6.3 HW3

Question 34

1. Let S_1 and S_2 be two nonempty subsets in a metric space with $S_1 \cap \overline{S_2} = \overline{S_1} \cap S_2 = \emptyset$. If $A \subseteq S_1 \cup S_2$ is a connected set, then either $A \subseteq S_1$ or $A \subseteq S_2$.

Proof. Assume we have $A \not\subseteq S_1$ and $A \not\subseteq S_2$. Then we know $A \cap S_1$ and $A \cap S_2$ are both non-empty. We know

$$A = (A \cap S_1) \cup (A \cap S_2) \tag{6.82}$$

We wish to show

$$A \cap S_1 \text{ and } A \cap S_2 \text{ are separated.} \tag{6.83}$$

Notice

$$\overline{A \cap S_1} \subseteq \overline{S_1} \tag{6.84}$$

Then because $\overline{S_1}$ and S_2 are disjoint, we know $\overline{A \cap S_1}$ and S_2 are disjoint. Then because $A \cap S_2 \subseteq S_2$, we know $\overline{A \cap S_1}$ and $A \cap S_2$ are disjoint. Similarly, notice

$$\overline{A \cap S_2} \subseteq \overline{S_2} \tag{6.85}$$

Then because $\overline{S_2}$ and S_1 are disjoint, we know $\overline{A \cap S_2}$ and S_1 are disjoint. Then because $A \cap S_1 \subseteq S_1$, we know $\overline{A \cap S_2}$ and $A \cap S_1$ are disjoint.

We have proved $A \cap S_1$ and $A \cap S_2$ are separated, which CaC to A is connected. ∎

Question 35

2. If A_1 and A_2 are two nonempty and connected sets with $A_1 \cap A_2 \neq \varnothing$. Prove or disprove that

(a) $A_1 \cap A_2$ is connected

(b) $A_1 \cup A_2$ is connected

Proof. Let $A_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and let $A_2 = \{(0, y) \in \mathbb{R}^2 : y \in \mathbb{R}\}$.

To show A_1, A_2 are both connected, we can show they are both path connected.

Let $(\cos\alpha, \sin\alpha) \in A_1$ and $(\cos\beta, \sin\beta) \in A_1$. Define $f : [0, 1] \to A_1$ by

$$f(x) = (\cos(\alpha + x(\beta - \alpha)), \sin(\alpha + x(\beta - \alpha))) \tag{6.86}$$

Clearly f is continuous, and $f(0) = (\cos\alpha, \sin\alpha), f(1) = (\cos\beta, \sin\beta)$.

Let $(0, z) \in A_2$ and $(0, y) \in A_2$. Define $g : [0, 1] \to A_2$ by

$$g(x) = (0, z + x(y - z)) \tag{6.87}$$

Clearly g is continuous and $g(0) = (0, z), g(1) = (0, y)$.

We see $A_1 \cap A_2 = \{(0, 1), (0, -1)\}$. We see $\{(0, 1)\}$ and $\{(0, -1)\}$ are separated, because they are close.

For (b), see next three Theorems. ■

Theorem 6.3.1. (Connected) If A is disconnected, then A can be partitioned into two non-empty disjoint relatively open subsets.

Proof. Because A is disconnected, we know $A = E \cup F$ for some pair E, F of separated sets. We wish to prove E, F are both relatively open to A.

Because $\overline{E} \cap F = \varnothing$, we know the closure of E in subspace topology of A is disjoint to F. Then with respect to A, we can see $\overline{E} \subseteq F^c = E$, so we know E is relatively closed to A.

Similarly, because $\overline{F} \cap E = \varnothing$, we know the closure of F in subspace topology of A is disjoint to E. Then with respect to A, we can see $\overline{F} \subseteq E^c = F$, so we know F is relatively closed to A.

Then because E and F are both relatively closed to A and $A = E \cup F$ where E, F are disjoint, we know E and F are both relatively open to A. ■

Theorem 6.3.2. (Subspace Topology) Let Y be a subspace of (X, d), let $E \subseteq X$, and let $p \in Y$. We have

$$p \text{ is an interior point of } E \text{ in } X \implies p \text{ is an interior point of } E \cap Y \text{ in } Y \tag{6.88}$$

$$Y \cap E^\circ \text{ in } X \text{ is a subset of the interior of } E \cap Y \text{ in } Y \tag{6.89}$$

$$E \text{ is open in } X \implies E \cap Y \text{ is open in } Y \tag{6.90}$$

where the converse may not hold true.

