5.11 Taylor's Formula

Theorem 5.11.1. (Taylor's Formula) For each nice function $g : [0,1] \to \mathbb{R}$ and $m \in \mathbb{N}$, we have

$$g(1) - g(0) = \sum_{k=1}^{m-1} \frac{g^{(k)}(0)}{k!} + \frac{g^{(m)}(c)}{m!} \text{ for some } c \in (0,1)$$

and

$$g(1) - g(0) - \sum_{k=1}^{m-1} \frac{g^{(k)}(0)}{k!} = \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{m-1}} g^{(m)}(t_m) dt_m dt_{m-1} \cdots dt_1$$

Proof. If $m = 1$, then the proof is trivial by MVT and FTC.

Suppose the Theorem hold true for m, we prove the Theorem hold true for $m+1$.

We first prove

$$g(1) - g(0) - \sum_{k=1}^{m} \frac{g^{(k)}(0)}{k!} = \int_0^1 \int_0^{t_1} \cdots \int_0^{t_m} g^{(m+1)}(t_{m+1}) dt_{m+1} \cdots dt_1$$

By induction hypothesis,

$$\begin{aligned} g(1) - g(0) - \sum_{k=1}^{m} \frac{g^{(k)}(0)}{k!} &= g(1) - g(0) - \sum_{k=1}^{m-1} \frac{g^{(k)}(0)}{k!} - \frac{g^{(m)}(0)}{m!} \\ &= \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{m-1}} g^{(m)}(t_m) dt_m dt_{m-1} \cdots dt_1 - \frac{g^{(m)}(0)}{m!} \\ &= \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{m-1}} \big(g^{(m)}(t_m) - g^{(m)}(0)\big) dt_m dt_{m-1} \cdots dt_1 \\ &\overset{\text{FTC}}{=} \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{m-1}} \int_0^{t_m} g^{(m+1)}(t_{m+1}) dt_{m+1} dt_m dt_{m-1} \cdots dt_1 \text{ (done)} \end{aligned}$$

We now prove

$$g(1) - g(0) - \sum_{k=1}^{m} \frac{g^{(k)}(0)}{k!} = \frac{g^{(m+1)}(c)}{(m+1)!}$$

By induction hypothesis,

$$\begin{aligned}
g(1) - g(0) - \sum_{k=1}^{m} \frac{g^{(k)}(0)}{k!} &= \int_0^1 \int_0^{t_1} \cdots \int_0^{t_m} g^{(m+1)}(t_{m+1}) dt_{m+1} \cdots dt_1 \\
&\overset{\text{Fubini}}{=} \int_0^1 \int_{t_{m+1}}^1 \int_{t_{m+1}}^{t_1} \cdots \int_{t_{m+1}}^1 g^{(m+1)}(t_{m+1}) dt_m \cdots dt_1 dt_{m+1} \\
&= \int_0^1 g^{(m+1)}(t_{m+1}) \int_{t_{m+1}}^1 \int_{t_{m+1}}^{t_1} \cdots \int_{t_{m+1}}^{t_{m-1}} dt_m \cdots dt_1 dt_{m+1} \\
&= \int_0^1 g^{(m+1)}(t_{m+1}) F(t_{m+1}) dt_{m+1} \\
&\overset{\text{MVT}}{=} g^{(m+1)}(c) \int_0^1 F(t_{m+1}) dt_{m+1} = \frac{g^{(m+1)}(c)}{(m+1)!} \text{ (done)}
\end{aligned}$$

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Theorem 5.11.2. (Taylor's Formula) Given $f : \mathbb{R}^n \to \mathbb{R}$ such that $f \in \mathcal{C}^m(S)$ for some $S \overset{\text{open,convex}}{=} \mathbb{R}^n$, for each $a, b \in S$, if we define $g : [0,1] \to \mathbb{R}$ by

$$g(t) \triangleq f(a + (b-a)t)$$

then there exists $\theta \in (0,1)$ such that

$$f(b) - f(a) = \sum_{k=1}^{m-1} \frac{1}{k!} f^{(k)}(a; b-a) + \frac{1}{m!} f^{(m)}(g(\theta); b-a)$$

Proof. Observe

$$\begin{aligned}
f(b) - f(a) &= g(1) - g(0) \\
&= \sum_{k=1}^{m-1} \frac{1}{k!} g^{(k)}(0) + g^m(\theta) \text{ for some } \theta
\end{aligned}$$

