Calculus HW3

Date: Mar 13 Made by Eric

1.

1.(a)

Proof. $PV = mRT$

$P = mrTV^{-1}$

$V = \frac{mR}{P}T$

$T = \frac{V}{mR}P$

$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = (-mrTV^{-2})(\frac{mR}{P})(\frac{V}{mR}) = \frac{-mrT}{PV} = -1$

■

1.(b)

Proof. $P = \frac{mR}{V}T$

$V = \frac{mR}{P}T$

$T\frac{\partial P}{\partial T}\frac{\partial V}{\partial T} = T(\frac{mR}{V})(\frac{mR}{P}) = mR\frac{mRT}{PV} = mR$ ■

2.

2.(a)

$f(x,t) = \tan^{-1}(x\sqrt{t})$

Proof. $f_x = \frac{1}{1+tx^2}\sqrt{t}$

$f_t = \frac{1}{1+tx^2}\frac{x}{2\sqrt{t}}$ ■

2.(b)

$f(x,y) = \int_y^x \cos(t^2)dt$

Proof. $f_x = \cos(x^2)$

$f = -\int_x^y \cos(t^2)dt$

$f_y = -\cos(y^2)$ ■

2.(c)

$f(\mathbf{x}) = \sqrt{x_1^2 + \cdots + x_n^2}$

Proof. Let $1 \leq i \leq n$

$f_i = \frac{x_i}{f(\mathbf{x})}$ ∎

2.(d)

$u(x, y) = f(\frac{x}{y})$

Proof. $u_x = f'(\frac{x}{y})\frac{1}{y}$

$u_y = f'(\frac{x}{y})\frac{-x}{y^2}$ ∎

3.

Proof. Notice $\forall i : 1 \leq i \leq n, \frac{\partial u}{\partial x_i} = a_i u$

Then $\forall i : 1 \leq i \leq n, \frac{\partial^2 u}{\partial x_i^2} = a_i^2 u$

$\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = \sum_{i=1}^n a_i^2 u = (\sum_{i=1}^n a_i^2)u = u$ ∎

4.

Proof. We now prove f_x exist on $\mathbb{R}^2$

We view f_x as the ordinal derivative of $g(x) = f(x, y)$ where y is fixed

Notice the product of two differentiable function is a differentiable function, and the composition of two differentiable function is a differentiable function

Let $h(x) = x^2 + y^2$ and $r(x) = \sin(\frac{1}{x})$

Notice r and h are both differentiable over $\mathbb{R} \setminus \{0\}$

Notice $g = h(r \circ h)$, so we know g is differentiable over $\mathbb{R} \setminus \{0\}$

Now we check if g is differentiable at 0, more precisely, if $\lim_{h\to 0} \frac{g(h)-g(0)}{h}$ exists when y is fixed at 0

$\lim_{h\to 0} \frac{g(h)-g(0)}{h} = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} h \sin(\frac{1}{h^2}) = 0$ (done)

We now prove f_y exist on $\mathbb{R}^2$

We view f_y as the ordinal derivative of $g(y) = f(x, y)$ where x is fixed

Notice the product of two differentiable function is a differentiable function, and the composition of two differentiable function is a differentiable function

Let $h(y) = x^2 + y^2$ and $r(y) = \sin(\frac{1}{y})$

Notice r and h are both differentiable over $\mathbb{R} \setminus \{0\}$

Notice $g = h(r \circ h)$, so we know g is differentiable over $\mathbb{R} \setminus \{0\}$

Now we check g is differentiable at 0, more precisely, if $\lim_{h\to 0} \frac{g(h)-g(0)}{h}$ exists when x is fixed at 0

$\lim_{h\to 0} \frac{g(h)-g(0)}{h} = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} h\sin(\frac{1}{h^2}) = 0$ (done)

Notice $f_x = 2x\sin(\frac{1}{x^2+y^2}) + \frac{-2x}{x^2+y^2}\cos(\frac{1}{x^2+y^2})$ when $(x,y) \neq (0,0)$

Setting $y = 0$, we see $\lim_{x\to 0^-} f_x = \infty$ by direct computation, so f_x is discontinuous at $(0,0)$

Notice $f_y = 2y\sin(x^2+y^2) + \frac{-2y}{x^2+y^2}\cos(\frac{1}{x^2+y^2})$ when $(x,y) \neq (0,0)$

Setting $x = 0$, we see $\lim_{y\to 0^-} f_y = \infty$ by direct computation, so f_y is discontinuous at $(0,0)$

Notice $f_x(0,y) = 0$ and $f_x(0,0) = 0$, so $f_{xy}(0,0) = 0$

Notice $f_y(x,0) = 0$ and $f_y(0,0) = 0$, so $f_{yx}(0,0) = 0$

■

5.

