Midterm 4

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Goal: $f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y)$ — (*)

we set $\lambda = \frac{k}{2^n}$, and $2^n \geq k \geq 0$. (k is interger)

and do mathmatic induction

case $n=0 \Rightarrow \lambda = 0$ or 1

case $n=1 \Rightarrow \lambda = 0$ or 1 or $\frac{1}{2}$, $0,1$ is trivial)

$\lambda = \frac{1}{2}$, Suppose the result is proved for $n \leq r$

and consider $\lambda = \frac{k}{2^{r+1}}$, If k is even, say $k = 2I$

then $\frac{k}{2^{r+1}} = \frac{I}{2^r}$, suppose k is odd. Then $1 \leq k \leq 2^{r+1}-1$

and so the numbers $I = \frac{k-1}{2}$ and $m = \frac{k+1}{2}$ are integers

with $0 \leq I < m \leq 2^r$, we now can write $\lambda = \frac{s+t}{2}$

where $s = \frac{k-1}{2^{r+1}} = \frac{I}{2^r}$ and $t = \frac{k+1}{2^{r+1}} = \frac{m}{2^r}$

$\Rightarrow \lambda x + (1-\lambda)y = \frac{[sx+(1-s)y]+[tx+(1-t)y]}{2}$

then we have

$$\begin{aligned} f(\lambda x+(1-\lambda)y) &\leq \frac{f(sx+(1-s)y)+f(tx+(1-t)y)}{2} \\ &\leq \frac{sf(x)+(1-s)f(y)+tf(x)+(1-t)f(y)}{2} \\ &= \left(\frac{s+t}{2}\right)f(x)+\left(1-\frac{s+t}{2}\right)f(y) \\ &= \lambda f(x)+(1-\lambda)f(y) \end{aligned}$$

the induction complete

Now for each fixed x and $y \Rightarrow$ (*) is continue function of λ. Hence the set on which this inequalty hold (the inverse image of the closed set $[0,\infty)$ under the mapping $\lambda \mapsto \lambda f(x)+(1-\lambda)f(y)-f(\lambda x+(1-\lambda)y)$) is a closed set. Since it contains all points $\frac{k}{2^n}$, $0 \leq k \leq n$, $n = 1,2,3,\cdots$, it must contains the closure of this set of points, it must contain all of $[0,1]$

$\Rightarrow f$ is convex