Chapter 3

Scripts

3.1 Script 3

A **primary decomposition** of an ideal I is an expression of I as a finite intersection of primary ideals

$$I = \bigcap_{i=1}^{n} Q_i$$

Moreover, if $\sqrt{Q_i}$ are all distinct and

$$\bigcap_{j \neq i} Q_j \not\subseteq Q_i \text{ for all } i$$

then we say the primary decomposition is **irredundant**.

Theorem 3.1.1. (First uniqueness theorem for irredundant primary decomposition) Given some irredundant primary decomposition $I = \bigcap_{i=1}^{n} Q_i$, we have

$$\left\{ \sqrt{Q_i} : 1 \leq i \leq n \right\} = \operatorname{Spec}(R) \cap \left\{ \sqrt{(I : x)} \subseteq R : x \in R \right\} \tag{3.1}$$

Proof. Before showing that both sides of Equation 3.1 are subsets of each other, we first make the following observation. For all $x \in R$, clearly

$$(I : x) = \left(\bigcap Q_i : x \right) = \bigcap (Q_i : x)$$

Therefore,

$$\sqrt{(I : x)} = \bigcap \sqrt{(Q_i : x)} = \bigcap_{k : x \notin Q_k} \sqrt{Q_k} \tag{3.2}$$

where the last equality is justified by

$$x \in Q_i \implies (Q_i : x) = R, \quad \text{and } x \notin Q_i \implies \sqrt{(Q_i : x)} = \sqrt{Q_i}$$

We now prove that the left hand side of Equation 3.1 is a subset of the right hand side. Fix i. By irredundancy of the decomposition, there exists some $x \in R$ such that x belongs to all Q_j except Q_i. This x by Equation 3.2 must satisfies

$$\sqrt{Q_i} = \sqrt{(I : x)}$$

Noting that $\sqrt{Q_i}$ must be prime due to Q_i being primary, we have shown the left hand side of Equation 3.1 is a indeed a subset of the right hand side.

Now, suppose for some $x \in R$ that $\sqrt{(I : x)}$ is prime. Because prime ideal must be proper, we know there must exists some k such that $x \notin Q_k$. By Equation 3.2, to finish the proof, we only need to show $\sqrt{Q_k} \subseteq \sqrt{(I : x)}$ for some k such that $x \notin Q_k$. Assume not for a contradiction. Then for all k such that $x \notin Q_k$, there exists $y_k \in \sqrt{Q_k}$ such that $y_k \notin \sqrt{(I : x)}$. The product of these y_k is an element of $\bigcap \sqrt{Q_k}$, thus an element of $\sqrt{(I : x)}$. This with $\sqrt{(I : x)}$ being prime shows that $y_k \in \sqrt{(I : x)}$ for some k, a contradiction. ■

Because of Theorem 3.1.1, we may well define the following notions. Given some decomposable ideal I, we say the prime ideals $\{\sqrt{Q_1}, \dots, \sqrt{Q_n}\}$ **belong** to I, and if $\sqrt{Q_i}$ is a minimal element of $\{\sqrt{Q_1}, \dots, \sqrt{Q_n}\}$, then we say $\sqrt{Q_i}$ is an **isolated** prime ideal belonging to I.

Theorem 3.1.2. (Proposition 4.6) Let I be a decomposable ideal. Any prime ideal $P \supseteq I$ contains an isolated prime ideal belonging to I.

Proof. Let

$$\bigcap_{i=1}^{n} Q_i = I \subseteq P$$

We have

$$\bigcap_{i=1}^{n} \sqrt{Q_i} = \sqrt{\bigcap_{i=1}^{n} Q_i} \subseteq \sqrt{P} = P$$

Because P is prime, we see that there must exists some $i \in \{1, \dots, n\}$ such that $\sqrt{Q_i} \subseteq P$, otherwise, we may construct some $\prod x_i \in \bigcap \sqrt{Q_i} \setminus P$ by selecting $x_i \in \sqrt{Q_i} \setminus P$. If $\sqrt{Q_i}$ is isolated, we are done. If not, then there exists some isolated ideal $\sqrt{Q_j}$ such that $\sqrt{Q_j} \subseteq \sqrt{Q_i}$ and we are done. ■