5.(a)

Proof.

$$\frac{\partial \frac{1}{r(\mathbf{x})}}{\partial x_i} = \frac{d\frac{1}{r(\mathbf{x})}}{dr(\mathbf{x})}\frac{\partial r(\mathbf{x})}{\partial x_i} = -r^{-2}(\mathbf{x})\frac{\partial r(\mathbf{x})}{\partial x_i} = -r^{-2}(\mathbf{x})\frac{2x_i}{2\sqrt{x_1^2+\cdots+x_n^2}} = \frac{-x_i}{r^3(\mathbf{x})}$$

■

5.(b)

Proof.

$$\frac{\partial r^m(\mathbf{x})}{\partial x_i} = \frac{dr^m(\mathbf{x})}{dr(\mathbf{x})}\frac{\partial r(\mathbf{x})}{\partial x_i} = mr^{m-1}(\mathbf{x})\frac{\partial r(\mathbf{x})}{\partial x_i} = mr^{m-1}(\mathbf{x})\frac{2x_i}{2\sqrt{x_1^2+\cdots+x_n^2}} = mr^{m-2}(\mathbf{x})x_i$$

■

6.

Proof. Write $z = f(x_1, \dots, x_n) = g(h(x_1, \dots, x_n))$

Write $y = h(x_1, \dots, x_n)$, so $z = g(y)$

Then

$$\frac{\partial f}{\partial x_i} = \frac{\partial z}{\partial x_i} = \frac{\partial z}{\partial y}\frac{\partial y}{\partial x_i} = g'(y)\frac{\partial y}{\partial x_i} = g'(y)h_i(x_1, \dots, x_n) = g'(h(x_1, \dots, x_n))h_i(x_1, \dots, x_n) \tag{1}$$

g is differentiable at $\mathbf{a}$ and $h_i(\mathbf{a})$ exists shows that $\frac{\partial f}{\partial x_i}(\mathbf{a})$

Substituting $(x_1, \dots, x_n) = \mathbf{a}$, we have

$$\frac{\partial f}{\partial x_i}(\mathbf{a}) = g'(h(\mathbf{a}))h_i(a)$$

■

7.

Proof. Notice $\frac{df(x,a)}{da} = f_y$

So $f(x, a) = \int_0^a f_y(x, y)dy + C$

Then obviously $\forall x \in [1, 1], \forall a \in (1, 1), f(x, a)$ is bounded by $\pm 2aM + C$

■

8.

8.(a)

Proof.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$\frac{\partial u}{\partial x} = 1 + 2\sqrt{\frac{y}{x}}$

$\frac{\partial u}{\partial y} = 1 + 2\sqrt{\frac{x}{y}}$

$\frac{dx}{dt} = 3t^2$

$\frac{dy}{dt} = -t^{-2}$

$$\frac{du}{dt} = (1 + 2\sqrt{\frac{y}{x}})3t^2 + (1 + 2\sqrt{\frac{x}{y}})(-t^{-2})$$

■

8.(b)

Proof.
$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$\frac{\partial u}{\partial x} = y + z$

$\frac{\partial u}{\partial y} = x + z$

$\frac{\partial u}{\partial z} = x + y$

$\frac{dx}{dt} = 2t$

$\frac{dy}{dt} = -2t + 1$

$\frac{dz}{dt} = -2(1 - t)$

$$\frac{du}{dt} = (y + z)2t + (x + z)(-2t + 1) - 2(x + y)(1 - t)$$

■

8.(c)

Proof. $\frac{\partial u}{\partial x} = z^2 y \sec(xy)\tan(xy)$

$\frac{\partial u}{\partial y} = z^2 x \sec(xy)\tan(xy)$

$\frac{\partial u}{\partial z} = 2x \sec(xy)$

$\frac{\partial x}{\partial s} = 2t$

$\frac{\partial x}{\partial t} = 2s$

$\frac{\partial y}{\partial s} = 1$

$\frac{\partial y}{\partial t} = -2t$

$\frac{\partial z}{\partial s} = 2ts$

$\frac{\partial z}{\partial t} = s^2$

(i)

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial u}{\partial s} = z^2 y \sec(xy)\tan(xy)2t + z^2 x \sec(xy)\tan(xy) + 2x\sec(xy)2ts$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$$

$$\frac{\partial u}{\partial t} = z^2 y \sec(xy)\tan(xy)2s + z^2 x \sec(xy)\tan(xy) + 2x\sec(xy)s^2$$

$$\frac{\partial}{\partial s}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial s\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial^2 x}{\partial s\partial t} + \frac{\partial^2 u}{\partial s\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial^2 y}{\partial s\partial t} + \frac{\partial^2 z}{\partial s\partial y}\frac{\partial z}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial^2 z}{\partial s\partial t}$$

$\frac{\partial^2 u}{\partial s\partial x} = \frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial s}$

$\frac{\partial^2 u}{\partial s\partial y} = \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial s}$

$\frac{\partial^2 u}{\partial s\partial z} = \frac{\partial^2 u}{\partial z^2}\frac{\partial z}{\partial s}$

$\frac{\partial^2 u}{\partial x^2} = z^2 y^2(\tan^2(xy)\sec(xy) + \sec^2(xy))$

$\frac{\partial^2 u}{\partial y^2} = z^2 x^2(\tan^2(xy)\sec(xy) + \sec^2(xy))$

$\frac{\partial^2 u}{\partial z^2} = 0$

$\frac{\partial^2 x}{\partial s\partial t} = 2$

$\frac{\partial^2 y}{\partial s\partial t} = 0$

$\frac{\partial^2 z}{\partial s\partial t} = 2s$

■

8.(d)

Proof.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial w}\frac{\partial w}{\partial r} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t}\frac{\partial t}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial w}\frac{\partial w}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}\frac{\partial t}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial w}\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}\frac{\partial t}{\partial r}$$

■

9.

