Chapter 5

Advanced Calculus HW

5.1 HW1

Question 18

1. Prove that the following statements are equivalent: for a given sequence $\{x_n\}$,
   (a) for every $0 < \epsilon \in \mathbb{Q}$, there exists $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ whenever $n \geq N$.
   (b) for every $0 < \epsilon \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ whenever $n \geq N$.

Proof. From (b) to (a), just observe $\mathbb{Q} \subseteq \mathbb{R}$ and we are done.Now we prove from (a) to (b).

Because $\mathbb{Q}$ is dense in $\mathbb{R}$, for all $\epsilon \in \mathbb{R}^+$, we can pick $\epsilon' \in \mathbb{Q}$ such that $0 < \epsilon' < \epsilon$. By (a), we know there exists $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon' < \epsilon$ whenever $n \geq N$. This finish the proof. ∎

Question 19

2. Let $\{x_n\}_{n=1}^{\infty}$ be a monotone increasing sequence such that

$$x_{n+1} - x_n \leq \frac{1}{n}.$$

Determine whether the sequence converges. (If yes, prove it; if not, disprove it or give a counterexample.)

Proof. No, consider p-series. The sequence $\{x_n\}_{n=1}^{\infty}$ defined by $x_i := \sum_{j=1}^{i} \frac{1}{j}$ is monotone increasing, satisfying the desired property, and from our knowledge, diverge. ∎

Question 20

Let $M_{n\times m}$ be the collection of all $n \times m$ matrices with real entries. Define a function $\|\cdot\| : M_{n\times m} \to \mathbb{R}$ by

$$\|A\| = \sup\left\{\frac{\|Ax\|_2}{\|x\|_2} : x \in \mathbb{R}^m, x \neq 0\right\},$$

where we recall that $\|\cdot\|_2$ is the 2-norm on Euclidean space given by

$$\|Ax\|_2 = \left(\sum_{i=1}^{k} x_i^2\right)^{1/2} \qquad \text{if } x \in \mathbb{R}^k.$$

Show that:

1. $\|A\| = \sup\{\|Ax\|_2 : x \in \mathbb{R}^m, \|x\|_2 = 1\} = \inf\{M \in \mathbb{R} : \|Ax\|_2 \le M\|x\|_2 \, \forall x \in \mathbb{R}^m\}$.
2. $\|Ax\|_2 \le \|A\|\|x\|_2$ for all $x \in \mathbb{R}^m$.
3. $\|\cdot\|$ defines a norm on $M_{n\times m}$.

Proof. In this proof, we use $|\cdot|$ to denote $\|\cdot\|_2$, and if we write x without specification, it belong to $\mathbb{R}^m$

We first show $\|A\| = \sup\{|Ax| : |x| = 1\}$

Assume $\sup\{\frac{|Ax|}{|x|} : x \neq 0\} = \|A\| > \sup\{|Ax| : |x| = 1\}$.Then we know $\sup\{|Ax| : |x| = 1\}$ is not an upper bound of $\{\frac{|Ax|}{|x|} : x \neq 0\}$, so we know there exists $x \in \mathbb{R}^m$ such that $\frac{|Ax|}{|x|} > \sup\{|Ay| : |y| = 1\}$.

Define $\hat{x} := \frac{x}{|x|}$. We have $\frac{|Ax|}{|x|} = |\frac{Ax}{|x|}| = |A\hat{x}| \le \sup\{|Ay| : |y| = 1\}$, since $|\hat{x}| = |\frac{x}{|x|}| = \frac{|x|}{|x|} = 1 \implies |A\hat{x}| \in \{|Ay| : |y| = 1\}$. This CaC .

Assume $\sup\{\frac{|Ax|}{|x|} : x \neq 0\} = \|A\| < \sup\{|Ax| : |x| = 1\}$. Then we know $\sup\{\frac{|Ax|}{|x|} : x \neq 0\}$ is not an upper bound of $\sup\{|Ax| : |x| = 1\}$, so we know there exists $\hat{x} \in \mathbb{R}^m : |\hat{x}| = 1$ such that $|A\hat{x}| > \sup\{\frac{|Ay|}{|y|} : y \neq 0\}$.

