5. Holomorphic vector bundles

Def: Let X be a complex manifold. A holomorphic vector bundle of rank r on X is a complex manifold E together with holomorphic map $\pi: E \longrightarrow X$ and a structure of $\mathbb{C}$-vector space on each fiber $\pi^{-1}(p)$ such that there exists an open covering $X = \bigcup_i U_i$ and biholomorphic maps $\psi_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{C}^r$ (called a trivialization) s.t.

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[r] \arrow[dr, "\pi"'] & U_i \times \mathbb{C}^r \arrow[d] \\ & U_i \end{tikzcd}$$

and $\psi_i(p,-) : \pi^{-1}(p) \xrightarrow{\sim} \mathbb{C}^r$ are $\mathbb{C}$-linear maps $\forall p \in U_i$.

The compositions $\psi_{ij} := \psi_i \circ \psi_j^{-1} : U_i \cap U_j \times \mathbb{C}^r \longrightarrow U_i \cap U_j \times \mathbb{C}^r$ are called the transition maps, which can be regard as a map

$$\psi_{ij} : U_i \cap U_j \longrightarrow GL_r(\mathbb{C}).$$

They satisfy the cocycle condition

$$\psi_{ij}\, \psi_{jk}\, \psi_{ki} = \mathrm{Id}$$

A holomorphic vector bundle of rank 1 is called a holomorphic line bundle.

We use $\mathcal{O}_X$ to denote the trivial line bundle $X \times \mathbb{C}$.

Any holomorphic vector bundles are determined by their transition maps:

$$E \cong \coprod_i U_i \times \mathbb{C}^r \Big/ \begin{array}{l} (p_i, v_i) \sim (p_j, v_j) \\ \Leftrightarrow (p_j, \psi_{ij} v_j) = (p_i, v_i) \end{array}$$

Hence

$$\left\{\begin{array}{l}\text{holomorphic vector bundle} \\ \text{of rank } r \text{ on } X\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\psi_{ij} : U_i \cap U_j \longrightarrow GL_r(\mathbb{C}) \\ \text{s.t. } \psi_{ij}\psi_{jk}\psi_{ki} = \mathrm{id}\end{array}\right\}$$

<u>Def</u> : Let $\pi_1 : E_1 \to X$, $\pi_2 : E_2 \to X$ be holomorphic vector bundles on a complex manifold X. A <u>vector bundle homomorphism</u> from E_1 to E_2 is a holomorphic map $f : E_1 \to E_2$ s.t.

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr] & & E_2 \arrow[dl] \\ & X & \end{tikzcd}$$

and $f|_{\pi_1^{-1}(p)} : \pi_1^{-1}(p) \longrightarrow \pi_2^{-1}(p)$ is a $\mathbb{C}$-linear map whose rank is independent of p.

Two holomorphic vector bundles on X are said to be <u>isomorphic</u> if there exists a bijective vector bundle homomorphism between them.

With respect to a common trivialization $\psi_i : \pi_1^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{C}^{r_1}$, $\psi_i' : \pi_2^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{C}^{r_2}$, for a vector bundle homomorphism $f : E \to F$, we define

$$f_i := \psi_i' \circ f \circ \psi_i^{-1} : U_i \times \mathbb{C}^{r_1} \longrightarrow U_i \times \mathbb{C}^{r_2}$$

which satisfy

$$f_i \psi_{ij} = \psi_{ij}' f_j$$

Conversely, any such a collection of such f_i's with constant rank induces a vector bundle homomorphism. In particular

$$\frac{\left\{\begin{array}{l}\text{holomorphic vector bundle} \\ \text{of rank } r \text{ on } X\end{array}\right\}}{\text{isomorphisms}} \longleftrightarrow \frac{\left\{\begin{array}{l}\psi_{ij} : U_i \cap U_j \times \mathbb{C}^r \to U_i \cap U_j \times \mathbb{C}^r \\ \text{s.t. } \psi_{ij}\psi_{jk}\psi_{ki} = id\end{array}\right\}}{\sim}$$

where $(\psi_{ij}) \sim (\psi'_{ij})$ iff $\exists$ bijective f_i's s.t.

$$f_i \psi_{ij} = \psi'_{ij} f_j$$

We have the following dictionary between operations on holomorphic vector bundles and the corresponding transition maps

$$E_1 \oplus E_2 \longleftrightarrow \psi_{ij} \oplus \psi'_{ij}$$

$$E_1 \otimes E_2 \longleftrightarrow \psi_{ij} \otimes \psi'_{ij}$$

$$\Lambda^{rk(E)} E \longleftrightarrow \det(\psi_{ij})$$

$$E^* \longleftrightarrow (\psi_{ij}^t)^{-1}$$

<u>Def</u>: Given a holomorphic map $f : X \to Y$ and a holomorphic vector bundle $\pi : E \to Y$ with transition maps (ψ_{ij}). The <u>pullback</u> of E via f is the holomorphic vector bundle f^*E determined by the transition maps $(\psi_{ij} \circ f)$.

