Deadline : 2023/04/12, 17:00.

1. Find an equation of the osculating circle of the curve $y = x^4 - x^2$ at the origin.

2. A disk of radius 1 is rotating in the counterclockwise direction at a constant angular speed ω. A particle starts at the center of the disk and moves toward the edge along a fixed radius so that its position at time t, $t \geq 0$, is given by $\mathbf{r}(t) = t\mathbf{R}(t)$, where

$$\mathbf{R}(t) = \langle \cos \omega t, \sin \omega t \rangle.$$

   (a) Show that the velocity $\mathbf{v}$ of the particle is

$$\mathbf{v} = \langle \cos \omega t, \sin \omega t \rangle + t\mathbf{v}_d$$

   where $\mathbf{v}_d = \mathbf{R}'(t)$ is the velocity of a point on the edge of the disk.

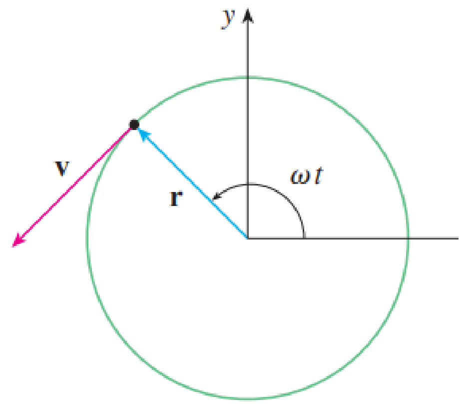
   (b) Show that the acceleration $\mathbf{a}$ of the particle is

$$\mathbf{a} = 2\mathbf{v}_d + t\mathbf{a}_d$$

   where $\mathbf{a}_d = \mathbf{R}''(t)$ is the acceleration of a point on the edge of the disk. The extra term $2\mathbf{v}_d$ is called the *Coriolis acceleration*; it is the result of the interaction of the rotation of the disk and the motion of the particle. One can obtain a physical demonstration of this acceleration by walking toward the edge of a moving merry-go-round.

   (c) Determine the Coriolis acceleration of a particle that moves on a rotating disk according to the equation

$$\mathbf{r}(t) = \langle e^{-t} \cos \omega t, e^{-t} \sin \omega t \rangle.$$



3.

A particle P moves with constant angular speed ω around a circle whose center is at the origin and whose radius is R. The particle is said to be in *uniform circular motion*. Assume that the motion is counterclockwise and that the particle is at the point $(R, 0)$ when $t = 0$. The position vector at time $t \geq 0$ is $\mathbf{r}(t) = \langle R\cos\omega t, R\sin\omega t\rangle$.

(a) Find the velocity vector $\mathbf{v}$ and show that $\mathbf{v} \cdot \mathbf{r} = 0$. Conclude that $\mathbf{v}$ is tangent to the circle and points in the direction of the motion.

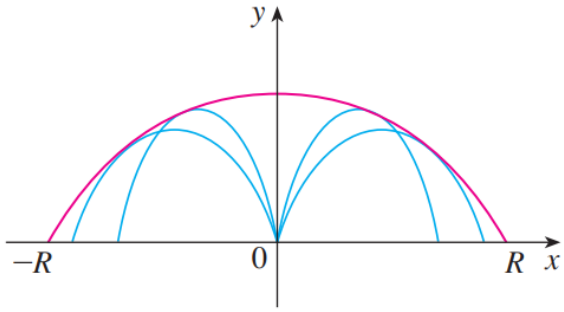
(b) Show that the speed $|\mathbf{v}|$ of the particle is the constant ωR. The *period* T of the particle is the time required for one complete revolution. Conclude that

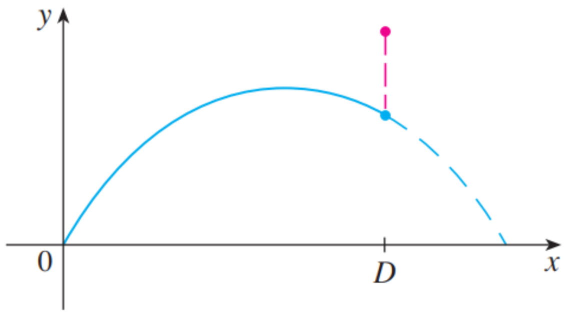
$$T = \frac{2\pi R}{|\mathbf{v}|} = \frac{2\pi}{\omega}.$$

(c) Find the acceleration vector $\mathbf{a}$. Show that it is proportional to $\mathbf{r}$ and that it points toward the origin. An acceleration with this property is called a *centripetal acceleration*. Show that the magnitude of the acceleration vector is $|\mathbf{a}| = R\omega^2$.

