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NCKU 112.1
Note for Probability Theory

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Contents

Chapter 1

σ-Algebra

1.1

Definition 1.1.1. (Definition of Measure Space) A measure space is a triple $(\Omega, \mathcal{G}, \mu)$ where

$$\Omega \text{ is a set} \tag{1.1}$$

$$\mathcal{G} \text{ is a } \sigma\text{-algebra over } \Omega \tag{1.2}$$

$$\mu \text{ is a measure on } (\Omega, \mathcal{G}) \tag{1.3}$$

Definition 1.1.2. (Definition of σ-Algebra) We say $\mathcal{G} \subseteq \mathcal{P}(\Omega)$ is a σ-algebra if

$$\Omega \in \mathcal{G} \tag{1.4}$$

$$X \in \mathcal{G} \implies \Omega \setminus X \in \mathcal{G} \text{ (Closed under complement)} \tag{1.5}$$

$$\mathcal{A} \subseteq \mathcal{G} \text{ and } |\mathcal{A}| \leq |\mathbb{N}| \implies \bigcup \mathcal{A} \in \mathcal{G} \text{ (Closed under countable union)} \tag{1.6}$$

From now, we denote $\Omega \setminus X$ by X^c. We say $\mathcal{G} = \{\varnothing, \Omega\}$ is the trivial σ-algebra on Ω.

Theorem 1.1.3. (Basic Property of σ-Algebra) Let $(\Omega, \mathcal{G})$ be a σ-algebra. Then we have

$$\varnothing \in \mathcal{G} \tag{1.7}$$

$$\mathcal{A} \subseteq \mathcal{G} \implies \bigcap \mathcal{A} \in \mathcal{G} \tag{1.8}$$

$$A, B \in \mathcal{G} \implies A \setminus B \in \mathcal{G} \tag{1.9}$$

Proof. Observe $\varnothing = \Omega^c$, and observe $\bigcap \mathcal{A} = (\bigcup_{X \in \mathcal{A}} X^c)^c$, and observe $A \setminus B = A \cup B^c$ ∎

Theorem 1.1.4. (Intersection of σ-Algebras is a σ-Algebra) Let S be a set of σ-algebra over Ω, then $\bigcap S$ is a σ-algebra.

Proof. missed ■

The following concerning measure

Definition 1.1.5. (Definition of a Measure) Let $\mathcal{G}$ be a σ-algebra over Ω. Function $\mu : \mathcal{G} \to \mathbb{R}$ is called a measure if

$$\forall E \in \mathcal{G}, \mu(E) \geq 0 \text{ (Nonnegative)} \tag{1.10}$$

$$\mu(\varnothing) = 0 \tag{1.11}$$

$$F \subseteq \mathcal{G} \text{ and } |F| \leq |\mathbb{N}| \implies \mu(\bigcup F) = \sum_{X \in F} \mu(X) \text{ (Countable additivity)} \tag{1.12}$$

The following concern generating a σ-Algebra from a set of subsets of sample space.

Theorem 1.1.6. (Representation of σ-Algebra) Let M be a countable partition of Ω. Then the set

$$\{\bigcup N : N \in \mathcal{P}(M)\} \tag{1.13}$$

is a σ-algebra

Proof. missed ■

Definition 1.1.7. (Definition of Generating σ-Algebra) Let $\mathcal{F} \subseteq \mathcal{P}(\Omega)$. The σ-algebra generated by $\mathcal{F}$ is defined to be the smallest σ-algebra that contain $\mathcal{F}$

Theorem 1.1.8. (Definition of Generating σ-Algebra) Let $\mathcal{F} \subseteq \mathcal{P}(\Omega)$. The smallest σ-algebra containing $\mathcal{F}$ consists precisely of set taking countable operation of complement countable operation.

Proof. need verified ■

Theorem 1.1.9. (Representation of σ-Algebra) Let M be a countable partition of Ω. Then the σ-algebra

$$\{\bigcup N : N \in \mathcal{P}(M)\} \tag{1.14}$$

is the σ-algebra generate by M

Proof. need verified ■

Theorem 1.1.10. (Representation of σ-Algebra) Let M be a countable partition of Ω. Then the σ-algebra

$$\{\bigcup N : N \in \mathcal{P}(M)\} \tag{1.15}$$

contain no proper subset of element of M

Proof. need verified ■

The following concern a class of measure space, called probability space.

Definition 1.1.11. (Definition of Probability Space) A probability space is a triple $(\Omega, \mathcal{G}, P)$ where

$$\Omega \text{ is a set called } \textit{sample space} \tag{1.16}$$

$$\mathcal{G} \text{ is a } \sigma\text{-algebra over } \Omega \text{ called event space} \tag{1.17}$$

$$P : \Omega \to [0,1] \text{ is a measure called probability measure} \tag{1.18}$$

where Ω is a set, called sample space, $\mathcal{G}$ is a σ-algebra over Ω, called event space and $P : \Omega \to [0,1]$ is called probability measure.

