5.10 HW6

Question 39

1. Let $u(x,y) = \dfrac{x^4+y^4}{x}$, $v(x,y) = \sin x + \cos y$ and f be a function that maps (x,y) to (u,v). Find the point (x,y) where we can solve for x,y in terms of u,v. Also, find $\dfrac{\partial x}{\partial u}, \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial u}, \dfrac{\partial y}{\partial v}$ at $f\left(\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

Proof. Compute

$$[df_{(x,y)}] = \begin{bmatrix} 3x^2 & \frac{4y^3}{x} \\ \cos x & -\sin y \end{bmatrix}$$

By Differentiability Theorem, it is now clear that f is continuously differentiable everywhere on its domain $\{(x,y) \in \mathbb{R}^2 : x \neq 0\}$.

Compute the determinant

$$\det\left(df_{(x,y)}\right) = -3x^2 \sin y - \frac{4y^3 \cos x}{x}$$

Now by Inverse Function Theorem, the set of points (x,y) where we can solve for x,y in terms of u,v is exactly

$$\{(x,y) \in \mathbb{R}^2 : 3x^2 \sin y + \frac{4y^3}{x} \cos x \neq 0 \text{ and } x \neq 0\}$$

Observe

$$[df_{(\frac{\pi}{2},\frac{\pi}{2})}] = \begin{bmatrix} \frac{3\pi^2}{4} & \pi^2 \\ 0 & -1 \end{bmatrix}$$

This implies the local inverse is

$$[d(f^{-1})_{f(\frac{\pi}{2},\frac{\pi}{2})}] = [df_{(\frac{\pi}{2},\frac{\pi}{2})}]^{-1} = \begin{bmatrix} \frac{4}{3\pi^2} & \frac{4}{3} \\ 0 & -1 \end{bmatrix}$$

We now have

$$\frac{\partial u}{\partial x} = \frac{4}{3\pi^2} \text{ and } \frac{\partial u}{\partial y} = \frac{4}{3} \text{ and } \frac{\partial v}{\partial x} = 0 \text{ and } \frac{\partial v}{\partial y} = -1$$

■

Question 40

2. Let $f : \mathbb{R}^4 \to \mathbb{R}^2$ be given by $f(x,y,u,v) = (xu + yv^2, xv^3 + y^2u^6)$.

   (a) Show that x, y can be solved in terms of u, v for (u,v) near $(1,-1)$ and (x,y) near $(1,-1)$.

   (b) From (a), If we write $x = g_1(u,v)$, $y = g_2(u,v)$ for (u,v) near $(1,-1)$ and let $g = (g_1, g_2)$, Find $Dg(u,v)$. (You don't need to calculate explicitly.)

Proof. Compute

$$[df] = \begin{bmatrix} u & v^2 & x & 2yv \\ v^3 & 2yu^6 & 6u^5y^2 & 3xv^2 \end{bmatrix}$$

By Differentiability Theorem, it is now clear that f is continuously differentiable on $\mathbb{R}^4$.

Compute

$$[df_{(1,-1,1,-1)}] = \begin{bmatrix} 1 & 1 & 1 & 2 \\ -1 & -2 & 6 & 3 \end{bmatrix}$$

It is clear that the left Jacobian $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ is invertible. Now by Implicit Function Theorem, we can conclude (a), and if we write the implicit function by g we have

$$[dg_{(u,v)}] = -\begin{bmatrix} u & v^2 \\ v^3 & 2yu^6 \end{bmatrix}^{-1} \circ \begin{bmatrix} x & 2yv \\ 6u^5y^2 & 3xv^2 \end{bmatrix} = \begin{bmatrix} \frac{6u^5v^2y^2 - 2u^6xy}{v^5 - 2u^7y} & \frac{3v^4x - 4u^6vy^2}{v^5 - 2u^7y} \\ \frac{v^3x - 6u^6y^2}{v^5 - 2u^7y} & \frac{-3uv^2x + 2v^4y}{v^5 - 2u^7y} \end{bmatrix}$$

$\blacksquare$

Question 41

3. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(x, y, z) = (xe^y + ye^z, xe^z + ze^y)$.

   (a) Show that y, z can be solved in terms of x for x near -1 and (y, z) near $(1, 1)$.

