3.6 HW4

Question 37

3. Show that the equation of the tangent plane of a surface which is the graph of a differentiable function $z = f(x, y)$, at the point $p_0 = (x_0, y_0)$, is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Recall the definition of the differential df of a function $f\colon R^2 \to R$ and show that the tangent plane is the graph of the differential df_p.

Proof. The question first ask us to prove

$$T_{p_0}(S) = \{(x, y, z) \in \mathbb{R}^3 : z = f(x_0, y_0) + (x - x_0)\partial_x f(p_0) + (y - y_0)\partial_y f(p_0)\}$$

By premise of the question, we know there exists a global chart $\mathbf{x}$

$$\mathbf{x}(x, y) \triangleq (x, y, f(x, y))$$

Compute

$$d\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \partial_x f & \partial_y f \end{bmatrix}$$

This tell us

$$\begin{aligned} T_{p_0}(S) &= (x_0, y_0, f(x_0, y_0)) + \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ \partial_x f(p_0) \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \partial_y f(p_0) \end{bmatrix}\right) \\ &= \{(x, y, z) \in \mathbb{R}^3 : z = f(x_0, y_0) + (x - x_0)\partial_x f(p_0) + (y_{y0})\partial_y f(p_0)\} \text{ (done)} \end{aligned}$$

Compute

$$df_{p_0} = \begin{bmatrix} \partial_x f(p_0) & \partial_y f(p_0) \end{bmatrix}$$

Then we see

$$T_{p_0}(S) = (x_0, y_0, f(x_0, y_0)) + \{(x, y, df_{p_0}(x, y)) \in \mathbb{R}^3 : (x, y) \in \mathbb{R}^2\}$$

where the right hand side is the graph of df_{p_0} when the origin is set to be $(p_0, f(p_0))$

∎

Question 38

5. If a coordinate neighborhood of a regular surface can be parametrized in the form

$$\mathbf{x}(u, v) = \alpha_1(u) + \alpha_2(v),$$

where α_1 and α_2 are regular parametrized curves, show that the tangent planes along a fixed coordinate curve of this neighborhood are all parallel to a line.

Proof. WOLG, suppose the coordinate curve $\gamma : I \to S$ is

$$\gamma(t) = \mathbf{x}(u_0, t)$$

Compute

$$d\mathbf{x} = \begin{bmatrix} \alpha_1'(u) & \alpha_2'(v) \end{bmatrix} \text{ and } d\mathbf{x}_{\gamma(t)} = \begin{bmatrix} \alpha_1'(u_0) & \alpha_2'(v) \end{bmatrix}$$

Then see that

$$T_{\gamma(t)}(S) = \text{span}(\alpha_1'(u_0), \alpha_2'(t))$$

Since $\alpha_1'(u_0)$ is fixed, we see $T_{\gamma(t)}(S)$ are all parallel to $\alpha_1'(u_0)$. ■

Question 39

10. (*Tubular Surfaces.*) Let $\alpha\colon I \to R^3$ be a regular parametrized curve with nonzero curvature everywhere and arc length as parameter. Let

$$\mathbf{x}(s, v) = \alpha(s) + r(n(s)\cos v + b(s)\sin v), \quad r = \text{const.} \neq 0, s \in I,$$

be a parametrized surface (the *tube* of radius r around α), where n is the normal vector and b is the binormal vector of α. Show that, when $\mathbf{x}$ is regular, its unit normal vector is

2. Regular Surfaces

$$N(s, v) = -(n(s)\cos v + b(s)\sin v).$$

Proof. Use Frenet-Serret Formula to compute

$$d\mathbf{x} = \left[\alpha' + r\big((-\kappa T - \tau B)\cos v + \tau N \sin v\big) \quad r(-N\sin v + B\cos v)\right]$$

