NCKU 112.2
Geometry 1

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Contents

Chapter 1

HW

1.1 HW1

In this section, we will use I to denote an **bounded open interval**. By a **curve** in $\mathbb{R}^n$, we mean a function form an open interval I to $\mathbb{R}^n$. We say a curve is **differentiable** if it is infinitely differentiable (smooth). That is, $\gamma^{(n)}(t)$ exists and are continuous for all $n \in \mathbb{N}$ and $t \in I$.

We say a differentiable curve $\gamma : I \to \mathbb{R}^n$ is **regular** if $\gamma'(t) \neq 0$ for all $t \in I$. We say a differentiable curve $\gamma : I \to \mathbb{R}^n$ is a **parametrized by arc-length** if $|\gamma'(t)| = 1$ for all $t \in I$.

For a regular curve γ, we say $\gamma'(t)$ is the **tangent vector** of γ at t, and we define the **unit tangent vector** T by

$$T(t) \triangleq \frac{\gamma'(t)}{|\gamma'(t)|}$$

We say $\gamma''(t)$ is the **oriented curvature** (normal vector) of γ at t, and we define the **unit normal vector** N by

$$N(t) \triangleq \frac{T'(t)}{|T'(t)|} = \frac{\gamma''(t)}{|\gamma''(t)|}$$

With some messy computation, we can observe

$$T'(t) = \frac{\gamma''(t)}{|\gamma'(t)|}$$

Some interesting facts can be observed from what we have deduced.

(a) γ', γ'' always exists.

(b) γ is parametrized by arc-length $\implies \gamma' \perp \gamma''$

(c) γ is parametrized by arc-length $\implies \gamma$ is regular

(d) T and T' exists at $t \iff \gamma$ is regular at t

(e) $T = \gamma' \iff \gamma$ is parametrized by arc-length

(f) N exists at $t \iff \gamma''(t) \neq 0 \iff \kappa(t) \neq 0$

(g) N and T' point to the same direction γ''.

(h) $|T'| = \kappa \iff \gamma$ is paramterized by arc-length

(i) $\gamma \perp \gamma'$ and $\gamma'' \perp \gamma'''$ are generally false even for curve γ paramterized by arc-length.

(j) Given a curve γ parametrized by arc-length

$$\begin{aligned} \gamma \text{ is a straight line on } [a,b] &\iff \gamma' \text{ and } T \text{ are constant on } (a,b) \\ &\iff \gamma''(t) = 0 \text{ on } (a,b) \\ &\iff \kappa(t) = 0 \text{ on } (a,b) \\ &\iff T'(t) = 0 \text{ on } (a,b) \end{aligned}$$

Notice that the last fact is false if γ is not paramtetrized by arc-length, since γ can move in the straight line with changing speed γ'.

Given a curve γ, if $T(t)$ and $N(t)$ exists (regular and non-zero curvature), we define its **binormal** vector by

$$B(t) = T(t) \times N(t)$$

Fix t. We say

$$\{T(t), N(t), B(t)\} \text{ form a } \textbf{positively oriented orthonormal basis} \text{ of } \mathbb{R}^3$$

This basis in general is constantly changing, yet always form an orthonormal basis.

Also, we say

$$\text{span}\Big(T(t), N(t)\Big) \text{ is the } \textbf{osculating plane} \text{ of } \gamma \text{ at } t$$

Suppose γ is parametrized by arc-length and always has non-zero curvature. With some geometric intuition, one shall note that $|T'|$ measure how curved γ is and that $|B'|$ measure how fast γ leave the osculating plane.

Because $|B| = 1$ is a constant, we can deduce

$$B' \perp B$$

and the computation

$$B' = T' \times N + T \times N' = T \times N'$$

give us

$$B' \perp T$$

This ultimately show us

$$B', N, T' \text{ are all parallel where } N, T' \text{ even point to the same direction}$$

Notice that if we parametrize the curve with opposite direction, then

(a) T, γ' change direction

(b) N, γ'' keep the same direction

(c) B change direction

(d) B' keep the same direction

Now, for a curve γ parametrized by arc-length, we define the **curvature** κ and **torsion** τ of γ by

$$\kappa(t) = |\gamma''(t)| \text{ and } \tau(t) = \frac{B'(t)}{N(t)}$$

With unfortunately heavy computation, we can verify that the definition of curvature must stay in the framework of curve parametrized by arc-length, otherwise we will be given two different values of curvature of two curves that are equivalent in the sense of sets.

