NCKU 112.2
Geometry 1

Eric Liu

Contents

Chapter 1

HW

1.1 HW1

In this section, by a **curve** in $\mathbb{R}^n$, we mean a function form an open interval I to $\mathbb{R}^n$. We say a curve is **differentiable** if it is infinitely differentiable (smooth). That is, $\gamma^{(n)}(t)$ exists for all $n \in \mathbb{N}$ and for all $t \in I$. Clearly, for each differentiable curve γ, the function $\gamma^{(n)} : I \to \mathbb{R}$ (also a curve) must be continuous. We say a differentiable curve $\gamma : I \to \mathbb{R}^n$ is **regular** if $\gamma'(t) \neq 0$ for all $t \in I$. We say a differentiable curve $\gamma : I \to \mathbb{R}^n$ is a **parametrized by arc-length** if $|\gamma'(t)| = 1$ for all $t \in I$.

Trick to parametrize by arc-length.

Given a regular curve $\gamma : I \to \mathbb{R}^n$ and fix $t_0 \in I$. We use

$$s(t) = \int_{t_0}^{t} |\gamma'(x)|\, dx$$

to define the arc-length of γ from $\gamma(t_0)$ to $\gamma(t)$. Because γ is regular, by FTC, it is clear that s is one-to-one.

Let $t(s)$ be the inverse of s. Define

$$\beta(s) \triangleq \alpha(t(s))$$

We have by Chain rule

$$\begin{aligned}\beta'(s) &= t'(s)\alpha'(t(s)) \\ &= \frac{\alpha'(t(s))}{s'(t)} \\ &= \frac{\alpha'(t(s))}{|\alpha'(t(s))|}\end{aligned}$$

Now, β is clearly a regular curve, and

$$\int_0^x |\beta'(s)|\, ds = x$$

Now, suppose a curve $\gamma(s)$ is parametrized by arc-length. We see that for all $s \in I$

$$\frac{d}{ds}(\gamma' \cdot \gamma')(s) = 2(\gamma'' \cdot \gamma')(s)$$

Then because γ' is constant 1, this implies for all s

$$\gamma''(s) \perp \gamma'(s) \tag{1.1}$$

This let us naturally define the **curvature** κ of γ by

$$\kappa(s) = |\gamma''(s)|$$

It is clear that if γ is linear (a straight line), then the curvature $\kappa(s)$ is 0 for all s.

For a regular curve γ, we define its **unit tangent** by

$$T(t) = \frac{\gamma'(t)}{|\gamma'(t)|}$$

and we define its **unit normal** by

$$N(t) = \frac{T'(t)}{|T'(t)|}$$

and define its **binormal** vector by

$$B(t) = T(t) \times N(t)$$

Notice that we have $T(t) \perp N(t)$ from Equation 1.1. Fix t_0. We say

$$\{T(t), N(t), B(t)\} \text{ form a } \textbf{positively oriented orthonormal basis} \text{ of } \mathbb{R}^3$$

This basis in general is constantly changing, yet always form an orthonormal basis.

In a geoemtric sense, we shall note that the curve $\gamma : I \to \mathbb{R}^3$ stay on a plane if and only if B is a constant (does not change orientation).

In the same spirit, T' measure how curved a curve is. (Notice that $|T'|$ generally is not a constant unlike $|T|$ and $|N|$). Again, in the same spirit, B' measure how fast γ leave the plane (osculating plane) spanned by T and N.

