NCKU 112.2
Geometry 1

Eric Liu

Contents

Chapter 1

Curve

1.1 Frenet Trihedron

In this section, we are given a smooth curve $\alpha(s) \in \mathbb{R}^3$ parametrized by arc-length and a smooth curve $\beta(t) \in \mathbb{R}^3$ with unknown parametrization. We seek to

(a) define Frenet Trihedron for α.

(b) prove Frenet-Serret Formula.

(c) give identity of torsion of α.

(d) give identity of curvature of β.

in particular, at the end of this section, using (d), we prove that the curvature of a plane curve $\beta(t) = (x, y)$ with unknown parametrization is exactly

$$\kappa(t) = \frac{|x'y'' - x''y'|}{\big((x')^2 + (y')^2\big)^{\frac{3}{2}}}$$

(a): Define Frenet Trihedron for α

We define the **tangent vector** T of α at each s by

$$T(s) \triangleq \alpha'(s)$$

and we define its **normal vector** N by

$$N(s) \triangleq \frac{T'(s)}{|T'(s)|}$$

Note that N is well defined if and only if $\alpha''(s) \neq 0$. We define **binormal vector** B by

$$B(s) \triangleq T(s) \times N(s)$$

Note that from $B' \perp B$ and $B' = \big(T \times N'\big) \perp T$, we have $B' /\!/ N$. This then justify our later definition of **torsion**.

We define **curvature** $\kappa(s)$ and **torsion** $\tau(s)$ by

$$\kappa(s) \triangleq |T'(s)| \text{ and } \tau(s) \triangleq \frac{B'(s)}{N(s)}$$

(b): Frenet-Serret Formula

Theorem 1.1.1. (Frenet-Serret Formula) Given a smooth curve $\alpha(s)$ parametrized by arc-length, if $T'(s) \neq 0$, we have the following

$$\begin{cases} T' = \kappa N \\ N' = -\kappa T - \tau B \\ B' = \tau N \end{cases}$$

Proof. The first and the third equations follows from that of definition. Now, see

$$\begin{aligned} N' &= (B \times T)' \\ &= B' \times T + B \times T' \\ &= \tau N \times T + B \times \kappa N \\ &= \tau(-B) + \kappa(-T) = -\kappa T - \tau B \end{aligned}$$

■

Note that in $\mathbb{R}^2$, the Frenet-Serret Formula still holds, in the sense

$$\begin{cases} T' = \kappa N \\ N' = -\kappa T \end{cases}$$

This can be proved by setting the ambient space to be $\mathbb{R}^3$.

(c): Identity of Torsion of $\alpha(s)$

Theorem 1.1.2. (Identity of Torsion) Given a smooth curve $\alpha(s) \in \mathbb{R}^3$ parametrized by arc-length, if $\kappa(s) \neq 0$, we have

$$\tau(s) = -\frac{\alpha'(s) \times \alpha''(s) \cdot \alpha'''(s)}{\big(\alpha''(s)\big)^2}$$

Proof. By definition

$$\alpha'(s) = T(s)$$

Compute

$$\alpha''(s) = T'(s) = -\kappa N(s)$$

Compute

$$\alpha''' = (-\kappa N)' = -\kappa' N + \kappa^2 T + \kappa\tau B$$

Compute

$$\alpha' \times \alpha'' = T \times (-\kappa N) = -\kappa B$$

Compute

$$\begin{aligned}\alpha' \times \alpha'' \cdot \alpha''' &= (-\kappa B) \cdot (-\kappa N + \kappa^2 T + \kappa\tau B)\\ &= -\kappa^2\tau\end{aligned}$$

The result then follows. ∎

(d): Identity of Curvature of $\beta(t)$

Theorem 1.1.3. (Identity of Curvature) Given a smooth curve $\beta(t) \in \mathbb{R}^3$ with unknown parametrization, we have

$$\kappa(t) = \frac{|\beta'(t) \times \beta''(t)|}{|\beta'(t)|^3}$$

Proof. Define s (arc-length) by

$$s(t) = \int_{t_0}^{t} |\beta'(t')|\, dt'$$

We have $\frac{ds}{dt} = |\beta'(t)|$. This then give us

$$\kappa(t) = \left|\frac{dT}{ds}\right| = \left|\frac{dT}{dt}\right| \cdot \left|\frac{dt}{ds}\right| = \left|\frac{dT}{dt}\right| \cdot \frac{1}{|\beta'(t)|}$$

We then can reduce the problem into

$$\text{proving } \left|\frac{dT}{dt}\right| = \frac{|\beta'(t) \times \beta''(t)|}{|\beta'(t)|^2}$$

Note that we have

(a) $\frac{ds}{dt} = |\beta'(t)|$

(b) $T(t) = \frac{\beta'(t)}{|\beta'(t)|}$

This then let us compute

$$T'(t) = \frac{|\beta'(t)|\,\beta''(t) - \beta'(t)\frac{d^2s}{(dt)^2}}{|\beta'(t)|^2}$$

and give us

$$\beta''(t) = \frac{d^2s}{dt^2}$$

Then we can deduce

$$\begin{aligned}\beta'(t) \times \beta''(t) &= \Big(\frac{ds}{dt}T(t)\Big) \times \Big(\frac{d^2s}{dt^2}T(t) + \frac{ds}{dt}T'(t)\Big)\\ &= (\frac{ds}{dt})^2 \cdot \Big(T(t) \times T'(t)\Big)\\ &= |\beta'(t)|^2 \cdot (\pm\,|T'(t)|)\end{aligned}$$

This then give us

$$|T'(t)| = \frac{|\beta'(t) \times \beta''(t)|}{|\beta'(t)|^2} \text{ (done)}$$

■

Corollary 1.1.4. (Curvature of Plane Curves with unknown parametrization) Given a smooth curve $\beta(t) = (x, y) \in \mathbb{R}^3$ with unknown parametrization, we have

$$\kappa(t) = \frac{|x'y'' - x''y'|}{\big((x')^2 + (y')^2\big)^{\frac{3}{2}}}$$

1.2 Fundamental Theorem of Local Curves

Prerequisite facts:

(a) Suppose $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $f : I \to \mathbb{R}^n$ has a limit at $t_0 \in I$. We have

$$\lim_{t\to t_0} T(f(t)) = T\big(\lim_{t\to t_0} f(t)\big)$$

Theorem 1.2.1. (Rigid motion on Local space curves) Let

(a) I be a bounded open interval

(b) $\gamma : I \to \mathbb{R}^3$ be a smooth curve such that $\kappa_\gamma(t) \neq 0$ for all $t \in I$

(c) $\rho \in L(\mathbb{R}^3, \mathbb{R}^3)$ be an orthogonal linear transformation with positive determinant

(d) $c \in \mathbb{R}^3$ be a vector in $\mathbb{R}^3$

(e) $\alpha : I \to \mathbb{R}^3$ be defined by $\alpha(t) \triangleq \rho(\gamma(t))$

(f) $\beta : I \to \mathbb{R}^3$ be defined by $\beta(t) \triangleq \alpha(t) + c$

We have

$$\begin{cases} \kappa_\gamma(t) = \kappa_\alpha(t) = \kappa_\beta(t) \\ \tau_\gamma(t) = \tau_\alpha(t) = \tau_\beta(t) \end{cases} \qquad (t \in I)$$

Proof. We first show

$$(\rho v) \times (\rho w) = \rho(v \times w) \qquad (v, w \in \mathbb{R}^3)$$

Fix $v, w \in \mathbb{R}^3$. We reduce the problem into proving

$$(\rho v) \times (\rho w) \cdot z = \rho(v \times w) \cdot z \qquad (z \in \mathbb{R}^3)$$

Observe

$$\begin{aligned} (\rho v) \times (\rho w) \cdot z &= \begin{vmatrix} \rho v & \rho w & \rho(\rho^{-1}(z)) \end{vmatrix} \\ &= \begin{vmatrix} v & w & \rho^{-1}(z) \end{vmatrix} \\ &= v \times w \cdot \rho^{-1}(z) \\ &= \rho(v \times w) \cdot z \text{ (done)} \end{aligned}$$

We first prove

$$\kappa_\alpha(t) = \kappa_\gamma(t) \qquad (t \in I)$$

Note that γ'' exists, so we can compute

$$\begin{aligned}
\kappa_\alpha &= \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} \\
&= \frac{|(\rho\gamma)' \times (\rho\gamma)''|}{|(\rho\gamma)'|^3} \\
&= \frac{|\rho\gamma' \times \rho\gamma''|}{|\rho\gamma'|^3} \\
&= \frac{|\rho(\gamma' \times \gamma'')|}{|\rho\gamma'|^3} \\
&= \frac{|\gamma' \times \gamma''|}{|\gamma'|^3} = \kappa_\gamma \text{ (done)}
\end{aligned}$$

We now prove

$$\tau_\alpha(t) = \tau_\gamma(t) \qquad (t \in I)$$

Compute

$$\begin{aligned}
\tau_\alpha &= -\frac{\alpha' \times \alpha'' \cdot \alpha'''}{|\alpha' \times \alpha''|} \\
&= -\frac{\rho\gamma' \times \rho\gamma'' \cdot \rho\gamma'''}{|\rho\gamma' \times \rho\gamma''|} \\
&= -\frac{\gamma' \times \gamma'' \cdot \gamma'''}{|\gamma' \times \gamma''|} = \tau_\gamma \text{ (done)}
\end{aligned}$$

■

Theorem 1.2.2. (Fundamental Theorem of Local Curves) Let

(a) I be a bounded open interval

(b) $\kappa : I \to \mathbb{R}^+$ be a smooth function

(c) $\tau : I \to \mathbb{R}$ be a smooth function

And, let E be the set of all space curves γ such that

(a) γ has domain I

(b) $|\gamma'(s)| = 1$

(c) $\kappa_\gamma(s) = \kappa(s)$

(d) $\tau_\gamma(s) = \tau(s)$

The following statement hold true.

(a) E is non-empty. **(existence part)**

(b) For each two $\gamma, \alpha \in E$, there exists an orthogonal linear transformation $\rho \in L(\mathbb{R}^3, \mathbb{R}^3)$ with positive determinant and a vector $c \in \mathbb{R}^3$ such that $\gamma(s) = \rho \circ \alpha(s) + c$ for all $s \in I$. **(uniqueness part)**

Proof. We first prove

$$E \text{ is non-empty}$$

Fix $s_0 \in I$, and fix

$$\begin{bmatrix} T(s_0) \\ N(s_0) \\ B(s_0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.1}$$

Consider the system of differential equation

$$\begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} T' \\ N' \\ B' \end{bmatrix}$$

This is a first order, linear system of differential equation (3, if you wish), which has a unique solution $\{T, N, B : I \to \mathbb{R}^3\}$ with given initial condition (Equation 1.1).

We claim

$$\gamma(s) \triangleq \gamma(s_0) + \int_{s_0}^{s} T(s')ds' \in E$$

Note that the solution $\{T, N, B : I \to \mathbb{R}^3\}$ must satisfy

$$\begin{aligned}
\frac{d}{ds}\langle T, N\rangle &= \kappa\langle N, N\rangle - \kappa\langle T, T\rangle - \tau\langle T, B\rangle \\
\frac{d}{ds}\langle T, B\rangle &= \kappa\langle N, B\rangle + \tau\langle T, N\rangle \\
\frac{d}{ds}\langle N, B\rangle &= -\kappa\langle T, B\rangle - \tau\langle B, B\rangle + \tau\langle N, N\rangle \\
\frac{d}{ds}\langle T, T\rangle &= 2\kappa\langle T, N\rangle \\
\frac{d}{ds}\langle N, N\rangle &= -2\kappa\langle T, N\rangle - 2\tau\langle B, N\rangle \\
\frac{d}{ds}\langle B, B\rangle &= 2\tau\langle N, B\rangle
\end{aligned}$$

which is another linear first order system of differential equation, which has unique solution when initial condition are given.

It is easy to check that

$$\langle T,T\rangle \triangleq \langle N,N\rangle \triangleq \langle B,B\rangle \triangleq 1 \text{ and } \langle T,N\rangle \triangleq \langle T,B\rangle \triangleq \langle N,B\rangle = 0 \tag{1.2}$$

is a solution.

Now, note that our unique solution $\{T, N, B\}$ given before has initial condition coincide with that of 1.2. Then it follows from the uniqueness of solution of linear first order system of differential equations that $\{T(s), N(s), B(s)\}$ is an orthonormal basis for all $s \in I$.

Since $\gamma'(s) = T(s)$, it is now clear that $\kappa_\gamma = \kappa$ and $\tau_\gamma = \tau$. (done)

Fix $\gamma, \alpha \in E$ and fix $s_0 \in I$. There clearly exists a rigid motion M such that if we denote

$$\overline{\alpha}(s) \triangleq M \circ \alpha(s)$$

Then

$$\gamma(s_0) = \overline{\alpha}(s_0) \text{ and } \begin{cases} T_\gamma(s_0) = T_{\overline{\alpha}}(s_0) \\ N_\gamma(s_0) = N_{\overline{\alpha}}(s_0) \\ B_\gamma(s_0) = B_{\overline{\alpha}}(s_0) \end{cases}$$

We only wish to prove

$$T_\gamma(s) = T_{\overline{\alpha}}(s) \qquad (s \in I)$$

Denote

$$T \triangleq T_\gamma \text{ and } \overline{T} \triangleq T_{\overline{\alpha}} \text{ and also similarly for } N, B$$

Compute

$$\begin{aligned}
&\frac{d}{ds}\Big(\big|T - \overline{T}\big|^2 + \big|N - \overline{N}\big|^2 + \big|B - \overline{B}\big|^2 \Big) \\
=&2\Big(\langle T' - \overline{T}', T - \overline{T}\rangle + \langle N' - \overline{N}', N - \overline{N}\rangle + \langle B' - \overline{B}', B - \overline{B}\rangle \Big) \\
=&2\Big(\kappa\langle N - \overline{N}, T - \overline{T}\rangle - \kappa\langle T - \overline{T}, N - \overline{N}\rangle - \tau\langle B - \overline{B}, N - \overline{N}\rangle + \tau\langle N - \overline{N}, B - \overline{B}\rangle \Big) \\
=&0
\end{aligned}$$

This then implies

$$\left|T-\overline{T}\right|^2+\left|N-\overline{N}\right|^2+\left|B-\overline{B}\right|^2 \text{ is a fixed constant on } I$$

Note that

$$T(s_0)=\overline{T}(s_0) \text{ and } N(s_0)=\overline{N}(s_0) \text{ and } B(s_0)=\overline{B}(s_0)$$

Then we know that fixed constant is exactly 0. This then let us deduce

$$T=\overline{T} \text{ on } I \quad \text{(done)}$$

$\blacksquare$

1.3 Isoperimetric Inequality and Four-Vertex Theorem

In this section, we are given a smooth plane curve $\alpha : [a, b] \to \mathbb{R}^2$ parametrized by arc-length. We say

(a) α is a **closed curve** if $\alpha(a) = \alpha(b)$.

(b) α is a **simple closed curve** if α is closed and $\alpha(t_1) \neq \alpha(t_2)$ for all $t_1, t_2 \in [a, b)$.

(c) α is **positively oriented** if $\alpha' \times \alpha''$ is always positive.

(d) α is **convex** the trace $\alpha([a, b])$ always entirely lies on the same side of the closed half-plane determined by $T(s)$ for all s

(e) α has **vertex** $\alpha(s_0)$ if $\kappa'(s_0) = 0$

The interior D of a piece-wise smooth, simple closed plane curve $\alpha : [a, b] \to \mathbb{R}^2$ can be assigned a number to represent its area

$$A(D) = \iint_D 1 dxdy \tag{1.3}$$

that match with our geometric intuition, in the sense that if one compute the area of a rectangle or that of a circle, using Formula 1.3, then one obtain number same as the number obtained using elementary geometric way (height $\times$ width, etc).

