

Suns

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# Chapter 1

# Groups

## 1.1 Definition of groups

Let  $M$  be a set equipped with a binary operation  $M \times M \rightarrow M$ . We say  $M$  is a **monoid** if the binary operation is associative and there exists a two-sided identity  $e \in M$ .

### Example 1.1.1: Identity of monoids is required to be two-sided

Defining  $(x, y) \mapsto y$ , we see that the operation is associative and every element is a left identity, but no element is a right identity unless  $|M| = 1$ . This is an example why identity must be two-sided.

Because the identity of a monoid is defined to be two-sided, clearly it must be unique. Suppose every element of monoid  $M$  has a left inverse. Fix  $x \in M$ . Let  $x^{-1} \in M$  be a left inverse of  $x$ . To see that  $x^{-1}$  is also a right inverse of  $x$ , let  $(x^{-1})^{-1} \in M$  be a left inverse of  $x^{-1}$  and use

$$(x^{-1})^{-1} = (x^{-1})^{-1}e = (x^{-1})^{-1}(x^{-1}x) = ((x^{-1})^{-1}x^{-1})x = ex = x$$

to deduce

$$xx^{-1} = (x^{-1})^{-1}x^{-1} = e$$

In other words, if we require every element of a monoid  $M$  to have a left inverse, then immediately every left inverse upgrades to a right inverse. In such case, we call  $M$  a **group**. Notice that inverses of elements of a group are clearly unique.

### Example 1.1.2: Another group axioms

Another set of axioms for group is to require the operation to be associative, identity to be left, and the existence of left inverses. Under such condition, we see that if an

element  $y$  is **idempotent**, then it must be identity, since  $y = (y^{-1}y)y = y^{-1}y = e$ . Because of such, we see a left inverse is also a right inverse, since  $(xx^{-1})(xx^{-1}) = xex^{-1} = xx^{-1}$ . This then shows that the left identity is also a right identity, since  $xe = x(x^{-1}x) = x$ .

### Example 1.1.3: Group criteria for finite set

If the underlying set is finite, then weaker assumption can be made. For example, suppose only that the binary operation to be associative and that both cancellation holds:

$$au = aw \implies u = w \quad \text{and} \quad ua = wa \implies u = w$$

Because the set is finite, for all  $a$ , we may attach it with an natural number  $n(a)$  such that  $a^{n(a)+1} = a$ . Clearly,

$$aa^{n(a)}b = a^{n(a)+1}b = ab = ab^{n(b)+1} = ab^{n(b)}b$$

This then by cancellation laws implies  $a^{n(a)} = b^{n(b)}$ , which can be easily checked to be the identity.

### Example 1.1.4: Finite subset closed under binary operation forms a subgroup

Let  $S \subseteq G$  be a finite subset closed under the binary operation. Because  $S$  is finite, for all  $a \in S$ , there exists  $n(a) \in \mathbb{N}$  such that

$$a^{n(a)}a = a$$

Multiplying both side with  $a^{-1}$ , we see that  $a^{n(a)} = e$  and  $a^{-1} = a^{n(a)} \in S$ .

### Example 1.1.5: Euler's totient function

By [example 1.1.3](#), we see that the set of nonzero integer relatively prime to  $n$  and modulo  $n$  forms a group under multiplication modulo  $n$ , called the **multiplicative group of integer modulo  $n$** , or equivalently the unit group of the ring  $\mathbb{Z}_n$ . This immediately shows that

$$a^{\varphi(a)} \equiv 1 \pmod{n}$$

for all  $a \in \mathbb{N}$  coprime with  $n$ , where the **totient function**  $\varphi(a)$  is the number of natural numbers smaller than  $a$  and coprime with  $a$ . We now have **Fermat's little**

**theorem** as a special case.

Unlike the category of monoids, the category of groups behaves much better. Given two groups  $G, H$  and a function  $\varphi : G \rightarrow H$ , if  $\varphi$  respects the binary operation, then  $\varphi$  also respects the identity:

$$e_H = (\varphi(x)^{-1})\varphi(x) = (\varphi(x)^{-1})\varphi(xe_G) = (\varphi(x)^{-1}\varphi(x))\varphi(e_G) = \varphi(e_G)$$

which implies that  $\varphi$  must also respect inverse. In such case, we call  $\varphi$  a **group homomorphism**. In this note, by a **subgroup**  $H$  of  $G$ , we mean an injective group homomorphism  $H \hookrightarrow G$ . Clearly, a subset of  $G$  forms a subgroup if and only if it is closed under both the binary operation and inverse. Note that one of the key basic property of subgroup  $H \subseteq G$  is that if  $g \notin H$ , then  $hg \notin H$ , since otherwise  $g = h^{-1}hg \in H$ .

Let  $S$  be a subset of  $G$ . The group of **words** in  $S$ :

$$\{s_1^{\epsilon_1} \cdots s_n^{\epsilon_n} \in G : n \in \mathbb{N} \cup \{0\} \text{ and } s_i \in S \text{ and } \epsilon_i = \pm 1\}$$

is clearly the smallest subgroup of  $G$  containing  $S$ . We say this subgroup is **generated** by  $S$ . If  $G$  is generated by a single element, we say  $G$  is **cyclic**. Let  $x \in G$ . The **order** of  $G$  is the cardinality of  $G$ , and the order of  $x$  is the cardinality of the cyclic subgroup  $\langle x \rangle \subseteq G$ , or equivalently the infimum of the set of natural numbers  $n$  that makes  $x^n = e$ . Clearly, finite cyclic groups of order  $n$  are all isomorphic to  $\mathbb{Z}_n$ .

Let  $G$  be a group and  $H$  a subgroup of  $G$ . The **right cosets**  $Hx$  are defined by  $Hx \triangleq \{hx \in G : h \in H\}$ . Clearly, when we define an equivalence relation in  $G$  by setting:

$$x \sim y \iff xy^{-1} \in H$$

the equivalence class  $[x]$  coincides with the right coset  $Hx$ . Note that if we partition  $G$  using **left cosets**, the equivalence relation being  $x \sim y \iff x^{-1}y \in H$ , then the two partitions need not to be identical.

#### Example 1.1.6: An non-normal subgroup

Let  $H \triangleq \{e, (1, 2)\} \subseteq S_3$ . The right cosets are

$$H(2, 3) = \{(2, 3), (1, 2, 3)\} \quad \text{and} \quad H(1, 3) = \{(1, 3), (1, 3, 2)\}$$

while the left cosets being

$$(2, 3)H = \{(2, 3), (1, 3, 2)\} \quad \text{and} \quad (1, 3)H = \{(1, 3), (1, 2, 3)\}$$

However, as one may verify, we have a well-defined bijection  $xH \mapsto Hx^{-1}$  between the sets of left cosets and right cosets of  $H$ . Therefore, we may define the **index**  $[G : H]$  of  $H$  in  $G$  to be the cardinality of the collection of left cosets of  $H$ , without falling into the discussion of left and right. Moreover, let  $K$  be a subgroup of  $H$ , by axiom of choice, clearly we have:

$$[G : K] = [G : H] \cdot [H : K]$$

which gives **Lagrange's theorem**

$$o(G) = [G : H] \cdot o(H)$$

as a corollary.

#### Example 1.1.7: Isomorphic subgroups can have different indices

Note that every nontrivial subgroups of  $\mathbb{Z}$  are isomorphic, yet they are of distinct index.

#### Example 1.1.8: Order formula

Let  $H, K \leq G$ . We have

$$|HK| \cdot o(H \cap K) = o(H) \cdot o(K)$$

since the natural map from the left coset space  $K \diagup H \cap K$  to the left coset space  $HK \diagup H$  forms a well-defined bijection. We also have

$$[G : H \cap K] \leq [G : H] \cdot [G : K]$$

because the natural map from the left coset space  $G \diagup H \cap K$  to the product  $G \diagup H \times G \diagup K$  of left coset spaces forms a well defined injection. Moreover, because

$$[G : H \cap K] = [G : H] \cdot [H : H \cap K] = [G : K] \cdot [K : H \cap K]$$

we see that

$$\text{lcm}([G : H], [G : K]) \leq [G : H \cap K]$$

if both  $[G : H]$  and  $[G : K]$  are finite.

## 1.2 Group action

Let  $G$  be a group and  $X$  a set. If we say  $G$  **acts on  $X$  from left**, we are defining a function  $G \times X \rightarrow X$  such that

- (i)  $e \cdot x = x$  for all  $x \in X$ .
- (ii)  $(gh) \cdot x = g \cdot (h \cdot x)$  for all  $g, h \in G$ .

Note that there is a difference between left action and right action, as  $gh$  means  $g \circ h$  in left action and means  $h \circ g$  in right action. Because groups admit inverses, a  $G$ -action is in fact a group homomorphism  $G \rightarrow \text{Sym}(X)$ .

Let  $x \in X$ . We call the set  $\Gamma \triangleq \{g \cdot x \in X : g \in G\}$  the **orbit** of  $x$ . Clearly the set  $\text{Stab}(x)$  of all elements of  $G$  that fixes  $x$  forms a group, called the **stabilizer subgroup**. Consider the action left, and let  $G/\text{Stab}(x)$  denote the left coset space. The fact that the obvious mapping between  $G/\text{Stab}(x)$  and  $\Gamma$  forms a bijection is called the **orbit-stabilizer theorem**, which relates the index of  $\text{Stab}(x)$  and the length of  $\Gamma$ :

$$[G : \text{Stab}(x)] = |\Gamma|$$

The two most important group action are **left multiplication** and **left conjugation**:

$$g \cdot A \triangleq \{ga \in G : a \in A\} \quad \text{and} \quad g \cdot A \triangleq \{gag^{-1} \in G : a \in A\}$$

on the power set of  $G$ . Clearly, orbit of a subgroup under left multiplication is its left coset space, while orbit of a subgroup under conjugation are isomorphic groups that have the same index: If  $K = gHg^{-1}$ , then

$$xH \mapsto xg^{-1}K$$

forms a well-defined bijection between left coset spaces of  $H$  and  $K$ .

### Example 1.2.1: Isomorphic subgroups don't need to have the same index

Isomorphic subgroups can have different indices. Consider subgroups of  $\mathbb{Z}$ .

**Theorem 1.2.2. (Cauchy's theorem for finite group)** Let  $p \mid o(G)$ . Then the number of elements of order divided by  $p$  is a positive multiple of  $p$ .

