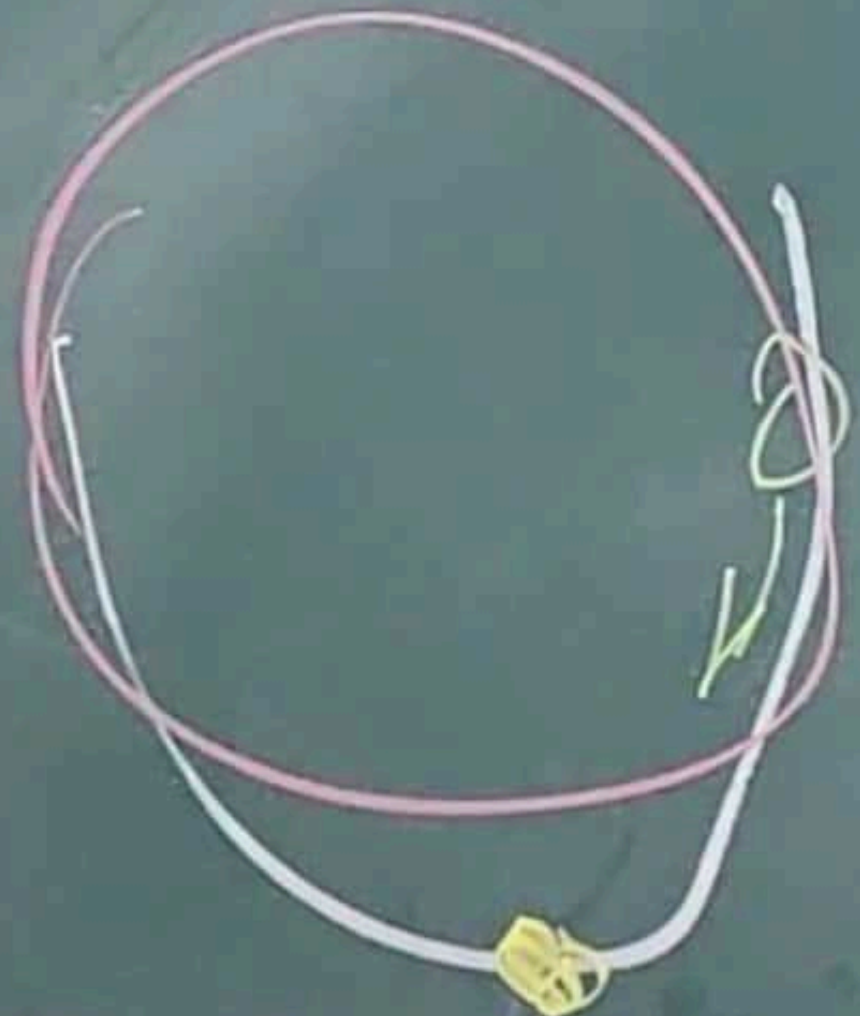
Quiz ① $\rho = \text{const},$

$= \rho(r)$

$= \rho(x, y, z)$

②

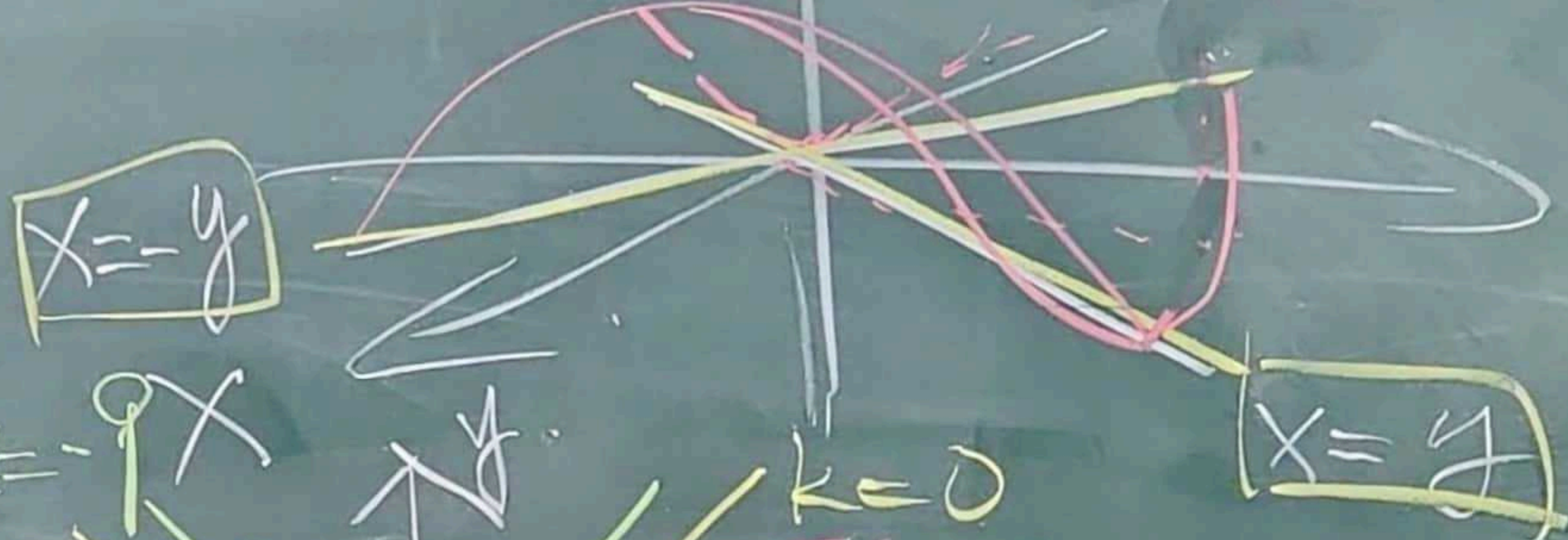


$k = ?$ Find max

