※ 注意：請於試卷上「非選擇題作答區」內依序作答，並應註明作答之大題及其題號。

Instructions.

- There are two problems in two pages.
- In a problem, if an exercise depends on the conclusions of other exercises that precede it, you may assume these conclusions without solving them.

Problem 1 (80 points). Let m and n be two positive integers. The $\mathbf{C}$-vector space of matrices of size $m \times n$ with coefficients in $\mathbf{C}$ is denoted by $M_{m,n}(\mathbf{C})$. We also set $M_n(\mathbf{C}) = M_{n,n}(\mathbf{C})$.

The aim of this problem is to prove the following statement.

Theorem. *Let m, n and r be positive integers with $r \le m \le n$. Let $V \subset M_{m,n}(\mathbf{C})$ be a $\mathbf{C}$-linear subspace. Assume that every matrix A in V satisfies $\operatorname{rank} A \le r$. Then*

$$\dim V \le nr.$$

(1) Show that it suffices to prove the theorem for $m = n$.

(2) Assume that $m = n$. Show that we can assume that V contains the block matrix

$$R = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

where I_r is the identity matrix of rank r.

From now on, we assume that $m = n$, and that $R \in V$.

(3) Let

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \in V$$

be a block matrix in V with $M_{11} \in M_r(\mathbf{C})$. Show that

$$M_{22} = 0 \quad \text{and} \quad M_{21}M_{12} = 0.$$

(Hint: you may consider the $(r+1) \times (r+1)$ minors of $M + tR$ for $t \in \mathbf{C}$.)

(4) Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{pmatrix} \in V, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & 0 \end{pmatrix} \in V$$

be two block matrices with $A_{11}, B_{11} \in M_r(\mathbf{C})$. Show that

$$A_{21}B_{12} + B_{21}A_{12} = 0.$$

(5) Let $\phi : V \to M_{r,n}(\mathbf{C})$ be the map sending a matrix $M \in V$ to its first r rows. Define the $\mathbf{C}$-linear subspace

$$W = \left\{ \begin{pmatrix} 0 & 0 \\ A_{21} & 0 \end{pmatrix} \in V \;\middle|\; A_{21} \in M_{n-r,r}(\mathbf{C}) \right\} \subset V,$$

and let $s = \dim W$. Show that

$$\dim \phi(V) \leq nr - s,$$

by considering the map

$$\psi : W \to M_{r,n}(\mathbb{C})^{\vee}$$
$$\begin{pmatrix} 0 & 0 \\ A_{21} & 0 \end{pmatrix} \mapsto T_{A_{21}}$$

to the dual of $M_{r,n}(\mathbb{C})$, where $T_{A_{21}}$ is the linear form defined by

$$T_{A_{21}}(B_{11}, B_{12}) = \mathrm{Tr}(A_{21}B_{12})$$

for every block matrix $(B_{11}, B_{12}) \in M_{r,n}(\mathbb{C})$ with $B_{11} \in M_r(\mathbb{C})$.

(6) Conclude that

$$\dim V \leq nr.$$

(7) Show that the inequality in the theorem is optimal. More precisely, for all positive integers m, n and r with $r \leq m \leq n$, construct $V \subset M_{m,n}(\mathbb{C})$ as in the theorem such that

$$\dim V = nr.$$

Problem 2 (20 points). Let V be a nonzero vector space over a field F. Let

$$B : V \times V \to F$$

be a non-degenerate symmetric bilinear form on V, and let

$$q : V \to F$$
$$v \mapsto B(v, v)$$

be the associated quadratic form. For every $x \in F$, we say that q represents x if $q(v) = x$ for some nonzero $v \in V$.

(1) Suppose that q represents 0. Show that q represents every element of F. (Hint: Consider $q(cv + w)$ with $c \in F$ and some suitable $w \in V$.)

(2) Show that B extends to a non-degenerate symmetric bilinear form on $V \oplus F$ whose associated quadratic form represents every element of F.

Notation: $\mathbf{R}$ is the set of real numbers, and $\mathbf{C}$ is the set of complex numbers. If $F = \mathbf{R}$ or $\mathbf{C}$, denote by $M_n(F)$ the $n \times n$ matrices with entries in F. If $A \in M_{m\times n}(F)$, denote by $A^t \in M_{n\times m}(F)$ the transpose of A. Denote by I_n the $n \times n$ identity matrix and 0_n the $n \times n$ zero matrix.

Problem 1 (10pts). Let $\mathrm{i} = \sqrt{-1} \in \mathbf{C}$ be a root of $X^2 + 1$. Let

$$v_1 = (1, 0, -\mathrm{i}), \quad v_2 = (1 + \mathrm{i}, 1 - \mathrm{i}, 1), \quad v_3 = (\mathrm{i}, \mathrm{i}, \mathrm{i}).$$

Show that $\{v_1, v_2, v_3\}$ is a basis of $\mathbf{C}^3$ and express the vector $v_4 = (1, 0, 1)$ as a linear combination of v_1, v_2 and v_3, namely find $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{C}$ such that $v_4 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$.

