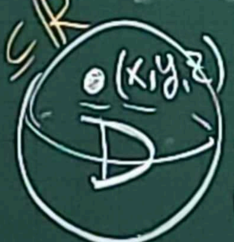
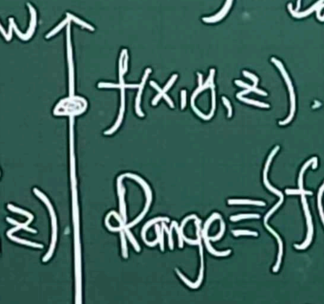
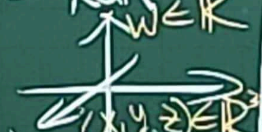
Def. A fn f of 3 variables $f: \underset{\subset \mathbb{R}^3}{\text{Dom}} \longrightarrow \underset{\subset \mathbb{R}}{\text{Ran}}$



$\xrightarrow{f}$

$w = f(x,y,z)$



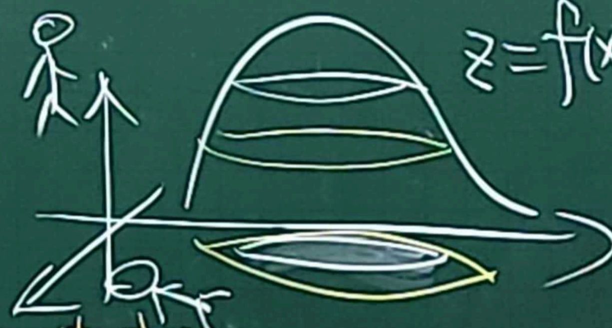


Range $= \{ f(x,y,z) \mid (x,y,z) \in D \}$

Def A fn of n variables

$f: \underset{\subseteq \mathbb{R}^n}{\text{Dom}} \longrightarrow \underset{\subseteq \mathbb{R}}{\text{Range}}$.

$(x_1, x_2, \cdots, x_n) \longmapsto y = f(x_1, \cdots, x_n)$.



⟨Ex⟩ Find domain and Range of $f(x,y,z) = \ln(z-y) + xy\sin z + \sqrt{4-x^2-y^2-z^2}$

⟨Ex⟩ Find the level surfaces of the fn $w = f(x,y,z) = x^2+y^2+z^2$

Graph $f(x,y,z) = k$ (constant)

