Chapter 1

General Analysis HW

1.1 HW1

Question 1

Show $\mathbb{R}^n$ is complete.

Proof. Let $\mathbf{x}_k$ be an arbitrary Cauchy sequence in $\mathbb{R}^n$. We are required to show $\mathbf{x}_k$ converge in $\mathbb{R}^n$. For each k, denote $\mathbf{x}_k$ by $(x_{(1,k)},\dots,x_{(n,k)})$. We claim that for each $i\in\{1,\dots,n\}$

$$x_{(i,k)} \text{ is a Cauchy sequence}$$

Fix i and $\epsilon>0$. To show $x_{(i,k)}$ is a Cauchy sequence, we are required to find $N\in\mathbb{N}$ such that for all $r,m\geq N$ we have

$$\left|x_{(i,r)}-x_{(i,m)}\right|\leq\epsilon$$

Because $\mathbf{x}_k$ is a Cauchy sequence in $\mathbb{R}^n$, we know there exists $N\in\mathbb{N}$ such that for all $r,m\geq N$, we have

$$|\mathbf{x}_r-\mathbf{x}_m|<\epsilon$$

Fix such N and arbitrary $r,m\geq M$. Observe

$$\left|x_{(i,r)}-x_{(i,m)}\right|\leq\sqrt{\sum_{j=1}^n\left|x_{(j,r)}-x_{(j,m)}\right|^2}=|\mathbf{x}_r-\mathbf{x}_m|<\epsilon$$

We have proved that for each $i\in\{1,\dots,n\}$, the real sequence $x_{(i,k)}$ is Cauchy. We now claim that for each $i\in\{1,\dots,n\}$, we have

$$\limsup_{r\to\infty}x_{(i,r)}\in\mathbb{R}\text{ and }\lim_{k\to\infty}x_{(i,k)}=\limsup_{r\to\infty}x_{(i,r)}$$

Again fix i. Because $x_{(i,k)}$ is a Cauchy sequence, we know there exists some N such that for all $r,m \geq N$, we have

$$\left|x_{(i,r)} - x_{(i,m)}\right| < 1$$

This implies that for all $r \geq N$, we have

$$x_{(i,r)} < x_{(i,N)} + 1 \tag{1.1}$$

Equation 1.1 then tell us

$$x_{(i,N)} + 1 \text{ is an upper bound of } \{x_{(i,r)} : r \geq N\}$$

Then by definition of sup, we have

$$\sup\{x_{(i,r)} : r \geq N\} \leq x_{(i,N)} + 1 \in \mathbb{R}$$

This then implies $\limsup_{r\to\infty} x_{(i,r)} \in \mathbb{R}$. We now prove

$$\lim_{k\to\infty} x_{(i,k)} = \limsup_{r\to\infty} x_{(i,r)} \tag{1.2}$$

Fix $\epsilon > 0$. We are required to find N such that

$$\forall k \geq N, \left|x_{(i,k)} - \limsup_{r\to\infty} x_{(i,r)}\right| \leq \epsilon$$

Because $\{x_{(i,k)}\}_{k\in\mathbb{N}}$ is a Cauchy sequence, we can let N_0 satisfy

$$\forall k,m \geq N_0, \left|x_{(i,k)} - x_{(i,m)}\right| < \frac{\epsilon}{2}$$

Because $\sup\{x_{(i,k)} : k \geq N'\} \searrow \limsup_{r\to\infty} x_{(i,r)}$ as $N' \to \infty$, we know there exists $N_1 > N_0$ such that

$$\limsup_{r\to\infty} x_{(i,r)} - \frac{\epsilon}{2} < \sup\{x_{(i,k)} : k \geq N_0\} \leq \limsup_{r\to\infty} x_{(i,r)} + \frac{\epsilon}{2}$$

