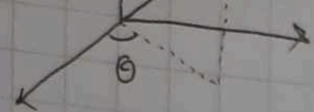


$0 \le \theta \le 2\pi$ $\quad y = \rho \sin\varphi \sin\theta$ $\quad |\partial(\rho,\varphi,\theta)| = \rho \sin\varphi$

$z = \rho \cos\varphi$

$$\iiint_T f(x,y,z)\,dx\,dy\,dz = \iiint_{S(\rho,\varphi,\theta)} \tilde{f}(\rho,\varphi,\theta)\,\rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta$$

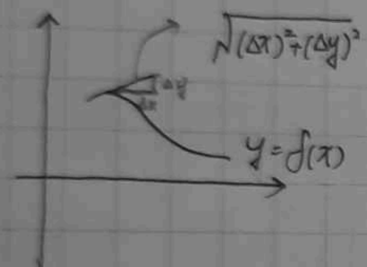
eg. Calculate the mass M of a solid ball of radius 1 with density $\lambda(x,y,z) = k(x^2+y^2+z^2)$

7/5

20分

$$\iiint_T x^2+y^2+z^2 = \int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^4 \sin\varphi\,d\rho\,d\varphi\,d\theta = \frac{2\pi}{5}\int_0^{\pi}\sin\varphi\,d\varphi = \frac{4\pi}{5}$$

阿忘記乘k了. 期中考到這(不調分)



$$\sqrt{(\Delta x)^2+(\Delta y)^2} = \sqrt{(\Delta x)^2+(f')^2(\Delta x)^2} = \sqrt{1+(f')^2}\,\Delta x$$

$$= \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

If $\gamma(t) = (x(t), y(t))$, $a \le t \le b$, $\sqrt{(\Delta x)^2+(\Delta y)^2} = \sqrt{(\frac{\Delta x}{\Delta t})(\Delta t)^2 + (\frac{\Delta y}{\Delta t})(\Delta t)^2} = \sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,\Delta t$

$$S = \int_a^b \sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,dt = \int_a^b \|\gamma'(t)\|\,dt$$

If $C = \{\gamma(t) = (x(t), y(t), z(t)),\ a \le t \le b\}$, $S(t) = \int_a^b \sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2+(\frac{dz}{dt})^2}\,dt$

$$= \int_a^b \|\gamma'(t)\|\,dt \quad , \quad \frac{ds}{dt} = \|\gamma'(t)\|$$

$$\int_C f(\gamma(t))\,ds = \int_a^b f(\gamma(t))\,\|\gamma'(t)\|\,dt$$

$\int_C f(\gamma(t))\,ds = \int_a^b f(\gamma(t))\,|\gamma'(t)|\,dt$

ex. $C=\{(x,y)\mid x=a\cos t,\ y=a\sin t,\ 0\le t\le\pi\}$. Find $\int_C x^2y\,dt$

$\int_C x^2y\,ds = \int_0^{\pi} a^2\cos^2 t\cdot a\sin t\sqrt{(-a\sin t)^2+(a\cos t)^2}\,dt$

let $u=\cos t$, $du=-\sin t$

$= \int_0^{\pi} a^4\cos^2 t\sin t\,dt$

$= -\int_1^{-1} u^2\,du = \frac{u^3}{3}\Big|_{-1}^{1} = \frac{2}{3}a^4$

ex. Find the arc length of $C: x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$

$x^2+y^2=b^2 \Rightarrow x=b\cos\theta,\ y=b\sin\theta$

let $x^{\frac{1}{3}}=a^{\frac{1}{3}}\cos\theta$, $y^{\frac{1}{3}}=a^{\frac{1}{3}}\sin\theta$, $0\le\theta\le\pi$

$L=\int_0^{2\pi}\sqrt{(-a^{\frac{1}{3}}\sin\theta)^2+(a^{\frac{1}{3}}\cos\theta)^2}\,d\theta = a^{\frac{1}{3}}\cdot 2\pi$

ex. $A(0,0,0)$, $B(1,2,3)$, $C=\overline{AB}$. Find $\int_C (x^2+y^3)\,ds$

$\gamma(t)$: $x=t$, $y=2t$, $z=3t$, $0\le t\le 1$

$\int_0^1 (t^2+(2t)^3)\sqrt{1+4+9}\,dt = \int_0^1 \sqrt{14}\,(t^2+8t^3)\,dt = \frac{7}{3}\sqrt{14}$

ex. $\int_C \dfrac{x^2\,dy - y^2\,dx}{x^{\frac{5}{3}}+y^{\frac{5}{3}}}$, $C=\{(a\cos^3 t,\ a\sin^3 t) : 0\le t\le\frac{\pi}{2}\}$

