Deadline : 2024/5/13, 17:00.

1. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x,\, y) = (xy,\, x\cos y,\, x\sin y) \quad , \quad g(u,\, v,\, w) = uv + vw + wu$$

   Calculate $(Df)(1,\, 0)$, $(Dg)(0,\, 1,\, 0)$, and $[D(g \circ f)](1,\, 0)$.

2. (a) Let $(X,\, \|-\|_X)$, $(Y,\, \|-\|_Y)$ be normed spaces and U be an open subset of X. If $f : U \to Y$ is differentiable at some $x_0 \in U$, prove that $(Df)(x_0)$ is **uniquely** determined by f.
   (Hint: If L_1 and L_2 are derivatives of f at x_0, write out the definition. Also U is open means you can find a small ball $B(x_0;\, r) \subset U$ with some $r > 0$, this r can do some work in your above definition.)

   (b) If U is **NOT** supposed to be open, does the result of (a) remain true? Prove it or give a counterexample.

3. For a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, prove that $\pm\dfrac{\nabla f}{\|\nabla f\|_{\mathbb{R}^n}}$ is the direction in which f increases/decreases most rapidly.

4. (a) If E is an open and connected subset in $\mathbb{R}^n$, prove that E is path-connected.

   (b) Let E be a open and connected subset of $\mathbb{R}^n$ and $f : E \to \mathbb{R}^m$ is differentiable. Show that if $(Df)(x)$ is a zero matrix for all $x \in E$, then f can only be a constant on E.

   (c) If E is changed to be open and convex, does the result of (b) remain true? Prove it or give a counterexample.

5. (Mean Value Theorem in multivariable Calculus)

   (a) Let U be an open subset of $\mathbb{R}^n$ and $f = (f_1, ..., f_n) : U \to \mathbb{R}^m$ be a differentiable function. If for $x,\, y \in U$, the line segment L joining x and y lies in U, prove that there exists $c_1, ..., c_m \in L$ such that

$$f_i(y) - f_i(x) = [(Df_i)(c_i)](y - x),\ i = 1, ..., m$$

   (b) If for $x,\, y \in U$, L does **NOT** lie in U, the above theorem may not hold. Give a counterexample.

6. (a) Let $f(x,\, y) = \begin{cases} \dfrac{x^4 + 2x^2y^2 - y^4}{x^2 + y^2} & ,\, (x,\, y) \neq (0,\, 0) \\ 0 & ,\, (x,\, y) = (0,\, 0) \end{cases}$, use $\varepsilon - \delta$ definition to check that f is differentiable at $(0,\, 0)$.

(b) Let $g(x,\, y) = \begin{cases} \dfrac{2x^3 + x^2y + 2xy^2 - y^3}{x^2 + y^2} & ,\, (x,\, y) \neq (0,\, 0) \\ 0 & ,\, (x,\, y) = (0,\, 0) \end{cases}$, prove that g is **NOT** differentiable at $(0,\, 0)$.

(c) Calculate the value of $\dfrac{\partial}{\partial y}\left(\dfrac{\partial g}{\partial x}\right)$ and $\dfrac{\partial}{\partial x}\left(\dfrac{\partial g}{\partial y}\right)$ at $(0,\, 0)$.