NCKU 112.2
Miscellaneous Facts

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Contents

Chapter 1

General Topology

1.1 Directed Sets

Axiom 1.1.1. (Axioms in Order Theory) Given an relation $(X, \leq)$, and suppose $x, y, z \in X$.

(a) $x \leq x$ (Reflexive)

(b) $x \leq y \leq z \implies x \leq z$ (Transitive)

(c) $x \leq y$ and $y \leq x \implies x = y$ (Antisymmetric)

(d) $x \leq y$ or $y \leq x$ (Connected)

(e) $\forall x, y \in X, \exists z \in X, x \leq z$ and $y \leq z$ (Directed)

We say $(X, \leq)$ form a

(a) total order if it is reflexive, transitive, antisymmetric and connected.

(b) partial order if it is reflexive, transitive and antisymmetric.

(c) preorder if it is reflexive and transitive.

(d) directed set if it is reflexive, transitive and directed.

Theorem 1.1.2. (Why is it called Preorder) Given a preorder $(X, \leq)$, the relation $\sim$ defined by

$$x \sim y \iff x \leq y \text{ and } y \leq x$$

is an equivalence relation and if we define $\leq^e$ on the equivalence class by

$$\exists x \in A, y \in B, x \leq y \implies A \leq^e B$$

Then $\leq^e$ is a partial order. Moreover, if the preorder $\leq$ is directed, then $\leq^e$ is also directed.

Proof. We first show $\sim$ is an equivalence relation. Because preoder is reflexive, we see

$$\forall x \in X, x \leq x \text{ which implies } \forall x \in X, x \sim x$$

For symmetry, it is easy to see

$$x \sim y \implies x \leq y \text{ and } y \leq x \implies y \sim x$$

For transitive, see

$$\begin{aligned} x \sim y \text{ and } y \sim z &\implies x \leq y \text{ and } y \leq x \text{ and } y \leq z \text{ and } z \leq y \\ &\implies x \leq z \text{ and } z \leq x \implies x \sim z \text{ (done)} \end{aligned}$$

We now show $\leq^e$ is a partial order. Reflexive property and Transitive property of $\leq^e$ follow from that of $\leq$. Suppose $A \leq^e B$ and $B \leq^e A$, where $x_1, x_2 \in A, y_1, y_2 \in B$ satisfy $x_1 \leq y_1$ and $y_2 \leq x_2$. Because $x_1, x_2 \in A$ and $y_1, y_2 \in B$, we have

$$x_1 \leq x_2 \text{ and } x_2 \leq x_1 \text{ and } y_1 \leq y_2 \text{ and } y_2 \leq y_1$$

Then because $\leq$ satisfy transitive, we have

$$\begin{cases} x_2 \leq x_1 \leq y_1 \implies x_2 \leq y_1 \\ y_1 \leq y_2 \leq x_2 \implies y_1 \leq x_2 \end{cases}$$

This tell us

$$x_2 \sim y_1$$

which implies $A = B$, thus proving $\leq^e$ is antisymmetirc. (done)

Lastly, we show $\leq$ is directed $\implies$ $\leq^e$ is directed. Let A, B be two arbitrary equivalence class. We wish to find an equivalence class T such that

$$A \leq^e T \text{ and } B \leq^e T$$

Let a, b respectively be an arbitrary element of A, B. Because $\leq$ is directed, we know there exists $c \in X$ such that

$$a \leq c \text{ and } b \leq c$$

We immediately see

$$A \leq^e [c] \text{ and } B \leq^e [c] \text{ (done)}$$

∎

Corollary 1.1.3. (Chunk Structure of Preorder) Given two equivalence class A, B, we have

$$A \leq^e B \implies \forall x \in A, y \in B, x \leq y$$

Proof. Because $A \leq^e B$, we know

$$\exists x_0 \in A, y_0 \in B, x_0 \leq y_0$$

Then by definition of $\sim$, we have

$$x \leq x_0 \leq y_0 \leq y$$

This give us

$$x \leq y$$

■

Definition 1.1.4. (Definition of Maximal element in Preorder) Let $(I, \leq)$ be a preorder. We say $m \in I$ is a maximal element if

$$\forall y \in I, m \leq y \implies y \leq m$$

Theorem 1.1.5. (In Preorder, Maximal element form an Equivalence class) Let $(I, \leq)$ be a preorder, and $m \in I$ be a maximal element. Then

$$\forall x \in [m], x \text{ is a maximal element}$$

Proof. Arbitrarily pick an element x in $[m]$. Suppose

$$x \leq y$$

By definition of $\sim$, we have

$$m \leq x \leq y$$

Thus $m \leq y$. Then because m is maximal, we know $y \leq m$. This now give us

$$y \leq m \leq x$$

■

Notice that in partially ordered set, where anti-symmetric property is true, the definition of maximal element $m \in I$ falls into

$$\forall y \in I, m \leq y \implies y = m$$

Definition 1.1.6. (Definition of Greatest element in Preorder) Let $(I, \leq)$ be a preorder. We say $x \in I$ is a greatest element if

$$\forall y \in I, y \leq x$$

Theorem 1.1.7. (In Directed Set, Maximal element is the Greatest) Suppose $(I, \leq)$ is a directed set.

$$x \in I \text{ is a maximal element } \implies x \in I \text{ is the greatest element}$$

Proof. Arbitrarily pick an element $y \in I$. Because I is directed, we see there exists an element z such that

$$y \leq z \text{ and } x \leq z$$

Then because x is maximal, we know

$$y \leq z \leq x$$

This shows

$$y \leq x$$

■

Theorem 1.1.8. (Sufficient Condition for Preorder to become Directed)

$$(I, \leq) \text{ is a preorder and has a greatest element } x \implies I \text{ is a directed set}$$

Proof. Given arbitrary two element $y, z \in I$, we see $y \leq x$ and $z \leq x$. ■

Example 1 (Partial Order that is Directed)

$$X = \{a, b, c\} \text{ and } a \leq c \text{ and } b \leq c$$

Example 2 (Partial Order that is Not Directed)

$$X = \{a, b, c\} \text{ and } a \leq b \text{ and } a \leq c$$

Example 3 (Partial Order that is Directed)

$$X = \mathbb{Z}_0^+ \text{ and } \forall x, y \in \mathbb{N}, x \leq y \iff y - x | 2 \text{ and } x \leq y$$
$$\text{and } \forall x \in \mathbb{N}, x \leq 0$$

Example 4 (Partial Order that is not Directed)

$$X = \mathbb{N} \text{ and } \forall x, y \in \mathbb{N}, x \leq y \iff y - x | 2 \text{ and } x \leq y$$

Example 5 (Directed Set that is not Partially Ordered)

$$\begin{aligned} X = \{a, b, c\} &\text{ and } a \leq b \text{ and } b \leq a \\ &\text{ and } a \leq c \text{ and } b \leq c \end{aligned}$$

Example 6 (Preorder that is Neither Directed nor Partially Ordered)

$$\begin{aligned} X = \{a, b, c, d\} &\text{ and } a \leq b \text{ and } b \leq a \\ &\text{ and } a \leq c \text{ and } b \leq c \\ &\text{ and } a \leq d \text{ and } b \leq d \end{aligned}$$

Example 7 (Directed Sets)

X is a metric space and $x \leq y \iff d(y, x_0) \leq d(x, x_0)$ where x_0 is a fixed point in X

Notice that this directed set is generally not antisymmetric, meaning it generally isn't a partial order. Also, notice that x_0 is the greatest element. Also, this order is connected, meaning if we take equivalence class on it, it become a total order.

Lastly, notice that if we remove x_0, X can still be directed, say if $X = \mathbb{R}^2$ and x_0 is the origin.

Example 8 (Directed Sets)

Suppose X, Y are both directed sets. We see $X \times Y$ is a directed set if we define

$$(x, y) \leq (a, b) \iff x \leq a \text{ and } y \leq b$$

Example 9 (Partial Order)

Every collection of sets is a partial order if we define

$$A \leq B \iff A \subseteq B$$

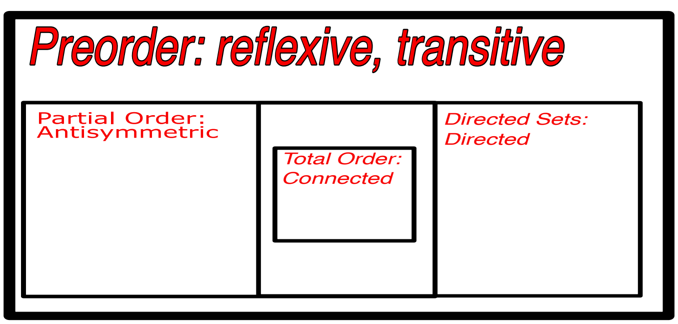
Also, every collection of sets form a partial order if we define

$$A \leq B \iff A \supseteq B$$

Example 10 (Directed Sets)

Suppose (X, τ) is a topological space and $x \in X$. Then all of τ, neighborhoods of x and open neighborhoods of x form directed sets under $\subseteq$, since X is open.

Also, τ, neighborhoods of x and open neighborhoods of x form directed sets under $\supseteq$, because intersection of two open set is again an open set and intersection of two neighborhood is again a neighborhood.



Definition 1.1.9. (Definition of Cofinal) Given a directed set $\mathcal{D}$, a subset $\mathcal{D}' \subseteq \mathcal{D}$ is called cofinal if

$$\forall d \in \mathcal{D}, \exists e \in \mathcal{D}', d \leq e$$

Theorem 1.1.10. (Cofinal Subset is a Directed Set with Original Order) Given a directed set $\mathcal{D}$

$$\mathcal{D}' \subseteq \mathcal{D} \text{ is cofinal} \implies \mathcal{D}' \text{ is a directed set}$$

Proof. Arbitrarily pick two $a, b \in \mathcal{D}'$. Because $\mathcal{D} \ni a, b$ is directed, we know

$$\exists c \in \mathcal{D}, a \leq c \text{ and } b \leq c$$

Then because $\mathcal{D}'$ is cofinal in $\mathcal{D}$, we know

$$\exists d \in \mathcal{D}', c \leq d$$

Then because transitivity of directed set, our proof is finished, as we have found an element d in $\mathcal{D}'$ that is greater than the arbitrary picked elements $a, b \in \mathcal{D}'$. ∎

1.2 Net

Definition 1.2.1. **(Subnet)** Given a net $w : \mathcal{D} \to X$ and $v : \mathcal{E} \to X$ and a function $h : \mathcal{E} \to \mathcal{D}$ we say v is a subnet of w if

$$\begin{cases} \forall e, e' \in \mathcal{E}, e \leq e' \implies h(e) \leq h(e')(\text{monotone}) \\ h[E] \text{ is cofinal in } \mathcal{D} \\ v = w \circ h \end{cases}$$

Definition 1.2.2. **(Net convergence)** We say the net $w : \mathcal{D} \to X$ converge to x, $w \to x$ if

Theorem 1.2.3. **($w \to x \implies v \to x$)** Suppose v is a subnet of w, we have

$$w \to x \implies v \to x$$

Proof. ∎

Theorem 1.2.4. ()

Definition 1.2.5. ()

Chapter 2

Metric Space

2.1

Chapter 3

Calculus

3.1 Examples for uniform convergence

Theorem 3.1.1. (Test Example) The sequence

$$f_n(x) = \frac{x^2}{x^2 + (1-nx)^2} \text{ is not equicontinuous on } [0, 1]$$

Proof. Notice that

$$f_n(\frac{1}{n}) = 1 \text{ and } f_n(0) = 0$$

Then for all δ, we see that if n is large enough

$$\text{then } \left|\frac{1}{n} - 0\right| < \delta \text{ and } \left|f_n(\frac{1}{n}) - f_n(0)\right| = 1$$

