§2. "Some" linear algebras

Let V be a finite dimensional real vector space.

Def: An almost complex structure on V is an endomorphism $J: V \to V$ s.t. $J^2 = -Id_V$.

Prop: (V, J) can be regarded as a complex vector space.

Pf: We define the $\mathbb{C}$-action on V by

$$(a + \sqrt{-1}\, b) \cdot v := a \cdot v + b J(v)$$

Then it is straightforward to check

$$z_1 (z_2 \cdot v) = (z_1 \cdot z_2) \cdot v \qquad \square$$

Cor: If V carries an almost complex structure, then $\dim_{\mathbb{R}}(V)$ is even.

Define $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^{\dim_{\mathbb{R}} V}$ and extend J $\mathbb{C}$-linearly:

$$J_{\mathbb{C}}(v \otimes z) := J(v) \otimes z$$

Prop: $J_{\mathbb{C}}$ has $\pm\sqrt{-1}$ as the only eigenvalues

Pf: $$\begin{aligned} J_{\mathbb{C}}(v) = \lambda v &\Leftrightarrow J_{\mathbb{C}}^2(v) = \lambda J_{\mathbb{C}}(v) \\ &\Leftrightarrow -v = \lambda J_{\mathbb{C}}(v) \\ &\Leftrightarrow -v = \lambda^2 v \\ &\Leftrightarrow \lambda^2 = -1 \\ &\Leftrightarrow \lambda = \pm\sqrt{-1} \end{aligned} \qquad \square$$

Define

$$V^{1,0} := \{ v \in V_{\mathbb{C}} : J_{\mathbb{C}}(v) = \sqrt{-1}\, v \}$$

$$V^{0,1} := \{ v \in V_{\mathbb{C}} : J_{\mathbb{C}}(v) = -\sqrt{-1}\, v \}$$

<u>Prop</u>: The map

$$v \longmapsto \frac{1}{2}\left(v - \sqrt{-1} J_{\mathbb{C}}(v),\, v + \sqrt{-1} J_{\mathbb{C}}(v) \right)$$

gives an isomorphism $V_{\mathbb{C}} \cong V^{10} \oplus V^{01}$ and complex conjugation gives $V^{1,0} \cong V^{0,1}$ as $\mathbb{R}$-vector spaces

<u>Pf</u>: The first assertion is clear. For the second assertion,

$$\overline{\left(v - \sqrt{-1} J_{\mathbb{C}}(v) \right)} = \bar{v} + \sqrt{-1}\, \overline{J_{\mathbb{C}}(v)} \underset{\substack{\uparrow \\ \text{since } J \text{ is real}}}{=} \bar{v} + \sqrt{-1} J_{\mathbb{C}}(\bar{v})$$

□

<u>Prop</u>: $(V, J) \cong (V^{1,0}, \sqrt{-1})$ as almost complex vector spaces.

<u>Pf</u>: We need to check that the map

$$\pi^{1,0} : v \mapsto \frac{1}{2}\left(v - \sqrt{-1} J(v) \right)$$

gives the desired isomorphism.

Clearly, this map is injective.

We have

$$\begin{aligned} \sqrt{-1}\, \pi^{1,0}(v) &= \frac{1}{2} \sqrt{-1} \left(v - \sqrt{-1} J(v) \right) \\ &= \frac{1}{2}\left(\sqrt{-1}\, v + J(v) \right) \\ &= \frac{1}{2}\left(J(v) - \sqrt{-1} J(J(v)) \right) = \pi^{1,0}(J(v)) \end{aligned}$$

□

Define

$$\Lambda^{p,q} V := \Lambda^p V^{1,0} \otimes_{\mathbb{C}} \Lambda^q V^{0,1}$$

Prop: We have

① $\Lambda^k V_{\mathbb{C}} \cong \bigoplus_{p+q=k} \Lambda^{p,q} V$

② $\overline{\Lambda^{p,q} V} = \Lambda^{q,p} V$

Pf: Choose a $\mathbb{C}$-basis $e_1, \cdots, e_n$ for $V^{1,0}$. Then $\bar{e}_1, \cdots, \bar{e}_n$ is a $\mathbb{C}$-basis for $V^{0,1}$ and