We wish to show

$$-(N\cos v + B\sin v) \perp d\mathbf{x}(\mathbb{R}^2)$$

We then only have to prove

$$\begin{aligned} &\big(N\cos v + B\sin v\big) \perp -N\sin v + B\cos v \\ \text{and } &\big(N\cos v + B\sin v\big) \perp \alpha' + r\big((-\kappa T - \tau B)\cos v + \tau N\sin v\big) \end{aligned}$$

Because $\{T, N, B\}$ form an othothonormal basis and α' is just T, we have

$$(N\cos v + B\sin v)\cdot(-N\sin v + B\cos v) = -(\cos v\sin v) + \sin v\cos v = 0$$

and have

$$(N\cos v + B\sin v)\cdot\Big(\alpha' + r\big((-\kappa T - \tau B)\cos v + \tau N\sin v\big)\Big)$$
$$= r\tau\cos v\sin v - r\tau\sin v\cos v = 0 \text{ (done)}$$

■

Question 40

13. A *critical point* of a differentiable function $f\colon S \to R$ defined on a regular surface S is a point $p \in S$ such that $df_p = 0$.

***a.** Let $f\colon S \to R$ be given by $f(p) = |p - p_0|$, $p \in S$, $p_0 \notin S$ (cf. Exercise 5, Sec. 2-3). Show that $p \in S$ is a critical point of f if and only if the line joining p to p_0 is normal to S at p.

b. Let $h\colon S \to R$ be given by $h(p) = p \cdot v$, where $v \in R^3$ is a unit vector (cf. Example 1, Sec. 2-3). Show that $p \in S$ is a critical point of f if and only if v is a normal vector of S at p.

Proof. Compute

$$dh_p(\alpha'(0)) = \frac{d}{dt}h(\alpha(t))|_{t=0} = v\cdot\alpha'(0)$$

In other words,

$$dh_p(w) = v\cdot w$$

This implies

$$dh_p(l) = 0, \forall l \in T_p(S) \iff v \perp T_p(S) \iff v \text{ is a normal vector of } S \text{ at } p$$

■

Question 41

15. Show that if all normals to a connected surface pass through a fixed point, the surface is contained in a sphere.

Proof. WOLG, suppose the fixed point is the origin. Let S be the regular surface. We are given the fact that any local chart $\mathbf{x}(u,v)$ satisfy the equation

$$\mathbf{x}(u,v) = f_{\mathbf{x}}(u,v)N(u,v) \tag{3.2}$$

where $f_{\mathbf{x}}(u,v)$ is a scalar-valued function and $N(u,v)$ is the unit normal vector of S at $\mathbf{x}(u,v)$. We first show that

the distance of the surface to the origin is locally a constant

In other words, we wish to prove that any local chart $\mathbf{x}(u,v)$ satisfy

$f_{\mathbf{x}}$ is constant on domain of $\mathbf{x}$

Doing partial derivative on $N \cdot N = 1$, we see that

$$\partial_u N \perp N \text{ and } \partial_v N \perp N$$

This implies

$$\partial_u N, \partial_v N \in T_p(S)$$

We know

$$\partial_u \mathbf{x}, \partial_v \mathbf{x} \in T_p(S)$$

Now, doing partial derivative on both side of (3.2), we see

$$\partial_u \mathbf{x} = (\partial_u f_{\mathbf{x}})N + f_{\mathbf{x}}(\partial_u N) \text{ and } \partial_v \mathbf{x} = (\partial_v f_{\mathbf{x}})N + f_{\mathbf{x}}(\partial_v N)$$

and see

$$(\partial_u f_{\mathbf{x}})N = \partial_u \mathbf{x} - f_{\mathbf{x}}(\partial_u N) \in T_p(S) \text{ and } (\partial_v f_{\mathbf{x}})N = \partial_v \mathbf{x} - f_{\mathbf{x}}(\partial_v N) \in T_p(S)$$

Then because $N \notin T_p(S)$, we can conclude $\partial_u f_{\mathbf{x}} = \partial_v f_{\mathbf{x}} = 0$. This establish that $f_{\mathbf{x}}$ is a constant. (done)

Note that the surface is connected, this implies that the distance of the surface to the origin is globally a constant, which implies the surface is contained in a sphere.

