Analysis (Honors) I, Fall 2023
Final Exam Solutions

1. (18 points; 3 points each) Determine whether each of the following statements is true or false. Write down a short explanation for each of your answers (a few sentences suffice).

   (a) $\mathbb{R}$ cannot be written as a countable union of Cantor sets.

   Solution. True. Since $\mathbb{R}$ is a complete metric space, if it is a union of countably many closed sets, then at least one of them has nonempty interior. However, Cantor sets are closed and have empty interior. $\square$

   (b) Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be Riemann integrable. Let $\overline{F}(y)$ and $\underline{F}(y)$ be the upper and lower slice integrals of f with respect to x. If $F(y)$ is any function that equals $\overline{F}(y)$ and $\underline{F}(y)$ for almost every y then the Riemann integral of F equals those of $\overline{F}$ and $\underline{F}$.

   Solution. False. Take $f \equiv 0$. Then $\overline{F}$ and $\underline{F}$ are also constant 0. Take $F = \chi_{\mathbb{Q}\cap[c,d]}$. Then F equals 0 almost everywhere, but F is not even Rieamnn integrable. $\square$

   (c) If $S \subseteq \mathbb{R}^d$ is open and bounded then χ_S is Riemann integrable.

   Solution. False. χ_S is Riemann integrable if and only if ∂S has measure zero. For instance, if we consider $d = 1$, and $S = F^c$, where F is a Fat Cantor set, then $\partial S = F$, which does not have measure zero, but S is open and bounded. $\square$

   (d) If $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous and if $E \subseteq \mathbb{R}$ is measurable then $f(E)$ is also measurable.

   Solution. False. Consider $f(x) = x+g(x)$, where g is the Cantor function. We can extend the domain of f such that it is uniformly continuous (since f is Hölder continuous). If $N \subseteq \mathbb{R}$ is not measurable, then $f(f^{-1}(N)) = N$ is not measurable, but $f^{-1}(N) \subseteq C$ is measurable. $\square$

   (e) If (f_n) is a sequence of nonnegative measurable functions then

   $$\limsup_{n\to\infty} \int f_n \leq \int \limsup_{n\to\infty} f_n.$$

   Solution. False. Take $f_n = \chi_{[n,n+1]}$. Then the left side is 1 but the right side is 0. $\square$

   (f) Suppose that f and g are Lebesgue integrable functions such that $\int_E f = \int_E g$ for all measurable $E \subseteq \mathbb{R}^d$. Then $f = g$ almost everywhere.

   Solution. True. Suppose not. Then either $\{f > g\}$ or $\{f < g\}$ has positive measure. Without loss of generality assume the former has positive measure. Then there exists n such that $E := \{f \geq g + 1/n\}$ has positive measure. Then $\int_E f \geq \int_E (g + 1/n) = \int_E f + \frac{1}{n} m(E)$, a contradiction. So $f = g$ a.e. $\square$

2. (15 points) Suppose that $f : [0,1] \to \mathbb{R}$ is a continuous functions such that

$$\int_0^1 f(x)e^{-nx}\, \mathrm{d}x = 0 \quad \text{for all } n = 0, 1, 2, \ldots.$$

Show that $f(x) = 0$ for all $x \in [0,1]$.

Solution. We can either apply Stone–Weierstrass or the Weierstrass approximation theorem (after a change of variables). We use the former one here. Define $\mathcal{A}$ to be the algebra generated by $\{e^{-nx} : n \geq 0\}$. Clearly it vanishes nowhere and it separate points (since e^{-x} is strictly decreasing). So $\mathcal{A}$ is dense in $C^0([0,1])$. Therefore, for the given f and $\varepsilon > 0$, there exists $g \in A$ such that $\|f - g\| < \varepsilon$. From the assumption we have $\int_0^1 f(x)g(x)\, \mathrm{d}x = 0$. So

$$\int_0^1 f(x)^2\, \mathrm{d}x \leq \left| \int_0^1 f(x)g(x)\, \mathrm{d}x \right| + \int_0^1 |f(x)| \cdot |f(x) - g(x)|\, \mathrm{d}x \leq \varepsilon \|f\|_{L^1}.$$

