4.3 PDE HW 3

Question 44

Find a solution of

$$\begin{cases} u_t = u_{xx} \\ u(x,0) = x^2 \end{cases}$$

Proof. Clearly $u = x^2 + 2t$ suffices. ■

Question 45

Consider the ODE

$$\begin{cases} u'' + u' = f \\ u'(0) = u(0) = \frac{1}{2}(u'(l) + u(l)) \end{cases}$$

where f is given.

(a) Is the solution unique?

(b) Does a solution necessarily exist, or is there a condition that f must satisfy for existence?

Proof. The solution space of linear homogeneous ODE $u'' + u' = 0$ is spanned by e^{-x} and constant. If we add in the initial condition $u'(0) = u(0)$, then the solution space become the subspace spanned by $e^{-x} - 2$. One can check that if $u \in \text{span}(e^{-x} - 2)$, then

$$u(0) = \frac{1}{2}(u'(l) + u(l)) \text{ for all } l \in \mathbb{R}$$

We now know the solution of the original ODE is not unique, since any solution added by $e^{-x} - 2$ is again a solution.

Integrating both side on $[0, l]$, we see that given the boundary conditions, f must satisfy

$$\begin{aligned} \int_0^l f(x)dx &= \int_0^l u'' + u' dx \\ &= u(l) + u'(l) - u(0) - u'(0) = 0 \end{aligned}$$

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Question 46

Find the regions in the xy plane where the equation

$$(1+x)u_{xx} + 2xyu_{xy} - y^2u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch them.

Proof. The discriminant is exactly

$$\begin{aligned}(xy)^2 - (1+x)(-y^2) &= x^2y^2 + xy^2 + y^2 \\ &= y^2(x^2+x+1) \\ &= y^2[(x+\frac{1}{2})^2 + \frac{3}{4}]\end{aligned}$$

It then follows that the equation is parabolic if and only if $y = 0$, and hyperbolic if and only if $y \neq 0$.

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