5.4 HW6

Question 62

4. Use the stereographic projection (cf. Exercise 16, Sec. 2-2) to show that the sphere is locally conformal to a plane.

Proof. We are given $f : \mathbb{R}^2 \to S^2$

$$f(x,y) = \frac{(2x, 2y, x^2+y^2-1)}{x^2+y^2+1}$$

We claim

$$df_{(x,y)}u \cdot df_{(x,y)}v = \frac{4(u \cdot v)}{(x^2+y^2+1)^2} \text{ for all } u, v \in \mathbb{R}^2$$

We reduce the problem into proving

$$\begin{cases} df_{(x,y)}e_1 \cdot df_{(x,y)}e_1 = \frac{4}{(x^2+y^2+1)^2} \\ df_{(x,y)}e_1 \cdot df_{(x,y)}e_2 = 0 \\ df_{(x,y)}e_2 \cdot df_{(x,y)}e_2 = \frac{4}{(x^2+y^2+1)^2} \end{cases}$$

Compute

$$df_{(x,y)} = \frac{1}{(x^2+y^2+1)^2} \begin{bmatrix} 2(y^2-x^2+1) & -4xy \\ -4xy & 2(x^2-y^2+1) \\ -4x & -4y \end{bmatrix}$$

The rest then follows from direct computation. (done)

Note that we actually miss the north pole. To prove that S^2 is locally conformal to $\mathbb{R}^2$ at the north pole, we just have to do the same computation with

$$g(x,y) = \frac{(2x, 2y, 1-x^2-y^2)}{x^2+y^2+1}$$

■

Question 63

5. Let $\alpha_1 : I \to R^3$, $\alpha_2 : I \to R^3$ be regular parametrized curves, where the parameter is the arc length. Assume that the curvatures k_1 of α_1 and k_2 of α_2 satisfy $k_1(s) = k_2(s) \neq 0$, $s \in I$. Let

$$\mathbf{x}_1(s, v) = \alpha_1(s) + v\alpha_1'(s),$$
$$\mathbf{x}_2(s, v) = \alpha_2(s) + v\alpha_2'(s)$$

be their (regular) tangent surfaces (cf. Example 5, Sec. 2-3) and let V be a neighborhood of (s_0, v_0) such that $\mathbf{x}_1(V) \subset R^3$, $\mathbf{x}_2(V) \subset R^3$ are regular surfaces (cf. Prop. 2, Sec. 2-3). Prove that $\mathbf{x}_1 \circ \mathbf{x}_2^{-1} : \mathbf{x}_2(V) \to \mathbf{x}_1(V)$ is an isometry.

Proof. Compute

$$\begin{aligned}
(\mathbf{x}_1)_s &= \alpha_1' + v\alpha_1'' \\
(\mathbf{x}_1)_v &= \alpha_1' \\
(\mathbf{x}_2)_s &= \alpha_2' + v\alpha_2'' \\
(\mathbf{x}_2)_v &= \alpha_2'
\end{aligned}$$

Using the fact $\alpha' = T$ and $\alpha'' = \kappa N$ compute

$$E_1 = 1 + v^2\kappa^2 \text{ and } F_1 = 1 \text{ and } G_1 = 1$$
$$E_2 = 1 + v^2\kappa^2 \text{ and } F_2 = 1 \text{ and } G_2 = 1$$

This implies that $\mathbf{x}_1 \circ \mathbf{x}_2^{-1} : \mathbf{x}_2(V) \to \mathbf{x}_1(V)$ is an isometry. ∎

Question 64

***6.** Let $\alpha: I \to R^3$ be a regular parametrized curve with $k(t) \neq 0, t \in I$. Let $\mathbf{x}(t, v)$ be its tangent surface. Prove that, for each $(t_0, v_0) \in I \times (R - \{0\})$, there exists a neighborhood V of (t_0, v_0) such that $\mathbf{x}(V)$ is isometric to an open set of the plane (*thus, tangent surfaces are locally isometric to planes*).