Now, notice that we already have $T' = \kappa N$ and $B' = \tau N$, and by basic identity, we have $N = B \times T$.

Then with some computation, we have the **Frenet Formula**

$$\begin{cases} T' = \kappa N \\ N' = B' \times T + B \times T' = -\tau B - \kappa T \\ B' = \tau N \end{cases}$$

Given two vectors $u, v \in \mathbb{R}^n$, we use **dot product**

$$u \cdot v = u_1 v_1 + \cdots + u_n v_n$$

to denote the Euclidean inner product, and we use **length**

$$|u| = \sqrt{\sum_{k=1}^{n} u_k^2}$$

to denote the Euclidean norm. Note that we clearly have

$$|u| = \sqrt{u \cdot u}$$

Given three vectors $u, v, w \in \mathbb{R}^3$, we define **cross product** by

$$\begin{aligned} u \times v &\triangleq \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1) \end{aligned}$$

With some simple computation, we have the following identity

(a) $u \times v = -v \times u$ (anti-commutative)

(b) $(au + w) \times v = a(u \times v) + w \times v$ (Linearity)

(c) $u \times (aw + v) = a(u \times w) + u \times v$

(d) $u \times v = 0 \iff u = cv$ for some $c \in \mathbb{R}$

(e) $(u \times v) \cdot w = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

(f) $(u \times v) \cdot v = 0 = (u \times v) \cdot u$

(g) $u \times v \perp u$ and $u \times v \perp v$

(h) $u \perp v \implies |u \times v| = |u| \cdot |v|$

(i) $(u \times v) \times w = (u \cdot w)v - (v \cdot w)u$

All proofs except that of the last identity are merely manipulation of determinant. A simple proof of the last identity follows from the fact both side are linear in all u, v, w, and the observation

$$(e_1 \times e_2) \times e_3 = 0 = (e_1 \cdot e_3)e_2 - (e_2 \cdot e_3)e_1$$

Theorem 1.1.1. (Differentiate the Dot Product) Given two parametrized curves $u, v : (a, b) \to \mathbb{R}^n$, such that u, v are differentiable at $t \in (a, b)$. We have

$$\frac{d}{dt}\big(u(t) \cdot v(t)\big) = u'(t) \cdot v(t) + u(t) \cdot v'(t)$$

Proof.

$$\begin{aligned} \frac{d}{dt}\big(u(t) \cdot v(t)\big) &= \frac{d}{dt}\sum_{k=1}^{n} u_k(t)v_k(t) \\ &= \sum_{k=1}^{n} \frac{d}{dt} u_k(t)v_k(t) \\ &= \sum_{k=1}^{n} u_k'(t)v_k(t) + u_k(t)v_k'(t) \\ &= \sum_{k=1}^{n} u_k'(t)v_k(t) + \sum_{k=1}^{n} u_k(t)v_k'(t) \\ &= u'(t) \cdot v(t) + u(t) \cdot v'(t) \end{aligned}$$

■

Theorem 1.1.2. (Differentiate the Cross Product) Given two curves $u, v : (a, b) \to \mathbb{R}^3$, such that u, v are differentiable at $t \in (a, b)$. We have

$$\frac{d}{dt}\big(u(t) \times v(t)\big) = u'(t) \times v(t) + u(t) \times v'(t)$$

Proof.

$$\begin{aligned}
\frac{d}{dt}\big(u(t)\times v(t)\big) =& \frac{d}{dt}(u_2v_3-u_3v_2, u_3v_1-u_1v_3, u_1v_2-u_2v_1) \\
=&(u_2'v_3+u_2v_3'-u_3'v_2-u_3v_2', \\
&u_3'v_1+u_3v_1'-u_1'v_3-u_1v_3', \\
&u_1'v_2+u_1v_2'-u_2'v_1-u_1v_2') \\
=&(u_2'v_3-u_3'v_2, u_3'v_1-u_1'v_3, u_1'v_2-u_2'v_1) \\
+&(u_2v_3'-u_3v_2', u_3v_1'-u_1v_3', u_1v_2'-u_1v_2') \\
=&u'\times v+u\times v'
\end{aligned}$$

∎

For the result above, sometimes we write

$$(u\cdot v)'=u'\cdot v+u\cdot v' \text{ and } (u\times v)'=u'\times v+u\times v'$$

Trick to parametrize by arc-length.

Given a regular curve $\gamma : I \to \mathbb{R}^n$ and fix $t_0 \in I$. We use

$$s(t)=\int_{t_0}^{t}|\gamma'(x)|\,dx$$

to define the arc-length of γ from $\gamma(t_0)$ to $\gamma(t)$. Because γ is regular, by FTC, it is clear that s is one-to-one.