Since $\varepsilon > 0$ is arbitrary, $\|f\|_{L^2} = 0$, which implies that f has to be 0 everywhere. □

3. (20 points total)

   (a) (12 points) Recall that a Banach space is a complete normed vector space. Let X be a Banach space with norm $\|\cdot\|$. Fix $x_0 \in X$. Suppose that $\Phi : \overline{B(x_0, r)} \to X$ is a function such that we can write $\Phi = I + \Psi$, where I is the identity map, and Ψ satisfies the following: there exists $\gamma \in (0,1)$ such that

   $$\|\Psi(x_1) - \Psi(x_2)\| \leq \gamma \|x_1 - x_2\| \quad \text{for all } x_1, x_2 \in \overline{B(x_0, r)}.$$

   Show the following: if $R = (1-\gamma)r$, then for any $y \in \overline{B(\Phi(x_0), R)}$, there is a unique $x \in \overline{B(x_0, r)}$ satisfying $\Phi(x) = y$.

   Solution. We shift the points x_0 and y_0 to 0 be redefining Φ. For $x \in \overline{B(0, r)}$, let

   $$\tilde{\Phi}(x) = \Phi(x + x_0) - \Phi(x_0) = x + \Psi(x + x_0) - \Psi(x_0).$$

   Then $\tilde{\Phi}(0) = 0$. Fix $y \in \overline{B(0, R)}$. Consider $T : \overline{B(0, r)} \to X$, given by

   $$T(x) = x - (\tilde{\Phi}(x) - y).$$

   Then T maps $\overline{B(0, r)}$ to itself: indeed,

   $$\begin{aligned} \|T(x)\| &= \|x - (\tilde{\Phi}(x) - y)\| \\ &= \|\Psi(x_0) - \Psi(x_0 + x) + y\| \\ &\leq \|\Psi(x_0 + x) - \Psi(x_0)\| + \|y\| \\ &\leq \gamma\|x\| + R \leq r. \end{aligned}$$

Note also that T is a contraction: for $x_1, x_2 \in \overline{B(0,r)}$,

$$\|T(x_2) - T(x_1)\| = \|\Psi(x_2 + x_0) - \Psi(x_1 + x_0)\| \leq \gamma\|x_2 - x_1\|.$$

Since $\overline{B(0,r)}$ is a closed subset in the complete space X, it is also complete. By the Banach fixed point theorem, T has a unique fixed point $x^* \in \overline{B(0,r)}$. Note that $T(x^*) = x^*$ if and only if $\tilde{\Phi}(x^*) = y$, and this implies our desired conclusion. $\square$

(b) (8 points) Let $K : [0,1]^2 \to \mathbb{R}$ be a nonconstant continuous function. Write $M = \|K\|_\infty$. Let $g \in C^0([0,1], \mathbb{R})$ be a continuous function such that $\|g\|_\infty < \frac{1}{8M}$. Show that the equation

$$u(x) = g(x) + \int_0^1 K(x,y)(u(y))^2 \, \mathrm{d}y$$

has a unique solution $u \in C([0,1])$ satisfying $\|u\|_\infty \leq \frac{1}{4M}$.

