7

FG-homomorphisms

For groups and vector spaces, the 'structure-preserving' functions are, respectively, group homomorphisms and linear transformations. The analogous functions for FG-modules are called FG-homomorphisms, and we introduce these in this chapter.

FG-homomorphisms

7.1 Definition

Let V and W be FG-modules. A function $\vartheta: V \to W$ is said to be an *FG-homomorphism* if ϑ is a linear transformation and

$$(vg)\vartheta = (v\vartheta)g \quad \text{for all } v \in V,\ g \in G.$$

In other words, if ϑ sends v to w then it sends vg to wg.

Note that if G is a finite group and $\vartheta: V \to W$ is an FG-homomorphism, then for all $v \in V$ and $r = \sum_{g\in G} \lambda_g g \in FG$, we have

$$(vr)\vartheta = (v\vartheta)r$$

since

$$(vr)\vartheta = \sum_{g\in G} \lambda_g (vg)\vartheta = \sum_{g\in G} \lambda_g (v\vartheta)g = (v\vartheta)r.$$

The next result shows that FG-homomorphisms give rise to FG-submodules in a natural way.

7.2 Proposition

Let V and W be FG-modules and let ϑ: $V \to W$ be an FG-homomorphism. Then Ker ϑ *is an FG-submodule of V, and* Im ϑ *is an FG-submodule of W.*

Proof First note that Ker ϑ is a subspace of V and Im ϑ is a subspace of W, since ϑ is a linear transformation.

Let $v \in$ Ker ϑ and $g \in G$. Then

$$(vg)\vartheta = (v\vartheta)g = 0g = 0,$$

so $vg \in$ Ker ϑ. Therefore Ker ϑ is an FG-submodule of V.

Now let $w \in$ Im ϑ, so that $w = v\vartheta$ for some $v \in V$. For all $g \in G$,

$$wg = (v\vartheta)g = (vg)\vartheta \in \text{Im}\,\vartheta,$$

and so Im ϑ is an FG-submodule of W.

■

7.3 Examples

(1) If ϑ: $V \to W$ is defined by $v\vartheta = 0$ for all $v \in V$, then ϑ is an FG-homomorphism, and Ker $\vartheta = V$, Im $\vartheta = \{0\}$.

(2) Let $\lambda \in F$, and define ϑ: $V \to V$ by $v\vartheta = \lambda v$ for all $v \in V$. Then ϑ is an FG-homomorphism. Provided $\lambda \neq 0$, we have Ker $\vartheta = \{0\}$, Im $\vartheta = V$.

(3) Suppose that G is a subgroup of S_n. Let $V = \text{sp}\,(v_1, \ldots, v_n)$ be the permutation module for G over F (see Definition 4.10), and let $W = \text{sp}\,(w)$ be the trivial FG-module (see Definition 4.8). We construct an FG-homomorphism ϑ from V to W. Define

$$\vartheta\colon \sum_{i=1}^{n} \lambda_i v_i \to \left(\sum_{i=1}^{n} \lambda_i\right) w \quad (\lambda_i \in F).$$

Thus $v_i\vartheta = w$ for all i. Then ϑ is a linear transformation, and for all $v = \sum \lambda_i v_i \in V$ and all $g \in G$, we have

$$(vg)\vartheta = \left(\sum \lambda_i v_{ig}\right)\vartheta = \left(\sum \lambda_i\right)w,$$

and

$$(v\vartheta)g = \left(\sum \lambda_i\right)wg = \left(\sum \lambda_i\right)w.$$

Therefore ϑ is an FG-homomorphism. Here,

$$\operatorname{Ker}\vartheta = \left\{\sum_{i=1}^{n} \lambda_i v_i \colon \sum_{i=1}^{n} \lambda_i = 0\right\},$$
$$\operatorname{Im}\vartheta = W.$$

By Proposition 7.2, Ker ϑ is an FG-submodule of the permutation module V.

Isomorphic *FG*-modules

7.4 Definition

Let V and W be FG-modules. We call a function $\vartheta\colon V \to W$ an *FG-isomorphism* if ϑ is an FG-homomorphism and ϑ is invertible. If there is such an FG-isomorphism, then we say that V and W are *isomorphic FG-modules* and write $V \cong W$.

In the next result, we check that if $V \cong W$ then $W \cong V$.

7.5 Proposition

If $\vartheta\colon V \to W$ is an FG-isomorphism, then the inverse $\vartheta^{-1}\colon W \to V$ is also an FG-isomorphism.

