臺灣大學數學系112學年度碩士班甄試試題

科目：線性代數

2022.10.20

Notation: $\mathbb{R}$ is the set of real numbers and $\mathbb{C}$ is the set of complex numbers. If $F = \mathbb{R}$ or $\mathbb{C}$ and n is a positive integer, we denote by $\mathrm{M}_n(F)$ the set of $n \times n$ matrices with entries in F and by I_n the identity matrix in $\mathrm{M}_n(F)$.

Problem 1 (15 pts)**.** Let

$$\mathbf{v}_1 = (1, 2, 0, 4),\ \mathbf{v}_2 = (-1, 1, 3, -3),\ \mathbf{v}_3 = (0, 1, -5, -2),\ \mathbf{v}_4 = (-1, -9, -1, -4)$$

be vectors in $\mathbb{R}^4$. Let W_1 be the subspace spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ and let W_2 be the subspace spanned by $\mathbf{v}_3$ and $\mathbf{v}_4$. Find the dimension and a basis of $W_1 \cap W_2$.

Problem 2. Let

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 3 & -4 & 1 \\ 3 & -8 & 5 \end{pmatrix}.$$

(1) (10pts) Find an invertible matrix $Q \in \mathrm{M}_3(\mathbb{C})$ such that

$$Q^{-1}AQ = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 12 \\ 0 & 1 & 1 \end{pmatrix}.$$

(2) (15pts) Find an invertible matrix $P \in \mathrm{M}_3(\mathbb{C})$ such that $P^{-1}AP$ is a diagonal matrix.

Problem 3. For any $A \in \mathrm{M}_2(\mathbb{C})$, define

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.$$

(1) (5pts) Evaluate $\sin \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$.

(2) (15pts) Prove or disprove: There exists $A \in \mathrm{M}_2(\mathbb{R})$ such that

$$\sin A = \begin{pmatrix} 1 & 2022 \\ 0 & 1 \end{pmatrix}.$$

Problem 4 (20pts)**.** Let $A = (a_{ij}) \in \mathrm{M}_n(\mathbb{C})$ and let $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ be roots of characteristic polynomial of A (counted with multiplicity). Show that

$$AA^* = A^*A \text{ if and only if } \sum_{1 \le i,j \le n} |a_{ij}|^2 = \sum_{k=1}^{n} |\lambda_k|^2.$$

Problem 5 (20pts)**.** Let $A, B \in \mathrm{M}_n(\mathbb{C})$. Suppose that all of the eigenvalues of A and B are positive real numbers. If $A^4 = B^4$, prove that $A = B$.

Problem 2. Cyclic subspace, cyclic vector, companion matrix

(1)

$$\det(A - xI) = \left|\begin{pmatrix} -x & 0 & 0 \\ 1 & -x & 12 \\ 0 & 1 & 1-x \end{pmatrix}\right| = -x^3 + x^2 + 12x = -x(x-4)(x+3)$$

eigenvalue of A is $0, 4, -3$, corresp. vector is

$$\Big(0, \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{v_1}\Big), \quad \Big(4, \underbrace{\begin{pmatrix} -7 \\ 1 \\ 29 \end{pmatrix}}_{v_2}\Big), \quad \Big(-3, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}_{v_3}\Big), \quad \text{let } v = v_1 + v_2 + v_3,$$

Then $\{v, Tv, T^2v\} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 4 & 16 \\ 1 & -3 & 9 \end{pmatrix}$ (use $\{v_1, v_2, v_3\}$ basis), which det is not zero, Thus v is a cyclic vector, Moreover, resp. this basis,

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 12 \\ 0 & 1 & 1 \end{pmatrix}_* \qquad Q = \begin{pmatrix} -6 & -28 & -112 \\ 3 & 1 & 25 \\ 31 & 113 & 473 \end{pmatrix}$$

(2) Diagonal

Use previous result, $\lambda_A = 0, -3, 4$, so $P = \begin{pmatrix} 1 & -7 & 0 \\ 1 & 1 & 1 \\ 1 & 29 & 1 \end{pmatrix}$

$$P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

Problem 1 $v \in W_1 \cap W_2 \iff \exists\, a, b, c, d \in \mathbb{F}$ such that

$$a\begin{pmatrix} 1 \\ 2 \\ 0 \\ 4 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \\ 3 \\ -3 \end{pmatrix} = c\begin{pmatrix} 0 \\ 1 \\ -5 \\ -2 \end{pmatrix} + d\begin{pmatrix} -1 \\ -9 \\ -1 \\ -4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 0 & -1 \\ 2 & 1 & 1 & -9 \\ 0 & 3 & -5 & -1 \\ 4 & -3 & -2 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \\ -c \\ -d \end{pmatrix} = \vec{0} \Rightarrow \begin{pmatrix} a \\ b \\ -c \\ -d \end{pmatrix} \in \ker\begin{pmatrix} 1 & -1 & 0 & -1 \\ 2 & 1 & 1 & -9 \\ 0 & 3 & -5 & -1 \\ 4 & -3 & -2 & -4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 3 & 1 & -7 \\ 0 & 3 & -5 & -1 \\ 0 & 1 & -2 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 3 & 1 & -7 \\ 0 & 0 & -6 & 6 \\ 0 & 0 & -7/3 & 7/3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a-b-d \\ 3b-c-7d \\ c-d \\ 0 \end{pmatrix} = \vec{0} \Rightarrow \begin{matrix} a=b+d \\ 3b=c+7d \\ c=d \end{matrix} \Rightarrow \begin{pmatrix} a \\ b \\ -c \\ -d \end{pmatrix} = \begin{pmatrix} b+c \\ 8/3c \\ -c \\ -c \end{pmatrix}$$

$$\Rightarrow b\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c\begin{pmatrix} 1 \\ 8/3 \\ -1 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} a \\ b \\ -c \\ -d \end{pmatrix} \in \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 8 \\ -3 \\ -3 \end{pmatrix} \right\}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 3 \\ 8 \end{pmatrix} \qquad a = x+3y \quad b = 8y$$

$$\Rightarrow W_1 \cap W_2 = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} -5 \\ 14 \\ 24 \\ -12 \end{pmatrix} \right\}, \ \dim W_1 \cap W_2 = 2$$

Prob. 3

(1) Note that $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n\times 3 \\ 0 & 1 \end{pmatrix}$, so $\sin\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \begin{pmatrix} 1 & (2n+1)\times 3 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} & 3\times \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \\ 0 & \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \end{pmatrix} = \begin{pmatrix} \sin 1 & 3\cos 1 \\ 0 & \sin 1 \end{pmatrix} \#$$

(2) Prove,

Note that when $\lambda_1 \neq 0$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\sin A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} \sin\lambda_1 & \lambda_1 \cos 1 \\ 0 & \sin\lambda_1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if $\sin A = \begin{pmatrix} 1 & 2022 \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} \sin\lambda_1 & \lambda_1 \cos 1 \\ 0 & \sin\lambda_1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 2022 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a\sin\lambda_1 + c\lambda_1\cos 1, & b\sin\lambda_1 + d\lambda_1\cos 1 \\ c\sin\lambda_1, & d\sin\lambda_1 \end{pmatrix} = \begin{pmatrix} a, & 2022a+b \\ c, & 2022c+d \end{pmatrix}$$

if $\sin\lambda_1 = 1 \quad c=0 \quad b=0 \quad d = \frac{2022}{\lambda_1 \cos 1} \quad a=1$

then $\begin{pmatrix} 1\times 1+0 & 2022 \\ 0 & \dfrac{2022\times 1}{\lambda_1\cos 1}\end{pmatrix} \overset{\text{left = right}}{=} \begin{pmatrix} 1 & 2022 \\ 0 & \dfrac{2022\times 1}{\lambda_1\cos 1}\end{pmatrix}$

and $\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{2022}{\lambda_1\cos 1}\end{pmatrix}$ is invertible.

Thus, if $A = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{2022}{\lambda_1\cos 1}\end{pmatrix}^{-1}\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \dfrac{2022}{\lambda_1\cos 1}\end{pmatrix}$, $\sin A = \begin{pmatrix} 1 & 2022 \\ 0 & 1\end{pmatrix}$

Prob. 4 Frobenius norm and trace of A^*A

Note that if $A = (a_1, \cdots, a_n)$ $A^*A = \begin{pmatrix} \bar a_1 \\ \vdots \\ \bar a_n\end{pmatrix}(a_1, \cdots, a_n)$, so we have $\operatorname{trace}(A^*A) = \|A\|_2^2$, where $\|A\|_2 = (\Sigma_{i,j} |a_{ij}|^2)^{1/2}$.

Note that $\forall A \in M_n(\mathbb{C})$ $U^*AU = \Sigma$, Σ is a upper-triangular, $A = U\Sigma U^*$

so
$$\|A\|_2^2 = \operatorname{trace}(A^*A) = \operatorname{trace}(U\Sigma^*U^*U\Sigma U^*)$$
$$= \operatorname{trace}(\Sigma^*\Sigma) = \|\Sigma\|_2^2 \geq \Sigma_{k=1}^n |\lambda_k|^2$$

Thus $\|A\|_2^2 = \Sigma_{k=1}^n |\lambda_k|^2$ iff Σ is a diagonal matrix, That is, A is normal.

Prob. 5 Appendix: Higham "Func. of Matrix"

$\Rightarrow$ From left Def. 1.1 and 1.4, $f: z \to z^{\frac{1}{4}}$ can be defined on spectral of A, so $\exists\, p \in F[x]$ such that $f(A) = p(A)$

Let $C = p(A)$, A and C commute, $A^{\frac{1}{4}}$, has positive spectral

unique

use "simultaneous triangularization", $A^{-1}C$ has positive spectrum and $(A^{-1}C)^4 = I$, so $A^{-1}C$ is diagonalizable, having 4^{th} roots of unity $\Rightarrow A = C$

only positive root is 1

Definition 1.1.[1] *The function f is said to be defined on the spectrum of A if the values*
$$f^{(j)}(\lambda_i), \quad j = 0{:}\, n_i - 1, \quad i = 1{:}\, s$$
exist. These are called the values of the function f on the spectrum of A.

In most cases of practical interest f is given by a formula, such as $f(t) = e^t$. However, the following definition of $f(A)$ requires only the values of f on the spectrum of A; it does not require any other information about f. Indeed any $\sum_{i=1}^s n_i$ arbitrary numbers can be chosen and assigned as the values of f on the spectrum of A. It is only when we need to make statements about global properties such as continuity that we will need to assume more about f.

Definition 1.2 (matrix function via Jordan canonical form). *Let f be defined on the spectrum of $A \in \mathbb{C}^{n\times n}$ and let A have the Jordan canonical form (1.2). Then*
$$f(A) := Zf(J)Z^{-1} = Z\operatorname{diag}(f(J_k))Z^{-1}, \tag{1.3}$$
where
$$f(J_k) := \begin{bmatrix} f(\lambda_k) & f'(\lambda_k) & \cdots & \dfrac{f^{(m_k-1)}(\lambda_k)}{(m_k-1)!} \\ & f(\lambda_k) & \ddots & \vdots \\ & & \ddots & f'(\lambda_k) \\ & & & f(\lambda_k)\end{bmatrix}. \tag{1.4}$$

Definition 1.4 (matrix function via Hermite interpolation). *Let f be defined on the spectrum of $A \in \mathbb{C}^{n\times n}$ and let ψ be the minimal polynomial of A. Then $f(A) := p(A)$, where p is the polynomial of degree less than*
$$\sum_{i=1}^s n_i = \deg\psi$$
that satisfies the interpolation conditions
$$p^{(j)}(\lambda_i) = f^{(j)}(\lambda_i), \qquad j = 0{:}\, n_i - 1, \quad i = 1{:}\, s. \tag{1.7}$$
There is a unique such p and it is known as the Hermite interpolating polynomial.

An example is useful for clarification. Consider $f(t) = \sqrt{t}$ and
$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3\end{bmatrix}.$$
The eigenvalues are 1 and 4, so $s = 2$ and $n_1 = n_2 = 1$. We take $f(t)$ as the principal branch $t^{1/2}$ of the square root function and find that the required interpolant satisfying $p(1) = f(1) = 1$ and $p(4) = f(4) = 2$ is
$$p(t) = f(1)\frac{t-4}{1-4} + f(4)\frac{t-1}{4-1} = \frac{1}{3}(t+2).$$
Hence
$$f(A) = p(A) = \frac{1}{3}(A + 2I) = \frac{1}{3}\begin{bmatrix} 4 & 2 \\ 1 & 5\end{bmatrix}.$$
It is easily checked that $f(A)^2 = A$. Note that the formula $A^{1/2} = (A+2I)/3$ holds more generally for any diagonalizable $n \times n$ matrix A having eigenvalues 1 and/or 4

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In the following, for a linear map $f : V \to V$, $\ker f$ and $\operatorname{im} f$ denote the kernel and the image of f, respectively.

