Chapter 11

Complex Analysis

11.1 Cauchy Integral Theorem

Abstract

Note that in this section, when we talk about derivative of function defined on subset of real line, we do consider one-sided derivative, i.e., for $\gamma : [a,b] \to \mathbb{C}$ to be C^1, the limit of $\frac{\gamma(a+h)-\gamma(a)}{h}$ as $h \searrow 0$ must exist.

Let $[a,b] \subseteq \mathbb{R}$ be some compact interval. We say $\gamma : [a,b] \to \mathbb{C}$ is a **parametrization** if

(a) $\gamma(x) \neq \gamma(y)$ unless $x = a$ and $y = b$.

(b) There exists some partition $\{a = c_0 < \cdots < c_N = b\}$ such that $\gamma|_{[c_n,c_{n+1}]} : [c_n, c_{n+1}] \to \mathbb{C}$ are C^1 wish non-vanishing derivative.

A parametrization $\gamma : [a,b] \to \mathbb{C}$ is said to be **closed** if $\gamma(a) = \gamma(b)$. Two parametrizations $\gamma : [a,b] \to \mathbb{C}, \alpha : [c,d] \to \mathbb{C}$ are said to be **equivalent** if there exists some C^1 bijection $s : [a,b] \to [c,d]$ such that

$$\gamma(t) = \alpha(s(t)) \text{ and } s'(t) > 0 \text{ for all } t \in [a,b]$$

Inverse Function Theorem shows that our definition for parametrization equivalence is indeed an equivalence relation. We then can define **contour** to be the equivalence class of parametrizations. Immediately, we see that all parametrization of a contour have the same image and if any of them is closed, then all of them are closed. This allow us to talk about the image of a contour and if a contour is closed. If we define **length** for parametrization $\gamma : [a,b] \to \mathbb{C}$ to be $\int_a^b \gamma'(t)dt$, then a change of variables shows that all parametrizations in $[\gamma]$ have the same length as γ. This allow us to define the **length for contour**. Now, given some parametrization $\gamma : [a,b] \to \mathbb{C}$ and some continuous complex-valued function f defined on the image $\gamma([a,b])$, we can define its **contour integral** by

$$\int_\gamma f(z)dz \triangleq \int_a^b f(\gamma(t))\gamma'(t)dt$$

Again the change of variables shows that our definition is well defined for contours. Similar to the real case, we have the estimation

$$\left| \int_\gamma f dz \right| \leq LM \tag{11.1}$$

where L is the length of γ and M is the maximum of $|f|$ on γ. We can also generalize Part 2 of Fundamental Theorem of Calculus to contour integral: If $D \subseteq \mathbb{C}$ is open, $f : D \to \mathbb{C}$ is continuous, and $F : D \to \mathbb{C}$ satisfy $F'(z) = f(z)$ for all $z \in D$, then for all contour $\gamma : [a,b] \to D$, we have

$$\int_\gamma f dz = F(\gamma(b)) - F(\gamma(a))$$

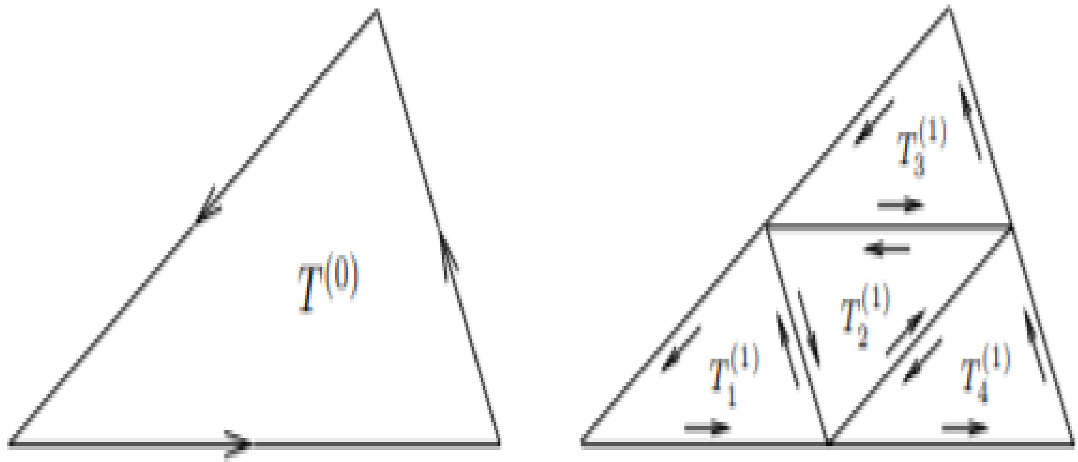
We are now ready to state Cauchy's Integral Theorem for triangles. Note that term "closed triangle" as a set include both its interior area and boundary. For example, a closed triangle can be

$$\{x + iy \in \mathbb{C} : x \in [0,1] \text{ and } y \in [0,x]\}$$

Theorem 11.1.1. (Cauchy's Integral Theorem for triangles) If $D \subseteq \mathbb{C}$ is open, $f : D \to \mathbb{C}$ is holomorphic and D contain some closed triangle T, then

$$\int_{\partial T} f dz = 0$$

Proof. Denote T by $T^{(0)}$. Construct triangles $T_1^{(1)}, T_2^{(1)}, T_3^{(1)}, T_4^{(1)}$ as in the figure below.



