Introduction to Probability Theory

The Binomial Asset Pricing Model

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- In the binomial asset pricing model, we model stock prices in discrete time, assuming that $t = 0$ as the initial step with the initial stock price S_0.
- At time $t = 1$, the stock price will change to one of two possible values: uS_0 and dS_0 with

$$0 < d < 1 < 1 + r < u,$$

where $r > 0$ is the interest rate in a money market for both borrowing and lending, such that, \$1 dollar out/in the money market becomes $\$(1 + r)$ in the next period.

- So, change of the stock price from S_0 to dS_0 represents a downward movement, and change of the stock price from S_0 to uS_0 represents an upward movement.
- We shall use the notation S_1 to represent the stock price at $t = 1$. Then, S_1 can be either uS_0 or dS_0.

- At $t = 2$, the binomial asset pricing model repeats the pattern of itself at $t = 1$.
- The stock price will change, from S_1, to one of two possible values: uS_1 and dS_1 with

$$0 < d < 1 < 1 + r < u,$$

making the possible stock prices at $t = 2$ to be either $u^2 S_0$, or udS_0, or $d^2 S_0$.

- At $t = 3$, the model allows stock prices to appear only in the following set:

$$\{u^3 S_0,\ u^2 dS_0,\ ud^2 S_0,\ d^3 S_0\}.$$

- Continuing this process, the binomial asset pricing model is said to have a discrete time horizon

$$\Lambda = \{0, 1, 2, \ldots, \mathrm{T}, \mathrm{T} + 1, \ldots\}.$$

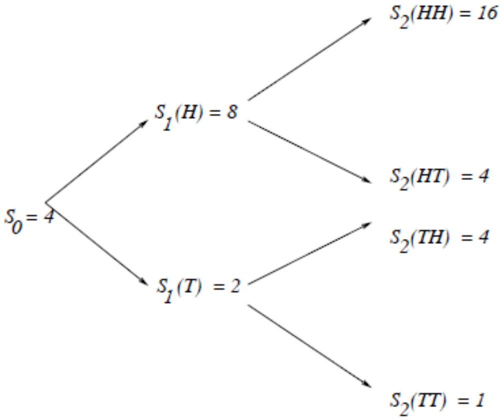
- Of course, stock price movements are much more complicated than indicated by the binomial asset pricing model.
- The model is still very useful in practice, however, because, with a sufficiently large number n of steps, the possible number of price values is astronomical.
- Moreover, if the actual time interval distance between two timescales is very small, this can almost be considered a continuous time model.

- Let us imagine that, at $t = 1$, we are tossing a coin, and when we get a "Head," the stock price moves up (from the price of it at the previous stage), but when we get a "Tail," the price moves down.
- Whether we shall get a head or a tail is said to be random, because that depends on chance.
- One has to be very careful about the word "random." It does not imply "completely unknown."
- Rather, we understand completely that the set of possible results consists of either head or tail, but we are not sure, at a particular toss of coin, which one would actually happen.

- After the second toss, the price will be one of:

$$S_2(HH) = u^2 S_0,\ S_2(HT) = udS_0,\ S_2(TH) = udS_0,\ S_2(TT) = d^2 S_0.$$

- For example, the binomial tree of stock prices with $S_0 = 4,\ u = 1/d = 2$ can be drawn as follows:



- Let us assume that the third toss is the last one and denote all possible outcomes of the three tosses by

$$\Omega_3 = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

- The time horizon is the set $\Lambda = \{0, 1, 2, 3\}$ with the terminating time $\mathrm{T} = 3$.
- The set Ω_3 is called the sample space for the experiment, and the elements $\omega = \omega_1\omega_2\omega_3$ of Ω are called sample points, where we denote the k^{th} component of ω by ω_k, $k = 1, 2, 3$.
- For example, when $\omega = HHT$, we have

$$\omega_1 = H,\ \omega_2 = H,\ \omega_3 = T.$$

- In general, we may consider an experiment of tossing a coin for infinitely many times. In that case, the terminating time $\mathrm{T} = \infty$, and the sample space is a space of infinite sequences, each component of which is either H or T.

