Eric's note on Complex Geometry

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Chapter 1

Single Variable Complex Analysis

1.1 Quick Recap

Given a complex-valued function $u+iv = f(x+iy)$ defined on some neighborhood of $z \in \mathbb{C}$, we say f is **complex-differentiable** at z if there exists some complex number denoted by $f'(x)$ such that

$$\frac{f(z+h)-f(z)-f'(z)h}{h} \to 0 \text{ as } h \to 0; h \in \mathbb{C}$$

If f is complex-differentiable at z, then obviously f satisfies the **Cauchy-Riemann equation**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ at } z$$

The converse hold true under an extra condition. If u, v satisfy the Cauchy-Riemann equation at z and are real-differentiable at z, then from a direct estimation, f is also complex-differentiable at z. Given open $U \subseteq \mathbb{C}$ and some complex-valued function $f : U \to \mathbb{C}$, we say f is **holomorphic** if f is complex-differentiable on all $z \in U$. In this note, if we say $\gamma : [a, b] \to \mathbb{C}$ is C^1, we mean there exists some C^1 map $\widetilde{\gamma} : (a - \epsilon, b + \epsilon) \to \mathbb{C}$ such that $\gamma = \widetilde{\gamma}|_{[a,b]}$. By a **contour**, precisely, we mean a map $\gamma : [a, b] \to \mathbb{C}$ such that for a finite set of points $\{a = x_0 < x_1 < \cdots < x_n < b = x_{n+1}\}$, the maps $\gamma|_{[x_i,x_{i+1}]}$ are C^1 and

$$\gamma'(t) \neq 0, \quad \text{for all } t \in [x_i, x_{i+1}]$$

Given some contour $\gamma : [a, b] \to \mathbb{C}$ and some z that does not lie in the image of γ, we define the **winding number** of z with respect to γ to be

$$\text{Ind}_\gamma(z) \triangleq \frac{1}{2\pi i} \int_\gamma \frac{d\xi}{\xi - z}$$

If the contour $\gamma : [a, b] \to \mathbb{C}$ is **closed**[1], by setting

$$f(t) \triangleq \frac{1}{2\pi i} \int_a^{a+t} \frac{\gamma'(s)}{\gamma(s) - z} ds$$

and by noting that $\frac{d}{dt}[e^{-2\pi i f(t)}(\gamma(t) - z)]$ is zero everywhere, we see that the winding number $\mathrm{Ind}_\gamma(z)$ is indeed an integer as expected. Moreover, because Ind_γ is continuous on $\mathbb{C} \setminus \{\gamma(t) : t \in [a, b]\}$ [2], we see Ind_γ is constant on each connected component of $\mathbb{C} \setminus \{\gamma(t) : t \in [a, b]\}$. Now, by a **domain**, we mean an nonempty open connected subset of $\mathbb{C}$. Finally, we may state our version of **Cauchy's Integral Theorem**.

Theorem 1.1.1. (Cauchy's Integral Theorem) Given some domain D, some holomorphic function $f : D \to \mathbb{C}$, and some closed contour $\gamma : [a, b] \to D$ that does not wind around any point in $D \setminus \{\gamma(t) : t \in [a, b]\}$, we have

$$\int_\gamma f = 0$$

Cauchy's Integral Theorem is the cornerstone of complex analysis. Its proof fundamentally relies on triangulation and its special case for triangles. For brevity, the proof is presented here. Note that when integrating along the boundary of a disk, the orientation matters unless the integral equals 0. To simplify matters, we adopt the universal convention that integration is always performed counterclockwise. Now, by a geometric arguments using 'cuts', we have **Cauchy's Integral Formula**, stating that if f is holomorphic on $|z - z_0| < r$, then

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_\epsilon(z_0)} \frac{f(\xi)}{\xi - z} d\xi, \quad \text{for all } \epsilon < r$$

This with the uniform convergence of

$$\frac{f(\xi)}{\xi - z} = \sum_{n=0}^\infty \frac{f(\xi)(z - z_0)^n}{(\xi - z_0)^{n+1}} \text{ on } \partial B_\epsilon(z_0)$$

shows that holomorphic functions are locally power series

$$f(z) = \sum_{n=0}^\infty \frac{1}{2\pi i} \Big(\int_{\partial B_\epsilon(z_0)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \Big) (z - z_0)^n, \quad \text{for all } \epsilon < r \text{ and } z \in B_\epsilon(z_0)$$

[1] $\gamma(a) = \gamma(b)$.

