Chapter 1

General Analysis HW

1.1 Brunn-Minkowski Inequality

Abstract

This HW assignment require us to prove the Brunn-Minkowski Inequality. Note that in this HW, we use bold face $\mathbf{x}$ to denote $(x_1, \dots, x_d)$ element of $\mathbb{R}^d$. Also, throughout this HW, we shall suppose $|A| > 0$ and $|A|, |B| < \infty$; otherwise, the proof would be trivial.

We first introduce some notation. Given two sets $A, B \subseteq \mathbb{R}^d$ and a point $\mathbf{x} \in \mathbb{R}^d$, we write

$$A + \mathbf{x} \triangleq \{\mathbf{a} + \mathbf{x} \in \mathbb{R}^d : \mathbf{a} \in A\}$$

and write

$$A + B \triangleq \{\mathbf{a} + \mathbf{b} \in \mathbb{R}^d : \mathbf{a} \in A \text{ and } \mathbf{b} \in B\}$$

Note that elementary set theory tell us

$$(A + \mathbf{x}) + (B + \mathbf{y}) = (A + B) + (\mathbf{x} + \mathbf{y}) \tag{1.1}$$

Theorem 1.1.1. (Brunn-Minkowski Inequality for rectangles) Suppose A, B are two **rectangles**, i.e., A is of the form $\prod_{j=1}^{d} [x_j, y_j]$, and so is B. We have

$$|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}} \leq |A + B|^{\frac{1}{d}}$$

Proof. Because Lebesgue measure is translation invariant, by Equation 1.1, we can WLOG suppose

$$A = \prod_{j=1}^{d} [0, a_j] \text{ and } B = \prod_{j=1}^{d} [0, b_j]$$

It is clear that

$$A + B = \prod_{j=1}^{d} [0, a_j + b_j]$$

By direct computation, we know that

$$|A + B| = \prod_{j=1}^{d} (a_j + b_j) \text{ and } |A| = \prod_{j=1}^{d} a_j \text{ and } |B| = \prod_{j=1}^{d} b_j$$

Then by AM-GM inequality, we have

$$\frac{|A|^{\frac{1}{d}}}{|A + B|^{\frac{1}{d}}} = \Big(\prod_{j=1}^{d} \frac{a_j}{a_j + b_j}\Big)^{\frac{1}{d}} \leq \frac{1}{d} \sum_{j=1}^{d} \frac{a_j}{a_j + b_j}$$

Similarly, by AM-GM inequality, we have

$$\frac{|B|^{\frac{1}{d}}}{|A + B|^{\frac{1}{d}}} = \Big(\prod_{j=1}^{d} \frac{b_j}{a_j + b_j}\Big)^{\frac{1}{d}} \leq \frac{1}{d} \sum_{j=1}^{d} \frac{b_j}{a_j + b_j}$$

Adding two inequalities together, we now have

$$\frac{|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}}{|A + B|^{\frac{1}{d}}} \leq \frac{1}{d} \sum_{j=1}^{d} \frac{a_j + b_j}{a_j + b_j} = 1$$

The result then follows from multiplying both side with $|A + B|^{\frac{1}{d}}$. ∎

Theorem 1.1.2. (Brunn-Minkowski Inequality for finite union of non-overlapping of rectangles) Suppose A is a union of a finite collection of non-overlapping rectangles, and the same holds for B. We have

$$|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}} \leq |A + B|^{\frac{1}{d}}$$

Proof. We prove by induction on k, the sum of the rectangles in A and B. The base case $k = 2$ have been proved by Theorem 1.1.1. Suppose the proposition hold true when $k \leq r$. Let $k = r + 1$. Because the rectangles in A are non-overlapping, by a translation and renaming axis if necessary, we can suppose the following proposition.

Proposition 1: Both $A^+ \triangleq A \cap \{\mathbf{x} \in \mathbb{R}^d : x_1 \geq 0\}$ and $A^- \triangleq A \cap \{\mathbf{x} \in \mathbb{R}^d : x_1 \leq 0\}$ are unions of a finite collection of non-overlapping rectangles, with each collection containing at least one fewer rectangle than A.

Proposition 1 holds because, if we write $A = A_1 \cup \cdots \cup A_m$, where $A_1, \ldots, A_m$ are non-overlapping rectangles, then by translation and remaining axis, we can suppose that A_1, A_2 lie in distinct closed subspace, either $\{\mathbf{x} \in \mathbb{R}^d : x_1 \geq 0\}$ or $\{\mathbf{x} \in \mathbb{R}^d : x_1 \leq 0\}$, while for all $n \geq 3$, $A_n \cap \{\mathbf{x} \in \mathbb{R}^d : x_1 \geq 0\}$ is either empty or also a rectangle.

