## 6.2 HW2

**Question 24**

Let $A \subseteq \mathbb{R}^n$ be an open set and $B \subset \mathbb{R}^n$ be any set. Then the set

$$A + B \equiv \{a + b : a \in A \text{ and } b \in B\}$$

is open.

*Proof.* Notice

$$A + B = \bigcup \{\{a + b : a \in A\} : b \in B\} \tag{6.19}$$

We only have to prove for all $b \in B$, the set $A + b := \{a + b : a \in A\}$ is open.

Fix $b$. Arbitrarily pick $a + b \in A + b$. Because $A$ is open, we know there exists $r \in \mathbb{R}^+$ such that

$$B_r(a) \subseteq A \tag{6.20}$$

Let

$$B_r(a) + b := \{x + b : x \in B_r(a)\} \tag{6.21}$$

We now prove

$$B_r(a) + b = B_r(a + b) \tag{6.22}$$

Arbitrarily pick $x + b \in B_r(a) + b$. We have

$$|(x + b) - (a + b)| = |x - a| < r \tag{6.23}$$

We have proved $B_r(a) + b \subseteq B_r(a + b)$. Arbitrarily pick $y \in B_r(a + b)$. Let $z = y - b$. We have

$$y = z + b \text{ and } |z - a| = |y - (a + b)| < r \tag{6.24}$$

The latter tell us $z \in B_r(a)$, so we have

$$y = z + b \in B_r(a) + b \tag{6.25}$$

Because $y$ is arbitrarily picked from $B_r(a+b)$, we have proved $B_r(a+b) \subseteq B_r(a)+b$ (done)

Notice that $r$ is selected to satisfy

$$B_r(a) \subseteq A \tag{6.26}$$

and it is clear that

$$B_r(a) + b \subseteq A + b \tag{6.27}$$

So we have

$$B_r(a + b) = B_r(a) + b \subseteq A + b \tag{6.28}$$

Notice $a + b$ is arbitrarily picked from $A + b$. Our proof is done (done) ■

## Question 25

Show that every open set in $\mathbb{R}$ is the union of at most countable collection of disjoint open intervals.

*Proof.* Because $\mathbb{Q}$ is countable and dense in $\mathbb{R}$, so in Theorem 2.7.1, we have proved the set

$$\mathcal{O} := \{B_r(p) : p \in \mathbb{Q} \text{ and } r \in \mathbb{Q}^+\} \tag{6.29}$$

is a countable base. Notice that every open ball $B_r(p)$ can be expressed as $(p - r, p + r)$, so we know $\mathcal{O}$ is a countable collection of open interval.

Let $A$ be an open set. We wish to prove $A$ can be expressed as union of a countable collection of disjoint open interval.

Because $\mathcal{O}$ is a base, for each $a \in A$, we can find $B_{r_a}(p_a)$, such that

$$a \in B_{r_a}(p_a) \subseteq A \tag{6.30}$$

Collect all such open ball for each points in $A$ and call this collection $\mathcal{O}'$.

Define a relation $\sim$ on $\mathcal{O}'$ by

$$(a, b) \sim (c, d) \text{ if } \exists S \in \mathcal{P}(\mathcal{O}') \text{ such that} \tag{6.31}$$

$$\bigcup S \text{ is an open interval and } (a, b) \subseteq \bigcup S \text{ and } (c, d) \subseteq \bigcup S \tag{6.32}$$

We wish to prove $\sim$ is an equivalence relation.

To show $(a_1, b_1) \sim (a_1, b_1)$, use $S = \{(a_1, b_1)\}$. To show $(a_1, b_1) \sim (a_2, b_2) \implies (a_2, b_2) \sim (a_1, b_1)$, use the same $S$. It is left to prove

$$(a_1, b_1) \sim (a_2, b_2) \text{ and } (a_2, b_2) \sim (a_3, b_3) \implies (a_1, b_1) \sim (a_3, b_3) \tag{6.33}$$

Let $S_1 \in \mathcal{P}(\mathcal{O}')$ be from $(a_1, b_1) \sim (a_2, b_2)$, and let $S_2 \in \mathcal{P}(\mathcal{O}')$ be from $(a_2, b_2) \sim (a_3, b_3)$.

