Line bundle, divisors, linear system

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Vector bundle

Definition 1.1 Given a smooth manifold M with dimenssion n. We define the tangent bundle to be

$$TM = \bigcup_{p\in M} T_pM$$

Now given a map

$$\pi : TM \longrightarrow M,\ (p,v) \in \{p\} \times T_pM \mapsto p$$

This makes TM into a C^∞ *vector bundle* over M.

Definition 1.2 (1) Given any map $\pi : E \to M$ we call the preimage $\pi^{-1}(\{p\})$ of a point p the fiber at p, denoted by E_p.
(2) For any two maps $\pi : E \to M$ and $\pi^{'} : E^{'} \to M$, a map $\phi : E \to E^{'}$ is said to be *fiber preserving* if $\phi(E_p) \subset E^{'}_p$ for all $p \in M$.

Proposition 1.3 $\phi : E \to E^{'}$ is fiber preserving if and only if the diagram

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr] & & E^{'} \arrow[dl] \\ & M & \end{tikzcd}$$

commutes.

Proof. First, we assume ϕ is fiber preserving, we have to check $\pi^{'} \circ \phi = \pi$, note that for all $x \in E$ there exists one and only one $p \in M$ such that $x \in E_p$. Hence

$$\pi^{'}(\phi(x)) = p = \pi(x)$$

Conversely, let $x \in E_p$, then $\pi(x) = p = \pi^{'}(\phi(x))$, hence $\phi(x) \in E^{'}_p$. □

Definition 1.4 A smooth map $\pi : E \to M$ between two manifolds is said to be *locally trivial* of rank r if
(i) Each E_p is a vector space of dimension r.
(ii) For all $p \in M$ there is a open nbd U of p and a fiber preserving diffeomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^r$ such that for all $q \in U$ the map

$$\phi|_{E_q} : E_q \longrightarrow \{q\} \times \mathbb{R}^r$$

is an isomorphism as vector sapce. Such an open set U is called a trivializing open set for E, and ϕ is called a trivialization of E over U. The collection $\{(U, \phi)\}$, with $\{U\}$ an open cover of M, is called a local trivialization for E, and $\{U\}$ is called a trivializing open cover of M for E.
Definition 1.5 A smooth *vector bundle rank r* is a triple (E, M, π) consisting of two manifolds E and M and a surjective smooth map $\pi : E \to M$ of locally trivial of rank r. E is called the *total space* and M is called the *base space*. We say that E is a *vector bundle* over M.

We may compare this notion with the *vector field*. Recall : A vector field on $X = \mathbb{R}^n$ is a map $F : X \to \mathbb{R}^n$. This vector field is said to be C^k if F is C^k. Therefore we may say a *vector field* on X is a way to give a vector on each $p \in X$. Now we can say a vector bundle is a way to given a vector space on each $p \in M$.

Example: *Product bundle* Let $E = M \times \mathbb{R}^n$, $\pi : E \to M$ be the projection map. Then E is a vector bundle over M of rank r. By the condition of (1.4 (ii)) we see that every vector bundle with rank r is "locally" isomorphic to the product bundle $M \times \mathbb{R}^r$.

Definition 1.6 A *line bundle* is a vector bundle of rank 1.

We give a sheaf version of the *vector bundle*.

Definition 1.7 Let X be a scheme, a *vector bundle* of rank n over X is a scheme Y and a morphism $Y \to X$ with the data consisting of open cover U_i of X and for each i we have an isomorphism

$$\psi_i : f^{-1}(U_i) \longrightarrow \mathbb{A}^n_{U_i} = U_i \times_{\mathbb{Z}} \mathbb{A}^n = \mathrm{Spec}\mathcal{O}_X(U_i)[x_1, ..., x_n]$$

such that for all i, j and any open affine subset $V = \mathrm{Spec}\ A \subset U_i \cap U_j$ the automorphism $\psi = \psi_j \circ \psi_i^{-1}$ of $\mathbb{A}^n_V = \mathrm{Spec}A[x_1, ..., x_n]$ is given by a linear automorphism

$$\theta : A[x_1, ..., x_n] \longrightarrow A[x_1, ..., x_n]$$

i.e. $\theta(a) = a$ for all $a \in A$ and $\theta(x_i) = \sum a_{ij}x_j$ for some $a_{ij} \in A$.

