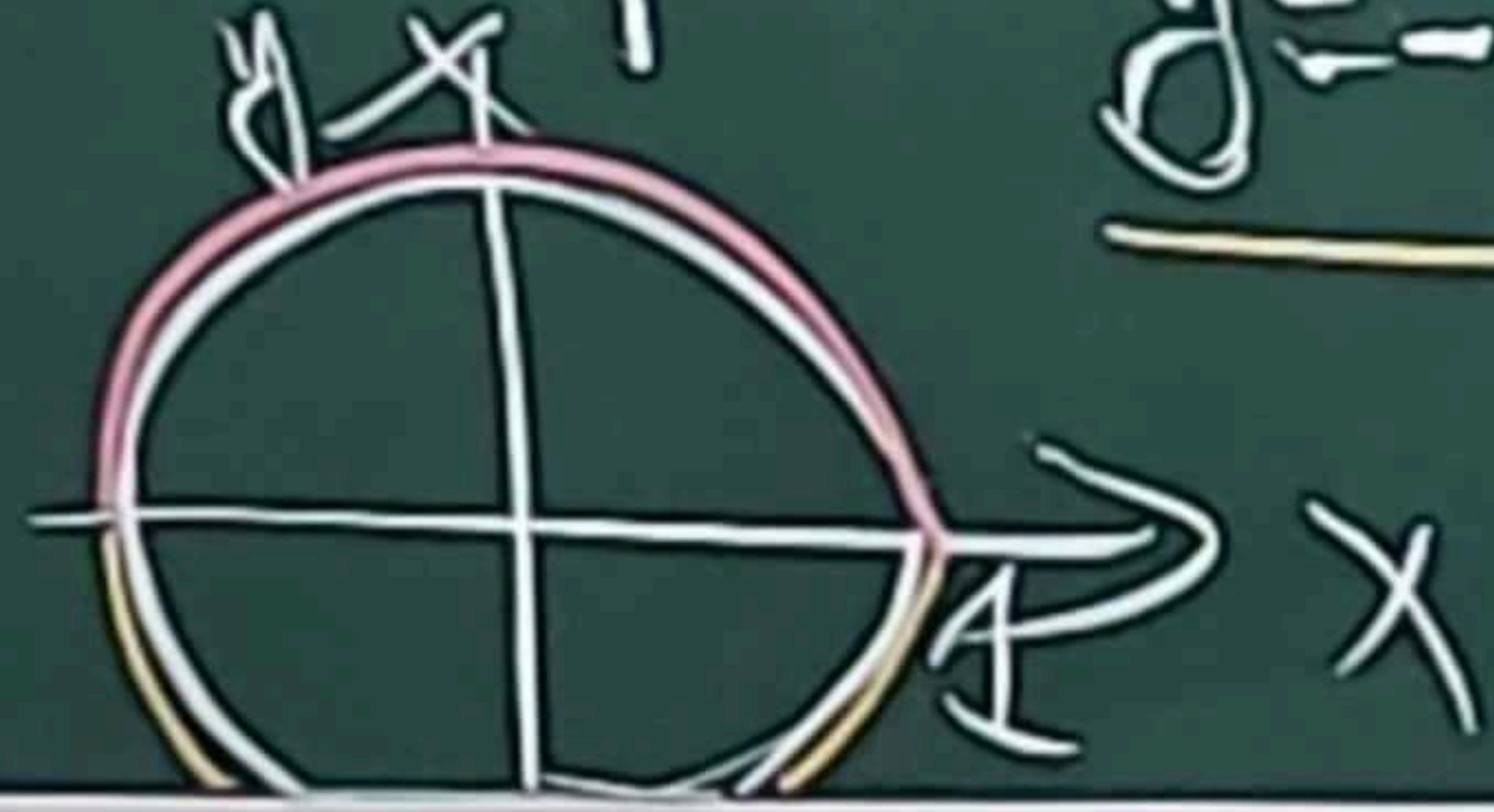
Implicit Differentiation

$y = f(x) \Rightarrow \frac{dy}{dx} = f'(x)$

Suppose $F(x, y) = 0$

(Ex) $x^2 + y^2 = 1 \Rightarrow \begin{array}{l} y = +\sqrt{1-x^2} = f(x) \\ y = -\sqrt{1-x^2} = g(x) \end{array}$



