

Commutative Algebra Final Project

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Chapter 1

Main Body

1.1 Topology of Zariski

Let X be a topological space. We say X is **irreducible** if X can not be written as union of two proper closed subset of X .

Equivalent Definition 1.1.1. (Irreducible topology) Given topological space X , the followings are equivalent:

- (i) X is irreducible.
- (ii) Every nonempty open subset of X is dense.
- (iii) Every two nonempty open subset of X have nonempty intersection.

Proof.

(i) \implies (ii): If nonempty open $Y \subseteq X$ is not dense, then \overline{Y} is proper closed and $X = \overline{Y} \cup (X - \overline{Y})$.

(ii) \implies (iii): Assume for a contradiction $Y_1, Y_2 \subseteq X$ are nonempty open that have empty intersection. One get a contradiction from $\overline{Y_1} = X$ and $Y_1 \subseteq X - Y_2$.

(iii) \implies (i): Assume for a contradiction X is reducible, says, $X = Y_1 \cup Y_2$. One get a contradiction from noting $X - Y_1$ and $X - Y_2$ have empty intersection. ■

If we say a subset of some topological space is **irreducible**, we mean that that subset when given the subspace topology is irreducible. Let X be some arbitrary topological space. In other words, $Y \subseteq X$ is irreducible if and only if Y can not be written as $Y = (F_1 \cap Y) \cup (F_2 \cap Y)$ where $F_1, F_2 \subseteq X$ are closed and don't contain Y .

Theorem 1.1.2. (Irreducibility and Closure) Given topological spaces $Y \subseteq X$,

$$Y \text{ is irreducible} \iff \overline{Y} \text{ is irreducible}$$

Proof. Just observe that given two closed sets $F_1, F_2 \subseteq X$, the statement

$$\overline{Y} = (F_1 \cap \overline{Y}) \cup (F_2 \cap \overline{Y}) \text{ and } F_1 \cup F_2 \text{ doesn't cover } \overline{Y}$$

is equivalent to

$$Y = (F_1 \cap Y) \cup (F_2 \cap Y) \text{ and } F_1 \cup F_2 \text{ doesn't cover } Y.$$

■

By Zorn's Lemma¹, every $x \in X$ has some not necessarily unique maximal irreducible set Y containing x . We call these maximal irreducible sets **irreducible components of X** . Because **closure of an irreducible set is also irreducible**, irreducible components are closed.

Let X be a topological space. We say X is **Noetherian** if its closed sets satisfy descending chain condition. Clearly, subspace Y of Noetherian space X is also Noetherian².

Equivalent Definition 1.1.3. (Noetherian topology) Given topological space X , the followings are equivalent:

- (i) X is Noetherian
- (ii) Open sets of X satisfy ascending chain condition.
- (iii) Every subset of X is compact.

Proof. (i) \implies (ii) is clear.

(ii) \implies (iii): Assume for a contradiction $Y \subseteq X$ has an open cover \mathcal{U} that has no finite subcover. Fix $E_1 \in \mathcal{U}$. Because \mathcal{U} has no finite subcover, there exists $E_2 \in \mathcal{U}$ such that $E_2 \cap Y - (E_1 \cap Y)$ is nonempty, i.e., $E_1 \cup E_2$ strictly contain E_1 . Again, because \mathcal{U} has no finite subcover, there exists $E_3 \in \mathcal{U}$ such that $E_3 \cap Y - ((E_1 \cup E_2) \cap Y)$ is nonempty. Repeating the same process, we construct a non-stopping ascending chain of open sets: $E_1 \subset E_1 \cup E_2 \subset E_1 \cup E_2 \cup E_3 \subset \dots$, a contradiction.

(iii) \implies (i): Let F_n be a descending chain of closed subsets of X . Clearly $X - F_n$ is an ascending chain and forms an open cover of $X - \bigcap F_n$. It follows from compactness of $X - \bigcap F_n$ that descending chain F_n must stop permanently at some point. ■

¹Clearly $\{x\}$ is irreducible. Let E_n be an ascending chain of irreducible set containing x . To see its upper bound is also irreducible, assume it isn't and cause a contradiction by showing E_n are reducible for large n .

²Given $X_1 \cap Y \supseteq X_2 \cap Y \supseteq \dots$, we have $X_n \cap Y = X_1 \cap \dots \cap X_n \cap Y$, and the descending chain $X_1 \supseteq X_1 \cap X_2 \supseteq \dots$ must stop.

As we shall see, most of the topologies studied in this note are Noetherian, and indeed they have nice relationship with irreducibility.

Theorem 1.1.4. (Topological form of Lasker-Noether Theorem) If topological space X is Noetherian, then for every nonempty closed $V \subseteq X$, there always exists unique irreducible closed $V_1, \dots, V_m \subseteq X$ such that

$$V = V_1 \cup \dots \cup V_m \quad (1.1)$$

and every V_i is necessary in the sense that $\bigcup_{j \neq i} V_j \neq V$.

Proof. Note that if V can be written as some finite union of irreducible closed subsets, then we may delete the unnecessary terms so that every term in the union is necessary. We now prove that every V can be written as a finite union of irreducible closed subset.

Let \mathcal{S} be the collection of nonempty closed subset of X that can not be written as finite union of irreducible closed subsets. Assume for a contradiction that \mathcal{S} is nonempty. Because X is Noetherian, \mathcal{S} have some minimal element, say Y . Because Y is reducible, we may write $Y = Y_1 \cup Y_2$, where Y_1 and Y_2 are both closed and contained strictly by Y . By minimality of $Y \in \mathcal{S}$, Y_1 and Y_2 can both be written as finite union of irreducible closed subset, a contradiction³.

It remains to prove the uniqueness of expression of V , so suppose

$$V = V_1 \cup \dots \cup V_m = W_1 \cup \dots \cup W_r$$

where V_i, W_j are all irreducible closed and "necessary". For each i , because V_i is irreducible and $V_i = \bigcup (W_j \cap V_i)$, we have $V_i \subseteq W_j$ for some fixed j . The same argument yields $W_j \subseteq V_p$ for some fixed p . It then follows from the "necessity" that $V_i = W_j$. This implies uniqueness. ■

Let X be a Noetherian topological space. **Topological form of Lasker-Noether Theorem** tell us that X , uniquely, can be written as $X = V_1 \cup \dots \cup V_m$ where V_i are all irreducible closed and necessary in this union. One may conjecture that these V_i are exactly the irreducible components of X . This is true indeed. Fix j . To see V_j is an irreducible component, let W be some irreducible component containing V_j . We are required to show $W = V_j$. Because $W = \bigcup (W \cap V_i)$ and W is an irreducible component⁴, we must have $W \subseteq V_l$ for some l . It then follows from the "necessity" that $j = l$ and $V_j = W$. Conversely, let W be some irreducible component of X . We are required to show W is one of V_i . Because W is irreducible

³Because $Y = Y_1 \cup Y_2$.

⁴So $W \cap V_i$ are all closed. Recall that irreducible components are closed, since closure of irreducible set is also irreducible.

and $W = \bigcup (W \cap V_i)$, we have $W \subseteq V_i$ for some i , which by maximality of W implies $W = V_i$.

