Calculus HW5

Date: Mar 13 Made by Eric

1.

Proof.

$$S = \{(x,y) \in \mathbb{R}^2 | 0 \le x \le y \le 1\} \tag{1}$$

$$T(S) = \{(u,v) \in \mathbb{R}^2 | 0 \le u^2 \le v \le 1, 0 \le u\} \tag{2}$$

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2.

2.(a)

Proof.

$$I := \int\int_R \sin(9x^2+4y^2)dA = \int\int_S \sin(u^2+v^2)\frac{1}{6}dudv \tag{3}$$

where

$$S = \{(u,v) | v^2+u^2 \le 1, 0 < v, 0 < u\} \tag{4}$$

$$\int\int_S \sin(u^2+v^2)\frac{1}{6}dudv = \frac{1}{6}\int_0^1 \int_0^{\frac{\pi}{2}} \sin(r^2)rd\theta dr \tag{5}$$

$$I = \frac{\pi}{12}\int_0^1 \sin(r^2)rdr \tag{6}$$

$$I = \frac{\pi}{24}\int_0^1 \sin(z)dz = \frac{\pi}{24}\cos(z)|_{z=0}^1 = \frac{\pi}{24}(\cos(1)-1) \tag{7}$$

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2.(b)

Proof.

$$I := \int\int_R e^{x+y}dydx = \int_{-1}^1\int_{-1}^1 e^u\frac{1}{2}dudv \tag{8}$$

where u is given by $x+y$ and v is given by $x-y$.

$$I = \int_{-1}^1 e^u du = e^1 - e^{-1} \tag{9}$$

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3.

3.(a)

$$I := \int\int_D \frac{1}{(x^2+y^2)^{\frac{n}{2}}} dA = \int_0^{2\pi}\int_r^R u^{-n+1} du d\theta \tag{10}$$

$$I = 2\pi \frac{1}{-n+2} u^{-n+2}|_{u=r}^R = \frac{2\pi}{-n+2}(R^{-n+2} - r^{-n+2}) \text{ If } n \neq 2 \tag{11}$$

$$I = 2\pi \ln(u)|_r^R = 2\pi(\ln(R) - \ln(r)) \text{ if } n = 2 \tag{12}$$

3.(b)

If $n < 2$, then $\lim_{r\to 0^+} I = \frac{2\pi}{-n+2}(R^{-n+2})$ exists.

If $n = 2$, then $\lim_{r\to 0^+} I = \infty$ doesn't exist.

If $n > 2$, then $\lim_{r\to 0^+} I = -\infty$ doesn't exist.

3.(c)

$$I := \int\int_E \frac{1}{(x^2+y^2+z^2)^{\frac{n}{2}}} dV = \int_r^R \int_0^{\pi}\int_0^{2\pi} \rho^{-n+1} \sin\phi d\theta d\phi d\rho \tag{13}$$

$$I = 2\pi \int_r^R \rho^{-n+1} \int_0^{\pi} \sin(\phi) d\phi d\rho \tag{14}$$

$$I = 4\pi \int_r^R \rho^{-n+1} = \frac{4\pi}{-n+2}\rho^{-n+2}|_{\rho=r}^R = \frac{4\pi}{-n+2}(R^{-n+2} - r^{-n+2}) \text{ If } n \neq 2 \tag{15}$$

$$I = 4\pi(\ln(R) - \ln(r)) \text{ If } n = 2 \tag{16}$$

3.(d)

If $n < 2$, then $\lim_{r\to 0^+} I = \frac{4\pi}{-n+2} R^{-n+2}$ exists.

If $n = 2$, then $\lim_{r\to 0^+} I = \infty$ doesn't exist.

If $n > 2$, then $\lim_{r\to 0^+} I = -\infty$ doesn't exist.

4.

$$I := \int\int_R [[x+y]] dA \tag{17}$$

Let $u = x + y$ and $v = x - y$. So $|\mathbf{J}| = \frac{1}{2}$

$$1 \leq \frac{u+v}{2} = x \leq 3 \text{ and } 2 \leq \frac{u-v}{2} = y \leq 5 \iff 3 \leq u \leq 8 \text{ and } -4 \leq v \leq 1 \tag{18}$$

$$I = \int_{-4}^{1}\int_{3}^{8} [[u]]dudv = \int_{-4}^{1} (3+4+5+6+7)dv = 125 \tag{19}$$

5.

$$I := \int_0^1 \int_0^1 e^{\max\{x^2,y^2\}} dydx = \int\int_R e^{\max\{x^2,y^2\}} dA \tag{20}$$

where $R = \{(x,y)|0 \leq x \leq 1, 0 \leq y \leq 1\}$. We partition R into two disjoint subsets.

$$R = A \cup B = \{(x,y) \in R|x \leq y\} \cup \{(x,y) \in R|y < x\} \tag{21}$$

And compute I by

$$I = \int\int_A e^{\max\{x^2,y^2\}} dydx + \int\int_B e^{\max\{x^2,y^2\}} dydx \tag{22}$$

$$I = \int\int_A e^{y^2} dydx + \int\int_B e^{x^2} dydx \tag{23}$$

Notice that when we compute the integral, we should ignore the boundary of the area, so

$$I = \int_0^1 \int_0^y e^{y^2} dxdy + \int_0^1 \int_0^x e^{x^2} dydx \tag{24}$$

$$I = \int_0^1 ye^{y^2} dy + \int_0^1 xe^{x^2} dx \tag{25}$$

$$I = \int_0^1 e^u \frac{du}{2} + \int_0^1 e^m \frac{dm}{2} \tag{26}$$

where $u = y^2$ and $m = x^2$

$$I = e - 1 \tag{27}$$

6.

