Linear Algebra – Final Exam
Jun. 15, 2022

1. (10%) Show that if
$$\begin{bmatrix} A & I \\ O & A^T \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{x}} \\ \boldsymbol{r} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix}$$
Then $\hat{\boldsymbol{x}}$ is a least squares solution of the system $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{r}$ is the residual vector.

2. (10%) Let A be a nonsingular $n \times n$ matrix. For each vector $\boldsymbol{x} \in R^n$, we define
$$\|\boldsymbol{x}\|_A = \|A\boldsymbol{x}\|_2$$
Show that this equation defines a norm on R^n.

3. (10%) Let $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k, \boldsymbol{x}_{k+1}, \ldots, \boldsymbol{x}_n\}$ be an orthonormal basis for an inner product space V. Let S_1 be the subspace of V spanned by $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k$, and let S_2 be the subspace spanned by $\boldsymbol{x}_{k+1}, \ldots, \boldsymbol{x}_n$. Show that $S_1 \perp S_2$.

4. (10%) Use the Gram-Schmidt process to transform the basis $\{(1,2,-1), (1,3,0), (4,1,0)\}$ into an orthonormal basis under the Euclidean inner product in R^3.

5. (10%) Given that the characteristic polynomial of a matrix A is $p(\lambda) = (\lambda+1)(\lambda-2)^2(\lambda+3)^2$, find $\det(A^{-1})$.

6. (10%) Let A be an $n \times n$ stochastic matrix and let $\boldsymbol{e}$ be the vector in R^n whose entries are all equal to 1. Show that $\boldsymbol{e}$ is an eigenvector of A^T.

7. (10%) Determine whether the point (1, -1) corresponds to a local minimum, local maximum, or saddle point for the function $f(x, y) = \dfrac{y}{x^2} - \dfrac{x}{y^2} + xy$.

8. (15%) Find the Cholesky decomposition LL^T for the following matrix. You need to show the detailed process.
$$\begin{bmatrix} 1 & -3 & 0 \\ -3 & 11 & -2 \\ 0 & -2 & 4 \end{bmatrix}$$

9. (15%) Prove that if A is a symmetric matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the singular values of A are $|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|$.