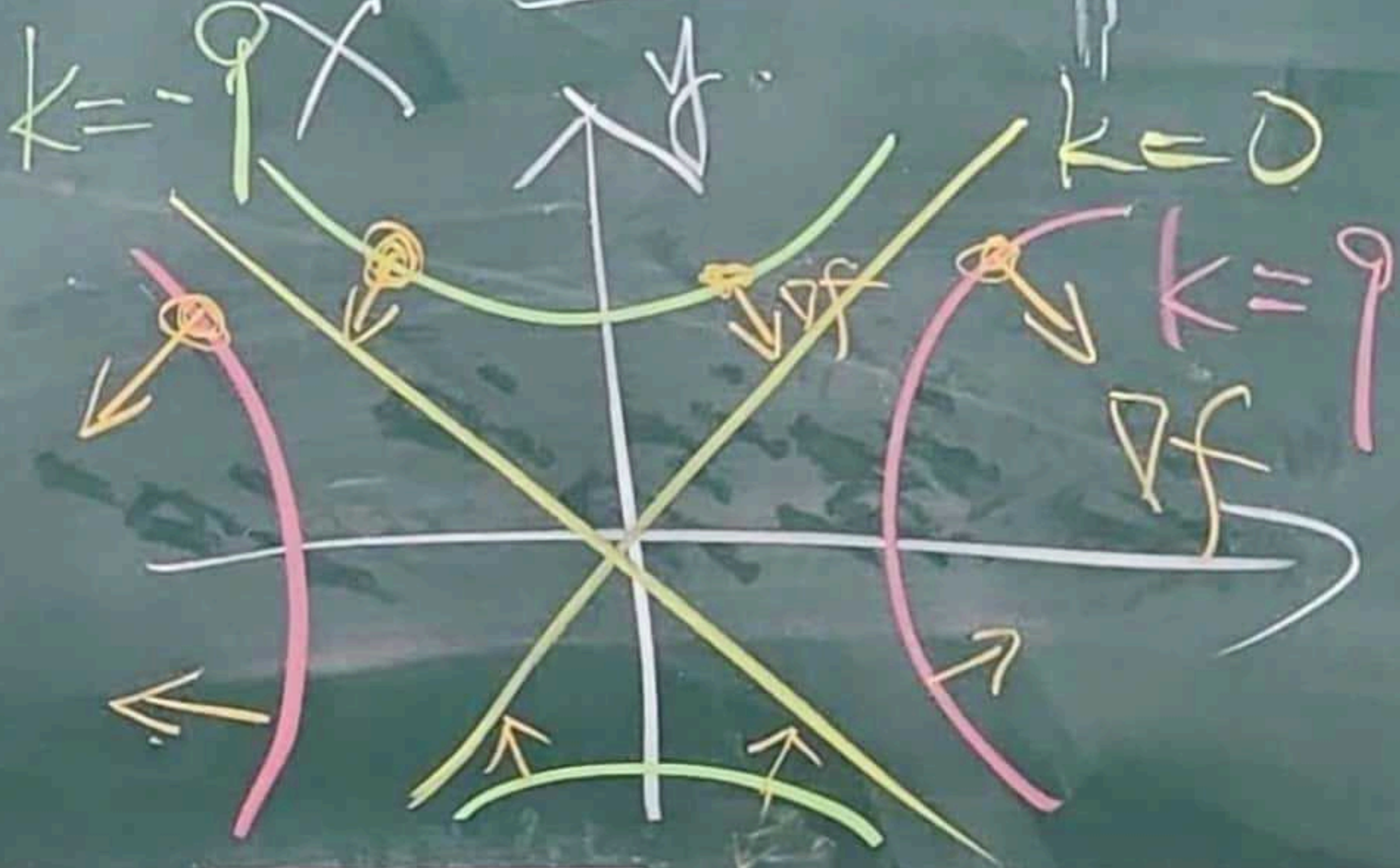
⟨Ex⟩ $f(x,y) = x^2 - y^2$ height

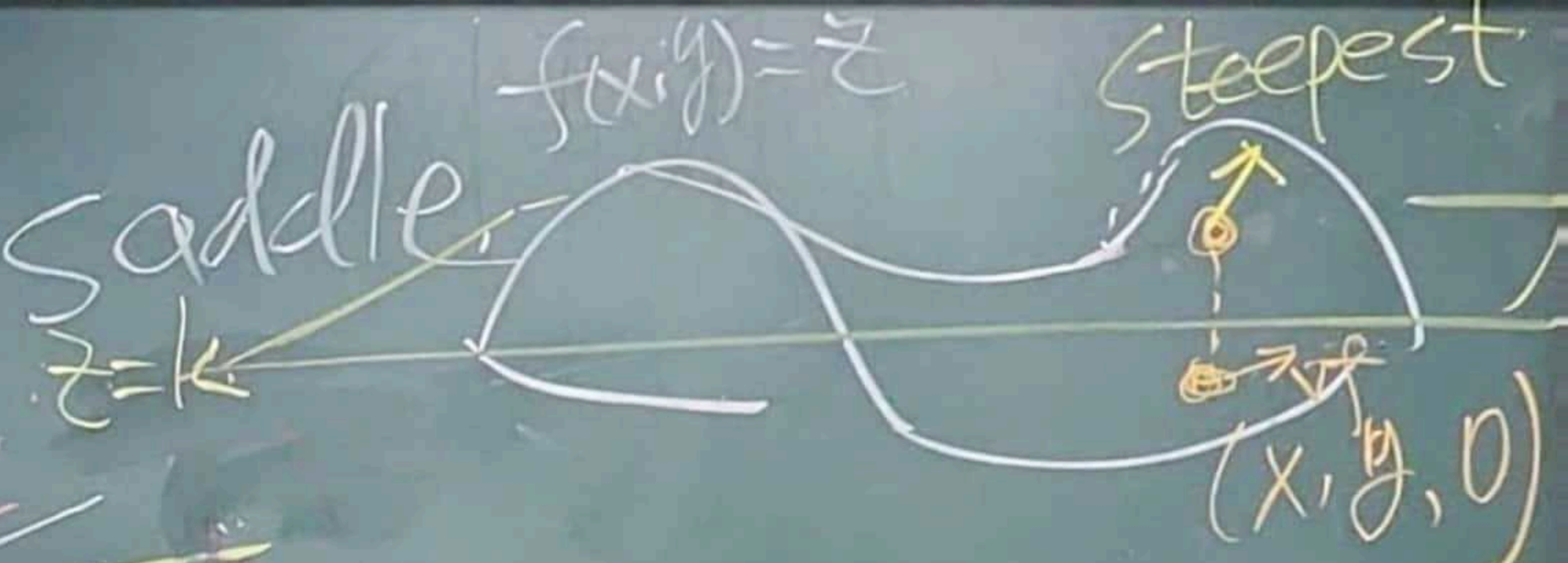
level curves $f(x,y) = x^2 - y^2 = \boxed{k}$