Given two vectors $u, v \in \mathbb{R}^n$, we use **dot product**

$$u \cdot v = u_1 v_1 + \cdots + u_n v_n$$

to denote the Euclidean inner product, and we use **length**

$$|u| = \sqrt{\sum_{k=1}^{n} u_k^2}$$

to denote the Euclidean norm. Note that we clearly have

$$|u| = \sqrt{u \cdot u}$$

Theorem 1.1.1. **(Differentiate the Dot Product)** Given two parametrized curves $u, v : (a, b) \to \mathbb{R}^n$, such that u, v are differentiable at $t \in (a, b)$. We have

$$\frac{d}{dt}\big(u(t) \cdot v(t)\big) = u'(t) \cdot v(t) + u(t) \cdot v'(t)$$

Proof.

$$\begin{aligned}\frac{d}{dt}\big(u(t)\cdot v(t)\big) &= \frac{d}{dt}\sum_{k=1}^{n} u_k(t)v_k(t)\\ &= \sum_{k=1}^{n}\frac{d}{dt}u_k(t)v_k(t)\\ &= \sum_{k=1}^{n} u_k'(t)v_k(t)+u_k(t)v_k'(t)\\ &= \sum_{k=1}^{n} u_k'(t)v_k(t)+\sum_{k=1}^{n} u_k(t)v_k'(t)\\ &= u'(t)\cdot v(t)+u(t)\cdot v'(t)\end{aligned}$$

■

For the result above, sometimes we write

$$(u\cdot v)' = u'\cdot v + u\cdot v'$$

Question 1: 1-2: 2

Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is the point of the trace of α closest to the origin and $\alpha'(t_0)\neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

Proof. Define $g : I \to \mathbb{R}$ by

$$g(t) \triangleq |\alpha(t)|^2 = (\alpha\cdot\alpha)(t)$$

Notice that

$$g'(t) = (2\alpha'\cdot\alpha)(t) \text{ if exists}$$

From premise, we know g attains minimum at t_0. This tell us

$$0 = g'(t_0) = (2\alpha'\cdot\alpha)(t_0)$$

Then, we can deduce

$$\alpha'(t_0)\cdot\alpha(t_0) = 0$$

This implies $\alpha(t_0)\perp\alpha'(t_0)$. ■

Question 2: 1-2: 5

Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve, with $\alpha''(t) \neq 0$ for all $t \in I$. Show that $|\alpha(t)|$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

Proof. We wish to prove

$$\exists \beta \in \mathbb{R}^*, \forall t \in I, |\alpha(t)| = \beta \iff \forall t \in I, (\alpha \cdot \alpha')(t) = 0$$

Define $g : I \to \mathbb{R}$ by

$$g(t) \triangleq |\alpha(t)|^2 = (\alpha \cdot \alpha)(t)$$

Notice that

$$g'(t) = (2\alpha' \cdot \alpha)(t) \tag{1.2}$$

$(\longrightarrow)$

From premise, g is a constant on I. This implies $g'(t) = 0$ for all $t \in I$. Then, from Equation 1.2, we see

$$(\alpha \cdot \alpha')(t) = 0 \text{ for all } t \in I$$

$(\longleftarrow)$

Again, from Equation 1.2, we deduce

$$\forall t \in I, (\alpha \cdot \alpha')(t) = 0 \implies \forall t \in I, g'(t) = 0$$

This implies g is a constant, which implies $|\alpha|$ is a constant, that is

$$\exists \beta \in \mathbb{R}, |\alpha(t)| = \beta$$

Lastly, we have to show

$$\beta \neq 0$$

Assume $\beta = 0$. Then, we see $\alpha(t) = 0$ for all $t \in I$. This implies $\alpha''(t) = 0$ for all $t \in I$, which CaC to the premise. (done) ■

Question 3: 1-3:2

2. A circular disk of radius 1 in the plane xy rolls without slipping along the x axis. The figure described by a point of the circumference of the disk is called a *cycloid* (Fig. 1-7).

