NCKU 112.2
Geometry 1

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Chapter 1

Curve

1.1 Frenet Trihedron

In this section, we are given a smooth curve $\alpha(s) \in \mathbb{R}^3$ parametrized by arc-length and a smooth curve $\beta(t) \in \mathbb{R}^3$ with unknown parametrization. We seek to

(a) define Frenet Trihedron for α.

(b) prove Frenet-Serret Formula.

(c) give identity of torsion of α.

(d) give identity of curvature of β.

in particular, at the end of this section, using (d), we prove that the curvature of a plane curve $\beta(t) = (x, y)$ with unknown parametrization is exactly

$$\kappa(t) = \frac{|x'y'' - x''y'|}{\big((x')^2 + (y')^2\big)^{\frac{3}{2}}}$$

(a): Define Frenet Trihedron for α

We define the **tangent vector** T of α at each s by

$$T(s) \triangleq \alpha'(s)$$

and we define its **normal vector** N by

$$N(s) \triangleq \frac{T'(s)}{|T'(s)|}$$

Note that N is well defined if and only if $\alpha''(s) \neq 0$. We define **binormal vector** B by

$$B(s) \triangleq T(s) \times N(s)$$

Note that from $B' \perp B$ and $B' = \big(T \times N'\big) \perp T$, we have $B' /\!/ N$. This then justify our later definition of **torsion**.

We define **curvature** $\kappa(s)$ and **torsion** $\tau(s)$ by

$$\kappa(s) \triangleq |T'(s)| \text{ and } \tau(s) \triangleq \frac{B'(s)}{N(s)}$$

(b): Frenet-Serret Formula

Theorem 1.1.1. (Frenet-Serret Formula) Given a smooth curve $\alpha(s)$ parametrized by arc-length, if $T'(s) \neq 0$, we have the following

$$\begin{cases} T' = \kappa N \\ N' = -\kappa T - \tau B \\ B' = \tau N \end{cases}$$

Proof. The first and the third equations follows from that of definition. Now, see

$$\begin{aligned} N' &= (B \times T)' \\ &= B' \times T + B \times T' \\ &= \tau N \times T + B \times \kappa N \\ &= \tau(-B) + \kappa(-T) = -\kappa T - \tau B \end{aligned}$$

■

Note that in $\mathbb{R}^2$, the Frenet-Serret Formula still holds, in the sense

$$\begin{cases} T' = \kappa N \\ N' = -\kappa T \end{cases}$$

This can be proved by setting the ambient space to be $\mathbb{R}^3$.

(c): Identity of Torsion of $\alpha(s)$

Theorem 1.1.2. (Identity of Torsion) Given a smooth curve $\alpha(s) \in \mathbb{R}^3$ parametrized by arc-length, if $\kappa(s) \neq 0$, we have

$$\tau(s) = -\frac{\alpha'(s) \times \alpha''(s) \cdot \alpha'''(s)}{\big(\alpha''(s)\big)^2}$$

Proof. By definition

$$\alpha'(s) = T(s)$$

Compute

$$\alpha''(s) = T'(s) = -\kappa N(s)$$

Compute

$$\alpha''' = (-\kappa N)' = -\kappa' N + \kappa^2 T + \kappa\tau B$$

Compute

$$\alpha' \times \alpha'' = T \times (-\kappa N) = -\kappa B$$

Compute

$$\begin{aligned}\alpha' \times \alpha'' \cdot \alpha''' &= (-\kappa B) \cdot (-\kappa N + \kappa^2 T + \kappa\tau B) \\ &= -\kappa^2 \tau\end{aligned}$$

The result then follows. ■

(d): Identity of Curvature of $\beta(t)$

Theorem 1.1.3. (Identity of Curvature) Given a smooth curve $\beta(t) \in \mathbb{R}^3$ with unknown parametrization, we have

$$\kappa(t) = \frac{|\beta'(t) \times \beta''(t)|}{|\beta'(t)|^3}$$

Proof. Define s (arc-length) by

$$s(t) = \int_{t_0}^{t} |\beta'(t')| \, dt'$$

We have $\frac{ds}{dt} = |\beta'(t)|$. This then give us

$$\kappa(t) = \left|\frac{dT}{ds}\right| = \left|\frac{dT}{dt}\right| \cdot \left|\frac{dt}{ds}\right| = \left|\frac{dT}{dt}\right| \cdot \frac{1}{|\beta'(t)|}$$

We then can reduce the problem into

$$\text{proving } \left|\frac{dT}{dt}\right| = \frac{|\beta'(t) \times \beta''(t)|}{|\beta'(t)|^2}$$

Note that we have

(a) $\frac{ds}{dt} = |\beta'(t)|$

(b) $T(t) = \frac{\beta'(t)}{|\beta'(t)|}$

This then let us compute

$$T'(t) = \frac{|\beta'(t)|\,\beta''(t) - \beta'(t)\frac{d^2s}{(dt)^2}}{|\beta'(t)|^2}$$

and give us

$$\beta''(t) = \frac{d^2s}{dt^2}$$

Then we can deduce

$$\begin{aligned}\beta'(t) \times \beta''(t) &= \Big(\frac{ds}{dt}T(t)\Big) \times \Big(\frac{d^2s}{dt^2}T(t) + \frac{ds}{dt}T'(t)\Big)\\ &= (\frac{ds}{dt})^2 \cdot \Big(T(t) \times T'(t)\Big)\\ &= |\beta'(t)|^2 \cdot (\pm\,|T'(t)|)\end{aligned}$$

This then give us

$$|T'(t)| = \frac{|\beta'(t) \times \beta''(t)|}{|\beta'(t)|^2} \text{ (done)}$$

∎

Corollary 1.1.4. (Curvature of Plane Curves with unknown parametrization) Given a smooth curve $\beta(t) = (x, y) \in \mathbb{R}^3$ with unknown parametrization, we have

$$\kappa(t) = \frac{|x'y'' - x''y'|}{\big((x')^2 + (y')^2\big)^{\frac{3}{2}}}$$

1.2 Fundamental Theorem of Local Curves

Prerequisite facts:

(a) Suppose $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $f : I \to \mathbb{R}^n$ has a limit at $t_0 \in I$. We have

$$\lim_{t \to t_0} T(f(t)) = T\big(\lim_{t \to t_0} f(t)\big)$$

Theorem 1.2.1. (Rigid motion on Local space curves) Let

(a) I be a bounded open interval

(b) $\gamma : I \to \mathbb{R}^3$ be a smooth curve such that $\kappa_\gamma(t) \neq 0$ for all $t \in I$

(c) $\rho \in L(\mathbb{R}^3, \mathbb{R}^3)$ be an orthogonal linear transformation with positive determinant

(d) $c \in \mathbb{R}^3$ be a vector in $\mathbb{R}^3$

(e) $\alpha : I \to \mathbb{R}^3$ be defined by $\alpha(t) \triangleq \rho(\gamma(t))$

(f) $\beta : I \to \mathbb{R}^3$ be defined by $\beta(t) \triangleq \alpha(t) + c$

We have

$$\begin{cases} \kappa_\gamma(t) = \kappa_\alpha(t) = \kappa_\beta(t) \\ \tau_\gamma(t) = \tau_\alpha(t) = \tau_\beta(t) \end{cases} \qquad (t \in I)$$

Proof. We first show

$$(\rho v) \times (\rho w) = \rho(v \times w) \qquad (v, w \in \mathbb{R}^3)$$

Fix $v, w \in \mathbb{R}^3$. We reduce the problem into proving

$$(\rho v) \times (\rho w) \cdot z = \rho(v \times w) \cdot z \qquad (z \in \mathbb{R}^3)$$

Observe

$$\begin{aligned} (\rho v) \times (\rho w) \cdot z &= \begin{vmatrix} \rho v & \rho w & \rho(\rho^{-1}(z)) \end{vmatrix} \\ &= \begin{vmatrix} v & w & \rho^{-1}(z) \end{vmatrix} \\ &= v \times w \cdot \rho^{-1}(z) \\ &= \rho(v \times w) \cdot z \text{ (done)} \end{aligned}$$

We first prove

$$\kappa_\alpha(t) = \kappa_\gamma(t) \qquad (t \in I)$$

Note that γ'' exists, so we can compute

$$\begin{aligned}\kappa_\alpha &= \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} \\ &= \frac{|(\rho\gamma)' \times (\rho\gamma)''|}{|(\rho\gamma)'|^3} \\ &= \frac{|\rho\gamma' \times \rho\gamma''|}{|\rho\gamma'|^3} \\ &= \frac{|\rho(\gamma' \times \gamma'')|}{|\rho\gamma'|^3} \\ &= \frac{|\gamma' \times \gamma''|}{|\gamma'|^3} = \kappa_\gamma \text{ (done)}\end{aligned}$$

We now prove

$$\tau_\alpha(t) = \tau_\gamma(t) \qquad (t \in I)$$

Compute

$$\begin{aligned}\tau_\alpha &= -\frac{\alpha' \times \alpha'' \cdot \alpha'''}{|\alpha' \times \alpha''|} \\ &= -\frac{\rho\gamma' \times \rho\gamma'' \cdot \rho\gamma'''}{|\rho\gamma' \times \rho\gamma''|} \\ &= -\frac{\gamma' \times \gamma'' \cdot \gamma'''}{|\gamma' \times \gamma''|} = \tau_\gamma \text{ (done)}\end{aligned}$$

■

Theorem 1.2.2. (Fundamental Theorem of Local Curves) Let

(a) I be a bounded open interval

(b) $\kappa : I \to \mathbb{R}^+$ be a smooth function

(c) $\tau : I \to \mathbb{R}$ be a smooth function

And, let E be the set of all space curves γ such that

(a) γ has domain I

(b) $|\gamma'(s)| = 1$

(c) $\kappa_\gamma(s) = \kappa(s)$

(d) $\tau_\gamma(s) = \tau(s)$

The following statement hold true.

(a) E is non-empty. **(existence part)**

(b) For each two $\gamma, \alpha \in E$, there exists an orthogonal linear transformation $\rho \in L(\mathbb{R}^3, \mathbb{R}^3)$ with positive determinant and a vector $c \in \mathbb{R}^3$ such that $\gamma(s) = \rho \circ \alpha(s) + c$ for all $s \in I$. **(uniqueness part)**

Proof. We first prove

E is non-empty

Fix $s_0 \in I$, and fix

$$\begin{bmatrix} T(s_0) \\ N(s_0) \\ B(s_0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.1}$$

Consider the system of differential equation

$$\begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} T' \\ N' \\ B' \end{bmatrix}$$

This is a first order, linear system of differential equation (3, if you wish), which has a unique solution $\{T, N, B : I \to \mathbb{R}^3\}$ with given initial condition (Equation 1.1).

We claim

$$\gamma(s) \triangleq \gamma(s_0) + \int_{s_0}^{s} T(s')ds' \in E$$

Note that the solution $\{T, N, B : I \to \mathbb{R}^3\}$ must satisfy

$$\begin{aligned}
\frac{d}{ds}\langle T, N\rangle &= \kappa\langle N, N\rangle - \kappa\langle T, T\rangle - \tau\langle T, B\rangle \\
\frac{d}{ds}\langle T, B\rangle &= \kappa\langle N, B\rangle + \tau\langle T, N\rangle \\
\frac{d}{ds}\langle N, B\rangle &= -\kappa\langle T, B\rangle - \tau\langle B, B\rangle + \tau\langle N, N\rangle \\
\frac{d}{ds}\langle T, T\rangle &= 2\kappa\langle T, N\rangle \\
\frac{d}{ds}\langle N, N\rangle &= -2\kappa\langle T, N\rangle - 2\tau\langle B, N\rangle \\
\frac{d}{ds}\langle B, B\rangle &= 2\tau\langle N, B\rangle
\end{aligned}$$

which is another linear first order system of differential equation, which has unique solution when initial condition are given.

It is easy to check that

$$\langle T, T\rangle \triangleq \langle N, N\rangle \triangleq \langle B, B\rangle \triangleq 1 \text{ and } \langle T, N\rangle \triangleq \langle T, B\rangle \triangleq \langle N, B\rangle = 0 \tag{1.2}$$

is a solution.

Now, note that our unique solution $\{T, N, B\}$ given before has initial condition coincide with that of 1.2. Then it follows from the uniqueness of solution of linear first order system of differential equations that $\{T(s), N(s), B(s)\}$ is an orthonormal basis for all $s \in I$.

Since $\gamma'(s) = T(s)$, it is now clear that $\kappa_\gamma = \kappa$ and $\tau_\gamma = \tau$. (done)

Fix $\gamma, \alpha \in E$ and fix $s_0 \in I$. There clearly exists a rigid motion M such that if we denote

$$\overline{\alpha}(s) \triangleq M \circ \alpha(s)$$

Then

$$\gamma(s_0) = \overline{\alpha}(s_0) \text{ and } \begin{cases} T_\gamma(s_0) = T_{\overline{\alpha}}(s_0) \\ N_\gamma(s_0) = N_{\overline{\alpha}}(s_0) \\ B_\gamma(s_0) = B_{\overline{\alpha}}(s_0) \end{cases}$$

We only wish to prove

$$T_\gamma(s) = T_{\overline{\alpha}}(s) \qquad (s \in I)$$

Denote

$$T \triangleq T_\gamma \text{ and } \overline{T} \triangleq T_{\overline{\alpha}} \text{ and also similarly for } N, B$$

Compute

$$\begin{aligned}
&\frac{d}{ds}\Big(\big|T - \overline{T}\big|^2 + \big|N - \overline{N}\big|^2 + \big|B - \overline{B}\big|^2 \Big) \\
=&2\Big(\langle T' - \overline{T}', T - \overline{T}\rangle + \langle N' - \overline{N}', N - \overline{N}\rangle + \langle B' - \overline{B}', B - \overline{B}\rangle \Big) \\
=&2\Big(\kappa\langle N - \overline{N}, T - \overline{T}\rangle - \kappa\langle T - \overline{T}, N - \overline{N}\rangle - \tau\langle B - \overline{B}, N - \overline{N}\rangle + \tau\langle N - \overline{N}, B - \overline{B}\rangle \Big) \\
=&0
\end{aligned}$$

This then implies

$$\left|T-\overline{T}\right|^2+\left|N-\overline{N}\right|^2+\left|B-\overline{B}\right|^2 \text{ is a fixed constant on } I$$

Note that

$$T(s_0)=\overline{T}(s_0) \text{ and } N(s_0)=\overline{N}(s_0) \text{ and } B(s_0)=\overline{B}(s_0)$$

Then we know that fixed constant is exactly 0. This then let us deduce

$$T=\overline{T} \text{ on } I \quad \text{(done)}$$

■

1.3 Isoperimetric Inequality and Four-Vertex Theorem

In this section, we are given a smooth plane curve $\alpha : [a,b] \to \mathbb{R}^2$ parametrized by arc-length. We say

(a) α is a **closed curve** if $\alpha(a) = \alpha(b)$.

(b) α is a **simple closed curve** if α is closed and $\alpha(t_1) \neq \alpha(t_2)$ for all $t_1, t_2 \in [a,b)$.

(c) α is **positively oriented** if $\alpha' \times \alpha''$ is always positive.

(d) α is **convex** the trace $\alpha([a,b])$ always entirely lies on the same side of the closed half-plane determined by $T(s)$ for all s

(e) α has **vertex** $\alpha(s_0)$ if $\kappa'(s_0) = 0$

The interior D of a piece-wise smooth, simple closed plane curve $\alpha : [a,b] \to \mathbb{R}^2$ can be assigned a number to represent its area

$$A(D) = \iint_D 1 dxdy \tag{1.3}$$

that match with our geometric intuition, in the sense that if one compute the area of a rectangle or that of a circle, using Formula 1.3, then one obtain number same as the number obtained using elementary geometric way (height $\times$ width, etc).

Note that by Green's Theorem, we know $A(D)$ equals to

(a) $\int_a^b xy' dt$

(b) $-\int_a^b yx' dt$

(c) $\frac{1}{2}\int_a^b xy' - yx' dt = \frac{1}{2}\int_a^b (x,y) \times (x',y') dt$

Theorem 1.3.1. (Isoperimetric Inequality) Given a piece-wise C^1 simple closed plane curve $\alpha : [0,l] \to \mathbb{R}^2$, where D is the interior of α and $A \triangleq A(D)$, we have

(a) $4\pi A \leq l^2$

(b) $4\pi A = l^2 \iff \alpha$ is a parametriation of S^1

Proof. We first show

$$4\pi A \leq l^2$$

Express $\alpha = (x, y)$. Define

$$r = \frac{1}{2}\Big(\sup_{[0,l]} x - \inf_{[0,l]} x \Big)$$

WOLG, suppose $\sup_{[0,l]} x = r$. Now, positively parametrize S^1 by $(\overline{x}, \overline{y})$ where

$$\overline{x} \triangleq x$$

Observe

$$\begin{aligned}
A + \pi r^2 &= \oint_{\alpha} x dy - \oint_{S^1} y dx \\
&= \int_0^l \big(xy' - x'\overline{y} \big) dt \\
&= \int_0^l (x, -\overline{y}) \cdot (y', x') dt \\
&\leq \int_0^l |(x, -\overline{y})| \cdot |(y', x')| \, dt \qquad \text{(Cauchy-Schwarz)} \\
&= \int_0^l \sqrt{x^2 + (-\overline{y})^2} \sqrt{(y')^2 + (x')^2} dt \\
&= \int_0^l \sqrt{x^2 + \overline{y}^2} dt \\
&= \int_0^l r dt = lr
\end{aligned}$$

Using AM-GM inequality, we now have

$$\sqrt{A \pi r^2} \leq \frac{A + \pi r^2}{2} \leq \frac{lr}{2}$$

This then implies

$$4\pi A \leq l^2 \text{ (done)}$$

It is easy to see that the equality hold true when α is a parametrization of S^1. We prove

$$4\pi A = l^2 \implies \alpha \text{ is a parametrization of } S^1 \text{ of radius } \frac{l}{2\pi}$$

We first prove

$$4\pi A = l^2 \implies x = \frac{l}{2\pi} y'$$

Note that $4\pi A = l^2$ implies the Cauchy-Schwarz inequality

$$(x, -\overline{y}) \cdot (y', x') \leq |(x, -\overline{y})| \cdot |(y', x')|$$

must hold equal. This then give us

$$(x, -\overline{y}) \,/\!\!/\, (y', x')$$

Also, note that $4\pi A = l^2$ implies the AM-GM inequality

$$\sqrt{A\pi r^2} \leq \frac{A + \pi r^2}{2}$$

must also hold equal. This then give us

$$A = \pi r^2 \text{ and } l = 2\pi r$$

Now, because

(a) (x, y) is parametrization by arc-length

(b) $|(x, -\overline{y})| = |(\overline{x}, \overline{y})| = r$

(c) $r = \frac{l}{2\pi}$

We have

$$x = \frac{l}{2\pi} y' \text{ (done)} \tag{1.4}$$

Similar argument now, WOLG, applies to

$$4\pi A = l^2 \implies y = \frac{l}{2\pi} x'$$

We now conclude

$$4\pi A = l^2 \implies |(x, y)| = \frac{l}{2\pi} |(y', x')| = \frac{l}{2\pi} \text{ (done)}$$

■

For next Theorem, we expand the definition of closed curve. We say

(a) α is a **closed curve** if $\alpha^k(a) = \alpha^k(b)$ for all $k \in \mathbb{Z}_0^+$

Lemma 1.3.2. If $\alpha : [0, l] \to \mathbb{R}^2$ is a smooth, convex, simple closed plane curve, then for all $A, B, C \in \mathbb{R}$, we have

$$\int_0^l \Big(Ax + By + C \Big) d\kappa = 0$$

Proof. Because α is closed, it is clear that

$$\int_0^l C d\kappa = 0$$

Let $\theta : [0, l] \to [0, 2\pi]$ satisfy

$$(x', y') = (\cos\theta, \sin\theta)$$

It is easy to check

(a) $\kappa = \theta'$

(b) $(x'', y'') = \kappa(-y', x')$

We now see

$$\begin{aligned}\int_0^l A x d\kappa &= A \int_0^l x\kappa' ds \\ &= -A \int_0^l x'\kappa ds \\ &= A \int_0^l y'' ds = 0\end{aligned}$$

and

$$\begin{aligned}\int_0^l B y d\kappa &= B \int_0^l y\kappa' ds \\ &= - \int_0^l y'\kappa ds \\ &= -B \int_0^l x'' ds = 0\end{aligned}$$

∎

Theorem 1.3.3. (Four-Vertex Theorem for Convex Curve) If $\alpha : [0, l] \to \mathbb{R}^2$ is a smooth, convex, simple closed plane curve, then

$$\alpha \text{ has at least four distinct vertices}$$

Proof. Because κ is continuous on $[0, l]$, by EVT, we know there exists p, q such that

$$\kappa(p) = \max_{[0,l]} \kappa \text{ and } \kappa(q) = \min_{[0,l]} \kappa$$

Because $\kappa(p), \kappa(q)$ are local (global, in fact) extremum, we know

$$\kappa'(p) = \kappa'(q) = 0$$

Note that if $p = q$, then the proof become trivial. We have thus obtained two distinct vertices, namely, p and q.

WOLG, suppose $p < q$. Let L be the unique line containing $\alpha(p)$ and $\alpha(q)$. We claim

$$\alpha[(p,q)] \text{ and } \alpha\Big[[0,l] \setminus [p,q]\Big] \text{ are on different side of } L$$

(done)

Note that

(a) $p < q$

(b) $\kappa(p)$ is a maximum

(c) $\kappa(q)$ is a minimum

Assume $\kappa' \leq 0$ on (p,q) and $\kappa' \geq 0$ on $[p,q]^c$. WOLG, suppose $\alpha[(p,q)]$ is on the left side of L. Let $A, B, C \in \mathbb{R}$ satisfy

(a) $L = \{(x,y) : Ax + By + C = 0\}$

(b) $A \in \mathbb{R}^+$

Now, because $\alpha[(p,q)]$ is completely on the left side of L, we see

$$\int_p^q \big(Ax + By + C\big) d\kappa > 0 \qquad (\because \kappa', Ax + By + C < 0)$$

and because $\alpha[[p,q]^c]$ is completely on the right side of L, we see

$$\int_0^p \big(Ax + By + C\big) d\kappa + \int_q^l \big(Ax + By + C\big) d\kappa > 0 \quad (\because \kappa', Ax + By + C > 0)$$

This now let us deduce

$$\int_0^l \big(Ax + By + C\big) d\kappa > 0 \text{ CaC}$$

WOLG, suppose $\kappa'(s) > 0$ for some $s \in (p,q)$. Because

(a) $\kappa(p)$ is a maximum

(b) $\kappa(q)$ is a minimum

we know there exists ϵ such that

$$\kappa(p+\epsilon) < 0 \text{ and } \kappa(q-\epsilon) < 0$$

Then by IVT, we see

$$\kappa'(s_1) = \kappa'(s_2) = 0 \text{ for some distinct } s_1, s_2 \in (p,q)$$

$\blacksquare$

Chapter 2

HW

2.1 HW1

Question 1: 1-2: 2

Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is the point of the trace of α closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

Proof. Define $g : I \to \mathbb{R}$ by

$$g(t) \triangleq |\alpha(t)|^2 = (\alpha \cdot \alpha)(t)$$

Notice that

$$g'(t) = (2\alpha' \cdot \alpha)(t) \text{ if exists}$$

From premise, we know g attains minimum at t_0. This tell us

$$0 = g'(t_0) = (2\alpha' \cdot \alpha)(t_0)$$

Then, we can deduce

$$\alpha'(t_0) \cdot \alpha(t_0) = 0$$

This implies $\alpha(t_0) \perp \alpha'(t_0)$. ∎

Question 2: 1-2: 5

Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve, with $\alpha''(t) \neq 0$ for all $t \in I$. Show that $|\alpha(t)|$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

Proof. We wish to prove

$$\exists \beta \in \mathbb{R}^*, \forall t \in I, |\alpha(t)| = \beta \iff \forall t \in I, (\alpha \cdot \alpha')(t) = 0$$

Define $g : I \to \mathbb{R}$ by

$$g(t) \triangleq |\alpha(t)|^2 = (\alpha \cdot \alpha)(t)$$

Notice that

$$g'(t) = (2\alpha' \cdot \alpha)(t) \tag{2.1}$$

$(\longrightarrow)$

From premise, g is a constant on I. This implies $g'(t) = 0$ for all $t \in I$. Then, from Equation 2.1, we see

$$(\alpha \cdot \alpha')(t) = 0 \text{ for all } t \in I$$

$(\longleftarrow)$

Again, from Equation 2.1, we deduce

$$\forall t \in I, (\alpha \cdot \alpha')(t) = 0 \implies \forall t \in I, g'(t) = 0$$

This implies g is a constant, which implies $|\alpha|$ is a constant, that is

$$\exists \beta \in \mathbb{R}, |\alpha(t)| = \beta$$

Lastly, we have to show

$$\beta \neq 0$$

Assume $\beta = 0$. Then, we see $\alpha(t) = 0$ for all $t \in I$. This implies $\alpha''(t) = 0$ for all $t \in I$, which CaC to the premise. (done) ∎

Question 3: 1-3:2

2. A circular disk of radius 1 in the plane xy rolls without slipping along the x axis. The figure described by a point of the circumference of the disk is called a *cycloid* (Fig. 1-7).

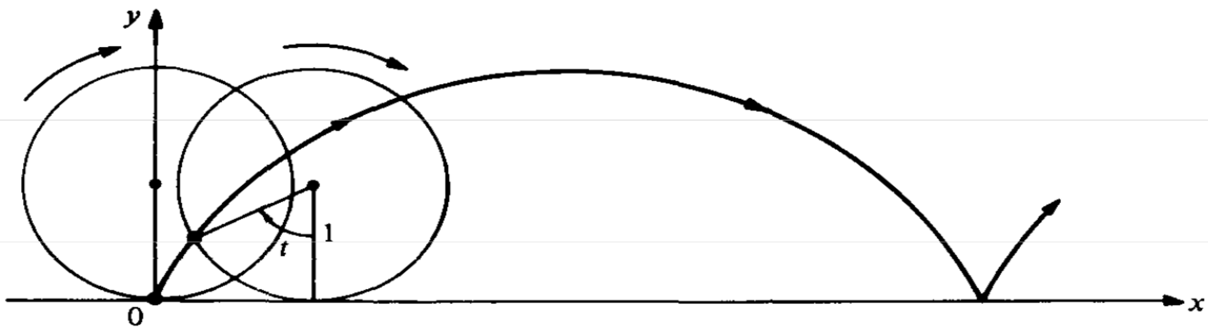


Figure 1-7. The cycloid.

***a.** Obtain a parametrized curve $\alpha: R \longrightarrow R^2$ the trace of which is the cycloid, and determine its singular points.

b. Compute the arc length of the cycloid corresponding to a complete rotation of the disk.

Proof. The solution of the question **a** is

$$\alpha(t) = (t - \sin t, 1 - \cos t)$$

Compute

$$\alpha'(t) = (1 - \cos t, \sin t)$$

and compute

$$|\alpha'(t)| = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} = \sqrt{2} \cdot \sqrt{1 - \cos t}$$

This implies the singular points are

$$\{2n\pi : n \in \mathbb{Z}\}$$

The solution of the question **b** is then

$$\begin{aligned}\int_0^{2\pi} |\alpha'(t)|\, dt &= \sqrt{2}\int_0^{2\pi} \sqrt{1-\cos t} dt \\ &= \sqrt{2}\int_0^{2\pi} \sqrt{2}\left|\sin\frac{t}{2}\right| dt \\ &= 2\int_0^{2\pi} \left|\sin\frac{t}{2}\right| dt \\ &= 4\int_0^{\pi} \sin(\frac{t}{2}) dt \\ &= -8\cos\frac{t}{2}\Big|_0^{\pi}\end{aligned}$$

■

Question 4: 1-3:4

4. Let $\alpha\colon (0, \pi) \longrightarrow R^2$ be given by

$$\alpha(t) = \left(\cos t, \cos t + \log \tan \frac{t}{2}\right),$$

where t is the angle that the y axis makes with the vector $\alpha(t)$. The trace of α is called the *tractrix* (Fig. 1-9). Show that

19:

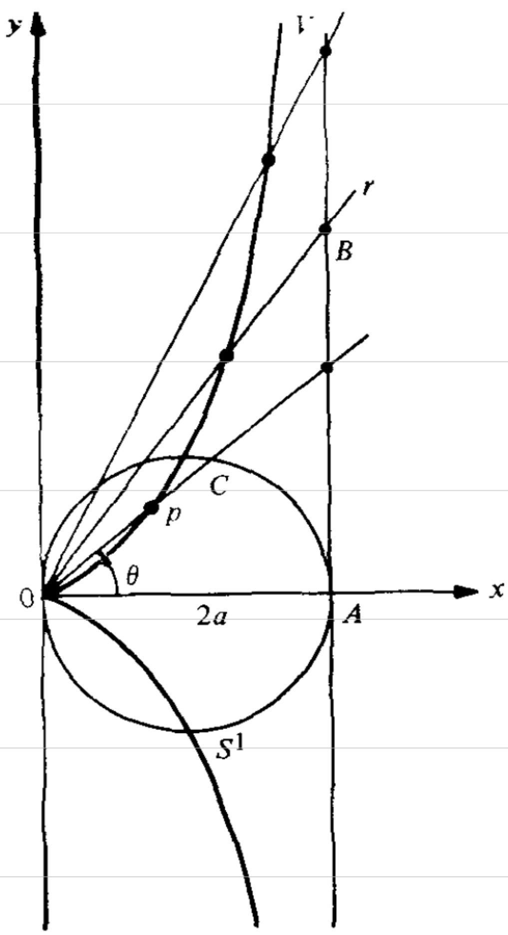


Figure 1-8. The cissoid of Diocles.

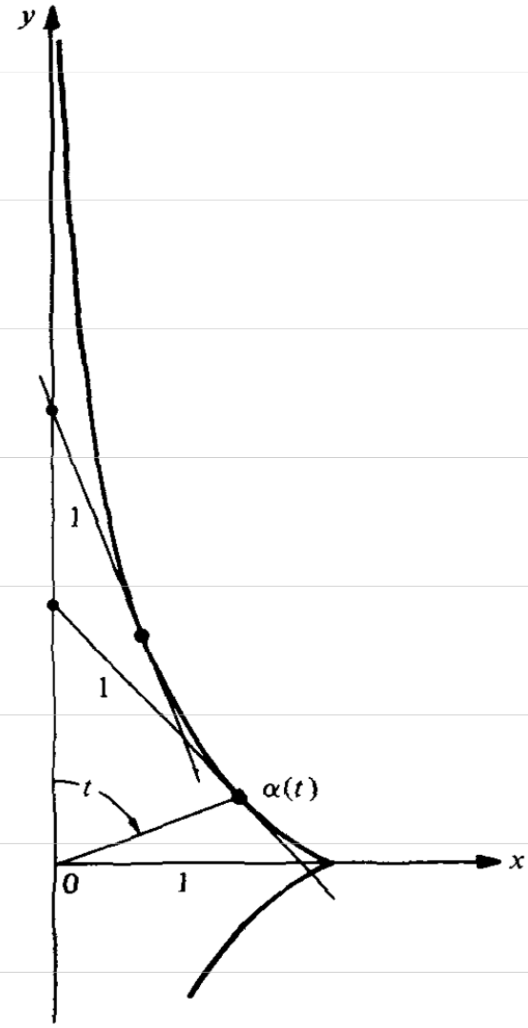


Figure 1-9. The tractrix.

a. α is a differentiable parametrized curve, regular except at $t = \pi/2$.

b. The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.

Typo correction: $\alpha(t) = (\sin t, \cos t + \ln \tan \frac{t}{2})$

Proof. **(a)**

Notice that the interval I is $(0, \pi)$. It is clear that

(a) $\sin t$ is smooth on $\mathbb{R}$

(b) $\cos t$ is smooth on $\mathbb{R}$

(c) $\ln t$ is smooth on $\mathbb{R}$ $\tan\frac{t}{2}$ is smooth on I

Then it follows that α is a differentiable curve.

Compute

$$\alpha'(t) = (\cos t, -\sin t + \frac{1}{\tan\frac{t}{2}} \cdot \sec^2\frac{t}{2} \cdot \frac{1}{2})$$

Because $\cos t = \alpha_1'(t)$ is 0 on I only when $t = \frac{\pi}{2}$, we know α is regular on I except possibly at $t = \frac{\pi}{2}$.

Compute

$$\alpha'(\frac{\pi}{2}) = (0, -1 + \frac{1}{1} \cdot 2 \cdot \frac{1}{2}) = (0, 0)$$

We now conclude α is regular on I except $\frac{\pi}{2}$.

(b)
A useful Identity give us

$$\alpha'(t) = (\cos t, -\sin t + \csc t)$$

From the following facts

(a) the first argument of the segment is from 0 to $\sin t = \alpha(t)$

(b) $\alpha_x'(t) = \cos t$

(c) $\frac{\sin t}{\cos t} = \tan t$

We conclude that the length of the segment is

$$\begin{aligned}|\tan t| \cdot |\alpha'(t)| &= |\tan t| \cdot \sqrt{\cos^2 t + \sin^2 t - 2\sin t \csc t + \csc^2 t} \\ &= |\tan t| \cdot \sqrt{1 - 2 + \csc^2 t} \\ &= |\tan t| \cdot \sqrt{\csc^2 t - 1} = |\tan t| \cdot \sqrt{\cot^2 t} = 1\end{aligned}$$

■

Question 5

7. A map $\alpha: I \longrightarrow R^3$ is *called a curve of class* C^k if each of the coordinate functions in the expression $\alpha(t) = (x(t), y(t), z(t))$ has continuous derivatives up to order k. If α is merely continuous, we say that α is of class C^0. A curve α is called *simple* if the map α is one-to-one. Thus, the curve in Example 3 of Sec. 1-2 is not simple.

Let $\alpha: I \longrightarrow R^3$ be a simple curve of class C^0. We say that α has a *weak tangent* at $t = t_0 \in I$ if the line determined by $\alpha(t_0 + h)$ and $\alpha(t_0)$ has a limit position when $h \longrightarrow 0$. We say that α has a *strong tangent* at $t = t_0$ if the line determined by $\alpha(t_0 + h)$ and $\alpha(t_0 + k)$ has a limit position when $h, k \longrightarrow 0$. Show that

a. $\alpha(t) = (t^3, t^2)$, $t \in R$, has a weak tangent but not a strong tangent at $t = 0$.

***b.** If $\alpha: I \longrightarrow R^3$ is of class C^1 and regular at $t = t_0$, then it has a strong tangent at $t = t_0$.

c. The curve given by

$$\alpha(t) = \begin{cases} (t^2, t^2), & t \geq 0, \\ (t^2, -t^2), & t \leq 0, \end{cases}$$

is of class C^1 but not of class C^2. Draw a sketch of the curve and its tangent vectors.

Proof. **(a)** Let $v = (0, 1)$. Compute

$$\frac{\alpha(t) - \alpha(0)}{|\alpha(t) - \alpha(0)|} \cdot v = \frac{t^2}{\sqrt{t^6 + t^4}} = \frac{1}{\sqrt{t^2 + 1}} \to 1 \text{ as } t \to 1$$

This implies α has a weak tangent at $t = 0$. Now, if α has a strong tangent, we must have

$$\frac{\alpha(h) - \alpha(-h)}{2h} \cdot v \to 1 \text{ or } \to -1$$

But this is clearly not the case as

$$\frac{\alpha(h) - \alpha(-h)}{2h} \cdot v = 0 \text{ for all } h > 0$$

So we have the conclusion that α has no strong tangent at 0.

(b) By MVT, for each h, k there exists a set of real numbers $\{c_x, c_y, c_z\}$ between $t+h$ and $t+k$ such that

$$\frac{\alpha(t_0+h)-\alpha(t_0+k)}{h-k} = \Big(x'(c_x), y'(c_y), z'(c_z)\Big)$$

Then because

$$h, k \to 0 \implies t_0+h, t_0+k \to t_0 \implies c_x, c_y, c_z \to t_0$$

Then from the fact α is of class C^1 (x', y', z' are all continuous), we can now deduce

$$\frac{\alpha(t_0+h)-\alpha(t_0+k)}{h-k} \to \alpha'(t_0) \text{ as } h, k \to 0 \tag{2.2}$$

Now, because $\alpha'(t_0) \neq 0$ as α is regular, we see

$$\lim_{h,k\to 0} \frac{\alpha(t_0+h)-\alpha(t_0+k)}{h-k} \cdot \alpha'(t_0) = |\alpha'(t_0)|^2$$

This then implies

$$\lim_{h,k\to 0} \frac{\alpha(t_0+h)-\alpha(t_0+k)}{|\alpha(t_0+h)-\alpha(t_0+k)|} \cdot \frac{\alpha'(t_0)}{|\alpha'(t_0)|} = 1$$

which implies the "strong tangent" must always converge to $\alpha'(t_0)$.

Notice that the last implication is backed by Equation 2.2

(c)

From

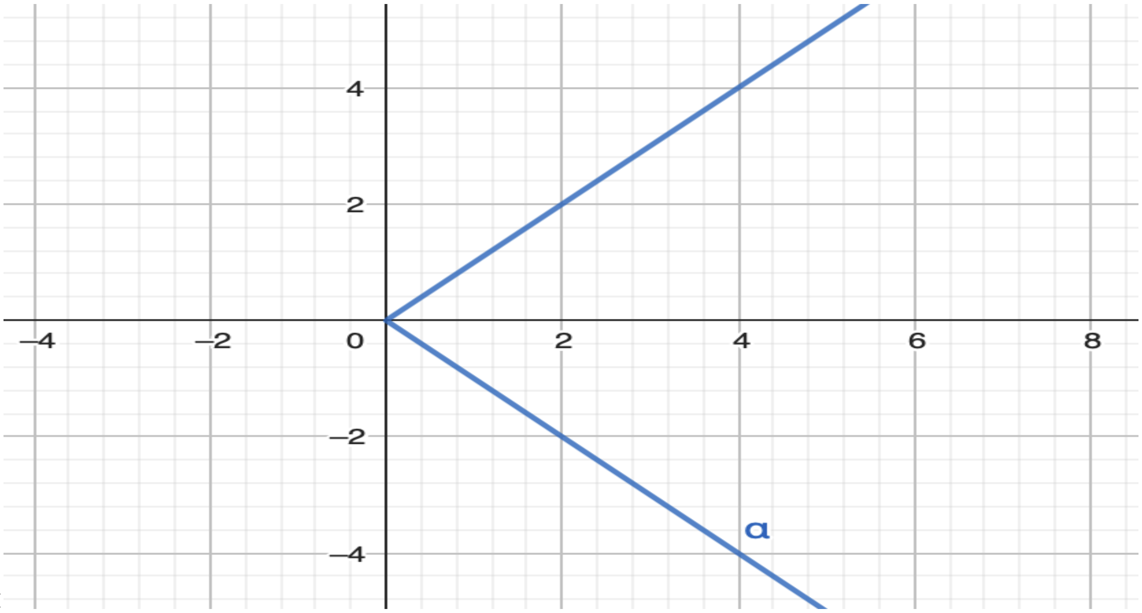
$$\alpha(t) = \left(t^2, \begin{cases} t^2 & \text{if } t \geq 0 \\ -t^2 & \text{if } t \leq 0 \end{cases}\right)$$

Compute

$$\alpha'(t) = \left(2t, \begin{cases} 2t & \text{if } t \geq 0 \\ -2t & \text{if } t \leq 0 \end{cases}\right)$$

Notice that the derivative at $t = 0$ is computed from definition instead of product rule.