Let $t(s)$ be the inverse of s. Define

$$\beta(s)\triangleq\alpha(t(s))$$

We have by Chain rule

$$\begin{aligned}
\beta'(s)&=t'(s)\alpha'(t(s)) \\
&=\frac{\alpha'(t(s))}{s'(t)} \\
&=\frac{\alpha'(t(s))}{|\alpha'(t(s))|}
\end{aligned}$$

Now, β is clearly a regular curve, and

$$\int_0^x |\beta'(s)|\, ds = x$$

Question 1: 1-2: 2

Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is the point of the trace of α closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

Proof. Define $g : I \to \mathbb{R}$ by

$$g(t) \triangleq |\alpha(t)|^2 = (\alpha \cdot \alpha)(t)$$

Notice that

$$g'(t) = (2\alpha' \cdot \alpha)(t) \text{ if exists}$$

From premise, we know g attains minimum at t_0. This tell us

$$0 = g'(t_0) = (2\alpha' \cdot \alpha)(t_0)$$

Then, we can deduce

$$\alpha'(t_0) \cdot \alpha(t_0) = 0$$

This implies $\alpha(t_0) \perp \alpha'(t_0)$. ∎

Question 2: 1-2: 5

Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve, with $\alpha''(t) \neq 0$ for all $t \in I$. Show that $|\alpha(t)|$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

Proof. We wish to prove

$$\exists \beta \in \mathbb{R}^*, \forall t \in I, |\alpha(t)| = \beta \iff \forall t \in I, (\alpha \cdot \alpha')(t) = 0$$

Define $g : I \to \mathbb{R}$ by

$$g(t) \triangleq |\alpha(t)|^2 = (\alpha \cdot \alpha)(t)$$

Notice that

$$g'(t) = (2\alpha' \cdot \alpha)(t) \tag{1.1}$$

$(\longrightarrow)$

From premise, g is a constant on I. This implies $g'(t) = 0$ for all $t \in I$. Then, from Equation 1.1, we see

$$(\alpha \cdot \alpha')(t) = 0 \text{ for all } t \in I$$

$(\longleftarrow)$

Again, from Equation 1.1, we deduce

$$\forall t \in I, (\alpha \cdot \alpha')(t) = 0 \implies \forall t \in I, g'(t) = 0$$

This implies g is a constant, which implies $|\alpha|$ is a constant, that is

$$\exists \beta \in \mathbb{R}, |\alpha(t)| = \beta$$

Lastly, we have to show

$$\beta \neq 0$$

Assume $\beta = 0$. Then, we see $\alpha(t) = 0$ for all $t \in I$. This implies $\alpha''(t) = 0$ for all $t \in I$, which CaC to the premise. (done) ■

Question 3: 1-3:2

2. A circular disk of radius 1 in the plane xy rolls without slipping along the x axis. The figure described by a point of the circumference of the disk is called a *cycloid* (Fig. 1-7).

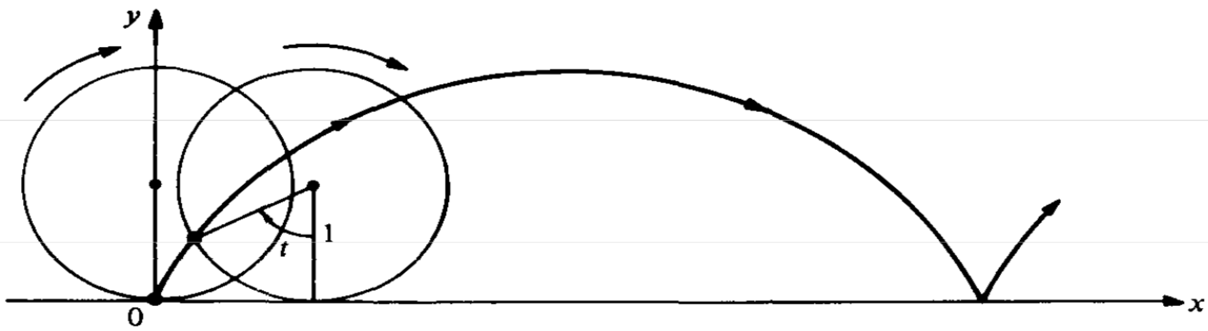


Figure 1-7. The cycloid.