■

Theorem 3.1.2. (Test Example) Prove

$$\frac{x}{1+nx^2} \text{ uniformly converge on } \mathbb{R}$$

Proof. It is clear that $\frac{x}{1+nx^2}$ pointwise converge to 0. Because $\frac{x}{1+nx^2}$ is an odd function, fixing ϵ, we only wish to find N such that

$$\forall x > 0, \forall n > N, \frac{x}{1+nx^2} < \epsilon$$

Observe

$$\begin{aligned} \frac{x}{1+nx^2} < \epsilon &\iff x < \epsilon(1+nx^2) \\ &\iff \frac{x-\epsilon}{\epsilon x^2} < n \end{aligned}$$

Notice that $\frac{x-\epsilon}{\epsilon x^2}$ is bounded since it is continuous and converge to 0 as $x \to \infty$. ■

3.2 Test Example

Theorem 3.2.1. (Cauchy-Schwarz Inequality for Integral) Let $\mathscr{R}\Big([a,b]\Big)$ be the space of Riemann-Integrable functions on $[a,b]$. It is clear that $\mathscr{R}\Big([a,b]\Big)$ is a vector space over $\mathbb{R}$. Define $\langle\cdot,\cdot\rangle$ on $\mathscr{R}\Big([a,b]\Big)$ by

$$\langle f,g\rangle = \int_a^b f(x)g(x)dx$$

It is easy to show

(a) $\forall f \in \mathscr{R}\Big([a,b]\Big), \langle f,f\rangle \geq 0$ (non-negativity)

(b) $\forall f,g \in \mathscr{R}\Big([a,b]\Big), \langle f,g\rangle = \langle g,f\rangle$ (Symmetry)

(c) $\forall f,g,h \in \mathscr{R}\Big([a,b]\Big), \forall c \in \mathbb{R}, \langle cf+g,h\rangle = c\langle f,h\rangle + \langle g,h\rangle$ (Linearity in first argument)

This make $\langle\cdot,\cdot\rangle$ a **positive semi-definite Hermitian form**. We shall prove Cauchy-Schwarz Inequality hold for positive semi-definite Hermitian form. That is, we shall prove

$$\forall f,g \in \mathscr{R}\Big([a,b]\Big), \langle f,g\rangle \leq \|f\|\cdot\|g\|$$

Proof. ■

Theorem 3.2.2. (Application) Given $f \in \mathscr{R}\Big([a,b]\Big)$ such that

(a) $f(a) = 0 = f(b)$

(b) $\int_a^b f^2(x)dx = 1$

(c) f is continuously differentiable on (a,b)

(d) $f' \in \mathscr{R}\Big([a,b]\Big)$

We have

$$\int_a^b xf(x)f'(x) = \frac{-1}{2}$$

and have

$$\int_a^b \big(f'(x)\big)^2dx \cdot \int_a^b \big(xf(x)\big)^2dx > \frac{1}{4}$$

Proof. Notice that

$$\frac{d}{dx}xf^2(x) = f^2(x) + 2xf(x)f'(x)$$

Then by Integral by Part (We have to check $\big(xf^2(x)\big)'(t) = f^2(t) + 2tf(t)f'(t)$ for all $t \in (a,b)$, and we have to check $xf^2(x)$ is continuous on $[a,b]$), we have

$$1 = \int_a^b f^2(x)dx = xf^2(x)\Big|_a^b - \int_a^b 2xf(x)f'(x)dx$$

Then because $f(b) = f(a) = 0$, we see

$$2\int_a^b xf(x)f'(x)dx = -1$$

We wish to show

$$\|f'\|^2 \cdot \|xf(x)\|^2 > \frac{1}{4} = \Big(\langle f', xf(x)\rangle\Big)^2$$

It is clear that $\geq$ is valid from Cauchy-Schwarz Inequality. We have to prove $\neq$. In other words, we have to prove

$$f' \text{ and } xf(x) \text{ are linearly independent}$$

Assume f' and $xf(x)$ are linearly dependent. Then

$$\exists c \in \mathbb{R}, \forall x \in [a,b], f'(x) = cxf(x)$$

The solution for this first order linear homogeneous ODE is

$$f(x) = Ae^{\frac{cx^2}{2}} \text{ where } A \in \mathbb{R} \text{ depends on } f(a) \text{ and } f(b)$$

Then because $f(a) = f(b) = 0$, we see $A = 0$. Then $\int_a^b f^2(x)dx = 0$ CaC ■

Theorem 3.2.3. (Example) Given $G, g, \alpha : [a,b] \to \mathbb{R}$, suppose

(a) $G'(x) = g(x)$ for all $x \in (a,b)$ (G is differentiable on (a,b))

(b) G is continuous on $[a,b]$

(c) α increase on $[a,b]$

(d) g is properly Riemann-Integrable on $[a,b]$

Prove

$$\int_a^b \alpha(x)g(x)dx = \alpha G\Big|_a^b - \int_a^b G(x)d\alpha$$

Proof. ■

3.3 Dini's Theroem

Theorem 3.3.1. **(Dini's Theorem)** Given a topological space X and a sequence of functions $f_n : X \to \mathbb{R}$, suppose

(a) X is compact

(b) f_n is continuous

(c) $f_n \to f$ pointwise

(d) f is continuous

(e) $f_n(x) \leq f_{n+1}(x)$ for all $x \in X$

Then

$$f_n \to f \text{ uniformly}$$

Proof. Define $g_n : X \to \mathbb{R}$

$$g_n = f - f_n$$

We reduce the problem into

$$\text{proving } g_n \to 0 \text{ uniformly}$$

Notice that we have the property

(a) $g_n(x) \geq g_{n+1}(x)$ for all $x \in X$

(b) g_n is continuous

(c) $g_n \to 0$ pointwise

Fix ϵ. We wish

$$\text{to find } N \text{ such that } \forall n > N, \forall x \in X, g_n(x) < \epsilon$$

Define $E_n \subseteq X$ by

$$E_n = \{x \in X : g_n(x) < \epsilon\}$$

Because g_n is continuous and $E_n = g_n^{-1}\Big[(-\infty, \epsilon)\Big]$, we know

$$E_n \text{ is open for all } n \in \mathbb{N}$$

We first prove

$$\{E_n\}_{n\in\mathbb{N}} \text{ is an open cover of } X$$

Fix $y \in X$. We wish

$$\text{to find } n \text{ such that } y \in E_n$$

Because $g_n(y) \to 0$, this is clear. (done)

We now prove

$$\{E_n\}_{n\in\mathbb{N}} \text{ is ascending}$$

Fix $n \in \mathbb{N}$. We wish

$$\text{to prove } E_n \subseteq E_{n+1}$$

Because $g_n(x) \geq g_{n+1}(x)$ for all $x \in X$ and $E_n = g_n^{-1}\Big[(-\infty, \epsilon)\Big]$ by definition, we see

$$y \in E_n \implies g_{n+1}(y) < g_n(y) < \epsilon \implies y \in E_{n+1} \text{ (done)}$$

Now, because X is compact and $\{E_n\}_{n\in\mathbb{N}}$ is an open cover of X, we know

$$\text{there exists } N \text{ such that } X \subseteq \bigcup_{k=1}^{N} E_k = E_N \tag{3.1}$$

It is clear such N works. (done) ■

Chapter 4

Multi-Variable Calculus

4.1

Chapter 5

HW

5.1 HW1

Question 1

1. Let $f_k : [0,1] \to \mathbb{R}$ be given by

$$f_k(x) = \begin{cases} 0, & \text{if } \frac{1}{k} \leq x \leq 1, \\ -kx+1, & \text{if } 0 \leq x < \frac{1}{k}. \end{cases}$$

   (a) Does $\{f_k\}_{k=1}^{\infty}$ converge pointwise on $[0,1]$? If so, find f such that $f_k \to f$ pointwise on $[0,1]$.

   (b) Does f_k converge uniformly on $[0,1]$?

Proof. **(a)** We claim

$$f_k \to f \text{ pointwise on } [0,1] \text{ where } f(x) = \begin{cases} 0 & \text{if } x \in (0,1] \\ 1 & \text{if } x = 0 \end{cases}$$

Because $\forall k \in \mathbb{N}, f_k(0) = 1$, it is clear $f_k(0) \to f(0)$. Now, let $x \in (0,1]$. We reduce our problem into proving

$$f_k(x) \to 0 \text{ as } k \to \infty$$

By definition, we have

$$\forall n > \frac{1}{x}, f_n(x) = 0 \text{ (done)}$$

Above is true since $n > \frac{1}{x} \implies \frac{1}{n} < x$.

b No. It is easy to show that f_k are all continuous and that f is discontinuous at 0. This let us deduce that the convergence is not uniform, since if it is, the function f should have been continuous. ■

Question 2

2. Let $f_k : [0,1] \to \mathbb{R}$ be given by $f_k(x) = x^k$.

   (a) Does $\{f_k\}_{k=1}^{\infty}$ converge pointwise on $[0,1]$? If so, find f such that $f_k \to f$ pointwise on $[0,1]$.

   (b) Does f_k converge uniformly on $[0,1]$?

   (c) For any $a \in (0,1)$, Does f_k converge uniformly on $[0,a]$?

Proof. **(a)** We claim

$$f_k \to f \text{ pointwise on } [0,1] \text{ where } f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \in [0,1) \end{cases}$$

Because $f_k(1) = 1$ for all $k \in \mathbb{N}$, it is clear $f_k(1) \to f(1)$. Now, let $x \in (0,1]$. We reduce our problem into proving

$$f_k(x) \to 0 \text{ as } k \to \infty$$

Fix ϵ. We wish

$$\text{to find } N \text{ such that } \forall n > N, f_n(x) < \epsilon$$

We claim

$$N > \log_x \epsilon \text{ works}$$

Fix $n > N$. Because $x < 1$, we see

$$f_n(x) = x^n < x^N < \epsilon \text{ (done)}$$

b No. It is easy to show that f_k are all continuous and that f is discontinuous at 1. This let us deduce that the convergence is not uniform, since if it is, the function f should have been continuous.

(c) Yes. Fix ϵ and $a \in (0,1)$. We wish to

$$\text{find } N \text{ such that } \forall n > N, \forall x \in [0,a], f_n(x) \le \epsilon$$

We claim

$$N > \log_a \epsilon \text{ works}$$

Observe

$$\forall n > N, \forall x \in [0, a], f_n(x) = x^n \le a^n \le a^N < \epsilon vdon$$

■

Question 3

3. Let $f_k : \mathbb{R} \to \mathbb{R}$ be given by $f_k(x) = \frac{\sin x}{k}$.

   (a) Does $\{f_k\}_{k=1}^{\infty}$ converge pointwise on $\mathbb{R}$? If so, find f such that $f_k \to f$ pointwise on $\mathbb{R}$.

   (b) Does f_k converge uniformly on $\mathbb{R}$?

Proof. We show

$$f_k \to 0 \text{ uniformly}$$

Remark: Notice that the 0 above is the function that map all reals to 0.

Fix ϵ.

$$\text{find } N \text{ such that } \forall n > N, \|f_n - 0\|_\infty \le \epsilon$$

We claim

$$N > \frac{1}{\epsilon} \text{ works}$$

Using the fact $|\sin x| \le 1$ for all $x \in \mathbb{R}$, we can deduce

$$\forall n > N, \forall x \in \mathbb{R}, |f_n(x)| = \left|\frac{\sin x}{n}\right| \le \frac{1}{n} < \frac{1}{N} < \epsilon$$

This then implies $\|f_n - 0\|_\infty \le \epsilon$ (done) .

