Infinite Dimensional Vector Space

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January 30, 2023

1 finitely generated but how

Before we prove at the end of this section, that every vector space, finitely generated or not, have a basis, we first give the definition of finitely generated and show that it is well defined.

Definition 1. *a vector space V is finitely generated if and only if*
$\exists S^{\not\infty} : \{u_i | i \in I\} \subseteq V$ *such* $\forall v \in V, v = \sum_{i \in I} \alpha_i u_i$

We now try to prove this is well defined.

Theorem 1.1. *definition of finitely generated space is well defined.*

Proof. Let there be a vector space V be both finitely generated and infinitely generated. We know there exists a finite subset generate V since V is finitely generated, but this CaC to the fact that there exists no finite subset generate V. So, a vector space V can not be both finitely generated and infinitely generated. ■

2 it actually does not generate the whole space

Let's consider a vector space V, where V contains all functions from $\mathbb{N}$ to $\mathbb{Q}$ and V is over $\mathbb{Q}$. V indeed forms a vector space since it is closed under addition and scaler multiplication, where scaler identity is just 1, and vector identity is f, where $f(x) = 0, \forall x \in \mathbb{N}$.*(1)

The vectors in V, although written in the form of functions, is actually equivalent to all rational sequences of index set $\mathbb{N}$. To prove this, we can form a vector space V_1 contains all rational sequences of index set $\mathbb{N}$, and set the isomorphic function T, where T is from V to V_1 and $(T(f))_i = f(i)$. The fact that T is an isomorphism can be easily checked.*(2)

One may realize that V_1 have some extraordinary properties that two or three dimensional Euclidean space don't have. Let's say, there is finite subset $S^{\not\infty}$ of V_1 generate the whole space. We can prove this cause a contradiction and show V_1 is infinitely generated.

Theorem 2.1. *V_1 is infinitely generated.*

Proof. Let $|S| = n$. We show there exists a vector e_i, where $i \in \{x \in \mathbb{N} | 1 \le x \le n+1\}$, and e_i is the sequence such $(e_i)_i = 1$ and $(e_i)_j = 0, \forall j \ne i$, such e_i can not be generated by S. This completes the proof since if V is generated by S, there should exists no vector that can not be generated by S.

First consider if e_1 is in Span(S). If not, our proof is done. If true, we express e_1 in the form $\sum_{i=1}^{n} \alpha_i u_i$, where $u_i \in S, \forall i \in \{x \in \mathbb{N} | 1 \le x \le n\}$. There exists some $i \in \{x \in \mathbb{N} | 1 \le x \le n\}$ such $\alpha_i \ne 0$, otherwise $\sum_{i=1}^{n} \alpha_i u_i = 0 \ne e_1$.

Let S_1 be $(S - \{u_i\}) \cup \{e_1\}$. We can prove Span($S_1$)=Span(S). Span($S_1$) $\subseteq$ Span(S), since $\forall v \in S_1, v \in$ Span(S). Span(S) $\subseteq$ Span(S_1), since $\forall u_j \in S, u_j \in S_1$ where $j \ne i$, and $u_i = \frac{1}{\alpha_i}(e_1 - \sum_{k=1, k \ne i}^{n} \alpha_k u_k)$ shows $u_i \in$ Span(S_1).

Then we consider if e_2 is in Span(S_1). If not, $e_2 \notin$ Span(S_1)=Span(S), our proof is done. If true, we do the exact same thing as above and have a S_2.

Say, we have "true" for consecutive n times in our algorithm. Our proof is still not done. Yet, we have $S_n : \{e_i | 1 \leq i \leq n\}$, and Span(S)=Span($S_n$). $e_{n+1} \notin$ Span(S_n), our proof is done. ∎

Following the proof above, even though V_1 is not finitely generated, one may naively argue that the infinite set $S : \{e_i | i \in \mathbb{N}\}$ generate V_1. However, this is false. We see the sequence c, where $c_i = 1, \forall i \in \mathbb{N}$ can not be generated by S, although c is probably the most imaginable vector to be "generated".

