HWs

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Chapter 1

General Analysis HW

1.1 Brunn-Minkowski Inequality

Abstract

This HW assignment require us to prove Brunn-Minkowski Inequality. Note that in this HW, we use bold face $\mathbf{x}, \mathbf{y}, \mathbf{z}$ to denote elements of $\mathbb{R}^d$, and we use the notation $(\mathbf{x}_1, \dots, \mathbf{x}_d) = \mathbf{x}$. Also, we shall suppose throughout this HW, WOLG, $|A| > 0$ and $|A|, |B| < \infty$, otherwise the proof is trivial.

We first introduce some notation. Given two sets $A, B \subseteq \mathbb{R}^d$ and a point $p \in \mathbb{R}^d$, we write

$$A + p \triangleq \{a + p \in \mathbb{R}^d : a \in A\}$$

and write

$$A + B \triangleq \{a + b \in \mathbb{R}^d : a \in A \text{ and } b \in B\}$$

Note that elementary set theory tell us

$$(A + p) + (B + q) = (A + B) + (p + q) \tag{1.1}$$

Theorem 1.1.1. (Brunn-Minkowski Inequality for Bricks) Suppose A, B are two **bricks**, i.e., A is of the form $\prod_{j=1}^{d} [x_j, y_j]$, and so is B. We have

$$|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}} \leq |A + B|^{\frac{1}{d}}$$

Proof. Because Lebesgue measure is translation invariant, by Equation 1.1, we can WOLG suppose

$$A = \prod_{j=1}^{d}[0, a_j] \text{ and } B = \prod_{j=1}^{d}[0, b_j]$$

It is clear that

$$A + B = \prod_{j=1}^{d}[0, a_j + b_j]$$

Now, by direct computation, we know that

$$|A + B| = \prod_{j=1}^{d}(a_j + b_j) \text{ and } |A| = \prod_{j=1}^{d} a_j \text{ and } |B| = \prod_{j=1}^{d} b_j$$

Note that by AM-GM inequality, we have

$$\frac{|A|^{\frac{1}{d}}}{|A + B|^{\frac{1}{d}}} = \Big(\prod_{j=1}^{n}\frac{a_j}{a_j + b_j}\Big)^{\frac{1}{n}} \leq \frac{1}{d}\sum_{j=1}^{d}\frac{a_j}{a_j + b_j}$$

Similarly by AM-GM inequality, we have

$$\frac{|B|^{\frac{1}{d}}}{|A + B|^{\frac{1}{d}}} = \Big(\prod_{j=1}^{n}\frac{b_j}{a_j + b_j}\Big)^{\frac{1}{n}} \leq \frac{1}{d}\sum_{j=1}^{d}\frac{b_j}{a_j + b_j}$$

Adding two inequalities together, now have

$$\frac{|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}}{|A + B|^{\frac{1}{d}}} \leq \frac{1}{d}\sum_{j=1}^{d}\frac{a_j + b_j}{a_j + b_j} = 1$$

The result then follows from multiplying both side with $|A + B|^{\frac{1}{d}}$. ■

Theorem 1.1.2. (Brunn-Minkowski Inequality for finite union of non-overlapping of bricks) Suppose A is a union of a finite collection of non-overlapping brick and so is B. We have

$$|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}} \leq |A + B|^{\frac{1}{d}}$$

Proof. We prove by induction on the sum k of the amount of bricks consisting A and the amount of bricks consisting B. The base case $k = 2$ have been proved by Theorem 1.1.1. Suppose the proposition hold true when $k \leq r$. Let $k = r + 1$. Because the bricks consisting of A are non-overlapping, by a translation (and renaming axis if necessary), we can suppose the following proposition.

Proposition 1: Both $A^+ \triangleq A \cap \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x}_1 \geq 0\}$ and $A^- \triangleq A \cap \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x}_1 \leq 0\}$ are union of collection of non-overlapping bricks, with each collection containing at least one fewer brick than A.

Proposition 1 hold because if we write $A = A_1 \cup \cdots \cup A_m$ where $A_1, \ldots, A_m$ are non-overlapping bricks, then by translation and remaining axis, we can suppose A_1, A_2 lie in distinct closed subspace, either $\{\mathbf{x} \in \mathbb{R}^d : \mathbf{x}_1 \geq 0\}$ or $\{\mathbf{x} \in \mathbb{R}^d : \mathbf{x}_1 \leq 0\}$, while for all $n \geq 3$, $A_n \cap \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x}_1 \geq 0\}$ is either empty or also a brick.

Now, note that $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(t) \triangleq \left| \Big(B + (t, 0, \ldots, 0) \Big) \cap \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x}_1 \geq 0\} \right|$$

is clearly a continuous function. (If B consists of p bricks, then h can be written as a finite sum of continuous function with compact support, $\sum_{k=1}^{p} h_k$) Then by IVT, we can translate B to let B satisfy

$$\frac{|A^+|}{|A|} = \frac{|B^+|}{|B|} \text{ where } B^+ \triangleq B \cap \{\mathbf{x} \in \mathbb{R}^d : x_1 \geq 0\} \tag{1.2}$$

Define $B^- \triangleq B \cap \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x}_1 \leq 0\}$. With reason similar to that of Proposition 1, we know B^+ and B^- are both union of collection of non-overlapping bricks, with each collection containing bricks no more than B. Therefore, with Proposition 1, we can deduce that the sum of the amount of bricks consisting A^+ (resp. A^-) and the amount bricks consisting B^+ (resp. B^-) is at least one fewer than $r+1$. Then because the proposition hold true for $k \leq r$, we now have

$$\left|A^+\right|^{\frac{1}{d}} + \left|B^+\right|^{\frac{1}{d}} \leq \left|A^+ + B^+\right|^{\frac{1}{d}} \text{ and } \left|A^-\right|^{\frac{1}{d}} + \left|B^-\right|^{\frac{1}{d}} \leq \left|A^- + B^-\right|^{\frac{1}{d}}$$

Note that for each $\mathbf{x}$ in the interior of $A^+ + B^+$, we must have $\mathbf{x}_1 > 0$, and for each $\mathbf{y}$ in the interior of $A^- + B^-$, we must have $\mathbf{y}_1 < 0$. This implies that $(A^+ + B^+)$ and $(A^- + B^-)$ are non-overlapping. Now, because

$$A + B = (A^+ + B^+) \cup (A^- + B^-)$$

if we define $\rho \triangleq \frac{|A^+|}{|A|}$, from Equation 1.2 we can finally deduce

$$\begin{aligned}
|A + B| &= \left|A^+ + B^+\right| + \left|A^- + B^-\right| \\
&\geq \Big(\left|A^+\right|^{\frac{1}{d}} + \left|B^+\right|^{\frac{1}{d}} \Big)^d + \Big(\left|A^-\right|^{\frac{1}{d}} + \left|B^-\right|^{\frac{1}{d}} \Big)^d \\
\left(\because \frac{|A^-|}{|A|} = \frac{|B^-|}{|B|} = 1 - \rho\right) \quad &= \Big((\rho\,|A|)^{\frac{1}{d}} + (\rho\,|B|)^{\frac{1}{d}} \Big)^d + \Big(((1-\rho)\,|A|)^{\frac{1}{d}} + ((1-\rho)\,|B|)^{\frac{1}{d}} \Big)^d \\
&= (|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}})^d
\end{aligned}$$

This then give us the desired inequality

$$|A+B|^{\frac{1}{d}} \geq |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}$$

■

Theorem 1.1.3. (Brunn-Minkowski Inequality for bounded open set) Suppose A, B are both bounded open subset of $\mathbb{R}^d$. We have

$$|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}} \leq |A+B|^{\frac{1}{d}}$$

Proof. Note that $A+B$ is also open since it is a union of the open sets:

$$A+B = \bigcup_{\mathbf{b} \in B} A + \mathbf{b}$$

It then follows that $A+B$ is Lebesgue measurable, so it makes sense for us to write $|A+B|$. By a proposition taught in class, we can let

$$A = \bigcup_{n \in \mathbb{N}} K_{n,a} \text{ and } B = \bigcup_{n \in \mathbb{N}} K_{n,b}$$

Fix arbitrary $\mathbf{x} \in A+B$. Let $\mathbf{a} \in A, \mathbf{b} \in B$ satisfy $\mathbf{x} = \mathbf{a} + \mathbf{b}$. Because $A = \bigcup K_{n,a}$ and $B = \bigcup K_{n,b}$, we know there exists $j_a, j_b \in \mathbb{N}$ such that $\mathbf{a} \in K_{j_a,a}$ and $\mathbf{b} \in K_{j_b,b}$. WOLG, suppose $j_a \geq j_b$. Now, because

$$\mathbf{x} = \mathbf{a} + \mathbf{b} \in \Big(\bigcup_{n=1}^{j_a} K_{n,a} \Big) + \Big(\bigcup_{n=1}^{j_a} K_{n,b} \Big)$$

and $\mathbf{x}$ is arbitrary selected from $A+B$, we have proved

$$\Big(\bigcup_{n=1}^{N} K_{n,a} \Big) + \Big(\bigcup_{n=1}^{N} K_{n,b} \Big) \nearrow A+B \text{ as } N \to \infty$$

This together with Theorem 1.1.2 then give us the desired inequality

$$\begin{aligned} |A+B|^{\frac{1}{d}} &= \lim_{N \to \infty} \left| \Big(\bigcup_{n=1}^{N} K_{n,a} \Big) + \Big(\bigcup_{n=1}^{N} K_{n,b} \Big) \right|^{\frac{1}{d}} \\ &\geq \lim_{N \to \infty} \left| \bigcup_{n=1}^{N} K_{n,a} \right|^{\frac{1}{d}} + \left| \bigcup_{n=1}^{N} K_{n,b} \right|^{\frac{1}{d}} \\ &= |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}} \end{aligned}$$

■

Theorem 1.1.4. (Brunn-Minkowski Inequality for compact set) Suppose A, B are both compact subset of $\mathbb{R}^d$. We have

$$|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}} \leq |A+B|^{\frac{1}{d}}$$

Proof. For each $\epsilon > 0$, define

$$A_\epsilon \triangleq \{\mathbf{x} \in \mathbb{R}^d : d(\mathbf{x}, A) < \epsilon\} \text{ and } B_\epsilon \triangleq \{\mathbf{x} \in \mathbb{R}^d : d(\mathbf{x}, B) < \epsilon\}$$

To see A_ϵ is open, observe that if $\mathbf{x} \in A_\epsilon$, then for all $\mathbf{y}$ in the open ball $d(\mathbf{x}, \mathbf{y}) < \frac{\epsilon - d(\mathbf{x}, A)}{2}$, we can pick some $\mathbf{z} \in A$ satisfying $d(\mathbf{x}, \mathbf{z}) < d(\mathbf{x}, A) + \frac{\epsilon}{2}$ to have

$$\begin{aligned}
d(\mathbf{y}, A) &\leq d(\mathbf{y}, \mathbf{z}) \\
&\leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{x}, \mathbf{y}) \\
&\leq d(\mathbf{x}, \mathbf{z}) + \frac{\epsilon - d(\mathbf{x}, A)}{2} \\
&\leq d(\mathbf{x}, \mathbf{z}) - d(\mathbf{x}, A) + \frac{\epsilon}{2} < \epsilon \text{ implying } \mathbf{y} \in A_\epsilon
\end{aligned}$$

Similar argument shows that B_ϵ are open. To see $A_\epsilon \searrow A$, note that for all $\mathbf{x} \notin A$, because $d(\mathbf{z}, \mathbf{x})$ is a function continuous in the variable $\mathbf{z}$ and A is compact, by EVT we have

$$d(\mathbf{x}, A) = d(\mathbf{x}, \mathbf{z}) > 0 \text{ for some } \mathbf{z} \in A$$

Note that the inequality hold because $\mathbf{z} \in A \implies \mathbf{x} \neq \mathbf{z}$. Similar argument shows that $B_\epsilon \searrow B$. We now prove

$$A + B = \bigcap_{\epsilon > 0} A_\epsilon + B_\epsilon \tag{1.3}$$

It is clear that

$$A + B \subseteq \bigcap_{\epsilon > 0} A_\epsilon + B_\epsilon \tag{1.4}$$

Fix arbitrary $\mathbf{z} \in \bigcap_{\epsilon > 0} A_\epsilon + B_\epsilon$. For all $n \in \mathbb{N}$, by definition there exists $\mathbf{a}_n \in A_{\frac{1}{n}}$ and $\mathbf{b}_n \in B_{\frac{1}{n}}$ such that $\mathbf{z} = \mathbf{a}_n + \mathbf{b}_n$. Bolzano-Weierstrass Theorem tell us that there exists convergent subsequence $\mathbf{a}_{n_k}$. Applying Bolzano-Weierstrass Theorem again, we see that there exists convergent subsequence $\mathbf{b}_{n_{k_j}}$. It is clear that $\mathbf{a}_{n_{k_j}}$ also converge. For brevity, we denote them simply by $\mathbf{a}_{n_k}$ and $\mathbf{b}_{n_k}$, and we denote their limit by

$$\mathbf{a} = \lim_{k \to \infty} \mathbf{a}_{n_k} \text{ and } \mathbf{b} = \lim_{k \to \infty} \mathbf{b}_{n_k}$$

We now shows that

$$\mathbf{a} \in A \tag{1.5}$$

Assume $a \notin A$ for a contradiction. By EVT, $d(\mathbf{a}, A) = d(\mathbf{a}, \mathbf{a}')$ for some $\mathbf{a}' \in A$. Note that $d(\mathbf{a}, \mathbf{a}') > 0$ because $\mathbf{a} \notin A \implies \mathbf{a} \neq \mathbf{a}'$. We have shown $d(\mathbf{a}, A) > 0$. Let K satisfy $d(\mathbf{a}, \mathbf{a}_{n_k}) < \frac{d(\mathbf{a},A)}{2}$ for all $k > K$. Select $m > K$ so that $\frac{1}{n_m} < \frac{d(\mathbf{a},A)}{2}$. Then because $d(\mathbf{a}_{n_m}, A) < \frac{1}{n_m} < \frac{d(\mathbf{a},A)}{2}$, we can select $\mathbf{a}'' \in A$ such that $d(\mathbf{a}_{n_m}, \mathbf{a}'') < \frac{d(\mathbf{a},A)}{2}$. This then give us

$$d(\mathbf{a}, A) \leq d(\mathbf{a}, \mathbf{a}'') \leq d(\mathbf{a}, \mathbf{a}_{n_m}) + d(\mathbf{a}_{n_m}, \mathbf{a}'') < \frac{d(\mathbf{a}, A)}{2} + \frac{d(\mathbf{a}, A)}{2}$$

which is clearly impossible. We have proved $\mathbf{a} \in A$. Similar arguments shows that $\mathbf{b} \in B$. Now, since $\mathbf{z} = \mathbf{a}_{n_k} + \mathbf{b}_{n_k}$ for all k, we see

$$\mathbf{z} = \lim_{k\to\infty} \mathbf{a}_{n_k} + \mathbf{b}_{n_k} = \lim_{k\to\infty} \mathbf{a}_{n_k} + \lim_{k\to\infty} \mathbf{b}_{n_k} = \mathbf{a} + \mathbf{b} \in A + B$$

Because $\mathbf{z}$ is arbitrarily selected from $\bigcap_{\epsilon>0} A_\epsilon + B_\epsilon$. We have in fact proved

$$\bigcap_{\epsilon>0} A_\epsilon + B_\epsilon \subseteq A + B$$

which together with Equation 1.4 implies Equation 1.3. With Equation 1.3 established, we can now apply Theorem 1.1.3 to have the desired inequality

$$\begin{aligned}
|A+B|^{\frac{1}{d}} &= \Big(\lim_{\epsilon\to 0} |A_\epsilon + B_\epsilon|\Big)^{\frac{1}{d}} \\
&= \lim_{\epsilon\to 0} |A_\epsilon + B_\epsilon|^{\frac{1}{d}} \\
&\geq \lim_{\epsilon\to 0} |A_\epsilon|^{\frac{1}{d}} + |B_\epsilon|^{\frac{1}{d}} = |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}
\end{aligned}$$

■

Before we proceed to the develop the remaining Brunn-Minkowski Inequality, we first have to prove that Lebesgue measure is **inner regular**.

Theorem 1.1.5. (Lebesgue measure is inner regular) If $A \subseteq \mathbb{R}^d$ is Lebesgue measurable, then

$$|A| = \sup\{|K| : K \subseteq A \text{ is compact }\}$$

Proof. Because A is measurable, we know $A\cap\overline{B_n(\mathbf{0})}$ is measurable for all $n\in\mathbb{N}$. Therefore, for all $n\in\mathbb{N}$, $(A\cap\overline{B_n(\mathbf{0})})^c$ is measurable. Then by definition, there exists open O_n containing $(A\cap\overline{B_n(\mathbf{0})})^c$, such that $|O_n\setminus(A\cap B_n(\mathbf{0}))^c|<\frac{1}{n}$. Now, for each $n\in\mathbb{N}$, define closed set $K_n\triangleq O_n^c$. We then have

$$K_n\subseteq A\cap\overline{B_n(\mathbf{0})}$$

and

$$(A\cap\overline{B_n(\mathbf{0})})\setminus K_n=A\cap\overline{B_n(\mathbf{0})}\cap O_n=O_n\setminus(A\cap B_n(\mathbf{0}))^c$$

This then give us

$$\left|(A\cap\overline{B_n(\mathbf{0})})\setminus K_n\right|=|O_n\setminus(A\cap B_n(\mathbf{0}))^c|<\frac{1}{n}$$

Note that because $K_n\subseteq B_n(\mathbf{0})$ is bounded and closed, by Hiene-Borel, we know K_n is compact. Lastly, to close out the proof, we are required to show $|K_n|\to|A|$ as $n\to\infty$. Note that $|A\cap B_n(\mathbf{0})|\nearrow|A|$ as $n\to\infty$ because $A\cap B_n(\mathbf{0})\nearrow A$ as $n\to\infty$. Then because $|A\cap B_n(\mathbf{0})|\geq|K_n|\geq|A\cap B_n(\mathbf{0})|-\frac{1}{n}$, we see that $|K_n|\to|A|$ by squeeze Theorem. ■

Theorem 1.1.6. (Brunn-Minkowski Inequality for measurable set) Suppose A,B are measurable subset of $\mathbb{R}^d$ and $A+B$ is also measurable. We have

$$|A|^{\frac{1}{d}}+|B|^{\frac{1}{d}}\leq|A+B|^{\frac{1}{d}}$$

Proof. Because Lebesgue measure is inner regular and A,B are of finite measure, for each $n\in\mathbb{N}$, we can let A_n,B_n each be compact subset of A,B such that $|A|-|A_n|<\frac{1}{n}$ and $|B|-|B_n|<\frac{1}{n}$. It then follows from Theorem 1.1.4 that

$$|A+B|^{\frac{1}{d}}\geq\lim_{n\to\infty}|A_n+B_n|^{\frac{1}{d}}\geq\lim_{n\to\infty}|A_n|^{\frac{1}{d}}+|B_n|^{\frac{1}{d}}=|A|^{\frac{1}{d}}+|B|^{\frac{1}{d}}$$

■

1.2 HW1

Question 1

Show $\mathbb{R}^n$ is complete.

Proof. Let $\mathbf{x}_k$ be an arbitrary Cauchy sequence in $\mathbb{R}^n$. We are required to show $\mathbf{x}_k$ converge in $\mathbb{R}^n$. For each k, denote $\mathbf{x}_k$ by $(x_{(1,k)},\dots,x_{(n,k)})$. We claim that for each $i\in\{1,\dots,n\}$

$$x_{(i,k)} \text{ is a Cauchy sequence}$$

Fix i and $\epsilon>0$. To show $x_{(i,k)}$ is a Cauchy sequence, we are required to find $N\in\mathbb{N}$ such that for all $r,m\geq N$ we have

$$\left|x_{(i,r)}-x_{(i,m)}\right|\leq\epsilon$$

Because $\mathbf{x}_k$ is a Cauchy sequence in $\mathbb{R}^n$, we know there exists $N\in\mathbb{N}$ such that for all $r,m\geq N$, we have

$$|\mathbf{x}_r-\mathbf{x}_m|<\epsilon$$

Fix such N and arbitrary $r,m\geq M$. Observe

$$\left|x_{(i,r)}-x_{(i,m)}\right|\leq\sqrt{\sum_{j=1}^n\left|x_{(j,r)}-x_{(j,m)}\right|^2}=|\mathbf{x}_r-\mathbf{x}_m|<\epsilon$$

We have proved that for each $i\in\{1,\dots,n\}$, the real sequence $x_{(i,k)}$ is Cauchy. We now claim that for each $i\in\{1,\dots,n\}$, we have

$$\limsup_{r\to\infty}x_{(i,r)}\in\mathbb{R} \text{ and } \lim_{k\to\infty}x_{(i,k)}=\limsup_{r\to\infty}x_{(i,r)}$$

Again fix i. Because $x_{(i,k)}$ is a Cauchy sequence, we know there exists some N such that for all $r,m\geq N$, we have

$$\left|x_{(i,r)}-x_{(i,m)}\right|<1$$

This implies that for all $r\geq N$, we have

$$x_{(i,r)}<x_{(i,N)}+1 \tag{1.6}$$

Equation 1.6 then tell us

$$x_{(i,N)}+1 \text{ is an upper bound of } \{x_{(i,r)}:r\geq N\}$$

Then by definition of sup, we have

$$\sup\{x_{(i,r)} : r \geq N\} \leq x_{(i,N)} + 1 \in \mathbb{R}$$

This then implies $\limsup_{r\to\infty} x_{(i,r)} \in \mathbb{R}$. We now prove

$$\lim_{k\to\infty} x_{(i,k)} = \limsup_{r\to\infty} x_{(i,r)} \tag{1.7}$$

Fix $\epsilon > 0$. We are required to find N such that

$$\forall k \geq N, \left| x_{(i,k)} - \limsup_{r\to\infty} x_{(i,r)} \right| \leq \epsilon$$

Because $\{x_{(i,k)}\}_{k\in\mathbb{N}}$ is a Cauchy sequence, we can let N_0 satisfy

$$\forall k, m \geq N_0, \left| x_{(i,k)} - x_{(i,m)} \right| < \frac{\epsilon}{2}$$

Because $\sup\{x_{(i,k)} : k \geq N'\} \searrow \limsup_{r\to\infty} x_{(i,r)}$ as $N' \to \infty$, we know there exists $N_1 > N_0$ such that

$$\limsup_{r\to\infty} x_{(i,r)} - \frac{\epsilon}{2} < \sup\{x_{(i,k)} : k \geq N_0\} \leq \limsup_{r\to\infty} x_{(i,r)} + \frac{\epsilon}{2}$$

Then because $\limsup_{r\to\infty} x_{(i,r)} - \frac{\epsilon}{2}$ is strictly smaller than the smallest upper bound of $\{x_{(i,k)} : k \geq N_1\}$, we see $\limsup_{n\to\infty} x_{(i,r)} - \frac{\epsilon}{2}$ is not an upper bound of $\{x_{(i,k)} : k \geq N_1\}$. This implies the existence of some N such that $N \geq N_1$ and

$$\limsup_{r\to\infty} x_{(i,r)} - \frac{\epsilon}{2} < x_{(i,N)} \leq \limsup_{r\to\infty} x_{(i,r)} + \frac{\epsilon}{2}$$

Now observe that for all $k \geq N$, because $N \geq N_1 \geq N_0$

$$\limsup_{r\to\infty} x_{(i,r)} - \epsilon < x_{(i,N)} - \frac{\epsilon}{2} < x_{(i,k)} < x_{(i,N)} + \frac{\epsilon}{2} \leq \limsup_{r\to\infty} x_{(i,r)} + \epsilon$$

This implies for all $k \geq N$, we have

$$\left| x_{(i,k)} - \limsup_{r\to\infty} x_{(i,r)} \right| \leq \epsilon$$

We have just proved Equation 1.7. Lastly, to close out the proof, we show

$$\lim_{k\to\infty} \mathbf{x}_k = \left(\lim_{k\to\infty} x_{(1,k)}, \dots, \lim_{k\to\infty} x_{(n,k)} \right) \tag{1.8}$$

Fix $\epsilon > 0$. For each $i \in \{1, \dots, n\}$, let N_i satisfy

$$\forall r \geq N_i, \left| x_{(i,r)} - \lim_{k\to\infty} x_{(i,k)} \right| \leq \frac{\epsilon}{\sqrt{n}}$$

Observe that for all $r \geq \max_{i \in \{1,\dots,n\}} N_i$, we have

$$\begin{aligned}\left|\mathbf{x}_r - \left(\lim_{k\to\infty} x_{(1,k)},\dots,\lim_{k\to\infty} x_{(n,k)}\right)\right| &= \sqrt{\sum_{i=1}^n \left|x_{(i,r)} - \lim_{k\to\infty} x_{(i,k)}\right|^2} \\ &\leq \sqrt{\sum_{i=1}^n \frac{\epsilon^2}{n}} = \epsilon\end{aligned}$$

We have proved Equation 1.8.

∎

Question 2

Show $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. Fix $x \in \mathbb{R}$ and $\epsilon > 0$. To show $\mathbb{Q}$ is dense in $\mathbb{R}$, we have to find $q \in \mathbb{Q}$ such that $|x - q| < \epsilon$.

Let $m \in \mathbb{N}$ satisfy $\frac{1}{m} < \epsilon$. Let n be the largest integer such that $n \leq mx$. Because n is the largest integer such that $n \leq mx$, we know $mx - n < 1$, otherwise we can deduce $n + 1 \leq mx$, which is impossible, since $n + 1$ is an integer and n is the largest integer such that $n \leq mx$. We now see that

$$\frac{n}{m} \in \mathbb{Q} \text{ and } \left|x - \frac{n}{m}\right| = \frac{mx - n}{m} < \frac{1}{m} < \epsilon$$

∎

Theorem 1.2.1. (Distance Formula) Given two subsets A, B of a metric space, we have

$$d(A, B) = \inf_{b \in B} d(A, b)$$

Proof. Fix arbitrary $b \in B$. It is clear that

$$d(A, B) \leq d(A, b)$$

It then follows $d(A, B) \leq \inf_{b \in B} d(A, b)$. Fix arbitrary $a \in A$ and $b_0 \in B$. Observe that

$$d(a, b_0) \geq d(A, b_0) \geq \inf_{b \in B} d(A, b)$$

It then follows $\inf_{b \in B} d(A, b) \leq d(A, B)$. ∎

Question 3

Let E_1, E_2 be non-empty sets in $\mathbb{R}^n$ with E_1 closed and E_2 compact. Show that there are points $x_1 \in E_1$ and $x_2 \in E_2$ such that

$$d(E_1, E_2) = |x_1 - x_2|$$

Deduce that $d(E_1, E_2)$ is positive if such E_1, E_2 are disjoint.

Proof. Because

(a) $f(x) \triangleq d(E_1, x)$ is a continuous function on $\mathbb{R}^n$.

(b) E_2 is compact.

It now follows by EVT there exists some $x_2 \in E_2$ such that

$$d(E_1, x_2) = \min_{x \in E_2} d(E_1, x) = \inf_{x \in E_2} d(E_1, x) = d(E_1, E_2)$$

where the last equality is proved above. We can now reduce the problem into finding x_1 in E_1 such that

$$d(x_1, x_2) = d(E_1, x_2)$$

For each $n \in \mathbb{N}$, let t_n satisfy

$$t_n \in E_1 \text{ and } d(t_n, x_2) < d(E_1, x_2) + \frac{1}{n}$$

Clearly, t_n is a bounded sequence. Then by Bolzano-Weierstrass Theorem, there exists a convergence subsequence t_{n_k}. Now, because E_1 is closed, we know

$$x_1 \triangleq \lim_{k \to \infty} t_{n_k} \in E_1$$

It then follows from the function $f(x) \triangleq d(x, x_2)$ being continuous on $\mathbb{R}^n$ such that

$$d(x_1, x_2) = \lim_{k \to \infty} d(t_{n,k}, x_2) = d(E_1, x_2)$$

∎

Question 4

Prove that the distance between two nonempty, compact, disjoint sets in $\mathbb{R}^n$ is positive.

Proof. The proof follows from the result in last question while acknowledging compact is closed. ∎

Question 5

Prove that if f is continuous on $[a,b]$, then f is Riemann-integrable on$[a,b]$.

Proof. Let $\overline{\int_a^b} f dx$ and $\underline{\int_a^b} f dx$ respectively denote the upper and lower Darboux sums. We prove that

$$\overline{\int_a^b} f dx = \underline{\int_a^b} f dx$$

Fix ϵ. We reduce the problem into proving the existence of some partition $\{a = x_0, x_1, \dots, x_n = b\}$ such that

$$\sum_{i=1}^n \Big[M_i - m_i\Big](x_i - x_{i-1}) \leq \epsilon$$

where

$$M_i \triangleq \sup_{t\in[x_{i-1},x_i]} f(t) \text{ and } m_i \triangleq \inf_{t\in[x_{i-1},x_i]} f(t)$$

Because f is continuous on the compact interval $[a,b]$, we know f is uniformly continuous on $[a,b]$. Let δ satisfy

$$|x-y| < \delta \text{ and } x,y \in [a,b] \implies |f(x) - f(y)| < \frac{\epsilon}{b-a}$$

Let n satisfy $\frac{b-a}{n} < \delta$. We claim the partition

$$\{a = x_0, x_1, \dots, x_n = b\} \text{ where } x_i \triangleq a + \frac{i(b-a)}{n} \text{ suffices}$$

Now, by EVT, we know that for each i, there exists some $t_{i,M}, t_{i,m} \in [x_{i-1}, x_i]$ such that

$$f(t_{i,m}) = m_i \text{ and } f(t_{i,M}) = M_i$$

Then because

$$|t_{i,m} - t_{i,M}| \leq x_i - x_{i-1} \leq \frac{b-a}{n} < \delta$$

We know $M_i - m_i < \frac{\epsilon}{b-a}$. This now give us

$$\begin{aligned}
\sum_{i=1}^n \Big[M_i - m_i\Big](x_i - x_{i-1}) &< \sum_{i=1}^n \frac{\epsilon}{(b-a)}(x_i - x_{i-1}) \\
&= \frac{\epsilon}{b-a}\sum_{i=1}^n (x_i - x_{i-1}) \\
&= \frac{\epsilon}{b-a}(b-a) = \epsilon
\end{aligned}$$

$\blacksquare$

Question 6

Find $\limsup_{n\to\infty} E_n$ and $\liminf_{n\to\infty} E_n$ where

$$E_n \triangleq \begin{cases} [\frac{-1}{n}, 1] & \text{if } n \text{ is odd} \\ [-1, \frac{1}{n}] & \text{if } n \text{ is even} \end{cases}$$

Proof. Fix arbitrary $n \in \mathbb{N}$. Let $p, q \geq n$ respectively be odd and even. We see

$$[0,1] \subseteq E_p \text{ and } [-1,0] \subseteq E_q$$

This now implies

$$[-1,1] \subseteq \bigcup_{k\geq n} E_k$$

Then because n is arbitrary, it follows

$$\limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} E_k = [-1,1]$$

Again, fix arbitrary $n \in \mathbb{N}$ and $\epsilon > 0$. Let p, q respectively be even and odd integers greater than $\max\{n, \frac{1}{\epsilon}\}$. We now see

$$\epsilon \notin [-1, \frac{1}{p}] = E_p \text{ and } -\epsilon \notin [\frac{-1}{q}, 1] = E_q$$

Because ϵ is arbitrary and clearly $0 \in E_k$ for all k, we now see

$$\bigcap_{k\geq n} E_k = \{0\}$$

Then because n is arbitrary, we see

$$\liminf_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} \bigcap_{k\geq n} E_k = \{0\}$$

$\blacksquare$

Question 7

Show that

$$(\limsup_{n\to\infty} E_n)^c = \liminf_{n\to\infty}(E_n)^c$$

and

$$E_n \searrow E \text{ or } E_n \nearrow E \implies \limsup_{n\to\infty} E_n = \liminf_{n\to\infty} E_n = E$$

Proof. Fix arbitrary $x \in (\limsup_{n\to\infty} E_n)^c$. We can deduce

$$\exists n, x \notin \bigcup_{k\geq n} E_k$$

This implies

$$\exists n, x \in \bigcap_{k\geq n} E_k^c$$

Then we see

$$x \in \bigcup_{n=1}^{\infty} \bigcap_{k\geq n} E_k^c = \liminf_{n\to\infty} E_n^c$$

We have proved $(\limsup_{n\to\infty} E_n)^c \subseteq \liminf_{n\to\infty} E_n^c$. We now prove the converse. Fix arbitrary $x \in \liminf_{n\to\infty} E_n^c$. We can deduce

$$\exists n, x \in \bigcap_{k\geq n} E_k^c$$

This implies

$$\exists n, x \notin \bigcup_{k\geq n} E_k$$

Then we see

$$x \notin \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} E_k = \limsup_{n\to\infty} E_n$$

∎

Theorem 1.2.2. (Equivalent Definition for Limit Superior) If we let E be the set of subsequential limits of a_n

$$E \triangleq \{L \in \overline{\mathbb{R}} : L = \lim_{k\to\infty} a_{n_k} \text{ for some } n_k\}$$

The set E is non-empty and

$$\max E = \limsup_{n\to\infty} a_n$$

Proof. Let $n_1 \triangleq 1$. Recursively, because

$$\sup_{j\geq n_k} a_k \geq \limsup_{n\to\infty} a_n > \limsup_{n\to\infty} a_n - \frac{1}{k} \text{ for each } k$$

We can let n_{k+1} be the smallest number such that

$$a_{n_{k+1}} > \limsup_{n\to\infty} a_n - \frac{1}{k}$$

It is straightforward to check $a_{n_k} \to \limsup_{n\to\infty} a_n$ as $k \to \infty$. Note that no subsequence can converge to $\limsup_{n\to\infty} a_n + \epsilon$ because there exists N such that $\sup_{k\geq N} a_k < \limsup_{n\to\infty} a_n + \epsilon$. ∎

Question 8

Show that

$$\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n$$

Proof. Note that $-a_{n_k}$ converge if and only if a_{n_k} converge. Then if we respectively define E and E^- to be the set of subsequential limits of a_n and $-a_n$, we see

$$E^- = \{-L \in \mathbb{R} : L \in E\}$$

We now see

$$\limsup_{n\to\infty}(-a_n) = \max E^- = -\min E = -\liminf_{n\to\infty} a_n$$

∎

Question 9

Show that

$$\limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \tag{1.9}$$

Proof. Fix arbitrary ϵ. Let N_a, N_b respectively satisfy

$$\sup_{n\geq N_a} a_n \leq \limsup_{n\to\infty} a_n + \frac{\epsilon}{2} \text{ and } \sup_{n\geq N_b} b_n \leq \limsup_{n\to\infty} b_n + \frac{\epsilon}{2}$$

Let $N \triangleq \max\{N_a, N_b\}$. We now see that

$$\limsup_{n\to\infty}(a_n+b_n) \leq \sup_{n\geq N}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n + \epsilon$$

The result then follows from ϵ being arbitrary. ■

Question 10

$$a_n, b_n \text{ is bounded non-negative } \implies \limsup_{n\to\infty}(a_nb_n) \leq (\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n) \tag{1.10}$$

Proof. There are three cases we should consider

(a) Both $\limsup_{n\to\infty} a_n$ and $\limsup_{n\to\infty} b_n$ equal 0.