Now, it is clear that x', y' are continuous. This implies $\alpha \in C^1$. Yet, we see y' is not differentiable at $t = 0$. This implies $\alpha \notin C^2$.

The sketch:  ■

Question 6

***8.** Let $\alpha: I \longrightarrow R^3$ be a differentiable curve and let $[a, b] \subset I$ be a closed interval. For every *partition*

$$a = t_0 < t_1 < \cdots < t_n = b$$

of $[a, b]$, consider the sum $\sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})| = l(\alpha, P)$, where P stands for the given partition. The norm $|P|$ of a partition P is defined as

$$|P| = \max(t_i - t_{i-1}), \; i = 1, \ldots, n.$$

Geometrically, $l(\alpha, P)$ is the length of a polygon inscribed in $\alpha([a, b])$ with vertices in $\alpha(t_i)$ (see Fig. 1-12). The point of the exercise is to show that the arc length of $\alpha([a, b])$ is, in some sense, a limit of lengths of inscribed polygons.

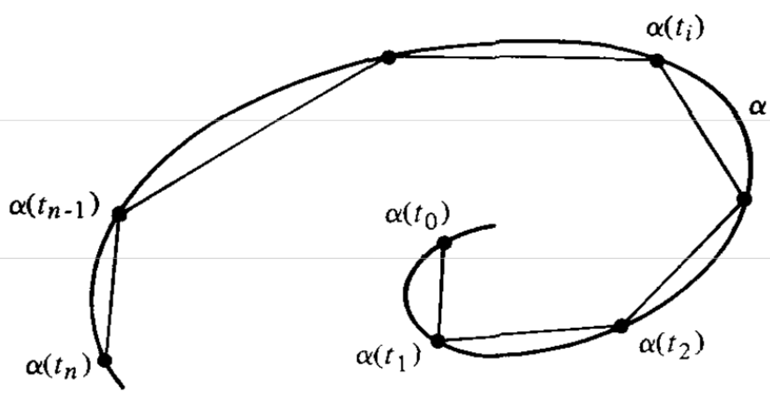


Figure 1-12

Prove that given $\epsilon > 0$ there exists $\delta > 0$ such that if $|P| < \delta$ then

$$\left| \int_a^b |\alpha'(t)|\, dt - l(\alpha, P) \right| < \epsilon.$$

Proof. We first prove

$$\int_a^b |\alpha'(t)|\, dt \geq l(\alpha, P)$$

By FTC, we have

$$\begin{aligned} |\alpha(t_i) - \alpha(t_{i-1})| &= \left| \int_{t_{i-1}}^{t_i} \alpha'(t) dt \right| \\ &\leq \int_{t_{i-1}}^{t_i} |\alpha'(t)| \, dt \end{aligned}$$

This then implies

$$l(\alpha, P) = \sum |\alpha(t_i) - \alpha(t_{i-1})| \leq \sum \int_{t_{i-1}}^{t_i} |\alpha'(t)| \, dt = \int_a^b |\alpha'(t)| \, dt \text{ (done)}$$

We have reduced the problem into

$$\text{finding } \delta \text{ such that } \forall P : |P| < \delta, \int_a^b |\alpha'(t)| \, dt - l(\alpha, P) < \epsilon$$

Because α' is uniformly continuous on $[a, b]$ ($\because$ continuous function on compact domain is uniformly continuous), we know there exists δ' such that

$$|\alpha'(s) - \alpha'(t)| < \frac{\epsilon}{2(b-a)} \text{ if } |s - t| < \delta'$$

We claim

$$\text{such } \delta' \text{ works}$$

Let $|P| < \delta$, and let $s_i \in [t_{i-1}, t_i]$. Because $|s_i - t_i| < \delta$, we have

$$|\alpha'(s_i) - \alpha'(t_i)| < \frac{\epsilon}{2(b-a)} \tag{2.3}$$

This give us

$$|\alpha'(s_i)| < |\alpha'(t_i)| + \frac{\epsilon}{2(b-a)}$$

Now, we can deduce

$$\begin{aligned}
\int_{t_{i-1}}^{t_i} |\alpha'(s)|\, ds &\leq |\alpha'(t_i)|\, \Delta t_i + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&= \int_{t_{i-1}}^{t_i} |\alpha'(t_i)|\, dt + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&= \left| \int_{t_{i-1}}^{t_i} \alpha'(t_i) dt \right| + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&= \left| \int_{t_{i-1}}^{t_i} \alpha'(t_i) - \alpha'(t) dt + \int_{t_{i-1}}^{t_i} \alpha'(t) dt \right| + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&\leq \left| \int_{t_{i-1}}^{t_i} \alpha'(t_i) - \alpha'(t) dt \right| + \left| \int_{t_{i-1}}^{t_i} \alpha'(t) dt \right| + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&\leq \frac{\epsilon}{2(b-a)}\Delta t_i + |\alpha(t_i) - \alpha(t_{i-1})| + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&= |\alpha(t_i) - \alpha(t_{i-1})| + \frac{\epsilon}{b-a}\Delta t_i
\end{aligned}$$

Notice that the last inequality follows from Equation 2.3. The long deduction above then give us

$$\begin{aligned}
\int_a^b |\alpha'(t)|\, dt &\leq \sum |\alpha(t_i) - \alpha(t_{i-1})| + \frac{\epsilon}{b-a}(b-a) \\
&= l(\alpha, P) + \epsilon
\end{aligned}$$

Then we have

$$\int_a^b |\alpha'(t)|\, dt - l(\alpha, P) \leq \epsilon \text{ (done)}$$

$\blacksquare$

Question 7

9. a. Let $\alpha: I \longrightarrow R^3$ be a curve of class C^0 (cf. Exercise 7). Use the approximation by polygons described in Exercise 8 to give a reasonable definition of arc length of α.

b. (*A Nonrectifiable Curve.*) The following example shows that, with any reasonable definition, the arc length of a C^0 curve in a closed interval may be unbounded. Let $\alpha: [0, 1] \longrightarrow R^2$ be given as $\alpha(t) = (t, t\sin(\pi/t))$ if $t \neq 0$, and $\alpha(0) = (0, 0)$. Show, geometrically, that the arc length of the portion of the curve corresponding to $1/(n+1) \leq t \leq 1/n$ is at least $2/(n+\frac{1}{2})$. Use this to show that the length of the curve in the interval $1/N \leq t \leq 1$ is greater than $2\sum_{n=1}^{N} 1/(n+1)$, and thus it tends to infinity as $N \longrightarrow \infty$.

Proof. **(a)** Suppose $I = [a, b]$. Define arc length by

$$\sup_P l(P, \alpha) \text{ where sup runs over all partition } P \text{ of } [a, b]$$

(b)

Geometrically, we know the arc length of the portion of the curve corresponding to $t \in [\frac{1}{n+1}, \frac{1}{n}]$ must be greater than

$$\left|\alpha(\frac{1}{n}) - \alpha(\frac{1}{n+\frac{1}{2}})\right| + \left|\alpha(\frac{1}{n+1}) - \alpha(\frac{1}{n+\frac{1}{2}})\right| \tag{2.4}$$

WOLG of n being odd or even, Compute

$$\begin{aligned}
\left|\alpha(\frac{1}{n}) - \alpha(\frac{1}{n+\frac{1}{2}})\right| &= \left|(\frac{1}{n}, 0) - (\frac{1}{n+\frac{1}{2}}, \frac{1}{n+\frac{1}{2}})\right| \\
&= \sqrt{(\frac{1}{n} - \frac{1}{n+\frac{1}{2}})^2 + (\frac{1}{n+\frac{1}{2}})^2} \\
&= \sqrt{\frac{1}{n^2} - \frac{4}{n(2n+1)} + \frac{8}{(2n+1)^2}} \\
&= \sqrt{\frac{(2n+1)^2 - 4n(2n+1) + 8n^2}{n^2(2n+1)^2}} \\
&= \sqrt{\frac{4n^2+1}{n^2(2n+1)^2}} \\
&= \frac{\sqrt{4n^2+1}}{n(2n+1)} \geq \frac{\sqrt{4n^2}}{n(2n+1)} = \frac{2}{2n+1}
\end{aligned}$$

and compute

$$\begin{aligned}
\left|\alpha(\frac{1}{n+\frac{1}{2}}) - \alpha(\frac{1}{n})\right| &= \left|(\frac{1}{n+\frac{1}{2}}, \frac{1}{n+\frac{1}{2}}) - (\frac{1}{n+1}, 0)\right| \\
&= \sqrt{(\frac{1}{n+1} - \frac{1}{n+\frac{1}{2}})^2 + (\frac{1}{n+\frac{1}{2}})^2} \\
&= \sqrt{\frac{1}{(n+1)^2} - \frac{4}{(n+1)(2n+1)} + \frac{8}{(2n+1)^2}} \\
&= \sqrt{\frac{(2n+1)^2 - 4(n+1)(2n+1) + 8(n+1)^2}{(n+1)^2(2n+1)^2}} \\
&= \sqrt{\frac{4n^2+8n+5}{(n+1)^2(2n+1)^2}} \\
&\geq \frac{\sqrt{4n^2+8n+4}}{(n+1)(2n+1)} = \frac{2}{2n+1}
\end{aligned}$$

From the computation and Equation 2.4, it is now clear that the arc length of the portion of the curve corresponding to $t \in [\frac{1}{n+1}, \frac{1}{n}]$ is at least $\frac{2}{n+\frac{1}{2}}$. With simple addition, this then implies the arc length of the curve in the interval $[\frac{1}{N}, 1]$ is at least

$$\sum_{n=1}^{N-1} \frac{2}{2n+1} = 2\sum_{n=1}^{N-1} \frac{1}{2n+1}$$

The number is clearly greater than

$$2\sum_{n=1}^{N-1} \frac{1}{2n+2}$$

which equals to

$$\sum_{n=1}^{N-1} \frac{1}{n+1}$$

The series diverge to $+\infty$ as N to ∞. ∎

Theorem 2.1.1. (Integrating the Dot Product) Given a curve $u : [a, b] \to \mathbb{R}^n$ and a vector $v \in \mathbb{R}^n$, suppose

(a) u is differentiable on (a, b)

(b) u is continuous on $[a, b]$

We have

$$\int_a^b u'(t) \cdot v dt = \Big(\int_a^b u'(t) dt \Big) \cdot v = \big(u(b) - u(a) \big) \cdot v$$

Proof.

$$\begin{aligned}
\int_a^b u'(t) \cdot v dt &= \int_a^b \sum_{k=1}^n u_k'(t) \cdot v_k dt \\
&= \sum_{k=1}^n \int_a^b u_k'(t) \cdot v_k dt \\
&= \sum_{k=1}^n v_k \int_a^b u_k'(t) dt \\
&= v \cdot \Big(\int_a^b u'(t) dt \Big)
\end{aligned}$$

∎

Question 8

10. (*Straight Lines as Shortest.*) Let $\alpha: I \longrightarrow R^3$ be a parametrized curve. Let $\{a, b\} \subset I$ and set $\alpha(a) = p$, $\alpha(b) = q$.

a. Show that, for any constant vector v, $|v| = 1$,

$$(q - p) \cdot v = \int_a^b \alpha'(t) \cdot v \, dt \leq \int_a^b |\alpha'(t)| \, dt.$$

b. Set

$$v = \frac{q - p}{|q - p|}$$

and show that

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)| \, dt;$$

that is, the curve of shortest length from $\alpha(a)$ to $\alpha(b)$ is the straight line joining these points.

Proof. **(a)**

The first equality

$$(q - p) \cdot v = \int_a^b \alpha'(t) \cdot v dt$$

follows directly from Theorem 2.1.1.

Now, by Cauchy-Schwarz inequality, we have

$$|\alpha'(t) \cdot v| \leq |\alpha'(t)| \cdot |v|$$

This then give us

$$\alpha'(t) \cdot v \leq |\alpha'(t) \cdot v| \leq |\alpha'(t)| \cdot |v| = |\alpha'(t)|$$

We now have

$$\int_a^b \alpha'(t) \cdot v \leq |\alpha'(t)| \, dt$$

as desired.

(b)
The first inequality tell us that if v is a constant and $|v| = 1$, we have

$$(q-p) \cdot v \leq \int_a^b |\alpha'(t)|\, dt$$

If $v = \frac{q-p}{|q-p|}$, it is clear that v is a constant and $|v| = 1$, and at the same time, we have

$$(q-p) \cdot v = \frac{(q-p) \cdot (q-p)}{|q-p|} = \frac{|q-p|^2}{|q-p|} = |q-p|$$

We now have

$$|q-p| = (q-p) \cdot v \leq \int_a^b |\alpha'(t)|\, dt$$

from the first inequality ■

Question 9

1. **Check whether the following bases are positive:**
a. **The basis $\{(1, 3), (4, 2)\}$ in R^2.**
b. **The basis $\{(1, 3, 5), (2, 3, 7), (4, 8, 3)\}$ in R^3.**

Proof. Compute

$$\begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} = -10$$

and compute

$$\begin{vmatrix} 1 & 2 & 4 \\ 3 & 3 & 8 \\ 5 & 7 & 3 \end{vmatrix} = -9$$

Both bases are negatively oriented. ■

Question 10

***2.** **A plane P contained in R^3 is given by the equation $ax + by + cz + d = 0$. Show that the vector $v = (a, b, c)$ is perpendicular to the plane and that $|d|/\sqrt{a^2 + b^2 + c^2}$ measures the distance from the plane to the origin $(0, 0, 0)$.**

Proof. Arbitrarily pick two points u, w in P. We wish to show

$$v \cdot (u - w) = 0$$

Because $v = (a, b, c)$ and

$$\begin{cases} au_1 + bu_2 + cu_3 = -d aw_1 + bw_2 + cw_3 = -d \end{cases}$$

We see

$$\begin{aligned} v \cdot (u - w) &= a(u_1 - w_2) + b(u_2 - w_2) + c(u_3 - w_2) \\ &= (-d) - (-d) = 0 \text{ (done)} \end{aligned}$$

To measure the distance between P and the origin, we wish to find a vector u such that $u \perp P$ and $u \in P$. We know that u must be linearly dependent with $v = (a, b, c)$, since the dimension of $P^\perp$ is 1. Then, we can write

$$u = c_0(a, b, c) \text{ for some } c_0 \in \mathbb{R}$$

Because $u \in P$, we know

$$c_0 a^2 + c_0 b^2 + c_0 c^2 + d = 0$$

This tell us

$$c_0 = \frac{-d}{a^2 + b^2 + c^2}$$

We now see that the distance $|u|$ between P and origin is

$$|u| = |c_0| \cdot \sqrt{a^2 + b^2 + c^2} = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$$

■

Question 11

***3. Determine the angle of intersection of the two planes $5x + 3y + 2z - 4 = 0$ and $3x + 4y - 7z = 0$.**

Proof. From last question, we know the two vectors u, v that are respectively perpendicular to $P : 5x + 3y + 2z - 4 = 0$ and $Q : 3x + 4y - 7z = 0$ respectively have the direction

$$(5, 3, 2) \text{ and } (3, 4, -7)$$

Then, we see the angle of the intersection are

$$\arccos \frac{5 \cdot 3 + 3 \cdot 4 + 2 \cdot (-7)}{\sqrt{5^2 + 3^2 + 2^2}\sqrt{3^2 + 4^2 + 7^2}} = \arccos \frac{13}{\sqrt{38}\sqrt{71}}$$

Notice that this angle is smaller than $\frac{\pi}{2}$ as we intend it to be. ■

Question 12

***6.** **Given two nonparallel planes $a_i x + b_i y + c_i z + d_i = 0$, $i = 1, 2$, show that their line of intersection may be parametrized as**

$$x - x_0 = u_1 t, \qquad y - y_0 = u_2 t, \qquad z - z_0 = u_3 t,$$

where (x_0, y_0, z_0) belongs to the intersection and $u = (u_1, u_2, u_3)$ is the vector product $u = v_1 \wedge v_2$, $v_i = (a_i, b_i, c_i)$, $i = 1, 2$.

Proof. Let $v = (x, y, z)$ be a point on the line of intersection. We see the vector $v-(x_0, y_0, z_0)$ lies on both planes, and thus must be perpendicular to $(a_1, b_1, c_1) = v_1$ and $(a_2, b_2, c_2) = v_2$ thus satisfying

$$v - (x_0, y_0, z_0) = tv_1 \times v_2 = tu \text{ for some } t \in \mathbb{R}$$

sine in $\mathbb{R}^3$, the only direction perpendicular to both v_1, v_2 is $v_1 \times v_2$. We can rewrite the above equation of course into

$$x - x_0 = u_1 t, y - y_0 = u_2 t, z - z_0 = u_3 t$$

■

2.2 HW2

Question 13

1. Given the parametrized curve (helix)

$$\alpha(s) = \left(a \cos \frac{s}{c}, a \sin \frac{s}{c}, b \frac{s}{c}\right), \qquad s \in R,$$

where $c^2 = a^2 + b^2$,

a. Show that the parameter s is the arc length.

b. Determine the curvature and the torsion of α.

c. Determine the osculating plane of α.

d. Show that the lines containing $n(s)$ and passing through $\alpha(s)$ meet the z axis under a constant angle equal to $\pi/2$.

e. Show that the tangent lines to α make a constant angle with the z axis.

Proof. **(a)** By computation

$$\alpha'(s) = (\frac{-a}{c} \sin \frac{s}{c}, \frac{a}{c} \cos \frac{s}{c}, \frac{b}{c})$$

So

$$|\alpha'(s)| = \sqrt{\frac{a^2+b^2}{c^2}} = 1 \qquad (\because \ \sin^2 + \cos^2 = 1 \ )$$

This shows α is parametrized by arc-length.

(b) By computation

$$\alpha''(s) = \left(\frac{-a}{c^2} \cos \frac{s}{c}, \frac{-a}{c^2} \sin \frac{s}{c}, 0\right)$$

Then because α is parametrized by arc-length, we have

$$\begin{aligned}\kappa(s) = |\alpha''(s)| &= \sqrt{\frac{a^2}{c^4}} \\ &= \frac{|a|}{c^2}\end{aligned}$$

By computation

$$\alpha'''(s) = \Big(\frac{a}{c^3}\sin\frac{s}{c}, \frac{-a}{c^3}\cos\frac{s}{c}, 0\Big)$$

Then using the identity of torsion, we have

$$\begin{aligned}
\tau(s) &= -\frac{-\big(\alpha'(s)\times\alpha''(s)\big)\cdot\alpha'''(s)}{\left|\kappa(s)\right|^2} \\
&= -\frac{\frac{a^2 b}{c^6}}{\frac{a^2}{c^4}} \\
&= \frac{b}{-c^2}
\end{aligned}$$

(c) Fix s. Define a set A by

$$A = \operatorname{span}\Big(\alpha'(s), \alpha''(s)\Big)$$

The osculating plane of α at s is then exactly

$$\{a + \alpha(s) : a \in A\}$$

(d) Because $\alpha''(s)$ by our computation is valued 0 in z-opponent, we know if the line containing N and passing through α meet the z axis, it must be under a constant angle equal to $\frac{\pi}{2}$. (use dot product to check this fact.).

Now, we only have to prove that the line does meet the z-axis. See that

$$\alpha + c^2\alpha'' = \Big(0, 0, b\frac{s}{c}\Big)$$

and we are done.
(e) Observe that

$$\alpha' \cdot \Big(0, 0, 1\Big) = \frac{b}{c} \text{ is a constant}$$

This together with the fact $|\alpha'|$ is a constant show that the angle between the tangent to α and z-axis is a constant.

$\blacksquare$

Question 14

*2. Show that the torsion τ of α is given by

$$\tau(s) = -\frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{|k(s)|^2}.$$

Theorem 2.2.1. (Identity of Torsion) Given a parametrized by arc-length cruve $\alpha : I \to \mathbb{R}^3$, we have

$$\tau(s) = -\frac{\big(\alpha'(s) \times \alpha''(s)\big) \cdot \alpha'''(s)}{\kappa^2(s)}$$

Proof. Because α is parametrized by arc-length, we have

$$\alpha'(s) = T(s)$$

We first show

$$\alpha''(s) = \kappa(s)N(s) \tag{2.5}$$

Compute

$$\begin{aligned} N(s) &= \frac{T'(s)}{|T'(s)|} \\ &= \frac{\alpha''(s)}{|\alpha''(s)|} = \frac{\alpha''(s)}{\kappa(s)} \text{ (done)} \end{aligned}$$

We now show

$$\alpha'''(s) = \kappa(s)\big(-(\tau B)(s) - (\kappa T)(s)\big) + \kappa'(s)N(s)$$

By Equation 2.5 and Frenet Formula, we have

$$\begin{aligned} \alpha'''(s) &= \kappa'(s)N(s) + \kappa(s)N'(s) \\ &= \kappa'(s)N(s) + \kappa(s)\big(-(\tau B)(s) - (\kappa T)(s)\big) \text{ (done)} \end{aligned}$$

Lastly, we verify

$$\begin{aligned} -\frac{\big(\alpha'(s) \times \alpha''(s)\big) \cdot \alpha'''(s)}{\kappa^2(s)} &= -\frac{(T \times \kappa N) \cdot \Big(\kappa\big(-\tau B - \kappa T\big) + \kappa' N\Big)}{\kappa^2} \\ &= -\frac{-\kappa^2 \tau (T \times N) \cdot B}{\kappa^2} \qquad (\because T \times N \cdot (T \text{ or } N) = 0) \\ &= \tau \end{aligned}$$

■

Question 15

3. Assume that $\alpha(I) \subset R^2$ (i.e., α is a plane curve) and give k a sign as in the text. Transport the vectors $t(s)$ parallel to themselves in such a way that the origins of $t(s)$ agree with the origin of R^2; the end points of $t(s)$ then describe a parametrized curve $s \longrightarrow t(s)$ called the *indicatrix of tangents* of α. Let $\theta(s)$ be the angle from e_1 to $t(s)$ in the orientation of R^2. Prove (a) and (b) (notice that we are assuming that $k \neq 0$).

a. The indicatrix of tangents is a regular parametrized curve.

b. $dt/ds = (d\theta/ds)n$, that is, $k = d\theta/ds$.

Proof. **(a)**

The indicatrix of tangents $\gamma : I \to \mathbb{R}^2$ is defined by

$$\gamma = \frac{\alpha'(s)}{|\alpha'(s)|}$$

Express $\alpha'(s)$ by

$$\alpha' \triangleq (x, y)$$

To show γ is regular. We wish to show

$$\gamma'(s) \neq 0 \text{ for all } s \in I$$

Express γ by

$$\gamma = \frac{(x, y)}{\sqrt{x^2 + y^2}}$$

Then, we see the x-component of $\gamma'(s)$ is

$$\gamma'(s)\Big|_x = \frac{x'y^2}{(x^2 + y^2)^{\frac{3}{2}}}$$

With similar computation on the y-component, we now arrive at

$$\gamma'(s) = \frac{\left(x'y^2, y'x^2\right)}{(x^2 + y^2)^{\frac{3}{2}}}$$

Now, for a contradiction, Assume $\gamma'(s) = 0$ for some s. Then one of the three things below must happen

(a) $x' = y' = 0$

(b) $y^2 = x^2 = 0$

(c) $x' = x^2 = 0$ WLOG

Because $(x, y) = \alpha'$ and α is parametrized by arc-length and curvature is non-zero by premise, we know it can not happen $\alpha'' = (x', y') = 0$.

Because $(x, y) = \alpha'$ and α is parametrized by arc-length, we also know it can not happen $\alpha' = (x, y) = 0$.

Now, we are given the hypothesis $x' = x^2 = 0$. Because α is parametrized by arc-length, from $x = 0$, we know $y = \pm 1$. Then because $|\alpha'|$ is constant, we can deduce

$$\begin{aligned} 0 &= (x', y') \cdot (x, y) \\ &= (0, y') \cdot (0, \pm 1) \end{aligned}$$

This show us $y' = 0$, which is impossible, since if $(x', y') = 0$ then the curvature is 0 CaC (done)

(b) The functions $\theta : [0, l] \to \mathbb{R}$, is defined by

$$T = (x, y) \triangleq (\cos\theta, \sin\theta)$$

By Frenet Formula, we have

$$\kappa N = T' = \theta'(-\sin\theta, \cos\theta) \tag{2.6}$$

Because $|(-\sin\theta, \cos\theta)| = 1$ and $(-\sin\theta, \cos\theta) \cdot T = 0$ and

$$\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = 1$$

we can identify $(-\sin\theta, \cos\theta) = N$. Then from Equation 2.6, we now can deduce

$$\kappa = \theta'$$

■

Question 16

6. A *translation* by a vector v in R^3 is the map $A: R^3 \to R^3$ that is given by $A(p) = p + v$, $p \in R^3$. A linear map $\rho: R^3 \to R^3$ is an *orthogonal transformation* when $\rho u \cdot \rho v = u \cdot v$ for all vectors $u, v \in R^3$. A *rigid motion* in R^3 is the result of composing a translation with an orthogonal transformation with positive determinant (this last condition is included because we expect rigid motions to preserve orientation).

a. Demonstrate that the norm of a vector and the angle θ between two vectors, $0 \leq \theta \leq \pi$, are invariant under orthogonal transformations with positive determinant.

b. Show that the vector product of two vectors is invariant under orthogonal transformations with positive determinant. Is the assertion still true if we drop the condition on the determinant?

c. Show that the arc length, the curvature, and the torsion of a parametrized curve are (whenever defined) invariant under rigid motions.

Proof. Let A be a translation and ρ be an orthogonal transformation.

(a) Observe

$$\begin{aligned}
\|v\| &= \sqrt{v \cdot v} \\
&= \sqrt{\rho v \cdot \rho v} \\
&= \|\rho v\|
\end{aligned}$$

Because θ is given by

$$\theta = \arccos \frac{v \cdot w}{|v| \cdot |w|}$$

and because norm is invariant under orthogonal transformation, from the definition of orthogonal transformation, we now see

$$\begin{aligned}
\theta &= \arccos \frac{v \cdot w}{|v| \cdot |w|} \\
&= \arccos \frac{\rho v \cdot \rho w}{|v| \cdot |w|} \\
&= \arccos \frac{\rho v \cdot \rho w}{|\rho v| \cdot |\rho w|} = \theta_\rho
\end{aligned}$$

where θ_ρ is the angle between ρv and ρw.

(b) Fix $v, w \in \mathbb{R}^3$ and a positive determinant orthogonal transformation ρ. We wish to show

$$\rho v \times \rho w = \rho(v \times w)$$

We can reduce the problem into proving

$$\rho v \times \rho w \cdot z = \rho(v \times w) \cdot z \text{ for all } z \in \mathbb{R}^3$$

Fix $z \in \mathbb{R}^3$. Because ρ has non-zero determinant, we know there exists $z' \in \mathbb{R}^3$ such that

$$\rho z' = z$$

Now, because orthogonal transformation has determinant ± 1 and we have know ρ has positive determinant, we know

$$\begin{aligned}
\rho v \times \rho w \cdot z &= \rho v \times \rho w \cdot \rho z' \\
&= \begin{vmatrix} \rho v \\ \rho w \\ \rho z' \end{vmatrix} \\
&= \begin{vmatrix} \rho v & \rho w & \rho z' \end{vmatrix} \qquad (\because \det A = \det A^t) \\
&= |\rho| \cdot \begin{vmatrix} v & w & z' \end{vmatrix} \qquad (\because \det A \det B = \det AB) \\
&= \begin{vmatrix} v \\ w \\ z' \end{vmatrix} \qquad (\because \det \rho = 1) \\
&= v \times w \cdot z' \\
&= \rho(v \times w) \cdot \rho z' \\
&= \rho(v \times w) \cdot z \text{ (done)}
\end{aligned}$$

The assertion is clearly false if the determinant is negative. One can check $v = (1, 0, 0)$ and $w = (0, 1, 0)$ and $\rho(x, y, z) = (-x, y, z)$.

(c) We first show arc length is invariant under rigid motion. We first show

$$\text{arc length is invariant under orthogonal transformation}$$

To show such, we only have to show

$$|(\rho \circ \gamma)'| = |\gamma'|$$

■

Fix $y \in I$. We have

$$|\gamma'(y)| = \left|\lim_{t\to y} \frac{\gamma(t)-\gamma(y)}{t-y}\right| = \lim_{t\to y}\left|\frac{\gamma(t)-\gamma(y)}{t-y}\right|$$

Notice that in above deduction, we exchange limit and norm. Such exchange hold true because the function inside is continuous.

Similarly, we have

$$|(\rho\circ\gamma)'(y)| = \lim_{t\to y}\left|\frac{\rho\circ\gamma(t)-\rho\circ\gamma(y)}{t-y}\right|$$

Then, we can reduce the problem into

$$\text{proving } |\gamma(t)-\gamma(y)| = |\rho\circ\gamma(t)-\rho\circ\gamma(y)|$$

Because $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ is an orthogonal transformation (a linear transformation too), we can deduce

$$\begin{aligned} |\rho\circ\gamma(t)-\rho\circ\gamma(y)| &= \big|\rho\big(\gamma(t)-\gamma(y)\big)\big| \\ &= |\gamma(t)-\gamma(y)| \ \ \text{(done)} \end{aligned}$$

We have proved arc-length is invariant under orthogonal transformation. With some simple computation, it is clear that arc-length is invariant under translation. This let us conclude arc length is invariant under rigid motion.

Now, to show curvature and torsion are also invariant under rigid motion. We first recall the following identities for curve parametrzied by arc-length

$$\kappa = |\gamma''| \text{ and } \tau = -\frac{\gamma'\times\gamma''\cdot\gamma'''}{\kappa^2}$$

We now prove

curvature is invariant under rigid motion

Notice that γ' is invariant under translation, so in fact, we only have to prove

curvature is invariant under orthogonal transformation

Observe

$$|\gamma''(y)| = \left|\lim_{t\to y} \frac{\gamma'(t)-\gamma'(y)}{t-y}\right| = \lim_{t\to y}\left|\frac{\gamma'(t)-\gamma'(y)}{t-y}\right|$$

and

$$|(\rho \circ \gamma)''(y)| = \lim_{t \to y} \left| \frac{(\rho \circ \gamma)'(t) - (\rho \circ \gamma)'(y)}{t - y} \right|$$

We can now reduce the problem into proving

$$|\gamma'(t) - \gamma'(y)| = |(\rho \circ \gamma)'(t) - (\rho \circ \gamma)'(y)|$$

Because ρ is a linear transformation, we can compute

$$\begin{aligned}
(\rho \circ \gamma)'(t) &= \lim_{u \to t} \frac{\rho \circ \gamma(u) - \rho \circ \gamma(t)}{u - t} \\
&= \lim_{u \to t} \rho\Big(\frac{\gamma(u) - \gamma(t)}{u - t}\Big) \\
&= \rho \lim_{u \to t} \Big(\frac{\gamma(u) - \gamma(t)}{u - t}\Big) = \rho \circ \gamma'(t)
\end{aligned} \tag{2.7}$$

We now using the fact norm is invariant under orthogonal transformation to compute

$$\begin{aligned}
|(\rho \circ \gamma)'(t) - (\rho \circ \gamma)'(y)| &= |\rho \circ \gamma'(t) - \rho \circ \gamma'(y)| \\
&= \big|\rho\big(\gamma'(t) - \gamma'(y)\big)\big| \\
&= |\gamma'(t) - \gamma'(y)| \ \text{(done)}
\end{aligned}$$

Now, notice that in Equation 2.7, we just proved

$$(\rho\gamma)' = \rho\gamma'$$

Iterating the same argument, we can show

$$\begin{aligned}
(\rho\gamma)'' &= \big((\rho\gamma)'\big)' \\
&= (\rho\gamma')' \\
&= \rho\gamma''
\end{aligned}$$

and also show

$$\begin{aligned}
(\rho\gamma)''' &= \big((\rho\gamma)''\big)' \\
&= (\rho\gamma'')' \\
&= \rho\gamma'''
\end{aligned}$$

We now using the fact that $|\rho| = 1$ to compute

$$\begin{aligned}
(\rho\gamma)' \times (\rho\gamma)'' \cdot (\rho\gamma)''' &= \big|(\rho\gamma)' \quad (\rho\gamma)'' \quad (\rho\gamma)'''\big| \\
&= \big|\rho\gamma' \quad \rho\gamma'' \quad \rho\gamma'''\big| \\
&= \big|\rho\left[\gamma' \quad \gamma'' \quad \gamma'''\right]\big| \\
&= |\rho| \cdot \big|\gamma' \quad \gamma'' \quad \gamma'''\big| \\
&= \gamma' \times \gamma'' \cdot \gamma'''
\end{aligned}$$

Above computation with identity of torsion and the fact curvature is invariant under orthogonal transformation with positive determinant then show that torsion is also invariant under orthogonal transformation with positive determinant.

Because $(\gamma + c)' = \gamma'$, together with what we have proved, it is easy to check torsion is also invariant under rigid motion.

Question 17

9. Given a differentiable function $k(s)$, $s \in I$, show that the parametrized plane curve having $k(s) = k$ as curvature is given by

$$\alpha(s) = \left(\int \cos\theta(s)\,ds + a, \int \sin\theta(s)\,ds + b \right),$$

where

$$\theta(s) = \int k(s)\,ds + \varphi,$$

and that the curve is determined up to a translation of the vector (a, b) and a rotation of the angle φ.

Proof. By Fundamental Theorem of Local Curves (you can think of our application as identifying $\tau = 0$), we know if such curve exists then it is unique up to translation and rotation. This reduced our proof into showing

$$\alpha \text{ has curvature } \kappa$$

Compute

$$\alpha' = (\cos\theta, \sin\theta)$$

This shows that α is parametrized by arc-length, and shows that we can compute

$$\begin{aligned} |\alpha''| &= |\theta'(-\sin\theta, \cos\theta)| \\ &= |\theta'| = |\kappa| = \kappa \text{ (done)} \end{aligned}$$

■

Question 18

11. One often gives a plane curve in polar coordinates by $\rho = \rho(\theta)$, $a \leq \theta \leq b$.

a. Show that the arc length is

$$\int_a^b \sqrt{\rho^2 + (\rho')^2}\, d\theta,$$

where the prime denotes the derivative relative to θ.

b. Show that the curvature is

$$k(\theta) = \frac{2(\rho')^2 - \rho\rho'' + \rho^2}{\{(\rho')^2 + \rho^2\}^{3/2}}.$$

Proof. **(a)**
Parametrize by

$$\alpha(\theta) \triangleq (x, y) \triangleq (r\cos\theta, r\sin\theta)$$

where $r(\theta)$ is a function. With respect to θ, we compute

$$\begin{aligned}(x')^2 + (y')^2 &= (r'\cos\theta - r\sin\theta)^2 + (r'\sin\theta + r\cos\theta)^2 \\ &= (r')^2\cos^2\theta + r^2\sin^2\theta + (r')^2\sin^2\theta + r^2\cos^2\theta \qquad (\because \text{ elimination }) \\ &= r^2 + (r')^2\end{aligned}$$

We now see that the arc-length can be computed by

$$\begin{aligned}\int_a^b |\alpha'(\theta)|\, d\theta &= \int_a^b \sqrt{(x')^2 + (y')^2} d\theta \\ &= \int_a^b \sqrt{r^2 + (r')^2} d\theta\end{aligned}$$

(b)

Recall that

$$\kappa(t) = \frac{x'y'' - x''y'}{\left((x')^2 + (y')^2\right)^{\frac{3}{2}}}$$

plugin

$$\begin{aligned}(x', y') &= (r'\cos\theta - r\sin\theta, r'\sin\theta + r\cos\theta) \\ (x'', y'') &= (r''\cos\theta - 2r'\sin\theta - r\cos\theta, r''\sin\theta + 2r'\cos\theta - r\sin\theta)\end{aligned}$$

To compute

$x'y'' = r'r''\cos\theta\sin\theta + 2(r')^2\cos^2\theta - rr'\cos\theta\sin\theta - rr''\sin^2\theta - 2rr'\cos\theta\sin\theta + r^2\sin^2\theta$
$x''y' = r'r''\sin\theta\cos\theta - 2(r')^2\sin^2\theta - rr'\cos\theta\sin\theta + rr''\cos^2\theta - 2rr'\cos\theta\sin\theta - r^2\cos^2\theta$

Eliminating the odd terms and using $\cos^2 + \sin^2 = 1$, we now compute

$$\begin{aligned}\kappa &= \frac{x'y'' - x''y'}{\left((x')^2 + (y')^2\right)^{\frac{3}{2}}} \\ &= \frac{2(r')^2 - 2rr'' + r^2}{\left(r^2 + (r')^2\right)^{\frac{3}{2}}}\end{aligned}$$

■

Question 19

17. In general, a curve α is called a *helix* if the tangent lines of α make a constant angle with a fixed direction. Assume that $\tau(s) \neq 0$, $s \in I$, and prove that:

***a.** α is a helix if and only if $k/\tau = \text{const.}$

***b.** α is a helix if and only if the lines containing $n(s)$ and passing through $\alpha(s)$ are parallel to a fixed plane.

***c.** α is a helix if and only if the lines containing $b(s)$ and passing through $\alpha(s)$ make a constant angle with a fixed direction.

d. The curve

$$\alpha(s) = \left(\frac{a}{c}\int \sin\theta(s)\,ds, \frac{a}{c}\int \cos\theta(s)\,ds, \frac{b}{c}s\right),$$

where $a^2 = b^2 + c^2$, is a helix, and that $k/\tau = b/a$.

