17

Normal subgroups and lifted characters

If N is a normal subgroup of the finite group G, and $N \neq \{1\}$, then the factor group G/N is smaller than G. The characters of G/N should therefore be easier to find than the characters of G. In fact, we can use the characters of G/N to get some of the characters of G, by a process which is known as lifting. Thus, normal subgroups help us to find characters of G. In the opposite direction, it is also true that the character table of G enables us to find the normal subgroups of G; in particular, it is easy to tell from the character table whether or not G is simple.

The linear characters of G (i.e. the characters of degree 1) are obtained by lifting the irreducible characters of G/N in the case where N is the derived subgroup of G. (The derived subgroup is defined below in Definition 17.7.) The linear characters, in turn, can be used to get new irreducible characters from a given irreducible character, in a way which we shall describe.

Lifted characters

We begin by constructing a character of G from a character of G/N.

17.1 Proposition
Assume that $N \lhd G$, and let $\tilde{\chi}$ be a character of G/N. Define χ: $G \to \mathbb{C}$ by

$$\chi(g) = \tilde{\chi}(Ng) \quad (g \in G).$$

Then χ is a character of G, and χ and $\tilde{\chi}$ have the same degree.

Proof Let $\tilde{\rho}$: $G/N \to \mathrm{GL}(n, \mathbb{C})$ be a representation of G/N with character $\tilde{\chi}$. The function ρ: $G \to \mathrm{GL}(n, \mathbb{C})$ which is given by the composition

$$g \to Ng \to (Ng)\tilde{\rho} \quad (g \in G)$$

is a homomorphism from G to GL $(n, \mathbb{C})$. Thus ρ is a representation of G. The character χ of ρ satisfies

$$\chi(g) = \mathrm{tr}\,(g\rho) = \mathrm{tr}\,((Ng)\tilde{\rho}) = \tilde{\chi}(Ng)$$

for all $g \in G$. Moreover, $\chi(1) = \tilde{\chi}(N)$, so χ and $\tilde{\chi}$ have the same degree.

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17.2 Definition
If $N \lhd G$ and $\tilde{\chi}$ is a character of G/N, then the character χ of G which is given by

$$\chi(g) = \tilde{\chi}(Ng) \quad (g \in G)$$

is called the *lift* of $\tilde{\chi}$ to G.

17.3 Theorem
Assume that $N \lhd G$. By associating each character of G/N with its lift to G, we obtain a bijective correspondence between the set of characters of G/N and the set of characters χ of G which satisfy $N \leqslant \mathrm{Ker}\ \chi$. Irreducible characters of G/N correspond to irreducible characters of G which have N in their kernel.

Proof If $\tilde{\chi}$ is a character of G/N, and χ is the lift of $\tilde{\chi}$ to G, then $\tilde{\chi}(N) = \chi(1)$. Also, if $k \in N$ then

$$\chi(k) = \tilde{\chi}(Nk) = \tilde{\chi}(N) = \chi(1),$$

so $N \leqslant \mathrm{Ker}\ \chi$.

Now let χ be a character of G with $N \leqslant \mathrm{Ker}\ \chi$. Suppose that $\rho: G \to$ GL $(n, \mathbb{C})$ is a representation of G with character χ. If $g_1, g_2 \in G$ and $Ng_1 = Ng_2$ then $g_1 g_2^{-1} \in N$, so $(g_1 g_2^{-1})\rho = I$, and hence $g_1\rho = g_2\rho$. We may therefore define a function $\tilde{\rho}: G/N \to$ GL $(n, \mathbb{C})$ by

$$(Ng)\tilde{\rho} = g\rho \quad (g \in G).$$

Then for all $g, h \in G$ we have

$$((Ng)(Nh))\tilde{\rho} = (Ngh)\tilde{\rho} = (gh)\rho = (g\rho)(h\rho)$$
$$= ((Ng)\tilde{\rho})((Nh)\tilde{\rho}),$$

so $\tilde{\rho}$ is a representation of G/N. If $\tilde{\chi}$ is the character of $\tilde{\rho}$ then

$$\tilde{\chi}(Ng) = \chi(g) \quad (g \in G).$$

Thus χ is the lift of $\tilde{\chi}$.