Remark: Notice that it is of course possible that $\|f_N\|_\infty = \epsilon$. This is why you shouldn't always set the goal by proving strict inequality when proving convergence. That maybe "technically cool" if you catch my drift, but it is just unnecessary and stupid. ■

Question 4

4. Let f_n be integrable on $[0,1]$ and $f_n \to f$ uniformly on $[0,1]$. Show that if $b_n \nearrow 1$ as $n \to \infty$, then

$$\lim_{n\to\infty} \int_0^{b_n} f_n(x)\,dx = \int_0^1 f(x)\,dx$$

Proof. Because f_n is Riemann-integrable on $(0,1)$ and $f_n \to f$ uniformly on $(0,1)$. We know f is Riemann-integrable on $(0,1)$ and

$$\int_0^1 f_n dx \to \int_0^1 f(x) dx \text{ as } n \to \infty$$

Then

$$\lim_{n\to\infty} \int_0^{b_n} f_n dx = \lim_{n\to\infty} \Big(\int_0^1 f_n dx - \int_{b_n}^1 f_n dx \Big) = \int_0^1 f dx - \lim_{n\to\infty} \int_{b_n}^1 f_n dx$$

This let us reduce the problem into proving

$$\int_{b_n}^1 f_n dx \to 0 \text{ as } n \to \infty$$

Fix ϵ. We wish

$$\text{to find } N \text{ such that } \forall n > N, \left| \int_{b_n}^1 f_n dx \right| \leq \epsilon$$

Because each $f_n : [0,1] \to \mathbb{R}$ is bounded (f_n is integrable), and $f_n \to f$ uniformly. We know f_n are uniformly bounded (This will be *fully* justified in the proof for Question 7). Then, we know there exists M such that

$$M > \sup_n (\sup_{[0,1]} |f_n|)$$

Because $b_n \nearrow 1$. We know

$$\exists N, \forall n > N, |b_n - 1| < \frac{\epsilon}{M}$$

We claim

$$\text{such } N \text{ works}$$

Let $n > N$. See

$$\begin{aligned}
\left| \int_{b_n}^1 f_n dx \right| &\leq \int_{b_n}^1 |f_n| \, dx \\
&\leq \int_{1-\frac{\epsilon}{M}}^1 |f_n| \, dx \\
&\leq \int_{1-\frac{\epsilon}{M}}^1 M dx = \epsilon \text{ (done)}
\end{aligned}$$

■

Lemma 5.1.1. (product of uniformly convergent sequence is uniformly convergent on bounded domain) Given

(a) $f_n \to f$ and $g_n \to g$ uniformly on I

(b) f, g are bounded on I

Then

$$f_n g_n \to fg \text{ on } I$$

Proof. Observe

$$\begin{aligned}
|(f_n g_n)(x) - (fg)(x)| &= \left| \big((f_n - f)g_n\big)(x) + \big(f(g_n - g)\big)(x) \right| \\
&\leq |(f_n - f)(x)| \cdot |g_n(x)| + |f(x)| \cdot |(g_n - g)(x)|
\end{aligned}$$

Notice that there exists M globally greater than both $|g_n|$ and $|f|$, and that $(f_n - f)(x)$ and $(g_n - g)(x)$ both uniformly converge to 0 and we are done. ■

Question 5

5. If f is continuous on $[0,1]$ and if

$$\int_0^1 f(x)\, x^n\, dx = 0 \quad (n = 0, 1, 2, \dots)$$

Prove that $f(x) = 0$ on $[0,1]$. Hint: The integral of the product of f with any polynomial is zero. Use the Weierstrass theorem to show that $\int_0^1 f^2(x)\, dx = 0$

Proof. By Stone-Weierstrass Theorem, there exists a sequence of polynomial $P_k \to f$ uniformly. Because each polynomial is an finite linear combination of $x^n \quad (n = 0, 1, 2, \dots)$, from premise we can deduce

$$\int_0^1 f P_k dx = 0 \text{ for all } k \in \mathbb{N}$$

Because f is continuous on the compact domain $[0,1]$ and $P_n \to f$. It is easy to see that f and P_n satisfy the hypothesis of Lemma 5.1.1. Then, we see

$$f P_n \to f^2 \text{ uniformly}$$

This then let us deduce

$$\int_0^1 f^2 dx = 0$$

Assume $f(x) \neq 0$ for some $x \in [0,1]$, in the aiming for a contradiction. Because f^2 is continuous at x ($\because f$ is continuous at x). We know there exists δ such that

$$\inf_{[x-\delta, x+\delta]} f^2 = \alpha > 0$$

for some appropriate α, says, $\alpha = \frac{f^2(x)}{2}$.

Now, because $f^2 \geq 0$, we have

$$\int_0^1 f^2 dt \geq \int_{x-\delta}^{x+\delta} f^2 dt \geq 2\delta\alpha > 0 \text{ CaC } \text{ to } \int_0^1 f^2 dt = 0$$

■

Question 6

6. Show that if $\{f_n\}$ is a sequence of continuous functions on E such that converges uniformly to f, then f is continuous on E.

Proof. Click the following hyperlink (Theorem 5.3.1) ■

Question 7

7. Prove that if f_n is bounded on E, $\forall n \in \mathbf{N}$ and f_n converges uniformly to a bounded function f on E, then $\{f_n\}$ is uniformly bounded on E.

Proof. We first prove

$$f \text{ is bounded}$$

Assume f is not bounded. Let $p \in E$, we know there exists sequence $x_n \subseteq E$ such that $d(f(x_n), p) \to \infty$. Now, for arbitrary $k \in \mathbb{N}$, we see

$$d(f(x_n), p) \leq d(f_k(x_n), f(x_n)) + d(f_k(x_n), p)$$

Then because $f_k(x_n) \to f(x_n)$ uniformly, this give us

$$d(f_k(x_n), p) \geq d(f(x_n, p)) - d(f_k(x_n), f(x_n)) \to \infty$$

This implies f_k is unbounded CaC . (done)

We now prove

$$f_n \text{ is uniformly bounded}$$

Let $p \in E$ and $M \in \mathbb{R}^+$ satisfy

$$f[E] \subseteq B_M(p)$$

Because $\|f_n - f\|_\infty \to 0$, we know there exists $L \in \mathbb{R}^+$ such that $\|f_n - f\|_\infty < L$ for all $n \in \mathbb{N}$. We claim

$$\bigcup_{n \in \mathbb{N}} f[E] \subseteq B_{M+L}(p)$$

Fix $n \in \mathbb{N}$ and $x \in E$. We wish to show

$$d(f_n(x), p) < M + L$$

Observe

$$d(f_n(x), p) \leq d(f_n(x), f(x)) + d(f(x), p) < L + M \text{ (done)}$$

$\blacksquare$

Question 8

8. Let $f_k : [0,1] \to \mathbb{R}$ be a sequence of functions such that

   (1) $|f_k(x)| \leq M_1$ for all $k \in \mathbb{N}$ and $x \in [0,1]$,

   (2) $|f_k'(x)| \leq M_2$ for all $k \in \mathbb{N}$ and $x \in [0,1]$.

   for some positive M_1, M_2.

   (a) Prove that there exists a subsequence of $\{f_k\}_{k=1}^{\infty}$ which converges uniformly on $[0,1]$.

   (b) If the assumption (1) is omitted, can $\{f_k\}_{k=1}^{\infty}$ still have a convergent subsequence? If yes, prove it; If not, give an counterexample.

   (c) Show that the assumption (1) can be replaced by $f_k(0) = 0$ for all $k \in \mathbb{N}$.

Proof. **(a)** The assumption (1) implies f_k is pointwise bounded. We first show

$$f_k \text{ are equicontinuous}$$

Fix ϵ. We wish to find δ such that

$$\forall n \in \mathbb{N}, \forall x, y \in [0,1], |x - y| < \delta \implies |f_n(x) - f_n(y)| \leq \epsilon$$

We claim

$$\delta < \frac{\epsilon}{M_2} \text{ works}$$

Fix $n \in \mathbb{N}$ and $x, y \in [0,1]$ such that $|x - y| < \delta$. By Lagrange's MVT, we see

$$\frac{|f_k(x) - f_k(y)|}{|x - y|} \leq M_2$$

Then

$$|f_k(x) - f_k(y)| \leq M_2 \cdot |x - y| \leq M_2 \cdot \delta = \epsilon \text{ (done)}$$

(b). No. Consider $f_k(x) = x + k$. It is clear that $f_k(x)$ has no even pointwise convergent sequence, as for all x_0, the sequence $f_k(x_0)$ diverge.

(c) Suppose we are given assumption (2). It suffice to show that

$$\forall k \in \mathbb{N}, f_k(0) = 0 \implies \exists M_1 \in \mathbb{R}^+, \forall k \in \mathbb{N}, \forall x \in [0,1], |f_k(x)| \leq M_1$$

We claim

$$M_1 = M_2 \text{ works}$$

Fix $k \in \mathbb{N}$ and $x \in [0,1]$. By FTC and assumption two, we see

$$|f_k(x)| = \left| \int_0^x f_k' dt \right| \leq \int_0^x |f_k'| \, dt \leq \int_0^1 |f_k'| \, dt \leq \int_0^1 M_2 dt = M_2 = M_1 \text{ (done)}$$

■

5.2 Limit Interchange

Given an arbitrary set X and a complete metric space $(\overline{Y}, d)$, in Section **??**, we have proved that the set of functions with the following properties

(a) boundedness

(b) unboundedness

are respectively closed under uniform convergence. In next section (Section 5.3), we will prove that the following three properties

(a) continuity

(b) uniform continuity

(c) K-Lipschitz continuity

are again all closed under uniform convergence, where the proof for continuity is closed under uniform convergence use Theorem 5.2.1 as a lemma.

Here, we prove

(a) convergent of sequences

in, of course, complete metric space, is also closed under uniform convergence.

The reason we require the codomain $\overline{Y}$ of sequence to be complete is explained in the last paragraph of Section **??**. An example of such beautiful closure is lost if the codmain (Y, d) is not complete is $Y = \mathbb{R}^*$ and $a_{n,k} = \frac{1}{n} + \frac{1}{k}$.

Theorem 5.2.1. (Change Order of Limit Operations: Part 1) Given a double sequence $a_{n,k}$ whose codomain is (Y, d). Suppose

(a) $a_{n,k} \to a_{\bullet,k}$ uniformly as $n \to \infty$

(b) $a_{n,k} \to A_n$ pointwise as $k \to \infty$.

