Eric's note on Complex Geometry

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Chapter 1

Complex Analysis of Single Variable

1.1 Complex Differentiation

Abstract

This is a short section introducing the idea of complex-differentiable function and prove some of their basic properties, i.e., Cauchy Riemann Criteria and Product and Quotient Rule for Complex-differentiable Function.

Given a complex-valued function f defined on some open subset of $\mathbb{C}$ containing z, we say f is **complex-differentiable at** z if there exists some complex number denoted by $f'(z)$ such that

$$\frac{f(z+h)-f(z)-f'(z)h}{h} \to 0 \text{ as } h \to 0; h \in \mathbb{C}$$

Immediately, one can see that a complex-differentiable function when viewed as a function between $\mathbb{R}^2$ is differentiable with derivative

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \text{ where } f'(z) = a + bi \tag{1.1}$$

With the form of derivative in mind, one may conjecture that complex-differentiable is a 'stricter' condition than merely differentiable when regarded as function between $\mathbb{R}^2$. This is exactly true. Consider the following example.

Example 1 (A non complex-differentiable function)

$$f(z) \triangleq \overline{z}$$

This is a linear function with matrix representation

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

which doesn't fit the necessary form in Equation 1.1.

Theorem 1.1.1. (Cauchy Riemann Criteria) Given a complex-valued function f defined on some open subset U of $\mathbb{C}$ containing z, if we write

$$f(x + yi) = u(x, y) + iv(x, y)$$

where $u, v : U \to \mathbb{R}$, then the following two statements are equivalent.

(a) f is complex differentiable at z.

(b) u, v are differentiable at z when U is viewed as subsets of $\mathbb{R}^2$ and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} \text{ at } z$$

Proof. (a) to (b) is an immediate result of Chain Rule and matrix representation of $f'(z)$. Suppose (b) is true. Let $h = h_1 + ih_2$ and $z = x + yi$. Because u, v are differentiable at (x, y), by the matrix representation of derivative, we have

$$\begin{aligned}
\frac{f(z+h) - f(z)}{h} &= \frac{u(x + h_1, y + h_2) - u(x, y)}{h_1 + ih_2} + i\frac{v(x + h_1, y + h_2) - v(x, y)}{h_1 + ih_2} \\
&= \frac{h_1 u_x + h_2 u_y + ih_1 v_x + ih_2 v_y + o(|h|)}{h_1 + ih_2} \\
&= \frac{(h_1 + ih_2)u_x + i(h_1 + ih_2)v_x + o(|h|)}{h_1 + ih_2} \\
&= u_x + iv_x + \frac{o(|h|)}{h_1 + ih_2} \to u_x + iv_x
\end{aligned}$$

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Theorem 1.1.2. (Product and Quotient Rule for Complex-differentiable Function) Given two function f, g complex-differentiable at z, we have

$$(fg)'(z) = f'(z)g(z) + f(z)g'(z)$$

and if $g(z) \neq 0$, we also have

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}$$

Proof. Observe

$$\begin{aligned}
&\frac{f(z+h)g(z+h) - f(z)g(z)}{h} \\
=&f(z+h)\left[\frac{g(z+h) - g(z)}{h}\right] + g(z)\left[\frac{f(z+h) - f(z)}{h}\right] \to f'(z)g(z) + f(z)g'(z)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\frac{f(z+h)}{g(z+h)} - \frac{f(z)}{g(z)}}{h} &= \frac{f(z+h)g(z) - f(z)g(z+h)}{g(z+h)g(z)h} \\
&= \frac{1}{g(z+h)g(z)}\left[g(z)\left(\frac{f(z+h) - f(z)}{h}\right) - f(z)\left(\frac{g(z+h) - g(z)}{h}\right)\right] \\
&\to \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}
\end{aligned}$$

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1.2 Uniform Convergence and Differentiation

Abstract

This is a section discussing the relationship between uniform convergence and differentiation, which heavily rely on the usage of MVT, and is used to prove Analytic function is smooth.

Before stating Theorem 1.2.1, let's see three examples why we don't (can't) use the hypothesis: $f_n \to f$ uniformly in our statement of Theorem 1.2.1

Example 2 (Differentiable functions are NOT closed under uniform convergence)

$$X = [-1, 1] \text{ and } f(x) = |x|$$

By Weierstrass approximation Theorem, there is a sequence of polynomials (differentiable) that uniformly converge to f, which is not differentiable at 0.

Example 3 (Derivative won't necessarily converge to the right place)

$$X = \mathbb{R} \text{ and } f_n(x) = \frac{\sin nx}{\sqrt{n}}$$

Compute

$$f'(x) = 0 \text{ and } f_n'(x) = \sqrt{n} \cos nx$$

Example 4 (Derivative won't necessarily converge to the right place)

$$X = \mathbb{R} \text{ and } f_n(x) = \frac{x}{1 + nx^2}$$

Compute

$$f = \tilde{0} \text{ and } f_n'(0) = 1$$

Informally speaking, these examples together with the fact Riemann integral are closed under uniform convergence should give you some ideas that differentiation and integration although are operations inverse to each other, are NOT symmetric. There is a certain hierarchy on continuous functions on a fixed compact interval. Thus, we have the next Theorem in its form. Note that in application, the next Theorem only require us to prove f_n' uniformly converge, and doesn't require us to prove to where does it converge.

Theorem 1.2.1. (Uniform Convergence and Differentiation) Given a bounded interval $[a,b]$ and some sequence of function $f_n : [a,b] \to \mathbb{R}$ such that

(a) f_n' uniformly converge on (a,b)

(b) f_n are continuous on $[a,b]$

(c) $f_n(x_0) \to L$ for some $x_0 \in [a,b]$

Then

(a) f_n uniformly converge on $[a,b]$

(b) and

$$\Big(\lim_{n\to\infty} f_n\Big)'(x_0) = \lim_{n\to\infty} f_n'(x_0) \text{ on } (a,b)$$

Proof. We first prove

$$f_n \text{ uniformly converge on } [a,b] \tag{1.2}$$

Fix ϵ. We wish

$$\text{to find } N \text{ such that } \|f_n - f_m\|_\infty \leq \epsilon \text{ for all } n,m > N$$

Because $f_n(x_0)$ converge, and f_n' uniformly converge, we know there exists N such that

$$\begin{cases} |f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2} \\ \|f_n' - f_m'\|_\infty < \frac{\epsilon}{2(b-a)} \end{cases} \quad \text{for all } n,m > N \tag{1.3}$$

We claim

$$\text{such } N \text{ works}$$

Fix $x \in [a,b]$ and $n,m > N$. We first show

$$\big|(f_n - f_m)(x) - (f_n - f_m)(x_0)\big| \leq \frac{\epsilon}{2}$$

Because $(f_n - f_m)' = f_n' - f_m'$, by MVT and Equation 1.3, we can deduce

$$\begin{aligned} \big|(f_n - f_m)(x) - (f_n - f_m)(x_0)\big| &= \Big|\big[(f_n - f_m)'(t)\big](x - x_0)\Big| \text{ for some } t \text{ between } x, x_0 \\ &< \frac{\epsilon}{2(b-a)} \cdot |x - x_0| \\ &\leq \frac{\epsilon}{2(b-a)} \cdot (b-a) = \frac{\epsilon}{2} \quad \big(\because x, x_0 \in [a,b]\big) \text{ (done)} \end{aligned}$$

Now, by Equation 1.3, we have

$$\begin{aligned}\big|(f_n - f_m)(x)\big| &\leq \big|(f_n - f_m)(x) - (f_n - f_m)(x_0)\big| + \big|(f_n - f_m)(x_0)\big| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ (done)}\end{aligned}$$

Let $f : [a, b] \to \mathbb{R}$ be the limit of f_n. It remains to prove

$$f'(x) = \lim_{n\to\infty} f_n'(x) \text{ on } (a, b) \tag{1.4}$$

Fix $x \in (a, b)$ and define $\varphi, \varphi_n : [a, b] \setminus x \to \mathbb{R}$ by

$$\varphi(t) \triangleq \frac{f(t) - f(x)}{t - x} \text{ and } \varphi_n(t) \triangleq \frac{f_n(t) - f_n(x)}{t - x}$$

It is clear that $\varphi_n \to \varphi$ pointwise on $[a, b] \setminus x$. We now show

$$\varphi_n \to \varphi \text{ uniformly on } [a, b] \setminus x$$

Fix ϵ. We have

$$\text{to find } N \text{ such that } |\varphi_n(t) - \varphi_m(t)| \leq \epsilon \text{ for all } n, m > N \text{ and } t \in [a, b] \setminus x$$

Because f_n' uniformly converge on (a, b), we know there exists N such that

$$\|f_n' - f_m'\|_\infty \leq \epsilon \text{ for all } n, m > N \tag{1.5}$$

We claim

$$\text{such } N \text{ works}$$

Fix $n, m > N$ and $t \in [a, b] \setminus x$. We wish to prove

$$|\varphi_n(t) - \varphi_m(t)| \leq \epsilon$$

Because $(f_n - f_m)' = f_n' - f_m'$, by MVT and Equation 1.5, we can deduce

$$\begin{aligned}|\varphi_n(t) - \varphi_m(t)| &\leq \left|\frac{f_n(t) - f_n(x)}{t - x} - \frac{f_m(t) - f_m(x)}{t - x}\right| \\ &= \left|\frac{(f_n - f_m)(t) - (f_n - f_m)(x)}{t - x}\right| \\ &= \big|(f_n' - f_m')(t_0)\big| \text{ for some } t_0 \text{ between } t, x \\ &\leq \epsilon \text{ (done)}\end{aligned}$$

Note that

$$\lim_{n\to\infty}\lim_{t\to x}\varphi_n(t) = \lim_{n\to\infty} f'(x) \text{ exists}$$

We can now exchange the limit and see that the derivative of f at x exists.

$$\begin{aligned} f'(x) = \lim_{t\to x}\varphi(t) &= \lim_{t\to x}\lim_{n\to\infty}\varphi_n(t) \\ &= \lim_{n\to\infty}\lim_{t\to x}\varphi_n(t) = \lim_{n\to\infty} f_n'(x) \text{ (done)} \end{aligned}$$

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Theorem 1.2.2. (Uniform Convergence and Holomorphic) Given a real number r and some sequence of function $f_n : \overline{D_r(z_0)} \to \mathbb{C}$ such that