1. Let V be a finite-dimensional complex inner product space. Let $d : V \to V$ be a linear map satisfying $d^2 = 0$. Let $\delta : V \to V$ be the adjoint of d and $\Delta = d\delta + \delta d$. Prove the following.

   (a) [5%] $d\delta x = 0$ implies that $\delta x = 0$, and $\delta d x = 0$ implies that $dx = 0$, for all $x \in V$.

   (b) [10%] $\ker \Delta = \ker d \cap \ker \delta$.

   (c) [10%] There is the orthogonal decomposition $V = \ker \Delta \oplus \operatorname{im} d \oplus \operatorname{im} \delta$.

   (d) [5%] There is the orthogonal decomposition $\ker d = \ker \Delta \oplus \operatorname{im} d$.

2. [10%] Let $V = \mathbb{R}^n$ be the space of column vectors, and M a positive definite symmetric $n \times n$ real matrix. Suppose the matrix $A \in M_n(\mathbb{R})$ satisfies $MAM^{-1} = A^t$. Show that there exists $P \in M_n(\mathbb{R})$ satisfying $P^tMP = I_n$ such that $P^{-1}AP$ is diagonal. (Here B^t denotes the transpose of the matrix B.)

3. (a) [10%] Let M be an invertible $n \times n$ complex matrix. Prove that there exists an invertible matrix A such that $A^2 = M$.

   (b) [10%] Let $n \geq 2$ and N be an $n \times n$ matrix over a field such that $N^n = 0$ but $N^{n-1} \neq 0$. Prove that there is no square matrix B such that $B^2 = N$.

4. [20%] Let V be a vector space over a field F and $u_1, \cdots, u_n \in V$ are linearly independent. Show that, for any $v_1, \cdots, v_n \in V$, $u_1 + \alpha v_1, \cdots, u_n + \alpha v_n$ are linearly independent for all but finitely many values of $\alpha \in F$.

5. [20%] Let P be an $n \times n$ matrix with coefficients in a field. Suppose $\operatorname{rank}(P) + \operatorname{rank}(I_n - P) = n$. Prove that $P^2 = P$.

Prob. 1 direct sum

(a) Since $0 = \langle d\delta x, x\rangle = \langle \delta x, \delta x\rangle$, $\delta x = 0$. Similarly,

$$0 = \langle \delta d x, x\rangle = \langle dx, dx\rangle, \quad dx = 0.$$

(b) if $x \in \ker\Delta = \ker(d\delta + \delta d)$, $d\delta x + \delta d x = 0 \Rightarrow d^2\delta x + d\delta d x = 0$

$\Rightarrow d\delta d x = 0 \Rightarrow dx = 0 \Rightarrow x \in \ker d$.

Note that $d^2 = 0$ mean $\delta^2 = 0$ $\langle d^2 x, x\rangle = \langle x, \delta^2 x\rangle \equiv 0$

Similar $d\delta x + \delta d x = 0 \Rightarrow \delta d\delta x + \delta^2 dx = 0 \Rightarrow \delta d \delta x = 0 \Rightarrow \delta x = 0$ $x \in \ker\delta$

$\Rightarrow \ker\Delta \subseteq \ker\delta \cap \ker d$, opposite, if $x \in \ker\delta\cap\ker d$, then

$\Delta x = d\delta x + \delta d x = 0 \Rightarrow x \in \ker\Delta$ *

(c) First, we show that $\ker\Delta \perp \operatorname{Im} d$ & $\ker\Delta \perp \operatorname{Im}\delta$ & $\operatorname{Im} d \perp \operatorname{Im}\delta$

If $x \in \ker\Delta$, $y \in \operatorname{Im} d \Rightarrow y = dv$ for some v,

$$\langle x, y\rangle = \langle x, dv\rangle = \langle \overset{=0}{\delta x}, v\rangle = 0 \quad \text{(previous result)}$$

If $x \in \ker\Delta$, $y \in \operatorname{Im}\delta \Rightarrow y = \delta v$ for some v,

$$\langle x, y\rangle = \langle x, \delta v\rangle = \langle \overset{=0}{dx}, v\rangle = 0 \quad \text{(previous result)}$$

If $x \in \operatorname{Im}\delta$, $y \in \operatorname{Im} d$, $x = \delta v_1$, $y \in dv_2$ for some v_1, v_2

$$\langle x, y\rangle = \langle \delta v_1, dv_2\rangle = \langle v_1, d^2 v_2\rangle = 0 \quad (d^2 = 0)$$

Next, we claim that $\forall\, v \in V$, $V = V_1 + V_2 + V_3$, $V_1 \in \ker\Delta$ $V_2 \in \operatorname{Im} d$ $V_3 \in \operatorname{Im}\delta$

Note that Δ is self-adjoint, so $V = \ker\Delta \oplus \left(\bigoplus_{\lambda\neq 0} E_\lambda\right)$, then for any $x \in \bigoplus_{\lambda\neq 0} E_\lambda$ $x = \sum_{\lambda_i} x_{\lambda_i}$, $x_{\lambda_i} \in E_{\lambda_i}$, $x = \sum_{\lambda_i} \frac{1}{\lambda_i}\Delta x_{\lambda_i} \in \operatorname{Im}\Delta$

so $\bigoplus_{\lambda\neq 0} E_\lambda \subseteq \operatorname{Im}\Delta$, By Rank-Kernel theorem, $\dim \bigoplus_{\lambda\neq 0} E_\lambda = \dim \operatorname{Im}\Delta$,

so $\bigoplus_{\lambda\neq 0} E_\lambda = \operatorname{Im}\Delta$. That mean $V = \ker\Delta \oplus \operatorname{Im}\Delta$, However,

$$V = \underbrace{V_1}_{\ker\Delta} + \underbrace{V_2}_{\operatorname{Im}\Delta} = \underbrace{V_1}_{\ker\Delta} + \underbrace{\Delta V_3}_{V_2 = \Delta V_3} = \underbrace{V_1}_{\in\ker\Delta} + \underbrace{d\delta V_3}_{\in \operatorname{Im} d} + \underbrace{\delta d V_3}_{\in\operatorname{Im}\delta} \in \ker\Delta \oplus \operatorname{Im} d \oplus \operatorname{Im}\delta$$

(d) We have showed that $\ker\Delta \perp \operatorname{Im} d$, since $\forall\ v \in \operatorname{Im} d$, $v = dv_0$ for some v_0, $dv = d^2 v_0 = 0$, so $\operatorname{Im} d \subseteq \ker d$. Moreover, we have $\ker\Delta \oplus \operatorname{Im} d \subseteq \ker d$ and since V is finite demension,

$$V = \operatorname{Im}\delta \oplus (\operatorname{Im}\delta)^{\perp} = \operatorname{Im}\delta \oplus \ker d = \operatorname{Im}\delta \oplus \ker\Delta \oplus \operatorname{Im} d, \text{ so}$$

$$\dim(\ker d) = \dim(\ker\Delta \oplus \operatorname{Im} d)$$

Thus $\ker d = \ker\Delta \oplus \operatorname{Im} d$.

Problem 2 Self-adjoint, positive root

Note that $MA = A^t M$ mean $(MA)^t = A^t M^t = A^t M = MA$

MA is self-adjoint, Moreover, $A = M^{-1}(MA) = M^{-\frac{1}{2}} M^{-\frac{1}{2}} S M^{-\frac{1}{2}} M^{\frac{1}{2}}$

Note that $M^{-\frac{1}{2}} S M^{-\frac{1}{2}}$ is self-adjoint, so A is diagonal and can be written as $A = M^{-\frac{1}{2}} P_1 D P_1^{-1} M^{\frac{1}{2}}$, Let R be $M^{-\frac{1}{2}} P_1$, then $R^{-1} A R = D$ and

$$R^t M R = P_1^t M^{-\frac{1}{2}} M M^{-\frac{1}{2}} P_1 = P^t P_1 = I$$

(Since $M^{-\frac{1}{2}} S M^{-\frac{1}{2}}$ is self-adjoint
$P^t = P^{-1}$

Prob. 3 nilpotent
(a)

8.21 Description of operators on complex vector spaces

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T. Then

(a) $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$;

(b) each $G(\lambda_j, T)$ is invariant under T;

(c) each $(T - \lambda_j I)|_{G(\lambda_j, T)}$ is nilpotent.

8.31 Identity plus nilpotent has a square root

Suppose $N \in \mathcal{L}(V)$ is nilpotent. Then $I + N$ has a square root.

Let λ_i be distinct eig-value of T, then by 8.21, each

$$T - \lambda_j I \big|_{G(\lambda_j, T)} = N_{\lambda_j} \Rightarrow T = I + \frac{1}{\lambda_j} N_{\lambda_j} \rightarrow \lambda_j \neq 0 \text{ since } T \text{ inver.}$$

By 8.31, $\sqrt{T}$ exist on $G(\lambda_j, T)$, say $\sqrt{T}_{\lambda_j}$. By 8.21 again,

$V = \bigoplus_{i=1} G(\lambda_i, T)$, $v = \sum_{i=1} v_i$ $v_i \in G(\lambda_i, T)$, so we define

$$Rv = \sqrt{T}_{\lambda_1} v_1 + \cdots + \sqrt{T}_{\lambda_j} v_j$$

Then

$$R^2 v = R(Rv) = R\left(\sqrt{T}_{\lambda_1} v_1 + \cdots + \sqrt{T}_{\lambda_j} v_j\right)$$

$$= (\sqrt{T}_{\lambda_1})^2 v_1 + \cdots + (\sqrt{T}_{\lambda_j})^2 v_j = T_{\lambda_1} v_1 + \cdots + T_{\lambda_j} v_j = Tv$$

(b)

Suppose not, $\exists\ B$ s.t. $B^2 = N$, Then $B^{2n} = N^n = 0$, Thus min-poly $\mid x^{2n}$, Moreover, min-poly | char-poly implies $B^n = 0$,

$\because\ 2n-2 = n + n-2 \geq n$, $B^{2n-2} = 0$, but $N^{n-1} = B^{2n-2} \neq 0 \Rightarrow \Leftarrow$.

Problem 5.

Goal : $P(I-P)v = 0\ \ \forall\ v \in V$, note that rank P + ker$P = n$,

so ker P = rank $(I-P)$, but $\forall\ v \in \ker P$, $v = v - Pv = (I-P)v$, so

ker $P \subseteq$ rank$(I-P)$, That induces that $\ker P = \operatorname{Im}(I-P)$, That is, $\forall v$

$$P(I-P)v = 0 \Rightarrow (P-P^2)v = 0 \Rightarrow Pv = P^2 v \ \#$$

Prob. 4

Let $\dim V = m$, $n \leq m$, extend $u_1, \cdots, u_n$ to a basis $\{u_i\} \sqcup \{z_j\}$.

To test $\{u_i + \alpha v_i\}_{i=1}^n$ linear indep. Let $x_i \in F$ such that

$$x_1(u_1 + \alpha v_1) + x_2(u_2 + \alpha v_2) + \cdots + x_n(u_n + \alpha v_n) = 0 \quad \text{—— } (*)$$

Since $\{u_i\} \sqcup \{z_j\}$ is a basis, $v_i = \sum_{j=1}^n y_{ij} u_j + \sum_{j=n+1}^m y_{ij} z_j$

So $(*)$ become

$$(x_1 + \alpha y_{11} x_1 + \alpha y_{21} x_2 + \cdots + \alpha y_{n1} x_n) u_1 + (\alpha y_{12} x_1 + x_2 + \alpha y_{22} x_2 + \cdots + \alpha y_{n2} x_n) u_2 +$$

$$\cdots + (\alpha y_{1n} x_1 + \alpha y_{2n} x_2 + \cdots + x_n + \alpha x_n y_{nn}) u_n = 0$$

$$\Rightarrow \overbrace{\begin{pmatrix} 1+\alpha y_{11}, & \alpha y_{21}, & \alpha y_{31}, & \cdots, & \alpha y_{n1} \\ \alpha y_{12}, & 1+\alpha y_{22}, & \alpha y_{32}, & \cdots, & \alpha y_{n2} \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha y_{1n}, & \alpha y_{1n}, & \cdots & , & 1+\alpha y_{nn} \end{pmatrix}}^{A} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Then the problem become " How many α such that $\ker A \neq 0$, Note that $A = I + \alpha \underbrace{\begin{pmatrix} y_{11} & y_{21} & \cdots & y_{n1} \\ \vdots & \vdots & & \vdots \\ y_{1n} & y_{1n} & & y_{nn} \end{pmatrix}}_{B} = I + \alpha B$, Hence, when we discuss $\ker A \neq \{0\} \iff$ Does 0 is a eig-value of $I + \alpha B$, Let $\{\lambda_i\}_{i=1}$ be distinct eig-value of B, then eig-value of $A = \{1 + \alpha \lambda_i\}_{i=1}$, so $1 + \alpha \lambda_i = 0$ for some $i \iff \alpha = \frac{-1}{\lambda_i}$, $\lambda_i \neq 0$, That complete proof.

