Deadline : 2024/06/12, 22:00.

1. (**Monotone Convergence Theorem for general functions**) Let $\{f_k\}$ be a sequence of measurable functions on a measurable set E. Show that

   (a) If $f_k \nearrow f$ *a.e.* on E and there exists $\phi \in L(E)$ such that $f_k \geq \phi$ *a.e.* on E for all k, then $\int_E f_k \to \int_E f$.

   (b) If $f_k \searrow f$ *a.e.* on E and there exists $\phi \in L(E)$ such that $f_k \leq \phi$ *a.e.* on E for all k, then $\int_E f_k \to \int_E f$.

2. (**Uniform Convergence Theorem for general functions**) Let $f_k \in L(E)$ for $k \in \mathbb{N}$ and let $\{f_k\}$ converge uniformly to f on E with $|E| < +\infty$. Prove that $f \in L(E)$ and $\int_E f_k \to \int_E f$.

3. (**Fatou's Lemma for general functions**) Let $\{f_k\}$ be a sequence of measurable functions on a measurable set E. Prove that if there exists $\phi \in L(E)$ such that $f_k \geq \phi$ *a.e.* on E for all k, then

$$\int_E (\liminf_{k\to\infty} f_k) \leq \liminf_{k\to\infty} \int_E f_k.$$

4. (**Bounded Convergence Theorem for general functions**) Let $\{f_k\}$ be a sequence of measurable functions on a measurable set E such that $f_k \to f$ *a.e.* in E. Prove that if $|E| < +\infty$ and there is a finite constant M such that $|f_k| \leq M$ *a.e.* in E, then $\int_E f_k \to \int_E f$.

5. (**General Lebesgue's Dominated Convergence Theorem**) Let $\{f_k\}$ be a sequence of measurable functions on a measurable set E such that $f_k \to f$ *a.e.* in E. Prove that if there exists $g_k \geq 0$, $g_k \in L(E)$ such that $|f_k| \leq g_k$ *a.e.* in E for all k and $g_k \to g$ *a.e.* for some $g \in L(E)$ with $\int_E g_k \to \int_E g$, then $\int_E f_k \to \int_E f$.

   Hint: Consider $|f_k| \leq g_k \in L(E) \Rightarrow g_k + f_k \geq 0$ and $g_k - f_k \geq 0$, apply Fatou's Lemma.