(a) f_n' uniformly converge on $D_r(z_0)$

(b) f_n are continuous on $\overline{D_r(z_0)}$

(c) $f_n(v) \to L$ for some $v \in \overline{D_r(z_0)}$

Then

(a) f_n uniformly converge on $\overline{D_r(z_0)}$

(b) and

$$\left(\lim_{n\to\infty} f_n\right)'(z) = \lim_{n\to\infty} f_n'(z) \text{ on } D_r(z_0)$$

Proof. We first prove

$$f_n \text{ uniformly converge on } \overline{D_r(z_0)} \tag{1.6}$$

Fix ϵ. We wish

$$\text{to find } N \text{ such that } \|f_n - f_m\|_\infty \leq \epsilon \text{ for all } n, m > N$$

Because $f_n(v)$ converge, and f_n' uniformly converge, we know there exists N such that

$$\begin{cases} |f_n(v) - f_m(v)| < \frac{\epsilon}{2} \\ \|f_n' - f_m'\|_\infty < \frac{\epsilon}{8r} \end{cases} \quad \text{for all } n, m > N \tag{1.7}$$

We claim

$$\text{such } N \text{ works}$$

Fix $z \in \overline{D_r(z_0)}$ and $n, m > N$. We first show

$$\big|(f_n - f_m)(z) - (f_n - f_m)(z_0)\big| \leq \frac{\epsilon}{2}$$

Denote $f_n - f_m : \overline{D_r(z_0)} \to \mathbb{C}$ by g. Because

$$|g(z) - g(z_0)| \leq \Big|\text{Re}\Big(g(z) - g(z_0)\Big)\Big| + \Big|\text{Im}\Big(g(z) - g(z_0)\Big)\Big|$$

WOLG, we only have to prove

$$\Big|\text{Re}\Big(g(z) - g(z_0)\Big)\Big| \leq \frac{\epsilon}{4}$$

Because $\overline{D_r(z_0)}$ is convex, we can define $h : [0,1] \to \mathbb{R}$ by

$$h(t) \triangleq \text{Re}\Big(g(tz + (1-t)z_0)\Big)$$

By Chain Rule and matrix representation of derivative, we see that for all $t \in (0,1)$

$$\begin{aligned} h'(t) = ac - bd \text{ where } z_0 - z = a + bi \\ \text{and } g'(tz + (1-t)(z_0)) = c + di \end{aligned}$$

Because $|a + bi| \leq r$ and $|c + di| \leq \frac{\epsilon}{8r}$ by Equation 1.7, if we use MVT, we see that

$$\begin{aligned} \Big|\text{Re}\Big(g(z) - g(z_0)\Big)\Big| = |h(1) - h(0)| &= |h(t)| \text{ for some } t \in (0,1) \\ &= |ac| + |bd| \leq \frac{\epsilon}{4} \text{ (done)} \end{aligned}$$

Now, by Equation 1.7, we have

$$\begin{aligned} \big|(f_n - f_m)(z)\big| &\leq \big|(f_n - f_m)(z) - (f_n - f_m)(v)\big| + \big|(f_n - f_m)(v)\big| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ (done)} \end{aligned}$$

Let $f : \overline{D_r(z_0)} \to \mathbb{C}$ be the limit of f_n. It remains to prove

$$f'(z) = \lim_{n\to\infty} f_n'(z) \text{ on } D_r(z_0) \tag{1.8}$$

Fix $z \in D_r(z_0)$ and define $\varphi, \varphi_n : \overline{D_r(z_0)} \setminus z \to \mathbb{R}$ by

$$\varphi(u) \triangleq \frac{f(u) - f(z)}{u - z} \text{ and } \varphi_n(u) \triangleq \frac{f_n(u) - f_n(z)}{u - z}$$

It is clear that $\varphi_n \to \varphi$ pointwise on $\overline{D_r(z_0)} \setminus z$. We now show

$$\varphi_n \to \varphi \text{ uniformly on } \overline{D_r(z_0)} \setminus z$$

Fix ϵ. We have

$$\text{to find } N \text{ such that } |\varphi_n(t) - \varphi_m(t)| \leq \epsilon \text{ for all } n, m > N \text{ and } t \in \overline{D_r(z_0)} \setminus z$$

Because f_n' uniformly converge on $D_r(z_0)$, we know there exists N such that

$$\|f_n' - f_m'\|_\infty \leq \frac{\epsilon}{4} \text{ for all } n, m > N \tag{1.9}$$

We claim

$$\text{such } N \text{ works}$$

Fix $n, m > N$ and $u \in \overline{D_r(z_0)}$. We wish to prove

$$|\varphi_n(u) - \varphi_m(u)| \leq \epsilon$$

Denote $f_n - f_m : \overline{D_r(z_0)} \to \mathbb{C}$ by g. Because

$$|\varphi_n(u) - \varphi_m(u)| = \left| \frac{g(u) - g(z)}{u - z} \right| \leq \frac{\left| \text{Re}\Big(g(u) - g(z)\Big) \right|}{|u - z|} + \frac{\left| \text{Re}\Big(g(u) - g(z)\Big) \right|}{|u - z|}$$

WOLG, we only have to prove

$$\frac{\left| \text{Re}\Big(g(u) - g(z)\Big) \right|}{|u - z|} \leq \frac{\epsilon}{2}$$

Again, define $h : [0, 1] \to \mathbb{R}$ by

$$h(t) \triangleq \text{Re}\Big(g(tu + (1 - t)z_0)\Big)$$

Then by Chain Rule and matrix representation of derivative, we see that for all $t \in (0, 1)$

$$\begin{aligned} h'(t) = ac - bd &\text{ where } u - z = a + bi \\ &\text{ and } g'(tu + (1 - t)(z)) = c + di \end{aligned}$$

Now, by MVT and Equation 1.9, we can deduce

$$\begin{aligned} \frac{\left| \text{Re}\Big(g(u) - g(z)\Big) \right|}{|u - z|} = \frac{|h(1) - h(0)|}{|u - z|} &= \frac{|h'(t)|}{|a + bi|} \text{ for some } t \in (0, 1) \\ &= \frac{|ac| + |bd|}{|a + bi|} \leq |c| + |d| \leq \frac{\epsilon}{2} \text{ (done)} \end{aligned}$$

Note that

$$\lim_{n\to\infty}\lim_{t\to x}\varphi_n(t) = \lim_{n\to\infty} f'(x) \text{ exists}$$

We can now exchange the limit and see that the derivative of f at x exists.

$$\begin{aligned} f'(x) = \lim_{t\to x}\varphi(t) &= \lim_{t\to x}\lim_{n\to\infty}\varphi_n(t) \\ &= \lim_{n\to\infty}\lim_{t\to x}\varphi_n(t) = \lim_{n\to\infty} f_n'(x) \text{ (done)} \end{aligned}$$

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1.3 Basic Technique on Sequence and Series

Abstract

This section prove some basic result on sequence and series, which will be heavily used in next section on analytic functions and Chapter: Beauty. Although written in an almost glossary form, we present the Theorems in a structural order based on the necessity of notion of absolute convergence and limit superior. Note that in this section, z, v, w always represent complex numbers, and a, b, c always represent real numbers.

Theorem 1.3.1. (Weierstrass M-test) Given sequences $f_n : X \to \mathbb{C}$, and suppose

$$\forall n \in \mathbb{N}, \forall x \in X, |f_n(x)| \leq M_n$$

Then

$$\sum_{n=1}^{\infty} M_n \text{ converge } \implies \sum_{n=1}^{\infty} f_n \text{ uniformly converge}$$

Proof. The proof follows from noting

$$\forall x \in X, \left| \sum_{k=m}^{n} f_k(x) \right| \leq \sum_{k=m}^{n} |f_k(x)| \leq \sum_{k=m}^{n} M_k$$

∎

Note that in our proof of Weierstrass M-test, we reduce the proof for uniform convergence into uniform Cauchy, which is a technique we shall also use later in Abel's test for uniform convergence. We now prove summation by part, which is a result hold in all fields, and is the essence of the proof of Dirichlet's test and Abel's test for uniform convergence.

Theorem 1.3.2. (Summation by Part)

$$\begin{aligned} f_n g_n - f_m g_m &= \sum_{k=m}^{n-1} f_k \Delta g_k + g_k \Delta f_k + \Delta f_k \Delta g_k \\ &= \sum_{k=m}^{n-1} f_k \Delta g_k + g_{k+1} \Delta f_k \end{aligned}$$

Proof. The proof follows induction which is based on

$$f_m g_m + f_m \Delta g_m + g_m \Delta f_m + \Delta f_m \Delta g_m = f_{m+1} g_{m+1}$$

∎

Theorem 1.3.3. (Dirichlet's Test) Suppose

(a) $a_n \to 0$ monotonically.

(b) $\sum_{n=1}^{N} z_n$ is bounded.

We have

$$\sum a_n z_n \text{ converge}$$

Proof. Define $Z_n \triangleq \sum_{n=1}^{N} z_n$ and let M bound $|Z_n|$. Using summation by part by letting $f_k = a_k$ and $g_k = Z_{k-1}$, we have

$$\begin{aligned}
\left|\sum_{k=m}^{n} a_k z_k\right| &= \left|a_{n+1}Z_n - a_m Z_{m-1} - \sum_{k=m}^{n} Z_k(a_{k+1}-a_k)\right| \\
&\leq |a_{n+1}Z_n| + |a_m Z_{m-1}| + \left|\sum_{k=m}^{n} Z_k(a_{k+1}-a_k)\right| \\
(\because a_n \text{ is monotone }) \qquad &\leq M\Big(|a_{n+1}| + |a_m| + |a_{n+1}-a_m|\Big)
\end{aligned}$$

■

Theorem 1.3.4. (Abel's Test for Uniform Convergence) Suppose $g_n : X \to \mathbb{R}$ is a uniformly bounded pointwise monotone sequence. Then given a sequence $f_n : X \to \mathbb{R}$,

$$\sum f_n \text{ uniformly converge } \implies \sum f_n g_n \text{ uniformly converge}$$

Proof. Define $R_n \triangleq \sum_{k=n}^{\infty} f_k$. Let M uniformly bound g_n. Because $R_n \to 0$ uniformly, we can let N satisfy

$$\forall n \geq N, \forall x \in X, |R_n(x)| < \frac{\epsilon}{6M}$$

Then for all $n, m \geq N$, using summation by part, we have

$$\begin{aligned}
\left|\sum_{k=m}^{n} f_k g_k\right| &= \left|\sum_{k=m}^{n} g_k \Delta R_k\right| \\
&\leq |R_{n+1}g_{n+1}| + |R_{m+1}g_{m+1}| + \sum_{k=m}^{n} |R_{k+1}\Delta g_k| \\
(\because g_n \text{ is pointwise monotone }) \qquad &\leq |R_{n+1}g_{n+1}| + |R_{m+1}g_{m+1}| + \frac{\epsilon}{6M}|g_{n+1}-g_m| \leq \epsilon
\end{aligned}$$

■

Although the proofs of Dirichlet's test and Abel's test for uniform convergence are quite similar, one should note that the "ways" summation by part is applied are slightly different, as one use $R_n \triangleq \sum_{k=n}^{\infty} f_k$ instead of $\sum_{k=1}^{n} f_k$, like $Z_n \triangleq \sum_{j=1}^{n} z_j$. As corollaries of Dirichlet's test, one have the famous **alternating series test** and Abel's test for complex series.