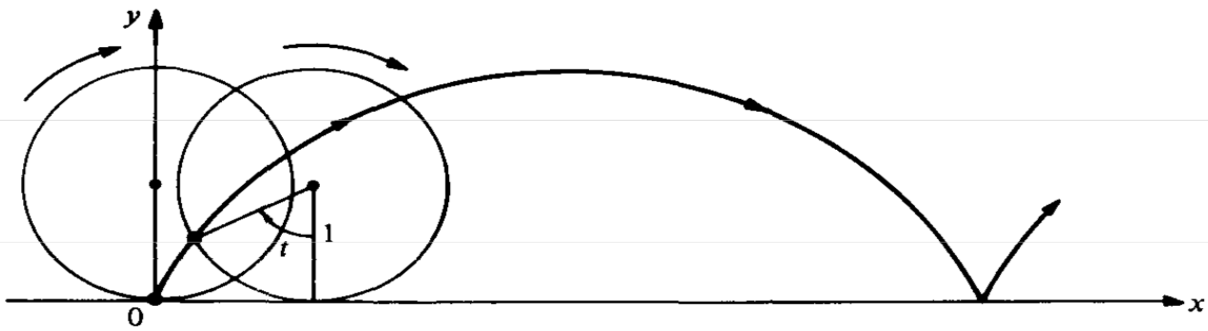


Figure 1-7. The cycloid.

***a.** Obtain a parametrized curve $\alpha: R \longrightarrow R^2$ the trace of which is the cycloid, and determine its singular points.

b. Compute the arc length of the cycloid corresponding to a complete rotation of the disk.

Proof. The solution of the question **a** is

$$\alpha(t) = (t - \sin t, 1 - \cos t)$$

Compute

$$\alpha'(t) = (1 - \cos t, \sin t)$$

and compute

$$|\alpha'(t)| = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} = \sqrt{2} \cdot \sqrt{1 - \cos t}$$

This implies the singular points are

$$\{2n\pi : n \in \mathbb{Z}\}$$

The solution of the question **b** is then

$$\begin{aligned}
\int_0^{2\pi} |\alpha'(t)|\, dt &= \sqrt{2}\int_0^{2\pi} \sqrt{1-\cos t} dt \\
&= \sqrt{2}\int_0^{2\pi} \sqrt{2}\left|\sin\frac{t}{2}\right| dt \\
&= 2\int_0^{2\pi} \left|\sin\frac{t}{2}\right| dt \\
&= 4\int_0^{\pi} \sin(\frac{t}{2}) dt \\
&= -8\cos\frac{t}{2}\Big|_0^{\pi}
\end{aligned}$$

■

Question 4: 1-3:4

4. Let $\alpha\colon (0,\pi) \longrightarrow R^2$ be given by

$$\alpha(t) = \left(\cos t, \cos t + \log \tan\frac{t}{2}\right),$$

where t is the angle that the y axis makes with the vector $\alpha(t)$. The trace of α is called the *tractrix* (Fig. 1-9). Show that

19:

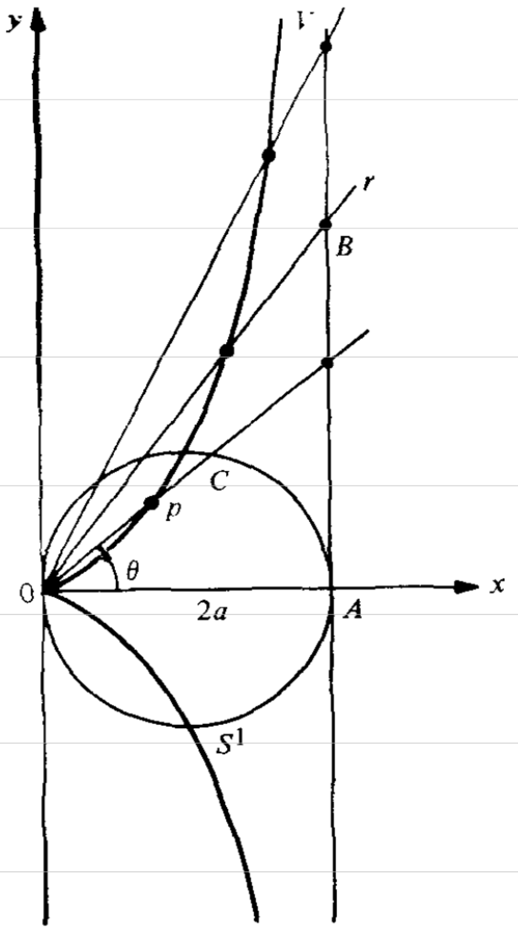


Figure 1-8. The cissoid of Diocles.

