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PREFACE TO FOURTH EDITION

The fifth edition of this text has a number of changes from the fourth edition. A new chapter on computer science connections to enumeration has been added (much of its contents appeared in the similar chapter in the second edition that was dropped in subsequent editions). There is a new 'Prelude' at the beginning of the text on the game of Mastermind, which serves as a friendly warm-up to combinatorial logic. There is a new section in the Network Algorithms chapter on the Transportation Problem. A new appendix on computational complexity was added. Finally, two pages about chromatic polynomials were added to the section on graph coloring. Dozens of additional exercises were incorporated. The numbers were changed in dozens of other easy counting exercises to reduce the use by students of solutions passed down from previous years. However, only minor changes have been made to the rest of the text.

A webpage with corrections to the fifth edition will be maintained at the author's website, which can be found by typing 'Alan Tucker' into a search engine such as Google. Instructors using this text are encouraged to e-mail the author at tucker@ams.stonybrook.edu, with corrections or comments. Thank you.

Part I Teaching Tips

1. Preparing to teach an applied combinatorics course for the first time.

The good news first. There is no challenging new material an instructor needs to learn to teach an undergraduate applied combinatorics course. Most chapters in this book are built around a few simple formulas or principles. This is a problem-solving text, not a theory text. Advance preparation consists primarily of working out for oneself problems in the book. The principal attribute required of an instructor for this course is a disciplined, logical mind.

Now the bad news. There is much that occurs in the classroom in this course for which an instructor cannot prepare. The critical tasks are: a) determining the reason underlying a student's wrong answer or misconception about a counting principle; and b) being able to spot alternate forms of an answer--expressions which look very different from the standard answer but still are correct. Some exercises have 4 or 5 equally correct analyses yielding different looking expressions. The combination of "debugging!' mistakes and unexpected forms of an answer arise in combination when a student uses an original analysis but makes mistakes along the way.

In this course, the job of the instructor is not so much to present a right analysis of problems as to help students learn how to devise a right analysis for themselves. This means, answering student questions (especially about homework problems) should be a major part of the class time. Trying to answer all these questions promptly in class is clearly fraught with danger. The best way, time permitting, to answer a "why can't you do it this way" question is for the instructor to work slowly through the problem, starting from first principles and using a small concrete example, and follow the student's suggested line of attack until a fault is found or a valid, correct answer is obtained. The instructor should encourage other students to participate in the analysis. Sometimes the instructor may get stuck and should quickly say that he/she needs time to think about the question in order to give a clearer answer. Don't try to hide the fact that the material can be tricky for the instructor, too.

One of the best ways to prepare for unforeseen questions about homework is to try to solve exercises in advance many different ways. An even better tactic is to encourage students to come to the instructor's office to ask some of their questions about homework before it is discussed in class. The same questions inevitably come up again in class, and now the instructor is all set with pretested, clear answers.

2. General teaching suggestions

This text is designed to be a resource for an enjoyable teaching experience for the instructor in combinatorial problem-solving—which in turn usually guarantees an enjoyable learning experience for the students. Happily, most mathematicians enjoy solving combinatorial problems. An applied combinatorics course using this text is ideally suited to a style of interactive mathematics teaching that gives students a more active and, hopefully, more rewarding role in the classroom. The following principles underlie this style of teaching.

First, introductory courses in any mathematical subject, especially courses serving many non-math majors, should try to leave students with a favorable attitude toward the subject and the learning experience. No matter how important the mathematical methods, students often avoid using them, or learning more about them, if they disliked the way they were taught.

Second, all applied mathematics courses should motivate topics with applications and develop concepts intuitively at first. In a first undergraduate course in an applied subject, a problem-solving approach is desirable (similar to the standard problem-solving approach in calculus). An instructor's enthusiasm for an elegant theorem may influence some students to study the mathematics for its own sake, but first of all, an instructor's outlook in class should reflect the needs and point of view of the majority of students, typically a problem-solving point of view.

Third, instructors should seek to develop good rapport with their students. They should foster student interest in the subject with their enthusiasm for teaching the material and with their encouraging answers to student's questions (even silly questions). If students believe that the instructor is interested in their ideas and questions, they are likely to give the instructor constructive feedback that will make the course even better.

3. Goals of this text

This text introduces students to combinatorial problem-solving. The most important technique for a combinatorial problem-solver is simply disciplined, logical analysis of "word" problems. Such reasoning is the foundation for building simple mathematical models of problems– models implicit in counting expressions built of sums and products of binomial coefficients, in generating function models, in recurrence relation models, or in graph models. The way for a student to learn this logical thinking is by working many, many exercises. Students will use this reasoning often, consciously and unconsciously, in computer science, in operations research, and in probability and statistics. A logical mind will serve a person well in any field. To some, such a goal makes this text little more than an IQ enrichment book. <u>Whatever the interpretation, the goal of developing such logical reasoning in a problem-solving framework should be foremost in the instructor's mind at all times.</u>

A secondary goal of this text is to introduce students to the basic concepts and tools of enumeration and graph theory: counting methods such as generating functions, recurrence relations, and the inclusion-exclusion formula, and basic types of graphs, such as planar graphs and trees, and the uses of graphs in computer science and operations research. Several of these topics have special pedagogical value. For example, recurrence relations develop recursive reasoning (so important in computer science). The inclusion-exclusion formula is a grand exercise in applied set theory. Polya's formula is a practical (and fairly painless) introduction to group theory.

Finally this text seeks to achieve these learning objectives in as appealing a manner as possible, with real-world problems, card problems and logical puzzles. This writer's experiences in teaching applied combinatorics consistently re-affirm the importance of giving problems interesting settings.

4. Student Background

Many students find the transition from calculus courses to a combinatorics course very frustrating. They have become conditioned to technical "plug in" problems, applying a given integration technique to various functions. They are not accustomed to having to figure out how a problem should be solved. A combinatorics instructor often hears the complaint from students that the textbook is unfair because the homework exercises are so different from the worked-out examples- they expect that the homework problems should be solvable by mimicking calculations in the examples. Related difficulties may arise for math majors used to theorem-proving courses. They may have trouble developing their own simple models for analyzing word problems.

Conversely, there are some students for whom combinatorial problem-solving comes very easily. However, as in most other math courses, students who coast along will see their grades deteriorate when other students catch and surpass them by dint of hard work.

5. Note-Taking

A correct solution to most combinatorics problems is often "obvious" once seen. As a consequence, students often copy down little or no explanation of the reasoning behind an answer. Consider the problem, how many different non-empty subsets (of any size) are there of the integers 1, 2, . . ., n. The answer is 2^n-1, which can be explained: any subset (including the empty set) can be represented as an n-digit binary sequence with the i-th digit = 1 if and only if i is in the subset- and there are 2^n n-digit binary sequences and hence 2^n subsets, 2^n-1 of which are non-empty. If a student understands the binary sequence model quickly, then the 2^n -1 answer will be obvious and the student may write down only the numerical answer with no explanation, or possibly the phase "model as binary sequence." But several days later when students read over their notes, the expression 2^n-1 may now mean nothing, nor may the phrase "model as binary sequence."

The instructor at first must remind students to be careful to take good notes, and the instructor should pause at times to allow students the chance to write out explanations. When the answer to a problem is an expression involving products and/or sums of binomial coefficients, etc., it is helpful to annotate the answer by writing a brief descriptive phrase for each term (e.g., “choose which 4 of the 6 apples”) with an arrow from the phrase to that term in the expression.

6. Homework Assignments and Reviewing for Tests

In this course, the problems are the message. Exams are meant primarily to measure a student's proficiency at solving problems. Most of the course should be centered around the homework and how a student learns from homework mistakes. If for each section, there were a homework assignment, class discussion of the assignment, and finally a second assignment on the material, then counting the homework grade on the second assignment would be reasonable. This writer often creates a de facto second assignment by going over half of the exercises in an assignment in one class and then letting students have extra time to go back and rework, if necessary, the other exercises and hand in the whole assignment in the next class.

The most important aspect of discussing students' homework solutions is the wrong answers. The old adage "learn from one's mistakes" is the essence of modeling. Mathematicians do not divine right answers. They have ideas and sort through them in an evolutionary way until all the steps needed for an answer are appropriately assembled. Along the way, wrong ideas must be corrected and refined. Typically, it is by finding errors in an initial guess that one gains the insight that leads to the correct analysis. Students learn only a modest amount from seeing right answers. The instructor should help students learn to find right answers from initial wrong answers. (Getting students to propose answers that are likely to be wrong for constructive analysis is a tricky business that requires good student-teacher rapport.)

Reviewing for tests is an important learning period in the course. The week before a test, students are eager to solve lots and lots of practice problems to prepare for tests. Students discover that they now can solve problems that tricked them earlier. It is helpful to hand out a copy of last year's test (or a realistic sample test).

PART II: COMMENTARY ON TEXT

Prelude: The Game of Mastermind

The game of Mastermind was introduced in the 1970's. It involves determining a sequence of 4 colored pegs, chosen from 6 colors (repetition is allowed). Guesses at a solution are scored with two markers that tell (i) how many pegs in the guess are correct; and (ii) how many pegs are part of the solution but are in the wrong position in the guessed solution. Given a set of guesses and their scores, students are asked to reason out what the correct solution must be. The combinatorial logic involved is solving such problems is the core of the reasoning skills used throughout this text.

This author uses challenging Mastermind exercises as supplementary problems that are posted on the course website once a week (Thursdays at 10 pm) with extra credit points on the next exam awarded to the first 10 students to submit solutions.

Chapter 1: Elements of Graph Theory

This chapter introduces students to graphs. A relatively informal approach is used, which avoids the litany of definitions and theorems found in most introductions to graph theory. Section 1.1 illustrates the diverse uses of graph models. Section 1.2 uses the question of when are two graphs isomorphic to explore the intrinsic structure of graphs. Section 1.3 presents an important simple theorem about graphs. Section 1.4 examines an important graph property, planarity.

The goal of this chapter is to develop:
(i) familiarity with graphs;
(ii) facility with constructing graph models; and
(iii) some 'common sense' about exploring and analyzing simple graph properties.

This is the most important chapter about graphs, and all its sections should be covered. Although graphs are assuming an ever-growing role in computer science and have always been important in operations research, some students will never have seen graphs before. The whole concept of a non-numerical mathematical model will be new to them. The examples and exercises are generally easier than the problems in later chapters. This chapter's problems aim to illustrate simple uses of graphs and to build a general familiarity among students with graph models.

The problem-solving analysis required in this chapter is efficient, but ad hoc, logical examination of possibilities, a crucial part of the combinatorial reasoning in later counting chapters. Even small graphs are such complicated structures that brute force enumeration of possible subgraphs to, say, verify a particular graph property is normally infeasible. Mathematical insights are required to simplify and efficiently sort through the possibilities.

Section 1.1: Graph Models

The section quickly introduces the concept of a graph (vertices, edges, adjacency relation, and little else). Note that, as defined in this book, graphs cannot have loops or duplicate edges. Example 4 introduces an exploratory but disciplined analysis of an edge covering problem. The unexpected relation between edges covers in Example 4 and independent sets in Example 5 should be emphasized. While Example 6 introduces a directed graph problem, it should be noted that directed graphs get very modest treatment in this text. Many other good examples of graph models may be found in the exercises.

Exercises: Exercises 1-6, 21, 26, 27 present other graph model problems. The other exercises mimic and extend the models introduced in the section's examples.

Section 1.2: Isomorphism

This section examines the structure of graphs, via the problem solving vehicle of asking whether two graphs are really different or are just "rearrangements" of each other. To answer such questions, it is necessary to begin introducing some graph terminology, such as, degree of a vertex and complete graph.

The informal use of some graph terms and the arguments using these terms in Chapters 1 and 2 may be too lacking in rigor for some instructors. However the machinery needed to make all definitions and proofs rigorous would greatly slow down the pace of the course and sidetrack the purposes of the course. Chapters 3 and 4 have more rigorous presentations. The introduction of the complement of a graph is an important secondary theme in this section.

Exercises: Exercise 14 anticipates the parity Corollary in the following section.

Section 1.3: A Simple Counting Formula

The theorem and corollary in this section are the only general results known to apply to all graphs (by 'general', we mean without specially defined terms). The corollary is used to prove the nonexistence of certain types of graphs, as demonstrated in Example 2.

Example 4 is a very appealing application of the parity Corollary although it requires a little time to present. The author finds that it is well received by students– the time to present it is worth the effort.

Exercises: Exercise 8 is a true real-world scheduling application of the parity Corollary. In the early 1970’s, the NFL owners once designed scheduling parameters for pro football games that violated the Corollary.

Section 1.4: Planar Graphs

Planar graphs are of practical importance because they arise so often in operations research problems. Planar graphs are also of historical importance in graph theory because much of the development of the subject centered around the question, are all planar graphs 4-colorable. This section surveys some ways of determining whether a graph is planar. At first, an ad hoc geometric approach is used. Next Kuratowski's forbidden configuration is tried. Then some theory is developed: Euler's formula for planar graphs and a powerful corollary. From this corollary, many graphs can be shown to be non-planar with a simple algebraic test. This section is a case study in the power of a little good theory over ad hoc arguments.

Exercises: The exercises develop several extensions of the theory and other basic concepts of planar graphs, e.g., properties of dual graphs in Exercise 10 and Platonic graphs in Exercise 24. Variations of Exercise 7 are a favorite for this author on exams. Exercises 25 and 26 integrate ideas in section 1.3 and section 1.4.

Supplement: Supplementary Exercises

This section presents a collection of additional graph theory problems. The problems examine a variety of graph terms, such as strong connectivity, cut-sets, and automorphisms. Exercise 5 uses the Pigeonhole Principle. Exercise 20 is a cute logical puzzle that can be analyzed with the help of graphs. Exercise 30 is a well-known Ramsey theory question, whose solution (given in Part 4 of this Manual) is famous for its elusive simplicity. Exercise 32 is a 'trick' problem whose answer is a set of isolated vertices.

Chapter 2: Covering Circuits and Coloring

This chapter presents three topics in graph theory, Euler cycles, Hamiltonian circuits, and graph coloring. These subjects are all over 100 years old, originally arising as games or semi-recreational mathematics, but have extensive uses today in operation research. Logical arguments are more carefully constructed and stated in this chapter, although only four theorems, the Euler Cycle Theorem, the Hamilton Tournament Theorem, the Polygonal Triangulation Theorem and the Five Color Theorem, are actually proved. The three topics presented here are inherently appealing because of their recreational flavor. Further, what is fun is also seen to have practical applications. With this motivation, students are asked to perform the careful logical analysis of the possibilities required in ad hoc testing for Hamiltonian circuits and coloring. Section 2.4 on coloring theory can be skipped or covered briefly in most courses.

Section 2.1: Euler Cycles

Euler cycles are a pictorial topic with a simple, complete theory. This section presents an application to street sweeping and uses it to develop an alternate proof of the Euler cycle theorem.

Exercises: There are several fairly easy variations of the Euler cycle theorem in the exercises: Exercises 8, 12, 13, 17a and 18. Exercise 19 is a famous application of Euler cycles.

Section 2.2: Hamilton Circuits

The simple theory of Euler cycles is now replaced by the more normal graphtheoretic situation of complicated ad hoc arguments, here needed to determine whether a graph has a Hamilton circuit. This section's focus is on thorough analyses to prove a particular graph cannot have a Hamilton circuit. The section present four theorems, only one of which is proved, about Hamilton paths and circuits. The section ends with an example (new to this edition) applying Hamilton paths to the construction of Gray codes.

Exercises: The graph in Exercise 4o (it also appears in Exer. 1o in section.2.3) is called Petersen's graph; it has many "'bad" properties. Exercises 9 and 10 present some "tricks" that can be used to prove quickly that certain graphs cannot have a Hamiltonian circuit. Exercise 15 about a knight's tour is a well-known result but finding the tour is tricky. The following heuristic works: starting from any vertex, for the next move always choose to go to the square from which the number of possible subsequent moves is minimal. (Note: Ball and Coxeter's *Mathematical Recreations and Essays*, U. of Toronto Press, p.l75-184, has a fascinating discussion of the history of the knight's tour problem, including contributions of Euler and DeMoivre).

Section 2.3: Graph Coloring

This section presents logical ad hoc arguments for determining the chromatic number of a graph and illustrates the practical uses of coloring with examples in operations research and computer science. Although the garbage collection model takes a whole page to explain, students have always found it very interesting.

Exercises: The trick to showing the graph in Exercise 1q cannot be 3-colored is first showing that a 3-coloring of the left half (vertices a,b,c,d,e,f,g,h) forces a and h to have different colors (as if they were joined by an edge), and similarly for i and p in the right half. Exercise 7 is a good review of geography. Exercises 10-14 are simple color modeling problems.

Section 2.4: Coloring Theorems

This section starts with an interesting problem in geometry. Then presents some results in coloring theory, which was initially motivated by a century of research on the Four-Color problem. It finishes with new material about chromatic polynomials. The Five Color Theorem, a baby sister of the famous Four Color Theorem, is proved. The original false 1876 proof of the Four Color Problem by Kempe used the same argument that works successfully in the Five Color Theorem proof. Namely, in Figure 2.15 suppose e is color 2 and G-x is 4-colored; try a 1-3 interchange at a and a 1-4 interchange at a, if both fail (there is a 1-3 path from a to c and a 1-4 path from a to d), now try a 2-3 interchange at e and a 2-4 interchange at b to make e 3 and b 4 and permit x to get color 2: the error is that the 2-3 interchange at e may break the 1-3 path from a to c so that the 2-4 interchange at b may change d to 2.

<u>Exercises</u>: Virtually all the exercises in this section are proofs.

Appendix: Graph Model for Instant Insanity

This clever model speaks for itself. It is possibly the most powerful use of a graph model known to this author—powerful in the sense of organizing the information in a combinatorially rich situation so efficiently that the problem can be solved by inspection in a few minutes once the graph representation of the puzzle is constructed.

There are enough components to the model that a good 20 minutes should be allocated to present the model and its analysis. Rather than using the sample cubes in the text, the instructor should buy his or her own set of cubes and solve them in class. To keep the cubes distinct, number some corner of each cube 1 through 4. Practice arranging the cubes into the correct pile a few times before class.

Sometimes the author presents this model at the end of the last class to end the course on a high note. The class usually bursts out in applause when the puzzle is solved.

Chapter 3: Trees and Searching

This chapter begins with some terminology and basic theorems about trees. The rest of the chapter applies trees to various searching problems. The trees are used to build search algorithms and to analyze these procedures. Section 3.2 introduces spanning trees and the two most common types of searches, depth-first search and breadth-first search. Section 3.3 illustrates two different uses of trees in attacking complex operations research problems, the Branch and Bound method and heuristic approximation. The sample operations research problem chosen here is the famous Traveling Salesperson Problem. Section 3.4 uses trees to analyze the behavior of several sorting algorithms.

This chapter is essential material for computer scientists (although much of it may also be covered in computer science courses).

Section 3.1: Properties of Trees

This section begins with a collection of tree-related definitions. The beginning of the chapter has been re-worked with a new theorem about the equivalence of different definitions of a tree. (These definitions are collected together in the Glossary at the end of text.) There are four theorems giving formulas involving various parameters of trees. Theorem 4 is used in Section 3.4 to show that any algorithm for sorting n items needs at least log (n!) [= O(n $\log_2 n$)] comparisons in the worst case. Theorem 5 presents an interesting way to associate a unique numerical sequence with a labeled tree.

Exercises: Exercises 3-18 are theory problems and exercises 19-27 involve applications.

Section 3.2: Enumeration with Trees

This section presents depth-first and breadth-first search methods with recreational examples and introduces the concept of a spanning tree. Tree traversals are also discussed.

Exercises: Exercise 30 presents a binary sequence characterization of binary trees. Exercise 32 presents a one-pass algorithm for finding an Euler cycle in a directed graph.

Section 3.3: The Traveling Salesperson Problem

This section first presents the Branch and Bound technique, a tree-based procedure to search for optimal solutions in combinatorial operations research problems. The problem used to illustrate the technique is the famous Traveling Salesperson Problem. Such complicated operations research problems are often attacked with heuristic procedures that yield near optimal solutions.

The second part of this section presents a tree-based heuristic for the Traveling Salesperson Problem. The proof bounding the accuracy of this heuristic, while basically straightforward, is a bit difficult for students to follow and may be skipped.

Exercises: Exercise 5 presents an important production scheduling problem.

Section 3.4: Tree Analysis of Sorting Algorithms

Binary trees are a convenient model for representing most sorting algorithms. They are also a necessary tool for analyzing the efficiency of sorting algorithms. This section concludes with a sorting procedure using partially ordered trees, called heaps. Rather than sort the whole list at once, a heap sort picks out one element at a time from a heap. Although most computer science students will have studied this material before, this brief revisit with its tree approach should be valuable to them.

Chapter 4: Network Algorithms

The theory of network flows is one of the great accomplishments in graph theory, both in terms of mathematical elegance and practical uses. This chapter first "warms up" for the augmenting flow algorithm with sections on shortest path and minimal spanning tree algorithms. The main part of this chapter is section 4.3 which introduces the basic theory of network flows. Section 4.4 applies flows to the combinatorial theory of matching. Search trees underlie the shortest path, maximal flow and matching algorithms. Section 4.5 (new) uses trees in a more central fashion to solve the Transportation Problem. An important pedagogical aspect of network flows is the use of an algorithm to prove a theorem (the Max Flow-Min Cut Theorem); the proof is literally constructed by the augmenting flow algorithm. This interplay of algorithms and theory is typical of much recent theory in computer science and operations research.

Section 4.1: Shortest Paths

Dijstra's shortest path algorithm is presented. It is a good example of operations research algorithms and serves to prepare students for the labeling procedure used in the augmenting flow algorithm in section 4.3.

Section 4.2: Minimal Spanning Trees

The two standard algorithms for finding a minimal spanning tree are presented and the validity of one, Prim's algorithm, is proved. Minimal spanning tree algorithms are somewhat like Euler cycles; the results in both case are uncharacteristically (for graph theory) simple. If time is tight, one can skip the proof of Prim's algorithm.

Section 4.3: Network Flows

This long section develops the basic theory of network flows. The section begins by introducing the terminology of network flows and basic properties of flows and cuts. The presentation here is mathematically rigorous. Corollary 2a should be carefully explained, for it is the key to understanding the flow algorithm. Next, an intuitive but faulty approach to flow building is illustrated. This is followed by the correct approach, the Augmenting Flow Algorithm of Ford and Fulkerson. Finally, several basic variations of network flows are modeled by standard single source-single sink flows.

Exercises: Exercises 12 and 13 present another basic variation on flows, capacitated vertices. Exercises 1-17 are computational problems; exercises 18-40 are theory problems.

Section 4.4: Algorithmic Matching

The theory of network flows is applied in this section to solve matching problems and to prove the two fundamental theorems of matching theory. Matching problems are an application of flows in which negative labeling arises naturally. The theory in this section should be skipped for students with limited mathematical maturity; the first two pages of section 4.3 should then be treated as just one more example of network flows. Example 4 is perhaps the most involved example in the text and illustrates the complex models that can be made with network flows. On the other hand, the problem solved here- the 'mathematical' (early) elimination of teams in a league championship- is very interesting.

Exercises: Exercises 1-14 are computational problems; exercises 15-24 are theory problems.

Section 4.5: The Transportation Problem (New)

This section presents the classical operations research analysis of the Transportation Problem with an emphasis on the underlying bipartite graph and the central role of trees in finding an initial solution and finding improved solutions. However, in operations research texts trees are studied from a linear algebraic viewpoint—in the constraint equations, the columns associated with edges of a tree form a basis for the solution space. (Historical Note: the author's father, A W Tucker, who was trained as a topologist but became one of the leaders in developing mathematical programming, was first attracted to the latter subject by the transportation problem with its rich graph-theoretic structure.)

Exercises. Most of the exercises is examples of transportation problems.

Chapter 5: General Counting Methods for Arrangements and Selections

This is the most important chapter in the text. It introduces general problem-solving in combinatorial enumeration. There is no theory, and only a few basic formulas: just lots of examples to help prepare students to do their own problem-solving. Some students will have difficulty solving problems whose analysis does not mimic an example in the text. It is virtually impossible to teach students the right way to do problems in advance. Rather, much of the learning will occur after assignments are done and homework is discussed in class.

As students worry about more complicated constraints found later in the chapter, they may become so uncertain of themselves that they no longer can solve easier problems. It is helpful to establish some personal rapport with students if uncertainty is setting in. Tell them this material is tricky for professors too (it is!) and encourage them to come to office hours with their questions.

Students easily misinterpret problems, and even extensive discussion of an exercise in class may not clear up the confusion. Only after the student slowly explains the way he or she analyzed the problem, can the instructor be in a position to help. If some students come to see the instructor with questions on the homework before the homework is discussed in class, then the instructor will be better prepared to help others with their questions during the class.

It is extremely important to stress the value of learning from one's mistakes. A good analogy is learning to throw darts. If you throw just one dart and it hits the bull's eye, you were lucky but have learned little about controlling the flight of a dart. But if you start with a poor throw and successively improve your aim until you hit the bull's eye, then you have really learned to throw darts.

The progression of increasingly more involved problems from Section 5.1 through 5.5 is the standard development found in most combinatorics texts. To lighten the possible tedium of combinatorial word problems, humorous settings are used in Section 5.1 and 5.2, e.g., Professor Mindthumper and names from Tolkien's "Lord of the Rings". The instructor may want to continue the levity by changing the setting of subsequent examples in the book, e.g., how many distributions are there of 20 identical administrators into . . .

Section 5.5 on binomial identities may be skipped. Note that the old section 5.6 in the fourth edition has been moved to section 10.1 in this edition.

Note that the solutions to selected exercises appear in a Supplement at the end of the chapter. The set of solved exercises is: Section 5.1- 13, 30, 33, Section 5.2- 45, 51, 55, 59, 65, 77, Section 5.3- 25, 28, 29, 31, 34, Section 5.4- 27, 28, 47, 52, 59, 61, 64.

Section 5.1: Two Basic Counting Principles

The Addition and Multiplication Principles, for breaking problems into disjoint or sequential subproblems, are the fundamental building blocks of all combinatorial enumeration. Later problems in this chapter will involve repeated, intermixed use of these two principles. Even though combinatorics is full of cute short solutions, the best way first to approach any counting problem is by using these two principles to produce a thorough case-by-case decomposition into small, manageable subproblems. Incomplete and incorrect decompositions are the cause of most counting errors. Part d) of Example 4 illustrates a common error that should be emphasized. It is hard for students to anticipate such mistakes. Instead, they must learn from their mistakes. This is why a large amount of class time in the course, in this writer's opinion, should be devoted to discussing homework, i.e., discussing the faults in wrong solutions so that students will be able to avoid them in the future.

<u>Exercises</u>: Exercises 30-31, 33, 34, 44 and 45 are primarily modeling or interpretation problems; the counting itself is easy once one knows what to count. Example 49 presents the game of Swap, with n white pegs and n blue pegs separated by a space; whites move left either one step (to an open space) or two steps (jumping over a blue peg to a following open space); blues move right in a similar fashion. For n=3, the game is played: BBB ->_WWW, BB_B <- WWW, BBWB_ <- WW, BBWB -> W_W, BB -> W_WBW, B -> _WBWBW, _B <- WB <- WB <- W (3 successive moves),WB -> WB -> WB -> _, W_ <- WB <- WBB, WWWB -> _BB, WWW_BBB. This game should be explained in class, before it is assigned.

Section 5.2: Simple Arrangements and Selections

Simple arrangements and selections are the small, manageable subproblems into which complex counting problems are commonly decomposed. This section introduces basic types of unconstrained and constrained arrangement and selection problems. The most common error in enumerating unordered sets is addressed in the Set Composition Principle and is demonstrated in Example 5d. This mistake of implicitly ordering the same set several different ways (picking a first part and then a second part of a set different ways) will be made over and over again in homework solutions. Example 6 is a counting problem that arises in quality control probabilities. Example 8 presents a measure of voting power, which has important uses in game theory and politics.

<u>Exercises</u>: The exercises in this section contain many different types of constrained arrangement and selection problems. The worked examples are of little help in attacking many of the later problems. Some students will get overwhelmed with the amount of ingenuity required and the variety of these exercises. Exercise 55 is the famous Birthday Paradox. Exercises 60 and 61 introduce two important combinatorial problems in sampling theory. Exercise 65a, counting poker hands with exactly one pair, is a pet problem of this author; it has many possible right and wrong analyses.

Section 5.3: Arrangement and Selections with Repetition

Most students have seen the formula for arrangements with repetition, but the formula for selections with repetition is usually new. This section introduces problems involving simple constraints with repetition.