Assume F_x, F_y conti, $\frac{\partial F}{\partial y} \neq 0$

$\xRightarrow{\text{Implicit Fn Thm}}$ y is implicitly defined as a diff. fn of x; i.e. $\exists f$ s.t. $y = f(x)$

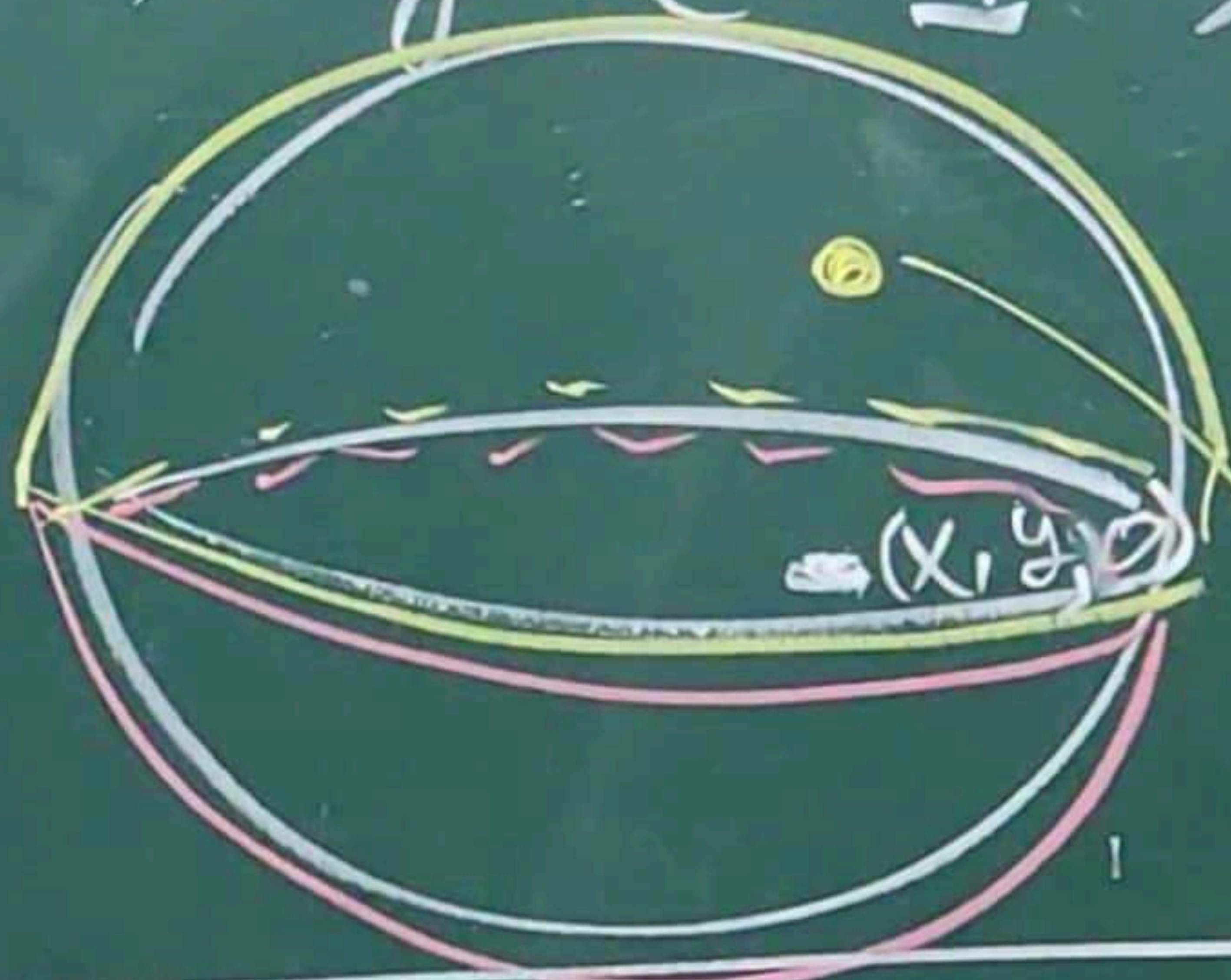
$\Rightarrow \frac{d}{dx} F(x, f(x)) = 0$

$\Rightarrow F_x \cdot \underbrace{\frac{dx}{dx}}_{1} + F_y \frac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} \qquad \frac{\partial F}{\partial y} \neq 0$$

(Ex) $x^2+y^2+z^2=1 \Rightarrow z=\sqrt{1-x^2-y^2}$

$z=-\sqrt{1-x^2-y^2}$



Suppose $F(x,y,z)=0$, $\frac{\partial F}{\partial z} \neq 0$, F_x, F_y, F_z conti.