Theorem 1.3.5. (Abel's Test for Complex Series) Suppose

(a) $\sum z_n$ converge.

(b) b_n is a bounded monotone sequence.

We have

$$\sum z_n b_n \text{ converge}$$

Proof. Denote $B \triangleq \lim_{n\to\infty} b_n$. By Dirichlet's Test, we know $\sum z_n(b_n - B)$ converge. The proof now follows form noting

$$\sum z_n b_n = \sum z_n(b_n - B) + B\sum z_n$$

∎

We now introduce the idea of absolute convergence, which we shall use throughout the remaining of the section. By a **permutation** $\sigma : E \to E$ on some set E, we merely mean σ is a bijective function. We say $\sum z_n$ **absolutely converge** if $\sum |z_n|$ converge, and say $\sum z_n$ **unconditionally converge** if for all permutation $\sigma : \mathbb{N} \to \mathbb{N}$, the series $\sum z_{\sigma(n)}$ converge and converge to the same value.

Theorem 1.3.6. (Absolutely Convergent Series Unconditionally Converge)

$$\sum z_n \text{ absolutely converge } \implies \sum z_n \text{ unconditionally converge}$$

Proof. The fact $\sum z_n$ converge follows from noting

$$\left|\sum_{k=n}^{m} z_k\right| \leq \sum_{k=n}^{m} |z_k| \leq \sum_{k=n}^{\infty} |z_k|$$

Now, fix ϵ and permutation σ. Let N_1 and N_2 satisfy

$$\sum_{n=N_1}^{\infty} |z_n| < \frac{\epsilon}{2} \text{ and } \left|\sum_{n=N}^{\infty} z_n\right| < \frac{\epsilon}{2} \text{ for all } N > N_2$$

Let $M \triangleq \max\{N_1, N_2\}$. Observe that for all $N > \max_{1\leq r\leq M} \sigma^{-1}(r)$, we have

$$\left|\sum z_n - \sum_{n=1}^{N} z_{\sigma(n)}\right| \leq \left|\sum_{n=M+1}^{\infty} z_n\right| + \sum_{n=M+1}^{\infty} |z_n| < \epsilon$$

■

Theorem 1.3.7. (Riemann Rearrangement Theorem) If $\sum a_n$ converge but not absolutely, then for each $L \in \overline{\mathbb{R}}$, there exists a permutation σ such that

$$\sum a_{\sigma(n)} = L$$

Proof. Define a_n^+ and a_n^- by

$$a_n^+ \triangleq \max\{a_n, 0\} \text{ and } a_n^- \triangleq \min\{a_n, 0\}$$

Because

$$\sum (a_n^+ + a_n^-) \text{ converge but } \sum (a_n^+ - a_n^-) = \infty$$

We know

$$\sum a_n^+ = \sum (-a_n^-) = \infty$$

WOLG, (why?), fix $L \in \mathbb{R}$ and suppose $a_n \neq 0$ for all n. Let $A = B = L$, and let two increasing sequence $\sigma^+, \sigma^- : \mathbb{N} \to \mathbb{N}$ satisfy

$$\sigma^+(k+1) = \min\{n \in \mathbb{N} : a_n > 0 \text{ and } n > \sigma^+(k)\}$$

and similar for σ^-. Now, recursively define p_k, q_k by

$$p_1 \text{ is the smallest number such that } \sum_{n=1}^{p_1} a_{\sigma^+(n)} \geq A \tag{1.10}$$

$$q_1 \text{ is the smallest number such that } \sum_{n=1}^{p_1} a_{\sigma^+(n)} + \sum_{n=1}^{q_1} a_{\sigma^-(n)} \leq B \tag{1.11}$$

$$p_{k+1} \text{ is the smallest number such that } \sum_{n=1}^{p_{k+1}} a_{\sigma^+(n)} + \sum_{n=1}^{q_k} a_{\sigma^-(n)} \geq A \tag{1.12}$$

$$q_{k+1} \text{ is the smallest number such that } \sum_{n=1}^{p_{k+1}} a_{\sigma^+(n)} + \sum_{n=1}^{q_{k+1}} a_{\sigma^-(n)} \leq B \tag{1.13}$$

We then define σ by

$$\sigma^+(1), \dots, \sigma^+(p_1), \sigma^-(1), \dots, \sigma^-(q_1), \sigma^+(p_1+1), \dots, \sigma^+(p_2), \sigma^-(q_1+1), \dots, \sigma^-(q_2), \dots$$

It then follows from

$$\left|\sum_{n=1}^{p} a_{\sigma^+}(n) + \sum_{n=1}^{q_k} a_{\sigma^-}(n) - L\right| \leq \min\{a_{\sigma^+(p_{k+1})}, |a_{\sigma^-(q_k)}|\} \text{ for all } p_k \leq p \leq p_{k+1}$$

and $a_n \to 0$ that $\sum a_{\sigma(n)} = L$. ■

Note that the method we deploy in the proof of Riemann rearrangement Theorem can be used to control the sequence to have arbitrary large set of subsequential limits by modifying the number of A, B in Equation (4.1), (4.2), (4.3) and (4.4).

Using Riemann rearrangement Theorem and equation

$$\max_{1\leq r\leq d} |x_n| \leq |\mathbf{x}| \leq \sum_{r=1}^{d} |x_r|$$

we can now generalize and strengthen Theorem 1.3.6 to

$$\begin{aligned}
\sum \mathbf{x}_n \text{ absolutely converge } &\iff \sum_n x_{n,r} \text{ absolutely converge for all } r \\
&\iff \sum_n x_{n,r} \text{ unconditionally converge for all } r \\
&\iff \sum \mathbf{x}_n \text{ unconditionally converge}
\end{aligned}$$

With this in mind, we can now well state the Fubini's Theorem for Double Series.

Theorem 1.3.8. (Fubini's Theorem for Double Series) If

$$\sum_n \sum_k |z_{n,k}| \text{ converge}$$

Then

$$\sum_{n,k} |z_{n,k}| \text{ converge and } \sum_{n,k} z_{n,k} = \sum_n \sum_k z_{n,k} = \sum_k \sum_n z_{n,k}$$

Proof. The fact $\sum z_{n,k}$ absolutely converge follow from

$$\sum_{n=1}^{N} \sum_{k=1}^{N} |z_{n,k}| \leq \sum_n \sum_k |z_{n,k}| \text{ for all } N$$

WOLG, it remains to prove

$$\sum_{n,k} z_{n,k} = \sum_n \sum_k z_{n,k}$$

Because $\sum_n \sum_k |z_{n,k}|$ converge, we can reduce the problem into proving the same statement for nonnegative series $a_{n,k}$. (why?)

$$\sum_n \sum_k |a_{n,k}| \text{ converge } \implies \sum_{n,k} a_{n,k} = \sum_n \sum_k a_{n,k}$$

Because

$$\sum_{n=1}^N \sum_{k=1}^N a_{n,k} \le \sum_{n=1}^N \sum_k a_{n,k} \le \sum_n \sum_k a_{n,k} \text{ for all } N$$

we see

$$\sum_{n,k} a_{n,k} \le \sum_n \sum_k a_{n,k}$$

It remains to prove

$$\sum_{n,k} a_{n,k} \ge \sum_n \sum_k a_{n,k}$$

Fix N and ϵ. We reduce the problem into proving

$$\sum_{n,k} a_{n,k} \ge \sum_{n=1}^N \sum_k a_{n,k} - \epsilon$$

Let K satisfy

$$\text{For all } 1 \le n \le N, \sum_{k=K+1}^{\infty} a_{n,k} < \frac{\epsilon}{N}$$

It then follows

$$\sum_{n,k} a_{n,k} \ge \sum_{n=1}^N \sum_{k=1}^K a_{n,k} \ge \sum_{n=1}^N \sum_k a_{n,k} - \epsilon \text{ (done)}$$

$\blacksquare$

Example 5 (Counter-Example for Fubini's Theorem for Double Series)

$$a_{n,k} \triangleq \begin{cases} 1 & \text{if } n = k \\ -1 & \text{if } n = k+1 \\ 0 & \text{if otherwise} \end{cases}$$

$$\sum |a_{n,k}| = \infty \text{ and } \sum_n \sum_k a_{n,k} = 1 \text{ and } \sum_k \sum_n a_{n,k} = 0$$

Theorem 1.3.9. (Merten's Theorem for Cauchy Product) Suppose

(a) $\sum_{n=0}^{\infty} z_n$ converge absolutely

(b) $\sum_{n=0}^{\infty} z_n = Z$

(c) $\sum_{n=0}^{\infty} v_n = V$

(d) $w_n = \sum_{k=0}^{n} z_k v_{n-k}$

Then we have

$$\sum_{n=0}^{\infty} w_n = ZV$$

Proof. We prove

$$\left| V\sum_{n=0}^{N} z_n - \sum_{n=0}^{N} w_n \right| \to 0 \text{ as } N \to \infty$$

Compute

$$\begin{aligned} V\sum_{n=0}^{N} z_n - \sum_{n=0}^{N} w_n &= \sum_{n=0}^{N} z_n (V - \sum_{k=0}^{N-n} v_k) \\ &= \sum_{n=0}^{N} z_n \sum_{k=N-n+1}^{\infty} v_k \end{aligned}$$

Because $\sum_{k=n}^{\infty} v_k \to 0$ as $n \to \infty$, we know there exists M such that

$$\left| \sum_{k=n}^{\infty} v_k \right| < M \text{ for all } n$$

Let N_0 satisfy

$$\sum_{n=N_0+1}^{\infty} |z_n| < \frac{\epsilon}{2M}$$

Let $N_1 > N_0$ satisfy

$$\left| \sum_{k=N-N_0+1}^{\infty} v_k \right| < \frac{\epsilon}{2(N_0+1)\sum_n |z_n|} \text{ for all } N > N_1$$

Now observe that for all $N > N_1$

$$\left| \sum_{n=0}^{N} z_n \Big(\sum_{k=N-n+1}^{\infty} v_k \Big) \right| \le \sum_{n=0}^{N_0} |z_n| \left| \sum_{k=N-n+1}^{\infty} v_k \right| + \sum_{n=N_0+1}^{N} |z_n| \left| \sum_{k=N-n+1}^{\infty} v_k \right| < \epsilon \text{ (done)}$$

■

We first define the **limit superior** by

$$\limsup_{n\to\infty} a_n \triangleq \lim_{n\to\infty} (\sup_{k\ge n} a_k)$$

Note that $\limsup_{n\to\infty} a_n$ must exists because $(\sup_{k\ge n} a_k)_n$ is a decreasing sequence.