***a.** Obtain a parametrized curve $\alpha: R \longrightarrow R^2$ the trace of which is the cycloid, and determine its singular points.

b. Compute the arc length of the cycloid corresponding to a complete rotation of the disk.

Proof. The solution of the question **a** is

$$\alpha(t) = (t - \sin t, 1 - \cos t)$$

Compute

$$\alpha'(t) = (1 - \cos t, \sin t)$$

and compute

$$|\alpha'(t)| = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} = \sqrt{2} \cdot \sqrt{1 - \cos t}$$

This implies the singular points are

$$\{2n\pi : n \in \mathbb{Z}\}$$

The solution of the question **b** is then

$$\begin{aligned}
\int_0^{2\pi} |\alpha'(t)|\, dt &= \sqrt{2}\int_0^{2\pi} \sqrt{1-\cos t} dt \\
&= \sqrt{2}\int_0^{2\pi} \sqrt{2}\left|\sin\frac{t}{2}\right| dt \\
&= 2\int_0^{2\pi} \left|\sin\frac{t}{2}\right| dt \\
&= 4\int_0^{\pi} \sin(\frac{t}{2}) dt \\
&= -8\cos\frac{t}{2}\Big|_0^{\pi}
\end{aligned}$$

■

Question 4: 1-3:4

4. Let $\alpha: (0, \pi) \longrightarrow R^2$ be given by

$$\alpha(t) = \left(\cos t, \cos t + \log \tan \frac{t}{2}\right),$$

where t is the angle that the y axis makes with the vector $\alpha(t)$. The trace of α is called the *tractrix* (Fig. 1-9). Show that

19:

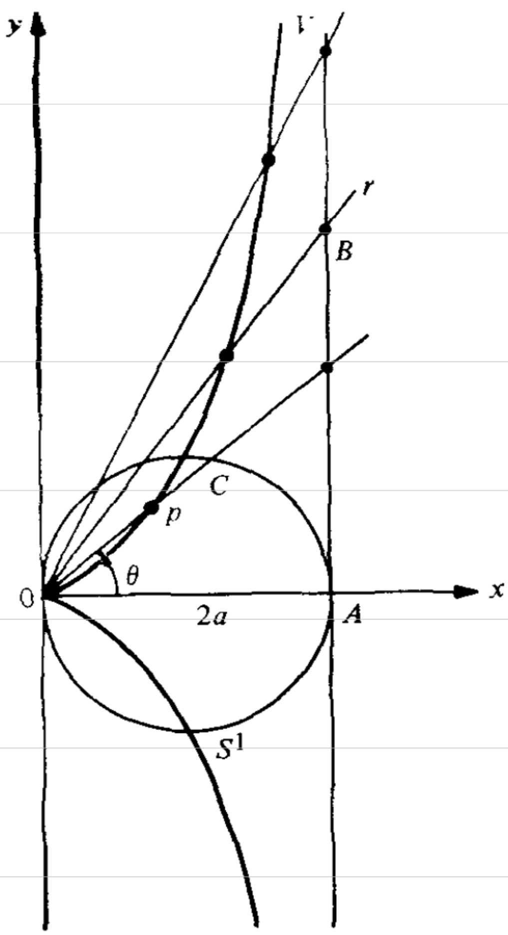


Figure 1-8. The cissoid of Diocles.

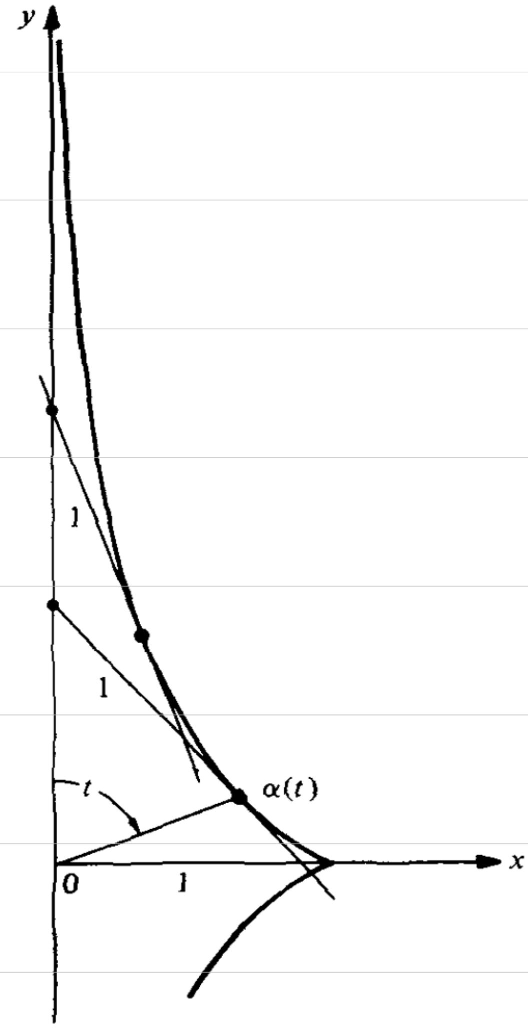


Figure 1-9. The tractrix.

a. α is a differentiable parametrized curve, regular except at $t = \pi/2$.

b. The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.

