RTFT HW2

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Problem A. Let G be a finite group and let $\tau : G \to GL_2(\mathbb{C})$ be a representation of degree 2. Suppose that there exist elements $g, h \in G$ so that the matrices $\tau(g)$ and $\tau(h)$ do NOT commute. Prove that the representation τ is irreducible.

Proof. Let $V = \mathbb{C}^2$ be a $\mathbb{C}[G]$ module defined by $\forall g \in G, \forall v \in V, g(v) = \tau(g)v$

Assume τ is reducible

Then, there exists a proper non-trivial submodule $W \subseteq V$

We now prove $dim(W) = 1$

$dim(W) \neq 0$, since W it non-trivial

Assume $dim(W) = 2$

$W \subseteq V \implies W = V$, CaC (done)

By Maschke's Theorem, there exists a sub-module $W' \subseteq V$, such that $V = W \bigoplus W'$

$dim(W) = 1$ and $V = W \bigoplus W' \implies dim(W') = 1$

Let $\{w_c\}$ be a basis of W and $\{w'_c\}$ be a basis of W'

So $\{w_c, w'_c\}$ is a basis of V, we from now denote $\{w_c, w'_c\} = \alpha$

We now prove $\forall g \in G, [g]_\alpha = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}, \exists (a, d) \in \mathbb{C}$

$\forall w \in W, [w]_\alpha = \begin{bmatrix} x \\ 0 \end{bmatrix}, \exists x \in \mathbb{C}$

$\forall g \in G, \forall w \in W, g(w) \in W \implies \forall g \in G, \forall w \in W, [g]_\alpha [w]_\alpha = [w_1]_\alpha, \exists w_1 \in W \implies \forall g \in G, \forall w \in W, [g]_\alpha [w]_\alpha = \begin{bmatrix} x \\ 0 \end{bmatrix}, \exists x \in \mathbb{C}, \implies \forall g \in G, [g]_\alpha = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}, \exists (a, b, d) \in \mathbb{C}$ **(i)**

$\forall w' \in W', [w']_\alpha = \begin{bmatrix} 0 \\ y \end{bmatrix}, \exists y \in \mathbb{C}$

$\forall g \in G, \forall w' \in W', g(w') \in W' \implies \forall g \in G, \forall w' \in W', [g]_\alpha [w']_\alpha = [w'_1]_\alpha, \exists w'_1 \in W' \implies \forall g \in G, \forall w' \in W', [g]_\alpha [w']_\alpha = \begin{bmatrix} 0 \\ y \end{bmatrix}, \exists y \in \mathbb{C}, \implies$

$\forall g \in G, [g]_\alpha = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}, \exists (a,d) \in \mathbb{C}$ (proven with **(i)**) (done)

$\forall g \in G, \exists (a,d) \in \mathbb{C}, [g]_\alpha = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \implies \forall g_1, g_2 \in G, \exists (a_1, d_1, a_2, d_2) \in \mathbb{C}, [g_1]_\alpha [g_2]_\alpha = \begin{bmatrix} a_1 & 0 \\ 0 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & 0 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 a_2 & 0 \\ 0 & d_1 d_2 \end{bmatrix} = \begin{bmatrix} a_2 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ 0 & d_1 \end{bmatrix} = [g_2]_\alpha [g_1]_\alpha$

So $\forall g_1, g_2 \in G, [g_1 g_2]_\alpha = [g_1]_\alpha [g_2]_\alpha = [g_2]_\alpha [g_1]_\alpha = [g_2 g_1]_\alpha$

Let $E = \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}$

Notice $\forall g_1, g_2 \in G, [g_1 g_2]_\alpha = [I_V]_E^\alpha \tau(g_1 g_2) [I_V]_\alpha^E$, and $\forall g_1, g_2 \in G, [g_2 g_1]_\alpha = [I_V]_E^\alpha \tau(g_2 g_1) [I_V]_\alpha^E$

So $\forall g_1, g_2 \in G, [I_V]_E^\alpha \tau(g_1 g_2) [I_V]_\alpha^E = [g_1 g_2]_\alpha = [g_2 g_1]_\alpha = [I_V]_E^\alpha \tau(g_2 g_1) [I_V]_\alpha^E$

In short, $\forall g_1, g_2 \in G, [I_V]_E^\alpha \tau(g_1 g_2) [I_V]_\alpha^E = [I_V]_E^\alpha \tau(g_2 g_1) [I_V]_\alpha^E$

Doing multiplication with $[I_V]_\alpha^E$ and $[I_V]_E^\alpha$, we have $\forall g_1, g_2 \in G, [I_V]_\alpha^E [I_V]_E^\alpha \tau(g_1 g_2) [I_V]_\alpha^E [I_V]_E^\alpha = [I_V]_\alpha^E [I_V]_E^\alpha \tau(g_2 g_1) [I_V]_\alpha^E [I_V]_E^\alpha$ **(ii)**

Notice $\forall g_1, g_2 \in G, [I_V]_\alpha^E [I_V]_E^\alpha \tau(g_1 g_2) [I_V]_\alpha^E [I_V]_E^\alpha = [I_V]_E^E \tau(g_1 g_2) [I_V]_E^E = I_2 \tau(g_1 g_2) I_2 = \tau(g_1 g_2)$ **(iii)**

And also notice, $\forall g_1, g_2 \in G, [I_V]_\alpha^E [I_V]_E^\alpha \tau(g_2 g_1) [I_V]_\alpha^E [I_V]_E^\alpha = [I_V]_E^E \tau(g_2 g_1) [I_V]_E^E = I_2 \tau(g_2 g_1) I_2 = \tau(g_2 g_1)$ **(iv)**

Combine **(ii)(iii)(iv)**, we see $\forall g_1, g_2 \in G, \tau(g_1 g_2) = \tau(g_2 g_1) \implies \tau(g_1)\tau(g_2) = \tau(g_2)\tau(g_1)$ CaC to the premise that there exists two elements in the image of τ such that they do not commute.

