<Ex> 1000^{999} , 999^{1000}

which one is bigger?

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} \doteq e$$

$$\left(\frac{1000^{99}}{999^{100}}\right)^{-1} = \left(\frac{1}{999}\left[\frac{(1+999)}{999}\right]^{999}\right)^{-1}$$

$$= \left(\frac{1}{999}\left(\frac{1}{999}+1\right)^{999}\right)^{-1}$$

$\to \approx e$

999^{1000} is bigger $\approx \left(\frac{e}{999}\right)^{-1} < 1$ $>$

黃聖堯

$$\left(\frac{1000}{999}\right)^{999} \times \frac{1}{999}$$

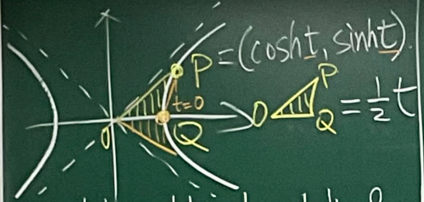
$$= \left(\frac{1+999}{999}\right)^{999} \times \frac{1}{999}$$

$$= \left(\frac{1}{999}+1\right)^{999} \times \frac{1}{999}$$

$$\approx e \times \frac{1}{999} < 1$$

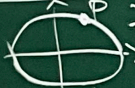
$$\Rightarrow 1000^{999} < 999^{1000}$$

林子閔



sinh t, cosh t = hyperbolic fns

⟨Ex⟩



$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

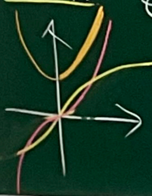
$P=(a\cos\theta, b\sin\theta),\ 0\le\theta\le 2\pi$

§6.7 Hyperbolic fns.

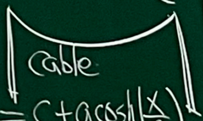
$\sinh x \equiv \frac{e^x-e^{-x}}{2}$, $\operatorname{csch} x \equiv \frac{2}{e^x-e^{-x}}$

$\cosh x \equiv \frac{e^x+e^{-x}}{2}$, $\operatorname{sech} x \equiv \frac{2}{e^x+e^{-x}}$

$\tanh x = \frac{e^x-e^{-x}}{e^x+e^{-x}}$, $\coth x \equiv \frac{e^x+e^{-x}}{e^x-e^{-x}}$



⟨Ex⟩



$y = c + a\cosh\left(\frac{x}{a}\right)$

Derivatives of Hyperbolic fns

$\frac{d}{dx}\sinh x = \frac{d}{dx}\left(\frac{1}{2}(e^x-e^{-x})\right) = \frac{1}{2}(e^x+e^{-x}) = \cosh x$

$\frac{d}{dx}\cosh x = \frac{d}{dx}\left(\frac{1}{2}(e^x+e^{-x})\right) = \frac{1}{2}(e^x-e^{-x}) = \sinh x$

$\frac{d}{dx}\tanh x = \frac{d}{dx}\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right) = \operatorname{sech}^2 x$

$\frac{d}{dx}\operatorname{csch} x = \frac{d}{dx}\frac{1}{\sinh x} = \frac{-1}{\sinh^2 x}\frac{d}{dx}\sinh x = \frac{-\cosh x}{\sinh^2 x}$

$= -\coth x \operatorname{csch} x$

$\frac{d}{dx}\operatorname{sech} x =$

$\frac{d}{dx}\coth$

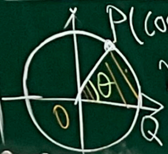
Identities:

$\sinh(-x) = -\sinh x$; $\cosh(-x) = \cosh(x)$

$\cosh^2 x - \sinh^2 x = 1$, $1-\tanh^2 x = \operatorname{sech}^2 x$

$\sinh(x+y) = \sinh x\cosh y + \cosh x \sinh y$

$\cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y$



$x^2+y^2=1$

$\triangle OPQ = \frac{1}{2}\theta$

sinθ, cosθ

circular fns

Inverse Hyperbolic fns. $y=\tanh^{-1}x \Leftrightarrow \tanh y = x$

$y=\sinh^{-1}x \Leftrightarrow \sinh y = x$,

$y=\cosh^{-1}x \Leftrightarrow \cosh y = x$, $\boxed{y\ge 0}$

$\sinh^{-1}x = \ln(x+\sqrt{x^2+1})$, $x\in\mathbb{R}$

$\cosh^{-1}x = \ln(x+\sqrt{x^2-1})$, $x\ge 1$

$\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

Caution: $\frac{d}{dx}\sinh x = h\cosh x$

show $\sinh^{-1}x = \ln(x+\sqrt{x^2+1})$.