Typo correction: $\alpha(t) = (\sin t, \cos t + \ln \tan \frac{t}{2})$

Proof. **(a)**
Notice that the interval I is $(0, \pi)$. It is clear that

(a) $\sin t$ is smooth on $\mathbb{R}$

(b) $\cos t$ is smooth on $\mathbb{R}$

(c) $\ln t$ is smooth on $\mathbb{R}$ $\tan \frac{t}{2}$ is smooth on I

Then it follows that α is a differentiable curve.

Compute

$$\alpha'(t) = (\cos t, -\sin t + \frac{1}{\tan \frac{t}{2}} \cdot \sec^2 \frac{t}{2} \cdot \frac{1}{2})$$

Because $\cos t = \alpha_1'(t)$ is 0 on I only when $t = \frac{\pi}{2}$, we know α is regular on I except possibly at $t = \frac{\pi}{2}$.

Compute

$$\alpha'(\frac{\pi}{2}) = (0, -1 + \frac{1}{1} \cdot 2 \cdot \frac{1}{2}) = (0, 0)$$

We now conclude α is regular on I except $\frac{\pi}{2}$.

(b)
A useful Identity give us

$$\alpha'(t) = (\cos t, -\sin t + \csc t)$$

From the following facts

(a) the first argument of the segment is from 0 to $\sin t = \alpha(t)$

(b) $\alpha_x'(t) = \cos t$

(c) $\frac{\sin t}{\cos t} = \tan t$

We conclude that the length of the segment is

$$\begin{aligned} |\tan t| \cdot |\alpha'(t)| &= |\tan t| \cdot \sqrt{\cos^2 t + \sin^2 t - 2 \sin t \csc t + \csc^2 t} \\ &= |\tan t| \cdot \sqrt{1 - 2 + \csc^2 t} \\ &= |\tan t| \cdot \sqrt{\csc^2 t - 1} = |\tan t| \cdot \sqrt{\cot^2 t} = 1 \end{aligned}$$

■

Question 5

7. A map $\alpha: I \longrightarrow R^3$ is *called a curve of class* C^k if each of the coordinate functions in the expression $\alpha(t) = (x(t), y(t), z(t))$ has continuous derivatives up to order k. If α is merely continuous, we say that α is of class C^0. A curve α is called *simple* if the map α is one-to-one. Thus, the curve in Example 3 of Sec. 1-2 is not simple.

Let $\alpha: I \longrightarrow R^3$ be a simple curve of class C^0. We say that α has a *weak tangent* at $t = t_0 \in I$ if the line determined by $\alpha(t_0 + h)$ and $\alpha(t_0)$ has a limit position when $h \longrightarrow 0$. We say that α has a *strong tangent* at $t = t_0$ if the line determined by $\alpha(t_0 + h)$ and $\alpha(t_0 + k)$ has a limit position when $h, k \longrightarrow 0$. Show that

a. $\alpha(t) = (t^3, t^2)$, $t \in R$, has a weak tangent but not a strong tangent at $t = 0$.

***b.** If $\alpha: I \longrightarrow R^3$ is of class C^1 and regular at $t = t_0$, then it has a strong tangent at $t = t_0$.

c. The curve given by

$$\alpha(t) = \begin{cases} (t^2, t^2), & t \geq 0, \\ (t^2, -t^2), & t \leq 0, \end{cases}$$

is of class C^1 but not of class C^2. Draw a sketch of the curve and its tangent vectors.

