# 6

# Group algebras

The group algebra of a finite group $G$ is a vector space of dimension $|G|$ which also carries extra structure involving the product operation on $G$. In a sense, group algebras are the source of all you need to know about representation theory. In particular, the ultimate goal of representation theory – that of understanding all the representations of finite groups – would be achieved if group algebras could be fully analysed. Group algebras are therefore of great interest.

After defining the group algebra of $G$, we shall use it to construct an important faithful representation, known as the regular representation of $G$, which will be explored in greater detail later on.

## The group algebra of *G*

Let $G$ be a finite group whose elements are $g_1, \ldots, g_n$, and let $F$ be $\mathbb{R}$ or $\mathbb{C}$.

We define a vector space over $F$ with $g_1, \ldots, g_n$ as a basis, and we call this vector space $FG$. Take as the elements of $FG$ all expressions of the form

$$\lambda_1 g_1 + \ldots + \lambda_n g_n \quad (\text{all } \lambda_i \in F).$$

The rules for addition and scalar multiplication in $FG$ are the natural ones: namely, if

$$u = \sum_{i=1}^{n} \lambda_i g_i \text{ and } v = \sum_{i=1}^{n} \mu_i g_i$$

are elements of $FG$, and $\lambda \in F$, then

$$u + v = \sum_{i=1}^{n} (\lambda_i + \mu_i) g_i \text{ and } \lambda u = \sum_{i=1}^{n} (\lambda \lambda_i) g_i.$$

With these rules, $FG$ is a vector space over $F$ of dimension $n$, with basis $g_1$, . . . , $g_n$. The basis $g_1$, . . . , $g_n$ is called the *natural* basis of $FG$.

*6.1 Example*

Let $G = C_3 = \langle a\colon a^3 = e \rangle$. (To avoid confusion with the element 1 of $F$, we write $e$ for the identity element of $G$, in this example.) The vector space $\mathbb{C}G$ contains

$$u = e - a + 2a^2 \text{ and } v = \tfrac{1}{2}e + 5a.$$

We have

$$u + v = \tfrac{3}{2}e + 4a + 2a^2, \; \tfrac{1}{3}u = \tfrac{1}{3}e - \tfrac{1}{3}a + \tfrac{2}{3}a^2.$$

Sometimes we write elements of $FG$ in the form

$$\sum_{g \in G} \lambda_g g \quad (\lambda_g \in F).$$

Now, $FG$ carries more structure than that of a vector space – we can use the product operation on $G$ to define multiplication in $FG$ as follows:

$$\begin{aligned} \left(\sum_{g \in G} \lambda_g g\right)\left(\sum_{h \in G} \mu_h h\right) &= \sum_{g,h \in G} \lambda_g \mu_h (gh) \\ &= \sum_{g \in G} \sum_{h \in G} (\lambda_h \mu_{h^{-1}g}) g \end{aligned}$$

where all $\lambda_g, \mu_h \in F$.

*6.2 Example*

If $G = C_3$ and $u$, $v$ are the elements of $\mathbb{C}G$ which appear in Example 6.1, then

$$\begin{aligned} uv &= (e - a + 2a^2)(\tfrac{1}{2}e + 5a) \\ &= \tfrac{1}{2}e + 5a - \tfrac{1}{2}a - 5a^2 + a^2 + 10a^3 \\ &= \tfrac{21}{2}e + \tfrac{9}{2}a - 4a^2. \end{aligned}$$

*6.3 Definition*

The vector space $FG$, with multiplication defined by

$$\left(\sum_{g \in G} \lambda_g g\right)\left(\sum_{h \in G} \mu_h h\right) = \sum_{g,h \in G} \lambda_g \mu_h (gh)$$

($\lambda_g, \mu_h \in F$), is called the *group algebra* of $G$ over $F$.

The group algebra $FG$ contains an identity for multiplication, namely the element $1e$ (where 1 is the identity of $F$ and $e$ is the identity of $G$). We write this element simply as 1.

*6.4 Proposition*

*Multiplication in $FG$ satisfies the following properties, for all $r, s, t \in FG$ and $\lambda \in F$:*

(1) $rs \in FG$;
(2) $r(st) = (rs)t$;
(3) $r1 = 1r = r$;
(4) $(\lambda r)s = \lambda(rs) = r(\lambda s)$;
(5) $(r + s)t = rt + st$;
(6) $r(s + t) = rs + rt$;
(7) $r0 = 0r = 0$.

