Deadline : 2024/4/29, 17:00.

1. If $\mathcal{X}$ and $\mathcal{Y}$ are normed vector space, we denote the space of all bounded linear maps from $\mathcal{X}$ to $\mathcal{Y}$ by $L(\mathcal{X},\mathcal{Y})$. Show that if $\mathcal{Y}$ is complete, then so is $L(\mathcal{X},\mathcal{Y})$.

2. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed vector space and $T:\mathcal{X}\to\mathcal{Y}$ be a linear map. Then show that T is bounded if and only if it is continuous.

3. (Refer problem 10 of ch6) Let $1\le p,q\le+\infty$ such that $1/p+1/q=1$. For $f\in\mathcal{R}[a,b]$, we define

$$\|f\|_p=\left\{\int_a^b |f(x)|^p\,dx\right\}^{1/p},\ \|f\|_\infty=\sup_{x\in[a,b]}|f(x)|$$

You may assume Young's inequality $ab\le a^p/p+b^q/q$ is true, where $a,b\ge 0$. Then show that $\forall\, f,g\in\mathcal{R}[a,b]$, we have

   (a) Holder's inequality : $\|fg\|_1\le\|f\|_p\,\|g\|_q$

   (b) Minkowski inequality : $\|f+g\|_p\le\|f\|_p+\|g\|_p$.

4. Let E be a compact set and K be a real valued function continuous on E. Define a linear map $A:\mathcal{R}(E)\to\mathcal{R}(E)$ by $(Af)(t)=K(t)f(t)$, $\forall\, t\in E$. Show that

   (a) A is bounded, i.e. $\exists\, M\ge 0$ such that $\|Af\|_2\le M\,\|f\|_2\,,\ \forall f\in\mathcal{R}(E)$

   (b) If we define operator norm $\|A\|=\sup\{\|Af\|_2:\|f\|_2=1\}$, then $\|A\|=\|a\|_\infty$.

5. Let $\mathcal{C}[0,1]$ be a normed vector space with sup-norm. Define $T:\mathcal{C}[0,1]\to\mathcal{C}[0,1]$ by

$$(Tf)(x)=\int_0^x f(t)\,dt.$$

Show that T is linear, continuous, and find $\|T\|$

6. Let $T(x,y)=(2x+y,x+2y)$ be a map on $\mathbb{R}^2$. Show T linear, bounded, and find $\|T\|$.

7. Let $T:\mathbb{R}^n\to\mathbb{R}^n$ be a linear operator with $\|T\|<1$. Show that $T_k=1+T+...+T^{k-1}$ converges to a linear operator S and $S\circ(1-T)=(1-T)\circ S=1$.

8. Two norms $\|\cdot\|$ and $\|\cdot\|'$ on $\mathcal{X}$ are said to be equivalent if $\exists\, c_1,c_2>0$ such that $c_1\,\|x\|\le\|x\|'\le c_2\,\|x\|$, $\forall x\in X$. Show that if $\mathcal{X}$ is a finite-dimensional vector space, then all norm on $\mathcal{X}$ are equivalent. Hint : Use basis, and the fact that unit ball in $\mathcal{X}$ isometric to unit ball in $\mathbb{R}^n$.