We have now established that the function which sends each character of G/N to its lift to G is a bijection between the set of characters of G/N and the set of characters of G which have N in their kernel. It remains to show that irreducible characters correspond to irreducible characters. To see this, let U be a subspace of $\mathbb{C}^n$, and note that

$$u(g\rho) \in U \text{ for all } u \in U \Leftrightarrow u(Ng)\tilde{\rho} \in U \text{ for all } u \in U.$$

Thus, U is a $\mathbb{C}G$-submodule of $\mathbb{C}^n$ if and only if U is a $\mathbb{C}(G/N)$-submodule of $\mathbb{C}^n$. The representation ρ is therefore irreducible if and only if the representation $\tilde{\rho}$ is irreducible. Hence χ is irreducible if and only if $\tilde{\chi}$ is irreducible.

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If we know the character table of G/N for some normal subgroup N of G, then Theorem 17.3 enables us to write down as many irreducible characters of G as there are irreducible characters of G/N.

17.4 Example

Let $G = S_4$ and

$$N = V_4 = \{1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\},$$

so that $N \lhd G$ (see Example 12.20). If we put $a = N(1\ 2\ 3)$ and $b = N(1\ 2)$ then

$$G/N = \langle a, b\rangle \text{ and } a^3 = b^2 = N, b^{-1}ab = a^{-1},$$

so $G/N \cong D_6$. We know from Example 16.3(1) that the character table of G/N is

	N	$N(1\ 2)$	$N(1\ 2\ 3)$
$\tilde{\chi}_1$	1	1	1
$\tilde{\chi}_2$	1	-1	1
$\tilde{\chi}_3$	2	0	-1

To calculate the lift χ of a character $\tilde{\chi}$ of G/N, we note that

$$\chi((1\ 2)(3\ 4)) = \tilde{\chi}(N) \quad \text{since } (1\ 2)(3\ 4) \in N,$$

$$\chi((1\ 2\ 3\ 4)) = \tilde{\chi}(N(1\ 3)) \quad \text{since } N(1\ 2\ 3\ 4) = N(1\ 3).$$

Hence the lifts of $\tilde{\chi}_1, \tilde{\chi}_2, \tilde{\chi}_3$ are χ_1, χ_2, χ_3, which are given by

	1	$(1\ 2)$	$(1\ 2\ 3)$	$(1\ 2)(3\ 4)$	$(1\ 2\ 3\ 4)$
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0

Then χ_1, χ_2, χ_3 are irreducible characters of G, since $\tilde{\chi}_1$, $\tilde{\chi}_2$, $\tilde{\chi}_3$ are irreducible characters of G/N.

Finding normal subgroups

The character table contains accessible information about the structure of a group, as our next two propositions will demonstrate. First we shall show how to find all the normal subgroups of G, once the character table of G is known. Recall that we can easily locate the kernel of an irreducible character χ from the character table, since

$$\operatorname{Ker}\chi = \{g \in G: \chi(g) = \chi(1)\}$$

(see Definition 13.13). Also Ker $\chi \lhd G$. Of course, any subgroup which is the intersection of the kernels of irreducible characters is a normal subgroup too. The following proposition shows that *every* normal subgroup arises in this way.

17.5 Proposition
If $N \lhd G$ then there exist irreducible characters $\chi_1, \ldots, \chi_s$ of G such that

$$N = \bigcap_{i=1}^{s} \operatorname{Ker} \chi_i.$$

Proof If g belongs to the kernel of each irreducible character of G, then $\chi(g) = \chi(1)$ for all characters χ, so $g = 1$ by Proposition 15.5. Hence the intersection of the kernels of all the irreducible characters of G is $\{1\}$.

Now let $\tilde{\chi}_1, \ldots, \tilde{\chi}_s$ be the irreducible characters of G/N. By the above observation,

$$\bigcap_{i=1}^{s} \operatorname{Ker} \tilde{\chi}_i = \{N\}.$$

For $1 \leqslant i \leqslant s$, let χ_i be the lift to G of $\tilde{\chi}_i$. If $g \in \operatorname{Ker} \chi_i$ then

$$\tilde{\chi}_i(N) = \chi_i(1) = \chi_i(g) = \tilde{\chi}_i(Ng),$$

and so $Ng \in \operatorname{Ker} \tilde{\chi}_i$. Therefore if $g \in \bigcap \operatorname{Ker} \chi_i$ then $Ng \in \bigcap \operatorname{Ker} \tilde{\chi}_i = \{N\}$, and so $g \in N$. Hence

$$N = \bigcap_{i=1}^{s} \operatorname{Ker} \chi_i.$$

■

It is particularly easy to tell from the character table of G whether or not G is simple:

17.6 Proposition
The group G is not simple if and only if

$$\chi(g) = \chi(1)$$

for some non-trivial irreducible character χ of G, and some non-identity element g of G.

