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Project Proposal: A deep and comprehensive study of equivalent characterizations of basic notions in Analysis

Abstract:

We will list and prove the equivalency of the important characterizations of some of the most basic notions in General Topology and Analysis as many as possible.

Introduction:

This study stemmed from Professor Feng's “Advanced Calculus” class, in which we were taught some fundamental notions in Analysis, including metric topology, theory of differentiation, Darboux integral, and Lebesgue integral. The class is somewhat taught from the Analysts’ angle in the sense that if one can use a definition of a concept to prove some result using that definition, one won’t be bothered to find more characterizations, which, as we all know, can gain us a deeper insight into the concept and help us prove more result.

This study addresses such an issue. We will bring in more mathematical objects (or concepts) helpful for further study that weren’t taught in the class and try to list all the “important” characterizations of them. The objects planned to be studied will be listed below. By “important,” we mean that the characterization either brings a deeper insight into the object or makes proving further results easier.

Objectives:

We will first go straight to the study of General Topology and give the following concepts many more definitions than one mainly learned in the respective class.

A. Interior
B. Closed
C. Closure

D. Boundary
E. Basis
F. Subbase
G. Continuity
H. Subspace Topology
I. Product Topology
J. Box Topology and its Relationship with Product Topology
K. Connectedness
L. Compactness

In particular, we shall prove the Alexander Subbase Theorem and that the product topology of n-Euclidean Space is identical to the topology generated by the metric. Next, we will discuss the topology of extended reals and extended complex numbers for the sake of a slicker treatment of Lebesgue measurability. Finally, we study the idea of homotopy and end our investigation of General Topology.

We then embarked on the study of General Analysis. As taught in class, the Borel sigma algebra is a collection of sets that can be formed from open sets through countable union, countable intersection, and relative complement. This is a, however, "vague" definition, as one is forced to use the idea of ordinal to construct Borel sigma-algebra, while the idea of ordinal wasn't taught in class. For this reason, some authors prefer to define the Borel sigma-algebra as the smallest sigma-algebra containing the collection of open sets. We will construct the first definition and prove the equivalency of the two definitions. Lastly, we will study the question: When we have a subspace of some topological space X, what is the relationship between the Borel sigma-algebra of the subspace and that of X?

Finally, we work out as many equivalent characterizations as possible for Lebesgue measures and integrals. If time permits, we try to give as many equivalent constructions of variants of the following pathological objects as possible.

A. Peano space-filling curve
B. Topologist's Sine curve
C. Long Line
D. Cantor-Lebesgue function
E. Volterra's function
F. Weierstrass' function