<u>Exercises</u>: The first 17 exercises are similar to the examples. The remaining exercises involve more complex constraints and, in general, are quite difficult word problems.

Section 5.4: Distributions

This section presents equivalent distribution models for arrangements and selections with repetition. The section extends and compounds the examples in the previous section. There are more complex constraints, identical and distinct objects are intermixed, certain consecutive pairs are forbidden, and the model of integer solutions of an equation is introduced. The equation model is central to the development of generating functions in the next chapter.

This section is the most difficult part of the course for most students. For several sections, students have been faced with increasingly complicated problems. At each stage, they had to move on without time to master fully the current problems. Assigning some review exercises from previous sections along with new problems is worth considering here. To assist students, a table at the end of the section summarizes different counting formulas for ordered and unordered sets.

Exercises: Exercises 31-34 provide straightforward, yet valuable, practice in translating between different arrangement and selection models. Example 47-49 are pet problems of the author (variants of which always appear on his tests). Example 51 presents an important alternate way to approach non-consecutivity problems (the method in Example 9 is more flexible, e.g., easily accommodating the added constraint of at least two symbols between each vowel).

Section 5.5: Binomial Coefficients

This section begins with an explanation of how binomial coefficients get their name, i.e., how the $C(n,k)$'s arise in the binomial expansion. Then the symmetry relation and Pascal's recurrence relation are given. Many courses will stop at this stop. The rest of the section concerns binomial identities. Although binomial identities do not appear to be problem-solving, the combinatorial reasoning used to verify them has much in common with the reasoning used in previous sections. Moreover, as shown in Example 4 and 5, identity (8) provides a convenient way to evaluate combinatorial sums. This summation technique is used in section 7.4 to solve a special class of recurrence relations.

Chapter 6: Generating Functions

Generating functions are probably the most important enumeration model in combinatorics. In contrast to the unstructured problem solving of Chapter 5, this chapter attacks problems with a very specific, well-structured model, that of polynomial multiplication. Section 6.1 explains how ordinary generating functions can model problems of selection with restricted repetition. The polynomial model is the sole message of this section. Section 6.2 is concerned with techniques for evaluating coefficients in given generating functions. These first two sections are the core material of this chapter. Many courses will cover just these two sections and possibly also section 6.4. Section 6.3 looks at partitions and their generating functions, as well as Ferrers graphs of partitions. Section 6.4 introduces exponential generating functions. The modeling performed by these functions is a bit more difficult to understand. In many undergraduate courses, it would be better to concentrate on a solid mastery of ordinary generating functions. Section 6.5 shows how summation problems can be solved with generating functions. This section is needed for section 7.5 on generating function methods for solving recurrence relations.

Section 6.1: Generating Function Models

In this section, students are shown how to build generating function models for selection problems with restricted types of repetition. The section starts with a review of the combinatorial explanation of the Binomial Theorem given in section 5.5. However, the Binomial Theorem can easily be explained at this point from scratch, if section 5.5 was skipped. Another polynomial expansion problem is restated as a combinatorics problem. Then the direction of the modeling is reversed, and combinatorial problems are restated as polynomial multiplication problems.

The three key concepts in this section are: a) polynomial multiplication generates a set of formal products, as in expansion (3); b) these formal products are characterized as a sum of exponents; and c) the number of such sums of exponents is just an integer-solution-of-equation problem, and conversely, any integer-solution-ofequation problem can be viewed as a sum of exponents problem. Any one of these three concepts may prove to be a major stumbling block for students. The best cure is for the instructor and a student to go slowly over the modeling process with a sample problem in both directions, from polynomial expansion to counting problem and from counting problem to polynomial expansion.

<u>Exercise</u>: The first 19 exercises are fairly straightforward. Exercises 20 and 21 are important examples of faulty generating functions. Even the very last exercises on multivariate generating functions are not hard.

Section 6.2: Calculating Coefficients

This section presents the necessary algebraic manipulation for evaluating particular coefficients of common types of generating functions. As these techniques are demonstrated, their combinatorial interpretations are also discussed. Letting rote algebraic manipulations do the work of the combinatorial analyses in Chapter 5 is a good example of the algebraic spirit of modern mathematics. The last page on combinatorial identities can easily be skipped.

<u>Exercises</u>: Exercises 11 and 15 present variations of the expansion formulas given in this section. Exercises 38-42, and 44 cover the basic properties of probability generating functions. Exercises 43 and 44 present examples of the "chain rule"- for generating functions. Exercise 44 is a basic result of queuing theory.

Section 6.3: Partitions

A brief discussion of partitions is given, primarily as an example of infinite generating functions. The section finishes with the Ferrers graph diagram for representing partitions and partition identities.

<u>Exercises</u>: Exercise 13 is an interesting use of Ferrers graphs, to prove that integer multiplication is commutative (this same problem is given as an induction proof exercise in Appendix A.2).

Section 6.4: Exponential Generating Functions

This section begins with a careful discussion of a sample exponential generating function model. The fact that each formal product now contributes a value of $n!/e_1! \quad e_r!$ to the coefficient makes exponential models much harder for students to understand. The topic is a nice extension of the ordinary generating function model introduced in the beginning of this chapter.

Exercises: Exercises 21 and 22 continue the probability generating function exercises in section 6.2.

Section 6.5: A Summation Method

This section shows how to build a generating function with coefficient an equal to a given combinatorial expression. Then the "summation operator" 1/(l-x) is used to sum such summands. The material in this section is used for solving recurrence relations in sections 7.4 and 7.5.

Chapter 7: Recurrence Relations.

A recurrence relation incorporates in a formal mathematical equation the combinatorial reasoning for decomposing a counting problem into similar subproblems. While recurrence relations are a fundamental tool in computer science for analyzing recursive algorithms, such relations have much wider applicability in all applied combinatorics than is generally realized. If enough parameters are used, virtually any counting problem can be modeled with a recurrence relation. This chapter's primary objective is teaching students how to build recurrence relation models. Its secondary objective is to present a brief survey of methods for closed form solutions to recurrence relations. Solving the relations is of limited importance because: a) an applied combinatorics course normally emphasizes the modeling side of problem-solving; b) medium-sized problems are readily solved numerically by iteration, once a recurrence relation is available; and c) the solution techniques, with the exception of generating function methods, are of limited general pedagogical value; one simply determines parameters in "cookbook" families of solutions.

Section 7.1: Recurrence Relation Models

This section is the heart of the chapter, and may be the only section some courses cover in this chapter. A variety of examples of recurrence relations are given. The section concludes with a discussion of the connection between recurrence relations and difference equations. This last material can be skipped; it raises an ancillary issue that can partially sidetrack students who should be concentrating on finding recurrence relation models. Like generating functions, recurrence models may confound some students at first. However, again a little private help should quickly straighten things out. A warning about compound interest relations: Many students have yet to worry about savings accounts, mortgages, etc., and so they will not see the great practical importance in these relations that the instructor does.

Exercises: Exercise 7 is the original Fibonacci problem about multiplying rabbits, the problem after which the Fibonacci relation is named; assigning it is a must (unless covered in class). Exercises 28 through 37 involve primarily multivariate relations and simultaneous relations. Students should be told that, when in doubt, it helps to add additional variables or supplementary equations. Exercises 46 and 47 are quite tricky.

Section 7.2: Divide-and-Conquer Relations

This section presents "cookbook" solutions to a class of recurrence relations that arise frequently in the analysis of recursive algorithms. Example 3 demonstrates, in a simple form, the idea behind Strassen's famous fast matrix multiplication (that multiplies two n x n matrices with less than n^3 multiplications).

Section 7.3: Solutions of Linear Recurrence Relations

Linear recurrence relations are the simplest form of recurrence relations. Even if this section is not assigned, the instructor should spend a few minutes in class sketching the form of solution to linear recurrence relations. The key pedagogical points in this section are the similarity of the forms of solution for linear recurrence relations and linear differential equations, and that the general solution to a linear recurrence relation is really a family of solutions, one for each possible set of initial conditions. Example 4 is fairly complicated and the instructor may want to skip it (to avoid scaring students).

Section 7.4: Solutions of Inhomogeneous Recurrence Relations

This section presents solution techniques for inhomogeneous first-order linear recurrence relations: $a_n = ka_{n-1}$ 1 $+f(n)$. If $k \neq 1$, a_n will be essentially the same type of function as $f(n)$. If $k = 1$, the relation is just a summation problem. In the latter case, it is necessary to have previously discussed summation, either using binomial identities (section 5.5) or generating functions (section 6.5).

Section 7.5: Solutions with Generating Functions

This section shows how recurrence relations can be converted into equations for associated generating functions. This conversion is the discrete counterpart to transform techniques for solving differential equations. While important, this material is difficult for most undergraduates who are not upper-division math majors. It could require upwards of a week of the course to present this section properly to other students. Examples 2 and 5 use partial fraction decompositions (a technique that, at best, students saw briefly in calculus and quickly forgot); to determine coefficients in partial fraction expansions, consider using the Heavyside method (discussed in most calculus texts).

Chapter 8: Inclusion – Exclusion Formula

In this chapter, the counting principle of complementation, that the number of objects without property A is the total number minus the number with property A, is generalized to situations involving many properties. Many problems presented in Chapters 5 and 6 can be solved more easily now. The "theory" of this chapter consists of working out in general set-theoretic terms a formula. expressed in terms of the sizes of various set intersections, for the number of objects that have none of a group of properties. To solve a given problem with this formula, a student must: a) define a group of properties; and b) count the number of objects with various subsets of the properties– these latter counting problems are usually simple Chapter 5-type counting problems. The last section develops a nice mini-theory for solving arrangement problems with restricted positions. This material is not difficult, but time constraints may force the last section to be skipped.

Section 8.1: Counting with Venn Diagrams

This section slowly generalizes the principle of complementation from one property to two properties and then to three properties. Some students already have trouble in the 2-property case with defining the two properties that are not to hold for the elements to be counted. This is a matter of logical negation in set theory and propositional logic. A second problem is trusting in a settheoretic formula. In Chapter 5, similar problems had to be carefully thought out and broken into appropriate subcases. Now students are conditioned to such an approach rather than "plugging into" a formula.

<u>Exercises</u>: Exercises 33-36, like Example 6, cannot be solved by inclusion-exclusion but rather require ad hoc Boolean algebra analysis. Exercises 20, 24, 27, and 28 ask for unions instead of intersections. The counting subproblems in Exercises 12, 27, 28, and 29 require combinatorial insights developed in Chapter 5.

Section 8.2: Inclusion-Exclusion Formula

Now the students should be ready for the general inclusion-exclusion formula. Some students will already have forgotten the necessary Chapter 5-type counting methods needed for solving some of the subproblems arising in the use of the inclusionexclusion formula. A few students may be bothered by the generality of the summation notation in the formula. The proof of the general Inclusion-Exclusion formula can be sketched as a generalization of the 3-set proof in section 8.1. Theorem 2, at the end of the section, can be skipped for most audiences.

<u>Exercises</u>: Exercise 14 has tricky wording; it is really a selection-with-repetition problem with at least one of each type. Exercise 15 is a nice variation on the standard inclusion-exclusion type of problem (here the 'answer' is given and N must be determined). Exercise 25a looks like a derangement problem but is not (there are n -1 properties, not n properties). Exercise 30 is a tricky "recursive" inclusionexclusion problem (see its solution in Part 4 of this Manual). Exercises 31-34 involve a new type of combinatorics problem, namely, combinatorial sets whose elements are themselves combinatorial sets; e.g., Exercise 31 looks at sequences of distinct subsets.

Section 8.3: Restricted Positions and Rook Polynomials

This section on rook polynomials presents a clever model for restricted position problems using chessboards and generating function models. Each step of the theory is easy to follow but the final result is far from obvious. The case of non-disjoint subboards (starting after Example 1) may be skipped.

Exercises: Exercises 2c,d,e, 8b, and 10 do not decompose into disjoint subboards (but 8 is not hard to do by inspection).

Chapter 9: Polya's Enumeration Formula

This chapter discusses a counting problem in applied group theory, a problem in which generating functions play an important role. This chapter has the only extensive theoretical development in the enumeration half of this book. The basic definitions and theoretical development are presented by examples without formal proofs. The emphasis is on understanding the theory by using it to solve problems. One of the pedagogical goals of this chapter is to present groups in a practical and natural setting (as opposed to the more formal setting in a modern algebra course). Polya's formula is usually not discussed in sophomore/junior-level applied combinatorics courses, because of its abstract foundations in group theory. However, the presentation "by example" used here is much easier to follow than the standard treatment. This concrete approach follows very closely how G. Polya taught this topic to Stanford undergraduates in 1969 (a course for which this author had the privilege to be the grader). Burnside's Lemma is the only part of this section that has been difficult for students at the author's institution.

Section 9.1: Equivalence and Symmetry Groups

This section introduces the sample problem of two colorings of the corners of a square. It is most helpful to have a physical model (say, of tinker toys) of a square to show how motions transform one coloring pattern of the square's comers into another pattern. The concepts of an equivalence relation and a group are presented and linked together. It is important here to emphasize the difference between a motion's permutation of the corners of the square and a motion's induced permutation of the colorings of the corners-- the latter permutations generate the equivalence classes we want to count. Examples 2, 3, and 4 about enumerating symmetries can be left for out-of-class reading, or covered lightly.

Exercises: Exercise 4 is a valuable warm-up for cycle decompositions in sections 9.3 and 9.4. Exercise 12 on the non-commutativity of symmetries should be assigned or covered in class. Exercises 16-25 involve basic group theory.

Section 9.2: Burnside's Lemma

The section on Bumside's Lemma is possibly the most difficult section in the book. A heuristic argument for this algebraic lemma is presented. The approach here now works well for the author. The key idea is that with "multiplicities" counted, every equivalence class will have size 8 for the square coloring problem, illustrated with the color class of C_{10}. If most students have previously had a modern algebra, a proof of the lemma should be given. In any case, after about 20-30 minutes of theoretical discussion, the instructor should turn to applying the lemma's formula. Students should not worry about remembering the explanation of the lemma.

Exercises: Exercise 11 is a cute (and tricky) combinatorics problem. Exercise 13 continues the theoretical development of the theory exercises in section 9.1.

Section 9.3: The Cycle Index

The section is a self-explanatory development of the role of the cycle structure representation of a motion in counting the number of colorings left fixed by a motion. It is desirable to get maximum student participation in filling in the various entries in the table. The examples in this section are easy for students to read out of class, but one should be covered in class.

Exercises: Exercise 8 presents a constraint most easily handled with the inclusion-exclusion formula.

Section 9.4: Polya's Formula

This section extends the development in section 9.3 to obtain the final Polya's formula. If Example 4 is discussed, a physical model of the cube is desirable.

Chapter 10 Computer Sciences Approaches to Enumeration

This chapter looks at enumeration from three computer science-based points of view, which range from very concrete to very abstract. Section 10.1 presents sequel algorithms to enumerate all permutations or *r*- combinations of a given set of elements. Section 10.2 introduces formal languages and shows how a combinatorial collection can be viewed as the sentences of a properly defined grammar. Section 10.3 introduces finite-state machines and looks at the problem of designing a finite-state machine to recognize sequences that are members of a given combinatorial collection.

Section 10.1: Generating Permutations and Combinations

This section gives algorithms for listing lexicographically permutations and *r*-combinations of an *n*-set. Then it illustrates how to use these algorithms to enumerate outcomes in earlier counting problems. This material can be integrated into the first two sections of Chapter 5. Writing out these algorithms precisely in some programming language is important for computer science students.

Linking combinatorial analysis with programming has an important dividend: it is often helpful to think of building a formula for an enumeration problem as a dynamic process, as if one were a computer that was printing out page after page of all possible outcomes. The fundamental question for the computer, what outcome to print next (e.g., what is the next outcome in this subclass, or is this subclass of outcomes completed and if so, what subclass should be listed next), generally requires the same type of analysis as needed to obtain a formula to count all outcomes.

Exercises: Exercises 9 and 10 about finding a given permutation's or combination's position in a lexicographic list are good problems for advanced students.

Section 10.2: Formal Languages and Grammars

This section introduces the formal structure of languages and grammars and then designs grammars whose sentences are a given combinatorial collection. The sentences can also be viewed as the leaves in the tree whose internal vertices are production rules in the associated grammar. Just enumeration algorithms can motivate the type of thinking needed to produce a formula to count all outcomes, grammars can help develop counting formulas. Grammars inherently involve a construction of the deserved combinatorial collection out of building blocks. These building blocks are typically the same building blocks arising in a counting formula.

Exercises. The exercises all involve building grammars for specified combinatorial collections.

Section 10.3: Finite-State Machines

This brief section takes a more theoretical approach to enumeration, by building finite-state machines to recognize sequences in a given combinatorial collection. This section would normally be skipped. Finite-state machines are much more complicated to design than grammars. Further, finite-finite machines are a basic topic that is carefully treated in theory of computation courses in computer science. However, most mathematics majors never will take such a course and finite-state machines (and automata theory) are one of the most mathematically rich areas of theoretical computer science.

Chapter 11 Games with Graphs

This chapter develops the theory of progressively finite games and applies this theory to find winning strategies in certain 2-person games. The ultimate goal is a winning strategy for the game of Nim. The two sections on progressively finite games, while recreational in spirit, contain some interesting theory. The definitions and proofs are all based on recursive constructions and thus have substantial pedagogical value, especially for computer scientists.

Section 11.1: Progressively Finite Games

This section introduces the basic concepts of progressively finite games: kernels, levels, and Grundy functions. Mie recursive constructions need to be constantly illustrated with specific examples. The existential proof of a kernel-based winning strategy and the inductive construction of a unique kernel are friendly examples of important proof techniques.

Exercises: Exercises 14 and 16 contain two important theoretical extensions (needless to say, the proofs are quite difficult).

Section 11.2: Nim-type Games

This section introduces the complicating generalization of direct sums of games, and with it the operation of digital sum. Notation gets quite involved and the proof of the Theorem is beyond the grasp of most undergraduates. However by learning how to play winning Nim, they get a feeling for the idea behind the proof.

Exercises: Exercise 6 is a concrete example of part b (the omitted part) of the proof of the Theorem.

Appendices

Appendices 1 and 3 have background material on set theory and probability. Appendix 2 presents induction, which should be review material but unfortunately is not for many students. Appendix 4 presents the Pigeonhole Principle, an important topic in combinatorial theory. Appendix 5 (new) gives a brief overview of computational complexity and NP-completeness. It is intended that these appendices will not normally be covered in class, although exercises from them might be assigned.

A.1: Set Theory

This section presents basic set-theoretic terms, notation, and operations. The set complementation laws (deMorgan's Laws) are stressed, since they will be used many times in chapters 5 and 8. Example I illustrates the 'real-world' issue of how much information is needed to solve a counting problem. This problem, like the problem of counting imprecise sets, is raised as a reminder to students that questions must be well posed before one can employ the combinatorial reasoning taught in this text.

Exercises: Note that in Exercise 2, some of the requested subsets cannot be generated (parts d and e). Exercise 5a consciously has contradictory data.

A.2: Mathemstical Induction

Anyone who takes a course in discrete mathematics should be exposed to induction arguments. While students may be asked to do few induction proofs themselves in this course, they at least should be developing a familiarity with induction arguments for later courses in computer science and other areas of discrete mathematics. If an instructor wants to put more emphasis on induction, extra examples (chosen from the exercises) can be done in class, and one or two induction exercises from this section can be assigned every week for the first month. Induction arguments are used in this text to prove Euler's formula for planar graphs (section 1.4), the 5-color theorem (section 2.4), a bound on the height of m-ary trees (section 3.1), the validity of Prim's minimal spanning tree algorithm (section 4.2), and several theorems in chapter 10, Games with Graphs.

Exercises: Exercise 15 proves the commutativity of integer multiplication, a well-known fact that few can prove (this fact also has a simple combinatorial proof, see Exercise 13 in section 6.3). Exercises 24 and 25 are classic examples of faulty induction arguments.

A.3: A Little Probability

The objective of this section is to present the definition of probability used in this text, namely, the fraction of 'favorable' outcomes. However, in giving this definition, we immediately raise the problem of distinguishing outcomes, or elementary events, from compound events. This problem becomes fairly subtle when identical objects are involved. A point that can cause some confusion is that in probability problems, objects are always distinguishable (even if they have the same shape and color), whereas in combinatorics one allows identical objects. Note that identical objects arise implicitly in some probability problems, such as counting all sequences with k (identical) heads and n-k (identical) tails when a coin is flipped n (distinguishable) times. Probabilities are used in Chapter 5 to give interesting settings to counting problems. Some advanced exercises in section 6.2 and 6.4 examine basic properties of probability generating functions.

A.4: Pigeonhole Principle

This brief section on the Pigeonhole Principle is independent of material in the rest of the text. The idea can be explained in half a minute in any subsequent course that may need it. However it is a traditional combinatorics topic whose advanced theory has blossomed in recent years. Some instructors may find this section appealing because Pigeonhole exercises require a type of creative abstract reasoning generally missing in the rest of the book. For a fuller, yet still elementary, discussion of the Pigeonhole Principle, see R. Brualdi, *Introduction to Combinatorics*, 3rd ed. (North Holland, New York, 1998).

Exercises: Exercise 18 is a famous Ramsey-theory question with a slick proof.

A.5 Computational Complexity and NP-Completeness

Many of graph problems studied in chapters 1 and 2 are NP-complete, e.g., edge coverings, existence of a Hamiltonian circuit, and minimal colorings. The instances studied of those problems were snall-sized and chosen to be tractable to ad hoc combinatorial analysis. It is important for students to be appeal how incredibly difficulty it is to solve large instances of such problems. This appendix, which was requested by several reviewers, provides the necessary background to appreciate what is means to say that a problem is NP-complete.

PART III: THREE SAMPLE COURSE SYLLABI

1. One-semester 14-week Course with 40% graph theory and 60% counting.

Week I- Sections 1.1, 1.2, 1.3

Week II- Sections 1.4, 2.1

Week III- Sections 2.2, 2.3

Week IV- Sections 3.1, 3.2

Week V- Sections 3.3, 3.4

Week VI- Review and Test 1
Note: extensive time for review before a test can be profitably used by students to work lots of problems from old tests or review sheets.

Week VII- Sections 5.1, 5.2

Week VIII- Sections 5.3, 5.4

Week IX- Sections 5.5, 6.1

Week X- Sections 6.2, 6.4

Week XI- Review and Test 2 (see note above with Test 1)

Week XII- Sections 7.1, 7.2, 7.3

Week XIII- Sections 8.1, 8.2

Week XIV- Sections 8.3 and Review

2. One-semester 15-week Course emphasizing Graph Theory or One-quarter Graph Theory Course (using first ten weeks)

Week I- Sections 1.1, 1.2

Week II Sections 1.3, 1.4,

Week III- Sections 2.1, 2.2

Week IV- Sections 2.3, 2.4

Week V- Review & Test
Note: extensive time for review before a test can be profitably used by students to work lots of problems from old tests or review sheets.

Week VI- Sections 3.1, 3.2, 3.5

Week VII- Sections 3.3, 3.4

Week VIII- Sections 4.1, 4.3 (up to Max Flow/Min Cut Algorithm)

Week IX- Sections 4.3, 4.4

Week X- Review & Test

Week XI- Sections 5.1, 5.2

Week XII- Sections 5.3, 5.4

Week XIII- Sections 6.1, 6.2

Week XIV- Sections 7.1, 7.2, 7.3

Week XV- Sections 8.1, 8.2

3. One-Semester Undergraduate Course emphasizing Counting or One-Quarter Enumerative Combinatorics Course (using first 10 weeks)

Weeks I - VIII follow Weeks VII - XIV in Course 1 (syllabus above), except in Week VII cover 7.5 and 8.1 and in Week VIII cover 8.2, 8.3

Weeks IX- Sections 9.1, 9.2, 9.3

Week X- Section 9.4 and Review

Week XI- Test and Section l.1

Weeks XI - XV follow approximately Weeks I - IV in Course 1 above, with time at end for review.

Part IV Solutions to Problems

Prelude Solutions

1. R Bu G Y;   2. Y W G R or W R Bu R;   3. Y Bu Bu W;   4. G Bk G Y or G W G Bu;

5. Bu Bk Bk Bu or Bk Bu Bu Bk;   6. Many possible guesses: possibilities are

Bu Bu Bu Bk or Bk Bu Bu Bu or W Y W W   7. G Y R Bk;   8. Bu O W Bk or Bu Bu Y O.

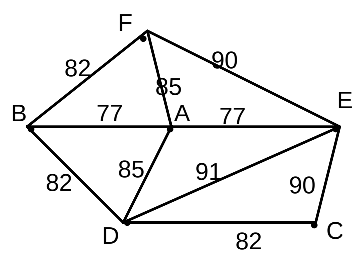
9. W W R Bu, W Bu R Bu, W Bu R W, Bk Bu Bk R, Bk Bk R Bu, Bu Bk R Bu, Bu Bu Bk R;

10. Bu O W Bk or Bu Bu Y O   11. O R Y Bu P;   12. W O G O Bu,

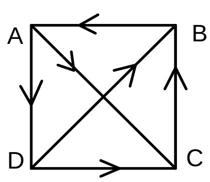
13a) Three black and one white, b) 14;   15. 9;   17. 1040;

19. Two of one color and one of a second color.

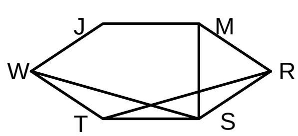
Chapter One Solutions

Section 1.1: 1a) 

b) 2 (C,D), (C,E), c) yes, several routes.

2a) 

b) ADCB, BADC, CBAD, DBAC, DCBA;   3a) a 5-circuit;

4a) 

b) 2 days, Mary,   c) yes- (John,Mary) or (John,Wendy);

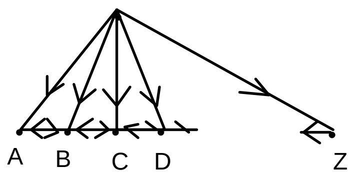
5a) many possibilities,   b) min = 4, several possibilities

6a) 5 vertices,   b) 9 vertices;

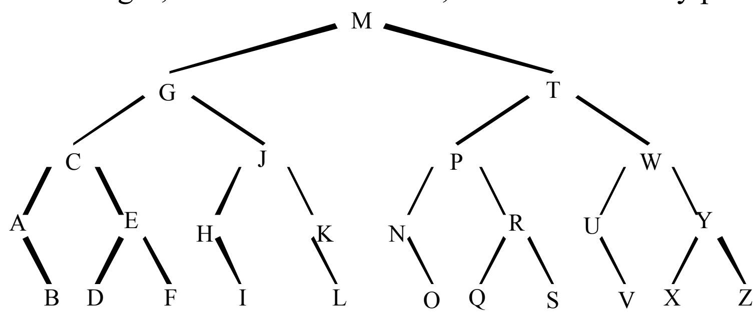
7. {*A*-*a*, *B*-*b*, *C*-*d*, *D*-*c*} (and other possibilities), b) *A* and *C* each can only fill job *b*;

9a) not possible (odd number of vertices;   c) *a*,*c*,*e* must be matched with just *b* and *e*;

Solutions

10a) 

b) add edges from c, k, and t to top vertex, shortest path has 11 edges, between z and n or o; 11. Many possibilities;

12a) 

b) 1;

13a) vertex = variety of chipmunk, if A splits into B and C, then make edges $(A,{}^{\to}C)$ and $(A,{}^{\to}C)$; b) 7 splits; 14. many possibilities;

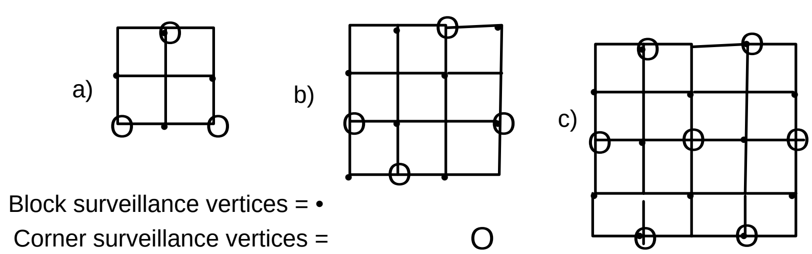
15a) 4 other pairs, b) $\{b,e\}$, $\{c,h\}$, $\{e,j\}$. $\{h,j\}$, $\{b,j\}$, $\{c,j\}$, $\{c,k\}$. $\{b,k\}$;

16a) (a,b), (a,g) and (c,d), (d,e) and (c,d), (e,f) and (d,e), (e,f),

b) $\{b,g\}$, $\{b,f\}$, $\{c,f\}$, $\{c,e\}$, $\{d,f\}$, c) 3 (the edges incident to any vertex);

17a) and b) C, E, c, d; 20. $\{c,e,k\}$,$\{b,h,i\}$;

19. minimal block surveillance- 6, minimal corner surveillance- 3;

20. 