$k = 0 \iff x = \pm y$



$y \nearrow = \langle f_x, f_y, ? \rangle$

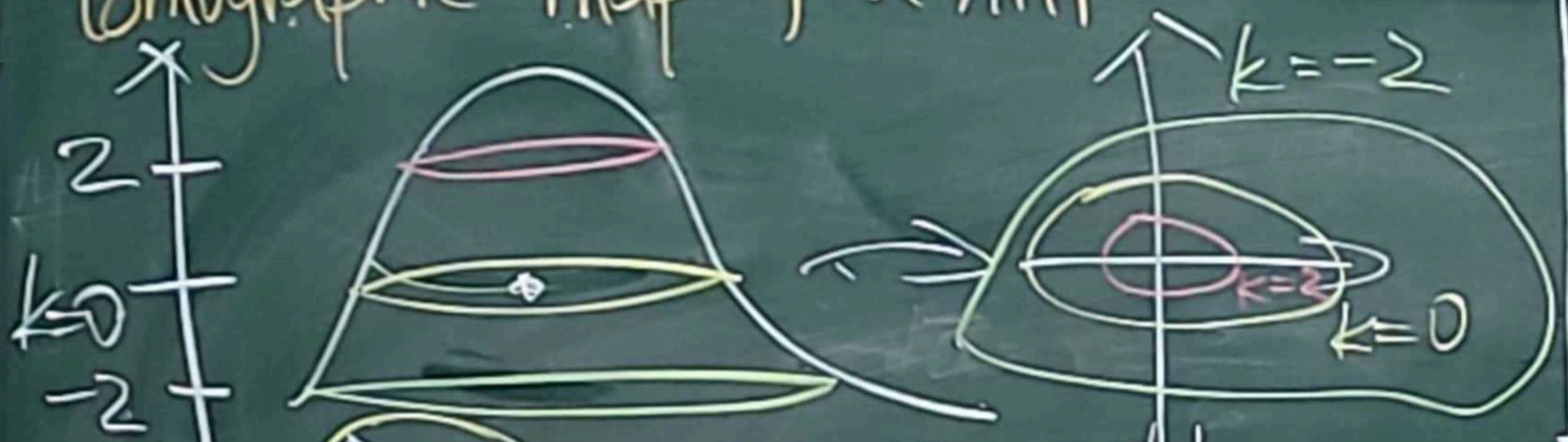


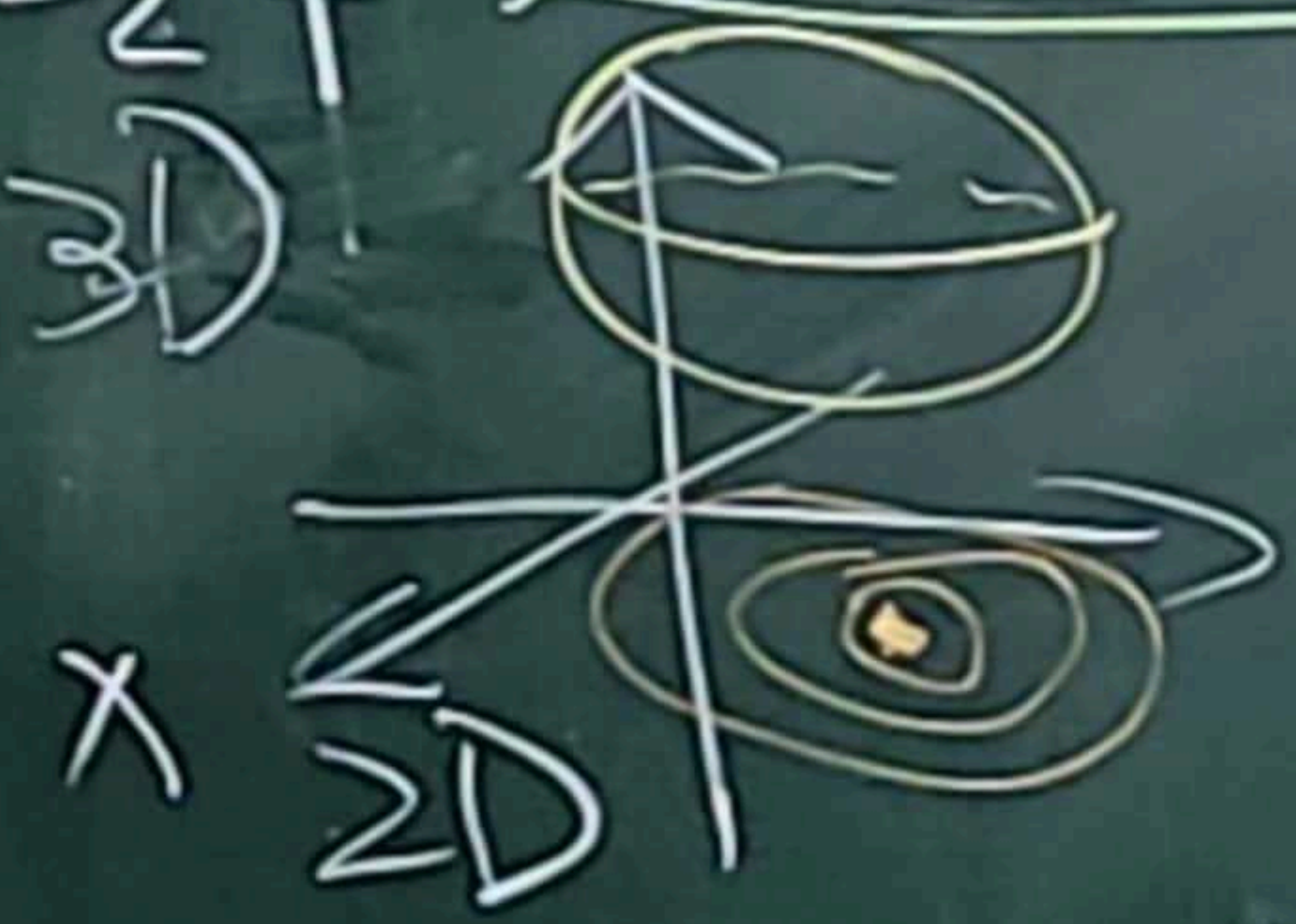


Gradient vector