Remark 1.8 There is a one-to-one correspondence between *vector bundle* rank n and $locally\ free\ sheaf$ of rank n.

Divisors

Weil divisors

Lemma 2.0 Let X be scheme, Y be an irreducible closed of codimension 1, then for each open affine $U = \mathrm{Spec}A$ with $Y \cap U \neq \emptyset$, $Y \cap U$ is irreducible of codimension 1.

Proof. Suppose not, let $\mathfrak{p}$ be the generic point of $Y \cap U$ and $\mathfrak{q}$ be another nontrivial prime ideal contained in $\mathfrak{p}$, let $\xi = \mathfrak{q}$, $\eta = \mathfrak{p}$ denote the points in the topological space X, then $\mathrm{cl}_U(\eta) \subset \mathrm{cl}_U(\xi)$ which mean take the closure of η, ξ in U. Thus $\mathrm{cl}_X(\eta) = Y \subset \mathrm{cl}_X(\xi)$, since Y is of codim=1, so $\mathrm{cl}_X(\xi) = X$ and therefore $\mathrm{cl}_U(\xi) = V(\mathfrak{q}) = U$ hence $\mathfrak{q} = 0$ which gives a contracdiction. $\square$

We assume all scheme are noetherian integral separated and regular in codimension 1.

Definition 2.1 (1) A prime divisor of X is a integra closed subscheme of codimension 1, and we let $PD(X)$ be the set of all prime divisor of X.
(2) A Weil divisor of X is an element of the free abelian group $\mathrm{Div}X = \mathbb{Z}^{\oplus PD(X)}$, we write the element $D \in \mathrm{Div}X$ as

$$D = \sum n_Y Y,\ n_Y \in \mathbb{Z}$$

D is said to be effective if $n_Y \geq 0$ for all prime divisor Y.

Remark 2.2 (1) For each open affine $U = \mathrm{Spec}A$ and prime divisor Y, if $Y \cap U \neq \emptyset$, then $Y \cap U$ is proper irreducible closed with codimension 1 in U, otherwise, the generic point of Y is correspondence to the zero ideal of A, hence it is equal to the generic point of X. This gives a contracdiction.
(2) $\mathcal{O}_\eta$ is a DVR (η is the generic point of Y). Since we have assume X is regular in codimension 1, so it is enough to show the local ring $\mathcal{O}_\eta$ has Krull dimension 1. Claim: $\mathrm{codim}(Y, X) = \dim\mathcal{O}_\eta$
May assume $X = \mathrm{Spec}A$ is affine. Then $Y = V(\mathfrak{p})$ with $\mathrm{ht}(\mathfrak{p}) = 1$ then $\mathcal{O}_\eta = A_\mathfrak{p}$ and $\dim A_\mathfrak{p} = 1$.
(3) Let $K = \mathcal{O}_\xi$ be the function field of X, where ξ is the generic point of X, and $\mathrm{Frac}(\mathcal{O}_\eta) = K$, so $\mathcal{O}_\eta$ is a discrete valuation ring of K, since X is separated, so its discrete valuation is uniquely determined, denoted by v_Y. Therefore $\mathcal{O}_\eta = \{f \in K \mid v_Y(f) \geq 0\}$
(4) We may define a discrete valuation on K, $\mathrm{ord} : K \to \mathbb{Z}$ defined by

$$\text{write } f = ct^n,\ t \text{ is the uniformnizer of } \mathcal{O}_\eta \text{ then } \mathrm{ord}(f) = n$$

Since the valuation v_Y is unique, hence $v_Y = \mathrm{ord}$.

Lemma 2.3 For each $f \in K^*$, there is only finitely many prime divisor Y such that $v_Y(f) \neq 0$.