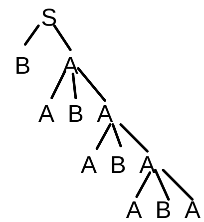
21a) 5: squares (2,4), (3,4), (4,4), (5,4), (8,4); b) 8; 22. 3;

23a) (i)$\{b,c\}$, (ii) $\{A,a,B,b,D,e\}$

b) (i) $\{a,d\}$, (ii) $\{C,c,d,E\}$;

26a) Directed graph with vertices named 0, 1, 2, ..., 16 and edges from i to i+1, i+2, i+3;

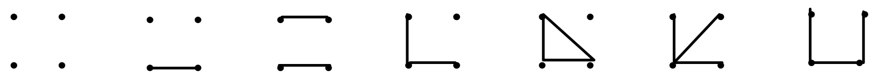
b) Winning positions are 4, 8, 12, 16; 27a)



b) not possible.

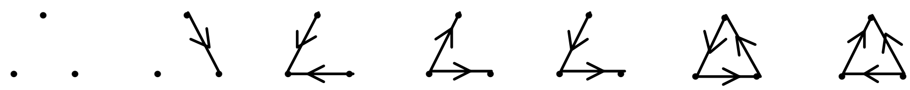
Section 1.2:

1.



plus complements of first 4 graphs

2.



3a) and b) Many possibilities;

4. No, the left graph has circuits of length 5 (e.g., (*a*,*b*,*c*,*d*,*e*,*a*)) while all circuits have even length in right graph; 5a) yes, *a*-1, *b*-5, *c*-4, *d*-3, *e*-2, *f*-6, b) yes, *a*-6, *b*-1, *c*-3, *d*-5, *e*-2, *f*-4, c) yes, *a*-3, *b*-4, *c*-7, *d*-1, *e*-6, *f*-8, *g*-2, *h*-5, *i*-9, d) no, right graph has one more edge than left graph, e) no, subgraphs of vertices of degree 3 do not match

f) yes, *a*-7, *b*-3, *c*-5, *d*-4, *e*-1, *f*-2, *g*-6, g) no, degree-2 vertex in a triangle only in right graph,

h) yes, *a*-1, *b*-3, *c*-5, *d*-2, *e*-4, f-6, i) no, degree-2 vertices are adjacent only in left graph,

j) no, degree-3 vertices are adjacent only in left graph,

k) yes, *a*-1, *b*-8, *c*-4, *d*-7, *e*-6, *f*-9, *g*-11, *h*-10, *i*-2, *j*-3, *k*-5, l) yes, *a*-1, *b*-2, *c*-3, etc.;

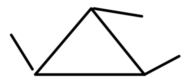
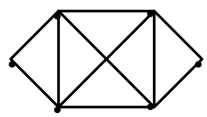
6a) no, only vertices 2,5 of degree 2, b) yes, *a*-5, *b*-6, *c*-2, *d*-1,

e-4, *f*-3, c) no, complement of left graph is 2 4-circuits, d) no, 5-degree and degree-3 vertices form a triangle only in right graph, e) No, only right graph has triangles,

f) yes, *a*-2, *b*-3, *c*-5, *d*-7, *e*-6, *f*-8, *g*-1, *h*-4, *i*-9, g) no, only left graph has triangles,

h) no, degrees do not match; 7. Graphs 1-6, 13-18, 31-36, 37-42 mutually isomorphic, 7-12, 19-24, 25-30 isomorphic; 8. Graphs 1-8, 9-16, 17-24 mutually isomorphic, other two distinct; 9. no, building on the isomorphism in Example 2, b → c but 5 ← 2;

10a) no, only vertex 5 has all edges directed inward, b) No, a vertex with out-degree 2, 6, has an edge going to a vertex, 1, with out-degree 2, but no such vertex in left graph;

12. no; 13. three in each; 14a)  b) no, deg-5 vertex adjacent to all others, deg-4 vertex adjacent to all but deg-1 vertex, and then deg-3 vertex cannot be adjacent to deg-1 or deg-2 vertices, c) 

Section 1.3: 1a) 12, b) 9, c) 8 or 10 or 20 or 40;

2a) vertex = person, edge = do not know each other, yes, b) vertex = person, edge = joining siblings, yes, c) no model, no, d) vertex = vertex in given graph, edge = joining *a* and *b* if *b* is adjacent to neighbor of *a*, i.e., there is a path of length two joining a and b— note that this "graph" can have loops (is a multigraph, see Section 2.1) but loops contribute 2, not 1, to a vertex's degree— no;

3. 12; 4. n is odd and so parity of degrees is preserved in complement— Answer: n - 1;

5. Solve for n in terms of m in the formula $m = n(n\ \ 1)/2$; 6. $n(n - 1)/4$;

7. if v vertices, then $e = \frac{1}{2}vp$ edges (where $\frac{1}{2}v$ is an integer since p is odd);

8. Make graph with vertex = team, edge = game. Within a given conference, the subgraph consists of 13 vertices, each of degree 11— contradicting Corollary;

9. sum of in-degrees (or out-degrees) = number of edges, since sum counts each edge once;

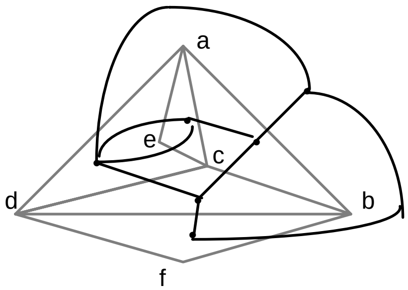
10a) (A,Z)-(C,X)-(E,Y)-(G,X)-(F,W)-(H,U)-(I,V)-(M,M)

b) (A,Z)-(B,X)-(C,Y)-(D,W)-(C,U)-(B,V)-(A,T)-(B,R)-(C,S)-(M,M);

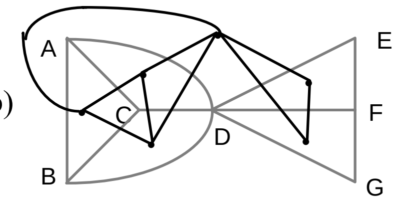
12. No way to get to such a vertex; 13a) no, b) yes, c) no;

14a) yes, {a,b,c,d}, {e,f,g,h}, b) no;   15. By same reasoning as in Parallel Climbers Puzzle, x and y must be in same component of G, and so adding edge (x,y) changes nothing.

Section 1.4: 1a)



b)



3a) yes,   b) no, delete a and (b,c), bipartition in $K_{3,3}$: {c,e,g},{d,f,h},   c) yes

d) no, delete (a,e), K_5 = {a,b,c,d,f},   e) no, delete (b,c), (e,f):{a,e,f},{b,c,d},

f) no, delete (b,j), (e,g): {a,d,h},{c,f,i},   g) no, many possible $K_{3,3}$,

h) no, delete (a,b), (b,c), (d,e), (f,g): {a,d,e},{c,f,g},   i) no, delete (d,e): {a,c,e},{b,f,g},

j) delete (f,g): {a,c,e},{b,d,h},   k) no, delete (a,b), (b,e), (c,f): {a,e,g},{d,f,h};

l) no, delete edges (c,d), (i,g):{a,h,j,{b,e,f };

4. interchange positions of c and d and of e and f in Figure1.19b;   5a) $n \leq 4$, b) r or $s \leq 2$;

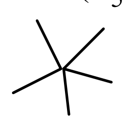
6. $K_{r,s,t}$ is planar if either i) two of r,s,t equal 1, or ii) r,s, and t are all ≤ 2;

7a) possible, $\mathbf{r} = 8$,   b) possible, $\mathbf{e} = 12$, c) not possible, $\mathbf{v} = 6$,   d) not possible, $\mathbf{r} = 10$,

e) possible, $\mathbf{v} = 7$,   f) possible, $\mathbf{e} = 12$, $\mathbf{r} = 8$,   g) not possible (parity violation),

h) possible, $\mathbf{e} = 10$, $\mathbf{v} = 7$,   i) possible, $\mathbf{e} = 20$, $\mathbf{r} = 10$,   j) not possible;

8. $n = 8$;   9a) degree of vertices in K_5 is 4 $\Rightarrow$ degree of vertices in $L(K_5)$ is 2 x (4 - 1) = 6,

$L(K_5)$ has $\mathbf{v} = 10$ and $\mathbf{e} = (1/2)\Sigma\deg = 3\mathbf{v} = 30 \Rightarrow \mathbf{e} > 3\mathbf{v} - 6$; b) 

10d) K_4;   11a) new circuit length = sum of two circuit lengths – 2x(no. of common edges in the two circuits, b) circuit length = sum of number of boundary edges of each enclosed region minus 2x(number of edges interior to circuit);   12a) 1, b) 1, c) 1, d) 2;

13a) $K_{3,3}$ and K_5 are critical nonplanar; 14a) if not triangular, an interior chord can be drawn; b) $3\mathbf{r}$ = sum of boundaries of regions $= 2\mathbf{e} \Rightarrow \frac{2}{3}\ \mathbf{e} = \mathbf{r} = \mathbf{e} - \mathbf{v} + 2 \Rightarrow$ $\mathbf{e} = 3\mathbf{v} - 6 = 3n - 6$ and $\mathbf{r} = \frac{3}{2}\ \mathbf{e} = \frac{3}{2}(3n - 6)$; 15a) $\mathbf{r} = \mathbf{e} - \mathbf{v} + \mathbf{c} + 1$

b) Using a), Corollary becomes $\mathbf{e} \leq 3\mathbf{v} - 3\mathbf{c} - 3$ $(\leq 3\mathbf{v} - 6)$; 16. The number of edges in G and $\overline{G}$ is 55 (=1/2 sum of degrees = 1/2(11x10)), but by Corollary $\mathbf{e} \leq 3(11) - 6 = 27$, and so if both are planar, G and $\overline{G}$ together can have at most 2x27 = 54 edges;

18a) If false, deg ≥ 6 and so $6\mathbf{v} \leq$ sum of degrees $= 2\mathbf{e} \leq 2(3\mathbf{v} - 6) = 6\mathbf{v} - 12$— impossible;

19. If false, deg ≥ 5 and so $5\mathbf{v} \leq$ sum of degrees $= 2\mathbf{e} \leq 2(3\mathbf{v} - 6) = 6\mathbf{v} - 12$, that is, $5\mathbf{v} \leq 6\mathbf{v} - 12$ or $12 \leq \mathbf{v}$— impossible; 22b) $\mathbf{s}$ = no. of squares, $\mathbf{v} - \mathbf{s}$ = no. of hexagons, $4\mathbf{s} + 6(\mathbf{v} - \mathbf{s})$ = sum of degrees $= 2\mathbf{e} \leq 2(3\mathbf{v} - 6) = 6\mathbf{v} - 12 \Rightarrow 4\mathbf{s} - 6\mathbf{s} \leq -12$ or $\mathbf{s} \geq 6$,

c) replace inequalities by equalities in b). 24d) (d_1,d_2) are (3,3)- K_4, (3,4)- cube, (4,3)- octahedron, (3,5)- dodecahedron, (5,3)- icosahedron;

25. $\mathbf{v} = p + 2l$, $\mathbf{e} = \frac{1}{2}\Sigma\text{deg} = 2p + 3l$, Answer $= \mathbf{r} - 1 = \mathbf{e} - \mathbf{v} + 1 = p+l+1$;

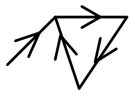
26. $\mathbf{v} \leq 2$ x (no. of pairs of mutually intersecting circles) = 12, $\mathbf{e} = \frac{1}{2}\Sigma\text{deg} = 2\mathbf{v}$, $\mathbf{r} = \mathbf{e} - \mathbf{v} + 2 = 2\mathbf{v} - \mathbf{v} + 2 = \mathbf{v} + 2 \leq 14$.

Supplement: 1. vertex = committee, edge = committee overlap (person). Graph is K_7 with $\mathbf{e} = \frac{1}{2}\Sigma\text{deg} = \frac{1}{2}(6\mathbf{v}) = 21$; 3. $n = 12$; 4. $\mathbf{e} \geq 11$;

5. If two vertices of degree 0, done; if one vertex of degree 0 ignore, have n vertices and degrees range from 1 to $n - 1$, that is, n numbers assuming $n - 1$ values $\Rightarrow$ some value must be repeated; 6. use same argument as in 5a); 7. G is union of circuits;

8. Use same argument as in 5a) but at i-th vertex on path pick an edge that does not go to

i-1, i-2, .. , i-(d-1)-th vertices; 9a) yes, trace out any sequence of edges and eventually a vertex z will be repeated, between first and second visit to z a circuit is formed,

b) no, e.g.,  10. If G is not a complete graph, there exists a pair u, v of non-adjacent vertices in G. Since G is connected, one can find a shortest path between u and v:

$u = x_1, x_2, x_3, , , x_k = v$. Then $x = x_2$, $y = x_1$, and $z = x_3$;

11. Vertices in different components of G are directly adjacent in $\overline{G}$, vertices in same component are joined in $\overline{G}$ by a path of length 2 via any vertex in other component of G;

12. *if*: a, b in different components of G-x, then x's removal disconnects G, *only if*: pick a, b in different components of G-x; 14a) possible, b) not possible;

15. *if*: suppose not strongly connected with no path from a to b– let V_1 consist of a and all vertices that can be reached by a directed path from a, V_2 is other vertices, *only if*: obvious;

16. With such a circuit, the removal of any edge-- on or not on the circuit—would not disconnect the graph; 17. Same reasoning as in Exer. 16.;

18. sketch of proof: find a circuit in G and direct edges consistently to yield initial strongly connected subgraph, then successively add (and consistently direct) side path from a vertex in the current strongly connected subgraph to another vertex in this subgraph;

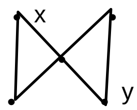
19a) yes, see Exer. 7 in Section 1.2, b) yes, odd number of vertices of odd degree

c) no; $\mathbf{e} = \frac{1}{2}\Sigma\deg = 13 \Rightarrow \mathbf{e} > 3\mathbf{v} - 6$; 20. Megabucks shakes 3 hands;

21. consider a vertex y which beat x, (i.e., $(\overrightarrow{y, x})$); if y beat every competitor that x then y would have a greater score than x— not possible and so for some w, $(\overrightarrow{x, w})$ and $(\overrightarrow{w, y})$;

22. if no directed circuits, them some vertex x_0 has rank 0 (see Exercise 9a) and so all other vertices have edges directed to x_0 and hence have rank ≥ 0; remove x_0 and remaining graph must have a vertex x_1 of rank 0; and continue process ;

23a) repeatedly remove side circuits until trail from x to y has no repeated vertices,

b) 

c) similar argument to part a), d) if odd number of edges of cycle are partitioned into circuits (part d)), then some circuit must have odd number of edges;

24a) Four mutually intersecting circles, b) Position circles at vertices in Fig. 1.3,

c) Two sets of 3 concentric circles, each circle in one set intersects all circles in other set;

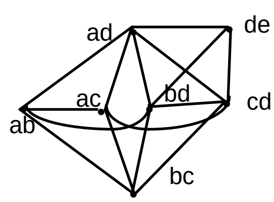
25a) (h,a), (h,g), b) circuit starts on one side of cut-set S and each time it crosses S it must cross back to finish on side where it started- even number of edges of S;

27. Following hint, edges in two complete subgraphs are maximized if one subgraph is K_{n-1} and other is isolated vertex (K_1)- K_{n-1} has $\mathbf{e} = \frac{1}{2}\Sigma\deg = \frac{1}{2}(n-1)(n-2)$. With more edges, graph must be connected; 28. Most edges in a bipartite n-vertex graph occurs if $n/2$ vertices in each part and all possible joining edges- $K_{n/2,n/2}$- which has $\mathbf{e} = (\frac{1}{2}n)^2 = \frac{1}{2}n^2$;

30. Pick any vertex, call it x. Assume by symmetry that a majority (at least 3) of edges incident at x are red, let 3 red edges from x go to vertices a, b, c, if there is a red edge between any two of a, b, c then red triangle (with x) results, otherwise a, b, c form blue triangle;

31a) 5-circuit or 3-edge path, b) By hint, if G has n vertices, number of edges in G = $\frac{1}{2}$(number of edges in K_n) $= \frac{1}{2}(\frac{1}{2}n(n-1)) = \frac{1}{4}n(n-1)$. n and n-1 are not each divisible by 2, and so either n or n-1 must be divisible by 4, i.e., $n = 4k$ or $n = 4k + 1$; 32. 7 isolated vertices;

33. *if*: obvious, *only if*: let x_n be vertex with 0 out-degree (if no such vertex, there is a directed circuit- Exer. 59), move x_n from graph and let x_{n-1} be vertex with 0 out-degree in remaining graph, continue indexing in this fashion; 34. Edge cover n, indep. set m.

35a) 

b) each edge of K_n is incident to n-2 other edges at each endvertex, 2(n-2) incidences in all,

c) can be no vertices of degree 0 or 1 and so all degrees ≥ 2; since $\mathbf{e} = \frac{1}{2}\Sigma\deg$, then to have $\mathbf{e} = \mathbf{v}$ (so that G and $L(G)$ to have same number of vertices), all deg = 2 ⇒ G is any circuit;

36. edges incident to a common vertex form a complete subgraph in $L(G)$, and each edge has two endvertices and the corresponding vertex in $L(G)$ is in two such complete subgraphs;

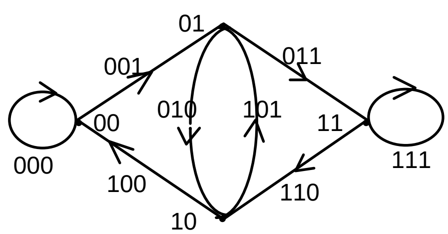
37a) *b* <--> *c*, *a* and *d* fixed, b) none, c) *a* <--> *c*, *e* <--> *f*, *b* and *d* fixed;

38a) and b)) are each bipartite graphs with different numbers of vertices on each side;

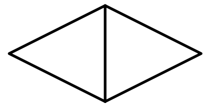
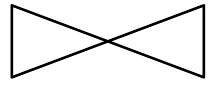
39. Sketch of proof: show that each vertex in $(C_1 \cup C_2) - (C_1 \cap C_2)$ has even degree and then repeatedly trace a path (without repeating any edge) until a vertex x is revisited and remove the circuit formed between the first and second visit to x; 40. sketch of proof: to maximize number of edges without a triangle build a complete bipartite graph; 2n-vertex complete bipartite graph has most edges with n vertices on each side, yielding n^2 edges.

Chapter Two Solutions

Section 2.1: 1a) many possibilities, b) many possibilities with *b* and *f* as end vertices; 2a) *n* odd, b) yes, K_2; c) *r* and *s* even 3. yes, make degree even at all other vertices and degree also even at new vertex (or else graph would have only one odd-degree vertex, violating Corollary in Section 1.3); 4. 2 times (to link two pairs of the 6 odd-degree vertices; 5a) No, once bridge crossed there is no way back to starting vertex, b) many possibilities, e.g., a 10-edge path; 6. for, the 0-edge path starting and ending at the single vertex fulfills all the requirements of an Euler cycle;
7. an isolated vertex added to a connected graph with even degrees now has an Euler cycle but is not connected; 8. add *k* edges joining pairs of odd-degree vertices so that the resulting graph has all even degrees, build Euler cycle, then remove *k* added edges from Euler cycle leaving *k* trails covering all edges of original graph;
9. build a graph with a vertex for each racer and an edge for each race, an Euler trail corresponds to a sequence of races in which each racer is in two consecutive races; this graph has the desired Euler trail because only A and F have odd degree;
10. No, build graph with vertex for each chessboard square and an edge joining two squares that are linked by a knight's move, the 8 vertices corresponding to squares on edge of board one away from corner have degree 3; 11. a set of deadheading edges must have one edge at each odd-degree vertex, joining these edges at odd-degree vertices together (without changing the parity of the degree of other vertices) requires a set of paths;
12a) trace a directed path (without repeated any edge) and must terminate at vertex you started, then add side tours by same process as in undirected case, b) at each vertex, pair off incoming and outgoing edges, forming a set of cycles, by connectedness of graph the cycles overlap and can be combined into one large cycle; c) many possibilities

13. A directed graph has an Euler trail if and only if at all but two vertices indegree = outdegree and at those two, indegree and outdegree differ by one; *proof*: add on extra edge so that indegree = outdegree at two unbalanced vertices and resulting graph has Euler cycle, remove added edge yielding desired Euler trail;   14. Euler cycle contains a trail from any vertex to any other vertex and every trail contains a path (earlier exercise);
15. no such Euler cycle containing all vertices;   16a) if edge E is incident to two vertices with degrees d, d' (both even degree), then E is incident to $(d\text{-}1) + (d'\text{-}1) = d+d' - 2$ other edges, and associated vertex for E in line graph has degree $d+d' - 2$ (even degree),
b) If every vertex has odd degree in G, then $d+d' - 2$ (as defined in part a)) will be even;
17a) many possibilities,   b) if at every stage graph of remaining edges is connected, then when just before using last edge E before being forced to stop at starting point, E is only edge remaining and once taken there are no remaining edges, c) applies to Euler trails;   18a) by using each edge twice, each vertex must have even degree,
b) sketch of proof: this rule can been shown to guarantee that Fleury's algorithm is being used (Exercise 17);   19.a) 

b) concatenating the first digits of each node on an Euler cycle produces desired sequence,
c) many possibilities, e.g., cycle 00-00-01-11-11-10-01 (-00) produces 00011101;

Section 2.2: 1. many possibilities, e.g., a)  , b) 
2. straightforward, many possibilities;   3. *a-g-c-b-f-e-i-k-h-d-j-a*;
4. each graph has many Hamilton paths-

Solutions

a) path: *b-a-c-e-d*; no circuit: rule 1 at *b*, *c*, *d* yields 3 edges at *a* and *e*,

b) path: *a-d-g-h-e-b-c-f-i*, no circuit: rule 1 at *d*, *e*, *f*, by symmetry at *b* choose *b-a* and delete *b-c* (rule 3), at *c* by rule 1 must use *c-a*, and now *a* has degree 3,

c) path: *f-a-d-g-b-e-h-c*, no circuit: rule 1 at *d*, *f* forces subcircuit *a-d-g-f-a*,

d) path: *7-1-2-3-4-5-6*; no circuit: rule 1 at *4*, must use 1-6- or else subcircuit 6-3-4-5-6, by same reasoning, must use 1-7 and 1-8 – 3 edges must be used at 1;

e) path: *a-b-g-i-f-e-d-h-c*, no circuit: rule 1 at *a* and *f*; now *b,h,i,e* can each take one more edge but *c*, *g*, *d* need 6 edges collectively connected to *b,h,i,e*;

f) path: *b-c-d-e-n-i-f-h-g-a-m-j-l-k*, delete at *a*, by sym., (*a*,*b*) => subcircuit (*b-c-e-d-b*);

g) path: *a-b-c-d-e-f-l-k-j-i-h-g-m*; no circuit: at *m*,by symmetry use *k-m-i*, delete *m-g*; at *j*, by rule 2, must use exactly one of *k-j* and *j-i*; by symmetry use *j-k* (and *j-d*) delete *j-i*; at *k*, delete *e-k* and *k-l*; forces *d-e-f* and *f-l-a*; delete *d-c* and *f-a*; subcircuit *a-b-c-i-m-k-j-d-e-f-l-g-a*; h) path: *d-i-e-j-h-c-g-k-f-a-b*, no circuit: three cases at *j*: 1) use *e-j-f* and delete *j-h, j-f*, forcing *i-h-c-g-k*; deleting at *c, b-c, c-d*; forcing subcircuit *k-g-c-h-i-d-a-b -k*; other two cases left to reader; i) path: *j-c-a-b-d-i-f-h-g-e*, no circuit: to visit *a*, *b*, require subpath *c-a-b-d* or *c-b-a-d* (or same path is reversed order), same argument applies to *f*,*i* and to *g*,*e*, combining these 3 types of subpaths yields a circuit excluding *j*;

j) path: *l-j-a-b-c-f-e-d-g-h-i-k-m*; no circuit: same reasoning as in Example 3,

k) path: *a-m-b-f-e-j-k-g-h-c-n-d-i*; no circuit: by rule 1 use *a-m-b*, *c-n-d*, *g-k-j*; at *g* by symmetry use *g-f*, delete *g-h*, forcing *i-h-c*; at *c*, delete *c-b*, forcing *b-f*; at *f*, delete *e-f*, forcing *j-e-a*, and subciruit *a-e-j-k-g-f-m-a*; l) path: *e-a-b-c-k-j-l-o-i-h-g-f-n-m-d*; no circuit: three cases at *o*, just 1) use *n-o-g* and delete *o-i*, *o-l*; use *m-l-j-i-k*; 3 edges at *j*;

m) path: *f-a-c-e-b-d-g-h-i*, no circuit: at *e*, 1) deleting *e-b*,*e-c* forces subcircuit *a-c-d-b-a*, 2) deleting *e-f*, *e-g* forces subcircuit *a-f-h-g-d-i-a*, 3) deleting *e-b*, *e-f* (or by symmetry *e-c*, *e-g*) and using *e-c*, *e-g* forces path *d-b-a-f-h*, delete *a-c*, *a-i* (rule 3), forces *c-d-i-h* (and more) using 3 edges at *d*, and finally 4) deleting *e-b*, *e-g* (or by symmetry *e-c*, *e-f*) and using *e-c*, *e-f* forces *a-b-d-g*, delete *d-c*, *d-i* (rule 3), forces *a-i*, *a-c* using 3 edges at *a*,

n) *g-a-b-d-c-f-e-j-k-q-p-o-n-l-m-h-i*, no circuit: use *h-i-j* (rule 1), two choices at *h* (vertical edge or *h-g*)- vertical edge, by symmetry choose *h-b* and delete *h-g*, *h-m* (rule 3), by rule 1 at *g* and *m* use *a-g-l-m-o*, delete *l-n*, at *n* use *o-n-q*, delete *o-p*, at *p* use *q-p-j*, delete *q-k* and *j-k* (now *k* has degree 1), similar reasoning when *h-g* is used at *h*;

o) path: *a-b-c-d-e-j-g-i-f-h*; no circuit: either 2 ro 4 edges used between outer pentagon (*a*, *b*, *c*, *d*, *e*) and inner star (*f*, *g*, *h*, *i*, *j*): case 1- 2 edges used: at least one edge of outer pentagon not used (rule 2), by symmetry *a-e* not used ⇒ use *b-a-f*, *d-e-j*; delete *b-g*, *c-h*, *d-i* (case 1 assumption) ⇒ use remaining edges at *b*, *c*, *d*, *g*, *g*, *h*, *i* ⇒ 3 edges forced to be used at *f* and *j*; case 2- 4 edges used: by symmetry assume *a-f* not used ⇒ by rule 1 use *e-a-b* and *h-f-i* ⇒ 2 edges used at *e* and *b*- delete *e-d* and delete *b-c* ⇒ at *c* and *d* must use *i-d-c-h* ⇒ subcircuit *i-d-c-h-f-i*,

p) path: *d-a-h-k-i-e-b-f-c-g-j*; no circuit: 3 cases at *f*: case 1) use *b-f-i* ⇒ delete *f-c*, *f-i* ⇒ use *a-c-g*, *g-j-k* ⇒ delete *g-h* ⇒ use *a-h-k* forming subcircuit *a-c-g-j-k-h-a*, case 2) use *b-f-c* ⇒ delete *f-i*, *f-j* ⇒ use *e*-i-*k*, *k-j-g* ⇒ delete *k-d*, *k-h* ⇒ use *e-d-a*, *g-h-a* ⇒ subcircuit *a-h-g-j-k-i-e-d-a*, case 3) use *b-f-j* ⇒ delete *f-c*, *f-i* ⇒ use *a-c*-g, *e-i-k*; now two subcases at *b*: 3a) use *b-a* ⇒ delete *b-c*, *a-d*, *a-h* ⇒ use *k-d-e* ⇒ subcircuit *d-e-i-k-d*, 3b) use *b-e* ⇒ delete *b-a* and *e-d* ⇒ use *a-d-k* ⇒ delete *a-h* ⇒ use *g-h-k* ⇒ subcircuit *a-c-g-h-k-d-a*;

5. path: *m-b-c-d-e-q-a-l-k-j-o-p-n-f-g-h-i*, no circuit: by symmetry at *p*, any delete any edge, choose *p-m*, at *m* and *c* rule 1 forces subcircuit *m-b-c-d-m*;

