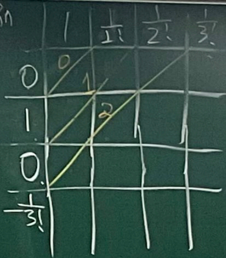
⟨Ex⟩ Find the 1st 3 terms in the Maclaurin series for (a) $e^x \sin x$ (b) $\tan x$

Sol: (A) by definition

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

(B) $e^x = 1 + x + \frac{1}{2!}x^2 + \cdots$

$\times)$ $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$



$0 = 0 + 0$

$1 = 1 + 0 = 0 + 1$

$2 = 2 + 0 = 1 + 1 = 0 + 2$

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots}{1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots}$$

perform the division of polynomials (or series)

3

(1)

(2)

(3)

If

§11.11 Applications of Taylor Polynomials

Approximating Fn by polynomials

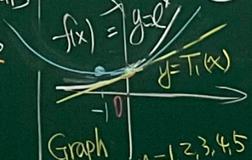
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n =$$

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

$T_n(x) \equiv$ Approximation $\quad R_n(x) =$ error

$y = T_1(x) = f(a) + f'(a)(x-a)$ 1st degree Taylor polynomial

⟨Ex⟩ $f(x) = e^x = \underbrace{\boxed{1+x}}_{T_1(x)} + \frac{x^2}{2!} + \cdots$



$T_2(x) = 1 + x + \frac{x^2}{2!}$

$T_2(0) = 1$

Graph $T_n(x)$ for $n = 1, 2, 3, 4, 5$

$T_2(x) = \frac{1}{2}(x^2 + 2x + 2) = \frac{1}{2}(x+1)^2 + \frac{1}{2}$

$T_n(x) \to f(x) \iff R_n(x) \to 0$

(1

K=

=

~

3 methods for estimating error

(1) Use Graphing device

(2) Use Alternating series estimation

i.e. $|S - S_n| < b_{n+1}$

(3) Use Taylor's Ineq. (Thm 11.10.9)

If $|f^{(n+1)}(x)| \le M \Rightarrow |R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$

⟨EX⟩ mass at rest $= m_0$, C = light-speed

mass with velocity $m = \frac{m_0}{\sqrt{1-\frac{V^2}{C^2}}}$ ~ 1 $\frac{V^2}{C^2} \propto$ small

Kinetic energy = total energy − energy at rest

$K = mC^2 - m_0C^2$

$= \left(\frac{1}{\sqrt{1-V^2/C^2}} - 1\right)C^2 m_0 = m_0C^2\left[\left(1-\frac{V^2}{C^2}\right)^{-\frac{1}{2}} - 1\right]$

Use binomial series

$(1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots$

$K = m_0C^2\left[\left(\cancel{1} + \frac{1}{2}\left(\frac{V^2}{C^2}\right) + \frac{3}{8}\left(\frac{V^2}{C^2}\right)^2 + \frac{5}{16}\left(\frac{V^2}{C^2}\right)^3 + \cdots\right) - \cancel{1}\right]$

$= m_0\cancel{C^2}\left(\frac{1}{2}\frac{V^2}{\cancel{C^2}} + \underbrace{\frac{3}{8}\frac{V^4}{C^{\cancel{4}2}} + \frac{5}{16}\frac{V^6}{C^{\cancel{6}4}} + \cdots}_{\text{small}}\right)$

$\approx \frac{1}{2}m_0V^2 + \text{small}\left(\frac{1}{C^2}\right)$

$\therefore K \approx \frac{1}{2}m_0V^2$ (Newton)

Estimate $K - \frac{1}{2}m_0V^2$ when

$|v| \le 100\ m/s = 360\ km/h$

$f(x) \equiv m_0C^2\left[(1+x)^{-\frac{1}{2}} - 1\right]$, $x = \frac{-V^2}{C^2}$

Compute $|f''(x)| = \left|\frac{3}{4}m_0C^2(1+x)^{-\frac{5}{2}}\right|$

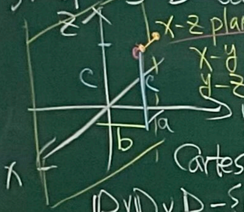
$= \frac{3}{4}m_0C^2(1+x)^{-\frac{5}{2}} \le M$ for $|x| \le \frac{10^4}{C^2}$

Taylor ineq. $\Rightarrow |R_1(x)| \le \frac{M}{2}x^2 \le (4.17\times10^{-10})m_0$

March 08 Midterm (I): 13:10 ~ 15:00

Ch 12 Vectors and the Geometry of Space

§12.1 3-D Coordinates systems



Cartesian product

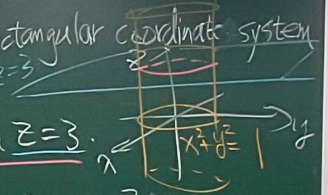
$\mathbb{R}\times\mathbb{R}\times\mathbb{R} = \{(x,y,z) \mid x,y,z\in\mathbb{R}\}$

$P\in\mathbb{R}^3 \longleftrightarrow (x,y,z)\in\mathbb{R}^3$

3-dim rectangular coordinate system

⟨Ex⟩ Graph

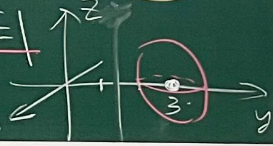
(a) $x^2+y^2=1$, $z=3$



(b) $x^2+(y-3)^2+z^2=1$

center (0,3,0)

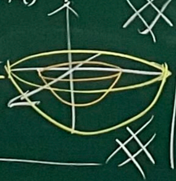
radius = 1



Eqn of sphere $(x-a)^2+(y-b)^2+(z-c)^2=r^2$

center=(a,b,c), radius=r

⟨Ex⟩ $\begin{cases} 1\le x^2+y^2+z^2\le 4 \\ z\le 0 \end{cases}$



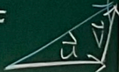
§12.2 Vectors

vector $\overset{\text{or}}{=}$ displacement $\overset{\text{or}}{=}$ velocity $\overset{\text{or}}{=}$ force:

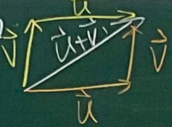
magnitude & direction

$\vec{v}=\overrightarrow{AB}$

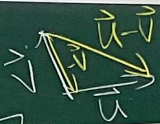
Def: vector addition $\vec{u}+\vec{v}=$



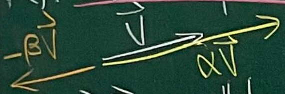
Parallelogram Law



$\vec{u}+\vec{v}=\vec{v}+\vec{u}$



Def. Scaler multiplication. $(-1)\vec{v}=-\vec{v}$



Def $\vec{u},\vec{v}$ = parallel if $\exists c \in \mathbb{R}$ s.t. $\vec{u}=c\vec{v}$

Def. Difference $\vec{u}-\vec{v}=\vec{u}+(-\vec{v})=\vec{u}+(-1)\vec{v}$

Def $\vec{a}=\vec{OP}=\langle a_1\ a_2\ a_3\rangle$ position vector

$A=(x_1, y_1, z_1)$, $B=(x_1+a_1, y_1+a_2, z_1+a_3) = \langle x_2\ y_2\ z_2\rangle$

$\Rightarrow \vec{a}=\vec{AB}=\langle a_1\ a_2\ a_3\rangle$

$\Leftrightarrow \vec{a}=\vec{AB}=\langle x_2-x_1, y_2-y_1, z_2-z_1\rangle$

$\vec{AB}$ = representation of $\vec{a}$