[2] One may prove this continuity by direct estimation.

Because all power series converge uniformly on disk with radius strictly smaller than its convergence radius, we may differentiate term by term and have **Taylor's Theorem for power series**

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

If $D \subseteq \mathbb{C}$ is a domain and $f : D \to \mathbb{C}$ is holomorphic, Taylor's Theorem for power series tell us that $\{z \in D : f^{(n)}(z) = 0 \text{ for all } n \geq 0\}$ is not only closed in D but also open, and thus equals to D if proved nonempty. One particularly weak condition for T to be nonempty is that $f \equiv 0$ on some $S \subseteq D$, and S has a limit point in D. This result is commonly referred to as **Identity Theorem**. By an **entire** function, we mean a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ defined on the whole complex plane. Obviously, for all $r > 0$ and $z \in B_r(0)$, we have

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \quad \text{where } c_n = \frac{1}{2\pi i} \int_{\partial B_r(0)} \frac{f(\xi)}{\xi^{n+1}} d\xi$$

If f is bounded, then direct estimations show that $c_n = 0$ for all $n > 0$. This result is commonly referred to as **Liouville's Theorem**. Suppose f is holomorphic on some annulus $r < |z - z_0| < R$. Cauchy's integral theorem, Cauchy's integral formula and a geometric argument using 'cuts' give us

$$f(z) = \frac{1}{2\pi i}\Big(\int_{\partial B_{R-\epsilon}(z_0)} \frac{f(\xi)}{\xi - z} d\xi - \int_{\partial B_{r+\epsilon}(z_0)} \frac{f(\xi)}{\xi - z} d\xi \Big)$$

This with the uniform convergence

$$\frac{f(\xi)}{\xi - z} = \sum_{n=0}^{\infty} \frac{f(\xi)(z - z_0)^n}{(\xi - z_0)^{n+1}} \text{ on } \partial B_{R-\epsilon}(z_0)$$

and the uniform convergence

$$\frac{f(\xi)}{z - \xi} = \sum_{n=0}^{\infty} \frac{f(\xi)(\xi - z_0)^n}{(z - z_0)^{n+1}} \text{ on } \partial B_{r+\epsilon}(z_0)$$

shows that

$$\begin{aligned} f(z) = &\sum_{n=0}^{\infty} \frac{1}{2\pi i}\Big(\int_{\partial B_{R-\epsilon}(z_0)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \Big)(z - z_0)^n \\ &+ \sum_{n=1}^{\infty} \frac{1}{2\pi i}\Big(\int_{\partial B_{r+\epsilon}(z_0)} (\xi - z_0)^{n-1} f(\xi) d\xi \Big)(z - z_0)^{-n} \end{aligned}$$

Because the integrands $(\xi - z_0)^k f(\xi)$ have no singularities on the annulus, again, we may apply a geometric argument using 'cuts' to simplify the expression into its **Laurent series**:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

where

$$c_n = \frac{1}{2\pi i} \int_{\partial B_\epsilon(z_0)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \text{ for all } r < \epsilon < R \text{ and } n \in \mathbb{Z} \tag{1.1}$$

If f is defined and holomorphic on some deleted neighborhood $B_\epsilon(z_0) \setminus \{z_0\}$ but not defined on $\{z_0\}$, we say z_0 is an **isolated singularity** of f, and we write

$$\text{Res}(f, z_0) \triangleq c_{-1}$$

to denote the **residue** of f at z_0. Let $D \subseteq \mathbb{C}$ be a simply connected domain, and suppose holomorphic f is defined on D except at some finite numbers of singularities. Let $\gamma : [a, b] \to D$ be some **simple**[3] closed contour, so that the image of γ is a Jordan curve, and we may apply the **Jordan Curve Theorem** to distinguish between the interior and the exterior of γ. A simple closed contour γ is **positively oriented** if the winding number is positive in the region enclosed by γ. We may now state our version of **Cauchy's Residue Theorem**.

