Deadline : 2023/12/04, 17:00.

The definition of continuity of function follows from textbook(W.Rudin)

1. Let $(X, d), (Y, d)$ be metric spaces; let $f : X \to Y$. Show that f is continuous on X if and only if for every open set V, $f^{-1}(V)$ is open.

2. Let X and Y be metric space; let $f : X \to Y$. Show that the following statements are equivalent:

    1. f is continuous.
    2. For every subset A of X, one has $f(\overline{A}) \subseteq \overline{f(A)}$.
    3. For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.

3. Suppose that E is a nonempty subset of $\mathbb{R}$, let $a \in E$, and that $f : E \to \mathbb{R}$. If f has the property that if $\{x_n\} \in E$ and x_n converge to a, then $f(x_n) \to f(a)$ as $n \to \infty$. Is f continuous at a ?Justify your result.

4. Let $f : M \to N$ be a map from metric M to another metric space N with the property that if a sequence $\{p_n\}$ in M converges, then the sequence $\{f(p_n)\}$ in N also converges. Is f continuous ? Justify your result.

5. Suppose f is a real function defined on $\mathbb{R}$ which satisfies

$$\lim_{h\to 0}[f(x+h) - f(x-h)] = 0$$

   for every $x \in \mathbb{R}$. Does this imply that f is continuous ?

6. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : I(\subseteq \mathbb{R}) \to \mathbb{R}$ be a map. Please state the definition of following nouns.
   (There is no standard answer, you can write down all equivalent statements as possible as you can)

    1. f is continuous on $\mathbb{R}$.
    2. f is uniformly continuous on $\mathbb{R}$.
    3. f is Lipschitz continuous on $\mathbb{R}$.
    4. g is absolutely continuous on I.

7. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : I(\subseteq \mathbb{R}) \to \mathbb{R}$ be a map. Prove or disprove following statement.

    (i) if f is continuous on $\mathbb{R}$, then f is uniformly continuous on $\mathbb{R}$.
    (ii) if f is uniformly continuous on $\mathbb{R}$, then f is continuous on $\mathbb{R}$.
    (iii) if g is continuous on I and I is compact, then g is uniformly continuous on I.
    (iv) if f is continuous on $\mathbb{R}$, then f is Lipschitz continuous on $\mathbb{R}$.
    (v) if f is Lipschitz continuous on $\mathbb{R}$, then f is continuous on $\mathbb{R}$.
    (vi) if g is Lipschitz continuous on I, then g is uniformly continuous on I.
    (vii) if g is uniformly continuous on I, then g is Lipschitz continuous on I.
    (viii) if g is absolutely continuous on I, then g is uniformly continuous on I.

(ix) if g is uniformly continuous on I, then g is absolutely continuous on I.

(x) if g is absolutely continuous on I, then g is Lipschitz continuous on I.

(xi) if g is Lipschitz continuous on I, then g is absolutely continuous on I.

8. Suppose that $f : [0, \infty) \to \mathbb{R}$ is continuous and there is an $L \in \mathbb{R}$ such that $f(x) \to L$ as $x \to \infty$. Prove that f is uniformly continuous on $[0, \infty)$.

9. Show that if f is monotone on an interval I, then f has at most countably many points of discontinuity on I.

Extra question
(If you finish there problems and want to obtain extra points, please email
symmetrickelly@gmail.com)

0.1 The sufficient/necessary condition of continuous functions/ uniformly continuous

10. Suppose $f : \mathbb{R} \to \mathbb{R}$ is uniformly continous. Show that f must be bounded by a linear function, i.e, there exist constants A, B s.t $|f(x)| \leq A + B|x|$ for all x.

11. Prove/disprove $f(x, y) = \frac{1}{(1-xy)^2}$ is uniformly continuous on $[0, 1] \times [0, 1] \backslash \{(1, 1)\}$

12. Prove that $f(x) = \frac{1}{x^2+1}$ is uniformly continuous on $\mathbb{R}$.

13. problem 6,19 of chapter 4 of Principles of Mathematical Analysis(third edition)

0.2 the lemma about continuous extensions

14. problem 5,13 of chapter 4 of Principles of Mathematical Analysis(third edition)

15. Please state Tietze Extension Theorem

0.3 the lemma about how to determine value of continuous func.

16. problem 4 of chapter 4 of Principles of Mathematical Analysis(third edition)

0.4 the lemma about separation by continuous func.

17. problem 20,22 of chapter 4 of Principles of Mathematical Analysis(third edition)

18. Please state the Urysohn's lemma and graph it

0.5 the relation between convex func. and continuous func.

19. problem 23,24 of chapter 4 of Principles of Mathematical Analysis(third edition)

0.6 Another

20. problem 18,26 of chapter 4 of Principles of Mathematical Analysis(third edition)