(b) Between $\limsup_{n\to\infty} a_n$ and $\limsup_{n\to\infty} b_n$, only one of them equals 0.

(c) Neither $\limsup_{n\to\infty} a_n$ nor $\limsup_{n\to\infty} b_n$ equals to 0.

In the first case, because a_n, b_n are both non-negative, we can deduce

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$$

which implies

$$\limsup_{n\to\infty}(a_nb_n) = \lim_{n\to\infty} a_nb_n = 0 = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$$

For second case, WOLG, suppose $\limsup_{n\to\infty} a_n = 0$. Fix arbitrary ϵ. We can let N satisfy

$$\sup_{n\geq N} a_n < \frac{\epsilon}{\sup_{n\in\mathbb{N}} b_n}$$

Since for all $n \geq N$, we have

$$a_n b_n \leq \frac{b_n \epsilon}{\sup_{k \in \mathbb{N}} b_k} \leq \epsilon$$

We now see

$$\limsup_{n\to\infty}(a_n b_n) \leq \sup_{n \geq N} a_n b_n \leq \epsilon$$

The result

$$\limsup_{n\to\infty} a_n b_n = 0 = \limsup_{n\to\infty} a_n \limsup_{n\to\infty} b_n$$

then follows from ϵ being arbitrary.

Lastly, for the last case, let N_a, N_b respectively satisfy

$$\sup_{n \geq N_a} a_n \leq \limsup_{n\to\infty} a_n \sqrt{1+\epsilon} \text{ and } \sup_{n \geq N_b} b_n \leq \limsup_{n\to\infty} b_n \sqrt{1+\epsilon}$$

Let $N \triangleq \max\{N_a, N_b\}$, because for each $n \geq N$, we have

$$a_n b_n \leq (\sup_{k \geq N_a} a_k)(\sup_{k \geq N_b} b_k) \leq (1+\epsilon)(\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n)$$

It then follows that

$$\limsup_{n\to\infty}(a_n b_n) \leq \sup_{n \geq N}(a_n b_n) \leq (1+\epsilon)(\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n)$$

The result then follows from ϵ being arbitrary. ∎

Question 11

Show that if either a_n or b_n converge, the equalities in Equation 1.9 and Equation 1.10 both hold true.

Proof. WOLG, suppose $\lim_{n\to\infty} a_n = L \in \mathbb{R}$. We then see

$$(a_{n_k} + b_{n_k}) \text{ converge } \iff b_{n,k} \text{ converge}$$

Let $E_{a,b}$ and E_b respectively be the set of subsequential limits of $(a_n + b_n)$ and b_n. We now have

$$E_{a,b} = \{L + L_b \in \mathbb{R} : L_b \in E_b\}$$

This give us

$$\limsup_{n\to\infty}(a_n+b_n)=\max E_{a,b}=L+\max E_b=\limsup_{n\to\infty} a_n+\limsup_{n\to\infty} b_n$$

Now, additionally, suppose a_n, b_n are both bounded and nonnegative. Again because

$$a_{n_k}b_{n,k} \text{ converge } \iff b_{n,k} \text{ converge}$$

We see

$$E_{a,b}=\{L(L_b)\in\mathbb{R}:L_b\in E_b\}$$

This give us

$$\limsup_{n\to\infty}(a_nb_n)=\max E_{a,b}=L\max E_b=(\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n)$$

■

Question 12

Give example for which inequality in Equation 1.9 and Equation 1.10 are not equalities.

Proof. If

$$a_n\triangleq\begin{cases}1 & \text{if } n \text{ is odd}\\ -1 & \text{if } n \text{ is even}\end{cases} \text{ and } b_n\triangleq\begin{cases}-1 & \text{if } n \text{ is odd}\\ 1 & \text{if } n \text{ is even}\end{cases}$$

we have

$$\limsup_{n\to\infty}(a_n+b_n)=0<2=\limsup_{n\to\infty} a_n+\limsup_{n\to\infty} b_n$$

Let $L>1$ and

$$a_n\triangleq\begin{cases}L-\frac{1}{k} & \text{if } n=2k-1\\ (L-\frac{1}{k})^{-1} & \text{if } n=2k\end{cases} \text{ and } b_n\triangleq\begin{cases}(L-\frac{1}{k})^{-1} & \text{if } n=2k-1\\ (L-\frac{1}{k}) & \text{if } n=2k\end{cases}$$

We have

$$\limsup_{n\to\infty} a_nb_n=1<L^2=\limsup_{n\to\infty} a_n\limsup_{n\to\infty} b_n$$

■

Question 13

Give an example of a decreasing sequence of nonempty closed sets in $\mathbb{R}^n$ whose intersection is empty.

Proof.

$$F_n \triangleq [n, \infty) \text{ suffices}$$

■

Question 14

Given an example of two disjoint, nonempty closed sets in E_1 and E_2 in $\mathbb{R}^n$ for which $d(E_1, E_2) = 0$.

Proof. Let

$$E_1 \triangleq \{n - \frac{1}{n} \in \mathbb{R} : n \in \mathbb{N} \text{ and } n \geq 2\} \text{ and } E_2 \triangleq \{n - \frac{1}{2n} \in \mathbb{R} : n \in \mathbb{N} \text{ and } n \geq 2\}$$

To see $E_1 \cap E_2 = \varnothing$, suppose $n - \frac{1}{n} = k - \frac{1}{2k}$ where n, k are two natural numbers greater than 2. We then see $\frac{1}{n} - \frac{1}{2k} = n - k$, which is impossible, since

$$\left| \frac{1}{n} - \frac{1}{2k} \right| < \max\{\frac{1}{2k}, \frac{1}{n}\} < 1$$

The fact E_1, E_2 are closed follows from both of them being totally disconnected. Now observe that for all ϵ, there exists large enough n such that

$$(n + \frac{1}{n}) - (n + \frac{1}{2n}) < \frac{1}{n} < \epsilon$$

This implies $d(E_1, E_2) = 0$. ■

Question 15

If f is defined and uniformly continuous on E, show there is a function $\overline{f}$ defined and continuous on $\overline{E}$ such that $\overline{f} = f$ on E.

Proof. Define $\overline{f}$ on E by $\overline{f} = f$. For each $x \in \overline{E} \setminus E$, associate x with a sequence $t_{n,x}$ in E converging to x. We now claim that for each $x \in \overline{E} \setminus E$ the limit

$$\lim_{n \to \infty} f(t_{n,x}) \text{ converge in } \mathbb{R}$$

Fix ϵ. Because f is uniformly continuous on E, we know there exists δ such that

$$a, b \in E \text{ and } |a-b| \leq \delta \implies |f(a) - f(b)| < \epsilon$$

Because $t_{n,x}$ converge, we know $t_{n,x}$ is Cauchy, then we know there exists N such that $|t_{n,x} - t_{m,x}| < \delta$ for all $n, m > N$, we then see that for all $n, m > N$, we have

$$|f(t_{n,x}) - f(t_{m,x})| < \epsilon$$

This implies $\{f(t_{n,x})\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, thus converge in $\mathbb{R}$.

Define

$$\overline{f}(x) \triangleq \lim_{n \to \infty} f(t_{n,x}) \text{ for all } x \in \overline{E} \setminus E$$

We are required to show $\overline{f}$ is also continuous on $\overline{E} \setminus E$. Fix ϵ and $x \in \overline{E} \setminus E$. Let δ satisfy

$$a, b \in E \text{ and } |a-b| \leq \delta \implies |f(a) - f(b)| < \frac{\epsilon}{3}$$

We claim

$$\sup_{t \in B_{\frac{\delta}{2}}(x) \cap \overline{E}} \left| \overline{f}(t) - \overline{f}(x) \right| \leq \epsilon$$

Fix $t \in B_{\frac{\delta}{2}}(x) \cap \overline{E}$. There are two possibilities

(a) $t \in E$

(b) $t \in \overline{E} \setminus E$

If $t \in E$, let n satisfy

$$\left| f(t_{n,x}) - \overline{f}(x) \right| < \frac{\epsilon}{3} \text{ and } |t_{n,x} - x| < \frac{\delta}{2}$$

Because

$$|t_{n,x} - t| \leq |t_{n,x} - x| + |t - x| < \delta$$

we can deduce $|f(t_{n,x}) - f(t)| < \frac{\epsilon}{3}$. This now give us

$$\left| f(t) - \overline{f}(x) \right| \leq |f(t_{n,x}) - f(t)| + \left| f(t_{n,x}) - \overline{f}(x) \right| < \epsilon$$

If $t \in \overline{E} \setminus E$. Write $y = t$ and let $t_{n,y}$ be the associated sequence in E. Because $y \in B_{\frac{\delta}{2}}(x)$, we know there exists $t_{n,y}$ such that

$$t_{n,y} \in B_{\frac{\delta}{2}}(x) \text{ and } \left| f(t_{n,y}) - \overline{f}(y) \right| < \frac{\epsilon}{3}$$

Again, let m satisfy

$$t_{m,x} \in B_{\frac{\delta}{2}}(x) \text{ and } \left|f(t_{m,x}) - \overline{f}(x)\right| < \frac{\epsilon}{3}$$

We know $|t_{n,y} - t_{m,x}| \leq \delta$ because they both belong to $B_{\frac{\delta}{2}}(x)$. We can now deduce

$$\left|\overline{f}(y) - \overline{f}(x)\right| = \left|\overline{f}(y) - f(t_{n,y})\right| + |f(t_{n,y}) - f(t_{m,x})| + \left|f(t_{m,x}) - \overline{f}(x)\right| < \epsilon$$

which finish the proof. ■

Question 16

If f is defined and uniformly continuous on a bounded set E, show that f is bounded on E.

Proof. By last question, we can extend f to a continuous $\overline{f}$ onto $\overline{E}$. Now because $\overline{E}$ is compact and $\left|\overline{f}\right|$ is continuous on $\overline{E}$, by EVT, there exists $a \in \overline{E}$ such that

$$\sup_{x \in E} |f(x)| \leq \max_{x \in \overline{E}} |f(x)| = f(a)$$

■

1.3 HW2

Question 17

Construct a two-dimensional Cantor set in the unit square $[0, 1]^2$ as follows. Subdivide the square into nine equal parts and keep only the four closed corner squares, removing the remaining region (which form a cross). Then repeat this process in a suitably scaled version for the remaining squares, ad infinitum. Show that the resulting set is perfect, has plane measure zero, and equals $\mathcal{C} \times \mathcal{C}$.

Proof. Let $\mathcal{C}'_n \subseteq \mathbb{R}^2$ be the result after the nth stage of removal, and let $\mathcal{C}_n \subseteq \mathbb{R}$ be the result after the nth stage of removal in the construction of the classical ternary Cantor set. It is clear that

$$\mathcal{C}'_n = \mathcal{C}_n \times \mathcal{C}_n \text{ for all } n$$

It then follows

$$\bigcap_n \mathcal{C}'_n = \bigcap_n \mathcal{C}_n \times \mathcal{C}_n = \mathcal{C} \times \mathcal{C}$$

The fact that $\mathcal{C} \times \mathcal{C}$ has plane measure zero follows from Lemma 1.3.1. Fix $(a, b) \in \mathcal{C} \times \mathcal{C}$. Because $\mathcal{C}$ is perfect, there exists some $b' \in \mathcal{C}$ such that

$$0 < |b' - b| < \epsilon$$

To see that $\mathcal{C}'$ is perfect, one see that

$$(a, b) \neq (a, b') \text{ and } (a, b') \in \mathcal{C}' \times \mathcal{C}' \text{ and } |(a, b) - (a, b')| = |b' - b| < \epsilon$$

■

Question 18

Construct a subset of $[0, 1]$ in the same manner as the Cantor set, except that at the kth stage, each interval removed has length $\delta 3^{-k}, 0 < \delta < 1$. Show that the resulting set is perfect, has measure $1 - \delta$, and contains no intervals.

Proof. Let $\mathcal{C}'_n \subseteq \mathbb{R}$ be the result after the nth stage of removal according to the description. Clearly, each $\mathcal{C}'_n$ has 2^n amount of connected component, we then can compute the length of $\mathcal{C}' \triangleq \bigcap \mathcal{C}'_n$ to be

$$1 - \sum_{k=1}^{\infty} 2^{k-1} \delta 3^{-k} = 1 - \frac{\frac{\delta}{3}}{1 - \frac{2}{3}} = 1 - \delta$$

Since each $\mathcal{C}_n'$ has 2^n amount of connected component of equal length and $\mathcal{C}_n' \subseteq [0,1]$, we know the length of each connected component of $\mathcal{C}_n'$ must not be greater than $\frac{1}{2^n}$. It then follows that no interval $[a, a+h]$ can be contained by all $\mathcal{C}_n'$ because if $[a, a+h]$ is a subset of some connected component of $\mathcal{C}_k'$ of some k, then the measure $h = |[a, a+h]|$ must be smaller than $\frac{1}{2^k}$, which is false when k is large enough. ∎

Question 19

If E_k is a sequence of sets with $\sum |E_k|_e < \infty$, show that $\limsup_{n\to\infty} E_n$ has measure zero.

Proof. Note that

$$\sum_{k=N}^{\infty} |E_k|_e = \left(\sum_{k=1}^{\infty} |E_k|_e - \sum_{k=1}^{N-1} |E_k|_e \right) \to 0 \text{ as } N \to \infty$$

and note for all N we have

$$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \subseteq \bigcup_{k=N}^{\infty} E_k$$

Then for arbitrary ϵ, if we let N satisfy $\sum_{k=N}^{\infty} |E_k|_e < \epsilon$, we see that

$$\left| \limsup_{n\to\infty} E_n \right|_e = \left| \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \right|_e \leq \left| \bigcup_{k=N}^{\infty} E_k \right|_e \leq \sum_{k=N}^{\infty} |E_k|_e < \epsilon$$

∎

Question 20

If E_1, E_2 are measurable, show that

$$|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|$$

Proof. Observe the following expression of each set in disjoint union

(a) $E_1 = (E_1 \setminus E_2) \sqcup (E_1 \cap E_2)$

(b) $E_2 = (E_2 \setminus E_1) \sqcup (E_1 \cap E_2)$

(c) $E_1 \cup E_2 = (E_1 \setminus E_2) \sqcup (E_1 \cap E_2) \sqcup (E_2 \setminus E_1)$

It now follows

$$\begin{aligned}|E_1 \cup E_2| + |E_1 \cap E_2| &= |E_1 \setminus E_2| + |E_1 \cap E_2| + |E_1 \cap E_2| + |E_2 \setminus E_1| \\ &= |E_1| + |E_2|\end{aligned}$$

■

Lemma 1.3.1. Given two subsets Z_1, Z_2 of $\mathbb{R}$, if $|Z_1| = 0$, then $|Z_1 \times Z_2| = 0$.

Proof. Let $A_n \triangleq [n, n+1)$. Because

$$Z_1 \times Z_2 = \bigsqcup_{n \in \mathbb{Z}} Z_1 \times (A_n \cap Z_2)$$

To show $|Z_1 \times Z_2| = 0$, we only have to $|Z_1 \times (A_n \cap Z_2)| = 0$ for all $n \in \mathbb{Z}$. In other words, we can WOLG suppose Z_2 is bounded.

Now, fix ϵ. We are required to find an countable closed cube cover $Q_n \times C_n$ for $Z_1 \times Z_2$ such that $\sum_n |Q_n \times C_n| < \epsilon$. Let $C_n = C$ for all n where C is a compact interval containing Z_2, and let Q_n be a countable compact interval cover for Z_1 such that $\sum |Q_n| < \frac{\epsilon}{|C|}$. It then follows $\sum_n |Q_n \times C_n| = \sum_n |Q_n| \, |C| < \epsilon$. ■

Theorem 1.3.2. (Product of Finite Measure Set) If E_1 and E_2 are measurable subset of $\mathbb{R}$ and $|E_1|, |E_2| < \infty$, then $E_1 \times E_2$ is measurable in $\mathbb{R}^2$ and

$$|E_1 \times E_2| = |E_1| \, |E_2|$$

Proof. Write $E_1 \triangleq H_1 \sqcup Z_1$ and $E_2 \triangleq H_2 \sqcup Z_2$ where $H_1, H_2 \in F_\sigma$ and $|H_1| = |E_1|$ and $|H_2| = |E_2|$. Now observe

$$E_1 \times E_2 = (H_1 \times H_2) \cup (Z_1 \times E_2) \cup (E_1 \times Z_2)$$

Note that if we write $H_1 = \bigcap F_{1,n}$ and $H_2 = \bigcap F_{2,n}$, we see $H_1 \times H_2 = \bigcap F_{1,n} \times F_{2,n} \in F_\sigma$ in $\mathbb{R}^2$, it now follows from Lemma 1.3.1 that $E_1 \times E_2$ is measurable.

Now, let S_n be a decreasing sequence of open set containing E_1 such that $|S_n \setminus E_1| < \frac{1}{n}$, and let T_n be a decreasing sequence of open set containing E_2 such that $|T_n \setminus E_2| < \frac{1}{n}$. In other words,

$$E_1 = S \setminus Z_1 \text{ and } E_2 = T \setminus Z_2 \text{ where } \begin{cases} S \triangleq \bigcap S_n \\ T \triangleq \bigcap T_n \\ |Z_1| = |Z_2| = 0 \end{cases}$$

We now have

$$S \times T = (E_1 \times E_2) \cup (Z_1 \times E_2) \cup (E_1 \times Z_2)$$

This then implies $|S \times T| = |S \times T|_e \leq |E_1 \times E_2|_e + |Z_1 \times E_2|_e + |E_1 \times Z_2|_e = |E_1 \times E_2|$, where the last inequality follows from Lemma 1.3.1. The reverse inequality is clear, since $E_1 \times E_2 \subseteq S \times T$. We have proved $|E_1 \times E_2| = |S \times T|$.

Now, for each n, write

$$S_n = \bigcup_{k \in \mathbb{N}} I_{k,S_n} \text{ and } T_n = \bigcup_{k \in \mathbb{N}} I_{k,T_n}$$

where $(I_{k,S_n})_k$ and $(I_{k,T_n})_k$ are non-overlapping compact interval. It then follows that

$$|S_n \times T_n| = \sum_{i,j} |I_{i,S_n} \times I_{j,T_n}| = \sum_{i,j} |I_{i,S_n}| \times |I_{j,T_n}| = \sum_i |I_{i,S_n}| \sum_j |I_{j,T_n}| = |S_n| \, |T_n|$$

Write $S \triangleq \bigcap_{n \in \mathbb{N}} S_n$ and $T \triangleq \bigcap_{n \in \mathbb{N}} T_n$. Because

(a) Each $S_n \times T_n$ is open.

(b) $|S_n \times T_n| = |S_n| \, |T_n|$ is bounded ($\because |S_n| \searrow |E_1| < \infty$).

(c) $S_n \times T_n \searrow S \times T$

We can now deduce

$$\begin{aligned} |E_1 \times E_2| = |S \times T| &= \lim_{n \to \infty} |S_n \times T_n| \\ &= \lim_{n \to \infty} |S_n| \, |T_n| \\ &= |E_1| \, |E_2| \end{aligned}$$

■

Question 21

If E_1 and E_2 are measurable subset of $\mathbb{R}$, then $E_1 \times E_2$ is a measurable subset of $\mathbb{R}^2$ and $|E_1 \times E_2| = |E_1| \, |E_2|$

Proof. Define

$$A_n \triangleq \{x \in \mathbb{R} : n \leq x < n + 1\}$$

Because

$$E_1 = \bigcup_{n\in\mathbb{Z}} E_1 \cap A_n \text{ and } E_2 = \bigcup_{k\in\mathbb{Z}} E_2 \cap A_k$$

We can deduce

$$E_1 \times E_2 = \bigcup_{n,k\in\mathbb{Z}} (E_1 \cap A_n) \times (E_2 \cap A_k)$$

Note that Theorem 1.3.2 tell us $(E_1 \cap A_n) \times (E_2 \cap A_k)$ is measurable, which implies $E_1 \times E_2$ is measurable. Theorem 1.3.2 also tell us $|(E_1 \cap A_n) \times (E_2 \cap A_k)| = |E_1 \cap A_n|\,|E_2 \cap A_k|$, which allow us to deduce

$$\begin{aligned}|E_1 \times E_2| = \sum_{n,k\in\mathbb{Z}} |(E_1 \cap A_n) \times (E_2 \cap A_k)| &= \sum_{n,k\in\mathbb{Z}} |E_1 \cap A_n|\,|E_2 \cap A_k| \\ &= \sum_{n\in\mathbb{Z}} |E_1 \cap A_n| \sum_{k\in\mathbb{Z}} |E_2 \cap A_k| = |E_1|\,|E_2|\end{aligned}$$

■

Question 22

Give an example that shows that the image of a measurable set under a continuous transformation may not be measurable. See also Exercise 10 of Chapter 7.

Proof. Consider the Cantor-Lebesgue function denoted by $f : [0,1] \to [0,1]$ and denote the classical ternary Cantor set by $\mathcal{C}$. Let V be a Vitali set contained by $[0,1]$. Because $f(\mathcal{C}) = [0,1]$, we know there exists $E \subseteq \mathcal{C}$ such that $f(E) = V$. Such E is measurable since $|E|_e \leq |\mathcal{C}| = 0$, yet its continuous image $V = f(E)$ is by definition non-measurable. ■

Question 23

Show that there exists disjoint $E_1, E_2, \ldots$ such that $|\bigcup E_k|_e < \sum |E_k|_e$ with strict inequality.

Proof. Let V be a Vitali Set contained by $[0,1]$. Enumerate $[0,1] \cap \mathbb{Q}$ by x_n. Define

$$E_n \triangleq \{v + x_n \in \mathbb{R} : v \in V\} \text{ for all } n$$

The sequence E_n is disjoint, since if $p \in E_n \cap E_m$, then there exists some pair v_n, v_m belong to V such that

$$v_n + x_n = p = v_m + x_m \tag{1.11}$$

which is impossible, since Equation 1.11 implies that $v_n \neq v_m$ and v_n, v_m are of difference of a rational number.

Now, note that for arbitrary n and $v \in V$, because $v \in V \subseteq [0,1]$ and $x_n \in [0,1]$, we have $v + x_n \in [0,2]$. This implies

$$\bigsqcup_n E_n \subseteq [0,2] \text{ and } \left|\bigsqcup_n E_n\right|_e \leq 2$$

Because V is non-measurable by definition, we know $|V|_e > 0$, and since outer measure is translation invariant, we can now deduce

$$\sum_n |E_n|_e = \sum_n |V|_e = \infty > 2 \geq \left|\bigsqcup_n E_n\right|_e$$

■

Question 24

Show that there exists decreasing sequence E_k of sets such that

(a) $E_k \searrow E$.

(b) $|E_k|_e < \infty$.

(c) $\lim_{k\to\infty} |E_k|_e > |E|_e$

Proof. Let V be a Vitali Set contained by $[0,1]$. Enumerate $[0,1] \cap \mathbb{Q}$ by x_n. Define

$$V + x_n \triangleq \{v + x_n \in \mathbb{R} : v \in V\}$$

In last question, we have proved that $V + x_n$ is pairwise disjoint. Define for all $n \in \mathbb{N}$

$$E_n \triangleq \bigsqcup_{k \geq n} V + x_k$$

Observe that

$$E_k \searrow \bigcap E_n = \varnothing$$

which implies $|\bigcap E_n|_e = 0$, but

$$\lim_{n\to\infty} |E_n|_e = \lim_{n\to\infty} \left|\bigsqcup_{k\geq n} V + x_k\right| \geq \lim_{n\to\infty} |V + x_n| = |V| > 0$$

■

Question 25

Let Z be a subset of $\mathbb{R}$ with measure zero. Show that the set $\{x^2 : x \in Z\}$ also has measure zero.

Proof. Fix $Z_n \triangleq Z \cap [-n, n]$. Since

$$\left|\{x^2 : x \in Z\}\right| \leq \sum_{n=1}^{\infty} \left|\{x^2 : x \in Z_n\}\right|_e$$

We only have to prove

$$\left|\{x^2 : x \in Z_n\}\right|_e = 0 \text{ for all } n$$

Fix ϵ, n. Let I_k be a compact interval cover of Z_n such that $\sum |I_k| < \frac{\epsilon}{2n}$. We shall suppose $I_k \subseteq [-n, n]$, since if not, we can just let $I_k' \triangleq I_k \cap [-n, n]$.

Now, define

$$I_k^2 \triangleq \{x^2 : x \in I_k\}$$

Clearly, I_k^2 are all compact intervals, and if we write $I_k \triangleq [a_k, b_k]$, we have the following inequalities

$$\begin{cases} 0 \leq a_k \leq b_k \implies \left|I_k^2\right| = b_k^2 - a_k^2 = |I_k|\,(b_k + a_k) \leq 2n\,|I_k| \\ a_k \leq 0 \leq b_k \implies \left|I_k^2\right| = \max\{a_k^2, b_k^2\} \leq (b_k - a_k)^2 = |I_k|\,(b_k - a_k) \leq 2n\,|I_k| \\ a_k \leq b_k \leq 0 \implies \left|I_k^2\right| = a_k^2 - b_k^2 = |I_k|\,(-a_k - b_k) \leq 2n\,|I_k| \end{cases}$$

Note that $\{I_k^2\}_{k \in \mathbb{N}}$ is a compact interval cover of $\{x^2 : x \in Z_n\}$, we now see

$$\left|\{x^2 : x \in Z_n\}\right|_e \leq \sum_k \left|I_k^2\right| \leq 2n \sum |I_k| < \epsilon$$

■

1.4 HW3

Question 26

Let f be a simple function, taking its distinct values on disjoint sets $E_1, \dots, E_N$. Show that f is measurable if and only if $E_1, \dots, E_N$ are measurable.

Proof. WOLG, let f take value a_n on E_n and

$$a_1 < a_2 < \cdots < a_N$$

If $E_1, \dots, E_N$ are all measurable, we see that for each $a \in \mathbb{R}$

$$\{f \geq a\} = \{f \geq a_n\} = E_n \sqcup \cdots \sqcup E_N \text{ is measurable}$$

where n is the smallest integer such that $a_n \geq a$. We have prove the if part. To see the only if part hold true, observe that for all $n \in \{1, \dots, N-1\}$

$$E_n = \{f \geq a_n\} \setminus \{f \geq a_{n+1}\} \text{ is measurable}$$

and

$$E_N = \{f \geq a_N\} \text{ is measurable}$$

■

Question 27

Let f be defined and measurable on $\mathbb{R}^n$. If T is a nonsingular linear transformation of $\mathbb{R}^n$, show that $f(T\mathbf{x})$ is measurable. (If $E_1 = \{\mathbf{x} : f(\mathbf{x}) > a\}$, and $E_2 = \{\mathbf{x} : f(T\mathbf{x}) > a\}$, show that $E_2 = T^{-1}E_1$)

Proof. Fix $a \in \mathbb{R}$. We are required to show

$$\{\mathbf{x} : f(T\mathbf{x}) > a\} \text{ is measurable}$$

Because f is measurable, we know $\{\mathbf{x} : f(\mathbf{x}) > a\}$ is measurable. The proof then follows from noting

$$\{\mathbf{x} : f(T\mathbf{x}) > a\} = T^{-1}\Big(\{\mathbf{x} : f(\mathbf{x}) > a\}\Big)$$

and the fact that $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ as a linear transformation preserve measurability. ■

Question 28

Give an example to show that $\varphi \circ f$ may not be measurable if $\varphi, f : \mathbb{R} \to \mathbb{R}$ are measurable and finite. (Let F be the Cantor-Lebesgue function and let f be its inverse suitably defined. Let φ be the characteristic function of a set of measure zero whose image under F is not measurable.) Show that the same may be true even if f is continuous. (Let $g(x) = x + F(x)$ and consider $f = g^{-1}$)

Proof. Let $F : [0,1] \to [0,1]$ be the Cantor-Lebesgue function, $\mathcal{C} \subseteq [0,1]$ be the classical ternary Cantor set. Note that $F(\mathcal{C}) = [0,1]$. By axiom of choice, we can let $\mathcal{C}'$ be some subset of $\mathcal{C}$ such that $F|_{\mathcal{C}'} : \mathcal{C}' \to [0,1]$ is a bijection. We can now define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) \triangleq \begin{cases} (F|_{\mathcal{C}'})^{-1}(x) & \text{if } x \in [0,1] \\ x & \text{if } x \notin [0,1] \end{cases}$$

$f : \mathbb{R} \to \mathbb{R}$ is measurable because f is increasing. Let V be a non-measurable set contained by $[0,1]$, and let $E \triangleq f(V)$. Define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(x) \triangleq \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$$

Note that E is measurable because

$$V \subseteq [0,1] \implies E = f(V) = (F|_{\mathcal{C}'})^{-1}(V) \subseteq \mathcal{C}'$$

It then follows that $\varphi : \mathbb{R} \to \mathbb{R}$ is measurable. Lastly, to see $\varphi \circ f : \mathbb{R} \to \mathbb{R}$ is not measurable, observe that

$$(\varphi \circ f)^{-1}(\{1\}) = f^{-1}(E) = V \text{ is not measurable}$$

where the last inequality follows since $f|_V : V \to E$ is a bijection.