Then because $\limsup_{r\to\infty} x_{(i,r)} - \frac{\epsilon}{2}$ is strictly smaller than the smallest upper bound of $\{x_{(i,k)} : k \geq N_1\}$, we see $\limsup_{n\to\infty} x_{(i,r)} - \frac{\epsilon}{2}$ is not an upper bound of $\{x_{(i,k)} : k \geq N_1\}$. This implies the existence of some N such that $N \geq N_1$ and

$$\limsup_{r\to\infty} x_{(i,r)} - \frac{\epsilon}{2} < x_{(i,N)} \leq \limsup_{r\to\infty} x_{(i,r)} + \frac{\epsilon}{2}$$

Now observe that for all $k \geq N$, because $N \geq N_1 \geq N_0$

$$\limsup_{r\to\infty} x_{(i,r)} - \epsilon < x_{(i,N)} - \frac{\epsilon}{2} < x_{(i,k)} < x_{(i,N)} + \frac{\epsilon}{2} \leq \limsup_{r\to\infty} x_{(i,r)} + \epsilon$$

This implies for all $k \geq N$, we have

$$\left| x_{(i,k)} - \limsup_{r\to\infty} x_{(i,r)} \right| \leq \epsilon$$

We have just proved Equation 1.2. Lastly, to close out the proof, we show

$$\lim_{k\to\infty} \mathbf{x}_k = \left(\lim_{k\to\infty} x_{(1,k)}, \dots, \lim_{k\to\infty} x_{(n,k)} \right) \tag{1.3}$$

Fix $\epsilon > 0$. For each $i \in \{1, \dots, n\}$, let N_i satisfy

$$\forall r \geq N_i, \left| x_{(i,r)} - \lim_{k\to\infty} x_{(i,k)} \right| \leq \frac{\epsilon}{\sqrt{n}}$$

Observe that for all $r \geq \max_{i\in\{1,\dots,n\}} N_i$, we have

$$\begin{aligned} \left| \mathbf{x}_r - \left(\lim_{k\to\infty} x_{(1,k)}, \dots, \lim_{k\to\infty} x_{(n,k)} \right) \right| &= \sqrt{\sum_{i=1}^{n} \left| x_{(i,r)} - \lim_{k\to\infty} x_{(i,k)} \right|^2} \\ &\leq \sqrt{\sum_{i=1}^{n} \frac{\epsilon^2}{n}} = \epsilon \end{aligned}$$

We have proved Equation 1.3.

■

Question 2

Show $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. Fix $x \in \mathbb{R}$ and $\epsilon > 0$. To show $\mathbb{Q}$ is dense in $\mathbb{R}$, we have to find $q \in \mathbb{Q}$ such that $|x - q| < \epsilon$.

Let $m \in \mathbb{N}$ satisfy $\frac{1}{m} < \epsilon$. Let n be the largest integer such that $n \leq mx$. Because n is the largest integer such that $n \leq mx$, we know $mx - n < 1$, otherwise we can deduce $n + 1 \leq mx$, which is impossible, since $n + 1$ is an integer and n is the largest integer such that $n \leq mx$. We now see that

$$\frac{n}{m} \in \mathbb{Q} \text{ and } \left| x - \frac{n}{m} \right| = \frac{mx - n}{m} < \frac{1}{m} < \epsilon$$

■

Theorem 1.1.1. (Distance Formula) Given two subsets A, B of a metric space, we have

$$d(A, B) = \inf_{b\in B} d(A, b)$$

Proof. Fix arbitrary $b \in B$. It is clear that

$$d(A, B) \leq d(A, b)$$

It then follows $d(A, B) \leq \inf_{b \in B} d(A, b)$. Fix arbitrary $a \in A$ and $b_0 \in B$. Observe that

$$d(a, b_0) \geq d(A, b_0) \geq \inf_{b \in B} d(A, b)$$

It then follows $\inf_{b \in B} d(A, b) \leq d(A, B)$. ■

Question 3

Let E_1, E_2 be non-empty sets in $\mathbb{R}^n$ with E_1 closed and E_2 compact. Show that there are points $x_1 \in E_1$ and $x_2 \in E_2$ such that

$$d(E_1, E_2) = |x_1 - x_2|$$

Deduce that $d(E_1, E_2)$ is positive if such E_1, E_2 are disjoint.

