1.6 HW5

Question 40

(a) Let E be a measurable subset of $\mathbb{R}^2$ such that for almost every $x \in \mathbb{R}$, $\{y \in \mathbb{R} : (x, y) \in E\}$ has $\mathbb{R}^1$-measure zero. Show that E has measure zero and that for almost every $y \in \mathbb{R}^1$, $\{x \in \mathbb{R} : (x, y) \in E\}$ has measure zero.

(b) Let $f(x, y)$ be nonnegative and measurable in $\mathbb{R}^2$. Suppose that for almost every $x \in \mathbb{R}^1$, $f(x, y)$ is finite for almost every y. Show that for almost every $y \in \mathbb{R}^1$, $f(x, y)$ is finite for almost every x.

Proof. **(a)**

By Tonelli's Theorem

$$\begin{aligned} |E| = \int_{\mathbb{R}^2} \chi_E d(x,y) &= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \chi_E dy \Big) dx \\ &= \int_{\mathbb{R}} |\{y \in \mathbb{R} : (x,y) \in E\}| \, dx = 0 \end{aligned}$$

where the last equality follows from the premise that $\{y \in \mathbb{R} : (x, y) \in E\}$ has $\mathbb{R}^1$-measure zero for almost every x. Again, by Tonelli's Theorem,

$$\begin{aligned} \int_{\mathbb{R}} |\{x \in \mathbb{R} : (x,y) \in E\}| \, dy &= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \chi_E dx \Big) dy \\ &= \int_{\mathbb{R}^2} \chi_E d(x,y) = 0 \end{aligned}$$

This implies

$$\int_{\mathbb{R}} |\{x \in \mathbb{R} : (x,y) \in E\}| \, dx = 0 \text{ for almost every } y$$

This is true if and only if

$$\{x \in \mathbb{R} : (x,y) \in E\} \text{ has measure zero for almost every } y$$

(b)

Let $F \triangleq \{(x, y) \in \mathbb{R}^2 : f(x, y) = \infty\}$. Observe

$$F = \bigcap_{n \in \mathbb{N}} \{(x,y) \in \mathbb{R}^2 : f(x,y) > n\}$$

It follows from f is measurable that F is measurable. By premise,

$$|\{y \in \mathbb{R} : (x,y) \in F\}| = 0 \text{ for almost every } x$$

We have shown F satisfies the hypothesis E satisfies. It then follows from part **(a)** that

$$|\{x \in \mathbb{R} : (x,y) \in F\}| = 0 \text{ for almost every } y$$

That is, for almost every $y \in \mathbb{R}$, $f(x,y) < \infty$ for almost every x. ■

Question 41

Let f be measurable and period $1 : f(t+1) = f(t)$. Suppose that there is a finite c such that

$$\int_0^1 |f(a+t) - f(b+t)|\, dt \leq c$$

for all a and b. Show that $f \in L[0,1]$. (Set $a = x, b = -x$, integrate with respect to x, and make the change of variables $\xi = x + t, \eta = -x + t$)

Proof. Consider the linear transformation

$$T = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Note that

$$T^{-1}(1,0) = (\frac{1}{2}, \frac{1}{2}) \text{ and } T^{-1}(0,1) = (\frac{-1}{2}, \frac{1}{2})$$

It follows that $T^{-1}([0,1]^2) \subseteq [-1,1] \times [0,1]$. Then by a change of variable and Tonelli's Theorem

$$\begin{aligned} \int_{[0,1]^2} |f(x) - f(t)|\, d(x,t) &\leq |\det T| \int_{[-1,1]\times[0,1]} |f(x+t) - f(-x+t)|\, d(x,t) \\ &= 2 \int_{-1}^1 \int_0^1 |f(x+t) - f(-x+t)|\, dt dx \\ &\leq 2 \int_{-1}^1 c dx = 4c \end{aligned}$$

We have shown $f(x) - f(t) \in L([0,1]^2)$. It then follows from Fubini's Theorem

$$\int_0^1 |f(x) - f(t)|\, dx \in \mathbb{R} \text{ for almost every } t \in [0,1]$$

