## 7.10 PDE HW 10

**Question 174**

Consider

$$\sum_{n=0}^{\infty}(-1)^n x^{2n}$$

(a) Does it converge pointwise in $(-1,1)$?

(b) Does it converge uniformly in $(-1,1)$?

(c) Does it converge in $L^2$ in $(-1,1)$?

*Proof.* Geometric series as such obviously converges to

$$\sum_{n=0}^{\infty}(-1)^n x^{2n} = \sum_{n=0}^{\infty}(-x^2)^n = \frac{1}{1+x^2}$$

Note that the remainder can also be computed by

$$\sup_{x\in(-1,1)}\left|\sum_{n=N}^{\infty}(-x^2)^n\right| = \sup_{x\in(-1,1)}\left|\frac{-x^{2N}}{1+x^2}\right| = \frac{1}{2} \text{ for all } N$$

It follows that the series does NOT converge uniformly on $(-1,1)$. Compute

$$\begin{aligned}\int_{-1}^{1}\Big(\sum_{n=N}^{\infty}(-x^2)^n\Big)^2 dx &= \int_{-1}^{1}\Big(\frac{-x^{2N}}{1+x^2}\Big)^2 dx \\ &= \int_{-1}^{1}\frac{x^{4N}}{(1+x^2)^2}dx \\ &\leq \int_{-1}^{1} x^{4N}dx = \frac{x^{4N+1}}{4N+1}\Big|_{x=-1}^{1} \to 0\end{aligned}$$

It follows that the series does converge in $L^2$. ■

**Question 175**

(Term by Term integration)

(a) If $f(x)$ is a piecewise continuous function in $[-l,l]$, show that its definite integral $F(x)=\int_{-l}^{x} f(s)ds$ has a full Fourier series that converges pointwise.

(b) Write this convergent series for $F(x)$ explicitly in terms of the Fourier coefficients $a_0, a_n, b_n$ of $f(x)$ where $a_0 = 0$. (Hint: Apply a convergence Theorem. Write the formulas for the coefficients and integrate by parts.)

*Proof.* Part (a) follows from observing $F' = f$ is pointwise continuous so that the classical Fourier series of $F$ converges to $F$ by Theorem 4 in the textbook.

Write

$$f(x) = \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{l}) + b_n \sin(\frac{n\pi x}{l})$$

Note that the definition of piece wise continuity in this book implies boundedness on compact domain, and note that each term

$$\sum_{n=1}^{N} a_n \cos(\frac{n\pi x}{l}) + b_n \sin(\frac{n\pi x}{l})$$

is obviously bounded on $[-l, l]$. Then because $f$ is bounded on $[-l, l]$, we know

$$\sum_{n=1}^{N} a_m \cos(\frac{n\pi x}{l}) + b_n \sin(\frac{n\pi x}{l}) \text{ is uniformly bounded on } [-l, l]$$

Therefore, we may apply DCT to compute

$$\begin{aligned}
F(x) &= \int_{-l}^{x} \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi s}{l}) + b_n \sin(\frac{n\pi s}{l}) ds \\
&= \sum_{n=1}^{\infty} \int_{-l}^{x} a_n \cos(\frac{n\pi s}{l}) + b_n \sin(\frac{n\pi s}{l}) ds \\
&= \sum_{n=1}^{\infty} \frac{a_n l}{n\pi} \sin(\frac{n\pi x}{l}) + \frac{-b_n l}{n\pi} \cos(\frac{n\pi x}{l}) + \frac{-(-1)^n b_n l}{n\pi}
\end{aligned}$$

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