$\{e_{i_1} \wedge \cdots \wedge e_{i_p}\}_{i_1 < \cdots < i_p}$ is a $\mathbb{C}$-basis for $\Lambda^{p,0} V$

$\{\bar{e}_{j_1} \wedge \cdots \wedge \bar{e}_{j_q}\}_{j_1 < \cdots < j_q}$ is a $\mathbb{C}$-basis for $\Lambda^{0,q} V$

Then

$\{e_{i_1} \wedge \cdots \wedge e_{i_p} \otimes \bar{e}_{j_1} \wedge \cdots \wedge \bar{e}_{j_q}\}_{\substack{i_1 < \cdots < i_p \\ j_1 < \cdots < j_q}}$ is a $\mathbb{C}$-basis for $\Lambda^{p,q} V$

We then map $\Lambda^{p,q} V$ to $\Lambda^k V_{\mathbb{C}}$ by

$$e_{i_1} \wedge \cdots \wedge e_{i_p} \otimes \bar{e}_{j_1} \wedge \cdots \wedge \bar{e}_{j_q} \longmapsto e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge \bar{e}_{j_1} \wedge \cdots \wedge \bar{e}_{j_q}$$

which is clearly injective. Since

$$\dim_{\mathbb{C}} \bigoplus_{p+q=k} \Lambda^{p,q} V = \sum_{p+q=k} \dim_{\mathbb{C}} \Lambda^{p,q} V = \sum_{p+q=k} \binom{n}{p}\binom{n}{q} = \binom{2n}{k}$$

This gives ①.

② is obvious. $\square$

This gives the projections

$$\pi^{p,q} : \Lambda^{\bullet} V_{\mathbb{C}} \longrightarrow \Lambda^{p,q} V$$

Hermitian structure

Let (V, g) be a finite dimensional real vector space. equipped with an inner product g

Def: An almost complex structure J on V is said to be compatible with g if

$$g(J(v_1), J(v_2)) = g(v_1, v_2)$$

$\forall\ v_1, v_2 \in V$.

Given a compatible J, we define the fundamental form

$$\omega(v_1, v_2) := g(J(v_1), v_2)$$

Prop: $\omega \in \Lambda^{1,1} V^* \cap \Lambda^2 V^*$

Pf: First of all, ω is real since g, J are. Also

$$\begin{aligned}
\omega(v_1, v_2) &= g(J(v_1), v_2) \\
&= g(v_2, J(v_1)) \\
&= g(-J(J(v_2)), J(v_1)) \\
&= -g(J(v_2), v_1) \\
&= -\omega(v_2, v_1)
\end{aligned}$$

$\Rightarrow\ \omega \in \Lambda^2 V^*$.

If $v_1, v_2 \in V^{1,0}$, then

$$\omega(v_1, v_2) = g(\sqrt{-1}\, v_1, v_2) = \sqrt{-1}\, g(v_1, v_2)$$

But g is symmetry, $\omega(v_1, v_2) = 0$. Similarly, if

$v_1, v_2 \in V^{0,1}$, we have $\omega(v_1, v_2) = 0$

Hence $\omega \in \Lambda^{1,1} V^*$ $\square$

Given (V, g, J), we define

$$h := g - \sqrt{-1}\,\omega$$

<u>Prop</u>: h is a Hermitian structure on (V, J)

<u>Pf</u>: We have $h(v,v) = g(v,v) > 0$ as long as $v \neq 0$.

