A point z_0 is an isolated singularity of a function f if z_0 does not belong to the domain of f and f is holomorphic on the punctured disc defined by $0 < |z - z_0| < R$ for some $R > 0$.

It is known that if z_0 is an isolated singularity of f, then f has the Laurent expansion at z_0:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n,$$

or equivalently:

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

(1) There are three types of isolated singularities: removable singularities, poles, and essential singularities. Provide the definition of each type and give an example for each.

(2) State the definition of the residue of a function at an isolated singularity.

(3) If z_0 is a simple pole of f, prove that:

$$\operatorname{res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

(4) If $f(z) = \dfrac{q(z)}{p(z)}$, where p and q are holomorphic, and z_0 is a simple zero of p, prove that:

$$\operatorname{res}(f, z_0) = \frac{q(z_0)}{p'(z_0)}.$$

(5) If z_0 is a pole of f of order m, prove that:

$$\operatorname{res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} g(z),$$

where $g(z) = (z - z_0)^m f(z)$.

(6) Find the residues of the following functions at the indicated points:

(a) $\dfrac{z}{(2 - 3z)(4z + 3)}$ at $z = \frac{2}{3}$,

(b) $\dfrac{z - \frac{1}{6}z^3 - \sin z}{z^8}$ at $z = 0$.

In this section, we aim to compute certain definite integrals using the Cauchy's residue theorem.

(1) Evaluate $\displaystyle\int_{-\pi}^{\pi} \frac{d\theta}{5+3\cos\theta}$ using the following steps:

(a) Consider the closed contour $\gamma : [-\pi, \pi] \to \mathbb{C}$ defined by $\gamma(\theta) = e^{i\theta}$. Then express I as a contour integral:

$$I = 2i \int_{\gamma} \frac{dz}{3z^2 + 10z + 3}.$$

(b) Use Cauchy's residue theorem to compute I.

(2) Evaluate the improper integral $I = \displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$ using the following steps:

(a) Find all poles of $\dfrac{1}{z^2+2z+2}$.

(b) Let $S_R : [0, \pi] \to \mathbb{C}$ be the semicircle $S_R(t) = Re^{it}$ and $L_R : [-R, R] \to \mathbb{C}$ be the line segment $L_R(t) = t$. Then, for R large enough, the closed contour $\gamma_R = S_R + L_R$ contains the pole $-1+i$ of the function $\dfrac{1}{z^2+2z+2}$. Compute the contour integral $\displaystyle\int_{\gamma_R} \frac{dz}{z^2+2z+2}$ using the residue theorem.

(c) Prove that:

$$\left| \int_{S_R} \frac{dz}{z^2+2z+2} \right| \leq \frac{2\pi}{R}.$$

(d) Prove that:

$$I = \lim_{R\to\infty} \int_{L_R} \frac{dz}{z^2+2z+2},$$

and find I.

(3) Use the same approach to evaluate:

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+2x^2+1}.$$