$$= \int_0^{\frac{\pi}{2}} \frac{3a^2\cos^7 t\,a\sin^3 t + 3a^2\sin^7 t\,a\cos^2 t}{a^{\frac{5}{3}}\cos^5 t + a^{\frac{5}{3}}\sin^5 t} = 3a^{\frac{4}{3}}\int_0^{\frac{\pi}{2}}\cos^2 t\sin^2 t\,dt$$

$$= \frac{3}{4}a^{\frac{4}{3}}\int_0^{\frac{\pi}{2}}\sin^2(2\theta)\,d\theta = \frac{3}{16}a^{\frac{4}{3}}\int_0^{\pi}\sin^2\theta\,d\theta = \frac{3}{16}\pi a^{\frac{4}{3}}$$

$\frac{dx}{dt} = -3a\cos^2 t\sin t$, $\frac{dy}{dt} = -3a\sin^2 t\cos t$

$L=\int_0^{2\pi}\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,dt = \int_0^{2\pi} 3a\sqrt{\cos^2 t\cdot\sin^2 t\,(\cos^2 t+\sin^2 t)}\,dt$

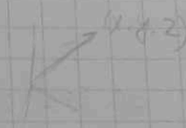
$= 3a\int_0^{2\pi}|\cos t\cdot\sin t|\,dt = 12a\int_0^{\frac{\pi}{2}}\cos t\sin t\,dt = 6a$

Def. $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ gradient $\Rightarrow \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$ f: scalar fn

$V = (V_1, V_2, V_3)$, $\nabla \cdot V = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$ divergens 這樣拼嗎?不知道

$$\nabla \times V = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}, \frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}, \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

ex. $V = (x, y, z)$. $\nabla V = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$



$$\nabla \times V = \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}, \frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}, \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = (0, 0, 0)$$

ex. $V(x, y, z) = (wy, wx, 0)$

$\nabla \cdot V = 0$, $\nabla \times V = (0, 0, 2w)$

Basic identies: If f is a scalar fn. then $(\nabla \times \nabla) f = \nabla \times \underbrace{(\nabla f)}_{\text{vector}} = \vec{0}$

pf. $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$

$$= \nabla \times (\nabla f) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \partial_x f & \partial_y f & \partial_z f \end{vmatrix} = (\partial_{yz} f - \partial_{yz} f, \partial_{xz} f - \partial_{xz} f, \partial_{xy} f - \partial_{xy} f)$$
$$= (0, 0, 0)$$

Let $V = (V_1, V_2, V_3)$ and f be a scalar fn. then

(i) $\nabla (\underbrace{\nabla \times V}_{\text{vector}}) = 0$

$$A \cdot (W \times V) = \begin{vmatrix} A_1 & A_2 & A_3 \\ W_1 & W_2 & W_3 \\ V_1 & V_2 & V_3 \end{vmatrix}$$

(ii) $\nabla (fV) = \overbrace{(\nabla f)}^{\text{vector}} \cdot V + f(\nabla \cdot V)$

(iii) $\nabla \times (fV) = (\nabla f) \times V + f(\nabla \times V)$

(i) $\nabla (\nabla \times V) = \begin{vmatrix} \partial x & \partial y & \partial z \\ \partial x & \partial y & \partial z \\ V_1 & V_2 & V_3 \end{vmatrix} = 0$

(ii) $\nabla (fV_1, fV_2, fV_3) = \partial_x (fV_1) + \partial_y (fV_2) + \partial_z (fV_3)$

$$= \partial_x f V_1 + f \partial_x V_1 + \partial_y f V_2 + f \partial_y V_2 + \partial_z f V_3 + f \partial_z V_3$$

$$= (\nabla f) \cdot V + f(\nabla \cdot V)$$

(iii) $\nabla \times (fV) = \nabla \times (fV_1, fV_2, fV_3)$

$$= \left(\frac{\partial (fV_3)}{\partial y} - \frac{\partial (fV_2)}{\partial z}, \frac{\partial (fV_1)}{\partial z} - \frac{\partial (fV_3)}{\partial x}, \frac{\partial (fV_2)}{\partial x} - \frac{\partial (fV_1)}{\partial y}\right) \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ fV_1 & fV_2 & fV_3 \end{vmatrix}$$

這題不會考

Def. The Laplacian operator

$$\Delta = \nabla^2 = \nabla \cdot \nabla \qquad \Delta f = \nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

ex. $f(x.y.z) = x^2+y^2+z^2 = 6$

Recall the work, $\Delta W = F \cdot \Delta d$, $C: \gamma(t) = (x(t).\ y(t).\ z(t))$.