Also,

$$\begin{aligned} \overline{h(v_1, v_2)} &= g(v_1, v_2) + \sqrt{-1}\,\omega(v_1, v_2) \\ &= g(v_2, v_1) - \sqrt{-1}\,\omega(v_2, v_1) \\ &= h(v_2, v_1) \end{aligned}$$

and

$$\begin{aligned} h(J(v_1), v_2) &= g(J(v_1), v_2) - \sqrt{-1}\,\omega(J(v_1), v_2) \\ &= \omega(v_1, v_2) + \sqrt{-1}\, g(v_1, v_2) \\ &= \sqrt{-1}\, h(v_1, v_2) \end{aligned}$$

$\square$

On the other hand, we can extend g by

$$g_{\mathbb{C}}(z_1 \cdot v_1, z_2 \cdot v_2) = z_1 \bar{z}_2\, g(v_1, v_2)$$

This gives Hermitian structure on $V_{\mathbb{C}}$.

<u>Prop</u>: The decomposition $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ respect $g_{\mathbb{C}}$

<u>Pf</u>: For any $v_1, v_2 \in V$,

$$v_1 - \sqrt{-1}\, J(v_1) \in V^{1,0}$$
$$v_2 + \sqrt{-1}\, J(v_2) \in V^{0,1}$$

$$\rightsquigarrow \; g_{\mathbb{C}}\left(v_1 - \sqrt{-1}\, J(v_1),\, v_2 + \sqrt{-1}\, J(v_2)\right)$$
$$= g(v_1, v_2) - g\left(\sqrt{-1}\, J(v_1), \sqrt{-1}\, J(v_2)\right)$$
$$- \sqrt{-1}\, g(v_1, J(v_2)) - \sqrt{-1}\, g(J(v_1), v_2)$$
$$= g(v_1, v_2) - g(v_1, v_2) - \sqrt{-1}\left(-\omega(v_1, v_2) + \omega(v_1, v_2)\right) = 0 \qquad \square$$

Prop: The map
$$v \mapsto \frac{1}{2}\left(v - \sqrt{-1}\, J(v)\right)$$
gives an isometry $\left(V, \frac{1}{2}h\right) \cong \left(V^{1,0}, g_{\mathbb{C}}|_{V^{1,0}}\right)$

Pf:
$$g_{\mathbb{C}}\left(\tfrac{1}{2}(v_1 - \sqrt{-1}\, J(v_1)), \tfrac{1}{2}(v_2 - \sqrt{-1}\, J(v_2))\right)$$
$$= \frac{1}{4}\left(g(v_1, v_2) + g(J(v_1), J(v_1)) + \sqrt{-1}\left(g(v_1, J(v_2)) - g(J(v_1), v_2)\right)\right)$$
$$= \frac{1}{4}\left(2g(v_1, v_2) + \sqrt{-1}\, 2\omega(v_1, v_2)\right)$$
$$= \frac{1}{2} h(v_1, v_2) \qquad \square$$

The inner product g induces an inner product on $\Lambda^k V$ as follows: Choose an orthonormal basis $e_1, \cdots, e_n$ for V. Then we declare $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}_{i_1 < \cdots < i_k}$ to be an orthonormal basis for $\Lambda^k V$. This gives g on $\Lambda^k V$.

Let $vol = e_1 \wedge \cdots \wedge e_n \in \Lambda^n V$ be a volume form on V.
We define the Hodge star operator $* : \Lambda^k V \longrightarrow \Lambda^{n-k} V$ by

$$\alpha \wedge *\beta = g(\alpha,\beta)\,\mathrm{vol}$$

<u>Prop</u>: For any inner product space (V,g) of dimension n, we have

① If $\{i_1,\cdots,i_k,j_1,\cdots,j_{n-k}\} = \{1,\cdots,n\}$,

$$*(e_{i_1}\wedge\cdots\wedge e_{i_k}) = \mathrm{sgn}(i_1,\cdots,i_k,j_1,\cdots,j_{n-k})\, e_{j_1}\wedge\cdots\wedge e_{j_{n-k}}$$