For the second part. Define $g : [0,1] \to [0,2]$ by

$$g(x) \triangleq x + F(x)$$

Because $F : [0,1] \to [0,1]$ is increasing, we may deduce

$$x < y \text{ and } x, y \in [0,1] \implies x + F(x) < y + F(y)$$

This implies g is strictly increasing. Note that g is continuous because g is the addition of two continuous function, and note that $g(0) = 0, g(1) = 2$. This allow us to deduce $g : [0,1] \to [0,2]$ is a bijection. Now, observe that $[0,1] \setminus \mathcal{C}$ is a countable union of disjoint open interval. For each connected components $I \subseteq [0,1] \setminus \mathcal{C}$, because F maps I to some constant, we see $g(I)$ is also an interval with the same length $|g(I)| = I$. Then from $|[0,1] \setminus \mathcal{C}| = 1$, we can deduce $|g([0,1] \setminus \mathcal{C})| = 1$, which implies $g(\mathcal{C}) = 1$. We then can let V be some non-measurable set contained by $g(\mathcal{C})$. Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) \triangleq \begin{cases} g^{-1}(x) & \text{if } x \in [0,2] \\ \frac{x}{2} & \text{if } x \notin [0,2] \end{cases}$$

$h : \mathbb{R} \to \mathbb{R}$ is measurable because it is increasing. Let $E \triangleq h(V)$. We see $E \subseteq \mathcal{C}$, which implies E is measurable, so when we define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(x) \triangleq \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

we see $\varphi : \mathbb{R} \to \mathbb{R}$ is also measurable. Lastly, to see $\varphi \circ h : \mathbb{R} \to \mathbb{R}$ is not measurable, observe

$$(\varphi \circ h)^{-1}(\{1\}) = h^{-1}(E) = V$$

■

Question 29

(a) Show that the limit of a decreasing (increasing) sequence of functions upper (lower) semicontinuous at $\mathbf{x}_0$ is upper (lower) semicontinuous at $\mathbf{x}_0$. In particular, the limit of a decreasing (increasing) sequence of functions continuous at $\mathbf{x}_0$ is upper (lower) semicontinuous at $\mathbf{x}_0$.

(b) Let f be upper semicontinuous and less than ∞ on $[a,b]$. Show that there exists continuous f_k on $[a,b]$ such that $f_k \searrow f$. (First show that there exist continuous f_k on $[a,b]$ such that $f_k \searrow f$)

Proof. **(a)** Let $f_n \searrow f$ and f_n be upper semicontinuous at $\mathbf{x}_0$. Fix ϵ. Because $f_n(\mathbf{x}_0) \searrow f(\mathbf{x}_0)$, we know there exists some N such that $f_N(\mathbf{x}_0) < f(\mathbf{x}_0) + \epsilon$. Because $f \leq f_N$ everywhere and f_N is upper semicontinuous at $\mathbf{x}_0$, we have

$$\limsup_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) \leq \limsup_{\mathbf{x} \to \mathbf{x}_0} f_N(\mathbf{x}) \leq f_N(\mathbf{x}_0) < f(\mathbf{x}_0) + \epsilon$$

Because ϵ can be arbitrary small, we have shown

$$\limsup_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) \leq f(\mathbf{x}_0)$$

i.e., f is also upper semicontinuous. The second part of question (a) ask us to prove the same thing with stronger hypothesis that f_n are continuous at $\mathbf{x}_0$, which implies f_n are upper semicontinuous at $\mathbf{x}_0$, so it immediately follows from what we have proved.

(b) Define $f_n : [a,b] \to [-\infty, \infty]$ by

$$f_n(x) \triangleq \sup_{p\in[a,b]} \Big(f(p) - n\,|x-p| \Big)$$

Fix n. We first show that $f_n : [a,b] \to [-\infty,\infty]$ is continuous. Fix ϵ, and let $x, y \in [a,b]$ satisfy $|x-y| < \frac{\epsilon}{n}$. For all $p \in [a,b]$, we may use reverse triangle inequality to show that

$$\Big|\Big(f(p) - n\,|x-p| \Big) - \Big(f(p) - n\,|y-p| \Big)\Big| = n\,\Big|\Big(|x-p| - |y-p| \Big)\Big| \le n\,|x-y| < \epsilon$$

This implies

$$f(p) - n\,|x-p| + \epsilon > f(p) - n\,|y-p| > f(p) - n\,|x-p| - \epsilon$$

Taking supremum on both side, we have

$$f_n(x) + \epsilon = \sup_{p\in[a,b]} \Big(f(p) - n\,|x-p| \Big) + \epsilon \ge \sup_{p\in[a,b]} \Big(f(p) - n\,|y-p| \Big) = f_n(y)$$

In summary,

$$f_n(x) + \epsilon \ge f_n(y) \ge f_n(x) - \epsilon$$

Because ϵ is arbitrary, we have shown f_n is (uniform) continuous. We now show f_n is decreasing. Fix $x \in [a,b]$ and $n < m$. Observe that for all $p \in [a,b]$

$$f(p) - m\,|x-p| < f(p) - n\,|x-p|$$

Taking supremum on both side, we have

$$f_m(x) = \sup_{p\in[a,b]} \Big(f(p) - m\,|x-p| \Big) \le \sup_{p\in[a,b]} \Big(f(p) - n\,|x-p| \Big) = f_n(x)$$

Because x is arbitrary, this implies f_n is indeed decreasing. Lastly, we show $f_n \to f$. Fix $x_0 \in [a,b]$ and ϵ. Because f is finite and upper semicontinuous on $[a,b]$, we may let $M \in \mathbb{R}$ be the upper bound of f on $[a,b]$ and let δ satisfy

$$\sup_{[x_0-\delta, x_0+\delta]} f(x) \le f(x_0) + \epsilon$$

Let $N > \frac{M-(f(x_0)+\epsilon)}{\delta}$. If $p \in [x_0 - \delta, x_0 + \delta]$, then $f(p) - N|p - x_0| \leq f(p) \leq f(x_0) + \epsilon$. If $p \notin [x_0 - \delta, x_0 + \delta]$, then

$$f(p) - N|p - x_0| \leq M - N|p - x_0| < f(x_0) + \epsilon$$

Letting p run through $[a, b]$, we now see

$$f_N(x_0) = \sup_{p\in[a,b]} \Big(f(p) - N|p - x_0| \Big) \leq f(x_0) + \epsilon$$

Because ϵ and x_0 are arbitrary, and f_n is decreasing, we have shown $f_n \to f$. ■

Question 30

Let f_k be a sequence of measurable function defined on a measurable set E with finite measure. If $|f_k(\mathbf{x})| \leq M_{\mathbf{x}} < \infty$ for all k and for each $\mathbf{x} \in E$, show that given $\epsilon > 0$, there exists closed $F \subseteq E$ and finite M such that $|E - F| < \epsilon$ and $|f_k(\mathbf{x})| \leq M$ for all k and $\mathbf{x} \in F$.

Proof. Define for all $n \in \mathbb{N}$

$$E_n \triangleq \bigcap_{k=1}^{\infty} \{f_k \leq n\}$$

Because f_k are measurable on E, we know E_n are measurable. Because for all $\mathbf{x} \in E, \sup_{n\in\mathbb{N}} |f_n(\mathbf{x})| < \infty$, we see that $E_n \nearrow E$. Then because E is of finite measure, we know there exists some N such that

$$|E \setminus E_N| < \frac{\epsilon}{2}$$

Because E_N is measurable, we know there exists some closed $F \subseteq E_N$ such that

$$|E_N \setminus F| < \frac{\epsilon}{2}$$

It then follows that

$$|E \setminus F| < \epsilon$$

and for all $\mathbf{x} \in F$,

$$\mathbf{x} \in F \implies \mathbf{x} \in E_N \implies |f_k(\mathbf{x})| < N \text{ for all } k \in \mathbb{N}$$

■

Question 31

If f is measurable on E, define $\omega_f(a) \triangleq |\{f > a\}|$ for $a \in \mathbb{R}$. If $f_k \nearrow f$, show $\omega_{f_k} \nearrow \omega_f$. If $f_k \xrightarrow{m} f$, show that $\omega_{f_k} \to \omega_f$ at each point of continuity of ω_f. (For the second part, show that if $f_k \xrightarrow{m} f$, then $\limsup_{k\to\infty} \omega_{f_k}(a) \leq \omega_f(a-\epsilon)$ and $\liminf_{k\to\infty} \omega_{f_k}(a) \geq \omega_f(a+\epsilon)$ for all $\epsilon > 0$).

Proof. Suppose $f_n \nearrow f$. We know that for all $a \in \mathbb{R}$

$$\{f_n > a\} \nearrow \{f > a\}$$

This implies

$$\omega_{f_n}(a) = |\{f_n > a\}| \nearrow |\{f > a\}| = \omega_f(a)$$

Because a is arbitrary, we have shown $\omega_{f_n} \nearrow \omega_f$.

Suppose $f_n \xrightarrow{m} f$. Fix ϵ. Observe that for all $a \in \mathbb{R}$ and $n \in \mathbb{N}$

$$\{f_n > a\} \subseteq \{|f_n - f| > \epsilon\} \cup \{f > a - \epsilon\}$$

Measuring both side, we get

$$\omega_{f_n}(a) = |\{f_n > a\}| \leq |\{|f_n - f| > \epsilon\}| + |\{f > a - \epsilon\}| = |\{|f_n - f| < \epsilon\}| + \omega_f(a-\epsilon)$$

Then because $f_n \xrightarrow{m} f$ implies $|\{|f_n - f| < \epsilon\}| \to 0$, if we take limit superior on both side, we get

$$\limsup_{n\to\infty} \omega_{f_n}(a) \leq \omega_f(a-\epsilon)$$

Again, observe that for all $a \in \mathbb{R}$ and $n \in \mathbb{N}$

$$\{f > a + \epsilon\} \subseteq \{|f_n - f| > \epsilon\} \cup \{f_n > a\}$$

Measuring both side, we get

$$\omega_f(a+\epsilon) = |\{f > a + \epsilon\}| \leq |\{|f_n - f| > \epsilon\}| + |\{f_n > a\}| = |\{|f_n - f| > \epsilon\}| + \omega_{f_n}(a)$$

Then because $f_n \xrightarrow{m} f$ implies $|\{|f_n - f| < \epsilon\}| \to 0$, if we take limit inferior on both side, we get

$$\omega_f(a+\epsilon) \leq \liminf_{n\to\infty} \omega_{f_n}(a)$$

Now, if ω_f is continuous at a, we have

$$\omega_f(a+\epsilon) \searrow \omega_f(a) \text{ and } \omega_f(a-\epsilon) \nearrow \omega_f(a) \text{ as } \epsilon \searrow 0$$

This then implies

$$\liminf_{n\to\infty} \omega_{f_n}(a) = \limsup_{n\to\infty} \omega_{f_n}(a) = \omega_f(a)$$

as we wished. ■

Question 32

If f is measurable and finite almost everywhere on $[a,b]$, show that given $\epsilon > 0$, there is a continuous g on $[a,b]$ such that $|\{f \neq g\}| < \epsilon$. Formulate and prove a similar result in $\mathbb{R}^n$ by combining Lusin's Theorem with the Tietze extension Theorem.

Proof. Let $E \triangleq \{x \in [a,b] : f(x) \in \mathbb{R}\}$. E is measurable because f is measurable on $[a,b]$. It is clear that f is indeed measurable on E. By Lusin's Theorem, there exists some closed set $F \subseteq E$ such that $|E \setminus F| < \epsilon$ and $f|_F : F \to \mathbb{R}$ is continuous. Because F is compact, (bounded by $[a,b]$), Tietze extension Theorem give us some continuous $g : [a,b] \to \mathbb{R}$ such that $g = f$ on F. It then follows that

$$\{f \neq g\} \subseteq [a,b] \setminus F$$

which give us the desired estimation

$$|\{f \neq g\}| \leq (b-a) - |F| = |E| - |F| < \epsilon$$

We may formulate the same result by

If f is measurable and finite almost everywhere on some compact $K \subseteq \mathbb{R}^d$, then for all ϵ, there exists continuous g on K such that $|\{f \neq g\}| < \epsilon$.

and give exactly the same argument to prove it. ■

1.5 HW4

Chapter 2

Complex Analysis HW

2.1 HW1

Theorem 2.1.1.

$$(1+i)^n, \frac{(1+i)^n}{n}, \frac{n!}{(1+i)^n} \text{ all diverge as } n \to \infty$$

Proof. Note that

$$|(1+i)^n| = 2^{\frac{n}{2}} \to \infty \text{ as } n \to \infty$$

This implies $(1+i)$ is unbounded, thus diverge.

Note that

$$\left|\frac{(1+i)^n}{n}\right| = \frac{(\sqrt{2})^n}{n}$$

Observe

$$\begin{aligned}\frac{(\sqrt{2})^n}{n} = \frac{[(\sqrt{2}-1)+1]^n}{n} &= \frac{\sum_{k=0}^n \binom{n}{k}(\sqrt{2}-1)^k}{n} \\ &\geq \frac{\binom{n}{2}(\sqrt{2}-1)^2}{n} = (n-1)[\frac{(\sqrt{2}-1)^2}{2}] \to \infty \text{ as } n \to \infty\end{aligned}$$

This implies $\frac{(1+i)^n}{n}$ is unbounded, thus diverge.

Note that

$$\left|\frac{n!}{(1+i)^n}\right| = \frac{n!}{(\sqrt{2})^n}$$

Note that for all $k \geq 8$, we have

$$\frac{k}{\sqrt{2}} \geq \frac{\sqrt{8}}{\sqrt{2}} = 2$$

This implies

$$\frac{n!}{(\sqrt{2})^n} = \prod_{k=1}^{n} \frac{k}{\sqrt{2}} = \frac{7!}{(\sqrt{2})^7} \prod_{k=8}^{n} \frac{k}{\sqrt{2}} \geq \frac{7!}{(\sqrt{2})^7} \prod_{k=8}^{n} 2 \geq \frac{7! 2^{n-8+1}}{(\sqrt{2})^7} \to \infty$$

which implies $\frac{n!}{(1+i)^n}$ is unbounded, thus diverge. ∎

Theorem 2.1.2.

$$n! z^n \text{ converge} \iff z = 0$$

Proof. If $z = 0$, then $n!z^n = 0$ for all n, which implies $n!z^n \to 0$. Now, suppose $z \neq 0$. Let $M \in \mathbb{N}$ satisfy $|z| > \frac{1}{M}$. Observe

$$|n!z^n| = \left| \prod_{k=1}^{n} kz \right| = \left| \prod_{k=1}^{2M-1} kz \right| \left| \prod_{k=2M}^{n} kz \right| \geq \left| \prod_{k=1}^{2M-1} kz \right| \left| \prod_{k=2M}^{n} 2Mz \right| \geq \left| \prod_{k=1}^{2M-1} kz \right| 2^{n-2M+1} \to \infty$$

This implies $n!z^n$ is unbounded, thus diverge. ∎

Theorem 2.1.3.

$$u_n \to u \implies v_n \triangleq \sum_{k=1}^{n} \frac{u_k}{n} \to u$$

Proof. Because

$$\sum_{k=1}^{n} \frac{u_k}{n} = \sum_{k \leq \sqrt{n}} \frac{u_k}{n} + \sum_{\sqrt{n} < k \leq n} \frac{u_k}{n}$$

It suffices to prove

$$\sum_{k \leq \sqrt{n}} \frac{u_k}{n} \to 0 \text{ and } \sum_{\sqrt{n} < k \leq n} \frac{u_k}{n} \to u \text{ as } n \to \infty$$

Because u_n converge, we can let M bound $|u_n|$. Observe

$$\left| \sum_{k \leq \sqrt{n}} \frac{u_k}{n} \right| \leq \sum_{k \leq \sqrt{n}} \left| \frac{u_k}{n} \right| \leq \sum_{k \leq \sqrt{n}} \frac{M}{n} \leq \frac{M\sqrt{n}}{n} = \frac{M}{\sqrt{n}} \to 0 \text{ as } n \to 0 \text{ (done)}$$

Because

$$\sum_{\sqrt{n}<k\leq n}\frac{u_k}{n}=\frac{n-\lceil\sqrt{n}\rceil+1}{n}\sum_{\sqrt{n}<k\leq n}\frac{u_k}{n-\lceil\sqrt{n}\rceil+1}$$

and

$$\lim_{n\to\infty}\frac{n-\lceil\sqrt{n}\rceil+1}{n}=1$$

We can reduce the problem into proving

$$\lim_{n\to\infty}\sum_{\sqrt{n}<k\leq n}\frac{u_k}{n-\lceil\sqrt{n}\rceil+1}=u$$

Fix ϵ. Let N satisfy that for all $n\geq N$, we have $|u_n-u|<\epsilon$. Then for all $n\geq N^2$, we have

$$\begin{aligned}\left|\left(\sum_{\sqrt{n}<k\leq n}\frac{u_k}{n-\lceil\sqrt{n}\rceil+1}\right)-u\right|&=\left|\sum_{\sqrt{n}<k\leq n}\frac{u_k-u}{n-\lceil\sqrt{n}\rceil+1}\right|\\&\leq\sum_{\sqrt{n}<k\leq n}\frac{|u_k-u|}{n-\lceil\sqrt{n}\rceil+1}\\&\leq\sum_{\sqrt{n}<k\leq n}\frac{\epsilon}{n-\lceil\sqrt{n}\rceil+1}=\epsilon\text{ (done)}\end{aligned}$$

■

2.2 Exercise 1

Let R be a complex algebra with 1_A and $a \in R$. Given a complex polynomial

$$f(Z) = a_0 + a_1 Z + \cdots + a_n Z^n,$$

we define the evaluation of f at a by

$$f(a) = a_0 1_A + a_1 a + \cdots + a_n a^n.$$

Question 33

Let $R = \mathbb{C}$ and $a = 1 + i$. Given $f(Z) = Z^3$. Evaluate $f(a)$.

Proof. $f(a) = (1+i)^3 = 2i(1+i) = -2 + 2i$ ∎

Question 34

Let $R = M_{2\times 2}(\mathbb{C})$ be the algebra of 2×2 complex matrices. Take

$$a = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

and $g(Z) = 3 + 2Z$. Evaluate $g(a)$.

Proof.

$$g(a) = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 2 & 5 \end{bmatrix}$$

∎

Question 35

Let R be the algebra of complex valued periodic functions of period 2π, i.e., $a \in R$ is a continuous function $a : \mathbb{R} \to \mathbb{C}$ so that $a(x + 2\pi) = a(x)$. Let $e(x) = \cos x + i \sin x$ and

$$h(Z) = 1 + Z + Z^2 + \cdots + Z^9.$$

Find $h(e)$.

Proof. Note that

$$\begin{aligned}(\cos x + i\sin x)(\cos y + i \sin y) &= (\cos x \cos y - \sin x \sin y) + i(\sin x \cos y + \cos x \sin y) \\ &= \cos(x+y) + i \sin(x+y)\end{aligned}$$

This give us

$$h(e) = \sum_{k=0}^{9} \cos(kx) + i \sin(kx)$$

$\blacksquare$

2.3 HW2

Theorem 2.3.1. (Root Test is Stronger Than Ratio Test)

$$\liminf_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| \leq \liminf_{n\to\infty}\sqrt[n]{|z_n|} \leq \limsup_{n\to\infty}\sqrt[n]{|z_n|} \leq \limsup_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right|$$

Proof. Fix ϵ and WOLG suppose $\liminf_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| > 0$. We prove

$$\liminf_{n\to\infty}\sqrt[n]{|z_n|} \geq \liminf_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| - \epsilon$$

Let $\alpha \in \mathbb{R}$ satisfy

$$\liminf_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| - \epsilon < \alpha < \liminf_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right|$$

Let N satisfy

$$\text{For all } n \geq N, \left|\frac{z_{n+1}}{z_n}\right| > \alpha$$

We then see

$$\sqrt[N+n]{|z_{N+n}|} \geq \sqrt[N+n]{|z_N|\,\alpha^n} = \alpha\left(\frac{|z_N|^{\frac{1}{N+n}}}{\alpha^{\frac{N}{N+n}}}\right) \to \alpha \text{ as } n \to \infty \text{ (done)}$$

The proof for the other side is similar. ∎

Question 36

Find the radius of convergence of the following series:

(a) $\sum \frac{z^n}{n}$.

(b) $\sum \frac{z^n}{n!}$.

(c) $\sum n! z^n$.

(d) $\sum n^k z^n$ where k is a positive integer.

(e) $\sum z^{n!}$.

Proof. We know

$$n^{\frac{1}{n}} = e^{\frac{\ln n}{n}} \to 1 \text{ as } n \to \infty \tag{2.1}$$

Equation 2.1 implies $n^{\frac{-1}{n}} \to 1$ as $n \to \infty$ and that $\sum \frac{z^n}{n}$ has radius of convergence 1. Equation 2.1 also implies $n^{\frac{k}{n}} \to 1$ and $\sum n^k z^n$ has radius of convergence 1.

We know

$$\lim_{n\to\infty} \frac{(n+1)!}{n!} = \infty$$

Then Theorem 2.3.1 tell us

$$\lim_{n\to\infty} (n!)^{\frac{1}{n}} = \infty \tag{2.2}$$

which implies that $\sum n! z^n$ has radius of convergence 0 and $\sum \frac{z^n}{n!}$ has radius of convergence ∞. Note that

$$\sum z^{n!} = \sum a_n z^n \text{ where } a_n = \begin{cases} 1 & \text{if } n = k! \text{ for some } k \\ 0 & \text{otherwise} \end{cases}$$

It is then clear that

$$\limsup_{n\to\infty} (a_n)^{\frac{1}{n}} = 1$$

and the radius is 1. ■

Question 37

The 0th order Bessel function $J_0(z)$ is defined by the power series

$$J_0(z) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2} \frac{z^n}{2n}$$

Find its radius of convergence.

Proof. Equation 2.1 and Equation 2.2 tell us

$$\lim_{n\to\infty} (2n(n!)^2)^{\frac{1}{n}} = \infty$$

which implies that the radius of convergence of $J_0(z)$ is ∞. ■

Theorem 2.3.2. (Abel's Test for Power Series) Suppose $a_n \to 0$ monotonically and $\sum a_n z^n$ has radius of convergence R .

$$\text{The power series } \sum a_n z^n \text{ at least converge on } \overline{D_R(0)} \setminus \{R\}$$

Proof. Note that

$$\sum \frac{a_n}{R^n} z^n \text{ has radius of convergence } R$$

Fix $z \in \overline{D_R(0)} \setminus \{R\}$. Note that

$$\left| \sum_{n=0} \frac{z^n}{R^n} \right| = \left| \frac{1 - (\frac{z}{R})^{N+1}}{1 - \frac{z}{R}} \right| \leq \frac{2}{\left|1 - \frac{z}{R}\right|} \text{ for all } N$$

It then follows from Dirichlet's Test that $\sum a_n (\frac{z}{R})^n$ converge. ∎

Question 38

Suppose that $\sum a_n z^n$ has radius of convergence R and let C be the circle $\{z \in \mathbb{C} : |z| = R\}$. Prove or disprove

(a) If $\sum a_n z^n$ converge at every point on C, except possibly one, then it converges absolutely every where on C

Proof. Consider $a_n \triangleq \frac{1}{n}$ for all $n \in \mathbb{N}$ and $a_0 \triangleq 1$. Then $\sum a_n z^n$ has convergence radius 1. Since $a_n \searrow 0$, it follows from Theorem 2.3.2, $\sum a_n z^n$ converge everywhere on $C \setminus \{1\}$. Observe that when $z = 1$, the series is just harmonic series, which diverge. ∎

Question 39

If $\sum a_n z^n$ has radius of convergence R, find the radius of convergence of

(a) $\sum n^3 a_n z^n$.

(b) $\sum a_n z^{3n}$.

(c) $\sum a_n^3 z^n$

Proof. Since $(n^3)^{\frac{1}{n}} \to 1$, we know $\sum n^3 a_n z^n$ also had radius of convergence R. We claim that the series $\sum a_n z^{3n}$ has convergence radius $R^{\frac{1}{3}}$. If $|z| < R^{\frac{1}{3}}$, then $\left|z^3\right| < R$ and thus

$$\sum a_n (z^3)^n \text{ converge}$$

and if $|z| > R^{\frac{1}{3}}$, then $\left|z^3\right| > R$ and

$$\sum a_n (z^3)^n \text{ diverge}$$

We have proved that $\sum a_n z^{3n}$ has convergence radius $R^{\frac{1}{3}}$.

Note that given a sub-sequence $|a_{n_k}|^{\frac{1}{n_k}}$,

$$|a_{n_k}|^{\frac{1}{n_k}} \text{ converge in extended reals if and only if } |a_{n_k}|^{\frac{3}{n_k}} \text{ converge in extended reals}$$

and if the former converge to L, then the latter converge to L^3. It now follows that

$$\limsup_{n\to\infty} |a_n^3| = (\limsup_{n\to\infty} |a_n|)^3 = \frac{1}{R^3}$$

It now follows that $\sum a_n^3 z^n$ has convergence radius R^3. ∎

Theorem 2.3.3. (Summation by Part)

$$\begin{aligned} f_n g_n - f_m g_m &= \sum_{k=m}^{n-1} f_k \Delta g_k + g_k \Delta f_k + \Delta f_k \Delta g_k \\ &= \sum_{k=m}^{n-1} f_k \Delta g_k + g_{k+1} \Delta f_k \end{aligned}$$

Proof. The proof follows induction which is based on

$$f_m g_m + f_m \Delta g_m + g_m \Delta f_m + \Delta f_m \Delta g_m = f_{m+1} g_{m+1}$$

∎

Question 40

Prove that, for $z \neq 1$

$$\sum_{n=1}^{k} \frac{z^n}{n} = \frac{z}{1-z} \Big(\sum_{n=1}^{k-1} \frac{1}{n(n+1)} - \sum_{n=1}^{k-1} \frac{z^n}{n(n+1)} + \frac{1-z^k}{k} \Big)$$

Show that the series $\sum \frac{z^n}{n}$ and $\sum \frac{z^n}{n(n+1)}$ have radius of convergence 1; that the latter series converge everywhere on $|z| = 1$, while the former converges everywhere on $|z| = 1$ except $z = 1$.

Proof. We prove by induction. The base case $k = 1$ is trivial. Suppose the equality hold when $k = m$. The difference of the left hand side is clearly $\frac{z^{m+1}}{m+1}$, and the difference of the right hand side is

$$\frac{z}{1-z}\left(\frac{1}{m(m+1)}-\frac{z^m}{m(m+1)}+\frac{1-z^{m+1}}{m+1}-\frac{1-z^m}{m}\right)$$
$$=\frac{z}{1-z}\cdot\frac{1-z^m+m-mz^{m+1}+(m+1)z^m-(m+1)}{m(m+1)}$$
$$=\frac{z}{1-z}\cdot\frac{-mz^{m+1}+mz^m}{m(m+1)}=\frac{z(z^m-z^{m+1})}{(1-z)(m+1)}=\frac{z^{m+1}}{m+1}$$

The fact that both series have radius of convergence 1 follows from $n^{\frac{1}{n}}\to 1$. Both of them converge on $\overline{D_1(0)}\setminus\{1\}$ by Theorem 2.3.2. The former clearly diverge at $z=1$, since it would be a harmonic series, and the latter converge at $z=1$ by comparison test with $\frac{1}{n(n+1)}\leq\frac{1}{n^2}$ for all $n\in\mathbb{N}$. ■

Question 41

Suppose that the power series $\sum a_n z^n$ has a recurring sequence of coefficients; that is $a_{n+k}=a_n$ for some fixed positive integer k and all n. Prove that the series converge for $|z|<1$ to a rational function $\frac{p(z)}{q(z)}$ where p,q are polynomials, and the roots of q are all on the unit circle. What happens if $a_{n+k}=\frac{a_n}{k}$ instead?

Proof. Let

$$L^-\triangleq\min\{|a_0|,\dots,|a_{k-1}|\}\text{ and }L^+\triangleq\max\{|a_0|,\dots,|a_{k-1}|\}$$

We see that

$$1=\liminf_{n\to\infty}(L^-)^{\frac{1}{n}}\leq\lim_{n\to\infty}|a_n|^{\frac{1}{n}}\leq\limsup_{n\to\infty}(L^+)^{\frac{1}{n}}=1$$

It then follows that $\sum a_n z^n$ has convergence radius 1. Now observe that for $|z|<1$, we have

$$z^k\sum_{n=0}^{\infty}a_nz^n=\sum_{n=k}^{\infty}a_nz^n=\sum_{n=0}^{\infty}a_nz^n-\sum_{n=0}^{k-1}a_nz^n$$

This now implies

$$\sum_{n=0}^{\infty}a_nz^n=\frac{\sum_{n=0}^{k-1}a_nz^n}{1-z^k}$$

Since $q(z) = 1 - z^k$, clearly the roots are all on the unit circle. Suppose now $b_n \triangleq a_n$ for all $n < k$ and $b_{n+k} \triangleq \frac{b_n}{k}$ for all $n \geq k$. We then have

$$b_n = \frac{a_n}{k^{q(n)}} \text{ where } q \text{ is the largest integer such that } qk \leq n$$

Note that $n - q(n)$ is always smaller than k. It then follows that

$$(k^{q(n)})^{\frac{1}{n}} = k^{\frac{q(n)}{n}} = k \cdot k^{\frac{q(n)-n}{n}} \to k$$

We then see that

$$\limsup_{n\to\infty} |b_n|^{\frac{1}{n}} = \frac{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}}{k} = \frac{1}{k}$$

It then follows that $\sum b_n z^n$ has convergence radius k. Now observe that for $|z| < k$, we have

$$z^k \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} b_n z^{n+k} = \frac{1}{k} \sum_{n=k}^{\infty} b_n z^n = \frac{1}{k}\Big(\sum_{n=0}^{\infty} b_n z^n - \sum_{n=0}^{k-1} b_n z^n\Big)$$

This now implies

$$\sum_{n=0}^{\infty} b_n z_n = \frac{\sum_{n=0}^{k-1} b_n z^n}{k(\frac{1}{k} - z^k)}$$

$\blacksquare$

2.4 Exercises 2

Let (M, d) be a metric space, $x \in M$ and F a subset of M.

Question 42

Prove that the following statements are equivalent

(a) There exists a sequence $\{x_n\}$ in F with $x_n \neq x$ so that $\lim_{n\to\infty} x_n = x$.

(b) For any ϵ, the intersection of $B'_\epsilon(x) \triangleq \{y \in M : 0 < d(x, y) < \epsilon\}$ and F are non-empty.

Proof. If (a) is true, then for all ϵ there exists some $x_n \in F$ such that $d(x_n, x) < \epsilon$. Because $x_n \neq x$, we know that $0 < d(x_n, x)$. This now implies $x_n \in B'_\epsilon(x) \cap F$.

If (b) is true, then for all n, we simply select a point in $x_n \in B'_{\frac{1}{n}}(x) \cap F$. After such selection, we see that $x_n \neq x$ and for all ϵ, if $n > \frac{1}{\epsilon}$, then $x_n \in B'_\epsilon(x) \cap F$. ■

Question 43

Prove that the following statements are equivalent

(a) F contain all its limit point.

(b) $U = M \setminus F$ is open.

Proof. If (a) is true, then for all $p \in U$, we know that p is not a limit point of F, then from the first question, we know that there exists ϵ such that $B'_\epsilon(x) \cap F = \varnothing$. Because $x \in U = M \setminus F$ also does not belong x, we also know that $B_\epsilon(x) \cap F = \varnothing$. This then implies that $B_\epsilon(x) \subseteq U$, since $U = M \setminus F$. We have proved that U is open.

