RTFT HW5

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Problem A.

Notice that $\forall x \in Q_8, |x^{Q_8}| = \frac{|Q_8|}{|C_{Q_8}(x)|} = \frac{8}{|C_{Q_8}(x)|}$ and $\langle x \rangle \leq C_{Q_8}(x) \leq Q_8$ and $|\langle i \rangle| = |\langle j \rangle| = |\langle k \rangle| = 4$.

From the above fact, we can deduce ($|C_{Q_8}(i)| = 4$ and $|i^{Q_8}| = 2$) or ($|C_{Q_8}(i)| = 8$ and or $|i^{Q_8}| = 1$).

Notice that $jij^{-1} = (-k)(-j) = -i$, so $-i \in i^{Q_8}$, which enable us to deduce $i^{Q_8} = \{\pm i\}$.

From the above fact, we can deduce ($|C_{Q_8}(j)| = 4$ and $|j^{Q_8}| = 2$) or ($|C_{Q_8}(j)| = 8$ and or $|j^{Q_8}| = 1$).

Notice that $iji^{-1} = k(-i) = -j$, so $-j \in i^{Q_8}$, which enable us to deduce $j^{Q_8} = \{\pm j\}$

From the above fact, we can deduce ($|C_{Q_8}(k)| = 4$ and $|k^{Q_8}| = 2$) or ($|C_{Q_8}(k)| = 8$ and or $|k^{Q_8}| = 1$).

Notice that $jkj^{-1} = i(-j) = -k$, so $-k \in k^{Q_8}$, which enable us to deduce $k^{Q_8} = \{\pm k\}$

$Q_8 \setminus (i^{Q_8} \cup j^{Q_8} \cup k^{Q_8}) = \{\pm e\}$

Notice $\{e\}$ is a conjuacy class.

So the conjuacy classes of Q_8 is $Q_8 = \{\pm i\} \cup \{\pm j\} \cup \{\pm k\} \cup \{e\} \cup \{-e\}$

Problem B

In this problem, we use $\langle a, b | ab = ba, a^2 = b^2 = e \rangle$ as our notation for $V_4 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$

We first decompose the regular $\mathbb{C}V_4$-module **(from now denoted V)** into product of irreducible submodules.

Notice that V_4 is abelian, so we know all irreducible $\mathbb{C}G$-submodules are one-dimensional. In other word, we will see $V = \text{span}(v_1) \oplus \text{span}(v_2) \oplus \text{span}(v_3) \oplus \text{span}(v_4)$, where $\text{span}(v_i)$ are submodule for $1 \leq i \leq 4$

Notice that we only have to show that $av_i = \lambda_1 v_i$ and $bv_i = \lambda_2 v_i, \exists \lambda_1, \lambda_2 \in \mathbb{C}$, then the fact that $ab(v_i) = \lambda_1\lambda_2 v_i$ and span(v_i) is a submodule follow immediately.

Obseve the following

$$a(e+a+b+ab) = e+a+b+ab \tag{1}$$

$$b(e+a+b+ab) = e+a+b+ab \tag{2}$$

So we know span$(e+a+b+ab)$ is a submodule of $\mathbb{C}G$.

Observe the following

$$a(e-a+b-ab) = -(e-a+b-ab) \tag{3}$$

$$b(e-a+b-ab) = e-a+b-ab \tag{4}$$

So we know span$(e-a+b-ab)$ is a submodule of $\mathbb{C}G$.

Observe the following

$$a(e+a-b-ab) = e+a-b-ab \tag{5}$$

$$b(e+a-b-ab) = -(e+a-b-ab) \tag{6}$$

So we know span$(e+a-b-ab)$ is a submodule of $\mathbb{C}G$.

Observe the following

$$a(e-a-b+ab) = -(e-a-b+ab) \tag{7}$$

$$b(e-a-b+ab) = -(e-a-b+ab) \tag{8}$$

So we know span$(e-a-b+ab)$ is a submodule of $\mathbb{C}G$.

Let $v_1 = e+a+b+ab$ and $v_2 = e-a+b-ab$ and $v_3 = e+a-b-ab$ and $v_4 = e-a-b+ab$.

The four vector v_1, v_2, v_3, v_4 are linearly independent because the matrix $[v_1, v_2, v_3, v_4]_E$ where $E = \{e, a, b, ab\}$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \tag{9}$$

have determinant 16.

So we can write $V =$ span$(v_1) \oplus$ span$(v_2) \oplus$ span$(v_3) \oplus$span(v_4).

We know $\{v_i\}$ is a basis of span(v_i), and we denote $\alpha_i = \{v_i\}$.

Notice that if $\frac{av_i}{v_i} = 1 \neq -1 = \frac{av_j}{v_j}$, then we see that T :span$(v_i) \to$span(v_j) defined by $v_i \mapsto \lambda v_j$ is **never** an $\mathbb{C}V_4$-isomorphism, since $T(av_i) = \lambda v_j \neq -\lambda v_j = aT(v_i)$. Combined with the same argument on b, we see none of span(v_i) are isomorphic to any of each other. Because all of them are of one-dimensional, we can observe the equation (1) to (8) and write the following table

	e	a	b	ab
χ_1	1	1	1	1
χ_2	1	-1	1	-1
χ_3	1	1	-1	-1
χ_4	1	-1	-1	1

Problem C

1.

Let $n =$dim(V).

