HWs

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Contents

Chapter 1

PDE LAST

1.1 Cheat Sheet

Consider the **Dirichlet** eigenproblem

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(l) = 0 \end{cases}$$

It is clear that 0 is NOT an eigenvalue. Suppose $\lambda = \beta^2 \in \mathbb{C}$ is an eigenvalue. We see that X must take the form

$$X = A\cos(\beta x) + B\sin(\beta x)$$

The initial condition implies

$$\begin{bmatrix} 1 & 0 \\ \cos(\beta l) & \sin(\beta l) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

This implies that

$$\beta = \frac{n\pi}{l} \text{ and } X = \sin(\frac{n\pi x}{l}) \text{ for } n \geq 1$$

Consider the **Neumann** eigenproblem

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(l) = 0 \end{cases}$$

It is clear that 0 is an eigenvalue with eigenspace spanned by constant function. Suppose

$\lambda = \beta^2 \in \mathbb{C}$ is an eigenvalue. We see that X must take the form

$$X = A\cos(\beta x) + B\sin(\beta x)$$

The initial condition then implies

$$\begin{bmatrix} 0 & \beta \\ -\beta\sin(\beta l) & \beta\cos(\beta l) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

This implies that

$$\beta = \frac{n\pi}{l} \text{ and } X = \cos(\frac{n\pi x}{l}) \text{ for } n \geq 1$$

Consider the **mixed** eigenproblem

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X'(l) = 0 \end{cases}$$

It is clear that 0 is Not an eigenvalue. Suppose $\lambda = \beta^2 \in \mathbb{C}$ is an eigenvalue. We see that X must take the form

$$X = A\cos(\beta x) + B\sin(\beta x)$$

The initial condition implies

$$\begin{bmatrix} 1 & 0 \\ -\beta\sin(\beta l) & \beta\cos(\beta l) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

This implies

$$\beta = \frac{(n+\frac{1}{2})\pi}{l} \text{ and } X_n = \sin(\frac{(n+\frac{1}{2})\pi x}{l}) \text{ for } n \geq 1$$

The **Fourier sine series of φ on** $(0, l)$ is

$$\varphi(x) = \sum_{n=0}^{\infty} A_n \sin(\frac{n\pi x}{l})$$

The **Fourier cosine series of φ on** $(0, l)$ is

$$\varphi(x) = \sum_{n=0}^{\infty} A_n \cos(\frac{n\pi x}{l})$$

The **Fourier full series of φ on** $(-l, l)$ us

$$\varphi(x) = \sum_{n=0}^{\infty} A_n \cos(\frac{n\pi x}{l}) + B_n \sin(\frac{n\pi x}{l})$$

The **Fourier complex series of φ on** $(-l, l)$ **is**

$$\varphi(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{-in\pi x}{l}}$$

Let $n, m \neq 0$. To compute the sine series, we have

$$\int_0^l \sin(\frac{n\pi x}{l}) \sin(\frac{m\pi x}{l}) dx = \begin{cases} \frac{l}{2} & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Let $n, m \neq 0$. To compute the cosine series, we have

$$\int_0^l \cos(\frac{n\pi x}{l}) \cos(\frac{m\pi x}{l}) dx = \begin{cases} \frac{l}{2} & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Let $n, m \neq 0$. To compute the full series, we have

$$\int_{-l}^l \sin(\frac{n\pi x}{l}) \sin(\frac{m\pi x}{l}) dx = \begin{cases} l & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

and

$$\int_{-l}^l \cos(\frac{n\pi x}{l}) \cos(\frac{m\pi x}{l}) dx = \begin{cases} l & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Let $n, m \in \mathbb{Z}$. To compute the complex series, we have

$$\int_{-l}^l e^{\frac{i(n-m)\pi x}{l}} dx = \begin{cases} 2l & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Let $n, m \geq 0$. To compute the weird series, we have

$$\int_0^l \sin(\frac{(n+\frac{1}{2})\pi x}{l}) \sin(\frac{(m+\frac{1}{2})\pi x}{l}) dx = \begin{cases} \frac{l}{2} & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

and

$$\int_0^l \cos(\frac{(n+\frac{1}{2})\pi x}{l}) \cos(\frac{(m+\frac{1}{2})\pi x}{l}) dx = \begin{cases} \frac{l}{2} & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

1.2 5.2 Even, Odd, Periodic and Complex Functions

Question 1

Show that $\cos x + \cos(\alpha x)$ is periodic if α is a rational number. What is its period? Show that $\cos x + \cos(\sqrt{2}x)$ is not periodic.

Proof. If $\alpha \in \mathbb{Q}$, we may write

$$\alpha = \frac{q}{p} \text{ where } p \in \mathbb{N}$$

we then see $\cos x + \cos(\alpha x)$ has period $2p\pi$. Assume $\cos x + \cos(\sqrt{2}x)$ have period r. Then

$$2 = \cos(r) + \cos(\sqrt{2}r)$$

which is easily seen impossible by noting for this to be true we must have $r = 2n\pi$. ■

Question 2

Prove

(a) If φ is an odd function, its full Fourier series on $(-l, l)$ has only sine terms.

(b) Also, if φ is an even function, its full Fourier series on $(-l, l)$ has only cosine terms. (Hint: Don't use the series directly. Use the formulas for the coefficients to show that every second coefficient vanishes.)

Proof. Write

$$\varphi = \sum_{n=0}^{\infty} A_n \cos(\frac{n\pi x}{l}) + B_n \sin(\frac{n\pi x}{l})$$

Compute

$$A_n l = \int_{-l}^{l} \varphi(x) \cos(\frac{n\pi x}{l}) dx = 0 \text{ for all } n \geq 1$$

if φ is odd. Compute

$$B_n l = \int_{-l}^{l} \varphi(x) \sin(\frac{n\pi x}{l}) dx = 0 \text{ for all } n \geq 1$$

if φ is even. ■

Question 3

Let $\varphi(x)$ be a function of period π. If $\varphi(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$ for all x, find the odd coefficients.

Proof. Because φ has period π, we have

$$\sum_{n=1}^{\infty} a_n \sin(nx) = \varphi(x) = \varphi(x+\pi) = \sum_{n=1}^{\infty} a_n \sin(nx + n\pi)$$
$$= \sum_{n=1}^{\infty} (-1)^n a_n \sin(nx)$$

For this to be true, we must have $a_{\text{odd}} = 0$. ∎

Question 4

Find the full Fourier series of e^x on $(-l, l)$ in its real and complex forms. (Hint: It is convenient to find the complex form first)

Proof. Write

$$e^x = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$$

Compute

$$\begin{aligned} c_n &= \frac{1}{2l} \int_{-l}^{l} e^x e^{\frac{-in\pi x}{l}} dx \\ &= \frac{1}{2l} \int_{-l}^{l} e^{\frac{(l-in\pi)x}{l}} dx \\ &= \frac{1}{2l} \cdot \frac{le^{\frac{(l-in\pi)x}{l}}}{l - in\pi} \Bigg|_{x=-l}^{l} \\ &= \frac{l(e^{l-in\pi} - e^{-(l-in\pi)})}{2l(l-in\pi)} \\ &= \frac{(-1)^n (e^l - e^{-l})}{2(l-in\pi)} = \frac{(-1)^n}{(l-in\pi)} \sinh(l) \end{aligned}$$

We now have

$$
\begin{aligned}
e^x &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(l-in\pi)} \sinh(l) e^{\frac{in\pi x}{l}} \\
&= \frac{\sinh(l)}{l} + \sum_{n=1}^{\infty} \Big[\frac{(-1)^n \sinh(l)}{(l-in\pi)} e^{\frac{in\pi x}{l}} + \frac{(-1)^n \sinh(l)}{(l+in\pi)} e^{\frac{-in\pi x}{l}} \Big] \\
&= \frac{\sinh(l)}{l} + \sum_{n=1}^{\infty} (-1)^n \sinh(l) \Big[\frac{\cos(\frac{n\pi x}{l}) + i\sin(\frac{n\pi x}{l})}{l-in\pi} + \frac{\cos(\frac{n\pi x}{l}) - i\sin(\frac{n\pi x}{l})}{l+in\pi} \Big] \\
&= \frac{\sinh(l)}{l} \\
&+ \sum_{n=1}^{\infty} (-1)^n \sinh(l) \cdot \frac{(l+in\pi)\big(\cos(\frac{n\pi x}{l}) + i\sin(\frac{n\pi x}{l})\big) + (l-in\pi)\big(\cos(\frac{n\pi x}{l}) - i\sin(\frac{n\pi x}{l})\big)}{l^2+n^2\pi^2} \\
&= \frac{\sinh(l)}{l} + \sum_{n=1}^{\infty} \frac{(-1)^n \sinh(l) \big[2l\cos(\frac{n\pi x}{l}) - 2n\pi \sin(\frac{n\pi x}{l})\big]}{l^2+n^2\pi^2} \\
&= \sum_{n=0}^{\infty} \frac{2(-1)^n \sinh(l)}{l^2+n^2\pi^2} \Big[l\cos(\frac{n\pi x}{l}) - n\pi \sin(\frac{n\pi x}{l}) \Big]
\end{aligned}
$$

■

Question 5

Repeat the last exercise for $\cosh(x)$

Proof. Recall from last exercise that

$$e^x = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sinh(l)}{l-in\pi} e^{\frac{in\pi x}{l}}$$

This give us

$$\cosh(x) = \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}\Big(\sum_{n=-\infty}^{\infty} \frac{(-1)^n \sinh(l)}{l-in\pi} e^{\frac{in\pi x}{l}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sinh(l)}{l-in\pi} e^{\frac{-in\pi x}{l}} \Big)$$

■

Question 6

Repeat the last exercise for $\sin x$. Assume that l is not an integer multiple of π. (Hint: First find the series for e^{ix})

Proof. ■

1.3 5.3 Orthogonality and General Fourier Series

In this section, we are concerned with the eigenproblem

$$X'' + \lambda X = 0 \text{ on } (a,b)$$

with a pair of boundary conditions

$$\begin{cases} \alpha_1 X(a) + \beta_1 X(b) + \gamma_1 X'(a) + \delta_1 X'(b) = 0 \\ \alpha_2 X(a) + \beta_2 X(b) + \gamma_2 X'(a) + \delta_2 X'(b) = 0 \end{cases}$$

where the coefficients α, β, γ and δ are all real. A pair of boundary condition is called **symmetric** if every pair of function $f, g : (a,b) \to \mathbb{C}$ that satisfies the boundary conditions also satisfies

$$f'(x)\overline{g(x)} - f(x)\overline{g'(x)}\Big|_{x=a}^{b} = 0$$

Immediately, we see that if the boundary condition is symmetric, two eigenfunction X_1, X_2 with distinct eigenvalues are either orthogonal or have conjugate eigenvalues since

$$\begin{aligned}(\lambda_1 - \overline{\lambda_2}) \int_a^b X_1 \overline{X_2} dx &= \int_a^b (-X_1'' \overline{X_2} + X_1 \overline{X_2''}) dx \\ &= (-X_1' \overline{X_2} + X_1 \overline{X_2'})\Big|_{x=a}^{b} = 0\end{aligned}$$

The second equality is often referred to as **Green's second identity**.

$$(\lambda_1 - \lambda_2) \int XY = \int X''Y - XY'' = X'Y - XY'\Big|_a^b = 0$$

Question 7

Consider $u_{tt} = c^2 u_{xx}$ for $0 < x < l$, with the boundary conditions $u(0,t) = u_x(l,t) = 0$, and the initial condition $u(x,0) = x, u_t(x,0) = 0$. Find the solution explicitly in series form.

Proof. The eigenproblems are

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X'(l) = 0 \end{cases} \quad \text{and} \quad \begin{cases} T'' + c^2 \lambda T = 0 \end{cases}$$

Direct computation shows that $\lambda \neq 0$. If $\lambda = \beta^2$, and $X = A\cos(\beta x) + B\sin(\beta x)$, then BCs give

$$\begin{bmatrix} 1 & 0 \\ -\beta\sin(\beta l) & \beta\cos(\beta l) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

This implies

$$\beta = \frac{(n+\frac{1}{2})\pi}{l} \text{ and } X_n = \sin(\frac{(n+\frac{1}{2})\pi x}{l})$$

and $T_n = A_n\cos(c\beta t) + B_n\sin(c\beta t)$. We now may write

$$u = \sum_{n=0}^{\infty}\Big(A_n\cos(\frac{(n+\frac{1}{2})c\pi t}{l}) + B_n\sin(\frac{(n+\frac{1}{2})c\pi t}{l})\Big)\sin(\frac{(n+\frac{1}{2})\pi x}{l})$$

Compute

$$u_t = \sum_{n=0}^{\infty}\frac{(n+\frac{1}{2})c\pi}{l}\Big(-A_n\sin(\frac{(n+\frac{1}{2})c\pi t}{l}) + B_n\cos(\frac{(n+\frac{1}{2})c\pi t}{l})\Big)\sin(\frac{(n+\frac{1}{2})\pi x}{l})$$

Plugin the ICs, we now have

$$x = \sum_{n=0}^{\infty} A_n\sin(\frac{(n+\frac{1}{2})\pi x}{l})$$

and

$$0 = \sum_{n=0}^{\infty}\frac{(n+\frac{1}{2})c\pi}{l}B_n\sin(\frac{(n+\frac{1}{2})\pi x}{l})$$

Because

$$\int_0^l \sin(\frac{(n+\frac{1}{2})\pi x}{l})\sin(\frac{(m+\frac{1}{2})\pi x}{l})dx = \begin{cases} \frac{l}{2} & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

for $n, m \geq 0$, we have

$$\frac{(-1)^n l^2}{(n+\frac{1}{2})^2\pi^2} = \int_0^l x\sin(\frac{(n+\frac{1}{2})\pi x}{l})dx = \frac{lA_n}{2}$$

Thus,

$$u = \sum_{n=0}^{\infty}\frac{2(-1)^n l}{(n+\frac{1}{2})^2\pi^2}\cos(\frac{(n+\frac{1}{2})c\pi t}{l})\sin(\frac{(n+\frac{1}{2})\pi x}{l})$$

$\blacksquare$

Question 8

Consider the problem $u_t = ku_{xx}$ for $0 < x < l$, with the boundary conditions $u(0,t) = U, u_x(l,t) = 0$, and the initial condition $u(x,0) = 0$ where U is a constant.

(a) Find the solution in series form. (Hint: Consider $u(x,t) - U$)

(b) Using a direct argument, show that the series converges for $t > 0$.

(c) If ϵ is a given margin of error, estimate how long a time is required for the value $u(l,t)$ at the endpoint to be approximated by the constant U within the error ϵ. (Hint: It is an alternating series with first U, so that the error is less than the next term)

Proof. Let $v \triangleq u - U$. We have

$$\begin{cases} v_t = kv_{xx} \text{ for } x \in (0,l) \textbf{ (Homogeneous DE)} \\ v(0,t) = v_x(l,t) = 0 \textbf{ (BC)} \\ v(x,0) = -U \textbf{ (IC)} \end{cases}$$

The eigenproblems are then

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X'(l) = 0 \end{cases} \quad \text{and} \quad \begin{cases} T' + k\lambda T = 0 \end{cases}$$

Thus,

$$v = \sum_{n=1}^{\infty} A_n e^{\frac{-(n+\frac{1}{2})^2\pi^2 kt}{l^2}} \sin\left(\frac{(n+\frac{1}{2})\pi x}{l}\right)$$

The IC becomes,

$$-U = \sum_{n=1}^{\infty} A_n \sin\left(\frac{(n+\frac{1}{2})\pi x}{l}\right)$$

We then have

$$v = \sum_{n=1}^{\infty} \frac{-2U}{(n+\frac{1}{2})\pi} e^{\frac{-(n+\frac{1}{2})^2\pi^2 kt}{l^2}} \sin\left(\frac{(n+\frac{1}{2})\pi x}{l}\right)$$

The series absolutely converge because it is eventually smaller than a geometric series. One may use root test to check this fact. ■

Question 9

Find the complex eigenvalues of the first-derivative operator $\frac{d}{dx}$ subject to the single boundary condition $X(0) = X(1)$. Are the eigenfunctions orthogonal on the interval $(0, 1)$?

Proof. We are solving the eigenproblem

$$\begin{cases} X' + \lambda X = 0 \\ X(0) = X(1) \end{cases}$$

Clearly, the solution X must take the form $X = e^{-\lambda x}$. The boundary conditions then implies

$$e^{-\lambda} = X(1) = X(0) = 1$$

which implies

$$\lambda = 2ni\pi \text{ for } n \in \mathbb{Z}$$

Compute for distinct n, m

$$\langle X_n, X_m \rangle = \int_0^1 e^{-2(n-m)i\pi x} dx = \frac{e^{-2(n-m)i\pi x}}{-2(n-m)i\pi}\Big|_{x=0}^1 = 0$$

So the eigenfunctions are indeed orthogonal. ∎

Question 10

Show by direct integration that the eigenfunctions associated with Robin BSc, namely

$$\varphi_n(x) = \cos(\beta_n x) + \frac{a_0}{\beta_n}\sin(\beta_n x) \text{ where } \lambda_n = \beta_n^2$$

are mutually orthogonal on $0 \leq x \leq l$, where β_n are the positive roots of

$$(\beta^2 - a_0 a_l)\tan(\beta l) = (a_0 + a_l)\beta$$

Proof. ∎

Question 11

Show that the boundary conditions

$$X(b) = \alpha X(a) + \beta X'(a) \text{ and } X'(b) = \gamma X(a) + \delta X'(a)$$

on an interval $a \le x \le b$ are symmetric if and only if $\alpha\delta - \beta\gamma = 1$.

Proof. Compute

$$
\begin{aligned}
(-X_1'X_2 + X_1X_2')\Big|_{x=a}^{b} &= -X_1'(b)X_2(b) + X_1(b)X_2'(b) \\
&\quad + X_1'(a)X_2(a) - X_1(a)X_2'(a) \\
&= -\Big(\gamma X_1(a) + \delta X_1'(a)\Big)\Big(\alpha X_2(a) + \beta X_2'(a)\Big) \\
&\quad + \Big(\alpha X_1(a) + \beta X_1'(a)\Big)\Big(\gamma X_2(a) + \delta X_2'(a)\Big) \\
&\quad + X_1'(a)X_2(a) - X_1(a)X_2'(a) \\
&= (-\delta\alpha + \beta\gamma + 1)\big(X_1'(a)X_2(a) - X_1(a)X_2'(a)\big)
\end{aligned}
$$

■

Question 12

(The Gram-Schmidt orthogonalization procedure) If $X_1, X_2, \ldots$ is any sequence (finite or infinite) of linearly independent vectors in any vector space with an inner, it can be replace by a sequence of linear combinations that are mutually orthogonal. The idea is that at each step one subtracts off the components parallel to the previous vectors. The procedure is as follows. First, we let $Z_1 = \frac{X_1}{\|X_1\|}$. Second, we define

$$Y_2 \triangleq X_2 - (X_2, Z_1)Z_1 \text{ and } Z_2 \triangleq \frac{Y_2}{\|Y_2\|}$$

Third, we define

$$Y_3 \triangleq X_3 - (X_3, Z_2)Z_2 - (X_3, Z_1)Z_1 \text{ and } Z_3 \triangleq \frac{Y_3}{\|Y_3\|}$$

and so on.

(a) Show that all vectors $Z_1, Z_2, Z_3, \ldots$ are orthogonal to each other.

(b) Apply the procedure to the pair of functions $\cos(x) + \cos(2x)$ and $3\cos(x) - 4\cos(2x)$ in the interval $(0, \pi)$ to get an orthogonal pair.

Proof. Compute

$$\begin{aligned}\|\cos(x)+\cos(2x)\|^2 &= \int_0^\pi (\cos(x)+\cos(2x))^2 dx \\ &= \int_0^\pi \frac{\cos(x)+\cos(2x)+\cos(3x)+\cos(4x)+2}{2} dx \\ &= \pi\end{aligned}$$

Compute

$$\begin{aligned}\langle \cos(x)+\cos(2x), 3\cos(x)-4\cos(2x)\rangle &= 3\langle \cos(x),\cos(x)\rangle - 4\langle \cos(2x),\cos(2x)\rangle \\ &= \frac{-\pi}{2}\end{aligned}$$

Thus an orthogonal pair can be

$$\cos(x)+\cos(2x) \text{ and } \frac{7}{2}(\cos(x)-\cos(2x))$$

■

Question 13

(a) Show that the condition $f(x)f'(x)|_a^b \leq 0$ is valid for any function that satisfies the Dirichlet, Neumann, or periodic boundary conditions.

(b) Show that it is also valid for Robin BCs provided that the constants a_0 and a_l are positive.

Proof. Part (a) is clear. Note that the periodic boundary conditions mean

$$f(b) = f(a) \text{ and } f'(b) = f'(a)$$

The Robin boundary conditions means

$$X' - a_0 X = 0 \text{ at } 0 \text{ and } X' + a_l X = 0 \text{ at } l$$

Compute

$$X'(b)X(b) - X'(a)X(a) = -a_l X^2(b) - a_0 X^2(a) \leq 0$$

■

Question 14

Use Green's first identity to prove if the boundary conditions is symmetric and

$$f(x)f'(x)\Big|_{x=a}^{b} \leq 0$$

for any real-valued f satisfying BCs, then there is no negative eigenvalue.

Proof. Note that the symmetry the boundary conditions have nothing to do here. Product rule give us

$$\int_a^b X''X dx = XX'\Big|_{x=a}^{b} - \int_a^b (X')^2 dx$$

The right hand side by definition is non-positive. The proof then follows from noting if X is of negative eigenvalue, then the left hand side is positive. ∎

1.4 5.4 Completeness

Theorem 1.4.1. (Infinite number of eigenvalues) Consider the eigenproblem

$$X'' + \lambda X = 0 \text{ on } (a, b)$$

with a pair of symmetric BCs. Let $f : [a, b] \to \mathbb{R}$ be a real-valued function. We have

(a) Any two eigenfunction that correspond to distinct eigenvalues are orthogonal.

(b) The set of eigenvalues is exactly a real sequence λ_n converging to ∞, and every eigenspace contain at least one real eigenfunction.

> Let $X_n : [a, b] \to \mathbb{R}$ be a sequence of real-valued eigenfunction corresponding to distinct eigenvalues. We say (X_n) is an orthogonal set, since they are indeed orthogonal to each other. For all $n \in \mathbb{N}$, we define the Fourier coefficient A_n by
>
> $$A_n \triangleq \frac{(f, X_n)}{\|X_n\|^2} = \frac{\int_a^b f(x)\overline{X_n(x)}dx}{\int_a^b |X_n(x)|^2 \, dx}$$

(c) If f is C^2 on $[a, b]$ and also satisfies the symmetric BCs, then the Fourier series converge $\sum A_n X_n$ converge to f uniformly on $[a, b]$.

(d) If $f \in L^2(a, b)$, then $\sum A_n X_n \to f$ in L^2

$$\left\| f - \sum_{n=1}^{N} A_n X_n \right\| \to 0 \text{ as } N \to \infty$$

(e) If $f' : [a, b] \to \mathbb{R}$ is piecewise continuous, then the classical Fourier series (full or sine or cosine) converge to f on (a, b).

Theorem 1.4.2. (Least Square Approximation) Fix N.

$$\left\| f - \sum_{n=1}^{N} c_n X_n \right\|$$

as a function in $c_1, \ldots, c_N$ has a minimum, which happen exactly when $c_n = A_n$.

Proof. Compute

$$\begin{aligned}
\left\| f - \sum_{n=1}^{N} c_n X_n \right\|^2 &= \int_a^b \left| f - \sum_{n=1}^{N} c_n X_n \right|^2 dx \\
&= \int_a^b \Big(f - \sum_{n=1}^{N} c_n X_n \Big)^2 dx \\
&= \int_a^b f^2 - 2 \sum_{n=1}^{N} c_n f X_n + \Big(\sum_{n=1}^{N} c_n X_n \Big)^2 dx \\
&= \|f\|^2 - 2c_n(f, X_n) + \sum_{n=1}^{N} c_n^2 \|X_n\|^2 \\
&= \|f\|^2 + \sum_{n=1}^{N} \|X_n\|^2 \left[c_n - \frac{(f, X_n)}{\|X_n\|^2} \right]^2 - \sum_{n=1}^{N} \frac{(f, X_n)^2}{\|X_n\|^2}
\end{aligned}$$

which obviously have a minimum when $c_n = \frac{(f,X_n)}{\|X_n\|^2} = A_n$. ■

Note that the proof above moreover gives the **Bessel's Inequality**

$$\|f\|^2 - \sum_{n=1}^{N} A_n^2 \|X_n\|^2 = \|f\|^2 - \sum_{n=1}^{N} \frac{(f, X_n)^2}{\|X_n\|^2} = \left\| f - \sum_{n=1}^{N} c_n X_n \right\|^2 \geq 0$$

That is,

$$\sum_{n=1}^{N} A_n^2 \|X_n\|^2 \leq \|f\|^2$$

It also shows that

$$\left\| f - \sum_{n=1}^{N} A_n X_n \right\| \to 0 \iff \|f\|^2 = \sum_{n=1}^{\infty} A_n^2 \|X_n\|^2$$

The expression on the right hand side is often called **Parseval equality**.

Question 15

Consider

$$\sum_{n=0}^{\infty}(-1)^n x^{2n}$$

(a) Does it converge pointwise in $(-1,1)$?

(b) Does it converge uniformly in $(-1,1)$?

(c) Does it converge in L^2 in $(-1,1)$?

Proof. Geometric series as such obviously converges to

$$\sum_{n=0}^{\infty}(-1)^n x^{2n} = \sum_{n=0}^{\infty}(-x^2)^n = \frac{1}{1+x^2}$$

Note that the remainder can also be computed by

$$\sup_{x\in(-1,1)}\left|\sum_{n=N}^{\infty}(-x^2)^n\right| = \sup_{x\in(-1,1)}\left|\frac{-x^{2N}}{1+x^2}\right| = \frac{1}{2} \text{ for all } N$$

It follows that the series does NOT converge uniformly on $(-1,1)$. Compute

$$\begin{aligned}\int_{-1}^{1}\Big(\sum_{n=N}^{\infty}(-x^2)^n\Big)^2 dx &= \int_{-1}^{1}\Big(\frac{-x^{2N}}{1+x^2}\Big)^2 dx \\ &= \int_{-1}^{1}\frac{x^{4N}}{(1+x^2)^2}dx \\ &\le \int_{-1}^{1} x^{4N}dx = \frac{x^{4N+1}}{4N+1}\Big|_{x=-1}^{1} \to 0\end{aligned}$$

It follows that the series does converge in L^2. ■

Question 16

Find the sine series of the function $\cos(x)$ on the interval $(0,\pi)$. For each x in $[-\pi,\pi]$, what is the sum of the series?

Proof. Write

$$\cos(x) \sim \sum_{n=1}^{\infty} A_n \sin(nx)$$

Compute the coefficients

$$A_n = \frac{\int_0^\pi \cos(x)\sin(nx)dx}{\int_0^\pi \sin^2(nx)dx} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{4n}{\pi(n^2-1)} & \text{if } n \text{ is even} \end{cases}$$

The Fourier series by oddness and smoothness of $\cos(x)$ converge to

$$\sum A_n \sin(nx) = \begin{cases} \cos(x) & \text{if } x \in (0,\pi) \\ 0 & \text{if } x = 0, \pm\pi \\ -\cos(x) & \text{if } x \in (-\pi, 0) \end{cases}$$

∎

Question 17

Let

$$\varphi(x) \triangleq \begin{cases} -1-x & \text{if } x \in (-1,0) \\ 1-x & \text{if } x \in (0,1) \end{cases}$$

(a) Find the full Fourier series of φ in the interval $(-1,1)$.

(b) Find the first three nonzero terms explicitly.

(c) Does it converge in L^2?

(d) Does it converge pointwise?

(e) Does it converge uniformly to φ in the interval $(-1,1)$?

Proof. Define

$$\psi(x) \triangleq \begin{cases} -1 & \text{if } x \in (-1,0) \\ 1 & \text{if } x \in (0,1) \end{cases}$$

Write

$$\psi(x) \sim \sum_{n=0}^{\infty} A_n \cos(n\pi x) + B_n \sin(n\pi x)$$

It is clear that $A_0 = 0$. Compute

$$A_n = \frac{\int_{-1}^1 \psi(x)\cos(n\pi x)dx}{2} = 0$$

Compute

$$B_n = \frac{2\int_0^1 \sin(n\pi x)dx}{2} = \frac{\cos(n\pi x)}{-n\pi}\Big|_{x=0}^1 = \frac{(-1)^n - 1}{-n\pi}$$

Therefore

$$\psi(x) \sim \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{-n\pi} \sin(n\pi x)$$

Write

$$x \sim \sum_{n=0}^{\infty} A_n \cos(n\pi x) + B_n \sin(n\pi x)$$

It is clear that $A_n = 0$. Compute

$$B_n = \frac{\int_{-1}^1 x \sin(n\pi x)dx}{2} = \frac{-2(-1)^n}{n\pi}$$

In conclusion

$$\begin{aligned}\varphi(x) = \psi(x) - x &\sim \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{-n\pi} - \frac{-2(-1)^n}{n\pi}\right] \sin(n\pi x) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n + 1}{n\pi} \sin(n\pi x)\end{aligned}$$

The series converge in L^2 since $\varphi \in L^2$. Because φ is smooth on $(-1, 0)$ and $(0, 1)$, the series converge to φ at least on $(-1, 0) \cup (0, 1)$.

We claim that the series does NOT converge uniformly. Because none of the partial sum has uniform distance with φ smaller than 1. Observe

$$\lim_{x\to 0} \Big(\sum_{n=1}^{N} \frac{(-1)^n + 1}{n\pi} \sin(n\pi x) = 0\Big) \text{ and } \lim_{x\to 0} \varphi(x)$$

If the convergence is uniform, then the former expression should have converge to the latter as $N \to \infty$. ■

Question 18

Let $f(x)$ be a function on $(-l,l)$ that has a continuous derivative and satisfies the periodic BCs.

$$f(-l) = f(l) \text{ and } f'(-l) = f'(l)$$

Let a_n and b_n be the Fourier coefficients of $f(x)$, and let a'_n and b'_n be the Fourier coefficients of its derivative $f'(x)$. Show that

$$a'_n = \frac{n\pi b_n}{l} \text{ and } b'_n = \frac{-n\pi a_n}{l} \text{ for } n \neq 0$$

(Hint: Write the formulas for a'_n and b'_n and integrate by parts.) This means that the Fourier series of $f'(x)$ is what you would obtain as if you differentiated term by term. It does not mean that the differentiated series converges.

Proof. Compute

$$\begin{aligned} a'_n &= \frac{1}{l}\int_{-l}^{l} f'(x)\cos(\frac{n\pi x}{l})dx \\ &= \frac{1}{l}\Big(f(x)\cos(\frac{n\pi x}{l})\Big|_{x=-l}^{l} + \frac{n\pi}{l}\int_{-l}^{l} f(x)\sin(\frac{n\pi x}{l})dx\Big) = \frac{n\pi b_n}{l} \end{aligned}$$

Compute

$$\begin{aligned} b'_n &= \frac{1}{l}\int_{-l}^{l} f'(x)\sin(\frac{n\pi x}{l})dx \\ &= \frac{1}{l}\Big(f(x)\sin(\frac{n\pi x}{l})\Big|_{x=-l}^{l} - \frac{n\pi}{l}\int_{-l}^{l} f(x)\cos(\frac{n\pi x}{l})\Big) = \frac{-n\pi a_n}{l} \end{aligned}$$

■

Question 19

(Term by Term integration)

(a) If $f(x)$ is a piecewise continuous function in $[-l,l]$, show that its definite integral $F(x) = \int_{-l}^{x} f(s)ds$ has a full Fourier series that converges pointwise.

(b) Write this convergent series for $F(x)$ explicitly in terms of the Fourier coefficients a_0, a_n, b_n of $f(x)$ where $a_0 = 0$. (Hint: Apply a convergence Theorem. Write the formulas for the coefficients and integrate by parts.)

Proof. Part (a) follows from observing $F' = f$ is pointwise continuous so that the classical Fourier series of F converges to F by Theorem 4 in the textbook.

Write

$$f(x) = \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{l}) + b_n \sin(\frac{n\pi x}{l})$$

Note that the definition of piece wise continuity in this book implies boundedness on compact domain, and note that each term

$$\sum_{n=1}^{N} a_n \cos(\frac{n\pi x}{l}) + b_n \sin(\frac{n\pi x}{l})$$

is obviously bounded on $[-l, l]$. Then because f is bounded on $[-l, l]$, we know

$$\sum_{n=1}^{N} a_m \cos(\frac{n\pi x}{l}) + b_n \sin(\frac{n\pi x}{l}) \text{ is uniformly bounded on } [-l, l]$$

Therefore, we may apply DCT to compute

$$\begin{aligned} F(x) &= \int_{-l}^{x} \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi s}{l}) + b_n \sin(\frac{n\pi s}{l}) ds \\ &= \sum_{n=1}^{\infty} \int_{-l}^{x} a_n \cos(\frac{n\pi s}{l}) + b_n \sin(\frac{n\pi s}{l}) ds \\ &= \sum_{n=1}^{\infty} \frac{a_n l}{n\pi} \sin(\frac{n\pi x}{l}) + \frac{-b_n l}{n\pi} \cos(\frac{n\pi x}{l}) + \frac{-(-1)^n b_n l}{n\pi} \end{aligned}$$

■

Question 20

Start with the Fourier sine series of $f(x) = x$ on the interval $(0, l)$. Apply Parseval's equality. Find the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Proof. Set

$$X_n \triangleq \sin(\frac{n\pi x}{l})$$

Compute the coefficients

$$A_n = \frac{2}{l} \int_0^l x \sin(\frac{n\pi x}{l}) dx = \frac{-2l(-1)^n}{n\pi}$$

Parseval's equality is now

$$\frac{l^3}{3} = \|f\|^2 = \sum_{n=1}^{\infty} |A_n|^2 \|X_n\|^2 = \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2} \cdot \frac{l}{2}$$

This give us

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

■

Question 21

Find the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^6}$$

Proof. Consider the full Fourier series of $f(x) = x^3$ on $(-\pi, \pi)$. Let

$$x^3 \sim A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) + B_n \sin(nx)$$

Because x^3 is odd, we have

$$A_0 = A_n = 0$$

Compute

$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^3 \sin(nx) = (-1)^n \left[\frac{12\pi}{n^3} - \frac{2\pi^2}{n}\right]$$

The rest of the computation follows from Parseval's equality and computing the sum of $\frac{1}{n^4}$ by considering the cosine Fourier series of x^2 on $(0, \pi)$. ■

Question 22

Let $\varphi(x) = |x|$ in $(-\pi, \pi)$. If we approximate it by the function

$$f(x) = \frac{1}{2}a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x$$

what choice of coefficients will minimize the L^2 error.

Proof. Compute

$$\frac{1}{2}a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{\pi}{2}$$

Compute

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(x)dx = \frac{-4}{\pi}$$

Compute

$$a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(2x)dx = 0$$

Because $|x|$ is even, obviously $b_1 = b_2 = 0$. ■

Question 23

Here is a general method to calculate the normalizing constants. Let $X(x,\lambda)$ be a family of real solutions of the ODE

$$-X'' = \lambda X$$

which depends in a smooth manner of λ as well as on x.

(a) Find the ODE satisfied by $\frac{\partial X}{\partial \lambda}$.

(b) Apply Green's second identity to the pair of functions X and $\frac{\partial X}{\partial \lambda}$ in order to obtain a formula for $\int_a^b X^2 dx$ in terms of the boundary values.

(c) As an example, use the result of part (b) and the Dirichlet boundary conditions to compute $\int_0^l \sin^2(\frac{m\pi x}{l})dx$.

Proof. ■

1.5 5.5 Completeness and the Gibbs Phenomenon

Question 24

Prove the Schwarz Inequality

$$|(f,g)| \le \|f\|_2 \cdot \|g\|_2$$

for any pair of functions.

Proof. Schwarz inequality for integral is a corollary of Holder's inequality

$$|(f,g)| = \left| \int f\overline{g}dx \right| \le \int |fg|\, dx = \|fg\|_1 \le \|f\|_2 \cdot \|g\|_2$$

∎

Question 25

Prove the inequality

$$[f(l) - f(0)]^2 \le l \int_0^l (f'(x))^2 dx$$

for any continuous $f : [0,l] \to \mathbb{R}$ that is C^1 on $(0,l)$.

Hint: Use Schwarz inequality with the pair $f'(x)$ and 1.

Proof. Because f is differentiable on $(0,l)$ and continuous on the boundary, we may use the Fundamental Theorem of Calculus and apply the Schwarz inequality for integral to the pair f' and 1 to deduce

$$[f(l) - f(0)]^2 = |(f',1)|^2 \le \|f'\|_2^2 \cdot \|1\|_2^2 = l \int_0^l (f'(x))^2 dx$$

∎

Question 26

Let

$$\int_{-\pi}^{\pi} \left[|f(x)|^2 + |g(x)|^2 \right] dx$$

be finite, where

$$g(x) \triangleq \frac{f(x)}{e^{ix}-1}.$$

Let c_n be the coefficients of the full complex Fourier series of f. Show that

$$\sum_{n=-N}^{N} c_n \to 0$$

Proof. Compute

$$\begin{aligned}
\sum_{n=-N}^{N} c_n &= \sum_{n=-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \sum_{n=-N}^{N} e^{-inx} dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cdot \frac{e^{-i(N+1)x} - e^{iNx}}{e^{ix}-1} dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) \Big[e^{-i(N+1)x} - e^{iNx} \Big] dx \\
&= \frac{(g, X_{N+1}) - (g, X_{-N})}{2\pi}
\end{aligned}$$

This give us

$$\left| \sum_{n=-N}^{N} c_n \right| \leq \frac{|(g, X_{N+1})| + |(g, X_{-N})|}{2\pi} = |A_{N+1}| + |A_{-N}|$$

Because g is L^2, the proof then follows from the Parseval's inequality

$$2\pi \sum_{n=-\infty}^{\infty} |A_n|^2 = \sum_{n=-\infty}^{\infty} |A_n|^2 \, \|X_n\|^2 = \|g\|^2 \in \mathbb{R}$$

■

Question 27

Show that if f is C^1 on $[-\pi, \pi]$ and satisfies the periodic BC and if $\int_{-\pi}^{\pi} f(x)dx = 0$, then

$$\int_{-\pi}^{\pi} |f|^2 \, dx \leq \int_{-\pi}^{\pi} |f'|^2 \, dx$$

Hint: Use Parseval's equality.

Proof. Because f is C^1 on $[-\pi, \pi]$, we know $f, f' \in L^2([-\pi, \pi])$. Therefore, if we write

$$f \sim A_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

and

$$f' \sim A_0' + \sum_{n=1}^{\infty} a_n' \cos(nx) + b_n' \sin(nx)$$

We know

$$\|f\|^2 = |A_0|^2 \, 2\pi + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

and

$$\|f'\|^2 = |A_0'|^2 \, 2\pi + \pi \sum_{n=1}^{\infty} ({a_n'}^2 + {b_n'}^2)$$

Compute

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx = 0$$

and

$$A_0' = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x)dx = \frac{1}{2\pi}[f(\pi) - f(-\pi)] = 0$$

Compute

$$\begin{aligned} a_n' &= \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \cos(nx) dx \\ &= \frac{1}{\pi} \left[f(x) \cos(nx) \Big|_{x=-\pi}^{\pi} + n \int_{-\pi}^{\pi} f(x) \sin(nx) dx \right] \\ &= n b_n \end{aligned}$$

and

$$
\begin{aligned}
b_n' &= \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin(nx)dx \\
&= \frac{1}{\pi}\Big[f(x)\sin(nx)\Big|_{x=-\pi}^{\pi} - n\int_{-\pi}^{\pi} f(x)\cos(nx)dx\Big] \\
&= -na_n
\end{aligned}
$$

In conclusion

$$
\|f\|^2 = \pi\sum_{n=1}^{\infty}(a_n^2+b_n^2) \leq \pi\sum_{n=1}^{\infty} n^2(a_n^2+b_n^2) = \|f'\|^2
$$

■

Question 28

Prove the validity of the Fourier series solution of the diffusion equation on $(0, l)$ with

$$
\begin{cases}
u_t = ku_{xx} \ \textbf{(Homogeneous DE)} \\
u_x(x,0) = u_x(x,l) = 0 \ \textbf{(Neumann BC)} \\
u(x,0) = \varphi(x) \ \textbf{(IC)}
\end{cases}
$$

where φ is continuous with a piecewise continuous derivative. That is, prove that the series truly converges to the solution.

Proof. ■

1.6 6.1 Laplace's Equation

In two dimensions, we always set

$$x \triangleq r\cos\theta \text{ and } y \triangleq r\sin\theta$$

In three dimensions, we always set

$$x \triangleq r\sin\theta\cos\phi \text{ and } y \triangleq r\sin\theta\sin\phi \text{ and } z = r\cos\theta$$

Theorem 1.6.1.

$$\Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Proof. Compute

$$\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

And compute its inverse

$$\begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \frac{-\sin\theta}{r} & \frac{\cos\theta}{r} \end{bmatrix}$$

Compute

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} \\ &= \cos\theta\frac{\partial}{\partial r} + \frac{-\sin\theta}{r}\frac{\partial}{\partial \theta} \end{aligned}$$

And compute

$$\begin{aligned} \frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} \\ &= \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \end{aligned}$$

Compute

$$\begin{aligned} \frac{\partial^2}{\partial x^2} &= \left[\cos\theta\frac{\partial}{\partial r} + \frac{-\sin\theta}{r}\frac{\partial}{\partial \theta}\right]^2 \\ &= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{-2\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} \\ &+ \frac{\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} \end{aligned}$$

Compute

$$
\begin{aligned}
\frac{\partial^2}{\partial y^2} &= \left[\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right]^2 \\
&= \sin\theta^2\frac{\partial^2}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} \\
&+ \frac{-\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial r} + \frac{-\sin\theta\cos\theta}{r}\frac{\partial}{\partial\theta}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Delta_2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \\
&= \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r}
\end{aligned}
$$

■

Theorem 1.6.2.

$$
\Delta_3 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}
$$

Proof. Note that if we set $s = r\sin\theta$, then

$$
x = s\cos\phi \text{ and } y = s\sin\phi
$$

And

$$
s = r\sin\theta \text{ and } z = r\cos\theta
$$

Note that our chain of change of variables is

$$
\begin{bmatrix} x \\ y \\ z \end{bmatrix} \leftarrow \begin{bmatrix} \phi \\ s \\ z \end{bmatrix} \leftarrow \begin{bmatrix} \phi \\ r \\ \theta \end{bmatrix}
$$

These with the same procedure in computation of Laplace in polar coordinate give us

$$
u_{xx} + u_{yy} = u_{ss} + \frac{u_s}{s} + \frac{u_{\phi\phi}}{s^2}
$$

and

$$
u_{ss} + u_{zz} = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2}
$$

This give us

$$u_{xx} + u_{yy} + u_{zz} = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} + \frac{u_s}{s} + \frac{u_{\phi\phi}}{s^2}$$

Compute

$$\begin{bmatrix} \frac{\partial \phi}{\partial \phi} & \frac{\partial \phi}{\partial r} & \frac{\partial \phi}{\partial \theta} \\ \frac{\partial s}{\partial \phi} & \frac{\partial s}{\partial r} & \frac{\partial s}{\partial \theta} \\ \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\theta & r\cos\theta \\ 0 & \cos\theta & -r\sin\theta \end{bmatrix}$$

This give us

$$\begin{bmatrix} \frac{\partial \phi}{\partial \phi} & \frac{\partial \phi}{\partial s} & \frac{\partial \phi}{\partial z} \\ \frac{\partial r}{\partial \phi} & \frac{\partial r}{\partial s} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial \phi} & \frac{\partial \theta}{\partial s} & \frac{\partial \theta}{\partial z} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\theta & \cos\theta \\ 0 & \frac{\cos\theta}{r} & \frac{-\sin\theta}{r} \end{bmatrix}$$

Compute

$$\begin{aligned} u_s &= u_\phi \frac{\partial \phi}{\partial s} + u_r \frac{\partial r}{\partial s} + u_\theta \frac{\partial \theta}{\partial s} \\ &= \sin\theta u_r + \frac{\cos\theta}{r} u_\theta \\ &= \frac{s}{r} u_r + \frac{\cos\theta}{r} u_\theta \end{aligned}$$

In conclusion, writing everything in spherical coordinate,

$$\begin{aligned} \Delta_3 &= u_{xx} + u_{yy} + u_{zz} \\ &= u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} + \frac{u_s}{s} + \frac{u_{\phi\phi}}{s^2} \\ &= u_{rr} + \frac{u_{\theta\theta}}{r^2} + \frac{u_{\phi\phi}}{r^2\sin^2\theta} + \frac{2}{r} u_r + \frac{\cot\theta}{r^2} u_\theta \end{aligned}$$

∎

Question 29

Find the solutions that depend only on r of the equation

$$u_{xx} + u_{yy} + u_{zz} = k^2 u$$

where k is a positive constant.

Proof. If u depend only on r, then the equation can be rewritten into

$$u_{rr} + \frac{2}{r}u_r = k^2 u$$

If we set $u = \frac{v}{r}$, we have

$$u' = \frac{v'}{r} - \frac{v}{r^2} \text{ and } u'' = \frac{v''}{r} - \frac{2v'}{r^2} + \frac{2v}{r^3}$$

Therefore,

$$\begin{aligned} 0 &= u'' + \frac{2}{r}u' - k^2 u \\ &= \frac{v'' - k^2 v}{r} \end{aligned}$$

This implies

$$v = Ae^{kr} + Be^{-kr}$$

and

$$u = \frac{Ae^{kr} + Be^{-kr}}{r}$$

■

Question 30

Solve

$$u_{xx} + u_{yy} + u_{zz} = 0$$

in the spherical shell $0 < a < r < b$ with boundary condition $u = A$ on $r = a$ and $u = B$ on $r = b$.

Proof. Because u vanish on the whole boundary, we may assume u depend only r. Therefore, the Laplacian equation in the spherical coordinate can be simplified into

$$0 = u_{rr} + \frac{2u_r}{r}$$

If we write $v = u_r$, then we can further simplify

$$v' + \frac{2v}{r} = 0$$

The solution of this ODE is clearly

$$u_r = v = \frac{c}{r^2}$$

Therefore,

$$u = \frac{C_1}{r} + C_2$$

Plugging in the boundary condition, we may solve for the two unknowns

$$u = \frac{(A-B)ab}{r(b-a)} + \frac{-Aa+bB}{b-a}$$

■

Question 31

Solve the Poisson Equation

$$\begin{cases} u_{xx} + u_{yy} = 1 \text{ in } r < a \\ u = 0 \text{ on } r = a \end{cases}$$

Proof. Write the Poisson equation in polar coordinate

$$u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = 1$$

Because $u(a, \theta)$ for all θ, we may suppose u is independent of θ. Therefore, the Poisson equation in polar coordinate simplifies to

$$u_{rr} + \frac{u_r}{r} = 1$$

The solution space of this ODE is exactly

$$\left\{ \frac{r^2}{4} + C_1 \ln r + C_2 : C_1, C_2 \in \mathbb{R} \right\}$$

Let

$$u = \frac{r^2}{4} + C_1 \ln r + C_2$$

Because u is finite on $r = 0$, we must have $C_1 = 0$. It then follows from $u = 0$ for $r = a$ that

$$u = \frac{r^2}{4} - \frac{a^2}{4}$$

■

Question 32

Solve the Poisson Equation

$$\begin{cases} u_{xx} + u_{yy} = 1 \text{ in the annulus } 0 < a < r < b \\ u = 0 \text{ for } r = a \text{ or } b \end{cases}$$

Proof. Write the Poisson equation in polar coordinate

$$u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = 1$$

Because $u(a, \theta) = 0$ for all θ, we may suppose u depend only on r. Therefore, the Poisson equation may be simplified into

$$u_{rr} + \frac{u_r}{r} = 1$$

The solution space of this ODE is exactly

$$\left\{ \frac{r^2}{4} + C_1 \ln r + C_2 : C_1, C_2 \in \mathbb{R} \right\}$$

Let

$$u = \frac{r^2}{4} + C_1 \ln r + C_2$$

Plugging in the boundary conditions, we may solve for the two unknowns C_1, C_2

$$u = \frac{r^2}{4} + \frac{b^2 - a^2}{4(\ln a - \ln b)} \ln r + \frac{b^2 \ln a - a^2 \ln b}{4(\ln b - \ln a)}$$

■

Question 33

Prove the uniqueness of the Dirichlet problem

$$\Delta u = f \text{ in } D, u = g \text{ on } \operatorname{bdy} D$$

by the energy method. That is, after subtracting two solutions $w = u - v$, multiply the Laplace equation for w by w itself and use the divergence Theorem.

Proof. Set $w \triangleq u - v$, so that $\Delta w = 0$ in D and $w = 0$ on boundary of D. Applying Green's first identity to w and w, we have

$$\iint_{\operatorname{bdy} D} w \frac{\partial w}{\partial \mathbf{n}} dS = \iiint_D \nabla w \cdot \nabla w d\mathbf{x} + \iiint_D w \Delta w d\mathbf{x}$$

Because w is constant on the boundary and Δw is zero in D, we know the left hand side and the second term of the right hand side is also zero. This leave us

$$\iiint_D |\nabla w|^2 \, d\mathbf{x} = 0$$

which implies $\nabla w = \mathbf{0}$ in D. This together with the Dirichlet boundary condition implies $w \equiv 0$. ■

Question 34

Show that there is no solution of

$$\Delta u = f \text{ in } D, \frac{\partial u}{\partial \mathbf{n}} = g \text{ on } \operatorname{bdy} D$$

in three dimension, unless

$$\iiint_D f d\mathbf{x} = \iint_{\operatorname{bdy} D} g dS$$

(Hint: Integrate the equation.) Also show that analogue in one and two dimensions.

Proof. ■

1.7 6.2 Rectangles and Cubes

Question 35

Solve $u_{xx} + u_{yy} = 0$ in the rectangle $0 < x < a, 0 < y < b$ with the following boundary conditions:

$$u_x = -a \text{ on } x = 0 \text{ and } u_x = 0 \text{ on } x = a$$
$$u_y = b \text{ on } y = 0 \text{ and } u_y = 0 \text{ on } y = b$$

(Hint: Note that the necessary condition of Exercise 6.1.11 is satisfied. A shortcut is to guess that the solution might be a quadratic polynomial in x and y)

$$\boxed{u = \frac{x^2}{2} - \frac{y^2}{2} - ax + by}$$

Proof. Setting $u = XY$, we are required to solve the eigenproblem

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = -a \text{ and } X'(a) = 0 \end{cases} \quad \text{and} \quad \begin{cases} Y'' - \lambda Y = 0 \\ Y'(0) = b \text{ and } Y'(b) = 0 \end{cases}$$

Obviously, 0 is NOT an eigenvalue. Let $\lambda = \beta^2 \in \mathbb{C}^*$. We see

$$X = A\cos(\beta x) + B\sin(\beta x)$$

Plugging the initial condition, we see

$$\begin{bmatrix} 0 & \beta \\ -\beta\sin(\beta a) & \beta\cos(\beta a) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -a \\ 0 \end{bmatrix}$$

■

Question 36

Prove that the eigenfunctions $\{\sin my \sin nz\}$ are orthogonal on the square $\{0 < y < \pi, 0 < z < \pi\}$.