Problem 2 (15 pts). Let

$$\mathbf{v}_1 = (0, 3, 3, 1),\ \mathbf{v}_2 = (2, 1, -3, 7),\ \mathbf{v}_3 = (1, 8, 6, 6),\ \mathbf{v}_4 = (1, 10, -4, 2)$$

be vectors in $\mathbf{R}^4$. Let $W_1 = \mathrm{span}_{\mathbf{R}}\{\mathbf{v}_1, \mathbf{v}_2\}$ and let $W_2 = \mathrm{span}_{\mathbf{R}}\{\mathbf{v}_3, \mathbf{v}_4\}$. Find the dimension and a basis of $W_1 \cap W_2$.

Problem 3 (25 pts). Let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix} \in M_{2\times 3}(\mathbf{R}).$$

(1) (15pts) Find an orthogonal matrix $P \in M_3(\mathbf{R})$ such that $P^{-1}A^tAP$ is a diagonal matrix.

(2) (10pts) Find the singular value decomposition of A. In other words, factorize $A = U\Sigma V^t$, where $U \in M_3(\mathbf{R})$ and $V \in M_3(\mathbf{R})$ are orthogonal matrices and $\Sigma \in M_{2\times 3}(\mathbf{R})$ is of the form

$$\Sigma = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \end{pmatrix}, \quad \lambda_1 \geq \lambda_2 \geq 0$$

Problem 4 (15pts). Let $V = M_3(\mathbf{C})$ be a 9-dimension vector space over $\mathbf{C}$ and let

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}.$$

Define the linear transformation $T : V \to V$ by

$$T(B) = AB - BA.$$

(1) (5pts) Find the dimension of $\mathrm{Ker}\, T$.

(2) (10pts) Show that T is diagonalizable.

Problem 5 (15pts). Let $A, B \in M_n(\mathbf{R})$. Prove that $\mathrm{rank}\, A + \mathrm{rank}\, B \leq n$ if and only if there exists an invertible matrix $X \in M_n(\mathbf{R})$ such that $AXB = 0_n$.

Problem 6 (20pts). Let A and B be elements in $M_n(\mathbf{C})$. Suppose that

$$AB - BA = c \cdot (A - B)$$

for some non-zero $c \in \mathbf{C}$. Prove that there exists an invertible matrix $P \in M_n(\mathbf{C})$ such that $P^{-1}AP$ and $P^{-1}BP$ are upper-triangular matrices with the same diagonal entries.

Notation: We denote by $\mathbf{C}$ the set of complex numbers. For any positive integer n, we denote by $\mathbf{C}^n$ the n-dimensional column vector spaces over $\mathbf{C}$; let I_n be the identity matrix in $\mathrm{M}_n(\mathbf{C})$.

Problem 1 (15pts). Let $T : \mathbf{C}^4 \to \mathbf{C}^3$ be the linear transformation defined by $T(v) = A \cdot v$, where

$$A = \begin{pmatrix} 5 & -3 & 1 & 2 \\ -1 & 3 & 3 & -2 \\ 1 & 0 & 1 & 0 \end{pmatrix} \in \mathrm{M}_{3\times 4}(\mathbf{C}).$$

(1) (5 pts) Find the rank and the nullity of T.
(2) (10pts) Find a base of $\mathrm{Ker}\, T$ (the kernel of T).

Problem 2 (15pts). For any complex number $a \in \mathbf{C}$, let V_a be the subspace spanned by the row vectors

$$(2, -5, a),\ (1, a, -4),\ (a, -1, -2).$$

Determine all possible values $a \in \mathbf{C}$ such that $\dim_{\mathbf{C}} V_a = 2$.

Problem 3 (25pts). Let

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 2 & -1 & 2 \\ 2 & -2 & 5 \end{pmatrix}.$$

(1) (15pts) Find an invertible matrix $P \in \mathrm{M}_3(\mathbf{C})$ such that $P^{-1}AP$ is a diagonal matrix.
(2) (10pts) Find an invertible matrix $Q \in \mathrm{M}_3(\mathbf{C})$ such that

$$Q^{-1}AQ = \begin{pmatrix} 0 & 0 & -4 \\ 1 & 0 & 1 \\ 0 & 1 & 4 \end{pmatrix}.$$

Problem 4 (15pts). Let $A \in \mathrm{M}_n(\mathbf{C})$ be a Hermitian matrix $\iff A = A^*$.

(1) (5 pts) Show that $\mathrm{Ker}\, A \cap \mathrm{Im}\, A = \{0\}$.
(2) (10pts) If $A^3 = 2A^2 + 2A$, show that $A = 0$.

