Chapter 2

Complex Analysis HW

2.1 HW1

Theorem 2.1.1.

$$(1+i)^n, \frac{(1+i)^n}{n}, \frac{n!}{(1+i)^n} \text{ all diverge as } n \to \infty$$

Proof. Note that

$$|(1+i)^n| = 2^{\frac{n}{2}} \to \infty \text{ as } n \to \infty$$

This implies $(1+i)$ is unbounded, thus diverge.

Note that

$$\left|\frac{(1+i)^n}{n}\right| = \frac{(\sqrt{2})^n}{n}$$

Observe

$$\begin{aligned}\frac{(\sqrt{2})^n}{n} = \frac{[(\sqrt{2}-1)+1]^n}{n} &= \frac{\sum_{k=0}^{n}\binom{n}{k}(\sqrt{2}-1)^k}{n} \\ &\geq \frac{\binom{n}{2}(\sqrt{2}-1)^2}{n} = (n-1)[\frac{(\sqrt{2}-1)^2}{2}] \to \infty \text{ as } n \to \infty\end{aligned}$$

This implies $\frac{(1+i)^n}{n}$ is unbounded, thus diverge.

Note that

$$\left|\frac{n!}{(1+i)^n}\right| = \frac{n!}{(\sqrt{2})^n}$$

Note that for all $k \geq 8$, we have

$$\frac{k}{\sqrt{2}} \geq \frac{\sqrt{8}}{\sqrt{2}} = 2$$

This implies

$$\frac{n!}{(\sqrt{2})^n} = \prod_{k=1}^{n} \frac{k}{\sqrt{2}} = \frac{7!}{(\sqrt{2})^7} \prod_{k=8}^{n} \frac{k}{\sqrt{2}} \geq \frac{7!}{(\sqrt{2})^7} \prod_{k=8}^{n} 2 \geq \frac{7! 2^{n-8+1}}{(\sqrt{2})^7} \to \infty$$

which implies $\frac{n!}{(1+i)^n}$ is unbounded, thus diverge. ■

Theorem 2.1.2.

$$n! z^n \text{ converge } \iff z = 0$$

Proof. If $z = 0$, then $n!z^n = 0$ for all n, which implies $n!z^n \to 0$. Now, suppose $z \neq 0$. Let $M \in \mathbb{N}$ satisfy $|z| > \frac{1}{M}$. Observe

$$|n!z^n| = \left| \prod_{k=1}^{n} kz \right| = \left| \prod_{k=1}^{2M-1} kz \right| \left| \prod_{k=2M}^{n} kz \right| \geq \left| \prod_{k=1}^{2M-1} kz \right| \left| \prod_{k=2M}^{n} 2Mz \right| \geq \left| \prod_{k=1}^{2M-1} kz \right| 2^{n-2M+1} \to \infty$$

This implies $n!z^n$ is unbounded, thus diverge. ■

Theorem 2.1.3.

$$u_n \to u \implies v_n \triangleq \sum_{k=1}^{n} \frac{u_k}{n} \to u$$

Proof. Because

$$\sum_{k=1}^{n} \frac{u_k}{n} = \sum_{k \leq \sqrt{n}} \frac{u_k}{n} + \sum_{\sqrt{n} < k \leq n} \frac{u_k}{n}$$

It suffices to prove

$$\sum_{k \leq \sqrt{n}} \frac{u_k}{n} \to 0 \text{ and } \sum_{\sqrt{n} < k \leq n} \frac{u_k}{n} \to u \text{ as } n \to \infty$$

Because u_n converge, we can let M bound $|u_n|$. Observe

$$\left| \sum_{k \leq \sqrt{n}} \frac{u_k}{n} \right| \leq \sum_{k \leq \sqrt{n}} \left| \frac{u_k}{n} \right| \leq \sum_{k \leq \sqrt{n}} \frac{M}{n} \leq \frac{M\sqrt{n}}{n} = \frac{M}{\sqrt{n}} \to 0 \text{ as } n \to 0 \text{ (done)}$$

Because

$$\sum_{\sqrt{n}<k\leq n} \frac{u_k}{n} = \frac{n - \lceil\sqrt{n}\rceil + 1}{n} \sum_{\sqrt{n}<k\leq n} \frac{u_k}{n - \lceil\sqrt{n}\rceil + 1}$$

and

$$\lim_{n\to\infty} \frac{n - \lceil\sqrt{n}\rceil + 1}{n} = 1$$

We can reduce the problem into proving

$$\lim_{n\to\infty} \sum_{\sqrt{n}<k\leq n} \frac{u_k}{n - \lceil\sqrt{n}\rceil + 1} = u$$

Fix ϵ. Let N satisfy that for all $n \geq N$, we have $|u_n - u| < \epsilon$. Then for all $n \geq N^2$, we have

$$\begin{aligned}
\left| \left(\sum_{\sqrt{n}<k\leq n} \frac{u_k}{n - \lceil\sqrt{n}\rceil + 1} \right) - u \right| &= \left| \sum_{\sqrt{n}<k\leq n} \frac{u_k - u}{n - \lceil\sqrt{n}\rceil + 1} \right| \\
&\leq \sum_{\sqrt{n}<k\leq n} \frac{|u_k - u|}{n - \lceil\sqrt{n}\rceil + 1} \\
&\leq \sum_{\sqrt{n}<k\leq n} \frac{\epsilon}{n - \lceil\sqrt{n}\rceil + 1} = \epsilon \text{ (done)}
\end{aligned}$$

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