$$\Omega_\infty = \{\omega = (\omega_1, \omega_2, \ldots, \omega_n, \ldots) \mid \omega_i = H \vee T\}.$$

- The stock price S_k at time k depends on the sample points of the sample space.
- In the experiment of tossing a coin for three times, the time horizon is $\{0, 1, 2, \mathrm{T} = 3\}$ so there are four stock prices S_0, S_1, S_2, S_3, the value of which depends on the sample point $\omega \in \Omega_3$.
- For example, the stock price at $t = 1$ can be denoted as

$$S_1(HHH) = S_1(HHT) = S_1(HTH) = S_1(HTT) = uS_0;$$

and

$$S_1(THH) = S_1(THT) = S_1(TTH) = S_1(TTT) = dS_0,$$

which depends on ω_1 of $\omega = \omega_1\omega_2\omega_3$ at $t = 1$.

- Similarly, the stock price at $t = 2$ can be denoted as

$$S_2(HHH) = S_2(HHT) = u^2 S_0;$$

$$S_1(HTH) = S_1(HTT) = S_2(THH) = S_2(THT) = udS_0;$$

and

$$S_2(TTH) = S_2(TTT) = d^2 S_0,$$

depending on $\omega_1\omega_2$ of $\omega = \omega_1\omega_2\omega_3$ at $t = 2$.

- Moreover, the stock price at $t = 3$ can be denoted as

$$S_3(HHH) = u^3 S_0;$$

$$S_3(HHT) = S_3(HTH) = S_3(THH) = u^2 d S_0;$$
$$S_3(TTH) = S_2(THT) = S_3(HTT) = u d^2 S_0;$$

and

$$S_3(TTT) = d^3 S_0,$$

depending on all three components of $\omega = \omega_1 \omega_2 \omega_3$ at $t = 3$.

- Finally, the initial stock price S_0 is a constant, which can be written as

$$S_0(\omega) = S_0, \quad \forall \omega \in \Omega_3.$$

- The four stock prices S_0, S_1, S_2, S_3 are called random variables defined on the sample space Ω_3, which is a function sending each sample point $\omega \in \Omega_3$ to a real number.

Stochastic process $X(t), \ t \in \Lambda$

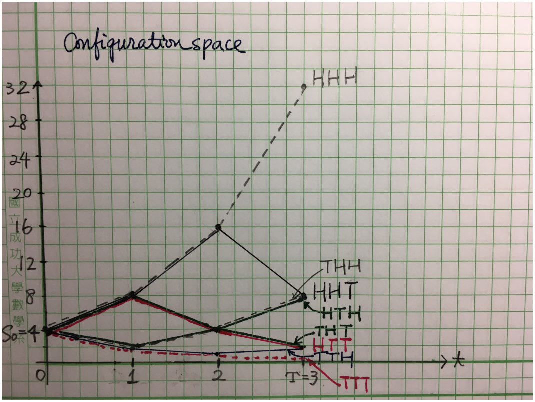
- In the binomial asset pricing model of 3 stages, the time horizon $\Lambda = \{0, 1, 2, 3\}$. The collection of random variables $X(t)_{t \in \Lambda} = (S_0, S_1, S_2, S_3)$ is said to form a stochastic process on the sample space Ω_3.
- That is, the stochastic process $X(t)_{t \in \Lambda} = (S_0, S_1, S_2, S_3)$ sends each sample point $\omega \in \Omega_3$ into a sequence of four real values:

$$X(t)_{t \in \Lambda}(\omega) = (S_0(\omega), S_1(\omega), S_2(\omega), S_3(\omega)), \ \omega \in \Omega_3.$$

- There are 8 such sequences in this example, each corresponding to a sample point in Ω_3.
- The space consisting of all such "sample sequences" can be drawn in the space $\Lambda \times \mathbb{R}$ (time space $\times$ state space) as trajectories. Such a space of trajectories is usually called as the "configuration space" or "foreground space," whereas the sample space Ω_3 is called the "background space."
- Therefore, a stochastic process sends each sample point in the background space to a trajectory in the foreground space.

