Calculus HW3

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1.

Proof. $y = x^4 - x^2$

$y' = 4x^3 - 2x$

$y'' = 12x^2 - x$

$\kappa = \frac{|y''|}{(1+(y')^2)^{\frac{3}{2}}}$

$\kappa(0) = \frac{2}{1} = 2$

The radius is then $\frac{1}{2}$

$y' = 0 \iff x = 0$ or $\pm\frac{1}{\sqrt{2}}$

$\forall x \in (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), y < 0 = y(0)$, since $y = (x^2 - 1)x^2$

So $y(0)$ is a local maximum

Then the circle is below the curve

Equation of the circle: $(y + \frac{1}{2})^2 + x^2 = \frac{1}{4}$

■

2.

2.(a)

Proof. $r(t) = tR(t)$

$v(t) = r'(t) = R(t) + tR'(t) = (\cos\omega t, \sin\omega t) + t\mathbf{v}_d$

■

2.(b)

Proof. $a(t) = v'(t) = \frac{d}{dt}(R(t) + tR'(t)) = R'(t) + R'(t) + tR''(t) = 2R'(t) + tR''(t) = 2v_d + a_d$ ■

2.(c)

Proof. $r(t) = (e^{-t}\cos\omega t, e^{-t}\sin\omega t)$

$R(t) = (\frac{e^{-t}\cos\omega t}{t}, \frac{e^{-t}\sin\omega t}{t})$

$v_d = 2R'(t)$

$R'(t) = (\frac{\frac{de^{-t}\cos\omega t}{dt}t - e^{-t}\cos\omega t}{t^2}, \frac{\frac{de^{-t}\sin\omega t}{dt}t - e^{-t}\sin\omega t}{t^2})$

$t(\frac{d}{dt}e^{-t}\cos\omega t) = t(-e^{-t}\cos\omega t - e^{-t}\sin\omega t)$

$t(\frac{d}{dt}e^{-t}\sin\omega t) = t(-e^{-t}\sin\omega t + e^{-t}\cos\omega t)$

$v_d = (2\frac{t(-e^{-t}\cos\omega t - e^{-t}\sin\omega t) - e^{-t}\cos\omega t}{t^2}, 2\frac{t(-e^{-t}\sin\omega t + e^{-t}\cos\omega t) - e^{-t}\sin\omega t}{t^2})$ ■

3.

3.(a)

Proof. $r(t) = (R\cos\omega t, R\sin\omega t)$

$v = r'(t) = (-\omega R\sin\omega t, \omega R\cos\omega t)$

$v \cdot r = \sqrt{(-\omega R^2 + \omega R^2)\cos\omega t\sin\omega t} = 0$

We see $v = \omega\begin{bmatrix}\cos\frac{\pi}{2} & -\sin\frac{\pi}{2}\\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2}\end{bmatrix} r$

So v indeed point counter clockwise ■

3.(b)

Proof. If $\omega < 0$, we can find t small enough to see the particle is moving clockwise
$|v| = \sqrt{\omega^2R^2(\sin^2\omega t + \cos^2\omega t)} = \sqrt{\omega^2R^2} = \omega R$

The period of time T to finish one revolution is the distance, circumference $2\pi R$, divided by the speed $|v|$

So $T = \frac{2\pi R}{|v|} = \frac{2\pi R}{\omega R} = \frac{2\pi}{\omega}$ ■

3.(c)

Proof. $a(t) = v'(t) = (-\omega^2R\cos\omega t, -\omega^2R\sin\omega t) = -\omega^2r(t)$

From equation above, it clearly point toward the origin

$|a| = \omega^2|r| = \omega^2R$

■

3.(d)

Proof. $|F| = m|a| = m\omega^2 R = m\frac{\omega^2 R^2}{R} = m\frac{|v|^2}{R}$ ■

4.

4.(a)

Proof. $r(t) = \langle (v_0 \cos\alpha)t, (v_0 \sin\alpha)t - \frac{1}{2}gt^2 \rangle$

$r'_y(t) = v_0 \sin\alpha - gt$

The maximum height is at $t = \frac{v_0 \sin\alpha}{g}$

$r_y(\frac{v_0 \sin\alpha}{g}) = \frac{v_0 \sin^2\alpha}{g} - \frac{1}{2}g\frac{v_0^2 \sin^2\alpha}{g^2} = \frac{v_0^2 \sin^2\alpha}{2g}$

The maximum of the maximum height then is when $\sin\alpha$ is the maximum, that is $\alpha = \frac{\pi}{2}$

So the maximum of the maximum height is $\frac{v_0^2}{2g}$ ■

4.(b),(c)

Enclosed hand writing

5.

