Chapter 5

Differential Geometry HW

5.1 HW1

Abstract

In this HW, we give precise definition to $\mathbb{P}^n$ and $\mathbb{R}P^n$, and we rigorously show

(a) $\mathbb{R}P^n$ has a smooth structure.

(b) $\mathbb{P}^n$ is homeomorphic to $\mathbb{R}P^n$

(c) $\mathbb{P}^n$ has a smooth structure.

We also solved the other two questions. Note that in this PDF, brown text is always a clickable hyperlink reference.

Define an equivalence relation on $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ by

$$\mathbf{x} \sim \mathbf{y} \overset{\triangle}{\iff} \mathbf{y} = \lambda \mathbf{x} \text{ for some } \lambda \in \mathbb{R}^*$$

Let $\mathbb{R}P^n \triangleq (\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}) \setminus \sim$ be the quotient space and let

$$V_i \triangleq \{\mathbf{x} \in \mathbb{R}^{n+1} : \mathbf{x}^i \neq 0\} \text{ for each } 1 \leq i \leq n+1$$

By definition, it is clear that

$$\text{either } \pi^{-1}(\pi(\mathbf{x})) \subseteq V_i \text{ or } \pi^{-1}(\pi(\mathbf{x})) \subseteq V_i^c$$

Then if we define $\phi_i : V_i \to \mathbb{R}^n$ by

$$\phi_i(\mathbf{x}) \triangleq \left(\frac{\mathbf{x}^1}{\mathbf{x}^i}, \dots, \frac{\mathbf{x}^{i-1}}{\mathbf{x}^i}, \frac{\mathbf{x}^{i+1}}{\mathbf{x}^i}, \dots, \frac{\mathbf{x}^{n+1}}{\mathbf{x}^i}\right)$$

because $\pi(\mathbf{x}) = \pi(\mathbf{y}) \implies \phi_i(\mathbf{x}) = \phi_i(\mathbf{y})$, we can well induce a map

$$\Phi_i : U_i \triangleq \pi(V_i) \subseteq \mathbb{R}P^n \to \mathbb{R}^n; \pi(\mathbf{x}) \mapsto \phi_i(\mathbf{x})$$

Note that one has the equation

$$\Phi_i(\pi(\mathbf{x})) = \phi_i(\mathbf{x}) \text{ for all } \mathbf{x} \in V_i$$

Theorem 5.1.1. (Real Projective Space with a differentiable atlas) We have

$$\mathbb{R}P^n \text{ with atlas } \big\{(U_i, \Phi_i) : 1 \leq i \leq n+1\big\} \text{ is a differentiable manifold}$$

Proof. We are required to prove

(a) (U_i, Φ_i) are all charts.

(b) $\big\{(U_i, \Phi_i) : 1 \leq i \leq n+1\big\}$ form a differentiable atlas.

(c) $\mathbb{R}P^n$ is Hausdorff.

(d) $\mathbb{R}P^n$ is second-countable.

Because $\pi^{-1}(U_i) = V_i$ and V_i is clearly open in $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$, we know $U_i \subseteq \mathbb{R}P^n$ is open. Note that clearly, $\Phi_i(U_i) = \mathbb{R}^n$. To show (U_i, Φ_i) is a chart, it remains to show that Φ_i is a homeomorphism between U_i and $\mathbb{R}^n$. It is straightforward to check Φ_i is one-to-one on U_i. This implies Φ_i is a bijective between U_i and $\mathbb{R}^n$.

Fix open $E \subseteq \mathbb{R}^n$. We see

$$\pi^{-1}(\Phi_i^{-1}(E)) = \phi_i^{-1}(E)$$

Then because $\phi_i : V_i \to \mathbb{R}^n$ is clearly continuous, we see $\phi_i^{-1}(E)$ is open in $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$, and it follows from definition of quotient topology $\Phi_i^{-1}(E) \subseteq \mathbb{R}P^n$ is open. Then because U_i is open in $\mathbb{R}P^n$, we see $\Phi_i^{-1}(E)$ is open in U_i. We have proved $\Phi_i : U_i \to \mathbb{R}^n$ is continuous.