Note that by Green's Theorem, we know $A(D)$ equals to

(a) $\int_a^b xy' dt$

(b) $-\int_a^b yx' dt$

(c) $\frac{1}{2}\int_a^b xy' - yx' dt = \frac{1}{2}\int_a^b (x, y) \times (x', y') dt$

Theorem 1.3.1. (Isoperimetric Inequality) Given a piece-wise C^1 simple closed plane curve $\alpha : [0, l] \to \mathbb{R}^2$, where D is the interior of α and $A \triangleq A(D)$, we have

(a) $4\pi A \leq l^2$

(b) $4\pi A = l^2 \iff \alpha$ is a parametriation of S^1

Proof. We first show

$$4\pi A \leq l^2$$

Express $\alpha = (x, y)$. Define

$$r = \frac{1}{2}\Big(\sup_{[0,l]} x - \inf_{[0,l]} x\Big)$$

WOLG, suppose $\sup_{[0,l]} x = r$. Now, positively parametrize S^1 by $(\overline{x}, \overline{y})$ where

$$\overline{x} \triangleq x$$

Observe

$$\begin{aligned}
A + \pi r^2 &= \oint_\alpha x dy - \oint_{S^1} y dx \\
&= \int_0^l \big(xy' - x'\overline{y}\big) dt \\
&= \int_0^l (x, -\overline{y}) \cdot (y', x') dt \\
&\leq \int_0^l |(x, -\overline{y})| \cdot |(y', x')| \, dt \qquad \text{(Cauchy-Schwarz)} \\
&= \int_0^l \sqrt{x^2 + (-\overline{y})^2} \sqrt{(y')^2 + (x')^2} dt \\
&= \int_0^l \sqrt{x^2 + \overline{y}^2} dt \\
&= \int_0^l r dt = lr
\end{aligned}$$

Using AM-GM inequality, we now have

$$\sqrt{A\pi r^2} \leq \frac{A + \pi r^2}{2} \leq \frac{lr}{2}$$

This then implies

$$4\pi A \leq l^2 \text{ (done)}$$

It is easy to see that the equality hold true when α is a parametrization of S^1. We prove

$$4\pi A = l^2 \implies \alpha \text{ is a parametrization of } S^1 \text{ of radius } \frac{l}{2\pi}$$

We first prove

$$4\pi A = l^2 \implies x = \frac{l}{2\pi} y'$$

Note that $4\pi A = l^2$ implies the Cauchy-Schwarz inequality

$$(x, -\overline{y}) \cdot (y', x') \le |(x, -\overline{y})| \cdot |(y', x')|$$

must hold equal. This then give us

$$(x, -\overline{y}) /\!/ (y', x')$$

Also, note that $4\pi A = l^2$ implies the AM-GM inequality

$$\sqrt{A\pi r^2} \le \frac{A + \pi r^2}{2}$$

must also hold equal. This then give us

$$A = \pi r^2 \text{ and } l = 2\pi r$$

Now, because

(a) (x, y) is parametrization by arc-length

(b) $|(x, -\overline{y})| = |(\overline{x}, \overline{y})| = r$

(c) $r = \frac{l}{2\pi}$

We have

$$x = \frac{l}{2\pi} y' \text{ (done)} \tag{1.4}$$

Similar argument now, WOLG, applies to

$$4\pi A = l^2 \implies y = \frac{l}{2\pi} x'$$

We now conclude

$$4\pi A = l^2 \implies |(x, y)| = \frac{l}{2\pi} |(y', x')| = \frac{l}{2\pi} \text{ (done)}$$

■

For next Theorem, we expand the definition of closed curve. We say

(a) α is a **closed curve** if $\alpha^k(a) = \alpha^k(b)$ for all $k \in \mathbb{Z}_0^+$

Lemma 1.3.2. If $\alpha : [0, l] \to \mathbb{R}^2$ is a smooth, convex, simple closed plane curve, then for all $A, B, C \in \mathbb{R}$, we have

$$\int_0^l \Big(Ax + By + C \Big) d\kappa = 0$$

Proof. Because α is closed, it is clear that

$$\int_0^l C d\kappa = 0$$

Let $\theta : [0,l] \to [0,2\pi]$ satisfy

$$(x', y') = (\cos\theta, \sin\theta)$$

It is easy to check

(a) $\kappa = \theta'$

(b) $(x'', y'') = \kappa(-y', x')$

We now see

$$\begin{aligned}\int_0^l A x d\kappa &= A\int_0^l x\kappa' ds \\ &= -A\int_0^l x'\kappa ds \\ &= A\int_0^l y'' ds = 0\end{aligned}$$

and

$$\begin{aligned}\int_0^l B y d\kappa &= B\int_0^l y\kappa' ds \\ &= -\int_0^l y'\kappa ds \\ &= -B\int_0^l x'' ds = 0\end{aligned}$$

■

Theorem 1.3.3. (Four-Vertex Theorem for Convex Curve) If $\alpha : [0,l] \to \mathbb{R}^2$ is a smooth, convex, simple closed plane curve, then

$$\alpha \text{ has at least four distinct vertices}$$

Proof. Because κ is continuous on $[0,l]$, by EVT, we know there exists p, q such that

$$\kappa(p) = \max_{[0,l]} \kappa \text{ and } \kappa(q) = \min_{[0,l]} \kappa$$

Because $\kappa(p), \kappa(q)$ are local (global, in fact) extremum, we know

$$\kappa'(p) = \kappa'(q) = 0$$

Note that if $p = q$, then the proof become trivial. We have thus obtained two distinct vertices, namely, p and q.

WOLG, suppose $p < q$. Let L be the unique line containing $\alpha(p)$ and $\alpha(q)$. We claim

$$\alpha[(p,q)] \text{ and } \alpha\Big[[0,l] \setminus [p,q]\Big] \text{ are on different side of } L$$

(done)

Note that

(a) $p < q$

(b) $\kappa(p)$ is a maximum

(c) $\kappa(q)$ is a minimum

Assume $\kappa' \leq 0$ on (p,q) and $\kappa' \geq 0$ on $[p,q]^c$. WOLG, suppose $\alpha[(p,q)]$ is on the left side of L. Let $A, B, C \in \mathbb{R}$ satisfy

(a) $L = \{(x,y) : Ax + By + C = 0\}$

(b) $A \in \mathbb{R}^+$

Now, because $\alpha[(p,q)]$ is completely on the left side of L, we see

$$\int_p^q \big(Ax + By + C\big) d\kappa > 0 \qquad (\because \kappa', Ax + By + C < 0)$$

and because $\alpha[[p,q]^c]$ is completely on the right side of L, we see

$$\int_0^p \big(Ax + By + C\big) d\kappa + \int_q^l \big(Ax + By + C\big) d\kappa > 0 \quad (\because \kappa', Ax + By + C > 0)$$

This now let us deduce

$$\int_0^l \big(Ax + By + C\big) d\kappa > 0 \text{ CaC}$$

WOLG, suppose $\kappa'(s) > 0$ for some $s \in (p,q)$. Because

(a) $\kappa(p)$ is a maximum

(b) $\kappa(q)$ is a minimum

we know there exists ϵ such that

$$\kappa(p+\epsilon) < 0 \text{ and } \kappa(q-\epsilon) < 0$$

Then by IVT, we see

$$\kappa'(s_1) = \kappa'(s_2) = 0 \text{ for some distinct } s_1, s_2 \in (p, q)$$

$\blacksquare$

Chapter 2

HW

2.1 HW1

Question 1: 1-2: 2

Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is the point of the trace of α closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

Proof. Define $g : I \to \mathbb{R}$ by

$$g(t) \triangleq |\alpha(t)|^2 = (\alpha \cdot \alpha)(t)$$

Notice that

$$g'(t) = (2\alpha' \cdot \alpha)(t) \text{ if exists}$$

From premise, we know g attains minimum at t_0. This tell us

$$0 = g'(t_0) = (2\alpha' \cdot \alpha)(t_0)$$

Then, we can deduce

$$\alpha'(t_0) \cdot \alpha(t_0) = 0$$

This implies $\alpha(t_0) \perp \alpha'(t_0)$. ∎

Question 2: 1-2: 5

Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve, with $\alpha''(t) \neq 0$ for all $t \in I$. Show that $|\alpha(t)|$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

Proof. We wish to prove

$$\exists \beta \in \mathbb{R}^*, \forall t \in I, |\alpha(t)| = \beta \iff \forall t \in I, (\alpha \cdot \alpha')(t) = 0$$

Define $g : I \to \mathbb{R}$ by

$$g(t) \triangleq |\alpha(t)|^2 = (\alpha \cdot \alpha)(t)$$

Notice that

$$g'(t) = (2\alpha' \cdot \alpha)(t) \tag{2.1}$$

$(\longrightarrow)$

From premise, g is a constant on I. This implies $g'(t) = 0$ for all $t \in I$. Then, from Equation 2.1, we see

$$(\alpha \cdot \alpha')(t) = 0 \text{ for all } t \in I$$

$(\longleftarrow)$

Again, from Equation 2.1, we deduce

$$\forall t \in I, (\alpha \cdot \alpha')(t) = 0 \implies \forall t \in I, g'(t) = 0$$

This implies g is a constant, which implies $|\alpha|$ is a constant, that is

$$\exists \beta \in \mathbb{R}, |\alpha(t)| = \beta$$

Lastly, we have to show

$$\beta \neq 0$$

Assume $\beta = 0$. Then, we see $\alpha(t) = 0$ for all $t \in I$. This implies $\alpha''(t) = 0$ for all $t \in I$, which CaC to the premise. (done) ■

Question 3: 1-3:2

2. A circular disk of radius 1 in the plane xy rolls without slipping along the x axis. The figure described by a point of the circumference of the disk is called a *cycloid* (Fig. 1-7).

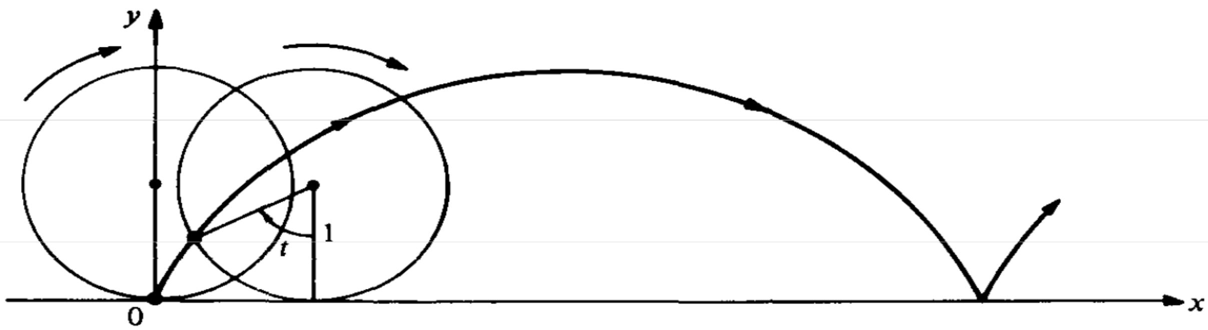


Figure 1-7. The cycloid.

***a.** Obtain a parametrized curve $\alpha: R \longrightarrow R^2$ the trace of which is the cycloid, and determine its singular points.

b. Compute the arc length of the cycloid corresponding to a complete rotation of the disk.

Proof. The solution of the question **a** is

$$\alpha(t) = (t - \sin t, 1 - \cos t)$$

Compute

$$\alpha'(t) = (1 - \cos t, \sin t)$$

and compute

$$|\alpha'(t)| = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} = \sqrt{2} \cdot \sqrt{1 - \cos t}$$

This implies the singular points are

$$\{2n\pi : n \in \mathbb{Z}\}$$

The solution of the question **b** is then

$$\begin{aligned}
\int_0^{2\pi} |\alpha'(t)|\, dt &= \sqrt{2}\int_0^{2\pi} \sqrt{1-\cos t}dt \\
&= \sqrt{2}\int_0^{2\pi} \sqrt{2}\left|\sin\frac{t}{2}\right| dt \\
&= 2\int_0^{2\pi} \left|\sin\frac{t}{2}\right| dt \\
&= 4\int_0^{\pi} \sin(\frac{t}{2})dt \\
&= -8\cos\frac{t}{2}\Big|_0^{\pi}
\end{aligned}$$

■

Question 4: 1-3:4

4. Let $\alpha\colon (0,\pi) \longrightarrow R^2$ be given by

$$\alpha(t) = \left(\cos t, \cos t + \log\tan\frac{t}{2}\right),$$

where t is the angle that the y axis makes with the vector $\alpha(t)$. The trace of α is called the *tractrix* (Fig. 1-9). Show that

19:

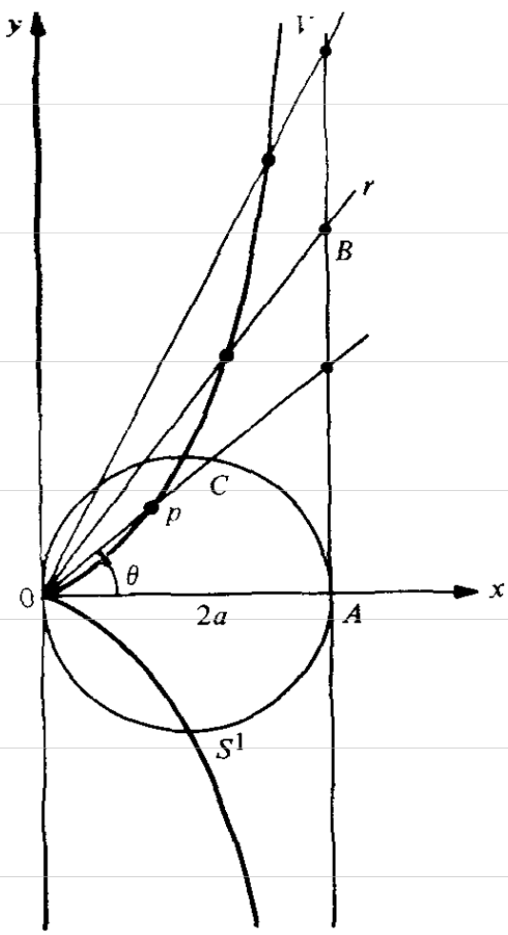


Figure 1-8. The cissoid of Diocles.

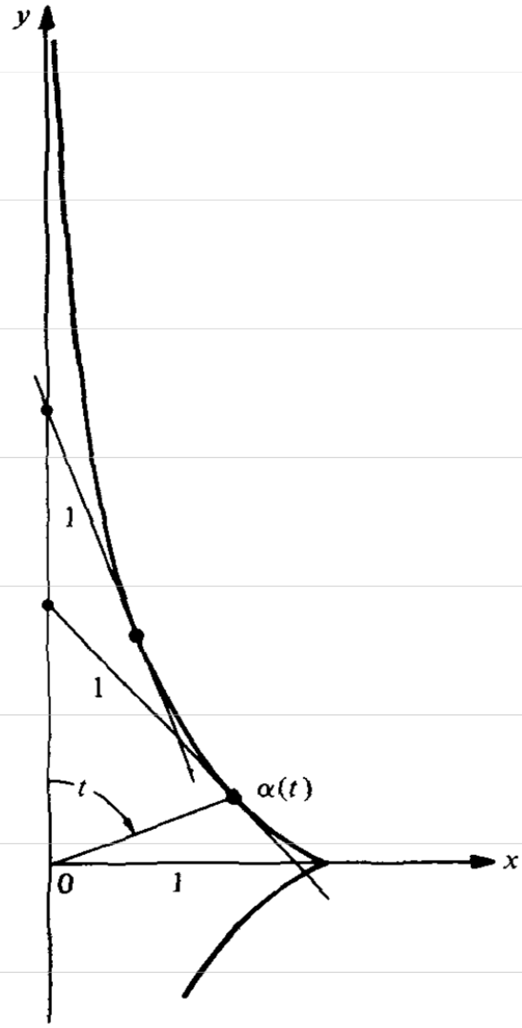


Figure 1-9. The tractrix.

a. α is a differentiable parametrized curve, regular except at $t = \pi/2$.

b. The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.

Typo correction: $\alpha(t) = (\sin t, \cos t + \ln \tan \frac{t}{2})$

Proof. **(a)**
Notice that the interval I is $(0, \pi)$. It is clear that

(a) $\sin t$ is smooth on $\mathbb{R}$

(b) $\cos t$ is smooth on $\mathbb{R}$

(c) $\ln t$ is smooth on $\mathbb{R}$ $\tan\frac{t}{2}$ is smooth on I

Then it follows that α is a differentiable curve.

Compute

$$\alpha'(t) = (\cos t, -\sin t + \frac{1}{\tan\frac{t}{2}} \cdot \sec^2\frac{t}{2} \cdot \frac{1}{2})$$

Because $\cos t = \alpha_1'(t)$ is 0 on I only when $t = \frac{\pi}{2}$, we know α is regular on I except possibly at $t = \frac{\pi}{2}$.

Compute

$$\alpha'(\frac{\pi}{2}) = (0, -1 + \frac{1}{1} \cdot 2 \cdot \frac{1}{2}) = (0, 0)$$

We now conclude α is regular on I except $\frac{\pi}{2}$.

(b)
A useful Identity give us

$$\alpha'(t) = (\cos t, -\sin t + \csc t)$$

From the following facts

(a) the first argument of the segment is from 0 to $\sin t = \alpha(t)$

(b) $\alpha_x'(t) = \cos t$

(c) $\frac{\sin t}{\cos t} = \tan t$

We conclude that the length of the segment is

$$\begin{aligned}
|\tan t| \cdot |\alpha'(t)| &= |\tan t| \cdot \sqrt{\cos^2 t + \sin^2 t - 2\sin t \csc t + \csc^2 t} \\
&= |\tan t| \cdot \sqrt{1 - 2 + \csc^2 t} \\
&= |\tan t| \cdot \sqrt{\csc^2 t - 1} = |\tan t| \cdot \sqrt{\cot^2 t} = 1
\end{aligned}$$

■

Question 5

7. A map $\alpha: I \longrightarrow R^3$ is *called a curve of class* C^k if each of the coordinate functions in the expression $\alpha(t) = (x(t), y(t), z(t))$ has continuous derivatives up to order k. If α is merely continuous, we say that α is of class C^0. A curve α is called *simple* if the map α is one-to-one. Thus, the curve in Example 3 of Sec. 1-2 is not simple.

Let $\alpha: I \longrightarrow R^3$ be a simple curve of class C^0. We say that α has a *weak tangent* at $t = t_0 \in I$ if the line determined by $\alpha(t_0 + h)$ and $\alpha(t_0)$ has a limit position when $h \longrightarrow 0$. We say that α has a *strong tangent* at $t = t_0$ if the line determined by $\alpha(t_0 + h)$ and $\alpha(t_0 + k)$ has a limit position when $h, k \longrightarrow 0$. Show that

a. $\alpha(t) = (t^3, t^2)$, $t \in R$, has a weak tangent but not a strong tangent at $t = 0$.

***b.** If $\alpha: I \longrightarrow R^3$ is of class C^1 and regular at $t = t_0$, then it has a strong tangent at $t = t_0$.

c. The curve given by

$$\alpha(t) = \begin{cases} (t^2, t^2), & t \geq 0, \\ (t^2, -t^2), & t \leq 0, \end{cases}$$

is of class C^1 but not of class C^2. Draw a sketch of the curve and its tangent vectors.