Proof Suppose there is a non-trivial irreducible character χ such that $\chi(g) = \chi(1)$ for some non-identity element g. Then $g \in \operatorname{Ker} \chi$, so $\operatorname{Ker} \chi \neq \{1\}$. If ρ is a representation of G with character χ, then $\operatorname{Ker} \chi = \operatorname{Ker} \rho$ by Theorem 13.11(2). Since χ is non-trivial and irreducible, $\operatorname{Ker} \rho \neq G$; hence $\operatorname{Ker} \chi \neq G$. Thus $\operatorname{Ker} \chi$ is a normal subgroup of G which is not equal to $\{1\}$ of G, and so G is not simple.

Conversely, suppose that G is not simple, so that there is a normal subgroup N of G with $N \neq \{1\}$ and $N \neq G$. Then by Proposition 17.5, there is an irreducible character χ of G such that $\operatorname{Ker} \chi$ is not $\{1\}$ or G. As $\operatorname{Ker} \chi \neq G$, χ is non-trivial; and taking $1 \neq g \in \operatorname{Ker} \chi$, we have $\chi(g) = \chi(1)$.

■

Linear characters

Recall that a linear character of a group is a character of degree 1. We shall show how to find all linear characters of any group G, since the first move in constructing the character table of G is often to write down the linear characters. As a preliminary step, it is necessary to determine the derived subgroup of G, which is defined in the following way.

17.7 Definition
For a group G, let G' be the subgroup of G which is generated by all elements of the form

$$g^{-1}h^{-1}gh \quad (g, h \in G).$$

Then G' is called the *derived subgroup* of G.

We abbreviate $g^{-1}h^{-1}gh$ as $[g, h]$. Thus

$$G' = \langle [g, h]\colon\ g, h \in G \rangle.$$

17.8 Examples
(1) If G is abelian then $[g, h] = 1$ for all $g, h \in G$, so $G' = \{1\}$.

(2) Let $G = S_3$. Clearly $[g, h]$ is always an even permutation, so $G' \leqslant A_3$. If $g = (1\ 2)$ and $h = (2\ 3)$ then $[g, h] = (1\ 2\ 3)$. Hence $G' = \langle (1\ 2\ 3) \rangle = A_3$.

We are going to show that $G' \lhd G$ and that the linear characters of G are the lifts to G of the irreducible characters of G/G'. One step is provided by the following proposition.

17.9 Proposition
If χ is a linear character of G, then $G' \leqslant \operatorname{Ker}\chi$.

Proof Let χ be a linear character of G. Then χ is a homomorphism from G to the multiplicative group of non-zero complex numbers. Therefore, for all g, $h \in G$,

$$\chi(g^{-1}h^{-1}gh) = \chi(g)^{-1}\chi(h)^{-1}\chi(g)\chi(h) = 1.$$

Hence $G' \leqslant \operatorname{Ker}\chi$.

■

Next, we explore some group-theoretic properties of the derived subgroup.

17.10 Proposition
Assume that $N \lhd G$.
(1) *$G' \lhd G$.*
(2) *$G' \leqslant N$ if and only if G/N is abelian. In particular, G/G' is abelian.*

Proof (1) Note that for all $a, b, x \in G$, we have

$$x^{-1}(ab)x = (x^{-1}ax)(x^{-1}bx), \text{ and}$$
$$x^{-1}a^{-1}x = (x^{-1}ax)^{-1}.$$

Now G' consists of products of elements of the form $[g, h]$ and their inverses. Therefore, to prove that $G' \lhd G$ it is sufficient by the first sentence to prove that $x^{-1}[g, h]x \in G'$ for all $g, h, x \in G$. But

$$\begin{aligned} x^{-1}[g, h]x &= x^{-1}g^{-1}h^{-1}ghx \\ &= (x^{-1}gx)^{-1}(x^{-1}hx)^{-1}(x^{-1}gx)(x^{-1}hx) \\ &= [x^{-1}gx, x^{-1}hx]. \end{aligned}$$

Therefore $G' \lhd G$.

