4.5 HW5

Question 50

2. Show that if a surface is tangent to a plane along a curve, then the points of this curve are either parabolic or planar.

Proof. Suppose the curve is $\alpha(s)$. We see that

$$(N \circ \alpha)(s) \text{ stay constant}$$

Differentiation give us

$$dN_{\alpha(s)}(\alpha'(s)) = 0$$

Then for each p that lies in $\alpha(I)$, we see

$$dN_p \text{ is not full rank}$$

This then show us

$$\det(dN_p) = 0$$

and give us the conclusion. ■

Question 51

3. Let $C \subset S$ be a regular curve on a surface S with Gaussian curvature $K > 0$. Show that the curvature k of C at p satisfies

$$|k| \geq \min(|k_1|, |k_2|),$$

where k_1 and k_2 are the principal curvatures of S at p.

Proof. Because $K > 0$, we know the principal curvatures κ_1, κ_2 satisfy

$$0 < \kappa_1 \leq \kappa_2 \text{ or } \kappa_2 \leq \kappa_1 < 0$$

WOLG, suppose $0 < \kappa_1 \leq \kappa_2$. Let $\alpha : (-\epsilon, \epsilon) \to C$ be an arc-length parametrization passing through p at $\alpha(0)$. Let θ be the angle between $N_\alpha(p)$ and $N(p)$. We know

$$\kappa \cos\theta = \kappa_{\alpha,p}$$

Because $0 < \kappa_1 \leq \kappa_2$, we can deduce

$$\kappa_{\alpha,p} \geq \kappa_1 = \min(|\kappa_1|, |\kappa_2|) > 0$$

This then implies

$$|\kappa| \geq |\kappa| |\cos\theta| = |\kappa\cos\theta| = |\kappa_{\alpha,p}| \geq \min(|\kappa_1|, |\kappa_2|)$$

■

Question 52

5. Show that the mean curvature H at $p \in S$ is given by

$$H = \frac{1}{\pi}\int_0^\pi k_n(\theta)\, d\theta,$$

where $k_n(\theta)$ is the normal curvature at p along a direction making an angle θ with a fixed direction.

Proof. Let e_1, e_2 be the principal direction. Suppose the fixed direction is $\cos\theta_0 e_1 + \sin\theta_0 e_2$. Define $\alpha : [0, \pi] \to T_p(S)$ by

$$\alpha(\theta) = \cos(\theta_0 + \theta)e_1 + \sin(\theta_0 + \theta)e_2$$

Compute $\kappa_n(\theta)$ by

$$\begin{aligned}\kappa_n(\alpha(\theta)) &= \mathrm{II}_p(\cos(\theta_0 + \theta)e_1 + \sin(\theta_0 + \theta)e_2) \\ &= \kappa_1 \cos^2(\theta_0 + \theta) + \kappa_2 \sin^2(\theta_0 + \theta)\end{aligned}$$

This then give us

$$\begin{aligned}\frac{1}{\pi}\int_0^\pi \kappa_n(\theta)d\theta &= \frac{1}{\pi}\int_0^\pi \kappa_1 \cos^2(\theta_0 + \theta) + \kappa_2 \sin^2(\theta_0 + \theta)d\theta \\ &= \frac{1}{\pi}(\frac{\kappa_1\pi}{2} + \frac{\kappa_2\pi}{2}) = H\end{aligned}$$

■

Question 53

6. Show that the sum of the normal curvatures for any pair of orthogonal directions, at a point $p \in S$, is constant.

Proof. Suppose e_1, e_2 are principal direction, and express the pair v_1, v_2 of orthogonal directions by $v_1 = \cos\theta e_1 + \sin\theta e_2$ and $v_2 = \cos(\theta + \frac{\pi}{2})e_1 + \sin(\theta + \frac{\pi}{2})e_2$. We have

$$\begin{aligned} \kappa_{v_1} &= \mathrm{II}_p(\cos\theta e_1 + \sin\theta e_2) \\ &= \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta \end{aligned}$$

and have

$$\begin{aligned} \kappa_{v_2} &= \mathrm{II}_p(\cos(\theta + \frac{\pi}{2})e_1 + \sin(\theta + \frac{\pi}{2})e_2) \\ &= \mathrm{II}_p(\sin\theta e_1 + \cos\theta e_2) \\ &= \kappa_1 \sin^2\theta + \kappa_2 \cos^2\theta \end{aligned}$$

This then give us

$$\kappa_{v_1} + \kappa_{v_2} = \kappa_1 + \kappa_2 = \text{const.}$$

■

Question 54

11. Let p be an elliptic point of a surface S, and let r and r' be conjugate directions at p. Let r vary in $T_p(S)$ and show that the minimum of the angle of r with r' is reached at a unique pair of directions in $T_p(S)$ that are symmetric with respect to the principal directions.