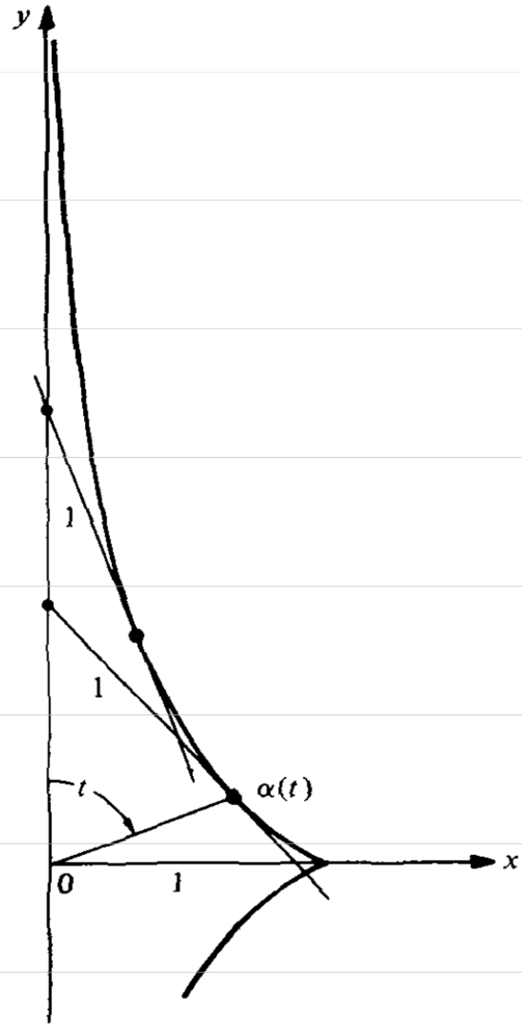


Figure 1-9. The tractrix.

a. α is a differentiable parametrized curve, regular except at $t = \pi/2$.

b. The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.

Typo correction: $\alpha(t) = (\sin t, \cos t + \ln \tan \frac{t}{2})$

Proof. **(a)**
Notice that the interval I is $(0, \pi)$. It is clear that

(a) $\sin t$ is smooth on $\mathbb{R}$

(b) $\cos t$ is smooth on $\mathbb{R}$

(c) $\ln t$ is smooth on $\mathbb{R}$ $\tan\frac{t}{2}$ is smooth on I

Then it follows that α is a differentiable curve.

Compute

$$\alpha'(t) = (\cos t, -\sin t + \frac{1}{\tan\frac{t}{2}} \cdot \sec^2\frac{t}{2} \cdot \frac{1}{2})$$

Because $\cos t = \alpha_1'(t)$ is 0 on I only when $t = \frac{\pi}{2}$, we know α is regular on I except possibly at $t = \frac{\pi}{2}$.

Compute

$$\alpha'(\frac{\pi}{2}) = (0, -1 + \frac{1}{1} \cdot 2 \cdot \frac{1}{2}) = (0, 0)$$

We now conclude α is regular on I except $\frac{\pi}{2}$.

