Friedberg Ch6.7

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Theorems

Theorem 1. *Let V and W be respectively n and m-dimensional inner product space, and let $T : V \to W$ be a linear transformation of rank r*

There exists orthonormal basis $\alpha = \{v_1, \dots, v_n\}$ for V and orthonormal basis $\beta = \{w_1, \dots, w_m\}$ for W and positive sacler $\sigma_1, \dots, \sigma_r$ such that

$$T(v_i) = \begin{cases} \sigma_i w_i & \text{if } i \leq r \\ 0 & \text{if } i > r \end{cases}$$

v_i is an eigenvector of T^*T corresponding to eigenvalue σ_i^2 if $i \leq r$ and to eigenvalue 0 if $i > r$

Proof. We first construct α, β and positive scalers.

We start with Basis α. By Rank-Nullity Theorem, we know $\dim(N(T)) = n - r$, so we can have an orthonormal basis $\{v_{r+1}, \dots, v_n\}$ for $N(T)$. Orthonormally expand $\{v_{r+1}, \dots, v_n\}$ to an orthonormal basis $\{v_1, \dots, v_n\}$ for V, which is the desired α.

We now show that $\{T(v_1), \dots, T(v_r)\}$ is orthogonal and linearly independent, so that later on we can orthonormally expand this set to basis β for W.

Let $j, k \leq r$. $\langle T(v_j), T(v_k) \rangle = \langle T \rangle$

$T(v_j) \neq 0$, since if $T(v_j) = 0$, U and $N(T)$ do not form a direct sum

Define $w_j = \frac{1}{\|T(v_j)\|} T(v_j)$, so $\|w_j\| = 1$

Then $T(v_j) = \|T(v_j)\| w_j$, so we also defined $\sigma_j = \|T(v_j)\|$ implicitly in the last line

Extend $\{w_1, \dots, w_r\}$ to a basis of W and orthogonalize and normalize the basis and we have the desired $\{w_1, \dots, w_m\}$ (done)

Now we prove v_i is an eigenvector of T^*T corresponding to eigenvalue σ_i^2 if $i \leq r$ and to eigenvalue 0 if $i > r$

Let $j > r$

$T^*T(v_j) = T^*(0) = 0$

Let $j \leq r$
$T^*T(v_j) = T^*(\sigma_j w_j) = \sigma_j T^*(w_j) = \sigma_j \sum_{i=1}^{n} \langle T^*(w_j), v_i \rangle v_i$

$= \sigma_j \sum_{i=1}^{n} \langle w_j, T(v_i) \rangle v_i = \sigma_j^2 v_j$ (done)

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