Representation Theory of Finite Group HW1

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Problem A

$F_n = \langle \alpha, \beta | \alpha^2 = \beta^2 = 1, (\alpha\beta)^n = 1 \rangle$

We know $\langle r, s | r^n = 1 = s^2, srs^{-1} = r^{-1} \rangle$ is a presentation of D_n

Let ψ be defined by $\alpha \mapsto rs, \beta \mapsto s$

We prove ψ preserve the defining relation of F_n

Proof. $s^2 = 1$ and $srs^{-1} = r^{-1} \implies srs = r^{-1}$

$\psi(\alpha)^2 = (rs)^2 = r(srs) = rr^{-1} = 1$

$s^2 = 1 \implies \psi(\beta)^2 = s^2 = 1$

$(\psi(\alpha)\psi(\beta))^n = (rss)^n = r^n = 1$

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So there exists an unique homomorphism $\phi : F_n \to D_n$ lifted by ψ

We prove ϕ is onto.

Proof. $\phi(\alpha\beta) = \phi(\alpha)\phi(\beta) = rss = r$

$\phi(\beta) = s$

The generators of D_n belong to $\phi[F_n] \implies D_n \subseteq \phi[F_n]$

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We prove $|P| = 8 = Q_8$, then ϕ is thus bijective, implicating that ϕ is an isomorphism from P to Q_8

Proof. Every element $w \in F_n$ by definition can be expressed as one of the following four forms.

(i) $w = \alpha^{c_1}\beta^{c_2}\dots\alpha^{c_n}$

(ii) $w = \alpha^{c_1}\beta^{c_2}\dots\alpha^{c_{n_1}}\beta^{c_n}$

(iii) $w = \beta^{c_1}\alpha^{c_2}\dots\beta^{c_n}$

(iv) $w = \beta^{c_1}\alpha^{c_2}\dots\beta^{c_{n-1}}\alpha^{c_n}$

Becasue $\beta^2 = 1$, we know $\beta^{c_i} = \beta$ if c_i is odd, $\beta^{c_i} = 1$ if c_i is even.

Same goes α

So every $w \neq 1 \in F_n$ starts either with α or with β, ends either with α or with β, and contain "isolated" terms α and β in the middle.

More precisely, w is in one of the following four forms.

(i) $w = \alpha\beta\dots\alpha$

(ii) $w = \alpha\beta\dots\alpha\beta$

(iii) $w = \beta\alpha\dots\beta$

(iv) $w = \beta\alpha\dots\beta\alpha$

In forms **(i)** and **(ii)**, $w = (\alpha\beta)^p\alpha^q, \exists\, q \in \mathbb{Z}$

If $q = 0$, w is in **(i)** form. If not, w is in **(ii)** form.

Notice in forms **(iii)** or **(iv)**, $w = (\beta\alpha)^m\alpha^k, \exists m, n \in \mathbb{Z}$, and $(\beta\alpha)\alpha = \beta = \alpha(\alpha\beta)$,so $w = (\beta\alpha)^m\alpha^k = \alpha^p(\alpha\beta)^q, \exists p, q \in \mathbb{Z}$

Because $(\alpha\beta)^n = 1 = \alpha^2$, we can set $0 \leq p \leq n-1, 0 \leq q \leq 1$, WOLG.

Till here, we have already turn every element $w \in F_n$ into the form $(\alpha\beta)^p\alpha^q, \exists 0 \leq p \leq n-1, 0 \leq q \leq 1$

We now show $p = 0, q = 0$ is the only possibility such $(\alpha\beta)^p\alpha^q = 1$, then $(\alpha\beta)^{d_1}\alpha^{d_2} = (\alpha\beta)^{d_3}\alpha^{d_4} \iff (\alpha\beta)^{d_1-d_3}\alpha^{d_2-d_4} = 1 \iff d_1 = d_3, d_2 = d_4$, the form $(\alpha\beta)^p\alpha^q$ have exactly $2n$ number amount of distinct elements.