The problem can now be reduced into proving

$$g^{(k)}(t) = f^{(k)}(g(t); b-a) \text{ for all } k \in \{1, \dots, m\}$$

We prove by induction, for base case, compute

$$\begin{aligned}
g'(t) &= \sum_{k=1}^{n} [(\partial_k f)(a + (b-a)t)] \cdot (b-a)_k \\
&= f'(a; b-a)
\end{aligned}$$

Suppose

$$g^{(k)}(t) = f^{(k)}(g(t); b-a)$$
$$= \sum_{j_1,\dots,j_k \in \{1,\dots,n\}} \Big[[(\partial_{j_1,\dots,j_k} f)\big(a + (b-a)\mathbf{i})] \cdot \prod_{i=1}^{k} (b-a)_{j_i}\Big]$$

This give us

$$\begin{aligned}
g^{(k+1)}(t) &= \sum_{j_1\dots,j_k \in \{1,\dots,n\}} \frac{d}{dt}\Big[[(\partial_{j_1,\dots,j_k} f)\big(a + (b-a)t\big)] \cdot \prod_{i=1}^{k} (b-a)_{j_i}\Big] \\
&= \sum_{j_1\dots,j_k \in \{1,\dots,n\}} \Big[[\frac{d}{dt}(\partial_{j_1,\dots,j_k} f)\big(a + (b-a)t\big)] \cdot \prod_{i=1}^{k} (b-a)_{j_i}\Big] \\
&= \sum_{j_1\dots,j_k \in \{1,\dots,n\}} \Big[[\sum_{j_{k+1} \in \{1,\dots,n\}} (\partial_{j_1,\dots,j_k,j_{k+1}} f)\big(a + (b-a)t\big) \cdot (b-a)_{j_{k+1}}] \cdot \prod_{i=1}^{k} (b-a)_{j_i}\Big] \\
&= \sum_{j_1\dots,j_k,j_{k+1} \in \{1,\dots,n\}} \Big[[(\partial_{j_1,\dots,j_k,j_{k+1}} f)\big(a + (b-a)t\big)] \cdot \prod_{i=1}^{k+1} (b-a)_{j_i}\Big] \\
&= f^{(k+1)}(g(t); b-a)
\end{aligned}$$

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Note that our definition is

$$f^{(k)}(z;c) = \sum_{j_1,\dots,j_k \in \{1,\dots,n\}} \Big[[\partial_{j_1,\dots,j_k} f(z)] \cdot \prod_{i=1}^{k} (c)_{j_i}\Big]$$

When $k = 1$, then

$$f^1(z;c) = \sum_{j=1}^{n} \partial_j f(z) \cdot c_j = \nabla f(z) \cdot c = df_z(c)$$

5.12 Fourier Stuff

Definition 5.12.1. (Definition of Trigonometric Polynomials) By a trigonometric polynomials, we mean a function $P : \mathbb{R} \to \mathbb{C}$ of the form

$$P(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \text{ where } a_n, b_n \text{ are complex numbers}$$

Note that

$$P(x) = \sum_{-N}^{N} c_n e^{inx} \text{ where } c_n = \frac{a_n}{2} + \frac{b_n}{2i} \text{ and } c_{-n} = \frac{a_n}{2} - \frac{b_n}{2i} \text{ for each } n \in \mathbb{N}$$

Definition 5.12.2. (Definition of inner products and norms) Given a function f, g defined on $[-\pi, \pi]$, we say

$$\langle f, g \rangle \triangleq \frac{1}{2\pi} \int_{-\pi}^{\pi} f \overline{g} dx$$

and

$$\|f\|_2 \triangleq \sqrt{\langle f, f \rangle}$$

It is clear that $\{e^{inx}\}_{n \in \mathbb{Z}}$ is orthonormal in $L^2(\mu)$. From now, we will write $\phi_n \triangleq e^{inx}$.

Definition 5.12.3. (Definition of Fourier coefficients) For each $n \in \mathbb{Z}$, the n-th Fourier coefficients of f is

$$c_n \triangleq \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx = \langle f, \phi_n \rangle$$

and we write

$$s_N(f) = \sum_{-N}^{N} c_n e^{inx} = \sum_{-N}^{N} c_n \phi_n$$

Theorem 5.12.4. (Special-case of Riesz-Fischer's Theorem) If f is Riemann-integrable on $[-\pi, \pi]$, then

$$\lim_{N \to \infty} \|f - s_N\|_2 = 0$$

Theorem 5.12.5. (Parseval's Identity) Suppose f, g are Riemann-integrable and $f \sim c_n$ and $g \sim \gamma_n$. We have

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f \overline{g} dx = \sum_{-\infty}^{\infty} c_n \overline{\gamma_n}$$