{3}-1}\cdot\frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$= 48\pi - 96\alpha + 64\sqrt[4]{12} + 8\sqrt[4]{12}\left(\sqrt{3}-1\right) - 8\sqrt[4]{12}\left(\sqrt{3}+1\right) = 48\pi + 48\sqrt[4]{12} - 96\sin^{-1}\!\left(\frac{\sqrt{3}-1}{2}\right)$$

$$\approx 204.16 \text{ m}^2$$

49. $L = \displaystyle\int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^{\pi} \sqrt{(2\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta$

$$= \int_0^{\pi} \sqrt{4(\cos^2\theta + \sin^2\theta)}\, d\theta = \int_0^{\pi} \sqrt{4}\, d\theta = \left[2\theta\right]_0^{\pi} = 2\pi$$

As a check, note that the curve is a circle of radius 1, so its circumference is $2\pi(1) = 2\pi$.

50. $L = \displaystyle\int_0^{\pi/2} \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^{\pi/2} \sqrt{(e^{\theta/2})^2 + \left(\tfrac{1}{2}e^{\theta/2}\right)^2}\, d\theta = \int_0^{\pi/2} \sqrt{(e^{\theta/2})^2\left(1 + \tfrac{1}{4}\right)}\, d\theta$

$$= \sqrt{\frac{5}{4}}\int_0^{\pi/2} \left|e^{\theta/2}\right| d\theta = \frac{\sqrt{5}}{2}\int_0^{\pi/2} e^{\theta/2}\, d\theta = \frac{\sqrt{5}}{2}\left[2e^{\theta/2}\right]_0^{\pi/2} = \sqrt{5}\left(e^{\pi/4} - 1\right)$$

51. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_0^{2\pi} \sqrt{(\theta^2)^2 + (2\theta)^2}\,d\theta = \int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2}\,d\theta$

$= \int_0^{2\pi} \sqrt{\theta^2(\theta^2 + 4)}\,d\theta = \int_0^{2\pi} \theta\sqrt{\theta^2 + 4}\,d\theta$

Now let $u = \theta^2 + 4$, so that $du = 2\theta\,d\theta$ $\left[\theta\,d\theta = \frac{1}{2}\,du\right]$ and

$$\int_0^{2\pi} \theta\sqrt{\theta^2+4}\,d\theta = \int_4^{4\pi^2+4} \tfrac{1}{2}\sqrt{u}\,du = \tfrac{1}{2}\cdot\tfrac{2}{3}\left[u^{3/2}\right]_4^{4(\pi^2+1)} = \tfrac{1}{3}[4^{3/2}(\pi^2+1)^{3/2} - 4^{3/2}] = \tfrac{8}{3}[(\pi^2+1)^{3/2} - 1]$$

52. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_0^{2\pi} \sqrt{[2(1+\cos\theta)]^2 + (-2\sin\theta)^2}\,d\theta = \int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\,d\theta$

$= \int_0^{2\pi} \sqrt{8 + 8\cos\theta}\,d\theta = \sqrt{8}\int_0^{2\pi} \sqrt{1+\cos\theta}\,d\theta = \sqrt{8}\int_0^{2\pi} \sqrt{2\cdot\frac{1}{2}(1+\cos\theta)}\,d\theta$

$= \sqrt{8}\int_0^{2\pi} \sqrt{2\cos^2\frac{\theta}{2}}\,d\theta = \sqrt{8}\,\sqrt{2}\int_0^{2\pi} \left|\cos\frac{\theta}{2}\right|\,d\theta = 4\cdot 2\int_0^{\pi} \cos\frac{\theta}{2}\,d\theta$ [by symmetry]

$= 8\left[2\sin\frac{\theta}{2}\right]_0^{\pi} = 8(2) = 16$

53. The blue section of the curve $r = 3 + 3\sin\theta$ is traced from $\theta = -\pi/2$ to $\theta = \pi$.

$r^2 + (dr/d\theta)^2 = (3 + 3\sin\theta)^2 + (3\cos\theta)^2 = 9 + 18\sin\theta + 9\sin^2\theta + 9\cos^2\theta = 18 + 18\sin\theta$

$L = \int_{-\pi/2}^{\pi} \sqrt{18 + 18\sin\theta}\,d\theta = \sqrt{18}\int_{-\pi/2}^{\pi} \sqrt{1+\sin\theta}\,d\theta = \sqrt{18}\int_{-\pi/2}^{\pi} \sqrt{\frac{(1+\sin\theta)(1-\sin\theta)}{1-\sin\theta}}\,d\theta$

$= \sqrt{18}\int_{-\pi/2}^{\pi} \sqrt{\frac{1-\sin^2\theta}{1-\sin\theta}}\,d\theta = \sqrt{18}\int_{-\pi/2}^{\pi} \sqrt{\frac{\cos^2\theta}{1-\sin\theta}}\,d\theta = \sqrt{18}\int_{-\pi/2}^{\pi} \frac{|\cos\theta|}{\sqrt{1-\sin\theta}}\,d\theta$

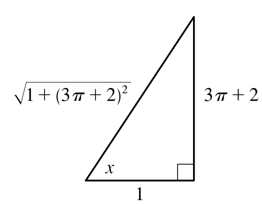
$= \sqrt{18}\left(\int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{\sqrt{1-\sin\theta}}\,d\theta + \int_{\pi/2}^{\pi} \frac{-\cos\theta}{\sqrt{1-\sin\theta}}\,d\theta\right)$

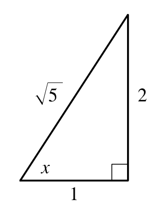
$\overset{s}{=} \sqrt{18}\left(\int_{-1}^{1} \frac{1}{\sqrt{1-u}}\,du - \int_{1}^{0} \frac{1}{\sqrt{1-u}}\,du\right) = \sqrt{18}\left(\left[-2(1-u)^{1/2}\right]_{-1}^{1} - \left[-2(1-u)^{1/2}\right]_{1}^{0}\right)$

$= \sqrt{18}\,(2\sqrt{2} + 2) = 12 + 6\sqrt{2}$

54. The blue section of the curve $r = \theta + 2$ is traced from $\theta = 0$ to $\theta = 3\pi$.

$r^2 + (dr/d\theta)^2 = (\theta + 2)^2 + (1)^2 = (\theta + 2)^2 + 1$





$L = \int_0^{3\pi} \sqrt{(\theta+2)^2 + 1}\,d\theta$

$= \int_{\tan^{-1}2}^{\tan^{-1}(3\pi+2)} \sqrt{\tan^2 x + 1}\,\sec^2 x\,dx$ $\left[\begin{array}{l}\theta + 2 = \tan x,\\ d\theta = \sec^2 x\,dx\end{array}\right]$

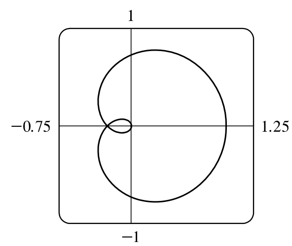
$= \int_{\tan^{-1}2}^{\tan^{-1}(3\pi+2)} \sqrt{\sec^2 x}\,\sec^2 x\,dx = \int_{\tan^{-1}2}^{\tan^{-1}(3\pi+2)} |\sec x|\sec^2 x\,dx$

$= \int_{\tan^{-1}2}^{\tan^{-1}(3\pi+2)} \sec^3 x\,dx = \frac{1}{2}\Big[\sec x\tan x + \ln|\sec x + \tan x|\Big]_{\tan^{-1}2}^{\tan^{-1}(3\pi+2)}$ [by Example 7.2.8]

$= \frac{1}{2}\left[\sqrt{1+(3\pi+2)^2}\,(3\pi+2) + \ln\left(\sqrt{1+(3\pi+2)^2} + 3\pi + 2\right)\right] - \frac{1}{2}\left[\sqrt{5}\,(2) + \ln\left(\sqrt{5}+2\right)\right]$ $[\approx 64.12]$

55. The curve $r = \cos^4(\theta/4)$ is completely traced with $0 \le \theta \le 4\pi$.

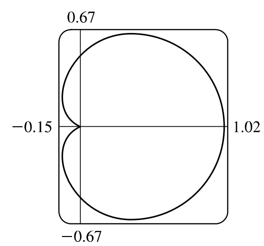
$$r^2 + (dr/d\theta)^2 = [\cos^4(\theta/4)]^2 + \left[4\cos^3(\theta/4)\cdot(-\sin(\theta/4))\cdot\tfrac{1}{4}\right]^2$$
$$= \cos^8(\theta/4) + \cos^6(\theta/4)\sin^2(\theta/4)$$
$$= \cos^6(\theta/4)[\cos^2(\theta/4) + \sin^2(\theta/4)] = \cos^6(\theta/4)$$



$$L = \int_0^{4\pi} \sqrt{\cos^6(\theta/4)}\,d\theta = \int_0^{4\pi} \left|\cos^3(\theta/4)\right|\,d\theta$$
$$= 2\int_0^{2\pi} \cos^3(\theta/4)\,d\theta \quad [\text{since } \cos^3(\theta/4) \ge 0 \text{ for } 0 \le \theta \le 2\pi] \quad = 8\int_0^{\pi/2} \cos^3 u\,du \quad \left[u = \tfrac{1}{4}\theta\right]$$
$$= 8\int_0^{\pi/2}(1-\sin^2 u)\cos u\,du = 8\int_0^1 (1-x^2)\,dx \quad \begin{bmatrix} x = \sin u, \\ dx = \cos u\,du \end{bmatrix}$$
$$= 8\left[x - \tfrac{1}{3}x^3\right]_0^1 = 8\left(1 - \tfrac{1}{3}\right) = \tfrac{16}{3}$$

56. The curve $r = \cos^2(\theta/2)$ is completely traced with $0 \le \theta \le 2\pi$.

$$r^2 + (dr/d\theta)^2 = [\cos^2(\theta/2)]^2 + \left[2\cos(\theta/2)\cdot(-\sin(\theta/2))\cdot\tfrac{1}{2}\right]^2$$
$$= \cos^4(\theta/2) + \cos^2(\theta/2)\sin^2(\theta/2)$$
$$= \cos^2(\theta/2)[\cos^2(\theta/2) + \sin^2(\theta/2)]$$
$$= \cos^2(\theta/2)$$



$$L = \int_0^{2\pi} \sqrt{\cos^2(\theta/2)}\,d\theta = \int_0^{2\pi} |\cos(\theta/2)|\,d\theta = 2\int_0^{\pi} \cos(\theta/2)\,d\theta \qquad [\text{since } \cos(\theta/2) \ge 0 \text{ for } 0 \le \theta \le \pi]$$
$$= 4\int_0^{\pi/2} \cos u\,du \quad \left[u = \tfrac{1}{2}\theta\right] \quad = 4\big[\sin u\big]_0^{\pi/2} = 4(1-0) = 4$$

57. The graph is symmetric about the polar axis and touches the pole when $r = \cos(\theta/5) = 0$, that is, when $\theta/5 = \pi/2$ or $\theta = 5\pi/2$. $|r|$ is a maximum when $\theta = 0$, so the top half of the red section of the curve is traced starting from the polar axis $\theta = 0$ to $\theta = \pi$. Half of the blue section of the curve is then traced from $\theta = \pi$ to $\theta = 5\pi/2$ (the pole), and, by symmetry, the entire blue section is traced from $\theta = \pi$ to $\theta = 5\pi/2 + (5\pi/2 - \pi) = 4\pi$.

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \left(\cos\frac{\theta}{5}\right)^2 + \left(-\frac{1}{5}\sin\frac{\theta}{5}\right)^2 = \cos^2\left(\frac{\theta}{5}\right) + \frac{1}{25}\sin^2\left(\frac{\theta}{5}\right)$$

Thus, $L = \displaystyle\int_\pi^{4\pi} \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_\pi^{4\pi} \sqrt{\cos^2\left(\frac{\theta}{5}\right) + \frac{1}{25}\sin^2\left(\frac{\theta}{5}\right)}\,d\theta.$

58. $r = \dfrac{\sin\theta}{\theta}$ is positive and decreasing in the θ-interval $(0, \pi)$, so a portion of the red curve is traced out between $\theta = 0$ and $\theta = \pi/2$, followed by the blue section of the curve from $\theta = \pi/2$ to $\theta = 3\pi/2$.

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \left(\frac{\sin\theta}{\theta}\right)^2 + \left(\frac{\theta\cdot\cos\theta - \sin\theta\cdot 1}{\theta^2}\right)^2 = \frac{\sin^2\theta}{\theta^2} + \frac{(\theta\cos\theta - \sin\theta)^2}{\theta^4}$$

Thus, $L = \displaystyle\int_{\pi/2}^{3\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_{\pi/2}^{3\pi/2} \sqrt{\frac{\sin^2\theta}{\theta^2} + \frac{(\theta\cos\theta - \sin\theta)^2}{\theta^4}}\,d\theta.$

59. One loop of the curve $r = \cos 2\theta$ is traced with $-\pi/4 \le \theta \le \pi/4$.

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \cos^2 2\theta + (-2\sin 2\theta)^2 = \cos^2 2\theta + 4\sin^2 2\theta = 1 + 3\sin^2 2\theta \quad\Rightarrow$$

$$L = \int_{-\pi/4}^{\pi/4} \sqrt{1 + 3\sin^2 2\theta}\, d\theta \approx 2.4221.$$

60. $r^2 + \left(\dfrac{dr}{d\theta}\right)^2 = \tan^2\theta + (\sec^2\theta)^2 \quad\Rightarrow\quad L = \displaystyle\int_{\pi/6}^{\pi/3} \sqrt{\tan^2\theta + \sec^4\theta}\, d\theta \approx 1.2789$

61. The curve $r = \sin(6\sin\theta)$ is completely traced with $0 \le \theta \le \pi$. $\quad r = \sin(6\sin\theta) \quad\Rightarrow$

$$\frac{dr}{d\theta} = \cos(6\sin\theta)\cdot 6\cos\theta, \text{ so } r^2 + \left(\frac{dr}{d\theta}\right)^2 = \sin^2(6\sin\theta) + 36\cos^2\theta\cos^2(6\sin\theta) \quad\Rightarrow$$

$$L = \int_0^{\pi} \sqrt{\sin^2(6\sin\theta) + 36\cos^2\theta\,\cos^2(6\sin\theta)}\, d\theta \approx 8.0091.$$

62. The curve $r = \sin(\theta/4)$ is completely traced with $0 \le \theta \le 8\pi$. $\quad r = \sin(\theta/4) \quad\Rightarrow\quad \dfrac{dr}{d\theta} = \dfrac{1}{4}\cos(\theta/4)$, so

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \sin^2(\theta/4) + \tfrac{1}{16}\cos^2(\theta/4) \quad\Rightarrow\quad L = \int_0^{8\pi} \sqrt{\sin^2(\theta/4) + \tfrac{1}{16}\cos^2(\theta/4)}\, d\theta \approx 17.1568.$$

63. $r = 2\cos\theta \quad\Rightarrow\quad x = r\cos\theta = 2\cos^2\theta,\ y = r\sin\theta = 2\sin\theta\,\cos\theta = \sin 2\theta \quad\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos 2\theta}{2\cdot 2\cos\theta\,(-\sin\theta)} = \frac{\cos 2\theta}{-\sin 2\theta} = -\cot 2\theta$$

When $\theta = \dfrac{\pi}{3}$, $\dfrac{dy}{dx} = -\cot\left(2\cdot\dfrac{\pi}{3}\right) = \cot\dfrac{\pi}{3} = \dfrac{1}{\sqrt{3}}$. [*Another method:* Use Equation 3.]

64. $r = 2 + \sin 3\theta \quad\Rightarrow\quad x = r\cos\theta = (2 + \sin 3\theta)\cos\theta,\ y = r\sin\theta = (2 + \sin 3\theta)\sin\theta \quad\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(2 + \sin 3\theta)\cos\theta + \sin\theta(3\cos 3\theta)}{(2 + \sin 3\theta)(-\sin\theta) + \cos\theta(3\cos 3\theta)}$$

When $\theta = \dfrac{\pi}{4}$, $\dfrac{dy}{dx} = \dfrac{\left(2 + \sin\frac{3\pi}{4}\right)\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\left(3\cos\frac{3\pi}{4}\right)}{\left(2 + \sin\frac{3\pi}{4}\right)\left(-\sin\frac{\pi}{4}\right) + \cos\frac{\pi}{4}\left(3\cos\frac{3\pi}{4}\right)} = \dfrac{\left(2 + \frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\cdot 3\left(-\frac{\sqrt{2}}{2}\right)}{\left(2 + \frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2}\cdot 3\left(-\frac{\sqrt{2}}{2}\right)}$

$$= \frac{\sqrt{2} + \frac{1}{2} - \frac{3}{2}}{-\sqrt{2} - \frac{1}{2} - \frac{3}{2}} = \frac{\sqrt{2} - 1}{-\sqrt{2} - 2}, \text{ or, equivalently, } 2 - \frac{3}{2}\sqrt{2}.$$

65. $r = 1/\theta \quad\Rightarrow\quad x = r\cos\theta = (\cos\theta)/\theta,\ y = r\sin\theta = (\sin\theta)/\theta \quad\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta(-1/\theta^2) + (1/\theta)\cos\theta}{\cos\theta(-1/\theta^2) - (1/\theta)\sin\theta}\cdot\frac{\theta^2}{\theta^2} = \frac{-\sin\theta + \theta\cos\theta}{-\cos\theta - \theta\sin\theta}$$

When $\theta = \pi$, $\dfrac{dy}{dx} = \dfrac{-0 + \pi(-1)}{-(-1) - \pi(0)} = \dfrac{-\pi}{1} = -\pi$.

66. $r = \sin\theta + 2\cos\theta \quad\Rightarrow\quad x = r\cos\theta = \sin\theta\cos\theta + 2\cos^2\theta,\ y = r\sin\theta = \sin^2\theta + 2\sin\theta\cos\theta \quad\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\sin\theta\cos\theta - 2\sin^2\theta + 2\cos^2\theta}{-\sin^2\theta + \cos^2\theta - 4\sin\theta\cos\theta}$$

When $\theta = \dfrac{\pi}{2}$, $\dfrac{dy}{dx} = \dfrac{2(1)(0) - 2(1) + 2(0)}{-1 + 0 - 4(1)(0)} = \dfrac{-2}{-1} = 2$.

67. $r = \cos 2\theta \quad\Rightarrow\quad x = r\cos\theta = \cos 2\theta\,\cos\theta,\ y = r\sin\theta = \cos 2\theta\,\sin\theta \quad\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos 2\theta\,\cos\theta + \sin\theta\,(-2\sin 2\theta)}{\cos 2\theta\,(-\sin\theta) + \cos\theta\,(-2\sin 2\theta)}$$

When $\theta = \frac{\pi}{4}$, $\dfrac{dy}{dx} = \dfrac{0\big(\sqrt{2}/2\big) + \big(\sqrt{2}/2\big)(-2)}{0\big(-\sqrt{2}/2\big) + \big(\sqrt{2}/2\big)(-2)} = \dfrac{-\sqrt{2}}{-\sqrt{2}} = 1.$

68. $r = 1 + 2\cos\theta \quad\Rightarrow\quad x = r\cos\theta = (1 + 2\cos\theta)\cos\theta,\ y = r\sin\theta = (1 + 2\cos\theta)\sin\theta \quad\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(1 + 2\cos\theta)\cos\theta + \sin\theta\,(-2\sin\theta)}{(1 + 2\cos\theta)(-\sin\theta) + \cos\theta\,(-2\sin\theta)}$$

When $\theta = \frac{\pi}{3}$, $\dfrac{dy}{dx} = \dfrac{2\left(\frac{1}{2}\right) + \big(\sqrt{3}/2\big)\big(-\sqrt{3}\big)}{2\big(-\sqrt{3}/2\big) + \frac{1}{2}\big(-\sqrt{3}\big)} \cdot \dfrac{2}{2} = \dfrac{2 - 3}{-2\sqrt{3} - \sqrt{3}} = \dfrac{-1}{-3\sqrt{3}} = \dfrac{\sqrt{3}}{9}.$

69. $r = \sin\theta \quad\Rightarrow\quad x = r\cos\theta = \sin\theta\cos\theta,\ y = r\sin\theta = \sin^2\theta \quad\Rightarrow\quad \frac{dy}{d\theta} = 2\sin\theta\cos\theta = \sin 2\theta = 0 \quad\Rightarrow$

$2\theta = 0$ or $\pi \quad\Rightarrow\quad \theta = 0$ or $\frac{\pi}{2} \quad\Rightarrow\quad$ horizontal tangent at $(0, 0)$, and $\left(1, \frac{\pi}{2}\right)$.

$\frac{dx}{d\theta} = -\sin^2\theta + \cos^2\theta = \cos 2\theta = 0 \quad\Rightarrow\quad 2\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2} \quad\Rightarrow\quad \theta = \frac{\pi}{4}$ or $\frac{3\pi}{4} \quad\Rightarrow\quad$ vertical tangent at $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$

and $\left(\frac{1}{\sqrt{2}}, \frac{3\pi}{4}\right)$.

70. $r = 1 - \sin\theta \quad\Rightarrow\quad x = r\cos\theta = \cos\theta\,(1 - \sin\theta),\ \ y = r\sin\theta = \sin\theta\,(1 - \sin\theta) \quad\Rightarrow$

$\frac{dy}{d\theta} = \sin\theta\,(-\cos\theta) + (1 - \sin\theta)\,\cos\theta = \cos\theta\,(1 - 2\sin\theta) = 0 \quad\Rightarrow\quad \cos\theta = 0$ or $\sin\theta = \frac{1}{2} \quad\Rightarrow$

$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$, or $\frac{3\pi}{2} \quad\Rightarrow\quad$ horizontal tangent at $\left(\frac{1}{2}, \frac{\pi}{6}\right)$, $\left(\frac{1}{2}, \frac{5\pi}{6}\right)$, and $\left(2, \frac{3\pi}{2}\right)$.

$$\begin{aligned} \tfrac{dx}{d\theta} &= \cos\theta\,(-\cos\theta) + (1 - \sin\theta)(-\sin\theta) = -\cos^2\theta - \sin\theta + \sin^2\theta = 2\sin^2\theta - \sin\theta - 1 \\ &= (2\sin\theta + 1)(\sin\theta - 1) = 0 \quad\Rightarrow \end{aligned}$$

$\sin\theta = -\frac{1}{2}$ or $1 \quad\Rightarrow\quad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$, or $\frac{\pi}{2} \quad\Rightarrow\quad$ vertical tangent at $\left(\frac{3}{2}, \frac{7\pi}{6}\right)$, $\left(\frac{3}{2}, \frac{11\pi}{6}\right)$, and $\left(0, \frac{\pi}{2}\right)$.

Note that the tangent is vertical, not horizontal, when $\theta = \frac{\pi}{2}$, since

$$\lim_{\theta\to(\pi/2)^-} \frac{dy/d\theta}{dx/d\theta} = \lim_{\theta\to(\pi/2)^-} \frac{\cos\theta\,(1 - 2\sin\theta)}{(2\sin\theta + 1)(\sin\theta - 1)} = \infty \text{ and } \lim_{\theta\to(\pi/2)^+} \frac{dy/d\theta}{dx/d\theta} = -\infty.$$

71. $r = 1 + \cos\theta \quad\Rightarrow\quad x = r\cos\theta = \cos\theta\,(1 + \cos\theta),\ \ y = r\sin\theta = \sin\theta\,(1 + \cos\theta) \quad\Rightarrow$

$\frac{dy}{d\theta} = (1 + \cos\theta)\,\cos\theta - \sin^2\theta = 2\cos^2\theta + \cos\theta - 1 = (2\cos\theta - 1)(\cos\theta + 1) = 0 \quad\Rightarrow\quad \cos\theta = \frac{1}{2}$ or $-1 \quad\Rightarrow$

$\theta = \frac{\pi}{3}, \pi$, or $\frac{5\pi}{3} \quad\Rightarrow\quad$ horizontal tangent at $\left(\frac{3}{2}, \frac{\pi}{3}\right)$, $(0, \pi)$, and $\left(\frac{3}{2}, \frac{5\pi}{3}\right)$.