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1. Let A be a 4×4 real symmetric matrix. Suppose that 1 and 2 are eigenvalues of A and the eigenspace for the eigenvalue 2 is 3-dimensional. Assume that $(1, -1, -1, 1)^t$ is an eigenvector for the eigenvalue 1. (Here v^t denotes the transpose of v.)
   (a) Find an orthonormal basis for the eigenspace for the eigenvalue 2 of A. (**10 points.**)
   (b) Find Av, where $v = (1, 0, 0, 0)^t$. (**10 points.**)
2. Let A be a real $n \times n$ matrix. Prove that
$$\operatorname{rank}(A^2) - \operatorname{rank}(A^3) \leq \operatorname{rank}(A) - \operatorname{rank}(A^2).$$
(**10 points.**)
3. Let V be a vector space of finite dimension over $\mathbb{R}$ and S, T, and U be subspaces of V. Prove or disprove (by giving counterexamples) the following statements:
   (a) $\dim(S+T) = \dim S + \dim T - \dim(S \cap T)$. (**10 points.**)
   (b) $\dim(S+T+U) = \dim S + \dim T + \dim U - \dim(S \cap T) - \dim(T \cap U) - \dim(U \cap S) + \dim(S \cap T \cap U)$. (**10 points.**)
4. (a) Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$. Compute $\exp A$. (**10 points.**)
   (b) Prove that $\det(\exp A) = \exp(\operatorname{tr} A)$ for $A \in M(n, \mathbb{C})$. (**10 points.**)
   (c) Prove or disprove (by giving counterexamples) that if A is nilpotent, then so is $\exp A - I_n$. Here a matrix M is said to be nilpotent if $M^k = 0$ for some positive integer k and I_n is the identity matrix of size n. (**10 points.**)
5. Let U and V be finite-dimensional vector spaces, and U^* and V^* be their dual spaces, respectively. For a linear transformation $T : U \to V$, define $T^* : V^* \to U^*$ by $(T^* f)(u) = f(Tu)$ for $f \in V^*$ and $u \in U$.
   (a) Prove that T is injective if and only if T^* is surjective. (**10 points.**)
   (b) Prove that T is surjective if and only if T^* is injective. (**10 points.**)

Prob. 1 Symmetric matrix

(1) Note that $\exists\ Q$ s.t. $QQ^t = I_4$ and $QAQ^t = D$, so if $(1, [1,-1,-1,1])$, then $\left(2, \operatorname{span}\left\{\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right\}\right)$ obviously

(2)

$$A\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = A\left(\frac{1}{4}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} + \frac{1}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right)$$

$$= 1\times\frac{1}{4}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix} + 2\times\frac{1}{2}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} + 2\times\frac{1}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}7/4\\1/4\\1/4\\-1/4\end{pmatrix}$$

Prob. 2 Frobenius inequality rank

We show more general case

$$\operatorname{rank}(A^{k+1}) - \operatorname{rank}(A^{k+2}) \le \operatorname{rank}(A^k) - \operatorname{rank}(A^{k+1})$$

Note that $\operatorname{Im}(A^{k+1}) \subseteq \operatorname{Im}(A^k)$, so there exist G (vec. space) such that

$\Rightarrow$ $\operatorname{Im}(A^{k+1}) \oplus G = \operatorname{Im} A^k$, More, $A(\operatorname{Im} A^k) = \operatorname{Im}(A^{k+1})$, so

$$\operatorname{Im}(A^{k+1}) = A(\operatorname{Im}(A^{k+1})) + AG = \operatorname{Im}(A^{k+2}) + AG$$

so

$$\begin{aligned}\operatorname{Rank}(A^{k+1}) &= \operatorname{Rank}(A^{k+2}) + \operatorname{Rank}(AG) - \operatorname{Rank}(A^{k+2} \cap AG) \\ &\le \operatorname{Rank}(A^{k+2}) + \operatorname{Rank}(AG) \\ &\le \operatorname{Rank}(A^{k+2}) + \operatorname{Rank} G \le \operatorname{Rank}(A^{k+2}) + \operatorname{Rank}(A^k) - \operatorname{Rank}(A^{k+1})\end{aligned}$$

$$\Rightarrow \operatorname{Rank}(A^{k+1}) - \operatorname{Rank}(A^{k+2}) \le \operatorname{Rank}(A^k) - \operatorname{Rank}(A^{k+1}) \ \#$$

Prob. 3 Prove

(A) Let B be a basis of $S \cap T$, Then extend B to be $A \sqcup B$ ($A \cap B = \phi$), a basis of S and $B \sqcup C$ ($B \cap C = \phi$), a basis of T. We claim that $A \sqcup B \sqcup C$ is a basis of $S+T$. $\operatorname{Span}(A \sqcup B \sqcup C) = S+T$ is obviously, to see it is independent, let α_i $1 \le i \le |A \cup B \cup C|$, $\alpha_1 V_1 + \alpha_2 V_2 + \cdots + \alpha_n V_n = 0$ (with $n = |A \cup B \cup C|$)

if $\exists\ \alpha_j \neq 0$ for some j such that $(*) = 0$, Then $\exists\ \alpha_{j_1} V_{j_1} \in A$

and $\alpha_{j_2} \alpha_{j_2} \in C$ (Since $A \cup B$ & $B \cup C$ are linear indep.), so there is

$x \in \langle A \rangle \cap \langle B \cup C \rangle \Rightarrow x \in S \cap T$ ($S = \text{Span}(A \cup B)$ $T = \text{Span}(B \cup C)$

$x \in \langle A \rangle \cap \langle B \rangle \Rightarrow x = 0$, a contradiction $\Rightarrow A \cup B \cup C$ linear ind.

$\Rightarrow$

$$\dim S + \dim T = |A \cup B| + |B \cup C| = |A| + |B| + |B| + |C|$$

$$= |A \cup B \cup C| + |B| = \dim(S+T) + \dim(S \cap T) \ \#$$

(b) disprove, Let $S = \langle s_1 \rangle$ $T = \langle t_1 \rangle$ $U = \langle s_1 + t_1 \rangle$

$$\dim(S + T + U) = \dim(S+T) = 2$$

$$\dim S + \dim T + \dim U - \dim(S \cap T) - \dim(T \cap U) - \dim(S \cap U)$$

$$+ \dim(S \cap T \cap U) = 1 + 1 + 1 - 0 - 0 - 0 - 0 = 3 \ \#$$

Prob 4. exp A

(a)

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},$$

$$\exp A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e & e \\ 0 & e \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -e & 2e \\ -e & e \end{pmatrix} = \begin{pmatrix} 0 & e \\ -e & 2e \end{pmatrix}$$

(b) Let $A = U \Sigma U^{-1}$, Where Σ is upper-triangular matrix

$\exp A = \exp U \Sigma U^{-1} = U \exp \Sigma\, U^{-1} \Rightarrow \det(\exp A) = \det(\exp \Sigma)$

Note that Σ^n is also a upper-tri matrix and entries on diagonal is

eig-value, entries of Σ^n is λ_i^n, Then

$$\det(\exp A) = \det(\exp \Sigma) = \exp\left(\Sigma_i \lambda_i\right) = \exp(\operatorname{tr} A) \ \#$$

(c) Since $M^k = 0$ for some $k \in \mathbb{N}$, so $\exp(M) = \sum_{i=0}^{k} \frac{1}{i!} M^i$

Then $\exp(M) - I = M\left(I_n + \frac{1}{2!} M + \cdots + \frac{1}{k!} M^{k-1}\right)$

$$\left(\exp(M) - I\right)^k = M^k \left(I_n + \frac{1}{2!} M + \cdots + \frac{1}{k!} M^{k-1}\right)^k = 0$$

So $\exp(M) - I$ is also a nilpotent.

Prob. 5 Relation between Range T, $\ker T^*$, M°

Let M be a set of vec. space V, The annihilator of M is defined by

$$M^\circ = \{ f \in V^* : f(M) = 0 \}$$

Note that

$$\dim M^\circ = \dim V^* - \dim T^*$$
$$= \dim V - \dim M \quad \text{For } \dim V < \infty$$

Note that

$$\ker(T^*) = \{ f \in V^* : f(Tu) \equiv 0 \}$$
$$= \{ f \in V^* : f(\operatorname{Im} T) \equiv 0 \}$$
$$= (\operatorname{Im} T)^\circ$$

$$\operatorname{Im}(T^*) = \{ g \in U^* : g = f(Tu) \text{ for some } f \in V^* \}$$

If $g \in \operatorname{Im} T^*$, $\forall v \in \ker T$, $g(v) = f(Tv) = f(0) = 0$

so $g \in (\ker T)^\circ$, For the reverse inclusion, Let $g \in (\ker T)^\circ$,

Goal : $\exists f \in V^*$ such that $g = f \circ T$. Let $U = \ker T \oplus S$,

$S = \langle \beta_1, \cdots, \beta_n \rangle$, define $f(T\beta_i) = g(\beta_i)$ (Note that $\{T\beta_i\}_{i=1}^{n}$ linear ind.

Then $g = f \circ T$ on U, $g \in \operatorname{Im} T^*$.

Thus, we have (1) $\operatorname{Im} T^* = (\ker T)^\circ$ (2) $\ker T^* = (\operatorname{Im} T)^\circ$

(3) $\dim U - \dim S = \dim S^\circ$

So

$$\dim(\operatorname{Im} T^*) = \dim(\ker T)^\circ = \dim U - \dim \ker T$$

$$\Rightarrow \dim(\operatorname{Im} T^*) + \dim \ker T = \dim U = \dim U^*$$

and

$$\dim \ker T^* = \dim(\operatorname{Im} T)^\circ = \dim V - \dim \operatorname{Im} T$$

$$\Rightarrow \dim \ker T^* + \dim \operatorname{Im} T = \dim V$$

T is injective $\Leftrightarrow$ $\dim \ker T = 0$ $\Leftrightarrow$ $\dim \operatorname{Im} T^* = \dim U^*$

$\Leftrightarrow$ T^* is surjective

T is surjective $\Leftrightarrow$ $\dim \operatorname{Im} T = \dim V$ $\Leftrightarrow$ $\dim \ker T^* = 0$

$\Leftrightarrow$ T^* is injective.