$\blacksquare$

Problem B. Consider S_3 acting on $V = C_3 = span(\{e_1, e_2, e_3\})$, the permutation module. We know that $span(\{e_1, e_2, e_3\})$ is a 1-dim sub-module of V, and we extend it to obtain a basis $\alpha = \{e_1 + e_2 + e_3, e_2, e_3\}$ for V

(a) Write down the matrices $[g]_\alpha$ for all $g \in S_3$

$[e]_\alpha = I_3$

$$[(1,2)]_\alpha = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[(1,3)]_\alpha = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$
$$[(2,3)]_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$
$$[(1,2,3)]_\alpha = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$
$$[(1,3,2)]_\alpha = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

(b) Find another basis β for V such that for every $g \in S_3$, the matrix $[g]_\beta$ looks like $[g]_\beta = \begin{bmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$

Let $\beta = \{e_1 + e_2 + e_3, e_1 - e_2, e_2 - e_3\}$

$[e]_\beta = I_3$

$$[(1,2)]_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$
$$[(1,3)]_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$
$$[(2,3)]_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$
$$[(1,2,3)]_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$
$$[(1,3,2)]_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

(c) Show that the map $\phi : S_3 \to GL_2(\mathbb{C})$ defined by $g \mapsto Z_g$, where Z_g is the submatrix consisting of the second and the third rows and columns of $[g]_\beta$ you found in (b), gives an irreducible representation of S_3.

Proof. Let L be the FG-module given rise to by ϕ

Assume ϕ is reducible **(i)**

Let W be a proper nontrivial FG-submodule of L
ϕ is of degree 2

If $dim(W) = 2$, W is not proper

If $dim(W) = 0$, W is trivial

So $dim(W) = 1$

Let γ be a basis of W

Write $\gamma = \{\begin{bmatrix} a \\ b \end{bmatrix}\}$

Assume $a = 0$

$b \neq 0$, other wise $dim(W) = 0$

Then we see $(1,2)\begin{bmatrix} 0 \\ b \end{bmatrix} = \begin{bmatrix} b \\ b \end{bmatrix} \notin W$ CaC

Assume $a \neq 0$

Write $b = na$

Then we see $(1,2)\begin{bmatrix} a \\ na \end{bmatrix} = \begin{bmatrix} -a \\ (n+1)a) \end{bmatrix}$

So $\frac{n+1}{-1} = n$, which give us $n = \frac{-1}{2}$

Notcie $(1,2)\begin{bmatrix} a \\ \frac{-1}{2}a \end{bmatrix} = \begin{bmatrix} \frac{-3}{2}a \\ \frac{-1}{2}a \end{bmatrix} \notin W$ CaC then CaC to **(i)** ■

Probelm C. In this problem we provide a counterexample of Maschke's Theorem when the group is infinite. Consider $\phi : \mathbb{Z} \to GL_2(\mathbb{C})$ the function given by

$$\phi(k) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, \forall k \in \mathbb{Z}$$

(a) Show that ϕ is a representation for the infinite group $\mathbb{Z}$ (Equivalently, you can prove that $\mathbb{C}^2$, viewed as column vectors, is a left $\mathbb{Z}$-module by the action induced from the representation)

Proof. Let $a, b \in \mathbb{Z}$

$\phi(a)\phi(b) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} = \phi(a+b)$ ■

(b) Show that ϕ is reducible (there exists a sub-representation or submodule)

Proof. Let $W = span(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$

We prove W is a submodule

$$\forall a \in \mathbb{Z}, \forall \begin{bmatrix} n \\ 0 \end{bmatrix} \in W, a \begin{bmatrix} n \\ 0 \end{bmatrix} = \phi(a) \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ 0 \end{bmatrix} \in W \qquad \blacksquare$$

(c) Show that ϕ is indecomposable

Proof. Assume ϕ is decomposable

Let W be the submodule $span(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$ from the last question.

So there exists W', such that $W \bigoplus W' = \mathbb{C}^2$

$dim(W') = dim(\mathbb{C}^2) - dim(W) = 2 - 1 = 1$

We now prove $W' = span(\begin{bmatrix} x \\ y \end{bmatrix}) \implies y \neq 0$

Assume $y = 0$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} \in span(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = W$ CaC (done)

Then we write $W' = span(\begin{bmatrix} n \\ 1 \end{bmatrix}), \exists n \in \mathbb{C}$

$$\phi(1) \begin{bmatrix} n \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} n \\ 1 \end{bmatrix} = \begin{bmatrix} n+1 \\ 1 \end{bmatrix} \notin span(\begin{bmatrix} n \\ 1 \end{bmatrix}) = W' \text{ CaC} \qquad \blacksquare$$