Proof. **(a)** Let $v = (0, 1)$. Compute

$$\frac{\alpha(t) - \alpha(0)}{|\alpha(t) - \alpha(0)|} \cdot v = \frac{t^2}{\sqrt{t^6 + t^4}} = \frac{1}{\sqrt{t^2 + 1}} \to 1 \text{ as } t \to 1$$

This implies α has a weak tangent at $t = 0$. Now, if α has a strong tangent, we must have

$$\frac{\alpha(h) - \alpha(-h)}{2h} \cdot v \to 1 \text{ or } \to -1$$

But this is clearly not the case as

$$\frac{\alpha(h) - \alpha(-h)}{2h} \cdot v = 0 \text{ for all } h > 0$$

So we have the conclusion that α has no strong tangent at 0.

(b) By MVT, for each h, k there exists a set of real numbers $\{c_x, c_y, c_z\}$ between $t+h$ and $t+k$ such that

$$\frac{\alpha(t_0+h)-\alpha(t_0+k)}{h-k} = \Big(x'(c_x), y'(c_y), z'(c_z)\Big)$$

Then because

$$h, k \to 0 \implies t_0+h, t_0+k \to t_0 \implies c_x, c_y, c_z \to t_0$$

Then from the fact α is of class C^1 (x', y', z' are all continuous), we can now deduce

$$\frac{\alpha(t_0+h)-\alpha(t_0+k)}{h-k} \to \alpha'(t_0) \text{ as } h, k \to 0 \tag{2.2}$$

Now, because $\alpha'(t_0) \neq 0$ as α is regular, we see

$$\lim_{h,k\to 0} \frac{\alpha(t_0+h)-\alpha(t_0+k)}{h-k} \cdot \alpha'(t_0) = |\alpha'(t_0)|^2$$

This then implies

$$\lim_{h,k\to 0} \frac{\alpha(t_0+h)-\alpha(t_0+k)}{|\alpha(t_0+h)-\alpha(t_0+k)|} \cdot \frac{\alpha'(t_0)}{|\alpha'(t_0)|} = 1$$

which implies the "strong tangent" must always converge to $\alpha'(t_0)$.

Notice that the last implication is backed by Equation 2.2

(c)

From

$$\alpha(t) = \left(t^2, \begin{cases} t^2 & \text{if } t \geq 0 \\ -t^2 & \text{if } t \leq 0 \end{cases}\right)$$

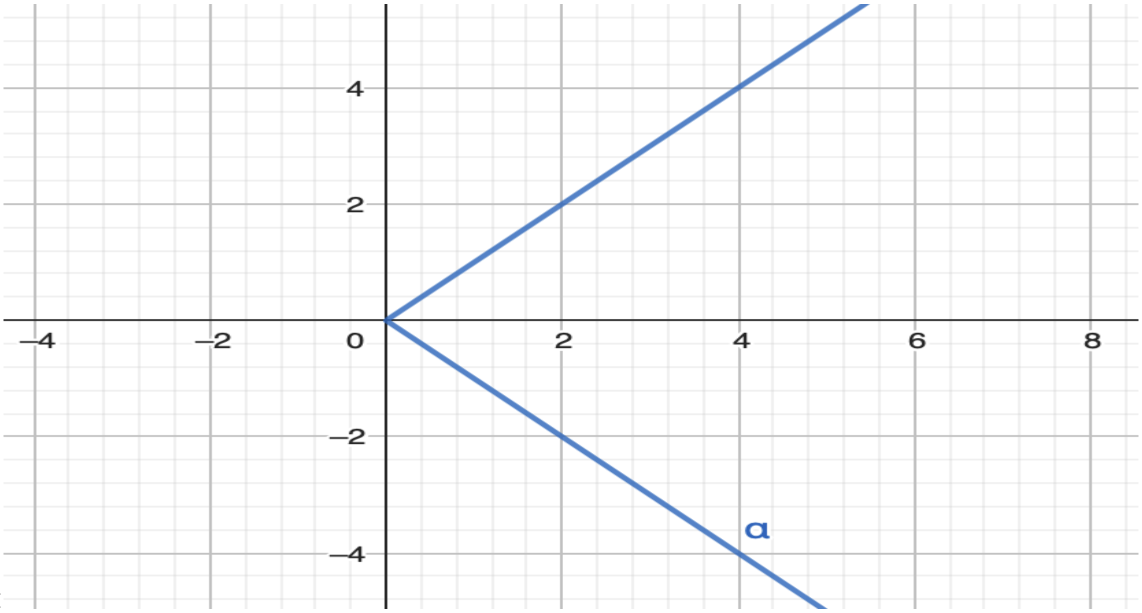
Compute

$$\alpha'(t) = \left(2t, \begin{cases} 2t & \text{if } t \geq 0 \\ -2t & \text{if } t \leq 0 \end{cases}\right)$$

Notice that the derivative at $t=0$ is computed from definition instead of product rule.

Now, it is clear that x', y' are continuous. This implies $\alpha \in C^1$. Yet, we see y' is not differentiable at $t = 0$. This implies $\alpha \notin C^2$.

The sketch:



■

Question 6

***8.** Let $\alpha: I \longrightarrow R^3$ be a differentiable curve and let $[a, b] \subset I$ be a closed interval. For every *partition*

$$a = t_0 < t_1 < \cdots < t_n = b$$

of $[a, b]$, consider the sum $\sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})| = l(\alpha, P)$, where P stands for the given partition. The norm $|P|$ of a partition P is defined as

$$|P| = \max(t_i - t_{i-1}), \; i = 1, \ldots, n.$$

Geometrically, $l(\alpha, P)$ is the length of a polygon inscribed in $\alpha([a, b])$ with vertices in $\alpha(t_i)$ (see Fig. 1-12). The point of the exercise is to show that the arc length of $\alpha([a, b])$ is, in some sense, a limit of lengths of inscribed polygons.

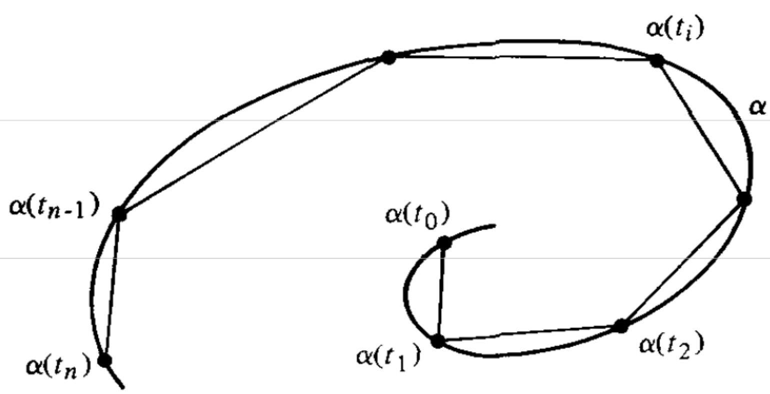


Figure 1-12

Prove that given $\epsilon > 0$ there exists $\delta > 0$ such that if $|P| < \delta$ then

$$\left| \int_a^b |\alpha'(t)|\, dt - l(\alpha, P) \right| < \epsilon.$$

Proof. We first prove

$$\int_a^b |\alpha'(t)|\, dt \geq l(\alpha, P)$$

By FTC, we have

$$\begin{aligned}|\alpha(t_i) - \alpha(t_{i-1})| &= \left|\int_{t_{i-1}}^{t_i} \alpha'(t)dt\right| \\ &\leq \int_{t_{i-1}}^{t_i} |\alpha'(t)|\, dt\end{aligned}$$

This then implies

$$l(\alpha, P) = \sum |\alpha(t_i) - \alpha(t_{i-1})| \leq \sum \int_{t_{i-1}}^{t_i} |\alpha'(t)|\, dt = \int_a^b |\alpha'(t)|\, dt \text{ (done)}$$

We have reduced the problem into

$$\text{finding } \delta \text{ such that } \forall P : |P| < \delta, \int_a^b |\alpha'(t)|\, dt - l(\alpha, P) < \epsilon$$

Because α' is uniformly continuous on $[a, b]$ ($\because$ continuous function on compact domain is uniformly continuous), we know there exists δ' such that

$$|\alpha'(s) - \alpha'(t)| < \frac{\epsilon}{2(b-a)} \text{ if } |s - t| < \delta'$$

We claim

$$\text{such } \delta' \text{ works}$$

Let $|P| < \delta$, and let $s_i \in [t_{i-1}, t_i]$. Because $|s_i - t_i| < \delta$, we have

$$|\alpha'(s_i) - \alpha'(t_i)| < \frac{\epsilon}{2(b-a)} \tag{2.3}$$

This give us

$$|\alpha'(s_i)| < |\alpha'(t_i)| + \frac{\epsilon}{2(b-a)}$$

Now, we can deduce

$$\begin{aligned}
\int_{t_{i-1}}^{t_i} |\alpha'(s)|\, ds &\leq |\alpha'(t_i)|\, \Delta t_i + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&= \int_{t_{i-1}}^{t_i} |\alpha'(t_i)|\, dt + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&= \left| \int_{t_{i-1}}^{t_i} \alpha'(t_i) dt \right| + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&= \left| \int_{t_{i-1}}^{t_i} \alpha'(t_i) - \alpha'(t) dt + \int_{t_{i-1}}^{t_i} \alpha'(t) dt \right| + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&\leq \left| \int_{t_{i-1}}^{t_i} \alpha'(t_i) - \alpha'(t) dt \right| + \left| \int_{t_{i-1}}^{t_i} \alpha'(t) dt \right| + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&\leq \frac{\epsilon}{2(b-a)}\Delta t_i + |\alpha(t_i) - \alpha(t_{i-1})| + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&= |\alpha(t_i) - \alpha(t_{i-1})| + \frac{\epsilon}{b-a}\Delta t_i
\end{aligned}$$

Notice that the last inequality follows from Equation 2.3. The long deduction above then give us

$$\begin{aligned}
\int_a^b |\alpha'(t)|\, dt &\leq \sum |\alpha(t_i) - \alpha(t_{i-1})| + \frac{\epsilon}{b-a}(b-a) \\
&= l(\alpha, P) + \epsilon
\end{aligned}$$

Then we have

$$\int_a^b |\alpha'(t)|\, dt - l(\alpha, P) \leq \epsilon \text{ (done)}$$

■

Question 7

9. a. Let $\alpha: I \longrightarrow R^3$ be a curve of class C^0 (cf. Exercise 7). Use the approximation by polygons described in Exercise 8 to give a reasonable definition of arc length of α.

b. (*A Nonrectifiable Curve.*) The following example shows that, with any reasonable definition, the arc length of a C^0 curve in a closed interval may be unbounded. Let $\alpha: [0, 1] \longrightarrow R^2$ be given as $\alpha(t) = (t, t\sin(\pi/t))$ if $t \neq 0$, and $\alpha(0) = (0, 0)$. Show, geometrically, that the arc length of the portion of the curve corresponding to $1/(n+1) \leq t \leq 1/n$ is at least $2/(n+\frac{1}{2})$. Use this to show that the length of the curve in the interval $1/N \leq t \leq 1$ is greater than $2\sum_{n=1}^{N} 1/(n+1)$, and thus it tends to infinity as $N \longrightarrow \infty$.

Proof. **(a)** Suppose $I = [a, b]$. Define arc length by

$$\sup_P l(P, \alpha) \text{ where sup runs over all partition } P \text{ of } [a, b]$$

(b)

Geometrically, we know the arc length of the portion of the curve corresponding to $t \in [\frac{1}{n+1}, \frac{1}{n}]$ must be greater than

$$\left|\alpha(\frac{1}{n}) - \alpha(\frac{1}{n+\frac{1}{2}})\right| + \left|\alpha(\frac{1}{n+1}) - \alpha(\frac{1}{n+\frac{1}{2}})\right| \tag{2.4}$$

WOLG of n being odd or even, Compute

$$\begin{aligned}
\left|\alpha(\frac{1}{n}) - \alpha(\frac{1}{n+\frac{1}{2}})\right| &= \left|(\frac{1}{n}, 0) - (\frac{1}{n+\frac{1}{2}}, \frac{1}{n+\frac{1}{2}})\right| \\
&= \sqrt{(\frac{1}{n} - \frac{1}{n+\frac{1}{2}})^2 + (\frac{1}{n+\frac{1}{2}})^2} \\
&= \sqrt{\frac{1}{n^2} - \frac{4}{n(2n+1)} + \frac{8}{(2n+1)^2}} \\
&= \sqrt{\frac{(2n+1)^2 - 4n(2n+1) + 8n^2}{n^2(2n+1)^2}} \\
&= \sqrt{\frac{4n^2+1}{n^2(2n+1)^2}} \\
&= \frac{\sqrt{4n^2+1}}{n(2n+1)} \geq \frac{\sqrt{4n^2}}{n(2n+1)} = \frac{2}{2n+1}
\end{aligned}$$

and compute

$$\begin{aligned}
\left|\alpha(\frac{1}{n+\frac{1}{2}}) - \alpha(\frac{1}{n})\right| &= \left|(\frac{1}{n+\frac{1}{2}}, \frac{1}{n+\frac{1}{2}}) - (\frac{1}{n+1}, 0)\right| \\
&= \sqrt{(\frac{1}{n+1} - \frac{1}{n+\frac{1}{2}})^2 + (\frac{1}{n+\frac{1}{2}})^2} \\
&= \sqrt{\frac{1}{(n+1)^2} - \frac{4}{(n+1)(2n+1)} + \frac{8}{(2n+1)^2}} \\
&= \sqrt{\frac{(2n+1)^2 - 4(n+1)(2n+1) + 8(n+1)^2}{(n+1)^2(2n+1)^2}} \\
&= \sqrt{\frac{4n^2+8n+5}{(n+1)^2(2n+1)^2}} \\
&\geq \frac{\sqrt{4n^2+8n+4}}{(n+1)(2n+1)} = \frac{2}{2n+1}
\end{aligned}$$

From the computation and Equation 2.4, it is now clear that the arc length of the portion of the curve corresponding to $t \in [\frac{1}{n+1}, \frac{1}{n}]$ is at least $\frac{2}{n+\frac{1}{2}}$. With simple addition, this then implies the arc length of the curve in the interval $[\frac{1}{N}, 1]$ is at least

$$\sum_{n=1}^{N-1} \frac{2}{2n+1} = 2\sum_{n=1}^{N-1} \frac{1}{2n+1}$$

The number is clearly greater than

$$2\sum_{n=1}^{N-1} \frac{1}{2n+2}$$

which equals to

$$\sum_{n=1}^{N-1} \frac{1}{n+1}$$

The series diverge to $+\infty$ as N to ∞. ∎

Theorem 2.1.1. (Integrating the Dot Product) Given a curve $u : [a, b] \to \mathbb{R}^n$ and a vector $v \in \mathbb{R}^n$, suppose

(a) u is differentiable on (a, b)

(b) u is continuous on $[a, b]$

We have

$$\int_a^b u'(t) \cdot v dt = \Big(\int_a^b u'(t) dt \Big) \cdot v = \big(u(b) - u(a) \big) \cdot v$$

Proof.

$$\begin{aligned} \int_a^b u'(t) \cdot v dt &= \int_a^b \sum_{k=1}^n u_k'(t) \cdot v_k dt \\ &= \sum_{k=1}^n \int_a^b u_k'(t) \cdot v_k dt \\ &= \sum_{k=1}^n v_k \int_a^b u_k'(t) dt \\ &= v \cdot \Big(\int_a^b u'(t) dt \Big) \end{aligned}$$

∎

Question 8

10. (*Straight Lines as Shortest.*) Let $\alpha: I \to R^3$ be a parametrized curve. Let $[a, b] \subset I$ and set $\alpha(a) = p$, $\alpha(b) = q$.

a. Show that, for any constant vector v, $|v| = 1$,

$$(q-p)\cdot v = \int_a^b \alpha'(t)\cdot v\, dt \leq \int_a^b |\alpha'(t)|\, dt.$$

b. Set

$$v = \frac{q-p}{|q-p|}$$

and show that

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)|\, dt;$$

that is, the curve of shortest length from $\alpha(a)$ to $\alpha(b)$ is the straight line joining these points.

Proof. **(a)**

The first equality

$$(q-p)\cdot v = \int_a^b \alpha'(t)\cdot v dt$$

follows directly from Theorem 2.1.1.

Now, by Cauchy-Schwarz inequality, we have

$$|\alpha'(t)\cdot v| \leq |\alpha'(t)|\cdot|v|$$

This then give us

$$\alpha'(t)\cdot v \leq |\alpha'(t)\cdot v| \leq |\alpha'(t)|\cdot|v| = |\alpha'(t)|$$

We now have

$$\int_a^b \alpha'(t)\cdot v \leq |\alpha'(t)|\, dt$$

as desired.

(b)
The first inequality tell us that if v is a constant and $|v| = 1$, we have

$$(q-p) \cdot v \leq \int_a^b |\alpha'(t)| \, dt$$

If $v = \frac{q-p}{|q-p|}$, it is clear that v is a constant and $|v| = 1$, and at the same time, we have

$$(q-p) \cdot v = \frac{(q-p) \cdot (q-p)}{|q-p|} = \frac{|q-p|^2}{|q-p|} = |q-p|$$

We now have

$$|q-p| = (q-p) \cdot v \leq \int_a^b |\alpha'(t)| \, dt$$

from the first inequality ■

Question 9

1. Check whether the following bases are positive:
a. The basis $\{(1, 3), (4, 2)\}$ in R^2.
b. The basis $\{(1, 3, 5), (2, 3, 7), (4, 8, 3)\}$ in R^3.