Proof. $r(t)$ lies in a plane if and only if $\exists m, p, q, d \in \mathbb{R}, \forall t \in \mathbb{R}, m(a_1t^2 + b_1t + c_1) + p(a_2t^2 + b_2t + c_2) + q(a_3t^2 + b_3t + c_3) - d = 0$ **The equation**

$m(a_1t^2 + b_1t + c_1) + p(a_2t^2 + b_2t + c_2) + q(a_3t^2 + b_3t + c_3) - d = 0 \iff (ma_1 + pa_2 + qa_3)t^2 + (mb_1 + pb_2 + qb_3)t + (mc_1 + pc_2 + qc_3 - d) = 0$

Write $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} m \\ p \\ q \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$
$(ma_1 + pa_2 + qa_3)t^2 + (mb_1 + pb_2 + qb_3)t + (mc_1 + pc_2 + qc_3 - d) = d_1t^2 + d_2t + (d_3 - d)$

So $\exists m, p, q, d \in \mathbb{R}, \forall t \in \mathbb{R}, m(a_1t^2 + b_1t + c_1) + p(a_2t^2 + b_2t + c_2) + q(a_3t^2 + b_3t + c_3) - d = 0 \iff \exists m, p, q, d \in \mathbb{R}, \forall t \in \mathbb{R}, d_1t^2 + d_2t + (d_3 - d) = 0$

We want to find $d_1 = d_2 = 0$, where (m, p, q) is nontrivial

So we can pick $d = d_3$, which give us $\forall t \in\ R, d_1t^2 + d_2t + d_3 - d = 0t^2 + 0td_3 - d = 0$

If $\{\begin{bmatrix}a_1\\b_1\\c_1\end{bmatrix},\begin{bmatrix}a_2\\b_2\\c_2\end{bmatrix},\begin{bmatrix}a_3\\b_3\\c_3\end{bmatrix}\}$ is linearly dependent, we find nontrivial m, p, q, such that $\begin{bmatrix}d_1\\d_2\\d_3\end{bmatrix} = 0$

If $\{\begin{bmatrix}a_1\\b_1\\c_1\end{bmatrix},\begin{bmatrix}a_2\\b_2\\c_2\end{bmatrix},\begin{bmatrix}a_3\\b_3\\c_3\end{bmatrix}\}$ is linearly independent, the image is the whole $\mathbb{R}^3$. We find m, p, q, such that $\begin{bmatrix}d_1\\d_2\\d_3\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

■

6.

6.(a)

$\lim_{(x,y)\to(0,0)} \frac{x^2+xy^2}{x^4+y^2}$

Proof. Approach from $y = x$

we see $\lim_{(x,y)\to(0,0)} \frac{x^2+xy^2}{x^4+y^2} = \lim_{x\to 0} \frac{x^3+x^2}{x^4+x^2} = \infty$

So not exists ■

6.(b)

$\lim_{(x,y)\to(1,1)} \frac{y-x}{1-y+\ln x}$

Proof. Approach from $y = x$

We have $\lim_{(x,y)\to(1,1)} \frac{y-x}{1-y+\ln x} = \lim_{(x,y)\to(1,1)} \frac{0}{1-x+\ln x} = 0$

Approach from $y = 2x - 1$

We have $\lim_{(x,y)\to(1,1)} \frac{y-x}{1-y+\ln x} = \lim_{x\to 1} \frac{x-1}{2-2x+\ln x} = \lim_{x\to 1} \frac{1}{-2+\frac{1}{x}} = -1$

By Queation 8, it does not exist

■

6.(c)

$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

Proof. Approach from $y = mx$

For all m, We have $\lim_{(x,y)\to(0,0)} = \frac{(m^2+1)x^2}{\sqrt{(m^2+1)x^2+1}-1} = \lim_{(x,y)\to(0,0)} \frac{2(m^2+1)x}{\frac{2(m^2+1)x}{2\sqrt{(m^2+1)x^2+1}}}$

$= \lim_{(x,y)\to(0,0)} 2\sqrt{(m^2+1)x^2+1} = 2$ ■

6.(d)

$\lim_{(x,y)\to(0,0)} xy \sin \frac{1}{x^2+y^2}$

Proof. $\lim_{(x,y)\to(0,0)} xy = 0$

Noted $\sin$ is bounded above by 1 and below by -1

So multiplying an element of the image of $\sin$ will only make xy closer to 0

Then the limit value of $xy \sin \frac{1}{x^2+y^2}$ is 0

■

6.(e)

$\lim_{(x,y)\to(6,3)} xy \cos x - 2y$

Proof. $\lim_{(x,y)\to(6,3)} xy \cos x - 2y = \lim_{(x,y)\to(6,3)} x \lim_{(x,y)\to(6,3)} y \lim_{(x,y)\to(6,3)} \cos x - 2y$

$= 6 * 3 * 1 = 18$ ■

7.