Define

$$S_3 := S_1 \cup S_2 \tag{6.34}$$

Because $(a_2, b_2) \subseteq \bigcup S_1 \cap \bigcup S_2$, we know $\bigcup S_1 \cap \bigcup S_2$ are not disjoint. Then because union of two intersecting open interval is again an open interval, we know $\bigcup S_3$ is an open interval.

Deduce

$$(a_1, b_1) \subseteq \bigcup S_1 \subseteq \bigcup S_3 \tag{6.35}$$

and deduce

$$(a_3, b_3) \subseteq \bigcup S_2 \subseteq \bigcup S_3 \text{ (done)} \tag{6.36}$$

Let $E$ be the collection of union of each equivalent class of $\sim$ on $\mathcal{O}'$. Because $\mathcal{O}'$ is countable, we know $E$ is countable. Every element of $\mathcal{O}'$ is a subset of $A$ by definition, so we know $\bigcup E = \bigcup \mathcal{O}' \subseteq A$. Every element of $A$ is in some $O \in \mathcal{O}'$ by definition, so we have $A \subseteq \bigcup \mathcal{O}' = \bigcup E$. It is only left to prove each member in $E$ is an open interval and each two members of $E$ are disjoint.

We first prove disjoint.

Let $B, C \in E$ and let $B', C'$ be equivalent class that satisfy

$$B = \bigcup B' \text{ and } C = \bigcup C' \tag{6.37}$$

Assume $B, C$ is not disjoint, say, $x \in B \cap C$, we have

$$\exists (a, b) \in B', x \in (a, b) \subseteq B \tag{6.38}$$

and have

$$\exists (c, d) \in C', x \in (c, d) \subseteq C \tag{6.39}$$

We then can see $(a, b)$ and $(c, d)$ intersect, then we can use $S = \{(a, b), (c, d)\}$ to show $(a, b) \sim (c, d)$ CaC (done) .

We now prove open interval.

Let $B \in E$, and let $B'$ be equivalent class that satisfy $B = \bigcup B'$. We wish to prove $B = (\inf B, \sup B)$ where negative and positive infinity is taken into account.

Because $B'$ contain only open sets, we know $B$ is open. Then we know $\sup B$ and $\inf B$ if exists, is not in $B$. To see such, just observe every open ball centering $\sup B$ has a number greater than all numbers in $B$, and similarly for $\inf B$ vice versa.

We have proved $B \subseteq (\inf B, \sup B)$.

We only have to prove every point in $(\inf B, \sup B)$ is in $B$, where infimum and supremum can be negative or positive infinity.

Assume $\exists x \in (\inf B, \sup B), x \notin B$. We have

$$\exists (y_1, z_1) \in B', (y_1, z_1) \subseteq (\inf B, x) \text{ and } \exists (y_2, z_2) \in B', (y_2, z_2) \subseteq (x, \sup B) \tag{6.40}$$

Because $(y_1, z_1) \sim (y_2, z_2)$ there exists $S \in \mathcal{O}'$ such that $(y_1, z_1) \subseteq \bigcup S$ and $\bigcup S$ is an open interval.

If $S$ contain any element $(v, w)$ not in $B'$, we can use $S$ to show $(v, w) \sim (y_2, z_2)$, causing a contradiction. We have proved $S$ is a subset of $B'$.

Now, Because $x \in \bigcup S$, we know there must exists some interval in $S \subseteq B'$ containing $x$ CaC to $x \notin B$ (done) (done) ■

**Question 26**

Let $A \subseteq B \subseteq \mathbb{R}$. Suppose that $A$ is a dense subset of $B$.
1. Prove that $B \subseteq \overline{A}$.
2. If $B$ is closed, determine whether $B = \overline{A}$.

*Proof.* $A$ is a dense subset of $B$ means that every point $b \in B \setminus A$ is a limit point of $A$ in the scope of $B$. Notice that $b$ is a limit point of $A$ in the scope of $B$ also means $b$ is a limit point in the scope of $\mathbb{R}$. Then we have proved $B \setminus A \subseteq A'$. It follows

$$B = A \cup (B \setminus A) \subseteq A \cup A' = \overline{A} \tag{6.41}$$

If $B$ is closed, we have

$$\overline{A} \subseteq \overline{B} = B \tag{6.42}$$

Then because $B \subseteq \overline{A}$, we have $B = \overline{A}$

■

**Definition 0.1.** A metric space $X$ is *sequentially compact* if every sequence of points in $X$ has a convergent sub-sequence converging to a point in $X$.