Proof. Compute

$$\begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} = -10$$

and compute

$$\begin{vmatrix} 1 & 2 & 4 \\ 3 & 3 & 8 \\ 5 & 7 & 3 \end{vmatrix} = -9$$

Both bases are negatively oriented. ■

Question 10

***2. A plane P contained in R^3 is given by the equation $ax + by + cz + d = 0$. Show that the vector $v = (a, b, c)$ is perpendicular to the plane and that $|d|/\sqrt{a^2 + b^2 + c^2}$ measures the distance from the plane to the origin $(0, 0, 0)$.**

Proof. Arbitrarily pick two points u, w in P. We wish to show

$$v \cdot (u - w) = 0$$

Because $v = (a, b, c)$ and

$$\begin{cases} au_1 + bu_2 + cu_3 = -d aw_1 + bw_2 + cw_3 = -d \end{cases}$$

We see

$$\begin{aligned} v \cdot (u - w) &= a(u_1 - w_2) + b(u_2 - w_2) + c(u_3 - w_2) \\ &= (-d) - (-d) = 0 \text{ (done)} \end{aligned}$$

To measure the distance between P and the origin, we wish to find a vector u such that $u \perp P$ and $u \in P$. We know that u must be linearly dependent with $v = (a, b, c)$, since the dimension of $P^{\perp}$ is 1. Then, we can write

$$u = c_0(a, b, c) \text{ for some } c_0 \in \mathbb{R}$$

Because $u \in P$, we know

$$c_0 a^2 + c_0 b^2 + c_0 c^2 + d = 0$$

This tell us

$$c_0 = \frac{-d}{a^2 + b^2 + c^2}$$

We now see that the distance $|u|$ between P and origin is

$$|u| = |c_0| \cdot \sqrt{a^2 + b^2 + c^2} = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$$

■

Question 11

***3. Determine the angle of intersection of the two planes $5x + 3y + 2z - 4 = 0$ and $3x + 4y - 7z = 0$.**

Proof. From last question, we know the two vectors u, v that are respectively perpendicular to $P : 5x + 3y + 2z - 4 = 0$ and $Q : 3x + 4y - 7z = 0$ respectively have the direction

$$(5, 3, 2) \text{ and } (3, 4, -7)$$

Then, we see the angle of the intersection are

$$\arccos \frac{5 \cdot 3 + 3 \cdot 4 + 2 \cdot (-7)}{\sqrt{5^2 + 3^2 + 2^2}\sqrt{3^2 + 4^2 + 7^2}} = \arccos \frac{13}{\sqrt{38}\sqrt{71}}$$

Notice that this angle is smaller than $\frac{\pi}{2}$ as we intend it to be. ■

Question 12

***6.** **Given two nonparallel planes $a_i x + b_i y + c_i z + d_i = 0$, $i = 1, 2$, show that their line of intersection may be parametrized as**

$$x - x_0 = u_1 t, \qquad y - y_0 = u_2 t, \qquad z - z_0 = u_3 t,$$

where (x_0, y_0, z_0) belongs to the intersection and $u = (u_1, u_2, u_3)$ is the vector product $u = v_1 \wedge v_2$, $v_i = (a_i, b_i, c_i)$, $i = 1, 2$.

Proof. Let $v = (x, y, z)$ be a point on the line of intersection. We see the vector $v-(x_0, y_0, z_0)$ lies on both planes, and thus must be perpendicular to $(a_1, b_1, c_1) = v_1$ and $(a_2, b_2, c_2) = v_2$ thus satisfying

$$v - (x_0, y_0, z_0) = tv_1 \times v_2 = tu \text{ for some } t \in \mathbb{R}$$

sine in $\mathbb{R}^3$, the only direction perpendicular to both v_1, v_2 is $v_1 \times v_2$. We can rewrite the above equation of course into

$$x - x_0 = u_1 t, y - y_0 = u_2 t, z - z_0 = u_3 t$$

■

2.2 HW2

Question 13

1. Given the parametrized curve (helix)

$$\alpha(s) = \left(a \cos \frac{s}{c}, a \sin \frac{s}{c}, b\frac{s}{c}\right), \qquad s \in R,$$

where $c^2 = a^2 + b^2$,

a. Show that the parameter s is the arc length.

b. Determine the curvature and the torsion of α.

c. Determine the osculating plane of α.

d. Show that the lines containing $n(s)$ and passing through $\alpha(s)$ meet the z axis under a constant angle equal to $\pi/2$.

e. Show that the tangent lines to α make a constant angle with the z axis.

Proof. **(a)** By computation

$$\alpha'(s) = (\frac{-a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c})$$

So

$$|\alpha'(s)| = \sqrt{\frac{a^2+b^2}{c^2}} = 1 \qquad (\because \ \sin^2 + \cos^2 = 1\ )$$

This shows α is parametrized by arc-length.

(b) By computation

$$\alpha''(s) = \left(\frac{-a}{c^2}\cos\frac{s}{c}, \frac{-a}{c^2}\sin\frac{s}{c}, 0\right)$$

Then because α is parametrized by arc-length, we have

$$\kappa(s) = |\alpha''(s)| = \sqrt{\frac{a^2}{c^4}} = \frac{|a|}{c^2}$$

By computation

$$\alpha'''(s) = \Big(\frac{a}{c^3}\sin\frac{s}{c}, \frac{-a}{c^3}\cos\frac{s}{c}, 0\Big)$$

Then using the identity of torsion, we have

$$\begin{aligned}
\tau(s) &= -\frac{-\big(\alpha'(s)\times\alpha''(s)\big)\cdot\alpha'''(s)}{|\kappa(s)|^2} \\
&= -\frac{\frac{a^2 b}{c^6}}{\frac{a^2}{c^4}} \\
&= \frac{b}{-c^2}
\end{aligned}$$

(c) Fix s. Define a set A by

$$A = \operatorname{span}\Big(\alpha'(s), \alpha''(s)\Big)$$

The osculating plane of α at s is then exactly

$$\{a + \alpha(s) : a \in A\}$$

(d) Because $\alpha''(s)$ by our computation is valued 0 in z-opponent, we know if the line containing N and passing through α meet the z axis, it must be under a constant angle equal to $\frac{\pi}{2}$. (use dot product to check this fact.).

Now, we only have to prove that the line does meet the z-axis. See that

$$\alpha + c^2\alpha'' = \Big(0, 0, b\frac{s}{c}\Big)$$

and we are done.
(e) Observe that

$$\alpha' \cdot \Big(0, 0, 1\Big) = \frac{b}{c} \text{ is a constant}$$

This together with the fact $|\alpha'|$ is a constant show that the angle between the tangent to α and z-axis is a constant.

$\blacksquare$

Question 14

***2.** Show that the torsion τ of α is given by

$$\tau(s) = -\frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{|k(s)|^2}.$$

Theorem 2.2.1. (Identity of Torsion) Given a parametrized by arc-length cruve $\alpha : I \to \mathbb{R}^3$, we have

$$\tau(s) = -\frac{\big(\alpha'(s) \times \alpha''(s)\big) \cdot \alpha'''(s)}{\kappa^2(s)}$$

Proof. Because $\underline{\alpha \text{ is parametrized by arc-length}}$, we have

$$\alpha'(s) = T(s)$$

We first show

$$\alpha''(s) = \kappa(s)N(s) \tag{2.5}$$

Compute

$$\begin{aligned} N(s) &= \frac{T'(s)}{|T'(s)|} \\ &= \frac{\alpha''(s)}{|\alpha''(s)|} = \frac{\alpha''(s)}{\kappa(s)} \text{ (done)} \end{aligned}$$

We now show

$$\alpha'''(s) = \kappa(s)\big(-(\tau B)(s) - (\kappa T)(s)\big) + \kappa'(s)N(s)$$

By Equation 2.5 and Frenet Formula, we have

$$\begin{aligned} \alpha'''(s) &= \kappa'(s)N(s) + \kappa(s)N'(s) \\ &= \kappa'(s)N(s) + \kappa(s)\big(-(\tau B)(s) - (\kappa T)(s)\big) \text{ (done)} \end{aligned}$$

Lastly, we verify

$$\begin{aligned} -\frac{\big(\alpha'(s) \times \alpha''(s)\big) \cdot \alpha'''(s)}{\kappa^2(s)} &= -\frac{(T \times \kappa N) \cdot \Big(\kappa\big(-\tau B - \kappa T\big) + \kappa' N\Big)}{\kappa^2} \\ &= -\frac{-\kappa^2 \tau (T \times N) \cdot B}{\kappa^2} \qquad (\because T \times N \cdot (T \text{ or } N) = 0) \\ &= \tau \end{aligned}$$

■

Question 15

3. Assume that $\alpha(I) \subset R^2$ (i.e., α is a plane curve) and give k a sign as in the text. Transport the vectors $t(s)$ parallel to themselves in such a way that the origins of $t(s)$ agree with the origin of R^2; the end points of $t(s)$ then describe a parametrized curve $s \longrightarrow t(s)$ called the *indicatrix of tangents* of α. Let $\theta(s)$ be the angle from e_1 to $t(s)$ in the orientation of R^2. Prove (a) and (b) (notice that we are assuming that $k \neq 0$).

a. The indicatrix of tangents is a regular parametrized curve.

b. $dt/ds = (d\theta/ds)n$, that is, $k = d\theta/ds$.

Proof. **(a)**

The indicatrix of tangents $\gamma : I \to \mathbb{R}^2$ is defined by

$$\gamma = \frac{\alpha'(s)}{|\alpha'(s)|}$$

Express $\alpha'(s)$ by

$$\alpha' \triangleq (x, y)$$

To show γ is regular. We wish to show

$$\gamma'(s) \neq 0 \text{ for all } s \in I$$

Express γ by

$$\gamma = \frac{(x, y)}{\sqrt{x^2 + y^2}}$$

Then, we see the x-component of $\gamma'(s)$ is

$$\gamma'(s)\Big|_x = \frac{x'y^2}{(x^2 + y^2)^{\frac{3}{2}}}$$

With similar computation on the y-component, we now arrive at

$$\gamma'(s) = \frac{\left(x'y^2, y'x^2\right)}{(x^2 + y^2)^{\frac{3}{2}}}$$

Now, for a contradiction, Assume $\gamma'(s) = 0$ for some s. Then one of the three things below must happen

(a) $x' = y' = 0$

(b) $y^2 = x^2 = 0$

(c) $x' = x^2 = 0$ WLOG

Because $(x, y) = \alpha'$ and α is parametrized by arc-length and curvature is non-zero by premise, we know it can not happen $\alpha'' = (x', y') = 0$.

Because $(x, y) = \alpha'$ and α is parametrized by arc-length, we also know it can not happen $\alpha' = (x, y) = 0$.

Now, we are given the hypothesis $x' = x^2 = 0$. Because α is parametrized by arc-length, from $x = 0$, we know $y = \pm 1$. Then because $|\alpha'|$ is constant, we can deduce

$$\begin{aligned} 0 &= (x', y') \cdot (x, y) \\ &= (0, y') \cdot (0, \pm 1) \end{aligned}$$

This show us $y' = 0$, which is impossible, since if $(x', y') = 0$ then the curvature is 0 CaC (done)

(b) The functions $\theta : [0, l] \to \mathbb{R}$, is defined by

$$T = (x, y) \triangleq (\cos\theta, \sin\theta)$$

By Frenet Formula, we have

$$\kappa N = T' = \theta'(-\sin\theta, \cos\theta) \tag{2.6}$$

Because $|(-\sin\theta, \cos\theta)| = 1$ and $(-\sin\theta, \cos\theta) \cdot T = 0$ and

$$\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = 1$$

we can identify $(-\sin\theta, \cos\theta) = N$. Then from Equation 2.6, we now can deduce

$$\kappa = \theta'$$

■

Question 16

6. A *translation* by a vector v in R^3 is the map $A: R^3 \longrightarrow R^3$ that is given by $A(p) = p + v$, $p \in R^3$. A linear map $\rho: R^3 \longrightarrow R^3$ is an *orthogonal transformation* when $\rho u \cdot \rho v = u \cdot v$ for all vectors $u, v \in R^3$. A *rigid motion* in R^3 is the result of composing a translation with an orthogonal transformation with positive determinant (this last condition is included because we expect rigid motions to preserve orientation).

a. Demonstrate that the norm of a vector and the angle θ between two vectors, $0 \leq \theta \leq \pi$, are invariant under orthogonal transformations with positive determinant.

b. Show that the vector product of two vectors is invariant under orthogonal transformations with positive determinant. Is the assertion still true if we drop the condition on the determinant?

c. Show that the arc length, the curvature, and the torsion of a parametrized curve are (whenever defined) invariant under rigid motions.

Proof. Let A be a translation and ρ be an orthogonal transformation.

(a) Observe

$$\begin{aligned} \|v\| &= \sqrt{v \cdot v} \\ &= \sqrt{\rho v \cdot \rho v} \\ &= \|\rho v\| \end{aligned}$$

Because θ is given by

$$\theta = \arccos \frac{v \cdot w}{|v| \cdot |w|}$$

and because norm is invariant under orthogonal transformation, from the definition of orthogonal transformation, we now see

$$\begin{aligned} \theta &= \arccos \frac{v \cdot w}{|v| \cdot |w|} \\ &= \arccos \frac{\rho v \cdot \rho w}{|v| \cdot |w|} \\ &= \arccos \frac{\rho v \cdot \rho w}{|\rho v| \cdot |\rho w|} = \theta_\rho \end{aligned}$$

where θ_ρ is the angle between ρv and ρw.

(b) Fix $v, w \in \mathbb{R}^3$ and a positive determinant orthogonal transformation ρ. We wish to show

$$\rho v \times \rho w = \rho(v \times w)$$

We can reduce the problem into proving

$$\rho v \times \rho w \cdot z = \rho(v \times w) \cdot z \text{ for all } z \in \mathbb{R}^3$$

Fix $z \in \mathbb{R}^3$. Because ρ has non-zero determinant, we know there exists $z' \in \mathbb{R}^3$ such that

$$\rho z' = z$$

Now, because orthogonal transformation has determinant ± 1 and we have know ρ has positive determinant, we know

$$\begin{aligned}
\rho v \times \rho w \cdot z &= \rho v \times \rho w \cdot \rho z' \\
&= \begin{vmatrix} \rho v \\ \rho w \\ \rho z' \end{vmatrix} \\
&= \begin{vmatrix} \rho v & \rho w & \rho z' \end{vmatrix} \qquad (\because \det A = \det A^t) \\
&= |\rho| \cdot \begin{vmatrix} v & w & z' \end{vmatrix} \qquad (\because \det A \det B = \det AB) \\
&= \begin{vmatrix} v \\ w \\ z' \end{vmatrix} \qquad (\because \det \rho = 1) \\
&= v \times w \cdot z' \\
&= \rho(v \times w) \cdot \rho z' \\
&= \rho(v \times w) \cdot z \text{ (done)}
\end{aligned}$$

The assertion is clearly false if the determinant is negative. One can check $v = (1, 0, 0)$ and $w = (0, 1, 0)$ and $\rho(x, y, z) = (-x, y, z)$.

(c) We first show arc length is invariant under rigid motion. We first show

arc length is invariant under orthogonal transformation

To show such, we only have to show

$$|(\rho \circ \gamma)'| = |\gamma'|$$

■

Fix $y \in I$. We have

$$|\gamma'(y)| = \left|\lim_{t \to y} \frac{\gamma(t) - \gamma(y)}{t - y}\right| = \lim_{t \to y} \left|\frac{\gamma(t) - \gamma(y)}{t - y}\right|$$

Notice that in above deduction, we exchange limit and norm. Such exchange hold true because the function inside is continuous.

Similarly, we have

$$|(\rho \circ \gamma)'(y)| = \lim_{t \to y} \left|\frac{\rho \circ \gamma(t) - \rho \circ \gamma(y)}{t - y}\right|$$

Then, we can reduce the problem into

$$\text{proving } |\gamma(t) - \gamma(y)| = |\rho \circ \gamma(t) - \rho \circ \gamma(y)|$$

Because $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ is an orthogonal transformation (a linear transformation too), we can deduce

$$\begin{aligned} |\rho \circ \gamma(t) - \rho \circ \gamma(y)| &= \big|\rho\big(\gamma(t) - \gamma(y)\big)\big| \\ &= |\gamma(t) - \gamma(y)| \ \ \text{(done)} \end{aligned}$$

We have proved arc-length is invariant under orthogonal transformation. With some simple computation, it is clear that arc-length is invariant under translation. This let us conclude arc length is invariant under rigid motion.