(b)
A useful Identity give us

$$\alpha'(t) = (\cos t, -\sin t + \csc t)$$

From the following facts

(a) the first argument of the segment is from 0 to $\sin t = \alpha(t)$

(b) $\alpha_x'(t) = \cos t$

(c) $\frac{\sin t}{\cos t} = \tan t$

We conclude that the length of the segment is

$$\begin{aligned}
|\tan t| \cdot |\alpha'(t)| &= |\tan t| \cdot \sqrt{\cos^2 t + \sin^2 t - 2\sin t \csc t + \csc^2 t} \\
&= |\tan t| \cdot \sqrt{1 - 2 + \csc^2 t} \\
&= |\tan t| \cdot \sqrt{\csc^2 t - 1} = |\tan t| \cdot \sqrt{\cot^2 t} = 1
\end{aligned}$$

■

Question 5

7. A map $\alpha: I \longrightarrow R^3$ is *called a curve of class* C^k if each of the coordinate functions in the expression $\alpha(t) = (x(t), y(t), z(t))$ has continuous derivatives up to order k. If α is merely continuous, we say that α is of class C^0. A curve α is called *simple* if the map α is one-to-one. Thus, the curve in Example 3 of Sec. 1-2 is not simple.

Let $\alpha: I \longrightarrow R^3$ be a simple curve of class C^0. We say that α has a *weak tangent* at $t = t_0 \in I$ if the line determined by $\alpha(t_0 + h)$ and $\alpha(t_0)$ has a limit position when $h \longrightarrow 0$. We say that α has a *strong tangent* at $t = t_0$ if the line determined by $\alpha(t_0 + h)$ and $\alpha(t_0 + k)$ has a limit position when $h, k \longrightarrow 0$. Show that

a. $\alpha(t) = (t^3, t^2)$, $t \in R$, has a weak tangent but not a strong tangent at $t = 0$.

***b.** If $\alpha: I \longrightarrow R^3$ is of class C^1 and regular at $t = t_0$, then it has a strong tangent at $t = t_0$.

c. The curve given by

$$\alpha(t) = \begin{cases} (t^2, t^2), & t \geq 0, \\ (t^2, -t^2), & t \leq 0, \end{cases}$$

is of class C^1 but not of class C^2. Draw a sketch of the curve and its tangent vectors.

Proof. **(a)**

$$\frac{\alpha(t)}{t} \to (0,0) \text{ as } t \to 0^-$$

$$\frac{\alpha(h)-\alpha(k)}{h-k} = \frac{\left(h^3-k^3, h^2-k^2\right)}{h-k} = \left(h^2+hk+k^2, h+k\right) \to$$

(b) By MVT, for each h, k there exists a set of real numbers $\{c_x, c_y, c_z\}$ between $t+h$ and $t+k$ such that

$$\frac{\alpha(t_0+h)-\alpha(t_0+k)}{h-k} = \left(x'(c_x), y'(c_y), z'(c_z)\right)$$

Then because

$$h, k \to 0 \implies t_0 + h, t_0 + k \to t_0 \implies c_x, c_y, c_z \to t_0$$

Then from the fact α is of class C^1 (x', y', z' are all continuous), we can now deduce

$$\frac{\alpha(t_0 + h) - \alpha(t_0 + k)}{h - k} \to \alpha'(t_0) \text{ as } h, k \to 0$$

Now, because $\alpha'(t_0) \neq 0$ as α is regular, we see

$$\lim_{h,k \to 0} \frac{\alpha(t_0 + h) - \alpha(t_0 + k)}{h - k} \cdot \alpha'(t_0) = |\alpha'(t_0)|^2$$

(c)