Theorem 1.1.2. (Cauchy's Residue Theorem) Let $D \subseteq \mathbb{C}$ be a simply connected domain, and let $\gamma : [a, b] \to D$ be some positively oriented simple closed contour. If f is is defined and holomorphic on D except at a finite set of points $\{z_1, \ldots, z_n\}$ that are all enclosed by γ, then

$$\int_\gamma f(\xi) d\xi = 2\pi i \sum_{j=1}^{n} \text{Res}(f, z_j)$$

There are many distinct rigorous proofs for Cauchy's residue Theorem. None of them are trivial by some geometric argument. Again, for brevity, we present a proof here using Green's Theorem. Suppose z_0 is an isolated singularity of f, and c_n are the coefficients of the Laurent series of f about z_0. There are three types of singularities depending on c_n. If $c_n = 0$ for all $n < 0$, then z_0 is said to be a **removable singularity**. By direct estimation[4], we see that if f is bounded on some deleted neighborhood of z_0, then z_0 is removable. This recognition is called **Riemann's removable singularity Theorem**. If there exists some sequence n_k of integers that converges to $-\infty$ such that $c_{n_k} \neq 0$ for all k, then we say z_0

[3] By simple, we mean $\gamma(t) = \gamma(s)$ if and only if $|t - s| = |a - b|$

[4] For each $n < 0$, let $\epsilon \to 0$ in Equation 1.1.

is an **essential singularity**. The last type of singularities is perhaps the most interesting. If there exists some $m < 0$ such that $c_m \neq 0$ and $c_n = 0$ for all $n < m$, we say z_0 is a **pole** of f with multiplicity m. In such case, obviously we may define some g by

$$g(z) \triangleq (z - z_0)^m f(z) \text{ for all } z \neq z_0$$

so that z_0 is merely a removable singularity of g, and $g(z_0) \neq 0$ after the removal. Now, because g is continuous at z_0, we see f is nonzero on some neighborhood around z_0, and we may compute on that neighborhood:

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} - \frac{m}{z - z_0} \text{ for all } z \neq z_0$$

This implies

$$\text{Res}\left(\frac{f'}{f}, z_0\right) = -m$$

Similarly, if z_0 is a **zero**[5] of f with multiplicity k, we may define g by

$$g(z) \triangleq (z - z_0)^{-k} f(z)$$

and compute

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} + \frac{k}{z - z_0} \text{ for all } z \neq z_0 \text{ in some neighborhood of } z_0$$

to deduce

$$\text{Res}\left(\frac{f'}{f}, z_0\right) = k$$

These observations together with Cauchy's Residue Theorem now give us the **Argument Principle**. Given simply connected domain $D \subseteq \mathbb{C}$, positively oriented simple closed contour $\gamma : [a, b] \to D$ and some f **meromorphic**[6] on D, if f has neither zeros nor poles on the image of γ, then

$$\int_\gamma \frac{f'(\xi)}{f(\xi)} d\xi = 2\pi i(Z - P)$$

[5]By z_0 being a zero of f with multiplicity k, we mean f is holomorphically defined on some neighborhood of z_0, $f(z_0) = 0$, and k is the smallest integer such that $c_k \neq 0$ where $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$. Note that zeros and poles are dual to each other.

[6]By f being meromorphic on open U, we mean that f is holomorphic on U except on a finite set of poles.

where Z and P are respectively the numbers of zeros and poles enclosed by γ counted with multiplicity.

Now, let D be some simply connected domain, let $\gamma : [a, b] \to D$ be some positively oriented simple closed contour, and let $f, g : D \to \mathbb{C}$ be two holomorphic function. If we require that $|g| < |f|$ on the image of γ, then obviously neither f, $f + g$ nor $1 + \frac{g}{f}$ can have a zero on the image of γ, so after we compute

$$\frac{(1 + \frac{g}{f})'}{1 + \frac{g}{f}} = \frac{(f + g)'}{f + g} - \frac{f'}{f}$$

we may apply the argument principle to f and g to conclude

$$Z_{f+g} - Z_f = \int_\gamma \frac{(1 + \frac{g}{f})'}{1 + \frac{g}{f}} d\xi$$

where Z_{f+g} and Z_f are the numbers of zeros of $f + g$ and f enclosed by γ counted with multiplicity. Moreover, if we define $\widetilde{\gamma} : [a, b] \to \mathbb{C}$ by

$$\widetilde{\gamma}(t) \triangleq 1 + \frac{g(t)}{f(t)}$$

we see

$$\int_{\widetilde{\gamma}} \frac{d\xi}{\xi} = \int_\gamma \frac{(1 + \frac{g}{f})'}{1 + \frac{g}{f}} d\xi = Z_{f+g} - Z_f$$