Proof. Let p, q be a pair of integer such that $p \neq m$ and $q \neq n$. Because $\{\sin(ix)\}_{i\in\mathbb{Z}}$ is an orthogonal system on $(0, \pi)$, we may compute

$$\begin{aligned}
&\int_0^\pi \int_0^\pi \sin(my)\sin(nz)\sin(py)\sin(qz)dydz \\
&= \int_0^\pi \sin(my)\sin(py)dy \int_0^\pi \sin(nz)\sin(qz)dz \\
&= 0 \cdot 0 = 0
\end{aligned}$$

This shows that $\{\sin(my)\sin(nz)\}$ are indeed orthogonal on the square $\{0 < y < \pi, 0 < z < \pi\}$. Note that we used the Fubini's theorem in our first equality, and the reason we can use the Fubini's theorem is that the function $\sin(my)\sin(nz)\sin(py)\sin(qz)$ is bounded by 1 on the bounded domain $[0, \pi]^2$. ■

Question 37

Find the harmonic function $u(x, y)$ in the square $D = \{0 < x < \pi, 0 < y < \pi\}$ with the boundary conditions:

$$\begin{aligned}
u_y &= 0 \text{ for } y = 0 \text{ and } \pi \\
u &= 0 \text{ for } x = 0 \\
u &= \cos^2 y = \frac{1 + \cos 2y}{2} \text{ for } x = \pi
\end{aligned}$$

Proof. Note that there is only one non-homogeneous BC, so there is no need to split up the solution. Let $u = XY$. We are required to first solve the following eigenproblem

$$\begin{cases} Y'' + \lambda Y = 0 \\ Y'(0) = Y'(\pi) = 0 \end{cases}$$

This is the Neumann eigenproblem, whose solution is

$$Y_n(y) = \cos(ny) \text{ and } n \in \mathbb{Z}_0^+; \lambda = n^2$$

We are now required to solve the ODE

$$\begin{cases} X_n'' - \lambda_n X_n = 0 \\ X(0) = 0 \end{cases}$$

The solution is

$$X_0 = A_0 x \text{ and } X_n = A_n \sinh(nx) \text{ for } n \geq 1$$

We now have

$$u = A_0 x + \sum_{n=1}^{\infty} A_n \sinh(nx)\cos(ny)$$

It remains to solve for A_n

$$\cos^2(y) = A_0\pi + \sum_{n=1}^{\infty} A_n \sinh(n\pi)\cos(ny)$$

Luckily, they can be written down explicitly

$$A_0 = \frac{1}{\pi^2}\int_0^{\pi} \cos^2(y)dy = \frac{1}{2\pi}$$

And

$$\begin{aligned} A_n &= \frac{2}{\pi\sinh(n\pi)}\int_0^{\pi} \cos^2(y)\cos(ny)dy \\ &= \frac{2}{\pi\sinh(n\pi)}\int_0^{\pi} \frac{\cos(2y)+1}{2} \\ &= \frac{1}{\pi\sinh(n\pi)}\int_0^{\pi} \cos(2y)\cos(ny)dy \\ &= \begin{cases} \frac{1}{2\sinh(2\pi)} & \text{if } n = 2 \\ 0 & \text{if } n \neq 2 \end{cases} \end{aligned}$$

In conclusion,

$$u = \frac{x}{2\pi} + \frac{\sinh(2x)\cos(2y)}{2\sinh(2\pi)}$$

$\blacksquare$

1.8 6.3 Poisson Formula

Consider

$$\begin{cases} \Delta u = 0 \text{ for } r < a \\ u = h(\theta) \text{ for } r = a \end{cases}$$

Rewrite Laplacian in polar coordinate

$$\Delta u = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2}$$

Let's make the separation of variable

$$u(r,\theta) \triangleq R(r)\Theta(\theta)$$

The Laplace Differential Equation then can be rewritten into

$$0 = R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta)$$

Divide both side $\frac{r^2}{R(r)\Theta(\theta)}$, we have

$$\frac{r^2R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda \text{ for some constant } \lambda$$

We are now interested in solving the two eigenproblems

$$\begin{cases} \Theta'' + \lambda\Theta = 0 \\ \Theta(\theta + 2\pi) = \Theta(\theta) \end{cases} \quad \text{and} \quad \begin{cases} r^2R'' + rR' - \lambda R = 0 \end{cases}$$

The solution of the first eigenproblem is

$$\lambda_n = n^2 \text{ and } \Theta_n = A_n\cos(n\theta) + B_n\sin(n\theta) \text{ for } n \in \mathbb{Z}_0^+$$

The second eigenproblem is of Euler type and clearly admits all eigenvalues $\lambda_n = n^2$

$$R_n = C_nr^n + D_nr^{-n} \text{ and } R_0 = D_0\ln r + C_0$$

Now, because Laplace Equation satisfy maximum principle and

$$\ln r \to -\infty \text{ and } r^{-n} \to \infty \text{ as } r \to 0^+$$

We can just write

$$R_n = C_n r^n \text{ and } R_0 = C_0$$

Therefore, summing up all the eigenfunctions, we have

$$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos(n\theta) + B_n \sin(n\theta))$$

Theorem 1.8.1. (Poisson Formula) The solution of the homogeneous Dirichlet problem for circle

$$\begin{cases} \nabla u = 0 \text{ for } r < a \\ u = h(\theta) \text{ for } r = a \end{cases}$$

is

$$u(r,\theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2} d\phi$$

Theorem 1.8.2. (Maximum Principle) Let $D \subseteq \mathbb{R}^n$ be connected, bounded and open. Let continuous $u : \overline{D} \to \mathbb{R}$ be harmonic in D. If u is non-constant, then

Both the maximum and the minimum of u does not appear in D.

Theorem 1.8.3. (Two Dimensional Mean Value Property of Harmonic function) Let $B_r(\mathbf{x}) \subseteq \mathbb{R}^2$ be an open ball. Given continuous $u : \overline{B_r(\mathbf{x})} \to \mathbb{R}$ harmonic inside, we have

$$u(\mathbf{x}) = \frac{1}{2\pi} \int_0^{2\pi} h(\theta) d\theta$$

where $h = u|_{\partial B_r(\mathbf{x})}$

Question 38

Suppose that u is a harmonic function in the disk $D = \{r < 2\}$ and that

$$u = 3\sin 2\theta + 1 \text{ for } r = 2$$

Without finding the solution, answer the following the questions.

(a) Find the maximum value of u in $\overline{D}$.

(b) Calculate the value of u at the origin.

Proof. Because u attain its maximum only on the boundary, we know

$$\max_{\overline{D}} u = 4$$

Using mean value property, we also know

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi}(3\sin(2\theta)+1)d\theta = 1$$

■

Question 39

Solve $u_{xx}+u_{yy}=0$ in the disk $\{r<a\}$ with the boundary condition

$$u = 1 + 3\sin\theta \text{ on } r = a$$

Proof. Recall that by separation of variables, we may write

$$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n(A_n\cos(n\theta)+B_n\sin(n\theta))$$

This give us

$$1+3\sin\theta = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} a^n(A_n\cos(n\theta)+B_n\sin(n\theta)) \text{ for } \theta\in(-\pi,\pi)$$

In other words,

$$A_0 = 2 \text{ and } B_1 = \frac{3}{a} \text{ and the rest are all zero}$$

This then give us

$$u(r,\theta) = 1 + \frac{3r\sin(\theta)}{a}$$

■

Plugging the boundary condition $u=h$ for $r=a$, we have

$$h(\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} a^n(A_n\cos(n\theta)+B_n\sin(n\theta))$$

This allow us to solve for A_0, A_n, B_n

$$A_0 = \frac{1}{\pi}\int_0^{2\pi} h(\phi)d\phi \text{ and } A_n = \frac{1}{a^n\pi}\int_0^{2\pi} h(\phi)\cos(n\phi)d\phi$$
$$B_n = \frac{1}{a^n\pi}\int_0^{2\pi} h(\phi)\sin(n\phi)d\varphi$$

Although the coefficients may seem daunting, the original u can actually be simplified as follows

$$u = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)d\varphi + \sum_{n=1}^{\infty}\frac{r^n}{a^n\pi}\int_0^{2\pi} h(\phi)\Big[\cos(n\phi)\cos(n\theta) + \sin(n\phi)\sin(n\theta)\Big]d\phi$$
$$= \frac{1}{2\pi}\int_0^{2\pi} h(\varphi)\Big[1 + 2\sum_{n=1}^{\infty}\frac{r^n}{a^n}\cos(n(\theta-\phi))\Big]d\phi$$

1.9 7.1 Green's First Identity

The divergence theorem states that given some vector field $\mathbf{F}$, we have

$$\int_D \nabla \cdot \mathbf{F} d\mathbf{x} = \int_{\partial D} \mathbf{F} \cdot \mathbf{n} dS$$

From product rule, we may deduce

$$\begin{aligned}\nabla \cdot (v \nabla u) &= \nabla v \cdot \nabla u + v(\nabla \cdot \nabla u)\\ &= \nabla v \cdot \nabla u + v \nabla^2 u\end{aligned}$$

This with divergence theorem give us

$$\int_{\partial D} v \frac{\partial u}{\partial \mathbf{n}} dS = \int_D \nabla v \cdot \nabla u d\mathbf{x} + \int_D v \nabla^2 u d\mathbf{x}$$

which is called **Green's first identity**. Immediately, we can use Green's first identity to prove the uniqueness for Dirichlet's problem and Neumann problem. Let u be the difference of two solutions, so u is harmonic with Dirichlet (or Neumann) boundary condition. Applying Green's first identity to u, u, we then have

$$0 = \int_{\partial D} u \frac{\partial u}{\partial \mathbf{n}} dS = \int_D |\nabla u|^2 d\mathbf{x}$$

This then implies u is a constant.

Also, note that the divergence theorem implies that the problem

$$\begin{cases} \nabla^2 u = f \text{ in } D \\ \frac{\partial u}{\partial \mathbf{n}} = g \text{ on } \partial D \end{cases}$$

has solution only if the integrals of f and g are equal, since

$$\int_D f = \int_D \nabla \cdot (\nabla u) = \int_{\partial D} (\nabla u) \cdot \mathbf{n} = \int_{\partial D} \frac{\partial u}{\partial \mathbf{n}} = \int_{\partial D} g$$

Chapter 2

General Analysis HW

2.1 Brunn-Minkowski Inequality

Abstract

This HW assignment require us to prove the Brunn-Minkowski Inequality. Note that in this HW, we use bold face $\mathbf{x}$ to denote $(x_1,\dots,x_d)$ element of $\mathbb{R}^d$. Also, throughout this HW, we shall suppose $|A|>0$ and $|A|,|B|<\infty$; otherwise, the proof would be trivial.

We first introduce some notation. Given two sets $A,B\subseteq\mathbb{R}^d$ and a point $\mathbf{x}\in\mathbb{R}^d$, we write

$$A+\mathbf{x}\triangleq\{\mathbf{a}+\mathbf{x}\in\mathbb{R}^d:\mathbf{a}\in A\}$$

and write

$$A+B\triangleq\{\mathbf{a}+\mathbf{b}\in\mathbb{R}^d:\mathbf{a}\in A\text{ and }\mathbf{b}\in B\}$$

Note that elementary set theory tell us

$$(A+\mathbf{x})+(B+\mathbf{y})=(A+B)+(\mathbf{x}+\mathbf{y})\tag{2.1}$$

Theorem 2.1.1. (Brunn-Minkowski Inequality for rectangles) Suppose A,B are two **rectangles**, i.e., A is of the form $\prod_{j=1}^d[x_j,y_j]$, and so is B. We have

$$|A|^{\frac{1}{d}}+|B|^{\frac{1}{d}}\leq|A+B|^{\frac{1}{d}}$$

Proof. Because Lebesgue measure is translation invariant, by Equation 2.1, we can WLOG suppose

$$A = \prod_{j=1}^{d} [0, a_j] \text{ and } B = \prod_{j=1}^{d} [0, b_j]$$

It is clear that

$$A + B = \prod_{j=1}^{d} [0, a_j + b_j]$$

By direct computation, we know that

$$|A + B| = \prod_{j=1}^{d} (a_j + b_j) \text{ and } |A| = \prod_{j=1}^{d} a_j \text{ and } |B| = \prod_{j=1}^{d} b_j$$

Then by AM-GM inequality, we have

$$\frac{|A|^{\frac{1}{d}}}{|A+B|^{\frac{1}{d}}} = \Big(\prod_{j=1}^{d} \frac{a_j}{a_j + b_j} \Big)^{\frac{1}{d}} \leq \frac{1}{d} \sum_{j=1}^{d} \frac{a_j}{a_j + b_j}$$

Similarly, by AM-GM inequality, we have

$$\frac{|B|^{\frac{1}{d}}}{|A+B|^{\frac{1}{d}}} = \Big(\prod_{j=1}^{d} \frac{b_j}{a_j + b_j} \Big)^{\frac{1}{d}} \leq \frac{1}{d} \sum_{j=1}^{d} \frac{b_j}{a_j + b_j}$$

Adding two inequalities together, we now have

$$\frac{|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}}{|A+B|^{\frac{1}{d}}} \leq \frac{1}{d} \sum_{j=1}^{d} \frac{a_j + b_j}{a_j + b_j} = 1$$

The result then follows from multiplying both side with $|A + B|^{\frac{1}{d}}$. ■

Theorem 2.1.2. (Brunn-Minkowski Inequality for finite union of non-overlapping of rectangles) Suppose A is a union of a finite collection of non-overlapping rectangles, and the same holds for B. We have

$$|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}} \leq |A + B|^{\frac{1}{d}}$$

Proof. We prove by induction on k, the sum of the rectangles in A and B. The base case $k = 2$ have been proved by Theorem 2.1.1. Suppose the proposition hold true when $k \leq r$. Let $k = r + 1$. Because the rectangles in A are non-overlapping, by a translation and renaming axis if necessary, we can suppose the following proposition.

Proposition 1: Both $A^+ \triangleq A \cap \{\mathbf{x} \in \mathbb{R}^d : x_1 \geq 0\}$ and $A^- \triangleq A \cap \{\mathbf{x} \in \mathbb{R}^d : x_1 \leq 0\}$ are unions of a finite collection of non-overlapping rectangles, with each collection containing at least one fewer rectangle than A.

Proposition 1 holds because, if we write $A = A_1 \cup \cdots \cup A_m$, where $A_1, \dots, A_m$ are non-overlapping rectangles, then by translation and remaining axis, we can suppose that A_1, A_2 lie in distinct closed subspace, either $\{\mathbf{x} \in \mathbb{R}^d : x_1 \geq 0\}$ or $\{\mathbf{x} \in \mathbb{R}^d : x_1 \leq 0\}$, while for all $n \geq 3$, $A_n \cap \{\mathbf{x} \in \mathbb{R}^d : x_1 \geq 0\}$ is either empty or also a rectangle.

Note that $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(t) \triangleq \left| \Big(B + (t, 0, \dots, 0) \Big) \cap \{\mathbf{x} \in \mathbb{R}^d : x_1 \geq 0\} \right|$$

is clearly an increasing continuous function such that

$$h(-M) = 0 \text{ and } h(M) = |B| \text{ for some } M > 0$$

Then by IVT, we can translate B to let B satisfy

$$\frac{|B^+|}{|B|} = \frac{|A^+|}{|A|} \text{ where } B^+ \triangleq B \cap \{\mathbf{x} \in \mathbb{R}^d : x_1 \geq 0\} \tag{2.2}$$

Define $B^- \triangleq B \cap \{\mathbf{x} \in \mathbb{R}^d : x_1 \leq 0\}$. With reasoning similar to that of Proposition 1, we know that B^+ and B^- are both unions of collections of non-overlapping rectangles, with each collection consisting of no more rectangles than B. Therefore, by Proposition 1, we can deduce that the sum of the number of rectangles in A^+ and B^+ is at least one fewer than $r + 1$, and the same holds for the sum of the number of rectangles in A^- and B^-. Then, because the proposition holds true for $k \leq r$, we now have

$$\left|A^+\right|^{\frac{1}{d}} + \left|B^+\right|^{\frac{1}{d}} \leq \left|A^+ + B^+\right|^{\frac{1}{d}} \text{ and } \left|A^-\right|^{\frac{1}{d}} + \left|B^-\right|^{\frac{1}{d}} \leq \left|A^- + B^-\right|^{\frac{1}{d}}$$

Note that for each $\mathbf{x}$ in the interior of $A^+ + B^+$, we must have $x_1 > 0$, and for each $\mathbf{y}$ in the interior of $A^- + B^-$, we must have $y_1 < 0$. This implies that $(A^+ + B^+)$ and $(A^- + B^-)$ are non-overlapping. Now, because

$$A + B = (A^+ + B^+) \cup (A^- + B^-)$$

if we define $\rho \triangleq \frac{|A^+|}{|A|}$, from Equation 2.2 we can finally deduce

$$
\begin{aligned}
|A+B| &= \left|A^+ + B^+\right| + \left|A^- + B^-\right| \\
&\geq \left(\left|A^+\right|^{\frac{1}{d}} + \left|B^+\right|^{\frac{1}{d}}\right)^d + \left(\left|A^-\right|^{\frac{1}{d}} + \left|B^-\right|^{\frac{1}{d}}\right)^d \\
\left(\because \frac{|A^-|}{|A|} = \frac{|B^-|}{|B|} = 1-\rho\right) \quad &= \left((\rho\,|A|)^{\frac{1}{d}} + (\rho\,|B|)^{\frac{1}{d}}\right)^d + \left(((1-\rho)\,|A|)^{\frac{1}{d}} + ((1-\rho)\,|B|)^{\frac{1}{d}}\right)^d \\
&= (|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}})^d
\end{aligned}
$$

which give us the desired inequality

$$|A+B|^{\frac{1}{d}} \geq |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}$$

■

Theorem 2.1.3. (Brunn-Minkowski Inequality for bounded open set) Suppose A, B are both bounded open subset of $\mathbb{R}^d$. We have

$$|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}} \leq |A+B|^{\frac{1}{d}}$$

Proof. Note that $A+B$ is open since it is a union of the open sets:

$$A+B = \bigcup_{\mathbf{b} \in B} A + \mathbf{b}$$

It follows that $A+B$ is Lebesgue measurable, so it makes sense for us to write $|A+B|$. By a proposition taught in class, we can let

$$A = \bigcup_{n \in \mathbb{N}} K_{n,a} \text{ and } B = \bigcup_{n \in \mathbb{N}} K_{n,b}$$

where $(K_{n,a})$ are non-overlapping rectangles, and so are $(K_{n,b})$. It is clear that

$$\left(\bigcup_{n=1}^{N} K_{n,a}\right) + \left(\bigcup_{n=1}^{N} K_{n,b}\right) \nearrow A+B \text{ as } N \to \infty$$

This together with Theorem 2.1.2 give us the desired inequality

$$
\begin{aligned}
|A+B|^{\frac{1}{d}} &= \lim_{N \to \infty} \left|\left(\bigcup_{n=1}^{N} K_{n,a}\right) + \left(\bigcup_{n=1}^{N} K_{n,b}\right)\right|^{\frac{1}{d}} \\
&\geq \lim_{N \to \infty} \left|\bigcup_{n=1}^{N} K_{n,a}\right|^{\frac{1}{d}} + \left|\bigcup_{n=1}^{N} K_{n,b}\right|^{\frac{1}{d}} = |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}
\end{aligned}
$$

■

Theorem 2.1.4. (Brunn-Minkowski Inequality for compact set) Suppose A, B are both compact subset of $\mathbb{R}^d$. We have

$$|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}} \leq |A+B|^{\frac{1}{d}}$$

Proof. For each $\epsilon > 0$, define

$$A_\epsilon \triangleq \{\mathbf{x} \in \mathbb{R}^d : d(\mathbf{x}, A) < \epsilon\} \text{ and } B_\epsilon \triangleq \{\mathbf{x} \in \mathbb{R}^d : d(\mathbf{x}, B) < \epsilon\}$$

To see A_ϵ is open, observe that if $\mathbf{x} \in A_\epsilon$, then for all $\mathbf{y}$ in the open ball $\{\mathbf{y} : d(\mathbf{x}, \mathbf{y}) < \frac{\epsilon - d(\mathbf{x}, A)}{2}\}$ centering $\mathbf{x}$, we can pick some $\mathbf{z} \in A$ satisfying $d(\mathbf{x}, \mathbf{z}) < d(\mathbf{x}, A) + \frac{\epsilon}{2}$ to have

$$\begin{aligned} d(\mathbf{y}, A) &\leq d(\mathbf{y}, \mathbf{z}) \\ &\leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{x}, \mathbf{y}) \\ &\leq d(\mathbf{x}, \mathbf{z}) + \frac{\epsilon - d(\mathbf{x}, A)}{2} \\ &\leq d(\mathbf{x}, \mathbf{z}) - d(\mathbf{x}, A) + \frac{\epsilon}{2} < \epsilon \text{ implying } \mathbf{y} \in A_\epsilon \end{aligned}$$

Similar argument shows that B_ϵ are open. To see $A_\epsilon \searrow A$, note that for all $\mathbf{x} \notin A$, because $d(\mathbf{x}, \mathbf{z})$ is a function continuous in the variable $\mathbf{z}$ and A is compact, by EVT we have

$$d(\mathbf{x}, A) = d(\mathbf{x}, \mathbf{z}) > 0 \text{ for some } \mathbf{z} \in A$$

where the inequality holds because $\mathbf{z} \in A \implies \mathbf{x} \neq \mathbf{z}$. Similar argument shows that $B_\epsilon \searrow B$. We now prove

$$A + B = \lim_{\epsilon \to 0} A_\epsilon + B_\epsilon \tag{2.3}$$

It is clear that

$$A + B \subseteq \bigcap_{\epsilon > 0} A_\epsilon + B_\epsilon \tag{2.4}$$

Fix an arbitrary $\mathbf{z} \in \bigcap_{\epsilon > 0} A_\epsilon + B_\epsilon$. For all $n \in \mathbb{N}$, by definition there exists $\mathbf{a}_n \in A_{\frac{1}{n}}$ and $\mathbf{b}_n \in B_{\frac{1}{n}}$ such that $\mathbf{z} = \mathbf{a}_n + \mathbf{b}_n$. By the Bolzano-Weierstrass Theorem, there exists convergent subsequence $\mathbf{a}_{n_k}$. Applying Bolzano-Weierstrass Theorem again, we find that there exists convergent subsequence $\mathbf{b}_{n_{k_j}}$. Clearly, $\mathbf{a}_{n_{k_j}}$ also converge. For brevity, we denote these subsequences simply by $\mathbf{a}_{n_k}$ and $\mathbf{b}_{n_k}$, and we denote their limit by

$$\mathbf{a} = \lim_{k \to \infty} \mathbf{a}_{n_k} \text{ and } \mathbf{b} = \lim_{k \to \infty} \mathbf{b}_{n_k}$$

We now shows that

$$\mathbf{a} \in A$$

Assume $\mathbf{a} \notin A$ for a contradiction. By EVT, $d(\mathbf{a}, A) = d(\mathbf{a}, \mathbf{a}')$ for some $\mathbf{a}' \in A$. Note that $d(\mathbf{a}, \mathbf{a}') > 0$ because $\mathbf{a} \notin A \implies \mathbf{a} \neq \mathbf{a}'$. We have shown $d(\mathbf{a}, A) = d(\mathbf{a}, \mathbf{a}') > 0$.

Let m be large enough so that $\frac{1}{n_m} < \frac{d(\mathbf{a},A)}{2}$. Since $d(\mathbf{a}_{n_m}, A) < \frac{1}{n_m} < \frac{d(\mathbf{a},A)}{2}$, we can select $\mathbf{a}'' \in A$ such that $d(\mathbf{a}_{n_m}, \mathbf{a}'') < \frac{d(\mathbf{a},A)}{2}$. This give us

$$d(\mathbf{a}, A) \leq d(\mathbf{a}, \mathbf{a}'') \leq d(\mathbf{a}, \mathbf{a}_{n_m}) + d(\mathbf{a}_{n_m}, \mathbf{a}'') < \frac{d(\mathbf{a}, A)}{2} + \frac{d(\mathbf{a}, A)}{2}$$

This results in $d(\mathbf{a}, A) < d(\mathbf{a}, A)$, a contradiction. We have proved $\mathbf{a} \in A$. Similar arguments shows that $\mathbf{b} \in B$.

Now, since $\mathbf{z} = \mathbf{a}_{n_k} + \mathbf{b}_{n_k}$ for all k, we see

$$\mathbf{z} = \lim_{k\to\infty} \mathbf{a}_{n_k} + \mathbf{b}_{n_k} = \lim_{k\to\infty} \mathbf{a}_{n_k} + \lim_{k\to\infty} \mathbf{b}_{n_k} = \mathbf{a} + \mathbf{b} \in A + B$$

Because $\mathbf{z}$ is arbitrarily selected from $\bigcap_{\epsilon>0} A_\epsilon + B_\epsilon$. We have in fact proved

$$\bigcap_{\epsilon>0} A_\epsilon + B_\epsilon \subseteq A + B$$

which together with Equation 2.4 implies Equation 2.3. With Equation 2.3 established, we can now apply Theorem 2.1.3 to have the desired inequality

$$\begin{aligned}
|A + B|^{\frac{1}{d}} &= \left(\lim_{\epsilon\to 0} |A_\epsilon + B_\epsilon| \right)^{\frac{1}{d}} \\
&= \lim_{\epsilon\to 0} |A_\epsilon + B_\epsilon|^{\frac{1}{d}} \\
&\geq \lim_{\epsilon\to 0} |A_\epsilon|^{\frac{1}{d}} + |B_\epsilon|^{\frac{1}{d}} = |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}
\end{aligned}$$

■

Before we proceed to develop the remaining Brunn-Minkowski Inequality, we first have to prove that Lebesgue measure is **inner regular**.

Theorem 2.1.5. (Lebesgue measure is inner regular) If $A \subseteq \mathbb{R}^d$ is Lebesgue measurable, then

$$|A| = \sup\{|K| : K \text{ is some compact subset of } A\}$$

Proof. Because A is measurable, we know $A \cap \overline{B_n(\mathbf{0})}$ is measurable for all $n \in \mathbb{N}$. Therefore, for all $n \in \mathbb{N}$, $(A \cap \overline{B_n(\mathbf{0})})^c$ is measurable. Then by definition, there exists open O_n

containing $(A \cap \overline{B_n(\mathbf{0})})^c$, such that $|O_n \setminus (A \cap B_n(\mathbf{0}))^c| < \frac{1}{n}$. Now, for each $n \in \mathbb{N}$, define closed set $K_n \triangleq O_n^c$. We then have

$$K_n \subseteq A \cap \overline{B_n(\mathbf{0})}$$

and

$$(A \cap \overline{B_n(\mathbf{0})}) \setminus K_n = A \cap \overline{B_n(\mathbf{0})} \cap O_n = O_n \setminus (A \cap B_n(\mathbf{0}))^c$$

This then give us

$$\left|(A \cap \overline{B_n(\mathbf{0})}) \setminus K_n\right| = |O_n \setminus (A \cap B_n(\mathbf{0}))^c| < \frac{1}{n}$$

Note that because $K_n \subseteq B_n(\mathbf{0})$ is bounded and closed, by Hiene-Borel, we know K_n is compact. Lastly, to close out the proof, we are required to show $|K_n| \to |A|$ as $n \to \infty$. Note that $\left|A \cap \overline{B_n(\mathbf{0})}\right| \nearrow |A|$ as $n \to \infty$ because $A \cap \overline{B_n(\mathbf{0})} \nearrow A$ as $n \to \infty$. Then because $\left|A \cap \overline{B_n(\mathbf{0})}\right| \geq |K_n| \geq \left|A \cap \overline{B_n(\mathbf{0})}\right| - \frac{1}{n}$, we see that $|K_n| \to |A|$ by squeeze Theorem. ■

Theorem 2.1.6. (Brunn-Minkowski Inequality for measurable set) Suppose A, B are measurable subset of $\mathbb{R}^d$ and $A + B$ is also measurable. We have

$$|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}} \leq |A + B|^{\frac{1}{d}}$$

Proof. Because Lebesgue measure is inner regular and A, B are of finite measure, for each $n \in \mathbb{N}$, we can let A_n, B_n each be compact subset of A, B such that $|A| - |A_n| < \frac{1}{n}$ and $|B| - |B_n| < \frac{1}{n}$. It then follows from Theorem 2.1.4 that

$$|A + B|^{\frac{1}{d}} \geq \lim_{n \to \infty} |A_n + B_n|^{\frac{1}{d}} \geq \lim_{n \to \infty} |A_n|^{\frac{1}{d}} + |B_n|^{\frac{1}{d}} = |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}$$

■

2.2 HW1

Question 40

Show $\mathbb{R}^n$ is complete.

Proof. Let $\mathbf{x}_k$ be an arbitrary Cauchy sequence in $\mathbb{R}^n$. We are required to show $\mathbf{x}_k$ converge in $\mathbb{R}^n$. For each k, denote $\mathbf{x}_k$ by $(x_{(1,k)},\dots,x_{(n,k)})$. We claim that for each $i\in\{1,\dots,n\}$

$$x_{(i,k)} \text{ is a Cauchy sequence}$$

Fix i and $\epsilon>0$. To show $x_{(i,k)}$ is a Cauchy sequence, we are required to find $N\in\mathbb{N}$ such that for all $r,m\geq N$ we have

$$\left|x_{(i,r)}-x_{(i,m)}\right|\leq\epsilon$$

Because $\mathbf{x}_k$ is a Cauchy sequence in $\mathbb{R}^n$, we know there exists $N\in\mathbb{N}$ such that for all $r,m\geq N$, we have

$$\left|\mathbf{x}_r-\mathbf{x}_m\right|<\epsilon$$

Fix such N and arbitrary $r,m\geq M$. Observe

$$\left|x_{(i,r)}-x_{(i,m)}\right|\leq\sqrt{\sum_{j=1}^{n}\left|x_{(j,r)}-x_{(j,m)}\right|^2}=|\mathbf{x}_r-\mathbf{x}_m|<\epsilon$$

We have proved that for each $i\in\{1,\dots,n\}$, the real sequence $x_{(i,k)}$ is Cauchy. We now claim that for each $i\in\{1,\dots,n\}$, we have

$$\limsup_{r\to\infty}x_{(i,r)}\in\mathbb{R} \text{ and } \lim_{k\to\infty}x_{(i,k)}=\limsup_{r\to\infty}x_{(i,r)}$$

Again fix i. Because $x_{(i,k)}$ is a Cauchy sequence, we know there exists some N such that for all $r,m\geq N$, we have

$$\left|x_{(i,r)}-x_{(i,m)}\right|<1$$

This implies that for all $r\geq N$, we have

$$x_{(i,r)}<x_{(i,N)}+1 \tag{2.5}$$

Equation 2.5 then tell us

$$x_{(i,N)}+1 \text{ is an upper bound of } \{x_{(i,r)}:r\geq N\}$$

Then by definition of sup, we have

$$\sup\{x_{(i,r)} : r \geq N\} \leq x_{(i,N)} + 1 \in \mathbb{R}$$

This then implies $\limsup_{r\to\infty} x_{(i,r)} \in \mathbb{R}$. We now prove

$$\lim_{k\to\infty} x_{(i,k)} = \limsup_{r\to\infty} x_{(i,r)} \tag{2.6}$$

Fix $\epsilon > 0$. We are required to find N such that

$$\forall k \geq N, \left| x_{(i,k)} - \limsup_{r\to\infty} x_{(i,r)} \right| \leq \epsilon$$

Because $\{x_{(i,k)}\}_{k\in\mathbb{N}}$ is a Cauchy sequence, we can let N_0 satisfy

$$\forall k, m \geq N_0, \left| x_{(i,k)} - x_{(i,m)} \right| < \frac{\epsilon}{2}$$

Because $\sup\{x_{(i,k)} : k \geq N'\} \searrow \limsup_{r\to\infty} x_{(i,r)}$ as $N' \to \infty$, we know there exists $N_1 > N_0$ such that

$$\limsup_{r\to\infty} x_{(i,r)} - \frac{\epsilon}{2} < \sup\{x_{(i,k)} : k \geq N_0\} \leq \limsup_{r\to\infty} x_{(i,r)} + \frac{\epsilon}{2}$$

Then because $\limsup_{r\to\infty} x_{(i,r)} - \frac{\epsilon}{2}$ is strictly smaller than the smallest upper bound of $\{x_{(i,k)} : k \geq N_1\}$, we see $\limsup_{n\to\infty} x_{(i,r)} - \frac{\epsilon}{2}$ is not an upper bound of $\{x_{(i,k)} : k \geq N_1\}$. This implies the existence of some N such that $N \geq N_1$ and

$$\limsup_{r\to\infty} x_{(i,r)} - \frac{\epsilon}{2} < x_{(i,N)} \leq \limsup_{r\to\infty} x_{(i,r)} + \frac{\epsilon}{2}$$

Now observe that for all $k \geq N$, because $N \geq N_1 \geq N_0$

$$\limsup_{r\to\infty} x_{(i,r)} - \epsilon < x_{(i,N)} - \frac{\epsilon}{2} < x_{(i,k)} < x_{(i,N)} + \frac{\epsilon}{2} \leq \limsup_{r\to\infty} x_{(i,r)} + \epsilon$$

This implies for all $k \geq N$, we have

$$\left| x_{(i,k)} - \limsup_{r\to\infty} x_{(i,r)} \right| \leq \epsilon$$

We have just proved Equation 2.6. Lastly, to close out the proof, we show

$$\lim_{k\to\infty} \mathbf{x}_k = \left(\lim_{k\to\infty} x_{(1,k)}, \dots, \lim_{k\to\infty} x_{(n,k)} \right) \tag{2.7}$$

Fix $\epsilon > 0$. For each $i \in \{1, \dots, n\}$, let N_i satisfy

$$\forall r \geq N_i, \left| x_{(i,r)} - \lim_{k\to\infty} x_{(i,k)} \right| \leq \frac{\epsilon}{\sqrt{n}}$$

Observe that for all $r \geq \max_{i\in\{1,\dots,n\}} N_i$, we have

$$\begin{aligned}\left|\mathbf{x}_r - \left(\lim_{k\to\infty} x_{(1,k)}, \dots, \lim_{k\to\infty} x_{(n,k)}\right)\right| &= \sqrt{\sum_{i=1}^{n} \left|x_{(i,r)} - \lim_{k\to\infty} x_{(i,k)}\right|^2} \\ &\leq \sqrt{\sum_{i=1}^{n} \frac{\epsilon^2}{n}} = \epsilon\end{aligned}$$

We have proved Equation 2.7.

■

Question 41

Show $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. Fix $x \in \mathbb{R}$ and $\epsilon > 0$. To show $\mathbb{Q}$ is dense in $\mathbb{R}$, we have to find $q \in \mathbb{Q}$ such that $|x - q| < \epsilon$.

Let $m \in \mathbb{N}$ satisfy $\frac{1}{m} < \epsilon$. Let n be the largest integer such that $n \leq mx$. Because n is the largest integer such that $n \leq mx$, we know $mx - n < 1$, otherwise we can deduce $n + 1 \leq mx$, which is impossible, since $n + 1$ is an integer and n is the largest integer such that $n \leq mx$. We now see that

$$\frac{n}{m} \in \mathbb{Q} \text{ and } \left|x - \frac{n}{m}\right| = \frac{mx - n}{m} < \frac{1}{m} < \epsilon$$

■

Theorem 2.2.1. (Distance Formula) Given two subsets A, B of a metric space, we have

$$d(A, B) = \inf_{b\in B} d(A, b)$$

Proof. Fix arbitrary $b \in B$. It is clear that

$$d(A, B) \leq d(A, b)$$

It then follows $d(A, B) \leq \inf_{b\in B} d(A, b)$. Fix arbitrary $a \in A$ and $b_0 \in B$. Observe that

$$d(a, b_0) \geq d(A, b_0) \geq \inf_{b\in B} d(A, b)$$

It then follows $\inf_{b\in B} d(A, b) \leq d(A, B)$. ■

Question 42

Let E_1, E_2 be non-empty sets in $\mathbb{R}^n$ with E_1 closed and E_2 compact. Show that there are points $x_1 \in E_1$ and $x_2 \in E_2$ such that

$$d(E_1, E_2) = |x_1 - x_2|$$

Deduce that $d(E_1, E_2)$ is positive if such E_1, E_2 are disjoint.

Proof. Because

(a) $f(x) \triangleq d(E_1, x)$ is a continuous function on $\mathbb{R}^n$.

(b) E_2 is compact.

It now follows by EVT there exists some $x_2 \in E_2$ such that

$$d(E_1, x_2) = \min_{x \in E_2} d(E_1, x) = \inf_{x \in E_2} d(E_1, x) = d(E_1, E_2)$$

where the last equality is proved above. We can now reduce the problem into finding x_1 in E_1 such that

$$d(x_1, x_2) = d(E_1, x_2)$$

For each $n \in \mathbb{N}$, let t_n satisfy

$$t_n \in E_1 \text{ and } d(t_n, x_2) < d(E_1, x_2) + \frac{1}{n}$$

Clearly, t_n is a bounded sequence. Then by Bolzano-Weierstrass Theorem, there exists a convergence subsequence t_{n_k}. Now, because E_1 is closed, we know

$$x_1 \triangleq \lim_{k \to \infty} t_{n_k} \in E_1$$

It then follows from the function $f(x) \triangleq d(x, x_2)$ being continuous on $\mathbb{R}^n$ such that

$$d(x_1, x_2) = \lim_{k \to \infty} d(t_{n,k}, x_2) = d(E_1, x_2)$$

■

Question 43

Prove that the distance between two nonempty, compact, disjoint sets in $\mathbb{R}^n$ is positive.

Proof. The proof follows from the result in last question while acknowledging compact is closed. ■

Question 44

Prove that if f is continuous on $[a,b]$, then f is Riemann-integrable on$[a,b]$.

Proof. Let $\overline{\int_a^b} f dx$ and $\underline{\int_a^b} f dx$ respectively denote the upper and lower Darboux sums. We prove that

$$\overline{\int_a^b} f dx = \underline{\int_a^b} f dx$$

Fix ϵ. We reduce the problem into proving the existence of some partition $\{a = x_0, x_1, \dots, x_n = b\}$ such that

$$\sum_{i=1}^n \Big[M_i - m_i\Big](x_i - x_{i-1}) \leq \epsilon$$

where

$$M_i \triangleq \sup_{t\in[x_{i-1},x_i]} f(t) \text{ and } m_i \triangleq \inf_{t\in[x_{i-1},x_i]} f(t)$$

Because f is continuous on the compact interval $[a,b]$, we know f is uniformly continuous on $[a,b]$. Let δ satisfy

$$|x-y| < \delta \text{ and } x,y \in [a,b] \implies |f(x) - f(y)| < \frac{\epsilon}{b-a}$$

Let n satisfy $\frac{b-a}{n} < \delta$. We claim the partition

$$\{a = x_0, x_1, \dots, x_n = b\} \text{ where } x_i \triangleq a + \frac{i(b-a)}{n} \text{ suffices}$$

Now, by EVT, we know that for each i, there exists some $t_{i,M}, t_{i,m} \in [x_{i-1}, x_i]$ such that

$$f(t_{i,m}) = m_i \text{ and } f(t_{i,M}) = M_i$$

Then because

$$|t_{i,m} - t_{i,M}| \leq x_i - x_{i-1} \leq \frac{b-a}{n} < \delta$$

We know $M_i - m_i < \frac{\epsilon}{b-a}$. This now give us

$$\begin{aligned}
\sum_{i=1}^n \Big[M_i - m_i\Big](x_i - x_{i-1}) &< \sum_{i=1}^n \frac{\epsilon}{(b-a)}(x_i - x_{i-1}) \\
&= \frac{\epsilon}{b-a} \sum_{i=1}^n (x_i - x_{i-1}) \\
&= \frac{\epsilon}{b-a}(b-a) = \epsilon
\end{aligned}$$

■

Question 45

Find $\limsup_{n\to\infty} E_n$ and $\liminf_{n\to\infty} E_n$ where

$$E_n \triangleq \begin{cases} [\frac{-1}{n}, 1] & \text{if } n \text{ is odd} \\ [-1, \frac{1}{n}] & \text{if } n \text{ is even} \end{cases}$$

Proof. Fix arbitrary $n \in \mathbb{N}$. Let $p, q \geq n$ respectively be odd and even. We see

$$[0, 1] \subseteq E_p \text{ and } [-1, 0] \subseteq E_q$$

This now implies

$$[-1, 1] \subseteq \bigcup_{k \geq n} E_k$$

Then because n is arbitrary, it follows

$$\limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} E_k = [-1, 1]$$

Again, fix arbitrary $n \in \mathbb{N}$ and $\epsilon > 0$. Let p, q respectively be even and odd integers greater than $\max\{n, \frac{1}{\epsilon}\}$. We now see

$$\epsilon \notin [-1, \frac{1}{p}] = E_p \text{ and } -\epsilon \notin [\frac{-1}{q}, 1] = E_q$$

Because ϵ is arbitrary and clearly $0 \in E_k$ for all k, we now see

$$\bigcap_{k\geq n} E_k = \{0\}$$

Then because n is arbitrary, we see

$$\liminf_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} \bigcap_{k\geq n} E_k = \{0\}$$

■

Question 46

Show that

$$(\limsup_{n\to\infty} E_n)^c = \liminf_{n\to\infty} (E_n)^c$$

and

$$E_n \searrow E \text{ or } E_n \nearrow E \implies \limsup_{n\to\infty} E_n = \liminf_{n\to\infty} E_n = E$$

Proof. Fix arbitrary $x \in (\limsup_{n\to\infty} E_n)^c$. We can deduce

$$\exists n, x \notin \bigcup_{k\geq n} E_k$$

This implies

$$\exists n, x \in \bigcap_{k\geq n} E_k^c$$

Then we see

$$x \in \bigcup_{n=1}^{\infty} \bigcap_{k\geq n} E_k^c = \liminf_{n\to\infty} E_n^c$$

We have proved $(\limsup_{n\to\infty} E_n)^c \subseteq \liminf_{n\to\infty} E_n^c$. We now prove the converse. Fix arbitrary $x \in \liminf_{n\to\infty} E_n^c$. We can deduce

$$\exists n, x \in \bigcap_{k\geq n} E_k^c$$

This implies

$$\exists n, x \notin \bigcup_{k\geq n} E_k$$

Then we see

$$x \notin \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} E_k = \limsup_{n\to\infty} E_n$$

■

Theorem 2.2.2. (Equivalent Definition for Limit Superior) If we let E be the set of subsequential limits of a_n

$$E \triangleq \{L \in \overline{\mathbb{R}} : L = \lim_{k\to\infty} a_{n_k} \text{ for some } n_k\}$$

The set E is non-empty and

$$\max E = \limsup_{n\to\infty} a_n$$

Proof. Let $n_1 \triangleq 1$. Recursively, because

$$\sup_{j\geq n_k} a_k \geq \limsup_{n\to\infty} a_n > \limsup_{n\to\infty} a_n - \frac{1}{k} \text{ for each } k$$

We can let n_{k+1} be the smallest number such that

$$a_{n_{k+1}} > \limsup_{n\to\infty} a_n - \frac{1}{k}$$

It is straightforward to check $a_{n_k} \to \limsup_{n\to\infty} a_n$ as $k \to \infty$. Note that no subsequence can converge to $\limsup_{n\to\infty} a_n + \epsilon$ because there exists N such that $\sup_{k\geq N} a_k < \limsup_{n\to\infty} a_n + \epsilon$. ∎

Question 47

Show that

$$\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n$$

Proof. Note that $-a_{n_k}$ converge if and only if a_{n_k} converge. Then if we respectively define E and E^- to be the set of subsequential limits of a_n and $-a_n$, we see

$$E^- = \{-L \in \mathbb{R} : L \in E\}$$

We now see

$$\limsup_{n\to\infty}(-a_n) = \max E^- = -\min E = -\liminf_{n\to\infty} a_n$$

∎

Question 48

Show that

$$\limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \tag{2.8}$$

Proof. Fix arbitrary ϵ. Let N_a, N_b respectively satisfy

$$\sup_{n\geq N_a} a_n \leq \limsup_{n\to\infty} a_n + \frac{\epsilon}{2} \text{ and } \sup_{n\geq N_b} b_n \leq \limsup_{n\to\infty} b_n + \frac{\epsilon}{2}$$

Let $N \triangleq \max\{N_a, N_b\}$. We now see that

$$\limsup_{n\to\infty}(a_n+b_n) \leq \sup_{n\geq N}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n + \epsilon$$

The result then follows from ϵ being arbitrary. ■

Question 49

$$a_n, b_n \text{ is bounded non-negative } \implies \limsup_{n\to\infty}(a_nb_n) \leq (\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n) \tag{2.9}$$

Proof. There are three cases we should consider

(a) Both $\limsup_{n\to\infty} a_n$ and $\limsup_{n\to\infty} b_n$ equal 0.

(b) Between $\limsup_{n\to\infty} a_n$ and $\limsup_{n\to\infty} b_n$, only one of them equals 0.

(c) Neither $\limsup_{n\to\infty} a_n$ nor $\limsup_{n\to\infty} b_n$ equals to 0.

In the first case, because a_n, b_n are both non-negative, we can deduce

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$$

which implies

$$\limsup_{n\to\infty}(a_nb_n) = \lim_{n\to\infty} a_nb_n = 0 = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$$

For second case, WOLG, suppose $\limsup_{n\to\infty} a_n = 0$. Fix arbitrary ϵ. We can let N satisfy

$$\sup_{n\geq N} a_n < \frac{\epsilon}{\sup_{n\in\mathbb{N}} b_n}$$

Since for all $n \geq N$, we have

$$a_n b_n \leq \frac{b_n \epsilon}{\sup_{k \in \mathbb{N}} b_k} \leq \epsilon$$

We now see

$$\limsup_{n\to\infty}(a_n b_n) \leq \sup_{n \geq N} a_n b_n \leq \epsilon$$

The result

$$\limsup_{n\to\infty} a_n b_n = 0 = \limsup_{n\to\infty} a_n \limsup_{n\to\infty} b_n$$

then follows from ϵ being arbitrary.

Lastly, for the last case, let N_a, N_b respectively satisfy

$$\sup_{n \geq N_a} a_n \leq \limsup_{n\to\infty} a_n \sqrt{1+\epsilon} \text{ and } \sup_{n \geq N_b} b_n \leq \limsup_{n\to\infty} b_n \sqrt{1+\epsilon}$$

Let $N \triangleq \max\{N_a, N_b\}$, because for each $n \geq N$, we have

$$a_n b_n \leq (\sup_{k \geq N_a} a_k)(\sup_{k \geq N_b} b_k) \leq (1+\epsilon)(\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n)$$

It then follows that

$$\limsup_{n\to\infty}(a_n b_n) \leq \sup_{n \geq N}(a_n b_n) \leq (1+\epsilon)(\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n)$$

The result then follows from ϵ being arbitrary. ∎

Question 50

Show that if either a_n or b_n converge, the equalities in Equation 2.8 and Equation 2.9 both hold true.

Proof. WOLG, suppose $\lim_{n\to\infty} a_n = L \in \mathbb{R}$. We then see

$$(a_{n_k} + b_{n_k}) \text{ converge } \iff b_{n,k} \text{ converge}$$

Let $E_{a,b}$ and E_b respectively be the set of subsequential limits of $(a_n + b_n)$ and b_n. We now have

$$E_{a,b} = \{L + L_b \in \mathbb{R} : L_b \in E_b\}$$

This give us

$$\limsup_{n\to\infty}(a_n+b_n)=\max E_{a,b}=L+\max E_b=\limsup_{n\to\infty}a_n+\limsup_{n\to\infty}b_n$$

Now, additionally, suppose a_n, b_n are both bounded and nonnegative. Again because

$$a_{n_k}b_{n,k} \text{ converge } \iff b_{n,k} \text{ converge}$$

We see

$$E_{a,b}=\{L(L_b)\in\mathbb{R}:L_b\in E_b\}$$

This give us

$$\limsup_{n\to\infty}(a_nb_n)=\max E_{a,b}=L\max E_b=(\limsup_{n\to\infty}a_n)(\limsup_{n\to\infty}b_n)$$

■

Question 51

Give example for which inequality in Equation 2.8 and Equation 2.9 are not equalities.

Proof. If

$$a_n\triangleq\begin{cases}1 & \text{if } n \text{ is odd}\\ -1 & \text{if } n \text{ is even}\end{cases} \text{ and } b_n\triangleq\begin{cases}-1 & \text{if } n \text{ is odd}\\ 1 & \text{if } n \text{ is even}\end{cases}$$

we have

$$\limsup_{n\to\infty}(a_n+b_n)=0<2=\limsup_{n\to\infty}a_n+\limsup_{n\to\infty}b_n$$

Let $L>1$ and

$$a_n\triangleq\begin{cases}L-\frac{1}{k} & \text{if } n=2k-1\\ (L-\frac{1}{k})^{-1} & \text{if } n=2k\end{cases} \text{ and } b_n\triangleq\begin{cases}(L-\frac{1}{k})^{-1} & \text{if } n=2k-1\\ (L-\frac{1}{k}) & \text{if } n=2k\end{cases}$$

We have

$$\limsup_{n\to\infty}a_nb_n=1<L^2=\limsup_{n\to\infty}a_n\limsup_{n\to\infty}b_n$$

■

Question 52

Give an example of a decreasing sequence of nonempty closed sets in $\mathbb{R}^n$ whose intersection is empty.

Proof.

$$F_n \triangleq [n, \infty) \text{ suffices}$$

■

Question 53

Given an example of two disjoint, nonempty closed sets in E_1 and E_2 in $\mathbb{R}^n$ for which $d(E_1, E_2) = 0$.

Proof. Let

$$E_1 \triangleq \{n - \frac{1}{n} \in \mathbb{R} : n \in \mathbb{N} \text{ and } n \geq 2\} \text{ and } E_2 \triangleq \{n - \frac{1}{2n} \in \mathbb{R} : n \in \mathbb{N} \text{ and } n \geq 2\}$$

To see $E_1 \cap E_2 = \varnothing$, suppose $n - \frac{1}{n} = k - \frac{1}{2k}$ where n, k are two natural numbers greater than 2. We then see $\frac{1}{n} - \frac{1}{2k} = n - k$, which is impossible, since

$$\left| \frac{1}{n} - \frac{1}{2k} \right| < \max\{\frac{1}{2k}, \frac{1}{n}\} < 1$$

The fact E_1, E_2 are closed follows from both of them being totally disconnected. Now observe that for all ϵ, there exists large enough n such that

$$(n + \frac{1}{n}) - (n + \frac{1}{2n}) < \frac{1}{n} < \epsilon$$

This implies $d(E_1, E_2) = 0$. ■

Question 54

If f is defined and uniformly continuous on E, show there is a function $\overline{f}$ defined and continuous on $\overline{E}$ such that $\overline{f} = f$ on E.