From

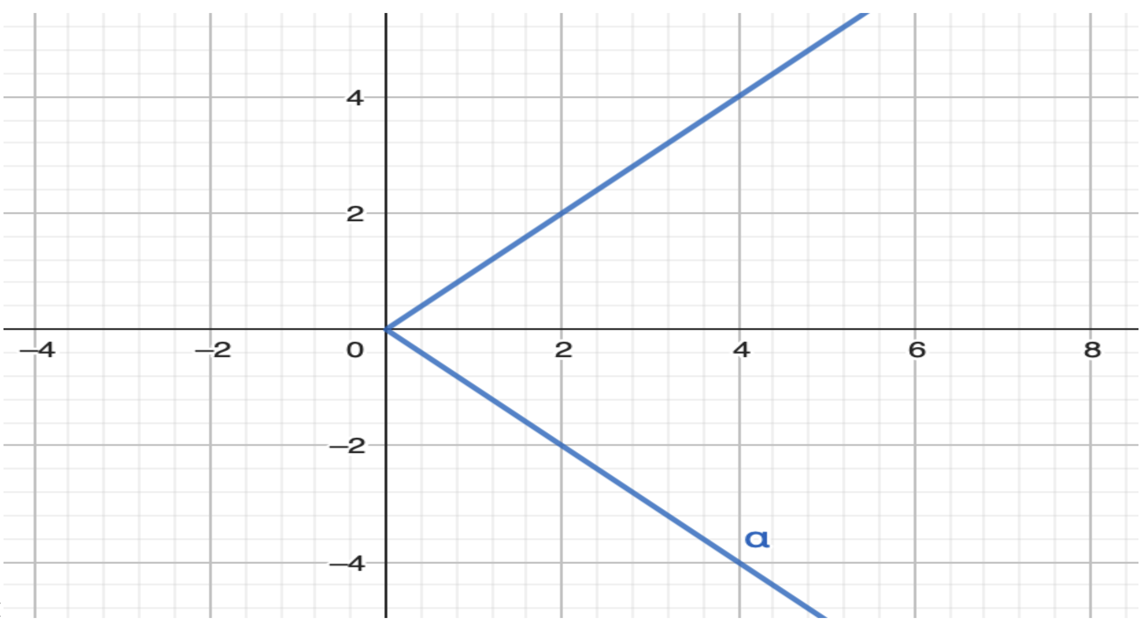
$$\alpha(t) = \left(t^2, \begin{cases} t^2 & \text{if } t \geq 0 \\ -t^2 & \text{if } t \leq 0 \end{cases}\right)$$

Compute

$$\alpha'(t) = \left(2t, \begin{cases} 2t & \text{if } t \geq 0 \\ -2t & \text{if } t \leq 0 \end{cases}\right)$$

Notice that the derivative at $t = 0$ is computed from definition instead of product rule.

Now, it is clear that x', y' are continuous. This implies $\alpha \in C^1$. Yet, we see y' is not differentiable at $t = 0$. This implies $\alpha \notin C^2$.

The sketch: 

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Theorem 1.1.2. (MVT for curve) Given a curve $\alpha : [a, b] \to \mathbb{R}^n$ such that

(a) α is differentaible on (a, b)

(b) α is continuous on $[a, b]$

there exists $\xi \in (a, b)$ such that

$$|\alpha(b) - \alpha(a)| \le |\alpha'(\xi)| (b - a)$$

Proof. Define $\phi : [a, b] \to \mathbb{R}$ by

$$\phi(t) = \alpha(t) \cdot \big(\alpha(b) - \alpha(a)\big)$$

Clearly ϕ satisfy the hypothesis of Lagrange's MVT, then we know there exists $\xi \in (a, b)$ such that

$$\phi(b) - \phi(a) = \phi'(\xi) \cdot (b - a)$$

Written the equation in α, we have

$$|\alpha(b) - \alpha(a)|^2 = (b - a)\alpha'(\xi) \cdot \big(\alpha(b) - \alpha(a)\big)$$

Notice that Cauchy-Schwarz inequality give us

$$\begin{aligned} (b - a) |\alpha'(\xi)| \cdot |\alpha(b) - \alpha(a)| &\ge (b - a) \left|\alpha'(\xi) \cdot \big(\alpha(b) - \alpha(a)\big)\right| \\ &= |\alpha(b) - \alpha(a)|^2 \end{aligned}$$

This then implies

$$(b - a) |\alpha'(\xi)| \ge |\alpha(b) - \alpha(a)|$$

■

Corollary 1.1.3. (Mean Value Inequality) Given a curve $\alpha : [a, b] \to \mathbb{R}^n$ such that

(a) α is differentiable on (a, b)