*Proof.* The set  $X$  of  $p$ -tuples  $(x_1, \dots, x_p)$  that satisfies  $x_1 \cdots x_p = e$  clearly has cardinality  $o(G)^{p-1}$ . Consider the group action  $\mathbb{Z}_p \rightarrow \text{Sym}(X)$  defined by

$$g \cdot (x_1, \dots, x_p) \triangleq (x_p, x_1, \dots, x_{p-1}), \quad \text{where } \mathbb{Z}_p = \langle g \rangle$$

Notice that  $x^p = e$  if and only if  $(x, \dots, x) \in X$ . Therefore the number of cardinality 1 orbit equals to number of solution to  $x^p = e$ . By **orbit-stabilizer theorem**, an orbit in  $X$  either has cardinality  $p$  or 1. Therefore, we may write

$$p \mid o(G)^{p-1} = m + kp$$

with  $m$  the number of cardinality 1 orbits and  $k$  the number of cardinality  $p$  orbits. Clearly we have  $p \mid m$ , as desired. ■

## 1.3 Normal subgroups

Because the inverse of an injective group homomorphism forms a group homomorphism, we know  $\text{Aut}(G)$  forms a group. We say  $\phi \in \text{Aut}(G)$  is an **inner automorphism** if  $\phi$  takes the form  $x \mapsto gxg^{-1}$  for some fixed  $g \in G$ . We say two elements  $x, y \in G$  are **conjugated** if there exists some inner automorphism that maps  $x$  to  $y$ . Clearly conjugacy forms an equivalence relation. We call its classes **conjugacy classes**.

From the point of view of inner automorphism, we see that it is well-defined whether an element  $g \in G$  **normalize** a subset  $S \subseteq G$ :

$$gSg^{-1} = S$$

independent of left and right. Because of the independence, For each subset  $S \subseteq G$ , we see that the set of elements  $g \in G$  that normalize  $S$  forms a group, called the **normalizer** of  $S$ , in fact the stabilizer subgroup  $\text{Stab}(S)$  under the conjugacy action.

### Example 1.3.1: Conjugation can send subgroups to proper subgroup

Consider  $G \triangleq \text{GL}_2(\mathbb{R})$  and consider:

$$H \triangleq \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(\mathbb{R}) : n \in \mathbb{Z} \right\} \quad \text{and} \quad g \triangleq \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(\mathbb{R})$$

Note that  $gHg^{-1} < H$ .

Given  $x, y \in G$ , we often write

$$[x, y] \triangleq xyx^{-1}y^{-1} \quad \text{or} \quad [x, y] \triangleq x^{-1}y^{-1}xy$$

and call  $[x, y]$  the **commutator** of  $x$  and  $y$ . The differences again lies in sides. Taking the first definition, we see that  $[x, y] \in H$  if and only if  $Hxy = Hyx$ , while the second definition leads us to  $[x, y] \in H \iff xyH = yxH$ . However, because  $[x, y]$  in first definition is just  $[x^{-1}, y^{-1}]$  in second definition, if  $H, K \leq G$ , then the set  $[H, K]$  is defined the same using either the first or the second definition. In general,  $[H, K]$  doesn't form a group. In fact, we clearly have  $[H, K] = [K, H]^{-1}$ .

**Equivalent Definition 1.3.2. (Normal subgroups)** Let  $N \leq G$ . We say  $N$  is a **normal subgroup** of  $G$  if any of the followings hold true:

- (i)  $\phi(N) \subseteq N$  for all  $\phi \in \text{Inn}(G)$
- (ii)  $\phi(N) = N$  for all  $\phi \in \text{Inn}(G)$

- (iii)  $xN = Nx$  for all  $x \in G$ .
- (iv) The set of all left cosets of  $N$  equals the set of all right cosets of  $N$ .
- (v)  $N$  is a union of conjugacy classes.
- (vi)  $[N, G] \subseteq N$ .
- (vii) For all  $x, y \in G$ , we have  $xy \in N \iff yx \in N$ .

*Proof.* (i)  $\implies$  (ii): Let  $\phi \in \text{Inn}(G)$ . By premise,  $\phi(N) \subseteq N$  and  $\phi^{-1}(N) \subseteq N$ . Applying  $\phi$  to both side of  $\phi^{-1}(N) \subseteq N$ , we have  $\phi(N) \subseteq N \subseteq \phi(N)$ , as desired.

(ii)  $\implies$  (iii): Consider the automorphisms:

$$\phi_{L,x}(g) = xg \quad \text{and} \quad \phi_{L,x^{-1}}(g) = x^{-1}g \quad \text{and} \quad \phi_{R,x}(g) = gx$$

Because  $\phi_{L,x^{-1}} \circ \phi_{R,x} \in \text{Inn}(G)$ , by premise we have:

$$xN = \phi_{L,x}(N) = \phi_{L,x} \circ \phi_{L,x^{-1}} \circ \phi_{R,x}(N) = \phi_{R,x}(N) = Nx$$

(iii)  $\implies$  (iv) is clear. (iv)  $\implies$  (iii): Let  $x \in G$ . By premise, there exists some  $y \in G$  that makes  $xN = Ny$ . Let  $x = ny$ . The proof then follows from noting

$$xN = Ny = N(n^{-1}x) = Nx$$

(iii)  $\implies$  (v): Let  $n \in N$  and  $x \in G$ . We are required to show  $xnx^{-1} \in N$ . Because  $xN = Nx$ , we know  $xn = \tilde{n}x$  for some  $\tilde{n} \in N$ . This implies

$$xnx^{-1} = \tilde{n}xx^{-1} = \tilde{n} \in N$$

(v)  $\implies$  (vi): Fix  $n \in N$  and  $x \in G$ . By premise,  $xn^{-1}x^{-1} \in N$ . Therefore,  $n(xn^{-1}x^{-1}) \in N$ , as desired.

(vi)  $\implies$  (vii): Let  $xy \in N$ . To see  $yx$  also belong to  $N$ , observe:

$$(xy)^{-1}(yx) = (xy)^{-1}x^{-1}xyx = [xy, x] \in N$$

(viii)  $\implies$  (i): Let  $n \in N$  and  $x \in G$ . Because  $(nx)x^{-1} = n \in N$ , by premise we have  $x^{-1}nx \in N$ , as desired. ■

### Example 1.3.3: Hamiltonian group

A group is said to be **Dedekind** if all of its subgroups are normal. Clearly every abelian group is Dedekind. Non-abelian Dedekind groups are called **Hamiltonian**.

The simplest Hamiltonian group is the **quaternion group**  $Q_8$ , which is the group of the quaternions under multiplication:

$$Q_8 \triangleq \{1, i, j, k, -1, -i, -j, -k\}$$

Note that  $Q_8$  is Hamiltonian because every nontrivial element is of order 4.

Given  $H \leq G$ , clearly,  $N_G(H)$  is the largest subgroups of  $G$  in which  $H$  is normal. Let  $N \trianglelefteq G$ . We say a group homomorphism  $\pi : G \rightarrow G/N$  satisfies the **universal property of quotient group**  $G/N$  if

- (i)  $\pi$  vanishes on  $N$ . (**Group condition**)
- (ii) For all group homomorphism  $f : G \rightarrow H$  that vanishes on  $N$  there exist a unique group homomorphism  $\tilde{f} : G/N \rightarrow H$  that makes the diagram:

$$\begin{array}{ccc} G & \xrightarrow{\pi} & G/N \\ & \searrow f & \downarrow \tilde{f} \\ & & H \end{array}$$

commute. (**Universality**)

**Theorem 1.3.4. (The first isomorphism theorem for groups)** The group homomorphism  $\pi : G \rightarrow G/N$  is always surjective with kernel  $N$ . Let  $f : G \rightarrow H$  be a group homomorphism. Then  $\ker f$  is normal in  $G$ , and the induced homomorphism  $\tilde{f} : G/\ker f \rightarrow H$  is injective.

*Proof.* They are consequences of the construction. ■

Let  $S \subseteq G$ , and consider its **centralizer**  $C_G(S) \triangleq \{g \in G : gs = sg \text{ for all } s \in S\}$ . To see that  $C_G(S) \trianglelefteq N_G(S)$ , one simply observe that  $C_G(S)$  is the kernel of the conjugacy action  $N_G(S) \longrightarrow \text{Sym}(S)$ . Clearly centralizers are always abelian. We call the centralizer of the whole group  $Z(G) \triangleq C_G(G)$  **center**. Because every element of the center form a single conjugacy classes, by **orbit-stabilizer theorem**, we have the **class equation**:

$$o(G) = o(Z(G)) + \sum [G : C_G(x)]$$

where  $x$  runs through conjugacy classes outside of  $Z(G)$ . From class equation, we immediately see that **finite  $p$ -groups** must have nontrivial centers.

### Example 1.3.5: Infinite $p$ -group

The **Prüfer group**  $\mathbb{Z}(p^\infty)$  is defined by

$$\mathbb{Z}(p^\infty) \triangleq \{\exp(2\pi im/p^n) \in \mathbb{C} : m \in \mathbb{Z} \text{ and } n \in \mathbb{N}\}$$

It is an infinite group whose every nontrivial element has order  $p^n$  for some  $n \in \mathbb{N}$ .

Given a subgroup  $H \leq G$ , when we consider the left multiplicative  $G$ -action  $\varphi : G \rightarrow \text{Sym}(G/H)$  on the left coset space  $G/H$ , clearly we have

$$\ker \varphi = \bigcap_{g \in G} H^g, \quad \text{where } H^g \triangleq gHg^{-1}$$

We call  $\text{Core}(H) \triangleq \ker \varphi$  the **core** of  $H$ , which is clearly the greatest subgroup of  $H$  that is normal in  $G$ . If  $G$  is finite, then because by **first isomorphism theorem** we know that  $G/\text{Core}(H)$  is a subgroup of  $\text{Sym}(G/H)$ , which implies that  $[G : \text{Core}(H)]$  divides  $[G : H]!$ .

**Theorem 1.3.6. (Property of finite group)** Let  $G$  be a finite group and  $p$  the smallest prime that divides  $o(G)$ . Then any subgroup  $H \leq G$  of index  $p$  is normal in  $G$ .

*Proof.* We are required to show that the  $\text{Core}(H) = H$ . Because  $H$  is of index  $p$ , we know  $[G : \text{Core}(H)]$  divides  $p!$ . This by minimality of  $p$  implies  $\text{Core}(H) = H$ . ■

Let  $N \trianglelefteq G$ . Because  $xy \in N$  if and only if  $yx \in N$ , regardless of the definition, we see that

$$[g, h] \in N \iff gN, hN \in G/N \text{ commutes}$$

Therefore, the factor group  $G/N$  is abelian if and only if  $[G, G] \subseteq N$ . In this note, we use  $G'$  to denote the **commutator subgroup** of  $G$ , the subgroup generated by  $[G, G]$ . From our observation, clearly  $G'$  is the smallest normal subgroup that makes the quotient abelian. In fact, any subgroup  $H$  containing  $G'$  is normal, since if  $ghg^{-1}h^{-1} \in H$ , then we clearly have  $ghg^{-1} \in H$ . We call  $G/G'$  the **abelianization** of  $G$ .