We see $|A\hat{x}| > \sup\{\frac{|Ay|}{|y|} : y \neq 0\} \ge \frac{|A\hat{x}|}{|\hat{x}|} = \frac{|A\hat{x}|}{1} = |A\hat{x}|$ CaC (done)

Observe $\inf\{M \in \mathbb{R} : \|Ax\|_2 \le M\|x\|_2 \, \forall x \in \mathbb{R}^m\} = \inf\{c \in \mathbb{R} : \forall x \neq 0, c \ge \frac{|Ax|}{|x|}\}$, since

$\forall M, |A\mathbf{0}| \le M|\mathbf{0}|$.

Observe that $\{c \in \mathbb{R} : \forall x \neq 0, c \ge \frac{|Ax|}{|x|}\}$ is the set of upper bound of $\{\frac{|Ax|}{|x|} : |x| \neq 0\}$, so $\inf\{c \in \mathbb{R} : \forall x, c \ge \frac{|Ax|}{|x|}\} = \|A\| = \sup\{\frac{|Ax|}{|x|} : |x| \neq 0\}$.

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Proof. In this proof, we use $|\cdot|$ to denote $\|\cdot\|_2$, and if we write x without specification, it belong to $\mathbb{R}^m$

If $x = 0$, then we trivially have $|Ax| = |0| = 0 \le \|A\||x| = 0$, so from now, we only have to consider $x \neq 0$.

If $x \neq 0$, we have $|Ax| \le \|A\||x| \iff \frac{|Ax|}{|x|} \le \|A\| = \sup\{\frac{|Ax|}{|x|} : x \neq 0\}$, trivially true. ■

Proof. In this proof, we use $|\cdot|$ to denote $\|\cdot\|_2$, and if we write x without specification, it belong to $\mathbb{R}^m$

For non-negativity, observe $\forall x \neq 0, \frac{|Ax|}{|x|} \ge 0 \implies \|A\| = \sup\{\frac{|Ax|}{|x|} : x \in \mathbb{R}^m, x \neq 0\} \ge 0$.

For definite-positive, observe $A = 0 \implies \forall x \neq 0, \frac{|Ax|}{|x|} = \frac{|0|}{|x|} = 0 \implies \|A\| = 0$. Also, if $A \neq 0$, we can pick a column, say p-th column, that contain non-zero entry. We see the vector $e \in \mathbb{R}^m$ where the only non-zero entry is the p-th entry being 1 satisfy $|Ae| > 0$, thus $\frac{|Ae|}{|e|} > 0$. Because $e \neq 0$, we see $\|A\| = \sup\{\frac{|Ax|}{|x|} : x \in \mathbb{R}^m, x \neq 0\} \ge \frac{|Ae|}{|e|} > 0$.

For absolute-homogenity, let $c \in \mathbb{R}$ and $A \in M_{n\times m}$. We wish to prove $\|cA\| = \sup\{\frac{|cAx|}{|x|} : x \neq 0\} = |c|\sup\{\frac{|Ax|}{|x|} : x \neq 0\} = |c|\|A\|$. Notice that $\frac{|cAx|}{|x|} = \frac{|c||Ax|}{|x|}$, so we only have to prove the more general statement : $c > 0 \implies c\sup X = \sup\{cx : x \in X\}$. Notice that $\forall x \in X, c\sup X \ge cx$, so we have $c\sup X \ge \sup\{cx : x \in X\}$. If $c\sup X$ is not the smallest upper bound, we see there exists cx such that $c\sup X < cx$, and we see $\sup X < x$, causing a contradiction, so we do have $c\sup X = \sup\{cx : x \in X\}$ (done)

For triangle-inequality, first observe $\frac{|(A+B)x|}{|x|} = \frac{|Ax+Bx|}{|x|} \le \frac{|Ax|+|Bx|}{|x|} = \frac{|Ax|}{|x|} + \frac{|Bx|}{|x|}$. Assume $\|A+B\| > \|A\| + \|B\|$.

Because $\|A\| + \|B\|$ is not an upper bound of $\{\frac{|(A+B)x|}{|x|} : x \neq 0\}$, we know there exists x' such that $\frac{|(A+B)x'|}{|x'|} > \|A\| + \|B\|$. Further, by definition of $\|A\|, \|B\|$, we have

$$\frac{|(A+B)x'|}{|x'|} > \frac{|Ax'|}{|x'|} + \frac{|Bx'|}{|x'|} \text{ CaC} \tag{5.1}$$

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Question 21

Suppose that $S_1, S_2, \ldots, S_n$ are sets in $\mathbb{R}$ and

$$S = \bigcup_{i=1}^{n} S_i.$$

Define $B_i = \sup S_i$ for $i = 1, \ldots, n$.

