# Theory of Numbers

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# Contents

# Chapter 1

# Groups

## 1.1 Isomorphism theorems

Let $M$ be a set equipped with a binary operation $M \times M \to M$. We say $M$ is a **monoid** if the binary operation is associative and there exists a two-sided identity $e \in M$.

**Example 1.1.1.** Defining $(x, y) \mapsto y$, we see that the operation is associative and every element is a left identity, but no element is a right identity unless $|M| = 1$. This is an example why identity must be two-sided.

Because the identity of a monoid is defined to be two-sided, clearly it must be unique. Suppose every element of monoid $M$ has a left inverse. Fix $x \in M$. Let $x^{-1} \in M$ be a left inverse of $x$. To see that $x^{-1}$ is also a right inverse of $x$, let $(x^{-1})^{-1} \in M$ be a left inverse of $x$ and use

$$(x^{-1})^{-1} = (x^{-1})^{-1}e = (x^{-1})^{-1}(x^{-1}x) = ((x^{-1})^{-1}x^{-1})x = ex = x$$

to deduce

$$xx^{-1} = (x^{-1})^{-1}x^{-1} = e$$

In other words, if we require every element of a monoid $M$ to has a left inverse, then immediately every left inverse upgrades to a right inverse. In such case, we call $M$ a **group**. Notice that inverses of elements of a group are clearly unique.

Unlike the category of monoids, the category of groups behaves much better. Given two groups $G, H$ and a function $\varphi : G \to H$, if $\varphi$ respects the binary operation, then $\varphi$ also respects the identity:

$$e_H = (\varphi(x)^{-1})\varphi(x) = (\varphi(x)^{-1})\varphi(xe_G) = (\varphi(x)^{-1}\varphi(x))\varphi(e_G) = \varphi(e_G)$$

which implies that $\varphi$ must also respect inverse. In such case, we call $\varphi$ a **group homomorphism**. Given a subset $H \subseteq G$ closed under the binary operation, if $H$ forms a group itself, then since the set inclusion $H \hookrightarrow G$ forms a group homomorphism, we have $e_H = e_G$, and thus $x^{-1}$ in $H, G$ are the same element.

In this note, by a **subgroup** $H$ of $G$, we mean an injective group homomorphism $H \hookrightarrow G$. Clearly, a subset of $G$ forms a subgroup if and only it is closed under both the binary operation and inverse. Note that one of the key basic property of subgroup $H \subseteq G$ is that if $g \notin H$, then $hg \notin H$, since otherwise $g = h^{-1}hg \in H$.

Let $S$ be a subset of $G$. The group of **words** in $S$ is clearly the smallest subgroup of $G$ containing $S$. We say this subgroup is **generated** by $S$. If $G$ is generated by a single element, we say $G$ is **cyclic**. Let $x \in G$. The **order** of $G$ is the cardinality of $G$, and the order of $x$ is the cardinality of the cyclic subgroup $\langle x \rangle \subseteq G$, or equivalently the infimum of the set of natural numbers $n$ that makes $x^n = e$. Clearly, finite cyclic groups of order $n$ are all isomorphic to $\mathbb{Z}_n$.

Let $G$ be a group and $H$ a subgroup of $G$. The **right cosets** $Hx$ are defined by $Hx \triangleq \{hx \in G : h \in H\}$. Clearly, when we define an equivalence relation in $G$ by setting:

$$x \sim y \overset{\triangle}{\iff} xy^{-1} \in H$$

the equivalence class $[x]$ coincides with the right coset $Hx$. Note that if we partition $G$ using **left cosets**, the equivalence relation being $x \sim y \iff x^{-1}y \in H$, then the two partitions need not to be identical.