Proof. We first prove the first statement. Let $\{x_n\}$ be a sequence in Y that converge to p. Because p is an interior point of E in X, and $\{x_n\}$ is in X, as $Y \subseteq X$, we know there exists N such that

$$n > N \implies x_n \in E \tag{6.91}$$

Notice $x_n \in Y$, and we are done.

For a nontrivial example of the converse of the first statement may not hold true, let $E = (0, 2)$, let $Y = \{1\} \cup (2, 3)$. We see 1 is an interior point of E in $\mathbb{R}$, but 1 isn't an interior point of $\{1\} = E \cap Y$ in Y.
The second and the third statement follows from the first statement. ∎

Theorem 6.3.3. (Union of Connected Sets that have Nonempty Intersection is Connected) Let $\mathcal{F}$ be a class of connected sets. We have

$$\bigcap \mathcal{F} \neq \varnothing \implies \bigcup \mathcal{F} \text{ is connected} \tag{6.92}$$

Proof. Assume $\bigcup \mathcal{F}$ is not connected. Let

$$\bigcup \mathcal{F} = A \dot{\cup} B \text{ and } A \neq \varnothing \neq B \tag{6.93}$$

And let A, B be relatively open to $\bigcup \mathcal{F}$. We know $\bigcap \mathcal{F}$ must intersect with either A, or B, or both.

WOLG, let

$$A \cap \bigcap \mathcal{F} \neq \varnothing \tag{6.94}$$

Because B is non-empty and $B \subseteq \bigcup \mathcal{F}$, we know B must intersect with some $F_n \in \mathcal{F}$. Notice that because $A \cap \bigcap \mathcal{F} \neq \varnothing$, we have $A \cap F_n \neq \varnothing$. Then by Theorem 6.3.2, we see $A \cap F_n$ and $B \cap F_n$ are both relatively open to F_n, while $F_n = (A \cap F_n) \dot{\cup} (B \cap F_n)$ CaC ∎

Question 36

3. Let $\{A_k\}_{k=1}^{\infty}$ be a family of connected subsets of M, and suppose that A is a connected subset of M such that $A_k \cap A \neq \emptyset$ for all $k \in \mathbb{N}$. Show that the union $(\bigcup_{k \in \mathbb{N}} A_k) \cup A$ is also connected.

Proof. Assume $\bigcup_{k \in \mathbb{N}} A_k \cup A$ is not connected. Let $\bigcup_{k \in \mathbb{N}} A_k \cup A$ be partitioned into two non-empty E, F relatively open to $\bigcup_{k \in \mathbb{N}} A_k \cup A$.

If E, F both intersect with A, observe that A can be partitioned into two non-empty $E \cap A$ and $F \cap A$, which are relatively open to A, causing a contradiction to A is connected.

Then, we only have to consider when only one of E, F intersect with A. WOLG, let E intersect with A.

We know F must intersect with some A_n. From last question, we know $A_n \cup A$ is connected. Notice that $A_n \cup A$ can be partitioned into two non-empty $E \cap (A \cup A_n)$ and $F \cap (A \cup A_n)$, and they are relatively open to $A \cup A_n$ CaC to $A_n \cup A$ is connected. ■

Question 37

4. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence, and define $s_n = \frac{1}{n}\sum_{k=1}^{n} a_k$. Prove or disprove that
   (a) If a_k converges, then s_n converges.

   (b) If s_n converges, then a_k converges.

   (c) Let $t = \frac{(2n-1)a_1+(2n-3)a_2+\ldots+3a_{n-1}+a_n}{n^2}$. Assume a_k converges to a. Does t_n also converge to a?

Proof. First notice

$$|s_n - L| = |\frac{\sum_{k=1}^{n} a_k}{n} - L| = |\frac{(\sum_{k=1}^{n} a_k) - nL}{n}| \tag{6.95}$$

$$= |\frac{\sum_{k=1}^{n} a_k - L}{n}| \tag{6.96}$$

$$= \frac{1}{n}|\sum_{k=1}^{n} a_k - L| \tag{6.97}$$

$$\leq \frac{1}{n}\sum_{k=1}^{n} |a_k - L| \tag{6.98}$$

We prove $\lim_{k\to\infty} a_k = L \implies \lim_{k\to\infty} s_k = L$.