1. Show that $\sup S = \max\{B_1, B_2, \ldots, B_n\}$.
2. If S is the union of an infinite collection of S_i, find the relation between $\sup S$ and B_i.

Proof. Let $\sup S_j = B_j = \max\{B_1, \ldots, B_n\}$. We first show $\sup S_j$ is an upper bound of S.

By definition, we have $\forall x \in S_j, x \leq \sup S_j$ and have $\forall i \neq j, \forall x \in S_i, x \leq \sup S_i \leq \sup S_j$, so we have $\forall x \in S, \exists k \in \{1, \ldots, n\}, x \in S_k \implies x \leq \sup S_k \leq \sup S_j$ (done)

We now show $\sup S_j$ is the least upper bound of S.

Assume there exists an upper bound of S smaller than $\sup S_j$. Denote that upper bound y. Because y is smaller than $\sup S_j$, we know y is not an upper bound of S_j, so we know there is a number $z \in S_j$ greater than y. Observe that the fact y is an upper bound of S implies y is greater than or equal to $z \in S_j \subseteq S$ CaC (done) ∎

Proof. We prove $\sup S = \sup\{B_i\}$.

Notice $\sup S$ is an upper bound of S_i, so we have $\forall i, \sup S > \sup S_i = B_i$. This means $\sup S$ is an upper bound of $\{B_i\}$. We have proved $\sup S \geq \sup\{B_i\}$. Assume $\sup S > \sup\{B_i\}$. Then because $\sup\{B_i\}$ is not an upper bound of S, we know there exists $s \in S$ such that $s > \sup\{B_i\}$. But because $S = \bigcup\{S_i\}$, we know $\exists S_j, s \in S_j$, which give us $s \leq \sup S_j = B_j \leq \sup\{B_i\}$ CaC (done) ∎

Question 22

Let A be a non-empty set of $\mathbb{R}$ which is bounded below. Define the set $-A$ by

$$-A \equiv \{-x \in \mathbb{R} : x \in A\}.$$

Prove that

$$\inf(A) = -\sup(-A).$$

Proof. Observe $\forall x \in -A, \sup(-A) \geq x \implies \forall a \in A, -\sup(-A) \leq a$. So $-\sup(-A)$ is an lower bound of A. Assume $-\sup(-A)$ is not the greatest lower bound of A (greatest lower bound exists because bounded below and completeness). Let $b > -\sup(-A)$ be another lower bound of A. We have $-b < \sup(-A)$, so we know $-b$ is not an upper bound of $-A$, then we know $\exists x \in -A, -b < x$. Then we know $\exists a \in A, -b < -a$, which implies $\exists a \in A, b > a$, but b is an lower bound of A CaC ■

Question 23

1. Let A, B be non-empty subsets of $\mathbb{R}$. Define $A + B$ as

$$A + B = \{x + y : x \in A, y \in B\}.$$

Justify if the following statements are true or false by providing a proof for the true statements and giving a counter-example for the false ones.

(a) $\sup(A + B) = \sup A + \sup B$.

(b) $\inf(A + B) = \inf A + \inf B$.

(c) $\sup(A \cap B) \leq \min\{\sup A, \sup B\}$.

(d) $\sup(A \cap B) = \min\{\sup A, \sup B\}$.

(e) $\sup(A \cup B) \geq \max\{\sup A, \sup B\}$.