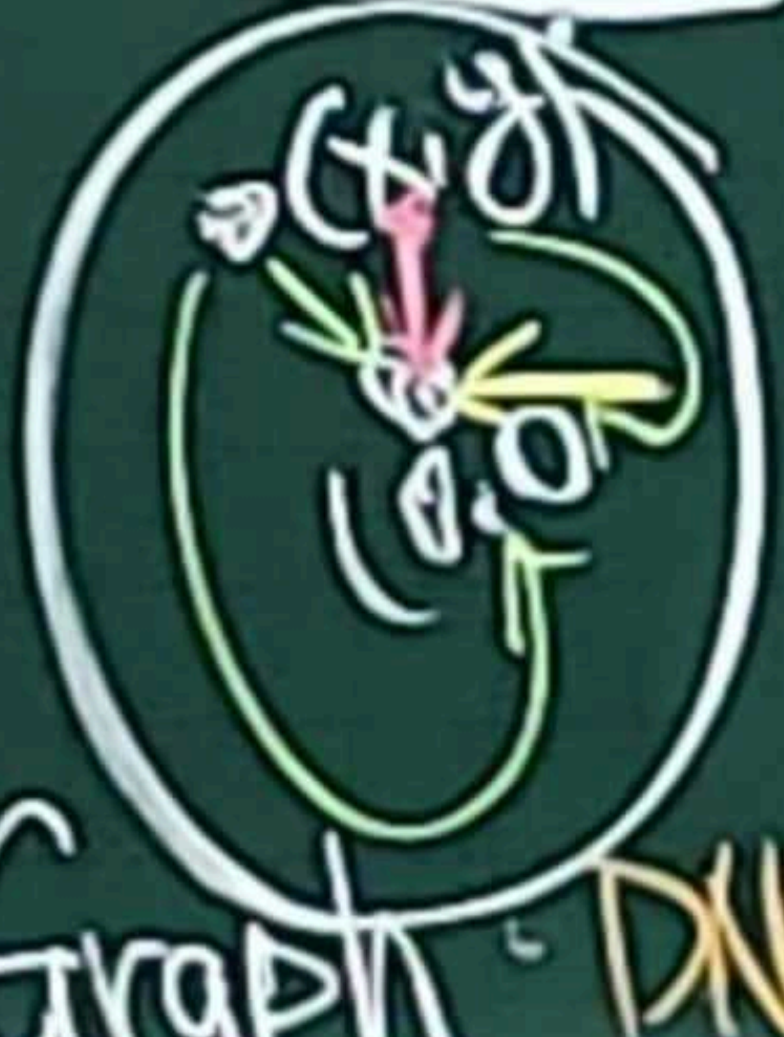
surface

§14.2 Limits and Continuity

⟨Ex⟩ $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2} \overset{?}{=} \lim\limits_{z\to 0} \dfrac{\sin z}{z}$