Problem 5 (15pts). Let $A \in \mathrm{M}_n(\mathbf{C})$ such that $A^n = 0$ but $A^{n-1} \neq 0$.

(1) (7pts) Show that there exists $v \in \mathbf{C}^n$ such that $\{v, Av, A^2v, \ldots, A^{n-1}v\}$ is a basis of $\mathbf{C}^n$.
(2) (8pts) If $B \in \mathrm{M}_n(\mathbf{C})$ such that $AB = BA$, prove that

$$B = a_0 + a_1A + a_2A^2 + \cdots a_{n-1}A^{n-1}$$

for some $a_0, \ldots, a_{n-1} \in \mathbf{C}$.

Problem 6 (15pts). Let $A, B \in \mathrm{M}_n(\mathbf{C})$. Suppose that the eigenvalues of A, B are all non-negative real numbers and that $\mathrm{null}(A) = \mathrm{null}(A^2)$ and $\mathrm{null}(B) = \mathrm{null}(B^2)$. If $A^4 = B^4$, prove that $A = B$.
(Recall that $\mathrm{null}(A)$:=the nullity of A = the dimension of the kernel of A)

試題隨卷繳回

Linear Algebra

1. **(20 points.)** Let $A, B \in M_{n\times n}(F)$ be two $n \times n$ matrices over a field F.
   (a) Prove that $\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$.
   (b) Prove that $\operatorname{rank}(A) + \operatorname{rank}(B) \leq \operatorname{rank}(AB) + n$.
2. **(15 points.)** Let A be an $n \times n$ matrix over $\mathbb{C}$ of the form

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_0 \end{pmatrix}.$$

   Define $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$ and $\omega = e^{2\pi i/n}$. Prove that

$$\det A = \prod_{j=0}^{n-1} f(\omega^j).$$

3. **(15 points.)** Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V over $\mathbb{C}$ and $f(x) \in \mathbb{C}[x]$ be a polynomial. Prove that the linear transformation $f(T)$ is invertible if and only if $f(x)$ and the minimal polynomial T have no common roots.
4. **(15 points.)** Let $v_1, \ldots, v_k$ be eigenvectors corresponding to k distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ of a linear operator T on a vector space V. Prove that the T-cyclic subspace generated by $v = v_1 + \cdots + v_k$ has dimension k.
5. **(15 points.)** Let $T : V \to V$ be a linear operator on a finite-dimensional inner product space V over $\mathbb{R}$ and T^* be its adjoint. Suppose that $T^* = T^3$. Prove that T^2 is diagonalizable over $\mathbb{R}$.
6. **(20 points.)** Let V be a vector space of dimension n over a field F. Determine the dimension over F of the vector space of multilinear alternating functions $f : V \times \cdots \times V \to F$ (k copies of V).

試題隨卷繳回

- Unless otherwise specified, everything is over $\mathbb{R}$.
- The ordinary inner product of $\mathbb{R}^n$ is denoted by $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$.
- $\mathcal{S}_n$ is the space of $n \times n$ square matrices.
- $\mathcal{P}$ is the vector space of polynomials of one variable x with real coefficients.
- Dual space V^* of real vector space V is $\{\alpha \mid \alpha : V \to \mathbb{R}, \alpha \text{ is linear}\}$.

(1) [16%] $V \subset \mathbb{R}^4$ is a subspace span by $\vec{\mathbf{u}} = [1\ -4\ 8\ 3]^{\mathrm{t}}$ and $\vec{\mathbf{v}} = [2\ -2\ 10\ 3]^{\mathrm{t}}$. Define a linear transformation $T : V \to V$ by

$$\begin{aligned} T(\vec{\mathbf{u}}) &= 5\,\vec{\mathbf{u}} + 2\,\vec{\mathbf{v}} \\ T(\vec{\mathbf{v}}) &= 7\,\vec{\mathbf{u}} + \vec{\mathbf{v}} \end{aligned}$$

The induced inner product of V from $\mathbb{R}^4$ is defined by $<\vec{\mathbf{x}}, \vec{\mathbf{y}}> = \vec{\mathbf{x}} \cdot \vec{\mathbf{y}}$, $\vec{\mathbf{x}}, \vec{\mathbf{y}} \in V$. Is T self-adjoint with respect to $<\ ,\ >$? Demonstrate your answer.