Proof. WOLG, suppose $0 < \kappa_1 \leq \kappa_2$. Express

$$r = \cos\theta e_1 + \sin\theta e_2 \text{ and } r' = \cos\theta' e_1 + \sin\theta' e_2$$

Now compute

$$\begin{aligned} \langle dN_p(r), r' \rangle &= \langle -\kappa_1 \cos\theta e_1 - \kappa_2 \sin\theta e_2, \cos\theta' e_1 + \sin\theta' e_2 \rangle \\ &= -\kappa_1 \cos\theta \cos\theta' - \kappa_2 \sin\theta \sin\theta' \end{aligned}$$

This give us the constraint

$$\kappa_1 \cos\theta\cos\theta' + \kappa_2 \sin\theta\sin\theta' = 0$$

and we are required to find the extremum of

$$\cos(\theta - \theta') = \cos\theta\cos\theta' + \sin\theta\sin\theta'$$

We use the method of Lagarange multiplier. Define

$$\begin{aligned} f(\theta,\theta') &\triangleq \cos\theta\cos\theta' + \sin\theta\sin\theta' \\ g(\theta,\theta') &\triangleq \kappa_1\cos\theta\cos\theta' + \kappa_2\sin\theta\sin\theta' \end{aligned}$$

We are required to

$$\text{maximize or minimize } f \text{ subjecting to the constraint } g = 0$$

Compute

$$\begin{aligned} \nabla f &= \Big(-\sin\theta\cos\theta' + \cos\theta\sin\theta', -\cos\theta\sin\theta' + \sin\theta\cos\theta' \Big) \\ \nabla g &= \Big(-\kappa_1\sin\theta\cos\theta' + \kappa_2\cos\theta\sin\theta', -\kappa_1\cos\theta\sin\theta' + \kappa_2\sin\theta\cos\theta' \Big) \end{aligned}$$

It is now easy and straight forward to check that

$$\begin{aligned} \cos\theta &= \sqrt{\frac{\kappa_1}{\kappa_1+\kappa_2}} \text{ and } \sin\theta = \sqrt{\frac{\kappa_2}{\kappa_1+\kappa_2}} \\ \cos\theta' &= \sqrt{\frac{\kappa_1}{\kappa_1+\kappa_2}} \text{ and } \sin\theta' = -\sqrt{\frac{\kappa_2}{\kappa_1+\kappa_2}} \end{aligned}$$

is a non-trivial solution of $\nabla f = \lambda\nabla g$.

This then let us conclude

$$r \in \operatorname{span}(\sqrt{\kappa_1}e_1 + \sqrt{\kappa_2}e_2) \text{ and } r' \in \operatorname{span}(\sqrt{\kappa_1}e_1 - \sqrt{\kappa_2}e_2)$$

which is the so called "symmetry". ■

Question 55

***18.** Let $\lambda_1, \ldots, \lambda_m$ be the normal curvatures at $p \in S$ along directions making angles $0, 2\pi/m, \ldots, (m-1)2\pi/m$ with a principal direction, $m > 2$. Prove that

$$\lambda_1 + \cdots + \lambda_m = mH,$$

where H is the mean curvature at p.

Proof. Let $v_1, \ldots, v_m$ be the directions of $\lambda_1, \ldots, \lambda_m$. We have

$$v_k = \cos(\frac{k(2\pi)}{m})e_1 + \sin(\frac{k(2\pi)}{m})e_2$$

Observe

$$\begin{aligned}\lambda_k = \mathrm{II}_p(v_k) &= \mathrm{II}_p(\cos(\frac{2\pi k}{m})e_1 + \sin(\frac{2\pi k}{m})e_2)\\ &= \kappa_1 \cos^2(\frac{2\pi k}{m}) + \kappa_2 \sin^2(\frac{2\pi k}{m})\end{aligned}$$

Now compute using elementary identity

$$\begin{aligned}\sum_{k=1}^m \lambda_k &= \sum_{k=1}^m \kappa_1 \cos^2(\frac{2\pi k}{m}) + \kappa_2 \sin^2(\frac{2\pi k}{m})\\ &= \kappa_1 \frac{m}{2} + \kappa_2 \frac{m}{2} = mH\end{aligned}$$

∎

Question 56

1. Show that at the origin $(0,0,0)$ of the hyperboloid $z = axy$ we have $K = -a^2$ and $H = 0$.

Proof. We have the global chart

$$\mathbf{x}(x,y) = (x, y, axy)$$

Compute

$$\mathbf{x}_x = (1, 0, ay) \text{ and } \mathbf{x}_y = (0, 1, ax)$$

Because $N \perp \mathbf{x}_x, \mathbf{x}_y$, we have

$$N(x, y) = \frac{(-ay, -ax, 1)}{\sqrt{a^2(x^2 + y^2) + 1}}$$

Define 2 curves that passing through origin from different direction

$$\alpha(t) \triangleq (t, 0, 0) \text{ and } \beta(t) \triangleq (0, t, 0)$$

Compute

$$N \circ \alpha(t) = \left(0, \frac{-at}{\sqrt{a^2t^2 + 1}}, \frac{1}{\sqrt{a^2t^2 + 1}}\right) \text{ and } N \circ \beta(t) = \left(\frac{-at}{\sqrt{a^2t^2 + 1}}, 0, \frac{1}{\sqrt{a^2t^2 + 1}}\right)$$