Define $\Psi_i : \mathbb{R}^n \to V_i$ by

$$\Psi(\mathbf{x}^1, \dots, \mathbf{x}^n) = (\mathbf{x}^1, \dots, \mathbf{x}^{i-1}, 1, \mathbf{x}^i, \dots, \mathbf{x}^n)$$

Observe that for all $\mathbf{x} \in \Phi_i(U_i)$, we have

$$\Phi_i^{-1}(\mathbf{x}) = \pi(\Psi_i(\mathbf{x}))$$

It then follows from $\Psi_i : \mathbb{R}^n \to V_i$ and $\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{R}P^n$ are continuous that $\Phi_i^{-1} : \mathbb{R}^n \to \mathbb{R}P^n$ is continuous.

We have proved that (Ψ_i, U_i) are all charts. Now, because V_i clearly cover $\mathbb{R}^{n+1}$, we know U_i also cover $\mathbb{R}P^n$. We have proved $\{(U_i, \Phi) : 1 \le i \le n+1\}$ form an atlas. The fact $\mathbb{R}P^n$ is second-countable follows.

Fix $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \Phi_i(U_i \cap U_j)$. We compute

$$\begin{aligned}\Phi_j \circ \Phi_i^{-1}(\mathbf{x}^1, \dots, \mathbf{x}^n) &= \Phi_j\Big([(\mathbf{x}^1, \dots, \mathbf{x}^{i-1}, 1, \mathbf{x}^i, \mathbf{x}^{i+1}, \dots, \mathbf{x}^n)]\Big) \\ &= \begin{cases} (\frac{\mathbf{x}^1}{\mathbf{x}^j}, \dots, \frac{\mathbf{x}^{j-1}}{\mathbf{x}^j}, \frac{\mathbf{x}^{j+1}}{\mathbf{x}^j}, \dots, \frac{\mathbf{x}^{i-1}}{\mathbf{x}^j}, \frac{1}{\mathbf{x}^j}, \frac{\mathbf{x}^i}{\mathbf{x}^j}, \dots, \frac{\mathbf{x}^n}{\mathbf{x}^j}) & \text{if } j < i \\ (\frac{\mathbf{x}^1}{\mathbf{x}^{j-1}}, \dots, \frac{\mathbf{x}^{i-1}}{\mathbf{x}^{j-1}}, \frac{1}{\mathbf{x}^{j-1}}, \frac{\mathbf{x}^i}{\mathbf{x}^{j-1}}, \dots, \frac{\mathbf{x}^{j-2}}{\mathbf{x}^{j-1}}, \frac{\mathbf{x}^j}{\mathbf{x}^{j-1}}, \dots, \frac{\mathbf{x}^n}{\mathbf{x}^{j-1}}) & \text{if } j > i \end{cases}\end{aligned}$$

This implies our atlas is indeed differentiable.

Before we prove $\mathbb{R}P^n$ is Hausdorff, we first prove that $\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{R}P^n$ is an open mapping. Let $U \subseteq \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ be open. Observe that

$$\pi^{-1}(\pi(U)) = \{t\mathbf{x} \in \mathbb{R}^{n+1} : t \neq 0 \text{ and } \mathbf{x} \in U\}$$

Fix $t_0\mathbf{x} \in \pi^{-1}(\pi(U))$. Let $B_\epsilon(\mathbf{x}) \subseteq U$. Observe that

$$B_{|t_0|\epsilon}(t_0\mathbf{x}) \subseteq t_0 B_\epsilon(\mathbf{x}) \subseteq t_0 U \subseteq \pi^{-1}(\pi(U))$$

This implies $\pi^{-1}(\pi(U))$ is open. (done)

Now, because π is open, to show $\mathbb{R}P^n$ is Hausdorff, we only have to show

$$R_\pi \triangleq \{(\mathbf{x}, \mathbf{y}) \in (\mathbb{R}^{n+1} \setminus \{\mathbf{0}\})^2 : \pi(\mathbf{x}) = \pi(\mathbf{y})\} \text{ is closed}$$

Define $f : (\mathbb{R}^{n+1} \setminus \{\mathbf{0}\})^2 \to \mathbb{R}$ by

$$f(\mathbf{x}, \mathbf{y}) \triangleq \sum_{i \neq j} (\mathbf{x}^i \mathbf{y}^j - \mathbf{x}^j \mathbf{y}^i)^2$$

Note that f is clearly continuous and $f^{-1}(0) = R_\pi$, which finish the proof. ∎

Alternatively, we can characterize $\mathbb{R}P^n$ by identifying the antipodal pints on $S^n \triangleq \{\mathbf{x} \in \mathbb{R}^{n+1} : |\mathbf{x}| = 1\}$ as one point

$$\mathbf{x} \sim \mathbf{y} \overset{\triangle}{\iff} \mathbf{x} = \mathbf{y} \text{ or } \mathbf{x} = -\mathbf{y}$$

and let $\mathbb{P}^n \triangleq S^n \backslash \sim$ be the quotient space.