(2) [16%] $\mathcal{P}_3 \equiv \{f(x) \in \mathcal{P} \mid \deg(f(x)) \leq 3\}$. Let $\mathcal{P}_3^*$ be the dual space of $\mathcal{P}_3$. For any $a \in \mathbb{R}$, define $\widehat{a} \in \mathcal{P}_3^*$ by $\widehat{a}(f(x)) = f(a)$ and $\mathrm{d}\hat{a} \in \mathcal{P}_3^*$ by $\mathrm{d}\hat{a}(f(x)) = f'(a)$.

a. Find the basis $\phi_{-1}(x)$, $\phi_0(x)$, $\phi_d(x)$, $\phi_1(x)$ of $\mathcal{P}_3$ such that $\widehat{-1}$, $\widehat{0}$, $\mathrm{d}\hat{0}$, $\widehat{1}$ are their corresponding dual basis.

b. Define $I \in \mathcal{P}_3^*$ by $I(f(x)) = \int_{-1}^{1} f(x)dx$. Find α, β, γ, $\epsilon \in \mathbb{R}$ such that
$$I = \alpha\,\widehat{-1} + \beta\,\widehat{0} + \gamma\,\mathrm{d}\hat{0} + \epsilon\,\widehat{1}$$

c. If there is $f(x) \in \mathcal{P}_3$ such that $f(-1) = -2$, $f(0) = 2$, $f'(0) = \pi$, $f(1) = -6$, evaluate $\int_{-1}^{1} f(x)dx$.

(3) [16%] $\Gamma = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \in \mathcal{S}_n$. $\mathcal{C}_n = \{X \mid X\Gamma = \Gamma X\}$ is a subspace of $\mathcal{S}_n$. Determine $\dim \mathcal{C}_n$ and find a basis of $\mathcal{C}_n$.

(4) [16%] $A \in \mathcal{S}_n$. Define m_{ij} to be the determinant of the submatrix formed by deleting the i-th row and j-th column of A. Define the classical adjont matrix $\operatorname{adj} A = [(-1)^{i+j} m_{ji}]$. Suppose A is not invertible, show that rank of $\operatorname{adj} A$ is ≤ 1. When is the rank of $\operatorname{adj} A = 1$?

(5) [16%] If $A = [a_{ij}] \in \mathcal{S}_n$ is positive definite, show that $\det A \leq a_{11}a_{22}\cdots a_{nn}$.

(6) [20%] $A \in \mathcal{S}_n(\mathbb{C})$. Over $\mathbb{C}$, show the following two statements are equivalent.

a. The characteristic polynomial of A is equal to minimal polynomial of A.

b. For any $X \in \mathcal{S}_n(\mathbb{C})$ satisfies $XA = AX$, X is a polynomial of A.

試題隨卷繳回

(1) (20 points) Let V_1 be the $\mathbb{R}$-*linear span* of functions: $\sin^i x \cdot \cos^j x$, $i, j = 0, ..., n$. Let V_2 be the $\mathbb{R}$-*linear span* of functions: $\sin kx$. $\cos kx$, $k = 0, ..., n$. Determine the dimensions of V_1 and V_2 and prove your assertion. Is it true that $V_1 = V_2$? Prove or disprove it.

(2) (15 points) Let $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a *linear transformation* and let id be the identity map sending every $v \in \mathbb{R}^n$ to v. Prove that there exist $C > 0$ such that for all $t \in \mathbb{R}$, $|t| > C$, the map $id + t \cdot \varphi$ is surjective.

(2) (15 points) Let $A := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $B := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$,

$$V = \{v \in \mathbb{C}^4 \mid A \cdot v = \lambda_a \cdot v, B \cdot v = \lambda_b \cdot v, \text{ for some } \lambda_a, \lambda_v \in \mathbb{C}\}.$$

Find a basis of V.

(4) (15 points) Let A be an $n \times n$ *diagonal* matrix with diagonal entries $A_{11}, ..., A_{nn}$. Show that the *linear span* W of A^k, $k = 0, 1, ...$, is of dimension n if and only if $A_{ii} \neq A_{jj}$ for different i and j.

(5) (15 points) Suppose φ and g are $\mathbb{R}$-linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $g \circ \varphi = \varphi^2 \circ g$ and g is *injective*. Show that φ and φ^2 have the same kernel (null-space), image, *eigenvalues* and *eigenspaces*.

(6) Prove or disprove the following statements (10 points for each).
Let $Q : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a quadratic form.

(a) Let $\mathbb{Z}^n \subset \mathbb{R}^n$ denote the subset consisting of vectors with integer coordinates. Then Q is *positive definite* if and only if $Q(v) > 0$ for all $v \in \mathbb{Z}^n$.

(b) There is some $n \times n$ matrix A such that $Q(v) = v^t \cdot A^t \cdot A \cdot v$, for all $v \in \mathbb{R}^n$. Here, B^t denotes the *transpose* of B.

試題隨卷繳回

※ 注意：全部題目均請作答於試卷內之「非選擇題作答區」，請標明題號依序作答。

- Unless otherwise specified, everything is over $\mathbb{R}$.
- The ordinary inner product of $\mathbb{R}^n$ is denoted by $\vec{u} \cdot \vec{v}$.
- $\mathcal{M}_{m\times n}$ is the space of $m \times n$ matrices; $f_M(t) = \det(tI_n - M)$ is the characteristic polynomial of M; $\operatorname{im} A$ is the image of A; $\ker A$ is the kernel of A; $V^\perp$ is the normal space of V. Parallelepiped = 平行六面體.
- Duel space V^* of real vector space V is $\{\alpha \mid \alpha : V \to \mathbb{R}, \alpha \text{ is linear}\}$.