If (b) is true, then for arbitrary $p \notin F$, we know there exists some ϵ such that $B_\epsilon(x)$ is disjoint with F. Because $B'_\epsilon(x)$ is a subset of $B_\epsilon(x)$, we can deduce that $B_\epsilon(x) \cap F = \varnothing$, which from the first question implies that p is not a limit point of F. Because p is arbitrary selected from $M \setminus F$, we have proved that none of the points in $M \setminus F$ is a limit point of F. This implies that if F has any limit point, then F must contain that limit point. ■

Question 44

Prove the following statements

(a) M and $\varnothing$ are closed.

(b) The intersection of any family of closed subsets of M is closed.

(c) The union of finitely many closed subsets of M is closed.

Proof. It is clear that M is open and trivially true that $\varnothing$ is open. It then follows from the second question that M and $\varnothing$ are both closed.

Let (F_α) be a collection of closed subsets of M. Arbitrary select a limit point x of $\bigcap F_\alpha$. Let $\{x_n\}$ be a sequence in $\bigcap F_\alpha$ with $x_n \neq x$ so that $\lim_{n\to\infty} x_n = x$. Arbitrary select β. Note that $\{x_n\}$ is also a sequence in F_β that converge to x with $x_n \neq x$. This now implies that x is a limit point of F_β. Then because F_β is closed, we see that $x \in F_\beta$. Now, since β is arbitrary selected, we see $x \in \bigcap_\alpha F_\alpha$. Because x is arbitrary, we have proved $\bigcap F_\alpha$ contained all its limit points.

Let $\{F_1, \dots, F_N\}$ be a collection of closed subsets of M. Let x be an arbitrary limit point of $\bigcup_{n=1}^N F_n$. Let $\{x_n\}$ be a sequence in $\bigcup_{n=1}^N F_n$ with $x_n \neq x$ converging to x. It is clear that there must exists some $j \in \{1, \dots, N\}$ such that F_j contain infinite terms of $\{x_n\}$, i.e., there exists a subsequence x_{n_k} such that $x_{n_k} \in F_j$ for all k. Because $\lim_{k\to\infty} x_{n_k} = \lim_{n\to\infty} x_n = x$, we now see that x is a limit point of F_j. It then follows from F_j being closed that $x \in F_j \subseteq \bigcup_{n=1}^N F_n$. Because x is arbitrary, we have proved that $\bigcup_{n=1}^N F_n$ is closed. ■

2.5 Exercise 3

Question 45

Provide a counterexample to the following statement: Suppose

$$f(z) = u(x,y) + iv(x,y)$$

is defined in a neighborhood of $z_0 = a + ib$. If the partial derivatives of u and v exist at (a,b) and satisfy the Cauchy-Riemann equations $u_x(a,b) = v_y(a,b)$ and $u_y(a,b) = -v_x(a,b)$, then f is holomorphic at z_0.

Proof. WOLG, let $a = b = 0$ and define

$$u(x,y) \triangleq \begin{cases} x & \text{if } y = 0 \\ y & \text{if } x = 0 \\ 0 & \text{if otherwise} \end{cases} \quad \text{and } v(x,y) \triangleq \begin{cases} -x & \text{if } y = 0 \\ y & \text{if } x = 0 \\ 0 & \text{if otherwise} \end{cases}$$

It is clear that

$$u_x = 1 = v_y \text{ and } u_y = 1 = -v_x \text{ at } (0,0)$$

but

$$\lim_{t\to 0; t\in\mathbb{R}} \frac{f(t) - f(0)}{t - 0} = \lim_{t\to 0; t\in\mathbb{R}} \frac{t - it}{t} = 1 - i$$

together with

$$\lim_{t\to 0; t\in\mathbb{R}} \frac{f(t+it) - f(0)}{t + it} = \lim_{t\to 0; t\in\mathbb{R}} \frac{0}{t + it} = 0$$

shows that f is not holomorphic at $(0,0)$. ∎

Question 46

Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Suppose that f is differentiable at (a,b) and that $f'(x) = 0$ for all $x \in (a,b)$. Prove that f is a constant function.

Proof. Assume $f(x) \neq f(y)$ for some $x \neq y \in [a,b]$. By MVT, we then see there exists some t between x, y (thus $t \in (a,b)$) such that $f'(t) = \frac{f(y)-f(x)}{y-x} \neq 0$, which is impossible. CaC ∎

Question 47

Let $B = B_R(x_0)$ be the open ball in $\mathbb{R}^n$ centered at x_0 with radius $R > 0$. Prove that if $f : B \to \mathbb{R}$ is a differentiable function such that $\nabla f = 0$ on B, then f is a constant function.

Proof. Let $\mathbf{x}, \mathbf{y}$ be two points in B. We are required to show $f(\mathbf{x}) = f(\mathbf{y})$. Define $g : [0, 1] \to B$ by

$$g(t) \triangleq f(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))$$

Note that g is well-defined since B is convex. Because f is differentiable, we have

$$g'(t) = \nabla f(\mathbf{x} + y(\mathbf{y} - \mathbf{x})) \cdot (\mathbf{y} - \mathbf{x}) = 0$$

It then follows from the last question that

$$f(\mathbf{y}) = g(1) = g(0) = f(\mathbf{x})$$

■

Question 48

Let U be an open subset of $\mathbb{R}^n$. A function $f : U \to \mathbb{R}$ is called **locally constant** if, for each $x \in U$, there exists an open neighborhood W of x such that $W \subseteq U$ and $f : W \to \mathbb{R}$ is constant on W. Prove that f is locally constant function if and only if $\nabla f = 0$ on U.

Proof. The if part follows from the last question by taking some small enough r such that $B_r(x) \subseteq U$. We now prove the only if part. Fix arbitrary $x \in U$. Because f is locally constant at x, we know there exists some $B_r(x)$ such that f is constant on $B_r(x)$. Therefore, we can let $c \in \mathbb{R}$ satisfy

$$f(y) = c \text{ for all } y \in B_r(x)$$

To see $\nabla f(x) = 0$, just observe that for arbitrary axis $\mathbf{j}$

$$f_{\mathbf{j}}(x) = \lim_{t \to 0} \frac{f(x + t\mathbf{j}) - f(x)}{t} = 0$$

since $f(x + t\mathbf{j}) = c = f(x)$ as long as $|t| < r$. Because $\mathbf{j}$ is arbitrary, it then follows that $\nabla f(x) = 0$, and because x is arbitrary selected from U, we have proved ∇f is 0 on U. ■

Question 49

Let D be an open, connected subset of $\mathbb{R}^n$. Prove that if $f : D \to \mathbb{R}$ is a locally constant function, then f is a constant function.

Proof. Observe that for all $p \in D$, f is constant on some neighborhood around p, thus continuous at p. We have shown $f : D \to \mathbb{R}$ is continuous. Fix $p \in D$, and let $c \triangleq f(p)$. Because $\{c\}$ is closed in $\mathbb{R}$ and $f : D \to \mathbb{R}$ is continuous, we know $f^{-1}(\{c\})$ is closed in D. We now show $f^{-1}(\{c\})$ is open in D. Fix arbitrary $q \in f^{-1}(\{c\})$. Because $f : D \to \mathbb{R}$ is locally constant, we know there exists some r such that $B_r(q) \subseteq D$ and f sends $B_r(q)$ to $f(q) = c$. It follows that $B_r(q) \subseteq f^{-1}(\{c\})$. Because q is arbitrary selected from $f^{-1}(\{c\})$, we have shown $f^{-1}(\{c\})$ is open in D.

In conclusion, we have shown $f^{-1}(\{c\})$ is both open and closed in D. It then follows from D being connected that $f^{-1}(\{c\}) = D$ or $\varnothing$. Because $p \in f^{-1}(\{c\})$, we can deduce $f^{-1}(\{c\}) = D$, i.e., f send all points in D to c, a constant function. ■

2.6 HW 3

Question 50

Let $\mathbb{C}_\pi \triangleq \{z \in \mathbb{C} : z \notin \mathbb{R}_0^-\}$. Prove that $\mathbb{C}_\pi$ is a domain. Define $r : \mathbb{C}_\pi \to \mathbb{C}$ by $(r(z))^2 = z$ and $\operatorname{Re} r(z) > 0$. Prove that r is continuous on $\mathbb{C}_\pi$ and $r'(z) = \frac{1}{2r(z)}$.

Proof. It is clear that $\mathbb{C}_\pi$ is non-empty and open. To see $\mathbb{C}_\pi$ is path-connected, observe that for all point $x + iy \in \mathbb{C}_\pi$, we can join $x + iy$ with 1 linearly by defining $\gamma : [0, 1] \to \mathbb{C}_\pi$ by

$$\gamma(t) \triangleq 1 + t(x + iy - 1)$$

We have proved $\mathbb{C}_\pi$ is a domain. Note that

$$\mathbb{C}_\pi = \{a + bi \in \mathbb{C} : a > 0 \text{ and } b \in (-\pi, \pi)\}$$

and the exact definition of $r : \mathbb{C}_\pi \to \mathbb{C}$ is

$$r(e^{a+bi}) \triangleq e^{\frac{a+bi}{2}}$$

This implies r is continuous. Compute

$$1 = \frac{d}{dz} z = \frac{d}{dz}(r(z))^2 = 2r(z)r'(z)$$

This give us $r'(z) = \frac{1}{2r(z)}$. ∎

Theorem 2.6.1. (Conjugated Polynomial)

$$\overline{z^n} \text{ is holomorphic at } 0 \text{ for all } n > 1$$

Proof. If we write

$$u + iv = \overline{(x + iy)^n} = (x - iy)^n$$

Because $n > 1$, we see from binomial Theorem that $u \in \mathbb{R}[x, y]$ is a polynomial with two indeterminate x, y whose terms all have degree greater than 1. Thus, both $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ are polynomial with two indeterminate x, y whose terms all have degree greater than 0. This implies

$$\frac{\partial u}{\partial x}(0) = \frac{\partial u}{\partial y}(0) = 0$$

Similar arguments shows that

$$\frac{\partial v}{\partial x}(0) = \frac{\partial v}{\partial y}(0) = 0$$

Because $u, v \in \mathbb{R}[x, y]$ are real multivariate polynomial, they are obviously real differentiable. Finally, we can use Cauchy-Riemann criteria to conclude that $\overline{z^n} = u + iv$ is holomorphic at 0. ∎

Question 51

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial. Prove that the function $g : \mathbb{C} \to \mathbb{C}$ defined by

$$g(z) \triangleq \overline{f(\overline{z})}$$

is holomorphic everywhere, but the function $h : \mathbb{C} \to \mathbb{C}$ defined by

$$h(z) \triangleq \overline{f(z)}$$

is holomorphic at 0 if and only if $f'(0) = 0$.

Proof. We can write

$$f(z) \triangleq \sum_{n=0}^{N} c_n z^n$$

and compute

$$g(z) = \sum_{n=0}^{N} \overline{c_n} z^n$$

We have shown $g : \mathbb{C} \to \mathbb{C}$ is a polynomial. It follows that g is holomorphic on $\mathbb{C}$. Compute

$$h(z) = \sum_{n=2}^{N} \overline{c_n z^n} + \overline{c_1 z} + \overline{c_0} \text{ and } f'(0) = c_1$$

Theorem 2.6.1 shows that

$$\sum_{n=2}^{N} \overline{c_n z^n} + \overline{c_0} \text{ is holomorphic at } 0$$

Note that $\overline{z}$ is not holomorphic at 0 since if we write $u + iv = \overline{z}$, then

$$\frac{\partial u}{\partial x} = 1 \neq -1 = \frac{\partial v}{\partial y}$$

The claim "h is holomorphic at 0 if and only if $f'(0) = 0$" then follows. ■

Question 52

Define

(a) $u, v : \mathbb{R}^2 \to \mathbb{R}$ by

$$u(x,y) = x^3 - 3xy^2 \text{ and } v(x,y) = 3x^2y - y^3$$

(b) $u, v : \{(x,y) \in \mathbb{R}^2 : x > 0\} \to \mathbb{R}$ by

$$u(x,y) = \frac{\ln(x^2+y^2)}{2} \text{ and } v(x,y) = \sin^{-1}(\frac{y}{\sqrt{x^2+y^2}})$$

Verify the Cauchy-Riemann Equation for these functions and state a complex function shows real and imaginary parts are u, v.

Proof. For (a), compute

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 3x^2 - 2y^2 \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -6xy$$

and observe

$$u + iv = (x+iy)^3$$

For (b), compute

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{x}{x^2+y^2} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{y}{x^2+y^2}$$

and observe

$$x + iy = e^{2u+iv}$$

which implies the function map z to $(\operatorname{Log} z) - \frac{\ln|z|}{2}$. ∎

Question 53

Let $f(z) = \sqrt{|xy|}$. Show that f satisfy the Cauchy-Riemann equation at 0, yet $f'(0)$ does not exists. Explain why.

Proof. Observe that

$$f(x) = f(iy) = 0 \text{ for all } x, y \in \mathbb{R}$$

We then have Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$$

Then if f is holomorphic at 0, we should have $f'(0) = 0$, but we can compute

$$\lim_{t\to 0; t\in\mathbb{R}^+} \frac{f(t+ti) - f(0)}{t+ti} = \lim_{t\to 0; t\in\mathbb{R}^+} \frac{t}{t+ti} = \frac{1}{1+i} \neq 0$$

which implies f is not holomorphic at 0. The reason that f satisfy the Cauchy-Riemann equation yet isn't holomorphic is that its real part when treated as a function from $\mathbb{R}^2$ to $\mathbb{R}$ is not differentiable at 0, as we have shown. (Note that $f = \operatorname{Re} f$) ∎

Question 54

Suppose that $f(z) = \sum a_n z^n$ is entire and satisfy

$$f' = f \text{ and } f(0) = 1$$

Find a_n. Show that

$$f(a+b) = f(a)f(b) \text{ for all } a, b \in \mathbb{C}$$

and compute $f(1)$ to five decimal points.

Proof. $f(0) = 1$ implies $a_0 = 1$. $f' = f$ implies $(n+1)a_{n+1} = a_n$, which give us

$$a_n = \frac{1}{n!} \text{ for all } n \geq 0$$

Fix $a, b \in \mathbb{C}$. Define $g : \mathbb{C} \to \mathbb{C}$ by

$$g(z) \triangleq f(a+b-z)f(z)$$

Compute

$$\begin{aligned} g'(z) &= -f'(a+b-z)f(z) + f(a+b-z)f'(z) \\ &= -f(a+b-z)f(z) + f(a+b-z)f(z) = 0 \end{aligned}$$

This implies g is constant. We then can deduce

$$f(a)f(b) = g(a) = g(0) = f(a+b)f(0) = f(a+b)$$

We know

$$f(1) = \sum_{n=0} \frac{1}{n!} = 2.7182818.....$$

∎

Chapter 3

PDE intro

3.1 1.2 First Order Linear Equations

(Principle of Geometric Method) Given a first order homogeneous linear PDE with the form

$$u_x + g(x,y)u_y = 0$$

We know if a curve $\gamma(x) = (x,y)$ satisfy

$$\gamma'(x) = c_x(1, g(x,y)) \text{ for some } c_x$$

Then

$$(u \circ \gamma)'(x) = 0 \text{ for all } x$$

Since

$$\gamma'(x) = (1, \frac{dy}{dx})$$

To find γ, we only wish to solve

$$\frac{dy}{dx} = g(x,y)$$

Question 55

Solve

$$(1 + x^2)u_x + u_y = 0$$

Proof. The ODE

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

has general solution $y = \arctan x + C$, so

$$u(x,y) = f(y - \arctan x)$$

■

Question 56

Solve

$$\begin{cases} yu_x + xu_y = 0 \\ u(0,y) = e^{-y^2} \end{cases}$$

In which region of the xy plane is the solution uniquely determined?

Proof. We are required to solve the ODE

$$\frac{dy}{dx} = \frac{x}{y}$$

Separating the variables and integrate

$$\int yy'dx = \int xdx \implies \frac{y^2}{2} = \frac{x^2}{2} + C$$

This now implies

$$u(x,y) = f(y^2 - x^2)$$

Initial Condition then give us

$$u = e^{x^2-y^2}$$

■

Question 57

Solve the equation

$$u_x + u_y = 1$$

Proof. Clearly $u = \frac{x}{2} + \frac{y}{2}$ is a solution. We now solve the PDE

$$v_x + v_y = 0$$

Solving the ODE

$$\frac{dy}{dx} = 1$$

we have

$$y = x + C$$

Thus the general solution is

$$u = \frac{x}{2} + \frac{y}{2} + f(y - x)$$

■

Question 58

Solve

$$\begin{cases} u_x + u_y + u = e^{x+2y} \\ u(x, 0) = 0 \end{cases}$$

Proof. Let $\gamma(x) = x + C$, we have

$$(u \circ \gamma)' + (u \circ \gamma) = e^{3x+2C}$$

We now solve the ODE

$$y' + y = e^{3x+2C}$$

The particular solution is clearly

$$y = \frac{1}{4}e^{3x+2C}$$

Thus the general solution is

$$y = \frac{e^{3x+2C}}{4} + \tilde{C}e^{-x}$$

We then now know

$$(u \circ \gamma)(x) = \frac{e^{3x+2C}}{4} + \tilde{C}e^{-x} \tag{3.1}$$

In other words,

$$u(x, x+C) = \frac{e^{3x+2C}}{4} + \tilde{C}e^{-x}$$

Putting in the initial conditions, we have

$$0 = u(-C, 0) = \frac{e^{-C}}{4} + \tilde{C}e^{C}$$

This implies

$$\tilde{C} = \frac{-e^{-2C}}{4}$$

Putting the back into Equation 3.1, we have

$$u(x, x+C) = \frac{e^{3x+2C} - e^{-x-2C}}{4}$$

So

$$u(x, y) = \frac{e^{3x+2(y-x)} - e^{-x-2(y-x)}}{4}$$

■

Question 59

Use the coordinate method to solve the equation

$$u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2$$

Proof. Let

$$\begin{cases} \xi \triangleq 2x - y \\ \eta \triangleq x + 2y \end{cases}$$

We see

$$\begin{cases} u_\xi = \frac{2}{5}u_x - \frac{1}{5}u_y \\ u_\eta = \frac{1}{5}u_x + \frac{2}{5}u_y \end{cases}$$

We then can rewrite

$$5u_\eta + \xi u = \xi\eta \tag{3.2}$$

which clearly have particular solution

$$u = \eta - \frac{5}{\xi}$$

To solve the linear homogeneous PDE

$$5u_\eta + \xi u = 0$$

Observe that for all fixed ξ, the PDE is just an ODE whose solution is exactly $u = C_\xi e^{\frac{-\xi\eta}{5}}$. We now know the general solution for PDE 3.2 is exactly

$$u = \eta - \frac{5}{\xi} + f(\xi)e^{\frac{-\xi\eta}{5}}$$

Then

$$u = x + 2y - \frac{5}{2x - y} + e^{\frac{-(2x-y)(x+2y)}{5}} f(2x - y)$$

$\blacksquare$

3.2 1.4 Initial and Boundary Condition

Question 60

Find a solution of

$$\begin{cases} u_t = u_{xx} \\ u(x,0) = x^2 \end{cases}$$

Proof. Clearly $u = x^2 + 2t$ suffices. ∎

3.3 1.5 Well Posed Problems

Given a vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$, Divergence Theorem shows

$$\iiint_D \nabla \cdot F dV = \iint_{\text{bdy } D} F \cdot \mathbf{n} dS$$

Then if F is the gradient of some scalar field $f : \mathbb{R}^3 \to \mathbb{R}$, we have

$$\iiint_D \Delta f dV = \iint_{\text{bdy } D} \frac{\partial f}{\partial \mathbf{n}} dS$$

Question 61

Consider the ODE

$$\begin{cases} u'' + u = 0 \\ u(0) = 0 \text{ and } u(L) = 0 \end{cases}$$

Is the solution unique? Does the answer depend on L?

Proof. We know the general solution space is exactly spanned by $\cos x$ and $\sin x$. Because

(a) $u(0) = 0$.

(b) $\sin 0 = 0$

(c) $\cos 0 = 1$

we know the solution of our original ODE must be of the form

$$u(x) = C \sin x$$

This implies that the solution is unique if and only if $2\pi \not\equiv L \pmod{2\pi}$ ■

Question 62

Consider the ODE

$$\begin{cases} u'' + u' = f \\ u'(0) = u(0) = \frac{1}{2}(u'(l) + u(l)) \end{cases}$$

where f is given.

(a) Is the solution unique?

(b) Does a solution necessarily exist, or is there a condition that f must satisfy for existence?

Proof. The solution space of linear homogeneous ODE $u'' + u' = 0$ is spanned by e^{-x} and constant. If we add in the initial condition $u'(0) = u(0)$, then the solution space become the subspace spanned by $e^{-x} - 2$. One can check that if $u \in \text{span}(e^{-x} - 2)$, then

$$u(0) = \frac{1}{2}(u'(l) + u(l)) \text{ for all } l \in \mathbb{R}$$

We now know the solution of the original ODE is not unique, since any solution added by $e^{-x} - 2$ is again a solution.

Integrating both side on $[0, l]$, we see that given the boundary conditions, f must satisfy

$$\begin{aligned} \int_0^l f(x)dx &= \int_0^l u'' + u'dx \\ &= u(l) + u'(l) - u(0) - u'(0) = 0 \end{aligned}$$

■

Question 63

Consider the Neumann problem

$$\Delta u = f(x, y, z) \text{ in } D \text{ and } \frac{\partial u}{\partial n} = 0 \text{ on } \text{ bdy } D$$

(a) What can we add to any solution to get another solution?

(b) Use the divergence Theorem and the PDE to show that

$$\iiint_D f(x, y, z)dxdydz = 0$$

is a necessary condition for the Neumann problem to have a solution.

Proof. Clearly, constants suffices, and observe

$$\iiint_D fdxdydx = \iiint_D \Delta udxdydz = \iiint_D \nabla \cdot (\nabla u)dxdydz = \iint_{\text{bdy } D} \nabla u \cdot \mathbf{n}dS = 0$$

■

Question 64

Consider the equation

$$u_x + yu_y = 0$$

with the boundary condition $u(x,0) = \varphi(x)$.

(a) $\varphi(x) = x \implies$ no solution exists

(b) $\varphi(x) = 1 \implies$ multiple solutions exist.

Proof. Using the geometric method, we see the characteristic curve is exactly $y = \tilde{C}e^x$. Thus the general solution is of the form

$$u(x,y) = f(e^{-x}y)$$

The boundary condition implies

$$\varphi(x) = u(x,0) = f(0)$$

The result then follows. ∎

3.4 1.6 Types of Second-Order Equations

Consider the constant coefficient Linear PDE

$$u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0$$

Ignoring the term with order less than 2, we have

$$u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} = 0$$

Or equivalently

$$(\partial_x + a_{12}\partial_y)^2u + (a_{22} - a_{12}^2)\partial_{yy}u = 0$$

Now, if we set

$$x \triangleq \xi \text{ and } y \triangleq a_{12}\xi + (\left|a_{22} - a_{12}^2\right|)^{\frac{1}{2}}\eta$$

we see that

$$\begin{cases} a_{22} - a_{12}^2 > 0 \implies u_{\xi\xi} + u_{\eta\eta} = 0 \quad \text{(Elliptic)} \\ a_{22} - a_{12}^2 = 0 \implies u_{\xi\xi} = 0 \quad \text{(Parabolic)} \\ a_{22} - a_{12}^2 < 0 \implies u_{\xi\xi} - u_{\eta\eta} = 0 \quad \text{(Hyperbolic)} \end{cases}$$

More generally, if we are given

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} = 0$$

then the discriminant is exactly

$$a_{11}a_{22} - a_{12}^2$$

Question 65

What is the type of each of the following equations.

(a) $u_{xx} - u_{xy} + u_{yy} + \cdots + u = 0$.

(b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$

Proof. The discriminant for (a) and (b) are respectively $\frac{3}{4}$ are 0, thus elliptic and parabolic. ∎

Question 66

Find the regions in the xy plane where the equation

$$(1+x)u_{xx} + 2xyu_{xy} - y^2u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch them.

Proof. The discriminant is exactly

$$\begin{aligned}(xy)^2 - (1+x)(-y^2) &= x^2y^2 + xy^2 + y^2 \\ &= y^2(x^2+x+1) \\ &= y^2[(x+\frac{1}{2})^2 + \frac{3}{4}]\end{aligned}$$

It then follows that the equation is parabolic if and only if $y = 0$, and hyperbolic if and only if $y \neq 0$.

■

Question 67

Reduce the elliptic equation

$$u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u = 0$$

to the form

$$v_{xx} + v_{yy} + cv = 0$$

by a change of dependent variable

$$u \triangleq ve^{\alpha x + \beta y}$$

then a change of scale

$$y' = \gamma y$$

Proof. The original elliptic equation can be written in the form

$$e^{\alpha x+\beta y}\Big[v_{xx} + 2\alpha v_x + \alpha^2 v + 3(v_{yy} + 2\beta v_y + \beta^2 v) - 2(v_x + \alpha v) + 24(v_y + \beta v) + 5v\Big] = 0$$

which implies

$$v_{xx} + 3v_{yy} + (2\alpha - 2)v_x + (6\beta + 24)v_y + (\alpha^2 + 3\beta^2 - 2\alpha + 24\beta + 5)v = 0$$

Letting $\alpha \triangleq 1$ and $\beta \triangleq -4$, we see

$$v_{xx} + 3v_{yy} + \tilde{c}v = 0$$

Letting $y \triangleq \sqrt{3}y'$, we then see

$$v_{xx} + v_{y'y'} + \tilde{c}v = 0$$

■

Question 68

Consider the equation $3u_y + u_{xy} = 0$.

(a) What is its type?

(b) Find the general solution. (Hint: Substitute $v = u_y$).

(c) With the auxiliary conditions $u(x,0) = e^{-3x}$ and $u_y(x,0) = 0$, does a solution exist? Is it unique?

Proof. Since the discriminant is exactly $\frac{-1}{4}$, the type is hyperbolic. Letting $v \triangleq u_y$, we see

$$3v + v_x = 0$$

This PDE on each horizontal line is an ODE and thus have the solution

$$u_y = v = f(y)e^{-3x}$$

Then u must be of the form

$$u = F(y)e^{-3x} + g(x)$$

Applying the initial condition $u_y(x,0) = 0$, we see

$$f(0)e^{-3x} = u_y(x,0) = 0$$

which implies $f(0) = 0$. Now apply another initial condition $u(x,0) = e^{-3x}$.

$$F(0)e^{-3x} + g(x) = u(x,0) = e^{-3x}$$

We then see the solutions exist and are not unique, since

$$\begin{cases} F(y) = 1 \\ g(x) = 0 \end{cases} \quad \text{and} \quad \begin{cases} F(y) = y^2 \\ g(x) = e^{-3x} \end{cases}$$

are both solutions. ■

3.5 2.1 The Wave Equation

Abstract

In this section, $c \in \mathbb{R}^*$.

Theorem 3.5.1. (General Solution of The Wave Equation) The general solution of the wave equation

$$u_{tt} = c^2 u_{xx}$$

is of the form

$$u = f(x+ct) + g(x-ct)$$

Proof. Observe that

$$0 = u_{tt} - c^2 u_{xx} = (\partial_t + c\partial_x)(\partial_t - c\partial_x)u$$

If we let $v = u_t - cu_x$, then we must have $v_t + cv_x = 0$. We know the general solution of v is $v = g(x-ct)$. We have reduce the problem into solving

$$u_t - cu_x = g(x-ct) \tag{3.3}$$

Now observe that for all $w : \mathbb{R} \to \mathbb{R}$

$$(\partial_t - c\partial_x)\big(w(x-ct)\big) = 2cw'(x-ct)$$

We then see the particular solution for Equation 3.3 is

$$u = \frac{1}{2c} G(x-ct)$$

and the general solution is then

$$u = f(x+ct) + \frac{1}{2c} G(x-ct)$$

∎

Theorem 3.5.2. (IVP for The Wave Equation) The Initial Value Problem

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

has exactly one solution

$$u(x,t) = \frac{1}{2}[\varphi(x+ct) + \varphi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds$$

Proof. Write $u(x,t) = f(x+ct) + g(x-ct)$. By initial condition, we know

$$f(x) + g(x) = \varphi(x) \text{ and } f'(x) - g'(x) = \frac{\psi(x)}{c}$$

Differentiating the former, we also have

$$f'(x) + g'(x) = \varphi'(x)$$

This then give us

$$f'(x) = \frac{\varphi'(x)}{2} + \frac{\psi(x)}{2c} \text{ and } g'(x) = \frac{\varphi'(x)}{2} - \frac{\psi(x)}{2c}$$

It now follows that

$$f(s) = \frac{\varphi(s)}{2} + \frac{1}{2c}\int_0^s \psi(x)dx + A \text{ and } g(s) = \frac{\varphi(s)}{2} - \frac{1}{2c}\int_0^s \psi(x)dx + B$$

Note that since $f(x) + g(x) = \varphi(x)$, we know $B = -A$.

We now have

$$\begin{aligned} u(x,t) &= f(x+ct) + g(x-ct) \\ &= \frac{\varphi(x+ct) + \varphi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x)dx \end{aligned}$$

■

Question 69

Solve

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = e^x \\ u_t(x,0) = \sin x \end{cases}$$

Proof. Let

$$u \triangleq f(x+ct) + g(x-ct)$$

Plugging the initial conditions, we know

$$\begin{cases} f(x) + g(x) = e^x \\ f'(x) - g'(x) = \frac{\sin x}{c} \end{cases}$$

This give us

$$f'(x) = \frac{e^x + \frac{\sin x}{c}}{2} \text{ and } g'(x) = \frac{e^x - \frac{\sin x}{c}}{2}$$

Then by FTC, we have

$$\begin{cases} f(x) = \frac{e^x - \frac{\cos x}{c}}{2} + f(0) - \frac{1}{2} + \frac{1}{2c} \\ g(x) = \frac{e^x + \frac{\cos x}{c}}{2} + g(0) - \frac{1}{2} - \frac{1}{2c} \end{cases}$$

Note that $f(0) + g(0) = e^0 = 1$, which cancel the constant terms in u, i.e.

$$\begin{aligned} u &= f(x+ct) + g(x-ct) \\ &= \frac{e^{x+ct} + e^{x-ct} + \frac{-\cos(x+ct)+\cos(x+ct)}{c}}{2} \end{aligned}$$

■

Question 70

If both φ and ψ are odd functions of x, show that the solution of $u(x,t)$ of the wave equation is also odd in x for all t.

Proof. Suppose

$$u = f(x+ct) + g(x-ct)$$

We have

$$\begin{cases} f + g = \varphi \\ f' - g' = \frac{\psi}{c} \end{cases}$$

This give us

$$f' = \frac{\varphi' + \frac{\psi}{c}}{2} \text{ and } g' = \frac{\varphi' - \frac{\psi}{c}}{2}$$

and

$$\begin{cases} f(x) = \frac{1}{2}\int_0^x (\varphi' + \frac{\psi}{c})ds + f(0) \\ g(x) = \frac{1}{2}\int_0^x (\varphi' - \frac{\psi}{c})ds + g(0) \end{cases}$$

that is

$$\begin{cases} f(x) = f(0) + \frac{1}{2}\Big[\varphi(x) - \varphi(0)\Big] + \frac{1}{2c}\int_0^x \psi(s)ds \\ g(x) = g(0) + \frac{1}{2}\Big[\varphi(x) - \varphi(0)\Big] - \frac{1}{2c}\int_0^x \psi(s)ds \end{cases}$$

Noting $f + g = \varphi$, we now have

$$\begin{aligned} u(x,t) &= f(x+ct) + g(x-ct) \\ &= \frac{\varphi(x+ct) + \varphi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds \end{aligned}$$

and

$$\begin{aligned} u(-x,t) &= \frac{\varphi(-x+ct) + \varphi(-x-ct)}{2} + \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(s)ds \\ &= \frac{-\varphi(x-ct) - \varphi(x+ct)}{2} - \frac{1}{2c}\int_{x+ct}^{x-ct} \psi(-u)du \quad (\because u = -s) \\ &= \frac{-\varphi(x-ct) - \varphi(x+ct)}{2} + \frac{1}{2c}\int_{x+ct}^{x-ct} \psi(u)du \quad (\because \psi \text{ is odd }) \\ &= \frac{-\varphi(x-ct) - \varphi(x+ct)}{2} - \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(u)du = -u(x,t) \end{aligned}$$

■

Question 71

A spherical wave is a solution of the three-dimensional wave equation of the form $u(r,t)$, where r is the distance to the origin (the spherical coordinate). The wave equation takes the form

$$u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right)$$

(a) Change variables $v = ru$ to get the equation for $v : v_{tt} = c^2 v_{rr}$.