∎

Question 42

18. Prove that if a regular surface S meets a plane P in a single point p, then this plane coincides with the tangent plane of S at p.

Proof. WOLG, suppose P is the x, y-plane, p is the origin and $S \subseteq \{(x, y, z) \in \mathbb{R}^3 : z \geq 0\}$ (This can be achieved by a rigid motion). We wish to show

$$T_p(S) = P$$

Because $\dim(T_p S) = \dim(P) = 2$, we can reduce the problem into proving

$$T_p(S) \subseteq P$$

Fix $w \in T_p(S)$. We reduce the problem into

$$\text{proving } w \in P$$

Let $\alpha : (-\epsilon, \epsilon) \to S$ satisfy

$$\alpha(0) = 0 \text{ and } \alpha'(0) = w$$

Express

$$\alpha(t) \triangleq (x, y, z)$$

Because S is above P, the x, y-plane, we know z attain minimum at 0. This implies $z'(0) = 0$, and implies $\alpha'(0) = (x'(t), y'(0), 0) \in P$ (the x, y-plane) (done) ■

Question 43

***20.** Show that the perpendicular projections of the center $(0,0,0)$ of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

onto its tangent planes constitute a regular surface given by

$$\{(x,y,z)\in R^3;(x^2+y^2+z^2)^2=a^2x^2+b^2y^2+c^2z^2\}-\{(0,0,0)\}.$$

Proof. Let S be the ellipsoid. Let

$$E\triangleq\{(x,y,z)\in\mathbb{R}^3:(x^2+y^2+z^2)^2=a^2x^2+b^2y^2+c^2z^2\}\setminus O$$

Let E' be the perpendicular projections of O of S onto the tangent planes of S. We are required to prove

$$E'=E$$

We first prove

$$E'\subseteq E$$

Fix $p_0=(x_0,y_0,z_0)\in S$. We first prove

$$T_{p_0}(S)=\{(x,y,z)\in\mathbb{R}^3:\frac{xx_0}{a^2}+\frac{yy_0}{b^2}+\frac{zz_0}{c^2}=1\}\tag{3.3}$$

Fix

$$f(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$$

We have $S = f^{-1}(1)$. Compute

$$\nabla f(x,y,z) = (\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2})$$

Then because $T_{p_0}(S) \perp \nabla f(p_0)$, we have

$$T_{p_0}(S) = \{(x,y,z) \in \mathbb{R}^3 : \frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = 1\} \text{ (done)}$$

We know that the line L passing through O and perpendicular to $T_{p_0}(S)$ can be parametrized by

$$L(t) = t\nabla f(p_0) \equiv t(\frac{x_0}{a^2}, \frac{y_0}{b^2}, \frac{z_0}{c^2})$$

Compute

$$L(t) \in T_{p_0}(S) \implies t(\frac{x_0^2}{a^4} + \frac{y_0^2}{b^4} + \frac{z_0^2}{c^4}) = 1$$

This implies

$$t = (\frac{x_0^2}{a^4} + \frac{y_0^2}{b^4} + \frac{z_0^2}{c^4})^{-1} \text{ and } q = t(\frac{x_0}{a^2}, \frac{y_0}{b^2}, \frac{z_0}{c^2})$$

where $q \in T_{p_0}(S)$ is the point to which O is perpendicular projected.