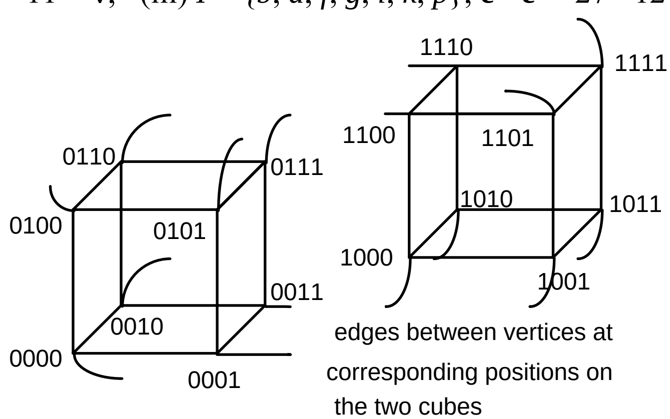
6. at *b* must use *b-g* or else subcircuit *a-b-c-h-g-f-a* is forced, also at *b* use *b-a* (symmetric

to *b*-*c*) and delete *b*-*c* and use *c*-*d*, delete *a*-*e* and use *d*-*e*-*j*, delete *f*-*g* (rule 2) and use *f*-*j*, now have subcircuit *a*-*b*-*g*-*h*-*c*-*d*-*e*-*j*-*f*-*a*; 7a) rule 1 at *f*, *h*, *j* forces subcircuit *e*-*f*-*g*-*h*-*i*-*j*-*e*, b) by symmetry at *p*, use *m*-*p*-*n* and delete *p*-*o* forcing *k*-*o*-*i*, at *m* and *n*, if both use edges going up (to *g*) then subcircuit *p*-*m*-*g*-*n*-*p*, if both *m*, *n* use edge down then subcircuit *p*-*m*-*k*-*o*-*i*-*n*-*p*, so by symmetry use *m*-*g* and *n*-*i*, delete *i*-*h*, *i*-*c*, *i*-*j* forcing *g*-*h*-*b*-*c*-*d*-*j*-*k*, plus *k*-*o*-*i*-*n*-*p*-*m*-*g*, yields a subcircuit;

8a) graph has just three 4-circuits which cannot be split evenly between inside and outside, b) graph has three 6-circuits and two 3-circuits, 6-circuits dominate and cannot be split, c) graph has nine 4-circuits which cannot be split evenly;

9a) Hamilton circuit alternates between red and blue vertices and so must have equal numbers of each, b) follows from part a), c) (i), (ii) have odd number of vertices, (iii) has 9 vertices in one part and 7 vertices in other part of bipartition;

10a) **e** - **e**' is number of edges that are available for a Hamilton circuit, must be enough to visit all vertices or else no Hamilton circuit can exist, b) if two vertices *x, y* of *I* are adjacent, then edge (*x*,*y*) contributes to count for both deg(*x*)-2 and deg(*y*)-2, c)(i) *I* = {*a*, *c*, *e*, *g*, *i*, *n*), **e** - **e**' = 24 - 12 = 12 < 13 = **v**, (ii) *I* = {*a*, *e*, *f*, *g*, *k*}, **e** - **e**' = 18 - 8 = 10 < 11 = **v**, (iii) *I* = {*b*, *d*, *f*, *g*, *i*, *k*, *p*}, **e** - **e**' = 27 - 12 = 15 < 16 = **v**;

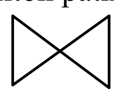
11a) 

b) many possibilities: 1-0000, 2-0001, 3-0011, 4-0010, 5-0110, 6-0111, 7-0101, 8-0100, 9-1100, 10-1101, 11-1111, 12-1110, 13-1010, 14-1011, 15-1001, 16-1000;

12. Follows immediately from interpretation of vertices of hypercube as binary n-tuples and x adjacent to y if they differ in one just bit. 13. See graph in Odd Solutions of text; 14. Exer 7 in Section 1.2 has all graphs (see Exer.12 in Section 1.2), all have Ham. circuits. 15. many solutions but very tedious, the following heuristic works: at each stage, look at all possible squares (not yet visited) a knight's move from current square and move to a square with the minimum number of possible squares for the next move;
16. associated graph has vertices = desks and edges = moves and is bipartite (see Exer. 9); solution forms a set of circuits which must use same number of 'red' and 'blue' vertices (see Exer . 9)— impossible since 25 vertices total;
17a) *rook*: starting at upper left corner, go down first column, square by square, at bottom of first column move right to bottom of second column, move up second column, square by square, continue going down and up successive columns finishing at top square of right column from which one moves back to top square of left column, *king*: similar to rook, except in columns 2, 3, . . . , n-1, avoid top square, when top square of right column reached, move left along top squares of each column to return to top left square,
b) *rook*: not possible (variation on reasoning in Exer. 14), *king*:(1,1)-(2,1)-(2,2)-(3,1)...(n,2), now go left and right covering rows 3 through n avoiding left square in each row except row n, finishes at (n,1) and now move up first column to starting square;
18. The graph is bipartite with 15 vertices, 8 on one side and 7 of the other, the middle inside cube and any corner cube are on different sides, but any Hamilton path would have to start and end on the side with 8 vertices; 19a) $n!$, b) form the Hamilton circuits by placing vertices in a circle: first circuit formed by joining consecutive vertices, second circuit formed by joining vertices two positions apart (one intervening vertex), third circuit formed by joining vertices 3 positions apart, and so on (no subcircuits because n prime);

c) form complete graph with professors as vertices, answer: 8 days, using part b) to form 8 Hamilton circuits out of the edges of K_{17};   20a) result immediate, b) initially form circuit of edges in $L(G)$ corresponding to edges on Hamilton circuit in G, if consecutive edges e, e' have common endvertex x, then any or all edges incident to x can be inserted between e and e' in the circuit in $L(G)$; c) many possibilities, e.g., 

21. if x, y nonadjacent, direct all edges inward at both x and y, now x and y can only be on a Hamilton path if they are the last vertex on the path- cannot be two last vertices;

22. Use the order of the vertices on a Hamilton path (guaranteed by Theorem 4) to provide the ranking;   23. many possibilities, e.g., 

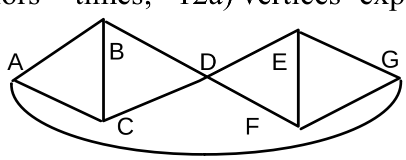
24a) proof by induction on n, easily verified for $n = 3$; assume true for n-1 and consider graph G with n vertices; observe that G is either K_n (which obviously has a Hamilton circuit) or must have a vertex x with $n/2 \le \deg(x) \le n$-2— if not, since G has $\binom{n}{2}+2$ edges, it can have at most one vertex y of degree $< n/2$ but if all other vertices have degree n-1, then these other vertices must all be adjacent to y and so $\deg(y) = n$-1; so the desired x exists and it can be checked that G-x has at least $\binom{n-1}{2}+2$ edges and so by induction G-x has a Hamilton circuit H', further x must be adjacent to two consecutive vertices in H' and can be inserted between them to form a Hamilton circuit in G;   b) consider K_{n-1} plus an edge from one vertex in K_{n-1} to an nth vertex- clearly no Hamilton circuit.

Section 2.3: 1a) 3, has K_3, b) 4, an attempt to 3-color outer 6 vertices gives b, d, f different colors so that c requires a fourth color, c) 2,   d) 4, a, b, e, g form a K_4, e) 2,   f) 3, odd circuit,   g) 4, isomorphic to graph in 1b),   h) 5, a,b,c,d,f form a K_5,

Solutions

i) 4, *a,c,g,i* form a K_4, j) 2, k) 2, l) 4, an attempt to 3-color vertices clockwise around the circle starting at *a* forces *j* to have same color as *a* m) 4, graph has 5-wheel, n) 4, graph has 5-wheel, o) 3, outer pentagon (odd-length circuit) cannot be 2-colored, p) 4, an attempt to 3-color the sequence of vertices *a, e, b, f, c, g, d* fails, q) 4, a 3-coloring forces *a* and *h* to have different colors, similarly for *i* and *p* so that *a, h, i, p* act like a K_4; 2a) 3, b) 3, c) 4; 3. (b), (g), (p), (q); 4a) $\{a, e, g\}$,$\{a, e, h\}$, $\{b, f, g\}$, $\{c, e, h\}$, b) *b, d, f*; 5a) $\{a,b,c\}$,$\{d,e,f\}$,$\{g,h\}$, b) $\{a,b,e\}$, $\{c,d\}$, $\{f,g\}$, c) $\{a,b,c\}$, $\{d,e,f\}$, $\{g,h,i\}$, $\{j,k\}$; 6a) Day 1: (a,b), (c,e), (d,f), day 2: (b,c), (a,d), (e,f), Day 3: (c,d), (b,e), (a,f), Day 4: (d,e), (a,c), (b,f), Day 5: (e,a), (b,d), (c,f), c) Edge chrom. no. of $K_n = n$ (*n* odd), $= n-1$ (*n* even). 7. no, Nevada (or Kentucky or West Virginia) and its neighboring states have duals that form odd-length wheels;

8a) assume map with *n*-1 circles has regions 2-colored, now in the interior of the *n*th circle complement the two colors (replace interior disk by 'negative' of current interior), b) give regions inside even number of circles one color and regions inside odd number of circles the other color; 9. 3; 10. vertices = animals, edges = incompatible animals, colors = open areas; 11. vertices = classes, edges = student in both classes, colors = times; 12a) vertices=experiments, edges=overlap in time, colors=observatories,

b) 

, 4 colors (with 3 colors, *A*, *D*, *G* must be same color);

13. vertices = ships, edges = overlapping visit, colors = piers; 14. vertices = banquets, edges = overlapping rooms, colors = days of the week 15a) yes, b) yes;

16. adjacent vertices A, B must have different level numbers, since one beat the other and so the winner must be at a higher level.

Section 2.4: 1. proceed as in Theorem 5 using induction and the fact that there is a vertex x of degree 5, but since 6 colors are available, x is immediately colored with a color different from those used by its 5 neighbors; 2. if bipartite, length of shortest circuit ≥ 4, then $\mathbf{e} \leq 2v - 4 = 12$; 3. The chromatic no. of G is the maximum of the chromatic nos. of its components; 4. False, e.g., any $K_{m,n}$ for $m, n \geq 3$; **5.** If the edge chromatic number is 2, the max degree id 2. If the vertex chromatic number is 2, then the graph is bipartite. Combining these two facts, each component of the graph must be a path or an even-length circuit. 6. Use induction argument with a splitting edge e as in the proof of Theorem 1, each of the two subgraphs, by induction, has at least two vertices of degree 2, although possibly one such vertex in each subgraph is incident to edge e (cannot be both endvertices of e or subgraph would not be triangulated), and so combined original graph has at least two vertices of degree 2;

7a) 3-color one triangle and then extend by successively coloring a vertex that lies on a triangle with 2 previously colored vertices, b) process in part a) yields unique coloring;

8a) the set of vertices of a given color form an independent set and so each color class has size $\leq q$; then (no. of colors)x(size of a color size) $\geq$ (no. of vertices), b) largest possible size of an independent set in a graph consists of a given vertex and all vertices not its neighbors, thus $q \leq (n\text{-}d)$ and result follows from part a), c) $K_{3,3}$ or any $K_{n,n}$;

9a) if not connected, then k colors are needed to color one of G's components and removing a vertex from another component will not reduce $\chi(G)$, b) if $\deg(x) \leq k\text{-}2$, then if G-x could be k-1 colored, also G could be k-1 colored by giving x one of the k-1 colors not used by one of x's k-2 (or less) neighbors, c) if x disconnects so that G-x has components G' and G' then one of $G'\cup x$ or $G''\cup x$ requires k colors and removing a vertex from the other component will not reduce $\chi(G)$; 10. follows immediately;

11a) for $n = 1$, $\chi(G) + \chi(\overline{G}) = 2$; assume for n-1 and consider an n-vertex graph G; by induction we can color G-x and $\overline{G}$ -x, for any given vertex x, with a total of n colors for both graphs (possibly less); x has a total of n-1 edges in G and $\overline{G}$ and so is not adjacent to one of the n colors classes in one of G or $\overline{G}$ and can be added to that color class, although in the other graph x may require an additional color— for a total of n+1, as required,
b) $\chi(\overline{G}) \geq$ size of largest complete subgraph in $\overline{G} = q$, size of largest independent set in G, and by Exercise 8a, $\chi(G)q \geq n$, c) square both sides of inequality, new inequality follows immediately from b) and fact that $a^2 + b^2 \geq 2ab$; 12. from Exercise 9b from Section 1.4, any circuit enclosing regions with even boundaries has even length, and so all circuits have even length and then by Theorem 2 of Section 1.3, graph is bipartite, i.e., 2-colorable;
13. $(k^2 - 6k + 8) = 0$ when $k = 4$ and so $P_k(G) = 0$ for $k = 4$, but the Four Color Theorem says that any planar graph can be 4-colored (i.e., has a positive number of 4-colorings);
14a) (i) $k(k\text{-}1)^3$, (ii) $k(k\text{-}1)(k\text{-}2)(k\text{-}3)$, (iii) $k(k\text{-}1)^3(k\text{-}2)$, (iv) $k[(k\text{-}1)^4\text{-} (k\text{-}1)^2\text{+} (k\text{-}1)(k\text{-}2)^2]$, v) $k(k\text{-}1)^2(k\text{-}2)\text{+}k(k\text{-}1)(k\text{-}2)^3$; 15. *if*: label with numbers that are the length of the longest path starting at the vertex, adjacent vertices must have diff. length longest paths since edge $(x,\vec{y})$ implies that x's longest path will be greater than y's longest path length,
only if: let colors be numbers 0, 1, . . , k-1 and direct edges from largest to smaller numbers:
16. mimic proof of Theorem 5, if x's 4 neighbors all have different colors do same 1-3 interchange between colors of two opposite neighbors of x, if interchange fails then do 2-4 interchange as in proof; 17. remove one vertex on each odd-circuit, the two vertices chosen to be nonadjacent (if this nonadjacency were not possible, there would have to be additional triangles); then the resulting graph has no odd circuits and can be 2-colored, now use third color for the two removed vertices;

18. The new graph G_{k+1} has a vertex adjacent to every possible combination of one vertex in each of k copies of G_k; if k colors are needed to color G_k, then in some vertex in G_{k+1} is adjacent to vertices with k different colors and so G_{k+1} requires k+1 colors.

Chapter 2 Supplement-- Instant Insanity: 1. G - 3 - B - 4 - G, R 1-loop, W 2-loop and G - 4 - R - 2 - W - 1 - B - 3 - G; 2. G - 2 - R - 4 - W - 1 - B - 3 - G with either G - 4 - R - 3 - W - 2 - B - 1 - G or G - 1 - B - 2 - G, R 3-loop, W 4-loop;

3. A, D or A, F or B, C or B, E or C,E or D, F, where A: G - 1 - R - 3 - W - 4 - B - 2 - G,

B: G - 2 - R - 4 - W - 1 - B - 3 - G, C: G - 1 - R- 3 - B - 4 - W - 2 - G,

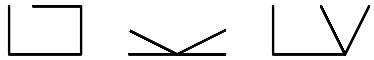
D: G - 2 - R - 3 - B - 1 - W - 4 - G, E: G - 4 - W - 3 - R - 1 - B - 2 - G,

F: G - 2 - W - 4 - R - 1 - B - 3 - G; 5a) yes, a Hamilton circuit is a factor,

b) having an Euler circuit has no relation to having a factor;

6a) (a, b, e, g), (c, d, h, f), b) none, c) (a, b, d, h, l), (e, f, g), (i, j, k).

Chapter Three Solutions

Section 3.1: 1a)  b) 

c)  2. 21

3. no odd circuits $\Rightarrow$ bipartite (Theorem 2 of Sect. 1.3) $\Rightarrow$ 2-colorable; 4. no circuits $\Rightarrow$ no $K_{3,3}$ or K_5 configuration (or just draw tree as in Figure 3.1 in planar fashion);

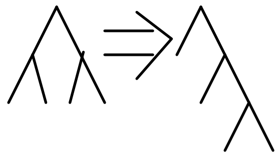
5a) no circuits means unique path from root to each vertex (two paths would yield a circuit), b) since G is connected, a subset of edges can be chosen to form a tree containing all vertices of G, *but* this subgraph has one fewer edges than vertices (by Theorem 1) and if G contained other edges, besides for those in this tree, it would have as many vertices as edges, c) if G has a circuit, removal of an edge on circuit does not disconnect $G \Rightarrow G$ has no circuits; now use part a); 6. $\mathbf{r}$ =1, and so $1 = \mathbf{r} = \mathbf{e} - \mathbf{v} + 2 \Rightarrow \mathbf{e} = \mathbf{v} - 1$;

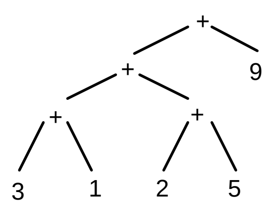
7. Start at any vertex and trace a trail (no repeated edges); since no vertex ever visited twice (if so, circuit would result) and graph is finite, trail must end at a vertex x of degree 1, now start trail-building again at x to get a second vertex of degree 1;

8. parts of Corollary all follow immediately from the formulas $n = l + i$ and $n = mi + 1$; e.g. for part b), given l, $l + i = n = mi + 1 \Rightarrow l - 1 = mi - i \Rightarrow i = (l - 1)/(m - 1)$ and $n = l + i = l + (l - 1)/(m - 1) = (ml - 1)/(m - 1)$; 9. height $h \Rightarrow$ each path to a leaf passes through at most h internal vertices with a choice of m children to go to at each internal vertex, for a total of at most m^h paths to a leaf; 10. $l \leq m^h \Rightarrow$ (by Corollary part b) $n = (ml - 1)/(m - 1) \leq (m^{h+1} - 1)/(m - 1)$; 11. i/n = (by Corollary part c) $\{(n - 1)/m\}/n = \frac{1}{m}\{1 - 1/n\} = 1/m$; 13. Largest n-1, smallest 2; 14. Since adding any edge (x,y) forms a circuit, there is already a path between x and y; so the graph is connected and circuit-free;

15. A tree is a 2-colorable (Exer. 3) and so at least n/2 vertices are in one color class;

16. $\mathbf{e} = \mathbf{v} - t$: for each tree, $\mathbf{e}_i = \mathbf{v}_i - 1$ and summing $\mathbf{e} = \Sigma\mathbf{e}_i = \Sigma(\mathbf{v}_i - 1) = n - t$;

17. Unbalancing a tree:  makes sum of level numbers larger and so smallest sum occurs when binary tree is balanced in which case each level number is $\lfloor \log_2 l \rfloor$ or $\lfloor \log_2 l \rfloor + 1$, and sum of level numbers is at least $l \lfloor \log_2 l \rfloor$;

18. If x is a center of T with neighbors $y_1, y_2, \ldots, y_m$ and two of the subtrees rooted at the y_i's contain leaves at the maximum level k, then x is the unique center (replacing x by any of the y_i's would yield a tree with height k+1; if only subtree of y_h has leaf at level k (leaves on other subtrees at smaller levels), then y_h could also be a center- some other y_g must have a leaf at level k-1 in T, now level k in new tree rooted at y_h, or else y_h, not x, is the unique center of T; 19a) internal vertices are +'s 

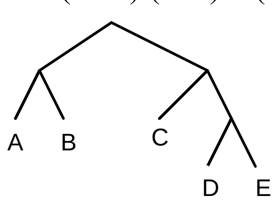
b) $\lceil \log_2 100 \rceil = 7$; 20. sequential search- check 1st item, then 2nd item, then 3rd item, etc.

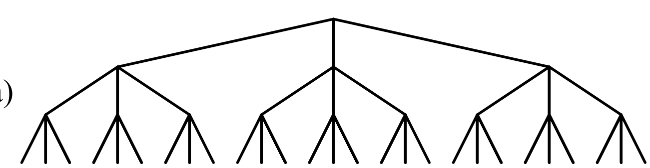
21. leaves = letters, each leaf = n-digit binary sequence $\Rightarrow$ 2^n letters;

22. the tournament winner retains his/her new can and so number of matches is n - 1;

23a) 24, b) 16, c) 12, d) 8 losers tournaments;

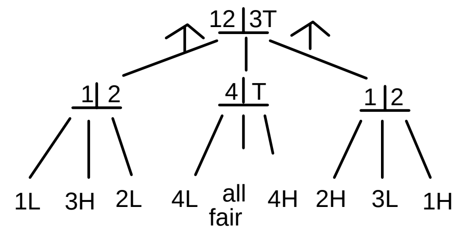
24. $n = (ml-1)/(m-1) = (5^7 - 1)/4 = 19531$, ans: \$5859.30; 25a) $\lfloor \log_2 \frac{n+1}{2} \rfloor = \lfloor \log_2(n+1) \rfloor - 1$,

b) 

26a) 

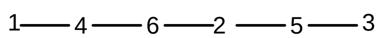
weigh 9 against 9, if right side rises up, take 9 on right side and apply weighings in Example 4; if left side rises, do same for left side, if balanced take other 9 with 3 vs. 3,

b) Inductively follow procedure in part a), initially dividing coins into 3 piles of 3^{n-1} and putting a pile on each side of scales; 27a)

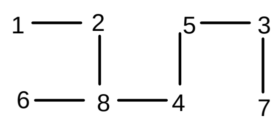


b) first weighing is either one coin on either side or two coins on either side, in either case some outcome has more than 3 possibilities that must be distinguished in the one additional weighing (impossible, for example with one on each side, there are 5 possibilities if the scales balance; 28. By Theorem 3, i = *mi* +1, *m* even => *n* odd.

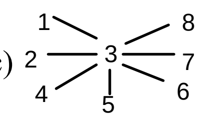
29a)



b)

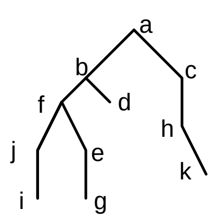


c)

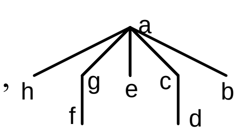


Section 3.2: 1a) any 8-vertex path, b) many possibilities, e.g., *a-b-d-c-h-k-i-g-e-f-j*, c) many possibilities, e.g., *a-b-c-d-e-f*-g-*h*; d) many possibilities, e.g., any 6-vertex path; 2a) a star (a vertex with 7 edges to other vertices) b) many possibilities,

e.g.,

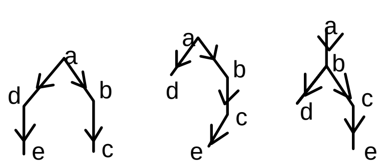


c) many possibilities, e.g.,



d) a 5-edge path centered at the root vertex with an additional pendant edge at the root; 3a) all trees on 5 vertices (see solution to Exercise 1b) in Section 3.1), b) all trees on 4 vertices (see solution to Exercise 1a) in Section 3.1),

c)



, places of *d* and *c* can be interchanged in last tree;

4. connected (lots of spanning trees); 5. 4 components, x_{17}, x_{19}, x_{23} isolated vertices,

depth-first spanning tree for other component has path x_2-x_4-x_6-x_3-x_9-x_{12}-x_8-x_{10}-x_5-x_{15}-x_{20}-x_{14}-x_{16}-x_{18}-x_{22}-x_{24}-x_{26}-x_{13}, plus edges from x_{20} to x_{25}, from x_{14} to x_7 to x_{21}, and from x_{22} to x_{11}; 6. subgraph generated by edges will be a set of spanning trees, since it has no circuits; but a spanning forest of k trees on n vertices has n-k edges ⇒ one tree;

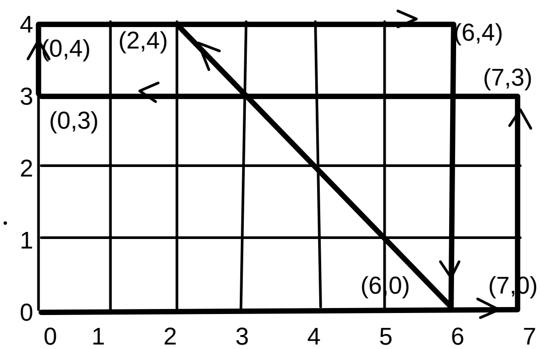
7. graph has no circuit, otherwise a second tree could be formed;

9a) if not all vertices reached, some reached vertex would have an edge to an unreached vertex, but a depth-first search would use such an edge, b) immediate;

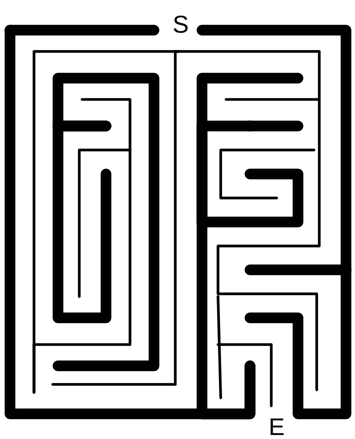
10a) breadth-first spanning tree finds shortest paths to all vertices, minimizing height

b) any tree with 2 or more edges; 11. if C has no edge of a spanning tree T, removal of C could not disconnect graph (spanning tree's edges connected graph);

12. several depth-first search paths, b) 4; 13.



14.



15. (0,0)-(0,4)-(4,0)-(4,4) —now 2 qts in 10-qt pitcher; 16a)(0,0)-(5,0)-(2,3)-(2,0)-(0,2)-(5,2)-(4,3), b) (0,0-(0,5)-(7,5);

17. right-hand-wall rule is same as depth-first search which takes leftmost branch at every intersection (corner); 18. Several possible solutions, one way continues search in

Example 2: (2,4)-(2,0)-(0,2)-(7,2)-(5,4)-(5,0)-(1,4); 19. Start with *D* (dog), *G* (goat), *T* (tin can), and *b* (boatsman) on near shore; take *G* across and return; then take *D* across and bring *G* back; then take *T* across, return, and take *G*;

20. Let wives be *A, B, C* and associated husbands be *a*, *b*, *c*. First *Cc* cross and *C* returns; then *ab* cross and *a* returns; then *BC* cross and *Bb* return; then *AB* cross and *c* returns; then *ab* cross and *a* returns; finally *ac* cross; 21. Not possible, 22. Start with *ABCDabcd* on near shore; *abc* across, *a* return, *ad* across yields *ABCD*/*abc*d; *a* returns, *BCD* cross yields *Aa*/*BCDbcd*; *Bb* return, *AB* across, *c* return and *abc* across completes crossing;

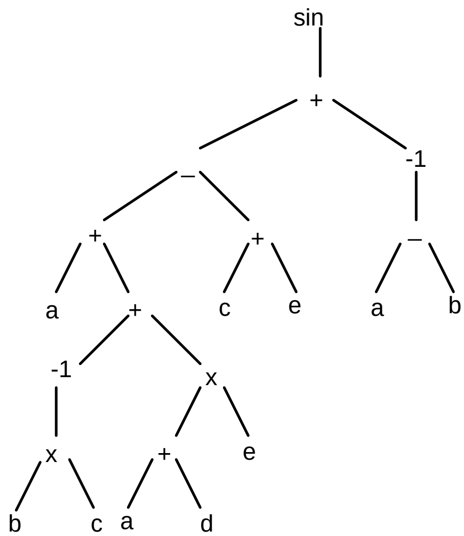
23 Let C_1 be the one cannibal that can row. Begin crossings as in part a), then when $M_1M_2C_1C_2/M_3C_3$ on near shore, perform round trip of M_1C_1 across and M_3C_3 return to obtain $M_2M_3C_2C_3/M_1C_1$. Now continue as in part a) with indices 1 and 3 interchanged.

24. start both 4-min and 7-min hourglasses; after 4 min (when smaller hourglass empties), restart 4-min hourglass; similarly, after 7-min, restart 7-min hourglass; at 8 min, turn 7-min hourglass which will run for one more min;

25. A and B cross (2 min.) and A returns (1 min.); next C and D cross (10 min.) and B returns (2 min.); finally A and B cross again (2 min.);

26a) *a*,*b*,*d*,*e*,*h*,*l*,*i*,*f*,*c*,*g*,*j*,*k* and *d*,*l*,*h*,*i*,*e*,*f*,*b*,*j*,*k*,*g*,*c*,*a*, b) *a*,*c*,*g*,*f*,*m*,*l*,*b*,*e*,*k*,*j*,*d*,*i*,*h* and *g*,*m*,*l*,*f*,*c*, *k*,*j*,*e*,*i*,*h*,*d*,*b*,*a*, c) *H*, *D*,*B*,*A*,*C*,*F*,*E*,*G*,*L*,*J*,*I*,*K*,*N*,*M* and *A*,*C*,*B*,*E*,*G*,*F*,*D*,*I*,*K*,*J*,*M*,*N*,*L*,*H*;

27.