(c) $A_n \to A$

Then we can deduce

$$\lim_{k\to\infty} a_{\bullet,k} \text{ exists and } \lim_{k\to\infty} a_{\bullet,k} = A$$

In other words, we can switch the order of limit operations

$$\lim_{k\to\infty}\lim_{n\to\infty} a_{n,k} = \lim_{n\to\infty}\lim_{k\to\infty} a_{n,k}$$

Proof. We wish to prove

$$a_{\bullet,k} \to A \text{ as } k \to \infty$$

Fix ϵ. Because $a_{n,k} \to a_{\bullet,k}$ uniformly and $A_n \to A$ as $n \to \infty$, we know there exists m such that

$$d(A_m, A) < \frac{\epsilon}{3} \text{ and } \forall k \in \mathbb{N}, d(a_{m,k}, a_{\bullet,k}) < \frac{\epsilon}{3} \tag{5.1}$$

Then because $a_{m,k} \to A_m$ as $k \to \infty$, we know there exists K such that

$$\forall k > K, d(a_{m,k}, A_m) < \frac{\epsilon}{3} \tag{5.2}$$

We now claim

$$\forall k > K, d(a_{\bullet,k}, A) < \epsilon$$

The claim is true since by Equation 5.1 and Equation 5.2, we have

$$\forall k > K, d(a_{\bullet,k}, A) \leq d(a_{\bullet,k}, a_{m,k}) + d(a_{m,k}, A_m) + d(A_m, A) < \epsilon \text{ (done)}$$

■

Theorem 5.2.2. (Change Order of Limit Operations: Part 2) Given a double sequence $a_{n,k}$ whose codomain is (Y, d). Suppose

(a) $a_{n,k} \to a_{\bullet,k}$ uniformly as $n \to \infty$

(b) $a_{n,k} \to A_n$ pointwise as $k \to \infty$

(c) $a_{\bullet,k} \to A$ as $k \to \infty$

Then we can deduce

$$A_n \text{ converge and } A_n \to A$$

Proof. Fix ϵ. We wish to find N such that

$$\forall n > N, d(A_n, A) < \epsilon$$

Because $a_{n,k} \to a_{\bullet,k}$ uniformly as $n \to \infty$, we can let N satisfy

$$\forall n > N, \forall k \in \mathbb{N}, d(a_{n,k}, a_{\bullet,k}) < \frac{\epsilon}{3} \tag{5.3}$$

We claim

$$\text{such } N \text{ works}$$

Arbitrarily pick $n > N$. Because $a_{\bullet,k} \to A$, and because $a_{n,k} \to A_n$, we know there exists j such that

$$d(a_{\bullet,j}, A) < \frac{\epsilon}{3} \text{ and } d(a_{n,j}, A_n) < \frac{\epsilon}{3} \tag{5.4}$$

From Equation 5.3 and Equation 5.4, we now have

$$d(A_n, A) \leq d(A_n, a_{n,j}) + d(a_{n,j}, a_{\bullet,j}) + d(a_{\bullet,j}, A) < \epsilon \text{ (done)}$$

∎

In summary of Theorem 5.2.1 and Theorem 5.2.2, given a double sequence $a_{n,k}$ converging both side

(a) $a_{n,k} \to a_{\bullet,k}$ pointwise as $n \to \infty$

(b) $a_{n,k} \to a_{n,\bullet}$ pointwise as $k \to \infty$

As long as

(a) one side of convergence is uniform

(b) between two sequence $\{a_{\bullet,k}\}_{k\in\mathbb{N}}$ and $\{a_{n,\bullet}\}_{n\in\mathbb{N}}$, one of them converge, say, to A

Then the other sequence also converge, and the limit is also A.

It is at this point, we shall introduce two other terminologies. Suppose f_n is a sequence of functions from an arbitrary set X to a metric space Y. We say f_n is **pointwise Cauchy** if for all fixed $x \in X$, the sequence $f_n(x)$ is Cauchy. We say f_n is **uniformly Cauchy** if for all ϵ, there exists $N \in \mathbb{N}$ such that

$$\forall n, m > N, \forall x \in X, d\big(f_n(x), f_m(x)\big) < \epsilon$$

In last Section (Section **??**), we define the **uniform metric** d_∞ on X^Y by

$$d_\infty(f, g) = \sup_{x\in X} d\big(f(x), g(x)\big)$$

and say that $f_n \to f$ uniformly if and only if $f_n \to f$ in (X^Y, d_∞). Similar to this clear fact, we have

$$f_n \text{ is uniformly Cauchy} \iff f_n \text{ is Cauchy in } (X^Y, d_\infty)$$

It should be very easy to verify that if f_n uniformly converge, then f_n is uniformly Cauchy, and just like sequences in metric space, the converse hold true if and only if the space $\big(X^Y, d_\infty\big)$ is complete. In Theorem 5.2.3, we give a necessary and sufficient condition for $\big(X^Y, d_\infty\big)$ to be complete.

Theorem 5.2.3. (Space of functions $\big(X^Y, d_\infty\big)$ is Complete iff Y is Complete) Given an arbitrary set X and a metric space (Y, d), we have

$$\text{the extended metric space } \big(X^Y, d_\infty\big) \text{ is complete } \iff Y \text{ is complete}$$

Proof. $(\longleftarrow)$

Suppose f_n is uniformly Cauchy. We wish

$$\text{to construct a } f : X \to Y \text{ such that } f_n \to f \text{ uniformly}$$

Because f_n is uniformly Cauchy, we know that for all $x \in X$, the sequence $f_n(x)$ is Cauchy in (Y, d). Then because Y is complete, we can define $f : X \to Y$ by

$$f(x) = \lim_{n\to\infty} f_n(x)$$

We claim

$$\text{such } f \text{ works, i.e. } f_n \to f \text{ uniformly}$$

Fix ϵ. We wish

$$\text{to find } N \in \mathbb{N} \text{ such that for all } n > N \text{ and } x \in X \text{ we have } d\big(f_n(x), f(x)\big) < \epsilon$$

Because f_n is uniformly Cauchy, we know there exists N such that

$$\forall n, m > N, \forall x \in X, d\big(f_n(x), f_m(x)\big) < \frac{\epsilon}{2} \tag{5.5}$$

We claim

$$\text{such } N \text{ works}$$

Assume there exists $n > N$ and $x \in X$ such that $d\big(f_n(x), f(x)\big) \geq \epsilon$. Because $f_k(x) \to f(x)$ as $k \to \infty$, we know

$$\exists m \in \mathbb{N}, d\big(f_m(x), f(x)\big) < \frac{\epsilon}{2} \tag{5.6}$$

Then from Equation 5.5 and Equation 5.6, we can deduce

$$\epsilon \leq d\big(f_n(x), f(x)\big) \leq d\big(f(x), f_m(x)\big) + d\big(f_n(x), f_m(x)\big) < \epsilon \text{ CaC (done)}$$

$(\longrightarrow)$

Let K be the set of constant functions in X^Y. We first prove

$$K \text{ is closed}$$

Arbitrarily pick $f \in K^c$. We wish

$$\text{to find } \epsilon \in \mathbb{R}^+ \text{ such that } B_\epsilon(f) \in K^c$$

Because f is not a constant function, we know there exists $x_1, x_2 \in X$ such that

$$d\big(f(x_1), f(x_2)\big) > 0$$

We claim that

$$\epsilon = \frac{d\big(f(x_1), f(x_2)\big)}{3} \text{ works}$$

Arbitrarily pick $g \in B_\epsilon(f)$. We wish

$$\text{to show } g \in K^c$$

Notice the triangle inequality

$$3\epsilon = d\big(f(x_1), f(x_2)\big) \leq d\big(f(x_1), g(x_1)\big) + d\big(g(x_1), g(x_2)\big) + d\big(g(x_2), f(x_2)\big) \tag{5.7}$$

Also, because $g \in B_\epsilon(f)$, we have

$$\forall x \in X, d\big(f(x), g(x)\big) < \epsilon \tag{5.8}$$

Then by Equation 5.7 and Equation 5.8, we see

$$d\big(g(x_1), g(x_2)\big) > \epsilon$$

This then implies g is not a constant function. (done)

Now, Because by premise (X^Y, d_∞) is complete, and we have proved K is closed in (X^Y, d_∞), we know K is complete. Then, we resolve the whole problem into proving

$$Y \text{ is isometric to } K$$

Define $\sigma : Y \to K$ by

$$y \mapsto \tilde{y} \text{ where } \forall x \in X, \tilde{y}(x) = y$$

It is easy to verify σ is an isometry. (done) ■

Corollary 5.2.4. (Space of Bounded functions $\big(B(X,Y),d_\infty\big)$ is Complete iff Y is Complete)

$$\big(B(X,Y),d_\infty\big) \text{ is complete } \iff Y \text{ is complete}$$

Proof. ($\longleftarrow$)

By Theorem 5.2.3, the space (X^Y,d_∞) is complete. Then because $B(X,Y)$ is closed in (X^Y,d_∞), we know $B(X,Y)$ is complete.

($\longrightarrow$)

Notice that the set of constant function K is a subset of the galaxy $B(X,Y)$. The whole proof in Theorem 5.2.3 works in here too. ■

Remember in the beginning of this section we say we will prove convergent sequences in Y is closed under uniform convergence if Y is complete. The proof of this result relies on Theorem 5.2.3.

Now, before we actually prove convergence sequences are closed under uniform convergence if codomain (Y,d) is complete (Theorem 5.2.6), we will state and prove Weierstrass M-test (Theorem 5.2.5), which concerns the uniform convergence of series of complex functions.

Theorem 5.2.5. (Weierstrass M-test) Given sequences $f_n : X \to \mathbb{C}$, and suppose

$$\forall n \in \mathbb{N}, \forall x \in X, |f_n(x)| \leq M_n \tag{5.9}$$

Then

$$\sum_{n=1}^{\infty} M_n \text{ converge } \implies \sum_{n=1}^{\infty} f_n \text{ uniformly converge}$$

Proof. Because $(\mathbb{C}, \|\cdot\|_2)$ is complete, by Corollary 5.2.4, we only wish to prove

$$\sum_{k=1}^{n} f_k \;_{n\in\mathbb{N}} \quad \text{is uniformly Cauchy}$$

Fix ϵ. We wish

$$\text{to find } N \text{ such that } \forall n,m > N, \forall x \in X, \left|\sum_{k=n}^{m} f_k(x)\right| < \epsilon$$

Because $\sum_{n=1}^{\infty} M_n$ converge, we know there exists N such that

$$\forall n, m > N, \sum_{k=n}^{m} M_k < \epsilon$$

We claim

$$\text{such } N \text{ works}$$

By Premise 5.9, we have

$$\forall n, m > N, \forall x \in X, \left| \sum_{k=n}^{m} f_k(x) \right| \leq \sum_{k=n}^{m} |f_k(x)| \leq \sum_{k=n}^{m} M_k < \epsilon$$

■

Theorem 5.2.6. (Convergent Sequences are Closed under Uniform Convergence if Codomain $\big(Y, d\big)$ is Complete) Given a complete metric space $\big(Y, d\big)$, let $\mathcal{C}_{\mathbb{N}}^{Y}$ be the set of convergent sequences in Y.

$$Y \text{ is complete} \implies \mathcal{C}_{\mathbb{N}}^{Y} \text{ is closed under uniform convergent}$$

Proof. Let $a_{n,k} \to a_{\bullet,k}$ uniformly as $n \to \infty$ where for all $n, k \in \mathbb{N}, a_{n,k} \in Y$ and let $A_n = \lim_{k\to\infty} a_{n,k}$ for all $n \in \mathbb{N}$.

$$\text{to prove } a_{\bullet,k} \text{ converge}$$

By Theorem 5.2.2, we can reduce the problem to

$$\text{proving } A_n \text{ converge}$$

Then because Y is complete, we can then reduce the problem into proving

$$A_n \text{ is Cauchy}$$

Fix ϵ. We wish to find N such that

$$\forall n, m > N, d(A_n, A_m) < \epsilon$$

Because $a_{n,k} \to a_{\bullet,k}$ uniformly, we can find N such that

$$\forall n, m > N, d_{\infty}(\{a_{n,k}\}_{k\in\mathbb{N}}, \{a_{m,k}\}_{k\in\mathbb{N}}) < \frac{\epsilon}{3} \tag{5.10}$$

We claim

$$\text{such } N \text{ works}$$

Arbitrarily pick $n, m > N$. We wish to prove

$$d(A_n, A_m) < \epsilon$$

Because $a_{n,k} \to A_n$ and $a_{m,k} \to A_m$ as $k \to \infty$, we can find j such that

$$d(a_{n,j}, A_n) < \frac{\epsilon}{3} \text{ and } d(a_{m,j}, A_m) < \frac{\epsilon}{3} \tag{5.11}$$

Then from Equation 5.10 and Equation 5.11, we can deduce

$$d(A_n, A_m) \leq d(A_n, a_{n,j}) + d(a_{n,j}, a_{m,j}) + d(a_{m,j}, A_m) < \epsilon \text{ (done)}$$

■

5.3 Closed under Uniform Convergence

The end goal for this section is to prove that the following properties

(a) continuity

(b) uniform continuity

(c) K-Lipschitz Continuity

Theorem 5.3.1. (Uniform Limit Theorem) Given a sequence of function f_n from a topological space (X, τ) to a metric space (Y, d), suppose

(a) $f_n \to f$ uniformly as $n \to \infty$

(b) f_n is continuous for all $n \in \mathbb{N}$

Then f is also continuous.