$\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2} =$

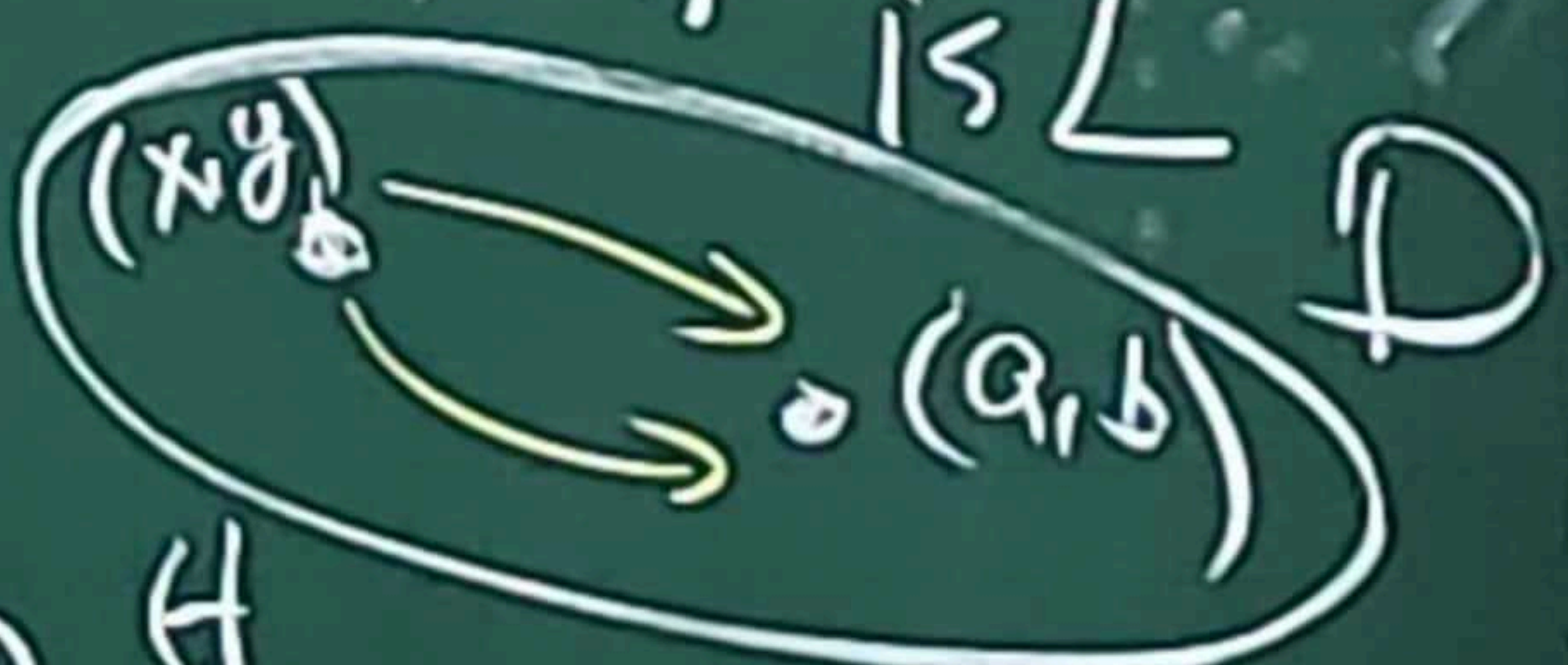
Graph



$x=0$, $\lim\limits_{y\to 0} \dfrac{-y^2}{y^2} = -1$

$y=0$, $\lim\limits_{x\to 0} \dfrac{x^2}{x^2} = 1$ Graph DNE

Def The limit of $f(x,y)$ as (x,y) approaches (a,b) is L



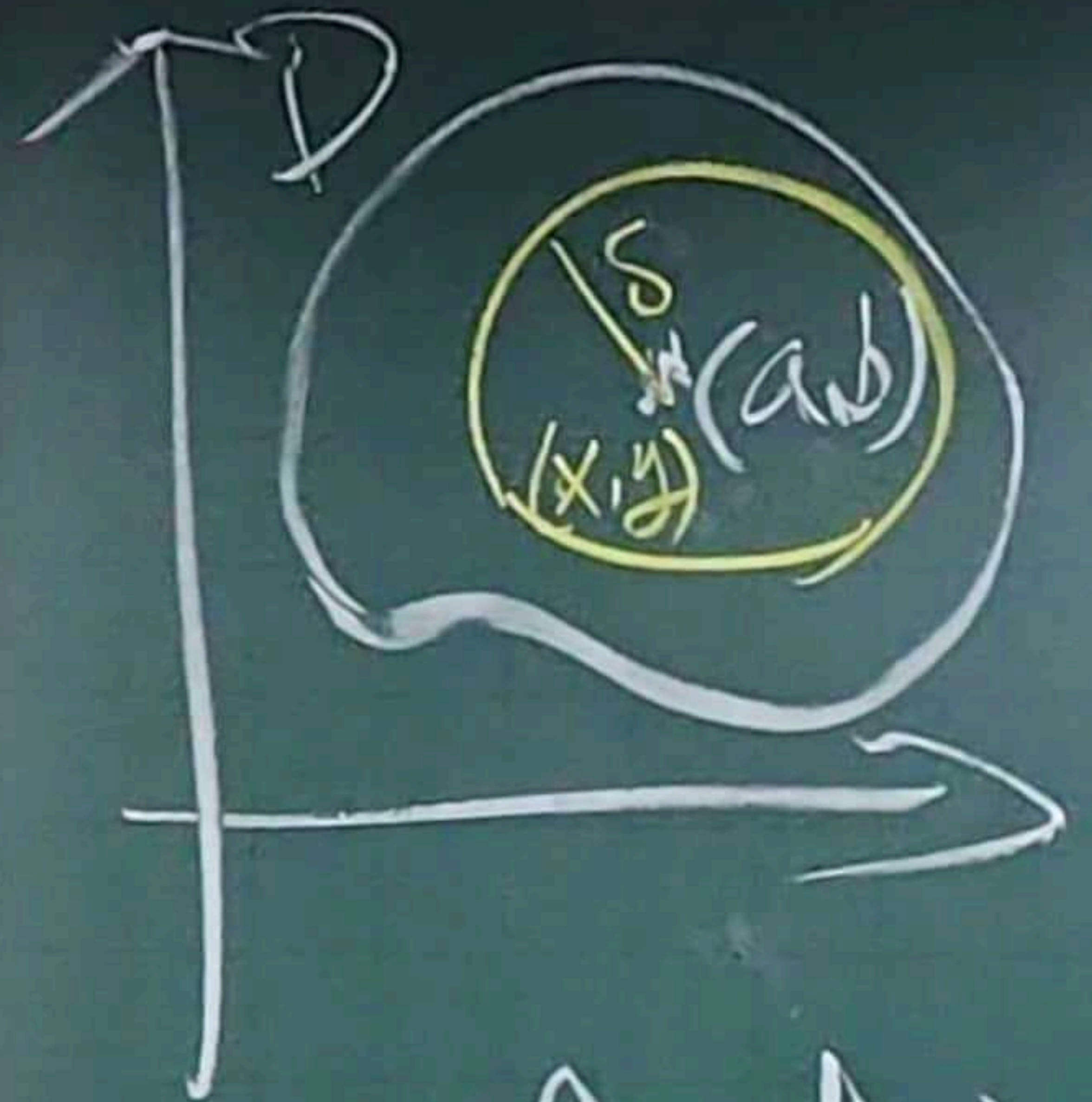
$\Longleftrightarrow \lim\limits_{(x,y)\to(a,b)} f(x,y) = L$

