7-(10)

Since $f'(x) = \sin\left(\frac{1}{x}\right) - \frac{\cos\left(\frac{1}{x}\right)}{x}$ is not bound on $(0,1)$,

f is not lip. conti,

(19) Just need to remember this,

↓ 最後一題

proof is in theorem 4.19, An Introduction to Analysis by William R wade,

7-(11)

True,

(power

$\Rightarrow$ 假設的強弱 (ability of assumption)

Lip $\geq$ absolutely conti $\geq$ u.c $\geq$ c.

18)

Without loss of generality (W.L.O.G), $L = 0$,

Goal: Given $\epsilon > 0$, $\exists\ \delta > 0$ s.t. if $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$,

$\because f \to L$ as $x \to \infty$, $\exists\ N \in \mathbb{N}$ s.t. $|f(x)| < \epsilon$ as $x \geq N$,

Note that f is u.c on $[0, N]$, $\exists\ \delta_1 > 0$ s.t. $(x, y) \in [0, N] \Rightarrow |f(x) - f(y)| < \epsilon$

Then $\forall\ x, y \in [0, \infty)$, take $\delta = \delta_1$, we have following discussion,

$x, y \in [0, N]$, $|f(x) - f(y)| < \epsilon$ by our choice,

$x, y \in [0, \infty) \setminus [0, N]$, $|f(x) - f(y)| \leq |f(x)| + |f(y)| < 2\epsilon$

$x \in [0, N]$, $y \in [0, \infty) \setminus [0, N]$, $|f(x) - f(y)| \leq |f(x) - f(N)| + |f(N)| + |f(y)|$

$\leq 3\epsilon$.

So f is u.c by choose $\delta = \delta_1$ #

(9)

Let $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0, \end{cases}$ f is conti on $[0,1]$,

so f is u.c.

Goal: f is not a.c.

(反證) Suppose f is a.c. Let $\epsilon = 1$, $\exists \delta > 0$ s.t. $\Downarrow \forall \sum_k |y_k - x_k| < \delta$, then

$\sum_k |f(y_k) - f(x_k)| < 1$.

Goal: find $\{(x_k, y_k)\}_{k=1}^{n}$ s.t. $\sum_k |y_k - x_k| < \delta$ but

$\sum_k |f(y_k) - f(x_k)| > 1$,

Let $\alpha_n = 2n\pi$, $\beta_n = \left(2n\pi + \frac{\pi}{2}\right)$ for $n \geq 1$

$$\sum_{n=1}^{\infty} \left(\frac{1}{\alpha_n} - \frac{1}{\beta_n}\right) = \sum_{n=1}^{\infty} \left(\frac{1}{2n\pi} - \frac{1}{2n\pi + \frac{\pi}{2}}\right)$$

$$= \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{2n\pi\left(2\pi n + \frac{\pi}{2}\right)} < \infty \text{ by lim test, with } a_n = \frac{1}{n^2}$$

$\because \sum_{n=1}^{\infty} \left|\frac{1}{\alpha_n} - \frac{1}{\beta_n}\right| < \infty$ $\therefore \exists N \in \mathbb{N}$ s.t. $\sum_{n=N+1}^{\infty} \left|\frac{1}{\alpha_n} - \frac{1}{\beta_n}\right| < \delta$,

Here $f\left(\frac{1}{\alpha_n}\right) = \frac{1}{2\pi n} \sin(2\pi n) = 0$, $f\left(\frac{1}{\beta_n}\right) = \frac{1}{2\pi n + \frac{\pi}{2}} \sin\left(2n\pi + \frac{\pi}{2}\right)$

$$= \frac{1}{2n\pi + \pi/2}$$

$\Rightarrow \sum_{n=1}^{\infty} |f(\alpha_n) - f(\beta_n)| = \sum_{n=1}^{\infty} \frac{1}{2n\pi + \pi/2} = \infty$ by lim test with $a_n = \frac{1}{n}$

Thus, $\sum_{n=N+1}^{\infty} \frac{1}{2\pi n + \pi/2} = \infty$, We choose $\tilde{N}$ s.t. $\sum_{n=N+1}^{\tilde{N}} \frac{1}{(\sim)} > 1$ #

7

11) False, Let $f: \mathbb{R} \to \mathbb{R}$ by $x \to x^2$ ($f(x) = x^2$)

Goal: Show that f is not u.c.

Suppose f is u.c. Given $\epsilon = 1$, $\exists\ \delta > 0$ s.t. if $|x-y| < \delta$,

$|f(x) - f(y)| < 1$, That is,

$$1 > |x^2 - y^2| = |x-y||x+y| > \delta |x+y|$$

Take $|x+y| > \frac{1}{\delta}$, we obtain a contradiction,

12) True, By def, that is easy to see.

