Isomorphism Theorems

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Definitions and Theorems

Theorem 1. *Let G and G' be two groups, and $\phi : G \to G'$ be a homomorphism with kernel K.*

There is an isomorphism $\mu : G/K \to \phi[G]$ such that $\phi(x) = \mu(Kx)$

Proof. Let $Kx = Ky$

$Kx = Ky \implies xy^{-1} \in K \implies \phi(xy^{-1}) = 1 \implies \phi(x) = \phi(y)$

$\mu(Kx) = \phi(x) = \phi(y) = \mu(Ky)$

μ is well defined.

$\forall Kx, Ky \in G/K, \mu(Kx)\mu(Ky) = \phi(x)\phi(y) = \phi(xy) = \mu((Kx)(Ky))$

μ is at least a homomorphism.

$\forall \phi(x) \in \phi[G], \mu(Kx) = \phi(x)$

$\mu(Kx) = \mu(Ky) \implies \phi(x) = \phi(y) \implies \phi(xy^{-1}) = 1 \implies xy^{-1} \in K, \implies Kx = Ky$

μ is an isomorphism.

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Theorem 2. *Let N be a normal subgroup of a group G. Let ϕ be a map from the set of normal subgroups of G containing N to the set of normal subgroup of G/N be defined by $\phi(L) = \{Nl | l \in L\}$ is one-to-one and onto.*

Proof.

We claims **(i)** ϕ indeed maps the normal subgroup of G containing N to the normal subgroup of G/N

Let $N \subseteq M \trianglelefteq G$

$\phi(M) = \{Nm | m \in M\}$

$\forall Nm_1, Nm_2 \in \{Nm | m \in M\}, (Nm_1)(Nm_2) = Nm_1m_2 \in \{Nm | m \in M\}$, since M is a subgroup itself.

$\{Nm|m \in M\}$ is closed under.

$1 \in M \implies N \in \{Nm|m \in M\}$

Identity is included in $\{Nm|m \in M\}$

$\forall Nm_1 \in \{Nm|m \in M\}, m_1^{-1} \in M \implies Nm_1^{-1} \in setNm|m \in M$

Inverses are included in $\{Nm|m \in M\}$

$\{Nm|m \in M\}$ is at least a subgroup.

$\forall Nx \in G/N, \forall Nm_1 \in \{Nm|m \in M\}, xm_1x^{-1} \in M \implies (Nx)(Nm_1)(Nx^{-1}) \in \{Nm|m \in M\}$

$\{Nm|m \in M\}$ is a normal subgroup.

Claim **(i)** is proven.

Now we embark on to prove ϕ is one-to-one and onto.

Let $K \trianglelefteq G/N$

$K = \{Na, Nb, \dots\}$

We claim that **(ii)** $\bigcup K$ is a normal subgroup containing N

$\forall x, y \in \bigcup K, (Nx)(Ny) \in K$, since K is a subgroup itself.

$(Nx)(Ny) \in K \implies Nxy \in K \implies xy \in \bigcup K$

$\bigcup K$ is closed under.

$1 \in N \subseteq \bigcup K$

Identity is included in $\bigcup K$

$\forall x \in \bigcup K, Nx^{-1} \in K \implies x^{-1} \in \bigcup K$

Inverses are included in $\bigcup K$

$\bigcup K$ is at least a subgroup.

$\forall x \in G, y \in \bigcup K, (Nx)(Ny)(Nx)^{-1} \in K$, since K it self is a normal subgroup.

So $Nxyx^{-1} \in K \implies xyx^{-1} \in \bigcup K$

$\bigcup K$ is a normal subgroup.

$N \in K \implies N \subseteq \bigcup K$

Claim **(ii)** is proven.

We claim **(iii)** $\phi(\bigcup K) = K$

$\phi(\bigcup K) = \{Nx | x \in \bigcup K\}$

$\forall x \in \bigcup K, Nx \in K \implies \{Ny | y \in \bigcup K\} \subseteq K$

$\forall Nx \in K, x \in Nx \subseteq \bigcup K \implies K \subseteq \{Nx | x \in \bigcup K\}$

$\phi(\bigcup K) = \{Nx | x \in \bigcup K\} = K$

Claim **(iii)** is proven.

So, ϕ is at least onto.

Let H, R be two normal subgroups of G containing N.