Proof. Because

(a) $f(x) \triangleq d(E_1, x)$ is a continuous function on $\mathbb{R}^n$.

(b) E_2 is compact.

It now follows by EVT there exists some $x_2 \in E_2$ such that

$$d(E_1, x_2) = \min_{x \in E_2} d(E_1, x) = \inf_{x \in E_2} d(E_1, x) = d(E_1, E_2)$$

where the last equality is proved above. We can now reduce the problem into finding x_1 in E_1 such that

$$d(x_1, x_2) = d(E_1, x_2)$$

For each $n \in \mathbb{N}$, let t_n satisfy

$$t_n \in E_1 \text{ and } d(t_n, x_2) < d(E_1, x_2) + \frac{1}{n}$$

Clearly, t_n is a bounded sequence. Then by Bolzano-Weierstrass Theorem, there exists a convergence subsequence t_{n_k}. Now, because E_1 is closed, we know

$$x_1 \triangleq \lim_{k \to \infty} t_{n_k} \in E_1$$

It then follows from the function $f(x) \triangleq d(x, x_2)$ being continuous on $\mathbb{R}^n$ such that

$$d(x_1, x_2) = \lim_{k \to \infty} d(t_{n,k}, x_2) = d(E_1, x_2)$$

■

Question 4

Prove that the distance between two nonempty, compact, disjoint sets in $\mathbb{R}^n$ is positive.

Proof. The proof follows from the result in last question while acknowledging compact is closed. ∎

Question 5

Prove that if f is continuous on $[a, b]$, then f is Riemann-integrable on$[a, b]$.

Proof. Let $\overline{\int_a^b} f dx$ and $\underline{\int_a^b} f dx$ respectively denote the upper and lower Darboux sums. We prove that

$$\overline{\int_a^b} f dx = \underline{\int_a^b} f dx$$

Fix ϵ. We reduce the problem into proving the existence of some partition $\{a = x_0, x_1, \dots, x_n = b\}$ such that

$$\sum_{i=1}^{n} \Big[M_i - m_i\Big](x_i - x_{i-1}) \leq \epsilon$$

where

$$M_i \triangleq \sup_{t \in [x_{i-1}, x_i]} f(t) \text{ and } m_i \triangleq \inf_{t \in [x_{i-1}, x_i]} f(t)$$

Because f is continuous on the compact interval $[a, b]$, we know f is uniformly continuous on $[a, b]$. Let δ satisfy

$$|x - y| < \delta \text{ and } x, y \in [a, b] \implies |f(x) - f(y)| < \frac{\epsilon}{b - a}$$

Let n satisfy $\frac{b-a}{n} < \delta$. We claim the partition

$$\{a = x_0, x_1, \dots, x_n = b\} \text{ where } x_i \triangleq a + \frac{i(b - a)}{n} \text{ suffices}$$

Now, by EVT, we know that for each i, there exists some $t_{i,M}, t_{i,m} \in [x_{i-1}, x_i]$ such that

$$f(t_{i,m}) = m_i \text{ and } f(t_{i,M}) = M_i$$

Then because

$$|t_{i,m} - t_{i,M}| \leq x_i - x_{i-1} \leq \frac{b - a}{n} < \delta$$

We know $M_i - m_i < \frac{\epsilon}{b-a}$. This now give us

$$\begin{aligned}\sum_{i=1}^{n}\Big[M_i - m_i\Big](x_i - x_{i-1}) &< \sum_{i=1}^{n}\frac{\epsilon}{(b-a)}(x_i - x_{i-1}) \\ &= \frac{\epsilon}{b-a}\sum_{i=1}^{n}(x_i - x_{i-1}) \\ &= \frac{\epsilon}{b-a}(b-a) = \epsilon\end{aligned}$$