Theorem 2.2. *Let $S = \{e_i | i \in \mathbb{N}\}$, c be the sequence such $c_k = 1, \forall k \in \mathbb{N}$. $c \notin$ Span(S).*

Proof. $\forall e_i, \lim_{k\to\infty} (e_i)_k = 0$. If $c = \sum_{i\in I} \alpha_i e_i$, where I is a finite subset of $\mathbb{N}$. $\lim_{k\to\infty} c_k = \lim_{k\to\infty} (\sum_{i\in I} \alpha_i e_i)_k = \sum_{i\in I} \alpha_i \lim_{k\to\infty} (e_i)_k = \sum_{i\in I} \alpha_i 0 = 0$. Yet, $\lim_{k\to\infty} c_k = 1$, CaC. ∎

A good naive question may be raised as follows, "If we add c into S, would that generate the whole space?". The answer is a concrete "No". We see the sequence d, where $d_k = k, \forall k \in \mathbb{N}$, can not be generated by $S \cup \{c\}$.

Theorem 2.3. *Let d be the sequence such $d_k = k, \forall k \in \mathbb{N}$. $d \notin$ Span($S \cup \{c\}$).*

Proof. All sequences in $S \cup \{c\}$ is convergent. If $d = \sum_{i\in I} \alpha_i u_i$, where I is the finite subset of the index set of $S \cup \{c\}$, d will also be convergent. Yet d is divergent, CaC. ∎

Now, we seem to have a problem in finding basis for V_1, but what we sure can tell is that if V_1 do have a basis β, β would be infinite. To find explicit expression of β is rather difficult. It is only wise to first ask a simpler question: Is V_1 countable or not. If V_1 is countable, obviously $\beta \subseteq V_1$ is also countable.

Theorem 2.4. *V_1 is uncountable.*

Proof. Let there be a sequence x of some sequences in V_1. If x is countable imply there exists a sequence u in V_1 but not in x, V_1 is uncountable. (This is exactly the same as diagonal proof, only in English)

Let $u_i = (x_i)_i + 1$. $u \neq x_i, \forall i \in \mathbb{N}$, our proof is done. ∎

Although it is easy to prove every basis of finitely generated space is of the same size, we never prove all basis of infinitely generated space has the same cardinal number.

Theorem 2.5. *Every basis has the same cardinal number.*

Proof. https://en.wikipedia.org/wiki/SchrderBernstein_theorem
https://en.wikipedia.org/wiki/Cardinal_number
https://math.stackexchange.com/questions/52667/proof-that-two-bases-of-a-vector-space-have-the-same-
∎

However, the fact that every basis has the same cardinal number can not help us prove or disprove β is countable or uncountable.

Theorem 2.6. *Let T_0 be a linearly independent set of a finitely generated space U_0, where α is a basis of U_0. $|T_0| = |\alpha|$ implies that T_0 span U_0.*

Proof. **Mark0:** introduce *replacement algorithm*, where we are given a vector set X and a linearly independent set Y.

step(0): return set RS is now X.
step(1): find a vector, denoted v, in Y, but not in RS, and is in *span*(X). If none, go to step(5).

step(2): express v as a linear combination of vectors in X.
step(3): find a vector, denoted u, in RS but not in Y, that take place in the linear combination of step(2). This is always doable, since if not, then v is a linear combination of vectors in Y, CaC.
step(4): return set RS is now $(X - \{u\}) \cup \{v\}$, go to step 1.
step(5): terminate and return RS.

Let X_1 be the return set of X went through the algorithm with arbitrary Y. $span(X_1) = span(X_0)$, since step 4 is the only step that alter the return set from X, but $span(X) = span((X - \{u\}) \cup \{v\})$. This can be easily checked by checking all vectors of $(X - \{u\}) \cup \{v\}$ is in X and all vectors of X is in $(X - \{u\}) \cup \{v\}$.

Mark1: $|\alpha|$ must be finite.

We prove if $|\alpha|$ is infinite, it CaC to that U_0 is finitely generated.
Let a finite subset X generate U_0. We input X and α to *replacement algorithm*, and see the return set have the property $span(X_1) = span(X)$ and $|X_1| < |\alpha|$. Then we can arbitrarily pick a vector that is in α but $\notin span(X_1)$, and our proof is done.

Mark2: Let $|T_0| = |\alpha| = n$ where n is just a natural number. We show $span(T_0) = span(\alpha) = U_0$.

Let $T_{al} = RS$ be the return set of $\alpha = X$ and $T_0 = Y$. We show $T_{al} = T_0$, so $U_0 = span(\alpha) = span(T_{al}) = span(T_0)$.

Since α generate U_0, all vectors in $Y = T_0$ is in $span(X = \alpha)$, so the algorithm only terminate when no vector in $Y = T_0$ is not in $RS = T_{al}$. Thus $T_0 \subseteq T_{al}$, where $|T_0| = |T_{al}|$. This show $T_{al} = T_0$. Our proof is done.

