Friedberg Ch6.5

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In this note, V always stand for a vector space over $\mathbb{F}$, V^- stands for a finite dimensional vector space over $\mathbb{F}$, and T is always a linear operator on V^-

Definition

Definition 1. *Let T satisfy $\forall x \in V^-, \|T(x)\| = \|x\|$*

$$T(x) \textit{ is an } \textbf{\textit{unitary operator}} \textit{ if } \mathbb{F} = \mathbb{C}$$
$$T(x) \textit{ is an } \textbf{\textit{orthogonal operator}} \textit{ if } \mathbb{F} = \mathbb{R}$$

Lemma 1. *Let U be self-adjoint*

$$\forall x \in V, \langle x, U(x) \rangle = 0 \implies U = 0$$

Proof. Pick an orthonormal basis β that diagonalize U

Let $\beta_i \in \beta$, and write $U(\beta_i) = \lambda_i \beta_i$

$0 = \langle \beta_i, U(\beta_i) \rangle = \lambda_i \langle \beta_i, \beta_i \rangle \implies \lambda_i = 0$ ∎

Theorem 2. *Let T be unitary and β be an orthonormal basis for V^-*

$$\textbf{\textit{(i)}}\ TT^* = T^*T = I$$
$$\textbf{\textit{(ii)}}\ \langle T(x), T(y) \rangle = \langle x, y \rangle$$
$$\textbf{\textit{(iii)}}\ T(\beta) \textit{ is an orhonormal basis for } V^-$$

Proof. $\langle x, x \rangle = \|x\|^2 = \|T(x)\|^2 = \langle T(x), T(x) \rangle = \langle T^*T(x), x \rangle \implies \langle (I - T^*T)x, x \rangle = 0$

Because $I - T^*T$ is self adjoint, so by Lemma 1, $I = T^*T$ **(i)**

$\langle T(x), T(y) \rangle = \langle T^*T(x), y \rangle = \langle x, y \rangle$ **(ii)**

Let $\beta_i, \beta_j \in \beta$

$\langle T(\beta_i), T(\beta_j) \rangle = \langle T^*T(\beta_i), \beta_j \rangle = \langle \beta_i, \beta_j \rangle = 0$ **(iii)**

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