

NCKU 112.1
Discrete Math

Eric Liu

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Chapter 0

General Counting Methods for Arrangements and Selection

0.1 Practical Identity

Theorem 0.1.1. (Fundamental Identity) We have

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad (1)$$

Proof. Notice that $\binom{n}{k}$ represent amount of ways to pick k numbers from the set $\{1, \dots, n\}$. There are two possibilities:

$$\begin{cases} 1 \text{ is picked} \\ 1 \text{ is not picked} \end{cases} \quad (2)$$

The amount of ways to pick k numbers from the set $\{1, \dots, n\}$ when 1 is mandatory to pick is $\binom{n-1}{k-1}$, and the amount of ways when 1 is mandatory not to pick is $\binom{n-1}{k}$ ■

The above identity is the most important in the sense that it allow us to deduce other identities with induction.

Theorem 0.1.2. (First Identity) We have

$$\binom{n}{k} \binom{k}{m} = \binom{n}{m} \binom{n-m}{k-m}$$

Proof. Consider possibility of picking subsets $I \subseteq \{1, \dots, n\}$ of cardinality k and subsets $I_1 \subseteq I$ of cardinality m . The amount of possibilities equals to $\binom{n}{k} \binom{k}{m}$.

We can first pick the subset I_1 which has $\binom{n}{m}$ possibilities. We can then pick the subset I by picking $k - m$ amount of numbers in $\{1, \dots, n\} \setminus I_1$ and add I_1 to have I . There are $\binom{n-m}{k-m}$ ways to do such, as $|\{1, \dots, n\} \setminus I_1| = n - m$. ■

Theorem 0.1.3. (Second Identity) We have

$$\sum_{k=0}^n \binom{n}{k} = 2^n$$

Proof. We have

$$\begin{aligned} 2^n &= (1 + 1)^n \\ &= \sum_{k=0}^n \binom{n}{k} 1^k 1^n \\ &= \sum_{k=0}^n \binom{n}{k} \end{aligned}$$

■

The first identity rely on the usage of intuition of picking r objects from n distinct objects.

The second identity use binomial theorem.

Theorem 0.1.4. (Identity When Both Arguments are Growing) We have

$$\sum_{k=0}^r \binom{n+k}{k} = \binom{n+r+1}{r}$$

Proof. Base case: $r = 0$

$$\begin{aligned} \sum_{k=0}^r \binom{n+k}{k} &= \sum_{k=0}^0 \binom{n}{0} \\ &= 1 \\ &= \binom{n+1}{0} \\ &= \binom{n+r+1}{r} \end{aligned}$$

Induction case: suppose

$$\sum_{k=0}^s \binom{n+k}{k} = \binom{n+s+1}{s}$$

Observe

$$\begin{aligned} \binom{n+s+2}{s+1} &= \binom{n+s+1}{s+1} + \binom{n+s+1}{s} \\ &= \sum_{k=0}^s \binom{n+k}{k} + \binom{n+s+1}{s+1} \\ &= \sum_{k=0}^{s+1} \binom{n+k}{k} \end{aligned}$$

■

Corollary 0.1.5. (Putting at most r different things in n barrels) We have

$$\sum_{k=0}^r H_k^n = \sum_{k=0}^r \binom{(n-1)+k}{k} = \binom{n+r}{r} = H_r^{n+1}$$

Theorem 0.1.6. (Identity When Only The Larger Argument is Growing) We have

$$\sum_{k=0}^{n-r} \binom{r+k}{r} = \binom{n+1}{r+1}$$

Proof. Base case: $n = r$

$$\sum_{k=0}^{n-r} \binom{r+k}{r} = \binom{r}{r} = 1 = \binom{r+1}{r+1} = \binom{n+1}{r+1}$$

Induction case: Suppose

$$\sum_{k=0}^{n-s} \binom{s+k}{s} = \binom{n+1}{s+1}$$

Observe

$$\begin{aligned}
\binom{n+2}{s+1} &= \binom{n+1}{s+1} + \binom{n+1}{s} \\
&= \sum_{k=0}^{n-s} \binom{s+k}{s} + \binom{n+1}{s} \\
&= \sum_{k=0}^{n+1-s} \binom{s+k}{s}
\end{aligned}$$

■

The above two identities can be applied when the arguments are growing, notice that the second identity is an identity when the smaller argument is growing.

Theorem 0.1.7. (Fifth Identity) We have

$$\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$$

Proof. Imagine an $n \times n$ grid. Imagine we wish to go from the lower left corner to top right corner without going left or low.