Proof. Define $\overline{f}$ on E by $\overline{f} = f$. For each $x \in \overline{E} \setminus E$, associate x with a sequence $t_{n,x}$ in E converging to x. We now claim that for each $x \in \overline{E} \setminus E$ the limit

$$\lim_{n \to \infty} f(t_{n,x}) \text{ converge in } \mathbb{R}$$

Fix ϵ. Because f is uniformly continuous on E, we know there exists δ such that

$$a, b \in E \text{ and } |a-b| \leq \delta \implies |f(a) - f(b)| < \epsilon$$

Because $t_{n,x}$ converge, we know $t_{n,x}$ is Cauchy, then we know there exists N such that $|t_{n,x} - t_{m,x}| < \delta$ for all $n, m > N$, we then see that for all $n, m > N$, we have

$$|f(t_{n,x}) - f(t_{m,x})| < \epsilon$$

This implies $\{f(t_{n,x})\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, thus converge in $\mathbb{R}$.

Define

$$\overline{f}(x) \triangleq \lim_{n\to\infty} f(t_{n,x}) \text{ for all } x \in \overline{E} \setminus E$$

We are required to show $\overline{f}$ is also continuous on $\overline{E} \setminus E$. Fix ϵ and $x \in \overline{E} \setminus E$. Let δ satisfy

$$a, b \in E \text{ and } |a-b| \leq \delta \implies |f(a) - f(b)| < \frac{\epsilon}{3}$$

We claim

$$\sup_{t \in B_{\frac{\delta}{2}}(x) \cap \overline{E}} \left|\overline{f}(t) - \overline{f}(x)\right| \leq \epsilon$$

Fix $t \in B_{\frac{\delta}{2}}(x) \cap \overline{E}$. There are two possibilities

(a) $t \in E$

(b) $t \in \overline{E} \setminus E$

If $t \in E$, let n satisfy

$$\left|f(t_{n,x}) - \overline{f}(x)\right| < \frac{\epsilon}{3} \text{ and } |t_{n,x} - x| < \frac{\delta}{2}$$

Because

$$|t_{n,x} - t| \leq |t_{n,x} - x| + |t - x| < \delta$$

we can deduce $|f(t_{n,x}) - f(t)| < \frac{\epsilon}{3}$. This now give us

$$\left|f(t) - \overline{f}(x)\right| \leq |f(t_{n,x}) - f(t)| + \left|f(t_{n,x}) - \overline{f}(x)\right| < \epsilon$$

If $t \in \overline{E} \setminus E$. Write $y = t$ and let $t_{n,y}$ be the associated sequence in E. Because $y \in B_{\frac{\delta}{2}}(x)$, we know there exists $t_{n,y}$ such that

$$t_{n,y} \in B_{\frac{\delta}{2}}(x) \text{ and } \left|f(t_{n,y}) - \overline{f}(y)\right| < \frac{\epsilon}{3}$$

Again, let m satisfy

$$t_{m,x} \in B_{\frac{\delta}{2}}(x) \text{ and } \left|f(t_{m,x}) - \overline{f}(x)\right| < \frac{\epsilon}{3}$$

We know $|t_{n,y} - t_{m,x}| \leq \delta$ because they both belong to $B_{\frac{\delta}{2}}(x)$. We can now deduce

$$\left|\overline{f}(y) - \overline{f}(x)\right| = \left|\overline{f}(y) - f(t_{n,y})\right| + |f(t_{n,y}) - f(t_{m,x})| + \left|f(t_{m,x}) - \overline{f}(x)\right| < \epsilon$$

which finish the proof. ■

Question 55

If f is defined and uniformly continuous on a bounded set E, show that f is bounded on E.

Proof. By last question, we can extend f to a continuous $\overline{f}$ onto $\overline{E}$. Now because $\overline{E}$ is compact and $\left|\overline{f}\right|$ is continuous on $\overline{E}$, by EVT, there exists $a \in \overline{E}$ such that

$$\sup_{x \in E} |f(x)| \leq \max_{x \in \overline{E}} |f(x)| = f(a)$$

■

2.3 HW2

Question 56

Construct a two-dimensional Cantor set in the unit square $[0,1]^2$ as follows. Subdivide the square into nine equal parts and keep only the four closed corner squares, removing the remaining region (which form a cross). Then repeat this process in a suitably scaled version for the remaining squares, ad infinitum. Show that the resulting set is perfect, has plane measure zero, and equals $\mathcal{C} \times \mathcal{C}$.

Proof. Let $\mathcal{C}'_n \subseteq \mathbb{R}^2$ be the result after the nth stage of removal, and let $\mathcal{C}_n \subseteq \mathbb{R}$ be the result after the nth stage of removal in the construction of the classical ternary Cantor set. It is clear that

$$\mathcal{C}'_n = \mathcal{C}_n \times \mathcal{C}_n \text{ for all } n$$

It then follows

$$\bigcap_n \mathcal{C}'_n = \bigcap_n \mathcal{C}_n \times \mathcal{C}_n = \mathcal{C} \times \mathcal{C}$$

The fact that $\mathcal{C} \times \mathcal{C}$ has plane measure zero follows from Lemma 2.3.1. Fix $(a,b) \in \mathcal{C} \times \mathcal{C}$. Because $\mathcal{C}$ is perfect, there exists some $b' \in \mathcal{C}$ such that

$$0 < |b' - b| < \epsilon$$

To see that $\mathcal{C}'$ is perfect, one see that

$$(a,b) \neq (a,b') \text{ and } (a,b') \in \mathcal{C}' \times \mathcal{C}' \text{ and } |(a,b) - (a,b')| = |b' - b| < \epsilon$$

∎

Question 57

Construct a subset of $[0,1]$ in the same manner as the Cantor set, except that at the kth stage, each interval removed has length $\delta 3^{-k}, 0 < \delta < 1$. Show that the resulting set is perfect, has measure $1 - \delta$, and contains no intervals.

Proof. Let $\mathcal{C}'_n \subseteq \mathbb{R}$ be the result after the nth stage of removal according to the description. Clearly, each $\mathcal{C}'_n$ has 2^n amount of connected component, we then can compute the length of $\mathcal{C}' \triangleq \bigcap \mathcal{C}'_n$ to be

$$1 - \sum_{k=1}^{\infty} 2^{k-1} \delta 3^{-k} = 1 - \frac{\frac{\delta}{3}}{1 - \frac{2}{3}} = 1 - \delta$$

Since each $\mathcal{C}'_n$ has 2^n amount of connected component of equal length and $\mathcal{C}'_n \subseteq [0,1]$, we know the length of each connected component of $\mathcal{C}'_n$ must not be greater than $\frac{1}{2^n}$. It then follows that no interval $[a, a+h]$ can be contained by all $\mathcal{C}'_n$ because if $[a, a+h]$ is a subset of some connected component of $\mathcal{C}'_k$ of some k, then the measure $h = |[a, a+h]|$ must be smaller than $\frac{1}{2^k}$, which is false when k is large enough. ■

Question 58

If E_k is a sequence of sets with $\sum |E_k|_e < \infty$, show that $\limsup_{n\to\infty} E_n$ has measure zero.

Proof. Note that

$$\sum_{k=N}^{\infty} |E_k|_e = \left(\sum_{k=1}^{\infty} |E_k|_e - \sum_{k=1}^{N-1} |E_k|_e \right) \to 0 \text{ as } N \to \infty$$

and note for all N we have

$$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \subseteq \bigcup_{k=N}^{\infty} E_k$$

Then for arbitrary ϵ, if we let N satisfy $\sum_{k=N}^{\infty} |E_k|_e < \epsilon$, we see that

$$\left| \limsup_{n\to\infty} E_n \right|_e = \left| \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \right|_e \le \left| \bigcup_{k=N}^{\infty} E_k \right|_e \le \sum_{k=N}^{\infty} |E_k|_e < \epsilon$$

■

Question 59

If E_1, E_2 are measurable, show that

$$|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|$$

Proof. Observe the following expression of each set in disjoint union

(a) $E_1 = (E_1 \setminus E_2) \sqcup (E_1 \cap E_2)$

(b) $E_2 = (E_2 \setminus E_1) \sqcup (E_1 \cap E_2)$

(c) $E_1 \cup E_2 = (E_1 \setminus E_2) \sqcup (E_1 \cap E_2) \sqcup (E_2 \setminus E_1)$

It now follows

$$
\begin{aligned}
|E_1 \cup E_2| + |E_1 \cap E_2| &= |E_1 \setminus E_2| + |E_1 \cap E_2| + |E_1 \cap E_2| + |E_2 \setminus E_1| \\
&= |E_1| + |E_2|
\end{aligned}
$$

∎

Lemma 2.3.1. Given two subsets Z_1, Z_2 of $\mathbb{R}$, if $|Z_1| = 0$, then $|Z_1 \times Z_2| = 0$.

Proof. Let $A_n \triangleq [n, n+1)$. Because

$$
Z_1 \times Z_2 = \bigsqcup_{n \in \mathbb{Z}} Z_1 \times (A_n \cap Z_2)
$$

To show $|Z_1 \times Z_2| = 0$, we only have to $|Z_1 \times (A_n \cap Z_2)| = 0$ for all $n \in \mathbb{Z}$. In other words, we can WOLG suppose Z_2 is bounded.

Now, fix ϵ. We are required to find an countable closed cube cover $Q_n \times C_n$ for $Z_1 \times Z_2$ such that $\sum_n |Q_n \times C_n| < \epsilon$. Let $C_n = C$ for all n where C is a compact interval containing Z_2, and let Q_n be a countable compact interval cover for Z_1 such that $\sum |Q_n| < \frac{\epsilon}{|C|}$. It then follows $\sum_n |Q_n \times C_n| = \sum_n |Q_n| \, |C| < \epsilon$. ∎

Theorem 2.3.2. (Product of Finite Measure Set) If E_1 and E_2 are measurable subset of $\mathbb{R}$ and $|E_1|, |E_2| < \infty$, then $E_1 \times E_2$ is measurable in $\mathbb{R}^2$ and

$$
|E_1 \times E_2| = |E_1| \, |E_2|
$$

Proof. Write $E_1 \triangleq H_1 \sqcup Z_1$ and $E_2 \triangleq H_2 \sqcup Z_2$ where $H_1, H_2 \in F_\sigma$ and $|H_1| = |E_1|$ and $|H_2| = |E_2|$. Now observe

$$
E_1 \times E_2 = (H_1 \times H_2) \cup (Z_1 \times E_2) \cup (E_1 \times Z_2)
$$

Note that if we write $H_1 = \bigcap F_{1,n}$ and $H_2 = \bigcap F_{2,n}$, we see $H_1 \times H_2 = \bigcap F_{1,n} \times F_{2,n} \in F_\sigma$ in $\mathbb{R}^2$, it now follows from Lemma 2.3.1 that $E_1 \times E_2$ is measurable.

Now, let S_n be a decreasing sequence of open set containing E_1 such that $|S_n \setminus E_1| < \frac{1}{n}$, and let T_n be a decreasing sequence of open set containing E_2 such that $|T_n \setminus E_2| < \frac{1}{n}$. In other words,

$$
E_1 = S \setminus Z_1 \text{ and } E_2 = T \setminus Z_2 \text{ where } \begin{cases} S \triangleq \bigcap S_n \\ T \triangleq \bigcap T_n \\ |Z_1| = |Z_2| = 0 \end{cases}
$$

We now have

$$S \times T = (E_1 \times E_2) \cup (Z_1 \times E_2) \cup (E_1 \times Z_2)$$

This then implies $|S \times T| = |S \times T|_e \leq |E_1 \times E_2|_e + |Z_1 \times E_2|_e + |E_1 \times Z_2|_e = |E_1 \times E_2|$, where the last inequality follows from Lemma 2.3.1. The reverse inequality is clear, since $E_1 \times E_2 \subseteq S \times T$. We have proved $|E_1 \times E_2| = |S \times T|$.

Now, for each n, write

$$S_n = \bigcup_{k \in \mathbb{N}} I_{k,S_n} \text{ and } T_n = \bigcup_{k \in \mathbb{N}} I_{k,T_n}$$

where $(I_{k,S_n})_k$ and $(I_{k,T_n})_k$ are non-overlapping compact interval. It then follows that

$$|S_n \times T_n| = \sum_{i,j} |I_{i,S_n} \times I_{j,T_n}| = \sum_{i,j} |I_{i,S_n}| \times |I_{j,T_n}| = \sum_i |I_{i,S_n}| \sum_j |I_{j,T_n}| = |S_n| \, |T_n|$$

Write $S \triangleq \bigcap_{n \in \mathbb{N}} S_n$ and $T \triangleq \bigcap_{n \in \mathbb{N}} T_n$. Because

(a) Each $S_n \times T_n$ is open.

(b) $|S_n \times T_n| = |S_n| \, |T_n|$ is bounded ($\because |S_n| \searrow |E_1| < \infty$).

(c) $S_n \times T_n \searrow S \times T$

We can now deduce

$$\begin{aligned} |E_1 \times E_2| = |S \times T| &= \lim_{n \to \infty} |S_n \times T_n| \\ &= \lim_{n \to \infty} |S_n| \, |T_n| \\ &= |E_1| \, |E_2| \end{aligned}$$

■

Question 60

If E_1 and E_2 are measurable subset of $\mathbb{R}$, then $E_1 \times E_2$ is a measurable subset of $\mathbb{R}^2$ and $|E_1 \times E_2| = |E_1| \, |E_2|$

Proof. Define

$$A_n \triangleq \{x \in \mathbb{R} : n \leq x < n + 1\}$$

Because

$$E_1 = \bigcup_{n\in\mathbb{Z}} E_1 \cap A_n \text{ and } E_2 = \bigcup_{k\in\mathbb{Z}} E_2 \cap A_k$$

We can deduce

$$E_1 \times E_2 = \bigcup_{n,k\in\mathbb{Z}} (E_1 \cap A_n) \times (E_2 \cap A_k)$$

Note that Theorem 2.3.2 tell us $(E_1 \cap A_n) \times (E_2 \cap A_k)$ is measurable, which implies $E_1 \times E_2$ is measurable. Theorem 2.3.2 also tell us $|(E_1 \cap A_n) \times (E_2 \cap A_k)| = |E_1 \cap A_n| \, |E_2 \cap A_k|$, which allow us to deduce

$$\begin{aligned}|E_1 \times E_2| = \sum_{n,k\in\mathbb{Z}} |(E_1 \cap A_n) \times (E_2 \cap A_k)| &= \sum_{n,k\in\mathbb{Z}} |E_1 \cap A_n| \, |E_2 \cap A_k| \\ &= \sum_{n\in\mathbb{Z}} |E_1 \cap A_n| \sum_{k\in\mathbb{Z}} |E_2 \cap A_k| = |E_1| \, |E_2|\end{aligned}$$

■

Question 61

Give an example that shows that the image of a measurable set under a continuous transformation may not be measurable. See also Exercise 10 of Chapter 7.

Proof. Consider the Cantor-Lebesgue function denoted by $f : [0,1] \to [0,1]$ and denote the classical ternary Cantor set by $\mathcal{C}$. Let V be a Vitali set contained by $[0,1]$. Because $f(\mathcal{C}) = [0,1]$, we know there exists $E \subseteq \mathcal{C}$ such that $f(E) = V$. Such E is measurable since $|E|_e \leq |\mathcal{C}| = 0$, yet its continuous image $V = f(E)$ is by definition non-measurable. ■

Question 62

Show that there exists disjoint $E_1, E_2, \dots$ such that $|\bigcup E_k|_e < \sum |E_k|_e$ with strict inequality.

Proof. Let V be a Vitali Set contained by $[0,1]$. Enumerate $[0,1] \cap \mathbb{Q}$ by x_n. Define

$$E_n \triangleq \{v + x_n \in \mathbb{R} : v \in V\} \text{ for all } n$$

The sequence E_n is disjoint, since if $p \in E_n \cap E_m$, then there exists some pair v_n, v_m belong to V such that

$$v_n + x_n = p = v_m + x_m \tag{2.10}$$

which is impossible, since Equation 2.10 implies that $v_n \neq v_m$ and v_n, v_m are of difference of a rational number.

Now, note that for arbitrary n and $v \in V$, because $v \in V \subseteq [0,1]$ and $x_n \in [0,1]$, we have $v + x_n \in [0,2]$. This implies

$$\bigsqcup_n E_n \subseteq [0,2] \text{ and } \left| \bigsqcup_n E_n \right|_e \leq 2$$

Because V is non-measurable by definition, we know $|V|_e > 0$, and since outer measure is translation invariant, we can now deduce

$$\sum_n |E_n|_e = \sum_n |V|_e = \infty > 2 \geq \left| \bigsqcup_n E_n \right|_e$$

■

Question 63

Show that there exists decreasing sequence E_k of sets such that

(a) $E_k \searrow E$.

(b) $|E_k|_e < \infty$.

(c) $\lim_{k\to\infty} |E_k|_e > |E|_e$

Proof. Let V be a Vitali Set contained by $[0,1]$. Enumerate $[0,1] \cap \mathbb{Q}$ by x_n. Define

$$V + x_n \triangleq \{v + x_n \in \mathbb{R} : v \in V\}$$

In last question, we have proved that $V + x_n$ is pairwise disjoint. Define for all $n \in \mathbb{N}$

$$E_n \triangleq \bigsqcup_{k \geq n} V + x_k$$

Observe that

$$E_k \searrow \bigcap E_n = \varnothing$$

which implies $|\bigcap E_n|_e = 0$, but

$$\lim_{n\to\infty} |E_n|_e = \lim_{n\to\infty} \left| \bigsqcup_{k\geq n} V + x_k \right| \geq \lim_{n\to\infty} |V + x_n| = |V| > 0$$

■

Question 64

Let Z be a subset of $\mathbb{R}$ with measure zero. Show that the set $\{x^2 : x \in Z\}$ also has measure zero.

Proof. Fix $Z_n \triangleq Z \cap [-n, n]$. Since

$$\left|\{x^2 : x \in Z\}\right| \leq \sum_{n=1}^{\infty} \left|\{x^2 : x \in Z_n\}\right|_e$$

We only have to prove

$$\left|\{x^2 : x \in Z_n\}\right|_e = 0 \text{ for all } n$$

Fix ϵ, n. Let I_k be a compact interval cover of Z_n such that $\sum |I_k| < \frac{\epsilon}{2n}$. We shall suppose $I_k \subseteq [-n, n]$, since if not, we can just let $I'_k \triangleq I_k \cap [-n, n]$.

Now, define

$$I_k^2 \triangleq \{x^2 : x \in I_k\}$$

Clearly, I_k^2 are all compact intervals, and if we write $I_k \triangleq [a_k, b_k]$, we have the following inequalities

$$\begin{cases} 0 \leq a_k \leq b_k \implies \left|I_k^2\right| = b_k^2 - a_k^2 = |I_k|\,(b_k + a_k) \leq 2n\,|I_k| \\ a_k \leq 0 \leq b_k \implies \left|I_k^2\right| = \max\{a_k^2, b_k^2\} \leq (b_k - a_k)^2 = |I_k|\,(b_k - a_k) \leq 2n\,|I_k| \\ a_k \leq b_k \leq 0 \implies \left|I_k^2\right| = a_k^2 - b_k^2 = |I_k|\,(-a_k - b_k) \leq 2n\,|I_k| \end{cases}$$

Note that $\{I_k^2\}_{k \in \mathbb{N}}$ is a compact interval cover of $\{x^2 : x \in Z_n\}$, we now see

$$\left|\{x^2 : x \in Z_n\}\right|_e \leq \sum_k \left|I_k^2\right| \leq 2n \sum |I_k| < \epsilon$$

$\blacksquare$

2.4 HW3

Question 65

Let f be a simple function, taking its distinct values on disjoint sets $E_1, \dots, E_N$. Show that f is measurable if and only if $E_1, \dots, E_N$ are measurable.

Proof. WOLG, let f take value a_n on E_n and

$$a_1 < a_2 < \cdots < a_N$$

If $E_1, \dots, E_N$ are all measurable, we see that for each $a \in \mathbb{R}$

$$\{f \geq a\} = \{f \geq a_n\} = E_n \sqcup \cdots \sqcup E_N \text{ is measurable}$$

where n is the smallest integer such that $a_n \geq a$. We have prove the if part. To see the only if part hold true, observe that for all $n \in \{1, \dots, N-1\}$

$$E_n = \{f \geq a_n\} \setminus \{f \geq a_{n+1}\} \text{ is measurable}$$

and

$$E_N = \{f \geq a_N\} \text{ is measurable}$$

■

Question 66

Let f be defined and measurable on $\mathbb{R}^n$. If T is a nonsingular linear transformation of $\mathbb{R}^n$, show that $f(T\mathbf{x})$ is measurable. (If $E_1 = \{\mathbf{x} : f(\mathbf{x}) > a\}$, and $E_2 = \{\mathbf{x} : f(T\mathbf{x}) > a\}$, show that $E_2 = T^{-1}E_1$)

Proof. Fix $a \in \mathbb{R}$. We are required to show

$$\{\mathbf{x} : f(T\mathbf{x}) > a\} \text{ is measurable}$$

Because f is measurable, we know $\{\mathbf{x} : f(\mathbf{x}) > a\}$ is measurable. The proof then follows from noting

$$\{\mathbf{x} : f(T\mathbf{x}) > a\} = T^{-1}\Big(\{\mathbf{x} : f(\mathbf{x}) > a\}\Big)$$

and the fact that $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ as a linear transformation preserve measurability. ■

Question 67

Give an example to show that $\varphi \circ f$ may not be measurable if $\varphi, f : \mathbb{R} \to \mathbb{R}$ are measurable and finite. (Let F be the Cantor-Lebesgue function and let f be its inverse suitably defined. Let φ be the characteristic function of a set of measure zero whose image under F is not measurable.) Show that the same may be true even if f is continuous. (Let $g(x) = x + F(x)$ and consider $f = g^{-1}$)

Proof. Let $F : [0,1] \to [0,1]$ be the Cantor-Lebesgue function, $\mathcal{C} \subseteq [0,1]$ be the classical ternary Cantor set. Note that $F(\mathcal{C}) = [0,1]$. By axiom of choice, we can let $\mathcal{C}'$ be some subset of $\mathcal{C}$ such that $F|_{\mathcal{C}'} : \mathcal{C}' \to [0,1]$ is a bijection. We can now define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) \triangleq \begin{cases} (F|_{\mathcal{C}'})^{-1}(x) & \text{if } x \in [0,1] \\ x & \text{if } x \notin [0,1] \end{cases}$$

$f : \mathbb{R} \to \mathbb{R}$ is measurable because f is increasing. Let V be a non-measurable set contained by $[0,1]$, and let $E \triangleq f(V)$. Define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(x) \triangleq \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$$

Note that E is measurable because

$$V \subseteq [0,1] \implies E = f(V) = (F|_{\mathcal{C}'})^{-1}(V) \subseteq \mathcal{C}'$$

It then follows that $\varphi : \mathbb{R} \to \mathbb{R}$ is measurable. Lastly, to see $\varphi \circ f : \mathbb{R} \to \mathbb{R}$ is not measurable, observe that

$$(\varphi \circ f)^{-1}(\{1\}) = f^{-1}(E) = V \text{ is not measurable}$$

where the last inequality follows since $f|_V : V \to E$ is a bijection.

For the second part. Define $g : [0,1] \to [0,2]$ by

$$g(x) \triangleq x + F(x)$$

Because $F : [0,1] \to [0,1]$ is increasing, we may deduce

$$x < y \text{ and } x, y \in [0,1] \implies x + F(x) < y + F(y)$$

This implies g is strictly increasing. Note that g is continuous because g is the addition of two continuous function, and note that $g(0) = 0, g(1) = 2$. This allow us to deduce $g : [0,1] \to [0,2]$ is a bijection. Now, observe that $[0,1] \setminus \mathcal{C}$ is a countable union of disjoint open interval. For each connected components $I \subseteq [0,1] \setminus \mathcal{C}$, because F maps I to some constant, we see $g(I)$ is also an interval with the same length $|g(I)| = I$. Then from $|[0,1] \setminus \mathcal{C}| = 1$, we can deduce $|g([0,1] \setminus \mathcal{C})| = 1$, which implies $g(\mathcal{C}) = 1$. We then can let V be some non-measurable set contained by $g(\mathcal{C})$. Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) \triangleq \begin{cases} g^{-1}(x) & \text{if } x \in [0,2] \\ \frac{x}{2} & \text{if } x \notin [0,2] \end{cases}$$

$h : \mathbb{R} \to \mathbb{R}$ is measurable because it is increasing. Let $E \triangleq h(V)$. We see $E \subseteq \mathcal{C}$, which implies E is measurable, so when we define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(x) \triangleq \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

we see $\varphi : \mathbb{R} \to \mathbb{R}$ is also measurable. Lastly, to see $\varphi \circ h : \mathbb{R} \to \mathbb{R}$ is not measurable, observe

$$(\varphi \circ h)^{-1}(\{1\}) = h^{-1}(E) = V$$

■

Question 68

(a) Show that the limit of a decreasing (increasing) sequence of functions upper (lower) semicontinuous at $\mathbf{x}_0$ is upper (lower) semicontinuous at $\mathbf{x}_0$. In particular, the limit of a decreasing (increasing) sequence of functions continuous at $\mathbf{x}_0$ is upper (lower) semicontinuous at $\mathbf{x}_0$.

(b) Let f be upper semicontinuous and less than ∞ on $[a,b]$. Show that there exists continuous f_k on $[a,b]$ such that $f_k \searrow f$. (First show that there exist continuous f_k on $[a,b]$ such that $f_k \searrow f$)

Proof. **(a)** Let $f_n \searrow f$ and f_n be upper semicontinuous at $\mathbf{x}_0$. Fix ϵ. Because $f_n(\mathbf{x}_0) \searrow f(\mathbf{x}_0)$, we know there exists some N such that $f_N(\mathbf{x}_0) < f(\mathbf{x}_0) + \epsilon$. Because $f \leq f_N$ everywhere and f_N is upper semicontinuous at $\mathbf{x}_0$, we have

$$\limsup_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) \leq \limsup_{\mathbf{x} \to \mathbf{x}_0} f_N(\mathbf{x}) \leq f_N(\mathbf{x}_0) < f(\mathbf{x}_0) + \epsilon$$

Because ϵ can be arbitrary small, we have shown

$$\limsup_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) \leq f(\mathbf{x}_0)$$

i.e., f is also upper semicontinuous. The second part of question (a) ask us to prove the same thing with stronger hypothesis that f_n are continuous at $\mathbf{x}_0$, which implies f_n are upper semicontinuous at $\mathbf{x}_0$, so it immediately follows from what we have proved.

(b) Define $f_n : [a,b] \to [-\infty, \infty]$ by

$$f_n(x) \triangleq \sup_{p \in [a,b]} \Big(f(p) - n\,|x-p| \Big)$$

It is clear that $f_n : [a,b] \to [-\infty,\infty]$ is decreasing. Fix $x \in [a,b]$. We now show

$$f_n(x) \searrow f(x)$$

Fix ϵ. Because f is upper semi-continuous at x, there exists δ such that

$$\sup_{y \in [x-\delta, x+\delta]} f(y) \leq f(x) + \epsilon$$

Because f is upper semi-continuous on compact $[a,b]$, we know f is bounded. Let M bound $|f|$. Let $N \geq \frac{2M}{\delta}$. We see that if $y \in [x-\delta, x+\delta]$, then

$$f(y) - N\,|x-y| \leq f(y) \leq f(x) + \epsilon$$

and if $y \notin [x-\delta, x+\delta]$, then

$$f(y) - N\,|x-y| \leq f(y) - 2M \leq f(x) + \epsilon$$

Fix n. We now show $f_n : [a,b] \to [-\infty,\infty]$ is continuous. Fix ϵ, we claim

$$|x-y| < \frac{\epsilon}{n} \implies |f_n(x) - f_n(y)| \leq \epsilon$$

Observe

$$\left| \Big(f(p) - n\,|x-p| \Big) - \Big(f(p) - n\,|y-p| \Big) \right| = n \left| \Big(|x-p| - |y-p| \Big) \right| \leq n\,|x-y| < \epsilon$$

This implies

$$f(p) - n\,|x-p| + \epsilon > f(p) - n\,|y-p| > f(p) - n\,|x-p| - \epsilon$$

Taking supremum on both side, we have

$$f_n(x) + \epsilon = \sup_{p \in [a,b]} \Big(f(p) - n\,|x-p| \Big) + \epsilon \geq \sup_{p \in [a,b]} \Big(f(p) - n\,|y-p| \Big) = f_n(y)$$

In summary,

$$f_n(x) + \epsilon \geq f_n(y) \geq f_n(x) - \epsilon$$

$\blacksquare$

Question 69

(a) Shows $f_n \nearrow f \implies \omega_{f_n} \nearrow \omega_f$.

(b) Shows that if ω_f is continuous at α and $f_n \xrightarrow{m} f$, then $\omega_{f_n}(\alpha) \to \omega_f(\alpha)$.

Proof. For **part (a)**, observe for all $\alpha \in \mathbb{R}$

$$\{f_n > \alpha\} \nearrow \{f > \alpha\}$$

This implies

$$\omega_{f_n}(a) = |\{f_n > a\}| \nearrow |\{f > a\}| = \omega_f(a)$$

For **part (b)**, fix ϵ and observe

$$\{f_n > \alpha\} \subseteq \{f > \alpha - \epsilon\} \cup \{|f_n - f| > \epsilon\}$$

This implies

$$\omega_{f_n}(\alpha) \leq \omega_f(\alpha - \epsilon) + |\{|f_n - f| > \epsilon\}|$$

Then because $|\{|f_n - f| > \epsilon\}| \to 0$, taking limit we have

$$\limsup_{n\to\infty} \omega_{f_n}(\alpha) \leq \omega_f(\alpha - \epsilon)$$

Again, observe

$$\{f_n > \alpha\} \cup \{|f_n - f| > \epsilon\} \supseteq \{f > \alpha + \epsilon\}$$

This implies

$$\omega_{f_n}(\alpha) + |\{|f_n - f| > \epsilon\}| \geq \omega_f(\alpha + \epsilon)$$

Then because $|\{|f_n - f| > \epsilon\}| \to 0$, taking limit we have

$$\liminf_{n\to\infty} \omega_{f_n}(\alpha) \geq \omega_f(\alpha + \epsilon)$$

■

Question 70

If f is measurable and finite almost everywhere on $[a, b]$, show that given $\epsilon > 0$, there is a continuous g on $[a, b]$ such that $|\{f \neq g\}| < \epsilon$. Formulate and prove a similar result in $\mathbb{R}^n$ by combining Lusin's Theorem with the Tietze extension Theorem.

Proof. Let $E \triangleq \{x \in [a,b] : f(x) \in \mathbb{R}\}$. E is measurable because f is measurable on $[a,b]$. It is clear that f is indeed measurable on E. By Lusin's Theorem, there exists some closed set $F \subseteq E$ such that $|E \setminus F| < \epsilon$ and $f|_F : F \to \mathbb{R}$ is continuous. Because F is compact, (bounded by $[a,b]$), Tietze extension Theorem give us some continuous $g : [a,b] \to \mathbb{R}$ such that $g = f$ on F. It then follows that

$$\{f \neq g\} \subseteq [a,b] \setminus F$$

which give us the desired estimation

$$|\{f \neq g\}| \leq (b-a) - |F| = |E| - |F| < \epsilon$$

We may formulate the same result by

> If f is measurable and finite almost everywhere on some compact $K \subseteq \mathbb{R}^d$, then for all ϵ, there exists continuous g on K such that $|\{f \neq g\}| < \epsilon$.

and give exactly the same argument to prove it. ■

2.5 HW4

Theorem 2.5.1. (Monotone Convergence Theorem) Let f_n be a sequence of measurable functions on E:

(a) If $f_n \nearrow f$ a.e. on E and there exists $\varphi \in L(E)$ such that $f_n \geq \varphi$ a.e. on E for all n, then $\int_E f_n \to \int_E f$.

(b) If $f_n \searrow f$ a.e. on E and there exists $\varphi \in L(E)$ such that $f_n \leq \varphi$ a.e. on E for all n, then $\int_E f_n \to \int_E f$.

Question 71

Show that Monotone convergence Theorem may fail if we drop the hypothesis that f_n is dominated by φ. Show that Uniform convergence Theorem may fail if we drop the hypothesis that domain is of finite measure.

Proof. For failure of first part of monotone convergence Theorem, define $f_n : (0, \infty) \to \mathbb{R}$ by

$$f_n(x) \triangleq \begin{cases} 1 & \text{if } x < n \\ -\infty & \text{if } x \geq n \end{cases}$$

Observe that $f_n \nearrow 1$ but $\int_0^\infty f_n = -\infty$ does not converge to $\int_0^\infty 1 = \infty$. For failure of the second part of monotone convergence Theorem, define $f_n : (0, \infty) \to \mathbb{R}$ by

$$f_n \triangleq \mathbf{1}_{(n,\infty)}$$

Observe that $f_n \searrow 0$ but $\int_0^\infty f_n = \infty$ does not converge to 0. For failure of uniform convergence Theorem, consider $\frac{1}{n} \to 0$ uniformly on $\mathbb{R}$ but

$$\int_{\mathbb{R}} \frac{1}{n} dx = \infty \text{ does not converge to } \int_{\mathbb{R}} 0 dx = 0$$

∎

Question 72

If $f \in L(0, 1)$, show that $x^n f(x) \in L(0, 1)$ for all $n \in \mathbb{N}$ and $\int_0^1 x^n f(x) dx \to 0$.

Proof. Because $f(x)$ and x^n are both measurable on $(0, 1)$, we know $x^n f(x)$ is measurable on $(0, 1)$. Because $0 < x < 1$, if $f(x)$ is finite, then $x^n f(x) \to 0$ as $n \to \infty$. Because f is integrable on $(0, 1)$, we know that f is finite almost everywhere on $(0, 1)$. We now see that

$x^n f(x)$ converge to 0 almost everywhere on $(0,1)$. Again because $0 < x < 1$, we see that $|x^n f(x)| \leq |f(x)|$. In other words, $x^n f(x)$ is dominated by $|f| \in L(0,1)$. We now can use dominated convergence Theorem to deduce $\int_0^1 x^n f(x) \to 0$. ■

Question 73

Let $f : (0,1)^2 \to \mathbb{R}$ satisfy

(a) $f(x,y)$ is always integrable in y on $(0,1)$.

(b) $\frac{\partial f}{\partial x}$ exists and is bounded on $(0,1)^2$.

Show that $\frac{\partial f}{\partial x}$ is a measurable function in y for all x and

$$\frac{d}{dx}\int_0^1 f(x,y)dy = \int_0^1 \frac{\partial}{\partial x} f(x,y)dy$$

Proof. For all $n \geq 2$, define $g_n : (0, 1-\frac{1}{n}) \times (0,1)$ by

$$g_n(x,y) \triangleq \frac{f(x+\frac{1}{n},y) - f(x,y)}{\frac{1}{n}}$$

It is clear that for all $n \geq 2$

$$\frac{\partial f}{\partial x}\Big|_{(0,1-\frac{1}{n})\times(0,1)} = \lim_{k\to\infty}\left(g_{n+k}\Big|_{(0,1-\frac{1}{n})\times(0,1)}\right)$$

This implies that

$$\frac{\partial f}{\partial x} \text{ is measurable on } (0,1-\frac{1}{n}) \times (0,1) \text{ for all } n \geq 2$$

It then follows that $\frac{\partial f}{\partial x}$ is measurable on $(0,1)^2$. Observe that for all $x_0 \in (0,1)$, we have

$$\{y \in (0,1) : \frac{\partial f}{\partial x}(x_0,y) > a\} = \{(x,y) \in (0,1)^2 : \frac{\partial f}{\partial x}(x,y) > a\} \cap (\{x_0\} \times (0,1))$$

It follows that $\frac{\partial f}{\partial x}$ is measurable in y for all x. Fix $x \in (0,1)$. By MVT, we know that for all y and h (small enough to make the following express make sense) we have

$$\frac{f(x+h,y) - f(x,y)}{h} = \frac{\partial f}{\partial x}(x+t,y) < M \text{ for some } t$$

where M is the constant that bounds $\frac{\partial f}{\partial x}$ on $(0,1)^2$. It then follows from DCT (dominated by M on $(0,1)$) that

$$\begin{aligned}
\frac{d}{dx}\int_0^1 f(x,y)dy &= \lim_{h\to 0}\frac{\int_0^1 f(x+h,y)dy - \int_0^1 f(x,y)dy}{h}\\
&= \lim_{h\to 0}\int_0^1 \frac{f(x+h,y)-f(x,y)}{h}dy\\
&= \int_0^1 \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}dy\\
&= \int_0^1 \frac{\partial f}{\partial x}(x,y)dy
\end{aligned}$$

■

Question 74

Suppose $p > 0$ and $\int_E |f - f_n|^p \to 0$. Show that $f_n \xrightarrow{m} f$ (and thus have a almost everywhere convergent subsequence).

Proof. Fix ϵ. The proof follows from Tchebysheff's inequality,

$$\left|\left\{\,|f_n - f| > \epsilon\right\}\right| = \left|\left\{\,|f_n - f|^p > \epsilon^p\right\}\right| \geq \frac{\int_E |f - f_n|^p}{\epsilon^p} \to 0$$

■

Question 75

If $p > 0$, $\int_E |f - f_n|^p \to 0$ and $\int_E |f_n|^p \leq M$ for all n, show that $\int_E |f|^p \leq M$.

Proof. By the last question, there exist some subsequence f_{n_k} converge to f almost everywhere. It then follows that $|f_{n_k}|^p \to |f|^p$ almost everywhere. We then have from Fatou's Lemma that

$$\int_E |f|^p \leq \liminf_{k\to\infty}\int_E |f_{n_k}|^p \leq M$$

■

Question 76

Give an example of a bounded continuous f on $(0,\infty)$ such that $\lim_{x\to\infty} f(x) = 0$ but $f \notin L^p((0,\infty))$.

Proof. Let $f(x) \triangleq \frac{1}{\ln(x+2)}$. It is clear that f is bounded continuous on $(0, \infty)$ and converge to 0 at infinity. Note that doing a change of variables $t = \ln(x+2)$, we have $dt = \frac{dx}{x+2} = \frac{dx}{e^t}$. Then for all $p > 0$, we have

$$\begin{aligned}\int_0^\infty f^p(x)dx &= \int_0^\infty \frac{1}{(\ln(x+2))^p}dx \\ &= \int_{\ln 2}^\infty \frac{e^t}{t^p}dt\end{aligned}$$

which diverge since the integrand itself converge to ∞. ■

Question 77

If $\int_A f = 0$ for every measurable subset A of measurable set E, show that $f = 0$ almost everywhere on E.

Proof. Observe that for all $n \in \mathbb{N}$

$$0 = \int_{\{f > \frac{1}{n}\}} f dx \geq \int_{\{f > \frac{1}{n}\}} \frac{1}{n} dx = \frac{\left|\{f > \frac{1}{n}\}\right|}{n}$$

This implies that $\left|\{f > \frac{1}{n}\}\right| = 0$ for all $n \in \mathbb{N}$. Again observe for all $n \in \mathbb{N}$

$$0 = \int_{\{f < \frac{-1}{n}\}} f dx \leq \int_{\{f < \frac{-1}{n}\}} \frac{-1}{n} dx = \frac{-\left|\{f < \frac{-1}{n}\}\right|}{n}$$

This implies that $\left|\{f < \frac{-1}{n}\}\right| = 0$ for all $n \in \mathbb{N}$. It then follows from $\{f > \frac{1}{n}\} \cup \{f < \frac{-1}{n}\} \nearrow \{f \neq 0\}$ that $|\{f \neq 0\}| = 0$, i.e., $f = 0$ almost everywhere on E. ■

2.6 HW5

Question 78

(a) Let E be a measurable subset of $\mathbb{R}^2$ such that for almost every $x \in \mathbb{R}$, $\{y \in \mathbb{R} : (x, y) \in E\}$ has $\mathbb{R}^1$-measure zero. Show that E has measure zero and that for almost every $y \in \mathbb{R}^1$, $\{x \in \mathbb{R} : (x, y) \in E\}$ has measure zero.

(b) Let $f(x, y)$ be nonnegative and measurable in $\mathbb{R}^2$. Suppose that for almost every $x \in \mathbb{R}^1$, $f(x, y)$ is finite for almost every y. Show that for almost every $y \in \mathbb{R}^1$, $f(x, y)$ is finite for almost every x.

Proof. **(a)**

By Tonelli's Theorem

$$|E| = \int_{\mathbb{R}^2} \chi_E d(x,y) = \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \chi_E dy \Big) dx$$
$$= \int_{\mathbb{R}} |\{y \in \mathbb{R} : (x,y) \in E\}| \, dx = 0$$

where the last equality follows from the premise that $\{y \in \mathbb{R} : (x,y) \in E\}$ has $\mathbb{R}^1$-measure zero for almost every x. Again, by Tonelli's Theorem,

$$\int_{\mathbb{R}} |\{x \in \mathbb{R} : (x,y) \in E\}| \, dy = \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \chi_E dx \Big) dy$$
$$= \int_{\mathbb{R}^2} \chi_E d(x,y) = 0$$

This implies

$$\int_{\mathbb{R}} |\{x \in \mathbb{R} : (x,y) \in E\}| \, dx = 0 \text{ for almost every } y$$

This is true if and only if

$$\{x \in \mathbb{R} : (x,y) \in E\} \text{ has measure zero for almost every } y$$

(b)

Let $F \triangleq \{(x,y) \in \mathbb{R}^2 : f(x,y) = \infty\}$. Observe

$$F = \bigcap_{n \in \mathbb{N}} \{(x,y) \in \mathbb{R}^2 : f(x,y) > n\}$$

It follows from f is measurable that F is measurable. By premise,

$$|\{y \in \mathbb{R} : (x,y) \in F\}| = 0 \text{ for almost every } x$$

We have shown F satisfies the hypothesis E satisfies. It then follows from part **(a)** that

$$|\{x \in \mathbb{R} : (x,y) \in F\}| = 0 \text{ for almost every } y$$

That is, for almost every $y \in \mathbb{R}$, $f(x,y) < \infty$ for almost every x. ■

Question 79

Let f be measurable and period $1 : f(t+1) = f(t)$. Suppose that there is a finite c such that

$$\int_0^1 |f(a+t) - f(b+t)|\, dt \le c$$

for all a and b. Show that $f \in L[0,1]$. (Set $a = x, b = -x$, integrate with respect to x, and make the change of variables $\xi = x + t, \eta = -x + t$)

Proof. Consider the linear transformation

$$T = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Note that

$$T^{-1}(1,0) = (\frac{1}{2}, \frac{1}{2}) \text{ and } T^{-1}(0,1) = (\frac{-1}{2}, \frac{1}{2})$$

It follows that $T^{-1}([0,1]^2) \subseteq [-1,1] \times [0,1]$. Then by a change of variable and Tonelli's Theorem

$$\begin{aligned}
\int_{[0,1]^2} |f(x) - f(t)|\, d(x,t) &\le |\det T| \int_{[-1,1]\times[0,1]} |f(x+t) - f(-x+t)|\, d(x,t) \\
&= 2 \int_{-1}^1 \int_0^1 |f(x+t) - f(-x+t)|\, dtdx \\
&\le 2 \int_{-1}^1 cdx = 4c
\end{aligned}$$

We have shown $f(x) - f(t) \in L([0,1]^2)$. It then follows from Fubini's Theorem

$$\int_0^1 |f(x) - f(t)|\, dx \in \mathbb{R} \text{ for almost every } t \in [0,1]$$

Fix some $t \in [0,1]$ such that $\int_0^1 |f(x)-f(t)|\,dx \in \mathbb{R}$. Because $[0,1]$ is not null, $f(t) \neq \pm\infty$ obviously. Define $h_t : [0,1] \to [-\infty,\infty]$ by

$$h_t(x) \triangleq f(x) - f(t)$$

It then follows from $f(x) = h_t(x) + f(t)$ that

$$\begin{aligned}\int_0^1 |f(x)|\,dx &\leq \int_0^1 |h_t(x)|\,dx + \int_0^1 |f(t)|\,dx \\ &\leq \int_0^1 |f(x)-f(t)|\,dx + |f(t)| \in \mathbb{R}\end{aligned}$$

We have shown $f \in L([0,1])$

■

Question 80

(a) If f is nonnegative and measurable on E and $\omega(y) = |\{\mathbf{x} \in E : f(\mathbf{x}) > y\}|\,, y > 0$, use Tonelli's Theorem to prove that $\int_E f = \int_{\mathbb{R}^+} \omega(y)dy$. (By definition of the integral $\int_E f = |R(f,E)| = \iint_{R(f,E)} d\mathbf{x}dy$. Use $\{\mathbf{x} \in E : f(\mathbf{x}) \geq y\} = \{\mathbf{x} : (x,y) \in R(f,E)\}$, and recall that $\omega(y) = |\{\mathbf{x} \in E : f(\mathbf{x}) \geq y\}|$ unless y is a point of discontinuity of ω)

(b) Deduce from the special case the general formula

$$\int_E f^p = p \int_{\mathbb{R}^+} y^{p-1}\omega(y)dy \qquad (f \geq 0, 0 < p < \infty)$$

Proof. Define

$$\begin{aligned}R(f,E)_y &\triangleq \{\mathbf{x} \in E : (\mathbf{x},y) \in R(f,E)\} \\ &= \{\mathbf{x} \in E : y < f(\mathbf{x})\}\end{aligned}$$

Thus,

$$\omega(y) = |R(f,E)_y|$$

Let $E \subseteq \mathbb{R}^n$. Part **(a)** then follows from applying Tonelli's Theorem to compute

$$\begin{aligned}\int_E f = |R(f,E)| &= \int_{\mathbb{R}^n \times \mathbb{R}^+} \chi_{R(f,E)} d(\mathbf{x},y) \qquad \text{Recall } y > 0 \\ &= \int_{\mathbb{R}^+} \Big[\int_{\mathbb{R}^n} \chi_{R(f,E)} d\mathbf{x}\Big] dy \\ &= \int_{\mathbb{R}^+} |R(f,E)_y|\,dy = \int_{\mathbb{R}^+} \omega(y)dy\end{aligned}$$

Define $\omega_p : \mathbb{R}^+ \to [0,\infty]$ by

$$\omega_p(y) \triangleq |\{\mathbf{x} \in E : y < f^p(\mathbf{x})\}|$$

It is clear that for all fixed y,

$$\{\mathbf{x} \in E : y < f^p(\mathbf{x})\} = \{\mathbf{x} \in E : y^{\frac{1}{p}} < f(\mathbf{x})\}$$

It follows that

$$\omega_p(y) = \omega(y^{\frac{1}{p}}) \text{ for all } y$$

We may now use the result in part (a) to deduce

$$\begin{aligned}\int_E f^p = \int_{\mathbb{R}^+} \omega_p(z)dz &= \int_{\mathbb{R}^+} \omega(z^{\frac{1}{p}})dz \\ &= \int_{\mathbb{R}^+} \omega(y)dy^p = p\int_{\mathbb{R}^+} y^{p-1}\omega(y)dy\end{aligned}$$

■

Question 81

For $f \in L(\mathbb{R})$, define the Fourier transform $\widehat{f} : \mathbb{R} \to \mathbb{C}$ of f by

$$\widehat{f}(x) \triangleq \frac{1}{2\pi}\int_{\mathbb{R}} f(t)e^{-ixt}dt$$

Show that if f and g belong to $L(\mathbb{R})$, then

$$\widehat{(f*g)}(x) = 2\pi\widehat{f}(x)\widehat{g}(x)$$

Proof. Because $f,g \in L(\mathbb{R})$, we know $|f| * |g| \in L(\mathbb{R})$. Thus, by Tonelli's Theorem

$$\begin{aligned}\int_{\mathbb{R}^2} \left| f(s)e^{-ixs}g(t-s)e^{-ix(t-s)} \right| d(s,t) &= \int_{\mathbb{R}^2} |f(s)g(t-s)|\, d(s,t) \\ &= \int_{\mathbb{R}} (|f| * |g|)(t)dt \in \mathbb{R}\end{aligned}$$

Therefore, we may use Fubini's Theorem to compute

$$\begin{aligned}\widehat{f*g}(x) &= \frac{1}{2\pi}\int_{\mathbb{R}}(f*g)(t)e^{-ixt}dt \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f(s)g(t-s)ds\Big)e^{-ixt}dt \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2} f(s)e^{-ixs}g(t-s)e^{-ix(t-s)}d(s,t) \\ (\xi = s \text{ and } \eta = t-s \text{ and } \det T = 1) \quad &= \frac{1}{2\pi}\int_{\mathbb{R}^2} f(\xi)e^{-ix\xi}g(\eta)e^{-ix\eta}d(\xi,\eta) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} f(\xi)e^{-ix\xi}d\xi\int_{\mathbb{R}} g(\eta)e^{-ix\eta}d\eta = 2\pi\widehat{f}(x)\widehat{g}(x)\end{aligned}$$

∎

Question 82

Let F be a closed subset of $\mathbb{R}$ and let $\delta(x) = d(x,F)$ be the corresponding distance function. If $\lambda > 0$ and f is nonnegative and integrable over complement of F, prove that the function

$$\int_{\mathbb{R}} \frac{\delta^{\lambda}(y)f(y)}{|x-y|^{1+\lambda}}dy$$

is integrable over F.