Proof. Parametrize α by arc-length s in an open neighborhood around t_0. We can now reparametrize $\mathbf{x}(V)$ by

$$\mathbf{x}(s, v) = \alpha(s) + v\alpha'(s)$$

By Fundamental Theorem of Local Curves, we know there exists a smooth arc-length parametrized curve $\beta : I \to \mathbb{R}^3$ such that β and α has the same curvature everywhere and has zero torsion everywhere. Because β has zero torsion everywhere, we know $B'_\beta = 0$ everywhere. In other words, B_β is constant on I. This implies the range of T_β is contained by a 2-dimensional subspace of $\mathbb{R}^3$. Let $p = \beta(t_0)$, and let such subspace be W. We see that

$$\beta(s) = \int_{t_0}^{s} T_\beta(s')ds' \in p + W$$

This implies that β is contained by the plane $p + W$.

Define $\mathbf{x}_2 : V \to \mathbb{R}^3$ by

$$\mathbf{x}_2(s, v) = \beta(s) + v\beta'(s)$$

Because $\beta'(s) \in W$, we see $\mathbf{x}_2(V)$ lies in $p + W$. It now follows from the last question that $\mathbf{x}(V)$ is isometric $\mathbf{x}_2(V)$. ∎

Question 65

***8.** Let $G\colon R^3 \to R^3$ be a map such that

$$|G(p) - G(q)| = |p - q| \quad \text{for all } p, q \in R^3$$

(that is, G is a *distance-preserving* map). Prove that there exists $p_0 \in R^3$ and a linear isometry (cf. Exercise 7) F of the vector space R^3 such that

$$G(p) = F(p) + p_0 \quad \text{for all } p \in R^3.$$

Proof. We claim one of such p_0 is

$$p_0 \triangleq G(0)$$

We reduce the problem into proving

$$F(p) \triangleq G(p) - G(0) \text{ is a linear isometry of } \mathbb{R}^3$$

Note that

$$F(0) = 0 \text{ and } |F(p)| = |G(p) - G(0)| = |p|$$

This by Mazur-Ulam Theorem implies F is affine. ∎

Question 66

10. Let S be a surface of revolution. Prove that the rotations about its axis are isometries of S.

Proof. We are given charts $\mathbf{x}, \overline{\mathbf{x}} : I \times (0, 2\pi) \to S$

$$\mathbf{x}(t,\theta) = \Big(\cos\theta f(t), \sin\theta f(t), g(t)\Big)$$
$$\overline{\mathbf{x}}(t,\theta) = \Big(\cos(\theta+\theta_0) f, \sin(\theta+\theta_0) f, g\Big)$$

It is clear that if we rotate S about z-axis counterclockwise of θ_0 degree, then the restriction of the action ϕ is exactly $\overline{\mathbf{x}} \circ \mathbf{x}^{-1}$. The problem is now reduced to proving

$$E = \overline{E} \text{ and } F = \overline{F} \text{ and } G = \overline{G}$$

Compute

$$\mathbf{x}_t = \Big(\cos\theta f', \sin\theta f', g'\Big)$$
$$\mathbf{x}_\theta = \Big(-\sin\theta f, \cos\theta f, 0\Big)$$
$$\overline{\mathbf{x}}_t = \Big(\cos(\theta+\theta_0) f', \sin(\theta+\theta_0) f', g'\Big)$$
$$\overline{\mathbf{x}}_\theta = \Big(-\sin(\theta+\theta_0) f, \cos(\theta+\theta_0) f, 0\Big)$$

This give us

$$E = (f')^2 + (g')^2 \text{ and } F = 0 \text{ and } G = f^2$$
$$\overline{E} = (f')^2 + (g')^2 \text{ and } \overline{F} = 0 \text{ and } \overline{G} = f^2$$

■

Question 67

14. We say that a differentiable map $\varphi\colon S_1 \to S_2$ *preserves angles* when for every $p \in S_1$ and every pair $v_1, v_2 \in T_p(S_1)$ we have

$$\cos(v_1, v_2) = \cos(d\varphi_p(v_1), d\varphi_p(v_2)).$$

Prove that φ is locally conformal if and only if it preserves angles.