The amount of path that cross $(0, n)$ is $\binom{n}{0}^2$ and the amount of path that cross $(1, n-1)$ is $\binom{n}{1}^2, \dots$

■

Theorem 0.1.8. (Identity When Multiplying Two Binomial where the Sum of the Smaller Arguments is Fixed)

$$\sum_{k=0}^r \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}$$

Proof. The right hand side is to pick r objects from $m+n$ distinct object. The left hand side is also the same, but done so by first picking from m then from n

■

Corollary 0.1.9. (identity When Multiplying Two Binomials where the Gap of the Smaller Arguments is Fixed)

$$\sum_{k=0}^m \binom{m}{k} \binom{n}{r+k} = \binom{m+n}{m+r}$$

Proof.

$$\begin{aligned}
 \sum_{k=0}^m \binom{m}{k} \binom{n}{r+k} &= \sum_{k=0}^m \binom{m}{m-k} \binom{n}{r+k} \\
 &= \sum_{u=0}^m \binom{m}{u} \binom{n}{m+r-u} \text{ where } u = m - k \\
 &= \binom{m+n}{m+r}
 \end{aligned}$$

■

Chapter 1

Generating Function

1.1 Modeling of Generating Function

Chapter 6 has 3 question, 6.4 has 1 question.

Theorem 1.1.1. (Putting Same Object into Distinct Barrels) Given

$$\sum_{i=1}^n e_i = r$$

There are

$$H_r^n := \binom{n+r-1}{n-1} = \binom{n+r-1}{r} \text{ amount of solutions}$$

1.2 Calculation of Generating Function

Theorem 1.2.1. (Important Identity) We have

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$$\frac{1}{(1-x)^n} = \left(\sum_{k=0}^{\infty} x^k\right)^n = \sum_{k=0}^{\infty} H_k^n x^k = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

$$\sum_{k=0}^n x^k = \frac{1-x^{n+1}}{1-x}$$

1.3 Exponential Generating Function

Theorem 1.3.1. (r Arrangement with and without Repetition of n objects)

This means the numbers of ways of selecting r object from distinct n object and rearrange them

Theorem 1.3.2. (Taylor Expansion)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Theorem 1.3.3. (The form of r -Term of Generating Function)

$$\left(\sum_{e_1+e_2+e_3=r} \frac{r!}{e_1!e_2!e_3!} \right) \frac{x^r}{r!}$$

Chapter 2

Recurrence Relation

2.1 Solution of Homogeneous Relation

Theorem 2.1.1. (Solution of Second order Homogeneous Recurrence Relation)

Let α, β be two constant. Suppose

$$a_n = \alpha a_{n-1} + \beta a_{n-2}$$

We have

$$\exists A, B, a_n = A\left(\frac{\alpha + \sqrt{\alpha^2 + 4\beta}}{2}\right)^n + B\left(\frac{\alpha - \sqrt{\alpha^2 + 4\beta}}{2}\right)^n$$

Proof. Let b_n and c_n be sequence that satisfy the recurrence relation. We see

$$(b + c)_n = b_n + c_n \tag{2.1}$$

$$= \alpha b_{n-1} + \beta b_{n-2} + \alpha c_{n-1} + \beta c_{n-2} \tag{2.2}$$

$$= \alpha(b + c)_{n-1} + \beta(b + c)_{n-2} \tag{2.3}$$

and

$$\gamma b_n = \gamma(b_{n-1} + b_{n-2}) \tag{2.4}$$

$$= \gamma b_{n-1} + \gamma b_{n-2} \tag{2.5}$$

We see that the set of sequences satisfying the recurrence relation is a vector space.

This vector space is of dimension 2, as the sequence is determined by the initial condition (a_1, a_2) .

We wish to find a basis, in which the elements are geometric series.

Suppose a geometric series $a_n = x^n$ satisfy the recurrence relation. Then we have

$$x^n = \alpha x^{n-1} + \beta x^{n-2}$$

This give us

$$x^2 - \alpha x - \beta = 0$$

Solving the equation, we have

$$x = \frac{\alpha \pm \sqrt{\alpha^2 + 4\beta}}{2}$$

Then each sequence satisfying the recurrence relation can be written in the form

$$a_n = A\left(\frac{\alpha + \sqrt{\alpha^2 + 4\beta}}{2}\right)^n + B\left(\frac{\alpha - \sqrt{\alpha^2 + 4\beta}}{2}\right)^n$$