Proof. **(a)**

$$\frac{\alpha(t)}{t} \to (0,0) \text{ as } t \to 0^-$$

$$\frac{\alpha(h) - \alpha(k)}{h - k} = \frac{\left(h^3 - k^3, h^2 - k^2\right)}{h - k} = \left(h^2 + hk + k^2, h + k\right) \to$$

(b) By MVT, for each h, k there exists a set of real numbers $\{c_x, c_y, c_z\}$ between $t + h$ and $t + k$ such that

$$\frac{\alpha(t_0 + h) - \alpha(t_0 + k)}{h - k} = \left(x'(c_x), y'(c_y), z'(c_z)\right)$$

Then because

$$h, k \to 0 \implies t_0 + h, t_0 + k \to t_0 \implies c_x, c_y, c_z \to t_0$$

Then from the fact α is of class C^1 (x', y', z' are all continuous), we can now deduce

$$\frac{\alpha(t_0 + h) - \alpha(t_0 + k)}{h - k} \to \alpha'(t_0) \text{ as } h, k \to 0$$

Now, because $\alpha'(t_0) \neq 0$ as α is regular, we see

$$\lim_{h,k \to 0} \frac{\alpha(t_0 + h) - \alpha(t_0 + k)}{h - k} \cdot \alpha'(t_0) = |\alpha'(t_0)|^2$$

(c)

From

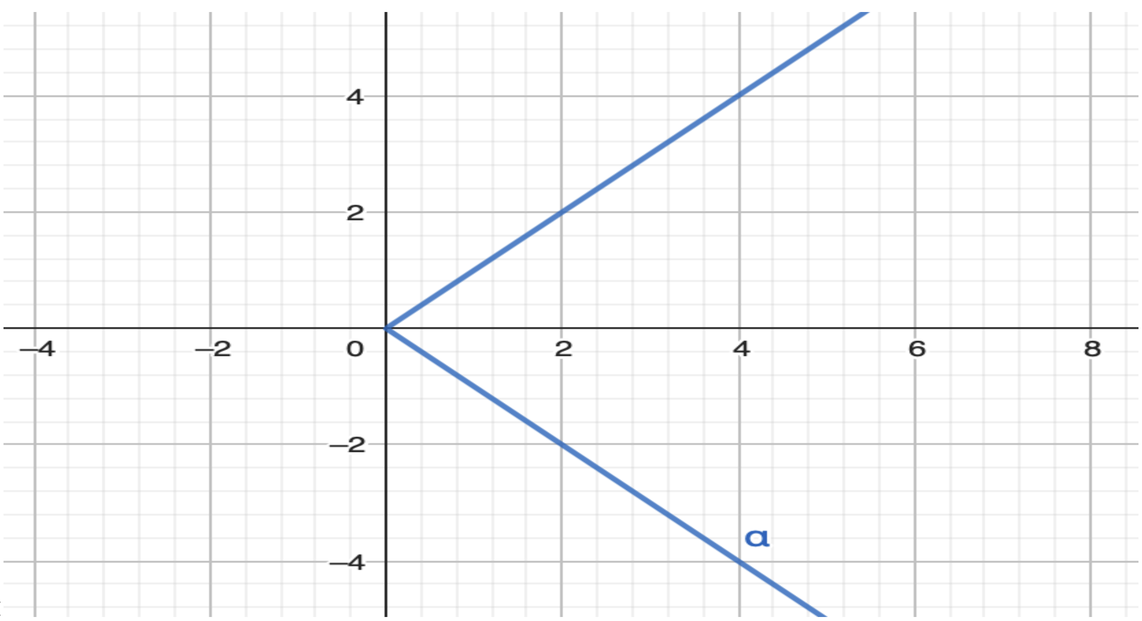
$$\alpha(t) = \left(t^2, \begin{cases} t^2 & \text{if } t \geq 0 \\ -t^2 & \text{if } t \leq 0 \end{cases}\right)$$

Compute

$$\alpha'(t) = \left(2t, \begin{cases} 2t & \text{if } t \geq 0 \\ -2t & \text{if } t \leq 0 \end{cases}\right)$$

Notice that the derivative at $t = 0$ is computed from definition instead of product rule.

Now, it is clear that x', y' are continuous. This implies $\alpha \in C^1$. Yet, we see y' is not differentiable at $t = 0$. This implies $\alpha \notin C^2$.