② $\langle \alpha, *\beta\rangle = (-1)^{k(n-k)}\langle *\alpha,\beta\rangle \quad \forall \alpha\in\Lambda^k V$

③ $(*|_{\Lambda^k V})^2 = (-1)^{k(n-k)}$

④ $*$ is an isometry on $(\Lambda^\bullet V, g)$

<u>Pf</u>: Ex. □

Given $(V,g,J) \rightsquigarrow \omega$ is a 2-form. We use the volume form

$$\mathrm{vol} = n!\,\omega^n \in \Lambda^{2n} V^*$$

<u>Def</u>: Given (V,g,J). The <u>Lefschetz operator</u> $L:\Lambda^\bullet V^*\to\Lambda^\bullet V^*$ is defined by

$$L(\alpha) := \omega\wedge\alpha \quad \left(\text{so } \Lambda^{p,q}V^* \xrightarrow{L} \Lambda^{p+1,q+1}V^*\right)$$

We define the <u>dual Lefschetz operator</u> $\Lambda:\Lambda^\bullet V^*\to\Lambda^\bullet V^*$ by

$$g(\Lambda\alpha,\beta) = g(\alpha, L\beta)$$

Prop: We have $\Lambda = *^{-1} L *$. In particular, $\Lambda : \Lambda^k V^* \to \Lambda^{k-2} V^*$.

Pf: $g(\Lambda\alpha,\beta)\,\mathrm{vol} = g(\alpha, L\beta)\,\mathrm{vol} = g(L\beta,\alpha)\,\mathrm{vol} = L\beta \wedge *\alpha = \beta \wedge L*\alpha$

$$= \beta \wedge * *^{-1} L * \alpha = g(\beta, *^{-1} L * \alpha)\,\mathrm{vol}$$

$$= g(*^{-1} L * \alpha, \beta)\,\mathrm{vol}$$

Hence $\Lambda : \Lambda^k V^* \xrightarrow{*} \Lambda^{2n-k} V^* \xrightarrow{L} \Lambda^{2n-k+2} V^* \xrightarrow{*^{-1}} \Lambda^{k-2} V^*$ □

We extend g on $\Lambda^\bullet V^*$ to $g_{\mathbb{C}}$ on $\Lambda^\bullet V_{\mathbb{C}}^*$ by

$$g_{\mathbb{C}}(z_1\alpha_1, z_2\alpha_2) = z_1 \bar{z}_2\, g(\alpha_1, \alpha_2)$$

Extend $*$ $\mathbb{C}$-linearly to $* : \Lambda^k V_{\mathbb{C}}^* \longrightarrow \Lambda^{2n-k} V_{\mathbb{C}}^*$. The

$$\alpha \wedge * \overline{\beta} = g_{\mathbb{C}}(\alpha, \beta)\,\mathrm{vol}$$

Prop: ① The decomposition $\Lambda^k V^* = \bigoplus_{p+q=k} \Lambda^{p,q} V^*$ respects $g_{\mathbb{C}}$

② $* : \Lambda^{p,q} V^* \longrightarrow \Lambda^{n-q,n-p} V^*$

③ $\Lambda : \Lambda^{p,q} V^* \longrightarrow \Lambda^{p-1,q-1} V^*$

Pf: Ex. □

We define the counting operator $H : \Lambda^\bullet V^* \longrightarrow \Lambda^\bullet V^*$ by

$$H\big|_{\Lambda^k V^*} := (k-n)\,\mathrm{id}_{\Lambda^k V^*}$$

<u>Thm</u>: We have

① $[H, L] = 2L$

② $[H, \Lambda] = -2\Lambda$

③ $[L, \Lambda] = H$

<u>Pf</u>: $(HL - LH)\alpha = (k+2-n)\omega\wedge\alpha - (k-n)\omega\wedge\alpha = 2\omega\wedge\alpha = 2L\alpha$

$(H\Lambda - \Lambda H)\alpha = (k-2-n)\Lambda\alpha - (k-n)\Lambda\alpha = -2\Lambda\alpha$

This proves ①, ②. For ③, we use induction on $2n = \dim_{\mathbb{R}} V$.