Arbitrarily pick $R \in \mathbb{R}^+$. We wish to find N such that

$$k > N \implies |s_k - L| < R \tag{6.99}$$

Because $\lim_{n\to\infty} a_n = L$, we know there exists N_0 such that

$$n > N_0 \iff |a_n - L| < \frac{R}{2} \tag{6.100}$$

Let

$$H = \sum_{k=1}^{N_0} |a_k - L| \text{ and } m = \frac{2}{R}(H - N_0 R) \tag{6.101}$$

We wish to prove

$$n > N_0 + m \implies |s_n - L| < R \tag{6.102}$$

Let $n = N_0 + u$ where $u > m$. Observe

$$|s_n - L| \leq \frac{1}{n}\sum_{k=1}^{n}|a_k - L| = \frac{\sum_{k=1}^{n}|a_k - L|}{N_0 + u} \tag{6.103}$$

$$= \frac{\sum_{k=1}^{N_0}|a_k - L| + \sum_{k=N_0+1}^{N_0+u}|a_k - L|}{N_0 + u} \tag{6.104}$$

$$\leq \frac{H + u\frac{R}{2}}{N_0 + u} \tag{6.105}$$

and Observe

$$u > m = \frac{2}{R}(H - N_0R) \implies \frac{Ru}{2} > H - N_0R \tag{6.106}$$

$$\implies N_0R + Ru > H + \frac{Ru}{2} \tag{6.107}$$

$$\implies R > \frac{H + \frac{Ru}{2}}{N_0 + u} \tag{6.108}$$

In other words, we have

$$n > N_0 + m \implies u > m \implies |s_n - L| \leq \frac{H + \frac{Ru}{2}}{N_0 + u} < R \text{ (done)} \tag{6.109}$$

For (b), we raise an counter-example. Let

$$a_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \tag{6.110}$$

To see $\{a_n\}$ does not converge to any number, consider the sub-sequences

$$\{a_{n_k}\} \text{ where } n_k = 2k - 1 \text{ and } \{a_{n_u}\} \text{ where } n_u = 2u \tag{6.111}$$

We have

$$\forall k, a_{n_k} = 1 \text{ and } \forall u, a_{n_u} = 0 \tag{6.112}$$

Then we see

$$\lim_{k\to\infty} a_{n_k} = 1 \text{ and } \lim_{u\to\infty} a_{n_u} = 0 \tag{6.113}$$

If $\{a_n\}$ converge, then these two sub-sequence should converge to the same number.

We wish to show

$$\lim_{n\to\infty} s_n = \frac{1}{2} \tag{6.114}$$

With simple logical computation, we have

$$s_n = \begin{cases} \frac{n-1}{2n} & \text{if } n \text{ is odd} \\ \frac{1}{2} & \text{if } n \text{ is even} \end{cases} \tag{6.115}$$

Notice

$$\frac{n-1}{2n} = \frac{1}{2} - \frac{1}{2n} \tag{6.116}$$

Fix $\epsilon \in \mathbb{R}^+$. We see that

$$n > \frac{1}{2\epsilon} \implies |s_n - \frac{1}{2}| = \begin{cases} \frac{1}{2n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} < \epsilon \tag{6.117}$$

Lastly, we prove

$$\lim_{n\to\infty} a_n = a \implies \lim_{n\to\infty} t_n = a \tag{6.118}$$

First notice

$$\begin{aligned}
|t_n - a| &= |\frac{(2n-1)a_1 + (2n-3)a_2 + \cdots + a_n}{n^2} - a| && (6.119)\\
&= |\frac{1}{n^2}[(2n-1)(a_1 - a) + (2n-3)(a_2 - a) + \cdots + (a_n - a)]| && (6.120)\\
&= \frac{1}{n^2}|\sum_{k=1}^{n}(2n-2k+1)(a_k - a)| && (6.121)\\
&\leq \frac{1}{n^2}\sum_{k=1}^{n}|(2n-2k+1)(a_k - a)| && (6.122)\\
&= \frac{1}{n^2}\sum_{k=1}^{n}(2n-2k+1)|a_k - a| && (6.123)\\
&\leq \frac{1}{n^2}\sum_{k=1}^{n}(2n-1)|a_k - a| = \frac{2n-1}{n^2}\sum_{k=1}^{n}|a_k - a| && (6.124)
\end{aligned}$$

Arbitrarily pick $\epsilon \in \mathbb{R}^+$. Because $\lim_{n\to\infty} a_n = a$. We know there exists N_0 such that

$$n > N_0 \implies |a_n - a| < \frac{\epsilon}{4} \implies (\frac{2n-1}{n})|a_n - a| = (2 - \frac{1}{n})|a_n - a| < 2|a_n - a| < \frac{\epsilon}{2} \tag{6.125}$$