$$\Delta W = W(t+h) - W(t)$$

$$= F(\gamma(t))(\gamma(t+h)-\gamma(t)) \Rightarrow \frac{W(t+h)}{h} = F(\gamma(t)) \cdot \frac{\gamma(t+h)-\gamma(t)}{h}$$

$$\overset{h\to 0}{\Rightarrow} W'(t) = F(\gamma(t)) \cdot \gamma'(t)$$

$$W = \int_a^b W'(t)\,dt = W(\gamma(b)) - W(\gamma(a)) = W(B) - W(A) = \int_a^b F(\gamma(t)) \cdot \gamma'(t)\,dt.$$

ex. Let $F(x.y.z) = (xy.\ 2.\ 4z)$, $C: \gamma(t) = (\cos t.\ \sin t.\ t)$. $0 \le t \le 2\pi$

$$W = \int_0^{2\pi} F(\gamma(t)) \cdot \gamma'(t)\,dt = \int_0^{2\pi} -xy\sin t + 2\cos t + 4t\,dt$$

$$= \int_0^{2\pi} -\cos t \sin^2 t + 2\cos t + 4t\,dt = 8\pi^2$$

Def. (Line Integral). Let $h(x.y.z) = (h_1.\ h_2\ h_3)$

$C: \gamma(u) = (x(u),\ y(u).\ z(u))$. $u \in [a.b]$

The line integral of h over C is $\oint_C h(\gamma) \cdot d\gamma = \int_a^b h(\gamma(u)) \cdot \gamma'(u)\,du$

Thm. $\int_C h(\gamma)\,d\gamma = \int_a^b h(\gamma(u))\gamma'(u)\,du$. is invariant of change of variables

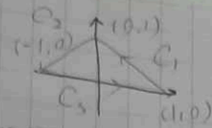
Pf. Let $\varphi: [c.d] \to [a.b]$ onto s.t $\varphi(c) = a$. $\varphi(d) = b$

$C: \gamma(u)$. $a \le u \le b$, $C: \gamma(\varphi(t))$ $c \le t \le d$

$$\int_a^b h(\gamma(u))\gamma'(u)\,du = \int_c^d h(\gamma(\varphi(t))\varphi'(t)\,dt. \qquad u = \varphi(t),\ \frac{du}{dt} = \varphi'(t) \Rightarrow du = \varphi'(t)\,dt$$

$$= \int_c^d h(R(t))R'(t)\,dt = \int_c^d h(R(t))\gamma'(\varphi(t))\varphi'(t)\,dt = \int_a^b h(\gamma(u))\gamma'(u)\,du$$

ex. $h(x,y) = (e^{y}, -\sin\pi x)$



, Find $\int_C h(r)\cdot dr$

$\int_{C_1} h(r)\, dr = \int_0^1 (e^{t}, -\sin\pi(1-t))\cdot(-1,1)\, dt$

$= \int_0^1 -e^{t} - \sin\pi(1-t)\, dt = 1-e-\frac{2}{\pi}$ $\quad C_1: \gamma\ [0,1] \xrightarrow{\gamma} (1-t, t)$

$\int_{C_2} h(r)\, dr = \int_0^1 (e^{1-t}, -\sin\pi(-t))\cdot(-1,-1)\, dt$

$= \int_0^1 -e^{1-t} + \sin(-\pi t)\, dt = 1-e-\frac{2}{\pi}$ $\quad C_2: \gamma\ [0,1] \longrightarrow (-t, 1-t)$

$\int_{C_3} h(r)\, dr = \int_0^1 (1, -\sin\pi(-1+2t))\cdot(2,0)\, dt$ $\quad C_3: \gamma\ [0,1] \to (-1+2t, 0)$

$= \int_0^1 2\, dt = 2$

$\int_C h(r)\cdot dr = \int_{C_1} h(r)\cdot dr + \int_{C_2} h(r)\cdot dr + \int_{C_3} h(r)\cdot dr = 4-2e-\frac{4}{\pi}$

ex. $A = (xy, yz, xz)$, $C = \{ x=t,\ y=t^2,\ z=t^3,\ 0\le t\le 1\}$, Find $\int_C A\cdot dr$

$\int_C A\cdot dr = \int_0^1 (t^3, t^5, t^4)(1, 2t, 3t^2)\, dt$

$= \int_0^1 t^3 + 2t^6 + 3t^6\, dt = \frac{1}{4} + \frac{2}{7} + \frac{3}{7} = \frac{27}{28}$

ex. $\xrightarrow{F} \boxed{m}$, $C: \gamma(t): [t_1, t_2] \longrightarrow \mathbb{R}^3$, Find $\int_C F\cdot dr$