Now, to show curvature and torsion are also invariant under rigid motion. We first recall the following identities for curve parametrzied by arc-length

$$\kappa = |\gamma''| \ \text{ and } \tau = -\frac{\gamma' \times \gamma'' \cdot \gamma'''}{\kappa^2}$$

We now prove

curvature is invariant under rigid motion

Notice that γ' is invariant under translation, so in fact, we only have to prove

curvature is invariant under orthogonal transformation

Observe

$$|\gamma''(y)| = \left|\lim_{t \to y} \frac{\gamma'(t) - \gamma'(y)}{t - y}\right| = \lim_{t \to y} \left|\frac{\gamma'(t) - \gamma'(y)}{t - y}\right|$$

and

$$|(\rho \circ \gamma)''(y)| = \lim_{t \to y} \left| \frac{(\rho \circ \gamma)'(t) - (\rho \circ \gamma)'(y)}{t - y} \right|$$

We can now reduce the problem into proving

$$|\gamma'(t) - \gamma'(y)| = |(\rho \circ \gamma)'(t) - (\rho \circ \gamma)'(y)|$$

Because ρ is a linear transformation, we can compute

$$\begin{aligned}
(\rho \circ \gamma)'(t) &= \lim_{u \to t} \frac{\rho \circ \gamma(u) - \rho \circ \gamma(t)}{u - t} \\
&= \lim_{u \to t} \rho\Big(\frac{\gamma(u) - \gamma(t)}{u - t}\Big) \\
&= \rho \lim_{u \to t} \Big(\frac{\gamma(u) - \gamma(t)}{u - t}\Big) = \rho \circ \gamma'(t)
\end{aligned} \tag{2.7}$$

We now using the fact norm is invariant under orthogonal transformation to compute

$$\begin{aligned}
|(\rho \circ \gamma)'(t) - (\rho \circ \gamma)'(y)| &= |\rho \circ \gamma'(t) - \rho \circ \gamma'(y)| \\
&= \big|\rho\big(\gamma'(t) - \gamma'(y)\big)\big| \\
&= |\gamma'(t) - \gamma'(y)| \ \text{(done)}
\end{aligned}$$

Now, notice that in Equation 2.7, we just proved

$$(\rho\gamma)' = \rho\gamma'$$

Iterating the same argument, we can show

$$\begin{aligned}
(\rho\gamma)'' &= \big((\rho\gamma)'\big)' \\
&= (\rho\gamma')' \\
&= \rho\gamma''
\end{aligned}$$

and also show

$$\begin{aligned}
(\rho\gamma)''' &= \big((\rho\gamma)''\big)' \\
&= (\rho\gamma'')' \\
&= \rho\gamma'''
\end{aligned}$$

We now using the fact that $|\rho| = 1$ to compute

$$\begin{aligned}
(\rho\gamma)' \times (\rho\gamma)'' \cdot (\rho\gamma)''' &= \begin{vmatrix}(\rho\gamma)' & (\rho\gamma)'' & (\rho\gamma)'''\end{vmatrix} \\
&= \begin{vmatrix}\rho\gamma' & \rho\gamma'' & \rho\gamma'''\end{vmatrix} \\
&= \big|\rho \begin{bmatrix}\gamma' & \gamma'' & \gamma'''\end{bmatrix}\big| \\
&= |\rho| \cdot \begin{vmatrix}\gamma' & \gamma'' & \gamma'''\end{vmatrix} \\
&= \gamma' \times \gamma'' \cdot \gamma'''
\end{aligned}$$

Above computation with identity of torsion and the fact curvature is invariant under orthogonal transformation with positive determinant then show that torsion is also invariant under orthogonal transformation with positive determinant.

Because $(\gamma + c)' = \gamma'$, together with what we have proved, it is easy to check torsion is also invariant under rigid motion.

Question 17

9. Given a differentiable function $k(s)$, $s \in I$, show that the parametrized plane curve having $k(s) = k$ as curvature is given by

$$\alpha(s) = \left(\int \cos \theta(s)\, ds + a, \int \sin \theta(s)\, ds + b \right),$$

where

$$\theta(s) = \int k(s)\, ds + \varphi,$$

and that the curve is determined up to a translation of the vector (a, b) and a rotation of the angle φ.

Proof. By Fundamental Theorem of Local Curves (you can think of our application as identifying $\tau = 0$), we know if such curve exists then it is unique up to translation and rotation. This reduced our proof into showing

$$\alpha \text{ has curvature } \kappa$$

Compute

$$\alpha' = (\cos\theta, \sin\theta)$$

This shows that α is parametrized by arc-length, and shows that we can compute

$$\begin{aligned} |\alpha''| &= |\theta'(-\sin\theta, \cos\theta)| \\ &= |\theta'| = |\kappa| = \kappa \text{ (done)} \end{aligned}$$

■

Question 18

11. One often gives a plane curve in polar coordinates by $\rho = \rho(\theta)$, $a \le \theta \le b$.

a. Show that the arc length is

$$\int_a^b \sqrt{\rho^2 + (\rho')^2}\, d\theta,$$

where the prime denotes the derivative relative to θ.

b. Show that the curvature is

$$k(\theta) = \frac{2(\rho')^2 - \rho\rho'' + \rho^2}{\{(\rho')^2 + \rho^2\}^{3/2}}.$$

Proof. **(a)**
Parametrize by

$$\alpha(\theta) \triangleq (x, y) \triangleq (r\cos\theta, r\sin\theta)$$

where $r(\theta)$ is a function. With respect to θ, we compute

$$\begin{aligned}(x')^2 + (y')^2 &= (r'\cos\theta - r\sin\theta)^2 + (r'\sin\theta + r\cos\theta)^2 \\ &= (r')^2\cos^2\theta + r^2\sin^2\theta + (r')^2\sin^2\theta + r^2\cos^2\theta \qquad (\because \text{ elimination }) \\ &= r^2 + (r')^2\end{aligned}$$

We now see that the arc-length can be computed by

$$\begin{aligned}\int_a^b |\alpha'(\theta)|\, d\theta &= \int_a^b \sqrt{(x')^2 + (y')^2} d\theta \\ &= \int_a^b \sqrt{r^2 + (r')^2} d\theta\end{aligned}$$

(b)

Recall that

$$\kappa(t) = \frac{x'y'' - x''y'}{\left((x')^2 + (y')^2\right)^{\frac{3}{2}}}$$

plugin

$$\begin{aligned}(x', y') &= (r'\cos\theta - r\sin\theta, r'\sin\theta + r\cos\theta) \\ (x'', y'') &= (r''\cos\theta - 2r'\sin\theta - r\cos\theta, r''\sin\theta + 2r'\cos\theta - r\sin\theta)\end{aligned}$$

To compute

$$x'y'' = r'r''\cos\theta\sin\theta + 2(r')^2\cos^2\theta - rr'\cos\theta\sin\theta - rr''\sin^2\theta - 2rr'\cos\theta\sin\theta + r^2\sin^2\theta$$
$$x''y' = r'r''\sin\theta\cos\theta - 2(r')^2\sin^2\theta - rr'\cos\theta\sin\theta + rr''\cos^2\theta - 2rr'\cos\theta\sin\theta - r^2\cos^2\theta$$

Eliminating the odd terms and using $\cos^2 + \sin^2 = 1$, we now compute

$$\begin{aligned}\kappa &= \frac{x'y'' - x''y'}{\left((x')^2 + (y')^2\right)^{\frac{3}{2}}} \\ &= \frac{2(r')^2 - 2rr'' + r^2}{\left(r^2 + (r')^2\right)^{\frac{3}{2}}}\end{aligned}$$

■

Question 19

17. In general, a curve α is called a *helix* if the tangent lines of α make a constant angle with a fixed direction. Assume that $\tau(s) \neq 0$, $s \in I$, and prove that:

***a.** α is a helix if and only if $k/\tau = \text{const}$.

***b.** α is a helix if and only if the lines containing $n(s)$ and passing through $\alpha(s)$ are parallel to a fixed plane.

***c.** α is a helix if and only if the lines containing $b(s)$ and passing through $\alpha(s)$ make a constant angle with a fixed direction.

d. The curve

$$\alpha(s) = \left(\frac{a}{c}\int \sin\theta(s)\,ds, \frac{a}{c}\int \cos\theta(s)\,ds, \frac{b}{c}s\right),$$

where $a^2 = b^2 + c^2$, is a helix, and that $k/\tau = b/a$.

Proof. **(a)** $(\longrightarrow)$

Because α is a helix, we know there exists fixed unit $a \in \mathbb{R}^3$ and $b \in \mathbb{R}$ such that

$$\alpha' \cdot a = b \text{ for all } s$$

This then implies

$$\alpha'' \cdot a = 0 \text{ for all } s$$

which implies

$$N \cdot a = 0$$

since N is parallel with α''. Because $\{T, N, B\}$ is an orthonormal basis, this $(N \cdot a = 0)$ together with a being unit then tell us we can express a by

$$a = T\cos\theta + B\sin\theta \text{ for some fixed } \theta \in \mathbb{R}$$

We now have the information $T\cos\theta + B\sin\theta$ is a constant function in s. Then, using Frenet Formula, we can deduce

$$0 = (T\cos\theta + B\sin\theta)' = \kappa N\cos\theta + \tau N\sin\theta$$

This them implies

$$\frac{\kappa}{\tau} = \frac{-\sin\theta}{\cos\theta} \text{ is a constant since } \theta \text{ is fixed.}$$

Notice that $\cos\theta \neq 0$ because $\tau \neq 0$ for all s.

$(\longleftarrow)$

Define $\theta \in \mathbb{R}$ by

$$\theta = \arctan\frac{-\kappa}{\tau}$$

We wish to show

$$a = T\cos\theta + B\sin\theta \text{ suffice}$$

Because we have

$$T \cdot a = \cos\theta$$

We only wish to show

$$a \text{ is a constant function in } s$$

Because $\theta = \arctan\frac{-\kappa}{\tau}$, we know

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{-\kappa}{\tau}$$

This then tell us

$$\tau\sin\theta + \kappa\cos\theta = 0$$

and implies

$$\tau N\sin\theta + \kappa N\cos\theta = 0$$

Then

$$
\begin{aligned}
a' &= (T\cos\theta + N\sin\theta)' \\
&= \kappa N\cos\theta + \tau N\sin\theta = 0
\end{aligned}
$$

This implies a is indeed a constant. (done)

(b)

$(\longrightarrow)$

Let $a \in \mathbb{R}^3$ be the unit vector such that

$$T \cdot a \text{ is fixed}$$

We now see

$$N \cdot a = (T \cdot a)' = 0$$

This implies

$$\text{the plane } \{a\}^\perp \text{ suffice}$$

$(\longleftarrow)$

Observe that

$$
\begin{aligned}
0 = N \cdot a &= \frac{T'}{|T'|} \cdot a \\
&\implies T' \cdot a = 0 \\
&\implies T \cdot a \text{ is fixed}
\end{aligned}
$$

(c)

$(\longrightarrow)$

Because $T \cdot a$ is fixed, we can deduce

$$\kappa N \cdot a = T' \cdot a = 0$$

Now observe from Frenet Formula that

$$(B \cdot a)' = -\tau N \cdot a = 0$$

This implies $B \cdot a$ is fixed.

$(\longleftarrow)$

Because $B \cdot a$ is fixed, we can deduce

$$\begin{aligned} 0 = (B \cdot a)' &= -\tau N \cdot a \\ \implies N \cdot a &= 0 \end{aligned}$$

The proof then now follows from the result of **(b)**.

(d)

First we have to notice the fucking typo correction $\frac{\kappa}{\tau} = \frac{a}{b}$.

Compute

$$\begin{aligned} \alpha'(s) &= \Big(\frac{a}{c}\sin\theta, \frac{a}{c}\cos\theta, \frac{b}{c}\Big) \\ \alpha''(s) &= \Big(\theta'\frac{a}{c}\cos\theta, \theta'\frac{-a}{c}\sin\theta, 0\Big) \\ \alpha'''(s) &= \Big(\theta''\frac{a}{c}\cos\theta + (\theta')^2\frac{-a}{c}\sin\theta, \theta''\frac{-a}{c}\sin\theta + (\theta')^2\frac{-a}{c}\cos\theta, 0\Big) \end{aligned}$$

This give us

$$\begin{aligned} &\alpha' \times \alpha'' \cdot \alpha''' \\ &= \frac{b}{c}\Big[\theta'\theta''\frac{-a^2}{c^2}\cos\theta\sin\theta + (\theta')^3\frac{-a^2}{c^2}\cos^2\theta - \theta'\theta''\frac{-a^2}{c}\sin\theta\cos\theta - (\theta')^3\frac{a^2}{c^2}\sin^2\theta\Big] \\ &= \frac{b}{c}\Big((\theta')^3\frac{-a^2}{c^2}\Big) = \frac{-a^2b}{c^3}(\theta')^3 \end{aligned}$$

And give us

$$\kappa = \theta'\frac{a}{c}$$

We now compute

$$\begin{aligned} \frac{\kappa}{\tau} &= \frac{\kappa}{-\frac{\alpha' \times \alpha'' \cdot \alpha'''}{\kappa^2}} \\ &= \frac{-\kappa^3}{\alpha' \times \alpha'' \cdot \alpha'''} \\ &= \frac{-(\theta')^3\frac{a^3}{c^3}}{\frac{-a^2b}{c^3}(\theta')^3} = \frac{a}{b} \end{aligned}$$

■

Question 20

3. Compute the curvature of the ellipse

$$x = a\cos t, \qquad y = b\sin t, \qquad t \in [0, 2\pi],\ a \neq b,$$

and show that it has exactly four vertices, namely, the points $(a, 0)$, $(-a, 0)$, $(0, b)$, $(0, -b)$.

Proof. Compute

$$\begin{cases} x' = -a\sin t \text{ and } x'' = -a\cos t \\ y' = b\cos t \text{ and } y'' = -b\sin t \end{cases}$$

Plugging the curvature formula

$$\kappa = \frac{x'y'' - x''y'}{\left((x')^2 + (y')^2\right)^{\frac{3}{2}}}$$

We now have

$$\kappa = \frac{ab}{\left(a^2\sin^2 t + b^2\cos^2 t\right)^{\frac{3}{2}}}$$

Compute

$$\kappa' = (2a^2\sin t\cos t - 2b^2\sin t\cos t)(a^2\sin^2 t + b^2\cos^2 t)^{\frac{-5}{2}} \cdot (ab)$$

We see that

$$\kappa' = 0 \iff \sin 2t = 0$$

This only happens when $t \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$ where 2π is just 0 in the sense of parametrizeation of closed curve. We have shown there are exactly four vertices

$$(a, 0), (-a, 0), (0, b), (0, -b)$$

■

Question 21

***4.** Let C be a plane curve and let T be the tangent line at a point $p \in C$. Draw a line L parallel to the normal line at p and at a distance d of p (Fig. 1-36). Let h be the length of the segment determined on L by C and T (thus, h is the "height" of C relative to T). Prove that

$$|k(p)| = \lim_{d \to 0} \frac{2h}{d^2},$$

where $k(p)$ is the curvature of C at p.

Proof. WOLG, let $p = (0,0)$, T be the x-axis and some neighborhood around p be above T. Positively oriented parametrize C by arc-length using (x,y) and $(x,y)(0) = (0,0)$. Using Taylor Theorem about $y(0)$, we see

$$y(s) = y(0) + y'(0)s + \frac{y''(0)}{2}s^2 + R_y \text{ where } \frac{R_y}{s^2} \to 0 \text{ as } s \to 0$$

Because T the tangent is the x-axis, we know $x''(0) = 0 \quad (\because N = (0,1))$. This tell us

$$\begin{aligned} |\kappa(0)| &= \sqrt{(x'')^2(0) + (y'')^2(0)} \\ &= y''(0) \qquad (\because N = (0,1)) \end{aligned}$$

■

By our setting $(x,y)(0) = (0,0)$, we see

$$y(0) = y'(0) = 0 \qquad (\because (x', y') = (1,0))$$

We now see

$$y''(0) = \frac{2\big(y(s) - R_y\big)}{s^2} \text{ for all } s \neq 0$$

This tell us

$$y''(0) = \lim_{s \to 0} \frac{2\big(y(s) - R_y\big)}{s^2} = \lim_{s \to 0} \frac{2y(s)}{s^2}$$

Using Taylor Theorem about $x(0)$, we see

$$x(s) = x(0) + x'(0)s + R_x \text{ where } \frac{R_x}{s} \to 0 \text{ as } x \to 0$$

Because $x(0) = 0$ and $x'(0) = 1$, we see

$$\lim_{s\to 0} \frac{x(s)}{s} = \lim_{s\to 0} \frac{s + R_x}{s} = 1$$

This now give us

$$|\kappa(0)| = y''(0) = \lim_{s\to 0} \frac{2y(s)}{s^2} = \lim_{s\to 0} \frac{2y(s)}{x^2(s)} = \lim_{d\to 0} \frac{2h}{d^2}$$

Question 22

6. Let $\alpha(s)$, $s \in [0, l]$ be a closed convex plane curve positively oriented. The curve

$$\beta(s) = \alpha(s) - rn(s),$$

where r is a positive constant and n is the normal vector, is called a *parallel* curve to α (Fig. 1-37). Show that

a. Length of β = length of $\alpha + 2\pi r$.

b. $A(\beta) = A(\alpha) + rl + \pi r^2$.

c. $k_\beta(s) = k_\alpha(s)/(1 + r)$.

Proof. **(a)**

Using Frenet Formula to compute

$$\beta'(s) = \alpha'(s) + r\kappa T(s)$$

Because α is parametrized by arc-length, we now know

$$|\beta'| = |(1 + r\kappa)\alpha'| = |1 + r\kappa| = 1 + r\kappa$$

This now give us

$$\int_0^l |\beta'|\, ds = l + r\int_0^l \kappa ds$$

Because a closed convex curve must also be simple (Sec. 5-7, Prop. 1), we now can deduce

$$\begin{aligned} \text{Length of } \beta &= l + r\int_0^l \kappa ds \\ &= \text{Length of } \alpha + r(2\pi) \end{aligned}$$

(b)

Set

$$\alpha = (x, y) \text{ and } \beta = (x - rN_1, y - rN_2)$$

Because

$$\beta' = (1 + r\kappa)\alpha'$$

We know

$$\beta' = \Big((1 + r\kappa)x', (1 + r\kappa)y'\Big)$$

Now, we use Green's Theorem to compute the Area

$$\begin{aligned} A(\beta) &= \frac{1}{2}\int_0^l (x - rN_1)(1 + r\kappa)y' - (y - rN_2)(1 + r\kappa)x' ds \\ &= \frac{1}{2}\int_0^l (xy' - yx')ds + \frac{r}{2}\int_0^l (\kappa xy' + \kappa x'y)ds \\ &+ \frac{r}{2}\int_0^l -(N_1y' + N_2x')ds + \frac{r^2}{2}\int_0^l (-N_1y'\kappa + N_2x'\kappa)ds \end{aligned}$$

Notice that by Frenet Formula, we have

$$N' = -\kappa(x', y')$$

so in fact we know

$$\kappa xy' + \kappa x'y = N' \cdot (-y, x)$$

Now using integral by part and the fact $\alpha = (x, y)$ is closed, we know

$$\begin{aligned} \int_0^l (\kappa xy' + \kappa x'y)ds &= \int_0^l N' \cdot (-y, x)ds \\ &= \int_0^l N \cdot (-y', x')ds \end{aligned}$$

Then now we have

$$\frac{r}{2}\int_0^l (\kappa xy' + \kappa x'y)ds + \frac{r}{2}\int_0^l -N_1y' + N_2x'ds = \frac{r}{2}\int_0^l 2N \cdot (-y', x')ds$$

Using positive orientation and the fact $|N| = 1 = |(-y', x')|$ to identify that $N = (-y', x')$, we now have

$$\frac{r}{2}\int_0^l 2N \cdot (-y', x')ds = rl$$

and have

$$\frac{r^2}{2}\int_0^l (-N_1 y'\kappa + N_2 x'\kappa)ds = \frac{r^2}{2}\int_0^l \kappa ds = r^2\pi$$

since (x, y) is simple closed. This finishes the proof.

