5.4 HW2

Question 99

Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the map

$$F(x, y) \triangleq (x, y, xy) = (u, v, w)$$

Let $p = (x, y) \in \mathbb{R}^2$. Compute $F_*(\frac{\partial}{\partial x}|_p)$ as a linear combination of $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w}$

Proof. For all $f \in C^\infty(\mathbb{R}^3)$, we have

$$\frac{\partial f \circ F}{\partial x}(x, y) = \frac{\partial f}{\partial u}(F(p)) + \frac{\partial f}{\partial w}(F(p))y$$

Why

This then give us

$$F_*\Big(\frac{\partial}{\partial x}\Big|_{(x,y)}\Big) = \frac{\partial}{\partial u} + y\frac{\partial}{\partial w}$$

8

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Question 100

Let G be a lie group with multiplication map $\mu : G \times G \to G$ and identity element e. Show that differential $\mu_{*,(e,e)} : \cancel{T_{(e,e)}G} \times G \to T_eG$ of μ at identity is

$$\mu_{*,(e,e)}(X_e, Y_e) = X_e + Y_e$$

Note that $T_{(p,q)}M \times N$ is isomorphic to $T_pM \oplus T_qN$ via the differential of two projection $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$.

Proof. We first justify the notation of writing tangent vectors in $T_{(e,e)}G \times G$ as (X_e, Y_e), and the proof will follow. Consider the projection $\pi_1 : G \times G \to G$ and $\pi_2 : G \times G \to G$

$$\pi_1(g, h) \triangleq g \text{ and } \pi_2(g, h) \triangleq h$$

Consider charts $(U, \varphi), (V, \psi)$ for G centering e. We can induce a chart $(U \times V, \Phi)$ for $G \times G$ centering e by

$$\Phi(g, h) \triangleq (\varphi(g), \psi(h))$$

In local coordinate, we have

$$\pi_1(\mathbf{x}^1, \ldots, \mathbf{x}^n, \mathbf{y}^1, \ldots, \mathbf{y}^n) = (\mathbf{x}^1, \ldots, \mathbf{x}^n) \text{ and } \pi_2(\mathbf{x}^1, \ldots, \mathbf{x}^n, \mathbf{y}^1, \ldots, \mathbf{y}^n) = (\mathbf{y}^1, \ldots, \mathbf{y}^n)$$

In abuse of notation, this give us

$$(\pi_1)_*\Big(\sum_{i=1}^n v^i\frac{\partial}{\partial \mathbf{x}^i}+w^i\frac{\partial}{\partial \mathbf{y}^i}\Big)=\sum_{i=1}^n v^i\frac{\partial}{\partial \mathbf{x}^i} \text{ and } (\pi_2)_*\Big(\sum_{i=1}^n v^i\frac{\partial}{\partial \mathbf{x}^i}+w^i\frac{\partial}{\partial \mathbf{y}^i}\Big)=\sum_{i=1}^n w^i\frac{\partial}{\partial \mathbf{y}^i}$$

This then give us a vector space isomorphism between $T_{(e,e)}G\times G$ and $T_eG\oplus T_eG$, on which our notation stand. Now, let $\gamma:(-\epsilon,\epsilon)\to G$ be a smooth curve centering e such that $\gamma'(0)=X_e$. Define another smooth curve $\alpha:(-\epsilon,\epsilon)\to G\times G$ in $G\times G$ by

$$\alpha(t)\triangleq(\gamma(t),e)$$

Because $\pi_2\circ\alpha$ is constant and $\pi_1\circ\alpha=\gamma$, we now see

$$(\pi_1)_{*,(e,e)}(\alpha'(0))=(\pi_1\circ\alpha)'(0)=\gamma'(0) \text{ and } (\pi_2)_{*,(e,e)}(\alpha'(0))=0$$

This implies $\alpha'(0)=(X_e,0)$. Compute

$$\mu\circ\alpha(t)=\gamma(t)+e=\gamma(t)$$

We now can deduce

$$\mu_{*,(e,e)}(X_e,0)=\mu_{*,(e,e)}(\alpha'(0))=(\mu\circ\alpha)'(0)=\gamma'(0)=X_e$$

Similar procedure can be applied to show

$$\mu_{*,(e,e)}(0,Y_e)=Y_e$$

It now follows from linearity of $\mu_{*,(e,e)}$ that

$$\mu_{*,(e,e)}(X_e,Y_e)=X_e+Y_e$$

10

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Question 101

Let $S^2=\{(x,y,z)\in\mathbb{R}^3:x^2+y^2+z^2=1\}$ be the sphere in $\mathbb{R}^3$. Consider the function $h:S^2\to\mathbb{R}$ defined by

$$h(x,y,z)\triangleq z$$

Find the critical points of h.