$\frac{dx}{d\theta} = -(1 + \cos\theta)\sin\theta - \cos\theta\sin\theta = -\sin\theta\,(1 + 2\cos\theta) = 0 \quad\Rightarrow\quad \sin\theta = 0$ or $\cos\theta = -\frac{1}{2} \quad\Rightarrow$

$\theta = 0, \pi, \frac{2\pi}{3}$, or $\frac{4\pi}{3} \quad\Rightarrow\quad$ vertical tangent at $(2, 0)$, $\left(\frac{1}{2}, \frac{2\pi}{3}\right)$, and $\left(\frac{1}{2}, \frac{4\pi}{3}\right)$.

Note that the tangent is horizontal, not vertical when $\theta = \pi$, since $\displaystyle\lim_{\theta\to\pi} \frac{dy/d\theta}{dx/d\theta} = 0$.

72. $r = e^\theta \quad\Rightarrow\quad x = r\cos\theta = e^\theta\cos\theta,\ \ y = r\sin\theta = e^\theta\sin\theta \quad\Rightarrow$

$\frac{dy}{d\theta} = e^\theta\sin\theta + e^\theta\cos\theta = e^\theta(\sin\theta + \cos\theta) = 0 \quad\Rightarrow\quad \sin\theta = -\cos\theta \quad\Rightarrow\quad \tan\theta = -1 \quad\Rightarrow$

$\theta = -\frac{1}{4}\pi + n\pi$ [n any integer] $\quad\Rightarrow\quad$ horizontal tangents at $\left(e^{\pi(n - 1/4)}, \pi\left(n - \frac{1}{4}\right)\right)$.

[continued]

$\frac{dx}{d\theta} = e^\theta \cos\theta - e^\theta \sin\theta = e^\theta(\cos\theta - \sin\theta) = 0 \quad\Rightarrow\quad \sin\theta = \cos\theta \quad\Rightarrow\quad \tan\theta = 1 \quad\Rightarrow$

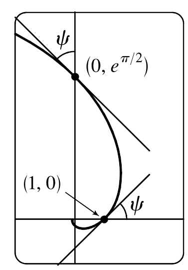
$\theta = \frac{1}{4}\pi + n\pi$ [n any integer] $\Rightarrow$ vertical tangents at $\left(e^{\pi(n+1/4)}, \pi\left(n+\frac{1}{4}\right)\right)$.

73. $\tan\psi = \tan(\phi - \theta) = \dfrac{\tan\phi - \tan\theta}{1 + \tan\phi\,\tan\theta} = \dfrac{\dfrac{dy}{dx} - \tan\theta}{1 + \dfrac{dy}{dx}\tan\theta} = \dfrac{\dfrac{dy/d\theta}{dx/d\theta} - \tan\theta}{1 + \dfrac{dy/d\theta}{dx/d\theta}\tan\theta}$

$$= \frac{\dfrac{dy}{d\theta} - \dfrac{dx}{d\theta}\tan\theta}{\dfrac{dx}{d\theta} + \dfrac{dy}{d\theta}\tan\theta} = \frac{\left(\dfrac{dr}{d\theta}\sin\theta + r\cos\theta\right) - \tan\theta\left(\dfrac{dr}{d\theta}\cos\theta - r\sin\theta\right)}{\left(\dfrac{dr}{d\theta}\cos\theta - r\sin\theta\right) + \tan\theta\left(\dfrac{dr}{d\theta}\sin\theta + r\cos\theta\right)} = \frac{r\cos\theta + r\cdot\dfrac{\sin^2\theta}{\cos\theta}}{\dfrac{dr}{d\theta}\cos\theta + \dfrac{dr}{d\theta}\cdot\dfrac{\sin^2\theta}{\cos\theta}}$$

$$= \frac{r\cos^2\theta + r\sin^2\theta}{\dfrac{dr}{d\theta}\cos^2\theta + \dfrac{dr}{d\theta}\sin^2\theta} = \frac{r}{dr/d\theta}$$

74. (a) $r = e^\theta \quad\Rightarrow\quad dr/d\theta = e^\theta$, so by Exercise 73, $\tan\psi = r/e^\theta = 1 \quad\Rightarrow$
$\psi = \arctan 1 = \frac{\pi}{4}$.



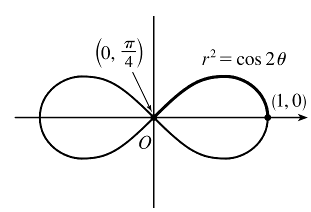
(b) The Cartesian equation of the tangent line at $(1, 0)$ is $y = x - 1$, and that of the tangent line at $(0, e^{\pi/2})$ is $y = e^{\pi/2} - x$.

(c) Let a be the tangent of the angle between the tangent and radial lines, that is, $a = \tan\psi$. Then, by Exercise 73, $a = \dfrac{r}{dr/d\theta} \quad\Rightarrow\quad \dfrac{dr}{d\theta} = \dfrac{1}{a}r \quad\Rightarrow$
$r = Ce^{\theta/a}$ (by Theorem 9.4.2).

75. (a) From (10.2.9),

$$S = \int_a^b 2\pi y\sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\,d\theta$$
$$= \int_a^b 2\pi y\sqrt{r^2 + (dr/d\theta)^2}\,d\theta \qquad \text{[from the derivation of Equation 10.4.6]}$$
$$= \int_a^b 2\pi r\sin\theta\sqrt{r^2 + (dr/d\theta)^2}\,d\theta$$

(b) The curve $r^2 = \cos 2\theta$ goes through the pole when $\cos 2\theta = 0 \quad\Rightarrow$
$2\theta = \frac{\pi}{2} \quad\Rightarrow\quad \theta = \frac{\pi}{4}$. We'll rotate the curve from $\theta = 0$ to $\theta = \frac{\pi}{4}$ and double this value to obtain the total surface area generated.



$r^2 = \cos 2\theta \quad\Rightarrow\quad 2r\,\dfrac{dr}{d\theta} = -2\sin 2\theta \quad\Rightarrow\quad \left(\dfrac{dr}{d\theta}\right)^2 = \dfrac{\sin^2 2\theta}{r^2} = \dfrac{\sin^2 2\theta}{\cos 2\theta}$.

$$S = 2\int_0^{\pi/4} 2\pi\sqrt{\cos 2\theta}\,\sin\theta\sqrt{\cos 2\theta + \left(\sin^2 2\theta\right)/\cos 2\theta}\,d\theta = 4\pi\int_0^{\pi/4}\sqrt{\cos 2\theta}\,\sin\theta\sqrt{\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}}\,d\theta$$
$$= 4\pi\int_0^{\pi/4}\sqrt{\cos 2\theta}\,\sin\theta\,\frac{1}{\sqrt{\cos 2\theta}}\,d\theta = 4\pi\int_0^{\pi/4}\sin\theta\,d\theta = 4\pi\big[-\cos\theta\big]_0^{\pi/4} = -4\pi\left(\tfrac{\sqrt{2}}{2} - 1\right) = 2\pi\left(2 - \sqrt{2}\right)$$

76. (a) Rotation around $\theta = \frac{\pi}{2}$ is the same as rotation around the y-axis, that is, $S = \int_a^b 2\pi x\,ds$ where $ds = \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$ for a parametric equation, and for the special case of a polar equation, $x = r\cos\theta$ and

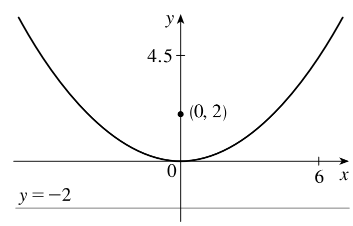
$ds = \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\, d\theta = \sqrt{r^2 + (dr/d\theta)^2}\, d\theta$ [see the derivation of Equation 10.4.6]. Therefore, for a polar equation rotated around $\theta = \frac{\pi}{2}$, $S = \int_a^b 2\pi r \cos\theta \sqrt{r^2 + (dr/d\theta)^2}\, d\theta$.

(b) As in the solution for Exercise 75(b), we can double the surface area generated by rotating the curve from $\theta = 0$ to $\theta = \frac{\pi}{4}$ to obtain the total surface area.

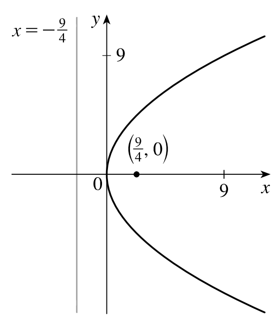
$$\begin{aligned} S &= 2\int_0^{\pi/4} 2\pi\sqrt{\cos 2\theta}\cos\theta\sqrt{\cos 2\theta + (\sin^2 2\theta)/\cos 2\theta}\, d\theta \\ &= 4\pi\int_0^{\pi/4}\sqrt{\cos 2\theta}\cos\theta\sqrt{\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}}\, d\theta \\ &= 4\pi\int_0^{\pi/4}\sqrt{\cos 2\theta}\cos\theta\,\frac{1}{\sqrt{\cos 2\theta}}\, d\theta = 4\pi\int_0^{\pi/4}\cos\theta\, d\theta \\ &= 4\pi\big[\sin\theta\big]_0^{\pi/4} = 4\pi\left(\frac{\sqrt{2}}{2} - 0\right) = 2\sqrt{2}\,\pi \end{aligned}$$

10.5 Conic Sections

1. $x^2 = 8y$ and $x^2 = 4py \;\Rightarrow\; 4p = 8 \;\Leftrightarrow\; p = 2$. The vertex is $(0,0)$, the focus is $(0,2)$, and the directrix is $y = -2$.

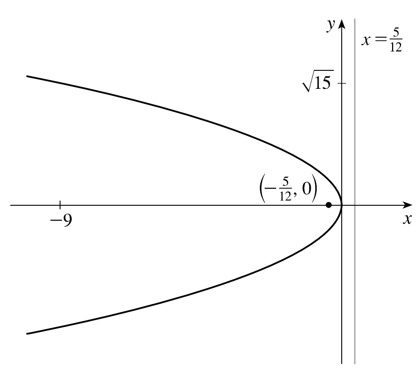


2. $9x = y^2$, so $4p = 9 \;\Leftrightarrow\; p = \frac{9}{4}$. The vertex is $(0,0)$, the focus is $\left(\frac{9}{4}, 0\right)$, and the directrix is $x = -\frac{9}{4}$.



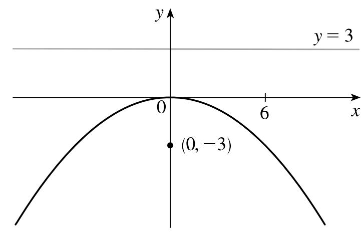
3. $5x + 3y^2 = 0 \;\Leftrightarrow\; y^2 = -\frac{5}{3}x$, so $4p = -\frac{5}{3} \;\Leftrightarrow\; p = -\frac{5}{12}$.

The vertex is $(0,0)$, the focus is $\left(-\frac{5}{12}, 0\right)$, and the directrix is $x = \frac{5}{12}$.

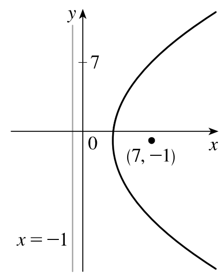


4. $x^2 + 12y = 0 \quad \Leftrightarrow \quad x^2 = -12y$, so $4p = -12 \quad \Leftrightarrow \quad p = -3$.

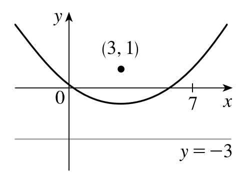
The vertex is $(0, 0)$, the focus is $(0, -3)$, and the directrix is $y = 3$.



5. $(y + 1)^2 = 16(x - 3)$, so $4p = 16 \quad \Leftrightarrow \quad p = 4$. The vertex is $(3, -1)$, the focus is $(3 + 4, -1) = (7, -1)$, and the directrix is $x = 3 - 4 = -1$.



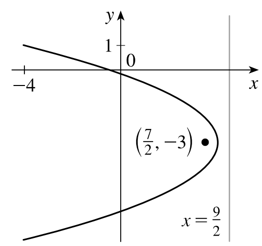
6. $(x - 3)^2 = 8\,(y + 1)$, so $4p = 8 \quad \Leftrightarrow \quad p = 2$. The vertex is $(3, -1)$, the focus is $(3, -1 + 2) = (3, 1)$, and the directrix is $y = -1 - 2 = -3$.



7. $y^2 + 6y + 2x + 1 = 0 \quad \Leftrightarrow \quad y^2 + 6y = -2x - 1 \quad \Leftrightarrow$

$y^2 + 6y + 9 = -2x + 8 \quad \Leftrightarrow \quad (y + 3)^2 = -2(x - 4)$, so $4p = -2 \quad \Leftrightarrow$

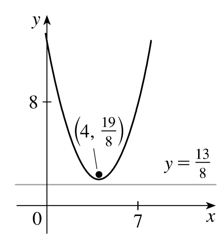
$p = -\frac{1}{2}$. The vertex is $(4, -3)$, the focus is $\left(\frac{7}{2}, -3\right)$, and the directrix is $x = \frac{9}{2}$.



8. $2x^2 - 16x - 3y + 38 = 0 \quad \Leftrightarrow \quad 2x^2 - 16x = 3y - 38 \quad \Leftrightarrow$

$2(x^2 - 8x + 16) = 3y - 38 + 32 \quad \Leftrightarrow \quad 2(x - 4)^2 = 3y - 6 \quad \Leftrightarrow$

$(x - 4)^2 = \frac{3}{2}(y - 2)$, so $4p = \frac{3}{2} \quad \Leftrightarrow \quad p = \frac{3}{8}$. The vertex is $(4, 2)$, the focus is $\left(4, \frac{19}{8}\right)$, and the directrix is $y = \frac{13}{8}$.

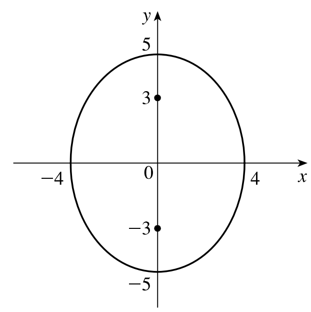


9. The equation has the form $y^2 = 4px$, where $p < 0$. Since the parabola passes through $(-1, 1)$, we have $1^2 = 4p(-1)$, so $4p = -1$ and an equation is $y^2 = -x$ or $x = -y^2$. $\quad 4p = -1$, so $p = -\frac{1}{4}$ and the focus is $\left(-\frac{1}{4}, 0\right)$ while the directrix is $x = \frac{1}{4}$.

10. The vertex is $(2, -2)$, so the equation is of the form $(x - 2)^2 = 4p(y + 2)$, where $p > 0$. The point $(0, 0)$ is on the parabola, so $4 = 4p(2) \quad \Rightarrow \quad 4p = 2$. Thus, an equation is $(x - 2)^2 = 2(y + 2)$. $\quad 4p = 2$, so $p = \frac{1}{2}$ and the focus is $\left(2, -\frac{3}{2}\right)$ while the directrix is $y = -\frac{5}{2}$.

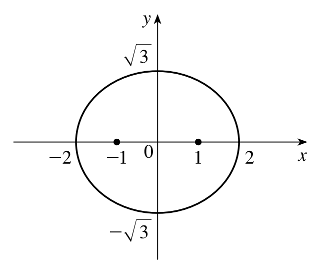
11. $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1 \quad \Rightarrow \quad a = \sqrt{25} = 5,\, b = \sqrt{16} = 4,$

$c = \sqrt{a^2 - b^2} = \sqrt{25 - 16} = \sqrt{9} = 3$. The ellipse is centered at $(0, 0)$ with vertices $(0, \pm 5)$. The foci are $(0, \pm 3)$.



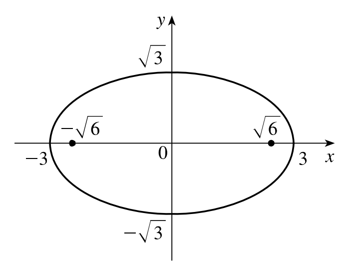
12. $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1 \quad \Rightarrow \quad a = \sqrt{4} = 2,\, b = \sqrt{3},$

$c = \sqrt{a^2 - b^2} = \sqrt{4 - 3} = \sqrt{1} = 1$. The ellipse is centered at $(0, 0)$ with vertices $(\pm 2, 0)$. The foci are $(\pm 1, 0)$.



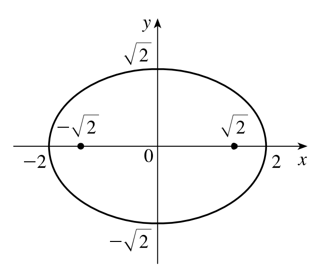
13. $x^2 + 3y^2 = 9 \quad \Leftrightarrow \quad \dfrac{x^2}{9} + \dfrac{y^2}{3} = 1 \quad \Rightarrow \quad a = \sqrt{9} = 3,\, b = \sqrt{3},$

$c = \sqrt{a^2 - b^2} = \sqrt{9 - 3} = \sqrt{6}$. The ellipse is centered at $(0, 0)$ with vertices $(\pm 3, 0)$. The foci are $\left(\pm\sqrt{6}, 0\right)$.



14. $x^2 = 4 - 2y^2 \quad \Leftrightarrow \quad x^2 + 2y^2 = 4 \quad \Leftrightarrow \quad \dfrac{x^2}{4} + \dfrac{y^2}{2} = 1 \quad \Rightarrow$

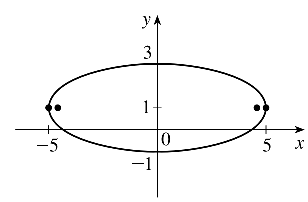
$a = \sqrt{4} = 2,\, b = \sqrt{2},\, c = \sqrt{a^2 - b^2} = \sqrt{4 - 2} = \sqrt{2}$. The ellipse is centered at $(0, 0)$ with vertices $(\pm 2, 0)$. The foci are $\left(\pm\sqrt{2}, 0\right)$.



15. $4x^2 + 25y^2 - 50y = 75 \quad \Leftrightarrow \quad 4x^2 + 25(y^2 - 2y + 1) = 75 + 25 \quad \Leftrightarrow$

$4x^2 + 25(y - 1)^2 = 100 \quad \Leftrightarrow \quad \dfrac{x^2}{25} + \dfrac{(y - 1)^2}{4} = 1 \quad \Rightarrow \quad a = \sqrt{25} = 5,$

$b = \sqrt{4} = 2,\, c = \sqrt{25 - 4} = \sqrt{21}$. The ellipse is centered at $(0, 1)$ with vertices $(\pm 5, 1)$. The foci are $\left(\pm\sqrt{21}, 1\right)$.

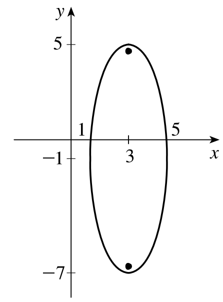


16. $9x^2 - 54x + y^2 + 2y + 46 = 0 \quad \Leftrightarrow$

$9(x^2 - 6x + 9) + y^2 + 2y + 1 = -46 + 81 + 1 \quad \Leftrightarrow$

$9(x-3)^2 + (y+1)^2 = 36 \quad \Leftrightarrow \quad \dfrac{(x-3)^2}{4} + \dfrac{(y+1)^2}{36} = 1 \quad \Rightarrow$

$a = \sqrt{36} = 6$, $b = \sqrt{4} = 2$, $c = \sqrt{36-4} = \sqrt{32} = 4\sqrt{2}$. The ellipse is centered at $(3, -1)$ with vertices $(3, 5)$ and $(3, -7)$. The foci are $\left(3, -1 \pm 4\sqrt{2}\right)$.



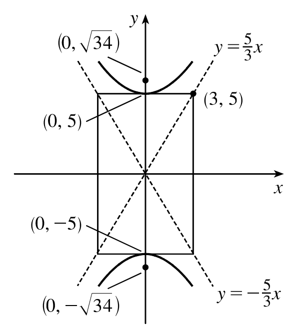
17. The center is $(0, 0)$, $a = 3$, and $b = 2$, so an equation is $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$. $c = \sqrt{a^2 - b^2} = \sqrt{5}$, so the foci are $\left(0, \pm\sqrt{5}\right)$.

18. The ellipse is centered at $(2, 1)$, with $a = 3$ and $b = 2$. An equation is $\dfrac{(x-2)^2}{9} + \dfrac{(y-1)^2}{4} = 1$. $c = \sqrt{a^2 - b^2} = \sqrt{5}$, so the foci are $\left(2 \pm \sqrt{5}, 1\right)$.

19. $\dfrac{y^2}{25} - \dfrac{x^2}{9} = 1 \quad \Rightarrow \quad a = 5, b = 3, c = \sqrt{a^2 + b^2} = \sqrt{25 + 9} = \sqrt{34} \quad \Rightarrow$

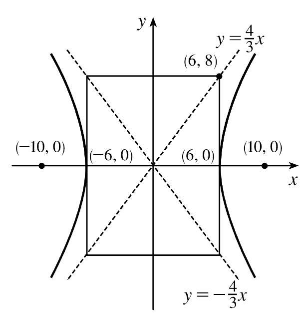
center $(0, 0)$, vertices $(0, \pm 5)$, foci $\left(0, \pm\sqrt{34}\right)$, asymptotes $y = \pm\frac{5}{3}x$.

Note: It is helpful to draw a $2a$-by-$2b$ rectangle whose center is the center of the hyperbola. The asymptotes are the extended diagonals of the rectangle.



20. $\dfrac{x^2}{36} - \dfrac{y^2}{64} = 1 \quad \Rightarrow \quad a = 6, b = 8, c = \sqrt{a^2 + b^2} = \sqrt{36 + 64} = 10 \quad \Rightarrow$

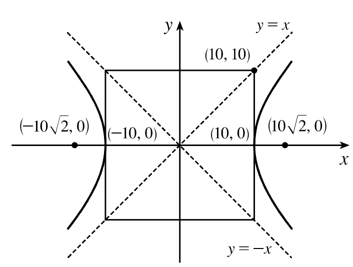
center $(0, 0)$, vertices $(\pm 6, 0)$, foci $(\pm 10, 0)$, asymptotes $y = \pm\frac{8}{6}x = \pm\frac{4}{3}x$



21. $x^2 - y^2 = 100 \quad \Leftrightarrow \quad \dfrac{x^2}{100} - \dfrac{y^2}{100} = 1 \quad \Rightarrow \quad a = b = 10$,

$c = \sqrt{100 + 100} = 10\sqrt{2} \quad \Rightarrow \quad$ center $(0, 0)$, vertices $(\pm 10, 0)$,

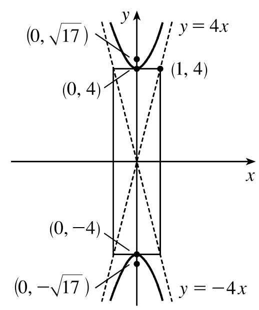
foci $\left(\pm 10\sqrt{2}, 0\right)$, asymptotes $y = \pm\frac{10}{10}x = \pm x$



22. $y^2 - 16x^2 = 16 \quad \Leftrightarrow \quad \dfrac{y^2}{16} - \dfrac{x^2}{1} = 1 \quad \Rightarrow \quad a = 4, b = 1,$

$c = \sqrt{16+1} = \sqrt{17} \quad \Rightarrow \quad$ center $(0,0)$, vertices $(0, \pm 4)$,

foci $\left(0, \pm\sqrt{17}\right)$, asymptotes $y = \pm\frac{4}{1}x = \pm 4x$

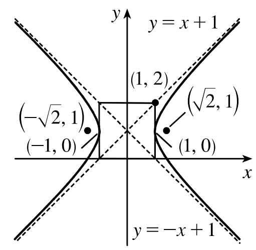


23. $x^2 - y^2 + 2y = 2 \quad \Leftrightarrow \quad x^2 - (y^2 - 2y + 1) = 2 - 1 \quad \Leftrightarrow$

$\dfrac{x^2}{1} - \dfrac{(y-1)^2}{1} = 1 \quad \Rightarrow \quad a = b = 1, c = \sqrt{1+1} = \sqrt{2} \quad \Rightarrow$

center $(0,1)$, vertices $(\pm 1, 1)$, foci $\left(\pm\sqrt{2}, 1\right)$,

asymptotes $y - 1 = \pm\frac{1}{1}x = \pm x$.