Solution. Consider $X = C^0([0,1])$. Let

$$\Phi(u)(x) = u(x) - \int_0^1 K(x,y)u(y)^2 \, \mathrm{d}y.$$

Then the corresponding Ψ is

$$\Psi(u)(x) = -\int_0^1 K(x,y)u(y)^2 \, \mathrm{d}y.$$

Choose $x_0 = 0$, and $y_0 = \Phi(0) = 0$. Let $r = 1/(4M)$. Then for any $u_1, u_2 \in \overline{B(0,r)}$,

$$\|\Psi(u_2) - \Psi(u_1)\|_\infty \leq \int_0^1 |K(x,y)||u_2^2 - u_1^2|(y) \, \mathrm{d}y \leq 2Mr\|u_2 - u_1\|_\infty \leq \frac{1}{2}\|u_2 - u_1\|_\infty.$$

By (a), for all $g \in B(0,R)$, where $R = r/2 = 1/(8M)$, the equation

$$u(x) - \int_0^1 K(x,y)(u(y))^2 \, \mathrm{d}y = g(x)$$

has a unique solution $u \in \overline{B(0,r)}$. $\square$

4. (14 points total) Let $A \subseteq [0,1]^2$ be the set of all $(x,y) \in \mathbb{Q}^2 \cap [0,1]^2$ such that whenever they are written as $x = p/q$ and $y = r/s$ (in the lowest terms) then $q = s$. Define $f(x,y) = \chi_A(x,y)$ for all $(x,y) \in [0,1]^2$.

(a) (6 points) Prove that A is dense in $[0,1]^2$ but any line parallel to the coordinate axes contains at most a finite subset of A.

Solution. The key is to observe the following: if p_k denote the k-th prime, and if we define

$$A_k = \left\{ \left(\frac{n}{p_k}, \frac{m}{p_k} \right) : n, m = 1, 2, \ldots, p_k - 1 \right\},$$

then $A = \bigcup_{k=1}^{\infty} A_k$. Since there are infinitely many prime, for any given $\varepsilon > 0$, there exists a prime p_k such that $p_k^{-1} < \varepsilon$, and any ball with radius $\sqrt{2}\varepsilon$ must intersect a point in A_k. This proves that A is dense in $[0,1]^2$. Clearly, any line parallel to the coordinate axes intersect at most finitely many points in A, because such a line either intersect no point in A, or its x-intercept or y-intercept is of the form a/p_k for some prime k, and its intersection with A has at most p_k many points. $\square$

(b) (2 points) Show that

$$\int_0^1 \left(\int_0^1 f(x,y)\ \mathrm{d}y \right)\ \mathrm{d}x = \int_0^1 \left(\int_0^1 f(x,y)\ \mathrm{d}x \right)\ \mathrm{d}y = 0.$$

Solution. From (a) we know that

$$\int_0^1 f(x,y)\ \mathrm{d}y = \int_0^1 f(x,y)\ \mathrm{d}x = 0.$$

This implies the result. $\square$

(c) (6 points) Is f Riemann integrable? Is f Lebesgue integrable? Prove your assertions.

Solution. f is not Riemann integrable, since A is dense in $[0,1]^2$ (the proof is the same as that $\chi_{\mathbb{Q}\cap[0,1]}$ is not Riemann integrable). f is Lebesgue integrable because A is countable and has measure zero, so $f = 0$ a.e., which is Lebesgue integrable. $\square$

5. (15 points, 5 points each) Recall that a topological space is separable if it contains a countable dense subset. Are the following metric spaces separable? Explain your answers.

(a) $C_b^0(\mathbb{R},\mathbb{R})$, the space of all bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}$, under the sup-norm.

Solution. It is not separable. Let F be any continuous function such that it is 0 outside $[0,1]$ and $F(1/2) = 1$ (for instance, consider a tent functions). For any subset of positive integers $I \subseteq \mathbb{N}$, we define

$$f_I(x) = \sum_{n\in I} F(x+n).$$

Observe that whenever $I \neq J$ then $\|f_I - f_J\| = 1$. Also, $(f_I)_{I\subseteq\mathbb{N}}$ is uncountable. If the space has a dense subset A, then each ball $B(f_I, 1/2)$ must contain at least one element of A, and these elements must be all distinct, which implies A is uncountable. So the space cannot be separable. $\square$

(b) $L^1(\mathbb{R}^d)$ under the L^1-norm.