Therefore, once we note that $\widetilde{\gamma}$ as a simple closed contour lies in $\mathrm{Re}(z) > 0$ [7], and note that $\log(z)$ is well-defined on $\mathrm{Re}(z) > 0$ with derivative $\frac{1}{z}$[8], we may finally use Fundamental Theorem of Calculus to conclude the **Rouché's Theorem**:

$$Z_f = Z_{f+g}$$

We close this section by proving the **Open Mapping Theorem** and the **Maximum Modulus Principle**. Let $U \subseteq \mathbb{C}$ be some domain, and let $f : U \to \mathbb{C}$ be holomorphic and non-constant. Because open disks are also domains, and because by Identity Theorem, f can not be constant on any open disks contained by U, to prove f is an open map, we only have to prove $f(U)$ is open, without loss of generality.

Fix $w_0 = f(z_0)$, and define $g : U \to \mathbb{C}$ by

$$g(z) \triangleq f(z) - w_0$$

[7]This is because $|g| < |f|$ on γ.

[8]by real inverse function theorem.

Because g is non-constant, by Identity Theorem, there exists some closed disk $K \subseteq U$ that centers z_0 and contains no zeros of g other than z_0. Defining

$$r \triangleq \min_{\partial K} |g| > 0 \text{ and } D \triangleq B_r(w_0)$$

we may reduce the proof into proving $D \subseteq f(K^\circ)$. For each $w_1 \neq w_0 \in D$, define $h : U \to \mathbb{C}$ by

$$h(z) \triangleq f(z) - w_1$$

Observing

$$|g(z)| \geq r > |w_0 - w_1| = |h(z) - g(z)| \text{ for all } z \in \partial K$$

we may use Rouché's Theorem to conclude that h indeed have some zero in K°. We have shown $f(U)$ is indeed open, which implies $\{|f(z)| : z \in U\}$ is open in $\mathbb{R}$, giving maximum modulus principle as a corollary.

1.2 Riemann Mapping Theorem

1.3 Proof of Cauchy's Integral and Residue Theorem

Chapter 2

Several Complex Variables

2.1 Untitled

Given some open subset U of $\mathbb{C}^n$, and some complex-valued function $f : U \to \mathbb{C}$, we say f is **holomorphic** in z_1 if for all $(z_2, \dots, z_n) \in \mathbb{C}^{n-1}$, the function defined by $z_1 \mapsto f(z_1, \dots, z_n)$ is holomorphic. In the context of theory of several complex variables, we use the notation $\epsilon \triangleq (\epsilon_1, \dots, \epsilon_n)$ to denote a tuple of positive real numbers, and we use the notation $B_\epsilon(a) \subseteq \mathbb{C}^n$ to denote **polydisc**

$$\{z \in \mathbb{C}^n : |z_i - a_i| < \epsilon_i \text{ for all } i\}$$

By repeatedly applying single variable version of Cauchy's integral formula, we see that if $f : U \to \mathbb{C}$ is holomorphic in all of its variables and $\overline{B_\epsilon(a)} \subseteq U$, then for all $z \in B_\epsilon(a)$, we have **Cauchy's integral formula in several variables**:

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\xi_1 - a_1| = \epsilon_1} \cdots \int_{|\xi_n - a_n| = \epsilon_n} \frac{f(\xi_1, \dots, \xi_n)}{(\xi_1 - z_1) \cdots (\xi_n - z_n)} d\xi_n \cdots d\xi_1$$