(b) α is continuous on $[a, b]$

we have

$$|\alpha(b) - \alpha(a)| \le (b - a) \sup_{(a,b)} |\alpha'|$$

Question 6

***8.** Let $\alpha: I \longrightarrow R^3$ be a differentiable curve and let $[a, b] \subset I$ be a closed interval. For every *partition*

$$a = t_0 < t_1 < \cdots < t_n = b$$

of $[a, b]$, consider the sum $\sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})| = l(\alpha, P)$, where P stands for the given partition. The norm $|P|$ of a partition P is defined as

$$|P| = \max(t_i - t_{i-1}), \; i = 1, \ldots, n.$$

Geometrically, $l(\alpha, P)$ is the length of a polygon inscribed in $\alpha([a, b])$ with vertices in $\alpha(t_i)$ (see Fig. 1-12). The point of the exercise is to show that the arc length of $\alpha([a, b])$ is, in some sense, a limit of lengths of inscribed polygons.

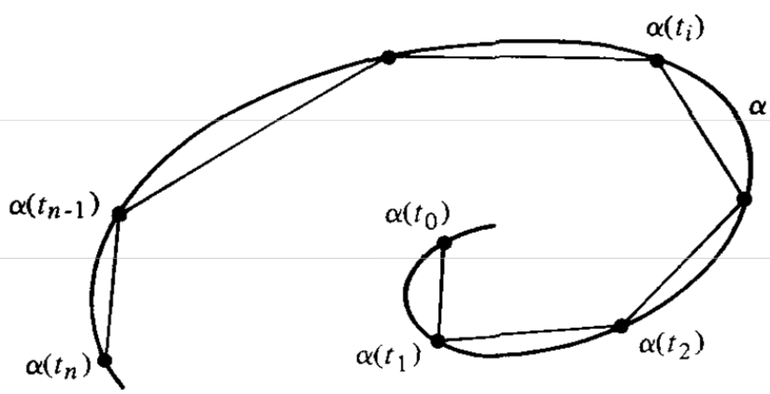

Figure 1-12

Prove that given $\epsilon > 0$ there exists $\delta > 0$ such that if $|P| < \delta$ then

$$\left| \int_a^b |\alpha'(t)|\, dt - l(\alpha, P) \right| < \epsilon.$$

Proof. We first prove

$$\int_a^b |\alpha'(t)|\, dt \geq l(\alpha, P)$$

By FTC, we have

$$\begin{aligned}|\alpha(t_i) - \alpha(t_{i-1})| &= \left|\int_{t_{i-1}}^{t_i} \alpha'(t)dt\right| \\ &\le \int_{t_{i-1}}^{t_i} |\alpha'(t)|\, dt\end{aligned}$$

This then implies

$$l(\alpha, P) = \sum |\alpha(t_i) - \alpha(t_{i-1})| \le \sum \int_{t_{i-1}}^{t_i} |\alpha'(t)|\, dt = \int_a^b |\alpha'(t)|\, dt \text{ (done)}$$

We have reduced the problem into

$$\text{finding } \delta \text{ such that } \forall P : |P| < \delta, \int_a^b |\alpha'(t)|\, dt - l(\alpha, P) < \epsilon$$

Because α' is uniformly continuous on $[a, b]$ ($\because$ continuous function on compact domain is uniformly continuous), we know there exists δ' such that

$$|\alpha'(s) - \alpha'(t)| < \frac{\epsilon}{2(b-a)} \text{ if } |s - t| < \delta'$$