This then give us

$$dN_0(\alpha'(0)) = (N \circ \alpha)'(0) = (0, -a, 0) \text{ and } dN_0(\beta'(0)) = (N \circ \beta)'(0) = (-a, 0, 0)$$

Note that $\alpha'(0) = (1, 0, 0)$ and $\beta'(0) = (0, 1, 0)$. We now see dN_0 have action

$$(1, 0, 0) \mapsto (0, -a, 0) \text{ and } (0, 1, 0) \mapsto (-a, 0, 0)$$

Then we can compute the eigenvalues of dN_0 to be a and $-a$. In other words, the principal curvatures at 0 are a and $-a$, which give us $K = -a^2$ and $H = 0$. ■

Question 57

5. Consider the parametrized surface (Enneper's surface)

$$\mathbf{x}(u, v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right)$$

and show that

a. The coefficients of the first fundamental form are

$$E = G = (1 + u^2 + v^2)^2, \quad F = 0.$$

b. The coefficients of the second fundamental form are

$$e = 2, \quad g = -2, \quad f = 0.$$

c. The principal curvatures are

$$k_1 = \frac{2}{(1 + u^2 + v^2)^2}, \quad k_2 = -\frac{2}{(1 + u^2 + v^2)^2}.$$

d. The lines of curvature are the coordinate curves.

e. The asymptotic curves are $u + v = \text{const.}$, $u - v = \text{const.}$

Proof. **(a)** Compute

$$\begin{aligned} \mathbf{x}_u &= (1 - u^2 + v^2, 2uv, 2u) \\ \mathbf{x}_v &= (2uv, 1 - v^2 + u^2, -2v) \end{aligned}$$

This give us

$$\begin{aligned} E &= (1 - u^2 + v^2)^2 + 4u^2v^2 + 4u^2 = (1 + u^2 + v^2)^2 \\ G &= (1 - v^2 + u^2)^2 + 4u^2v^2 + rv^2 = (1 + u^2 + v^2)^2 \\ F &= 2uv(1 - u^2 + v^2 + 1 - v^2 + u^2) - 4uv = 0 \end{aligned}$$

(b) Compute

$$\begin{aligned} \mathbf{x}_{uu} &= (-2u, 2v, 2) \\ \mathbf{x}_{vv} &= (2u, -2v, -2) \\ \mathbf{x}_{uv} &= (2v, 2u, 0) \end{aligned}$$

Compute

$$\sqrt{EG - F^2} = (1 + u^2 + v^2)^2$$

Compute

$$\begin{aligned} \left|\mathbf{x}_u \ \ \mathbf{x}_v \ \ \mathbf{x}_{uu}\right| &= 2(1 + u^2 + v^2)^2 \\ \left|\mathbf{x}_u \ \ \mathbf{x}_v \ \ \mathbf{x}_{vv}\right| &= -2(1 + u^2 + v^2)^2 \\ \left|\mathbf{x}_u \ \ \mathbf{x}_v \ \ \mathbf{x}_{uv}\right| &= 0 \end{aligned}$$

This then give us

$$e = 2 \text{ and } g = -2 \text{ and } f = 0$$

(c) Compute

$$K = \frac{eg - f^2}{EG - F^2} = \frac{-4}{(1 + u^2 + v^2)^4}$$

and

$$H = \frac{Ge + gE - 2fF}{2(EG - F^2)} = 0$$

This tell us $-\kappa_1 = \kappa_2 = \sqrt{K}$ and

$$\kappa_1 = \frac{-2}{(1 + u^2 + v^2)^2} \text{ and } \kappa_2 = \frac{2}{(1 + u^2 + v^2)^2}$$

(d) Given $\alpha(t) = \mathbf{x}(u(t), v(t))$. We know

$$\alpha \text{ is a line of curvature } \iff \mathrm{II}_p(\alpha') = (\kappa_1 \text{ or } \kappa_2) \left|\alpha'\right|^2$$

Plugin the first fundamental form and second fundamental form, we now know that α is a line of curvature if and only if

$$e(u')^2 + 2f(u')(v') + g(v')^2 = (\kappa_1 \text{ or } \kappa_2)\big(E(u')^2 + 2F(u')(v') + G(v')^2\big)$$

We have already known the value of the coefficients and the value of κ_1, κ_2, so we can deduce α is a line of curvature if and only if

$$2(u')^2 - 2(v')^2 = \pm 2\big((u')^2 + (v')^2\big)$$

The solution of this equation is clearly

$$u' = 0 \text{ or } v' = 0$$

This implies that

$$\alpha \text{ is a line of curvature} \iff u'=0 \text{ or } v'=0 \iff \alpha \text{ is a coordinate curve}$$