Let $\phi(H) = \phi(R)$

$\forall h \in H, Nh \in \phi(H) = \phi(R) \implies Nh = Nr, \exists r \in R \implies hr^{-1} \in N \subseteq R \implies h \in R$

WOLG, $\forall r \in R, r \in H$

$R = H$

ϕ is indeed one-to-one. ■

Definition 1. *Let A and B be two subgroup of G.*

$AB = \{ab | a \in A, b \in B\}$

Theorem 3. *Let G be a group in which H is a subgroup and N is a normal subgroup. NH is a group.*

Proof. $\forall n_1h_1, n_2h_2 \in NH, n_1h_1n_2h_2 = n_1(h_1n_2)h_2 = n_1(n_3h_1)h_2, \exists n_3 \in N \implies n_1h_1n_2h_2 = (n_1n_3)(h_1h_2) \in NH$

NH is closed under.

$1 = 11 \in NH$

Identity is included in NH

Let $n_1h_1 \in NH$

Pick $n_2 = h_1^{-1}n_1^{-1}h_1 \in N$

$n_2h_1^{-1} \in NH$

$n_1h_1n_2h_1^{-1} = n_1h_1h_1^{-1}n_1^{-1}h_1h_1^{-1} = 1$

Inverses are included in NH

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Theorem 4. *Let G be a group in which H is a subgroup and N is a normal subgroup. $N \trianglelefteq NH$ and $H \cap N \trianglelefteq H$*

Proof. Because $NH \subseteq G$, so $\forall nh \in NH, (nh)N = N(nh)$

Let $U = H \cap N$

Let $h \in H, u \in U \subseteq H$

$huh^{-1} \in H$

Because $u \in N$, so $huh^{-1} \in N$

So $huh^{-1} \in U$

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Theorem 5. *Let G be a group in which H is a subgroup and N is a normal subgroup.*

$(HN)/N \simeq H/(H \cap N)$

Proof. Let $U = H \cap N$

Let $\phi : (HN)/N \simeq H/(H \cap N)$ be defined by $\phi((hn)N) = hU$

$\forall (h_1n_1)N, (h_2n_2)N \in (HN)/N, \phi((h_1n_1)N)\phi((h_2n_2)N) = (h_1U)(h_2U) = (h_1h_2)U = \phi((h_1h_2n_1n_2)N) = \phi((h_1h_2)N(h_2n_2)N)$

ϕ is at least a homomorphism.

$\forall hU \in H/(H \cap N), \phi((h1)N) = hU$

ϕ is onto.

Let $\phi((h_1n_1)N) = \phi((h_2n_2)N)$

so $h_1 = h_2$

$h_1 = h_2 \implies (h_1 n_1)N = h_1 N = h_2 N = (h_2 n_2)N$

ϕ is one-to-one.

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Theorem 6. *Let G be a group in which H and K are two normal subgroup, where $K \leq H$.*

$(H/K) \trianglelefteq (G/K)$

Proof. $\forall hK \in H/K, \forall gK \in G/K, ghg^{-1} \in H \implies (gK)(hK)(g^{-1}K) \in H/K$ ■

Theorem 7. *Let G be a group in which H and K are two normal subgroup, where $K \leq H$.*

$G/H \simeq (G/K)/(H/K)$

Proof. Let $U = H/K$

Let $\phi : G/H \to (G/K)/(H/K)$ be defined by $\phi(gH) = (gK)U$

$\forall g_1H, g_2H \in G/H, \phi(g_1H)\phi(g_2H) = (g_1K)U(g_2K)U = ((g_1K)(g_2)K)U = (g_1g_2K)U = \phi(g_1g_2H) = \phi((g_1H)(g_2H))$

ϕ is at least a homomorphism.

$\forall (gK)U \in (G/K)/(H/K), \phi(gH) = (gK)U$

ϕ is onto.

Let $\phi(g_1H) = \phi(g_2H)$

So $(g_1K)U = (g_2K)U$

So $((g_1K)(g_2K)^{-1})U = U$

$U = ((g_1K)(g_2K)^{-1})U = ((g_1K)(g_2^{-1}K))U = (g_1g_2^{-1}K)U$

So $g_1g_2^{-1}K \in U = H/K$

So $g_1g_2^{-1}K = hK, \exists h \in H$

Because $K \subseteq H$, so $g_1g_2^{-1}K = hK, \exists h \in H \implies g_1g_2^{-1} \in H \implies g_1H = g_2H$

ϕ is one-to-one.

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