When $n=1$, $V \cong \mathbb{R}^2$ and we have

$$\Lambda^0 V^* = \mathbb{R}$$
$$\Lambda^1 V^* = V^*$$
$$\Lambda^2 V^* = \mathbb{R}\cdot\omega$$

If $\alpha \in \Lambda^0 V^*$, α is a scalar.

$[L, \Lambda]\alpha = -\Lambda L\alpha = -\alpha(*^{-1}L*)\omega = -\alpha(*^{-1}\omega) = -\alpha$

If $\alpha \in \Lambda^1 V^*$, $[L,\Lambda]\alpha = 0$ by degree reason.

Finally, $[L,\Lambda]\omega = L\Lambda\omega = L(*^{-1}\omega) = \omega$

This proves ③ for $n=1$. For a general (V, g, J), we decompose it as

$$(V, g, J) \cong (V_1, g_1, J_1) \oplus (V_2, g_2, J_2)$$

Then $\omega = \omega_1 \oplus \omega_2$ and hence $L = L_1 \oplus L_2$ and $\Lambda = \Lambda_1 \oplus \Lambda_2$. Recall that

$$\Lambda^\bullet V^* \cong \Lambda^\bullet V_1^* \otimes \Lambda^\bullet V_2^*$$

$$\begin{aligned}
\Rightarrow [L,\Lambda](\alpha_1\otimes\alpha_2) &= (L_1\oplus L_2)(\Lambda_1\alpha_1\otimes\alpha_2 + \alpha_1\otimes\Lambda_2\alpha_2) \\
&\quad - (\Lambda_1\oplus\Lambda_2)(L_1\alpha_1\otimes\alpha_2 + \alpha_1\otimes L_2\alpha_2) \\
&= L_1\Lambda_1\alpha_1\otimes\alpha_2 + \cancel{\Lambda_1\alpha_1\otimes L_2\alpha_2} + \cancel{L_1\alpha_1\otimes\Lambda_2\alpha_2} \\
&\quad + \alpha_1\otimes L_2\Lambda_2\alpha_2 - \Lambda_1L_1\alpha_1\otimes\alpha_2 - \cancel{L_1\alpha_1\otimes\Lambda_2\alpha_2} \\
&\quad - \cancel{\Lambda_1\alpha_1\otimes L_2\alpha_2} - \alpha_1\otimes\Lambda_2L_2\alpha_2 \\
&= [L_1,\Lambda_1]\alpha_1\otimes\alpha_2 + \alpha_1\otimes[L_2,\Lambda_2]\alpha_2
\end{aligned}$$

Suppose $\alpha_i \in \Lambda^{k_i}V_i^*$. Then by induction hypothesis,

$$\begin{aligned}
[L,\Lambda](\alpha_1\otimes\alpha_2) &= (k_1-n_1)\alpha_1\otimes\alpha_2 + (k_2-n_2)\alpha_1\otimes\alpha_2 \\
&= (k-n)\,\alpha_1\otimes\alpha_2
\end{aligned}$$

$\square$

<u>Cor</u>: The action of L, Λ, H defines an $\mathfrak{sl}_2$-representation on $\Lambda^\bullet V^*$.

<u>Cor</u>: $[L^i,\Lambda]\alpha = i(k-n+i-1)L^{i-1}\alpha$, $\forall\, \alpha\in\Lambda^k V^*$

<u>Pf</u>: Ex.

<u>Def</u>: An element $\alpha\in\Lambda^k V^*$ is called <u>primitive</u> if $\Lambda\alpha = 0$.
Let $P^k \subset \Lambda^k V^*$ be the space of primitive k-forms

<u>Thm</u>: Let (V, g, J) as before. We have

① $\Lambda^k V^* = \bigoplus_{i\geq 0} L^i(P^{k-2i})$ (Lefschetz decomposition)

and the decomposition respect g.