   (b) From (a), If we write $(y, z) = g(x)$ for x near -1, Find $g'(x)$. (You don't need to calculate explicitly.)

Proof. Compute

$$[df] = \begin{bmatrix} e^y & xe^y + e^z & ye^z \\ e^z & ze^y & xe^z + e^y \end{bmatrix}$$

By Differentiability Theorem, it is now clear that f is continuously differentiable on $\mathbb{R}^3$. Compute

$$[df_{(-1,1,1)}] = \begin{bmatrix} e & 0 & e \\ e & e & 0 \end{bmatrix}$$

It is clear that the right Jacobian $\begin{bmatrix} 0 & e \\ e & 0 \end{bmatrix}$ is invertible. Now, by Implicit Function Theorem, we can conclude (a), and if we write the implicit function by g, we have

$$[dg_x] = -\begin{bmatrix} xe^y + e^z & ye^z \\ ze^y & xe^z \end{bmatrix}^{-1} \circ \begin{bmatrix} e^y \\ e^z \end{bmatrix}$$

■

Question 42

4. Suppose $f : \mathbb{R} \to \mathbb{R}^n$ is differentiable, such that $|f(t)| = 1$ for every $t \in \mathbb{R}$.

   (a) Prove that $f'(t) \cdot f(t) = 0$. (Here "$\cdot$" is the standard inner product in $\mathbb{R}^n$)

   (b) In fact, this result has a geometric interpretation. For example, in $\mathbb{R}^2$, the function $f(t) = (\cos t, \sin t)$ satisfies $|f(t)| = 1$. Draw the graph of $f(t)$ and $f'(t)$ on $\mathbb{R}^2$, what do you discover?

Proof. If we define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(t) \triangleq f(t) \cdot f(t)$$

We have

$$g \text{ is constant } 1 \text{ and } g' = 0$$

It remains to prove

$$g'(t) = 2f'(t) \cdot f(t)$$

Observe

$$g(t) = \sum_{k=1}^{n} f_k^2(t)$$

This implies

$$g'(t) = \sum_{k=1}^{n} 2f_k(t)f_k'(t) = 2f'(t) \cdot f(t) \text{ (done)}$$

From the graph below, we see that $f(t)$ and $f'(t)$ orthogonal. ∎

Question 43

5. (Second Derivative Test) Let V be an open subset of $\mathbb{R}^2$ and $(a, b) \in V$, and suppose that $f : V \to \mathbb{R}$ satisfy $\nabla f(a, b) = 0$. Suppose also that $f \in \mathcal{C}^2(V)$, and set $D = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2$. Prove the following statements:

   (a) If $D > 0$ and $f_{xx}(a, b) > 0$, then f admits a local minimum at (a, b).

   (b) If $D > 0$ and $f_{xx}(a, b) < 0$, then f admits a local maximum at (a, b).

   (c) If $D < 0$, then f is a saddle point at (a, b).

Definition 5.10.1. (Definition of Hessian) For all $p \in V$, we define

$$H(p) \triangleq \begin{bmatrix} f_{xx}(p) & f_{xy}(p) \\ f_{yx}(p) & f_{yy}(p) \end{bmatrix} \text{ and } D(p) \triangleq (f_{xx} \cdot f_{yy} - f_{xy}^2)(p)$$

Note that $f \in \mathcal{C}^2(V) \implies f_{xy} = f_{yx}$ on $V \implies H$ is symmetric on V.

Then by Spectral Theorem, we know

$$H \text{ has an orthonormal basis consisting of eigenvectors}$$

Definition 5.10.2. (Definition of Saddle) Given real-valued function f exists on an open set $V \subseteq \mathbb{R}^2$ containing p and is differentiable at p, we say f is a saddle point at p if

$$df_p = 0 \text{ and } f \text{ is neither a local minimum nor a local maximum at } p$$

Lemma 5.10.3. Given a $\alpha(t)$ that lies in V and satisfy $\alpha'' = 0$. We have

$$(f \circ \alpha)'' = \alpha' \cdot H\alpha'$$

Proof. Using Chain Rule and the property of gradient, we can write

$$(f \circ \alpha)' = \nabla f(\alpha) \cdot \alpha'$$

This give us

$$\begin{aligned}(f \circ \alpha)'' &= (\nabla f(\alpha))' \cdot \alpha' + \nabla f(\alpha) \cdot \alpha'' \\ &= (\nabla f(\alpha))' \cdot \alpha'\end{aligned}$$

