Chapter 3

PDE intro HW

3.1 HW1

Theorem 3.1.1.

$$\text{Show } u \mapsto u_x + uu_y \text{ is non-linear}$$

Proof. See that

$$2u \mapsto 2u_x + 4uu_y \neq 2(u_x + uu_y) \tag{3.1}$$

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Theorem 3.1.2.

$$\text{Solve } (1 + x^2)u_x + u_y = 0$$

Proof. The characteristic curve has the derivative

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

The solution to this ODE is

$$y = \arctan x + C$$

We now see that the solution to the PDE in Equation 3.1 is

$$u = f\big((\arctan x) - y\big) \text{ where } f : \mathbb{R} \to \mathbb{R} \text{ is an arbitrary smooth function}$$

A characteristic curve is as followed.

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[3]: plot(arctan(x), (x, -10, 10), title='Plot of y = arctan(x)', legend_label='y = arctan(x)')
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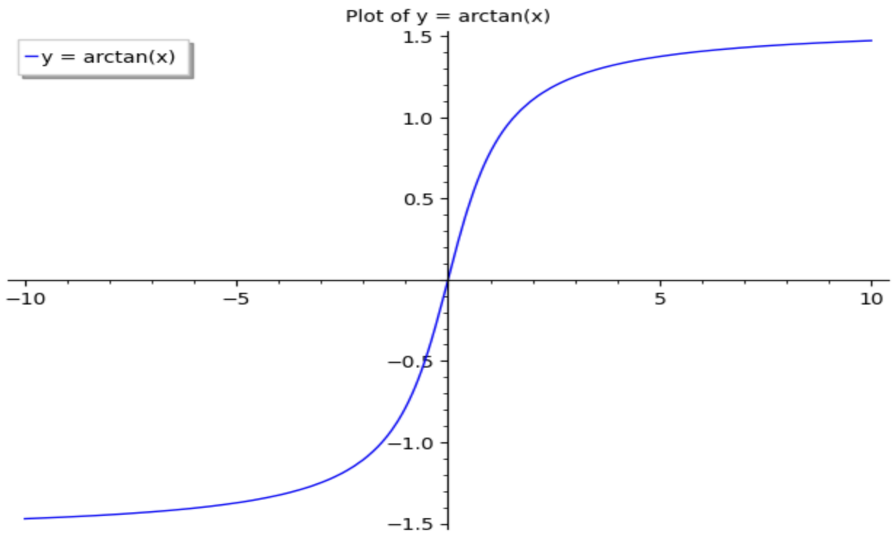
[3]:



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Theorem 3.1.3.

$$\text{Solve } au_x + bu_y + cu = 0 \tag{3.2}$$

Proof. Fix

$$\begin{cases} x' \triangleq ax + by \\ y' \triangleq bx - ay \end{cases}$$

This map is clearly a diffeomorphism. Compute

$$\begin{cases} u_x = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = au_{x'} + bu_{y'} \\ u_y = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} = bu_{x'} - au_{y'} \end{cases}$$

Plugging it back into the PDE in Equation 3.2, we have

$$cu + (a^2 + b^2)u_{x'} = 0 \tag{3.3}$$

If $c = a^2 + b^2 = 0$, then all smooth functions are solution. If $a^2 + b^2 = 0$ but $c \neq 0$, then clearly the only solution is $u = \tilde{0}$. If $a^2 + b^2 \neq 0$ but $c = 0$, then $u_{x'} = \tilde{0}$, which implies $u = f(y')$ where $y' = bx - ay$ and f can be arbitrary smooth function.

Now, suppose $a^2 + b^2 \neq 0 \neq c$, note that the PDE in Equation 3.3 is just an ODE of the form

$$y + \frac{a^2 + b^2}{c} y' = 0$$

The general solution to this ODE is

$$y = Ce^{\frac{-ct}{a^2+b^2}}$$

In other words, the general solution of the PDE in Equation 3.3 is

$$u = Ce^{\frac{-cx'}{a^2+b^2}} = Ce^{\frac{-c(ax+by)}{a^2+b^2}}$$

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