In this note, we use multi-index notation $\alpha = (\alpha_1, \dots, \alpha_n)$, and adapt the notation

$$\alpha + 1 \triangleq (\alpha_1 + 1, \dots, \alpha_n + 1)$$

For each $\xi \in \partial B_\epsilon(a)$, because we have the absolute convergence

$$\sum_{\alpha_1} \cdots \sum_{\alpha_n} \left| \frac{(z-a)^\alpha}{(\xi - a)^{\alpha+1}} \right| \in \mathbb{R}$$

by Fubini's Theorem, for each sequence α^k that runs through $\mathbb{N}^n$, we have

$$\sum_k \frac{(z-\alpha)^{\alpha^k}}{(\xi - a)^{\alpha^k + 1}} = \sum_{\alpha_1} \cdots \sum_{\alpha_n} \frac{(z-a)^\alpha}{(\xi - a)^{\alpha+1}}$$

so it make sense to write the multiple series into a single series

$$\sum_{\alpha} \frac{(z-a)^{\alpha}}{(\xi-a)^{\alpha+1}} \triangleq \sum_{\alpha_1} \cdots \sum_{\alpha_n} \frac{(z-a)^{\alpha}}{(\xi-a)^{\alpha+1}}$$

Now, because the convergence of $\sum_{\alpha} \frac{f(\xi)(z-a)^{\alpha}}{(\xi-a)^{\alpha}}$ is uniform on $\partial B_{\epsilon}(a)$, we have

$$f(z) = \frac{1}{(2\pi i)^n} \sum_{\alpha} c_{\alpha}(z-a)^{\alpha}$$

where

$$c_{\alpha} = \int_{|\xi_1-a_1|=\epsilon_1} \cdots \int_{|\xi_n-a_n|=\epsilon_n} \frac{f(\xi)}{(\xi-a)^{\alpha+1}} d\xi$$

2.2 Hartog's Theorem on separate holomorphicity

2.3 Bijective holomorphic map is biholomorphic

Chapter 3

Some Linear Algebra

3.1 Complexification

Let V be a finite dimensional real vector space. By **complexification** $V_{\mathbb{C}}$ of V, formally, we mean the complex vector space $V \times V$ defined by

$$(a+bi)(v_1,v_2) \triangleq (av_1 - bv_2, av_2 + bv_1)$$

To make visual computation easier, we use the notation $v_1 + iv_2$ to denote $(v_1,v_2) \in V_{\mathbb{C}}$. Note that if V has a $\mathbb{R}$-basis $\{v_1,\dots,v_n\}$, then clearly $V_{\mathbb{C}}$ has $\mathbb{C}$-basis $\{v_1,\dots,v_n\}$, so we know $\text{Dim}_{\mathbb{C}}(V_{\mathbb{C}}) = \text{Dim}_{\mathbb{R}}(V)$. By the term **complex conjugation** on $V_{\mathbb{C}}$, we mean the $\mathbb{R}$-linear map

$$\overline{v_1 + iv_2} \triangleq v_1 - iv_2$$

which is not $\mathbb{C}$-linear. If W is another real vector space, and $f : V \to W$ is some $\mathbb{R}$-linear map, then the **complexified map** $f_{\mathbb{C}} : V_{\mathbb{C}} \to W_{\mathbb{C}}$ is the $\mathbb{C}$-linear map defined by

$$f_{\mathbb{C}}(v_1 + iv_2) \triangleq f(v_1) + if(v_2)$$

Moreover, if g is an inner product on V, we may also extend g onto $V_{\mathbb{C}}$ by defining

$$g_{\mathbb{C}}(u_1 + iu_2, v_1 + iv_2) \triangleq g(u_1,v_1) + g(u_2,v_2) + i[g(u_2,v_1) - g(u_1,v_2)]$$

By an **almost complex** structure on V, we mean some $\mathbb{R}$-linear map $J \in \text{End}(V)$ whose squared is the negate of the identity map of V. Clearly, if J is an almost complex structure on V, then we may make V into a complex vector space by defining

$$(a+bi)v \triangleq av + bJ(v)$$

Notably, after this assignment of $\mathbb{C}$-action, we see that $\text{Dim}_{\mathbb{R}}(V)$ must be even, since if $\{v_1,\dots,v_n\}$ form a $\mathbb{C}$-basis for V, then $\{v_1, J(v_1),\dots,v_n, J(v_n)\}$ form a $\mathbb{R}$-basis for V.

