國立成功大學114學年度「碩士班」甄試入學考試

線性代數

Notation:

- $M_{n\times n}(\mathbb{F})$: the set of all $n \times n$ matrices over the field $\mathbb{F}$
- I_n: the $n \times n$ identity matrix
- $\mathrm{End}(V)$: the set of all linear transformations from V to itself
- A^*: the conjugate transpose of the matrix A
- $\ker(\alpha)/\operatorname{im}(\alpha)/\operatorname{tr}(\alpha)$: kernel/image/trace of α

(1) Let $S = \{2, 3, 5, 6, 7\}$. Let V be the vector space of all functions $S \to \mathbb{R}$ with $(f+g)(x) = f(x) + g(x)$ and $(cf)(x) = cf(x)$ for $f, g \in V, c \in \mathbb{R}$.

(a) (8%) V is a k-dimensional vector space over $\mathbb{R}$, $k =$?

(b) (8%) Let $f_i(x) = x^i$. Is $\{f_1, f_2, \ldots, f_k\}$ a basis for V?

(2) Let V be the vector space of all polynomials with real coefficients satisfying $\deg(f(x)) < n$. Let $T \in \mathrm{End}(V)$ defined by $T(f(x)) = x^2(f(x+1) - f(x) - f'(x))$.

(a) (10%) In the case $n = 5$, find all eigenvectors of T.

(b) (10%) In the general case, find all eigenvalues of T. Is T diagonalizable?

(3) (10%) Let $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 0 & 1 & 2 \end{bmatrix}$, compute A^{100}.

(4) (12%) Let V be a 2-dimensional vector space over $\mathbb{F}$ and $\alpha \in \mathrm{End}(V)$, $\alpha^2 \neq 0$. Show that $V = \ker(\alpha) \oplus \operatorname{im}(\alpha)$. (Hint: consider minimal polynomial)

(5) Let $V = M_{4\times 4}(\mathbb{C})$, define $\langle A, B\rangle = \operatorname{tr}(AB^*)$.

(a) (10%) Show that $\langle \cdot, \cdot \rangle$ defines an inner product on V over $\mathbb{C}$.

(b) (10%) Let W be the subspace of V consisting of all skew-symmmetric matrices (i.e. $A = -A^T$). Find an orthonormal basis for W.

(6) (10%) Let V be an n-dimensional vector space over $\mathbb{F}$. Let $\alpha \in \mathrm{End}(V)$ for which there exists a set S of $n+1$ eigenvectors satisfying the condition that every subset of size n is a basis for V. Show that $\alpha = cI_n$ for some constant c.

(7) (12%) Let $A \in M_{n\times n}(\mathbb{R})$ satisfy $A^2 + I_n = 0$. Show that n is even, and there exists $P \in M_{n\times n}(\mathbb{R})$ such that $P^{-1}AP = \begin{bmatrix} 0 & -I_{n/2} \\ I_{n/2} & 0 \end{bmatrix}$.

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- Show all your work and justify all your answers.
- $\mathbb{R}$ denotes the field of real numbers, and n denotes a positive integer.

1. (12 points) Let A be an $n \times n$ real matrix whose (i, j) entry is

$$A_{ij} = \begin{cases} j, & \text{if } i \le j, \\ 0, & \text{otherwise,} \end{cases}$$

   where $i, j = 1, \ldots, n$. Find the inverse of A.

2. (12 points) Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$. For $(x_1, y_1), (x_2, y_2) \in V$ and $a \in \mathbb{R}$, define

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 y_2) \quad \text{and} \quad a(x_1, y_1) = (ax_1, y_1).$$

   Is V a vector space over $\mathbb{R}$ with these operations?

3. (15 points) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the function defined by

$$T(x, y, z) = (2x - y, 3y - 2z, x + y - z)$$

   for $(x, y, z) \in \mathbb{R}^3$. Prove that T is a linear transformation. Is T one-to-one?

4. (15 points) Let $V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid 2x_1 - x_2 + 3x_3 - x_4 = 0\}$. Find a basis β for V such that $(2, 1, -1, 0) \in \beta$.