Configurations in the Foreground Space

$X(t)_{t\in\Lambda} = (S_0, S_1, S_2, S_3)(\omega)$ where $\omega \in$
$\Omega_3 = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$



General Definition of a stochastic process $X(t), \ t \in \Lambda$

- In mathematical notation, a stochastic process is defined as a family of random variables $X(t)_{t\in\Lambda}$ indexed by $t \in \Lambda$ such that
  - For each fixed $t \in \Lambda$ in the time horizon, it is a random variable defined on Ω;
  - For each sample point $\omega \in \Omega$, the family of random variables $X(t)_{t\in\Lambda}$ send this ω to a family of real values. The family of values, since it is associated with ω, is often denoted by $\omega(t), \ t \in \Lambda$. If Λ is a finite set such as $\{0, 1, 2, 3\}$, $\omega(t), \ t \in \Lambda$ is a vector in $\mathbb{R}^4$. If $\Lambda = \mathbb{N}$, $\omega(t), \ t \in \Lambda$ is an infinite sequence. If $\Lambda = [a, b]$, an interval, then $\omega(t), \ t \in \Lambda$ is a real-valued function defined on $[a, b]$.

- Equivalently, a stochastic process $X(t)_{t\in\Lambda}$ on a sample space Ω can be viewed as a real-valued function X on a product space $\Lambda \times \Omega$, written as

$$X : (t, \omega) \in \Lambda \times \Omega \quad \mapsto \quad \mathbb{R}$$

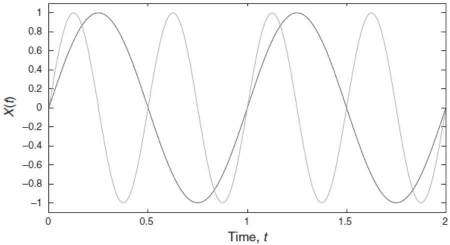
Example of a stochastic process $X(t), t \in \Lambda = [0,2]$, $\Omega = \{H, T\}$ with continuous trajectories

- In this process, the background space is $\Omega = \{H, T\}$
- The process is specified by the foreground space:

$$H \mapsto x_H(t) = \sin(2\pi t); \;\; T \mapsto x_T(t) = \sin(4\pi t), \;\; t \in \Lambda = [0,2]$$

- For each $t_0 \in [0,2]$, for example, $t_0 = \frac{2}{3}$, it is a Bernoulli random variable on $\Omega = \{H, T\}$ in a way such that

$$t_0(H) = \frac{2}{3}(H) = \sin(\frac{4\pi}{3}) = -\frac{\sqrt{3}}{2}; t_0(T) = \frac{2}{3}(T) = \sin(\frac{8\pi}{3}) = \frac{\sqrt{3}}{2}$$



Homework Exercise #1: Probability space

Let S_0 be the initial stock price at $t = 0$. Assume that, at each step, the stock price will either move up by a factor $u > 1$ multiplied with its value at the previous step when a coin is tossed to get a head; or move down by a factor $0 < d < 1$ when the coin turns out to be a tail. For example, when the first toss is H, the stock price will be uS_0 at time $t = 1$. Otherwise, dS_0. Write down the stochastic process $\{X(t)_{t\in\Lambda}\}$ corresponding to the $5-$stage binomial asset pricing model up to $t = 5$.

- Write down the background sample space for the background experiment.
- Write down the foreground configuration space. Try to display all members in the foreground space by trajectories.
- The set of all possible real values in the process X: $\{X(t,\xi)|t \in \Lambda,\ \xi \in \Omega\}$ is called the state space and each value in the state space is called the state of the process. Write down the state space Θ of the process. Order the states in the State space by the frequency of visits for all $\xi \in \Omega$ at all time $t \in \Lambda$.

