RTFT Ch8 Maschke's Theorem

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In this note, G is always a group.

In this note, V is always a vector space.

Theorems

Theorem 1. *Let $V = \mathbb{F}[G]$, and U be a $\mathbb{F}G$-submodule of V*

There exists a $\mathbb{F}G$-submodule of W, such that $V = U \oplus W$

Proof. Arbitrarily pick subspace W_0 such that $U \oplus W_0 = V$

Let $\phi : V \to U$ be defined by $u + w_0 \mapsto u$

Let $\tau : V \to V$ be defined by $\tau v = \frac{1}{|G|}\sum_{g\in G} g^{-1}\phi g v$

We now prove τ is an $\mathbb{F}G$-homomorphism

Let $c \in \mathbb{F}, v, v' \in V, h \in G$

$\tau(cv + v') = \frac{1}{|G|}\sum_{g\in G} g^{-1}\phi g(cv + v') = \frac{1}{|G|}\sum_{g\in G} g^{-1}\phi(cgv + gv')$

$= \frac{1}{|G|}\sum_{g\in G} g^{-1}(c\phi gv + \phi gv') = \frac{1}{|G|}\sum_{g\in G} cg^{-1}\phi gv + g^{-1}\phi gv'$

$= \frac{1}{|G|}c\sum_{g\in G} g^{-1}\phi gv + \frac{1}{|G|}\sum_{g\in G} g^{-1}\phi gv' = c\tau v + \tau v'$

$\tau hv = \frac{1}{|G|}\sum_{g\in G} g^{-1}\phi ghv = \frac{1}{|G|}\sum_{gh\in G} hh^{-1}g^{-1}\phi ghv =$

$\frac{1}{|G|}\sum_{gh\in G} h(gh)^{-1}\phi(gh)v = h\frac{1}{|G|}\sum_{r\in G} r\phi r^{-1}v = h\tau v$ (done)

We now prove $R(\tau) = U$

Notice $R(\phi) = U$

By definition of $\tau : v \mapsto \frac{1}{|G|}\sum_{g\in G} g^{-1}\phi gv$, we know $R(\tau) \subseteq \bigcup_{g\in G} g[R(\phi)] = \bigcup_{g\in G} g[U]$

Because U is an submodule, we know $\forall g \in G, g[U] \subseteq U$

This give us $R(\tau) \subseteq U$

Let $u \in U$

Because U is g-invariant, so $gu \in U$, which give us $\phi gu = gu$, since U is a submodule

$\tau u = \frac{1}{|G|}\sum_{g\in G} g^{-1}\phi gu = \frac{1}{|G|}\sum_{g\in G} g^{-1}gu = \frac{1}{|G|}\sum_{g\in G} u = u$

So $U \subseteq R(\tau)$ (done)

We now prove τ is a projection

Let $v \in V$

Write $u = \tau v$

Because $R(\tau) = U$, we know $u \in U$

Because U is a submodule, we know $gu \in U$

$\phi gu = gu$, since ϕ is a projection onto U

$\tau^2 v = \tau u = \frac{1}{|G|}\sum_{g\in G} g^{-1}\phi gu = \frac{1}{|G|}\sum_{g\in G} g^{-1}gu = u = \tau v$ (done)

So $V = N(\tau) \oplus R(\tau) = N(\tau) \oplus U$, where $N(\tau)$ is a submodule, since τ is a $\mathbb{F}G$-homomorphism

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Exercises

1.

Let $G = \langle x | x^3 = e\rangle$ and let $V = \mathbb{C}[G]$ with basis v_1, v_2, defined by $xv_1 = v_2, xv_2 = -v_1 - v_2$

Write V into a direct sum of two $\mathbb{F}G$-submodule

Proof. $V = span[2v_1 + (1+\sqrt{3}i)v_2] \oplus span[2v_1 + (1-\sqrt{3}i)v_2]$

Let $U = span[2v_1 + (1+\sqrt{3}i)v_2]$ and $W = span[2v_1 + (1-\sqrt{3}i)v_2]$

Let $u = 2v_1 + (1+\sqrt{3}i)v_2$ and $w = 2v_1 + (1-\sqrt{3}i)v_2$

Check $xu \in U$ and $xw \in W$

Then we see $x^2u \in U$ and $x^2w \in W$

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2.

Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$

Express group algebra $\mathbb{R}G$ as a direct sum of four 1-dimensional $\mathbb{R}G$-module

Proof. Represent the $\mathbb{R}G$ with the basis $\alpha = \{(0,0),(1,0),(0,1),(1,1)\}$

$[(0,0)]_\alpha = I_4$

$$[(1,0)]_\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$[(0,1)]_\alpha = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$[(1,1)]_\alpha = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbb{R}G = span(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}) \oplus span(\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}) \oplus span(\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}) \oplus span(\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix})$$ ■

3.

Find a group G, a $\mathbb{C}G$-module V and a $\mathbb{C}G$-homomorphism $\phi : V \to V$ such that

$$V \neq N(\phi) \oplus R(\phi)$$

Proof. Let G be any group and $V = \mathbb{C}^2$ with the structure $\forall g \in G, v \in V, gv = v$, that is, every g-action is a trivial linear transformation

Let $T \in GL(V)$

We see $gTv = Tv = Tgv$

So T is a $\mathbb{C}G$-homomorphism

Let T be defined by $av_1 + bv_2 \mapsto av_2$

$N(T) = span(v_2)$

$R(T) = span(v_2)$ ■

7.

Let G be a finite simple group

Prove there exists a faithful irreducible $\mathbb{C}G$-module

Proof. Let G be a finite simple group

Let $V = \mathbb{C}G$ Algebra

Let $g \in G$, where $g \neq e$

$g(1g^{-1}) = 1e \neq 1g^{-1}$, so g-action is not trivial

This tell us V is faithful

Write V into $V = U_1 \oplus \cdots \oplus U_r$, where $\forall 1 \leq i \leq r, U_i$ is a irreducible submodule

Let $g \in G$, where $g \neq e$

Assume $\forall u \in \bigcup_{i=1}^{r} U_i, gu = u$

$\forall v \in V, gv = g(\sum_{i=1}^{r} u_i) = \sum_{i=1}^{r} u_i = v$

g have trivial g-action CaC to that V is faithful

We pick $t \in \bigcup_{i=1}^{r} U_i, gt \neq t$

We know $\exists 1 \leq k \leq r, t \in U_k$

Fix such k and we let $N = \{h \in G | hu_k = u_k, \forall u_k \in U_k\}$

We now prove $N \trianglelefteq G$

Let $h, l \in N, s \in G, u \in U_k$

$(hl)u = hlu = hu = u \implies hl \in N$

$eu = u \implies e \in N$

$h^{-1}u = h^{-1}(hu) = (h^{-1}h)u = u \implies h^{-1} \in N$

Because U_k is a submodule, we know that U_k is s-invariant

So $s^{-1}u \in U_k$

This give us $shs^{-1}u = s[h(s^{-1}u)] = s[s^{-1}u] = u$

So $shs^{-1} \in N$ (done)

Because G is simple, either $N = \{e\}$ or $N = G$

We know $gt \neq t$, where $t \in U_k$, so $g \notin N$, which give us $N \neq G$

$N = \{e\}$ tell us U_k is faithful

So U_k is the faithful irreducible $\mathbb{C}G$-module we are looking for

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