Suns

Eric Liu

Contents

Chapter 1

HWs

1.1 HW1

For question 1, recall that by class equation, p-group has nontrivial center.

Question 1

Show that

(i) If $H/Z(H)$ is cyclic, then H is abelian.

(ii) If H is of order p^2, then H is abelian.

From now on, suppose G is non-abelian with order p^3.

(iii) $o(Z(G)) = p$.

(iv) $Z(G) = G^{(1)}$.

Proof. Let $a, b \in H$ and $H/Z(H) \triangleq \langle hZ(H) \rangle$. Write $a \triangleq h^n z_1$ and $b \triangleq h^m z_2$. Because $z_1, z_2 \in Z(H)$, we may compute:

$$ab = h^n z_1 h^m z_2 = h^{n+m} z_1 z_2 = ba$$

as desired. Let $o(H) = p^2$. Because H is a p-group, we know the center of H is nontrivial. Therefore the order of its center is $\in \left\{p, p^2, \right\}$. To see that its order isn't p, just observe that if so, then by (i), H is abelian, which contradicts to $o(Z(H)) = p$. We have shown $o(Z(H)) = p^2 = o(H)$, so H is abelian.

Because G is non-abelian with order p^3, we know $o(Z(G)) \in \left\{p, p^2\right\}$. Part (i) tell us that $o(Z(G)) \neq p^2$, so $o(Z(G)) = p$.

We now prove $Z(G) = G^{(1)}$. Because $o(Z(G)) = p$, by part (ii) we know $G \diagup Z(G)$ is abelian, which implies $G^{(1)} \leq Z(G)$, which implies $G^{(1)}$ is either trivial or equal to $Z(G)$. Because G is non-abelian, we know $G^{(1)}$ is nontrivial. Therefore $G^{(1)} = Z(G)$, as desired. ■

Question 2

(i) Let M, N be two normal subgroups of G with $MN = G$. Prove that

$$G \diagup (M \cap N) \cong (G \diagup M) \times (G \diagup N)$$

(ii) Let H, K be two distinct subgroups of G of index 2. Prove that $H \cap K$ is a normal subgroup with index 4 and $G \diagup (H \cap K)$ is not cyclic.

Proof. (i) requires us to prove the map $G \diagup (M \cap N) \to (G \diagup M) \times (G \diagup N)$ defined by

$$g(M \cap N) \mapsto (gM, gN) \tag{1.1}$$

forms a well-defined group isomorphism. The map is indeed well-defined. Let $gM \triangleq M$ and $gN \triangleq N$, to prove the map is injective, we are required to prove $g(M \cap N) = M \cap N$, which is clearly true.

It remains to show the map is surjective. Fix $g, h \in G$. We are required to show the existence of some element $k \in G$ such that $kM = gM$ and $kN = hN$. Because $G = MN$, we may write $g \triangleq mn$ and $h \triangleq \widetilde{m}\widetilde{n}$. We then clearly have

$$gM = nM = \widetilde{m}nM \quad \text{ and } \quad hN = \widetilde{m}N = \widetilde{m}nN$$

In other words, $k \triangleq \widetilde{m}n$ suffices.

(ii): Because H, K are both of index 2 in G, we know they are both normal in G, so by second isomorphism theorem, HK forms a subgroup of G.

Because $H \neq K$, we know HK properly contains H. This implies $[HK : H] > 1$. Now, since

$$2 = [G : H] = [G : HK] \cdot [HK : H]$$

we see that we must have $G = HK$. Note that $H \cap K$ is normal since it is the intersection of normal subgroups. The rest then follows from part (i):

$$G \diagup (H \cap K) \cong (G \diagup H) \times (G \diagup K) \cong C_2 \times C_2$$

■

Question 3

Let G be a group of order pq, where $p > q$ are prime.

(i) Show that there exists a unique subgroup of order p.

(ii) Suppose $a \in G$ with $o(a) = p$. Show that $\langle a \rangle \subseteq G$ is normal and for all $x \in G$, we have $x^{-1}ax = a^i$ for some $0 < i < p$.

