Friedberg Ch 6.1

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In this note, V always stand for a vector space over $\mathbb{F}$

Definition and Theorem

Definition 1. *An complex **inner product** on V, is a function from $V \times V$ to $\mathbb{C}$, satisfy the following*

$$\textbf{(a) } \langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$$
$$\textbf{(b) } \langle cx, y\rangle = c\langle x, y\rangle$$
$$\textbf{(c) } \overline{\langle x, y\rangle} = \langle y, x\rangle$$
$$\textbf{(d) } \forall x \neq 0, \langle x, x\rangle \in \mathbb{R}^+$$

Definition 2. *A vector space V equipped with an inner product on V, is an **inner product space***

Definition 3. *Let $A \in M_{n\times n}(\mathbb{F})$. The **conjugate transpose** A^* of A is defined by*

$$A^*_{i,j} = \overline{A_{j,i}}$$

Theorem 1. *Let V be an inner product space, $x, y, z \in V$, and $c \in \mathbb{F}$*

(a) $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$
(b) $\langle x, cy\rangle = \overline{c}\langle x, y\rangle$
(c) $\langle x, 0\rangle = \langle 0, x\rangle = 0$
(d) $\langle x, x\rangle = 0 \iff x = 0$
(e) $\forall x \in V, \langle x, y\rangle = \langle x, z\rangle \implies y = z$

Proof. **(a)**

$$\langle x, y+z\rangle = \overline{\langle y+z, x\rangle} = \overline{\langle y, x\rangle + \langle z, x\rangle} = \overline{\langle y, x\rangle} + \overline{\langle z, x\rangle} = \langle x, y\rangle + \langle x, z\rangle$$

(b)

$$\langle x, cy\rangle = \overline{\langle cy, x\rangle} = \overline{c\langle y, x\rangle} = \overline{c}\,\overline{\langle y, x\rangle} = \overline{c}\langle x, y\rangle$$

(c)

$$\langle 0, x\rangle = 0\langle 1, x\rangle = 0 = \overline{0}\langle x, 1\rangle = \langle x, 0\rangle$$

(d)

$(\longleftarrow)$

By **(c)**

$(\longrightarrow)$

By definition

(e)

$\langle x, y\rangle = \langle x, z\rangle \implies 0 = \langle x, y\rangle - \langle x, z\rangle = \langle x, y\rangle + \overline{-1}\langle x, z\rangle = \langle x, y - z\rangle$

$\langle y - z, y - z\rangle = 0 \implies y - z = 0 \implies y = z$

■

Definition 4. *Let V be an inner product space. The* ***norm*** *or* ***length*** *$\|x\|$ of x, is defined by*

$$\|x\| = \sqrt{\langle x, x\rangle}$$

Theorem 2. *Let V be an inner product space, $x, y \in V$, and $c \in \mathbb{F}$*

(a) $\|cx\| = |c| * \|x\|$

(b) $\|x\| = 0 \iff x = 0$

(c) $|\langle x, y\rangle| \leq \|x\| * \|y\|$

(d) $\|x + y\| \leq \|x\| + \|y\|$

Proof. **(a)**

$\|cx\| = \sqrt{\langle cx, cx\rangle} = \sqrt{c\bar{c}\langle x, x\rangle} = |c|\sqrt{\langle x, x\rangle} = |c| \cdot \|x\|$

(b)

$(\longleftarrow)$

$\|0\| = \sqrt{\langle 0, 0\rangle} = \sqrt{0} = 0$

$(\longrightarrow)$

$\|x\| = 0 \implies \sqrt{\langle x, x\rangle} = 0 \implies \langle x, x\rangle = 0 \implies x = 0$

(c)

Let $z = y - \frac{\langle y,x\rangle}{\|x\|^2}x$

So $\langle z, x\rangle = \langle y - \frac{\langle y,x\rangle}{\|x\|^2}x, x\rangle = \langle y, x\rangle - \frac{\langle y,x\rangle}{\|x\|^2}\langle x, x\rangle = \langle y, x\rangle - \langle y, x\rangle = 0$

Notice $y = \frac{\langle y,x\rangle}{\|x\|^2}x + z$

$\langle x, y\rangle\langle y, x\rangle = \langle x, \frac{\langle y,x\rangle}{\|x\|^2}x\rangle\langle\frac{\langle y,x\rangle}{\|x\|^2}x, x\rangle = \overline{\frac{\langle y,x\rangle}{\|x\|^2}}\langle x, x\rangle\frac{\langle y,x\rangle}{\|x\|^2}\langle x, x\rangle =$