■

Question 6

Find $\limsup_{n\to\infty} E_n$ and $\liminf_{n\to\infty} E_n$ where

$$E_n \triangleq \begin{cases}[\frac{-1}{n}, 1] & \text{if } n \text{ is odd} \\ [-1, \frac{1}{n}] & \text{if } n \text{ is even}\end{cases}$$

Proof. Fix arbitrary $n \in \mathbb{N}$. Let $p, q \geq n$ respectively be odd and even. We see

$$[0, 1] \subseteq E_p \text{ and } [-1, 0] \subseteq E_q$$

This now implies

$$[-1, 1] \subseteq \bigcup_{k\geq n} E_k$$

Then because n is arbitrary, it follows

$$\limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty}\bigcup_{k\geq n} E_k = [-1, 1]$$

Again, fix arbitrary $n \in \mathbb{N}$ and $\epsilon > 0$. Let p, q respectively be even and odd integers greater than $\max\{n, \frac{1}{\epsilon}\}$. We now see

$$\epsilon \notin [-1, \frac{1}{p}] = E_p \text{ and } -\epsilon \notin [\frac{-1}{q}, 1] = E_q$$

Because ϵ is arbitrary and clearly $0 \in E_k$ for all k, we now see

$$\bigcap_{k\geq n} E_k = \{0\}$$

Then because n is arbitrary, we see

$$\liminf_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} \bigcap_{k\geq n} E_k = \{0\}$$

■

Question 7

Show that

$$(\limsup_{n\to\infty} E_n)^c = \liminf_{n\to\infty} (E_n)^c$$

and

$$E_n \searrow E \text{ or } E_n \nearrow E \implies \limsup_{n\to\infty} E_n = \liminf_{n\to\infty} E_n = E$$

Proof. Fix arbitrary $x \in (\limsup_{n\to\infty} E_n)^c$. We can deduce

$$\exists n, x \notin \bigcup_{k\geq n} E_k$$

This implies

$$\exists n, x \in \bigcap_{k\geq n} E_k^c$$

Then we see

$$x \in \bigcup_{n=1}^{\infty} \bigcap_{k\geq n} E_k^c = \liminf_{n\to\infty} E_n^c$$

We have proved $(\limsup_{n\to\infty} E_n)^c \subseteq \liminf_{n\to\infty} E_n^c$. We now prove the converse. Fix arbitrary $x \in \liminf_{n\to\infty} E_n^c$. We can deduce

$$\exists n, x \in \bigcap_{k\geq n} E_k^c$$

This implies

$$\exists n, x \notin \bigcup_{k\geq n} E_k$$

Then we see

$$x \notin \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k = \limsup_{n \to \infty} E_n$$

$\blacksquare$

Theorem 1.1.2. (Equivalent Definition for Limit Superior) If we let E be the set of subsequential limits of a_n

$$E \triangleq \{L \in \overline{\mathbb{R}} : L = \lim_{k \to \infty} a_{n_k} \text{ for some } n_k\}$$

The set E is non-empty and

$$\max E = \limsup_{n \to \infty} a_n$$

Proof. Let $n_1 \triangleq 1$. Recursively, because

$$\sup_{j \geq n_k} a_k \geq \limsup_{n \to \infty} a_n > \limsup_{n \to \infty} a_n - \frac{1}{k} \text{ for each } k$$

We can let n_{k+1} be the smallest number such that

$$a_{n_{k+1}} > \limsup_{n \to \infty} a_n - \frac{1}{k}$$

It is straightforward to check $a_{n_k} \to \limsup_{n \to \infty} a_n$ as $k \to \infty$. Note that no subsequence can converge to $\limsup_{n \to \infty} a_n + \epsilon$ because there exists N such that $\sup_{k \geq N} a_k < \limsup_{n \to \infty} a_n + \epsilon$. $\blacksquare$