$\sinh y = x = \frac{e^y - e^{-y}}{2}$

黃聖堯

$e^y - 2x - e^{-y} = 0$

$e^{2y} - 2xe^y - 1 = 0$

$e^y = \frac{2x\pm\sqrt{4x^2+4}}{2} = x\pm\sqrt{x^2+1}$

$e^y > 0$ (負不合)

$e^y = x+\sqrt{x^2+1}$ $y=\ln(x+\sqrt{x^2+1})$

$\cosh x = \frac{e^x+e^{-x}}{2} = y$

$\Rightarrow \frac{e^y+e^{-y}}{2} = x$ $e^y = u$

梁廣宥

$u + \frac{1}{u} = 2x$

$e^y > 0$

$u^2 - 2xu + 1 = 0$

$y=\ln u = \ln\frac{2x\pm\sqrt{4x^2-4}}{2} \overset{?}{\Rightarrow} \ln x+\sqrt{x^2-1}$

Derivatives of Inverse hyperbolic fns

$\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}$ 林子陵

$\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}}$ 陳嵩富

$\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}$

王冠章

林子陵

$\frac{d}{dx}(\sinh^{-1}x)$

$y=\sinh^{-1}x$: $\boxed{x=\sinh y}$

$\frac{d}{dx}\sinh y = \frac{d}{dx}\sinh(\sinh^{-1}x)$

$= \frac{d}{dx}x = 1$

$\Rightarrow \frac{d}{dy}\sinh y \frac{dy}{dx} = \cosh y \frac{d}{dx}(\sinh^{-1}x) = 1$

$\frac{d}{dx}\sinh^{-1}x = \frac{1}{\cosh y}$

$= \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$

$\frac{d}{dx}(\cosh^{-1}x) = \frac{d}{dx}\ln(x+\sqrt{x^2-1})$

陳嵩富

(1) $= \frac{1}{x+\sqrt{x^2-1}}\left(1+\frac{x}{\sqrt{x^2-1}}\right)$

(2)

$\frac{d}{dx}\cosh(\cosh^{-1}x) = \frac{d}{dx}\cosh y = \frac{d}{dx}x$

$\sinh(\cosh^{-1}x)\cdot\frac{d}{dx}\cosh^{-1}x = 1$

$\frac{d}{dx}\cosh^{-1}x = \frac{1}{\sinh(\cosh^{-1}x)} =$

$c^2 - sh^2 = 1$

$sh = \sqrt{x^2-1}$

§6.8 Indeterminate Forms and L'Hopital's Rule

Def: If $f(x)\to 0$, $g(x)\to 0$ as $x\to a$

then $\lim_{x\to a}\frac{f(x)}{g(x)}$ = indeterminate form of type $\frac{0}{0}$

If $f(x)\to\infty$ (or $-\infty$) & $g(x)\to\infty$ (or $-\infty$) as $x\to a$, then $\lim_{x\to a}\frac{f(x)}{g(x)}$ = indeterminate form of type $\frac{\infty}{\infty}$

L'Hopital's Rule:

Suppose that f & g are diff. and $g'(x)\neq 0$ nearby $x=a$. Suppose that $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ or that

$\lim_{x\to a} f(x)=\pm\infty$, $\lim_{x\to a} g(x)=\pm\infty$.

$\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists (or is ∞ or $-\infty$)

$$\Rightarrow \boxed{\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}}$$

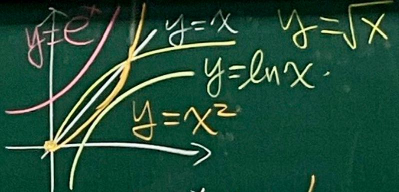
⟨Ex⟩ $\lim_{x\to 0}\frac{\sin x}{x}\left(\frac{0}{0}\right) \overset{\text{L'Hopital}}{=} \lim_{x\to 0}\frac{\cos x}{1} = 1$ #