5. (15 points) Let V be the real vector space of all $n \times n$ real matrices, and let $A \in V$. Suppose that W is the subspace of V spanned by the set $\{A^i \mid i \text{ is a non-negative integer}\}$, where A^0 is defined to be the $n \times n$ identity matrix. Prove that $\dim(W) \le n$.

6. (15 points) Let V be a finite-dimensional complex inner product space, and let T be a positive definite linear operator on V. Prove that $T = S^*S$ for some invertible linear operator S on V. Here S^* denotes the adjoint of S.

7. (16 points) Find the Jordan canonical form of the real matrix $\begin{bmatrix} 4 & -3 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

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1. (10 points) Let A, B be two $m \times n$ matrix. Show that $|\operatorname{rank}(A) - \operatorname{rank}(B)| \leq \operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$.

2. (16 points) Let A be an $n \times n$ matrix and $r_k = \operatorname{rank}(A^k)$.

   (a). Show that $\lim_{k\to\infty} r_k$ exist.

   (b). If $r_3 \neq r_4$, Is A diagonalizable? Show your answer.

3. (8 points) Let $A = [a_{ij}]$ be an $n \times n$ matrix satisfying the condition that each a_{ij} is either equal to 1 or to -1. Show that $\det(A)$ is an integer multiple of 2^{n-1}.

4. (16 points) Let S, T be linear operator on V such that $S^2 = S$. Show that the range of S is invariant under T if and only if $STS = TS$. Show that both the range and null space of S are invariant under T if and only if $ST = TS$.

5. (20 points) Define a real vector space $V = \{f(x) \mid f(x) = ax^2 + bx + c, a, b, c \in \mathbb{R}\}$, with inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$.

   (a). Find an orthonormal basis for V.

   (b). Using (a), find $f \in V$ to maximize $f(\frac{1}{2})$ subject to the constraint $\langle f, f\rangle = 1$.

6. (16 points) Let $A = \begin{bmatrix} -2 & 3 & 1 \\ 0 & a & 3 \\ 0 & -3 & 4-a \end{bmatrix}$. Find the condition of a such that A is diagonalizable over real number.

7. (14 points) Let A be an $n \times n$ real symmetric matrix. Show that the matrix $A^2 + A + I$ is positive-definite.

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In this test, all vector spaces are finite dimensional over $\mathbb{C}$.

1. (15 points) Let T be a linear operator on a vector space V. Prove that T is diagonalizable if and only if its minimal polynomial is square-free.

2. (15 points) Let V be a vector space. A linear operator S on V is semisimple if for every S-invariant subspace W of V there exists an S-invariant subspace W' of V such that $V = W \oplus W'$. Prove that every diagonalizable operator on V is semisimple, and deduce that every linear operator T on V can be decomposed uniquely as $T = S + N$, where S is semisimple, N is nilpotent, and $SN = NS$.

3. (15 points) Let T and U be normal operators on an inner product space V such that $TU = UT$. Prove that $UT^* = T^*U$, where T^* is the adjoint of T.

4. (15 points) Let T and U be Hermitian operators on an inner product space $(V, \langle \cdot, \cdot \rangle)$ such that $\langle T(x), x \rangle > 0$ for all nonzero $x \in V$. Prove that UT is diagonalizable and has only real eigenvalues.

5. (15 points) Find the total number of distinct equivalence classes of congruent $n \times n$ real symmetric matrices and justify your answer.

6. (15 points) Let A be an $n \times n$ complex matrix, t be a variable, and I be the identity matrix. Prove that
$$\det(I - tA) = \exp\left(- \sum_{i \geq 1} \frac{\operatorname{tr}(A^i) t^i}{i} \right).$$

7. (10 points) Let $A = (a_{i,j})$ be a $2n \times 2n$ matrix such that $A^T = -A$. The Phaffian of A is defined as
$$\operatorname{pf}(A) := \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1), \sigma(2i)},$$
where S_{2n} is the symmetric group of order $2n$ and $\operatorname{sgn}(\sigma)$ is the signature of σ. Prove that for any $2n \times 2n$ matrix B,
$$\operatorname{pf}(BAB^T) = \det(B) \operatorname{pf}(A).$$