$\int_C F\cdot dr = \int_{t_1}^{t_2} F(\gamma(t))\cdot\gamma'(t)\, dt$ $\quad v(t) = \gamma'(t),\ a(t) = v'(t) = \gamma''(t)$

$= \int_{t_1}^{t_2} m\, a(t)\cdot v(t)\, dt$

$= m\int_{t_1}^{t_2} v'(t)\cdot v(t)\, dt$ $\quad |v(t)|^2\ \frac{d}{dt}(u_1^2(t) + u_2^2(t))$

$= m\int_{t_1}^{t_2} \frac{1}{2}\frac{d}{dt}|v(t)|^2$ $\quad = 2u_1(t)u_1'(t) + 2u_2(t)u_2'(t)$

$\quad = 2(u_1(t), u_2(t))\cdot(u_1, u_2)'$

$= \frac{m}{2}|v(t)|^2 \Big|_{t_1}^{t_2} = \frac{m}{2}v^2(t_2) - \frac{m}{2}v^2(t_1)$ $\quad = 2U(t)\cdot U'(t)$

ex. $A = (y^2+z^2, z^2+x^2, x^2+y^2)$, $C_1: \overline{ab}$, $a(0,0,0)$, $b(1,1,1)$, $C_2: \gamma(t) = (t, t^2, t^3)$, $0\le t\le 1$

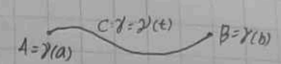
(1) $\int_{C_1} A\, dr = \int_0^1 (2t^2, 2t^2, 2t^2)(1,1,1)\, dt = \int_0 6t^2\, dt = 2$

(2) $\int_{C_2} A\, dr = \int_0^1 (t^4+t^6, t^2+t^6, t^2+t^4)(1, 2t, 3t^2)\, dt$

$= \int_0^1 t^4 + t^6 + 2t^3 + 2t^7 + 3t^4 + 3t^6\, dt = \frac{297}{140}$

Thm (Fundamental thm of line integral) : Let $C: \gamma = \gamma(t),\ t \in [a.b]$ be a piece-wise smooth curve that begins at $A = \gamma(a)$ and ends at $\gamma(b) = B$

If $f \in C'(\Omega)$, $\Omega > C$, then $\int_C \nabla f(\gamma) \cdot d\gamma = f(B) - f(A)$



pf. If C is smooth $\int_C \nabla f(\gamma) d\gamma = \int_a^b \nabla f(\gamma(t)) \cdot \gamma'(t)\, dt = \int_a^b \frac{d}{dt}(f(\gamma(t)))dt$

$= f(\gamma(b)) - f(\gamma(a)) = f(B) - f(A)$

If $C = C_1 \cup C_2 \cdots \cup C_n$, $\int_C \nabla f(\gamma) d\gamma = \sum_{i=1}^{n} \int_{C_i} \nabla f(\gamma) d\gamma = [f(\gamma(a_1)) - f(\gamma(a_0))] + [f(\gamma(a_2)) - f(\gamma(a_1))]$

$\cdots + [f(\gamma(a_n)) - f(\gamma(a_{n-1}))] = f(\gamma(a_n)) - f(\gamma(a_0)) = f(B) - f(A)$

Remark : If the curve is closed (ie. $B = A$), then $\oint_C \nabla f(\gamma) d\gamma = 0$

ex. Let $F(x.y.z) = -k \frac{\ell}{|\ell|^3} = -k \frac{(x.y.z)}{(x^2+y^2+z^2)^{\frac{3}{2}}}$, $\ell = (x.y.z)$, Find $\int_C F(\gamma) \cdot d\gamma$.

where C is any curve from $(0.3.0)$ to $(4.3.0)$

$\int_C \nabla f(\gamma) \cdot d\gamma = f(4.3.0) - f(0.3.0)$, $\frac{\partial}{\partial x}\left(\frac{1}{|\ell|}\right) = \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}} \cdot 2x = -\frac{x}{|\ell|^3}$

$= \frac{k}{|\ell(4.3.0)|} - \frac{k}{|\ell(0.3.0)|} = k\left(\frac{1}{5} - \frac{1}{3}\right) = -\frac{2k}{15}$ $\qquad \nabla\left(\frac{1}{|\ell|}\right) = -\frac{(x.y.z)}{|\ell|^3} = -\frac{\ell}{|\ell|^3}$

Def. A force field F that is a gradient field $F = \nabla f$ is called a conservative field.