We close this section by introducing the idea of variety dimension in a purely topological setting. Let X be a topological space. We define the **dimension of X** to be the supremum of all integers n such that there exists a chain $\emptyset = Z_0 \subset Z_1 \subset \cdots \subset Z_n$ of distinct irreducible closed subsets of X . Immediately we see same wanted property of our definition for dimension: Given subspace $Y \subseteq X$, we have $\text{topdim}(Y) \leq \text{topdim}(X)$.⁵

⁵To prove this, you will need: Y_i closed in $Y \implies Y_i = \overline{Y_i} \cap Y$.

1.2 Affine Variety

Let k be some field. We use the notation \mathbb{A}^n to denote the **affine n -space** over k , the Cartesian product k^n . Given some collection $S \subseteq k[x_1, \dots, x_n]$ of polynomials, we use notation $V(S) = \{a \in \mathbb{A}^n : F(a) = 0 \text{ for all } F \in S\}$ to denote its **zero set (locus)**, and given $X \subseteq \mathbb{A}^n$, we use notation $I(X)$ to denote the ideal of polynomials that vanish on X . If $X \subseteq \mathbb{A}^n$ is the locus of some $S \subseteq k[x_1, \dots, x_n]$, we say X is **algebraic**. Clearly this give rise to a topology on \mathbb{A}^n , the **Zariski topology**, in which $X \subseteq \mathbb{A}^n$ is closed if and only if X is algebraic⁶. We often call irreducible closed subset of \mathbb{A}^n **affine variety**. Clearly, we have:

$$\overline{X} = V(I(X)), \quad \text{for all } X \subseteq \mathbb{A}^n \quad (1.2)$$

Already, at this early stage, we have the prime-irreducibility correspondence:

Theorem 1.2.1. (Prime-irreducibility correspondence) For all $X \subseteq \mathbb{A}^n$,

$$X \text{ is irreducible} \iff I(X) \text{ is prime} \quad (1.3)$$

Proof. Before the proof, we first make the remark that clearly for all $F, G \in k[x_1, \dots, x_n]$, we have:

$$FG \in I(X) \iff X = (X \cap V(F)) \cup (X \cap V(G)) \quad (1.4)$$

(\implies) in [Equation 1.3](#) is then an immediate consequence of (\implies) in [Equation 1.4](#).

We now prove (\impliedby) in [Equation 1.3](#). Assume for a contradiction that $X = (X \cap V_1) \cup (X \cap V_2)$ for some V_1, V_2 such that neither of them covers X . Clearly, there exists some $F \in I(V_1) - I(X)$. This F satisfies $I(V_1) - I(X)$. Similar argument gives us some $G \in I(V_2) - I(X)$, but $FG \in I(X)$, a contradiction to $I(X)$ being prime. ■

Corollary 1.2.2. (Irreducible criteria in affine spaces) For all $X \subseteq \mathbb{A}^n$,

$$X \text{ is irreducible} \iff \overline{X} \text{ is irreducible}$$

Proof. If X is empty, then the assertion holds true trivially. We from now suppose X is nonempty. If X is irreducible, then $I(\overline{X}) = I(X)$ is prime. If \overline{X} is irreducible, then $I(X) = I(\overline{X})$ is prime. ■

Because [Hilbert's Basis Theorem](#) only require the underlying ring of polynomials to be Noetherian, which all fields are, we can already deduce some important properties from the

⁶Note that $V(S_1) \cup V(S_2) = V(\{f_1 f_2 : f_1 \in S_1, f_2 \in S_2\})$ and $\bigcap V(S_\alpha) = V(\bigcup S_\alpha)$

hypothesis that k is a field, albeit a rather weak hypothesis. For example, one may show \mathbb{A}^n as a topological space is Noetherian⁷, every closed $X \subseteq \mathbb{A}^n$ is of the form $X = V(f_1, \dots, f_m)$ for some $f_1, \dots, f_m \in k[x_1, \dots, x_n]$,⁸ and \mathbb{A}^n is irreducible if, moreover, k is infinite:

Theorem 1.2.3. (Polynomial Identity Principal) Let k be an infinite field and $F \neq 0 \in k[x_1, \dots, x_n]$. There exists some nonzero $\mathbf{a} \in \mathbb{A}^n$ such that $F(\mathbf{a}) \neq 0$.

Proof. We prove by induction on n . Base case $n = 1$ follows from fundamental theorem of algebra and taking algebraic closure \bar{k} . Suppose the assertion hold true for $n - 1$, and write

$$F(x_1, \dots, x_n) = \sum_{i=0}^d G_i(x_1, \dots, x_{n-1})x_n^i, \quad \text{where } d = \deg_{x_n} F$$

By inductive hypothesis, there exists some nonzero $(a_1, \dots, a_{n-1}) \in \mathbb{A}^{n-1}$ such that

$$G_d(a_1, \dots, a_{n-1}) \neq 0$$

It then follows from the base case that there exists some nonzero $x_n \in k$ such that $F(x_1, \dots, x_n) \neq 0$. ■

Corollary 1.2.4. (Affine space is irreducible if the underlying field is infinite) If field k is infinite, then \mathbb{A}^n as a topological space is irreducible.

Proof. Suppose $\mathbb{A}^n = V(F_1, \dots, F_r) \cup V(G_1, \dots, G_m)$. Clearly $F_1 \cdots F_r G_1 \cdots G_m$ vanishes on \mathbb{A}^n . This implies $F_1 \cdots F_r G_1 \cdots G_m = 0 \in k[x_1, \dots, x_n]$, which implies one of them is zero. Says, $F_1 = 0$. In such case, we see $V(F_1, \dots, F_r) = V(F_2, \dots, F_r)$ and

$$\mathbb{A}^n = V(F_2, \dots, F_r) \cup V(G_1, \dots, G_m)$$

Repeating the same argument, we see that either F_1, \dots, F_r are all zeros or G_1, \dots, G_m are all zeros. ■

Most of the time we require k to be closed. In such case, we have the powerful **Nullstellensatz**:

$$I(V(I)) = \sqrt{I}, \quad \text{for all ideal } I \subseteq k[x_1, \dots, x_n]$$

As **Theorem 1.2.1** have already shown, functor:

$$\{X \subseteq \mathbb{A}^n : X \text{ is a variety.}\} \xrightarrow{I} \text{Spec}(k[x_1, \dots, x_n])$$

is injective. **Nullstellensatz** moreover shows that this functor is surjective. In particular, I forms a bijection between points in \mathbb{A}^n with $\text{Max}(k[x_1, \dots, x_n])$ ⁹.

⁷Given descending chain of closed subset $X_0 \supset X_1 \supset \cdots \supset X_m$ of \mathbb{A}^n , we must have $I(X_0) \subset I(X_1) \subset \cdots \subset I(X_m)$ where the inequalities hold true because of **Equation 1.2**. Now recall that **Hilbert's Basis Theorem** guarantees $k[x_1, \dots, x_n]$ is Noetherian.

⁸For its reason, consider the third equivalent definition of Noetherian module.