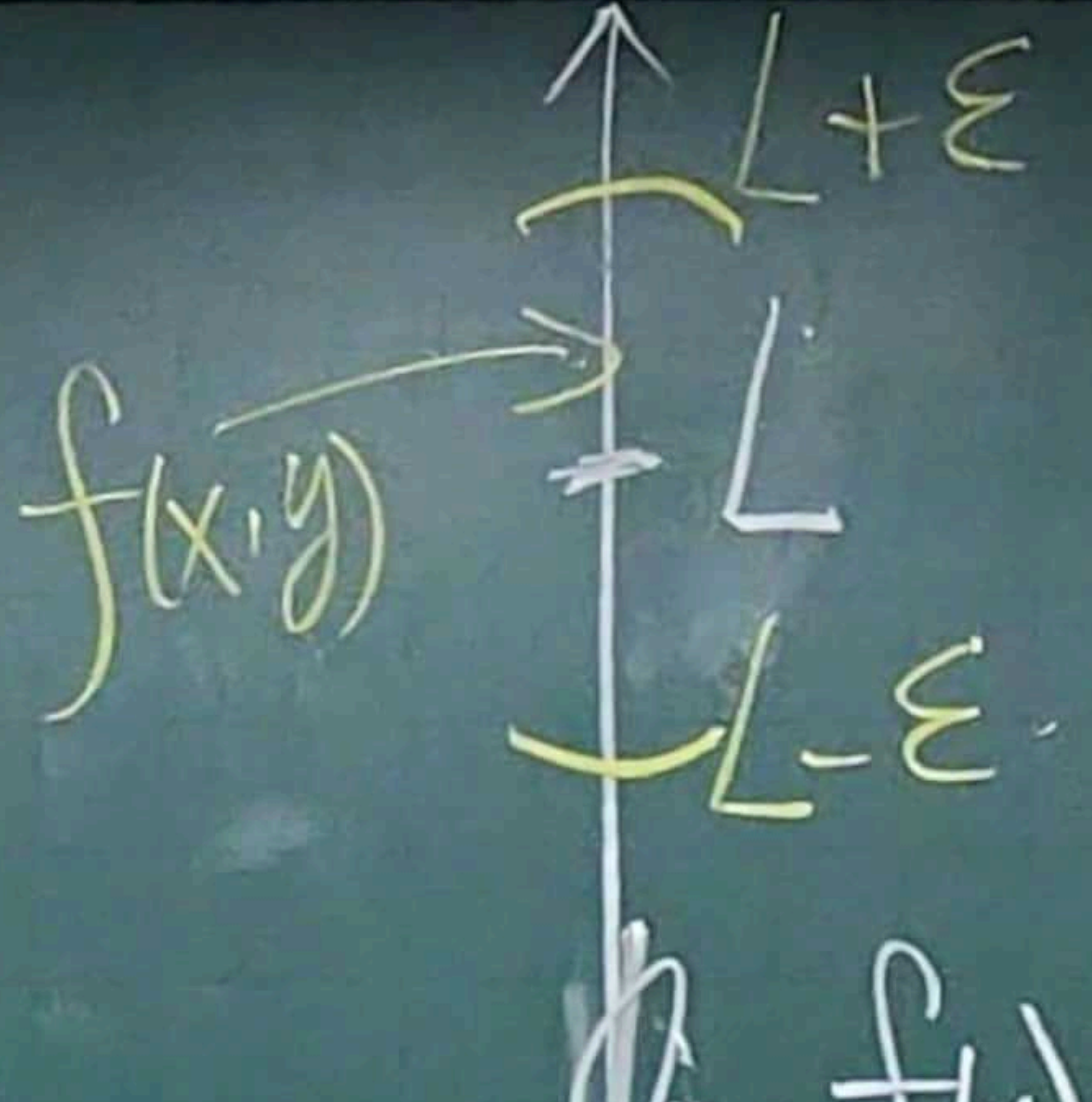
$\Longleftrightarrow$ if for every $\varepsilon>0$, $\exists\, \delta>0$, s.t.