Proof Certainly ϑ^{-1} is an invertible linear transformation, so we need only show that ϑ^{-1} is an FG-homomorphism. For $w \in W$ and $g \in G$,

$$
\begin{aligned}
((w\vartheta^{-1})g)\vartheta &= ((w\vartheta^{-1})\vartheta)g \quad \text{as } \vartheta \text{ is an } FG\text{-homomorphism} \\
&= wg \\
&= ((wg)\vartheta^{-1})\vartheta.
\end{aligned}
$$

Hence $(w\vartheta^{-1})\, g = (wg)\vartheta^{-1}$, as required.

■

Suppose that ϑ: $V \to W$ is an FG-isomorphism. Then we may use ϑ and ϑ^{-1} to translate back and forth between the isomorphic FG-modules V and W, and prove that V and W share the same structural properties. We list some examples below:

(1) dim V = dim W (since $v_1, \ldots, v_n$ is a basis of V if and only if $v_1\vartheta, \ldots, v_n\vartheta$ is a basis of W);
(2) V is irreducible if and only if W is irreducible (since X is an FG-submodule of V if and only if $X\vartheta$ is an FG-submodule of W);
(3) V contains a trivial FG-submodule if and only if W contains a trivial FG-submodule (since X is a trivial FG-submodule of V if and only if $X\vartheta$ is a trivial FG-submodule of W).

Just as we often regard isomorphic groups as being identical, we frequently disdain to distinguish between isomorphic FG-modules. For the moment, though, we continue simply to emphasize the similarity between isomorphic FG-modules. In the next result, we show that isomorphic FG-modules correspond to equivalent representations.

7.6 Theorem
Suppose that V is an FG-module with basis $\mathscr{B}$, and W is an FG-module with basis $\mathscr{B}'$. Then V and W are isomorphic if and only if the representations

$$\rho\colon g \to [g]_{\mathscr{B}} \text{ and } \sigma\colon g \to [g]_{\mathscr{B}'}$$

are equivalent.

Proof We first establish the following fact:

(7.7) The FG-modules V and W are isomorphic if and only if there are a basis $\mathscr{B}_1$ of V and a basis $\mathscr{B}_2$ of W such that

$$[g]_{\mathscr{B}_1} = [g]_{\mathscr{B}_2} \quad \text{for all } g \in G.$$

To see this, suppose first that ϑ is an FG-isomorphism from V to W, and let $v_1, \ldots, v_n$ be a basis $\mathscr{B}_1$ of V; then $v_1\vartheta, \ldots, v_n\vartheta$ is a basis $\mathscr{B}_2$ of W. Let $g \in G$. Since $(v_i g)\vartheta = (v_i\vartheta)\, g$ for each i, it follows that $[g]_{\mathscr{B}_1} = [g]_{\mathscr{B}_2}$.

Conversely, suppose that $v_1, \ldots, v_n$ is a basis $\mathscr{B}_1$ of V and $w_1, \ldots, w_n$ is a basis $\mathscr{B}_2$ of W such that $[g]_{\mathscr{B}_1} = [g]_{\mathscr{B}_2}$ for all $g \in G$. Let ϑ be the invertible linear transformation from V to W for which $v_i\vartheta = w_i$ for all i. Let $g \in G$. Since $[g]_{\mathscr{B}_1} = [g]_{\mathscr{B}_2}$, we deduce that $(v_i g)\vartheta = (v_i\vartheta)\, g$ for all i, and hence ϑ is an FG-isomorphism. This completes the proof of (7.7).

Now assume that V and W are isomorphic FG-modules. By (7.7), there are a basis $\mathscr{B}_1$ of V and a basis $\mathscr{B}_2$ of W such that $[g]_{\mathscr{B}_1} = [g]_{\mathscr{B}_2}$ for all $g \in G$. Define a representation ϕ of G by $\phi: g \to [\, g]_{\mathscr{B}1}$. Then by Theorem 4.12(1), ϕ is equivalent to both ρ and σ. Hence ρ and σ are equivalent.

Conversely, suppose that ρ and σ are equivalent. Then by Theorem 4.12(2), there is a basis $\mathscr{B}''$ of V such that $g\sigma = [g]_{\mathscr{B}''}$ for all $g \in G$; that is, $[g]_{\mathscr{B}'} = [g]_{\mathscr{B}''}$ for all $g \in G$. Therefore V and W are isomorphic FG-modules, by (7.7).