Let

$$H = \sum_{k=1}^{N_0}|a_k - a| \tag{6.126}$$

We have

$$|t_n - a| \le \frac{2n-1}{n^2}\sum_{k=1}^{n}|a_k - a| = \frac{2n-1}{n^2}\sum_{k=1}^{N_0}|a_k - a| + \frac{2n-1}{n^2}\sum_{k=N_0+1}^{n}|a_k - a| \tag{6.127}$$

$$= \frac{2n-1}{n^2}H + \frac{2n-1}{n^2}\sum_{k=N_0+1}^{n}|a_k - a| \tag{6.128}$$

$$= \frac{2n-1}{n^2}H + \frac{1}{n}\sum_{k=N_0+1}^{n}\frac{2n-1}{n}|a_k - a| \tag{6.129}$$

$$\le \frac{2n-1}{n^2}H + \frac{1}{n}(n - N_0)\frac{\epsilon}{2} \tag{6.130}$$

Then if we let

$$X_n := \frac{2n-1}{n^2}H + \frac{(n-N_0)\epsilon}{2n} \tag{6.131}$$

For all $n > N_0$, we have

$$|t_n - a| < X_n \tag{6.132}$$

Notice

$$X_n = \frac{2H}{n} - \frac{H}{n^2} + \frac{\epsilon}{2} - \frac{N_0\epsilon}{2n} \tag{6.133}$$

So we have

$$\lim_{n\to\infty} X_n = \frac{\epsilon}{2} \tag{6.134}$$

Then we know there exists some N_1 such that

$$n > N_1 \implies |X_n - \frac{\epsilon}{2}| < \frac{\epsilon}{2} \implies |t_n - a| \le X_n < \epsilon \text{ (done)} \tag{6.135}$$

■

Question 38

5. If $a_k > 0$ for all $k \in \mathbb{N}$, prove that

$$\liminf_{k\to\infty}\frac{a_{k+1}}{a_k} \le \liminf_{k\to\infty}\sqrt[k]{a_k} \le \limsup_{k\to\infty}\sqrt[k]{a_k} \le \limsup_{k\to\infty}\frac{a_{k+1}}{a_k}$$

Moreover, find a sequence $\{a_k\}_{k=1}^{\infty}$ such that

$$\limsup_{k\to\infty}\sqrt[k]{a_k} < \limsup_{k\to\infty}\frac{a_{k+1}}{a_k}$$

Proof. The fact

$$\liminf_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \sqrt[n]{a_n} \tag{6.136}$$

follows from definition.

We first prove

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} \sqrt[n]{a_n} \tag{6.137}$$

Let

$$\alpha = \liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \tag{6.138}$$

Notice that $\{a_k\}$ being positive give us $\alpha \geq 0$ and $0 \leq \liminf_{n\to\infty} \sqrt[n]{a_n}$. If $\alpha = 0$, the proof is done trivially. We only have to consider when α is positive.

Arbitrarily pick positive β smaller than α:

$$\beta < \alpha = \liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \tag{6.139}$$

Then we know there exists N such that

$$\forall n \geq N, \frac{a_{n+1}}{a_n} > \beta \tag{6.140}$$

This implies

$$\forall k, a_{N+k} > \beta^k a_N \tag{6.141}$$

Then for all $n > N$, we have

$$\sqrt[n]{a_n} > \sqrt[n]{\beta^{n-N} a_N} = \beta \sqrt[n]{\beta^{-N} a_N} \tag{6.142}$$

Because

$$\lim_{n\to\infty} \beta \sqrt[n]{\beta^{-N} a_N} = \beta \tag{6.143}$$

We see

$$\liminf_{n\to\infty} \sqrt[n]{a_n} \geq \beta \tag{6.144}$$

Notice that β is arbitrarily pick from $\{x \in \mathbb{R} : 0 \leq x < \alpha\}$, so we have in fact proved

$$0 \leq x < \alpha \implies x \leq \liminf_{n\to\infty} \sqrt[n]{a_n} \tag{6.145}$$

If $\liminf_{n\to\infty} \sqrt[n]{a_n} < \alpha$, there should exists $x < \alpha$ such that $\liminf_{n\to\infty} \sqrt[n]{a_n} < x$, which we have prove is impossible. (done)

We now prove

$$\limsup_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} \tag{6.146}$$

Let $\gamma = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$. If $\gamma = \infty$, the proof is done trivially. We only have to consider when $\gamma < \infty$.