European Call Option with strike price $K > 0$ and Expiration at time 1

- This option confers the right to buy the stock at time 1 for K dollars.
- In the money value, equivalently, if the stock price S_1 at time 1 is greater than K, the option is worth $S_1 - K > 0$ at time 1.
- Otherwise, if $S_1 \leq K$, it is worth 0.
- Since the stock price S_1 is random, the value for the call option at time 1 is thus random.
- We denote the value (payoff) of this option at expiration to be

$$\begin{aligned} V_1(\omega) &= V_1(\omega_1) \\ &= (S_1(\omega) - K)^+ \\ &= \max\{S_1(\omega) - K, 0\}, \ \forall \omega = \omega_1\omega_2\omega_3 \in \Omega_3 \end{aligned}$$

Risk Premium and Hedging

- When a trader sells an European call option, a "short position" in stock is created, because he/she now has an obligation to sell the stock to the option-buyer at the strike price K.
- The seller could incur an infinite loss, when the stock price at time 1 becomes arbitrarily high.
- For this reason, the seller collects "risk premium" at time 0, denoted by V_0, from the buyer in order to justify his/her future risk at time 1.
- Moreover, to reduce the risk of adverse price movements (for selling a call option, the adverse price movement means that the stock price goes higher), one needs to "hedge" the short position in stock by taking the opposite position (buying stock) for Δ_0 shares.

Arbitrage v.s. Hedging

- You can use the proceeds V_0 of the sale of the option for buying Δ_0 shares of stocks, and then borrow if necessary at interest rate r (from the money market) to complete the purchase.
- If V_0 is more than necessary to buy the Δ_0 shares of stock, you invest the residual money at interest rate r.
- In either case, you will have $V_0 - \Delta_0 S_0$ dollars invested in the money market, where this quantity might be negative to indicate "borrowing."
- You will also own Δ_0 shares of stock.
- As a whole, you have created a "portfolio at time 0" consisting of shorting an European call option, owning Δ_0 shares of stock, with some cashes $V_0 - \Delta_0 S_0$ in the money market, "without" investing any money from his/her own pocket.

Arbitrage v.s. Hedging

- "Arbitrage" is the method of buying one asset in one market and selling simultaneously the same or similar asset in another one, in order to make a profit (now or future) from the difference in price in the two places.
- The portfolio specified above in some cases can earn you money, though it may not be "risk free." It is thus a kind of "arbitrage" since you have invested nothing.
- At time 1, the value of the portfolio is a random variable

$$-(S_1(\omega) - K)^+ + \Delta_0 S_1(\omega) + (1+r)(V_0 - \Delta_0 S_0)$$

where the negative sign in $-(S_1(\omega) - K)^+$ indicates that, since the option has been already sold to the third party, any positive value of the option becomes your obligation.

The value of the portfolio at time 1

$$-(S_1(\omega) - K)^+ + \Delta_0 S_1(\omega) + (1+r)(V_0 - \Delta_0 S_0)$$

- To arbitrage, you want to price the risk premium V_0 and determine Δ_0, both are unknowns, based on the interest rate r, current stock price, the striking price K, so that the value of your portfolio at time 1 is always non-negative regardless the stock price at time 1.
- In the binomial asset pricing model of three stages, there are only two values for $S_1(\omega)$ (though there are 8 sample points in Ω_3), either it be $S_1(\omega) = uS_0$ or $S_1(\omega) = dS_0$.
- We have thus two linear inequalities in two unknowns V_0, Δ_0:

$$\begin{aligned}
\Delta_0 S_1(H) + (1+r)(V_0 - \Delta_0 S_0) &\geq V_1(H) = (S_1(H) - K)^+ \\
\Delta_0 S_1(T) + (1+r)(V_0 - \Delta_0 S_0) &\geq V_1(T) = (S_1(T) - K)^+
\end{aligned}$$