⁹Using **the weak form of Nullstellensatz**, if $\mathfrak{m} \subseteq k[x_1, \dots, x_n]$ is maximal, then one construct isomorphism $\phi : k[x_1, \dots, x_n]/\mathfrak{m} \rightarrow k$ and set $a_i \triangleq x_i$ to see $\phi([f]) = f(a)$ and $\{a\} = V(\mathfrak{m})$

1.3 Projective variety

Given some ring A and monoid M , by a **M -grading on A** , we mean a collection $(A_m)_{m \in M}$ of subgroup of the additive group of A such that $A = \bigoplus A_m$ ¹⁰ and $A_m A_n \subseteq A_{m+n}$ ¹¹ for all $m, n \in M$. Fix $a \in A$. If $a = a_{m_1} + \cdots + a_{m_r}$ for $a_{m_i} \in A_{m_i}$, we say a_{m_i} are the **homogeneous components of a** , and if $r = 1$ we say a is a **homogeneous element**. Note that there is ambiguity whether 0 should be considered homogeneous. If one consider 0 homogeneous, then 0 should have arbitrary degree.

We say an ideal $I \subseteq A$ is **homogeneous** if I have a set of generators that are all homogeneous, or equivalently, if $I = \bigoplus I \cap A_n$. Note that the sum, product, and intersection of homogeneous ideals are homogeneous. Clearly, if $M = \mathbb{Z}$ or \mathbb{Z}_0^+ , then A_0 forms a subring of A ¹²; A_i all form obvious A_0 -modules; and radical of homogeneous ideal remain homogeneous¹³.

Let k be some field. Clearly, we may define on $k^{n+1} - \{0\}$ an equivalence relation by setting

$$a \sim b \stackrel{\Delta}{\iff} a = \lambda b, \quad \text{for some } \lambda \in k$$

Similar to the affine n -space, we use the notation \mathbb{P}^n to represent the set of equivalence class of k^{n+1} . Clearly, when we give the polynomial ring $k[x_0, \dots, x_n]$ the obvious grading, a polynomial $f \in k[x_0, \dots, x_n]$ is homogeneous in the graded sense if and only if it is homogeneous in the usual sense¹⁴. Suppose $f \in k[x_0, \dots, x_n]$. Even though the value of f on \mathbb{P}^n is not well-defined, if f is homogeneous then indeed it is well-defined whether $f(p) = 0$ for fixed $p \in \mathbb{P}^n$, so it make sense for us to talk about the **(projective) algebraic set** $V(S) = \{p \in \mathbb{P}^n : f(p) = 0 \text{ for all } f \in S\}$ for every collection $S \subseteq k[x_1, \dots, x_{n+1}]$ of homogeneous polynomial. Again this give rise to **Zariski topology** on \mathbb{P}^n where $E \subseteq \mathbb{P}^n$ is closed if and only if E is algebraic, and again we say $X \subseteq \mathbb{P}^n$ is a **projective variety** if X is irreducible closed.

Let S_1 and $S_2 \subseteq k[x_0, \dots, x_n]$ be two collections of homogeneous polynomials that generate the same ideal. Clearly we have $V(S_1) = V(S_2)$. Therefore, given some homogeneous ideal $J \subseteq k[x_0, \dots, x_n]$, we may well define algebraic $V(J)$. Trivially, for each $X \subseteq \mathbb{P}^n$, the ideal generated by homogeneous $F \in k[x_0, \dots, x_n]$ that vanishes on X , called the **defining**

¹⁰The direct sum is a direct sum of groups.

¹¹You may interpret $A_m A_n$ as $\{a_m a_n \in A : a_m \in A_m, a_n \in A_n\}$ here.

¹²To see $1 \in A_0$, consider the highest graded component of 1 and $1 = 1 \cdot 1$.

¹³If $(x_1 + \cdots + x_r)^l \in I$ with x_r highest grade and I homogeneous, then since the highest grade term of $(x_1 + \cdots + x_r)^l$ is x_r^l , we have $x_r \in \sqrt{I}$, which implies $x_1 + \cdots + x_{r-1} \in \sqrt{I}$.

¹⁴i.e., every term has the same degree.

ideal of X if X is closed, is homogeneous, and moreover, contains only polynomials that "vanishes on X in the obvious sense."

Immediately, we see that projective space have lots of similar property with affine space, for example, similar to [Equation 1.2](#), we have

$$\overline{X} = V(I(X)), \quad \text{for all } X \subseteq \mathbb{P}^n$$

And similar to the [fact that if the underlying field is infinite, the affine space is irreducible](#), we have:

Theorem 1.3.1. (Projective space is irreducible if the underlying field is infinite) If field k is infinite, then \mathbb{P}^n is irreducible.

Proof. Suppose $\mathbb{P}^n = V(F_1, \dots, F_r) \cup V(G_1, \dots, G_m)$. Clearly $F_1 \cdots F_r G_1 \cdots G_m$ vanishes on \mathbb{P}^n . This implies $F_1 \cdots F_r G_1 \cdots G_m$ vanishes on $\mathbb{A}^{n+1} - \{\mathbf{0}\}$. Now, if we were able to prove $F_1 \cdots F_r G_1 \cdots G_m$ also vanishes $\mathbf{0}$, thus whole \mathbb{A}^{n+1} , then we may use the same argument in [Corollary 1.2.4](#). To see that $F_1 \cdots F_r G_1 \cdots G_m$ also vanishes at $\mathbf{0}$, just observe that because $F_1 \cdots F_r G_1 \cdots G_m$ is homogeneous, if it doesn't vanish at $\mathbf{0}$, then $F_1, \dots, F_r, G_1, \dots, G_m$ are all nonzero constant, which is clearly impossible. ■

Also we have the prime-irreducibility correspondence, whose proof in the projective setting requires some more algebraic effort.

Theorem 1.3.2. (Projective prime-irreducibility correspondence) For all $X \subseteq \mathbb{P}^n$,

$$X \text{ is irreducible} \iff I(X) \text{ is prime}$$

Proof. (\implies): Let $F, G \in k[x_0, \dots, x_n]$ satisfy $FG \in I(X)$. We are required to prove one of them belongs to $I(X)$. Let F_1, \dots, F_d and G_1, \dots, G_r be the (nonzero) homogeneous component of F and G with $\deg(F_i) < \deg(F_{i+1}), \deg(G_i) < \deg(G_{i+1})$ for all i . Because $F_d G_r$ is exactly the highest degree term of $FG \in I(X)$ and because $I(X)$ is homogeneous, we know $F_d G_d \in I(X)$. Because X is irreducible and because $X = (X \cap V(F_d)) \cup (X \cap V(G_r))$, we now see $F_d \in I(X)$ or $G_r \in I(X)$.

WLOG, let $F_d \in I(X)$. There are two cases: either G_r is in $I(X)$ or not. We first prove the case when G_r isn't in $I(X)$.

Because $F_{d-1} G_r + F_d G_{r-1}$ is either zero or the second higher degree term of $FG \in I(X)$, we know $F_{d-1} G_r + F_d G_{r-1} \in I(X)$. This together with $F_d \in I(X)$ implies $F_{d-1} G_r \in I(X)$. It then follows from the irreducibility of X and $X = (X \cap V(F_{d-1})) \cup (X \cap V(G_r))$ that $F_{d-1} \in I(X)$. Similar arguments applies to show in order that $F_{d-2}, \dots, F_1 \in I(X)$, finishing the proof of the case $G_r \notin I(X)$.

