Alternating series Test

$\sum a_n = \sum (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - \cdots \cdots S$

$\{b_n\} > 0, \searrow 0$

$\Rightarrow \sum a_n (= \sum (-1)^{n+1} b_n)$ conv. and

$|S - S_n| = \left| \sum_{k=1}^{\infty} (-1)^{k+1} b_k - \sum_{k=1}^{n} (-1)^{k+1} b_k \right|$

Error $\leq b_{n+1}$

pf $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \underbrace{1 - \frac{1}{2}}_{S_2} + \underbrace{\frac{1}{3} - \frac{1}{4}}_{S_4} + \underbrace{\frac{1}{5} - \frac{1}{6}}_{S_6} + \underbrace{\frac{1}{7} - \frac{1}{8}}_{S_8} + \cdots$

$\frac{1}{n} \to b_n$

$\{S_{2n}\} \nearrow$ $\quad$ $\{S_{2n-1}\} \searrow$

$S = \underbrace{1}_{S_1} \underbrace{- \frac{1}{2} + \frac{1}{3}}_{S_3,\ <0} \underbrace{- \frac{1}{4} + \frac{1}{5}}_{S_5,\ <0} \underbrace{- \frac{1}{6} + \frac{1}{7}}_{S_7,\ <0} \underbrace{- \frac{1}{8} + \frac{1}{9}}_{S_9,\ <0} - \cdots$

prove that $S_{2n} < S_{2k-1} \quad \forall n, k$

$S_2 < S_4 < S_6 < \cdots < S_{2n} \cdots < \cdots S_{2k-1} < \cdots < S_5 < S_3 < S_1$ ※

Rearrangement $\quad \sum \frac{1}{n} = \infty$

$\{1, 3, 5, 7, \cdots\}$

$\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \cdots\}$

$\sum (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

王冠章

$a_n \quad 1 + \frac{1}{3} + \frac{1}{5} + \cdots = \infty \ (*) \Leftarrow \forall n,\ a_n > b_n$ by Comparison Test

$b_n \quad \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots = \infty \ (**)$

$= \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots\right)$

Since $** \to \infty$ as $n \to \infty$

thus $* \to \infty$ as $n \to \infty$ ※

Thm: $\sum a_n = A.C. \Rightarrow \sum a_n \equiv S = \sum a_{f(n)}$

$\forall f(n)$ $\quad$ rearrangement

⟨Ex⟩ $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cancel{\frac{1}{6}}+\frac{1}{7}-\frac{1}{8}+\cdots = S$ $(*)$

$+)\ \frac{1}{2}-\frac{1}{4}+\cancel{\frac{1}{6}}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\frac{1}{14}-\frac{1}{16}+\cdots = \frac{1}{2}S$

$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots\cdots = \frac{3}{2}S$

= rearrangement of $(*)$ (Riemann)

Q: $\exists$ a rearrangement $f(n)$ s.t.

$$\sum a_{f(n)} = \pi, e \quad \text{for } \sum a_n = \sum (-1)^{n+1}\frac{1}{n}$$

§11.7 Strategy for Testing Series

1° If $\sum \frac{1}{n^p}$ (p-series) $\Rightarrow$ conv. $p>1$, div. $p\le 1$

2° If $\sum ar^{n-1}$ or $\sum ar^n$

$\Rightarrow$ Geometric series conv. $|r|<1$, div. $|r|\ge 1$

3° $\sum a_n \overset{a_n>0}{\sim}$ p-series, Geometric series $\Rightarrow$ Comparison Test

4° If $\lim a_n \neq 0 \Rightarrow \sum a_n$ div.

5° If $\sum (-1)^{n-1} b_n$ or $\sum (-1)^n b_n \Rightarrow$ Alternating Series Test

6° ratio Test

7° root Test

8° If $a_n = f(n)$ and $\int_1^\infty f(x)dx$ easy $\Rightarrow$ Integral Test

[redacted]

Test for Conv. or div.

⟨Ex⟩ $\sum \frac{n-1}{2n+1}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n-1}{2n+1} = \frac{1}{2} \Rightarrow \text{div.}$

⟨Ex⟩ $\sum_{n=10}^{\infty} \frac{\sqrt{n^3+1}}{3n^2+4n+2} > \sum_{n=10}^{\infty} b_n = \sum_{n=10}^{\infty} \frac{\sqrt{n^3}}{5n^2} = \sum_{n=10}^{\infty} \frac{1}{5n^{\frac{1}{2}}}$

(p-series) div.