Express $q = (x,y,z)$. Now, we can reduce the problem into proving

$$(x^2 + y^2 + z^2)^2 = a^2x^2 + b^2y^2 + c^2z^2$$

Compute

$$(x^2 + y^2 + z^2)^2 = \Big(t^2(\frac{x_0^2}{a^4} + \frac{y_0^2}{b^4} + \frac{z_0^2}{c^2})\Big)^2$$

Compute

$$a^2x^2 + b^2y^2 + c^2z^2 = t^2(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2}) = t^2 \quad (\because \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = 1)$$

We reduce the problem into proving

$$t^2(\frac{x_0^2}{a^4} + \frac{y_0^2}{b^4} + \frac{z_0^2}{c^4})^2 = 1$$

Note that $t = (\frac{x_0^2}{a^4} + \frac{y_0^2}{b^4} + \frac{z_0^2}{c^4})^{-1}$ and we are done. (done)

It remains to prove

$$E \subseteq E'$$

Fix $(x_1, y_1, z_1) \in S_1$. Let

$$r \triangleq \sqrt{a^2x_1^2 + b^2y_1^2 + c^2z_1^2}$$

We claim

$$(x_1, y_1, z_1) \text{ is the projection of } O \text{ onto the tangent plane of } S \text{ at } (\frac{a^2x_1}{r}, \frac{b^2y_1}{r}, \frac{c^2z_1}{r})$$

It is easily checked that $(\frac{a^2x_1}{r}, \frac{b^2y_1}{r}, \frac{c^2z_1}{r}) \in S$. Let $p = (\frac{a^2x_1}{r}, \frac{b^2y_1}{r}, \frac{c^2z_1}{r})$. Using (3.3), we have

$$T_p(S) = \{(x, y, z) \in \mathbb{R}^3 : \frac{xx_1 + yy_1 + zz_1}{r} = 1\}$$

It is now very clear that

$$(x_1, y_1, z_1) \text{ as a vector is perepndicular to } T_p(S)$$

and we can use the fact $(x_1, y_1, z_1) \in E$ to compute

$$\frac{x_1^2 + y_1^2 + z_1^2}{r} = \frac{\sqrt{a^2x_1^2 + b^2y_1^2 + c^2z_1^2}}{r} = 1$$

This conclude $(x_1, y_1, z_1) \in T_p(S)$. (done) ∎

Question 44

1. Compute the first fundamental forms of the following parametrized surfaces where they are regular:

a. $\mathbf{x}(u, v) = (a \sin u \cos v, b \sin u \sin v, c \cos u)$; ellipsoid.

b. $\mathbf{x}(u, v) = (au \cos v, bu \sin v, u^2)$; elliptic paraboloid.

c. $\mathbf{x}(u, v) = (au \cosh v, bu \sinh v, u^2)$; hyperbolic paraboloid.

d. $\mathbf{x}(u, v) = (a \sinh u \cos v, b \sinh u \sin v, c \cosh u)$; hyperboloid of two sheets.

Proof. Let $\alpha'(0) \in T_p(S)$ and express $\alpha(t) = \mathbf{x}(u(t), v(t))$. We have

$$\begin{aligned} I_p(\alpha'(0)) &= \alpha'(0) \cdot \alpha'(0) \\ &= \Big(u'(0)\partial_u \mathbf{x}(p) + v'(0)\partial_u \mathbf{x}(p)\Big) \cdot \Big(u'(0)\partial_u \mathbf{x}(p) + v'(0)\partial_u \mathbf{x}(p)\Big) \\ &= |\mathbf{x}_u(p)|^2 (u'(0))^2 + 2(\partial_u \mathbf{x}(p) \cdot \partial_v \mathbf{x}(p))u'(0)v'(0) + |\mathbf{x}_v(p)|^2 (v'(0))^2 \\ &\triangleq E(u'(0))^2 + 2Fu'(0)v'(0) + G(v'(0))^2 \end{aligned}$$

From now on, we compute only E, F, G.