(f) $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

Proof. We prove $\sup(A + B) = \sup A + \sup B$. For all $a + b \in A + B$, we by definition have $a \leq \sup A, b \leq \sup B$, so we have $a + b \leq \sup A + \sup B$. This prove $\sup A + \sup B$ is an upper bound of $A + B$. Assume there exists an upper bound x of $A + B$ smaller than $\sup A + \sup B$. We have $x - \sup B < \sup A$, so we know $x - \sup B$ is not an upper bound of A. Then we know $\exists a' \in A, x - \sup B < a'$. This implies $x - a' < \sup B$, so we know $x - a'$ is not an upper bound of B. Then we know there exists $b' \in B$ such that $x - a' < b'$. This implies $x < a' + b' \in A + B$ CaC to x is an upper bound of $A + B$ ■

Proof. We prove $\inf(A + B) = \inf A + \inf B$. For all $a + b \in A + B$, we by definition have $\inf A \leq a, \inf B \leq b$, so we have $\inf A + \inf B \leq a + b$. This prove $\inf A + \inf B$ is an lower bound of $A + B$. Assume there exists an lower bound x of $A + B$ greater than $\inf A + \inf B$. We have $x - \inf A > \inf B$, so we know $x - \inf A$ is not an lower bound of B. Then we know $\exists b' \in B, x - \inf A > b'$. This implies $x - b' > \inf A$, so we know $x - b'$ is not an lower bound of A. Then we know $\exists a' \in A, x - b' > a'$. So we know $x < a' + b' \in A + B$ CaC to x is an lower bound of $A + B$ ■

Proof. We prove $\sup(A \cap B) \leq \min\{\sup A, \sup B\}$.

WOLG, let $\sup A \leq \sup B$. By definition $x \in A \cap B \implies x \in A \implies x \leq \sup A$, so we know $\sup A$ is an upper bound of $A \cap B$. This implies $\sup A \cap B \leq \sup(A \cap B)$ ■

Proof. We show $\sup(A \cap B) = \min\{\sup A, \sup B\}$ is not always correct. Let $A = [0, 2]$ and $B = [0, 1] \cup [3, 4]$. We have $\sup(A \cap B = [0, 1]) = 1 \neq \min\{\sup A = 2, \sup B = 4\}$ ■

Proof. $\sup A \cup B$ is an upper bound of both A and B, so $\sup A \cup B > \sup A$ and $\sup A \cup B > \sup B$

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Proof. We prove $\sup(A \cup B) = \max\{\sup A, \sup B\}$. WOLG, let $\sup A \geq \sup B$. Assume $\sup B \leq \sup A < \sup(A \cup B)$. Let x be a number between $\sup A$ and $\sup A \cup B$(x exists since it can be $\frac{\sup A + \sup(A \cup B)}{2}$). Because $x < \sup(A \cup B)$, we know x is not an upper bound of $A \cup B$. By definition, we know there exists $z \in A \cup B$ such that $x < z$. We know either $z \in A$ or $z \in B$, but we see $z \in A \implies z \leq \sup A < x$ and we see $z \in B \implies z \leq \sup B \leq \sup A < x$ CaC ■

Question 24

7. Let $S \subseteq \mathbb{R}$ be bounded below and non-empty. Show that

$$\inf S = \sup\{x \in \mathbb{R} : x \text{ is a lower bound for } S\}.$$

Proof. Denote $B = \{x \in R : x \text{ is a lower bound for } S\}$. Assume $\inf S > \sup B$. Let $\inf S > x > \sup B$. Notice $\inf S > x$ implies there is a lower bound b of S greater than x. Observe $b > x$ and $b \in B \implies x$ is not an upper bound of B CaC $x > \sup B$.

Assume $\inf S < \sup B$. Let $\inf S < x < \sup B$. Notice $\inf S < x$ implies there exists $s' \in S$ such that $s' < x$, and notice $x > \sup B$ implies there exists $b' \in B$ such that $b' > x$. We see $b' > x > s'$ while b' is an lower bound of S CaC ■

Question 25

8. Let f be a continuous function on $\mathbb{R}$ and D is a dense subset in $\mathbb{R}$. Prove that:

1. $\sup_{x \in D} f(x) = \sup_{x \in \mathbb{R}} f(x)$.

2. There exists a sequence $\{x_n\}_{n=1}^{\infty}$ in D such that

$$\lim_{n\to\infty} f(x_n) = \sup_{x\in\mathbb{R}} f(x).$$

Proof. Because $D \subseteq \mathbb{R}$, we must have $\sup\{f(x) : x \in D\} \leq \sup\{f(x) : x \in \mathbb{R}\}$, since any upper bound of the latter will be one of the former.