The sketch: 

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Theorem 1.1.3. (MVT for curve) Given a curve $\alpha : [a, b] \to \mathbb{R}^n$ such that

(a) α is differentaible on (a, b)

(b) α is continuous on $[a, b]$

there exists $\xi \in (a, b)$ such that

$$|\alpha(b) - \alpha(a)| \leq |\alpha'(\xi)| (b - a)$$

Proof. Define $\phi : [a, b] \to \mathbb{R}$ by

$$\phi(t) = \alpha(t) \cdot \big(\alpha(b) - \alpha(a)\big)$$

Clearly ϕ satisfy the hypothesis of Lagrange's MVT, then we know there exists $\xi \in (a, b)$ such that

$$\phi(b) - \phi(a) = \phi'(\xi) \cdot (b - a)$$

Written the equation in α, we have

$$|\alpha(b) - \alpha(a)|^2 = (b - a)\alpha'(\xi) \cdot \big(\alpha(b) - \alpha(a)\big)$$

Notice that Cauchy-Schwarz inequality give us

$$\begin{aligned} (b - a) |\alpha'(\xi)| \cdot |\alpha(b) - \alpha(a)| &\geq (b - a) \left|\alpha'(\xi) \cdot \big(\alpha(b) - \alpha(a)\big)\right| \\ &= |\alpha(b) - \alpha(a)|^2 \end{aligned}$$

This then implies

$$(b - a) |\alpha'(\xi)| \geq |\alpha(b) - \alpha(a)|$$

■

Corollary 1.1.4. (Mean Value Inequality) Given a curve $\alpha : [a, b] \to \mathbb{R}^n$ such that

(a) α is differentiable on (a, b)

(b) α is continuous on $[a, b]$

we have

$$|\alpha(b) - \alpha(a)| \leq (b - a) \sup_{(a,b)} |\alpha'|$$

Question 6

***8.** Let $\alpha: I \longrightarrow R^3$ be a differentiable curve and let $[a, b] \subset I$ be a closed interval. For every *partition*

$$a = t_0 < t_1 < \cdots < t_n = b$$

of $[a, b]$, consider the sum $\sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})| = l(\alpha, P)$, where P stands for the given partition. The norm $|P|$ of a partition P is defined as

$$|P| = \max(t_i - t_{i-1}), \ i = 1, \ldots, n.$$

Geometrically, $l(\alpha, P)$ is the length of a polygon inscribed in $\alpha([a, b])$ with vertices in $\alpha(t_i)$ (see Fig. 1-12). The point of the exercise is to show that the arc length of $\alpha([a, b])$ is, in some sense, a limit of lengths of inscribed polygons.

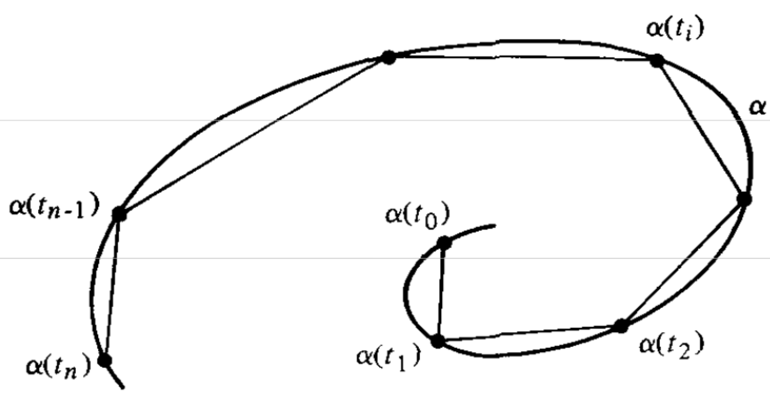

Figure 1-12

Prove that given $\epsilon > 0$ there exists $\delta > 0$ such that if $|P| < \delta$ then

$$\left| \int_a^b |\alpha'(t)| \, dt - l(\alpha, P) \right| < \epsilon.$$

Proof. We first prove

$$\int_a^b |\alpha'(t)| \, dt \geq l(\alpha, P)$$

By FTC, we have

$$\begin{aligned}|\alpha(t_i)-\alpha(t_{i-1})| &= \left|\int_{t_{i-1}}^{t_i}\alpha'(t)dt\right| \\ &\le \int_{t_{i-1}}^{t_i}|\alpha'(t)|\,dt\end{aligned}$$

This then implies

$$l(\alpha,P)=\sum|\alpha(t_i)-\alpha(t_{i-1})|\le\sum\int_{t_{i-1}}^{t_i}|\alpha'(t)|\,dt=\int_a^b|\alpha'(t)|\,dt \text{ (done)}$$

We have reduced the problem into

$$\text{finding } \delta \text{ such that } \forall P:|P|<\delta,\int_a^b|\alpha'(t)|\,dt-l(\alpha,P)<\epsilon$$