Now, suppose V is a finite dimensional real vector space with an almost complex structure J, and consider the complexified map $J_{\mathbb{C}} \in \text{End}_{\mathbb{C}}(V_{\mathbb{C}})$. Routine computation shows that the squared of $J_{\mathbb{C}}$ is also the negate of the identity map of $V_{\mathbb{C}}$. Let $V^{1,0}, V^{0,1}$ be the eigenspace of $V_{\mathbb{C}}$ with respect to function $J_{\mathbb{C}} \in \text{End}_{\mathbb{C}}(V_{\mathbb{C}})$ eigenvalues $\pm i$. Computing for all $v \in V_{\mathbb{C}}$ that

$$v = \Big(\frac{v - iJ_{\mathbb{C}}(v)}{2}\Big) + \Big(\frac{v + iJ_{\mathbb{C}}(v)}{2}\Big) \in V^{1,0} \oplus V^{0,1}$$

we see that $V_{\mathbb{C}}$ is the direct sum of $V^{1,0}$ and $V^{0,1}$. Moreover, because the complex conjugation on $V_{\mathbb{C}}$ is an $\mathbb{R}$-isomorphism between $V^{1,0}$ and $V^{0,1}$[1], we see

$$\text{Dim}_{\mathbb{C}}(V^{1,0}) = \text{Dim}_{\mathbb{C}}(V^{0,1}) = \frac{\text{Dim}_{\mathbb{C}}(V_{\mathbb{C}})}{2}$$

This together with the clearly injective $\mathbb{R}$-linear map $\pi^{1,0} : V \to V^{1,0}$ defined by

$$\pi^{1,0}(v) \triangleq \frac{v - iJ(v)}{2}$$

shows[2] that V and $V^{1,0}$ are $\mathbb{R}$-isomorphic.[3] We adapt the notation

$$\bigwedge^{p,q} V \triangleq \bigwedge^{p} V^{1,0} \otimes_{\mathbb{C}} \bigwedge^{q} V^{0,1}$$

and refer to elements α of $\bigwedge^{p,q} V$ as **bidegree** (p, q).

Theorem 3.1.1. (Decomposition of wedge product of $V_{\mathbb{C}}$ into bidegrees) There exists a $\mathbb{C}$-linear isomorphism

$$\bigwedge^{k} V_{\mathbb{C}} \cong \bigoplus_{p+q=k} \bigwedge^{p,q} V$$

Proof. Let $\{e_1, \dots, e_n\}$ be a $\mathbb{C}$-basis for $V^{1,0}$. Fix some $p + q = k$. We know

$$\Big\{e_{i_1} \wedge \cdots \wedge e_{i_p} \otimes \overline{e_{j_1}} \wedge \cdots \wedge \overline{e_{j_q}} : 1 \le i_1 < \cdots < i_p \le n \text{ and } 1 \le j_1 < \cdots < j_q \le n\Big\}$$

forms a $\mathbb{C}$-basis for $\bigwedge^{p,q} V$. Because $\{e_1, \dots, e_n, \overline{e_1}, \dots, \overline{e_n}\}$ forms a $\mathbb{C}$-basis for $V_{\mathbb{C}}$, the $\mathbb{C}$-linear map $\varphi_{p,q} : \bigwedge^{p,q} V \to \bigwedge^{k} V_{\mathbb{C}}$ determined by

$$e_{i_1} \wedge \cdots \wedge e_{i_p} \otimes \overline{e_{j_1}} \wedge \cdots \wedge \overline{e_{j_q}} \mapsto e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge \overline{e_{j_1}} \wedge \cdots \wedge \overline{e_{j_q}}$$

is injective. One may now check that the $\mathbb{C}$-linear map

$$\bigoplus_{p+q=k} \varphi_{p,q} : \bigoplus_{p+q=k} \bigwedge^{p,q} V \to \bigwedge^{k} V_{\mathbb{C}}$$

forms a $\mathbb{C}$-linear isomorphism. ■

[1] This is a result of routine computation.