$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$

tomographic map of a hill

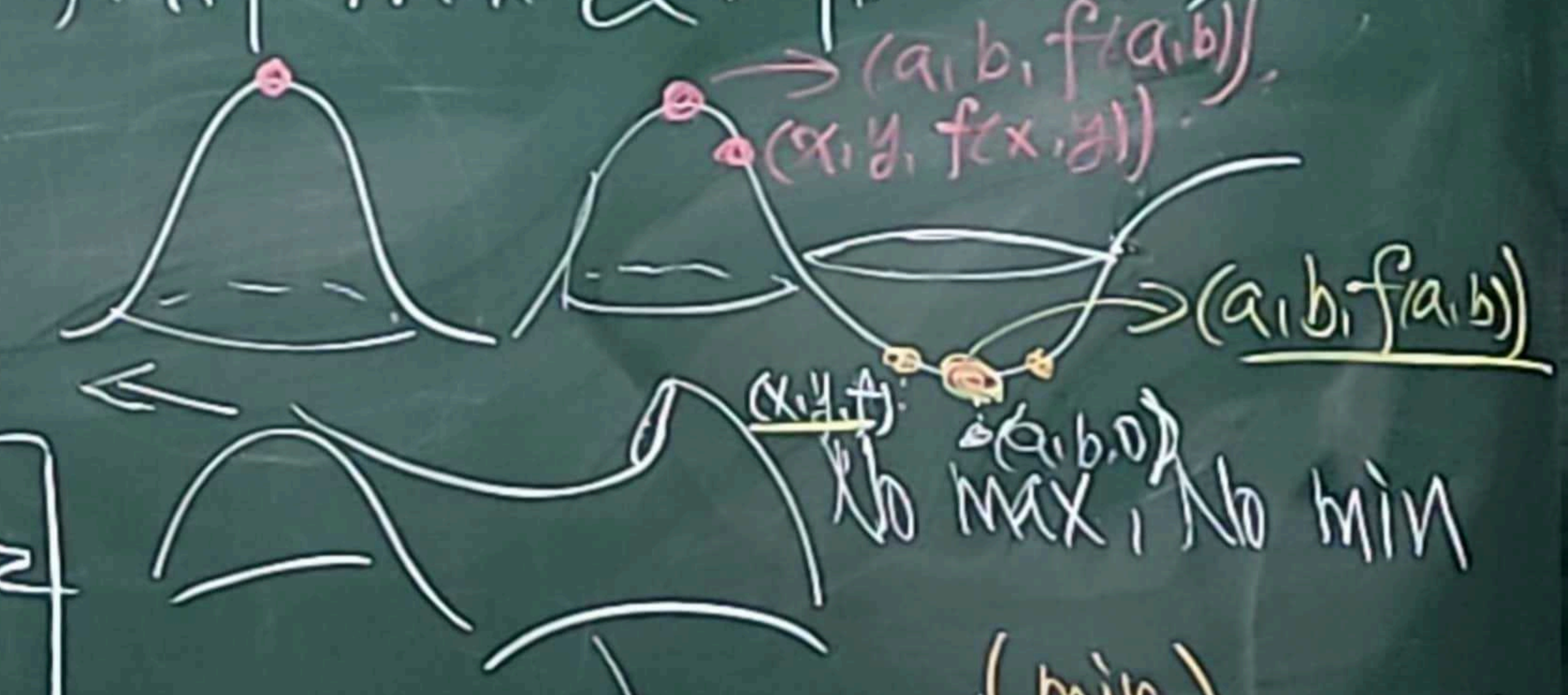




4D sphere

$$x^2+y^2+z^2+w^2=r^2$$