Because α' is uniformly continuous on $[a,b]$ ($\because$ continuous function on compact domain is uniformly continuous), we know there exists δ' such that

$$|\alpha'(s)-\alpha'(t)|<\frac{\epsilon}{2(b-a)} \text{ if } |s-t|<\delta'$$

We claim

$$\text{such } \delta' \text{ works}$$

Let $|P|<\delta$, and let $s_i\in[t_{i-1},t_i]$. Because $|s_i-t_i|<\delta$, we have

$$|\alpha'(s_i)-\alpha'(t_i)|<\frac{\epsilon}{2(b-a)} \tag{1.2}$$

This give us

$$|\alpha'(s_i)|<|\alpha'(t_i)|+\frac{\epsilon}{2(b-a)}$$

Now, we can deduce

$$\begin{aligned}
\int_{t_{i-1}}^{t_i} |\alpha'(s)|\, ds &\leq |\alpha'(t_i)|\, \Delta t_i + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&= \int_{t_{i-1}}^{t_i} |\alpha'(t_i)|\, dt + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&= \left| \int_{t_{i-1}}^{t_i} \alpha'(t_i) dt \right| + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&= \left| \int_{t_{i-1}}^{t_i} \alpha'(t_i) - \alpha'(t) dt + \int_{t_{i-1}}^{t_i} \alpha'(t) dt \right| + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&\leq \left| \int_{t_{i-1}}^{t_i} \alpha'(t_i) - \alpha'(t) dt \right| + \left| \int_{t_{i-1}}^{t_i} \alpha'(t) dt \right| + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&\leq \frac{\epsilon}{2(b-a)}\Delta t_i + |\alpha(t_i) - \alpha(t_{i-1})| + \frac{\epsilon}{2(b-a)}\Delta t_i \\
&= |\alpha(t_i) - \alpha(t_{i-1})| + \frac{\epsilon}{b-a}\Delta t_i
\end{aligned}$$

Notice that the last inequality follows from Equation 1.2. The long deduction above then give us

$$\begin{aligned}
\int_a^b |\alpha'(t)|\, dt &\leq \sum |\alpha(t_i) - \alpha(t_{i-1})| + \frac{\epsilon}{b-a}(b-a) \\
&= l(\alpha, P) + \epsilon
\end{aligned}$$

Then we have

$$\int_a^b |\alpha'(t)|\, dt - l(\alpha, P) \leq \epsilon \text{ (done)}$$

■

Question 7

9. a. Let $\alpha: I \longrightarrow R^3$ be a curve of class C^0 (cf. Exercise 7). Use the approximation by polygons described in Exercise 8 to give a reasonable definition of arc length of α.

b. (*A Nonrectifiable Curve.*) The following example shows that, with any reasonable definition, the arc length of a C^0 curve in a closed interval may be unbounded. Let α: $[0, 1] \longrightarrow R^2$ be given as $\alpha(t) = (t, t\sin(\pi/t))$ if $t \neq 0$, and $\alpha(0) = (0, 0)$. Show, geometrically, that the arc length of the portion of the curve corresponding to $1/(n+1) \leq t \leq 1/n$ is at least $2/(n+\frac{1}{2})$. Use this to show that the length of the curve in the interval $1/N \leq t \leq 1$ is greater than $2\sum_{n=1}^{N} 1/(n+1)$, and thus it tends to infinity as $N \longrightarrow \infty$.

Proof. **(a)** Suppose $I = [a, b]$. Define arc length by

$$\sup_P l(P, \alpha) \text{ where sup runs over all partition } P \text{ of } [a, b]$$

(b)

Geometrically, we know the arc length of the portion of the curve corresponding to $t \in [\frac{1}{n+1}, \frac{1}{n}]$ must be greater than

$$\left|\alpha(\frac{1}{n}) - \alpha(\frac{1}{n+\frac{1}{2}})\right| + \left|\alpha(\frac{1}{n+1}) - \alpha(\frac{1}{n+\frac{1}{2}})\right|$$

This

■

Question 8

10. (*Straight Lines as Shortest.*) Let $\alpha: I \longrightarrow R^3$ be a parametrized curve. Let $\{a, b\} \subset I$ and set $\alpha(a) = p$, $\alpha(b) = q$.

a. Show that, for any constant vector v, $|v| = 1$,

$$(q - p) \cdot v = \int_a^b \alpha'(t) \cdot v \, dt \leq \int_a^b |\alpha'(t)| \, dt.$$

b. Set

$$v = \frac{q - p}{|q - p|}$$

and show that

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)| \, dt;$$

that is, the curve of shortest length from $\alpha(a)$ to $\alpha(b)$ is the straight line joining these points.