2.3 HW2

Theorem 2.3.1. (Root Test is Stronger Than Ratio Test)

$$\liminf_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| \le \liminf_{n\to\infty}\sqrt[n]{|z_n|} \le \limsup_{n\to\infty}\sqrt[n]{|z_n|} \le \limsup_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right|$$

Proof. Fix ϵ and WOLG suppose $\liminf_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| > 0$. We prove

$$\liminf_{n\to\infty}\sqrt[n]{|z_n|} \ge \liminf_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| - \epsilon$$

Let $\alpha \in \mathbb{R}$ satisfy

$$\liminf_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| - \epsilon < \alpha < \liminf_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right|$$

Let N satisfy

$$\text{For all } n \ge N, \left|\frac{z_{n+1}}{z_n}\right| > \alpha$$

We then see

$$\sqrt[N+n]{|z_{N+n}|} \ge \sqrt[N+n]{|z_N|\,\alpha^n} = \alpha\Big(\frac{|z_N|^{\frac{1}{N+n}}}{\alpha^{\frac{N}{N+n}}}\Big) \to \alpha \text{ as } n \to \infty \text{ (done)}$$

The proof for the other side is similar. ∎

Question 29

Find the radius of convergence of the following series:

(a) $\sum \frac{z^n}{n}$.

(b) $\sum \frac{z^n}{n!}$.

(c) $\sum n! z^n$.

(d) $\sum n^k z^n$ where k is a positive integer.

(e) $\sum z^n$.

Proof. We know

$$n^{\frac{1}{n}} = e^{\frac{\ln n}{n}} \to 1 \text{ as } n \to \infty \tag{2.1}$$

Equation 2.1 implies $n^{\frac{-1}{n}} \to 1$ as $n \to \infty$ and that $\sum \frac{z^n}{n}$ has radius of convergence 1. Equation 2.1 also implies $n^{\frac{k}{n}} \to 1$ and $\sum n^k z^n$ has radius of convergence 1.

We know

$$\lim_{n\to\infty} \frac{(n+1)!}{n!} = \infty$$

Then Theorem 2.3.1 tell us

$$\lim_{n\to\infty} (n!)^{\frac{1}{n}} = \infty \tag{2.2}$$

which implies that $\sum n! z^n$ has radius of convergence 0 and $\sum \frac{z^n}{n!}$ has radius of convergence ∞. Note that

$$\sum z^{n!} = \sum a_n z^n \text{ where } a_n = \begin{cases} 1 & \text{if } n = k! \text{ for some } k \\ 0 & \text{otherwise} \end{cases}$$

It is then clear that

$$\limsup_{n\to\infty} (a_n)^{\frac{1}{n}} = 1$$

and the radius is 1. ■

Question 30

The 0th order Bessel function $J_0(z)$ is defined by the power series

$$J_0(z) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2} \frac{z^n}{2n}$$

Find its radius of convergence.

Proof. Equation 2.1 and Equation 2.2 tell us

$$\lim_{n\to\infty} (2n(n!)^2)^{\frac{1}{n}} = \infty$$

which implies that the radius of convergence of $J_0(z)$ is ∞. ■

Theorem 2.3.2. (Abel's Test for Power Series) Suppose $a_n \to 0$ monotonically and $\sum a_n z^n$ has radius of convergence R .

$$\text{The power series } \sum a_n z^n \text{ at least converge on } \overline{D_R(0)} \setminus \{R\}$$

Proof. Note that

$$\sum \frac{a_n}{R^n} z^n \text{ has radius of convergence } R$$

Fix $z \in \overline{D_R(0)} \setminus \{R\}$. Note that

$$\left| \sum_{n=0} \frac{z^n}{R^n} \right| = \left| \frac{1 - (\frac{z}{R})^{N+1}}{1 - \frac{z}{R}} \right| \leq \frac{2}{\left|1 - \frac{z}{R}\right|} \text{ for all } N$$

It then follows from Dirichlet's Test that $\sum a_n (\frac{z}{R})^n$ converge. ∎

Question 31

Suppose that $\sum a_n z^n$ has radius of convergence R and let C be the circle $\{z \in \mathbb{C} : |z| = R\}$. Prove or disprove

(a) If $\sum a_n z^n$ converge at every point on C, except possibly one, then it converges absolutely every where on C

Proof. Consider $a_n \triangleq \frac{1}{n}$ for all $n \in \mathbb{N}$ and $a_0 \triangleq 1$. Then $\sum a_n z^n$ has convergence radius 1. Since $a_n \searrow 0$, it follows from Theorem 2.3.2, $\sum a_n z^n$ converge everywhere on $C \setminus \{1\}$. Observe that when $z = 1$, the series is just harmonic series, which diverge. ∎

Question 32

If $\sum a_n z^n$ has radius of convergence R, find the radius of convergence of

(a) $\sum n^3 a_n z^n$.

(b) $\sum a_n z^{3n}$.

(c) $\sum a_n^3 z^n$

Proof. Since $(n^3)^{\frac{1}{n}} \to 1$, we know $\sum n^3 a_n z^n$ also had radius of convergence R. We claim that the series $\sum a_n z^{3n}$ has convergence radius $R^{\frac{1}{3}}$. If $|z| < R^{\frac{1}{3}}$, then $\left|z^3\right| < R$ and thus

$$\sum a_n (z^3)^n \text{ converge}$$

and if $|z| > R^{\frac{1}{3}}$, then $\left|z^3\right| > R$ and

$$\sum a_n (z^3)^n \text{ diverge}$$

We have proved that $\sum a_n z^{3n}$ has convergence radius $R^{\frac{1}{3}}$.