Theorem 1.3.10. (Equivalent Definition for Limit Superior) If we let E be the set of subsequential limits of a_n

$$E \triangleq \{L \in \overline{\mathbb{R}} : L = \lim_{k\to\infty} a_{n_k} \text{ for some } n_k\}$$

The set E is non-empty and

$$\max E = \limsup_{n\to\infty} a_n$$

Proof. Let $n_1 \triangleq 1$. Recursively, because

$$\sup_{j\ge n_k} a_k \ge \limsup_{n\to\infty} a_n > \limsup_{n\to\infty} a_n - \frac{1}{k} \text{ for each } k$$

We can let n_{k+1} be the smallest number such that

$$a_{n_{k+1}} > \limsup_{n\to\infty} a_n - \frac{1}{k}$$

It is straightforward to check $a_{n_k} \to \limsup_{n\to\infty} a_n$ as $k \to \infty$. Note that no subsequence can converge to $\limsup_{n\to\infty} a_n + \epsilon$ because there exists N such that $\sup_{k\ge N} a_k < \limsup_{n\to\infty} a_n + \epsilon$. ■

We can now state the **limit comparison test** as follows. Given a positive sequence b_n,

$$\limsup_{n\to\infty}\frac{|z_n|}{b_n}\in\mathbb{R} \text{ and } \sum b_n \text{ converge } \implies \sum z_n \text{ absolutely converge}$$
$$\liminf_{n\to\infty}\frac{b_n}{|z_n|}>0 \text{ and } \sum z_n \text{ diverge } \implies \sum b_n \text{ diverge}$$

Theorem 1.3.11. (Geometric Series)

$$|z|<1 \implies \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

Proof. The proof follows from noting

$$(1-z)\sum_{n=0}^{N} z^n = 1-z^{N+1}\to 1 \text{ as } N\to\infty$$

∎

Theorem 1.3.12. (Ratio and Root Test)

$$\limsup_{n\to\infty}\sqrt[n]{|z_n|}<1 \text{ or } \limsup_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right|<1 \implies \sum z_n \text{ absolutely converge}$$
$$\liminf_{n\to\infty}\sqrt[n]{|z_n|}>1 \text{ or } \liminf_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right|>1 \implies \sum z_n \text{ diverge}$$

Proof. The convergent part follows from comparison to an appropriate geometric series and the diverge part follows from noting $|z_n|$ does not converge to 0. ∎

Theorem 1.3.13. (Root Test is Stronger Than Ratio Test)

$$\liminf_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right|\le\liminf_{n\to\infty}\sqrt[n]{|z_n|}\le\limsup_{n\to\infty}\sqrt[n]{|z_n|}\le\limsup_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right|$$

Proof. Fix ϵ and WLOG suppose $\liminf_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right|>0$. We prove

$$\liminf_{n\to\infty}\sqrt[n]{|z_n|}\ge\liminf_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right|-\epsilon$$

Let $\alpha\in\mathbb{R}$ satisfy

$$\liminf_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right|-\epsilon<\alpha<\liminf_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right|$$

Let N satisfy

$$\text{For all } n \geq N, \left|\frac{z_{n+1}}{z_n}\right| > \alpha$$

We then see

$$\sqrt[N+n]{|z_{N+n}|} \geq \sqrt[N+n]{|z_N|\,\alpha^n} = \alpha\Big(\frac{|z_N|^{\frac{1}{N+n}}}{\alpha^{\frac{N}{N+n}}}\Big) \to \alpha \text{ as } n \to \infty \text{ (done)}$$

The proof for the other side is similar. ■

Theorem 1.3.14. (Root Test Trick) For all $k \in \mathbb{N}$

$$\limsup_{n\to\infty} |z_{n+k}|^{\frac{1}{n}} = \limsup_{n\to\infty} |z_n|^{\frac{1}{n}}$$

Proof. This is a direct corollary of equivalent definition for limit superior. ■

Lastly, we prove Cauchy's condensation Test, whose existence is almost solely for investigating p-Series.

Theorem 1.3.15. (Cauchy's Condensation Test) Suppose $a_n \searrow 0$. We have

$$\sum_{n=0}^{\infty} 2^n a_{2^n} \text{ converge } \iff \sum_{n=1}^{\infty} a_n \text{ converge}$$

Proof. Observe that for all $N \in \mathbb{N}$

$$\sum_{n=0}^{N} 2^n a_{2^n} \geq \sum_{n=0}^{N}\sum_{k=1}^{2^n} a_{2^n+k-1} = \sum_{n=1}^{2^{N+1}-1} a_n$$

and

$$2\sum_{n=1}^{2^N-1} a_n = 2\sum_{n=1}^{N}\sum_{k=0}^{2^{n-1}-1} a_{2^{n-1}+k} \geq 2\sum_{n=1}^{N} 2^{n-1}a_{2^n} = \sum_{n=1}^{N} 2^n a_{2^n}$$

■

Theorem 1.3.16. (p-Series)

$$\sum_{n=1}^{\infty}\frac{1}{n^p} \text{ converge } \iff p > 1$$

Proof. Observe that

$$\sum_{n=0}^{\infty} 2^n \frac{1}{(2^n)^p} = \sum_{n=0}^{\infty}(2^{1-p})^n$$

The result then follows from Cauchy's Condensation Test and geometric series. ■

1.4 Analytic Functions

Abstract

This section introduces the concept of analytic functions and proves some of their basic properties, including the Identity Theorem. We will rely on the tools developed in the previous section on sequences and series. Note that throughout this section, z will always denote a complex number.

In this section, by a **power series**, we mean a pair (z_0, c_n) where $z_0 \in \mathbb{C}$ is called the **center** of power series, and $c_n \in \mathbb{C}$ are the coefficients sequence. By **radius of convergence**, we mean a unique $R \in \mathbb{R}_0^+ \cup \infty$ such that

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n \begin{cases} \text{converge absolutely} & \text{if } |z - z_0| < R \\ \text{diverge} & \text{if } |z - z_0| > R \end{cases}$$

Such R always exist (and is unique, the uniqueness can be checked without computing the actual value of R) and is exactly

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{c_n}} \tag{1.14}$$

This result is called **Cauchy-Hadamard Theorem** and is proved by applying Root Test to $\sum c_n (z - z_0)^n$. Note that Cauchy-Hadamard Theorem does not tell us whether a power series converges at points of boundary of disk of convergence. It require extra works to determine if the power series converge at boundary.

Theorem 1.4.1. (Abel's Test for Power Series) Suppose $a_n \to 0$ monotonically and $\sum a_n z^n$ has radius of convergence R .

$$\text{The power series } \sum a_n z^n \text{ at least converge on } \overline{D_R(0)} \setminus \{R\}$$

Proof. Note that

$$\sum \frac{a_n}{R^n} z^n \text{ has radius of convergence } R$$

Fix $z \in \overline{D_R(0)} \setminus \{R\}$. Note that

$$\left| \sum_{n=0} \frac{z^n}{R^n} \right| = \left| \frac{1 - (\frac{z}{R})^{N+1}}{1 - \frac{z}{R}} \right| \leq \frac{2}{|1 - \frac{z}{R}|} \text{ for all } N$$

It then follows from Dirichlet's Test that $\sum a_n (\frac{z}{R})^n$ converge. ∎

Example 6 (Discussion of Convergence on Boundary)

$$f_q(z) = \sum_{n=0}^{\infty} n^q z^n \text{ provided } q \in \mathbb{R}$$

It is clear that f_q has convergence radius 1 for all $q \in \mathbb{R}$. For boundary, we have

$$\begin{cases} q < -1 \implies f_q \text{ converge on } S^1 \\ -1 \leq q < 0 \implies f_q \text{ converge on } S^1 \setminus \{1\} \\ 0 \leq q \implies f_q \text{ diverge on } S^1 \end{cases}$$

Note that

(a) At $z = 1$, the discussion is just p-series.

(b) $n^q \searrow 0$ if and only if $q < 0$; and if $n^q \searrow 0$, then the series converge by Abel's test for power series.

(c) If $q \geq 0$, $n^q z^n$ does not converge to 0 on $S^1 \setminus \{1\}$

Notice that the fact $\sum c_n(z-z_0)^n$ absolutely converge in $D_R(z_0)$ implies the convergence is uniform on all $\overline{D_{R-\epsilon}(z_0)}$ by M-Test. However, on $D_R(z_0)$, the convergence is not always uniform.

Example 7 (Failure of Uniform Convergence on $D_R(z_0)$)

$$f(z) = \sum_{n=0}^{\infty} z^n$$

Note $R = 1$. Use Geometric series formula to show $f(z) = \frac{1}{1-z}$ on $D_1(0)$. It is then clear that f is unbounded on $D_1(0)$ while all partial sums $\sum_{k=0}^{n} z^k$ is bounded on $D_1(0)$.

We now introduce some terminologies. We say a complex function f is **analytic at** $z_0 \in \mathbb{C}$ if f there exists a power series (z_0, c_n) whose convergence radius is greater than 0 and f agrees with $\sum_{n=0}^{\infty} c_n(z - z_0)^n$ on $D_R(z_0)$ for some R (of course, such R must not be strictly greater than the radius of convergence of (a, c_n)). It shall be quite clear that if f, g are both analytic at $z \in \mathbb{C}$ with radius $R_f \leq R_g$, then by Merten's Theorem for Cauchy product, $f + g$ and fg are analytic at z with radius at least R_f. We now investigate deeper into analytic functions.