if $(x,y)\in D$ and $0<|(x,y)-(a,b)|<\delta \Longrightarrow |f(x,y)-L|<\varepsilon$

$\Longleftrightarrow \lim\limits_{\substack{x\to a\\ y\to b}} f(x,y) = L$. $\qquad |(x-a,y-b)| \equiv \sqrt{(x-a)^2+(y-b)^2}$

$\Longleftrightarrow f(x,y)\to L$ as $(x,y)\to(a,b)$





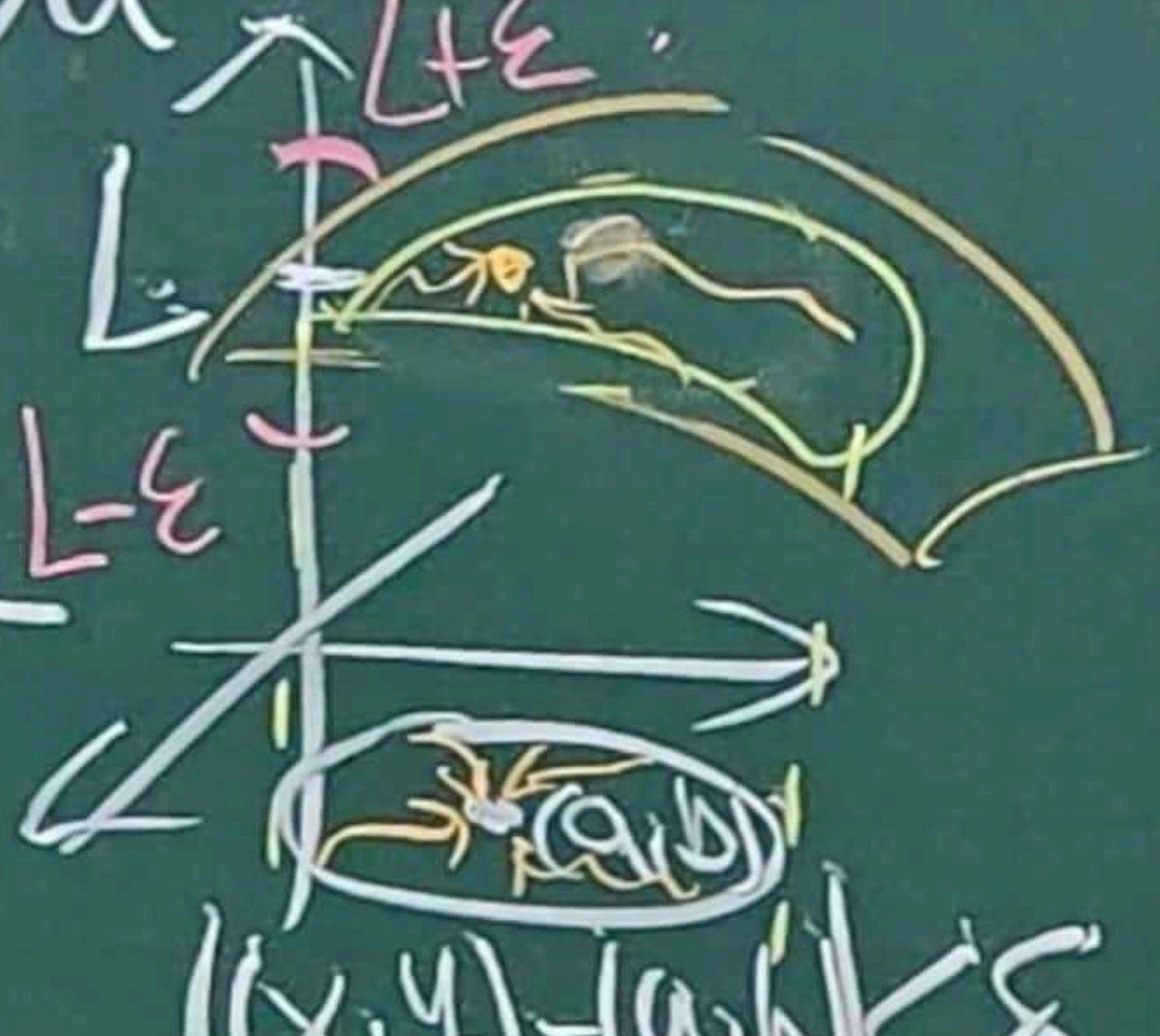
Prop. If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along C_1

$f(x,y) \to L_2$ 〃 〃 C_2

and $L_1 \neq L_2 \Rightarrow \lim_{(x,y)\to(a,b)} f(x,y)$ DNE

Recall $\lim_{x\to a} f(x) = L \Leftrightarrow \lim_{x\to a^{\pm}} f(x) = L$.