For the case of $G_r \in I(X)$, just observe that $G_r \in I(X) \implies (F - F_d)(G - G_r) \in I(X)$, and repeat the arguments all over again.

(\Leftarrow): Assume for a contradiction that $X = (X \cap V_1) \cup (X \cap V_2)$ for some V_1, V_2 such that neither of them covers X . Clearly, there exists some homogeneous $F \in I(V_1)$ that doesn't vanish on the whole X . This F satisfies $I(V_1) - I(X)$. Similar argument gives us some homogeneous $G \in I(V_2) - I(X)$, but $FG \in I(X)$, a contradiction to $I(X)$ being prime. ■

Corollary 1.3.3. (Irreducible criteria in projective space) For all $X \subseteq \mathbb{P}^n$,

$$X \text{ is irreducible} \iff \overline{X} \text{ is irreducible}$$

Proof. If X is empty, then the assertion holds true trivially. We from now suppose X is nonempty. If X is irreducible, then $I(\overline{X}) = I(X)$ is prime. If \overline{X} is irreducible, then $I(X) = I(\overline{X})$ is prime. ■

Let $S \subseteq k[x_0, \dots, x_n]$ be some homogeneous ideal or collection of homogeneous polynomials. If necessary, we often distinguish the two vanishing sets in affine and projective setting by writing:

$$V_{\mathbb{P}}(S) \triangleq \{[x_0 : \dots : x_n] \in \mathbb{P}^n : F(x_0, \dots, x_n) = 0 \text{ for all } F \in S\}$$

and writing:

$$V_{\mathbb{A}}(S) \triangleq \{(x_0, \dots, x_n) \in \mathbb{A}^{n+1} : F(x_0, \dots, x_n) = 0 \text{ for all } F \in S\}.$$

Theorem 1.3.4. (Vanishing sets of collections of homogeneous polynomials are cones) Let $\pi : (\mathbb{A}^{n+1} - \{\mathbf{0}\}) \rightarrow \mathbb{P}^n$ be the quotient map. Given any collection $S \subseteq k[x_0, \dots, x_n]$ of homogeneous polynomials, $V_{\mathbb{A}}(S)$ equals to either $\pi^{-1}(V_{\mathbb{P}}(S))$ or $\pi^{-1}(V_{\mathbb{P}}(S)) \cup \{\mathbf{0}\}$.

Proof. It is trivial to check:

$$\pi(V_{\mathbb{A}}(S)) = V_{\mathbb{P}}(S) \tag{1.5}$$

Equation 1.5 finishes the proof because it implies $V_{\mathbb{A}}(S) - \{\mathbf{0}\} \subseteq \pi^{-1}(V_{\mathbb{P}}(S))$ and implies $\pi^{-1}(V_{\mathbb{P}}(S)) \subseteq V_{\mathbb{A}}(S)$. ■

Let $S \subseteq k[x_0, \dots, x_n]$ be a collection of homogeneous polynomials. We often call $V_{\mathbb{A}}(S)$ **cone**. This notion helps us prove the following result.

Theorem 1.3.5. (Zariski topology on projective space is the quotient topology) The quotient map $\pi : (\mathbb{A}^{n+1} - \{\mathbf{0}\}) \rightarrow \mathbb{P}^n$ is a topological quotient map.

Proof. To prove that π is a topological quotient, we need to show that π is surjective, continuous, and every subset of \mathbb{P}^n that has closed preimage is closed. Clearly π is surjective. To see π is continuous, just observe for every collection S of homogeneous polynomials, we have

$$\pi(V_{\mathbb{A}}(S)) = V_{\mathbb{P}}(S) \quad (1.6)$$

Note that if the underlying field is finite, then the Zariski topologies are discrete so the assertion holds trivially¹⁵. We from now suppose the underlying field k is infinite. Suppose $E \subseteq \mathbb{P}^n$ have closed preimage $\pi^{-1}(E) = V_{\mathbb{A}}(T) - \{\mathbf{0}\}$, where $T \subseteq k[x_0, \dots, x_n]$. We show

$$E = V_{\mathbb{P}}(\{F_i \in k[x_0, \dots, x_n] : F_i \text{ is the homogeneous component of some } F \in T.\})$$

The " \supseteq " holds true trivially. We now show the " \subseteq ". Fix $F \in T$ with homogeneous decomposition $F = \sum F_i$. We are required to prove F_i all vanish on E . Fix $[x_0 : \dots : x_n] \in E$. We are required to prove all F_i vanish at $[x_0 : \dots : x_n]$. Define formal polynomial $p(y) \in k[y]$ by:

$$p(y) \triangleq \sum F_i(x_0, \dots, x_n)y^i = \sum F_i(yx_0, \dots, yx_n)$$

Assume for a contradiction that there exists some F_i such that $F_i(x_0, \dots, x_n) \neq 0$. Because k is infinite, by **polynomial identity principal**, there exists nonzero $y' \in k$ such that $p(y') \neq 0$. However, because by definition $(y'x_0, \dots, y'x_n) \in \pi^{-1}(E)$ and because F vanish on $\pi^{-1}(E)$, we also have:

$$0 = F(y'x_0, \dots, y'x_n) = \sum F_i(y'x_0, \dots, y'x_n) = p(y')$$

a contradiction. ■

We now enter the better behaved and smaller category: when the underlying field is algebraically closed. In particular, we prove the famous **projective Nullstellensatz**:

Theorem 1.3.6. (Homogeneous or Projective Nullstellensatz) Given algebraically closed field k and homogeneous ideal $\mathfrak{a} \subseteq k[x_0, \dots, x_n]$, if $V(\mathfrak{a})$ is nonempty¹⁶, then

$$I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$$

Proof. Clearly we have:

$$\sqrt{\mathfrak{a}} \subseteq I(V(\mathfrak{a}))$$

¹⁵Per this guy on MO: <https://mathoverflow.net/questions/16146/elementary-proof-that-projective-space-is-a-quotient>

¹⁶This implies \mathfrak{a} contains no nonzero constant.