A. [15%] 是非題. 若錯誤, 需說明原因或給出反例. 本題答案須寫在答案簿最前面.

1. There is a linear transformation $A : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\operatorname{im} A = \ker A$.
2. $A \in \mathcal{M}_{n\times n}$. Suppose $A^2 = A$ then $\ker A = (\operatorname{im} A)^\perp$.
3. For any A, B, $C \in \mathcal{M}_{n\times n}$, $\operatorname{tr}(ABC) = \operatorname{tr}(CBA)$.
4. The matrix representation A of an adjoint transformation satisfies $A^t = A$.
5. Symmetric matrix A is positive definite if and only if all its diagonal elements are positive.

B. [85%] 計算/證明題。(6A) 和 (6B) 只選擇一題作答，兩題皆答，以先寫者計算。

(1) [15%] Find all Jordan canonical forms for square matrices in $\mathcal{M}_{n\times n}$, $n \leq 6$, with minimal polynomial $(t-1)^2(t+1)^2$.

(2) [15%] For A, $B \in \mathcal{M}_{m\times n}$, show that $f_{BA^t}(t) = f_{B^tA}(t) \cdot t^{m-n}$.

(3) [15%] Consider $V = \{A \mid AX = XA, \text{for any } X \in \mathcal{M}_{n\times n}\} \subset \mathcal{M}_{n\times n}$. Show that V is an one dimensional subspace of $\mathcal{M}_{n\times n}$.

(4) [15%] $A \in \mathcal{M}_{n\times n}$. Suppose $(t^2+1) | f_A(t)$, are there $\vec{u}$, $\vec{v} \in \mathbb{R}^n$ such that $A\vec{u} = \vec{v}$ and $A\vec{v} = -\vec{u}$? Prove or disprove it.

(5) [15%] U is a subspace of a finite dimensional vector space V. Consider $D_U \subset V^*$ defined by $\{\alpha \in V^* \mid U \text{ is a subspace of } \ker \alpha\}$. Show that D_U is a subspace of dimension $\dim V - \dim U$.

(6A) [10%] Show the volume V of the parallelepiped span by $\vec{u}$, $\vec{v}$, $\vec{w} \in \mathbb{R}^n$ satisfies

$$\begin{aligned} V^2 =& ||\vec{u}||^2\, ||\vec{v}||^2\, ||\vec{w}||^2 \;+\; 2\,(\vec{u}\cdot\vec{v})\,(\vec{v}\cdot\vec{w})\,(\vec{w}\cdot\vec{u}) \\ & - ||\vec{u}||^2\,(\vec{v}\cdot\vec{w})^2 - ||\vec{v}||^2\,(\vec{w}\cdot\vec{u})^2 - ||\vec{w}||^2\,(\vec{u}\cdot\vec{v})^2 \end{aligned}$$

(6B) [10%] Following diagram of vector spaces and linear transformations satisfies

(a) $\ker f_{i+1} = \operatorname{im} f_i$, $\ker g_{i+1} = \operatorname{im} g_i$, $i = 0, 1, 2, 3, 4$.

(b) $\alpha_{i+1} \circ f_i = g_i \circ \alpha_i$, $i = 1, 2, 3, 4$.

$$\begin{tikzcd} \{0_{A_1}\} \arrow[r, "f_0"] & A_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1\ \cong"] & B_1 \arrow[r, "f_2"] \arrow[d, "\alpha_2\ \cong"] & C_1 \arrow[r, "f_3"] \arrow[d, "\alpha_3"] & D_1 \arrow[r, "f_4"] \arrow[d, "\alpha_4\ \cong"] & E_1 \arrow[r, "f_5"] \arrow[d, "\alpha_5\ \cong"] & \{0_{E_1}\} \\ \{0_{A_2}\} \arrow[r, "g_0"] & A_2 \arrow[r, "g_1"] & B_2 \arrow[r, "g_2"] & C_2 \arrow[r, "g_3"] & D_2 \arrow[r, "g_4"] & E_2 \arrow[r, "g_5"] & \{0_{E_1}\} \end{tikzcd}$$

Show that if α_1, α_2, α_4, α_5 are isomorphisms, then α_3 is an isomorphism.

試題隨卷繳回

- Unless otherwise specified, everything is over $\mathbb{R}$.
- The ordinary inner product of $\mathbb{R}^n$ is denoted by $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$.
- $\mathcal{M}_{m\times n}$ is the space of $m \times n$ matrices; $f_M(t) = \mathsf{det}(tI_n - M)$ is the characteristic polynomial of M; $\operatorname{im} A$ is the image of A; $\ker A$ is the kernel of A; $V^{\perp}$ is the normal space of V. Parallelepiped = 平行六面體.
- Duel space V^* of real vector space V is $\{\alpha \mid \alpha : V \to \mathbb{R}, \alpha \text{ is linear}\}$.

