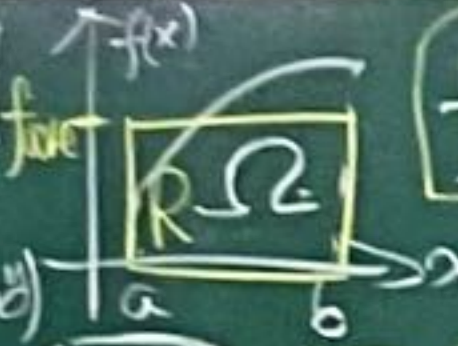
Recall



$$\boxed{f_{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$|R|$ $\quad$ $\|\Omega|$

$z=f(x,y)$

$$\boxed{f_{ave}} = \frac{1}{|R|}\iint_R f(x,y)\,dA$$

$\| V|$

重

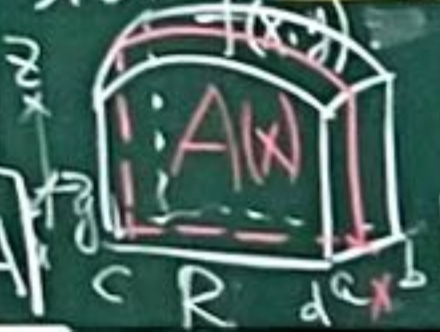
Properties of Double Integrals

(a) $\iint_R [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$

(b) $\iint_R [c f(x,y)]\,dA = c\iint_R f(x,y)\,dA$

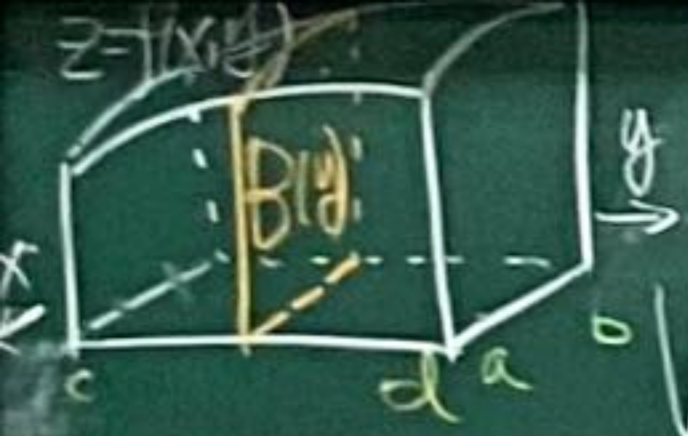
(c) If $f \geq g$ on R, $\Rightarrow \iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$

§15.2 Iterated Integrals 逐次



$A(x) = \int_c^d f(x,y)\,dy$ (Area)

Volume $V = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$



$$B(y)=\int_a^b f(x,y)\,dx$$

$$V=\int_c^d\left[\int_a^b f(x,y)\,dx\right]dy$$

Fubini Thm If $f=$ conti on $R=[a,b]\times[c,d]$

$$\Rightarrow \iint_R f(x,y)\,dA=\int_a^b\left(\int_c^d f\,dy\right)dx=\int_c^d\int_a^b f\,dx\,dy$$

Prop. $\displaystyle\iint_R g(x)h(y)\,dA=\int_a^b g(x)\,dx\int_c^d h(y)\,dy$

where $R=[a,b]\times[c,d]$

Pf $\displaystyle\iint_R g(x)h(y)\,dA \overset{\text{Fubini}}{=} \int_a^b\int_c^d g(x)h(y)\,dy\,dx$