Proof. We are required to show

$$\int_F \left|\int_{\mathbb{R}} \frac{\delta^{\lambda}(y)f(y)}{|x-y|^{1+\lambda}}dy\right| dx \in \mathbb{R}$$

We know

$$\int_F \left|\int_{\mathbb{R}} \frac{\delta^{\lambda}(y)f(y)}{|x-y|^{1+\lambda}}dy\right| dx \leq \int_F\int_{\mathbb{R}} \frac{|\delta^{\lambda}(y)f(y)|}{|x-y|^{1+\lambda}}dydx$$

Because $\delta(y) = 0$ when $y \in F$, we may compute

$$\int_F\int_{\mathbb{R}} \frac{|\delta^{\lambda}(y)f(y)|}{|x-y|^{1+\lambda}}dydx = \int_F\int_{\mathbb{R}\setminus F} \frac{|\delta^{\lambda}(y)f(y)|}{|x-y|^{1+\lambda}}dydx$$

We have reduced the problem into proving

$$\int_F\int_{\mathbb{R}\setminus F} \frac{|\delta^{\lambda}(y)f(y)|}{|x-y|^{1+\lambda}}dydx \in \mathbb{R} \tag{2.11}$$

Now, by Tonelli's Theorem,

$$\int_F \int_{\mathbb{R}\setminus F} \frac{\left|\delta^\lambda(y)f(y)\right|}{|x-y|^{1+\lambda}} dydx = \int_{\mathbb{R}\setminus F} \int_F \frac{\left|\delta^\lambda(y)f(y)\right|}{|x-y|^{1+\lambda}} dxdy$$
$$= \int_{\mathbb{R}\setminus F} \delta^\lambda(y)\,|f(y)| \int_F \frac{dx}{|x-y|^{1+\lambda}} dy$$

By definition of δ, we have

$$x \in F \implies |x-y| \geq \delta(y)$$

Therefore,

$$\int_F \int_{\mathbb{R}\setminus F} \frac{\left|\delta^\lambda(y)f(y)\right|}{|x-y|^{1+\lambda}} dydx = \int_{\mathbb{R}\setminus F} \delta^\lambda(y)\,|f(y)| \int_F \frac{dx}{|x-y|^{1+\lambda}} dy$$
$$\leq \int_{\mathbb{R}\setminus F} \delta^\lambda(y)\,|f(y)| \int_{|x-y|\geq\delta(y)} \frac{dx}{|x-y|^{1+\lambda}} dy$$

Then by two changes of variables, we have

$$\int_F \int_{\mathbb{R}\setminus F} \frac{\left|\delta^\lambda(y)f(y)\right|}{|x-y|^{1+\lambda}} dydx \leq \int_{\mathbb{R}\setminus F} \delta^\lambda(y)\,|f(y)| \int_{|x-y|\geq\delta(y)} \frac{dx}{|x-y|^{1+\lambda}} dy$$
$$\leq \int_{\mathbb{R}\setminus F} \delta^\lambda(y)\,|f(y)| \int_{\delta(y)}^\infty \frac{2dt}{t^{1+\lambda}} dy$$
$$= 2\int_{\mathbb{R}\setminus F} \delta^\lambda(y)\,|f(y)| \left[\frac{t^{-\lambda}}{-\lambda}\right]\Bigg|_{t=\delta(y)}^\infty dy$$
$$= \frac{2}{\lambda}\int_{\mathbb{R}\setminus F} |f(y)|\,dy \in \mathbb{R}$$

We have proved Equation 2.11, thus proving the whole proposition. ■

Question 83

Use Fubini's Theorem to prove that

$$\int_{\mathbb{R}^n} e^{-|\mathbf{x}|^2} d\mathbf{x} = \pi^{\frac{n}{2}}$$

Proof. We shall prove by induction. If $n = 1$, we have

$$\int_{\mathbb{R}} e^{-x^2} dx = 2\int_0^\infty e^{-x^2} dx$$

Let $I \triangleq \int_{\mathbb{R}} e^{-x^2} dx$. We wish to show $I^2 = \pi$. Compute

$$\begin{aligned}
I^2 &= 4 \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy \\
&= 4 \int_0^\infty \Big(\int_0^\infty e^{-y^2} dy \Big) e^{-x^2} dx \\
&= 4 \int_0^\infty \Big(\int_0^\infty e^{-(x^2+y^2)} dy \Big) dx \\
&= 4 \int_0^\infty \Big(\int_0^\infty e^{-x^2(1+s^2)} x ds \Big) dx \\
(\because \text{ Tonelli's Theorem }) \quad &= 4 \int_0^\infty \int_0^\infty e^{-x^2(1+s^2)} x dx ds \\
&= 4 \int_0^\infty \Big[\frac{e^{-x^2(1+s^2)}}{-2(1+s^2)} \Big] \Big|_{x=0}^\infty ds \\
&= 2 \int_0^\infty \frac{1}{1+s^2} ds \\
&= 2 \arctan(s) \Big|_{s=0}^\infty = \pi
\end{aligned}$$

Note that in the forth equality, we take the change of variables $s = \frac{y}{x}$. Although such change of variable is not justified for Lebesgue integral, it is for Riemann integral, and the improper Riemann integral here coincides with the Lebesgue integral, thus justifying the forth equality. Now, suppose the proposition hold true for $n < r$. Let $n = r$. By Tonelli's Theorem, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{-|\mathbf{x}|^2} d\mathbf{x} &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} e^{-(|\mathbf{y}|^2 + z^2)} d\mathbf{y} dz \\
&= \int_{\mathbb{R}^{n-1}} e^{-|\mathbf{y}|^2} d\mathbf{y} \int_{\mathbb{R}} e^{-z^2} dz = \pi^{\frac{n-1}{2}} \cdot \pi^{\frac{1}{2}} = \pi^{\frac{n}{2}}
\end{aligned}$$

■

Chapter 3

Complex Analysis HW

3.1 HW1

Theorem 3.1.1.

$$(1+i)^n, \frac{(1+i)^n}{n}, \frac{n!}{(1+i)^n} \text{ all diverge as } n \to \infty$$

Proof. Note that

$$|(1+i)^n| = 2^{\frac{n}{2}} \to \infty \text{ as } n \to \infty$$

This implies $(1+i)$ is unbounded, thus diverge.

Note that

$$\left|\frac{(1+i)^n}{n}\right| = \frac{(\sqrt{2})^n}{n}$$

Observe

$$\begin{aligned}\frac{(\sqrt{2})^n}{n} = \frac{[(\sqrt{2}-1)+1]^n}{n} &= \frac{\sum_{k=0}^{n}\binom{n}{k}(\sqrt{2}-1)^k}{n} \\ &\geq \frac{\binom{n}{2}(\sqrt{2}-1)^2}{n} = (n-1)[\frac{(\sqrt{2}-1)^2}{2}] \to \infty \text{ as } n \to \infty\end{aligned}$$

This implies $\frac{(1+i)^n}{n}$ is unbounded, thus diverge.

Note that

$$\left|\frac{n!}{(1+i)^n}\right| = \frac{n!}{(\sqrt{2})^n}$$

Note that for all $k \geq 8$, we have

$$\frac{k}{\sqrt{2}} \geq \frac{\sqrt{8}}{\sqrt{2}} = 2$$

This implies

$$\frac{n!}{(\sqrt{2})^n} = \prod_{k=1}^{n} \frac{k}{\sqrt{2}} = \frac{7!}{(\sqrt{2})^7} \prod_{k=8}^{n} \frac{k}{\sqrt{2}} \geq \frac{7!}{(\sqrt{2})^7} \prod_{k=8}^{n} 2 \geq \frac{7!2^{n-8+1}}{(\sqrt{2})^7} \to \infty$$

which implies $\frac{n!}{(1+i)^n}$ is unbounded, thus diverge. ∎

Theorem 3.1.2.

$$n!z^n \text{ converge} \iff z = 0$$

Proof. If $z = 0$, then $n!z^n = 0$ for all n, which implies $n!z^n \to 0$. Now, suppose $z \neq 0$. Let $M \in \mathbb{N}$ satisfy $|z| > \frac{1}{M}$. Observe

$$|n!z^n| = \left| \prod_{k=1}^{n} kz \right| = \left| \prod_{k=1}^{2M-1} kz \right| \left| \prod_{k=2M}^{n} kz \right| \geq \left| \prod_{k=1}^{2M-1} kz \right| \left| \prod_{k=2M}^{n} 2Mz \right| \geq \left| \prod_{k=1}^{2M-1} kz \right| 2^{n-2M+1} \to \infty$$

This implies $n!z^n$ is unbounded, thus diverge. ∎

Theorem 3.1.3.

$$u_n \to u \implies v_n \triangleq \sum_{k=1}^{n} \frac{u_k}{n} \to u$$

Proof. Because

$$\sum_{k=1}^{n} \frac{u_k}{n} = \sum_{k \leq \sqrt{n}} \frac{u_k}{n} + \sum_{\sqrt{n} < k \leq n} \frac{u_k}{n}$$

It suffices to prove

$$\sum_{k \leq \sqrt{n}} \frac{u_k}{n} \to 0 \text{ and } \sum_{\sqrt{n} < k \leq n} \frac{u_k}{n} \to u \text{ as } n \to \infty$$

Because u_n converge, we can let M bound $|u_n|$. Observe

$$\left| \sum_{k \leq \sqrt{n}} \frac{u_k}{n} \right| \leq \sum_{k \leq \sqrt{n}} \left| \frac{u_k}{n} \right| \leq \sum_{k \leq \sqrt{n}} \frac{M}{n} \leq \frac{M\sqrt{n}}{n} = \frac{M}{\sqrt{n}} \to 0 \text{ as } n \to 0 \text{ (done)}$$

Because

$$\sum_{\sqrt{n}<k\leq n} \frac{u_k}{n} = \frac{n-\lceil\sqrt{n}\rceil+1}{n} \sum_{\sqrt{n}<k\leq n} \frac{u_k}{n-\lceil\sqrt{n}\rceil+1}$$

and

$$\lim_{n\to\infty} \frac{n-\lceil\sqrt{n}\rceil+1}{n} = 1$$

We can reduce the problem into proving

$$\lim_{n\to\infty} \sum_{\sqrt{n}<k\leq n} \frac{u_k}{n-\lceil\sqrt{n}\rceil+1} = u$$

Fix ϵ. Let N satisfy that for all $n \geq N$, we have $|u_n - u| < \epsilon$. Then for all $n \geq N^2$, we have

$$\begin{aligned}
\left|\left(\sum_{\sqrt{n}<k\leq n} \frac{u_k}{n-\lceil\sqrt{n}\rceil+1}\right) - u\right| &= \left|\sum_{\sqrt{n}<k\leq n} \frac{u_k - u}{n-\lceil\sqrt{n}\rceil+1}\right| \\
&\leq \sum_{\sqrt{n}<k\leq n} \frac{|u_k - u|}{n-\lceil\sqrt{n}\rceil+1} \\
&\leq \sum_{\sqrt{n}<k\leq n} \frac{\epsilon}{n-\lceil\sqrt{n}\rceil+1} = \epsilon \text{ (done)}
\end{aligned}$$

■

3.2 Exercise 1

Let R be a complex algebra with 1_A and $a \in R$. Given a complex polynomial

$$f(Z) = a_0 + a_1 Z + \cdots + a_n Z^n,$$

we define the evaluation of f at a by

$$f(a) = a_0 1_A + a_1 a + \cdots + a_n a^n.$$

Question 84

Let $R = \mathbb{C}$ and $a = 1 + i$. Given $f(Z) = Z^3$. Evaluate $f(a)$.

Proof. $f(a) = (1+i)^3 = 2i(1+i) = -2 + 2i$ ■

Question 85

Let $R = M_{2\times 2}(\mathbb{C})$ be the algebra of 2×2 complex matrices. Take

$$a = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

and $g(Z) = 3 + 2Z$. Evaluate $g(a)$.

Proof.

$$g(a) = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 2 & 5 \end{bmatrix}$$

■

Question 86

Let R be the algebra of complex valued periodic functions of period 2π, i.e., $a \in R$ is a continuous function $a : \mathbb{R} \to \mathbb{C}$ so that $a(x + 2\pi) = a(x)$. Let $e(x) = \cos x + i \sin x$ and

$$h(Z) = 1 + Z + Z^2 + \cdots + Z^9.$$

Find $h(e)$.

Proof. Note that

$$\begin{aligned}(\cos x + i \sin x)(\cos y + i \sin y) &= (\cos x \cos y - \sin x \sin y) + i(\sin x \cos y + \cos x \sin y) \\ &= \cos(x+y) + i \sin(x+y)\end{aligned}$$

This give us

$$h(e) = \sum_{k=0}^{9} \cos(kx) + i\sin(kx)$$

$\blacksquare$

3.3 HW2

Theorem 3.3.1. (Root Test is Stronger Than Ratio Test)

$$\liminf_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| \leq \liminf_{n\to\infty}\sqrt[n]{|z_n|} \leq \limsup_{n\to\infty}\sqrt[n]{|z_n|} \leq \limsup_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right|$$

Proof. Fix ϵ and WOLG suppose $\liminf_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| > 0$. We prove

$$\liminf_{n\to\infty}\sqrt[n]{|z_n|} \geq \liminf_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| - \epsilon$$

Let $\alpha \in \mathbb{R}$ satisfy

$$\liminf_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| - \epsilon < \alpha < \liminf_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right|$$

Let N satisfy

$$\text{For all } n \geq N, \left|\frac{z_{n+1}}{z_n}\right| > \alpha$$

We then see

$$\sqrt[N+n]{|z_{N+n}|} \geq \sqrt[N+n]{|z_N|\,\alpha^n} = \alpha\Big(\frac{|z_N|^{\frac{1}{N+n}}}{\alpha^{\frac{N}{N+n}}}\Big) \to \alpha \text{ as } n \to \infty \text{ (done)}$$

The proof for the other side is similar. ∎

Question 87

Find the radius of convergence of the following series:

(a) $\sum \frac{z^n}{n}$.

(b) $\sum \frac{z^n}{n!}$.

(c) $\sum n! z^n$.

(d) $\sum n^k z^n$ where k is a positive integer.

(e) $\sum z^{n!}$.

Proof. We know

$$n^{\frac{1}{n}} = e^{\frac{\ln n}{n}} \to 1 \text{ as } n \to \infty \tag{3.1}$$

Equation 3.1 implies $n^{\frac{-1}{n}} \to 1$ as $n \to \infty$ and that $\sum \frac{z^n}{n}$ has radius of convergence 1. Equation 3.1 also implies $n^{\frac{k}{n}} \to 1$ and $\sum n^k z^n$ has radius of convergence 1.

We know

$$\lim_{n\to\infty} \frac{(n+1)!}{n!} = \infty$$

Then Theorem 3.3.1 tell us

$$\lim_{n\to\infty} (n!)^{\frac{1}{n}} = \infty \tag{3.2}$$

which implies that $\sum n! z^n$ has radius of convergence 0 and $\sum \frac{z^n}{n!}$ has radius of convergence ∞. Note that

$$\sum z^{n!} = \sum a_n z^n \text{ where } a_n = \begin{cases} 1 & \text{if } n = k! \text{ for some } k \\ 0 & \text{otherwise} \end{cases}$$

It is then clear that

$$\limsup_{n\to\infty} (a_n)^{\frac{1}{n}} = 1$$

and the radius is 1. ■

Question 88

The 0th order Bessel function $J_0(z)$ is defined by the power series

$$J_0(z) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2} \frac{z^n}{2n}$$

Find its radius of convergence.

Proof. Equation 3.1 and Equation 3.2 tell us

$$\lim_{n\to\infty} (2n(n!)^2)^{\frac{1}{n}} = \infty$$

which implies that the radius of convergence of $J_0(z)$ is ∞. ■

Theorem 3.3.2. (Abel's Test for Power Series) Suppose $a_n \to 0$ monotonically and $\sum a_n z^n$ has radius of convergence R .

$$\text{The power series } \sum a_n z^n \text{ at least converge on } \overline{D_R(0)} \setminus \{R\}$$

Proof. Note that

$$\sum \frac{a_n}{R^n} z^n \text{ has radius of convergence } R$$

Fix $z \in \overline{D_R(0)} \setminus \{R\}$. Note that

$$\left| \sum_{n=0} \frac{z^n}{R^n} \right| = \left| \frac{1 - (\frac{z}{R})^{N+1}}{1 - \frac{z}{R}} \right| \leq \frac{2}{\left|1 - \frac{z}{R}\right|} \text{ for all } N$$

It then follows from Dirichlet's Test that $\sum a_n (\frac{z}{R})^n$ converge. ∎

Question 89

Suppose that $\sum a_n z^n$ has radius of convergence R and let C be the circle $\{z \in \mathbb{C} : |z| = R\}$. Prove or disprove

(a) If $\sum a_n z^n$ converge at every point on C, except possibly one, then it converges absolutely every where on C

Proof. Consider $a_n \triangleq \frac{1}{n}$ for all $n \in \mathbb{N}$ and $a_0 \triangleq 1$. Then $\sum a_n z^n$ has convergence radius 1. Since $a_n \searrow 0$, it follows from Theorem 3.3.2, $\sum a_n z^n$ converge everywhere on $C \setminus \{1\}$. Observe that when $z = 1$, the series is just harmonic series, which diverge. ∎

Question 90

If $\sum a_n z^n$ has radius of convergence R, find the radius of convergence of

(a) $\sum n^3 a_n z^n$.

(b) $\sum a_n z^{3n}$.

(c) $\sum a_n^3 z^n$

Proof. Since $(n^3)^{\frac{1}{n}} \to 1$, we know $\sum n^3 a_n z^n$ also had radius of convergence R. We claim that the series $\sum a_n z^{3n}$ has convergence radius $R^{\frac{1}{3}}$. If $|z| < R^{\frac{1}{3}}$, then $\left|z^3\right| < R$ and thus

$$\sum a_n (z^3)^n \text{ converge}$$

and if $|z| > R^{\frac{1}{3}}$, then $\left|z^3\right| > R$ and

$$\sum a_n (z^3)^n \text{ diverge}$$

We have proved that $\sum a_n z^{3n}$ has convergence radius $R^{\frac{1}{3}}$.

Note that given a sub-sequence $|a_{n_k}|^{\frac{1}{n_k}}$,

$$|a_{n_k}|^{\frac{1}{n_k}} \text{ converge in extended reals if and only if } |a_{n_k}|^{\frac{3}{n_k}} \text{ converge in extended reals}$$

and if the former converge to L, then the latter converge to L^3. It now follows that

$$\limsup_{n\to\infty} |a_n^3| = (\limsup_{n\to\infty} |a_n|)^3 = \frac{1}{R^3}$$

It now follows that $\sum a_n^3 z^n$ has convergence radius R^3. ∎

Theorem 3.3.3. (Summation by Part)

$$\begin{aligned} f_n g_n - f_m g_m &= \sum_{k=m}^{n-1} f_k \Delta g_k + g_k \Delta f_k + \Delta f_k \Delta g_k \\ &= \sum_{k=m}^{n-1} f_k \Delta g_k + g_{k+1} \Delta f_k \end{aligned}$$

Proof. The proof follows induction which is based on

$$f_m g_m + f_m \Delta g_m + g_m \Delta f_m + \Delta f_m \Delta g_m = f_{m+1} g_{m+1}$$

∎

Question 91

Prove that, for $z \neq 1$

$$\sum_{n=1}^{k} \frac{z^n}{n} = \frac{z}{1-z} \Big(\sum_{n=1}^{k-1} \frac{1}{n(n+1)} - \sum_{n=1}^{k-1} \frac{z^n}{n(n+1)} + \frac{1-z^k}{k} \Big)$$

Show that the series $\sum \frac{z^n}{n}$ and $\sum \frac{z^n}{n(n+1)}$ have radius of convergence 1; that the latter series converge everywhere on $|z| = 1$, while the former converges everywhere on $|z| = 1$ except $z = 1$.

Proof. We prove by induction. The base case $k = 1$ is trivial. Suppose the equality hold when $k = m$. The difference of the left hand side is clearly $\frac{z^{m+1}}{m+1}$, and the difference of the right hand side is

$$\begin{aligned}
&\frac{z}{1-z}\Big(\frac{1}{m(m+1)}-\frac{z^m}{m(m+1)}+\frac{1-z^{m+1}}{m+1}-\frac{1-z^m}{m}\Big)\\
&=\frac{z}{1-z}\cdot\frac{1-z^m+m-mz^{m+1}+(m+1)z^m-(m+1)}{m(m+1)}\\
&=\frac{z}{1-z}\cdot\frac{-mz^{m+1}+mz^m}{m(m+1)}=\frac{z(z^m-z^{m+1})}{(1-z)(m+1)}=\frac{z^{m+1}}{m+1}
\end{aligned}$$

The fact that both series have radius of convergence 1 follows from $n^{\frac{1}{n}}\to 1$. Both of them converge on $\overline{D_1(0)}\setminus\{1\}$ by Theorem 3.3.2. The former clearly diverge at $z=1$, since it would be a harmonic series, and the latter converge at $z=1$ by comparison test with $\frac{1}{n(n+1)}\leq\frac{1}{n^2}$ for all $n\in\mathbb{N}$. ■

Question 92

Suppose that the power series $\sum a_n z^n$ has a recurring sequence of coefficients; that is $a_{n+k}=a_n$ for some fixed positive integer k and all n. Prove that the series converge for $|z|<1$ to a rational function $\frac{p(z)}{q(z)}$ where p,q are polynomials, and the roots of q are all on the unit circle. What happens if $a_{n+k}=\frac{a_n}{k}$ instead?

Proof. Let

$$L^-\triangleq\min\{|a_0|,\dots,|a_{k-1}|\}\text{ and }L^+\triangleq\max\{|a_0|,\dots,|a_{k-1}|\}$$

We see that

$$1=\liminf_{n\to\infty}(L^-)^{\frac{1}{n}}\leq\lim_{n\to\infty}|a_n|^{\frac{1}{n}}\leq\limsup_{n\to\infty}(L^+)^{\frac{1}{n}}=1$$

It then follows that $\sum a_n z^n$ has convergence radius 1. Now observe that for $|z|<1$, we have

$$z^k\sum_{n=0}^{\infty}a_nz^n=\sum_{n=k}^{\infty}a_nz^n=\sum_{n=0}^{\infty}a_nz^n-\sum_{n=0}^{k-1}a_nz^n$$

This now implies

$$\sum_{n=0}^{\infty}a_nz^n=\frac{\sum_{n=0}^{k-1}a_nz^n}{1-z^k}$$

Since $q(z) = 1 - z^k$, clearly the roots are all on the unit circle. Suppose now $b_n \triangleq a_n$ for all $n < k$ and $b_{n+k} \triangleq \frac{b_n}{k}$ for all $n \geq k$. We then have

$$b_n = \frac{a_n}{k^{q(n)}} \text{ where } q \text{ is the largest integer such that } qk \leq n$$

Note that $n - q(n)$ is always smaller than k. It then follows that

$$(k^{q(n)})^{\frac{1}{n}} = k^{\frac{q(n)}{n}} = k \cdot k^{\frac{q(n)-n}{n}} \to k$$

We then see that

$$\limsup_{n\to\infty} |b_n|^{\frac{1}{n}} = \frac{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}}{k} = \frac{1}{k}$$

It then follows that $\sum b_n z^n$ has convergence radius k. Now observe that for $|z| < k$, we have

$$z^k \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} b_n z^{n+k} = \frac{1}{k} \sum_{n=k}^{\infty} b_n z^n = \frac{1}{k}\Big(\sum_{n=0}^{\infty} b_n z^n - \sum_{n=0}^{k-1} b_n z^n \Big)$$

This now implies

$$\sum_{n=0}^{\infty} b_n z_n = \frac{\sum_{n=0}^{k-1} b_n z^n}{k(\frac{1}{k} - z^k)}$$

■

3.4 Exercises 2

Let (M, d) be a metric space, $x \in M$ and F a subset of M.

Question 93

Prove that the following statements are equivalent

(a) There exists a sequence $\{x_n\}$ in F with $x_n \neq x$ so that $\lim_{n\to\infty} x_n = x$.

(b) For any ϵ, the intersection of $B'_\epsilon(x) \triangleq \{y \in M : 0 < d(x,y) < \epsilon\}$ and F are non-empty.

Proof. If (a) is true, then for all ϵ there exists some $x_n \in F$ such that $d(x_n, x) < \epsilon$. Because $x_n \neq x$, we know that $0 < d(x_n, x)$. This now implies $x_n \in B'_\epsilon(x) \cap F$.

If (b) is true, then for all n, we simply select a point in $x_n \in B'_{\frac{1}{n}}(x) \cap F$. After such selection, we see that $x_n \neq x$ and for all ϵ, if $n > \frac{1}{\epsilon}$, then $x_n \in B'_\epsilon(x) \cap F$. ■

Question 94

Prove that the following statements are equivalent

(a) F contain all its limit point.

(b) $U = M \setminus F$ is open.

Proof. If (a) is true, then for all $p \in U$, we know that p is not a limit point of F, then from the first question, we know that there exists ϵ such that $B'_\epsilon(x) \cap F = \varnothing$. Because $x \in U = M \setminus F$ also does not belong x, we also know that $B_\epsilon(x) \cap F = \varnothing$. This then implies that $B_\epsilon(x) \subseteq U$, since $U = M \setminus F$. We have proved that U is open.

If (b) is true, then for arbitrary $p \notin F$, we know there exists some ϵ such that $B_\epsilon(x)$ is disjoint with F. Because $B'_\epsilon(x)$ is a subset of $B_\epsilon(x)$, we can deduce that $B_\epsilon(x) \cap F = \varnothing$, which from the first question implies that p is not a limit point of F. Because p is arbitrary selected from $M \setminus F$, we have proved that none of the points in $M \setminus F$ is a limit point of F. This implies that if F has any limit point, then F must contain that limit point. ■

Question 95

Prove the following statements

(a) M and $\varnothing$ are closed.

(b) The intersection of any family of closed subsets of M is closed.

(c) The union of finitely many closed subsets of M is closed.

Proof. It is clear that M is open and trivially true that $\varnothing$ is open. It then follows from the second question that M and $\varnothing$ are both closed.

Let (F_α) be a collection of closed subsets of M. Arbitrary select a limit point x of $\bigcap F_\alpha$. Let $\{x_n\}$ be a sequence in $\bigcap F_\alpha$ with $x_n \neq x$ so that $\lim_{n\to\infty} x_n = x$. Arbitrary select β. Note that $\{x_n\}$ is also a sequence in F_β that converge to x with $x_n \neq x$. This now implies that x is a limit point of F_β. Then because F_β is closed, we see that $x \in F_\beta$. Now, since β is arbitrary selected, we see $x \in \bigcap_\alpha F_\alpha$. Because x is arbitrary, we have proved $\bigcap F_\alpha$ contained all its limit points.

Let $\{F_1, \dots, F_N\}$ be a collection of closed subsets of M. Let x be an arbitrary limit point of $\bigcup_{n=1}^N F_n$. Let $\{x_n\}$ be a sequence in $\bigcup_{n=1}^N F_n$ with $x_n \neq x$ converging to x. It is clear that there must exists some $j \in \{1, \dots, N\}$ such that F_j contain infinite terms of $\{x_n\}$, i.e., there exists a subsequence x_{n_k} such that $x_{n_k} \in F_j$ for all k. Because $\lim_{k\to\infty} x_{n_k} = \lim_{n\to\infty} x_n = x$, we now see that x is a limit point of F_j. It then follows from F_j being closed that $x \in F_j \subseteq \bigcup_{n=1}^N F_n$. Because x is arbitrary, we have proved that $\bigcup_{n=1}^N F_n$ is closed. ■

3.5 Exercise 3

Question 96

Provide a counterexample to the following statement: Suppose

$$f(z) = u(x, y) + iv(x, y)$$

is defined in a neighborhood of $z_0 = a + ib$. If the partial derivatives of u and v exist at (a, b) and satisfy the Cauchy-Riemann equations $u_x(a, b) = v_y(a, b)$ and $u_y(a, b) = -v_x(a, b)$, then f is holomorphic at z_0.

Proof. WOLG, let $a = b = 0$ and define

$$u(x, y) \triangleq \begin{cases} x & \text{if } y = 0 \\ y & \text{if } x = 0 \\ 0 & \text{if otherwise} \end{cases} \quad \text{and } v(x, y) \triangleq \begin{cases} -x & \text{if } y = 0 \\ y & \text{if } x = 0 \\ 0 & \text{if otherwise} \end{cases}$$

It is clear that

$$u_x = 1 = v_y \text{ and } u_y = 1 = -v_x \text{ at } (0, 0)$$

but

$$\lim_{t \to 0; t \in \mathbb{R}} \frac{f(t) - f(0)}{t - 0} = \lim_{t \to 0; t \in \mathbb{R}} \frac{t - it}{t} = 1 - i$$

together with

$$\lim_{t \to 0; t \in \mathbb{R}} \frac{f(t + it) - f(0)}{t + it} = \lim_{t \to 0; t \in \mathbb{R}} \frac{0}{t + it} = 0$$

shows that f is not holomorphic at $(0, 0)$. ∎

Question 97

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose that f is differentiable at (a, b) and that $f'(x) = 0$ for all $x \in (a, b)$. Prove that f is a constant function.

Proof. Assume $f(x) \neq f(y)$ for some $x \neq y \in [a, b]$. By MVT, we then see there exists some t between x, y (thus $t \in (a, b)$) such that $f'(t) = \frac{f(y)-f(x)}{y-x} \neq 0$, which is impossible. CaC ∎

Question 98

Let $B = B_R(x_0)$ be the open ball in $\mathbb{R}^n$ centered at x_0 with radius $R > 0$. Prove that if $f : B \to \mathbb{R}$ is a differentiable function such that $\nabla f = 0$ on B, then f is a constant function.

Proof. Let $\mathbf{x}, \mathbf{y}$ be two points in B. We are required to show $f(\mathbf{x}) = f(\mathbf{y})$. Define $g : [0, 1] \to B$ by

$$g(t) \triangleq f(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))$$

Note that g is well-defined since B is convex. Because f is differentiable, we have

$$g'(t) = \nabla f(\mathbf{x} + y(\mathbf{y} - \mathbf{x})) \cdot (\mathbf{y} - \mathbf{x}) = 0$$

It then follows from the last question that

$$f(\mathbf{y}) = g(1) = g(0) = f(\mathbf{x})$$

■

Question 99

Let U be an open subset of $\mathbb{R}^n$. A function $f : U \to \mathbb{R}$ is called **locally constant** if, for each $x \in U$, there exists an open neighborhood W of x such that $W \subseteq U$ and $f : W \to \mathbb{R}$ is constant on W. Prove that f is locally constant function if and only if $\nabla f = 0$ on U.

Proof. The if part follows from the last question by taking some small enough r such that $B_r(x) \subseteq U$. We now prove the only if part. Fix arbitrary $x \in U$. Because f is locally constant at x, we know there exists some $B_r(x)$ such that f is constant on $B_r(x)$. Therefore, we can let $c \in \mathbb{R}$ satisfy

$$f(y) = c \text{ for all } y \in B_r(x)$$

To see $\nabla f(x) = 0$, just observe that for arbitrary axis $\mathbf{j}$

$$f_{\mathbf{j}}(x) = \lim_{t \to 0} \frac{f(x + t\mathbf{j}) - f(x)}{t} = 0$$

since $f(x + t\mathbf{j}) = c = f(x)$ as long as $|t| < r$. Because $\mathbf{j}$ is arbitrary, it then follows that $\nabla f(x) = 0$, and because x is arbitrary selected from U, we have proved ∇f is 0 on U. ■

Question 100

Let D be an open, connected subset of $\mathbb{R}^n$. Prove that if $f : D \to \mathbb{R}$ is a locally constant function, then f is a constant function.

Proof. Observe that for all $p \in D$, f is constant on some neighborhood around p, thus continuous at p. We have shown $f : D \to \mathbb{R}$ is continuous. Fix $p \in D$, and let $c \triangleq f(p)$. Because $\{c\}$ is closed in $\mathbb{R}$ and $f : D \to \mathbb{R}$ is continuous, we know $f^{-1}(\{c\})$ is closed in D. We now show $f^{-1}(\{c\})$ is open in D. Fix arbitrary $q \in f^{-1}(\{c\})$. Because $f : D \to \mathbb{R}$ is locally constant, we know there exists some r such that $B_r(q) \subseteq D$ and f sends $B_r(q)$ to $f(q) = c$. It follows that $B_r(q) \subseteq f^{-1}(\{c\})$. Because q is arbitrary selected from $f^{-1}(\{c\})$, we have shown $f^{-1}(\{c\})$ is open in D.

In conclusion, we have shown $f^{-1}(\{c\})$ is both open and closed in D. It then follows from D being connected that $f^{-1}(\{c\}) = D$ or $\varnothing$. Because $p \in f^{-1}(\{c\})$, we can deduce $f^{-1}(\{c\}) = D$, i.e., f send all points in D to c, a constant function. ■

3.6 HW 3

Question 101

Let $\mathbb{C}_\pi \triangleq \{z \in \mathbb{C} : z \notin \mathbb{R}_0^-\}$. Prove that $\mathbb{C}_\pi$ is a domain. Define $r : \mathbb{C}_\pi \to \mathbb{C}$ by $(r(z))^2 = z$ and $\operatorname{Re} r(z) > 0$. Prove that r is continuous on $\mathbb{C}_\pi$ and $r'(z) = \frac{1}{2r(z)}$.

Proof. It is clear that $\mathbb{C}_\pi$ is non-empty and open. To see $\mathbb{C}_\pi$ is path-connected, observe that for all point $x + iy \in \mathbb{C}_\pi$, we can join $x + iy$ with 1 linearly by defining $\gamma : [0, 1] \to \mathbb{C}_\pi$ by

$$\gamma(t) \triangleq 1 + t(x + iy - 1)$$

We have proved $\mathbb{C}_\pi$ is a domain. Note that

$$\mathbb{C}_\pi = \{a + bi \in \mathbb{C} : a > 0 \text{ and } b \in (-\pi, \pi)\}$$

and the exact definition of $r : \mathbb{C}_\pi \to \mathbb{C}$ is

$$r(e^{a+bi}) \triangleq e^{\frac{a+bi}{2}}$$

This implies r is continuous. Compute

$$1 = \frac{d}{dz} z = \frac{d}{dz}(r(z))^2 = 2r(z)r'(z)$$

This give us $r'(z) = \frac{1}{2r(z)}$. ∎

Theorem 3.6.1. (Conjugated Polynomial)

$$\overline{z^n} \text{ is holomorphic at 0 for all } n > 1$$

Proof. If we write

$$u + iv = \overline{(x + iy)^n} = (x - iy)^n$$

Because $n > 1$, we see from binomial Theorem that $u \in \mathbb{R}[x, y]$ is a polynomial with two indeterminate x, y whose terms all have degree greater than 1. Thus, both $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ are polynomial with two indeterminate x, y whose terms all have degree greater than 0. This implies

$$\frac{\partial u}{\partial x}(0) = \frac{\partial u}{\partial y}(0) = 0$$

Similar arguments shows that

$$\frac{\partial v}{\partial x}(0) = \frac{\partial v}{\partial y}(0) = 0$$

Because $u, v \in \mathbb{R}[x, y]$ are real multivariate polynomial, they are obviously real differentiable. Finally, we can use Cauchy-Riemann criteria to conclude that $\overline{z^n} = u + iv$ is holomorphic at 0. ∎

Question 102

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial. Prove that the function $g : \mathbb{C} \to \mathbb{C}$ defined by

$$g(z) \triangleq \overline{f(\overline{z})}$$

is holomorphic everywhere, but the function $h : \mathbb{C} \to \mathbb{C}$ defined by

$$h(z) \triangleq \overline{f(z)}$$

is holomorphic at 0 if and only if $f'(0) = 0$.

Proof. We can write

$$f(z) \triangleq \sum_{n=0}^{N} c_n z^n$$

and compute

$$g(z) = \sum_{n=0}^{N} \overline{c_n} z^n$$

We have shown $g : \mathbb{C} \to \mathbb{C}$ is a polynomial. It follows that g is holomorphic on $\mathbb{C}$. Compute

$$h(z) = \sum_{n=2}^{N} \overline{c_n z^n} + \overline{c_1 z} + \overline{c_0} \text{ and } f'(0) = c_1$$

Theorem 3.6.1 shows that

$$\sum_{n=2}^{N} \overline{c_n z^n} + \overline{c_0} \text{ is holomorphic at } 0$$

Note that $\overline{z}$ is not holomorphic at 0 since if we write $u + iv = \overline{z}$, then

$$\frac{\partial u}{\partial x} = 1 \neq -1 = \frac{\partial v}{\partial y}$$

The claim "h is holomorphic at 0 if and only if $f'(0) = 0$" then follows. ■

Question 103

Define

(a) $u, v : \mathbb{R}^2 \to \mathbb{R}$ by

$$u(x,y) = x^3 - 3xy^2 \text{ and } v(x,y) = 3x^2y - y^3$$

(b) $u, v : \{(x,y) \in \mathbb{R}^2 : x > 0\} \to \mathbb{R}$ by

$$u(x,y) = \frac{\ln(x^2+y^2)}{2} \text{ and } v(x,y) = \sin^{-1}(\frac{y}{\sqrt{x^2+y^2}})$$

Verify the Cauchy-Riemann Equation for these functions and state a complex function shows real and imaginary parts are u, v.

Proof. For (a), compute

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 3x^2 - 2y^2 \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -6xy$$

and observe

$$u + iv = (x+iy)^3$$

For (b), compute

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{x}{x^2+y^2} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{y}{x^2+y^2}$$

and observe

$$x + iy = e^{2u+iv}$$

which implies the function map z to $(\operatorname{Log} z) - \frac{\ln|z|}{2}$. ■

Question 104

Let $f(z) = \sqrt{|xy|}$. Show that f satisfy the Cauchy-Riemann equation at 0, yet $f'(0)$ does not exists. Explain why.

Proof. Observe that

$$f(x) = f(iy) = 0 \text{ for all } x, y \in \mathbb{R}$$

We then have Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$$

Then if f is holomorphic at 0, we should have $f'(0) = 0$, but we can compute

$$\lim_{t\to 0; t\in\mathbb{R}^+} \frac{f(t+ti) - f(0)}{t+ti} = \lim_{t\to 0; t\in\mathbb{R}^+} \frac{t}{t+ti} = \frac{1}{1+i} \neq 0$$

which implies f is not holomorphic at 0. The reason that f satisfy the Cauchy-Riemann equation yet isn't holomorphic is that its real part when treated as a function from $\mathbb{R}^2$ to $\mathbb{R}$ is not differentiable at 0, as we have shown. (Note that $f = \operatorname{Re} f$) ■

Question 105

Suppose that $f(z) = \sum a_n z^n$ is entire and satisfy

$$f' = f \text{ and } f(0) = 1$$

Find a_n. Show that

$$f(a+b) = f(a)f(b) \text{ for all } a, b \in \mathbb{C}$$

and compute $f(1)$ to five decimal points.

Proof. $f(0) = 1$ implies $a_0 = 1$. $f' = f$ implies $(n+1)a_{n+1} = a_n$, which give us

$$a_n = \frac{1}{n!} \text{ for all } n \geq 0$$

Fix $a, b \in \mathbb{C}$. Define $g : \mathbb{C} \to \mathbb{C}$ by

$$g(z) \triangleq f(a+b-z)f(z)$$

Compute

$$\begin{aligned} g'(z) &= -f'(a+b-z)f(z) + f(a+b-z)f'(z) \\ &= -f(a+b-z)f(z) + f(a+b-z)f(z) = 0 \end{aligned}$$

This implies g is constant. We then can deduce

$$f(a)f(b) = g(a) = g(0) = f(a+b)f(0) = f(a+b)$$

We know

$$f(1) = \sum_{n=0} \frac{1}{n!} = 2.7182818.....$$

■

Proof. Let $z = x + iy$ be on the two horizontal sides of the square, so from $|y| \geq \frac{1}{2}$, we have

$$\begin{aligned}
|\cot(\pi z)| &= \left| \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right| \\
&\leq \left| \frac{|e^{i\pi z}| + |e^{-i\pi z}|}{|e^{i\pi z}| - |e^{-i\pi z}|} \right| \\
&= \left| \frac{e^{-\pi y} + e^{\pi y}}{e^{-\pi y} - e^{\pi y}} \right| = |\coth(\pi y)| \leq \coth(\frac{\pi}{2})
\end{aligned}$$

Let $z = \pm(N + \frac{1}{2}) + it$ be on the two vertical sides of the square. Note that because N is even, we have

$$\begin{aligned}
|\cot(\pi z)| &= \left| \frac{\cos(\pi(N + \frac{1}{2}) + \pi i t)}{\sin(\pi(N + \frac{1}{2}) + \pi i t)} \right| \\
&= \left| \frac{\cos(\frac{\pi}{2} + \pi i t)}{\sin(\frac{\pi}{2} + \pi i t)} \right| \\
&= \left| \frac{\sin(\pi i t)}{\cos(\pi i t)} \right| \\
&= \left| \frac{\sinh(\pi t)}{\cosh(\pi t)} \right| \\
&= |\tanh(\pi t)| = \left| \frac{1 - e^{-2\pi t}}{1 + e^{-2\pi t}} \right| \leq 1
\end{aligned}$$

We have shown the existence of some constant M such that $|\cot(\pi z)| \leq M$ for all $z \in C_N$ for all even N. ■

Chapter 4

PDE intro

4.1 1.2 First Order Linear Equations

(Principle of Geometric Method) Given a first order homogeneous linear PDE with the form

$$u_x + g(x,y)u_y = 0$$

We know if a curve $\gamma(x) = (x,y)$ satisfy

$$\gamma'(x) = c_x(1, g(x,y)) \text{ for some } c_x$$

Then

$$(u \circ \gamma)'(x) = 0 \text{ for all } x$$

Since

$$\gamma'(x) = (1, \frac{dy}{dx})$$

To find γ, we only wish to solve

$$\frac{dy}{dx} = g(x,y)$$

Question 106

Solve

$$(1+x^2)u_x + u_y = 0$$

Proof. The ODE

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

has general solution $y = \arctan x + C$, so

$$u(x,y) = f(y - \arctan x)$$

∎

Question 107

Solve

$$\begin{cases} yu_x + xu_y = 0 \\ u(0,y) = e^{-y^2} \end{cases}$$

In which region of the xy plane is the solution uniquely determined?

Proof. We are required to solve the ODE

$$\frac{dy}{dx} = \frac{x}{y}$$

Separating the variables and integrate

$$\int yy'dx = \int xdx \implies \frac{y^2}{2} = \frac{x^2}{2} + C$$

This now implies

$$u(x,y) = f(y^2 - x^2)$$

Initial Condition then give us

$$u = e^{x^2-y^2}$$

∎

Question 108

Solve the equation

$$u_x + u_y = 1$$

Proof. Clearly $u = \frac{x}{2} + \frac{y}{2}$ is a solution. We now solve the PDE

$$v_x + v_y = 0$$

Solving the ODE

$$\frac{dy}{dx} = 1$$

we have

$$y = x + C$$

Thus the general solution is

$$u = \frac{x}{2} + \frac{y}{2} + f(y - x)$$

■

Question 109

Solve

$$\begin{cases} u_x + u_y + u = e^{x+2y} \\ u(x,0) = 0 \end{cases}$$

Proof. Let $\gamma(x) = x + C$, we have

$$(u \circ \gamma)' + (u \circ \gamma) = e^{3x+2C}$$

We now solve the ODE

$$y' + y = e^{3x+2C}$$

The particular solution is clearly

$$y = \frac{1}{4}e^{3x+2C}$$

Thus the general solution is

$$y = \frac{e^{3x+2C}}{4} + \tilde{C}e^{-x}$$

We then now know

$$(u \circ \gamma)(x) = \frac{e^{3x+2C}}{4} + \tilde{C}e^{-x} \tag{4.1}$$

In other words,

$$u(x, x + C) = \frac{e^{3x+2C}}{4} + \tilde{C}e^{-x}$$

Putting in the initial conditions, we have

$$0 = u(-C, 0) = \frac{e^{-C}}{4} + \tilde{C}e^{C}$$

This implies

$$\tilde{C} = \frac{-e^{-2C}}{4}$$

Putting the back into Equation 4.1, we have

$$u(x, x + C) = \frac{e^{3x+2C} - e^{-x-2C}}{4}$$

So

$$u(x, y) = \frac{e^{3x+2(y-x)} - e^{-x-2(y-x)}}{4}$$

■

Question 110

Use the coordinate method to solve the equation

$$u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2$$

Proof. Let

$$\begin{cases} \xi \triangleq 2x - y \\ \eta \triangleq x + 2y \end{cases}$$

We see

$$\begin{cases} u_\xi = \frac{2}{5}u_x - \frac{1}{5}u_y \\ u_\eta = \frac{1}{5}u_x + \frac{2}{5}u_y \end{cases}$$

We then can rewrite

$$5u_\eta + \xi u = \xi\eta \tag{4.2}$$

which clearly have particular solution

$$u = \eta - \frac{5}{\xi}$$

To solve the linear homogeneous PDE

$$5u_\eta + \xi u = 0$$

Observe that for all fixed ξ, the PDE is just an ODE whose solution is exactly $u = C_\xi e^{\frac{-\xi\eta}{5}}$. We now know the general solution for PDE 4.2 is exactly

$$u = \eta - \frac{5}{\xi} + f(\xi)e^{\frac{-\xi\eta}{5}}$$

Then

$$u = x + 2y - \frac{5}{2x - y} + e^{\frac{-(2x-y)(x+2y)}{5}} f(2x - y)$$

$\blacksquare$

4.2 1.4 Initial and Boundary Condition

Question 111

Find a solution of

$$\begin{cases} u_t = u_{xx} \\ u(x,0) = x^2 \end{cases}$$

Proof. Clearly $u = x^2 + 2t$ suffices. ∎

4.3 1.5 Well Posed Problems

Given a vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$, Divergence Theorem shows

$$\iiint_D \nabla \cdot F dV = \iint_{\text{bdy } D} F \cdot \mathbf{n} dS$$

Then if F is the gradient of some scalar field $f : \mathbb{R}^3 \to \mathbb{R}$, we have

$$\iiint_D \Delta f dV = \iint_{\text{bdy } D} \frac{\partial f}{\partial \mathbf{n}} dS$$

Question 112

Consider the ODE

$$\begin{cases} u'' + u = 0 \\ u(0) = 0 \text{ and } u(L) = 0 \end{cases}$$

Is the solution unique? Does the answer depend on L?

Proof. We know the general solution space is exactly spanned by $\cos x$ and $\sin x$. Because

(a) $u(0) = 0$.

(b) $\sin 0 = 0$

(c) $\cos 0 = 1$

we know the solution of our original ODE must be of the form

$$u(x) = C \sin x$$

This implies that the solution is unique if and only if $2\pi \not\equiv L \pmod{2\pi}$ ■

Question 113

Consider the ODE

$$\begin{cases} u'' + u' = f \\ u'(0) = u(0) = \frac{1}{2}(u'(l) + u(l)) \end{cases}$$

where f is given.

(a) Is the solution unique?

(b) Does a solution necessarily exist, or is there a condition that f must satisfy for existence?

Proof. The solution space of linear homogeneous ODE $u'' + u' = 0$ is spanned by e^{-x} and constant. If we add in the initial condition $u'(0) = u(0)$, then the solution space become the subspace spanned by $e^{-x} - 2$. One can check that if $u \in \text{span}(e^{-x} - 2)$, then

$$u(0) = \frac{1}{2}(u'(l) + u(l)) \text{ for all } l \in \mathbb{R}$$

We now know the solution of the original ODE is not unique, since any solution added by $e^{-x} - 2$ is again a solution.

Integrating both side on $[0, l]$, we see that given the boundary conditions, f must satisfy

$$\begin{aligned}\int_0^l f(x)dx &= \int_0^l u'' + u'dx \\ &= u(l) + u'(l) - u(0) - u'(0) = 0\end{aligned}$$

■

Question 114

Consider the Neumann problem

$$\Delta u = f(x, y, z) \text{ in } D \text{ and } \frac{\partial u}{\partial n} = 0 \text{ on } \text{ bdy } D$$

(a) What can we add to any solution to get another solution?

(b) Use the divergence Theorem and the PDE to show that

$$\iiint_D f(x, y, z)dxdydz = 0$$

is a necessary condition for the Neumann problem to have a solution.

Proof. Clearly, constants suffices, and observe

$$\iiint_D fdxdydx = \iiint_D \Delta udxdydz = \iiint_D \nabla \cdot (\nabla u)dxdydz = \iint_{\text{bdy } D} \nabla u \cdot \mathbf{n}dS = 0$$

■

Question 115

Consider the equation

$$u_x + y u_y = 0$$

with the boundary condition $u(x, 0) = \varphi(x)$.

(a) $\varphi(x) = x \implies$ no solution exists

(b) $\varphi(x) = 1 \implies$ multiple solutions exist.

Proof. Using the geometric method, we see the characteristic curve is exactly $y = \tilde{C}e^x$. Thus the general solution is of the form

$$u(x, y) = f(e^{-x}y)$$

The boundary condition implies

$$\varphi(x) = u(x, 0) = f(0)$$

The result then follows. ∎

4.4 1.6 Types of Second-Order Equations

Consider the constant coefficient Linear PDE

$$u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0$$

Ignoring the term with order less than 2, we have

$$u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} = 0$$

Or equivalently

$$(\partial_x + a_{12}\partial_y)^2u + (a_{22} - a_{12}^2)\partial_{yy}u = 0$$

Now, if we set

$$x \triangleq \xi \text{ and } y \triangleq a_{12}\xi + (|a_{22} - a_{12}^2|)^{\frac{1}{2}}\eta$$

we see that

$$\begin{cases} a_{22} - a_{12}^2 > 0 \implies u_{\xi\xi} + u_{\eta\eta} = 0 \quad \text{(Elliptic)} \\ a_{22} - a_{12}^2 = 0 \implies u_{\xi\xi} = 0 \quad \text{(Parabolic)} \\ a_{22} - a_{12}^2 < 0 \implies u_{\xi\xi} - u_{\eta\eta} = 0 \quad \text{(Hyperbolic)} \end{cases}$$

More generally, if we are given

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} = 0$$

then the discriminant is exactly

$$a_{11}a_{22} - a_{12}^2$$

Question 116

What is the type of each of the following equations.

(a) $u_{xx} - u_{xy} + u_{yy} + \cdots + u = 0$.

(b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$

Proof. The discriminant for (a) and (b) are respectively $\frac{3}{4}$ are 0, thus elliptic and parabolic. ∎

Question 117

Find the regions in the xy plane where the equation

$$(1+x)u_{xx} + 2xyu_{xy} - y^2 u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch them.

Proof. The discriminant is exactly

$$\begin{aligned}(xy)^2 - (1+x)(-y^2) &= x^2y^2 + xy^2 + y^2 \\ &= y^2(x^2+x+1) \\ &= y^2[(x+\frac{1}{2})^2 + \frac{3}{4}]\end{aligned}$$

It then follows that the equation is parabolic if and only if $y=0$, and hyperbolic if and only if $y \neq 0$.

■

Question 118

Reduce the elliptic equation

$$u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u = 0$$

to the form

$$v_{xx} + v_{yy} + cv = 0$$

by a change of dependent variable

$$u \triangleq ve^{\alpha x + \beta y}$$

then a change of scale

$$y' = \gamma y$$

Proof. The original elliptic equation can be written in the form

$$e^{\alpha x+\beta y}\Big[v_{xx} + 2\alpha v_x + \alpha^2 v + 3(v_{yy} + 2\beta v_y + \beta^2 v) - 2(v_x + \alpha v) + 24(v_y + \beta v) + 5v\Big] = 0$$

which implies

$$v_{xx} + 3v_{yy} + (2\alpha - 2)v_x + (6\beta + 24)v_y + (\alpha^2 + 3\beta^2 - 2\alpha + 24\beta + 5)v = 0$$

Letting $\alpha \triangleq 1$ and $\beta \triangleq -4$, we see

$$v_{xx} + 3v_{yy} + \tilde{c}v = 0$$

Letting $y \triangleq \sqrt{3}y'$, we then see

$$v_{xx} + v_{y'y'} + \tilde{c}v = 0$$

∎

Question 119

Consider the equation $3u_y + u_{xy} = 0$.