**Question 27**

Let $A$ and $B$ be subsets of a metric space $(M, d)$ and denote $\text{cl}(A) = \overline{A}$. Show that
1. $\text{cl}(\text{cl}(A)) = \text{cl}(A)$.
2. $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$.
3. $\text{cl}(A \cap B) \subseteq \text{cl}(A) \cap \text{cl}(B)$. Find an example such that $\text{cl}(A \cap B) \subsetneq \text{cl}(A) \cap \text{cl}(B)$.

*Proof.* Notice that closure is the smallest closed set containing the set.

Because $\overline{A}$ is a closed set, we know the smallest closed set containing $\overline{A}$ is $\overline{A}$ itself.

Deduce

$$A \subseteq A \cup B \implies \overline{A} \subseteq \overline{A \cup B} \tag{6.43}$$

and deduce

$$B \subseteq A \cup B \implies \overline{B} \subseteq \overline{A \cup B} \tag{6.44}$$

so deduce

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cup B} \tag{6.45}$$

Notice that $\overline{A} \cup \overline{B}$ is a closed set containing both $A$ and $B$, thus containing $A \cup B$, so because $\overline{A \cup B}$ by definition is the smallest closed set containing $A \cup B$, we have

$$\overline{A \cup B} \subseteq \overline{A} \cup \overline{B} \tag{6.46}$$

Notice that $\overline{A}$ and $\overline{B}$ are both closed set containing $A \cap B$. Then because $\overline{A \cap B}$ is the smallest closed set containing $A \cap B$. We have

$$\overline{A \cap B} \subseteq \overline{A} \text{ and } \overline{A \cap B} \subseteq \overline{B} \tag{6.47}$$

Then have

$$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B} \tag{6.48}$$

Let $A = (0,1)$ and $B = (1,2)$. We have

$$\overline{A \cap B} = \overline{\varnothing} = \varnothing \text{ and } \overline{A} \cap \overline{B} = [0,1] \cap [1,2] = \{1\} \tag{6.49}$$

■

**Question 28**

Let $K$ be a sequentially compact set in a metric space $(M,d)$ and let $F \subseteq K$ be closed. Prove that $F$ is sequentially compact.

*Proof.* Let $\{x_i\}_{i=1}^{\infty}$ be a sequence in $F$

$$\{x_i\}_{i=1}^{\infty} \subseteq F \subseteq K \tag{6.50}$$

Because $K$ is sequentially compact, we know there exists a sub-sequence $\{x_{n_i}\}_{i=1}^{\infty}$ converge to some point $a$

$$\lim_{i \to \infty} \{x_{n_i}\} = a \tag{6.51}$$

We only have to prove $a$ is in $F$.

Assume $a \notin F$. Because $F$ is closed, we know $F^c$ is open. Then $a \in F^c$ tell us there exists an open ball $B_r(a)$ contain no point in $F$, thus disjoint to $\{x_{n_i}\}$ CaC

■

**Definition 0.2.** The discrete metric $d$ on a set $X$ is defined by

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

for any $x, y \in X$. In this case $(X, d)$ is called a discrete metric space.

**Question 29**

Let $(M, d)$ be a metric space with discrete metric $d$. Prove that every compact set in $M$ is finite.

*Proof.* Notice we have

$$\forall p \in M, B_1(p) = \{p\} \tag{6.52}$$

So we know no set in $M$ has a limit point. Let $K \subseteq M$ be compact. By Theorem 2.7.4, we know $K$ is limit point compact. Assume $K$ is infinite. Then $K$ has a limit point CaC . ■

**Question 30**

Let $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ be a convergent sequence with the limit $x$. Prove that the set $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact.

*Proof.* Let

$$E \subseteq \{x_n : n \in \mathbb{N}\} \cup \{x\} \tag{6.53}$$

If $E$ is infinite, we can regard $E \setminus \{x\}$ as a sub-sequence of $\{x_n\}$. Then because $\{x_n\}_{n=1}^{\infty}$ converge to $x$, we know $E \setminus \{x\}$ converge to $x$. Then we know $x$ is a limit point of $E \setminus \{x\}$ thus a limit point of $E$. We have proved every infinite subset of $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ has a limit point $x$ in $\{x_n : n \in \mathbb{N}\} \cup \{x\}$. By Theorem 2.7.4, $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact. ■

**Question 31**

1. Let $A$ and $B$ be two subsets of a metric space $(M, d)$. The distance between $A$ and $B$ is defined by

$$d(A, B) \equiv \inf\{d(x, y) : x \in A, y \in B\}.$$

   (a) Give an example of two disjoint, nonempty, closed sets $A$ and $B$ in $\mathbb{R}^n$ for which $d(A, B) = 0$.