Find a basis α for V such that $[g]_\alpha$ is a diagonal matrix, and denote

$$[g]_\alpha = \begin{bmatrix} \omega_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \omega_n \end{bmatrix}$$

Because $[gx]_\alpha = [xg]_\alpha$, we see $\forall i, j, \omega_i([x]_\alpha)_{i,j} = \omega_j([x]_\alpha)_{i,j}$. Then $\forall i, j, \omega_i = \omega_j$

So we can write $[g]_\alpha = \omega_i I_n, \exists i$

Then $|\chi(g)| = |n\omega_i| = n$

2.

Because χ is an irreducible character, we know $\langle \chi, \chi \rangle = 1$. That is $1 = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\chi(g)}$.

$$1 = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\chi(g)} \implies 1 = \frac{1}{|G|} \sum_{g \in G \setminus Z(G)} \chi(g)\overline{\chi(g)} + \frac{1}{|G|} \sum_{g \in Z(G)} \chi(g)\overline{\chi(g)} \tag{10}$$

Then

$$|G| = \sum_{g \in G \setminus Z(G)} \chi(g)\overline{\chi(g)} + \sum_{g \in Z(G)} \chi(g)\overline{\chi(g)} = \sum_{g \in G \setminus Z(G)} \chi(g)\overline{\chi(g)} + |Z(G)|n^2 \tag{11}$$

$$|G| \geq |Z(G)|n^2 \implies n^2 \leq \frac{|G|}{|Z(G)|} \tag{12}$$

Problem D

Notice that the rows of the character table form an orthonormal basis of space of class functions, so we deduce the following equations

$$\frac{\alpha_1^2+\alpha_2^2}{12}+\frac{\alpha_3^2+\alpha_4^2}{6}+\frac{\alpha_5^2+\alpha_6^2}{4}=1 \tag{13}$$

$$\frac{\beta_1^2+\beta_2^2}{12}+\frac{\beta_3^2+\beta_4^2}{6}+\frac{\beta_5^2+\beta_6^2}{4}=1 \tag{14}$$

$$\frac{\alpha_1\beta_1+\alpha_2\beta_2}{12}+\frac{\alpha_3\beta_3+\alpha_4\beta_4}{6}+\frac{\alpha_5\beta_5+\alpha_6\beta_6}{4}=0 \tag{15}$$

$$(\alpha_1+\alpha_2)+2(\alpha_3+\alpha_4)+3(\alpha_5+\alpha_6)=0 \tag{16}$$

$$(\alpha_1-\alpha_2)+2(-\alpha_3+\alpha_4)+3i(\alpha_5-\alpha_6)=0 \tag{17}$$

$$(\alpha_1+\alpha_2)+2(\alpha_3+\alpha_4)+3(-\alpha_5-\alpha_6)=0 \tag{18}$$

$$(\alpha_1-\alpha_2)+2(-\alpha_3+\alpha_4)+3i(-\alpha_5+\alpha_6)=0 \tag{19}$$

$$(\beta_1+\beta_2)+2(\beta_3+\beta_4)+3(\beta_5+\beta_6)=0 \tag{20}$$

$$(\beta_1-\beta_2)+2(-\beta_3+\beta_4)+3i(\beta_5-\beta_6)=0 \tag{21}$$

$$(\beta_1+\beta_2)+2(\beta_3+\beta_4)+3(-\beta_5-\beta_6)=0 \tag{22}$$

$$(\beta_1-\beta_2)+2(-\beta_3+\beta_4)+3i(-\beta_5+\beta_6)=0 \tag{23}$$

From equation (16)−(18), we can tell $\alpha_6=-\alpha_5$, and from equation (17)−(19), we can tell $\alpha_6=\alpha_5$. So we know $\alpha_5=\alpha_6=0$

Then from equation (16)−(17), we know $\alpha_2=-2\alpha_3$ and from equation (16)+(17), we know $\alpha_1=-2\alpha_4$.

Then from equation (13), we know $\frac{\alpha_3^2+\alpha_4^2}{2}=1$.

From equation (20)−(22), we can tell $\beta_6=-\beta_5$, and from equation (21)−(23), we can tell $\beta_6=\beta_5$. So we know $\beta_5=\beta_6=0$

Then from equation (20)−(21), we know $\beta_2=-2\beta_3$ and from equation ()+(17), we know $\beta_1=-2\beta_4$.

Then from equation (14), we know $\frac{\beta_3^2+\beta_4^2}{2}=1$.

Then from equation (15), we know $\frac{\alpha_3\beta_3+\alpha_4\beta_4}{2}=0$.

To solve the quadratic equation $\begin{cases}\alpha_3^2+\alpha_4^2=2\\ \beta_3^2+\beta_4^2=2\\ \alpha_3\beta_3+\alpha_4\beta_4=0\end{cases}$

We simply transfer it to the language of $\|(\alpha_3, \alpha_4)\| = \sqrt{2} = \|(\beta_3, \beta_4)\|$ and $(\alpha_3, \alpha_4) \perp (\beta_3, \beta_4)$ and realize that χ_5 and χ_6 are linearly independent and $\alpha_1, \alpha_2 \in \mathbb{R}^+$, to have $\chi_5 = \begin{bmatrix} 2 \\ 2 \\ -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}^t$

$\chi_6 = \begin{bmatrix} 2 \\ -2 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}^t$