Proof. $(\longleftarrow)$

Fix $p \in S$. Let e_1, e_2 be an orthonormal basis of T_pS. By premise,

$$\langle d\phi_p e_1, d\phi_p e_2\rangle = 0$$

Express

$$\langle d\phi_p e_1, d\phi_p e_1\rangle = \lambda_1$$
$$\langle d\phi_p e_2, d\phi_p e_2\rangle = \lambda_2$$

Let

$$v_1 = e_1 \text{ and } v_2 = \cos\theta e_1 + \sin\theta e_2$$

Then by premise

$$\cos\theta = \frac{\lambda_1 \cos\theta}{\sqrt{\lambda_1}\sqrt{\cos^2\theta\lambda_1 + \sin^2\theta\lambda_2}}$$

Note that the denominator will never be zero because $d\phi_p$ is full rank by premise.

We can now deduce

$$\lambda_1 = \cos^2\theta\lambda_1 + \sin^2\theta\lambda_2$$

This implies

$$(1 - \cos^2\theta)\lambda_1 = \sin^2\theta\lambda_2$$

which implies $\lambda_1 = \lambda_2$. We now see that for all $c_1, c_2, d_1, d_2 \in \mathbb{R}$, we have

$$\begin{aligned}\langle c_1e_1 + c_2e_2, d_1e_1 + d_2e_2\rangle &= c_1d_1 + c_2d_2\\ &= \frac{1}{\lambda_1}\langle d\phi_p(c_1e_1 + c_2e_2), d\phi_p(d_1e_1 + d_2e_2)\rangle\end{aligned}$$

$(\longrightarrow)$

Fix $p \in S$. Let e_1, e_2 be an orthonormal basis of T_pS. By premise, we see

$$\begin{aligned}\langle d\phi_p e_1, d\phi_p e_2 \rangle = \lambda^2 \langle e_1, e_2 \rangle = 0\\ \langle d\phi_p e_1, d\phi_p e_1 \rangle = \lambda^2 \langle e_1, e_1 \rangle = \lambda^2\\ \langle d\phi_p e_2, d\phi_p e_2 \rangle = \lambda^2 \langle e_2, e_2 \rangle = \lambda^2\end{aligned}$$

Express $v_1, v_2 \in T_pS$

$$v_1 \triangleq c_1 e_1 + c_2 e_2 \text{ and } v_2 \triangleq d_1 e_1 + d_2 e_2$$

Using the equations set we have, we see

$$\begin{aligned}|d\phi_p v_1| &= |c_1 d\phi_p e_1 + c_2 d\phi_p e_2|\\ &= \sqrt{|c_1 d\phi_p e_1|^2 + |c_2 d\phi_p e_2|^2} = \lambda \sqrt{c_1^2 + c_2^2} = \lambda |v_1|\end{aligned}$$

Similarly,

$$|d\phi_p v_2| = \lambda |v_2|$$

Observe

$$\begin{aligned}\cos(v_1, v_2) &= \frac{\langle v_1, v_2 \rangle}{|v_1| |v_2|}\\ &= \frac{\langle d\phi_p v_1, d\phi_p v_2 \rangle}{\lambda^2 |v_1| |v_2|}\\ &= \frac{\langle d\phi_p v_1, d\phi_p v_2 \rangle}{|d\phi_p v_1| |d\phi_p v_2|} = \cos(d\phi_p v_1, d\phi_p v_2)\end{aligned}$$

$\blacksquare$

Question 68

15. Let $\varphi: R^2 \to R^2$ be given by $\varphi(x, y) = (u(x, y), v(x, y))$, where u and v are differentiable functions that satisfy the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x.$$

Show that φ is a local conformal map from $R^2 - Q$ into R^2, where $Q = \{(x, y) \in R^2; u_x^2 + u_y^2 = 0\}$.