(c)

Recall that

$$\kappa(a, b) = \frac{a'b'' - a''b'}{\left((a')^2 + (b')^2\right)^{\frac{3}{2}}}$$

We use this formula on β to compute

$$\begin{aligned}
\kappa_\beta &= \frac{(1 + r\kappa)x'\big((1 + r\kappa)y'\big)' - \big((1 + r\kappa)x'\big)'(1 + r\kappa)y'}{(1 + r\kappa)^3} \\
&= \frac{(1 + r\kappa)^2(x'y'' - x''y')}{(1 + r\kappa)^3} \\
&= \frac{x'y'' - x''y'}{1 + r\kappa} = \frac{\kappa}{1 + r\kappa} \qquad (\because (x')^2 + (y')^2 = 1)
\end{aligned}$$

■

Question 23

8. ***a.** Let $\alpha(s)$, $s \in [0, l]$, be a plane simple closed curve. Assume that the curvature $k(s)$ satisfies $0 < k(s) \le c$, where c is a constant (thus, α is less curved than a circle of radius $1/c$). Prove that

$$\text{length of } \alpha \ge \frac{2\pi}{c}.$$

b. In part a replace the assumption of being simple by "α has rotation index N." Prove that

$$\text{length of } \alpha \ge \frac{2\pi N}{c}.$$

Proof. **(a)**

Because α is simple closed and $\kappa \le c$, we know

$$cl = \int_0^l c ds \ge \int_0^l \kappa ds = 2\pi$$

This then implies

$$\text{Length of } \alpha = l \ge \frac{2\pi}{c}$$

(b)

Because α has rotation index N and $\kappa \le c$, we know

$$cl = \int_0^l c ds \ge \int_0^l \kappa ds = N2\pi$$

This the implies

$$\text{Length of } \alpha = l \ge \frac{2\pi N}{c}$$

■

Question 24

***11.** Given a nonconvex simple closed plane curve C, we can consider its *convex hull* H (Fig. 1-39), that is, the boundary of the smallest convex set containing the interior of C. The curve H is formed by arcs of C and by the segments of the tangents to C that bridge "the nonconvex gaps" (Fig. 1-39). It can be proved that H is a C^1 closed convex curve. Use this to show that, in the isoperimetric problem, we can restrict ourselves to convex curves.

Figure 1-39

Proof. Suppose we have proved that a convex closed curve must satisfy the isoperimetric inequality. Let C be an arbitrary closed plane curve, and let H be its convex hull. Now, because straight line is the shortest curve between two point and because we know H, a convex curve, must satisfy isoperimetric inequality, we now see

$$4\pi A(C) \leq 4\pi A(H) \leq l_H^2 \leq l_C^2$$

If the equality hold true, we can deduce from $l_H = l_C$ that $H = C$ and use the argument for isoperimetric inequality of convex curve to argue that $C = H$ must be a circle. ■

Question 25

3. Show that the two-sheeted cone, with its vertex at the origin, that is, the set $\{(x, y, z) \in R^3; x^2 + y^2 - z^2 = 0\}$, is not a regular surface.

Proof. Let $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 0\}$. It is clear S contain $(0, 0, 0)$. To show S is not regular, we only wish to find a neighborhood V around $(0, 0, 0)$ in S such that V can not expressed as graph of differentiable functions from $\mathbb{R}^2$ to $\mathbb{R}$. This is trivially true, as all neighborhood ought to contain some open ball $B_\epsilon(0)$, and in this open ball, if we fix, say $(x, y) \in B_\epsilon(0)$ such that $(x, y, \sqrt{x^2 + y^2}) \in B_\epsilon(0)$, we see that $z = -\sqrt{x^2 + y^2}$ is also in $B_\epsilon(0)$ and S. The same argument applies to when (x, z) and (y, z) are fixed. ■

Question 26

6. Give another proof of Prop. 1 by applying Prop. 2 to $h(x, y, z) = f(x, y) - z$.

Proof. Because f is differentiable, we see f_x, f_y are all continuous on U. This then implies

$$h_x(x,y,z) = f_x(x,y), h_y(x,y,z) = f_y(x,y), h_z = -1 \text{ are all continuous on } U$$

We have shown h is differentiable. Now that observe

$$h(x,y,z) = 0 \implies (x,y,z) = (x,y,f(x,y))$$

The converse of course hold true. This then implies

$$f[U] = h^{-1}[0]$$

Fix arbitrary $(x,y) \in U$. We see

$$\mathbf{d}h\big(x,y,f(x,y)\big) = \begin{bmatrix} h_x & h_y & h_z \end{bmatrix}\Big|_{(x,y,f(x,y))} = \begin{bmatrix} f_x(x,y) & f_y(x,y) & -1 \end{bmatrix}$$

which is clearly not onto. This show

$$(x,y,f(x,y)) \text{ is not a critical point}$$

Because $(x,y) \in U$ is arbitrary, we have shown $f[U]$ contain no critical point. Now it follows 0 is a regular value and $f[U] = h^{-1}[0]$ is a regular surface. ∎

Question 27

7. Let $f(x, y, z) = (x + y + z - 1)^2$.

a. Locate the critical points and critical values of f.

b. For what values of c is the set $f(x, y, z) = c$ a regular surface?

c. Answer the questions of parts a and b for the function $f(x, y, z) = xyz^2$.

Proof. **(a)**

Compute

$$f_x = f_y = f_z = 2(x+y+z-1)$$

This implies the set of critical points are

$$\{(x,y,z) \in \mathbb{R}^3 : x+y+z = 1\}$$

Then it follows from simple computation the set of critical values is exactly

$$\{0\}$$

(b)

For all $c > 0$, the set $f^{-1}[c]$ is a regular surface, and for all $c < 0$, the set $f^{-1}[c]$ is empty (thus trivially regular).

(c)

Compute

$$f_x = yz^2 \text{ and } f_y = xz^2 \text{ and } f_z = 2xyz$$

This implies the set of critical points is

$$\{(x, y, z) : z = 0 \text{ or } x = y = 0\}$$

With simple computation, we see the set of critical values is exactly

$$\{0\}$$

The set of regular values are exactly $\mathbb{R}^*$, so all $c \neq 0$ suffice. ■

Question 28

8. **Let $\mathbf{x}(u, v)$ be as in Def. 1. Verify that $d\mathbf{x}_q \colon R^2 \to R^3$ is one-to-one if and only if**

$$\frac{\partial \mathbf{x}}{\partial u} \wedge \frac{\partial \mathbf{x}}{\partial v} \neq 0.$$

Proof. Note that

$$dx_q = \begin{bmatrix} \partial_u x & \partial_v x \end{bmatrix}$$

This give us

$$dx_q : \mathbb{R}^2 \to \mathbb{R}^3 \text{ is one-to-one } \iff \partial_u x, \partial_v x \in \mathbb{R}^3 \text{ is linearly independent everywhere}$$

Then we can reduce the problem into proving

$$\partial_u x, \partial_v x \in \mathbb{R}^3 \text{ is linearly independent everywhere } \iff \partial_u x \times \partial_v x \neq 0 \text{ everywhere}$$

This then follows from Theorem 2.2.2 at the next page, as one can see that each component of the output of cross product is exactly the three determinant. ■

Theorem 2.2.2. (Computation to check Linearly Independence)

$$\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \end{bmatrix} \text{ is linearly independent } \iff \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} \neq 0 \text{ or } \begin{vmatrix} v_2 & w_2 \\ v_3 & w_3 \end{vmatrix} \neq 0 \text{ or } \begin{vmatrix} v_1 & w_1 \\ v_3 & w_3 \end{vmatrix} \neq 0$$

Proof. $(\longleftarrow)$

Assume v, w are linearly dependent. Fix $w_k = cv_k$. We see

$$\begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} = cv_1v_2 - cv_1v_2 = 0 \text{ CaC}$$

$(\longrightarrow)$

Assume all determinant are 0. Pick k such that v_k is non-zero. Define

$$c \triangleq \frac{w_k}{v_k}$$

WOLG, suppose

$$w_1 = cv_1 \text{ and } v_1 \neq 0$$

We then can deduce

$$\begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} \implies cv_1v_2 = v_1w_2 \implies w_2 = cv_2$$

The same arguement implies $w_3 = cv_3$ CaC ■

2.3 HW3

Question 29

12. Show that $\mathbf{x}\colon U \subset R^2 \to R^3$ given by

$$\mathbf{x}(u, v) = (a \sin u \cos v, b \sin u \sin v, c \cos u), \quad a, b, c \neq 0,$$

where $0 < u < \pi, 0 < v < 2\pi$, is a parametrization for the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Describe geometrically the curves $u = \text{const.}$ on the ellipsoid.

Proof. We are required to show

(a) range of $\mathbf{x}$ lies in the ellipsoid

(b) $\mathbf{x}$ is smooth

(c) $d\mathbf{x}$ is one-to-one everywhere on $U \triangleq (0, \pi) \times (0, 2\pi)$

(d) $\mathbf{x}$ is a homeomorphism

Compute

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \cos^2 u = 1$$

This shows that the range of $\mathbf{x}$ indeed lies in the ellipsoid.

It is clear that $\mathbf{x}$ is smooth.

Compute

$$d\mathbf{x} = \begin{bmatrix} a \cos u \cos v & -a \sin u \sin v \\ b \cos u \sin v & b \sin u \cos v \\ -c \sin u & 0 \end{bmatrix}$$

Then compute

$$\frac{\partial(y, z)}{\partial(u, v)} = bc \sin^2 u \cos v \text{ and } \frac{\partial(x, z)}{\partial(u, v)} = -ac \sin^2 u \sin v$$

Because $u \in (0,\pi)$ and $v \in (0,2\pi)$, and $b,c \neq 0$, we now can deduce

$$\frac{\partial(y,z)}{\partial(u,v)} = 0 \iff v \in \{\frac{\pi}{2}, \frac{3}{2}\pi\}$$
$$\frac{\partial(y,z)}{\partial(u,v)} = 0 \iff v = \pi$$

This then let us deduce

$$d\mathbf{x} \text{ is one-to-one everywhere on } (0,\pi)\times(0,2\pi)$$

Traditionally, the function arctan is defined on $\mathbb{R}$ and have codomaiin $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Deduce first from the z-component of $\mathbf{x}$. We see

$$\mathbf{x}^{-1}(x,y,z) = \Big(\arccos\frac{z}{c}, \begin{cases} \arctan\frac{ay}{bx} & \text{if } x,y \in \mathbb{R}^+ \text{ (first quadrant)} \\ \frac{\pi}{2} & \text{if } x = 0, y \in \mathbb{R}^+ \\ \frac{3\pi}{2} + \arctan\frac{ay}{bx} & \text{if } x \in \mathbb{R}^-, y \in \mathbb{R}^+ \text{ (second quadrant)} \\ \pi + \arctan\frac{ay}{bx} & \text{if } x \in \mathbb{R}^-, y \in \mathbb{R}^-_0 \text{ (third quadrant)} \\ \frac{3\pi}{2} & \text{if } x = 0, y \in \mathbb{R}^- \\ \frac{4\pi}{2} + \arctan\frac{ay}{bx} & \text{if } x \in \mathbb{R}^+, y \in \mathbb{R}^- \text{ (forth quadrant)} \end{cases}\Big)$$

Now it follows that $\mathbf{x}$ is indeed a homeomorphism.

When u is fixed, the image is an oval missing a point $(a\sin u, 0, c\cos u)$ floating in air (contained by $\{(x,y,c_0) \in \mathbb{R}^3 : (x,y) \in \mathbb{R}^2\}$ where $c_0 = c\cos u$ is fixed) ■

Definition 2.3.1. (Definition of regular plane curve) We say $C \subseteq \mathbb{R}^2$ is a regular plane curve if for all $p \in C$ there exists

(a) an open neighborhood $p \in V \subseteq \mathbb{R}^2$

(b) an open set $U \subseteq \mathbb{R}$

(c) a function $\mathbf{x} : U \to V \cap C$

such that $\mathbf{x}$ satisfy

(a) $\mathbf{x}$ is smooth

(b) $\mathbf{x}$ is a homoeomorphism between U and $V \cap C$

(c) $d\mathbf{x}_q \in L(\mathbb{R}, \mathbb{R}^2)$ is one-to-one for all $q \in U$

Definition 2.3.2. (Definition of regular space curve) We say $C \subseteq \mathbb{R}^3$ is a regular space curve if for all $p \in C$ there exists

(a) an open neighborhood $p \in V \subseteq \mathbb{R}^3$

(b) an open set $U \subseteq \mathbb{R}$

(c) a function $\mathbf{x} : U \to V \cap C$

such that $\mathbf{x}$ satisfy

(a) $\mathbf{x}$ is smooth

(b) $\mathbf{x}$ is a homoeomorphism between U and $V \cap C$

(c) $d\mathbf{x}_q \in L(\mathbb{R}, \mathbb{R}^3)$ is one-to-one for all $q \in U$

Question 30

17. Define a regular curve in analogy with a regular surface. Prove that

a. The inverse image of a regular value of a differentiable function

$$f : U \subset R^2 \to R$$

is a regular plane curve. Give an example of such a curve which is not connected.

b. The inverse image of a regular value of a differentiable map

$$F : U \subset R^3 \to R^2$$

is a regular curve in R^3. Show the relationship between this proposition and the classical way of defining a curve in R^3 as the intersection of two surfaces.

***c. The set $C = \{(x, y) \in R^2; x^2 = y^3\}$ is not a regular curve.**

Proof. **(a)**

Suppose $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a smooth function and c is a regular value. We wish to prove

$$C \triangleq f^{-1}[c] \text{ is a regular plane curve}$$

Fix $p \in f^{-1}[c]$. We wish

$$\text{to find a local parametrization } \mathbf{x} : I \subseteq \mathbb{R} \to C \text{ around } p$$

Because c is a regular value, we know df_p is one-to-one. Then, WOLG, we can let $\partial_y F(p) \neq c$. Define $F : U \to \mathbb{R}^2$ by

$$F(x, y) \triangleq \Big(x, f(x, y)\Big)$$

Compute

$$dF = \begin{bmatrix} 1 & 0 \\ \partial_x f & \partial_y f \end{bmatrix}$$

It is now clear that $\det(dF_p) \neq 0$. Now, because f is smooth, we can use inverse function Theorem and obtain a diffeoemorphism F between open neighborhood around p and open neighborhoood around $f(p)$. Now, note that $f[C] = \{c\}$. This tell us

$$F[C] \subseteq \{(x, c) \in \mathbb{R}^2 : x \in \mathbb{R}\}$$

we now claim

$$\mathbf{x}(u) \triangleq F^{-1}(u, c) \text{ is the desired local parametrization around } p$$

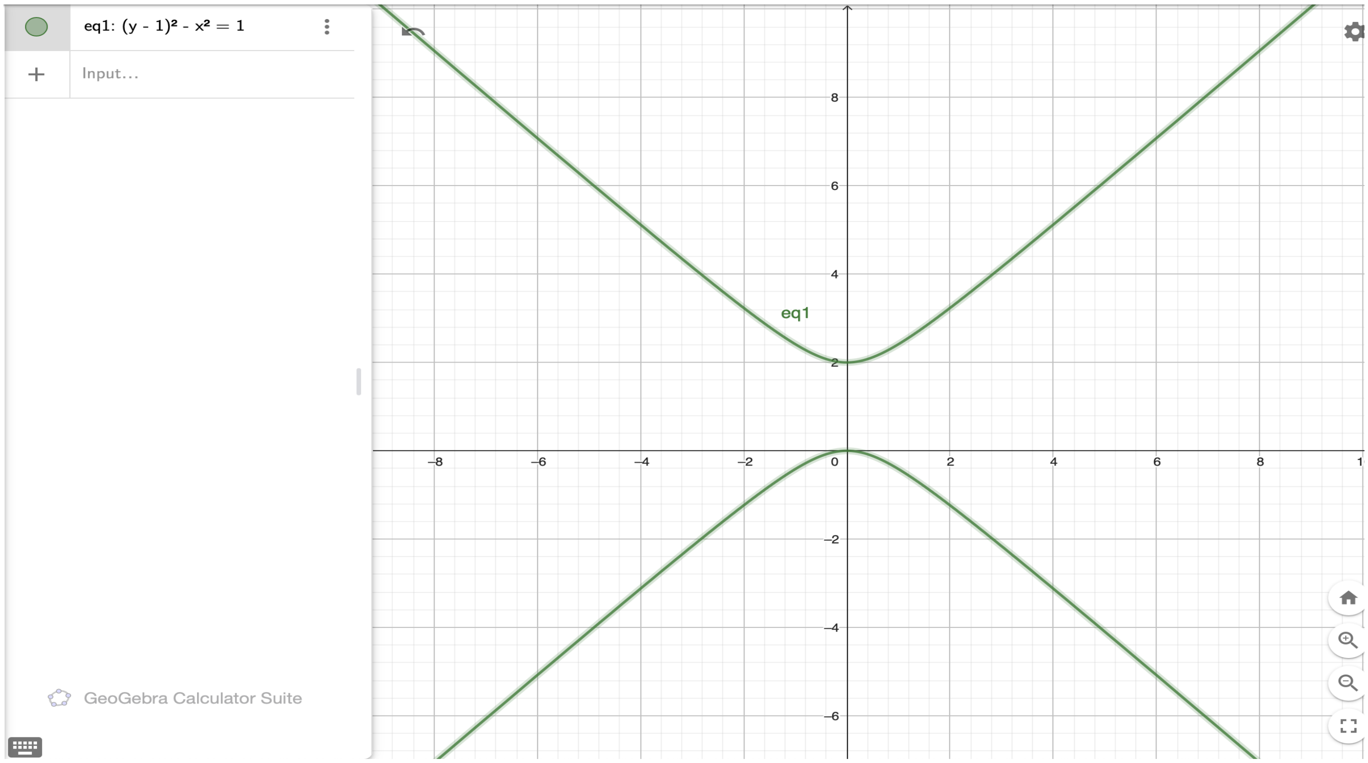
The fact that $\mathbf{x}$ is smooth and homeomorphism follows from

(a) F is a diffeomorphism around p

(b) $\mathbf{x}$ can be identified as restriction of F^{-1}

Note that $d(F^{-1})_p = \big(dF_{F^{-1}(p)}\big)^{-1} \neq 0$. Now, because $\mathbf{x}$ is restriction of F^{-1}, we see $d\mathbf{x}$ must not be 0 around p. (done)

An example is $f(x, y) = (y - 1)^2 - x^2$.