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科目：線性代數

2018.10.19

1. Find all possible Jordan forms for 8×8 real matrices having $x^2(x-2)^3$ as minimal polynomial. (**20 points.**)
2. Let V be a vector space over a field $\mathbb{F}$ of infinite elements, and let $v_1, \ldots, v_n$ be vectors in V, where n is a positive integer. Suppose that $v_0 + zv_1 + \ldots + z^n v_n = 0$ for infinitely many z in $\mathbb{F}$. Prove that all v_i's are zero. (**20 points.**)
3. Let $V = M(n, \mathbb{R})$ be the vector space of all $n \times n$ matrices and $f : V \to \mathbb{R}$ be a linear transformation. Assume that $f(AB) = f(BA)$ for all $A, B \in V$ and $f(I_n) = n$, where I_n is the identity matrix in V. Prove that f is the trace function. (**20 points.** *Hint*: Consider the cases $A = E_{ij}$ and $B = E_{k\ell}$ for various E_{ij} and $E_{k\ell}$. Here E_{ij} denotes the matrix whose (i, j)-entry is 1 and whose other entries are 0.)
4. Let V be an n-dimensional vector space over $\mathbb{R}$ and $B : V \times V \to \mathbb{R}$ be a symmetric bilinear form on V. (*Symmetric* means $B(u, v) = B(v, u)$ for all $u, v \in V$. *Bilinear* means that B is linear in each of the two variables.)
   (a) Let W be a vector subspace of V and let
   $$W^\perp = \{u \in V : B(u, v) = 0 \text{ for all } v \in W\}.$$
   Prove that if $\dim W = m$, then $\dim W^\perp \geq n - m$. (**10 points.** *Hint*: Choose a basis $\{v_1, \ldots, v_m\}$ for W and consider the map
   $$u \longmapsto (B(u, v_1), \ldots, B(u, v_m))$$
   from V into $\mathbb{R}^m$.)
   (b) Prove that $V = W \oplus W^\perp$ if and only if the restriction of B to W is non-degenerate. (*Nondegenerate* means that $v = 0$ is the only vector of W such that $B(u, v) = 0$ for all $u \in W$.) (**15 points.**)
   (c) Prove that if B is nondegenerate on V, then there is a nonnegative integer p with $p \leq n$ and a basis $\{v_1, \ldots, v_n\}$ such that
   $$B(v_i, v_j) = \begin{cases} 1, & \text{if } 1 \leq i = j \leq p, \\ -1, & \text{if } p+1 \leq i = j \leq n, \\ 0, & \text{if } i \neq j. \end{cases}$$
   (**15 points.**)

Prob. 1 Jordan Form

$$\begin{pmatrix} \begin{smallmatrix}2&0&0\\0&2&0\\0&0&2\end{smallmatrix} & & X \\ & \begin{smallmatrix}2&1&0\\0&2&1\\0&0&2\end{smallmatrix} & \\ X & & \begin{smallmatrix}0&1\\0&0\end{smallmatrix}\end{pmatrix}、\begin{pmatrix} \begin{smallmatrix}2&1&0\\0&2&1\\0&0&2\end{smallmatrix} & & X \\ & \begin{smallmatrix}2&1&0\\0&2&1\\0&0&2\end{smallmatrix} & \\ X & & \begin{smallmatrix}0&1\\0&0\end{smallmatrix}\end{pmatrix}、\begin{pmatrix} 2 & & & X \\ & \begin{smallmatrix}2&1&0\\0&2&1\\0&0&2\end{smallmatrix} & & \\ & & \begin{smallmatrix}0&1\\0&0\end{smallmatrix} & \\ X & & & \begin{smallmatrix}0&1\\0&0\end{smallmatrix}\end{pmatrix}$$

$$\begin{pmatrix} 0 & & & X \\ & \begin{smallmatrix}2&1&0\\0&2&1\\0&0&2\end{smallmatrix} & & \\ & & \begin{smallmatrix}0&1\\0&0\end{smallmatrix} & \\ X & & & \begin{smallmatrix}0&1\\0&0\end{smallmatrix}\end{pmatrix}、\begin{pmatrix} \begin{smallmatrix}2&0&0\\0&2&0\\0&0&0\end{smallmatrix} & & X \\ & \begin{smallmatrix}2&1&0\\0&2&1\\0&0&2\end{smallmatrix} & \\ X & & \begin{smallmatrix}0&1\\0&0\end{smallmatrix}\end{pmatrix}、\begin{pmatrix} \begin{smallmatrix}2&0&0\\0&0&0\\0&0&0\end{smallmatrix} & & X \\ & \begin{smallmatrix}2&1&0\\0&2&1\\0&0&2\end{smallmatrix} & \\ X & & \begin{smallmatrix}0&1\\0&0\end{smallmatrix}\end{pmatrix}$$

Prob. 2 Fundamental theorem of Algebra

If some $v_1 \neq 0 \Rightarrow$ some entry of $v_1 \neq 0$, by FT of Algebra, $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z\, a_0 = 0$ only have most n roots, That is contradiction.

Prob. 3 Trace func.

Since f is linear, if we show that $f(E_{ij}) = 0$ for any $i \neq j$, and $f(E_{ii}) = 1$, then f must be the trace func.

$\Rightarrow$ Consider $f(E_{ij} E_{kl})$, we have following case.

$i = l$, $j = k$, but $i \neq j$. then

$$\begin{aligned} f(E_{ij} E_{kl}) &= f(E_{il}) \\ f(E_{kl} E_{ij}) &= f(E_{kj}) \end{aligned} \Rightarrow f(E_{il}) = f(E_{kj}),$$

Since $f(I_n) = n$,
$n = n f(E_{ii})$
$\Rightarrow f(E_{ii}) = 1$

$i = l$, $j \neq k$

$$\begin{aligned} f(E_{ij} E_{kl}) &= f(0) = \underset{\substack{\uparrow \\ f \text{ is linear}}}{0} \\ f(E_{kl} E_{ij}) &= f(E_{kj}) \end{aligned} \Rightarrow f(E_{kj}) = 0$$

Thus, f is the trace func.

Prob. 4 Bilinear symmetric func.

(a) Choose a basis of W, say $\{v_1, \cdots, v_m\}$, Consider the map $\hat{B}(u)$

$V \to \mathbb{R}^m$ by $v \longmapsto (B(v, v_1), \cdots, B(v, v_m))$, Then by Rank - Null Theorem $\dim V = \dim \ker \hat{B} + \dim \operatorname{Im} \hat{B}$, so

$$\dim W^{\perp} = \dim \ker \hat{B} = \dim V - \dim \operatorname{Im} \hat{B}$$
$$\geq \dim V - \dim \mathbb{R}^m$$
$$= n - m.$$

(b) If $V = W \oplus W^{\perp}$, $v \in W$ such that $B|_W(u, v) = 0$ for all $u \in W$ mean $v \in W^{\perp}$, but $W \cap W^{\perp} = \{0\}$, so $v = 0$.

If $B|_W$ is non-degenerate, then $W \cap W^{\perp} = \{0\}$ obviously, Note that $W + W^{\perp} \leq V$, but $\dim W + W^{\perp} = \dim W + \dim W^{\perp}$ large then n (by (a)), so $\dim V = \dim W + W^{\perp} = n$. Thus $V = W + W^{\perp}$.

Hoffman, Linear Algebra : P. 369.

(c)

First, we show that $\exists$ a basis of V, say $\{\alpha_i\}_{i=1}^{n}$, such that $f(\alpha_i, \alpha_j) = 0$ for $i \neq j$

We use the induction on n, if $B(u, v) \equiv 0$ or $\dim V = 1$, then the result is obvious true. Suppose that is true for $n < k$

When $n = k$, First, we find a vector α s.t. $f(\alpha, \alpha) \neq 0$, if no this vector exist, note that $f(\alpha, \beta) = \frac{1}{4}[f(\alpha+\beta, \alpha+\beta) - f(\alpha-\beta, \alpha-\beta)]$.

Thus, if $f(\alpha, \alpha) = 0$ $\forall \alpha \in V \Rightarrow f(\alpha, \beta) \equiv 0$ for any $\alpha, \beta \in V$, $\Rightarrow\Leftarrow$

Hence, $\exists$ α s.t. $f(\alpha, \alpha) \neq 0$. Let $W = \langle \alpha \rangle$ and $W^{\perp} = \langle \alpha \rangle^{\perp}$

We claim that $V = W \oplus W^{\perp}$, First, if $\beta \in W \cap W^{\perp}$, $\beta = c\alpha$ and $f(\alpha, \beta) = 0$, $f(\alpha, c\alpha) = c f(\alpha, \alpha) = 0 \Rightarrow c = 0 \Rightarrow \beta = 0$.

Next, if $\gamma \in V$, let $\beta = \gamma - \frac{f(\alpha, \gamma)}{f(\alpha, \alpha)} \alpha$

$$f(\alpha, \beta) = f(\alpha, \gamma) - \frac{f(\alpha, \gamma)}{f(\alpha, \alpha)} f(\alpha, \alpha) = 0$$

$\beta \in W^{\perp}$, Then $\gamma = \frac{f(\alpha, \gamma)}{f(\alpha, \alpha)} \alpha + \beta \in W + W^{\perp}$, so $V = W \oplus W^{\perp}$

By induction, $\exists \{\alpha_2, \cdots \alpha_n\}$ such that $f(\alpha_i, \alpha_j) = 0$, $W^{\perp} = \langle \{\alpha_i\}_{i=2}^{n} \rangle$

Take $\langle \alpha_1, \alpha_2, \cdots, \alpha_n \rangle$, Then $V = \langle \alpha_1, \cdots, \alpha_n \rangle$ #

Use this basis, let $\beta_i = \alpha_i / |f(\alpha_i, \alpha_i)|^{1/2}$. Since $B(u,v)$ is non-degenerate, $f(\alpha_i, \alpha_i) \neq 0$, otherwise, $f(\beta, \alpha_i) = 0 \Rightarrow \alpha_i = 0 \Rightarrow \Leftarrow$.

Then $\langle \beta_1, \cdots, \beta_n \rangle$ is the basis we want.

$$\left(f(\beta_i, \beta_i) = \frac{1}{|f(\alpha_i, \alpha_i)|} f(\alpha_i, \alpha_i) = \pm 1 \right)$$

By rearrange $\langle \beta_1, \cdots, \beta_n \rangle$ to $\langle \hat{\beta}_1, \cdots, \hat{\beta}_n \rangle$, we can find $p \leq n$ such that $f(\beta_i, \beta_i) = 1$ if $1 \leq i \leq p$; $f(\beta_i, \beta_i) = -1$ if $p+1 \leq i \leq n$.

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科目：線性代數

2017.10.20

(1) (20 points) Let $A = \begin{pmatrix} -1 & 3 & -2 \\ 2 & 3 & 0 \\ 11 & -6 & 7 \end{pmatrix}$. Find the lower triangular Jordan canonical form of A. Please compute $\exp(tA)$ and derive the general solution to $x'(t) = A\, x(t)$, where $x(t)$ is a 3-dimensional column vector.

(2) (20 points) Let V be an n-dimensional complex vector space, and $T : V \to V$ be an invertible linear map such that $T^2 = 1$. (a) Show that T is diagonalizable, (b) Let S be the vector space of linear transformations from V to V that commute with T. Please express $\dim_{\mathbb{C}} S$ in terms of n and the trace of T.

(3) (20 points) Let $A = (A_{ij})$ be a real invertible skew-symmetric $2n \times 2n$ matrix.
(a) Show that all eigenvalues of A are pure imaginary.
(b) Define the Pfaffian $Pf(A)$ of A by

$$Pf(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) A_{\sigma(1),\sigma(2)} A_{\sigma(3),\sigma(4)} \cdots A_{\sigma(2n-1),\sigma(2n)} .$$

Let B be any real $2n \times 2n$ matrix. Show that $Pf(BAB^T) = Pf(A)\det(B)$.
(c) Assuming the fact that there exists a real orthogonal $2n \times 2n$ matrix O such that

$$OAO^T = \mathrm{diag}\left\{ \begin{pmatrix} 0 & m_1 \\ -m_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & m_2 \\ -m_2 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & m_n \\ -m_n & 0 \end{pmatrix} \right\},$$

where $m_i \in \mathbb{R}$ for $i = 1, \cdots, n$. Show that $\det(A) = Pf(A)^2$.

(4) (20 points) Let $A, B \in M_n(\mathbb{C})$ be $n \times n$ complex matrices. Show that A and B are simultaneously triangularizable (*i.e.* there exists an invertible matrix $P \in GL_n(\mathbb{C})$ such that PAP^{-1} and PBP^{-1} are both upper triangular) if A and B commute.
Hint: Let λ be one of the eigenvalues of A. Try to show $B(\ker(A - \lambda I)) \subset \ker(A - \lambda I)$.

(5) (20 points) Show that

$$\begin{vmatrix} X_0 & X_1 & X_2 & \dots & X_{n-1} \\ X_{n-1} & X_0 & X_1 & \dots & X_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_1 & X_2 & X_3 & \dots & X_0 \end{vmatrix} = \prod_{j=0}^{n-1} \left(\sum_{k=0}^{n-1} \zeta^{jk} X_k \right)$$

where ζ is a primitive n-th root of unity.

Hint: You may first compute, for example, $\begin{pmatrix} X_0 & X_1 & X_2 & X_3 \\ X_3 & X_0 & X_1 & X_2 \\ X_2 & X_3 & X_0 & X_1 \\ X_1 & X_2 & X_3 & X_0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 & \zeta^3 \\ 1 & \zeta^2 & \zeta^4 & \zeta^6 \\ 1 & \zeta^3 & \zeta^6 & \zeta^9 \end{pmatrix}$.