<u>Rmk</u>: The fiber f^*E_x of f^*E is canonically isomorphic to $E_{f(x)}$.

<u>Example</u>: Let

$$\mathcal{O}_{\mathbb{P}^n}(-1) := \{([z], w) \in \mathbb{P}^n \times \mathbb{C}^{n+1} : w \in [z]\}$$

Under the projection $\pi : \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathbb{P}^n$, $\mathcal{O}_{\mathbb{P}^n}(-1)$ is a holomorphic line bundle on $\mathbb{P}^n$, called the tautological line bundle. Indeed, on the standard chart $U_i = \{z_i \neq 0\}$, we trivialize $\pi^{-1}(U_i)$ by

$$\psi_i : ([z], w) \longmapsto ([z], w_i)$$

The transition map ψ_{ij} is given by

$$\psi_{ij} : ([z], t) \mapsto \left([z], t\left(\frac{z_0}{z_j}, \cdots, \frac{z_n}{z_j}\right)\right) \mapsto \left([z], \frac{z_i}{z_j} t\right)$$

The way to remember $\mathcal{O}_{\mathbb{P}^n}(-1)$ is that 'The line over the line is the line itself', i.e.

$$\begin{array}{rcccc} \text{as a fiber} \longrightarrow & \ell & \subset & \mathcal{O}_{\mathbb{P}^n}(-1) \\ & \downarrow & & \downarrow \\ \text{as a point} \longrightarrow & \ell & \in & \mathbb{P}^n \end{array}$$

<u>Def</u>: We define $\mathcal{O}_{\mathbb{P}^n}(1) := \mathcal{O}_{\mathbb{P}^n}(-1)^*$ and $\mathcal{O}_{\mathbb{P}^n}(\pm k) := \mathcal{O}_{\mathbb{P}^n}(\pm 1)^{\otimes k}$ for $k > 0$. We put $\mathcal{O}_{\mathbb{P}^n}(0) := \mathcal{O}_{\mathbb{P}^n}$. If E is any holomorphic vector bundle on $\mathbb{P}^n$, we denote by $E(k)$ the holomorphic vector bundle $E \otimes \mathcal{O}_{\mathbb{P}^n}(k)$.

For a complex manifold X, let $\mathrm{Pic}(X)$ be the set of holomorphic line bundles on X modulo isomorphism.

Prop: $\mathrm{Pic}(X)$ is an abelian group under tensor product with identity being $\mathcal{O}_X$ and inverse of $L \in \mathrm{Pic}(X)$ is L^*.

Pf: Clearly, tensor product of two holomorphic line bundles is still a holomorphic line bundles. Moreover, $\mathcal{O}_X$ has cocycle (1), so $(\psi_{ij}) \otimes (1) = (\psi_{ij})$ $\forall$ cocycle (ψ_{ij}). Hence $L \otimes \mathcal{O}_X = L$. Finally, if (ψ_{ij}) is a cocycle corresponds to L, then (ψ_{ij}^{-1}) is a cocycle corresponds to L^*. Therefore

$$(\psi_{ij}) \otimes (\psi_{ij}^{-1}) = (\psi_{ij} \cdot \psi_{ij}^{-1}) = (1),$$

meaning that $L \otimes L^* = \mathcal{O}_X$. $\square$

Given a holomorphic vector bundle $\pi : E \to X$, we define

$$\Gamma(X,E) := \{ s : X \xrightarrow{\text{holo}} E \mid \pi \circ s = \mathrm{id}_X \},$$

the space of global sections of E. Given $s \in \Gamma(X,E)$, we can write

$$\psi_i \circ s|_{U_i} = \sum_{\alpha=1}^{r} s_i^{(\alpha)} 1_\alpha$$

where $\{1_\alpha\}_{\alpha=1}^{r}$ is the standard basis for $\mathbb{C}^r$ and $s_i^{(\alpha)} : U_i \to \mathbb{C}$ are some holomorphic functions. Then

$$\sum_{\alpha=1}^{r} s_i^{(\alpha)} 1_\alpha = (\psi_i \circ \psi_j^{-1}) \left(\sum_{\beta=1}^{r} s_j^{(\beta)} 1_\beta \right) = \sum_{\alpha,\beta=1}^{r} s_j^{(\beta)} \psi_{ij}^{(\beta\alpha)} 1_\alpha$$

$$\Rightarrow \quad s_i^{(\alpha)} = \sum_{\alpha=1}^{r} s_j^{(\beta)} \psi_{ij}^{(\beta\alpha)} \quad \text{—— } (*)$$