Proof. The third Sylow theorem stated that the number n_p of Sylow p-subgroups satisfies

$$n_p \equiv 1 \pmod{p} \quad \text{and} \quad n_p \mid q$$

Because $p > q$, together they implies $n_p = 1$. We have shown that G has a unique Sylow p-subgroup, as desired. Clearly, $\langle a \rangle$ is that Sylow p-subgroup, so second Sylow theorem shows that $\langle a \rangle$ is normal. Because $\langle a \rangle$ is normal, we clearly have $x^{-1}ax = a^i$ for some $0 \leq i < p$. The fact that $i \neq 0$ is clear, since otherwise we would have $ax = x$. ■

Question 4

Let H, K be two subgroups of G of finite indices m, n. Show that

$$\operatorname{lcm}(m, n) \leq [G : H \cap K] \leq mn$$

Proof. Let $\Omega_{H\cap K}, \Omega_H$, and Ω_K respectively denote the set of left cosets of $H \cap K$, H, and K. The map $\Omega_{H\cap K} \to \Omega_H \times \Omega_K$ defined by

$$g(H \cap K) \mapsto (gH, gK) \tag{1.2}$$

is well defined since

$$g(H \cap K) = l(H \cap K) \implies g^{-1}l \in H \cap K \implies gH = lH \text{ and } gK = lK$$

Such set map is injective since if $gH = lH$ and $gK = lK$, then $g^{-1}l \in H$ and $g^{-1}l \in K$, which implies $g(H \cap K) = l(H \cap K)$. From the injectivity of map 1.2, we have shown $[G : H \cap K]$ indeed have upper bound mn.

Because

$$[G : H \cap K] = [G : H] \cdot [H : H \cap K] = [G : K] \cdot [K : H \cap K]$$

we know both n and m divides $[G : H \cap K]$, which gives the desired lower bound $\operatorname{lcm}(m, n)$. ■

Question 5

(i) Let G be a group, $H \leq G$, and $x \in G$ of finite order. Prove that if k is the smallest natural number that makes $x^k \in H$, then $k \mid o(x)$.

(ii) Let G be a group and N a normal subgroup of G. Prove that

$$o(gN) = \inf \left\{ k \in \mathbb{N} : g^k \in N \right\}, \quad \text{where } \inf \varnothing = \infty$$

(iii) Let G be a finite group, H, N two subgroups of G with N normal. Show that if $o(H)$ and $[G : N]$ are coprime, then $H \leq N$.

Proof. (i): Let $a \triangleq qk + r \in \mathbb{N}$ with $0 \leq r < k$. If $x^a \in H$, then $x^r = x^a \cdot (x^k)^{-q} \in H$, which implies $r = 0$. We have shown that k divides all natural numbers a that makes $x^a \in H$, which includes $o(x)$.

(ii): This is a simple observation that $(gN)^k = g^k N \in N \iff g^k \in N$.

(iii): By second isomorphism theorem, we know $[HN : N] = [H : H \cap N]$ which divides both $o(H)$ and $[G : N]$. This by their coprimality implies $[H : H \cap N] = 1$, which shows that $H \leq N$. ∎

Question 6

Let G be a finite group with Sylow p-subgroup P and normal subgroup N. Let

$$o(PN) \triangleq p^a m \quad \text{and} \quad o(N) \triangleq p^b n, \quad \text{where } p \nmid m, n$$

Show that $P \cap N$ forms a Sylow p-subgroup of N, and use such to deduce N have index p^{a-b} in PN.

Proof. By second isomorphism theorem, we have

$$o(PN) \cdot o(P \cap N) = o(P) \cdot o(N)$$

Clearly P must also be a p-Sylow subgroup of PN, so $o(P) = p^a$. This gives us

$$o(P \cap N) = p^b$$

which implies that $P \cap N$ is a Sylow p-subgroup of N. We now have

$$[PN : N] = \frac{o(PN)}{o(N)} = \frac{o(P)}{o(P \cap N)} = p^{a-b}$$

∎

Question 7

Let G be a finite group. Prove that if $H \leq G$ is Hall and $N \trianglelefteq G$, then both $H \cap N \leq N$ and $HN/N \leq G/N$ are Hall.

Proof. To prove that $H \cap N \leq N$ is Hall, we are required to prove $o(H \cap N)$ and $[N : H \cap N]$ are coprime. Recall that by second isomorphism theorem, we have

$$[N : H \cap N] = [HN : H]$$

which implies that $[N : H \cap N]$ is a factor of $[G : H]$. Clearly $o(H \cap N)$ is a factor of $o(H)$. It now follows from the coprimality of the pair $[G : H], o(H)$ that $[N : H \cap N], o(H \cap N)$ are coprime.