(b) Solve for v and thereby solve the spherical wave equation.

(c) Solve it with the initial condition $u(r,0) = \varphi(r), u_t(r,0) = \psi(r)$, taking both $\varphi(r)$ and $\psi(r)$ to be even functions of r.

Proof. If we let $v = ru$, then

$$v_{tt} = ru_{tt} \text{ and } v_{rr} = ru_{rr} + 2u_r$$

This then give us

$$v_{tt} = ru_{tt} = rc^2\left(u_{rr} + \frac{2}{r}u_r\right) = c^2(ru_{rr} + 2u_r) = c^2 v_{rr}$$

The general solution of v is

$$v = f(r+ct) + g(r-ct)$$

and thus

$$u(r,t) = \frac{f(ct+r) + g(r-ct)}{r}$$

Plugging the initial conditions, we have

$$\frac{f(r)+g(r)}{r} = u(r,0) = \varphi(r) \text{ and } \frac{c[f'(r)-g'(r)]}{r} = u_t(r,0) = \psi(r)$$

In other words,

$$\begin{cases} f(r)+g(r) = r\varphi(r) \\ f'(r)-g'(r) = \frac{r\psi(r)}{c} \end{cases}$$

Differentiating the first equation, we have

$$f'(r) + g'(r) = \varphi(r) + r\varphi'(r)$$

We now can solve f', g'

$$f'(r) = \frac{\varphi(r) + r\varphi'(r) + \frac{r\psi(r)}{c}}{2} \text{ and } g'(r) = \frac{\varphi(r) + r\varphi'(r) - \frac{r\psi(r)}{c}}{2}$$

We now have

$$\begin{aligned} f(r) &= f(1) + \int_1^r f'(s)ds \\ &= f(1) + \Big[\frac{s\varphi(s)}{2}\Big]\Big|_{s=1}^r + \frac{1}{2c}\int_1^r s\psi(s)ds \end{aligned}$$

and

$$\begin{aligned} g(r) &= g(1) + \int_1^r g'(s)ds \\ &= g(1) + \Big[\frac{s\varphi(s)}{2}\Big]\Big|_{s=1}^r - \frac{1}{2c}\int_1^r s\psi(s)ds \end{aligned}$$

Noting that $f(1)+g(1) = 1\varphi(1)$, we can cancel these terms and get

$$\begin{aligned} u(r,t) &= \frac{f(r+ct)+g(r-ct)}{r} \\ &= \frac{(r+ct)\varphi(r+ct) + (r-ct)\varphi(r-ct)}{2r} + \frac{1}{2cr}\int_1^{r+ct} s\varphi(s)ds - \frac{1}{2cr}\int_1^{r-ct} s\varphi(s)ds \\ &= \frac{(r+ct)\varphi(r+ct) + (r-ct)\varphi(r-ct)}{2r} + \frac{1}{2cr}\int_{r-ct}^{r+ct} s\varphi(s)ds \end{aligned}$$

$\blacksquare$

Question 72

Solve

$$\begin{cases} u_{xx} + u_{xt} - 20u_{tt} = 0 \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

Proof. The PDE can be write in the form of

$$(\partial_x + 5\partial_t)(\partial_x - 4\partial_t)u = 0$$

which have the general solution

$$u(x,t) = f(5x - t) + g(4x + t)$$

We now see

$$f(5x) + g(4x) = \varphi(x) \text{ and } -f'(5x) + g'(4x) = \psi(x)$$

This then give us

$$f'(5x) = \frac{(\varphi' - 4\psi)(x)}{9} \text{ and } g'(4x) = \frac{(\varphi' + 5\psi)(x)}{9}$$

and thus

$$\begin{aligned} f(x) &= f(0) + \int_0^x f'(s)ds \\ &= f(0) + \int_0^x \frac{(\varphi' - 4\psi)(\frac{s}{5})}{9} ds \\ &= f(0) + \frac{5}{9}\Big[\varphi(\frac{x}{5}) - \varphi(0)\Big] - \frac{4}{9}\int_0^x \psi(\frac{s}{5})ds \end{aligned}$$

and similarly

$$\begin{aligned} g(x) &= g(0) + \int_0^x g'(s)ds \\ &= g(0) + \int_0^x \frac{(\varphi' + 5\psi)(\frac{s}{4})}{9} ds \\ &= g(0) + \frac{4}{9}\Big[\varphi(\frac{x}{4}) - \varphi(0)\Big] + \frac{5}{9}\int_0^x \psi(\frac{s}{4})ds \end{aligned}$$

Noting that $f(0) + g(0) = u(0,0) = \psi(0)$, we now have

$$\begin{aligned} u(x,t) &= f(5x-t) + g(4x+t) \\ &= \frac{5\varphi(\frac{5x-t}{5}) + 4\varphi(\frac{4x+t}{4})}{9} - \frac{4}{9}\int_0^{5x-t} \psi(\frac{s}{5})ds + \frac{5}{9}\int_0^{4x+t} \psi(\frac{s}{4})ds \end{aligned}$$

■

Question 73

Find the general solution of

$$3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x+t)$$

Proof. Clearly, we can rewrite the equation to be

$$(3\partial_x + \partial_t)(\partial_x + 3\partial_t)u = \sin(x+t)$$

If we let $v = u_x + 3u_t$, then we have

$$3v_x + v_t = \sin(x+t)$$

Define

$$\gamma(x) \triangleq (x, \frac{x}{3} + c)$$

We then see

$$(v \circ \gamma)'(x) = \frac{\sin(x + \frac{x}{3} + c)}{3}$$

which implies

$$v(x, \frac{x}{3} + c) = v \circ \gamma(x) = \frac{\cos(\frac{4x}{3} + c)}{-4} + C_c$$

and thus

$$\begin{aligned} v(x,t) &= \frac{\cos(\frac{4x}{3} + t - \frac{x}{3})}{-4} + f(t - \frac{x}{3}) \\ &= \frac{\cos(x+t)}{-4} + f(3t - x) \end{aligned}$$

It remains to solve

$$u_x + 3u_t = \frac{\cos(x+t)}{-4} + f(3t - x)$$

Now define

$$\gamma(x) \triangleq (x, 3x + c)$$

We then see

$$(u \circ \gamma)'(x) = \frac{\cos(4x + c)}{-4} + f(8x + 3c)$$

which implies

$$u(x, 3x + c) = \frac{\sin(4x + c)}{-16} + \frac{F(8x + 3c)}{8} + u(0, c) + f(c)$$

and thus

$$u(x, t) = \frac{\sin(x + t)}{-16} + \tilde{F}(-x + 3t) + g(t - 3x)$$

where g is the initial condition. ■

3.6 2.2 Causality and Energy

Question 74

Show that the wave equation has the following invariant properties

(a) Any translate $u(x-y,t)$ where y is fixed, is also a solution.

(b) Any derivative, say u_x, is also a solution.

(c) The dilated function $u(ax,at)$ is also a solution.

Proof. The first property follows from direct computation, the second property follows from $0_x = 0$ and the third property follows from observing $v \triangleq u(ax,at)$ satisfy $v_{tt} = a^2u_{tt} = a^2c^2u_{xx} = v_{xx}$. ∎

Question 75

If $u(x,t)$ satisfy the wave equation $u_{tt} = u_{xx}$, prove the identity

$$u(x+h,t+k) + u(x-h,t-k) = u(x+k,t+h) + u(x-k,t-h)$$

Proof. Define $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(x) \triangleq u(x,0) \text{ and } \psi(x) \triangleq u_t(x,0)$$

We then know that

$$\begin{aligned} u(x,t) &= \frac{\varphi(x+t)+\varphi(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \psi(s)ds \\ &\triangleq \frac{A(x,t)+B(x,t)+C(x,t)}{2} \end{aligned}$$

where

$$\begin{cases} A(x,t) \triangleq \varphi(x+t) \\ B(x,t) \triangleq \varphi(x-t) \\ C(x,t) \triangleq \int_{x-t}^{x+t} \psi(s)ds \end{cases}$$

The proof then follows from

$$\begin{aligned} A(x+h,t+k) = A(x+k,t+h) &\text{ and } A(x-h,t-k) = A(x-k,t-h) \\ B(x+h,t+k) = B(x-k,t-h) &\text{ and } B(x-h,t-k) = B(x+k,t+h) \\ C(x+h,t+k) = C(x+k,t+h) &\text{ and } C(x-h,t-k) = C(x-k,t-h) \end{aligned}$$

∎

Question 76

Suppose that we have the PDE of damped string

$$\begin{cases} \rho u_{tt} - T u_{xx} + r u_t = 0 \text{ where } r > 0 \\ u(x,0) = 0 \text{ if } |x| > N \end{cases}$$

Show that if we define the energy $E(t)$ of this system by

$$E(t) = \int_{-\infty}^{\infty} (\rho u_t^2 + T u_x^2) dx$$

Then the energy decrease as time goes.

Proof. Because u is smooth, we have

$$\begin{aligned} E'(t) &= \int_{-\infty}^{\infty} (\rho u_t^2 + T u_x^2)_t dx \\ &= \int_{-\infty}^{\infty} (2\rho u_t u_{tt} + 2T u_x u_{xt}) dx \\ &= \int_{-\infty}^{\infty} [2u_t(T u_{xx} - r u_t) + 2T u_x u_{xt}] dx \\ &= \int_{-\infty}^{\infty} [2T(u_t u_x)_x - 2r u_t^2] dx \\ &= 2T u_t t_x \Big|_{x=0}^{\infty} - \int_{-\infty}^{\infty} 2r u_t^2 \\ &= -\int_{-\infty}^{\infty} 2r u_t^2 \leq 0 \end{aligned}$$

■

3.7 2.3 The Diffusion Equation

In this section, we shall first set

$$\Omega \triangleq (0,l) \text{ and } \Gamma \triangleq \Omega \times \{0\} \cup \partial\Omega \times [0,T]$$

and write

$$\Omega_T \triangleq \Omega \times (0,T)$$

We suppose $u : \overline{\Omega_T} \to \mathbb{R}$ satisfy

$$u \in C^2(\Omega \times (0,T])$$

If u achieve a maximum on $\Omega \times (0,T]$, then at that point u must have

$$u_t \geq 0 \text{ and } u_{xx} \leq 0$$

Theorem 3.7.1. **(Weak Maximum Principle)** If

$$u_t - ku_{xx} \leq 0 \text{ on } \Omega \times (0,T] \tag{3.4}$$

then u must achieve its maximum at Γ.

Proof. Because Γ is compact, we know there exists a maximum M of u on Γ. Fix ϵ and define $v : \overline{\Omega_T} \to \mathbb{R}$

$$v(x,t) \triangleq u(x,t) + \epsilon x^2$$

Because

$$u(x,t) \leq \max_{\overline{\Omega_T}} v - \epsilon x^2 \text{ for all } (x,t) \in \overline{\Omega_T}$$

we can reduce the problem into proving

$$\max_{\overline{\Omega_T}} v \leq M + \epsilon l^2 \text{ for all } p \in F$$

From Equation 3.4, we can deduce

$$v_t - kv_{xx} = u_t - ku_{xx} - 2k\epsilon < 0$$

Because v is continuous, we know v attain its maximum at some point. Now, with diffusion inequality, we can deduce

(a) The maximum of v must not be in Ω_T, otherwise at that point $v_t = 0$ and $v_{xx} \leq 0$ yield a contradiction.

(b) The maximum of v must also not be in the top edge $\partial\Omega_T \setminus \Gamma$, otherwise $v_t \geq 0$ and $v_{xx} \leq 0$ yield a contradiction.

We have proved that v can only attain maximum at some point $(x_0, t_0) \in F_0$, and it follows that

$$\max_{(x,t)\in F} v(x,t) = v(x_0,t_0) = u(x_0,t_0) + \epsilon x_0^2 \leq M + \epsilon l^2 \text{ (done)}$$

■

Corollary 3.7.2. (Weak Minimum Principle) The minimum of u must also happen on F_0.

Now, consider the Dirichlet's problem

$$\begin{cases} u_t - ku_{xx} = f(x,t) \text{ for } 0 < x < l \text{ and } t > 0 \\ u(x,0) = \varphi(x) \text{ for } 0 \leq x \leq l \\ u(0,t) = g(t) \text{ and } u(l,t) = h(t) \text{ for } t \geq 0 \end{cases} \tag{3.5}$$

Note that for all T, because the difference w of two solution u_1, u_2 for Dirichlet's function must satisfy

$$\begin{cases} w_t = kw_{xx} \text{ on } \overline{\Omega_T} \setminus \Gamma \\ w(x,0) = w(0,t) = 0 \text{ for any } 0 \leq x \leq l \text{ and } 0 \leq t \leq T \end{cases}$$

By minimum and maximum principle we can deduce $w = 0$ on Ω, and thus $u_1 = u_2$ on F. It then follows that $u_1 = u_2$ on $[0,l] \times [0,\infty)$.

Theorem 3.7.3. (Uniqueness for the Dirichlet's problem for the diffusion equation with Energy Method) If $u_1, u_2 : [0,l] \times [0,\infty)$ are both solution of the Dirichlet's problem, then $u_1 = u_2$.

Proof. Define $w : [0,l] \times [0,\infty) \to \mathbb{R}$ by $w = u_1 - u_2$. Multiplying w with $(w_t - kw_{xx})$, we see that for all $x \in (0,l)$ and $t > 0$,

$$0 = (w_t - kw_{xx})w = \left(\frac{w^2}{2}\right)_t + (-kw_x w)_x + kw_x^2$$

Because $w(0,t) = w(l,t) = 0$ for all t, it follows that for all $t > 0$

$$\begin{aligned} 0 &= \int_0^l \left[\left(\frac{w^2}{2}\right)_t + (-kw_x w)_x + kw_x^2\right] dx \\ &= \int_0^l \left[\left(\frac{w^2}{2}\right)_t + kw_x^2\right] dx \end{aligned}$$

which implies

$$I'(t) \leq 0 \text{ if we define } I : [0,\infty) \to \mathbb{R} \text{ by } I(t) \triangleq \int_0^l \Big(\frac{w^2}{2}\Big) dx$$

Because $I(0) = 0$ by definition and $I(t)$ are integrals of non-negative functions, we can deduce I is identically 0. The desired result $w(x,t) = 0$ for all $x, t \in [0,l] \times [0,\infty)$ then follows. ■

Now, consider Dirichlet's problem with different initial conditions $\varphi_1, \varphi_2 : [0,l] \to \mathbb{R}$, and suppose $u_1, u_2 : [0,l] \times [0,\infty)$ are corresponding solutions. The maximum and minimum principle give us a L^∞ estimation for stability

$$\max_{[0,l]\times[0,\infty)} |u_1 - u_2| \leq \max_{[0,l]} |\varphi_1 - \varphi_2|$$

While the energy method give us a L^2 estimation for stability: For all $t \geq 0$,

$$\int_0^l \Big(\frac{w^2(x,t)}{2}\Big) dx = I(t) \leq I(0) = \int_0^l \Big(\frac{w^2(x,0)}{2}\Big) = \int_0^l \frac{(\varphi_1 - \varphi_2)^2}{2} dx$$

Question 77

Consider the diffusion equation

$$\begin{cases} u_t = u_{xx} \text{ on } (0,1) \times (0,\infty) \\ u(0,t) = u(1,t) = 0 \text{ for all } t > 0 \\ u(x,0) = 1 - x^2 \end{cases}$$

(a) Show that $u(x,t) > 0$ for all $(x,t) \in (0,1) \times (0,\infty)$.

(b) Define $\mu : (0,\infty) \to \mathbb{R}$ by $\mu(t) \triangleq \max_{x\in[0,1]} u(x,t)$. Show that μ is a decreasing function.

Proof. The proof of (a) follows from a loose application of the strong minimum principle.

The proof of (b) follows from noting $v(x,t) \triangleq u(x, t + t_0) : [0,1] \times [0,\infty)$ also is a solution of the diffusion equation and application of maximum principle on v. ■

Question 78

Consider the Dirichlet's problem

$$\begin{cases} u_t = u_{xx} \text{ on } (0,1)\times(0,\infty) \\ u(0,t) = u(1,t) = 0 \text{ for all } t \geq 0 \\ u(x,0) = 4x(1-x) \end{cases}$$

Show that

(a) $0 < u(x,t) < 1$ for all $t > 0$ and $0 < x < 1$.

(b) $u(x,t) = u(1-x,t)$ for all $t \geq 0$ and $0 \leq x \leq 1$.

(c) Use the energy method to show that $\int_0^1 u^2 dx$ is a strictly decreasing function of t.

Proof. (a) follows from application of strong maximum and minimum principle. (b) follows from uniqueness of Dirichlet's problem and the observation that $u(1-x,t)$ is also a solution.

The energy method give us

$$\frac{d}{dt}\int_0^1 \frac{u^2}{4}dx = -\int_0^1 u_x^2 dx \leq 0 \text{ for all } t > 0$$

and (c) follows. ■

Question 79

Verify that

$$u = -2xt - x^2 \text{ is a solution of } u_t = xu_{xx}$$

and find the location of maximum of t in the close rectangle $\{-2 \leq x \leq 2, 0 \leq t \leq 1\}$.

Proof. Write

$$u = -(x+t)^2 + t^2$$

It follows that the maximum occurs at $t = -x = 1$. ■

Question 80

Prove the comparison principle for the diffusion equation: If u and v are two solutions and

$$u \leq v \text{ for } t = 0, x = 0, x = l$$

then

$$u \leq v \text{ on } [0, l] \times [0, \infty)$$

Proof. This follows from application of the minimum principle on $v - x$. ■

Question 81

Suppose

$$\begin{cases} u_t - ku_{xx} = f \\ v_t - kv_{xx} = g \end{cases} \quad \text{and } f \leq g$$

and suppose

$$u \leq v \text{ at } x = 0, x = l \text{ and } t = 0$$

Prove that

$$u \leq v \text{ on } [0, l] \times [0, \infty)$$

Proof. Let $w \triangleq u - v : \overline{\Omega_T} \to \mathbb{R}$. It is clear that

$$w_t - kw_{xx} \leq 0 \text{ on } \overline{\Omega_T} \setminus \Gamma$$

It then follows that w attain its maximum on Γ, which must not be greater than 0. ■

3.8 2.4 Diffusion on the whole line

In this section, we are concerned with solving the following initial value problem (**Cauchy problem**)

$$\begin{cases} u_t = ku_{xx} \text{ on } \mathbb{R} \times (0, \infty) \\ \lim_{t \to 0} u(x,t) = \varphi(x) \text{ for all specified } x \end{cases}$$

We shall mostly express our answer with function erf : $\mathbb{R} \to \mathbb{R}$

$$\text{erf}(x) \triangleq \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2} dp$$

Theorem 3.8.1. (Solution of Dirac Initial Condition) If φ is defined to be

$$\varphi(x) \triangleq \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

then a solution is

$$Q(x,t) = \frac{1}{2} + \frac{\text{erf}(\frac{x}{\sqrt{4kt}})}{2} \tag{3.6}$$

Proof. Note that our version of diffusion equation admits dilated solutions. This inspire us to guess

$$Q(x,t) \triangleq g\Big(\frac{x}{\sqrt{4kt}}\Big)$$

Direct computation yields

$$Q_t = \frac{-x}{2\sqrt{4k}t^{\frac{3}{2}}} g'\Big(\frac{x}{\sqrt{4kt}}\Big) \text{ and } Q_{xx} = g''\Big(\frac{x}{\sqrt{4kt}}\Big)\frac{1}{4kt}$$

If we let $p = \frac{x}{\sqrt{4kt}}$, we now have

$$Q_t = \frac{-pg'(p)}{2t} \text{ and } Q_{xx} = \frac{g''(p)}{4kt}$$

Plugging this back to diffusion equation and canceling the common terms, we have

$$\frac{g''(p)}{2} + pg'(p) = 0$$

The general solution to this ODE is

$$g(p) = c_1 \,\text{erf}(p) + c_2$$

In other words,

$$Q(x,t) = c_1 \operatorname{erf}(\frac{x}{\sqrt{4kt}}) + c_2$$

Plugging this back to the initial condition, we see

$$Q(x,t) = \frac{1 + \operatorname{erf}(\frac{x}{\sqrt{4kt}})}{2}$$

■

Differentiating Equation 3.6 with respect to x, we have another solution

$$S(x,t) \triangleq \frac{1}{2\sqrt{\pi kt}} e^{-\frac{x^2}{4kt}}$$

Solution S is often called the **fundamental solution**, since for all initial condition φ that have compact support, we gain a solution to the initial value problem by

$$u(x,t) \triangleq (S * \varphi)(x,t)$$

where

$$(S * \varphi)(x,t) = \int_{\mathbb{R}} S(x-y,t)\varphi(y)dy$$

This is true because if we define $F(x,y,t) = Q(x-y,t)$, we have

$$\begin{aligned}
u(x,t) &= \int_{\mathbb{R}} F_x(x,y,t)\varphi(y)dy \\
&= \int_{\mathbb{R}} -F_y(x,y,t)\varphi(y)dy \\
&= -F(x,y,t)\varphi(y)|_{y=-\infty}^{\infty} + \int_{\mathbb{R}} F(x,y,t)\varphi'(y)dy \\
&= \int_{\mathbb{R}} Q(x-y,t)\varphi'(y)dy
\end{aligned}$$

and thus

$$\begin{aligned}
\text{For all } x, \lim_{t\to 0} u(x,t) &= \int_{\mathbb{R}} \lim_{t\to 0} Q(x-y,t)\varphi'(y)dy \\
&= \int_{-\infty}^{x} \varphi'(y)dy = \varphi(x)
\end{aligned}$$

Question 82

Solve the diffusion equation with the initial condition

$$\varphi(x) \triangleq \begin{cases} 1 & \text{if } |x| < l \\ 0 & \text{if } |x| > l \end{cases}$$

Proof. Define

$$F(x,y,t) \triangleq Q(x-y,t)$$

The solution is exactly

$$\begin{aligned} u(x,t) &= (S * \varphi)(x,t) \\ &= \int_{\mathbb{R}} S(x-y,t)\varphi(y)dy \\ &= \int_{-l}^{l} S(x-y,t)dy \\ &= \int_{-l}^{l} F_x(x,y,t)dy \\ &= \int_{-l}^{l} -F_y(x,y,t)dy = F(x,y,t)\Big|_{y=l}^{-l} = Q(x+l,t) - Q(x-l,t) = \frac{\operatorname{erf}(\frac{x+l}{\sqrt{4kt}}) - \operatorname{erf}(\frac{x-l}{\sqrt{4kt}})}{2} \end{aligned}$$

■

Question 83

Solve the diffusion equation with the initial condition

$$\varphi(x) \triangleq \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Proof. Define

$$F(x,y,t) \triangleq Q(x-y,t)$$

The solution is exactly

$$
\begin{aligned}
u(x,t) &= (S * \varphi)(x,t) \\
&= \int_{\mathbb{R}} S(x-y,t)\varphi(y)dy \\
&= \int_0^\infty e^{-y} S(x-y,t)dy \\
&= \frac{1}{2\sqrt{t\pi k}} \int_0^\infty e^{\frac{-(x-y)^2}{4kt} - y} dy \\
&= \frac{1}{2\sqrt{t\pi k}} \int_0^\infty e^{\frac{-(y-(x-2kt))^2 + (x-2kt)^2 - x^2}{4kt}} dy \\
&= \frac{e^{kt-x}}{2\sqrt{t\pi k}} \int_0^\infty e^{\frac{-(y-(x-2kt))^2}{4kt}} dy \\
&= \frac{e^{kt-x}}{\sqrt{\pi}} \int_{\frac{2kt-x}{2\sqrt{kt}}}^\infty e^{-s^2} ds \quad (\because s = \frac{y-(x-2kt)}{2\sqrt{kt}}) \\
&= \frac{e^{kt-x}}{\sqrt{\pi}} \Big(\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \operatorname{erf}(\frac{2kt-x}{2\sqrt{kt}}) \Big) \\
&= \frac{e^{kt-x}}{2} \Big[1 - \operatorname{erf}\Big(\frac{2kt-x}{2\sqrt{kt}} \Big) \Big]
\end{aligned}
$$

■

Question 84

Show that for any fixed $\delta > 0$

$$\max_{\delta \leq |x| < \infty} S(x,t) \to 0 \text{ as } t \to 0$$

Proof. Note that for all fixed $t > 0$,

$$\max_{\delta \leq |x| < \infty} S(x,t) = \max_{\delta \leq |x| < \infty} \frac{1}{2\sqrt{kt\pi}} e^{\frac{-x^2}{4kt}} = \frac{1}{2\sqrt{kt\pi}} e^{\frac{-\delta^2}{4kt}}$$

The proof then follows from noting $e^{\frac{-1}{t}} = o(\sqrt{t})$. ■

Question 85

Let $\varphi(x)$ be a continuous function such that $|\varphi(x)| \leq Ce^{ax^2}$. Show that formula

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt}} \varphi(y)dy$$

for diffusion equation make sense for $0 < t < \frac{1}{4ak}$ but not necessarily for larger t.

Proof. Because φ is continuous, we know

$$e^{\frac{-(x-y)^2}{4kt}} \varphi(y) \text{ is at least measurable in } y \text{ on } \mathbb{R}$$

We now see that if $0 < t < \frac{1}{4ak}$, then

$$\int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4kt}} \varphi(y)dy \leq C \int_{\mathbb{R}} e^{ay^2+b(x-y)^2} dy < \infty \text{ where } b < -a$$

If $t \geq \frac{1}{4ak}$, and we take $\varphi = Ce^{ay^2}$, then we have

$$\int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4kt}} \varphi(y)dy = C \int_{\mathbb{R}} e^{ay^2+b(x-y)^2} dy = \infty$$

because $b \geq -a$.

■

Question 86

Prove the uniqueness of the diffusion problem with Neumann boundary conditions:

$$\begin{aligned} &u_t - ku_{xx} = f(x,t) \text{ for } 0 < x < l, t > 0 \\ &u(x,0) = \varphi(x) \\ &u_x(0,t) = g(t) \text{ and } u_x(l,t) = h(t) \end{aligned}$$

Proof. The proof follows from energy method. ■

Question 87

Solve the diffusion equation with constant dissipation:

$$\begin{aligned} &u_t - ku_{xx} + bu = 0 \text{ for } -\infty < x < \infty \\ &u(x,0) = \varphi(x) \end{aligned}$$

where $b > 0$ is a constant. (Hint: Make the change of variables $u(x,t) = e^{-bt}v(x,t)$)

Proof. If we make the change of variables $v(x,t) \triangleq e^{bt}u(x,t)$, then

$$v_t = e^{bt}(u_t + bu) \text{ and } v_{xx} = e^{bt}u_{xx}$$

It then follows that

$$v_t - kv_{xx} = e^{bt}(u_t + bu - ku_{xx}) = 0$$

The initial condition for v is

$$v(x,0) = u(x,0) = \varphi(x)$$

Then we know

$$v(x,t) = \int_{\mathbb{R}} S(x-y,t)\varphi(y)dy$$

It then follows that

$$u(x,t) = e^{-bt}\int_{\mathbb{R}} S(x-y,t)\varphi(y)dy$$

■

Question 88

Solve the diffusion equation with variable dissipation :

$$\begin{aligned} &u_t - ku_{xx} + bt^2u = 0 \text{ for } -\infty < x < \infty \\ &u(x,0) = \varphi(x) \end{aligned}$$

where $b > 0$ is a constant. (Hint: Make the change of variables $u(x,t) = e^{\frac{-bt^3}{3}}v(x,t)$)

Proof. If we make the change of variables $v(x,t) \triangleq e^{\frac{bt^3}{3}}u(x,t)$, then

$$v_t = e^{\frac{bt^3}{3}}(bt^2u + u_t) \text{ and } v_{xx} = e^{\frac{bt^3}{3}}(u_{xx})$$

It then follows that

$$v_t - kv_{xx} = e^{\frac{bt^3}{3}}(u_t - ku_{xx} + bt^2u) = 0$$

The initial condition for v is

$$v(x,0) = u(x,0) = \varphi(x)$$

It then follows

$$v(x,t) = \int_{\mathbb{R}} S(x-y,t)\varphi(y)dy$$

and

$$u(x,t) = e^{\frac{-bt^3}{3}} \int_{\mathbb{R}} S(x-y,t)\varphi(y)dy$$

■

Question 89

Solve the heat equation with convection:

$$\begin{aligned} &u_t - ku_{xx} + Vu_x = 0 \text{ for } -\infty < x < \infty \\ &u(x,0) = \varphi(x) \end{aligned}$$

(Hint: Go to a moving frame of reference by substituting $y = x - Vt$)

Proof. If we define $v(x,t) \triangleq u(x+Vt,t)$, then

$$v_u = u_t + Vu_x \text{ and } v_{xx} = u_{xx}$$

It then follows that

$$v_t - kv_{xx} = u_t - ku_{xx} + Vu_x = 0$$

Note that v has the initial condition

$$v(x,0) = u(x,0) = \varphi(x)$$

So we have

$$v(x,t) = \int_{\mathbb{R}} S(x-y,t)\varphi(y)dy$$

It then follows

$$u(x,t) = u(x-Vt+Vt,t) = v(x-Vt,t) = \int_{\mathbb{R}} S(x-Vt-y,t)\varphi(y)dy$$

■

Question 90

Show that $S_2(x,y,t) \triangleq S(x,t)S(y,t)$ satisfy the diffusion equation $S_t = k(S_{xx} + S_{yy})$.

Deduce that $S_2(x,y,t)$ is the source function for two-dimensional diffusion.