Obviously, we may parametrize the boundaries of these triangles so that

$$\int_{\partial T^{(0)}} f dz = \sum_{n=1}^{4} \int_{\partial T_n^{(1)}} f dz$$

Taking absolute value on both side, we deduce

$$\left| \int_{\partial T^{(0)}} f dz \right| \leq 4 \left| \int_{\partial T_j^{(1)}} f dz \right| \text{ for some } j \in \{1, 2, 3, 4\}$$

Denote $T_j^{(1)}$ by $T^{(1)}$. Repeating this process, we obtain a decreasing sequence of triangles

$$T^{(0)} \supseteq T^{(1)} \supseteq \cdots \supseteq T^{(n)} \supseteq \cdots$$

with the property that

$$\left| \int_{\partial T^{(0)}} f dz \right| \leq 4^n \left| \int_{\partial T^{(n)}} f dz \right| \tag{11.2}$$

Let $d^{(n)}$ and $p^{(n)}$ denote the diameter and perimeter of $T^{(n)}$ for all $n \in \mathbb{Z}_0^+$. Some tedious effort shows that

$$d^{(n)} = 2^{-n} d^{(0)} \text{ and } p^{(n)} = 2^{-n} p^{(0)} \tag{11.3}$$

Theorem 2.3.2 implies

$$\bigcap_{n \in \mathbb{N}} T^{(n)} = \{z_0\} \text{ for some } z_0 \in D$$

Because f is holomorphic at z_0, we may write $f : D \to \mathbb{C}$ by

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + o(z - z_0)(z - z_0)$$

Clearly the first two terms have antiderivatives. Using Equation 11.2 and Equation 11.3, we may now estimate

$$\begin{aligned}
\left|\int_{\partial T^{(0)}} f(z)dz\right| \leq 4^n \left|\int_{\partial T^{(n)}} f(z)dz\right| &= 4^n \left|\int_{\partial T^{(n)}} o(z - z_0)(z - z_0)dz\right| \\
&\leq 4^n p^{(n)} d^{(n)} \max_{z \in \partial T^{(n)}} |o(z - z_0)| \\
&= p^{(0)} d^{(0)} \max_{z \in \partial T^{(n)}} |o(z - z_0)| \to 0 \text{ as } n \to \infty
\end{aligned}$$

■

By $D \subseteq \mathbb{C}$ being **star-convex with center** z_*, we mean that for all $z \in D$, the contour $\gamma : [0, 1] \to \mathbb{C}$ defined by

$$\gamma(t) \triangleq z_* + t(z - z_*)$$

satisfy $\gamma([0, 1]) \subseteq D$.

Theorem 11.1.2. (Existence of antiderivative on star-convex domain) Suppose $D \subseteq \mathbb{C}$ is open and star-convex with centre z_*. If $f : D \to \mathbb{C}$ is holomorphic, then $F : D \to \mathbb{C}$ defined by

$$F(z) \triangleq \int_\gamma f(w)dw \text{ where } \gamma : [0, 1] \to D \text{ is defined by } \gamma(t) \triangleq z_* + t(z - z_*)$$

is an antiderivative of f.

Proof. Fix $z_0 \in D$. Because D is open, there exists some open ball $B_\epsilon(z_0)$ small enough to be contained by D. For all $z \in B_\epsilon(z_0)$, the closed triangle T specified by the vertices $\{z_*, z, z_0\}$ is contained by D, since all $p \in T$ lies in some line segment joining z_* and w where w is some point that lies in the line segment joining z and z_0. We then can apply Cauchy's Integral Theorem for triangles to have the estimate

$$\begin{aligned}
\left|\frac{F(z) - F(z_0)}{z - z_0} - f(z_0)\right| &= \left|\frac{\int_\gamma [f(w) - f(z_0)]dw}{z - z_0}\right| \\
&\leq \max_{w \in \gamma} |f(w) - f(z_0)| \to 0 \text{ as } z \to z_0
\end{aligned}$$

where γ is the line segment traveling from z_0 to z. ■

At this point, it is appropriate for us to define **the winding number** $w(\gamma, z_0)$ **of a contour** $\gamma : [a, b] \to \mathbb{C}$ **around some point** $z_0 \notin \gamma$ by

$$w(\gamma, z_0) \triangleq \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0} dz$$