The method to show that $HN/N \leq G/N$ is Hall is similar. Again, by second isomorphism theorem,

$$o(HN/N) = \frac{o(HN)}{o(N)} = \frac{o(H)}{o(H \cap N)}$$

we see $o(HN/N)$ is a factor of $o(H)$. Now, since

$$\left[\frac{G}{N} : \frac{HN}{N}\right] = [G : HN]$$

we see that $[G/N : HN/N]$ is a factor of $[G : HN]$, which is a factor of $[G : H]$. It then follows from coprimality of the pair $[G : H], o(H)$ that $o(HN/N), [G/N : HN/N]$ are coprime. ■

1.2 HW2

Question 1

Prove that if p is a prime and $o(G) = p^\alpha$ with $\alpha \in \mathbb{N}$, then every subgroup H of index p is normal. Deduce that every group of order p^2 has a normal subgroup of order p.

Proof. The first part is a special case of the general result that proper subgroups of finite subgroups of smallest possible index is normal, which is a result of consideration **cores of subgroups**.

Let $\varphi : G \to \text{Bij}(G/H)$ be the left multiplicative action of G on the left coset space G/H. First isomorphism theorem give us

$$[G : \ker(\varphi)] = o(\text{Im}(\varphi)) \mid o\left(\text{Bij}(G/H)\right) = p!$$

Because G is a p-group, this then implies $[G : \ker(\varphi)] \in \{1, p\}$. The fact that H is proper forces $[G : \ker(\varphi)] = p$. Clearly we have $\ker(\varphi) \leq H$. Combining them, we see $H = \ker(\varphi)$, as desired.

Let $o(G) \triangleq p^2$. We have already shown that every subgroup of G of order p is normal. To see that G has a subgroup of order p, just consider its center. If G is non-abelian, then its center is a subgroup of order p. If G is abelian, then $G \cong C_{p^2}$ or $C_p \times C_p$, and they clearly both have subgroup of order p.

Remark: In fact, Wielandt's proof for Sylow's theorem shows that in general, given $o(G) = p_1^{d_1} \cdots p_n^{d_n}$, for all $e_i \leq d_i$, G has a subgroup of order $p_i^{e_i}$. ∎

Question 2

Let G be a group of odd order. Prove that for any nontrivial $x \in G$, we have $\text{Cl}(x) \neq \text{Cl}(x^{-1})$.

Proof. Assume for a contradiction that $\text{Cl}(x) = \text{Cl}(x^{-1})$. Because

$$(gxg^{-1})^{-1} = gx^{-1}g^{-1} \in \text{Cl}(x^{-1}) = \text{Cl}(x)$$

we know the function

$$\varphi : \text{Cl}(x) \to \text{Cl}(x) \quad \text{ and } \quad \varphi(g) \triangleq g^{-1}$$

is well-defined on $\text{Cl}(x)$. Since $\text{Cl}(x) = \text{Cl}(x^{-1})$, we know φ is surjective. Clearly φ is injective, so it is bijective. Because $o(G)$ is odd, we know $\varphi(g) \neq g$ for all $g \in \text{Cl}(x)$, since otherwise, we would have an element of order 2. We have shown that

$$g \sim h \overset{\triangle}{\iff} \varphi(g) = h$$

is an equivalence relation defined on $\mathrm{Cl}(x)$ whose equivalence classes all have cardinality 2. This implies that $\mathrm{Cl}(x)$ has even cardinality, which is impossible, since by orbit-stabilizer theorem, the cardinality of $\mathrm{Cl}(x)$ equals to the order of $C_G(x) \leq G$. ■

Question 3

Let $o(G) = p^n$ with $n \geq 3$ and $o(Z(G)) = p$. Prove that G has a conjugacy class of size p.

Proof. Class equation stated that

$$o(G) = o(Z(G)) + \sum |\mathrm{Cl}(x)|$$

and the orbit-stabilizer theorem shows that $|\mathrm{Cl}(x)|$ is of order powers of p. If they are of p-powers ≥ 2, then we see

$$0 \equiv o(G) \equiv o(Z(G)) + \sum |\mathrm{Cl}(x)| \equiv p \pmod p$$

, a contradiction. ■

Question 4

Prove that if the center of G is of index n, then every conjugacy class has at most n elements.

Proof. Let $x \in G$. Because $Z(G) \leq C_G(x)$, by orbit-stabilizer theorem, we have:

$$|\mathrm{Cl}(x)| = [G : C_G(x)] \leq [G : Z(G)] = n$$

■

Question 5

Let $H, K \leq G$ be two finite subgroups. Show that

$$|KH| = \frac{o(H)o(K)}{o(H \cap K)} \tag{1.3}$$

Remark: The hint give a rigorous proof, but I prefer a heuristic one.