This reduce the problem into proving

$$(\nabla f(\alpha))' \cdot \alpha' = \alpha' \cdot H\alpha'$$

Express

$$\nabla f(\alpha) = \Big((\partial_x f)(\alpha), (\partial_y f)(\alpha)\Big)$$

Compute

$$\big(\nabla f(\alpha)\big)' = \Big(\nabla(\partial_x f)(\alpha) \cdot \alpha', \nabla(\partial_y f)(\alpha) \cdot \alpha'\Big)$$

Note that

$$\nabla(\partial_x f) = (\partial_{xx} f, \partial_{xy} f) \text{ and } \nabla(\partial_y f) = (\partial_{yx} f, \partial_{yy} f)$$

Express $\alpha = (\alpha_1, \alpha_2)$. Compute

$$\begin{aligned}\nabla(\partial_x f)(\alpha) \cdot \alpha' &= \partial_{xx} f(\alpha) \cdot \alpha_1' + \partial_{xy} f(\alpha) \cdot \alpha_2' \\ \nabla(\partial_y f)(\alpha) \cdot \alpha' &= \partial_{yx} f(\alpha) \cdot \alpha_1' + \partial_{yy} f(\alpha) \cdot \alpha_2'\end{aligned}$$

We can now compute

$$\begin{aligned}\big(\nabla f(\alpha)\big)' \cdot \alpha' &= \Big((\alpha_1')\partial_{xx} f(\alpha) + (\alpha_2')\partial_{xy} f(\alpha), (\alpha_1)'\partial_{yx} f(\alpha) + (\alpha_2')\partial_{yy} f(\alpha)\Big) \cdot \alpha' \\ &= (\alpha_1')^2 \partial_{xx} f(\alpha) + 2(\alpha_1'\alpha_2')\partial_{xy} f(\alpha) + (\alpha_2')^2 \partial_{yy} f(\alpha) \\ &= \alpha' \cdot H\alpha' \text{ (done)}\end{aligned}$$

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Lemma 5.10.4. (Property of H)

(a) If $D > 0$ and $f_{xx} > 0$, then H is positive definite.

(b) If $D > 0$ and $f_{xx} < 0$, then H is negative definite.

(c) If $D < 0$, then H has a positive and a negative eigenvalue.

Proof. Observe that $\det(H) = D$. This let us write the characteristic polynomial of H by

$$t^2 - (\partial_{xx} f + \partial_{yy} f)t + D$$

Then we know the eigenvalues are

$$t = \frac{(\partial_{xx} f + \partial_{yy} f) \pm \sqrt{(\partial_{xx} f + \partial_{yy} f)^2 - 4D}}{2}$$

Now, if

$$\partial_{xx} f \partial_{yy} f - (\partial_{xy} f)^2 = D > 0$$

Then we have

$$\partial_{xx} f \partial_{yy} f > 0$$

Moreover, if $\partial_{xx} f > 0$, we can deduce

$$\partial_{xx} f + \partial_{yy} f > 0$$

Then from the formula of the eigenvalues t, because $D > 0$, regardless of the sign of the square-root term, we have

$$t > 0$$

Note that $t \in \mathbb{R}$ by Spectral Theorem.

Similarly, if $D > 0$ and $\partial_{xx} f < 0$, then $t < 0$, and if $D < 0$, then the two eigenvalues are of opposite signs and non-zero.

Now, if $D > 0$ and $f_{xx} > 0$, the two eigenvalues of H are all positive. Fix the orthonormal basis $\{e_1, e_2\}$ corresponding to eigenvalues $t_1, t_2 > 0$. We see that for all $c_1e_1 + c_2e_2 \neq 0 \in \mathbb{R}^2$

$$(c_1e_1 + c_2e_2) \cdot H(c_1e_2 + c_2e_2) = t_1c_1^2 + t_2c_2^2 > 0$$

Similarly, if $D > 0$ and $f_{xx} < 0$, the two eigenvalues of H are negative, which leads to