A. [15%] 是非題. 若錯誤, 需說明原因或給出反例. <u>本題答案須寫在答案簿最前面</u>.

1. There is a linear transformation $A : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\operatorname{im} A = \ker A$.
2. $A \in \mathcal{M}_{n\times n}$. Suppose $A^2 = A$ then $\ker A = (\operatorname{im} A)^{\perp}$.
3. For any A, B, $C \in \mathcal{M}_{n\times n}$, $\mathsf{tr}(ABC) = \mathsf{tr}(CBA)$.
4. The matrix representation A of an self-adjoint transformation satisfies $A^{\mathsf{t}} = A$.
5. Symmetric matrix A is positive definite if and only if all its diagonal elements are positive.

B. [85%] 計算/證明題。(6A) 和 (6B) 只選擇一題作答，兩題皆答，以先寫者計算。

(1) [15%] Find all Jordan canonical forms for square matrices in $\mathcal{M}_{n\times n}$, $n \le 6$, with minimal polynomial $(t-1)^2(t+1)^2$.

(2) [15%] For A, $B \in \mathcal{M}_{m\times n}$, show that $f_{BA^{\mathsf{t}}}(t) = f_{B^{\mathsf{t}}A}(t) \cdot t^{m-n}$.

(3) [15%] Consider $V = \{A \mid AX = XA, \text{for any } X \in \mathcal{M}_{n\times n}\} \subset \mathcal{M}_{n\times n}$. Show that V is an one dimensional subspace of $\mathcal{M}_{n\times n}$.

(4) [15%] $A \in \mathcal{M}_{n\times n}$. Suppose $(t^2+1)|f_A(t)$, are there nonzero $\vec{\mathbf{u}}$, $\vec{\mathbf{v}} \in \mathbb{R}^n$ such that $A\vec{\mathbf{u}} = \vec{\mathbf{v}}$ and $A\vec{\mathbf{v}} = -\vec{\mathbf{u}}$? Prove or disprove it.

(5) [15%] U is a subspace of a finite dimensional vector space V. Consider $D_U \subset V^*$ defined by $\{\alpha \in V^* \mid U \text{ is a subspace of } \ker\alpha\}$. Show that D_U is a subspace of dimension $\dim V - \dim U$.

(6A) [10%] Show the volume V of the parallelepiped span by $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{w}} \in \mathbb{R}^n$ satisfies

$$\begin{aligned} V^2 =& ||\vec{\mathbf{u}}||^2\, ||\vec{\mathbf{v}}||^2\, ||\vec{\mathbf{w}}||^2 \;+\; 2\,(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})\,(\vec{\mathbf{v}} \cdot \vec{\mathbf{w}})\,(\vec{\mathbf{w}} \cdot \vec{\mathbf{u}}) \\ &- ||\vec{\mathbf{u}}||^2\,(\vec{\mathbf{v}} \cdot \vec{\mathbf{w}})^2 - ||\vec{\mathbf{v}}||^2\,(\vec{\mathbf{w}} \cdot \vec{\mathbf{u}})^2 - ||\vec{\mathbf{w}}||^2\,(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})^2 \end{aligned}$$

(6B) [10%] Following diagram of vector spaces and linear transformations satisfies

(a) $\ker f_{i+1} = \operatorname{im} f_i$, $\ker g_{i+1} = \operatorname{im} g_i$, $i = 0, 1, 2, 3, 4$.

(b) $\alpha_{i+1} \circ f_i = g_i \circ \alpha_i$, $i = 1, 2, 3, 4$.

$$\begin{array}{ccccccccccccc} \{0_{A_1}\} & \overset{f_0}{\to} & A_1 & \overset{f_1}{\to} & B_1 & \overset{f_2}{\to} & C_1 & \overset{f_3}{\to} & D_1 & \overset{f_4}{\to} & E_1 & \overset{f_5}{\to} & \{0_{E_1}\} \\ & & \alpha_1 \downarrow\cong & & \alpha_2 \downarrow\cong & & \alpha_3 \downarrow & & \alpha_4 \downarrow\cong & & \alpha_5 \downarrow\cong & & \\ \{0_{A_2}\} & \overset{g_0}{\to} & A_2 & \overset{g_1}{\to} & B_2 & \overset{g_2}{\to} & C_2 & \overset{g_3}{\to} & D_2 & \overset{g_4}{\to} & E_2 & \overset{g_5}{\to} & \{0_{E_1}\} \end{array}$$

Show that if α_1, α_2, α_4, α_5 are isomorphisms, then α_3 is an isomorphism.

Notice: You *must* show all your *work* in order to receive full credit.