Immediately, we see that our definition satisfy our geometric intuition in the sense that the circle $\gamma : [0, 2\pi] \to \mathbb{C}$ is defined by

$$\gamma(t) \triangleq z_0 + e^{it}$$

have winding number

$$w(\gamma, z_0) = \frac{1}{2\pi i} \int_0^{2\pi} e^{-it} i e^{it} dt = 1$$

Moreover, we expect any closed contour $\gamma : [a, b] \to \mathbb{C}$ to have integer-valued winding number. This is true. Consider $f : [a, b] \to \mathbb{C}$ defined by

$$f(t) \triangleq \frac{1}{2\pi i} \int_a^t \frac{\gamma'(s)}{\gamma(s) - z_0} ds$$

One may check by direct computation that

$$\frac{d}{dt} e^{-f(t)} (\gamma(t) - z_0) \equiv 0$$

It then follows from γ being closed that

$$e^{-f(a)} = e^{-f(b)}$$

which implies

$$w(\gamma, z_0) = f(b) = f(a) + 2\pi i n = 2\pi i n$$

Given some contour $\gamma : [a, b] \to \mathbb{C}$, if we define $g : \mathbb{C} \setminus \gamma \to \mathbb{C}$ by

$$g(z) \triangleq w(\gamma, z)$$

we see that g is continuous, since if $z_0 \notin \gamma$, we may find $D_r(z_0)$ disjoint with γ and obtain the estimate

$$\begin{aligned}|w(\gamma,z_0)-w(\gamma,z_1)| &= \frac{1}{2\pi}\left|\int_\gamma \Big[\frac{1}{z-z_0}-\frac{1}{z-z_1}\Big]dz\right| \\ &= \frac{1}{2\pi}\left|\int_\gamma \frac{z_1-z_0}{(z-z_0)(z-z_1)}dz\right| \\ &\leq \frac{L\,|z_1-z_0|}{r^2\pi} \text{ where } L \text{ is the length of } \gamma\end{aligned}$$

as long as $|z_0-z_1|<\frac{r}{2}$. The continuity of g together with the fact that g can only be integer-valued implies that g is constant on any connected component of $\mathbb{C}\setminus\gamma$. We may now finally state our version of Cauchy Integral Theorem.

Theorem 11.1.3. (Cauchy Integral Theorem) Suppose $D\subseteq\mathbb{C}$ is open and $f:D\to\mathbb{C}$ is holomorphic. If $\gamma:[a,b]\to\mathbb{C}$ is a closed contour lying in D such that $w(\gamma,z)=0$ for all $z\notin D$, then

$$\int_\gamma f dz = 0$$

Proof. As Prof Frank remarked, the proof is omitted here for being too long and tricky. ■

Theorem 11.1.4. (Cauchy Integral Formula) Let $U\subseteq\mathbb{C}$ be open, D be an closed disk contained by U, and C be a closed contour running through the boundary of D counterclockwise. If $f:U\to\mathbb{C}$ is holomorphic and $a\in D^\circ$, then

$$f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

Proof. Fix ϵ. Let δ satisfy

$$|z-a|\leq\delta \implies |f(z)-f(a)|\leq\epsilon$$

With a geometric argument using Cauchy Integral Theorem, one have

$$\int_C \frac{f(z)}{z-a}dz = \int_\gamma \frac{f(z)}{z-a}dz$$

where $\gamma:[0,2\pi]\to D^\circ$ is defined by

$$\gamma(t)\triangleq a+\delta e^{it}$$

The proof then follows from the estimation

$$\begin{aligned}
\left|f(a) - \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz\right| &= \left|f(a) - \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz\right| \\
&= \left|f(a) - \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+\delta e^{it})}{\delta e^{it}} i\delta e^{it} dt\right| \\
&= \frac{1}{2\pi}\left|\int_0^{2\pi} f(a+\delta e^{it}) - f(a)dt\right| \leq \epsilon
\end{aligned}$$

■

Theorem 11.1.5. (Holomorphic functions are analytic) Let $U \subseteq \mathbb{C}$ be an open, D be an closed disk contained by U and centering a with radius R. Let C be a closed contour running through the boundary of D counterclockwise. If we define for all $n \geq 0$

$$c_n \triangleq \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

then the power series

$$\sum_{n=0}^{\infty} c_n(z-a)^n$$

agrees with f on D°

Proof. Let $z \in D^\circ$. By Cauchy's Integral Formula, we have

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw \\
&= \frac{1}{2\pi i}\int_C f(w)\Big[\frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \cdots + \frac{(z-a)^m}{(w-a)^{m+1}} + \frac{(z-a)^{m+1}}{(w-a)^{m+1}(w-z)}\Big]dw \\
&= \sum_{n=0}^{m} c_n(z-a)^n + \frac{1}{2\pi i}\int_C \frac{(z-a)^{m+1}}{(w-a)^{m+1}(w-z)}dw
\end{aligned}$$

The proof then follows from noting $\left|\frac{z-a}{w-a}\right| < 1$ and direct estimation of Equation 11.1. ■