$$(c_1e_1 + c_2e_2) \cdot H(c_1e_1 + c_2e_2) = t_1c_1^2 + t_2c_2^2 < 0$$

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Proof. **($D>0$ and $\partial_{xx}f(a,b)>0$ Case)** Because $f\in\mathcal{C}^2(V)$, we know there exists $B_\epsilon(a,b)$ such that D and $\partial_{xx}f$ are positive on $B_\epsilon(a,b)$. We claim

$$f(a,b)=\min_{(x,y)\in B_\epsilon(a,b)} f(x,y)$$

Fix $(c,d)\in B_\epsilon(a,b)$. We reduce the problem into proving

$$f(a,b)\le f(c,d)$$

Let $\gamma:[-1,1]\to B_\epsilon(a,b)$ be the line joining $(2a-c,2b-d)$ and (c,d)

$$\gamma(t)\triangleq\Big(a+(c-a)t,b+(d-b)t\Big)$$

Note that $\gamma''=0$ and $\gamma(0)=(a,b)$. Now by Lemma 5.10.3 and Lemma 5.10.4, we have

$$(f\circ\gamma)''>0 \text{ on } [-1,1]$$

This implies $f\circ\gamma$ is convex on $[-1,1]$. Compute using $\nabla f(a,b)=0$

$$(f\circ\gamma)'(0)=df_{(a,b)}(c-a,d-b)=\nabla f(a,b)\cdot(c-a,d-b)=0$$

This then implies

$$(f\circ\gamma)(0) \text{ is minimum on } [-1,1]\quad\text{(done)}$$

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Proof. **($D>0$ and $\partial_{xx}f(a,b)<0$ Case)** Because $f\in\mathcal{C}^2(V)$, we know there exists $B_\epsilon(a,b)$ such that D is positive and $\partial_{xx}f$ is negative on $B_\epsilon(a,b)$. We claim

$$f(a,b)=\max_{(x,y)\in B_\epsilon(a,b)} f(x,y)$$

Fix $(c,d)\in B_\epsilon(a,b)$. We reduce the problem into proving

$$f(a,b)\ge f(c,d)$$

Let $\gamma:[-1,1]\to B_\epsilon(a,b)$ be the line joining $(2a-c,2b-d)$ and (c,d)

$$\gamma(t)\triangleq\Big(a+(c-a)t,b+(d-b)t\Big)$$

Note that $\gamma''=0$ and $\gamma(0)=(a,b)$. Now by Lemma 5.10.3 and Lemma 5.10.4, we have

$$(f\circ\gamma)''<0 \text{ on } [-1,1]$$

This implies $f\circ\gamma$ is concave on $[-1,1]$. Compute using $\nabla f(a,b)=0$

$$(f\circ\gamma)'(0)=df_{(a,b)}(c-a,d-b)=\nabla f(a,b)\cdot(c-a,d-b)=0$$

This then implies

$$(f\circ\gamma)(0) \text{ is maximum on } [-1,1]\quad\text{(done)}$$

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Proof. **($D < 0$ Case)** From $\nabla f(a,b) = 0$, we know $df_{(a,b)} = 0$. This reduce the problem into proving

$$(a,b) \text{ is not a local extremum}$$

Fix ϵ such that $D < 0$ on $B_\epsilon(a,b)$. We prove

$$\min_{(x,y)\in B_\epsilon(a,b)} f(x,y) < f(a,b) < \max_{(x,y)\in B_\epsilon(a,b)} f(x,y)$$

Fix two curve $\alpha, \beta : (-\epsilon, \epsilon) \to \mathbb{R}^2$ by

$$\alpha(t) \triangleq (a,b) + te_1 \text{ and } \beta(t) \triangleq (a,b) + te_2$$

where e_1, e_2 are the eigenvectors of $H(p)$, e_1 correspond to the positive eigenvalue and e_2 correspond to the negative eigenvalue.

