Deadline : 2023/10/30, 17:00.

If you want to finish this homework, please recall Abel's formula and Dirichlet's test.

1. Suppose that $\sum_{k=1}^{\infty} a_k$ converges and that $b_k \searrow b$ as $k \to \infty$. Prove that $\sum_{k=1}^{\infty} a_k b_k$ converges.

2. Show that under the hypotheses of Dirichlet's test,

$$\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty} s_k (b_k - b_{k+1}),$$

   where $s_k = \sum_{j=1}^{k} a_j$.

3. Suppose that $\sum_{k=1}^{\infty} a_k$ converges. Prove that if $b_k \nearrow \infty$ and $\sum_{k=1}^{\infty} a_k b_k$ converges, then

$$b_m \sum_{k=m}^{\infty} a_k \to 0$$

   as $m \to \infty$.

4. Suppose that $a_k > 0$ and $\sum_{k=1}^{\infty} a_k$ converges. Prove that there exist b_k such that $\lim_{k\to\infty} \frac{b_k}{a_k} = \infty$ and $\sum_{k=1}^{\infty} b_k$ converges.

5. Suppose that $a_k > 0$ and $\sum_{k=1}^{\infty} a_k$ diverges. Prove that there exist b_k such that $\lim_{k\to\infty} \frac{b_k}{a_k} = 0$ and $\sum_{k=1}^{\infty} b_k$ diverges.

6. Prove that

$$\sum_{k=1}^{\infty} a_k \cos(kx)$$

   converges for every $x \in (0, 2\pi)$ and every $a_k \searrow 0$. What happens when $x = 0$?

7. Prove that

$$\sum_{k=1}^{\infty} a_k \sin((2k+1)x)$$

   converges for every $x \in \mathbb{R}$ and every $a_k \searrow 0$.

8. Suppose that $\sum_{k=1}^{\infty} a_k^2$ and $\sum_{k=1}^{\infty} b_k^2$ converges. Prove that the following series

   1. $\sum_{k=1}^{\infty} |a_k b_k|$.

   2. $\sum_{k=1}^{\infty} (a_k + b_k)^2$.

   3. $\sum_{k=1}^{\infty} \frac{|a_k|}{k}$

   converge.

9. Does the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k \ln(k+1)}{k}$$

   converge? Does it converge absolutely? Justify your answer.

10. Find all values of $p \in \mathbb{R}$ make following series converges absolutely.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\ln^p(k)}$$

11. Let a_k and b_k be real sequences. Decide which of the following statements are true and which are false. Prove the true ones and given counterexamples to the false ones.

    1. if $a_k \searrow 0$, as $k \to \infty$, and $\sum_{k=1}^{\infty} b_k$ converges conditionally, then $\sum_{k=1}^{\infty} a_k b_k$ converges.

    2. if $a_k \to 0$, as $k \to \infty$, then $\sum_{k=1}^{\infty} (-1)^k a_k$ converges.

    3. if $a_k \to 0$, as $k \to \infty$, and $a_k \geq 0$ for all $k \in \mathbb{N}$, then $\sum_{k=1}^{\infty} (-1)^k a_k$ converges.

    4. if $a_k \to 0$, as $k \to \infty$, and $\sum_{k=1}^{\infty} (-1)^k a_k$ converges, then $a_k \searrow 0$ as $k \to \infty$.

Extra question

(If you finish there problems and want to obtain extra points, please email symmetrickelly@gmail.com)

This week don't have any extra question. But please review all of your proof of homework $1 \sim 3$ and discuss it with others. By the way, the extra question is already open to everyone who want to get bonus.

After correcting homework 2, none can get A+ on question 8. Most of the shortcomings are as follows:

1. Lack of detail in the writing (please double-check many properties of your example/ counterexample).
2. Confused logical reasoning or your goal (practice consistently using 'because..., ...' to achieve the desired results).
3. Inappropriate use of textbook theorems (please be mindful of using conditions).
4. Insufficiently clever methods."

"In addition, when writing proofs, please emulate the proofs in Book "William R Wade's "An Introduction to Analysis". Clearly state all matters, provide details, maintain a continuous logical reasoning process, and ensure that the text is well spaced and legible for clarity. Thank you."