§14.7 Max & Min values

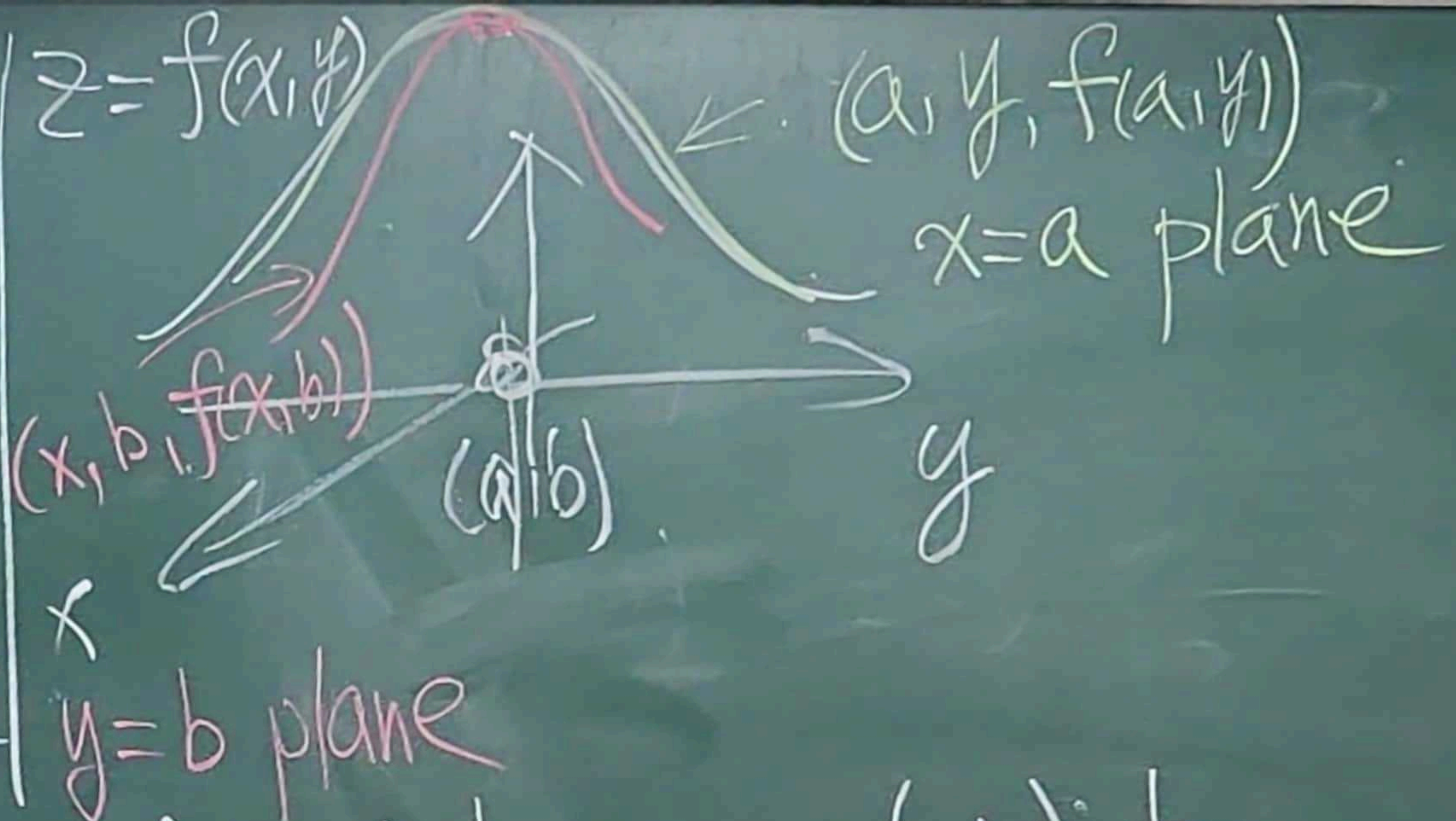


No max, No min

Def f has (⇒) a local max (min) at (a, b)
if $f(x,y) \le f(a,b)$ when (x, y) is near