(d) Suppose that the particle has mass m. Show that the magnitude of the force $\mathbf{F}$ that is required to produce this motion, called a *centripetal force*, is

$$|\mathbf{F}| = \frac{m|\mathbf{v}|^2}{R}.$$





4.

A projectile is fired from the origin with angle of elevation α and initial speed v_0. Assuming that air resistance is negligible and that the only force acting on the projectile is gravity, g, we showed in Example 13.4.5 that the position vector of the projectile is

$$\mathbf{r}(t) = \langle (v_0 \cos \alpha)t, (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \rangle.$$

We also showed that the maximum horizontal distance of the projectile is achieved when $\alpha = 45°$ and in this case the range is $R = v_0^2/g$.

(a) At what angle should the projectile be fired to achieve maximum height and what is the maximum height?

(b) Fix the initial speed v_0 and consider the parabola $x^2 + 2Ry - R^2 = 0$, whose graph is shown in the figure. Show that the projectile can hit any target inside or on the boundary of the region bounded by the parabola and the x-axis, and that it can't hit any target outside this region.

(c) Suppose that the gun is elevated to an angle of inclination α in order to aim at a target that is suspended at a height h directly over a point D units downrange. The target is released at the instant the gun is fired. Show that the projectile always hits the target, regardless of the value v_0, provided the projectile does not hit the ground "before" D.

5. Show that the curve with vector equation

$$\mathbf{r}(t) = \langle a_1 t^2 + b_1 t + c_1, a_2 t^2 + b_2 t + c_2, a_3 t^2 + b_3 t + c_3 \rangle$$

lies in a plane and find an equation of the plane.

6. Determine whether the following limit exists, also find the limit if it exists.

   (a) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2 + xy^2}{x^4 + y^2}$

   (b) $\displaystyle \lim_{(x,y)\to(1,1)} \frac{y - x}{1 - y + \ln x}$

   (c) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$

   (d) $\displaystyle \lim_{(x,y)\to(0,0)} xy \sin \frac{1}{x^2 + y^2}$

   (e) $\displaystyle \lim_{(x,y)\to(6,3)} xy \cos(x - 2y)$

7. Determine the set of the points at which the function continuous.

   (a) $\displaystyle f(x, y) = \frac{\sin(xy)}{e^x - y^2}$

   (b) $f(x, y) = \begin{cases} \dfrac{x^2 y^2}{2x^2 + y^2}, & (x, y) \neq (0, 0), \\ 1, & (x, y) = (0, 0). \end{cases}$

8. Prove that if $\displaystyle \lim_{(x,y)\to(a,b)} f(x, y)$ exists, then the limit is unique.

9. Prove that $f(x, y) = \max(x, y)$ is continuous on $\mathbb{R}^2$

10. Define $f(x, y) = \|(x, y)\|$ on $\mathbb{R}^2$ and let $g(x) : \mathbb{R} \to \mathbb{R}$ be a bounded function.

    (a) Prove that f is continuous.

    (b) Prove that $g \circ f$ is bounded on $\mathbb{R}^2$.

    (c) Suppose that g is a positive and decreasing function. Prove that $\displaystyle \lim_{\|(x,y)\|\to\infty} g \circ f(x, y)$ exists.

    (d) Suppose that $h : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a function of two variables and $M = \displaystyle \sup_{(x,y)\in D} h(x, y)$. Prove that that exist sequences $\{x_n\}$ and $\{y_n\}$ such that $(x_n, y_n) \in D$ for every $n \in \mathbb{N}$ and $\displaystyle \lim_{n\to\infty} h(x_n, y_n) = M$.

(e) Let $K = \sup_{x \in \mathbb{R}} g(x)$. Prove or disprove whether there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n \to \infty} g \circ f(x_n, y_n) = K$.