■

Corollary 2.1.2. (Solution of Homogeneous Recurrence Relation) Suppose

$$a_n = \sum_{k=1}^m c_k a_{n-k}$$

The exact solution is clearly determined by the first a_1, \dots, a_m . Then we know

Set of solutions is a m -dimensional vector space

If lucky, the basis can be gained by solving the equation

$$\alpha^m = \sum_{k=1}^m c_k \alpha^{k-1}$$

to have basis

$$\{\{\alpha_1^n\}_{n \in \mathbb{N}}, \{\alpha_2^n\}_{n \in \mathbb{N}}, \dots, \{\alpha_m^n\}_{n \in \mathbb{N}}\}$$

Then we can express the general solution by

$$a_n = \sum_{k=1}^m d_k \alpha_k^n$$

2.2 Solution of Inhomogeneous Recurrence Relation

Theorem 2.2.1. (General Solution of Inhomogeneous Recurrence Relation) Consider the recurrence relation a_n

$$a_n = \sum_{k=1}^m c_k a_{n-k} + f(n)$$

If we have a particular solution a_n^* , and a solution to b_n to the homogeneous recurrence relation

$$b_n = \sum_{k=1}^m c_k b_{n-k}$$

Then $(a + b)_n$ is a solution. To guess a particular solution a_n^* , we first have to observe the form of $f(n)$. If $f(n)$ is a polynomial of degree p , then we can guess

$$a_n^* = A_p n^p + A_{p-1} n^{p-1} + \cdots + A_1 n + A_0$$

2.3 Solution with Generating Function

Question 1

Let $a_{-1} = 0$. Find a generating function

$$g(x) = \sum_{k=0} a_k x^k$$

that satisfy the recurrence relation

$$\forall n \geq 1, a_n = a_{n-1} + n \quad (2.6)$$

Proof. Multiplying Equation 2.6 on both side with x^n , we have

$$a_n x^n = a_{n-1} x^n + n x^n$$

Then we can rewrite $g(x)$ into

$$\begin{aligned} g(x) &= \sum_{k=0} a_k x^k \\ &= a_0 + \sum_{k=1} a_k x^k \\ &= a_0 + \sum_{k=1} (a_{k-1} x^k + k x^k) \\ &= a_0 + x \sum_{k=1} a_{k-1} x^{k-1} + \sum_{k=1} k x^k \\ &= a_0 + x \sum_{k=1} a_{k-1} x^{k-1} + \sum_{k=1} k x^k \\ &= a_0 + x \sum_{k=0} a_k x^k + \sum_{k=1} \binom{k}{1} x^k \\ &= a_0 + x g(x) + \sum_{k=0} \binom{k+1}{1} x^{k+1} \\ &= a_0 + x g(x) + x \sum_{k=0} \binom{k+1}{k} x^k \\ &= a_0 + x g(x) + x \sum_{k=0} H_k^2 x^k \\ &= a_0 + x g(x) + \frac{x}{(1-x)^2} \end{aligned}$$

In summary

$$g(x) = a_0 + xg(x) + \frac{x}{(1-x)^2}$$

This give us

$$g(x) = \frac{a_0}{1-x} + \frac{x}{(1-x)^3}$$

Then we see

$$\begin{aligned} g(x) &= a_0 \sum_{k=0}^{\infty} x^k + x \sum_{k=0}^{\infty} H_k^3 x^k \\ &= a_0 \sum_{k=0}^{\infty} x^k + \sum_{k=0}^{\infty} \binom{k+2}{k} x^{k+1} \\ &= a_0 x_0 + \left(\sum_{k=1}^{\infty} a_0 x^k + \binom{k+1}{k-1} x^k \right) \\ &= a_0 x_0 + \sum_{k=1}^{\infty} \left(a_0 + \binom{k+1}{k-1} \right) x^k \\ &= a_0 x_0 + \sum_{k=1}^{\infty} \left(a_0 + \binom{k+1}{2} \right) x^k \\ &= \sum_{k=0}^{\infty} \left(a_0 + \binom{k+1}{2} \right) x^k \end{aligned}$$