$f(a,b) =$ local max (min) value of f.

f has an absolute max (min) at (a,b)
if $f(x,y) \leq (\geq) f(a,b)$ $\forall (x,y) \in \mathrm{Dom}$

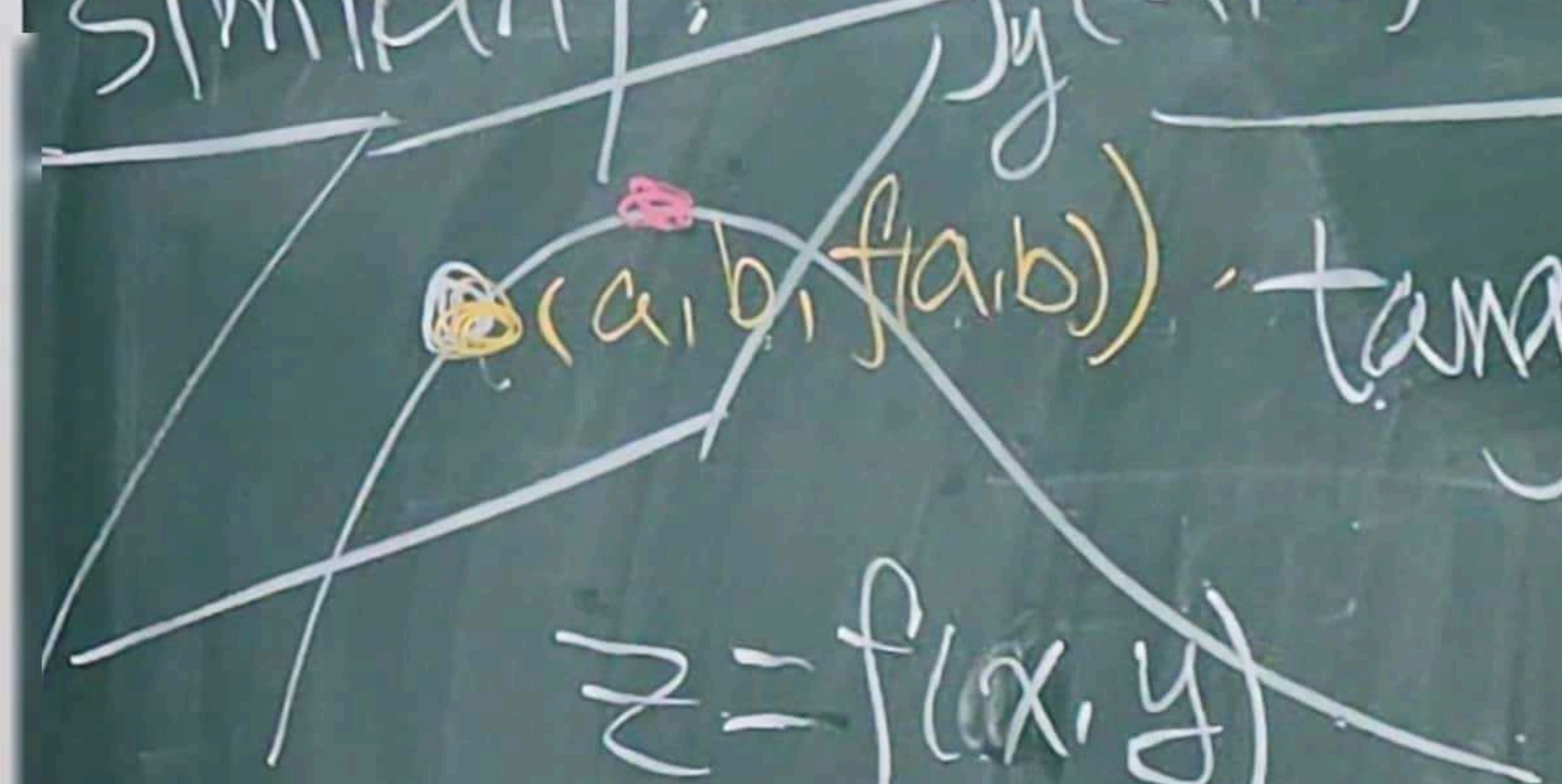


Thm If f has a local max or min at (a,b) and f_x f_y exist
$\Rightarrow f_x(a,b) = 0 = f_y(a,b)$

$g(x) \equiv f(x,b)$ has a max (min) at $x=a$

$\Rightarrow g'(x)\big|_{x=a} = f_x(a,b) = 0$

Similarly. $f_y(a,b)=0$ ※※



tangent plane.