Proof. Let U =Spec$A$ be an open affine subsect of X on which f is regular. i.e. $f \in A$. Let $Z = X \setminus U$, since X is noetherian so Z has only contains finitely many irreducible components of X and hence contains only finitely many primes divisors of X. Therefore it is enough to show $v_Y(f) \neq 0$ for finitely many prime divisor Y with $U \cap Y \neq \emptyset$. Since f is regular on U hence $v_Y(f) \geq 0$ for such prime divisor Y. Note that

$$v_Y(f) > 0 \Longleftrightarrow f \in (t) = \mathfrak{p}_Y = \eta \Longleftrightarrow (f) \subset \mathfrak{p}\ \forall \mathfrak{p} \in Y \cap U \Longleftrightarrow Y \cap U \subset V(f)$$

Since f is nonzero, and the unique generic point of U is the zero ideal hence $V(f)$ is proper, A is Noetherian so $V(f)$ contains only finitely irreducible closed subset of U with codimension 1. $\square$

Definition 2.4 (1) Let $f \in K^*$, we define the divisor of f to be

$$(f) = \sum v_Y(f) Y$$

This is well-defined by Lemma 2.3. Any divisor which is equal to the divisor of an elemnet of K^* is called principal divisor.
(2) Let $f \in K^*$ we say f is a zero along Y of order $v_Y(f)$ if $v_Y(f) > 0$, pole along Y with order $-v_Y(f)$ if $v_Y(f) < 0$.
(3) Let $D_1, D_2 \in$Div$X$, they are said to be linear equivalent if $D_1 - D_2$ is principal, we define the divisor class group by ClX =DivX/H where H is the subgroup of *principal divisor*. ($H \leq$Div$X$, $(f/g) = (f) - (g)$)

Remark 2.5 ClX is an invariant of schemes. In general, it is not easy to caculate.

Proposition 2.6 Let A be a Noetherian integral domain. Then A is an UFD iff X =Spec$A$ is normal and Cl$X = 0$ (We say a scheme is normal if its local ring $\mathcal{O}_x$ is integrally closed for all $x \in X$)

Proof. First, we note that (1) UFD are integrally closed hence their localization are integrally closed so X is normal. (2) A Noetherian domain is UFD iff every prime ideal of height 1 is principal.([Matsumura Thm 47 p.149]) Thus we just need to show the following statement: If A is a domain, then every prime ideal of height 1 is principal iff Cl$X = 0$.
Assume every prime ideal of height 1 is principal, consider a prime divisor $Y \subset X$, then Y associted with a prime ideal $\mathfrak{p}$ which is the generic point of Y, Y is codimension 1 hence ht$(\mathfrak{p}) = 1$, hence $\mathfrak{p} = (f)$ for some $f \in A$, hence it is a unifornizer of the DVR $A_\mathfrak{p}$ so $v_Y(f) = 1$. Let $\mathfrak{q}$ be another prime ideal of height 1, then it is principal, say $\mathfrak{q} = (g)$, if $f \in (g)$ then $\mathfrak{p} \subset \mathfrak{q}$, so $\mathfrak{q} = \mathfrak{p}$. Hence $v_Z(f) = 1$ iff $Z = Y$, $v_Z(f) = 0$ otherwise. Therefore $Y = v_Y(f)Y$. Thus every prime divisor Y is principal, hence Cl$X = 0$.
Conversely, we assume Cl$X = 0$, let $\mathfrak{p} \in X$ with ht$(\mathfrak{p}) = 1$, let $Y = V(\mathfrak{p})$ then there exists $f \in K^*$ such that $(f) = Y$. Claim :$f \in A$ and $\mathfrak{p} = (f)$
$Y = (f)$ so $v_Y(f) = 1 \Rightarrow f \in A_\mathfrak{p}$ and $(f) = \mathfrak{p}A_\mathfrak{p}$. Let $\mathfrak{q}$ be another prime ideal with height 1, $Z = V(\mathfrak{q})$ be a prime divisor of X, then $v_Z(f) = 0$ hence $f \in A_\mathfrak{q}$. Thus

$$f \in \textstyle\bigcap_{ht(\mathfrak{q})=1} A_\mathfrak{q} = A \text{ ([Matsumura] Thm38 p.132)}$$

Let $g \in \mathfrak{p} = (f)$ then $v_Y(g) \geq 1$ and $v_Z(g) \leq 0$ for all prime divisor $Z \neq Y$ ($\because g \in A$). Hence

$$v_Z(g/f) = v_Z(g) - v_Z(f) = v_Z(g) \geq 0 \text{ for all } Z \neq Y,\ v_Y(g/f) = v_Y(g) - 1 \geq 0$$

hence $g/f \in A$, and therefore $g \in fA = (f)$, hence $\mathfrak{p}$ is principal. $\square$

Example If $X = \mathbb{A}_k^n$ =Spec$k[x_1, ..., x_n]$, then Cl$X = 0$.