■

Question 2

Use generating function to solve the recurrence relation

$$\forall n \geq 2, a_n = a_{n-1} + a_{n-2}$$

where the initial condition is

$$a_0 = a_1 = 1$$

Suppose

$$g(x) = \sum_{k=0}^{\infty} a_k x^k$$

satisfy the recurrence relation. Then we have

$$\begin{aligned}
g(x) - a_0 - a_1x &= \sum_{k=2}^{\infty} a_k x^k \\
&= \sum_{k=2}^{\infty} (a_{k-1} + a_{k-2}) x^k \\
&= x \sum_{k=2}^{\infty} a_{k-1} x^{k-1} + x^2 \sum_{k=2}^{\infty} a_{k-2} x^{k-2} \\
&= x \sum_{k=1}^{\infty} a_k x^k + x^2 \sum_{k=0}^{\infty} a_k x^k \\
&= x(g(x) - 1) + x^2 g(x)
\end{aligned}$$

Then we have

$$(1 - x - x^2)g(x) = 1$$

Notice

$$1 - x - x^2 = -\left(x - \frac{-1 + \sqrt{5}}{2}\right)\left(x - \frac{-1 - \sqrt{5}}{2}\right)$$

Then

$$\begin{aligned}
g(x) &= \frac{-1}{\left(x - \frac{-1 + \sqrt{5}}{2}\right)\left(x - \frac{-1 - \sqrt{5}}{2}\right)} \\
&= \frac{\frac{-1}{\sqrt{5}}}{x - \frac{-1 + \sqrt{5}}{2}} + \frac{\frac{1}{\sqrt{5}}}{x - \frac{-1 - \sqrt{5}}{2}} \\
&= \frac{\frac{1}{\sqrt{5}}}{\frac{-1 + \sqrt{5}}{2} - x} + \frac{\frac{-1}{\sqrt{5}}}{\frac{-1 - \sqrt{5}}{2} - x} \\
&= \frac{\frac{2}{\sqrt{5}(-1 + \sqrt{5})}}{1 - \frac{2}{-1 + \sqrt{5}}x} + \frac{\frac{-2}{\sqrt{5}(-1 - \sqrt{5})}}{1 - \frac{2}{-1 - \sqrt{5}}x} \\
&= \frac{2}{5 - \sqrt{5}} \sum_{k=0}^{\infty} \left(\frac{2}{-1 + \sqrt{5}}x\right)^k + \frac{-2}{-5 - \sqrt{5}} \sum_{k=0}^{\infty} \left(\frac{2}{-1 - \sqrt{5}}x\right)^k
\end{aligned}$$

Chapter 3

HW

3.1 HW3

Question 3

How many numbers between 0 and 10000 have a sum of digit

(a) Equal to 7?

(b) Less than or equal to 7?

(c) Equal to 13

Proof. Represent the number by

$$e_0e_1e_2e_3e_4$$

where if the number is 237, we have

$$e_0 = 0 \text{ and } e_1 = 0 \text{ and } e_2 = 2 \text{ and } e_3 = 3 \text{ and } e_4 = 7$$

And of course we have the constraint

$$0 \leq e_1, e_2, e_3, e_4 \leq 9 \text{ and } e_0 \in \{0, 1\} \text{ and } e_0 = 1 \implies e_1 = e_2 = e_3 = e_4 = 0$$

Then the first question is equivalent to asking the amount of solution of

$$e_0 + e_1 + e_2 + e_3 + e_4 = 7$$

If $e_1 = 1$, clearly there is no solution. If $e_0 = 0$, there exists $H_7^4 = \binom{10}{7} = 120$ amount of solutions.

The second question is equivalent to asking the amount of solution of

$$e_0 + e_1 + e_2 + e_3 + e_4 \leq 7$$

If $e_0 = 1$, clearly there exists only one solution. If $e_0 = 0$, there exists

$$\sum_{k=0}^7 H_k^4 = \sum_{k=0}^7 \binom{k+3}{k} = \binom{11}{7} = 330$$

So

The amount of solutions are 331

The third question is equivalent to asking the amount of solution of

$$e_0 + e_1 + e_2 + e_3 + e_4 = 13$$

Clearly, we can not have $e_0 = 1$, so our question has become

$$e_1 + e_2 + e_3 + e_4 = 13$$

where the constrain is

$$0 \leq e_1, e_2, e_3, e_4 \leq 9$$

If we remove the 9 upper bound constrain, the amount of solutions is then

$$H_{13}^4 = 560$$

Adding the constrain back, we need to remove those solutions that doesn't satisfy the constrain, i.e. $e_j > 9$ for some $j \in \{1, 2, 3, 4\}$.