■

7.8 Example

Let $G = \langle a : a^3 = 1 \rangle$, a cyclic group of order 3, and let W denote the regular FG-module. Then $1, a, a^2$ is a basis of W; call it $\mathscr{B}'$. We have

$$[1]_{\mathscr{B}'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, [a]_{\mathscr{B}'} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$[a^2]_{\mathscr{B}'} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Compare the FG-module V defined in Example 4.11, with basis v_1, v_2, v_3 such that

$$v_1 a = v_2,\ v_2 a = v_3,\ v_3 a = v_1.$$

Writing $\mathscr{B}$ for the basis v_1, v_2, v_3 of V, we have

$$[g]_{\mathscr{B}} = [g]_{\mathscr{B}'} \quad \text{for all } g \in G.$$

According to (7.7), the FG-modules V and W are therefore isomorphic. Indeed, the function

$$\vartheta\colon \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 \to \lambda_1 1 + \lambda_2 a + \lambda_3 a^2 \quad (\lambda_i \in F)$$

is an FG-isomorphism from V to W.

7.9 Example

Let $G = D_8 = \langle a, b\colon a^4 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$. In Example 3.4(1) we encountered two equivalent representations ρ and σ of G, where

$$a\rho = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, b\rho = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$a\sigma = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, b\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let V be the $\mathbb{C}G$-module with basis v_1, v_2 for which

$$v_1 a = v_2,\ v_1 b = v_1,$$

$$v_2 a = -v_1,\ v_2 b = -v_2$$

(see Example 4.5(1)), and, in a similar way, let W be the $\mathbb{C}G$-module with basis w_1, w_2 for which

$$\begin{aligned} w_1 a &= \mathrm{i}w_1, & w_1 b &= w_2, \\ w_2 a &= -\mathrm{i}w_2, & w_2 b &= w_1 \end{aligned}$$

Thus, if we write $\mathscr{B}$ for the basis v_1, v_2 of V and $\mathscr{B}'$ for the basis w_1, w_2 of W, then for all $g \in G$ we have

$$\rho\colon g \to [g]_{\mathscr{B}} \text{ and } \sigma\colon g \to [g]_{\mathscr{B}'}.$$

According to Theorem 7.6, the $\mathbb{C}G$-modules V and W are isomorphic, since ρ and σ are equivalent. To verify this directly, let $\vartheta\colon V \to W$ be the invertible linear transformation such that

$$\vartheta\colon v_1 \to w_1 + w_2,$$

$$v_2 \to \mathrm{i}w_1 - \mathrm{i}w_2.$$

Then $(v_j a)\vartheta = (v_j\vartheta)a$ and $(v_j b)\vartheta = (v_j\vartheta)b$ for $j = 1, 2$, and hence ϑ is a $\mathbb{C}G$-isomorphism from V to W. (Compare Example 3.4(1).)

Direct sums

We conclude the chapter with a discussion of direct sums of FG-modules, and we show that these give rise to FG-homomorphisms.

Let V be an FG-module, and suppose that

$$V = U \oplus W,$$

where U and W are FG-submodules of V. Let $u_1, \ldots, u_m$ be a basis $\mathscr{B}_1$ of U, and $w_1, \ldots, w_n$ be a basis $\mathscr{B}_2$ of W. Then by (2.9), $u_1, \ldots, u_m, w_1, \ldots, w_n$ is a basis $\mathscr{B}$ of V, and for $g \in G$,

$$[g]_{\mathscr{B}} = \left(\begin{array}{c|c} [g]_{\mathscr{B}_1} & 0 \\ \hline 0 & [g]_{\mathscr{B}_2} \end{array}\right).$$

More generally, if $V = U_1 \oplus \ldots \oplus U_r$, a direct sum of FG-submodules U_i, and $\mathscr{B}_i$ is a basis of U_i, then we can amalgamate $\mathscr{B}_1, \ldots, \mathscr{B}_r$ to obtain a basis $\mathscr{B}$ of V, and for $g \in G$,

$$[g]_{\mathscr{B}} = \begin{pmatrix} [g]_{\mathscr{B}_1} & & 0 \\ & \ddots & \\ 0 & & [g]_{\mathscr{B}_r} \end{pmatrix}. \tag{7.10}$$

The next result shows that direct sums give rise naturally to FG-homomorphisms.

7.11 Proposition
Let V be an FG-module, and suppose that

$$V = U_1 \oplus \ldots \oplus U_r$$

where each U_i is an FG-submodule of V. For $v \in V$, we have $v = u_1 + \ldots + u_r$ for unique vectors $u_i \in U_i$, and we define $\pi_i \colon V \to V$ $(1 \leqslant i \leqslant r)$ by setting

$$v\pi_i = u_i.$$

Then each π_i is an FG-homomorphism, and is also a projection of V.