Fix some $t \in [0,1]$ such that $\int_0^1 |f(x) - f(t)|\, dx \in \mathbb{R}$. Because $[0,1]$ is not null, $f(t) \neq \pm\infty$ obviously. Define $h_t : [0,1] \to [-\infty, \infty]$ by

$$h_t(x) \triangleq f(x) - f(t)$$

It then follows from $f(x) = h_t(x) + f(t)$ that

$$\begin{aligned}\int_0^1 |f(x)|\, dx &\leq \int_0^1 |h_t(x)|\, dx + \int_0^1 |f(t)|\, dx \\ &\leq \int_0^1 |f(x) - f(t)|\, dx + |f(t)| \in \mathbb{R}\end{aligned}$$

We have shown $f \in L([0,1])$

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Question 42

(a) If f is nonnegative and measurable on E and $\omega(y) = |\{\mathbf{x} \in E : f(\mathbf{x}) > y\}|, y > 0$, use Tonelli's Theorem to prove that $\int_E f = \int_{\mathbb{R}^+} \omega(y)dy$. (By definition of the integral $\int_E f = |R(f,E)| = \iint_{R(f,E)} d\mathbf{x}dy$. Use $\{\mathbf{x} \in E : f(\mathbf{x}) \geq y\} = \{\mathbf{x} : (x,y) \in R(f,E)\}$, and recall that $\omega(y) = |\{\mathbf{x} \in E : f(\mathbf{x}) \geq y\}|$ unless y is a point of discontinuity of ω)

(b) Deduce from the special case the general formula

$$\int_E f^p = p \int_{\mathbb{R}^+} y^{p-1} \omega(y) dy \qquad (f \geq 0, 0 < p < \infty)$$

Proof. Define

$$\begin{aligned}R(f,E)_y &\triangleq \{\mathbf{x} \in E : (\mathbf{x}, y) \in R(f,E)\} \\ &= \{\mathbf{x} \in E : y < f(\mathbf{x})\}\end{aligned}$$

Thus,

$$\omega(y) = |R(f,E)_y|$$

Let $E \subseteq \mathbb{R}^n$. Part **(a)** then follows from applying Tonelli's Theorem to compute

$$\begin{aligned}\int_E f = |R(f,E)| &= \int_{\mathbb{R}^n \times \mathbb{R}^+} \chi_{R(f,E)} d(\mathbf{x}, y) \qquad \text{Recall } y > 0 \\ &= \int_{\mathbb{R}^+} \Big[\int_{\mathbb{R}^n} \chi_{R(f,E)} d\mathbf{x} \Big] dy \\ &= \int_{\mathbb{R}^+} |R(f,E)_y|\, dy = \int_{\mathbb{R}^+} \omega(y) dy\end{aligned}$$

Define $\omega_p : \mathbb{R}^+ \to [0, \infty]$ by

$$\omega_p(y) \triangleq |\{\mathbf{x} \in E : y < f^p(\mathbf{x})\}|$$

It is clear that for all fixed y,

$$\{\mathbf{x} \in E : y < f^p(\mathbf{x})\} = \{\mathbf{x} \in E : y^{\frac{1}{p}} < f(\mathbf{x})\}$$

It follows that

$$\omega_p(y) = \omega(y^{\frac{1}{p}}) \text{ for all } y$$

We may now use the result in part (a) to deduce

$$\begin{aligned}\int_E f^p = \int_{\mathbb{R}^+} \omega_p(z)dz &= \int_{\mathbb{R}^+} \omega(z^{\frac{1}{p}})dz \\ &= \int_{\mathbb{R}^+} \omega(y)dy^p = p\int_{\mathbb{R}^+} y^{p-1}\omega(y)dy\end{aligned}$$

