4.4 PDE HW 4

Question 58

Solve

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = e^x \\ u_t(x,0) = \sin x \end{cases}$$

Proof. Let

$$u \triangleq f(x+ct) + g(x-ct)$$

Plugging the initial conditions, we know

$$\begin{cases} f(x) + g(x) = e^x \\ f'(x) - g'(x) = \frac{\sin x}{c} \end{cases}$$

This give us

$$f'(x) = \frac{e^x + \frac{\sin x}{c}}{2} \text{ and } g'(x) = \frac{e^x - \frac{\sin x}{c}}{2}$$

Then by FTC, we have

$$\begin{cases} f(x) = \frac{e^x - \frac{\cos x}{c}}{2} + f(0) - \frac{1}{2} + \frac{1}{2c} \\ g(x) = \frac{e^x + \frac{\cos x}{c}}{2} + g(0) - \frac{1}{2} - \frac{1}{2c} \end{cases}$$

Note that $f(0) + g(0) = e^0 = 1$, which cancel the constant terms in u, i.e.

$$\begin{aligned} u &= f(x+ct) + g(x-ct) \\ &= \frac{e^{x+ct} + e^{x-ct} + \frac{-\cos(x+ct)+\cos(x-ct)}{c}}{2} \end{aligned}$$

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Question 59

Solve

$$\begin{cases} u_{xx} + u_{xt} - 20u_{tt} = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

Proof. The PDE can be write in the form of

$$(\partial_x + 5\partial_t)(\partial_x - 4\partial_t)u = 0$$

which have the general solution

$$u(x,t) = f(5x - t) + g(4x + t)$$

We now see

$$f(5x) + g(4x) = \phi(x) \text{ and } -f'(5x) + g'(4x) = \psi(x)$$

This then give us

$$f'(5x) = \frac{(\phi' - 4\psi)(x)}{9} \text{ and } g'(4x) = \frac{(\phi' + 5\psi)(x)}{9}$$

and thus

$$\begin{aligned} f(x) &= f(0) + \int_0^x f'(s)ds \\ &= f(0) + \int_0^x \frac{(\phi' - 4\psi)(\frac{s}{5})}{9} ds \\ &= f(0) + \frac{5}{9}\Big[\phi(\frac{x}{5}) - \phi(0)\Big] - \frac{4}{9}\int_0^x \psi(\frac{s}{5})ds \end{aligned}$$

and similarly

$$\begin{aligned} g(x) &= g(0) + \int_0^x g'(s)ds \\ &= g(0) + \int_0^x \frac{(\phi' + 5\psi)(\frac{s}{4})}{9} ds \\ &= g(0) + \frac{4}{9}\Big[\phi(\frac{x}{4}) - \phi(0)\Big] + \frac{5}{9}\int_0^x \psi(\frac{s}{4})ds \end{aligned}$$

Noting that $f(0) + g(0) = u(0,0) = \psi(0)$, we now have

$$\begin{aligned} u(x,t) &= f(5x - t) + g(4x + t) \\ &= \frac{5\phi(\frac{5x-t}{5}) + 4\phi(\frac{4x+t}{4})}{9} - \frac{4}{9}\int_0^{5x-t} \psi(\frac{s}{5})ds + \frac{5}{9}\int_0^{4x+t} \psi(\frac{s}{4})ds \end{aligned}$$

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