(a) What is its type?

(b) Find the general solution. (Hint: Substitute $v = u_y$).

(c) With the auxiliary conditions $u(x,0) = e^{-3x}$ and $u_y(x,0) = 0$, does a solution exist? Is it unique?

Proof. Since the discriminant is exactly $\frac{-1}{4}$, the type is hyperbolic. Letting $v \triangleq u_y$, we see

$$3v + v_x = 0$$

This PDE on each horizontal line is an ODE and thus have the solution

$$u_y = v = f(y)e^{-3x}$$

Then u must be of the form

$$u = F(y)e^{-3x} + g(x)$$

Applying the initial condition $u_y(x,0) = 0$, we see

$$f(0)e^{-3x} = u_y(x,0) = 0$$

which implies $f(0) = 0$. Now apply another initial condition $u(x,0) = e^{-3x}$.

$$F(0)e^{-3x} + g(x) = u(x,0) = e^{-3x}$$

We then see the solutions exist and are not unique, since

$$\begin{cases} F(y) = 1 \\ g(x) = 0 \end{cases} \quad \text{and} \quad \begin{cases} F(y) = y^2 \\ g(x) = e^{-3x} \end{cases}$$

are both solutions. ∎

4.5 2.1 The Wave Equation

Abstract

In this section, $c \in \mathbb{R}^*$.

Theorem 4.5.1. (General Solution of The Wave Equation) The general solution of the wave equation

$$u_{tt} = c^2 u_{xx}$$

is of the form

$$u = f(x+ct) + g(x-ct)$$

Proof. Observe that

$$0 = u_{tt} - c^2 u_{xx} = (\partial_t + c\partial_x)(\partial_t - c\partial_x)u$$

If we let $v = u_t - cu_x$, then we must have $v_t + cv_x = 0$. We know the general solution of v is $v = g(x-ct)$. We have reduce the problem into solving

$$u_t - cu_x = g(x-ct) \tag{4.3}$$

Now observe that for all $w : \mathbb{R} \to \mathbb{R}$

$$(\partial_t - c\partial_x)\big(w(x-ct)\big) = 2cw'(x-ct)$$

We then see the particular solution for Equation 4.3 is

$$u = \frac{1}{2c} G(x-ct)$$

and the general solution is then

$$u = f(x+ct) + \frac{1}{2c} G(x-ct)$$

∎

Theorem 4.5.2. (IVP for The Wave Equation) The Initial Value Problem

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

has exactly one solution

$$u(x,t) = \frac{1}{2}[\varphi(x+ct) + \varphi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds$$

Proof. Write $u(x,t) = f(x+ct) + g(x-ct)$. By initial condition, we know

$$f(x) + g(x) = \varphi(x) \text{ and } f'(x) - g'(x) = \frac{\psi(x)}{c}$$

Differentiating the former, we also have

$$f'(x) + g'(x) = \varphi'(x)$$

This then give us

$$f'(x) = \frac{\varphi'(x)}{2} + \frac{\psi(x)}{2c} \text{ and } g'(x) = \frac{\varphi'(x)}{2} - \frac{\psi(x)}{2c}$$

It now follows that

$$f(s) = \frac{\varphi(s)}{2} + \frac{1}{2c}\int_0^s \psi(x)dx + A \text{ and } g(s) = \frac{\varphi(s)}{2} - \frac{1}{2c}\int_0^s \psi(x)dx + B$$

Note that since $f(x) + g(x) = \varphi(x)$, we know $B = -A$.

We now have

$$\begin{aligned} u(x,t) &= f(x+ct) + g(x-ct) \\ &= \frac{\varphi(x+ct) + \varphi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x)dx \end{aligned}$$

■

Question 120

Solve

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = e^x \\ u_t(x,0) = \sin x \end{cases}$$

Proof. Let

$$u \triangleq f(x+ct) + g(x-ct)$$

Plugging the initial conditions, we know

$$\begin{cases} f(x) + g(x) = e^x \\ f'(x) - g'(x) = \frac{\sin x}{c} \end{cases}$$

This give us

$$f'(x) = \frac{e^x + \frac{\sin x}{c}}{2} \text{ and } g'(x) = \frac{e^x - \frac{\sin x}{c}}{2}$$

Then by FTC, we have

$$\begin{cases} f(x) = \frac{e^x - \frac{\cos x}{c}}{2} + f(0) - \frac{1}{2} + \frac{1}{2c} \\ g(x) = \frac{e^x + \frac{\cos x}{c}}{2} + g(0) - \frac{1}{2} - \frac{1}{2c} \end{cases}$$

Note that $f(0) + g(0) = e^0 = 1$, which cancel the constant terms in u, i.e.

$$\begin{aligned} u &= f(x+ct) + g(x-ct) \\ &= \frac{e^{x+ct} + e^{x-ct} + \frac{-\cos(x+ct)+\cos(x+ct)}{c}}{2} \end{aligned}$$

■

Question 121

If both φ and ψ are odd functions of x, show that the solution of $u(x,t)$ of the wave equation is also odd in x for all t.

Proof. Suppose

$$u = f(x+ct) + g(x-ct)$$

We have

$$\begin{cases} f + g = \varphi \\ f' - g' = \frac{\psi}{c} \end{cases}$$

This give us

$$f' = \frac{\varphi' + \frac{\psi}{c}}{2} \text{ and } g' = \frac{\varphi' - \frac{\psi}{c}}{2}$$

and

$$\begin{cases} f(x) = \frac{1}{2}\int_0^x (\varphi' + \frac{\psi}{c}) ds + f(0) \\ g(x) = \frac{1}{2}\int_0^x (\varphi' - \frac{\psi}{c}) ds + g(0) \end{cases}$$

that is

$$\begin{cases} f(x) = f(0) + \frac{1}{2}\Big[\varphi(x) - \varphi(0)\Big] + \frac{1}{2c}\int_0^x \psi(s) ds \\ g(x) = g(0) + \frac{1}{2}\Big[\varphi(x) - \varphi(0)\Big] - \frac{1}{2c}\int_0^x \psi(s) ds \end{cases}$$

Noting $f+g=\varphi$, we now have

$$\begin{aligned} u(x,t) &= f(x+ct)+g(x-ct) \\ &= \frac{\varphi(x+ct)+\varphi(x-ct)}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds \end{aligned}$$

and

$$\begin{aligned} u(-x,t) &= \frac{\varphi(-x+ct)+\varphi(-x-ct)}{2}+\frac{1}{2c}\int_{-x-ct}^{-x+ct}\psi(s)ds \\ &= \frac{-\varphi(x-ct)-\varphi(x+ct)}{2}-\frac{1}{2c}\int_{x+ct}^{x-ct}\psi(-u)du \quad (\because u=-s) \\ &= \frac{-\varphi(x-ct)-\varphi(x+ct)}{2}+\frac{1}{2c}\int_{x+ct}^{x-ct}\psi(u)du \quad (\because \psi \text{ is odd }) \\ &= \frac{-\varphi(x-ct)-\varphi(x+ct)}{2}-\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(u)du = -u(x,t) \end{aligned}$$

■

Question 122

A spherical wave is a solution of the three-dimensional wave equation of the form $u(r,t)$, where r is the distance to the origin (the spherical coordinate). The wave equation takes the form

$$u_{tt}=c^2\left(u_{rr}+\frac{2}{r}u_r\right)$$

(a) Change variables $v=ru$ to get the equation for $v: v_{tt}=c^2v_{rr}$.

(b) Solve for v and thereby solve the spherical wave equation.

(c) Solve it with the initial condition $u(r,0)=\varphi(r), u_t(r,0)=\psi(r)$, taking both $\varphi(r)$ and $\psi(r)$ to be even functions of r.

Proof. If we let $v=ru$, then

$$v_{tt}=ru_{tt} \text{ and } v_{rr}=ru_{rr}+2u_r$$

This then give us

$$v_{tt}=ru_{tt}=rc^2\left(u_{rr}+\frac{2}{r}u_r\right)=c^2(ru_{rr}+2u_r)=c^2v_{rr}$$

The general solution of v is

$$v = f(r+ct) + g(r-ct)$$

and thus

$$u(r,t) = \frac{f(ct+r)+g(r-ct)}{r}$$

Plugging the initial conditions, we have

$$\frac{f(r)+g(r)}{r} = u(r,0) = \varphi(r) \text{ and } \frac{c[f'(r)-g'(r)]}{r} = u_t(r,0) = \psi(r)$$

In other words,

$$\begin{cases} f(r)+g(r) = r\varphi(r) \\ f'(r)-g'(r) = \frac{r\psi(r)}{c} \end{cases}$$

Differentiating the first equation, we have

$$f'(r)+g'(r) = \varphi(r) + r\varphi'(r)$$

We now can solve f', g'

$$f'(r) = \frac{\varphi(r)+r\varphi'(r)+\frac{r\psi(r)}{c}}{2} \text{ and } g'(r) = \frac{\varphi(r)+r\varphi'(r)-\frac{r\psi(r)}{c}}{2}$$

We now have

$$\begin{aligned} f(r) &= f(1) + \int_1^r f'(s)ds \\ &= f(1) + \left[\frac{s\varphi(s)}{2}\right]\Big|_{s=1}^r + \frac{1}{2c}\int_1^r s\psi(s)ds \end{aligned}$$

and

$$\begin{aligned} g(r) &= g(1) + \int_1^r g'(s)ds \\ &= g(1) + \left[\frac{s\varphi(s)}{2}\right]\Big|_{s=1}^r - \frac{1}{2c}\int_1^r s\psi(s)ds \end{aligned}$$

Noting that $f(1)+g(1) = 1\varphi(1)$, we can cancel these terms and get

$$\begin{aligned} u(r,t) &= \frac{f(r+ct)+g(r-ct)}{r} \\ &= \frac{(r+ct)\varphi(r+ct)+(r-ct)\varphi(r-ct)}{2r} + \frac{1}{2cr}\int_1^{r+ct} s\varphi(s)ds - \frac{1}{2cr}\int_1^{r-ct} s\varphi(s)ds \\ &= \frac{(r+ct)\varphi(r+ct)+(r-ct)\varphi(r-ct)}{2r} + \frac{1}{2cr}\int_{r-ct}^{r+ct} s\varphi(s)ds \end{aligned}$$

■

Question 123

Solve

$$\begin{cases} u_{xx} + u_{xt} - 20u_{tt} = 0 \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

Proof. The PDE can be write in the form of

$$(\partial_x + 5\partial_t)(\partial_x - 4\partial_t)u = 0$$

which have the general solution

$$u(x,t) = f(5x - t) + g(4x + t)$$

We now see

$$f(5x) + g(4x) = \varphi(x) \text{ and } -f'(5x) + g'(4x) = \psi(x)$$

This then give us

$$f'(5x) = \frac{(\varphi' - 4\psi)(x)}{9} \text{ and } g'(4x) = \frac{(\varphi' + 5\psi)(x)}{9}$$

and thus

$$\begin{aligned} f(x) &= f(0) + \int_0^x f'(s)ds \\ &= f(0) + \int_0^x \frac{(\varphi' - 4\psi)(\frac{s}{5})}{9} ds \\ &= f(0) + \frac{5}{9}\Big[\varphi(\frac{x}{5}) - \varphi(0)\Big] - \frac{4}{9}\int_0^x \psi(\frac{s}{5})ds \end{aligned}$$

and similarly

$$\begin{aligned} g(x) &= g(0) + \int_0^x g'(s)ds \\ &= g(0) + \int_0^x \frac{(\varphi' + 5\psi)(\frac{s}{4})}{9} ds \\ &= g(0) + \frac{4}{9}\Big[\varphi(\frac{x}{4}) - \varphi(0)\Big] + \frac{5}{9}\int_0^x \psi(\frac{s}{4})ds \end{aligned}$$

Noting that $f(0)+g(0)=u(0,0)=\psi(0)$, we now have

$$\begin{aligned}
u(x,t) &= f(5x-t)+g(4x+t) \\
&= \frac{5\varphi(\frac{5x-t}{5})+4\varphi(\frac{4x+t}{4})}{9} - \frac{4}{9}\int_0^{5x-t}\psi(\frac{s}{5})ds + \frac{5}{9}\int_0^{4x+t}\psi(\frac{s}{4})ds
\end{aligned}$$

■

Question 124

Find the general solution of

$$3u_{tt}+10u_{xt}+3u_{xx}=\sin(x+t)$$

Proof. Clearly, we can rewrite the equation to be

$$(3\partial_x+\partial_t)(\partial_x+3\partial_t)u=\sin(x+t)$$

If we let $v=u_x+3u_t$, then we have

$$3v_x+v_t=\sin(x+t)$$

Define

$$\gamma(x)\triangleq(x,\frac{x}{3}+c)$$

We then see

$$(v\circ\gamma)'(x)=\frac{\sin(x+\frac{x}{3}+c)}{3}$$

which implies

$$v(x,\frac{x}{3}+c)=v\circ\gamma(x)=\frac{\cos(\frac{4x}{3}+c)}{-4}+C_c$$

and thus

$$\begin{aligned}
v(x,t) &= \frac{\cos(\frac{4x}{3}+t-\frac{x}{3})}{-4}+f(t-\frac{x}{3}) \\
&= \frac{\cos(x+t)}{-4}+f(3t-x)
\end{aligned}$$

It remains to solve

$$u_x+3u_t=\frac{\cos(x+t)}{-4}+f(3t-x)$$

Now define

$$\gamma(x) \triangleq (x, 3x + c)$$

We then see

$$(u \circ \gamma)'(x) = \frac{\cos(4x + c)}{-4} + f(8x + 3c)$$

which implies

$$u(x, 3x + c) = \frac{\sin(4x + c)}{-16} + \frac{F(8x + 3c)}{8} + u(0, c) + f(c)$$

and thus

$$u(x, t) = \frac{\sin(x + t)}{-16} + \tilde{F}(-x + 3t) + g(t - 3x)$$

where g is the initial condition. ■

4.6 2.2 Causality and Energy

Question 125

Show that the wave equation has the following invariant properties

(a) Any translate $u(x-y,t)$ where y is fixed, is also a solution.

(b) Any derivative, say u_x, is also a solution.

(c) The dilated function $u(ax,at)$ is also a solution.

Proof. The first property follows from direct computation, the second property follows from $0_x = 0$ and the third property follows from observing $v \triangleq u(ax,at)$ satisfy $v_{tt} = a^2 u_{tt} = a^2c^2 u_{xx} = v_{xx}$. ∎

Question 126

If $u(x,t)$ satisfy the wave equation $u_{tt} = u_{xx}$, prove the identity

$$u(x+h,t+k) + u(x-h,t-k) = u(x+k,t+h) + u(x-k,t-h)$$

Proof. Define $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(x) \triangleq u(x,0) \text{ and } \psi(x) \triangleq u_t(x,0)$$

We then know that

$$\begin{aligned} u(x,t) &= \frac{\varphi(x+t) + \varphi(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \psi(s)ds \\ &\triangleq \frac{A(x,t) + B(x,t) + C(x,t)}{2} \end{aligned}$$

where

$$\begin{cases} A(x,t) \triangleq \varphi(x+t) \\ B(x,t) \triangleq \varphi(x-t) \\ C(x,t) \triangleq \int_{x-t}^{x+t} \psi(s)ds \end{cases}$$

The proof then follows from

$$\begin{aligned} A(x+h,t+k) = A(x+k,t+h) &\text{ and } A(x-h,t-k) = A(x-k,t-h) \\ B(x+h,t+k) = B(x-k,t-h) &\text{ and } B(x-h,t-k) = B(x+k,t+h) \\ C(x+h,t+k) = C(x+k,t+h) &\text{ and } C(x-h,t-k) = C(x-k,t-h) \end{aligned}$$

∎

Question 127

Suppose that we have the PDE of damped string

$$\begin{cases} \rho u_{tt} - T u_{xx} + r u_t = 0 \text{ where } r > 0 \\ u(x,0) = 0 \text{ if } |x| > N \end{cases}$$

Show that if we define the energy $E(t)$ of this system by

$$E(t) = \int_{-\infty}^{\infty} (\rho u_t^2 + T u_x^2) dx$$

Then the energy decrease as time goes.

Proof. Because u is smooth, we have

$$\begin{aligned} E'(t) &= \int_{-\infty}^{\infty} (\rho u_t^2 + T u_x^2)_t dx \\ &= \int_{-\infty}^{\infty} (2\rho u_t u_{tt} + 2T u_x u_{xt}) dx \\ &= \int_{-\infty}^{\infty} [2u_t(T u_{xx} - r u_t) + 2T u_x u_{xt}] dx \\ &= \int_{-\infty}^{\infty} [2T(u_t u_x)_x - 2r u_t^2] dx \\ &= 2T u_t t_x \Big|_{x=0}^{\infty} - \int_{-\infty}^{\infty} 2r u_t^2 \\ &= - \int_{-\infty}^{\infty} 2r u_t^2 \le 0 \end{aligned}$$

■

4.7 2.3 The Diffusion Equation

In this section, we shall first set

$$\Omega \triangleq (0,l) \text{ and } \Gamma \triangleq \Omega \times \{0\} \cup \partial\Omega \times [0,T]$$

and write

$$\Omega_T \triangleq \Omega \times (0,T)$$

We suppose $u : \overline{\Omega_T} \to \mathbb{R}$ satisfy

$$u \in C^2(\Omega \times (0,T])$$

If u achieve a maximum on $\Omega \times (0,T]$, then at that point u must have

$$u_t \geq 0 \text{ and } u_{xx} \leq 0$$

Theorem 4.7.1. (Weak Maximum Principle) If

$$u_t - ku_{xx} \leq 0 \text{ on } \Omega \times (0,T] \tag{4.4}$$

then u must achieve its maximum at Γ.

Proof. Because Γ is compact, we know there exists a maximum M of u on Γ. Fix ϵ and define $v : \overline{\Omega_T} \to \mathbb{R}$

$$v(x,t) \triangleq u(x,t) + \epsilon x^2$$

Because

$$u(x,t) \leq \max_{\overline{\Omega_T}} v - \epsilon x^2 \text{ for all } (x,t) \in \overline{\Omega_T}$$

we can reduce the problem into proving

$$\max_{\overline{\Omega_T}} v \leq M + \epsilon l^2 \text{ for all } p \in F$$

From Equation 4.4, we can deduce

$$v_t - kv_{xx} = u_t - ku_{xx} - 2k\epsilon < 0$$

Because v is continuous, we know v attain its maximum at some point. Now, with diffusion inequality, we can deduce

(a) The maximum of v must not be in Ω_T, otherwise at that point $v_t = 0$ and $v_{xx} \leq 0$ yield a contradiction.

(b) The maximum of v must also not be in the top edge $\partial\Omega_T \setminus \Gamma$, otherwise $v_t \geq 0$ and $v_{xx} \leq 0$ yield a contradiction.

We have proved that v can only attain maximum at some point $(x_0, t_0) \in F_0$, and it follows that

$$\max_{(x,t)\in F} v(x,t) = v(x_0,t_0) = u(x_0,t_0) + \epsilon x_0^2 \leq M + \epsilon l^2 \text{ (done)}$$

■

Corollary 4.7.2. (Weak Minimum Principle) The minimum of u must also happen on F_0.

Now, consider the Dirichlet's problem

$$\begin{cases} u_t - ku_{xx} = f(x,t) \text{ for } 0 < x < l \text{ and } t > 0 \\ u(x,0) = \varphi(x) \text{ for } 0 \leq x \leq l \\ u(0,t) = g(t) \text{ and } u(l,t) = h(t) \text{ for } t \geq 0 \end{cases} \tag{4.5}$$

Note that for all T, because the difference w of two solution u_1, u_2 for Dirichlet's function must satisfy

$$\begin{cases} w_t = kw_{xx} \text{ on } \overline{\Omega_T} \setminus \Gamma \\ w(x,0) = w(0,t) = 0 \text{ for any } 0 \leq x \leq l \text{ and } 0 \leq t \leq T \end{cases}$$

By minimum and maximum principle we can deduce $w = 0$ on Ω, and thus $u_1 = u_2$ on F. It then follows that $u_1 = u_2$ on $[0,l] \times [0,\infty)$.

Theorem 4.7.3. (Uniqueness for the Dirichlet's problem for the diffusion equation with Energy Method) If $u_1, u_2 : [0,l] \times [0,\infty)$ are both solution of the Dirichlet's problem, then $u_1 = u_2$.

Proof. Define $w : [0,l] \times [0,\infty) \to \mathbb{R}$ by $w = u_1 - u_2$. Multiplying w with $(w_t - kw_{xx})$, we see that for all $x \in (0,l)$ and $t > 0$,

$$0 = (w_t - kw_{xx})w = \left(\frac{w^2}{2}\right)_t + (-kw_x w)_x + kw_x^2$$

Because $w(0,t) = w(l,t) = 0$ for all t, it follows that for all $t > 0$

$$\begin{aligned} 0 &= \int_0^l \left[\left(\frac{w^2}{2}\right)_t + (-kw_x w)_x + kw_x^2\right] dx \\ &= \int_0^l \left[\left(\frac{w^2}{2}\right)_t + kw_x^2\right] dx \end{aligned}$$

which implies

$$I'(t) \le 0 \text{ if we define } I : [0,\infty) \to \mathbb{R} \text{ by } I(t) \triangleq \int_0^l \Big(\frac{w^2}{2}\Big) dx$$

Because $I(0) = 0$ by definition and $I(t)$ are integrals of non-negative functions, we can deduce I is identically 0. The desired result $w(x,t) = 0$ for all $x, t \in [0,l] \times [0,\infty)$ then follows. ■

Now, consider Dirichlet's problem with different initial conditions $\varphi_1, \varphi_2 : [0,l] \to \mathbb{R}$, and suppose $u_1, u_2 : [0,l] \times [0,\infty)$ are corresponding solutions. The maximum and minimum principle give us a L^∞ estimation for stability

$$\max_{[0,l]\times[0,\infty)} |u_1 - u_2| \le \max_{[0,l]} |\varphi_1 - \varphi_2|$$

While the energy method give us a L^2 estimation for stability: For all $t \ge 0$,

$$\int_0^l \Big(\frac{w^2(x,t)}{2}\Big) dx = I(t) \le I(0) = \int_0^l \Big(\frac{w^2(x,0)}{2}\Big) = \int_0^l \frac{(\varphi_1 - \varphi_2)^2}{2} dx$$

Question 128

Consider the diffusion equation

$$\begin{cases} u_t = u_{xx} \text{ on } (0,1) \times (0,\infty) \\ u(0,t) = u(1,t) = 0 \text{ for all } t > 0 \\ u(x,0) = 1 - x^2 \end{cases}$$

(a) Show that $u(x,t) > 0$ for all $(x,t) \in (0,1) \times (0,\infty)$.

(b) Define $\mu : (0,\infty) \to \mathbb{R}$ by $\mu(t) \triangleq \max_{x\in[0,1]} u(x,t)$. Show that μ is a decreasing function.

Proof. The proof of (a) follows from a loose application of the strong minimum principle.

The proof of (b) follows from noting $v(x,t) \triangleq u(x, t + t_0) : [0,1] \times [0,\infty)$ also is a solution of the diffusion equation and application of maximum principle on v. ■

Question 129

Consider the Dirichlet's problem

$$\begin{cases} u_t = u_{xx} \text{ on } (0,1) \times (0,\infty) \\ u(0,t) = u(1,t) = 0 \text{ for all } t \geq 0 \\ u(x,0) = 4x(1-x) \end{cases}$$

Show that

(a) $0 < u(x,t) < 1$ for all $t > 0$ and $0 < x < 1$.

(b) $u(x,t) = u(1-x,t)$ for all $t \geq 0$ and $0 \leq x \leq 1$.

(c) Use the energy method to show that $\int_0^1 u^2 dx$ is a strictly decreasing function of t.

Proof. (a) follows from application of strong maximum and minimum principle. (b) follows from uniqueness of Dirichlet's problem and the observation that $u(1-x,t)$ is also a solution.

The energy method give us

$$\frac{d}{dt}\int_0^1 \frac{u^2}{4} dx = -\int_0^1 u_x^2 dx \leq 0 \text{ for all } t > 0$$

and (c) follows. ∎

Question 130

Verify that

$$u = -2xt - x^2 \text{ is a solution of } u_t = xu_{xx}$$

and find the location of maximum of t in the close rectangle $\{-2 \leq x \leq 2, 0 \leq t \leq 1\}$.

Proof. Write

$$u = -(x+t)^2 + t^2$$

It follows that the maximum occurs at $t = -x = 1$. ∎

Question 131

Prove the comparison principle for the diffusion equation: If u and v are two solutions and

$$u \leq v \text{ for } t = 0, x = 0, x = l$$

then

$$u \leq v \text{ on } [0, l] \times [0, \infty)$$

Proof. This follows from application of the minimum principle on $v - x$. ■

Question 132

Suppose

$$\begin{cases} u_t - ku_{xx} = f \\ v_t - kv_{xx} = g \end{cases} \quad \text{and } f \leq g$$

and suppose

$$u \leq v \text{ at } x = 0, x = l \text{ and } t = 0$$

Prove that

$$u \leq v \text{ on } [0, l] \times [0, \infty)$$

Proof. Let $w \triangleq u - v : \overline{\Omega_T} \to \mathbb{R}$. It is clear that

$$w_t - kw_{xx} \leq 0 \text{ on } \overline{\Omega_T} \setminus \Gamma$$

It then follows that w attain its maximum on Γ, which must not be greater than 0. ■

4.8 2.4 Diffusion on the whole line

In this section, we are concerned with solving the following initial value problem (**Cauchy problem**)

$$\begin{cases} u_t = ku_{xx} \text{ on } \mathbb{R} \times (0, \infty) \\ \lim_{t\to 0} u(x,t) = \varphi(x) \text{ for all specified } x \end{cases}$$

We shall mostly express our answer with function erf : $\mathbb{R} \to \mathbb{R}$

$$\operatorname{erf}(x) \triangleq \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2} dp$$

Theorem 4.8.1. (Solution of Dirac Initial Condition) If φ is defined to be

$$\varphi(x) \triangleq \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

then a solution is

$$Q(x,t) = \frac{1}{2} + \frac{\operatorname{erf}(\frac{x}{\sqrt{4kt}})}{2} \tag{4.6}$$

Proof. Note that our version of diffusion equation admits dilated solutions. This inspire us to guess

$$Q(x,t) \triangleq g\Big(\frac{x}{\sqrt{4kt}}\Big)$$

Direct computation yields

$$Q_t = \frac{-x}{2\sqrt{4k}t^{\frac{3}{2}}} g'\Big(\frac{x}{\sqrt{4kt}}\Big) \text{ and } Q_{xx} = g''\Big(\frac{x}{\sqrt{4kt}}\Big)\frac{1}{4kt}$$

If we let $p = \frac{x}{\sqrt{4kt}}$, we now have

$$Q_t = \frac{-pg'(p)}{2t} \text{ and } Q_{xx} = \frac{g''(p)}{4kt}$$

Plugging this back to diffusion equation and canceling the common terms, we have

$$\frac{g''(p)}{2} + pg'(p) = 0$$

The general solution to this ODE is

$$g(p) = c_1 \operatorname{erf}(p) + c_2$$

In other words,

$$Q(x,t) = c_1 \operatorname{erf}(\frac{x}{\sqrt{4kt}}) + c_2$$

Plugging this back to the initial condition, we see

$$Q(x,t) = \frac{1 + \operatorname{erf}(\frac{x}{\sqrt{4kt}})}{2}$$

■

Differentiating Equation 4.6 with respect to x, we have another solution

$$S(x,t) \triangleq \frac{1}{2\sqrt{\pi kt}} e^{-\frac{x^2}{4kt}}$$

Solution S is often called the **fundamental solution**, since for all initial condition φ that have compact support, we gain a solution to the initial value problem by

$$u(x,t) \triangleq (S * \varphi)(x,t)$$

where

$$(S * \varphi)(x,t) = \int_{\mathbb{R}} S(x-y,t)\varphi(y)dy$$

This is true because if we define $F(x,y,t) = Q(x-y,t)$, we have

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}} F_x(x,y,t)\varphi(y)dy \\ &= \int_{\mathbb{R}} -F_y(x,y,t)\varphi(y)dy \\ &= -F(x,y,t)\varphi(y)|_{y=-\infty}^{\infty} + \int_{\mathbb{R}} F(x,y,t)\varphi'(y)dy \\ &= \int_{\mathbb{R}} Q(x-y,t)\varphi'(y)dy \end{aligned}$$

and thus

$$\begin{aligned} \text{For all } x, \lim_{t\to 0} u(x,t) &= \int_{\mathbb{R}} \lim_{t\to 0} Q(x-y,t)\varphi'(y)dy \\ &= \int_{-\infty}^{x} \varphi'(y)dy = \varphi(x) \end{aligned}$$

Question 133

Solve the diffusion equation with the initial condition

$$\varphi(x) \triangleq \begin{cases} 1 & \text{if } |x| < l \\ 0 & \text{if } |x| > l \end{cases}$$

Proof. Define

$$F(x, y, t) \triangleq Q(x - y, t)$$

The solution is exactly

$$\begin{aligned} u(x,t) &= (S * \varphi)(x,t) \\ &= \int_{\mathbb{R}} S(x-y,t)\varphi(y)dy \\ &= \int_{-l}^{l} S(x-y,t)dy \\ &= \int_{-l}^{l} F_x(x,y,t)dy \\ &= \int_{-l}^{l} -F_y(x,y,t)dy = F(x,y,t)\Big|_{y=l}^{-l} = Q(x+l,t) - Q(x-l,t) = \frac{\operatorname{erf}(\frac{x+l}{\sqrt{4kt}}) - \operatorname{erf}(\frac{x-l}{\sqrt{4kt}})}{2} \end{aligned}$$

■

Question 134

Solve the diffusion equation with the initial condition

$$\varphi(x) \triangleq \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Proof. Define

$$F(x, y, t) \triangleq Q(x - y, t)$$

The solution is exactly

$$
\begin{aligned}
u(x,t) &= (S * \varphi)(x,t) \\
&= \int_{\mathbb{R}} S(x-y,t)\varphi(y)dy \\
&= \int_0^\infty e^{-y} S(x-y,t)dy \\
&= \frac{1}{2\sqrt{t\pi k}} \int_0^\infty e^{\frac{-(x-y)^2}{4kt} - y} dy \\
&= \frac{1}{2\sqrt{t\pi k}} \int_0^\infty e^{\frac{-(y-(x-2kt))^2 + (x-2kt)^2 - x^2}{4kt}} dy \\
&= \frac{e^{kt-x}}{2\sqrt{t\pi k}} \int_0^\infty e^{\frac{-(y-(x-2kt))^2}{4kt}} dy \\
&= \frac{e^{kt-x}}{\sqrt{\pi}} \int_{\frac{2kt-x}{2\sqrt{kt}}}^\infty e^{-s^2} ds \quad (\because s = \frac{y-(x-2kt)}{2\sqrt{kt}}) \\
&= \frac{e^{kt-x}}{\sqrt{\pi}} \left(\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \operatorname{erf}(\frac{2kt-x}{2\sqrt{kt}}) \right) \\
&= \frac{e^{kt-x}}{2} \left[1 - \operatorname{erf}\left(\frac{2kt-x}{2\sqrt{kt}} \right) \right]
\end{aligned}
$$

■

Question 135

Show that for any fixed $\delta > 0$

$$\max_{\delta \le |x| < \infty} S(x,t) \to 0 \text{ as } t \to 0$$

Proof. Note that for all fixed $t > 0$,

$$\max_{\delta \le |x| < \infty} S(x,t) = \max_{\delta \le |x| < \infty} \frac{1}{2\sqrt{kt\pi}} e^{\frac{-x^2}{4kt}} = \frac{1}{2\sqrt{kt\pi}} e^{\frac{-\delta^2}{4kt}}$$

The proof then follows from noting $e^{\frac{-1}{t}} = o(\sqrt{t})$. ■

Question 136

Let $\varphi(x)$ be a continuous function such that $|\varphi(x)| \leq Ce^{ax^2}$. Show that formula

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt}} \varphi(y)dy$$

for diffusion equation make sense for $0 < t < \frac{1}{4ak}$ but not necessarily for larger t.

Proof. Because φ is continuous, we know

$$e^{\frac{-(x-y)^2}{4kt}} \varphi(y) \text{ is at least measurable in } y \text{ on } \mathbb{R}$$

We now see that if $0 < t < \frac{1}{4ak}$, then

$$\int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4kt}} \varphi(y)dy \leq C \int_{\mathbb{R}} e^{ay^2+b(x-y)^2} dy < \infty \text{ where } b < -a$$

If $t \geq \frac{1}{4ak}$, and we take $\varphi = Ce^{ay^2}$, then we have

$$\int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4kt}} \varphi(y)dy = C \int_{\mathbb{R}} e^{ay^2+b(x-y)^2} dy = \infty$$

because $b \geq -a$.

■

Question 137

Prove the uniqueness of the diffusion problem with Neumann boundary conditions:

$$\begin{aligned} &u_t - ku_{xx} = f(x,t) \text{ for } 0 < x < l, t > 0 \\ &u(x,0) = \varphi(x) \\ &u_x(0,t) = g(t) \text{ and } u_x(l,t) = h(t) \end{aligned}$$

Proof. The proof follows from energy method. ■

Question 138

Solve the diffusion equation with constant dissipation:

$$\begin{aligned} &u_t - ku_{xx} + bu = 0 \text{ for } -\infty < x < \infty \\ &u(x,0) = \varphi(x) \end{aligned}$$

where $b > 0$ is a constant. (Hint: Make the change of variables $u(x,t) = e^{-bt}v(x,t)$)

Proof. If we make the change of variables $v(x,t) \triangleq e^{bt}u(x,t)$, then

$$v_t = e^{bt}(u_t + bu) \text{ and } v_{xx} = e^{bt}u_{xx}$$

It then follows that

$$v_t - kv_{xx} = e^{bt}(u_t + bu - ku_{xx}) = 0$$

The initial condition for v is

$$v(x,0) = u(x,0) = \varphi(x)$$

Then we know

$$v(x,t) = \int_{\mathbb{R}} S(x-y,t)\varphi(y)dy$$

It then follows that

$$u(x,t) = e^{-bt}\int_{\mathbb{R}} S(x-y,t)\varphi(y)dy$$

■

Question 139

Solve the diffusion equation with variable dissipation :

$$\begin{aligned} &u_t - ku_{xx} + bt^2u = 0 \text{ for } -\infty < x < \infty \\ &u(x,0) = \varphi(x) \end{aligned}$$

where $b > 0$ is a constant. (Hint: Make the change of variables $u(x,t) = e^{\frac{-bt^3}{3}}v(x,t)$)

Proof. If we make the change of variables $v(x,t) \triangleq e^{\frac{bt^3}{3}}u(x,t)$, then

$$v_t = e^{\frac{bt^3}{3}}(bt^2u + u_t) \text{ and } v_{xx} = e^{\frac{bt^3}{3}}(u_{xx})$$

It then follows that

$$v_t - kv_{xx} = e^{\frac{bt^3}{3}}(u_t - ku_{xx} + bt^2u) = 0$$

The initial condition for v is

$$v(x,0) = u(x,0) = \varphi(x)$$

It then follows

$$v(x,t) = \int_{\mathbb{R}} S(x-y,t)\varphi(y)dy$$

and

$$u(x,t) = e^{\frac{-bt^3}{3}} \int_{\mathbb{R}} S(x-y,t)\varphi(y)dy$$

■

Question 140

Solve the heat equation with convection:

$$\begin{aligned} &u_t - ku_{xx} + Vu_x = 0 \text{ for } -\infty < x < \infty \\ &u(x,0) = \varphi(x) \end{aligned}$$

(Hint: Go to a moving frame of reference by substituting $y = x - Vt$)

Proof. If we define $v(x,t) \triangleq u(x+Vt,t)$, then

$$v_u = u_t + Vu_x \text{ and } v_{xx} = u_{xx}$$

It then follows that

$$v_t - kv_{xx} = u_t - ku_{xx} + Vu_x = 0$$

Note that v has the initial condition

$$v(x,0) = u(x,0) = \varphi(x)$$

So we have

$$v(x,t) = \int_{\mathbb{R}} S(x-y,t)\varphi(y)dy$$

It then follows

$$u(x,t) = u(x - Vt + Vt, t) = v(x - Vt, t) = \int_{\mathbb{R}} S(x - Vt - y, t)\varphi(y)dy$$

■

Question 141

Show that $S_2(x,y,t) \triangleq S(x,t)S(y,t)$ satisfy the diffusion equation $S_t = k(S_{xx} + S_{yy})$.

Deduce that $S_2(x,y,t)$ is the source function for two-dimensional diffusion.

Proof. We have

$$(S_2)_t(x,y,t) = S_t(x,t)S(y,t) + S(x,t)S_t(y,t)$$

and

$$(S_2)_{xx} = S_{xx}(x,t)S(y,t) \text{ and } (S_2)_{yy} = S(x,t)S_{yy}(y,t)$$

This then give us

$$(S_2)_t - k(S_2)_{xx} - k(S_2)_{yy} = S(y,t)[S_t(x,t) - S_{xx}(x,t)] + S(x,t)[S_t(y,t) - S_{yy}(y,t)] = 0$$

To see that S_2 is indeed fundamental solution, observe

$$\begin{aligned}
\iint_{\mathbb{R}^2} S_2(x-r,y-s,0)\varphi(r,s)drds &= \iint_{\mathbb{R}^2} S(x-r,0)S(y-s,0)\varphi(r,s)drds \\
&= \int_{\mathbb{R}} S(x-r,0)\int_{\mathbb{R}} S(y-s,0)\varphi(r,s)dsdr \\
&= \int_{\mathbb{R}} S(x-r,0)\varphi(r,y)dr \\
&= \varphi(x,y)
\end{aligned}$$

■

Chapter 5

PDE intro 2

5.1 3.1 Diffusion on the half line

Consider the following **Dirichlet boundary condition problem**

$$\begin{cases} v_t - kv_{xx} = 0 \text{ for } x \in (0,\infty) \textbf{ (Homogeneous DE)} \\ v(x,0) = \varphi(x) \textbf{ (IC)} \\ v(0,t) = 0 \textbf{ (Dirichlet BC)} \end{cases}$$

Define $\varphi_{\text{odd}} : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_{\text{odd}}(x) \triangleq \begin{cases} \varphi(x) & \text{if } x > 0 \\ -\varphi(x) & \text{if } x < 0 \end{cases}$$

It then follows that φ_{odd} is an odd function, and we can solve the Cauchy problem with respect to this initial condition φ_{odd} and have the solution

$$u(x,t) \triangleq \int_{-\infty}^{\infty} S(x-y,t)\varphi_{\text{odd}}(y)dy$$

Now, because

$$S(x,t) = \frac{1}{2\sqrt{\pi kt}} e^{\frac{-x^2}{4kt}} \text{ is clearly even in } x$$

We can deduce

$$\begin{aligned} u(-x,t) &= \int_{-\infty}^{\infty} S(-x-y,t)\varphi_{\text{odd}}(y)dy \\ &= -\int_{-\infty}^{\infty} S(x+y,t)\varphi_{\text{odd}}(-y)dy \quad (\because S \text{ is even and } \varphi_{\text{odd is odd}}) \\ &= -\int_{-\infty}^{\infty} S(x-r)\varphi_{\text{odd}}(r)dr = -u(x,t) \quad (\because r=-y) \end{aligned}$$

In other words, we have deduced that u is an odd function in x. It then follows that $u(0,t) = -u(-0,t) = 0$. Then we see that the restriction $v \triangleq u|_{(\mathbb{R}^+)^2}$ form a solution of the Dirichlet boundary condition problem. In particular, we can express v in a form without usage of φ_{odd} if we consider

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} S(x-y,t)\varphi_{\text{odd}}(y)dy \\ &= \int_{0}^{\infty} S(x-y,t)\varphi(y)dy + \int_{-\infty}^{0} S(x-y,t)(-\varphi(-y))dy \\ &= \int_{0}^{\infty} [S(x-y,t) - S(x+y,t)]\varphi(y)dy \\ &= \frac{1}{\sqrt{4\pi kt}}\int_{0}^{\infty} [e^{\frac{-(x-y)^2}{4kt}} - e^{\frac{-(x+y)^2}{4kt}}]\varphi(y)dy \end{aligned}$$

Now, consider the following **Neumann boundary condition problem**

$$\begin{cases} w_t - kw_{xx} = 0 \text{ for } x \in (0,\infty) \ \textbf{(Homogeneous DE)} \\ w(x,0) = \varphi(x) \ \textbf{(IC)} \\ w_x(0,t) = 0 \ \textbf{(Neumann BC)} \end{cases}$$

Define $\varphi_{\text{even}} : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_{\text{even}}(x) \triangleq \begin{cases} \varphi(x) & \text{if } x > 0 \\ \varphi(-x) & \text{if } x < 0 \end{cases}$$

It then follows that φ_{even} is an even function, and we can solve the Cauchy problem with respect to this initial condition φ_{even} and have the solution

$$u(x,t) \triangleq \int_{-\infty}^{\infty} S(x-y,t)\varphi_{\text{even}}(y)dy$$

Again because S is even in x, we can deduce

$$\begin{aligned}
u(-x,t) &= \int_{-\infty}^{\infty} S(-x-y,t)\varphi_{\text{even}}(y)dy \\
&= \int_{-\infty}^{\infty} S(-x-y,t)\varphi_{\text{even}}(-y)dy \\
(\because z=-y) \qquad &= -\int_{\infty}^{-\infty} S(-x+z,t)\varphi_{\text{even}}(z)dz = u(x,t)
\end{aligned}$$

Now, we have proved that u is even in x. This then give $u_x(0,t)=0$, and solve the **Neumann problem** by letting $w \triangleq u|_{(\mathbb{R}^+)^2}$. In particular, we can express u in a form without usage of φ_{even} if we consider

$$\begin{aligned}
u(x,t) &= \int_{-\infty}^{\infty} S(x-y,t)\varphi_{\text{even}}(y)dy \\
&= \int_{0}^{\infty} S(x-y,t)\varphi(y)dy + \int_{-\infty}^{0} S(x-y,t)\varphi(-y)dy \\
&= \int_{0}^{\infty} [S(x-y,t)+S(x+y,t)]\varphi(y)dy \\
&= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} [e^{\frac{-(x-y)^2}{\sqrt{4kt}}} + e^{\frac{-(x+y)^2}{\sqrt{4kt}}}]\varphi(y)dy
\end{aligned}$$

Question 142

Solve

$$\begin{aligned}
&u_t = ku_{xx} \\
&u(x,0) = e^{-x} \\
&u(0,t) = 0
\end{aligned}$$

on the half line $0 < x < \infty$

Proof. Extend the initial condition to

$$\varphi_{\text{odd}}(x) \triangleq \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -e^{x} & \text{if } x < 0 \end{cases}$$

We then solve

$$\begin{aligned}
u(x,t) &= \frac{1}{2\sqrt{\pi kt}}\int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4kt}}\varphi_{\text{odd}}(y)dy \\
&= \frac{1}{2\sqrt{\pi kt}}\Big[\int_0^\infty e^{\frac{-(x-y)^2}{4kt}}e^{-y}dy - \int_{-\infty}^0 e^{\frac{-(x-y)^2}{4kt}}e^{y}dy\Big] \\
&= \frac{1}{2\sqrt{\pi kt}}\Big[\int_0^\infty e^{\frac{-(y-(x-2kt))^2-4ktx+4k^2t^2}{4kt}}dy - \int_{-\infty}^0 e^{\frac{-(y-(x+2kt))^2+4ktx+4k^2t^2}{4kt}}dy\Big] \\
&= \frac{1}{2\sqrt{\pi kt}}\Big[e^{-x+kt}\int_0^\infty e^{\frac{-(y-(x-2kt))^2}{4kt}}dy - e^{x+kt}\int_{-\infty}^0 e^{\frac{-(y-(x+2kt))^2}{4kt}}dy\Big] \\
&= \frac{1}{\sqrt{\pi}}\Big[e^{-x+kt}\int_{\frac{2kt-x}{2\sqrt{kt}}}^\infty e^{-p^2}dp - e^{x+kt}\int_{-\infty}^{\frac{-2kt-x}{2\sqrt{kt}}} e^{-p^2}dp\Big] \\
&= \frac{1}{\sqrt{\pi}}\Big[e^{-x+kt}\Big(\frac{\sqrt{\pi}}{2} - \frac{2}{\sqrt{\pi}}\operatorname{erf}(\frac{2kt-x}{2\sqrt{kt}})\Big) - e^{x+kt}\Big(\frac{\sqrt{\pi}}{2} - \frac{2}{\sqrt{\pi}}\operatorname{erf}(\frac{2kt+x}{2\sqrt{kt}})\Big)\Big]
\end{aligned}$$

■

Question 143

Solve

$$\begin{aligned}
u_t &= ku_{xx} \\
u(x,0) &= 0 \\
u(0,t) &= 1
\end{aligned}$$

on the half line $0 < x < \infty$.

Proof. It is clear that if a function $v(x,t)$ satisfy the diffusion equation and the initial and boundary condition

$$v(x,0) = -1 \text{ and } v(0,t) = 0$$

then $u \triangleq v+1$ is a desired solution. Note that v is just

$$v(x,t) = \int_{\mathbb{R}} S(x-y,t)\varphi_{\text{odd}}(y)dy$$

where

$$\varphi_{\text{odd}}(y) = \begin{cases} -1 & \text{if } y > 0 \\ 0 & \text{if } y = 0 \\ 1 & \text{if } y < 0 \end{cases}$$

■

Question 144

Consider the following problem with a Robin boundary condition:

$$\begin{aligned}
&u_t = ku_{xx} \text{ on the half line } 0 < x < \infty \\
&u(x,0) = x \\
&u_x(0,t) - 2u(0,t) = 0
\end{aligned}$$

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) \triangleq \begin{cases} x & \text{if } x \geq 0 \\ x + 1 - e^{2x} & \text{if } x < 0 \end{cases}$$

and let

$$v(x,t) \triangleq \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt}} f(y)dy$$

(a) What PDE and initial condition does $v(x,t)$ satisfy for $-\infty < x < \infty$?

(b) Let $w = v_x - 2v$. What PDE and initial condition does $w(x,t)$ satisfy $-\infty < x < \infty$?

(c) Show that $f'(x) - 2f(x)$ is an odd function.

(d) Show that w is an odd function of x.

(e) Deduce that v satisfy the Robin condition.

Proof. v satisfy the initial condition: $v(x,0) = f(x)$, and w satisfy the initial conditions

$$w(x,0+) = v_x(x,0+) - 2v(x,0+) = f'(x) - 2f(x)$$

Note that the initial condition for w is $\varphi(x) = f'(x) - 2f(x)$ is odd. It then follows that

$$w(x,t) = \int_{\mathbb{R}} S(x-y,t)\varphi(y)dy \text{ is odd in } x$$

To see v satisfy the Robin condition, observe

$$v_x(0,t) - 2v(0,t) = w(0,t) = 0$$

∎

Question 145

Generalize the method of the last exercises to the case of general initial data $\varphi(x)$ and arbitrary constant coefficient for $u(0,t)$ in the boundary condition.

Proof. We are required to solve

$$\begin{cases} u_t - ku_{xx} = 0 \text{ for } x \in (0,\infty) \textbf{ (Homogeneous DE)} \\ u(x,0) = \varphi(x) \textbf{ (IC)} \\ u_x(0,t) - cu(0,t) = 0 \textbf{ (Robin BC)} \text{ where } c > 0 \text{ is some constant} \end{cases}$$

If function $f : \mathbb{R}^* \to \mathbb{R}$ satisfy

(a) $f(x) \triangleq \varphi(x)$ for $x > 0$

(b) $f'(x) - cf(x)$ is odd for $x \neq 0$

then the function

$$u(x,t) \triangleq \int_{\mathbb{R}} S(x-y,t)f(y)dy \text{ for } x \in \mathbb{R}$$

suffice initial condition. To see that u satisfy the Robin boundary condition, observe that $u_x - cu$ is a solution to the diffusion equation with initial condition

$$\begin{aligned}(u_x - cu)(x,0) &= \lim_{h\to 0} \frac{u(x+h,0) - u(x,0)}{h} - cu(x,0) \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - cf(x) \\ &= f'(x) - cf(x) \text{ for } x \in \mathbb{R}\end{aligned}$$

which with Theorem of uniqueness of solution implies

$$(u_x - cu)(x,t) = \int_{\mathbb{R}} S(x-y,t)[f'(y) - cf(y)]dy$$

It then follows from $f' - cf$ is odd that $(u_x - cu)$ is odd in x, and thus $(u_x - cu)(0,t) = 0$. ■

5.2 3.2 Reflection of waves

We now consider the **Dirichlet's problem for wave on the half line** $(0,\infty)$

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0 \text{ for } x \in (0,\infty) \textbf{ (Homogeneous DE)} \\ v(x,0) = \varphi(x), v_t(x,0) = \psi(x) \textbf{ (IC)} \\ v(0,t) = 0 \textbf{ (BC)} \end{cases}$$

One can check that if we again extend $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}$ to odd function $\varphi_{\text{odd}}, \psi_{\text{odd}} : \mathbb{R} \to \mathbb{R}$

$$\varphi_{\text{odd}}(x) \triangleq \begin{cases} \varphi(x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\varphi(-x) & \text{if } x < 0 \end{cases} \text{ and } \psi_{\text{odd}}(x) \triangleq \begin{cases} \psi(x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\psi(-x) & \text{if } x < 0 \end{cases}$$

and solve the **Cauchy's problem for wave on the whole line** with respect to them

$$u(x,t) \triangleq \frac{\varphi_{\text{odd}}(x+ct) + \varphi_{\text{odd}}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{odd}}(y)dy$$

then its restriction $v \triangleq u|_{[0,\infty)\times\mathbb{R}}$ is again a solution to the Dirichlet's problem for wave on the half line, where the boundary condition follows from u being odd in x as easily checked.

Consider also the **Neumann problem for wave on half line**

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0 \text{ for } x \in (0,\infty) \textbf{ (Homogeneous DE)} \\ v(x,0) = \varphi(x), v_t(x,0) = \psi(x) \textbf{ (IC)} \\ v_x(0,t) = 0 \textbf{ (BC)} \end{cases}$$

Question 146

Solve the Neumann problem for the wave equation on the half-line $0 < x < \infty$.

Proof. Define

$$\varphi_{\text{even}}(x) \triangleq \begin{cases} \varphi(x) & \text{if } x > 0 \\ \varphi(-x) & \text{if } x < 0 \end{cases} \text{ and } \psi_{\text{even}}(x) \triangleq \begin{cases} \psi(x) & \text{if } x > 0 \\ \psi(-x) & \text{if } x < 0 \end{cases}$$

and

$$u(x,t) \triangleq \frac{\varphi_{\text{even}}(x+ct) + \varphi_{\text{even}}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{even}}(s)ds$$

■

Question 147

Solve

$$\begin{cases} u_{tt} = 4u_{xx} \text{ for } x \in (0, \infty) \textbf{ (Homogeneous DE)} \\ u(x,0) = 1, u_t(x,0) = 0 \textbf{ (IC)} \\ u(0,t) = 0 \textbf{ (Dirichlet BC)} \end{cases}$$

using the reflection method. The solution has a singularity. Find its location.