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Question 43

For $f \in L(\mathbb{R})$, define the Fourier transform $\widehat{f} : \mathbb{R} \to \mathbb{C}$ of f by

$$\widehat{f}(x) \triangleq \frac{1}{2\pi}\int_{\mathbb{R}} f(t)e^{-ixt}dt$$

Show that if f and g belong to $L(\mathbb{R})$, then

$$\widehat{(f * g)}(x) = 2\pi\widehat{f}(x)\widehat{g}(x)$$

Proof. Because $f, g \in L(\mathbb{R})$, we know $|f| * |g| \in L(\mathbb{R})$. Thus, by Tonelli's Theorem

$$\begin{aligned}\int_{\mathbb{R}^2} \left|f(s)e^{-ixs}g(t-s)e^{-ix(t-s)}\right| d(s,t) &= \int_{\mathbb{R}^2} |f(s)g(t-s)|\, d(s,t) \\ &= \int_{\mathbb{R}} (|f| * |g|)(t)dt \in \mathbb{R}\end{aligned}$$

Therefore, we may use Fubini's Theorem to compute

$$\begin{aligned}
\widehat{f * g}(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} (f*g)(t)e^{-ixt}dt \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f(s)g(t-s)ds\Big)e^{-ixt}dt \\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2} f(s)e^{-ixs}g(t-s)e^{-ix(t-s)}d(s,t) \\
(\xi = s \text{ and } \eta = t-s \text{ and } \det T = 1) \qquad &= \frac{1}{2\pi}\int_{\mathbb{R}^2} f(\xi)e^{-ix\xi}g(\eta)e^{-ix\eta}d(\xi,\eta) \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} f(\xi)e^{-ix\xi}d\xi \int_{\mathbb{R}} g(\eta)e^{-ix\eta}d\eta = 2\pi\widehat{f}(x)\widehat{g}(x)
\end{aligned}$$

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Question 44

Let F be a closed subset of $\mathbb{R}$ and let $\delta(x) = d(x,F)$ be the corresponding distance function. If $\lambda > 0$ and f is nonnegative and integrable over complement of F, prove that the function

$$\int_{\mathbb{R}} \frac{\delta^\lambda(y)f(y)}{|x-y|^{1+\lambda}}dy$$

is integrable over F.

Proof. We are required to show

$$\int_F \left|\int_{\mathbb{R}} \frac{\delta^\lambda(y)f(y)}{|x-y|^{1+\lambda}}dy\right| dx \in \mathbb{R}$$

We know

$$\int_F \left|\int_{\mathbb{R}} \frac{\delta^\lambda(y)f(y)}{|x-y|^{1+\lambda}}dy\right| dx \le \int_F\int_{\mathbb{R}} \frac{|\delta^\lambda(y)f(y)|}{|x-y|^{1+\lambda}}dydx$$

Because $\delta(y) = 0$ when $y \in F$, we may compute

$$\int_F\int_{\mathbb{R}} \frac{|\delta^\lambda(y)f(y)|}{|x-y|^{1+\lambda}}dydx = \int_F\int_{\mathbb{R}\setminus F} \frac{|\delta^\lambda(y)f(y)|}{|x-y|^{1+\lambda}}dydx$$

We have reduced the problem into proving

$$\int_F\int_{\mathbb{R}\setminus F} \frac{|\delta^\lambda(y)f(y)|}{|x-y|^{1+\lambda}}dydx \in \mathbb{R} \tag{1.11}$$

Now, by Tonelli's Theorem,

$$\begin{aligned}\int_F \int_{\mathbb{R}\setminus F} \frac{\left|\delta^\lambda(y) f(y)\right|}{|x-y|^{1+\lambda}} dy dx &= \int_{\mathbb{R}\setminus F} \int_F \frac{\left|\delta^\lambda(y) f(y)\right|}{|x-y|^{1+\lambda}} dx dy \\ &= \int_{\mathbb{R}\setminus F} \delta^\lambda(y) \left|f(y)\right| \int_F \frac{dx}{|x-y|^{1+\lambda}} dy\end{aligned}$$