Theorem 1.4.2. (Term by Term Differentiation) Given a power series (z_0, c_n) of convergence radius $R > 0$, if we define $f : D_R(z_0) \to \mathbb{C}$ by

$$f(z) \triangleq \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

Then f is holomorphic on $D_R(z_0)$ and its derivative at z_0 is also a power series with radius of convergence R

$$f'(z) = \sum_{n=0}^{\infty} (n+1) c_{n+1} (z - z_0)^n$$

Proof. Because $(n+1)^{\frac{1}{n}} \to 1$, we can use Theorem 1.3.14 to deduce

$$\limsup_{n\to\infty} \left((n+1)\left|c_{n+1}\right|\right)^{\frac{1}{n}} = \limsup_{n\to\infty} \left|c_{n+1}\right|^{\frac{1}{n}} = \limsup_{n\to\infty} \left|c_n\right|^{\frac{1}{n}}$$

which implies that the power series $\sum_{n=0}^{\infty}(n+1)c_{n+1}(z-z_0)^n$ is of radius of convergence R. We now prove

$$f'(z) = \sum_{n=0}^{\infty} (n+1) c_{n+1} (z - z_0)^n \text{ on } D_R(z_0)$$

Define $f_m : D_R(z_0) \to \mathbb{C}$ by

$$f_m(z) \triangleq \sum_{n=0}^{m} c_n (z - z_0)^n$$

Observe

(a) $f_m \to f$ pointwise on $D_R(a)$

(b) $f_m'(z) = \sum_{n=0}^{m-1}(n+1)c_{n+1}(z-z_0)^n$ for all m

Fix $z \in D_R(z_0)$. Proposition (b) allow us to reduce the problem into proving

$$f'(z) = \lim_{m\to\infty} f_m'(z) \text{ on } D_R(a) \tag{1.15}$$

Let $z \in D_r(z_0)$ where $r < R$. With proposition (a) in mind, to show Equation 1.15, by Theorem 1.2.2, we only have to prove f_m' uniformly converge on $D_r(z_0)$, which follows from M-Test and the fact that $\sum_{n=0}^{\infty}(n+1)c_{n+1}(z-z_0)^n$ absolutely converge on $D_R(z_0)$. (done) ∎

Suppose

$$f(z) \triangleq \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

Now by repeatedly applying Theorem 1.4.2, we see

$$f^{(k)}(z) = \sum_{n=0}^{\infty} (n+k)\cdots(n+1) c_{n+k} (z - z_0)^n \text{ for all } k \in \mathbb{Z}_0^+ \tag{1.16}$$

This then give us

$$c_k = \frac{f^{(k)}(z_0)}{k!} \text{ for all } k \in \mathbb{Z}_0^+ \tag{1.17}$$

and

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots \text{ on } D_R(z_0) \tag{1.18}$$

Equation 1.18 is often called the **Taylor expansion of f at** z_0. Notably, Equation 1.17 tell us that if f is constant 0, then $c_n = 0$ for all n.

Example 8 (Smooth but not Analytic Function)

$$f(x) = \begin{cases} e^{\frac{-1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Use induction to show that

$$f^{(k)}(x) = P_k(\frac{1}{x}) e^{-(\frac{1}{x})^2} \quad \exists P_k \in \mathbb{R}[x^{-1}], \forall k \in \mathbb{Z}_0^+, \forall x \in \mathbb{R}^*$$

and again use induction to show that

$$f^{(k)}(0) = 0 \quad \forall k \in \mathbb{Z}_0^+$$

The trick to show $f^{(k)}(0) = 0$ is let $u = \frac{1}{x}$.

Now, with Theorem 1.4.2, we see that f is not analytic at 0.

Example 9 (Bump Function)

$$f(x) = \begin{cases} e^{\frac{-1}{1-x^2}} & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

Use the same trick (but more advanced) to show f is smooth, and note that f is not analytic at ± 1.

Now, it comes an interesting question. Given a complex-valued function f analytic at z_0 with radius R, and suppose $z_1 \in D_R(z_0)$.

(a) Is f also analytic at z_1?

(b) What do we know about the radius of convergence of f at z_1?

(c) Suppose f is indeed analytic at z_1. It is trivial to see that the power series $(z_0, c_{0;n})$ and $(z_1, c_{1;n})$ must agree in the intersection of their convergence disks, and because f is given, we by Theorem 1.4.2 and Equation 1.17, have already known the value of $c_{1;n}$. Can we verify that the power series $(z_0, c_{0;n})$ and $(z_1, c_{1;n})$ do indeed agree with each other on the common convergence interval?

Taylor's Theorem for power series give satisfying answers to these problems.

Theorem 1.4.3. (Taylor's Theorem for Power Series) Given a function f analytic at z_0 with radius R, and suppose $z_1 \in D_R(z_0)$. Then

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_1)}{k!}(z - z_1)^k \text{ on } D_{R-|z_1-z_0|}(z_1)$$

Proof. WOLG, let $z_0 = 0$. Suppose z satisfy $|z - z_1| < R - |z_1|$. By Equation 1.17, we can compute

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(z - z_1 + z_1)^k \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \sum_{n=0}^{k} \binom{k}{n} (z - z_1)^n z_1^{k-n} \\ &= \sum_{k=0}^{\infty} \sum_{n=0}^{k} \frac{f^{(k)}(0)}{k!} \binom{k}{n} (z - z_1)^n z_1^{k-n} \end{aligned}$$

Note that

$$\begin{aligned}
\sum_{k=0}^{\infty}\left|\sum_{n=0}^{\infty}\frac{f^{(k)}(0)}{k!}\binom{k}{n}(z-z_1)^n z_1^{k-n}\right| &\leq \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\left|\frac{f^{(k)}(0)}{k!}\right|\binom{k}{n}|z-z_1|^n\cdot|z_1|^{k-n} \\
&= \sum_{k=0}^{\infty}\left|\frac{f^{(k)}(0)}{k!}\right|\sum_{n=0}^{\infty}\binom{k}{n}|z-z_1|^n\cdot|z_1|^{k-n} \\
&= \sum_{k=0}^{\infty}\left|\frac{f^{(k)}(0)}{k!}\right|\big(|z-z_1|+|z_1|\big)^k
\end{aligned}$$

is a convergent series, by Cauchy-Hadamard Theorem and $|z-z_1|+|z_1|<R$; thus, we can use Fubini's Theorem for double series to deduce

$$\begin{aligned}
\sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{f^{(k)}(0)}{k!}\binom{k}{n}(z-z_1)^n z_1^{k-n} &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{f^{(k)}(0)}{k!}\binom{k}{n}(z-z_1)^n z_1^{k-n} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}\binom{k}{n}(z-z_1)^n z_1^{k-n} \\
&= \sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{f^{(k)}(0)}{k!}\binom{k}{n}(z-z_1)^n z_1^{k-n} \\
&= \sum_{n=0}^{\infty}\Big[\sum_{k=n}^{\infty}\frac{f^{(k)}(0)}{k!}\binom{k}{n}z_1^{k-n}\Big](z-z_1)^n
\end{aligned}$$

We have reduced the problem into proving

$$\sum_{k=n}^{\infty}\frac{f^{(k)}(0)}{k!}\binom{k}{n}z_1^{k-n}=\frac{f^{(n)}(z_1)}{n!}$$

Because z_1 is in $D_R(0)$, by Equation 1.16 and Equation 1.17, we can compute

$$\begin{aligned}
f^{(n)}(z_1) &= \sum_{k=0}^{\infty}(k+n)\cdots(k+1)\cdot\frac{f^{(n+k)}(0)}{(n+k)!}z_1^k \\
&= \sum_{k=n}^{\infty}(k)\cdots(k-n+1)\cdot\frac{f^{(k)}(0)}{k!}\cdot z_1^{k-n} \\
&= \sum_{k=n}^{\infty}\frac{f^{(k)}(0)}{(k-n)!}z_1^{k-n}
\end{aligned}$$

We now have

$$\frac{f^{(n)}(z_1)}{n!} = \sum_{k=n}^{\infty} \frac{f^{(k)}(0)}{n!(k-n)!} z_1^{k-n} = \sum_{k=n}^{\infty} \frac{f^{(k)}(0)}{k!} \binom{k}{n} z_1^{k-n} \text{ (done)}$$

■

Lastly, to close this section, we prove the Identity Theorem, which is extremely useful in complex analysis.

Theorem 1.4.4. (Identity Theorem) Given two analytic complex-valued function $f, g : D \to \mathbb{C}$ defined on some open connected $D \subseteq \mathbb{C}$, if f, g agree on some subset $S \subseteq D$ such that S has a limit point in D, then f, g agree on the whole region D.

Proof. Define

$$T \triangleq \{z \in D : f^{(k)}(z) = g^{(k)}(z) \text{ for all } k \geq 0\}$$

Since D is connected, we can reduce the problem into proving T is non-empty, open and closed in D. Let c be a limit point of S in D. We first show

$$c \in T$$

Assume $c \notin T$. Let m be the smallest integer such that $f^{(m)}(c) \neq g^{(m)}(c)$. We can write the Taylor expansion of $f - g$ at c by

$$\begin{aligned}(f-g)(z) &= (z-c)^m \Big[\frac{(f-g)^{(m)}(c)}{m!} + \frac{(f-g)^{(m+1)}(c)}{(m+1)!}(z-c) + \cdots\Big] \\ &\triangleq (z-c)^m h(z)\end{aligned}$$

Clearly, $h(c) \neq 0$. Now, because h is continuous at c (h is a well-defined power series at c with radius greater than 0), we see h is non-zero on some $B_\epsilon(c)$, which is impossible, since $(f-g) \equiv 0$ on $S \setminus \{c\}$ implies $h = 0$ on $S \setminus \{c\}$. CaC (done)

Fix $z \in T$. Because f, g are analytic at z and $f^{(k)}(z) = g^{(k)}(z)$ for all k, we see $f - g$ is constant 0 on some open disk $B_\epsilon(z)$. We have proved that T is open. To see T is closed in D, one simply observe that

$$T = \bigcap_{k \geq 0} \{z \in D : (f-g)^{(k)}(z) = 0\}$$

and $(f-g)^{(k)}$ is continuous on D. (done) ■

1.5 Cauchy Integral Theorem

Abstract

Note that in this section, when we talk about derivative of function defined on subset of real line, we do consider one-sided derivative, i.e., for $\gamma : [a, b] \to \mathbb{C}$ to be C^1, the limit of $\frac{\gamma(a+h)-\gamma(a)}{h}$ as $h \searrow 0$ must exist.

Let $[a, b] \subseteq \mathbb{R}$ be some compact interval. We say $\gamma : [a, b] \to \mathbb{C}$ is a **parametrization** if

(a) $\gamma(x) \neq \gamma(y)$ unless $x = a$ and $y = b$.

(b) There exists some partition $\{a = c_0 < \cdots < c_N = b\}$ such that $\gamma|_{[c_n,c_{n+1}]} : [c_n, c_{n+1}] \to \mathbb{C}$ are C^1 wish non-vanishing derivative.

