Deadline : 2022/11/09, 17:00.

Definition 1. *Let A be a subset of $\mathbb{R}$, we say that*

*(1) A is bounded above if there is a number $M \in \mathbb{R}$ such that $a \leq M$ for all $a \in A$. We call such M an **upper bound** (上界) of A.*

*(2) A is bounded below if there is a number $N \in \mathbb{R}$ such that $a \geq N$ for all $a \in A$. We call such N an **lower bound** (下界) of A.*

Definition 2. *Let f be bounded on $[a, b]$, we say that M is a **supremum** (最小上界) of f on $[a, b]$, denoted by $\sup\limits_{x\in[a,b]} f(x)$, if M satisfying*

(i) M is an upper bound of f on $[a, b]$, i.e. $f(x) \leq M$ for all $x \in [a, b]$.

(ii) if M_1 is an upper bound of f on $[a, b]$, then $M \leq M_1$.

Definition 3. *Let f be bounded on $[a, b]$, we say that N is a **infimum** (最大下界) of f on $[a, b]$, denoted by $\inf\limits_{x\in[a,b]} f(x)$, if N satisfying*

(i) N is an lower bound of f on $[a, b]$, i.e. $f(x) \geq N$ for all $x \in [a, b]$.

(ii) if N_1 is an lower bound of f on $[a, b]$, then $N \geq N_1$.

Definition 4.

*(1) Let P be a finite collection of points that satisfies $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. We call such P a **partition** (分割) of $[a, b]$.*

(2) The norm (mesh size) of a partition P is defined by

$$|P| = \max_{1\leq k\leq n} \triangle x_k \quad \text{where } \triangle x_k = x_k - x_{k-1}.$$

*(3) If P_1 and P_2 are two partitions of $[a, b]$ and $P_1 \subseteq P_2$, then P_2 is called a **refinement** (細分) of P_1.*

(4) If P_1 and P_2 are two partitions of $[a, b]$, then $P_1 \cup P_2$ is called a common refinement of P_1 and P_2.

Definition 5. *Suppose that f is bounded on $[a,b]$ and $P = \{x_0, x_1, \cdots, x_n\}$ is a partition of $[a,b]$. Let $m_i = \inf\limits_{x\in[x_{i-1},x_i]} f(x)$ and $M_i = \sup\limits_{x\in[x_{i-1},x_i]} f(x)$. We say that*

$$\sum_{i=1}^{n} m_i(x_i - x_{i-1}) \text{ is the lower sum of } f \text{ for } P, \text{ denoted by } L(f,P),$$
$$\sum_{i=1}^{n} M_i(x_i - x_{i-1}) \text{ is the upper sum of } f \text{ for } P, \text{ denoted by } U(f,P).$$

Definition 6. *Let f be a bounded function on $[a,b]$. We say that f is* ***Darboux integrable*** *(達布可積) on $[a,b]$ if $\sup\limits_P L(f,P) = \inf\limits_P U(f,P)$. We call the number* ***the Darboux integral*** *(達布積分) of f on $[a,b]$ and denoted by $\displaystyle\int_a^b f(x)\,dx$.*

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1. Find the upper sum $U(f,P)$ and the lower sum $L(f,P)$ of the given function f on the given interval with the given partition P,

   (a) $f(x) = x$ on $[0,2]$, $\quad P = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$.

   (b) $f(x) = \begin{cases} 3x-1, & 0 \le x \le 2, \\ -4x+2, & 2 < x \le 4. \end{cases} \qquad P = \{0,1,2,3,4\}$.

2. Let f be bounded on $[a,b]$, P_1 and P_2 be two partitions of $[a,b]$ such that P_2 is a refinement of P_1. Prove that $U(f,P_1) \ge U(f,P_2)$ and $L(f,P_1) \le L(f,P_2)$.

3. Suppose f is bounded on $[a,b]$. Prove that f is Darboux integral on $[a,b]$ if and only if for every $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$.

4. Suppose that a function f is bounded and integrable on $[a,b]$ and $[c,d] \subseteq [a,b]$. Prove that f is integrable on $[c,d]$.

5. Let $f(x)$ be a bounded function defined as

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [a,b], \\ 0, & x \notin \mathbb{Q} \cap [a,b]. \end{cases}$$

   Prove that f is not integrable on $[a,b]$.

6. Compute
$$\int_{-2}^{2} \sin(x^3)\sqrt{\frac{1}{x^6+3x^4-x^2+5}}\,dx.$$

7. Suppose that $g : \mathbb{R} \to [a,b]$ is a continuous function and f is integrable on $[a,b]$. Prove that
$$F(x) := \int_a^{g(x)} f(t)\,dt$$
is continuous on $\mathbb{R}$.

   (**Hint**: Express $F(x)$ as a composite function of $g(x)$ and $G(x) := \int_a^x f(t)\,dt$.)

8. Evaluate the following limit.

   (a) $\displaystyle\lim_{n\to\infty} \frac{1}{n^{16}} \sum_{i=1}^{n} i^{15}$

   (b) $\displaystyle\lim_{n\to\infty} n^{-\frac{3}{2}} \sum_{i=1}^{n} \sqrt{i}$

   (c) $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{2n} \frac{i^2}{n^3}$

   (d) $\displaystyle\lim_{n\to\infty} \sum_{i=n}^{2n} \frac{1}{i}$

9. Use u-substitution to evaluate the indefinite integral

   (a) $\int e^{\cos x} \sin x\,dx$

   (b) $\int e^{-x}[1+\cos(e^{-x})]\,dx$

   (c) $\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx$

   (d) $\int \frac{e^x}{\sqrt{1+e^x}}\,dx$

10. Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

   (a) $y = x+1$, $y = 0$, $x = 0$, $x = 2$; about the x-axis.

   (b) $x = 2\sqrt{y}$, $x = 0$, $y = 9$; about the y-axis.

   (c) $y = 2x$, $y = x^2$; about the x-axis.

   (d) $x = 0$, $x = 9 - y^2$; about $x = -1$.