$z=f(x,y)$

$z-f(a,b)=\underbrace{f_x(a,b)}_{=0}(x-a)+\underbrace{f_y(a,b)}_{=0}(y-b)$

$z=f(a,b)$ horizontal plane

<Ex> $f(x,y)=x^2-y^2 \Rightarrow \begin{cases} f_x(x,y)=0 \\ f_y(x,y)=0 \end{cases}$

$\Rightarrow \begin{cases} 2x=0 \\ -2y=0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \end{cases}$

<Ex> $f(x)=x^2 \Rightarrow f'(0)=0$ and $f(0)=\min$

$f(x,y)=x^2+y^2 \Rightarrow \begin{cases} f_x(0,0)=0 \\ f_y(0,0)=0 \end{cases}$ and $f(0,0)=\min$

$f(x)=x^3 \Rightarrow f'(0)=0$ and $f(0)\neq \min$ nor max

$f(x,y)=x^2-y^2 \Rightarrow \begin{cases} f_x(0,0)=0 \\ f_y(0,0)=0 \end{cases}$ and $f(0,0)\neq \min$ nor max

Recall

For $y=f(x)$ we use 1st-derivative test, or 2nd derivative test to determine min or max

Def (a,b) = critical pt of f if

$f_x(a,b)=0=f_y(a,b)$ or $\begin{matrix} f_x(a,b) \\ f_y(a,b) \end{matrix}$ DNE

<Ex> $f(x,y)=x^2+y^2-2x-6y+14$

Find critical pts, local min (max), absolute min (max)

Sol: $\begin{cases} f_x(x,y) = 0 \\ f_y(x,y) = 0 \end{cases}$ critical pts

f_x & f_y exist $\forall (x,y)$

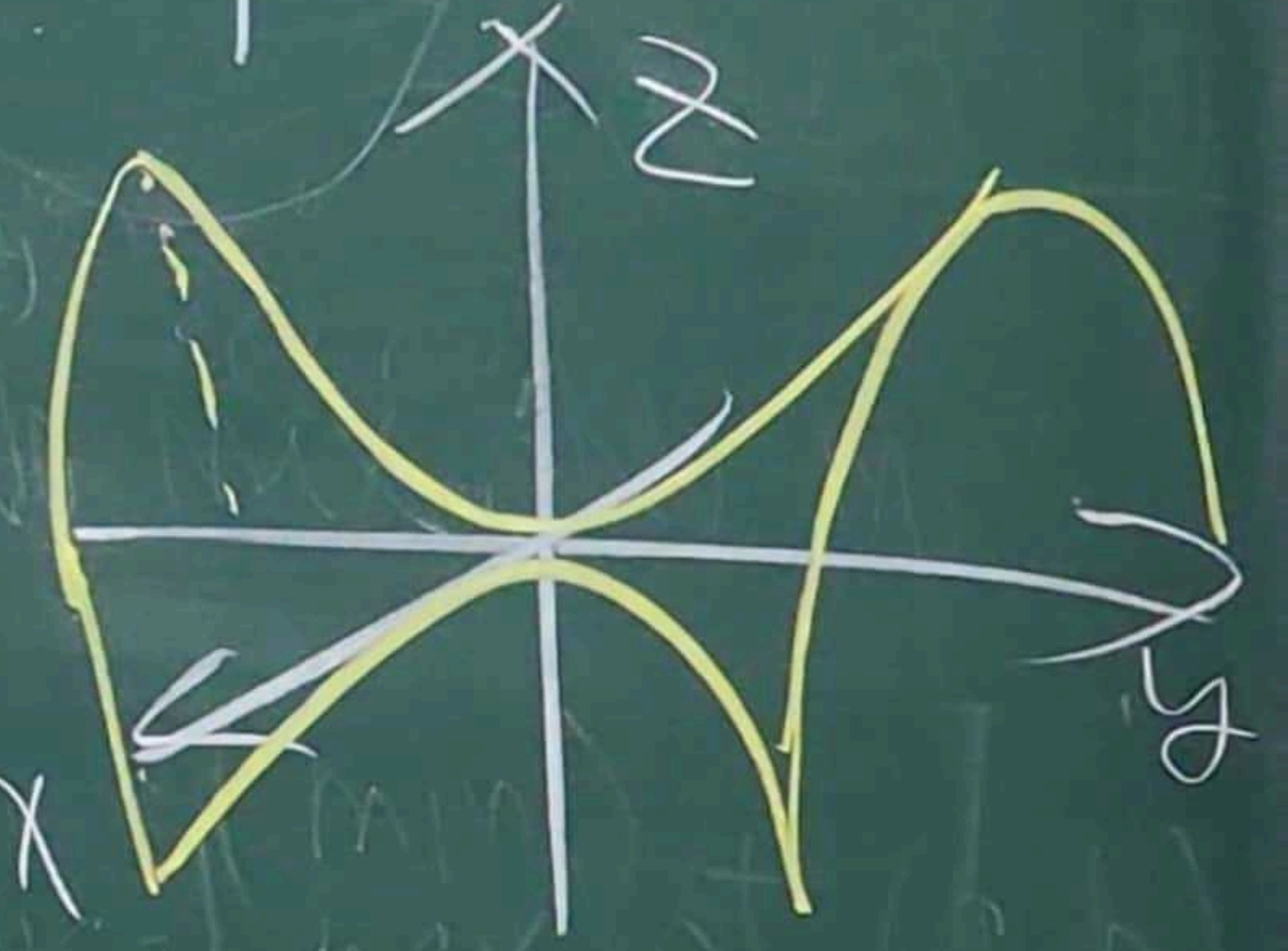
$f(x,y) = \underline{(x-1)^2 + (y-3)^2 + 4} \geq 4$

$\Rightarrow f(1,3) = 4$ ~~local min~~ absolute min

No local max Graph

⟨Ex⟩ Find extreme values of $f(x,y) = y^2 - x^2$

$\Rightarrow \begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$ critical pt of f.



