Friedberg Ch6.4

Date: Mar 28 Made by Eric

In this note, V always stand for a vector space over $\mathbb{F}$, V^- stands for a finite dimensional vector space over $\mathbb{F}$, and T is always a linear operator on V^-

Definitions

Definition 1. *Let $A \in M_{n\times n}(\mathbb{F})$*

T *is **self-adjoint** if* $T^* = T$
A *is **self-adjoint** if* $A^* = A$

Definition 2. *Let $A \in M_{n\times n}(\mathbb{F})$*

T *is **normal** if* $T \circ T^*$ *is self-adjoint, that is* $T \circ T^* = T^* \circ T$
A *is **normal** if* AA^* *is self-adjoint,that is* $AA^* = A^*A$

Theorem 1. *Let T be normal*

(i) $\|T(x)\| = \|T^*(x)\|$
(ii) $\forall c \in \mathbb{F}, T - cI_V$ *is normal*
(iii) $T(x) = \lambda x \implies T^*(x) = \overline{\lambda}(x)$
(iv) *If there exists x_1, x_2 two eigenvectors of T corresponding to distinct eigenvalues λ_1, λ_2, then $x_1 \perp x_2$*

Proof. **(i)**
$\|T(x)\|^2 = \langle T(x), T(x)\rangle = \langle T^*(T(x)), x\rangle = \langle T(T^*(x)), x\rangle = \langle T^*(x), T^*(x)\rangle = \|T^*(x)\|^2$

(ii)

$(T - cI)[(T^* - \overline{c}I)(v)]$

$= (T - cI)[T^*(v)] - \overline{c}(T - cI)(v)$

$= T(T^*(v)) - cT^*(v) - \overline{c}T(v) + \overline{c}cv$

$= T^*(T(v)) - cT^*(v) - \overline{c}T(v) + \overline{c}cv$

$= (T^* - \overline{c}I)(T(v)) - c(T^* - \overline{c}I)(v)$

$= (T^* - \overline{c}I)(T(v) - cv)$

$= (T^* - \overline{c}I)[(T - cI)(v)]$

(iii)

Let $U = T - \lambda I$

$\|T^*(x) - \overline{\lambda}x\| = \|(T-\lambda I)^*(x)\| = \|U^*(x)\| = \|U(x)\| = \|(T-\lambda I)(x)\| = \|0\| = 0$

So $T^*(x) - \overline{\lambda}x = 0$

(iv)

$\lambda_1\langle x_1, x_2\rangle = \langle \lambda_1 x_1, x_2\rangle = \langle T(x_1), x_2\rangle = \langle x_1, T^*(x_2)\rangle = \langle x_1, \overline{\lambda_2} x_2\rangle = \lambda_2\langle x_1, x_2\rangle$

$(\lambda_2 - \lambda_1)\langle x_1, x_2\rangle = 0$ ∎

Theorems

Theorem 2. *Let V^- be over $\mathbb{R}$ or $\mathbb{C}$, and let the characteristic polynomial f_T of T splits*

There exists an orthonormal basis β for V^-, such that $[T]_\beta$ is an upper triangular matrix

Proof. We prove by induction

Base Step: This is true when $dim(V^-) = 1$

Every basis contain only one vector, which can be made orthonormal by normalization, and apparently, β, the basis normalized, satisfy that $[T]_\beta$ is an upper triangular matrix

Induction Step: This is true when $dim(V^-) = n \longrightarrow$ This is true when $dim(V^-) = n + 1$

Let z' be an eigenvector of T corresponding to λ

We know $(T - \lambda I)(z') = 0$

So $\forall v \in V^-, 0 = \langle (T-\lambda I)(z'), v\rangle = \langle z', (T-\lambda I)^*(v)\rangle = \langle z', (T^* - \overline{\lambda} I)(v)\rangle$

So $z' \perp R(T^* - \overline{\lambda} I)$

Then $rank(T^* - \overline{\lambda} I) < dim(V)$, which tell us that $N(T^* - \overline{\lambda} I)$ is non-trivial

So there exists eigenvector z of T^* corresponding to $\overline{\lambda}$

Let $W = \{z\}^\perp$

We now prove W is T-invariant

$\forall w \in W, \langle T(w), z \rangle = \langle w, T^*(z) \rangle = \langle w, 0 \rangle = 0$ (done)

Let f_{T_W} be the characteristic polynomial of T_W

f_{T_W} divides f_T tell us that f_{T_W} also split

Obviously, $dim(W) = n$

By the premise, we have an orthonormal basis β' of W, such that $[T_W]_{\beta'}$ is an upper triangular matrix