Clearly if $e_j > 9$, then no other digit would be greater than 9. The amount of solutions that should be removed are

$$4(H_{13-10}^3 + H_{13-11}^3 + H_{13-12}^3 + H_{13-13}^3) = 4\left(\sum_{k=0}^3 H_k^3\right) = 4\left(\sum_{k=0}^3 \binom{k+2}{k}\right) = 4\binom{6}{3} = 80$$

Then the amount of solutions are

$$560 - 80 = 480 \text{ ways}$$

■

Question 4

Evaluate

$$\sum_{k=1}^n \binom{n}{k} \binom{n}{k-1}$$

Proof.

$$\begin{aligned} \sum_{k=1}^n \binom{n}{k} \binom{n}{k-1} &= \sum_{k=1}^n \binom{n+1}{k} \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} - \binom{n+1}{0} - \binom{n+1}{n+1} \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} - 2 \\ &= 2^{n+1} - 2 \end{aligned}$$

■

Question 5

Show that the generating function for the number of integer solutions to

$$e_1 + e_2 + e_3 + e_4 = r, 0 \leq e_1 \leq e_2 \leq e_3 \leq e_4$$

is

$$(1 + x + x^2 + \cdots) (1 + x^2 + x^4 + \cdots) (1 + x^3 + x^6 + \cdots) (1 + x^4 + x^8 + \cdots)$$

Proof. Define

$$d_1 := e_2 - e_1 \text{ and } d_2 := e_3 - e_2 \text{ and } d_3 := e_4 - e_3$$

Then we have

$$e_2 = e_1 + d_1 \text{ and } e_3 = e_1 + d_1 + d_2 \text{ and } e_4 = e_1 + d_1 + d_2 + d_3$$

The question is thus reduced to finding the generating function for

$$4e_1 + 3d_1 + 2d_2 + d_3 = r, \{e_1, d_1, d_2, d_3\} \in \mathbb{N} \cup \{0\}$$

Which is

$$(1 + x^4 + x^8 + \cdots) (1 + x^3 + x^6 + \cdots) (1 + x^2 + x^4 + \cdots) (1 + x + x^2 + \cdots)$$

■

Question 6

Use the equation

$$\frac{(1-x^2)^n}{(1-x)^n} = (1+x)^n$$

to show that

$$\sum_{k=0}^{\frac{m}{2}} (-1)^k \binom{n}{k} \binom{n+m-2k-1}{n-1} = \binom{n}{m} \quad m \leq n \text{ and } m \text{ even}$$

Proof. Observe that

$$\binom{n}{m} \text{ is the coefficient of } x^m \text{ in } (1+x)^n$$

So we only have to prove that

$$\sum_{k=0}^{\frac{m}{2}} (-1)^k \binom{n}{k} \binom{n+m-2k-1}{n-1} \text{ is the coefficient of } x^m \text{ in } \frac{(1-x^2)^n}{(1-x)^n}$$

Observe that

$$(1-x^2)^n = \sum_{k=0}^n (-1)^k \binom{n}{k} x^{2k}$$

and that

$$\frac{1}{(1-x)^n} = \sum_{u=0}^{\infty} \binom{u+n-1}{u} x^u$$

Then because m is even, we can compute the coefficient by summing $k \in [0, \frac{m}{2}]$ and $u = m - 2k$ from $(2k + u = m)$, which tell us that the coefficient is

$$\sum_{k=0}^{\frac{m}{2}} (-1)^k \binom{n}{k} \binom{m-2k+n-1}{m-2k}$$

which equals to

$$\sum_{k=0}^{\frac{m}{2}} (-1)^k \binom{n}{k} \binom{n+m-2k-1}{n-1}$$

■

Question 7

Show that

$$2(1-x)^{-3} \left[(1-x)^{-3} + (1+x)^{-3} \right]$$

is the generating function for the number of ways to toss r identical dice and obtain even sum.

Proof. The generating function of ways to toss r identical dice is

$$\left(\sum_{k=1}^6 x^k \right)^r$$

which equals to

$$\left(\frac{1-x^7}{1-x} \right)^r$$

Then the generating function for the number of ways to toss r identical dice and obtain even sum is

$$\frac{1}{2} \left[\left(\frac{1-x^7}{1-x} \right)^r - \left(\frac{1+x^7}{1+x} \right)^r \right]$$

■

3.2 HW4

Question 8

How many 10-letter words are there in which each of the letters e,n,r,s occur

- (a) At most once?
- (b) At least once?