Proof. Fix $x \in X$, and let $x_k \to x$. We wish to prove

$$f(x_k) \to f(x)$$

Because $f_n \to f$ uniformly as $n \to \infty$, we know

$$f_n(x_k)_{k\in\mathbb{N}} \to f(x_k)_{k\in\mathbb{N}} \text{ uniformly as } n \to \infty \tag{5.12}$$

Also, because for each $n \in \mathbb{N}$, the function f_n is continuous at x, we know

$$\forall n \in \mathbb{N}, f_n(x_k) \to f_n(x) \text{ as } k \to \infty \tag{5.13}$$

Then because $f_n \to f$ pointwise, we know

$$f_n(x) \to f(x) \tag{5.14}$$

Now, because Equation 5.12, Equation 5.13 and Equation 5.14, by Theorem 5.2.1, we have

$$\lim_{k\to\infty} f(x_k) = \lim_{k\to\infty}\lim_{n\to\infty} f_n(x_k) = \lim_{n\to\infty}\lim_{k\to\infty} f_n(x_k) = \lim_{n\to\infty} f_n(x) = f(x) \text{ (done)}$$

∎

Suppose X is a compact Hausdroff space, with Theorem **??**, we can now say that the set $\mathcal{C}(X)$ of complex-valued continuous functions on X

Theorem 5.3.2. (Uniformly Continuous functions are Closed under Uniform Convergence) Given a sequence of functions f_n from a metric space (X, d_X) to metric space (Y, d_Y), suppose

(a) $f_n \to f$ uniformly

(b) f_n is uniformly continuous for all $n \in \mathbb{N}$

Then f is also uniformly continuous

Proof. Fix ϵ. We wish

$$\text{to find } \delta \text{ such that } \forall x, y \in X, d_X(x,y) < \delta \implies d_Y\big(f(x), f(y)\big) < \epsilon$$

Because $f_n \to f$ uniformly, we know there exists $m \in \mathbb{N}$ such that

$$\forall x \in X, d_Y\big(f_m(x), f(x)\big) < \frac{\epsilon}{3} \tag{5.15}$$

Because f_m is uniformly continuous, we know

$$\exists \delta, \forall x, y \in X, d_X(x,y) < \delta \implies d_Y\big(f_m(x), f_m(y)\big) < \frac{\epsilon}{3} \tag{5.16}$$

We claim

$$\text{such } \delta \text{ works}$$

Let $x, y \in X$ satisfy $d_X(x,y) < \delta$. We wish

$$\text{to prove } d_Y\big(f(x), f(y)\big) < \epsilon$$

From Equation 5.15 and Equation 5.16, we have

$$d_Y\big(f(x), f(y)\big) \leq d_Y\big(f(x), f_m(x)\big) + d_Y\big(f_m(x), f_m(y)\big) + d_Y\big(f_m(y), f(y)\big) = \epsilon \text{ (done)}$$

∎

Theorem 5.3.3. **(K-Lipschitz functions are Closed under Uniform Convergence)** Given a sequence of functions f_n from metric space (X, d_X) to metric space (Y, d_Y), suppose

(a) $f_n \to f$ uniformly as $n \to \infty$

(b) f_n is K-Lipschtize continuous for all $n \in \mathbb{N}$

Then f is also K-Lipschtize continuous.

Proof. Arbitrarily pick $x, y \in X$, to show f is K-Lipschtize continuous, we wish

$$\text{to show } d_Y\big(f(x), f(y)\big) \leq K d_X(x,y)$$

Fix ϵ. We reduce the problem into proving

$$d_Y\big(f(x), f(y)\big) < K d_X(x,y) + \epsilon$$

Because $f_n \to f$ uniformly as $n \to \infty$, we know there exists m such that

$$\forall z \in X, d_Y\big(f(z), f_m(z)\big) < \frac{\epsilon}{2} \tag{5.17}$$

Because f_m is K-Lispchitz continuous, we know

$$d_Y\big(f_m(x), f_m(y)\big) \leq K d_X(x, y) \tag{5.18}$$

Now, from Equation 5.18 and Equation 5.17, we now see

$$d_Y\big(f(x), f(y)\big) \leq d_Y\big(f(x), f_m(x)\big) + d_Y\big(f_m(x), f_m(y)\big) + d_Y\big(f_m(y), f(y)\big) < K d_X(x, y) + \epsilon$$

■

An example of sequences of Lipschitz continuous functions with unbounded Lipschitz constant can uniformly converge to a non-Lipschitz continuous function is given below

Example 11 (Lipschitz functions with Unbounded Lipschitz constant Uniformly Converge to a non-Lipschitz function)

$$X = [0, 1] \text{ and } f_n(x) = \sqrt{x + \frac{1}{n}}$$

5.4 HW2

Question 9

1. Suppose f is Riemann integrable on $[0, A]$ for all $A < \infty$, and $f(x) \to 1$ as $x \to \infty$. Prove that
$$\lim_{t\to 0^+} t\int_0^\infty e^{-tx} f(x)\,dx = 1$$

Proof. We can reduce the problem into proving
$$\lim_{t\to 0^+}\int_0^\infty te^{-tx}f(x)dx - 1 = 0$$
Notice that for each $t > 0$, we have
$$1 = \int_0^\infty te^{-tx}dx$$
This then give us
$$\begin{aligned}\int_0^\infty e^{-tx}f(x)dx - 1 &= \int_0^\infty te^{-tx}f(x)dx - \int_0^\infty te^{-tx}dx\\ &= \int_0^\infty te^{-tx}\big[f(x)-1\big]dx\end{aligned}$$
Define $g(x) \triangleq f(x) - 1$. Because $f \to 1$ at ∞, we know $g \to 0$ at infinity. We now reduce the problem into proving
$$\lim_{t\to 0^+}\int_0^\infty te^{-tx}g(x)dx = 0$$
Note that with simple computation
$$\int_0^\infty te^{-tx}dx \text{ exists for all } t \in \mathbb{R}^+$$
Then because we have $te^{-tx} \sim te^{-tx}f(x)$ as $x \to \infty$, we see
$$\int_0^\infty te^{-tx}g(x)dx \text{ exists for all } t \in \mathbb{R}^+ \text{ by Integral Test and Limit Comparison Test}$$
Fix ϵ. We now reduce the problem into proving
$$\text{finding } \delta \text{ such that } \left|\int_0^\infty te^{-tx}g(x)dx\right| \le \epsilon \text{ for all } t \in (0,\delta)$$

Let A be large enough such that $g(x)$ is $\frac{\epsilon}{2}$-close to 0 whenever $x \geq A$. Note that g is bounded on $[A, \infty)$ and bounded on $[0, A]$ because g is integrable on $[0, A]$. Now, let $M > \sup_{\mathbb{R}^+} |g|$. We claim

$$\delta = \frac{-\ln(1 - \frac{\epsilon}{2M})}{A} \text{ works}$$

Observe

$$\begin{aligned}
\left| \int_0^\infty te^{-tx} g(x) dx \right| &\leq \left| \int_0^A te^{-tx} g(x) dx \right| + \left| \int_A^\infty te^{-tx} g(x) dx \right| \\
&\leq \int_0^A te^{-tx} \left| g(x) \right| dx + \int_A^\infty te^{-tx} \left| g(x) \right| dx \\
&\leq M \int_0^A te^{-tx} dx + \frac{\epsilon}{2} \int_A^\infty te^{-tx} dx \\
&\leq -Me^{-tx} \Big|_{x=0}^{A} + \frac{\epsilon}{2} \int_0^\infty te^{-tx} dx \\
&\leq M(1 - e^{-tA}) + \frac{\epsilon}{2} \\
&\leq M(1 - e^{-\delta A}) + \frac{\epsilon}{2} \\
&= M(1 - e^{\ln(1 - \frac{\epsilon}{2M})}) + \frac{\epsilon}{2} = \epsilon \text{ (done)}
\end{aligned}$$

■

Question 10

2. For $\delta \in (0, \pi)$ we define $f(x) = \begin{cases} 1 & , |x| \le \delta \\ 0 & , \delta < |x| \le \pi \end{cases}$, also $f(x+2\pi) = f(x)$ for all x.

   (a) Compute the Fourier coefficients of f.

   (b) Conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2}$.

   (c) Deduce from Parseval's theorem that $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} = \frac{\pi - \delta}{2}$.

   (d) Let $\delta \to 0$ and prove that $\displaystyle\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$.

   (e) Put $\delta = \dfrac{\pi}{2}$ in (c), what do you discover?

Proof. **(a)**

For $n \neq 0$, compute

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \\ &= \frac{1}{2\pi}\int_{-\delta}^{\delta} \cos(-nx) + i\sin(nx)dx \\ &= \frac{1}{2\pi}\int_{-\delta}^{\delta} \cos(-nx)dx \quad (\because \text{ sin is odd function }) \\ &= \frac{1}{2\pi} \cdot \frac{\sin(-nx)}{-n}\Big|_{x=-\delta}^{\delta} = \frac{\sin(n\delta)}{n\pi} \end{aligned}$$

Compute

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{\delta}{\pi}$$

(b)

Note that $c_{-n} = c_n$. We then deduce that the Fourier Series of f is

$$\frac{\delta}{\pi} + \sum_{n=1}^{\infty} \frac{2\sin(n\delta)}{n\pi} e^{-inx}$$

Because f is constant around 0, it is clearly Lipschitz at 0. We now deduce

$$1 = f(0) = \frac{\delta}{\pi} + \sum_{n=1}^{\infty} \frac{2\sin(n\delta)}{n\pi}$$

This implies

$$\sum_{n=1}^{\infty} \frac{2\sin(n\delta)}{n\pi} = 1 - \frac{\delta}{\pi}$$

which implies

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = (1 - \frac{\delta}{\pi}) \cdot \frac{\pi}{2} = \frac{\pi - \delta}{\pi} \cdot \frac{\pi}{2} = \frac{\pi - \delta}{2}$$

(c)

Parseval's Theorem says

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \sum_{-\infty}^{\infty} |c_n|^2$$

Clearly f is Riemann-Integrable. Plugin our setting, we see

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2 \, dx = \frac{\delta}{\pi}$$

and

$$\begin{aligned}\sum_{-\infty}^{\infty} |c_n|^2 &= (\frac{\delta}{\pi})^2 + \sum_{n=1}^{\infty} 2\Big(\frac{\sin(n\delta)}{n\pi}\Big)^2 \\ &= (\frac{\delta}{\pi})^2 + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2}\end{aligned}$$

This let us deduce

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2} = \frac{\pi^2}{2} \cdot (\frac{\delta}{\pi} - \frac{\delta^2}{\pi^2}) = \frac{\pi\delta - \delta^2}{2}$$

So

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} = \frac{\pi - \delta}{2}$$

(d)