② $P^k = 0$ for $k > n$

③ $L^{n-k} : P^k \longrightarrow \Lambda^{2n-k} V^*$ is injective for $k \le n$

④ $L^{n-k} : \Lambda^k V^* \longrightarrow \Lambda^{2n-k} V^*$ is bijective for $k \le n$

⑤ $P^k = \{\alpha \in \Lambda^k V^* : L^{n-k+1} \alpha = 0\}$ for $k \le n$

Pf : ① : We know that $\Lambda^\bullet V_{\mathbb{C}}^*$ is a finite dimensional sl_2-representation and so

$$\Lambda^\bullet V^* = \bigoplus_\alpha W_\alpha \quad \leftarrow \text{irreducible } sl_2\text{-representations}$$

For any $v \in W_i$, $\Lambda^i v = 0$ for some $i >> 0$ by degree reason

Hence each W_α has a primitive subspace $\mathbb{R} \cdot v_\alpha$ ($\Lambda v_\alpha = 0$)

Consider the subspace $\mathbb{R}\langle v_\alpha, Lv_\alpha, L^2 v_\alpha, \cdots\rangle \subset W_\alpha$. By the previous corollary, it is a subrepresentation of W_α. Since W_α is irreducible, we have

$$W_\alpha = \mathbb{R}\langle v_\alpha, Lv_\alpha, L^2 v_\alpha, \cdots\rangle$$

As a whole,

$$\Lambda^\bullet V^* = \bigoplus_\alpha \mathbb{R}\langle v_\alpha, Lv_\alpha, L^2 v_\alpha, \cdots\rangle$$

Taking the degree k part, we obtain the decomposition.

For orthogonality, we have

$$\begin{aligned} g(L^i v_1, L^j v_2) &= g(L^{i-1} v_1, \Lambda L^j v_2) \qquad (i > j) \\ &= \text{const} \cdot g(L^{i-1} v_1, L^{j-1} v_2) \\ &\;\vdots \\ &= \text{const} \cdot g(L^{i-j} v_1, v_2) \\ &= \text{const} \cdot g(L^{i-j-1} v_1, \Lambda v_2) = 0 \end{aligned}$$

②: Given $\alpha \in P^k$ with $k>n$, let $i \geq 0$ be the minimal integer s.t. $L^i \alpha = 0$. If $i>0$, then

$$0 = [L^i, \Lambda]\alpha = i(k-n+i-1)L^{i-1}\alpha$$

$$k>n \text{ and } i>0 \Rightarrow L^{i-1}\alpha = 0,$$

violating minimality of i, so $i=0$, meaning $\alpha = 0$.

③: If $\alpha \neq 0 \in P^k$, $k \leq n$, let $i>0$ be the minimal integer s.t. $L^i\alpha = 0$. Then

$$0 = [L^i, \Lambda]\alpha = i(k-n+i-1)L^{i-1}\alpha$$

By minimality of i, $L^{i-1}\alpha \neq 0$ and so $k-n+i-1=0$

Hence $i = n-k+1$ and so $L^{n-k}\alpha \neq 0$, proving injectivity

④: Follows from ①, ②, ③.

⑤: In ③, we have proved that $P^k \subset \ker(L^{n-k+1})$.

Conversely, if $L^{n-k+1}\alpha = 0$, then

$$L^{n-k+2}\Lambda\alpha = [L^{n-k+2}, \Lambda]\alpha = (n-k+2)L^{n-k+1}\alpha = 0$$

But by ④, L^{n-k+2} is injective on $\Lambda^{k-2}V^*$, we can only have $\Lambda\alpha = 0$, i.e. $\alpha \in P^k$ □