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1. Find the inverse of
$$\begin{pmatrix} 2 & 0 & 0 & 3 & 0 \\ 0 & 2 & 0 & 0 & 8 \\ 0 & 0 & 9 & 0 & 0 \\ 3 & 0 & 0 & 5 & 0 \\ 0 & 4 & 0 & 0 & 17 \end{pmatrix}$$
(15 points)

2. Show that for any $A \in \mathbb{R}^{m\times n}$, $\operatorname{rank}(A^T A) = \operatorname{rank}(A)$. (15 points)

3. Let $\mathcal{P}_2$ be the real vector space of real quadratic polynomials (polynomials of degree at most 2). Find an orthonormal basis for $\mathcal{P}_2$ with respect to the inner product $\langle f, g\rangle = f(-1)g(-1) + f(0)g(0) + f(1)g(1)$. (You do not need to show that it is truely an inner product.) (15 points)

4. For real t show that
$$e^{\begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$
(15 points)

5. (a) Find the matrix $P \in \mathbb{R}^{3\times 3}$ such that $x \mapsto Px$ is the orthogonal projection of $\mathbb{R}^3$ onto $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. (10 points)

   (b) Find $\min_{x\in\mathbb{R}^2} \left\| \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} x \right\|$. (10 points)

6. (a) Let $A \in \mathbb{C}^{n\times n}$. Prove that if $x^* A x \geq 0$ for all $x \in \mathbb{C}^n$, then A is Hermitian. (10 points)

   (b) Let $A \in \mathbb{R}^{n\times n}$. Is it true that $x^T A x \geq 0$ for all $x \in \mathbb{R}^n$ implies A is symmetric? (10 points)

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1. Find e^A, where $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$. (15 points)

2. Let $T_j : \mathbb{R}^2 \to \mathbb{R}^2$, $j = 1, 2$, be a rotation by some angle θ_j about some point $x_j \in \mathbb{R}^2$. Show that if $\theta_1 + \theta_2 \notin \{2k\pi \mid k \in \mathbb{Z}\}$, then the composition T_2T_1, is also a rotation about some point. (15 points)

3. Let A be a real skew-symmetric matrix, that is, $A^t = -A$. Prove the following statements.

   (a) Each eigenvalue of A is either 0 or a purely imaginary number. (10 points)

   (b) The rank of A is even. (10 points)

4. Let $C([-\pi, \pi])$ be the space of real continuous functions on $[-\pi, \pi]$ with inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$. Find an orthonormal basis for the subspace $W = \operatorname{span}(1, x, \sin x)$. (15 points)

5. Let $M_{2\times2}$ be the space of 2×2 real matrices. Consider the linear operator S on $M_{2\times2}$ defined by

$$S(X) = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} X + X \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix},$$

   where $a, b, c, d \in \mathbb{R}$.

   (a) Write down the representative matrix of S with respect to the basis

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$ (10 points)

   (b) Give the necessary and sufficient condition on a, b, c, d so that S is invertible. (10 points)

6. Let A be an $n \times n$ (real or complex) matrix. Show that if A is nilpotent (i.e. $A^k = 0$ for some $k \in \mathbb{N}$), then $I - A$ is invertible, where I is the $n \times n$ identity matrix. (15 points)

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【基礎數學】: Part I. 線性代數

Linear Algebra

In the following, $F^{m\times n}$ denotes the class of all $m \times n$ matrices with entries in the field F, where $F = \mathbb{R}$ or $\mathbb{C}$. Vectors in F^n will be regarded as column vectors. We say a matrix $A \in \mathbb{R}^{n\times n}$ is symmetry if $A^T = A$, a matrix $A \in \mathbb{C}^{n\times n}$ is Hermitian if $A^* = \overline{A}^T = A$. A matrix $A \in \mathbb{R}^{n\times n}$ is positive definite if $x^T A x > 0$ for all nonzero $x \in \mathbb{R}^n$.