Compute using Lemma 5.10.4

$$(f \circ \alpha)''(0) > 0 \text{ and } (f \circ \beta)''(0) < 0$$

Because $(f \circ \alpha)''$ and $(f \circ \beta)''$ is continuous, we know $(f \circ \alpha)'' > 0$ and $(f \circ \beta)'' < 0$ on a small enough interval I containing 0. In other words

$$(f \circ \alpha) \text{ is strictly convex and } (f \circ \beta) \text{ is strictly concave on } I$$

Again, we know $(f \circ \alpha)'(0) = \nabla f(a,b) \cdot e_1 = 0$ and, similarly, $(f \circ \beta)'(0) = 0$. We can now deduce

$$\begin{cases} f \circ \alpha \\ f \circ \beta \end{cases} \text{reach} \begin{cases} \text{strict minimum} \\ \text{strict maximum} \end{cases} \text{at } p \text{ on } I$$

The former implies $f(a,b) < \max_{(x,y)\in B_\epsilon(a,b)} f(x,y)$, and the latter implies $f(a,b) > \min_{(x,y)\in B_\epsilon(a,b)} f(x,y)$. (done) ∎

Question 44

6. Let f be a nonnegative and measurable function on E. Prove that

$$\int_E f(x)\,dx = \sup\left[\sum_j \left(\inf_{x\in E_j} f(x)\right)|E_j|\right]$$

where the supremum is taken over all decompositions $E = \bigcup_j E_j$ of E into the union of a finite number of disjoint measurable sets E_j.

Proof. We first show

$$\int_E f(x)dx \geq \sup\left[\sum_j \left(\inf_{x\in E_j} f(x)\right)|E_j|\right]$$

Fix a finite disjoint measurable decomposition E_j of E. We reduce the problem into proving

$$\int_E f(x)dx \geq \sum_j \left(\inf_{x\in E_j} f(x)\right)|E_j|$$

Define

$$s_0 \triangleq \sum_j \left(\inf_{x\in E_j} f(x)\right)\mathbf{1}_{E_j}$$

Because E_j are disjoint, we know

$$s_0(x) = \inf_{t\in E_j} f(t) \text{ if } x \in E_j$$

This implies s_0 is simple. Because E_j, are all measurable, we can deduce s_0 is measurable. In conclusion, $s_0 : E \to \mathbb{R}$ is a simple measurable function.

Given arbitrary $x \in E$, we know x must belong to one and only one of the decomposition. Express $x \in E_i$. This give us

$$s_0(x) = \inf_{t\in E_i} f(t) \leq f(x)$$

Because x is arbitrary, we can conclude $s \leq f$ on E.

Question 45

7. If $\{f_k\}_{k\in\mathbb{N}}$ is a sequence of nonnegative and measurable functions on E, prove that

$$\int_E \left(\sum_{k=1}^{\infty} f_k(x)\right) dx = \sum_{k=1}^{\infty}\left(\int_E f_k(x)\,dx\right)$$

Proof. Fix

$$f_n \triangleq \sum_{k=1}^{n} f_k \text{ and } f \triangleq \sum_{k=1}^{\infty} f_k$$

Because f_k are non-negative, we know f_n are pointwise increasing. Clearly $f_n \to f$ pointwise. Note that f_n are all measurable, since measurable functions are closed under addition.

Now by Lebesgue monotone convergence Theorem, we have

$$\begin{aligned}\int_E \Big(\sum_{k=1}^{\infty} f_k\Big) dx = \int_E f dx = \lim_{n\to\infty}\int_E f_n dx = \lim_{n\to\infty}\int_E \sum_{k=1}^{n} f_k dx = \lim_{n\to\infty}\sum_{k=1}^{n}\int_E f_k dx \\ = \sum_{k=1}^{\infty}\int_E f_k(x) dx\end{aligned}$$

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Question 46

8. Let f be nonnegative and measurable on E, prove that

$$\int_E f(x)\,dx = 0 \iff \exists Z \subset E \text{ such that } |Z| = 0 \text{ and } f(x) = 0 \text{ on } E\backslash Z$$

Proof. Note that if $|E| = 0$, then $\int_E f dx = 0$, and we can just let $Z = E$. We suppose $|E| > 0$ from now.