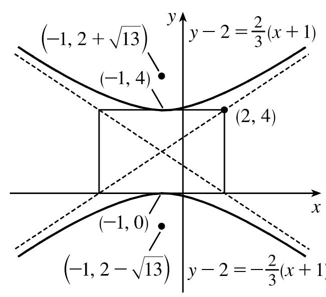
24. $9y^2 - 4x^2 - 36y - 8x = 4 \quad \Leftrightarrow$

$9(y^2 - 4y + 4) - 4(x^2 + 2x + 1) = 4 + 36 - 4 \quad \Leftrightarrow$

$9(y-2)^2 - 4(x+1)^2 = 36 \quad \Leftrightarrow \quad \dfrac{(y-2)^2}{4} - \dfrac{(x+1)^2}{9} = 1 \quad \Rightarrow$

$a = 2, b = 3, c = \sqrt{4+9} = \sqrt{13} \quad \Rightarrow \quad$ center $(-1, 2)$, vertices

$(-1, 2 \pm 2)$, foci $\left(-1, 2 \pm \sqrt{13}\right)$, asymptotes $y - 2 = \pm\frac{2}{3}(x+1)$.



25. The hyperbola has vertices $(\pm 3, 0)$, which lie on the x-axis, so the equation has the form $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with $a = 3$. The point $(5, 4)$ is on the hyperbola, so $\dfrac{5^2}{3^2} - \dfrac{4^2}{b^2} = 1 \quad \Rightarrow \quad \dfrac{25}{9} - \dfrac{16}{b^2} = 1 \quad \Rightarrow \quad \dfrac{16}{9} = \dfrac{16}{b^2} \quad \Rightarrow \quad b^2 = 9 \quad \Rightarrow \quad b = 3$. Thus, an equation is $\dfrac{x^2}{9} - \dfrac{y^2}{9} = 1$. Now, $c^2 = a^2 + b^2 = 9 + 9 = 18 \quad \Rightarrow \quad c = \sqrt{18} = 3\sqrt{2}$, so the foci are $\left(\pm 3\sqrt{2}, 0\right)$, while the asymptotes are $y = \pm\dfrac{a}{b}x = \pm\dfrac{3}{3}x = \pm x$.

26. The hyperbola has vertices $(0, \pm 2)$, which lie on the y-axis, so the equation has the form $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$ with $a = 2$. The point $(8, 6)$ is on the hyperbola, so $\dfrac{6^2}{2^2} - \dfrac{8^2}{b^2} = 1 \quad \Rightarrow \quad 9 - \dfrac{64}{b^2} = 1 \quad \Rightarrow \quad 8 = \dfrac{64}{b^2} \quad \Rightarrow \quad b^2 = 8 \quad \Rightarrow \quad b = 2\sqrt{2}$. Thus, an equation is $\dfrac{y^2}{4} - \dfrac{x^2}{8} = 1$. Now, $c^2 = a^2 + b^2 = 4 + 8 = 12 \quad \Rightarrow \quad c = \sqrt{12} = 2\sqrt{3}$, so the foci are $\left(0, \pm 2\sqrt{3}\right)$, while the asymptotes are $y = \pm\dfrac{a}{b}x = \pm\dfrac{2}{2\sqrt{2}}x = \pm\dfrac{1}{\sqrt{2}}x$.

27. $4x^2 = y^2 + 4 \quad \Leftrightarrow \quad 4x^2 - y^2 = 4 \quad \Leftrightarrow \quad \dfrac{x^2}{1} - \dfrac{y^2}{4} = 1$. This is an equation of a *hyperbola* with vertices $(\pm 1, 0)$. The foci are at $\left(\pm\sqrt{1+4}, 0\right) = \left(\pm\sqrt{5}, 0\right)$.

28. $4x^2 = y + 4 \quad \Leftrightarrow \quad x^2 = \frac{1}{4}(y+4)$. This is an equation of a *parabola* with $4p = \frac{1}{4}$, so $p = \frac{1}{16}$. The vertex is $(0, -4)$ and the focus is $\left(0, -4 + \frac{1}{16}\right) = \left(0, -\frac{63}{16}\right)$.

29. $x^2 = 4y - 2y^2 \quad \Leftrightarrow \quad x^2 + 2y^2 - 4y = 0 \quad \Leftrightarrow \quad x^2 + 2(y^2 - 2y + 1) = 2 \quad \Leftrightarrow \quad x^2 + 2(y-1)^2 = 2 \quad \Leftrightarrow$

$\dfrac{x^2}{2} + \dfrac{(y-1)^2}{1} = 1$. This is an equation of an *ellipse* with vertices at $\left(\pm\sqrt{2}, 1\right)$. The foci are at $\left(\pm\sqrt{2-1}, 1\right) = (\pm 1, 1)$.

30. $y^2 - 2 = x^2 - 2x \quad \Leftrightarrow \quad y^2 - x^2 + 2x = 2 \quad \Leftrightarrow \quad y^2 - (x^2 - 2x + 1) = 2 - 1 \quad \Leftrightarrow \quad \dfrac{y^2}{1} - \dfrac{(x-1)^2}{1} = 1$. This is an equation of a *hyperbola* with vertices $(1, \pm 1)$. The foci are at $\left(1, \pm\sqrt{1+1}\right) = \left(1, \pm\sqrt{2}\right)$.

31. $3x^2 - 6x - 2y = 1 \quad \Leftrightarrow \quad 3x^2 - 6x = 2y + 1 \quad \Leftrightarrow \quad 3(x^2 - 2x + 1) = 2y + 1 + 3 \quad \Leftrightarrow \quad 3(x-1)^2 = 2y + 4 \quad \Leftrightarrow$ $(x-1)^2 = \frac{2}{3}(y+2)$. This is an equation of a *parabola* with $4p = \frac{2}{3}$, so $p = \frac{1}{6}$. The vertex is $(1, -2)$ and the focus is $\left(1, -2 + \frac{1}{6}\right) = \left(1, -\frac{11}{6}\right)$.

32. $x^2 - 2x + 2y^2 - 8y + 7 = 0 \quad \Leftrightarrow \quad (x^2 - 2x + 1) + 2(y^2 - 4y + 4) = -7 + 1 + 8 \quad \Leftrightarrow \quad (x-1)^2 + 2(y-2)^2 = 2 \quad \Leftrightarrow$

$\dfrac{(x-1)^2}{2} + \dfrac{(y-2)^2}{1} = 1$. This is an equation of an *ellipse* with vertices at $\left(1 \pm \sqrt{2}, 2\right)$. The foci are at $\left(1 \pm \sqrt{2-1}, 2\right) = (1 \pm 1, 2)$.

33. The parabola with vertex $(0, 0)$ and focus $(1, 0)$ opens to the right and has $p = 1$, so its equation is $y^2 = 4px$, or $y^2 = 4x$.

34. The parabola with focus $(0, 0)$ and directrix $y = 6$ has vertex $(0, 3)$ and opens downward, so $p = -3$ and its equation is $(x-0)^2 = 4p(y-3)$, or $x^2 = -12(y-3)$.

35. The distance from the focus $(-4, 0)$ to the directrix $x = 2$ is $2 - (-4) = 6$, so the distance from the focus to the vertex is $\frac{1}{2}(6) = 3$ and the vertex is $(-1, 0)$. Since the focus is to the left of the vertex, $p = -3$. An equation is $y^2 = 4p(x+1) \quad \Rightarrow$ $y^2 = -12(x+1)$.

36. The parabola with vertex $(2, 3)$ and focus $(2, -1)$ opens downward and has $p = -1 - 3 = -4$, so its equation is $(x-2)^2 = 4p(y-3)$, or $(x-2)^2 = -16(y-3)$.

37. The parabola with vertex $(3, -1)$ having a horizontal axis has equation $[y - (-1)]^2 = 4p(x-3)$. Since it passes through $(-15, 2)$, $(2+1)^2 = 4p(-15-3) \quad \Rightarrow \quad 9 = 4p(-18) \quad \Rightarrow \quad 4p = -\frac{1}{2}$. An equation is $(y+1)^2 = -\frac{1}{2}(x-3)$.

38. The parabola with vertical axis and passing through $(0, 4)$ has equation $y = ax^2 + bx + 4$. It also passes through $(1, 3)$ and $(-2, -6)$, so

$$\begin{cases} 3 = a + b + 4 \\ -6 = 4a - 2b + 4 \end{cases} \Rightarrow \begin{cases} -1 = a + b \\ -10 = 4a - 2b \end{cases} \Rightarrow \begin{cases} -1 = a + b \\ -5 = 2a - b \end{cases}$$

Adding the last two equations gives us $3a = -6$, or $a = -2$. Since $a + b = -1$, we have $b = 1$, and an equation is $y = -2x^2 + x + 4$.

39. The ellipse with foci $(\pm 2, 0)$ and vertices $(\pm 5, 0)$ has center $(0, 0)$ and a horizontal major axis, with $a = 5$ and $c = 2$, so $b^2 = a^2 - c^2 = 25 - 4 = 21$. An equation is $\dfrac{x^2}{25} + \dfrac{y^2}{21} = 1$.

40. The ellipse with foci $\left(0, \pm\sqrt{2}\right)$ and vertices $(0, \pm 2)$ has center $(0, 0)$ and a vertical major axis, with $a = 2$ and $c = \sqrt{2}$, so $b^2 = a^2 - c^2 = 4 - 2 = 2$. An equation is $\dfrac{x^2}{2} + \dfrac{y^2}{4} = 1$.

41. Since the vertices are $(0, 0)$ and $(0, 8)$, the ellipse has center $(0, 4)$ with a vertical axis and $a = 4$. The foci at $(0, 2)$ and $(0, 6)$ are 2 units from the center, so $c = 2$ and $b = \sqrt{a^2 - c^2} = \sqrt{4^2 - 2^2} = \sqrt{12}$. An equation is $\dfrac{(x-0)^2}{b^2} + \dfrac{(y-4)^2}{a^2} = 1 \;\Rightarrow\; \dfrac{x^2}{12} + \dfrac{(y-4)^2}{16} = 1$.

42. Since the foci are $(0, -1)$ and $(8, -1)$, the ellipse has center $(4, -1)$ with a horizontal axis and $c = 4$. The vertex $(9, -1)$ is 5 units from the center, so $a = 5$ and $b = \sqrt{a^2 - c^2} = \sqrt{5^2 - 4^2} = \sqrt{9}$. An equation is $\dfrac{(x-4)^2}{a^2} + \dfrac{(y+1)^2}{b^2} = 1 \;\Rightarrow\; \dfrac{(x-4)^2}{25} + \dfrac{(y+1)^2}{9} = 1$.

43. An equation of an ellipse with center $(-1, 4)$ and vertex $(-1, 0)$ is $\dfrac{(x+1)^2}{b^2} + \dfrac{(y-4)^2}{4^2} = 1$. The focus $(-1, 6)$ is 2 units from the center, so $c = 2$. Thus, $b^2 + 2^2 = 4^2 \;\Rightarrow\; b^2 = 12$, and the equation is $\dfrac{(x+1)^2}{12} + \dfrac{(y-4)^2}{16} = 1$.

44. Foci $F_1(-4, 0)$ and $F_2(4, 0) \;\Rightarrow\; c = 4$ and an equation is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. The ellipse passes through $P(-4, 1.8)$, so $2a = |PF_1| + |PF_2| \;\Rightarrow\; 2a = 1.8 + \sqrt{8^2 + (1.8)^2} \;\Rightarrow\; 2a = 1.8 + 8.2 \;\Rightarrow\; a = 5$.
$b^2 = a^2 - c^2 = 25 - 16 = 9$ and the equation is $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$.

45. An equation of a hyperbola with vertices $(\pm 3, 0)$ is $\dfrac{x^2}{3^2} - \dfrac{y^2}{b^2} = 1$. Foci $(\pm 5, 0) \;\Rightarrow\; c = 5$ and $3^2 + b^2 = 5^2 \;\Rightarrow\; b^2 = 25 - 9 = 16$, so the equation is $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$.

46. An equation of a hyperbola with vertices $(0, \pm 2)$ is $\dfrac{y^2}{2^2} - \dfrac{x^2}{b^2} = 1$. Foci $(0, \pm 5) \;\Rightarrow\; c = 5$ and $2^2 + b^2 = 5^2 \;\Rightarrow\; b^2 = 25 - 4 = 21$, so the equation is $\dfrac{y^2}{4} - \dfrac{x^2}{21} = 1$.

47. The center of a hyperbola with vertices $(-3, -4)$ and $(-3, 6)$ is $(-3, 1)$, so $a = 5$ and an equation is $\dfrac{(y-1)^2}{5^2} - \dfrac{(x+3)^2}{b^2} = 1$. Foci $(-3, -7)$ and $(-3, 9) \;\Rightarrow\; c = 8$, so $5^2 + b^2 = 8^2 \;\Rightarrow\; b^2 = 64 - 25 = 39$ and the equation is $\dfrac{(y-1)^2}{25} - \dfrac{(x+3)^2}{39} = 1$.

48. The center of a hyperbola with vertices $(-1, 2)$ and $(7, 2)$ is $(3, 2)$, so $a = 4$ and an equation is $\dfrac{(x-3)^2}{4^2} - \dfrac{(y-2)^2}{b^2} = 1$. Foci $(-2, 2)$ and $(8, 2) \;\Rightarrow\; c = 5$, so $4^2 + b^2 = 5^2 \;\Rightarrow\; b^2 = 25 - 16 = 9$ and the equation is $\dfrac{(x-3)^2}{16} - \dfrac{(y-2)^2}{9} = 1$.

49. The center of a hyperbola with vertices $(\pm 3, 0)$ is $(0, 0)$, so $a = 3$ and an equation is $\dfrac{x^2}{3^2} - \dfrac{y^2}{b^2} = 1$.

Asymptotes $y = \pm 2x \quad\Rightarrow\quad \dfrac{b}{a} = 2 \quad\Rightarrow\quad b = 2(3) = 6$ and the equation is $\dfrac{x^2}{9} - \dfrac{y^2}{36} = 1$.

50. The center of a hyperbola with foci $(2, 0)$ and $(2, 8)$ is $(2, 4)$, so $c = 4$ and an equation is $\dfrac{(y-4)^2}{a^2} - \dfrac{(x-2)^2}{b^2} = 1$.

The asymptote $y = 3 + \frac{1}{2}x$ has slope $\frac{1}{2}$, so $\dfrac{a}{b} = \dfrac{1}{2} \quad\Rightarrow\quad b = 2a$ and $a^2 + b^2 = c^2 \quad\Rightarrow\quad a^2 + (2a)^2 = 4^2 \quad\Rightarrow$

$5a^2 = 16 \quad\Rightarrow\quad a^2 = \frac{16}{5}$ and so $b^2 = 16 - \frac{16}{5} = \frac{64}{5}$. Thus, an equation is $\dfrac{(y-4)^2}{16/5} - \dfrac{(x-2)^2}{64/5} = 1$.

51. In Figure 8, we see that the point on the ellipse closest to a focus is the closer vertex (which is a distance $a - c$ from it) while the farthest point is the other vertex (at a distance of $a + c$). So for this lunar orbit, $(a - c) + (a + c) = 2a = (1728 + 110) + (1728 + 314)$, or $a = 1940$; and $(a + c) - (a - c) = 2c = 314 - 110$, or $c = 102$. Thus, $b^2 = a^2 - c^2 = 3{,}753{,}196$, and the equation is $\dfrac{x^2}{3{,}763{,}600} + \dfrac{y^2}{3{,}753{,}196} = 1$.

52. (a) Choose V to be the origin, with x-axis through V and F. Then F is $(p, 0)$, A is $(p, 5)$, so substituting A into the equation $y^2 = 4px$ gives $25 = 4p^2$ so $p = \frac{5}{2}$ and $y^2 = 10x$.

(b) $x = 11 \quad\Rightarrow\quad y = \sqrt{110} \quad\Rightarrow\quad |CD| = 2\sqrt{110}$

53. (a) Set up the coordinate system so that A is $(-320, 0)$ and B is $(320, 0)$.

$|PA| - |PB| = (1200)(300) = 360{,}000 \text{ m} = 360 \text{ km} = 2a \quad\Rightarrow\quad a = 180$, and $c = 320$ so

$b^2 = c^2 - a^2 = 70{,}000 \qquad\Rightarrow\quad \dfrac{x^2}{32{,}400} - \dfrac{y^2}{70{,}000} = 1$.

(b) Due north of $B \quad\Rightarrow\quad x = 320 \quad\Rightarrow\quad \dfrac{(320)^2}{32{,}400} - \dfrac{y^2}{70{,}000} = 1 \quad\Rightarrow\quad y = \dfrac{3{,}500}{9} \approx 389$ km

54. $|PF_1| - |PF_2| = \pm 2a \quad\Leftrightarrow\quad \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a \quad\Leftrightarrow$

$\sqrt{(x+c)^2 + y^2} = \sqrt{(x-c)^2 + y^2} \pm 2a \quad\Leftrightarrow\quad (x+c)^2 + y^2 = (x-c)^2 + y^2 + 4a^2 \pm 4a\sqrt{(x-c)^2 + y^2} \quad\Leftrightarrow$

$4cx - 4a^2 = \pm 4a\sqrt{(x-c)^2 + y^2} \quad\Leftrightarrow\quad c^2x^2 - 2a^2cx + a^4 = a^2(x^2 - 2cx + c^2 + y^2) \quad\Leftrightarrow$

$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2) \quad\Leftrightarrow\quad b^2x^2 - a^2y^2 = a^2b^2$ [where $b^2 = c^2 - a^2$] $\quad\Leftrightarrow\quad \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

55. The function whose graph is the upper branch of this hyperbola is concave upward. The function is

$y = f(x) = a\sqrt{1 + \dfrac{x^2}{b^2}} = \dfrac{a}{b}\sqrt{b^2 + x^2}$, so $y' = \dfrac{a}{b}x(b^2 + x^2)^{-1/2}$ and

$y'' = \dfrac{a}{b}\left[(b^2 + x^2)^{-1/2} - x^2(b^2 + x^2)^{-3/2}\right] = ab(b^2 + x^2)^{-3/2} > 0$ for all x, and so f is concave upward.

56. We can follow exactly the same sequence of steps as in the derivation of Formula 4, except we use the points $(1, 1)$ and $(-1, -1)$ in the distance formula (first equation of that derivation) so $\sqrt{(x-1)^2 + (y-1)^2} + \sqrt{(x+1)^2 + (y+1)^2} = 4$

will lead (after moving the second term to the right, squaring, and simplifying) to $2\sqrt{(x+1)^2+(y+1)^2} = x+y+4$, which, after squaring and simplifying again, leads to $3x^2 - 2xy + 3y^2 = 8$.

57. (a) If $k > 16$, then $k - 16 > 0$, and $\dfrac{x^2}{k} + \dfrac{y^2}{k-16} = 1$ is an *ellipse* since it is the sum of two squares on the left side.

(b) If $0 < k < 16$, then $k - 16 < 0$, and $\dfrac{x^2}{k} + \dfrac{y^2}{k-16} = 1$ is a *hyperbola* since it is the difference of two squares on the left side.

(c) If $k < 0$, then $k - 16 < 0$, and there is *no curve* since the left side is the sum of two negative terms, which cannot equal 1.

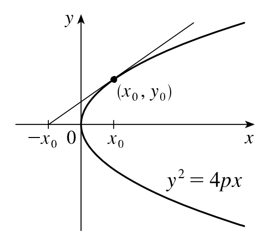
(d) In case (a), $a^2 = k$, $b^2 = k - 16$, and $c^2 = a^2 - b^2 = 16$, so the foci are at $(\pm 4, 0)$. In case (b), $k - 16 < 0$, so $a^2 = k$, $b^2 = 16 - k$, and $c^2 = a^2 + b^2 = 16$, and so again the foci are at $(\pm 4, 0)$.

58. (a) $y^2 = 4px \quad\Rightarrow\quad 2yy' = 4p \quad\Rightarrow\quad y' = \dfrac{2p}{y}$, so the tangent line is

$$y - y_0 = \frac{2p}{y_0}(x - x_0) \quad\Rightarrow\quad yy_0 - y_0^2 = 2p(x - x_0) \quad\Leftrightarrow$$

$$yy_0 - 4px_0 = 2px - 2px_0 \quad\Rightarrow\quad yy_0 = 2p(x + x_0).$$

(b) The x-intercept is $-x_0$.



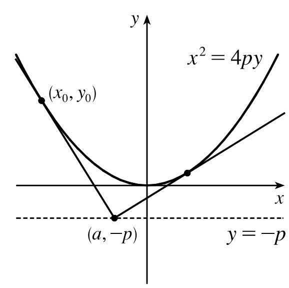
59. $x^2 = 4py \quad\Rightarrow\quad 2x = 4py' \quad\Rightarrow\quad y' = \dfrac{x}{2p}$, so the tangent line at (x_0, y_0) is $y - \dfrac{x_0^2}{4p} = \dfrac{x_0}{2p}\,(x - x_0)$. This line passes through the point $(a, -p)$ on the directrix, so $-p - \dfrac{x_0^2}{4p} = \dfrac{x_0}{2p}\,(a - x_0) \quad\Rightarrow\quad -4p^2 - x_0^2 = 2ax_0 - 2x_0^2 \quad\Leftrightarrow$

$x_0^2 - 2ax_0 - 4p^2 = 0 \quad\Leftrightarrow\quad x_0^2 - 2ax_0 + a^2 = a^2 + 4p^2 \quad\Leftrightarrow$

$(x_0 - a)^2 = a^2 + 4p^2 \quad\Leftrightarrow\quad x_0 = a \pm \sqrt{a^2 + 4p^2}$. The slopes of the tangent lines at $x = a \pm \sqrt{a^2 + 4p^2}$ are $\dfrac{a \pm \sqrt{a^2 + 4p^2}}{2p}$, so the product of the two slopes is

$$\frac{a + \sqrt{a^2 + 4p^2}}{2p} \cdot \frac{a - \sqrt{a^2 + 4p^2}}{2p} = \frac{a^2 - (a^2 + 4p^2)}{4p^2} = \frac{-4p^2}{4p^2} = -1,$$

showing that the tangent lines are perpendicular.



60. Without a loss of generality, let the ellipse, hyperbola, and foci be as shown in the figure.

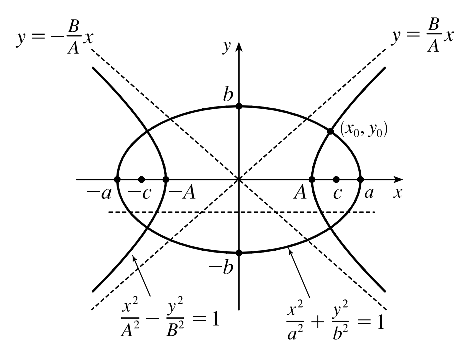
The curves intersect (eliminate y^2) $\Rightarrow$

$$B^2\left(\frac{x^2}{A^2} - \frac{y^2}{B^2}\right) + b^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = B^2 + b^2 \quad\Rightarrow$$

$$\frac{B^2x^2}{A^2} + \frac{b^2x^2}{a^2} = B^2 + b^2 \quad\Rightarrow\quad x^2\left(\frac{B^2}{A^2} + \frac{b^2}{a^2}\right) = B^2 + b^2 \quad\Rightarrow$$

$$x^2 = \frac{B^2 + b^2}{\dfrac{a^2B^2 + b^2A^2}{A^2a^2}} = \frac{A^2a^2(B^2 + b^2)}{a^2B^2 + b^2A^2}.$$

Similarly, $y^2 = \dfrac{B^2b^2(a^2 - A^2)}{b^2A^2 + a^2B^2}$.