Proof. **(a)** $(\longrightarrow)$

Because α is a helix, we know there exists fixed unit $a \in \mathbb{R}^3$ and $b \in \mathbb{R}$ such that

$$\alpha' \cdot a = b \text{ for all } s$$

This then implies

$$\alpha'' \cdot a = 0 \text{ for all } s$$

which implies

$$N \cdot a = 0$$

since N is parallel with α''. Because $\{T, N, B\}$ is an orthonormal basis, this $(N \cdot a = 0)$ together with a being unit then tell us we can express a by

$$a = T\cos\theta + B\sin\theta \text{ for some fixed } \theta \in \mathbb{R}$$

We now have the information $T\cos\theta + B\sin\theta$ is a constant function in s. Then, using Frenet Formula, we can deduce

$$0 = (T\cos\theta + B\sin\theta)' = \kappa N\cos\theta + \tau N\sin\theta$$

This them implies

$$\frac{\kappa}{\tau} = \frac{-\sin\theta}{\cos\theta} \text{ is a constant since } \theta \text{ is fixed.}$$

Notice that $\cos\theta \neq 0$ because $\tau \neq 0$ for all s.

$(\longleftarrow)$

Define $\theta \in \mathbb{R}$ by

$$\theta = \arctan\frac{-\kappa}{\tau}$$

We wish to show

$$a = T\cos\theta + B\sin\theta \text{ suffice}$$

Because we have

$$T \cdot a = \cos\theta$$

We only wish to show

$$a \text{ is a constant function in } s$$

Because $\theta = \arctan\frac{-\kappa}{\tau}$, we know

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{-\kappa}{\tau}$$

This then tell us

$$\tau\sin\theta + \kappa\cos\theta = 0$$

and implies

$$\tau N\sin\theta + \kappa N\cos\theta = 0$$

Then

$$\begin{aligned} a' &= (T\cos\theta + N\sin\theta)' \\ &= \kappa N\cos\theta + \tau N\sin\theta = 0 \end{aligned}$$

This implies a is indeed a constant. (done)

(b)

$(\longrightarrow)$

Let $a \in \mathbb{R}^3$ be the unit vector such that

$$T \cdot a \text{ is fixed}$$

We now see

$$N \cdot a = (T \cdot a)' = 0$$

This implies

$$\text{the plane } \{a\}^\perp \text{ suffice}$$

$(\longleftarrow)$

Observe that

$$\begin{aligned} 0 = N \cdot a &= \frac{T'}{|T'|} \cdot a \\ &\implies T' \cdot a = 0 \\ &\implies T \cdot a \text{ is fixed} \end{aligned}$$

(c)

$(\longrightarrow)$

Because $T \cdot a$ is fixed, we can deduce

$$\kappa N \cdot a = T' \cdot a = 0$$

Now observe from Frenet Formula that

$$(B \cdot a)' = -\tau N \cdot a = 0$$

This implies $B \cdot a$ is fixed.

$(\longleftarrow)$

Because $B \cdot a$ is fixed, we can deduce

$$\begin{aligned} 0 = (B \cdot a)' &= -\tau N \cdot a \\ \implies N \cdot a &= 0 \end{aligned}$$

The proof then now follows from the result of **(b)**.

(d)

First we have to notice the fucking typo correction $\frac{\kappa}{\tau} = \frac{a}{b}$.

Compute

$$\begin{aligned} \alpha'(s) &= \Big(\frac{a}{c}\sin\theta, \frac{a}{c}\cos\theta, \frac{b}{c}\Big) \\ \alpha''(s) &= \Big(\theta'\frac{a}{c}\cos\theta, \theta'\frac{-a}{c}\sin\theta, 0\Big) \\ \alpha'''(s) &= \Big(\theta''\frac{a}{c}\cos\theta + (\theta')^2\frac{-a}{c}\sin\theta, \theta''\frac{-a}{c}\sin\theta + (\theta')^2\frac{-a}{c}\cos\theta, 0\Big) \end{aligned}$$

This give us

$$\begin{aligned} &\alpha' \times \alpha'' \cdot \alpha''' \\ &= \frac{b}{c}\Big[\theta'\theta''\frac{-a^2}{c^2}\cos\theta\sin\theta + (\theta')^3\frac{-a^2}{c^2}\cos^2\theta - \theta'\theta''\frac{-a^2}{c}\sin\theta\cos\theta - (\theta')^3\frac{a^2}{c^2}\sin^2\theta\Big] \\ &= \frac{b}{c}\Big((\theta')^3\frac{-a^2}{c^2}\Big) = \frac{-a^2b}{c^3}(\theta')^3 \end{aligned}$$

And give us

$$\kappa = \theta'\frac{a}{c}$$

We now compute

$$\begin{aligned} \frac{\kappa}{\tau} &= \frac{\kappa}{-\frac{\alpha' \times \alpha'' \cdot \alpha'''}{\kappa^2}} \\ &= \frac{-\kappa^3}{\alpha' \times \alpha'' \cdot \alpha'''} \\ &= \frac{-(\theta')^3\frac{a^3}{c^3}}{\frac{-a^2b}{c^3}(\theta')^3} = \frac{a}{b} \end{aligned}$$

■

Question 20

3. Compute the curvature of the ellipse

$$x = a\cos t, \qquad y = b\sin t, \qquad t \in [0, 2\pi],\ a \neq b,$$

and show that it has exactly four vertices, namely, the points $(a, 0)$, $(-a, 0)$, $(0, b)$, $(0, -b)$.

Proof. Compute

$$\begin{cases} x' = -a\sin t \text{ and } x'' = -a\cos t \\ y' = b\cos t \text{ and } y'' = -b\sin t \end{cases}$$

Plugging the curvature formula

$$\kappa = \frac{x'y'' - x''y'}{\left((x')^2 + (y')^2\right)^{\frac{3}{2}}}$$

We now have

$$\kappa = \frac{ab}{\left(a^2\sin^2 t + b^2\cos^2 t\right)^{\frac{3}{2}}}$$

Compute

$$\kappa' = (2a^2\sin t\cos t - 2b^2\sin t\cos t)(a^2\sin^2 t + b^2\cos^2 t)^{\frac{-5}{2}} \cdot (ab)$$

We see that

$$\kappa' = 0 \iff \sin 2t = 0$$

This only happens when $t \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$ where 2π is just 0 in the sense of parametrizeation of closed curve. We have shown there are exactly four vertices

$$(a, 0), (-a, 0), (0, b), (0, -b)$$

■

Question 21

***4.** Let C be a plane curve and let T be the tangent line at a point $p \in C$. Draw a line L parallel to the normal line at p and at a distance d of p (Fig. 1-36). Let h be the length of the segment determined on L by C and T (thus, h is the "height" of C relative to T). Prove that

$$|k(p)| = \lim_{d \to 0} \frac{2h}{d^2},$$

where $k(p)$ is the curvature of C at p.

Proof. WOLG, let $p = (0,0)$, T be the x-axis and some neighborhood around p be above T. Positively oriented parametrize C by arc-length using (x, y) and $(x, y)(0) = (0, 0)$. Using Taylor Theorem about $y(0)$, we see

$$y(s) = y(0) + y'(0)s + \frac{y''(0)}{2}s^2 + R_y \text{ where } \frac{R_y}{s^2} \to 0 \text{ as } s \to 0$$

Because T the tangent is the x-axis, we know $x''(0) = 0 \quad (\because N = (0, 1))$. This tell us

$$\begin{aligned} |\kappa(0)| &= \sqrt{(x'')^2(0) + (y'')^2(0)} \\ &= y''(0) \qquad (\because N = (0,1)) \end{aligned}$$

■

By our setting $(x, y)(0) = (0, 0)$, we see

$$y(0) = y'(0) = 0 \qquad (\because (x', y') = (1, 0))$$

We now see

$$y''(0) = \frac{2\big(y(s) - R_y\big)}{s^2} \text{ for all } s \neq 0$$

This tell us

$$y''(0) = \lim_{s \to 0} \frac{2\big(y(s) - R_y\big)}{s^2} = \lim_{s \to 0} \frac{2y(s)}{s^2}$$

Using Taylor Theorem about $x(0)$, we see

$$x(s) = x(0) + x'(0)s + R_x \text{ where } \frac{R_x}{s} \to 0 \text{ as } x \to 0$$

Because $x(0) = 0$ and $x'(0) = 1$, we see

$$\lim_{s\to 0} \frac{x(s)}{s} = \lim_{s\to 0} \frac{s + R_x}{s} = 1$$

This now give us

$$|\kappa(0)| = y''(0) = \lim_{s\to 0} \frac{2y(s)}{s^2} = \lim_{s\to 0} \frac{2y(s)}{x^2(s)} = \lim_{d\to 0} \frac{2h}{d^2}$$

Question 22

6. Let $\alpha(s)$, $s \in [0, l]$ be a closed convex plane curve positively oriented. The curve

$$\beta(s) = \alpha(s) - rn(s),$$

where r is a positive constant and n is the normal vector, is called a *parallel* curve to α (Fig. 1-37). Show that

a. Length of β = length of $\alpha + 2\pi r$.

b. $A(\beta) = A(\alpha) + rl + \pi r^2$.

c. $k_\beta(s) = k_\alpha(s)/(1 + r)$.

Proof. **(a)**

Using Frenet Formula to compute

$$\beta'(s) = \alpha'(s) + r\kappa T(s)$$

Because α is parametrized by arc-length, we now know

$$|\beta'| = |(1 + r\kappa)\alpha'| = |1 + r\kappa| = 1 + r\kappa$$

This now give us

$$\int_0^l |\beta'|\, ds = l + r\int_0^l \kappa ds$$

Because a closed convex curve must also be simple (Sec. 5-7, Prop. 1), we now can deduce

$$\begin{aligned}\text{Length of } \beta = l + r\int_0^l \kappa ds \\ = \text{ Length of } \alpha + r(2\pi)\end{aligned}$$

(b)

Set

$$\alpha = (x, y) \text{ and } \beta = (x - rN_1, y - rN_2)$$

Because

$$\beta' = (1 + r\kappa)\alpha'$$

We know

$$\beta' = \Big((1 + r\kappa)x', (1 + r\kappa)y'\Big)$$

Now, we use Green's Theorem to compute the Area

$$\begin{aligned}
A(\beta) &= \frac{1}{2}\int_0^l (x - rN_1)(1 + r\kappa)y' - (y - rN_2)(1 + r\kappa)x' ds \\
&= \frac{1}{2}\int_0^l (xy' - yx')ds + \frac{r}{2}\int_0^l (\kappa xy' + \kappa x'y)ds \\
&+ \frac{r}{2}\int_0^l -(N_1 y' + N_2 x')ds + \frac{r^2}{2}\int_0^l (-N_1 y'\kappa + N_2 x'\kappa)ds
\end{aligned}$$

Notice that by Frenet Formula, we have

$$N' = -\kappa(x', y')$$

so in fact we know

$$\kappa xy' + \kappa x'y = N' \cdot (-y, x)$$

Now using integral by part and the fact $\alpha = (x, y)$ is closed, we know

$$\begin{aligned}
\int_0^l (\kappa xy' + \kappa x'y)ds &= \int_0^l N' \cdot (-y, x)ds \\
&= \int_0^l N \cdot (-y', x')ds
\end{aligned}$$

Then now we have

$$\frac{r}{2}\int_0^l (\kappa xy' + \kappa x'y)ds + \frac{r}{2}\int_0^l -N_1 y' + N_2 x' ds = \frac{r}{2}\int_0^l 2N \cdot (-y', x')ds$$

Using positive orientation and the fact $|N| = 1 = |(-y', x')|$ to identify that $N = (-y', x')$, we now have

$$\frac{r}{2}\int_0^l 2N \cdot (-y', x')ds = rl$$

and have

$$\frac{r^2}{2}\int_0^l (-N_1 y'\kappa + N_2 x'\kappa)ds = \frac{r^2}{2}\int_0^l \kappa ds = r^2\pi$$

since (x, y) is simple closed. This finishes the proof.

(c)

Recall that

$$\kappa(a, b) = \frac{a'b'' - a''b'}{\big((a')^2 + (b')^2\big)^{\frac{3}{2}}}$$

We use this formula on β to compute

$$\begin{aligned}
\kappa_\beta &= \frac{(1 + r\kappa)x'\big((1 + r\kappa)y'\big)' - \big((1 + r\kappa)x'\big)'(1 + r\kappa)y'}{(1 + r\kappa)^3} \\
&= \frac{(1 + r\kappa)^2(x'y'' - x''y')}{(1 + r\kappa)^3} \\
&= \frac{x'y'' - x''y'}{1 + r\kappa} = \frac{\kappa}{1 + r\kappa} \qquad (\because (x')^2 + (y')^2 = 1)
\end{aligned}$$

■

Question 23

8. ***a.** Let $\alpha(s)$, $s \in [0, l]$, be a plane simple closed curve. Assume that the curvature $k(s)$ satisfies $0 < k(s) \leq c$, where c is a constant (thus, α is less curved than a circle of radius $1/c$). Prove that

$$\text{length of } \alpha \geq \frac{2\pi}{c}.$$

b. In part a replace the assumption of being simple by "α has rotation index N." Prove that

$$\text{length of } \alpha \geq \frac{2\pi N}{c}.$$

Proof. (a)

Because α is simple closed and $\kappa \leq c$, we know

$$cl = \int_0^l c ds \geq \int_0^l \kappa ds = 2\pi$$

This then implies

$$\text{Length of } \alpha = l \geq \frac{2\pi}{c}$$

(b)

Because α has rotation index N and $\kappa \leq c$, we know

$$cl = \int_0^l c ds \geq \int_0^l \kappa ds = N2\pi$$

This the implies

$$\text{Length of } \alpha = l \geq \frac{2\pi N}{c}$$

■

Question 24

***11.** Given a nonconvex simple closed plane curve C, we can consider its *convex hull* H (Fig. 1-39), that is, the boundary of the smallest convex set containing the interior of C. The curve H is formed by arcs of C and by the segments of the tangents to C that bridge "the nonconvex gaps" (Fig. 1-39). It can be proved that H is a C^1 closed convex curve. Use this to show that, in the isoperimetric problem, we can restrict ourselves to convex curves.

Figure 1-39

Proof. Suppose we have proved that a convex closed curve must satisfy the isoperimetric inequality. Let C be an arbitrary closed plane curve, and let H be its convex hull. Now, because straight line is the shortest curve between two point and because we know H, a convex curve, must satisfy isoperimetric inequality, we now see

$$4\pi A(C) \leq 4\pi A(H) \leq l_H^2 \leq l_C^2$$

If the equality hold true, we can deduce from $l_H = l_C$ that $H = C$ and use the argument for isoperimetric inequality of convex curve to argue that $C = H$ must be a circle. ∎

Question 25

3. Show that the two-sheeted cone, with its vertex at the origin, that is, the set $\{(x, y, z) \in R^3; x^2 + y^2 - z^2 = 0\}$, is not a regular surface.

Proof. Let $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 0\}$. It is clear S contain $(0, 0, 0)$. To show S is not regular, we only wish to find a neighborhood V around $(0, 0, 0)$ in S such that V can not expressed as graph of differentiable functions from $\mathbb{R}^2$ to $\mathbb{R}$. This is trivially true, as all neighborhood ought to contain some open ball $B_\epsilon(0)$, and in this open ball, if we fix, say $(x, y) \in B_\epsilon(0)$ such that $(x, y, \sqrt{x^2 + y^2}) \in B_\epsilon(0)$, we see that $z = -\sqrt{x^2 + y^2}$ is also in $B_\epsilon(0)$ and S. The same argument applies to when (x, z) and (y, z) are fixed. ∎

Question 26

6. Give another proof of Prop. 1 by applying Prop. 2 to $h(x, y, z) = f(x, y) - z$.

Proof. Because f is differentiable, we see f_x, f_y are all continuous on U. This then implies

$$h_x(x,y,z) = f_x(x,y), h_y(x,y,z) = f_y(x,y), h_z = -1 \text{ are all continuous on } U$$

We have shown h is differentiable. Now that observe

$$h(x,y,z) = 0 \implies (x,y,z) = (x,y,f(x,y))$$

The converse of course hold true. This then implies

$$f[U] = h^{-1}[0]$$

Fix arbitrary $(x,y) \in U$. We see

$$\mathbf{d}h\big(x,y,f(x,y)\big) = \begin{bmatrix} h_x & h_y & h_z \end{bmatrix}\Big|_{(x,y,f(x,y))} = \begin{bmatrix} f_x(x,y) & f_y(x,y) & -1 \end{bmatrix}$$

which is clearly not onto. This show

$$(x,y,f(x,y)) \text{ is not a critical point}$$

Because $(x,y) \in U$ is arbitrary, we have shown $f[U]$ contain no critical point. Now it follows 0 is a regular value and $f[U] = h^{-1}[0]$ is a regular surface. ∎

Question 27

7. Let $f(x, y, z) = (x + y + z - 1)^2$.

a. Locate the critical points and critical values of f.

b. For what values of c is the set $f(x, y, z) = c$ a regular surface?

c. Answer the questions of parts a and b for the function $f(x, y, z) = xyz^2$.

Proof. **(a)**

Compute

$$f_x = f_y = f_z = 2(x+y+z-1)$$

This implies the set of critical points are

$$\{(x,y,z) \in \mathbb{R}^3 : x+y+z = 1\}$$

Then it follows from simple computation the set of critical values is exactly

$$\{0\}$$

(b)

For all $c > 0$, the set $f^{-1}[c]$ is a regular surface, and for all $c < 0$, the set $f^{-1}[c]$ is empty (thus trivially regular).

(c)

Compute

$$f_x = yz^2 \text{ and } f_y = xz^2 \text{ and } f_z = 2xyz$$

This implies the set of critical points is

$$\{(x, y, z) : z = 0 \text{ or } x = y = 0\}$$

With simple computation, we see the set of critical values is exactly

$$\{0\}$$

The set of regular values are exactly $\mathbb{R}^*$, so all $c \neq 0$ suffice. ■

Question 28

8. Let $\mathbf{x}(u, v)$ be as in Def. 1. Verify that $d\mathbf{x}_q \colon R^2 \to R^3$ is one-to-one if and only if

$$\frac{\partial \mathbf{x}}{\partial u} \wedge \frac{\partial \mathbf{x}}{\partial v} \neq 0.$$

Proof. Note that

$$dx_q = \begin{bmatrix} \partial_u x & \partial_v x \end{bmatrix}$$

This give us

$$dx_q : \mathbb{R}^2 \to \mathbb{R}^3 \text{ is one-to-one} \iff \partial_u x, \partial_v x \in \mathbb{R}^3 \text{ is linearly independent everywhere}$$

Then we can reduce the problem into proving

$$\partial_u x, \partial_v x \in \mathbb{R}^3 \text{ is linearly independent everywhere} \iff \partial_u x \times \partial_v x \neq 0 \text{ everywhere}$$

This then follows from Theorem 2.2.2 at the next page, as one can see that each component of the output of cross product is exactly the three determinant. ■

Theorem 2.2.2. (Computation to check Linearly Independence)

$$\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \end{bmatrix} \text{ is linearly independent } \iff \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} \neq 0 \text{ or } \begin{vmatrix} v_2 & w_2 \\ v_3 & w_3 \end{vmatrix} \neq 0 \text{ or } \begin{vmatrix} v_1 & w_1 \\ v_3 & w_3 \end{vmatrix} \neq 0$$

Proof. $(\longleftarrow)$

Assume v, w are linearly dependent. Fix $w_k = cv_k$. We see

$$\begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} = cv_1v_2 - cv_1v_2 = 0 \text{ CaC}$$

$(\longrightarrow)$

Assume all determinant are 0. Pick k such that v_k is non-zero. Define

$$c \triangleq \frac{w_k}{v_k}$$

WOLG, suppose

$$w_1 = cv_1 \text{ and } v_1 \neq 0$$

We then can deduce

$$\begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} \implies cv_1v_2 = v_1w_2 \implies w_2 = cv_2$$

The same arguement implies $w_3 = cv_3$ CaC ■

2.3 HW3

Question 29

12. Show that $\mathbf{x}: U \subset R^2 \to R^3$ given by

$$\mathbf{x}(u, v) = (a \sin u \cos v, b \sin u \sin v, c \cos u), \quad a, b, c \neq 0,$$

where $0 < u < \pi$, $0 < v < 2\pi$, is a parametrization for the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Describe geometrically the curves $u = \text{const.}$ on the ellipsoid.

Proof. We are required to show

(a) range of $\mathbf{x}$ lies in the ellipsoid

(b) $\mathbf{x}$ is smooth

(c) $d\mathbf{x}$ is one-to-one everywhere on $U \triangleq (0, \pi) \times (0, 2\pi)$

(d) $\mathbf{x}$ is a homeomorphism

Compute

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \cos^2 u = 1$$

This shows that the range of $\mathbf{x}$ indeed lies in the ellipsoid.

It is clear that $\mathbf{x}$ is smooth.

Compute

$$d\mathbf{x} = \begin{bmatrix} a \cos u \cos v & -a \sin u \sin v \\ b \cos u \sin v & b \sin u \cos v \\ -c \sin u & 0 \end{bmatrix}$$

Then compute

$$\frac{\partial(y, z)}{\partial(u, v)} = bc \sin^2 u \cos v \text{ and } \frac{\partial(x, z)}{\partial(u, v)} = -ac \sin^2 u \sin v$$

Because $u \in (0,\pi)$ and $v \in (0,2\pi)$, and $b, c \neq 0$, we now can deduce

$$\frac{\partial(y,z)}{\partial(u,v)} = 0 \iff v \in \{\frac{\pi}{2}, \frac{3}{2}\pi\}$$
$$\frac{\partial(y,z)}{\partial(u,v)} = 0 \iff v = \pi$$

This then let us deduce

$$d\mathbf{x} \text{ is one-to-one everywhere on } (0,\pi) \times (0,2\pi)$$

Traditionally, the function arctan is defined on $\mathbb{R}$ and have codomaiin $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Deduce first from the z-component of $\mathbf{x}$. We see

$$\mathbf{x}^{-1}(x,y,z) = \Big(\arccos \frac{z}{c}, \begin{cases} \arctan \frac{ay}{bx} & \text{if } x, y \in \mathbb{R}^+ \text{ (first quadrant)} \\ \frac{\pi}{2} & \text{if } x = 0, y \in \mathbb{R}^+ \\ \frac{3\pi}{2} + \arctan \frac{ay}{bx} & \text{if } x \in \mathbb{R}^-, y \in \mathbb{R}^+ \text{ (second quadrant)} \\ \pi + \arctan \frac{ay}{bx} & \text{if } x \in \mathbb{R}^-, y \in \mathbb{R}^-_0 \text{ (third quadrant)} \\ \frac{3\pi}{2} & \text{if } x = 0, y \in \mathbb{R}^- \\ \frac{4\pi}{2} + \arctan \frac{ay}{bx} & \text{if } x \in \mathbb{R}^+, y \in \mathbb{R}^- \text{ (forth quadrant)} \end{cases} \Big)$$

Now it follows that $\mathbf{x}$ is indeed a homeomorphism.

When u is fixed, the image is an oval missing a point $(a \sin u, 0, c \cos u)$ floating in air (contained by $\{(x,y,c_0) \in \mathbb{R}^3 : (x,y) \in \mathbb{R}^2\}$ where $c_0 = c \cos u$ is fixed) ∎

Definition 2.3.1. (Definition of regular plane curve) We say $C \subseteq \mathbb{R}^2$ is a regular plane curve if for all $p \in C$ there exists

(a) an open neighborhood $p \in V \subseteq \mathbb{R}^2$

(b) an open set $U \subseteq \mathbb{R}$

(c) a function $\mathbf{x} : U \to V \cap C$

such that $\mathbf{x}$ satisfy

(a) $\mathbf{x}$ is smooth

(b) $\mathbf{x}$ is a homoeomorphism between U and $V \cap C$

(c) $d\mathbf{x}_q \in L(\mathbb{R}, \mathbb{R}^2)$ is one-to-one for all $q \in U$

Definition 2.3.2. (Definition of regular space curve) We say $C \subseteq \mathbb{R}^3$ is a regular space curve if for all $p \in C$ there exists

(a) an open neighborhood $p \in V \subseteq \mathbb{R}^3$

(b) an open set $U \subseteq \mathbb{R}$

(c) a function $\mathbf{x} : U \to V \cap C$

such that $\mathbf{x}$ satisfy

(a) $\mathbf{x}$ is smooth

(b) $\mathbf{x}$ is a homoeomorphism between U and $V \cap C$

(c) $d\mathbf{x}_q \in L(\mathbb{R}, \mathbb{R}^3)$ is one-to-one for all $q \in U$

Question 30

17. Define a regular curve in analogy with a regular surface. Prove that

a. The inverse image of a regular value of a differentiable function

$$f: U \subset R^2 \to R$$

is a regular plane curve. Give an example of such a curve which is not connected.

b. The inverse image of a regular value of a differentiable map

$$F: U \subset R^3 \to R^2$$

is a regular curve in R^3. Show the relationship between this proposition and the classical way of defining a curve in R^3 as the intersection of two surfaces.

***c. The set $C = \{(x, y) \in R^2; x^2 = y^3\}$ is not a regular curve.**

Proof. **(a)**

Suppose $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a smooth function and c is a regular value. We wish to prove

$$C \triangleq f^{-1}[c] \text{ is a regular plane curve}$$

Fix $p \in f^{-1}[c]$. We wish

$$\text{to find a local parametrization } \mathbf{x} : I \subseteq \mathbb{R} \to C \text{ around } p$$

Because c is a regular value, we know df_p is one-to-one. Then, WOLG, we can let $\partial_y F(p) \neq c$. Define $F : U \to \mathbb{R}^2$ by

$$F(x,y) \triangleq \Big(x, f(x,y)\Big)$$

Compute

$$dF = \begin{bmatrix} 1 & 0 \\ \partial_x f & \partial_y f \end{bmatrix}$$

It is now clear that $\det(dF_p) \neq 0$. Now, because f is smooth, we can use inverse function Theorem and obtain a diffeoemorphism F between open neighborhood around p and open neighborhoood around $f(p)$. Now, note that $f[C] = \{c\}$. This tell us

$$F[C] \subseteq \{(x,c) \in \mathbb{R}^2 : x \in \mathbb{R}\}$$

we now claim

$$\mathbf{x}(u) \triangleq F^{-1}(u,c) \text{ is the desired local parametrization around } p$$

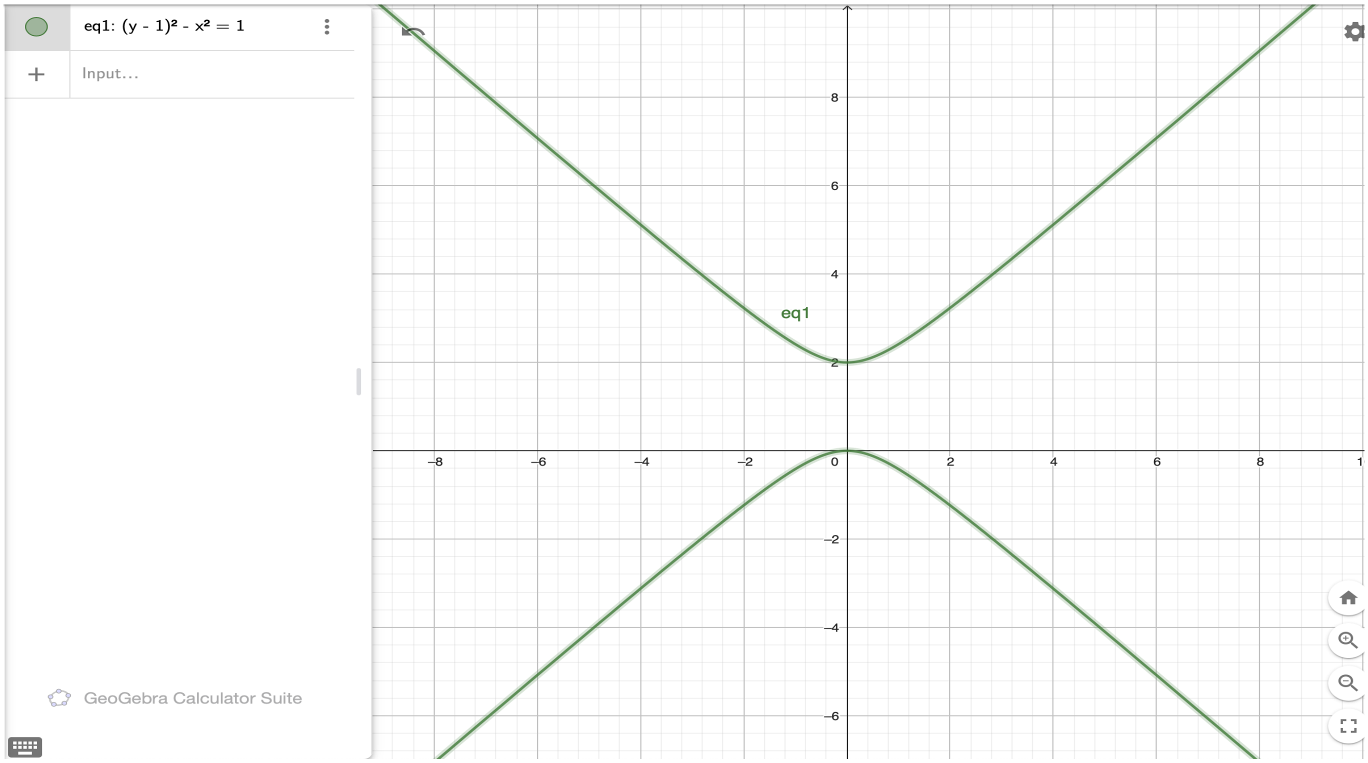
The fact that $\mathbf{x}$ is smooth and homeomorphism follows from

(a) F is a diffeomorphism around p

(b) $\mathbf{x}$ can be identified as restriction of F^{-1}

Note that $d(F^{-1})_p = \big(dF_{F^{-1}(p)}\big)^{-1} \neq 0$. Now, because $\mathbf{x}$ is restriction of F^{-1}, we see $d\mathbf{x}$ must not be 0 around p. (done)

An example is $f(x,y) = (y-1)^2 - x^2$.



(b)

Suppose $F : U \subseteq \mathbb{R}^3 \to \mathbb{R}^2$ is a smooth function and (c_0, c_1) is a regular value. We wish to prove

$$C \triangleq F^{-1}[(c_0, c_1)] \text{ is a regular space curve}$$

Fix $p \in F^{-1}[(c_0, c_1)]$. We wish

$$\text{to find a local parametrization } \mathbf{x} : I \subseteq \mathbb{R} \to C \text{ around } p$$

Define $G : U \to \mathbb{R}^3$ by

$$G(x, y, z) \triangleq \Big(x, F(x, y, z)\Big)$$

Compute

$$dG = \begin{bmatrix} 1 & 0 & 0 \\ \partial_x F_1 & \partial_y F_1 & \partial_z F_1 \\ \partial_x F_2 & \partial_y F_2 & \partial_z F_2 \end{bmatrix}$$

Because p is a regular point of F, we can WOLG, suppose

$$\det\big(dG_p\big) = \det\Big(\frac{\partial(F_1, F_2)}{\partial(y, z)}\Big|_p\Big) \neq 0$$

This Then, by Inverse function Theorem, G is locally a diffeomorphism around p. We now see

$$\mathbf{x}(t) \triangleq G^{-1}(t, c_0, c_1) \text{ is the desired local parametrization around } p \text{ (done)}$$

Suppose we are given two function $A, B : \mathbb{R}^3 \to \mathbb{R}$, and suppose $A^{-1}[c_0], B^{-1}[c_1]$ are two surfaces. Define $F : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$F(p) \triangleq (A(p), B(p))$$

We see that

$$\text{the intersection } A^{-1}[c_0] \cap B^{-1}[c_1] \text{ is exactly } F^{-1}[(c_0, c_1)]$$

(c)

Assume for a contradiction, C is a regular curve. Note that $(0,0) \in C$. We know there exists an open-neighborhood $N \subseteq \mathbb{R}^2$ around $(0,0)$ such that $N \cap C$ is the graph of some differentiable function in x or y. However, this is impossible, since if one view $N \cap C$ as a function in x, the function $y = x^{\frac{2}{3}}$ is not differetaible at $x = 0$, and one can not even view $N \cap C$ as a function in y as each y correspond to two x, namely $x = \pm y^{\frac{3}{2}}$. CaC

■

Question 31

2. Let $S \subset R^3$ be a regular surface and $\pi : S \to R^2$ be the map which takes each $p \in S$ into its orthogonal projection over $R^2 = \{(x, y, z) \in R^3; z = 0\}$. Is π differentiable?

Proof. Yes. Fix p in S. We wish to prove

$$\pi \text{ is differentiable at } p \text{ in the sense of manifold}$$

Let $\mathbf{x}_1 : U_1 \subseteq \mathbb{R}^2 \to V_1 \cap S \subseteq \mathbb{R}^3$ be a local parametrization around p. Define a local parametrization $\mathbf{x}_2 : U_2 \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ around $\pi(p)$ by

$$\mathbf{x}_2 \triangleq \mathbf{id}_{U_2}$$

We are require to prove

$$\mathbf{x}_2^{-1} \circ \pi \circ \mathbf{x}_1 \text{ is differentiable at } \mathbf{x}_1^{-1}(p)$$

Notice that

(a) $\mathbf{x}_1 : U_1 \to \mathbb{R}^3$ is differentiable at p by definition

(b) $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is clearly differntiable, with derivative $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(c) $\mathbf{x}_2^{-1} = \mathbf{id}_{U_2} : U_2 \to \mathbb{R}^2$ is clearly differntiable.

This shows that $\mathbf{x}_2^{-1} \circ \pi \circ \mathbf{x}_1$ is differentiable at p. (done) ∎

Question 32

3. Show that the paraboloid $z = x^2 + y^2$ is diffeomorphic to a plane.

Proof. Let S be the paraboloid. We show

$$S \text{ is diffeomorphic to } \{(x, y, 0) : x, y \in \mathbb{R}\}$$

Define

$$\pi : S \to \{(x, y, 0) : x, y \in \mathbb{R}\} \text{ by } \pi(x, y, z) \triangleq (x, y, 0)$$

We wish to show

$$\pi \text{ and } \pi^{-1} \text{ is differentiable everywhere in the sense of manifold}$$

Define global parametrizaitons $\mathbf{x}_1$ of S and global parametrization $\mathbf{x}_2$ of $\{(x, y, 0) : x, y \in \mathbb{R}\}$ by

(a) $\mathbf{x}_1 : \mathbb{R}^2 \to S$ and $\mathbf{x}_1(x, y) \triangleq (x, y, x^2 + y^2)$

(b) $\mathbf{x}_2 : \mathbb{R}^2 \to \{(x, y, 0) : x, y \in \mathbb{R}\}$ and $\mathbf{x}_2 \triangleq \mathbf{id}_{\mathbb{R}^2}$

We now reword the problem into proving

$$\mathbf{x}_2^{-1} \circ \pi \circ \mathbf{x}_1 : \mathbb{R}^2 \to \mathbb{R}^2 \text{ and } \mathbf{x}_1^{-1} \circ \pi^{-1} \circ \mathbf{x}_2 : \mathbb{R}^2 \to \mathbb{R}^2 \text{ both are differentiable}$$

Because $\pi, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_2^{-1}$ are clearly differentiable, we only have to prove

$\pi^{-1} : \{(x, y, 0) : x, y \in \mathbb{R}\} \to \mathbb{R}^3$ is differentiable and $\mathbf{x}_1^{-1} : \mathbb{R}^3 \cap S \to \mathbb{R}^2$ is differnetiable on S

Observe

$$\pi^{-1}(x, y, 0) \equiv (x, y, x^2 + y^2) \text{ and } \mathbf{x}_1(x, y, z) \equiv (x, y)$$

It is now clear that π^{-1} and $\mathbf{x}_1^{-1}$ are both differentiable. (done) ∎

Question 33

6. Prove that the definition of a differentiable map between surfaces does not depend on the parametrizations chosen.

Proof. Suppose we are given a map $\pi : S_1 \to S_2$ differentiable in the sense of manifold. We know for arbitrary p in S_1, there exists

(a) $\mathbf{x}_1 : U_1 \to S_1$ (a local parametrization around p)

(b) $\mathbf{x}_2 : U_2 \to S_2$ (a local parametrization around $\pi(p)$)

such that

$$\left(\mathbf{x}_2^{-1} \circ \pi \circ \mathbf{x}_1\right) : U_1 \subseteq \mathbb{R}^2 \to U_2 \subseteq \mathbb{R}^2 \text{ is a diffeomorphism} \tag{2.8}$$

Now, fix two arbitrary

(a) $\mathbf{x}_1' : U_1' \to S_1$ (a local parametrization around p)

(b) $\mathbf{x}_2' : U_2' \to S_2$ (a local parametrization around $\pi(p)$)

We are required to prove (Note that the domain of each composited function may be smaller, but this does not undermine the validity of our argument, since we only care about the differentiablity at p)

$$\left((\mathbf{x}_2')^{-1} \circ \pi \circ \mathbf{x}_1'\right) : U_1' \subseteq \mathbb{R}^2 \to U_2' \subseteq \mathbb{R}^2 \text{ is a diffeomorphism}$$

Note that

$$\begin{aligned}(\mathbf{x}_2')^{-1} \circ \pi \circ \mathbf{x}_1' &= (\mathbf{x}_2')^{-1} \circ \left(\mathbf{x}_2 \circ \mathbf{x}_2^{-1}\right) \circ \pi \circ \left(\mathbf{x}_1 \circ \mathbf{x}_1^{-1}\right) \circ \mathbf{x}_1' \\ &= (\mathbf{x}_2')^{-1} \circ \mathbf{x}_2 \circ \mathbf{x}_2^{-1} \circ \pi \circ \mathbf{x}_1 \circ \mathbf{x}_1^{-1} \circ \mathbf{x}_1' \\ &= h_2 \circ \mathbf{x}_2^{-1} \circ \pi \circ \mathbf{x}_1 \circ h_1\end{aligned}$$

where

$$\begin{cases} h_1 \equiv \mathbf{x}_1^{-1} \circ \mathbf{x}_1' : U_1' \to U_1 \\ h_2 \equiv (\mathbf{x}_2')^{-1} \circ \mathbf{x}_2 : U_2 \to U_2' \end{cases} \quad \text{are changes of coordinate}$$

Now, because changes of coordinates are diffeomorhphism, and $\mathbf{x}_2^{-1} \circ \pi \circ \mathbf{x}_1$ is diffeomorphism by Equation 2.8 (definition), we see that

$$(\mathbf{x}_2')^{-1} \circ \pi \circ \mathbf{x}_1' = h_2 \circ \left(\mathbf{x}_2^{-1} \circ \pi \circ \mathbf{x}_1\right) \circ h_1 \text{ is a diffeomorphism at } (\mathbf{x}_1')^{-1}(p)$$

This then conclude the proof, since p is arbitrary picked from S_1. (done) ∎

Definition 2.3.3. (Definition of Differentiable function on a regular curve) Given two regular curve C_1, C_2, we say the function $f : C_1 \to C_2$ is differentiable at p if for all local parametrizations $\mathbf{x}_1 : I \to C_1 \ni p, \mathbf{x}_2 : I \to C_2 \ni f(p)$, we have

$$\mathbf{x}_2^{-1} \circ f \circ \mathbf{x}_1 \text{ is differentaible as a real to real function}$$

Question 34

9. a. **Define the notion of differentiable function on a regular curve. What does one need to prove for the definition to make sense? Do not prove it now. If you have not omitted the proofs in this section, you will be asked to do it in Exercise 15.**

b. **Show that the map $E\colon R \to S^1 = \{(x, y) \in R^2; x^2 + y^2 = 1\}$ given by**

$$E(t) = (\cos t, \sin t), \quad t \in R,$$

is differentiable (geometrically, E "wraps" R around S^1).