If $q = 0$, $1 = (\alpha\beta)^p\alpha^q \implies (\alpha\beta)^p = 1 \implies p = 0$, since $(\alpha\beta)$ is of order n

Assume $1 = (\alpha\beta)^p\alpha$

Let C_α be defined by $C_\alpha(x) = \alpha x\alpha$

Let C_β be defined by $C_\beta(x) = \beta x\beta$

$C_\alpha(1) = \alpha\alpha = 1 = \beta\beta = C_\beta(1)$

$C_\alpha(1 = (\alpha\beta)^p\alpha) = \alpha(\alpha\beta)^p\alpha\alpha = \alpha(\alpha\beta)(\alpha\beta)^{p-1} = \beta(\alpha\beta)^{p-1} = 1$

$C_\beta(1 = \beta(\alpha\beta)^{p-1}) = \beta\beta(\alpha\beta)^{p-1}\beta = (\alpha\beta)^{p-2}(\alpha\beta)\beta = (\alpha\beta)^{p-2}\alpha = 1$

So, $\cdots \circ C_\beta \circ C_\alpha((\alpha\beta)^p\alpha = 1) = (\alpha \text{ or } \beta) = 1$ CaC

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Probelem B

$P = \langle a, b | a^4 = 1 = b^4, a^2 = b^2, ba = a^{-1}b \rangle$

Let ψ be defined by $a \mapsto i, b \mapsto j$

We prove ψ preserve the definig relation in P

Proof. $(\psi(a))^4 = i^4 = 1$

$(\psi(b))^4 = j^4 = 1$

$(\psi(a)^2) = i^2 = -1 = j^2 = (\psi(b))^2$

$\psi(b)\psi(a) = ji = -k = (-i)j = (\psi(a))^{-1}\psi(b)$ ■

So there exists an unique homomorphism $\phi : P \to Q_8$ lifted by ψ

We prove ϕ is onto.

Proof. $1 = i^4$
$-1 = i^2$
$i = i$
$-i = i^3$
$j = j$
$-j = j^3$
$k = ij$
$-k = ji$

■

We prove $|P| = 8 = Q_8$, then ϕ is thus bijective, implicating that ϕ is an isomorphism from P to Q_8

Proof. Because $ba = a^{-1}b$ and $a^4 = 1 = b^4$, we can turn every elements in P into the form $a^pb^q, \exists 0 \leq p, q \leq 3$

Using $a^2 = b^2$, we can have following:

$a^2b = b^3$

$ab^2 = a^3$

$a^2b^2 = 1$

$a^3b = ab^3$

$a^3b^2 = a$

$a^2b^3 = b$

$a^3b^3 = ab$

and obviously $b^2 = a^2$

We only have to show $S := \{1, a, b, ab, a^3, b^3, ab^3, a^2\}$ (S is the elements on the RHS of the equations above) contains 8 distinct elements.

Clearly, $\{1, a, b\}$ are unique elements

Assume $a = ab$ or $b = ab$, then $1 = b$ or $1 = a$ CaC

Assume $1 = ab$

$ba = a^{-1}b \implies aba = aa^{-1}b \implies a = b$ CaC

So $\{1, a, b, ab\}$ are all distinct elements.

And we see a^3, b^3, ab^3 are respectively the inverse elements of a, b, ab, so $\{1, a, b, ab, a^3, b^3, ab^3\}$ are all distinct elements.

Lastly, we see order(a)=order(b)=order(ab)=4, and order(a^2)=2, so $\{1, a, b, ab, a^3, b^3, ab^3, a^2\}$ are all distinct elements. ■

Problem C

$\tilde{D_n} = \{\begin{bmatrix} \pm 1 & k \\ 0 & 1 \end{bmatrix} | k \in \mathbb{Z}_n\}$

$D_n = \langle r, s | r^n = 1 = s^2, srs^{-1} = r^{-1} \rangle$

Let ψ be defined by $\psi(r) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\psi(s) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$

We prove ψ preserve the defining relation of D_n

Proof. We prove $\forall 0 < k < n, \psi(r)^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, by inducion. **(i)**

$$\text{Base step: } \psi(r)^1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

This is trivial.

$$\text{Induction step: } \psi(r)^{k-1} = \begin{bmatrix} 1 & k-1 \\ 0 & 1 \end{bmatrix} \implies \psi(r)^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

$\psi(r)^k = \psi(r)^{k-1}\psi(r) = \begin{bmatrix} 1 & k-1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

So $\psi(r)^n = \psi(r)^{n-1}\psi(r) = \begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} = I_2$

$\psi(s)^2 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}^2 = I_2$

$\psi(s)\psi(r)\psi(s)^{-1} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} =$
$\phi(r)^{-1}$

■

So there exists an unique homomorphism $\phi : D_n \to \tilde{D_n}$ lifted by ψ

We prove ϕ is onto.

Proof. $\forall k \in \mathbb{Z}_n$, $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = \phi(r)^k$ have alreay been proven in **(i)**.
$\forall k \in \mathbb{Z}_n$, $\begin{bmatrix} -1 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -k+1 \\ 0 & 1 \end{bmatrix} = \phi(s)\phi(r)^{-k+1}$ ■

Clearly, $|\tilde{D_n}| = 2n = |D_n|$, implicating that ϕ is an isomorphism from $\tilde{D_n} = D_n$.