28. if more than one circuit results, then some edge *e* is on one circuit and not on other circuit, but removal of *e* leaves a circuit— impossible since *n*-1-edge spanning subgraph of an *n*-vertex graph must be a tree (see solution of Exer. 5b) of Section 3.1);

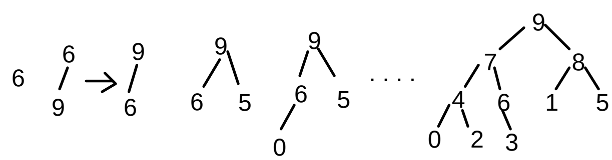
Section 3.3: 1. cost is 11: 1-3-2-4-1; 2. cost is 20: 1-4-2-6-3-5-1;

4. min tour is 1-4-3-2-5-6-1 with cost of 18. An "ideal" tour, call it S, would use the two cheapest edges at each vertex, i.e., in each row; this ideal tour costs 1/2(sum of the 2 cheapest edges at each vertex) = 16 1/2 for Figure 3.23 [divide by 2 since each edge counted twice]; however this set of edges uses 3 edges in first column and in third column; a feasible tour would have to use at least 2 second-choice entries producing a new "ideal" cost of 17 1/2, i.e., at least 18; 5. cost is 14: T_1-T_5-T_4-T_3T_2-T_1;

6. equivalent to traveling salesperson problem except subcircuits allowed, cost is 16: 1-2, 2-1, 3-4,4-3 (use subset of 0 entries in Fig. 3.16); 7a) 1-4-3-2, b) 1-4-5-3-2-6-1, c) 2-3-5-1-4-2; 8b) (i) 1-6-3-2-4-5-1, (ii) 6-5-2-3-4-1; 9.

$$\begin{array}{ccc} 2 & 1 & 2. \\ 2 & 2 & 2 \\ 2 & 1 & 2 \end{array}$$

Section 3.4: 1. Outcomes are ordered lists; 2.At each stage one compares first (smallest) remaining item on each list and selects the smaller to be the next item on the merged list; 3. A binary comparison tree has $n!$ outcomes or leaves (as noted at the beginning of this section); so average number of comparisons = average leaf level = (by Exer. 13 in Section 3.1) $\log_2 n! = n \log_2 n$; 4a) Assume a balanced heap (a heap that is a balanced tree) on the first k items has been built; attach (k+1)-st item (new leaf) as the child of an internal vertex with one child or else as the child of a former leaf at the largest level; now repeatedly compare new item with its parent and interchange the two items if parent is smaller; b) 1 part a)



5. If the initial heap is balanced (as described in Exer. 4a), then largest level number is $\log_2 n$ and the number of comparisons to adjust the heap each time the root is removed will equal the largest level number, i.e., $\log_2 n$; for n iterations of removing the root and readjusting the heap, there will be $n\log_2 n$ comparisons; constructing initial heap is similar;

6. In case of a tie on a comparison, either item can be considered the smaller item;

9b) $O(n\log_2 n)$;

Chapter Four Solutions

Section 4.1: 1. length 14: *c-d-h-k-j-m*; 2a) 19: *a-f-g-l-q-v-w-x-y*, b) 21: *d-c-h-m-r*, c) 15: *e-j-i-h-g*; 3a) 31: *L-c-d-f-g-k-W*, b) 32: *L-b-h-j-m-W*, c) 13: *L-a-c-d-f-g-k-W*, d) *L-c-d-f-g-k-W*, 4 paths; 4a) R=57: *L-c-d-f-g-k-W*, b) R=63: *L-c-d-f-i-k-W*;

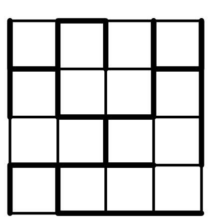
5a) 5: *L-b-h-j-m-W*, b) 6: *L-c-d-f-g-k-W*, c) 6: *L-c-e-g-j-m-W*;

6. 31: *L-c-d-f-g-k-W*; 7. If algorithm has found shortest paths to all vertices $\leq m$ units from a, then a vertex x will be distance m+1 from a if and only if the length $k(y,x)$ from a vertex y (y is closer to a than x) plus the distance $d(y)$ of y from a equals m+1— this is exactly the test that the algorithm performs; 8. Induct on k (i.e., shortest paths using the first k vertices as middle vertices on these paths), follows immediately; 9 Many possible examples E.g., a directed circuit from a to b to c to a with two edges of length 1 and the third edge of length –3; 10a) If multiple shortest paths, choice of which path depends on which vertex is used for first label (when more than one choice possible); order vertices so that desired vertex is first label in all cases where there is a choice;

11. Define a tree by letting the first label of a vertex be its parent;

Section 4.2: 1a) 59: path *L-a-c-d-h-f-g-i-k-W* plus edges (b,d), (d,e), (g,j), (k,l), (k,m), b) 60: replace (k,W) by (m,W), c) L,j (other possibilities), d) Modify part b) by deleting (a,c), (g,i), and adding (a,b), (g,k); 2. Same answer as in 1a)

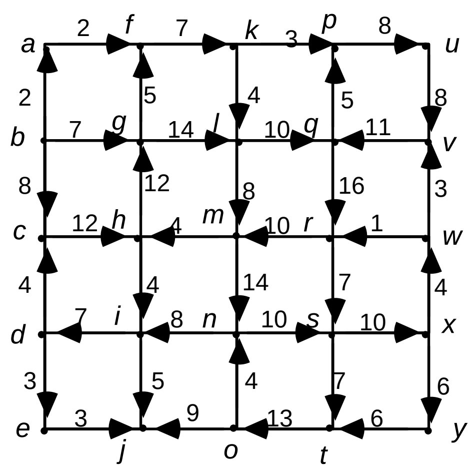
3ab) 39: path *N-b-c-d-e-g-j-m-R* plus edges (d,h), (e,f), (f,i), (j,k); 4. cost is 63;

5. 

6. Prim's algorithm gives minimal spanning tree and it will proceed in a unique fashion if all edges have distinct costs; 7. Let initial edge be prescribed edge and then continue with Prim's algorithm; 8. modification: add longest edge that does not form circuit; 10. See solution of Exer. 26 in Section 3.2; 11a) If the set of shortest edges do not form a circuit and they are not all in T', then add an omitted one and remove a longer edge in the resulting circuit (like in proof of Prim's algorithm) to obtain a shorter minimal spanning tree— impossible, b) Same reasoning as a), c) *if*: part b) is property of minimal spanning tree used to prove validity of Prim's algorithm; *only if*: verified in part b);

Section 4.3: 1. max flow = 13, $P=\{a,b,c\}$; 2a) max flow = 19, $P=\{a,b,d\}$, b) max flow = 24, $P=\{a,b,c\}$, c) max flow = 14, $P=\{a\}$; 3. max flow = 50, $\overline{P} = \{f,z\}$; 4a) max flow = 19, $\overline{P} = \{W\}$, d) Impossible, more flow must go into h than can leave h;

5a) max flow = 13, $P= \{a,b,c\}$, c)



6. Yes, build successive flow paths by choosing leftmost unsaturated edge leaving each vertex; 7. Set capacities of edges out of a and into z equal to 100 (equivalent to unlimited flow), max flow = 150, $P=\{a,b,c,d\}$; 8a) not possible, cut with $P=\{a,b,c,d,e,g\}$has capacity 50, b) now possible; 9. 5, Build paths by choosing leftmost unused edge leaving each vertex; 10. 15, Send 3 messengers along each path used in single-messenger problem; 11a) 3 (3 edges leaving L), b) 15, quintuple flow in part a); 12. split b, c,d each into

two vertices, one x_i for incoming edges and one x_o for outgoing edges with an edge of capacity 5 joining the two vertices; max flow = 12, P={a,b_i,b_o,c_i, c_o, d_i, e};

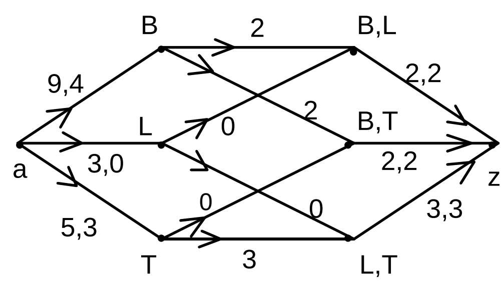
13. see Exer. 12. for modeling the vertex capacity constraint; max flow = 40, $\overline{P}$ = {f_i, f_o, g_o, z}; 14. 3; 15. max flow = 2100, route 400 on a_0-d_2-z_3-z_4, 300 on a_0-c_2-z_4, 400 on a_0-b_1-z_3-z_4, 200 on a_0-b_1-c_2-d_3-z_4, 400 on a_0-a_1-d_3-z_4, 400 on a_0-a_1-b_2-z_4; 16. max flow = 2200 (1100 leaving a on Monday and on Wednesday); 17. max flow = 36; 19a) Algorithm tries to reduce flow in incoming edge before adding flow to outgoing edge, b) yes; 21. define tree with a vertex's parent being its first label; 22a) No flow in only edges into a or only edges out of z, b) Maximum a-z flows need not violate condition (b) but maximum z-a flows could; 23a) Put flow of 1 in the two edges between c and e; other edges no flow, b) Take flow in a); and add a 2-unit flow path a-d-f-z; 26. Number of edge checks ≤ 2x(number of edges), at most, every edge is checked once at each endvertex; 27. If each flow path crosses a cut once, then the capacity of the cut = sum of values of such flow paths = value of flow; 28. A set of edges disconnecting a from z obviously cuts all flow paths from a to z; a min cut must be a cutset since it disconnects flow paths between P and $\overline{P}$ and no subset of min cut does this; 29a) This is max flow-min cut theorem for model in Example 6, b) Make edge capacities 1 and build vertex constraints of 1 (see Exer. 12), result is max flow-min cut theorem for this network; 30a) edge sets where in-degree = out-degree at each intermediate vertex, 31. One needs to build an initial feasible flow (requires at least one edge entering and one edge leaving each vertex, except a and z), then use same algorithm to reduce, instead of increase flow along a flow path; 32a) Have source with edge of capacity 1 going into

each vertex on one side of bipartite, have a similar sink for other set of vertices, and make edge capacities 0 in bipartite graph; 33a) In step 2a of augmenting flow algorithm, incoming flow cannot be reduced below its lower bound value, otherwise use same algorithm;

Section 4.4: 1a) several possibilities, see b), b) using pairings *A-G*, *Lo-J*, first labels are (except for *a*, all second labels are 1): *Bo-a*$^+$, *F-Bo*$^+$, *G-Bo*$^+$, *C-F*$^-$, *A-G*$^-$, *J-C*$^+$, *Bi-A*$^+$, *Lo-J*$^-$, *D-Bi*$^-$, *La-Lo*$^+$, *z-La*$^+$, new matching *A-G*, *D-Bi*, *Lo-La*, *Bo-F*, *C-J*;

2a) several possibilities, b) *E-a*$^+$, *e-E*$^+$, *f-E*$^+$, *A-e*$^-$, *D-f*$^-$, *a-A*$^+$, *c-A*$^+$, *d-D*$^+$, *z-a*$^+$; new matching: *A-c*, *B-b*, *C-a*, *D-f*, *E-e*; 3. Give edges from *a* and into *z* the appropriate supplies and demands. Middle edges still ∞, one solution: *A-Bi* (3 dates), *D-Bi* (2), *Bo-F* (1), *Lo-F* (2), *C-F* (1), *A-G* (1), *Bo-G* (2), *C-J* (3), *Lo-J* (2), *D-La* (3); 4. *A* - *Bi* & *G*, *Bo* -*F* & *G*, *C* - *F* & *J*, *D* - Bi & *La*, *Lo* - *J* & *La*; 5. no, schools with demand of 7 Ph.D.'s can only hire at most 6 (one from each univ.)

6. yes



7.

7a) Can be co-champions with either Lions or Tigers, b) Not possible; Lions and Tigers must play each other 3 times but each team can only win once;

8. Only possible to be co-champions, jointly with the other three teams.

9. Only possible to be co-champions, jointly with the other three teams.

10. Not possible; Lions and Tigers play each other 3 times, but each can only win once;

11ab) make complete bipartite graph, each vertex of degree *n*, by Example 3 there is a pairing for the first night, remove these edges, now again by Example 3, there is a pairing for the second night, continue in this fashion;

12a) none, b) many possibilities (easily solved by inspection);

13. Start with a standard set-of-distinct-representatives matching network; rplace the source a with 3 sources (one for each university), each with capacity $m/3$ and unit capacity edges to each university's graduates; 14. Make a bipartite-type matching network with X-vertices representing rows and Y-vertices representing columns; the edge from a to X-vertex i has a **lower** bound of (integer) k and an upper bound of k+1 if the sum of the entries in the i-th row is between k and k+1; the edge from Y-vertex j to z is similarly defined; and the edge (i,j) has a lower bound of (integer) k and upper bound of k+1 is entry i,j in the matrix has a value between k and k+1;

15. Necessary and sufficient condition: for any set S of vertices, $|S| \leq |s(S)|$, where $s(S)$ is the set of successors of vertices in S (vertices with an edge coming in from a vertex in S); this condition guarantees a complete matching in hinted bipartite graph, which corresponds to a set of edges in original graph with one edge out of and one edge into each vertex;

17. for each left-side vertex x, form a set of x's (right-side) neighbors;

18. the capacity of the cut is increased if vertex (t_i, t_j) is in P, since then the edge $((t_i, t_j), \vec{z})$ is in the cut;

Section 4.5

1a) $x_{12} = 30$, $x_{21} = 20$, $x_{22} = 10$, b) $x_{11} = 20$, $x_{12} = 10$, $x_{22} = 30$; 2a) $x_{12} = 30$, $x_{13} = 20$, $x_{21} = 20$, b) $x_{11} = 20$, $x_{12} = 30$, $x_{23} = 20$; 3a) $x_{11} = 30$, $x_{22} = 30$, $x_{31} = 10$, $x_{32} = 20$, b) $x_{12} = 30$, $x_{21} = 10$, $x_{22} = 20$, $x_{31}.= 30$; 4a) $x_{11} = 10$, $x_{12} = 20$, $x_{22} = 20$, $x_{23} = 10$, $x_{31} = 30$, b) $x_{12} = 30$, $x_{21} = 20$, $x_{22} = 10$, $x_{31}.= 20$, $x_{33} = 10$; 5a) $x_{13} =30$, $x_{21} =20$, $x_{23} =10$, $x_{32} =20$,

x_{33} =10, b) x_{12} =20, x_{13} =10, x_{21} =20, x_{23} =10, x_{33} =30;   6a)x_{12} = 10, x_{13} = 30, x_{21} = 30, x_{31} = 20, x_{33} = 10, x_{34} = 20, b) x_{13} = 40, x_{21} = 30, x_{31} = 20, x_{32} = 10, x_{34} = 20;

7a) x_{11} =30, x_{23} =30, x_{32} =10, x_{33} =20, x_{42} =30, b) x_{13} =30, x_{21} =30, x_{31} =10, x_{33} =20, x_{42} =30.

Chapter Five Solutions

Section 5.1: 1a) 20, 56, b) 6, 15; 2a) 41, b) 21, c) 36; 3. 7 x 10 x 6 x 4; 4. 10^{30};

5. 26^5, 26 x 25 x 24 x 23 x 22; 6. *n* x (*n* - 1); 7a) 9 + 7 + 5, b) 9 x 7 x 5,

c) 3 x {9 x 8 x (7 + 5) + 7 x 6 x (9 + 5) + 5 x 4 x (9 + 7)}; 8a) 14, b) 196, c) 192, d) 122;

9a) 4 x 47 = 188, b) 1 x 48 + 12 x 47 = 612; 10. 4 x 6 - 1; 11. 12; 12. 5^2 x 26^2,

26^2 x 21^2; 13. See Supplement at end of Chapter 5; 14. 6 x 5; 15. 52 x 48/52 x 51;

16a) 36^2, b) 1/36, $(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 5^2 + 4^2 + 3^2 + 2^2 + 1^2)/36^2$ = 146/1296,

d) $\frac{1}{2}(1 - \frac{146}{1296})$; 17a) 10 x 9 x 8 x 8, 10 x 9 x 8^{n-2}, b) 9 x 8 x 1/9 x 8 x 8,

c) 9 x 8 x 7 x 1)/9 x 9 x 8 x 8; 18a) 2 x 26^3 x 10^3, b) $(26 + 26^2 + 26^3)$ x (2 x 10 + 3 x 10^2 + 4 x 10^3); 19. 6 x 5 x 4 - 5 x 4 x 3; 20. $(11^2 + 10^2)/21^2$; 21. 6 x 5 x 4 - 4 x 3 x 2 or 2 x 3 x 4 x 3 + 3 x 2 x 4; 22. 3 x 2^9; 23a) 9^2 x 8 x 7, b) 7 x 8^3, c) 9^2 x 8 x 7 - 7^2 x 6 x 5;

24. 10^8; 25. 10 x 9 x $(2^5 - 2)/2$; 26. 4 x 3 x $8^2/10^4$ - first position 8 and then position 9; 27. 4^5 - 3 x 4^2; 28a) 7^{20}, b) 20 x 19 x 5^{18}; 29. 15 x 10 x (14 + 9);

30. See Supplement at end of Chapter 5; 31. 4 x 10^3; 32. (3 + 3 + 1)/80;

33. See Supplement at end of Chapter 5; 34. $(n + 1)^4$ - 1; 35. 2 x 25/50 x 49;

36. 2 x 85/100 x 99; 37. (3 x 3 x 2 + 3 x 3!)/6^3 - pick the possible smallest value (3 choices), count ways of picking middle value and then arranging them (3x2 possibilities when middle value same as largest or smallest); 38. 2 x 3 x 3 x 2 x 4 - 6; 39. 9 x 7 x 5 x 3;

40. 5+ 5^2+ 5^3+ 5^4+ 5^5; 41. 16x 8 x 7/2, *n* x*m* x(*n* + *m* - 2)/2; 42. 64^2- (4 x 4 +24x 6 +36x 9)/2, *n* x *m* (4 x 4 + 2(*n* + *m* - 4) x 6 + (*m* - 2) x (*n* - 2) x 9)/2; 43. $\frac{1}{2}$ {28 x (64 - 22) (Queen on edge of board) + 20 x (64 - 24) (Queen one away from edge of board) + 12 x (64 - 26) + 4 x (64- 28)}; 44. 2^7 - 1 - 7; 45. 2^{10}– 1, each friend is or is not in the subset,

minus case of no one invited; 46. $(1 + 2 \ldots + 8)^2$; 47. 2 x 3 x 6.

Section 5.2: 1. 52! 2. $P(26,8)$; 3. $P(15,7)$; 4. 8!, $2 \times 4!^2$; 5. $P(8,5)$;
6. $C(26,6)$; 7. $C(12,5)$, $C(10,3)$; 8. $C(13,5)$, $C(18,3) \times C(5,2)/C(13,5)$;
9. $[C(9,7) + C(9,8) + C(9,9)]/2^9$; 10. 2 x 4!/6!; 11a) C(10,2) x C(8,2) x C(6,2) x C(4,2),
b) 5!x 5!; 12a) C(14,3)x C(11,5), b) C(14,7)/2; 13. $5 \times C(4,2) \times 24^2$; 14a) $C(10,8)$, b) 2 x 8,
c) $C(5,3)$; 15. $C(n,8) \times 2^{n-8}$; 16a) $48/C(52,5)$, b) $13 \times 48/C(52,5)$, c) $C(13,2) \times C(4,2)^2 \times 44/C(52,5)$, d) $13 \times C(4,3) \times 12 \times C(4,2)/C(52,5)$, e) $10 \times 4^5/C(52,5)$, f) $4^5 \times C(13,5)/C(52,5)$;
17a) $C(4,2) \times \{C(6,4) + C(6,5) + C(6,6)\} + C(4,3) \times C(6,6)$, b) C(9,3) + C(9,4) + C(9,5),
c) C(10,5) – C(7,2), d) $\{C(4,2) \times C(6,2) - 3 \times 5\} + \{C(4,1) \times C(6,3) - C(5,2)\} + C(6,4)$;
18.a) $30 \times C(20,2)/C(50,3)$, b) $\{30 \times C(20,2) + C(20,3)\}/C(60,3)$;
19.a) $\{C(4,2) \times 9^2 + C(4,3) \times 9 + C(4,4)\}/10^4$, b) $C(10,2) \times (2^4 - 2)/10^4$, **c)** $C(10,4)/10^4$;
20a) $8!^3$, b) $3^3/8^3$; 21. 3! x 6! x 8! x 5!; 22. $13!^4/52!/4!^{13}$; 23. $C(21,5) \times 10!$, $2 \times 5!^2/10!$;
24. $1 - P(10,9)/10^9$; 25a) 2 x 5!/6! = 1/3, b) $C(6,2) \times 4!/6! = 1/2$ 26. $P(20,10)$;
27a) $C(n,3)$, b) $C(n-m,3) + m \times C(n-m,2)$; 28. $C(3,1) \times C(5,3)$; 29. 12
30. $P(7,4)$; 31a) 1-vote 1/6, 2-vote 1/2, b) 1-vote 1/6, 2-vote 1/3,
c) 1-vote 468/7!, 2-vote 1056/7!, d) 1-vote 396/7!, 2-vote 864/7!; 32. $C(13,3) \times C(4,2)^3 \times 44$;
33a) 26!/2, b) $26!/2^2$; c) C(26,5) x 21!; 34. C(5,2) x C(5,3) + C(5,3) x C(5,2) +
C(5,4) x C(5,1) + C(5,5); 35. C(5,3) x P(6,3) x P(21,3) + C(5,4) x P(6,4) x P(21,2) + 21 x 6!;
36. 2 x C(5,3) + 5C(4,2); 37. 4 x 3! x C(7,2) x C(5,2) x 3!; 38. 11!/4! - 10!/4
39. $C(30,5)^{12}/C(360,60)$; 40. $C(12,8) \times 30^8/C(360,8)$; 41a) {(3 x 4 + 2 x 5) x 8!}/10!,
b) {1 x 9 x 8! + 8 x 8 x 8!}/10!; 42a) $4 \times C(13,2) \times C(13,1)^3/C(52,5)$,

b) $\{4 \times 13^3 \times C(13,3) + C(4,2) \times C(13,2)^2 \times 13^2\}/C(52,6)$;

43a) $[C(4,1)^4 \times C(36,1) + 4 \times C(4,2) \times C(4,1)^3]/C(52,5)$, b) $\{C(26,5) + C(13,1)^2 \times C(26,3) + C(13,2)^2 \times C(26,1)\}/C(52,5)$; 44. $C(n,k) \times 10^{n-k}$; 45. See Supplement at end of Chapter 5; 46. $m = 12$; 47. $\{C(30,2) \times C(28,2) \times C(26,2) \times C(24,2)\}/4!$; 48a) 1/4, b) $C(6,3) \times 3! \times 5!/8!$; 49. $4 \times 3 \times 8!/2!2!$; 50. $5!/2! \times 7!/2!2!$; 51a) see Supplement at end of Chapter 5;

52. $C(k - 1,2) \times (20 - k)/C(20,4)$; 53. $C(64,8) \times C(56,8)$, $C(32,4) \times C(28,4)$;

54. $8! \times [C(21,8) + 5 \times C(21,7) + C(5,2) \times C(21,6) + C(5,3) \times C(21,5)]$;

55. See Supplement at end of Chapter 5; 56. $2 \times 2\ (n-k-1) \times (n-3)!$; 57. $C(6,3) \times C(4,2) + 2 \times [C(6,2) \times C(4,2) + C(6,3) \times C(4,1)] + 6 \times C(4,2) + C(6,2) \times C(4,1) + C(6,3)]$;

58. $C(5,2) \times C(11,3) + 5 \times \{C(11,4) - C(8,4)\} + C(3,3) \times C(8,2) + C(3,2) \times C(8,3)$;

59. See Supplement at end of Chapter 5; 60a) $1 - C(40,5)/C(50,5)$, b) $1 - \{C(40,10) + 10 \times C(40,9)\}/C(50,10)$, c) $P(40,k - 1) \times 10/P(50,k)$, d) $C(40,k - 10) \times (k - 1)! \times 10/P(50,k)$;

61a) $C(10,2) \times C(k-10,18)/C(k,20)$, b) $k = 100$; 62. $7 \times 6 \times 4$.