(e) Given $\alpha(t)=\mathbf{x}(u(t),v(t))$. Observe

$$\begin{aligned}\alpha \text{ is an asymptotic curve} &\iff \mathrm{II}_p(\alpha')=0\\ &\iff e(u')^2+2f(u')(v')+g(v')^2=0\\ &\iff (u')^2-(v')^2=0\\ &\iff u'=v' \text{ or } u'=-v'\\ &\iff (u+v)'=0 \text{ or } (u-v)'=0\\ &\iff u+v=\text{const. or } u-v=\text{const.}\end{aligned}$$

■

Question 58

6. (*A Surface with $K \equiv -1$; the Pseudosphere.*)

***a.** Determine an equation for the plane curve C, which is such that the segment of the tangent line between the point of tangency and some line r in the plane, which does not meet the curve, is constantly equal to 1 (this curve is called the *tractrix*; see Fig. 1-9).

b. Rotate the tractrix C about the line r; determine if the "surface" of revolution thus obtained (the *pseudosphere*; see Fig. 3-22) is regular and find out a parametrization in a neighborhood of a regular point.

c. Show that the Gaussian curvature of any regular point of the pseudosphere is -1.

Proof. **(a)** In HW1, we have seen that the following plane curve $\alpha:(0,\frac{\pi}{2})\to\mathbb{R}^2$ satisfy the desired condition

$$\alpha(t)=\left(\sin t, \cos t+\ln(\tan\frac{t}{2})\right)$$

See at the end of this HW a proof that α satisfy the desired condition.

From now, we let $C = \alpha\Big((0, \frac{\pi}{2})\Big)$. Note that C is regular, and the rotation of C is only the upper half of the usual Pseudosphere. To see a picture of our pseudosphere, see Fig. 3-22, and note that our pseudosphere have empty intersection with the x, y-plane.

(b) We here give a more abstract result. Suppose

$$\gamma(t) = \Big(f(t), g(t)\Big) \text{ with } f \neq 0 \text{ every where}$$

with domain $I \overset{\text{open}}{\subseteq} \mathbb{R}$ is a regular parametrized smooth curve. We claim $\mathbf{x} : (0, 2\pi) \times I$

$$\mathbf{x}(\theta, t) \triangleq \Big(f(t)\cos\theta, f(t)\sin\theta, g(t)\Big)$$

is a regular chart.

It is clear that $\mathbf{x}$ is smooth. We now show $d\mathbf{x}$ is one-to-one on $(0, 2\pi) \times I$. Compute

$$\mathbf{x}_\theta = \Big(-f\sin\theta, f\cos\theta, 0\Big) \text{ and } \mathbf{x}_t = (f'\cos\theta, f'\sin\theta, g')$$

Assume $d\mathbf{x}$ is not one-to-one. We then can deduce

$$fg'\cos\theta = fg'\sin\theta = ff' = 0 \text{ for some } (t, \theta)$$

Fix such (t, θ). Because $f \neq 0$, we then can deduce

$$f'(t) = g'(t)\cos\theta = g'(t)\sin\theta = 0$$

Because $\cos\theta$ and $\sin\theta$ can not be both 0, we then can deduce

$$f'(t) = g'(t) = 0$$

which CaC to the premise γ is a regular parametrization.

Lastly, we are required to show

$$\mathbf{x}^{-1} \text{ is continuous}$$

Express

$$\mathbf{x}(\theta, t) \triangleq \Big(x, y, z\Big)(\theta, t)$$

We wish to show

$$\theta, t \text{ are continuous functions in } (x, y, z)$$

Because $z(\theta, t) = g(t)$, we know

$$t = g^{-1}(z)$$

This implies t is a continuous function in (x, y, z).

Compute

$$\theta = \begin{cases} \arctan \frac{y}{x} & \text{if } x, y \in \mathbb{R}^+ \text{ (first quadrant)} \\ \frac{\pi}{2} & \text{if } x = 0, y \in \mathbb{R}^+ \\ \frac{3\pi}{2} + \arctan \frac{y}{x} & \text{if } x \in \mathbb{R}^-, y \in \mathbb{R}^+ \text{ (second quadrant)} \\ \pi + \arctan \frac{y}{x} & \text{if } x \in \mathbb{R}^-, y \in \mathbb{R}_0^- \text{ (third quadrant)} \\ \frac{3\pi}{2} & \text{if } x = 0, y \in \mathbb{R}^- \\ \frac{4\pi}{2} + \arctan \frac{y}{x} & \text{if } x \in \mathbb{R}^+, y \in \mathbb{R}^- \text{ (forth quadrant)} \end{cases}$$

This implies θ is a continuous function in (x, y, z). (done)

(c) Let S be surface of revolution of C. We are given the chart

$$\mathbf{x}(\theta, t) = \Big(\sin t \cos\theta, \sin t \sin\theta, \cos t + \ln(\tan \frac{t}{2})\Big)$$

Note that $\mathbf{x}(\theta, t)$ dose not cover all of S, and

$$\mathbf{y}(\theta, t) = \Big(\sin t \cos(\theta + \frac{\pi}{2}), \sin t \sin(\theta + \frac{\pi}{2}), \cos t + \ln(\tan \frac{t}{2})\Big)$$

cover the rest of S. Note that $\mathbf{y}$ is merely a rotation of $\mathbf{x}$, so if we show that $\mathbf{x}(U)$ has constant Gauss curvature -1, then the proof is finished.