By definition of δ, we have

$$x \in F \implies |x-y| \geq \delta(y)$$

Therefore,

$$\begin{aligned}\int_F \int_{\mathbb{R}\setminus F} \frac{\left|\delta^\lambda(y) f(y)\right|}{|x-y|^{1+\lambda}} dy dx &= \int_{\mathbb{R}\setminus F} \delta^\lambda(y) \left|f(y)\right| \int_F \frac{dx}{|x-y|^{1+\lambda}} dy \\ &\leq \int_{\mathbb{R}\setminus F} \delta^\lambda(y) \left|f(y)\right| \int_{|x-y|\geq \delta(y)} \frac{dx}{|x-y|^{1+\lambda}} dy\end{aligned}$$

Then by two changes of variables, we have

$$\begin{aligned}\int_F \int_{\mathbb{R}\setminus F} \frac{\left|\delta^\lambda(y) f(y)\right|}{|x-y|^{1+\lambda}} dy dx &\leq \int_{\mathbb{R}\setminus F} \delta^\lambda(y) \left|f(y)\right| \int_{|x-y|\geq \delta(y)} \frac{dx}{|x-y|^{1+\lambda}} dy \\ &\leq \int_{\mathbb{R}\setminus F} \delta^\lambda(y) \left|f(y)\right| \int_{\delta(y)}^\infty \frac{2dt}{t^{1+\lambda}} dy \\ &= 2 \int_{\mathbb{R}\setminus F} \delta^\lambda(y) \left|f(y)\right| \left[\frac{t^{-\lambda}}{-\lambda}\right]\Bigg|_{t=\delta(y)}^\infty dy \\ &= \frac{2}{\lambda} \int_{\mathbb{R}\setminus F} \left|f(y)\right| dy \in \mathbb{R}\end{aligned}$$

We have proved Equation 1.11, thus proving the whole proposition. ■

Question 45

Use Fubini's Theorem to prove that

$$\int_{\mathbb{R}^n} e^{-|\mathbf{x}|^2} d\mathbf{x} = \pi^{\frac{n}{2}}$$

Proof. We shall prove by induction. If $n = 1$, we have

$$\int_{\mathbb{R}} e^{-x^2} dx = 2 \int_0^\infty e^{-x^2} dx$$

Let $I \triangleq \int_{\mathbb{R}} e^{-x^2} dx$. We wish to show $I^2 = \pi$. Compute

$$\begin{aligned}
I^2 &= 4\int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy \\
&= 4\int_0^\infty \Big(\int_0^\infty e^{-y^2} dy\Big) e^{-x^2} dx \\
&= 4\int_0^\infty \Big(\int_0^\infty e^{-(x^2+y^2)} dy\Big) dx \\
&= 4\int_0^\infty \Big(\int_0^\infty e^{-x^2(1+s^2)} x ds\Big) dx \\
(\because \text{ Tonelli's Theorem }) \quad &= 4\int_0^\infty \int_0^\infty e^{-x^2(1+s^2)} x dx ds \\
&= 4\int_0^\infty \Big[\frac{e^{-x^2(1+s^2)}}{-2(1+s^2)}\Big]\Big|_{x=0}^\infty ds \\
&= 2\int_0^\infty \frac{1}{1+s^2} ds \\
&= 2\arctan(s)\Big|_{s=0}^\infty = \pi
\end{aligned}$$

Note that in the forth equality, we take the change of variables $s = \frac{y}{x}$. Although such change of variable is not justified for Lebesgue integral, it is for Riemann integral, and the improper Riemann integral here coincides with the Lebesgue integral, thus justifying the forth equality. Now, suppose the proposition hold true for $n < r$. Let $n = r$. By Tonelli's Theorem, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{-|\mathbf{x}|^2} d\mathbf{x} &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} e^{-(|\mathbf{y}|^2+z^2)} d\mathbf{y} dz \\
&= \int_{\mathbb{R}^{n-1}} e^{-|\mathbf{y}|^2} d\mathbf{y} \int_{\mathbb{R}} e^{-z^2} dz = \pi^{\frac{n-1}{2}} \cdot \pi^{\frac{1}{2}} = \pi^{\frac{n}{2}}
\end{aligned}$$

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