We now prove $I(V(\mathfrak{a})) \subseteq \sqrt{\mathfrak{a}}$. Because **affine Nullstellensatz** said that $\sqrt{\mathfrak{a}} = I(V_{\mathbb{A}}(\mathfrak{a}))$, we only have to prove $I(V_{\mathbb{P}}(\mathfrak{a})) \subseteq I(V_{\mathbb{A}}(\mathfrak{a}))$. Fixing homogeneous $F \in I(V_{\mathbb{P}}(\mathfrak{a}))$ and $\mathbf{x} \in V_{\mathbb{A}}(\mathfrak{a})$, we are required to prove F vanishes at \mathbf{x} . If F is zero, there is nothing to prove, so we from now on suppose F is nonzero. Because by premise $V_{\mathbb{P}}(\mathfrak{a})$ is nonempty, we know F can not be nonzero constant. In other words, homogeneous F have positive degree. **Theorem 1.3.4** said that $V_{\mathbb{A}}(\mathfrak{a})$ either equals to $\pi^{-1}(V_{\mathbb{P}}(\mathfrak{a}))$ or $\pi^{-1}(V_{\mathbb{P}}(\mathfrak{a})) \cup \{0\}$. If $\mathbf{x} = \mathbf{0}$, then F vanishes at \mathbf{x} because F is homogeneous with positive degree. If $\mathbf{x} \in \pi^{-1}(V_{\mathbb{P}}(\mathfrak{a}))$, then F vanishes at \mathbf{x} because it vanishes at $\pi(\mathbf{x})$. ■

Theorem 1.3.7. (Special case of projective Nullstellensatz) Given algebraically closed field k and homogeneous ideal $\mathfrak{a} \subseteq k[x_0, \dots, x_n]$, the followings are equivalents:

- (i) $V_{\mathbb{P}}(\mathfrak{a})$ is empty.
- (ii) $\sqrt{\mathfrak{a}} = k[x_0, \dots, x_n]$ or $\sqrt{\mathfrak{a}} = \{F \in k[x_0, \dots, x_n] : F(0) = 0\}$.
- (iii) For some $d > 0$, we have $\{F \in k[x_0, \dots, x_n] : \deg(F) = d\} \subseteq \mathfrak{a}$.

Proof. (i) \implies (ii): **Theorem 1.3.4** implies $V_{\mathbb{A}}(\mathfrak{a})$ either is empty or contains only the origin. This together with **affine Nullstellensatz** proves (ii).

(ii) \implies (i): **Affine Nullstellensatz** said that $I(V_{\mathbb{A}}(\mathfrak{a})) = \sqrt{\mathfrak{a}}$. By premise no matter what $\sqrt{\mathfrak{a}}$ actually is, it must contains $\{x_0, \dots, x_n\}$. This implies $V_{\mathbb{A}}(\mathfrak{a}) \subseteq \{0\}$, which proves (i) by **Theorem 1.3.4**. ■

1.4 Morphism

By a **quasi affine (or projective) variety** $X \subseteq \mathbb{A}^n$ (resp. \mathbb{P}^n), we mean X is a subset of some variety $V \subseteq \mathbb{A}^n$ (resp. \mathbb{P}^n) and X is open in V .

Equivalent Definition 1.4.1. (Quasi variety) Given $X \subseteq \mathbb{A}^n$ (or \mathbb{P}^n), if $X \subseteq V$ satisfy that V is irreducible closed and X is nonempty open in V , then $V = \overline{X}$.

Proof. Clearly $\overline{X} \subseteq V$. We now prove $V \subseteq \overline{X}$, which follows from $\overline{X} \cap V = \text{cl}_V(X) = V$, where the last equality holds true because by **equivalent definition of irreducibility**, X is dense in V . ■

Given some quasi affine variety $X \subseteq \mathbb{A}^n$ and $f : X \rightarrow k$, we say f is **regular at** $p \in X$ if there exist $U \subseteq X$ open in X containing p and exist $g, h \in k[x_1, \dots, x_n]$ such that h nonzero on U and $f = g/h$ on U . In the projective setting, says $X \subseteq \mathbb{P}^n$ is a quasi projective variety and f maps X into k , we say f is **regular at** $p \in X$ if there exists $U \subseteq X$ containing p and open in X such that there exists homogeneous $g, h \in k[x_0, \dots, x_n]$ such that h nonzero on U , $f = g/h$ on U , and g, h have same degree. Given a quasi variety X , we say $f : X \rightarrow k$ is **regular** if it is regular at each points of X , and we denote the **ring of regular function on** X by $\mathcal{O}(X)$.¹⁷

Theorem 1.4.2. (Regular functions are continuous) Given quasi variety $X \subseteq \mathbb{A}^n$ (or \mathbb{P}^n), if $f : X \rightarrow k$ is regular, then f is continuous.

Proof. Because every closed set in k is either finite or the whole space¹⁸, we only have to show $f^{-1}(a)$ is closed for fixed $a \in k$. Let $(U_i, g_i, h_i)_{i \in I}$ cover X , where h_i nonzero and $f = g_i/h_i$ on U_i . Because X is coherent¹⁹ with $(U_i)_I$, we only have to show $U_i \cap f^{-1}(a)$ is closed in U_i for fixed $i \in I$, which follows from noting $U_i \cap f^{-1}(a) = V(g - ah) \cap U_i$. ■

Let X be a quasi variety with $p \in X$. Consider the collection of all pairs $(p \in U \subseteq X, f : U \rightarrow k)$, where f is regular, and the collection of all $(U \subseteq X, f : U \rightarrow k)$, where U is nonempty and f is regular, because **regular functions are continuous** and **nonempty open**

¹⁷It is IMPORTANT to deduce that regular functions indeed forms a ring here, since it is impossible to later check transitivity of **equivalence 1.7** purely topologically, without using the algebraic result of one regular functions minus another is still regular, thus continuous. I guess the moral of the story here is that one must remember \mathbb{A}^1 is a field, and thus have a field structure, which shall be used when topological method doesn't work; or that if something forms an algebraic structure, then utilize that structure as much as possible. Indeed, we are talking about ALGEBRAIC geometry here.

¹⁸To see such, consider **Hilbert Basis Theorem** and fundamental theorem of algebra.

¹⁹You may Google what does **coherent topology** mean. To see that every topological space X and any of its open cover U_i are always coherent, just observe for any $Z \subseteq X$, we have $Z = \bigcup (U_i \cap Z)$ and $X - Z = \bigcup (U_i - Z)$.

subset of irreducible space is always dense, we may well define two equivalence relation respectively on these two collection by:

$$(U, f) \sim (O, g) \stackrel{\Delta}{\iff} f = g \text{ on } U \cap O \quad (1.7)$$

²⁰to form the ring $\mathcal{O}_{p,X}$ of **germs**²¹ of regular functions on X near p and the **function field** $K(X)$ of X .

Let $(U, f) \in K(X)$. If $p \in U$ and f doesn't vanish at p , then f^{-1} is well-defined and regular on some smaller neighborhood of p . This implies that indeed $K(X)$ is a field, and that the units of $\mathcal{O}_{p,X}$ are exactly those who doesn't vanish at p , which implies the non-units, those vanish at p , forms an ideal of $\mathcal{O}_{p,X}$, justifying the colloquial convention of calling $\mathcal{O}_{p,X}$ the **local ring of p on X** ²². Clearly, the obvious mappings of $\mathcal{O}(X) \longrightarrow \mathcal{O}_{p,X} \longrightarrow K(X)$ are well-defined injective ring homomorphism, and they will be how we treat one as subset of another.

Let X, Y be two quasi varieties. We say $\varphi : X \rightarrow Y$ is a **quasi variety morphism** if φ is continuous and for every $(U, f) \in K(Y)$, the function $f \circ \varphi : \varphi^{-1}(U) \rightarrow k$ is regular. If we say $\varphi : X \rightarrow Y$ is a **quasi variety isomorphism**, we mean that φ is bijective with its inverse being also a quasi variety morphism. Clearly, if $\varphi : X \rightarrow Y$ is a quasi variety isomorphism, then $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ as ring are isomorphic; for each $p \in X$, $\mathcal{O}_{p,X}$ and $\mathcal{O}_{\varphi(p),Y}$ as ring are isomorphic; and $K(X)$ and $K(Y)$ as ring are isomorphic. Because of such, we often say $\mathcal{O}(X)$, $\mathcal{O}_{p,X}$, and $K(X)$ are **invariant up to isomorphisms**.