Fix ϵ. Because $\int_0^{\infty} \frac{\sin^2}{x^2} dx$ absolutely converge, we can find R satisfying

$$\left| \int_R^{\infty} \frac{\sin^2 x}{x^2} dx \right| < \frac{\epsilon}{3} \text{ and } R > \frac{3}{\epsilon}$$

Define $\delta_N \triangleq \frac{R}{N}$. Partition $[0, R]$ by $\{0, R(\frac{1}{N}), R(\frac{2}{N}), \dots, R\}$. We see

$$\sum_{n=1}^{N} \frac{\sin^2(n\delta_N)}{(n\delta_N)^2} \delta_N \text{ is a Riemann Sum of Norm } |\delta_N|$$

This implies that there exists N_0 such that

$$\forall N > N_0, \left| \sum_{n=1}^{N} \frac{\sin^2(n\delta_N)}{(n\delta_N)^2} \delta_N - \int_0^R \frac{\sin^2 x}{x^2} dx \right| < \frac{\epsilon}{3}$$

Fix $N > N_0$. Observe that

$$\begin{aligned}
\left| \sum_{n=N+1}^{\infty} \frac{\sin^2(n\delta_N)}{(n\delta_N)^2} \delta_N \right| &\leq \sum_{n=N+1}^{\infty} \frac{\sin^2(n\delta_N)}{n^2\delta_N} \\
&\leq \sum_{n=N+1}^{\infty} \frac{1}{n^2\delta_N} \\
&= \frac{1}{\delta_N} \sum_{n=N+1}^{\infty} \frac{1}{n^2} \\
&\leq \frac{1}{\delta_N} \int_N^{\infty} \frac{1}{x^2} dx \quad (\because \frac{1}{x^2} \searrow) \\
&= \frac{1}{N\delta_N} = \frac{1}{R} < \frac{\epsilon}{3}
\end{aligned}$$

We now see

$$\begin{aligned}
&\left|\sum_{n=1}^{\infty}\frac{\sin^2(n\delta_N)}{(n\delta_N)^2}\delta_N-\int_0^{\infty}\frac{\sin^2 x}{x^2}dx\right| \\
\leq&\left|\sum_{n=1}^{N}\frac{\sin^2(n\delta_N)}{(n\delta_N)^2}\delta_N-\int_0^{R}\frac{\sin^2 x}{x^2}dx\right|+\left|\sum_{n=N+1}^{\infty}\frac{\sin^2(n\delta_N)}{(n\delta_N)^2}\delta_N\right|+\left|\int_R^{\infty}\frac{\sin^2 x}{x^2}dx\right| \\
\leq&\frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon
\end{aligned}$$

This now implies, for all ϵ, we can find R and a threshold N_0 corresponding to R such that

$$\forall N>N_0,\left|\sum_{n=1}^{\infty}\frac{\sin^2(n\delta_N)}{(n\delta_N)^2}\delta_N-\int_0^{\infty}\frac{\sin^2 x}{x^2}dx\right|\leq\epsilon$$

Then for $\epsilon_k=\frac{1}{k}$, we can find a sequence of real number $\delta_k\triangleq\frac{R_k}{N_k}\to 0$ such that

$$\left|\sum_{n=1}^{\infty}\frac{\sin^2(n\delta_k)}{(n\delta_k)^2}\delta_k-\int_0^{\infty}\frac{\sin^2 x}{x^2}dx\right|\leq\frac{1}{k}$$

Because we know

$$\sum_{n=1}^{\infty}\frac{\sin^2(n\delta_k)}{(n\delta_k)^2}\delta_k=\frac{\pi-\delta_k}{2}$$

We now see for each ϵ', because $\delta_k\to 0$, we can find k large enough such that

$$\begin{aligned}
\left|\int_0^{\infty}\frac{\sin^2 x}{x^2}dx-\frac{\pi}{2}\right|&=\left|\int_0^{\infty}\frac{\sin^2 x}{x^2}dx-\frac{\pi-\delta_k}{2}-\frac{\delta_k}{2}\right| \\
&\leq\left|\int_0^{\infty}\frac{\sin^2 x}{x^2}dx-\sum_{n=1}^{\infty}\frac{\sin^2(n\delta_k)}{(n\delta_k)^2}\delta_k\right|+\frac{\delta_k}{2} \\
&\leq\frac{1}{k}+\frac{\delta_k}{2}<\epsilon'
\end{aligned}$$

(e)

Put $\delta=\frac{\pi}{2}$. We have

$$\begin{aligned}
\frac{\pi}{4}&=\frac{\pi-\delta}{2} \\
&=\frac{2}{\pi}\cdot\sum_{n=1}^{\infty}\frac{\sin^2\frac{\pi}{2}n}{n^2}
\end{aligned}$$

This then implies

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

$\blacksquare$

Question 11

3. If $f(x) = (\pi - |x|)^2$ on $[-\pi, \pi]$, prove that $f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$, and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad , \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Proof. Compute Fourier coefficient

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\pi - |x|)^2 dx = \frac{\pi^2}{3}$$

and

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\pi - |x|)^2 e^{-inx} dx \\ &= \frac{2}{n^2} \end{aligned}$$

Note that f is an even function, that $f'(x) = 2(x - \pi)$ on $(0, \pi]$ and that

$$f'(0) = \lim_{x \to 0^+} \frac{(\pi - |x|)^2 - \pi^2}{x} = \lim_{x \to 0^+} \frac{-2\pi x + x^2}{x} = -2\pi$$

This now let us deduce

$$|f'| \leq 2\pi \text{ on } [-\pi, \pi]$$

Which implies f is 2π-Lipschitz on $[-\pi, \pi]$. This tell us that the Fourier Series $s_N(f; x)$ converge to f, meaning

$$\begin{aligned} f(x) = \sum_{-\infty}^{\infty} c_n e^{-inx} &= \frac{\pi^2}{3} + \sum_{-\infty, n \neq 0}^{\infty} \frac{2}{n^2} \cdot \big(\cos(-nx) + i \sin(-nx)\big) \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4 \cos nx}{n^2} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} \end{aligned}$$

We now can deduce

$$\begin{aligned} f(0) &= \pi^2 \\ &= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} \end{aligned}$$

This then implies

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Because f is continuous on $[-\pi, \pi]$, we know f is Riemann-Integrable. Then Parseval's Theorem assert

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2 \, dx = \sum_{-\infty}^{\infty} |c_n|^2$$

Now compute

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2 \, dx &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - |x|)^4 dx \\ &= \frac{1}{\pi} \int_{0}^{\pi} (\pi - x)^4 dx \quad (\because (\pi - |x|)^4 \text{ is even }) \\ &= \frac{1}{\pi} \cdot \frac{(\pi - x)^5}{-5} \Big|_{x=0}^{\pi} = \frac{\pi^4}{5} \end{aligned}$$

and

$$\sum_{-\infty}^{\infty} |c_n|^2 = \frac{\pi^4}{9} + 2\sum_{n=1}^{\infty} \frac{4}{n^4}$$

This now implies

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{8} \cdot (\frac{\pi^4}{5} - \frac{\pi^4}{9}) = \frac{\pi^4}{90}$$

■

Question 12

4. Let $K_N(x) = \dfrac{1}{N+1} \displaystyle\sum_{n=0}^{N} \frac{\sin(n+\frac{1}{2})x}{\sin(x/2)}$, show that

   (a) $K_N(x) = \dfrac{1}{N+1} \cdot \dfrac{1-\cos(N+1)x}{1-\cos x}$.

   (b) $K_N \geq 0$.

   (c) $\dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} K_N(x)\, dx = 1$.

   (d) If $0 < \delta \leq |x| \leq \pi$ then $K_N(x) \leq \dfrac{1}{N+1} \cdot \dfrac{2}{1-\cos\delta}$.

   (e) Let $s_N(f;\, x)$ be the N-th partial sum of the Fourier series of f, consider the arithmedic means

   $$\sigma_N = \frac{s_0 + s_1 + ... + s_N}{N+1}$$

   Prove that $\sigma_N(f;\, x) = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(x-t) K_N(t)\, dt$.

   (f) Use (e)'s result to prove the Fejer's Theorem:

   If f is continuous with period 2π, then $\sigma_N(f;\, x) \to f(x)$ uniformly on $[-\pi, \pi]$.

Proof. Proving (a) can be reduced to proving

$$\sum_{n=0}^{N} \frac{\sin(n+\frac{1}{2})x}{\sin(\frac{x}{2})} = \frac{1-\cos(N+1)x}{1-\cos x}$$

Using $\sin\alpha\sin\beta = \frac{1}{2}\Big(\cos(\alpha-\beta) - \cos(\alpha+\beta)\Big)$ to compute

$$\begin{aligned}
\sum_{n=0}^{N} \frac{\sin(n+\frac{1}{2})x}{\sin(\frac{x}{2})} &= \frac{1-\cos x}{1-\cos x} \sum_{n=0}^{N} \frac{\sin(n+\frac{1}{2})x}{\sin(\frac{x}{2})} \\
&= \frac{2\sin^2(\frac{x}{2})}{1-\cos x} \sum_{n=0}^{N} \frac{\sin(n+\frac{1}{2})x}{\sin(\frac{x}{2})} \\
&= \frac{1}{1-\cos x} \sum_{n=0}^{N} 2\sin(\frac{x}{2})\sin(n+\frac{1}{2})x \\
&= \frac{1}{1-\cos x} \sum_{n=0}^{N} \cos(-nx) - \cos(n+1)x \\
&= \frac{1}{1-\cos x} \sum_{n=0}^{N} \cos(nx) - \cos(n+1)x = \frac{1-\cos(N+1)x}{1-\cos x} \text{ (done)}
\end{aligned}$$

(b)

Notice that $\cos x < 1$ and $\cos(N+1)x \leq 1 \quad (\because K_N$ is only well defined on $(0, 2\pi)$). This then implies

$$1 - \cos x > 0 \text{ and } 1 - \cos(N+1)x \geq 0$$

Then we can deduce

$$K_N(x) = \frac{1}{N+1} \cdot \frac{1-\cos(N+1)x}{1-\cos x} \geq 0$$

(c)

We first compute the Dirchlet Kernel D_N

$$\begin{aligned}
D_N(x) &= \sum_{-N}^{N} e^{-inx} \\
&= \frac{e^{i(-N)x} - e^{i(N+1)x}}{1-e^{ix}} \\
&= \frac{e^{i(-N-\frac{1}{2})x} - e^{i(N+\frac{1}{2})x}}{e^{i\frac{-1}{2}x} - e^{i\frac{1}{2}x}} \\
&= \frac{2i\sin((-N-\frac{1}{2})x)}{2i\sin(\frac{-1}{2}x)} = \frac{\sin(N+\frac{1}{2})x}{\sin(\frac{x}{2})}
\end{aligned}$$

and

$$D_N(x) = \sum_{-N}^{N} e^{-inx} = 1 + 2\sum_{n=1}^{N} \cos nx$$

Now we can compute

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)dx &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{N+1}\sum_{n=0}^{N} \frac{\sin(n+\frac{1}{2})x}{\sin\frac{x}{2}}dx \\
&= \frac{1}{2\pi(N+1)}\sum_{n=0}^{N}\int_{-\pi}^{\pi} D_n(x)dx \\
&= \frac{1}{2\pi(N+1)}\sum_{n=0}^{N}\int_{-\pi}^{\pi}(1+2\sum_{k=1}^{n}\cos kx)dx \\
&= \frac{1}{2\pi(N+1)}\sum_{n=0}^{N} 2\pi = 1 \quad (\because \int_{-\pi}^{\pi}\cos kx = 0)
\end{aligned}$$

(d)