黃聖亮

⟨Ex⟩ $\sum n e^{-n^2} \Rightarrow \text{conv.}$

陳嵩富

$-\frac{1}{2}\int_1^{\infty} -2x e^{-x^2} dx \Big|_1^{\infty} = -\frac{1}{2} e^{-x^2} \Big|_1^{\infty}$

$= 0 - \left(\frac{-1}{2e}\right) = \frac{1}{2e} \Rightarrow \text{conv}$

王冠章

⟨Ex⟩ $\sum (-1)^n \frac{n^3}{n^4+1} = a_n$

let $b_n = |a_n|$

$\lim_{n\to\infty} \frac{n^3}{n^4+1} = \lim_{n\to\infty} \frac{1}{n+\frac{1}{n^3}} = 0$

$b_n = \frac{n^3}{n^4+1}$

$= \frac{1}{n+\frac{1}{n^3}} > \frac{1}{n+1+\frac{1}{(n+1)^3}} = b_{n+1}$

by Alternating Series Test $\Rightarrow$ Convergent

⟨Ex⟩ $\sum \frac{2^k}{k!}$

$\frac{a_{k+1}}{a_k} = \frac{\frac{2^{k+1}}{(k+1)!}}{\frac{2^k}{k!}} = \frac{2}{k+1}$

黃頎粄

$\because \lim_{k\to\infty} \frac{2}{k+1} = 0$, $\therefore \sum \frac{2^k}{k!}$ conv. (Ratio Test)

⟨Ex⟩ $\sum \frac{1}{2+3^n} =$

$\frac{1}{2+3^n} < \frac{1}{3^n} \quad \forall n \in \mathbb{Z}.$

$\sum \left(\frac{1}{3}\right)^n \Rightarrow$ conv (Geo Series: $|r|<1$)

$\therefore \sum \frac{1}{2+3^n}$ conv.

(By comparison test.)

§11.8. Power Series.

$\sum C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$

C_n = coefficients.

Def: a fn $f(x) \equiv \sum C_n x^n$ if conv.

More generally,

$\sum C_n (x-a)^n \equiv C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots$

power series in $(x-a)$ (or about a)

⟨EX⟩ Find x s.t. $\underbrace{\sum_{n=1}^{\infty} n!\, x^n}_{(*)}$ conv.

Sol. By ratio Test, Given $x \neq 0$

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!\, x^{n+1}}{n!\, x^n}\right| = (n+1)\underbrace{|x|}_{\text{given}} \to \cancel{(L<1)}\ \infty$

$(*)$ div. for all $x \neq 0$

$(*)\big|_{x=0} = 0$

※

⟨EX⟩ Find x s.t. $\sum \frac{(x-3)^n}{n}$ conv.

Sol: Use ratio test.

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(x-3)^{n+1}}{n+1}}{\frac{(x-3)^n}{n}}\right| = \left|\frac{n}{n+1}(x-3)\right| \xrightarrow[n\to\infty]{} \boxed{|x-3| \equiv L < 1}$

$|x-3| < 1 \iff 2 < x < 4$

$\sum \frac{(x-3)^n}{n} = \begin{cases} \text{conv. if } |x-3|<1 \\ \text{div. if } |x-3|>1 \end{cases}$

$\sum \frac{(x-3)^n}{n}\Big|_{x=4} = \sum \frac{1}{n}$ div. (Integral Test)

$\sum \frac{(x-3)^n}{n}\Big|_{x=2} = \sum \frac{(-1)^n}{n}$ conv. Alternating Series Test

※

$\sum \frac{(x-3)^n}{n} = \begin{cases} \text{conv. for } 2 \leq x < 4 \\ \text{div. for } x<2,\ x \geq 4 \end{cases}$

※

農曆=舊曆=陰曆

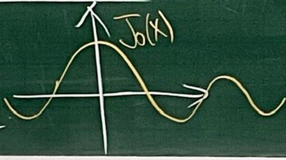
Lunar New Year $\overset{?}{=}$ Chinese New Year

湯若望

Kepler Newton

(Ex). Bessel fns

$$J_0(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$



By Ratio Test

$$\frac{a_{n+1}}{a_n}=\frac{\dfrac{(-1)^{n+1}x^{2(n+1)}}{2^{2(n+1)}((n+1)!)^2}}{\dfrac{(-1)^n x^{2n}}{2^{2n}(n!)^2}}=\frac{(-1)x^2}{2^2(n+1)^2}\xrightarrow{(n\to\infty)}0<1$$

$\therefore J_0(x)$ conv. $\forall x$

$$\frac{(n!)^2}{((n+1)!)^2}=\frac{(\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdots\cancel{n})^2}{(\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdots\cancel{n}\cdot(n+1))^2}=\frac{1}{(n+1)^2}$$

Graph Partial Sums:

$S_0(x)=1$, $S_1(x)=1-\frac{x^2}{4}$, $S_2(x)=1-\frac{x^2}{4}+\frac{x^4}{64}$

$S_3(x)=1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2364}$, $S_4(x)=\cdots$

and $J_0(x)$

Thm: For $\underbrace{\sum c_n(x-a)^n}_{(*)}$, $\exists$ 3 possibilities

(i) $(*)$ conv. only at $x=a$

(ii) $(*)$ conv. $\forall x$

(iii) $\exists R>0$, s.t. $(*)=\begin{cases}\text{conv. if } |x-a|<R\\ \text{div. if } |x-a|>R\end{cases}$

Def such $R\equiv$ radius of convergence (R.C.)

(i) $R=0$

(ii) $R=\infty$

Interval of convergence: $(a-R,\ a,\ a+R)$