   (b) Let $A, B$ be nonempty sets in $\mathbb{R}^n$ with $A$ closed and $B$ compact. Show that there are points $a \in A$ and $b \in B$ such that $d(a, b) = |a - b|$. Deduce that $d(A, B)$ is positive if such $A, B$ are disjoint.

*Proof.* Let $n = 1$. Let

$$A = \{a_k = k - 1 + \sum_{i=1}^{k-1} 2^{-i-1} : k \in \mathbb{N}\} \tag{6.54}$$

Let

$$B = \{b_k = k - \sum_{i=1}^{k-1} 2^{-i-1} : u \in \mathbb{N}\} \tag{6.55}$$

Notice the pattern

$$a_1 = 0 < \frac{1}{2} < 1 = b_1 \tag{6.56}$$
$$a_2 = 1 + \frac{1}{4} < 1 + \frac{1}{2} < 2 - \frac{1}{4} = b_2 \tag{6.57}$$
$$a_3 = 2 + \frac{1}{4} + \frac{1}{8} < 2 + \frac{1}{2} < 3 - \frac{1}{4} - \frac{1}{8} = b_3 \tag{6.58}$$
$$\vdots \tag{6.59}$$

and compute

$$b_k - a_k = 2^{1-k} \tag{6.60}$$

First observe for each non-negative integer $m$m the interval $[m, m+1]$ contain exactly one point from $A$ and contain exactly one point from $B$. To see $A$ and $B$ are disjoint, observe that because $A, B$ contain only non-negative numbers, so every point is in $[m, m+1]$ for some non-negative integer $m$, and we know each interval only contain $a_{m+1}$ from $A$ and $b_{m+1}$ from $B$ where $b_{m+1} - a_{m+1} = 2^{-m}$.

We now show $A, B$ are both closed. For negative $-x \in \mathbb{R}$, observe $B_x(-x)$ intersect with neither $A$ nor $B$. We know $B_1(0)$ does not intersect with $B$ and $0 \in A$. For positive $x$ not in $A$, let $m = \lfloor x \rfloor$. We know point closest to $x$ is either $a_m, a_{m+1}$ or $a_{m+2}$. Then we can let $r = \min\{|x - a_m|, |x - a_{m+1}|, |x - a_{m+2}|\}$, and see $B_r(x)$ does not intersect with $A$.For positive $x$ not in $B$, let $m = \lfloor x \rfloor$. We know point closest to $x$ is either $b_m, b_{m+1}$ or $b_{m+2}$. Then we can let $r = \min\{|x - b_m|, |x - b_{m+1}|, |x - b_{m+2}|\}$, and see $B_r(x)$ does not intersect with $B$.

Lastly, to prove $d(A, B) = 0$ is to prove for each positive real $r$, there exists a pair of $a, b$ such that $|a - b| < r$. Observe

$$b_k - a_k = 2^{1-k} < r \iff 1 - k < \log_2 r \iff k > 1 - \log_2 r \tag{6.61}$$

And we are done, by picking great enough $k$.

Now we do (b).

Because

$$\{d(a,b) : a \in A, b \in B\} = \bigcup \{\{d(a,b) : a \in A\} : b \in B\} \tag{6.62}$$

so we have

$$\begin{aligned} d(A,B) &= \inf\{d(a,b) : a \in A, b \in B\} &(6.63)\\ &= \inf \bigcup \{\{d(a,b) : a \in A\} : b \in B\} &(6.64)\\ &= \inf\{\inf\{d(a,b) : a \in A\} : b \in B\} &(6.65)\\ &= \inf\{d(A,\{b\}) : b \in B\} &(6.66)\end{aligned}$$

where the third equality hold true by the fourth question of HW1.

We first prove if $d(A,B) = 0$, then $A$ and $B$ intersect.