Question 8

Show that

$$\limsup_{n \to \infty} (-a_n) = -\liminf_{n \to \infty} a_n$$

Proof. Note that $-a_{n_k}$ converge if and only if a_{n_k} converge. Then if we respectively define E and E^- to be the set of subsequential limits of a_n and $-a_n$, we see

$$E^- = \{-L \in \mathbb{R} : L \in E\}$$

We now see

$$\limsup_{n \to \infty} (-a_n) = \max E^- = -\min E = -\liminf_{n \to \infty} a_n$$

$\blacksquare$

Question 9

Show that

$$\limsup_{n\to\infty}(a_n+b_n)\leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \tag{1.4}$$

Proof. Fix arbitrary ϵ. Let N_a, N_b respectively satisfy

$$\sup_{n\geq N_a} a_n \leq \limsup_{n\to\infty} a_n + \frac{\epsilon}{2} \text{ and } \sup_{n\geq N_b} b_n \leq \limsup_{n\to\infty} b_n + \frac{\epsilon}{2}$$

Let $N \triangleq \max\{N_a, N_b\}$. We now see that

$$\limsup_{n\to\infty}(a_n+b_n)\leq \sup_{n\geq N}(a_n+b_n)\leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n + \epsilon$$

The result then follows from ϵ being arbitrary. ■

Question 10

$$a_n, b_n \text{ is bounded non-negative } \implies \limsup_{n\to\infty}(a_nb_n)\leq (\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n) \tag{1.5}$$

Proof. There are three cases we should consider

(a) Both $\limsup_{n\to\infty} a_n$ and $\limsup_{n\to\infty} b_n$ equal 0.

(b) Between $\limsup_{n\to\infty} a_n$ and $\limsup_{n\to\infty} b_n$, only one of them equals 0.

(c) Neither $\limsup_{n\to\infty} a_n$ nor $\limsup_{n\to\infty} b_n$ equals to 0.

In the first case, because a_n, b_n are both non-negative, we can deduce

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$$

which implies

$$\limsup_{n\to\infty}(a_nb_n) = \lim_{n\to\infty} a_nb_n = 0 = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$$

For second case, WOLG, suppose $\limsup_{n\to\infty} a_n = 0$. Fix arbitrary ϵ. We can let N satisfy

$$\sup_{n\geq N} a_n < \frac{\epsilon}{\sup_{n\in\mathbb{N}} b_n}$$

Since for all $n \geq N$, we have

$$a_n b_n \leq \frac{b_n \epsilon}{\sup_{k \in \mathbb{N}} b_k} \leq \epsilon$$

We now see

$$\limsup_{n \to \infty} (a_n b_n) \leq \sup_{n \geq N} a_n b_n \leq \epsilon$$

The result

$$\limsup_{n \to \infty} a_n b_n = 0 = \limsup_{n \to \infty} a_n \limsup_{n \to \infty} b_n$$

then follows from ϵ being arbitrary.

Lastly, for the last case, let N_a, N_b respectively satisfy

$$\sup_{n \geq N_a} a_n \leq \limsup_{n \to \infty} a_n \sqrt{1+\epsilon} \text{ and } \sup_{n \geq N_b} b_n \leq \limsup_{n \to \infty} b_n \sqrt{1+\epsilon}$$

Let $N \triangleq \max\{N_a, N_b\}$, because for each $n \geq N$, we have

$$a_n b_n \leq (\sup_{k \geq N_a} a_k)(\sup_{k \geq N_b} b_k) \leq (1+\epsilon)(\limsup_{n \to \infty} a_n)(\limsup_{n \to \infty} b_n)$$

It then follows that

$$\limsup_{n \to \infty} (a_n b_n) \leq \sup_{n \geq N} (a_n b_n) \leq (1+\epsilon)(\limsup_{n \to \infty} a_n)(\limsup_{n \to \infty} b_n)$$

The result then follows from ϵ being arbitrary. ■

Question 11

Show that if either a_n or b_n converge, the equalities in Equation 1.4 and Equation 1.5 both hold true.