$\xrightarrow[\text{Fn Thm}]{\text{Implicit}}$ z is implicit defined as a fn $z=f(x,y)$

s.t. $F(x,y,f(x,y))=0$

$$\frac{\partial}{\partial x} F(x,y,f(x,y)) = 0$$

$$\Rightarrow F_x \underbrace{\frac{dx}{dx}}_{=1} + F_y \underbrace{\frac{\partial y}{\partial x}}_{=0} + F_z \frac{\partial f(x,y)}{\partial x} = 0$$

$\Rightarrow F_x + F_z \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$, $F_z \neq 0$

(F_x = F sub x)

(F^x = F sup x)

($\frac{1}{2}$ = one over two = a half)

Analogously, $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$, $F_z \neq 0$

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<Ex> Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3+y^3+z^3+6xyz=1$

$\Updownarrow$

Let $F(x,y,z) = x^3+y^3+z^3+6xyz-1=0$

Sol: Ⓐ compute $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$, $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$

Ⓑ $z = f(x,y) \Rightarrow$ compute $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$

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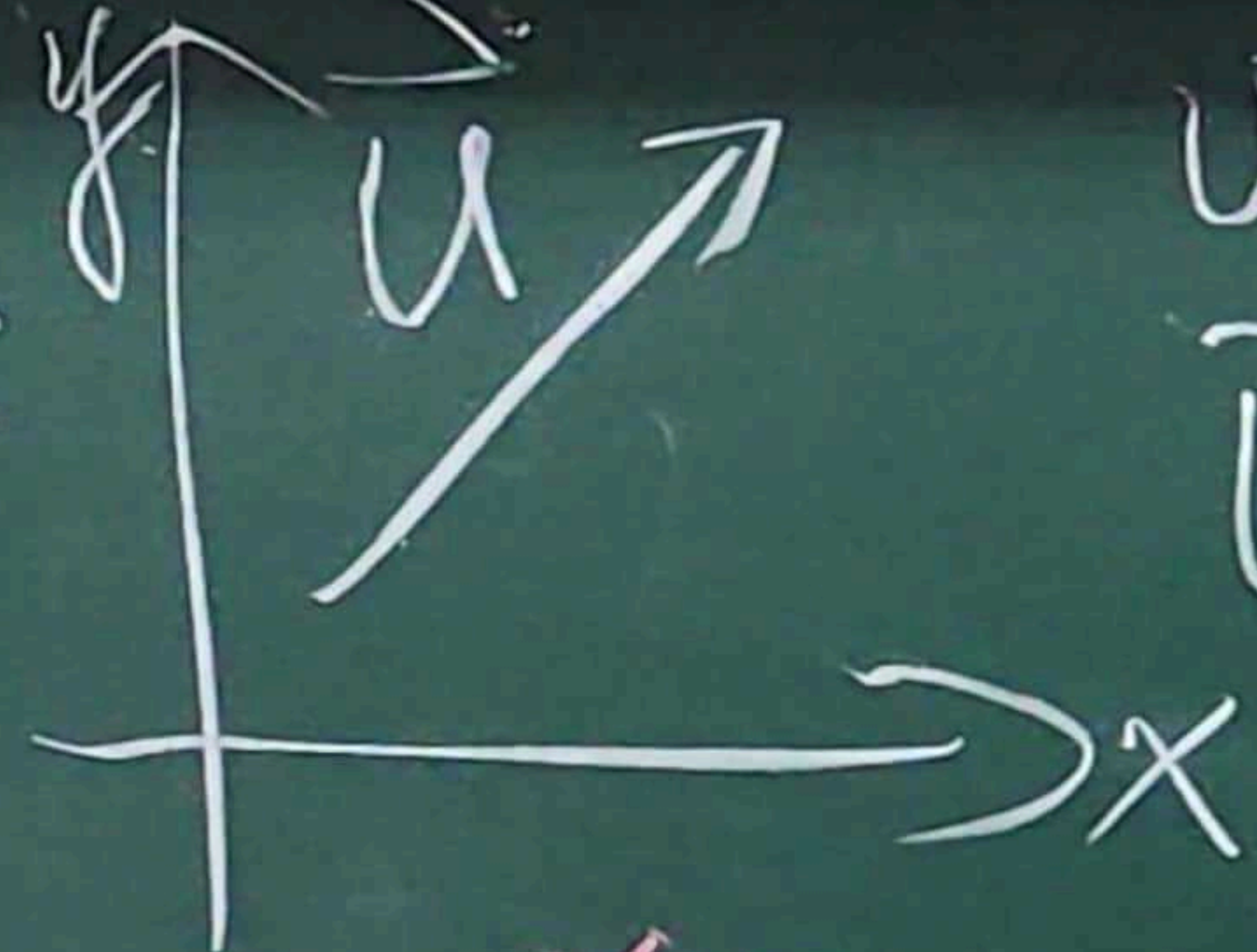
§14.6 Directional derivatives & Gradient Vector