Using the decomposition

$$\Lambda^k V_{\mathbb{C}}^* \cong \bigoplus_{p+q=k} \Lambda^{p,q} V^*$$

we can define $P^{p,q} := P^{p+q} \cap \Lambda^{p,q} V^*$. so that

$$P^k = \bigoplus_{p+q=k} P^{p,q}$$

Since L and Λ are real, the relation $\overline{\Lambda^{p,q} V^*} = \Lambda^{q,p} V^*$ implies

$$\overline{P^{p,q}} = P^{q,p}$$

<u>Def</u>: Let (V, g, J) as before. The Hodge-Riemann pairing $Q : \Lambda^k V \times \Lambda^k V \longrightarrow \mathbb{R}$ is defined by

$$Q(\alpha, \beta) := (-1)^{\frac{k(k-1)}{2}} \alpha \wedge \beta \wedge \omega^{n-k}$$

<u>Thm</u>: Let (V, g, J) as before. Then

① $Q(\Lambda^{p,q} V^*, \Lambda^{p',q'} V^*) = 0$ if $(p,q) \neq (q', p')$

② $\sqrt{-1}^{\,p-q} Q(\alpha, \bar{\alpha}) = (n-(p+q))! \, g_{\mathbb{C}}(\alpha, \alpha) > 0$

$\forall \alpha \in P^{p,q} \setminus \{0\}$ and $p+q \leq n$.

<u>Pf</u>: ①: Since ω is of $(1,1)$-type,

$$Q(\alpha, \beta) \neq 0 \Rightarrow p + p' = k = q + q'$$

Together with $p+q = k = p'+q'$, we can only have $(p,q) = (q', p')$

②: We compute

$$\begin{aligned} Q(\alpha, \bar{\alpha}) \operatorname{vol} &= (-1)^{\frac{k(k-1)}{2}} \alpha \wedge \bar{\alpha} \wedge \omega^{n-k} \\ &= (-1)^{\frac{k(k-1)}{2}} \alpha \wedge L^{n-k} \bar{\alpha} \\ &= (-1)^{\frac{k(k-1)}{2}} g_{\mathbb{C}}(\alpha, \beta) \operatorname{vol} \end{aligned}$$

where $k = p+q$ and $* \bar{\beta} = L^{n-k} \bar{\alpha}$. We have

$$*^2 \bar{\beta} = (-1)^k \bar{\beta}$$

We want compute $* L^{n-k}$.

We define $\mathbb{I} : \Lambda^\bullet V_\mathbb{C} \longrightarrow \Lambda^\bullet V_\mathbb{C}$ by

$$\mathbb{I}\big|_{\Lambda^{p,q}V} := \sqrt{-1}^{\,p-q}.$$

In other words,

$$\mathbb{I} = \sum_{p,q} \sqrt{-1}^{\,p-q} \pi^{p,q}$$

where $\pi^{p,q} : \Lambda^\bullet V \longrightarrow \Lambda^{p,q} V$ is the canonical projection.

Prop: For $\alpha \in P^k$, we have

$$* L^j \alpha = (-1)^{\frac{k(k+1)}{2}} \frac{j!}{(n-k-j)!} L^{n-k-j} \mathbb{I} \alpha$$

Pf: Too long, omitted.

Using this, we have

$$*^2 \overline{\beta} = * L^{n-k} \overline{\alpha} = (-1)^{\frac{k(k+1)}{2}} (n-k)! \sqrt{-1}^{\,q-p} \overline{\alpha}$$

$$\Rightarrow \quad \beta = (-1)^{k+\frac{k(k+1)}{2}} (n-k)! \sqrt{-1}^{\,p-q} \alpha$$

As a whole

$$Q(\alpha, \overline{\alpha}) = (-1)^{k+\frac{k(k+1)}{2}+\frac{k(k-1)}{2}} (n-k)! \sqrt{-1}^{\,q-p} g_\mathbb{C}(\alpha, \overline{\alpha})$$

$$\Rightarrow \sqrt{-1}^{\,p-q} Q(\alpha, \overline{\alpha}) = (n-k)! \, g_\mathbb{C}(\alpha, \overline{\alpha}) > 0$$