Proof. Fix $t_0 \in \mathbb{R}$. We wish to prove

$$E \text{ is differentaible at } t_0 \text{ in the sense of manifold}$$

Locally parametize t_0 and $E(t_0)$ by

$$\mathbf{x}_1(t) \triangleq t \text{ and } \mathbf{x}_2(t) \triangleq (\cos t, \sin t)$$

Now, see that

$$\mathbf{x}_2^{-1} \circ E \circ \mathbf{x}_1(t) = t \text{ is clearly differentiable} \quad \text{(done)}$$

∎

Question 35

14. Let $A \subset S$ be a subset of a regular surface S. Prove that A is itself a regular surface if and only if A is open in S; that is, $A = U \cap S$, where U is an open set in R^3.

Proof. $(\longleftarrow)$

Fix $a \in A$. Because $a \in S$, we know there exists a local parametrization $\mathbf{x}_1 : E \to V \cap S$ around a where V is open in . Suppose

$$E' \triangleq \mathbf{x}_1^{-1}[U \cap V \cap S]$$

Now, from

(a) $U \cap V \cap S$ is open in $V \cap S$ $\qquad$ ($\because U$ is open in $\mathbb{R}^3$)

(b) $\mathbf{x}_1$ is a homeomorphism between E and $V \cap S$

(c) $\mathbf{x}_1$ is smooth

(d) $d\mathbf{x}_1$ is one-to-one for all $p \in E$

(e) $E' \subseteq E$

We see

(a) E' is open in $\mathbb{R}^2$

(b) the restriction $\mathbf{x}_1|_{E'}$ is a local parametrizaiton around a contained by A

Because a is arbitrary, this established that A is a regular surface.

$(\longrightarrow)$

Suppose A is a regular surface. Fix arbitrary $a \in A$. Using Proposition 2.2.3 in Do Carmo, WOLG, we can suppose there exists a chart $\mathbf{x} : U \to \overline{V} \cap S$ around a where $\overline{V}$ is open in $\mathbb{R}^3$ such that

$$\mathbf{x}(x, y) = (x, y, f(x, y)) \text{ for some smooth } f$$

Because each curve in A lies in S and A is itself a regular surface, we see that

$$T_a(A) = T_a(S)$$

This tell us the restriction

$$\mathbf{x}|_A \text{ is one-to-one}$$

Note that $\mathbf{x}$ is smooth. Now by inverse function theorem, it follows that there exists V open in $\mathbb{R}^3$ such that

$$\mathbf{x}|_{A \cap V} \text{ is a chart around } a$$

Now, define $W \triangleq V \cap \overline{V}$. Now, identify

$$\mathbf{x}|_{A\cap W} \text{ is a chart in both } A \text{ and } S$$

We now see

$$A \cap W \text{ is an open neighborhood around } a \text{ in topology of } S$$

This shows that $a \in A^\circ$ in topology of S. Because a is arbitrary, it follows that A is open in S. ∎

Question 36

***16.** Let $R^2 = \{(x, y, z) \in R^3; z = -1\}$ be identified with the complex plane $\mathbb{C}$ by setting $(x, y, -1) = x + iy = \zeta \in \mathbb{C}$. Let $P: \mathbb{C} \to \mathbb{C}$ be the complex polynomial

$$P(\zeta) = a_0\zeta^n + a_1\zeta^{n-1} + \cdots + a_n, \quad a_0 \neq 0, a_i \in \mathbb{C}, i = 0, \dots, n.$$

Denote by π_N the stereographic projection of $S^2 = \{(x, y, z) \in R^3; x^2 + y^2 + z^2 = 1\}$ from the north pole $N = (0, 0, 1)$ onto R^2. Prove that the map $F: S^2 \to S^2$ given by

$$\begin{aligned} &F(p) = \pi_N^{-1} \circ P \circ \pi_N(p), \quad \text{if } p \in S^2 - \{N\}, \\ &F(N) = N \end{aligned}$$

is differentiable.

Proof. Note that

$$\pi_N(x,y,z) = \left(\frac{2x}{1-z}, \frac{2y}{1-z}\right) \text{ and } \pi_N^{-1}(u,v) = \left(\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{u^2+v^2-4}{u^2+v^2+4}\right)$$

and that

$$P(x,y,-1) = \Big(Q(x,y), T(x,y), -1\Big) \text{ for some } Q, T \in \mathbb{R}[x,y]$$

It is then now clear that

$$F \text{ is differentiable on } S^2 \setminus N$$

Note that

$$\pi_S(x,y,z) = (\frac{2x}{1+z}, \frac{2y}{1+z}) \text{ and } \pi_S^{-1}(u,v) = (\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{-u^2-v^2+4}{u^2+v^2+4})$$

Note that

$$\pi_S^{-1} \circ \pi_S \circ F \circ \pi_S^{-1} \circ \pi_S \text{ coincide with } F \text{ on } S^2 \setminus S$$

Then, we can reduce proving F is differentiable at N into

$$\text{proving } \pi_S^{-1} \circ \pi_S \circ F \circ \pi_S^{-1} \circ \pi_S \text{ is differentiable at } N$$

Compute

$$\begin{aligned}\pi_N \circ \pi_S^{-1}(u,v) &= \pi_N(\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{-u^2-v^2+4}{u^2+v^2+4}) \\ &= (\frac{4u}{u^2+v^2}, \frac{4v}{u^2+v^2})\end{aligned}$$

Compute

$$\pi_S \circ \pi_N^{-1}(u,v) = (\frac{4u}{u^2+v^2}, \frac{4v}{u^2+v^2})$$

Note that if we identify $\mathbb{R}^2 \setminus 0$ by $\mathbb{C}^*$, we see

$$\pi_S \circ \pi_N^{-1}(z) = \pi_N \circ \pi_S^{-1}(z) = \frac{4z}{|z^2|} = \frac{4z}{z\overline{z}} = \frac{4}{\overline{z}} \qquad (z \in \mathbb{C}^*)$$

Fix $z \in \mathbb{C}^*$. Compute

$$\begin{aligned}\pi_S \circ F \circ \pi_S^{-1}(z) &= \pi_S \circ \pi_N^{-1} \circ P \circ \pi_N \circ \pi_S^{-1}(z) \\ &= \pi_S \circ \pi_N^{-1} \circ P(\frac{4}{\overline{z}}) \\ &= \pi_S \circ \pi_N^{-1}\Big(a_0(\frac{4}{\overline{z}})^n + \cdots + a_n\Big) \\ &= \frac{4}{\overline{a_0(\frac{4}{\overline{z}})^n + \cdots + a_n}} \\ &= \frac{4}{\overline{a_0}(\frac{4}{z})^n + \cdots + \overline{a_n}} \\ &= \frac{4z^n}{\overline{a_n}z^n + \cdots + \overline{a_0}4^n}\end{aligned}$$

Compute

$$\pi_S \circ F \circ \pi_S^{-1}(0) = \pi_S \circ F(N) = \pi_S(N) = 0$$

Then because $a_0 \neq 0$. It is now clear that

$$\pi_S \circ F \circ \pi_S^{-1}(z) = \frac{4z^n}{\overline{a_n} z^n + \cdots + \overline{a_0} 4^n} \qquad (z \in \mathbb{C})$$

which is differentiable at 0. Now, we have

(a) $\pi_S : S^2 \setminus S \to \mathbb{C}$ is differentiable on $S^2 \setminus S$

(b) $\pi_S \circ F \circ \pi_S^{-1} : \mathbb{C} \to \mathbb{C}$ is differnetiable on $\mathbb{C}$

(c) $\pi_S^{-1} : \mathbb{C} \to S^2 \setminus S$ is differntiable on $\mathbb{C}$

We can deduce

$$\pi_S^{-1} \circ \left(\pi_S \circ F \circ \pi_S^{-1} \right) \circ \pi_S \text{ is differentiable on } S^2 \setminus S \text{ (done)}$$

Note that we used Theorem 2.3.4 in our proof. ∎

Theorem 2.3.4. (Composition of Differentiable functions is differentiable) Given three regular surfaces $\{S_1, S_2, S_3\}$, two differentiable functions $f_1 : S_1 \to S_2$ and $f_2 : S_2 \to S_3$, we see

$$f_2 \circ f_1 \text{ is differentiable on } S_1$$

Proof. Fix $p_1 \in S_1$. We wish to prove

$$f_2 \circ f_1 \text{ is differentiable at } p_1$$

Set

(a) $p_2 \triangleq f_1(p_1)$

(b) $p_3 \triangleq f_2(p_2)$

Let

(a) $\mathbf{x}_1 : U_1 \to V_1 \cap S_1 \ni p_1$ be a local parametrzaiton

(b) $\mathbf{x}_2 : U_2 \to V_2 \cap S_2 \ni p_2$ be a local parametrizaiton

(c) $\mathbf{x}_3 : U_3 \to V_3 \cap S_3 \ni p_3$ be a local parametrizaiton

We wish to prove

$$\mathbf{x}_3^{-1} \circ f_2 \circ f_1 \circ \mathbf{x}_1 \text{ is differentiable at } p_1$$

Observe that

$$\begin{aligned} \mathbf{x}_3^{-1} \circ f_2 \circ f_1 \circ \mathbf{x}_1 &= \mathbf{x}_3^{-1} \circ f_2 \circ \mathbf{x}_2 \circ \mathbf{x}_2^{-1} \circ f_1 \circ \mathbf{x}_1 \\ &= \left(\mathbf{x}_3^{-1} \circ f_2 \circ \mathbf{x}_2\right) \circ \left(\mathbf{x}_2^{-1} \circ f_1 \circ \mathbf{x}_1\right) \text{ is differentiable } \text{ (done)} \end{aligned}$$

■

Chapter 3

Surface

3.1 Prerequisite

Theorem 3.1.1. (Inverse function Theorem) Given a map f from E open in $\mathbb{R}^n$ to $\mathbb{R}^n$, if

(a) f is continuously differentiable on E

(b) df_a is one-to-one

(c) $a \in E$

Then there exists

(a) $U \subseteq E$ open in E where $a \in U$

(b) $V \subseteq \mathbb{R}^n$ open in $\mathbb{R}^n$

Such that

$$f|_U : U \to V \text{ is a diffeomorphism}$$

Theorem 3.1.2. (Implicit function Theorem) Given a map f from an open neighborhood around $(a, b) \in E \subseteq \mathbb{R}^{n+m} \to \mathbb{R}^n$ such that

(a) f is continuously differentiable on E

(b) the linear transformation $df_{(a,b)}|_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one

(c) $f(a, b) = 0$

Then there exists open neighborhood U around $(a, b) \in U \subseteq \mathbb{R}^{n+m}$ and open neighborhood V around $b \in V \subseteq \mathbb{R}^m$ such that we can uniquely define a function $g : V \to U$ by

$$f(g(y), y) = 0 \text{ for all } y \in V$$

and g is continuously differentiable with $dg_b = -(df_{(a,b)}|_{\mathbb{R}^n})^{-1} \circ df_{(a,b)}|_{\mathbb{R}^m}$

3.2 Equivalent Definition of Regular Surface

Definition 3.2.1. (Definition of Regular Surface: Local Parametrization) We say a set $S \subseteq \mathbb{R}^3$ is a regular surface if for all $p \in S$ there exists

(a) an open neighborhood $p \in V \subseteq \mathbb{R}^3$

(b) an open set $U \subseteq \mathbb{R}^2$

(c) a function $\mathbf{x} : U \to V \cap S$

such that $\mathbf{x}$ satisfy

(a) $\mathbf{x}$ is smooth

(b) $\mathbf{x}$ is a homemomorphism between U and $V \cap S$

(c) $d\mathbf{x}_q \in L(\mathbb{R}^2, \mathbb{R}^3)$ is one-to-one for all $q \in U$

Definition 3.2.2. (Definition of Regular Surface: Implicit function) We say a set $S \subseteq \mathbb{R}^3$ is a regular surface if for all $p \in S$ there exists

(a) an open neighborhood $p \in V \subseteq \mathbb{R}^3$

(b) a function $F : V \to \mathbb{R}$

such that F satisfy

(a) dF_q is onto for all $q \in V \cap S$

(b) F is smooth on V

(c) $\exists c_0 \in \mathbb{R}, V \cap S = \Big\{(x, y, z) \in V : F(x, y, z) = c_0\Big\}$

We now verify the equivalency between the two definitions.

Theorem 3.2.3. (Implicit function definition $\longrightarrow$ Local parametrization definition)

Proof. Fix $p \in S$. We are given an open neighborhood $p \in V \subseteq \mathbb{R}^3$ and a function $F : V \to \mathbb{R}$ such that, WOLG,

(a) dF_q is onto for all $q \in V \cap S$

(b) F is smooth on V

(c) $V \cap S = \Big\{(x, y, z) \in V : F(x, y, z) = 0\Big\}$

We wish to find

$$\text{a local parametrization } \mathbf{x} \text{ around } p$$

Define $f : V \to \mathbb{R}^3$ by

$$f(x, y, z) = (x, y, F(x, y, z))$$

Because $dF_p \neq 0$, WOLG, we can suppose $\partial_z F(p) \neq 0$. Now, see

$$\det(df) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \partial_x F & \partial_y F & \partial_z F \end{vmatrix} \neq 0$$

Now, note that

(a) f is smooth on V $\quad (\because\ F$ is smooth on $V\ )$

(b) df_p is one-to-one $\quad (\because \partial_z F(p) \neq 0)$

Then, by inverse function Theorem, we know

$$f\big|_{V'} \to U' \text{ is a local diffeomorphism around } V' \ni p \text{ and around } U' \ni f(p)$$

Let $U \subseteq \{(x, y, 0) : (x, y) \in \mathbb{R}^2\}$ be an open-neighborhood around $f(p)$ contained by U'. We claim

$$\mathbf{x} \triangleq f^{-1}\big|_U \text{ suffices}$$

Note that $\mathbf{x}$ is well-defined since $U \subseteq U'$ and f is a bijective between V' and U'.

Also, note that $\mathbf{x}$ do maps points in U to points in $V \cap S$, since $V \cap S = \Big\{(x, y, z) \in V : F(x, y, z) = 0\Big\}$.

Suppose that

$$\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} \end{bmatrix} \triangleq df_\alpha^{-1} = \left(df_{f^{-1}(\alpha)}\right)^{-1} \text{ for all } \alpha \in U$$

We clearly have

$$d\mathbf{x} = \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \\ \alpha_{3,1} & \alpha_{3,2} \end{bmatrix}$$

Now, one can use the premise

$$dF_q \text{ is onto for all } q \in V \cap S$$

to check $\mathbf{x}$ do satisfy the regular condition. (Compute $\det(df_\alpha^{-1})$ using co-factor formula on the third column) (done) ■

Definition 3.2.4. (Definition of Regular surface: Monge Patches) We say a set $S \subseteq \mathbb{R}^3$ is a regular surface if for all $p \in S$ there exists some open neighborhood $p \in V \subseteq \mathbb{R}^3$ such that

> $V \cap S$ can be expressed as the graph of some smooth function $f : O \subseteq \mathbb{R}^2 \to \mathbb{R}$ in the sense that one of the followings hold

(a) $V \cap S = \{(x, y, f) : (x, y) \in O\}$ for some smooth $f : O \subseteq \mathbb{R}^2 \to \mathbb{R}$

(b) $V \cap S = \{(x, f, z) : (x, z) \in O\}$ for some smooth $f : O \subseteq \mathbb{R}^2 \to \mathbb{R}$

(c) $V \cap S = \{(f, y, z) : (y, z) \in O\}$ for some smooth $f : O \subseteq \mathbb{R}^2 \to \mathbb{R}$

3.3 Differentiable Mapping Between Regular Surfaces

Theorem 3.3.1. (Change of Parameter is a diffeomorphism) Let p be a point of a regular surface S, and let $\mathbf{x}_1 : U_1 \to S$ and $\mathbf{x}_2 : U_2 \to S$ be two charts containing p. Define $W \triangleq \mathbf{x}_1(U_1) \cap \mathbf{x}_2(U_2) \cap S$. Then

The **change of coordinate** $h \triangleq \mathbf{x}_2^{-1} \circ \mathbf{x}_1 : \mathbf{x}_1^{-1}(W) \to \mathbf{x}_2^{-1}(W)$ is a diffoeomorphism

Proof. Note the symmetry of h and h^{-1}. WOLG, we only have to prove

$$h : \mathbf{x}_1^{-1}(W) \to \mathbf{x}_2^{-1}(W) \text{ is differentiable}$$

Fix $r \in \mathbf{x}_1^{-1}(W)$. We prove

$$h \text{ is differentiable at } r$$

Define $q \triangleq h(r)$. Express $\mathbf{x}_2$ by

$$\mathbf{x}_2(u,v) = (x,y,z)$$

Because $(d\mathbf{x}_2)_q$ is one-to-one, WOLG, we have

$$\begin{vmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \end{vmatrix} (q) \neq 0$$

Now, define $F : \mathbf{x}_2^{-1}(W) \times \mathbb{R} \to \mathbb{R}^3$ by

$$F(u,v,t) \triangleq \Big(x(u,v), y(u,v), z(u,v) + t \Big) \tag{3.1}$$

Compute

$$\det(dF_{(q,0)}) = \begin{vmatrix} \partial_u x & \partial_v x & 0 \\ \partial_u y & \partial_v y & 0 \\ \partial_u z & \partial_v z & 1 \end{vmatrix} (q,0) \neq 0$$

Then by Inverse function Theorem, we see that there exists an open neighborhood $M \subseteq \mathbb{R}^3$ around $F(q,0)$ such that F^{-1} exists and is differentiable on M.

Now, from Equation 3.1, note that

$$F(u,v,0) = \mathbf{x}_2(u,v)$$

Recall the definition h, we now have

$$h = \mathbf{x}_2^{-1} \circ \mathbf{x}_1 = F^{-1} \circ \mathbf{x}_1 \text{ on } \mathbf{x}_1^{-1}(M)$$

Then because

(a) $\mathbf{x}_1$ is differentiable at r

(b) F^{-1} is differentiable at $F(q,0) = \mathbf{x}_2(q) = \mathbf{x}_1(r)$

We see h is indeed differentiable at r (done) ■

Now, given two regular surfaces S_1, S_2, a set $E \subseteq S_1$ open in S_1, and a function $f : E \to S_2$, we can say f is **differentiable at** $p \in E$ if there exists some local charts $\mathbf{x}_1 : U_1 \to S_1, \mathbf{x}_2 : U_2 \to S_2$, containing $p, f(p)$ such that

(a) there exists ϵ small enough such that $B_\epsilon(\mathbf{x}_1^{-1}(p)) \subseteq U_1$ and $\mathbf{x}_2^{-1} \circ f \circ \mathbf{x}_1$ is defined on $B_\epsilon(\mathbf{x}_1^{-1}(p))$

(b) $\mathbf{x}_2^{-1} \circ f \circ \mathbf{x}_1 : B_\epsilon(\mathbf{x}_1^{-1}(p)) \to U_2$ is differentiable at $\mathbf{x}_1^{-1}(p)$

One can check that if $f : E \to S_2$ is differetnaible at p, then $f : E \to S_2$ must be continuous at p. Also, using Theorem 3.3.1, one can check that if f is differnetiable at p, then every local charts $\mathbf{x}_3 : U_3 \to S_1, \mathbf{x}_4 : U_4 \to S_2$ containing $p, f(p)$ satisfy

$$\mathbf{x}_4^{-1} \circ f \circ \mathbf{x}_3 \text{ is differentiable at } \mathbf{x}_3^{-1}(p)$$

Thus, if we wish to check whether some function $f : E \to S_2$ is differentiable at p, every pair of local charts respectively containing $p, f(p)$ can be used.

Theorem 3.3.2. (Composition of Differentiable functions is differentiable) Given three regular surfaces S_1, S_2, S_3, two differentiable functions $f_1 : S_1 \to S_2$ and $f_2 : S_2 \to S_3$, we see

$$f_2 \circ f_1 \text{ is differentiable on } S_1$$

Proof. Fix $p_1 \in S_1$. We wish to prove

$$f_2 \circ f_1 \text{ is differentiable at } p_1$$

Set

$$p_2 \triangleq f_1(p_1) \text{ and } p_3 \triangleq f_2(p_2)$$

Let

(a) $\mathbf{x}_1 : U_1 \to S_1 \ni p_1$ be a local chart

(b) $\mathbf{x}_2 : U_2 \to S_2 \ni p_2$ be a local chart

(c) $\mathbf{x}_3 : U_3 \to S_3 \ni p_3$ be a local chart

We wish to prove

$$\mathbf{x}_3^{-1} \circ f_2 \circ f_1 \circ \mathbf{x}_1 \text{ is differentiable at } p_1$$

Observe that

$$\begin{aligned}
\mathbf{x}_3^{-1} \circ f_2 \circ f_1 \circ \mathbf{x}_1 &= \mathbf{x}_3^{-1} \circ f_2 \circ \mathbf{x}_2 \circ \mathbf{x}_2^{-1} \circ f_1 \circ \mathbf{x}_1 \\
&= \left(\mathbf{x}_3^{-1} \circ f_2 \circ \mathbf{x}_2\right) \circ \left(\mathbf{x}_2^{-1} \circ f_1 \circ \mathbf{x}_1\right) \text{ is differentiable } \ \text{(done)}
\end{aligned}$$

■

3.4 Equivalent Definition of Tangent Plane

Definition 3.4.1. (Definition of Tangent Plane: Space of Tangent Vectors) Given a regular surface S and $p \in S$, we define **tangent plane** T_pS **to** S **at** p by

$$T_pS = \{\alpha'(0) \in \mathbb{R}^3 | \ \alpha : I \to S \text{ is a smooth curve passing through } p \text{ at } \alpha(0) \ \}$$

Definition 3.4.2. (Definition of Tangent Plane: Local Parametrization) Given a regular surface S, that $p \in S$ and a local parametrzaiton $\mathbf{x} : (u, v) \mapsto (x, y, z)$ around p such that $\mathbf{x}(q) = p$, we define **tangent plane** T_pS **to** S **at** p by

$$T_pS = d\mathbf{x}(\mathbb{R}^2)$$

Theorem 3.4.3. (Equivalence of Definitions) Given a regular surface S, $p \in S$ and a local chart $\mathbf{x} : U \to S$ containing p such that $\mathbf{x}(q) = p$, we have

$$d\mathbf{x}_q(\mathbb{R}^2) = \{\alpha'(0) \in \mathbb{R}^3 | \alpha : (-\epsilon, \epsilon) \to S \text{ satisfy } \alpha(0) = p\}$$

Proof. We first prove

$$d\mathbf{x}_q(\mathbb{R}^2) \subseteq \{\alpha'(0) \in \mathbb{R}^3 | \alpha : (-\epsilon, \epsilon) \to S \text{ satisfy } \alpha(0) = p\}$$

Fix $w \in d\mathbf{x}_q(\mathbb{R}^2)$. We reduce the problem into

$$\text{constructing smooth } \alpha : (-\epsilon, \epsilon) \to S \text{ such that } \alpha(0) = p \text{ and } \alpha'(0) = w$$

We know there exists $v \in \mathbb{R}^2$ such that $d\mathbf{x}_q(v) = w$. Define $\gamma : (-\epsilon, \epsilon) \to U$ by

$$\gamma(t) = tv + q$$

We claim

$$\alpha = \mathbf{x} \circ \gamma : (-\epsilon, \epsilon) \to S \text{ suffices}$$

Note that $\mathbf{x} \circ \gamma(0) = \mathbf{x}(q) = p$ and

$$(\mathbf{x} \circ \gamma)'(0) = d\mathbf{x}_q(\gamma'(0)) = d\mathbf{x}_q(v) = w \text{ (done)}$$

We now prove

$$\{\alpha'(0) \in \mathbb{R}^3 | \alpha : (-\epsilon, \epsilon) \to S \text{ satisfy } \alpha(0) = p\} \subseteq d\mathbf{x}_q(\mathbb{R}^2)$$

Fix a smooth curve $\alpha : (-\epsilon, \epsilon) \to S$ passing through p at $\alpha(0)$. We wish

$$\text{to show } \alpha'(0) \in d\mathbf{x}_q(\mathbb{R}^2)$$

Define $\beta : (-\epsilon, \epsilon) \to U$ by

$$\beta(t) = \mathbf{x}^{-1} \circ \alpha(t)$$

We claim

$$\alpha'(0) = d\mathbf{x}_q(\beta'(0))$$

Compute using Chain Rule

$$\beta'(0) = (d\mathbf{x}^{-1})_p \circ \alpha'(0)$$

Note that $(d\mathbf{x}^{-1})_p = (d\mathbf{x}_q)^{-1}$. This then give us

$$d\mathbf{x}_q(\beta'(0)) = d\mathbf{x}_q \circ (d\mathbf{x}^{-1})_p \circ \alpha'(0) = \alpha'(0) \text{ (done)}$$

■

Now, given a differentiable function f between two regular surface S_1, S_2, we can define the **derivative** df of f

Definition 3.4.4. (Derivative of Differntiable mapping between regular surface) Given a differentiable mapping $f : S_1 \to S_2$, we define $df_p : T_pS_1 \to T_{f(p)}S_2$ by

$$df_p(\alpha'(0)) \triangleq (f \circ \alpha)'(0)$$

Theorem 3.4.5. (Derivative is well-defined and linear)

Proof. Fix local charts $\mathbf{x}(u, v)$ containing p and $\overline{\mathbf{x}}(\overline{u}, \overline{v})$ containing $f(p)$. Express f locally in coordinate

$$f(u, v) = (f_1(u, v), f_2(u, v))$$

Express $\alpha(t)$ in the form

$$\alpha(t) = (u(t), v(t))$$

We now have

$$(f \circ \alpha)(t) = (f_1(u, v), f_2(u, v))(t)$$

Compute

$$(f \circ \alpha)'(0) = \Big((\frac{\partial f_1}{\partial u}, \frac{\partial f_1}{\partial v})(p) \cdot \alpha'(0), (\frac{\partial f_2}{\partial u}, \frac{\partial f_2}{\partial v})(p) \cdot \alpha'(0) \Big) \tag{3.2}$$

This shows that $(f \circ \alpha)'(0)$ is only determined by $\alpha'(0)$ and f. Thus, our definition is well defined, and from (3.2), it is easily checked that df_p is linear. ■

3.5 First Fundamental Form

Definition 3.5.1. (First Fundamental Form) Given a regular surface S and $p \in S$, by **the first fundamental form of** S **at** p, we mean a function $I_p : T_p(S) \to \mathbb{R}$ defined by

$$I_p(w) \triangleq w \cdot w$$

Given a local chart $\mathbf{x}(u,v) : U \to S$, we can define three real valued- function $E, F, G : U \to \mathbb{R}$ by

$$\begin{aligned} E(u,v) &\triangleq |\partial_u \mathbf{x}(u,v)|^2 \\ F(u,v) &\triangleq \partial_u \mathbf{x}(u,v) \cdot \partial_v \mathbf{x}(u,v) \\ G(u,v) &\triangleq |\partial_v \mathbf{x}(u,v)|^2 \end{aligned}$$

These three functions help us compute the arc-length of a smooth curve that lies in $\mathbf{x}(U)$. Suppose $\alpha(t)$ lies in $\mathbf{x}(U)$, and express

$$\alpha(t) = \mathbf{x}(u(t), v(t))$$

Note that the expression give us

$$\alpha'(t) = d\mathbf{x}(u'(t), v'(t)) = u'(t)\partial_u \mathbf{x}(u(t), v(t)) + v'(t)\partial_v \mathbf{x}(u(t), v(t)) \text{ where } q = \alpha(t)$$

We see that

$$\begin{aligned} \int_0^r |\alpha'(t)|\, dt &= \int_0^r \sqrt{\alpha'(t) \cdot \alpha'(t)} dt \\ &= \int_0^r \sqrt{\big(u'(t)\partial_u \mathbf{x}(q) + v'(t)\partial_v \mathbf{x}(q)\big) \cdot \big(u'(t)\partial_u \mathbf{x}(q) + v'(t)\partial_v \mathbf{x}(q)\big)} dt \\ &= \int_0^r \sqrt{E(q)\big(u'(t)\big)^2 + 2F(q)u'(t)v'(t) + G(q)\big(v'(t)\big)^2} dt \end{aligned}$$

These three functions also help us define and compute the area of some bounded region Q that lies in $\mathbf{x}(U)$. Suppose we define the area of Q by

$$A(Q) = \iint_Q |\mathbf{x}_u \times \mathbf{x}_v|\, dudv$$

One can rigorously by computation in component to justify the following

$$|v \times w|^2 + (v \cdot w)^2 = (\sin^2\theta + \cos^2\theta)\, |v|^2 |w|^2 = |v|^2 |w|^2$$

This then give us

$$\iint_Q |\mathbf{x}_u \times \mathbf{x}_v|\, dudv = \iint_Q \sqrt{|\mathbf{x}_u|^2 |\mathbf{x}_v|^2 - (\mathbf{x}_u \cdot \mathbf{x}_v)^2} dudv$$
$$= \iint_Q \sqrt{EG - F^2} dudv$$

Also, note that the angle of two coordinates can also be computed by

$$\cos\theta = \frac{\mathbf{x}_u \cdot \mathbf{x}_v}{|\mathbf{x}_u|\,|\mathbf{x}_v|} = \frac{F}{\sqrt{EG}}$$

!!!!! Show that given a stereographical projection $\mathbf{x} : \mathbb{R}^2 \times 0 \to S^2 \setminus N$ preserve angle. Prove that given $v, w \in \mathbb{R}^2$ starting at $p \in \mathbb{R}^2$, we have

$$\frac{d\mathbf{x}_q(v) \cdot d\mathbf{x}_q(w)}{|d\mathbf{x}_q(v)|\,|d\mathbf{x}_q(w)|} = v \cdot w$$

So, we can show that $\mathbf{x} : \mathbb{R}^2 \to S^2 \setminus N$ satisfy

$$d\mathbf{x}(v) \cdot d\mathbf{x}(w) = C(v \cdot w) \text{ where } C \text{ is a constant}$$

is conformal (preserving angle).

3.6 HW4

Question 37

3. Show that the equation of the tangent plane of a surface which is the graph of a differentiable function $z = f(x, y)$, at the point $p_0 = (x_0, y_0)$, is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Recall the definition of the differential df of a function $f\colon R^2 \to R$ and show that the tangent plane is the graph of the differential df_p.

Proof. The question first ask us to prove

$$T_{p_0}(S) = \{(x, y, z) \in \mathbb{R}^3 : z = f(x_0, y_0) + (x - x_0)\partial_x f(p_0) + (y - y_0)\partial_y f(p_0)\}$$

By premise of the question, we know there exists a global chart $\mathbf{x}$

$$\mathbf{x}(x, y) \triangleq (x, y, f(x, y))$$

Compute

$$d\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \partial_x f & \partial_y f \end{bmatrix}$$

This tell us

$$\begin{aligned} T_{p_0}(S) &= (x_0, y_0, f(x_0, y_0)) + \text{span}(\begin{bmatrix} 1 \\ 0 \\ \partial_x f(p_0) \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \partial_y f(p_0) \end{bmatrix}) \\ &= \{(x, y, z) \in \mathbb{R}^3 : z = f(x_0, y_0) + (x - x_0)\partial_x f(p_0) + (y_{y0})\partial_y f(p_0)\} \text{ (done)} \end{aligned}$$

Compute

$$df_{p_0} = \begin{bmatrix} \partial_x f(p_0) & \partial_y f(p_0) \end{bmatrix}$$

Then we see

$$T_{p_0}(S) = (x_0, y_0, f(x_0, y_0)) + \{(x, y, df_{p_0}(x, y)) \in \mathbb{R}^3 : (x, y) \in \mathbb{R}^2\}$$

where the right hand side is the graph of df_{p_0} when the origin is set to be $(p_0, f(p_0))$

■

Question 38

5. If a coordinate neighborhood of a regular surface can be parametrized in the form

$$\mathbf{x}(u, v) = \alpha_1(u) + \alpha_2(v),$$

where α_1 and α_2 are regular parametrized curves, show that the tangent planes along a fixed coordinate curve of this neighborhood are all parallel to a line.

Proof. WOLG, suppose the coordinate curve $\gamma : I \to S$ is

$$\gamma(t) = \mathbf{x}(u_0, t)$$

Compute

$$d\mathbf{x} = \begin{bmatrix} \alpha_1'(u) & \alpha_2'(v) \end{bmatrix} \text{ and } d\mathbf{x}_{\gamma(t)} = \begin{bmatrix} \alpha_1'(u_0) & \alpha_2'(v) \end{bmatrix}$$

Then see that

$$T_{\gamma(t)}(S) = \operatorname{span}(\alpha_1'(u_0), \alpha_2'(t))$$

Since $\alpha_1'(u_0)$ is fixed, we see $T_{\gamma(t)}(S)$ are all parallel to $\alpha_1'(u_0)$. ■

Question 39

10. (*Tubular Surfaces.*) Let $\alpha\colon I \to R^3$ be a regular parametrized curve with nonzero curvature everywhere and arc length as parameter. Let

$$\mathbf{x}(s, v) = \alpha(s) + r(n(s)\cos v + b(s)\sin v), \qquad r = \text{const.} \neq 0, s \in I,$$

be a parametrized surface (the *tube* of radius r around α), where n is the normal vector and b is the binormal vector of α. Show that, when $\mathbf{x}$ is regular, its unit normal vector is

2. Regular Surfaces

$$N(s, v) = -(n(s)\cos v + b(s)\sin v).$$

Proof. Use Frenet-Serret Formula to compute

$$d\mathbf{x} = \left[\alpha' + r\big((-\kappa T - \tau B)\cos v + \tau N \sin v\big) \quad r(-N\sin v + B\cos v)\right]$$

We wish to show

$$-(N\cos v + B\sin v) \perp d\mathbf{x}(\mathbb{R}^2)$$

We then only have to prove

$$\begin{aligned} &\big(N\cos v + B\sin v\big) \perp -N\sin v + B\cos v \\ \text{and } &\big(N\cos v + B\sin v\big) \perp \alpha' + r\big((-\kappa T - \tau B)\cos v + \tau N\sin v\big) \end{aligned}$$

Because $\{T, N, B\}$ form an othothonormal basis and α' is just T, we have

$$(N\cos v + B\sin v)\cdot(-N\sin v + B\cos v) = -(\cos v\sin v) + \sin v\cos v = 0$$

and have

$$
\begin{aligned}
&(N\cos v + B\sin v)\cdot\Big(\alpha' + r\big((-\kappa T - \tau B)\cos v + \tau N \sin v\big)\Big)\\
&= r\tau\cos v\sin v - r\tau\sin v\cos v = 0 \text{ (done)}
\end{aligned}
$$

$\blacksquare$

Question 40

13. A *critical point* of a differentiable function $f\colon S \to R$ defined on a regular surface S is a point $p \in S$ such that $df_p = 0$.

***a.** Let $f\colon S \to R$ be given by $f(p) = |p - p_0|$, $p \in S$, $p_0 \notin S$ (cf. Exercise 5, Sec. 2-3). Show that $p \in S$ is a critical point of f if and only if the line joining p to p_0 is normal to S at p.

b. Let $h\colon S \to R$ be given by $h(p) = p\cdot v$, where $v \in R^3$ is a unit vector (cf. Example 1, Sec. 2-3). Show that $p \in S$ is a critical point of f if and only if v is a normal vector of S at p.

Proof. Compute

$$dh_p(\alpha'(0)) = \frac{d}{dt}h(\alpha(t))|_{t=0} = v\cdot\alpha'(0)$$

In other words,

$$dh_p(w) = v\cdot w$$

This implies

$$dh_p(l) = 0, \forall l \in T_p(S) \iff v \perp T_p(S) \iff v \text{ is a normal vector of } S \text{ at } p$$

$\blacksquare$

Question 41

15. Show that if all normals to a connected surface pass through a fixed point, the surface is contained in a sphere.

Proof. WOLG, suppose the fixed point is the origin. Let S be the regular surface. We are given the fact that any local chart $\mathbf{x}(u,v)$ satisfy the equation

$$\mathbf{x}(u,v) = f_{\mathbf{x}}(u,v)N(u,v) \tag{3.3}$$

where $f_{\mathbf{x}}(u,v)$ is a scalar-valued function and $N(u,v)$ is the unit normal vector of S at $\mathbf{x}(u,v)$. We first show that

the distance of the surface to the origin is locally a constant

In other words, we wish to prove that any local chart $\mathbf{x}(u,v)$ satisfy

$f_{\mathbf{x}}$ is constant on domain of $\mathbf{x}$

Doing partial derivative on $N \cdot N = 1$, we see that

$$\partial_u N \perp N \text{ and } \partial_v N \perp N$$

This implies

$$\partial_u N, \partial_v N \in T_p(S)$$

We know

$$\partial_u \mathbf{x}, \partial_v \mathbf{x} \in T_p(S)$$

Now, doing partial derivative on both side of (3.3), we see

$$\partial_u \mathbf{x} = (\partial_u f_{\mathbf{x}})N + f_{\mathbf{x}}(\partial_u N) \text{ and } \partial_v \mathbf{x} = (\partial_v f_{\mathbf{x}})N + f_{\mathbf{x}}(\partial_v N)$$

and see

$$(\partial_u f_{\mathbf{x}})N = \partial_u \mathbf{x} - f_{\mathbf{x}}(\partial_u N) \in T_p(S) \text{ and } (\partial_v f_{\mathbf{x}})N = \partial_v \mathbf{x} - f_{\mathbf{x}}(\partial_v N) \in T_p(S)$$

Then because $N \notin T_p(S)$, we can conclude $\partial_u f_{\mathbf{x}} = \partial_v f_{\mathbf{x}} = 0$. This establish that $f_{\mathbf{x}}$ is a constant. (done)

Note that the surface is connected, this implies that the distance of the surface to the origin is globally a constant, which implies the surface is contained in a sphere.