(1) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $U = \{X \in \mathbb{R}^{2\times 2} : AX = XA\}$, find the dimension of U. (20 points)

(2) Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $\theta \in [0, 2\pi]$.

a. Show that $A^n = \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix}$ for all $n \in \mathbb{N}$. (10 points)

b. Calculate A^{-n} for all $n \in \mathbb{N}$. (10 points)

(3) Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix.

a. Show that all eigenvalues of A are real. (10 points)

b. If (λ_1, y_1) and (λ_2, y_2) are two eigenpairs of A with $\lambda_1 \neq \lambda_2$, show that $\langle y_1, y_2 \rangle = 0$. (10 points)

(4) Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be symmetric positive definite matrix, show that

a. $a_{ii} > 0$ for all $1 \le i \le n$. (10 points)

b. $a_{ii}a_{jj} > a_{ij}^2$ for all $i \neq j$. (10 points)

(5) Let $w_i \in \mathbb{R}$, $1 \le i \le 4$ and $A = \begin{bmatrix} w_1w_1 & w_1w_2 & w_1w_3 & w_1w_4 \\ w_2w_1 & w_2w_2 & w_2w_3 & w_2w_4 \\ w_3w_1 & w_3w_2 & w_3w_3 & w_3w_4 \\ w_4w_1 & w_4w_2 & w_4w_3 & w_4w_4 \end{bmatrix}$ with $w_1^2 + w_2^2 + w_3^2 + w_4^2 = 1$.

a. Find all eigenvalues of A and its algebraic multiplicity. (10 points)

b. Calculate $\det(I_4 - 2A)$. (10 points)

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【基礎數學】：Part I. 線性代數

In the following, $F^{m\times n}$ denotes the class of all $m \times n$ matrices with entries in the field F, where $F = \mathbb{R}$ or $\mathbb{C}$. Vectors in F^n will be regarded as column vectors. $F^{m\times n}$ and F^n are vector spaces over F in the canonical way.

Justify all your answers for the problems below.

1. Let $W \subset \mathbb{R}^4$ be the space of solutions of the system of linear equations $AX = 0$, where $A = \begin{bmatrix} 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{bmatrix}$. Find a basis for W. (15 points)

2. Let L be the line $y = mx$ in $\mathbb{R}^2$, where $m \in \mathbb{R}$. Find the matrix $A \in \mathbb{R}^{2\times 2}$ so that $x \mapsto Ax$ is the orthogonal projection onto L. (15 points)

3. Compute $\det(M)$, where M is the following $n \times n$ tridiagonal matrix:

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}. \text{ (15 points)}$$

4. Suppose $n \geq m$. Let $v_1, \ldots, v_m$ be linearly independent vectors in $\mathbb{C}^n$, and $K_1, \ldots, K_m$ be linear subspaces of $\mathbb{C}^n$. Let A be the subspace of $\mathbb{C}^{n\times n}$ containing all matrices M such that $Mv_j \in K_j$ for $j = 1, 2, \ldots, m$. Find $\dim(A)$ (in terms of $n, \dim(K_1), \ldots, \dim(K_m)$). (20 points)

5. Let $P = \begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix} \in \mathbb{R}^{2\times 2}$. Give the necessary and sufficient condition on a, b such that $\lim_{n\to\infty} P^n$ exists. (20 points)

6. Find a nonsingular $Q \in \mathbb{C}^{3\times 3}$ such that $A = QJQ^{-1}$, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, and J is the Jordan form of A. (15 points)

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【基礎數學】: Part I. 線性代數

Note: $\mathbb{R}$ denotes the field of real numbers, and n denotes a positive integer.

1. (10%) Is there a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1,-2,0) = (1,1)$, $T(3,-5,1) = (2,3)$, and $T(-1,3,1) = (3,0)$? *Justify your answer.*

2. (15%) Let V be the vector space of all polynomials of degree at most n with real coefficients. For $i = 0, 1, \ldots, n$, let $p_i(x) = x^i + x^{i+1} + \cdots + x^n \in V$. Show that $\{p_0(x), p_1(x), \ldots, p_n(x)\}$ is a basis for V.