$$\overset{(b)}{=} \int_a^b g(x)\boxed{\int_c^d h(y)\,dy}\,dx$$

$$\overset{(b)}{=} \int_a^b g(x)\,dx\int_c^d h(y)\,dy$$

#

§15.3 Double Integrals over general regions

Suppose

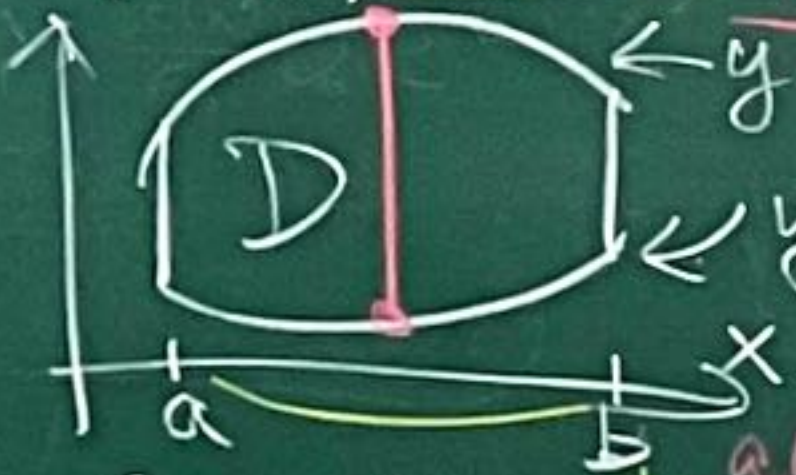
$= [a,b] \times [c,d]$

Def $F(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0 & (x,y) \notin D \end{cases}$

$$\iint_{\Omega} f(x,y)\,dA = \iint_{R} F(x,y)\,dA$$

Def D = region of type I if

$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$



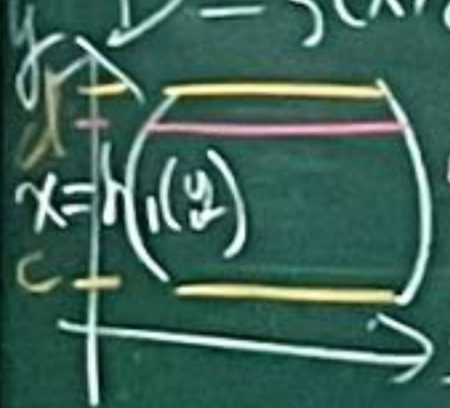
$\leftarrow y = g_2(x)$

$\leftarrow y = g_1(x)$.

$$\Rightarrow \iint_{D} f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Def D = type II region if

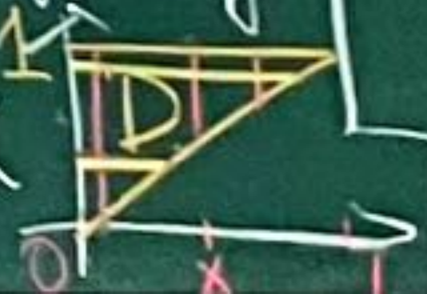
$D \equiv \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$



$x = h_2(y)$

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

<Ex> Evaluate $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$



$$\overset{\text{Fubini}}{=} \iint_D \sin(y^2)\,dA \underset{\text{Fubini}}{=} \int_0^1 \int_0^y \sin(y^2)\,dx\,dy$$

$$= \int_0^1 \sin(y^2)\, y\,dy$$

$$= \int_0^1 \sin(y^2)\, d\left(\tfrac{1}{2}y^2\right)$$

$$=$$

Prop: $f, g : D \longrightarrow \mathbb{R}$ conti,

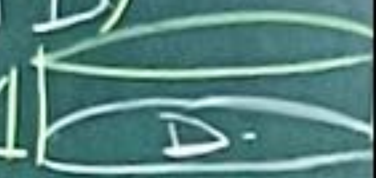
(a) $\iint_D [f+g]\,dA = \iint_D f\,dA + \iint_D g\,dA$

(b) $\iint_D cf\,dA = c\iint_D f\,dA$

(c) $f \geq g$ on $D \Rightarrow \iint_D f\,dA \geq \iint_D g\,dA$

(d) $D = D_1 \cup D_2$ and $D_1 \cap D_2 = \emptyset$
$\Rightarrow \iint_D f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA$

(e) $\iint_D 1\,dA = 1 \cdot A(D)$ (area of D)
$= V(D \times [0,1])$



(f) If $m \leq f \leq M$ on $D \Rightarrow \underbrace{m|D|}_{\text{Area}} \leq \iint_D f\,dA \leq M|D|$