A simple example of a σ-algebra is

$$\Omega_2 = \{HH, HT, TH, TT\}, \mathcal{G} = \{\varnothing, X, \{HT, HH\}, \{TH, TT\}\} \tag{1.19}$$

Notice in this example, Ω is ought to be interpreted as tossing two coins and $\mathcal{G}$ is to observe the first coin is head or tail.

To expand the first example, we have another simple example:

$$\Omega_3 = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \tag{1.20}$$

Define

$$A_H := \{HHH, HHT, HTH, HTT\} \text{ and } A_T := \{THH, THT, TTH, TTT\} \tag{1.21}$$

which is the information of tossing head or tail on first try.

Notice $A_T = A_H^c$. Define

$$A_{HH} := \{HHH, HHT\} \text{ and } A_{HT} := \{HTH, HTT\} \tag{1.22}$$

$$A_{TH} := \{THH, THT\} \text{ and } A_{TT} := \{TTH, TTT\} \tag{1.23}$$

so we have

$$A_H = A_{HH} \cup A_{HT} \text{ and } A_T = A_{TH} \cup A_{TT} \tag{1.24}$$

Then we can define

$$\mathcal{G} := \{\bigcup N : N \in \mathcal{P}(M)\} \tag{1.25}$$

where $M = \{A_{HH}, A_{HT}, A_{TH}, A_{TT}\}$

Notice we can define four σ-algebras by

$$\mathcal{F}_0 = \{\varnothing, \Omega\}, \mathcal{F}_1 = \{\varnothing, \Omega, A_T, A_H\}, \mathcal{F}_2 = \mathcal{G}, \mathcal{F}_3 = \mathcal{P}(\Omega) \tag{1.26}$$

then we have

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \tag{1.27}$$

The following concern Borel σ-algebra

Definition 1.1.12. (Definition of Borel-Algebra) The Borel-Algebra on $\mathbb{R}$, which we denote $\mathcal{B}(\mathbb{R})$ is the σ-algebra generated by all open interval of $\mathbb{R}$.

Some members of $\mathcal{B}(\mathbb{R})$:

$$(b, a), (a, \infty), \mathbb{R} \tag{1.28}$$

$$(b, a] = (b, \infty) \setminus (a, \infty) \tag{1.29}$$
$$[a, \infty) = \mathbb{R} \setminus (-\infty, a) \tag{1.30}$$
$$[a, b] = [a, \infty) \setminus (b, \infty) \tag{1.31}$$
$$\{a\} = \mathbb{R} \setminus (-\infty, a) \cup (a, \infty) \tag{1.32}$$

Some members of $\mathcal{B}(\mathbb{R})$: $(b, a), (a, \infty), (b, a], (-\infty, a), [a, \infty), [a, b]$

Definition 1.1.13. (Definition of a Random Variable) We say the function from Ω to $\mathbb{R}$ is a random variable.

We now define 3 random variable for example from the last example of σ-algebra.

Let $S_0, u, d \in \mathbb{R}^+$ and let $d < 1 < u$. We define three random variables S_1, S_2, S_3 on Ω_3

$$S_1(\omega) = \begin{cases} uS_0 & \text{if } \omega \in A_H \\ dS_0 & \text{if } \omega \in A_T \end{cases} \quad S_2(\omega) = \begin{cases} u^2 S_0 & \text{if } \omega \in A_{HH} \\ udS_0 & \text{if } \omega \in A_{HT} \cup A_{TH} \\ d^2 S_0 & \text{if } \omega \in A_{TT} \end{cases} \tag{1.33}$$

$$S_3(\omega) = \begin{cases} u^3 S_0 & \text{if } \omega \in \{HHH\} \\ u^2 d S_0 & \text{if } \omega \in \{HHT, HTH, THH\} \\ ud^2 S_0 & \text{if } \omega \in \{HTT, THT, TTH\} \\ d^3 S_0 & \text{if } \omega \in \{TTT\} \end{cases} \tag{1.34}$$

Often, we just use S to denote $S(\omega)$.

Theorem 1.1.14. (Construct σ-Algebra with Random Variable) Let X be a random variable on Ω. We define

$$X^{-1}[B] = \{\omega \in \Omega : X(\omega) \in B\} \tag{1.35}$$

and define the σ-algebra $\sigma(X)$ by

$$\sigma(X) = \{X^{-1}[B] : B \in \mathcal{B}(\mathbb{R})\} \tag{1.36}$$

We can verify $\sigma(X)$ is a σ-algebra.

Proof. missed ■

Notice $\sigma(S_1) = \mathcal{F}_1, \sigma(S_2) = \mathcal{F}_2, \sigma(S_3) = \mathcal{F}_3$