Solution. Yes, it is separable. First, continuous functions with support in $[-n,n]$ for some n is dense in $L^1(\mathbb{R}^d)$, and the space of continuous functions with support in $[-n,n]$ has a countable dense set (polynomials with rational coefficients), by the Weierstrass approximation theorem. You can fill in the details yourself. $\square$

(c) $L^\infty([0,1])$ under the L^∞-norm.

Solution. Same as (a). We define $F_n = \chi_{[1/(n+1),1/n]}$ instead, and $f_I = \sum_{n\in I} F_n$. The argument is the same. $\square$

6. (18 points) Let $E \subseteq \mathbb{R}^d$ be a measurable set with $m(E) < \infty$.

(a) (2 points) Let $g : E \to \mathbb{R}$ be measurable. Show that there are at most countable many $M \in \mathbb{R}$ such that

$$m(\{x \in E : g(x) = M\}) > 0.$$

Solution. For any sequence (M_n) in E, one has

$$m(E) \geq \sum_{n=1}^{\infty} m(\{x \in E : g(x) = M_n\}).$$

If there are uncountably many M such that $m(\{x \in E : g(x) = M\}) > 0$, then taking sup over all sequences, the right side converges to the uncountable sum

$$\sum_{M\in\mathbb{R}} m(\{x \in E : g(x) = M\}),$$

which must be ∞, since a nonnegative uncountable sum is finite only if there are at most countably many nonzero terms. But this contradicts $m(E) < \infty$.

If you do not want to make use of the fact about uncountable sum, one can also argue as follows. Fix $\varepsilon > 0$. Then there are at most finitely many M such that $m(\{x \in E : g(x) = M\}) \geq \varepsilon$. (Why?) Taking union over positive rational ε, the set of M such that $m(\{x \in E : g(x) = M\}) > 0$ is at most countable. $\square$

(b) (11 points) Let (f_n) be a sequence of Lebesgue integrable functions defined on E such that $f_n \to f$ almost everywhere, where f is also Lebesgue integrable. Prove that the following are equivalent.

i. $\displaystyle\lim_{M\to\infty} \sup_{n\geq 1} \int_{\{|f_n|\geq M\}} |f_n| = 0.$

ii. $\displaystyle\lim_{n\to\infty} \int |f - f_n| = 0.$

iii. $\displaystyle\lim_{n\to\infty} \int |f_n| \to \int |f|.$

Solution. Assume that (f_n) satisfies i. For any M, define the truncation

$$f_n^{(M)} = \begin{cases} -M & \text{if } f_n \leq -M, \\ M & \text{if } f_n \geq M, \\ f_n & \text{otherwise.} \end{cases}$$

Defining $f^{(M)}$ analogously for f, we have $f_n^{(M)} \to f^{(M)}$ a.e. Therefore by the bounded convergence theorem,

$$\int |f_n^{(M)} - f^{(M)}| \to 0 \quad \text{as } n \to \infty.$$

Note that

$$|f_n - f_n^{(M)}| \le |f_n| \mathbf{1}_{\{|f_n| \ge M\}}, \quad |f - f^{(M)}| \le |f| \mathbf{1}_{\{|f| \ge M\}},$$

and hence

$$\begin{aligned}\limsup_{n\to\infty} \int |f_n - f| &\le \limsup_{n\to\infty} \int |f_n - f_n^{(M)}| + \int |f - f^{(M)}| \\ &\le \sup_n \int |f_n| \mathbf{1}_{\{|f_n| \ge M\}} + \int |f| \mathbf{1}_{\{|f| \ge M\}}.\end{aligned}$$

By i., the first term goes to 0 as $M \to \infty$. The second integral also goes to 0 as $M \to \infty$ by dominated convergence: because $\int |f| < \infty$ and

$$|f| \mathbf{1}_{\{|f| \ge M\}} \le |f|, \quad |f| \mathbf{1}_{\{|f| \ge M\}} \to 0 \quad \text{as } M \to \infty.$$

This implies ii.