Proof. Because $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ satisfy Cauchy-Riemann equations, we can express

$$[d\phi_p] = \begin{bmatrix} \lambda_1 & \lambda_2 \\ -\lambda_2 & \lambda_1 \end{bmatrix} \text{ where } u_x = \lambda_1 \text{ and } u_y = \lambda_2$$

Because $\lambda_1^2 + \lambda_2^2 = u_x^2 + u_y^2$ is differentiable and non-zero on $\mathbb{R}^2 \setminus Q$, we can reduce the problem into proving

$$\langle d\phi_p v_1, d\phi_p v_2 \rangle = (\lambda_1^2 + \lambda_2^2)\langle v_1, v_2 \rangle \text{ for all } v_1, v_2 \in \mathbb{R}^2$$

Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$. We reduce the problem into proving

$$\begin{cases} \langle d\phi_p e_1, d\phi_p e_1 \rangle = (\lambda_1^2 + \lambda_2^2)\langle e_1, e_1 \rangle = \lambda_1^2 + \lambda_2^2 \\ \langle d\phi_p e_1, d\phi_p e_2 \rangle = (\lambda_1^2 + \lambda_2^2)\langle e_1, e_2 \rangle = 0 \\ \langle d\phi_p e_2, d\phi_p e_2 \rangle = (\lambda_1^2 + \lambda_2^2)\langle e_2, e_2 \rangle = \lambda_1^2 + \lambda_2^2 \end{cases}$$

Compute

$$\langle d\phi_p e_1, d\phi_p e_1 \rangle = |(\lambda_1, -\lambda_2)|^2 = \lambda_1^2 + \lambda_2^2$$

Compute

$$\langle d\phi_p e_2, d\phi_p e_2 \rangle = |(\lambda_2, \lambda_1)|^2 = \lambda_1^2 + \lambda_2^2$$

Compute

$$\langle d\phi_p e_1, d\phi_p e_2 \rangle = \lambda_1 \lambda_2 - \lambda_2 \lambda_1 = 0 \text{ (done)}$$

∎

Question 69

2. Show that if $\mathbf{x}$ is an isothermal parametrization, that is, $E = G = \lambda(u, v)$ and $F = 0$, then

$$K = -\frac{1}{2\lambda}\Delta(\log\lambda),$$

where $\Delta\varphi$ denotes the Laplacian $(\partial^2\varphi/\partial u^2) + (\partial^2\varphi/\partial v^2)$ of the function φ. Conclude that when $E = G = (u^2 + v^2 + c)^{-2}$ and $F = 0$, then $K = \text{const.} = 4c$.

Proof. We first show

$$\text{If } F = 0 \text{ then } K = \frac{-1}{2\sqrt{EG}}\Big[\Big(\frac{E_v}{\sqrt{EG}}\Big)_v + \Big(\frac{G_u}{\sqrt{EG}}\Big)_u\Big]$$

Because $F = 0$, we can compute

$$\Gamma^1_{11} = \frac{E_u}{2E}$$
$$\Gamma^2_{11} = \frac{E_v}{-2G}$$
$$\Gamma^1_{12} = \frac{E_v}{2E}$$
$$\Gamma^2_{12} = \frac{G_u}{2G}$$
$$\Gamma^1_{22} = \frac{G_u}{-2E}$$
$$\Gamma^2_{22} = \frac{G_v}{2G}$$

We have the Gauss Formula

$$K = \frac{(\Gamma^2_{12})_u - (\Gamma^2_{11})_v + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{21} - \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{11}\Gamma^2_{12}}{-E}$$