Proof. Let K acts on the left coset spaces G/H by left multiplication. Because $kH = H$ if and only if $k \in H$, we know

$$\text{Stab}_K(H) = K \cap H$$

Therefore, by orbit-stabilizer theorem, we have

$$\frac{o(K)}{o(H \cap K)} = |\{kH \in G/H : k \in K\}|$$

Define an equivalence class in K by setting $k \sim \widetilde{k} \overset{\triangle}{\iff} kH = \widetilde{k}H$. Pick a representative element our of each class and collect them into a set T. Clearly

$$|T| = |\{kH \in G/H : k \in K\}|$$

Therefore, to prove equation 1.3, we only have to show the natural map

$$T \times H \to KH \quad \text{and} \quad (k, h) \mapsto kh$$

is bijective. Let $kh = \widetilde{k}\widetilde{h}$. To show the natural map is injective, we are required to show $k = \widetilde{k}$ and $h = \widetilde{h}$. Because $k^{-1}\widetilde{k} = \widetilde{h}^{-1}h \in H$, we know $kH = \widetilde{k}H$, which implies $k = \widetilde{k}$, which then implies $h = \widetilde{h}$, as desired.

Fix $kh \in KH$. By definition, there exists some $\widetilde{k} \in T$ such that $kh \in kH = \widetilde{k}H$. Therefore, there exists some $\widetilde{h} \in H$ such that $kh = \widetilde{k}\widetilde{h}$. We have shown that the natural map is surjective, as desired. ∎

Question 6

Let G be a non-abelian group of order 21. Prove that $Z(G) = 1$.

Proof. Because G is non-abelian, we know $o(Z(G)) \in \{1, 3, 7\}$. To see that $o(Z(G)) \notin \{3, 7\}$, just observe that if so, then part (i) of the first question of HW1 implies that G is abelian, a contradiction. ∎

Question 7

Find all finite groups which have exactly two conjugacy classes.

Proof. Let G be a finite group that has exactly two conjugacy classes. One of the conjugacy class is $\{e\}$. Let a be an element of the other class. By class equation and orbit-stabilizer theorem, we have

$$o(G) - 1 = |\text{Cl}(a)| \text{ divides } o(G)$$

which forces $o(G) = 2$ and $G \cong C_2$. ∎

Question 8

Let $H < G$. Show that

$$\bigcup_{g \in G} gHg^{-1} \neq G$$

Proof. The question as stated is still wrong. (Flipping the statement to its contraposition won't make what's wrong right, they are equivalent). Consider

$$G \triangleq \mathrm{GL}_n(\mathbb{C}) \quad \text{and} \quad H \triangleq \{A \in \mathrm{GL}_n(\mathbb{C}) : A \text{ upper triangular. }\}$$

Jordan form shows that G is indeed the union of conjugation of H.

Remark: The question may be fixed by adding the hypothesis $o(G) < \infty$. See this MSE post:

https://math.stackexchange.com/questions/374078/reference-for-a-finite-group-cannot-be-union-of-conjugates-of-a-proper-subgroup ■

1.3 HW3

Question 1

Let $o(G) = 60$. Show that if G is simple, then G must have exactly 24 elements of order 5 and 20 elements of order 3.

Proof. By Sylow, we have

$$n_5 \equiv 1 \pmod 5 \quad \text{and} \quad n_5 \mid 12$$

which by simplicity of G implies $n_5 = 6$. The same argument gives us $n_3 \in \{4, 10\}$. To see $n_3 \neq 4$, just recall that second sylow stated that conjugacy action $G \longrightarrow \text{Sym}(\text{Syl}_3(G)) \cong S_{n_3}$ is nontrivial, and is therefore injective by simplicity of G. We now see that $n_3 = 4$ is too small to satisfies

$$o(G) = 60 \mid n_3!$$

Remark: In fact $G \cong A_5$. ∎

Question 2

Let $o(G) = pqr$ with $p < q < r$ prime. Prove that G has a normal Sylow r-subgroup H.

Proof. By Sylow and counting arguments we know $1 \in \{n_p, n_q, n_r\}$. Therefore, if neither of n_p and n_q is 1, we are done. Suppose $1 \in \{n_p, n_q\}$. Either way, we get a normal subgroup N such that $o(G/N) \in \{qr, pr\}$, and by Question 3 of HW 1, we also get a normal $H/N \in \text{Syl}_r(G/N)$. This give us a characteristic $K \in \text{Syl}_r(H)$, which is normal in G, since

$$K \text{ char } H \trianglelefteq G \implies K \trianglelefteq G$$

∎

Question 3

Let $o(G) = p^3 q$ with p, q prime. Show that one of the followings statement is true:

(i) G has a normal Sylow p-subgroup.

