3

Group representations

A representation of a group G gives us a way of visualizing G as a group of matrices. To be precise, a representation is a homomorphism from G into a group of invertible matrices. We set out this idea in more detail, and give some examples of representations. We also introduce the concept of equivalence of representations, and consider the kernel of a representation.

Representations

Let G be a group and let F be $\mathbb{R}$ or $\mathbb{C}$. Recall from the first chapter that GL (n, F) denotes the group of invertible $n \times n$ matrices with entries in F.

3.1 Definition

A *representation* of G over F is a homomorphism ρ from G to GL (n, F), for some n. The *degree* of ρ is the integer n.

Thus if ρ is a function from G to GL (n, F), then ρ is a representation if and only if

$$(gh)\rho = (g\rho)(h\rho) \quad \text{for all } g, h \in G.$$

Since a representation is a homomorphism, it follows that for every representation ρ: $G \to$ GL (n, F), we have

$$1\rho = I_n, \text{ and}$$

$$g^{-1}\rho = (g\rho)^{-1} \quad \text{for all } g \in G,$$

where I_n denotes the $n \times n$ identity matrix.

3.2 Examples

(1) Let G be the dihedral group $D_8 = \langle a, b \colon a^4 = b^2 = 1,\ b^{-1}ab = a^{-1} \rangle$. Define the matrices A and B by

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and check that

$$A^4 = B^2 = I,\ B^{-1}AB = A^{-1}.$$

It follows (see Example 1.4) that the function $\rho \colon G \to \mathrm{GL}\,(2, F)$ which is given by

$$\rho \colon a^i b^j \to A^i B^j \quad (0 \leqslant i \leqslant 3,\ 0 \leqslant j \leqslant 1)$$

is a representation of D_8 over F. The degree of ρ is 2.

The matrices $g\rho$ for g in D_8 are given in the following table:

g	1	a	a^2	a^3
$g\rho$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

g	b	ab	a^2b	a^3b
$g\rho$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(2) Let G be any group. Define $\rho \colon G \to \mathrm{GL}\,(n, F)$ by

$$g\rho = I_n \quad \text{for all } g \in G,$$

where I_n is the $n \times n$ identity matrix, as usual. Then

$$(gh)\rho = I_n = I_n I_n = (g\rho)(h\rho)$$

for all $g, h \in G$, so ρ is a representation of G. This shows that every group has representations of arbitrarily large degree.

Equivalent representations

We now look at a way of converting a given representation into another one.

Let $\rho: G \to \mathrm{GL}(n, F)$ be a representation, and let T be an invertible $n \times n$ matrix over F. Note that for all $n \times n$ matrices A and B, we have

$$(T^{-1}AT)(T^{-1}BT) = T^{-1}(AB)T.$$

We can use this observation to produce a new representation σ from ρ; we simply define

$$g\sigma = T^{-1}(g\rho)T \quad \text{for all } g \in G.$$

Then for all $g, h \in G$,

$$\begin{aligned}(gh)\sigma &= T^{-1}((gh)\rho)T \\ &= T^{-1}((g\rho)(h\rho))T \\ &= T^{-1}(g\rho)T \cdot T^{-1}(h\rho)T \\ &= (g\sigma)(h\sigma),\end{aligned}$$

and so σ is, indeed, a representation.

3.3 Definition

Let $\rho: G \to \mathrm{GL}(m, F)$ and $\sigma : G \to \mathrm{GL}(n, F)$ be representations of G over F. We say that ρ is *equivalent* to σ if $n = m$ and there exists an invertible $n \times n$ matrix T such that for all $g \in G$,

$$g\sigma = T^{-1}(g\rho)T.$$

Note that for all representations ρ, σ and τ of G over F, we have (see Exercise 3.4):

(1) ρ is equivalent to ρ;
(2) if ρ is equivalent to σ then σ is equivalent to ρ;
(3) if ρ is equivalent to σ and σ is equivalent to τ, then ρ is equivalent to τ.

In other words, equivalence of representations is an equivalence relation.

3.4 Examples

(1) Let $G = D_8 = \langle a,\ b\colon a^4 = b^2 = 1,\ b^{-1}ab = a^{-1}\rangle$, and consider the representation ρ of G which appears in Example 3.2(1). Thus $a\rho = A$ and $b\rho = B$, where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Assume that $F = \mathbb{C}$, and define

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix}.$$

Then

$$T^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix}.$$

In fact, T has been constructed so that $T^{-1}AT$ is diagonal; we have

$$T^{-1}AT = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \quad T^{-1}BT = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and so we obtain a representation σ of D_8 for which

$$a\sigma = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \quad b\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The representations ρ and σ are equivalent.

(2) Let $G = C_2 = \langle a\colon a^2 = 1\rangle$ and let

$$A = \begin{pmatrix} -5 & 12 \\ -2 & 5 \end{pmatrix}.$$

Check that $A^2 = I$. Hence $\rho\colon 1 \to I,\ a \to A$ is a representation of G. If

$$T = \begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix},$$

then

$$T^{-1}AT = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and so we obtain a representation σ of G for which

$$1\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, a\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and σ is equivalent to ρ.

There are two easily recognized situations where the only representation which is equivalent to ρ is ρ itself; these are when the degree of ρ is 1, and when $g\rho = I_n$ for all g in G. However, there are usually lots of representations which are equivalent to ρ.