Proof. We have

$$(S_2)_t(x,y,t) = S_t(x,t)S(y,t) + S(x,t)S_t(y,t)$$

and

$$(S_2)_{xx} = S_{xx}(x,t)S(y,t) \text{ and } (S_2)_{yy} = S(x,t)S_{yy}(y,t)$$

This then give us

$$(S_2)_t - k(S_2)_{xx} - k(S_2)_{yy} = S(y,t)[S_t(x,t) - S_{xx}(x,t)] + S(x,t)[S_t(y,t) - S_{yy}(y,t)] = 0$$

To see that S_2 is indeed fundamental solution, observe

$$\begin{aligned}
\iint_{\mathbb{R}^2} S_2(x-r, y-s, 0)\varphi(r,s)drds &= \iint_{\mathbb{R}^2} S(x-r,0)S(y-s,0)\varphi(r,s)drds \\
&= \int_{\mathbb{R}} S(x-r,0)\int_{\mathbb{R}} S(y-s,0)\varphi(r,s)dsdr \\
&= \int_{\mathbb{R}} S(x-r,0)\varphi(r,y)dr \\
&= \varphi(x,y)
\end{aligned}$$

■

Chapter 4

PDE intro 2

4.1 3.1 Diffusion on the half line

Consider the following **Dirichlet boundary condition problem**

$$\begin{cases} v_t - kv_{xx} = 0 \text{ for } x \in (0, \infty) \textbf{ (Homogeneous DE)} \\ v(x,0) = \varphi(x) \textbf{ (IC)} \\ v(0,t) = 0 \textbf{ (Dirichlet BC)} \end{cases}$$

Define $\varphi_{\text{odd}} : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_{\text{odd}}(x) \triangleq \begin{cases} \varphi(x) & \text{if } x > 0 \\ -\varphi(x) & \text{if } x < 0 \end{cases}$$

It then follows that φ_{odd} is an odd function, and we can solve the Cauchy problem with respect to this initial condition φ_{odd} and have the solution

$$u(x,t) \triangleq \int_{-\infty}^{\infty} S(x-y,t)\varphi_{\text{odd}}(y)dy$$

Now, because

$$S(x,t) = \frac{1}{2\sqrt{\pi kt}} e^{\frac{-x^2}{4kt}} \text{ is clearly even in } x$$

We can deduce

$$\begin{aligned} u(-x,t) &= \int_{-\infty}^{\infty} S(-x-y,t)\varphi_{\text{odd}}(y)dy \\ &= -\int_{-\infty}^{\infty} S(x+y,t)\varphi_{\text{odd}}(-y)dy \quad (\because S \text{ is even and } \varphi_{\text{odd is odd}}) \\ &= -\int_{-\infty}^{\infty} S(x-r)\varphi_{\text{odd}}(r)dr = -u(x,t) \quad (\because r=-y) \end{aligned}$$

In other words, we have deduced that u is an odd function in x. It then follows that $u(0,t) = -u(-0,t) = 0$. Then we see that the restriction $v \triangleq u|_{(\mathbb{R}^+)^2}$ form a solution of the Dirichlet boundary condition problem. In particular, we can express v in a form without usage of φ_{odd} if we consider

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} S(x-y,t)\varphi_{\text{odd}}(y)dy \\ &= \int_{0}^{\infty} S(x-y,t)\varphi(y)dy + \int_{-\infty}^{0} S(x-y,t)(-\varphi(-y))dy \\ &= \int_{0}^{\infty} [S(x-y,t) - S(x+y,t)]\varphi(y)dy \\ &= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} [e^{\frac{-(x-y)^2}{4kt}} - e^{\frac{-(x+y)^2}{4kt}}]\varphi(y)dy \end{aligned}$$

Now, consider the following **Neumann boundary condition problem**

$$\begin{cases} w_t - kw_{xx} = 0 \text{ for } x \in (0,\infty) \ \textbf{(Homogeneous DE)} \\ w(x,0) = \varphi(x) \ \textbf{(IC)} \\ w_x(0,t) = 0 \ \textbf{(Neumann BC)} \end{cases}$$

Define $\varphi_{\text{even}} : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_{\text{even}}(x) \triangleq \begin{cases} \varphi(x) & \text{if } x > 0 \\ \varphi(-x) & \text{if } x < 0 \end{cases}$$

It then follows that φ_{even} is an even function, and we can solve the Cauchy problem with respect to this initial condition φ_{even} and have the solution

$$u(x,t) \triangleq \int_{-\infty}^{\infty} S(x-y,t)\varphi_{\text{even}}(y)dy$$

Again because S is even in x, we can deduce

$$\begin{aligned}
u(-x,t) &= \int_{-\infty}^{\infty} S(-x-y,t)\varphi_{\text{even}}(y)dy \\
&= \int_{-\infty}^{\infty} S(-x-y,t)\varphi_{\text{even}}(-y)dy \\
(\because z=-y) \qquad &= -\int_{\infty}^{-\infty} S(-x+z,t)\varphi_{\text{even}}(z)dz = u(x,t)
\end{aligned}$$

Now, we have proved that u is even in x. This then give $u_x(0,t) = 0$, and solve the **Neumann problem** by letting $w \triangleq u|_{(\mathbb{R}^+)^2}$. In particular, we can express u in a form without usage of φ_{even} if we consider

$$\begin{aligned}
u(x,t) &= \int_{-\infty}^{\infty} S(x-y,t)\varphi_{\text{even}}(y)dy \\
&= \int_{0}^{\infty} S(x-y,t)\varphi(y)dy + \int_{-\infty}^{0} S(x-y,t)\varphi(-y)dy \\
&= \int_{0}^{\infty} [S(x-y,t) + S(x+y,t)]\varphi(y)dy \\
&= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} [e^{\frac{-(x-y)^2}{\sqrt{4kt}}} + e^{\frac{-(x+y)^2}{\sqrt{4kt}}}]\varphi(y)dy
\end{aligned}$$

Question 91

Solve

$$\begin{aligned}
&u_t = ku_{xx} \\
&u(x,0) = e^{-x} \\
&u(0,t) = 0
\end{aligned}$$

on the half line $0 < x < \infty$

Proof. Extend the initial condition to

$$\varphi_{\text{odd}}(x) \triangleq \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -e^{x} & \text{if } x < 0 \end{cases}$$

We then solve

$$
\begin{aligned}
u(x,t) &= \frac{1}{2\sqrt{\pi kt}}\int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4kt}}\varphi_{\text{odd}}(y)dy\\
&= \frac{1}{2\sqrt{\pi kt}}\Big[\int_0^\infty e^{\frac{-(x-y)^2}{4kt}}e^{-y}dy - \int_{-\infty}^0 e^{\frac{-(x-y)^2}{4kt}}e^{y}dy\Big]\\
&= \frac{1}{2\sqrt{\pi kt}}\Big[\int_0^\infty e^{\frac{-(y-(x-2kt))^2-4ktx+4k^2t^2}{4kt}}dy - \int_{-\infty}^0 e^{\frac{-(y-(x+2kt))^2+4ktx+4k^2t^2}{4kt}}dy\Big]\\
&= \frac{1}{2\sqrt{\pi kt}}\Big[e^{-x+kt}\int_0^\infty e^{\frac{-(y-(x-2kt))^2}{4kt}}dy - e^{x+kt}\int_{-\infty}^0 e^{\frac{-(y-(x+2kt))^2}{4kt}}dy\Big]\\
&= \frac{1}{\sqrt{\pi}}\Big[e^{-x+kt}\int_{\frac{2kt-x}{2\sqrt{kt}}}^\infty e^{-p^2}dp - e^{x+kt}\int_{-\infty}^{\frac{-2kt-x}{2\sqrt{kt}}} e^{-p^2}dp\Big]\\
&= \frac{1}{\sqrt{\pi}}\Big[e^{-x+kt}\Big(\frac{\sqrt{\pi}}{2}-\frac{2}{\sqrt{\pi}}\operatorname{erf}(\frac{2kt-x}{2\sqrt{kt}})\Big) - e^{x+kt}\Big(\frac{\sqrt{\pi}}{2}-\frac{2}{\sqrt{\pi}}\operatorname{erf}(\frac{2kt+x}{2\sqrt{kt}})\Big)\Big]
\end{aligned}
$$

■

Question 92

Solve

$$
\begin{aligned}
u_t &= ku_{xx}\\
u(x,0) &= 0\\
u(0,t) &= 1
\end{aligned}
$$

on the half line $0 < x < \infty$.

Proof. It is clear that if a function $v(x,t)$ satisfy the diffusion equation and the initial and boundary condition

$$v(x,0) = -1 \text{ and } v(0,t) = 0$$

then $u \triangleq v + 1$ is a desired solution. Note that v is just

$$v(x,t) = \int_{\mathbb{R}} S(x-y,t)\varphi_{\text{odd}}(y)dy$$

where

$$\varphi_{\text{odd}}(y) = \begin{cases} -1 & \text{if } y > 0 \\ 0 & \text{if } y = 0 \\ 1 & \text{if } y < 0 \end{cases}$$

■

Question 93

Consider the following problem with a Robin boundary condition:

$$\begin{aligned}&u_t = ku_{xx} \text{ on the half line } 0 < x < \infty\\&u(x,0) = x\\&u_x(0,t) - 2u(0,t) = 0\end{aligned}$$

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) \triangleq \begin{cases} x & \text{if } x \geq 0 \\ x + 1 - e^{2x} & \text{if } x < 0 \end{cases}$$

and let

$$v(x,t) \triangleq \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt}} f(y)dy$$

(a) What PDE and initial condition does $v(x,t)$ satisfy for $-\infty < x < \infty$?

(b) Let $w = v_x - 2v$. What PDE and initial condition does $w(x,t)$ satisfy $-\infty < x < \infty$?

(c) Show that $f'(x) - 2f(x)$ is an odd function.

(d) Show that w is an odd function of x.

(e) Deduce that v satisfy the Robin condition.

Proof. v satisfy the initial condition: $v(x,0) = f(x)$, and w satisfy the initial conditions

$$w(x, 0+) = v_x(x, 0+) - 2v(x, 0+) = f'(x) - 2f(x)$$

Note that the initial condition for w is $\varphi(x) = f'(x) - 2f(x)$ is odd. It then follows that

$$w(x,t) = \int_{\mathbb{R}} S(x-y,t)\varphi(y)dy \text{ is odd in } x$$

To see v satisfy the Robin condition, observe

$$v_x(0,t) - 2v(0,t) = w(0,t) = 0$$

∎

Question 94

Generalize the method of the last exercises to the case of general initial data $\varphi(x)$ and arbitrary constant coefficient for $u(0,t)$ in the boundary condition.

Proof. We are required to solve

$$\begin{cases} u_t - ku_{xx} = 0 \text{ for } x \in (0,\infty) \textbf{ (Homogeneous DE)} \\ u(x,0) = \varphi(x) \textbf{ (IC)} \\ u_x(0,t) - cu(0,t) = 0 \textbf{ (Robin BC)} \text{ where } c > 0 \text{ is some constant} \end{cases}$$

If function $f : \mathbb{R}^* \to \mathbb{R}$ satisfy

(a) $f(x) \triangleq \varphi(x)$ for $x > 0$

(b) $f'(x) - cf(x)$ is odd for $x \neq 0$

then the function

$$u(x,t) \triangleq \int_{\mathbb{R}} S(x-y,t)f(y)dy \text{ for } x \in \mathbb{R}$$

suffice initial condition. To see that u satisfy the Robin boundary condition, observe that $u_x - cu$ is a solution to the diffusion equation with initial condition

$$\begin{aligned} (u_x - cu)(x,0) &= \lim_{h\to 0} \frac{u(x+h,0) - u(x,0)}{h} - cu(x,0) \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - cf(x) \\ &= f'(x) - cf(x) \text{ for } x \in \mathbb{R} \end{aligned}$$

which with Theorem of uniqueness of solution implies

$$(u_x - cu)(x,t) = \int_{\mathbb{R}} S(x-y,t)[f'(y) - cf(y)]dy$$

It then follows from $f' - cf$ is odd that $(u_x - cu)$ is odd in x, and thus $(u_x - cu)(0,t) = 0$. ■

4.2 3.2 Reflection of waves

We now consider the **Dirichlet's problem for wave on the half line** $(0, \infty)$

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0 \text{ for } x \in (0, \infty) \textbf{ (Homogeneous DE)} \\ v(x,0) = \varphi(x), v_t(x,0) = \psi(x) \textbf{ (IC)} \\ v(0,t) = 0 \textbf{ (BC)} \end{cases}$$

One can check that if we again extend $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}$ to odd function $\varphi_{\text{odd}}, \psi_{\text{odd}} : \mathbb{R} \to \mathbb{R}$

$$\varphi_{\text{odd}}(x) \triangleq \begin{cases} \varphi(x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\varphi(-x) & \text{if } x < 0 \end{cases} \text{ and } \psi_{\text{odd}}(x) \triangleq \begin{cases} \psi(x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\psi(-x) & \text{if } x < 0 \end{cases}$$

and solve the **Cauchy's problem for wave on the whole line** with respect to them

$$u(x,t) \triangleq \frac{\varphi_{\text{odd}}(x+ct) + \varphi_{\text{odd}}(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{\text{odd}}(y) dy$$

then its restriction $v \triangleq u|_{[0,\infty) \times \mathbb{R}}$ is again a solution to the Dirichlet's problem for wave on the half line, where the boundary condition follows from u being odd in x as easily checked.

Consider also the **Neumann problem for wave on half line**

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0 \text{ for } x \in (0, \infty) \textbf{ (Homogeneous DE)} \\ v(x,0) = \varphi(x), v_t(x,0) = \psi(x) \textbf{ (IC)} \\ v_x(0,t) = 0 \textbf{ (BC)} \end{cases}$$

Question 95

Solve the Neumann problem for the wave equation on the half-line $0 < x < \infty$.

Proof. Define

$$\varphi_{\text{even}}(x) \triangleq \begin{cases} \varphi(x) & \text{if } x > 0 \\ \varphi(-x) & \text{if } x < 0 \end{cases} \text{ and } \psi_{\text{even}}(x) \triangleq \begin{cases} \psi(x) & \text{if } x > 0 \\ \psi(-x) & \text{if } x < 0 \end{cases}$$

and

$$u(x,t) \triangleq \frac{\varphi_{\text{even}}(x+ct) + \varphi_{\text{even}}(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{\text{even}}(s) ds$$

■

Question 96

Solve

$$\begin{cases} u_{tt} = 4u_{xx} \text{ for } x \in (0, \infty) \textbf{ (Homogeneous DE)} \\ u(x,0) = 1, u_t(x,0) = 0 \textbf{ (IC)} \\ u(0,t) = 0 \textbf{ (Dirichlet BC)} \end{cases}$$

using the reflection method. The solution has a singularity. Find its location.

Proof. Define

$$\varphi(x) \triangleq \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases} \text{ and } \psi(x) \triangleq 0$$

We are required to solve the following Dirichlet's problem for wave equation

$$u_{tt} = 4u_{xx} \text{ for } -\infty < x < \infty$$
$$u(x,0) = \varphi(x) \text{ and } u_t(x) = \psi(x)$$

The solution is exactly

$$\begin{aligned} u(x,t) &= \frac{\varphi(x+2t) + \varphi(x-2t)}{2} + \int_{x-2t}^{x+2t} \psi(s)ds \\ &= \frac{\varphi(x+2t) + \varphi(x-2t)}{2} \\ &= \begin{cases} 1 & \text{if } x - 2t > 0 \\ 0 & \text{if } x + 2t > 0 > x - 2t \\ -1 & \text{if } 0 > x + 2t \end{cases} \end{aligned}$$

On the half line, the solution is

$$u(x,t) = \begin{cases} 1 & \text{if } x - 2t > 0 \\ 0 & \text{if } x - 2t < 0 \end{cases}$$

So the singularity is exactly on the line $x - 2t = 0$ ■

Question 97

Solve

$$\begin{cases} u_{tt} = c^2 u_{xx} \text{ for } x \in (0,\infty), t \in [0,\infty) \textbf{ (Homogeneous DE)} \\ u(x,0) = 0, u_t(x,0) = V \textbf{ (IC)} \\ au_x(0,t) + u_t(0,t) = 0 \textbf{ (BC)} \end{cases}$$

Proof. Define

$$v(x,t) \triangleq au_x(x,t) + u_t(x,t)$$

Compute

$$v(0,t) = 0$$

Compute

$$\begin{aligned} v(x,0) &= au_x(x,0) + u_t(x,0) \\ &= 0 + V = V \end{aligned}$$

Compute

$$\begin{aligned} v_t(x,0) &= au_{xt}(x,0) + u_{tt}(x,0) \\ &= a(u_t(x,0))_x + c^2 u_{xx}(x,0) = 0 \end{aligned}$$

Then by reflection method, we see

$$v(x,t) \triangleq \frac{\varphi(x+ct) + \varphi(x-ct)}{2}$$

where

$$\varphi(x) \triangleq \begin{cases} V & \text{if } x > 0 \\ -V & \text{if } x < 0 \end{cases}$$

which implies

$$v(x,t) = \begin{cases} V & \text{if } x - ct > 0 \\ 0 & \text{if } x + ct > 0 > x - ct \\ -V & \text{if } 0 > x + ct \end{cases}$$

We are now required to solve

$$au_x + u_t = \begin{cases} V & \text{if } x - ct > 0 \\ 0 & \text{if } x - ct < 0 \end{cases}$$

on the first quadrant. Geometric method (If we require u to be continuous on the singularity) then shows

$$u(x,t) \triangleq \begin{cases} Vt & \text{if } x-ct>0 \\ \frac{V}{a-c}(at-x) & \text{if } x-ct<0 \end{cases}$$

■

Question 98

Find $u(\frac{2}{3},2), u(\frac{1}{4},\frac{7}{2})$ if

$$\begin{cases} u_{tt}=u_{xx} \text{ for } x\in(0,1) \textbf{ (Homogeneous DE)} \\ u(x,0)=x^2(1-x), u_t(x,0)=(1-x)^2 \textbf{ (IC)} \\ u(0,t)=u(1,t)=0 \textbf{ (BC)} \end{cases}$$

Proof. Extend the IC "oddly". With some tedious effort, we see

$$u(\frac{2}{3},2)=\frac{4}{27} \text{ and } u(\frac{1}{4},\frac{7}{2})=\frac{-1}{48}$$

■

Question 99

Solve

$$\begin{cases} u_{tt}=9u_{xx} \text{ for } x\in(0,\frac{\pi}{2}) \textbf{ (Homogeneous DE)} \\ u(x,0)=\cos x, u_t(x,0)=0 \textbf{ (IC)} \\ u_x(0,t)=0 \text{ and } u(\frac{\pi}{2},t)=0 \textbf{ (BC)} \end{cases}$$

Proof. Define

$$\varphi(x)\triangleq\cos x \text{ if } x\in(\frac{-\pi}{2},\pi)$$

and let φ have period $\frac{3\pi}{2}$. The solution is

$$u=\frac{\varphi(x+3t)+\varphi(x-3t)}{2}$$

■

4.3 3.3 Diffusion with a source

If we consider the non homogeneous diffusion equation

$$\begin{aligned} &u_t - ku_{xx} = f(x,t) \text{ for } -\infty < x < \infty, t > 0 \\ &u(x,0) = \varphi(x) \end{aligned}$$

we have the following

Theorem 4.3.1. (Diffusion with a source) If $f, \varphi : \mathbb{R} \to \mathbb{R}$ are smooth function tend to 0 as $|x| \to \infty$, then

$$u(x,t) = \int_{\mathbb{R}} S(x-y,t)\varphi(y)dy + \int_0^t \int_{\mathbb{R}} S(x-y,t-s)f(y,s)dyds$$

is a solution to the diffusion equation

$$\begin{aligned} &u_t - ku_{xx} = f(x,t) \text{ for } -\infty < x < \infty, t > 0 \\ &u(x,0+) = \varphi(x) \text{ for } -\infty < x < \infty \end{aligned}$$

Proof. It is clear that u satisfy the initial condition, and its first term satisfy the homogeneous diffusion equation. We only have to show

$$v(x,t) \triangleq \int_0^t \int_{\mathbb{R}} S(x-t,t-s)f(y,s)dyds \text{ satisfy } v_t - kv_{xx} = f(x,t)$$

Now compute

$$\begin{aligned} v_t(t,x) &= \frac{\partial}{\partial t}\Big(\int_0^t \int_{\mathbb{R}} S(x-y,t-s)f(y,s)dyds\Big) \\ &= \int_{\mathbb{R}} S(x-y,0+)f(y,t)dy + \int_0^t \int_{\mathbb{R}} S_t(x-y,t-s)f(y,s)dyds \\ &= f(x,t) + \int_0^t \int_{\mathbb{R}} kS_{xx}(x-y,t-s)f(y,s)dyds \\ &= f(x,t) + k\frac{\partial^2}{(\partial x)^2}\int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds = f(x,t) + kv_{xx}(x,t) \end{aligned}$$

We have proved $v_t - kv_{xx} = f(x,t)$. (Note that the partial derivative with respect to x in the third line is with respect to the first component while in the forth line is with respect to the actual x) ■

For source on the half line

$$\begin{aligned}
&v_t - kv_{xx} = f(x,t) \text{ for } 0 < x < \infty, 0 < t < \infty \\
&v(x, 0+) = \varphi(x) \text{ for } 0 < x < \infty \\
&v(0+, t) = h(t) \text{ for } 0 < t < \infty
\end{aligned}$$

If such v exists and moreover we let $V(x,t) \triangleq v(x,t) - h(t)$, then we see V satisfy

$$\begin{aligned}
&V_t - kV_{xx} = f(x,t) - h'(t) \text{ for } 0 < x < \infty, 0 < t < \infty \\
&V(x, 0+) = \varphi(x) - h(0) \text{ for } 0 < x < \infty \\
&V(0+, t) = 0 \text{ for } 0 < t < \infty
\end{aligned}$$

Such V can be solved with a reflection.

Duhamel's principle basically says that if you differentiate a convolution $Z(t)$ between kernel S and another function $Y(t)$, where S is dependent on t, then $Z'(t) = AZ(t) + Y(t)$ where $\frac{d}{dt}S = AS$.

Question 100

Solve the inhomogeneous diffusion equation on the half-line with Dirichlet boundary condition:

$$\begin{cases} u_t - ku_{xx} = f(x,t) \text{ for } x \in (0, \infty) \textbf{ (Non-homogeneous DE)} \\ u(x,0) = \varphi(x) \textbf{ (IC)} \\ u(0,t) = 0 \textbf{ (BC)} \end{cases}$$

using the method of reflection.

Proof. Define

$$f_{\text{odd}}(x,t) \triangleq \begin{cases} f(x,t) & \text{if } x > 0 \\ -f(-x,t) & \text{if } x < 0 \end{cases} \text{ and } \varphi_{\text{odd}}(x) \triangleq \begin{cases} \varphi(x) & \text{if } x > 0 \\ -\varphi(-x) & \text{if } x < 0 \end{cases}$$

The formula

$$u(x,t) = \int_{\mathbb{R}} S(x-y,t)\varphi_{\text{odd}}(y)dy + \int_0^t \int_{\mathbb{R}} S(x-y,t-s) f_{\text{odd}}(y,s)dyds$$

then satisfy

$$\begin{cases} u_t - ku_{xx} = f_{\text{odd}}(x,t) \textbf{ (Non-homogeneous DE)} \\ u(x,0) = \varphi_{\text{odd}}(x) \textbf{ (IC)} \\ u(x,t) = -u(-x,t) \text{ for all } x \in \mathbb{R}^* \textbf{ (BC for restriction)} \end{cases}$$

It then follows that the restriction of u on the half line is a solution to the original problem.

■

Question 101

Solve the completely inhomogeneous diffusion equation problem on the half-line

$$\begin{cases} v_t - kv_{xx} = f(x,t) \text{ for } x \in (0,\infty) \textbf{ (Non-homogeneous DE)} \\ v(x,0) = \varphi(x) \textbf{ (IC)} \\ v_x(0,t) = h(t) \textbf{ (BC)} \end{cases}$$

by carrying out the subtraction method begun in the text.

Proof. Define

$$w(x,t) \triangleq v(x,t) - xh(t)$$

We see

$$\begin{cases} w_t - kw_{xx} = f(x,t) - xh'(t) \text{ for } x \in (0,\infty) \textbf{ (Non-homogeneous DE)} \\ w(x,0) = \varphi(x) - xh(0) \textbf{ (IC)} \\ w_x(0,t) = 0 \textbf{ (Good BC)} \end{cases}$$

Define

$$g_{\text{even}} \triangleq \begin{cases} f(x,t) - xh'(t) & \text{if } x > 0 \\ f(-x,t) + xh'(t) & \text{if } x < 0 \end{cases} \text{ and } \psi_{\text{even}}(x) \triangleq \begin{cases} \varphi(x) - xh(0) & \text{if } x > 0 \\ \varphi(-x) + xh(0) & \text{if } x < 0 \end{cases}$$

We then see

$$w(x,t) = \int_{\mathbb{R}} S(x-y,t)\psi_{\text{even}}(y)dy + \int_0^t \int_{\mathbb{R}} S(x-y,t-s)g_{\text{even}}(y)dyds$$

■

Question 102

Solve the inhomogeneous Neumann diffusion problem on the half-line

$$\begin{cases} w_t - kw_{xx} = 0 \text{ for } x \in (0,\infty) \textbf{ (Homogeneous DE)} \\ w_x(0,t) = h(t) \textbf{ (Non-homogeneous BC)} \\ w(x,0) = \varphi(x) \textbf{ (IC)} \end{cases}$$

by the subtraction method indicated in the text.

Proof. Suppose w is a solution to our problem. If we define

$$u(x,t) \triangleq w(x,t) - xh(t) \text{ for } x \in (0,\infty)$$

We see that u satisfy

$$\begin{cases} u_t - ku_{xx} = -xh'(t) \text{ for } x \in (0,\infty) \textbf{ (Non-homogeneous DE)} \\ u_x(0,t) = 0 \textbf{ (Good BC)} \\ u(x,0) = \varphi(x) - xh(0) \textbf{ (IC)} \end{cases}$$

Define

$$f_{\text{even}}(x,t) \triangleq \begin{cases} -xh'(t) & \text{if } x > 0 \\ xh'(t) & \text{if } x < 0 \end{cases}$$

And define

$$\varphi_{\text{even},*} \triangleq \begin{cases} \varphi(x) - xh(0) & \text{if } x > 0 \\ \varphi(-x) + xh(0) & \text{if } x < 0 \end{cases}$$

And define

$$u_{\text{even}}(x,t) \triangleq \int_{\mathbb{R}} S(x-y)\varphi_{\text{even},*}(y)dy + \int_0^t \int_{\mathbb{R}} S(x-y,t-s)f_{\text{even}}(y,s)dyds$$

It then follows that u_{even} solve the non-homogeneous DE and IC. To see that $u_x(0,t) = 0$, one simply observe that u is even in x. The solution to the original problem is then

$$w(x,t) \triangleq u_{\text{even}}(x,t) + xh(t) \text{ for } x \in (0,\infty)$$

■

4.4 3.4 Waves with a source

We first offer a formula

$$F(t) \triangleq \int_{s_0}^{t} f(t,s)ds \implies F'(t) = f(t,t) + \int_{s_0}^{t} f_t(t,s)ds$$

Theorem 4.4.1. (Waves with a source) Consider the non-homogeneous wave equation

$$\begin{aligned} &u_{tt} - c^2 u_{xx} = f(x,t) \text{ for } -\infty < x < \infty \\ &u(x,0) = \varphi(x) \\ &u_t(x,0) = \psi(x) \end{aligned}$$

The solution is

$$u(x,t) = \frac{1}{2}[\varphi(x+ct) + \varphi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds$$

Proof. It is easily checked that $u(x,0) = \varphi(x)$. We now compute

$$\begin{aligned} u_t(x,t) &= \frac{1}{2}[\psi(x+ct) - \psi(x-ct)] \\ &\quad + \frac{1}{2c}\Big[\int_x^x f(y,t)dy + c\int_0^t f(x+c(t-s),s) - f((x-c(t-s)),s)ds\Big] \end{aligned}$$

which give us $u_t(x,0+) = \psi(x)$. ■

Note that the solution immediately implies the stability in the following form

$$|(u_1 - u_2)(x,t)| \leq \|\varphi_1 - \varphi_2\|_\infty + t\|\psi_1 - \psi_2\|_\infty + \frac{1}{2c} \cdot \frac{2ct^2}{2} \cdot \|f_1 - f_2\|_{\infty,T}$$

where

$$\|f_1 - f_2\|_{\infty,T} = \max_{0 \leq t \leq T, x \in \mathbb{R}} |(f_1 - f_2)(x,t)|$$

Question 103

Solve

$$\begin{cases} u_{tt} - c^2 u_{xx} = xt \ \textbf{(Non-homogeneous DE)} \\ u(x,0) = 0, u_t(x,0) = 0 \ \textbf{(IC)} \end{cases}$$

Proof.

$$u = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} ysdyds$$
$$= \frac{xt^3}{6}$$

■

Question 104

Solve

$$\begin{cases} u_{tt} - c^2 u_{xx} = e^{ax} \text{ (Non-homogeneous DE)} \\ u(x,0) = 0, u_t(x,0) = 0 \text{ (IC)} \end{cases}$$

Proof.

$$u = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} e^{ay} dyds$$
$$= \frac{e^{a(x+ct)} - 2e^{ax} + e^{a(x-ct)}}{2ac}$$

■

Question 105

Solve

$$\begin{cases} u_{tt} - c^2 u_{xx} = \cos x \text{ (Non-homogeneous DE)} \\ u(x,0) = \sin x, u_t(x,0) = 1 + x \text{ (IC)} \end{cases}$$

Proof.

$$u = \frac{\sin(x+ct) + \sin(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} (1+s)ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \cos y dyds$$
$$= \frac{\sin(x+ct) + \sin(x-ct)}{2} + (x+1)t + \frac{2\cos x - \cos(x+ct) - \cos(x-ct)}{2c^2}$$

■

Question 106

Given the wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx} \text{ for } x \in (0, \infty) \textbf{ (Homogeneous DE)} \\ u(x,0) = 0 \textbf{ (IC)} \\ u(0,t) = h(t) \textbf{ (BC)} \end{cases}$$

Show that the solution is

$$u(x,t) \triangleq \begin{cases} h(t - \frac{x}{c}) & \text{if } x < ct \\ 0 & \text{if } x \geq ct \end{cases}$$

Proof. Check manually. ■

Question 107

Solve

$$\begin{cases} u_{tt} = c^2 u_{xx} \text{ for } x \in (0, \infty) \textbf{ (Homogeneous DE)} \\ u(x,0) = x, u_t(x,0) = 0 \textbf{ (IC)} \\ u(0,t) = t^2 \textbf{ (BC)} \end{cases}$$

Proof. If we define

$$w = u - t^2$$

we see that w satisfy

$$\begin{cases} w_{tt} - c^2 w_{xx} = -2 \text{ for } x \in (0, \infty) \textbf{ (Non-homogeneous DE)} \\ w(x,0) = x, w_t(x,0) = 0 \textbf{ (IC)} \\ w(0,t) = 0 \textbf{ (Dirichlet BC)} \end{cases}$$

then if we define

$$\varphi_{\text{odd}}(x) \triangleq x \text{ and } f_{\text{odd}}(x,t) \triangleq \begin{cases} -2 & \text{if } x > 0 \\ 2 & \text{if } x < 0 \end{cases}$$

and

$$\begin{aligned} w(x,t) &\triangleq \frac{\varphi(x+ct) + \varphi(x-ct)}{2} + \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f_{\text{odd}}(y,s) dyds \\ &= x + \begin{cases} -t^2 & \text{if } x - ct > 0 \\ \frac{x^2}{c^2} - \frac{2tx}{c} & \text{if } x - ct < 0 \end{cases} \end{aligned}$$

Note that we only consider when $x \geq 0$. This then give us

$$u(x,t) = \begin{cases} x & \text{if } x - ct > 0 \\ x + (t - \frac{x}{c})^2 & \text{if } x - ct < 0 \end{cases}$$

$\blacksquare$

4.5 3.5 Diffusion Revisited

4.6 Cheat Sheet

The most fundamental wave equation is

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \text{ for } x \in \mathbb{R} \textbf{ (Non-homogeneous DE)} \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \textbf{ (IC)} \end{cases}$$

The formula for this equation is

$$u(x,t) \triangleq \frac{\varphi(x+ct) - \varphi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds \tag{4.1}$$

It is easy to see that Formula 4.1 agree with the formula we have for solving homogeneous wave equation. Sometimes, the question deforms, and ask you to solve

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \text{ for } x \in (0,\infty) \textbf{ (Homogeneous DE)} \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \textbf{ (IC)} \\ u(0,t) = 0 \text{ or } u_t(0,t) = 0 \textbf{ (BC)} \end{cases}$$

If the boundary condition is Dirichlet, i.e., $u(0,t) = 0$, we simply extend φ and ψ in odd fashion. If the boundary condition is Neumann, i.e., $u_t(0,t) = 0$, we extend φ and ψ in even fashion.

The most fundamental diffusion equation is

$$\begin{cases} u_t - k u_{xx} = f(x,t) \text{ for } x \in \mathbb{R} \textbf{ (Non-homogeneous DE)} \\ u(x,0) = \varphi(x) \textbf{ (IC)} \end{cases}$$

The formula for this equation is

$$u(x,t) \triangleq \int_{\mathbb{R}} S(x-y,t)\varphi(y)dy + \int_0^t \int_{\mathbb{R}} S(x-y,t-s)f(y,s)dyds$$

Sometimes the question deform and ask you to solve

$$\begin{cases} u_t - k u_{xx} = f(x,t) \text{ for } x \in (0,\infty) \textbf{ (Non-homogeneous DE)} \\ u(x,0) = \varphi(x) \textbf{ (IC)} \\ u(0,t) = 0 \text{ or } u_x(0,t) = 0 \text{ or } u_x(0,t) = h(t) \textbf{ (BC)} \end{cases}$$

If BC is Dirichlet, we simply extend f, φ in odd fashion. If BC is Neumann, we simply extend f, φ in even fashion. If BC is $u_x(0,t) = h(t)$, we define $w = u - xh(t)$, and do odd extension to solve w.