(a) Compute

$$\begin{aligned} \partial_u \mathbf{x} &= (a \cos u \cos v, b \cos u \sin v, -c \sin u) \\ \text{and } \partial_v \mathbf{x} &= (-a \sin u \sin v, b \sin u \cos v, 0) \end{aligned}$$

This give us

$$\begin{aligned} E &= |\partial_u \mathbf{x}|^2 = \cos^2 u(a^2 \cos^2 v + b^2 \sin^2 v) + c^2 \sin^2 u \\ F &= \partial_u \mathbf{x} \cdot \partial_v \mathbf{x} = (-a^2 + b^2) \cos u \sin u \cos v \sin v \\ G &= |\partial_v \mathbf{x}|^2 = \sin^2 u(a^2 \sin^2 v + b^2 \cos^2 v) \end{aligned}$$

(b) Compute

$$\partial_u \mathbf{x} = (a\cos v, b\sin v, 2u) \text{ and } \partial_v \mathbf{x} = (-au\sin v, bu\cos v, 0)$$

This give us

$$\begin{aligned}
E &= |\partial_u \mathbf{x}|^2 = a^2\cos^2 v + b^2\sin^2 v + 4u^2 \\
F &= \partial_u \mathbf{x} \cdot \partial_v \mathbf{x} = \cos v \sin v(-a^2 + b^2)u \\
G &= |\partial_v \mathbf{x}|^2 = u^2(a^2\sin^2 v + b^2\cos^2 v)
\end{aligned}$$

(c) Compute

$$\partial_u \mathbf{x} = (a\cosh v, b\sinh v, 2u) \text{ and } \partial_v \mathbf{x} = (au\sinh v, bu\cosh v, 0)$$

This give us

$$\begin{aligned}
E &= |\partial_u \mathbf{x}|^2 = a^2\cosh^2 v + b^2\sinh^2 v + 4u^2 \\
F &= \partial_u \mathbf{x} \cdot \partial_v \mathbf{x} = (a^2 + b^2)u\cosh v \sinh v \\
G &= |\partial_v \mathbf{x}|^2 = u^2(a^2\sinh^2 v + b^2\cosh^2 v)
\end{aligned}$$

(d) Compute

$$\partial_u \mathbf{x} = (a\cosh u\cos v, b\cosh u\sin v, c\sinh u) \text{ and } \partial_v \mathbf{x} = (-a\sinh u\sin v, b\sinh u\cos v, 0)$$

This give us

$$\begin{aligned}
E &= |\partial_u \mathbf{x}|^2 = \cosh^2 u(a^2\cos^2 v + b^2\sin^2 v) + c^2\sinh^2 u \\
F &= \partial_u \mathbf{x} \cdot \partial_v \mathbf{x} = \cosh u \sinh u \cos v \sin v(b^2 - a^2) \\
G &= |\partial_v \mathbf{x}|^2 = \sinh^2 u(a^2\sin^2 v + b^2\cos^2 v)
\end{aligned}$$

■

Question 45

3. Obtain the first fundamental form of the sphere in the parametrization given by stereographic projection (cf. Exercise 16, Sec. 2-2).

Proof. We are given

$$\mathbf{x}(u,v) = \left(\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{2(u^2+v^2)}{u^2+v^2+4}\right)$$

Note that in Question 2.5.1, we have already given the complete formula of $I_p(\alpha'(0))$. We only have to compute E, F, G.

$$\partial_u \mathbf{x} = \Big(\frac{4(-u^2+v^2+4)}{(u^2+v^2+4)^2}, \frac{-8vu}{(u^2+v^2+4)^2}, \frac{16u}{(u^2+v^2+4)^2}\Big)$$
$$\partial_v \mathbf{x} = \Big(\frac{-8uv}{(u^2+v^2+4)^2}, \frac{4(-v^2+u^2+4)}{(u^2+v^2+4)^2}, \frac{16v}{(u^2+v^2+4)^2}\Big)$$

Then compute

$$E = \frac{16u^4+32u^2v^2+128u^2+16v^4+128v^2+256}{(u^2+v^2+4)^4}$$
$$F = 0$$
$$G = \frac{16v^4+32u^2v^2+128v^2+16u^4+128u^2+256}{(u^2+v^2+4)^4}$$