Theorem 5.1.2. (Equivalent Definitions of Real Projective Space)

$$\mathbb{R}P^n \text{ and } \mathbb{P}^n \text{ are homeomorphic}$$

Proof. Define $F : \mathbb{P}^n \to \mathbb{R}P^n$ by

$$\{\mathbf{x}, -\mathbf{x}\} \mapsto \{\lambda \mathbf{x} : \lambda \in \mathbb{R}^*\}$$

It is straightforward to check that F is well-defined and bijective. Define $f : S^n \to \mathbb{R}P^n$ by

$$f = \pi \circ \mathbf{id}$$

where $\mathbf{id} : S^n \to \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ and $\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{R}P^n$ are continuous. Check that

$$f = F \circ p$$

where $p : S^n \to \mathbb{P}^n$ is the quotient mapping. It now follows from the universal property that F is continuous, and since $\mathbb{P}^n$ is compact and $\mathbb{R}P^n$ is Hausdorff, it also follows that F is a homeomorphism between $\mathbb{R}P^n$ and $\mathbb{P}^n$. ■

Knowing that $F : \mathbb{P}^n \to \mathbb{R}P^n$ is a homeomorphism and $\mathbb{R}P^n$ is a smooth manifold, we see that $\mathbb{P}^n$ is Hausdorff and second-countable, and if we define the atlas

$$\{(F^{-1}(U_i), \Phi_i \circ F) : 1 \leq i \leq n+1\}$$

We see this atlas is indeed smooth, since

$$(\Phi_i \circ F) \circ (\Phi_j \circ F)^{-1} = \Phi_i \circ \Phi_j^{-1}$$

Question 73

Let X be a set equipped with

(a) a collection $(U_\alpha)_{\alpha \in I}$ of subsets that covers X.

(b) a collection of bijection $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ that maps U_α to an open subset $\phi_\alpha(U_\alpha)$ of $\mathbb{R}^n$.

(c) For each $\alpha, \beta \in I$, the set $\phi_\alpha(U_\alpha \cap U_\beta)$ is open.

(d) For each $\alpha, \beta \in I, \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\beta \cap U_\alpha) \to \phi_\beta(U_\alpha, U_\beta)$ is smooth.

Give X a topology so that X is a smooth manifold.

Proof. If we define $E \subseteq X$ is open if and only if

$$\phi_\alpha(U_\alpha \cap E) \text{ is open for all } \alpha$$

we see that given arbitrary collection of open sets $(E_j)_{j\in J}$, we have

$$\phi_\alpha(U_\alpha \cap \bigcup_{j\in J} E_j) = \bigcup_{j\in J} \phi_\alpha(U_\alpha \cap E_j) \text{ for all } \alpha \in I$$

thus open sets are closed under union, and given two open sets E_1, E_2, we have

$$\phi_\alpha(U_\alpha \cap E_1 \cap E_2) \subseteq \phi_\alpha(U_\alpha \cap E_1) \cap \phi_\alpha(U_\alpha \cap E_2) \text{ for all } \alpha \in I$$

Note that if $\mathbf{x} \in \phi_\alpha(U_\alpha \cap E_1) \cap \phi_\alpha(U_\alpha \cap E_2)$, then there exists $p_1 \in U_\alpha \cap E_1$ and $p_2 \in U_\alpha \cap E_2$ such that $\phi_\alpha(p_1) = \phi_\alpha(p_2) = \mathbf{x}$. Because ϕ_α is one-to-one, we can deduce $p_1 = p_2 \in E_2$, it then follows

$$\mathbf{x} = \phi(p_1) \in \phi_\alpha(U_\alpha \cap E_1 \cap E_2)$$

We now see

$$\phi_\alpha(U_\alpha \cap E_1) \cap \phi_\alpha(U_\alpha \cap E_2) \subseteq \phi_\alpha(U_\alpha \cap E_1 \cap E_2) \text{ for all } \alpha \in I$$

We have proved that our topology on X is well-defined.