If $B$ is finite, then $0 = d(A,B) = \min\{d(A,\{b\} : b \in B)\}$, so we know there exists $b \in B$ such that $0 = d(A,\{b\}) = \inf\{d(a,b) : a \in A\}$. Then we can deduce every open ball $B_r(b)$ contain a point $a \in A$, since there exists $a$ such that $d(a,b) < r$. We have proved $b$ is a limit point of $A$, then because $A$ is closed, we know $b \in A$.

If $B$ is infinite, then $0 = \inf\{d(A,\{b\}) : b \in B\}$. For each $n$, we then know there exists $b_n$ such that $d(A,\{b_n\}) < \frac{1}{n}$.

Construct an infinite sequence $\{b_n\}_{n\in\mathbb{N}}$ by picking $b_n$ such that $d(A,\{b_n\}) < \frac{1}{n}$. We can see our method for picking $b_n$ will give us an infinite sequence as every $b$ satisfy $d(A,\{b\}) > 0$. By Theorem 2.7.4, we know $B$ is limit point compact, so we know $\{b_n\}_{n\in\mathbb{N}}$ has a limit point in $B$.

Denote the limit point for $\{b_n\}_{n\in\mathbb{N}}$ as $p$. We now show $p \in A$. Assume $p \in A^c$. Because $A$ is closed, we know there exists an open ball $B_r(p)$ disjoint with $A$. Then we know $d(A,\{p\}) \geq r$.

Notice that if we fix $b' \in B$. We have triangle inequality

$$d(A,\{p\}) \leq d(A,\{b'\}) + d(b',p)* \tag{6.67}$$

Because

$$d(A,\{b'\}) + d(b',p) = \inf\{d(a,b') : a \in A\} + d(b',p) = \inf\{d(a,b') + d(b',p) : a \in A\} \tag{6.68}$$

where for each $a$, we have

$$d(A,\{p\}) \leq d(a,p) \leq d(a,b') + d(b',p) \tag{6.69}$$

Then from

$$\forall b \in B, r \leq d(A,\{p\}) \leq d(A,\{b\}) + d(\{b\},\{p\}) \tag{6.70}$$

we have

$$\forall b \in B[d(A,\{b\}) < \frac{r}{2} \implies d(\{b\},\{p\}) > \frac{r}{2}] \tag{6.71}$$

Then we see if $n > \frac{2}{r}$, then $b_n$ is not contained by the open ball $B_{\frac{r}{2}}(p)$. In other words, the open ball $B_{\frac{r}{2}}(p)$ that center a limit point $p$ contain at most $\lfloor \frac{2}{r} \rfloor$ amount of point in $\{b_n\}_{n\in\mathbb{N}}$ CaC (done)

Lastly, we prove there exists $a \in A, b \in B$ such that $d(A,B) = d(a,b)$. If $A$ and $B$ are not disjoint, the fact $\exists a \in A, \exists b \in B, d(A,B) = |a-b|$ is clear by picking $a = b$. We only have to consider when $A$ and $B$ are disjoint.

Let $d(A,B) = u$. We know $u > 0$, because of the result we have proved.

We have

$$\inf\{d(A,\{b\}) : b \in B\} = d(A,B) = u \tag{6.72}$$

At this stage, we wish to prove $\exists b \in B$ such that $d(A,\{b\}) = u$. Assume there does not exist such $b$

We know, for each $n \in \mathbb{N}$, there exists a point $b_n$ such that

$$d(A,\{b_n\}) < u + \frac{1}{n} \tag{6.73}$$

Because $\forall b \in B, d(A,\{b\}) > u$, we know if we collect one $b_n$ that satisfy $d(A,\{b_n\}) < u + \frac{1}{n}$ for each $n \in \mathbb{N}$, the sequence $\{b_n\}_{n\in\mathbb{N}}$ is infinite.

Then by Theorem 2.7.4, we know there exists a limit point $p \in B$ for $\{b_n\}_n \in \mathbb{N}$.

We know $d(A,\{p\}) > u$ by our assumption.

Let $m = d(A,\{p\}) > u$. Then from

$$\forall b \in B, m = d(A,\{p\}) < d(A,\{b\}) + d(b,p) \text{ This was proved at * } \tag{6.74}$$

we have

$$\forall b \in B, d(A,\{b\}) < u + \frac{1}{n} \implies d(b,p) > m - u - \frac{1}{n} \tag{6.75}$$

Let $k$ be a natural such that $u + \frac{1}{k} < m$. For each natural $t$ greater than $k$, we have

$$d(b_t, p) > m - u - \frac{1}{t} > m - u - \frac{1}{k} \tag{6.76}$$

In other words, the open ball $B_{m-u-\frac{1}{k}}(p)$ centering a limit point contain at most $k$ amount of point in $\{b_n\}_{n\in\mathbb{N}}$ CaC (done)

Let $b \in B$ satisfy $d(A, \{b\}) = u$. At the final stage, we wish to prove $\exists a \in A, d(a, b) = u$.

