HW1

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Every time the symbol ε is used in this home work, it is a positive real number.

Every time the symbol N or $(N_i, \forall i \in \mathbb{N})$ is used in this home work, it is a natural number.

1

Suppose that $\{a_n\}$ converge to 0, and $\{b_n\}$ is bounded. Prove that $\{a_n b_n\}$ converge to 0

Proof. From premise, $\lim_{n\to\infty} a_n = 0$, and that $\exists M \in \mathbb{R}, \forall i, |b_i| < M$

$\lim_{n\to\infty} a_n = 0 \iff \forall \varepsilon, \exists N, \forall i > N, |a_i| < \varepsilon$

So, $\forall \varepsilon, \exists N_1, \forall i > N_1, |a_i| < \frac{\varepsilon}{M} \implies |a_i b_i| = |a_i||b_i| < \frac{\varepsilon}{M} M = \varepsilon$

So, $\forall \varepsilon, \exists N_1, \forall i > N_1 \to |a_i b_i| < \varepsilon$, which give us $\{a_n b_n\}$ converge to 0.

$\blacksquare$

2

Let S be a non-empty subset of $\mathbb{R}$ which is bounded above. Set $s = sup(S)$. Show that there exists a sequence $\{a_n\}$ in S which converge to s

Proof. Let $\{c_n\}$ be a monotonic increasing sequence in S and containing all elements of S.

Because $\{c_n\}$ is monotonic increasing and is bounded above by premise, $\{c_n\}$ converges.

Assume $\lim_{i\to\infty} c_i = u < s$

Then $\exists N, \forall i > N, |c_i - u| < \frac{s-u}{2} \implies c_i - u < \frac{s-u}{2} \implies c_i < \frac{s+u}{2} < s$

Because $\{c_n\}$ is monotonic increasing, $\forall j \leq N, \forall i > N, c_j \leq c_i < \frac{s+u}{2}$

Then $\forall k, c_k < \frac{s+u}{2} < s$ CaC to that $s = sup(S)$.

Assume $\lim_{i\to\infty} c_i = u > s$

Then $\exists N, \forall i > N, |c_i - u| < \frac{u-s}{2} \implies \frac{s-u}{2} < c_i - u \implies s < \frac{s+u}{2} < c + i$, CaC to that $s = sup(S)$

So $\lim_{i \to \infty} c_i = s$

■

3

Suppose that $\sum_{n=1}^{\infty} a_n^2, \sum_{n=1}^{\infty} b_n^2$ converges. Prove that the following series converge.

(a) $\sum_{n=1}^{\infty} |a_n b_n|$

(b) $\sum_{n=1}^{\infty} (a_n + b_n)^2$

(c) $\sum_{n=1}^{\infty} \frac{|a_n|}{n}$

3.(a)

Proof.

We claim that **(i)** $\forall 1 \leq n, 0 \leq |a_n b_n| \leq a_n^2$ or $0 \leq |a_n b_n| \leq b_n^2$

Cleary, $\forall n, |a_n| \leq |b_n|$ or $|b_n| \leq |a_n|$

$|a_n| \leq |b_n| \implies |a_n b_n| = |a_n||b_n| \leq |b_n||b_n| = |b_n^2|$

$|b_n| \leq |a_n| \implies |a_n b_n| = |a_n||b_n| \leq |a_n||a_n| = |a_n^2|$

Claim **(i)** is proven.

So from claim **(i)**, $\forall 1 \leq n, |a_n b_n| \leq a_n^2 + b_n^2$

So $\sum_{n=1}^{\infty} a_n^2 + b_n^2 = \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2$ is an upper bound of the series $\sum_{n=1}^{\infty} |a_n b_n|$, since every term of the latter is smaller than the corresponding term of the former.

Clearly, $\sum_{n=1}^{\infty} |a_n b_n|$ is monotonic increasing, since every term is non-negative.

So $\sum_{n=1}^{\infty} |a_n b_n|$ must converge.

■

3.(b)

Proof. $\sum_{n=1}^{\infty} |a_n b_n|$ converge $\implies \sum_{n=1}^{\infty} 2a_n b_n$ converge $\implies \sum_{n=1}^{\infty} 2a_n b_n + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2$ converge $\implies \sum_{n=1}^{\infty} a_n^2 + 2a_n b_n + b_n^2$ converge $\implies$

$\sum_{n=1}^{\infty}(a_n + b_n)^2 = \sum_{n=1}^{\infty} a_n^2 + 2a_nb_n + b_n^2$ converge.

■

3.(c)

Proof.