(2) Let $g, h \in G$. We have

$$ghg^{-1}h^{-1} \in N \Leftrightarrow Ngh = Nhg \Leftrightarrow (Ng)(Nh) = (Nh)(Ng).$$

Hence $G' \leqslant N$ if and only if G/N is abelian. Since we have proved that $G' \lhd G$, we deduce that G/G' is abelian.

■

It follows from Proposition 17.10 that G' is the smallest normal subgroup of G with abelian factor group.

Given the derived subgroup G', we can obtain the linear characters of G by applying the next theorem.

17.11 Theorem

The linear characters of G are precisely the lifts to G of the irreducible characters of G/G'. In particular, the number of distinct linear characters of G is equal to $|G/G'|$, and so divides $|G|$.

Proof Let $m = |G/G'|$. Since G/G' is abelian, Theorem 9.8 shows that G/G' has exactly m irreducible characters $\tilde{\chi}_1, \ldots, \tilde{\chi}_m$, all of degree 1. The lifts $\chi_1, \ldots, \chi_m$ of these characters to G also have degree 1, and by Theorem 17.3

they are precisely the irreducible characters χ of G such that $G' \leqslant \operatorname{Ker} \chi$. In view of Proposition 17.9, the characters $\chi_1, \ldots, \chi_m$ are therefore all the linear characters of G.

■

17.12 Example

Let $G = S_n$. We shall show that $G' = A_n$. If $n = 1$ or 2 then S_n is abelian, so $G' = \{1\} = A_n$. We proved that $S_3' = A_3$ in Example 17.8(2), so we assume that $n \geqslant 4$.

As $S_n/A_n \cong C_2$, we have $G' \leqslant A_n$ by Proposition 17.10(2). If $g = (1\ 2)$, $h = (2\ 3)$ and $k = (1\ 2)(3\ 4)$, then

$$[g, h] = (1\ 2\ 3),\ [h, k] = (1\ 4)(2\ 3).$$

Since $G' \lhd G$, all the elements in $(1\ 2\ 3)^G$ and $(1\ 4)(2\ 3)^G$ belong to G'. Therefore, by Theorem 12.15, G' contains all 3-cycles and all elements of cycle-shape (2, 2). But every product of two transpositions is equal to the identity, a 3-cycle or an element of cycle-shape (2, 2); and A_n consists of permutations, each of which is the product of an even number of transpositions. Therefore $A_n \leqslant G'$. We have now proved that $G' = A_n$.

17.13 Example

We find the linear characters of S_n ($n \geqslant 2$). From the last example, we know that $S_n' = A_n$. Since $S_n/S_n' = \{A_n, A_n(1\ 2)\} \cong C_2$, the group S_n/S_n' has two linear characters $\tilde{\chi}_1$ and $\tilde{\chi}_2$, where

$$\tilde{\chi}_1(A_n(1\ 2)) = 1,$$

$$\tilde{\chi}_2(A_n(1\ 2)) = -1.$$

Therefore by Theorem 17.11, S_n has exactly two linear characters χ_1, χ_2, which are given by

$$\chi_1 = 1_{S_n},$$
$$\chi_2(g) = \begin{cases} 1, & \text{if } g \in A_n, \\ -1, & \text{if } g \notin A_n. \end{cases}$$

Not only are the linear characters of G important in being irreducible characters, but they can also be used to construct new irreducible characters from old, as the next result shows.

17.14 Proposition
Suppose that χ is a character of G and λ is a linear character of G. Then the product $\chi\lambda$, defined by

$$\chi\lambda(g) = \chi(g)\lambda(g) \quad (g \in G)$$

is a character of G. Moreover, if χ is irreducible, then so is $\chi\lambda$.

Proof Let ρ: $G \to \mathrm{GL}\,(n, \mathbb{C})$ be a representation with character χ. Define $\rho\lambda$: $G \to \mathrm{GL}\,(n, \mathbb{C})$ by

$$g(\rho\lambda) = \lambda(g)(g\rho) \quad (g \in G).$$

Thus $g(\rho\lambda)$ is the matrix $g\rho$ multiplied by the complex number $\lambda(g)$. Since ρ and λ are homomorphisms it follows easily that $\rho\lambda$ is a homomorphism. The matrix $g(\rho\lambda)$ has trace $\lambda(g)$ tr $(g\rho)$, which is $\lambda(g)\chi(g)$. Hence $\rho\lambda$ is a representation of G with character $\chi\lambda$.