Cartier divisors

In this section, we want to generlize the notion "divisor" to arbitrary scheme. Before we define the *Cartier divisor*, we have to give a brief introduction to *total quotient ring*. Recall that we can define qoutient ring on an integral domain, now we want to define an analogous concept to arbitrary rings.

Definition 2.6.1 Let A be a ring, S be the set of non-zero divisors of A, then we define the *total quotient ring* of A to be $S^{-1}A$. Note that if A is an integral domain, then $S^{-1}A$ =Frac(A)

Proposition 2.6.2 Let A be a ring, K be the total quotient ring of A then
(a) $A \longrightarrow K$ is injective
(b) Every element in K is either zero divisor or an unit

Proof. [AtM] Ch3 Execise 9 □

Definition 2.7 Let X be a scheme, for each open set U, let $S(U)$ denote the set of elements of the ring $\mathcal{O}_X(U)$ which are not zero divisor in the local ring $\mathcal{O}_x$ for all $x \in U$. Then $U \mapsto S(U)^{-1}\Gamma(U, \mathcal{O}_X)$ form a presheaf of X, we define the *sheaf of total quotient rings* to be the associated sheaf of $U \mapsto S(U)^{-1}\Gamma(U, \mathcal{O}_X)$.

Remark 2.8 The sheaf of total quotient ring replace the notion of function field on integral scheme.

Definition 2.9 (1) A *Cartier divisor* of a scheme X is a global section of the quotient sheaf $\mathscr{K}^*/\mathcal{O}_X^*$, where $\mathscr{K}^*$ denotes the invertible elements in $\mathscr{K}$. Recall the definition of quotient sheaf, it is the sheaf associated to the presheaf $U \mapsto \mathscr{K}(U)^*/\mathcal{O}_X(U)^*$, hence every global section is locally comes from $\overline{f_i} \in \mathscr{K}(U_i)^*/\mathcal{O}_X(U_i)^*,\ f_i \in \mathscr{K}(U_i)^*,\ (f_i/f_j)|_{U_i \cap U_j} \in \mathcal{O}(U_i \cap U_j)^*\ \forall\, i,j$. Thus we may represent a *Cartier divisor* by $\{(f_i, U_i)\}$ where U_i is an open cover of X.
(2) A *Caetier divisor* is said to be principal if it is in the image of $\Gamma(X, \mathscr{K}^*) \to \Gamma(X, \mathscr{K}^*/\mathcal{O}_X^*)$. More precisely, it is in the image of the morphism

$$\Gamma(X, \mathscr{K}^*) \to \Gamma(X, \mathscr{K}^*)/\Gamma(X, \mathcal{O}_X^*) \to \Gamma(X, \mathscr{K}^*/\mathcal{O}_X^*)$$

Hence every principal Cartier divisor can be represent by a single element $f \in \Gamma(X, \mathscr{K}^*)$.
(3) We let CaClX denote the group of *Cartier divisor* quotient the subgroup of principal *Cartier divisor*.

It is natural to ask when does the *Cartier divisor* equal to the *Weil divisor* whenever *Weil divisor* is defined. Here we give a proposition.

Proposition 2.10 Let X be an integral, separated noetherian scheme with every local ring is UFD (in which case we say X is locally factorial). Then Div$X \cong$ CaDiv$X$, and furthermore, the principal Weil divisors are correspondence to principle Cartier divisor under this isomorphism.

