6.9 PDE HW 9

Question 163

Find the full Fourier series of e^x on $(-l, l)$ in its real and complex forms. (Hint: It is convenient to find the complex form first)

Proof. Write

$$e^x = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$$

Compute

$$\begin{aligned}
c_n &= \frac{1}{2l}\int_{-l}^{l} e^x e^{\frac{-in\pi x}{l}} dx \\
&= \frac{1}{2l}\int_{-l}^{l} e^{\frac{(l-in\pi)x}{l}} dx \\
&= \frac{1}{2l} \cdot \frac{le^{\frac{(l-in\pi)x}{l}}}{l-in\pi}\bigg|_{x=-l}^{l} \\
&= \frac{l(e^{l-in\pi} - e^{-(l-in\pi)})}{2l(l-in\pi)} \\
&= \frac{(-1)^n(e^l - e^{-l})}{2(l-in\pi)} = \frac{(-1)^n}{(l-in\pi)}\sinh(l)
\end{aligned}$$

We now have

$$
\begin{aligned}
e^x &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(l-in\pi)} \sinh(l) e^{\frac{in\pi x}{l}} \\
&= \frac{\sinh(l)}{l} + \sum_{n=1}^{\infty} \Big[\frac{(-1)^n \sinh(l)}{(l-in\pi)} e^{\frac{in\pi x}{l}} + \frac{(-1)^n \sinh(l)}{(l+in\pi)} e^{\frac{-in\pi x}{l}} \Big] \\
&= \frac{\sinh(l)}{l} + \sum_{n=1}^{\infty} (-1)^n \sinh(l) \Big[\frac{\cos(\frac{n\pi x}{l}) + i\sin(\frac{n\pi x}{l})}{l-in\pi} + \frac{\cos(\frac{n\pi x}{l}) - i\sin(\frac{n\pi x}{l})}{l+in\pi} \Big] \\
&= \frac{\sinh(l)}{l} \\
&+ \sum_{n=1}^{\infty} (-1)^n \sinh(l) \cdot \frac{(l+in\pi)\big(\cos(\frac{n\pi x}{l}) + i\sin(\frac{n\pi x}{l})\big) + (l-in\pi)\big(\cos(\frac{n\pi x}{l}) - i\sin(\frac{n\pi x}{l})\big)}{l^2+n^2\pi^2} \\
&= \frac{\sinh(l)}{l} + \sum_{n=1}^{\infty} \frac{(-1)^n \sinh(l) \big[2l\cos(\frac{n\pi x}{l}) - 2n\pi \sin(\frac{n\pi x}{l}) \big]}{l^2+n^2\pi^2} \\
&= \sum_{n=0}^{\infty} \frac{2(-1)^n \sinh(l)}{l^2+n^2\pi^2} \Big[l\cos(\frac{n\pi x}{l}) - n\pi \sin(\frac{n\pi x}{l}) \Big]
\end{aligned}
$$

■

Question 164

Find the complex eigenvalues of the first-derivative operator $\frac{d}{dx}$ subject to the single boundary condition $X(0) = X(1)$. Are the eigenfunctions orthogonal on the interval $(0, 1)$?

Proof. We are solving the eigenproblem

$$
\begin{cases} X' + \lambda X = 0 \\ X(0) = X(1) \end{cases}
$$

Clearly, the solution X must take the form $X = e^{-\lambda x}$. The boundary conditions then implies

$$
e^{-\lambda} = X(1) = X(0) = 1
$$

which implies

$$
\lambda = 2ni\pi \text{ for } n \in \mathbb{Z}
$$

Compute for distinct n, m

$$\langle X_n, X_m \rangle = \int_0^1 e^{-2(n-m)i\pi x} dx = \frac{e^{-2(n-m)i\pi x}}{-2(n-m)i\pi}\bigg|_{x=0}^{1} = 0$$

So the eigenfunctions are indeed orthogonal. ■