To see ii. implies iii., use the triangle inequality.

Last, suppose that iii. holds. Define $f_n^{(M)}$ and $f^{(M)}$ a bit differently: set $f_n^{(M)} = f_n \mathbf{1}_{\{|f_n| < M\}}$ and $f^{(M)} = f \mathbf{1}_{\{|f| < M\}}$. Then as long as $m(\{|f| = M\}) = 0$, we still have $f_n^{(M)} \to f^{(M)}$ a.e. (Check this!) Write

$$\int (|f_n| - |f|) = \int (|f_n| - |f_n^{(M)}|) + \int (|f_n^{(M)}| - |f^{(M)}|) + \int (|f^{(M)}| - |f|).$$

Taking $n \to \infty$ and using iii. and bounded convergence, we find for such an M,

$$\lim_{n\to\infty} \int (|f_n| - |f_n^{(M)}|) = \int (|f| - |f^{(M)}|).$$

As $M \to \infty$, the right side converges to 0, so given $\varepsilon > 0$ we can choose M such that

$$\limsup_{n\to\infty} \left| \int (|f_n| - |f_n^{(M)}|) \right| < \varepsilon.$$

(Here, we used the fact that there are at most countably many M such that $m(\{|f| = M\}) > 0$, so that such a choice of M exists.) Therefore, $|\int (|f_n| - |f_n^{(M)}|)| < \varepsilon$ for all but finitely many n. Using the integrability of the f_n, we may further increase M to make all these terms less than ε for all n:

$$\sup_n \left| \int (|f_n| - |f_n^{(M)}|) \right| < \varepsilon.$$

Of course, the left hand side is just

$$\sup_n \left| \int_{\{|f_n| \ge M\}} |f_n| \right|,$$

which shows i. $\square$

Remark: The condition i. is also known as uniform integrability. Sometimes it is a very useful condition for us to verify that the limit of the integrals equals the integral of the limit. You may also find the fact that ii. and iii. are equivalent interesting. Of course, there are some restrictions, such as $m(E) < \infty$ and we have to first know that the limiting function is integrable. For an easier application of this result, see the next part.

(c) (5 points) Let (h_n) be a sequence of Lebesgue integrable functions defined on E. Suppose that $h_n \to h$ almost everywhere, and there exists $\varepsilon > 0$ such that

$$\sup_{n\geq 1} \int |h_n|^{1+\varepsilon} < \infty.$$

Prove that h is Lebesgue integrable and $\int h_n \to \int h$ as $n \to \infty$.

Solution. We first show that (h_n) satisfies i. Let $M > 0$. Then for all n,

$$\int_{\{|h_n|\geq M\}} |h_n| \leq \frac{1}{M^\varepsilon} \int_{\{|h_n|\geq M\}} |h_n|^{1+\varepsilon} \leq \frac{C}{M^\varepsilon},$$

which goes to 0 uniformly in n. Hence (h_n) satisfies i. Note that this also implies (h_n) is integrable:

$$\int |h_n| = \int_{\{|h_n|<M\}} |h_n| + \int_{\{|h_n|\geq M\}} |h_n| \leq Mm(E) + \frac{C}{M^\varepsilon} < \infty.$$

Next we show h is integrable: by Fatou,

$$\begin{aligned}\int |h| \leq \liminf_{n\to\infty} \int |h_n| &\leq \sup_n \int \max\{|h_n|^{1+\varepsilon}, 1\} \\ &\leq m(E) + \sup_n \int |h_n|^{1+\varepsilon} < \infty.\end{aligned}$$

Hence by (b) we have $\int |h_n - h| \to 0$, and the result follows from the triangle inequality. $\square$