(b)

Suppose $F : U \subseteq \mathbb{R}^3 \to \mathbb{R}^2$ is a smooth function and (c_0, c_1) is a regular value. We wish to prove

$$C \triangleq F^{-1}[(c_0, c_1)] \text{ is a regular space curve}$$

Fix $p \in F^{-1}[(c_0, c_1)]$. We wish

$$\text{to find a local parametrization } \mathbf{x} : I \subseteq \mathbb{R} \to C \text{ around } p$$

Define $G : U \to \mathbb{R}^3$ by

$$G(x, y, z) \triangleq \Big(x, F(x, y, z)\Big)$$

Compute

$$dG = \begin{bmatrix} 1 & 0 & 0 \\ \partial_x F_1 & \partial_y F_1 & \partial_z F_1 \\ \partial_x F_2 & \partial_y F_2 & \partial_z F_2 \end{bmatrix}$$

Because p is a regular point of F, we can WOLG, suppose

$$\det\big(dG_p\big) = \det\Big(\frac{\partial(F_1, F_2)}{\partial(y, z)}\Big|_p\Big) \neq 0$$

This Then, by Inverse function Theorem, G is locally a diffeomorphism around p. We now see

$$\mathbf{x}(t) \triangleq G^{-1}(t, c_0, c_1) \text{ is the desired local parametrization around } p \text{ (done)}$$

Suppose we are given two function $A, B : \mathbb{R}^3 \to \mathbb{R}$, and suppose $A^{-1}[c_0], B^{-1}[c_1]$ are two surfaces. Define $F : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$F(p) \triangleq (A(p), B(p))$$

We see that

$$\text{the intersection } A^{-1}[c_0] \cap B^{-1}[c_1] \text{ is exactly } F^{-1}[(c_0, c_1)]$$

(c)

Assume for a contradiction, C is a regular curve. Note that $(0, 0) \in C$. We know there exists an open-neighborhood $N \subseteq \mathbb{R}^2$ around $(0, 0)$ such that $N \cap C$ is the graph of some differentiable function in x or y. However, this is impossible, since if one view $N \cap C$ as a function in x, the function $y = x^{\frac{2}{3}}$ is not differetaible at $x = 0$, and one can not even view $N \cap C$ as a function in y as each y correspond to two x, namely $x = \pm y^{\frac{3}{2}}$. CaC

∎

Question 31

2. Let $S \subset R^3$ be a regular surface and $\pi : S \to R^2$ be the map which takes each $p \in S$ into its orthogonal projection over $R^2 = \{(x, y, z) \in R^3; z = 0\}$. Is π differentiable?

Proof. Yes. Fix p in S. We wish to prove

$$\pi \text{ is differentiable at } p \text{ in the sense of manifold}$$

Let $\mathbf{x}_1 : U_1 \subseteq \mathbb{R}^2 \to V_1 \cap S \subseteq \mathbb{R}^3$ be a local parametrization around p. Define a local parametrization $\mathbf{x}_2 : U_2 \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ around $\pi(p)$ by

$$\mathbf{x}_2 \triangleq \mathbf{id}_{U_2}$$

We are require to prove

$$\mathbf{x}_2^{-1} \circ \pi \circ \mathbf{x}_1 \text{ is differentiable at } \mathbf{x}_1^{-1}(p)$$

Notice that

(a) $\mathbf{x}_1 : U_1 \to \mathbb{R}^3$ is differentiable at p by definition

(b) $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is clearly differntiable, with derivative $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(c) $\mathbf{x}_2^{-1} = \mathbf{id}_{U_2} : U_2 \to \mathbb{R}^2$ is clearly differntiable.

This shows that $\mathbf{x}_2^{-1} \circ \pi \circ \mathbf{x}_1$ is differentiable at p. (done) ■

Question 32

3. Show that the paraboloid $z = x^2 + y^2$ is diffeomorphic to a plane.

Proof. Let S be the paraboloid. We show

$$S \text{ is diffeomorphic to } \{(x, y, 0) : x, y \in \mathbb{R}\}$$

Define

$$\pi : S \to \{(x, y, 0) : x, y \in \mathbb{R}\} \text{ by } \pi(x, y, z) \triangleq (x, y, 0)$$

We wish to show

$$\pi \text{ and } \pi^{-1} \text{ is differentiable everywhere in the sense of manifold}$$

Define global parametrizaitons $\mathbf{x}_1$ of S and global parametrization $\mathbf{x}_2$ of $\{(x, y, 0) : x, y \in \mathbb{R}\}$ by

(a) $\mathbf{x}_1 : \mathbb{R}^2 \to S$ and $\mathbf{x}_1(x, y) \triangleq (x, y, x^2 + y^2)$

(b) $\mathbf{x}_2 : \mathbb{R}^2 \to \{(x, y, 0) : x, y \in \mathbb{R}\}$ and $\mathbf{x}_2 \triangleq \mathbf{id}_{\mathbb{R}^2}$

We now reword the problem into proving

$$\mathbf{x}_2^{-1} \circ \pi \circ \mathbf{x}_1 : \mathbb{R}^2 \to \mathbb{R}^2 \text{ and } \mathbf{x}_1^{-1} \circ \pi^{-1} \circ \mathbf{x}_2 : \mathbb{R}^2 \to \mathbb{R}^2 \text{ both are differentiable}$$

Because $\pi, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_2^{-1}$ are clearly differentiable, we only have to prove

$$\pi^{-1} : \{(x, y, 0) : x, y \in \mathbb{R}\} \to \mathbb{R}^3 \text{ is differentiable and } \mathbf{x}_1^{-1} : \mathbb{R}^3 \cap S \to \mathbb{R}^2 \text{ is differnetiable on } S$$

Observe

$$\pi^{-1}(x, y, 0) \equiv (x, y, x^2 + y^2) \text{ and } \mathbf{x}_1(x, y, z) \equiv (x, y)$$

It is now clear that π^{-1} and $\mathbf{x}_1^{-1}$ are both differentiable. (done) ■

Question 33

6. Prove that the definition of a differentiable map between surfaces does not depend on the parametrizations chosen.

Proof. Suppose we are given a map $\pi : S_1 \to S_2$ differentiable in the sense of manifold. We know for arbitrary p in S_1, there exists

(a) $\mathbf{x}_1 : U_1 \to S_1$ (a local parametrization around p)

(b) $\mathbf{x}_2 : U_2 \to S_2$ (a local parametrization around $\pi(p)$)

such that

$$\left(\mathbf{x}_2^{-1} \circ \pi \circ \mathbf{x}_1\right) : U_1 \subseteq \mathbb{R}^2 \to U_2 \subseteq \mathbb{R}^2 \text{ is a diffeomorphism} \tag{2.8}$$

Now, fix two arbitrary

(a) $\mathbf{x}_1' : U_1' \to S_1$ (a local parametrization around p)

(b) $\mathbf{x}_2' : U_2' \to S_2$ (a local parametrization around $\pi(p)$)

We are required to prove (Note that the domain of each composited function may be smaller, but this does not undermine the validity of our argument, since we only care about the differentiablity at p)

$$\left((\mathbf{x}_2')^{-1} \circ \pi \circ \mathbf{x}_1'\right) : U_1' \subseteq \mathbb{R}^2 \to U_2' \subseteq \mathbb{R}^2 \text{ is a diffeomorphism}$$

Note that

$$\begin{aligned}
(\mathbf{x}_2')^{-1} \circ \pi \circ \mathbf{x}_1' &= (\mathbf{x}_2')^{-1} \circ \left(\mathbf{x}_2 \circ \mathbf{x}_2^{-1}\right) \circ \pi \circ \left(\mathbf{x}_1 \circ \mathbf{x}_1^{-1}\right) \circ \mathbf{x}_1' \\
&= (\mathbf{x}_2')^{-1} \circ \mathbf{x}_2 \circ \mathbf{x}_2^{-1} \circ \pi \circ \mathbf{x}_1 \circ \mathbf{x}_1^{-1} \circ \mathbf{x}_1' \\
&= h_2 \circ \mathbf{x}_2^{-1} \circ \pi \circ \mathbf{x}_1 \circ h_1
\end{aligned}$$

where

$$\begin{cases} h_1 \equiv \mathbf{x}_1^{-1} \circ \mathbf{x}_1' : U_1' \to U_1 \\ h_2 \equiv (\mathbf{x}_2')^{-1} \circ \mathbf{x}_2 : U_2 \to U_2' \end{cases} \quad \text{are changes of coordinate}$$

Now, because changes of coordinates are diffeomorhphism, and $\mathbf{x}_2^{-1} \circ \pi \circ \mathbf{x}_1$ is diffeomorphism by Equation 2.8 (definition), we see that

$$(\mathbf{x}_2')^{-1} \circ \pi \circ \mathbf{x}_1' = h_2 \circ \left(\mathbf{x}_2^{-1} \circ \pi \circ \mathbf{x}_1\right) \circ h_1 \text{ is a diffeomorphism at } (\mathbf{x}_1')^{-1}(p)$$

This then conclude the proof, since p is arbitrary picked from S_1. (done) ∎

Definition 2.3.3. (Definition of Differentiable function on a regular curve) Given two regular curve C_1, C_2, we say the function $f : C_1 \to C_2$ is differentiable at p if for all local parametrizations $\mathbf{x}_1 : I \to C_1 \ni p, \mathbf{x}_2 : I \to C_2 \ni f(p)$, we have

$$\mathbf{x}_2^{-1} \circ f \circ \mathbf{x}_1 \text{ is differentaible as a real to real function}$$

Question 34

9. a. **Define the notion of differentiable function on a regular curve. What does one need to prove for the definition to make sense? Do not prove it now. If you have not omitted the proofs in this section, you will be asked to do it in Exercise 15.**

b. **Show that the map $E\colon R \to S^1 = \{(x, y) \in R^2;\, x^2 + y^2 = 1\}$ given by**

$$E(t) = (\cos t, \sin t), \qquad t \in R,$$

is differentiable (geometrically, E "wraps" R around S^1).

Proof. Fix $t_0 \in \mathbb{R}$. We wish to prove

$$E \text{ is differentaible at } t_0 \text{ in the sense of manifold}$$

Locally parametize t_0 and $E(t_0)$ by

$$\mathbf{x}_1(t) \triangleq t \text{ and } \mathbf{x}_2(t) \triangleq (\cos t, \sin t)$$

Now, see that

$$\mathbf{x}_2^{-1} \circ E \circ \mathbf{x}_1(t) = t \text{ is clearly differentiable} \quad \text{(done)}$$

∎

Question 35

14. Let $A \subset S$ be a subset of a regular surface S. Prove that A is itself a regular surface if and only if A is open in S; that is, $A = U \cap S$, where U is an open set in R^3.

Proof. $(\longleftarrow)$

Fix $a \in A$. Because $a \in S$, we know there exists a local parametrization $\mathbf{x}_1 : E \to V \cap S$ around a where V is open in . Suppose

$$E' \triangleq \mathbf{x}_1^{-1}[U \cap V \cap S]$$

Now, from

(a) $U \cap V \cap S$ is open in $V \cap S$ $\qquad (\because U$ is open in $\mathbb{R}^3)$

(b) $\mathbf{x}_1$ is a homeomorphism between E and $V \cap S$

(c) $\mathbf{x}_1$ is smooth

(d) $d\mathbf{x}_1$ is one-to-one for all $p \in E$

(e) $E' \subseteq E$

We see

(a) E' is open in $\mathbb{R}^2$

(b) the restriction $\mathbf{x}_1|_{E'}$ is a local parametrizaiton around a contained by A

Because a is arbitrary, this established that A is a regular surface.

$(\longrightarrow)$

Suppose A is a regular surface. Fix arbitrary $a \in A$. Using Proposition 2.2.3 in Do Carmo, WOLG, we can suppose there exists a chart $\mathbf{x} : U \to \overline{V} \cap S$ around a where $\overline{V}$ is open in $\mathbb{R}^3$ such that

$$\mathbf{x}(x, y) = (x, y, f(x, y)) \text{ for some smooth } f$$

Because each curve in A lies in S and A is itself a regular surface, we see that

$$T_a(A) = T_a(S)$$

This tell us the restriction

$$\mathbf{x}|_A \text{ is one-to-one}$$

Note that $\mathbf{x}$ is smooth. Now by inverse function theorem, it follows that there exists V open in $\mathbb{R}^3$ such that

$$\mathbf{x}|_{A \cap V} \text{ is a chart around } a$$

Now, define $W \triangleq V \cap \overline{V}$. Now, identify

$$\mathbf{x}|_{A\cap W} \text{ is a chart in both } A \text{ and } S$$

We now see

$$A \cap W \text{ is an open neighborhood around } a \text{ in topology of } S$$

This shows that $a \in A^\circ$ in topology of S. Because a is arbitrary, it follows that A is open in S. ∎

Question 36

***16.** Let $R^2 = \{(x, y, z) \in R^3; z = -1\}$ be identified with the complex plane $\mathbb{C}$ by setting $(x, y, -1) = x + iy = \zeta \in \mathbb{C}$. Let $P\colon \mathbb{C} \to \mathbb{C}$ be the complex polynomial

$$P(\zeta) = a_0\zeta^n + a_1\zeta^{n-1} + \cdots + a_n, \quad a_0 \neq 0, a_i \in \mathbb{C}, i = 0, \ldots, n.$$

Denote by π_N the stereographic projection of $S^2 = \{(x, y, z) \in R^3; x^2 + y^2 + z^2 = 1\}$ from the north pole $N = (0, 0, 1)$ onto R^2. Prove that the map $F\colon S^2 \to S^2$ given by

$$\begin{aligned} &F(p) = \pi_N^{-1} \circ P \circ \pi_N(p), \quad \text{if } p \in S^2 - \{N\}, \\ &F(N) = N \end{aligned}$$

is differentiable.