Note that $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(t) \triangleq \left| \Big(B + (t, 0, \ldots, 0) \Big) \cap \{\mathbf{x} \in \mathbb{R}^d : x_1 \geq 0\} \right|$$

is clearly an increasing continuous function such that

$$h(-M) = 0 \text{ and } h(M) = |B| \text{ for some } M > 0$$

Then by IVT, we can translate B to let B satisfy

$$\frac{|B^+|}{|B|} = \frac{|A^+|}{|A|} \text{ where } B^+ \triangleq B \cap \{\mathbf{x} \in \mathbb{R}^d : x_1 \geq 0\} \tag{1.2}$$

Define $B^- \triangleq B \cap \{\mathbf{x} \in \mathbb{R}^d : x_1 \leq 0\}$. With reasoning similar to that of Proposition 1, we know that B^+ and B^- are both unions of collections of non-overlapping rectangles, with each collection consisting of no more rectangles than B. Therefore, by Proposition 1, we can deduce that the sum of the number of rectangles in A^+ and B^+ is at least one fewer than $r+1$, and the same holds for the sum of the number of rectangles in A^- and B^-. Then, because the proposition holds true for $k \leq r$, we now have

$$\left|A^+\right|^{\frac{1}{d}} + \left|B^+\right|^{\frac{1}{d}} \leq \left|A^+ + B^+\right|^{\frac{1}{d}} \text{ and } \left|A^-\right|^{\frac{1}{d}} + \left|B^-\right|^{\frac{1}{d}} \leq \left|A^- + B^-\right|^{\frac{1}{d}}$$

Note that for each $\mathbf{x}$ in the interior of $A^+ + B^+$, we must have $x_1 > 0$, and for each $\mathbf{y}$ in the interior of $A^- + B^-$, we must have $y_1 < 0$. This implies that $(A^+ + B^+)$ and $(A^- + B^-)$ are non-overlapping. Now, because

$$A + B = (A^+ + B^+) \cup (A^- + B^-)$$

if we define $\rho \triangleq \frac{|A^+|}{|A|}$, from Equation 1.2 we can finally deduce

$$\begin{aligned}
|A+B| &= \left|A^+ + B^+\right| + \left|A^- + B^-\right| \\
&\geq \left(\left|A^+\right|^{\frac{1}{d}} + \left|B^+\right|^{\frac{1}{d}} \right)^d + \left(\left|A^-\right|^{\frac{1}{d}} + \left|B^-\right|^{\frac{1}{d}} \right)^d \\
\left(\because \frac{|A^-|}{|A|} = \frac{|B^-|}{|B|} = 1-\rho\right) \quad &= \left((\rho\,|A|)^{\frac{1}{d}} + (\rho\,|B|)^{\frac{1}{d}} \right)^d + \left(((1-\rho)\,|A|)^{\frac{1}{d}} + ((1-\rho)\,|B|)^{\frac{1}{d}} \right)^d \\
&= (|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}})^d
\end{aligned}$$

which give us the desired inequality

$$|A+B|^{\frac{1}{d}} \geq |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}$$

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Theorem 1.1.3. (Brunn-Minkowski Inequality for bounded open set) Suppose A, B are both bounded open subset of $\mathbb{R}^d$. We have

$$|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}} \leq |A+B|^{\frac{1}{d}}$$

Proof. Note that $A+B$ is open since it is a union of the open sets:

$$A + B = \bigcup_{\mathbf{b} \in B} A + \mathbf{b}$$

It follows that $A+B$ is Lebesgue measurable, so it makes sense for us to write $|A+B|$. By a proposition taught in class, we can let

$$A = \bigcup_{n \in \mathbb{N}} K_{n,a} \text{ and } B = \bigcup_{n \in \mathbb{N}} K_{n,b}$$

where $(K_{n,a})$ are non-overlapping rectangles, and so are $(K_{n,b})$. It is clear that

$$\left(\bigcup_{n=1}^{N} K_{n,a} \right) + \left(\bigcup_{n=1}^{N} K_{n,b} \right) \nearrow A + B \text{ as } N \to \infty$$

This together with Theorem 1.1.2 give us the desired inequality

$$\begin{aligned}
|A+B|^{\frac{1}{d}} &= \lim_{N\to\infty} \left| \left(\bigcup_{n=1}^{N} K_{n,a} \right) + \left(\bigcup_{n=1}^{N} K_{n,b} \right) \right|^{\frac{1}{d}} \\
&\geq \lim_{N\to\infty} \left| \bigcup_{n=1}^{N} K_{n,a} \right|^{\frac{1}{d}} + \left| \bigcup_{n=1}^{N} K_{n,b} \right|^{\frac{1}{d}} = |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}
\end{aligned}$$