Kernels of representations

We conclude the chapter with a discussion of the kernel of a representation $\rho\colon G \to \mathrm{GL}\,(n, F)$. In agreement with Definition 1.8, this consists of the group elements g in G for which $g\rho$ is the identity matrix. Thus

$$\operatorname{Ker}\rho = \{g \in G\colon\ g\rho = I_n\}.$$

Note that $\operatorname{Ker}\rho$ is a normal subgroup of G.

It can happen that the kernel of a representation is the whole of G, as is shown by the following definition.

3.5 Definition

The representation $\rho\colon G \to \mathrm{GL}\,(1, F)$ which is defined by

$$g\rho = (1) \quad \text{for all } g \in G,$$

is called the *trivial* representation of G.

To put the definition another way, the trivial representation of G is the representation where every group element is sent to the 1×1 identity matrix.

Of particular interest are those representations whose kernel is just the identity subgroup.

3.6 Definition

A representation $\rho\colon G \to \mathrm{GL}\,(n, F)$ is said to be *faithful* if $\operatorname{Ker}\rho = \{1\}$; that is, if the identity element of G is the only element g for which $g\rho = I_n$.

3.7 Proposition

A representation ρ of a finite group G is faithful if and only if $\operatorname{Im}\rho$ *is isomorphic to G.*

Proof We know that $\operatorname{Ker}\rho \lhd G$ and by Theorem 1.10, the factor group $G/\operatorname{Ker}\rho$ is isomorphic to $\operatorname{Im}\rho$. Therefore, if $\operatorname{Ker}\rho = \{1\}$ then $G \cong \operatorname{Im}\rho$. Conversely, if $G \cong \operatorname{Im}\rho$, then these two groups have the same (finite) order, and so $|\operatorname{Ker}\rho| = 1$; that is, ρ is faithful.

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3.8 Examples

(1) The representation ρ of D_8 given by

$$(a^i b^j)\rho = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^j$$

as in Example 3.2(1) is faithful, since the identity is the only element g which satisfies $g\rho = I$. The group generated by the matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is therefore isomorphic to D_8.

(2) Since $T^{-1}AT = I_n$ if and only if $A = I_n$, it follows that all representations which are equivalent to a faithful representation are faithful.

(3) The trivial representation of a group G if faithful if and only if $G = \{1\}$.

In Chapter 6 we shall show that every finite group has a faithful representation.

The basic problem of representation theory is to discover and understand representations of finite groups.

Summary of Chapter 3

1. A representation of a group G is a homomorphism from G into $\mathrm{GL}(n, F)$, for some n.
2. Representations ρ and σ of G are equivalent if and only if there exists an invertible matrix T such that for all $g \in G$,
$$g\sigma = T^{-1}(g\rho)T.$$
3. A representation is faithful if it is injective.

Exercises for Chapter 3

1. Let G be the cyclic group of order m, say $G = \langle a: a^m = 1 \rangle$. Suppose that $A \in \mathrm{GL}(n, \mathbb{C})$, and define $\rho: G \to \mathrm{GL}(n, \mathbb{C})$ by

$$\rho: a^r \to A^r \quad (0 \leqslant r \leqslant m-1).$$

Show that ρ is a representation of G over $\mathbb{C}$ if and only if $A^m = I$.

2. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{2\pi i/3} \end{pmatrix}, \; C = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

and let $G = \langle a: a^3 = 1 \rangle \cong C_3$. Show that each of the functions $\rho_j: G \to \mathrm{GL}(2, \mathbb{C})$ $(1 \leqslant j \leqslant 3)$, defined by

$$\rho_1: a^r \to A^r,$$

$$\rho_2: a^r \to B^r,$$

$$\rho_3: a^r \to C^r \quad (0 \leqslant r \leqslant 2),$$

is a representation of G over $\mathbb{C}$. Which of these representations are faithful?

3. Suppose that $G = D_{2n} = \langle a, b: a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle$, and $F = \mathbb{R}$ or $\mathbb{C}$. Show that there is a representation $\rho: G \to \mathrm{GL}(1, F)$ such that $a\rho = (1)$ and $b\rho = (-1)$.

4. Suppose that ρ, σ and τ are representations of G over F. Prove:

(1) ρ is equivalent to ρ;
(2) if ρ is equivalent to σ, then σ is equivalent to ρ;
(3) if ρ is equivalent to σ, and σ is equivalent to τ, then ρ is equivalent to τ.

5. Let $G = D_{12} = \langle a, b: a^6 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$. Define the matrices A, B, C, D over $\mathbb{C}$ by

$$A = \begin{pmatrix} e^{i\pi/3} & 0 \\ 0 & e^{-i\pi/3} \end{pmatrix},\ B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix},\ D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Prove that each of the functions ρ_k: $G \to \mathrm{GL}\,(2, \mathbb{C})$ ($k = 1, 2, 3, 4$), given by

$$\rho_1:\ a^r b^s \to A^r B^s,$$

$$\rho_2:\ a^r b^s \to A^{3r}(-B)^s,$$

$$\rho_3:\ a^r b^s \to (-A)^r B^s,$$

$$\rho_4:\ a^r b^s \to C^r D^s \quad (0 \leqslant r \leqslant 5, 0 \leqslant s \leqslant 1),$$

is a representation of G. Which of these representations are faithful? Which are equivalent?

6. Give an example of a faithful representation of D_8 of degree 3.
7. Suppose that ρ is a representation of G of degree 1. Prove that $G/\mathrm{Ker}\,\rho$ is abelian.
8. Let ρ be a representation of the group G. Suppose that g and h are elements of G such that $(g\rho)(h\rho) = (h\rho)(g\rho)$. Does it follow that $gh = hg$?