**Theorem 1.3.7. (Normal subgroup that intersects with commutator subgroup must be contained by the center)** Given normal subgroup  $N \trianglelefteq G$ , if  $G' \cap N = 1$ , then  $N \leq Z(G)$ .

*Proof.* Fix  $n \in N$  and  $g \in G$ . Because  $N$  is normal, we know  $gng^{-1}n^{-1} \in N$ . It then follows from  $N \cap G' = 1$  that  $gng^{-1}n^{-1} = e$ , which implies  $gn = ng$ . ■

### Example 1.3.8: Dihedral group

Fix  $n \geq 3$ . We write the **Dihedral group**  $D_n$  as

$$\{r_0, \dots, r_{n-1}, s_0, \dots, s_{n-1}\}$$

where

$$r_k \triangleq \begin{pmatrix} \cos \frac{2\pi k}{n} & -\sin \frac{2\pi k}{n} \\ \sin \frac{2\pi k}{n} & \cos \frac{2\pi k}{n} \end{pmatrix} \quad \text{and} \quad s_k \triangleq \begin{pmatrix} \cos \frac{2\pi k}{n} & \sin \frac{2\pi k}{n} \\ \sin \frac{2\pi k}{n} & -\cos \frac{2\pi k}{n} \end{pmatrix}$$

stand for rotation and symmetries. Easily, one then can check that if  $n$  is even, then  $Z(D_n) = \{r_0, r_{\frac{n}{2}}\}$ , and if  $n$  is odd, then  $Z(D_n) = \{r_0\}$

### Example 1.3.9: Criterion for abelianess

If  $G/Z(G)$  is cyclic, then for all  $a, b \in G$ , we may write  $a = g^n z_1$  and  $b = g^m z_2$  to see that

$$ab = g^n z_1 g^m z_2 = g^{n+m} z_1 z_2 = ba$$

and concludes that  $G$  is abelian. Such criteria can not be weaken to the case  $G/Z(G)$  cyclic. For example, consider  $D_4$ . Because  $o(Z(D_4)) = 2$ , we know  $D_4/Z(D_4)$  is abelian, but  $D_4$  isn't.

### Theorem 1.3.10. (Property of $p$ -groups)

- (i) Groups of order  $p^2$  are all abelian.
- (ii) Groups of order  $p^3$ , if not abelian, satisfies  $o(Z(G)) = p$  and  $Z(G) = G'$

*Proof.* (i): This follows from our criteria for abelianess and the fact that finite  $p$ -groups have nontrivial centers.

(ii):  $o(Z(G)) = p$  also follows from our criteria for abelianess and the fact that finite  $p$ -groups have nontrivial centers. Now, because  $G$  is non-abelian and because  $G/Z(G)$  is abelian, we see that we must have  $G' = Z(G)$ . ■

## 1.4 Second and third isomorphism theorem

**Theorem 1.4.1. (Second isomorphism theorem)** Let  $H \leq G$ . If  $K \leq N_G(H)$ , then their product:

$$HK \triangleq \{hk \in G : h \in H \text{ and } k \in K\}$$

forms a group (in fact, the subgroup generated by  $H \cup K$ ) and is defined independent of left and right. Moreover,  $H \trianglelefteq HK$  with  $hkH = Hk$ , and  $H \cap K \trianglelefteq K$  with

$$HK/H \cong K/H \cap K \quad \text{via} \quad kH \longleftrightarrow k(H \cap K)$$

*Proof.* To see  $HK \subseteq KH$ , simply observe  $hk = k(k^{-1}hk)$ . The converse inclusion is proved similarly. The fact that  $HK$  forms a group now follows. The rest are clear. ■

### Example 1.4.2: Product of two subgroups

In general, product of two subgroups needs not to form a group. For example, consider the product of the subgroup  $H$  generated by  $(1, 2) \in S_3$  and the subgroup  $K$  generated by  $(2, 3) \in S_3$ . Since  $(2, 3)(1, 2) \notin HK$ , we see  $HK$  isn't a group.

On the other hand, given two normal subgroups  $N, M \trianglelefteq G$ . By the **preserved-by-conjugations definition of normal subgroups**, clearly both  $NM$  and  $N \cap M$  are normal in  $G$ .

**Theorem 1.4.3. (Third isomorphism theorem and correspondence theorem)** Let  $N \trianglelefteq G$ . The canonical projection  $\pi : G \rightarrow G/N$  gives rise to a bijection between the set of subgroups of  $G$  that contains  $N$  and the set of subgroups of  $G/N$ . The bijection is moreover a bijection between the set of normal subgroups of  $G$  that contains  $N$  and the set of normal subgroups of  $G/N$ . In fact, given  $K \trianglelefteq G$  that contains  $N$ , if we identify  $K/N$  as a subgroup of  $G/N$  the natural way:

$$\begin{array}{ccc} K & \xrightarrow{\pi} & \frac{K}{N} \\ & \searrow & \downarrow \\ & & \frac{G}{N} \end{array}$$

then  $\frac{K}{N} \trianglelefteq \frac{G}{N}$  is the normal subgroup that corresponds to  $K \trianglelefteq G$ , and we have a natural isomorphism  $\frac{G}{K} \cong \frac{G}{N}$ .

*Proof.* Routine. ■

## 1.5 Symmetric groups

Given the **symmetric group**  $S_n$ , to define parity, the fastest way is to realize it as the **group of permutation matrix**, and then call those that have determinant 1 **even**, while those that have determinant  $-1$  **odd**. Clearly we have the **alternating group**

$$A_n \triangleq \{g \in S_n : \det(g) = 1\}$$

To see that  $[S_n : A_n] = 2$  for all  $n \geq 2$ , just consider that for any  $g \in S_n - A_n$ , we have a bijection between  $A_n$  and  $S_n - A_n$  defined by

$$a \mapsto ag$$

By direct observation, we see that every permutation has a fixed **cycle type**. Because

$$\sigma \circ (a_1, \dots, a_k) \circ \sigma^{-1} = (\sigma(a_1), \dots, \sigma(a_k))$$

we see that the conjugacy classes of  $S_n$  coincides with cycle type classes.

**Theorem 1.5.1.** ( $S_n$  has trivial center for  $n \geq 3$ )  $Z(S_n) = 1$  for all  $n \geq 3$ .

*Proof.* Let  $\tau \neq e \in S_n$ . Because  $\tau$  is nontrivial, we know  $\tau(i) = j$  for some  $i \neq j$ . Let  $\sigma(i) = i$  and  $\sigma(j) \neq j$ . We now see  $\sigma \circ \tau \circ \sigma^{-1} \neq \tau$ , as desired. ■

**Theorem 1.5.2.** ( $A_n$  are generated by 3-cycles for  $n \geq 3$ ) Given  $n \geq 3$ ,  $A_n$  is generated by 3-cycles.

*Proof.* Such is proved via induction on  $n$ . The base case is clear. We now prove the inductive case. Let  $\sigma \in A_n$ . By inductive hypothesis, if  $\sigma$  doesn't move  $n$ , then  $\sigma$  can be generated by 3-cycles. If  $\sigma$  move  $n$ , then because inverse of a 3-cycle is a 3-cycle, and because there exists a 3-cycle  $\tau$  such that  $\tau \circ \sigma$  fix  $n$ , we see  $\sigma$  can also be generated by 3-cycles. ■

**Theorem 1.5.3.** (Commutator subgroup of  $S_n$  is  $A_n$  for  $n \geq 3$ )  $S'_n = A_n$  for all  $n \geq 3$ .

*Proof.*  $S'_n \leq A_n$  follows from computation. Because  $A_n$  is generated by 3-cycles, to show  $S'_n = A_n$ , we only have to show  $S'_n$  contains all 3-cycles, which follows from computing

$$(a, b, c) = [(a, b), (a, c)]$$
■

**Theorem 1.5.4.** (Simplicity of  $A_n$ )

- (i)  $A_3$  is simple.

(ii)  $A_4$  is not simple.

(iii)  $A_n$  is simple for  $n \geq 5$ .

*Proof.* (i): Simplicity of  $A_3$  follows from  $o(A_3) = 3$ .

(ii): There are three ways to partition  $\{1, 2, 3, 4\}$  into 2 disjoint subsets, each of cardinality 2, i.e.,

$$\Pi_1 \triangleq \{\{1, 2\}, \{3, 4\}\} \quad \text{and} \quad \Pi_2 \triangleq \{\{1, 3\}, \{2, 4\}\} \quad \text{and} \quad \Pi_3 \triangleq \{\{1, 4\}, \{2, 3\}\}$$

Clearly,  $S_4$  acts on  $\{\Pi_1, \Pi_2, \Pi_3\}$ , that is,  $S_4 \longrightarrow \text{Sym}(\{\Pi_1, \Pi_2, \Pi_3\}) \cong S_3$ . Direct computation now shows that we have a surjective group homomorphism  $A_4 \twoheadrightarrow A_3$  with kernel  $\{e, (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}$ .

(iii): We first prove  $A_5$  is simple. Because  $A_5 \trianglelefteq S_5$  is normal, we know  $A_5$  is the union of the classes of cycle types

$$(1, 2, 3) \quad \text{and} \quad (1, 2)(3, 4) \quad \text{and} \quad (1, 2, 3, 4, 5)$$

Direct computation shows that the first cycle type class has 20 elements, that the second cycle type class has 15 elements, and that the third cycle type class has 24 elements. It now follows from [Lagrange's theorem](#) and from [normal subgroups are unions of conjugacy classes](#) that  $A_5$  has no proper nontrivial subgroups. ([done](#))

Let  $N \trianglelefteq A_n$ , with  $\tau \neq e \in N$  and  $n > 5$ . We are required to show  $N = A_n$ . Because  $n \geq 5$ , clearly for any pair of 3-cycles  $\{(a_1, a_2, a_3), (b_1, b_2, b_3)\}$ , there exists some  $\sigma \in A_n$  such that  $(b_1, b_2, b_3) = \sigma \circ (a_1, a_2, a_3) \circ \sigma^{-1}$ . In other words, the class of 3-cycles forms a conjugacy class of  $A_n$ . This together with normality  $N \trianglelefteq A_n$  and the fact that  $A_n$  is generated by set of 3-cycles reduce the problem into proving  $N$  contains a 3-cycle.