Proof. WOLG, suppose $\lim_{n \to \infty} a_n = L \in \mathbb{R}$. We then see

$$(a_{n_k} + b_{n_k}) \text{ converge} \iff b_{n,k} \text{ converge}$$

Let $E_{a,b}$ and E_b respectively be the set of subsequential limits of $(a_n + b_n)$ and b_n. We now have

$$E_{a,b} = \{L + L_b \in \mathbb{R} : L_b \in E_b\}$$

This give us

$$\limsup_{n\to\infty}(a_n+b_n)=\max E_{a,b}=L+\max E_b=\limsup_{n\to\infty}a_n+\limsup_{n\to\infty}b_n$$

Now, additionally, suppose a_n,b_n are both bounded and nonnegative. Again because

$$a_{n_k}b_{n,k}\text{ converge }\iff b_{n,k}\text{ converge}$$

We see

$$E_{a,b}=\{L(L_b)\in\mathbb{R}:L_b\in E_b\}$$

This give us

$$\limsup_{n\to\infty}(a_nb_n)=\max E_{a,b}=L\max E_b=(\limsup_{n\to\infty}a_n)(\limsup_{n\to\infty}b_n)$$

∎

Question 12

Give example for which inequality in Equation 1.4 and Equation 1.5 are not equalities.

Proof. If

$$a_n\triangleq\begin{cases}1&\text{if }n\text{ is odd}\\-1&\text{if }n\text{ is even}\end{cases}\text{ and }b_n\triangleq\begin{cases}-1&\text{if }n\text{ is odd}\\1&\text{if }n\text{ is even}\end{cases}$$

we have

$$\limsup_{n\to\infty}(a_n+b_n)=0<2=\limsup_{n\to\infty}a_n+\limsup_{n\to\infty}b_n$$

Let $L>1$ and

$$a_n\triangleq\begin{cases}L-\frac{1}{k}&\text{if }n=2k-1\\(L-\frac{1}{k})^{-1}&\text{if }n=2k\end{cases}\text{ and }b_n\triangleq\begin{cases}(L-\frac{1}{k})^{-1}&\text{if }n=2k-1\\(L-\frac{1}{k})&\text{if }n=2k\end{cases}$$

We have

$$\limsup_{n\to\infty}a_nb_n=1<L^2=\limsup_{n\to\infty}a_n\limsup_{n\to\infty}b_n$$

∎

Question 13

Give an example of a decreasing sequence of nonempty closed sets in $\mathbb{R}^n$ whose intersection is empty.

Proof.

$$F_n \triangleq [n, \infty) \text{ suffices}$$

■

Question 14

Given an example of two disjoint, nonempty closed sets in E_1 and E_2 in $\mathbb{R}^n$ for which $d(E_1, E_2) = 0$.

Proof. Let

$$E_1 \triangleq \{n - \frac{1}{n} \in \mathbb{R} : n \in \mathbb{N} \text{ and } n \geq 2\} \text{ and } E_2 \triangleq \{n - \frac{1}{2n} \in \mathbb{R} : n \in \mathbb{N} \text{ and } n \geq 2\}$$

To see $E_1 \cap E_2 = \varnothing$, suppose $n - \frac{1}{n} = k - \frac{1}{2k}$ where n, k are two natural numbers greater than 2. We then see $\frac{1}{n} - \frac{1}{2k} = n - k$, which is impossible, since

$$\left| \frac{1}{n} - \frac{1}{2k} \right| < \max\{\frac{1}{2k}, \frac{1}{n}\} < 1$$

The fact E_1, E_2 are closed follows from both of them being totally disconnected. Now observe that for all ϵ, there exists large enough n such that

$$(n + \frac{1}{n}) - (n + \frac{1}{2n}) < \frac{1}{n} < \epsilon$$

This implies $d(E_1, E_2) = 0$. ■

Question 15

If f is defined and uniformly continuous on E, show there is a function $\overline{f}$ defined and continuous on $\overline{E}$ such that $\overline{f} = f$ on E.