∎

Question 42

18. Prove that if a regular surface S meets a plane P in a single point p, then this plane coincides with the tangent plane of S at p.

Proof. WOLG, suppose P is the x, y-plane, p is the origin and $S \subseteq \{(x, y, z) \in \mathbb{R}^3 : z \geq 0\}$ (This can be achieved by a rigid motion). We wish to show

$$T_p(S) = P$$

Because $\dim(T_pS) = \dim(P) = 2$, we can reduce the problem into proving

$$T_p(S) \subseteq P$$

Fix $w \in T_p(S)$. We reduce the problem into

$$\text{proving } w \in P$$

Let $\alpha : (-\epsilon, \epsilon) \to S$ satisfy

$$\alpha(0) = 0 \text{ and } \alpha'(0) = w$$

Express

$$\alpha(t) \triangleq (x, y, z)$$

Because S is above P, the x, y-plane, we know z attain minimum at 0. This implies $z'(0) = 0$, and implies $\alpha'(0) = (x'(t), y'(0), 0) \in P$ (the x, y-plane) (done) ■

Question 43

***20.** Show that the perpendicular projections of the center $(0,0,0)$ of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

onto its tangent planes constitute a regular surface given by

$$\{(x,y,z)\in R^3;(x^2+y^2+z^2)^2=a^2x^2+b^2y^2+c^2z^2\}-\{(0,0,0)\}.$$

Proof. Let S be the ellipsoid. Let

$$E\triangleq\{(x,y,z)\in\mathbb{R}^3:(x^2+y^2+z^2)^2=a^2x^2+b^2y^2+c^2z^2\}\setminus O$$

Let E' be the perpendicular projections of O of S onto the tangent planes of S. We are required to prove

$$E'=E$$

We first prove

$$E'\subseteq E$$

Fix $p_0=(x_0,y_0,z_0)\in S$. We first prove

$$T_{p_0}(S)=\{(x,y,z)\in\mathbb{R}^3:\frac{xx_0}{a^2}+\frac{yy_0}{b^2}+\frac{zz_0}{c^2}=1\}\tag{3.4}$$

Fix

$$f(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$$

We have $S = f^{-1}(1)$. Compute

$$\nabla f(x, y, z) = (\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2})$$

Then because $T_{p_0}(S) \perp \nabla f(p_0)$, we have

$$T_{p_0}(S) = \{(x, y, z) \in \mathbb{R}^3 : \frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = 1\} \text{ (done)}$$

We know that the line L passing through O and perpendicular to $T_{p_0}(S)$ can be parametrized by

$$L(t) = t\nabla f(p_0) \equiv t(\frac{x_0}{a^2}, \frac{y_0}{b^2}, \frac{z_0}{c^2})$$

Compute

$$L(t) \in T_{p_0}(S) \implies t(\frac{x_0^2}{a^4} + \frac{y_0^2}{b^4} + \frac{z_0^2}{c^4}) = 1$$

This implies

$$t = (\frac{x_0^2}{a^4} + \frac{y_0^2}{b^4} + \frac{z_0^2}{c^4})^{-1} \text{ and } q = t(\frac{x_0}{a^2}, \frac{y_0}{b^2}, \frac{z_0}{c^2})$$

where $q \in T_{p_0}(S)$ is the point to which O is perpendicular projected.

Express $q = (x, y, z)$. Now, we can reduce the problem into proving

$$(x^2 + y^2 + z^2)^2 = a^2x^2 + b^2y^2 + c^2z^2$$

Compute

$$(x^2 + y^2 + z^2)^2 = \Big(t^2(\frac{x_0^2}{a^4} + \frac{y_0^2}{b^4} + \frac{z_0^2}{c^2})\Big)^2$$

Compute

$$a^2x^2 + b^2y^2 + c^2z^2 = t^2(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2}) = t^2 \quad (\because \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = 1)$$

We reduce the problem into proving

$$t^2(\frac{x_0^2}{a^4} + \frac{y_0^2}{b^4} + \frac{z_0^2}{c^4})^2 = 1$$

Note that $t = (\frac{x_0^2}{a^4} + \frac{y_0^2}{b^4} + \frac{z_0^2}{c^4})^{-1}$ and we are done. (done)

It remains to prove

$$E \subseteq E'$$

Fix $(x_1, y_1, z_1) \in S_1$. Let

$$r \triangleq \sqrt{a^2 x_1^2 + b^2 y_1^2 + c^2 z_1^2}$$

We claim

(x_1, y_1, z_1) is the projection of O onto the tangent plane of S at $(\dfrac{a^2 x_1}{r}, \dfrac{b^2 y_1}{r}, \dfrac{c^2 z_1}{r})$

It is easily checked that $(\frac{a^2 x_1}{r}, \frac{b^2 y_1}{r}, \frac{c^2 z_1}{r}) \in S$. Let $p = (\frac{a^2 x_1}{r}, \frac{b^2 y_1}{r}, \frac{c^2 z_1}{r})$. Using (3.4), we have

$$T_p(S) = \{(x, y, z) \in \mathbb{R}^3 : \frac{x x_1 + y y_1 + z z_1}{r} = 1\}$$

It is now very clear that

$$(x_1, y_1, z_1) \text{ as a vector is perepndicular to } T_p(S)$$

and we can use the fact $(x_1, y_1, z_1) \in E$ to compute

$$\frac{x_1^2 + y_1^2 + z_1^2}{r} = \frac{\sqrt{a^2 x_1^2 + b^2 y_1^2 + c^2 z_1^2}}{r} = 1$$

This conclude $(x_1, y_1, z_1) \in T_p(S)$. (done) ■

Question 44

1. Compute the first fundamental forms of the following parametrized surfaces where they are regular:

a. $\mathbf{x}(u,v) = (a \sin u \cos v, b \sin u \sin v, c \cos u)$; ellipsoid.

b. $\mathbf{x}(u,v) = (au \cos v, bu \sin v, u^2)$; elliptic paraboloid.

c. $\mathbf{x}(u,v) = (au \cosh v, bu \sinh v, u^2)$; hyperbolic paraboloid.

d. $\mathbf{x}(u,v) = (a \sinh u \cos v, b \sinh u \sin v, c \cosh u)$; hyperboloid of two sheets.

Proof. Let $\alpha'(0) \in T_p(S)$ and express $\alpha(t) = \mathbf{x}(u(t), v(t))$. We have

$$\begin{aligned}
I_p(\alpha'(0)) &= \alpha'(0) \cdot \alpha'(0) \\
&= \Big(u'(0)\partial_u \mathbf{x}(p) + v'(0)\partial_u \mathbf{x}(p)\Big) \cdot \Big(u'(0)\partial_u \mathbf{x}(p) + v'(0)\partial_u \mathbf{x}(p)\Big) \\
&= |\mathbf{x}_u(p)|^2 (u'(0))^2 + 2(\partial_u \mathbf{x}(p) \cdot \partial_v \mathbf{x}(p)) u'(0) v'(0) + |\mathbf{x}_v(p)|^2 (v'(0))^2 \\
&\triangleq E(u'(0))^2 + 2F u'(0) v'(0) + G(v'(0))^2
\end{aligned}$$

From now on, we compute only E, F, G.

(a) Compute

$$\begin{aligned}
\partial_u \mathbf{x} &= (a \cos u \cos v, b \cos u \sin v, -c \sin u) \\
\text{and } \partial_v \mathbf{x} &= (-a \sin u \sin v, b \sin u \cos v, 0)
\end{aligned}$$

This give us

$$\begin{aligned}
E &= |\partial_u \mathbf{x}|^2 = \cos^2 u (a^2 \cos^2 v + b^2 \sin^2 v) + c^2 \sin^2 u \\
F &= \partial_u \mathbf{x} \cdot \partial_v \mathbf{x} = (-a^2 + b^2) \cos u \sin u \cos v \sin v \\
G &= |\partial_v \mathbf{x}|^2 = \sin^2 u (a^2 \sin^2 v + b^2 \cos^2 v)
\end{aligned}$$

(b) Compute

$$\partial_u \mathbf{x} = (a\cos v, b\sin v, 2u) \text{ and } \partial_v \mathbf{x} = (-au\sin v, bu\cos v, 0)$$

This give us

$$\begin{aligned}
E &= |\partial_u \mathbf{x}|^2 = a^2\cos^2 v + b^2\sin^2 v + 4u^2 \\
F &= \partial_u \mathbf{x} \cdot \partial_v \mathbf{x} = \cos v \sin v(-a^2 + b^2)u \\
G &= |\partial_v \mathbf{x}|^2 = u^2(a^2\sin^2 v + b^2\cos^2 v)
\end{aligned}$$

(c) Compute

$$\partial_u \mathbf{x} = (a\cosh v, b\sinh v, 2u) \text{ and } \partial_v \mathbf{x} = (au\sinh v, bu\cosh v, 0)$$

This give us

$$\begin{aligned}
E &= |\partial_u \mathbf{x}|^2 = a^2\cosh^2 v + b^2\sinh^2 v + 4u^2 \\
F &= \partial_u \mathbf{x} \cdot \partial_v \mathbf{x} = (a^2 + b^2)u\cosh v \sinh v \\
G &= |\partial_v \mathbf{x}|^2 = u^2(a^2\sinh^2 v + b^2\cosh^2 v)
\end{aligned}$$

(d) Compute

$$\partial_u \mathbf{x} = (a\cosh u\cos v, b\cosh u\sin v, c\sinh u) \text{ and } \partial_v \mathbf{x} = (-a\sinh u\sin v, b\sinh u\cos v, 0)$$

This give us

$$\begin{aligned}
E &= |\partial_u \mathbf{x}|^2 = \cosh^2 u(a^2\cos^2 v + b^2\sin^2 v) + c^2\sinh^2 u \\
F &= \partial_u \mathbf{x} \cdot \partial_v \mathbf{x} = \cosh u\sinh u\cos v\sin v(b^2 - a^2) \\
G &= |\partial_v \mathbf{x}|^2 = \sinh^2 u(a^2\sin^2 v + b^2\cos^2 v)
\end{aligned}$$

■

Question 45

3. Obtain the first fundamental form of the sphere in the parametrization given by stereographic projection (cf. Exercise 16, Sec. 2-2).

Proof. We are given

$$\mathbf{x}(u, v) = \left(\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{2(u^2+v^2)}{u^2+v^2+4}\right)$$

Note that in Question 2.5.1, we have already given the complete formula of $I_p(\alpha'(0))$. We only have to compute E, F, G.

$$\partial_u \mathbf{x} = \Big(\frac{4(-u^2+v^2+4)}{(u^2+v^2+4)^2}, \frac{-8vu}{(u^2+v^2+4)^2}, \frac{16u}{(u^2+v^2+4)^2}\Big)$$

$$\partial_v \mathbf{x} = \Big(\frac{-8uv}{(u^2+v^2+4)^2}, \frac{4(-v^2+u^2+4)}{(u^2+v^2+4)^2}, \frac{16v}{(u^2+v^2+4)^2}\Big)$$

Then compute

$$E = \frac{16u^4+32u^2v^2+128u^2+16v^4+128v^2+256}{(u^2+v^2+4)^4}$$

$$F = 0$$

$$G = \frac{16v^4+32u^2v^2+128v^2+16u^4+128u^2+256}{(u^2+v^2+4)^4}$$

■

Question 46

7. The coordinate curves of a parametrization $\mathbf{x}(u, v)$ constitute a *Tchebyshef net* if the lengths of the opposite sides of any quadrilateral formed by them are equal. Show that a necessary and sufficient condition for this is

$$\frac{\partial E}{\partial v} = \frac{\partial G}{\partial u} = 0.$$

Proof. For each v, define

$$\alpha_v(t) = \mathbf{x}(t, v)$$

WOLG, we are require to show

$$\forall r \in \mathbb{R}^+, \int_0^r |\alpha_v'(t)|\, dt \text{ is a constant in } v \iff \frac{\partial E}{\partial v} = 0$$

$(\longleftarrow)$

Note that we have

$$u'(t) = 1 \text{ and } v'(t) = 0 \text{ if we write } \alpha_v(t) = \mathbf{x}(u(t), v(t))$$

Then, we have

$$\int_0^C |\alpha_v'(t)|\, dt = \int_0^C \sqrt{E(t,v)}dt$$

If $\frac{\partial E}{\partial v} = 0$, then for each v_1, v_2, we clearly have

$$\int_0^r \left|\alpha_{v_1}'(t)\right| dt = \int_0^r \sqrt{E(t,v_1)}dt = \int_0^r \sqrt{E(t,v_2)}dt = \int_0^r \left|\alpha_{v_2}'(t)\right| dt$$

$(\longrightarrow)$

Suppose for all $r \in \mathbb{R}^+$, the function $\int_0^r |\alpha_v'(t)|\, dt$ is a constant in v. We then can define $f : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$f(r) = \int_0^r |\alpha_v'(t)|\, dt \text{ for all } v$$

Differentiating f, we see

$$f'(r) = \sqrt{E(r,v)} \text{ for all } v$$

This tell us

$$E(r,v) \text{ is a constant in } v \text{ for all } r$$

which implies

$$\frac{\partial E}{\partial v} = 0 \text{ (done)}$$

■

Question 47

***8.** Prove that whenever the coordinate curves constitute a Tchebyshef net (see Exercise 7) it is possible to reparametrize the coordinate neighborhood in such a way that the new coefficients of the first fundamental form are

$$E = 1, \quad F = \cos\theta, \quad G = 1,$$

where θ is the angle of the coordinate curves.

Proof. Note that $\frac{\partial E}{\partial v} = \frac{\partial G}{\partial u} = 0$ implies

$$E \text{ stay constant in } v \text{ and } G \text{ stay constant in } u$$

In other words, E can be treated as a function of u and G can be treated as a function of v. Let

$$\overline{u}(u) = \int_0^u \sqrt{E(t)}dt \text{ and } \overline{v}(v) = \int_0^v \sqrt{G(t)}dt$$

Reparametrize by

$$\mathbf{x}(u(\overline{u}), v(\overline{v}))$$

By Chain Rule and Single-Variable Inverse Function Theorem, we have

$$\partial_{\overline{u}}\mathbf{x} = \frac{1}{\sqrt{E(u)}}\partial_u\mathbf{x} \text{ and } \partial_{\overline{v}}\mathbf{x} = \frac{1}{\sqrt{G(v)}}\partial_v\mathbf{x}$$

This give us

$$\overline{E} = \partial_{\overline{u}}\mathbf{x} \cdot \partial_{\overline{u}}\mathbf{x} = \frac{E(u)}{E(u)} = 1 \text{ and } \overline{G} = \partial_{\overline{v}}\mathbf{x} \cdot \partial_{\overline{v}}\mathbf{x} = \frac{G(v)}{G(v)} = 1$$

Now, by CS-inequality, we know $\overline{F} = \partial_{\overline{u}}\mathbf{x} \cdot \partial_{\overline{v}}\mathbf{x} \in (-1, 1)$. Then there must exists θ such that $\overline{F} = \cos\theta$. ∎

Question 48

11. Let S be a surface of revolution and C its generating curve (cf. Example 4, Sec. 2-3). Let s be the arc length of C and denote by $\rho = \rho(s)$ the distance to the rotation axis of the point of C corresponding to s.

a. (*Pappus' Theorem.*) Show that the area of S is

$$2\pi \int_0^l \rho(s)\, ds,$$

where l is the length of C.

b. Apply part a to compute the area of a torus of revolution.

Proof. **(a)**

Because S is a surface of revolution, we have an almost global chart

$$\mathbf{x}(\theta, s) = (\cos\theta f(s), \sin\theta f(s), g(s))$$

where

$$f(s) = \rho(s) \text{ and } (f')^2 + (g')^2 = 1$$

Compute

$$\partial_\theta \mathbf{x} = (-\sin\theta f(s), \cos\theta f(s), 0) \text{ and } \partial_s \mathbf{x} = (\cos\theta f'(s), \sin\theta f'(s), g'(s))$$

This let us compute

$$\begin{aligned} E &= |\partial_\theta \mathbf{x}|^2 = |f(s)|^2 = \rho(s)^2 \\ F &= \partial_\theta \mathbf{x} \cdot \partial_s \mathbf{x} = 0 \\ G &= |\partial_s \mathbf{x}|^2 = (f'(s))^2 + (g'(s))^2 = 1 \end{aligned}$$

Now we see that the area $A(S)$ of S is exactly

$$A(S) = \int_0^{2\pi} \int_0^l \sqrt{EG - F^2} ds d\theta = 2\pi \int_0^l \rho(s) ds$$

(b)
Note that

$$\big(f(s), g(s)\big) \triangleq (a + r\cos\frac{s}{r}, a + r\sin\frac{s}{r}) \text{ where } f(s) \equiv \rho(s)$$

satisfy all the condition. Then we can compute the surface area of the torus by

$$2\pi \int_0^{2\pi r} (a + r\cos(\frac{s}{r}))ds = 4\pi^2 ra$$

■

Question 49

14. (*Gradient on Surfaces.*) The *gradient* of a differentiable function $f: S \to R$ is a differentiable map grad $f: S \to R^3$ which assigns to each point $p \in S$ a vector grad $f(p) \in T_p(S) \subset R^3$ such that

$$\langle \operatorname{grad} f(p), v \rangle_p = df_p(v) \quad \text{for all } v \in T_p(S).$$

Show that

a. If E, F, G are the coefficients of the first fundamental form in a parametrization $\mathbf{x}: U \subset R^2 \to S$, then grad f on $\mathbf{x}(U)$ is given by

$$\operatorname{grad} f = \frac{f_u G - f_v F}{EG - F^2}\mathbf{x}_u + \frac{f_v E - f_u F}{EG - F^2}\mathbf{x}_v.$$

In particular, if $S = R^2$ with coordinates x, y,

$$\operatorname{grad} f = f_x e_1 + f_y e_2,$$

where $\{e_1, e_2\}$ is the canonical basis of R_2 (*thus, the definition agrees with the usual definition of gradient in the plane*).

b. If you let $p \in S$ be fixed and v vary in the unit circle $|v| = 1$ in $T_p(s)$, then $df_p(v)$ is maximum if and only if $v = \operatorname{grad} f / |\operatorname{grad} f|$ (*thus*, grad $f(p)$ *gives the direction of maximum variation of f at p*).

c. If grad $f \neq 0$ at all points of the *level curve* $C = \{q \in S; f(q) = \text{const.}\}$, then C is a regular curve on S and grad f is normal to C at all points of C.

Proof. (a)

Suppose $\nabla f = \frac{f_u G - f_v F}{EG - F^2}\mathbf{x}_u + \frac{f_v E - f_u F}{EG - F^2}$. First observe

$$\begin{aligned}
\langle \nabla f, \mathbf{x}_u \rangle &= \frac{f_u G - f_v F}{EG - F^2} E + \frac{f_v E - f_u F}{EG - F^2} F \\
&= \frac{f_u (GE - F^2)}{EG - F^2} \\
&= f_u = \frac{\partial}{\partial u}(f \circ \mathbf{x}) = df(\mathbf{x}_u)
\end{aligned}$$

The justification of the last inequality is as followed. Define

$$\alpha(t) = \mathbf{x}(u_0 + t, v_0)$$

We have

$$\alpha(0) = \mathbf{x}(u_0, v_0) \text{ and } \alpha'(0) = d\mathbf{x}_{(u_0,v_0)}(1,0) = \mathbf{x}_u(u_0, v_0)$$

Now

$$df_{\mathbf{x}(u_0,v_0)}(\mathbf{x}_u(u_0, v_0)) \overset{\text{def}}{=} \frac{d}{dt}(f \circ \alpha(t))\Big|_{t=0} = \frac{d}{dt}\big((f \circ \mathbf{x})(u_0 + t, v_0)\big) = \frac{\partial}{\partial u}(f \circ \mathbf{x})(u_0, v_0)$$

This justified the $\langle \nabla f, \mathbf{x}_u \rangle = df(\mathbf{x}_u)$.

Similarly, we have

$$\langle \nabla f, \mathbf{x}_v \rangle = df(\mathbf{x}_v)$$

Now, for all $w \in T_p(S)$, we see that

$$\begin{aligned}
\langle \nabla f(p), w \rangle &= \langle \nabla f(p), c_u \mathbf{x}_u(p) + c_v \mathbf{x}_v(p) \rangle \quad (\text{ for a unique pair } c_u, c_v \in \mathbb{R} \text{ }) \\
&= c_u \langle \nabla f(p), \mathbf{x}_u(p) \rangle + c_v \langle \nabla f(p), \mathbf{x}_v(p) \rangle \\
&= c_u df_p(\mathbf{x}_u(p)) + c_v df_p(\mathbf{x}_v(p)) \\
&= df_p(c_u \mathbf{x}_u(p) + c_v \mathbf{x}_v(p)) = df_p(w)
\end{aligned}$$

If $S = \mathbb{R}^2$ with coordinates x, y, we can easily compute

$$\mathbf{x}_x = (1, 0) \text{ and } \mathbf{x}_y = (0, 1)$$

and

$$E = 1 \text{ and } F = 0 \text{ and } G = 1$$

Then from the formula of ∇f we just derived, we have

$$\nabla f = f_x(1,0) + f_y(0,1) \equiv f_x e_1 + f_y e_2$$

(b)

By C-S inequality, we see that

$$df_p(v) \equiv \langle \nabla f(p), v\rangle \text{ reach maximum if and only if } v = c_0 \nabla f(p) \text{ for some positive } c_0$$

It then come very clear, under the constraint $|v| = 1$, that c_0 must be $\frac{1}{|\nabla f(p)|}$.

(c)

Let that constant be c_0, and let $p = \mathbf{x}(q) \in C$. Define $g : \mathbb{R}^{2+1} \to \mathbb{R}$ by

$$g(u,v,t) = f(\mathbf{x}(u,v)) - t$$

Check that

$$\partial_t g = -1 \neq 0$$

Then by Implicit function theorem, there exists a function $h : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ such that

$$h(q) = c_0 \text{ and } g(u,v,h(u,v)) = 0 \text{ for all } (u,v) \in U$$

We now see

$$C = (h \circ \mathbf{x}^{-1})^{-1}(c_0) \text{ which is a regular preimage}$$

This established that C is a regular curve.

Locally parametrize $\gamma(t) \subseteq C$. Because $f \circ \gamma$ stay constant, we see

$$0 = df_{\gamma(t)}(\gamma'(t)) = \langle \nabla f(\gamma(t)), \gamma'(t)\rangle$$

This then implies $\nabla f(\gamma(t))$ is perpendicular to $\gamma'(t)$, thus perpendicualr to C. ■

Chapter 4

Gauss Map

4.1 Equivalent Definitions of the Term "Orientable"

Only three results are important in this section

(a) Definition 4.1.2

(b) Preimage of regular value is orientable. (Theorem 4.1.4)

(c) The Gauss Map $N : S \to \mathbb{R}^3$ maps S into S^2 because $N(p)$ is unit for all $p \in S$, and because $T_pS \perp N(p) \perp T_{N(p)}S^2$, we see $T_{N(p)}S^2 = T_pS$, so one can just treat the derivative of $dN_p : T_pS \to T_{N(p)}S^2$ as a linear transformation $dN_p : T_pS \to T_pS$ that maps T_pS into itself.

Particularly, Theorem 4.1.4 give a simple proof that S^2 is orientable, and perhaps almost all questions in midterm 2 that ask you to show some regular surface is orientable can be very easily answered using Theorem 4.1.4.

In this section, by an **atlas** of a regular surface S, we mean a set L of parametrizatoin

$$L = \{\mathbf{x}_\lambda : U_\lambda \to S | \lambda \in \Lambda\}$$

such that L cover S

$$S \subseteq \bigcup_{\mathbf{x}_\lambda \in L} \mathbf{x}_\lambda(U_\lambda)$$

Definition 4.1.1. (First Definition: Consistently Oriented Atlas) We say S is orientable if S has at least one **consistently oriented atlas** L.

We say an atlas L of S is consistently orietned if every two $\mathbf{x}(u,v), \overline{\mathbf{x}}(\overline{u},\overline{v})$ that belongs L and has non-empty intersection E satisfy

$$\det\Big([df_{(u,v)}]\Big) > 0 \text{ for all } (u,v) \in \mathbf{x}^{-1}(E)$$

where the function f from $\mathbf{x}^{-1}(E)$ to $(\overline{\mathbf{x}})^{-1}(E)$ is defined by

$$f(u,v) \triangleq (\overline{\mathbf{x}})^{-1} \circ \mathbf{x}(u,v)$$

Definition 4.1.2. (Second Definition: Gauss Map) We say S is orientable if there exists a differentiable function N, which we call **Gauss map**, that maps each $p \in S$ to a unit vector $N(p) \in \mathbb{R}^3$ such that

$$N(p) \perp T_p(S)$$

Note that in Definition 4.1.1, $[df]$ is just

$$[df] = \frac{\partial(\overline{u},\overline{v})}{\partial(u,v)}$$

You will later use this fact.

Theorem 4.1.3. (Equivalence of Definition 4.1.1 and Definition 4.1.2)

Proof. Suppose S has an consistently oriented atlas L. We are required to

$$\text{define an } N : S \to \mathbb{R}^3 \text{ that satisfy Definition 4.1.2}$$

For each $p \in S$, arbitrarily pick a chart $\mathbf{x} \in L$ covering p. We claim

$$N(p) \triangleq \Big(\frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}\Big)(p) \text{ suffices}$$

We are required to show

(a) N is well defined

(b) N is differentiable

(c) $\forall p \in S, |N(p)| = 1$

It is easy to check (c). That's why (c) isn't colored.

Suppose $\overline{\mathbf{x}} \in L$ cover p. To show N is well defined, we have to show

$$\big(\overline{\mathbf{x}}_{\overline{u}} \times \overline{\mathbf{x}}_{\overline{v}}\big)(p) = c\big((\mathbf{x}_u \times \mathbf{x}_v)\big)(p) \text{ for some } c \in \mathbb{R}^+$$

Calculus 2 taught us (You shall blindly assume this is correct.)

$$\overline{\mathbf{x}}_{\overline{u}} \times \overline{\mathbf{x}}_{\overline{v}} = \big(\mathbf{x}_u \times \mathbf{x}_v\big)\det\Big(\frac{\partial(u,v)}{\partial(\overline{u},\overline{v})}\Big) \tag{4.1}$$

Note that the Definition 4.1.1 says $\det\Big(\frac{\partial(u,v)}{\partial(\overline{u},\overline{v})}\Big)$ is positive at p (in fact every where the two chart $\mathbf{x}, \overline{\mathbf{x}}$ intersect). This complete the proof of (a).

Fix $p \in S$. To show N is differentiable at p, we are required to show there exists $\mathbf{x}$ covering p such that

$$N \circ \mathbf{x} : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3 \text{ is differentiable}$$

This shall be clear once we write down the exact expression of $N \circ \mathbf{x}$

$$\text{for all } (u_0,v_0) \in U \text{ we have } N \circ \mathbf{x}(u_0,v_0) = \frac{\frac{\partial \mathbf{x}}{\partial u}(u_0,v_0) \times \frac{\partial \mathbf{x}}{\partial v}(u_0,v_0)}{\big|\frac{\partial \mathbf{x}}{\partial u}(u_0,v_0) \times \frac{\partial \mathbf{x}}{\partial v}(u_0,v_0)\big|}$$

This is clearly differnetiable. (done)

Now, given a differtiable unit normal vector field $N : S \to \mathbb{R}^3$, we are required

to construct a consistently oriented atlas L

Note that given a chart $\mathbf{x}(u,v) : U \to S$ containing p, if we define

$$E \triangleq \{(x,y) \in \mathbb{R}^2 : (y,x) \in U\}$$

(It is easy to see E is open by using the fact U is open) and define $\mathbf{x}' : E \to S$ by

$$\mathbf{x}'(u,v) \triangleq \mathbf{x}(v,u)$$

we have

$$\mathbf{x}'_u \times \mathbf{x}'_v = \mathbf{x}_v \times \mathbf{x}_u = -(\mathbf{x}_u \times \mathbf{x}_v)$$

It is clear that there exists an atlas L contained only connected charts. We can now use $\mathbf{x}'$ to replace the $\mathbf{x}$ that has the wrong direction, i.e.

$$\frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}(p) = -N(p)$$

and have an atlas L' that always give us the correct direction.

Note that there is a reason we require L to contain only connected charts, since if $\mathbf{x}(U)$ is disconnected for some $\mathbf{x}$, it is possible

$$\frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}(u_0, v_0) = N(\mathbf{x}(u_0, v_0)) \text{ and } \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}(u_1, v_1) = -N(\mathbf{x}(u_1, v_1))$$

which invalidate the whole proof.

The fact that all charts in L' has the same direction can now let us easily use Equation 4.1 to verify

$$\det\Big(\frac{\partial(u,v)}{\partial(\overline{u},\overline{v})}\Big) > 0 \text{ (done)}$$

■

Theorem 4.1.4. (Preimage of a Regular Value is orientible) If $f : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function and $r \in \mathbb{R}$ is a regular value of f, then

$$f^{-1}(r) \text{ is an orientable regular surface}$$

Proof. We claim

$$N(p) \triangleq \frac{\nabla f(p)}{|\nabla f(p)|} \text{ suffices}$$

The fact $\nabla f(p) \perp T_pS$ is proved in the last Midterm. The trick is let α be a curve that lies in $S \triangleq f^{-1}(r)$, and see

$$0 = (f \circ \alpha)'(s) = \nabla f(\alpha(s)) \cdot \alpha'(s)$$

Express $N(p)$ precisely,

$$N(x_0, y_0, z_0) = \Big(\frac{f_x}{\sqrt{f_x^2 + f_y^2 + f_z^2}}, \frac{f_y}{\sqrt{f_x^2 + f_y^2 + f_z^2}}, \frac{f_z}{\sqrt{f_x^2 + f_y^2 + f_z^2}}\Big)\Big|_{p=(x_0,y_0,z_0)}$$

It is then clear that $N : S \to \mathbb{R}^3$ is a restriction of the differentiable function $\frac{\nabla f}{|\nabla f|}$. Use the fact f is smooth to show $\frac{\nabla f}{|\nabla f|}$ is differentiable. This implies N is differentiable. ■

Reminder: If a function $f : S \to \mathbb{R}^3$ is the restriction of some differentiable function $g : \mathbb{R}^3 \to \mathbb{R}$, then f is differentiable. This can be directly proved using Chain Rule.

4.2 Second Fundamental Form

Given an inner product space $\Big(V, \langle\cdot,\cdot\rangle\Big)$, we say a linear transformation $T \in L(V,V)$ is **self-adjoint** if

$$\langle Tv, w\rangle = \langle v, Tw\rangle \text{ for all } v, w \in V$$

Definition 4.2.1. (Definition of Second Fundamental Form) We define the second fundamental form $\text{II}_p(v) : T_pS \to R$ on each point p of S by

$$\text{II}_p(v) \triangleq -\langle dN_p(v), v\rangle$$

We will later show $dN_p : T_pS \to T_pS$ is self-adjoint. This then implies that II_p is a quadratic form.

Definition 4.2.2. (Definition of Normal Curvature) Given a curve $\alpha(s)$ that lies in S passing through $p = \alpha(0)$, let κ be the curvature of C at p, and let

$$\cos\theta \triangleq \langle N_\alpha(p), N(p)\rangle$$

The normal curvature $\kappa_{\alpha,p}$ of $\alpha(s)$ at p is then defined by

$$\kappa_{\alpha,p} \triangleq \kappa\langle N_\alpha(p), N(p)\rangle = \kappa\cos\theta$$

Theorem 4.2.3. (Normal curvature depends only on $\alpha'(0)$) Suppose we are given a curve $\alpha(s)$ that lies in S passing through p. We have

$$\kappa_{\alpha,p} = \text{II}_p(\alpha'(0))$$

Proof. By definition, we have

$$\langle N(\alpha(s)), \alpha'(s)\rangle = 0 \text{ for all } s$$

Taking Differentiation

$$\langle N(\alpha(s)), \alpha''(s)\rangle + \langle dN_{\alpha(s)}(\alpha'(s)), \alpha'(s)\rangle = 0$$

This give us

$$\text{II}_{\alpha(s)}(\alpha'(s)) = \langle N(\alpha(s)), \alpha''(s)\rangle$$

Especially, let $s = 0$, we have

$$\text{II}_p(\alpha'(0)) = \langle N(p), \kappa N_\alpha(p)\rangle = \kappa\langle N_\alpha(p), N(p)\rangle = \kappa_{\alpha,p}$$

∎

One thing about Theorem 4.2.3 is that it show us that give two curve $\alpha(s), \beta(s)$ passing though $p = \alpha(0) = \beta(0)$, if two curves has the same tangent at 0, i.e.

$$\alpha'(0) = \beta'(0)$$

then we have

$$\kappa_{\alpha,p} = \mathrm{II}_p(\alpha'(0)) = \mathrm{II}_p(\beta'(0)) = \kappa_{\beta,p}$$

Theorem 4.2.4. **(dN_p is self-adjoint)** Given a $p \in S$, the linear transformation $dN_p : T_p(S) \to T_{N(p)}(S^2)$ is self-adjoint.

Proof. We first prove

$$\langle dN_p(\mathbf{x}_u), \mathbf{x}_v \rangle = \langle \mathbf{x}_u, dN_p(\mathbf{x}_v) \rangle$$

Note that

$$\langle N \circ \mathbf{x}, \mathbf{x}_u \rangle = \langle N \circ \mathbf{x}, \mathbf{x}_v \rangle = 0 \text{ on } U$$

Respectively take partial derivative with respect to v, u, we have

$$\langle dN_p(\mathbf{x}_v), \mathbf{x}_u \rangle + \langle N \circ \mathbf{x}, \mathbf{x}_{uv} \rangle = \langle dN_p(\mathbf{x}_u), \mathbf{x}_v \rangle + \langle N \circ \mathbf{x}, \mathbf{x}_{vu} \rangle = 0 \text{ on } U \text{ where } p = \mathbf{x}(u, v)$$

Note that $\mathbf{x}_{vu} = \mathbf{x}_{uv}$ and we are done. (done)

Observe

$$\begin{aligned}
&\langle dN_p(c_1\mathbf{x}_u + c_2\mathbf{x}_v), c_3\mathbf{x}_u + c_4\mathbf{x}_v \rangle \\
&= c_1c_3\langle dN_p(\mathbf{x}_u), \mathbf{x}_u \rangle + c_2c_4\langle dN_p(\mathbf{x}_v), \mathbf{x}_v \rangle + c_1c_4\langle dN_p(\mathbf{x}_u), \mathbf{x}_v \rangle + c_2c_3\langle dN_p(\mathbf{x}_v), \mathbf{x}_u \rangle \\
&= c_1c_3\langle dN_p(\mathbf{x}_u), \mathbf{x}_u \rangle + c_2c_4\langle dN_p(\mathbf{x}_v), \mathbf{x}_v \rangle + (c_1c_4 + c_2c_3)\langle dN_p(\mathbf{x}_u), \mathbf{x}_v \rangle \qquad (4.2) \\
&= \langle c_1\mathbf{x}_u + c_2\mathbf{x}_v, dN_p(c_3\mathbf{x}_u + c_4\mathbf{x}_v) \rangle \qquad (4.3)
\end{aligned}$$

Note that Equation 4.2 and Equation 4.3 use the violet result, and note that Equation 4.3 implies that dN_p is self-adjoint. ■

Because $dN_p : T_p(S) \to T_p(S)$ is self-adjoint, by spectral theorem, we know there exists an orthonormal basis $\{e_1, e_2\}$ of T_pS containing only eigenvectors. Express

$$dN_p(e_1) = -\kappa_1 e_1 \text{ and } dN_p(e_2) = -\kappa_2 e_2 \text{ and } \kappa_1 \geq \kappa_2$$

We say κ_1, κ_2 are the **principal curvature of** p **at** S and e_1, e_2 are the **principal direction of** p **at** S. Note that principal direction can change sign, yet the value of principal curvature remain the same.

Note that we have

$$\begin{aligned}\mathrm{II}_p(c_1e_1+c_2e_2) &= -dN_p(c_1e_1+c_2e_2)\cdot(c_1e_1+c_2e_2)\\ &= -(-c_1\kappa_1e_1-c_2\kappa_2e_2)\cdot(c_1e_1+c_2e_2)\\ &= c_1^2\kappa_1+c_2^2\kappa_2\end{aligned}$$

Suppose $\alpha'(0)=c_1e_1+c_2e_2$. By Theorem 4.2.3, we have

$$\begin{aligned}\kappa_{\alpha,p}=\mathrm{II}_p(c_1e_1+c_2e_2) &= c_1^2\kappa_1+c_2^2\kappa_2\\ &= c_1^2\kappa_1+(1-c_1^2)\kappa_2 \qquad (\because c_1^2+c_2^2=1)\\ &= \kappa_2+c_1^2(\kappa_1-\kappa_2)\end{aligned}$$

Then $\kappa_{\alpha,p}$ reach maximum when $\alpha'(0)=e_2$ and reach minimum when $\alpha'(0)=e_1$.

Suppose we are given a linear transformation $A\in L(\mathbb{R}^2,\mathbb{R}^2)$, and $\{e_1,e_2\},\{q_1,q_1\}$ are two different basis. One can check that

$$\det\Big([A]_{\{e_1,e_2\}}\Big)=\det\Big([A]_{\{q_1,q_2\}}\Big)\text{ and }\operatorname{tr}\Big([A]_{\{e_1,e_2\}}\Big)=\operatorname{tr}\Big([A]_{\{q_1,q_2\}}\Big)$$

Now when we are given dN_p, we can just say

$$\det\Big(dN_p\Big)=(-\kappa_1)(-\kappa_2)=\kappa_1\kappa_2\text{ and }\operatorname{tr}\Big(dN_p\Big)=-(\kappa_1+\kappa_2)$$

It shall be clear that if we define $\overline{N}=-N$, then $d\overline{N}_p=-dN_p$ and the principal curvature will change sign, while as $\det\Big(dN_p\Big)=\det\Big(d\overline{N}_p\Big)$.