Proof Clearly π_i is a linear transformation; and π_i is an FG-homomorphism, since for $v \in V$ with $v = u_1 + \ldots + u_r$ ($u_j \in U_j$ for all j), and $g \in G$, we have

$$(vg)\pi_i = (u_1 g + \ldots + u_r g)\pi_i = u_i g = (v\pi_i)g.$$

Also,

$$v\pi_i^2 = u_i\pi_i = u_i = v\pi_i,$$

so $\pi_i^2 = \pi_i$. Thus π_i is a projection (see Definition 2.30).

■

We now present a technical result concerning sums of irreducible FG-modules which will be used at a later stage.

7.12 Proposition
Let V be an FG-module, and suppose that

$$V = U_1 + \ldots + U_r,$$

where each U_i is an irreducible FG-submodule of V. Then V is a direct sum of some of the FG-submodules U_i.

Proof The idea is to choose as many as we can of the FG-submodules $U_1, \ldots, U_r$ so that the sum of our chosen FG-submodules is direct. To this end, choose a subset $Y = \{W_1, \ldots, W_s\}$ of $\{U_1, \ldots, U_r\}$ which has the properties that

$W_1 + \ldots + W_s$ is direct (i.e. equal to $W_1 \oplus \ldots \oplus W_s$), but

$W_1 + \ldots + W_s + U_i$ is not direct, if $U_i \notin Y$.

Let

$$W = W_1 + \ldots + W_s.$$

We claim that $U_i \subseteq W$ for all i. If $U_i \in Y$ this is clear, so assume that $U_i \notin Y$. Then $W + U_i$ is not a direct sum, so $W \cap U_i \neq \{0\}$. But $W \cap U_i$ is an FG-submodule of U_i, and U_i is irreducible; therefore $W \cap U_i = U_i$, and so $U_i \subseteq W$, as claimed.

Since $U_i \subseteq W$ for all i with $1 \leqslant i \leqslant r$, we have $V = W = W_1 \oplus \cdots \oplus W_s$, as required.

■

Finally, we remark that if $V_1, \cdots, V_r$ are FG-modules, then we can make the external direct sum $V_1 \oplus \cdots \oplus V_r$ (see Chapter 2) into an FG-module by defining

$$(v_1, \ldots, v_r)g = (v_1 g, \ldots, v_r g)$$

for all $v_i \in V_i$ $(1 \leqslant i \leqslant r)$ and all $g \in G$.

Summary of Chapter 7

1. If V and W are FG-modules and $\vartheta\colon V \to W$ is a linear transformation which satisfies

$$(vg)\vartheta = (v\vartheta)g$$

   for all $v \in V$, $g \in G$, then ϑ is an FG-homomorphism.

2. Kernels and images of FG-homomorphisms are FG-modules.
3. Isomorphic FG-modules correspond to equivalent representations.

Exercises for Chapter 7

1. Let U, V and W be FG-modules, and let $\vartheta\colon U \to V$ and $\phi\colon V \to W$ be FG-homomorphisms. Prove that $\vartheta\phi\colon U \to W$ is an FG-homomorphism.
2. Let G be the subgroup of S_5 which is generated by $(1\ 2\ 3\ 4\ 5)$. Prove that the permutation module for G over F is isomorphic to the regular FG-

module.

3. Assume that V is an FG-module. Prove that the subset

$$V_0 = \{v \in V: vg = v \text{ for all } g \in G\}$$

is an FG-submodule of V. Show that the function

$$\vartheta: v \to \sum_{g \in G} vg \quad (v \in V)$$

is an FG-homomorphism from V to V_0. Is it necessarily surjective?

4. Suppose that V and W are isomorphic FG-modules. Define the FG-submodules V_0 and W_0 of V and W as in Exercise 3. Prove that V_0 and W_0 are isomorphic FG-modules.

5. Let G be the subgroup of S_4 which is generated by (1 2) and (3 4). Is the permutation module for G over F isomorphic to the regular FG-module?

6. Let $G = C_2 = \langle x: x^2 = 1\rangle$.

   (a) Show that the function

   $$\vartheta: \alpha 1 + \beta x \to (\alpha - \beta)(1 - x) \quad (\alpha, \beta \in F)$$

   is an FG-homomorphism from the regular FG-module to itself.

   (b) Prove that $\vartheta^2 = 2\vartheta$.

   (c) Find a basis $\mathscr{B}$ of FG such that

   $$[\vartheta]_{\mathscr{B}} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$