The rest then follows from expressing both K and $\frac{-1}{2\sqrt{EG}}\Big[\Big(\frac{E_v}{\sqrt{EG}}\Big)_v + \Big(\frac{G_u}{\sqrt{EG}}\Big)_u\Big]$ in the form of $\frac{f}{g}, \frac{h}{k}$ where f, g, h, k are some product of the 0-th, first and second derivatives of E, F, G, and check $fk = gh$. (done)

Now, note that

$$(\ln\lambda)_u = \frac{\lambda_u}{\lambda} \text{ and } (\ln\lambda)_v = \frac{\lambda_v}{\lambda}$$

Using our formula, we see

$$\begin{aligned}
K &= \frac{-1}{2\sqrt{EG}}\Big[\Big(\frac{E_v}{\sqrt{EG}}\Big)_v + \Big(\frac{G_u}{\sqrt{EG}}\Big)_u\Big] \\
&= \frac{1}{-2\lambda}\Big[\Big(\frac{\lambda_u}{\lambda}\Big)_u + \Big(\frac{\lambda_v}{\lambda}\Big)_v\Big] \\
&= \frac{1}{-2\lambda}\Big((\ln\lambda)_{uu} + (\ln\lambda)_{vv}\Big) = \frac{1}{-2\lambda}\Delta(\ln\lambda)
\end{aligned}$$

If $\lambda = (u^2+v^2-c)^2$, we have

$$K = 4c$$

■

Question 70

3. Verify that the surfaces

$$\begin{aligned}
\mathbf{x}(u, v) &= (u\cos v, u\sin v, \log u), \qquad u > 0 \\
\bar{\mathbf{x}}(u, v) &= (u\cos v, u\sin v, v),
\end{aligned}$$

have equal Gaussian curvature at the points $\mathbf{x}(u, v)$ and $\bar{\mathbf{x}}(u, v)$ but that the mapping $\bar{\mathbf{x}} \circ \mathbf{x}^{-1}$ is not an isometry. This shows that the "converse" of the Gauss theorem is not true.

Proof. Compute

$$\begin{aligned}
\mathbf{x}_u &= (\cos v, \sin v, \frac{1}{u}) \text{ and } \mathbf{x}_v = (-u\sin v, u\cos v, 0) \\
\overline{\mathbf{x}}_u &= (\cos v, \sin v, 0) \text{ and } \overline{\mathbf{x}}_v = (-u\sin v, u\cos v, 1)
\end{aligned}$$

This give us

$$\begin{aligned}
E &= 1 + \frac{1}{u^2} \text{ and } F = 0 \text{ and } G = u^2 \\
\overline{E} &= 1 \text{ and } \overline{F} = 0 \text{ and } \overline{G} = 1 + u^2
\end{aligned}$$

The fact $E \neq \overline{E}$ implies that $\overline{\mathbf{x}} \circ \mathbf{x}^{-1}$ is not an isometry, since if $\phi \triangleq \overline{\mathbf{x}} \circ \mathbf{x}^{-1}$ is an isometry, then the local chart $\phi \circ \mathbf{x}$ would have have the same coefficients as $\mathbf{x}$, but $\phi \circ \mathbf{x} = \overline{\mathbf{x}}$.

Now compute

$$\mathbf{x}_{uu} = (0,0,-u^{-2}) \text{ and } \mathbf{x}_{uv} = (-\sin v, \cos v, 0) \text{ and } \mathbf{x}_{vv} = (-u\cos v, -u\sin v, 0)$$
$$\overline{\mathbf{x}}_{uu} = (0,0,0) \text{ and } \overline{\mathbf{x}}_{uv} = (-\sin v, \cos v, 0) \text{ and } \overline{\mathbf{x}}_{vv} = (-u\cos v, -u\sin v, 0)$$