Some definitions:

$$K\triangleq\kappa_1\kappa_2\text{ and }H\triangleq\frac{\kappa_1+\kappa_2}{2}$$

an **asymptotic direction** $v\in T_pS$ is a vector such that

$$\mathrm{II}_p(v)=0$$

For a reason of the naming, one can quickly check that

$$S\triangleq\{(x,y,0)\in\mathbb{R}^3:(x,y)\in\mathbb{R}^2\}\text{ contain only planar point}$$

and

$$S \triangleq \{(x,y,x^2+y^2) \in \mathbb{R}^3 : (x,y) \in \mathbb{R}^2\}$$

Given the standard chart of $\mathbb{T}^2$

$$\mathbf{x}(u,v) = \Big((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u\Big)$$

We have

$$N(u,v) = (\cos u \cos v, \cos u \sin v, \sin u)$$

4.3 Computation of Second Fundamental Form

(Motivation behind the definitions of e, f, g)

Suppose we are given a connected chart $\mathbf{x}(u,v) : U \to S$. We now define the following functions

(a) $N = \left(\frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}\right)$

(b) $N_u = \frac{\partial}{\partial u}(N \circ \mathbf{x})$ and $N_v = \frac{\partial}{\partial v}(N \circ \mathbf{x})$

Given a curve $\alpha(t)$ that lies in $\mathbf{x}(U)$, and let $\gamma(t) : (-\epsilon, \epsilon) \to U$ be a curve that satisfy

$$\alpha(t) = \mathbf{x} \circ \gamma(t)$$

Express

$$\gamma(t) = \big(u(t), v(t)\big)$$

We have

$$\alpha'(t) = d\mathbf{x}_{\gamma(t)}(\gamma'(t)) = \begin{bmatrix} \mathbf{x}_u & \mathbf{x}_v \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} = u'\mathbf{x}_u + v'\mathbf{x}_v$$

We also have

$$dN(\alpha') = (N \circ \alpha)'(t) = (N \circ \mathbf{x} \circ \gamma)'(t) = \begin{bmatrix} N_u & N_v \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} = u'N_u + v'N_v$$

Now observe

$$\begin{aligned} \mathrm{II}_p(\alpha') &= -\langle dN(\alpha'), \alpha' \rangle \\ &= -\langle u'N_u + v'N_v, u'\mathbf{x}_u + v'\mathbf{x}_v \rangle \\ &= e(u')^2 + 2fu'v' + g(v')^2 \end{aligned}$$

where

$$\begin{aligned} e &\triangleq -\langle N_u, \mathbf{x}_u \rangle = \langle N, \mathbf{x}_{uu} \rangle \\ f &\triangleq \frac{-1}{2}\Big(\langle N_v, \mathbf{x}_u \rangle + \langle N_u, \mathbf{x}_v \rangle\Big) = \langle N, \mathbf{x}_{uv} \rangle \\ g &\triangleq -\langle N_v, \mathbf{x}_v \rangle = \langle N, \mathbf{x}_{vv} \rangle \end{aligned}$$

Note that

$$\langle N_v, \mathbf{x}_u \rangle = -\langle N, \mathbf{x}_{uv} \rangle = -\langle N, \mathbf{x}_{vu} \rangle = \langle N_u, \mathbf{x}_v \rangle$$

(Computation of e, f, g) Note that

$$\begin{aligned} e &= \langle N, \mathbf{x}_{uu} \rangle \\ &= \langle \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}, \mathbf{x}_{uu} \rangle \\ &= \frac{\langle \mathbf{x}_u \times \mathbf{x}_v, \mathbf{x}_{uu} \rangle}{\sqrt{EG - F^2}} = \frac{1}{\sqrt{EG - F^2}} \begin{vmatrix} \mathbf{x}_u & \mathbf{x}_v & \mathbf{x}_{uu} \end{vmatrix} \end{aligned}$$

$$\begin{aligned} f &= \langle N, \mathbf{x}_{uv} \rangle \\ &= \frac{\langle \mathbf{x}_u \times \mathbf{x}_v, \mathbf{x}_{uv} \rangle}{\sqrt{EG - F^2}} = \frac{1}{\sqrt{EG - F^2}} \begin{vmatrix} \mathbf{x}_u & \mathbf{x}_v & \mathbf{x}_{uv} \end{vmatrix} \end{aligned}$$

$$\begin{aligned} g &= \langle N, \mathbf{x}_{vv} \rangle \\ &= \frac{\langle \mathbf{x}_u \times \mathbf{x}_v, \mathbf{x}_{vv} \rangle}{\sqrt{EG - F^2}} = \frac{1}{\sqrt{EG - F^2}} \begin{vmatrix} \mathbf{x}_u & \mathbf{x}_v & \mathbf{x}_{vv} \end{vmatrix} \end{aligned}$$

(How to use e, f, g, E, F, G to compute K) Note that $N_u \perp N \perp N_v$, so $N_u, N_v \in T_pS$. Then we can express

$$N_u = a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v \text{ and } N_v = a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v$$

We can now express e, f, g by

$$\begin{aligned} e &= -\langle N_u, \mathbf{x}_u \rangle = -\langle a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v, \mathbf{x}_u \rangle = -a_{11}E - a_{21}F \\ f &= -\langle N_v, \mathbf{x}_u \rangle = -\langle a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v, \mathbf{x}_u \rangle = -a_{12}E - a_{22}F \\ &= -\langle N_u, \mathbf{x}_v \rangle = -\langle a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v, \mathbf{x}_v \rangle = -a_{11}F - a_{21}G \\ g &= -\langle N_v, \mathbf{x}_v \rangle = -\langle a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v, \mathbf{x}_v \rangle = -a_{12}F - a_{22}G \end{aligned}$$

This give us

$$\begin{bmatrix} e & f \\ f & g \end{bmatrix} = -\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \tag{4.4}$$

At the same time, from the following algebraic deduction

$$\begin{aligned} dN(u'\mathbf{x}_u + v'\mathbf{x}_v) &= u'N_u + v'N_v \\ &= u'(a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v) + v'(a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v) \\ &= (u'a_{11} + v'a_{12})\mathbf{x}_u + (u'a_{21} + v'a_{22})\mathbf{x}_v \end{aligned}$$

we have

$$[dN]_{\mathbf{x}_u,\mathbf{x}_v} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{4.5}$$

Now using Equation 4.4,Equation 4.5 and the fact that the product of eigenvalues is the determinant (Diagonalize the diagonlizable matrix to check), we have

$$K = \det\Big([dN]_{\mathbf{x}_u,\mathbf{x}_v}\Big) = \frac{eg - f^2}{EG - F^2}$$

Note that the principal curvatures κ_1, κ_2 are the roots of the following polynomial

$$(a_{11} + t)(a_{22} + t) - a_{12}a_{21} = t^2 + (a_{11} + a_{22})t + a_{11}a_{22} - a_{12}a_{21}$$

This implies

$$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{a_{11} + a_{22}}{-2}$$

We can now see κ_1, κ_2 is the root of

$$t^2 - 2Ht + K = 0$$

This implies

$$\kappa = H \pm \sqrt{H^2 - K}$$

Now we can solve H in terms of e, f, g, E, F, G from Equation 4.4

$$H = \frac{Ge - 2fF + Eg}{2(EG - F^2)}$$

4.4 Dupin Indicatrix

Fix the orientation on an orientible surface S. Fix $p \in S$. Let $e_1, e_2 \in T_pS$ be the principal direction at p, with minimum and maximum normal curvature κ_1, κ_2. We have

$$\mathrm{II}_p(c_1e_1 + c_2e_2) = c_1^2\kappa_1 + c_2^2\kappa_2$$

Then if we classify, independently of the choice of the orientation, $p \in S$ by saying p is

(a) Elliptic if $\det\Big(dN_p\Big) > 0$

(b) Hyperbolic if $\det\Big(dN_p\Big) < 0$

(c) Parabolic if $\det\Big(dN_p\Big) = 0$ and $dN_p \neq 0$

(d) Planar if $dN_p = 0$

We see that

(a) $\mathrm{II}_p^{-1}(\pm 1)$ is an ellipse if p is elliptic.

(b) $\mathrm{II}_p^{-1}(\pm 1)$ is an hyperbola if p is hyperbolic.

(c) $\mathrm{II}_p^{-1}(\pm 1)$ is the union of two straight line if p is parabolic.

(d) $\mathrm{II}_p^{-1}(\pm 1) = \varnothing$ if p is planar.

We say $\mathrm{II}_p^{-1}(\pm 1)$ is the **Dupin indicatrix** of p.

We say $v \in T_pS$ is an **asymptotic direction** in T_pS if

$$\mathrm{II}_p(v) = 0$$

Note that

$$dN_p(v) = 0 \begin{cases} \Longrightarrow \\ \not\longleftarrow \end{cases} \mathrm{II}_p(v) = 0$$

and if

(a) p is elliptic, then the only asymptotic direction is 0.

(b) p is hyperbolic, then $c_1 e_2 + c_2 e_2$ is the asymptotic direction if and only if $c_1 = c_2 = 0$ or $\frac{c_2}{c_1} = \pm\sqrt{\frac{\kappa_2}{-\kappa_1}}$, and the set of asymptotic direction would be exactly the two asymptotic lines.

(c) p is parabolic, then the set of asymptotic direction will contain only the multiples of e_1 or e_2, depending on which of κ_1, κ_2 is 0.

(d) p is planar, than all $v \in T_p S$ is asymptotic direction.

4.5 HW5

Question 50

2. Show that if a surface is tangent to a plane along a curve, then the points of this curve are either parabolic or planar.

Proof. Suppose the curve is $\alpha(s)$. We see that

$$(N \circ \alpha)(s) \text{ stay constant}$$

Differentiation give us

$$dN_{\alpha(s)}(\alpha'(s)) = 0$$

Then for each p that lies in $\alpha(I)$, we see

$$dN_p \text{ is not full rank}$$

This then show us

$$\det(dN_p) = 0$$

and give us the conclusion. ■

Question 51

3. Let $C \subset S$ be a regular curve on a surface S with Gaussian curvature $K > 0$. Show that the curvature k of C at p satisfies

$$|k| \geq \min(|k_1|, |k_2|),$$

where k_1 and k_2 are the principal curvatures of S at p.

Proof. Because $K > 0$, we know the principal curvatures κ_1, κ_2 satisfy

$$0 < \kappa_1 \leq \kappa_2 \text{ or } \kappa_2 \leq \kappa_1 < 0$$

WOLG, suppose $0 < \kappa_1 \leq \kappa_2$. Let $\alpha : (-\epsilon, \epsilon) \to C$ be an arc-length parametrization passing through p at $\alpha(0)$. Let θ be the angle between $N_\alpha(p)$ and $N(p)$. We know

$$\kappa \cos\theta = \kappa_{\alpha,p}$$

Because $0 < \kappa_1 \leq \kappa_2$, we can deduce

$$\kappa_{\alpha,p} \geq \kappa_1 = \min(|\kappa_1|, |\kappa_2|) > 0$$

This then implies

$$|\kappa| \geq |\kappa| |\cos\theta| = |\kappa \cos\theta| = |\kappa_{\alpha,p}| \geq \min(|\kappa_1|, |\kappa_2|)$$

■

Question 52

5. Show that the mean curvature H at $p \in S$ is given by

$$H = \frac{1}{\pi}\int_0^\pi k_n(\theta)\,d\theta,$$

where $k_n(\theta)$ is the normal curvature at p along a direction making an angle θ with a fixed direction.

Proof. Let e_1, e_2 be the principal direction. Suppose the fixed direction is $\cos\theta_0 e_1 + \sin\theta_0 e_2$. Define $\alpha : [0, \pi] \to T_p(S)$ by

$$\alpha(\theta) = \cos(\theta_0 + \theta)e_1 + \sin(\theta_0 + \theta)e_2$$

Compute $\kappa_n(\theta)$ by

$$\begin{aligned}\kappa_n(\alpha(\theta)) &= \mathrm{II}_p(\cos(\theta_0 + \theta)e_1 + \sin(\theta_0 + \theta)e_2) \\ &= \kappa_1 \cos^2(\theta_0 + \theta) + \kappa_2 \sin^2(\theta_0 + \theta)\end{aligned}$$

This then give us

$$\begin{aligned}\frac{1}{\pi}\int_0^\pi \kappa_n(\theta)d\theta &= \frac{1}{\pi}\int_0^\pi \kappa_1 \cos^2(\theta_0 + \theta) + \kappa_2 \sin^2(\theta_0 + \theta)d\theta \\ &= \frac{1}{\pi}(\frac{\kappa_1\pi}{2} + \frac{\kappa_2\pi}{2}) = H\end{aligned}$$

■

Question 53

6. Show that the sum of the normal curvatures for any pair of orthogonal directions, at a point $p \in S$, is constant.

Proof. Suppose e_1, e_2 are principal direction, and express the pair v_1, v_2 of orthogonal directions by $v_1 = \cos\theta e_1 + \sin\theta e_2$ and $v_2 = \cos(\theta + \frac{\pi}{2})e_1 + \sin(\theta + \frac{\pi}{2})e_2$. We have

$$\begin{aligned}\kappa_{v_1} &= \mathrm{II}_p(\cos\theta e_1 + \sin\theta e_2)\\ &= \kappa_1\cos^2\theta + \kappa_2\sin^2\theta\end{aligned}$$

and have

$$\begin{aligned}\kappa_{v_2} &= \mathrm{II}_p(\cos(\theta + \frac{\pi}{2})e_1 + \sin(\theta + \frac{\pi}{2})e_2)\\ &= \mathrm{II}_p(\sin\theta e_1 + \cos\theta e_2)\\ &= \kappa_1\sin^2\theta + \kappa_2\cos^2\theta\end{aligned}$$

This then give us

$$\kappa_{v_1} + \kappa_{v_2} = \kappa_1 + \kappa_2 = \text{const.}$$

■

Question 54

11. Let p be an elliptic point of a surface S, and let r and r' be conjugate directions at p. Let r vary in $T_p(S)$ and show that the minimum of the angle of r with r' is reached at a unique pair of directions in $T_p(S)$ that are symmetric with respect to the principal directions.

Proof. WOLG, suppose $0 < \kappa_1 \le \kappa_2$. Express

$$r = \cos\theta e_1 + \sin\theta e_2 \text{ and } r' = \cos\theta' e_1 + \sin\theta' e_2$$

Now compute

$$\begin{aligned}\langle dN_p(r), r'\rangle &= \langle -\kappa_1\cos\theta e_1 - \kappa_2\sin\theta e_2, \cos\theta' e_1 + \sin\theta' e_2\rangle\\ &= -\kappa_1\cos\theta\cos\theta' - \kappa_2\sin\theta\sin\theta'\end{aligned}$$

This give us the constraint

$$\kappa_1 \cos\theta \cos\theta' + \kappa_2 \sin\theta \sin\theta' = 0$$

and we are required to find the extremum of

$$\cos(\theta - \theta') = \cos\theta \cos\theta' + \sin\theta \sin\theta'$$

We use the method of Lagarange multiplier. Define

$$\begin{aligned} f(\theta, \theta') &\triangleq \cos\theta \cos\theta' + \sin\theta \sin\theta' \\ g(\theta, \theta') &\triangleq \kappa_1 \cos\theta \cos\theta' + \kappa_2 \sin\theta \sin\theta' \end{aligned}$$

We are required to

$$\text{maximize or minimize } f \text{ subjecting to the constraint } g = 0$$

Compute

$$\begin{aligned} \nabla f &= \Big(-\sin\theta \cos\theta' + \cos\theta \sin\theta', -\cos\theta \sin\theta' + \sin\theta \cos\theta' \Big) \\ \nabla g &= \Big(-\kappa_1 \sin\theta \cos\theta' + \kappa_2 \cos\theta \sin\theta', -\kappa_1 \cos\theta \sin\theta' + \kappa_2 \sin\theta \cos\theta' \Big) \end{aligned}$$

It is now easy and straight forward to check that

$$\begin{aligned} \cos\theta &= \sqrt{\frac{\kappa_1}{\kappa_1 + \kappa_2}} \text{ and } \sin\theta = \sqrt{\frac{\kappa_2}{\kappa_1 + \kappa_2}} \\ \cos\theta' &= \sqrt{\frac{\kappa_1}{\kappa_1 + \kappa_2}} \text{ and } \sin\theta' = -\sqrt{\frac{\kappa_2}{\kappa_1 + \kappa_2}} \end{aligned}$$

is a non-trivial solution of $\nabla f = \lambda \nabla g$.

This then let us conclude

$$r \in \operatorname{span}(\sqrt{\kappa_1} e_1 + \sqrt{\kappa_2} e_2) \text{ and } r' \in \operatorname{span}(\sqrt{\kappa_1} e_1 - \sqrt{\kappa_2} e_2)$$

which is the so called "symmetry". ■

Question 55

***18.** Let $\lambda_1, \ldots, \lambda_m$ be the normal curvatures at $p \in S$ along directions making angles $0, 2\pi/m, \ldots, (m-1)2\pi/m$ with a principal direction, $m > 2$. Prove that

$$\lambda_1 + \cdots + \lambda_m = mH,$$

where H is the mean curvature at p.

Proof. Let $v_1, \ldots, v_m$ be the directions of $\lambda_1, \ldots, \lambda_m$. We have

$$v_k = \cos\left(\frac{k(2\pi)}{m}\right)e_1 + \sin\left(\frac{k(2\pi)}{m}\right)e_2$$

Observe

$$\begin{aligned}\lambda_k = \mathrm{II}_p(v_k) &= \mathrm{II}_p\left(\cos\left(\frac{2\pi k}{m}\right)e_1 + \sin\left(\frac{2\pi k}{m}\right)e_2\right) \\ &= \kappa_1 \cos^2\left(\frac{2\pi k}{m}\right) + \kappa_2 \sin^2\left(\frac{2\pi k}{m}\right)\end{aligned}$$

Now compute using elementary identity

$$\begin{aligned}\sum_{k=1}^{m} \lambda_k &= \sum_{k=1}^{m} \kappa_1 \cos^2\left(\frac{2\pi k}{m}\right) + \kappa_2 \sin^2\left(\frac{2\pi k}{m}\right) \\ &= \kappa_1 \frac{m}{2} + \kappa_2 \frac{m}{2} = mH\end{aligned}$$

∎

Question 56

1. Show that at the origin $(0, 0, 0)$ of the hyperboloid $z = axy$ we have $K = -a^2$ and $H = 0$.

Proof. We have the global chart

$$\mathbf{x}(x, y) = (x, y, axy)$$

Compute

$$\mathbf{x}_x = (1, 0, ay) \text{ and } \mathbf{x}_y = (0, 1, ax)$$

Because $N \perp \mathbf{x}_x, \mathbf{x}_y$, we have

$$N(x, y) = \frac{(-ay, -ax, 1)}{\sqrt{a^2(x^2 + y^2) + 1}}$$

Define 2 curves that passing through origin from different direction

$$\alpha(t) \triangleq (t, 0, 0) \text{ and } \beta(t) \triangleq (0, t, 0)$$

Compute

$$N \circ \alpha(t) = \left(0, \frac{-at}{\sqrt{a^2t^2 + 1}}, \frac{1}{\sqrt{a^2t^2 + 1}}\right) \text{ and } N \circ \beta(t) = \left(\frac{-at}{\sqrt{a^2t^2 + 1}}, 0, \frac{1}{\sqrt{a^2t^2 + 1}}\right)$$

This then give us

$$dN_0(\alpha'(0)) = (N \circ \alpha)'(0) = (0, -a, 0) \text{ and } dN_0(\beta'(0)) = (N \circ \beta)'(0) = (-a, 0, 0)$$

Note that $\alpha'(0) = (1, 0, 0)$ and $\beta'(0) = (0, 1, 0)$. We now see dN_0 have action

$$(1, 0, 0) \mapsto (0, -a, 0) \text{ and } (0, 1, 0) \mapsto (-a, 0, 0)$$

Then we can compute the eigenvalues of dN_0 to be a and $-a$. In other words, the principal curvatures at 0 are a and $-a$, which give us $K = -a^2$ and $H = 0$. ■

Question 57

5. Consider the parametrized surface (Enneper's surface)

$$\mathbf{x}(u, v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right)$$

and show that

a. The coefficients of the first fundamental form are

$$E = G = (1+u^2+v^2)^2, \quad F = 0.$$

b. The coefficients of the second fundamental form are

$$e = 2, \quad g = -2, \quad f = 0.$$

c. The principal curvatures are

$$k_1 = \frac{2}{(1+u^2+v^2)^2}, \quad k_2 = -\frac{2}{(1+u^2+v^2)^2}.$$

d. The lines of curvature are the coordinate curves.

e. The asymptotic curves are $u+v = \text{const.}$, $u-v = \text{const}$.

Proof. **(a)** Compute

$$\begin{aligned}
\mathbf{x}_u &= (1-u^2+v^2, 2uv, 2u) \\
\mathbf{x}_v &= (2uv, 1-v^2+u^2, -2v)
\end{aligned}$$

This give us

$$\begin{aligned}
E &= (1-u^2+v^2)^2 + 4u^2v^2 + 4u^2 = (1+u^2+v^2)^2 \\
G &= (1-v^2+u^2)^2 + 4u^2v^2 + rv^2 = (1+u^2+v^2)^2 \\
F &= 2uv(1-u^2+v^2+1-v^2+u^2) - 4uv = 0
\end{aligned}$$

(b) Compute

$$\begin{aligned}
\mathbf{x}_{uu} &= (-2u, 2v, 2) \\
\mathbf{x}_{vv} &= (2u, -2v, -2) \\
\mathbf{x}_{uv} &= (2v, 2u, 0)
\end{aligned}$$

Compute

$$\sqrt{EG-F^2} = (1+u^2+v^2)^2$$

Compute

$$\begin{aligned}
\begin{vmatrix}\mathbf{x}_u & \mathbf{x}_v & \mathbf{x}_{uu}\end{vmatrix} &= 2(1+u^2+v^2)^2 \\
\begin{vmatrix}\mathbf{x}_u & \mathbf{x}_v & \mathbf{x}_{vv}\end{vmatrix} &= -2(1+u^2+v^2)^2 \\
\begin{vmatrix}\mathbf{x}_u & \mathbf{x}_v & \mathbf{x}_{uv}\end{vmatrix} &= 0
\end{aligned}$$

This then give us

$$e = 2 \text{ and } g = -2 \text{ and } f = 0$$

(c) Compute

$$K = \frac{eg-f^2}{EG-F^2} = \frac{-4}{(1+u^2+v^2)^4}$$

and

$$H = \frac{Ge+gE-2fF}{2(EG-F^2)} = 0$$

This tell us $-\kappa_1 = \kappa_2 = \sqrt{K}$ and

$$\kappa_1 = \frac{-2}{(1+u^2+v^2)^2} \text{ and } \kappa_2 = \frac{2}{(1+u^2+v^2)^2}$$

(d) Given $\alpha(t) = \mathbf{x}(u(t), v(t))$. We know

$$\alpha \text{ is a line of curvature } \iff \mathrm{II}_p(\alpha') = (\kappa_1 \text{ or } \kappa_2)\left|\alpha'\right|^2$$

Plugin the first fundamental form and second fundamental form, we now know that α is a line of curvature if and only if

$$e(u')^2 + 2f(u')(v') + g(v')^2 = (\kappa_1 \text{ or } \kappa_2)\big(E(u')^2 + 2F(u')(v') + G(v')^2\big)$$

We have already known the value of the coefficients and the value of κ_1, κ_2, so we can deduce α is a line of curvature if and only if

$$2(u')^2 - 2(v')^2 = \pm 2\big((u')^2 + (v')^2\big)$$

The solution of this equation is clearly

$$u' = 0 \text{ or } v' = 0$$

This implies that

$$\alpha \text{ is a line of curvature} \iff u' = 0 \text{ or } v' = 0 \iff \alpha \text{ is a coordinate curve}$$

(e) Given $\alpha(t) = \mathbf{x}(u(t), v(t))$. Observe

$$\begin{aligned}
\alpha \text{ is an asymptotic curve} &\iff \mathrm{II}_p(\alpha') = 0 \\
&\iff e(u')^2 + 2f(u')(v') + g(v')^2 = 0 \\
&\iff (u')^2 - (v')^2 = 0 \\
&\iff u' = v' \text{ or } u' = -v' \\
&\iff (u+v)' = 0 \text{ or } (u-v)' = 0 \\
&\iff u + v = \text{const. or } u - v = \text{const.}
\end{aligned}$$

■

Question 58

6. (*A Surface with $K \equiv -1$; the Pseudosphere.*)

***a.** Determine an equation for the plane curve C, which is such that the segment of the tangent line between the point of tangency and some line r in the plane, which does not meet the curve, is constantly equal to 1 (this curve is called the *tractrix*; see Fig. 1-9).

b. Rotate the tractrix C about the line r; determine if the "surface" of revolution thus obtained (the *pseudosphere*; see Fig. 3-22) is regular and find out a parametrization in a neighborhood of a regular point.

c. Show that the Gaussian curvature of any regular point of the pseudosphere is -1.

Proof. **(a)** In HW1, we have seen that the following plane curve $\alpha : (0, \frac{\pi}{2}) \to \mathbb{R}^2$ satisfy the desired condition

$$\alpha(t) = \left(\sin t, \cos t + \ln(\tan \frac{t}{2})\right)$$

See at the end of this HW a proof that α satisfy the desired condition.

From now, we let $C = \alpha\Big((0,\frac{\pi}{2})\Big)$. Note that C is regular, and the rotation of C is only the upper half of the usual Pseudosphere. To see a picture of our pseudosphere, see Fig. 3-22, and note that our pseudosphere have empty intersection with the x,y-plane.

(b) We here give a more abstract result. Suppose

$$\gamma(t) = \Big(f(t), g(t)\Big) \text{ with } f \neq 0 \text{ every where}$$

with domain $I \overset{\text{open}}{\subseteq} \mathbb{R}$ is a regular parametrized smooth curve. We claim $\mathbf{x} : (0, 2\pi) \times I$

$$\mathbf{x}(\theta, t) \triangleq \Big(f(t)\cos\theta, f(t)\sin\theta, g(t)\Big)$$

is a regular chart.

It is clear that $\mathbf{x}$ is smooth. We now show $d\mathbf{x}$ is one-to-one on $(0, 2\pi) \times I$. Compute

$$\mathbf{x}_\theta = \Big(-f\sin\theta, f\cos\theta, 0\Big) \text{ and } \mathbf{x}_t = (f'\cos\theta, f'\sin\theta, g')$$

Assume $d\mathbf{x}$ is not one-to-one. We then can deduce

$$fg'\cos\theta = fg'\sin\theta = ff' = 0 \text{ for some } (t, \theta)$$

Fix such (t, θ). Because $f \neq 0$, we then can deduce

$$f'(t) = g'(t)\cos\theta = g'(t)\sin\theta = 0$$

Because $\cos\theta$ and $\sin\theta$ can not be both 0, we then can deduce

$$f'(t) = g'(t) = 0$$

which CaC to the premise γ is a regular parametrization.

Lastly, we are required to show

$$\mathbf{x}^{-1} \text{ is continuous}$$

Express

$$\mathbf{x}(\theta, t) \triangleq \Big(x, y, z\Big)(\theta, t)$$

We wish to show

$$\theta, t \text{ are continuous functions in } (x, y, z)$$

Because $z(\theta,t)=g(t)$, we know

$$t=g^{-1}(z)$$

This implies t is a continuous function in (x,y,z).

Compute

$$\theta=\begin{cases}\arctan\frac{y}{x} & \text{if } x,y\in\mathbb{R}^+ \text{ (first quadrant)}\\ \frac{\pi}{2} & \text{if } x=0, y\in\mathbb{R}^+\\ \frac{3\pi}{2}+\arctan\frac{y}{x} & \text{if } x\in\mathbb{R}^-, y\in\mathbb{R}^+ \text{ (second quadrant)}\\ \pi+\arctan\frac{y}{x} & \text{if } x\in\mathbb{R}^-, y\in\mathbb{R}_0^- \text{ (third quadrant)}\\ \frac{3\pi}{2} & \text{if } x=0, y\in\mathbb{R}^-\\ \frac{4\pi}{2}+\arctan\frac{y}{x} & \text{if } x\in\mathbb{R}^+, y\in\mathbb{R}^- \text{ (forth quadrant)}\end{cases}$$

This implies θ is a continuous function in (x,y,z). (done)

(c) Let S be surface of revolution of C. We are given the chart

$$\mathbf{x}(\theta,t)=\Big(\sin t\cos\theta,\sin t\sin\theta,\cos t+\ln(\tan\frac{t}{2})\Big)$$

Note that $\mathbf{x}(\theta,t)$ dose not cover all of S, and

$$\mathbf{y}(\theta,t)=\Big(\sin t\cos(\theta+\frac{\pi}{2}),\sin t\sin(\theta+\frac{\pi}{2}),\cos t+\ln(\tan\frac{t}{2})\Big)$$

cover the rest of S. Note that $\mathbf{y}$ is merely a rotation of $\mathbf{x}$, so if we show that $\mathbf{x}(U)$ has constant Gauss curvature -1, then the proof is finished.

Compute

$$\mathbf{x}_\theta=\Big(-\sin t\sin\theta,\sin t\cos\theta,0\Big)\text{ and }\mathbf{x}_t=\Big(\cos t\cos\theta,\cos t\sin\theta,-\sin t+\frac{\sec^2\frac{t}{2}}{2\tan\frac{t}{2}}\Big)$$

Simplify the z-component of $\mathbf{x}_t$

$$-\sin t+\frac{\sec^2\frac{t}{2}}{2\tan\frac{t}{2}}=-\sin t+\frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}=-\sin t+\frac{1}{\sin t}=\cos t\cot t \tag{4.6}$$

We now have

$$\mathbf{x}_\theta=\sin t\Big(-\sin\theta,\cos\theta,0\Big)\text{ and }\mathbf{x}_t=\cos t\Big(\cos\theta,\sin\theta,\cot t\Big)$$

We can now compute

$$
\begin{aligned}
E &= \sin^2 t \\
F &= 0 \\
G &= \cot^2 t
\end{aligned}
$$

Using the fact $N \perp \mathbf{x}_\theta, \mathbf{x}_t$ and Equation 4.6, we can conclude

$$
N \text{ is parallel with } \Big(\cot t \cos\theta, \cot t \sin\theta, -1 \Big)
$$

and conclude

$$
N = \Big(\cos t \cos\theta, \cos t \sin\theta, -\sin t \Big)
$$

Compute

$$
N_t = \Big(-\sin t \cos\theta, -\sin t \sin\theta, -\cos t \Big) \text{ and } N_\theta = \Big(-\cos t \sin\theta, \cos t \cos\theta, 0 \Big)
$$

Compute

$$
\begin{aligned}
e &= -N_\theta \cdot \mathbf{x}_\theta = -\sin t \cos t \\
f &= -N_\theta \cdot \mathbf{x}_t = 0 \\
g &= -N_t \cdot \mathbf{x}_t = \cot t
\end{aligned}
$$

Finally, we conclude

$$
K = \frac{eg - f^2}{EG - F^2} = \frac{-\sin t \cos t \cot t}{\sin^2 t \cot^2 t} = -1
$$

■

Lemma 4.5.1. (Umbilical Lemma)

$$
p \text{ is umbilical } \iff dN_p(v) \cdot N \times v = 0 \text{ for all } v \in T_pS
$$

Proof. From left to right is clear. We prove only from right to left. Fix arbitrary $v \in T_pS$. We know $\{N, v, N \times v\}$ is an orthogonal basis. Express $dN_p(v)$ in the form

$$
dN_p(v) = \lambda_1 v + \lambda_2 N + \lambda_3 N \times v
$$

Because $dN_p(v) \in T_pS$, we know $\lambda_2 = 0$. Using $dN_p(v) \cdot N \times v = 0$, we can further deduce $\lambda_3 = 0$. We now see $dN_p(v) = \lambda_1 v$. Because v is arbitrary, our proof is finished. ■

Question 59

20. Determine the umbilical points of the elipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Proof. **(Elipsoid Case)**

We claim

$p=(x,y,z)$ is umbilical $\iff$ For all $(v_1,v_2,v_3)\in T_pS$ we have the following three equations

$$\frac{-xv_2v_3}{a^2}(\frac{1}{b^2}-\frac{1}{c^2})+\frac{yv_1v_3}{b^2}(\frac{1}{a^2}-\frac{1}{c^2})+\frac{-zv_1v_2}{c^2}(\frac{1}{a^2}-\frac{1}{b^2})=0 \tag{4.7}$$

$$\frac{xv_1}{a^2}+\frac{yv_2}{b^2}+\frac{zv_3}{c^2}=0 \tag{4.8}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \tag{4.9}$$

It is clear that $(x,y,z)\in S$ if and only if Equation 4.8 and Equation 4.9 are both satisfied. The proof of our claim now can be reduced to proving p is umbilical if and only if Equation 4.7 is satisfied.

Using Gradient, It is easy to see that we have an orientation

$$N(x,y,z)=\frac{\left(\frac{x}{a^2},\frac{y}{b^2},\frac{z}{c^2}\right)}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}$$

Define $h:S\to\mathbb{R}$ by

$$h(x,y,z)\triangleq\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}$$

Fix an arc-length parametrized curve $\alpha:I\to S$, passing through $p=\alpha(0)$ and express

$$\alpha(s)=\Big(x(s),y(s),z(s)\Big)$$

We have

$$(h \circ \alpha)(N \circ \alpha) = \Big(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\Big)$$

Differentiating both side yield us

$$(h \circ \alpha)' N(\alpha) + h(\alpha) dN_\alpha(\alpha') = \Big(\frac{x'}{a^2}, \frac{y'}{b^2}, \frac{z'}{c^2}\Big)$$

and give us

$$h(p)dN_p(v) = \Big(\frac{x'}{a^2}, \frac{y'}{b^2}, \frac{z'}{c^2}\Big) - (h \circ \alpha)' N(p) \text{ where } v = \alpha'(0) = (x', y', z')$$

Note that $(h \circ \alpha)' N$ is parallel with N. Now by Lemma 4.5.1, we see p is umbilical if and only if for all α we have

$$\begin{vmatrix} \frac{x'}{a^2} & \frac{y'}{b^2} & \frac{z'}{c^2} \\ \frac{x}{a^2} & \frac{y}{b^2} & \frac{z}{c^2} \\ x' & y' & z' \end{vmatrix} = 0$$

Expand the above determinant and substitute (x', y', z') with (v_1, v_2, v_3). We see this is exactly Equation 4.7. (done)

Now, WOLG, suppose $0 < a < b < c$.

We claim

umbilical point will never be in the plane $z = 0$

Assume $z = 0$ and $p = (x, y, 0)$ is an umbilical point . For all $(v_1, v_2, v_3) \in T_pS$, we have

$$\frac{-xv_2v_3}{a^2}\Big(\frac{1}{b^2} - \frac{1}{c^2}\Big) + \frac{yv_1v_3}{b^2}\Big(\frac{1}{a^2} - \frac{1}{c^2}\Big) = 0 \tag{4.10}$$

$$\frac{xv_1}{a^2} + \frac{yv_2}{b^2} = 0 \tag{4.11}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{4.12}$$

Equation 4.12 implies that either $x \neq 0$ or $y \neq 0$. WOLG suppose $x \neq 0$. We can now fix $(v_1, v_2, v_3) \in T_pS$ such that $v_2v_3 \neq 0$.

By multiplying with $\frac{x}{v_3}$ on both side, Equation 4.10 now implies

$$\frac{x^2v_2}{a^2}\Big(\frac{1}{b^2} - \frac{1}{c^2}\Big) = \frac{xyv_1}{b^2}\Big(\frac{1}{a^2} - \frac{1}{c^2}\Big)$$

Note that Equation 4.11 give us $\frac{y}{b^2} = \frac{-xv_1}{a^2v_2}$. This then implies

$$\frac{x^2v_2}{a^2}(\frac{1}{b^2} - \frac{1}{c^2}) = \frac{xyv_1}{b^2}(\frac{1}{a^2} - \frac{1}{c^2}) = \frac{-v_1^2x^2}{a^2v_2}(\frac{1}{a^2} - \frac{1}{c^2})$$

We can now use $a < b < c$ to deduce

$$0 < \frac{x^2v_2^2}{a^2}(\frac{1}{b^2} - \frac{1}{c^2}) = \frac{-v_1^2x^2}{a^2}(\frac{1}{a^2} - \frac{1}{c^2}) < 0 \text{ CaC } \text{ (done)}$$

Lastly, we claim p is an umbilical point if and only if

$$x^2 = a^2\frac{b^2 - a^2}{c^2 - a^2} \text{ and } y = 0 \text{ and } z^2 = c^2\frac{c^2 - b^2}{c^2 - a^2}$$

Because $z \neq 0$. We can now replace Equation 4.7 with

$$\frac{-xzv_2v_3}{a^2c^2}(\frac{1}{b^2} - \frac{1}{c^2}) + \frac{yzv_1v_3}{b^2c^2}(\frac{1}{a^2} - \frac{1}{c^2}) + \frac{-z^2v_1v_2}{c^4}(\frac{1}{a^2} - \frac{1}{b^2}) = 0 \tag{4.13}$$

Equation 4.8 give us $\frac{z}{c^2} = \frac{-xv_1}{a^2v_3} + \frac{-yv_2}{b^2v_3}$. Substitute this into Equation 4.13 (Except the $\frac{z^2}{c^4}$ term), we have

$$\begin{aligned} &v_2^2\frac{yx}{b^2a^2}\left(\frac{1}{b^2} - \frac{1}{c^2}\right) - v_1^2\frac{yx}{b^2a^2}\left(\frac{1}{a^2} - \frac{1}{c^2}\right) \\ &+v_1v_2\left(\frac{x^2}{a^4}\left(\frac{1}{b^2} - \frac{1}{c^2}\right) - \frac{y^2}{b^4}\left(\frac{1}{a^2} - \frac{1}{c^2}\right) - \frac{z^2}{c^4}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)\right) = 0 \end{aligned} \tag{4.14}$$

Assume $x = 0$, we see

$$0 < \frac{z^2}{c^4}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \frac{-y^2}{b^4}\left(\frac{1}{a^2} - \frac{1}{c^2}\right) < 0 \text{ CaC}$$

Because $x \neq 0$, we know there exists (v_1, v_2, v_3) such that $v_1 = 0$ and $v_2 = 1$. Substituting this back into Equation 4.14, we can deduce

$$xy = 0$$

Because $x \neq 0$. We now know $y = 0$. Again substituting this back into Equation 4.14, we have

$$\frac{x^2}{a^4}\left(\frac{1}{b^2} - \frac{1}{c^2}\right) = \frac{z^2}{c^4}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)$$

Substituting $y = 0$ back into Equation 4.9, we now see

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$$

We can now solve for x^2, z^2 in terms of a, b, c, using only linear algebra. The solution is

$$x^2 = a^2\frac{b^2 - a^2}{c^2 - a^2} \text{ and } z^2 = c^2\frac{c^2 - b^2}{c^2 - a^2} \text{ (done)}$$

■

Question 60

22. (*The Hessian.*) Let $h: S \to R$ be a differentiable function on a surface S, and let $p \in S$ be a critical point of h (i.e., $dh_p = 0$). Let $w \in T_p(S)$ and let

$$\alpha: (-\epsilon, \epsilon) \to S$$

be a parametrized curve with $\alpha(0) = p, \alpha'(0) = w$. Set

$$H_p h(w) = \left.\frac{d^2(h \circ \alpha)}{dt^2}\right|_{t=0}.$$

a. Let $\mathbf{x}: U \to S$ be a parametrization of S at p, and show that (the fact that p is a critical point of h is essential here)

$$H_p h(u'\mathbf{x}_u + v'\mathbf{x}_v) = h_{uu}(p)(u')^2 + 2h_{uv}(p)u'v' + h_{vv}(p)(v')^2.$$

Conclude that $H_p h: T_p(S) \to R$ is a well-defined (i.e., it does not depend on the choice of $\mathbf{x}$) quadratic form on $T_p(S)$. $H_p h$ is called the *Hessian* of h at p.

b. Let $h: S \to R$ be the height function of S relative to $T_p(S)$; that is, $h(q) = \langle q - p, N(p)\rangle, q \in S$. Verify that p is a critical point of h and thus that the Hessian $H_p h$ is well defined. Show that if $w \in T_p(S)$, $|w| = 1$, then

$$H_p h(w) = \text{normal curvature at } p \text{ in the direction of } w.$$

Conclude that *the Hessian at* p *of the height function relative to* $\mathrm{T_p(S)}$ *is the second fundamental form of* S *at* p.