(ii) G has a normal Sylow q-subgroup.

(iii) $p = 2$, $q = 3$.

Proof. Suppose (i) and (ii) are both false. Then by sylow we have $n_p = q$ and $p < q$. Because $p < q$, applying sylow again we have $n_q \in \{p^2, p^3\}$. Because $n_p > 1$, by counting we see that $n_q \neq p^3$. Therefore $n_q = p^2$. Then by sylow, $p^2 = n_q \equiv 1 \pmod q$, which implies $q \mid (p-1)(p+1)$. Because $p < q$ and q is prime, we now see $q = p+1$, which can only happens if $p = 2$ and $q = 3$. ■

Question 4

Show that no group of order 30 is simple.

Proof. Consider n_3 and n_5. We have $n_5 \in \{1, 6\}$ and $n_3 \in \{1, 10\}$. If $n_5 \neq 1 \neq n_3$, then there are 24 elements of order 5 and 20 elements of order 3, impossible for a group of order 30. ■

Question 5

Let G be a finite group with sylow p-subgroup P and normal subgroup N. Show that $P \cap N \in \mathrm{Syl}_p(N)$ and that $PN/N \in \mathrm{Syl}_p(G/N)$.

Proof. Second isomorphism theorem implies that

$$o(PN) \cdot o(P \cap N) = o(P) \cdot o(N)$$

$P \in \mathrm{Syl}_p(PN)$ implies that $o(P)$ and $o(PN)$ have the same p-power. Together, they imply that $o(P \cap N)$ and $o(N)$ have the same p-power, which is only possible if $P \cap N \in \mathrm{Syl}_p(N)$.

$P \in \mathrm{Syl}_p(G)$ implies that $o(P)$ and $o(G)$ have the same p-power. Because $o(P \cap N)$ and $o(N)$ have the same p-power, we now see

$$\frac{o(PN)}{o(N)} = \frac{o(P)}{o(P \cap N)}$$

has the same p-power as $o(G)/o(N)$, which is only possible if $PN/N \in \mathrm{Syl}_p(G/N)$. ■

Question 6

Let G be a finite group, $H \leq G$ a subgroup with $[G : H] = n$. Show that:

(i) For all subgroup $K \leq G$, we have $[H : H \cap K] \leq [G : K]$.

(ii) $[H : H \cap H^g] \leq n$ for all $g \in G$.

(iii) If H is a maximal proper subgroup of G and H is abelian, show that $H \cap H^g \trianglelefteq G$ for all $g \notin H$.

(iv) Suppose that G is simple. If H is abelian and n is prime, then $H = 1$.

Proof. Let $H \diagup H \cap K$ and $G \diagup K$ denote left coset spaces. (i) is a consequence of verifying that the function

$$H \diagup H \cap K \longrightarrow G \diagup K; \quad h(H \cap K) \mapsto hK$$

is well-defined and injective. (ii) is then a corollary of (i).

We now prove (iii). Fix $g \notin H$. There are two cases: Either $H = H^g$ or $H \neq H^g$. For the first case, just observe that by maximality of H, we will have $N_G(H) = G$. We now claim that $H \neq H^g \implies H \cap H^g \leq Z(G)$. Because H is abelian, we know $H \cap H^g \leq Z(H)$. Clearly we also have $H \cap H^g \leq H^g = Z(H^g)$. We now have $H \cap H^g \leq Z(\langle H, H^g \rangle)$, where $\langle H, H^g \rangle = G$ by maximality of H, as desired.

We now prove (iv). Clearly the primality of n forces H to be a maximal proper subgroup of G. Therefore by (iii), simplicity of G forces $H \cap H^g = 1$ for all $g \notin H$. This by (ii) implies $o(H) \leq n$ and therefore $n \leq o(G) \leq n^2$. Write $o(G) \triangleq nk$ so $k \in \{1, \dots, n\}$. We wish to show $k = 1$. To see $k \neq n$, just recall that if so, then by first Question of HW1, G would be abelian, contradicting to its simplicity. To see $k \notin \{2, \dots, n-1\}$, just observe that if so, then G has a normal (proper) Sylow n-subgroup, contradicting to simplicity of G. ∎

Question 7

Let G be a finite group with $P \in \text{Syl}_p(G)$. Suppose $N \trianglelefteq G$ and $[G : N] = o(P) > 1$. Show that

(i) N is the subset of G consisting of all elements of order not divisible by p.