The famous “delta-hedging” formula

$$
\begin{aligned}
\Delta_0 S_1(H) + (1+r)(V_0 - \Delta_0 S_0) &\geq V_1(H) = (S_1(H) - K)^+ \\
\Delta_0 S_1(T) + (1+r)(V_0 - \Delta_0 S_0) &\geq V_1(T) = (S_1(T) - K)^+
\end{aligned}
$$

- In the equilibrium, the equality signs hold.
- By subtracting the two, we obtain the famous “delta-hedging” formula
$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}.$$
- The number of shares to hold for the hedging purpose is the derivative (in the sense of Calculus) of the value of the derivative security (European option) with respect to the value of underlined asset.
- After completing the calculation, we have the risk premium:
$$V_0 = \frac{1}{1+r}\left[\frac{1+r-d}{u-d} V_1(H) + \frac{u-(1+r)}{u-d} V_1(T)\right]$$

The arbitrage pricing: set the price for V_0

$$V_0 = \frac{1}{1+r}\left[\frac{1+r-d}{u-d}V_1(H) + \frac{u-(1+r)}{u-d}V_1(T)\right]$$

- To simplify the formula, let us define

$$\tilde{p} = \frac{1+r-d}{u-d}; \quad \tilde{q} = \frac{u-(1+r)}{u-d}.$$

  Then the risk premium can be represented as

$$V_0 = \underbrace{\frac{1}{1+r}}_{discount\ \ factor} \underbrace{[\tilde{p}V_1(H) + \tilde{q}V_1(T)]}_{risk\ \ neutral\ \ expectation}$$

- Since $\tilde{p}, \tilde{q} \in (0,1)$ and $\tilde{p} + \tilde{q} = 1$, they are "probabilities."
- They are called "risk neutral probability" because the probabilities have already taken into account of risk by comparing amongst various market factors u, d, r, while having nothing to do with the randomness of H and T.

Example

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}.$$

$$V_0 = \frac{1}{1+r}\left[\tilde{p}V_1(H) + \tilde{q}V_1(T)\right], \ \tilde{p} = \frac{1+r-d}{u-d}; \quad \tilde{q} = \frac{u-(1+r)}{u-d}.$$

- Let's assume that $d = 0.8, r = 0.05, u = 1.2, S_0 = 20, K = 17$. Then, $S_1(H) = 24, S_1(T) = 16$.
- The value for the option $V_1 = (S_1 - K)^+$ is a random variable with $V_1(H) = (24-17)^+ = 7, V_1(T) = (16-17)^+ = 0$. Moreover, the risk neutral probabilities: $\tilde{p} = \frac{1+0.05-0.8}{1.2-0.8} = \frac{5}{8}; \tilde{q} = \frac{3}{8}$.
- Compute the delta-hedging $\Delta_0 = \frac{7-0}{24-16} = \frac{7}{8}$.
- Compute the risk premium $V_0 = \frac{1}{1.05}[\frac{5}{8} \times 7 + \frac{3}{8} \times 0] = 4.166$.

Construct the portfolio

- Short the European call option of strike price $K = 17$ at risk premium $V_0 = \$4.166$.
- Long $\Delta_0 = 7/8$ shares of stock at \$20 for hedging, which accounts for \$17.5
- Borrow $\$17.5 - \$4.166 = \$13.334$ with interest rate $r = 0.05$ from the money market.