Prob. 1 Jordan Form

$$\det(A-\lambda) = \left|\begin{pmatrix} -1-\lambda & 3 & -2 \\ 2 & 3-\lambda & 0 \\ 11 & -6 & 7-\lambda \end{pmatrix}\right| = (7-\lambda)(\lambda^2-2\lambda-9) - 2(11\lambda-45)$$

$$= -22\lambda + 90 - \lambda^3 + 2\lambda^2 + 9\lambda + 7\lambda^2 - 14\lambda - 63$$

$$= -\lambda^3 + 9\lambda^2 - 27\lambda + 27$$

$$= -(\lambda-3)^3$$

$$\Rightarrow A-3I = \begin{pmatrix} -4 & 3 & -2 \\ 2 & 0 & 0 \\ 11 & -6 & 4 \end{pmatrix} \Rightarrow \ker(A-3I) = \left\langle \left\{ \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \right\} \right\rangle$$

$$(A-3I)V_2 = V_1 \Rightarrow V_2 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

$$(A-3I)V_3 = V_2 \Rightarrow V_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow A = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & -2 & 3 \end{pmatrix}^{-1}}_{Q^{-1}} \begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & -2 & 3 \end{pmatrix}}_{Q}$$

$$\Rightarrow \exp A = Q^{-1} \exp\begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \end{pmatrix} Q$$

$$= Q^{-1} \sum_{k=1}^{\infty} \frac{1}{k!} \begin{pmatrix} 3^k & 0 & 0 \\ \binom{k}{1}3^{k-1} & 3^k & 0 \\ \binom{k}{2}3^{k-2} & \binom{k}{1}3^{k-1} & 3^k \end{pmatrix} Q = Q^{-1} \begin{pmatrix} e^3 & 0 & 0 \\ e^3 & e^3 & 0 \\ \frac{e^3}{2} & e^3 & e^3 \end{pmatrix} Q$$

Prob. 2 Min-poly

(1) Since $T^2 = 1 \Rightarrow T^2 - 1 \equiv 0 \Rightarrow$ min-p of $T \mid x^2-1$

$\Rightarrow$ min-p of T is a product of linear form $(x-1)(x+1)$, $(x+1)$, $x-1$

$\Rightarrow$ T is diagonalizable (Since T is invertible $\lambda_T = 1$ or -1 only

(2) Find a basis of V such that $[T]_\beta$ is diagonal, Then

$ST = TS$ mean if $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, A、B、C、D are submatrix

Let $x = \dim(\ker(T-I))$, $y = \dim(\ker(T+I)) \Rightarrow x-y = \text{trace } T$

$$ST = TS \Leftrightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} A & -B \\ C & -D \end{pmatrix} = \begin{pmatrix} A & B \\ -C & -D \end{pmatrix} \Rightarrow \begin{cases} B = C = 0 \\ A, D \text{ arbitrary.} \end{cases}$$

Thus, $\dim S = x^2 + y^2$, Note that $x+y = n$ $x-y = \text{trace } T$

$$\Rightarrow \dim S = \frac{n^2 + (\text{trace } T)^2}{2}$$

Prob. 3 Skew symmetric matrix

(a) Let λ, v be a eign-value / vector respectively, Then

$$\lambda \langle v, v\rangle = \langle \lambda v, v\rangle = \langle Av, v\rangle = \langle v, -Av\rangle = \langle v, -\lambda v\rangle$$
$$= -\bar{\lambda}\langle v, v\rangle$$

$\Rightarrow \lambda = -\bar{\lambda} \Rightarrow \lambda$ is pure imaginary.

(b)

$$\text{Pf}(A) \equiv \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) A_{\sigma(1)\sigma(2)} A_{\sigma(3)\sigma(4)} \cdots A_{\sigma(2n-1)\sigma(2n)}$$

We use exterior algebra to show !!

Exterior algebra $\{x^1, \ldots, x^{2n}\}$, $x^i \wedge x^j = -x^j \wedge x^i$, $x^i \wedge x^i = 0$

Def $w_i = \sum_{j=1}^n a_{ij} x^j \Rightarrow A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$

$$\Rightarrow w_1 \wedge w_2 \wedge \cdots \wedge w_n = \det A \; x^1 \wedge x^2 \wedge \cdots \wedge x^n$$

Def $\Omega = \sum_{i<j} a_{ij} e_i \wedge e_j \Rightarrow \frac{1}{n!}(\overbrace{\Omega \wedge \Omega \wedge \cdots \wedge \Omega}^{n}) = \text{pf}(A)\, e_1 \wedge \cdots \wedge e_n$

or $\bar{\Omega} = \sum a_{ij} e_i \wedge e_j \Rightarrow (\bar{\Omega})^n = 2^n n!\, \text{pf}(A)\, e_1 \wedge \cdots \wedge e_n$

Thus,

$$A \to \Omega_A = \sum a_{ij} e_i \wedge e_j \xrightarrow{\wedge^n} 2^n n!\, \text{Pf}(A)\, e_1 \wedge \cdots \wedge e_{2n}$$

$$BAB^T \mapsto \sum_{ij} (BAB^T)_{ij} e_i \wedge e_j \to \sum_{ij} \Big(\sum_{k\ell} b_{ik} b_{j\ell} a_{k\ell}\Big) e_i \wedge e_j$$

Let $f_k = \sum_i B_{ik} e_i$, $f_\ell = \sum_j B_{j\ell} e_j$

$$\Rightarrow \sum_{ij} \Big(\sum_{k\ell} b_{ik} b_{j\ell} a_{k\ell}\Big) e_i \wedge e_j = \sum_{k\ell} a_{k\ell} \Big(\sum_{ij} b_{ik} b_{j\ell}\Big) e_i \wedge e_j$$

$= \sum_{kl} a_{kl} f_k \wedge f_l \xrightarrow{\wedge^n} 2^n n!\, Pf(A)\, f_1 \wedge \cdots \wedge f_{2n} = 2^n n!\, Pf(BAB^T)\, e_1 \wedge \cdots \wedge e_{2n}$

But $f_1 \wedge \cdots \wedge f_{2n} = \det(B)\, e_1 \wedge \cdots \wedge e_n$, so $Pf(A) \det B = Pf(BAB^T)$ #

(c) Skew - Symmetric Matrix $\rightarrow$ spectral theorem

$\Rightarrow \forall\ M_{2n}(\mathbb{R})$, $A^T = -A$ matrix, $A = Q \Sigma Q^T$, Q is orthogonal

And
$$\Sigma = \begin{bmatrix} \begin{matrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{matrix} & & X \\ & \ddots & \\ X & & \begin{matrix} 0 & \lambda_r \\ -\lambda_r & 0 \end{matrix} \end{bmatrix}$$, where λ_i is spectral of A

$\Rightarrow$ Note that $\det A = \det(Q \Sigma Q^T) = \det \Sigma = \lambda_1^2 \times \cdots \times \lambda_n^2$, where λ_i is positive part of eig-value

$\Rightarrow pf(A) = pf(Q \Sigma Q^T) = \det(Q)\, pf(\Sigma) = \lambda_1 \cdots \lambda_n \Rightarrow (pf(A))^2 = \det A$ #

Prob. 4 Simultaneously triangular Hoffman P. 207

First, we claim $B(\ker(A - \lambda I)) \subset \ker(A - \lambda I)$, if $v \in B(\ker(A - \lambda I))$,

$v = Bu$, $u \in \ker(A - \lambda I) \Rightarrow Av = ABu = BAu = B\lambda u = \lambda Bu = \lambda v \Rightarrow v \in \ker A - \lambda I$,

Second, Consider $A|_{\ker(A-\lambda I)}$ and $B|_{\ker(A-\lambda I)}$,

$\dim(\ker(A - \lambda I)) \neq 0$, if $\dim(\ker(A - \lambda I)) = \dim V$, Since any matrix can be

written in $Q \Sigma Q^{-1}$, Σ is upper triangular, Choose Q, each column vec.

of $Q \in \ker(A - \lambda I) \Rightarrow [A]_Q = \lambda I$, $B = \Sigma \Rightarrow P = Q$

if $\dim(\ker(A - \lambda I)) < \dim V$, Use induction, $A|_{\ker(A-\lambda I)}$ and $B|_{\ker(A-\lambda I)}$ can

be simultaneously triangularizable, More general. $B(\underbrace{\ker(A - \lambda I)^n}_{G(\lambda)}) \subseteq \ker(A - \lambda I)^n$

Note that $V = G(\lambda_1) \oplus \cdots \oplus G(\lambda_n)$, each $\dim G(\lambda_i) < \dim V$, so it can find

a basis β_{λ_i} such that $A|_{G(\lambda_i)}$ and $B|_{G(\lambda_i)}$ are simu - tri, Take $\beta = \sqcup_i \{\beta_i\}$,

we done.

Prob. 5 V - determinae

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_{n-1} & x_0 & \cdots & x_{n-2} \\ \vdots & \vdots & & \vdots \\ x_1 & x_2 & \cdots & x_0 \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \xi & \xi^2 \cdots & \xi^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \xi^{n-1} & \xi^{2n-2} & \xi^{(n-1)^2} \end{pmatrix} = \begin{pmatrix} A_1 & A_2 & \cdots & A_n \\ A_1 & \xi A_2 & & \xi^{n-1} A_n \\ \vdots & \vdots & & \vdots \\ A_1 & \xi^{n-1} A_2 & \cdots & \xi^{(n-1)^2} A_n \end{pmatrix}$$

where $A_i = (x_0, \cdots, x_{n-1}) \cdot (1, \xi^{i-1}, \cdots, \xi^{(i-1)(n-1)}) = \sum_{k=0}^{n-1} (\xi^{(i-1)k} \times x_k)$

So

$$\det \begin{pmatrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_{n-1} & x_0 & \cdots & x_{n-2} \\ \vdots & \vdots & & \vdots \\ x_1 & x_2 & \cdots & x_0 \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \xi & \xi^2 \cdots & \xi^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \xi^{n-1} & \xi^{2n-2} & \xi^{(n-1)^2} \end{pmatrix} = \det \begin{pmatrix} A_1 & A_2 & \cdots & A_n \\ A_1 & \xi A_2 & & \xi^{n-1} A_n \\ \vdots & \vdots & & \vdots \\ A_1 & \xi^{n-1} A_2 & \cdots & \xi^{(n-1)^2} A_n \end{pmatrix}$$

$$= \prod_{i=1}^{n} A_i \times \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \xi & \xi^2 \cdots & \xi^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \xi^{n-1} & \xi^{2n-2} & \xi^{(n-1)^2} \end{pmatrix} = \left(\prod_{i=1}^{n} \sum_{k=0}^{n-1} [\xi^{(i-1)k} \times x_k] \right) \times \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \xi & \xi^2 \cdots & \xi^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \xi^{n-1} & \xi^{2n-2} & \xi^{(n-1)^2} \end{pmatrix}$$

$$\Rightarrow \det \begin{pmatrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_{n-1} & x_0 & \cdots & x_{n-2} \\ \vdots & \vdots & & \vdots \\ x_1 & x_2 & \cdots & x_0 \end{pmatrix} = \prod_{j=0}^{n-1} \sum_{k=0}^{n-1} (\xi^{jk} x_k) \ \#$$

臺灣大學數學系 106 學年度碩士班甄試試題

科目：線性代數

2016.10.21

1. (20%) Let $A \in M_{n\times n}(F)$ where F is a field.
   (a) Show that if k is the largest integer such that some $k \times k$ submatrix of A has a nonzero determinant, then $\operatorname{rank}(A) = k$.
   (b) If A is nilpotent of index m (that is, $A^m = 0$ but $A^{m-1} \neq 0$), and if, for each vector v in F^n, W_v is defined to be the subspace generated by $v, Av, \ldots, A^{m-1}v$, how large can the dimension of W_v be? (Justify your answer.)

2. (30%)
   (a) Let $A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$. Find the general solution to the system of differential equations

$$\frac{dX}{dt} = AX, \text{ where } X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

   where for each i, $x_i(t)$ is a differentiable real-valued function of the real variable t.
   (b) Let V be the space of all real polynomials having degree less than 4 with the inner product $\langle f(x), g(x)\rangle = \int_{-1}^{1} f(t)g(t)dt$. Let T be a linear operator on V defined by $T(f(x)) = f'(x) + 3f(x)$.
   Use the Gram-Schmidt process to replace $\beta = \{1, 1+x, x+x^2, x^2+x^3\}$ by an orthonormal basis for V and find the matrix representation of the adjoint T^* of T in this orthonormal basis.

