Note for Euclidean Norm

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Definition

Definition 1. *An **Euclidean Domain** D is an integral domain equipped with an **Euclidean norm** $\gamma : D \setminus \{0\} \to \mathbb{Z}_0^+$ that satisfy*

For all nonzero $b \in D$, and for all $a \in D$, we can perform division algorithm to have

$$a = bq + r \tag{1}$$

where $r = 0$ or $\gamma(r) < \gamma(b)$

For all non-zero $a, b \in D$, we have

$$\gamma(a) \leq \gamma(ab) \tag{2}$$

Also, we say a have Euclidean norm $\gamma(a)$.

Theorem

Theorem 1. *Every Euclidean Domain is an PID.*

Proof. Notice that the trivial ideal $\{0\}$ is principal, so we only have to show that every non-trivial ideal in an Euclidean domain is a principal ideal. Arbitrarily pick an non-trivial ideal N from an Euclidean Domain D, and pick an element b from D of the smallest Euclidean norm. Arbitrarily pick an element a from N, and perform the division algorithm to have $a = bq + r$. Notice $r = a - bq \in N$, so because b is the element of the smallest Euclidean norm in N, we know $\gamma(r) < \gamma(b)$ is impossible, which let us deduce $r = 0$. This enable us to deduce $a = bq$. ∎

Corollary 1.1. *Every Euclidean Domain is an UFD.*

For this amazing corollary, we can use the notion of association class in the development of theorems of Euclidean Domain.

Theorem 2. *Every element in the same association class have the same Euclidean norm.*

Proof. Suppose $b = au$. Observe $\gamma(a) \leq \gamma(au) = \gamma(b) \leq \gamma(bu^{-1}) = \gamma(a)$ ∎

Corollary 2.1.

$$\gamma(a) < \gamma(ab) \iff b \text{ is not a unit} \tag{3}$$

Proof. From left to right, observe that if b is a unit, then $\gamma(a) = \gamma(ab)$ CaC. From right to left, assume $\gamma(a) = \gamma(ab)$, we see $\forall ax \in \langle a \rangle, ax = q(ab) + r$ where $r = a(x - qb) \implies \gamma(a) \leq \gamma(r)$ and $\gamma(r) \leq \gamma(ab) = \gamma(a)$, CaC unless $r = 0$. So $ab | ax$, then $\langle a \rangle \subseteq \langle ab \rangle$ and apparently $\langle ab \rangle \subseteq \langle a \rangle$. So $\langle ab \rangle = \langle a \rangle$, and this imply b is a unit CaC. ∎

Theorem 3. *In an Euclidean Domain, the elements of smallest Euclidean norm are precisely the units.*

Proof. For all non-zero $a \in D$, observe $\gamma(1) \leq \gamma(1a) = \gamma(a)$, and for all unit $u \in D$, observe $\gamma(u) \leq \gamma(uu^{-1}) = \gamma(1)$. ■