For ease in **Theorem 1.4.3**, we first introduce two notions. Let S be the collection of homogeneous elements of $k[x_0, \dots, x_n]$. Clearly, we may well-define mappings $\alpha : S \rightarrow k[x_1, \dots, x_n]$ and $\beta : k[x_1, \dots, x_n] \rightarrow S$ by

$$\alpha(F) \triangleq F(1, x_1, \dots, x_n) \text{ and } \beta(G) \triangleq x_0^{\deg(G)} g \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0} \right)$$

Let $F \in S$ and $G \in k[x_1, \dots, x_n]$. We refer to $\alpha(F)$ as the **dehomogenization** of F and denote it by F_* , and we refer to $\beta(G)$ as the **homogenization** of G and denote it by G^* . Actions of homogenization are actually easy to memorize. For example, homogenization send $x_2 - x_1^2$ to $x_0x_2 - x_1^2$, and send $x_2 - x_1^3$ to $x_2x_0^2 - x_1^3$. Dehomogenization sends $x_0x_2^2 + x_1^3$ to $x_2^2 + x_1^3$.

²⁰As noted before, to check transitivity, you must consider the regular function $f - h$.

²¹You may Google what does **germ** means. It is generic.

²²Recall that if the set of non-units forms an ideal, then the ring is local.

Theorem 1.4.3. (Standard embedding of \mathbb{A}^n into \mathbb{P}^n) Let $U \triangleq \mathbb{P}^n - V(x_0)$. The map $\varphi : U \rightarrow \mathbb{A}^n$ defined by

$$\varphi([x_0 : \cdots : x_n]) \triangleq \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0} \right)$$

is a well-defined homeomorphism, and moreover a quasi variety isomorphism if the underlying field k is infinite.

Proof. It is routine to check φ is well-defined and bijective. We now prove it is indeed bi-continuous. Let $Y \subseteq U$ be closed in U , and let \overline{Y} be the closure of Y in \mathbb{P}^n . By definition, there exists some collection T of homogeneous polynomials $F \in k[x_0, \dots, x_n]$ whose locus is exactly \overline{Y} . To see $\varphi(Y)$ is closed, just check $\varphi(Y)$ is the locus of the dehomogenizations of T . Conversely, given closed $W \subseteq \mathbb{A}^n$, we know there exists $T \subseteq k[x_1, \dots, x_n]$ whose locus is exactly W . To see $\varphi^{-1}(W)$ is closed, just check that $\varphi^{-1}(W)$ is the intersection between U and the locus of the homogenization of T . We have shown φ is indeed a homeomorphism.

From now on suppose k is infinite, and we wish to prove φ is moreover a quasi-variety isomorphism. To see U is indeed a quasi variety, note that **projective space is irreducible when underlying field is infinite** and that **nonempty subset of irreducible space is dense**.

Fix $(O, f) \in K(\mathbb{A}^n)$, where, WLOG, $f = F/G$ on O for some $F, G \in k[x_1, \dots, x_n]$ such that G nonzero on O . To see φ is indeed a morphism, just check $f \circ \varphi|_{\varphi^{-1}(O)} = (x_0^{\deg(G)} F^*) / (x_0^{\deg(F)} G^*)$. Fix $(O, f) \in K(U)$, where, WLOG, $f = F/G$ on O for some same degree homogeneous $F, G \in k[x_0, \dots, x_n]$ such that G nonzero on O . To see φ^{-1} is indeed a morphism, just check $f \circ \varphi^{-1}|_{\varphi(O)} = F_*/G_*$. ■

In fact, given quasi variety mapping φ whose target is affine, there is an if and only if test for whether φ is a quasi variety morphism.

Theorem 1.4.4. (If and only if test for quasi variety mapping whose target is affine) Given quasi variety mapping $\varphi : X \rightarrow Y \subseteq \mathbb{A}^n$,

$$\varphi \text{ is a quasi variety morphism.} \iff x_i \circ \varphi : X \rightarrow k \text{ are all regular.}$$

Proof. (\implies) follows from definition. We now prove (\impliedby). Clearly, because regular function on X is closed under addition and multiplication **as noted before**, for all $F \in k[x_1, \dots, x_n]$, the function $F \circ \varphi$ must be regular. It then follows from **continuity of regular functions** and

$$\varphi^{-1}(V(S) \cap Y) = \bigcap_{F \in S} (F \circ \varphi)^{-1}(0), \quad \text{for any } S \subseteq k[x_1, \dots, x_n]$$

that φ is continuous. Fix $(U, g) \in K(Y)$ and $p \in \varphi^{-1}(U)$. It remains to prove $g \circ \varphi : \varphi^{-1}(U) \rightarrow k$ is regular at p . Because $U \xrightarrow{g} k$ is regular, there exists open $O \subseteq U$ containing $\varphi(p)$ such that $g = \frac{h_1}{h_2}$ on O for some $h_1, h_2 \in k[x_1, \dots, x_n]$ where h_2 nonzero on whole O and degree of them being the same if X is projective. Clearly,

$$g \circ \varphi = \frac{h_1 \circ \varphi}{h_2 \circ \varphi}, \quad \text{on } \varphi^{-1}(O).$$

Because $h_1 \circ \varphi$ and $h_2 \circ \varphi$ are regular as function from X to k as we proved in the first paragraph of this proof, we know there exists open $V \subseteq \varphi^{-1}(O)$ containing p such that

$$h_1 \circ \varphi = \frac{F_1}{F_2} \text{ and } h_2 \circ \varphi = \frac{F_3}{F_4}, \quad \text{on } V$$

for some polynomials F_1, F_2, F_3, F_4 , with F_2, F_4 nonzero on V and the pairs having the same degree if X projective. This give us

$$g \circ \varphi = \frac{F_1 F_4}{F_2 F_3}, \quad \text{on } V$$

where $\deg(F_1 F_4) = \deg(F_2 F_3)$ if X is projective. We have shown that indeed $g \circ \varphi$ is regular at p . ■

Given an affine quasi variety $X \subseteq \mathbb{A}^n$, we use notation $\Gamma(X)$ to denote its **coordinate ring** $k[x_1, \dots, x_n]/I(X)$, which is clearly isomorphic to the ring of function $f : X \rightarrow k$ such that $f = F|_X$ for some $F \in k[x_1, \dots, x_n]$ ²³. From the second point of view, immediately we see that coordinate ring of affine quasi variety is in fact an integral domain²⁴. Later we will see more usage of the realization of $\Gamma(X)$ as ring of polynomial functions in the proof of [Theorem 1.4.5](#).

Theorem 1.4.5. (Algebraic structure of global ring, local ring, and function field of affine quasi variety) If $X \subseteq \mathbb{A}^n$ is an affine quasi variety, then

- (i) $\mathcal{O}(X) \cong \Gamma(X)$.
- (ii) For all $p \in X$, if we let $\mathfrak{m}_p \subseteq \Gamma(X)$ be the maximal ideal corresponding to p , then $\mathcal{O}_{p,X} \cong \Gamma(X)_{\mathfrak{m}_p}$.
- (iii) $K(X) \cong \text{Frac}(\Gamma(X))$.

