Recall $e = \lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}$ (*)

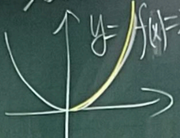
$\approx 2.7182\cdots$

⟨Ex⟩ 100^{99}, 99^{100} which one is bigger?

Sol $\frac{100^{99}}{99^{100}} = \frac{(99+1)^{99}}{99^{99}\cdot 99}$

big $= \left(1+\frac{1}{99}\right)^{99}\cdot\frac{1}{99} \underset{(*)}{\overset{n=99}{\approx}} 2.72\cdot\frac{1}{99} < 1$

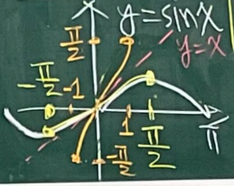
§6.6 Inverse Trigonometric Fns



$y=f(x)=x^2$, $f|_{\mathbb{R}}$ has no inverse.

$\Rightarrow f|_{[0,\infty)}$ is 1-1, hence

f has an inverse, $g(x)=\sqrt{x}$



$y=\sin x|_{[-\frac{\pi}{2},\frac{\pi}{2}]}$ is 1-1 $\Rightarrow \exists$ inverse

$\sin^{-1}x = y \Leftrightarrow \sin y = x$

$-\frac{\pi}{2}\le y\le\frac{\pi}{2}$, $-1\le x\le 1$

Cancelation: $\sin^{-1}(\sin x)=x$, $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$

$\sin(\sin^{-1}x)=x$, $-1\le x\le 1$

$\sin^{-1}$ = arcsine

For derivative, let $y=\sin^{-1}x$ (*)

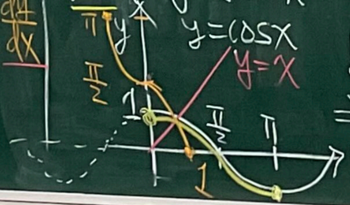
$\frac{d}{dx}\sin y = \frac{d}{dx}\sin(\sin^{-1}x) = \frac{d}{dx}x = 1$ Find $\frac{dy}{dx}$

Chain Rule $= \left(\frac{d}{dy}\sin y\right)\cdot\left(\frac{dy}{dx}\right) = \cos y\cdot\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} \underset{\sin y = x}{\overset{(*)}{=}} \frac{1}{\sqrt{1-x^2}}$, $-1<x<1$

Inverse cosine fn (arc cosine fn)

$\cos^{-1}x = y \Leftrightarrow x=\cos y$ and $0\le y\le\pi$



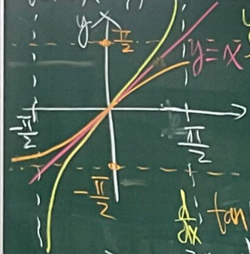
$y=\cos x|_{[0,\pi]}$ 1-1

$\Rightarrow \cos^{-1}x = y$

$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$

Inverse tangent fn (or arctan fn)

$$\tan^{-1}x = y \iff \underline{x = \tan y},\ -\frac{\pi}{2}<y<\frac{\pi}{2}$$



$$\frac{d}{dx}\tan y = \frac{d}{dx}\cancel{\tan\tan^{-1}}x = 1$$

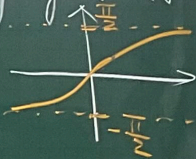
$$\Rightarrow \sec^2 y\cdot\frac{dy}{dx} = 1 \quad (*)$$

$$\Rightarrow \frac{d}{dx}(\tan^{-1}x) = \frac{dy}{dx} \overset{(*)}{=} \frac{1}{\sec^2 y} \overset{(**)}{=} \frac{1}{1+x^2}$$

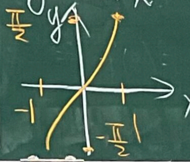
$$\tan y = \tan\tan^{-1}x = x$$

$$\sec^2 y = 1+\tan^2 y = 1+x^2 \quad (**)$$

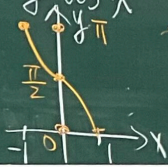
y = $\tan^{-1}x$



y = $\sin^{-1}x$



y = $\cos^{-1}x$



$y = \csc^{-1}x\ (|x|\geq 1) \iff \frac{1}{\sin y} = x$

$\iff \csc y = x$ and $y = (0,\frac{\pi}{2}]\cup(\pi,\frac{3}{2}\pi]$

$y = \sec^{-1}x\ (|x|\geq 1) \iff$

$\sec y = x$ and $y\in[0,\frac{\pi}{2})\cup[\pi,\frac{3}{2}\pi] \iff \frac{1}{\cos y} = x$

$y = \cot^{-1}x\ (x\in\mathbb{R}) \iff$

$\cot y = x$ and $y\in(0,\pi) \iff \frac{1}{\tan y} = x$

Graphs

Table of derivatives

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}},\quad \frac{d}{dx}(\csc^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}},\quad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2},\quad \frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$$

⟨Ex⟩ simplify $\cos(\tan^{-1}x)$

⟨Ex⟩ show $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

$\Updownarrow$

Recall $\tan(\theta + \frac{\pi}{2}) = -\cot(\theta)$

Integration Formula

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$$

⟨Ex⟩ $\int \frac{1}{\sqrt{1-a^2x^2}}\,dx =$ 梁廣宥

$u = ax \quad du = a\,dx$

$$\Rightarrow \frac{1}{a}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{a}(\sin^{-1}u + C) = \frac{1}{a}\sin^{-1}ax + C$$

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⟨Ex⟩ $\int \frac{1}{1+a^2x^2}\,dx = \int \frac{1}{1+u^2}\cdot\frac{1}{a}\,du$

$u = ax \quad \frac{du}{dx} = a \quad \frac{du}{a} = dx$

$$= \frac{\tan^{-1}u}{a} + C$$

§6.7. Hyperbolic Fns.

$\sinh x = \frac{1}{2}(e^x + e^{-x})$

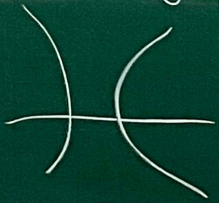
$\cosh x = \frac{1}{2}(e^x - e^{-x})$

$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

$\coth x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$\operatorname{sech} x = \frac{2}{e^x + e^{-x}}$

$\operatorname{csch} x = \frac{2}{e^x - e^{-x}}$



$\Rightarrow$

$\sinh^{-1}x =$

$\cosh^{-1}x =$

$\tanh^{-1}x =$

$\coth^{-1}x =$

$\operatorname{sech}^{-1}x =$

$\operatorname{csch}^{-1}x =$