(Thm 6.11 of Rudin) Let f be a bounded function on $[a,b]$, write $m \leq f \leq M$, such that $f \in \mathcal{R}(\varphi)$. Then $\phi(f) \in \mathcal{R}(\varphi)$ on $[a,b]$, if ϕ is continuous on $[m,M]$.

Strategy: By Thm 6.6 of Rudin, it is alternative to prove $g = \phi(f) \in \mathcal{R}(\varphi)$ by showing $\forall \varepsilon > 0, \exists\, \Gamma \subset [a,b]$ s.t. $U(\Gamma, g, \varphi) - L(\Gamma, g, \varphi) < \varepsilon$. It is clear that if f is continuous on $[a,b]$, then we can complete the proof by uniform continuity of g on $[a,b]$. Hence, the main difficulty would be how to estimate $(\, M(g) - m(g)\,)\, \Delta\varphi$, when f contains discontinuity of δ. Although the existence of R-S integral tolerate discontinuity of f and φ, it suffices to squeeze the size of $\Delta\varphi$ by another arbitrary parameter η.

proof: we may assume $\varphi(a) < \varphi(b)$, $m < M$, and $K = \sup_{[m,M]} \phi - \inf_{[m,M]} \phi > 0$, otherwise the theorem hold trivially. $\forall\, \varepsilon > 0$, let $\xi = \varepsilon / 2\,(\, \varphi(b) - \varphi(a)\,) > 0$, then we choose $\delta > 0$ by uniform continuity of ϕ on $[m,M]$, such that if $s, t \in [m,M]$ and $|s - t| < \delta$, then $|\phi(s) - \phi(t)| < \xi$. For the given $\delta > 0$, we choose $\eta = \varepsilon / 2K > 0$ (where K is finite, due to continuity of ϕ on $[m,M]$), then there exists a partition Γ of $[a,b]$ s.t $\sum_{i=1}^{n} (\, M_i(f) - m_i(f)\,) \Delta\varphi_i < \delta\eta$, since $f \in \mathcal{R}(\varphi)$. We divide the index $\{1, ..., n\}$ into $A = \{1 \leq i \leq n : M_i(f) - m_i(f) < \delta\}$ and $B = \{1, ..., n\} - A$. Since $M_i \geq m_i$ and $\Delta\varphi_i \geq 0$, $\forall\, i$, for $i \in B$, we have

$$\delta \sum_{i \in B} \Delta\varphi_i \leq \sum_{i \in B} (\, M_i(f) - m_i(f)\,) \Delta\varphi_i \leq \sum_{i=1}^{n} (\, M_i(f) - m_i(f)\,) \Delta\varphi_i < \delta\eta$$

further, since $M_i(g) - m_i(g) \leq K$, $\forall\, i$, we have

$$\sum_{i \in B} (\, M_i(g) - m_i(g)\,)\, \Delta\varphi_i \leq K \sum_{i \in B} \Delta\varphi_i < K\eta = \frac{\varepsilon}{2}$$

If $i \in A$, then $|f(x) - f(y)| \leq M_i(f) - m_i(f) < \delta$, and $|g(x) - g(y)| < \xi, \forall\, x, y \in [x_{i-1}, x_i]$, where $g = \phi(f)$ for simplicity. It follows that $M_i(g) - m_i(g) \leq \xi$, and

$$\sum_{i \in A} (\, M_i(g) - m_i(g)\,) \Delta\varphi_i \leq \xi \sum_{i \in A} \Delta\varphi_i \leq \frac{\varepsilon}{2}$$

We conclude the proof by Thm 6.6 that $\phi(f) \in \mathcal{R}(\varphi)$, since there exists a partition Γ of $[a,b]$ such that

$$U(\Gamma, \phi(f), \varphi) - L(\Gamma, \phi(f), \varphi) = \sum_{i=1}^{n} (\, M_i(g) - m_i(g)\,) \Delta\varphi_i < \varepsilon$$