Winter Break, Advanced Calculus I, Yung-fu Fang, 2024

Jan. 08 2024 Show All Work Name: Id: *Group*

1. State the definitions of the real numbers, a compact set, and a countable dense set. Let K be a compact set in a metric space X. Show that K has a countable dense subset.

2. State the definitions of a continuous function, a uniform continuous function, a differentiable function, a convergent sequence $\{a_n\}$, a convergent series $\sum a_n$, a pointwise convergent sequence of functions $\{f_n\}$, a uniform convergent sequence of functions $\{f_n\}$, an equicontinuous family of functions $\{f_n\}$, Riemann-Stieltjes integral $\int_a^b f d\alpha$.

3. State the Mean-Value Theorem, Fundamental Theorem of Calculus, Taylor's Theorem, and Weierstrass Theorem.

4. Let f be a continuous function on R^1, $f'(x)$ exists for all $x \neq 0$, and $f'(x) \to 3$ as $x \to 0$. Does it follow that $f'(0)$ exists? It it exists, then prove it; otherwise give an example.

5. Suppose that α increases on $[a, b]$, $a \leq x_0 \leq b$, α is continuous at x_0, $f(x_0) = 1$ and $f(x) = 0$ for $x \neq x_0$. Prove that $f \in R(\alpha)$ and $\int_a^b f d\alpha = 0$.

6. Let $a < s < b$, f be bounded on $[a.b]$, f continuous at s, and $\alpha(x) = 0$ for $x \leq s$, $= 1$ for $x > s$. Show that $\int_a^b f d\alpha = f(s)$.

7. Show that the sequence of functions $\{f_n(x) = \dfrac{x^2}{x^2 + (1 - nx)^2} : 0 \leq x \leq 1, n = 1, 2, 3, \cdots\}$ is pointwise convergent, but not equicontinuous.

8. Prove that every uniform convergent sequence of bounded functions is uniform bounded.

9. For $n = 1, 2, 3, \cdots$, $x \in R$, and $f_n(x) = \dfrac{x}{1 + nx^2}$. Show that $\{f_n\}$ converges uniformly to a function f and $f'(x) = \lim_{n\to\infty} f_n'(x)$ for $x \neq 0$. Find $f'(0)$ and $\lim_{n\to\infty} f_n'(0)$.

10. (a) Let f be continuous on R, $f(x) = 0$ for $x \in (-\infty, 0) \cup (1, \infty)$, $Q_n(x) = (1-x^2)^n$ for $n = 1, 2, 3, \cdots$. Show that $P_n(x) = \displaystyle\int_{-1}^{1} f(x+t)Q_n(t)\,dt$ is a polynomial.

(b) Prove that $(1-x^2)^n \geq 1 - nx^2$ on $[-1, 1]$.

11. Discuss the convergence and divergence of the power series $\displaystyle\sum_{k=1}^{\infty} k^q z^k$, where $z \in \mathbb{C}$.

11. Suppose f is a real, continuously differentiable function on $[a,\, b] f(a) = f(b) = 0$, and $\displaystyle\int_a^b f^2(x)dx = 1$. Prove that $\displaystyle\int_a^b xf(x)f'(x)dx = -\frac{1}{2}$ and that $\displaystyle\int_a^b [f'(x)]^2 dx \cdot \int_a^b x^2 f^2(x)dx > \frac{1}{4}$.

12. Suppose α increases monotonically on $[a, b]$, g is continuous, and $g(x) = G'(x)$ for $a \leq x \leq b$. Prove that $\displaystyle\int_a^b \alpha(x)g(x)dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G(x)d\alpha$.

Hint: Take g real, without loss of generality. Given a partition $P = \{x_0, x_1, ..., x_n.\}$, choose $t_i \in (x_{i-1}, x_i)$ so that $g(t_i)\Delta x_i = G(x_{i-1}) - G(x_i)$. Show that $\displaystyle\sum_{i=1}^{n} \alpha(x_i)g(t_i)\Delta x_i = G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^{n} G(x_{i-1})\Delta\alpha_i$.

13. State and prove the STONE-WEIERSTRASS THEOREM.

Have a Nice Winter Break!