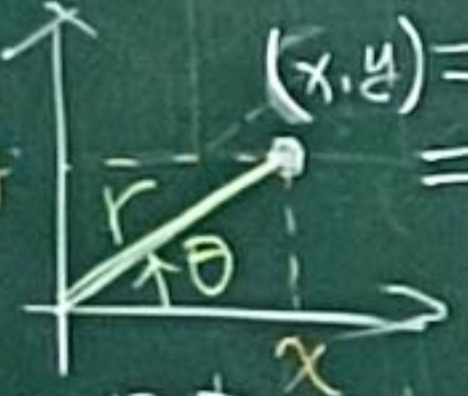
pf: by (c) $\underbrace{\iint_D m\,dA}_{= m|D|} \leq \iint_D f(x,y)\,dA \leq \underbrace{\iint_D M\,dA}_{= M|D|}$ ※

§15.4 Double Integrals in Polar Coordinates

To compute $\iint_R f(x,y)\,dA$

$(x,y) = (r\cos\theta, r\sin\theta)$.

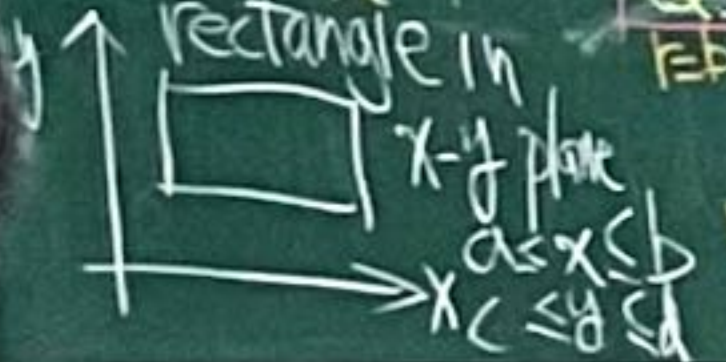
$= [r,\theta]$



R = polar rectangle.

$$a \le r \le b$$
$$\alpha \le \theta \le \beta$$

$$\Delta r = \frac{b-a}{m},\ \Delta\theta = \frac{\beta-\alpha}{n}.$$



Polar subrectangle

$$R_{ij} = \{(r,\theta) \mid r_{i-1} \le r \le r_i ,\ \theta_{j-1} \le \theta \le \theta_j\}$$

choose $r_j^* = \frac{1}{2}(r_{j-1} + r_j)$, $\theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j)$

$$|R_{ij}| = \frac{1}{2} r_i^2 \Delta\theta - \frac{1}{2} r_{i-1}^2 \Delta\theta$$

$$= \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta$$

$$= \frac{1}{2}(r_i + r_{i-1})\,\Delta r\,\Delta\theta \approx r_i^* \Delta r \Delta\theta$$

Riemann Sum of Double integral

$$\iint_D f(x,y)\,dA$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \underbrace{f(r_i^*\cos\theta_j^*,\ r_i^*\sin\theta_j^*)}_{=\,g(r_i^*,\,\theta_j^*)}\ \overbrace{r_i^*\Delta r\Delta\theta}^{|R_{ij}|\,=}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*, \theta_j^*)\,\Delta r\,\Delta\theta$$

Riemann Sum of g on $[a,b]\times[\alpha,\beta]$

Thus $\iint_D f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\,|R_{ij}|$

Change to Polar Coordinates.

$= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*, \theta_j^*)\,\Delta r\,\Delta\theta$

$= \int_\alpha^\beta \int_a^b g(r,\theta)\,dr\,d\theta$

$= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

If f = conti. on $R = \left\{(r,\theta) \,\middle|\, \begin{array}{l} 0 \le a \le r \le b \\ \alpha \le \theta \le \beta \\ 0 \le \beta-\alpha \le 2\pi \end{array}\right\}$