We claim that **(i)** $\forall 1 \leq n, \frac{|a_n|}{n} \leq \frac{1}{n^2}$ or $\frac{|a_n|}{n} \leq a_n^2$

Clearly, $\forall 1 \leq n, |a_n| \leq \frac{1}{n}$ or $\frac{1}{n} \leq |a_n|$

$|a_n| \leq \frac{1}{n} \implies \frac{|a_n|}{n} \leq \frac{1}{n^2}$

$\frac{1}{n} \leq |a_n| \implies \frac{|a_n|}{n} \leq |a_n^2|$

Claim **(i)** is proven.

From claim **(i)** $\forall 1 \leq n, \frac{|a_n|}{n} \leq \frac{1}{n^2} + a_n^2$

So $\sum_{n=1}^{\infty} \frac{1}{n^2} + a_n^2$ is an upper bound of $\sum_{n=1}^{\infty} \frac{|a_n|}{n}$, since every term of the latter is smaller than the corresponding term of the former.

Clearly, $\sum_{n=1}^{\infty} \frac{|a_n|}{n}$ is monotonic increasing.

So $\sum_{n=1}^{\infty} \frac{|a_n|}{n}$ must converge.

■

4

Determine Whether

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

converge.

Proof.

We claim that **(i)** $a_n = 2^k$, where $k = \sum_{j=1}^{n} \frac{1}{2^j}$

we prove claim **(i)** by induction.

Base step: $a_1 = 2^k$, where $k = \sum_{j=1}^{1} \frac{1}{2^j}$

$a_1 = 2^{\frac{1}{2}}$, where $\frac{1}{2} = \sum_{j=1}^{1} \frac{1}{2^j} = k$

Induction step: $a_u = 2^k$, where $k = \sum_{j=1}^{u} \frac{1}{2^j} \to a_{u+1} = 2^{\sum_{j=1}^{u+1} \frac{1}{2^j}}$

By premise, $a_{u+1} = \sqrt{2a_u}$

$a_{u+1} = \sqrt{2a_u} = \sqrt{2}\sqrt{a_u} = 2^{\frac{1}{2}} 2^{\frac{k}{2}} = 2^{\frac{1}{2}} 2^{\frac{1}{2}\sum_{j=1}^{u} \frac{1}{2^j}} = 2^{\frac{1}{2} + \sum_{j=1}^{u} \frac{1}{2^j 2}} = 2^{\frac{1}{2} + \sum_{j=2}^{u+1} \frac{1}{2^j}} = 2^{\sum_{j=1}^{u+1} \frac{1}{2^j}}$

Claim **(i)** is proven.

From claim **(i)**, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2^{\sum_{j=1}^{n} \frac{1}{2^j}} = 2^{\lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{2^j}} = 2^{\lim_{n\to\infty} 1 - \frac{1}{2^n}} = 2^1 = 2$

It converges. ■

5

Find the values of x for which the series converge, also find the sum of the series of those values of x.

(a) $\sum_{n=1}^{\infty} (-5)^n x^n$

(b) $\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n}$

(c) $\sum_{n=1}^{\infty} \frac{2^n}{x^n}$

5.(a)

Proof. $\sum_{n=1}^{\infty} (-5)^n x^n = \sum_{n=1}^{\infty} (-1)^n (5x)^n$

We claim that **(i)** the convergence interval is $(-\frac{1}{5}, \frac{1}{5})$

If $x \in (-\frac{1}{5}, \frac{1}{5})$, $5x \in (-1, 1)$, then by alternating series test $\sum_{n=1}^{\infty} (-1)^n (5x)^n$ converge.

If $x \notin (-\frac{1}{5}, \frac{1}{5})$, then $\lim_{n\to\infty} (-1)^n (5x)^n \neq 0$, which shows $\sum_{n=1}^{\infty} (-1)^n (5x)^n$ diverge.

Claim **(i)** is proven.

$\sum_{n=1}^{\infty} (-5)^n x^n = \sum_{n=1}^{\infty} (-1)^n (5x)^n = \sum_{n=1}^{\infty} (-5x)^n = \frac{-5x}{1+5x}$ ■

5.(b)

Proof. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n} = \sum_{n=1}^{\infty} (\frac{x-2}{3})^n$

We claim that **(i)** the convergence interval is $(-1, 5)$

If $x \in (-1, 5)$, then $\frac{x-2}{3} \in (-1, 1)$, then $\lim_{n\to\infty} \sqrt[n]{|(\frac{x-2}{3})^n|} < 1$, then by root test, $\sum_{n=1}^{\infty}(\frac{x-2}{3})^n$ converge.

If $x \notin (-1, 5)$, then $\frac{x-2}{3} \notin (-1, 1)$, then $\lim_{n\to\infty}(\frac{x-2}{3})^n \neq 0$ shows that $\sum_{n=1}^{\infty}(\frac{x-2}{3})^n$ diverge.

Claim **(i)** is proven.