Notice that $\{a_k\}$ being positive give us $\gamma \geq 0$. Arbitrarily pick positive δ greater than γ:

$$\delta > \gamma = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} \tag{6.147}$$

Then we know there exists N such that

$$\forall n \geq N, \frac{a_{n+1}}{a_n} < \delta \tag{6.148}$$

This implies

$$\forall k, a_{N+k} < \delta^k a_N \tag{6.149}$$

Then for all $n > N$, we have

$$\sqrt[n]{a_n} < \sqrt[n]{\delta^{n-N} a_N} = \delta \sqrt[n]{\delta^{-N} a_N} \tag{6.150}$$

Because

$$\lim_{n\to\infty} \delta \sqrt[n]{\delta^{-N} a_N} = \delta \tag{6.151}$$

We see

$$\limsup_{n\to\infty} \sqrt[n]{a_n} \leq \delta \tag{6.152}$$

Notice that δ is arbitrarily picked from $\{x \in \mathbb{R} : x > \gamma\}$, so we have in fact proved

$$x > \gamma \implies x \geq \limsup_{n\to\infty} \sqrt[n]{a_n} \tag{6.153}$$

If $\limsup_{n\to\infty} \sqrt[n]{a_n} > \gamma$, there should exists some $x > \gamma$ such that $\limsup_{n\to\infty} \sqrt[n]{a_n} > x$, which we have proved is impossible. (done)

Let

$$a_k = \begin{cases} 2 & \text{if } k \text{ is odd} \\ 1 & \text{if } k \text{ is even} \end{cases} \tag{6.154}$$

We see

$$\limsup_{n\to\infty} \sqrt[n]{a_n} = 1 < 2 \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} \tag{6.155}$$

■

Question 39

6. If $s_1 = \sqrt{2}$, and

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}}, \quad n = 1, 2, 3, \tag{6.156}$$

prove that s_n converge and bounded above by 2

Proof. We first prove $\{s_n\}$ increase monotonically by induction.

Base case:

$$s_1^2 = 2 < 2 + \sqrt{2 + \sqrt{\sqrt{2}}} = s_2^2 \implies s_1 < s_2 \tag{6.157}$$

Induction case: Let $s_k < s_{k+1}$. We wish to prove $s_{k+1} < s_{k+2}$. Because $s_{k+1} = \sqrt{2 + \sqrt{s_k}}$, we have

$$s_k \leq s_{k+1} = \sqrt{2 + \sqrt{s_k}} \tag{6.158}$$

Then we can deduce

$$s_k \leq \sqrt{2 + \sqrt{s_k}} \tag{6.159}$$

$$\implies \sqrt{s_k} \leq \sqrt{\sqrt{2 + \sqrt{s_k}}} \tag{6.160}$$

$$\implies 2 + \sqrt{s_k} \leq 2 + \sqrt{\sqrt{2 + \sqrt{s_k}}} \tag{6.161}$$

$$\implies \sqrt{2 + \sqrt{s_k}} \leq \sqrt{2 + \sqrt{\sqrt{2 + \sqrt{s_k}}}} \tag{6.162}$$

$$\implies s_{k+1} = \sqrt{2 + \sqrt{s_k}} \leq \sqrt{2 + \sqrt{s_{k+1}}} = s_{k+2} \text{ (done)} \tag{6.163}$$

We now prove $\{s_n\}$ is bounded above by 2 by induction.

Base case: $s_1 = \sqrt{2} < 2$.

Induction case: Let $s_k \leq 2$. We wish to prove $s_{k+1} \leq 2$. Observe

$$s_k \leq 2 \implies s_k \leq 4 \tag{6.164}$$

$$\implies \sqrt{s_k} \leq \sqrt{4} = 2 \tag{6.165}$$

$$\implies 2 + \sqrt{s_k} \leq 4 \tag{6.166}$$

$$\implies s_{k+1} = \sqrt{2 + \sqrt{s_k}} \leq \sqrt{4} = 2 \text{ (done)} \tag{6.167}$$

The fact that $\{s_n\}$ monotonically increase and bounded above tell us $\{s_n\}$ converge. ■

Question 40

7. Suppose $a_n > 0$ and $s_n = \sum_{k=1}^n a_k$. If s_n diverges, prove or disprove that $t_n = \sum_{k=1}^n \frac{a_k}{1+a_k}$ diverges. What can be said about

$$S_n = \sum_{k=1}^n \frac{a_k}{1+ka_k} \tag{6.168}$$

$$T_n = \sum_{k=1}^n \frac{a_k}{1+k^2a_k} \tag{6.169}$$

$$\text{If } s_n = \sum_{k=1}^n a_k \text{ converge, does } J_n = \sum_{k=1}^n ka_k \text{ converge} \tag{6.170}$$

Proof. We prove

$$t_n \text{ converge} \implies s_n \text{ converge} \tag{6.171}$$

Notice that

$$a_n = \frac{a_n + a_n^2}{1+a_n} = \frac{a_n}{1+a_n} + \frac{a_n^2}{1+a_n} \tag{6.172}$$

So we have

$$s_n = t_n + \sum_{k=1}^n \frac{a_n^2}{1+a_n} \tag{6.173}$$

Because t_n converge, above tell us we only have to prove $\sum_{k=1}^n \frac{a_n^2}{1+a_n}$ converge.