For 2D, $\lim_{(x,y)\to(a,b)} f(x,y) = L$



⟨Ex⟩ $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$

$\left|\frac{3x^2y}{x^2+y^2}\right| = 3\cdot \underbrace{\frac{x^2}{x^2+y^2}}_{\leq 1}|y| \leq 3|y| \xrightarrow[y\to 0]{} 0$

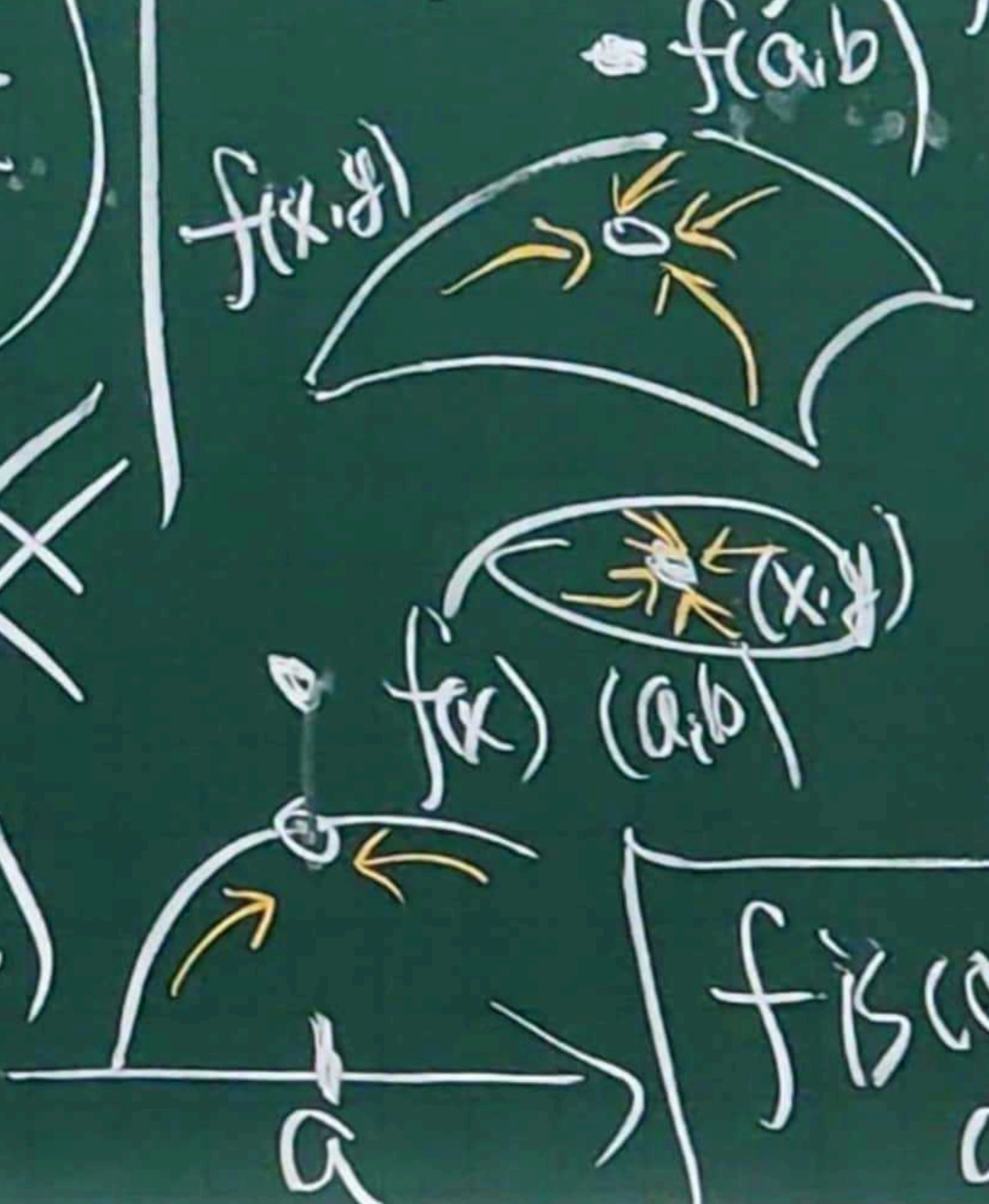
By squeezing thm,

$$|(x,y)-(0,0)| = \sqrt{x^2+y^2} < \delta \quad \left(\text{Choose } \delta = \frac{\varepsilon}{3}\right)$$

$$\left|\frac{3x^2y}{x^2+y^2} - \underset{=0}{L}\right| = 3\underbrace{\frac{x^2}{x^2+y^2}}_{\le 1}|y|$$