3. (20%) Let A be an $n \times n$ real matrix such that $A^2 = A$. Show that the trace of A is equal to the rank of A. Is A similar over $\mathbb{R}$ to a diagonal matrix? *Justify your answer.*

4. (20%) Let T be a linear operator on a finite-dimensional vector space such that $\operatorname{rank}(T^2) = \operatorname{rank}(T)$. Show that $\mathrm{N}(T) \cap \mathrm{R}(T) = \{0\}$. (Here $\mathrm{N}(T)$ and $\mathrm{R}(T)$ are the null space and the range of T respectively.)

5. (15%) Let V be the vector space of all polynomials of degree at most 3 with real coefficients. Let D be the linear operator on V defined by $D(p) = p'$ for $p \in V$. Find the Jordan form of D.

6. (20%) Let T and U be linear operators on an n-dimensional vector space V. Suppose that $\{v, T(v), \ldots, T^{n-1}(v)\}$ is a basis for V for some $v \in V$, and that $TU = UT$. Show that $U = p(T)$ for some polynomial p.

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【基礎數學】：Part II. 線性代數

Entrance exam for master degree program: Linear Algebra

1. (20 points) Consider the 5×5 real matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

   and the following problems concerning A.

   (a) Find an invertible matrix P such that PA is a row-reduced echelon matrix.

   (b) Find a basis for the row space W of A.

   (c) Find a basis for the vector space V of all 5×1 column matrices X such that $AX = \mathbf{0}$.

   (d) For what 5×1 column matrices Y does the equation $AX = Y$ has solutions?

2. (10 points) Let A be an $n \times n$ real matrix with transpose A^T. Prove that $\operatorname{rank}(A^T A) = \operatorname{rank} A$.

3. (10 points) Let V be the vector space of all real 3×3 matrices and let A be the diagonal matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

   Calculate the determinant of the linear transformation T on V defined by $T(X) = AX + XA$.

4. (20 points) Let A be an $n \times n$ orthogonal matrix, that is, A is a real $n \times n$ matrix with $A^T A = I$ where I is the $n \times n$ identity matrix.

   (a) Show that $\det A = \pm 1$.

   (b) Show that x and Ax have the same length for all $x \in \mathbb{R}^n$.

   (c) If λ is an eigenvalue of A, Prove that $|\lambda| = 1$.

   (d) If $n = 3$ and $\det A = 1$, prove that 1 is an eigenvalue of A.

5. (15 points) Show that all the eigenvalues of a real symmetric matrix are real, and that the eigenvectors are perpendicular to each other when they correspond to different eigenvalues.

6. (15 points) Consider the matrix

$$A = \begin{pmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix}.$$

   Find a Jordan form J of A and an invertible matrix Q such that $A = QJQ^{-1}$.

7. (10 points) Show that every matrix is similar to its transpose.

This exam has 7 questions, for a total of 100 points.

【基礎數學】：Part II. 線性代數

In what follows, $\mathbb{R}$ denotes the field of all real numbers, and $\mathrm{M}_{n\times n}(\mathbb{R})$ denotes the vector space of all $n\times n$ real matrices.

1. Let $P_2(\mathbb{R})$ be the vector space of all polynomials of degree at most 2 with real coefficients. Suppose $T\colon \mathrm{M}_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ is the linear transformation defined by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 3c + (a-2b)x + dx^2.$$

   (a) (10%) Find a basis for the null space of T and determine the dimension of the range of T.

   (b) (10%) Let $\gamma = \{1, x, x^2\}$, which is the standard ordered basis for $P_2(\mathbb{R})$. Find an ordered basis β for $\mathrm{M}_{2\times 2}(\mathbb{R})$ such that the matrix representation $[T]_\beta^\gamma$ of T in β and γ is

$$\begin{pmatrix} 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

2. Let n be a positive integer, and let $S_i = \{A \in \mathrm{M}_{n\times n}(\mathbb{R}) \mid A^t = (-1)^i A\}$ for $i = 1, 2$. Here A^t denotes the transpose of A.

   (a) (6%) Prove that S_i is a subspace of $\mathrm{M}_{n\times n}(\mathbb{R})$ for $i = 1, 2$.