63. $C(12,5) \times C(4,2) \times 2 \times C(7,2) \times C(5,2) \times C(3,2)$. 64a) 2^{11}, b) 2^{11};

65. See Supplement at end of Chapter 5; 66. $C(100,m + n)$;

67a) $C(45,3) + C(45,2) \times C(45,1)$, b) $3 \times C(30,3)$ (three integers with same value mod 3) $+ C(30,1)^3$ (each different value mod 3), c) as in b), break into cases on the values mod 4): $C(22,3) + C(23,2) \times 22 + C(23,2) \times 23 + 23 \times C(22,2) + 22^2 \times 23$;

68. $\{C(4,2) + 1\}/C(11,2)$, $\{4 \times 4 \times 3 + 2 \times 4 \times 2 \times 1\}/C(11,3)$; 69a) $(2 \times 2^3 + 3 \times 2^2)/2^8$, b) $\{(2 \times 2^3 + 3 \times 2^2) + (2^2 + 2 \times 2) + (2 \times 2 + 1) + 2 + 1\}/2^8$ 70. $P(C(5,3),3)$, $P(C(n,3),3)$;

71. $C(8,3)$, $C(8,3) - 8 \times 5$; 72. $C(11,5)$; 73. See Supplement of Selected Solutions at end of Chapter 5; 74. $C(12,5) \times C(11,1) \times C(2,1) \times C(5,5) + C(12,4) \times C(11,2) \times C(2,2) \times C(5,4)$; 75a) $C(10,6)$, b) $C(10,6) + 5 \times C(10,5) + 4 \times C(10,4) + C(10,3)$; 76. 2^{n-1};

77. See Supplement of Selected Solutions at end of Chapter 5.

Section 5.3: 1. 5!/1!2!2!; 2. 11!/4!4!2!1!; 3a) 3^7, b) $(7!/3!2!^2)/3^7$; 4. 3 x 6!/3!3! + $6!/2!^3$ + 3! x (6!/3!2!1!); 5. $C(8 + 3 - 1, 8)$; 6. $C(4 + 6 - 1, 4)$; 7. $C(9 + 3 - 1, 9)$, subtract cases where one party has a majority (8 or more) $C(9 + 3 - 1, 9) - 3 \times C(3 + 3 - 1, 3)$; 8a) 9! x [1/5!2!2! + 2/3!4!2! + 2/4!3!2! + 1/3!3!3!], b) $C(2 + 3 - 1, 2)$; 9a) $C(8 + 3 - 1, 8)$, b) $C(15 + 3 - 1, 15) - C(4 + 3 - 1, 4)$; 10.a) $C(5+3 -1, 5)\times C(3+3 -1, 3)$, b) $3^5[3 \times (6!/4!1!1!) + 3! \times (6!/3!2!1!) + (6!/2!2!2!)]$; 11**.** $C(5,2) \times C(8 + 2 - 1), 8)$; 12. $\Sigma_k C(3,k) \times C((10 - k) + 3 - 1, (10 - k))$; 13. 2 x 6!/2!2!1!1!; 14. $C(r + n - 1, r)$; 15. $C(10,6) \times [C(6,2) \times 8!/2!2!1!^4 + 6 \times 8!/3!1!^5]$; 16. $C(5,2) \times C(5,1) \times 9! \times \{3/7! + 3!/6!2! + 3!/5!3! + 3/5!2!2! + 3/4!4!/ 3!/4!3!2! + 1/3!^3\}$ + $C(5,4) \times C(5,2) \times 9! \times \{6/4! + P(6,2)/3!2! + C(6,3)/2!^3\}$; 17. consider the cases of: (i) 4 of one letter and 1 of another letter, (ii) 3 of one kind and 2 of another letter, (iii) 3 of one letter and 1 of two others, (iv) 2 of two letters and 1 of another, and (iii) 2 of one letter and 1 of three others- 2 x 3 x 5!/4!1! + 2 x 2 x 5!/3!2! + 2 x $C(3,2)$ x 5!/3!1!1! + $C(3,2)$ x 2 x 5!/2!2!1! + 3 x $5!/2!1!^3$; 18. $C(8 + 25 - 1, 8)$, pick on even number of odd numbers $[C(8 + 12 - 1, 8) + C(6 + 12 - 1, 6) \times C(2 + 13 - 1, 2) + C(4 + 12 - 1, 4) \times C(4 + 13 - 1, 4) + C(2 + 12 - 1, 2) \times C(6 + 13 - 1, 6) + C(8 + 13 - 1,8)]/C(8 + 25 - 1, 8)$; 19. 3 x 10!/2!4!; 20a) 13!/5!4!3!1!, b) 13!/9!3!1!; 21. 10!/2!2! – 9!/2!2!; 22a) 3 x 10!/4!3!2!1! 23a) 10!/4!3!2!, b) 2 x 10!/4!3!2! – 9!/4!2!2!; 24. $C(50,r+s)$; 25. see Supplement of Selected Solutions at end of Chapter 5; 26a) 4^9, b) Consider sequence of first finger's rings, a slash, then second finger's rings, a slash, etc.- 12!/3!; 27. Counts all (1,2,3)-sequences of length10 two ways: left side looks at all cases of k_1 1's, k_2 2's, and k_3 3's, while right side counts all (unrestricted)

10-digit sequences of 1's, 2's, 3's; 28. See Supplement of Selected Solutions at end of Chapter 5; 29. see Supplement of Selected Solutions at end of Chapter 5; 30. $(10!/2!^5)/5!$ 31. See Supplement of Selected Solutions at end of Chapter 5; 32. count outcomes with one pair of vowels consecutive and subtract off outcomes with 3 vowels in a row $3 \times [7!/2! - 3 \times 2 \times 6!/2!]$;

33. must have *b* or *d* after each *c* (except possibly last *c*); sum is over number of *b*'s following *c*'s (first sum when last *c* followed by *b* or *d*, second sum when last *c* is at end of sequence:

$\Sigma\, C(3,k)21!/7!k!(3-k)!(8-k)!(6 - (3-k))! + \Sigma\, C(2,k)21!/7!k!(2-k)!(8-k)!(6 - (2-k))!$

34. See Supplement of Selected Solutions at end of Chapter 5.

Section 5.4: 1a) $C(36 + 4 -1, 36)$, b) 1, c) $C(32 + 4 -1, 32)$; 2a) 4^{16},

b) $C(4,2) \times 16!/6!6!2!2!$, c) $16!/4!^4$; 3a) $C(13,5) \times C(13,2) \times C(13,3)^2/C(52,13)$,

b) $[13!/5!5!2!1! \times 39!/8!8!11!12!]/52!/13!^4$, c) $4 \times C(48,9)/C(52,13)$,

d) $4! \times (13!/4!3!^3)^4/52!/13!^4$; 4a) $C(7 + 3 - 1,7) \times C(6 + 3 - 1, 6) \times C(7 + 3 - 1, 7)$,

b) $C(7 + 3 - 1,7) \times C(6 + 3 - 1, 6) \times C(3 + 3 - 1, 3)$;

5. $C(8 + 4 - 1,8) \times C(2 + 4 - 1, 2) \times C(6 + 4 - 1, 6)$; 6. $3 \times C(21,14) \times 2^7$;

7. $C((5-2) + 4 - 1, (5-2)) \times 5!$ 8. $5! \times C(6 + 6 - 1, 6)$; 9a) $C(9,4) \times 8!/2! \times 5!/2!2!$,

b) $C(9,4) \times 8!/2!$; 10. $C((21-4) + 6 - 1, (21-4)) \times 21! \times 5!$;

11. $C((14-5) + 7 - 1, (14-5))/C(20,14)$; 12a) $C(28 + 5 - 1, 28)$, b) $C(28 - 1, 5 - 1)$,

c) $C(13 + 5 - 1, 13)$; 13. $C(15 + 3 - 1, 15)$; 14. add 'slack' variable: $C(100 - 1, 5 - 1)$;

15a) $[52!/13!^4]/4!$, b) $[52!/8!^3 7!^4]/3! \times 4!$;

16. This strategy gives each person a distinct first book and then possibly additional books. Since the set of books a person receives in not ordered, this strategy violates the Set Composition Principle in Section 5.2; 17. $1 - [10! - 10 \times 9 \times (10!/2!)]/10^{10}$;

18a) $P(n, k)$, b) $C(n, k)$; 19a) $4^3 \times C(9 + 4 - 1, 9)$, b) $P(4,3) \times C(9 + 4 - 1, 9)$,

c) Distribute teddies and fill out each child with lollipops: 4^3;

20. C(30,20) x C(20,15); 21. 4 x C(4 + 4 - 1,4);

22a) C(15 + 5 – 1,15), b) C(10 + 5 – 1, 10); 23. 2 x C(7 + 4 – 1, 7) - 1;

24. $C((7\text{-}3) + 5 - 1, (7\text{-}3)) \times 7!/13!/4!2!2!$; 25. $C((7\text{-}3) + 5 - 1, (7\text{-}3)) \times 7!/4!2!1!$

26. $\Sigma\ C(15\text{-}3k) + 4 - 1, (15\text{-}3k))$ 27. See Supplement of Selected Solutions at end of Chapter 5; 28. See Supplement of Selected Solutions at end of Chapter 5;

29. $[10!/2!^5]/5^{10}$; 30a) $\Sigma_{k=0}^{7} (35 - 5k + 1)$, b) $\Sigma_{k=0}^{7} (35 - 5k + 1) + \Sigma_{k=0}^{2} (10 - 5k + 1)$,

31a) distributions of 8 distinct items into 3 boxes, b) distributions 9 distinct items into 3 boxes with two items in the first box, etc. 32a) arrangements of length *n* of *n* letters with repetition, b) arrangements of 15 letters using 3 letters of each of 5 types;

33a) distributions of 6 identical objects into 31 boxes, $\Sigma_{i=1}^{31} x_i = 6$, b) distributions of 5 identical objects into 3 boxes with at most 5 objects in first box, etc., $\Sigma_{i=1}^{3} x_i = 5, x_1 \leq 5$, $x_2 \leq 4$, $x_3 \leq 2$; 34a) selection of 30 objects from 5 types, $\Sigma_{i=1}^{5} x_i = 30$, b) selection of 18 objects from 6 types with at least 2 of each type, $\Sigma_{i=1}^{6} x_i = 18, x_i \geq 2$;

35. $C(30 + 3 - 1, 30)$, $3 \times C((30\text{-}16) + 3 - 1, (30\text{-}16))$; 36a) $C(30 + 5 - 1, 30)$, b) $5 \times [C(32 + 4 - 1, 32) + C(31 + 4 - 1, 31)]$, c) $C(40 + 5 - 1, 40) - 5 \times C(19 + 5 - 1)$, d) $C(5,2) \times [\{\Sigma_{k=9}^{13} C((40\text{-}3k) + 2 - 1, (40\text{-}3k))\} - 2 - 2 \times C(4 + 2 - 1, 4)]$; 37a) $C(7 + 4 - 1, 7)$, b) $C(7 + 5 - 1, 7) + 1$, c) $C(13 + 4 - 1, 13) - 4 \times C(3 + 4 - 1)$; 38. $C(55 + 5 - 1, 55)$;

39. $\Sigma_{i=0}^{6} C(k + 2 - 1, k) \times C((12\text{-}2k) + 2 - 1, (12\text{-}2k))$; 40. $C(7 + 3 - 1,7) \times C(13 + 3 - 1, 13)$;

41. $\Sigma_{k=0}^{7} C(7 + 3 - 1, 7) \times C((20\text{-}k) + 4 - 1, (20\text{-}k))$; 42a) $C(20 + 5 - 1, 20) - 5 \times C(9 + 5 - 1, 9)$, b) $C(20 + 5 - 1, 20) - 5 \times C(11 + 5 - 1, 11) + C(5,2) \times C(2 + 5 - 1, 2)$,

c) $\Sigma_{i=0}^{6}$ $C((20\text{-}3k)+3\text{-}1, (20\text{-}3k))$; 43a) 5 x 6 x 8, b) 5 x 6 x 8 - 2; 44. 4!x7!/2!x C(5 + 3–1, 5);

45. 4! x C(4 + 5 - 1,4); 46. 8![C(21,8) + 5 x C(21,7)] + P(5,2) x P(21,6) x C(5 + 3 – 1, 5) + P(5,3) x P(21,5) x C(3 + 4 – 1, 3); 47. C(7 + 4 – 1,7) x 9!/6!; 48. 2 x C(1 + 3 – 1, 1) x 5!/2!; 49. C(3 + 4 – 1, 3) x 5!/3!; 50a) 9!/2!2! - 9!/4!, b) 2 x 5! x {*C*(3 + 5 - 1, 3) + *C*(4 + 5 - 1, 4) + *C*(5 + 5 - 1, 5)} c) 2 x 5! x {2 x *C*(5 + 5 - 1, 5) + 2 x *C*(4 + 5 - 1, 4) + *C*(3 + 5 - 1, 3)};

51a) 5! x *C*(9,5), b) 21! x 5! x *C*(22,5); 52. See Supplement of Selected Solutions at end of Chapter 5; 53. $\Sigma_{k=0}^{13}$ $C(13,k) \times C(39,13-k) \times C(26+k,k) \times C(26,13)$; 54. $16!/4!^4 \times 36!/9!^4 +$ $C(4,2) \times 16!/5!5!3!3! \times 36!/8!8!10!10! + 4 \times 3 \times !16!/5!4!3!3! \times 36!/8!9!9!10! + 4 \times 3 \times$ $16!/6!4!3!3! \times 36!/7!9!10!10! + 4 \times 16!/7!3!^3 \times 36!/6!10!^3$ 55. Σ 15!/*a*!*b*!*c*! summing over all *a*, *b*, *c*, 3-tuples where *a* + *b* + *c* = 15 with no letter greater than 7;

56a) $\Sigma_{k=0}^{10}$ $C((20\text{-}2k) + (m\text{-}2) - 1, (20\text{-}2k)$, b) $\Sigma_{k=0}^{10}$ $C(20,k) \times C(20 - k,k) \times (m\text{-}2)^{20\text{-}2k}$;

57. $3! \times 25!/16!8!1! + 3 \times 25!/14!7!4! \times 2^4 + 3 \times [25!/12!6!7! \times (2^7 - 2) - 25!/6!6!1!] +$ $25!/10!5!^3$; 58. $C(n,m) \times C(r + m - 1, r - n + m)$; 59. See Supplement of Selected Solutions at end of Chapter 5; 60. 13!/6!7! x [*C*(10 + 5 -1, 10) + 2 x 3 x *C*(11 + 4 - 1, 11) + 5 x *C*(12 + 3 - 1, 12) + 14]; 61. See Supplement of Selected Solutions at end of Chapter 5; 62. $C(40,k) \times C(13 - 1, k - 1) \times 13! \times 39!/52!$;

63. $C((n\text{-}2m) + (2m+2) - 1, (n\text{-}2m))$ {simplifies to $C(n + 1,2m + 1)$; 64. See Supplement of Selected Solutions at end of Chapter 5; 65. $\sum_{k=0}^{5} C(k + m - 1,k) \times C((r\text{-}k) + (n\text{-}m) - 1, (r\text{-}k))$;

64. $C(13,11) \times P(13,11)^4 \times [C(4,2) + (4!/2!) \times 2^2]$.

Section 5.5: 8. $n = 4$; 11b) $C(n + 1,2) + 6 \times C(n + 1,3) + 6 \times C(n + 1, 4)$;

12a) $72 \times C(n + 2,4)$, b) $4 \times n + 21 \times C(n + 1,2) + 18 \times C(n + 1,3)$,

c) $(n - 1) \times C(n + 1,2) - 2 \times C(n + 1,3)$; 13a) $2^n + n \times 2^{n-1}$, b) 1.5×2^n;

14a) 0, b) $n \times (n - 1) \times 2^{n-2}$, c) 3^n, d) $3 \times n \times 4^{n-1}$, e) 0, f) $(2^{n+1}1\text{-}1)/(n + 1)$, g) $(n + 1) \times 2^n$; 20a) $k = n/2$ (or nearest integer), b) $k = n$,

21. $\frac{1}{2} C(2n + 2, n + 1) - C(2n, n)$; 23. 0.

Chapter Six Solutions

Section 6.1: 1a) 7 products—$xxxx$, $x^3 11x$, $x^3 1x1$, $x^3 x11$, $1x^3 1x$, $1x^3 x1$, $xx^3 11$,

(b) 5 products—$1x^4$, xx^3, x^2x^2, x^3x, $1x^4$, (c) 7 products—$x^4 111$, $1x^4 11$, $1x^3 x1$, $1x^3 1x$, $x^3 1x1$,

$x^3 11x$, $11x^2x^2$, d) 15 products- $x^4 11$, $x^3 x1$, $x^3 1x$, $x^2x^2 1$, $x^2 xx$, $x^2 1x^2$, $xx^3 1$, $xx^2 x$, xxx^2,

$x1x^3$, $1x^4 1$, $1x^3 x$, $1x^2x^2$, $1xx^3$, $11x^4$; 2a) $(1 + x + x^2 + x^3 + x^4)^5$,

b) $(x + x^2 + x^3 + x^4)^3$, c) $(x^2 + x^4 + x^6 + x^8)(x^3 + x^5 + x^7)(x^2 + x^3 + \ldots + x^8)^2$,

d) $(1 + x + x^2 + \ldots)^4$, e) $(x + x^2 + x^3 + \ldots)^2(x + x^3 + x^5 + \ldots)(x + x^3)$;

3a) $(1+x + x^2 + x^3 + x^4 + x^5)(1 + x + x^2 + x^3 + x^4)^2$, b) $(x + x^2 + x^3 + x^4 + x^5)(x + x^2 + x^3 + x^4)$x

$(x + x^2 + \ldots + x^7 + x^8)$, c) $(1 + x + x^2 + \ldots)^4$, d) $(x + x^3 + x^5 + \ldots)^2(1 + x + x^2 + \ldots)^5$;

4a) $(1 + x + x^2 + x^3 + x^4 + x^5)^5$, b) $(x^3 + x^4 + x^5 + x^6)^4$, c) $(x + x^3 + x^5 + \ldots)^7$,

d) $(1 + x + \ldots + x^5)(1 + x + x^2 + \ldots)^2$; 5. $(1 + x + x^2 + \ldots)^2(1 + x)^2$, coef. of x^6;

6. $(1 + x + x^2 + \ldots)^5$; 7. $(x^3 + x^4 + x^5 + \ldots)^4$, coefficient of x^{18};

8a) $(1 + x + x^2 + \ldots)^4$, coefficient of x^{27}, b) $(x + x^2 + \ldots)^4$, coefficient of x^{27},

c) $(1 + x + x^2 + \ldots + x^{13})^4$,coefficient of x^{27}, 9. $(1 + x + x^2 + \ldots)^n$;

10. $(1 + x)^u(1 + x + x^2)^v(1 + x + x^2 + x^3)^w$; 11. $(1 + x^2 + x^4 + \ldots)(x + x^3 + x^5 + \ldots)$x

$(1 + x + x^2 + \ldots)^{n-2}$; 12. $(x^{r_1} + x^{r_1+2} + \ldots + x^{s_1})(x^{r_2} + x^{r_2+2} + \ldots + x^{s_2}))(+ x + x^2 + x^3)^{q-2}$;

13. $(x + x^2 + x^3 + x^4 + x^5 + x^6)^n$; 14a) $(x + x^3 + x^5)^3(x^2 + x^4 + x^6)^3$,

b) $\prod_{i=1}^{6} [(x + x^2 + x^3 + x^4 + x^5 + x^6) - x^i]$; 15. $(x^{-3} + x^{-2} + x^{-1} + 1 + x + x^2 + x^3)^4$;

16. $(1 + x + \ldots + x^9)^6$; 17. $(1 + x^5 + x^{10} + \ldots)^8$; 18a) coef. of x^{20} in $(x^2 + x^3 + \ldots + x^7)^5$,

18b) $x^{10}(1 + \ldots + x^5)^5$, corresponds to first picking 10 balls (two in each box);

19. $(1 + x + x^2 + \ldots)(x + x^2 + x^3 + \ldots)^4(1 + x + x^2 + \ldots)$,

coefficient of x^{15}, generally x^{n-5}; 20. r cannot appear in generating function;

21. cannot have a variable number of factors; 22. define $f_1, f_2, f_3\ f_4$ so that $e_1 = f_1,\ e_2 = f_1 + f_2,\ e_3 = f_1 + f_2 + f_3,\ e_4 = f_1 + f_2 + f_3 + f_4$, now $e_1 + e_2 + e_3 + e_4 = 4f_1 + 3f_2 + 2f_3 + 1f_4$;

23. $1 + x + x^2 + \ldots)(1 + x^5 + x^{10} + \ldots)(1 + x^{10} + x^{20} + \ldots)$;

24a) coef of x^{20} in $(1 + 5x)^{50}$, b) $(1 + 5x + C(5,2)x^2 + C(5,3)x^3)^{50}$;

25. $(1 + x + x^2 + \ldots)^5(1 + y + y^2 + y^3)^5$; 26a) $\{(x^3 + x^4 + x^5)(y^3 + y^4 + y^5)(z^3 + z^4 + z^5)\}^n$, b) $\{(x^3y^3 + x^4y^3 + x^5y^3 + x^4y^4 + x^5y^4 + x^5y^5)(z^3 + z^4 + z^5)\}^n$, c) $(x^3y^3 + x^4y^4 + x^5y^5)^3$ x $(x^3 + x^4 + x^5)^{n-2}(y^3 + y^4 + y^5)^{n-2}(z^3 + z^4 + z^5)^n$, d) $\{x^3z^4 + x^3z^5 + x^4z^3 + x^4z^5 + x^5z^3 + x^5z^4)$ x $(y^3 + y^4 + y^5)\}^n$; 27. $(xy + xz + yz)^8$; 28a) $(x_1 + x_2 + \ldots + x_m)^n$, b) $(1 + x_1{}^2 + \ldots + x_m{}^2)^n$,

c) $\prod_{i=1}^{n} \{1 + x_i + x_i{}^2 + \ldots + x_i{}^i\}$; 29. $C(p,n)$, $n!$

Section 6.2: 1. $C(9 + n - 1, 9)$; 2. $C((r - 35) + 7 - 1, (r - 35))$; 3. $C(m,8) + C(m,6) + C(m,4)$;

4. $C(7 + 6 - 1, 7) - C(2 + 6 - 1, 2)$; 5. $C(9 + 4 - 1,9) - 4 \times C(3 + 4 - 1, 3)$;

6. $C(21 + 3 - 1, 21) - C(6 + 3 - 1, 6) - C(5 + 3 - 1, 5)$; 7. $C(13 + 7 - 1, 13) - 7 \times C(8 + 7 - 1, 8) + C(7,2) \times C(3 + 7 - 1, 3)$; 8. $C(15 + 9 - 1, 15) - 9 \times C(10 + 9 - 1, 10) + C(9,2) \times C(5 + 9 - 1, 5) - C(9,3)$; 9. $C(14 + 4 - 1, 14) - 4 \times C(7 + 4 - 1, 7) + 6 \times 1$;

10. $C(24 + 6 - 1, 24) - 6 \times C(17 + 6 - 1, 17) + C(6,2) \times C(10 + 6 - 1, 10) - C(6,3) \times C(3 + 6 - 1,3)$;

11a) $C(10 + 10 - 1, 10)$, b) $C(10 + 4 - 1, 10) - 3 \times C(11 + 4 - 1, 11)$, c) 0,

d) $C(11 + 2 - 1, 11) + 3 \times C(12 + 2 - 1, 12)$, e) $b^{12} \times C((12,m)$;

12a) $(1 - x^{28})/(1 - x^4)$, b) $x^{20}(1 - x^{180})/(1 - x^{20})$; 13. 0;

14. $C(6 + 7 - 1, 6)$ 15a) 0, b) 1, c) $C(12 + 8 - 1, 12)$, d) $4^{12} \times C(12 + 5 - 1, 12)$,

e) $C(4 + 4 - 1, 4)$; 16. $10!/2!3!5!$; 17a) $(x^2 + x^3 + x^4 + \ldots)^3$, $C((10-6) + 3 - 1, (10-6))$,

b) $(1 + x + x^2)(1 + x + x^2 + \ldots)^2$, $C(10 + 3 - 1, 10) - C(7 + 3 - 1, 7)$,

c) $(1 + x^2 + x^4 + \ldots)(1 + x + x^2 + \ldots)^2$, $\Sigma\, C((10\text{-}2k) + 2 - 1, (10\text{-}2k))$;

18a) $C(r - 8, 8 - 1)$, b) $C(r/2 + 8 - 1, r/2)$, n even; 19. $C(12 + 5 - 1, 12) - 5 \times C(7 + 5 - 1, 7) + C(5,2) \times C(2 + 5 - 1, 2)$; 20. $C(10 + 5 - 1, 10) - 3 \times C(6 + 5 - 1, 6) + 3 \times C(2 + 5 - 1, 2)$;

21a) $C(15 + n - 1, 15) + C(10 + n - 1, 10)$, b) $C(n,15) + C(n,10)$;

22. $C(15 + 10 - 1, 15) - 10 \times C(9 + 10 - 1, 9) + C(10,2) \times C(3 + 10 - 1, 3)$;

23. $C(8 + 7 - 1, 8) - 7 \times C(3 + 7 - 1, 3)$; 24. $\sum_{i=1}^{13} (-1)^i \times C(50,i) \times C((40 - 3i) + 50 - 1, (40 - 3i))$;

25. $C(6 + 3 - 1, 6) - C(3 + 3 - 1, 3) - C(2 + 3 - 1, 2)$; 26. $[C(15 + 6 - 1, 15) - 6 \times C(9 + 6 - 1,9) + C(6,2) \times C(3 + 6 - 1, 3)]/2$; 27. $C(14 + 10 - 1, 14) - 4 \times C(10 + 10 - 1, 10) - 6 \times C(7 + 10 - 1, 7) - C(4,2) \times C(6 + 10 - 1, 6) + 4 \times 6 \times C(3 + 10 - 1, 3) + 4 \times C(2 + 10 - 1, 2) + C(6,2)$; 28. $\{C(17 + 9 - 1, 17) - 9 \times C(11 + 9 - 1, 11) + C(9,2) \times C(5 + 9 - 1, 5)\}/C(25,8)$;

29.$[C(10 + 4 - 1, 10) - 4 \times C(4 + 4 - 1, 4)] \times [C(15 + 4 - 1, 15) - 4 \times C(9 + 4 - 1, 9) + C(4,2) \times C(3 + 4 - 1, 3)]$; 30. $(1 - x - x^2 - x^3 - x^4 - x^5 - x^6)^{-1} = 1 + (x + x^2 + x^3 + x^4 + x^5 + x^6) + (x + x^2 + x^3 + x^4 + x^5 + x^6)^2 + \ldots$ 33a) $C(m + n, m + r)$, b) 0, if r odd, $C(n, r/2)$, r even, c) 3^n; 34a) $C(n_2 + 6, n_2) - C(n_1 + 6, n_1)$, b) $C(n + 1, m + 1)$; 35b) 2,

36. (5) is not defined for $x = -1$; 38c) $(\frac{1}{2} t)^5/(1 - t)^5$, d) $(pt)^m/(1 - qt)^{m-1}$; 39b) $n/2$, pn, 10, m/p; 40b) $pn + p^2 + p^2n^2$ or $pqn + p^2n^2$, 60; 37. $P_X(t) = (\frac{1}{2})^m\left(\frac{1 - (t/2)^{s+1}}{1 - (t/2)}\right)^m$.

Section 6.3: 1. a) 5 partitions- 4, 3+1, 2+1+1, 2+2, 1+1+1+1, b) 11 partitions- 6, 5+1, 4+2, 4+1+1, 3+3, 3+2+1, 3+1+1+1, 2+2+2, 2+2+1+1, 2+1+1+1+1, 1+1+1+1+1+1;

2a) $\frac{1}{1 - x^2}\ \frac{1}{1 - x^4}\ \frac{1}{1 - x^6}\ \ldots$, b) $(1 + x)(1 + x^3)(1 + x^5)\ldots$

3. $(1 + x + x^2 + x^3)(1 + x^2 + x^4 + x^6)(1 + x^3 + x^6 + x^9) \ldots$;

4a) $\dfrac{1}{(1 - x^2)(1 - x^3)(1 - x^7)}$, b) $\dfrac{x^{16}}{1 - x^2}(x^6 + x^9 + x^{12} + \ldots + x^{24})(1 + x^7 + x^{14})$;

5. $\dfrac{1}{(1 - x)(1 - x^5)(1 - x^{10})(1 - x^{25})}$; 6a) $\dfrac{1}{(1 - x)(1 - x^2)(1 - x^3)}$, b) $\dfrac{1}{(1 - x)(1 - x^2) \ldots (1 - x^n)}$;

7b) $(1 + x)(1 + x^2)(1 + x^3) \ldots = (\dfrac{1 - x^2}{1 - x})(\dfrac{1 - x^4}{1 - x^2})(\dfrac{1 - x^6}{1 - x^3})(\dfrac{1 - x^8}{1 - x^4}) \ldots$
= (after canceling) $\dfrac{1}{(1 - x)(1 - x^3)(1 - x^5)\ldots}$; 19a) multiply the partition generating function (given just before Example 1) times $\dfrac{1}{1 - x}$, b) $(x^3 + x^6)/\{(1-x^2)(1-x^4)(1-x^6)\}$, c)same as b); 21a) let the number of dots forming the first row and first column (= $2k$ -1 if first row and column have length k) in a self-conjugate Ferrers diagram be the length of the first row in the distinct, odd-parts diagram; delete the first row and column of the self-conjugate diagram and use the number of dots in the reduced self-conjugate diagram to define the second row of the distinct, odd-parts diagram, etc.

Section 6.4: 1. $(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!})^5$; 2. $(\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!})^6$; 3. $(1 + x)^5 e^{21x}$;

4a) e^x- 1$)^n$, b) $\Sigma\, (-1)^k \times C(n,k) \times r^{n-k}$; 5. $(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!})^{13}$, coefficient of $x^{13}/13!$;

6. $4^8 - 3^8 + 8 \times 3^7$; 7a) $1/2(3^r + 1)$, b) $1/4(3^r + 2 + (-1)^r)$, c) $3^r - 2 \times 2^r + 2$;

8. $(4^r - 3^r - 2^r + 1)/2$; 9a) $22^{10} + 4 \times P(10,1) \times 22^9 + 6 \times P(10,2) \times 22^8 + 4 \times P(10,3) \times 22^7 + P(10,4) \times 22^6$, b) $26^{10} - 4 \times 25^{10} + 6 \times 24^{10} - 4 \times 23^{10} + 22^{10}$;

10a) $3^r - 3 \times C(r,2) \times 2^{r-2} + 3 \times C(r,2) \times C(r - 2,2)$, b) $\frac{1}{4}(3^r - 4 \times 2^r + 6 + (-1)^r)$; 11. $\frac{1}{2}4^r$;

12. $(x + \frac{x^3}{1!2!} + \frac{x^4}{1!3!} + \frac{x^5}{1!4!} + \frac{x^5}{2!3!} + \frac{x^6}{2!4!} + \frac{x^7}{3!4!})e^{3x}$; 13a) $e^x - 2 \times e^{(n-1)x/n} + e^{(n-2)x/n}$,

b) $[n^r - 2(n - 1)^r + (n - 2)^r]/n^r$; 14. $C(n,m) \times (e^x - 1)^{n-m}$; 15a) $e^x - 1)/x$, b) $(1 - x)^{-1}$;

21. $e^{\mu(x-1)}$; 22c) $m_1 = n/2$, $m_2 = m_2^* = (n^2 - n)/4$, d) $m_1 = m_1^* = \mu$.

Section 6.5: 1a) $x/(1 - x)^2$, b) $13/(1 - x)$, c) $3x(1 + x)/(1 - x)^3$, d) $3x/(1 - x)^2 + 7/(1 - x)$,

e) $4!x^4/(1 - x)^5$; 2a) $C(n + 1,2)$, b) $13(n + 1)$, c) $\dfrac{(2n + 1)n(n + 1)}{2}$, d) $C(n + 1,2) + 7(n + 1)$,

e) $4! \times C(n + 1,5)$; 3. $x(3-x)/(1 - x)^3$; 4a) $r^2 = P(r,2) + P(r,1)$, $r^3 = P(r,3) + P(r,2) + P(r,1)$,

b) $\dfrac{9x^3}{(1 - x)^4} + \dfrac{8x^2}{(1 - x)^3} + \dfrac{2x}{(1 - x)^2}$; 5a) $(4x^2 - 3x + 1)/(1 - x)^3$, b) $\log_e(1-x)$;

6. $a_r - a_{r-1}$; 8. $\dfrac{h(x)}{1 - x^{-1}}$ or $\dfrac{xh(x)}{x - 1}$.