⟨Ex⟩ $\lim_{x\to 1}\frac{\ln x}{x-1}\left(\frac{0}{0}\right) \overset{\text{L'Hopital}}{=} \lim_{x\to 1}\frac{\frac{1}{x}}{1} = 1$ #

⟨Ex⟩ $\lim_{x\to\infty}\frac{\ln x}{x}\left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = 0$

⟨Ex⟩ $\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}\left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = 0$

⟨Ex⟩ $\lim_{x\to\infty}\frac{\ln x}{\sqrt[3]{x}} = 0$



⟨Ex⟩ $\lim_{x\to\infty}\frac{e^x}{x^2}\left(\frac{\infty}{\infty}\right) \overset{L}{=} \lim_{x\to\infty}\frac{e^x}{2x} \overset{L}{=} \lim_{x\to\infty}\frac{e^x}{2} = \infty$

⟨Ex⟩ $\lim_{x\to\pi^-}\frac{\sin x}{1-\cos x} \overset{L}{=} \lim_{x\to\pi^-}\frac{\cos x}{+\sin x} = -\infty$ (X)

$\left(\frac{0}{2}\right) = 0$

Indeterminate Products

Def If $f(x) \to 0$, $g(x) \to \infty$ (or $-\infty$) as $x \to a$

$$\lim_{x\to a}[f(x)\cdot g(x)] \; (0\cdot\infty) = \lim \frac{f(x)}{\frac{1}{g(x)}} \left(\frac{0}{0}\right) = \lim \frac{g(x)}{\frac{1}{f(x)}} \left(\frac{\infty}{\infty}\right)$$

indeterminate form of type $0\cdot\infty$

⟨Ex⟩ $\lim_{x\to 0^+} x\ln x \; (0\cdot\infty) = \lim_{x\to 0^+} \frac{(\ln x)'}{\left(\frac{1}{x}\right)'} \left(\frac{\infty}{\infty}\right)$

$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+}(-x) = 0$ Graph ※

⟨Ex⟩ $\lim_{x\to -\infty} x e^x \; (\infty\cdot 0) = \lim_{x\to -\infty} \frac{(e^x)'}{\left(\frac{1}{x}\right)'} \left(\frac{0}{0}\right)$

$= \lim_{x\to -\infty} \frac{e^x}{-\frac{1}{x^2}} = \lim_{x\to -\infty} -x^2 e^x$

or $= \lim_{x\to -\infty} \frac{(x)'}{(e^{-x})'} \left(\frac{\infty}{\infty}\right) = \lim_{x\to -\infty} \frac{1}{-e^{-x}} = 0$

Graph $f(x) = xe^x$

Def $f(x) \to \infty$, $g(x) \to \infty$ as $x \to a$

$\lim_{x\to a}[f(x) - g(x)]$ = indeterminate form of type $\infty - \infty$

⟨Ex⟩ $\lim_{n\to\infty}\left(\sqrt{n+1} - \sqrt{n}\right)$

$= \lim_{n\to\infty}\left(\sqrt{n+1} - \sqrt{n}\right)\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$

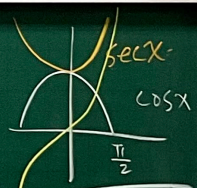
$= \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} \left(\frac{1}{\infty}\right) = 0$

陳怡杉

⟨Ex⟩ $\lim_{x\to \frac{\pi}{2}^-}(\sec x - \tan x)\cdot\frac{\sec x + \tan x}{\sec x + \tan x}$

$= \lim_{x\to\frac{\pi}{2}^-} \frac{\sec^2 x - \tan^2 x}{\sec x + \tan x} = \lim_{x\to\frac{\pi}{2}^-} \frac{1}{\underset{\infty}{\sec x} + \underset{\infty}{\tan x}} = 0$

$\frac{1}{\infty}$



Def $\lim_{x\to a}[f(x)]^{g(x)}$ is an indeterminate form of type

1° 0^0 if $f(x)\to 0$, $g(x)\to 0$

2° ∞^0 if $f(x)\to\infty$, $g(x)\to 0$

3° 1^∞ if $f(x)\to 1$, $g(x)\to\pm\infty$

鄒汶彣

Note $[f(x)]^{g(x)} = e^{g(x)\ln f(x)}$ $\quad \infty\cdot 0$, 1^∞

⟨Ex⟩ $\lim_{x\to 0^+} x^x = ?$ Graph