²³We call these functions **polynomial functions**.

²⁴If $FG = 0$ on $X \subseteq Y \subseteq \mathbb{A}^n$ where Y is a variety, then because Y is irreducible, one of $F^{-1}(0) \cap Y, G^{-1}(0) \cap Y$ must be whole Y .

Proof. Before the proof, we first remark that: If we fix p , regarding $\Gamma(X)$ either as the ring of polynomial function $f : X \rightarrow k$ or as the quotient $k[x_1, \dots, x_n]/I(X)$ makes no mathematical differences. In the former, $I(p) \subseteq \Gamma(X)$ is the ring of polynomial functions that vanishes at p , and in the latter, $I(p) \subseteq \Gamma(X)$ is the image of $I(p) \subseteq k[x_1, \dots, x_n]$ under the quotient map $k[x_1, \dots, x_n] \twoheadrightarrow k[x_1, \dots, x_n]/I(X)$. Either way, the $I(p) \subseteq \Gamma(X)$ is what we meant by \mathfrak{m}_p in (ii), and using the latter point of view with the bijection between \mathbb{A}^n and $\text{Max}(k[x_1, \dots, x_n])$, we have a bijection between X and $\text{Max}(\Gamma X)$: $p \leftrightarrow \mathfrak{m}_p = I(p)$.

Fix p . Let $\beta : \Gamma(X)_{\mathfrak{m}_p} \rightarrow \mathcal{O}_{p,X}$ be the obvious mapping²⁵: $f/g \mapsto [(U, f/g)]$ where $f, g : X \rightarrow k$ are polynomial function and $U = \{x \in X : g(x) \neq 0\}$. Clearly β is an injective ring homomorphism²⁶. To see β is surjective, just recall the definition of regular function. We have constructed isomorphism $\beta : \Gamma(X)_{\mathfrak{m}_p} \rightarrow \mathcal{O}_{p,X}$, proving (ii). The same argument works for constructing isomorphism $\text{Frac}(\Gamma X) \cong K(X)$, proving (iii).

Let $\alpha : \Gamma(X) \rightarrow \mathcal{O}(X)$ be the obvious mapping, which is clearly an injective ring homomorphism. It remains to prove α is surjective, which follows from checking the diagram:

$$\begin{array}{ccc} \Gamma(X) & \xrightarrow{\alpha} & \mathcal{O}(X) \\ \updownarrow & & \updownarrow \\ \bigcap_{\mathfrak{m} \in \text{Max}(\Gamma X)} (\Gamma X)_{\mathfrak{m}} & \longleftrightarrow & \bigcap_{p \in X} \mathcal{O}_{p,X} \end{array}$$

²⁷is indeed commutative, where the left and right sides are, respectively, subrings of $\text{Frac}(\Gamma X)$ and $K(X)$. ■

It may seem surprising that $\mathcal{O}(X) \cong \Gamma(X)$, since one might expect the former to be a localization of the latter, given that regular functions are defined locally as quotients of global polynomial functions. However, this apparent tension is actually a valuable opportunity to distinguish between the local and global perspectives: regularity is inherently a local notion. Indeed, by definition, a function $f : X \rightarrow k$ is regular if, locally, it can be written as a fraction g/h with $g, h \in \Gamma(X)$.

²⁵From β you can see that the done localization $\Gamma(X)_{\mathfrak{m}_p}$ is of geometrical nature, not algebraic.

²⁶Note that $[(U, f)] = 0 \in \mathcal{O}_{p,X}$ if and only if $f = 0$ on U , since if $f \neq 0$ on U , then $f + f \neq f$. This implies injectivity of β since $f^{-1}(0) \subseteq X$ is closed in X , containing $g^{-1}(k^\times) \subseteq X$, whose closure in X are X . ($\because Z \subseteq Y \subseteq X \implies \text{cl}_Z(Y) = \text{cl}_X(X) \cap Y$)

²⁷The left isomorphism is explain in this paragraph of Section ??, the bottom isomorphism is from (ii), and the right injective homomorphism is from [this paragraph](#) of this section.

In fact, if one attempts to show that $\mathcal{O}_{p,X}$ is the localization $(f \in \mathcal{O}(X) : f(p) \neq 0)^{-1} \mathcal{O}(X)$ —that is, to verify that the obvious map $\mathcal{O}(X) \rightarrow \mathcal{O}_{p,X}$ satisfies the universal property—without invoking [Theorem 1.4.5](#), one quickly encounters difficulties and will be forced to use isomorphism $\mathcal{O}(X) \cong \Gamma(X)$ ²⁸.

Now, let X, Y be two quasi varieties with Y affine. It is clear that the three structures $\mathcal{O}(X)$, $\mathcal{O}_{p,X}$, and $K(X)$ all admit obvious k -algebra structures, and from the proof of [Theorem 1.4.5](#), the ring isomorphism $\mathcal{O}(Y) \cong \Gamma(Y)$ is also a k -algebra isomorphism.

Theorem 1.4.6. (Induction forms a bijection between set of quasi varieties of morphism and set of k -algebra homomorphism from target's coordinate ring to source's global ring if target is affine) Let X, Y be two quasi varieties with $Y \subseteq \mathbb{A}^n$. Denoting the collection of quasi variety morphisms $X \rightarrow Y$ by $\text{Hom}(X, Y)$ and the collection of k -algebra homomorphism $\Gamma(Y) \rightarrow \mathcal{O}(X)$ by $\text{Hom}(\mathcal{O}(X), \Gamma(Y))$, clearly we can well define a map $\text{Hom}(X, Y) \xrightarrow{\alpha} \text{Hom}(\Gamma(Y), \mathcal{O}(X))$ by

$$\varphi \mapsto (\tilde{\varphi} : \Gamma(Y) \rightarrow \mathcal{O}(X)), \quad \text{where } \tilde{\varphi}(F) \triangleq F \circ \varphi.$$

Such α is bijective.

Proof. Fix $h \in \text{Hom}(\Gamma(Y), \mathcal{O}(X))$. Consider $x_i \in \Gamma(Y)$, $\xi_i \triangleq h(x_i) \in \mathcal{O}(X)$, and $\psi : X \rightarrow \mathbb{A}^n$ defined by $\psi(p) \triangleq (\xi_1(p), \dots, \xi_n(p))$. Because α is clearly injective²⁹, our end goal here is to show that $\psi \in \text{Hom}(X, Y)$ and $h = \tilde{\psi}$.

Because $Y = V(I(Y))$, to show that ψ at least forms a map from X to Y , we only have to show $F \circ \psi = 0$ for any fixed $F \in I(Y)$. Because h is a k -algebra homomorphism and F in k -algebra $\Gamma(Y)$ can be appropriately (See footnote) generated by x_i , we have $h(F) = F(h(x_1), \dots, h(x_n))$ ³⁰. This by definition of ξ and ψ give us

$$h(F) = F(\xi_1, \dots, \xi_n) = F \circ \psi \tag{1.8}$$

This together with the fact F as an element of $\Gamma(Y)$ is zero implies, indeed, $F \circ \psi = 0$.