Compute

$$\mathbf{x}_\theta = \Big(-\sin t \sin\theta, \sin t \cos\theta, 0\Big) \text{ and } \mathbf{x}_t = \Big(\cos t \cos\theta, \cos t \sin\theta, -\sin t + \frac{\sec^2 \frac{t}{2}}{2 \tan \frac{t}{2}}\Big)$$

Simplify the z-component of $\mathbf{x}_t$

$$-\sin t + \frac{\sec^2 \frac{t}{2}}{2 \tan \frac{t}{2}} = -\sin t + \frac{1}{2 \sin \frac{t}{2} \cos \frac{t}{2}} = -\sin t + \frac{1}{\sin t} = \cos t \cot t \tag{4.6}$$

We now have

$$\mathbf{x}_\theta = \sin t\Big(-\sin\theta, \cos\theta, 0\Big) \text{ and } \mathbf{x}_t = \cos t\Big(\cos\theta, \sin\theta, \cot t\Big)$$

We can now compute

$$\begin{aligned} E &= \sin^2 t \\ F &= 0 \\ G &= \cot^2 t \end{aligned}$$

Using the fact $N \perp \mathbf{x}_\theta, \mathbf{x}_t$ and Equation 4.6, we can conclude

$$N \text{ is parallel with } \Big(\cot t \cos\theta, \cot t \sin\theta, -1 \Big)$$

and conclude

$$N = \Big(\cos t \cos\theta, \cos t \sin\theta, -\sin t \Big)$$

Compute

$$N_t = \Big(-\sin t \cos\theta, -\sin t \sin\theta, -\cos t \Big) \text{ and } N_\theta = \Big(-\cos t \sin\theta, \cos t \cos\theta, 0 \Big)$$

Compute

$$\begin{aligned} e &= -N_\theta \cdot \mathbf{x}_\theta = -\sin t \cos t \\ f &= -N_\theta \cdot \mathbf{x}_t = 0 \\ g &= -N_t \cdot \mathbf{x}_t = \cot t \end{aligned}$$

Finally, we conclude

$$K = \frac{eg - f^2}{EG - F^2} = \frac{-\sin t \cos t \cot t}{\sin^2 t \cot^2 t} = -1$$

■

Lemma 4.5.1. (Umbilical Lemma)

$$p \text{ is umbilical } \iff dN_p(v) \cdot N \times v = 0 \text{ for all } v \in T_pS$$

Proof. From left to right is clear. We prove only from right to left. Fix arbitrary $v \in T_pS$. We know $\{N, v, N \times v\}$ is an orthogonal basis. Express $dN_p(v)$ in the form

$$dN_p(v) = \lambda_1 v + \lambda_2 N + \lambda_3 N \times v$$

Because $dN_p(v) \in T_pS$, we know $\lambda_2 = 0$. Using $dN_p(v) \cdot N \times v = 0$, we can further deduce $\lambda_3 = 0$. We now see $dN_p(v) = \lambda_1 v$. Because v is arbitrary, our proof is finished. ■

Question 59

20. Determine the umbilical points of the elipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Proof. **(Elipsoid Case)**

We claim

$p=(x,y,z)$ is umbilical $\iff$ For all $(v_1,v_2,v_3)\in T_pS$ we have the following three equations

$$\frac{-xv_2v_3}{a^2}(\frac{1}{b^2}-\frac{1}{c^2})+\frac{yv_1v_3}{b^2}(\frac{1}{a^2}-\frac{1}{c^2})+\frac{-zv_1v_2}{c^2}(\frac{1}{a^2}-\frac{1}{b^2})=0 \tag{4.7}$$

$$\frac{xv_1}{a^2}+\frac{yv_2}{b^2}+\frac{zv_3}{c^2}=0 \tag{4.8}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \tag{4.9}$$

It is clear that $(x,y,z)\in S$ if and only if Equation 4.8 and Equation 4.9 are both satisfied. The proof of our claim now can be reduced to proving p is umbilical if and only if Equation 4.7 is satisfied.

Using Gradient, It is easy to see that we have an orientation

$$N(x,y,z)=\frac{\left(\frac{x}{a^2},\frac{y}{b^2},\frac{z}{c^2}\right)}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}$$

Define $h:S\to\mathbb{R}$ by

$$h(x,y,z)\triangleq\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}$$

Fix an arc-length parametrized curve $\alpha:I\to S$, passing through $p=\alpha(0)$ and express

$$\alpha(s)=\Big(x(s),y(s),z(s)\Big)$$

We have

$$(h\circ\alpha)(N\circ\alpha)=\Big(\frac{x}{a^2},\frac{y}{b^2},\frac{z}{c^2}\Big)$$