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Theorem 1.1.4. (Brunn-Minkowski Inequality for compact set) Suppose A,B are both compact subset of $\mathbb{R}^d$. We have

$$|A|^{\frac{1}{d}}+|B|^{\frac{1}{d}}\leq |A+B|^{\frac{1}{d}}$$

Proof. For each $\epsilon>0$, define

$$A_\epsilon \triangleq \{\mathbf{x}\in\mathbb{R}^d : d(\mathbf{x},A)<\epsilon\} \text{ and } B_\epsilon \triangleq \{\mathbf{x}\in\mathbb{R}^d : d(\mathbf{x},B)<\epsilon\}$$

To see A_ϵ is open, observe that if $\mathbf{x}\in A_\epsilon$, then for all $\mathbf{y}$ in the open ball $\{\mathbf{y}:d(\mathbf{x},\mathbf{y})<\frac{\epsilon-d(\mathbf{x},A)}{2}\}$ centering $\mathbf{x}$, we can pick some $\mathbf{z}\in A$ satisfying $d(\mathbf{x},\mathbf{z})<d(\mathbf{x},A)+\frac{\epsilon}{2}$ to have

$$\begin{aligned}
d(\mathbf{y},A)&\leq d(\mathbf{y},\mathbf{z})\\
&\leq d(\mathbf{x},\mathbf{z})+d(\mathbf{x},\mathbf{y})\\
&\leq d(\mathbf{x},\mathbf{z})+\frac{\epsilon-d(\mathbf{x},A)}{2}\\
&\leq d(\mathbf{x},\mathbf{z})-d(\mathbf{x},A)+\frac{\epsilon}{2}<\epsilon \text{ implying } \mathbf{y}\in A_\epsilon
\end{aligned}$$

Similar argument shows that B_ϵ are open. To see $A_\epsilon \searrow A$, note that for all $\mathbf{x}\notin A$, because $d(\mathbf{x},\mathbf{z})$ is a function continuous in the variable $\mathbf{z}$ and A is compact, by EVT we have

$$d(\mathbf{x},A)=d(\mathbf{x},\mathbf{z})>0 \text{ for some } \mathbf{z}\in A$$

where the inequality holds because $\mathbf{z}\in A \implies \mathbf{x}\neq\mathbf{z}$. Similar argument shows that $B_\epsilon\searrow B$. We now prove

$$A+B=\lim_{\epsilon\to 0}A_\epsilon+B_\epsilon \tag{1.3}$$

It is clear that

$$A+B\subseteq\bigcap_{\epsilon>0}A_\epsilon+B_\epsilon \tag{1.4}$$

Fix an arbitrary $\mathbf{z}\in\bigcap_{\epsilon>0}A_\epsilon+B_\epsilon$. For all $n\in\mathbb{N}$, by definition there exists $\mathbf{a}_n\in A_{\frac{1}{n}}$ and $\mathbf{b}_n\in B_{\frac{1}{n}}$ such that $\mathbf{z}=\mathbf{a}_n+\mathbf{b}_n$. By the Bolzano-Weierstrass Theorem, there exists convergent subsequence $\mathbf{a}_{n_k}$. Applying Bolzano-Weierstrass Theorem again, we find that there exists convergent subsequence $\mathbf{b}_{n_{k_j}}$. Clearly, $\mathbf{a}_{n_{k_j}}$ also converge. For brevity, we denote these subsequences simply by $\mathbf{a}_{n_k}$ and $\mathbf{b}_{n_k}$, and we denote their limit by

$$\mathbf{a}=\lim_{k\to\infty}\mathbf{a}_{n_k} \text{ and } \mathbf{b}=\lim_{k\to\infty}\mathbf{b}_{n_k}$$

We now shows that

$$\mathbf{a}\in A$$

Assume $\mathbf{a} \notin A$ for a contradiction. By EVT, $d(\mathbf{a}, A) = d(\mathbf{a}, \mathbf{a}')$ for some $\mathbf{a}' \in A$. Note that $d(\mathbf{a}, \mathbf{a}') > 0$ because $\mathbf{a} \notin A \implies \mathbf{a} \neq \mathbf{a}'$. We have shown $d(\mathbf{a}, A) = d(\mathbf{a}, \mathbf{a}') > 0$.

Let m be large enough so that $\frac{1}{n_m} < \frac{d(\mathbf{a},A)}{2}$. Since $d(\mathbf{a}_{n_m}, A) < \frac{1}{n_m} < \frac{d(\mathbf{a},A)}{2}$, we can select $\mathbf{a}'' \in A$ such that $d(\mathbf{a}_{n_m}, \mathbf{a}'') < \frac{d(\mathbf{a},A)}{2}$. This give us

$$d(\mathbf{a}, A) \leq d(\mathbf{a}, \mathbf{a}'') \leq d(\mathbf{a}, \mathbf{a}_{n_m}) + d(\mathbf{a}_{n_m}, \mathbf{a}'') < \frac{d(\mathbf{a}, A)}{2} + \frac{d(\mathbf{a}, A)}{2}$$

This results in $d(\mathbf{a}, A) < d(\mathbf{a}, A)$, a contradiction. We have proved $\mathbf{a} \in A$. Similar arguments shows that $\mathbf{b} \in B$.