Thm. (Independent of path Thm) : Let $F \in C(\Omega)$. Then $\int_C F(\gamma) \cdot d\gamma$ is independent of path iff $F(\gamma) = \nabla f(\gamma)$ for some scalar fn f, ie. F is a conservative field.

pf. $(\Rightarrow)$ Def $f(0) = 0$, $f(x) = \int_C F(\gamma) \cdot d\gamma$, $C\ (0.0.0) \to X$

$\frac{\partial}{\partial x} f(x) = \lim_{h \to 0} \frac{f(x + h(1.0.0)) - f(x)}{h} = \lim_{h \to 0} \frac{\int_C F(\gamma) \cdot d\gamma}{h} = \lim_{h \to 0} \frac{F(x) \cdot h(1.0.0)}{h} = F_1(x)$

The conservation of Mechanical energy. Suppose $F = -\nabla U$

Then $\frac{1}{2} m |\gamma'(t)|^2 + U(\gamma(t)) = C$

pf $\frac{d}{dt}\left(\frac{1}{2} m |\gamma'(t)|^2\right) = m\gamma''(t) \cdot \gamma'(t) = ma \cdot \gamma'(t) = F \cdot \gamma'(t)$

$\frac{d}{dt}(U(\gamma(t))) = \nabla U(\gamma(t)) \cdot \gamma'(t) = -F \cdot \gamma'(t)$

$\frac{d}{dt}\left(\frac{1}{2} m |\gamma'(t)|^2 + U(\gamma(t))\right) = 0$

Consider $h(x,y,z) = (P,Q,R)$ $\quad C: \gamma(t) = (x(t), y(t), z(t))$, $t \in [a,b]$

Then $\int_C h(\gamma)\cdot d\gamma = \int_a^b (P,Q,R)\cdot(x'(t), y'(t), z'(t))\,dt =$

$= \int_a^b P x'(t)\,dt + \int_a^b Q y'(t)\,dt + \int_a^b R z'(t)\,dt$

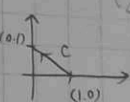
$\int_C P\,dx = \int_a^b P x'(t)\,dt$

$\int_C Q\,dy = \int_a^b Q y'(t)\,dt$

$\int_C R\,dz = \int_a^b R z'(t)\,dt$

So $\int_C h(\gamma)\cdot d\gamma = \int_C P\,dx + Q\,dy + R\,dz$

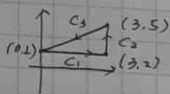
ex. $\int_C x^2 y\,dx + xy\,dy$, $C: \gamma(t) = (1-t, t)$. $t \in [0,1]$



$= \int_0^1 (1-t)^2\cdot t\cdot(-1) + (1-t)\cdot t\cdot 1\,dt$

$= \int_0^1 -t^3 + 2t^2 - t - t^2 + t\,dt = \int_0^1 -t^3 + t^2\,dt = -\frac{t^4}{4} + \frac{t^3}{3}\Big|_0^1 = \frac{1}{12}$

ex. $\int_C xy^2\,dx + xy^2\,dy$. $C = C_1 \cup C_2$



On C_1. $y=2$, $dy=0$. $\int_{C_1} xy^2\,dx + xy^2\,dy = \int_0^3 4x\,dx = 18$

On C_2. $x=3$. $dx=0$ $\int_{C_2} xy^2\,dx + xy^2\,dy = \int_2^5 3y^2\,dy = 117$

On C_3. $y = x+2$. $dx = dy$ $\int_{C_3} xy^2\,dx + xy^2\,dy = 2\int_0^3 x(x+2)^2\,dx = \frac{297}{2}$

Thm. Let $F = (M,N,P)$ with $M,N,P \in C'(D)$ where D is simple connected

then F is conservative ($F = \nabla f$) iff curl $F = 0$

pf. ($\Rightarrow$) $\nabla\times F = \nabla\times(\nabla f) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \partial_x f & \partial_y f & \partial_z f \end{vmatrix} = (\partial^2_{yz} f - \partial^2_{yz} f, \quad) = \vec{0}$

($\Leftarrow$) Stock's Thm. $\int_S \text{curl } F\cdot\vec{n}\,ds = \int_{\partial S} F\cdot d\gamma = 0$

$\vec{0} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = (\partial_y P - \partial_z N, \partial_z M - \partial_x P, \partial_x N - \partial_y M) \Longleftrightarrow \partial_y P = \partial_z N, \partial_z M = \partial_x P, \partial_x N = \partial_y M$

In two variables $F = (M,N)$

Then F is conservative iff $\partial_x N = \partial_y M$ $\begin{vmatrix} \partial & \partial \\ M & N \end{vmatrix}$

ex. $F = (4x^3 + 9x^2y^2, 6x^3y + 6y^5)$, Find f s.t $\nabla f = F$

$\partial_x(6x^3y + 6y^5) = 18x^2y = \partial_y(4x^3 + 9x^2y^2)$

$\partial_x f = 4x^3 + 9x^2y^2 \Rightarrow f(x,y) = x^4 + 3x^3y^2 + c(y)$

$\partial_y f = 6x^3y + c'(y) = 6x^3y + 6y^5 \Rightarrow c'(y) = 6y^5 \Rightarrow y = y^6 + C$