Recall $z=f(x,y)$

$$\boxed{f_x(x_0,y_0)=\lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} = \text{rate of change of } z \text{ in } x\text{-direction}}$$

$f_y(x_0,y_0)= \ldots$

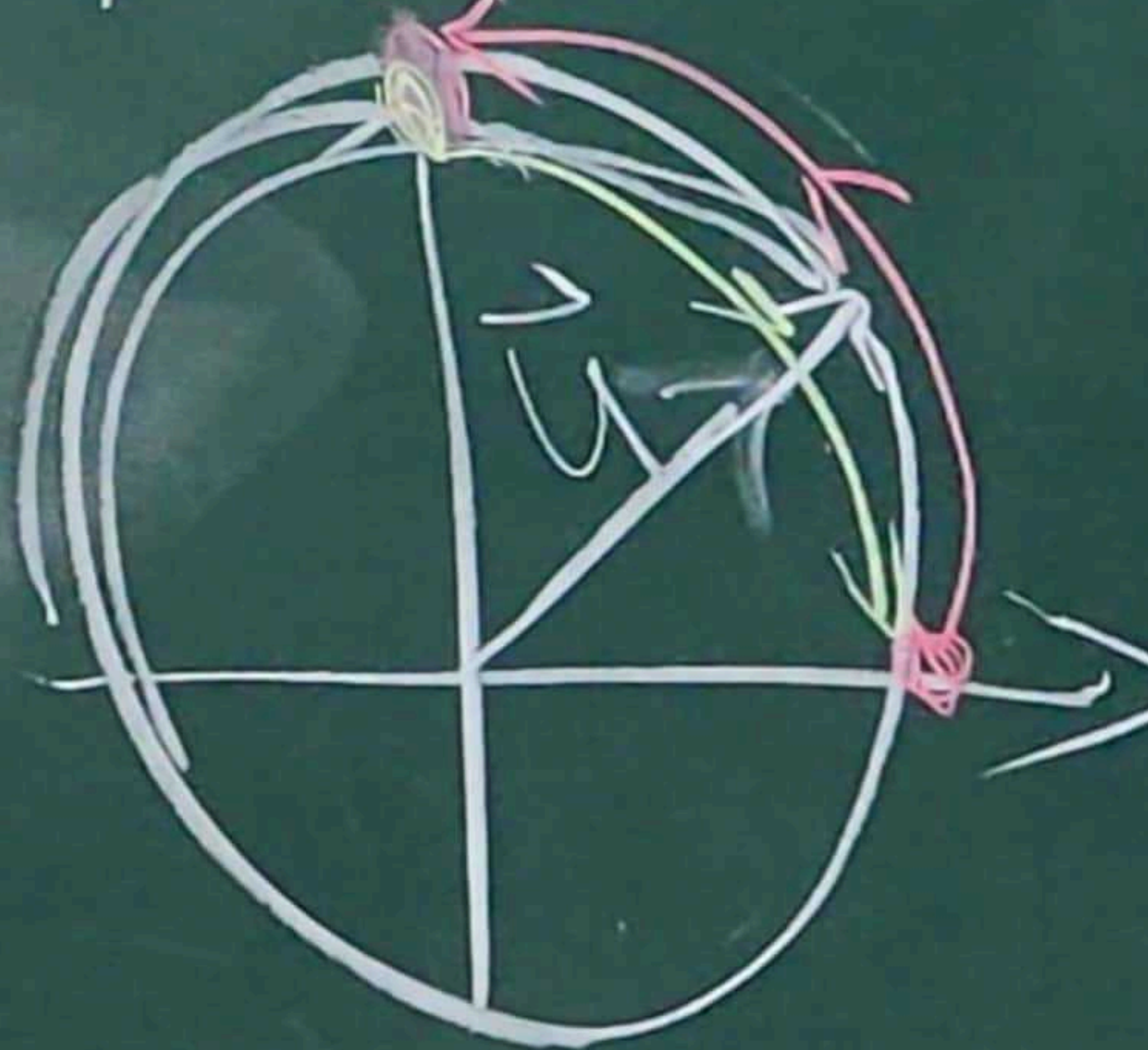
Def:



unit vector

$\vec{u}=\langle a,b\rangle=\langle\cos\theta,\sin\theta\rangle$

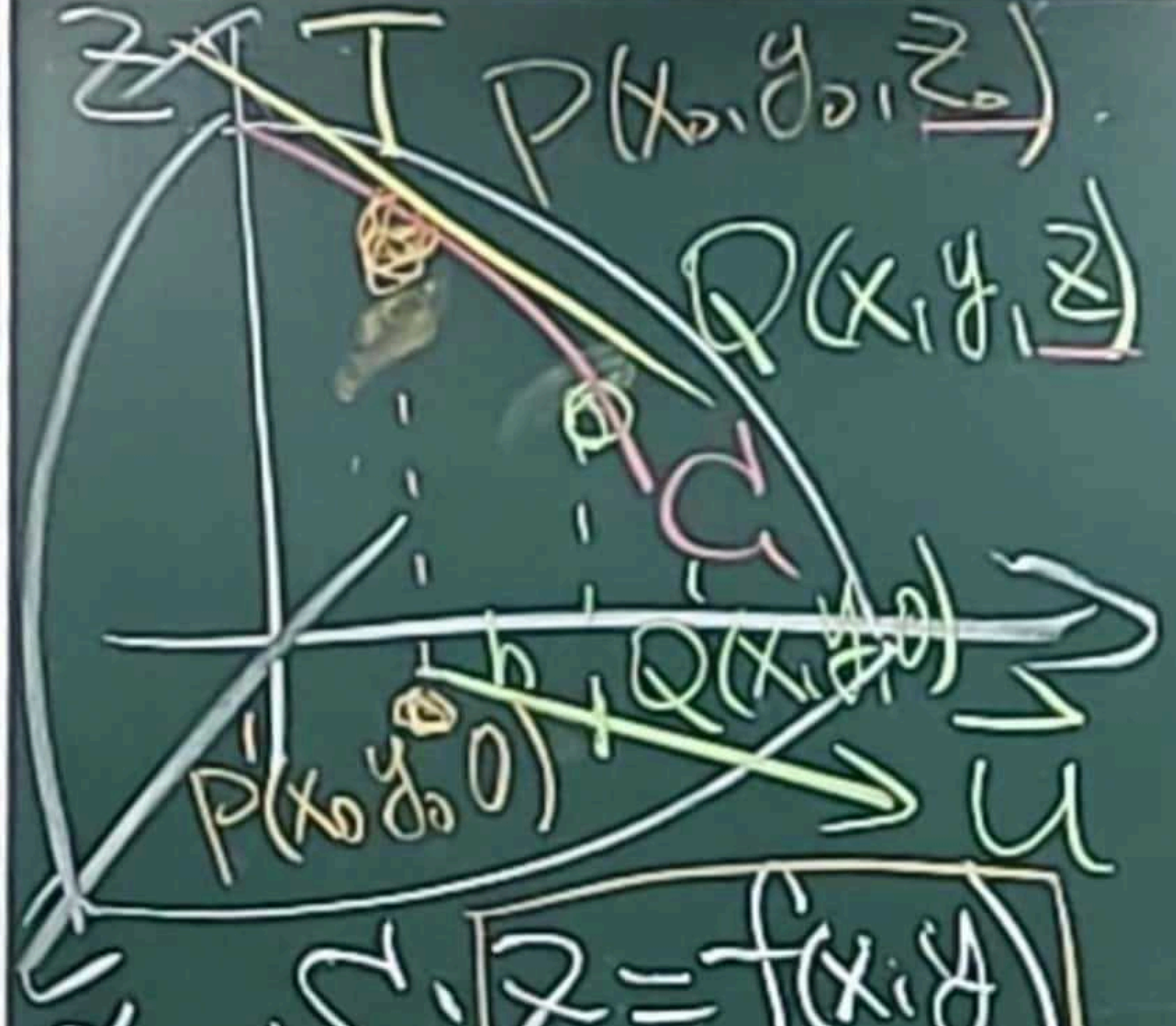
$a^2+b^2=1$



$\theta: 0 \to \frac{\pi}{2}$

$\langle \sin\theta, \cos\theta\rangle$

$\theta: 0 \to \frac{\pi}{2}$



$S: \boxed{z = f(x, y)}$

Find the rate of change of z in the direction of $\vec{u}$

$z_0 = f(x_0, y_0) \Rightarrow P(x_0, y_0, z_0) \in S$

The slope of the tangent line T to C at P

$=$ rate of change of z in the direction of $\vec{u}$

$\overrightarrow{P'Q'} = h\vec{u} = (ha, hb)$

$= \langle x - x_0, y - y_0 \rangle$

$\Rightarrow x = x_0 + ha, \; y = y_0 + hb$

and

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

Def **Directional derivative** of f at (x_0, y_0) in the direction of $\vec{u} = \langle a, b \rangle$, $|\vec{u}| = 1$

is $D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \dfrac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$

if the limit exists.

Note $D_{\vec{i}} f = f_x$ for $\vec{u} = \vec{i} = \langle 1, 0 \rangle$, $D_{\vec{j}} f = f_y$

Thm If f = diff. $\Rightarrow$ then the directional derivative in the direction $\vec{u} = \langle a, b \rangle$, $|\vec{u}| = 1$

$D_{\vec{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$

Recall If f_x, f_y = exists and conti $\Rightarrow$ f = diff.

pf. $D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \dfrac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$