Now for all $g \in G$, the complex number $\lambda(g)$ is a root of unity, so $\lambda(g)\overline{\lambda(g)} = 1$. Therefore

$$\begin{aligned} \langle \chi\lambda, \chi\lambda \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi(g)\lambda(g)\overline{\chi(g)\lambda(g)} \\ &= \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\chi(g)} = \langle \chi, \chi \rangle. \end{aligned}$$

By Theorem 14.20, it follows that $\chi\lambda$ is irreducible if and only if χ is irreducible.

■

The general case of a product of two characters will be discussed in Chapter 19.

Summary of Chapter 17

1. Characters of G/N correspond to characters χ of G for which $N \leqslant \operatorname{Ker} \chi$. The character of G which corresponds to the character $\tilde{\chi}$ of G/N is the *lift* of $\tilde{\chi}$, and is given by $\chi(g) = \tilde{\chi}(Ng)$ $(g \in G)$.
2. The normal subgroups of G can be found from the character table of G.
3. The linear characters of G are precisely the lifts to G of the irreducible characters of G/G'.

Exercises for Chapter 17

1. Let $G = Q_8 = \langle a, b\colon a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1}\rangle$.
   (a) Find the five conjugacy classes of G.
   (b) Find G', and construct all the linear characters of G.
   (c) Complete the character table of G.
   Compare your table with the character table of D_8 (Example 16.3(3)).

2. Let a and b be the following permutations in S_7:

$$a = (1\ 2\ 3\ 4\ 5\ 6\ 7),\ b = (2\ 3\ 5)(4\ 7\ 6).$$

   Let $G = \langle a, b\rangle$. Check that

$$a^7 = b^3 = 1,\ b^{-1}ab = a^2.$$

   (a) Show that G has order 21.
   (b) Find the conjugacy classes of G.
   (c) Find the character table of G.

3. Show that every group of order 12 has 3, 4 or 12 linear characters, and hence cannot be simple.
4. A certain group G of order 12 has precisely six conjugacy classes, with representatives $g_1, \ldots, g_6$ (where $g_1 = 1$), and has irreducible characters χ, ϕ with values as follows:

	g_1	g_2	g_3	g_4	g_5	g_6
χ	1	$-i$	i	1	-1	-1
ϕ	2	0	0	-1	-1	2

Use Proposition 17.14 to complete the character table of G. What are the sizes of the conjugacy classes of G?

5. The character table of D_8 is as shown (see Example 16.3(3)):

	1	a^2	a, a^3	b, a^2b	ab, a^3b
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

Express each normal subgroup of D_8 as an intersection of kernels of irreducible characters, as in Proposition 17.5.

6. You are given that the group

$$T_{4n} = \langle a, b \colon a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle$$

has order $4n$. (It is known as a *dicyclic group*.)

(a) Show that if ε is any $(2n)$th root of unity in $\mathbb{C}$, then there is a representation of T_{4n} over $\mathbb{C}$ which sends

$$a \to \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \; b \to \begin{pmatrix} 0 & 1 \\ \varepsilon^{n} & 0 \end{pmatrix}.$$

(b) Find all the irreducible representations of T_{4n}.

7. For $n \geqslant 1$, the group

$$U_{6n} = \langle a, b \colon a^{2n} = b^3 = 1, \; a^{-1}ba = b^{-1} \rangle$$

has order $6n$.

(a) Let $\omega = e^{2\pi i/3}$. Show that if ε is any $(2n)$th root of unity in $\mathbb{C}$, then there is a representation of U_{6n} over $\mathbb{C}$ which sends

$$a \to \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix}, \; b \to \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}.$$

(b) Find all the irreducible representations of U_{6n}.

8. Let n be an odd positive integer. The group

$$V_{8n} = \langle a, b \colon a^{2n} = b^4 = 1, \; ba = a^{-1}b^{-1}, \; b^{-1}a = a^{-1}b \rangle$$

has order $8n$.

(a) Show that if ε is any nth root of unity in $\mathbb{C}$, then there is a representation of V_{8n} over $\mathbb{C}$ which sends

$$a \to \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon^{-1} \end{pmatrix}, \; b \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(b) Find all the irreducible representations of V_{8n}.