Because t_n converge, we know

$$\lim_{n\to\infty} \frac{a_n}{1+a_n} = 0 \tag{6.174}$$

Assume $\lim_{n\to\infty} a_n \neq 0$. Then there exists ϵ such that

$$\forall N \in \mathbb{N}, \exists n > N, a_n > \epsilon \tag{6.175}$$

Notice

$$a_n > \epsilon \implies \frac{a_n}{1+a_n} = 1 - \frac{1}{1+a_n} > 1 - \frac{1}{1+\epsilon} \implies |\frac{a_n}{1+a_n} - 0| > 1 - \frac{1}{1+\epsilon} \tag{6.176}$$

In other words, there exists a sub-sequence $\{a_{f(n)}\}$ such that

$$\lim_{n\to\infty} \frac{a_{f(n)}}{1+a_{f(n)}} \neq 0 \text{ CaC} \tag{6.177}$$

We have proved $\lim_{n\to\infty} a_n = 0$. Then we know there exists N_1 such that

$$\forall n > N_1, a_n < 1 \tag{6.178}$$

In other words,

$$\forall n > N_1, \frac{a_n^2}{1+a_n} < \frac{a_n}{1+a_n} \tag{6.179}$$

By comparison test, our proof is done (done) .

We show

$$\text{It is possible } s_n \text{ diverge and } S_n \text{ converge.} \tag{6.180}$$

Let

$$a_k = \begin{cases} 1 & \text{if } \exists u \in \mathbb{N}, k = u^2 \\ \frac{1}{k^2} & \text{otherwise} \end{cases} \tag{6.181}$$

Clearly, $\lim_{n\to\infty} s_n = \infty$. Yet, we have

$$\lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} \frac{a_k}{1+ka_k} = \sum_{u=1} \frac{a_{u^2}}{1+u^2 a_{u^2}} + \sum_{k\in\mathbb{N}\setminus\{r^2:r\in\mathbb{N}\}} \frac{a_k}{1+ka_k} \tag{6.182}$$

$$= \sum_{u=1} \frac{1}{1+u^2} + \sum_{k\in\mathbb{N}\setminus\{r^2:r\in\mathbb{N}\}} \frac{\frac{1}{k^2}}{1+\frac{1}{k}} \tag{6.183}$$

$$= \sum_{u=1} \frac{1}{1+u^2} + \sum_{k\in\mathbb{N}\setminus\{r^2:r\in\mathbb{N}\}} \frac{1}{k(k+1)} \tag{6.184}$$

Notice that

$$\frac{1}{1+u^2} < \frac{1}{u^2} \text{ and } \frac{1}{k(k+1)} < \frac{1}{k^2} \tag{6.185}$$

Then by comparison test, we know

$$\text{both } \sum_{u=1} \frac{1}{1+u^2} \text{ and } \sum_{k\in\mathbb{N}\setminus\{r^2:r\in\mathbb{N}\}} \frac{1}{k(k+1)} \text{ converge} \tag{6.186}$$

Then we know

$$S_n \text{ also converge (done)} \tag{6.187}$$

Notice that for all n

$$k \geq 1 \implies 1 + k^2 a_k > 1 + k a_k \implies \frac{a_k}{1+k^2 a_k} < \frac{a_k}{1+ka_k} \implies T_n < S_n \tag{6.188}$$

Then because the term is non-negative, by comparison test, we know the example above also satisfy T_n converge while s_n diverge.
Notice that if we let $a_k = \frac{1}{k^2}$, then s_n converge and $J_n = \sum_{k=1}^{n} \frac{1}{k}$ diverge.

∎

Question 41

8. Assume $A \subset \mathbb{R}$ is compact and let $a \in A$. Suppose $\{a_n\}$ is a sequence in A such that every convergent sub-sequence of $\{a_n\}$ converges to a.

1. Does the sequence $\{a_n\}$ also converge to a?
2. Without the assumption that A is compact, does the sequence $\{a_n\}$ converge to a?

