- Unless otherwise specified, everything is over $\mathbb{R}$.
- The ordinary inner product of $\mathbb{R}^n$ is denoted by $\vec{u} \cdot \vec{v}$.
- $\mathcal{S}_n$ is the space of $n \times n$ square matrices.
- $\mathcal{P}$ is the vector space of polynomials of one variable x with real coefficients.
- Dual space V^* of real vector space V is $\{\alpha \mid \alpha : V \to \mathbb{R}, \alpha \text{ is linear}\}$.

(1) [16%] $V \subset \mathbb{R}^4$ is a subspace span by $\vec{u} = [1 \ -4 \ 8 \ 3]^{\text{t}}$ and $\vec{v} = [2 \ -2 \ 10 \ 3]^{\text{t}}$. Define a linear transformation $T : V \to V$ by
$$T(\vec{u}) = 5\,\vec{u} + 2\,\vec{v}$$
$$T(\vec{v}) = 7\,\vec{u} + \vec{v}$$
The induced inner product of V from $\mathbb{R}^4$ is defined by $< \vec{x}, \vec{y} > = \vec{x} \cdot \vec{y}$, $\vec{x}, \vec{y} \in V$. Is T self-adjoint with respect to $<\ ,\ >$? Demonstrate your answer.

(2) [16%] $\mathcal{P}_3 \equiv \{f(x) \in \mathcal{P} \mid \deg(f(x)) \le 3\}$. Let $\mathcal{P}_3^*$ be the dual space of $\mathcal{P}_3$. For any $a \in \mathbb{R}$, define $\widehat{a} \in \mathcal{P}_3^*$ by $\widehat{a}(f(x)) = f(a)$ and $\mathrm{d}\widehat{a} \in \mathcal{P}_3^*$ by $\mathrm{d}\widehat{a}(f(x)) = f'(a)$.

    a. Find the basis $\phi_{-1}(x)$, $\phi_0(x)$, $\phi_d(x)$, $\phi_1(x)$ of $\mathcal{P}_3$ such that $\widehat{-1}$, $\widehat{0}$, $\mathrm{d}\widehat{0}$, $\widehat{1}$ are their corresponding dual basis.

    b. Define $I \in \mathcal{P}_3^*$ by $I(f(x)) = \int_{-1}^{1} f(x)dx$. Find α, β, γ, $\epsilon \in \mathbb{R}$ such that
$$I = \alpha\,\widehat{-1} + \beta\,\widehat{0} + \gamma\,\mathrm{d}\widehat{0} + \epsilon\,\widehat{1}$$

    c. If there is $f(x) \in \mathcal{P}_3$ such that $f(-1) = -2$, $f(0) = 2$, $f'(0) = \pi$, $f(1) = -6$, evaluate $\int_{-1}^{1} f(x)dx$.

(3) [16%] $\Gamma = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \in \mathcal{S}_n$. $\mathcal{C}_n = \{X \mid X\Gamma = \Gamma X\}$ is a subspace of $\mathcal{S}_n$.
Determine $\dim \mathcal{C}_n$ and find a basis of $\mathcal{C}_n$.

(4) [16%] $A \in \mathcal{S}_n$. Define m_{ij} to be the determinant of the submatrix formed by deleting the i-th row and j-th column of A. Define the classical adjont matrix $\operatorname{adj} A = [(-1)^{i+j} m_{ji}]$. Suppose A is not invertible, show that rank of $\operatorname{adj} A$ is ≤ 1. When is the rank of $\operatorname{adj} A = 1$?

(5) [16%] If $A = [a_{ij}] \in \mathcal{S}_n$ is positive definite, show that $\det A \le a_{11}a_{22}\cdots a_{nn}$.

(6) [20%] $A \in \mathcal{S}_n(\mathbb{C})$. Over $\mathbb{C}$, show the following two statements are equivalent.

    a. The characteristic polynomial of A is equal to minimal polynomial of A.

    b. For any $X \in \mathcal{S}_n(\mathbb{C})$ satisfies $XA = AX$, X is a polynomial of A.

<div align="center">試題隨卷繳回</div>