We claim

$$\text{such } \delta' \text{ works}$$

Let $|P| < \delta$, and let $s_i \in [t_{i-1}, t_i]$. Because $|s_i - t_i| < \delta$, we have

$$|\alpha'(s_i) - \alpha'(t_i)| < \frac{\epsilon}{2(b-a)} \tag{1.3}$$

This give us

$$|\alpha'(s_i)| < |\alpha'(t_i)| + \frac{\epsilon}{2(b-a)}$$

Now, we can deduce

$$\begin{aligned}
\int_{t_{i-1}}^{t_i} |\alpha'(s)|\, ds &\leq |\alpha'(t_i)|\, \Delta t_i + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&= \int_{t_{i-1}}^{t_i} |\alpha'(t_i)|\, dt + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&= \left| \int_{t_{i-1}}^{t_i} \alpha'(t_i) dt \right| + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&= \left| \int_{t_{i-1}}^{t_i} \alpha'(t_i) - \alpha'(t) dt + \int_{t_{i-1}}^{t_i} \alpha'(t) dt \right| + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&\leq \left| \int_{t_{i-1}}^{t_i} \alpha'(t_i) - \alpha'(t) dt \right| + \left| \int_{t_{i-1}}^{t_i} \alpha'(t) dt \right| + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&\leq \frac{\epsilon}{2(b-a)}\Delta t_i + |\alpha(t_i) - \alpha(t_{i-1})| + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&= |\alpha(t_i) - \alpha(t_{i-1})| + \frac{\epsilon}{b-a}\Delta t_i
\end{aligned}$$

Notice that the last inequality follows from Equation 1.3. The long deduction above then give us

$$\begin{aligned}
\int_a^b |\alpha'(t)|\, dt &\leq \sum |\alpha(t_i) - \alpha(t_{i-1})| + \frac{\epsilon}{b-a}(b-a) \\
&= l(\alpha, P) + \epsilon
\end{aligned}$$

Then we have

$$\int_a^b |\alpha'(t)|\, dt - l(\alpha, P) \leq \epsilon \text{ (done)}$$

$\blacksquare$

Question 7

9. a. Let $\alpha: I \longrightarrow R^3$ be a curve of class C^0 (cf. Exercise 7). Use the approximation by polygons described in Exercise 8 to give a reasonable definition of arc length of α.

b. (*A Nonrectifiable Curve.*) The following example shows that, with any reasonable definition, the arc length of a C^0 curve in a closed interval may be unbounded. Let $\alpha: [0, 1] \longrightarrow R^2$ be given as $\alpha(t) = (t, t\sin(\pi/t))$ if $t \neq 0$, and $\alpha(0) = (0, 0)$. Show, geometrically, that the arc length of the portion of the curve corresponding to $1/(n+1) \leq t \leq 1/n$ is at least $2/(n+\frac{1}{2})$. Use this to show that the length of the curve in the interval $1/N \leq t \leq 1$ is greater than $2\sum_{n=1}^{N} 1/(n+1)$, and thus it tends to infinity as $N \longrightarrow \infty$.

Proof. **(a)** Suppose $I = [a, b]$. Define arc length by

$$\sup_P l(P, \alpha) \text{ where sup runs over all partition } P \text{ of } [a, b]$$

(b)

Geometrically, we know the arc length of the portion of the curve corresponding to $t \in [\frac{1}{n+1}, \frac{1}{n}]$ must be greater than

$$\left|\alpha(\frac{1}{n}) - \alpha(\frac{1}{n+\frac{1}{2}})\right| + \left|\alpha(\frac{1}{n+1}) - \alpha(\frac{1}{n+\frac{1}{2}})\right|$$

$\blacksquare$

Question 8

10. (*Straight Lines as Shortest.*) Let $\alpha: I \to R^3$ be a parametrized curve. Let $[a, b] \subset I$ and set $\alpha(a) = p$, $\alpha(b) = q$.

a. Show that, for any constant vector v, $|v| = 1$,

$$(q - p)\cdot v = \int_a^b \alpha'(t)\cdot v\, dt \leq \int_a^b |\alpha'(t)|\, dt.$$

b. Set

$$v = \frac{q - p}{|q - p|}$$

and show that

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)|\, dt;$$

that is, the curve of shortest length from $\alpha(a)$ to $\alpha(b)$ is the straight line joining these points.