[2] $\text{Dim}_{\mathbb{R}}(V) = \text{Dim}_{\mathbb{C}}(V_{\mathbb{C}}) = 2\,\text{Dim}_{\mathbb{C}}(V^{1,0}) = \text{Dim}_{\mathbb{R}}(V^{1,0})$

[3] Noting that the imaginary unit i forms an almost complex structure on $V^{1,0}$, one may even shows the isomorphism $\pi^{1,0}(J(v)) = i\pi^{1,0}(v)$.

3.2 Hodge star operator

Let (V, g) be an oriented d-dimensional real inner product space whose orientation is determined by the orthonormal basis $\{e_1, \ldots, e_d\}$, for each exterior power $\bigwedge^k V$, we may well define (?) an inner product g^k on $\bigwedge^k V$ such that for every orthonormal basis $\{e_1, \ldots, e_d\}$ of V, the basis $\{e_I \in \bigwedge^k V : I = \{i_1 < \cdots < i_k\}\}$ is orthonormal for $\bigwedge^k V$, and well define (?) the **Hodge star operator** $\star : \bigwedge^k V \to \bigwedge^{d-k} V$ so that

$$\alpha \wedge \star\beta = g^k(\alpha, \beta) \cdot e_1 \wedge \cdots \wedge e_d \text{ for all } \alpha, \beta \in \bigwedge^k V$$

Let (V, g) be a finite dimensional real inner product space, and let $J : V \to V$ be an almost complex structure compatible with g. Let $\omega \in \mathrm{Hom}^2(V \times V, \mathbb{R})$ be the fundamental form associated with (V, g, J). I read from your note and also the textbook that

$$\omega \in \bigwedge^{1,1} V^* \tag{3.1}$$

where $\bigwedge^{1,1} W$ is defined by

$$\bigwedge^{1,1} W \triangleq \bigwedge W^{1,0} \otimes_{\mathbb{C}} \bigwedge W^{0,1}$$

My question is: How does statement 3.1 make sense? Shouldn't $\bigwedge^{1,1} V^*$ be the space of bilinear maps

Thank you for your reply. I just realized that I wrongly identified $\bigwedge^{1,1} V^*$. But I still can't piece things together to show that

$$\bigwedge^{1,1} V^* \text{ is the space of} \tag{3.2}$$

$$g_{\mathbb{C}}(u_1 + iu_2, v_1 + iv_2) \triangleq g(u_1, v_1) + g(u_2, v_2) + i[g(u_2, v_1) - g(u_1, v_2)]$$

and

$$\omega_{\mathbb{C}}(u_1 + iu_2, v_1 + iv_2) \triangleq \omega(u_1, v_1) + \omega(u_2, v_2) + i[\omega(u_2, v_1) - \omega(u_1, v_2)]$$

$$\omega_{\mathbb{C}}(z_1 v_1, z_2 v_2) \triangleq z_1 \overline{z_2} \omega(v_1, v_2)$$

$$\begin{aligned}\bigwedge^{1,1} V^* &= \bigwedge^1 (V^*)^{1,0} \otimes_{\mathbb{C}} \bigwedge^1 (V^*)^{0,1} \\ &= (V^*)^{1,0} \otimes_{\mathbb{C}} (V^*)^{0,1} \\ &= \mathrm{Hom}^2([(V^*)^{1,0}]^* \times [(V^*)^{0,1}]^*, \mathbb{C}) \\ &= \mathrm{Hom}^2(V^{1,0} \times V^{0,1}, \mathbb{C})\end{aligned}$$

$$\mathrm{Hom}^2(\bigwedge (V^*)^{1,0} \times \bigwedge (V^*)^{0,1}, \mathbb{C})$$

by definition of tensor product? How cam an element $\omega \in \mathrm{Hom}^2(V \times V, \mathbb{R})$ be identified as an element of $\mathrm{Hom}^2(\bigwedge (V^*)^{1,0} \times \bigwedge (V^*)^{0,1}, \mathbb{C})$?