■

Question 46

7. The coordinate curves of a parametrization $\mathbf{x}(u, v)$ constitute a *Tchebyshef net* if the lengths of the opposite sides of any quadrilateral formed by them are equal. Show that a necessary and sufficient condition for this is

$$\frac{\partial E}{\partial v} = \frac{\partial G}{\partial u} = 0.$$

Proof. For each v, define

$$\alpha_v(t) = \mathbf{x}(t, v)$$

WOLG, we are require to show

$$\forall r \in \mathbb{R}^+, \int_0^r |\alpha_v'(t)|\, dt \text{ is a constant in } v \iff \frac{\partial E}{\partial v} = 0$$

$(\longleftarrow)$

Note that we have

$$u'(t) = 1 \text{ and } v'(t) = 0 \text{ if we write } \alpha_v(t) = \mathbf{x}(u(t), v(t))$$

Then, we have

$$\int_0^C |\alpha_v'(t)|\, dt = \int_0^C \sqrt{E(t,v)} dt$$

If $\frac{\partial E}{\partial v} = 0$, then for each v_1, v_2, we clearly have

$$\int_0^r \left|\alpha_{v_1}'(t)\right| dt = \int_0^r \sqrt{E(t,v_1)} dt = \int_0^r \sqrt{E(t,v_2)} dt = \int_0^r \left|\alpha_{v_2}'(t)\right| dt$$

$(\longrightarrow)$

Suppose for all $r \in \mathbb{R}^+$, the function $\int_0^r |\alpha_v'(t)|\, dt$ is a constant in v. We then can define $f : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$f(r) = \int_0^r |\alpha_v'(t)|\, dt \text{ for all } v$$

Differentiating f, we see

$$f'(r) = \sqrt{E(r,v)} \text{ for all } v$$

This tell us

$$E(r,v) \text{ is a constant in } v \text{ for all } r$$

which implies

$$\frac{\partial E}{\partial v} = 0 \text{ (done)}$$

$\blacksquare$

Question 47

***8.** Prove that whenever the coordinate curves constitute a Tchebyshef net (see Exercise 7) it is possible to reparametrize the coordinate neighborhood in such a way that the new coefficients of the first fundamental form are

$$E = 1, \quad F = \cos\theta, \quad G = 1,$$

where θ is the angle of the coordinate curves.

Proof. Note that $\frac{\partial E}{\partial v} = \frac{\partial G}{\partial u} = 0$ implies

$$E \text{ stay constant in } v \text{ and } G \text{ stay constant in } u$$

In other words, E can be treated as a function of u and G can be treated as a function of v. Let

$$\overline{u}(u) = \int_0^u \sqrt{E(t)}dt \text{ and } \overline{v}(v) = \int_0^v \sqrt{G(t)}dt$$

Reparametrize by

$$\mathbf{x}(u(\overline{u}), v(\overline{v}))$$

By Chain Rule and Single-Variable Inverse Function Theorem, we have

$$\partial_{\overline{u}}\mathbf{x} = \frac{1}{\sqrt{E(u)}}\partial_u \mathbf{x} \text{ and } \partial_{\overline{v}}\mathbf{x} = \frac{1}{\sqrt{G(v)}}\partial_v \mathbf{x}$$

This give us

$$\overline{E} = \partial_{\overline{u}}\mathbf{x} \cdot \partial_{\overline{u}}\mathbf{x} = \frac{E(u)}{E(u)} = 1 \text{ and } \overline{G} = \partial_{\overline{v}}\mathbf{x} \cdot \partial_{\overline{v}}\mathbf{x} = \frac{G(v)}{G(v)} = 1$$

Now, by CS-inequality, we know $\overline{F} = \partial_{\overline{u}}\mathbf{x} \cdot \partial_{\overline{v}}\mathbf{x} \in (-1, 1)$. Then there must exists θ such that $\overline{F} = \cos\theta$. ■

Question 48

11. Let S be a surface of revolution and C its generating curve (cf. Example 4, Sec. 2-3). Let s be the arc length of C and denote by $\rho = \rho(s)$ the distance to the rotation axis of the point of C corresponding to s.

a. (*Pappus' Theorem.*) Show that the area of S is

$$2\pi \int_0^l \rho(s)\, ds,$$

where l is the length of C.

b. Apply part a to compute the area of a torus of revolution.