(ii) If the elements of $G - N$ all has p-power order, then $P = N_G(P)$.

Proof. (i): Because P is p-sylow and $[G : N] = o(P)$, we know $p \nmid o(N)$. This implies that no element of N has order divisible by p. Let $g \in G$ with $p \nmid o(g)$. To see that $g \in N$, just observe that because $o(gN) \mid o(g)$ and $o(gN)$ is a power of p, we have $o(gN) = 1$.

(ii): Assume for a contradiction that $P < N_G(P)$. Then there exists some nontrivial $Q \in \text{Syl}_q(N_G(P))$ with $q \neq p$. Because $Q \leq N_G(P)$, we know $\langle [Q, P] \rangle \leq P$. Because of (i), $Q \leq N$, which by normality of N implies $\langle [Q, P] \rangle \leq N$. We now see

$$\langle [Q, P] \rangle \leq P \cap N = 1 \tag{1.4}$$

Let $y \in P - N$ and $x \in Q \leq N$ be nontrivial. Inequality 1.4 implies

$$(xy)^n = x^n y^n, \qquad \text{for all } n \in \mathbb{N}$$

We now see that $xy \in G - N$ has order divisible by pq, a contradiction to the premise. ■

1.4 HW4

Question 1

Show that the center of products is a product of centers:

$$Z(G_1) \times \cdots \times Z(G_n) = Z(G_1 \times \cdots \times G_n)$$

Deduce that a direct product of groups is abelian if and only if each of its factor is abelian.

Proof. The "$\subseteq$" is clear. To see that

$$g_1 \times \cdots \times g_n \in Z(G_1 \times \cdots \times G_n) \implies g_i \in Z(G_i)$$

just observe that if not, then

$$[g_1 \times \cdots \times g_n, e_1 \times \cdots \times x_i \times \cdots \times e_n] \neq e \in \prod G_j$$

The second part then follows from noting

$$Z(G_1 \times \cdots \times G_n) = G_1 \times \cdots \times G_n \iff Z(G_i) = G_i, \quad \text{for all } i$$

$\blacksquare$

Question 2

Let $G \triangleq A_1 \times \cdots \times A_n$ and $B_i \trianglelefteq A_i$ for all i. Prove that $B_1 \times \cdots \times B_n \trianglelefteq G$ and that

$$\frac{A_1 \times \cdots \times A_n}{B_1 \times \cdots \times B_n} = \frac{A_1}{B_1} \times \cdots \times \frac{A_n}{B_n}$$

Proof. Normality of $\prod B_i$ follows from computing:

$$(g_1, \dots, g_n)(b_1, \dots, b_n)(g_1, \dots, g_n)^{-1} = (g_1 b_1 g_1^{-1}, \dots, g_n b_n g_n^{-1}) \in \prod B_i$$

The second part require us to show that the map defined by:

$$\prod \left(\frac{A_i}{B_i} \right) \longrightarrow \frac{\prod A_i}{\prod B_i}; \quad \prod \left(\frac{a_i}{B_i} \right) \mapsto \frac{\prod a_i}{\prod B_i}$$

is a well-defined group isomorphism, which boils down to showing that it is (i) well-defined, (ii) actually a homomorphism, (iii) injective, and (iv) surjective. To see it is injective, just observe that if $\prod a_i \in \prod B_i$, then $a_i \in B_i$ for all i, and therefore $\prod \frac{a_i}{B_i} = e$. The rest are clear. $\blacksquare$

Question 3

Let G be a finite abelian group with $m \mid o(G)$. Show that G has a subgroup of order m.

Proof. Recall that primary decomposition form of structure theorem for finitely generated abelian groups stated that

$$G \cong G_{T_{p_1}} \times \cdots \times G_{T_{p_n}}$$

where the **torsion p-subgroup** G_{T_p} of G

$$G_{T_p} \triangleq \left\{x \in G : o(x) = p^k \text{ for some } k \geq 0\right\}$$

is:

$$G_{T_p} \cong C_{p^{d_1}} \times \cdots \times C_{p^{d_m}}$$

Because of such, we only have to prove that for all $0 \leq d \leq \sum_{i=1}^{m} d_i$, the group $C_{p^{d_1}} \times \cdots \times C_{p^{d_m}}$ has a subgroup of order p^d. This is clear, since we may find

$$d_1 + \cdots + d_k \leq d \leq d_1 + \cdots + d_{k+1}$$

and the subgroup

$$C_{p^{d_1}} \times \cdots \times C_{p^{d_k}} \times \langle x^{p^{d_{k+1}-r}} \rangle \times 1 \times \cdots \times 1$$

where $C_{p^{d_{k+1}}} \triangleq \langle x \rangle$ and $r \triangleq d - (d_1 + \cdots + d_k)$ suffices. ∎

Question 4

Show that the subgroups and quotients of a nilpotent group G are also nilpotent.