Proof.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos(\theta) + \frac{\partial u}{\partial y}\sin(\theta) \tag{2}$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\frac{\partial^2 u}{\partial x\partial y}\sin\theta\cos\theta \tag{3}$$

$$\frac{\partial^2 u}{\partial\theta^2} = \frac{\partial^2 u}{\partial\theta\partial x} r(-\sin\theta) + \frac{\partial u}{\partial x} r(-\cos\theta) + \frac{\partial^2 u}{\partial\theta\partial y} r\cos\theta + \frac{\partial u}{\partial y} r(-\sin\theta) \quad (4)$$

$$\frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = \frac{-1}{r}\left(\frac{\partial^2 u}{\partial\theta\partial x}\sin\theta - \frac{\partial^2 u}{\partial\theta\partial y}\cos\theta\right)$$

$\frac{\partial^2 u}{\partial\theta\partial x} = -r\frac{\partial^2 u}{\partial x^2}\sin\theta + r\frac{\partial^2 u}{\partial y\partial x}\cos\theta$

$\frac{\partial^2 u}{\partial\theta\partial y} = r\frac{\partial^2 u}{\partial y^2}\cos\theta - r\frac{\partial^2 u}{\partial x\partial y}\sin\theta$

$$\frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = \sin^2(\theta)\frac{\partial^2 u}{\partial x^2} + \cos^2(\theta)\frac{\partial^2 u}{\partial y^2} - 2\sin\theta\cos\theta\frac{\partial^2 u}{\partial y\partial x} \quad (5)$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = \frac{\partial^2 u}{\partial x^2}(\sin^2\theta + \cos^2\theta) + \frac{\partial^2 u}{\partial y^2}(\sin^2\theta + \cos^2\theta) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad (6)$$

■

10.

10.(a)

Proof.

$$z = x + y + 16 - 8(\sqrt{x} + \sqrt{y}) + 2\sqrt{xy}$$

$$\frac{\partial z}{\partial x} = 1 + \frac{\sqrt{y} - 4}{\sqrt{x}}$$

$$\frac{\partial z}{\partial y} = 1 + \frac{\sqrt{x} - 4}{\sqrt{y}}$$

$\frac{\partial z}{\partial x}(1,4) = -1$

$\frac{\partial z}{\partial y}(1,4) = \frac{-1}{2}$

The tangent plane at $(1,4,1)$ is

$$z - 1 = -(x-1) + \frac{1}{-2}(y-4)$$

■

10.(b)

Proof.
$$\frac{\partial z}{\partial x} = \cos(x\cos y)\cos y$$
$$\frac{\partial z}{\partial y} = -\cos(x\cos y)x\sin y$$

$\frac{\partial z}{\partial x}(1, \frac{1}{2}\pi) = 0$

$\frac{\partial z}{\partial y}(1, \frac{1}{2}\pi) = -1$

The tangent plane at $(1, \frac{1}{2}\pi, 0)$ is
$$z = -(y - \frac{1}{2}\pi)$$

■

11.

Proof. $\frac{\partial f}{\partial x} = 2x + 2yz$

$\frac{\partial f}{\partial y} = 2xz - z^2$

$\frac{\partial f}{\partial z} = -2yz + 2xy$

$\frac{\partial f}{\partial x}(1, 1, 2) = 6$

$\frac{\partial f}{\partial y}(1, 1, 2) = 0$

$\frac{\partial f}{\partial z}(1, 1, 2) = -2$

The direction vector **u** is $\frac{1}{\sqrt{14}}(2, 1, -3)$

The direction derivative is $\frac{1}{\sqrt{14}}(12 + 6)$

■

12.

Proof.
$$f_x = -2xe^y$$
$$f_y = e^y(y^2 + 2y - x^2)$$

$f_x = 0 = f_y \implies x = 0$ and $(y = 0$ or $-2)$

So the critical points are $(0, 0)$ or $(0, -2)$

$f_{xx} = -2e^y$

$f_{yy} = e^y(y^2 + 4y + 2 - x^2)$

$f_{xy} = -2xe^y$

$$D = -2e^{2y}[(y+2)^2 + x^2 - 6]$$

$\forall (a,b) \in \mathbb{R}^2, f_{xx}(a,b) < 0$

$D(0,0) = 4 > 0$

$D(0,2) < 0$

So $(0,2)$ is a saddle points and $(0,0)$ is a local maximum

■