**Example 1.1.2.** Let $H \triangleq \{e, (1,2)\} \subseteq S_3$. The right cosets are

$$H(2,3) = \{(2,3),(1,2,3)\} \quad \text{and} \quad H(1,3) = \{(1,3),(1,3,2)\}$$

while the left cosets being

$$(2,3)H = \{(2,3),(1,3,2)\} \quad \text{and} \quad (1,3)H = \{(1,3),(1,2,3)\}$$

■

However, as one may verify, we have a well-defined bijection $xH \mapsto Hx^{-1}$ between the sets of left cosets and right cosets of $H$. Therefore, we may define the **index** $|G : H|$ of $H$ in $G$ to be the cardinality of the collection of left cosets of $H$, without falling into the discussion of left and right. Moreover, by axiom of choice, there exists a set $T \subseteq G$ such that $|T \cap xH| = 1$ for all $x \in G$. Such $T$ clearly makes the set map $T \times H \to G$ defined by:

$$(t, h) \mapsto th$$

a bijection. This proves the **Lagrange's theorem**:

$$|G| = |G : H| \cdot |H|$$

Consider a group $G$ of prime order. If $x \neq e \in G$, then clearly the cyclic subgroup $\langle x \rangle$ must be $G$ by Lagrange's theorem.

Because the inverse of an injective group homomorphism forms a group homomorphism, we know the set $\mathrm{Aut}(G)$ of automorphisms of $G$ forms a group. We say $\phi \in \mathrm{Aut}(G)$ is an **inner automorphism** if $\phi$ takes the form $x \mapsto gxg^{-1}$ for some fixed $g \in G$. We say two elements $x, y \in G$ are **conjugated** if there exists some inner automorphism that maps $x$ to $y$. Clearly conjugacy forms a equivalence relation. We call its classes **conjugacy classes**.

**Example 1.1.3.** Consider $G \triangleq \mathrm{GL}_2(\mathbb{R})$ and consider:

$$H \triangleq \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}) : n \in \mathbb{Z} \right\} \quad \text{and} \quad g \triangleq \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$$

Note that $gHg^{-1} \subset H$. In other words, inner automorphisms can maps a subgroup $H$ into a subgroup strictly contained by $H$ if $G$ is infinite.

**Equivalent Definition 1.1.4. (Normal subgroups)** Let $G$ be a group and $N$ a subgroup. We say $N$ is a **normal subgroup** of $G$ if any of the followings hold true:

(i) $\phi(N) \subseteq N$ for all $\phi \in \mathrm{Inn}(G)$

(ii) $\phi(N) = N$ for all $\phi \in \mathrm{Inn}(G)$

(iii) $xN = Nx$ for all $x \in G$.

(iv) The set of all left cosets of $N$ equals the set of all right cosets of $N$.

(v) $N$ is a union of conjugacy classes.

(vi) For all $n \in N$ and $x \in G$, their **commutator** $nxn^{-1}x^{-1} \in G$ lies in $N$.

(vii) For all $x, y \in G$, we have $xy \in N \iff yx \in N$.

*Proof.* (i) $\implies$ (ii): Let $\phi \in \mathrm{Inn}(G)$. By premise, $\phi(N) \subseteq N$ and $\phi^{-1}(N) \subseteq N$. Applying $\phi$ to both side of $\phi^{-1}(N) \subseteq N$, we have $\phi(N) \subseteq N \subseteq \phi(N)$, as desired.

(ii) $\implies$ (iii): Consider the automorphisms:

$$\phi_{L,x}(g) = xg \quad \text{and} \quad \phi_{L,x^{-1}}(g) = x^{-1}g \quad \text{and} \quad \phi_{R,x}(g) = gx$$

Because $\phi_{L,x^{-1}} \circ \phi_{R,x} \in \mathrm{Inn}(G)$, by premise we have:

$$xN = \phi_{L,x}(N) = \phi_{L,x} \circ \phi_{L,x^{-1}} \circ \phi_{R,x}(N) = \phi_{R,x}(N) = Nx$$

(iii) $\Longrightarrow$ (iv) is clear. (iv) $\Longrightarrow$ (iii): Let $x \in G$. By premise, there exists some $y \in G$ that makes $xN = Ny$. Let $x = ny$. The proof then follows from noting

$$xN = Ny = N(n^{-1}x) = Nx$$

(iii) $\Longrightarrow$ (v): Let $n \in N$ and $x \in G$. We are required to show $xnx^{-1} \in N$. Because $xN = NX$, we know $xn = \widetilde{n}x$ for some $\widetilde{n} \in N$. This implies

$$xnx^{-1} = \widetilde{n}xx^{-1} = \widetilde{n} \in N$$

(v) $\Longrightarrow$ (vi): Fix $n \in N$ and $x \in G$. By premise, $xn^{-1}x^{-1} \in N$. Therefore, $n(xn^{-1}x^{-1}) \in N$, as desired.