[continued]

Next we find the slopes of the tangent lines of the curves: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \;\Rightarrow\; \dfrac{2x}{a^2}+\dfrac{2yy'}{b^2}=0 \;\Rightarrow\; \dfrac{yy'}{b^2}=-\dfrac{x}{a^2} \;\Rightarrow$ $y'_E=-\dfrac{b^2}{a^2}\dfrac{x}{y}$ and $\dfrac{x^2}{A^2}-\dfrac{y^2}{B^2}=1 \;\Rightarrow\; \dfrac{2x}{A^2}-\dfrac{2yy'}{B^2}=0 \;\Rightarrow\; \dfrac{yy'}{B^2}=\dfrac{x}{A^2} \;\Rightarrow\; y'_H=\dfrac{B^2}{A^2}\dfrac{x}{y}$. The product of the slopes at (x_0,y_0) is $y'_E\,y'_H=-\dfrac{b^2B^2x_0^2}{a^2A^2y_0^2}=-\dfrac{b^2B^2\left[\dfrac{A^2a^2(B^2+b^2)}{a^2B^2+b^2A^2}\right]}{a^2A^2\left[\dfrac{B^2b^2(a^2-A^2)}{b^2A^2+a^2B^2}\right]}=-\dfrac{B^2+b^2}{a^2-A^2}$. Since $a^2-b^2=c^2$ and $A^2+B^2=c^2$, we have $a^2-b^2=A^2+B^2 \;\Rightarrow\; a^2-A^2=b^2+B^2$, so the product of the slopes is -1, and hence, the tangent lines at each point of intersection are perpendicular.

61. $9x^2+4y^2=36 \;\Leftrightarrow\; \dfrac{x^2}{4}+\dfrac{y^2}{9}=1$. We use the parametrization $x=2\cos t$, $y=3\sin t$, $0\le t\le 2\pi$. The circumference is given by

$$\begin{aligned} L&=\textstyle\int_0^{2\pi}\sqrt{(dx/dt)^2+(dy/dt)^2}\,dt=\int_0^{2\pi}\sqrt{(-2\sin t)^2+(3\cos t)^2}\,dt\\ &=\textstyle\int_0^{2\pi}\sqrt{4\sin^2 t+9\cos^2 t}\,dt=\int_0^{2\pi}\sqrt{4+5\cos^2 t}\,dt\end{aligned}$$

Now use Simpson's Rule with $n=8$, $\Delta t=\dfrac{2\pi-0}{8}=\dfrac{\pi}{4}$, and $f(t)=\sqrt{4+5\cos^2 t}$ to get

$L\approx S_8=\frac{\pi/4}{3}\left[f(0)+4f\left(\frac{\pi}{4}\right)+2f\left(\frac{\pi}{2}\right)+4f\left(\frac{3\pi}{4}\right)+2f(\pi)+4f\left(\frac{5\pi}{4}\right)+2f\left(\frac{3\pi}{2}\right)+4f\left(\frac{7\pi}{4}\right)+f(2\pi)\right]\approx 15.9.$

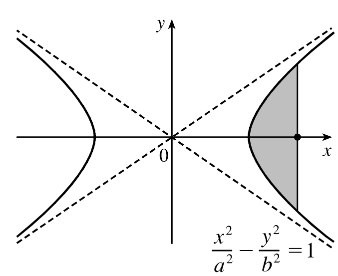
62. The length of the major axis is $2a$, so $a=\frac{1}{2}(1.18\times 10^{10})=5.9\times 10^9$. The length of the minor axis is $2b$, so $b=\frac{1}{2}(1.14\times 10^{10})=5.7\times 10^9$. An equation of the ellipse is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, or converting into parametric equations, $x=a\cos\theta$ and $y=b\sin\theta$. So

$$L=4\textstyle\int_0^{\pi/2}\sqrt{(dx/d\theta)^2+(dy/d\theta)^2}\,d\theta=4\int_0^{\pi/2}\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\,d\theta$$

Using Simpson's Rule with $n=10$, $\Delta\theta=\frac{\pi/2-0}{10}=\frac{\pi}{20}$, and $f(\theta)=\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$, we get

$$L\approx 4\cdot S_{10}=4\cdot\tfrac{\pi}{20\cdot 3}\left[f(0)+4f\left(\tfrac{\pi}{20}\right)+2f\left(\tfrac{2\pi}{20}\right)+\cdots+2f\left(\tfrac{8\pi}{20}\right)+4f\left(\tfrac{9\pi}{20}\right)+f\left(\tfrac{\pi}{2}\right)\right]\approx 3.64\times 10^{10}\text{ km}$$

63. $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1 \;\Rightarrow\; \dfrac{y^2}{b^2}=\dfrac{x^2-a^2}{a^2} \;\Rightarrow\; y=\pm\dfrac{b}{a}\sqrt{x^2-a^2}$.



$A=2\displaystyle\int_a^c\frac{b}{a}\sqrt{x^2-a^2}\,dx\overset{39}{=}\frac{2b}{a}\left[\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|\right]_a^c$

$=\dfrac{b}{a}\left[c\sqrt{c^2-a^2}-a^2\ln\left|c+\sqrt{c^2-a^2}\right|+a^2\ln|a|\right]$

Since $a^2+b^2=c^2$, $c^2-a^2=b^2$, and $\sqrt{c^2-a^2}=b$.

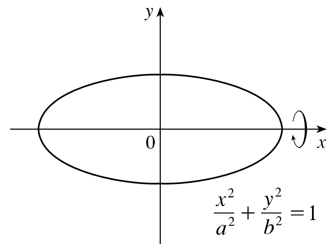
$=\dfrac{b}{a}\left[cb-a^2\ln(c+b)+a^2\ln a\right]=\dfrac{b}{a}\left[cb+a^2(\ln a-\ln(b+c))\right]$

$=b^2c/a+ab\ln[a/(b+c)]$, where $c^2=a^2+b^2$.

64. (a) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \quad\Rightarrow\quad \dfrac{y^2}{b^2}=\dfrac{a^2-x^2}{a^2} \quad\Rightarrow\quad y=\pm\dfrac{b}{a}\sqrt{a^2-x^2}.$

$$V=\int_{-a}^{a}\pi\left(\frac{b}{a}\sqrt{a^2-x^2}\right)^2dx=2\pi\frac{b^2}{a^2}\int_0^a(a^2-x^2)\,dx$$

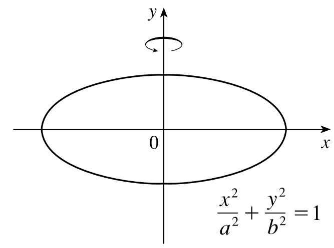
$$=\frac{2\pi b^2}{a^2}\left[a^2x-\tfrac{1}{3}x^3\right]_0^a=\frac{2\pi b^2}{a^2}\left(\frac{2a^3}{3}\right)=\frac{4}{3}\pi b^2a$$



(b) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \quad\Rightarrow\quad \dfrac{x^2}{a^2}=\dfrac{b^2-y^2}{b^2} \quad\Rightarrow\quad x=\pm\dfrac{a}{b}\sqrt{b^2-y^2}.$

$$V=\int_{-b}^{b}\pi\left(\frac{a}{b}\sqrt{b^2-y^2}\right)^2dy=2\pi\frac{a^2}{b^2}\int_0^b(b^2-y^2)\,dy$$

$$=\frac{2\pi a^2}{b^2}\left[b^2y-\tfrac{1}{3}y^3\right]_0^b=\frac{2\pi a^2}{b^2}\left(\frac{2b^3}{3}\right)=\frac{4}{3}\pi a^2b$$



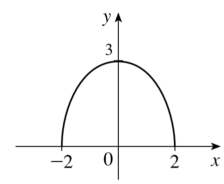
65. $9x^2+4y^2=36 \quad\Leftrightarrow\quad \dfrac{x^2}{4}+\dfrac{y^2}{9}=1 \quad\Rightarrow\quad a=3,\, b=2$. By symmetry, $\overline{x}=0$. By Example 2 in Section 7.3, the area of the top half of the ellipse is $\frac{1}{2}(\pi ab)=3\pi$. Solve $9x^2+4y^2=36$ for y to get an equation for the top half of the ellipse:

$9x^2+4y^2=36 \quad\Leftrightarrow\quad 4y^2=36-9x^2 \quad\Leftrightarrow\quad y^2=\frac{9}{4}(4-x^2) \quad\Rightarrow\quad y=\frac{3}{2}\sqrt{4-x^2}$. Now

$$\overline{y}=\frac{1}{A}\int_a^b\frac{1}{2}[f(x)]^2\,dx=\frac{1}{3\pi}\int_{-2}^{2}\frac{1}{2}\left(\frac{3}{2}\sqrt{4-x^2}\right)^2dx=\frac{3}{8\pi}\int_{-2}^{2}(4-x^2)\,dx$$

$$=\frac{3}{8\pi}\cdot 2\int_0^2(4-x^2)\,dx=\frac{3}{4\pi}\left[4x-\frac{1}{3}x^3\right]_0^2=\frac{3}{4\pi}\left(\frac{16}{3}\right)=\frac{4}{\pi}$$

so the centroid is $(0, 4/\pi)$.



66. (a) Consider the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ with $a>b$, so that the major axis is the x-axis. Let the ellipse be parametrized by $x=a\cos t,\ y=b\sin t,\ 0\le t\le 2\pi$. Then

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=a^2\sin^2t+b^2\cos^2t=a^2(1-\cos^2t)+b^2\cos^2t=a^2+(b^2-a^2)\cos^2t=a^2-c^2\cos^2t,$$

where $c^2=a^2-b^2$. Using symmetry and rotating the ellipse about the major axis gives us surface area

$$S=\int 2\pi y\,ds=2\int_0^{\pi/2}2\pi(b\sin t)\sqrt{a^2-c^2\cos^2t}\,dt=4\pi b\int_c^0\sqrt{a^2-u^2}\left(-\frac{1}{c}\,du\right)\qquad\left[\begin{array}{l}u=c\cos t\\ du=-c\sin t\,dt\end{array}\right]$$

$$=\frac{4\pi b}{c}\int_0^c\sqrt{a^2-u^2}\,du\overset{30}{=}\frac{4\pi b}{c}\left[\frac{u}{2}\sqrt{a^2-u^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right)\right]_0^c=\frac{2\pi b}{c}\left[c\sqrt{a^2-c^2}+a^2\sin^{-1}\left(\frac{c}{a}\right)\right]$$

$$=\frac{2\pi b}{c}\left[bc+a^2\sin^{-1}\left(\frac{c}{a}\right)\right]$$

(b) As in part (a),

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=a^2\sin^2t+b^2\cos^2t=a^2\sin^2t+b^2(1-\sin^2t)=b^2+(a^2-b^2)\sin^2t=b^2+c^2\sin^2t.$$

Rotating about the minor axis gives us

$$S=\int 2\pi x\,ds=2\int_0^{\pi/2}2\pi(a\cos t)\sqrt{b^2+c^2\sin^2 t}\,dt=4\pi a\int_0^c\sqrt{b^2+u^2}\left(\frac{1}{c}\,du\right)\qquad\left[\begin{array}{c}u=c\sin t\\ du=c\cos t\,dt\end{array}\right]$$

$$\overset{21}{=}\frac{4\pi a}{c}\left[\frac{u}{2}\sqrt{b^2+u^2}+\frac{b^2}{2}\ln\left(u+\sqrt{b^2+u^2}\right)\right]_0^c=\frac{2\pi a}{c}\left[c\sqrt{b^2+c^2}+b^2\ln\left(c+\sqrt{b^2+c^2}\right)-b^2\ln b\right]$$

$$=\frac{2\pi a}{c}\left[ac+b^2\ln\left(\frac{a+c}{b}\right)\right]$$

67. Differentiating implicitly, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1\;\Rightarrow\;\dfrac{2x}{a^2}+\dfrac{2yy'}{b^2}=0\;\Rightarrow\;y'=-\dfrac{b^2x}{a^2y}$ $[y\neq 0]$. Thus, the slope of the tangent line at P is $-\dfrac{b^2x_1}{a^2y_1}$. The slope of F_1P is $\dfrac{y_1}{x_1+c}$ and of F_2P is $\dfrac{y_1}{x_1-c}$. By the formula in Problem 21 in Problems Plus following Chapter 3, we have

$$\tan\alpha=\frac{\dfrac{y_1}{x_1+c}+\dfrac{b^2x_1}{a^2y_1}}{1-\dfrac{b^2x_1y_1}{a^2y_1(x_1+c)}}=\frac{a^2y_1^2+b^2x_1(x_1+c)}{a^2y_1(x_1+c)-b^2x_1y_1}=\frac{a^2b^2+b^2cx_1}{c^2x_1y_1+a^2cy_1}\qquad\left[\begin{array}{c}\text{using } b^2x_1^2+a^2y_1^2=a^2b^2,\\ \text{and } a^2-b^2=c^2\end{array}\right]$$

$$=\frac{b^2\left(cx_1+a^2\right)}{cy_1\left(cx_1+a^2\right)}=\frac{b^2}{cy_1}$$

and
$$\tan\beta=\frac{-\dfrac{b^2x_1}{a^2y_1}-\dfrac{y_1}{x_1-c}}{1-\dfrac{b^2x_1y_1}{a^2y_1(x_1-c)}}=\frac{-a^2y_1^2-b^2x_1(x_1-c)}{a^2y_1\,(x_1-c)-b^2x_1y_1}=\frac{-a^2b^2+b^2cx_1}{c^2x_1y_1-a^2cy_1}=\frac{b^2\left(cx_1-a^2\right)}{cy_1\left(cx_1-a^2\right)}=\frac{b^2}{cy_1}$$

Thus, $\alpha=\beta$.

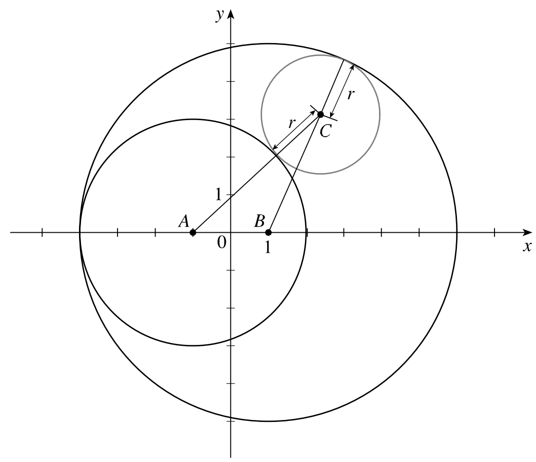
68. The slopes of the line segments F_1P and F_2P are $\dfrac{y_1}{x_1+c}$ and $\dfrac{y_1}{x_1-c}$, where P is (x_1,y_1). Differentiating implicitly, $\dfrac{2x}{a^2}-\dfrac{2yy'}{b^2}=0\;\Rightarrow\;y'=\dfrac{b^2x}{a^2y}\;\Rightarrow$ the slope of the tangent at P is $\dfrac{b^2x_1}{a^2y_1}$, so by the formula in Problem 21 in Problems Plus following Chapter 3,

$$\tan\alpha=\frac{\dfrac{b^2x_1}{a^2y_1}-\dfrac{y_1}{x_1+c}}{1+\dfrac{b^2x_1y_1}{a^2y_1(x_1+c)}}=\frac{b^2x_1(x_1+c)-a^2y_1^2}{a^2y_1(x_1+c)+b^2x_1y_1}=\frac{b^2(cx_1+a^2)}{cy_1(cx_1+a^2)}\quad\left[\begin{array}{c}\text{using } x_1^2/a^2-y_1^2/b^2=1,\\ \text{and } a^2+b^2=c^2\end{array}\right]=\frac{b^2}{cy_1}$$

and
$$\tan\beta=\frac{-\dfrac{b^2x_1}{a^2y_1}+\dfrac{y_1}{x_1-c}}{1+\dfrac{b^2x_1y_1}{a^2y_1(x_1-c)}}=\frac{-b^2x_1(x_1-c)+a^2y_1^2}{a^2y_1(x_1-c)+b^2x_1y_1}=\frac{b^2(cx_1-a^2)}{cy_1(cx_1-a^2)}=\frac{b^2}{cy_1}$$

So $\alpha=\beta$.

69. Let C be the center of a circle (gray) with radius r that is tangent to both black circles (see the figure). We wish to show that $AC+BC$ is constant for all values of r, that is, for any circle drawn tangent to both black circles. The smaller black circle has radius 3, so $AC = 3 + r$, and the larger black circle has radius 5, so $BC = 5 - r$. Hence, $AC + BC = 3 + r + 5 - r = 8$, which is a constant. Since the sum of the distances from C to $(-1, 0)$ and $(1, 0)$ is constant, the centers of all the circles lie on an ellipse with foci $(\pm 1, 0)$ $[c = 1]$. The sum of the distances from the foci to any point on the ellipse is $2a$, so $2a = 8 \Rightarrow a = 4$. Now, $c^2 = a^2 - b^2 \Rightarrow 1^2 = 4^2 - b^2 \Rightarrow b^2 = 15$. Thus, the ellipse has equation $\dfrac{x^2}{16} + \dfrac{y^2}{15} = 1$.



10.6 Conic Sections in Polar Coordinates

1. The directrix $x = 2$ is to the right of the focus at the origin, so we use the form with "$+e\cos\theta$" in the denominator. (See Theorem 6 and Figure 2.) $e = 1$ for a parabola, so an equation is $r = \dfrac{ed}{1 + e\cos\theta} = \dfrac{1 \cdot 2}{1 + 1\cos\theta} = \dfrac{2}{1 + \cos\theta}$.

2. The directrix $y = 6$ is above the focus at the origin, so we use the form with "$+e\sin\theta$" in the denominator. An equation of the ellipse is $r = \dfrac{ed}{1 + e\sin\theta} = \dfrac{\frac{1}{3} \cdot 6}{1 + \frac{1}{3}\sin\theta} = \dfrac{6}{3 + \sin\theta}$.

3. The directrix $y = -4$ is below the focus at the origin, so we use the form with "$-e\sin\theta$" in the denominator. An equation of the hyperbola is $r = \dfrac{ed}{1 - e\sin\theta} = \dfrac{2 \cdot 4}{1 - 2\sin\theta} = \dfrac{8}{1 - 2\sin\theta}$.

4. The directrix $x = -3$ is to the left of the focus at the origin, so we use the form with "$-e\cos\theta$" in the denominator. An equation of the hyperbola is $r = \dfrac{ed}{1 - e\cos\theta} = \dfrac{\frac{5}{2} \cdot 3}{1 - \frac{5}{2}\cos\theta} = \dfrac{15}{2 - 5\cos\theta}$.

5. The vertex $(2, \pi)$ is to the left of the focus at the origin, so we use the form with "$-e\cos\theta$" in the denominator. An equation of the ellipse is $r = \dfrac{ed}{1 - e\cos\theta}$. Using eccentricity $e = \dfrac{2}{3}$ with $\theta = \pi$ and $r = 2$, we get $2 = \dfrac{\frac{2}{3}d}{1 - \frac{2}{3}(-1)} \Rightarrow$

$2 = \dfrac{2d}{5} \Rightarrow d = 5$, so we have $r = \dfrac{\frac{2}{3}(5)}{1 - \frac{2}{3}\cos\theta} = \dfrac{10}{3 - 2\cos\theta}$.

6. The directrix $r = 4\csc\theta$ (equivalent to $r\sin\theta = 4$ or $y = 4$) is above the focus at the origin, so we will use the form with "$+e\sin\theta$" in the denominator. The distance from the focus to the directrix is $d = 4$, so an equation of the ellipse is

$r = \dfrac{ed}{1 + e\sin\theta} = \dfrac{(0.6)(4)}{1 + 0.6\sin\theta} \cdot \dfrac{5}{5} = \dfrac{12}{5 + 3\sin\theta}$.

7. The vertex $\left(3, \frac{\pi}{2}\right)$ is 3 units above the focus at the origin, so the directrix is 6 units above the focus ($d = 6$), and we use the form "$+e\sin\theta$" in the denominator. $e = 1$ for a parabola, so an equation is $r = \dfrac{ed}{1 + e\sin\theta} = \dfrac{1(6)}{1 + 1\sin\theta} = \dfrac{6}{1 + \sin\theta}$.

8. The directrix $r = -2\sec\theta$ (equivalent to $r\cos\theta = -2$ or $x = -2$) is left of the focus at the origin, so we will use the form with "$-e\cos\theta$" in the denominator. The distance from the focus to the directrix is $d = 2$, so an equation of the hyperbola is $r = \dfrac{ed}{1 - e\cos\theta} = \dfrac{2(2)}{1 - 2\cos\theta} = \dfrac{4}{1 - 2\cos\theta}$.

9. $r = \dfrac{3}{1 - \sin\theta}$, where $e = 1$, so the conic is a parabola. If $\sin\theta$ appears in the denominator, use 0 and π for θ. If $\cos\theta$ appears in the denominator, use $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ for θ. Thus, when $\theta = 0$ or π, $r = 3$. Hence, the equation is matched with graph VI.

10. $r = \dfrac{9}{1 + 2\cos\theta}$, where $e = 2 > 1$, so the conic is a hyperbola. When $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, $r = 9$. Hence, the equation is matched with graph III.

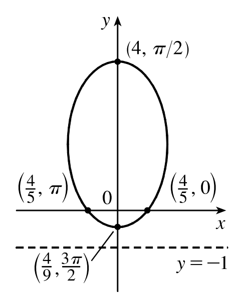
11. $r = \dfrac{12}{8 - 7\cos\theta} \cdot \dfrac{1/8}{1/8} = \dfrac{3/2}{1 - \frac{7}{8}\cos\theta}$, where $e = \dfrac{7}{8} < 1$, so the conic is an ellipse. When $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, $r = \frac{12}{8}$. Hence, the equation is matched with graph II.

12. $r = \dfrac{12}{4 + 3\sin\theta} \cdot \dfrac{1/4}{1/4} = \dfrac{3}{1 + \frac{3}{4}\sin\theta}$, where $e = \dfrac{3}{4} < 1$, so the conic is an ellipse. When $\theta = 0$ or π, $r = \frac{12}{4}$. Hence, the equation is matched with graph V.

13. $r = \dfrac{5}{2 + 3\sin\theta} \cdot \dfrac{1/2}{1/2} = \dfrac{5/2}{1 + \frac{3}{2}\sin\theta}$, where $e = \dfrac{3}{2} > 1$, so the conic is a hyperbola. When $\theta = 0$ or π, $r = \frac{5}{2}$. Hence, the equation is matched with graph IV.

14. $r = \dfrac{3}{2 - 2\cos\theta} \cdot \dfrac{1/2}{1/2} = \dfrac{3/2}{1 - \cos\theta}$, where $e = 1$, so the conic is a parabola. When $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, $r = \frac{3}{2}$. Hence, the equation is matched with graph I.

15. $r = \dfrac{4}{5 - 4\sin\theta} \cdot \dfrac{1/5}{1/5} = \dfrac{4/5}{1 - \frac{4}{5}\sin\theta}$, where $e = \frac{4}{5}$ and $ed = \frac{4}{5} \quad\Rightarrow\quad d = 1$.



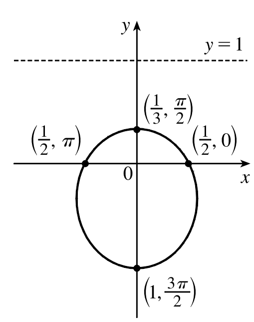
(a) Eccentricity $= e = \frac{4}{5}$

(b) Since $e = \frac{4}{5} < 1$, the conic is an ellipse.

(c) Since "$-\ e\sin\theta$" appears in the denominator, the directrix is below the focus at the origin, $d = |Fl| = 1$, so an equation of the directrix is $y = -1$.

(d) The vertices are $\left(4, \frac{\pi}{2}\right)$ and $\left(\frac{4}{9}, \frac{3\pi}{2}\right)$.