Proof. Define

$$\varphi(x) \triangleq \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases} \text{ and } \psi(x) \triangleq 0$$

We are required to solve the following Dirichlet's problem for wave equation

$$u_{tt} = 4u_{xx} \text{ for } -\infty < x < \infty$$
$$u(x,0) = \varphi(x) \text{ and } u_t(x) = \psi(x)$$

The solution is exactly

$$\begin{aligned} u(x,t) &= \frac{\varphi(x+2t) + \varphi(x-2t)}{2} + \int_{x-2t}^{x+2t} \psi(s)ds \\ &= \frac{\varphi(x+2t) + \varphi(x-2t)}{2} \\ &= \begin{cases} 1 & \text{if } x - 2t > 0 \\ 0 & \text{if } x + 2t > 0 > x - 2t \\ -1 & \text{if } 0 > x + 2t \end{cases} \end{aligned}$$

On the half line, the solution is

$$u(x,t) = \begin{cases} 1 & \text{if } x - 2t > 0 \\ 0 & \text{if } x - 2t < 0 \end{cases}$$

So the singularity is exactly on the line $x - 2t = 0$ ■

Question 148

Solve

$$\begin{cases} u_{tt} = c^2 u_{xx} \text{ for } x \in (0,\infty), t \in [0,\infty) \textbf{ (Homogeneous DE)} \\ u(x,0) = 0, u_t(x,0) = V \textbf{ (IC)} \\ au_x(0,t) + u_t(0,t) = 0 \textbf{ (BC)} \end{cases}$$

Proof. Define

$$v(x,t) \triangleq au_x(x,t) + u_t(x,t)$$

Compute

$$v(0,t) = 0$$

Compute

$$\begin{aligned} v(x,0) &= au_x(x,0) + u_t(x,0) \\ &= 0 + V = V \end{aligned}$$

Compute

$$\begin{aligned} v_t(x,0) &= au_{xt}(x,0) + u_{tt}(x,0) \\ &= a(u_t(x,0))_x + c^2 u_{xx}(x,0) = 0 \end{aligned}$$

Then by reflection method, we see

$$v(x,t) \triangleq \frac{\varphi(x+ct) + \varphi(x-ct)}{2}$$

where

$$\varphi(x) \triangleq \begin{cases} V & \text{if } x > 0 \\ -V & \text{if } x < 0 \end{cases}$$

which implies

$$v(x,t) = \begin{cases} V & \text{if } x - ct > 0 \\ 0 & \text{if } x + ct > 0 > x - ct \\ -V & \text{if } 0 > x + ct \end{cases}$$

We are now required to solve

$$au_x + u_t = \begin{cases} V & \text{if } x - ct > 0 \\ 0 & \text{if } x - ct < 0 \end{cases}$$

on the first quadrant. Geometric method (If we require u to be continuous on the singularity) then shows

$$u(x,t) \triangleq \begin{cases} Vt & \text{if } x-ct>0 \\ \frac{V}{a-c}(at-x) & \text{if } x-ct<0 \end{cases}$$

■

Question 149

Find $u(\frac{2}{3},2), u(\frac{1}{4},\frac{7}{2})$ if

$$\begin{cases} u_{tt}=u_{xx} \text{ for } x\in(0,1) \textbf{ (Homogeneous DE)} \\ u(x,0)=x^2(1-x), u_t(x,0)=(1-x)^2 \textbf{ (IC)} \\ u(0,t)=u(1,t)=0 \textbf{ (BC)} \end{cases}$$

Proof. Extend the IC "oddly". With some tedious effort, we see

$$u(\frac{2}{3},2)=\frac{4}{27} \text{ and } u(\frac{1}{4},\frac{7}{2})=\frac{-1}{48}$$

■

Question 150

Solve

$$\begin{cases} u_{tt}=9u_{xx} \text{ for } x\in(0,\frac{\pi}{2}) \textbf{ (Homogeneous DE)} \\ u(x,0)=\cos x, u_t(x,0)=0 \textbf{ (IC)} \\ u_x(0,t)=0 \text{ and } u(\frac{\pi}{2},t)=0 \textbf{ (BC)} \end{cases}$$

Proof. Define

$$\varphi(x)\triangleq \cos x \text{ if } x\in(\frac{-\pi}{2},\pi)$$

and let φ have period $\frac{3\pi}{2}$. The solution is

$$u=\frac{\varphi(x+3t)+\varphi(x-3t)}{2}$$

■

5.3 3.3 Diffusion with a source

If we consider the non homogeneous diffusion equation

$$\begin{aligned} &u_t - ku_{xx} = f(x,t) \text{ for } -\infty < x < \infty, t > 0 \\ &u(x,0) = \varphi(x) \end{aligned}$$

we have the following

Theorem 5.3.1. (Diffusion with a source) If $f, \varphi : \mathbb{R} \to \mathbb{R}$ are smooth function tend to 0 as $|x| \to \infty$, then

$$u(x,t) = \int_{\mathbb{R}} S(x-y,t)\varphi(y)dy + \int_0^t \int_{\mathbb{R}} S(x-y,t-s)f(y,s)dyds$$

is a solution to the diffusion equation

$$\begin{aligned} &u_t - ku_{xx} = f(x,t) \text{ for } -\infty < x < \infty, t > 0 \\ &u(x,0+) = \varphi(x) \text{ for } -\infty < x < \infty \end{aligned}$$

Proof. It is clear that u satisfy the initial condition, and its first term satisfy the homogeneous diffusion equation. We only have to show

$$v(x,t) \triangleq \int_0^t \int_{\mathbb{R}} S(x-t,t-s)f(y,s)dyds \text{ satisfy } v_t - kv_{xx} = f(x,t)$$

Now compute

$$\begin{aligned} v_t(t,x) &= \frac{\partial}{\partial t}\Big(\int_0^t \int_{\mathbb{R}} S(x-y,t-s)f(y,s)dyds\Big) \\ &= \int_{\mathbb{R}} S(x-y,0+)f(y,t)dy + \int_0^t \int_{\mathbb{R}} S_t(x-y,t-s)f(y,s)dyds \\ &= f(x,t) + \int_0^t \int_{\mathbb{R}} kS_{xx}(x-y,t-s)f(y,s)dyds \\ &= f(x,t) + k\frac{\partial^2}{(\partial x)^2} \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds = f(x,t) + kv_{xx}(x,t) \end{aligned}$$

We have proved $v_t - kv_{xx} = f(x,t)$. (Note that the partial derivative with respect to x in the third line is with respect to the first component while in the forth line is with respect to the actual x) ∎

For source on the half line

$$\begin{aligned}
&v_t - kv_{xx} = f(x,t) \text{ for } 0 < x < \infty, 0 < t < \infty \\
&v(x, 0+) = \varphi(x) \text{ for } 0 < x < \infty \\
&v(0+, t) = h(t) \text{ for } 0 < t < \infty
\end{aligned}$$

If such v exists and moreover we let $V(x,t) \triangleq v(x,t) - h(t)$, then we see V satisfy

$$\begin{aligned}
&V_t - kV_{xx} = f(x,t) - h'(t) \text{ for } 0 < x < \infty, 0 < t < \infty \\
&V(x, 0+) = \varphi(x) - h(0) \text{ for } 0 < x < \infty \\
&V(0+, t) = 0 \text{ for } 0 < t < \infty
\end{aligned}$$

Such V can be solved with a reflection.

Duhamel's principle basically says that if you differentiate a convolution $Z(t)$ between kernel S and another function $Y(t)$, where S is dependent on t, then $Z'(t) = AZ(t) + Y(t)$ where $\frac{d}{dt}S = AS$.

Question 151

Solve the inhomogeneous diffusion equation on the half-line with Dirichlet boundary condition:

$$\begin{cases}
u_t - ku_{xx} = f(x,t) \text{ for } x \in (0, \infty) \textbf{ (Non-homogeneous DE)} \\
u(x, 0) = \varphi(x) \textbf{ (IC)} \\
u(0, t) = 0 \textbf{ (BC)}
\end{cases}$$

using the method of reflection.

Proof. Define

$$f_{\text{odd}}(x,t) \triangleq \begin{cases} f(x,t) & \text{if } x > 0 \\ -f(-x,t) & \text{if } x < 0 \end{cases} \text{ and } \varphi_{\text{odd}}(x) \triangleq \begin{cases} \varphi(x) & \text{if } x > 0 \\ -\varphi(-x) & \text{if } x < 0 \end{cases}$$

The formula

$$u(x,t) = \int_{\mathbb{R}} S(x-y,t)\varphi_{\text{odd}}(y)dy + \int_0^t \int_{\mathbb{R}} S(x-y,t-s) f_{\text{odd}}(y,s)dyds$$

then satisfy

$$\begin{cases} u_t - ku_{xx} = f_{\text{odd}}(x,t) \ \textbf{(Non-homogeneous DE)} \\ u(x,0) = \varphi_{\text{odd}}(x) \ \textbf{(IC)} \\ u(x,t) = -u(-x,t) \text{ for all } x \in \mathbb{R}^* \ \textbf{(BC for restriction)} \end{cases}$$

It then follows that the restriction of u on the half line is a solution to the original problem. ∎

Question 152

Solve the completely inhomogeneous diffusion equation problem on the half-line

$$\begin{cases} v_t - kv_{xx} = f(x,t) \text{ for } x \in (0,\infty) \ \textbf{(Non-homogeneous DE)} \\ v(x,0) = \varphi(x) \ \textbf{(IC)} \\ v_x(0,t) = h(t) \ \textbf{(BC)} \end{cases}$$

by carrying out the subtraction method begun in the text.

Proof. Define

$$w(x,t) \triangleq v(x,t) - xh(t)$$

We see

$$\begin{cases} w_t - kw_{xx} = f(x,t) - xh'(t) \text{ for } x \in (0,\infty) \ \textbf{(Non-homogeneous DE)} \\ w(x,0) = \varphi(x) - xh(0) \ \textbf{(IC)} \\ w_x(0,t) = 0 \ \textbf{(Good BC)} \end{cases}$$

Define

$$g_{\text{even}} \triangleq \begin{cases} f(x,t) - xh'(t) & \text{if } x > 0 \\ f(-x,t) + xh'(t) & \text{if } x < 0 \end{cases} \text{ and } \psi_{\text{even}}(x) \triangleq \begin{cases} \varphi(x) - xh(0) & \text{if } x > 0 \\ \varphi(-x) + xh(0) & \text{if } x < 0 \end{cases}$$

We then see

$$w(x,t) = \int_{\mathbb{R}} S(x-y,t)\psi_{\text{even}}(y)dy + \int_0^t \int_{\mathbb{R}} S(x-y,t-s)g_{\text{even}}(y)dyds$$

∎

Question 153

Solve the inhomogeneous Neumann diffusion problem on the half-line

$$\begin{cases} w_t - kw_{xx} = 0 \text{ for } x \in (0,\infty) \textbf{ (Homogeneous DE)} \\ w_x(0,t) = h(t) \textbf{ (Non-homogeneous BC)} \\ w(x,0) = \varphi(x) \textbf{ (IC)} \end{cases}$$

by the subtraction method indicated in the text.

Proof. Suppose w is a solution to our problem. If we define

$$u(x,t) \triangleq w(x,t) - xh(t) \text{ for } x \in (0,\infty)$$

We see that u satisfy

$$\begin{cases} u_t - ku_{xx} = -xh'(t) \text{ for } x \in (0,\infty) \textbf{ (Non-homogeneous DE)} \\ u_x(0,t) = 0 \textbf{ (Good BC)} \\ u(x,0) = \varphi(x) - xh(0) \textbf{ (IC)} \end{cases}$$

Define

$$f_{\text{even}}(x,t) \triangleq \begin{cases} -xh'(t) & \text{if } x > 0 \\ xh'(t) & \text{if } x < 0 \end{cases}$$

And define

$$\varphi_{\text{even},*} \triangleq \begin{cases} \varphi(x) - xh(0) & \text{if } x > 0 \\ \varphi(-x) + xh(0) & \text{if } x < 0 \end{cases}$$

And define

$$u_{\text{even}}(x,t) \triangleq \int_{\mathbb{R}} S(x-y)\varphi_{\text{even},*}(y)dy + \int_0^t \int_{\mathbb{R}} S(x-y,t-s)f_{\text{even}}(y,s)dyds$$

It then follows that u_{even} solve the non-homogeneous DE and IC. To see that $u_x(0,t) = 0$, one simply observe that u is even in x. The solution to the original problem is then

$$w(x,t) \triangleq u_{\text{even}}(x,t) + xh(t) \text{ for } x \in (0,\infty)$$

∎

5.4 3.4 Waves with a source

We first offer a formula

$$F(t) \triangleq \int_{s_0}^{t} f(t,s)ds \implies F'(t) = f(t,t) + \int_{s_0}^{t} f_t(t,s)ds$$

Theorem 5.4.1. (Waves with a source) Consider the non-homogeneous wave equation

$$\begin{aligned} &u_{tt} - c^2 u_{xx} = f(x,t) \text{ for } -\infty < x < \infty \\ &u(x,0) = \varphi(x) \\ &u_t(x,0) = \psi(x) \end{aligned}$$

The solution is

$$u(x,t) = \frac{1}{2}[\varphi(x+ct) + \varphi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds$$

Proof. It is easily checked that $u(x,0) = \varphi(x)$. We now compute

$$\begin{aligned} u_t(x,t) = &\frac{1}{2}[\psi(x+ct) - \psi(x-ct)] \\ &+ \frac{1}{2c}\Big[\int_x^x f(y,t)dy + c\int_0^t f(x+c(t-s),s) - f((x-c(t-s)),s)ds\Big] \end{aligned}$$

which give us $u_t(x,0+) = \psi(x)$. ■

Note that the solution immediately implies the stability in the following form

$$|(u_1-u_2)(x,t)| \le \|\varphi_1 - \varphi_2\|_\infty + t\,\|\psi_1 - \psi_2\|_\infty + \frac{1}{2c}\cdot\frac{2ct^2}{2}\cdot\|f_1 - f_2\|_{\infty,T}$$

where

$$\|f_1 - f_2\|_{\infty,T} = \max_{0\le t\le T, x\in\mathbb{R}} |(f_1 - f_2)(x,t)|$$

The solution of the ODE

$$u'' + c^2 u = f(t), u(0) = c_0, u'(0), c_1$$

is

$$u(t) = c_0 \cos(ct) + \frac{c_1 \sin(ct)}{c} + \int_0^t \frac{\sin(c(t-s))f(s)ds}{c}$$

Question 154

Solve

$$\begin{cases} u_{tt} - c^2 u_{xx} = xt \text{ (Non-homogeneous DE)} \\ u(x,0) = 0, u_t(x,0) = 0 \text{ (IC)} \end{cases}$$

Proof.

$$\begin{aligned} u &= \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} ysdyds \\ &= \frac{xt^3}{6} \end{aligned}$$

■

Question 155

Solve

$$\begin{cases} u_{tt} - c^2 u_{xx} = e^{ax} \text{ (Non-homogeneous DE)} \\ u(x,0) = 0, u_t(x,0) = 0 \text{ (IC)} \end{cases}$$

Proof.

$$\begin{aligned} u &= \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} e^{ay}dyds \\ &= \frac{e^{ax}(e^{act} - e^{-act} - 2)}{2a^2c^2} \end{aligned}$$

■

Question 156

Solve

$$\begin{cases} u_{tt} - c^2 u_{xx} = \cos x \text{ (Non-homogeneous DE)} \\ u(x,0) = \sin x, u_t(x,0) = 1 + x \text{ (IC)} \end{cases}$$

Proof.

$$\begin{aligned} u &= \frac{\sin(x+ct)+\sin(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct}(1+s)ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \cos y dy ds \\ &= \frac{\sin(x+ct)+\sin(x-ct)}{2} + (x+1)t + \frac{2\cos x - \cos(x+ct) - \cos(x-ct)}{2c^2} \end{aligned}$$

■

Question 157

Given the wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx} \text{ for } x \in (0,\infty) \textbf{ (Homogeneous DE)} \\ u(x,0) = 0 \textbf{ (IC)} \\ u(0,t) = h(t) \textbf{ (BC)} \end{cases}$$

Show that the solution is

$$u(x,t) \triangleq \begin{cases} h(t-\frac{x}{c}) & \text{if } x < ct \\ 0 & \text{if } x \geq ct \end{cases}$$

Proof. Check manually. ■

Question 158

Solve

$$\begin{cases} u_{tt} = c^2 u_{xx} \text{ for } x \in (0,\infty) \textbf{ (Homogeneous DE)} \\ u(x,0) = x, u_t(x,0) = 0 \textbf{ (IC)} \\ u(0,t) = t^2 \textbf{ (BC)} \end{cases}$$

Proof. If we define

$$w = u - t^2$$

we see that w satisfy

$$\begin{cases} w_{tt} - c^2 w_{xx} = -2 \text{ for } x \in (0,\infty) \textbf{ (Non-homogeneous DE)} \\ w(x,0) = x, w_t(x,0) = 0 \textbf{ (IC)} \\ w(0,t) = 0 \textbf{ (Dirichlet BC)} \end{cases}$$

then if we define

$$\varphi_{\text{odd}}(x) \triangleq x \text{ and } f_{\text{odd}}(x,t) \triangleq \begin{cases} -2 & \text{if } x > 0 \\ 2 & \text{if } x < 0 \end{cases}$$

and

$$\begin{aligned} w(x,t) &\triangleq \frac{\varphi(x+ct)+\varphi(x-ct)}{2} + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f_{\text{odd}}(y,s)dyds \\ &= x + \begin{cases} -t^2 & \text{if } x-ct > 0 \\ \frac{x^2}{c^2} - \frac{2tx}{c} & \text{if } x-ct < 0 \end{cases} \end{aligned}$$

Note that we only consider when $x \geq 0$. This then give us

$$u(x,t) = \begin{cases} x & \text{if } x-ct > 0 \\ x + (t - \frac{x}{c})^2 & \text{if } x-ct < 0 \end{cases}$$

$\blacksquare$

5.5 3.5 Diffusion Revisited

5.6 Cheat Sheet

The most fundamental wave equation is

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \text{ for } x \in \mathbb{R} \textbf{ (Non-homogeneous DE)} \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \textbf{ (IC)} \end{cases}$$

The formula for this equation is

$$u(x,t) \triangleq \frac{\varphi(x+ct)+\varphi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds \tag{5.1}$$

It is easy to see that Formula 5.1 agree with the formula we have for solving homogeneous wave equation. Sometimes, the question deforms, and ask you to solve

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \text{ for } x \in (0,\infty) \textbf{ (Homogeneous DE)} \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \textbf{ (IC)} \\ u(0,t) = 0 \text{ or } u_t(0,t) = 0 \textbf{ (BC)} \end{cases}$$

If the boundary condition is Dirichlet, i.e., $u(0,t) = 0$, we simply extend φ and ψ in odd fashion. If the boundary condition is Neumann, i.e., $u_t(0,t) = 0$, we extend φ and ψ in even fashion.

The most fundamental diffusion equation is

$$\begin{cases} u_t - ku_{xx} = f(x,t) \text{ for } x \in \mathbb{R} \textbf{ (Non-homogeneous DE)} \\ u(x,0) = \varphi(x) \textbf{ (IC)} \end{cases}$$

The formula for this equation is

$$u(x,t) \triangleq \int_{\mathbb{R}} S(x-y,t)\varphi(y)dy + \int_0^t \int_{\mathbb{R}} S(x-y,t-s)f(y,s)dyds$$

Sometimes the question deform and ask you to solve

$$\begin{cases} u_t - ku_{xx} = f(x,t) \text{ for } x \in (0,\infty) \textbf{ (Non-homogeneous DE)} \\ u(x,0) = \varphi(x) \textbf{ (IC)} \\ u(0,t) = 0 \text{ or } u_x(0,t) = 0 \text{ or } u_x(0,t) = h(t) \textbf{ (BC)} \end{cases}$$

If BC is Dirichlet, we simply extend f, φ in odd fashion. If BC is Neumann, we simply extend f, φ in even fashion. If BC is $u_x(0,t) = h(t)$, we define $w = u - xh(t)$, and do odd extension to solve w.

j

Chapter 6

PDE 3

6.1 4.1 Separation of Variables, the Dirichlet Condition

Consider the wave equation on some finite interval with Dirichlet boundary condition

$$\begin{cases} u_{tt} = c^2 u_{xx} \text{ for } x \in (0, l) \textbf{ (Homogeneous DE)} \\ u(0, t) = u(l, t) = 0 \textbf{ (Dirichlet BC)} \end{cases}$$

If we suppose

$$u(x, t) = X(x)T(t)$$

where $X : [0, l] \to \mathbb{R}, T : \mathbb{R} \to \mathbb{R}$, we see from the wave equation that

$$T''(t)X(x) = c^2 X''(x)T(t)$$

WOLG, we can deduce

$$\frac{X''}{X} = \frac{T''}{c^2 T} = -\lambda$$

where λ is a constant since $\frac{X''}{X}$ only depend on x and $\frac{T''}{c^2 T}$ only depend on t. This then give us the ODEs

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(l) = 0 \end{cases} \quad \text{and } T'' + c^2 \lambda T = 0$$

For X to have a non-trivial solution, we must have $\lambda = (\frac{n\pi}{l})^2$. We now know the solution of this ODE is

$$T(t) \triangleq A\cos(\frac{cn\pi t}{l}) + B\sin(\frac{cn\pi t}{l}) \text{ and } X(x) \triangleq D\sin(\frac{n\pi x}{l})$$

Some tedious effort can be used to verify that

$$u(x,t) \triangleq \Big[A\cos(\frac{cn\pi t}{l}) + B\sin(\frac{cn\pi t}{l})\Big] D\sin(\frac{n\pi x}{l})$$

indeed satisfy the wave equation.

Now consider the heat equation on some finite interval with Dirichlet boundary condition

$$\begin{cases} u_t = ku_{xx} \text{ for } x \in (0,l) \textbf{ (Homogeneous DE)} \\ u(0,t) = u(l,t) = 0 \textbf{ (Dirichlet BC)} \end{cases}$$

If we suppose

$$u(x,t) = T(t)X(x)$$

where $X : [0,l] \to \mathbb{R}, T : [0,\infty) \to \mathbb{R}$, we see from the heat equation that

$$T'(t)X(x) = kT(t)X''(x)$$

WOLG we can deduce

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = -\lambda$$

which give us the following ODEs

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(l) = 0 \end{cases} \quad \text{and } T'(t) = -\lambda k T(t)$$

If we wish X to have a non-trivial solution, then we must require $\lambda = (\frac{n\pi}{l})^2$. We now can solve these ODEs to have

$$T(t) = Ae^{-(\frac{n\pi}{l})^2 kt} \text{ and } X(x) = B\sin(\frac{n\pi x}{l})$$

Some tedious effort can now be used to show that

$$u(x,t) = ABe^{-(\frac{n\pi}{l})^2 kt} \sin(\frac{n\pi x}{l})$$

indeed satisfy the heat equation.

Question 159

Consider waves in a resistant medium that satisfy the problem

$$\begin{cases} u_{tt} = c^2 u_{xx} - r u_t \text{ for } 0 < x < l \textbf{ (Homogeneous DE)} \\ u(0,t) = u(l,t) = 0 \textbf{ (Dirichlet BC)} \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \textbf{ (IC)} \end{cases}$$

where r is a constant, $0 < r < \frac{2\pi c}{l}$. Write down the series expansion of the solution.

Proof. Setting $u = XT$, we have the eigenproblem

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(l) = 0 \end{cases} \quad \text{and } T'' + rT' + c^2 \lambda T = 0$$

It is clear that λ is not an eigenvalue. If $\lambda = -\gamma^2$ is an eigenvalue, then X must take the form $X = Ae^{\gamma x} + Be^{-\gamma x}$, and boundary conditions implies

$$\begin{bmatrix} 1 & 1 \\ e^{\gamma l} & e^{-\gamma l} \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} = 0$$

This implies the eigenspace if exists is spanned by $e^{\gamma x} - e^{-\gamma x}$ and for the eigenspace to exist, we must have

$$e^{\gamma l} = e^{-\gamma l}$$

Then by periodical property of exp, we now have

$$\gamma_n = \frac{in\pi}{l} \text{ and } \lambda_n = \frac{n^2\pi^2}{l^2}$$

which allow us to rewrite the eigenfunction

$$\begin{aligned} X &= e^{\frac{in\pi x}{l}} - e^{\frac{-in\pi x}{l}} \\ &= 2i \sin(\frac{n\pi x}{l}) \end{aligned}$$

Suppose $T = e^{at}$. Plug in the DE, we have

$$(a^2 + ra + c^2\lambda)e^{at} = 0$$

which implies

$$\begin{aligned} a &= \frac{-r \pm \sqrt{r^2 - 4c^2\lambda}}{2} \\ &= \frac{-r \pm i\sqrt{4c^2\lambda - r^2}}{2} \end{aligned}$$

This give us

$$T = A_n e^{\frac{-rt}{2}} \cos\left(\frac{\sqrt{4c^2\frac{n^2\pi^2}{l^2} - r^2}t}{2}\right) + B_n e^{\frac{-rt}{2}} \sin\left(\frac{\sqrt{4c^2\frac{n^2\pi^2}{l^2} - r^2}t}{2}\right)$$

We now have

$$u = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right) e^{\frac{-rt}{2}} \left[A_n \cos\left(\frac{\sqrt{4c^2n^2\pi^2 - r^2l^2}}{2l}t\right) + B_n \sin\left(\frac{\sqrt{4c^2n^2\pi^2 - r^2l^2}}{2l}t\right) \right]$$

■

Question 160

Separate the variables for the equation

$$\begin{cases} tu_t = u_{xx} + 2u \ \textbf{(Non-homogeneous DE)} \\ u(0,t) = u(\pi,t) = 0 \ \textbf{(BC)} \end{cases}$$

Shows that there are infinite number of solutions that satisfy the initial condition $u(x,0) = 0$. So uniqueness is false for this question.

Proof. We have the eigenproblem

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(\pi) = 0 \end{cases} \quad \text{and } tT' + (\lambda - 2)T = 0$$

Same procedure shows that the eigenvalue is exactly n^2 where $n \in \mathbb{N}$ and

$$X = A_n \sin(nx)$$

This also let us solve

$$T = A_n t^{(2-n^2)}$$

■

6.2 4.2 The Neumann Condition

Question 161

Consider the equation

$$\begin{cases} u_{tt} = c^2 u_{xx} \text{ for } 0 < x < l \textbf{ (Homogeneous DE)} \\ u_x(0,t) = 0, u(l,t) = 0 \textbf{ (BC)} \end{cases}$$

(a) Shows that the eigenfunctions are $\cos \frac{(n+\frac{1}{2})\pi x}{l}$.

(b) Write the series expansion for a solution $u(x,t)$.

Proof. Again, setting $u \triangleq XT$, we have

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X(l) = 0 \end{cases} \quad \text{and } T'' + c^2 \lambda T = 0$$

It is clear that $\lambda = 0$ is not an eigenvalue. Suppose $-\lambda = \gamma^2$. Then X must take the form $Ae^{\gamma x} + Be^{-\gamma x}$. Plugging the initial conditions, we see

$$\begin{bmatrix} \gamma & -\gamma \\ e^{\gamma l} & e^{-\gamma l} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

This give us

$$\lambda = -\gamma^2 \text{ is an eigenvalue if and only if } \gamma(e^{-\gamma l} + e^{\gamma l}) = 0$$

If $\gamma(e^{-\gamma l} + e^{\gamma l}) = 0$, then by periodical property of exp, we see

$$\gamma = \frac{(2n+1)i\pi}{2l}$$

We now can solve

$$\begin{aligned} X &= Ae^{\frac{-(2n+1)i\pi x}{2l}} + Be^{\frac{(2n+1)i\pi x}{2l}} \\ &= A(e^{\frac{-(2n+1)i\pi x}{2l}} + e^{\frac{(2n+1)i\pi x}{2l}}) \\ &= 2A\cos(\frac{(n+\frac{1}{2})\pi x}{l}) \end{aligned}$$

with eigenvalue $\lambda_n = \frac{(2n+1)^2\pi^2}{4l^2}$. Note that the eigenvalues are all positive. We then can solve $T = B\cos(c\sqrt{\lambda}t) + C\sin(c\sqrt{\lambda}t)$ and solve

$$u = \sum_{n=1}^{\infty} \left[A_n \cos(\frac{(n+\frac{1}{2})\pi x}{l})\right]\left[B_n \cos(c\frac{(2n+1)\pi}{2l}t) + C_n \sin(c\frac{(2n+1)\pi}{2l}t)\right]$$

∎

Question 162

Solve the Schrodinger equation

$$\begin{cases} u_t = iku_{xx} \text{ for } 0 < x < l \textbf{ (Homogeneous DE)} \\ u_x(0,t) = u(l,t) = 0 \textbf{ (BC)} \end{cases}$$

for real $k \in (0, l)$.

Proof. Again we do the separation of the variables

$$u \triangleq T(t)X(x)$$

Some tedious efforts shows that u is a solution of this original question as long as X, T satisfy the following ODE

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) = X(l) = 0 \end{cases} \quad \text{and } T'(t) + \lambda ikT(t) = 0$$

where $\lambda \in \mathbb{C}$ is arbitrary constant. The solution of the second ODE is obviously

$$T(t) \triangleq Ce^{-\lambda ikt}$$

where $C \in \mathbb{C}$ is arbitrary constant. It remains to find what values can λ take so that X has non-trivial solutions. If $\lambda = 0$, then to satisfy the ordinary differential equation, the solution must take the forms $X = C + Dx$, where $C, D \in \mathbb{C}$ are arbitrary constant. Plugging the initial conditions, we see that $C = D = 0$. In other words, if $\lambda = 0$, then X can only be trivial. If $\lambda \neq 0$, ODE of X suggest that X must take the form

$$X \triangleq Ae^{\gamma x} + Be^{-\gamma x}$$

where $\gamma \in \mathbb{C}$ satisfy $\gamma^2 = -\lambda$. Plug in $X'(0) = 0$, we see

$$0 = \gamma(A - B)$$

which implies $A = B$. Plug in $X(l) = 0$, we see

$$0 = Ae^{\gamma l} + Be^{-\gamma l} = A(e^{\gamma l} + e^{-\gamma l})$$

Then for X to be non-trivial, we must have

$$e^{\gamma l} + e^{-\gamma l} = 0$$

By periodicity property of exponential function, we then can deduce

$$\gamma = \frac{i\pi(2n+1)}{l} \text{ and } \lambda = \frac{(2n+1)^2\pi^2}{l^2}$$

It then follows from $X = Ae^{\gamma x} + Be^{-\gamma x}$ that

$$\begin{aligned} X &= (A+B)\cos\left(\frac{\pi(2n+1)x}{l}\right) + (A-B)i\sin\left(\frac{\pi(2n+1)x}{l}\right) \\ &= (A+B)\cos\left(\frac{\pi(2n+1)x}{l}\right) \quad (\because A = B) \end{aligned}$$

■

Question 163

Consider diffusion inside an enclosed circular tube. Let its length (circumference) be $2l$. Let x denote the arc length parameter where $-l \le x \le l$. Then the concentration of the diffusing substance satisfies

$$\begin{cases} u_t = ku_{xx} \text{ for } -l \le x \le l \textbf{ (Homogeneous DE)} \\ u(-l,t) = u(l,t) \text{ and } u_x(-l,t) = u_x(l,t) \textbf{ (BC)} \end{cases}$$

These are called periodic boundary conditions.

(a) Show that the eigenvalues are $\lambda = (\frac{n\pi}{l})^2$ for $n \in \mathbb{Z}_0^+$.

(b) Show that the concentration is

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l}\right)e^{\frac{-n^2\pi^2 kt}{l^2}}$$

Proof. The eigenproblems are

$$\begin{cases} X'' + \lambda X = 0 \\ X(-l) = X(l) \\ X'(-l) = X'(l) \end{cases} \quad \text{and } T' + \lambda kT = 0$$

If $\lambda = 0$ is an eigenvalue, then X must take the form $C + Dx$, and the boundary conditions implies

$$\begin{bmatrix} 0 & 2l \\ 0 & 0 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

i.e., the eigenfunctions are constants. Note that if $\lambda = 0$, then T is also a constant. If $\lambda = -\gamma^2 \neq 0$ is an eigenvalue, then X must take the form $Ae^{\gamma x} + Be^{-\gamma x}$, and the boundary conditions implies

$$\begin{bmatrix} e^{-\gamma l} - e^{\gamma l} & e^{\gamma l} - e^{-\gamma l} \\ \gamma(e^{\gamma l} - e^{-\gamma l}) & \gamma(e^{\gamma l} - e^{-\gamma l}) \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = 0$$

which, to let X be non-trivial, implies

$$e^{\gamma l} = e^{-\gamma l}$$

By periodical property of exp, we now see

$$\gamma_n = \frac{in\pi}{l} \text{ and } \lambda_n = \frac{n^2\pi^2}{l^2}$$

for $n \in \mathbb{N}$, with eigenfunction

$$\begin{aligned} X &= Ae^{\frac{in\pi x}{l}} + Be^{\frac{-in\pi x}{l}} \\ &= A_n \sin(\frac{n\pi x}{l}) \end{aligned}$$

where $A = -B$ is required so that X is an eigenfunction. We now can solve $T = e^{-\lambda kt} = e^{\frac{-n^2\pi^2 kt}{l^2}}$. ■

6.3 Cheat Sheet

Consider the **Dirichlet** eigenproblem

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(l) = 0 \end{cases}$$

It is clear that 0 is NOT an eigenvalue. Suppose $\lambda = \beta^2 \in \mathbb{C}$ is an eigenvalue. We see that X must take the form

$$X = A\cos(\beta x) + B\sin(\beta x)$$

The initial condition implies

$$\begin{bmatrix} 1 & 0 \\ \cos(\beta l) & \sin(\beta l) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

This implies that

$$\beta = \frac{n\pi}{l} \text{ and } X = \sin(\frac{n\pi x}{l}) \text{ for } n \geq 1$$

Consider the **Neumann** eigenproblem

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(l) = 0 \end{cases}$$

It is clear that 0 is an eigenvalue with eigenspace spanned by constant function. Suppose $\lambda = \beta^2 \in \mathbb{C}$ is an eigenvalue. We see that X must take the form

$$X = A\cos(\beta x) + B\sin(\beta x)$$

The initial condition then implies

$$\begin{bmatrix} 0 & \beta \\ -\beta\sin(\beta l) & \beta\cos(\beta l) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

This implies that

$$\beta = \frac{n\pi}{l} \text{ and } X = \cos(\frac{n\pi x}{l}) \text{ for } n \geq 1$$

6.4 4.3 Robin Condition

In this section, we are concerned with the following eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) - a_0 X(0) = 0 \\ X'(l) + a_l X(l) = 0 \end{cases}$$

where $a_0 \neq 0 \neq a_l$. If $\lambda = 0$ is an eigenvalue, then X must take the form $C + Dx$. Plugging the initial condition, we see that

$$\begin{bmatrix} -a_0 & 1 \\ a_l & 1 + la_l \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = 0$$

This implies that

$$\lambda = 0 \text{ is an eigenvalue if and only if } a_0 + a_l = -la_0 a_l$$

with eigenspace $\text{span}(1 + a_0 x)$. If $\lambda = \beta^2 > 0$ is an eigenvalue where $\beta > 0$, then X must take the form $A\cos(\beta x) + B\sin(\beta x)$. Plugging the initial condition, we see that

$$\begin{bmatrix} -a_0 & \beta \\ a_l\cos(\beta l) - \beta\sin(\beta l) & \beta\cos(\beta l) + a_l\sin(\beta l) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

This implies that $\lambda = \beta^2$ is an eigenvalue if and only if

$$(a_0 + a_l)\beta\cos(\beta l) + (a_0 a_l - \beta^2)\sin(\beta l) = 0 \tag{6.1}$$

Suppose Equation 6.1 hold true. There are four possibilities regarding the signs of a_0 and a_l. Let's first suppose $a_0, a_l \in \mathbb{R}^+$. Again, there are two cases, either $l = \frac{(n+\frac{1}{2})\pi}{\sqrt{a_0 a_l}}$ or not. If $l = \frac{(n+\frac{1}{2})\pi}{\sqrt{a_0 a_l}}$, then we see $\beta = \sqrt{a_0 a_l}$ is a solution and this is the only solution that makes $\cos(\beta l) = 0$; in other words, β won't be $\frac{(n+\frac{1}{2})\pi}{l}$. Suppose $\beta \neq \sqrt{a_0 a_l}$. We then can rewrite Equation 6.1 as

$$\tan(\beta l) = \frac{(a_0 + a_l)\beta}{\beta^2 - a_0 a_l} \tag{6.2}$$

which clearly have infinite solutions and can be easily numerically approximated. We have solved the case when $l = \frac{(n+\frac{1}{2})\pi}{\sqrt{a_0 a_l}}$. Let's suppose $l \neq \frac{(n+\frac{1}{2})\pi}{\sqrt{a_0 a_l}}$. Note that we can deduce a contradiction from $\cos(\beta l) = 0$. This implies β won't be $\frac{(n+\frac{1}{2})\pi}{l}$ and again give us Equation 6.2. We have solved the case when $l \neq \frac{(n+\frac{1}{2})\pi}{\sqrt{a_0 a_l}}$. In conclusion, if $l = \frac{(n+\frac{1}{2})\pi}{\sqrt{a_0 a_l}}$, then the set of eigenvalue β is exactly $\sqrt{a_0 a_l}$ and all values that make Equation 6.2 well-defined and true, and if $l \neq \frac{(n+\frac{1}{2})\pi}{\sqrt{a_0 a_l}}$, then the set of eigenvalue β are exactly all values that make Equation 6.2 well-defined and true.

Now, let's suppose $a_0 \in \mathbb{R}^-$ and $a_l \in \mathbb{R}^+$ with addition hypothesis $a_0 + a_l > 0$. It is then easy to see the set of eigenvalues are exactly all values that make Equation 6.2 well-defined and true.

Consider the diffusion equation on finite interval with robin conditions

$$\begin{cases} u_t = k u_{xx} \text{ for } 0 < x < l \textbf{ (Homogeneous DE)} \\ (u_x - a_0 u)(0,t) = 0 \text{ and } (u_x + a_l u)(l,t) = 0 \textbf{ (Robin BC)} \end{cases}$$

Some tedious effort shows that $u \triangleq T(t)X(x)$ is a solution of the original equation as long as X, T satisfy the following ODE

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) - a_0 X(0) = 0 \\ X'(l) + a_l X(l) = 0 \end{cases} \quad \text{and } T'(t) + \lambda k T(t) = 0$$

Then T must be

$$T(t) \triangleq C e^{-\lambda k t}$$

where $C \in \mathbb{C}$ is arbitrary constant. It remains to find λ so that X has a non-trivial solution. If $\lambda = 0$, then X must be of the form

$$X(x) \triangleq A + Bx$$

where $A, B \in \mathbb{C}$ are constants. Initial conditions implies

$$B - a_0 A = 0 \text{ and } B + a_l(A + Bl) = 0$$

This then shows that X has a nontrivial solution if and only if

$$\begin{bmatrix} -a_0 & 1 \\ a_l & l a_l + 1 \end{bmatrix} \text{ is non-singular, i.e., } a_0 + a_l = -a_0 a_l l$$

If $\lambda \in \mathbb{C}^*$, then X must be of the form

$$X(x) \triangleq Ae^{\gamma x} + Be^{-\gamma x}$$

where $A, B \in \mathbb{C}$ are constants and γ satisfy $\gamma^2 = -\lambda$. Initial conditions at $x = 0$ implies

$$\gamma(A - B) - a_0(A + B) = X'(0) - a_0 X(0) = 0$$

which implies

$$\gamma = \frac{a_0(A+B)}{A-B} \text{ and } \lambda = \frac{-a_0^2(A+B)^2}{(A-B)^2}$$

Initial conditions at $x = l$ implies

$$e^{\gamma l}(\gamma A + a_l A) + e^{-\gamma l}(-\gamma B + a_l B) = X'(l) + a_l X(l) = 0$$

In summary

$$\begin{bmatrix} \gamma - a_0 & -\gamma - a_0 \\ e^{\gamma l}(\gamma + a_l) & e^{-\gamma l}(-\gamma + a_l) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

Then for X to have non-trivial solution, we must have

$$e^{-\gamma l}(-\gamma + a_l)(\gamma - a_0) + e^{\gamma l}(\gamma + a_l)(\gamma + a_0) = 0$$

This shows that

$$\lambda \text{ is an eigenvalue if and only if}$$

Question 164

Consider the eigenvalue problem with Robin BCs at both ends:

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) - a_0 X(0) = 0 \\ X'(l) + a_l X(l) = 0 \end{cases}$$

(a) Show that $\lambda = 0$ is an eigenvalue if and only if $a_0 + a_l = -a_0 a_l l$.

(b) Find the eigenfunctions corresponding to the zero eigenvalue.

Proof. Suppose $\lambda = 0$ is an eigenvalue. Then $X'' + \lambda X = 0$ implies $X = C + Dx$ where C, D are constants and they can not both be zero. The initial conditions can now be rewritten as

$$\begin{bmatrix} -a_0 & 1 \\ a_l & la_l + 1 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then because C, D can not both be zero, we see that the left matrix $\begin{bmatrix} -a_0 & 1 \\ a_l & la_l + 1 \end{bmatrix}$ must be singular. Computing the determinant, we now have $a_0 + a_l = -a_0 a_l l$.

Suppose $a_0 + a_l = -a_0 a_l l$. One can see that $1 + a_0 x$ spans the eigenspace. ■

Question 165

On the interval $0 \leq x \leq 1$ of length one, consider the eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) + X(0) = 0 \\ X(1) = 0 \end{cases}$$

(a) Find an eigenfunction with eigenvalue zero.

(b) Find an equation for the positive eigenvalues $\lambda = \beta^2$.

(c) Show graphically from part (b) that there are an infinite number of positive eigenvalues.

(d) Is there a negative eigenvalue?

Proof. If eigenvalue $\lambda = 0$ is zero, then X must take the form $C + Dx$, where C, D are constants that can not both be zero. The initial condition can now be rewritten as $C + D = 0$. In other words, $x - 1$ spans the eigenspace.

If eigenvalue is positive $\lambda = \beta^2$ where $\beta > 0$, then X must take the form

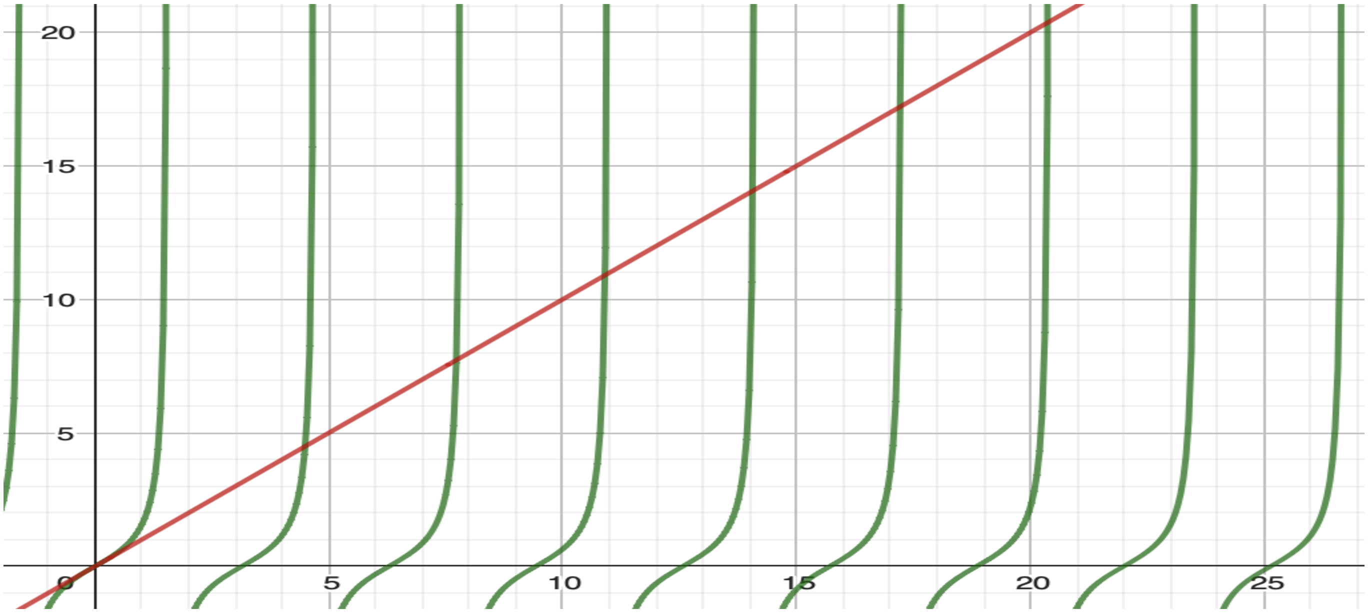
$$X = A\cos(\beta x) + B\sin(\beta x)$$

The initial condition can now be rewritten as

$$\begin{cases} \beta B + A = 0 \\ A\cos\beta + B\sin\beta = 0 \end{cases} \quad \text{or} \quad \begin{bmatrix} 1 & \beta \\ \cos\beta & \sin\beta \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

Then for λ to be an eigenvalue, we must have

$$\sin\beta = \beta\cos\beta$$



If eigenvalue is negative $\lambda = -\beta^2$ where $\beta > 0$, then X must take the form

$$X = A\cosh(\beta x) + B\sinh(\beta x)$$

The initial condition can now be rewritten as

$$\begin{cases} \beta B + A = 0 \\ A\cosh(\beta) + B\sinh(\beta) = 0 \end{cases} \quad \text{or} \quad \begin{bmatrix} 1 & \beta \\ \cosh(\beta) & \sinh(\beta) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

This implies

$$\beta = \tanh(\beta)$$

This equation have no positive solution β, so there are no negative eigenvalues. ■

Question 166

Let $a_0 < 0, a_l > -a_0$ and $a_0 + a_l < -a_0 a_l l$. Solve the wave equations on finite interval with robin conditions

$$\begin{cases} u_{tt} = c^2 u_{xx} \text{ for } 0 < x < l \textbf{ (Homogeneous DE)} \\ (u_x - a_0 u)(0,t) = 0 \text{ and } (u_x + a_l u)(l,t) = 0 \textbf{ (Robin BC)} \end{cases}$$

Proof. Again, setting $u = XT$, we have

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) - a_0 X(0) = 0 \\ X'(l) + a_l X(l) = 0 \end{cases} \quad \text{and } T'' + c^2\lambda T = 0$$

Because $a_0 + a_l < -a_0 a_l l$, we know 0 is not an eigenvalue. If $\lambda = \beta^2 > 0$ is an eigenvalue where $\beta > 0$, then X must take the form $A\cos(\beta x) + B\sin(\beta x)$. Plugging the initial condition, we see that

$$\begin{bmatrix} -a_0 & \beta \\ a_l\cos(\beta l) - \beta\sin(\beta l) & \beta\cos(\beta l) + a_l\sin(\beta l) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

This implies that $\lambda = \beta^2$ is an eigenvalue if and only if

$$(a_0 + a_l)\beta\cos(\beta l) + (a_0 a_l - \beta^2)\sin(\beta l) = 0 \tag{6.3}$$

It is then clear that the sets of positive eigenvalues is exactly

$$\left\{\lambda = \beta^2 \in \mathbb{R}^+ : \beta \neq \frac{(n+\frac{1}{2})\pi}{l} \text{ and } \tan(\beta l) = \frac{(a_0 + a_l)\beta}{\beta^2 - a_0 a_l}\right\}$$

If $\lambda = -\beta^2 < 0$ is an eigenvalue where $\beta > 0$, then X must take the form $A\cosh(\beta x) + B\sinh(\beta x)$. Plugging the initial condition, we see that

$$\begin{bmatrix} -a_0 & \beta \\ a_l\cosh(\beta l) + \beta\sinh(\beta l) & \beta\cosh(\beta l) + a_l\sinh(\beta l) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

This implies that $\lambda = -\beta^2$ is an eigenvalue if and only if

$$(a_0 + a_l)\beta\cosh(\beta l) + (a_0 a_l + \beta^2)\sinh(\beta l) = 0$$

It is then clear that the set of negative eigenvalues is exactly

$$\left\{\lambda = -\beta^2 \in \mathbb{R}^- : \beta^2 \neq -a_0 a_l \text{ and } \tanh(\beta l) = \frac{-(a_0 + a_l)\beta}{a_0 a_l + \beta^2}\right\}$$

We have conclude that there are countable amount of positive eigenvalues $(\lambda_n)_{n=1}$ and one negative eigenvalue λ_0. This give us

$$\begin{aligned} u = &\Big(A_0\cosh(c\sqrt{-\lambda_0}t) + B_0\sinh(c\sqrt{-\lambda_0}t)\Big)\Big(\sqrt{-\lambda_0}\cosh(\sqrt{-\lambda_0}x) + a_0\sinh(\sqrt{-\lambda}x)\Big) \\ &+ \sum_{n=1}^{\infty}\Big(A_n\cos(c\sqrt{\lambda_n}t) + B_n\sin(c\sqrt{\lambda_n}t)\Big)\Big(\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x) + a_0\sin(\sqrt{\lambda_n}x)\Big) \end{aligned}$$

■

Question 167

(a) Prove that the total energy is conserved for the wave equation with Dirichlet BCs, where the energy is defined to be

$$E = \frac{1}{2}\int_0^l \Big(\frac{u_t^2}{c^2} + u_x^2\Big)dx \tag{6.4}$$

(b) Do the same for the Neumann BCs.

(c) For the Robin BCs, show that

$$E_R = \frac{1}{2}\int_0^l \Big(\frac{u_t^2}{c^2} + u_x^2\Big)dx + \frac{a_l}{2}(u(l,t))^2 + \frac{a_0}{2}(u(0,t))^2$$

is conserved.

Proof. If energy $E(t)$ is defined as in Equation 6.4, then we have

$$\begin{aligned} E'(t) &= \int_0^l \Big(\frac{u_t u_{tt}}{c^2} + u_x u_{xt}\Big)dx \\ &= \int_0^l (u_t u_{xx} + u_x u_{xt})dx \\ &= \int_0^l (u_t u_x)_x dx \\ &= u_t u_x(l,t) - u_t u_x(0,t) \end{aligned}$$

If we have the Dirichlet BCs, we see that $u_t(l,t) = u_t(0,t) = 0$ by direct differentiation. If we have the Neumann BCs, we see that $u_x(l,t) = u_x(0,t) = 0$. We now compute

$$\begin{aligned} E_R'(t) &= u_t u_x(l,t) - u_t u_x(0,t) + a_l u u_t(l,t) + a_0 u u_t(0,t) \\ &= (u_x + a_l u)u_t(l,t) + (-u_x + a_0 u)u_t(0,t) \end{aligned}$$

which have 0 coefficients $(u_x + a_l u)(l,t)$ and $(-u_x + a_0 u)(0,t)$ by Robin conditions. ∎

Question 168

Consider a string that is fixed at the en $x = 0$ and is free at the end $x = l$ except that a load of given mass is attached to the right end.