Differentiating both side yield us

$$(h\circ\alpha)'N(\alpha)+h(\alpha)dN_\alpha(\alpha')=\Big(\frac{x'}{a^2},\frac{y'}{b^2},\frac{z'}{c^2}\Big)$$

and give us

$$h(p)dN_p(v)=\Big(\frac{x'}{a^2},\frac{y'}{b^2},\frac{z'}{c^2}\Big)-(h\circ\alpha)'N(p)\text{ where }v=\alpha'(0)=(x',y',z')$$

Note that $(h\circ\alpha)'N$ is parallel with N. Now by Lemma 4.5.1, we see p is umbilical if and only if for all α we have

$$\begin{vmatrix}\frac{x'}{a^2}&\frac{y'}{b^2}&\frac{z'}{c^2}\\ \frac{x}{a^2}&\frac{y}{b^2}&\frac{z}{c^2}\\ x'&y'&z'\end{vmatrix}=0$$

Expand the above determinant and substitute (x',y',z') with (v_1,v_2,v_3). We see this is exactly Equation 4.7. (done)

Now, WOLG, suppose $0<a<b<c$.

We claim

umbilical point will never be in the plane $z=0$

Assume $z=0$ and $p=(x,y,0)$ is an umbilical point . For all $(v_1,v_2,v_3)\in T_pS$, we have

$$\frac{-xv_2v_3}{a^2}\Big(\frac{1}{b^2}-\frac{1}{c^2}\Big)+\frac{yv_1v_3}{b^2}\Big(\frac{1}{a^2}-\frac{1}{c^2}\Big)=0 \tag{4.10}$$

$$\frac{xv_1}{a^2}+\frac{yv_2}{b^2}=0 \tag{4.11}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{4.12}$$

Equation 4.12 implies that either $x\neq 0$ or $y\neq 0$. WOLG suppose $x\neq 0$. We can now fix $(v_1,v_2,v_3)\in T_pS$ such that $v_2v_3\neq 0$.

By multiplying with $\frac{x}{v_3}$ on both side, Equation 4.10 now implies

$$\frac{x^2v_2}{a^2}\Big(\frac{1}{b^2}-\frac{1}{c^2}\Big)=\frac{xyv_1}{b^2}\Big(\frac{1}{a^2}-\frac{1}{c^2}\Big)$$

Note that Equation 4.11 give us $\frac{y}{b^2} = \frac{-xv_1}{a^2v_2}$. This then implies

$$\frac{x^2v_2}{a^2}(\frac{1}{b^2} - \frac{1}{c^2}) = \frac{xyv_1}{b^2}(\frac{1}{a^2} - \frac{1}{c^2}) = \frac{-v_1^2x^2}{a^2v_2}(\frac{1}{a^2} - \frac{1}{c^2})$$

We can now use $a < b < c$ to deduce

$$0 < \frac{x^2v_2^2}{a^2}(\frac{1}{b^2} - \frac{1}{c^2}) = \frac{-v_1^2x^2}{a^2}(\frac{1}{a^2} - \frac{1}{c^2}) < 0 \text{ CaC} \quad \text{(done)}$$

Lastly, we claim p is an umbilical point if and only if

$$x^2 = a^2\frac{b^2 - a^2}{c^2 - a^2} \text{ and } y = 0 \text{ and } z^2 = c^2\frac{c^2 - b^2}{c^2 - a^2}$$

Because $z \neq 0$. We can now replace Equation 4.7 with

$$\frac{-xzv_2v_3}{a^2c^2}(\frac{1}{b^2} - \frac{1}{c^2}) + \frac{yzv_1v_3}{b^2c^2}(\frac{1}{a^2} - \frac{1}{c^2}) + \frac{-z^2v_1v_2}{c^4}(\frac{1}{a^2} - \frac{1}{b^2}) = 0 \tag{4.13}$$

Equation 4.8 give us $\frac{z}{c^2} = \frac{-xv_1}{a^2v_3} + \frac{-yv_2}{b^2v_3}$. Substitute this into Equation 4.13 (Except the $\frac{z^2}{c^4}$ term), we have

$$\begin{aligned} &v_2^2\frac{yx}{b^2a^2}\left(\frac{1}{b^2} - \frac{1}{c^2}\right) - v_1^2\frac{yx}{b^2a^2}\left(\frac{1}{a^2} - \frac{1}{c^2}\right) \\ &+v_1v_2\left(\frac{x^2}{a^4}\left(\frac{1}{b^2} - \frac{1}{c^2}\right) - \frac{y^2}{b^4}\left(\frac{1}{a^2} - \frac{1}{c^2}\right) - \frac{z^2}{c^4}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)\right) = 0 \end{aligned} \tag{4.14}$$

Assume $x = 0$, we see

$$0 < \frac{z^2}{c^4}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \frac{-y^2}{b^4}\left(\frac{1}{a^2} - \frac{1}{c^2}\right) < 0 \text{ CaC}$$