- In the next period, the value of the portfolio:
  - If the stock goes up to \$24,

$$-V_1(H)+\Delta_0 S_1(H)+(1+r)(V_0-\Delta_0 S_0) = -7+\frac{7}{8}\times 24-1.05\times 13.334 = 0$$

  - If the stock moves down to \$16,

$$-V_1(T)+\Delta_0 S_1(T)+(1+r)(V_0-\Delta_0 S_0) = \frac{7}{8}\times 16-1.05\times 13.334 = 0$$

- Possible arbitrage: when $S_1 \geq 17$, the value of the option $V_1 = S_1 - 17 \geq 0$, $-(S_1 - 17) + \frac{7}{8}S_1 - 14 \geq 0 \Leftrightarrow 17 \leq S_1 \leq 24$. When $S_1 < 17$, $V_1 = 0$. Then, $\frac{7}{8}S_1 - 14 \geq 0 \Leftrightarrow S_1 \geq 16$.

Leverage effect

- It has been seen that, when $S_1 \in (16, 24)$, the option seller can arbitrage through the portfolio.
- How about the buyer of the option? He/she paid a premium of \$4.116 to acquire the right of buying the stock at the price \$17 at time 1.
- If $S_1 \leq 17 + 4.116 \times 1.05 = 21.32$, the buyer of the call option becomes a loser anyway.
- Otherwise, if $S_1 > 21.32$, he/she could win $S_1 - 21.32$. The higher S_1 is, the more he/she wins.
- It is then clear that the buyer of the option intended to bet on the "bullish" side of the stock.
- What attracts the investor to favour a call option more than the stock itself is the "leverage effect."
- The rate of return for buying the stock is $\frac{S_1 - 20}{20}$, whereas that for buying an option is $\frac{S_1 - 21.32}{4.116}$. Approximately, the rate of return for the latter is 5 times larger than that of the former.

European Call Option with strike price $K > 0$ and Expiration at time 2

- This option confers the right to buy the stock at time 2 for K dollars.
- At expiration $t = 2$, the payoff of this option is $V_2(\omega) = (S_2(\omega) - K)^+$, which is a random variable on $\omega = \omega_1\omega_2\omega_3 \in \Omega_3$ although the payoff value V_2 depends only on the first two tosses ω_1 and ω_2.
- The agent wants to determine the arbitrage price V_0 (risk premium) for this option at time zero and the delta-hedging Δ_0 to hedge the short position in stock.
- In doing so, the agent buys Δ_0 shares of stock, investing $V_0 - \Delta_0 S_0$ dollars in the money market to finance this.
- At time 1, the value of the portfolio (excluding the short position in the option since it is not mature yet and you have no obligation to pay anything) is a random variable

$$\Delta_0 S_1(\omega) + (1 + r)(V_0 - \Delta_0 S_0).$$

European Call Option with strike price $K > 0$ and Expiration at time 2

- Define the value of the portfolio (excluding the short position in the option) as a random variable X_1. That is,

$$X_1 = \Delta_0 S_1(\omega) + (1+r)(V_0 - \Delta_0 S_0)$$

- Since X_1 depends on the outcome of the first coin toss, there are two possible values for the portfolio at time 1.

$$\begin{aligned} X_1(H) &= \Delta_0 S_1(H) + (1+r)(V_0 - \Delta_0 S_0) \qquad (1) \\ X_1(T) &= \Delta_0 S_1(T) + (1+r)(V_0 - \Delta_0 S_0) \qquad (2) \end{aligned}$$

- With X_1 dollars (or equivalent value) on hand at time 1, the agent can readjust his/her portfolio by deciding to hedge, either $\Delta_1(H)$ or $\Delta_1(T)$ stocks, depending on the outcome ω_1. (with necessary finance $X_1 - \Delta_1 S_1$ from the money market).