Conversely, given a collection of holomorphic functions $(s_i^{(\alpha)} : U_i \to \mathbb{C})$ so that $(*)$ holds $\forall i, j$. We obtain a global section $s \in \Gamma(X, E)$ s.t.

$$\psi_i \circ s|_{U_i} = \sum_{\alpha=1}^{r} s_i^{(\alpha)} 1_\alpha \quad \forall i$$

<u>Example</u>: The transition map ψ_{ij} of $\mathcal{O}_{\mathbb{P}^n}(1)$ is given by $\frac{z_j}{z_i}$, i.e.

$$\psi_{ij} : 1 \longmapsto \frac{z_j}{z_i} 1$$

Consider the coordinate functions $s_i^k := \frac{z_k}{z_i} : U_i \to \mathbb{C}$. Then

$$s_i^k = \frac{z_k}{z_i} = \frac{z_j}{z_i} \frac{z_k}{z_j} = \psi_{ij} \cdot s_j^k$$

Hence $(s_i^k)_i$ are holomorphic sections of $\mathcal{O}_{\mathbb{P}^n}(1)$.
Since $(s_i^k)_i$ represents the homogeneous coordinate z_k, we simply say z_k's are holomorphic sections of $\mathcal{O}_{\mathbb{P}^n}(1)$.
More generally, this calculation shows that any homogeneous polynomial of degree d is a holomorphic section of $\mathcal{O}_{\mathbb{P}^n}(d)$ and actually, these are all! Hence

$$\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \cong \mathrm{Sym}^d(\mathbb{C}^{n+1})^*$$

and

$$\dim_{\mathbb{C}} \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \binom{n+d}{d}$$

<u>Def</u>: Let X be a complex manifold and $\{\varphi_i : U_i \to \mathbb{C}^n\}$ be an atlas for X. Define $\varphi_{ij} := \varphi_i \circ \varphi_j^{-1}$. The <u>holomorphic tangent bundle</u> T_X of X is the holomorphic vector bundle corresponds to the cocycle

$(J(\varphi_{ij}))_{i,j}$. We also define

$$\Omega_X := T_X^* \qquad (\text{holomorphic cotangent bundle})$$
$$\Omega_X^p := \Lambda^p \Omega_X \qquad (\text{bundle of holomorphic } p\text{-forms})$$
$$K_X := \Omega_X^n \qquad (\text{canonical bundle})$$

Prop: Let $Y \subset X$ be a complex submanifold. Then there is a canonical injection $T_Y \subset T_X|_Y$

Pf: Recall that if $\varphi_i : U_i \longrightarrow \mathbb{C}^n$ is a chart for X. Then $\varphi_i|_Y : U_i \cap Y \longrightarrow \{z \in \mathbb{C}^n : z_{m+1} = \cdots = z_n = 0\} \cong \mathbb{C}^m$ is a chart for Y. Then the cocycle (ψ_{ij}) of T_Y is given as a submatrix of $(J(\varphi_{ij})|_Y)$:

$$J(\varphi_{ij})|_Y = \begin{pmatrix} \psi_{ij} & * \\ 0 & \phi_{ij} \end{pmatrix}$$

This gives $T_Y \subset T_X|_Y$. $\square$

Def: Let $Y \subset X$ be a complex submanifold. The normal bundle $\mathcal{N}_{Y/X}$ is the quotient bundle

$$\mathcal{N}_{Y/X} := T_X|_Y / T_X$$

Prop: [Adjunction formula]

Let $Y \subset X$ be a complex submanifold. Then

$$K_Y \cong K_X|_Y \otimes \det(\mathcal{N}_{Y/X})$$

Pf: Recall that

$$J(\varphi_{ij})|_Y = \begin{pmatrix} \psi_{ij} & * \\ 0 & \phi_{ij} \end{pmatrix}$$

One checks that (ϕ_{ij}) is a cocycle and corresponds to the normal bundle $\mathcal{N}_{Y/X}$. The proposition then follows from

$$\det(J(\varphi_{ij}))|_Y = \det(J(\varphi_{ij})|_Y) = \det(\psi_{ij}) \cdot \det(\phi_{ij}). \qquad \square$$