Consider  $A_5$  as a subgroup of  $A_n$  that fixes a particular set of  $n - 5$  numbers. Then clearly we have  $N \cap A_5 \trianglelefteq A_5$ . Because  $A_5$  is simple and contains a 3-cycle, we now only have to show  $N \cap A_5 > 1$ , that is, to show  $N$  contains an element that fixes at least  $n - 5$  elements.

Let  $(a_1, a_2, a_3)$  be a 3-cycle such that  $\{\tau(a_1), \tau(a_2), \tau(a_3)\}, \{a_1, a_2, a_3\}$  overlap. We now see  $\tau \circ (a_1, a_2, a_3) \circ \tau^{-1} \circ (a_1, a_2, a_3)^{-1} \in N$  is an element that fixes at least  $n - 5$  elements. ■

**Theorem 1.5.5. (The only proper nontrivial normal subgroup of  $S_n$ )** For  $n \geq 5$ ,  $A_n$  is the only proper nontrivial normal subgroup of  $S_n$ .

*Proof.* Let  $N$  be a proper nontrivial subgroup of  $S_n$ . We are required to show  $N = A_n$ . Because  $[S_n : A_n] = 2$ , we only have to show  $A_n \leq N$ . This boils down to showing

$N \cap A_n > 1$ , since  $A_n$  is simple. Assume for a contradiction that  $N \cap A_n = 1$ . The contradiction  $N = 1$  then follows from  $A_n = S'_n$  and  $Z(S_n) = 1$ , since normal subgroup that intersects with commutator subgroup must be contained by the center.  $\blacksquare$

## 1.6 Finitely generated abelian group

**Equivalent Definition 1.6.1. (Recognition theorem)** Let  $N_1, \dots, N_k \trianglelefteq G$ . We say  $G$  is an **internal direct products** of  $N_i$  if any of the followings hold true:

- (i) The natural map  $N_1 \times \dots \times N_k \rightarrow G$  forms a group isomorphism.
- (ii)  $N_1 \cdots N_k = G$  and  $N_i \cap \prod_{j \neq i} N_j = \{e\}$  for all  $i$ .

*Proof.* (i)  $\implies$  (ii): Clearly we have  $N_1 \cdots N_k = G$ . Let  $n_2 \cdots n_k \in N_1$ . Because  $n_2 \cdots n_k$  is both the image of  $(n_2 \cdots n_k, e, \dots, e)$  and  $(e, n_2, \dots, n_k)$ , by injectivity of the natural map, we know  $n_2 = \dots = n_k = e$ .

(ii)  $\implies$  (i): We first need to show that the map forms a homomorphism, which boils down to showing that for  $i \neq j$ , we have  $[N_i, N_j] = 1$ , which follows from the premise and the fact that  $[N_i, N_j] \leq N_i \cap N_j$ . Surjectivity is clear. For injectivity, if  $n_1 \cdots n_k = e$ , then because  $n_2 \cdots n_k = n_1^{-1} \in N_1$ , we know  $n_2 \cdots n_k = e$  and  $n_1 = e$ . Continuing the process, we see  $n_1 = \dots = n_k = e$ . ■

### Example 1.6.2: The requirement in definitions of internal direct products for groups

Let  $G \triangleq \mathbb{Z}_4 \times \mathbb{Z}_2$ . Clearly the direct product of  $\langle(1, 0)\rangle$  and  $\langle(2, 0)\rangle$  is isomorphic to  $G$ , but they do not form an internal direct product of  $G$ . It is because of such, we must require  $N_1 \times \dots \times N_k$  not only isomorphic to  $G$ , but moreover the natural way in **definition of internal direct products for groups**.

**Theorem 1.6.3. (Change of basis for finitely generated abelian group)** Let  $k \in \mathbb{N}$ ,  $G = \langle x_1, \dots, x_k \rangle$  be abelian and  $c_1, \dots, c_k \in \mathbb{N}$  satisfies  $\gcd(c_1, \dots, c_k) = 1$ . Then there exists  $y_1, \dots, y_k \in G$  such that  $G = \langle y_1, \dots, y_k \rangle$  and  $y_1 = c_1 x_1 + \dots + c_k x_k$ .

*Proof.* Such is proved via induction on  $s \triangleq c_1 + \dots + c_k$ . The base case  $s = k$  is clear. We now prove the inductive case. Because  $\gcd(c_1, \dots, c_k) = 1$ , by changing the order if necessary, we may write  $c_1 > c_2$ . Now, because  $\gcd(c_1 - c_2, c_2, \dots, c_k) = 1$  and  $G = \langle x_1, x_2 - x_1, x_3, \dots, x_k \rangle$ , we see by inductive hypothesis that there exists  $y_1, \dots, y_k$  such that  $G = \langle y_1, \dots, y_k \rangle$  and

$$\begin{aligned} y_1 &= (c_1 - c_2)x_1 + c_2(x_2 + x_1) + \dots + c_k x_k \\ &= c_1 x_1 + \dots + c_k x_k \end{aligned}$$

as desired. ■

**Theorem 1.6.4. (Fundamental theorem for finitely generated abelian group, invariant factor form)** Let  $G$  be a finitely generated abelian group. Then we may write:

$$G \cong \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_s} \times \mathbb{Z}^r$$

for  $n_i \geq 2$ . Moreover, we know that

- (i) Such  $r$  is unique, called the **rank** of  $G$ .
- (ii) Under the assumption that  $n_i$  each is a power of some prime, the expression exists and is unique, called the **primary decomposition form**.
- (iii) Under the assumption that  $n_i \mid n_{i+1}$  for all  $i$ , the expression exists and is unique, called the **invariant factor form**.

*Proof.* We first show the **existence** via induction on the number of generators. The base case is clear. Suppose that the existence holds true for any abelian group that has a generating set of cardinality  $k - 1$ , and suppose that  $G$  has a generating set  $\{x_1, \dots, x_k\}$  where  $x_1$  has order smaller than any elements of any generating sets of cardinality  $k$ .

By inductive hypothesis, we only have to show that  $G$  is an internal direct product of  $\langle x_1 \rangle$  and  $\langle x_2, \dots, x_k \rangle$ . Assume not for a contradiction. Then there exists  $m_1 \neq 0$  such that  $m_1 x_1 + m_2 x_2 + \cdots + m_k x_k = 0$ . By possibly changing sign of some of the  $x_i$  and exchanging the order, we may suppose  $m_1 < o(x_1)$  and  $m_1, \dots, m_t \in \mathbb{N}$  and  $m_{t+1} = \cdots = m_k = 0$ .

Let  $c_i \triangleq \frac{m_i}{\gcd(m_1, \dots, m_t)}$  for all  $i \in \{1, \dots, t\}$ . By [theorem 1.6.3](#), we know there exists  $y_1, \dots, y_t \in G$  such that  $\langle y_1, \dots, y_t \rangle = \langle x_1, \dots, x_t \rangle$  with  $y_1 = c_1 x_1 + \cdots + c_t x_t$ . Clearly, we have  $G = \langle y_1, \dots, y_t, x_{t+1}, \dots, x_k \rangle$ . Compute

$$\gcd(m_1, \dots, m_t) y_1 = m_1 x_1 + \cdots + m_t x_t = 0$$

We now see  $o(y_1) \leq m_1 < o(x_1)$ , a contradiction to the choice that  $x_1$  has the smallest order. (**done**)

(i): Fix an expression of  $G$ , and let  $p$  be a prime that satisfies  $p \nmid n_i$  for all  $i$ . The rest then follows from checking that  $G/pG \cong \mathbb{Z}_p^s$ .

(ii): The existence follows from noting that

$$\gcd(m, n) = 1 \implies \mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$$

The uniqueness follows from noting that given a primary decomposition form whose  $p$ -part has the form  $\mathbb{Z}_{p^{n_1}} \times \cdots \times \mathbb{Z}_{p^{n_k}}$ , then the **torsion  $p$ -subgroup**  $G_{T_p}$  of  $G$

$$G_{T_p} \triangleq \{x \in G : o(x) = p^n \text{ for some } n \geq 0\}$$

is

$$G_{T_p} \cong \mathbb{Z}_{p^{n_1}} \times \cdots \times \mathbb{Z}_{p^{n_k}}$$

and that

$$\frac{p^{d-1}\mathbb{Z}_{p^{n_i}}}{p^d\mathbb{Z}_{p^{n_i}}} \cong \begin{cases} \mathbb{Z}_p & \text{if } d-1 < n_i \\ 0 & \text{if } d-1 \geq n_i \end{cases}$$

(iii): Both the existence and uniqueness of invariant factor form follows from that of primary decomposition form: Just consider that  $\mathbb{Z}_{n_s}$  can only be  $\mathbb{Z}_{p_1}^{d_1} \times \cdots \times \mathbb{Z}_{p_m}^{d_m}$ , where  $p_i$  non-repeatedly running through all the occurring prime with  $d_i$  being the highest exponential. ■

## 1.7 Sylow theorems

**Theorem 1.7.1. (Combinatorial facts)** Let  $p$  be prime. Then:

- (i) Given  $m \geq r \in \mathbb{N} \cup \{0\}$  with  $t \in \mathbb{N}$  coprime with  $p$ , the natural number  $\binom{p^m t}{p^r}$  has  $p$ -part  $p^{m-r}$ .
- (ii) We have

$$\binom{m}{n} \equiv \prod_{i=0}^k \binom{m_i}{n_i} \pmod{p}$$

when we write  $m = m_k p^k + \cdots + m_0 p^0$  and  $n = n_k p^k + \cdots + n_0 p^0$ . This is called **Lucas modulo binomial formula**.

*Proof.* (i) follows from noting that

$$\binom{p^m t}{p^r} = \frac{p^m t (p^m t - 1) \cdots (p^m t - (p^r - 1))}{p^r (p^r - 1) \cdots (p^r - (p^r - 1))}$$

and that for all  $i \in \{1, \dots, p^r - 1\}$ , the three natural number  $\{i, p^m t - i, p^r - i\}$  share the same  $p$ -part.