(a) Show that is satisfies the problem

$$\begin{cases} u_{tt} = c^2 u_{xx} \text{ for } 0 < x < l \textbf{ (Homogeneous DE)} \\ u(0,t) = 0, u_{tt}(l,t) = -ku_x(l,t) \textbf{ (BCs)} \end{cases}$$

(b) What is the eigenvalue problem in this case?

(c) Find the equation for the positive eigenvalues and find the eigenfunctions.

Proof. The eigenvalue problem in this case is

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0 \\ c^2 X''(l) + kX'(l) = 0 \end{cases}$$

If $\lambda = \beta^2$ where $\beta > 0$ is an eigenvalue, then X must take the form $A\cos(\beta x) + B\sin(\beta x)$, and boundary condition implies

$$\begin{bmatrix} 1 & 0 \\ -c^2\beta^2\cos(\beta l) - k\beta\sin(\beta l) & -c^2\beta^2\sin(\beta l) + k\beta\cos(\beta l) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

This implies the eigenspace is spanned by $\sin(\beta x)$ and the equation is

$$k\beta\cos(\beta l) = c^2\beta^2\sin(\beta l)$$

■

Question 169

Solve the eigenvalue problem $x^2u'' + 3xu' + \lambda u = 0$ for $1 < x < e$ with $u(1) = u(e) = 0$. Assume $\lambda > 1$.

Proof. We first find the solution space of the differential equation. Let u take the form $u = x^r$. Compute

$$\begin{aligned} 0 &= x^r\left[r(r-1) + 3r + \lambda\right] \\ &= x^r\left[(r+1)^2 + \lambda - 1\right] \end{aligned}$$

This implies

$$r = -1 + \pm i\sqrt{\lambda - 1}$$

and implies that the solution space is spanned by

$$\frac{\cos(\sqrt{\lambda - 1}\ln(x))}{x} \text{ and } \frac{\sin(\sqrt{\lambda - 1}\ln(x))}{x}$$

Let

$$u = A\frac{\cos(\sqrt{\lambda - 1}\ln(x))}{x} + B\frac{\sin(\sqrt{\lambda - 1}\ln(x))}{x}$$

Boundary condition then implies

$$\begin{bmatrix} 1 & 0 \\ \frac{\cos(\sqrt{\lambda-1})}{e} & \frac{\sin(\sqrt{\lambda-1})}{e} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

The eigenvalues greater than 1 are then $\lambda = n^2\pi^2 + 1$ with eigenfunction $\frac{\sin(\sqrt{\lambda-1}\ln(x))}{x}$ ■

6.5 5.1 The coefficient

When $n \neq m \in \mathbb{Z}^*$, we may compute

$$\begin{aligned}
\int_0^\pi \sin(nx)\sin(mx)dx &= \frac{1}{-4}\int_0^\pi (e^{inx} - e^{-inx})(e^{imx} - e^{-imx})dx \\
&= \frac{1}{-4}\int_0^\pi \left(e^{i(n+m)x} - e^{i(m-n)x} - e^{i(n-m)x} + e^{i(-n-m)x}\right)dx \\
&= \frac{1}{-4}\Big[\frac{e^{i(n+m)x} - e^{-i(n+m)x}}{i(n+m)} + \frac{e^{i(m-n)x} - e^{i(n-m)x}}{i(n-m)}\Big]\Big|_{x=0}^{\pi} = 0
\end{aligned}$$

If $n = m$, we may compute

$$\int_0^\pi \sin(nx)\sin(nx)dx = \frac{1}{-4}\int_0^\pi \left(e^{2inx} + e^{-2inx} - 2\right)dx = \frac{\pi}{2}$$

We then can conclude

$$\int_0^l \sin(\frac{n\pi x}{l})\sin(\frac{m\pi x}{l})dx = \frac{l}{\pi}\int_0^\pi \sin(nu)\sin(mu)du = \begin{cases} 0 & \text{if } n \neq m \\ \frac{l}{2} & \text{if } n = m \end{cases}$$

When $n \neq m \in \mathbb{Z}^*$, we may compute

$$\begin{aligned}
\int_0^\pi \cos(nx)\cos(mx)dx &= \frac{1}{4}\int_0^\pi (e^{inx} + e^{-inx})(e^{imx} + e^{-imx})dx \\
&= \frac{1}{4}\int_0^\pi \left(e^{i(n+m)x} + e^{i(m-n)x} + e^{i(n-m)x} + e^{-i(n+m)x}\right)dx \\
&= \frac{1}{4}\Big[\frac{e^{i(n+m)x} - e^{-i(n+m)x}}{i(n+m)} + \frac{e^{i(n-m)x} - e^{i(m-n)x}}{i(n-m)}\Big]\Big|_{x=0}^{\pi} = 0
\end{aligned}$$

If $n = m$, we may compute

$$\int_0^\pi \cos(nx)\cos(nx)dx = \frac{1}{4}\int_0^\pi \left(e^{2inx} + e^{-2inx} + 2\right)dx = \frac{\pi}{2}$$

We then can conclude

$$\int_0^l \cos(\frac{n\pi x}{l})\cos(\frac{m\pi x}{l})dx = \frac{l}{\pi}\int_0^\pi \cos(nu)\cos(mu)du = \begin{cases} 0 & \text{if } n \neq m \\ \frac{l}{2} & \text{if } n = m \end{cases}$$

This also give us

$$\int_{-l}^{l} \sin^2(\frac{n\pi x}{l})dx = \int_{-l}^{l} \cos^2(\frac{n\pi x}{l})dx = l$$
$$\int_{-l}^{l} \sin(\frac{n\pi x}{l})\sin(\frac{m\pi x}{l})dx = \int_{-l}^{l} \cos(\frac{n\pi x}{l})\cos(\frac{n\pi x}{l})dx = 0$$

$$\int_{-l}^{l} \cos^2(\frac{n\pi x}{l})dx = \int_{-l}^{l} \sin^2(\frac{n\pi x}{l})dx = l$$
$$\int_{0}^{l} \cos^2(\frac{n\pi x}{l})dx = \int_{0}^{l} \sin^2(\frac{n\pi x}{l})dx = \frac{l}{2}$$

Question 170

In the expansion $1 = \sum_{n\,\mathrm{odd}}(\frac{4}{n\pi})\sin(nx)$, valid for $x \in (0,\pi)$, put $x = \frac{\pi}{4}$ to calculate the sum

$$\begin{aligned}&(1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \cdots) + (\frac{1}{3} - \frac{1}{7} + \frac{1}{11} - \frac{1}{15} + \cdots)\\ &= 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots\end{aligned}$$

Proof. What? ■

Question 171

Let $\varphi(x) \equiv x^2$ for $0 \le x \le 1 = l$.

(a) Calculate its Fourier sine series.

(b) Calculate its Fourier cosine series.

Proof. Write

$$\varphi(x) = \sum_{n=0}^{\infty} B_n \sin(n\pi x)$$

and

$$\int_0^1 \varphi(x)\sin(n\pi x)dx = B_n \int_0^1 \sin^2(n\pi x)dx = \frac{B_n}{2}$$

This give us

$$
\begin{aligned}
\frac{B_n}{2} &= \int_0^1 x^2 \sin(n\pi x)dx \\
&= \frac{(-1)^n}{-n\pi} + \frac{2}{n\pi}\int_0^1 x\cos(n\pi x)dx \\
&= \frac{(-1)^n}{-n\pi} - \frac{2}{n^2\pi^2}\int_0^1 \sin(n\pi x)dx \\
&= \frac{(-1)^n}{-n\pi} + \frac{2(-1)^n - 2}{n^3\pi^3} \\
&= \frac{n^2\pi^2(-1)^{n+1} + 2(-1)^n - 2}{n^3\pi^3}
\end{aligned}
$$

and

$$
x^2 = \sum_{n=0}^{\infty} \frac{2\big(-2 + (-1)^n(2 - n^2\pi^2)\big)}{n^3\pi^3}\sin(n\pi x)
$$

Write

$$
\varphi(x) = \sum_{n=0}^{\infty} A_n \cos(n\pi x)
$$

and

$$
\int_0^1 \varphi(x)\cos(n\pi x)dx = A_n\int_0^1 \cos^2(n\pi x)dx = \frac{A_n}{2}
$$

This give us

$$
\begin{aligned}
\frac{A_n}{2} &= \int_0^1 x^2\cos(n\pi x)dx \\
&= \frac{-2}{n\pi}\int_0^1 x\sin(n\pi x)dx \\
&= \frac{-2}{n\pi}\Big(\frac{-(-1)^n}{n\pi} + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)dx\Big) = \frac{2(-1)^n}{n^2\pi^2}
\end{aligned}
$$

and $A_0 = \frac{2}{3}$. This give us

$$
\varphi(x) = \frac{2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2\pi^2}\cos(n\pi x)
$$

■

Question 172

Find the Fourier cosine series of the function $|\sin x|$ in the interval $(-\pi, \pi)$. Use it to find the sums

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} \text{ and } \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}$$

Proof. Write

$$|\sin x| = \sum_{n=0}^{\infty} A_n \cos(nx)$$

Compute

$$\begin{aligned}
A_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} A_0 \cos(0x) dx \\
&= \frac{1}{2\pi} \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} A_n \cos(nx) dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\sin x| \, dx = \frac{2}{\pi}
\end{aligned}$$

Compute for all $n \in \mathbb{N} \setminus \{1\}$

$$\begin{aligned}
A_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} A_n \cos^2(nx) dx \\
&= \frac{1}{\pi} \sum_{m=0}^{\infty} \int_{-\pi}^{\pi} A_m \cos(mx) \cos(nx) dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} \Big(\sum_{m=0}^{\infty} A_m \cos(mx) \Big) \cos(nx) dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} |\sin x| \cos(nx) dx \\
&= \frac{-1}{\pi} \Big(\frac{(-1)^{n+1}-1}{n+1} - \frac{(-1)^{n-1}-1}{n-1} \Big) = \frac{2((-1)^{n+1}-1)}{\pi(n^2-1)}
\end{aligned}$$

Compute

$$
\begin{aligned}
A_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} A_1\cos^2(x)dx \\
&= \frac{1}{\pi}\sum_{m=0}^{\infty}\int_{-\pi}^{\pi} A_m\cos(mx)\cos(x)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\Big(\sum_{m=0}^{\infty} A_m\cos(mx)\Big)\cos(x)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}|\sin(x)|\cos(x)dx = 0
\end{aligned}
$$

We have shown

$$
\begin{aligned}
|\sin(x)| &= \frac{2}{\pi} + \sum_{n=2}^{\infty}\frac{2((-1)^{n+1}-1)}{\pi(n^2-1)}\cos(nx) \\
&= \frac{2}{\pi} + \sum_{n \text{ even}}\frac{2((-1)^{n+1}-1)}{\pi(n^2-1)}\cos(nx) \\
&= \frac{2}{\pi} + \sum_{k\in\mathbb{N}}\frac{-4}{\pi(4k^2-1)}\cos(2kx)
\end{aligned}
$$

Plugging $x = 0$, we see

$$
\sum_{k\in\mathbb{N}}\frac{1}{4k^2-1} = \frac{1}{2}
$$

Plugging $x = \frac{\pi}{2}$, we see

$$
\begin{aligned}
1 &= \frac{2}{\pi} + \sum_{k\in\mathbb{N}}\frac{-4}{\pi(4k^2-1)}\cos(k\pi) \\
&= \frac{2}{\pi} + \sum_{k\in\mathbb{N}}\frac{-4}{\pi(4k^2-1)}(-1)^k
\end{aligned}
$$

That is

$$
\frac{\pi-2}{-4} = \sum_{k\in\mathbb{N}}\frac{(-1)^k}{4k^2-1}
$$

$\blacksquare$

Question 173

(a) Find the sine series of x^3 on $(0,l)$.

(b) Find the cosine series of x^4 on $(0,l)$.

Proof. Write

$$x^3 = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l})$$

Then for all $n \in \mathbb{N}$, we have

$$\begin{aligned}\int_0^l x^3 \sin(\frac{n\pi x}{l})dx &= \int_0^l \sum_{m=1}^{\infty} A_m \sin(\frac{m\pi x}{l})\sin(\frac{n\pi x}{l})dx \\ &= \frac{A_n l}{2}\end{aligned}$$

Compute

$$\int_0^l x^3 \sin(\frac{n\pi x}{l})dx = \frac{-l^4(-1)^n}{n\pi} + \frac{6l^4(-1)^n}{n^3\pi^3}$$

This give us

$$A_n = (-1)^n l^3\Big(\frac{-2}{n\pi} + \frac{12}{n^3\pi^3}\Big)$$

That is

$$x^3 = \sum_{n=1}^{\infty}(-1)^n l^3\Big(\frac{-2}{n\pi} + \frac{12}{n^3\pi^3}\Big)\sin(\frac{n\pi x}{l})$$

Integrating both side, we have

$$\begin{aligned}\frac{x^4}{4} + C &= \sum_{n=1}^{\infty}(-1)^n l^3\Big(\frac{-2}{n\pi} + \frac{12}{n^3\pi^3}\Big)\int \sin(\frac{n\pi x}{l})dx \\ &= \sum_{n=1}^{\infty}(-1)^n l^3\Big(\frac{-2}{n\pi} + \frac{12}{n^3\pi^3}\Big)\Big(\frac{-l\cos(\frac{n\pi x}{l})}{n\pi} + C_n\Big)\end{aligned}$$

Simplifying the result

$$x^4 = C_0 + \sum_{n=1}^{\infty}(-1)^n l^4\Big(\frac{8}{n^2\pi^2} + \frac{-48}{n^4\pi^4}\Big)\cos(\frac{n\pi x}{l})$$

Note that

$$\frac{l^5}{5} = \int_0^l x^4 dx = \int_0^l C_0 dx = lC_0$$

This give us

$$x^4 = \frac{l^4}{5} + \sum_{n=1}^{\infty} (-1)^n l^4 \Big(\frac{8}{n^2\pi^2} + \frac{-48}{n^4\pi^4} \Big) \cos(\frac{n\pi x}{l})$$

■

Question 174

Put $x = 0$ in the last question to compute

$$\sum_1 \frac{(-1)^n}{n^4}$$

Proof. Putting $x = 0$ and divided both side with l^4 in last question, we have

$$0 = \frac{1}{5} + \sum_{n=1}^{\infty} (-1)^n \Big(\frac{8}{n^2\pi^2} + \frac{-48}{n^4\pi^4} \Big)$$

This give us

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} &= \frac{\pi^4}{-48} \Big(\frac{-1}{5} - \sum_{n=1}^{\infty} \frac{8(-1)^n}{n^2\pi^2} \Big) \\
&= \frac{\pi^4}{-48} \Big(\frac{-1}{5} - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \Big) \\
&= \frac{\pi^4}{-48} \Big(\frac{-1}{5} + \frac{2}{3} \Big) = \frac{7\pi^4}{-720}
\end{aligned}$$

■

Question 175

Solve

$$\begin{cases} u_{tt} = c^2 u_{xx} \text{ for } 0 < x < \pi \textbf{ (Homogeneous DE)} \\ u_x(0,t) = u_x(\pi,t) = 0 \textbf{ (Newman BC)} \\ u(x,0) = 0, u_t(x,0) = \cos^2(x) \textbf{ (IC)} \end{cases}$$

Proof. We see that $u = XT$ satisfy the DE and BC as long as

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(\pi) = 0 \end{cases} \qquad \text{and } T'' + c^2 \lambda T = 0$$

for some constant $\lambda \in \mathbb{C}$. It is clear that for X, 0 is an eigenvalue with constant eigenfunction, and the rest of the eigenvalues are $\lambda_n = n^2$ with eigenfunction $\cos(nx)$. This give us

$$u = A_0(B_0 + C_0 t) + \sum_{n=1}^{\infty} A_n \cos(nx)\big(B_n \cos(cnt) + C_n \sin(cnt)\big)$$

A change of notation give us

$$u = A_0 + B_0 t + \sum_{n=1}^{\infty} A_n \cos(nx)\cos(cnt) + B_n \cos(nx)\sin(cnt)$$

Plug in the initial condition $u(x,0) = 0$, we see

$$0 = A_0 + \sum_{n=1}^{\infty} A_n \cos(nx)$$

Because constants have Fourier coefficient all 0 except the first term, we may now deduce $A_0 = A_n = 0$. We now may write

$$u = B_0 t + \sum_{n=1}^{\infty} B_n \cos(nx)\sin(cnt)$$

Compute

$$u_t = B_0 + \sum_{n=1}^{\infty} B_n cn \cos(nx)\cos(cnt)$$

Plug in the initial condition $\cos^2(x) = u_t(x,0)$, we see

$$\cos^2(x) = B_0 + \sum_{n=1}^{\infty} B_n cn \cos(nx)$$

We now compute the Fourier cosine series of $\cos^2(x)$ on $(0,\pi)$.

$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$

This give us $B_0 = \frac{1}{2}$ and

$$B_n \triangleq \begin{cases} \frac{1}{4c} & \text{if } n = 2 \\ 0 & \text{if } n \neq 2 \end{cases}$$

In summary

$$u = \frac{t}{2} + \frac{\cos(2x)\sin(2ct)}{4c}$$

■

Chapter 7

PDE HW

7.1 PDE HW 1

Theorem 7.1.1.

$$\text{Show } u \mapsto u_x + uu_y \text{ is non-linear}$$

Proof. See that

$$2u \mapsto 2u_x + 4uu_y \neq 2(u_x + uu_y) \tag{7.1}$$

■

Theorem 7.1.2.

$$\text{Solve } (1 + x^2)u_x + u_y = 0$$

Proof. The characteristic curve has the derivative

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

The solution to this ODE is

$$y = \arctan x + C$$

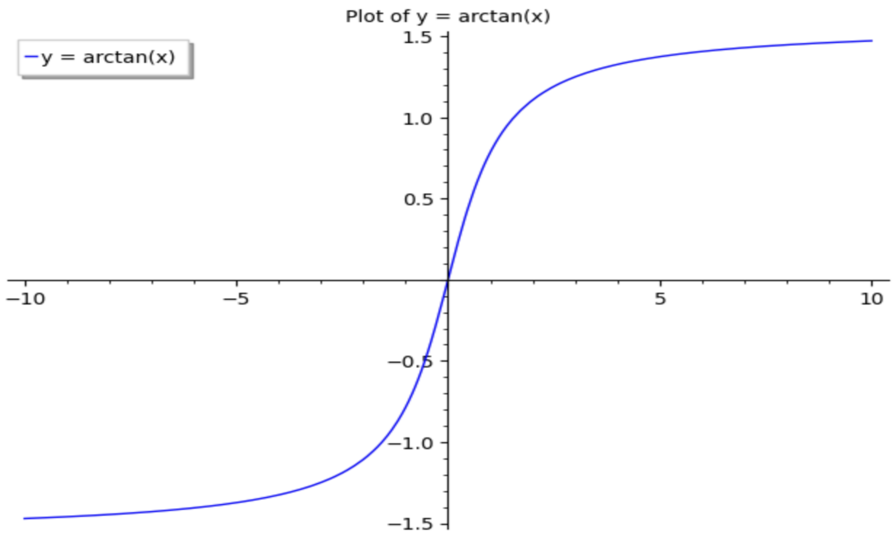
We now see that the solution to the PDE in Equation 7.1 is

$$u = f\big((\arctan x) - y\big) \text{ where } f : \mathbb{R} \to \mathbb{R} \text{ is an arbitrary smooth function}$$

A characteristic curve is as followed.

```
[3]: plot(arctan(x), (x, -10, 10), title='Plot of y = arctan(x)', legend_label='y = arctan(x)')
```

[3]:



■

Theorem 7.1.3.

$$\text{Solve } au_x + bu_y + cu = 0 \tag{7.2}$$

Proof. Fix

$$\begin{cases} x' \triangleq ax + by \\ y' \triangleq bx - ay \end{cases}$$

This map is clearly a diffeomorphism. Compute

$$\begin{cases} u_x = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = au_{x'} + bu_{y'} \\ u_y = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} = bu_{x'} - au_{y'} \end{cases}$$

Plugging it back into the PDE in Equation 7.2, we have

$$cu + (a^2 + b^2)u_{x'} = 0 \tag{7.3}$$

If $c = a^2 + b^2 = 0$, then all smooth functions are solution. If $a^2 + b^2 = 0$ but $c \neq 0$, then clearly the only solution is $u = \tilde{0}$. If $a^2 + b^2 \neq 0$ but $c = 0$, then $u_{x'} = \tilde{0}$, which implies $u = f(y')$ where $y' = bx - ay$ and f can be arbitrary smooth function.

Now, suppose $a^2 + b^2 \neq 0 \neq c$, note that the PDE in Equation 7.3 is just an ODE of the form

$$y + \frac{a^2 + b^2}{c} y' = 0$$

The general solution to this ODE is

$$y = Ce^{\frac{-ct}{a^2+b^2}}$$

In other words, the general solution of the PDE in Equation 7.3 is

$$u = Ce^{\frac{-cx'}{a^2+b^2}} = Ce^{\frac{-c(ax+by)}{a^2+b^2}}$$

■

7.2 PDE HW 2

Question 176

Consider hear flow in a long circular cylinder where the temperature depends only on t and on the distance r to the axis of the cylinder. Here $r = \sqrt{x^2+y^2}$ is the cylindrical coordinate. From the three dimensional hear equation derive the equation $u_t = k(u_{rr} + \frac{u_r}{r})$

Proof. Write the three dimensional hear equation by

$$u_t = k\Delta u$$

Note that the Laplacian Δu when written in cylindrical coordinate is

$$\Delta u = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} + u_{zz}$$

Because the premise says that u is constant in z and θ, we know $u_{\theta\theta} = u_{zz} = 0$

$$\Delta u = u_{rr} + \frac{u_r}{r}$$

This give us

$$u_t = k(u_{rr} + \frac{u_r}{r})$$

∎

7.3 PDE HW 3

Question 177

Find a solution of

$$\begin{cases} u_t = u_{xx} \\ u(x,0) = x^2 \end{cases}$$

Proof. Clearly $u = x^2 + 2t$ suffices. ■

Question 178

Consider the ODE

$$\begin{cases} u'' + u' = f \\ u'(0) = u(0) = \frac{1}{2}(u'(l) + u(l)) \end{cases}$$

where f is given.

(a) Is the solution unique?

(b) Does a solution necessarily exist, or is there a condition that f must satisfy for existence?

Proof. The solution space of linear homogeneous ODE $u'' + u' = 0$ is spanned by e^{-x} and constant. If we add in the initial condition $u'(0) = u(0)$, then the solution space become the subspace spanned by $e^{-x} - 2$. One can check that if $u \in \text{span}(e^{-x} - 2)$, then

$$u(0) = \frac{1}{2}(u'(l) + u(l)) \text{ for all } l \in \mathbb{R}$$

We now know the solution of the original ODE is not unique, since any solution added by $e^{-x} - 2$ is again a solution.

Integrating both side on $[0, l]$, we see that given the boundary conditions, f must satisfy

$$\begin{aligned} \int_0^l f(x)dx &= \int_0^l u'' + u'dx \\ &= u(l) + u'(l) - u(0) - u'(0) = 0 \end{aligned}$$

■

Question 179

Find the regions in the xy plane where the equation

$$(1+x)u_{xx} + 2xyu_{xy} - y^2u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch them.

Proof. The discriminant is exactly

$$\begin{aligned}(xy)^2 - (1+x)(-y^2) &= x^2y^2 + xy^2 + y^2 \\ &= y^2(x^2+x+1) \\ &= y^2[(x+\frac{1}{2})^2 + \frac{3}{4}]\end{aligned}$$

It then follows that the equation is parabolic if and only if $y = 0$, and hyperbolic if and only if $y \neq 0$.

■

7.4 PDE HW 4

Question 180

Solve

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = e^x \\ u_t(x,0) = \sin x \end{cases}$$

Proof. Let

$$u \triangleq f(x+ct) + g(x-ct)$$

Plugging the initial conditions, we know

$$\begin{cases} f(x) + g(x) = e^x \\ f'(x) - g'(x) = \frac{\sin x}{c} \end{cases}$$

This give us

$$f'(x) = \frac{e^x + \frac{\sin x}{c}}{2} \text{ and } g'(x) = \frac{e^x - \frac{\sin x}{c}}{2}$$

Then by FTC, we have

$$\begin{cases} f(x) = \frac{e^x - \frac{\cos x}{c}}{2} + f(0) - \frac{1}{2} + \frac{1}{2c} \\ g(x) = \frac{e^x + \frac{\cos x}{c}}{2} + g(0) - \frac{1}{2} - \frac{1}{2c} \end{cases}$$

Note that $f(0) + g(0) = e^0 = 1$, which cancel the constant terms in u, i.e.

$$\begin{aligned} u &= f(x+ct) + g(x-ct) \\ &= \frac{e^{x+ct} + e^{x-ct} + \frac{-\cos(x+ct) + \cos(x-ct)}{c}}{2} \end{aligned}$$

■

Question 181

Solve

$$\begin{cases} u_{xx} + u_{xt} - 20u_{tt} = 0 \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

Proof. The PDE can be write in the form of

$$(\partial_x + 5\partial_t)(\partial_x - 4\partial_t)u = 0$$

which have the general solution

$$u(x,t) = f(5x - t) + g(4x + t)$$

We now see

$$f(5x) + g(4x) = \varphi(x) \text{ and } -f'(5x) + g'(4x) = \psi(x)$$

This then give us

$$f'(5x) = \frac{(\varphi' - 4\psi)(x)}{9} \text{ and } g'(4x) = \frac{(\varphi' + 5\psi)(x)}{9}$$

and thus

$$\begin{aligned} f(x) &= f(0) + \int_0^x f'(s)ds \\ &= f(0) + \int_0^x \frac{(\varphi' - 4\psi)(\frac{s}{5})}{9} ds \\ &= f(0) + \frac{5}{9}\Big[\varphi(\frac{x}{5}) - \varphi(0)\Big] - \frac{4}{9}\int_0^x \psi(\frac{s}{5})ds \end{aligned}$$

and similarly

$$\begin{aligned} g(x) &= g(0) + \int_0^x g'(s)ds \\ &= g(0) + \int_0^x \frac{(\varphi' + 5\psi)(\frac{s}{4})}{9} ds \\ &= g(0) + \frac{4}{9}\Big[\varphi(\frac{x}{4}) - \varphi(0)\Big] + \frac{5}{9}\int_0^x \psi(\frac{s}{4})ds \end{aligned}$$

Noting that $f(0)+g(0)=u(0,0)=\psi(0)$, we now have

$$\begin{aligned} u(x,t) &= f(5x-t)+g(4x+t) \\ &= \frac{5\varphi(\frac{5x-t}{5})+4\varphi(\frac{4x+t}{4})}{9} - \frac{4}{9}\int_0^{5x-t}\psi(\frac{s}{5})ds + \frac{5}{9}\int_0^{4x+t}\psi(\frac{s}{4})ds \end{aligned}$$

■

7.5 PDE HW 5

Question 182

Solve

$$
\begin{aligned}
u_t &= k u_{xx} \\
u(x,0+) &= e^{-x} \\
u(0+,t) &= 0
\end{aligned}
$$

on the half line $0 < x < \infty$

Proof. Extend the initial condition to

$$
\varphi_{\text{odd}}(x) \triangleq \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -e^{x} & \text{if } x < 0 \end{cases}
$$

We then solve

$$
\begin{aligned}
u(x,t) &= \frac{1}{2\sqrt{\pi k t}} \int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4kt}} \varphi_{\text{odd}}(y) dy \\
&= \frac{1}{2\sqrt{\pi k t}} \Big[\int_0^\infty e^{\frac{-(x-y)^2}{4kt}} e^{-y} dy - \int_{-\infty}^0 e^{\frac{-(x-y)^2}{4kt}} e^{y} dy \Big] \\
&= \frac{1}{2\sqrt{\pi k t}} \Big[\int_0^\infty e^{\frac{-(y-(x-2kt))^2 - 4ktx + 4k^2t^2}{4kt}} dy - \int_{-\infty}^0 e^{\frac{-(y-(x+2kt))^2 + 4ktx + 4k^2t^2}{4kt}} dy \Big] \\
&= \frac{1}{2\sqrt{\pi k t}} \Big[e^{-x+kt} \int_0^\infty e^{\frac{-(y-(x-2kt))^2}{4kt}} dy - e^{x+kt} \int_{-\infty}^0 e^{\frac{-(y-(x+2kt))^2}{4kt}} dy \Big] \\
&= \frac{1}{\sqrt{\pi}} \Big[e^{-x+kt} \int_{\frac{2kt-x}{2\sqrt{kt}}}^\infty e^{-p^2} dp - e^{x+kt} \int_{-\infty}^{\frac{-2kt-x}{2\sqrt{kt}}} e^{-p^2} dp \Big] \\
&= \frac{1}{\sqrt{\pi}} \Big[e^{-x+kt} \Big(\frac{\sqrt{\pi}}{2} - \frac{2}{\sqrt{\pi}} \operatorname{erf}\big(\frac{2kt-x}{2\sqrt{kt}}\big) \Big) - e^{x+kt} \Big(\frac{\sqrt{\pi}}{2} - \frac{2}{\sqrt{\pi}} \operatorname{erf}\big(\frac{2kt+x}{2\sqrt{kt}}\big) \Big) \Big]
\end{aligned}
$$

∎

7.6 PDE HW 6

Question 183

Solve $u_{tt} = 4u_{xx}$ for $0 < x < \infty, u(0,t) = 0, u(x,0) = 1, u_t(x,0) = 0$ using the reflection method. The solution has a singularity find its location.

Proof. Define

$$\varphi(x) \triangleq \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases} \text{ and } \psi(x) \triangleq 0$$

We are required to solve the following Dirichlet's problem for wave equation

$$u_{tt} = 4u_{xx} \text{ for } -\infty < x < \infty$$
$$u(x,0) = \varphi(x) \text{ and } u_t(x) = \psi(x)$$

The solution is exactly

$$\begin{aligned} u(x,t) &= \frac{\varphi(x+2t) + \varphi(x-2t)}{2} + \int_{x-2t}^{x+2t} \psi(s)ds \\ &= \frac{\varphi(x+2t) + \varphi(x-2t)}{2} \\ &= \begin{cases} 1 & \text{if } x - 2t > 0 \\ 0 & \text{if } x + 2t > 0 > x - 2t \\ -1 & \text{if } 0 > x + 2t \end{cases} \end{aligned}$$

On the half line, the solution is

$$u(x,t) = \begin{cases} 1 & \text{if } x - 2t > 0 \\ 0 & \text{if } x - 2t < 0 \end{cases}$$

So the singularity is exactly on the line $x - 2t = 0$ ∎

Question 184

Solve the inhomogeneous Neumann diffusion problem on the half-line

$$\begin{cases} w_t - kw_{xx} = 0 \text{ for } x \in (0, \infty) \ \textbf{(Homogeneous DE)} \\ w_x(0,t) = h(t) \ \textbf{(Non-homogeneous BC)} \\ w(x,0) = \varphi(x) \ \textbf{(IC)} \end{cases}$$

by the subtraction method indicated in the text.

Proof. Suppose w is a solution to our problem. If we define

$$u(x,t) \triangleq w(x,t) - xh(t) \text{ for } x \in (0,\infty)$$

We see that u satisfy

$$\begin{cases} u_t - ku_{xx} = -xh'(t) \text{ for } x \in (0,\infty) \textbf{ (Non-homogeneous DE)} \\ u_x(0,t) = 0 \textbf{ (Good BC)} \\ u(x,0) = \varphi(x) - xh(0) \textbf{ (IC)} \end{cases}$$

Define

$$f_{\text{even}}(x,t) \triangleq \begin{cases} -xh'(t) & \text{if } x > 0 \\ xh'(t) & \text{if } x < 0 \end{cases}$$

And define

$$\varphi_{\text{even},*} \triangleq \begin{cases} \varphi(x) - xh(0) & \text{if } x > 0 \\ \varphi(-x) + xh(0) & \text{if } x < 0 \end{cases}$$

And define

$$u_{\text{even}}(x,t) \triangleq \int_{\mathbb{R}} S(x-y)\varphi_{\text{even},*}(y)dy + \int_0^t \int_{\mathbb{R}} S(x-y,t-s)f_{\text{even}}(y,s)dyds$$

It then follows that u_{even} solve the non-homogeneous DE and IC. To see that $u_x(0,t) = 0$, one simply observe that u is even in x. The solution to the original problem is then

$$w(x,t) \triangleq u_{\text{even}}(x,t) + xh(t) \text{ for } x \in (0,\infty)$$

∎

7.7 PDE HW 7

Question 185

Solve

$$\begin{cases} u_{tt} = c^2 u_{xx} \text{ for } x \in (0,\infty) \textbf{ (Homogeneous DE)} \\ u(x,0) = x, u_t(x,0) = 0 \textbf{ (IC)} \\ u(0,t) = t^2 \textbf{ (BC)} \end{cases}$$

Proof. If we define

$$w = u - t^2$$

we see that w satisfy

$$\begin{cases} w_{tt} - c^2 w_{xx} = -2 \text{ for } x \in (0,\infty) \textbf{ (Non-homogeneous DE)} \\ w(x,0) = x, w_t(x,0) = 0 \textbf{ (IC)} \\ w(0,t) = 0 \textbf{ (Dirichlet BC)} \end{cases}$$

then if we define

$$\varphi_{\text{odd}}(x) \triangleq x \text{ and } f_{\text{odd}}(x,t) \triangleq \begin{cases} -2 & \text{if } x > 0 \\ 2 & \text{if } x < 0 \end{cases}$$

and

$$\begin{aligned} w(x,t) &\triangleq \frac{\varphi(x+ct) + \varphi(x-ct)}{2} + \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f_{\text{odd}}(y,s)\,dyds \\ &= x + \begin{cases} -t^2 & \text{if } x - ct > 0 \\ \frac{x^2}{c^2} - \frac{2tx}{c} & \text{if } x - ct < 0 \end{cases} \end{aligned}$$

Note that we only consider when $x \geq 0$. This then give us

$$u(x,t) = \begin{cases} x & \text{if } x - ct > 0 \\ x + (t - \frac{x}{c})^2 & \text{if } x - ct < 0 \end{cases}$$

∎

7.8 PDE HW 8

Question 186

Solve the Schrodinger equation

$$\begin{cases} u_t = iku_{xx} \text{ for } 0 < x < l \textbf{ (Homogeneous DE)} \\ u_x(0,t) = u(l,t) = 0 \textbf{ (BC)} \end{cases}$$

for real $k \in (0, l)$.

Proof. Again we do the separation of the variables

$$u \triangleq T(t)X(x)$$

Some tedious efforts shows that u is a solution of this original question as long as X, T satisfy the following ODE

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) = X(l) = 0 \end{cases} \quad \text{and } T'(t) + \lambda ikT(t) = 0$$

where $\lambda \in \mathbb{C}$ is arbitrary constant. The solution of the second ODE is obviously

$$T(t) \triangleq Ce^{-\lambda ikt}$$

where $C \in \mathbb{C}$ is arbitrary constant. It remains to find what values can λ take so that X has non-trivial solutions. If $\lambda = 0$, then to satisfy the ordinary differential equation, the solution must take the forms $X = C + Dx$, where $C, D \in \mathbb{C}$ are arbitrary constant. Plugging the initial conditions, we see that $C = D = 0$. In other words, if $\lambda = 0$, then X can only be trivial. If $\lambda \neq 0$, ODE of X suggest that X must take the form

$$X \triangleq Ae^{\gamma x} + Be^{-\gamma x}$$

where $\gamma \in \mathbb{C}$ satisfy $\gamma^2 = -\lambda$. Plug in $X'(0) = 0$, we see

$$0 = \gamma(A - B)$$

which implies $A = B$. Plug in $X(l) = 0$, we see

$$0 = Ae^{\gamma l} + Be^{-\gamma l} = A(e^{\gamma l} + e^{-\gamma l})$$

Then for X to be non-trivial, we must have

$$e^{\gamma l} + e^{-\gamma l} = 0$$

By periodicity property of exponential function, we then can deduce

$$\gamma = \frac{i\pi(2n+1)}{l} \text{ and } \lambda = \frac{(2n+1)^2\pi^2}{l^2}$$

It then follows from $X = Ae^{\gamma x} + Be^{-\gamma x}$ that

$$\begin{aligned} X &= (A+B)\cos\left(\frac{\pi(2n+1)x}{l}\right) + (A-B)i\sin\left(\frac{\pi(2n+1)x}{l}\right) \\ &= (A+B)\cos\left(\frac{\pi(2n+1)x}{l}\right) \quad (\because A = B) \end{aligned}$$

$\blacksquare$

7.9 PDE HW 9

Question 187

Find the full Fourier series of e^x on $(-l,l)$ in its real and complex forms. (Hint: It is convenient to find the complex form first)

Proof. Write

$$e^x = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$$

Compute

$$\begin{aligned}
c_n &= \frac{1}{2l}\int_{-l}^{l} e^x e^{\frac{-in\pi x}{l}}\,dx \\
&= \frac{1}{2l}\int_{-l}^{l} e^{\frac{(l-in\pi)x}{l}}\,dx \\
&= \frac{1}{2l}\cdot \frac{le^{\frac{(l-in\pi)x}{l}}}{l-in\pi}\Big|_{x=-l}^{l} \\
&= \frac{l(e^{l-in\pi}-e^{-(l-in\pi)})}{2l(l-in\pi)} \\
&= \frac{(-1)^n(e^l-e^{-l})}{2(l-in\pi)} = \frac{(-1)^n}{(l-in\pi)}\sinh(l)
\end{aligned}$$

We now have

$$
\begin{aligned}
e^x &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(l-in\pi)} \sinh(l) e^{\frac{in\pi x}{l}} \\
&= \frac{\sinh(l)}{l} + \sum_{n=1}^{\infty} \Big[\frac{(-1)^n \sinh(l)}{(l-in\pi)} e^{\frac{in\pi x}{l}} + \frac{(-1)^n \sinh(l)}{(l+in\pi)} e^{\frac{-in\pi x}{l}} \Big] \\
&= \frac{\sinh(l)}{l} + \sum_{n=1}^{\infty} (-1)^n \sinh(l) \Big[\frac{\cos(\frac{n\pi x}{l}) + i\sin(\frac{n\pi x}{l})}{l-in\pi} + \frac{\cos(\frac{n\pi x}{l}) - i\sin(\frac{n\pi x}{l})}{l+in\pi} \Big] \\
&= \frac{\sinh(l)}{l} \\
&+ \sum_{n=1}^{\infty} (-1)^n \sinh(l) \cdot \frac{(l+in\pi)\big(\cos(\frac{n\pi x}{l}) + i\sin(\frac{n\pi x}{l})\big) + (l-in\pi)\big(\cos(\frac{n\pi x}{l}) - i\sin(\frac{n\pi x}{l})\big)}{l^2+n^2\pi^2} \\
&= \frac{\sinh(l)}{l} + \sum_{n=1}^{\infty} \frac{(-1)^n \sinh(l) \big[2l\cos(\frac{n\pi x}{l}) - 2n\pi \sin(\frac{n\pi x}{l})\big]}{l^2+n^2\pi^2} \\
&= \sum_{n=0}^{\infty} \frac{2(-1)^n \sinh(l)}{l^2+n^2\pi^2} \Big[l\cos(\frac{n\pi x}{l}) - n\pi \sin(\frac{n\pi x}{l}) \Big]
\end{aligned}
$$

■

Question 188

Find the complex eigenvalues of the first-derivative operator $\frac{d}{dx}$ subject to the single boundary condition $X(0) = X(1)$. Are the eigenfunctions orthogonal on the interval $(0,1)$?

Proof. We are solving the eigenproblem

$$
\begin{cases} X' + \lambda X = 0 \\ X(0) = X(1) \end{cases}
$$

Clearly, the solution X must take the form $X = e^{-\lambda x}$. The boundary conditions then implies

$$
e^{-\lambda} = X(1) = X(0) = 1
$$

which implies

$$
\lambda = 2ni\pi \text{ for } n \in \mathbb{Z}
$$

Compute for distinct n, m

$$\langle X_n, X_m \rangle = \int_0^1 e^{-2(n-m)i\pi x} dx = \frac{e^{-2(n-m)i\pi x}}{-2(n-m)i\pi}\bigg|_{x=0}^{1} = 0$$

So the eigenfunctions are indeed orthogonal. ■

7.10 PDE HW 10

Question 189

Consider

$$\sum_{n=0}^{\infty}(-1)^n x^{2n}$$

(a) Does it converge pointwise in $(-1,1)$?

(b) Does it converge uniformly in $(-1,1)$?

(c) Does it converge in L^2 in $(-1,1)$?

Proof. Geometric series as such obviously converges to

$$\sum_{n=0}^{\infty}(-1)^n x^{2n} = \sum_{n=0}^{\infty}(-x^2)^n = \frac{1}{1+x^2}$$

Note that the remainder can also be computed by

$$\sup_{x\in(-1,1)}\left|\sum_{n=N}^{\infty}(-x^2)^n\right| = \sup_{x\in(-1,1)}\left|\frac{-x^{2N}}{1+x^2}\right| = \frac{1}{2} \text{ for all } N$$

It follows that the series does NOT converge uniformly on $(-1,1)$. Compute

$$\begin{aligned}\int_{-1}^{1}\Big(\sum_{n=N}^{\infty}(-x^2)^n\Big)^2 dx &= \int_{-1}^{1}\Big(\frac{-x^{2N}}{1+x^2}\Big)^2 dx \\ &= \int_{-1}^{1}\frac{x^{4N}}{(1+x^2)^2}dx \\ &\leq \int_{-1}^{1} x^{4N}dx = \frac{x^{4N+1}}{4N+1}\bigg|_{x=-1}^{1} \to 0\end{aligned}$$

It follows that the series does converge in L^2. ■

Question 190

(Term by Term integration)

(a) If $f(x)$ is a piecewise continuous function in $[-l,l]$, show that its definite integral $F(x) = \int_{-l}^{x} f(s)ds$ has a full Fourier series that converges pointwise.

(b) Write this convergent series for $F(x)$ explicitly in terms of the Fourier coefficients a_0, a_n, b_n of $f(x)$ where $a_0 = 0$. (Hint: Apply a convergence Theorem. Write the formulas for the coefficients and integrate by parts.)

Proof. Part (a) follows from observing $F' = f$ is pointwise continuous so that the classical Fourier series of F converges to F by Theorem 4 in the textbook.

Write

$$f(x) = \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{l}) + b_n \sin(\frac{n\pi x}{l})$$

Note that the definition of piece wise continuity in this book implies boundedness on compact domain, and note that each term

$$\sum_{n=1}^{N} a_n \cos(\frac{n\pi x}{l}) + b_n \sin(\frac{n\pi x}{l})$$

is obviously bounded on $[-l, l]$. Then because f is bounded on $[-l, l]$, we know

$$\sum_{n=1}^{N} a_m \cos(\frac{n\pi x}{l}) + b_n \sin(\frac{n\pi x}{l}) \text{ is uniformly bounded on } [-l, l]$$

Therefore, we may apply DCT to compute

$$\begin{aligned} F(x) &= \int_{-l}^{x} \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi s}{l}) + b_n \sin(\frac{n\pi s}{l}) ds \\ &= \sum_{n=1}^{\infty} \int_{-l}^{x} a_n \cos(\frac{n\pi s}{l}) + b_n \sin(\frac{n\pi s}{l}) ds \\ &= \sum_{n=1}^{\infty} \frac{a_n l}{n\pi} \sin(\frac{n\pi x}{l}) + \frac{-b_n l}{n\pi} \cos(\frac{n\pi x}{l}) + \frac{-(-1)^n b_n l}{n\pi} \end{aligned}$$

$\blacksquare$

7.11 PDE HW 11

Question 191

Prove the Schwarz Inequality

$$|(f,g)| \leq \|f\|_2 \cdot \|g\|_2$$

for any pair of functions.

Proof. Schwarz inequality for integral is a corollary of Holder's inequality

$$|(f,g)| = \left| \int f\overline{g}dx \right| \leq \int |fg|\, dx = \|fg\|_1 \leq \|f\|_2 \cdot \|g\|_2$$

The following is a proof of Holder's inequality. Note that here we use $p = q = 2$. ∎

Theorem 7.11.1. (Holder's Inequality) Let f, g be two functions measurable on E. We have

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Proof. The proof is trivial if $p = 1$ or ∞, so we may suppose

$$1 < p < \infty$$

Now, if $\|f\|_p = 0$, then $f = 0$ almost everywhere, which renders the inequality trivial since $\|fg\|_1 = 0$. If $\|f\|_p = \infty$, the proof is again trivial. We may now suppose

$$\|f\|_p, \|g\|_q \in (0, \infty)$$

Define f_1, g_1 by

$$f_1 \triangleq \frac{f}{\|f\|_p} \text{ and } g_1 \triangleq \frac{g}{\|g\|_q}$$

Because $1 < p < \infty$, by Young's Inequality for product, we have

$$\begin{aligned} \|f_1 g_1\|_1 = \int_E |f_1 g_1| &\leq \int_E \left(\frac{|f_1|^p}{p} + \frac{|g_1|^q}{q} \right) \\ &= \frac{\|f_1\|_p^p}{p} + \frac{\|g_1\|_q^q}{q} = \frac{1}{p} + \frac{1}{q} = 1 \end{aligned}$$

The proof then follows the assumption $\|f\|_p, \|g\|_q \in (0, \infty)$ and

$$\|f_1 g_1\|_1 = \frac{\|fg\|_1}{\|f\|_p \|g\|_q}$$

∎

Question 192

Solve the Poisson Equation

$$\begin{cases} u_{xx} + u_{yy} = 1 \text{ in } r < a \\ u = 0 \text{ on } r = a \end{cases}$$

Proof. Write the Poisson equation in polar coordinate

$$u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = 1$$

Because $u(a, \theta)$ for all θ, we may suppose u is independent of θ. Therefore, the Poisson equation in polar coordinate simplifies to

$$u_{rr} + \frac{u_r}{r} = 1$$

The solution space of this ODE is exactly

$$\left\{ \frac{r^2}{4} + C_1 \ln r + C_2 : C_1, C_2 \in \mathbb{R} \right\}$$

Let

$$u = \frac{r^2}{4} + C_1 \ln r + C_2$$

Because u is finite on $r = 0$, we must have $C_1 = 0$. It then follows from $u = 0$ for $r = a$ that

$$u = \frac{r^2}{4} - \frac{a^2}{4}$$

∎

7.12 PDE HW 12

Question 193

Prove that the eigenfunctions $\{\sin my \sin nz\}$ are orthogonal on the square $\{0 < y < \pi, 0 < z < \pi\}$.

Proof. Let p, q be a pair of integer such that $p \neq m$ and $q \neq n$. Because $\{\sin(ix)\}_{i \in \mathbb{Z}}$ is an orthogonal system on $(0, \pi)$, we may compute

$$\begin{aligned}
&\int_0^\pi \int_0^\pi \sin(my)\sin(nz)\sin(py)\sin(qz)dydz \\
&= \int_0^\pi \sin(my)\sin(py)dy \int_0^\pi \sin(nz)\sin(qz)dz \\
&= 0 \cdot 0 = 0
\end{aligned}$$

This shows that $\{\sin(my)\sin(nz)\}$ are indeed orthogonal on the square $\{0 < y < \pi, 0 < z < \pi\}$. Note that we used the Fubini's theorem in our first equality, and the reason we can use the Fubini's theorem is that the function $\sin(my)\sin(nz)\sin(py)\sin(qz)$ is bounded by 1 on the bounded domain $[0, \pi]^2$. ∎

Chapter 8

Differential Geometry HW

8.1 HW1

Abstract

In this HW, we give precise definition to $\mathbb{P}^n$ and $\mathbb{R}P^n$, and we rigorously show

(a) $\mathbb{R}P^n$ has a smooth structure.

(b) $\mathbb{P}^n$ is homeomorphic to $\mathbb{R}P^n$

(c) $\mathbb{P}^n$ has a smooth structure.

We also solved the other two questions. Note that in this PDF, brown text is always a clickable hyperlink reference.

Define an equivalence relation on $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ by

$$\mathbf{x} \sim \mathbf{y} \overset{\triangle}{\iff} \mathbf{y} = \lambda \mathbf{x} \text{ for some } \lambda \in \mathbb{R}^*$$

Let $\mathbb{R}P^n \triangleq (\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}) \setminus \sim$ be the quotient space and let

$$V_i \triangleq \{\mathbf{x} \in \mathbb{R}^{n+1} : \mathbf{x}^i \neq 0\} \text{ for each } 1 \leq i \leq n+1$$

By definition, it is clear that

$$\text{either } \pi^{-1}(\pi(\mathbf{x})) \subseteq V_i \text{ or } \pi^{-1}(\pi(\mathbf{x})) \subseteq V_i^c$$

Then if we define $\varphi_i : V_i \to \mathbb{R}^n$ by

$$\varphi_i(\mathbf{x}) \triangleq \Big(\frac{\mathbf{x}^1}{\mathbf{x}^i},\dots,\frac{\mathbf{x}^{i-1}}{\mathbf{x}^i},\frac{\mathbf{x}^{i+1}}{\mathbf{x}^i},\dots,\frac{\mathbf{x}^{n+1}}{\mathbf{x}^i}\Big)$$

because $\pi(\mathbf{x}) = \pi(\mathbf{y}) \implies \varphi_i(\mathbf{x}) = \varphi_i(\mathbf{y})$, we can well induce a map

$$\Phi_i : U_i \triangleq \pi(V_i) \subseteq \mathbb{R}P^n \to \mathbb{R}^n; \pi(\mathbf{x}) \mapsto \varphi_i(\mathbf{x})$$

Note that one has the equation

$$\Phi_i(\pi(\mathbf{x})) = \varphi_i(\mathbf{x}) \text{ for all } \mathbf{x} \in V_i$$

Theorem 8.1.1. (Real Projective Space with a differentiable atlas) We have

$$\mathbb{R}P^n \text{ with atlas } \{(U_i,\Phi_i) : 1 \leq i \leq n+1\} \text{ is a differentiable manifold}$$

Proof. We are required to prove

(a) (U_i,Φ_i) are all charts.

(b) $\{(U_i,\Phi_i) : 1 \leq i \leq n+1\}$ form a differentiable atlas.

(c) $\mathbb{R}P^n$ is Hausdorff.

(d) $\mathbb{R}P^n$ is second-countable.

Because $\pi^{-1}(U_i) = V_i$ and V_i is clearly open in $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$, we know $U_i \subseteq \mathbb{R}P^n$ is open. Note that clearly, $\Phi_i(U_i) = \mathbb{R}^n$. To show (U_i,Φ_i) is a chart, it remains to show that Φ_i is a homeomorphism between U_i and $\mathbb{R}^n$. It is straightforward to check Φ_i is one-to-one on U_i. This implies Φ_i is a bijective between U_i and $\mathbb{R}^n$.