Proof. Define $\overline{f}$ on E by $\overline{f} = f$. For each $x \in \overline{E} \setminus E$, associate x with a sequence $t_{n,x}$ in E converging to x. We now claim that for each $x \in \overline{E} \setminus E$ the limit

$$\lim_{n \to \infty} f(t_{n,x}) \text{ converge in } \mathbb{R}$$

Fix ϵ. Because f is uniformly continuous on E, we know there exists δ such that

$$a,b \in E \text{ and } |a-b| \leq \delta \implies |f(a)-f(b)| < \epsilon$$

Because $t_{n,x}$ converge, we know $t_{n,x}$ is Cauchy, then we know there exists N such that $|t_{n,x}-t_{m,x}| < \delta$ for all $n,m > N$, we then see that for all $n,m > N$, we have

$$|f(t_{n,x}) - f(t_{m,x})| < \epsilon$$

This implies $\{f(t_{n,x})\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, thus converge in $\mathbb{R}$.

Define

$$\overline{f}(x) \triangleq \lim_{n\to\infty} f(t_{n,x}) \text{ for all } x \in \overline{E}\setminus E$$

We are required to show $\overline{f}$ is also continuous on $\overline{E}\setminus E$. Fix ϵ and $x \in \overline{E}\setminus E$. Let δ satisfy

$$a,b \in E \text{ and } |a-b| \leq \delta \implies |f(a)-f(b)| < \frac{\epsilon}{3}$$

We claim

$$\sup_{t\in B_{\frac{\delta}{2}}(x)\cap\overline{E}} \left|\overline{f}(t)-\overline{f}(x)\right| \leq \epsilon$$

Fix $t \in B_{\frac{\delta}{2}}(x)\cap\overline{E}$. There are two possibilities

(a) $t \in E$

(b) $t \in \overline{E}\setminus E$

If $t \in E$, let n satisfy

$$\left|f(t_{n,x})-\overline{f}(x)\right| < \frac{\epsilon}{3} \text{ and } |t_{n,x}-x| < \frac{\delta}{2}$$

Because

$$|t_{n,x}-t| \leq |t_{n,x}-x| + |t-x| < \delta$$

we can deduce $|f(t_{n,x})-f(t)| < \frac{\epsilon}{3}$. This now give us

$$\left|f(t)-\overline{f}(x)\right| \leq |f(t_{n,x})-f(t)| + \left|f(t_{n,x})-\overline{f}(x)\right| < \epsilon$$

If $t \in \overline{E}\setminus E$. Write $y=t$ and let $t_{n,y}$ be the associated sequence in E. Because $y \in B_{\frac{\delta}{2}}(x)$, we know there exists $t_{n,y}$ such that

$$t_{n,y} \in B_{\frac{\delta}{2}}(x) \text{ and } \left|f(t_{n,y})-\overline{f}(y)\right| < \frac{\epsilon}{3}$$

Again, let m satisfy

$$t_{m,x} \in B_{\frac{\delta}{2}}(x) \text{ and } \left|f(t_{m,x}) - \overline{f}(x)\right| < \frac{\epsilon}{3}$$

We know $|t_{n,y} - t_{m,x}| \leq \delta$ because they both belong to $B_{\frac{\delta}{2}}(x)$. We can now deduce

$$\left|\overline{f}(y) - \overline{f}(x)\right| = \left|\overline{f}(y) - f(t_{n,y})\right| + |f(t_{n,y}) - f(t_{m,x})| + \left|f(t_{m,x}) - \overline{f}(x)\right| < \epsilon$$

which finish the proof. ∎

Question 16

If f is defined and uniformly continuous on a bounded set E, show that f is bounded on E.

Proof. By last question, we can extend f to a continuous $\overline{f}$ onto $\overline{E}$. Now because $\overline{E}$ is compact and $\left|\overline{f}\right|$ is continuous on $\overline{E}$, by EVT, there exists $a \in \overline{E}$ such that

$$\sup_{x\in E} |f(x)| \leq \max_{x\in\overline{E}} |f(x)| = f(a)$$

∎