Proof. The generating function for (a) is

$$\left(\frac{x^0}{0!} + \frac{x^1}{1!}\right)^4 \times \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)^{22} = (x+1)^4 e^{22x} \quad (3.1)$$

$$= (x^4 + 4x^3 + 6x^2 + 4x + 1) \sum_{n=0}^{\infty} \frac{(22x)^n}{n!} \quad (3.2)$$

Then the 10th-coefficient is

$$x^4 \times \frac{(22x)^6}{6!} + 4x^3 \times \frac{(22x)^7}{7!} + 6x^2 \times \frac{(22x)^8}{8!} + 4x \times \frac{(22x)^9}{9!} + \frac{(22x)^{10}}{10!}$$

which is

$$(22^6 \times (7 \times 8 \times 9 \times 10) + 4 \times 22^7 \times 8 \times 9 \times 10 + 6 \times 22^8 \times 9 \times 10 + 4 \times 22^9 \times 10 + 22^{10}) \times \frac{x^{10}}{10!}$$

The generating function for (b) is

$$\begin{aligned} \left(\sum_{n=1}^{\infty} \frac{x^n}{n!}\right)^4 \times \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)^{22} &= (e^x - 1)^4 e^{22x} \\ &= (e^{4x} - 4e^{3x} + 6e^{2x} - 4e^x + 1)e^{22x} \\ &= e^{26x} - 4e^{25x} + 6e^{24x} - 4e^{23x} + e^{22x} \\ &= \sum_{n=0}^{\infty} \frac{(26x)^n}{n!} - \sum_{n=0}^{\infty} \frac{4(25x)^n}{n!} + \sum_{n=0}^{\infty} \frac{6(24x)^n}{n!} - \sum_{n=0}^{\infty} \frac{4(23x)^n}{n!} + \sum_{n=0}^{\infty} \frac{(22x)^n}{n!} \end{aligned}$$

Then the 10th-coefficient for (b) is

$$(26^{10} - 25^{10} \times 4 + 24^{10} \times 6 - 23^{10} \times 4 + 22^{10}) \times \frac{x^{10}}{10!}$$

■

Question 9

Find a recurrence relation for the number of n -digit ternary $(0, 1, 2)$ sequence in which no 1 appears anywhere to the right of any 2.

Proof. Consider an n -digit sequence $\{x_i\}_{i=1}^n$ with desired property:

$$\forall i, x_i \in \{0, 1, 2\} \text{ and } x_i = 2 \implies x_{i+1} \neq 1$$

If $x_n \in \{0, 2\}$, we see

$$\{x_i\}_{i=1}^{n-1} \text{ satisfy desired property} \implies \{x_i\}_{i=1}^n \text{ satisfy desired property}$$

Notice that the converse of the above statement is clearly true, so we have

$$\{x_i\}_{i=1}^{n-1} \text{ satisfy desired property} \iff \{x_i\}_{i=1}^n \text{ satisfy desired property}$$

If $x_n = 1$, we see

$$\{x_i\}_{i=1}^{n-1} \text{ satisfy desired property and } x_{n-1} \neq 2 \implies \{x_i\}_{i=1}^n \text{ satisfy desired property}$$

Again notice that the converse of the above statement is clearly true under the hypothesis $x_n = 1$

$$\{x_i\}_{i=1}^{n-1} \text{ satisfy desired property and } x_{n-1} \neq 2 \iff \{x_i\}_{i=1}^n \text{ satisfy desired property}$$

Also, if $x_{n-1} = 2$, we see

$$\{x_i\}_{i=1}^{n-1} \text{ satisfy desired property} \iff \{x_i\}_{i=1}^{n-2} \text{ satisfy desired property}$$

Now we reach to the recurrence relation

$$a_n = 2a_{n-1} + (a_{n-1} - a_{n-2}) = 3a_{n-1} - a_{n-2}$$

where the initial condition is clearly

$$a_1 = 3 \text{ and } a_2 = 3 + 3 + 2 = 8$$

■

Question 10

Find an solve a recurrence relation for the number of ways to make a pile of n chips using red, white, and blue chips and such that no two red chips are together.

Proof. If the last chip is of the color white or blue, it is clear that the pile is legal if and only if the rest of pile is legal. If the last chip is of color red, the pile is legal if and only if the rest of the pile is legal and the second last chip is of the color white or blue. Then we reach to the recurrence relation

$$a_n = 2a_{n-1} + 2a_{n-2}$$

The sequence that satisfy the above homogeneous is a vector space of dimension 2. Suppose a geometric sequence $a_n = \alpha^n$ is in the vector space. We have

$$\alpha^n = 2\alpha^{n-1} + 2\alpha^{n-2}$$

Solving the quadratic equation, we have

$$\alpha = 1 \pm \sqrt{3}$$

Then we see the set of sequence

$$\{\{(1 + \sqrt{3})^n\}_{n \in \mathbb{N}}, \{(1 - \sqrt{3})^n\}_{n \in \mathbb{N}}\}$$

form a basis of the vector space.