   (b) (12%) Prove that $\mathrm{M}_{n\times n}(\mathbb{R})$ is the direct sum of S_1 and S_2.

3. Consider the real matrix $A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & -1 & -1 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$.

   (a) (8%) Find the characteristic polynomial for A.

   (b) (12%) Find the minimal polynomial for A. Is A similar to a diagonal matrix? *Justify your answer.*

4. (12%) Let V be a finite-dimensional inner product space whose inner product is denoted by $\langle\cdot\,,\cdot\rangle$, and let T be a self-adjoint operator on V (that is, T is equal to its adjoint T^*). Prove that if $\langle v, T(v)\rangle = 0$ for all $v \in V$, then T is the zero linear operator.

5. (18%) Let T be a linear operator on a finite-dimensional complex vector space V. Suppose W is a T-invariant subspace of V and $W \neq V$. Prove that there exists a vector $v \in V \setminus W$ such that $T(v) - \lambda v \in W$ for some eigenvalue λ of T.

6. (12%) Let A be a 9×9 real matrix such that $A^6 + A^3 = A^5 + A^4$. Is A similar over $\mathbb{R}$ to a upper triangular matrix? *Justify your answer.*

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【基礎數學】：Part II. 線性代數

Notation

- n: a positive integer
- $M_{n\times n}(F)$: the set of all $n \times n$ matrices over the field F
- $\mathbb{R}$: the field of all real numbers
- $\mathbb{C}$: the field of all complex numbers
- A^*: the conjugate transpose of the matrix A

1. (12%) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the $\mathbb{R}$-linear map defined by

$$T(a,b,c) = (a-3b-2c, a+b, 3a+5b+c).$$

Find the rank of T, and find a basis for the null space of T.

2. (12%) Suppose W_1 and W_2 are the following subspaces of the real vector space $M_{3\times 3}(\mathbb{R})$:

$$W_1 = \left\{ \begin{pmatrix} a & 2a & b \\ b & c & 0 \\ 0 & 0 & d \end{pmatrix} : a,b,c,d \in \mathbb{R} \right\}, \quad W_2 = \left\{ \begin{pmatrix} a & b & 2a \\ b & 2c & d \\ 0 & d & 0 \end{pmatrix} : a,b,c,d \in \mathbb{R} \right\}.$$

Find the dimension of the subspace $W_1 + W_2$.

3. Consider the real matrix $A = \begin{pmatrix} 12 & -5 & -5 & 3 \\ 20 & -8 & -10 & 0 \\ 7 & -3 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

   (a) (10%) Find the characteristic polynomial of A.

   (b) (5%) Is A similar to a 4×4 diagonal matrix over $\mathbb{R}$? Justify your answer.

   (c) (5%) Is A similar to a 4×4 diagonal matrix over $\mathbb{C}$? Justify your answer.

4. (12%) Show that if A is a 3×3 real matrix, then A is similar to

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} 0 & \lambda & 0 \\ 1 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 0 & 0 & \lambda \\ 1 & 0 & \mu \\ 0 & 1 & \nu \end{pmatrix}$$

for some $\lambda, \mu, \nu \in \mathbb{R}$.

5. (12%) Let A be a 6×6 complex matrix such that $A^3 = 0$. Find all possible Jordan canonical forms of A.

6. (12%) Suppose $N \in M_{n\times n}(\mathbb{C})$ is normal, i.e., $N^*N = NN^*$. Show that N is self-adjoint if and only if all eigenvalues of N are real.

7. Let $\langle A, B\rangle$ be the trace of AB^* for all $A, B \in M_{n\times n}(\mathbb{C})$.

   (a) (10%) Show that $\langle \cdot, \cdot \rangle$ is an inner product on $M_{n\times n}(\mathbb{C})$.

   (b) (10%) Let $P \in M_{n\times n}(\mathbb{C})$ be invertible, and let T be the linear operator on $M_{n\times n}(\mathbb{C})$ defined by $T(A) = P^{-1}AP$. Find the adjoint of T with respect to the inner product $\langle \cdot, \cdot \rangle$.