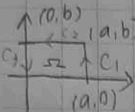
$\Rightarrow f(x,y) = x^4 + 3x^3y^2 + y^6 + C$

很有可能會考

Green's Thm. Let Ω be a simple closed region with smooth boundary

Let P and $Q \in C'(\Omega)$, then $\iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_C P\,dx + Q\,dy$

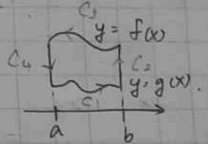
$\begin{vmatrix} \partial x & \partial y \\ P & Q \end{vmatrix}$ counter clock



pf. $\iint_\Omega \frac{\partial Q}{\partial x} dx\,dy = \int_0^b \int_0^a \frac{\partial Q}{\partial x} dx\,dy = \int_0^b Q(a,y) - Q(0,y)\,dy$

$= \int_{C_1} Q\,dy + \int_{C_3} Q\,dy = \int_C Q\,dy$

$\iint_\Omega \frac{\partial P}{\partial y} dx\,dy = \int_0^a \int_0^b \frac{\partial P}{\partial y} dy\,dx = \int_0^a P($



$\Rightarrow$ If y-simple. $\Omega = \left\{ \begin{array}{l} g(x) \le y \le f(x), \\ a \le x \le b \end{array} \right\}$

pf. $\iint_\Omega \frac{\partial P}{\partial y} dy\,dx = \int_a^b \int_{g(x)}^{f(x)} \frac{\partial P}{\partial y} dy\,dx = \int_a^b P(x, f(x)) - P(x, g(x))\,dx$

$\int_C P\,dx = \int_{C_1} P\,dx + \int_{C_2} P\,\overset{0}{dx} + \int_{C_3} P\,dx + \int_{C_4} P\,\overset{0}{dx}$

$= \int_a^b P(x, g(x))\,dx - \int_a^b P(x, f(x))\,dx = -\iint_\Omega \frac{\partial P}{\partial y} dx\,dy$

$\Rightarrow$ If x-simple

pf. $\iint_\Omega \frac{\partial Q}{\partial x} dx\,dy = \int_C Q\,dy$

Cor. Let $F = (P, Q)$, F is conservative ($\nabla f = F$) $\Leftrightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$\frac{\partial f}{\partial x} = P \qquad \frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}$

$\Rightarrow$

$\frac{\partial f}{\partial y} = Q \qquad \frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$

ex. $\int_C (3x^2 + y)\,dx + (2x + y^3)\,dy$, $C: x^2 + y^2 = a^2$. $a > 0$ $x = a\cos\theta$, $y = a\sin\theta$

$\int_C 3x^2y + \frac{y^2}{2} - x^2 + xy^2\,dx\,dy$ $\qquad dx = -a\sin\theta\,d\theta$, $dy = a\cos\theta\,d\theta$

① $= \int_0^{2\pi} (3a^2\cos^2\theta + a\sin\theta)(-a\sin\theta)\,d\theta$

$+ (2a\cos\theta + a^3\sin^3\theta)\,a\cos\theta\,d\theta$

$= a^2 \int_0^{2\pi} (-\sin^2\theta + 2\cos^2\theta)\,d\theta = a^2 \int_0^{2\pi} -1 + 3\cos^2\theta = -a^2 2\pi + 3a^2\pi = \pi a^2$

② $= \iint_\Omega \frac{\partial}{\partial x}(2x + y^3) - \frac{\partial}{\partial y}(3x^2 + y)\,dx\,dy = \iint_\Omega (2-1)\,dx\,dy = \pi a^2$

ex. Let Ω be a simple closed region Ω with piecewise smooth curve C. then $A(\Omega)$

$$A(\Omega) = \oint_C -y\,dx = \oint_C x\,dy = \frac{1}{2}\oint_C -y\,dx + x\,dy = \iint_\Omega 1\,dx\,dy$$

ex. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 、

考 $x = a\cos\theta$. $y = b\sin\theta$. $\int_0^{2\pi} a\cos\theta\, b\cos\theta\, d\theta = ab\int_0^{2\pi} \cos^2\theta\, d\theta$

ex. Let C_1 be a closed curve s.t $(0.0) \notin C_1$

Then $\oint_{C_1} \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = \begin{cases} 0 & \text{if } C_1 \text{ doesn't enclosed } (0.0) \\ 2\pi & C_1 \text{ enclosed } (0.0) \end{cases}$