(vi) $\Longrightarrow$ (vii): Let $xy \in N$. To see $yx$ also belong to $N$, observe:

$$(xy)^{-1}(yx) = (xy)^{-1}x^{-1}xyx = [xy, x] \in N$$

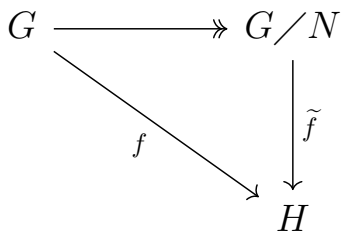
(viii) $\Longrightarrow$ (i): Let $n \in N$ and $x \in G$. Because $(nx)x^{-1} = n \in N$, by premise we have $x^{-1}nx \in N$, as desired. ■

From the point of view of inner automorphism, we see that it is well-defined whether an element $g \in G$ **normalize** a subset $S \subseteq G$, independent of left and right. Because of the independence, For each subset $S \subseteq G$, we see that the set of elements $g \in G$ that normalize $S$ forms a group, called the **normalizer** of $S$. Note that if $g$ normalize $S$, then $gS = Sg$.

As we shall show, normal subgroup are what we may perform quotient on in the category of groups. Let $G$ be a group and $N \subseteq G$ a subgroup. We say a group homomorphism $G \to G/N$ that vanishes on $N$ satisfies the **universal property of quotient group** $G/N$ if

(i) it vanishes on $N$. **(Group condition)**

(ii) for all group homomorphism $f : G \to H$ that vanishes on $N$ there exist a unique group homomorphism $\widetilde{f} : G/N \to H$ that makes the diagram:




commute. **(Universality)**

**Theorem 1.1.5. (The first isomorphism theorem for groups)** Let $N$ be a subgroup of $G$ and $j : G \to G/N$ satisfies the universal property. Then $j$ is surjective with kernel $N$.

*Proof.* ■

Because the kernel of a group homomorphism is clearly normal, if $N$ is not normal, then there can not be a pair $G \to G/N$ that satisfies the universal property. If any things, this is the "reason" why normal subgroups are what meant to be quotiented in the category of group.

**Corollary 1.1.6. (The first isomorphism theorem)**

*Proof.* ■

**Example 1.1.7.** $G \triangleq S_3$. $S \triangleq \langle(1,2)\rangle$ and $H \triangleq \langle(2,3)\rangle$. $SH$ doesn't form a group. $(2,3)(1,2) \notin SH$.

Second isomorphism theorem.
Third isomorphism theorem.
Correspondence theorem.
Because $\varphi \circ \phi_g \circ \varphi^{-1} = \phi_{\varphi(g)}$, we know $\mathrm{Inn}(G)$ forms a normal subgroup of $\mathrm{Aut}(G)$.

## 1.2 Group action

Let $G$ be a group and $X$ a set. If we say $G$ **acts** on $X$ we are defining a function $G \times X \to X$ such that

(i) $e \cdot x = x$ for all $x \in X$.

(ii) $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$.

Because groups admit inverses, a $G$-action is in fact a group homomorphism $G \to \mathrm{Sym}(X)$. The trivial action then correspond to the trivial group homomorphism. An action is **faithful** if it is injective.

Show that $Z(G) \subseteq \mathrm{Ker}\,\theta$ if and only if $\theta$ is faithful.

An action is **free** if $g \cdot x = x$ for a $x \in X$ implies $g = e$. Note that the isomorphism $\mathrm{Sym}(X) \to \mathrm{Sym}(X)$ is always injective but never free unless $|X| \leq 2$. The action is **transitive** if for any $x, y \in X$, there always exists some $g \in G$ such that $y = g \cdot x$. An action is **regular** if it is both free and transitive.