§15.6. Surface Area

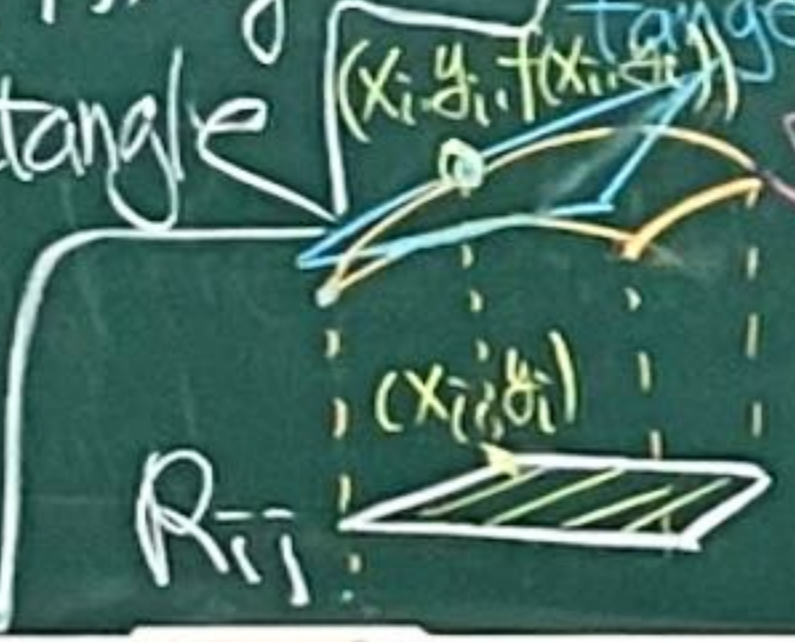
In §8.2 surface area of revolution

Let surface $S: z=f(x,y)$, f_x, f_y = conti.

$f(x,y)\ge 0$, domain D = rectangle

$D=\bigcup R_{ij}$

$|R_{ij}| = \Delta A = \Delta x\,\Delta y$



tangent plane $\Delta T_{ij} = |T_{ij}|$

$S_{ij} \to \Delta S_{ij} = |S_{ij}|$

Recall

$$\vec a\times\vec b = \begin{vmatrix} \vec i & \vec j & \vec k\\ \Delta x & 0 & f_x(x_i,y_j)\Delta x \\ 0 & \Delta y & f_y(x_i,y_j)\Delta y\end{vmatrix}$$

$$= -f_x(x_i,y_j)\underbrace{\Delta x\Delta y}_{\Delta A}\,\vec i - f_y(x_i,y_j)\Delta y\,\Delta x\,\vec j + \Delta x\Delta y\,\vec k$$

$$= [-f_x(x_i,y_j)\vec i - f_y(x_i,y_j)\vec j + \vec k]\,\Delta A$$

$$\Rightarrow |T_{ij}| = |\vec a\times\vec b| = |-f_x(x_i,y_j)\vec i - f_y(x_i,y_j)\vec j + \vec k|\,\Delta A = \sqrt{(f_x(x_i,y_j))^2+(f_y(x_i,y_j))^2+1}\;\Delta A$$

$$\Rightarrow \text{area of } S = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}|T_{ij}| = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}\sqrt{(f_x(x_i,y_j))^2+(f_y(x_i,y_j))^2+1}\;\Delta A$$

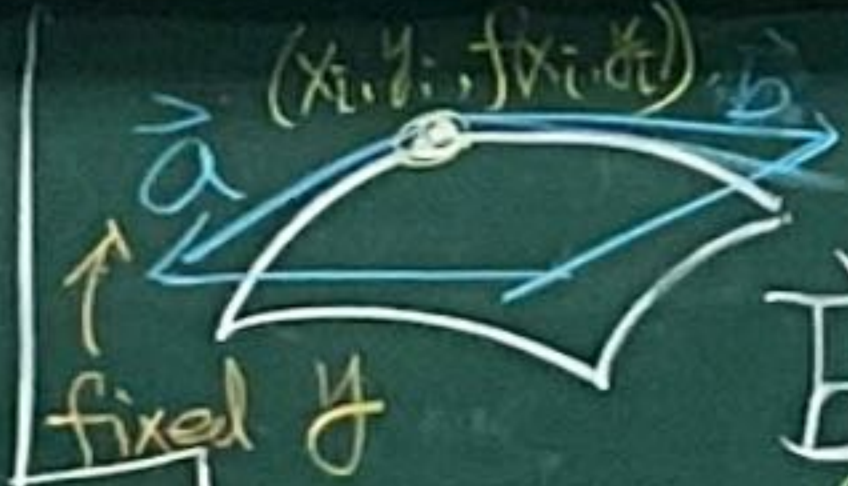
Riemann Sum $\iint_D \sqrt{(f_x)^2+(f_y)^2+1}\,dA$