The Value of the Portfolio at time 2

- The value of the portfolio at time 2

$$\underbrace{-(S_2(\omega) - K)^+}_{= -V_2(\omega)} + \Delta_1(\omega) S_2(\omega) + (1+r)(X_1(\omega) - \Delta_1(\omega) S_1(\omega)).$$

- There are four possible values of the Portfolio at time 2 and we want them to be all non-negative:

$$\begin{aligned}
-V_2(HH) + \Delta_1(H) S_2(HH) + (1+r)(X_1(H) - \Delta_1(H) S_1(H)) &\geq 0 \\
-V_2(HT) + \Delta_1(H) S_2(HT) + (1+r)(X_1(H) - \Delta_1(H) S_1(H)) &\geq 0 \\
-V_2(TH) + \Delta_1(T) S_2(TH) + (1+r)(X_1(T) - \Delta_1(T) S_1(T)) &\geq 0 \\
-V_2(TT) + \Delta_1(T) S_2(TT) + (1+r)(X_1(T) - \Delta_1(T) S_1(T)) &\geq 0
\end{aligned}$$

- There are 4 unknowns: the delta-hedging $\Delta_1(H)$ and $\Delta_1(T)$ at time 1; the delta-hedging Δ_0 at time 0 and the risk premium V_0, where

$$\begin{aligned}
X_1(H) &= \Delta_0 S_1(H) + (1+r)(V_0 - \Delta_0 S_0) \\
X_1(T) &= \Delta_0 S_1(T) + (1+r)(V_0 - \Delta_0 S_0)
\end{aligned}$$

Solving $\Delta_1(H)$ and $\Delta_1(T)$ at time 1; the delta-hedging Δ_0 at time 0 and the risk premium V_0

- Recall that the option expired at time 1, which leads to the two linear inequalities in two unknowns V_0, Δ_0:

$$
\begin{aligned}
\Delta_0 S_1(H) + (1+r)(V_0 - \Delta_0 S_0) &\geq V_1(H) = (S_1(H) - K)^+ \\
\Delta_0 S_1(T) + (1+r)(V_0 - \Delta_0 S_0) &\geq V_1(T) = (S_1(T) - K)^+
\end{aligned}
$$

- For the option expired at time 2, there are 4 equations and 4 unknowns:

$$
\begin{array}{ccc}
\Delta_1(H)S_2(HH) + (1+r)(X_1(H) - \Delta_1(H)S_1(H)) & \geq & V_2(HH) \\
\Delta_1(H)S_2(HT) + (1+r)(X_1(H) - \Delta_1(H)S_1(H)) & \geq & V_2(HT) \\
-------------------- & -- & ---- \\
\Delta_1(T)S_2(TH) + (1+r)(X_1(T) - \Delta_1(T)S_1(T)) & \geq & V_2(TH) \\
\Delta_1(T)S_2(TT) + (1+r)(X_1(T) - \Delta_1(T)S_1(T)) & \geq & V_2(TT)
\end{array}
$$

- Then, from $\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}$, we have solved

$$
\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)}; \ \Delta_1(T) = \frac{V_2(TH) - V_2(TT)}{S_2(TH) - S_2(TT)}
$$

Before solving the delta-hedging Δ_0 at time 0 and the risk premium V_0, solve $X_1(H)$ and $X_1(H)$ first

- Recall that the option expired at time 1, which leads to the two linear inequalities in two unknowns V_0, Δ_0:

$$
\begin{aligned}
\Delta_0 S_1(H) + (1+r)(V_0 - \Delta_0 S_0) &\geq V_1(H) = (S_1(H) - K)^+ \\
\Delta_0 S_1(T) + (1+r)(V_0 - \Delta_0 S_0) &\geq V_1(T) = (S_1(T) - K)^+
\end{aligned}
$$

- For the option expired at time 2, there are 4 equations and 4 unknowns:

$$
\begin{aligned}
\Delta_1(H) S_2(HH) + (1+r)(X_1(H) - \Delta_1(H) S_1(H)) &\geq V_2(HH) \\
\Delta_1(H) S_2(HT) + (1+r)(X_1(H) - \Delta_1(H) S_1(H)) &\geq V_2(HT) \\
----------------------- & -- \quad ---- \\
\Delta_1(T) S_2(TH) + (1+r)(X_1(T) - \Delta_1(T) S_1(T)) &\geq V_2(TH) \\
\Delta_1(T) S_2(TT) + (1+r)(X_1(T) - \Delta_1(T) S_1(T)) &\geq V_2(TT)
\end{aligned}
$$