考 $\frac{y}{x^2+y^2}dx + \frac{-x}{x^2+y^2}dy$

考

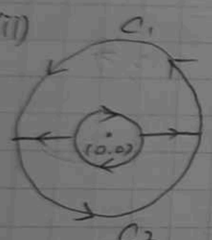
(i) $\begin{vmatrix} \partial x & \partial y \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = \iint_\Omega \partial_x\left(\frac{x}{x^2+y^2}\right) - \partial_y\left(\frac{-y}{x^2+y^2}\right) dx\,dy$

$\overset{\shortparallel}{0}$

$$\frac{-2x^2+2y^2}{(x^2+y^2)^2} = 忠合$$

考5題.
1題 conservative 線積分
1題 算面積(用 green Thm)

(ii)



$C = C_1 + C_2 - O_r$. $\int_{C_1} = \iint_{\Omega_1} = 0$, $\int_{C_2} = \iint_{\Omega_2} = 0$

$\Rightarrow \oint_C = -\int_{O_r} = \int_{O_2} \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$

$= \int_0^{2\pi} \frac{r^2}{r^2}(\sin^2\theta + \cos^2\theta)\,d\theta = 2\pi$

(下週一(11/13) 會重新猜題)

ex. $\oint_C xy^2\,dx + (x^2+y)\,dy = \iint_\Omega (2x - 2xy)\,dx\,dy$

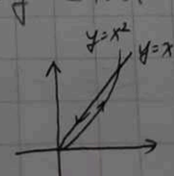
$= \int_0^1\int_0^{1-x} (2x - 2xy)\,dy\,dx = \int_0^1 2x(1-x) - x(1-x)^2\,dx = \frac{1}{4}$

考類似題目×2 (用 Def or Green Thm 算)

ex. $\oint_C (2y + \sqrt{9+x^2})\,dx + (5x + e^{\tan^{-1}y})\,dy$, $C: x^2+y^2 = a^2$

$\partial_x(5x + e^{\tan^{-1}y}) - \partial_y(2y + \sqrt{9+x^2}) = 3 \Rightarrow \iint_\Omega 3\,dx\,dy = 3\pi a^2$

(記得寫 By Green Thm)



ex. $\oint_C (2xy - x^2)\,dx + (x+y^2)\,dy$. C: bounded by $\begin{cases} y = x^2 \\ y = x \end{cases}$

$\int_0^1\int_{x^2}^{x} 1 - 2x\,dy\,dx = \int_0^1 (x^2 - x)(2x-1)\,dx$

$= \int_0 2x^3 - 3x^2 + x\,dx = \frac{2}{4} - 1 + \frac{1}{2} = 0$

ex. $C \begin{cases} x = \dfrac{3at}{1+t^3} \\ y = \dfrac{3at^2}{1+t^3} \end{cases}$ $a>0$. Ω bounded by C. Find area of Ω

$(1+t^3)x = 3at$, $1+t^3 = \dfrac{3at}{x}$, $y = \dfrac{3at^2}{\frac{3at}{x}} = xt$

$t = \dfrac{y}{x}$, $x = \dfrac{3\cdot\frac{y}{x}}{1+\frac{y^3}{x^3}} = \dfrac{3xy}{x^3+y^3}$

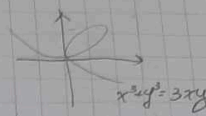
$x^3+y^3=3xy$

$$|\Omega| = \int_C x\,dy = -\int_C y\,dx = \frac{1}{2}\int_C x\,dx - y\,dy$$

$$= \int_0^\infty \frac{3at}{1+t^3}\, y'(t)\,dt = \text{他⓪去算}^2\text{看}.$$

ex. $C\begin{cases} x=\frac{3at}{1+t^3} \\ y=\frac{3at^2}{1+t^3} \end{cases}$ $a>0$. Ω bounded by C. Find area of Ω

$(1+t^3)x=3at$, $1+t^3=\frac{3at}{x}$, $y=\frac{3at^2}{\frac{3at}{x}}=xt$



$t=\frac{y}{x}$, $x=\frac{3\cdot\frac{y}{x}}{1+\frac{y^3}{x^3}}=\frac{3xy}{x^3+y^3}$

$|\Omega|=\int_C x\,dy=-\int_C y\,dx=\frac{1}{2}\int_C x\,dx-y\,dy$,

$\int_C x\,dy=3a\int_0^\infty \frac{t}{1+t^3}\,y'(t)\,dt=9a^2\int_0^\infty t^2(2-t^3)(1+t^3)^{-3}\,dt$ 難