16. $r = \dfrac{1}{2 + \sin\theta} \cdot \dfrac{1/2}{1/2} = \dfrac{1/2}{1 + \frac{1}{2}\sin\theta}$, where $e = \dfrac{1}{2}$ and $ed = \dfrac{1}{2} \quad\Rightarrow\quad d = 1$.



(a) Eccentricity $= e = \dfrac{1}{2}$

(b) Since $e = \dfrac{1}{2} < 1$, the conic is an ellipse.

(c) Since "$+e\sin\theta$" appears in the denominator, the directrix is above the focus at the origin, $d = |Fl| = 1$, so an equation of the directrix is $y = 1$.

(d) The vertices are $\left(\frac{1}{3}, \frac{\pi}{2}\right)$ and $\left(1, \frac{3\pi}{2}\right)$.

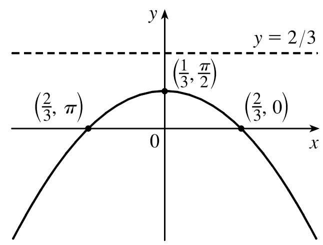
17. $r = \dfrac{2}{3+3\sin\theta}\cdot\dfrac{1/3}{1/3} = \dfrac{2/3}{1+1\sin\theta}$, where $e = 1$ and $ed = \frac{2}{3} \;\Rightarrow\; d = \frac{2}{3}$.

(a) Eccentricity $= e = 1$

(b) Since $e = 1$, the conic is a parabola.

(c) Since "$+\ e\sin\theta$" appears in the denominator, the directrix is above the focus at the origin. $d = |Fl| = \frac{2}{3}$, so an equation of the directrix is $y = \frac{2}{3}$.

(d) The vertex is at $\left(\frac{1}{3}, \frac{\pi}{2}\right)$, midway between the focus and directrix.



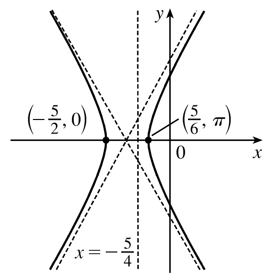
18. $r = \dfrac{5}{2-4\cos\theta}\cdot\dfrac{1/2}{1/2} = \dfrac{5/2}{1-2\cos\theta}$, where $e = 2$ and $ed = \dfrac{5}{2} \;\Rightarrow\; d = \dfrac{5}{4}$.

(a) Eccentricity $= e = 2$

(b) Since $e = 2 > 1$, the conic is a hyperbola.

(c) Since "$-e\cos\theta$" appears in the denominator, the directrix is to the left the focus at the origin. $d = |Fl| = \frac{5}{4}$, so an equation of the directrix is $x = -\frac{5}{4}$.

(d) The vertices are $\left(-\frac{5}{2}, 0\right)$ and $\left(\frac{5}{6}, \pi\right)$, so the center is midway between them, that is, $\left(\frac{5}{3}, \pi\right)$.



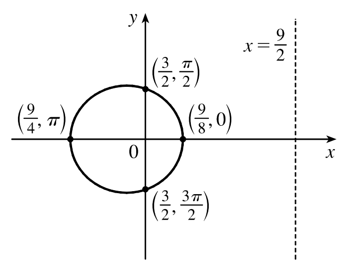
19. $r = \dfrac{9}{6+2\cos\theta}\cdot\dfrac{1/6}{1/6} = \dfrac{3/2}{1+\frac{1}{3}\cos\theta}$, where $e = \frac{1}{3}$ and $ed = \frac{3}{2} \;\Rightarrow\; d = \frac{9}{2}$.

(a) Eccentricity $= e = \frac{1}{3}$

(b) Since $e = \frac{1}{3} < 1$, the conic is an ellipse.

(c) Since "$+e\cos\theta$" appears in the denominator, the directrix is to the right of the focus at the origin. $d = |Fl| = \frac{9}{2}$, so an equation of the directrix is $x = \frac{9}{2}$.

(d) The vertices are $\left(\frac{9}{8}, 0\right)$ and $\left(\frac{9}{4}, \pi\right)$, so the center is midway between them, that is, $\left(\frac{9}{16}, \pi\right)$.



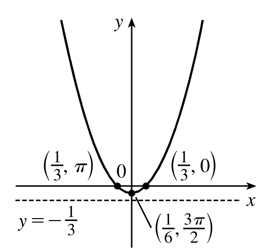
20. $r = \dfrac{1}{3-3\sin\theta}\cdot\dfrac{1/3}{1/3} = \dfrac{1/3}{1-1\sin\theta}$, where $e = 1$ and $ed = \dfrac{1}{3} \;\Rightarrow\; d = \dfrac{1}{3}$.

(a) Eccentricity $= e = 1$

(b) Since $e = 1$, the conic is a parabola.

(c) Since "$-e\sin\theta$" appears in the denominator, the directrix is below the focus at the origin, $d = |Fl| = \frac{1}{3}$, so an equation of the directrix is $y = -\frac{1}{3}$.

(d) The vertex is at $\left(\frac{1}{6}, \frac{3\pi}{2}\right)$, midway between the focus and the directrix.



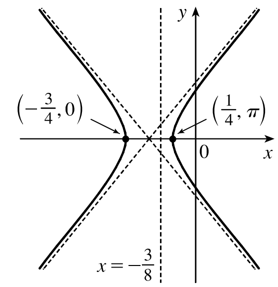
21. $r = \dfrac{3}{4 - 8\cos\theta} \cdot \dfrac{1/4}{1/4} = \dfrac{3/4}{1 - 2\cos\theta}$, where $e = 2$ and $ed = \frac{3}{4} \quad\Rightarrow\quad d = \frac{3}{8}$.

(a) Eccentricity $= e = 2$

(b) Since $e = 2 > 1$, the conic is a hyperbola.

(c) Since "$-e\cos\theta$" appears in the denominator, the directrix is to the left of the focus at the origin. $d = |Fl| = \frac{3}{8}$, so an equation of the directrix is $x = -\frac{3}{8}$.

(d) The vertices are $\left(-\frac{3}{4}, 0\right)$ and $\left(\frac{1}{4}, \pi\right)$, so the center is midway between them, that is, $\left(\frac{1}{2}, \pi\right)$.



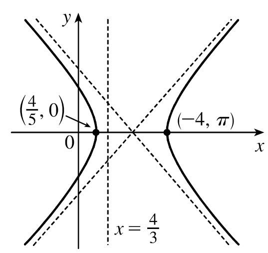
22. $r = \dfrac{4}{2 + 3\cos\theta} \cdot \dfrac{1/2}{1/2} = \dfrac{2}{1 + \frac{3}{2}\cos\theta}$, where $e = \dfrac{3}{2}$ and $ed = 2 \quad\Rightarrow\quad d = \dfrac{4}{3}$.

(a) Eccentricity $= e = \dfrac{3}{2}$

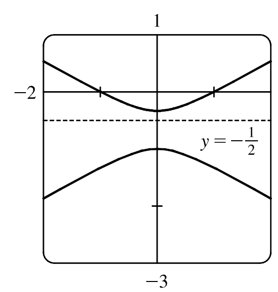
(b) Since $e = \dfrac{3}{2} > 1$, the conic is a hyperbola.

(c) Since "$+e\cos\theta$" appears in the denominator, the directrix is to the right of the focus at the origin. $d = |Fl| = \frac{4}{3}$, so an equation of the directrix is $x = \frac{4}{3}$.

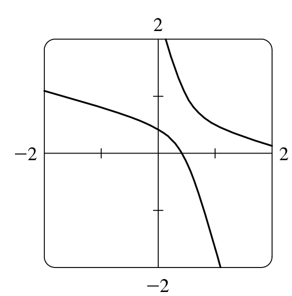
(d) The vertices are $\left(\frac{4}{5}, 0\right)$ and $(-4, \pi)$, so the center is midway between them, that is, $\left(\frac{8}{5}, 0\right)$.



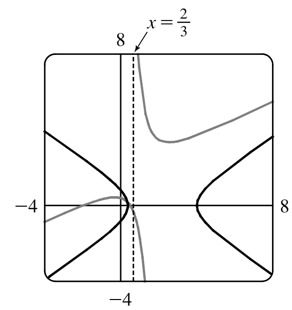
23. (a) $r = \dfrac{1}{1 - 2\sin\theta}$, where $e = 2$ and $ed = 1 \quad\Rightarrow\quad d = \frac{1}{2}$. The eccentricity $e = 2 > 1$, so the conic is a hyperbola. Since "$-e\sin\theta$" appears in the denominator, the directrix is below the focus at the origin. $d = |Fl| = \frac{1}{2}$, so an equation of the directrix is $y = -\frac{1}{2}$. The vertices are $\left(-1, \frac{\pi}{2}\right)$ and $\left(\frac{1}{3}, \frac{3\pi}{2}\right)$, so the center is midway between them, that is, $\left(\frac{2}{3}, \frac{3\pi}{2}\right)$.



(b) By the discussion that precedes Example 4, the equation is $r = \dfrac{1}{1 - 2\sin\left(\theta - \frac{3\pi}{4}\right)}$.



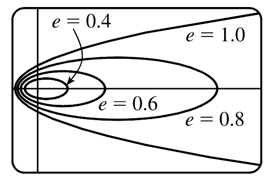
24. $r = \dfrac{4}{5 + 6\cos\theta} = \dfrac{4/5}{1 + \frac{6}{5}\cos\theta}$, so $e = \frac{6}{5}$ and $ed = \frac{4}{5} \;\Rightarrow\; d = \frac{2}{3}$.



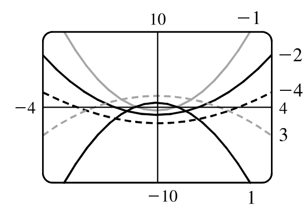
An equation of the directrix is $x = \frac{2}{3} \;\Rightarrow\; r\cos\theta = \frac{2}{3} \;\Rightarrow\; r = \dfrac{2}{3\cos\theta}$.

If the hyperbola is rotated about its focus (the origin) through an angle $\pi/3$, its equation is the same as that of the original, with θ replaced by $\theta - \frac{\pi}{3}$ (see Example 4), so $r = \dfrac{4}{5 + 6\cos\left(\theta - \frac{\pi}{3}\right)}$.

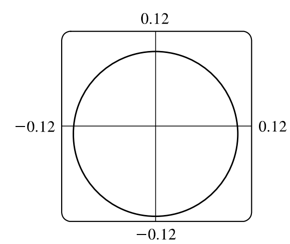
25. $r = \dfrac{e}{1 - e\cos\theta}$. For $e < 1$ the curve is an ellipse. It is nearly circular when e is close to 0. As e increases, the graph is stretched out to the right, and grows larger (that is, its right-hand focus moves to the right while its left-hand focus remains at the origin.) At $e = 1$, the curve becomes a parabola with focus at the origin.



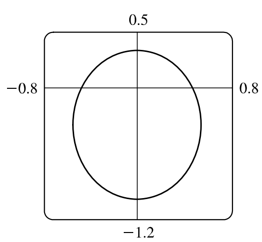
26. (a) $r = \dfrac{ed}{1 + e\sin\theta}$. The value of d does not seem to affect the shape of the conic (a parabola) at all, just its size, position, and orientation (for $d < 0$ it opens upward, for $d > 0$ it opens downward).



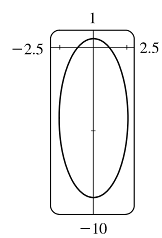
(b) We consider only positive values of e. When $0 < e < 1$, the conic is an ellipse. As $e \to 0^+$, the graph approaches perfect roundness and zero size. As e increases, the ellipse becomes more elongated, until at $e = 1$ it turns into a parabola. For $e > 1$, the conic is a hyperbola, which moves downward and gets broader as e continues to increase.



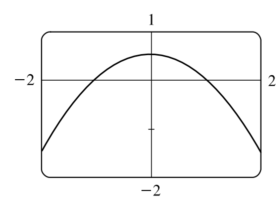
$e = 0.1$



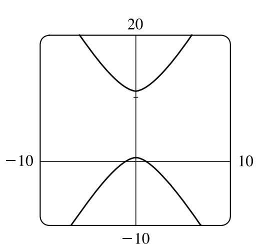
$e = 0.5$



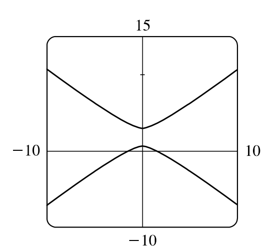
$e = 0.9$



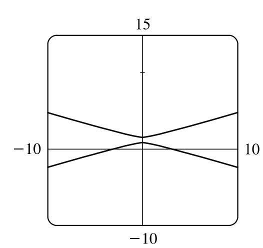
$e = 1$



$e = 1.1$



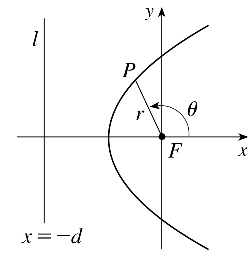
$e = 1.5$



$e = 10$

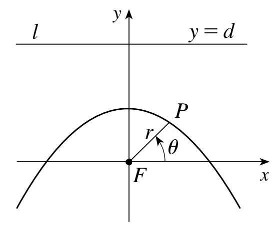
27. $|PF| = e\,|Pl| \;\Rightarrow\; r = e[d - r\cos(\pi - \theta)] = e(d + r\cos\theta) \;\Rightarrow$

$r(1 - e\cos\theta) = ed \;\Rightarrow\; r = \dfrac{ed}{1 - e\cos\theta}$



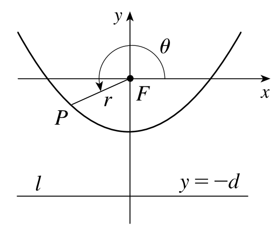
28. $|PF| = e\,|Pl| \;\Rightarrow\; r = e[d - r\sin\theta] \;\Rightarrow\; r(1 + e\sin\theta) = ed \;\Rightarrow$

$r = \dfrac{ed}{1 + e\sin\theta}$



29. $|PF| = e\,|Pl| \;\Rightarrow\; r = e[d - r\sin(\theta - \pi)] = e(d + r\sin\theta) \;\Rightarrow$

$r(1 - e\sin\theta) = ed \;\Rightarrow\; r = \dfrac{ed}{1 - e\sin\theta}$



30. The parabolas intersect at the two points where $\dfrac{c}{1 + \cos\theta} = \dfrac{d}{1 - \cos\theta} \;\Rightarrow\; \cos\theta = \dfrac{c - d}{c + d} \;\Rightarrow\; r = \dfrac{c + d}{2}$.

For the first parabola, $\dfrac{dr}{d\theta} = \dfrac{c\sin\theta}{(1 + \cos\theta)^2}$, so

$$\frac{dy}{dx} = \frac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta} = \frac{c\sin^2\theta + c\cos\theta(1 + \cos\theta)}{c\sin\theta\cos\theta - c\sin\theta(1 + \cos\theta)} = \frac{1 + \cos\theta}{-\sin\theta}$$

and similarly for the second, $\dfrac{dy}{dx} = \dfrac{1 - \cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1 + \cos\theta}$. Since the product of these slopes is -1, the parabolas intersect at right angles.

31. We are given $e = 0.093$ and $a = 2.28 \times 10^8$. By (7), we have

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta} = \frac{2.28 \times 10^8[1 - (0.093)^2]}{1 + 0.093\cos\theta} \approx \frac{2.26 \times 10^8}{1 + 0.093\cos\theta}$$

32. We are given $e = 0.048$ and $2a = 1.56 \times 10^9 \;\Rightarrow\; a = 7.8 \times 10^8$. By (7), we have

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta} = \frac{7.8 \times 10^8[1 - (0.048)^2]}{1 + 0.048\cos\theta} \approx \frac{7.78 \times 10^8}{1 + 0.048\cos\theta}$$

33. Here $2a =$ length of major axis $= 36.18$ AU $\;\Rightarrow\; a = 18.09$ AU and $e = 0.97$. By (7), the equation of the orbit is $r = \dfrac{18.09[1 - (0.97)^2]}{1 + 0.97\cos\theta} \approx \dfrac{1.07}{1 + 0.97\cos\theta}$. By (8), the maximum distance from the comet to the sun is $18.09(1 + 0.97) \approx 35.64$ AU or about 5.346 billion kilometers.

34. Here $2a =$ length of major axis $= 356.5$ AU $\Rightarrow$ $a = 178.25$ AU and $e = 0.9951$. By (7), the equation of the orbit is $r = \dfrac{178.25[1-(0.9951)^2]}{1+0.9951\cos\theta} \approx \dfrac{1.7426}{1+0.9951\cos\theta}$. By (8), the minimum distance from the comet to the sun is $178.25(1-0.9951) \approx 0.8734$ AU or about 131.01 million kilometers.

35. The minimum distance is at perihelion, where $4.6\times10^7 = r = a(1-e) = a(1-0.206) = a(0.794)$ $\Rightarrow$ $a = 4.6\times10^7/0.794$. So the maximum distance, which is at aphelion, is $r = a(1+e) = \left(4.6\times10^7/0.794\right)(1.206) \approx 7.0\times10^7$ km.

36. At perihelion, $r = a(1-e) = 4.43\times10^9$, and at aphelion, $r = a(1+e) = 7.37\times10^9$. Adding, we get $2a = 11.80\times10^9$, so $a = 5.90\times10^9$ km. Therefore $1+e = a(1+e)/a = \frac{7.37}{5.90} \approx 1.249$ and $e \approx 0.249$.

37. From Exercise 35, we have $e = 0.206$ and $a(1-e) = 4.6\times10^7$ km. Thus, $a = 4.6\times10^7/0.794$. From (7), we can write the equation of Mercury's orbit as $r = a\dfrac{1-e^2}{1+e\cos\theta}$. So since

$$\frac{dr}{d\theta} = \frac{a(1-e^2)e\sin\theta}{(1+e\cos\theta)^2} \quad\Rightarrow$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \frac{a^2(1-e^2)^2}{(1+e\cos\theta)^2} + \frac{a^2(1-e^2)^2\,e^2\sin^2\theta}{(1+e\cos\theta)^4} = \frac{a^2(1-e^2)^2}{(1+e\cos\theta)^4}\,(1+2e\cos\theta+e^2)$$

the length of the orbit is

$$L = \int_0^{2\pi}\sqrt{r^2+(dr/d\theta)^2}\,d\theta = a(1-e^2)\int_0^{2\pi}\frac{\sqrt{1+e^2+2e\cos\theta}}{(1+e\cos\theta)^2}\,d\theta \approx 3.6\times10^8 \text{ km}$$

This seems reasonable, since Mercury's orbit is nearly circular, and the circumference of a circle of radius a is $2\pi a \approx 3.6\times10^8$ km.

10 Review

TRUE-FALSE QUIZ

1. False. Consider the curve defined by $x = f(t) = (t-1)^3$ and $y = g(t) = (t-1)^2$. Then $g'(t) = 2(t-1)$, so $g'(1) = 0$, but its graph has a *vertical* tangent when $t = 1$. *Note:* The statement is true if $f'(1) \neq 0$ when $g'(1) = 0$.

2. False. If $x = f(t)$ and $y = g(t)$ are twice differentiable, then $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}$.

3. False. For example, if $f(t) = \cos t$ and $g(t) = \sin t$ for $0 \le t \le 4\pi$, then the curve is a circle of radius 1, hence its length is 2π, but $\int_0^{4\pi}\sqrt{[f'(t)]^2+[g'(t)]^2}\,dt = \int_0^{4\pi}\sqrt{(-\sin t)^2+(\cos t)^2}\,dt = \int_0^{4\pi}1\,dt = 4\pi$, since as t increases from 0 to 4π, the circle is traversed twice.

4. False. The speed of the particle with parametric equations $x = 3t + 1$, $y = 2t^2 + 1$ is $\sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{3^2 + (4t)^2}$. When $t = 3$, the speed $= \sqrt{9 + 16(3)^2} = \sqrt{153}$. However, $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4t}{3}$, so $\left.\dfrac{dy}{dx}\right|_{t=3} = \dfrac{4 \cdot 3}{3} = 4 \neq \sqrt{153}$.

5. False. If $(r, \theta) = (1, \pi)$, then $(x, y) = (-1, 0)$, so $\tan^{-1}(y/x) = \tan^{-1} 0 = 0 \neq \theta$. The statement is true for points in quadrants I and IV.

6. True. The curve $r = 1 - \sin 2\theta$ is unchanged if we rotate it through $180°$ about O because $1 - \sin 2(\theta + \pi) = 1 - \sin(2\theta + 2\pi) = 1 - \sin 2\theta$. So it's unchanged if we replace r by $-r$. (See the discussion after Example 8 in Section 10.3.) In other words, it's the same curve as $r = -(1 - \sin 2\theta) = \sin 2\theta - 1$.

7. True. The polar equation $r = 2$, the Cartesian equation $x^2 + y^2 = 4$, and the parametric equations $x = 2\sin 3t$, $y = 2\cos 3t$ $[0 \leq t \leq 2\pi]$ all describe the circle of radius 2 centered at the origin.

8. False. The first pair of equations, $x = t^2$ and $y = t^4$, gives the portion of the parabola $y = x^2$ with $x \geq 0$, whereas the second pair of equations, $x = t^3$ and $y = t^6$, traces out the whole parabola $y = x^2$.

9. True. $y^2 = 2y + 3x \quad \Leftrightarrow \quad (y-1)^2 = 3x + 1 = 3\left(x + \frac{1}{3}\right) = 4\left(\frac{3}{4}\right)\left(x + \frac{1}{3}\right)$, which is the equation of a parabola with vertex $\left(-\frac{1}{3}, 1\right)$ and focus $\left(-\frac{1}{3} + \frac{3}{4}, 1\right)$, opening to the right.

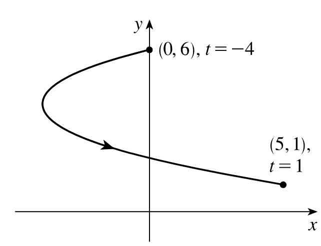
10. True. By rotating and translating the parabola, we can assume it has an equation of the form $y = cx^2$, where $c > 0$. The tangent at the point $\left(a, ca^2\right)$ is the line $y - ca^2 = 2ca(x - a)$; i.e., $y = 2cax - ca^2$. This tangent meets the parabola at the points $\left(x, cx^2\right)$ where $cx^2 = 2cax - ca^2$. This equation is equivalent to $x^2 = 2ax - a^2$ [since $c > 0$]. But $x^2 = 2ax - a^2 \quad \Leftrightarrow \quad x^2 - 2ax + a^2 = 0 \quad \Leftrightarrow \quad (x-a)^2 = 0 \quad \Leftrightarrow \quad x = a \quad \Leftrightarrow$ $\left(x, cx^2\right) = \left(a, ca^2\right)$. This shows that each tangent meets the parabola at exactly one point.

11. True. Consider a hyperbola with focus at the origin, oriented so that its polar equation is $r = \dfrac{ed}{1 + e\cos\theta}$, where $e > 1$. The directrix is $x = d$, but along the hyperbola we have $x = r\cos\theta = \dfrac{ed\cos\theta}{1 + e\cos\theta} = d\left(\dfrac{e\cos\theta}{1 + e\cos\theta}\right) \neq d$.

EXERCISES

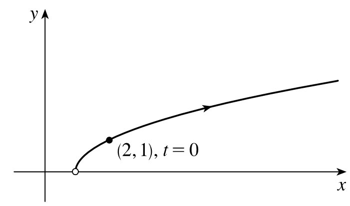
1. $x = t^2 + 4t$, $y = 2 - t$, $-4 \leq t \leq 1$.

Since $y = 2 - t$ and $-4 \leq t \leq 1$, we have $1 \leq y \leq 6$. $t = 2 - y$, so $x = (2-y)^2 + 4(2-y) = 4 - 4y + y^2 + 8 - 4y = y^2 - 8y + 12 \quad \Leftrightarrow$ $x + 4 = y^2 - 8y + 16 = (y-4)^2$. This is part of a parabola with vertex $(-4, 4)$, opening to the right.