3. (30%)
   (a) Let $A \in M_{n\times n}(\mathbb{R})$. Show that there exists an orthogonal matrix Q and a positive semi-definite symmetric matrix P such that $A = QP$.
   (b) Let V be a finite-dimensional vector space over $\mathbb{C}$ and T be a linear operator on V. Show that T is normal if and only if its adjoint $T^* = g(T)$ for some polynomial $g(x) \in \mathbb{C}[x]$.

4. (20 %) Let $T \in \operatorname{End}_{\mathbb{C}}(V)$ for a finite-dimensional $\mathbb{C}$-vector space V.
   (a) Show that we have an expression of T as $T = S + N$ with $S, N \in \operatorname{End}_{\mathbb{C}}(V)$, such that S is diagonalisable, N is nilpotent and $SN = NS$.
   (b) Show that both S and N are uniquely defined by these conditions.
   (c) Show that there is a polynomial $p(x) \in \mathbb{C}[x]$ with $p(0) = 0$ such that $S = p(T)$.

Prob. 1 Rank

(a) We claim that column (A) has k linear independent vectors,

Let $A \in M_{n\times n}(F)$, B_k is submatrix of A with $\det B_k \neq 0$, Then we focus on the corresp. column vec. of v_1 from $B_k \Rightarrow$

we note that $\{v_1, \cdots, v_k\}$ are linear indep. Otherwise, $v_1 \cdots v_k \in C(A)$

$\exists\, a_i$ ($\neq 0$ for all a_i) $1 \le i \le k$ s.t. $a_1 v_1 + \cdots + a_k v_k = 0 \Rightarrow \det(B_k) = 0$ since its column vec is linear dep. Thus $k \le \operatorname{rank}(A)$. For oppsite, if rank $A = k_1 > k$, then we can find $v_1, \cdots, v_{k_1}$ linear indep vec of column (A) such that

$$\begin{pmatrix} v_{11} & \cdots & v_{k_1 1} \\ \vdots & & \vdots \\ v_{1n} & \cdots & v_{k_1 n} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_{k_1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$(v_1 = (v_{11}, \cdots, v_{1n}))$

$\Leftrightarrow x_1 = x_2 = \cdots = x_{k_1} = 0$

so we at least have k_1 effective eqn such that $\left|\begin{pmatrix} v_{11} & \cdots & v_{k_1 1} \\ \vdots & & \vdots \\ v_{1j} & \cdots & v_{k_1 j} \end{pmatrix}\right| \neq 0$

That is a contradiction.

(b) Let $\beta = \{v, Av, \cdots, A^{m-1}v\}$ and a_i, $0 \le i \le m-1$ s.t. $\sum_{i=0}^{m-1} a_i A^i v = 0$

Apply A^{m-1} on $\sum_{i=0}^{m-1} a_i A^i v = 0 \Rightarrow a_0 A^{m-1} v = 0 \Rightarrow a_0$ if $A^{m-1} v \neq 0$.

But $A^{m-1} \neq 0$, we can find $v \in V$ s.t. $A^{m-1} v \neq 0$, Thus

$$\dim W_v = |\beta| = m \ \#$$

(For a_1, Apply A^{m-2} on $\sum_{i=1}^{m-1} a_i A v = 0 \Rightarrow a_1 A^{m-1} v = 0 \Rightarrow a_2 = 0$

$\Rightarrow$ For general, a_j, Apply $A^{m-1-j} \Rightarrow a_j A^{m-1} v = 0$

Prob. 2 Jordan Form

$$D_A = \underbrace{\begin{pmatrix} 1 & 2 & 2 \\ -1/3 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{A^{-1}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \underbrace{\begin{pmatrix} -3 & 6 & 6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix}}_{A}$$

$\Rightarrow$ The soln. of $\frac{dx}{dt} = AX \Rightarrow X = X_0 \exp(A) = X_0 A^{-1} \begin{pmatrix} e & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^2 \end{pmatrix} A$

(b) Gram - Schmidt

$$V_1 = \frac{1}{\sqrt{2}}, \quad V_2 = (1+x) - \langle 1+x, \tfrac{1}{\sqrt{2}} \rangle \tfrac{1}{\sqrt{2}} \Big/ \|V_2\| = \frac{3x}{\sqrt{6}}$$

$$V_3 = \frac{x^2 + x - \langle V_3, \frac{1}{\sqrt{2}} \rangle \frac{1}{\sqrt{2}} - \langle V_3, \frac{3}{\sqrt{6}} x \rangle \frac{3}{\sqrt{6}} x}{\|V_3\|} = \frac{3\sqrt{10}}{4} x^2 - \frac{\sqrt{10}}{4}$$

$$V_4 = \frac{x^3 + x^2 - [\frac{1}{3} + \frac{3}{5} x + x^2 - \frac{1}{3}]}{\|V_4\|} = \frac{x^3 - \frac{3}{5} x}{\|V_4\|}$$

$$= \frac{5\sqrt{14}}{4} x^3 - \frac{3\sqrt{14}}{4} x$$

$\frac{\sqrt{10}}{4}(3x^2-1)$ $\quad \frac{\sqrt{14}}{4}(5x^3-3x)$

$$\Rightarrow \beta = \{V_1, V_2, V_3, V_4\} = \left\{ \frac{1}{\sqrt{2}}, \frac{3x}{\sqrt{6}}, \frac{3\sqrt{10}}{4} x^2 - \frac{\sqrt{10}}{4}, \frac{5\sqrt{14}}{4} x^3 - \frac{3\sqrt{14}}{4} x \right\}$$

$$[T]_\beta = \begin{pmatrix} 3 & \sqrt{3} & 0 & \sqrt{7} \\ 0 & 3 & 4\sqrt{15} & 0 \\ 0 & 0 & 3 & 5\sqrt{14}/\sqrt{10} \\ 0 & 0 & 0 & 3 \end{pmatrix} \Rightarrow [T^*]_\beta = \begin{pmatrix} 3 & 0 & 0 & 0 \\ \sqrt{3} & 3 & 0 & 0 \\ 0 & 4\sqrt{15} & 3 & 0 \\ \sqrt{7} & 0 & \frac{5\sqrt{14}}{\sqrt{10}} & 3 \end{pmatrix}$$

Prob. 3 Singular value decomposition & Isometry decomposition

Note that $\forall A \in \mathcal{L}(U)$, $A = U\Sigma V^*$, where $UU^* = VV^* = I$

and $A = S\sqrt{AA^*}$, S is isometric, let $A = (UV)(V\Sigma V^*)$

$$\Rightarrow \begin{cases} Q = UV^* \\ P = V\Sigma V^* \end{cases} \quad \text{Since} \quad \begin{cases} UV^*(VU^*) = I \Rightarrow Q \text{ is othonomal} \\ V\Sigma V^* = (V\Sigma V^*)^* \text{ and each eig-value of } \Sigma \geq 0 \end{cases}$$

$\Rightarrow P$ is positive.

Moreover, if A is invertible, Q, V are unique

Note : if two positive semidefined matrix A, B such that $A^2 = B^2$, then $A = B$

Let $A = QP = UZ$, we claim that $P = Z$, note that since A is invertible, $P \cdot Z$ are also invertible, $QP = UZ \Rightarrow U^T Q = ZP^{-1}$

Because Q, U are othonomal, $ZP^{-1} = U^T Q$ is orthonomal, That is

$$I = (ZP^{-1})^* ZP^{-1} = (P^{-1})^* Z^* Z P^{-1} = P^{-1} Z^2 P^{-1} \Rightarrow Z^2 = P^2$$

By remark, $Z = P$.

Prob. 3. Normal, common vec and conjugate eig-value

(b) If $T^* = g(T)$ for some $g \in \mathbb{C}[x]$, Obviously $T^*T = TT^*$.

If $T^*T = TT^*$. By spectral theorem, $\exists$ $\lambda_1, \cdots, \lambda_n$ (distinct) and $v_1, \cdots, v_n$ ($\in E(\lambda_i)$) such that

$v = \sum_{i=1}^{n} v_i$

$Tv = \sum_{i=1}^{n} \lambda_i v_i$, Remember that T and T^* has common eig-vector and conjugate eig-value, So our goal is $T^*v = \sum_{i=1}^{n} \bar{\lambda}_i v_i = \sum_{i=1}^{n} g(T) v_i$

$\Rightarrow g(T) v_i = \bar{\lambda} v_i \Rightarrow g(\lambda) v_i = \bar{\lambda} v_i \Rightarrow g(\lambda) = \bar{\lambda}$

By Lagrange interpolation, we can find such $g(x)$ s.t. $g(\lambda) = \bar{\lambda}$

Then $T^* = g(T)$ #

Prob. 4
(a)

Theorem 12 (Primary Decomposition Theorem). *Let T be a linear operator on the finite-dimensional vector space V over the field F. Let p be the minimal polynomial for T,*

$$p = p_1^{r_1} \cdots p_k^{r_k}$$

where the p_i are distinct irreducible monic polynomials over F and the r_i are positive integers. Let W_i be the null space of $p_i(T)^{r_i}$, $i = 1, \ldots, k$. Then

(i) $V = W_1 \oplus \cdots \oplus W_k$;
(ii) *each W_i is invariant under T;*
(iii) *if T_i is the operator induced on W_i by T, then the minimal polynomial for T_i is $p_i^{r_i}$.*

Theorem 11. *Let T be a linear operator on a finite-dimensional space V. If T is diagonalizable and if $c_1, \ldots, c_k$ are the distinct characteristic values of T, then there exist linear operators $E_1, \ldots, E_k$ on V such that*

(i) $T = c_1E_1 + \cdots + c_kE_k$;
(ii) $I = E_1 + \cdots + E_k$;
(iii) $E_iE_j = 0$, $i \neq j$;
(iv) $E_i^2 = E_i$ (*E_i is a projection*);
(v) *the range of E_i is the characteristic space for T associated with c_i.*

Theorem 13. *Let T be a linear operator on the finite-dimensional vector space V over the field F. Suppose that the minimal polynomial for T decomposes over F into a product of linear polynomials. Then there is a diagonalizable operator D on V and a nilpotent operator N on V such that*

(i) $T = D + N$,
(ii) $DN = ND$.

The diagonalizable operator D and the nilpotent operator N are uniquely determined by (i) and (ii) and each of them is a polynomial in T.

Note that in PDT $\Rightarrow$ we find $E_1, \cdots, E_n$ such that $I = E_1 + \cdots + E_n$ and each $E_iE_j = 0$, $i \neq j$, E_i is a projection, Moreover, since $\mathbb{C}$ is algebra closed, min-poly of T is a product of linear form and $E_i = (T - C_i)^{t_i}$ if min-poly $= \prod_{i=1}^{n} (x - C_i)^{t_i}$, Let

$$S = C_1E_1 + C_2E_2 + \cdots + C_nE_n$$

By theorem 11 $\Rightarrow$ S is a diagonalizable operator

Let $H = T - S = (T - C_1)E_1 + (T - C_2)E_2 + \cdots + (T - C_n)E_n$

Note that range $E_i = \ker(T-c_i)^{t_i}$, so when n enough large, $N^n = (T-c_1)^n E_1 + (T-c_2)E_2 + \cdots + (T-c_n)^n E_n$
$= 0$
when $n \geq \max\{t_i : i=1,\dots,n\}$, Thus N is a Nilpotent.
Remark that each E_i is a polynomial form of T, so S and N commute $\Rightarrow SN = NS$

(b) Note that S and N are poly form of T, we claim that for any D, M such that $T = S + N = D + M$ and $DM = MD$, D is diag and M is nilpotent, then $S = D$, $N = M$.