13) Theorem 4.19 of W.Rudin,

(14)

Same example of (1),

15) True L.c > u.c > c By def, that is easy to see,

16) True L.c > u.c By def, that is easy to see.

17) Let $f(x) = \sqrt{|x|}$, $f: [0,1] \to \mathbb{R}$,

Obvious f is u.c by Prob. (7) - 13), $|f(x) - f(y)| < C|x-y|$

Note if f is L.c and f is differentiable, $|f'(x)| < C$.

$f'(x) = \frac{1}{2\sqrt{x}}$ is not bound on $(0,1)$, so f is not Lip.c.

18) True, (L.c) ; (a.c) > u.c > c, By def, easy

We show that $\bar{A} = A$, By elementary set theory, we have

Goal: $\forall x \in \bar{A}$, $x \in A$,

$$f(A) = f(f^{-1}(B)) \subseteq B.$$

Therefore, if $x \in \bar{A}$,

$$f(x) \in f(\bar{A}) \subseteq \overline{f(A)} \subset \bar{B} = B$$

so

$$x \in f^{-1}(B) = A. \quad \text{Thus } \bar{A} \subseteq A \Rightarrow \bar{A} = A$$

6.

1. 略

2. $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. if $|x-y| < \delta$ $(x, y \in \mathbb{R})$,

then $|f(x) - f(y)| < \epsilon$.

3.

$\exists C > 0$ s.t. $|f(x) - f(y)| < C|x-y|$ $\forall x, y \in \mathbb{R}$.

4.

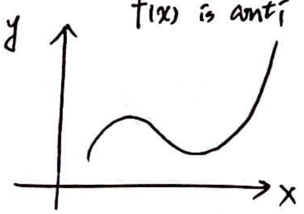
$\forall \epsilon > 0$, $\exists \delta > 0$ s.t. whenever a finite sequence of pairwise disjoint sub-intervals (x_k, y_k) of I with $x_k < y_k \in I$ satisfies

$\Sigma_k |y_k - x_k| < \delta$

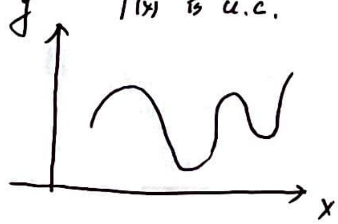
then

$\Sigma_k |f(y_k) - f(x_k)| < \epsilon$

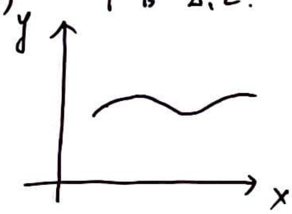
(1) $f(x)$ is conti



(2) $f(x)$ is u.c.



(3) f is L.c.



(2)

From (1), we known that f is continuous $\Leftrightarrow$ $\forall V \overset{open}{\subseteq} Y$, $f^{-1}(V)$ is open

We show that 1. $\Leftrightarrow$ 3. 1. (prop.) $\Rightarrow$ 2. $\Rightarrow$ 3.

$\forall B \overset{closed}{\subseteq} Y$, $Y \setminus B$ is open. $\because f$ is conti $\therefore f^{-1}(Y \setminus B)$ is open

then $f^{-1}(B) = f^{-1}(Y \setminus \{Y \setminus B\}) = X \setminus f^{-1}(Y \setminus B)$ is closed in X

or

$f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$ is open $\Rightarrow f^{-1}(B)$ is closed in X.

(There is the direction of assuming f is conti)

if $\forall B \overset{closed}{\subseteq} Y$, $f^{-1}(B)$ is closed, $\forall V \overset{open}{\subseteq} Y$, $Y \setminus V$ is closed

$\Rightarrow f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is closed $\Rightarrow f^{-1}(V)$ is open

(Goal: $\forall$ open set V, $f^{-1}(V)$ is open)

1 $\Rightarrow$ 2.

It is clear that $f(A) \subseteq \overline{f(A)}$, so we just need to discuss

$x \in \partial A$, ($\bar{A} = A \sqcup \partial A$)

Let $x \in \partial A$; U is an open set contain $f(x)$ (nighborhord), $\because f$ is conti,

$f^{-1}(V)$ is open set contain x, Since $x \in \partial A$, $\exists\, y \neq x$ s.t. $y \in A$,

$y \in f^{-1}(V)$, then $f(y) \in V$, so $f(x) \in \overline{f(A)}$, as desired.

2. $\Rightarrow$ 3.

Let B be closed in Y and let $A = f^{-1}(B)$

Goal: $A = f^{-1}(B)$ is closed,