7.11 PDE HW 11

Question 182

Prove the Schwarz Inequality

$$|(f,g)| \leq \|f\|_2 \cdot \|g\|_2$$

for any pair of functions.

Proof. Schwarz inequality for integral is a corollary of Holder's inequality

$$|(f,g)| = \left|\int f\overline{g}dx\right| \leq \int |fg|\, dx = \|fg\|_1 \leq \|f\|_2 \cdot \|g\|_2$$

The following is a proof of Holder's inequality. Note that here we use $p = q = 2$. ■

Theorem 7.11.1. (Holder's Inequality) Let f, g be two functions measurable on E. We have

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Proof. The proof is trivial if $p = 1$ or ∞, so we may suppose

$$1 < p < \infty$$

Now, if $\|f\|_p = 0$, then $f = 0$ almost everywhere, which renders the inequality trivial since $\|fg\|_1 = 0$. If $\|f\|_p = \infty$, the proof is again trivial. We may now suppose

$$\|f\|_p, \|g\|_q \in (0,\infty)$$

Define f_1, g_1 by

$$f_1 \triangleq \frac{f}{\|f\|_p} \text{ and } g_1 \triangleq \frac{g}{\|g\|_q}$$

Because $1 < p < \infty$, by Young's Inequality for product, we have

$$\begin{aligned}\|f_1g_1\|_1 = \int_E |f_1g_1| &\leq \int_E \left(\frac{|f_1|^p}{p} + \frac{|g_1|^q}{q}\right) \\ &= \frac{\|f_1\|_p^p}{p} + \frac{\|g_1\|_q^q}{q} = \frac{1}{p} + \frac{1}{q} = 1\end{aligned}$$

The proof then follows the assumption $\|f\|_p, \|g\|_q \in (0,\infty)$ and

$$\|f_1g_1\|_1 = \frac{\|fg\|_1}{\|f\|_p \|g\|_q}$$

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Question 183

Solve the Poisson Equation

$$\begin{cases} u_{xx} + u_{yy} = 1 \text{ in } r < a \\ u = 0 \text{ on } r = a \end{cases}$$

Proof. Write the Poisson equation in polar coordinate

$$u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = 1$$

Because $u(a, \theta)$ for all θ, we may suppose u is independent of θ. Therefore, the Poisson equation in polar coordinate simplifies to

$$u_{rr} + \frac{u_r}{r} = 1$$

The solution space of this ODE is exactly

$$\left\{ \frac{r^2}{4} + C_1 \ln r + C_2 : C_1, C_2 \in \mathbb{R} \right\}$$

Let

$$u = \frac{r^2}{4} + C_1 \ln r + C_2$$

Because u is finite on $r = 0$, we must have $C_1 = 0$. It then follows from $u = 0$ for $r = a$ that

$$u = \frac{r^2}{4} - \frac{a^2}{4}$$

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