Surface area of $S = |S|$

$$= \sum_{i,j}|S_{ij}| \approx \sum_{i,j}|T_{ij}|.$$

Def: $A(S)=|S| = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}|T_{ij}|$

Recall $y=f(x)$ $\quad |C| = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{1+|f'(x_i^*)|^2}\,\Delta x = \int_a^b\sqrt{(f'(x))^2+1}\,dx$



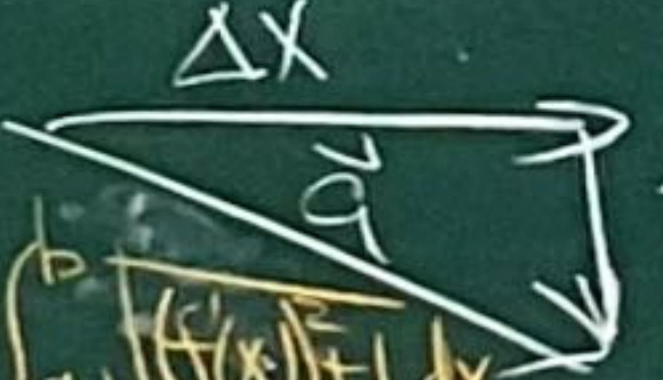
fixed y

$\vec a = (\Delta x, 0, f_x(x_i,y_j)\Delta x\,\vec k)$

$\vec b = (0, \Delta y, f_y(x_i,y_j)\Delta y\,\vec k)$

for fixed x

?



(Ex) Sphere $x^2+y^2+z^2=a^2$ Find $|S|$. $\quad D=\{x^2+y^2\le a^2\}$

$z=\sqrt{a^2-x^2-y^2}=f(x,y)$

Compute $f_x(x,y) = \dfrac{-x}{\sqrt{a^2-x^2-y^2}}$

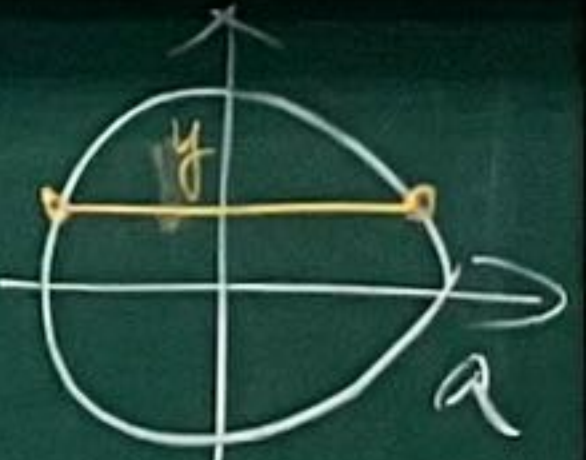
$f_y(x,y)=\dfrac{-y}{\sqrt{a^2-x^2-y^2}}$

$x^2+y^2\le a^2$

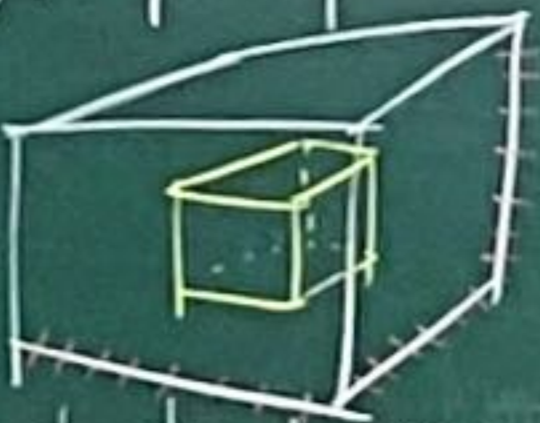
area of Sphere $2\iint_D\sqrt{(f_x)^2+(f_y)^2+1}\,dA$

$$=2\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\sqrt{\left(\frac{-x}{\sqrt{a^2-x^2-y^2}}\right)^2+\left(\frac{-y}{\sqrt{a^2-x^2-y^2}}\right)^2+1}\;dx\,dy$$