Assume $\sup\{f(x) : x \in D\} < \sup\{f(x) : x \in \mathbb{R}\}$. Because $\sup\{f(x) : x \in D\}$ is not an upper bound of $\{f(x) : x \in \mathbb{R}\}$, we know there exists $x' \in \mathbb{R}$ such that

$$f(x') > \sup\{f(x) : x \in D\} \tag{5.2}$$

We first construct a sequence $\{x_i\}_{i=1}^{\infty}$ in D such that

$$\lim_{i\to\infty} x_i = x' \text{ which reads } \forall\epsilon, \exists N, n > N \longrightarrow |x_n - x'| < \epsilon \tag{5.3}$$

By Axiom of Choice and the fact that D is dense in $\mathbb{R}$, we can pick $x_i \in (x' - \frac{1}{i}, x' + \frac{1}{i})$, so we have

$$\forall i, |x_i - x'| < \frac{1}{i} \tag{5.4}$$

For all ϵ, we can pick a natural $N > \epsilon$, so we have

$$n > N \implies |x_n - x'| < \frac{1}{n} < \frac{1}{N} < \frac{1}{\epsilon} \text{ (done)} \tag{5.5}$$

We now prove

$$\lim_{i\to\infty} f(x_i) = f(x') \text{ which reads } \forall\epsilon, \exists N, n > N \longrightarrow |f(x_n) - f(x')| < \epsilon \tag{5.6}$$

Because f is continuous, we have

$$\forall\epsilon, \exists\delta, \forall u \in \mathbb{R}, |u - x'| < \delta \implies |f(u) - f(x')| < \epsilon \tag{5.7}$$

Then for all ϵ, we can first let δ satisfy $|u - x| < \delta \implies |f(u) - f(x')| < \epsilon$. Then by the violet fact we can pick N such that

$$n > N \implies |x_n - x'| < \delta \implies |f(x_n) - f(x')| < \epsilon \text{ (done)} \tag{5.8}$$

Let $H = \sup\{f(x) : x \in D\}$. Now we prove

$$\lim_{i\to\infty} f(x_i) \leq H \tag{5.9}$$

Assume $f(x') = \lim_{i\to\infty} f(x_i) > H$. We know there exists N such that

$$n > N \longrightarrow |f(x_n) - f(x')| < |H - f(x')| = f(x') - H \tag{5.10}$$

The last equality hold true due to the premise equation (5.2). Notice

$$|f(x_n) - f(x')| < f(x') - H \implies H - f(x') < f(x_n) - f(x') \implies f(x_n) > H \tag{5.11}$$

so in fact we know there exists N such that

$$n > N \longrightarrow f(x_n) > H = \sup\{f(x) : x \in D\} \text{ CaC } \text{ (done)} \tag{5.12}$$

Now, using all our proven facts, we have

$$\sup\{f(x) : x \in D\} < f(x') = \lim_{i\to\infty} f(x_i) \leq H = \sup\{f(x) : x \in D\} \text{ CaC} \tag{5.13}$$

where the first inequality follows from premise equation (5.2) ■

Proof. Let $H = \sup\{f(x) : x \in D\}$. We first prove

$$\forall i \in \mathbb{N}, \{f(x) : x \in D \text{ and } H - f(x) < \frac{1}{i}\} \neq \varnothing \tag{5.14}$$

Assume there exists some $n \in \mathbb{N}$ such that the set is empty. We then have

$$\forall x \in D, H \geq f(x) + \frac{1}{n} \tag{5.15}$$

So we have

$$\forall x \in D, H - \frac{1}{2n} \geq f(x) + \frac{1}{2n} > f(x) \tag{5.16}$$

Then we see $H - \frac{1}{2n}$ is an upper bound of $\{f(x) : x \in D\}$ smaller than H CaC (done)

By Axiom of Choice, we can construct a sequence $\{x_i\}_{i=1}^{\infty}$ by picking $x_i : f(x_i) \in \{f(x) : x \in D \text{ and } H - f(x) < \frac{1}{i}\}$. Then we have

$$\forall \epsilon, n > [\frac{1}{\epsilon}] + 1 \implies n > \frac{1}{\epsilon} \implies |f(x_n) - H| < \frac{1}{n} < \epsilon \tag{5.17}$$

This written in limit sign is

$$\lim_{i\to\infty} f(x_i) = H = \sup\{f(x) : x \in D\} = \sup\{f(x) : x \in \mathbb{R}\} \tag{5.18}$$

■

5.2 HW2