Proof. Consider the atlas $\{(U,\varphi),(V,\psi)\}$ for S^2 where $U=S^2\setminus\{(0,0,1)\},V=S^2\setminus\{(0,0,-1)\}$ and

$$\varphi(x,y,z)\triangleq\Big(\frac{x}{1-z},\frac{y}{1-z}\Big) \text{ and } \psi(x,y,z)=\Big(\frac{x}{1+z},\frac{y}{1+z}\Big)$$

Some algebra trick and tedious efforts shows that this indeed form a smooth atlas and gives us their inverse

$$\varphi^{-1}(u,v) = \Big(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\Big)$$
$$\psi^{-1}(u,v) = \Big(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{1-u^2-v^2}{u^2+v^2+1}\Big)$$

Compute

$$[d(h\circ\varphi^{-1})_{(u,v)}] = \begin{bmatrix} \frac{-4u}{(u^2+v^2+1)^2} & \frac{-4v}{(u^2+v^2+1)^2} \end{bmatrix}$$
$$[d(h\circ\psi^{-1})_{(u,v)}] = \begin{bmatrix} \frac{4u}{(u^2+v^2+1)^2} & \frac{4v}{(u^2+v^2+1)^2} \end{bmatrix}$$

This then shows the set of critical points is exactly

$$\{\varphi^{-1}(0,0), \psi^{-1}(0,0)\} = \{(0,0,-1),(0,0,1)\}$$

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Question 102

A smooth map $f : M \to N$ is said to be **a traversal to an embedded submanifold** $S \subseteq N$ if for every point $p \in f^{-1}(S)$, we have

$$f_{*,p}(T_pM) + T_{f(p)}S = T_{f(p)}N$$

The goal of this exercise is to prove the Transversality Theorem: If a smooth map $f : M \to N$ is a traversal to an embedded submanifold S of codimension k in N, then $f^{-1}(S)$ is a regular submanifold of codimension k in M. Let $p \in f^{-1}(S)$ and $(U, \mathbf{x}^1, \ldots, \mathbf{x}^n)$ be an adapted chart for N centering $f(p)$ with respect to S. Define $g : U \to \mathbb{R}^k$ by

$$g(\mathbf{x}^1, \ldots, \mathbf{x}^n) \triangleq (\mathbf{x}^{n-k+1}, \ldots, \mathbf{x}^n)$$

(a) Show that $f^{-1}(U) \cap f^{-1}(S) = (g \circ f)^{-1}(\mathbf{0})$.

(b) Show that $f^{-1}(U) \cap f^{-1}(S)$ is a regular level set of the function $g \circ f : f^{-1}(U) \to \mathbb{R}^k$.

(c) Prove the Transversality Theorem.

Proof. At first, we shall point out that $g \circ f$ is a function defined only on $f^{-1}(U)$. (a) follows trivially from the fact that $(U, \mathbf{x}^1, \ldots \mathbf{x}^n)$ is an adapted chart. We now prove (b).

You still have to verify.

Fix arbitrary $p \in f^{-1}(U) \cap f^{-1}(S)$. Trivially, by definition of g,

$$T_{f(p)}S \subseteq \operatorname{Ker} g_{*,f(p)}$$

Explicit formula of g shows that $g_{*,f(p)}$ is a vector space ~~epimorphism~~ that maps $T_{f(p)}N$ into $T_{g\circ f(p)}\mathbb{R}^k$, which implies that $\operatorname{Ker} g_{*,f(p)}$ has dimension $n-k$, same as $T_{f(p)}S$ and give us

$$T_{f(p)}S = \operatorname{Ker} g_{*,f(p)}$$

It now follows from f being a traversal and $g_{*,f(p)}$ being surjective that

$$(g \circ f)_{*,p}(T_pM) = g_{*,f(p)} \circ f_{*,p}(T_pM) = \operatorname{Im} g_{*,f(p)} = T_{g\circ f(p)}\mathbb{R}^k$$

We have shown that $g \circ f$ is regular at p. (b) then follows from p is arbitrary selected from $f^{-1}(U) \cap f^{-1}(S)$.

Now, by Regular level set Theorem, we see that $f^{-1}(U) \cap f^{-1}(S)$ is an embedded submanifold of $f^{-1}(U)$ with codimension k. Because f is continuous, we know $f^{-1}(U)$ is open, thus an embedded submanifold of M with dimension m. It now follows that $f^{-1}(U) \cap f^{-1}(S)$ is an embedded submanifold of M with codimension k.

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(b) You have to say directly that $f^{-1}(U) \cap f^{-1}(S)$ is the level set of a regular value of $g \circ f$. Or maybe you can complete the proof by showing $f^{-1}(U) \cap f^{-1}(S)$ is connected.

7