5

FG-submodules and reducibility

We begin the study of *FG*-modules by introducing the basic building blocks of the theory – the irreducible *FG*-modules. First we require the notion of an *FG*-submodule of an *FG*-module.

Throughout, G is a group and F is $\mathbb{R}$ or $\mathbb{C}$.

FG-submodules

5.1 Definition
Let V be an *FG*-module. A subset W of V is said to be an *FG-submodule* of V if W is a subspace and $wg \in W$ for all $w \in W$ and all $g \in G$.

Thus an *FG*-submodule of V is a subspace which is also an *FG*-module.

5.2 Examples
(1) For every *FG*-module V, the zero subspace $\{0\}$, and V itself, are *FG*-submodules of V.

(2) Let $G = C_3 = \langle a\colon a^3 = 1\rangle$, and let V be the 3-dimensional *FG*-module defined in Example 4.11. Thus, V has basis v_1, v_2, v_3, and

$$v_1 1 = v_1,\ v_2 1 = v_2,\ v_3 1 = v_3,$$

$$v_1 a = v_2,\ v_2 a = v_3,\ v_3 a = v_1,$$

$$v_1 a^2 = v_3,\ v_2 a^2 = v_1,\ v_3 a^2 = v_2.$$

Put $w = v_1 + v_2 + v_3$, and let $W = \text{sp}\,(w)$, the 1-dimensional subspace spanned by w. Since

$$w1 = wa = wa^2 = w,$$

W is an FG-submodule of V. However, sp $(v_1 + v_2)$ is not an FG-submodule, since

$$(v_1 + v_2)a = v_2 + v_3 \notin \mathrm{sp}\,(v_1 + v_2).$$

Irreducible FG-modules

5.3 Definition

An FG-module V is said to be *irreducible* if it is non-zero and it has no FG-submodules apart from $\{0\}$ and V.

If V has an FG-submodule W with W not equal to $\{0\}$ or V, then V is *reducible*.

Similarly, a representation ρ: $G \to \mathrm{GL}\,(n, F)$ is *irreducible* if the corresponding FG-module F^n given by

$$vg = v(g\rho) \quad (v \in F^n,\ g \in G)$$

(see Theorem 4.4(1)) is irreducible; and ρ is *reducible* if F^n is reducible.

Suppose that V is a reducible FG-module, so that there is an FG-submodule W with $0 < \dim W < \dim V$. Take a basis $\mathscr{B}_1$ of W and extend it to a basis $\mathscr{B}$ of V. Then for all g in G, the matrix $[g]_{\mathscr{B}}$ has the form

$$\left(\begin{array}{c|c} X_g & 0 \\ \hline Y_g & Z_g \end{array}\right) \tag{5.4}$$

for some matrices X_g, Y_g and Z_g, where X_g is $k \times k$ ($k = \dim W$).

A representation of degree n is reducible if and only if it is equivalent to a representation of the form (5.4), where X_g is $k \times k$ and $0 < k < n$.

Notice that in (5.4), the functions $g \to X_g$ and $g \to Z_g$ are representations of G: to see this, let g, $h \in G$ and multiply the matrices $[g]_{\mathscr{B}}$ and $[h]_{\mathscr{B}}$ given by (5.4). Notice also that if V is reducible then $\dim V \geqslant 2$.

5.5 Examples

(1) Let $G = C_3 = \langle a: a^3 = 1\rangle$ and let V be the 3-dimensional FG-module with basis v_1, v_2, v_3 such that

$$v_1 a = v_2,\ v_2 a = v_3,\ v_3 a = v_1,$$

as in Example 4.11. We saw in Example 5.2(2) that V is a reducible FG-module, and has an FG-submodule $W = \mathrm{sp}\,(v_1 + v_2 + v_3)$. Let $\mathscr{B}$ be the basis $v_1 + v_2 + v_3, v_1, v_2$ of V. Then

$$[1]_{\mathscr{B}} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right),\ [a]_{\mathscr{B}} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 1 & -1 & -1 \end{array}\right),$$

$$[a^2]_{\mathscr{B}} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 1 & -1 & -1 \\ 0 & 1 & 0 \end{array}\right).$$

This reducible representation gives us two other representations: at the 'top left' we have the trivial representation and at the 'bottom right' we have the representation which is given by

$$1 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ a \to \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},\ a^2 \to \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

(2) Let $G = D_8$ and let $V = F^2$ be the 2-dimensional FG-module described in Example 4.5(1). Thus $G = \langle a, b\rangle$, and for all $(\lambda, \mu) \in V$ we have

$$(\lambda, \mu)a = (-\mu, \lambda), \quad (\lambda, \mu)b = (\lambda, -\mu).$$

We claim that V is an irreducible FG-module. To see this, suppose that there is an FG-submodule U which is not equal to V. Then $\dim U \leqslant 1$, so $U = \mathrm{sp}\,((\alpha, \beta))$ for some $\alpha, \beta \in F$. As U is an FG-module, $(\alpha, \beta)b$ is a scalar multiple of (α, β), and hence either $\alpha = 0$ or $\beta = 0$. Since $(\alpha, \beta)a$ is also a scalar multiple of (α, β), this forces $\alpha = \beta = 0$, so $U = \{0\}$. Consequently V is irreducible, as claimed.

Summary of Chapter 5

1. If V is an FG-module, and W is a subspace of V which is itself an FG-module, then W is an FG-submodule of V.
2. The FG-module V is irreducible if it is non-zero and the only FG-submodules are $\{0\}$ and V.

Exercises for Chapter 5

1. Let $G = C_2 = \langle a : a^2 = 1\rangle$, and let $V = F^2$. For $(\alpha, \beta) \in V$, define $(\alpha, \beta)1 = (\alpha, \beta)$ and $(\alpha, \beta)a = (\beta, \alpha)$. Verify that V is an FG-module and find all the FG-submodules of V.
2. Let ρ and σ be equivalent representations of the group G over F. Prove that if ρ is reducible then σ is reducible.
3. Which of the four representations of D_{12} defined in Exercise 3.5 are irreducible?
4. Define the permutations $a, b, c \in S_6$ by

$$a = (1\ 2\ 3),\ b = (4\ 5\ 6),\ c = (2\ 3)(4\ 5),$$

and let $G = \langle a, b, c\rangle$.

(a) Check that

$$a^3 = b^3 = c^2 = 1,\ ab = ba,$$
$$c^{-1}ac = a^{-1} \text{ and } c^{-1}bc = b^{-1}.$$

Deduce that G has order 18.

(b) Suppose that ε and η are complex cube roots of unity. Prove that there is a representation ρ of G over $\mathbb{C}$ such that

$$a\rho = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix},\ b\rho = \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix},\ c\rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(c) For which values of ε, η is ρ faithful?
(d) For which values of ε, η is ρ irreducible?

5. Let $G = C_{13}$. Find a $\mathbb{C}G$-module which is neither reducible nor irreducible.