First, since $DM = MD$, $T = D + M \Rightarrow D, M$ comute with T
$\Rightarrow D, M$ commute with S、N. More, $S - D = M - N$
$\Rightarrow$ Since D、S commute and diagonalizable $\Rightarrow D, S$ can be simulatations diagonal, Because M, N commute and nilpotent $\Rightarrow M - N$ is nilpotent
$\Rightarrow S - D = M - N$ is diagon and nilpotent
$\Rightarrow (S-D)^n = 0$ for some $n \in \mathbb{N}$, $\Rightarrow$ min-poly of $S-D \mid x^n$
$\Rightarrow$ Since $S-D$ diagonal, min-poly of $S-D$ is $h(x) = x$, That mean $S - D = 0 \Rightarrow S = D \Rightarrow M = N$

(c) $S = p(T)$ for some $p(x) \in \mathbb{C}[x]$ have proved in PDT, Let h be a min-poly of T (i.e. $h(T) = 0$), if $h(0) \neq 0$, then
$S = p(T) = p(T) - \frac{p(0)}{h(0)} h(T) \equiv \tilde{p}(T)$, $\tilde{p}(0) = p(0) - \frac{p(0)}{h(0)} h(0) = 0$
if $h(0) = 0$, That mean $h(x) = x^{t_1} \times (x-c_2)^{t_2} \times \cdots \times (x-c_n)^{t_n}$, Recall that $S = p(T) = \Sigma_{i=1}^n c_i E_i = \Sigma_{i=1}^n c_i f_i(T) g_i(T)$, where
$\Sigma_{i=1}^n f_i(x) g_i(x) = 1$; $f_i(x) \equiv \frac{h(x)}{(x-c_i)^{t_i}}$ $(c_1 = 0)$

Thus, if $h(0)=0$ $S = p(T) = \sum_{i=2}^{n} c_i f_i(T) g_i(T)$, $c_i \neq 0$ $i \neq 1$, Then
$p(x) = \sum_{i=2}^{n} c_i f_i(x) g_i(x)$, but $x^{t_1} \mid f_i(x)$ if $i \neq 1$, so $p(0) = 0$.

Note that PDT tell us that, if we know min-poly of T, say $h(x)$

$$h(x) = (x-c_1)^{t_1} \times (x-c_2)^{t_2} \times \cdots \times (x-c_n)^{t_n}.$$

Then $V = \ker(T-c_1)^{t_1} \oplus \ker(T-c_2)^{t_2} \oplus \cdots \oplus \ker(T-c_n)^{t_n}$

$I = E_1 + E_2 + \cdots + E_n$

$E_i = f_i(T) g_i(T)$, $f_i(x) = \dfrac{h(x)}{(x-c_i)^{t_i}}$, $\sum_{i=1}^{n} f_i(x) g_i(x) = 1$

$S = c_1 E_1 + \cdots + c_n E_n = \sum_{i=1}^{n} c_i f_i(T) g_i(T)$

$N = T - \sum_{i=1}^{n} c_i f_i(T) g_i(T)$ #

國立成功大學 112 學年度「碩士班」甄試入學考試

線性代數

1. (10 points) Let A, B be two $m \times n$ matrix. Show that $|\operatorname{rank}(A) - \operatorname{rank}(B)| \leq \operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$.

2. (16 points) Let A be an $n \times n$ matrix and $r_k = \operatorname{rank}(A^k)$.

   (a). Show that $\lim_{k\to\infty} r_k$ exist.

   (b). If $r_3 \neq r_4$, Is A diagonalizable? Show your answer.

3. (8 points) Let $A = [a_{ij}]$ be an $n \times n$ matrix satisfying the condition that each a_{ij} is either equal to 1 or to -1. Show that $\det(A)$ is an integer multiple of 2^{n-1}.

4. (16 points) Let S, T be linear operator on V such that $S^2 = S$. Show that the range of S is invariant under T if and only if $STS = TS$. Show that both the range and null space of S are invariant under T if and only if $ST = TS$.

5. (20 points) Define a real vector space $V = \{f(x) \mid f(x) = ax^2 + bx + c, a, b, c \in \mathbb{R}\}$, with inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$.

   (a). Find an orthonormal basis for V.

   (b). Using (a), find $f \in V$ to maximize $f(\frac{1}{2})$ subject to the constraint $\langle f, f\rangle = 1$.

6. (16 points) Let $A = \begin{bmatrix} -2 & 3 & 1 \\ 0 & a & 3 \\ 0 & -3 & 4-a \end{bmatrix}$. Find the condition of a such that A is diagonalizable over real number.

7. (14 points) Let A be an $n \times n$ real symmetric matrix. Show that the matrix $A^2 + A + I$ is positive-definite.

Prob. 1. Rank / null space

(a) Note $v \in R(A+B)$ $v=(A+B)\varphi$ for some $\varphi \in V$, then $v \in R(A)+R(B)$

so $\dim R(A+B) \le \dim R(A) + \dim R(B) - \dim(R(A) \cap R(B))$
$\le \dim R(A) + \dim R(B)$

$\Rightarrow \operatorname{rank}(A+B) \le \operatorname{rank} A + \operatorname{rank} B$

by (a) $\operatorname{rank}(A) = \operatorname{rank}(A+B-B) \le \operatorname{rank}(A+B) + \operatorname{rank}(-B)$
$= \operatorname{rank}(A+B) + \operatorname{rank}(B)$

$\Rightarrow \operatorname{rank}(A) - \operatorname{rank}(B) \le \operatorname{rank}(A+B)$

Similar $\operatorname{rank}(B) - \operatorname{rank}(A) \le \operatorname{rank}(A+B)$

$\Rightarrow |\operatorname{rank}(A) - \operatorname{rank}(B)| \le \operatorname{rank}(A+B)$

Prob. Decreasing of image

Since $R(A^{k+1}) \subseteq R(A^k)$, so $\operatorname{rank}(A^{k+1}) \le \operatorname{rank}(A^k)$

r_k is a positive decreasing seq, so it converge.

Note that $\operatorname{rank}(A)$ = Number of eig-value $\neq 0$, if A is diagonalizable, then nonzero eigenvalue of A also a nonzero of A^k and $\operatorname{rank}(A^k) = \operatorname{rank}(A)$, Thus, A is not diagonalizable if $r_3 \neq r_4$.

Prob. 3 Determine

Use elementary operater $A \to B$, entries of B are $\{0, 2, -2\}$

By def of det of B, $\det B = \sum_{\sigma \in S_{n-1}} a_{1\sigma(1)} \cdots a_{n-1\,\sigma(n-1)}$, then

each term is 0 or $k2^{n-1}$, $k \in \mathbb{Z}$. So $2^{n-1} \mid \det A = \det B$

Prob. 4 Projection

(a)

$\Rightarrow$ if $STS = TS$, Let $v \in R(S)$, $v = S\varphi$

$\Rightarrow Tv = TS\varphi = STS\varphi \in R(S) \Rightarrow T(R(S)) \subseteq R(S)$

$\Rightarrow$ if $T(R(S)) \subseteq R(S)$, then $TS\varphi = S\psi$ for some $\psi \in V$, then $STS\varphi = S^2\varphi = S\varphi = TS\varphi \Rightarrow STS = TS$

(b) If $TS = ST$, let $v \in R(S)$ and $u \in N(S)$

$Tv = TS\varphi = ST\varphi \in R(S) \Rightarrow T(R(S)) \subseteq R(S)$

$STu = TSu = T0 = 0 \Rightarrow Tu \in N(S) \Rightarrow T(N(S)) \subseteq N(S)$

If $T(R(S)) \subseteq R(S)$ & $T(N(S)) \subseteq N(S)$, Note that $\forall v \in V$

$v = (I-S)v + Sv \in N(S) + R(S)$

$T(I-S)v \in N(S)$ and $TSv \in R(S)$ Previous result

$\Rightarrow STv = ST((I-S)v + Sv) = STSv = TSv$ #

Prob. 5 Gram - smit

Note that $\{1, x, x^2\}$ is a basis of V $\quad x^2 - x + \frac{1}{6}$

$V_1 = 1 \quad V_2 = \dfrac{x - 1/2}{\|V_2\|} = 2\sqrt{3}x - \sqrt{3}$

$V_3 = \dfrac{x^2 - (\langle x^2, 1\rangle 1 + \langle x^2, V_2\rangle V_2)}{\|V_3\|} = \dfrac{x^2 - \frac{1}{3} - \frac{\sqrt{3}}{6}(2\sqrt{3}x - \sqrt{3})}{\|V_3\|}$

$= 6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}$

Let $f(x) = aV_1 + bV_2 + cV_3 \quad \langle f, f\rangle = 1 \Rightarrow a^2 + b^2 + c^2 = 1$

$f(1/2) = a - \frac{\sqrt{5}}{2}c \Rightarrow (1, 0, \frac{\sqrt{5}}{2}) \,/\!/\, (2a, 2b, 2c)$

and $a^2+b^2+c^2 = 1 \quad \lambda a = 1 \quad \lambda b = 0 \quad \lambda c = \frac{\sqrt{5}}{2}$

(λ must not be zero)

if $\lambda \neq 0 \Rightarrow b = 0 \quad a = 1/\lambda \quad c = \dfrac{\sqrt{5}}{2\lambda}$, Let $k = 1/\lambda$

$k^2 + \frac{5}{4}k^2 = 1 \quad k^2 = \frac{4}{9} \quad k = \pm\frac{2}{3} \quad \lambda = \pm\frac{3}{2} \quad a = \pm\frac{2}{3}$

$c = \pm\sqrt{5}/3 \quad (a, b, c) = (\frac{2}{3}, 0, \frac{\sqrt{5}}{3})$ or $(\frac{-2}{3}, 0, \frac{-\sqrt{5}}{3})$

Then extremum of $f(\frac{1}{2}) = \frac{-1}{6}$ & $\frac{1}{6}$, since $a^2+b^2+c^2 = 1$ is compact in $\mathbb{R}^3$, $f(\frac{1}{2})$ has a max on it $\Rightarrow \max f(\frac{1}{2}) = \frac{1}{6}$

$f = \frac{-2}{3} \times 1 + \frac{-\sqrt{5}}{3} \times (6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}) = -10x^2 + 10x - 7/3$ #

Prob. 6 symmetric matrix

By spectrum theorem, $A = \sigma^{-1} D \sigma$ for some D : diagonal,

Then $A^2 + A + I = \sigma^{-1}(D^2 + D + I)\sigma$ is symmetric and since

$\forall \lambda \in \mathbb{R}$, $\lambda^2 + \lambda + 1 > 0$, so $A^2 + A + I$ has positive eig-value,

$\Rightarrow$ A is positive defined.

Prob. 5 diagonalizable.

$$A = \begin{pmatrix} -2 & 3 & 1 \\ 0 & a & 3 \\ 0 & -3 & 4-a \end{pmatrix}$$

A is diagonalizable $\Leftrightarrow$ min-poly of A is linear product

Note, min-poly & char-poly has same root

$$\det(A - xI) = (-2-x)\begin{vmatrix} a-x & 3 \\ -3 & 4-a-x \end{vmatrix} = 0$$

$$= -(x+2)\left[(x-a)(x+a-4)+9\right] = 0$$

$$= -(x+2)\underline{(x^2 - 4x - a(a-4) + 9)} = 0$$

$\longrightarrow$ must be reducible

$\Rightarrow 16 - 4(-a(a-4)+9) > 0$ or $16 - 4(-a(a-4)+9) = 0$

$\Rightarrow a^2 - 4a - 5 > 0$ $\quad$ $-1 \quad 5$ $\quad$ $a^2 - 4a - 5 = 0$

$a > 5$ or $a = -1$ $\quad$ $(a-5)(a+1) = 0$ $\quad$ $a = -1$ or 5

must be diagonalizable

$$A = \begin{pmatrix} -2 & 3 & 1 \\ 0 & 5 & 3 \\ 0 & -3 & -1 \end{pmatrix} \text{ or } \begin{pmatrix} -2 & 3 & 1 \\ 0 & -1 & 3 \\ 0 & -3 & 5 \end{pmatrix}$$

$\lambda = -2, 2$ (rep) $\quad$ $\lambda = -2, 2$

$$\underset{(\text{for } a=5)}{A - 2I} = \begin{pmatrix} -4 & 3 & 1 \\ 0 & 3 & 3 \\ 0 & -3 & -3 \end{pmatrix} \Rightarrow \ker(A-2I) = 1 \neq 2 \Rightarrow A \text{ is not diagonalizable,}$$

$$\underset{(\text{for } a=-1)}{A - 2I} = \begin{pmatrix} -4 & 3 & 1 \\ 0 & -3 & 3 \\ 0 & -3 & 3 \end{pmatrix} \Rightarrow \ker(A-2I) = 1 \neq 2 \Rightarrow A \text{ is not diagonalizable,}$$

$\Rightarrow$ A is diagonalizable $\Leftrightarrow$ $a > 5$ or $a < -1$ #

國立成功大學 111 學年度「碩士班」研究生甄試入學考試

線性代數

In this test, all vector spaces are finite dimensional over $\mathbb{C}$.