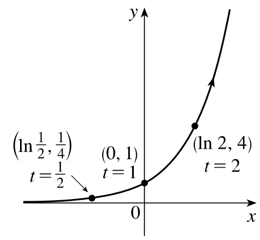
2. $x = 1 + e^{2t}$, $y = e^t$.

$x = 1 + e^{2t} = 1 + (e^t)^2 = 1 + y^2$, $y > 0$.



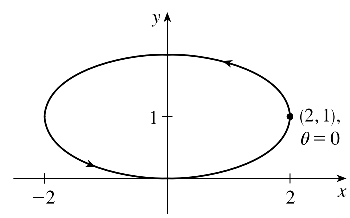
3. $x = \ln t, t > 0 \;\Rightarrow\; t = e^x$.

$y = t^2 = (e^x)^2 = e^{2x}$.

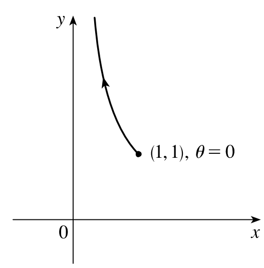


4. $x = 2\cos\theta$, $y = 1 + \sin\theta$, $\cos^2\theta + \sin^2\theta = 1 \;\Rightarrow$

$\left(\frac{x}{2}\right)^2 + (y-1)^2 = 1 \;\Rightarrow\; \frac{x^2}{4} + (y-1)^2 = 1$. This is an ellipse, centered at $(0, 1)$, with semimajor axis of length 2 and semiminor axis of length 1.



5. $y = \sec\theta = \frac{1}{\cos\theta} = \frac{1}{x}$. Since $0 \le \theta \le \pi/2$, $0 < x \le 1$ and $y \ge 1$. This is part of the hyperbola $y = 1/x$.



6. $x = 2 + 4\cos\pi t,\; y = -3 + 4\sin\pi t \;\Rightarrow\; \cos\pi t = \frac{x-2}{4},\; \sin\pi t = \frac{y+3}{4}.\; \cos^2\pi t + \sin^2\pi t = 1 \;\Rightarrow$

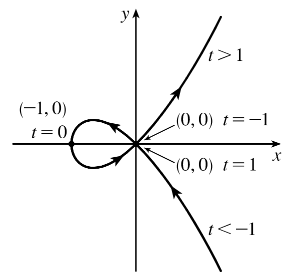
$\left(\frac{x-2}{4}\right)^2 + \left(\frac{y+3}{4}\right)^2 = 1$, so the motion of the particle takes place on the circle centered at $(2, -3)$ with radius 4. As t goes from 0 to 4, the particle starts at $(6, -3)$ and moves counterclockwise along the circle $(x-2)^2 + (y+3)^2 = 16$, tracing the circle twice.

7. Three different sets of parametric equations for the curve $y = \sqrt{x}$ are

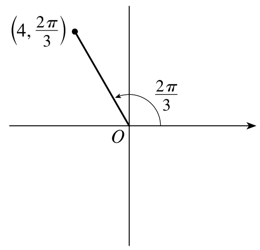
(i) $x = t,\;\; y = \sqrt{t}$
(ii) $x = t^4,\;\; y = t^2$
(iii) $x = \tan^2 t,\;\; y = \tan t,\;\; 0 \le t < \pi/2$

There are many other sets of equations that also give this curve.

8. For $t < -1$, $x > 0$ and $y < 0$ with x decreasing and y increasing. When $t = -1$, $(x, y) = (0, 0)$. When $-1 < t < 0$, we have $-1 < x < 0$ and $0 < y < 1/2$. When $t = 0$, $(x, y) = (-1, 0)$. When $0 < t < 1$, $-1 < x < 0$ and $-\frac{1}{2} < y < 0$. When $t = 1$, $(x, y) = (0, 0)$ again. When $t > 1$, both x and y are positive and increasing.



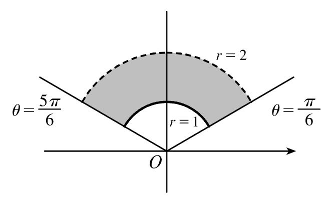
9. (a)



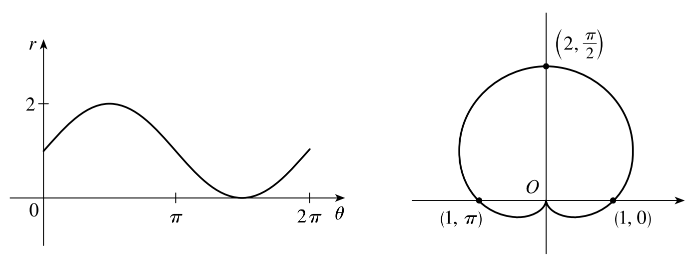
The Cartesian coordinates are $x = 4\cos\frac{2\pi}{3} = 4\left(-\frac{1}{2}\right) = -2$ and $y = 4\sin\frac{2\pi}{3} = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$, that is, the point $\left(-2, 2\sqrt{3}\right)$.

(b) Given $x = -3$ and $y = 3$, we have $r = \sqrt{(-3)^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$. Also, $\tan\theta = \dfrac{y}{x} \;\Rightarrow\; \tan\theta = \dfrac{3}{-3}$, and since $(-3, 3)$ is in the second quadrant, $\theta = \frac{3\pi}{4}$. Thus, one set of polar coordinates for $(-3, 3)$ is $\left(3\sqrt{2}, \frac{3\pi}{4}\right)$, and two others are $\left(3\sqrt{2}, \frac{11\pi}{4}\right)$ and $\left(-3\sqrt{2}, \frac{7\pi}{4}\right)$.

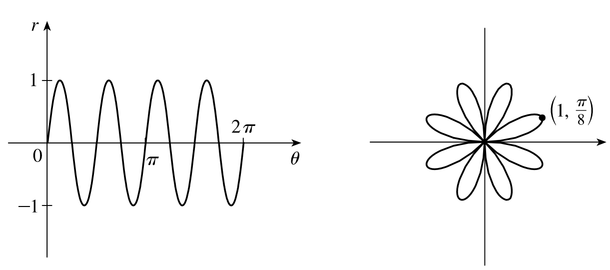
10. $1 \le r < 2$, $\frac{\pi}{6} \le \theta \le \frac{5\pi}{6}$



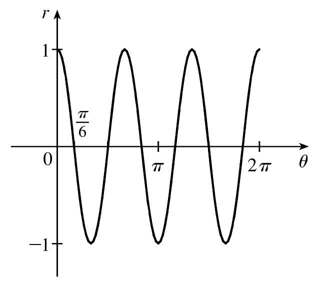
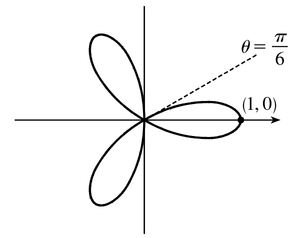
11. $r = 1 + \sin\theta$. This cardioid is symmetric about the $\theta = \pi/2$ axis.



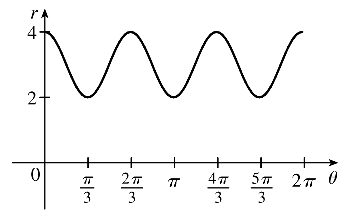
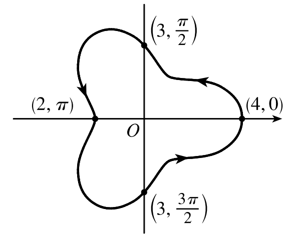
12. $r = \sin 4\theta$. This is an eight-leaved rose.



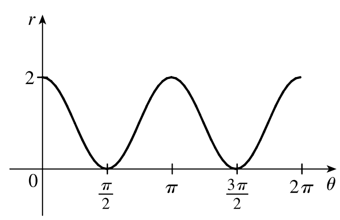
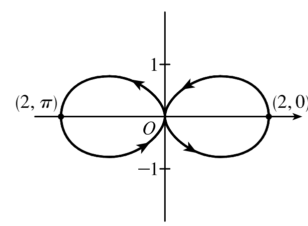
13. $r = \cos 3\theta$. This is a three-leaved rose. The curve is traced twice.

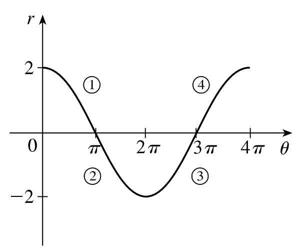
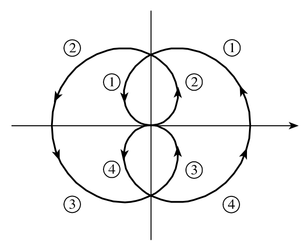
14. $r = 3 + \cos 3\theta$. The curve is symmetric about the horizontal axis.

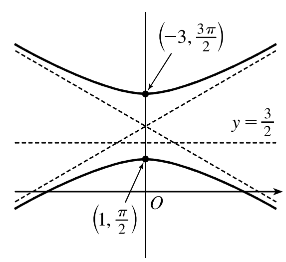
15. $r = 1 + \cos 2\theta$. The curve is symmetric about the pole and both the horizontal and vertical axes.

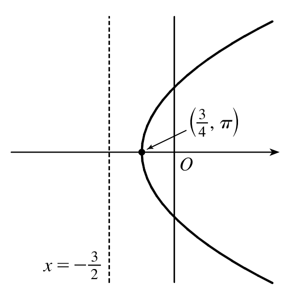
16. $r = 2\cos(\theta/2)$. The curve is symmetric about the pole and both the horizontal and vertical axes.

17. $r = \dfrac{3}{1 + 2\sin\theta} \;\Rightarrow\; e = 2 > 1$, so the conic is a hyperbola. $de = 3 \;\Rightarrow$ $d = \frac{3}{2}$ and the form "$+2\sin\theta$" imply that the directrix is above the focus at the origin and has equation $y = \frac{3}{2}$. The vertices are $\left(1, \frac{\pi}{2}\right)$ and $\left(-3, \frac{3\pi}{2}\right)$.



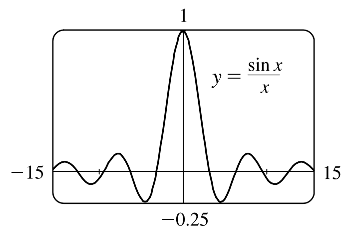
18. $r = \dfrac{3}{2 - 2\cos\theta} \cdot \dfrac{1/2}{1/2} = \dfrac{3/2}{1 - 1\cos\theta} \;\Rightarrow\; e = 1$, so the conic is a parabola. $de = \frac{3}{2} \;\Rightarrow\; d = \frac{3}{2}$ and the form "$-2\cos\theta$" imply that the directrix is to the left of the focus at the origin and has equation $x = -\frac{3}{2}$. The vertex is $\left(\frac{3}{4}, \pi\right)$.

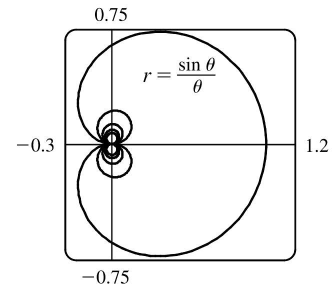


19. $x + y = 2 \quad \Leftrightarrow \quad r\cos\theta + r\sin\theta = 2 \quad \Leftrightarrow \quad r(\cos\theta + \sin\theta) = 2 \quad \Leftrightarrow \quad r = \dfrac{2}{\cos\theta + \sin\theta}$

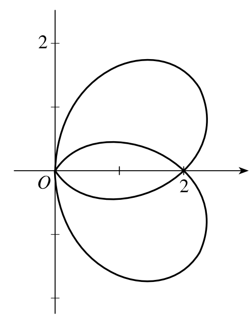
20. $x^2 + y^2 = 2 \quad \Rightarrow \quad r^2 = 2 \quad \Rightarrow \quad r = \sqrt{2}$. [$r = -\sqrt{2}$ gives the same curve.]

21. $r = (\sin\theta)/\theta$. As $\theta \to \pm\infty$, $r \to 0$. As $\theta \to 0$, $r \to 1$. In the first figure, there are an infinite number of x-intercepts at $x = \pi n$, n a nonzero integer. These correspond to pole points in the second figure.





22. $r = 2$ when $\theta = 0$ and when $\theta = 2\pi$. r has a maximum value of approximately 2.6 at about $\theta = \frac{\pi}{6}$ and a minimum value of approximately -2.6 at about $\theta = \frac{5\pi}{6}$. The graph touches the pole ($r = 0$) when $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. Since r is positive in the θ-intervals $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$ and negative in the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, the graph lies entirely in the first and fourth quadrants.



23. $x = \ln t$, $y = 1 + t^2$; $t = 1$. $\dfrac{dy}{dt} = 2t$ and $\dfrac{dx}{dt} = \dfrac{1}{t}$, so $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t}{1/t} = 2t^2$.

When $t = 1$, $(x, y) = (0, 2)$ and $dy/dx = 2$.

24. $x = t^3 + 6t + 1$, $y = 2t - t^2$; $t = -1$. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2 - 2t}{3t^2 + 6}$. When $t = -1$, $(x, y) = (-6, -3)$ and $\dfrac{dy}{dx} = \dfrac{4}{9}$.

25. $r = e^{-\theta} \quad \Rightarrow \quad y = r\sin\theta = e^{-\theta}\sin\theta$ and $x = r\cos\theta = e^{-\theta}\cos\theta \quad \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{-e^{-\theta}\sin\theta + e^{-\theta}\cos\theta}{-e^{-\theta}\cos\theta - e^{-\theta}\sin\theta} \cdot \frac{-e^{\theta}}{-e^{\theta}} = \frac{\sin\theta - \cos\theta}{\cos\theta + \sin\theta}.$$

When $\theta = \pi$, $\dfrac{dy}{dx} = \dfrac{0 - (-1)}{-1 + 0} = \dfrac{1}{-1} = -1$.

26. $r = 3 + \cos 3\theta \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \dfrac{-3\sin 3\theta\sin\theta + (3 + \cos 3\theta)\cos\theta}{-3\sin 3\theta\cos\theta - (3 + \cos 3\theta)\sin\theta}$.

When $\theta = \pi/2$, $\dfrac{dy}{dx} = \dfrac{(-3)(-1)(1) + (3 + 0)\cdot 0}{(-3)(-1)(0) - (3 + 0)\cdot 1} = \dfrac{3}{-3} = -1$.

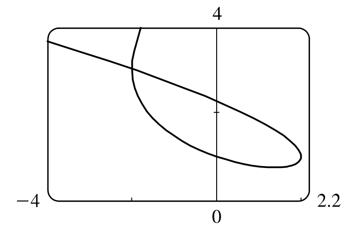
27. $x = t + \sin t$, $y = t - \cos t \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1 + \sin t}{1 + \cos t} \quad \Rightarrow$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{\dfrac{(1 + \cos t)\,\cos t - (1 + \sin t)(-\sin t)}{(1 + \cos t)^2}}{1 + \cos t} = \frac{\cos t + \cos^2 t + \sin t + \sin^2 t}{(1 + \cos t)^3} = \frac{1 + \cos t + \sin t}{(1 + \cos t)^3}$$

28. $x = 1 + t^2$, $y = t - t^3$. $\dfrac{dy}{dt} = 1 - 3t^2$ and $\dfrac{dx}{dt} = 2t$, so $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1-3t^2}{2t} = \frac{1}{2}t^{-1} - \frac{3}{2}t$.

$$\frac{d^2y}{dx^2} = \frac{d(dy/dx)/dt}{dx/dt} = \frac{-\frac{1}{2}t^{-2} - \frac{3}{2}}{2t} = -\tfrac{1}{4}t^{-3} - \tfrac{3}{4}t^{-1} = -\frac{1}{4t^3}(1+3t^2) = -\frac{3t^2+1}{4t^3}.$$

29. We graph the curve $x = t^3 - 3t$, $y = t^2 + t + 1$ for $-2.2 \le t \le 1.2$. By zooming in or using a cursor, we find that the lowest point is about $(1.4, 0.75)$. To find the exact values, we find the t-value at which $dy/dt = 2t + 1 = 0 \quad \Leftrightarrow \quad t = -\frac{1}{2} \quad \Leftrightarrow \quad (x, y) = \left(\frac{11}{8}, \frac{3}{4}\right)$.



30. We estimate the coordinates of the point of intersection to be $(-2, 3)$. In fact this is exact, since both $t = -2$ and $t = 1$ give the point $(-2, 3)$. So the area enclosed by the loop is

$$\begin{aligned}\int_{t=-2}^{t=1} y\,dx &= \int_{-2}^{1} (t^2 + t + 1)(3t^2 - 3)\,dt = \int_{-2}^{1} (3t^4 + 3t^3 - 3t - 3)\,dt \\ &= \left[\tfrac{3}{5}t^5 + \tfrac{3}{4}t^4 - \tfrac{3}{2}t^2 - 3t\right]_{-2}^{1} = \left(\tfrac{3}{5} + \tfrac{3}{4} - \tfrac{3}{2} - 3\right) - \left[-\tfrac{96}{5} + 12 - 6 - (-6)\right] = \tfrac{81}{20}\end{aligned}$$

31. $x = 2a\cos t - a\cos 2t \quad \Rightarrow \quad \dfrac{dx}{dt} = -2a\sin t + 2a\sin 2t = 2a\sin t(2\cos t - 1) = 0 \quad \Leftrightarrow$

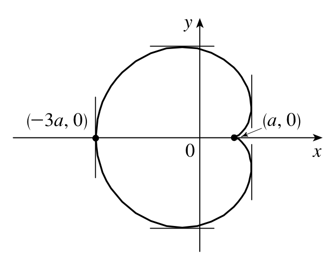
$\sin t = 0$ or $\cos t = \frac{1}{2} \quad \Rightarrow \quad t = 0, \frac{\pi}{3}, \pi$, or $\frac{5\pi}{3}$.

$y = 2a\sin t - a\sin 2t \quad \Rightarrow \quad \dfrac{dy}{dt} = 2a\cos t - 2a\cos 2t = 2a\left(1 + \cos t - 2\cos^2 t\right) = 2a(1 - \cos t)(1 + 2\cos t) = 0 \quad \Rightarrow$

$t = 0, \frac{2\pi}{3}$, or $\frac{4\pi}{3}$.

Thus the graph has vertical tangents where $t = \frac{\pi}{3}$, π and $\frac{5\pi}{3}$, and horizontal tangents where $t = \frac{2\pi}{3}$ and $\frac{4\pi}{3}$. To determine what the slope is where $t = 0$, we use l'Hospital's Rule to evaluate $\lim\limits_{t \to 0} \dfrac{dy/dt}{dx/dt} = 0$, so there is a horizontal tangent there.

t	x	y
0	a	0
$\frac{\pi}{3}$	$\frac{3}{2}a$	$\frac{\sqrt{3}}{2}a$
$\frac{2\pi}{3}$	$-\frac{1}{2}a$	$\frac{3\sqrt{3}}{2}a$
π	$-3a$	0
$\frac{4\pi}{3}$	$-\frac{1}{2}a$	$-\frac{3\sqrt{3}}{2}a$
$\frac{5\pi}{3}$	$\frac{3}{2}a$	$-\frac{\sqrt{3}}{2}a$



32. From Exercise 31, $x = 2a\cos t - a\cos 2t$, $y = 2a\sin t - a\sin 2t \quad \Rightarrow$

$$\begin{aligned}A &= 2\int_{\pi}^{0} (2a\sin t - a\sin 2t)(-2a\sin t + 2a\sin 2t)\,dt = 4a^2\int_0^{\pi} (2\sin^2 t + \sin^2 2t - 3\sin t\sin 2t)\,dt \\ &= 4a^2\int_0^{\pi} \left[(1 - \cos 2t) + \tfrac{1}{2}(1 - \cos 4t) - 6\sin^2 t\,\cos t\right] dt = 4a^2\left[t - \tfrac{1}{2}\sin 2t + \tfrac{1}{2}t - \tfrac{1}{8}\sin 4t - 2\sin^3 t\right]_0^{\pi} \\ &= 4a^2\left(\tfrac{3}{2}\right)\pi = 6\pi a^2\end{aligned}$$

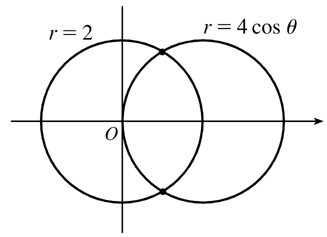
33. The curve $r^2 = 9\cos 5\theta$ has 10 "petals." For instance, for $-\frac{\pi}{10} \le \theta \le \frac{\pi}{10}$, there are two petals, one with $r > 0$ and one with $r < 0$.

$$A = 10\int_{-\pi/10}^{\pi/10} \tfrac{1}{2}r^2\,d\theta = 5\int_{-\pi/10}^{\pi/10} 9\cos 5\theta\,d\theta = 5\cdot 9\cdot 2\int_0^{\pi/10} \cos 5\theta\,d\theta = 18\big[\sin 5\theta\big]_0^{\pi/10} = 18$$

34. $r = 1 - 3\sin\theta$. The inner loop is traced out as θ goes from $\alpha = \sin^{-1}\left(\frac{1}{3}\right)$ to $\pi - \alpha$, so

$$A = \int_\alpha^{\pi-\alpha} \tfrac{1}{2}r^2\,d\theta = \int_\alpha^{\pi/2}(1-3\sin\theta)^2\,d\theta = \int_\alpha^{\pi/2}\left[1 - 6\sin\theta + \tfrac{9}{2}(1-\cos 2\theta)\right]d\theta$$
$$= \left[\tfrac{11}{2}\theta + 6\cos\theta - \tfrac{9}{4}\sin 2\theta\right]_\alpha^{\pi/2} = \tfrac{11}{4}\pi - \tfrac{11}{2}\sin^{-1}\left(\tfrac{1}{3}\right) - 3\sqrt{2}$$

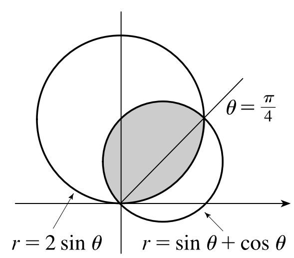
35. The curves intersect when $4\cos\theta = 2 \;\Rightarrow\; \cos\theta = \frac{1}{2} \;\Rightarrow\; \theta = \pm\frac{\pi}{3}$ for $-\pi \le \theta \le \pi$. The points of intersection are $\left(2, \frac{\pi}{3}\right)$ and $\left(2, -\frac{\pi}{3}\right)$.



36. The two curves clearly both contain the pole. For other points of intersection, $\cot\theta = 2\cos(\theta + 2n\pi)$ or $-2\cos(\theta + \pi + 2n\pi)$, both of which reduce to $\cot\theta = 2\cos\theta \;\Leftrightarrow\; \cos\theta = 2\sin\theta\cos\theta \;\Leftrightarrow\; \cos\theta(1 - 2\sin\theta) = 0 \;\Rightarrow$ $\cos\theta = 0$ or $\sin\theta = \frac{1}{2} \;\Rightarrow\; \theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ or $\frac{3\pi}{2} \;\Rightarrow\;$ intersection points are $\left(0, \frac{\pi}{2}\right)$, $\left(\sqrt{3}, \frac{\pi}{6}\right)$, and $\left(\sqrt{3}, \frac{11\pi}{6}\right)$.