Proof. Let H be a subgroup of G, and let

$$1 \triangleq G_{(n)} \trianglelefteq \cdots \trianglelefteq G_{(1)} \trianglelefteq G_{(0)} \triangleq G$$

be a central series. To see that

$$1 = G_{(n)} \cap H \trianglelefteq \cdots \trianglelefteq G_{(1)} \cap H \trianglelefteq H$$

forms a central series, just observe that since

$$[G_{(k)}, H] \subseteq [G_{(k)}, G] \subseteq G_{(k+1)}$$

We have

$$[H \cap G_{(k)}, H] \leq G_{(k+1)} \cap H$$

Let $N \trianglelefteq G$ and $\pi : G \to G/N$ be the canonical projection. It is clear that

$$1 = \pi(G_{(n)}) \trianglelefteq \cdots \trianglelefteq \pi(G_{(1)}) \trianglelefteq \pi(G_{(0)}) = G/N$$

forms a central series. ∎

Question 5

Show that if $G/Z(G)$ is nilpotent, then G is nilpotent.

Proof. Let

$$1 \triangleq H_0 \trianglelefteq \cdots \trianglelefteq H_n \triangleq G/Z(G)$$

be a central series, and let $\pi : G \to G/Z(G)$ be the canonical projection. The proof then follows from noting that we have the central series:

$$1 \trianglelefteq Z(G) \trianglelefteq \pi^{-1}(H_1) \trianglelefteq \cdots \trianglelefteq \pi^{-1}(H_n) = G$$

■

Question 6

Let $o(G) = pqr$ with $p < q < r$ prime. Show that G is solvable.

Proof. Recall that by Question 2 of HW 3, we have a characteristic $R \in \mathrm{Syl}_r(G)$, and that by Question 3 of HW 1, we have a normal $P \in \mathrm{Syl}_p(G/R)$. Let $\pi : G \to G/R$ be the canonical projection. We then see

$$1 \trianglelefteq R \trianglelefteq \pi^{-1}(P) \trianglelefteq G$$

is a desired normal series, since the factor groups can only be cyclic. ■

Theorem 1.4.1. (Proper subgroups of nilpotent groups satisfy normalizer condition) If G is nilpotent, then any $H < G$ satisfies normalizer condition.

Proof. Note that if H doesn't contain $Z(G)$, then the elements of $Z(G)$ that lies outside H complete the proof, so we only have to consider the case $Z(G) \leq H$.

This is proved by induction on nilpotency class n of G. The base case $n = 1$ is clear. The inductive case follows from third isomorphism theorem for groups and the observation $G/Z(G)$ has the nilpotent class one smaller than that of G. ■

Equivalent Definition 1.4.2. (Finite nilpotent group) Let G be a finite group. The followings are equivalent:

(i) G is nilpotent.

(ii) Proper subgroups of G satisfies normalizer condition.

(iii) Sylow subgroups of G are all normal.

(iv) G is the internal direct product of its Sylow subgroups.

Proof. (i) $\implies$ (ii): This is true even if G is infinite.

(ii) $\implies$ (iii): If G is a p-group, then the proof is trivial. Let G not be a p-group and let $P \in \text{Syl}_p(G)$. To see P is normal, just observe that since normalizers of Sylow subgroups don't satisfy normalizer condition, the normalizer of P must be G.

(iii) $\implies$ (iv): This follows from the definition of finite internal direct product.

(iv) $\implies$ (i): This follows from the fact that p-groups is nilpotent and that nilpotency is closed under taking finite direct product. ■

Question 7

Show that a finite group G is nilpotent if and only if every $a, b \in G$ that makes $\gcd(o(a), o(b)) = 1$ also makes $ab = ba$.

Proof. ($\implies$): Write $G \triangleq P_1 \times \cdots \times P_n$ with P_i sylow. Since

$$o\left((x_1, \dots, x_n)\right) = \prod o(x_i)$$

we know that if the orders of $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are coprime to each other, then for all i, we must have either $x_i = e$ or $y_i = e$. This implies $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ commute.