Suppose $0 < \delta \le |x| \le \pi$. Observe

$$\begin{aligned}
K_N(x) &= \frac{1}{N+1}\cdot\frac{1-\cos(N+1)x}{1-\cos x} \\
&\le \frac{1}{N+1}\cdot\frac{2}{1-\cos x} \quad (\because \cos x < 1) \\
&\le \frac{1}{N+1}\cdot\frac{2}{1-\cos\delta} \quad (\because 0 < \delta \le |x| \le \pi \implies \cos x \le \cos\delta < 1)
\end{aligned}$$

(e)

Compute

$$\begin{aligned}
\sigma_N(f;x) &= \frac{(s_0+\cdots+s_N)}{N+1}(f;x)\\
&= \frac{1}{N+1}\sum_{k=0}^{N} s_k(f;x)\\
&= \frac{1}{N+1}\sum_{k=0}^{N}\sum_{n=-k}^{k} c_n e^{inx}\\
&= \frac{1}{N+1}\sum_{k=0}^{N}\sum_{n=-k}^{k}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt e^{inx}\\
&= \frac{1}{N+1}\sum_{k=0}^{N}\frac{1}{2\pi}\sum_{n=-k}^{k}\int_{-\pi}^{\pi} f(t)e^{in(x-t)}dt\\
&= \frac{1}{N+1}\sum_{k=0}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\sum_{n=-k}^{k} e^{in(x-t)}dt\\
&= \frac{1}{N+1}\sum_{k=0}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_k(x-t)dt\\
&= \frac{1}{(N+1)2\pi}\sum_{k=0}^{N}\int_{x+\pi}^{x-\pi} -f(x-u)D_k(u)du \quad (\because u = x-t)\\
&= \frac{1}{(N+1)2\pi}\sum_{k=0}^{N}\int_{x-\pi}^{x+\pi} f(x-u)D_k(u)du\\
&= \frac{1}{(N+1)2\pi}\sum_{k=0}^{N}\int_{-\pi}^{\pi} f(x-u)D_k(u)du \quad (\because \text{ periodicity of } D_k \text{ and } f\ )\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\cdot\Big(\frac{1}{N+1}\sum_{k=0}^{N} D_k(u)\Big)du\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)K_N(u)du
\end{aligned}$$

(e)

Fix ϵ. We wish

to find N' such that for all $N > N'$ and $x \in \mathbb{R}$ we have $|\sigma_N(f;x) - f(x)| \le \epsilon$

Because f is continuous with period 2π, we know f is uniformly continuous on $\mathbb{R}$. We then can fix δ small enough such that

$$\sup_{|t|\leq\delta} |f(x-t)-f(x)| < \frac{\epsilon}{2}$$

Also, we can fix $M > \sup_{[-\pi,\pi]} |f|$. Define $Q_\delta = \frac{4M(\pi-\delta)}{\pi(1-\cos\delta)}$. We claim

$$N' > \frac{2Q_\delta}{\epsilon} \text{ works}$$

Fix $N > N'$ and $x \in \mathbb{R}$. Using $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)dt = 1$ and $K_N \geq 0$, see

(a) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)dt = 1$

(b) $K_N \geq 0$

(c) $\pi \geq |x| \geq \delta \implies K_N(x) \leq \frac{1}{N+1}\cdot\frac{2}{1-\cos\delta}$

$$\begin{aligned}
|\sigma_N(f;x)-f(x)| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)dt - f(x)\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)dt\right| \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-t)-f(x)|\, K_N(t)dt \\
&= \frac{1}{2\pi}\int_{-\delta}^{\delta} |f(x-t)-f(x)|\, K_N(t)dt + \frac{2(\pi-\delta)2M}{2\pi}\cdot\frac{1}{N+1}\cdot\frac{2}{1-\cos\delta} \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\epsilon}{2}K_N(t)dt + \frac{4M(\pi-\delta)}{(N+1)\pi(1-\cos\delta)} \\
&= \frac{\epsilon}{2} + \frac{4M(\pi-\delta)}{\frac{2Q_\delta}{\epsilon}\pi(1-\cos\delta)} = \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon \text{ (done)}
\end{aligned}$$

■

Question 13

5. If $f(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k$ is a power series with positive radius of convergence R, show that

$$f'(x) = \sum_{k=1}^{\infty} k a_k (x-x_0)^{k-1}$$

for $x \in (x_0 - R, x_0 + R)$.

Theorem 5.4.1. (Power Series are Smooth) Given a power series (a, c_n) of convergence radius R, if we define $f : D_R(a) \to \mathbb{C}$ by

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

Then

$$f \text{ is of class } C^\infty \text{ on } D_R(a) \text{ and } f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} c_n (z-a)^{n-k} \text{ on } D_R(a)$$

Proof. We prove by induction. Base case $k = 0$ is trivial. Fix $k \geq 0$. Suppose we have

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} c_n (z-a)^{n-k} \text{ on } D_R(a)$$

We are required to prove

$$f^{(k+1)}(z) = \sum_{n=k+1}^{\infty} \frac{n!}{(n-k-1)!} c_n (z-a)^{n-k-1} \text{ on } D_R(a)$$

Set f_m

$$f_m(z) \triangleq \sum_{n=k}^{k+m} \frac{n!}{(n-k)!} c_n (z-a)^{n-k}$$

We have

$$f_m \to f^{(k)} \text{ pointwise on } D_R(a) \text{ and } f_m'(z) = \sum_{n=k+1}^{k+m} \frac{n!}{(n-k-1)!} c_n (z-a)^{n-k-1} \tag{5.19}$$

We abstract our problem into proving

$$f_m' \to f^{(k+1)} \text{ pointwise on } D_R(a)$$

Fix $z_0 \in D_R(a)$. We only wish to prove

$$(f^{(k)})'(z_0) = \lim_{m\to\infty} f_m'(z_0)$$

Fix ϵ such that $|z_0 - a| < R - \epsilon$. By Equation 5.19, using Theorem 5.5.2 (Uniform Convergence and Differentiaiton). We only have to prove

$$f_m' \text{ uniformly converge on } \overline{D}_{R-\epsilon}$$

Note that

$$f_m'(z) = \sum_{n=0}^{m-1} \frac{(n+k+1)!}{n!} c_{n+k+1}(z-a)^n$$

so we can compute the radius of convergence for f_m'

$$\begin{aligned}\limsup_{n\to\infty} \sqrt[n]{\frac{(n+k+1)!}{n!}\,|c_{n+k+1}|} &= \limsup_{n\to\infty} \sqrt[n]{|c_{n+k+1}|} \\ &= \limsup_{n\to\infty} \sqrt[n]{|c_n|} = R\end{aligned}$$

Together by Cauchy-Hadamrd (absolute convergent on $a + R - \epsilon$) and M-test show that

$$\sum_{n=0}^{\infty} \frac{(n+k+1)!}{n!} c_{n+k+1}(z-a)^n \text{ uniformly converge on } \overline{D}_{R-\epsilon}(a) \quad \text{(done)}$$

$\blacksquare$

Question 14

6. Let $I \subseteq \mathbb{R}$ be a finite interval.

(a) Let $f_k : I \to \mathbb{R}$ be differentiable for all $k \in \mathbb{N}$, and $\{f'_k\}$ converges uniformly on I. Determine whether $\{f_k\}$ converges?

(b) Let $f_k : I \to \mathbb{R}$ be differentiable for all $k \in \mathbb{N}$, and $\{f_k\}$ converges uniformly on I. Determine whether f is differentiable?

Proof. **(a)** No. Let $f_k = k$. It is then a trivial counter example.

(b) No. Consider $|x|$. The function $|x|$ is continuous on $[-1, 1]$ but not differentiable on $x = 0$. By Weierstrass approximation Theorem, we know there exists a sequence of polynomials on $[-1, 1]$ uniformly converge to $|x|$, and they clearly all are differentiable. ■

Question 15

7. Let $f_k : [0,1] \to \mathbb{R}$ be differentiable on $(0,1)$, and f_k converges uniformly to f on $[0,1]$ for some $f : [0,1] \to \mathbb{R}$. Determine whether f'_k converges uniformly?

Proof. No. Consider

Example 12 (Derivative won't necessarily converge to the right place)

$$X = \mathbb{R} \text{ and } f_n(x) = \frac{\sin nx}{\sqrt{n}}$$

Compute

$$f'(x) = 0 \text{ and } f'_n(x) = \sqrt{n} \cos nx$$

$f_n'(0) \to \infty$ shows that f_n' doesn't even have to be pointwise convergence. Note that the fact f_k uniformly converge can be easily proved by choosing $n > \frac{1}{\epsilon^2}$ ■

Question 16

8. Let $f_k : I \to \mathbb{R}$ be Riemann integrable where $I \subseteq \mathbb{R}$ be a finite interval. Suppose f_k converges pointwise to a function $f : I \to \mathbb{R}$. Determine whether f is Riemann integrable on I?

Proof. No. Consider

Example 13 (Riemann-integrable functions Pointwise Converge to a Non-Riemann-integrable function)

$$X = [-1, 1] \text{ and } f_m(x) = \lim_{n\to\infty} (\cos m!\pi x)^{2n}$$

Because cos has range $[-1, 1]$, we know that

$$m!x \in \mathbb{Z} \iff f_m(x) \neq 0$$

This tell us that for all $x \in \mathbb{R} \setminus \mathbb{Q}$, $f_m(x) = 0$, and that for all $x \in \mathbb{Q}$, we have $f_n(x) = 1$ for large enough n, with some simple computation.

Then, we see that

$$f_n \to \mathbf{1}_{\mathbb{Q}} \text{ pointwise}$$

Now, notice that for all fixed m, if $m!x \in \mathbb{Z}$, we must have

$$x = \frac{p}{m!} \text{ for some } p \in \mathbb{Z}$$

Such x in bounded domain must then happen only finite amount of time. This show f_n are all continuous almost everywhere and thus integrable, while $\mathbf{1}_{\mathbb{Q}}$, the function to which they converge, is not, as it is discontinuous almost everywhere. ■

5.5 Uniform Convergence on Integration and Differentiation

Theorem 5.5.1. (Riemann-Integration and Uniform Convergence) Given a function $\alpha : [a,b] \to \mathbb{R}$ and a sequence of functions $f_n : [a,b] \to \mathbb{R}$ such that

(a) α increase on $[a,b]$

(b) $\int_a^b f_n d\alpha$ exists for all $n \in \mathbb{N}$

(c) $f_n \to f$ uniformly on $[a,b]$

Then

$$\lim_{n\to\infty} \int_a^b f_n d\alpha \text{ exists and } \int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha$$

Proof. We first prove

$$\int_a^b f d\alpha \text{ exists}$$

Fix ϵ. We wish to prove

$$\overline{\int_a^b} f d\alpha - \underline{\int_a^b} f d\alpha < \epsilon$$

Let $\epsilon_n = \|f_n - f\|_\infty$. Because $f_n \to f$ uniformly, we know

$$\text{there exists } n \in \mathbb{N} \text{ such that } \epsilon_n = \|f_n - f\|_\infty < \frac{\epsilon}{2\big[\alpha(b) - \alpha(a)\big]}$$

Because α increase, by definition of ϵ_n, we see

$$\int_a^b (f_n - \epsilon_n) d\alpha \leq \underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha \leq \int_a^b (f_n + \epsilon_n) d\alpha$$

Because $\epsilon_n < \frac{\epsilon}{2\big[\alpha(b)-\alpha(a)\big]}$, we now see

$$\begin{aligned}\overline{\int_a^b} f d\alpha - \underline{\int_a^b} f d\alpha &\leq \int_a^b (f_n + \epsilon_n) d\alpha - \int_a^b (f_n - \epsilon_n) d\alpha \\ &= \int_a^b (2\epsilon_n) d\alpha < 2\epsilon_n \cdot \big[\alpha(b) - \alpha(a)\big] = \epsilon \text{ (done)}\end{aligned}$$

We now prove

$$\int_a^b f_n d\alpha \to \int_a^b f d\alpha \text{ as } n \to \infty$$

Fix ϵ. We wish

$$\text{to find } N \text{ such that } \forall n > N, \left| \int_a^b f_n d\alpha - \int_a^b f d\alpha \right| < \epsilon$$

Recall the definition $\epsilon_n = \|f_n - f\|_\infty$. Because $\epsilon_n \to 0$, we know

$$\text{there exists } N \text{ such that } \forall n > N, \epsilon_n < \frac{\epsilon}{\alpha(b) - \alpha(a)} \tag{5.20}$$

We claim

$$\text{such } N \text{ works}$$

Fix $n > N$. From Equation 5.20, we see

$$\begin{aligned} \left| \int_a^b f_n d\alpha - \int_a^b f d\alpha \right| &= \left| \int_a^b (f_n - f) d\alpha \right| \\ &\leq \int_a^b |f_n - f| \, d\alpha \\ &\leq \int_a^b \epsilon_n d\alpha = \epsilon_n \big[\alpha(b) - \alpha(a)\big] < \epsilon \text{ (done)} \end{aligned}$$

■

Before the next Theorem, let's see three examples why this time we don't (can't) use the hypothesis: $f_n \to f$ uniformly.