$$\le 3|y| \le 3\sqrt{x^2+y^2} < \varepsilon \quad (<\varepsilon)$$

Def A fn $f(x,y)$ is called continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$



$$(x_1,\dots,x_n)\to(a_1,\dots,a_n) \iff \sqrt{(x_1-a_1)^2+\dots+(x_n-a_n)^2}\to 0$$

Recall $f(x)$ = conti. at a if $\lim_{x\to a} f(x) = f(a)$

f is continuous on D if f = conti. at every $(x,y)\in D$

(Ex) polynomial fns = conti.

A rational fns $\frac{P(x,y)}{g(x,y)}$ = conti. except $g(x,y) \neq 0$

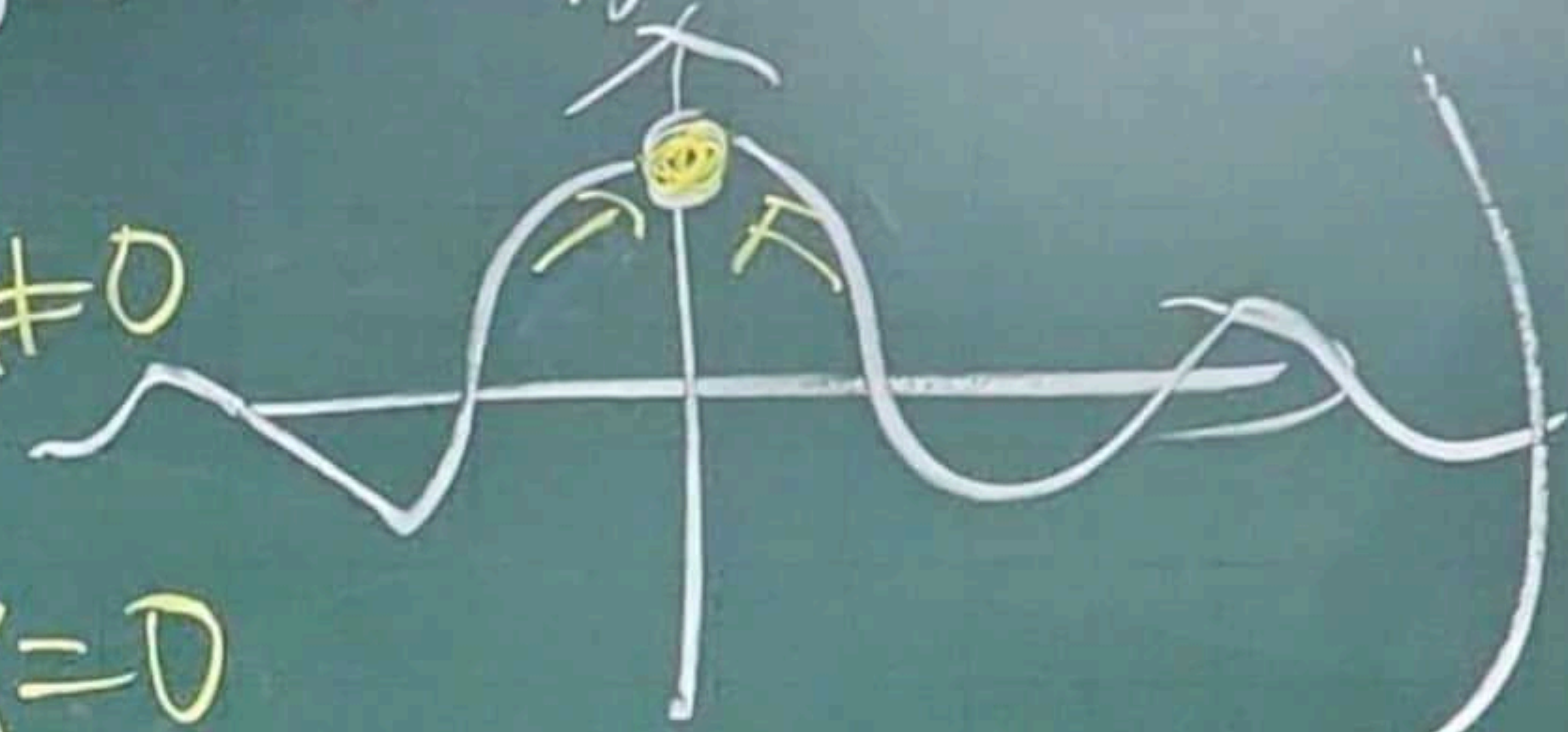
(Ex) $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$