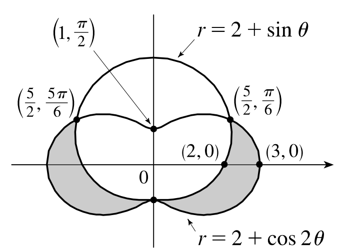
37. The curves intersect where $2\sin\theta = \sin\theta + \cos\theta \;\Rightarrow$ $\sin\theta = \cos\theta \;\Rightarrow \theta = \frac{\pi}{4}$, and also at the origin (at which $\theta = \frac{3\pi}{4}$ on the second curve).

$$A = \int_0^{\pi/4} \tfrac{1}{2}(2\sin\theta)^2\,d\theta + \int_{\pi/4}^{3\pi/4} \tfrac{1}{2}(\sin\theta + \cos\theta)^2\,d\theta$$
$$= \int_0^{\pi/4}(1 - \cos 2\theta)\,d\theta + \tfrac{1}{2}\int_{\pi/4}^{3\pi/4}(1 + \sin 2\theta)\,d\theta$$
$$= \left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4} + \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\cos 2\theta\right]_{\pi/4}^{3\pi/4} = \tfrac{1}{2}(\pi - 1)$$



38. $A = 2\int_{-\pi/2}^{\pi/6} \frac{1}{2}\left[(2 + \cos 2\theta)^2 - (2 + \sin\theta)^2\right]d\theta$

$$= \int_{-\pi/2}^{\pi/6}\left[4\cos 2\theta + \cos^2 2\theta - 4\sin\theta - \sin^2\theta\right]d\theta$$
$$= \left[2\sin 2\theta + \tfrac{1}{2}\theta + \tfrac{1}{8}\sin 4\theta + 4\cos\theta - \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/6}$$
$$= \tfrac{51}{16}\sqrt{3}$$



39. $x = 3t^2,\; y = 2t^3$.

$$L = \int_0^2 \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^2 \sqrt{(6t)^2 + (6t^2)^2}\,dt = \int_0^2 \sqrt{36t^2 + 36t^4}\,dt = \int_0^2 \sqrt{36t^2}\sqrt{1 + t^2}\,dt$$
$$= \int_0^2 6\,|t|\sqrt{1 + t^2}\,dt = 6\int_0^2 t\sqrt{1 + t^2}\,dt = 6\int_1^5 u^{1/2}\left(\tfrac{1}{2}du\right) \qquad \left[u = 1 + t^2,\, du = 2t\,dt\right]$$
$$= 6\cdot\tfrac{1}{2}\cdot\tfrac{2}{3}\left[u^{3/2}\right]_1^5 = 2(5^{3/2} - 1) = 2\left(5\sqrt{5} - 1\right)$$

40. $x = 2 + 3t, \quad y = \cosh 3t \quad \Rightarrow \quad (dx/dt)^2 + (dy/dt)^2 = 3^2 + (3\sinh 3t)^2 = 9(1 + \sinh^2 3t) = 9\cosh^2 3t$, so

$L = \int_0^1 \sqrt{9\cosh^2 3t}\,dt = \int_0^1 |3\cosh 3t|\ dt = \int_0^1 3\cosh 3t\,dt = \big[\sinh 3t\big]_0^1 = \sinh 3 - \sinh 0 = \sinh 3.$

41. $L = \int_\pi^{2\pi} \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_\pi^{2\pi} \sqrt{(1/\theta)^2 + (-1/\theta^2)^2}\,d\theta = \displaystyle\int_\pi^{2\pi} \frac{\sqrt{\theta^2+1}}{\theta^2}\,d\theta$

$$\overset{24}{=} \left[-\frac{\sqrt{\theta^2+1}}{\theta} + \ln\!\left(\theta + \sqrt{\theta^2+1}\right)\right]_\pi^{2\pi} = \frac{\sqrt{\pi^2+1}}{\pi} - \frac{\sqrt{4\pi^2+1}}{2\pi} + \ln\!\left(\frac{2\pi + \sqrt{4\pi^2+1}}{\pi + \sqrt{\pi^2+1}}\right)$$

$$= \frac{2\sqrt{\pi^2+1} - \sqrt{4\pi^2+1}}{2\pi} + \ln\!\left(\frac{2\pi + \sqrt{4\pi^2+1}}{\pi + \sqrt{\pi^2+1}}\right)$$

42. $L = \int_0^\pi \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_0^\pi \sqrt{\left[\sin^3\!\left(\tfrac13\theta\right)\right]^2 + \left[\sin^2\!\left(\tfrac13\theta\right)\cos\!\left(\tfrac13\theta\right)\right]^2}\,d\theta$

$$= \int_0^\pi \sqrt{\sin^6\!\left(\tfrac13\theta\right) + \sin^4\!\left(\tfrac13\theta\right)\cos^2\!\left(\tfrac13\theta\right)}\,d\theta$$

$$= \int_0^\pi \sqrt{\sin^4\!\left(\tfrac13\theta\right)\left[\sin^2\!\left(\tfrac13\theta\right) + \cos^2\!\left(\tfrac13\theta\right)\right]}\,d\theta = \int_0^\pi \sin^2\!\left(\tfrac13\theta\right)d\theta$$

$$= \int_0^\pi \tfrac12\left[1 - \cos\!\left(\tfrac23\theta\right)\right]d\theta = \left[\tfrac12\left(\theta - \tfrac32\sin\!\left(\tfrac23\theta\right)\right)\right]_0^\pi = \tfrac12\pi - \tfrac38\sqrt3$$

43. (a) $x = \frac12(t^2+3),\ y = 5 - \frac13 t^3$. $dx/dt = t$ and $dy/dt = -t^2$, so the speed at time t is the function

$v(t) = s'(t) = \sqrt{t^2 + (-t^2)^2}$. At the point $(6,-4)$, $y = 5 - \frac13 t^3 = -4 \quad\Rightarrow\quad 9 = \frac13 t^3 \quad\Rightarrow\quad t^3 = 27 \quad\Rightarrow\quad t = 3$.

Thus, the speed of the particle at the point $(6,-4)$ is $v(3) = \sqrt{3^2 + 3^4} = \sqrt{90} \approx 9.49\text{ m/s}$.

(b) To find the average speed of the particle for $0 \le t \le 8$, we find the total distance L traveled in this time, and divide it by the length of the interval. By Theorem 10.2.5,

$$L = \int_0^8 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^8 \sqrt{t^2 + t^4}\,dt = \int_0^8 t\sqrt{1+t^2}\,dt$$

$$= \tfrac12 \int_1^{65} \sqrt{u}\,du \qquad \left[u = 1 + t^2,\ du = 2t\,dt\right]$$

$$= \tfrac12\left[\tfrac23 u^{3/2}\right]_1^{65} = \tfrac13\left(65^{3/2} - 1\right)$$

Thus, the average speed is $\dfrac{L}{8} = \dfrac{1}{24}\left(65\sqrt{65} - 1\right) \approx 21.79\text{ m/s}$.

44. (a) We see from the figure in the text that the blue section of the curve $r = 2\cos^2(\theta/2)$ is traced from $\theta = \pi/2$ to $\theta = \pi$. Thus, its length is given by

$$L = \int_{\pi/2}^{\pi} \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_{\pi/2}^{\pi} \sqrt{[2\cos^2(\theta/2)]^2 + [-2\sin(\theta/2)\cos(\theta/2)]^2}\,d\theta$$

$$= \int_{\pi/2}^{\pi} \sqrt{4\cos^4(\theta/2) + 4\sin^2(\theta/2)\cos^2(\theta/2)}\,d\theta = \int_{\pi/2}^{\pi} |2\cos(\theta/2)|\sqrt{\cos^2(\theta/2) + \sin^2(\theta/2)}\,d\theta$$

$$= \int_{\pi/2}^{\pi} 2\cos(\theta/2)\cdot 1\,d\theta = \Big[4\sin(\theta/2)\Big]_{\pi/2}^{\pi} = 4\cdot 1 - 4\cdot\frac{1}{\sqrt2} = 4 - 2\sqrt2$$

(b) $A = \displaystyle\int_{\pi/2}^{\pi} \frac{1}{2} r^2\, d\theta = \int_{\pi/2}^{\pi} \frac{1}{2}[2\cos^2(\theta/2)]^2\, d\theta = \frac{1}{2}\int_{\pi/2}^{\pi} (1+\cos\theta)^2\ d\theta = \frac{1}{2}\int_{\pi/2}^{\pi} (1 + 2\cos\theta + \cos^2\theta)\, d\theta$

$$= \frac{1}{2}\int_{\pi/2}^{\pi} \left[1 + 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\right] d\theta = \frac{1}{2}\left[\theta + 2\sin\theta + \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{2}\left[\pi + 0 + \frac{1}{2}(\pi + 0) - \frac{\pi}{2} - 2 - \frac{1}{2}\left(\frac{\pi}{2} + 0\right)\right] = \frac{1}{2}\left(\frac{3\pi}{4} - 2\right) = \frac{3\pi}{8} - 1$$

45. $x = 4\sqrt{t},\ y = \dfrac{t^3}{3} + \dfrac{1}{2t^2},\ 1 \le t \le 4 \quad \Rightarrow$

$$S = \textstyle\int_1^4 2\pi y\sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_1^4 2\pi\left(\frac{1}{3}t^3 + \frac{1}{2}t^{-2}\right)\sqrt{\left(2/\sqrt{t}\right)^2 + (t^2 - t^{-3})^2}\, dt$$

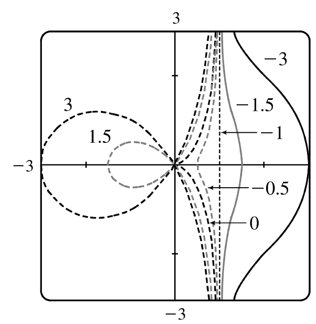
$$= 2\pi\textstyle\int_1^4 \left(\frac{1}{3}t^3 + \frac{1}{2}t^{-2}\right)\sqrt{(t^2 + t^{-3})^2}\, dt = 2\pi\int_1^4 \left(\frac{1}{3}t^5 + \frac{5}{6} + \frac{1}{2}t^{-5}\right) dt = 2\pi\left[\frac{1}{18}t^6 + \frac{5}{6}t - \frac{1}{8}t^{-4}\right]_1^4 = \frac{471{,}295}{1024}\pi$$

46. $x = 2 + 3t, \quad y = \cosh 3t \quad \Rightarrow \quad (dx/dt)^2 + (dy/dt)^2 = 3^2 + (3\sinh 3t)^2 = 9(1 + \sinh^2 3t) = 9\cosh^2 3t$, so

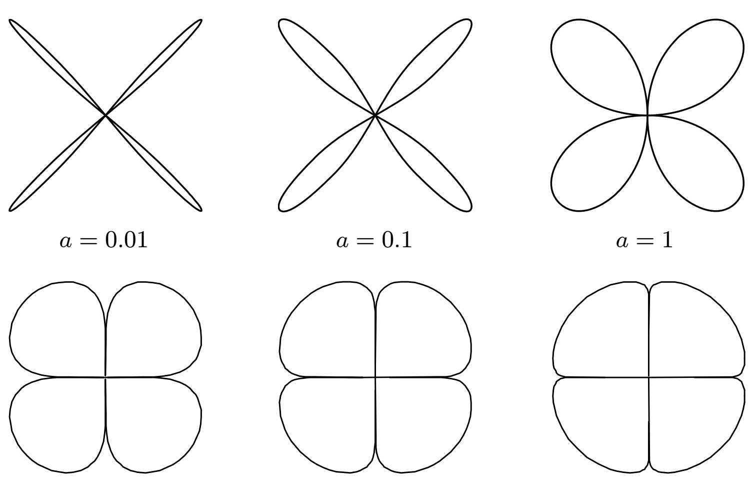
$$S = \textstyle\int_0^1 2\pi y\, ds = \int_0^1 2\pi\cosh 3t\sqrt{9\cosh^2 3t}\, dt = \int_0^1 2\pi\cosh 3t\,|3\cosh 3t|\ dt = \int_0^1 2\pi\cosh 3t \cdot 3\cosh 3t\, dt$$

$$= 6\pi\textstyle\int_0^1 \cosh^2 3t\, dt = 6\pi\int_0^1 \frac{1}{2}(1 + \cosh 6t)\, dt = 3\pi\left[t + \frac{1}{6}\sinh 6t\right]_0^1 = 3\pi\left(1 + \frac{1}{6}\sinh 6\right) = 3\pi + \frac{\pi}{2}\sinh 6$$

47. $x = \dfrac{t^2 - c}{t^2 + 1},\ y = \dfrac{t(t^2 - c)}{t^2 + 1}$. For all c except -1, the curve is asymptotic to the line $x = 1$. For $c < -1$, the curve bulges to the right near $y = 0$. As c increases, the bulge becomes smaller, until at $c = -1$ the curve is the straight line $x = 1$. As c continues to increase, the curve bulges to the left, until at $c = 0$ there is a cusp at the origin. For $c > 0$, there is a loop to the left of the origin, whose size and roundness increase as c increases. Note that the x-intercept of the curve is always $-c$.



48. For a close to 0, the graph of $r^a = |\sin 2\theta|$ consists of four thin petals. As a increases, the petals get wider, until as $a \to \infty$, each petal occupies almost its entire quarter-circle.



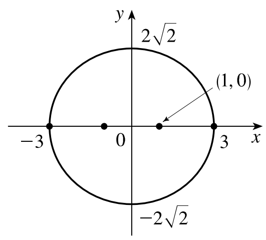
$a = 0.01$ $\quad a = 0.1$ $\quad a = 1$

$a = 5$ $\quad a = 10$ $\quad a = 25$

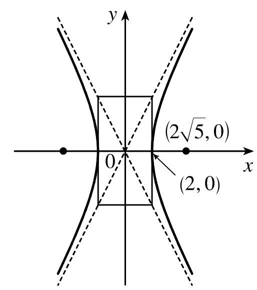
49. $\dfrac{x^2}{9}+\dfrac{y^2}{8}=1$ is an ellipse with center $(0,0)$.

$a=3,\ b=2\sqrt{2},\ c=1 \quad\Rightarrow$

foci $(\pm1,0)$, vertices $(\pm3,0)$.



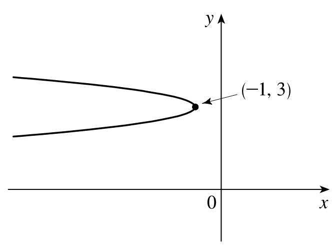
50. $4x^2-y^2=16 \quad\Leftrightarrow\quad \dfrac{x^2}{4}-\dfrac{y^2}{16}=1$ is a hyperbola with center $(0,0)$, vertices $(\pm2,0)$, $a=2$, $b=4$, $c=\sqrt{16+4}=2\sqrt{5}$, foci $\left(\pm2\sqrt{5},0\right)$ and asymptotes $y=\pm2x$.



51. $6y^2+x-36y+55=0 \quad\Leftrightarrow$

$6(y^2-6y+9)=-(x+1) \quad\Leftrightarrow$

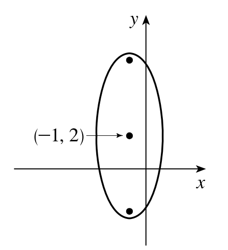
$(y-3)^2=-\frac{1}{6}(x+1)$, a parabola with vertex $(-1,3)$, opening to the left, $p=-\frac{1}{24} \quad\Rightarrow\quad$ focus $\left(-\frac{25}{24},3\right)$ and directrix $x=-\frac{23}{24}$.



52. $25x^2+4y^2+50x-16y=59 \quad\Leftrightarrow$

$25(x+1)^2+4(y-2)^2=100 \quad\Leftrightarrow$

$\frac{1}{4}(x+1)^2+\frac{1}{25}(y-2)^2=1$ is an ellipse centered at $(-1,2)$ with foci on the line $x=-1$, vertices $(-1,7)$ and $(-1,-3)$; $a=5$, $b=2 \quad\Rightarrow\quad c=\sqrt{21} \quad\Rightarrow$ foci $\left(-1,2\pm\sqrt{21}\right)$.



53. The ellipse with foci $(\pm4,0)$ and vertices $(\pm5,0)$ has center $(0,0)$ and a horizontal major axis, with $a=5$ and $c=4$, so $b^2=a^2-c^2=5^2-4^2=9$. An equation is $\dfrac{x^2}{25}+\dfrac{y^2}{9}=1$.

54. The distance from the focus $(2,1)$ to the directrix $x=-4$ is $2-(-4)=6$, so the distance from the focus to the vertex is $\frac{1}{2}(6)=3$ and the vertex is $(-1,1)$. Since the focus is to the right of the vertex, $p=3$. An equation is $(y-1)^2=4\cdot3[x-(-1)]$, or $(y-1)^2=12(x+1)$.

55. The center of a hyperbola with foci $(0,\pm4)$ is $(0,0)$, so $c=4$ and an equation is $\dfrac{y^2}{a^2}-\dfrac{x^2}{b^2}=1$.

The asymptote $y=3x$ has slope 3, so $\dfrac{a}{b}=\dfrac{3}{1} \quad\Rightarrow\quad a=3b$ and $a^2+b^2=c^2 \quad\Rightarrow\quad (3b)^2+b^2=4^2 \quad\Rightarrow$

$10b^2=16 \quad\Rightarrow\quad b^2=\frac{8}{5}$ and so $a^2=16-\frac{8}{5}=\frac{72}{5}$. Thus, an equation is $\dfrac{y^2}{72/5}-\dfrac{x^2}{8/5}=1$, or $\dfrac{5y^2}{72}-\dfrac{5x^2}{8}=1$.

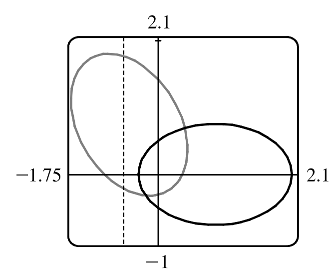
56. The ellipse with foci $(3,\pm2)$ has center $(3,0)$. $a=\frac{8}{2}=4$, $c=2 \quad\Rightarrow\quad b=\sqrt{a^2-c^2}=\sqrt{4^2-2^2}=\sqrt{12} \quad\Rightarrow$ an equation of the ellipse is $\dfrac{(x-3)^2}{12}+\dfrac{y^2}{16}=1$.

57. $x^2 + y = 100 \;\Leftrightarrow\; x^2 = -(y - 100)$ has its vertex at $(0, 100)$, so one of the vertices of the ellipse is $(0, 100)$. Another form of the equation of a parabola is $x^2 = 4p(y - 100)$ so $4p(y - 100) = -(y - 100) \;\Rightarrow\; 4p = -1 \;\Rightarrow\; p = -\frac{1}{4}$. Therefore the shared focus is found at $\left(0, \frac{399}{4}\right)$ so $2c = \frac{399}{4} - 0 \;\Rightarrow\; c = \frac{399}{8}$ and the center of the ellipse is $\left(0, \frac{399}{8}\right)$. So $a = 100 - \frac{399}{8} = \frac{401}{8}$ and $b^2 = a^2 - c^2 = \dfrac{401^2 - 399^2}{8^2} = 25$. So the equation of the ellipse is $\dfrac{x^2}{b^2} + \dfrac{\left(y - \frac{399}{8}\right)^2}{a^2} = 1 \;\Rightarrow$ $\dfrac{x^2}{25} + \dfrac{\left(y - \frac{399}{8}\right)^2}{\left(\frac{401}{8}\right)^2} = 1$, or $\dfrac{x^2}{25} + \dfrac{(8y - 399)^2}{160{,}801} = 1$.

58. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \;\Rightarrow\; \dfrac{2x}{a^2} + \dfrac{2y}{b^2}\dfrac{dy}{dx} = 0 \;\Rightarrow\; \dfrac{dy}{dx} = -\dfrac{b^2}{a^2}\dfrac{x}{y}$. Therefore $\dfrac{dy}{dx} = m \;\Leftrightarrow\; y = -\dfrac{b^2}{a^2}\dfrac{x}{m}$. Combining this condition with $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, we find that $x = \pm\dfrac{a^2 m}{\sqrt{a^2m^2 + b^2}}$. In other words, the two points on the ellipse where the tangent has slope m are $\left(\pm\dfrac{a^2m}{\sqrt{a^2m^2+b^2}}, \mp\dfrac{b^2}{\sqrt{a^2m^2+b^2}}\right)$. The tangent lines at these points have the equations $y \pm \dfrac{b^2}{\sqrt{a^2m^2+b^2}} = m\left(x \mp \dfrac{a^2m}{\sqrt{a^2m^2+b^2}}\right)$ or $y = mx \mp \dfrac{a^2m^2}{\sqrt{a^2m^2+b^2}} \mp \dfrac{b^2}{\sqrt{a^2m^2+b^2}} = mx \mp \sqrt{a^2m^2+b^2}$.

59. Directrix $x = 4 \;\Rightarrow\; d = 4$, so $e = \frac{1}{3} \;\Rightarrow\; r = \dfrac{ed}{1 + e\cos\theta} = \dfrac{4}{3 + \cos\theta}$.

60. $r = \dfrac{2}{4 - 3\cos\theta} = \dfrac{1/2}{1 - \frac{3}{4}\cos\theta} \;\Rightarrow\; e = \frac{3}{4}$ and $d = \frac{2}{3}$. The equation of the directrix is $x = -\frac{2}{3} \;\Rightarrow\; r = -2/(3\cos\theta)$. To obtain the equation of the rotated ellipse, we replace θ in the original equation with $\theta - \frac{2\pi}{3}$, and get $r = \dfrac{2}{4 - 3\cos\left(\theta - \frac{2\pi}{3}\right)}$.



61. See the end of the proof of Theorem 10.6.1. If $e > 1$, then $1 - e^2 < 0$ and Equations 10.6.4 become $a^2 = \dfrac{e^2d^2}{(e^2-1)^2}$ and $b^2 = \dfrac{e^2d^2}{e^2 - 1}$, so $\dfrac{b^2}{a^2} = e^2 - 1$. The asymptotes $y = \pm\dfrac{b}{a}x$ have slopes $\pm\dfrac{b}{a} = \pm\sqrt{e^2 - 1}$, so the angles they make with the polar axis are $\pm\tan^{-1}\left[\sqrt{e^2-1}\right] = \cos^{-1}(\pm 1/e)$.

62. (a) If (a, b) lies on the curve, then there is some parameter value t_1 such that $\dfrac{3t_1}{1 + t_1^3} = a$ and $\dfrac{3t_1^2}{1 + t_1^3} = b$. If $t_1 = 0$, the point is $(0, 0)$, which lies on the line $y = x$. If $t_1 \neq 0$, then the point corresponding to $t = \dfrac{1}{t_1}$ is given by $x = \dfrac{3(1/t_1)}{1 + (1/t_1)^3} = \dfrac{3t_1^2}{t_1^3 + 1} = b$, $y = \dfrac{3(1/t_1)^2}{1 + (1/t_1)^3} = \dfrac{3t_1}{t_1^3 + 1} = a$. So (b, a) also lies on the curve. [Another way to see this is to do part (e) first; the result is immediate.] The curve intersects the line $y = x$ when $\dfrac{3t}{1 + t^3} = \dfrac{3t^2}{1 + t^3} \;\Rightarrow$ $t = t^2 \;\Rightarrow\; t = 0$ or 1, so the points are $(0, 0)$ and $\left(\frac{3}{2}, \frac{3}{2}\right)$.

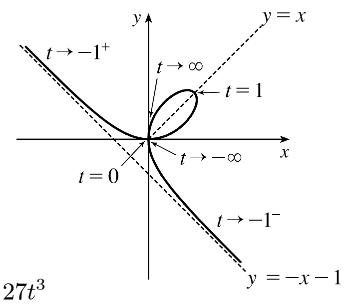
(b) $\dfrac{dy}{dt} = \dfrac{(1+t^3)(6t) - 3t^2(3t^2)}{(1+t^3)^2} = \dfrac{6t - 3t^4}{(1+t^3)^2} = 0$ when $6t - 3t^4 = 3t(2-t^3) = 0 \Rightarrow t = 0$ or $t = \sqrt[3]{2}$, so there are horizontal tangents at $(0,0)$ and $\left(\sqrt[3]{2}, \sqrt[3]{4}\right)$. Using the symmetry from part (a), we see that there are vertical tangents at $(0,0)$ and $\left(\sqrt[3]{4}, \sqrt[3]{2}\right)$.

(c) Notice that as $t \to -1^+$, we have $x \to -\infty$ and $y \to \infty$. As $t \to -1^-$, we have $x \to \infty$ and $y \to -\infty$. Also $y - (-x-1) = y + x + 1 = \dfrac{3t + 3t^2 + (1+t^3)}{1+t^3} = \dfrac{(t+1)^3}{1+t^3} = \dfrac{(t+1)^2}{t^2 - t + 1} \to 0$ as $t \to -1$. So $y = -x - 1$ is a slant asymptote.