A parametrization $\gamma : [a, b] \to \mathbb{C}$ is said to be **closed** if $\gamma(a) = \gamma(b)$. Two parametrizations $\gamma : [a, b] \to \mathbb{C}, \alpha : [c, d] \to \mathbb{C}$ are said to be **equivalent** if there exists some C^1 bijection $s : [a, b] \to [c, d]$ such that

$$\gamma(t) = \alpha(s(t)) \text{ and } s'(t) > 0 \text{ for all } t \in [a, b]$$

Inverse Function Theorem shows that our definition for parametrization equivalence is indeed an equivalence relation. We then can define **contour** to be the equivalence class of parametrizations. Immediately, we see that all parametrization of a contour have the same image and if any of them is closed, then all of them are closed. This allow us to talk about the image of a contour and if a contour is closed. If we define **length** for parametrization $\gamma : [a, b] \to \mathbb{C}$ to be $\int_a^b \gamma'(t)dt$, then a change of variables shows that all parametrizations in $[\gamma]$ have the same length as γ. This allow us to define the **length for contour**. Now, given some parametrization $\gamma : [a, b] \to \mathbb{C}$ and some continuous complex-valued function f defined on the image $\gamma([a, b])$, we can define its **contour integral** by

$$\int_\gamma f(z)dz \triangleq \int_a^b f(\gamma(t))\gamma'(t)dt$$

Again the change of variables shows that our definition is well defined for contours.

Similar to the real case, we have the estimation

$$\left| \int_\gamma f dz \right| \leq LM \tag{1.19}$$

where L is the length of γ and M is the maximum of $|f|$ on γ. We can also generalize Part 2 of Fundamental Theorem of Calculus to contour integral: If $D \subseteq \mathbb{C}$ is open, $f : D \to \mathbb{C}$ is continuous, and $F : D \to \mathbb{C}$ satisfy $F'(z) = f(z)$ for all $z \in D$, then for all contour $\gamma : [a, b] \to D$, we have

$$\int_\gamma f dz = F(\gamma(b)) - F(\gamma(a))$$

We are now ready to state Cauchy's Integral Theorem for triangles. Note that term "closed triangle" as a set include both its interior area and boundary. For example, a closed triangle can be

$$\{x + iy \in \mathbb{C} : x \in [0, 1] \text{ and } y \in [0, x]\}$$

Theorem 1.5.1. (Cauchy's Integral Theorem for triangles) If $D \subseteq \mathbb{C}$ is open, $f : D \to \mathbb{C}$ is holomorphic and D contain some closed triangle T, then

$$\int_{\partial T} f dz = 0$$

Proof. Denote T by $T^{(0)}$. Construct triangles $T_1^{(1)}, T_2^{(1)}, T_3^{(1)}, T_4^{(1)}$ as in the figure below.

triangle.png

Obviously, we may parametrize the boundaries of these triangles so that

$$\int_{\partial T^{(0)}} f dz = \sum_{n=1}^{4} \int_{\partial T_n^{(1)}} f dz$$

Taking absolute value on both side, we deduce

$$\left| \int_{\partial T^{(0)}} f dz \right| \leq 4 \left| \int_{\partial T_j^{(1)}} f dz \right| \text{ for some } j \in \{1, 2, 3, 4\}$$

Denote $T_j^{(1)}$ by $T^{(1)}$. Repeating this process, we obtain a decreasing sequence of triangles

$$T^{(0)} \supseteq T^{(1)} \supseteq \cdots \supseteq T^{(n)} \supseteq \cdots$$

with the property that

$$\left| \int_{\partial T^{(0)}} f dz \right| \leq 4^n \left| \int_{\partial T^{(n)}} f dz \right| \tag{1.20}$$

Let $d^{(n)}$ and $p^{(n)}$ denote the diameter and perimeter of $T^{(n)}$ for all $n \in \mathbb{Z}_0^+$. Some tedious effort shows that

$$d^{(n)} = 2^{-n} d^{(0)} \text{ and } p^{(n)} = 2^{-n} p^{(0)} \tag{1.21}$$

Theorem **??** implies

$$\bigcap_{n \in \mathbb{N}} T^{(n)} = \{z_0\} \text{ for some } z_0 \in D$$

Because f is holomorphic at z_0, we may write $f : D \to \mathbb{C}$ by

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + o(z - z_0)(z - z_0)$$

Clearly the first two terms have antiderivatives. Using Equation 1.20 and Equation 1.21, we may now estimate

$$\begin{aligned} \left| \int_{\partial T^{(0)}} f(z) dz \right| \leq 4^n \left| \int_{\partial T^{(n)}} f(z) dz \right| &= 4^n \left| \int_{\partial T^{(n)}} o(z - z_0)(z - z_0) dz \right| \\ &\leq 4^n p^{(n)} d^{(n)} \max_{z \in \partial T^{(n)}} |o(z - z_0)| \\ &= p^{(0)} d^{(0)} \max_{z \in \partial T^{(n)}} |o(z - z_0)| \to 0 \text{ as } n \to \infty \end{aligned}$$

■

By $D \subseteq \mathbb{C}$ being **star-convex with center** z_*, we mean that for all $z \in D$, the contour $\gamma : [0,1] \to \mathbb{C}$ defined by

$$\gamma(t) \triangleq z_* + t(z - z_*)$$

satisfy $\gamma([0,1]) \subseteq D$.

Theorem 1.5.2. (Existence of antiderivative on star-convex domain) Suppose $D \subseteq \mathbb{C}$ is open and star-convex with centre z_*. If $f : D \to \mathbb{C}$ is holomorphic, then $F : D \to \mathbb{C}$ defined by

$$F(z) \triangleq \int_\gamma f(w)dw \text{ where } \gamma : [0,1] \to D \text{ is defined by } \gamma(t) \triangleq z_* + t(z - z_*)$$

is an antiderivative of f.

Proof. Fix $z_0 \in D$. Because D is open, there exists some open ball $B_\epsilon(z_0)$ small enough to be contained by D. For all $z \in B_\epsilon(z_0)$, the closed triangle T specified by the vertices $\{z_*, z, z_0\}$ is contained by D, since all $p \in T$ lies in some line segment joining z_* and w where w is some point that lies in the line segment joining z and z_0. We then can apply Cauchy's Integral Theorem for triangles to have the estimate

$$\begin{aligned}\left|\frac{F(z) - F(z_0)}{z - z_0} - f(z_0)\right| &= \left|\frac{\int_\gamma [f(w) - f(z_0)]dw}{z - z_0}\right| \\ &\leq \max_{w \in \gamma} |f(w) - f(z_0)| \to 0 \text{ as } z \to z_0\end{aligned}$$

where γ is the line segment traveling from z_0 to z. ∎

At this point, it is appropriate for us to define **the winding number** $w(\gamma, z_0)$ **of a contour** $\gamma : [a,b] \to \mathbb{C}$ **around some point** $z_0 \notin \gamma$ by

$$w(\gamma, z_0) \triangleq \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0} dz$$

Immediately, we see that our definition satisfy our geometric intuition in the sense that the circle $\gamma : [0, 2\pi] \to \mathbb{C}$ is defined by

$$\gamma(t) \triangleq z_0 + e^{it}$$

have winding number

$$w(\gamma, z_0) = \frac{1}{2\pi i} \int_0^{2\pi} e^{-it} i e^{it} dt = 1$$

Moreover, we expect any closed contour $\gamma : [a,b] \to \mathbb{C}$ to have integer-valued winding number. This is true. Consider $f : [a,b] \to \mathbb{C}$ defined by

$$f(t) \triangleq \frac{1}{2\pi i}\int_a^t \frac{\gamma'(s)}{\gamma(s)-z_0}ds$$

One may check by direct computation that

$$\frac{d}{dt}e^{-2\pi i f(t)}(\gamma(t)-z_0) \equiv 0$$

It then follows from γ being closed that

$$e^{-2\pi i f(a)} = e^{-2\pi i f(b)}$$

which implies

$$w(\gamma, z_0) = f(b) = f(a) + n = n \in \mathbb{Z}$$

Given some contour $\gamma : [a,b] \to \mathbb{C}$, if we define $g : \mathbb{C} \setminus \gamma \to \mathbb{C}$ by

$$g(z) \triangleq w(\gamma, z)$$

we see that g is continuous, since if $z_0 \notin \gamma$, we may find $D_r(z_0)$ disjoint with γ and obtain the estimate

$$\begin{aligned}
|w(\gamma,z_0) - w(\gamma,z_1)| &= \frac{1}{2\pi}\left|\int_\gamma \Big[\frac{1}{z-z_0} - \frac{1}{z-z_1}\Big]dz\right| \\
&= \frac{1}{2\pi}\left|\int_\gamma \frac{z_1-z_0}{(z-z_0)(z-z_1)}dz\right| \\
&\leq \frac{L\,|z_1-z_0|}{r^2\pi} \text{ where } L \text{ is the length of } \gamma
\end{aligned}$$

as long as $|z_0 - z_1| < \frac{r}{2}$. The continuity of g together with the fact that g can only be integer-valued implies that g is constant on any connected component of $\mathbb{C} \setminus \gamma$. We may now finally state our version of Cauchy Integral Theorem.

Theorem 1.5.3. (Cauchy Integral Theorem) Suppose $D \subseteq \mathbb{C}$ is open and $f : D \to \mathbb{C}$ is holomorphic. If $\gamma : [a,b] \to \mathbb{C}$ is a closed contour lying in D such that $w(\gamma, z) = 0$ for all $z \notin D$, then

$$\int_\gamma f dz = 0$$

Proof. As Prof Frank remarked, the proof is omitted here for being too long and tricky. ∎

Theorem 1.5.4. (Cauchy Integral Formula) Let $U \subseteq \mathbb{C}$ be open, D be an closed disk contained by U, and C be a closed contour running through the boundary of D counterclockwise. If $f : U \to \mathbb{C}$ is holomorphic and $a \in D^\circ$, then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

Proof. Fix ϵ. Let δ satisfy

$$|z-a| \leq \delta \implies |f(z) - f(a)| \leq \epsilon$$

With a geometric argument using Cauchy Integral Theorem, one have

$$\int_C \frac{f(z)}{z-a}dz = \int_\gamma \frac{f(z)}{z-a}dz$$

where $\gamma : [0, 2\pi] \to D^\circ$ is defined by

$$\gamma(t) \triangleq a + \delta e^{it}$$

The proof then follows from the estimation

$$\begin{aligned}
\left| f(a) - \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz \right| &= \left| f(a) - \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz \right| \\
&= \left| f(a) - \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+\delta e^{it})}{\delta e^{it}} i\delta e^{it}dt \right| \\
&= \frac{1}{2\pi}\left| \int_0^{2\pi} f(a+\delta e^{it}) - f(a)dt \right| \leq \epsilon
\end{aligned}$$

∎

Theorem 1.5.5. (Holomorphic functions are analytic) Let $U \subseteq \mathbb{C}$ be an open, D be an closed disk contained by U and centering a with radius R. Let C be a closed contour running through the boundary of D counterclockwise. If we define for all $n \geq 0$

$$c_n \triangleq \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

then the power series

$$\sum_{n=0}^\infty c_n (z-a)^n$$

agrees with f on D°.