Fix open $E \subseteq \mathbb{R}^n$. We see

$$\pi^{-1}(\Phi_i^{-1}(E)) = \varphi_i^{-1}(E)$$

Then because $\varphi_i : V_i \to \mathbb{R}^n$ is clearly continuous, we see $\varphi_i^{-1}(E)$ is open in $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$, and it follows from definition of quotient topology $\Phi_i^{-1}(E) \subseteq \mathbb{R}P^n$ is open. Then because U_i is open in $\mathbb{R}P^n$, we see $\Phi_i^{-1}(E)$ is open in U_i. We have proved $\Phi_i : U_i \to \mathbb{R}^n$ is continuous.

Define $\Psi_i : \mathbb{R}^n \to V_i$ by

$$\Psi(\mathbf{x}^1,\dots,\mathbf{x}^n) = (\mathbf{x}^1,\dots,\mathbf{x}^{i-1},1,\mathbf{x}^i,\dots,\mathbf{x}^n)$$

Observe that for all $\mathbf{x} \in \Phi_i(U_i)$, we have

$$\Phi_i^{-1}(\mathbf{x}) = \pi(\Psi_i(\mathbf{x}))$$

It then follows from $\Psi_i : \mathbb{R}^n \to V_i$ and $\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{R}P^n$ are continuous that $\Phi_i^{-1} : \mathbb{R}^n \to \mathbb{R}P^n$ is continuous.

We have proved that (Ψ_i, U_i) are all charts. Now, because V_i clearly cover $\mathbb{R}^{n+1}$, we know U_i also cover $\mathbb{R}P^n$. We have proved $\{(U_i, \Phi) : 1 \leq i \leq n+1\}$ form an atlas. The fact $\mathbb{R}P^n$ is second-countable follows.

Fix $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \Phi_i(U_i \cap U_j)$. We compute

$$\begin{aligned}\Phi_j \circ \Phi_i^{-1}(\mathbf{x}^1, \dots, \mathbf{x}^n) &= \Phi_j\Big([(\mathbf{x}^1, \dots, \mathbf{x}^{i-1}, 1, \mathbf{x}^i, \mathbf{x}^{i+1}, \dots, \mathbf{x}^n)]\Big) \\ &= \begin{cases} (\frac{\mathbf{x}^1}{\mathbf{x}^j}, \dots, \frac{\mathbf{x}^{j-1}}{\mathbf{x}^j}, \frac{\mathbf{x}^{j+1}}{\mathbf{x}^j}, \dots, \frac{\mathbf{x}^{i-1}}{\mathbf{x}^j}, \frac{1}{\mathbf{x}^j}, \frac{\mathbf{x}^i}{\mathbf{x}^j}, \dots, \frac{\mathbf{x}^n}{\mathbf{x}^j}) & \text{if } j < i \\ (\frac{\mathbf{x}^1}{\mathbf{x}^{j-1}}, \dots, \frac{\mathbf{x}^{i-1}}{\mathbf{x}^{j-1}}, \frac{1}{\mathbf{x}^{j-1}}, \frac{\mathbf{x}^i}{\mathbf{x}^{j-1}}, \dots, \frac{\mathbf{x}^{j-2}}{\mathbf{x}^{j-1}}, \frac{\mathbf{x}^j}{\mathbf{x}^{j-1}}, \dots, \frac{\mathbf{x}^n}{\mathbf{x}^{j-1}}) & \text{if } j > i \end{cases}\end{aligned}$$

This implies our atlas is indeed differentiable.

Before we prove $\mathbb{R}P^n$ is Hausdorff, we first prove that $\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{R}P^n$ is an open mapping. Let $U \subseteq \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ be open. Observe that

$$\pi^{-1}(\pi(U)) = \{t\mathbf{x} \in \mathbb{R}^{n+1} : t \neq 0 \text{ and } \mathbf{x} \in U\}$$

Fix $t_0\mathbf{x} \in \pi^{-1}(\pi(U))$. Let $B_\epsilon(\mathbf{x}) \subseteq U$. Observe that

$$B_{|t_0|\epsilon}(t_0\mathbf{x}) \subseteq t_0 B_\epsilon(\mathbf{x}) \subseteq t_0 U \subseteq \pi^{-1}(\pi(U))$$

This implies $\pi^{-1}(\pi(U))$ is open. (done)

Now, because π is open, to show $\mathbb{R}P^n$ is Hausdorff, we only have to show

$$R_\pi \triangleq \{(\mathbf{x}, \mathbf{y}) \in (\mathbb{R}^{n+1} \setminus \{\mathbf{0}\})^2 : \pi(\mathbf{x}) = \pi(\mathbf{y})\} \text{ is closed}$$

Define $f : (\mathbb{R}^{n+1} \setminus \{\mathbf{0}\})^2 \to \mathbb{R}$ by

$$f(\mathbf{x}, \mathbf{y}) \triangleq \sum_{i \neq j} (\mathbf{x}^i \mathbf{y}^j - \mathbf{x}^j \mathbf{y}^i)^2$$

Note that f is clearly continuous and $f^{-1}(0) = R_\pi$, which finish the proof. ■

Alternatively, we can characterize $\mathbb{R}P^n$ by identifying the antipodal pints on $S^n \triangleq \{\mathbf{x} \in \mathbb{R}^{n+1} : |\mathbf{x}| = 1\}$ as one point

$$\mathbf{x} \sim \mathbf{y} \overset{\triangle}{\iff} \mathbf{x} = \mathbf{y} \text{ or } \mathbf{x} = -\mathbf{y}$$

and let $\mathbb{P}^n \triangleq S^n \backslash \sim$ be the quotient space.

Theorem 8.1.2. (Equivalent Definitions of Real Projective Space)

$$\mathbb{R}P^n \text{ and } \mathbb{P}^n \text{ are homeomorphic}$$

Proof. Define $F : \mathbb{P}^n \to \mathbb{R}P^n$ by

$$\{\mathbf{x}, -\mathbf{x}\} \mapsto \{\lambda \mathbf{x} : \lambda \in \mathbb{R}^*\}$$

It is straightforward to check that F is well-defined and bijective. Define $f : S^n \to \mathbb{R}P^n$ by

$$f = \pi \circ \mathbf{id}$$

where $\mathbf{id} : S^n \to \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ and $\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{R}P^n$ are continuous. Check that

$$f = F \circ p$$

where $p : S^n \to \mathbb{P}^n$ is the quotient mapping. It now follows from the universal property that F is continuous, and since $\mathbb{P}^n$ is compact and $\mathbb{R}P^n$ is Hausdorff, it also follows that F is a homeomorphism between $\mathbb{R}P^n$ and $\mathbb{P}^n$. ■

Knowing that $F : \mathbb{P}^n \to \mathbb{R}P^n$ is a homeomorphism and $\mathbb{R}P^n$ is a smooth manifold, we see that $\mathbb{P}^n$ is Hausdorff and second-countable, and if we define the atlas

$$\{(F^{-1}(U_i), \Phi_i \circ F) : 1 \leq i \leq n+1\}$$

We see this atlas is indeed smooth, since

$$(\Phi_i \circ F) \circ (\Phi_j \circ F)^{-1} = \Phi_i \circ \Phi_j^{-1}$$

Question 194

Let X be a set equipped with

(a) a collection $(U_\alpha)_{\alpha \in I}$ of subsets that covers X.

(b) a collection of bijection $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ that maps U_α to an open subset $\varphi_\alpha(U_\alpha)$ of $\mathbb{R}^n$.

(c) For each $\alpha, \beta \in I$, the set $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open.

(d) For each $\alpha, \beta \in I, \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\beta \cap U_\alpha) \to \varphi_\beta(U_\alpha, U_\beta)$ is smooth.

Give X a topology so that X is a smooth manifold.

Proof. If we define $E \subseteq X$ is open if and only if

$$\varphi_\alpha(U_\alpha \cap E) \text{ is open for all } \alpha$$

we see that given arbitrary collection of open sets $(E_j)_{j\in J}$, we have

$$\varphi_\alpha(U_\alpha \cap \bigcup_{j\in J} E_j) = \bigcup_{j\in J} \varphi_\alpha(U_\alpha \cap E_j) \text{ for all } \alpha \in I$$

thus open sets are closed under union, and given two open sets E_1, E_2, we have

$$\varphi_\alpha(U_\alpha \cap E_1 \cap E_2) \subseteq \varphi_\alpha(U_\alpha \cap E_1) \cap \varphi_\alpha(U_\alpha \cap E_2) \text{ for all } \alpha \in I$$

Note that if $\mathbf{x} \in \varphi_\alpha(U_\alpha \cap E_1) \cap \varphi_\alpha(U_\alpha \cap E_2)$, then there exists $p_1 \in U_\alpha \cap E_1$ and $p_2 \in U_\alpha \cap E_2$ such that $\varphi_\alpha(p_1) = \varphi_\alpha(p_2) = \mathbf{x}$. Because φ_α is one-to-one, we can deduce $p_1 = p_2 \in E_2$, it then follows

$$\mathbf{x} = \varphi(p_1) \in \varphi_\alpha(U_\alpha \cap E_1 \cap E_2)$$

We now see

$$\varphi_\alpha(U_\alpha \cap E_1) \cap \varphi_\alpha(U_\alpha \cap E_2) \subseteq \varphi_\alpha(U_\alpha \cap E_1 \cap E_2) \text{ for all } \alpha \in I$$

We have proved that our topology on X is well-defined.

Note that U_α is open in X follows from premise (c). Thus, if some $E \subseteq U_\alpha$ is open in U_α, then E is open in X and $\varphi_\alpha(E) = \varphi_\alpha(U_\alpha \cap E)$ is open. We have proved that $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ is an open mapping. The fact that $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ is continuous trivially follows from

(a) U_α is open in X.

(b) our definition of topology on X.

(c) $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ is a bijection.

We have proved that $(U_\alpha, \varphi_\alpha)$ are indeed charts. The fact that they form a smooth atlas follows from premise (a) and (d). ∎

Question 195

Let $\mathbb{R}$ be the real line with the differentiable structure given by the maximal atlas of the chart $(\mathbb{R}, \varphi = \mathbf{id} : \mathbb{R} \to \mathbb{R})$, where id is the identity map, and let $\mathbb{R}'$ be the real line with the differentiable structure given by the maximal atlas of the chart $(\mathbb{R}', \psi : \mathbb{R}' \to \mathbb{R})$, where $\psi(x) = x^{1/3}$.

(a) Show that these two differentiable structures are distinct.

(b) Show that there is a diffeomorphism between $\mathbb{R}$ and $\mathbb{R}'$. (Hint: The identity map $\mathbb{R} \to \mathbb{R}$ is not the desired diffeomorphism.)

Proof. To see these two smooth structure are distinct. Observe

$$\mathbf{id}^{-1} \circ \psi : \mathbb{R} \to \mathbb{R}; x \mapsto x^{\frac{1}{3}} \text{ is not smooth at } 0$$

We claim $\varphi : \mathbb{R} \mapsto \mathbb{R}'$ defined by

$$\varphi(x) \triangleq x^3 \text{ is a diffeomorphism}$$

It is clear that φ is a homeomorphism. To see φ is a smooth mapping from $\mathbb{R}$ to $\mathbb{R}'$, observe that

$$\psi \circ \varphi \circ \mathbf{id}^{-1}(x) = x$$

To see φ^{-1} is a smooth mapping from $\mathbb{R}'$ to $\mathbb{R}$, observe that

$$\mathbf{id} \circ \varphi \circ \psi^{-1}(x) = x$$

We have proved that φ is a diffeomorphism between $\mathbb{R}$ and $\mathbb{R}'$. ∎

8.2 Appendix

Theorem 8.2.1. (Homeomorphism between Compact Space and Hausdorff Space) Suppose

(a) X is compact.

(b) Y is Hausdorff.

(c) $f : X \to Y$ is a continuous bijective function.

Then

$$f \text{ is a homeomorphism between } X \text{ and } Y$$

Proof. Because closed subset of compact set is compact and continuous function send compact set to compact set, we see for each closed $E \subseteq X$, $f(E) \subseteq Y$ is compact. The result then follows from $f(E) \subseteq Y$ being closed since Y is Hausdorff. ∎

Theorem 8.2.2. (Hausdorff and Quotient) If $\pi : X \to Y$ is an open mapping, and we define

$$R_\pi \triangleq \{(x, y) \in X^2 : \pi(x) = \pi(y)\}$$

Then

$$R_\pi \text{ is closed } \iff Y \text{ is Hausdorff}$$

Proof. Suppose R_π is closed. Fix some x, y such that $\pi(x) \neq \pi(y)$. Because R_π is closed, we know there exists open neighborhood U_x, U_y such that $U_x \times U_y \subseteq (R_\pi)^c$. It is clear that $\pi(U_x), \pi(U_y)$ are respectively open neighborhood of $\pi(x)$ and $\pi(y)$. To see $\pi(U_x)$ and $\pi(U_y)$ are disjoint, assume that $\pi(a) \in \pi(U_x) \cap \pi(U_y)$. Let $a_x \in U_x$ and $a_y \in U_y$ satisfy $\pi(a_x) = \pi(a) = \pi(a_y)$, which is impossible because $(a_x, a_y) \in (R_\pi)^c$. CaC

Suppose Y is Hausdorff. Fix some x, y such that $\pi(x) \neq \pi(y)$. Let U_x, U_y be open neighborhoods of $\pi(x), \pi(y)$ separating them. Observe that $(x, y) \in \pi^{-1}(U_x) \times \pi^{-1}(U_y) \subseteq (R_\pi)^c$ ∎

8.3 Example: $S^1, \mathbb{R} \setminus \mathbb{Z}$ diffeomorphism

Equip $S^1 \triangleq \{(x,y) \in \mathbb{R}^2 : |(x,y)| = 1\}$ with the standard four projection chart smooth atlas as one of them being

$$V \triangleq \{(x,y) \in \mathbb{R}^2 : y > 0\} \text{ and } \varphi_V : V \to \mathbb{R}; (x,y) \mapsto x$$

Let $p : \mathbb{R} \to \mathbb{R} \setminus \mathbb{Z}$ be the quotient map and let

$$U_0 \triangleq p\Big((0,1)\Big) \text{ and } U_1 \triangleq p\Big((\frac{-1}{2}, \frac{1}{2})\Big)$$

which are both open as one can readily check. Define $\varphi_0 : U_0 \to (0,1)$ by

$$\varphi_0(p(t)) \triangleq t_0 \text{ where } t_0 \in (0,1) \text{ and } p(t_0) = p(t)$$

and $\varphi_1 : U_1 \to (-\frac{1}{2}, \frac{1}{2})$ by

$$\varphi_1(p(t)) \triangleq t_0 \text{ where } t_0 \in (-\frac{1}{2}, \frac{1}{2}) \text{ and } p(t_0) = p(t)$$

Clearly, the function $G : \mathbb{R} \setminus \mathbb{Z} \to S^1$ well-defined by $G(p(x)) \triangleq (\cos 2\pi x, \sin 2\pi x)$ is a homeomorphism, as one can check that

(a) G is a continuous bijection. (Using Universal property of quotient map)

(b) $\mathbb{R} \setminus \mathbb{Z}$ is compact. (by finite sub-cover definition)

(c) S^1 is Hausdorff.

We now compute that $\varphi_V \circ G \circ \varphi_0^{-1}$ is defined on whole $(0,1)$, and is exactly

$$\varphi_V \circ G \circ \varphi_0^{-1}(t) \triangleq \cos 2\pi t \text{ smooth}$$

8.4 HW2

Question 196

Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the map

$$F(x,y) \triangleq (x,y,xy) = (u,v,w)$$

Let $p = (x,y) \in \mathbb{R}^2$. Compute $F_*(\frac{\partial}{\partial x}|_p)$ as a linear combination of $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w}$

Proof. For all $f \in C^\infty(\mathbb{R}^3)$, we have

$$\frac{\partial f \circ F}{\partial x}(x,y) = \frac{\partial f}{\partial u}(F(p)) + \frac{\partial f}{\partial w}(F(p))y$$

This then give us

$$F_*\Big(\frac{\partial}{\partial x}\Big|_{(x,y)}\Big) = \frac{\partial}{\partial u} + y\frac{\partial}{\partial w}$$

$\blacksquare$

Question 197

Let G be a lie group with multiplication map $\mu : G \times G \to G$ and identity element e. Show that differential $\mu_{*,(e,e)} : T_{(e,e)}G \times G \to T_eG$ of μ at identity is

$$\mu_{*,(e,e)}(X_e, Y_e) = X_e + Y_e$$

Note that $T_{(p,q)}M \times N$ is isomorphic to $T_pM \oplus T_qN$ via the differential of two projection $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$.

Proof. We first justify the notation of writing tangent vectors in $T_{(e,e)}G \times G$ as (X_e, Y_e), and the proof will follow. Consider the projection $\pi_1 : G \times G \to G$ and $\pi_2 : G \times G \to G$

$$\pi_1(g,h) \triangleq g \text{ and } \pi_2(g,h) \triangleq h$$

Consider charts $(U, \varphi), (V, \psi)$ for G centering e. We can induce a chart $(U \times V, \Phi)$ for $G \times G$ centering e by

$$\Phi(g,h) \triangleq (\varphi(g), \psi(h))$$

In local coordinate, we have

$$\pi_1(\mathbf{x}^1, \dots, \mathbf{x}^n, \mathbf{y}^1, \dots, \mathbf{y}^n) = (\mathbf{x}^1, \dots, \mathbf{x}^n) \text{ and } \pi_2(\mathbf{x}^1, \dots, \mathbf{x}^n, \mathbf{y}^1, \dots, \mathbf{y}^n) = (\mathbf{y}^1, \dots, \mathbf{y}^n)$$

In abuse of notation, this give us

$$(\pi_1)_*\Big(\sum_{i=1}^n v^i\frac{\partial}{\partial \mathbf{x}^i}+w^i\frac{\partial}{\partial \mathbf{y}^i}\Big)=\sum_{i=1}^n v^i\frac{\partial}{\partial \mathbf{x}^i}\text{ and }(\pi_2)_*\Big(\sum_{i=1}^n v^i\frac{\partial}{\partial \mathbf{x}^i}+w^i\frac{\partial}{\partial \mathbf{y}^i}\Big)=\sum_{i=1}^n w^i\frac{\partial}{\partial \mathbf{y}^i}$$

This then give us a vector space isomorphism between $T_{(e,e)}G\times G$ and $T_eG\oplus T_eG$, on which our notation stand. Now, let $\gamma:(-\epsilon,\epsilon)\to G$ be a smooth curve centering e such that $\gamma'(0)=X_e$. Define another smooth curve $\alpha:(-\epsilon,\epsilon)\to G\times G$ in $G\times G$ by

$$\alpha(t)\triangleq(\gamma(t),e)$$

Because $\pi_2\circ\alpha$ is constant and $\pi_1\circ\alpha=\gamma$, we now see

$$(\pi_1)_{*,(e,e)}(\alpha'(0))=(\pi_1\circ\alpha)'(0)=\gamma'(0)\text{ and }(\pi_2)_{*,(e,e)}(\alpha'(0))=0$$

This implies $\alpha'(0)=(X_e,0)$. Compute

$$\mu\circ\alpha(t)=\gamma(t)+e=\gamma(t)$$

We now can deduce

$$\mu_{*,(e,e)}(X_e,0)=\mu_{*,(e,e)}(\alpha'(0))=(\mu\circ\alpha)'(0)=\gamma'(0)=X_e$$

Similar procedure can be applied to show

$$\mu_{*,(e,e)}(0,Y_e)=Y_e$$

It now follows from linearity of $\mu_{*,(e,e)}$ that

$$\mu_{*,(e,e)}(X_e,Y_e)=X_e+Y_e$$

■

Question 198

Let $S^2=\{(x,y,z)\in\mathbb{R}^3:x^2+y^2+z^2=1\}$ be the sphere in $\mathbb{R}^3$. Consider the function $h:S^2\to\mathbb{R}$ defined by

$$h(x,y,z)\triangleq z$$

Find the critical points of h.

Proof. Consider the atlas $\{(U,\varphi),(V,\psi)\}$ for S^2 where $U=S^2\setminus\{(0,0,1)\},V=S^2\setminus\{(0,0,-1)\}$ and

$$\varphi(x,y,z)\triangleq\Big(\frac{x}{1-z},\frac{y}{1-z}\Big)\text{ and }\psi(x,y,z)=\Big(\frac{x}{1+z},\frac{y}{1+z}\Big)$$

Some algebra trick and tedious efforts shows that this indeed form a smooth atlas and gives us their inverse

$$\varphi^{-1}(u,v) = \Big(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\Big)$$
$$\psi^{-1}(u,v) = \Big(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{1-u^2-v^2}{u^2+v^2+1}\Big)$$

Compute

$$[d(h\circ\varphi^{-1})_{(u,v)}] = \begin{bmatrix} \frac{-4u}{(u^2+v^2+1)^2} & \frac{-4v}{(u^2+v^2+1)^2} \end{bmatrix}$$
$$[d(h\circ\psi^{-1})_{(u,v)}] = \begin{bmatrix} \frac{4u}{(u^2+v^2+1)^2} & \frac{4v}{(u^2+v^2+1)^2} \end{bmatrix}$$

This then shows the set of critical points is exactly

$$\{\varphi^{-1}(0,0), \psi^{-1}(0,0)\} = \{(0,0,-1),(0,0,1)\}$$

■

Question 199

A smooth map $f: M \to N$ is said to be **a traversal to an embedded submanifold** $S \subseteq N$ if for every point $p \in f^{-1}(S)$, we have

$$f_{*,p}(T_pM) + T_{f(p)}S = T_{f(p)}N$$

The goal of this exercise is to prove the Transversality Theorem: If a smooth map $f: M \to N$ is a traversal to an embedded submanifold S of codimension k in N, then $f^{-1}(S)$ is a regular submanifold of codimension k in M. Let $p \in f^{-1}(S)$ and $(U, \mathbf{x}^1, \dots, \mathbf{x}^n)$ be an adapted chart for N centering $f(p)$ with respect to S. Define $g: U \to \mathbb{R}^k$ by

$$g(\mathbf{x}^1, \dots, \mathbf{x}^n) \triangleq (\mathbf{x}^{n-k+1}, \dots, \mathbf{x}^n)$$

(a) Show that $f^{-1}(U) \cap f^{-1}(S) = (g\circ f)^{-1}(\mathbf{0})$.

(b) Show that $f^{-1}(U) \cap f^{-1}(S)$ is a regular level set of the function $g \circ f : f^{-1}(U) \to \mathbb{R}^k$.

(c) Prove the Transversality Theorem.

Proof. At first, we shall point out that $g \circ f$ is a function defined only on $f^{-1}(U)$. (a) follows trivially from the fact that $(U, \mathbf{x}^1, \dots \mathbf{x}^n)$ is an adapted chart. We now prove (b).

Fix arbitrary $p \in f^{-1}(U) \cap f^{-1}(S)$. Trivially, by definition of g,

$$T_{f(p)}S \subseteq \operatorname{Ker} g_{*,f(p)}$$

Explicit formula of g shows that $g_{*,f(p)}$ is a vector space epimorphism that maps $T_{f(p)}N$ into $T_{g\circ f(p)}\mathbb{R}^k$, which implies that $\operatorname{Ker} g_{*,f(p)}$ has dimension $n-k$, same as $T_{f(p)}S$ and give us

$$T_{f(p)}S = \operatorname{Ker} g_{*,f(p)}$$

It now follows from f being a traversal and $g_{*,f(p)}$ being surjective that

$$(g \circ f)_{*,p}(T_pM) = g_{*,f(p)} \circ f_{*,p}(T_pM) = \operatorname{Im} g_{*,f(p)} = T_{g\circ f(p)}\mathbb{R}^k$$

We have shown that $g \circ f$ is regular at p. (b) then follows from p is arbitrary selected from $f^{-1}(U) \cap f^{-1}(S)$.

Now, by Regular level set Theorem, we see that $f^{-1}(U) \cap f^{-1}(S)$ is an embedded submanifold of $f^{-1}(U)$ with codimension k. Because f is continuous, we know $f^{-1}(U)$ is open, thus an embedded submanifold of M with dimension m. It now follows that $f^{-1}(U) \cap f^{-1}(S)$ is an embedded submanifold of M with codimension k. We have proved the Transversality Theorem. ■

8.5 Bundle

A **smooth real vector bundle of rank k over the smooth manifold** M is a smooth manifold E together with the surjective smooth map $\pi : E \to M$ such that

(a) Each fiber $\pi^{-1}(p)$ has a real k-dimensional vector space structure.

(b) For all $p \in M$, there exists some neighborhood U of p and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that Φ map the fiber $\pi^{-1}(p)$ vector space isomorphiscally to $\{p\} \times \mathbb{R}^k$.

Note that we often call E the **total space** and M the **base space**. The neighborhood U and the smooth diffeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ is often called the **smooth local trivialization**, and if there exists a global smooth trivialization, we say $(E, M, \pi : E \to M)$ is a **trivial bundle**. It is clear that tangent bundle TM is a smooth real vector bundle of rank m over the smooth manifold M where the induced chart $\Phi_m : \pi^{-1}(U_n) \to \mathbb{R}^{2m}$ are smooth local trivialization. If we are given a smooth right inverse $\sigma : M \to E$ of π

$$\pi \circ \sigma(p) = p \text{ for all } p \in M$$

we say σ is a **smooth section of the bundle** $\pi : E \to M$.

8.6 HW 3

Question 200

Let V be a finite dimensional vector space over $\mathbb{R}$. Show that for

$$\dim(V) < 4$$

Every non-zero element of $\bigwedge^2(V)$ can be expressed as a wedge product of two vectors in V. Give an example to show that this is not true if $\dim(V) = 4$.

Theorem 8.6.1. (Case of Zero and One Dimension) If

$$\dim(V) \leq 1$$

Then every non-zero element of $\bigwedge^2(V)$ can be expressed as a wedge product of two vectors in V.

Proof. Recall

$$\dim\left(\bigwedge^2(V)\right) = \binom{\dim(V)}{2} = 0$$

This implies $\bigwedge^2(V) = 0$. There does not exists non-zero element of $\bigwedge^2(V)$, rendering the proposition viciously true. ■

Theorem 8.6.2. (Case of Two Dimension) If

$$\dim(V) = 2$$

Then every non-zero element of $\bigwedge^2(V)$ can be expressed as a wedge product of two vectors in V.

Proof. Let $\{e_1, e_2\}$ be a basis for V. We have

$$\bigwedge^2(V) = \text{span}\{e_1 \wedge e_2\}$$

Therefore, for all $\omega \in \bigwedge^2(V)$, we have

$$\omega = c(e_1 \wedge e_2) = (ce_1) \wedge e_2 \text{ for some } c \in \mathbb{R}$$

Theorem 8.6.3. (Case of Three Dimensions) If

$$\{e_1, e_2, e_3\} \text{ is a basis for } V$$

Then every non-zero element of $\bigwedge^2(V)$ can be expressed as a wedge product of two vectors in V.

Proof. We know $\bigwedge^2(V)$ have the following basis

$$\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$$

Therefore, for arbitrary $\omega \in \bigwedge^2(V)$, we may express

$$\omega = \omega_1(e_1 \wedge e_2) + \omega_2(e_1 \wedge e_3) + \omega_3(e_2 \wedge e_3) \text{ for some } \omega_1, \omega_2, \omega_3 \in \mathbb{R}$$

Write $\mathbf{x} = (\omega_3, -\omega_2, \omega_1) \in \mathbb{R}^3$. By premise, $\mathbf{x} \neq \mathbf{0}$. Using Gram-Schmidt algorithm, we know there exists some $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$ such that

$$|\mathbf{a}| = |\mathbf{b}| = 1 \text{ and } \{\mathbf{x}, \mathbf{a}, \mathbf{b}\} \text{ are orthogonal}$$

The orthogonality of $\{\mathbf{x}, \mathbf{a}, \mathbf{b}\}$ implies

$$\mathbf{x} = c\mathbf{a} \times \mathbf{b} \text{ for some } c \in \mathbb{R}$$

Explicitly,

$$\begin{cases} \omega_1 = \mathbf{x}_3 = c(a_1b_2 - a_2b_1) \\ \omega_2 = -\mathbf{x}_2 = c(a_1b_3 - a_3b_1) \\ \omega_3 = \mathbf{x}_1 = c(a_2b_3 - a_3b_2) \end{cases}$$

We now see

$$\begin{aligned} &[c(a_1e_1 + a_2e_2 + a_3e_3)] \wedge (b_1e_1 + b_2e_2 + b_3e_3) \\ &= c(a_1b_2 - a_2b_1)(e_1 \wedge e_2) + c(a_1b_3 - a_3b_1)(e_1 \wedge e_3) + c(a_2b_3 - a_3b_2)(e_2 \wedge e_3) \\ &= \omega_1(e_1 \wedge e_2) + \omega_2(e_1 \wedge e_3) + \omega_3(e_2 \wedge e_3) = \omega \end{aligned}$$

We have shown

$$\omega = (ca_1e_2 + ca_2e_2 + ca_3e_3) \wedge (b_1e_1 + b_2e_2 + b_3e_3)$$

That is, ω can indeed be expressed as a wedge product of two vectors in V.

∎

Theorem 8.6.4. (Case of Four Dimensions) If

$$\{e_1, e_2, e_3, e_4\} \text{ is a basis for } V$$

Then $e_1 \wedge e_2 + e_3 \wedge e_4$ can not be expressed as a wedge product of two vectors in V.

Proof. Assume for a contradiction that for some $(a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4) \in \mathbb{R}^4$, we have

$$e_1 \wedge e_2 + e_3 \wedge e_4 = (a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4) \wedge (b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4) \tag{8.1}$$

Equating the coefficients of $e_1 \wedge e_2$, we have

$$a_1 b_2 - a_2 b_1 = 1$$

This implies that one of a_1, b_1 is non-zero. WLOG, suppose $a_1 \neq 0$. Now, equating the coefficients of $e_1 \wedge e_3$ and $e_1 \wedge e_4$, we have

$$a_1 b_3 - a_3 b_1 = 0 = a_1 b_4 - a_4 b_1$$

Dividing a_1, we may deduce

$$b_3 = \frac{a_3 b_1}{a_1} \text{ and } b_4 = \frac{a_4 b_1}{a_1}$$

Therefore, the coefficients of $e_3 \wedge e_4$ in the right side expression of Equation 8.1 is

$$a_3 b_4 - a_4 b_3 = \frac{a_3 a_4 b_1}{a_1} - \frac{a_4 a_3 b_1}{a_1} = 0$$

which does not equals to 1, the coefficient of $e_3 \wedge e_4$ in the left side expression of Equation 8.1. This cause a contradiction. ■

Question 201

Let α be the 1-form $dz + xdy$ on $\mathbb{R}^3$.

(a) Find a basis for $\operatorname{Ker} \alpha$.

(b) Compute $\alpha \wedge d\alpha$.

(c) Find the vector field R that satisfies $\alpha(R) = 1$ and $\iota_R d\alpha = 0$.

(d) Let R be the same vector filed in (c), and let $\varphi_t : \mathbb{R}^3 \to \mathbb{R}^3$ denote its flows. Compute $\mathcal{L}_R \alpha$ and $\varphi_t^* \alpha$ for all fixed t.

Theorem 8.6.5. (Part 1) For all $p = (p_1, p_2, p_3) \in \mathbb{R}^3$, the kernel of $\alpha_p : T_p\mathbb{R}^3 \to \mathbb{R}$ has the basis

$$\left\{ \frac{\partial}{\partial x}\Big|_p, \frac{\partial}{\partial y}\Big|_p - p_1 \frac{\partial}{\partial z}\Big|_p \right\}$$

Proof. Let

$$c_1 \frac{\partial}{\partial x}\Big|_p + c_2 \frac{\partial}{\partial y}\Big|_p + c_3 \frac{\partial}{\partial z}\Big|_p \in \operatorname{Ker} \alpha_p$$

Compute

$$0 = \alpha_p\Big(c_1\frac{\partial}{\partial x}\Big|_p + c_2\frac{\partial}{\partial y}\Big|_p + c_3\frac{\partial}{\partial z}\Big|_p\Big) = (dz + xdy)\Big(c_1\frac{\partial}{\partial x}\Big|_p + c_2\frac{\partial}{\partial y}\Big|_p + c_3\frac{\partial}{\partial z}\Big|_p\Big)$$
$$= c_3 + p_1c_2$$

This implies

$$\frac{\partial}{\partial x}\Big|_p, \frac{\partial}{\partial y}\Big|_p - p_1\frac{\partial}{\partial z}\Big|_p \in \operatorname{Ker}\alpha_p$$

Because

$$\alpha_p\frac{\partial}{\partial z}\Big|_p = 1$$

We know $\operatorname{Im}(\alpha_p) = \mathbb{R}$. Therefore,

$$\dim(\operatorname{Ker}\alpha_p) = 3 - \dim(\operatorname{Im}\alpha_p) = 2$$

It is clear that

$$\Big\{\frac{\partial}{\partial x}\Big|_p, \frac{\partial}{\partial y}\Big|_p - p_1\frac{\partial}{\partial z}\Big|_p\Big\} \subseteq \operatorname{Ker}\alpha_p \text{ is linearly independent}$$

It then follows that

$$\Big\{\frac{\partial}{\partial x}\Big|_p, \frac{\partial}{\partial y}\Big|_p - p_1\frac{\partial}{\partial z}\Big|_p\Big\} \text{ is indeed a basis for } \operatorname{Ker}\alpha_p$$

∎

Theorem 8.6.6. (Part 2)

$$\alpha \wedge d\alpha = dx \wedge dy \wedge dz$$

Proof. Compute

$$\begin{aligned} d\alpha &= d(dz + xdy) \\ &= d^2z + dx \wedge dy + xd^2y \\ &= dx \wedge dy \end{aligned}$$

Compute

$$\begin{aligned} \alpha \wedge d\alpha &= (dz + xdy) \wedge (dx \wedge dy) \\ &= dz \wedge dx \wedge dy + xdy \wedge dx \wedge dy \\ &= dx \wedge dy \wedge dz \end{aligned}$$

∎

Theorem 8.6.7. (Part 3)

$$R \triangleq \frac{\partial}{\partial z} \text{ is the unique vector filed that satisfies } \alpha(R) = 1 \text{ and } \iota_R d\alpha = 0$$

Proof. Suppose

$$R \triangleq R^1 \frac{\partial}{\partial x} + R^2 \frac{\partial}{\partial y} + R^3 \frac{\partial}{\partial z} \text{ satisfies } \alpha(R) = 1 \text{ and } \iota_R d\alpha = 0$$

For all $V \in \mathfrak{X}(\mathbb{R}^3)$, if we write

$$V = V^1 \frac{\partial}{\partial x} + V^2 \frac{\partial}{\partial y} + V^3 \frac{\partial}{\partial z}$$

Then

$$\begin{aligned} \begin{vmatrix} R^1 & V^1 \\ R^2 & V^2 \end{vmatrix} = \begin{vmatrix} dxR & dxV \\ dyR & dyV \end{vmatrix} &= (dx \wedge dy)(R, V) \\ &= d\alpha(R, V) = \iota_R d\alpha(V) = 0 \end{aligned} \tag{8.2}$$

If any of R^1 or R^2 is non-zero at some point $p \in \mathbb{R}^3$, by setting $V^1 = -R^2$ and $V_2 = R^1$ at p we have

$$\begin{vmatrix} R^1 & V^1 \\ R^2 & V^2 \end{vmatrix} \text{ is non-zero at } p$$

which contradicts to Equation 8.2. Therefore, we must have $R^1 = R^2 = 0$ on $\mathbb{R}^3$. We may now compute

$$1 = \alpha(R) = (dz + xdy)(R^3 \frac{\partial}{\partial z}) = R^3$$

We may now conclude

$$R = \frac{\partial}{\partial z}$$

■

Theorem 8.6.8. (Part 4) For all fixed t,

$$\varphi_t^* \alpha = \alpha$$

And

$$\mathcal{L}_R \alpha = 0$$

Proof. Fix t. Obviously,

$$\varphi_t(x,y,z) = (x,y,z+t)$$

Let $p = (x_0, y_0, z_0) \in \mathbb{R}^3$ and

$$v = v^1 \frac{\partial}{\partial x}\Big|_p + v^2 \frac{\partial}{\partial y}\Big|_p + v^3 \frac{\partial}{\partial z}\Big|_p \in T_p\mathbb{R}^3$$

Denote $(x_0, y_0, z_0 + t)$ by q. Compute

$$\begin{aligned}
(\varphi_t^*\alpha)_p(v) &= \alpha_q((\varphi_t)_{*,p}v) \\
&= \alpha_q\Big(v^1 \frac{\partial}{\partial x}\Big|_q + v^2 \frac{\partial}{\partial y}\Big|_q + v^3 \frac{\partial}{\partial z}\Big|_q\Big) \\
&= (dz + x_0 dy)\Big(v^1 \frac{\partial}{\partial x}\Big|_q + v^2 \frac{\partial}{\partial y}\Big|_q + v^3 \frac{\partial}{\partial z}\Big|_q\Big) \\
&= v^3 + x_0 v^2 \\
&= (dz + x_0 dy)\Big(v^1 \frac{\partial}{\partial x}\Big|_p + v^2 \frac{\partial}{\partial y}\Big|_p + v^3 \frac{\partial}{\partial z}\Big|_p\Big) \\
&= \alpha_p(v)
\end{aligned}$$

We have shown $(\varphi_t^*\alpha)_p = \alpha_p$. Because p is arbitrary, this implies $\varphi_t^*\alpha = \alpha$. We may now compute

$$\mathcal{L}_R\alpha = \lim_{t\to 0} \frac{(\varphi_t^*\alpha)_p - \alpha_p}{t} = \lim_{t\to 0}\frac{0}{t} = 0$$

∎

Question 202

Orient S^n in $\mathbb{R}^{n+1}$ as the boundary of the unit closed ball.

(a) Show that a volume form on S^n is

$$\omega = \sum_{i=1}^{n+1} (-1)^{i-1} \mathbf{x}^i d\mathbf{x}^1 \wedge \cdots \wedge \widehat{d\mathbf{x}^i} \wedge \cdots \wedge d\mathbf{x}^{n+1}$$

where the caret $\widehat{\ }$ over $d\mathbf{x}^i$ indicates that $d\mathbf{x}^i$ is to be omitted.

(b) Show that on S^2

$$\omega = \begin{cases} \frac{dy\wedge dz}{x} & \text{for } x \neq 0 \\ \frac{dz\wedge dx}{y} & \text{for } y \neq 0 \\ \frac{dx\wedge dy}{z} & \text{for } z \neq 0 \end{cases}$$

(c) Calculate $\int_{S_2} \omega$

Theorem 8.6.9. (Part 1) Show that a volume form on S^n is

$$\omega = \sum_{i=1}^{n+1} (-1)^{i-1} \mathbf{x}^i d\mathbf{x}^1 \wedge \cdots \wedge \widehat{d\mathbf{x}^i} \wedge \cdots \wedge d\mathbf{x}^{n+1}$$

where the caret $\widehat{}$ over $d\mathbf{x}^i$ indicates that $d\mathbf{x}^i$ is to be omitted.

Proof. Let $i : S^n \to \mathbb{R}^{n+1}$ be the inclusion map and define $V \in \mathfrak{X}(\mathbb{R}^{n+1})$ by

$$V_{\mathbf{y}} \triangleq \sum_{i=1}^{n+1} \mathbf{y}^i \frac{\partial}{\partial \mathbf{x}^i}\Big|_{\mathbf{y}}$$

So that V is nowhere tangent to S^n. By Proposition 15.21 of "Introduction to Smooth Manifold" by John Lee, we know

$$i^*(\iota_V d\mathbf{x}^1 \wedge \cdots \wedge d\mathbf{x}^{n+1}) \text{ is a volume form on } S^n$$

Compute

$$\begin{aligned} i^*(\iota_V d\mathbf{x}^1 \wedge \cdots \wedge d\mathbf{x}^{n+1}) &= i^*\Big(\sum_{i=1}^{n+1} (-1)^{i-1} V^i d\mathbf{x}^1 \wedge \cdots \wedge \widehat{d\mathbf{x}^i} \wedge \cdots \wedge d\mathbf{x}^{n+1} \Big) \\ &= \sum_{i=1}^{n+1} (-1)^{i-1} (V^i \circ i) d(\mathbf{x}^1 \circ i) \wedge \cdots \wedge \widehat{d\mathbf{x}^i} \wedge \cdots \wedge d(\mathbf{x}^{n+1} \circ i) \\ &= \sum_{i=1}^{n+1} (-1)^{i-1} V^i d\mathbf{x}^1 \wedge \cdots \wedge \widehat{d\mathbf{x}^i} \wedge \cdots \wedge d\mathbf{x}^{n+1} \\ &= \sum_{i=1}^{n+1} (-1)^{i-1} \mathbf{x}^i d\mathbf{x}^1 \wedge \cdots \wedge \widehat{d\mathbf{x}^i} \wedge \cdots \wedge d\mathbf{x}^{n+1} = \omega \end{aligned}$$

We have shown

$$\omega = i^*(\iota_V d\mathbf{x}^1 \wedge \cdots \wedge d\mathbf{x}^{n+1})$$

This implies ω is indeed a volume form on S^n. ■

Theorem 8.6.10. (Part 2) Show that on S^2,

$$\omega = \begin{cases} \frac{dy\wedge dz}{x} \text{ for } x \neq 0 \\ \frac{dz\wedge dx}{y} \text{ for } y \neq 0 \\ \frac{dx\wedge dy}{z} \text{ for } z \neq 0 \end{cases}$$

Proof. Define $f \in \Omega^0(\mathbb{R}^3)$ by

$$f(x,y,z) \triangleq \sqrt{x^2+y^2+z^2}$$

So we have

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

Let $i : S^2 \to \mathbb{R}^3$ be the inclusion map. Because $f \circ i : S^2 \to \mathbb{R}^3$ is constant 1, we may compute

$$\begin{aligned} 0 &= d(f\circ i) = d(i^* f) = i^*(df) = i^*\Big(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\Big) \\ &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \\ &= \frac{xdx + ydy + zdz}{\sqrt{x^2+y^2+z^2}} = xdx + ydy + zdz \end{aligned}$$

This give us

$$\begin{cases} dx = \frac{ydy+zdz}{-x} \text{ for } x \neq 0 \\ dy = \frac{xdx+zdz}{-y} \text{ for } y \neq 0 \\ dz = \frac{xdx+ydy}{-z} \text{ for } z \neq 0 \end{cases}$$

Therefore, for $x \neq 0$

$$\begin{aligned} \omega &= xdy\wedge dz - ydx \wedge dz + zdx \wedge dy \\ &= xdy \wedge dz - y(\frac{ydy+zdz}{-x}) \wedge dz + z(\frac{ydy+zdz}{-x}) \wedge dy \\ &= (x + \frac{y^2}{x} + \frac{z^2}{x})dy \wedge dz \\ &= \frac{(x^2+y^2+z^2)dy\wedge dz}{x} = \frac{dy\wedge dz}{x} \end{aligned}$$

Similarly, for $y \neq 0$

$$\begin{aligned}
\omega &= xdy \wedge dz - ydx \wedge dz + zdx \wedge dy \\
&= x(\frac{xdx + zdz}{-y}) \wedge dz - ydx \wedge dz + zdx \wedge (\frac{xdx + zdz}{-y}) \\
&= (\frac{x^2}{y} + y + \frac{z^2}{y})dz \wedge dx \\
&= \frac{(x^2 + y^2 + z^2)dz \wedge dx}{y} = \frac{dz \wedge dx}{y}
\end{aligned}$$

Lastly, for $z \neq 0$

$$\begin{aligned}
\omega &= xdy \wedge dz - ydx \wedge dz + zdx \wedge dy \\
&= xdy \wedge (\frac{xdx + ydy}{-z}) - ydx \wedge (\frac{xdx + ydy}{-z}) + zdx \wedge dy \\
&= (\frac{x^2}{z} + \frac{y^2}{z} + z)dx \wedge dy \\
&= \frac{(x^2 + y^2 + z^2)dx \wedge dy}{z} = \frac{dx \wedge dy}{z}
\end{aligned}$$

∎

Theorem 8.6.11. (Part 3) Orient S^n in $\mathbb{R}^{n+1}$ as the boundary of the unit closed ball.

$$\int_{S^2} \omega = 4\pi$$

Proof. Because

$$\omega = i^*(\iota_V d\mathbf{x}^1 \wedge \cdots \wedge d\mathbf{x}^{n+1})$$

And because

$$d\mathbf{x}^1 \wedge \cdots \wedge d\mathbf{x}^{n+1} \text{ is a positively oriented volume form on the unit closed ball}$$

We know ω as a volume form of S^n is also positively oriented. Therefore, when we consider the chart

$$U \triangleq \{(x, y, z) \in S^2 : z > 0\} \text{ and } \varphi(x, y, z) \triangleq (x, y)$$

And the chart

$$V \triangleq \{(x, y, z) \in S^2 : z < 0\} \text{ and } \psi(x, y, z) \triangleq (x, y)$$

According to our computation in part 2, we may integrate

$$\int_U \omega = \int_{\varphi(U)} \frac{1}{\sqrt{1-x^2-y^2}} dxdy$$
$$= \int_0^1 \int_0^{2\pi} \frac{r}{\sqrt{1-r^2}} d\theta dr = 2\pi \int_0^1 \frac{r}{\sqrt{1-r^2}} dr = 2\pi(-\sqrt{1-r^2})\Big|_{r=0}^{1} = 2\pi$$

And integrate

$$\int_V \omega = -\int_{\psi(V)} \frac{1}{-\sqrt{1-x^2-y^2}} dxdy = 2\pi$$

Therefore,

$$\int_{S^2} \omega = \int_U \omega + \int_V \omega = 4\pi$$

■

Question 203

Let M be a manifold of dimension n, and $\{U_i\}_{i\in I}$ be a countable open cover. Suppose that each U_i is diffeomorphic to $\mathbb{R}^n$ and all $U_{ij} \triangleq U_i \cap U_j$ and $U_{ijk} \triangleq U_i \cap U_j \cap U_k$ are either diffeomorphic to $\mathbb{R}^n$ or empty. Choose a total order $<$ on I, and consider the following sequence of real vector space

$$\mathcal{W}_1 = \prod_{i\in I} \mathbb{R} \xrightarrow{\lambda} \mathcal{W}_2 = \prod_{i<j\in I; U_{ij}\neq\varnothing} \mathbb{R} \xrightarrow{\mu} \mathcal{W}_3 = \prod_{i<j<k\in I; U_{ijk}\neq\varnothing} \mathbb{R}$$

where the linear maps are defined by

$$\lambda : (c_i)_{i\in I} \mapsto (c_i - c_j)_{i<j\in I; U_{ij}\neq\varnothing}$$
$$\mu : (c_{ij})_{i<j\in I; U_{ij}\neq\varnothing} \mapsto (c_{ij} + c_{jk} - c_{ik})_{i<j<k\in I; I_{ijk}\neq\varnothing}$$

which satisfies

$$\mu \circ \lambda = 0$$

Theorem 8.6.12. **(a)** Let α be a closed 1-form. Show that for each $i \in I$, we have $\alpha|_{U_i} = df_i$ for some smooth function $f_i : U_i \to \mathbb{R}$. Show that there exists a unique element (c_{ij}) in $\mathcal{W}_2$ with $f_i|_{U_{ij}} - f_j|_{U_{ij}} = c_{ij}$ for all $i < j, U_{ij} \neq \varnothing$. Show that $\mu((c_{ij})) = 0$

Proof. Because

(i) U_i is diffeomorphic to $\mathbb{R}^n$.

(ii) α is closed.

(iii) $H^1(\mathbb{R}^n) = 0$ by Poincare Lemma.

We know there exists some smooth $f_i : U_i \to \mathbb{R}$ such that $\alpha|_{U_i} = df_i$. Fix $i < j, U_{ij} \neq \varnothing$. On U_{ij}, we may compute

$$d(f_i - f_j) = df_i - df_j = \alpha - \alpha = 0$$

Which implies

$$f_i|_{U_{ij}} - f_j|_{U_{ij}} \text{ is some constant } c_{ij} \text{ on } U_{ij}$$

Lastly, to see $\mu((c_{ij})) = 0$. Fix $i < j < k, p \in U_{ijk}$, and compute

$$\begin{aligned} c_{ij} + c_{jk} - c_{ik} &= (f_i - f_j)(p) + (f_j - f_k)(p) - (f_i - f_k)(p) \\ &= (f_i - f_j - f_k + f_k - f_i)(p) = 0 \end{aligned}$$

■

Theorem 8.6.13. (b) Show that in Part (a), the element $(c_{ij}) + \operatorname{Im}\lambda \in \operatorname{Ker}\mu \diagup \operatorname{Im}\lambda$ is independent of the choice of f_i, and depend only on the cohomology class $[\alpha] \in H^1(M)$.

Proof. Let $\widehat{f_i} : U_i \to \mathbb{R}$ also satisfy $\alpha|_{U_i} = d\widehat{f_i}$, and again induce

$$\widehat{c_{ij}} \triangleq \widehat{f_i} - \widehat{f_j}$$

Because

$$d(f_i - \widehat{f_i}) = df_i - d\widehat{f_i} = \alpha - \alpha = 0$$

We know $f_i, \widehat{f_i}$ differ by some constant, which we denote

$$c_i \triangleq f_i - \widehat{f_i}$$

Now, compute

$$\begin{aligned} \lambda(c_i) &= (c_i - c_j)_{i<j\in I; U_{ij}\neq\varnothing} \\ &= (f_i - \widehat{f_i} - f_j + \widehat{f_j})_{i<j\in I; U_{ij}\neq\varnothing} \\ &= (c_{ij} - \widehat{c_{ij}})_{i<j\in I; U_{ij}\neq\varnothing} \end{aligned}$$

We have shown $(\widehat{c_{ij}})_{i<j\in I; U_{ij}\neq\varnothing}, (c_{ij})_{i<j\in I; U_{ij}\neq\varnothing}$ differ by $\lambda(c_i)$. That is, the map

$$\alpha \mapsto (c_{ij}) + \operatorname{Im}\lambda \in \frac{\operatorname{Ker}\mu}{\operatorname{Im}\lambda} \text{ is well-defined}$$

■

Using the machinery developed, we may now define a linear map $\Phi : H^1(M) \to \operatorname{Ker}\mu \big/ \operatorname{Im}\lambda$.

Theorem 8.6.14. **(c)** Show that Φ is injective.

Proof. Fix $[\alpha] \neq 0$. We are required to show the induced $(c_{ij}) \in \mathcal{W}_2$ does not belong to $\operatorname{Im}\lambda$. Assume for a contradiction that $(c_{ij}) = \lambda((c_i))$ for some $(c_i) \in \mathcal{W}_1$ ∎