(ii): Let  $M$  be a set of  $m$  elements. Partition  $M$  into  $m_i$  cycles of length  $p^i$ , i.e.,

$$M \triangleq \bigcup_{i=0}^k \bigcup_{j=1}^{m_i} \Gamma_{i,j}, \quad \text{where } |\Gamma_{i,j}| = p^i$$

Because of such, we see that  $M$  can be acted on by the group

$$G \triangleq \prod_{i=0}^k \overbrace{\mathbb{Z}_{p^i} \times \cdots \times \mathbb{Z}_{p^i}}^{m_i}$$

Clearly,  $G$  also acts on the set  $X$  of the set of subsets of  $M$  that has  $n$  elements. Because  $G$  is a  $p$ -group, **orbit-stabilizer theorem** tell us that

$$\binom{m}{n} = |X| \equiv |\text{Fix}(G)| \pmod{p}$$

where  $\text{Fix}(G)$  is the set of elements of  $X$  fixed by all  $g \in G$ . Because of uniqueness of representation in base  $p$ , we know that elements of  $\text{Fix}(G)$  are exactly those subsets of  $X$  that contains  $n_i$  cycles of length  $p^i$ . The proof now follows from directly computing that  $|\text{Fix}(G)| = \prod_{i=0}^k \binom{m_i}{n_i}$ . ■

**Theorem 1.7.2. (First and Third Sylow theorem, Wielandt's proof)** Let  $G$  be a finite group of order  $p^m t$  with  $\gcd(p, t) = 1$ . Let  $1 \leq r \leq m$ . Then the number  $n_p$  of  $p$ -subgroup with order  $p^r$  satisfies

$$n_p \equiv 1 \pmod{p}$$

*Proof.* Let  $X$  be the set of subset of  $G$  with cardinality  $p^r$ . Our goal is to find all elements of  $X$  that forms a group. Clearly we may define a left  $G$ -action on  $X$  by setting

$$g \cdot \{x_1, \dots, x_{p^r}\} \triangleq \{gx_1, \dots, gx_{p^r}\}$$

Let  $\Gamma$  be an orbit. If  $\Gamma$  contains a group, then we see that  $\Gamma$  is the left coset space of that group, containing exactly one group and satisfying  $|\Gamma| = p^{m-r}t$ . If  $\Gamma$  doesn't contain any group, there still exists some  $S \in \Gamma$  such that  $e \in S$ , and clearly we will have  $\text{Stab}(S) \subseteq S$ . Because  $S$  isn't a group, we see  $p^r = |S| > o(\text{Stab}(S))$ , which by **orbit-stabilizer theorem** implies that  $|\Gamma| = [G : \text{Stab}(S)] = p^{m-r+c}t$  for some  $c \geq 1$ .

In summary, by counting orbit, we have shown that:

$$\binom{p^m t}{p^r} = |X| = n_p p^{m-r} t + l p^{m-r+1} t, \quad \text{for some } l \in \mathbb{N}$$

Let  $ut \equiv 1 \pmod{p}$ . Recalling that  $\binom{p^m t}{p^r}$  has  $p$ -power  $p^{m-r}$ , it remains to show

$$u \cdot \frac{\binom{p^m t}{p^r}}{p^{m-r}} \equiv 1 \pmod{p}$$

which follows from noting:

$$u \cdot \frac{\binom{p^m t}{p^r}}{p^{m-r}} = ut \cdot \binom{p^m t - 1}{p^r - 1} \equiv \binom{p^m t - 1}{p^r - 1} \equiv 1 \pmod{p}$$

where the last equality follows from **Lucas modulo binomial formula** and the observation that

$$p^m t - 1 = t_k p^{m+k} + \dots + (t_0 - 1)p^m + (p - 1)p^{m-1} + \dots + (p - 1)p^0$$

where  $t = t_k p^k + \dots + t_0 p^0$  with  $t_0 > 0$ . ■

**Theorem 1.7.3. (Counting lemma for  $p$ -group)** Let  $G$  be a  $p$ -group acting on a finite set  $X$ . Then

$$|X| \equiv |\text{Fix}(G)| \pmod{p}$$

where  $\text{Fix}(G) \triangleq \{x \in X : gx = x \text{ for all } g \in G\}$ .

*Proof.* This is a consequence of **orbit-stabilizer theorem**. ■

#### Example 1.7.4: Normal subgroups of finite $p$ -groups intersect with centers

Let  $1 < N \trianglelefteq P$ , where  $P$  is a finite  $p$ -group. Let  $P$  acts on  $N$  by conjugation. By our counting lemma for  $p$ -group, we have

$$0 \equiv o(N) \equiv |\{n \in N : gng^{-1} = n \text{ for all } g \in P\}| \pmod{p}$$

Noting that the latter is the set  $N \cap Z(P)$ , we now see  $N \cap Z(P) > 1$ .

**Theorem 1.7.5. (Second Sylow theorem)** Sylow  $p$ -subgroups are conjugated to each other.

*Proof.* Let  $H$  and  $P$  be two Sylow  $p$ -subgroups of  $G$ , and let  $H$  acts on left coset space of  $P$  by left multiplication. Because  $P$  is Sylow, by counting lemma for  $p$ -group, we know the number of fixed points  $gP$  is nonzero. Let  $gP$  be a fixed point. We then see that, as desired,  $g^{-1}hg \in H$  for all  $h \in H$ , since  $hgP = gP$ . ■

Second Sylow theorem may be the most important one out of the three, as it suggests that there is only one conjugacy class within  $\text{Syl}_p(G)$ , i.e., it self, which, more importantly, shows that if a sylow  $p$ -group is unique, then it must be normal.

**Theorem 1.7.6. (Remaining part of third Sylow theorem)** Let  $G$  be a finite group, and let  $n_p$  be the number of Sylow  $p$ -subgroup of  $G$ . For all Sylow  $p$ -subgroup  $P$  of  $G$ , we have

$$n_p = [G : N(P)]$$

*Proof.* This is a consequence of second Sylow theorem and orbit stabilizer theorem, where we note that when  $G$  acts on  $\text{Syl}_p(G)$  by conjugation we have  $\text{Stab}(P) = N(P)$ . ■

**Corollary 1.7.7. (Application of Sylow theorems)** Let  $p, q, r$  be three distinct prime.

(i) Groups of order  $pq$  must be non-simple. Says,  $p < q$ . If  $p$

#### Example 1.7.8: Groups of order $pq$

Let  $o(G) = pq$  with  $p > q$  being prime. Because  $n_p \equiv 1 \pmod{p}$  and  $n_p \mid q$ , we see  $n_p = 1$ , which implies the non-simplicity of  $G$ .

Suppose  $q \nmid p - 1$ . Then from  $n_q \equiv 1 \pmod{q}$  and  $n_q \mid p$ , we see  $n_q = 1$ , which by recognition theorem implies  $G \cong \mathbb{Z}_p \times \mathbb{Z}_q$ .

### Example 1.7.9: Groups of order $p^2q$ must have a normal Sylow subgroup unless $p = 2$ and $q = 3$

Let  $o(G) = p^2q$ . If  $p > q$ , then clearly  $n_p$  can only be 1. If  $p < q$ , then we must have  $n_q \in \{1, p^2\}$ . Assume moreover that  $n_q = p^2$ . Then from  $n_q \equiv 1 \pmod{q}$ , we see  $q = p + 1$ , which can only happen if  $q = 3$  and  $p = 2$ .

### Example 1.7.10: Groups of order $p^a m$ with $p \nmid m$

Let  $o(G) = p^a m$  with  $a \in \mathbb{N}$  and  $p \nmid m$ . If  $G$  is simple, then  $o(G) \mid n_p$ !

### Example 1.7.11: Simple group of order 60 can only be $A_5$

Let  $o(G) = 60$  and  $G$  be simple. Simplicity of  $G$  implies  $n_2 \in \{3, 5, 15\}$ .

### Example 1.7.12

Let  $o(G) = pqr$  with  $p < q < r$  prime. We have  $1 \in \{n_p, n_q, n_r\}$ .

### Example 1.7.13: No group of order 132 is simple

## 1.8 Nilpotency and Solvability

More than often, we care about the existence of **central series**

$$1 \trianglelefteq \cdots \trianglelefteq A_{n-1} \trianglelefteq A_n \trianglelefteq A_{n+1} \trianglelefteq \cdots \trianglelefteq G$$

where we requires the successive quotient to be **central**, i.e.,  $[G, A_{n+1}] \leq A_n$ , or equivalently,  $A_{n+1}/A_n \leq Z(G/A_n)$ , or equivalently,  $xyA_n = yxA_n \in G/A_n$  if one of  $x, y$  is in  $A_{n+1}$ .

To construct one, one can consider the **upper central series**, defining  $G_{(n)} \triangleq [G, G_{(n-1)}]$  with  $G_{(0)} \triangleq G$ . This gives us

$$\cdots \trianglelefteq G_{(2)} \trianglelefteq G_{(1)} \trianglelefteq G$$

Note that

**Theorem 1.8.1. (Every subgroup of a nilpotent group is subnormal)** Let  $G$  be a nilpotent group with  $H \leq G$ . Then  $H$  is a subnormal subgroup of  $G$ .

*Proof.* Let

$$1 \triangleleft A_1 \triangleleft \cdots \triangleleft A_n = G$$

■

**Corollary 1.8.2. (Nilpotent group satisfies normalizer condition)** Let  $G$  be a nilpotent group. If  $H < G$ , then  $H < N_G(H)$ .

A group is said to be nilpotent if it admits a central series.

**Equivalent Definition 1.8.3. (Nilpotency for finite group)** Let  $G$  be a finite group. The followings are equivalent:

- (i)  $G$  is nilpotent.
- (ii) Sylow subgroups of  $G$  are all normal.
- (iii) Sylow subgroups of  $G$  are all normal and they form an inner direct product equal to  $G$ .
- (iv)

*Proof.*

■

## 1.9 Remarkable Exercises

### Question 1: Wording problem

Let  $n \in \mathbb{Z}$  satisfies

$$(ab)^n = a^n b^n \quad \text{and} \quad (ab)^{n-1} = a^{n-1} b^{n-1} \quad \text{and} \quad (ab)^{n+1} = a^{n+1} b^{n+1}$$

for all  $a, b \in G$ . Show that  $G$  is abelian.

*Proof.*

$$\begin{aligned} a^n b^n ab &= (ab)^n(ab) = (ab)^{n+1} = a^{n+1} b^{n+1} \\ \implies b^n a &= ab^n \\ \implies (a^{n-1} b^{n-1})ba &= a^n b^n = (a^{n-1} b^{n-1})ab \\ \implies ba &= ab \end{aligned}$$

■

### Question 2: Wording problem with finite group

Let  $G$  be a finite group whose order is not divisible by 3. Suppose

$$(ab)^3 = a^3 b^3, \quad \text{for all } a, b \in G$$

Show that  $G$  is abelian.