*Proof* (1) It follows immediately from the definition of $rs$ that $rs \in FG$.

(2) Let

$$r = \sum_{g \in G} \lambda_g g,\ s = \sum_{g \in G} \mu_g g,\ t = \sum_{g \in G} \nu_g g,$$

($\lambda_g, \mu_g, \nu_g \in F$). Then

$$\begin{aligned}
(rs)t &= \sum_{g,h,k \in G} \lambda_g \mu_h \nu_k (gh)k \\
&= \sum_{g,h,k \in G} \lambda_g \mu_h \nu_k g(hk) \\
&= r(st).
\end{aligned}$$

We leave the proofs of the other equations as easy exercises.

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In fact, any vector space equipped with a multiplication satisfying properties (1)–(7) of Proposition 6.4 is called an *algebra*. We shall be concerned only with group algebras, but it is worth pointing out that the axioms for an algebra mean that it is both a vector space and a ring.

## The regular *FG*-module

We now use the group algebra to define an important $FG$-module.

Let $V = FG$, so that $V$ is a vector space of dimension $n$ over $F$, where $n = |G|$. For all $u, v \in V$, $\lambda \in F$ and $g, h \in G$, we have

$$\begin{aligned}
&vg \in V, \\
&v(gh) = (vg)h, \\
&v1 = v, \\
&(\lambda v)g = \lambda(vg), \\
&(u + v)g = ug + vg,
\end{aligned}$$

by parts (1), (2), (3), (4) and (5) of Proposition 6.4, respectively. Therefore $V$ is an $FG$-module.

*6.5 Definition*

Let $G$ be a finite group and $F$ be $\mathbb{R}$ or $\mathbb{C}$. The vector space $FG$, with the natural multiplication $vg$ ($v \in FG$, $g \in G$), is called the *regular* $FG$-*module*.

The representation $g \to [g]_{\mathscr{B}}$ obtained by taking $\mathscr{B}$ to be the natural basis of $FG$ is called the *regular representation* of $G$ over $F$.

Note that the regular $FG$-module has dimension equal to $|G|$.

*6.6 Proposition*

*The regular FG-module is faithful.*

*Proof* Suppose that $g \in G$ and $vg = v$ for all $v \in FG$. Then $1g = 1$, so $g = 1$, and the result follows.

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*6.7 Example*

Let $G = C_3 = \langle a\colon a^3 = e\rangle$. The elements of $FG$ have the form

$$\lambda_1 e + \lambda_2 a + \lambda_3 a^2 \quad (\lambda_i \in F).$$

We have

$$\begin{aligned}
(\lambda_1 e + \lambda_2 a + \lambda_3 a^2)e &= \lambda_1 e + \lambda_2 a + \lambda_3 a^2,\\
(\lambda_1 e + \lambda_2 a + \lambda_3 a^2)a &= \lambda_3 e + \lambda_1 a + \lambda_2 a^2,\\
(\lambda_1 e + \lambda_2 a + \lambda_3 a^2)a^2 &= \lambda_2 e + \lambda_3 a + \lambda_1 a^2.
\end{aligned}$$

By taking matrices relative to the basis $e$, $a$, $a^2$ of $FG$, we obtain the regular representation of $G$:

$$e \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, a^2 \to \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

## ***FG*** acts on an ***FG*-module**

You will remember that an $FG$-module is a vector space over $F$, together with a multiplication $vg$ for $v \in V$ and $g \in G$ (and the multiplication satisfies various axioms). Now, it is sometimes helpful to extend the definition of the multiplication so that we have an element $vr$ of $V$ for all elements $r$ in the group algebra $FG$. This is done in the following natural way.

*6.8 Definition*
Suppose that $V$ is an $FG$-module, and that $v \in V$ and $r \in FG$; say $r = \sum_{g\in G} \mu_g g$ ($\mu_g \in F$). Define $vr$ by

$$vr = \sum_{g\in G} \mu_g (vg).$$

*6.9 Examples*
(1) Let $V$ be the permutation module for $S_4$, as described in Example 4.9. If

$$r = \lambda(1\ 2) + \mu(1\ 3\ 4) \quad (\lambda, \mu \in F)$$

then

$$v_1 r = \lambda v_1(1\ 2) + \mu v_1(1\ 3\ 4) = \lambda v_2 + \mu v_3,$$

$$v_2 r = \lambda v_1 + \mu v_2,$$

$$(2v_1 + v_2)r = \lambda v_1 + (2\lambda + \mu)v_2 + 2\mu v_3.$$

(2) If $V$ is the regular $FG$-module, then for all $v \in V$ and $r \in FG$, the element $vr$ is simply the product of $v$ and $r$ as elements of the group algebra, given by Definition 6.3.

