Deadline : 2023/03/08, 17:00.

1. Suppose that $\{a_n\}$ converges to 0 and $\{b_n\}$ is bounded. Prove that $\{a_nb_n\}$ converges.

2. Let $S$ be a nonempty subset of $\mathbb{R}$ which is bounded above. Set $s = \sup S$. Show that there exists a sequence $\{a_n\}$ in $S$ which converges to $s$.

   **Definition 1.** *Let $S$ be a nonempty subset of $\mathbb{R}$ which is bounded above, we say that $s$ is a* ***supremum*** *(最小上界) of $S$, denoted by $\sup S$, if $s$ satisfying*

   *(i) $s$ is an upper bound of $S$ on, i.e. $x \le s$ for all $x \in S$.*

   *(ii) if $s_1$ is an upper bound of $S$, then $s \le s_1$.*

3. Suppose that $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converges. Prove that the following series

$$(a) \sum_{n=1}^{\infty} |a_nb_n|, \qquad (b) \sum_{n=1}^{\infty} (a_n+b_n)^2, \qquad (c) \sum_{n=1}^{\infty} \frac{|a_n|}{n}$$

   converge.

4. Determine whether the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \cdots$$

   converges.

5. Find the value of $x$ for which the series converges, also find the sum of the series for those values of $x$.

$$(a) \sum_{n=1}^{\infty} (-5)^n x^n, \qquad (b) \sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n}, \qquad (c) \sum_{n=1}^{\infty} \frac{2^n}{x^n}.$$

6. Use Comparison Test to determine whether the series convergent or divergent

$$(a) \sum_{n=1}^{\infty} \frac{7n+2}{\sqrt{2n^3-1}}, \qquad (b) \sum_{n=1}^{\infty} ne^{-n^2}, \qquad (c) \sum_{n=1}^{\infty} \frac{2n!}{(2n)!}.$$

7. Let $F(x) = \int_0^x \frac{t}{1+t^2}\, dt$. Find the Taylor polynomial of degree $2n$ of $F(x)$ at 0.

8. Use the Maclaurin series for $f(x) = x\sin(x^2)$ to find $f^{(203)}(0)$.