Normalize z and we see $\beta' \cup \{z\}$ is the desired β ■

Theorem 3. *Let* $\mathbb{F} = \mathbb{C}$

T is normal if and only if there exists an orthonormal basis β of V^- consisting of eigenvectors

Proof. $(\longleftarrow)$

Notice $[T]_\beta$ and $[T^*]_\beta = ([T]_\beta)^*$is diagonal

So $[TT^*]_\beta = [T]_\beta[T^*]_\beta = [T^*]_\beta[T]_\beta = [T^*T]_\beta$

This give us $TT^* = T^*T$

$(\longrightarrow)$

Because $\mathbb{F} = \mathbb{C}$, the characteristic polynomial splits

Let β be an orthonormal basis of V^-, such that $[T]_\beta$ is an upper triangular matrix

Let $A = [T]_\beta$

Let $n = dim(V^-)$

We now prove β consists of eigenvectors

$T(v_1) = A_{1,1}v_1$

Because $T(v_2) = A_{1,2}v_1 + A_{2,2}v_2$, so $A_{1,2} = \langle T(v_2), v_1 \rangle = \langle v_2, T^*(v_1) \rangle = \langle v_2, \overline{\lambda_1}v_1 \rangle = 0$

Then $T(v_2) = A_{2,2}v_2$

Because $T(v_3) = A_{1,3}v_1 + A_{2,3}v_2 + A_{3,3}v_3$, so $A_{1,3} = \langle T(v_3), v_1 \rangle = \langle v_3, A_{1,1}v_1 \rangle = 0$ ..... (done)

■

Theorem 4. *Let T be a linear operator on V^- over $\mathbb{R}$ and $dim(V^-) = n$*

T is self-adjoint if and only if there exists an orthonormal basis β of V consisting of eigenvectors of T

Proof. $(\longleftarrow)$

$[T]_\beta$ is diagonal

and we know $\forall 1 \le j \le n, ([T]_\beta)_{j,j} = \lambda_j$, where λ_j is the eigenvalues corresponding to v_j

T is self-adjoint give us $T = T^*$, so $T^* \circ T = T \circ T = T \circ T^*$

So we know $T^*(v_j) = \overline{\lambda_j} v_j$

This give us $([T^*]_\beta)_{j,j} = \overline{\lambda_j} = \lambda_j$

Notice $[T^*]_\beta = ([T]_\beta)^*$ is also diagonal

So $[T^*]_\beta = [T]_\beta$

Then $T^* = T$

$(\longrightarrow)$

We now prove the characteristic polynomial f of T splits

Arbitrarily pick an orthonormal basis α of V^-, and let $A = [T]_\alpha$

Let $T_A : \mathbb{C}^n \to \mathbb{C}^n$ be defined by $T_A(v) = Av$

Let f_{T_A} be the characteristic polynomial of T_A

T_A is self-adjoint, since $[T_A]_E = A$, and $A^* = ([T]_\alpha)^* = [T^*]_\alpha = [T]_\alpha = A$

f_{T_A} split since it is over $\mathbb{C}$

Let v be a eigenvectors of T_A corresponding to λ

$\lambda v = T_A(v) = (T_A)^*(v) = \overline{\lambda} v$

So $\lambda = \overline{\lambda}$, then $\lambda \in \mathbb{R}$

Notice $T_A = L_A$

So T_A have the same characteristic polynomial with A

And A have the same characteristic polynomial with T, since $A = [T]_\alpha$

So T have the characteristic polynomial splits over $\mathbb{R}$ (done)

We then pick an orthonormal basis β for V^{-}, such that $[T]_\beta$ is upper triangular

Let $A = [T]_\beta$

We now prove A is diagonal

$A^* = ([T]_\beta)^* = [T^*]_\beta = [T]_\beta = A$

Because A is upper triangular, we know $A^* = A \implies A$ is diagonal (done)

■

Summary

All of following properties exist only in finite dimensional space, since its proof require matrix representation

(A)

Over $\mathbb{R}$ or $\mathbb{C}$

If the characteristic polynomial of T splits, than T can be expressed by an upper triangular matrix with an orthonormal basis.

(B)

Over $\mathbb{C}$

Normal is equivalent to orthonormally diagonalizable

(C)

Over $\mathbb{R}$

Normal and characteristic polynomial splits implies orthonormally diagonalizable

Self-adjoint is equivalent to orthonormally diagonalizable by any orthonormal basis

(D)

Let T be normal

(i) The adjoint of T and T transform a vector to two vector of same length

(ii) $\forall c \in \mathbb{F}, T - cI_V$ is normal

(iii) The adjoint of T have the same eigenspace as as T, but the corresponding eigenvalues is conjugate

(iv) Every eigenspace of T is perpendicular to each other

Exercises

2.