5.6 PDE HW 6

Question 105

Solve $u_{tt} = 4u_{xx}$ for $0 < x < \infty, u(0,t) = 0, u(x,0) = 1, u_t(x,0) = 0$ using the reflection method. The solution has a singularity find its location.

Proof. Define

$$\varphi(x) \triangleq \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases} \text{ and } \psi(x) \triangleq 0$$

We are required to solve the following Dirichlet's problem for wave equation

$$u_{tt} = 4u_{xx} \text{ for } -\infty < x < \infty$$
$$u(x,0) = \varphi(x) \text{ and } u_t(x) = \psi(x)$$

The solution is exactly

$$\begin{aligned} u(x,t) &= \frac{\varphi(x+2t) + \varphi(x-2t)}{2} + \int_{x-2t}^{x+2t} \psi(s)ds \\ &= \frac{\varphi(x+2t) + \varphi(x-2t)}{2} \\ &= \begin{cases} 1 & \text{if } x - 2t > 0 \\ 0 & \text{if } x + 2t > 0 > x - 2t \\ -1 & \text{if } 0 > x + 2t \end{cases} \end{aligned}$$

On the half line, the solution is

$$u(x,t) = \begin{cases} 1 & \text{if } x - 2t > 0 \\ 0 & \text{if } x - 2t < 0 \end{cases}$$

So the singularity is exactly on the line $x - 2t = 0$ ■

Question 106

Solve the inhomogeneous Neumann diffusion problem on the half-line

$$\begin{cases} w_t - kw_{xx} = 0 \text{ for } x \in (0,\infty) \textbf{ (Homogeneous DE)} \\ w_x(0,t) = h(t) \textbf{ (Non-homogeneous BC)} \\ w(x,0) = \varphi(x) \textbf{ (IC)} \end{cases}$$

by the subtraction method indicated in the text.

Proof. Suppose w is a solution to our problem. If we define

$$u(x,t) \triangleq w(x,t) - xh(t) \text{ for } x \in (0,\infty)$$

We see that u satisfy

$$\begin{cases} u_t - ku_{xx} = -xh'(t) \text{ for } x \in (0,\infty) \textbf{ (Non-homogeneous DE)} \\ u_x(0,t) = 0 \textbf{ (Good BC)} \\ u(x,0) = \varphi(x) - xh(0) \textbf{ (IC)} \end{cases}$$

Define

$$f_{\text{even}}(x,t) \triangleq \begin{cases} -xh'(t) & \text{if } x > 0 \\ xh'(t) & \text{if } x < 0 \end{cases}$$

And define

$$\varphi_{\text{even},*} \triangleq \begin{cases} \varphi(x) - xh(0) & \text{if } x > 0 \\ \varphi(-x) + xh(0) & \text{if } x < 0 \end{cases}$$

And define

$$u_{\text{even}}(x,t) \triangleq \int_{\mathbb{R}} S(x-y)\varphi_{\text{even},*}(y)dy + \int_0^t \int_{\mathbb{R}} S(x-y,t-s)f_{\text{even}}(y,s)dyds$$

It then follows that u_{even} solve the non-homogeneous DE and IC. To see that $u_x(0,t) = 0$, one simply observe that u is even in x. The solution to the original problem is then

$$w(x,t) \triangleq u_{\text{even}}(x,t) + xh(t) \text{ for } x \in (0,\infty)$$

■