Then

$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

(EX) Find the volume

Sol: $z=0 \Rightarrow x^2+y^2=1$

$D=\{(x,y) \mid x^2+y^2 \le 1\}$

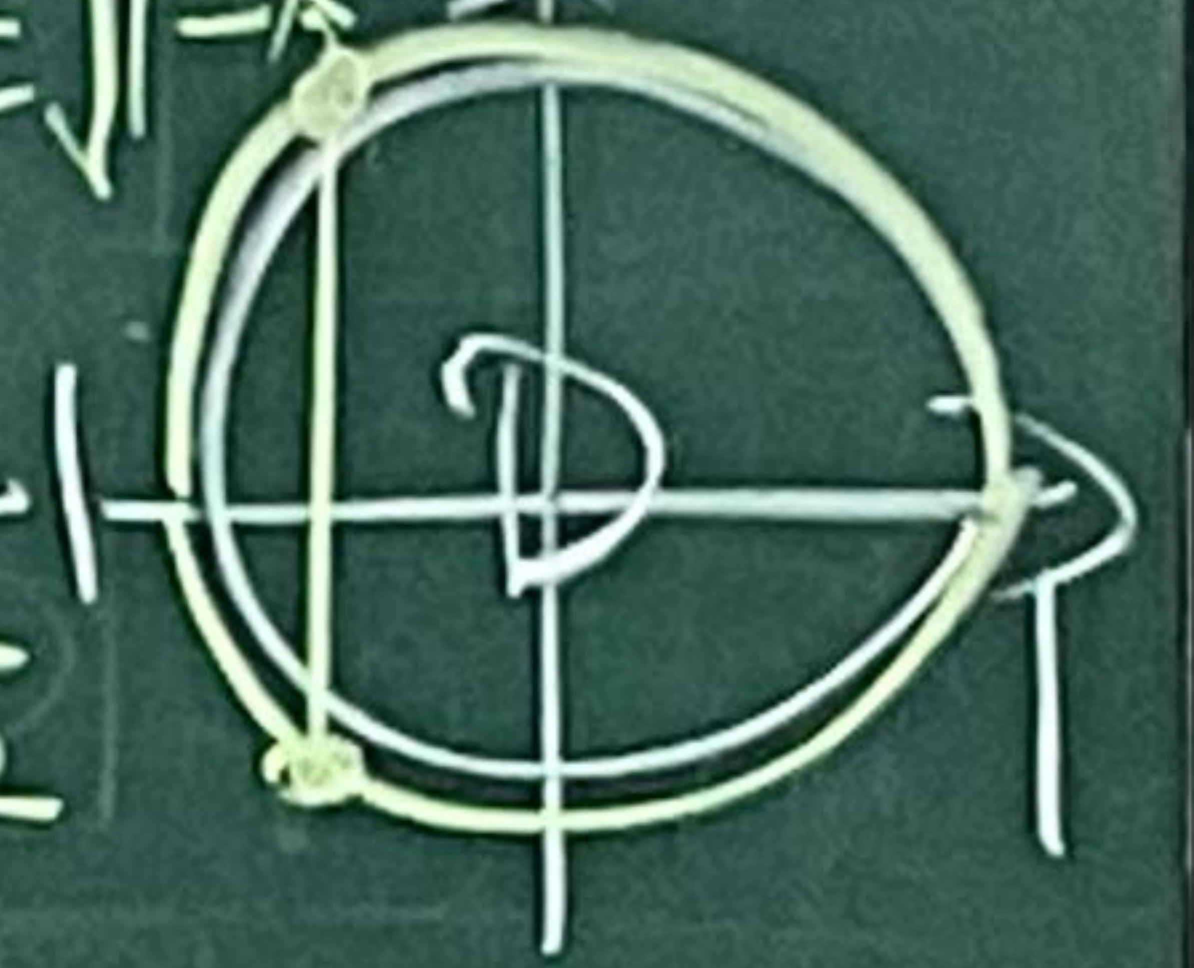
$=\{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$

Volume $= \iint_D (1-x^2-y^2)\,dA$

Polar coord. $= \int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta = ?$

(B) rectangular coordinates

$V=\iint_D (1-x^2-y^2)\,dA$



$y=-\sqrt{1-x^2}$

$=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1-x^2-y^2)\,dy\,dx = ?$