I realized I had the completely wrong understanding. Please confirm with me if my current understanding is correct. The statement

$$\omega \in \bigwedge^{1,1} V^* \tag{3.3}$$

actually means

$$\begin{aligned}\omega_{\mathbb{C}} \in \bigwedge^2 V_{\mathbb{C}}^* &= \text{ alternating subspace of } V_{\mathbb{C}}^* \otimes V_{\mathbb{C}}^* && (3.4)\\ &= \text{ alternating subspace of } \mathrm{Hom}^2(V_{\mathbb{C}} \times V_{\mathbb{C}}, \mathbb{C}) && (3.5)\\ &= \bigwedge^{2,0} V^* \oplus \bigwedge^{1,1} V^* \oplus \bigwedge^{0,2} V^* && (3.6)\end{aligned}$$

and that the components of $\omega_{\mathbb{C}}$ in $\bigwedge^{2,0} V^*$ and $\bigwedge^{0,2} V^*$ are both zero, where we define $\omega_{\mathbb{C}}$ by

$$\begin{aligned}&\omega_{\mathbb{C}}(a_1 v_1 + i b_1 v_2, a_2 w_1 + i b_2 w_2) && (3.7)\\ &\triangleq a_1 a_2 \omega(v_1, w_1) - b_1 b_2 \omega(v_2, w_2) + i[b_1 a_2 \omega(v_2, w_1) + a_1 b_2 \omega(v_1, w_2)] && (3.8)\end{aligned}$$

3.3 Fundamental form and complex inner product induced by compatible real inner product and almost complex structure

For each finite dimensional real inner product space (V, g), an almost complex structure $J : V \to V$ is said to be **compatible** with g if

$$g(J(v_1), J(v_2)) = g(v_1, v_2)$$

Given such compatible (V, g, J), one see that the decomposition $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ is orthogonal, and we associate it with its **fundamental form** $\omega \in \mathrm{Hom}^2(V \times V, \mathbb{R})$ by

$$\omega(v, w) \triangleq g(J(v), w)$$

It is easy to see ω is always alternating, and that $h \triangleq g - i\omega$ forms a complex inner product when V is viewed as a complex vector space.

Chapter 4

Complex Manifold

4.1 Definition and Examples

Given some topological manifold M, by a **holomorphic chart**, we mean an open subset $U \subseteq M$ and a topological embedding $\varphi : U \to \mathbb{C}^n$. For two complex charts φ_i and φ_j to be **compatible**, the function $\varphi_i \circ \varphi_j|_{\varphi_j(U_i \cap U_j)}$ has to be holomorphic. A collection of complex charts is said to be a **holomorphic atlas** if the collection covers the whole M and all pairs of the charts in the collection are compatible. A **complex structure** on M if exists, is just a maximal holomorphic atlas. By a **complex manifold**, we mean a topological manifold together with a complex structure.

The first example of complex manifold is the **complex projective space**. Define an equivalence relation on $\mathbb{C}^{n+1} \setminus \{0\}$ by

$$z \sim w \overset{\triangle}{\iff} z = \lambda w \text{ for some } \lambda \in \mathbb{C}^*$$

The complex projective space is defined to be the quotient topological space $\mathbb{P}^n \triangleq (\mathbb{C}^{n+1} \setminus \{0\}) / \sim$. We use the notation $(z_0 : \cdots : z_n)$ to denote elements of $\mathbb{P}^n$. Clearly,

$$U_i \triangleq \{(z_0 : \cdots : z_n) \in \mathbb{P}^n : z_i \neq 0\}$$

forms an open cover of $\mathbb{P}^n$, and the maps

$$\varphi_i : U_i \to \mathbb{C}^n, \quad (z_0 : \cdots : z_n) \mapsto (\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_n}{z_i})$$

forms a holomorphic atlas.

The second example is the **complex tori**. Define an equivalence relation on $\mathbb{C}^n$ by

$$z \sim w \overset{\triangle}{\iff} \text{For all } j \text{ there exists some } a, b \in \mathbb{Z} \text{ such that } z_j - w_j = a + ib$$

The complex tori is defined to be the quotient topological space $\mathbb{C}^n/\sim$. Let π be the quotient map. Clearly for all $z \in \mathbb{C}^n$, if we let $\epsilon = (\frac{1}{2}, \ldots, \frac{1}{2})$, then $\pi(B_\epsilon(z))$ is open in the tori.