考 1.2 線積分(參數化)
3 線積分(保守量場)
4 範圍有無包含原點的題目
5 算面積

$$\left(t^2(1+t^3)^{-1}\right)'=2t(1+t^3)^{-1}-3t^2\cdot t^2(1+t^3)^{-2}$$
$$=t(1+t^3)^{-2}[2+2t^3-3t^3]$$
= 靠腰他擦掉了

$\int_C -y\,dx=-\int_0^\infty y\,x'(t)\,dt=-9a^2\int_0^\infty t^2(1-3t^3)(1+t^3)^{-3}\,dt$ 難

所以用 $\frac{1}{2}\int_C x\,dy-y\,dx=\frac{9a^2}{2}\int_0^\infty[2-t^3-1+2t^3]\,t^2(1+t^3)^{-3}\,dt$

$=\frac{9a^2}{2}\int_0^\infty t^2(1+t^3)^{-2}\,dt=\frac{3}{2}a^2$ 要記得背這個.

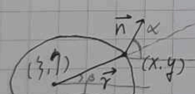
ex. $\int_A^B(3x^2+y)\,dx+(2y+x)\,dy$ 一定是保守量場.

$=f(B)-f(A)$ 證明與路徑無關. (有考一題類似的)

$(P,Q)=(3x^2+y,\ 2y+x)$, $\nabla f=(3x^2+y,\ 2y+x)$

$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=1-1=0$. $\Rightarrow$ is independent of path

ex.



$\vec{\gamma}=(x-\xi,\ y-\eta)$. Find $u(\xi,\eta)=\int_C\frac{\cos(\vec{\gamma}\cdot\vec{n})}{\gamma}\,ds$. $\gamma=|\vec{\gamma}|$

$\vec{\gamma}\cdot\vec{n}=|\vec{\gamma}||\vec{n}|\cos(\vec{\gamma}\cdot\vec{n})=|\vec{\gamma}|\cos(\vec{\gamma}\cdot\vec{n})$

$\cos(\vec{\gamma}\cdot\vec{n})=\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$ $(x-\xi=\gamma\cos\beta,\ y-\eta=\gamma\sin\beta)$

$=\frac{x-\xi}{\gamma}\cos\alpha+\frac{y-\eta}{\gamma}\sin\alpha$ $\qquad \vec{\gamma}=\left(\cos(\alpha+\frac{\pi}{2}),\ \sin(\alpha+\frac{\pi}{2})\right)$

$=\frac{x-\xi}{\gamma}\,dy-\frac{y-\eta}{\gamma}\,dx$ $\qquad =(-\sin\alpha,\ \cos\alpha)$

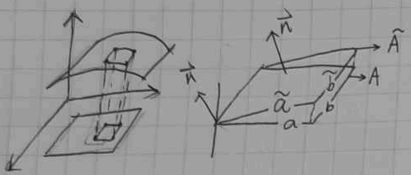
$u(\xi,\eta)=\int_C\underbrace{\frac{-(y-\eta)}{\gamma^2}}_{P}dx+\underbrace{\frac{(x-\xi)}{\gamma^2}}_{Q}dx$

$=0,\ (\xi,\eta)\notin\Omega$

$$\frac{\partial Q}{\partial x}=\gamma^{-2}-2\gamma^{-3}=\frac{\gamma^2-2(x-\xi)^2}{\gamma^4}=\frac{-(x-\xi)+(y-\eta)^2}{\gamma^4}$$
$$-\frac{\partial P}{\partial y}=\gamma^{-2}-2\gamma^{-4}(y-\eta)^2=\frac{(x-\xi)^2-(y-\eta)^2}{\gamma^4}\Rightarrow\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0$$

$u(\xi,\eta) = \int_C \frac{\cos(\vec{r},\vec{n})}{r} ds$, $r = |\vec{r}|$ $x = \xi + \epsilon\cos t$, $y = \eta + \epsilon \sin t$

$= \int_{O_\epsilon(\xi,\eta)} \frac{\cos(\vec{r},\vec{n})}{r} ds = \int_{O_\epsilon} \frac{1}{r^2} ds = \int_0^{2\pi} ds = 2\pi$ 不考



$\tilde{A} = \frac{1}{\cos\gamma} A$, $\vec{n}(0,0,1) = |\vec{n}| \cdot |(0,0,1)| \cos\gamma = \cos\gamma$

$= \frac{1}{\sqrt{1+f_x^2+f_y^2}}$

$F(x,y,z) = z - f(x,y) = 0$, $\nabla F = (-f_x, -f_y, 1)$,

$\vec{n} = \frac{(-f_x, -f_y, 1)}{\sqrt{1+f_x^2+f_y^2}}$

$\iint_S F(x,y,f(x,y))\, ds = \iint_\Omega F(x,y,f(x,y)) \sqrt{1+f_x^2+f_y^2}\, dA$