1. (15 points) Let T be a linear operator on a vector space V. Prove that T is diagonalizable if and only if its minimal polynomial is square-free.

2. (15 points) Let V be a vector space. A linear operator S on V is semisimple if for every S-invariant subspace W of V there exists an S-invariant subspace W' of V such that $V = W \oplus W'$. Prove that every diagonalizable operator on V is semisimple, and deduce that every linear operator T on V can be decomposed uniquely as $T = S + N$, where S is semisimple, N is nilpotent, and $SN = NS$.

3. (15 points) Let T and U be normal operators on an inner product space V such that $TU = UT$. Prove that $UT^* = T^*U$, where T^* is the adjoint of T.

4. (15 points) Let T and U be Hermitian operators on an inner product space $(V, \langle \cdot, \cdot \rangle)$ such that $\langle T(x), x \rangle > 0$ for all nonzero $x \in V$. Prove that UT is diagonalizable and has only real eigenvalues.

5. (15 points) Find the total number of distinct equivalence classes of congruent $n \times n$ real symmetric matrices and justify your answer.

6. (15 points) Let A be an $n \times n$ complex matrix, t be a variable, and I be the identity matrix. Prove that

$$\det(I - tA) = \exp\left(- \sum_{i \geq 1} \frac{\operatorname{tr}(A^i) t^i}{i} \right).$$

7. (10 points) Let $A = (a_{i,j})$ be a $2n \times 2n$ matrix such that $A^T = -A$. The Phaffian of A is defined as

$$\operatorname{pf}(A) := \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)},$$

where S_{2n} is the symmetric group of order $2n$ and $\operatorname{sgn}(\sigma)$ is the signature of σ. Prove that for any $2n \times 2n$ matrix B,

$$\operatorname{pf}(BAB^T) = \det(B) \operatorname{pf}(A).$$

Prob. 1

$\Leftarrow$ if T is diagonalizable, Then $T = \theta^{-1} D \theta$, Note that $p(T)$ can be rewritten by $\theta^{-1} p(D) \theta$, Let $p(t) = \prod_{i=1}^{n} (t - c_i)$, c_i is a eig-value of T, $\forall\, v \in V$, $v = \sum_{i=1}^{n} v_i$ $v_i \in \ker(T - c_i I)$ $p(T) v = \prod_{i=1}^{n} (T - c_i I) v$
$= \sum_{i=1}^{n} \left(\prod_{j=1}^{n} (T - c_j I) \right) v_i = 0$, so $p(T) v \equiv 0$
$\Rightarrow$ By definition of min-poly of T $\Rightarrow$ min-poly $\mid p$, so min-poly of T is square free.

$\Rightarrow$ Let W be the subspace spanned by all of the char-vector of T, and suppose $W \neq V$, We need follows lemma

Let $\dim V < \infty$ over $\mathbb{F}$, Let $p(t) = (x - c_1)^{r_1} \times \cdots \times (x - c_n)^{r_n}$ $c_i \in \mathbb{F}$.
Let $W < V$, $T(W) \subseteq W$, there exist a vec $\alpha \in V$ s.t.
(a) $\alpha \notin W$ $(T - c_i)\alpha \in W$, for some c_i

Thus, $\exists\ \alpha \in W$ and c_i of T s.t. $\beta = (T - c_i I)\alpha \in W$,
$\Rightarrow \beta = \beta_1 + \cdots + \beta_k$, $T\beta_i = c_i \beta_i$ $1 \le i \le k$, Thus,
$\forall\, h(x) \in \mathbb{F}[x]$, $h(T)\beta = h(c_1)\beta_1 + \cdots + h(c_k)\beta_k \in W$
Now $p = (x - c_i) q$, $q \in \mathbb{F}[x]$. Also
$\underline{q(x) - q(c_i) = (x - c_i) h}$ for some $h \in \mathbb{F}[x]$
$\Rightarrow \underline{[q(T) - q(c_i)]\alpha} = (T - c_i) h \alpha = h (T - c_i) \alpha \in W$
and $0 = p(T)\alpha = (T - c_i) q(T) \alpha$, $\underline{q(T)\alpha \text{ is eig-vec.}}$
$q(T)\alpha \in W \Rightarrow q(c_i)\alpha = q(T)\alpha - h(T - c_i)\alpha \in W$
Since $\alpha \notin W \Rightarrow \underline{q(c_i) = 0} \Rightarrow$ That is contradicts the

fact that p has distinct roots

Prob. 2 PDT, Semisimple operator,

Note if $T(W) \subseteq W$, then $V = W \oplus W'$ iff W is T-admissible

(if $T(W) \subseteq W$, $V = W \oplus W'$ $\quad \forall \beta \in V$, $\beta = w + w'$, $f(T)\beta = f(T)w + f(T)w'$

$\Rightarrow f(T)\beta \in W \Rightarrow \underbrace{f(T)\beta}_{\in W} = \underbrace{f(T)w}_{\in W}$ $\quad (f(T)w' = 0)$)

Note if min-poly of T is irr over F, T is semi-simple

Let $W \leq V$, $T(W) \subseteq W$ $\Rightarrow$ Test $f(T)\beta \in W \to f(T)\beta = f(T)\alpha$ for some $\alpha \in W$.

if $f(T)\beta = 0$, that is easy to see (p: min-poly of f)

if $f(T)\beta \neq 0$, $p \nmid f$ $\Rightarrow$ $\gcd(p, f) = 1$ $\Rightarrow \exists g_1 \cdot g_2$ $g_1 p + g_2 f = 1$

$\Rightarrow g_1(T)p(T) + g_2(T)f(T) = I$ $\Rightarrow$ $I = g_2(T)f(T)$

$\Rightarrow \beta = \underbrace{g_2(T)}_{g_2(T)(W) \subseteq W}\underbrace{f(T)\beta}_{\in W}$ $\Rightarrow \beta \in W$

Note T is semi-simple $\Leftrightarrow$ min-poly of T is a product of (distinct) irr poly

if T is semi-simple and $g^2 | p$ g is irr, let $W = \ker g(T)$

$T(W) \subseteq W$ obvious. $p = g^2 h \Rightarrow g(T)h(T) \neq 0$, $\exists \beta$ s.t. $g(T)h(T)\beta \neq 0$

$\Rightarrow$ but $g(T)h(T) \in W$ and if $gh(T)\beta = gh(T)\alpha$, $\alpha \in W$

$gh\alpha = hg\alpha = h(g\alpha) = 0 \longrightarrow W$ is not T-admissible $\to V \neq W \oplus W'$

if min-poly of T = product of (distinct) irr poly, let $W \leq V$, $T(W) \subseteq W$

Use PDT, $V = W_1 \oplus \cdots \oplus W_n$ $\quad W_i = \ker p_i(T)$, $T_i = T|_W$

$\Rightarrow W \cap W_i \leq W_i$ and $T_i(W \cap W_i) \subseteq W \cap W_i$, by previous note

$W_i = V_i \oplus W \cap W_i$, $V = (\oplus_{i=1}^n V_i) \oplus (\oplus_{i=1}^n W \cap W_i) = W' \oplus W$

Thus, recal that $\mathbb{C}$ is algebraically closed field, if min-poly of T is a product of distinct irr poly mean min-poly $= \prod_{i=1}^{n} (x-c_i)$

By theorem, T is diagonalizable

By PDT, $V = W_1 \oplus \cdots \oplus W_n$, $W_j = \ker (T-c_j)^{r_j}$ $\quad p(t) = \prod_{j=1}^{n} (t-c_j)^{r_j}$

So $I = E_1 + \cdots + E_n$, $E_j = (T-c_j)^{r_j}$ $\quad T = TE_1 + TE_2 + \cdots + TE_n$

and let $D = c_1 E_1 + \cdots + c_n E_n \Rightarrow$ By theorem D is diagonalizable,

therefore, by note, D is semi-simple.

Take $H = T - D = (T-c_1)E_1 + \cdots + (T-c_n)E_n$, Then

$$N^{\max r_j = r} = (T-c_1)^r E_1 + \cdots + (T-c_n)^r E_n = 0$$

Recall that E_j has a poly form of T, Hence D and N commute.

Prob. 4 Symmetric / positive

(a) diagonalizable, Since T is positive, $T = R^2$ for some Hermiton positive operator

then $UT = UR^2 = R^{-1}R\, URR$, where RUR is a Hermiton

so UT is similar to Hermitian operator, so UT is diagonalizable

(b) Moreover, UT has only real eigenvalue since RUR has only real eig-value.

Prob. 3 Simultaneous Diagonalizable

Since $UT = TU$, T and U can be diagonal use same P, i.e. $T = P^{-1} D_1 P$, $U = P^{-1} D_2 P$

Note that T, T^* use same eig-vec but conjugate eig-value, $T^* = P^{-1} D_1^* P$

Therefore, $UT^* = P^{-1} D_2 D_1^* P = P^{-1} D_1^* D_2 P = T^* U$ #

Prob. 6 ODE

Note that if we define $f(t) = \det (I - tA)$, $f(0) = 1$, use techonique in ODE, We show that $f'(t) + p(t) f(t) = 0$

$\Rightarrow \quad f(t) = f(0) \exp\left(\int_0^t p(s)\,ds \right), \quad p(t) = \sum_{i \geq 0} tr(A^{i+1}) t^i$ *

Note that $\det(I - tA) = (1-\lambda_1 t)(1-\lambda_2 t)\cdots(1-\lambda_n t)$, where λ_j

$1 \leq j \leq n = \dim V$ is eig-value of A (no need to be distinct).

$$\frac{df}{dt} = (-\lambda_1) \times \prod_{j \neq 1} (1-\lambda_j t) + (-\lambda_2) \prod_{j \neq 2} (1-\lambda_j t) + \cdots + (-\lambda_n) \prod_{j \neq n} (1-\lambda_j t)$$

$$= \frac{-\lambda_1}{1-\lambda_1 t} \prod_j (1-\lambda_j t) + \cdots + \frac{-\lambda_n}{(1-\lambda_n t)} \prod_j (1-\lambda_j t)$$

$$= \left(\sum_{j=1}^n \frac{-\lambda_j}{1-\lambda_j t} \right) f$$

Note that $trA = \lambda_1 + \cdots + \lambda_n$; $trA^i = \lambda_1^i + \cdots + \lambda_n^i$

$$= \left(\sum_{j=1}^n -\lambda_j \sum_{k=0}^\infty (\lambda_j t)^k \right) f$$

$\sum_{j=1}^n \left(\sum_{k=0}^\infty -\lambda_j^{k+1} t^k \right) = \sum_{k=0}^\infty -tr(A^{k+1}) t^k$

Note that it need constrain that $|\lambda_n| < 1$.

$$= \left(\sum_{k=0}^\infty -tr(A^{k+1}) t^k \right) f \Rightarrow \frac{df}{dt} + \left(\sum_{k=0}^\infty tr(A^{k+1}) t^k \right) f = 0 \quad *$$

Prob. 5 Congruent $n \times n$ real symmetric matrix (Sylvester's law of inertia)

Note : A, B is congruent if $A = P^t B P$ for some $P \in GL(\mathbb{R}^n)$

Sylvester's law of inertia state that

$A, B \in M_{n\times n}(\mathbb{R})$, A, B are symmetric, Then

A and B is congruent $\Leftrightarrow$ the number of negative / positive λ_j are same

If A is symmetric, $A = Q^T D Q$ for some Q is a othonomal matrix,

Since D (→ entrie is real.) can be rewritten as $W \Sigma W^T$, where Σ is a diagonal matrix

with entries $= +1, -1, 0$, W is a diagonal matrix with $w_{ii} = \sqrt{|D_{ii}|}$

So $A = Q^T W \Sigma W^T Q$ → A is congruent with Σ

→ if A is congruent with B ⇒ A, B has same number or $+, -$ eig-value.

⇒ Therefore, the different eg-class of congruent class is

$x + y + z = n$, $x =$ # of $+1$, # of -1, # of 0, $x, y, z \geq 0$

$\Rightarrow$ (x, y, z) has $\dfrac{(n+2)!}{n!\,2!} = \dfrac{(n+2)(n+1)}{2}$ #