Now, let

$$a_n = A(1 + \sqrt{3})^n + B(1 - \sqrt{3})^n$$

Because $a_1 = 3, a_2 = 8$ (\because simple computation), we solve the linear equation

$$\begin{cases} A(1 + \sqrt{3}) + B(1 - \sqrt{3}) = 3 \\ A(1 + \sqrt{3})^2 + B(1 - \sqrt{3})^2 = 8 \end{cases}$$

to have

$$A = \frac{1}{2} + \frac{\sqrt{3}}{3} \text{ and } B = \frac{1}{2} - \frac{\sqrt{3}}{3}$$

We now have the solution

$$a_n = \left(\frac{1}{2} + \frac{\sqrt{3}}{3}\right)(1 + \sqrt{3})^n + \left(\frac{1}{2} - \frac{\sqrt{3}}{3}\right)(1 - \sqrt{3})^n$$

■

Question 11

Find a general solution to

$$a_n - 5a_{n-1} + 6a_{n-2} = 2 + 3n$$

Proof. From the form of right hand side, we can guess that the form of particular solution is

$$a_n = xn + y$$

Then we can solve x, y from

$$\begin{aligned} & xn + y - 5(x(n-1) + y) + 6(x(n-2) + y) = 2 + 3n \\ \implies & n(x - 5x + 6x) + (y + 5x - 5y - 12x + 6y) = 3n + 2 \\ \implies & (2x)n + (-7x + 2y) = 3n + 2 \\ \implies & \begin{cases} 2x = 3 \\ -7x + 2y = 2 \end{cases} \\ \implies & x = \frac{3}{2} \text{ and } y = \frac{25}{4} \end{aligned}$$

Clearly, if b_n, c_n satisfy

$$\begin{cases} b_n - 5b_{n-1} + 6b_{n-2} = 0 \\ c_n - 5c_{n-1} + 6c_{n-2} = 3n + 2 \end{cases}$$

we have

$$(b + c)_n - 5(b + c)_{n-1} + 6(b + c)_{n-2} = 3n + 2$$

We have already solved a particular $c_n = \frac{3}{2}n + \frac{25}{4}$. We now solve for second order homogenous recurrence relation

$$b_n = 5b_{n-1} - 6b_{n-2}$$

Consider

$$\alpha^2 = 5\alpha - 6$$

We have

$$\alpha = 2 \text{ or } 3$$

We now see

$$a_n = A2^n + B3^n + \frac{3}{2}n + \frac{25}{4}$$

■

3.3 Final

Theorem 3.3.1. (5.5.25)

$$\binom{2n}{n}^2 = \sum_{k=0}^n \frac{(2n)!}{(k!)^2(n-k)!^2}$$

Proof. Expand

$$\binom{2n}{n}^2 = \sum_{k=0}^n \binom{n}{k}^2 \binom{2n}{n}$$

■

Theorem 3.3.2. (5.5.26) Given $n \leq m$

$$\sum_{k=0}^n \binom{n}{k} \binom{m}{k} = \binom{n+m}{m}$$

Proof. Consider walking from $(0,0)$ to (m,n) , and notice that we must cross the line $L : y + x = n$ ■

Theorem 3.3.3. (5.5.28) Show that

$$\sum_{j=0}^k \binom{n+k-j-1}{k-j} \binom{m+j-1}{j} = \binom{n+m+k-2}{k}$$

Proof. Notice the textbook is wrong on this one. Consider walking from $(0,0)$ to $(k, n-1+m-1)$, and notice that we must cross the line $L : y = n-1$ ■

Theorem 3.3.4. (6.4.7)

- (a) $\frac{3^r+1}{2}, g(x) = e^{2x}(\frac{e^x+e^{-x}}{2})$
- (b) $\frac{3^r+2+(-1)^r}{4}, g(x) = e^x(\frac{e^x+e^{-x}}{2})^2$
- (c) $3^r - 2^{r+1} + 1, g(x) = (e^x - 1)^2 e^x$

Theorem 3.3.5. (6.4.10)