Proof. Since UFD are integrally closed, so every local ring of X with dimension one is DVR. Thus we can talk about Weil divisors. Also note that $\mathscr{K}(U) = \mathcal{O}_\xi$ for all open affine U, by the glueing lemma of sheaves we get $\mathscr{K}$ is the costant sheaf of $K = \mathcal{O}_\xi$ which is the function field of X.
Now given a Cartier divisor f, represented by $\{(U_i, f_i)\},\ f_i \in K^*$, we define the associtaed Weil divisor as follows. For each prime divisor Y, we let the coefficient of Y to be $v_Y(f_i)$, where i is the index such that $U_i \cap Y \neq \emptyset$. Note that $v_Y(f_i/f_j) = 0$ whenever $v_Y(f_j),\ v_Y(f_j)$ is defined, since f_i/f_j is invertible in $\mathcal{O}(U_i \cap U_j)$. Thus this it is well-defined. Since X is Noetherian hence quasi-compact so such open cover U_i is finite, then by (2.3) the sum $D = \sum v_Y(f_i)Y$ is finite.
Conversely, given a Weil divisor $D = \sum n_Y Y$. For each $x \in X$, we define the divisor D_x on Spec$\mathcal{O}_x$ induced by D. For each prime divisor Y which containing x, let U =Spec$A$ be an affine open nbd of x. Then $\eta \in U$, hence $x \in V(\eta) = U \cap Y$. Therefore $\eta \subset x$ (as prime ideal of A). Thus $\mathcal{O}_x\eta = A_x\eta$ is a prime ideal of $\mathcal{O}_x = A_x$. Since η has height one in U, so $V(\eta)$ is an irreducible closed subset of codimension 1 in U, therefore $\mathfrak{p}_Y = A_x\eta$ also has height 1. Then we define $D_x = \sum n_Y V(\mathfrak{p}_Y)$, where Y run through all prime divisor

containing x. By (2.6), D_x is principal dor each x. Say $D_x = (f_x)$ for some $f_x \in K^*$. Since $D = \sum n_Y Y$ is finite sum, so there is only finitely many prime divisor Y with $x \notin Y$ such that the coefficient of $(f_x) \in$Div$X$ is differ to D. Thus for each x, there is an open nbd U_x of x such that

$$D|_{U_x} = \sum n_Y (Y \cap U_x) = (f_x)|_{U_x} = \sum v_Y(f_x)(Y \cap U_x)$$

Also note that $(f_x)|_{U_x \cap U_y} = (f_y)|_{U_x \cap U_y}$ hence $v_Y(f_x/f_y) = 0$. So f_x/f_y lies in every open affine subset of $U_x \cap U_y$, this implies $f_x/f_y \in \mathcal{O}(U_x \cap U_y)^*$. Hence $\{f_x, U_x)\}$ represent a cartier divisor of X. $\square$

Picard groups

Definition 3.1 An $\mathcal{O}_X$-module $\mathscr{F}$ on a ringed space X is said to be invertible is there is an open cover U_i of X, such that $\mathscr{F}|_{U_i} \cong \mathcal{O}_X|_{U_i}$.

Lemma 3.2 (Glueing lemma) Let U_i be an open cover of X, and $\mathscr{F}_i$ be a sheaf on each U_i, and for each i, j there is an isomorphism of sheaves

$$\varphi_{ij} : \mathscr{F}_i|_{U_i \cap U_j} \longrightarrow \mathscr{F}_j|_{U_i \cap U_j}$$

with the following properties:
(1) φ_{ii} is identity
(2) For each i, j, k $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_i \cap U_j \cap U_k$
Then there is a unique sheaf (up to isomorphism) $\mathscr{F}$ on X with isomorphism

$$\psi_i : \mathscr{F}|_{U_i} \longrightarrow \mathscr{F}_i$$

such that for each i, j, $\psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$. We say $\mathscr{F}$ is obtained by glueing the sheaves $\mathscr{F}_i$ via φ_{ij}.

Remark 3.2.1 In general, the gleing lemma is not applicable on invertible sheaves. We assume $\mathscr{L}$ is an invertible sheaves with an open cover U_i and isomorphism such that

$$g_i : \mathscr{L}|_{U_i} \longrightarrow \mathcal{O}_X|_{U_i}$$

But in general, $g_j \circ g_i^{-1} : \mathcal{O}_{U_i \cap U_j} \to \mathcal{O}_{U_i \cap U_j}$ is not identity on $U_i \cap U_j$, so the sheaf we obtaied by the glueing data $\{(\mathcal{O}_{U_i}, g_i)\}$ (which is $\mathscr{L}$) is not isomorphic to $\mathcal{O}$ (which is obtained by the glueing data $\{(\mathcal{O}_{U_i}, id_{U_i})\}$)

Proposition 3.3 Let $\mathscr{L}$ and $\mathscr{M}$ be two invertible sheaves on a ringed space X. Then $\mathscr{L} \otimes \mathscr{M}$ is invertible and there is an invertible sheaves $\mathscr{L}^{-1}$ such that $\mathscr{L} \otimes \mathscr{L}^{-1} \cong \mathcal{O}_X$.