Example 14 (Differentiable functions are NOT closed under uniform convergence)

$$X = [-1, 1] \text{ and } f(x) = |x|$$

By Weierstrass approximation Theorem, there is a sequence of polynomials (differentiable) that uniformly converge to f, which is not differntiable at 0.

Example 15 (Derivative won't necessarily converge to the right place)

$$X = \mathbb{R} \text{ and } f_n(x) = \frac{\sin nx}{\sqrt{n}}$$

Compute

$$f'(x) = 0 \text{ and } f_n'(x) = \sqrt{n}\cos nx$$

Example 16 (Derivative won't necessarily converge to the right place)

$$X = \mathbb{R} \text{ and } f_n(x) = \frac{x}{1+nx^2}$$

Compute

$$f = \tilde{0} \text{ and } f_n'(0) = 1$$

Informally speaking, these examples together with the fact integral are closed under uniform convergence (Theorem 5.5.1) should give you some ideas that differentiation and integration although are operations inverse to each other, are NOT symmetric. There is a certain hierarchy on continuous functions on a fixed compact interval. Thus, we have the next Theorem in its form.

Theorem 5.5.2. (Uniform Convergence and Differentiation) Given a sequence of function $f_n : [a, b] \to \mathbb{R}$ such that

(a) $f_n(x_0) \to L$ for some $x_0 \in [a, b]$

(b) f_n are differentiable on (a, b)

(c) f_n are continuous on $[a, b]$

(d) f_n' uniformly converge on (a, b)

Then there exists a function $f : [a, b] \to \mathbb{R}$ such that

$$\begin{aligned} f &\text{ is differentiable on } (a, b) \\ \text{and } f_n &\to f \text{ uniformly on } [a, b] \\ \text{and } f_n' &\to f' \text{ uniformly on } (a, b) \end{aligned}$$

Proof. We first prove

$$f_n \text{ uniformly converge on } [a, b] \tag{5.21}$$

Fix ϵ. We wish

$$\text{to find } N \text{ such that } \|f_n - f_m\|_\infty \leq \epsilon \text{ for all } n, m > N$$

Because $f_n(x_0)$ converge, and f_n' uniformly converge, we know there exists N such that

$$\begin{cases} |f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2} \\ \|f_n' - f_m'\|_\infty < \frac{\epsilon}{2(b-a)} \end{cases} \quad \text{for all } n, m > N \tag{5.22}$$

We claim

$$\text{such } N \text{ works}$$

Fix $x \in [a, b]$ and $n, m > N$. We need

$$\text{to show } |f_n(x) - f_m(x)| \leq \epsilon$$

We first prove

$$|f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| \leq \frac{\epsilon}{2}$$

Because $(f_n - f_m)' = f_n' - f_m'$, by MVT (Theorem **??**) and Equation 5.22, we can deduce

$$\begin{aligned} |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| &= |(f_n - f_m)(x) - (f_n - f_m)(x_0)| \\ &= \Big| \big[(f_n - f_m)'(t)\big](x - x_0) \Big| \text{ for some } t \text{ between } x, x_0 \\ &< \frac{\epsilon}{2(b-a)} \cdot |x - x_0| \\ &\leq \frac{\epsilon}{2(b-a)} \cdot (b - a) = \frac{\epsilon}{2} \;\; \big(\because x, x_0 \in [a, b]\big) \text{ (done)} \end{aligned}$$

Now, by Equation 5.22, we have

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ (done)} \end{aligned}$$

We claim

$$f(x) \triangleq \lim_{n \to \infty} f_n(x) \text{ for all } x \in [a, b] \text{ works} \tag{5.23}$$

We first show

$$f \text{ is differentiable on } (a, b)$$

Fix $x \in (a,b)$. We wish to prove

$$\lim_{t\to x} \frac{f(t)-f(x)}{t-x} \text{ exists}$$

Define $\phi : [a,b] \setminus x \to \mathbb{R}$ by

$$\phi(t) \triangleq \frac{f(t)-f(x)}{t-x}$$

We reduce our problem into proving

$$\lim_{t\to x} \phi(t) \text{ exists}$$

Set $\phi_n : [a,b] \setminus x \to \mathbb{R}$ by

$$\phi_n(t) \triangleq \frac{f_n(t)-f_n(x)}{t-x}$$

We first show

$$\phi_n \text{ uniformly converge on } [a,b] \setminus x \tag{5.24}$$

Fix ϵ. We have

$$\text{to find } N \text{ such that } |\phi_n(t)-\phi_m(t)| \leq \epsilon \text{ for all } n,m > N \text{ and } t \in [a,b] \setminus x$$

Because f_n' uniformly converge on $[a,b]$, we know there exists N such that

$$\|f_n' - f_m'\|_\infty \leq \epsilon \text{ for all } n,m > N \tag{5.25}$$

We claim

$$\text{such } N \text{ works}$$

Fix $n,m > N$ and $t \in [a,b] \setminus x$. We wish to prove

$$|\phi_n(t)-\phi_m(t)| \leq \epsilon$$

Because $(f_n - f_m)' = f_n' - f_m'$, by MVT (Theorem **??**) and Equation 5.25, we can deduce

$$\begin{aligned}
|\phi_n(t)-\phi_m(t)| &\leq \left| \frac{f_n(t)-f_n(x)}{t-x} - \frac{f_m(t)-f_m(x)}{t-x} \right| \\
&= \left| \frac{\big(f_n - f_m\big)(t) - \big(f_n - f_m\big)(x)}{t-x} \right| \\
&= \left| \big(f_n' - f_m'\big)(t_0) \right| \text{ for some } t_0 \text{ between } t,x \\
&\leq \epsilon \text{ (done)}
\end{aligned}$$

We now show

$$\phi_n \to \phi \text{ pointwise on } [a,b] \setminus x \tag{5.26}$$

Because $f_n \to f$ on $[a,b]$ by definition (Equation 5.23), (the convergence is in fact uniform as we have shown. This doesn't matter here tho), for each $t \in [a,b] \setminus x$, we can deduce

$$\lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty} \frac{f_n(t)-f_n(x)}{t-x} = \frac{f(t)-f(x)}{t-x} = \phi(t) \text{ (done)}$$

Now, by Equation 5.24 and Equation 5.26, we know

$$\phi_n \to \phi \text{ uniformly on } [a,b] \setminus x$$

Notice that because $f_n'(x)$ converge, we know

$$\lim_{n\to\infty} \lim_{t\to x} \phi_n(t) = \lim_{n\to\infty} f_n'(x) \text{ exists}$$

Then (Notice that the second equality below hold true because we have known $\lim_{n\to\infty} \lim_{t\to x} \phi_n(t)$ exists), we can finally deduce

$$\begin{aligned} \lim_{t\to x} \phi(t) &= \lim_{t\to x} \lim_{n\to\infty} \phi_n(t) \\ &= \lim_{n\to\infty} \lim_{t\to x} \phi_n(t) \\ &= \lim_{n\to\infty} f_n'(x) \text{ exists} \quad \text{(done)} \end{aligned}$$

Now, notice that $f'(x) = \lim_{t\to x} \phi(t)$, so in fact, we have just proved $f_n' \to f'$, and the convergence is uniform by premise. Also, the statement

$$f_n \to f \text{ uniformly on } [a,b]$$

has been proved, since we already have $f_n \to f$ by our setting (Equation 5.23) and we have proved such convergence is uniform (Equation 5.21). The proof is now completed. (done) ■

As Rudin remarked, a much shorter (and much more intuitive) proof can be given, if we require f' to be continuous on $[a,b]$.

Theorem 5.5.3. (Uniform Convergence and Differentiation: Weaker Version) Given a sequence of function $f_n : [a,b] \to \mathbb{R}$ such that

(a) $f_n(x_0) \to L$ for some $x_0 \in [a,b]$

(b) f_n are differentiable on (a,b)

(c) f'_n are continuous on $[a,b]$ (f'_n at a,b are one-sided)

(d) f_n are continuous on $[a,b]$

(e) f'_n uniformly converge on $[a,b]$

Then there exists a function $f:[a,b]\to\mathbb{R}$ such that

$$\begin{aligned} f &\text{ is differentiable on } (a,b) \\ \text{and } f_n &\to f \text{ uniformly on } [a,b] \\ \text{and } f'_n &\to f' \text{ uniformly on } (a,b) \end{aligned}$$

Proof. We claim

$$f(x)=\lim_{n\to\infty}\int_{x_0}^{x} f'_n(t)dt+L \text{ works}$$

Note that $\lim_{n\to\infty}\int_{x_0}^{x} f'_n(t)dt$ exists because f'_n uniformly converge (Theorem 5.5.1).

Because f'_n uniformly converge and are continuous on $[a,b]$, by ULT, we know

$$\int_{x_0}^{x}\lim_{n\to\infty} f'_n(t)dt+L \text{ exists}$$

and know

$$f(x)=\int_{x_0}^{x}\lim_{n\to\infty} f'_n(t)dt+L$$

By FTC, we see

$$f'(x)=\lim_{n\to\infty} f'_n(x) \text{ on } (a,b)$$

Such convergence is uniform by premise. To finish the proof, we now only have to prove

$$f_n\to f \text{ uniformly on } [a,b]$$

Fix ϵ. We wish

$$\text{to find } N \text{ such that } |f_n(x)-f(x)|\leq\epsilon \text{ for all } n>N \text{ and } x\in[a,b]$$

Because $f'_n\to f$ uniformly, and $f_n(x_0)\to L=f(x_0)$ (Check $L=f(x_0)$), we know there exists N such that

$$\begin{cases} \|f'_n-f\|_\infty<\frac{\epsilon}{2(b-a)} \\ |f_n(x_0)-f(x_0)|<\frac{\epsilon}{2} \end{cases} \quad \text{for all } n>N$$

We claim

$$\text{such } N \text{ works}$$

Fix $n > N$ and $x \in [a, b]$. Observe

$$\begin{aligned}
|f(x) - f_n(x)| &= \left| \int_{x_0}^{x} \big(f'(t) - f_n'(t)\big) dt + f(x_0) - f_n(x_0) \right| \\
&\leq \int_{x_0}^{x} |f'(t) - f_n'(t)| \, dt + |f(x_0) - f_n(x_0)| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ (done)}
\end{aligned}$$

■