"$4\pi a^2$"



§15.7. Triple Integrals.



rectangle box

$$B = \{(x,y,z) \mid a \le x \le b,\ c \le y \le d,\ r \le z \le s\} = [a,b]\times[c,d]\times[r,s]$$

$$B = \bigcup_{i=1}^{m}\bigcup_{j=1}^{n}\bigcup_{k=1}^{\ell} B_{ijk}.$$

$$B_{ijk} = [x_{i-1}, x_i]\times[y_{j-1}, y_j]\times[z_{k-1}, z_k].$$

$$[a,b] \longrightarrow \{x_0, x_1, \cdots, x_m\}$$

$$[c,d] \longrightarrow \{y_0, y_1, \cdots, y_n\}$$

$$[r,s] \longrightarrow \{z_0, z_1, \cdots, z_\ell\}$$

Volume of B_{ijk}

$$= \Delta V = \Delta x \Delta y \Delta z.$$

Triple Riemann Sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{\ell} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V \xrightarrow[m,n,\ell\to\infty]{} \iiint_B f(x,y,z)\,dV$$

Def the triple integral of $w = f(x,y,z)$ over B.

$$\iiint_B f(x,y,z)\,dV \equiv \lim_{m,n,\ell\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{\ell} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V$$

if the limit exists.

Fubini $f =$ conti. on $B = [a,b]\times[c,d]\times[r,s]$

Thm $$\iiint_B f(x,y,z)\,dV = \int_r^s\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz = \int_a^b\int_r^s\int_c^d f(x,y,z)\,dy\,dz\,dx = \cdots$$

〈Ex〉 In a movie theater. Assume that the average waiting time for tickets is 10 min.s and the average waiting for popcorn is 5 mins. Assuming that they are independent. Find the P that a moviegoer waits a total of less than 20 mins

Sol'n: Assume X = waiting time for tickets

Y = 〃 〃 popcorn

p.d.f. for waiting time (μ = mean)

$$f(t)=\begin{cases}0 & \text{for } t<0\\ \mu^{-1}e^{-t/\mu} & \text{for } t\geq 0\end{cases}$$

$\Rightarrow$ p.d.f.s are

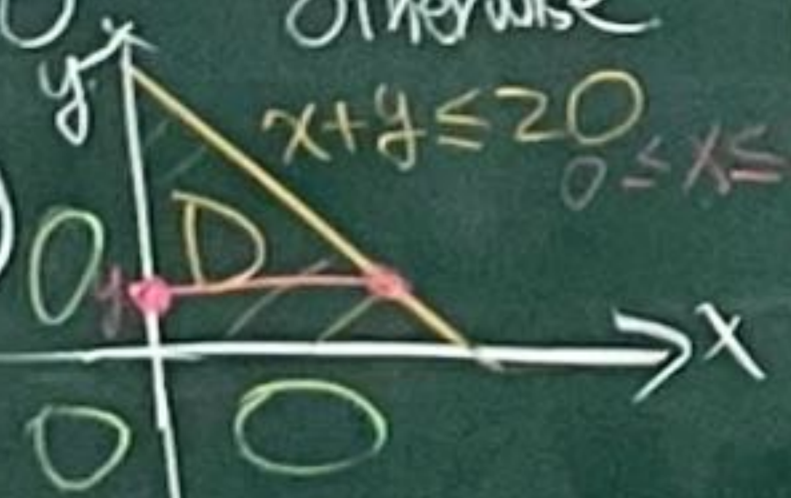
$$f_1(x)=\begin{cases}0 & x<0\\ 10^{-1}e^{-x/10} & x\geq 0\end{cases},\quad f_2(y)=\begin{cases}0, & y<0\\ 5^{-1}e^{-y/5}, & y\geq 0\end{cases}$$