- (a) $3^r - 3 \cdot 2^{r-2} \cdot \binom{r}{2} + 18 \binom{r}{4}, g(x) = (e^x - \frac{x^2}{2})^3$. The coefficient I compute is only for when $r > 6$. I'm not sure if the result remain valid for $0 \leq r \leq 6$

$$(b) \frac{3^k}{4} - 2^k + \frac{3}{2} + \frac{(-1)^k}{4} + \binom{0}{k}, g(x) = \left(\frac{e^x + e^{-x}}{2} - 1\right)^2 e^x$$

Theorem 3.3.6. (6.4.11)

$$\frac{4^r + \binom{0}{r}}{2}, g(x) = \left(\frac{e^{2x} + e^{-2x}}{2}\right) e^{2x}$$

Theorem 3.3.7. (Conjecture of Question 6.4.12) If we ask for the total number of 0s, 1s and 2s are even, the generating function should be

$$g(x) = \left(\frac{e^{3x} + e^{-3x}}{2}\right) e^x?$$

Theorem 3.3.8. (7.3.3)

$$(a) a_n = 3a_{n-1} + 4a_{n-2}, a_0 = a_1 = 1 \implies a_n = \frac{1}{10}4^{n+1} + \frac{-6}{10}(-1)^{n+1}$$

$$(b) a_n = a_{n-2}, a_0 = a_1 = 1 \implies a_n = 1$$

$$(c) a_n = 2a_{n-1} - a_{n-2}, a_0 = a_1 = 2 \implies \text{multiplicity is greater than 1, won't be in exam.}$$

$$(d) a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}, a_0 = a_1 = 1, a_2 = 2 \implies \text{multiplicity is greater than 1}$$

Theorem 3.3.9. (7.4.9)

$$(a) a_n = 3a_{n-1} - 2, a_0 = 0 \implies a_n = 1 - 3^n$$

$$(b) a_n = 2a_{n-1} + (-1)^n, a_0 = 2 \implies a_n = \frac{5}{3}2^n + \frac{(-1)^n}{3}$$

$$(c) a_n = 2a_{n-1} + n, a_0 = 1 \implies a_n = (-n - 2) + 3 \cdot 2^n$$

$$(d) a_n = 2a_{n-1} + 2n^2, a_0 = 3 \implies a_n = 15 \cdot 2^n + (-2n^2 - 8n - 12)$$

Theorem 3.3.10. (7.4.19)

$$(a) a_n^2 = 2a_{n-1}^2 + 1, a_0 = 1 \implies a_n = \sqrt{2^{n+1} - 1}$$

$$(b) a_n = -na_{n-1} + n!, a_0 = 1 \implies a_n = n! \left(\frac{1}{2} \cdot (-1)^n + \frac{1}{2}\right)$$

Hint for (b): Let $b_n = \frac{a_n}{n!}$

Theorem 3.3.11. (Conjecture) If the remaining term is of the form $r^n + n^m$, we should guess particular solution by

$$a_n = c_{m+1}r^n + \sum_{k=0}^m c_k n^k$$

Theorem 3.3.12. (7.5.3)

$$\forall n \geq 1, a_n = \sum_{i=0}^{n-1} a_i a_{n-1-i}, a_0 = 1 \implies g(x) = \frac{1}{\sqrt{1-x}}$$

$$\forall n \geq 3, a_n = \sum_{i=2}^{n-2} a_i a_{n-i}, a_0 = a_1 = a_2 = 1 \implies (g(x) - 1 - x)^2 = g(x) - x^2 - x - 1$$

$$\forall n \geq 2, a_n = \sum_{i=1}^{n-1} 2^i a_{n-i}, a_0 = a_1 = 1 \implies \frac{g(x) - 1}{1 - 2x} = g(x) - x - 1$$

Theorem 3.3.13. (7.5.8)

$$\forall n \geq 3, a_n = 3a_{n-1} - a_{n-3}, \begin{cases} a_0 = -1 \\ a_1 = 0 \\ a_2 = 0 \end{cases} \implies g(x) = \frac{-3x - 1}{x^3 - 3x + 1}$$

Theorem 3.3.14. (8.1.35)

25 people bring just salad and 5 bring just sandwiches

Theorem 3.3.15. (8.1.36)

2 people take Greek and Hebrew

Theorem 3.3.16. (8.3.2)

(a) $6x^4 + 20x^3 + 21x^2 + 8x + 1$

(b) $6x^4 + 17x^3 + 17x^2 + 7x + 1$

(c) $4x^3 + 12x^2 + 7x + 1$

(d) $4x^3 + 14x^2 + 8x + 1$

$$H_9^6 - (6H_9^5 - 15H_9^4 + 20H_9^3 - 15H_9^2 + 6H_9^1)$$