Proof. Let $z \in D^\circ$. By Cauchy's Integral Formula, we have

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw \\
&= \frac{1}{2\pi i}\int_C f(w)\Big[\frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \cdots + \frac{(z-a)^m}{(w-a)^{m+1}} + \frac{(z-a)^{m+1}}{(w-a)^{m+1}(w-z)}\Big]dw \\
&= \sum_{n=0}^{m} c_n (z-a)^n + \frac{1}{2\pi i}\int_C \frac{(z-a)^{m+1}}{(w-a)^{m+1}(w-z)}dw
\end{aligned}$$

The proof then follows from noting $\left|\frac{z-a}{w-a}\right| < 1$ and direct estimation of Equation 1.19. ∎

What follows from the fact holomorphic functions are analytic and Taylor's Theorem for Power Series is that if f is holomorphic on some open disk $D_{r+\epsilon}(a)$ and C is the closed contour running through the boundary of $D_r(a)$ counterclockwise, then we have a nice formula

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

which agrees with our Cauchy integral Formula.

In particular, we often encounter complex function $f : \mathbb{C} \to \mathbb{C}$ that is **entire**, i.e., f is holomorphic on the whole $\mathbb{C}$. Theorem 1.5.5 states that for all $R > 0$, if we let C be the closed contour running through the boundary of the open disk centering 0 with radius R counterclockwise, then

$$f = \sum_{n=0}^{\infty} c_{n,R} z^n \text{ on the open disk with radius } R$$

for $c_{n,R}$ given in Theorem 1.5.5. Let $R_1 < R_2$. Trivially,

$$\sum_{n=0}^{\infty} c_{n,R_1} z^n = \sum_{n=0}^{\infty} c_{n,R_2} z^n \text{ on the open disk with radius } R_1$$

It then follows from Identity Theorem that

$$c_{n,R_1} = c_{n,R_2} \text{ for all } n$$

Letting $R \to \infty$, we may now just write

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \text{ on } \mathbb{C}$$

Theorem 1.5.6. (Liouville's Theorem) If entire $f : \mathbb{C} \to \mathbb{C}$ is bounded, then f is constant.

Proof. Write

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \text{ on } \mathbb{C}$$

By Theorem 1.5.5,

$$c_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z^{n+1}} dz$$

where C_r is the closed contour running through the boundary of the open disk centering the origin with radius r. Let $n \geq 1$, and let M be an upper bound of $|f|$. The proof then following from letting $r \to \infty$ in the below estimation

$$|c_n| = \frac{1}{2\pi} \cdot (2\pi r) \cdot \max_{C_r} \left| \frac{f(z)}{z^{n+1}} \right| \leq \frac{M}{r^n}$$

■

1.6 Residue Formula

Abstract

Before we begin developing this section, we first give the following remark. Let

$$0 < R_1 < r_1 < r_2 < R_2 < \infty$$

Let C_1, C_2 respectively be closed contours running through the boundary of $D_{r_1}(0), D_{r_2}(0)$. If f is holomorphic on on $R_1 < |z| < R_2$, then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

by a simple geometric argument. Therefore, we may give the next Theorem a well-defined statement.

Theorem 1.6.1. (Laurent's Theorem) Suppose

$$0 \leq R_1 < r < R_2 \leq \infty$$

C_r is a closed contour running through the circle centering z_0 with radius r counterclockwise, and f is holomorphic on the annulus $R_1 < |z - z_0| < R_2$. If we define

$$c_n \triangleq \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{(z - z_0)^{n+1}} dz \text{ for all } n \in \mathbb{Z}$$

Then $\sum_{n \geq 0} c_n h^n$ converge for $|h| < R_2$, $\sum_{n<0} c_n h^n$ converge for $|h| > R_1$ and

$$f(z_0 + h) = \sum_{n \in \mathbb{Z}} c_n h^n \text{ on the annulus } R_1 < |h| < R_2$$

Proof. Fix some $h \in \mathbb{C}$ such that $R_1 < |h| < R_2$. Let r_1, r_2 satisfy

$$R_1 < r_1 < |h| < r_2 < R_2$$

And let C_{r_1}, C_{r_2} respectively be closed contours running through the circle centering z_0 with radius r_1, r_2 counterclockwise. We first show that

$$f(z_0 + h) = \frac{1}{2\pi i} \int_{C_{r_1}} \frac{f(z)}{z - (z_0 + h)} dz - \frac{1}{2\pi i} \int_{C_{r_2}} \frac{f(z)}{z - (z_0 + h)} dz$$

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With Laurent's Theorem, given holomorphic $f : D_\epsilon(z_0) \setminus \{z_0\} \to \mathbb{C}$, we may write

$$f(z) = \sum_{n \in \mathbb{Z}} c_n (z - z_0)^n \text{ for } 0 < |z - z_0| < \epsilon$$

If $c_n = 0$ for all $n < 0$, obviously we may define $f(z_0) \triangleq c_0$, so that f is still holomorphic after such extension. If this is the case, we say f has a **removable singularity at** z_0.

Theorem 1.6.2. (Riemann's removable singularity Theorem) Given holomorphic $f : D_\epsilon(z_0) \setminus \{z_0\} \to \mathbb{C}$

$$z_0 \text{ is a removable singularity of } f \iff f \text{ is bounded on some } D_\delta(z_0) \setminus \{z_0\}$$

Proof. From left to right is clear. Suppose f is bounded by M on $D_\delta(z_0) \setminus \{z_0\}$. Because

$$c_n = \frac{1}{2\pi i} \int_{C_r} f(z)(z - z_0)^{-n-1} dz$$

We may estimate

$$|c_n| \leq \frac{1}{2\pi}(2\pi r)(M r^{-n-1}) = M r^{-n}$$

The proof then follows from letting $n < 0$ and $r \to 0$. ∎

If for some $m < 0$, we have

$$c_m \neq 0 \text{ and } c_n = 0 \text{ for all } n \leq m$$

then we say f has a **pole of order** m **at** z_0. If a pole is of order 1, we say such pole is a **simple pole**.

Theorem 1.6.3. (Recognition of Pole) Given holomorphic $f : D_\epsilon(z_0) \setminus \{z_0\} \to \mathbb{C}$

$$f \text{ has a pole of order } m \text{ at } z_0 \iff \lim_{z \to z_0} (z - z_0)^m f(z) \in \mathbb{C}^*$$

Proof. From left to right is clear. From right to left, define $g : D_\epsilon(z_0) \setminus \{z_0\} \to \mathbb{C}$ by

$$g(z) \triangleq (z - z_0)^m f(z)$$

From premise, we may deduce g is bounded on some $D_\delta(z_0) \setminus \{z_0\}$. Therefore, by Theorem 1.6.2, g has a removable singularity at z_0. Thus, we may write

$$g(z) = \sum_{n=0}^{\infty} a_n z^n$$

where $a_0 \neq 0$ by premise. The proof then follows from writing f in terms of a_n. ∎

Writing holomorphic $f : D_\epsilon(z_0) \setminus \{z_0\} \to \mathbb{C}$ in terms of Laurent Series

$$f(z) = \sum_{n\in\mathbb{Z}} c_n(z - z_0)^n$$

We define the **residue of f at** z_0 to be

$$\text{Res}(f, z_0) \triangleq c_{-1}$$

If there are infinitely many $n < 0$ such that $c_n \neq 0$, we say f has a **essential singularity** at z_0. Obviously, if f has a removable singularity at z_0, then the residue of f at z_0 is 0. Immediately, from direct computation with Laurent expansion of $f : D_\epsilon(z_0) \setminus \{z_0\} \to \mathbb{C}$, we may conclude the following easy ways to compute residue for relatively simple function.

Theorem 1.6.4. (Calculation of Residue) If holomorphic $p, q : D_\epsilon(z_0) \to \mathbb{C}$ satisfies

$$p(z_0) \neq 0 \text{ and } q(z_0) = 0, q'(z_0) \neq 0$$

Then $f = \frac{p}{q}$ has a simple pole at z_0 and

$$\text{Res}(f, z_0) = \frac{p(z_0)}{q'(z_0)}$$

If $f : D_\epsilon(z_0) \setminus \{z_0\} \to \mathbb{C}$ has a pole of order m at z_0, then

$$\text{Res}(f, z_0) = \lim_{z\to z_0} \frac{1}{(m-1)!}\Big(\frac{d}{dz}\Big)^{m-1}(z - z_0)^m f(z)$$

Theorem 1.6.5. (Cauchy's Residue Theorem) Suppose that f is holomorphic in an open set containing a positively oriented Jordan curve γ and its interior, except for poles at the points $z_1, \dots, z_N$ inside γ. Then

$$\int_\gamma f(z)dz = 2\pi i \sum_{n=1}^{N} \text{Res}(f, z_n)$$

Using the residue theorem, we may compute the following integral that will definitely exists in context of Fourier analysis.

Example 10 (Poisson Kernel)

$$\int_0^\pi \frac{\log x}{1 + x^2} dx$$

Let $D \subseteq \mathbb{C}$ be open connected, $f : D \to \mathbb{C}$ be holomorphic. Suppose f does not vanish identically on D. Identity Theorem tell us that the set of **zeros**

$$Z \triangleq \{z \in D : f(z) = 0\}$$

must have no limit point in D. Therefore, for each zero $z_0 \in Z$, when we write locally

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

there exists some smallest $n \geq 1$ such that $c_n \neq 0$. We then say f has a **zero of order n at z_0**, and if $n = 1$, we say this zero is **simple**.