Proof. Express

$$\alpha(t) = \mathbf{x}(u(t), v(t))$$

We know

$$w = \alpha' = u'\mathbf{x}_u + v'\mathbf{x}_v$$

Compute

$$\begin{aligned}
H_p(u'\mathbf{x}_u + v'\mathbf{x}_v) = H_p h(w) &= (h \circ \alpha)''(0) \\
&= (h_u u' + h_v v')' \\
&= (h_{uu}u' + h_{uv}v')u' + h_u(u'') + (h_{vu}u' + h_{vv}v')v' + h_v(v'') \\
&= h_{uu}(u')^2 + 2h_{uv}(u')(v') + h_{vv}(v')^2 \quad (\because dh_p = 0 \implies h_u = h_v = 0)
\end{aligned}$$

We know $\{\mathbf{x}_u, \mathbf{x}_v\}$ is a basis of T_pS. Our computation now show us that for any given vector $c_1\mathbf{x}_u + c_2\mathbf{x}_v \in T_pS$, we have

$$\begin{aligned}
H_p(c_1\mathbf{x}_u + c_2\mathbf{x}_v) &= h_{uu}c_1^2 + 2h_{uv}c_1c_2 + h_{vv}c_2^2 \\
&= \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} h_{uu} & h_{uv} \\ h_{uv} & h_{vv} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}
\end{aligned}$$

This then implies H_p is a quadratic form on T_pS.

Note that the definition of H_ph doesn't take usage of $\mathbf{x}$. We merely defiend

$$H_ph(w) \triangleq (h \circ \alpha)''(0)$$

(b) Compute

$$\begin{aligned}
h_u &= \langle \mathbf{x}_u, N(p) \rangle = 0 \\
\text{and } h_v &= \langle \mathbf{x}_v, N(p) \rangle = 0
\end{aligned}$$

This implies $dh_p = 0$, so p is indeed a critical point of h.

Compute

$$\begin{aligned}
h_{uu} &= \langle \mathbf{x}_{uu}, N \rangle = e \\
h_{uv} &= \langle \mathbf{x}_{uv}, N \rangle = f \\
h_{vv} &= \langle \mathbf{x}_{vv}, N \rangle = g
\end{aligned}$$

This now give us

$$H_ph(w) = (u')^2 e + 2u'v'f + g(v')^2 = \mathrm{II}_p(w)$$

and conclude the desired result.

∎

The following is the reference of 3.3.6 **(a)**

Question 61: 1-3:4

4. Let $\alpha\colon (0, \pi) \longrightarrow R^2$ be given by

$$\alpha(t) = \left(\cos t, \cos t + \log \tan \frac{t}{2}\right),$$

where t is the angle that the y axis makes with the vector $\alpha(t)$. The trace of α is called the *tractrix* (Fig. 1-9). Show that

19:

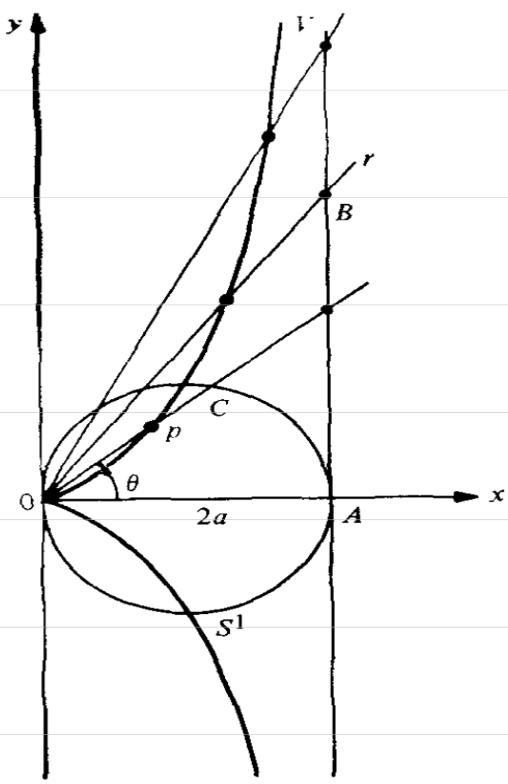


Figure 1-8. The cissoid of Diocles.

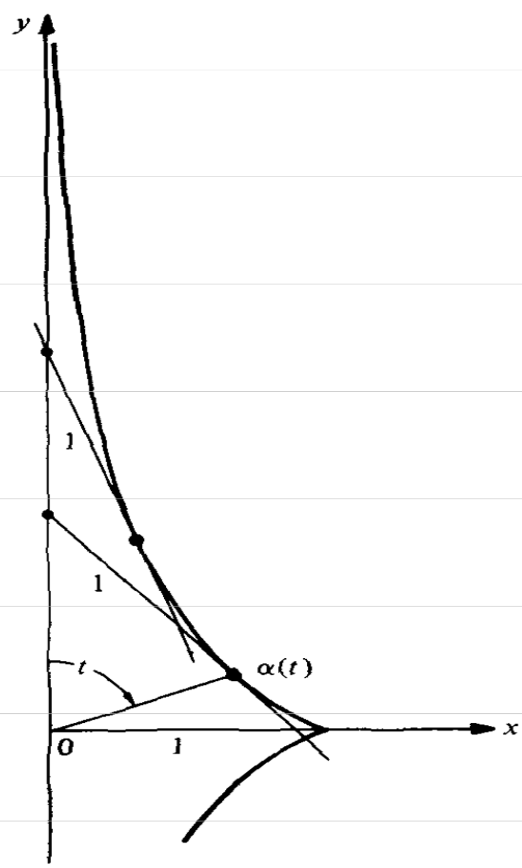


Figure 1-9. The tractrix.

a. α is a differentiable parametrized curve, regular except at $t = \pi/2$.

b. The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.

Typo correction: $\alpha(t) = (\sin t, \cos t + \ln \tan \frac{t}{2})$

Proof. **(a)**

Notice that the interval I is $(0, \pi)$. It is clear that

(a) $\sin t$ is smooth on $\mathbb{R}$

(b) $\cos t$ is smooth on $\mathbb{R}$

(c) $\ln t$ is smooth on $\mathbb{R}^+$

(d) $\tan \frac{t}{2}$ is smooth on I

Then it follows that α is a differentiable curve.

Compute

$$\alpha'(t) = (\cos t, -\sin t + \frac{1}{\tan \frac{t}{2}} \cdot \sec^2 \frac{t}{2} \cdot \frac{1}{2})$$

Because $\cos t = \alpha_1'(t)$ is 0 on I only when $t = \frac{\pi}{2}$, we know α is regular on I except possibly at $t = \frac{\pi}{2}$.

Compute

$$\alpha'(\frac{\pi}{2}) = (0, -1 + \frac{1}{1} \cdot 2 \cdot \frac{1}{2}) = (0, 0)$$

We now conclude α is regular on I except $\frac{\pi}{2}$.

(b)

A useful Identity give us

$$\alpha'(t) = (\cos t, -\sin t + \csc t)$$

From the following facts

(a) the first argument of the segment is from 0 to $\sin t = \alpha(t)$

(b) $\alpha_x'(t) = \cos t$

(c) $\frac{\sin t}{\cos t} = \tan t$

We conclude that the length of the segment is

$$\begin{aligned}
|\tan t| \cdot |\alpha'(t)| &= |\tan t| \cdot \sqrt{\cos^2 t + \sin^2 t - 2 \sin t \csc t + \csc^2 t} \\
&= |\tan t| \cdot \sqrt{1 - 2 + \csc^2 t} \\
&= |\tan t| \cdot \sqrt{\csc^2 t - 1} = |\tan t| \cdot \sqrt{\cot^2 t} = 1
\end{aligned}$$

■

Chapter 5

The Intrinsic Geometry of Surfaces

5.1 Isometry

Definition 5.1.1. (Definition of Isometry) Given two regular surface $S, \overline{S}$, we say map $\phi : S \to \overline{S}$ is an **isometry** if

(a) ϕ is a diffeomorphism

(b) $w_1 \cdot w_2 = d\phi_p(w_1) \cdot d\phi_p(w_2)$ for all $p \in S$ and $w_1, w_2 \in T_pS$

Suppose $q \in \overline{S}$ and $v_1, v_2 \in T_q\overline{S}$ and $\phi(p) = q$. One can check

$$\begin{aligned} v_1 \cdot v_2 &= d\phi_p \circ (d\phi_p)^{-1} v_1 \cdot d\phi_p \circ (d\phi_p)^{-1} v_2 \\ &= (d\phi_p)^{-1} v_1 \cdot (d\phi_p)^{-1} v_2 \\ &= d(\phi^{-1})_q v_1 \cdot d(\phi^{-1})_q v_2 \end{aligned}$$

This established that Isometry is a symmetric relation. Checking Isometry relation is also transitive is again straightforward. This established that Isometry is an equivalence relation.

We now can say that two regular surface $S, \overline{S}$ is **isometric** if there exists an isometry between them.

Theorem 5.1.2. (Isometric if and only if first fundamental forms are equivalent) Given a diffeomorphism $\phi : S \to \overline{S}$

$$\phi \text{ is an isometry} \iff I_p(w) = I_{\phi(p)}(d\phi_p w) \text{ for all } w \in T_pS$$

Proof. From left to right is straight forward. From right to left, fix $p \in S$ and $w_1, w_2 \in T_pS$. Observe

$$\begin{aligned}\langle w_1, w_2 \rangle &= \frac{1}{2}\Big(I_p(w_1+w_2) - I_p(w_1) - I_p(w_2)\Big) \\ &= \frac{1}{2}\Big(I_{\phi(p)}(d\phi_p(w_1+w_2)) - I_{\phi(p)}(d\phi_p w_1) - I_{\phi(p)}(d\phi_p w_2)\Big) \\ &= \frac{1}{2}\Big(I_{\phi(p)}(d\phi_p w_1 + d\phi_p w_2) - I_{\phi(p)}(d\phi_p w_1) - I_{\phi(p)}(d\phi_p w_2)\Big) \\ &= \langle d\phi_p w_1, d\phi_p w_2 \rangle\end{aligned}$$

■

Theorem 5.1.3. (Existence of Local Isometry) Given two regular charts $\mathbf{x}, \overline{\mathbf{x}}$ from U to $S, \overline{S}$ such that

$$E = \overline{E} \text{ and } F = \overline{F} \text{ and } G = \overline{G} \text{ on } U$$

We have

$$\overline{\mathbf{x}} \circ \mathbf{x}^{-1} : \mathbf{x}(U) \to \overline{\mathbf{x}}(U) \text{ is an isometry}$$

Proof. It is straightforward to check $\overline{\mathbf{x}} \circ \mathbf{x}^{-1} : \mathbf{x}(U) \to \overline{\mathbf{x}}(U)$ is a diffeomorphism, trivially using the chart $\mathbf{x}, \overline{\mathbf{x}}$.

Suppose that we are given a curve $\alpha(t) \triangleq \mathbf{x}(u(t), v(t))$. We wish to show

$$I_\alpha(\alpha') = I_{\phi(\alpha)}(d\phi_\alpha \alpha')$$

We know

$$I_\alpha(\alpha') = E(u')^2 + 2F(u')(v') + G(v')^2$$

Observe that

$$\phi \circ \alpha(t) = \overline{\mathbf{x}}(u(t), v(t))$$

This implies

$$\begin{aligned}I_{\phi(\alpha)}(d\phi_\alpha \alpha') &= \overline{E}(u')^2 + 2\overline{F}(u')(v') + \overline{G}(v')^2 \\ &= E(u')^2 + 2F(u')(v') + G(v')^2 = I_\alpha(\alpha') \text{ (done)}\end{aligned}$$

■

Definition 5.1.4. (Definition of Locally Isometric) We say S **is locally isometric** to $\overline{S}$ if for all $p \in S$, there exists an isometry from an open set containing p to $\overline{S}$. We say $S, \overline{S}$ are **locally isomteric**, if $S, \overline{S}$ are locally isomteric to each other.

We now extend our discussion from length to angle.

Definition 5.1.5. (Definition of Conformal Map) A diffeomorphism $\phi : S \to \overline{S}$ is a conformal map if for all $p \in S$ and $v_1, v_2 \in T_pS$, we have

$$\langle d\phi_p(v_1), d\phi_p(v_2)\rangle = \lambda^2(p)\langle v_1, v_2\rangle$$

where λ^2 is a nowhere-zero differentiable function on S.

One can check that the inverse of a conformal map is still conformal by following computation. Fix $q \triangleq \phi(p)$ and $w_1 \triangleq d\phi_p(v_1), w_2 \triangleq d\phi_p(v_2)$. We have

$$\begin{aligned}\langle d(\phi^{-1})_q w_1, d(\phi^{-1})_q w_2\rangle &= \langle (d\phi_p)^{-1}w_1, (d\phi_p)^{-1}w_2\rangle \\ &= \langle v_1, v_2\rangle \\ &= \frac{\lambda^2}{\lambda^2}\langle v_1, v_2\rangle \\ &= \frac{1}{\lambda^2}\langle d\phi_p v_1, d\phi_p v_2\rangle \\ &= \frac{1}{\lambda^2}\langle w_1, w_2\rangle\end{aligned}$$

This established that conformal is a symmetric relation. Checking conformal relation is also transitive is again straightforward. This established that conformal is an equivalence relation.

We now can say that two regular surface $S, \overline{S}$ are **conformal** if there exists an conformal map between them.

Theorem 5.1.6. (Existence of Local Conformality) Given $\mathbf{x} : U \to S$ and $\overline{\mathbf{x}} : U \to \overline{S}$ such that

$$E = \lambda^2\overline{E} \text{ and } F = \lambda^2\overline{F} \text{ and } G = \lambda^2\overline{G}$$

where λ^2 is a nowhere-zero differentaible function on U. Then the map

$$\overline{\mathbf{x}} \circ \mathbf{x}^{-1} : \mathbf{x}(U) \to \overline{\mathbf{x}}(U)$$

Proof. Fix $p \in \mathbf{x}(U)$ and $v_1, v_2 \in T_pS$. Compute

$$\langle v_1, v_2\rangle = \frac{1}{2}\Big(I_p(v_1 + v_2) - I_p(v_1) - I_p(v_2)\Big)$$

Express

$$v_1 \triangleq c_1\mathbf{x}_u + c_2\mathbf{x}_v \text{ and } v_2 \triangleq d_1\mathbf{x}_u + d_2\mathbf{x}_v$$

Continue the computation

$$\begin{aligned}\langle v_1, v_2\rangle &= \frac{1}{2}\Big((c_1+d_1)^2E + (c_1+d_1)(c_2+d_2)2F + (c_1+d_1)^2G - \cdots\Big)\\ &= \frac{\lambda^2}{2}\Big((c_1+d_1)^2\overline{E} + (c_1+d_1)(c_2+d_2)2\overline{F} + (c_1+d_1)^2\overline{G} + \cdots\Big)\\ &= \frac{1}{2}\Big(I_{\phi(p)}\big((c_1+d_1)\overline{\mathbf{x}}_u + (c_2+d_2)\overline{\mathbf{x}}_v\big) - I_{\phi(p)}(c_1\overline{\mathbf{x}}_u + c_2\overline{\mathbf{x}}_v) - I_{\phi(p)}(d_1\overline{\mathbf{x}}_u + d_2\overline{\mathbf{x}}_v)\Big)\\ &= \frac{1}{2}\Big(I_{\phi(p)}(d\phi_p v_1 + d\phi_p v_2) - I_{\phi(p)}(d\phi_p v_1) - I_{\phi(p)}(d\phi_p v_2)\Big)\\ &= \langle d\phi_p v_1, d\phi_p v_2\rangle\end{aligned}$$

■

A very important Theorem regarding isometry is as follow.

Theorem 5.1.7. (Isometry induce chart) Given a regular surface S, a chart $\mathbf{x}: U \to S$, an isometry ϕ from an open $V \subseteq \mathbf{x}(U)$ to another regular surface $\overline{S}$, the chart $\overline{\mathbf{x}} : \mathbf{x}^{-1}(V) \to \overline{S}$ defined by

$$\overline{\mathbf{x}}(u,v) \triangleq \phi \circ \mathbf{x}(u,v)$$

is regular and give us

$$\overline{E} = E \text{ and } \overline{F} = F \text{ and } \overline{G} = G$$

Proof. Observe

$$\overline{\mathbf{x}}_u = d\phi_p(\mathbf{x}_u) \text{ and } \overline{\mathbf{x}}_v = d\phi_p(\mathbf{x}_v)$$

Because $d\phi_p$ is full rank ($\because \phi$ is differentiable) and $\mathbf{x}_u, \mathbf{x}_v$ are linearly independent, we see $\overline{\mathbf{x}}_u, \overline{\mathbf{x}}_u$ are also linearly independent. This shows that $\overline{\mathbf{x}}$ is regular.

Now because ϕ is an isometry,

$$\overline{E} = \langle \overline{\mathbf{x}}_u, \overline{\mathbf{x}}_u\rangle = \langle d\phi_p\mathbf{x}_u, d\phi_p\mathbf{x}_u\rangle = \langle \mathbf{x}_u, \mathbf{x}_u\rangle = E$$

The proof for rest of the statement is similar. ■

5.2 Examples of Isometry and Conformal

Example 1 (Part of the Cylinder is Isometry to Rectangle)

$$U \triangleq \{(u,v) \in \mathbb{R}^2 : u \in (0,2\pi)\}$$

Define $\mathbf{x} : U \to \mathbb{R}^3$ and $\overline{\mathbf{x}} : U \to \mathbb{R}^3$ by, where w_1, w_2 are orthonormal

$$\mathbf{x}(u,v) \triangleq p_0 + uw_1 + vw_2 \text{ and } \overline{\mathbf{x}}(u,v) \triangleq (\cos u, \sin u, v)$$

Define $S, \overline{S}$ by

$$S \triangleq \mathbf{x}(U) \text{ and } \overline{S} = \overline{\mathbf{x}}(U)$$

It is clear that $S, \overline{S}$ are all regular surface, as a global coordinate are already given. Define $f : \overline{S} \to S$ by

$$f(p) \triangleq \mathbf{x} \circ (\overline{\mathbf{x}})^{-1}(p)$$

We claim

$$f \text{ is an isometry from } \overline{S} \text{ to } S$$

We first prove

$$f \text{ is a diffeomorphism}$$

Observe

$$\mathbf{x}^{-1} \circ f \circ \overline{\mathbf{x}} = \mathbf{id} \text{ and } (\overline{\mathbf{x}})^{-1} \circ f^{-1} \circ \mathbf{x} = \mathbf{id}$$

This conclude f is diffeomorphism. (done)

Fix $\phi \triangleq f^{-1} = \overline{\mathbf{x}} \circ \mathbf{x}^{-1}$. Suppose that we are given a curve $\alpha(t) \triangleq \mathbf{x}(u(t), v(t))$ lies in S. We wish to show

$$I_\alpha(\alpha') = I_{\phi(\alpha)}(d\phi_\alpha \alpha')$$

We know

$$I_\alpha(\alpha') = E(u')^2 + 2F(u')(v') + G(v')^2$$

Observe that

$$\phi \circ \alpha(t) = \overline{\mathbf{x}}(u(t), v(t))$$

This implies

$$I_{\phi\alpha}(d\phi_\alpha \alpha') = \overline{E}(u')^2 + 2\overline{F}(u')(v') + \overline{G}(v')^2$$

After checking that

$$\begin{aligned} E &= \overline{E} = 1 \\ F &= \overline{F} = 0 \\ G &= \overline{G} = 1 \text{ (done)} \end{aligned}$$

5.3 Christoffel symbols

Suppose S is an oriented regular surface. Given a local $\mathbf{x} : U \to S$ with the same orientation, because $\{\mathbf{x}_u, \mathbf{x}_v, N\}$ are pointwise an orthogonal basis of $\mathbb{R}^3$, we know there exists real-valued functions $\Gamma^1_{11}, \Gamma^1_{12}, \Gamma^1_{21}, \Gamma^1_{22}, \Gamma^2_{11}, \Gamma^2_{12}, \Gamma^2_{21}, \Gamma^2_{22}, a_{11}, a_{21}, a_{12}, a_{22}$ defined on U such that we have pointwise

$$\begin{aligned}
\mathbf{x}_{uu} &= \Gamma^1_{11}\mathbf{x}_u + \Gamma^2_{11}\mathbf{x}_v + eN \\
\mathbf{x}_{uv} &= \Gamma^1_{12}\mathbf{x}_u + \Gamma^2_{12}\mathbf{x}_v + fN \\
\mathbf{x}_{vu} &= \Gamma^1_{21}\mathbf{x}_u + \Gamma^2_{21}\mathbf{x}_v + fN \\
\mathbf{x}_{vv} &= \Gamma^1_{22}\mathbf{x}_u + \Gamma^2_{22}\mathbf{x}_v + gN \\
N_u &= a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v \\
N_v &= a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v
\end{aligned}$$

Because $\mathbf{x}$ is smooth, we know $\mathbf{x}_{uv} = \mathbf{x}_{vu}$.This tell us

$$\Gamma^1_{12} = \Gamma^1_{21} \text{ and } \Gamma^2_{12} = \Gamma^2_{21}$$

One can differentiate $\langle \mathbf{x}_u, \mathbf{x}_u \rangle$ with respect to u and observe

$$\Gamma^1_{11}E + \Gamma^2_{11}F = \langle \mathbf{x}_{uu}, \mathbf{x}_u \rangle = \frac{1}{2}E_u$$

Similarly, one can deduce

$$\begin{cases}
\Gamma^1_{11}E + \Gamma^2_{11}F & = \langle \mathbf{x}_{uu}, \mathbf{x}_u \rangle = \frac{1}{2}E_u \\
\Gamma^1_{12}E + \Gamma^2_{12}F & = \langle \mathbf{x}_{uv}, \mathbf{x}_u \rangle = \frac{1}{2}E_v \\
\Gamma^1_{21}F + \Gamma^2_{21}G & = \langle \mathbf{x}_{vu}, \mathbf{x}_v \rangle = \frac{1}{2}G_u \\
\Gamma^1_{22}F + \Gamma^2_{22}G & = \langle \mathbf{x}_{vv}, \mathbf{x}_v \rangle = \frac{1}{2}G_v
\end{cases}$$

Now, because

$$F_u = \frac{\partial}{\partial u}\langle \mathbf{x}_u, \mathbf{x}_v \rangle = \langle \mathbf{x}_{uu}, \mathbf{x}_v \rangle + \langle \mathbf{x}_u, \mathbf{x}_{uv} \rangle = \langle \mathbf{x}_{uu}, \mathbf{x}_v \rangle + \frac{1}{2}E_v$$

We have

$$\Gamma^1_{11}F + \Gamma^2_{11}G = \langle \mathbf{x}_{uu}, \mathbf{x}_v \rangle = F_u - \frac{1}{2}E_u \tag{5.1}$$

Again because

$$F_v = \frac{\partial}{\partial_v}\langle \mathbf{x}_u, \mathbf{x}_v\rangle = \langle \mathbf{x}_{uv}, \mathbf{x}_v\rangle + \langle \mathbf{x}_u, \mathbf{x}_{vv}\rangle = \frac{1}{2}G_u + \langle \mathbf{x}_u, \mathbf{x}_{vv}\rangle$$

We have

$$\Gamma^1_{22}E + \Gamma^2_{22}F = \langle \mathbf{x}_u, \mathbf{x}_{vv}\rangle = F_v - \frac{1}{2}G_u \tag{5.2}$$

In conclusion, we have the following three sets of equations

$$\begin{cases} E\Gamma^1_{11} + F\Gamma^2_{11} = \frac{1}{2}E_u \\ F\Gamma^1_{11} + G\Gamma^2_{11} = F_u - \frac{1}{2}E_u \end{cases}$$
$$\begin{cases} E\Gamma^1_{12} + F\Gamma^2_{12} = \frac{1}{2}E_v \\ F\Gamma^1_{12} + G\Gamma^2_{12} = \frac{1}{2}G_u \end{cases}$$
$$\begin{cases} F\Gamma^1_{22} + G\Gamma^2_{22} = \frac{1}{2}G_v \\ E\Gamma^1_{22} + F\Gamma^2_{22} = F_v - \frac{1}{2}G_u \end{cases}$$

Now, because $\sqrt{EG - F^2} = |\mathbf{x}_u \times \mathbf{x}_v| > 0$, we see the determinant $EG - F^2 \neq 0$. This implies that the unknowns $\Gamma^1_{11}, \Gamma^2_{11}, \Gamma^1_{12}, \Gamma^2_{12}, \Gamma^1_{22}, \Gamma^2_{22}$ can all be solved for.

Theorem 5.3.1. (Gauss Formula)

$$K = \frac{(\Gamma^2_{12})_u - (\Gamma^2_{11})_v + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{21} - \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{11}\Gamma^2_{12}}{-E}$$

Proof. ■

5.4 Vector Field

In this section, S is a regular surface with orientation N, $I \triangleq [0, l]$, $\alpha : I \to S$ is a smooth curve, and when α lies in coordinate chart $\mathbf{x} : U \to S$, α is expressed

$$\alpha(t) \triangleq \mathbf{x}(u(t), v(t))$$

Definition 5.4.1. (Definition of Vector Field along α) A **vector field w along α** is a function $w : I \to \mathbb{R}^3$ such that

$$w(t) \in T_{\alpha(t)}S \text{ for all } t \in I$$

It is clear that $\{w, N, N \times w\}$ form an orthogonal basis for each $t \in I$. We then can write

$$w'(t) = A(t)w + B(t)N + C(t)N \times w$$

and define the **covariant derivative** by

$$\frac{Dw}{dt}(t) \triangleq Aw + CN \times w = w' - \langle w', N \rangle$$

If w is unit, we see $A = 0$, and we shall denote C by $\left[\frac{Dw}{dt}\right]$

Definition 5.4.2. (Definition of Parallel Vector Field) A vector field w along α is said to be **parallel** if

$$\frac{Dw}{dt}(t) = 0 \text{ for all } t \in I$$

or, equivalently

$$w'(t) \perp T_{\alpha(t)}S \text{ for all } t \in I$$

Theorem 5.4.3. (Expected Property of Parallel Vector Field) If w, v are parallel vector fields along α, then the following real-valued functions

(a) $\langle w(t), v(t) \rangle$

(b) $|w(t)|$

(c) $|v(t)|$

(d) $\frac{\langle w(t), v(t) \rangle}{|w| \cdot |v|}$

are constants on I.

Proof. It is straightforward to check that we only have to prove $\langle w(t), v(t)\rangle$ is constant on I. Observe

$$\Big(\langle w(t), v(t)\rangle\Big)' = \langle w'(t), v(t)\rangle + \langle w(t), v'(t)\rangle$$

WOLG, it suffices to show

$$\langle w'(t), v(t)\rangle = 0 \text{ on } I$$

Because w is parallel, by definition, w' is orthogonal to T_pS. The conclusion then follows from the fact $v \in T_pS$. (done) ■

Lemma 5.4.4. Let a, b be two $\mathbb{R}$-valued differentiable functions on I such that

$$a^2 + b^2 = 1 \text{ on } I$$

Let $t_0 \in I, \phi_0 \in \mathbb{R}$ satisfy

$$a(t_0) = \cos\phi_0 \text{ and } b(t_0) = \sin\phi_0$$

The differentiable function

$$\phi \triangleq \phi_0 + \int_{t_0}^{t} (ab' - ba')dt$$

satisfy

$$a(t) = \cos\phi(t) \text{ and } b(t) = \sin\phi(t) \text{ on } I$$

Proof. It suffices to show

$$(a - \cos\phi)^2 + (b - \sin\phi)^2 = 0 \text{ on } I$$

Because $\big((a - \cos\phi)^2 + (b - \sin\phi)^2\big)\Big|_{t=t_0} = 0$, it suffices to show

$$\Big((a - \cos\phi)^2 + (b - \sin\phi)^2\Big)' = 0 \text{ on } I$$

Observe

$$\begin{aligned}(a - \cos\phi)^2 + (b - \sin\phi)^2 &= (a^2 + b^2) + (\cos^2\phi + \sin^2\phi) - 2a\cos\phi - 2b\sin\phi \\ &= 2 - 2a\cos\phi - 2b\sin\phi\end{aligned}$$

This reduce the problem into proving

$$(a\cos\phi + b\sin\phi)' = 0 \text{ on } I$$

Observe

$$
\begin{aligned}
(a\cos\phi + b\sin\phi)' &= a'\cos\phi + b'\sin\phi - a\phi'\sin\phi + b\phi'\cos\phi \\
&= (a' + b\phi')\cos\phi + (b' - a\phi')\sin\phi \\
&= (a' + b(ab' - ba'))\cos\phi + (b' - a(ab' - ba'))\sin\phi
\end{aligned}
$$

This reduce the problem into proving

$$
a' + b(ab' - ba') = b' - a(ab' - ba') = 0 \text{ on } I
$$

Note that

$$
2(aa' + bb') = (a^2 + b^2)' = 0 \text{ on } I
$$

This give us

$$
aa' = -bb'
$$

We now see

$$
\begin{aligned}
a' + b(ab' - ba') &= a' + abb' - a'b^2 \\
&= a' - a^2a' - a'b^2 \\
&= a' - a' = 0
\end{aligned}
$$

We now see

$$
\begin{aligned}
b' - a(ab' - ba') &= b' - a^2b' + aa'b \\
&= b' - a^2b' - b^2b' \\
&= b' - b' = 0 \text{ (done)}
\end{aligned}
$$

■

Given two vectors $v, w \in \mathbb{R}^3$, and let $\widehat{v} \triangleq \frac{v}{|v|}, \widehat{w} \triangleq \frac{w}{|w|}$. We can show that there exists ϕ such that

$$
\langle \widehat{v}, \widehat{w} \rangle = \cos\phi \text{ and } \langle (\widehat{v} \times \widehat{w}) \times \widehat{v}, \widehat{w} \rangle = \sin\phi
$$

Lemma 5.4.5.

5.5 Real Geodesic

Definition 5.5.1. (Definition of Parametrized Geodesic) Let $\alpha : I \to S$ be a smooth curve in S. By definition of tangent plane, α' is a vector field along α. If α' is differentiable and parallel, then we say α is a **parametrized geodesic**.

Note that α' is parallel if and only if α'' orthogonal to T_pS. Then if we are given the fact α is parametrized by arc-length, then α is a paremetrized geodesic if and only if N_α is parallel with N on I.

Let n be the normal of C

$$\begin{aligned} \kappa_C n = \frac{dT}{ds} &= 0T + \kappa_n N + \kappa_g N \times T \text{ where } N \text{ is the normal of } S \\ \implies \kappa_C^2 &= \kappa_n^2 + \kappa_g^2 \end{aligned}$$

Definition 5.5.2. (Definition of Geodesic) Let C be a connected regular curve in S. If for each $p \in C$, there exists an arc-length parametrization $\alpha : I \to C$ around p such that α''

Given a curve C (a subset) in S, we say C is a **geodesic** if N_C and N_S are parallel, where N_C is the normal of C and N_S is the normal of S.

5.6 HW6

Question 62

4. Use the stereographic projection (cf. Exercise 16, Sec. 2-2) to show that the sphere is locally conformal to a plane.

Proof. We are given $f : \mathbb{R}^2 \to S^2$

$$f(x,y) = \frac{(2x, 2y, x^2+y^2-1)}{x^2+y^2+1}$$

We claim

$$df_{(x,y)}u \cdot df_{(x,y)}v = \frac{4(u \cdot v)}{(x^2+y^2+1)^2} \text{ for all } u, v \in \mathbb{R}^2$$

We reduce the problem into proving

$$\begin{cases} df_{(x,y)}e_1 \cdot df_{(x,y)}e_1 = \frac{4}{(x^2+y^2+1)^2} \\ df_{(x,y)}e_1 \cdot df_{(x,y)}e_2 = 0 \\ df_{(x,y)}e_2 \cdot df_{(x,y)}e_2 = \frac{4}{(x^2+y^2+1)^2} \end{cases}$$

Compute

$$df_{(x,y)} = \frac{1}{(x^2+y^2+1)^2} \begin{bmatrix} 2(y^2-x^2+1) & -4xy \\ -4xy & 2(x^2-y^2+1) \\ 4x & 4y \end{bmatrix}$$

The rest then follows from direct computation. (done)

Note that we actually miss the north pole. To prove that S^2 is locally conformal to $\mathbb{R}^2$ at the north pole, we just have to do the same computation with

$$g(x,y) = \frac{(2x, 2y, 1-x^2-y^2)}{x^2+y^2+1}$$

∎

Question 63

5. Let $\alpha_1 : I \to R^3$, $\alpha_2 : I \to R^3$ be regular parametrized curves, where the parameter is the arc length. Assume that the curvatures k_1 of α_1 and k_2 of α_2 satisfy $k_1(s) = k_2(s) \neq 0$, $s \in I$. Let

$$\mathbf{x}_1(s, v) = \alpha_1(s) + v\alpha_1'(s),$$
$$\mathbf{x}_2(s, v) = \alpha_2(s) + v\alpha_2'(s)$$

be their (regular) tangent surfaces (cf. Example 5, Sec. 2-3) and let V be a neighborhood of (s_0, v_0) such that $\mathbf{x}_1(V) \subset R^3$, $\mathbf{x}_2(V) \subset R^3$ are regular surfaces (cf. Prop. 2, Sec. 2-3). Prove that $\mathbf{x}_1 \circ \mathbf{x}_2^{-1} : \mathbf{x}_2(V) \to \mathbf{x}_1(V)$ is an isometry.

Proof. Compute

$$\begin{aligned}
(\mathbf{x}_1)_s &= \alpha_1' + v\alpha_1'' \\
(\mathbf{x}_1)_v &= \alpha_1' \\
(\mathbf{x}_2)_s &= \alpha_2' + v\alpha_2'' \\
(\mathbf{x}_2)_v &= \alpha_2'
\end{aligned}$$

Using the fact $\alpha' = T$ and $\alpha'' = \kappa N$ compute

$$\begin{aligned}
E_1 = 1 + v^2\kappa^2 \text{ and } F_1 = 1 \text{ and } G_1 = 1 \\
E_2 = 1 + v^2\kappa^2 \text{ and } F_2 = 1 \text{ and } G_2 = 1
\end{aligned}$$

This implies that $\mathbf{x}_1 \circ \mathbf{x}_2^{-1} : \mathbf{x}_2(V) \to \mathbf{x}_1(V)$ is an isometry. ■

Question 64

***6.** Let $\alpha: I \to R^3$ be a regular parametrized curve with $k(t) \neq 0$, $t \in I$. Let $\mathbf{x}(t, v)$ be its tangent surface. Prove that, for each $(t_0, v_0) \in I \times (R - \{0\})$, there exists a neighborhood V of (t_0, v_0) such that $\mathbf{x}(V)$ is isometric to an open set of the plane (*thus, tangent surfaces are locally isometric to planes*).

Proof. Parametrize α by arc-length s in an open neighborhood around t_0. We can now reparametrize $\mathbf{x}(V)$ by

$$\mathbf{x}(s,v) = \alpha(s) + v\alpha'(s)$$

By Fundamental Theorem of Local Curves, we know there exists a smooth arc-length parametrized curve $\beta : I \to \mathbb{R}^3$ such that β and α has the same curvature everywhere and has zero torsion everywhere. Because β has zero torsion everywhere, we know $B'_\beta = 0$ everywhere. In other words, B_β is constant on I. This implies the range of T_β is contained by a 2-dimensional subspace of $\mathbb{R}^3$. Let $p = \beta(t_0)$, and let such subspace be W. We see that

$$\beta(s) = \int_{t_0}^{s} T_\beta(s')ds' \in p + W$$

This implies that β is contained by the plane $p + W$.