Because $x \neq 0$, we know there exists (v_1, v_2, v_3) such that $v_1 = 0$ and $v_2 = 1$. Substituting this back into Equation 4.14, we can deduce

$$xy = 0$$

Because $x \neq 0$. We now know $y = 0$. Again substituting this back into Equation 4.14, we have

$$\frac{x^2}{a^4}\left(\frac{1}{b^2} - \frac{1}{c^2}\right) = \frac{z^2}{c^4}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)$$

Substituting $y = 0$ back into Equation 4.9, we now see

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$$

We can now solve for x^2, z^2 in terms of a, b, c, using only linear algebra. The solution is

$$x^2 = a^2 \frac{b^2 - a^2}{c^2 - a^2} \text{ and } z^2 = c^2 \frac{c^2 - b^2}{c^2 - a^2} \text{ (done)}$$

■

Question 60

22. (*The Hessian.*) Let $h: S \to R$ be a differentiable function on a surface S, and let $p \in S$ be a critical point of h (i.e., $dh_p = 0$). Let $w \in T_p(S)$ and let

$$\alpha: (-\epsilon, \epsilon) \to S$$

be a parametrized curve with $\alpha(0) = p, \alpha'(0) = w$. Set

$$H_p h(w) = \left. \frac{d^2(h \circ \alpha)}{dt^2} \right|_{t=0}.$$

a. Let $\mathbf{x}: U \to S$ be a parametrization of S at p, and show that (the fact that p is a critical point of h is essential here)

$$H_p h(u'\mathbf{x}_u + v'\mathbf{x}_v) = h_{uu}(p)(u')^2 + 2h_{uv}(p)u'v' + h_{vv}(p)(v')^2.$$

Conclude that $H_p h: T_p(S) \to R$ is a well-defined (i.e., it does not depend on the choice of $\mathbf{x}$) quadratic form on $T_p(S)$. $H_p h$ is called the *Hessian* of h at p.

b. Let $h: S \to R$ be the height function of S relative to $T_p(S)$; that is, $h(q) = \langle q - p, N(p) \rangle, q \in S$. Verify that p is a critical point of h and thus that the Hessian $H_p h$ is well defined. Show that if $w \in T_p(S)$, $|w| = 1$, then

$$H_p h(w) = \text{normal curvature at } p \text{ in the direction of } w.$$

Conclude that *the Hessian at* p *of the height function relative to* $\mathrm{T_p(S)}$ *is the second fundamental form of* S *at* p.

Proof. Express

$$\alpha(t) = \mathbf{x}(u(t), v(t))$$

We know

$$w = \alpha' = u'\mathbf{x}_u + v'\mathbf{x}_v$$

Compute

$$\begin{aligned} H_p(u'\mathbf{x}_u + v'\mathbf{x}_v) = H_p h(w) &= (h \circ \alpha)''(0) \\ &= (h_u u' + h_v v')' \\ &= (h_{uu}u' + h_{uv}v')u' + h_u(u'') + (h_{vu}u' + h_{vv}v')v' + h_v(v'') \\ &= h_{uu}(u')^2 + 2h_{uv}(u')(v') + h_{vv}(v')^2 \quad (\because dh_p = 0 \implies h_u = h_v = 0) \end{aligned}$$

We know $\{\mathbf{x}_u, \mathbf{x}_v\}$ is a basis of T_pS. Our computation now show us that for any given vector $c_1\mathbf{x}_u + c_2\mathbf{x}_v \in T_pS$, we have

$$\begin{aligned} H_p(c_1\mathbf{x}_u + c_2\mathbf{x}_v) &= h_{uu}c_1^2 + 2h_{uv}c_1c_2 + h_{vv}c_2^2 \\ &= \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} h_{uu} & h_{uv} \\ h_{uv} & h_{vv} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \end{aligned}$$

This then implies H_p is a quadratic form on T_pS.

Note that the definition of H_ph doesn't take usage of $\mathbf{x}$. We merely defiend

$$H_ph(w) \triangleq (h \circ \alpha)''(0)$$

(b) Compute

$$\begin{aligned} h_u &= \langle \mathbf{x}_u, N(p) \rangle = 0 \\ \text{and } h_v &= \langle \mathbf{x}_v, N(p) \rangle = 0 \end{aligned}$$

This implies $dh_p = 0$, so p is indeed a critical point of h.

Compute

$$\begin{aligned} h_{uu} &= \langle \mathbf{x}_{uu}, N \rangle = e \\ h_{uv} &= \langle \mathbf{x}_{uv}, N \rangle = f \\ h_{vv} &= \langle \mathbf{x}_{vv}, N \rangle = g \end{aligned}$$

This now give us

$$H_ph(w) = (u')^2e + 2u'v'f + g(v')^2 = \mathrm{II}_p(w)$$

and conclude the desired result.