Assume $\forall a \in A, d(a, b) > u$. We know, for each $n \in \mathbb{N}$, there exists a point $a_n$ such that

$$d(a_n, b) < u + \frac{1}{n} \tag{6.77}$$

Because $\forall a \in A, d(a, b) > u$, we know if we we collect one $a_n$ that satisfy $d(a_n, b) < u + \frac{1}{n}$ for each $n \in \mathbb{N}$, the sequence $\{a_n\}_{n\in\mathbb{N}}$ is infinite.

Notice that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is also bounded as one can check $B_{2(u+1)}(a_1)$ contain all $\{a_n\}_{n\in\mathbb{N}}$, using $b$ as a "pivot".

Then because we are in $\mathbb{R}^n$ and $\{a_n\}_{n\in\mathbb{N}}$ is closed and bounded, we know $\{a_n\}_{n\in\mathbb{N}}$ is compact, then by Theorem 2.7.4, we know $\{a_n\}_{n\in\mathbb{N}}$ has a limit point in itself.

Let $a_k$ be a limit point of $\{a_n\}_{n\in\mathbb{N}}$.

Let $m = d(a_k, b) > u$. Observe that

$$\forall a_n, m = d(a_k, b) < d(b, a_n) + d(a_n, a_k) \tag{6.78}$$

give us

$$\forall a_n, d(a_n, b) < u + \frac{1}{n} \implies d(a_n, a_k) > m - u - \frac{1}{n} \tag{6.79}$$

Let $s \in \mathbb{N}$ be great enough so that $m - u - \frac{1}{s} > 0$. Then we see for each natural $w$ greater than $s$, we have

$$d(a_w, b) < u + \frac{1}{w} < u + \frac{1}{s} \tag{6.80}$$

so we have

$$d(a_w, a_k) > m - u - \frac{1}{s} \tag{6.81}$$

Then we see the open ball $B_{m-u-\frac{1}{s}}(a_k)$ contain at most $s$ amount of point in $\{a_n\}_{n\in\mathbb{N}}$ CaC (done) ■

**Question 32**

9. Let $(M, d)$ be a metric space.

   (a) Show that the union of a finite number of compact subsets of $M$ is compact.

   (b) Show that the intersection of an arbitrary collection of compact subsets of $M$ is compact.

*Proof.* Let $\mathcal{K} = \{K_1, \dots, K_n\}$ be a finite collection of compact subset of $M$. We wish to show $\bigcup \mathcal{K}$ is compact. Let $\mathcal{G}$ be an open cover for $\mathcal{K}$. For each $m \in \mathbb{N} : 1 \leq m \leq n$, because $K_m \subseteq \bigcup \mathcal{K}$, we know $\mathcal{G}$ is also an open cover for $K_m$. Then, we can pick a finite sub-cover $\mathcal{G}_m \subseteq \mathcal{G}$ for $K_m$. Collect all such finite sub-cover $\mathcal{G}_1, \dots, \mathcal{G}_m$. The union $\bigcup\{\mathcal{G}_1, \dots, \mathcal{G}_m\}$ is a finite open cover for $\bigcup \mathcal{K}$.

Notice that a compact set must be closed, and arbitrary collection of closed sets are closed. Then, we know the intersection of an arbitrary collection of compact set must be a closed set and a subset of every compact set in the collection. Then because closed subset of compact set is compact, we know the intersection is compact. ■

**Question 33**

10. A metric space $(M, d)$ is said to be *separable* if there is a countable subset $A$ which is dense in $M$. Show that every compact set is separable.

*Proof.* See the proof in Theorem 2.7.4, there is a summary where you can find the numbering of rigorous proof for each step. Some proof for corollary is omitted because they are immediate consequences of theorem before.

Because I use a program written by artificial intelligence to update my reference in latex code whenever I edit my latex code, some reference may be directed to a wrong theorem, but all the tools are there in Chapter 2. ■