(1) (20 points) Show that if $\mathbb{R}^n = W_1 \cup W_2 \cup \cdots \cup W_k \cup \cdots$, where each W_k is a subspace, then $\mathbb{R}^n = W_i$ holds for some i.

(2) (20 points) Let a, b, c, d, e, f be real numbers such that the quadratic form $Q(x,y,z) := ax^2 + by^2 + cz^2 + 2dxy + 2eyz + 2fxz$ is positive definite. Then the region bounded by the surface $Q(x,y,z) = 1$ has volume equals

$$\frac{4\pi}{3\sqrt{abc + 2def - ae^2 - bf^2 - cd^2}}.$$

(3) (15 points) The are infinitely many t in $\mathbb{R}$ such that the vectors $(t, 2t^2, 3t^3, 4t^4)$, $(t^2, 2t^3, 3t^4, 4)$, $(t^3, 2t^4, 3t, 4t^2)$, $(t^4, 2t, 3t^2, 4t^3)$ form a basis of $\mathbb{R}^4$.

(4) (15 points) Determine all values of $a, b, c, d, e, f \in \mathbb{R}$ such that the matrix $A := \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 2 & f \\ 0 & 0 & 0 & 2 \end{pmatrix}$ is *not* diagonalizable.

(5) (10 points) If a 5×5 matrix $A \in M_5(\mathbb{R})$ satisfies $A^7 = I_5$ (the identity matrix), then 1 is an eigenvalue of A.

(6) Prove or disprove the following statements (10 points for each).

(a) If $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear transformation with the null space (kernel) of dimension $n-1$, then there exists some $v \in \mathbb{R}^n$ and a non-zero $\lambda \in \mathbb{R}$ such that $\psi(v) = \lambda \cdot v$.

(b) If W_1 and W_2 are 8-dimensional subspaces of $\mathbb{R}^{10}$, then there exist $a_1, a_2, \cdots, a_{10}, b_1, b_2, \cdots, b_{10}, c_1, c_2, \cdots, c_{10}, d_1, d_2, \cdots, d_{10}$ in $\mathbb{R}$ such that the intersection $W_1 \cap W_2$ is the set of all vectors $(x_1, ..., x_{10})$ with $x_1, ..., x_{10}$ a solution to the system of equations

$$\begin{cases} a_1x_1 + a_2x_2 + \cdots + a_ix_i + \cdots + a_{10}x_{10} &= 0 \\ b_1x_1 + b_2x_2 + \cdots + b_ix_i + \cdots + b_{10}x_{10} &= 0 \\ c_1x_1 + c_2x_2 + \cdots + c_ix_i + \cdots + c_{10}x_{10} &= 0 \\ d_1x_1 + d_2x_2 + \cdots + d_ix_i + \cdots + d_{10}x_{10} &= 0. \end{cases}$$

試題隨卷繳回

GRADUATE ENTRANCE EXAM 2016: LINEAR ALGEBRA

Notation: R is the set of real numbers, and C is the set of complex numbers. If $F = \mathrm{R}$ or C, denote by $M_n(F)$ the $n \times n$ matrices with entries in F.

Problem 1 (10pts). Find all possible $a \in \mathrm{R}$ such that the vectors

$$(1, 3, a),\ (a, 4, 3),\ (0, a, 1) \in \mathrm{R}^3$$

are linearly dependent.

Problem 2 (10pts). Find a set of polynomials $p_0(t) = a$, $p_1(t) = b + ct$ and $p_2(t) = d + et + ft^2$ with coefficients $a, b, c, d, e, f \in \mathrm{R}$ so that $\{p_0, p_1, p_2\}$ is an orthonormal set of polynomials with respect to the inner product $\langle f, g\rangle = \int_0^2 f(t)g(t)dt$.

Problem 3 (20pts). Let

$$A = \begin{pmatrix} 1 & -3 & 0 \\ 3 & 4 & -3 \\ 3 & 3 & -2 \end{pmatrix} \in M_3(\mathrm{R}).$$

Find an invertible $P \in M_3(\mathrm{R})$ such that

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & -3 & 1 \end{pmatrix}.$$

Problem 4 (15pts). Let $V = M_3(\mathrm{C})$ be a 9-dimension vector space over C and let

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}.$$

Define the linear transformation $T : V \to V$ by

$$T(B) = ABA^{-1}.$$

Show that T is also diagonalizable.

Problem 5 (20pts). Let $A, B \in M_n(\mathrm{C})$. Suppose that eigenvalues of A and B are all real numbers and that $\operatorname{rank} A = \operatorname{rank} A^2$ and $\operatorname{rank} B = \operatorname{rank} B^2$. If A^3 is similar to B^3 (namely there exists an invertible $P \in M_n(\mathrm{C})$ such that $P^{-1}A^3P = B^3$), prove that A is similar to B.

