Deadline : 2024/3/11, 17:00.

1. Let $f_k : [0,1] \to \mathbb{R}$ be given by

$$f_k(x) = \begin{cases} 0, & \text{if } \frac{1}{k} \le x \le 1, \\ -kx+1, & \text{if } 0 \le x < \frac{1}{k}. \end{cases}$$

   (a) Does $\{f_k\}_{k=1}^{\infty}$ converge pointwise on $[0,1]$? If so, find f such that $f_k \to f$ pointwise on $[0,1]$.

   (b) Does f_k converge uniformly on $[0,1]$?

2. Let $f_k : [0,1] \to \mathbb{R}$ be given by $f_k(x) = x^k$.

   (a) Does $\{f_k\}_{k=1}^{\infty}$ converge pointwise on $[0,1]$? If so, find f such that $f_k \to f$ pointwise on $[0,1]$.

   (b) Does f_k converge uniformly on $[0,1]$?

   (c) For any $a \in (0,1)$, Does f_k converge uniformly on $[0,a]$?

3. Let $f_k : \mathbb{R} \to \mathbb{R}$ be given by $f_k(x) = \dfrac{\sin x}{k}$.

   (a) Does $\{f_k\}_{k=1}^{\infty}$ converge pointwise on $\mathbb{R}$? If so, find f such that $f_k \to f$ pointwise on $\mathbb{R}$.

   (b) Does f_k converge uniformly on $\mathbb{R}$?

4. Let f_n be integrable on $[0,1]$ and $f_n \to f$ uniformly on $[0,1]$. Show that if $b_n \nearrow 1$ as $n \to \infty$, then

$$\lim_{n\to\infty} \int_0^{b_n} f_n(x)\,dx = \int_0^1 f(x)\,dx$$

5. If f is continuous on $[0,1]$ and if

$$\int_0^1 f(x)\,x^n\,dx = 0 \ \ (n = 0, 1, 2, ...)$$

   Prove that $f(x) = 0$ on $[0,1]$. Hint: The integral of the product of f with any polynomial is zero. Use the Weierstrass theorem to show that $\int_0^1 f^2(x)\,dx = 0$

6. Show that if $\{f_n\}$ is a sequence of continuous functions on E such that converges uniformly to f, then f is continuous on E.

7. Prove that if f_n is bounded on E, $\forall n \in \mathbf{N}$ and f_n converges uniformly to a bounded function f on E, then $\{f_n\}$ is uniformly bounded on E.

8. Let $f_k : [0,1] \to \mathbb{R}$ be a sequence of functions such that

   (1) $|f_k(x)| \leq M_1$ for all $k \in \mathbb{N}$ and $x \in [0,1]$,

   (2) $|f_k'(x)| \leq M_2$ for all $k \in \mathbb{N}$ and $x \in [0,1]$.

   for some positive M_1, M_2.

   (a) Prove that there exists a subsequence of $\{f_k\}_{k=1}^{\infty}$ which converges uniformly on $[0,1]$.

   (b) If the assumption (1) is omitted, can $\{f_k\}_{k=1}^{\infty}$ still have a convergent subsequence? If yes, prove it; If not, give an counterexample.

   (c) Show that the assumption (1) can be replaced by $f_k(0) = 0$ for all $k \in \mathbb{N}$.