Chapter Seven Solutions

Section 7.1: 1. $a_n = 4a_{n-1}$, $a_1 = 4$; 2a) $a_n = a_{n-1} + a_{n-2} + a_{n-4}$, b) $a_5 = 10$;

3. $a_n = 2a_{n-1} + a_{n-2}$; 4a) $a_n = a_{n-3} + a_{n-5} + a_{n-10}$, b) $a_{12} = 1$;

5. $a_n = 2a_{n-1} + 2a_{n-5} + a_{n-10} + a_{n-25}$; 6a) $a_n = a_{n-1} + a_{n-2}$, b) $a_n = 2a_{n-1} + 2a_{n-2}$,

c) $a_n = a_{n-1} + 2a_{n-2} + 2a_{n-3} + 2a_{n-4} + \ldots$ 7. $a_n = a_{n-1} + a_{n-2}$; 9. $a_n = a_{n-1} + a_{n-2}$;

10. $a_n = a_{n-1} + a_{n-2}$; 11. $a_n = a_{n-1} + a_{n-2}$; 12. $a_n = a_{n-1} + 2n - 2$;

13a) $a_n = a_{n-1} + n$, $n > k$, $a_k = k+1$, b) 43; 15. $a_n = 1.06(a_{n-1} + 50)$;

16a) $a_n = 1.08a_{n-1} - 3000$, b) 21 years; 17. $a_n = 2a_{n-1} + 2a_{n-2} + 2a_{n-3}$.

18. $a_n = 2a_{n-1} + a_{n-2}$; 19a) $a_n = a_{n-1} + a_{n-3} + a_{n-5}$, b) $a_n = a_{n-1} + a_{n-3} + a_{n-5} - a_{n-6}$,

c) $a_n = a_{n-1} + a_{n-3} + a_{n-5} - a_{n-9}$ 20a) $a_n = 2a_{n-1} + a_{n-2}$, b) $a_n = a_{n-2} + 2a_{n-3} + 4a_{n-4}$,

c) $a_n = 3a_{n-1} - a_{n-4}$; 21. $a_n = a_{n-1} + 3^{n-1}$; 22. $a_n = 2a_{n-1} - a_{n-3} + 2^{n-3}$; (2nd and 3rd terms count sequences that start with 001 followed by $(n - 3)$-digit sequences with no consec. 0's;

23. $a_n = 2a_{n-1} + 4^{n-1}$ or $a_n = 4a_{n-1} + 2^{n-1}$; 24. $a_n = 2a_{n-1} + 2^{n-1}$; 25. $a_n = 3a_{n-1} - a_{n-3}$;

26. $a_n = 2a_{n-1} + 1$; 27. $a_n = (2n - 1)a_{n-1}$; 28. $a_{n,k} = \mathrm{a}_{n,k-1} + a_{n-1,k}$;

29. $a_{n,k} = a_{n,k-1} + a_{n-1,k-1} + a_{n-2,k-1} + a_{n-3,k-1}$; 30. $a_{n,k} = a_{n-2,k-1} + a_{n-4,k-1} + a_{n-6,k-1}$;

31. $a_{n,k} = \mathrm{a}_{n-1,k-1} + a_{n-k,k}$; 32. $a_{n,m,k} = a_{n-1,m,k-1} + a_{n-2,m,k-1} + a_{n-3,m,k-1} + a_{n-4,m-1,k-1}$;

33. $a_n = b_{n-1} + c_{n-1}$, $b_n = c_n = 2^{n-1} - b_{n-1} - c_{n-1}$;

34a) $a_n = 3a_{n-1} + b_{n-1}$, $b_n = 3b_{n-1} + a_{n-1}$, b) $a_n = 2a_{n-1} + 2b_{n-1}$, $b_n = 2b_{n-1} + 2a_{n-1}$,

c) $a_n = b_{n-1} + c_{n-1}$, $b_n = c_n = 4^{n-1} - b_{n-1} - c_{n-1}$; 35. a_n = such sequences starting with a 0, b_n = such sequences starting with a 1, c_n = n-digit binary sequences with no consecutive 0's, $a_n = b_{n-1} + c_{n-3}$, $b_n = a_{n-1} + b_{n-1}$, $c_n = c_{n-1} + c_{n-2}$;

36.$a_{n,k} = a_{n,k,0} + a_{n,k,1}$, $a_{n,k,0} = a_{n-1,k,1}$, $a_{n,k,1} = a_{n-1,k,0} + a_{n-1,k-1,1}$, where $a_{n,k,0}$ ($a_{n,k,1}$) = such sequences starting with a 1 (0); 37. p_n = ways to hand out a penny, nickel or dime on successive days with a penny on the first day, n_n and d_n are defined similarly, $p_n = n_{n-1} + d_{n-1}$, $n_n = p_{n-5} + d_{n-5}$, $d_n = p_{n-10} + n_{n-10}$;

38. $a_n = a_{n-2}a_2 + a_{n-4}a_4 + a_{n-6}a_6 + \ldots + a_2a_{n-2}$; 39. $a_n = a_{n-1}a_3 + a_{n-2}a_4 + a_{n-3}a_5 + \ldots + a_3a_{n-1}$;

40. $a_n = a_{n-1}a_1 + a_{n-2}a_2 + a_{n-3}a_3 + \ldots + a_1a_{n-1}$; 41a) 3, 0, b) $2n - 1$, 2, c) $3n^2 - 3n + 1$, $6n - 6$;
42. $\Delta p_n = \Delta f_n / f_n$; 43a) $a_n = 2a_{n-1}$, $a_1 = 1$, b) $a_n = 2^{n-1}$, c) the first k integers in the sequence will form a set of consecutive integers, the $(k+1)$-st integer can be the next larger or the next smaller number to extend this consecutive set; 44a) $f(n,k) = f(n - 1, k) + f(n - 2, k - 1)$;

45. $a_n = a_{n-1}$, n not a multiple of 5, $a_n = a_{n-1}+1$, n a multiple of 5 but not 10 or 25, $a_n=a_{n-1}+2$, n a multiple of 10 or 25; 46a) $a_n = a_0a_{n-1} + a_1a_{n-2} + \ldots + a_{n-1}a_0$, $a_0 = a_1 = 1$, b) $b_n = a_{n-1}$ (by part a);
47. $a_n = a_{n-3}$, n even, $a_n = a_{n-3} + \lfloor \frac{n+1}{4} \rfloor$; 48. $a_n = 3a_{n-1} + 2a_{n-2} + \ldots + 2a_1 + 1$, $n \geq 2$.

Section 7.2: 1a) $An - 5$, b) $An^{1/2} - 2n$, c) $A + 4n - \lfloor \log_2 n \rfloor$, d) $An - 2$, e) $An^4 - 5n/7$, f) $An^2 - 3n$;
2. $a_n = 2a_{n/2} + 1$, $a_n = n - 1$; 3a) $a_n = a_{n/10} + 1$, $a_n = \log_{10} n$, b) $a_n = 10a_{n/10} + 1$, $a_n = \frac{1}{9} n - \frac{1}{9}$; 4. $a_n = 2a_{n/2} + 50n$, $a_n = 50n\lfloor \log_2 n + 2 \rfloor$; 5. $a_n = 2a_{n/2} + n - 1$, $a_n = n \log_2 n - n + 1$; 6. $a_n = 2a_{n/2} + 100\lfloor \log_2 n \rfloor$, $a_n = 100[3n - \lfloor \ln_2(n-2) \rfloor]$;

8b) $a_n = 2a_{n/2} + 3$, c) $a_n = 2n - 3$; 9a) Pick largest from first half and largest from second half and compare, b) $a_n = 2a_{n/2} + 1$, c) $a_n = n - 1$.

Section 7.3: 1. $(1.08)^n \times 1000$; 2. $a_n = 2a_{n-1}$, $a_n = 3 \times 2^n$; 3a) $a_n = \frac{2}{5} 4^n + \frac{3}{5} (-1)^n$,

b) $a_n = 1$, c) $a_n = 2$, d) $a_n = \frac{1}{2}n^2 - \frac{1}{2}n + 1$; 4. $A_1 = \frac{1}{\sqrt{5}}(\frac{1}{2} + \frac{1}{2}\sqrt{5})$, $A_2 = \frac{1}{\sqrt{5}}(\frac{1}{2} - \frac{1}{2}\sqrt{5})$;

5. $a_n = 2a_{n-1} + a_{n-2}$, $a_n = \frac{2+\sqrt{2}}{4}(1+\sqrt{2})^n + \frac{2-\sqrt{2}}{4}(1-\sqrt{2})^n$;

6. $a_n = 2a_{n-1} + 2a_{n-2}$, $a_n = (1/6)(3+2 \times 3^{1/2})[(1+3^{1/2})^n + (1-3^{1/2})^n]$;

7. $p_n - p_{n-1} = 2(p_{n-1} - p_{n-2})$, $p_n = 3 \times 2^n - 2$; 11. $c_1 = 9$, $c_2 = -18$.

Section 7.4: 1a) $a_n = 3C(n,2) + 1$, b) $a_n = 2C(n+1,3) + 3$, c) $a_n = 6C(n+2,3) - 3C(n+1,2)$;

2. $a_n = a_{n-1} + 2$, $a_n = 2n$; 3. $a_n = a_{n-1} + 2C(n, 2) + n$, $a_n = 2C(n+1,3) + C(n+1,2)$;

4. $a_n = a_{n-1} + C(n,2) + 1$, $a_n = C(n+1,3) + n + 1$; 5. $a_n = a_{n-1} + \Sigma_{k=1}^{n-3} \{k \times (n-2-k) + 1\}$,

$a_n = C(n,4) + C(n,2) - n + 1$; 6. $a_n = -3(-1)^n + 2n + 6$; 7. $a_n = 1250(1.04)^n - 1250$;

8a) $a_n = -3200 \times 1.05^n + 200n + 4200$, b) $a_n = -3410 \times 1.05^n + 210n + 4410$;

9a) $-3^n + 1$, b) $\frac{5}{3}2^n + \frac{1}{3}(-1)^n$, c) $3 \times 2^n - n - 2$, d) $15 \times 2^n - 2n^2 - 8n - 12$;

0. $a_n = 3^n - \frac{1}{2}(n^2 + 3n)$; 11. $3 \times 2^n - 3n - 2$ 12. $a_n = 2a_{n-1} + 2^{n-1}$, $a_n = 2^n + n \times 2^{n-1}$;

13. $a_n = 2a_{n-1} - a_{n-2} + 10 \times 2^k$, $a_n = 960n - 20 + 40 \times 2^n$; 17. $A \times 2^n + B \times 2^n + \frac{3}{2}n + \frac{25}{4}$;

18. $c_1 = 7$, $c_2 = 10$, $c_3 = 12$, $c_4 = -19$; 19a) $a_n = 1$, b) $a_n = n!$, n even, $a_n = 0$, n odd.

Section 7.5: 1a) $g(x) - 1 = xg(x) + \frac{2x}{1-x}$, b) $g(x) - x - 1 = 3x(g(x) - 1) - 2x^2g(x) + \frac{2x^2}{1-x}$,

c) $g(x) - 1 = xg(x) + \frac{2x^2}{(1-x)^3}$, d) $g(x) - 1 = 2xg(x) + \frac{1}{1-2x} - 1$; 2a) $a_n = 2n + 1$,

b) $a_n = 2 \times 2^n - 2n - 1$, c) $a_n = 2 \times C(n+1, 3) + 1$, d) $a_n = 2^n + n \times 2^n$; 3a) $g(x) - 1 = xg(x)^2$,

b) $g(x) - 1 - x - x^2 = (g(x) - 1 - x)^2$, c) $g(x) - 1 - x = \frac{1}{1-2x}(g(x) - 1)$; 4a) $F_n(x) = xF_n(x)$

$- 2xF_{n-1}(x)$, $F_n(x) = \dfrac{(-2x)^n}{(1 - x)^n}$, b) $F_n(x) = 2F_{n-1}(x) - 3xF_n(x)$, $F_n(x) = \dfrac{2^n}{(1 + 3x)^n}$;

7. $a_n = a_2a_{n-1} + a_3a_{n-2} + . . . a_{n-1}a_2$, $n \geq 3$, $a_2 = 1$, $a_n = \dfrac{1}{n - 1} C(2n - 4, n - 2)$;

8. $a_n = 3a_{n-1} - a_{n-3}$, $a_0 = -1$, $a_1 = a_2 = 0$, $g(x) = \dfrac{1 + 3x}{1 - 3x + x^3}$;

9. $a_n = a_1a_{n-1} + a_2a_{n-2} + . . . a_{n-1}a_1$, $n \geq 2$, $a_1 = 1$, $a_n = \dfrac{1}{n} C(2n - 2, n - 1)$;

10. $a_{n,k} = a_{n,k-1} + a_{n-1,k}$, $F_n(x) = xF_n(x) + F_{n-1}(x)$, $F_n(x) = F_{n-1}(x)/(1 - x)$, $a_{n,k} = C(k + n - 1,k)$;

11. $a_{n,k} = pa_{n-1,k-1} + qa_{n-1,k}$, $F_n(x) = (q + px)^n$; 12a) $a_{n,k} = n \times a_{n-1,k-1}$ or

$a_{n,k} = a_{n-1,k} + k \times a_{n-1,k-1}$; 13. similar to recurrence relations in Example 5 except

with 4^{n-1} replacing 3^{n-1} and $a_1 = 0$ instead of $a_1 = 1$ (still $b_1 = c_1 = 1$), $a_n = \frac{2}{15}(4^n - 1)$, n even,

$a_n = \frac{3}{15}(4^n - 4)$, n odd; 14. $a_n = a_{n-1} + b_{n-1} + c_{n-1}$, $b_n = a_{n-1} + b_{n-1} + c_{n-1}$,

$c_n = a_{n-1} + b_{n-1} + c_{n-1}$, $a_n = b_n = c_n = 3^{n-1}$; 15a) $a_n = \Sigma\, C(n - 1, k - 1)a_{n-k}$, b) $g(x) = e^{e^x-1}$;

16a) $s_{n,r} = s_{n-1,r-1} - (r - 1)s_{n,r-1}$, $(1 + x)\dfrac{F_n(x)}{dx} = F_{n-1}(x)$.

Chapter Eight Solutions

Section 8.1: 1. $2 \times 5 \times 26^5 - 5^2 \times 26^4$; 2. $10^9 - 10!$; 3. $3^n - 3 \times 2^{n-1}$;

4. $1 - (9 + 1)/2^9$; 5. $\{C(52,5) - C(13,6) \times C(4,1)^6\}/C(52,6)$; 6. $1 - 1/n!$;

7. $4 \times (9^3 - 9 \times 8 \times C(3,2))$; 8. 40; 9. 600 - 200 - 200 - 150;

10. 10! - 2 x 9! + 8!; 11a) 200 - 70 - 100 + 30, b) 200 - 100 - 60; 12a) 2 x 24! - 22!,

b) 26! - 24! - 23! + 22!; 13. (200 + 500 – 100)/1000; 14. 6!/2! + 7!/3! – 5!;

15a) 200 - 3 x 80 + 3 x 30 - 15, b) 80 - 2 x 30 + 15;

16a) yes, 45- 20 = 25% like chess and tennis but not bridge and similarly 25% like chess and bridge but not tennis and 20% like all three- yields 70% who like chess, a contradiction.

b) 40%; 17. 30 - 15 - 10 - 6 + 5 + 3 + 2 - 1; 18. 280 - 140 - 56 - 40 + 28 + 20 + 8 - 4;

19. $3^{20} - 3 \times 2^{20} + 3 \times 1$; 20. $3 \times 9^n - 3 \times 8^n + 7^n$; 21. $6!/2!^3 - 3 \times 5!/2!^2 + 3 \times 4!/2! - 3!$;

22. $5! - 3 \times 2 \times 4! + 3 \times 2^2 \times 3! - 2^3 \times 2!$; 23. C(37,10) – [C(27,10) + C(25,10) + C(22,10)] + C(15,10) + C(12,10) + 1; 24. 3 x 5! – 3 x 4! + 3!;

25. C(52,6) – 3 x C(48,6) + 3 x C(44,6) – C(40,6); 26. 2 x 8!/4!2! + 10!/4!4! - [2 x 7!/4! + 5!/2!] + 4!; 27. $3 \times C(6,2) \times 4! - \{2 \times C(6,3) \times 3! + 6!/2!^2\} + C(6,4) \times 2!$;

28. $[2 \times C(11,4) \times 7!/2! + C(11,3) \times 8!/2!^2] - [C(11,6) \times 5! + C(11,5) \times 6!/2! + C(11,4) \times C(7,3) \times 4!] + C(11,7) \times 4!$; 29. $15!/3!^5 - 3 \times 5! \times 10!/2!^5 + 3 \times 5!^3 - 5!^3$;

30a) 0, b) 30; 31. 20; 32. $n = 100$; 33. $N(Y) = N(Y\text{-}K) + N(Y \cap K) = 50 + 20$;

34a) 39, b) 12; 35. 25, 5; 36. 2.

Section 8.2: 1. $10^m - 3 \times 9^m + 3 \times 8^m - 7^m$; 2. $6^9 - C(6,1) \times 5^9 + C(6,2) \times 4^9 - C(6,3) \times 3^9 + C(6,4) \times 2^9 - C(6,5)$; 3. 13 x C(48,4) - C(13,2) x 1;

4. $630 - (315 + 210 + 126 + 90) + (105 + 63 + 45 + 42 + 30 + 18) - (21 + 15 + 9 + 6) - 3$;

5a) $\{C(52,13) - C(4,1) \times C(39,13) + C(4,2) \times C(26,13) - C(4,3) \times C(13,13)\}/C(52,13)$,

b) $\{C(4,1) \times C(39,13) - C(4,2) \times C(26,13) + C(4,3) \times C(13,13)\}/C(52,13)$,

c) $\{C(52,13) - C(4,1) \times C(48,13) + C(4,2) \times C(44,13) - C(4,3) \times C(40,13) + C(36,13)\}/C(52,13)$;

6. $10!/2!^5 - C(5,1) \times 5 \times 8!/2!^4 + C(5,2) \times P(5,2) \times 6!/2!^3 - C(5,3) \times P(5,3) \times 4!/2!^2$

$+ C(5,4) \times P(5,4) - C(5,5) \times 5!$; 7. $9!/3!^3 - 3 \times 7!/3!^2 + 3 \times 5!/3! - 3!$;

8. $\Sigma (-1)^k \times C(n,k) \times (2n - k)!/2!^{n-k}$; 9. $26! - \{3 \times 23! + 24!\} + \{2 \times 20! + 2 \times 21!\} - 18!$;

10a) $C(28 + 4 - 1, 28) - C(4,1) \times C(17 + 4 - 1, 17) + C(4,2) \times C(6 + 4 - 1, 6)$,

b) $C(68 + 4 - 1, 68) - C(4,1) \times C(37 + 4 - 1, 37) + C(4,2) \times C(6 + 4 - 1, 6)$,

c) $C(28 + 4 - 1, 28) - C(22 + 4 - 1, 22) - C(17 + 4 - 1, 17) - C(12 + 4 - 1, 12) - C(6 + 4 - 1, 6)$

$+ C(11 + 4 - 1, 11) + C(6 + 4 - 1, 6) + 1$;

11. $C(24 + 6 - 1,24) - 3 \times C(17 + 6 - 1,17) + 3 \times C(10 + 6 - 1,10) - C(3 + 6 - 1,3)$;

12. Easier to solve without inclusion-exclusion (4 digits appear once or 1 digit appears 3 times or 1 digit appears 4 times): $P(10,4) + C(10,2) \times 4!/3!1! + 10$;

13. $C(11 + 4 - 1,11) - C(4,1) \times C(9 + 3 - 1,9) + C(4,2) \times 8$;

14. $\Sigma (-1)^k \times C(n,k) \times C(5(n-k), r)$; 15. 27; 16. $\approx 26!^2/e$;

17. $\approx 10!^2/e$; 18. assume n even: $\Sigma (-1)^k \times C(n/2,k) \times (n - k)!$; 19. $10!/2!^5 \times \{10!/2!^5 - C(5,1) \times 8!/2!^4 + C(5,2) \times 6!/2!^3 - C(5,3) \times 4!/2!^2 + C(5,4) - C(5,5)\}$; 20. $1 - 1/e$;

21. $15!/3!^5 - 5 \times 5 \times 12!/3!^4 + C(5,2) \times P(5,2) \times 9!/3!^3 - C(5,3) \times P(5,3) \times 6!/3!^2 + 5 \times 5! - 5!$;

22. $8^4 - C(6,1) \times 8^3 + C(6,2) \times 8^2 - \{(C(6,3) - 3) \times 8 + 3 \times 8^2) + \{C(6,4) - C(6,5) + C(6,6)\} \times 8$;

23a) $n^5 - C(5,1) \times n^4 + C(5,2) \times n^3 - C(5,3) \times n^2 + \{C(5,4) - C(5,5)\} \times n$; b) $n^5 - C(5,1) \times n^4 + C(5,2) \times n^3 - \{(C(5,3) - 1) \times n^2 + n^3\} + \{(C(5,4) - 2) \times n + 2 \times n^2\} - n$; c) $n^5 - C(7,1) \times n^4 + C(7,2) \times n^3 - \{(C(7,3) - 3) \times n^2 + 3 \times n^3\} + \{(C(7,4) - 14) \times n + 14 \times n^2\} - \{(C(7,5) - 2) \times n + 2$

x $n^2\}$ + $\{C(7,6) - C(7,7)\}$ x n 24. $5^5 - C(8,1) \times 5^4 + C(8,2) \times 5^3 - \{(C(8,3) - 4) \times 5^2 + 4 \times 5^3\}$ $+ \{(C(8,4) - 25) \times 5 + 25 \times 5^2\} - \{(C(8,5) - 4) \times 5 + 4 \times 5^2\} + (C(8,6) - C(8,7) + 1) \times 5$;

25a) $\Sigma\, (-1)^k \times C(n - 1,k) \times (n - k)!$; 26. D_n; 27. given that $C(2 + 6 - 1,2) = 21$, then $\Sigma\, (-1)^k \times C(6,k) \times (21-k)^n$; 28. $\Sigma\, (-1)^k \times C(n,k) \times 2^k \times (2n - 1 - k)!$;

29. $5!^n \times \{(2n - 1)! - \Sigma\, C(n,k) \times (2n - 1 - k)!\}$; 30. $9!/3!^3 - 3 \times \{8!/1!1!3!3! - 7!/1!3!3!\}$ $+ 3 \times \{7!/1!1!1!1!3! - 2 \times 6!/1!1!1!3! + 5!/1!1!3!\} - \{6! - 3 \times 5! + 3 \times 4! - 3!\}$;

31. $P(C(7,3),7) - 7 \times P(C(6,3),7) + C(7,2) \times P(C(5,3),7)$; 32. $P(C(6,2),4) – 6 \times P(C(5,2),4)$ $+ C(6,2) \times P(C(4,2),4)$; 33. $C(P(6,3),8) – 6 \times C(P(5,3),8) + C(6,2) \times C(P(4,3),8)$;

34. $m^6 - 6 \times m^5 + C(6,2) \times m^4 - C(6,3) \times m^3 + C(6,4) \times m^2 - C(6,5) \times m + C(6,6) \times m$;

35. $\{\Sigma\, (-1)^k \times C(n,k) \times (n - k)^r\}/n!$; 36c) $D_1 = 0$, $D_2 = 1$, $D_3 = 2$, $D_4 = 9$, $D_5 = 44$, $D_6 = 265$, $D_7 = 1854$, $D_8 = 14{,}833$, $D_9 = 133{,}496$, $D_{10} = 1{,}334{,}961$;

37. $\Sigma_{k=3}^{n-1}(-1)^{k+1} \times C(k,3) \times C(n,k) \times (n - k)^r$, $\Sigma_{k=3}^{n-1}(-1)^{k+1} \times C(k-1,2) \times C(n,k) \times (n - k)^r$;

38. $C(52,6) - C(4,2) \times C(26,6) + C(2,1) \times C(4,3) \times C(13,6)$; 39. $C(5,2) \times 9!/2!^3$ – $2 \times C(5,3) \times 8!/2!^2 + 3 \times C(5,4) \times 7!/2! - 4 \times 6!$; 47. $\Sigma_{k=0}^{n}\; k \times \{\Sigma_{j=k}^{n}(-1)^{j-k} \times C(j,k) \times n!/j!\}$.

Section 8.3: 1. 5 x 5 board with darkened squares on main diagonal;

2a) $1 + 8x + 21x^2 + 20x^3 + 6x^4$, b) $1 + 7x + 17x^2 + 17x^3 + 6x^4$,

c) $1 + 7x + 12x^2 + 4x^3$, d) $1 + 8x + 14x^2 + 4x^3$, e) $1 + 12x + 46x^2 + 56x^3$,

3. $5! – 8 \times 4! + 20 \times 3! – 16 \times 2! + 4 \times 1!$; 4. $5! – 8 \times 4! + 22 \times 3! – 24 \times 2! + 9 \times 1!$;

5. $7! - 9 \times 6! + 30 \times 5! - 46 \times 4! + 32 \times 3! - 8 \times 2!$;

6. $6! \times (6! - 7 \times 5! + 18 \times 4! - 21 \times 3! + 11 \times 2! - 2 \times 1!)/29^6$;

7. $5! - 7 \times 4! + 16 \times 3! - 13 \times 2! + 2 \times 1!$; 8a) 6/20, b) 10/20; 9. 3;

Solutions

+

10. $\Sigma\, C(n,k)^2 \times k! \times x^k$;   11a) 4 x 5 board with darkened squares in 4 positions just to right of main diagonal, b) $(x + 1)^4$, c) $\Sigma_{j=k}^{5}(-1)^{j-k} \times C(j,k) \times C(n - 1,j) \times (n - j)!$;

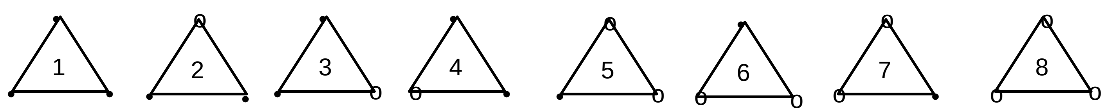
13. 2 x 2 array of darkened squares and "L" (a column of 3 squares beside a single square) both have $1 + 4x + 2x^2$;   14c) $\Sigma_{k=0}^{n} \{C(2n - k,k) + C(2n - k - 1, k - 1)\}x^k$.

Chapter Nine Solutions

Section 9.1: 1 (a) not symmetric, (b) yes, (c) not transitive, (d) not transitive, (e) not transitive; 2a) no, b) yes, 0, c) yes, 0, d) yes, 1, e) yes, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; 3a) 6 symmetries (as in Example 3), b) 4 symmetries, c) 1 symmetry (just the identity); 4a) $(abcd)$, b) $(ac)(bd)$, c) $(ad)(bc)$, d) $(a)(b)(c)(d)$, e) $(ab)(cd)$, f) $(ad)(bc)$, g) (1346)(257); 5a) all C_i left fixed,

b) $\begin{pmatrix} C_1C_2C_3C_4C_5C_6C_7C_8C_9C_{10}C_{11}C_{12}C_{13}C_{14}C_{15}C_{16} \\ C_1C_3C_4C_5C_2C_7C_8C_9C_6C_{11}C_{10}C_{13}C_{14}C_{15}C_{12}C_{16} \end{pmatrix}$,

d) $\begin{pmatrix} C_1C_2C_3C_4C_5C_6C_7C_8C_9C_{10}C_{11}C_{12}C_{13}C_{14}C_{15}C_{16} \\ C_1C_3C_2C_5C_4C_6C_9C_8C_7C_{11}C_{10}C_{15}C_{14}C_{13}C_{12}C_{16} \end{pmatrix}$,

6a) 

b) (i) (1)(234)(567)(8), (ii) (1)(2)(34)(57)(6)(8); 10a) π_1, b) π_7, c) π_2, d) π_2;

11a) a, b, c are rotations of 0°, 120°, and 240°, respectively d, e, f are flips around vertical axis, axis 30° clockwise of vertical, and axis 30° counterclockwise of vertical; row is first symmetry, column second symmetry:

	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	c	a	f	d	e
c	c	a	b	e	f	d
d	d	e	f	a	b	c
e	e	f	d	c	a	b
f	f	d	e	b	c	a

b) straightforward, c) let $a = \begin{pmatrix} 1\,2\,3\,4 \\ 1\,2\,3\,4 \end{pmatrix}$, $b = \begin{pmatrix} 1\,2\,3\,4 \\ 2\,1\,4\,3 \end{pmatrix}$, $c = \begin{pmatrix} 1\,2\,3\,4 \\ 3\,4\,1\,2 \end{pmatrix}$, $d = \begin{pmatrix} 1\,2\,3\,4 \\ 4\,3\,2\,1 \end{pmatrix}$,

	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

12. $\pi_2 \cdot \pi_5 = \pi_5 \cdot \pi_2$, in general, any rotation with any flip;

13. only left structure (right structure has 6 isomers); 20a) no, b) yes, c) no, d) yes;

21b) $\{\pi_1,\pi_3,\pi_5,\pi_6\}$ or $\{\pi_1,\pi_3,\pi_7,\pi_8\}$, c) in addition to subgroups in b) and G, G', G'', other subgroups are $\{\pi_1\pi_i\}$, for i = 3, 5, 6, 8; 22a) 2^{n^2}, b) $2^{C(n,2)}$.

