Deadline : 2022/11/23, 17:00.

1. Prove that for all $x > 0$ and all positive integers n,

$$e^x > 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

where $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$.

2. (a) Let f be differentiable on $(0, \infty)$ with $f(xy) = f(x) + f(y)$ for all $x, y > 0$. Prove that

$$f(x) = C \ln x \quad \text{for some constant } C.$$

Hint:

(i) Show that $f(1) = 0$.
(ii) Show that $f'(y) = \frac{C}{y}$ for any y and some constant C.
(iii) Let $g(x) = f(x) - C\ln x$ and consider $g'(x)$.

(b) Let f be differentiable on $\mathbb{R}$ with $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Prove that

$$f(x) \equiv 0 \text{ or } f(x) = e^{Cx} \quad \text{for some constant } C.$$

Hint:

(i) Show that $f(0) = 0$ or 1.
(ii) When $f(0) = 0$, show that $f(0) \equiv 0$.
(iii) When $f(0) = 1$, show that $f(x) \neq 0$ and $f(x) > 0$ for all x.
(iv) Show that $f'(x) = Cf(x)$ for some constant C.
(v) By using the result $\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}$ to complete this problem.

3. **The error function** The *error function*

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

is useed in probability, statistics, and engineering.

(a) Show that $\int_a^b e^{-t^2}\, dt = \frac{1}{2}\sqrt{\pi}[\operatorname{erf}(b) - \operatorname{erf}(a)]$.
(b) Show that the function $y = e^{x^2}\operatorname{erf}(x)$ satisfies the differential equation $y' = 2xy + \frac{2}{\sqrt{\pi}}$.

4. If $f(x) = 3 + x + e^x$, find $(f^{-1})'(4)$.

5. The geologist C. F. Richter defined the magnitude of an earthquake to be $\log_{10}(I/S)$, where I is the intensity of the quake (measured by the amplitude of a seismograph 100 km from the epicenter) and S is the intensity of a 'standard' earthquake (where the amplitude is only 1 micron $= 10^{-4}$ cm).

   The 1989 Loma Prieta earthquake that shook San Francisco had a magnitude of 7.1 on the Richter scale. The 1906 San Francisco earthquake was 16 times as intense. What was its magnitude on the Richter scale?

6. The work done by a gas when it expands from volume V_1 to volume V_2 is $W = \int_{V_1}^{V_2} P\,dV$, where $P = P(V)$ is the pressure as a function of the volume V. (See textbook Exercise 5.4.29.)

   Boyle's Law states that when a quantity of gas expands at constant temperature, $PV = C$, where C is a constant. If the initial volume is 600 cm^3 and the initial pressure is 150 kPa, find the work done by the gas when it expands at constant temperature to 1000 cm^3.

7. The half-life of cesium-137 is 30 years. Suppose we have a 100-mg sample.

   (a) Find the mass that remains after t years.

   (b) How muvh of the sample remains after 100 years?

   (c) After how long will only 1 mg remain?

8. A ladder 5 m long leans against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 m/s, how fast is the angle between the ladder and the wall changing when the bottom of the ladder is 3 m from the base of the wall?

9. Compute

$$(a)\ \lim_{x\to\infty} x \sin\frac{\pi}{x} \qquad (b)\ \lim_{x\to 1^+} \tan(\frac{\pi x}{2}) \ln x$$

$$(c)\ \lim_{x\to 1} x\left(\frac{x}{x-1} - \frac{1}{\ln x}\right) \qquad (d)\ \lim_{x\to 0^+} x^{\sqrt{x}}$$

$$(e)\ \lim_{x\to 0^+} (4x+1)^{\cot x} \qquad (f)\ \lim_{x\to\infty} (e^x + x)^{\frac{1}{x}}$$

$$(g)\ \lim_{x\to\infty} e^{-x^2} \int_0^x e^{t^2}\, dt$$

10. (a) Compute $\int (\ln x)^2\,dx$.

    (b) Let n be a positive integer. Show that

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx.$$

11. (**87' Calculus Exam**) Calculate

$$\int \frac{\cos x}{\sqrt{4-\cos^2 x}}\,dx.$$

12. (**90' Calculus Exam**)

    (a) Use integration by parts to show that if f has an inverse with continuous first derivative, then

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int x(f^{-1})'(x)\,dx.$$

    (b) If $\int f(x)\,dx = F(x)+C$, express $\int f^{-1}(x)\,dx$ in terms of F.

    (c) Calculate $\int \tan^{-1} x\,dx$.

13. Calculate

$(a)\ \int \frac{\sin^{-1}(\ln x)}{x}\,dx$ $\quad(b)\ \int \cos(\ln x)\,dx$ $\quad(c)\ \int_1^{2e} x^2(\ln x)^2\,dx$

$(d)\ \int_0^1 \ln(1+x^2)\,dx$ $\quad(e)\ \int \frac{x+4}{x^2+2x+5}\,dx$ $\quad(f)\ \int_0^a \frac{1}{(a^2+x^2)^{\frac{3}{2}}}\,dx,\ a>0$

$(g)\ \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx$ $\quad(h)\ \int_0^a x^2\sqrt{a^2-x^2}\,dx$ $\quad(i)\ \int \frac{x^2}{(3+4x-4x^2)^{\frac{3}{2}}}\,dx$