$\langle y,y\rangle = \langle \frac{\langle y,x\rangle}{\|x\|^2}x+z, \frac{\langle y,x\rangle}{\|x\|^2}x+z\rangle = \langle \frac{\langle y,x\rangle}{\|x\|^2}x, \frac{\langle y,x\rangle}{\|x\|^2}x\rangle + \langle z,z\rangle = \frac{\langle y,x\rangle\langle x,y\rangle}{\|x\|^4}\langle x,x\rangle + \langle z,z\rangle = \frac{\langle y,x\rangle\langle x,y\rangle}{\|x\|^2} + \langle z,z\rangle$

$\langle x,x\rangle\langle y,y\rangle = \langle y,x\rangle\langle x,y\rangle + \langle x,x\rangle\langle z,z\rangle$

So $\langle x,x\rangle\langle y,y\rangle \geq \langle y,x\rangle\langle x,y\rangle$

$\langle x,y\rangle\langle y,x\rangle \leq \langle x,x\rangle\langle y,y\rangle \implies \langle x,y\rangle\overline{\langle x,y\rangle} \leq \langle x,x\rangle\langle y,y\rangle$

$\langle x,y\rangle\overline{\langle x,y\rangle} \leq \langle x,x\rangle\langle y,y\rangle \implies \sqrt{\langle x,y\rangle\overline{\langle x,y\rangle}} \leq \sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle}$
$\sqrt{\langle x,y\rangle\overline{\langle x,y\rangle}} \leq \sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle} \implies |\langle x,y\rangle| \leq \|x\| * \|y\|$

(d)

$\langle x+y,x+y\rangle = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$

$|\langle x,y\rangle| \leq \|x\| * \|y\|$ and $|\langle y,x\rangle| \leq \|x\| * \|y\|$ give us $\langle x+y,x+y\rangle \leq \langle x,x\rangle + \langle y,y\rangle + 2\|x\| * \|y\| = (\sqrt{\langle x,x\rangle} + \sqrt{\langle y,y\rangle})^2$

So $\sqrt{\langle x+y,x+y\rangle} \leq \sqrt{\langle x,x\rangle} + \sqrt{\langle y,y\rangle}$

That is $\|x+y\| \leq \|x\| + \|y\|$

■

Definition 5. *Let V be an inner product space, and $x,y \in V$. x and y are **orthogonoal** if $\langle x,y\rangle = 0$. A subset S if **orthogonal** if every two distinct vector in S are orthogonal. x is a **unit vector** if $\|x\| = 1$. A subset S is **orthonormal** if it is orthogonal and consists of only unit vectors*

Exercises

1.

1.(a)

Proof. Yes ■

1.(b)

Proof. BLANK ■

1.(c)

Proof. No, $\langle u,cv\rangle = \overline{c}\langle u,v\rangle$ ■

1.(d)

Proof. No, $\forall c \in \mathbb{R}$, $\langle v, w\rangle = c\sum_{i=1}^{n} v_i w_i$ is an inner product space ■

1.(e)

Proof. No, it works in infinite dimensions too ■

1.(f)

Proof. No, every matrix have conjugate transpose ■

1.(g)

Proof. No, as long as $\langle x, y-z\rangle = 0$, that is $x \perp y - z$, $\langle x, y\rangle = \langle x, z\rangle$ ■

1.(h)

Proof. Yes, $\langle y, y\rangle = 0 \implies y = 0$ ■

2.

Proof. $\langle x, y\rangle = 8 + 5i$

$\|x\| = \sqrt{7}$

$\|y\| = \sqrt{14}$

$\|x + y\| = \sqrt{37}$

$|\langle x, y\rangle| = \sqrt{89} \leq \sqrt{98} = \|x\|\|y\|$

$\|x+y\| \leq \|x\| + \|y\| \iff \sqrt{37} \leq \sqrt{7} + \sqrt{14} \iff 37 \leq 21 + 2\sqrt{98} \iff 8 \leq \sqrt{98}$ ■

3.