Proof. Assume a_n does not converge to a. We know there exists ϵ such that there exists a sub-sequence $\{a_{n_k}\}_{k\in\mathbb{N}}$

$$\forall k \in \mathbb{N}, a_{n_k} \notin B_\epsilon(a) \tag{6.189}$$

Because A is (sequentially) compact, we know there must exist a sub-sequence $\{a_{n_{k_u}}\}_u \in \mathbb{N}$ such that

$$\{a_{n_{k_u}}\} \text{ converge} \tag{6.190}$$

Notice that

$$\{a_{n_{k_u}}\} \text{ is a sub-sequence of } \{a_n\} \tag{6.191}$$

So we know

$$\lim_{u\to\infty} a_{n_{k_u}} = a \text{ CaC } \text{ to } \forall n \in \mathbb{N}, a_{n_k} =\notin B_\epsilon(a) \tag{6.192}$$

Let $A = \mathbb{Q}$, and let

$$a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases} \tag{6.193}$$

and we see every convergent sub-sequence converge to 0 but a_n itself does not converge to 0 ■

Question 42

9. Suppose that $a_k \neq 0$ for large k and that

$$p = \lim_{k\to\infty} \frac{\ln\left(\frac{1}{|a_k|}\right)}{\ln(k)}$$

exists as an extended real number.

(a) If $p > 1$, then $\sum_{k=1}^{\infty} a_k$ converges absolutely.

(b) If $p < 1$, then $\sum_{k=1}^{\infty} a_k$ diverge

Proof. Let $p > \alpha > 1$. Then we know there exists N such that

$$\forall n > N, \frac{\ln(\frac{1}{|a_n|})}{\ln n} > \alpha > 1 \tag{6.194}$$

This give us

$$\forall n > N, \ln(\frac{1}{|a_n|}) > \alpha \ln n \tag{6.195}$$

This give us

$$\forall n > N, \frac{\ln(\frac{1}{|a_n|})}{\alpha} > \ln n \tag{6.196}$$

This give us

$$\forall n > N, \ln(|a_n|^{-\frac{1}{\alpha}}) > \ln n \tag{6.197}$$

This give us

$$\forall n > N, |a_n|^{\frac{-1}{\alpha}} > n \tag{6.198}$$

This give us

$$\forall n > N, |a_n| < n^{-\alpha} < n^{-1} \tag{6.199}$$

By comparison test, we are done.

Let $p < \beta < 1$. Then we know there exists N such that

$$\forall n > N, \frac{\ln(\frac{1}{|a_n|})}{\ln n} < \beta < 1 \tag{6.200}$$

This give us

$$\forall n > N, \frac{\ln(|a_n|^{-1})}{\beta} < \ln n \tag{6.201}$$

This give us

$$\forall n > N, \ln(|a_n|^{\frac{-1}{\beta}}) < \ln n \tag{6.202}$$

This give us

$$\forall n > N, |a_n|^{\frac{-1}{\beta}} < n \tag{6.203}$$

This give us

$$\forall n > N, |a_n| > n^{-\beta} > n^{-1} \tag{6.204}$$

By comparison test, we are done. ■

Question 43

10. Suppose that $f:\mathbb{R}\to(0,\infty)$ is differentiable, that $f(x)\to 0$ as $x\to\infty$, and that

$$\alpha = \lim_{x\to\infty}\frac{xf'(x)}{f(x)}$$

exists. If $\alpha<-1$, prove that

$$\sum_{k=1}^{\infty} f(k)$$

converges.

Proof. Let β satisfy

$$\alpha<\beta<-1 \tag{6.205}$$

Because

$$\lim_{x\to\infty}\frac{xf'(x)}{f(x)}=\alpha \tag{6.206}$$

We know there exists R such that

$$\forall x>R, \frac{xf'(x)}{f(x)}\le\beta \tag{6.207}$$

Then we have

$$\forall x>R, \frac{d}{dx}\ln f(x)=\frac{f'(x)}{f(x)}\le\frac{\beta}{x} \tag{6.208}$$

This tell us

$$\forall y>R, \int_R^y \frac{d}{dx}\ln f(x)dx\le\int_R^y\frac{\beta}{x}dx \tag{6.209}$$

Which means

$$\forall y>R, \ln(\frac{f(y)}{f(R)})=\ln f(y)-\ln f(R)\le\beta(\ln(y)-\ln(R))=\ln(\frac{y}{R})^{\beta} \tag{6.210}$$

This shows

$$\forall y>R, \frac{f(y)}{f(R)}\le(\frac{y}{R})^{\beta} \tag{6.211}$$

It means

$$\forall y>R, f(y)\le\frac{f(R)}{R^{\beta}}y^{\beta} \tag{6.212}$$

Notice $\beta < -1$, we know the series

$$\sum_{n>R} \frac{f(R)}{R^\beta} n^\beta = \frac{f(R)}{R^\beta} \sum_{n>R} n^\beta \text{ converge} \tag{6.213}$$