$\because \lim_{(x,y)\to(0,0)} f(x,y)$ DNE

$\Rightarrow f(x,y)$ = conti. in $D = \{(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}\}$.

(Ex) $f(x,y) = \begin{cases} \frac{3x^2y}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$

Eg $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$



$f(x,y)$ = conti. $\forall (x,y) \neq (0,0)$

$\lim_{(x,y)\to(0,0)} f(x,y) \overset{\text{check}}{=} 0 \overset{\text{by def}}{=} f(0,0)$ Thus f = conti. at $(x,y) = (0,0)$

$\therefore f$ = conti. on $\mathbb{R}^2$.

Def A fn $f(x,y,z)$ has a limit L as $(x,y,z) \to (a,b,c)$

$$\Longleftrightarrow \lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = L.$$

$\Longleftrightarrow$ For every $\varepsilon > 0$, $\exists\, \delta > 0$, s.t.

if $(x,y,z) \in$ Domain of f and $0 < \sqrt{(x-a)^2+(y-b)^2+(z-c)^2} < \delta$

then $|f(x,y,z) - L| < \varepsilon$

$|\vec{x}-\vec{a}| < \delta$

Def. The fn f is continuous at $\vec{a}$

if $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$.

§14.3 Partial Derivatives

T \ H	40	50	60
26			
28			
30			

$f(T,60) = g(T)$

$f(30,H) = G(H)$

T = temperature, H = Humidity

$I = f(T,H)$ humidex

$$g'(30) = \lim_{h\to 0} \frac{g(30+h)-g(30)}{h} = \lim_{h\to 0} \frac{f(30+h,60)-f(30,60)}{h}$$

$$\approx \frac{g(32)-g(30)}{2} = \frac{f(32,60)-f(30,60)}{2} = ①$$

or

$$\approx \frac{g(30)-g(28)}{2} = \frac{f(30,60)-f(28,60)}{2} = ②$$

$$g'(30) \approx \frac{1}{2}(① + ②) \approx 1.75$$

At $(T,H) = (30,60)$

$T \nearrow 1^\circ C \Rightarrow I \nearrow 1.75^\circ C$

Analogously compute $G'(60)$.

$\Rightarrow G'(60) \approx 0.3$

At $(T, H) = (30, 60)$ $H \uparrow 1\% \Rightarrow I \uparrow 0.3°C$

Def: partial derivatives of f w.r.t. x at (a, b)

$$f_x(a,b) = g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h} = \lim_{h \to 0} \frac{f(a+h, b) - f(a,b)}{h}$$

$f(x, b) \equiv g(x)$

Partial derivative of f w.r.t. y at (a, b)

$$f_y(a,b) = G'(b) = \lim_{h \to 0} \frac{G(b+h) - G(b)}{h}$$

$f(a, y) = G(y)$

$$= \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h}$$

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x}$$

$$= f_1 = D_1 f = D_x f = \partial_x f$$

$z = f(x, y)$