($\impliedby$): We need to show that Sylow subgroups of G are normal. Let $P_1, \dots, P_n$ each be a Sylow subgroup of G with distinct p. By premise, we have $P_k \leq N_G(P_1)$ for all $k \geq 2$. This then implies $G = N_G(P_1)$, as desired. ■

Question 8

Let $G \triangleq HK$ be finite and $S \leq G$ be a p-subgroup that contains some $P \in \text{Syl}_p(H)$ and $Q \in \text{Syl}_p(K)$. Show that

(i) S is p-Sylow in G.

(ii) $S = (S \cap H)(S \cap K)$

Proof. Because $P \cap Q \leq H \cap K$, we know p-part of

$$o(G) = \frac{o(H)o(K)}{o(H \cap K)}$$

is $\leq$ than p-part of

$$\frac{o(P)o(Q)}{o(P\cap Q)} = |PQ| \leq o(S)$$

which can only happen if $S \leq G$ is p-Sylow with $|PQ| = o(S)$. By definition (premise), $P \leq S \cap H \leq H$. Because S is a p-group, we know $S \cap H$ is also a p-group. $P \in \text{Syl}_p(H)$ then forces $S \cap H = P$. Similarly, we have $S \cap K = Q$. Now, to see $S = PQ$, just recall that $|PQ| = o(S)$ ∎

Theorem 1.4.3. (Every p-subgroup is contained by some Sylow p-subgroup) Let G be a finite group and $H \leq G$ a p-group. Then H must be contained by some Sylow p-subgroup of G.

Proof. Consider the conjugacy action $H \longrightarrow \text{Bij}\big(\text{Syl}_p(G)\big)$. First Sylow theorem and orbit-stabilizer theorem shows that there must be a singleton orbit. Let that singleton be P.

We claim $H \leq P$. Because $\{P\}$ is a singleton orbit of the conjugacy action, we know $H \leq N_G(P)$. Then by second isomorphism theorem, we see that HP is a group such that $HP/P \cong H/H \cap P$. This implies that HP is a p-group. The fact HP contains P forces $P = HP$, which implies $H \leq P$. ∎

Question 9

Let $M \trianglelefteq G$ and $N \trianglelefteq G$ with M, N finite and nilpotent. Prove that MN is nilpotent.

Proof. Let p be a prime. Let $M_p \in \text{Syl}_p(M)$ and $N_p \in \text{Syl}_p(N)$ be the unique p-sylow subgroups. Our goal is to show that

$$P \triangleq M_pN_p \text{ is the unique Sylow } p\text{-subgroup of } MN.$$

Because

$$M_p \operatorname{char} M \trianglelefteq G \quad \text{ and } \quad N_p \operatorname{char} N \trianglelefteq G$$

We know $M_p, N_p \trianglelefteq G$. This implies that $P = M_pN_p \trianglelefteq G$, which implies $P \trianglelefteq MN$. Therefore, we reduce the problem into showing that

$$P \in \text{Syl}_p(MN) \tag{1.5}$$

Before proving such, we first need to show that

$$M_p \cap N_p \text{ is a unique Sylow } p\text{-subgroup of } M \cap N$$

Let $S \in \text{Syl}_p(M \cap N)$. Since every p-subgroup is contained by some Sylow p-subgroup, from $S \leq M \cap N \leq M$, we know $S \leq M_p$. Similarly, we have $S \leq N_p$. Therefore, we have

$S \leq M_p \cap N_p$. This then implies $M_p \cap N_p$ is the unique Sylow p-subgroup of $M \cap N$, as desired.

We may now compute p-part:

$$|o(P)|_p = \left|\frac{o(M_p)o(N_p)}{o(M_p \cap N_p)}\right|_p = \left|\frac{o(M)o(N)}{o(M \cap N)}\right|_p = |o(MN)|_p$$

to see that indeed, $P \in \mathrm{Syl}_p(MN)$.

$\blacksquare$

Question 10

Let G be finite with $A, B \trianglelefteq G$ and $G \diagup A, G \diagup B$ solvable. Prove that $G \diagup (A \cap B)$ is solvable.

Proof. Consider the group homomorphism $\varphi : G \to G \diagup A \times G \diagup B$ defined by

$$\varphi(g) \triangleq (gA, gB)$$

Clearly, $\ker(\varphi) = A \cap B$, so by first isomorphism theorem, we have $G \diagup (A \cap B) \leq G \diagup A \times G \diagup B$. Solvability of $G \diagup (A \cap B)$ then follows from the fact that solvable groups are closed under finite direct product and taking subgroups.

Remark: As it turns out, finiteness of G is unnecessary. $\blacksquare$