Proof. The first part is trvial, we prove the second part. Let $\mathscr{L}^{-1} = \check{\mathscr{L}} = \mathscr{H}om(\mathscr{L}, \mathcal{O}_X)$, it is invertible clearly. Note that for each open set U, we may view $\mathscr{H}om(\mathscr{L}, \mathcal{O}_X)(U)$ =Hom$(\mathscr{L}, \mathcal{O}_X)$ as a product $\prod_{V \subseteq U} \mathrm{Hom}_{\mathcal{O}_X(V)}(\mathscr{L}(V), \mathcal{O}_X(V))$. Also by the unuversal property of tensor product, we have

$$(\mathscr{L}^{-1} \otimes \mathscr{L})(U) \to \mathcal{O}_X(U),\ (f_V) \otimes s \mapsto f_U(s)$$

Now we may choose an open affine cover such that $\mathscr{L}|_{U_i} \cong \mathcal{O}_{U_i}$. So every element of $(f_V) \in \mathrm{Hom}(\mathscr{L}|_{U_i}, \mathcal{O}_{U_i})$ is uniquely determined by f_U. Thus we consider the following statement: Let M be a free A-module of rank 1, then $\check{M} \otimes M \cong A$ via the morphism we constructed above. Write $M = Ae$, then every element in $\check{M} \otimes M$ can be written as

$$\sum (f_x \otimes x) = \sum (f_x \otimes a_x e) = (\sum a_x f_x) \otimes e$$

Then $(\sum a_x f_x)(e) = 0$ implies it is a zero function, this prove the morphism is injective. For each $a \in A$, let $f \in \check{M}$ such that $f(e) = a$, then $f \otimes e \mapsto a$. This proved the morphism of sheaves we constructed is an isomorphism. $\square$

Definition 3.4 For any ringed space X, we define the *Picard group* of X to be

$$\mathrm{Pic}X = (\{\textit{Isomorphism class of invertible sheaves}\}, \otimes)$$

Note that Pic$X$ is an abelian group.

Remark 3.3 We also can say that Pic$X$ is the group consist of isomorphism class of *line bundle*.

Definition 3.5 For each Cartier divisor $D = \{(U_i, f_i)\}$ on a scheme X, we define the subsheaf $\mathscr{L}(D)$ of $\mathscr{K}$ associated by D to be the glueing sheaf of $U_i \mapsto \mathcal{O}_{U_i} f_i^{-1}$. Since $f_i/f_j \in \mathcal{O}(U_i \cap U_j)^*$, so $\mathcal{O}_{U_i \cap U_j}(f_i^{-1})|_{U_i \cap U_j} = \mathcal{O}_{U_i \cap U_j}(f_j^{-1})|_{U_i \cap U_j}$ i.e. It is gluable.

Proposition 3.6 Let X be a scheme, then
(1) For any Cartier divisor D, $\mathscr{L}(D)$ is an invertible sheaf on X, and there is an one-to-one correspondence between Cartier divisors and invertible subsheaf of $\mathscr{K}$
(2) $\mathscr{L}(D_1 - D_2) \cong \mathscr{L}(D_1) \otimes \mathscr{L}(D_2)^{-1}$
(3) $D_1 - D_2$ is principal iff $\mathscr{L}(D_1) \cong \mathscr{L}(D_2)$