*Proof.* Because  $3 \nmid o(G)$ , we know if  $g^3 = e$ , then  $g = e$ . In other words, the endomorphism  $x \mapsto x^3$  is an automorphism. Because of such, we only have to prove  $a^3 b^3 = b^3 a^3$  for all  $a, b \in G$ . This then follows from

$$\begin{aligned} (ab)^3 &= a^3 b^3, \quad \text{for all } a, b \in G \\ \implies baba &= a^2 b^2, \quad \text{for all } a, b \in G \\ \implies (ba)^2 &= a^2 b^2, \quad \text{for all } a, b \in G \\ \implies (ab)^4 &= [(ab)^2]^2 = [b^2 a^2]^2 = a^4 b^4, \quad \text{for all } a, b \in G \\ \implies (ab)^4 &= a(ba)^3 b = ab^3 a^3 b, \quad \text{for all } a, b \in G \\ \implies a^3 b^3 &= b^3 a^3, \quad \text{for all } a, b \in G \end{aligned}$$

■

### Question 3: Wording problem

Let  $a, b \in G$  satisfies

$$o(a) = 5 \quad \text{and} \quad aba^{-1} = b^2$$

Find  $o(b)$ .

*Proof.*

$$\begin{aligned} aba^{-1} &= b^2 \\ \implies a^2ba^{-2} &= ab^2a^{-1} = (aba^{-1})^2 = b^4 \\ \implies b &= a^5a^{-5} = a^4b^2a^{-4} = (a^2ba^{-2})^2 = b^8 \end{aligned}$$

Therefore  $o(b) = 7$ . ■

## Chapter 2

# Linear algebra done Taiwan

# Chapter 3

## Temporary put HWs

### 3.1 Exercises

For question 1, recall that by class equation,  $p$ -group can not have trivial center, and recall that  $G/N$  is abelian if and only if  $[G, G] \leq N$ .

#### Question 4

Show that

- (i) If  $H/Z(H)$  is cyclic, then  $H$  is abelian.
- (ii) If  $H$  is of order  $p^2$ , then  $H$  is abelian.

From now on, suppose  $G$  is non-abelian with order  $p^3$ .

- (iii)  $|Z(G)| = p$ .
- (iv)  $Z(G) = [G, G]$ .

*Proof.* Let  $a, b \in H$  and  $H/Z(H) = \langle hZ \rangle$ . Write  $a = h^n z_1$  and  $b = h^m z_2$ . Because  $z_1, z_2 \in Z(H)$ , we may compute:

$$ab = h^n z_1 h^m z_2 = h^{n+m} z_1 z_2 = ba$$

as desired.

Let  $|H| = p^2$ . Because  $H$  is a  $p$ -group, we know  $Z(H)$  is nontrivial, therefore either  $|Z(H)| = p$  or  $|Z(H)| = p^2$ . To see the former is impossible, just observe that if so, then  $|H/Z(H)| = p$ , which implies  $H/Z(H)$  is cyclic, which by part (i) implies  $Z(H) = H$ .

Because  $G$  is non-abelian, we know  $|Z(G)| \neq p^3$ . Because  $G$  is a  $p$ -group, we know  $|Z(G)| \neq 1$ . Therefore, either  $|Z(G)| = p$  or  $|Z(G)| = p^2$ . Part (i) tell us that  $|Z(G)| \neq p^2$ , otherwise  $G$  is abelian, a contradiction. We have shown  $|Z(G)| = p$ , as desired.

We now prove  $Z(G) = [G, G]$ . Because  $|Z(G)| = p$ , by part (ii) we know  $G/Z(G)$  is abelian. This implies  $[G, G] \leq Z(G)$ , which implies  $[G, G]$  is either trivial or equal to  $Z(G)$ . Because  $G$  is non-abelian, we know  $[G, G]$  can not be trivial. This implies  $Z(G) = [G, G]$ , as desired. ■

## Question 5

- (i) Let  $M, N$  be two normal subgroups of  $G$  with  $MN = G$ . Prove that

$$G/(M \cap N) \cong (G/M) \times (G/N)$$

- (ii) Let  $H, K$  be two distinct subgroups of  $G$  of index 2. Prove that  $H \cap K$  is a normal subgroup with index 4 and  $G/(H \cap K)$  is not cyclic.

*Proof.* The map  $G/(M \cap N) \rightarrow (G/M) \times (G/N)$  defined by

$$g(M \cap N) \mapsto (gM, gN) \tag{3.1}$$

is clearly a well-defined group homomorphism, since if  $gM = hM$  and  $gN = hN$ , then  $gh^{-1} \in M$  and  $gh^{-1} \in N$ , which implies  $gh^{-1} \in M \cap N$ , which implies  $g(M \cap N) = h(M \cap N)$ . Let  $gM = M$  and  $gN = N$ . Then  $g \in M \cap N$  and  $g(M \cap N) = M \cap N$ . Therefore map 3.1 is also injective. It remains to show map 3.1 is surjective. Fix  $g, h \in G$ . Write  $g = mn$  and  $h = \tilde{m}\tilde{n}$ . Clearly  $gM = nM = \tilde{m}nM$  and  $hN = \tilde{m}N = \tilde{m}nN$ . This implies that mapping 3.1 maps  $\tilde{m}n$  to  $(gM, gN)$ , as desired.

Because  $H, K$  are both of index 2 in  $G$ , we know they are both normal in  $G$ . This by second isomorphism theorem implies  $HK$  forms a subgroup of  $G$ . Because  $H \neq K$ , we know  $HK$  properly contains  $H$ , which by finiteness of  $G$  implies the index of  $HK$  is strictly less than  $H$ , i.e.,  $HK = G$ . Note that  $H \cap K$  is normal since it is the intersection of normal subgroups. By part (i), we now have  $G/(H \cap K) \cong (G/H) \times (G/K) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ , which shows that  $H \cap K$  has index 4 and  $G/(H \cap K)$  is cyclic. ■

## Question 6

Let  $G$  be a group of order  $pq$ , where  $p > q$  are prime.

- (i) Show that there exists a unique subgroup of order  $p$ .
- (ii) Suppose  $a \in G$  with  $o(a) = p$ . Show that  $\langle a \rangle \subseteq G$  is normal and for all  $x \in G$ ,

we have  $x^{-1}ax = a^i$  for some  $0 < i < p$ .

*Proof.* The third Sylow theorem stated that the number  $n_p$  of Sylow  $p$ -subgroups satisfies

$$n_p \equiv 1 \pmod{p} \quad \text{and} \quad n_p \mid q$$

Because  $p > q$ , together they implies  $n_p = 1$ . Since Sylow  $p$ -subgroups of  $G$  are exactly subgroups of order  $p$ , we have proved (i). ■

The third Sylow theorem also stated that  $n_p = |G : N_G(P)|$  for any Sylow  $p$ -subgroup  $P \leq G$ . Therefore,  $N_G(\langle a \rangle) = G$ , i.e.,  $\langle a \rangle$  is normal in  $G$ . Fix  $x \in G$ . It remains to prove  $xax^{-1} \neq e$ , which is a consequence of the fact that conjugacy (automorphism) preserves order. ■

### Question 7

Let  $H, K$  be two subgroups of  $G$  of finite indices  $m, n$ . Show that

$$\text{lcm}(m, n) \leq |G : H \cap K| \leq mn$$

*Proof.* Let  $\Omega_{H \cap K}, \Omega_H$ , and  $\Omega_K$  respectively denote the set of left cosets of  $H \cap K, H$ , and  $K$ . The map  $\Omega_{H \cap K} \rightarrow \Omega_H \times \Omega_K$  defined by

$$g(H \cap K) \mapsto (gH, gK) \tag{3.2}$$

is well defined since

$$g(H \cap K) = l(H \cap K) \implies g^{-1}l \in H \cap K \implies gH = lH \text{ and } gK = lK$$

such set map is injective since if  $gH = lH$  and  $gK = lK$ , then  $g^{-1}l \in H \cap K$ , which implies  $g(H \cap K) = l(H \cap K)$ , as desired. From the injectivity of map 3.2, we have shown index of  $H \cap K$  indeed have upper bound  $mn$ .

Because

$$|G : H \cap K| = |G : H| \cdot |H : H \cap K| = |G : K| \cdot |K : H \cap K|$$

we know both  $n$  and  $m$  divides  $|G : H \cap K|$ , which gives the desired lower bound  $\text{lcm}(m, n)$ . ■

## Question 8

(i) Let  $G$  be a group,  $H \leq G$ , and  $x \in G$  of finite order. Prove that if  $k$  is the smallest natural number that makes  $x^k \in H$ , then  $k \mid o(x)$ .

(ii) Let  $G$  be a group and  $N$  a normal subgroup of  $G$ . Prove that

$$o(gN) = \inf \{k \in \mathbb{N} : g^k \in N\}, \quad \text{where } \inf \emptyset = \infty$$

(iii) Let  $G$  be a finite group,  $H, N$  two subgroups of  $G$  with  $N$  normal. Show that if  $o(H)$  and  $|G : N|$  are coprime, then  $H \leq N$ .

*Proof.* (i): Let  $a = qk + r \in \mathbb{N}$  with  $0 \leq r < k$ . If  $x^a \in H$ , then  $x^r = x^a \cdot (x^k)^{-q} \in H$ , which implies  $r = 0$ . We have shown that  $k$  divides all natural numbers  $a$  that makes  $x^a \in H$ , which includes  $o(x)$ .

(ii): This is a simple observation that  $(gN)^k = g^k N \in N \iff g^k \in N$ .

(iii): By second isomorphism theorem, we know  $|HN : N| = |H : H \cap N|$  which divides both  $o(H)$  and  $|G : N|$ . This by coprimality implies  $|H : H \cap N| = 1$ , which shows that  $H \leq N$ . ■

## Question 9

Let  $G$  be a finite group with Sylow  $p$ -subgroup  $P$  and normal subgroup  $N$ . Show that  $P \cap N$  forms a Sylow  $p$ -subgroup of  $N$ , and use such to deduce  $N$  have index  $p^{\nu_p(o(PN)) - \nu_p(o(N))}$  in  $PN$ .