$\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n} = \sum_{n=1}^{\infty}(\frac{x-2}{3})^n = \frac{\frac{x-2}{3}}{1-(\frac{x-2}{3})} = \frac{x-2}{5-x}$

■

5.(c)

Proof. $\sum_{n=1}^{\infty} \frac{2^n}{x^n} = \sum_{n=1}^{\infty}(\frac{2}{x})^n$

We claim that **(i)** the convergence interval is $(-\infty, -2) \cup (2, \infty)$

If $x \in (-\infty, -2) \cup (2, \infty)$, then $\frac{2}{x} \in (-1, 1)$, $\lim_{n\to\infty} \sqrt[n]{|(\frac{2}{x})^n|} < 1$, then by root test $\sum_{n=1}^{\infty}(\frac{2}{x})^n$ converge.

If $x \notin (-\infty, -2) \cup (2, \infty)$, then $\frac{2}{x} \notin (-1, 1)$, then $\lim_{n\to\infty}(\frac{2}{x})^n \neq 0$ shows that $\sum_{n=1}^{\infty}(\frac{2}{x})^n$ diverge.

Claim **(i)** is proven.

$\sum_{n=1}^{\infty} \frac{2^n}{x^n} = \sum_{n=1}^{\infty}(\frac{2}{x})^n = \frac{\frac{2}{x}}{1-\frac{2}{x}} = \frac{2}{x-2}$ ■

6

Use comparison test to determine whether the series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{7n+2}{\sqrt{2n^3-1}}$

(b) $\sum_{n=1}^{\infty} ne^{-n^2}$

(c) $\sum_{n=1}^{\infty} \frac{2n!}{(2n)!}$

6.(a)

Proof. $\lim_{n\to\infty} \frac{\frac{7n+2}{\sqrt{2n^3-1}}}{\frac{7n}{\sqrt{2n^3-1}}} = \lim_{n\to\infty} \frac{7n+2}{7n} = 1 \implies \sum_{n=1}^{\infty} \frac{7n+2}{\sqrt{2n^3-1}}$ converge $\longleftrightarrow$ $\sum_{n=1}^{\infty} \frac{7n}{\sqrt{2n^3-1}}$ converge.

$\lim_{n\to\infty} \frac{\frac{7n}{\sqrt{2n^3-1}}}{\frac{7n}{\sqrt{2n^3}}} = \lim_{n\to\infty} \frac{\sqrt{2n^3-1}}{\sqrt{2n^3}} = \sqrt{\lim_{n\to\infty} \frac{2n^3-1}{2n^3}} = \sqrt{1} = 1 \implies \sum_{n=1}^{\infty} \frac{7n}{\sqrt{2n^3-1}}$ converge $\longleftrightarrow$ $\sum_{n=1}^{\infty} \frac{7n}{\sqrt{2n^3}}$ converge.

$\sum_{n=1}^{\infty} \frac{7n}{\sqrt{2n^3}} = \frac{7}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverge by integral test. ($\frac{1}{\sqrt{n}} dn = 2\sqrt{n}$)

So $\sum_{n=1}^{\infty} \frac{7n+2}{\sqrt{2n^3-1}}$ diverge. ■

6.(b)

Proof. $\sum_{n=1}^{\infty} ne^{-n^2} = \sum_{n=1}^{\infty} e^{(\ln n)-n^2}$

$\lim_{n\to\infty} \sqrt[n]{|e^{(\ln n)-n^2}|} = \lim_{n\to\infty} e^{\frac{\ln n}{n} - n} = e^{-\infty} = 0$

So by root test, this series converge.

■

6.(c)

Proof. Set a_n to be $\frac{2n!}{(2n)!}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{2(n+1)!}{(2n+2)!}}{\frac{2n!}{(2n)!}} = \lim_{n\to\infty} \frac{n+1}{(2n+1)(2n+2)} = 0$

So by ratio test, the series converge.

■

7

Let $F(x) = \int_0^x \frac{t}{1+t^2} dt$, find the Taylor polynomial series of degree $2n$ of $F(x)$ at 0

Proof. Notcie $(\frac{1}{2}\ln(t^2+1))' = \frac{t}{t^2+1}$

So $F(x) = \frac{1}{2}\ln(x^2+1)$

So $F(x) = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n}}{n}$

So the desire polynomial series is $\frac{1}{2}\sum_{k=1}^{2n}(-1)^{k-1}\frac{x^{2k}}{k}$ ■

8

Use the Maclaurin series for $f(x) = x\sin(x^2)$ to find $f^{(203)}(0)$

Proof. $\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$

So $\sin x^2 = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{(2n+1)!}$

So $f(x) = x\sin x^2 = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+3}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$

So $\frac{f^{(203)}(0)}{203!}x^{203} = (-1)^{50}\frac{x^{4(50)+3}}{(2(50)+1)!}$

So $f^{(203)}(0) = (-1)^{50}\frac{203!}{101!}$

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