- In addition, from $V_0 = \frac{1}{1+r}\left[\tilde{p}V_1(H) + \tilde{q}V_1(T)\right]$, $\tilde{p} = \frac{1+r-d}{u-d}$; $\tilde{q} = \frac{u-(1+r)}{u-d}$,

$$
X_1(H) = \frac{1}{1+r}\left[\tilde{p}V_2(HH) + \tilde{q}V_2(HT)\right]; \; X_1(T) = \frac{1}{1+r}\left[\tilde{p}V_2(TH) + \tilde{q}V_2(TT)\right]
$$

Having obtained $X_1(H)$ and $X_1(H)$, we now solve the delta-hedging Δ_0 at time 0 and the risk premium V_0.

- Recall, from (1) and (2), that the value of the portfolio at time 1 is a random variable taking the following two values:

$$
\begin{aligned}
X_1(H) &= \Delta_0 S_1(H) + (1+r)(V_0 - \Delta_0 S_0) \\
X_1(T) &= \Delta_0 S_1(T) + (1+r)(V_0 - \Delta_0 S_0).
\end{aligned}
$$

- Also, recall that, $\Delta_0 = \frac{V_1(H)-V_1(T)}{S_1(H)-S_1(T)}$, and $V_0 = \frac{1}{1+r}\left[\tilde{p}V_1(H) + \tilde{q}V_1(T)\right]$, $\tilde{p} = \frac{1+r-d}{u-d}$; $\tilde{q} = \frac{u-(1+r)}{u-d}$, are solution formula for

$$
\begin{aligned}
\Delta_0 S_1(H) + (1+r)(V_0 - \Delta_0 S_0) &\geq V_1(H) = (S_1(H) - K)^+ \\
\Delta_0 S_1(T) + (1+r)(V_0 - \Delta_0 S_0) &\geq V_1(T) = (S_1(T) - K)^+
\end{aligned}
$$

- With the formula, we can now solve

$$
\Delta_0 = \frac{X_1(H) - X_1(T)}{S_1(H) - S_1(T)}; \;\; V_0 = \frac{1}{1+r}\left[\tilde{p}X_1(H) + \tilde{q}X_1(T)\right]
$$

$X_1(H)$, $X_1(T)$ can be viewed as the value of the option at time 1.

- For an option with maturity time 2, although the agent does not have to pay anything at time 1, the option still holds certain value for a possible trading at time 1.
- From the formula of arbitrage pricing,

$$\Delta_0 = \frac{X_1(H) - X_1(T)}{S_1(H) - S_1(T)}; \;\; V_0 = \frac{1}{1+r}\left[\tilde{p}X_1(H) + \tilde{q}X_1(T)\right],$$

it makes sense that we can think $X_1(H)$, $X_1(T)$ just as the value of the option at time 1 with which there is a chance to arbitrage at time 2.

Homework Exercise #2: Arbitrage pricing

The pattern of the binomial asset pricing model for an option which expires at time k with the strike price $K > 0$ and $0 < d < 1 + r < u$ emerging above persists, regardless of k. Let S_k be the stock price at time t, $t = 1, 2, \ldots, k$ and let V_t, $t = 0, 1, 2, \ldots, k$ denote the value of the option at time t, which is a random variable whose possible values depend on the first t coin tosses $\omega_1, \omega_2, \ldots, \omega_t$ of the sample point $\omega \in \Omega_k$.

- What is the value of the option, V_k, at the expiration time k?
- If the agent wants to hedge the short position of the option, how many shares of the stock does he/she has to purchase? Notice that the hedging is also a random variable depending on the outcome of the first $k - 1$ coin tosses.
- Write down the value of the option at $t = k - 1$.
- Explain how to iterate the above formula to price the risk premium at time 0.

Thank You!