Proof.

$$I := \int_0^t \int_0^x \int_0^y f(z)dzdydz = \int\int\int_E f(z)dV \tag{28}$$

where

$$E = \{(x,y,z)|z \leq y \leq x \leq t\} \tag{29}$$

$$I = \int_0^t \int \int_R f(z) dA dz \tag{30}$$

where

$$R = \{(x, y) | z \le y \le x \le t\} \tag{31}$$

$$I = \int_0^t \int_z^t \int_y^t f(z) dx dy dz \tag{32}$$

$$I = \int_0^t \int_z^t f(z)(t - y) dy dz \tag{33}$$

$$I = \int_0^t [f(z)t(t - z) - \int_z^t f(z) y dy] dz \tag{34}$$

$$I = \int_0^t f(z)t(t - z) - (f(z)\frac{y^2}{2})|_{y=z}^t dz \tag{35}$$

$$I = \int_0^t f(z)[t(t - z) - \frac{t^2 - z^2}{2}] dz \tag{36}$$

$$I = \int_0^t f(z) \frac{t^2 - 2tz + z^2}{2} dz = \frac{1}{2} \int_0^t (t - z)^2 f(z) dz \tag{37}$$

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7.

Proof. We show that

$$I := \int_0^1 \int_0^1 |\frac{y - x}{(x + y)^3}| dx dy \to \infty \tag{38}$$

Substitute x, y by $u = x + y, v = y - x$. Notice $|\mathbf{J}| = \frac{1}{2}$, so

$$I = \frac{1}{2} \int_0^2 \int_{-1}^1 |\frac{v}{u^3}| dv du \tag{39}$$

$$I = \frac{1}{2} \int_0^2 (\int_0^1 \frac{v}{u^3} dv + \int_{-1}^0 \frac{-v}{u^3} dv) du = \frac{1}{2} \int_0^2 \frac{1}{u^3} (\frac{1}{2} + \frac{1}{2}) du \tag{40}$$

$$I = \frac{1}{2} \int_0^2 \frac{1}{u^3} du = \frac{1}{-4} u^{-2}|_{u=0}^2 \to \infty \tag{41}$$

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8.

Proof. WOLG, assume there exists a point $(x_c, y_c) \in D$ such that $f(x_c, y_c) = L > 0$.

Let $N_2(f(x_c, y_c))$ be a neighborhood of $f(x_c, y_c)$ bounded by $\frac{L}{2}$ and $\frac{3L}{2}$. Because f is continuous, we know there is a neighborhood $N_1((x_c, y_c)) \subseteq D$ of (x_c, y_c) such that for all (x, y) in $N_1((x_c, y_c))$, we have $\frac{L}{2} < f(x, y) < \frac{3L}{2}$.

Then we see $\int\int_{N_1(x_c, y_c)} f(x, y)dA > 0$ CaC ■

9.

9.(a)

$$I := \int_C \mathbf{v} \cdot d\mathbf{r} = \int_a^b \mathbf{v} \cdot \mathbf{r}'(t)dt = \mathbf{v} \cdot \int_a^b \mathbf{r}'(t)dt = \mathbf{v} \cdot (\mathbf{r}(b) - \mathbf{r}(a)) \tag{42}$$

9.(b)

$$I := \int_C \mathbf{r} \cdot d\mathbf{r} = \int_C \mathbf{r} \cdot \mathbf{r}'(t)dt = \frac{1}{2}\int_C \frac{d}{dt}\mathbf{r} \cdot \mathbf{r}(t)dt \tag{43}$$

$$I = \frac{1}{2}\int_C d(\mathbf{r} \cdot \mathbf{r}(t)) = \frac{1}{2}(\mathbf{r} \cdot \mathbf{r}(b) - \mathbf{r} \cdot \mathbf{r}(a)) = \frac{1}{2}[|\mathbf{r}(b)|^2 - |\mathbf{r}(a)|^2] \tag{44}$$

10.

10.(a)

$$f = 3x + x^2y^2 \tag{45}$$

$$I := \int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \tag{46}$$

where

$$\mathbf{r}(t) = (t, \frac{1}{t}) \text{ and } a = 1 \text{ and } b = 4 \tag{47}$$

$$I = f(4, \frac{1}{4}) - f(1, 1) = 9 \tag{48}$$

10.(b)

$$f = xe^{xy} \tag{49}$$

$$I := \int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \tag{50}$$

where

$$a = 0 \text{ and } b = \frac{\pi}{2} \tag{51}$$

$$I = f(0, 2) - f(1, 0) = -e \tag{52}$$

10.(c)

$$f = x \sin y + y \cos z \tag{53}$$

$$I := \int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \tag{54}$$

where

$$a = 0 \text{ and } b = \frac{\pi}{2} \tag{55}$$

$$I = f(1, \frac{\pi}{2}, \pi) - f(0, 0, 0) = 1 - \frac{\pi}{2} \tag{56}$$