Noticing that [Equation 1.8](#) also holds true for any $F \in \Gamma(Y)$ ³¹ that's not in $I(Y)$, we conclude that if ψ really is a morphism, then we will have $h = \tilde{\psi}$. Therefore, it only remains to show ψ is indeed a morphism, which follows from $x_i \circ \psi = \xi_i \in \mathcal{O}(X)$ for all $x_i \in k[x_1, \dots, x_n]$ and [Test 1.4.4](#). ■

²⁸As the action of isomorphism α in [Theorem 1.4.5](#) suggest, one can really just write $\mathcal{O}(X) = \Gamma(X)$

²⁹A direct proof suffices. It is easy to construct $F \in k[x_1, \dots, x_n]$ for arbitrary $p_1 \neq p_2 \in \mathbb{A}^n$ that satisfies $F(p_1) \neq F(p_2)$.

³⁰For example, $h(x_1^2 + x_2) = (h(x_1))^2 + h(x_2)$. Note that the F on the left hand side is an element of $\Gamma(Y)$, and that the F on the right hand side is an actual polynomial.

³¹Perhaps here it will be better to write $k[x_1, \dots, x_n]$ in place of $\Gamma(Y)$.

Corollary 1.4.7. (Quasi affine varieties are isomorphic if and only if their coordinate ring are isomorphic as k -algebra) Let X, Y be two quasi affine variety. We have

$$X, Y \text{ isomorphic as quasi variety} \iff \Gamma(Y), \Gamma(X) \text{ isomorphic as } k\text{-algebra}.$$

Proof. If $\varphi : X \rightarrow Y$ is a quasi variety isomorphism, then $\tilde{\varphi} : \Gamma(Y) \rightarrow \Gamma(X)$ is a k -algebra isomorphism with inverse $\widetilde{\varphi^{-1}} : \Gamma(X) \rightarrow \Gamma(Y)$. If $\phi : \Gamma(Y) \rightarrow \Gamma(X)$ is a k -algebra isomorphism, then the unique morphism $\varphi : X \rightarrow Y$ that satisfies $\tilde{\varphi} = \phi$ is an isomorphism with the obvious inverse. ■

Chapter 2

Appendix

2.1 Hilbert's Nullstellensatz and basis theorem

Theorem 2.1.1. (Hilbert's Basis Theorem) If A is Noetherian, then the polynomial ring $A[x]$ is also Noetherian.

Proof. Let X be an ideal in $A[x]$. We are required to show that X is finitely generated. Let I be the ideal in A that contains exactly the leading coefficients of elements of X . Because A is Noetherian, we may let $I = \langle a_1, \dots, a_n \rangle$ and let $f_1, \dots, f_n \in X$ have leading coefficients a_1, \dots, a_n . Let $X' \triangleq \langle f_1, \dots, f_n \rangle \subseteq X$ and let $r \triangleq \max\{\deg(f_1), \dots, \deg(f_n)\}$.

We first show

$$X = \left(X \cap \langle 1, x, \dots, x^{r-1} \rangle \right) + X' \quad (2.1)$$

Let $f \in X$ with $\deg(f) = m$ and leading coefficients a . We wish to show $f \in (X \cap \langle 1, x, \dots, x^{r-1} \rangle) + X'$. Because $a \in I$, we may find some $u_i \in A$ such that $a = \sum u_i a_i$. Clearly, these u_i satisfy

$$f - \sum u_i f_i x^{m-\deg(f_i)} \in X, \quad \text{and} \quad \sum u_i f_i x^{m-\deg(f_i)} \in X'$$

and satisfy

$$\deg \left(f - \sum u_i f_i x^{m-\deg(f_i)} \right) < m$$

Proceeding this way, we end up with $f - g = h$ where $g \in X'$ and $h \in X \cap \langle 1, x, \dots, x^{r-1} \rangle$. We have proved [Equation 2.1](#). Now, because X' is finitely generated, to show X is finitely generated, it only remains to show the ideal $X \cap \langle 1, x, \dots, x^{r-1} \rangle$ is finitely generated, which follows immediately from noting $\langle 1, x, \dots, x^{r-1} \rangle$ as a module is Noetherian. ■

Theorem 2.1.2. (Weak form of Nullstellensatz) Given field k and finitely generated k -algebra B , if B is a field then it is a finite algebraic extension of k .

Proof. A proof can be found in the end of Chapter 5 of Atiyah-MacDonald. Another proof can be found in Chapter 7 of Atiyah-MacDonald, at page 82. ■

Theorem 2.1.3. (Hilbert's Nullstellensatz) Given algebraically closed field k and ideal $I \subseteq k[x_1, \dots, x_n]$. If we let V be the locus of I :

$$V \triangleq \{x \in k^n : F(x) = 0 \text{ for all } F \in I\}$$

and let J be the defining ideal of V :

$$J \triangleq \{F \in k[x_1, \dots, x_n] : F(x) = 0 \text{ for all } x \in V\}$$

then $J = \sqrt{I}$.

Proof. $\sqrt{I} \subseteq J$ is clear. Assume for a contradiction that $F \in J - \sqrt{I}$. Because $F \notin \sqrt{I}$, there exists some prime $\mathfrak{p} \subseteq k[x_1, \dots, x_n]$ that contains \sqrt{I} but does not contain F . Denote

$$B \triangleq k[x_1, \dots, x_n]_{/\mathfrak{p}} \text{ and } g \triangleq [F] \in B \text{ and } C \triangleq B_g$$

Let \mathfrak{m} be some maximal ideal of C . Because of the k -algebra homomorphism diagram:

$$k[x_1, \dots, x_n] \xrightarrow{\text{ring quotient}} B \xrightarrow{\text{localization}} C \xrightarrow{\text{ring quotient}} C_{/\mathfrak{m}} \quad (2.2)$$

We see that by **Hilbert Basis Theorem**, theorem ??, and equivalent definition of Noetherian, $C_{/\mathfrak{m}}$ is finitely generated over k , thus a finite algebraic extension of k by **weak form of Nullstellensatz**. Because k is algebraically closed, this implies $C_{/\mathfrak{m}} \cong k$.

Now, for each $1 \leq i \leq n$, let $t_i \in k \cong C_{/\mathfrak{m}}$ be the image of $x_i \in k[x_1, \dots, x_n]$ under the k -module homomorphism in **diagram 2.2**. Letting $t \triangleq (t_1, \dots, t_n) \in k^n$, it is easy to check¹ by direct computation that **diagram 2.2** have action $G \in k[x_1, \dots, x_n] \mapsto G(t) \in k$. Because $I \subseteq \mathfrak{p}$, by construction of B we see **diagram 2.2** maps every element of I to $0 \in k$. Yet, at the same time the image of F in C is a unit by construction of C , which implies the image of F in the quotient ring $C_{/\mathfrak{m}}$ is nonzero. We have shown $t \in V$ and $F(t) \neq 0$, a contradiction. ■

¹Recall $k[x_1, \dots, x_n] = \langle x_1, \dots, x_n \rangle$.