Problem 6 (25pts). Let A and B be elements in $M_n(\mathrm{C})$. If $A^2B + BA^2 = 2ABA$, show that $(AB - BA)^n = 0$.

試題隨卷繳回

(1) (15%) Let $V = \mathbb{R}^6$. Let W_1 be the subspace of V spanned by

$$(1,2,3,4,5,6), (3,4,6,7,9,10), (0,1,0,2,0,3), (1,-2,3,-4,5,-6),$$

and W_2 be the subspace of V spanned by

$$(1,1,1,2,2,3), (-2,0,-1,0,1,2), (1,0,1,0,2,0), (0,0,1,0,-2,-2).$$

Find the dimension of the subspace $W_1 \cap W_2$ and find a basis for this subspace.

(2) (15%) Let

$$C = \begin{bmatrix} -x & 1 & 3 & 1 & 2 \\ -2 & 0 & x & 2 & 2 \\ x & 0 & -2 & -3 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & x & -2 \end{bmatrix}.$$

Find an integer x such that all entries of the inverse of C are integers. For such x, find C^{-1}.

(3) (15%) Let V be the vector space of all $n \times n$ matrices over F. Let T be the linear operator on V defined by $T(A) = A^t$. Test T for diagonalizability, and if T is diagonalizable, find a basis for V such that the matrix representation of T is diagonal.

(4) (15%) Let V and W be F-vector spaces, and V^* and W^* be the dual space of V and W, respectively. Let $T : V \to W$ be a linear transformation. Define $T^* : W^* \to V^*$ by $T^*(f) = f \circ T$ for all $f \in W^*$. Show that T is onto if and only if T^* is one to one.

(5) (10%) Let A and B be $n \times n$ matrices over a field F. Show that if A is invertible, there are at most n scalars c in F such that $cA + B$ is not invertible.

(6) (15%)

(a) Let S and T be linear operators on a finite-dimensional vector space. If $p(t)$ is a polynomial such that $p(ST) = 0$, and if $q(t) = tp(t)$, show that $q(TS) = 0$.

(b) What is the relation between the minimal polynomials of ST and TS.

(7) (15%) Let V be a vector space with a basis $\{u_1, u_2, \ldots, u_n\}$. Let $\langle\ ,\ \rangle$ be an inner product on V. If $c_1, c_2, \ldots, c_n$ are any n scalars, show that there is exactly one vector v in V such that $\langle v, u_j \rangle = c_j$, $j = 1, 2, \ldots, n$.

試題隨卷繳回

Notation: $\mathbb{R}$ is the set of real numbers and $\mathbb{C}$ is the set of complex numbers.

Problem 1 (15 pts). Let $<,>$ be the standard inner product on $\mathbb{R}^3$ given by $<v,w>=a_1a_2+b_1b_2+c_1c_2$ if $v=(a_1,b_1,c_1)$ and $w=(a_2,b_2,c_2)$. Let W be the subspace in $\mathbb{R}^3$ given by

$$W=\{(x,y,z)\in\mathbb{R} \mid 2x+7y=0,\ x-2y+z=0\}.$$

Find an orthonormal basis of W. Namely, find a basis $\{w_1,w_2\}$ of W such that $<w_1,w_1>=<w_2,w_2>=1$ and $<w_1,w_2>=0$.

Problem 2 (20 pts). Let

$$A=\begin{pmatrix}1&2&2\\1&2&-1\\-1&1&4\end{pmatrix}.$$

(1) Compute the characteristic polynomial of A.
(2) Find an invertible $P\in M_3(\mathbb{R})$ such that $P^{-1}AP$ is diagonal.

Problem 3 (25pts). Let V be a finite dimensional vector space over $\mathbb{R}$ and let $A: V\to V$ be a $\mathbb{R}$-linear transformation. Prove that

(1) (10 pts) if $A^k=0$ for some positive integer k, then $I-A$ is invertible, where I is the identity map.
(2) (15 pts) V is generated by kernel of A^k and the image of A^k for some k. In other words, prove $V=\operatorname{Ker}A^k+\operatorname{Im}A^k$ for some k.

Problem 4 (20pts). Let $\mathrm{L}: M_2(\mathbb{R})\to M_2(\mathbb{R})$ be the linear transformation defined by

$$\mathrm{L}(X)=\begin{pmatrix}3&-1\\4&-2\end{pmatrix}X-X\begin{pmatrix}3&-1\\4&-2\end{pmatrix}$$

(1) Find the dimension of the kernel of L.
(2) Find a basis for the image of L.

Problem 5 (20 pts). If $A\in M_n(\mathbb{C})$ such that $AA^*=A^*A$ and $v\in\mathbb{C}^n$ is a column vector, prove that

(1) $A^2v=0$, then $Av=0$.
(2) If $A^kv=0$ for some $k\geq 1$, then $Av=0$.
(3) Show that the minimal polynomial of A has distinct roots.

試題隨卷繳回