Note that U_α is open in X follows from premise (c). Thus, if some $E \subseteq U_\alpha$ is open in U_α, then E is open in X and $\phi_\alpha(E) = \phi_\alpha(U_\alpha \cap E)$ is open. We have proved that $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is an open mapping. The fact that $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is continuous trivially follows from

(a) U_α is open in X.

(b) our definition of topology on X.

(c) $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a bijection.

We have proved that (U_α, ϕ_α) are indeed charts. The fact that they form a smooth atlas follows from premise (a) and (d). ■

Question 74

Let $\mathbb{R}$ be the real line with the differentiable structure given by the maximal atlas of the chart $(\mathbb{R}, \varphi = \mathbf{id} : \mathbb{R} \to \mathbb{R})$, where id is the identity map, and let $\mathbb{R}'$ be the real line with the differentiable structure given by the maximal atlas of the chart $(\mathbb{R}', \psi : \mathbb{R}' \to \mathbb{R})$, where $\psi(x) = x^{1/3}$.

(a) Show that these two differentiable structures are distinct.

(b) Show that there is a diffeomorphism between $\mathbb{R}$ and $\mathbb{R}'$. (Hint: The identity map $\mathbb{R} \to \mathbb{R}$ is not the desired diffeomorphism.)

Proof. To see these two smooth structure are distinct. Observe

$$\mathbf{id}^{-1} \circ \psi : \mathbb{R} \to \mathbb{R}; x \mapsto x^{\frac{1}{3}} \text{ is not smooth at } 0$$

We claim $\phi : \mathbb{R} \mapsto \mathbb{R}'$ defined by

$$\phi(x) \triangleq x^3 \text{ is a diffeomorphism}$$

It is clear that ϕ is a homeomorphism. To see ϕ is a smooth mapping from $\mathbb{R}$ to $\mathbb{R}'$, observe that

$$\psi \circ \phi \circ \mathbf{id}^{-1}(x) = x$$

To see ϕ^{-1} is a smooth mapping from $\mathbb{R}'$ to $\mathbb{R}$, observe that

$$\mathbf{id} \circ \phi \circ \psi^{-1}(x) = x$$

We have proved that ϕ is a diffeomorphism between $\mathbb{R}$ and $\mathbb{R}'$. ∎

5.2 Appendix

Theorem 5.2.1. (Homeomorphism between Compact Space and Hausdorff Space) Suppose

(a) X is compact.

(b) Y is Hausdorff.

(c) $f : X \to Y$ is a continuous bijective function.

Then

$$f \text{ is a homeomorphism between } X \text{ and } Y$$

Proof. Because closed subset of compact set is compact and continuous function send compact set to compact set, we see for each closed $E \subseteq X$, $f(E) \subseteq Y$ is compact. The result then follows from $f(E) \subseteq Y$ being closed since Y is Hausdorff. ■

Theorem 5.2.2. (Hausdorff and Quotient) If $\pi : X \to Y$ is an open mapping, and we define

$$R_\pi \triangleq \{(x, y) \in X^2 : \pi(x) = \pi(y)\}$$

Then

$$R_\pi \text{ is closed} \iff Y \text{ is Hausdorff}$$

Proof. Suppose R_π is closed. Fix some x, y such that $\pi(x) \neq \pi(y)$. Because R_π is closed, we know there exists open neighborhood U_x, U_y such that $U_x \times U_y \subseteq (R_\pi)^c$. It is clear that $\pi(U_x), \pi(U_y)$ are respectively open neighborhood of $\pi(x)$ and $\pi(y)$. To see $\pi(U_x)$ and $\pi(U_y)$ are disjoint, assume that $\pi(a) \in \pi(U_x) \cap \pi(U_y)$. Let $a_x \in U_x$ and $a_y \in U_y$ satisfy $\pi(a_x) = \pi(a) = \pi(a_y)$, which is impossible because $(a_x, a_y) \in (R_\pi)^c$. CaC

Suppose Y is Hausdorff. Fix some x, y such that $\pi(x) \neq \pi(y)$. Let U_x, U_y be open neighborhoods of $\pi(x), \pi(y)$ separating them. Observe that $(x, y) \in \pi^{-1}(U_x) \times \pi^{-1}(U_y) \subseteq (R_\pi)^c$ ■