(d) $\dfrac{dx}{dt} = \dfrac{(1+t^3)(3) - 3t(3t^2)}{(1+t^3)^2} = \dfrac{3 - 6t^3}{(1+t^3)^2}$ and from part (b) we have $\dfrac{dy}{dt} = \dfrac{6t - 3t^4}{(1+t^3)^2}$. So $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{t(2-t^3)}{1-2t^3}$.

Also $\dfrac{d^2y}{dx^2} = \dfrac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{dx/dt} = \dfrac{2(1+t^3)^4}{3(1-2t^3)^3} > 0 \Leftrightarrow t < \dfrac{1}{\sqrt[3]{2}}$.

So the curve is concave upward there and has a minimum point at $(0,0)$ and a maximum point at $\left(\sqrt[3]{2}, \sqrt[3]{4}\right)$. Using this together with the information from parts (a), (b), and (c), we sketch the curve.



(e) $x^3 + y^3 = \left(\dfrac{3t}{1+t^3}\right)^3 + \left(\dfrac{3t^2}{1+t^3}\right)^3 = \dfrac{27t^3 + 27t^6}{(1+t^3)^3} = \dfrac{27t^3(1+t^3)}{(1+t^3)^3} = \dfrac{27t^3}{(1+t^3)^2}$

and $3xy = 3\left(\dfrac{3t}{1+t^3}\right)\left(\dfrac{3t^2}{1+t^3}\right) = \dfrac{27t^3}{(1+t^3)^2}$, so $x^3 + y^3 = 3xy$.

(f) We start with the equation from part (e) and substitute $x = r\cos\theta$, $y = r\sin\theta$. Then $x^3 + y^3 = 3xy \Rightarrow$ $r^3\cos^3\theta + r^3\sin^3\theta = 3r^2\cos\theta\,\sin\theta$. For $r \neq 0$, this gives $r = \dfrac{3\cos\theta\,\sin\theta}{\cos^3\theta + \sin^3\theta}$. Dividing numerator and denominator by $\cos^3\theta$, we obtain $r = \dfrac{3\left(\dfrac{1}{\cos\theta}\right)\dfrac{\sin\theta}{\cos\theta}}{1 + \dfrac{\sin^3\theta}{\cos^3\theta}} = \dfrac{3\sec\theta\,\tan\theta}{1+\tan^3\theta}$.

(g) The loop corresponds to $\theta \in \left(0, \frac{\pi}{2}\right)$, so its area is

$$A = \int_0^{\pi/2} \frac{r^2}{2}\,d\theta = \frac{1}{2}\int_0^{\pi/2}\left(\frac{3\sec\theta\,\tan\theta}{1+\tan^3\theta}\right)^2 d\theta = \frac{9}{2}\int_0^{\pi/2}\frac{\sec^2\theta\,\tan^2\theta}{(1+\tan^3\theta)^2}\,d\theta = \frac{9}{2}\int_0^{\infty}\frac{u^2\,du}{(1+u^3)^2} \quad [\text{let } u = \tan\theta]$$

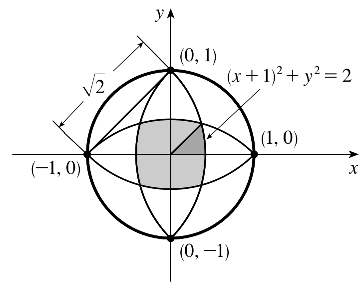
$$= \lim_{b\to\infty} \tfrac{9}{2}\left[-\tfrac{1}{3}(1+u^3)^{-1}\right]_0^b = \tfrac{3}{2}$$

(h) By symmetry, the area between the folium and the line $y = -x - 1$ is equal to the enclosed area in the third quadrant, plus twice the enclosed area in the fourth quadrant. The area in the third quadrant is $\frac{1}{2}$, and since $y = -x - 1 \Rightarrow$ $r\sin\theta = -r\cos\theta - 1 \Rightarrow r = -\dfrac{1}{\sin\theta + \cos\theta}$, the area in the fourth quadrant is

$$\frac{1}{2}\int_{-\pi/2}^{-\pi/4}\left[\left(-\frac{1}{\sin\theta + \cos\theta}\right)^2 - \left(\frac{3\sec\theta\,\tan\theta}{1+\tan^3\theta}\right)^2\right] d\theta \overset{\text{CAS}}{=} \frac{1}{2}.$$ Therefore, the total area is $\frac{1}{2} + 2\left(\frac{1}{2}\right) = \frac{3}{2}$.

PROBLEMS PLUS

1. See the figure. The circle with center $(-1,0)$ and radius $\sqrt{2}$ has equation $(x+1)^2+y^2=2$ and describes the circular arc from $(0,-1)$ to $(0,1)$. Converting the equation to polar coordinates gives us



$(r\cos\theta+1)^2+(r\sin\theta)^2=2 \quad\Rightarrow$

$r^2\cos^2\theta+2r\cos\theta+1+r^2\sin^2\theta=2 \quad\Rightarrow$

$r^2(\cos^2\theta+\sin^2\theta)+2r\cos\theta=1 \quad\Rightarrow\quad r^2+2r\cos\theta=1$. Using the quadratic formula to solve for r gives us

$r=\dfrac{-2\cos\theta\pm\sqrt{4\cos^2\theta+4}}{2}=-\cos\theta+\sqrt{\cos^2\theta+1}$ for $r>0$.

The darkest shaded region is $\frac{1}{8}$ of the entire shaded region A, so $\frac{1}{8}A=\int_0^{\pi/4}\frac{1}{2}r^2\,d\theta=\frac{1}{2}\int_0^{\pi/4}(1-2r\cos\theta)\,d\theta \quad\Rightarrow$

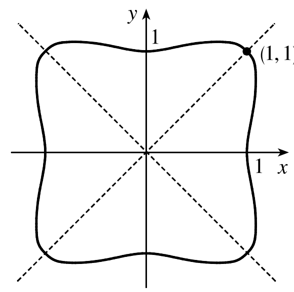
$$\begin{aligned}
\tfrac{1}{4}A&=\int_0^{\pi/4}\left[1-2\cos\theta\left(-\cos\theta+\sqrt{\cos^2\theta+1}\right)\right]d\theta=\int_0^{\pi/4}\left(1+2\cos^2\theta-2\cos\theta\sqrt{\cos^2\theta+1}\right)d\theta\\
&=\int_0^{\pi/4}\left[1+2\cdot\tfrac{1}{2}(1+\cos 2\theta)-2\cos\theta\sqrt{(1-\sin^2\theta)+1}\right]d\theta\\
&=\int_0^{\pi/4}(2+\cos 2\theta)\,d\theta-2\int_0^{\pi/4}\cos\theta\sqrt{2-\sin^2\theta}\,d\theta\\
&=\left[2\theta+\tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4}-2\int_0^{1/\sqrt{2}}\sqrt{2-u^2}\,du \qquad \begin{bmatrix}u=\sin\theta,\\ du=\cos\theta\,d\theta\end{bmatrix}\\
&=\left(\frac{\pi}{2}+\frac{1}{2}\right)-(0+0)-2\left[\frac{u}{2}\sqrt{2-u^2}+\sin^{-1}\frac{u}{\sqrt{2}}\right]_0^{1/\sqrt{2}} \qquad \begin{bmatrix}\text{Formula 30,}\\ a=\sqrt{2}\end{bmatrix}\\
&=\frac{\pi}{2}+\frac{1}{2}-2\left(\frac{1}{2\sqrt{2}}\cdot\frac{\sqrt{3}}{\sqrt{2}}+\frac{\pi}{6}\right)=\frac{\pi}{2}+\frac{1}{2}-\frac{1}{2}\sqrt{3}-\frac{\pi}{3}=\frac{\pi}{6}+\frac{1}{2}-\frac{1}{2}\sqrt{3}.
\end{aligned}$$

Thus, $A=4\left(\dfrac{\pi}{6}+\dfrac{1}{2}-\dfrac{1}{2}\sqrt{3}\right)=\dfrac{2\pi}{3}+2-2\sqrt{3}$.

2. (a) The curve $x^4+y^4=x^2+y^2$ is symmetric about both axes and about the line $y=x$ (since interchanging x and y does not change the equation) so we need only consider $y\geq x\geq 0$ to begin with. Implicit differentiation gives $4x^3+4y^3y'=2x+2yy' \quad\Rightarrow\quad y'=\dfrac{x(1-2x^2)}{y(2y^2-1)} \quad\Rightarrow\quad y'=0$ when $x=0$ and when $x=\pm\frac{1}{\sqrt{2}}$. If $x=0$, then $y^4=y^2 \quad\Rightarrow\quad y^2(y^2-1)=0 \quad\Rightarrow\quad y=0$ or ± 1. The point $(0,0)$ can't be a highest or lowest point because it is isolated. [If $-1<x<1$ and $-1<y<1$, then $x^4<x^2$ and $y^4<y^2 \quad\Rightarrow\quad x^4+y^4<x^2+y^2$, except for $(0,0)$.] If $x=\frac{1}{\sqrt{2}}$, then $x^2=\frac{1}{2}$, $x^4=\frac{1}{4}$, so $\frac{1}{4}+y^4=\frac{1}{2}+y^2 \quad\Rightarrow\quad 4y^4-4y^2-1=0 \quad\Rightarrow\quad y^2=\frac{4\pm\sqrt{16+16}}{8}=\frac{1\pm\sqrt{2}}{2}$. But $y^2>0$, so $y^2=\frac{1+\sqrt{2}}{2} \quad\Rightarrow\quad y=\pm\sqrt{\frac{1}{2}\left(1+\sqrt{2}\right)}$. Near the point $(0,1)$, the denominator of y' is positive and the numerator changes from negative to positive as x increases through 0, so $(0,1)$ is a local minimum point. At

$\left(\frac{1}{\sqrt{2}}, \sqrt{\frac{1+\sqrt{2}}{2}}\right)$, y' changes from positive to negative, so that point gives a maximum. By symmetry, the highest points on the curve are $\left(\pm\frac{1}{\sqrt{2}}, \sqrt{\frac{1+\sqrt{2}}{2}}\right)$ and the lowest points are $\left(\pm\frac{1}{\sqrt{2}}, -\sqrt{\frac{1+\sqrt{2}}{2}}\right)$.

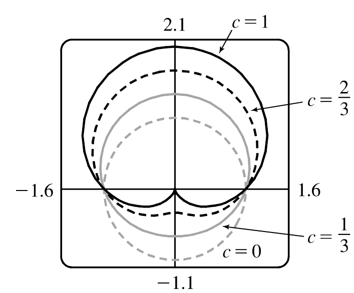
(b) We use the information from part (a), together with symmetry with respect to the axes and the lines $y = \pm x$, to sketch the curve.



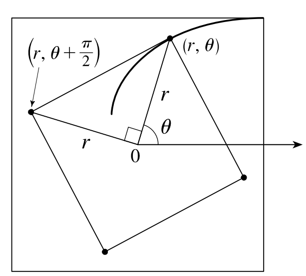
(c) In polar coordinates, $x^4 + y^4 = x^2 + y^2$ becomes $r^4\cos^4\theta + r^4\sin^4\theta = r^2$ or $r^2 = \dfrac{1}{\cos^4\theta + \sin^4\theta}$. By the symmetry shown in part (b), the area enclosed by the curve is $A = 8\displaystyle\int_0^{\pi/4} \frac{1}{2} r^2\, d\theta = 4\int_0^{\pi/4} \frac{d\theta}{\cos^4\theta + \sin^4\theta} \overset{\text{CAS}}{=} \sqrt{2}\,\pi$.

3. In terms of x and y, we have $x = r\cos\theta = (1 + c\sin\theta)\cos\theta = \cos\theta + c\sin\theta\cos\theta = \cos\theta + \frac{1}{2}c\sin 2\theta$ and $y = r\sin\theta = (1 + c\sin\theta)\sin\theta = \sin\theta + c\sin^2\theta$. Now $-1 \le \sin\theta \le 1 \quad\Rightarrow\quad -1 \le \sin\theta + c\sin^2\theta \le 1 + c \le 2$, so $-1 \le y \le 2$. Furthermore, $y = 2$ when $c = 1$ and $\theta = \frac{\pi}{2}$, while $y = -1$ for $c = 0$ and $\theta = \frac{3\pi}{2}$. Therefore, we need a viewing rectangle with $-1 \le y \le 2$.

To find the x-values, look at the equation $x = \cos\theta + \frac{1}{2}c\sin 2\theta$ and use the fact that $\sin 2\theta \ge 0$ for $0 \le \theta \le \frac{\pi}{2}$ and $\sin 2\theta \le 0$ for $-\frac{\pi}{2} \le \theta \le 0$. [Because $r = 1 + c\sin\theta$ is symmetric about the y-axis, we only need to consider $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.] So for $-\frac{\pi}{2} \le \theta \le 0$, x has a maximum value when $c = 0$ and then $x = \cos\theta$ has a maximum value of 1 at $\theta = 0$. Thus, the maximum value of x must occur on $\left[0, \frac{\pi}{2}\right]$ with $c = 1$. Then $x = \cos\theta + \frac{1}{2}\sin 2\theta \quad\Rightarrow$ $\frac{dx}{d\theta} = -\sin\theta + \cos 2\theta = -\sin\theta + 1 - 2\sin^2\theta \quad\Rightarrow\quad \frac{dx}{d\theta} = -(2\sin\theta - 1)(\sin\theta + 1) = 0$ when $\sin\theta = -1$ or $\frac{1}{2}$ [but $\sin\theta \ne -1$ for $0 \le \theta \le \frac{\pi}{2}$]. If $\sin\theta = \frac{1}{2}$, then $\theta = \frac{\pi}{6}$ and $x = \cos\frac{\pi}{6} + \frac{1}{2}\sin\frac{\pi}{3} = \frac{3}{4}\sqrt{3}$. Thus, the maximum value of x is $\frac{3}{4}\sqrt{3}$, and, by symmetry, the minimum value is $-\frac{3}{4}\sqrt{3}$. Therefore, the smallest viewing rectangle that contains every member of the family of polar curves $r = 1 + c\sin\theta$, where $0 \le c \le 1$, is $\left[-\frac{3}{4}\sqrt{3}, \frac{3}{4}\sqrt{3}\right] \times [-1, 2]$.



4. (a) Let us find the polar equation of the path of the bug that starts in the upper right corner of the square. If the polar coordinates of this bug, at a particular moment, are (r, θ), then the polar coordinates of the bug that it is crawling toward must be $\left(r, \theta + \frac{\pi}{2}\right)$. (The next bug must be the same distance from the origin and the angle between the lines joining the bugs to the pole must be $\frac{\pi}{2}$.) The Cartesian coordinates of the first bug are $(r\cos\theta, r\sin\theta)$ and for the second bug we have $x = r\cos\left(\theta + \frac{\pi}{2}\right) = -r\sin\theta$, $y = r\sin\left(\theta + \frac{\pi}{2}\right) = r\cos\theta$. So the slope of the line joining the bugs is



$\dfrac{r\cos\theta - r\sin\theta}{-r\sin\theta - r\cos\theta} = \dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$. This must be equal to the slope of the tangent line at (r,θ), so by

Equation 10.4.7 we have $\dfrac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta} = \dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$. Solving for $\dfrac{dr}{d\theta}$, we get

$\dfrac{dr}{d\theta}\sin^2\theta + \dfrac{dr}{d\theta}\sin\theta\cos\theta + r\sin\theta\cos\theta + r\cos^2\theta = \dfrac{dr}{d\theta}\sin\theta\cos\theta - \dfrac{dr}{d\theta}\cos^2\theta - r\sin^2\theta + r\sin\theta\cos\theta \Rightarrow$

$\dfrac{dr}{d\theta}\left(\sin^2\theta + \cos^2\theta\right) + r\left(\cos^2\theta + \sin^2\theta\right) = 0 \quad\Rightarrow\quad \dfrac{dr}{d\theta} = -r$. Solving this differential equation as a separable equation (as in Section 9.3), or using Theorem 9.4.2 with $k=-1$, we get $r = Ce^{-\theta}$. To determine C we use the fact that, at its starting position, $\theta = \frac{\pi}{4}$ and $r = \frac{1}{\sqrt{2}}a$, so $\frac{1}{\sqrt{2}}a = Ce^{-\pi/4} \quad\Rightarrow\quad C = \frac{1}{\sqrt{2}}ae^{\pi/4}$. Therefore, a polar equation of the bug's path is $r = \frac{1}{\sqrt{2}}ae^{\pi/4}e^{-\theta}$ or $r = \frac{1}{\sqrt{2}}ae^{(\pi/4)-\theta}$.

(b) The distance traveled by this bug is $L = \int_{\pi/4}^{\infty}\sqrt{r^2 + (dr/d\theta)^2}\,d\theta$, where $\dfrac{dr}{d\theta} = \dfrac{a}{\sqrt{2}}e^{\pi/4}(-e^{-\theta})$ and so

$r^2 + (dr/d\theta)^2 = \frac{1}{2}a^2e^{\pi/2}e^{-2\theta} + \frac{1}{2}a^2e^{\pi/2}e^{-2\theta} = a^2e^{\pi/2}e^{-2\theta}$. Thus

$$\begin{aligned} L &= \int_{\pi/4}^{\infty} ae^{\pi/4}e^{-\theta}\,d\theta = ae^{\pi/4}\lim_{t\to\infty}\int_{\pi/4}^{t} e^{-\theta}\,d\theta = ae^{\pi/4}\lim_{t\to\infty}\left[-e^{-\theta}\right]_{\pi/4}^{t} \\ &= ae^{\pi/4}\lim_{t\to\infty}\left[e^{-\pi/4} - e^{-t}\right] = ae^{\pi/4}e^{-\pi/4} = a \end{aligned}$$

5. Without loss of generality, assume the hyperbola has equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. Use implicit differentiation to get

$\dfrac{2x}{a^2} - \dfrac{2y\,y'}{b^2} = 0$, so $y' = \dfrac{b^2x}{a^2y}$. The tangent line at the point (c,d) on the hyperbola has equation $y - d = \dfrac{b^2c}{a^2d}(x-c)$.

The tangent line intersects the asymptote $y = \dfrac{b}{a}x$ when $\dfrac{b}{a}x - d = \dfrac{b^2c}{a^2d}(x-c) \quad\Rightarrow\quad abdx - a^2d^2 = b^2cx - b^2c^2 \quad\Rightarrow$

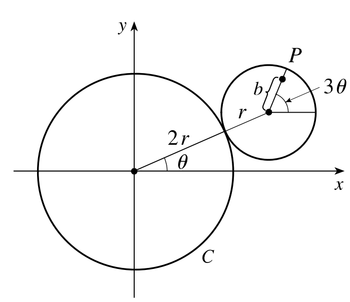
$abdx - b^2cx = a^2d^2 - b^2c^2 \quad\Rightarrow\quad x = \dfrac{a^2d^2 - b^2c^2}{b(ad-bc)} = \dfrac{ad+bc}{b}$ and the y-value is $\dfrac{b}{a}\dfrac{ad+bc}{b} = \dfrac{ad+bc}{a}$.

Similarly, the tangent line intersects $y = -\dfrac{b}{a}x$ at $\left(\dfrac{bc-ad}{b}, \dfrac{ad-bc}{a}\right)$. The midpoint of these intersection points is

$\left(\dfrac{1}{2}\left(\dfrac{ad+bc}{b} + \dfrac{bc-ad}{b}\right), \dfrac{1}{2}\left(\dfrac{ad+bc}{a} + \dfrac{ad-bc}{a}\right)\right) = \left(\dfrac{1}{2}\dfrac{2bc}{b}, \dfrac{1}{2}\dfrac{2ad}{a}\right) = (c,d)$, the point of tangency.

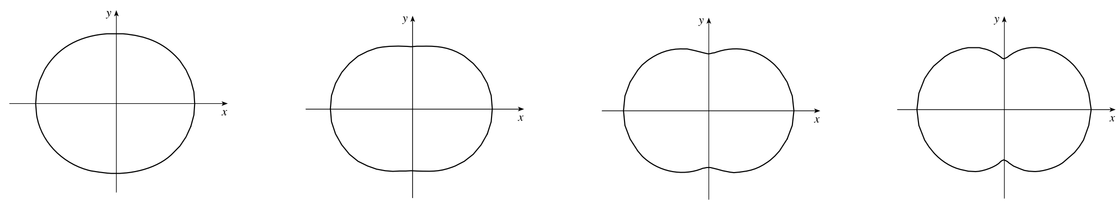
Note: If $y=0$, then at $(\pm a, 0)$, the tangent line is $x = \pm a$, and the points of intersection are clearly equidistant from the point of tangency.

6. (a) Since the smaller circle rolls without slipping around C, the amount of arc traversed on C ($2r\theta$ in the figure) must equal the amount of arc of the smaller circle that has been in contact with C. Since the smaller circle has radius r, it must have turned through an angle of $2r\theta/r = 2\theta$. In addition to turning through an angle 2θ, the little circle has rolled through an angle θ against C. Thus, P has turned through an angle of 3θ as shown in the figure. (If the little circle had turned through an angle of 2θ with its center pinned to the x-axis,



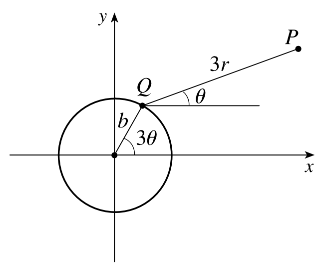
then P would have turned only 2θ instead of 3θ. The movement of the little circle around C adds θ to the angle.) From the figure, we see that the center of the small circle has coordinates $(3r\cos\theta, 3r\sin\theta)$. Thus, P has coordinates (x, y), where $x = b\cos 3\theta + 3r\cos\theta$ and $y = b\sin 3\theta + 3r\sin\theta$.

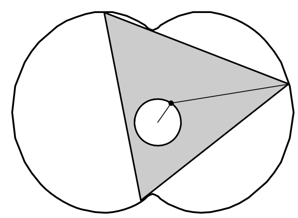
(b)



$b = \frac{1}{5}r$ $\qquad$ $b = \frac{2}{5}r$ $\qquad$ $b = \frac{3}{5}r$ $\qquad$ $b = \frac{4}{5}r$

(c) The diagram gives an alternate description of point P on the epitrochoid. Q moves around a circle of radius b, and P rotates one-third as fast with respect to Q at a distance of $3r$. Place an equilateral triangle with sides of length $3\sqrt{3}r$ so that its centroid is at Q and one vertex is at P. (The distance from the centroid to a vertex is $\frac{1}{\sqrt{3}}$ times the length of a side of the equilateral triangle.)





As θ increases by $\frac{2\pi}{3}$, the point Q travels once around the circle of radius b, returning to its original position. At the same time, P (and the rest of the triangle) rotate through an angle of $\frac{2\pi}{3}$ about Q, so P's position is occupied by another vertex. In this way, we see that the epitrochoid traced out by P is simultaneously traced out by the other two vertices as well. The whole equilateral triangle sits inside the epitrochoid (touching it only with its vertices) and each vertex traces out the curve once while the centroid moves around the circle three times.

(d) We view the epitrochoid as being traced out in the same way as in part (c), by a rotor for which the distance from its center to each vertex is $3r$, so it has radius $6r$. To show that the rotor fits inside the epitrochoid, it suffices to show that for any position of the tracing point P, there are no points on the opposite side of the rotor which are outside the epitrochoid. But the most likely case of intersection is when P is on the y-axis, so as long as the diameter of the rotor $\left(\text{which is } 3\sqrt{3}\,r\right)$ is less than the distance between the y-intercepts, the rotor will fit. The y-intercepts occur when $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2} \Rightarrow$ $y = -b + 3r$ or $y = b - 3r$, so the distance between the intercepts is $(-b + 3r) - (b - 3r) = 6r - 2b$, and the rotor will fit if $3\sqrt{3}\,r \le 6r - 2b \Leftrightarrow 2b \le 6r - 3\sqrt{3}\,r \Leftrightarrow b \le \frac{3}{2}\left(2 - \sqrt{3}\right)r$.