∎

The following is the reference of 3.3.6 **(a)**

Question 61: 1-3:4

4. Let $\alpha: (0, \pi) \longrightarrow R^2$ be given by

$$\alpha(t) = \left(\cos t, \cos t + \log \tan \frac{t}{2}\right),$$

where t is the angle that the y axis makes with the vector $\alpha(t)$. The trace of α is called the *tractrix* (Fig. 1-9). Show that

19:

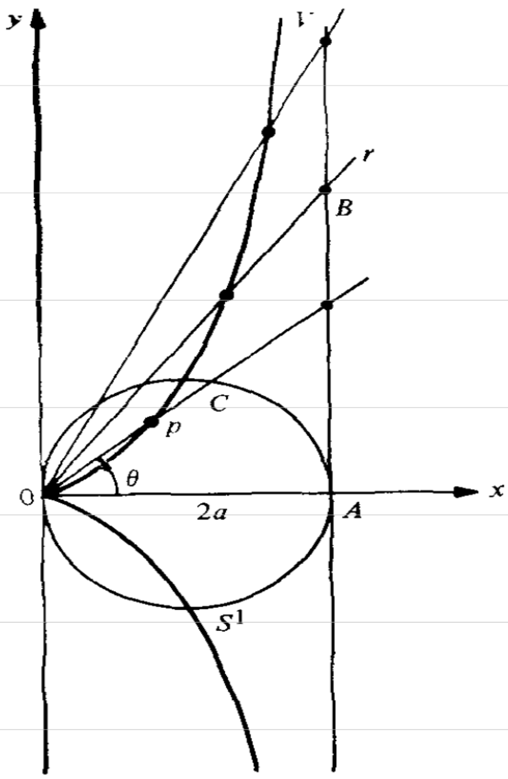

Figure 1-8. The cissoid of Diocles.

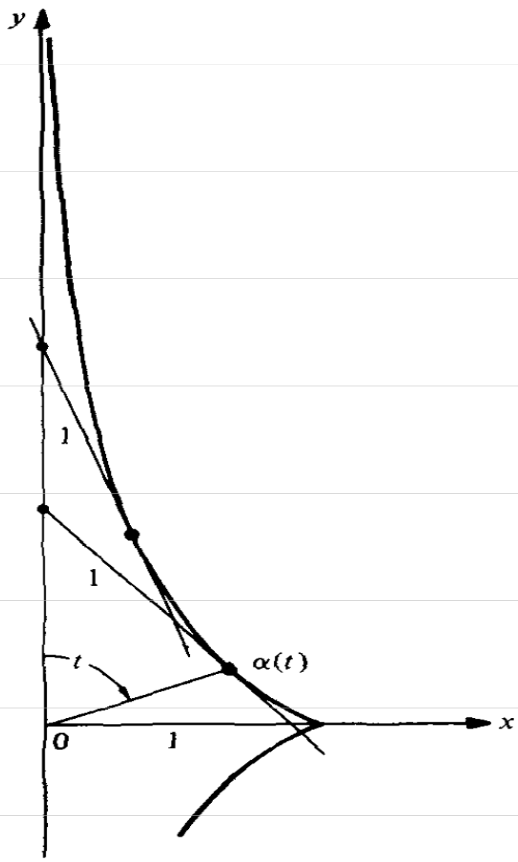

Figure 1-9. The tractrix.

a. α is a differentiable parametrized curve, regular except at $t = \pi/2$.

b. The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.

Typo correction: $\alpha(t) = (\sin t, \cos t + \ln \tan \frac{t}{2})$

Proof. **(a)**

Notice that the interval I is $(0, \pi)$. It is clear that

(a) $\sin t$ is smooth on $\mathbb{R}$

(b) $\cos t$ is smooth on $\mathbb{R}$

(c) $\ln t$ is smooth on $\mathbb{R}^+$

(d) $\tan \frac{t}{2}$ is smooth on I

Then it follows that α is a differentiable curve.

Compute

$$\alpha'(t) = (\cos t, -\sin t + \frac{1}{\tan \frac{t}{2}} \cdot \sec^2 \frac{t}{2} \cdot \frac{1}{2})$$

Because $\cos t = \alpha_1'(t)$ is 0 on I only when $t = \frac{\pi}{2}$, we know α is regular on I except possibly at $t = \frac{\pi}{2}$.

Compute

$$\alpha'(\frac{\pi}{2}) = (0, -1 + \frac{1}{1} \cdot 2 \cdot \frac{1}{2}) = (0, 0)$$

We now conclude α is regular on I except $\frac{\pi}{2}$.

(b)

A useful Identity give us

$$\alpha'(t) = (\cos t, -\sin t + \csc t)$$

From the following facts

(a) the first argument of the segment is from 0 to $\sin t = \alpha(t)$

(b) $\alpha_x'(t) = \cos t$

(c) $\frac{\sin t}{\cos t} = \tan t$

We conclude that the length of the segment is

$$\begin{aligned} |\tan t| \cdot |\alpha'(t)| &= |\tan t| \cdot \sqrt{\cos^2 t + \sin^2 t - 2 \sin t \csc t + \csc^2 t} \\ &= |\tan t| \cdot \sqrt{1 - 2 + \csc^2 t} \\ &= |\tan t| \cdot \sqrt{\csc^2 t - 1} = |\tan t| \cdot \sqrt{\cot^2 t} = 1 \end{aligned}$$

■