$f(0,y) = y^2 > 0 \ (y \neq 0)$

$f(x,0) = -x^2 < 0 \ (x \neq 0)$.

$\Rightarrow f(0,0) \neq$ min nor max

2nd Derivative Test

Suppose f_{xx}, f_{yy}, f_{xy} = conti near (a,b)

and $f_x(a,b) = f_y(a,b) = 0$. Let

$$D = D(a,b) = f_{xx}(a,b)\,f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2 = \left.\begin{vmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{vmatrix}\right|_{\substack{x=a \\ y=b}}$$

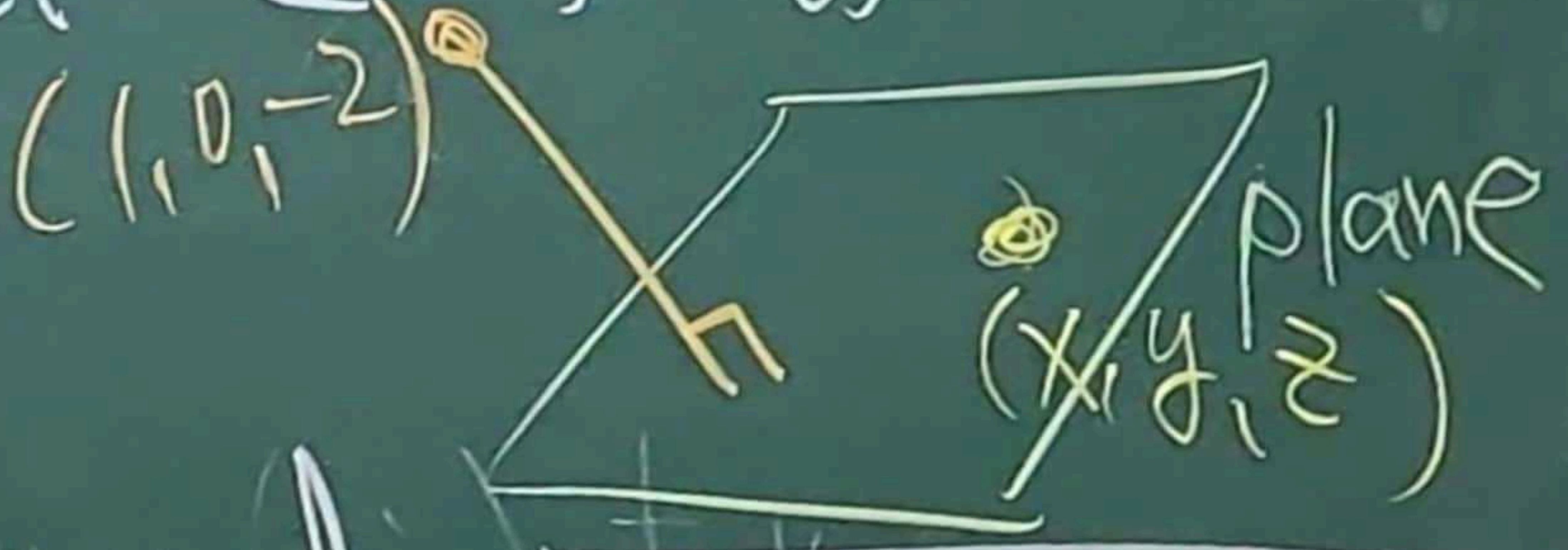
(a) $\boxed{D>0}$, and $f_{xx}(a,b)>0 \Rightarrow f(a,b)$ = local min

(b) $\boxed{D>0}$ and $f_{xx}(a,b)<0 \Rightarrow f(a,b)$ = local max

(c) $\boxed{D<0} \Rightarrow f(a,b) \neq$ local min or max

(Ex) Find the shortest distance from $\boxed{(1,0,-2)}$ to $\boxed{x+2y+z=4}$

Sol: Set $z = f(x,y) = 4-x-2y$



distance $d = \sqrt{(x-1)^2+(y-0)^2+(z+2)^2}$

$= \sqrt{(x-1)^2+y^2+(6-x-2y)^2} = f(x,y)$

min of $d(x,y) = d(a,b)$

min of $d^2(x,y) = d^2(a,b)$

Set $f(x,y) = d^2 = (x-1)^2 + y^2 + (6-x-2y)^2$

$\Rightarrow \begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} a = 11/6 \\ b = 5/3 \end{cases}$

Compute $f_{xx}(a,b)$, $f_{yy}(a,b)$, $f_{xy}(a,b) = f_{yx}(a,b)$

Compute $D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = 24 > 0$

By 2nd derivative test

$D(a,b) > 0$, $f_{xx}(a,b) > 0$,

$f(a,b) = \frac{25}{6}$ local min (absolute min)

$\Rightarrow d = \sqrt{f(a,b)} = \frac{5}{6}\sqrt{6}$ shortest distance