*Proof.* By second isomorphism theorem, we have

$$o(PN) \cdot o(P \cap N) = o(P) \cdot o(N)$$

Because  $P$  is Sylow with  $P \subseteq PN$ , we know

$$\nu_p(o(PN)) = \nu_p(o(P))$$

This shows that, indeed,  $P \cap N$  forms a Sylow  $p$ -subgroup of  $N$ :

$$\nu_p(o(P \cap N)) = \nu_p(o(N))$$

as desired. Because  $P \cap N \leq P$  and because  $P$  is Sylow, we know  $o(P \cap N)$  is a power of  $p$ . It then follows that:

$$|PN : N| = \frac{o(PN)}{o(N)} = \frac{o(P)}{o(P \cap N)} = p^{\nu_p(o(P)) - \nu_p(o(P \cap N))} = p^{\nu_p(o(PN)) - \nu_p(o(P))}$$

## Question 10

Prove that if  $H$  is a Hall subgroup of  $G$  and  $N \trianglelefteq G$ , then  $H \cap N$  is a Hall subgroup of  $N$  and  $HN/N$  is a Hall subgroup of  $G/N$ .

*Proof.* The facts that:

- (i) By second isomorphism theorem, we have  $|N : H \cap N| = |HN : H|$ , which divides  $|G : H|$ .
- (ii)  $o(H \cap N) \mid o(H)$ .
- (iii)  $o(H)$  and  $|G : H|$  are coprime.

implies  $o(H \cap N)$  and  $|N : H \cap N|$  are coprime, i.e.,  $H \cap N$  is Hall in  $N$ .

The facts that:

- (i)  $o(HN/N) = \frac{o(HN)}{o(N)} = \frac{o(H)}{o(H \cap N)}$  divides  $o(H)$ . (second isomorphism theorem)
- (ii)  $|(G/N) : (HN/N)| = |G : HN|$  divides  $|G : H|$ .
- (iii)  $o(H)$  and  $|G : H|$  are coprime.

implies  $o(HN/N)$  and  $|(G/N) : (HN/N)|$  are coprime, i.e.,  $HN/N$  is Hall in  $G/N$ .

## 3.2 Exercises II

### Question 11: subgroup of $p$ -group of index $p$ is normal

Prove that if  $p$  is a prime and  $o(G) = p^\alpha$  with  $\alpha \in \mathbb{N}$ , then every subgroup  $H$  of index  $p$  is normal.

Deduce that every group of order  $p^2$  has a normal subgroup of order  $p$ .

*Proof.* Let  $G$  acts on the left cosets spaces  $\Omega$  of  $H$ . We have a group homomorphism  $\varphi : G \rightarrow \text{Sym}(\Omega)$ . Clearly we have  $\ker \varphi \subseteq H$ . By first isomorphism theorem, we know

$$|G : \ker \varphi| = o(\text{Im } \varphi) \mid \text{Sym}(\Omega)$$

Noting that  $|\text{Sym } \Omega| = p!$ , we see  $\ker \varphi$  has index  $\leq p$ , which when combined with the fact  $\ker \varphi \subseteq H$  shows that  $H = \ker \varphi$ , as desired. ■

Suppose  $\alpha = 2$ . By first Sylow theorem, there is a subgroup of  $G$  of order  $p$ . This subgroup is normal from what we have just proved. ■

### Question 12

Let  $G$  be a group of odd order. Prove that for any  $x \neq e \in G$ , we have  $\text{Cl}(x) \neq \text{Cl}(x^{-1})$ .

*Proof.* Assume for a contradiction that  $\text{Cl}(x) = \text{Cl}(x^{-1})$ . Because  $(gxg^{-1})^{-1} = gx^{-1}g^{-1} \in \text{Cl}(x^{-1}) = \text{Cl}(x)$ , the inversion is well defined on  $\text{Cl}(x)$ , and moreover clearly bijective. Because  $o(G)$  is odd, we may pair up the elements of  $\text{Cl}(x)$  via inversion to see  $|\text{Cl}(x)|$  is even. This is impossible since by **orbit-stabilizer theorem**,  $|\text{Cl}(x)|$  is the index of some subgroup of  $G$ . ■

### Question 13

Let  $o(G) = p^n$  with  $n \geq 3$  and  $o(Z(G)) = p$ . Prove that  $G$  has a conjugacy class of size  $p$ .

*Proof.* Class equation stated that

$$o(G) = o(Z(G)) + \sum |\text{Cl}(x)| \quad (3.3)$$

and the **orbit stabilizer theorem** shows that  $|\text{Cl}(x)|$  is of order powers of  $p$ . If they are of  $p$ -powers  $\geq 2$ , then we see

$$0 \equiv o(G) \equiv p \equiv o(Z(G)) + \sum |\text{Cl}(x)| \pmod{p}$$

a contradiction. ■

### Question 14

Prove that if the center of  $G$  is of index  $n$ , then every conjugacy class has at most  $n$  elements.

*Proof.* Let  $x \in G$ . Because  $Z(G) \subseteq C_G(x)$ , by **orbit-stabilizer theorem**, we have:

$$|\text{Cl}(a)| = |G : C_G(a)| \leq |G : Z(G)| = n$$

### Question 15

Let  $H, K \subseteq G$  be two finite subgroups. Show that

$$|HK| = \frac{o(H)o(K)}{o(H \cap K)}$$

**Remark:** The hint give a rigorous proof, but I prefer a heuristic one.

*Proof.* Consider the right coset spaces  $\Omega \triangleq \{Hx : x \in G\}$ , and let  $K$  acts on  $\Omega$  by right multiplication. Because  $Hk = H$  if and only if  $k \in H$ , we know the stabilizer subgroup  $K_H$  is identical to  $K \cap H$ . Therefore, by **orbit-stabilizer theorem**, we have

$$\frac{o(K)}{o(H \cap K)} = |\{Hk : k \in K\}|$$

Define an equivalence class in  $K$  by setting  $k \sim \tilde{k} \iff Hk = H\tilde{k}$ . Pick a representative element our of each class and collect them into a set  $T$ . Clearly

$$|T| = |\{Hk : k \in K\}|$$

and we have a natural bijection  $H \times T \rightarrow HK$ . This finishes the proof. ■

### Question 16

Let  $G$  be a non-abelian group of order 21. Prove that  $Z(G) = 1$ .

*Proof.* If  $o(Z(G)) = 3$  or  $7$ , then because  $G/Z(G)$  is cylic ■

### Question 17

Find all finite groups which have exactly two conjugacy classes.

*Proof.* Let  $G$  be a finite group that has exactly two conjugacy classes. One of the conjugacy class is  $\{e\}$ . Let  $a$  be an element of the other class. By class equation and **orbit-stabilizer theorem**, we have

$$|G| - 1 = |\text{Cl}(a)| \mid o(G)$$

This implies  $|G| = 2$ , which implies  $G = \mathbb{Z}_2$ . ■

### Question 18

Let  $H$  be a subgroup of  $G$  and let

$$\bigcup_{g \in G} gHg^{-1} = G$$

Show that  $H = G$ .

### 3.3 Exercises III

#### Question 19

Let  $o(G) = 60$ . Show that if  $G$  is simple, then  $G$  must have exactly 24 elements of order 5 and 20 elements of order 3.

*Proof.* By **sylow**, we have

$$n_5 \equiv 1 \pmod{5} \quad \text{and} \quad n_5 \mid 12$$

which by simplicity of  $G$  implies  $n_5 = 6$ . The same argument gives us  $n_3 \in \{4, 10\}$ . To see  $n_3 \neq 4$ , just recall that **second sylow** stated that conjugacy action  $G \rightarrow \text{Sym}(\text{Syl}_3(G)) \cong S_{n_3}$  is nontrivial, and is therefore injective by simplicity of  $G$ . We now see that  $n_3 = 4$  is too small to satisfies

$$o(G) = 60 \mid n_3!$$

■

#### Question 20

Let  $o(G) = pqr$  with  $p < q < r$  prime. Prove that  $G$  has a normal Sylow  $r$ -subgroup  $H$ .

*Proof.* By sylow and counting arguments we know  $1 \in \{n_p, n_q, n_r\}$ . Therefore, if neither of  $n_p$  and  $n_q$  is 1, we are done. Suppose  $1 \in \{n_p, n_q\}$ . Either way, we get a normal subgroup  $N$  such that  $o(G/N) \in \{qr, pr\}$ . We also get a normal  $H/N \in \text{Syl}_r(G/N)$ . This give us a characteristic  $K \in \text{Syl}_r(H)$ , which is normal in  $G$ . ■

#### Question 21

Let  $o(G) = p^3q$  with  $p, q$  prime. Show that one of the followings statement is true:

- (i)  $G$  has a normal Sylow  $p$ -subgroup.
- (ii)  $G$  has a normal Sylow  $q$ -subgroup.
- (iii)  $p = 2, q = 3$ .

*Proof.* Suppose (i) and (ii) are both false. Then by **sylow** we have  $n_p = q$  and  $p < q$ . Because  $p < q$ , applying **sylow** again we have  $n_q \in \{p^2, p^3\}$ . Because  $n_p > 1$ , by counting we see that  $n_q \neq p^3$ . Therefore  $n_q = p^2$ . Then by **sylow**,  $p^2 = n_q \equiv 1 \pmod{q}$ , which implies  $q \mid (p-1)(p+1)$ . Because  $p < q$  and  $q$  is prime, we now see  $q = p+1$ , which can only happens if  $p = 2$  and  $q = 3$ . ■

## Question 22

Show that no group of order 30 is simple.

*Proof.* Consider  $n_3$  and  $n_5$ . We have  $n_5 \in \{1, 6\}$  and  $n_3 \in \{1, 10\}$ . If  $n_5 \neq 1 \neq n_3$ , then there are 24 elements of order 5 and 20 elements of order 3, impossible for a group of order 30. ■

## Question 23

Let  $G$  be a finite group with sylow  $p$ -subgroup  $P$  and normal subgroup  $N$ . Show that  $P \cap N$  is  $p$ -sylow in  $N$  and that  $PN/N$  is  $p$ -sylow in  $G/N$ .

*Proof.* Second isomorphism theorem implies that

$$o(PN) \cdot o(P \cap N) = o(P) \cdot o(N)$$

Because  $P$  is  $p$ -sylow, we know  $o(P)$  and  $o(PN)$  has the same  $p$ -power, which implies that  $o(P \cap N)$  and  $o(N)$  has the same  $p$ -power, as desired. Again counting the  $p$ -power of  $PN/N \subseteq G/N$ , we see  $PN/N$  is  $p$ -sylow. ■

## Question 24

Let  $G$  be a finite group,  $H \leq G$  a subgroup with  $[G : H] = n$ . Show that:

- (i) For all subgroup  $K \leq G$ , we have  $[H : H \cap K] \leq [G : K]$ .
- (ii)  $[H : H \cap H^g] \leq n$  for all  $g \in G$ .
- (iii) If  $H$  is a maximal proper subgroup of  $G$  and  $H$  is abelian, show that  $H \cap H^g \trianglelefteq G$  for all  $g \notin H$ .
- (iv) Suppose that  $G$  is simple. If  $H$  is abelian and  $n$  is prime, then  $H = 1$ .