Section 9.2: 1a) 24, b) 70; 2. 33; 3 . $\frac{1}{3}(3^{15} + 2 \text{ x } 3^5)$; 4. $\frac{1}{4}(2^{64} + 2 \text{ x } 2^{16} + 2^{32})$;

5. 51; 6a) 28; 7. $\frac{1}{2}(5^n + 5^{n/2})$ n even, and $\frac{1}{2}(5^n + 3 \text{ x } 5^{(n-1)/2})$ n even; 8. 280;

9a) in cycle form: $\pi_1 = (1)(2)(3)$, $\pi_2 = (12)(3)$, $\pi_3 = (13)(2)$, $\pi_4 = (1)(23)$, $\pi_5 = (123)$, $\pi_6 = (132)$, b) $\psi(\pi_1) = C(12 + 3 - 1, 12) = 91$, $\psi(\pi_2) = \psi(\pi_3) = \psi(\pi_4) = 7$, $\psi(\pi_5) = \psi(\pi_6) = 1$, answer: $\frac{1}{6}(91 + 7 + 7 + 7 + 1 + 1) = 19$; 10. $\frac{1}{2}(3 \text{ x } 2^{n-1})$, n even, $\frac{1}{2}(3 \text{ x } 2^{n-1} + 3 \text{ x } 2^{(n-1)/2})$, n odd; 11. $\psi(\pi_1) = 18$, $\psi(\pi_3) = 6$, $\psi(\pi_7) = \psi(\pi_8) = 12$, and other $\psi(\pi_i)$'s = 0, answer $\frac{1}{8}(18 + 6 + 12 + 12) = 6$; 12a) $\frac{1}{4}(455 + 2 \text{ x } 49 + 7) = 140$, b) $\frac{1}{4}(165 + 2 \text{ x } 25 + 5) = 55$;

13b) $\{\pi_1,\pi_7\}$, c) $\{\pi_1\pi_6\}$.

Section 9.3: 1. 55; 2a) 208, b) 136; 3a) 130, b) 92, c) cyclic color sequence on hexagon of R-W-B-R-W-W and R-B-W-R-W-W; 4a) 315, b) 2195, c) 5934, d) 16107;

5a) $\frac{1}{6}(m^4 + 2m^2 + 3m^3)$, b) $\frac{1}{8}(m^{12} + 2m^3 + 3m^6 + 2m^7)$, c) $\frac{1}{2}(m^5 + m^3)$, d) $\frac{1}{4}$ $(m^8 + 3m^4)$, e) $\frac{1}{12}(m^7 + 2m^2 + 2m^3 + 4m^4 + 3m^5)$, f) same as b).;

6a) $\frac{1}{8}(x_1^4 + 2x_4 + 3x_2^2 + 2x_1^2x_2)$, b) 21, c) $\frac{1}{8}(3^8 + 2 \text{ x } 3^2 + 3^4 + 4 \text{ x } 3^5) = 954$;

7a) $\frac{1}{6}(m^6 + 2m^2 + 3m^4)$, b) $\frac{1}{8}(m^{12} + 2m^3 + 3m^6 + 2m^7)$, c) $\frac{1}{2}(m^6 + m^4)$, d) $\frac{1}{4}(m^{10} + m^5 + m^6 + m^7)$, e) $\frac{1}{12}(m^{12} + 2m^2 + 2m^4 + m^6 + 6m^7)$, f) b) $\frac{1}{8}(m^{16} + 2m^4 + m^8 + 4m^9)$, 8a) (i) 2, (ii) 9, (iii) 258, b) (i) 1, (ii) 6, (iii) 147; 9. $\frac{1}{4}(7^4 + 3 \times 7^2) = 637$;

10a) $\frac{1}{24}(x_1^4 + 6x_4 + 3x_2^2 + 8x_1x_3 + 6x_1^2x_2)$, b) 15; 11a) $\frac{1}{2}(2^8 + 2^4) = 136$,

b) $\frac{1}{4}(2^8 + 2^4 + 0 + 2^4) = 72$; 13. $\psi(\pi_i)$ = number of cycles of length 1;

15a) $\frac{1}{p}(m^p + (p - 1) \times m)$, b) $\frac{1}{2p}(m^p + (p - 1) \times m + p \times m^{(p+1)/2})$.

Section 9.4: 1. $b^5 + b^4w + 2b^3w^2 + 2b^2w^3 + bw^4 + w^5$; 2a) $\frac{1}{6}\{(b + w)^6 + 2(b^6 + w^6)$ $+ 2(b^3 + w^3)^2 + (b^2 + w^2)^3\}$, 4, b) $\frac{1}{9}\{(b + w)^9 + 6(b^9 + w^9) + 2(b^3 + w^3)^3\}$, 10,

c) $\frac{1}{10}\{(b + w)^{10} + 4(b^{10} + w^{10}) + 4(b^5 + w^5)^2 + (b^2 + w^2)^5\}$, 12,

d) $\frac{1}{11}\{(b + w)^{11} + 10(b^{11} + w^{11})\}$, 15; 3. $b^4 + w^4 + r^4 + b^3w + b^3r + bw^3 + w^3r + br^3$ $+ wr^3 + 2b^2w^2 + 2b^2r^2 + 2w^2r^2 + 2b^2wr + 2bw^2r + 2bwr^2$; 4) $\frac{1}{4}\{(b + w)^{16} + 2(b^4 + w^4)^4$ $+ 2(b^2 + w^2)^8\}$; 5.(a) $\frac{1}{6}\{(b + w)^4 + 2(b^3 + w^3)(b + w) + 3(b^2 + w^2)(b + w)^2\}$,

b) $\frac{1}{8}\{(b + w)^{12} + 2(b^4 + w^4)^3 + 3(b^2 + w^2)^6 + 2(b^2 + w^2)^5(b + w)^2\}$; c) $\frac{1}{2}\{(b + w)^5 +$ $(b + w)(b^2 + w^2)^2\}$, d) $\frac{1}{4}\{(b + w)^8 + 3(b^2 + w^2)^4\}$, e) $\frac{1}{12}\{(b + w)^7 + 2(b^6 + w^6)(b + w)$ $+ 2(b^3 + w^3)^2(b + w) + 4(b^2 + w^2)^3(b + w) + 3(b^2 + w^2)^2(b + w)^3\}$, f) same as b);.

6a) and c) $\frac{1}{4}\{(b + w)^5 + 2(b + w)(b^4 + w^4) + (b + w)(b^2 + w^2)^2\}$, b) $\frac{1}{4}\{(b + w)^8 + 2(b^4 + w^4)^2 +$ $(b^2 + w^2)^4\}$; 7.(a) $\frac{1}{6}\{(b + w)^6 + 2(b^3 + w^3)^2 + 3(b^2 + w^2)^2(b + w)^2\}$, b) $\frac{1}{8}\{(b + w)^{12} +$ $2(b^4 + w^4)^3 + 3(b^2 + w^2)^6 + 2(b^2 + w^2)^5(b + w)^2$, c) $\frac{1}{2}\{(b + w)^6 + (b + w)^2(b^2 + w^2)^2\}$, d) $\frac{1}{4}\{(b + w)^{10} + (b^2 + w^2)^5\} + (b + w)^2(b^2 + w^2)^4 + (b + w)^4(b^2 + w^2)^3\}$, e) $\frac{1}{12}\{(b + w)^{12} + 2(b^6 +$ $w^6)^2 + 2(b^3 + w^3)^4 + (b^2 + w^2)^6 + 6(b^2 + w^2)^5(b + w)^2\}$,

f) $\frac{1}{8}\{(b + w)^{16} + 2(b^4 + w^4)^4 + (b^2 + w^2)^8 + 4(b^2 + w^2)^7(b + w)^2\}$;

8. $\frac{1}{24}\{(b + w)^{12} + 6(b^4+w^4)^3 + 3(b^2 + w^2)^6 + 6(b + w)^2(b^2 + w^2)^5$

$+ 8(b^3 + w^3)^4\}$, 13; 9a) $\frac{1}{12}\{(b + w)^4 + 8(b^3 + w^3)(b + w) + 3(b^2 + w^2)^2\}$,

b) $\frac{1}{24}\{(b + w)^6 + 6(b^4 + w^4)(b + w)^2 + 3(b^2 + w^2)^2(b + w)^2 + 6(b^2 + w^2)^3 + 8(b^3 + w^3)^2\}$;

10a) $b^2w^2r + b^2wr^2 + bw^2r^2$, b) no positive terms, c) $b^4w^4 + b^4r^4 + w^4r^4 + b^4w^3r$

$+ b^4w^2r^2 + b^4wr^3 + b^3w^4r + b^2w^4r^2 + bw^4r^3 + b^3wr^4 + b^2w^2r^4 + bw^3r^4 + b^3w^3r^2 +$

$b^3w^2r^3 + b^2w^3r^3$; 11. $\frac{1}{24}\{(b + w)^4 + 6(b^4 + b^4) + 8(b^3 + w^3)(b + w) + 3(b^2 + w^2)^2$

$+ 6(b^2 + w^2)(b + w)^2\}$; 13a) if not a cyclic rotation of all corners, the length of the cycle would have to divide p — impossible, b) $C(p,k)/p$;

14a) $\frac{1}{8}\{(1 + b + b^2)^4 + 2(1 + b^4 + b^8) + 3(1 + b^2 + b^4)^2 + 2(1 + b + b^2)^2(1 + b^2 + b^4)\}$,

$= 1 + b + 3b^2 + 3b^3 + 5b^4 + 3b^5 + 3b^6 + b^7 + b^8$, b) $\frac{1}{8}\{(r + w + rw)^4 + 2(r^4 + w^4 + r^4w^8)$

$+ 3(r^2 + w^2 + r^2w^2)^2 + 2(r + w + rw)^2(r^2 + w^2 + r^2w^2)\}$; 15a) 36, b) 216;

16. 11; 20. $a_{k,n}$ equals the number of partitions of n into k or less parts, which equals the number of partitions of n into parts of size k or less: $g_k(x) = \dfrac{1}{(1 - x)(1 - x^2)(1 - x^3) \ldots (1 - x^k)}$

Chapter Ten Solutions

Section 10.1: 1. 1234, 1243, 1324, 1342, etc., 2. *aceh, ache, aech, aehc*, etc.;

3. 1234, 1235, 1236, 1245, 1246, 1256, 1345, 1346,1356, 1456, 2345, 2346, 2356, 2456, 3456;

4. *adh, adj, ahj, dhj*; 5a) 123456789(10), (10)987654321, b) *abcd, wxyz*;

Section 10.2 1a) $s \Rightarrow 1s \Rightarrow 101s \Rightarrow 1011s \Rightarrow 101101s \Rightarrow 101101$, b) $s \Rightarrow 2s \Rightarrow 22s \Rightarrow 220t$ $\Rightarrow 2201t \Rightarrow 22010t \Rightarrow 220102t \Rightarrow 220102$, c) $s \Rightarrow 2s_2 \Rightarrow 20t_3 \Rightarrow 202t_4 \Rightarrow 2020t_5 \Rightarrow 20202$,

d) $s \Rightarrow t_3p_8 \Rightarrow c_{3s}c_{3h}c_{3c}p_8 \Rightarrow c_{3s}c_{3h}c_{3c}c_{8h}c_{8d}$, e) $s \Rightarrow 0s1 \Rightarrow 00s11 \Rightarrow 000s111 \Rightarrow 00001111$;

2. a) $s \Rightarrow tt \Rightarrow 0t \Rightarrow 001t \Rightarrow 0010$, b) $s \Rightarrow tt \Rightarrow 10tt \Rightarrow 100t \Rightarrow 1001$, c) $s \Rightarrow tt \Rightarrow 0t \Rightarrow$ $001t \Rightarrow 00110t \Rightarrow 001100$; 3. many possibilities, e.g., a) $s \Rightarrow tt \Rightarrow 01tt \Rightarrow 01t10t \Rightarrow$ $01010t \Rightarrow 010101$, b) $s \Rightarrow tt \Rightarrow 01tt \Rightarrow 010t \Rightarrow 01010t \Rightarrow 010101$;

4. s $\rightarrow 1t$, $t \rightarrow 0t$, $t \rightarrow 1t$, $t \rightarrow 0$; 5. $s \rightarrow tv$, $t \rightarrow 0t$, $t \rightarrow 0$, $v \rightarrow 1v$, $v \rightarrow 1$;

6. $s \rightarrow$ ttttt, $t \rightarrow 0$, $t \rightarrow 1$; 7a) $s \rightarrow pppp$, and $p \rightarrow c_j$, where c_j ranges over the six colors, b) $s \rightarrow Rppp$, $s \rightarrow pRpp$, $s \rightarrow ppRp$, $s \rightarrow pppR$, and $p \rightarrow c_j$, where c_j ranges over the five non-Red colors, c) same as b) but c_j ranges over all colors;

8a) $s \rightarrow f_{\text{i}}$, where i ranges over the 4 card values, $f_{\text{I}} \rightarrow c_{\text{is}}c_{\text{i}h}c_{\text{ic}}c_{\text{i}h}d_{\text{i}}$, where d_{i} ranges over the 48 cards whose value is not i, b) $s \rightarrow s_{\text{i}}$, i = the four suits, $s_{\text{i}} \rightarrow$ {all subsets of 5 cards in suit I}; 9. $s \rightarrow t_2u_2$, $s \rightarrow t_2v_2$, $s \rightarrow u_2v_2$, $s \rightarrow t_2$BC, $s \rightarrow$ Au_2C, $s \rightarrow$ ABv_2, $t_{\text{i}}, \rightarrow$ AA, $u_2 \rightarrow$ BB, $v_2 \rightarrow$ CC; 10. $s \rightarrow 0s'$, $s \rightarrow 1s''$, $s' \rightarrow 1s$, $s'' \rightarrow 0s$, $s \rightarrow 0t$, $s \rightarrow 1t$, $t \rightarrow 0t'$, $t \rightarrow 1t''$, $t' \rightarrow 1t$, $t'' \rightarrow 0t$, $t \rightarrow 0$, $t \rightarrow 1$;

11. $s \rightarrow$ Ts_2, $s \rightarrow$ Ht_2, $s_2 \rightarrow$ Ts_3, $s_2 \rightarrow$ Ht_3, $s_3 \rightarrow$ THH, $s_3 \rightarrow$ Ht_4, $s_2 \rightarrow$ Tt_3, $t_2 \rightarrow$ Tt_3, $t_3 \rightarrow$ Tt_4, $t_4 \rightarrow$ TH; 12.

Section 10.3: 1a) $(t_o/–)\to(s_1/1)\to(s_1/1)\to(t_1/1)\to(t_1/1)\to(s_2/2)\to(s_2/2)\to(t_2/2)$,

b) $(t_o/–)\to(s_1/1)\to(t_1/1)\to(s_2/2)\to(t_2/2)\to(s_3/3)\to(s_3/3)\to(s_3/3)$;

3a) $(t_o/–)\to(t_1/6)\to(t_1/\text{-})\to(t_1/\text{-})\to(t_1/3)\to(t_1/\text{-})\to(t_1/\text{-})\to(t_1/1)\to(t_1/\text{-})\to(t_1/\text{-})$,

b) $(t_o/–)\to(d_1/\text{-})\to(d_1/\text{-})\to(d_1/\text{-})\to(s_4/5)\to(t_1/4)\to(t_1/\text{-})\to(t_1/1)\to(t_1/\text{-})\to(t_1/\text{-})$,

c) $(t_o/–)\to(d_1/\text{-})\to(d_1/\text{-})\to(d_1/\text{-})\to(d_2/\text{-})\to(d_2/\text{-})\to(d_2/\text{-})\to(s_5'/6)\to(s_4/5)\to(t_1/4)$;

5. States of machine are t_i, i = -3, -2, -1, 0, 1, 2, 3; when j (=-1 or +1) is read in state t_i, next state is t_{i+j} and $i + j$ is printed 7. States of machine are t_i, i = 0, 1, 2, 3; 4, 5; when j is read in state t_i, next state is t_k and k is printed, where $k = i + j$ (mod 6):

9. From t_0, reading 0 yields (s/1) and 1 yields (u,0), from s, 0 yields (w/1) and 1 yields (s/0); from u, 0 yields (s/0) and 1 yields (s/1); from w, 0 yields (w/0) and 1 yields (w/1);

11a) starting from s, reading 0 yields (t,Y), 1 yields (r,N), and 2 yields (s,Y)l from t, any input yields (t,Y); from r, any input yields (r,N), b) Starting from s, reading R yields (t_2/N) and any non-R yields (s_2/N), from s_2, R yields (t_3/N) and non-R yields (s_3/N), from t_2,R yields (r,N) and non-R yields (t_3,N), from s_3, R yields (t_4/N) and non-R yields (s_4/N), from t_3, R yields (r/N) and non-R yields (t_4/N), from s_4, R yields (*end*,Y) and non-R yields (r,N), from t_4, R yields (r,N) and non-R yields (*end*,Y), and from r, R or non-R yields (r,N).

Chapter Eleven Solutions

Section 10.1: 1a) $\{a,c\}$ or $\{b,d\}$, b) f, c) no kernel, consider directed 5-circuit b, a, d, g, h, b, a- if a is K (kernel), then d not in K, then g in K, then h not in K, then b in K — impossible since a in K; similar sort of argument (also involving c, f, e) if a not in K;

2. kernel = {0, 6, 12, 18, 24), first player wins by moving to kernel;

3. {3, 4, 9, 11, 12, 16, 17, 21, 25, 26, 27, 31, 32, 36, over 40};

4. first player to 2, second player wins at 4, first player to 1 or 5, second player to 6, then first player to 7 or 8, second player wins at 9, first player to 11, second players wins at 16;

5. A goes to 2 , B must go to 4 or else A will win; 6a) (0,0), (1,2), (2,1), (3,5), (5,3), (4,7), (7,4), (6,10), b) (0,0), (0,6), (6,0), (1,2), (2,1), (1,8), (8,1), (2,7), (7,2), (3,5), (5,3), (5,9), (6,6), (7,8); 7. move to multiples of $5k + 1$ (initial position is win for second player);

8a) $g(a) = g(c) = 0$, $g(b) = g(d) = 1$ (or interchange values), b) no Grundy function,

c)

$x =$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
$g(x) =$	1	1	2	0	0	1	1	2	2	0	1	0	0	2	1	1	0	0	2	1	1	0	2	2

$x =$	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	over 40
$g(x) =$	1	0	0	0	2	1	1	0	0	2	1	1	0	2	2	1	1	0

9. by symmetry assume $g(a) = 0$, then $g(e) = 1$, then $g(d) = 0$, then $g(c) = 1$, then $g(e) = 0$, but now two kernel vertices are adjacent; 10. graph in Exercise 1b), for example;

11. S is a kernel if and only if all vertices not in S have an edge to a vertex in S while no vertex in S has an edge to a vertex in S, that is, if only if $W(S) = S$;

13. follows from parts a) and b) of Exercise 12 since $g(x) = k$ means there is a path of length k starting at x while $l(x)$ is length of longest path starting at x; longest path is at least length k (maybe there is another longer path starting at x);

15. suppose x and y are adjacent because there is an edge from x to y; then x must have a

larger level number than y and a different Grundy value from its successor y;

17. if there were an infinite number of vertices, then one of the finite number of starting vertices, call it x_i, must have an infinite number of vertices reachable from it, and one of the finite number of successors of x_1, call it x_2, must have an infinite number of vertices reachable from it, and one of the finite number of successors of x_2, call it x_3, must have an infinite number of vertices reachable from it, and so on, without end;

19. Let a = oooo, b = ooo, c = o_oo, d = oo_o, e = oo, f = o_o, g = o, h = _ (win);

$s(a) = \{b, c, d, e, f\}$, $s(b) = \{e, f, g\}$, $s(c) = s(d) = \{e, f, g\}$, $s(e) = \{g, h\}$, $s(f) = \{g\}$, $s(g) = h$, $s(h\} = \varnothing$; $g(f) = g(h) = 0$, $g(a) = g(g) = 1$, $g(e) = 2$, $g(b) = g(c) = g(d) = 3$.

Section 10.2: 1a) 7, remove 3 from 4th pile, b) 0, c) 4, remove 4 from 2nd, 3rd, or 4th pile, d) 0; 2a) 0, b) 2, remove 2 from 3rd pile, c) 2, remove 2 from 1st pile, d) 1, remove 1 from 3rd pile; 3a) 3, remove 3 from 3rd pile or 2 from 4th pile, b) 2, remove 2 from 3rd or 4th pile, c) 0, d) 0; 4a) 0, b) 2, remove 2 from pile 3, c) 2, remove 2 from pile 1, d) 1, remove 1 from pile 3; 5a) 0, b) 0, c) 1, remove 1 from pile 4, d) d, remove 1 from pile 3; 6a) remove 3 from pile 2 or 3, b) remove 2 from pile 2 or 3, c) remove 1 from pile 2 or 3; 7a) 3, add nickel to 3rd pile, b) (0, 0), (0, 4), (0, 6), (0, 9), (1, 1), (2, 2), (2, 5), (2, 8), (3, 3), (3, 7), (4, 4), (4, 6), (5, 5); 7. if $c_j = d_j$, then trivially $c' + c_j = c' + d_j$; if $c' + c_j = c' + d_j$, then c_j and d_j must have 1's in the same positions in their binary representations, that is, they must be equal; 11. Immediate that the proposed strategy works;

13a) remove 3 balls along one of the 3 lines formed by balls on one of the 3 side of the arrangement.

Appendices Solutions

Section A.1: 1a) 12, 27 b) 2, 3, 6, 7, 9, 12, 15, 17, 18, 21, 22, 24, 27,

c) 1, 4, 5, 8, 10, 11, 13, 14, 16, 19, 20, 23, 25, 26, 28, 29, d) all $1 \le k \le 29$ except 12, 27;

2a) $A \cap B \cap C$, b) $\overline{A \approx B \approx C}$, c) $A \cap B - C$, d) none, e) none, f) $\overline{A} \cup (A \cap B \cap C)$;

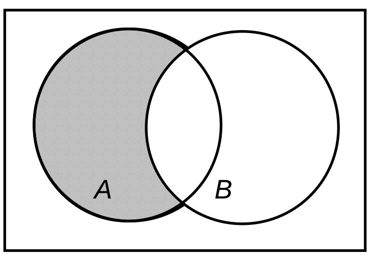
3. we are given that $N(\overline{R} \cap M) = 2$ as well as that $N(M) = N(R) = N(\overline{M}) = N(\overline{R}) = 4$;

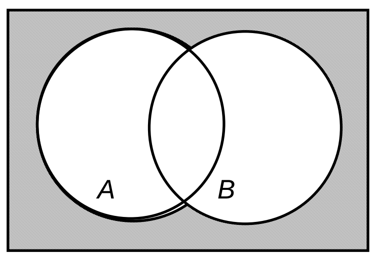
then $N(R \cap M) = N(M) - N(\overline{R} \cap M) = 4 - 2 = 2$, $N(\overline{R} \cap \overline{M}) = N(\overline{R}) - N(\overline{R} \cap M) = 4 - 2 = 2$,

and clearly $N(R \cap \overline{M}) = 2$; 4. 2 memory, rechargeable and 2 no-memory, non-rechargeable,

or 2 memory, non-rechargeable and 2 no-memory, rechargeable;

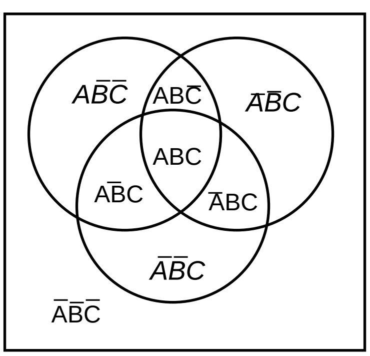
5a) impossible, b) yes, 20 - 8 - 8 = 4, c) 20 -15 = 5;

7a) 

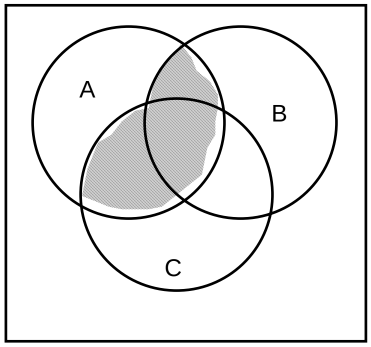
, b) 

c) $(\overline{A \approx B}) \cap (\overline{A \leftrightarrow B}) = (\overline{A \leftrightarrow B})$, see Figure A1.3, d) $A - (B - A) = A$;

8a) $\overline{A} \cup B$, b) $(A \cup B) \cap (A \cap B)$;

9. 

here all expression involve $\cap$, so that $A\overline{B}C = A \cap \overline{B} \cap C$;

10a) obvious b) 

, c) and d) unshaded part of A in b);

12a) a) and c) both equal *B*; 16. let A_1 = outcomes with 1 or 2 on first die, define A_2 similarly, then must count $N(A_1 \cup A_2) = 12 + 12 - 4 = 20$;

17a) $E = S \cup H \cup C$, $52^3 - 39^3$, b) $E = (S \cup H \cup C) \cap (\overline{S \leftrightarrow H \leftrightarrow C})$, $52^3 - 39^3 - 13^3$,

c) $E = (S \cap H \cap \overline{C}) \cup (S \cap \overline{H} \cap C) \cup (\overline{S} \cap H \cap C)$, $3 \times 13^2 \times 39$;

Section A.2: 23. one can only prove by induction a property that is a function of n; for example, one can prove that there are a finite number of binary sequence of length $\leq n$;

24. step (2) assumes proposition has been proved for $n = 2$— not true;

25. the initial step only assumes that $n \geq 1$, not $n \geq 2$, but for $n = 1$, a^{n-2} is undefined.

Section A.3: 1. 1/2; 2. prob(E_0) =1/8, prob(E_1) = 3/8, prob(E_2) = 3/8, prob(E_3) = 1/8;

3a) 1/6, b) 18/36 = 1/2, c) 3/36 = 1/12; 4. (3 x 6 + 1)/216; 5a) 1/6, b) 1/2;

6a) 2/(5 x 4), b) (4 x 3)/(5 x 4) = 3/5; 7a) 1/3 x 1/3 = 1/9, b) 2 x 1/3 x 2/3 = 4/9;

8. (2 + 6)/(5 x 4); 9. (2 x 2 x 2)/4! = 1/3; 10. $1 - 5^5/6^5$; 11. 1 - (50 + 40 - 20)/100 = 3/10; 12. if D_i = integers from 1 to 50 divisible by i (or equivalently, a multiple of i), numerator is $N(D_3 \cup D_4)$: (16 + 12 - 4)/50; 13. 2/3; 14. all sequences of 5 selections (as if eyes were open); 15a) all sequences with k tails, $k \geq 0$, and one head followed by a

head, b) all positive integers, c) all ordered pairs of positive integers, d) all sequences of k black balls, $k \geq 0$, followed by a red ball.

Section A.4: 1. $n + 1$; 4. $(8 + 10 + 3 \times 11) + 1 = 52$; 6. 6, since $C(5,3) = 10 < 12$;

12. consider the n-1 differences, modulo n-1, of the first number subtracted from each of the other n-1 integers; if any difference is 0 mod n-1, we are done; otherwise two of the differences, say, a_i-a_1 and a_j-a_1 must be the same mod n-1, implying a_i-a_j = 0 mod n-1;

15. Printer i is connected to computers i, $i + 1$, $i + 2$, $i + 3$, $i + 4$, $i + 5$;

16. 24 connections, each dummy must go to all 6 real printers- if dummy printer A connected to just 5 real printers, a set of calls to dummy A and those 5 real printers could not be handled.