Compare the next result with Proposition 6.4.

*6.10 Proposition*

*Suppose that $V$ is an $FG$-module. Then the following properties hold for all $u, v \in V$, all $\lambda \in F$ and all $r, s \in FG$:*

(1) $vr \in V$;
(2) $v(rs) = (vr)s$;
(3) $v1 = v$;
(4) $(\lambda v)r = \lambda(vr) = v(\lambda r)$;
(5) $(u + v)r = ur + vr$;
(6) $v(r + s) = vr + vs$;
(7) $v0 = 0r = 0$.

*Proof* All parts except (2) are straightforward, and we leave them to you. We shall give a proof of part (2), assuming the other parts.

Let $v \in V$, and let $r, s \in FG$ with

$$r = \sum_{g \in G} \lambda_g g, \; s = \sum_{h \in G} \mu_h h.$$

Then

$$\begin{aligned}
v(rs) &= v\left(\sum_{g,h} \lambda_g \mu_h (gh)\right) \\
&= \sum_{g,h} \lambda_g \mu_h (v(gh)) && \text{by (4) and (6)} \\
&= \sum_{g,h} \lambda_g \mu_h ((vg)h) \\
&= \left(\sum_g \lambda_g (vg)\right)\left(\sum_h \mu_h h\right) && \text{by (4), (5), (6)} \\
&= (vr)s.
\end{aligned}$$

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## Summary of Chapter 6

1. The group algebra $FG$ of $G$ over $F$ consists of all linear combinations of elements of $G$, and has a natural multiplication defined on it.
2. The vector space $FG$, with the natural multiplication $vg$ ($v \in FG$, $g \in G$) is the regular $FG$-module.
3. The regular $FG$-module is faithful.

## Exercises for Chapter 6

1. Suppose that $G = D_8 = \langle a, b\colon a^4 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$.

   (a) Let $x$ and $y$ be the following elements of $\mathbb{C}G$:

   $$x = a + 2a^2,\ y = b + ab - a^2.$$

   Calculate $xy$, $yx$ and $x^2$.

   (b) Let $z = b + a^2b$. Show that $zg = gz$ for all $g$ in $G$. Deduce that $zr = rz$ for all $r$ in $\mathbb{C}G$.
2. Work out matrices for the regular representation of $C_2 \times C_2$ over $F$.
3. Let $G = C_2$. For $r$ and $s$ in $\mathbb{C}G$, does $rs = 0$ imply that $r = 0$ or $s = 0$?
4. Assume that $G$ is a finite group, say $G = \{g_1, \dots, g_n\}$, and write $c$ for the element $\sum_{i=1}^{n} g_i$ of $\mathbb{C}G$.

   (a) Prove that $ch = hc = c$ for all $h$ in $G$.

   (b) Deduce that $c^2 = |G|c$.

   (c) Let $\vartheta\colon \mathbb{C}G \to \mathbb{C}G$ be the linear transformation sending $v$ to $vc$ for all $v$ in $\mathbb{C}G$. What is the matrix $[\vartheta]_{\mathscr{B}}$, where $\mathscr{B}$ is the basis $g_1, \dots, g_n$ *of* $\mathbb{C}$ *G?*
5. If $V$ is an $FG$-module, prove from the definition that

$$0r = 0 \text{ for all } r \in FG, \text{ and}$$

$$v0 = 0 \text{ for all } v \in V,$$

where the symbol 0 is used for the zero elements of $V$ and $FG$.

Show that for every finite group $G$, with $|G| > 1$, there exists an $FG$-module $V$ and elements $v \in V$, $r \in FG$ such that $vr = 0$, but neither $v$ nor $r$ is 0.

6. Suppose that $G = D_6 = \langle a, b: a^3 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$, and let $\omega = e^{2\pi i/3}$. Prove that the 2-dimensional subspace $W$ of $\mathbb{C}G$, defined by

$$W = \text{sp}\,(1 + \omega^2 a + \omega a^2,\ b + \omega^2 ab + \omega a^2 b),$$

is an irreducible $\mathbb{C}G$-submodule of the regular $\mathbb{C}G$-module.