Note that given a sub-sequence $|a_{n_k}|^{\frac{1}{n_k}}$,

$|a_{n_k}|^{\frac{1}{n_k}}$ converge in extended reals if and only if $|a_{n_k}|^{\frac{3}{n_k}}$ converge in extended reals and if the former converge to L, then the latter converge to L^3. It now follows that

$$\limsup_{n\to\infty} |a_n^3| = (\limsup_{n\to\infty} |a_n|)^3 = \frac{1}{R^3}$$

It now follows that $\sum a_n^3 z^n$ has convergence radius R^3. ■

Theorem 2.3.3. (Summation by Part)

$$\begin{aligned} f_n g_n - f_m g_m &= \sum_{k=m}^{n-1} f_k \Delta g_k + g_k \Delta f_k + \Delta f_k \Delta g_k \\ &= \sum_{k=m}^{n-1} f_k \Delta g_k + g_{k+1} \Delta f_k \end{aligned}$$

Proof. The proof follows induction which is based on

$$f_m g_m + f_m \Delta g_m + g_m \Delta f_m + \Delta f_m \Delta g_m = f_{m+1} g_{m+1}$$

■

Question 33

Prove that, for $z \neq 1$

$$\sum_{n=1}^{k} \frac{z^n}{n} = \frac{z}{1-z}\Big(\sum_{n=1}^{k-1} \frac{1}{n(n+1)} - \sum_{n=1}^{k-1} \frac{z^n}{n(n+1)} + \frac{1-z^k}{k}\Big)$$

Show that the series $\sum \frac{z^n}{n}$ and $\sum \frac{z^n}{n(n+1)}$ have radius of convergence 1; that the latter series converge everywhere on $|z| = 1$, while the former converges everywhere on $|z| = 1$ except $z = 1$.

Proof. We prove by induction. The base case $k = 1$ is trivial. Suppose the equality hold when $k = m$. The difference of the left hand side is clearly $\frac{z^{m+1}}{m+1}$, and the difference of the right hand side is

$$
\begin{aligned}
&\frac{z}{1-z}\left(\frac{1}{m(m+1)}-\frac{z^m}{m(m+1)}+\frac{1-z^{m+1}}{m+1}-\frac{1-z^m}{m}\right) \\
&=\frac{z}{1-z}\cdot\frac{1-z^m+m-mz^{m+1}+(m+1)z^m-(m+1)}{m(m+1)} \\
&=\frac{z}{1-z}\cdot\frac{-mz^{m+1}+mz^m}{m(m+1)}=\frac{z(z^m-z^{m+1})}{(1-z)(m+1)}=\frac{z^{m+1}}{m+1}
\end{aligned}
$$

The fact that both series have radius of convergence 1 follows from $n^{\frac{1}{n}}\to 1$. Both of them converge on $\overline{D_1(0)}\setminus\{1\}$ by Theorem 2.3.2. The former clearly diverge at $z=1$, since it would be a harmonic series, and the latter converge at $z=1$ by comparison test with $\frac{1}{n(n+1)}\leq\frac{1}{n^2}$ for all $n\in\mathbb{N}$. ■

Question 34

Suppose that the power series $\sum a_n z^n$ has a recurring sequence of coefficients; that is $a_{n+k}=a_n$ for some fixed positive integer k and all n. Prove that the series converge for $|z|<1$ to a rational function $\frac{p(z)}{q(z)}$ where p,q are polynomials, and the roots of q are all on the unit circle. What happens if $a_{n+k}=\frac{a_n}{k}$ instead?

Proof. Let

$$
L^-\triangleq\min\{|a_0|,\dots,|a_{k-1}|\}\text{ and }L^+\triangleq\max\{|a_0|,\dots,|a_{k-1}|\}
$$

We see that

$$
1=\liminf_{n\to\infty}(L^-)^{\frac{1}{n}}\leq\lim_{n\to\infty}|a_n|^{\frac{1}{n}}\leq\limsup_{n\to\infty}(L^+)^{\frac{1}{n}}=1
$$

It then follows that $\sum a_n z^n$ has convergence radius 1. Now observe that for $|z|<1$, we have

$$
z^k\sum_{n=0}^{\infty}a_n z^n=\sum_{n=k}^{\infty}a_n z^n=\sum_{n=0}^{\infty}a_n z^n-\sum_{n=0}^{k-1}a_n z^n
$$

This now implies

$$
\sum_{n=0}^{\infty}a_n z^n=\frac{\sum_{n=0}^{k-1}a_n z^n}{1-z^k}
$$

Since $q(z) = 1 - z^k$, clearly the roots are all on the unit circle. Suppose now $b_n \triangleq a_n$ for all $n < k$ and $b_{n+k} \triangleq \frac{b_n}{k}$ for all $n \geq k$. We then have

$$b_n = \frac{a_n}{k^{q(n)}} \text{ where } q \text{ is the largest integer such that } qk \leq n$$

Note that $n - q(n)$ is always smaller than k. It then follows that

$$(k^{q(n)})^{\frac{1}{n}} = k^{\frac{q(n)}{n}} = k \cdot k^{\frac{q(n)-n}{n}} \to k$$

We then see that

$$\limsup_{n\to\infty} |b_n|^{\frac{1}{n}} = \frac{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}}{k} = \frac{1}{k}$$

It then follows that $\sum b_n z^n$ has convergence radius k. Now observe that for $|z| < k$, we have

$$z^k \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} b_n z^{n+k} = \frac{1}{k} \sum_{n=k}^{\infty} b_n z^n = \frac{1}{k}\Big(\sum_{n=0}^{\infty} b_n z^n - \sum_{n=0}^{k-1} b_n z^n\Big)$$

This now implies

$$\sum_{n=0}^{\infty} b_n z_n = \frac{\sum_{n=0}^{k-1} b_n z^n}{k(\frac{1}{k} - z^k)}$$

$\blacksquare$