2.4 Exercises 2

Let (M, d) be a metric space, $x \in M$ and F a subset of M.

Question 35

Prove that the following statements are equivalent

(a) There exists a sequence $\{x_n\}$ in F with $x_n \neq x$ so that $\lim_{n\to\infty} x_n = x$.

(b) For any ϵ, the intersection of $B'_\epsilon(x) \triangleq \{y \in M : 0 < d(x, y) < \epsilon\}$ and F are non-empty.

Proof. If (a) is true, then for all ϵ there exists some $x_n \in F$ such that $d(x_n, x) < \epsilon$. Because $x_n \neq x$, we know that $0 < d(x_n, x)$. This now implies $x_n \in B'_\epsilon(x) \cap F$.

If (b) is true, then for all n, we simply select a point in $x_n \in B'_{\frac{1}{n}}(x) \cap F$. After such selection, we see that $x_n \neq x$ and for all ϵ, if $n > \frac{1}{\epsilon}$, then $x_n \in B'_\epsilon(x) \cap F$. ∎

Question 36

Prove that the following statements are equivalent

(a) F contain all its limit point.

(b) $U = M \setminus F$ is open.

Proof. If (a) is true, then for all $p \in U$, we know that p is not a limit point of F, then from the first question, we know that there exists ϵ such that $B'_\epsilon(x) \cap F = \varnothing$. Because $x \in U = M \setminus F$ also does not belong x, we also know that $B_\epsilon(x) \cap F = \varnothing$. This then implies that $B_\epsilon(x) \subseteq U$, since $U = M \setminus F$. We have proved that U is open.

If (b) is true, then for arbitrary $p \notin F$, we know there exists some ϵ such that $B_\epsilon(x)$ is disjoint with F. Because $B'_\epsilon(x)$ is a subset of $B_\epsilon(x)$, we can deduce that $B_\epsilon(x) \cap F = \varnothing$, which from the first question implies that p is not a limit point of F. Because p is arbitrary selected from $M \setminus F$, we have proved that none of the points in $M \setminus F$ is a limit point of F. This implies that if F has any limit point, then F must contain that limit point. ∎

Question 37

Prove the following statements

(a) M and $\varnothing$ are closed.

(b) The intersection of any family of closed subsets of M is closed.

(c) The union of finitely many closed subsets of M is closed.

Proof. It is clear that M is open and trivially true that $\varnothing$ is open. It then follows from the second question that M and $\varnothing$ are both closed.

Let (F_α) be a collection of closed subsets of M. Arbitrary select a limit point x of $\bigcap F_\alpha$. Let $\{x_n\}$ be a sequence in $\bigcap F_\alpha$ with $x_n \neq x$ so that $\lim_{n\to\infty} x_n = x$. Arbitrary select β. Note that $\{x_n\}$ is also a sequence in F_β that converge to x with $x_n \neq x$. This now implies that x is a limit point of F_β. Then because F_β is closed, we see that $x \in F_\beta$. Now, since β is arbitrary selected, we see $x \in \bigcap_\alpha F_\alpha$. Because x is arbitrary, we have proved $\bigcap F_\alpha$ contained all its limit points.

Let $\{F_1, \dots, F_N\}$ be a collection of closed subsets of M. Let x be an arbitrary limit point of $\bigcup_{n=1}^N F_n$. Let $\{x_n\}$ be a sequence in $\bigcup_{n=1}^N F_n$ with $x_n \neq x$ converging to x. It is clear that there must exists some $j \in \{1, \dots, N\}$ such that F_j contain infinite terms of $\{x_n\}$, i.e., there exists a subsequence x_{n_k} such that $x_{n_k} \in F_j$ for all k. Because $\lim_{k\to\infty} x_{n_k} = \lim_{n\to\infty} x_n = x$, we now see that x is a limit point of F_j. It then follows from F_j being closed that $x \in F_j \subseteq \bigcup_{n=1}^N F_n$. Because x is arbitrary, we have proved that $\bigcup_{n=1}^N F_n$ is closed. ■