Proof. **(a)**

Because S is a surface of revolution, we have an almost global chart

$$\mathbf{x}(\theta, s) = (\cos\theta f(s), \sin\theta f(s), g(s))$$

where

$$f(s) = \rho(s) \text{ and } (f')^2 + (g')^2 = 1$$

Compute

$$\partial_\theta \mathbf{x} = (-\sin\theta f(s), \cos\theta f(s), 0) \text{ and } \partial_s \mathbf{x} = (\cos\theta f'(s), \sin\theta f'(s), g'(s))$$

This let us compute

$$\begin{aligned} E &= |\partial_\theta \mathbf{x}|^2 = |f(s)|^2 = \rho(s)^2 \\ F &= \partial_\theta \mathbf{x} \cdot \partial_s \mathbf{x} = 0 \\ G &= |\partial_s \mathbf{x}|^2 = (f'(s))^2 + (g'(s))^2 = 1 \end{aligned}$$

Now we see that the area $A(S)$ of S is exactly

$$A(S) = \int_0^{2\pi} \int_0^l \sqrt{EG - F^2} ds d\theta = 2\pi \int_0^l \rho(s) ds$$

(b)
Note that

$$\big(f(s), g(s)\big) \triangleq (a + r\cos\frac{s}{r}, a + r\sin\frac{s}{r}) \text{ where } f(s) \equiv \rho(s)$$

satisfy all the condition. Then we can compute the surface area of the torus by

$$2\pi \int_0^{2\pi r} (a + r\cos(\frac{s}{r}))ds = 4\pi^2 ra$$

■

Question 49

14. (*Gradient on Surfaces.*) The *gradient* of a differentiable function $f\colon S \to R$ is a differentiable map grad $f\colon S \to R^3$ which assigns to each point $p \in S$ a vector grad $f(p) \in T_p(S) \subset R^3$ such that

$$\langle \operatorname{grad} f(p), v\rangle_p = df_p(v) \quad \text{for all } v \in T_p(S).$$

Show that

a. If E, F, G are the coefficients of the first fundamental form in a parametrization $\mathbf{x}\colon U \subset R^2 \to S$, then grad f on $\mathbf{x}(U)$ is given by

$$\operatorname{grad} f = \frac{f_u G - f_v F}{EG - F^2}\mathbf{x}_u + \frac{f_v E - f_u F}{EG - F^2}\mathbf{x}_v.$$

In particular, if $S = R^2$ with coordinates x, y,

$$\operatorname{grad} f = f_x e_1 + f_y e_2,$$

where $\{e_1, e_2\}$ is the canonical basis of R_2 (*thus, the definition agrees with the usual definition of gradient in the plane*).

b. If you let $p \in S$ be fixed and v vary in the unit circle $|v| = 1$ in $T_p(s)$, then $df_p(v)$ is maximum if and only if $v = \operatorname{grad} f/|\operatorname{grad} f|$ (*thus*, grad $f(p)$ *gives the direction of maximum variation of f at p*).

c. If grad $f \neq 0$ at all points of the *level curve* $C = \{q \in S; f(q) = \text{const.}\}$, then C is a regular curve on S and grad f is normal to C at all points of C.