$$\lim_{h\to 0} \frac{f(x_0+ha,\, y_0+hb)-f(x_0,\, y_0+hb)}{ha}\, a$$

$$+\lim_{h\to 0} \frac{f(x_0,\, y_0+hb)-f(x_0,\, y_0)}{hb}\, b$$

$$= f_x(x_0, y_0)\cdot a + f_y(x_0, y_0)\, b$$

<Ex> Find $D_{\vec{u}} f(x,y)$ if $f(x,y) = x^3 - 3xy + 4y^2$

and $\vec{u}$ = unit vector given by angle $\theta = \frac{\pi}{6}$

i.e, $\vec{u} = \langle \cos\frac{\pi}{6}, \sin\frac{\pi}{6} \rangle = \langle a, b \rangle$

Sol: $D_{\vec{u}} f(x,y) \equiv f_x(x,y)\, a + f_y(x,y)\, b =$

Notice that $D_{\vec{u}} f(x,y) = f_x(x,y)\, a + f_y(x,y)\, b$

$= \langle f_x(x,y), f_y(x,y) \rangle \cdot \langle a, b \rangle$

$= \langle f_x, f_y \rangle \cdot \vec{u}$ gradient of $f \equiv \nabla f$

Def gradient of f

$$\nabla f(x,y) \equiv \langle f_x(x,y), f_y(x,y) \rangle$$

$$= f_x(x,y)\vec{i} + f_y(x,y)\vec{j} = f_x\vec{i} + f_y\vec{j}$$

$$\Rightarrow \boxed{D_{\vec{u}}f(x,y) = \nabla f(x,y)\cdot\vec{u} = \nabla f\cdot\vec{u}}$$

$\langle$Ex$\rangle$ Find $D_{\vec{u}}f$ of $f(x,y) = x^2y^3 - 4y$ at $(2,-1)$ in the direction of $\vec{v} = 2\vec{i} + 5\vec{j}$

$4+25$

Sol: Compute $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{2}{\sqrt{29}}\vec{i} + \frac{5}{\sqrt{29}}\vec{j} = \langle a, b \rangle$$

Thus $D_{\vec{u}}f(x,y)\Big|_{\substack{x=2\\y=-1}} = \nabla f\cdot\vec{u}\Big|_{\substack{x=2\\y=-1}} =$