7.(a)

$f(x) = \frac{\sin xy}{e^x - y^2}$

Proof. Notice $\sin xy$,e^x and y^2 are all continuous functions

we give out only the uncontinuous point, that is, $e^x - y^2 = 0$. The rest of the points are all continuous.

Uncontinuous: $\{(x, \sqrt{e^x}) | x \in \mathbb{R}\}$ ■

7.(b)

Proof. Notice x^2, y^2, $2x^2$ and y^2 are all continuous function, and $\forall (x,y) \neq (0,0)$, $2x^2 + y^2 \neq 0$

So the only possible uncontinuous point is $(0,0)$, leaving the rest of the domain continuous

Approaching from $y = x$

We have $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{2x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^4}{3x^2} = 0 \neq 1$

So, at $(0,0)$, weather the limit exists or not, we know it is uncontinuous

∎

8.

Proof. Assume $\lim_{(x,y)\to(a,b)} f(x,y) = L$ and $\lim_{(x,y)} f(x,y) = M$ and $L < M$

Let $\epsilon = \frac{M-L}{2}$

Pick δ_L, such that $\forall |(x-a, y-b)| < \delta_L, |f(x,y) - L| < \epsilon$

$|f(x,y) - L| < \epsilon \implies f(x,y) < \frac{M+L}{2}$ **(i)**

Pick δ_M, such that $\forall |(x-a, y-b)| < \delta_M, |f(x,y) - M| < \epsilon$

$|f(x,y) - M| < \epsilon \implies f(x,y) > \frac{M+L}{2}$ **(ii)**

Arbitrarily pick $(x,y) < min(\delta_L, \delta_M)$, we CaC from **(i),(ii)** ∎

9.

Proof. $\forall (x,y) : y > x, f(x,y) = y$, which is continuous.

$\forall (x,y) : y < x, f(x,y) = x$, which is continuous.

We consider only $H := \{(x,x) | x \in \mathbb{R}\}$

Let $(a,a) \in H$

For each ϵ, we pick $\delta = \epsilon$

$\sqrt{(x-a)^2 + (y-a)^2} < \delta = \epsilon \implies a - \delta < x < a + \delta$ and $a - \delta < y < a + \delta \implies a - \delta < max(x,y) < a + \delta \implies |f(x,y) - f(a,a)| < \delta = \epsilon$

∎

10.

10.(a)

Proof. x^2, y^2, $h(s) = \sqrt{s}$, are all continuous function, and $x^2 + y^2 > 0$, yielding us that $h(x^2 + y^2)$ is continuous

∎

10.(b)

Proof. $\forall m \in \mathbb{R}, g(m) < n$

$\forall (x,y) \in \mathbb{R}^2, \exists m \in \mathbb{R}, f(x,y) = m$

$\forall (x,y) \in \mathbb{R}^2, \exists m \in \mathbb{R}, g \circ f(x,y) = g(m) < n$ ∎

10.(c)

Proof. Because g is bounded below by 0 and decreasing

$\lim_{n\to\infty} g(n)$ exists

Then $\lim_{\|(x,y)\|\to\infty} g \circ f(x,y) = \lim_{n\to\infty} g(n)$ exists

∎

10.(d)

Proof. Rearrange D into a sequence $\{(x,y)_i\}$ such that $\{h((x,y)_i)\}$ is an increasing function

Then we see $\lim_{i\to\infty}\{h((x,y)_i)\} = M$

∎

10.e

Proof. No, let $g(x) = \begin{cases} 1, x < 0 \\ -1, x \geq 0 \end{cases}$

Notice $im(f) \subseteq \mathbb{R}^+$

So $im(g \circ f) = -1$

Then obviously, $\forall \{x_n\}, \{y_n\}, \lim_{n\to\infty} g \circ f(x_n, y_n) \neq 1 = K$

∎