Let $z_0 \in \mathbb{C}$. Given some holomorphic $f : D_\epsilon(z_0) \setminus \{z_0\} \to \mathbb{C}$, we say f has a **removable singularity at z_0** if we may define $f(z_0)$ to be some complex number so that $f : D_\epsilon(z_0) \to \mathbb{C}$ is holomorphic. Suppose $f : D_\epsilon(z_0) \setminus \{z_0\} \to \mathbb{C}^*$. We say f has a **pole of order n at z_0** if the function $g : D_\epsilon(z_0) \to \mathbb{C}$ defined by

$$g(z) \triangleq \begin{cases} \frac{1}{f(z)} & \text{if } z \neq z_0 \\ 0 & \text{if } z = z_0 \end{cases}$$

is holomorphic with the order n at zero z_0. Obviously, there exists an unique holomorphic $h : D_\epsilon(z_0) \to \mathbb{C}$ such that

$$g(z) = (z - z_0)^n h(z)$$

It is clear that h does not vanish on $D_\epsilon(z_0)$. Therefore, we may write

$$G(z) = \frac{1}{h(z)}$$

so that $G : D_\epsilon(z_0) \to \mathbb{C}$ is holomorphic and write

$$f(z) = \frac{G(z)}{(z - z_0)^n} = \sum_{k=-n}^{\infty} a_k (z - z_0)^k$$

We then call

$$\sum_{k=-n}^{-1} a_k (z - z_0)^k$$

the **principal part of f at the pole** z_0 and

$$\mathrm{Res}_{z_0} f \triangleq a_{-1}$$

the **residue of f at the pole** z_0. It is clear from the Laurent expansion of f and direct computation that

$$\mathrm{Res}_{z_0} f =$$

1.7 Script

Question 1

There are three types of isolated singularities: removable singularity, poles and essential singularity. Provide the definition of each type and give an example.

Proof. Let $f : D_\epsilon(z_0) \setminus \{z_0\} \to \mathbb{C}$ be holomorphic so that f has an isolated singularity at z_0. By Laurent's Theorem, we may express f as a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

If $c_n = 0$ for all negative n, say

$$f(z) = (z - z_0)^2 + 7 \text{ on } D_\epsilon(z_0) \setminus \{z_0\}$$

we say f has a removable singularity at z_0. If there are only finite numbers of negative n such that $c_n \neq 0$, and m is the smallest integer such that

$$c_n \neq 0 \text{ for } n = m$$

we say f has a pole of order m at z_0. For example, f can be

$$f(z) = \frac{1}{z - z_0} \text{ on } D_\epsilon(z_0) \setminus \{z_0\}$$

If there are infinite number of negative n such that $c_n \neq 0$, for example,

$$f(z) = e^{\frac{1}{(z-z_0)}} = \sum_{n=0}^{\infty} \frac{1}{n!}(z - z_0)^{-n} \text{ on } D_\epsilon(z_0) \setminus \{z_0\}$$

we say f has an essential singularity at z_0. ■

Question 2

State the definition of the residue of a function at an isolated singularity.

Proof. Let $f : D_\epsilon(z_0) \setminus \{z_0\} \to \mathbb{C}$ be holomorphic so that f has an isolated singularity at z_0. By Laurent's Theorem, we may express f as a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

The residue of f at z_0 is defined to be c_{-1}.

$$\text{Res}(f, z_0) \triangleq c_{-1}$$

■

Question 3

If z_0 is a simple pole of f, prove that

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

Proof. Because z_0 is a simple pole of f, we may write

$$f(z) = [\operatorname{Res}(f, z_0)](z - z_0)^{-1} + \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

This give us

$$\begin{aligned}
\lim_{z \to z_0} (z - z_0) f(z) &= \lim_{z \to z_0} \operatorname{Res}(f, z_0) + \sum_{n=0}^{\infty} c_n (z - z_0)^{n+1} \\
&= \operatorname{Res}(f, z_0) + \sum_{n=0}^{\infty} \lim_{z \to z_0} c_n (z - z_0)^{n+1} \\
&= \operatorname{Res}(f, z_0)
\end{aligned}$$

∎

Question 4

If $f(z) = \frac{q(z)}{p(z)}$ where p and q are holomorphic, and z_0 is a simple zero of p, prove that

$$\operatorname{Res}(f, z_0) = \frac{q(z_0)}{p'(z_0)}$$

Proof. Because z_0 is a simple zero of p, we know

$$p'(z_0) \neq 0$$

If $q(z_0) \neq 0$, then f has a simple pole at z_0, and from result of last question we may compute

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} \frac{(z - z_0) q(z)}{p(z) - p(z_0)} = \frac{q(z_0)}{p'(z_0)}$$

∎

Question 5

If z_0 is a pole of f of order m, prove that

$$\text{Res}(f, z_0) = \lim_{z\to z_0} \frac{1}{(m-1)!}\Big(\frac{d}{dz}\Big)^{m-1} g(z)$$

where $g(z) = (z - z_0)^m f(z)$

Proof. Because z_0 is a pole of f of order m, we may write

$$f(z) = \sum_{n=-m}^{\infty} c_n (z - z_0)^n \text{ and } g(z) = \sum_{n=0}^{\infty} c_{n-m}(z - z_0)^n$$

Or write

$$g(z) = c_{-m} + c_{-m+1}(z - z_0) + \cdots + c_{-1}(z - z_0)^{m-1} + c_0(z - z_0)^m + \cdots$$

Compute

$$g^{(m-1)}(z) = \frac{(m-1)!}{0!}c_{-1} + \frac{m!}{1!}c_0(z - z_0) + \frac{(m+1)!}{2!}c_1(z - z_0)^2 + \cdots$$

This give us

$$\lim_{z\to z_0} \frac{1}{(m-1)!} g^{(m-1)}(z) = c_{-1} = \text{Res}(f, z_0)$$

■

Question 6

Find the residue of the following function at the indicated points:

(a) $\frac{z}{(2-3z)(4z+3)}$ at $z = \frac{2}{3}$.

(b) $\frac{z - \frac{1}{6}z^3 - \sin z}{z^8}$ at $z = 0$.

Proof. For the first function, observe

$$\lim_{z\to \frac{2}{3}} (z - \frac{2}{3}) \cdot \frac{z}{(2-3z)(4z+3)} = \lim_{z\to \frac{2}{3}} \frac{z}{-3(4z+3)} = \frac{2}{-51} \neq 0$$

which implies $z = \frac{2}{3}$ is a simple pole. Therefore, from the question earlier, we may deduce its residue is exactly $\frac{2}{-51}$. For the second function, we can just compute its Laurent series by

$$\frac{z-\frac{1}{6}z^3-\sin z}{z^8}=\frac{1}{z^7}-\frac{1}{6z^5}-\frac{1}{z^8}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}$$
$$=\frac{1}{5!}z^{-3}+\frac{-1}{7!}z^{-1}+\frac{1}{9!}z+\cdots$$

Therefore, its residue is $\frac{-1}{7!}$. ■

Question 7

Evaluate

$$\int_{-\pi}^{\pi}\frac{d\theta}{5+3\cos\theta}$$

Proof. Define $\gamma:[-\pi,\pi]\to\mathbb{C}$ by

$$\gamma(t)=e^{it}$$

We have

$$\begin{aligned}
I\triangleq -2i\int_\gamma\frac{dz}{3z^2+10z+3}&=\int_\gamma\frac{dz}{\frac{3i}{2}z^2+5iz+\frac{3}{2}i}\\
&=\int_\gamma\frac{1}{5+\frac{3}{2}(z+\frac{1}{z})}\cdot\frac{1}{iz}dz\\
&=\int_{-\pi}^{\pi}\frac{1}{5+\frac{3}{2}(e^{it}+e^{-it})}\cdot\frac{ie^{it}}{ie^{it}}dt\\
&=\int_{-\pi}^{\pi}\frac{1}{5+3\cos t}dt
\end{aligned}$$

In summary

$$\int_{-\pi}^{\pi}\frac{d\theta}{5+3\cos\theta}=-2i\int_\gamma\frac{dz}{3z^2+10z+3}$$

Note that

$$\frac{1}{3z^2+10+3}=\frac{1}{(3z+1)(z+3)}$$

have two simple poles $z=-3$ and $z=\frac{1}{-3}$, and have residue $\frac{1}{8}$ at pole $z=\frac{1}{-3}$. It then follows from residue theorem that

$$\int_{-\pi}^{\pi}\frac{d\theta}{5+3\cos\theta}=-2i\cdot 2\pi i\cdot\frac{1}{8}=\frac{\pi}{2}$$

■

Question 8

Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$$

Proof. Let $S_R : [0,\pi] \to \mathbb{C}$ be the semi-circle $S_R(t) \triangleq Re^{it}$, $L_R : [-R,R] \to \mathbb{C}$ be the line segment $L_R(t) \triangleq t$, and γ the wedge $\gamma = S_R + L_R$. By residue theorem,

$$\int_\gamma \frac{dz}{z^2+2z+2} = 2\pi i \cdot \frac{1}{2i} = \pi$$

for large enough R. Observe

$$\left| \int_{S_R} \frac{dz}{z^2+2z+2} \right| \le \pi R \cdot \max_{S_R} \frac{1}{|z+1-i| \cdot |z+1+i|} \le \frac{\pi R}{(R-\sqrt{2})^2}$$

This implies

$$\lim_{R\to\infty} \int_{S_R} \frac{dz}{z^2+2z+2} = 0$$

Therefore,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} &= \lim_{R\to\infty} \int_{L_R} \frac{dz}{z^2+2z+2} \\ &= \lim_{R\to\infty} \pi - \int_{S_R} \frac{dz}{z^2+2z+2} = \pi \end{aligned}$$

■

Question 9

Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+2x^2+1}$$

Proof. Let $S_R : [0,\pi] \to \mathbb{C}$ be the semi-circle $S_R(t) \triangleq Re^{it}$, $L_R : [-R,R] \to \mathbb{C}$ be the line segment $L_R(t) \triangleq t$, and γ the wedge $\gamma = S_R + L_R$. By residue theorem,

$$\int_\gamma \frac{dz}{z^4+2z^2+1} = 2\pi i \lim_{z\to i} \frac{d}{dz} \frac{1}{(z+i)^2} = \frac{\pi}{2}$$

for large enough R. Observe

$$\left|\int_{S_R} \frac{dz}{z^4+2z^2+1}\right| \leq \pi R \cdot \max_{S_R} \frac{1}{|z+i|^2 \cdot |z-i|^2} \leq \frac{\pi R}{(R-1)^4}$$

This implies

$$\lim_{R\to\infty} \int_{S_R} \frac{dz}{z^4+2z^2+1} = 0$$

Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dx}{x^4+2x^2+1} &= \lim_{R\to\infty} \int_{L_R} \frac{dz}{z^4+2z^2+1} \\ &= \lim_{R\to\infty} \frac{\pi}{2} - \int_{S_R} \frac{dz}{z^4+2z^2+1} = \frac{\pi}{2}\end{aligned}$$

$\blacksquare$