Proof. (1) Let $\mathscr{L}$ be the invertible subsheaf of $\mathscr{K}^*$ then there is an open cover U_i of X such that $\mathscr{L}|_{U_i} = \mathcal{O}_{U_i} g_i$ for some $g_i \in \mathscr{K}(U_i)$, and note that such g_i is a non zero divisor of $\mathscr{K}(U_i)$ hence it is an unit, also $\mathcal{O}_{U_i \cap U_j}(g_i)|_{U_i \cap U_j} = \mathcal{O}_{U_i \cap U_j}(g_j)|_{U_i \cap U_j}$ hence $g_i/g_j \in \mathcal{O}(U_i \cap U_j)^*$. Thus we obtained a Cartier divisor representedby $\{(U_i, g_i^{-1})\}$.
(2) Suppose $D_1 = \{(U_i, f_i)\}$, $D_2 = \{(U_i, g_i)\}$ (We may assumethe index and open cover are the same since we may intersect two open cover and get a common refinement of these two open cover) Then $D_1 - D_2 = \{(U_i, f_i/g_i)\}$ and therefor $\mathscr{L}(D_1 - D_2)$ is locally generated by $f_i^{-1} g_i$ which is clearly isomorphic to $\mathscr{L}(D_1) \otimes \mathscr{L}(D_2)^{-1}$.

(3) It suffieces to show D is principal iff $\mathscr{L}(D) \cong \mathcal{O}_X$. Let D represented by $\{(X, f)\}$ then

$$\mathscr{L}(D) = \mathcal{O}_X f^{-1} \cong \mathcal{O}_X$$

Conversely, given an invertible subsheaf $\mathcal{O}_X g,\ g \in \mathscr{K}^*(X)$, then $\{(X, g)\}$ is principal. □

Corollary 3.7 There is an injective group homomorphism CaCl$X \longrightarrow$ Pic$X$.

Proposition 3.8 If X is integral, then the morphism we constructed in (3.5 (1)) is surjective.

Proof. It suffices to show every invertible subsheaf is isomorphic to an subsheaf of $\mathscr{K}$. In this case $\mathscr{K}$ is the constant sheaf of $K =$ function field. Let $\mathscr{L}$ be an invertible sheaf. Consider the sheaf $\mathscr{K} \otimes \mathscr{L}$ and an open cover U_i such that $\mathscr{L}|_{U_i} \cong \mathcal{O}_{U_i}$ then

$$\mathscr{K}|_{U_i} \cong (\mathscr{K} \otimes \mathscr{L})|_{U_i}$$

So it is a constant sheaf on U_i. Since X is irreducible hence every open set in U is connected then by glueing lemma we get $\mathscr{K} \otimes \mathscr{L}$ is a constant sheaf of K. Thus we may embed $\mathscr{L} \hookrightarrow \mathscr{K} \otimes \mathscr{L} \cong \mathscr{K}$ by the natural map. □

Corollary 3.9 If X is noetherian separated integral and locally factorial, then Cl$X \cong$ Pic$X$.

Definition 3.10 A Cartier divisor $\{(U_i, f_i)\}$ is said to be $effective$ if $f_i \in \Gamma(U_i, \mathcal{O}_{U_i})$ for all i. In this case we define the associated locally principal closed subscheme, Y to be the closed subscheme defined by the sheaf of ideal $\mathscr{I}$ locally generated by f_i.

Proposition 3.11 Let D be an effective Cartier divisor on X, then the sheaf of ideal $\mathscr{I}_Y$ is isomorphic to $\mathscr{L}(-D)$, where Y is the associated locally principal closed subscheme.

Proof. $\mathscr{L}(-D)$ is locally generated by $f_i \in \mathcal{O}(U_i)$. □

Linear system

Definition 4.1 Let $\mathscr{L}$ be an invertible sheaf on an integral scheme X, let $0 \neq s \in \Gamma(X, \mathscr{L})$. We define the effective Cartier divisor $D = (s)_0$, the *divisor of zeros* of s. Over any open set where $\mathscr{L}$ is trivial, i.e. $\mathscr{L}|_U \cong \mathcal{O}_U$, let $\varphi : \mathscr{L}|_U \to \mathcal{O}_U$ be such isomorphism, then $\{(U, \varphi(s))\}$ determined an effective Cartier divisor on X. Note that if there are two isomorphism $\varphi_1, \varphi_2 : \mathscr{L}|_U \to \mathcal{O}_U$ then $\varphi_1 = u\varphi_2$ for some $u \in \mathcal{O}_U^*$. So this is well-defined.