Since X, Y = independent

$\Rightarrow$ the joint p.f. $f(x,y)=f_1(x)f_2(y)=\begin{cases}\frac{1}{50}e^{-\frac{x}{10}}e^{-\frac{y}{5}} & \text{for } x\geq 0,\ y\geq 0\\ 0, & \text{otherwise}\end{cases}$

Thus $P(X+Y\leq 20)=P((X,Y)\in D)$

$\equiv \iint_D f(x,y)\,dA = ?$



$0\leq x\leq 20-y$

$$\iint_D f(x,y)\,dA$$

$$=\int_0^{20}\int_0^{20-y}\frac{1}{50}e^{-\frac{x}{10}}e^{-\frac{y}{5}}\,dx\,dy$$

$$=\int_0^{20}-\frac{1}{5}e^{-\frac{y}{5}}e^{-\frac{x}{10}}\Big|_0^{20-y}\,dy$$

$$=\int_0^{20}-\frac{1}{5}e^{-\frac{y}{5}}\left(e^{-2+\frac{y}{10}}-1\right)dy$$

$$=\int_0^{20}-\frac{1}{5}e^{-2-\frac{y}{5}}+\frac{1}{5}e^{-\frac{y}{5}}\,dy$$

$=e^{-2-\frac{y}{5}}\Big|_0^{20}$

$-e^{-\frac{y}{5}}\Big|_0^{20}$

$=$

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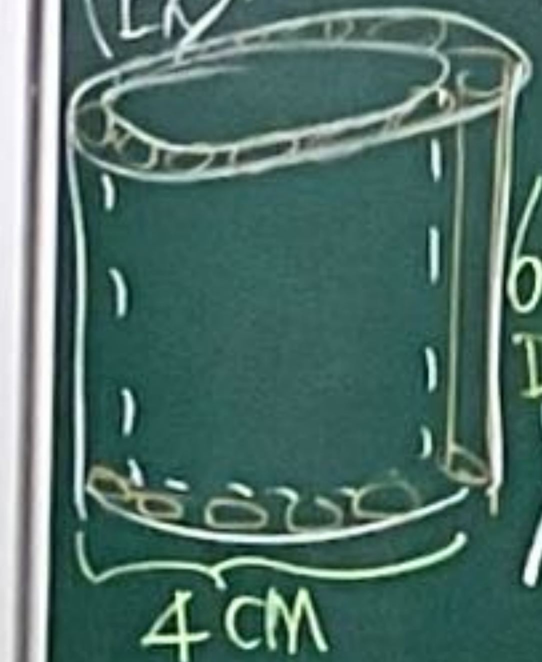
〈Ex〉 Roller bearing are produced with diameter 4 cm and length 6 cm.



Diameter X = normally distributed with mean 4 cm and standard deviation 0.01 cm

Length Y = normally distributed with mean 6 cm and standard deviation 0.01 cm

Assume X, Y = independent

Find the joint d.f. and graph it

Find the P that a bearing has either length or diameter that differs from the mean by more than 0.02 cm

Sol' p.d.f of N.D. $f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

μ = mean, σ = standard deviation

$$f(x,y)\overset{\text{independent}}{=\!=}f_1(x)f_2(y)=\frac{1}{0.01\sqrt{2\pi}}e^{\frac{-(x-4)^2}{2(0.01)^2}}\cdot\frac{1}{0.01\sqrt{2\pi}}\cdot e^{\frac{-(y-6)^2}{2(0.01)^2}}=\frac{1}{(0.01)^2 2\pi}e^{\frac{-(x-4)^2}{2(0.01)^2}}e^{\frac{-(y-6)^2}{2(0.01)^2}}$$

$$P(|X-4|>0.02 \text{ or } |Y-6|>0.02)$$

$$=1-P(|X-4|\leq 0.02 \text{ and } |Y-6|\leq 0.02)$$

$$P=1-\int_{3.98}^{4.02}\int_{5.98}^{6.02}f_1(x)f_2(y)\,dy\,dx\ ?$$

$$=1-\int_{3.98}^{4.02}\frac{1}{0.01\sqrt{2\pi}}e^{\frac{-(x-4)^2}{2(0.01)^2}}dx\int_{5.98}^{6.02}\frac{1}{0.01\sqrt{2\pi}}e^{\frac{-(y-6)^2}{2(0.01)^2}}dy=\cdots$$

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