Proof. Note that

$$\pi_N(x,y,z) = \left(\frac{2x}{1-z}, \frac{2y}{1-z}\right) \text{ and } \pi_N^{-1}(u,v) = \left(\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{u^2+v^2-4}{u^2+v^2+4}\right)$$

and that

$$P(x,y,-1) = \Big(Q(x,y), T(x,y), -1\Big) \text{ for some } Q, T \in \mathbb{R}[x,y]$$

It is then now clear that

$$F \text{ is differentiable on } S^2 \setminus N$$

Note that

$$\pi_S(x,y,z) = (\frac{2x}{1+z}, \frac{2y}{1+z}) \text{ and } \pi_S^{-1}(u,v) = (\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{-u^2-v^2+4}{u^2+v^2+4})$$

Note that

$$\pi_S^{-1} \circ \pi_S \circ F \circ \pi_S^{-1} \circ \pi_S \text{ coincide with } F \text{ on } S^2 \setminus S$$

Then, we can reduce proving F is differentiable at N into

$$\text{proving } \pi_S^{-1} \circ \pi_S \circ F \circ \pi_S^{-1} \circ \pi_S \text{ is differentiable at } N$$

Compute

$$\begin{aligned}\pi_N \circ \pi_S^{-1}(u,v) &= \pi_N(\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{-u^2-v^2+4}{u^2+v^2+4}) \\ &= (\frac{4u}{u^2+v^2}, \frac{4v}{u^2+v^2})\end{aligned}$$

Compute

$$\pi_S \circ \pi_N^{-1}(u,v) = (\frac{4u}{u^2+v^2}, \frac{4v}{u^2+v^2})$$

Note that if we identify $\mathbb{R}^2 \setminus 0$ by $\mathbb{C}^*$, we see

$$\pi_S \circ \pi_N^{-1}(z) = \pi_N \circ \pi_S^{-1}(z) = \frac{4z}{|z^2|} = \frac{4z}{z\overline{z}} = \frac{4}{\overline{z}} \qquad (z \in \mathbb{C}^*)$$

Fix $z \in \mathbb{C}^*$. Compute

$$\begin{aligned}\pi_S \circ F \circ \pi_S^{-1}(z) &= \pi_S \circ \pi_N^{-1} \circ P \circ \pi_N \circ \pi_S^{-1}(z) \\ &= \pi_S \circ \pi_N^{-1} \circ P(\frac{4}{\overline{z}}) \\ &= \pi_S \circ \pi_N^{-1}\Big(a_0(\frac{4}{\overline{z}})^n + \cdots + a_n\Big) \\ &= \frac{4}{\overline{a_0(\frac{4}{\overline{z}})^n + \cdots + a_n}} \\ &= \frac{4}{\overline{a_0}(\frac{4}{z})^n + \cdots + \overline{a_n}} \\ &= \frac{4z^n}{\overline{a_n}z^n + \cdots + \overline{a_0}4^n}\end{aligned}$$

Compute

$$\pi_S \circ F \circ \pi_S^{-1}(0) = \pi_S \circ F(N) = \pi_S(N) = 0$$

Then because $a_0 \neq 0$. It is now clear that

$$\pi_S \circ F \circ \pi_S^{-1}(z) = \frac{4z^n}{\overline{a_n}z^n + \cdots + \overline{a_0}4^n} \qquad (z \in \mathbb{C})$$

which is differentiable at 0. Now, we have

(a) $\pi_S : S^2 \setminus S \to \mathbb{C}$ is differentiable on $S^2 \setminus S$

(b) $\pi_S \circ F \circ \pi_S^{-1} : \mathbb{C} \to \mathbb{C}$ is differnetiable on $\mathbb{C}$

(c) $\pi_S^{-1} : \mathbb{C} \to S^2 \setminus S$ is differntiable on $\mathbb{C}$

We can deduce

$$\pi_S^{-1} \circ \left(\pi_S \circ F \circ \pi_S^{-1}\right) \circ \pi_S \text{ is differentiable on } S^2 \setminus S \text{ (done)}$$

Note that we used Theorem 2.3.4 in our proof. ■

Theorem 2.3.4. (Composition of Differentiable functions is differentiable) Given three regular surfaces $\{S_1, S_2, S_3\}$, two differentiable functions $f_1 : S_1 \to S_2$ and $f_2 : S_2 \to S_3$, we see

$$f_2 \circ f_1 \text{ is differentiable on } S_1$$

Proof. Fix $p_1 \in S_1$. We wish to prove

$$f_2 \circ f_1 \text{ is differentiable at } p_1$$

Set

(a) $p_2 \triangleq f_1(p_1)$

(b) $p_3 \triangleq f_2(p_2)$

Let

(a) $\mathbf{x}_1 : U_1 \to V_1 \cap S_1 \ni p_1$ be a local parametrzaiton

(b) $\mathbf{x}_2 : U_2 \to V_2 \cap S_2 \ni p_2$ be a local parametrizaiton

(c) $\mathbf{x}_3 : U_3 \to V_3 \cap S_3 \ni p_3$ be a local parametrizaiton

We wish to prove

$$\mathbf{x}_3^{-1} \circ f_2 \circ f_1 \circ \mathbf{x}_1 \text{ is differentiable at } p_1$$

Observe that

$$\begin{aligned}
\mathbf{x}_3^{-1} \circ f_2 \circ f_1 \circ \mathbf{x}_1 &= \mathbf{x}_3^{-1} \circ f_2 \circ \mathbf{x}_2 \circ \mathbf{x}_2^{-1} \circ f_1 \circ \mathbf{x}_1 \\
&= \left(\mathbf{x}_3^{-1} \circ f_2 \circ \mathbf{x}_2\right) \circ \left(\mathbf{x}_2^{-1} \circ f_1 \circ \mathbf{x}_1\right) \text{ is differentiable } \ \text{(done)}
\end{aligned}$$

■

Chapter 3

Surface

3.1 Prerequisite

Theorem 3.1.1. (Inverse function Theorem) Given a map f from E open in $\mathbb{R}^n$ to $\mathbb{R}^n$, if

(a) f is continuously differentiable on E

(b) df_a is one-to-one

(c) $a \in E$

Then there exists

(a) $U \subseteq E$ open in E where $a \in U$

(b) $V \subseteq \mathbb{R}^n$ open in $\mathbb{R}^n$

Such that

$$f|_U : U \to V \text{ is a diffeomorphism}$$

Theorem 3.1.2. (Implicit function Theorem) Given a map f from an open neighborhood around $(a, b) \in E \subseteq \mathbb{R}^{n+m} \to \mathbb{R}^n$ such that

(a) f is continuously differentiable on E

(b) the linear transformation $df_{(a,b)}|_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one

(c) $f(a, b) = 0$

Then there exists open neighborhood U around $(a, b) \in U \subseteq \mathbb{R}^{n+m}$ and open neighborhood V around $b \in V \subseteq \mathbb{R}^m$ such that we can uniquely define a function $g : V \to U$ by

$$f(g(y), y) = 0 \text{ for all } y \in V$$

and g is continuously differentiable with $dg_b = -(df_{(a,b)}|_{\mathbb{R}^n})^{-1} \circ df_{(a,b)}|_{\mathbb{R}^m}$

3.2 Equivalent Definition of Regular Surface

Definition 3.2.1. (Definition of Regular Surface: Local Parametrization) We say a set $S \subseteq \mathbb{R}^3$ is a regular surface if for all $p \in S$ there exists

(a) an open neighborhood $p \in V \subseteq \mathbb{R}^3$

(b) an open set $U \subseteq \mathbb{R}^2$

(c) a function $\mathbf{x} : U \to V \cap S$

such that $\mathbf{x}$ satisfy

(a) $\mathbf{x}$ is smooth

(b) $\mathbf{x}$ is a homemomorphism between U and $V \cap S$

(c) $d\mathbf{x}_q \in L(\mathbb{R}^2, \mathbb{R}^3)$ is one-to-one for all $q \in U$

Definition 3.2.2. (Definition of Regular Surface: Implicit function) We say a set $S \subseteq \mathbb{R}^3$ is a regular surface if for all $p \in S$ there exists

(a) an open neighborhood $p \in V \subseteq \mathbb{R}^3$

(b) a function $F : V \to \mathbb{R}$

such that F satisfy

(a) dF_q is onto for all $q \in V \cap S$

(b) F is smooth on V

(c) $\exists c_0 \in \mathbb{R}, V \cap S = \Big\{(x, y, z) \in V : F(x, y, z) = c_0\Big\}$

We now verify the equivalency between the two definitions.

Theorem 3.2.3. (Implicit function definition $\longrightarrow$ Local parametrization definition)

Proof. Fix $p \in S$. We are given an open neighborhood $p \in V \subseteq \mathbb{R}^3$ and a function $F : V \to \mathbb{R}$ such that, WOLG,

(a) dF_q is onto for all $q \in V \cap S$

(b) F is smooth on V

(c) $V \cap S = \Big\{(x, y, z) \in V : F(x, y, z) = 0\Big\}$

We wish to find

$$\text{a local parametrization } \mathbf{x} \text{ around } p$$

Define $f : V \to \mathbb{R}^3$ by

$$f(x,y,z) = (x,y,F(x,y,z))$$

Because $dF_p \neq 0$, WOLG, we can suppose $\partial_z F(p) \neq 0$. Now, see

$$\det(df) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \partial_x F & \partial_y F & \partial_z F \end{vmatrix} \neq 0$$

Now, note that

(a) f is smooth on V $\quad (\because\ F$ is smooth on $V\ )$

(b) df_p is one-to-one $\quad (\because \partial_z F(p) \neq 0)$

Then, by inverse function Theorem, we know

$$f\big|_{V'} \to U' \text{ is a local diffeomorphism around } V' \ni p \text{ and around } U' \ni f(p)$$

Let $U \subseteq \{(x,y,0) : (x,y) \in \mathbb{R}^2\}$ be an open-neighborhood around $f(p)$ contained by U'. We claim

$$\mathbf{x} \triangleq f^{-1}\big|_U \text{ suffices}$$

Note that $\mathbf{x}$ is well-defined since $U \subseteq U'$ and f is a bijective between V' and U'.

Also, note that $\mathbf{x}$ do maps points in U to points in $V \cap S$, since $V \cap S = \Big\{(x,y,z) \in V : F(x,y,z) = 0\Big\}$.

Suppose that

$$\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} \end{bmatrix} \triangleq df_\alpha^{-1} = \big(df_{f^{-1}(\alpha)}\big)^{-1} \text{ for all } \alpha \in U$$

We clearly have

$$d\mathbf{x} = \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \\ \alpha_{3,1} & \alpha_{3,2} \end{bmatrix}$$

Now, one can use the premise

$$dF_q \text{ is onto for all } q \in V \cap S$$

to check $\mathbf{x}$ do satisfy the regular condition. (Compute $\det(df_\alpha^{-1})$ using co-factor formula on the third column) (done) ■

Definition 3.2.4. (Definition of Regular surface: Monge Patches) We say a set $S \subseteq \mathbb{R}^3$ is a regular surface if for all $p \in S$ there exists some open neighborhood $p \in V \subseteq \mathbb{R}^3$ such that

> $V \cap S$ can be expressed as the graph of some smooth function $f : O \subseteq \mathbb{R}^2 \to \mathbb{R}$ in the sense that one of the followings hold

(a) $V \cap S = \{(x, y, f) : (x, y) \in O\}$ for some smooth $f : O \subseteq \mathbb{R}^2 \to \mathbb{R}$

(b) $V \cap S = \{(x, f, z) : (x, z) \in O\}$ for some smooth $f : O \subseteq \mathbb{R}^2 \to \mathbb{R}$

(c) $V \cap S = \{(f, y, z) : (y, z) \in O\}$ for some smooth $f : O \subseteq \mathbb{R}^2 \to \mathbb{R}$

3.3 Change of Parameter

Given two regular surfaces S_1, S_2, an open neighborhood $V_1 \subseteq \mathbb{R}^3$ around p, and a continuous function $f : V_1 \cap S_1 \to S_2$, we say f is differentiable at $p \in V_1$ if there exists local parametrizations

(a) $\mathbf{x}_1 : U_1 \subseteq \mathbb{R}^2 \to V_1' \cap S_1$ around p

(b) $\mathbf{x}_2 : U_2 \subseteq \mathbb{R}^2 \to V_2 \cap S_2$ around $f(p)$

such that

$$\mathbf{x}_2^{-1} \circ f \circ \mathbf{x}_1 \text{ is differentaible at } \mathbf{x}_1^{-1}(p)$$

Note the our definition of differentiablity of function between regular surface is well-defined, in the sense that if f differentiable, all local parametrizations suffice to check. This fact is backed by Theorem 3.3.1

Theorem 3.3.1. (Change of Parameter is a diffeomorphism) Let p be a point of a regular surface S, and let $\mathbf{x} : U_1 \to V_1 \cap S$ and $\mathbf{y} : U_2 \to V_2 \cap S$ be two local parametrization around p. Define $W \triangleq V_1 \cap V_2 \cap S$. Then

$$\text{The } \textbf{change of coordinate } h = \mathbf{x}^{-1} \circ \mathbf{y} : \mathbf{y}^{-1}(W) \to \mathbf{x}^{-1}(W) \text{ is a diffoeomorphism}$$

Proof. It is clear that h is a homeomorphism, since $\mathbf{x}, \mathbf{y}$ both are.

Note the symmetry $h^{-1} = \mathbf{y}^{-1} \circ \mathbf{x}$. WOLG, we only have to prove

$$h : \mathbf{y}^{-1}(W) \subseteq U_2 \to \mathbf{x}^{-1}(W) \subseteq U_1 \text{ is differentiable}$$

Fix $r \in \mathbf{y}^{-1}(W)$. We prove

$$h \text{ is differentiable at } r$$

Define $q \triangleq h(r) \in \mathbf{x}^{-1}(W) \subseteq \mathbb{R}^2$. Let's write $\mathbf{x}$ by

$$\mathbf{x}(u, v) = (x, y, z)$$

By definition, we know $d\mathbf{x}_q$ is one-to-one. This, WOLG, let us have

$$\begin{vmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \end{vmatrix} (q) \neq 0$$

Now, define $F : \mathbf{x}^{-1}(W) \times \mathbb{R} \to \mathbb{R}^3$ by

$$F(u, v, t) \triangleq \Big(x(u, v), y(u, v), z(u, v) + t \Big) \tag{3.1}$$

Compute

$$\det(dF_{(q,0)}) = \begin{vmatrix} \partial_u x & \partial_v x & 0 \\ \partial_u y & \partial_v y & 0 \\ \partial_u z & \partial_v z & 1 \end{vmatrix} (q,0) \neq 0$$

Then by Inverse function Theorem, we see that there exists an open neighborhood $M \subseteq \mathbb{R}^3$ around $F(q,0)$ such that F^{-1} exists and is differentiable on M.

Now, from Equation 3.1, note that

$$F(u,v,0) = \mathbf{x}(u,v)$$

Recall the definition h, we now have

$$h \equiv \mathbf{x}^{-1} \circ \mathbf{y} = F^{-1} \circ \mathbf{y}$$

Then because

(a) $\mathbf{y}$ is differentiable at r

(b) F^{-1} is differentiable at $F(q,0) = \mathbf{x}(q) = \mathbf{y}(r)$

We see h is indeed differentiable at r (done) ■

Theorem 3.3.2. (Composition of Differentiable functions is differentiable) Given three regular surfaces $\{S_1, S_2, S_3\}$, two differentiable functions $f_1 : S_1 \to S_2$ and $f_2 : S_2 \to S_3$, we see

$$f_2 \circ f_1 \text{ is differentiable on } S_1$$

Proof. Fix $p_1 \in S_1$. We wish to prove

$$f_2 \circ f_1 \text{ is differentiable at } p_1$$

Set

(a) $p_2 \triangleq f_1(p_1)$

(b) $p_3 \triangleq f_2(p_2)$

Let

(a) $\mathbf{x}_1 : U_1 \to V_1 \cap S_1 \ni p_1$ be a local parametrzaiton

(b) $\mathbf{x}_2 : U_2 \to V_2 \cap S_2 \ni p_2$ be a local parametrizaiton

(c) $\mathbf{x}_3 : U_3 \to V_3 \cap S_3 \ni p_3$ be a local parametrizaiton

We wish to prove

$$\mathbf{x}_3^{-1} \circ f_2 \circ f_1 \circ \mathbf{x}_1 \text{ is differentiable at } p_1$$

Observe that

$$\begin{aligned}\mathbf{x}_3^{-1} \circ f_2 \circ f_1 \circ \mathbf{x}_1 &= \mathbf{x}_3^{-1} \circ f_2 \circ \mathbf{x}_2 \circ \mathbf{x}_2^{-1} \circ f_1 \circ \mathbf{x}_1 \\ &= \left(\mathbf{x}_3^{-1} \circ f_2 \circ \mathbf{x}_2\right) \circ \left(\mathbf{x}_2^{-1} \circ f_1 \circ \mathbf{x}_1\right) \text{ is differentiable } \quad \text{(done)}\end{aligned}$$

$\blacksquare$

3.4 Equivalent Definition of Tangent Plane

Definition 3.4.1. (Definition of Tangent Plane: Space of Tangent Vectors) Given a regular surface S and $p \in S$, we define **tangent plane** T_pS **to** S **at** p by

$$T_pS = \{\alpha'(0) \in \mathbb{R}^3 | \, \alpha : I \to S \text{ is a smooth curve passing through } p \text{ at } \alpha(0) \ \}$$

Definition 3.4.2. (Definition of Tangent Plane: Local Parametrization) Given a regular surface S, that $p \in S$ and a local parametrzaiton $\mathbf{x} : (u, v) \mapsto (x, y, z)$ around p such that $\mathbf{x}(q) = p$, we define **tangent plane** T_pS **to** S **at** p by

$$T_pS = \text{span}(\partial_u \mathbf{x}(q), \partial_v \mathbf{x}(q))$$

3.5 Questions Regarding Manifold

Definition 3.5.1. (Definition of Topological Manifold? 1) Given a topological space (M, τ_M), we say M is a **n-manifold** if for each p in M there exists N_p such that

(a) $p \in N_p \subseteq M$

(b) N_p contain some U open in M

(c) N_p is homeomorphic to some E open in $\mathbb{R}^n$

Definition 3.5.2. (Definition of Topological Manifold? 2) Given a topological space (M, τ_M), we say M is a **n-manifold** if for each p in M there exists N_p such that

(a) $p \in N_p \subseteq M$

(b) N_p is open in M

(c) N_p is homeomorphic to some E open in $\mathbb{R}^n$

Definition 3.5.3. (Definition of Topological Manifold? 3) Given a topological space (M, τ_M), we say M is a **n-manifold** if for each p in M there exists N_p such that

(a) $p \in N_p \subseteq M$

(b) N_p is open in M

(c) N_p is homeomorphic to $\mathbb{R}^n$

Definition 3.5.4. (Definition of Topological Manifold? 4) Given a topological space (M, τ_M), we say M is a **n-manifold** if for each p in M there exists N_p such that

(a) $p \in N_p \subseteq M$

(b) N_p contain some U open in M

(c) N_p is homeomorphic to $\mathbb{R}^n$

Question: are the above 4 definitions all equivalent?

Definition 3.5.5. (Definition of Differentiable Manifold?) Given a 2-manifold M, we say M is a **differentiable manifold** if for each two chart $\mathbf{x}_1 : U_1 \to V_1 \subseteq M, \mathbf{x}_2 : U_2 \to V_2 \subseteq M$ such that

$$\mathbf{x}_1(U_1) \cap \mathbf{x}_2(U_2) \neq \varnothing$$

we have the statement

$$f \begin{cases} : \mathbf{x}_1^{-1}(V_1 \cap V_2) \to U_2 \\ q \mapsto \mathbf{x}_2^{-1} \circ \mathbf{x}_1(q) \end{cases} \quad \text{is differentiable}$$

$\boxed{\text{Question: In our class, we have proved that a regular surface is a differentiable manifold. Does there exists a 2-manifold that is NOT a differentiable manifold?}}$