Proof. $\langle f, g\rangle = 1$

$\|f\| = \sqrt{\frac{1}{3}}$

$\|g\| = \sqrt{\frac{1}{2}(e^2 - 1)}$

$\|f + g\| = \sqrt{\frac{1}{2}e^2 + \frac{11}{6}}$

$\langle f, g\rangle \leq \|f\| * \|g\| \iff 1 \leq \sqrt{\frac{1}{6}(e^2 - 1)} \iff 6 \leq e^2 - 1$

$\|f+g\| \leq \|f\|+\|g\| \iff \sqrt{\frac{1}{2}e^2+\frac{11}{6}} \leq \sqrt{\frac{1}{3}}+\sqrt{\frac{1}{2}(e^2-1)} \iff \frac{1}{2}e^2+\frac{11}{6} \leq \frac{1}{3}+\frac{1}{2}(e^2-1)2\sqrt{\frac{1}{6}(e^2-1)} \iff \frac{11}{6}-\frac{1}{3}+\frac{1}{2} \leq 2\sqrt{\frac{1}{6}(e^2-1)} \iff 1 \leq \sqrt{\frac{1}{6}(e^2-1)} \iff 7 \leq e^2$ ■

8.

8.(a)

Proof. $\langle (1,1),(1,1)\rangle = 1-1=0$ ■

8.(b)

Proof. $\langle A+C,B\rangle = tr((A+C)+B) = tr(A)+tr(C)+tr(B) \neq tr(A)+2tr(B)+tr(C) = tr(A+B)+tr(C+B) = \langle A,B\rangle+\langle C,B\rangle$ if $tr(B)\neq 0$ ■

8.(c)

Proof. Let $f = x^2+x+1$ and $g = x+1$

$\langle f,g\rangle = \frac{17}{6} \neq \frac{11}{6} = \langle g,f\rangle$ ■

10.

Proof. $\|x+y\|^2 = \langle x+y,x+y\rangle = \langle x,x\rangle+\langle x,y\rangle+\langle y,x\rangle+\langle y,y\rangle = \langle x,x\rangle+\langle y,y\rangle = \|x\|^2+\|y\|^2$ ■

12.

Proof. We prove by induction

$$\text{Base step: } \|a_1v_1+a_2v_2\|^2 = |a_1|^2\|v_1\|^2+|a_2|^2\|v_2\|^2$$

$\|a_1v_1+a_2v_2\|^2 = \langle a_1v_1+a_2v_2, a_1v_1+a_2v_2\rangle = \langle a_1v_1,a_1v_1\rangle+\langle a_2v_2,a_1v_1\rangle+\langle a_1v_1,a_2v_2\rangle+\langle a_2v_2,a_2v_2\rangle = |a_1|^2\langle v_1,v_1\rangle+a_2\overline{a_1}\langle v_2,v_1\rangle+a_1\overline{a_2}\langle v_1,v_2\rangle+|a_2|^2\langle v_2,v_2\rangle = |a_1|^2\|v_1\|^2+|a_2|^2\|v_2\|^2$

$$\text{Induction step:}$$
$$\|\sum_{i=1}^n a_iv_i\|^2 = \sum_{i=1}^n |a_i|^2\|v_i\|^2 \implies \|\sum_{i=1}^{n+1} a_iv_i\|^2 = \sum_{i=1}^{n+1}|a_i|^2\|v_i\|^2$$

$\langle \sum_{i=1}^n a_iv_i, a_{n+1}v_{n+1}\rangle = \sum_{i=1}^n\langle a_iv_i,a_{n+1}v_{n+1}\rangle = 0$

Then $\|\sum_{i=1}^{n+1} a_iv_i\|^2 = \|\sum_{i=1}^n a_iv_i+a_{n+1}v_{n+1}\|^2 = \|\sum_{i=1}^n a_iv_i\|^2+\|a_{n+1}v_{n+1}\|^2 = \sum_{i=1}^n |a_i|\|v_i\|^2+|a_{n+1}|^2\|v_{n+1}\|^2 = \sum_{i=1}^{n+1}|a_i|^2\|v_i\|^2$ ■

15.

Prove $\langle x, y\rangle = \|x\|\|y\|$ if and only if $x = ay, \exists a \in \mathbb{F}$

Proof. $(\longleftarrow)$

$\langle x, y\rangle = \langle ay, y\rangle = a\langle y, y\rangle = a\|y\|^2 = a\|y\|\|y\| = \|ay\|\|y\| = \|x\|\|y\|$

$(\longrightarrow)$

Let $a = \frac{\langle x,y\rangle}{\|y\|^2}$

Let $z = x - ay$

$\langle z, y\rangle = \langle x, y\rangle - a\langle y, y\rangle = \langle x, y\rangle - a\|y\|^2 = 0$

$x = ay + z \implies \|x\|^2 = \|ay + z\|^2 = \|ay\|^2 + \|z\|^2 \implies \|z\|^2 = \|x\|^2 - |a|^2\|y\|^2$

$a = \frac{\langle x,y\rangle}{\|y\|^2} = \frac{\|x\|\|y\|}{\|y\|^2} = \frac{\|x\|}{\|y\|}$

So $\|z\|^2 = \|x\|^2 - \|x\|^2 = 0$

Then $\langle z, z\rangle = 0$

So $z = 0$, then $x = ay$

■

20.