Now, since $\mathbf{z} = \mathbf{a}_{n_k} + \mathbf{b}_{n_k}$ for all k, we see

$$\mathbf{z} = \lim_{k\to\infty} \mathbf{a}_{n_k} + \mathbf{b}_{n_k} = \lim_{k\to\infty} \mathbf{a}_{n_k} + \lim_{k\to\infty} \mathbf{b}_{n_k} = \mathbf{a} + \mathbf{b} \in A + B$$

Because $\mathbf{z}$ is arbitrarily selected from $\bigcap_{\epsilon>0} A_\epsilon + B_\epsilon$. We have in fact proved

$$\bigcap_{\epsilon>0} A_\epsilon + B_\epsilon \subseteq A + B$$

which together with Equation 1.4 implies Equation 1.3. With Equation 1.3 established, we can now apply Theorem 1.1.3 to have the desired inequality

$$\begin{aligned}
|A + B|^{\frac{1}{d}} &= \left(\lim_{\epsilon\to 0} |A_\epsilon + B_\epsilon|\right)^{\frac{1}{d}} \\
&= \lim_{\epsilon\to 0} |A_\epsilon + B_\epsilon|^{\frac{1}{d}} \\
&\geq \lim_{\epsilon\to 0} |A_\epsilon|^{\frac{1}{d}} + |B_\epsilon|^{\frac{1}{d}} = |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}
\end{aligned}$$

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Before we proceed to develop the remaining Brunn-Minkowski Inequality, we first have to prove that Lebesgue measure is **inner regular**.

Theorem 1.1.5. (Lebesgue measure is inner regular) If $A \subseteq \mathbb{R}^d$ is Lebesgue measurable, then

$$|A| = \sup\{|K| : K \text{ is some compact subset of } A\}$$

Proof. Because A is measurable, we know $A \cap \overline{B_n(\mathbf{0})}$ is measurable for all $n \in \mathbb{N}$. Therefore, for all $n \in \mathbb{N}$, $(A \cap \overline{B_n(\mathbf{0})})^c$ is measurable. Then by definition, there exists open O_n

containing $(A \cap \overline{B_n(\mathbf{0})})^c$, such that $|O_n \setminus (A \cap B_n(\mathbf{0}))^c| < \frac{1}{n}$. Now, for each $n \in \mathbb{N}$, define closed set $K_n \triangleq O_n^c$. We then have

$$K_n \subseteq A \cap \overline{B_n(\mathbf{0})}$$

and

$$(A \cap \overline{B_n(\mathbf{0})}) \setminus K_n = A \cap \overline{B_n(\mathbf{0})} \cap O_n = O_n \setminus (A \cap B_n(\mathbf{0}))^c$$

This then give us

$$\left|(A \cap \overline{B_n(\mathbf{0})}) \setminus K_n\right| = |O_n \setminus (A \cap B_n(\mathbf{0}))^c| < \frac{1}{n}$$

Note that because $K_n \subseteq B_n(\mathbf{0})$ is bounded and closed, by Hiene-Borel, we know K_n is compact. Lastly, to close out the proof, we are required to show $|K_n| \to |A|$ as $n \to \infty$. Note that $\left|A \cap \overline{B_n(\mathbf{0})}\right| \nearrow |A|$ as $n \to \infty$ because $A \cap \overline{B_n(\mathbf{0})} \nearrow A$ as $n \to \infty$. Then because $\left|A \cap \overline{B_n(\mathbf{0})}\right| \geq |K_n| \geq \left|A \cap \overline{B_n(\mathbf{0})}\right| - \frac{1}{n}$, we see that $|K_n| \to |A|$ by squeeze Theorem. ∎

Theorem 1.1.6. (Brunn-Minkowski Inequality for measurable set) Suppose A, B are measurable subset of $\mathbb{R}^d$ and $A + B$ is also measurable. We have

$$|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}} \leq |A + B|^{\frac{1}{d}}$$

Proof. Because Lebesgue measure is inner regular and A, B are of finite measure, for each $n \in \mathbb{N}$, we can let A_n, B_n each be compact subset of A, B such that $|A| - |A_n| < \frac{1}{n}$ and $|B| - |B_n| < \frac{1}{n}$. It then follows from Theorem 1.1.4 that

$$|A + B|^{\frac{1}{d}} \geq \lim_{n \to \infty} |A_n + B_n|^{\frac{1}{d}} \geq \lim_{n \to \infty} |A_n|^{\frac{1}{d}} + |B_n|^{\frac{1}{d}} = |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}$$

∎