This give us

$$g = \overline{g} = 0$$

and because $F = \overline{F} = 0$, we can now deduce

$$K = \overline{K} = 0$$

using $K = \frac{eg}{EG}$. ■

Question 71

5. If the coordinate curves form a Tchebyshef net (cf. Exercises 7 and 8, Sec. 2-5), then $E = G = 1$ and $F = \cos\theta$. Show that in this case

$$K = -\frac{\theta_{uv}}{\sin\theta}.$$

Proof. $E = G = 1$ and $F = \cos\theta$ give us three linear system

$$\begin{cases} \Gamma^1_{11} + (\cos\theta)\Gamma^2_{11} = 0 \\ (\cos\theta)\Gamma^1_{11} + \Gamma^2_{11} = \theta_u(-\sin\theta) \end{cases} \quad \begin{cases} \Gamma^1_{12} + (\cos\theta)\Gamma^2_{12} = 0 \\ (\cos\theta)\Gamma^1_{12} + \Gamma^2_{12} = 0 \end{cases} \quad \begin{cases} \Gamma^1_{22} + (\cos\theta)\Gamma^2_{22} = \theta_v(-\sin\theta) \\ (\cos\theta)\Gamma^1_{22} + \Gamma^2_{22} = 0 \end{cases}$$

Solve them to get

$$\begin{cases} \Gamma^1_{11} = (\cot\theta)\theta_u \\ \Gamma^2_{11} = -(\csc\theta)\theta_u \end{cases} \quad \begin{cases} \Gamma^1_{12} = 0 \\ \Gamma^2_{12} = 0 \end{cases} \quad \begin{cases} \Gamma^1_{22} = (-\csc\theta)\theta_v \\ \Gamma^2_{22} = (\cot\theta)\theta_v \end{cases}$$

Now, we use Gauss formula

$$\begin{aligned} K &= \frac{(\Gamma^2_{12})_u - (\Gamma^2_{11})_v + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{21} - \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{11}\Gamma^2_{12}}{-E} \\ &= \big((-\csc\theta)\theta_u\big)_v + (\cot\theta)(\csc\theta)\theta_u\theta_v \\ &= -(\csc\theta)\theta_{uv} + -(\csc\theta)(\cot\theta)\theta_u\theta_v + (\cot\theta)(\csc\theta)\theta_u\theta_v \\ &= -(\csc\theta)\theta_{uv} = -\frac{\theta_{uv}}{\sin\theta} \end{aligned}$$

■

Question 72

8. Compute the Christoffel symbols for an open set of the plane

a. In Cartesian coordinates.

b. In polar coordinates.

Use the Gauss formula to compute K in both cases.

Proof. **(a)** We are given

$$\mathbf{x}(u,v) = (u,v,0)$$

which implies

$$E = G = 1 \text{ and } F = 0$$

This trivially give us

$$\Gamma^i_{jk} = 0 \text{ for all } i,j,k \quad \text{and } K = 0$$

(b) We are given

$$\mathbf{x}(u,v) = (v\cos u, v\sin u, 0)$$

which implies

$$E = v^2 \text{ and } F = 0 \text{ and } G = 1$$

Because $F = 0$, the solving of the 2-dimensional linear equations is trivial

$$\Gamma^1_{11} = \frac{E_u}{2E} = 0$$
$$\Gamma^2_{11} = \frac{-E_v}{2G} = -v$$
$$\Gamma^1_{12} = \frac{E_v}{2E} = \frac{1}{v}$$
$$\Gamma^2_{12} = \frac{G_u}{2G} = 0$$
$$\Gamma^1_{22} = \frac{G_u}{-2E} = 0$$
$$\Gamma^2_{22} = \frac{G_v}{2G} = 0$$

Then use Gauss Formula

$$K = \frac{(\Gamma_{12}^2)_u - (\Gamma_{11}^2)_v + \Gamma_{12}^1\Gamma_{11}^2 + \Gamma_{12}^2\Gamma_{12}^2 - \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{11}^1\Gamma_{12}^2}{-E}$$
$$= \frac{1-1}{-v^2} = 0$$

■