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Chapter 5

PDE 3

5.1 4.1 Separation of Variables, the Dirichlet Condition

Consider the wave equation on some finite interval with Dirichlet boundary condition

$$\begin{cases} u_{tt} = c^2 u_{xx} \text{ for } x \in (0,l) \ \textbf{(Homogeneous DE)} \\ u(0,t) = u(l,t) = 0 \ \textbf{(Dirichlet BC)} \end{cases}$$

If we suppose

$$u(x,t) = X(x)T(t)$$

where $X : [0,l] \to \mathbb{R}, T : \mathbb{R} \to \mathbb{R}$, we see from the wave equation that

$$T''(t)X(x) = c^2 X''(x)T(t)$$

WOLG, we can deduce

$$\frac{X''}{X} = \frac{T''}{c^2 T} = -\lambda$$

where λ is a constant since $\frac{X''}{X}$ only depend on x and $\frac{T''}{c^2T}$ only depend on t. This then give us the ODEs

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(l) = 0 \end{cases} \quad \text{and } T'' + c^2 \lambda T = 0$$

For X to have a non-trivial solution, we must have $\lambda = (\frac{n\pi}{l})^2$. We now know the solution of this ODE is

$$T(t) \triangleq A\cos(\frac{cn\pi t}{l}) + B\sin(\frac{cn\pi t}{l}) \text{ and } X(x) \triangleq D\sin(\frac{n\pi x}{l})$$

Some tedious effort can be used to verify that

$$u(x,t) \triangleq \Big[A\cos(\frac{cn\pi t}{l}) + B\sin(\frac{cn\pi t}{l})\Big] D\sin(\frac{n\pi x}{l})$$

indeed satisfy the wave equation.

Now consider the heat equation on some finite interval with Dirichlet boundary condition

$$\begin{cases} u_t = ku_{xx} \text{ for } x \in (0,l) \textbf{ (Homogeneous DE)} \\ u(0,t) = u(l,t) = 0 \textbf{ (Dirichlet BC)} \end{cases}$$

If we suppose

$$u(x,t) = T(t)X(x)$$

where $X : [0,l] \to \mathbb{R}, T : [0,\infty) \to \mathbb{R}$, we see from the heat equation that

$$T'(t)X(x) = kT(t)X''(x)$$

WOLG we can deduce

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = -\lambda$$

which give us the following ODEs

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(l) = 0 \end{cases} \quad \text{and } T'(t) = -\lambda k T(t)$$

If we wish X to have a non-trivial solution, then we must require $\lambda = (\frac{n\pi}{l})^2$. We now can solve these ODEs to have

$$T(t) = Ae^{-(\frac{n\pi}{l})^2 kt} \text{ and } X(x) = B\sin(\frac{n\pi x}{l})$$

Some tedious effort can now be used to show that

$$u(x,t) = ABe^{-(\frac{n\pi}{l})^2 kt}\sin(\frac{n\pi x}{l})$$

indeed satisfy the heat equation.

Question 108

Consider waves in a resistant medium that satisfy the problem

$$\begin{cases} u_{tt} = c^2 u_{xx} - r u_t \text{ for } 0 < x < l \textbf{ (Homogeneous DE)} \\ u(0,t) = u(l,t) = 0 \textbf{ (Dirichlet BC)} \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \textbf{ (IC)} \end{cases}$$

where r is a constant, $0 < r < \frac{2\pi c}{l}$. Write down the series expansion of the solution.

Question 109

Separate the variables for the equation

$$\begin{cases} t u_t = u_{xx} + 2u \textbf{ (Non-homogeneous DE)} \\ u(0,t) = u(\pi,t) = 0 \textbf{ (BC)} \end{cases}$$

Shows that there are infinite number of solutions that satisfy the initial condition $u(x,0) = 0$. So uniqueness is false for this question.

5.2 4.2 The Neumann Condition

Question 110

Consider the equation

$$\begin{cases} u_{tt} = c^2 u_{xx} \text{ for } 0 < x < l \textbf{ (Homogeneous DE)} \\ u_x(0,t) = 0, u(l,t) = 0 \textbf{ (BC)} \end{cases}$$

(a) Shows that the eigenfunctions are $\cos \frac{(n+\frac{1}{2})\pi x}{l}$.

(b) Write the series expansion for a solution $u(x,t)$.

Question 111

Solve the Schrodinger equation

$$\begin{cases} u_t = iku_{xx} \text{ for } 0 < x < l \textbf{ (Homogeneous DE)} \\ u_x(0,t) = u(l,t) = 0 \textbf{ (BC)} \end{cases}$$

for real $k \in (0,l)$.

Proof. Again we do the separation of the variables

$$u \triangleq T(t)X(x)$$

Some tedious efforts shows that u is a solution of this original question as long as X,T satisfy the following ODE

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) = X(l) = 0 \end{cases} \quad \text{and } T'(t) + \lambda ikT(t) = 0$$

where $\lambda \in \mathbb{C}$ is arbitrary constant. The solution of the second ODE is obviously

$$T(t) \triangleq Ce^{-\lambda ikt}$$

where $C \in \mathbb{C}$ is arbitrary constant. It remains to find what values can λ take so that X has non-trivial solutions. If $\lambda = 0$, then to satisfy the ordinary differential equation, the solution must take the forms $X = C + Dx$, where $C, D \in \mathbb{C}$ are arbitrary constant. Plugging the initial conditions, we see that $C = D = 0$. In other words, if $\lambda = 0$, then X can only be trivial. If $\lambda \neq 0$, ODE of X suggest that X must take the form

$$X \triangleq Ae^{\gamma x} + Be^{-\gamma x}$$

where $\gamma \in \mathbb{C}$ satisfy $\gamma^2 = -\lambda$. Plug in $X'(0) = 0$, we see

$$0 = \gamma(A - B)$$

which implies $A = B$. Plug in $X(l) = 0$, we see

$$0 = Ae^{\gamma l} + Be^{-\gamma l} = A(e^{\gamma l} + e^{-\gamma l})$$

Then for X to be non-trivial, we must have

$$e^{\gamma l} + e^{-\gamma l} = 0$$

By periodicity property of exponential function, we then can deduce

$$\gamma = \frac{i\pi(2n+1)}{l} \text{ and } \lambda = \frac{(2n+1)^2\pi^2}{l^2}$$

It then follows from $X = Ae^{\gamma x} + Be^{-\gamma x}$ that

$$\begin{aligned} X &= (A+B)\cos\left(\frac{\pi(2n+1)x}{l}\right) + (A-B)i\sin\left(\frac{\pi(2n+1)x}{l}\right) \\ &= (A+B)\cos\left(\frac{\pi(2n+1)x}{l}\right) \quad (\because A = B) \end{aligned}$$

■

Question 112

Consider diffusion inside an enclosed circular tube. Let its length (circumference) be $2l$. Let x denote the arc length parameter where $-l \leq x \leq l$. Then the concentration of the diffusing substance satisfies

$$\begin{cases} u_t = ku_{xx} \text{ for } -l \leq x \leq l \textbf{ (Homogeneous DE)} \\ u(-l,t) = u(l,t) \text{ and } u_x(-l,t) = u_x(l,t) \textbf{ (BC)} \end{cases}$$

These are called periodic boundary conditions.

(a) Show that the eigenvalues are $\lambda = (\frac{n\pi}{l})^2$ for $n \in \mathbb{Z}_0^+$.

(b) Show that the concentration is

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{l} + B_n\sin\frac{n\pi x}{l}\right)e^{\frac{-n^2\pi^2 kt}{l^2}}$$

5.3 4.3 Robin Condition

Consider the diffusion equation on finite interval with robin conditions

$$\begin{cases} u_t = ku_{xx} \text{ for } 0 < x < l \textbf{ (Homogeneous DE)} \\ (u_x - a_0 u)(0,t) = 0 \text{ and } (u_x + a_l u)(l,t) = 0 \textbf{ (Robin BC)} \end{cases}$$

Some tedious effort shows that $u \triangleq T(t)X(x)$ is a solution of the original equation as long as X, T satisfy the following ODE

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) - a_0 X(0) = 0 \\ X'(l) + a_l X(l) = 0 \end{cases} \quad \text{and } T'(t) + \lambda k T(t) = 0$$

Then T must be

$$T(t) \triangleq Ce^{-\lambda kt}$$

where $C \in \mathbb{C}$ is arbitrary constant. It remains to find λ so that X has a non-trivial solution. If $\lambda = 0$, then X must be of the form

$$X(x) \triangleq A + Bx$$

where $A, B \in \mathbb{C}$ are constants. Initial conditions implies

$$B - a_0 A = 0 \text{ and } B + a_l(A + Bl) = 0$$

This then shows that X has a nontrivial solution if and only if

$$\begin{bmatrix} -a_0 & 1 \\ a_l & la_l + 1 \end{bmatrix} \text{ is non-singular, i.e., } a_0 + a_l = -a_0 a_l l$$

If $\lambda \in \mathbb{C}^*$, then X must be of the form

$$X(x) \triangleq Ae^{\gamma x} + Be^{-\gamma x}$$

where $A, B \in \mathbb{C}$ are constants and γ satisfy $\gamma^2 = -\lambda$. Initial conditions at $x = 0$ implies

$$\gamma(A - B) - a_0(A + B) = X'(0) - a_0 X(0) = 0$$

which implies

$$\gamma = \frac{a_0(A+B)}{A-B} \text{ and } \lambda = \frac{-a_0^2(A+B)^2}{(A-B)^2}$$

Initial conditions at $x = l$ implies

$$e^{\gamma l}(\gamma A + a_l A) + e^{-\gamma l}(-\gamma B + a_l B) = X'(l) + a_l X(l) = 0$$

In summary

$$\begin{bmatrix} \gamma - a_0 & -\gamma - a_0 \\ e^{\gamma l}(\gamma + a_l) & e^{-\gamma l}(-\gamma + a_l) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

Then for X to have non-trivial solution, we must have

$$e^{-\gamma l}(-\gamma + a_l)(\gamma - a_0) + e^{\gamma l}(\gamma + a_l)(\gamma + a_0) = 0$$

Question 113

Consider the eigenvalue problem with Robin BCs at both ends:

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) - a_0 X(0) = 0 \\ X'(l) + a_l X(l) = 0 \end{cases}$$

(a) Show that $\lambda = 0$ is an eigenvalue if and only if $a_0 + a_l = -a_0 a_l l$.

(b) Find the eigenfunctions corresponding to the zero eigenvalue.

Proof. Suppose $\lambda = 0$ is an eigenvalue. Then $X'' + \lambda X = 0$ implies $X = C + Dx$ where C, D are constants and they can not both be zero. The initial conditions can now be rewritten as

$$\begin{bmatrix} -a_0 & 1 \\ a_l & la_l + 1 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then because C, D can not both be zero, we see that the left matrix $\begin{bmatrix} -a_0 & 1 \\ a_l & la_l + 1 \end{bmatrix}$ must be singular. Computing the determinant, we now have $a_0 + a_l = -a_0 a_l l$.

Suppose $a_0 + a_l = -a_0 a_l l$. One can see that $1 + a_0 x$ spans the eigenspace. ∎

Question 114

On the interval $0 \leq x \leq 1$ of length one, consider the eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) + X(0) = 0 \\ X(1) = 0 \end{cases}$$

(a) Find an eigenfunction with eigenvalue zero.

(b) Find an equation for the positive eigenvalues $\lambda = \beta^2$.

(c) Show graphically from part (b) that there are an infinite number of positive eigenvalues.

(d) Is there a negative eigenvalue?

Proof. If eigenvalue $\lambda = 0$ is zero, then X must take the form $C + Dx$, where C, D are constants that can not both be zero. The initial condition can now be rewritten as $C + D = 0$. In other words, $x - 1$ spans the eigenspace.

If eigenvalue is positive $\lambda = \beta^2$ where $\beta > 0$, then X must take the form

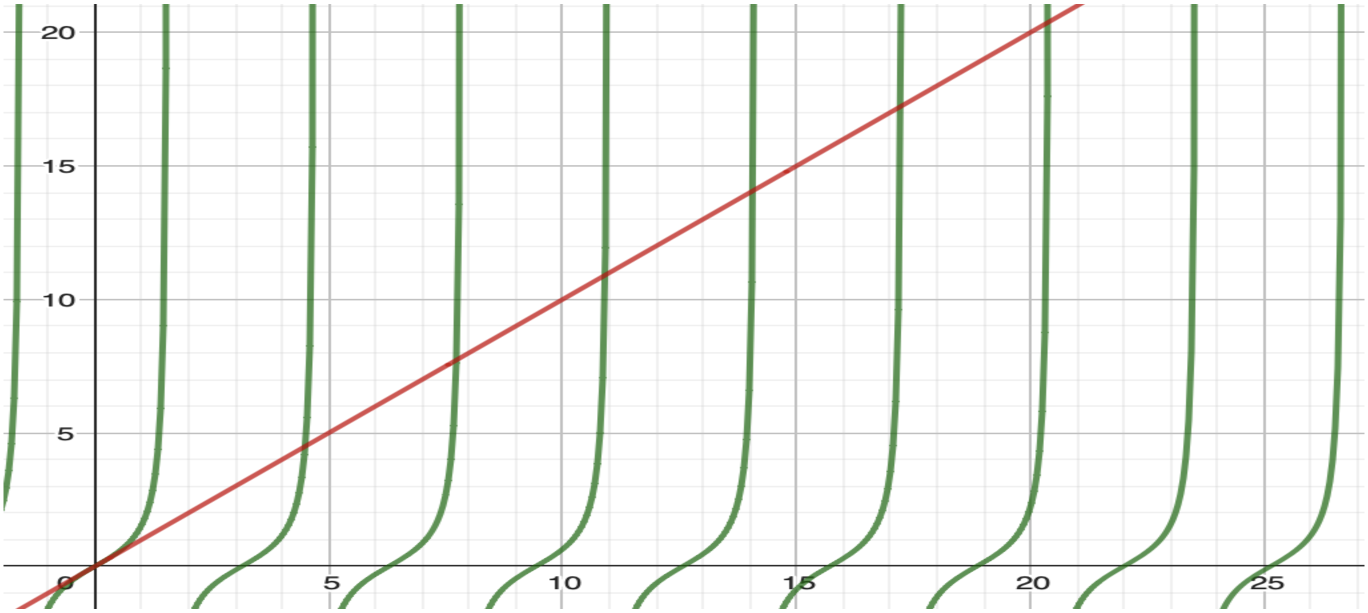
$$X = A\cos(\beta x) + B\sin(\beta x)$$

The initial condition can now be rewritten as

$$\begin{cases} \beta B + A = 0 \\ A\cos\beta + B\sin\beta = 0 \end{cases} \quad \text{or} \quad \begin{bmatrix} 1 & \beta \\ \cos\beta & \sin\beta \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

Then for λ to be an eigenvalue, we must have

$$\sin\beta = \beta\cos\beta$$



If eigenvalue is negative $\lambda = -\beta^2$ where $\beta > 0$, then X must take the form

$$X = A\cosh(\beta x) + B\sinh(\beta x)$$

The initial condition can now be rewritten as

$$\begin{cases} \beta B + A = 0 \\ A\cosh(\beta) + B\sinh(\beta) = 0 \end{cases} \quad \text{or} \quad \begin{bmatrix} 1 & \beta \\ \cosh(\beta) & \sinh(\beta) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

This implies

$$\beta = \tanh(\beta)$$

This equation have no positive solution β, so there are no negative eigenvalues. ■

Question 115

Let $a_0 < 0, a_l > -a_0$ and $a_0 + a_l < -a_0 a_l l$. Solve the wave equations on finite interval with robin conditions

$$\begin{cases} u_{tt} = c^2 u_{xx} \text{ for } 0 < x < l \ \textbf{(Homogeneous DE)} \\ (u_x - a_0 u)(0,t) = 0 \text{ and } (u_x + a_l u)(l,t) = 0 \ \textbf{(Robin BC)} \end{cases}$$

Chapter 6

PDE HW

6.1 PDE HW 1

Theorem 6.1.1.

$$\text{Show } u \mapsto u_x + uu_y \text{ is non-linear}$$

Proof. See that

$$2u \mapsto 2u_x + 4uu_y \neq 2(u_x + uu_y) \tag{6.1}$$

■

Theorem 6.1.2.

$$\text{Solve } (1 + x^2)u_x + u_y = 0$$

Proof. The characteristic curve has the derivative

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

The solution to this ODE is

$$y = \arctan x + C$$

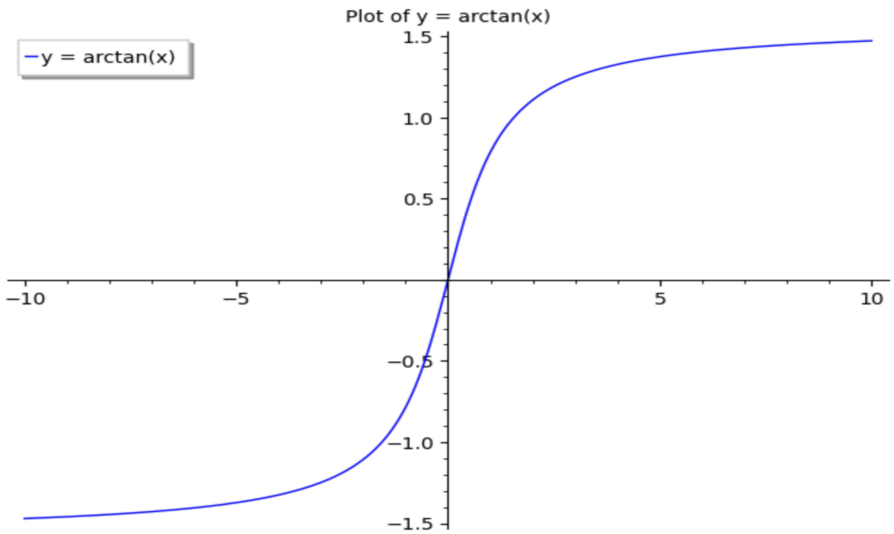
We now see that the solution to the PDE in Equation 6.1 is

$$u = f\big((\arctan x) - y\big) \text{ where } f : \mathbb{R} \to \mathbb{R} \text{ is an arbitrary smooth function}$$

A characteristic curve is as followed.

[3]:
```
plot(arctan(x), (x, -10, 10), title='Plot of y = arctan(x)', legend_label='y = arctan(x)')
```

[3]:



$\blacksquare$

Theorem 6.1.3.

$$\text{Solve } au_x + bu_y + cu = 0 \tag{6.2}$$

Proof. Fix

$$\begin{cases} x' \triangleq ax + by \\ y' \triangleq bx - ay \end{cases}$$

This map is clearly a diffeomorphism. Compute

$$\begin{cases} u_x = \dfrac{\partial u}{\partial x'}\dfrac{\partial x'}{\partial x} + \dfrac{\partial u}{\partial y'}\dfrac{\partial y'}{\partial x} = au_{x'} + bu_{y'} \\ u_y = \dfrac{\partial u}{\partial x'}\dfrac{\partial x'}{\partial y} + \dfrac{\partial u}{\partial y'}\dfrac{\partial y'}{\partial y} = bu_{x'} - au_{y'} \end{cases}$$

Plugging it back into the PDE in Equation 6.2, we have

$$cu + (a^2 + b^2)u_{x'} = 0 \tag{6.3}$$

If $c = a^2 + b^2 = 0$, then all smooth functions are solution. If $a^2 + b^2 = 0$ but $c \neq 0$, then clearly the only solution is $u = \tilde{0}$. If $a^2 + b^2 \neq 0$ but $c = 0$, then $u_{x'} = \tilde{0}$, which implies $u = f(y')$ where $y' = bx - ay$ and f can be arbitrary smooth function.

Now, suppose $a^2 + b^2 \neq 0 \neq c$, note that the PDE in Equation 6.3 is just an ODE of the form

$$y + \frac{a^2 + b^2}{c}y' = 0$$

The general solution to this ODE is

$$y = Ce^{\frac{-ct}{a^2+b^2}}$$

In other words, the general solution of the PDE in Equation 6.3 is

$$u = Ce^{\frac{-cx'}{a^2+b^2}} = Ce^{\frac{-c(ax+by)}{a^2+b^2}}$$

■

6.2 PDE HW 2

Question 116

Consider hear flow in a long circular cylinder where the temperature depends only on t and on the distance r to the axis of the cylinder. Here $r = \sqrt{x^2 + y^2}$ is the cylindrical coordinate. From the three dimensional hear equation derive the equation $u_t = k(u_{rr} + \frac{u_r}{r})$

Proof. Write the three dimensional hear equation by

$$u_t = k\Delta u$$

Note that the Laplacian Δu when written in cylindrical coordinate is

$$\Delta u = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} + u_{zz}$$

Because the premise says that u is constant in z and θ, we know $u_{\theta\theta} = u_{zz} = 0$

$$\Delta u = u_{rr} + \frac{u_r}{r}$$

This give us

$$u_t = k(u_{rr} + \frac{u_r}{r})$$

■

6.3 PDE HW 3

Question 117

Find a solution of

$$\begin{cases} u_t = u_{xx} \\ u(x,0) = x^2 \end{cases}$$

Proof. Clearly $u = x^2 + 2t$ suffices. ■

Question 118

Consider the ODE

$$\begin{cases} u'' + u' = f \\ u'(0) = u(0) = \frac{1}{2}(u'(l) + u(l)) \end{cases}$$

where f is given.

(a) Is the solution unique?

(b) Does a solution necessarily exist, or is there a condition that f must satisfy for existence?

Proof. The solution space of linear homogeneous ODE $u'' + u' = 0$ is spanned by e^{-x} and constant. If we add in the initial condition $u'(0) = u(0)$, then the solution space become the subspace spanned by $e^{-x} - 2$. One can check that if $u \in \text{span}(e^{-x} - 2)$, then

$$u(0) = \frac{1}{2}(u'(l) + u(l)) \text{ for all } l \in \mathbb{R}$$

We now know the solution of the original ODE is not unique, since any solution added by $e^{-x} - 2$ is again a solution.

Integrating both side on $[0, l]$, we see that given the boundary conditions, f must satisfy

$$\begin{aligned} \int_0^l f(x)dx &= \int_0^l u'' + u'dx \\ &= u(l) + u'(l) - u(0) - u'(0) = 0 \end{aligned}$$

■

Question 119

Find the regions in the xy plane where the equation

$$(1+x)u_{xx} + 2xyu_{xy} - y^2u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch them.

Proof. The discriminant is exactly

$$\begin{aligned}(xy)^2 - (1+x)(-y^2) &= x^2y^2 + xy^2 + y^2 \\ &= y^2(x^2 + x + 1) \\ &= y^2[(x + \frac{1}{2})^2 + \frac{3}{4}]\end{aligned}$$

It then follows that the equation is parabolic if and only if $y = 0$, and hyperbolic if and only if $y \neq 0$.

■

6.4 PDE HW 4

Question 120

Solve

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = e^x \\ u_t(x,0) = \sin x \end{cases}$$

Proof. Let

$$u \triangleq f(x+ct) + g(x-ct)$$

Plugging the initial conditions, we know

$$\begin{cases} f(x) + g(x) = e^x \\ f'(x) - g'(x) = \frac{\sin x}{c} \end{cases}$$

This give us

$$f'(x) = \frac{e^x + \frac{\sin x}{c}}{2} \text{ and } g'(x) = \frac{e^x - \frac{\sin x}{c}}{2}$$

Then by FTC, we have

$$\begin{cases} f(x) = \frac{e^x - \frac{\cos x}{c}}{2} + f(0) - \frac{1}{2} + \frac{1}{2c} \\ g(x) = \frac{e^x + \frac{\cos x}{c}}{2} + g(0) - \frac{1}{2} - \frac{1}{2c} \end{cases}$$

Note that $f(0) + g(0) = e^0 = 1$, which cancel the constant terms in u, i.e.

$$\begin{aligned} u &= f(x+ct) + g(x-ct) \\ &= \frac{e^{x+ct} + e^{x-ct} + \frac{-\cos(x+ct)+\cos(x-ct)}{c}}{2} \end{aligned}$$

∎

Question 121

Solve

$$\begin{cases} u_{xx} + u_{xt} - 20u_{tt} = 0 \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

Proof. The PDE can be write in the form of

$$(\partial_x + 5\partial_t)(\partial_x - 4\partial_t)u = 0$$

which have the general solution

$$u(x,t) = f(5x - t) + g(4x + t)$$

We now see

$$f(5x) + g(4x) = \varphi(x) \text{ and } -f'(5x) + g'(4x) = \psi(x)$$

This then give us

$$f'(5x) = \frac{(\varphi' - 4\psi)(x)}{9} \text{ and } g'(4x) = \frac{(\varphi' + 5\psi)(x)}{9}$$

and thus

$$\begin{aligned} f(x) &= f(0) + \int_0^x f'(s)ds \\ &= f(0) + \int_0^x \frac{(\varphi' - 4\psi)(\frac{s}{5})}{9} ds \\ &= f(0) + \frac{5}{9}\Big[\varphi(\frac{x}{5}) - \varphi(0)\Big] - \frac{4}{9}\int_0^x \psi(\frac{s}{5})ds \end{aligned}$$

and similarly

$$\begin{aligned} g(x) &= g(0) + \int_0^x g'(s)ds \\ &= g(0) + \int_0^x \frac{(\varphi' + 5\psi)(\frac{s}{4})}{9} ds \\ &= g(0) + \frac{4}{9}\Big[\varphi(\frac{x}{4}) - \varphi(0)\Big] + \frac{5}{9}\int_0^x \psi(\frac{s}{4})ds \end{aligned}$$

Noting that $f(0)+g(0)=u(0,0)=\psi(0)$, we now have

$$\begin{aligned} u(x,t) &= f(5x-t)+g(4x+t) \\ &= \frac{5\varphi(\frac{5x-t}{5})+4\varphi(\frac{4x+t}{4})}{9} - \frac{4}{9}\int_0^{5x-t}\psi(\frac{s}{5})ds + \frac{5}{9}\int_0^{4x+t}\psi(\frac{s}{4})ds \end{aligned}$$

■

6.5 PDE HW 5

Question 122

Solve

$$\begin{aligned}
u_t &= ku_{xx} \\
u(x,0+) &= e^{-x} \\
u(0+,t) &= 0
\end{aligned}$$

on the half line $0 < x < \infty$

Proof. Extend the initial condition to

$$\varphi_{\text{odd}}(x) \triangleq \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -e^{x} & \text{if } x < 0 \end{cases}$$

We then solve

$$\begin{aligned}
u(x,t) &= \frac{1}{2\sqrt{\pi kt}} \int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4kt}} \varphi_{\text{odd}}(y)dy \\
&= \frac{1}{2\sqrt{\pi kt}} \Big[\int_0^\infty e^{\frac{-(x-y)^2}{4kt}} e^{-y}dy - \int_{-\infty}^0 e^{\frac{-(x-y)^2}{4kt}} e^{y}dy \Big] \\
&= \frac{1}{2\sqrt{\pi kt}} \Big[\int_0^\infty e^{\frac{-(y-(x-2kt))^2-4ktx+4k^2t^2}{4kt}} dy - \int_{-\infty}^0 e^{\frac{-(y-(x+2kt))^2+4ktx+4k^2t^2}{4kt}} dy \Big] \\
&= \frac{1}{2\sqrt{\pi kt}} \Big[e^{-x+kt} \int_0^\infty e^{\frac{-(y-(x-2kt))^2}{4kt}} dy - e^{x+kt} \int_{-\infty}^0 e^{\frac{-(y-(x+2kt))^2}{4kt}} dy \Big] \\
&= \frac{1}{\sqrt{\pi}} \Big[e^{-x+kt} \int_{\frac{2kt-x}{2\sqrt{kt}}}^\infty e^{-p^2}dp - e^{x+kt} \int_{-\infty}^{\frac{-2kt-x}{2\sqrt{kt}}} e^{-p^2}dp \Big] \\
&= \frac{1}{\sqrt{\pi}} \Big[e^{-x+kt} \Big(\frac{\sqrt{\pi}}{2} - \frac{2}{\sqrt{\pi}} \operatorname{erf}(\frac{2kt-x}{2\sqrt{kt}}) \Big) - e^{x+kt} \Big(\frac{\sqrt{\pi}}{2} - \frac{2}{\sqrt{\pi}} \operatorname{erf}(\frac{2kt+x}{2\sqrt{kt}}) \Big) \Big]
\end{aligned}$$

■

6.6 PDE HW 6

Question 123

Solve $u_{tt} = 4u_{xx}$ for $0 < x < \infty, u(0,t) = 0, u(x,0) = 1, u_t(x,0) = 0$ using the reflection method. The solution has a singularity find its location.

Proof. Define

$$\varphi(x) \triangleq \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases} \text{ and } \psi(x) \triangleq 0$$

We are required to solve the following Dirichlet's problem for wave equation

$$u_{tt} = 4u_{xx} \text{ for } -\infty < x < \infty$$
$$u(x,0) = \varphi(x) \text{ and } u_t(x) = \psi(x)$$

The solution is exactly

$$\begin{aligned} u(x,t) &= \frac{\varphi(x+2t) + \varphi(x-2t)}{2} + \int_{x-2t}^{x+2t} \psi(s)ds \\ &= \frac{\varphi(x+2t) + \varphi(x-2t)}{2} \\ &= \begin{cases} 1 & \text{if } x - 2t > 0 \\ 0 & \text{if } x + 2t > 0 > x - 2t \\ -1 & \text{if } 0 > x + 2t \end{cases} \end{aligned}$$

On the half line, the solution is

$$u(x,t) = \begin{cases} 1 & \text{if } x - 2t > 0 \\ 0 & \text{if } x - 2t < 0 \end{cases}$$

So the singularity is exactly on the line $x - 2t = 0$ ∎

Question 124

Solve the inhomogeneous Neumann diffusion problem on the half-line

$$\begin{cases} w_t - kw_{xx} = 0 \text{ for } x \in (0, \infty) \textbf{ (Homogeneous DE)} \\ w_x(0,t) = h(t) \textbf{ (Non-homogeneous BC)} \\ w(x,0) = \varphi(x) \textbf{ (IC)} \end{cases}$$

by the subtraction method indicated in the text.

Proof. Suppose w is a solution to our problem. If we define

$$u(x,t) \triangleq w(x,t) - xh(t) \text{ for } x \in (0,\infty)$$

We see that u satisfy

$$\begin{cases} u_t - ku_{xx} = -xh'(t) \text{ for } x \in (0,\infty) \textbf{ (Non-homogeneous DE)} \\ u_x(0,t) = 0 \textbf{ (Good BC)} \\ u(x,0) = \varphi(x) - xh(0) \textbf{ (IC)} \end{cases}$$

Define

$$f_{\text{even}}(x,t) \triangleq \begin{cases} -xh'(t) & \text{if } x > 0 \\ xh'(t) & \text{if } x < 0 \end{cases}$$

And define

$$\varphi_{\text{even},*} \triangleq \begin{cases} \varphi(x) - xh(0) & \text{if } x > 0 \\ \varphi(-x) + xh(0) & \text{if } x < 0 \end{cases}$$

And define

$$u_{\text{even}}(x,t) \triangleq \int_{\mathbb{R}} S(x-y)\varphi_{\text{even},*}(y)dy + \int_0^t \int_{\mathbb{R}} S(x-y,t-s)f_{\text{even}}(y,s)dyds$$

It then follows that u_{even} solve the non-homogeneous DE and IC. To see that $u_x(0,t) = 0$, one simply observe that u is even in x. The solution to the original problem is then

$$w(x,t) \triangleq u_{\text{even}}(x,t) + xh(t) \text{ for } x \in (0,\infty)$$

■

6.7 PDE HW 7

Question 125

Solve

$$\begin{cases} u_{tt} = c^2 u_{xx} \text{ for } x \in (0,\infty) \textbf{ (Homogeneous DE)} \\ u(x,0) = x, u_t(x,0) = 0 \textbf{ (IC)} \\ u(0,t) = t^2 \textbf{ (BC)} \end{cases}$$

Proof. If we define

$$w = u - t^2$$

we see that w satisfy

$$\begin{cases} w_{tt} - c^2 w_{xx} = -2 \text{ for } x \in (0,\infty) \textbf{ (Non-homogeneous DE)} \\ w(x,0) = x, w_t(x,0) = 0 \textbf{ (IC)} \\ w(0,t) = 0 \textbf{ (Dirichlet BC)} \end{cases}$$

then if we define

$$\varphi_{\text{odd}}(x) \triangleq x \text{ and } f_{\text{odd}}(x,t) \triangleq \begin{cases} -2 & \text{if } x > 0 \\ 2 & \text{if } x < 0 \end{cases}$$

and

$$\begin{aligned} w(x,t) &\triangleq \frac{\varphi(x+ct) + \varphi(x-ct)}{2} + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f_{\text{odd}}(y,s)\,dyds \\ &= x + \begin{cases} -t^2 & \text{if } x - ct > 0 \\ \frac{x^2}{c^2} - \frac{2tx}{c} & \text{if } x - ct < 0 \end{cases} \end{aligned}$$

Note that we only consider when $x \geq 0$. This then give us

$$u(x,t) = \begin{cases} x & \text{if } x - ct > 0 \\ x + (t - \frac{x}{c})^2 & \text{if } x - ct < 0 \end{cases}$$

∎

6.8 PDE HW 8

Question 126

Solve the Schrodinger equation

$$\begin{cases} u_t = iku_{xx} \text{ for } 0 < x < l \textbf{ (Homogeneous DE)} \\ u_x(0,t) = u(l,t) = 0 \textbf{ (BC)} \end{cases}$$

for real $k \in (0, l)$.