Remark 4.1.1 If $Ae \to A$ is an isomorphism then $e \mapsto u$ which is an unit in A.

Definition 4.1.2 (1) A variety over an algebraically closed field k is an integral separated scheme X finite type over k, or equivalently, X is integral separated, quasi-compact and for each open affine U =Spec$A$, A is a finitely generated k-algebra. In addition, a projective variety over k is a projective scheme with above condition.

Definition 4.2 A curve over k is a variety of dimension 1. A curve is said to be nonsingular if all local rings are regular.

Remark 4.2.1 Every "k" are assumed to be algebraically closed.

Propositiom 4.3 Let X be a projective variety over k. Let D_0 be a divisor on X and let $\mathscr{L} = \mathscr{L}(D_0)$ be a line bundle of X. Then:
(1) for each $s \in \Gamma(X, \mathscr{L}) \setminus \{0\}$, $(s)_0$ is effective and linear equivalent to D_o.
(2) every divisor who linear equivalent to D_0 is of the form $(s)_0$ for some $s \in \Gamma(X, \mathscr{L}) \setminus \{0\}$.
(3) for any two $s, t \in \Gamma(X, \mathscr{L})$ with $(s)_0 = (t)_0$ iff there is an element $a \in k^*$ such that $s = at$.

Proof. (1) Since X is integral, we may identify $\mathscr{L}$ into a subsheaf of $\mathscr{K}$, which is a constant sheaf of the function field K. Therefore s is correspondence to a rational function $f \in K^*$. Let $D_0 = \{(U_i, f_i)\}$, then $\mathscr{L}$ is locally generated by f_i^{-1}. Somwe get an isomorphism $\varphi_i : \mathscr{L}|_{U_i} \to \mathcal{O}|_{U_i}$ by multiplying f_i, and hence $D = (s)_0 = \{(U_i, ff_i)\}$. Thus $D - D_0 = (f) = \{(X, f)\}$.
(2) We represent $D, D_0, (f)$ locally by $\{h_i\}$, $\{g_i\}$, $\{f_i\}$ where f_i is the restriction of f on each open cover. Since D is effective hence $h_i \in \mathcal{O}$. By the relation $D = D_o + (f)$ we see that $f_i g_i = h_i \in \mathcal{O}$, and therefore $f_i \in g_i^{-1}\mathcal{O} = \mathcal{L}(D_0)$, so h_i is obtained by the isomorphism $\mathscr{L}|_{U_i} \to \mathcal{O}_{U_i},\ x \mapsto g_i x$. Thus D is comes from the global section of $\mathscr{L}$. (f_i are come from the global section f.)
(3) Let $(s)_0, (t)_0$ represented bt $\{f_i\}$, $\{g_i\}$ locally, $(s)_0 = (t)_0$ means $f_i/g_i \in \mathcal{O}^*$ since $0 = (s)_0 - (t)_0$ is principal hence we may represent the Cartier divisor $\{f_i/g_i\}$ by a single element $a \in \Gamma(X, \mathcal{O}^*)$. Thus $a(t)_0 = (s)_0$ which implies $at = s$. Since X is projective over k, so $\Gamma(X, \mathcal{O}) = k$. □

Definition 4.4 A complete linear system on a nonsingular projective variety is defined as a set of all effective dvisors which linear equivalent to some given divisor D_0, denoted by $|D_0|$.

Remark 4.4.1 By (4.3) we see that $|D_0|$ is 1-1 correpondence to $(\Gamma(X, \mathscr{L}(D_0)) \setminus \{0\})/k^*$. This gives $|D_0|$ a structure of the set of closed point of a projective space over k. ($\Gamma(X, \mathscr{L})$ is a finite dimensional vector space over k)

Definition 4.5 A linear system $\mathfrak{d}$ is a subset of a complete linear system $|D_0|$ which correspondence to a subvector space $V \subset \Gamma(X, \mathscr{L}(D_0))$. More precisely, $V = \{s \in \Gamma(X, \mathscr{L}) \setminus \{0\} \mid (s)_0 \in \mathfrak{d}\} \cup \{0\}$, hence $\mathfrak{d} = \{(s)_0 \mid s \in V \setminus \{0\}\}$. And we define dim$\mathfrak{d}$=dim$_k V - 1$.