20.(a)

Proof. $\frac{1}{4}\|x + y\|^2 - \frac{1}{4}\|x - y\|^2 = \frac{1}{4}\langle x + y, x + y\rangle - \frac{1}{4}\langle x - y, x - y\rangle = \frac{1}{4}(\langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle) - \frac{1}{4}(\langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle) = \frac{1}{2}\langle x, y\rangle + \frac{1}{2}\langle y, x\rangle = \langle x, y\rangle$ ■

20.(b)

Proof. $\frac{1}{4}\sum_{k=1}^4 i^k\|x + i^k y\|^2 = \frac{1}{4}\sum_{k=1}^4 i^k\langle x + i^k y, x + i^k y\rangle = \frac{1}{4}[i\langle x + iy, x + iy\rangle - \langle x - y, x - y\rangle - i\langle x - iy, x - iy\rangle + \langle x + y, x + y\rangle] = \frac{1}{4}[\langle x + y, x + y\rangle - \langle x - y, x - y\rangle + \langle ix - y, x + iy\rangle - \langle ix + y, x - iy\rangle] = \frac{1}{4}\langle x + y, x + y\rangle - \frac{1}{4}\langle x - y, x - y\rangle + \frac{1}{4}[i\langle x, x + iy\rangle - \langle y, x + iy\rangle - i\langle x, x - iy\rangle - \langle y, x - iy\rangle] = \frac{1}{2}\langle x, y\rangle + \frac{1}{2}\langle y, x\rangle + \frac{i}{4}[\langle x, x + iy\rangle - \langle x, x - iy\rangle] - \frac{1}{4}[\langle y, x + iy\rangle + \langle y, x - iy\rangle] = \langle x, y\rangle + \frac{i}{4}\langle x, 2iy\rangle - \frac{1}{4}\langle y, 2x\rangle = \frac{1}{2}\langle x, y\rangle + \frac{1}{2}\langle y, x\rangle + \frac{1}{2}\langle x, y\rangle - \frac{1}{2}\langle y, x\rangle = \langle x, y\rangle$ ■

21.

21.(a)

Proof. $A_1^* = [\frac{1}{2}(A + A^*)]^* = \frac{1}{2}(A^* + A) = A_1$

$A_2^* = [\frac{1}{2i}(A - A^*)]^* = \frac{1}{-2i}(A^* - A) = \frac{1}{2i}(A - A^*) = A_2$ ∎

21.(b)

Proof. $A = B_1 + iB_2 \implies A^* = B_1^* - iB_2^* = B_1 - iB_2$

$A + A^* = 2B_1$

$A - A^* = 2iB_2$

$B_1 = \frac{1}{2}(A + A^*)$

$B_2 = \frac{1}{2i}(A - A^*)$ ∎

29.

Proof. $\langle x+z, y\rangle = [x+z, y] + i[x+z, iy] = [x, y] + [z, y] + i\{[x, iy] + [z, iy]\} = [x, y] + i[x, iy] + [z, y] + i[z, iy] = \langle x, y\rangle + \langle z, y\rangle$

$\langle cx, y\rangle = [cx, y] + i[cx, iy] = c[x, y] + ci[x, iy] = c\{[x, y] + i[x, iy]\} = c\langle x, y\rangle$

$\langle x, y\rangle - \overline{\langle y, x\rangle} = [x, y] + i[x, iy] - \overline{[y, x] + i[y, ix]} = [x, y] - [y, x] + i\{[x, iy] + [y, ix]\} = i\{[x, iy] + [y, ix]\} = i\{[x - y, iy] + [y - x, ix]\} = i\{[x - y, iy] - [x - y, ix]\} = i[x - y, i(y - x)] = -i[y - x, i(y - x)] = 0 \implies \langle x, y\rangle = \overline{\langle y, x\rangle}$

$\langle x, x\rangle = 0 \implies [x, x] + i[x, ix] = 0 \implies [x, x] = 0 \implies x = 0$

∎