Define $\mathbf{x}_2 : V \to \mathbb{R}^3$ by

$$\mathbf{x}_2(s,v) = \beta(s) + v\beta'(s)$$

Because $\beta'(s) \in W$, we see $\mathbf{x}_2(V)$ lies in $p + W$. It now follows from the last question that $\mathbf{x}(V)$ is isometric $\mathbf{x}_2(V)$. ∎

Question 65

***8.** Let $G\colon R^3 \to R^3$ be a map such that

$$|G(p) - G(q)| = |p - q| \quad \text{for all } p, q \in R^3$$

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(that is, G is a *distance-preserving* map). Prove that there exists $p_0 \in R^3$ and a linear isometry (cf. Exercise 7) F of the vector space R^3 such that

$$G(p) = F(p) + p_0 \quad \text{for all } p \in R^3.$$

Proof. We claim one of such p_0 is

$$p_0 \triangleq G(0)$$

We reduce the problem into proving

$$F(p) \triangleq G(p) - G(0) \text{ is a linear isometry of } \mathbb{R}^3$$

Note that

$$F(0) = 0 \text{ and } |F(p)| = |G(p) - G(0)| = |p|$$

This by Mazur-Ulam Theorem implies F is affine. ■

Question 66

10. Let S be a surface of revolution. Prove that the rotations about its axis are isometries of S.

Proof. We are given charts $\mathbf{x}, \overline{\mathbf{x}} : I \times (0, 2\pi) \to S$

$$\mathbf{x}(t,\theta) = \Big(\cos\theta f(t), \sin\theta f(t), g(t)\Big)$$
$$\overline{\mathbf{x}}(t,\theta) = \Big(\cos(\theta+\theta_0) f, \sin(\theta+\theta_0) f, g\Big)$$

It is clear that if we rotate S about z-axis counterclockwise of θ_0 degree, then the restriction of the action ϕ is exactly $\overline{\mathbf{x}} \circ \mathbf{x}^{-1}$. The problem is now reduced to proving

$$E = \overline{E} \text{ and } F = \overline{F} \text{ and } G = \overline{G}$$

Compute

$$\mathbf{x}_t = \Big(\cos\theta f', \sin\theta f', g'\Big)$$
$$\mathbf{x}_\theta = \Big(-\sin\theta f, \cos\theta f, 0\Big)$$
$$\overline{\mathbf{x}}_t = \Big(\cos(\theta+\theta_0) f', \sin(\theta+\theta_0) f', g'\Big)$$
$$\overline{\mathbf{x}}_\theta = \Big(-\sin(\theta+\theta_0) f, \cos(\theta+\theta_0) f, 0\Big)$$

This give us

$$E = (f')^2 + (g')^2 \text{ and } F = 0 \text{ and } G = f^2$$
$$\overline{E} = (f')^2 + (g')^2 \text{ and } \overline{F} = 0 \text{ and } \overline{G} = f^2$$

∎

Question 67

14. We say that a differentiable map $\varphi\colon S_1 \to S_2$ *preserves angles* when for every $p \in S_1$ and every pair $v_1, v_2 \in T_p(S_1)$ we have

$$\cos(v_1, v_2) = \cos(d\varphi_p(v_1), d\varphi_p(v_2)).$$

Prove that φ is locally conformal if and only if it preserves angles.

Proof. $(\longleftarrow)$

Fix $p \in S$. Let e_1, e_2 be an orthonormal basis of T_pS. By premise,

$$\langle d\phi_p e_1, d\phi_p e_2\rangle = 0$$

Express

$$\langle d\phi_p e_1, d\phi_p e_1\rangle = \lambda_1$$
$$\langle d\phi_p e_2, d\phi_p e_2\rangle = \lambda_2$$

Let

$$v_1 = e_1 \text{ and } v_2 = \cos\theta e_1 + \sin\theta e_2$$

Then by premise

$$\cos\theta = \frac{\lambda_1 \cos\theta}{\sqrt{\lambda_1}\sqrt{\cos^2\theta\lambda_1 + \sin^2\theta\lambda_2}}$$

Note that the denominator will never be zero because $d\phi_p$ is full rank by premise.

We can now deduce

$$\lambda_1 = \cos^2\theta\lambda_1 + \sin^2\theta\lambda_2$$

This implies

$$(1 - \cos^2\theta)\lambda_1 = \sin^2\theta\lambda_2$$

which implies $\lambda_1 = \lambda_2$. We now see that for all $c_1, c_2, d_1, d_2 \in \mathbb{R}$, we have

$$\begin{aligned}\langle c_1e_1 + c_2e_2, d_1e_1 + d_2e_2\rangle &= c_1d_1 + c_2d_2\\ &= \frac{1}{\lambda_1}\langle d\phi_p(c_1e_1 + c_2e_2), d\phi_p(d_1e_1 + d_2e_2)\rangle\end{aligned}$$

$(\longrightarrow)$

Fix $p \in S$. Let e_1, e_2 be an orthonormal basis of T_pS. By premise, we see

$$\begin{aligned}\langle d\phi_p e_1, d\phi_p e_2\rangle &= \lambda^2 \langle e_1, e_2\rangle = 0\\ \langle d\phi_p e_1, d\phi_p e_1\rangle &= \lambda^2 \langle e_1, e_1\rangle = \lambda^2\\ \langle d\phi_p e_2, d\phi_p e_2\rangle &= \lambda^2 \langle e_2, e_2\rangle = \lambda^2\end{aligned}$$

Express $v_1, v_2 \in T_pS$

$$v_1 \triangleq c_1e_1 + c_2e_2 \text{ and } v_2 \triangleq d_1e_1 + d_2e_2$$

Using the equations set we have, we see

$$\begin{aligned}|d\phi_p v_1| &= |c_1 d\phi_p e_1 + c_2 d\phi_p e_2|\\ &= \sqrt{|c_1 d\phi_p e_1|^2 + |c_2 d\phi_p e_2|^2} = \lambda\sqrt{c_1^2 + c_2^2} = \lambda\, |v_1|\end{aligned}$$

Similarly,

$$|d\phi_p v_2| = \lambda\, |v_2|$$

Observe

$$\begin{aligned}\cos(v_1, v_2) &= \frac{\langle v_1, v_2\rangle}{|v_1|\,|v_2|}\\ &= \frac{\langle d\phi_p v_1, d\phi_p v_2\rangle}{\lambda^2\,|v_1|\,|v_2|}\\ &= \frac{\langle d\phi_p v_1, d\phi_p v_2\rangle}{|d\phi_p v_1|\,|d\phi_p v_2|} = \cos(d\phi_p v_1, d\phi_p v_2)\end{aligned}$$

■

Question 68

15. Let $\varphi\colon R^2 \to R^2$ be given by $\varphi(x, y) = (u(x, y), v(x, y))$, where u and v are differentiable functions that satisfy the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x.$$

Show that φ is a local conformal map from $R^2 - Q$ into R^2, where $Q = \{(x, y) \in R^2; u_x^2 + u_y^2 = 0\}$.

Proof. Because $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ satisfy Cauchy-Riemann equations, we can express

$$[d\phi_p] = \begin{bmatrix} \lambda_1 & \lambda_2 \\ -\lambda_2 & \lambda_1 \end{bmatrix} \text{ where } u_x = \lambda_1 \text{ and } u_y = \lambda_2$$

Because $\lambda_1^2 + \lambda_2^2 = u_x^2 + u_y^2$ is differentiable and non-zero on $\mathbb{R}^2 \setminus Q$, we can reduce the problem into proving

$$\langle d\phi_p v_1, d\phi_p v_2 \rangle = (\lambda_1^2 + \lambda_2^2)\langle v_1, v_2 \rangle \text{ for all } v_1, v_2 \in \mathbb{R}^2$$

Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$. We reduce the problem into proving

$$\begin{cases} \langle d\phi_p e_1, d\phi_p e_1 \rangle = (\lambda_1^2 + \lambda_2^2)\langle e_1, e_1 \rangle = \lambda_1^2 + \lambda_2^2 \\ \langle d\phi_p e_1, d\phi_p e_2 \rangle = (\lambda_1^2 + \lambda_2^2)\langle e_1, e_2 \rangle = 0 \\ \langle d\phi_p e_2, d\phi_p e_2 \rangle = (\lambda_1^2 + \lambda_2^2)\langle e_2, e_2 \rangle = \lambda_1^2 + \lambda_2^2 \end{cases}$$

Compute

$$\langle d\phi_p e_1, d\phi_p e_1 \rangle = |(\lambda_1, -\lambda_2)|^2 = \lambda_1^2 + \lambda_2^2$$

Compute

$$\langle d\phi_p e_2, d\phi_p e_2 \rangle = |(\lambda_2, \lambda_1)|^2 = \lambda_1^2 + \lambda_2^2$$

Compute

$$\langle d\phi_p e_1, d\phi_p e_2 \rangle = \lambda_1 \lambda_2 - \lambda_2 \lambda_1 = 0 \text{ (done)}$$

∎

Question 69

2. Show that if $\mathbf{x}$ is an isothermal parametrization, that is, $E = G = \lambda(u, v)$ and $F = 0$, then

$$K = -\frac{1}{2\lambda}\Delta(\log \lambda),$$

where $\Delta\varphi$ denotes the Laplacian $(\partial^2\varphi/\partial u^2) + (\partial^2\varphi/\partial v^2)$ of the function φ. Conclude that when $E = G = (u^2 + v^2 + c)^{-2}$ and $F = 0$, then $K =$ const. $= 4c$.

Proof. We first show

$$\text{If } F = 0 \text{ then } K = \frac{-1}{2\sqrt{EG}}\Big[\Big(\frac{E_v}{\sqrt{EG}}\Big)_v + \Big(\frac{G_u}{\sqrt{EG}}\Big)_u\Big]$$

Because $F = 0$, we can compute

$$\Gamma^1_{11} = \frac{E_u}{2E}$$
$$\Gamma^2_{11} = \frac{E_v}{-2G}$$
$$\Gamma^1_{12} = \frac{E_v}{2E}$$
$$\Gamma^2_{12} = \frac{G_u}{2G}$$
$$\Gamma^1_{22} = \frac{G_u}{-2E}$$
$$\Gamma^2_{22} = \frac{G_v}{2G}$$

We have the Gauss Formula

$$K = \frac{(\Gamma^2_{12})_u - (\Gamma^2_{11})_v + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{21} - \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{11}\Gamma^2_{12}}{-E}$$

The rest then follows from expressing both K and $\frac{-1}{2\sqrt{EG}}\Big[\Big(\frac{E_v}{\sqrt{EG}}\Big)_v + \Big(\frac{G_u}{\sqrt{EG}}\Big)_u\Big]$ in the form of $\frac{f}{g}, \frac{h}{k}$ where f, g, h, k are some product of the 0-th, first and second derivatives of E, F, G, and check $fk = gh$. (done)

Now, note that

$$(\ln\lambda)_u = \frac{\lambda_u}{\lambda} \text{ and } (\ln\lambda)_v = \frac{\lambda_v}{\lambda}$$

Using our formula, we see

$$\begin{aligned}
K &= \frac{-1}{2\sqrt{EG}}\Big[\Big(\frac{E_v}{\sqrt{EG}}\Big)_v + \Big(\frac{G_u}{\sqrt{EG}}\Big)_u\Big] \\
&= \frac{1}{-2\lambda}\Big[\Big(\frac{\lambda_u}{\lambda}\Big)_u + \Big(\frac{\lambda_v}{\lambda}\Big)_v\Big] \\
&= \frac{1}{-2\lambda}\Big((\ln\lambda)_{uu} + (\ln\lambda)_{vv}\Big) = \frac{1}{-2\lambda}\Delta(\ln\lambda)
\end{aligned}$$

If $\lambda = (u^2+v^2-c)^2$, we have

$$K = 4c$$

■

Question 70

3. Verify that the surfaces

$$\begin{aligned}
\mathbf{x}(u,v) &= (u\cos v, u\sin v, \log u), \qquad u > 0 \\
\bar{\mathbf{x}}(u,v) &= (u\cos v, u\sin v, v),
\end{aligned}$$

have equal Gaussian curvature at the points $\mathbf{x}(u,v)$ and $\bar{\mathbf{x}}(u,v)$ but that the mapping $\bar{\mathbf{x}}\circ\mathbf{x}^{-1}$ is not an isometry. This shows that the "converse" of the Gauss theorem is not true.

Proof. Compute

$$\begin{aligned}
\mathbf{x}_u &= (\cos v, \sin v, \frac{1}{u}) \text{ and } \mathbf{x}_v = (-u\sin v, u\cos v, 0) \\
\overline{\mathbf{x}}_u &= (\cos v, \sin v, 0) \text{ and } \overline{\mathbf{x}}_v = (-u\sin v, u\cos v, 1)
\end{aligned}$$

This give us

$$\begin{aligned}
&E = 1 + \frac{1}{u^2} \text{ and } F = 0 \text{ and } G = u^2 \\
&\overline{E} = 1 \text{ and } \overline{F} = 0 \text{ and } \overline{G} = 1 + u^2
\end{aligned}$$

The fact $E \neq \overline{E}$ implies that $\overline{\mathbf{x}} \circ \mathbf{x}^{-1}$ is not an isometry, since if $\phi \triangleq \overline{\mathbf{x}} \circ \mathbf{x}^{-1}$ is an isometry, then the local chart $\phi \circ \mathbf{x}$ would have have the same coefficients as $\mathbf{x}$, but $\phi \circ \mathbf{x} = \overline{\mathbf{x}}$.

Now compute

$$\mathbf{x}_{uu} = (0, 0, -u^{-2}) \text{ and } \mathbf{x}_{uv} = (-\sin v, \cos v, 0) \text{ and } \mathbf{x}_{vv} = (-u\cos v, -u\sin v, 0)$$
$$\overline{\mathbf{x}}_{uu} = (0, 0, 0) \text{ and } \overline{\mathbf{x}}_{uv} = (-\sin v, \cos v, 0) \text{ and } \overline{\mathbf{x}}_{vv} = (-u\cos v, -u\sin v, 0)$$

This give us

$$g = \overline{g} = 0$$

and because $F = \overline{F} = 0$, we can now deduce

$$K = \overline{K} = 0$$

using $K = \frac{eg}{EG}$. ∎

Question 71

5. If the coordinate curves form a Tchebyshef net (cf. Exercises 7 and 8, Sec. 2-5), then $E = G = 1$ and $F = \cos\theta$. Show that in this case

$$K = -\frac{\theta_{uv}}{\sin\theta}.$$

Proof. $E = G = 1$ and $F = \cos\theta$ give us three linear system

$$\begin{cases} \Gamma^1_{11} + (\cos\theta)\Gamma^2_{11} = 0 \\ (\cos\theta)\Gamma^1_{11} + \Gamma^2_{11} = \theta_u(-\sin\theta) \end{cases} \quad \begin{cases} \Gamma^1_{12} + (\cos\theta)\Gamma^2_{12} = 0 \\ (\cos\theta)\Gamma^1_{12} + \Gamma^2_{12} = 0 \end{cases} \quad \begin{cases} \Gamma^1_{22} + (\cos\theta)\Gamma^2_{22} = \theta_v(-\sin\theta) \\ (\cos\theta)\Gamma^1_{22} + \Gamma^2_{22} = 0 \end{cases}$$

Solve them to get

$$\begin{cases} \Gamma^1_{11} = (\cot\theta)\theta_u \\ \Gamma^2_{11} = -(\csc\theta)\theta_u \end{cases} \quad \begin{cases} \Gamma^1_{12} = 0 \\ \Gamma^2_{12} = 0 \end{cases} \quad \begin{cases} \Gamma^1_{22} = (-\csc\theta)\theta_v \\ \Gamma^2_{22} = (\cot\theta)\theta_v \end{cases}$$

Now, we use Gauss formula

$$\begin{aligned} K &= \frac{(\Gamma^2_{12})_u - (\Gamma^2_{11})_v + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{21} - \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{11}\Gamma^2_{12}}{-E} \\ &= \big((-\csc\theta)\theta_u\big)_v + (\cot\theta)(\csc\theta)\theta_u\theta_v \\ &= -(\csc\theta)\theta_{uv} + -(\csc\theta)(\cot\theta)\theta_u\theta_v + (\cot\theta)(\csc\theta)\theta_u\theta_v \\ &= -(\csc\theta)\theta_{uv} = -\frac{\theta_{uv}}{\sin\theta} \end{aligned}$$

∎

Question 72

8. Compute the Christoffel symbols for an open set of the plane

a. In Cartesian coordinates.

b. In polar coordinates.

Use the Gauss formula to compute K in both cases.

Proof. **(a)** We are given

$$\mathbf{x}(u, v) = (u, v, 0)$$

which implies

$$E = G = 1 \text{ and } F = 0$$

This trivially give us

$$\Gamma^i_{jk} = 0 \text{ for all } i, j, k \quad \text{and } K = 0$$

(b) We are given

$$\mathbf{x}(u, v) = (v \cos u, v \sin u, 0)$$

which implies

$$E = v^2 \text{ and } F = 0 \text{ and } G = 1$$

Because $F = 0$, the solving of the 2-dimensional linear equations is trivial

$$\begin{aligned}
\Gamma^1_{11} &= \frac{E_u}{2E} = 0 \\
\Gamma^2_{11} &= \frac{-E_v}{2G} = -v \\
\Gamma^1_{12} &= \frac{E_v}{2E} = \frac{1}{v} \\
\Gamma^2_{12} &= \frac{G_u}{2G} = 0 \\
\Gamma^1_{22} &= \frac{G_u}{-2E} = 0 \\
\Gamma^2_{22} &= \frac{G_v}{2G} = 0
\end{aligned}$$

Then use Gauss Formula

$$\begin{aligned}
K &= \frac{(\Gamma_{12}^2)_u - (\Gamma_{11}^2)_v + \Gamma_{12}^1\Gamma_{11}^2 + \Gamma_{12}^2\Gamma_{12}^2 - \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{11}^1\Gamma_{12}^2}{-E} \\
&= \frac{1-1}{-v^2} = 0
\end{aligned}$$

■

5.7 HW7

Question 73

1. a. Show that if a curve $C \subset S$ is both a line of curvature and a geodesic, then C is a plane curve.

b. Show that if a (nonrectilinear) geodesic is a plane curve, then it is a line of curvature.

c. Give an example of a line of curvature which is a plane curve and not a geodesic.

Proof. **(a)** We are required to show

$$\tau \text{ is } 0 \text{ everywhere on } C$$

Frenet Equations give us

$$T' = \kappa_C N_C \text{ where } \kappa_C, N_C \text{ are the curvature and the normal of } C$$

Because C is a geodesic, we know N_C is parallel with N, where N is the normal of S. WOLG, we can let $N_C = N$. Now, because C is a line of curvature, we know

$$N_C' = N' = dN(\alpha') = \lambda T$$

where λ is the principal curvature and α is some arc-length parametrization of C.

Now by Frenet Equations, we have

$$\lambda T = N_C' = -\kappa_C T - \tau B$$

It then follows that $\tau = 0$. (done)

(b) Let α be an arc-length parametrization of C. Because C is a geodesic, again WOLG, we can let $N_C = N$. Then by Frenet equations, we have

$$\begin{aligned} dN(T) = N' = (N_C)' &= \kappa_C T + \tau B \\ &= \kappa_C T \quad (\tau \text{ is } 0, \text{ since } C \text{ is plane curve }) \end{aligned}$$

This implies that $\alpha' = T$ is an eigenvalue of dN, which implies C is a line of curvature.

(c) Consider

$$C \triangleq S^2 \cap \{(x, y, \sqrt{2}) \in \mathbb{R}^3 : (x, y) \in \mathbb{R}^2\}$$

C is a line of curvature, since every direction is principal direction on S^2.

C is not a geodesic, since the only geodesic on S^2 is the great circles, while C isn't.

To see that only great circles on S^2 are geodesics, one note that given $p \in S$ and $w \in T_pS$, there exists only one geodesic passing through p with direction $\frac{w}{|w|}$. ■

Question 74

7. Intersect the cylinder $x^2 + y^2 = 1$ with a plane passing through the x axis and making an angle θ, $0 < \theta < \pi/2$, with the xy plane.

a. Show that the intersecting curve is an ellipse C.

b. Compute the absolute value of the geodesic curvature of C in the cylinder at the points where C meets their principal axes.

Proof. **(a)** The plane can be characterized by $z = \tan\theta y$. Then C can be characterized by

$$\begin{cases} x^2 + y^2 = 1 \\ z = (\tan\theta) y \end{cases}$$

Clearly, we can parametrized C by

$$\alpha(t) = \Big(\cos t, \sin t, \tan\theta \sin t\Big)$$

Observe

$$\alpha(t) = (\cos t)v + (\sin t)w \text{ where } v = (1, 0, 0) \text{ and } w = (0, 1, \tan\theta)$$

This conclude that C is an ellipse.

(b) WOLG, we only have to compute κ_g for $\alpha(0)$ and $\alpha(\frac{\pi}{2})$.

Compute

$$\begin{cases} \alpha'(t) = \Big(-\sin t, \cos t, \tan\theta \cos t \Big) \\ \alpha''(t) = \Big(-\cos t, -\sin t, -\tan\theta \sin t \Big) \\ |\alpha' \times \alpha''| = \sec\theta \\ \kappa_C = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} \\ \kappa_C(0) = \cos^2\theta \text{ and } \kappa_C(\frac{\pi}{2}) = \sec\theta \\ \alpha'(0) = (0, 1, \tan\theta) = \sec\theta(0, \cos\theta, \sin\theta) \\ \alpha'(\frac{\pi}{2}) = (-1, 0, 0) \end{cases}$$

It is easily checked that the principal curvatures and directions at $\alpha(0) = (1, 0, 0)$ are

$$1 \text{ relative to } (0, 1, 0) \text{ and } 0 \text{ relative to } (0, 0, 1)$$

This implies κ_n at $\alpha(0)$ is $\cos^2\theta$, which implies $|\kappa_g| = \sqrt{\kappa_C^2 - \kappa_n^2} = 0$ at $\alpha(0)$.

Similarly, it is easily checked that the principal curvatures and directions at $\alpha(\frac{\pi}{2}) = (0, 1, \tan\theta)$ are

$$1 \text{ relative to } (1, 0, 0) \text{ and } 0 \text{ relative to } (0, 0, 1)$$

This implies κ_n at $\alpha(\frac{\pi}{2})$ is 1, which implies $|\kappa_g| = \sqrt{\kappa_C^2 - \kappa_n^2} = \sqrt{\sec^2\theta - 1} = \tan\theta$. ■

Question 75

***9.** Consider two meridians of a sphere C_1 and C_2 which make an angle φ at the point p_1. Take the parallel transport of the tangent vector w_0 of C_1, along C_1 and C_2, from the initial point p_1 to the point p_2 where the two

meridians meet again, obtaining, respectively, w_1 and w_2. Compute the angle from w_1 to w_2.

Proof. Because meridians are geodesic, we can speak of the following parametrization.

Let $\alpha : [0, l] \to S$ be the geodesic parametrization from p_1 to p_2 along C_1, and let $\overline{\alpha} : [0, l] \to S$ be the geodesic parametrization from p_1 to p_2 along C_2. Note that the angle between $\alpha'(0)$ and $(\overline{\alpha})'(0)$ is given ϕ by premise.

WLOG, we now write

$$\begin{aligned} w_0 &= \cos\theta_0 e_1 + \sin\theta e_2 \\ \alpha'(0) &= \cos(\theta_0 + \psi_0)e_1 + \sin(\theta_0 + \psi_0)e_2 \\ (\overline{\alpha})'(0) &= \cos(\theta_0 + \psi_0 + \phi_0)e_1 + \sin(\theta_0 + \psi_0 + \phi_0)e_2 \end{aligned}$$

where $e_1 = (1, 0, 0), e_2 = (0, 1, 0)$ and p_1 is the north pole.

Because parallel transport along geodesics preserves the angle between the vector and the speed of the geodesic, we know

$$\text{the angle from } w_1 \text{ to } \alpha'(l) \text{ is still } \psi_0$$

and

$$\text{the angle from } w_2 \text{ to } (\overline{\alpha})'(l) \text{ is still } \psi_0 + \phi_0$$

It is clear that

$$\begin{aligned}
\alpha'(l) &= -\alpha'(0) = \cos(\theta_0 + \psi_0 + \pi)e_1 + \sin(\theta_0 + \psi_0 + \pi)e_2 \\
(\overline{\alpha})'(l) &= -(\overline{\alpha})'(0) = \cos(\theta_0 + \psi_0 + \phi_0 + \pi)e_1 + \sin(\theta_0 + \psi_0 + \phi_0 + \pi)e_2
\end{aligned}$$

This give us

$$\begin{aligned}
w_1 &= \cos(\theta_0 + 2\psi_0 + \pi)e_1 + \sin(\theta_0 + 2\psi_0 + \pi)e_2 \\
w_2 &= \cos(\theta_0 + 2\psi_0 + 2\phi_0 + \pi)e_1 + \sin(\theta_0 + 2\psi_0 + 2\phi_0 + \pi)e_2
\end{aligned}$$

Then the angle from w_1 to w_2 is $2\phi_0$, where ϕ_0 is the angle C_1, C_2 make at the north pole p_1. ∎

Question 76

***10.** Show that the geodesic curvature of an oriented curve $C \subset S$ at a point $p \in C$ is equal to the curvature of the plane curve obtained by projecting C onto the tangent plane $T_p(S)$ along the normal to the surface at p.

Proof. Let α be a geodesic parametrization of C around $\alpha(0) = p$. The orthogonal projection β of α onto T_pS is

$$\beta(s) = \alpha(s) + \langle p - \alpha(s), N(p)\rangle N(p)$$

where $N(p)$ is the normal of S at p. Compute

$$\beta'(s) = \alpha'(s) - \langle \alpha'(s), N(p)\rangle N(p)$$

Compute

$$\beta''(s) = \alpha''(s) - \langle \alpha''(s), N(p) \rangle N(p)$$

This give us

$$\beta'(0) = \alpha'(0) \text{ and } \beta''(0) = \frac{D\alpha'}{ds}(0)$$

Then because $|\beta'(0)| = |\alpha'(0)| = 1$, the curvature κ of β at p is then

$$\begin{aligned}\kappa = \frac{|\beta'(0) \times \beta''(0)|}{|\beta'(0)|^3} = \left|\alpha'(0) \times \frac{D\alpha'}{ds}(0)\right| &= |\alpha'(0)| \cdot \left|\frac{D\alpha'}{ds}(0)\right| \sin\theta \\ &= \left|\frac{D\alpha'}{ds}(0)\right| \sin\theta\end{aligned}$$

where θ is the angle between $\alpha'(0)$ and $\frac{D\alpha'}{ds}(0)$. On the other hand, we know

$$\begin{aligned}\kappa_g(p) &= \langle \frac{D\alpha'}{ds}(0), N(p) \times \alpha'(0) \rangle \\ &= \left|\frac{D\alpha'}{ds}(0)\right| \cdot |N(p) \times \alpha'(0)| \cos\phi \\ &= \left|\frac{D\alpha'}{ds}(0)\right| \cos\phi\end{aligned}$$

where ϕ is the angle between $\frac{D\alpha'}{ds}(0)$ and $N(p) \times \alpha'(0)$.

Note that $N \times \alpha'(0), \alpha'(0), \frac{D\alpha'}{ds}(0)$ are all in T_pS, and the angle from $\alpha'(0)$ to $N \times \alpha'(0)$ is $\frac{\pi}{2}$. This implies $\theta = \phi + \frac{\pi}{2}$, and conclude the result. ■

Question 77

14. Let S be an oriented regular surface and let $\alpha\colon I \to S$ be a curve parametrized by arc length. At the point $p = \alpha(s)$ consider the three unit vectors (the *Darboux trihedron*) $T(s) = \alpha'(s)$, $N(s) =$ the normal vector to S at p, $V(s) = N(s) \wedge T(s)$. Show tbat

$$\frac{dT}{ds} = 0 + aV + bN,$$
$$\frac{dV}{ds} = -aT + 0 + cN,$$
$$\frac{dN}{ds} = -bT - cV + 0,$$

where $a = a(s)$, $b = b(s)$, $c = c(s)$, $s \in I$. The above formulas are the analogues of Frenet's formulas for the trihedron T, V, N. To establish the geometrical meaning of the coefficients, prove that

a. $c = -\langle dN/ds, V\rangle$; conclude from this that $\alpha(I) \subset S$ is a line of curvature if and only if $c \equiv 0$ ($-c$ is called the *geodesic torsion* of α; cf. Exercise 19, Sec. 3-2).

b. b is the normal curvature of $\alpha(I) \subset S$ at p.

c. a is the geodesic curvature of $\alpha(I) \subset S$ at p.

Proof. It is clear that $T, N, V = N \times T$ form an orthonormal basis. Write

$$\begin{bmatrix} T' \\ V' \\ N' \end{bmatrix} = M \begin{bmatrix} T \\ V \\ N \end{bmatrix}$$

where M is a 3×3-matrix, and we are required to prove

(a) $M_{k,k} = 0$ for all k

(b) $M_{i,j} = -M_{j,i}$ for all i, j

Note that $M_{1,1} = T' \times T$, $M_{2,2} = V' \times V$, $M_{3,3} = N' \times N$. (a) follows from the fact T, V, M

are all unit.

Because T, V, M are orthogonal, we know

$$\begin{aligned} M_{1,2} = T' \cdot V \text{ and } M_{2,1} = V' \cdot T \\ M_{1,3} = T' \cdot V \text{ and } M_{3,1} = N' \cdot T \\ M_{2,3} = V' \cdot N \text{ and } M_{3,2} = N' \cdot V \end{aligned}$$

(b) then follows from T, V, M are orthogonal, and the fact $(w_1 \cdot w_2)' = w_1' \cdot w_2 + w_1 \cdot w_2'$.

(a) We know $\alpha(I)$ is a line of curvature if and only if N' is parallel with T everywhere. It follows from $N' = -bT - cV$ that $c \equiv 0$ if and only if $\alpha(I)$ is a line of curvature.

(b) We know

$$\kappa_\alpha N_\alpha = \frac{dT}{ds} = aV + bN$$

Then the normal curvature κ_n is

$$\kappa_n = \kappa_\alpha \langle N_\alpha, N \rangle = b$$

(c) Note that

$$\frac{d\alpha'}{ds} = bN + aN \times T$$

This give us

$$\frac{D\alpha'}{ds} = aN \times T = aN \times \alpha'$$

which implies a is the geodesic curvature.

Question 78

17. Let $\alpha: I \to R^3$ be a curve parametrized by arc length s, with nonzero curvature and torsion. Consider the parametrized surface (Sec. 2-3)

$$\mathbf{x}(s, v) = \alpha(s) + vb(s), \quad s \in I, -\epsilon < v < \epsilon, \epsilon > 0,$$

where b is the binormal vector of α. Prove that if ϵ is small, $\mathbf{x}(I \times (-\epsilon, \epsilon)) = S$ is a regular surface over which $\alpha(I)$ is a geodesic (*thus, every curve is a geodesic on the surface generated by its binormals*).

Proof. Compute, using Frenet Equations

$$\mathbf{x}_s = T + v\tau N_\alpha$$
$$\mathbf{x}_v = B$$

This give us

$$\mathbf{x}_s \times \mathbf{x}_v = -N_\alpha + v\tau T$$

which give us

$$N(s, 0) = N_\alpha$$

This implies $\alpha(I)$ has the same normal as S, which implies $\alpha(I)$ is a geodesic. ■

Question 79

1. Let $S \subset R^3$ be a regular, compact, connected, orientable surface which is not homeomorphic to a sphere. Prove that there are points on S where the Gaussian curvature is positive, negative, and zero.

Proof. Because S is not homeomorphic to a sphere, we know $\iint_S K d\sigma = 2\pi\chi(S) \leq 0$. Then the proof reduce to proving

$$S \text{ is elliptic at some } p \in S$$

Let q be an arbitrary point in $\mathbb{R}^3$. Note that the function $f : \mathbb{R}^3 \to \mathbb{R}_0^+$ defined by

$$f(p) \triangleq |p - q|$$

is continuous. Then because S is compact, we know f attain a maximum r_0 at some $p \in S$. Denote the sphere centering q with radius $r \in \mathbb{R}^+$ by $S^2(r)$. Note that $S^2(r_0) = f^{-1}(r_0)$. This implies that S is "contained" in $S^2(r_0)$ and

$$p \in S \cap S^2(r_0)$$

Now, given an arbitrary normal section C of S at p. By Section 1.7, Question 4, we see C must have normal curvature greater than $r_0 > 0$ at p. (Note that if $r_0 = 0$, then $S = \{q\}$, which is not a regular surface.) Because C is arbitrary, we now see that the two principal curvature must be greater than $r_0 > 0$. This implies $K > 0$. (done)

∎

Question 80

2. Let T be a torus of revolution. Describe the image of the Gauss map of T and show, without using the Gauss-Bonnet theorem, that

$$\iint_T K\, d\sigma = 0.$$

Compute the Euler-Poincaré characteristic of T and check the above result with the Gauss-Bonnet theorem.

Proof. We are given the standard chart

$$\mathbf{x}(u,v) = \Big((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u\Big)$$

Some messy computation give us

$$N(u,v) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|} = (\cos u \cos v, \cos u \sin v, \sin u)$$

For each $\mathbf{x}(u_0, v_0) \in T$, there exists a circle

$$C = \{\mathbf{x}(u, v_0) : u \in [0, 2\pi]\}$$

containing $\mathbf{x}(u_0, v_0)$, and one can check that T is just the normal of such C.

Observe

$$\begin{aligned}\frac{d}{du}\mathbf{x}(u, v_0) &= \Big(-r\sin u \cos v_0, -r\sin u \sin v_0, r\cos u\Big)\\ \frac{d}{du}N(u, v_0) &= \Big(-\sin u \cos v_0, -\sin u \sin v_0, \cos u\Big)\end{aligned}$$

This then implies for all $\mathbf{x}(u, v) \in S$, one of the principal curvature is $\frac{1}{r}$.

Observe

$$\begin{aligned}\frac{d}{dv}\mathbf{x}(u_0, v) &= \Big(-(a + r\cos u_0)\sin v, (a + r\cos u_0)\cos v, 0\Big)\\ \frac{d}{dv}N(u_0, v) &= \Big(-\cos u_0 \sin v, \cos u_0 \cos v, 0\Big)\end{aligned}$$

Then then implies for all $\mathbf{x}(u, v)$, another principal curvature is $\frac{\cos u}{a+r\cos u}$. We now have

$$K(u, v) = \frac{\cos u}{r(a + r\cos u)}$$

Compute

$$\begin{aligned}\mathbf{x}_u &= (-r\sin u \cos v, -r\sin u \sin v, r\cos u)\\ \mathbf{x}_v &= (-(a + r\cos u)\sin v, (a + r\cos u)\cos v, 0)\\ E &= r^2 \text{ and } F = 0 \text{ and } G = (a + r\cos u)^2\\ EG - F^2 &= r^2(a + r\cos u)^2\end{aligned}$$

Note the symmetry

$$\begin{aligned}K(\frac{\pi}{2} - t, v) &= -K(\frac{\pi}{2} + t, v)\\ K(\frac{-\pi}{2} - t, v) &= -K(\frac{-\pi}{2} + t, v)\\ (EG - F^2)(\frac{\pi}{2} - t, v) &= (EG - F^2)(\frac{\pi}{2} + t, v)\\ (EG - F^2)(\frac{-\pi}{2} - t, v) &= (EG - F^2)(\frac{-\pi}{2} + t, v) \text{ for all } t \in [0, \frac{\pi}{2}], v \in [0, 2\pi]\end{aligned}$$

This give

$$\iint_R K d\sigma = \iint_{[0,2\pi]^2} K(u,v)\sqrt{EG - F^2} du dv = 0$$

It now follows from Gauss-Bonnet that $\chi(T) = 0$. ■

Question 81

4. Compute the Euler-Poincaré characteristic of

a. An ellipsoid.

***b.** The surface $S = \{(x, y, z) \in R^3; x^2 + y^{10} + z^6 = 1\}$.

Proof. **(a)** Given ellipsoid S

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

We can map S to S^2 by

$$f(x,y,z) = (ax, by, cz)$$

It is clear that $f : S \to S^2$ is continuous, one-to-one and onto, and admits an continuous inverse

$$f^{-1}(x,y,z) = (\frac{x}{a}, \frac{y}{b}, \frac{c}{z})$$

It is now established that f is a homeomorphism between S and S^2. Then we see ellipsoid have the same Euler-Poincare characteristic as S^2, i.e. 2.

To see $\chi(S^2) = 2$, one can use the triangulation $\{T_1, \dots, T_8\}$, in which each T_k is the intersection between one of the octant and S^2. We then have $F = 8$, $E = 12$, $V = 6$, so $\chi(S^2) = 8 - 12 + 6 = 2$.

(b) Map S to S^2 by

$$f(x,y,z) = (x, y^5, z^3)$$

It is clear that f continuous, one-to-one and onto, and admits a continuous inverse

$$f^{-1}(x,y,z) = \left(x, \begin{cases} y^{\frac{1}{5}} & \text{if } y \geq 0 \\ -(-y)^{\frac{1}{5}} & \text{if } y < 0 \end{cases}, \begin{cases} z^{\frac{1}{3}} & \text{if } z \geq 0 \\ -(-z)^{\frac{1}{3}} & \text{if } z < 0 \end{cases} \right)$$

It is now established that f is a homeomorphism between S and S^2. Then we see $\chi(S) = \chi(S^2) = 2$ ■

5.8 fdlskjf

Definition 5.8.1. (Isometry) Given a diffeomorphishm $\phi : S \to S^2$, we say ϕ is an **isometry** if for all $p \in S$ and w in T_pS, we have

$$I_p(w) = I_{\phi(p)}(d\phi_p w)$$

Definition 5.8.2. (Conformal) Given a diffeomorphishm $\phi : S \to S^2$, we say ϕ is a **conformal mapping** if there exists a differentiable $\mathbb{R}$-valued function λ non-vanishing on S such that for all $p \in S$ and $w_1, w_2 \in T_pS$, we have

$$\langle w_1, w_2 \rangle = \lambda(p)\langle d\phi_p w_1, d\phi_p w_2 \rangle$$

Definition 5.8.3. (Vector Fields) By an **vector field** along $\alpha : I \to S$, we mean a function $w : I \to \mathbb{R}^3$ such that

$$w(t) \in T_{\alpha(t)}S \text{ for all } t \in I$$

Definition 5.8.4. (Covariant Derivative) The corvariant derivaitve is the following function

$$\frac{Dw}{dt}(t) \triangleq w'(t) - \langle w'(t), N(\alpha(t))\rangle N(\alpha(t))$$

where N is the normal of S.

Definition 5.8.5. (Algebraic Value) If w is unit,

$$\left[\frac{Dw}{dt}(t)\right] \triangleq \langle w'(t), N \times w(t)\rangle$$

where N is the normal of S