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Theorem 2.7. *Let T_1 be a linearly independent set of a infinitely generated space U_1, where α is a basis of U_1. $|T_1| = |\alpha|$ does **not** imply that T_1 span U_1.*

Proof. **Mark0:** Polynomial space $\mathbb{R}[x]$ over $\mathbb{R}$ is an infinitely generated space.

Let X be a finite set that generate $\mathbb{R}[x]$, we see x^n where $n > deg(p), \forall p \in X$, is not in $span(X)$.

Mark1: $\{x^n | n \in \mathbb{N}\} = \alpha$ is a basis of $\mathbb{R}[x] = U_1$, and $|\{x^{2n} | n \in \mathbb{N}\}| = |\{x^n | n \in \mathbb{N}\}|$. Yet, $x \notin span(\{x^{2n} | n \in \mathbb{N}\})$.

■

The immediate consequence of **Theorem 2.7.** is that we can not possibly argue that β uncountable simply by raising S as an example of a linearly independent set that doesn't generate V_1, since there can happen the situation where two linearly independent set of the same cardinality is respectively a basis and not a basis. To now, we are not certain whether β is countable or not, and to know the answer, we need clues more than that S is countable and doesn't generate V_1.

3 yet every vector space has a basis

Before we finish our last step, we first claim that our result is based on the **axiom of choice**, which is used in the argument of Maximal principal.

Lemma 3.1. *Let X be a partially ordered set. If all chains in X have some upper bound, X has a maximal element.*

Proof. This is Zorn's Lemma, whose proof is omitted here. ■

Theorem 3.2. *For all vector space V, there exists a linearly independent set $T \subseteq V$, such no linearly independent set other than T contain T.*

Proof. Let X be the set of all linearly independent set in V, where X is partially ordered by inclusion.

For all chains C of (sets in X), we first prove $\bigcup C \in X$, that is, $\bigcup C$ is linearly independent.

Let $\bigcup C$ be linearly dependent, so $\exists v_0 \in \bigcup C, v_0 = \sum_{i \in I} \alpha_i v_i$, where $I : \{1, ..., n\}$.

Let $S_k = \{v_i | 0 \leq i \leq k\}$. We prove by induction that there exists a set N in C, such $S_n \subseteq N$, so N is linearly dependent, CaC to $N \in C \subseteq X$.

Base step: $\exists N_0$, such $S_0 \subseteq N_0$.

Assume there is no set N_0 in C, such $S_0 \subset N_0$. Then, there is no set $N_0 \in C$, such $v_0 \in N_0$, since $v_0 \in N_0 \implies S_0 \subseteq N_0$, and $S_0 \subseteq N_0 \implies v_0 \in N_0$. Then, $\forall N \in C, v_0 \notin N$, CaC to $v_0 \in \bigcup C$.

Induction step: $(\exists N_k \in C$, such $S_k \subseteq N_k) \rightarrow (\exists N_{k+1} \in C, S_{k+1} \subseteq N_{k+1})$.

If $v_{k+1} \in N_k$, $S_{k+1} \subseteq N_{k+1}$, our proof is done.

If not, assume $\forall N \in C, S_{k+1} \nsubseteq N$. Since C is a chain of sets ordered by inclusion, $\forall N \in C$, either $N \subseteq N_k$, or $N_k \subseteq N$.

If $\exists N, v_{k+1} \in N$, $N_k \subseteq N$, since $N \nsubseteq N_k$. In this case, our proof is done, since $S_{k+1} \subseteq N$, clearly.

If not, then $\forall N \in C, N \subseteq N_k$, then $v_{k+1} \notin N \in C, \forall N \in C$, then $v_{k+1} \notin \bigcup C$, CaC. So it can only be the previous case, our proof is done.

We notice $\forall N \in C$, $N \subseteq \bigcup C$, so $\bigcup C$ is a upper bound in X for all C.

Then by Zorn's Lemma, we will have a maximal linearly independent set T in X, such no other sets in X contain T.

∎

Theorem 3.3. *every vector space have a basis.*

Proof. We prove the maximal linearly independent set T from the previous theorem is a basis.

Assume T is not a basis, that is, $span(T) \neq V$. Then there exists some vectors v that is not in $span(T)$, pick such a vector v and we see $T \subset T \cup \{v\}$, and $T \cup \{v\}$ is linearly independent, CaC T is maximal linearly independent set. ∎

4 further reading

https://mathoverflow.net/questions/46063/explicit-hamel-basis-of-real-numbers