Then because $f(y)$ is always positive (given by the question), we know $\sum_{n=1}^{\infty} f(n)$ converge by comparison test. ∎

Lemma 6.3.4. (Bernoulli Inequality) Let $r \geq 1$ and $x \geq -1$. We have

$$(1+x)^r \geq 1 + rx \tag{6.214}$$

Proof. Let $r \geq 1$, and let

$$f(x) = (1+x)^r - 1 - rx \tag{6.215}$$

We wish to prove

$$\forall x \geq -1, f(x) \geq 0 \tag{6.216}$$

We now split the proof into two parts

$$\forall x \geq 0, f(x) \geq 0 \tag{6.217}$$
$$\forall x \in [-1, 0), f(x) \geq 0 \tag{6.218}$$

First notice by computation

$$f'(x) = r(1+x)^{r-1} - r = r((1+x)^{r-1} - 1) \tag{6.219}$$

And notice by some algebra

$$f'(x) \geq 0 \iff (1+x)^{r-1} \geq 1 \iff x \geq 0 \tag{6.220}$$

We first prove

$$\forall x \geq 0, f(x) \geq 0 \tag{6.221}$$

By computation,

$$f(0) = 0 \tag{6.222}$$

This give us

$$\forall x \geq 0, f(x) = f(x) - 0 = f(x) - f(0) = \int_0^x f'(t)dt \geq 0 \text{ (done)} \tag{6.223}$$

We now prove

$$\forall x \in [-1, 0), f(x) \geq 0 \tag{6.224}$$

Notice that above have shown

$$f'(x) \geq 0 \iff x \geq 0 \tag{6.225}$$

So we know

$$\forall x \in [-1,0), f'(x) < 0 \tag{6.226}$$

Observe that $f(x)$ is an polynomial, so we know $f(x)$ is continuous. Then if for some $y \in [-1,0)$, we have $f(y) < 0 = f(0)$, there must exists $u \in (y,0) \subseteq [-1,0)$ such that $f'(u) > 0$, which is impossible. ■

Question 44

11. Suppose that $\{a_n\}$ is a sequence of nonzero real numbers and that

$$p = \lim_{k\to\infty} k(1 - |\frac{a_{k+1}}{a_k}|)$$

exists as an extended real number. Prove that

$$\sum_{k=1}^{\infty} |a_k|$$

converges absolutely when $p > 1$.

Proof. Pick α that satisfy

$$\lim_{k\to\infty} k(1 - |\frac{a_{k+1}}{a_k}|) > \alpha > 1 \tag{6.227}$$

We know there exist N such that

$$\forall n > N, n(1 - |\frac{a_{n+1}}{a_n}|) > \alpha \tag{6.228}$$

With a little algebra,

$$\forall n > N, 1 - \frac{\alpha}{n} > |\frac{a_{n+1}}{a_n}| \tag{6.229}$$

Plugin Lemma 6.3.4 (Bernoulli Inequality) with $r = \alpha > 1$ and $x = \frac{-1}{n} \geq -1$. We have

$$\forall n > N, |\frac{a_{n+1}}{a_n}| < 1 + \frac{-\alpha}{n} \leq (1 - \frac{1}{n})^\alpha = (\frac{n-1}{n})^\alpha \tag{6.230}$$

Then we have

$$(*)\forall n > N, |a_{n+1}|n^\alpha < |a_n|(n-1)^\alpha \tag{6.231}$$

For each $n > N$, define

$$b_n = |a_{n+1}|n^\alpha \tag{6.232}$$

Then we have

$$|a_n|(n-1)^\alpha = b_{n-1} \tag{6.233}$$

So by (∗) we have

$$\forall n > N+1, b_n < b_{n-1} \tag{6.234}$$

This tell us $\{b_n\}_{n>N+1}$ is a decreasing sequence. Then we know $\{b_n : n > N+1\}$ is bounded above. More precisely, let

$$M = b_{N+1} \tag{6.235}$$

We have

$$\forall n > N+1, b_n < M \tag{6.236}$$

Recall the definition of b_n, we have

$$\forall n > N+1, |a_{n+1}|n^{\alpha} < M \tag{6.237}$$

In other words

$$\forall n > N+2, |a_n| < Mn^{-\alpha} \tag{6.238}$$

Notice that because $\alpha > 1$. We know

$$\sum_{n\in\mathbb{N}} Mn^{-\alpha} \text{ converge} \tag{6.239}$$

Then by comparison test, we know

$$\sum_{n\in\mathbb{N}} |a_n| \text{ converge} \tag{6.240}$$

■