Proof. (a)

Suppose $\nabla f = \frac{f_u G - f_v F}{EG - F^2}\mathbf{x}_u + \frac{f_v E - f_u F}{EG - F^2}$. First observe

$$\begin{aligned}
\langle \nabla f, \mathbf{x}_u \rangle &= \frac{f_u G - f_v F}{EG - F^2} E + \frac{f_v E - f_u F}{EG - F^2} F \\
&= \frac{f_u (GE - F^2)}{EG - F^2} \\
&= f_u = \frac{\partial}{\partial u}(f \circ \mathbf{x}) = df(\mathbf{x}_u)
\end{aligned}$$

The justification of the last inequality is as followed. Define

$$\alpha(t) = \mathbf{x}(u_0 + t, v_0)$$

We have

$$\alpha(0) = \mathbf{x}(u_0, v_0) \text{ and } \alpha'(0) = d\mathbf{x}_{(u_0,v_0)}(1, 0) = \mathbf{x}_u(u_0, v_0)$$

Now

$$df_{\mathbf{x}(u_0,v_0)}(\mathbf{x}_u(u_0, v_0)) \overset{\text{def}}{=} \frac{d}{dt}(f \circ \alpha(t))\Big|_{t=0} = \frac{d}{dt}\big((f \circ \mathbf{x})(u_0 + t, v_0)\big) = \frac{\partial}{\partial u}(f \circ \mathbf{x})(u_0, v_0)$$

This justified the $\langle \nabla f, \mathbf{x}_u \rangle = df(\mathbf{x}_u)$.

Similarly, we have

$$\langle \nabla f, \mathbf{x}_v \rangle = df(\mathbf{x}_v)$$

Now, for all $w \in T_p(S)$, we see that

$$\begin{aligned}
\langle \nabla f(p), w \rangle &= \langle \nabla f(p), c_u \mathbf{x}_u(p) + c_v \mathbf{x}_v(p) \rangle \quad (\text{ for a unique pair } c_u, c_v \in \mathbb{R} \text{ }) \\
&= c_u \langle \nabla f(p), \mathbf{x}_u(p) \rangle + c_v \langle \nabla f(p), \mathbf{x}_v(p) \rangle \\
&= c_u df_p(\mathbf{x}_u(p)) + c_v df_p(\mathbf{x}_v(p)) \\
&= df_p(c_u \mathbf{x}_u(p) + c_v \mathbf{x}_v(p)) = df_p(w)
\end{aligned}$$

If $S = \mathbb{R}^2$ with coordinates x, y, we can easily compute

$$\mathbf{x}_x = (1, 0) \text{ and } \mathbf{x}_y = (0, 1)$$

and

$$E = 1 \text{ and } F = 0 \text{ and } G = 1$$

Then from the formula of ∇f we just derived, we have

$$\nabla f = f_x(1,0) + f_y(0,1) \equiv f_x e_1 + f_y e_2$$

(b)

By C-S inequality, we see that

$$df_p(v) \equiv \langle \nabla f(p), v \rangle \text{ reach maximum if and only if } v = c_0 \nabla f(p) \text{ for some positive } c_0$$

It then come very clear, under the constraint $|v| = 1$, that c_0 must be $\frac{1}{|\nabla f(p)|}$.

(c)

Let that constant be c_0, and let $p = \mathbf{x}(q) \in C$. Define $g : \mathbb{R}^{2+1} \to \mathbb{R}$ by

$$g(u,v,t) = f(\mathbf{x}(u,v)) - t$$

Check that

$$\partial_t g = -1 \neq 0$$

Then by Implicit function theorem, there exists a function $h : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ such that

$$h(q) = c_0 \text{ and } g(u,v,h(u,v)) = 0 \text{ for all } (u,v) \in U$$

We now see

$$C = (h \circ \mathbf{x}^{-1})^{-1}(c_0) \text{ which is a regular preimage}$$

This established that C is a regular curve.

Locally parametrize $\gamma(t) \subseteq C$. Because $f \circ \gamma$ stay constant, we see

$$0 = df_{\gamma(t)}(\gamma'(t)) = \langle \nabla f(\gamma(t)), \gamma'(t) \rangle$$

This then implies $\nabla f(\gamma(t))$ is perpendicular to $\gamma'(t)$, thus perpendicualr to C. ■