$(\longleftarrow)$

Decompose E into arbitrary disjoint finite measurable union E_j. Because the decomposition is arbitrary, using the result of the sixth question, we only have to prove

$$\sum_j (\inf_{E_j} f)\,|E_j| = 0$$

We reduce the problem into proving

$$\forall j, |E_j| > 0 \implies \inf_{E_j} f = 0$$

Fix $|E_j| > 0$. Notice that $f = 0$ on Z^c. This allow us to reduce the problem into proving

$$|E_j \cap Z^c| > 0$$

Note that $|Z^c| = |E| - |Z| = |E|$. Observe

$$|E| \geq |E_j \cup Z^c| \geq |Z^c| = |E|$$

This implies $|E_j \cup Z^c| = |E|$. This give us

$$|E_j| + |Z^c| - |E_j \cap Z^c| = |E_j \cup Z^c| = |E|$$

Now because $|Z^c| = |E|$, we can deduce

$$|E_j \cap Z^c| = |E_j| > 0 \text{ (done)}$$

$(\longrightarrow)$

Fix

$$B_n \triangleq \{x \in E : f(x) \geq \frac{1}{n}\}$$

Because f is measurable, we know B_n are measurable. Note that each B_n must be of zero-measure, otherwise $\int_E f \geq \int_{B_n} f \geq \frac{|B_n|}{n} > 0$.

Now let

$$Z \triangleq \bigcup_{n \in \mathbb{N}} B_n$$

Observe

$$|Z| \leq \sum_{n \in \mathbb{N}} |B_n| = 0$$

and observe

$$x \in Z^c \implies \forall n \in \mathbb{N}, f(x) \notin B_n \implies \forall n \in \mathbb{N}, 0 \leq f(x) < \frac{1}{n} \implies f(x) = 0$$

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Question 47

9. Let $g(x) = \begin{cases} 0 & , 0 \le x \le 1/2 \\ 1 & , 1/2 < x \le 1 \end{cases}$ and $f_{2k}(x) = g(x)$, $f_{2k-1}(x) = g(1-x)$, $x \in [0,\ 1]$, $k \in \mathbb{N}$. Show that

$$\liminf_{n\to\infty} f_n(x) = 0 \text{ on } [0,\ 1]$$

but $\displaystyle\int_0^1 f_n(x)\,dx = \frac{1}{2}$. (This gives an example of strict inequality in **Fatou lemma**)

Proof. Compute

$$f_{2k}(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}] \\ 1 & \text{if } x \in (\frac{1}{2}, 1] \end{cases} \text{ and } f_{2k-1}(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}) \\ 0 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

Observe

$$\text{For all } x \in [0, \frac{1}{2}), \{f_{2k}(x)\}_{k\in\mathbb{N}} = 0 \implies \liminf_{n\to\infty} f_n = 0 \text{ on } [0, \frac{1}{2})$$

Observe

$$\{f_k(x)\}_{k\in\mathbb{N}} = 0 \implies \liminf_{n\to\infty} f_n = 0 \text{ on } \{\frac{1}{2}\}$$

Observe

$$\text{For all } x \in (\frac{1}{2}, 1], \{f_{2k-1}(x)\}_{k\in\mathbb{N}} = 0 \implies \liminf_{n\to\infty} f_n = 0 \text{ on } (\frac{1}{2}, 1]$$

This now conclude

$$\liminf_{n\to\infty} f_n = 0 \text{ on } [0, 1]$$

Compute

$$\int_0^1 f_{2k} dx = \int_{\frac{1}{2}}^1 1 dx = \frac{1}{2}$$

Compute

$$\int_0^1 f_{2k-1} dx = \int_0^{\frac{1}{2}} 1 dx = \frac{1}{2}$$

This conclude $\int_0^1 f_n dx = \frac{1}{2}$ for all $n \in \mathbb{N}$. ∎

Question 48

10. Let $f(x) = \begin{cases} 1/n & , |x| \leq n \\ 0 & , |x| > n \end{cases}$. Show that $f_n(x)$ uniformly converges to 0 on $\mathbb{R}$ but $\displaystyle\int_{-\infty}^{\infty} f_n(x)\, dx = 2$ for all $n \in \mathbb{N}$.

Proof. Fix ϵ. Let $N > \frac{1}{\epsilon}$. We now see that for each $n > N$, we have

$$\sup_{\mathbb{R}} |f| = \frac{1}{n} < \epsilon$$

This conclude that f_n uniformly converge to $\mathbb{R}$.

Compute

$$\int_{\mathbb{R}} f_n(x)dx = \int_{-n}^{n} \frac{1}{n} dx = 2$$

∎