The second remark gives an example of two distinct irredundant primary decomposition of an ideal. Let $\langle x^2, xy\rangle \subseteq \mathbb{F}[x, y]$. We have

$$\langle x^2, xy\rangle = \langle x\rangle \cap \langle x, y\rangle^2 = \langle x\rangle \cap \langle x^2, y\rangle$$

where

$$y \notin \langle x, y\rangle^2$$

Also, note from Theorem 3.1.2 that

$$\mathrm{Nil}(R) = \bigcap \text{ all minimal primes ideal belonging to } \{0\}$$

Theorem 3.1.3. (Set of zero-divisors is the union of all prime ideals belonging to $\{0\}$) If we let D be set of zero-divisors of R, then

$$D = \bigcup \{I \in \mathrm{Spec}(R) : I \text{ belongs to } \{0\}\}$$

Proof. Clearly,

$$D = \bigcup_{x \neq 0} \sqrt{(\{0\} : x)}$$

This together with Equation 3.2 shows that D is a subset of the union of all prime ideals belonging to $\{0\}$. The converse follows directly from Theorem 3.1.1. ■

We may generalize Theorem 3.1.3 as following. Let $I = \bigcap Q_i$ be an irredundant primary decomposition. Let $\pi : R \to R/I$ be the quotient map. Clearly $\{[0]\} = \bigcap \pi(Q_i)$ forms an irredundant primary decomposition. Therefore, Theorem 3.1.3 implies

$$\bigcup \sqrt{\pi(Q_i)} = \Big\{[x] \in R/I : xy \in I \text{ for some } y \neq 0\Big\}$$

which implies

$$\bigcup \sqrt{Q_i} = \Big\{x \in R : (I : x) \neq I\Big\}$$

Theorem 3.1.4. (Proposition 4.8) Let S be a multiplicatively closed subset of A, and let Q be a P-primary ideal.

$$S \cap P \neq \varnothing \implies S^{-1}Q = S^{-1}A$$

and

$$S \cap P = \varnothing \implies S^{-1}Q \text{ is } P\text{-primary and its contraction in } A \text{ is } Q$$

Proof. If $s \in S \cap P$, then $s^n \in Q$ for some $n > 0$, and $\frac{s^n}{1} \in S^{-1}Q$. Note that

$$\frac{s^n}{1} \cdot \frac{1}{s^n} = \frac{s^n}{s^n} = 1$$

Suppose $S \cap P = \varnothing$. Note that $S^{-1}Q = Q^e$, so to show the contraction of $S^{-1}Q$ is Q, we only have to show

$$Q^{ec} = Q \tag{3.3}$$

Obviously $Q \subseteq Q^{ec}$. We show the opposite. The second part of proposition 3.11 states that

$$Q^{ec} = \bigcup_{s \in S} (Q : s)$$

Because $Q \subseteq P$, if $as \in Q$, then $a \in Q$. Therefore, $a \in (Q : s) \implies a \in Q$. We have shown goal 3.3. Note that the fifth part of proposition 3.11 states that

$$\sqrt{S^{-1}Q} = S^{-1}\sqrt{Q} = S^{-1}P$$

It remains to show $S^{-1}Q$ is indeed primary. Let $\frac{ab}{ss'} = \frac{q}{s''} \in S^{-1}Q$. This implies $(abs'' - qss')t = 0$ for some $t \in S$, which implies $ab(s''t) \in Q$. Because S is closed under multiplication and $S \cap P = \varnothing$, we know $(s''t)^n \notin Q$ for all $n > 0$. This implies $ab \in Q$, which implies $a \in Q$ or some powers of b is an element of Q. We have shown either $\frac{a}{s} \in S^{-1}Q$ or some power of $\frac{b}{s''}$ belongs to $S^{-1}Q$. We have shown $S^{-1}Q$ is indeed primary. ∎