Proof. Again we do the separation of the variables

$$u \triangleq T(t)X(x)$$

Some tedious efforts shows that u is a solution of this original question as long as X, T satisfy the following ODE

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) = X(l) = 0 \end{cases} \quad \text{and } T'(t) + \lambda ikT(t) = 0$$

where $\lambda \in \mathbb{C}$ is arbitrary constant. The solution of the second ODE is obviously

$$T(t) \triangleq Ce^{-\lambda ikt}$$

where $C \in \mathbb{C}$ is arbitrary constant. It remains to find what values can λ take so that X has non-trivial solutions. If $\lambda = 0$, then to satisfy the ordinary differential equation, the solution must take the forms $X = C + Dx$, where $C, D \in \mathbb{C}$ are arbitrary constant. Plugging the initial conditions, we see that $C = D = 0$. In other words, if $\lambda = 0$, then X can only be trivial. If $\lambda \neq 0$, ODE of X suggest that X must take the form

$$X \triangleq Ae^{\gamma x} + Be^{-\gamma x}$$

where $\gamma \in \mathbb{C}$ satisfy $\gamma^2 = -\lambda$. Plug in $X'(0) = 0$, we see

$$0 = \gamma(A - B)$$

which implies $A = B$. Plug in $X(l) = 0$, we see

$$0 = Ae^{\gamma l} + Be^{-\gamma l} = A(e^{\gamma l} + e^{-\gamma l})$$

Then for X to be non-trivial, we must have

$$e^{\gamma l} + e^{-\gamma l} = 0$$

By periodicity property of exponential function, we then can deduce

$$\gamma = \frac{i\pi(2n+1)}{l} \text{ and } \lambda = \frac{(2n+1)^2\pi^2}{l^2}$$

It then follows from $X = Ae^{\gamma x} + Be^{-\gamma x}$ that

$$\begin{aligned} X &= (A+B)\cos\left(\frac{\pi(2n+1)x}{l}\right) + (A-B)i\sin\left(\frac{\pi(2n+1)x}{l}\right) \\ &= (A+B)\cos\left(\frac{\pi(2n+1)x}{l}\right) \quad (\because A = B) \end{aligned}$$

$\blacksquare$

Chapter 7

Differential Geometry HW

7.1 HW1

Abstract

In this HW, we give precise definition to $\mathbb{P}^n$ and $\mathbb{R}P^n$, and we rigorously show

(a) $\mathbb{R}P^n$ has a smooth structure.

(b) $\mathbb{P}^n$ is homeomorphic to $\mathbb{R}P^n$

(c) $\mathbb{P}^n$ has a smooth structure.

We also solved the other two questions. Note that in this PDF, brown text is always a clickable hyperlink reference.

Define an equivalence relation on $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ by

$$\mathbf{x} \sim \mathbf{y} \overset{\triangle}{\iff} \mathbf{y} = \lambda \mathbf{x} \text{ for some } \lambda \in \mathbb{R}^*$$

Let $\mathbb{R}P^n \triangleq (\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}) \setminus \sim$ be the quotient space and let

$$V_i \triangleq \{\mathbf{x} \in \mathbb{R}^{n+1} : \mathbf{x}^i \neq 0\} \text{ for each } 1 \leq i \leq n+1$$

By definition, it is clear that

$$\text{either } \pi^{-1}(\pi(\mathbf{x})) \subseteq V_i \text{ or } \pi^{-1}(\pi(\mathbf{x})) \subseteq V_i^c$$

Then if we define $\varphi_i : V_i \to \mathbb{R}^n$ by

$$\varphi_i(\mathbf{x}) \triangleq \Big(\frac{\mathbf{x}^1}{\mathbf{x}^i},\dots,\frac{\mathbf{x}^{i-1}}{\mathbf{x}^i},\frac{\mathbf{x}^{i+1}}{\mathbf{x}^i},\dots,\frac{\mathbf{x}^{n+1}}{\mathbf{x}^i}\Big)$$

because $\pi(\mathbf{x}) = \pi(\mathbf{y}) \implies \varphi_i(\mathbf{x}) = \varphi_i(\mathbf{y})$, we can well induce a map

$$\Phi_i : U_i \triangleq \pi(V_i) \subseteq \mathbb{R}P^n \to \mathbb{R}^n; \pi(\mathbf{x}) \mapsto \varphi_i(\mathbf{x})$$

Note that one has the equation

$$\Phi_i(\pi(\mathbf{x})) = \varphi_i(\mathbf{x}) \text{ for all } \mathbf{x} \in V_i$$

Theorem 7.1.1. (Real Projective Space with a differentiable atlas) We have

$$\mathbb{R}P^n \text{ with atlas } \big\{(U_i,\Phi_i) : 1 \leq i \leq n+1\big\} \text{ is a differentiable manifold}$$

Proof. We are required to prove

(a) (U_i,Φ_i) are all charts.

(b) $\big\{(U_i,\Phi_i) : 1 \leq i \leq n+1\big\}$ form a differentiable atlas.

(c) $\mathbb{R}P^n$ is Hausdorff.

(d) $\mathbb{R}P^n$ is second-countable.

Because $\pi^{-1}(U_i) = V_i$ and V_i is clearly open in $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$, we know $U_i \subseteq \mathbb{R}P^n$ is open. Note that clearly, $\Phi_i(U_i) = \mathbb{R}^n$. To show (U_i,Φ_i) is a chart, it remains to show that Φ_i is a homeomorphism between U_i and $\mathbb{R}^n$. It is straightforward to check Φ_i is one-to-one on U_i. This implies Φ_i is a bijective between U_i and $\mathbb{R}^n$.

Fix open $E \subseteq \mathbb{R}^n$. We see

$$\pi^{-1}(\Phi_i^{-1}(E)) = \varphi_i^{-1}(E)$$

Then because $\varphi_i : V_i \to \mathbb{R}^n$ is clearly continuous, we see $\varphi_i^{-1}(E)$ is open in $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$, and it follows from definition of quotient topology $\Phi_i^{-1}(E) \subseteq \mathbb{R}P^n$ is open. Then because U_i is open in $\mathbb{R}P^n$, we see $\Phi_i^{-1}(E)$ is open in U_i. We have proved $\Phi_i : U_i \to \mathbb{R}^n$ is continuous.

Define $\Psi_i : \mathbb{R}^n \to V_i$ by

$$\Psi(\mathbf{x}^1,\dots,\mathbf{x}^n) = (\mathbf{x}^1,\dots,\mathbf{x}^{i-1},1,\mathbf{x}^i,\dots,\mathbf{x}^n)$$

Observe that for all $\mathbf{x} \in \Phi_i(U_i)$, we have

$$\Phi_i^{-1}(\mathbf{x}) = \pi(\Psi_i(\mathbf{x}))$$

It then follows from $\Psi_i : \mathbb{R}^n \to V_i$ and $\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{R}P^n$ are continuous that $\Phi_i^{-1} : \mathbb{R}^n \to \mathbb{R}P^n$ is continuous.

We have proved that (Ψ_i, U_i) are all charts. Now, because V_i clearly cover $\mathbb{R}^{n+1}$, we know U_i also cover $\mathbb{R}P^n$. We have proved $\{(U_i, \Phi) : 1 \leq i \leq n+1\}$ form an atlas. The fact $\mathbb{R}P^n$ is second-countable follows.

Fix $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \Phi_i(U_i \cap U_j)$. We compute

$$\Phi_j \circ \Phi_i^{-1}(\mathbf{x}^1, \dots, \mathbf{x}^n) = \Phi_j\Big([(\mathbf{x}^1, \dots, \mathbf{x}^{i-1}, 1, \mathbf{x}^i, \mathbf{x}^{i+1}, \dots, \mathbf{x}^n)]\Big)$$
$$= \begin{cases} (\frac{\mathbf{x}^1}{\mathbf{x}^j}, \dots, \frac{\mathbf{x}^{j-1}}{\mathbf{x}^j}, \frac{\mathbf{x}^{j+1}}{\mathbf{x}^j}, \dots, \frac{\mathbf{x}^{i-1}}{\mathbf{x}^j}, \frac{1}{\mathbf{x}^j}, \frac{\mathbf{x}^i}{\mathbf{x}^j}, \dots, \frac{\mathbf{x}^n}{\mathbf{x}^j}) & \text{if } j < i \\ (\frac{\mathbf{x}^1}{\mathbf{x}^{j-1}}, \dots, \frac{\mathbf{x}^{i-1}}{\mathbf{x}^{j-1}}, \frac{1}{\mathbf{x}^{j-1}}, \frac{\mathbf{x}^i}{\mathbf{x}^{j-1}}, \dots, \frac{\mathbf{x}^{j-2}}{\mathbf{x}^{j-1}}, \frac{\mathbf{x}^j}{\mathbf{x}^{j-1}}, \dots, \frac{\mathbf{x}^n}{\mathbf{x}^{j-1}}) & \text{if } j > i \end{cases}$$

This implies our atlas is indeed differentiable.

Before we prove $\mathbb{R}P^n$ is Hausdorff, we first prove that $\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{R}P^n$ is an open mapping. Let $U \subseteq \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ be open. Observe that

$$\pi^{-1}(\pi(U)) = \{t\mathbf{x} \in \mathbb{R}^{n+1} : t \neq 0 \text{ and } \mathbf{x} \in U\}$$

Fix $t_0\mathbf{x} \in \pi^{-1}(\pi(U))$. Let $B_\epsilon(\mathbf{x}) \subseteq U$. Observe that

$$B_{|t_0|\epsilon}(t_0\mathbf{x}) \subseteq t_0 B_\epsilon(\mathbf{x}) \subseteq t_0 U \subseteq \pi^{-1}(\pi(U))$$

This implies $\pi^{-1}(\pi(U))$ is open. (done)

Now, because π is open, to show $\mathbb{R}P^n$ is Hausdorff, we only have to show

$$R_\pi \triangleq \{(\mathbf{x}, \mathbf{y}) \in (\mathbb{R}^{n+1} \setminus \{\mathbf{0}\})^2 : \pi(\mathbf{x}) = \pi(\mathbf{y})\} \text{ is closed}$$

Define $f : (\mathbb{R}^{n+1} \setminus \{\mathbf{0}\})^2 \to \mathbb{R}$ by

$$f(\mathbf{x}, \mathbf{y}) \triangleq \sum_{i \neq j} (\mathbf{x}^i \mathbf{y}^j - \mathbf{x}^j \mathbf{y}^i)^2$$

Note that f is clearly continuous and $f^{-1}(0) = R_\pi$, which finish the proof. ∎

Alternatively, we can characterize $\mathbb{R}P^n$ by identifying the antipodal pints on $S^n \triangleq \{\mathbf{x} \in \mathbb{R}^{n+1} : |\mathbf{x}| = 1\}$ as one point

$$\mathbf{x} \sim \mathbf{y} \overset{\triangle}{\iff} \mathbf{x} = \mathbf{y} \text{ or } \mathbf{x} = -\mathbf{y}$$

and let $\mathbb{P}^n \triangleq S^n \backslash \sim$ be the quotient space.

Theorem 7.1.2. (Equivalent Definitions of Real Projective Space)

$$\mathbb{R}P^n \text{ and } \mathbb{P}^n \text{ are homeomorphic}$$

Proof. Define $F : \mathbb{P}^n \to \mathbb{R}P^n$ by

$$\{\mathbf{x}, -\mathbf{x}\} \mapsto \{\lambda \mathbf{x} : \lambda \in \mathbb{R}^*\}$$

It is straightforward to check that F is well-defined and bijective. Define $f : S^n \to \mathbb{R}P^n$ by

$$f = \pi \circ \mathbf{id}$$

where $\mathbf{id} : S^n \to \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ and $\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{R}P^n$ are continuous. Check that

$$f = F \circ p$$

where $p : S^n \to \mathbb{P}^n$ is the quotient mapping. It now follows from the universal property that F is continuous, and since $\mathbb{P}^n$ is compact and $\mathbb{R}P^n$ is Hausdorff, it also follows that F is a homeomorphism between $\mathbb{R}P^n$ and $\mathbb{P}^n$. ∎

Knowing that $F : \mathbb{P}^n \to \mathbb{R}P^n$ is a homeomorphism and $\mathbb{R}P^n$ is a smooth manifold, we see that $\mathbb{P}^n$ is Hausdorff and second-countable, and if we define the atlas

$$\{(F^{-1}(U_i), \Phi_i \circ F) : 1 \leq i \leq n+1\}$$

We see this atlas is indeed smooth, since

$$(\Phi_i \circ F) \circ (\Phi_j \circ F)^{-1} = \Phi_i \circ \Phi_j^{-1}$$

Question 127

Let X be a set equipped with

(a) a collection $(U_\alpha)_{\alpha \in I}$ of subsets that covers X.

(b) a collection of bijection $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ that maps U_α to an open subset $\varphi_\alpha(U_\alpha)$ of $\mathbb{R}^n$.

(c) For each $\alpha, \beta \in I$, the set $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open.

(d) For each $\alpha, \beta \in I, \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\beta \cap U_\alpha) \to \varphi_\beta(U_\alpha, U_\beta)$ is smooth.

Give X a topology so that X is a smooth manifold.

Proof. If we define $E \subseteq X$ is open if and only if

$$\varphi_\alpha(U_\alpha \cap E) \text{ is open for all } \alpha$$

we see that given arbitrary collection of open sets $(E_j)_{j\in J}$, we have

$$\varphi_\alpha(U_\alpha \cap \bigcup_{j\in J} E_j) = \bigcup_{j\in J} \varphi_\alpha(U_\alpha \cap E_j) \text{ for all } \alpha \in I$$

thus open sets are closed under union, and given two open sets E_1, E_2, we have

$$\varphi_\alpha(U_\alpha \cap E_1 \cap E_2) \subseteq \varphi_\alpha(U_\alpha \cap E_1) \cap \varphi_\alpha(U_\alpha \cap E_2) \text{ for all } \alpha \in I$$

Note that if $\mathbf{x} \in \varphi_\alpha(U_\alpha \cap E_1) \cap \varphi_\alpha(U_\alpha \cap E_2)$, then there exists $p_1 \in U_\alpha \cap E_1$ and $p_2 \in U_\alpha \cap E_2$ such that $\varphi_\alpha(p_1) = \varphi_\alpha(p_2) = \mathbf{x}$. Because φ_α is one-to-one, we can deduce $p_1 = p_2 \in E_2$, it then follows

$$\mathbf{x} = \varphi(p_1) \in \varphi_\alpha(U_\alpha \cap E_1 \cap E_2)$$

We now see

$$\varphi_\alpha(U_\alpha \cap E_1) \cap \varphi_\alpha(U_\alpha \cap E_2) \subseteq \varphi_\alpha(U_\alpha \cap E_1 \cap E_2) \text{ for all } \alpha \in I$$

We have proved that our topology on X is well-defined.

Note that U_α is open in X follows from premise (c). Thus, if some $E \subseteq U_\alpha$ is open in U_α, then E is open in X and $\varphi_\alpha(E) = \varphi_\alpha(U_\alpha \cap E)$ is open. We have proved that $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ is an open mapping. The fact that $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ is continuous trivially follows from

(a) U_α is open in X.

(b) our definition of topology on X.

(c) $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ is a bijection.

We have proved that $(U_\alpha, \varphi_\alpha)$ are indeed charts. The fact that they form a smooth atlas follows from premise (a) and (d). ∎

Question 128

Let $\mathbb{R}$ be the real line with the differentiable structure given by the maximal atlas of the chart $(\mathbb{R}, \varphi = \mathbf{id} : \mathbb{R} \to \mathbb{R})$, where id is the identity map, and let $\mathbb{R}'$ be the real line with the differentiable structure given by the maximal atlas of the chart $(\mathbb{R}', \psi : \mathbb{R}' \to \mathbb{R})$, where $\psi(x) = x^{1/3}$.

(a) Show that these two differentiable structures are distinct.

(b) Show that there is a diffeomorphism between $\mathbb{R}$ and $\mathbb{R}'$. (Hint: The identity map $\mathbb{R} \to \mathbb{R}$ is not the desired diffeomorphism.)

Proof. To see these two smooth structure are distinct. Observe

$$\mathbf{id}^{-1} \circ \psi : \mathbb{R} \to \mathbb{R}; x \mapsto x^{\frac{1}{3}} \text{ is not smooth at } 0$$

We claim $\varphi : \mathbb{R} \mapsto \mathbb{R}'$ defined by

$$\varphi(x) \triangleq x^3 \text{ is a diffeomorphism}$$

It is clear that φ is a homeomorphism. To see φ is a smooth mapping from $\mathbb{R}$ to $\mathbb{R}'$, observe that

$$\psi \circ \varphi \circ \mathbf{id}^{-1}(x) = x$$

To see φ^{-1} is a smooth mapping from $\mathbb{R}'$ to $\mathbb{R}$, observe that

$$\mathbf{id} \circ \varphi \circ \psi^{-1}(x) = x$$

We have proved that φ is a diffeomorphism between $\mathbb{R}$ and $\mathbb{R}'$. ∎

7.2 Appendix

Theorem 7.2.1. (Homeomorphism between Compact Space and Hausdorff Space) Suppose

(a) X is compact.

(b) Y is Hausdorff.

(c) $f : X \to Y$ is a continuous bijective function.

Then

$$f \text{ is a homeomorphism between } X \text{ and } Y$$

Proof. Because closed subset of compact set is compact and continuous function send compact set to compact set, we see for each closed $E \subseteq X$, $f(E) \subseteq Y$ is compact. The result then follows from $f(E) \subseteq Y$ being closed since Y is Hausdorff. ∎

Theorem 7.2.2. (Hausdorff and Quotient) If $\pi : X \to Y$ is an open mapping, and we define

$$R_\pi \triangleq \{(x, y) \in X^2 : \pi(x) = \pi(y)\}$$

Then

$$R_\pi \text{ is closed} \iff Y \text{ is Hausdorff}$$

Proof. Suppose R_π is closed. Fix some x, y such that $\pi(x) \neq \pi(y)$. Because R_π is closed, we know there exists open neighborhood U_x, U_y such that $U_x \times U_y \subseteq (R_\pi)^c$. It is clear that $\pi(U_x), \pi(U_y)$ are respectively open neighborhood of $\pi(x)$ and $\pi(y)$. To see $\pi(U_x)$ and $\pi(U_y)$ are disjoint, assume that $\pi(a) \in \pi(U_x) \cap \pi(U_y)$. Let $a_x \in U_x$ and $a_y \in U_y$ satisfy $\pi(a_x) = \pi(a) = \pi(a_y)$, which is impossible because $(a_x, a_y) \in (R_\pi)^c$. CaC

Suppose Y is Hausdorff. Fix some x, y such that $\pi(x) \neq \pi(y)$. Let U_x, U_y be open neighborhoods of $\pi(x), \pi(y)$ separating them. Observe that $(x, y) \in \pi^{-1}(U_x) \times \pi^{-1}(U_y) \subseteq (R_\pi)^c$ ∎

7.3 Example: $S^1, \mathbb{R} \setminus \mathbb{Z}$ diffeomorphism

Equip $S^1 \triangleq \{(x,y) \in \mathbb{R}^2 : |(x,y)| = 1\}$ with the standard four projection chart smooth atlas as one of them being

$$V \triangleq \{(x,y) \in \mathbb{R}^2 : y > 0\} \text{ and } \varphi_V : V \to \mathbb{R}; (x,y) \mapsto x$$

Let $p : \mathbb{R} \to \mathbb{R} \setminus \mathbb{Z}$ be the quotient map and let

$$U_0 \triangleq p\Big((0,1)\Big) \text{ and } U_1 \triangleq p\Big((\frac{-1}{2}, \frac{1}{2})\Big)$$

which are both open as one can readily check. Define $\varphi_0 : U_0 \to (0,1)$ by

$$\varphi_0(p(t)) \triangleq t_0 \text{ where } t_0 \in (0,1) \text{ and } p(t_0) = p(t)$$

and $\varphi_1 : U_1 \to (-\frac{1}{2}, \frac{1}{2})$ by

$$\varphi_1(p(t)) \triangleq t_0 \text{ where } t_0 \in (-\frac{1}{2}, \frac{1}{2}) \text{ and } p(t_0) = p(t)$$

Clearly, the function $G : \mathbb{R} \setminus \mathbb{Z} \to S^1$ well-defined by $G(p(x)) \triangleq (\cos 2\pi x, \sin 2\pi x)$ is a homeomorphism, as one can check that

(a) G is a continuous bijection. (Using Universal property of quotient map)

(b) $\mathbb{R} \setminus \mathbb{Z}$ is compact. (by finite sub-cover definition)

(c) S^1 is Hausdorff.

We now compute that $\varphi_V \circ G \circ \varphi_0^{-1}$ is defined on whole $(0,1)$, and is exactly

$$\varphi_V \circ G \circ \varphi_0^{-1}(t) \triangleq \cos 2\pi t \text{ smooth}$$

7.4 HW2

Question 129

Let $F:\mathbb{R}^2\to\mathbb{R}^3$ be the map

$$F(x,y)\triangleq(x,y,xy)=(u,v,w)$$

Let $p=(x,y)\in\mathbb{R}^2$. Compute $F_*(\frac{\partial}{\partial x}|_p)$ as a linear combination of $\frac{\partial}{\partial u},\frac{\partial}{\partial v},\frac{\partial}{\partial w}$

Proof. For all $f\in C^\infty(\mathbb{R}^3)$, we have

$$\frac{\partial f\circ F}{\partial x}(x,y)=\frac{\partial f}{\partial u}(F(p))+\frac{\partial f}{\partial w}(F(p))y$$

This then give us

$$F_*\Big(\frac{\partial}{\partial x}\Big|_{(x,y)}\Big)=\frac{\partial}{\partial u}+y\frac{\partial}{\partial w}$$

■

Question 130

Let G be a lie group with multiplication map $\mu:G\times G\to G$ and identity element e. Show that differential $\mu_{*,(e,e)}:T_{(e,e)}G\times G\to T_eG$ of μ at identity is

$$\mu_{*,(e,e)}(X_e,Y_e)=X_e+Y_e$$

Note that $T_{(p,q)}M\times N$ is isomorphic to $T_pM\oplus T_qN$ via the differential of two projection $\pi_1:M\times N\to M$ and $\pi_2:M\times N\to N$.

Proof. We first justify the notation of writing tangent vectors in $T_{(e,e)}G\times G$ as (X_e,Y_e), and the proof will follow. Consider the projection $\pi_1:G\times G\to G$ and $\pi_2:G\times G\to G$

$$\pi_1(g,h)\triangleq g\text{ and }\pi_2(g,h)\triangleq h$$

Consider charts $(U,\varphi),(V,\psi)$ for G centering e. We can induce a chart $(U\times V,\Phi)$ for $G\times G$ centering e by

$$\Phi(g,h)\triangleq(\varphi(g),\psi(h))$$

In local coordinate, we have

$$\pi_1(\mathbf{x}^1,\dots,\mathbf{x}^n,\mathbf{y}^1,\dots,\mathbf{y}^n)=(\mathbf{x}^1,\dots,\mathbf{x}^n)\text{ and }\pi_2(\mathbf{x}^1,\dots,\mathbf{x}^n,\mathbf{y}^1,\dots,\mathbf{y}^n)=(\mathbf{y}^1,\dots,\mathbf{y}^n)$$

In abuse of notation, this give us

$$(\pi_1)_*\Big(\sum_{i=1}^n v^i\frac{\partial}{\partial \mathbf{x}^i}+w^i\frac{\partial}{\partial \mathbf{y}^i}\Big)=\sum_{i=1}^n v^i\frac{\partial}{\partial \mathbf{x}^i}\text{ and }(\pi_2)_*\Big(\sum_{i=1}^n v^i\frac{\partial}{\partial \mathbf{x}^i}+w^i\frac{\partial}{\partial \mathbf{y}^i}\Big)=\sum_{i=1}^n w^i\frac{\partial}{\partial \mathbf{y}^i}$$

This then give us a vector space isomorphism between $T_{(e,e)}G\times G$ and $T_eG\oplus T_eG$, on which our notation stand. Now, let $\gamma:(-\epsilon,\epsilon)\to G$ be a smooth curve centering e such that $\gamma'(0)=X_e$. Define another smooth curve $\alpha:(-\epsilon,\epsilon)\to G\times G$ in $G\times G$ by

$$\alpha(t)\triangleq(\gamma(t),e)$$

Because $\pi_2\circ\alpha$ is constant and $\pi_1\circ\alpha=\gamma$, we now see

$$(\pi_1)_{*,(e,e)}(\alpha'(0))=(\pi_1\circ\alpha)'(0)=\gamma'(0)\text{ and }(\pi_2)_{*,(e,e)}(\alpha'(0))=0$$

This implies $\alpha'(0)=(X_e,0)$. Compute

$$\mu\circ\alpha(t)=\gamma(t)+e=\gamma(t)$$

We now can deduce

$$\mu_{*,(e,e)}(X_e,0)=\mu_{*,(e,e)}(\alpha'(0))=(\mu\circ\alpha)'(0)=\gamma'(0)=X_e$$

Similar procedure can be applied to show

$$\mu_{*,(e,e)}(0,Y_e)=Y_e$$

It now follows from linearity of $\mu_{*,(e,e)}$ that

$$\mu_{*,(e,e)}(X_e,Y_e)=X_e+Y_e$$

■

Question 131

Let $S^2=\{(x,y,z)\in\mathbb{R}^3:x^2+y^2+z^2=1\}$ be the sphere in $\mathbb{R}^3$. Consider the function $h:S^2\to\mathbb{R}$ defined by

$$h(x,y,z)\triangleq z$$

Find the critical points of h.

Proof. Consider the atlas $\{(U,\varphi),(V,\psi)\}$ for S^2 where $U=S^2\setminus\{(0,0,1)\},V=S^2\setminus\{(0,0,-1)\}$ and

$$\varphi(x,y,z)\triangleq\Big(\frac{x}{1-z},\frac{y}{1-z}\Big)\text{ and }\psi(x,y,z)=\Big(\frac{x}{1+z},\frac{y}{1+z}\Big)$$

Some algebra trick and tedious efforts shows that this indeed form a smooth atlas and gives us their inverse

$$\varphi^{-1}(u,v) = \Big(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\Big)$$
$$\psi^{-1}(u,v) = \Big(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{1-u^2-v^2}{u^2+v^2+1}\Big)$$

Compute

$$[d(h\circ\varphi^{-1})_{(u,v)}] = \begin{bmatrix} \frac{-4u}{(u^2+v^2+1)^2} & \frac{-4v}{(u^2+v^2+1)^2} \end{bmatrix}$$
$$[d(h\circ\psi^{-1})_{(u,v)}] = \begin{bmatrix} \frac{4u}{(u^2+v^2+1)^2} & \frac{4v}{(u^2+v^2+1)^2} \end{bmatrix}$$

This then shows the set of critical points is exactly

$$\{\varphi^{-1}(0,0), \psi^{-1}(0,0)\} = \{(0,0,-1),(0,0,1)\}$$

■

Question 132

A smooth map $f : M \to N$ is said to be **a traversal to an embedded submanifold** $S \subseteq N$ if for every point $p \in f^{-1}(S)$, we have

$$f_{*,p}(T_pM) + T_{f(p)}S = T_{f(p)}N$$

The goal of this exercise is to prove the Transversality Theorem: If a smooth map $f : M \to N$ is a traversal to an embedded submanifold S of codimension k in N, then $f^{-1}(S)$ is a regular submanifold of codimension k in M. Let $p \in f^{-1}(S)$ and $(U, \mathbf{x}^1, \dots, \mathbf{x}^n)$ be an adapted chart for N centering $f(p)$ with respect to S. Define $g : U \to \mathbb{R}^k$ by

$$g(\mathbf{x}^1, \dots, \mathbf{x}^n) \triangleq (\mathbf{x}^{n-k+1}, \dots, \mathbf{x}^n)$$

(a) Show that $f^{-1}(U) \cap f^{-1}(S) = (g\circ f)^{-1}(\mathbf{0})$.

(b) Show that $f^{-1}(U) \cap f^{-1}(S)$ is a regular level set of the function $g \circ f : f^{-1}(U) \to \mathbb{R}^k$.

(c) Prove the Transversality Theorem.

Proof. At first, we shall point out that $g\circ f$ is a function defined only on $f^{-1}(U)$. (a) follows trivially from the fact that $(U, \mathbf{x}^1, \dots \mathbf{x}^n)$ is an adapted chart. We now prove (b).

Fix arbitrary $p \in f^{-1}(U) \cap f^{-1}(S)$. Trivially, by definition of g,

$$T_{f(p)}S \subseteq \operatorname{Ker} g_{*,f(p)}$$

Explicit formula of g shows that $g_{*,f(p)}$ is a vector space epimorphism that maps $T_{f(p)}N$ into $T_{g\circ f(p)}\mathbb{R}^k$, which implies that $\operatorname{Ker} g_{*,f(p)}$ has dimension $n-k$, same as $T_{f(p)}S$ and give us

$$T_{f(p)}S = \operatorname{Ker} g_{*,f(p)}$$

It now follows from f being a traversal and $g_{*,f(p)}$ being surjective that

$$(g \circ f)_{*,p}(T_pM) = g_{*,f(p)} \circ f_{*,p}(T_pM) = \operatorname{Im} g_{*,f(p)} = T_{g\circ f(p)}\mathbb{R}^k$$

We have shown that $g \circ f$ is regular at p. (b) then follows from p is arbitrary selected from $f^{-1}(U) \cap f^{-1}(S)$.

Now, by Regular level set Theorem, we see that $f^{-1}(U) \cap f^{-1}(S)$ is an embedded submanifold of $f^{-1}(U)$ with codimension k. Because f is continuous, we know $f^{-1}(U)$ is open, thus an embedded submanifold of M with dimension m. It now follows that $f^{-1}(U) \cap f^{-1}(S)$ is an embedded submanifold of M with codimension k. We have proved the Transversality Theorem. ■

7.5 Bundle

A **smooth real vector bundle of rank k over the smooth manifold** M is a smooth manifold E together with the surjective smooth map $\pi : E \to M$ such that

(a) Each fiber $\pi^{-1}(p)$ has a real k-dimensional vector space structure.

(b) For all $p \in M$, there exists some neighborhood U of p and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that Φ map the fiber $\pi^{-1}(p)$ vector space isomorphiscally to $\{p\} \times \mathbb{R}^k$.

Note that we often call E the **total space** and M the **base space**. The neighborhood U and the smooth diffeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ is often called the **smooth local trivialization**, and if there exists a global smooth trivialization, we say $(E, M, \pi : E \to M)$ is a **trivial bundle**. It is clear that tangent bundle TM is a smooth real vector bundle of rank m over the smooth manifold M where the induced chart $\Phi_m : \pi^{-1}(U_n) \to \mathbb{R}^{2m}$ are smooth local trivialization. If we are given a smooth right inverse $\sigma : M \to E$ of π

$$\pi \circ \sigma(p) = p \text{ for all } p \in M$$

we say σ is a **smooth section of the bundle** $\pi : E \to M$.