*Proof.* Let  $H/H \cap K$  and  $G/K$  denote left coset spaces. (i) is a consequence of verifying that the function

$$H/H \cap K \longrightarrow G/K; \quad h(H \cap K) \mapsto hK$$

is well-defined and injective. (ii) is then a corollary of (i).

We now prove (iii). Fix  $g \notin H$ . There are two cases: Either  $H = H^g$  or  $H \neq H^g$ . For the first case, just observe that by maximality of  $H$ , we will have  $N_G(H) = G$ . We now claim that  $H \neq H^g \implies H \cap H^g \subseteq Z(G)$ . Because  $H$  is abelian, we know  $H \cap H^g \leq Z(H)$ . Clearly we also have  $H \cap H^g \leq Z(H^g)$ . We now have  $H \cap H^g \leq Z(\langle H, H^g \rangle)$ , where

$\langle H, H^g \rangle = G$  by maximality of  $H$ , as desired.

We now prove (iv). Clearly the primality of  $n$  forces  $H$  to be a maximal proper subgroup of  $G$ . Therefore by (iii),  $H \cap H^g = 1$  for all  $g \notin H$ . This by (ii) implies  $n \leq o(G) \leq n^2$ . Write  $o(G) \triangleq nk$  so  $k \in \{1, \dots, n\}$ . We wish to show  $k = 1$ . To see  $k \neq n$ , just recall that if so, then  $G$  would be abelian, contradicting to its simplicity. To see  $k \notin \{2, \dots, n-1\}$ , just observe that if so, then the unique  $n$ -Sylow subgroup would be proper, contradicting to simplicity of  $G$ .  $\blacksquare$

### Question 25

Let  $G$  be a finite group with  $P \in \text{Syl}_p(G)$ . Suppose that  $N$  is a normal subgroup of  $G$  with  $[G : N] = o(P) > 1$ . Show that

- (i)  $N$  is the subset of  $G$  consisting of all elements of order not divisible by  $p$ .
- (ii) If the elements of  $G - N$  all has  $p$ -power order, then  $P = N_G(P)$ .

*Proof.* Because  $P$  is  $p$ -sylow and  $[G : N] = o(P)$ , we know  $p \nmid o(N)$ . This implies that no element of  $N$  has order divisible by  $p$ . Let  $g \in G$  with  $p \nmid o(g)$ . To see that  $g \in N$ , just observe that because  $o(gN) \mid o(g)$  and  $o(gN)$  is a power of  $p$ , we have  $o(gN) = 1$ .

Assume for a contradiction that  $P < N_G(P)$ . Then there exists some nontrivial sylow  $q$ -subgroup  $Q$  of  $N_G(P)$  with  $q \neq p$ . By definition we have  $[Q, P] \leq P$ . By (i),  $Q \leq N$ . Therefore we also have  $[Q, P] \leq N$ . Coprimality of orders of  $N$  and  $P$  now tell us that  $[Q, P] = 1$ . We now see that the product of two nontrivial elements  $x \in Q, y \in P$  has order divisible by  $pq$ , a contradiction to the premise.  $\blacksquare$

## 3.4 Exercises IV

### Question 26

Show that the center of products is a product of centers:

$$Z(G_1) \times \cdots \times Z(G_n) = Z(G_1 \times \cdots \times G_n)$$

Deduce that a direct product of groups is abelian if and only if each of its factor is abelian.

*Proof.* The " $\subseteq$ " is clear. To see that

$$g_1 \times \cdots \times g_n \in Z(G_1 \times \cdots \times G_n) \implies g_i \in Z(G_i)$$

just observe that if not, then

$$[g_1 \times \cdots \times g_n, e_1 \times \cdots \times e_i \times \cdots \times e_n] \neq e \in \prod G_j$$

The second part then follows from noting

$$Z(G_1 \times \cdots \times G_n) = G_1 \times \cdots \times G_n \iff Z(G_i) = G_i, \quad \text{for all } i$$

■

### Question 27

Let  $G \triangleq A_1 \times \cdots \times A_n$  and  $B_i \trianglelefteq A_i$  for all  $i$ . Prove that  $B_1 \times \cdots \times B_n \trianglelefteq G$  and that

$$\frac{A_1 \times \cdots \times A_n}{B_1 \times \cdots \times B_n} = \frac{A_1}{B_1} \times \cdots \times \frac{A_n}{B_n}$$

*Proof.*

$$(g_1, \dots, g_n)(b_1, \dots, b_n)(g_1, \dots, g_n)^{-1} = (g_1 b_1 g_1^{-1}, \dots, g_n b_n g_n^{-1}) \in \prod B_i$$

The second part require us to show that

$$\prod \left( \frac{A_i}{B_i} \right) \longrightarrow \frac{\prod A_i}{\prod B_i}; \quad \prod \left( \frac{a_i}{B_i} \right) \mapsto \frac{\prod a_i}{\prod B_i}$$

is a well-defined group isomorphism, which boils down to showing that it is (i) well-defined, (ii) actually a homomorphism, (iii) injective, and (iv) surjective. To see it is injective, just observe that if  $\prod a_i \in \prod B_i$ , then  $a_i \in B_i$  for all  $i$ , and therefore  $\prod \frac{a_i}{B_i} = e$ . The rest are clear.

### Question 28

Let  $G$  be a finite abelian group with  $m \mid o(G)$ . Show that  $G$  has a subgroup of order  $m$ .

*Proof.* This follows from noting that if  $o(a) = p^n$ , then  $o(a^{p^{n-d}}) = p^d$ . (Ans also structure theorem for finite abelian group) ■

### Question 29

Show that the subgroups and quotients of a nilpotent group  $G$  are also nilpotent.

*Proof.* Let  $H$  be a subgroup of  $G$ , and write

$$0 = G_{(n)} \trianglelefteq \cdots \trianglelefteq G_{(1)} \trianglelefteq G_{(0)} = G, \quad \text{with } G_{(k)} \triangleq [G, G_{(k-1)}]$$

To see that

$$0 \leq H_n \leq \cdots \leq H_1 \leq H$$

form a central series, where  $H_k \triangleq H \cap G_{(k)}$ , just observe that

$$[H, H \cap G_{(k)}] \leq H \text{ and } [H, H \cap G_{(k)}] \leq [G, G_{(k)}] \leq G_{(k-1)}$$

together implies

$$[H, H_k] \leq H \cap G_{(k-1)} = H_{k-1}$$

Let  $N$  be a normal subgroup of  $G$ , and let  $m \leq n$  be the largest number such that  $N \leq G_{(m)}$ . It is clear that

$$\frac{N}{N} \leq \frac{G_{(m)}}{N} \leq \cdots \leq \frac{G_{(1)}}{N} \leq \frac{G}{N}$$

form a central series. ■

### Question 30

Show that if  $G/Z(G)$  is nilpotent, then  $G$  is nilpotent.

*Proof.* Consider the central series

$$\frac{Z(G)}{Z(G)} \trianglelefteq \frac{G_1}{Z(G)} \trianglelefteq \cdots \trianglelefteq \frac{G_n}{Z(G)} = \frac{G}{Z(G)}$$

Clearly we have the central series

$$0 \trianglelefteq Z(G) \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$$

### Question 31

Let  $o(G) = pqr$  with  $p < q < r$  prime. Show that  $G$  is solvable.

*Proof.* Recall that we have a normal subgroup  $M \in \text{Syl}_r(G)$ . Then we have a normal subgroup  $\frac{H}{M} \in \text{Syl}_q(G)$ . Then  $1 \trianglelefteq M \trianglelefteq H \trianglelefteq G$  forms the desired series. ■

### Question 32

Show that a finite group  $G$  is nilpotent if and only if every  $a, b \in G$  that makes  $\gcd(o(a), o(b)) = 1$  also makes  $ab = ba$ .

*Proof.* ( $\implies$ ): Write  $G = P_1 \times \cdots \times P_n$  with  $P_i$  sylow. Clearly if the orders of  $(x_1, \dots, x_n)$  and  $(y_1, \dots, y_n)$  are coprime to each other, then for all  $i$ , we must have either  $x_i = e$  or  $y_i = e$ . This implies the commutativity.

( $\impliedby$ ): We need to show that Sylow subgroups of  $G$  are normal. Let  $P_1, \dots, P_n$  each be a Sylow subgroup of  $G$  with distinct  $p$ . By premise, we see that  $P_k \subseteq N_G(P_1)$  for all  $k \geq 2$ . This then implies  $G = N_G(P_1)$ , as desired. ■

### Question 33

Let  $G = HK$  be finite and  $S \leq G$  be a  $p$ -subgroup that contains some  $p$ -Sylow subgroup  $P$  of  $H$  and some  $p$ -Sylow subgroup  $Q$  of  $K$ . Show that

- (i)  $S$  is  $p$ -Sylow in  $G$ .
- (ii)  $S = (S \cap H)(S \cap K)$

*Proof.* Because  $P \cap Q \leq H \cap K$ , we know  $p$ -part of

$$o(G) = \frac{o(H)o(K)}{o(H \cap K)}$$

is smaller than

$$\frac{o(P)o(Q)}{o(P \cap Q)} = |PQ| \leq o(S)$$

which can only happen if  $S$  is Sylow with  $|PQ| = o(S)$ . By definition,  $P \leq S \cap H \leq H$ . Because  $S$  is a  $p$ -group, we know  $S \cap H$  is also a  $p$ -group. Sylowness of  $P \leq H$  then forces  $S \cap H = P$ . Similarly, we have  $S \cap K = Q$ . Now, to see  $S = PQ$ , just recall that  $|PQ| = o(S)$  ■

### Question 34

Let  $M \trianglelefteq G$  and  $N \trianglelefteq G$  with  $M, N$  finite and nilpotent. Prove that  $MN$  is nilpotent.

*Proof.* The proof follows from noting that if  $S \in \text{Syl}_p(MN)$ , then by earlier questions,  $S$  is uniquely determined by  $S = (M \cap S)(N \cap S)$  with  $M \cap S \in \text{Syl}_p(M)$  and  $N \cap S \in \text{Syl}_p(N)$  uniquely determined. ■

### Question 35

Let  $G$  be finite with  $A, B \trianglelefteq G$  and  $G/A, G/B$  solvable. Prove that  $G/(A \cap B)$  is solvable.

*Proof.*