NCKU 112.2
General Analysis

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Contents

Chapter 1

General Topology

1.1 Equivalent Characterizations of Basic Notions

Abstract

This section give a compact and comprehensive development of some of the most basic notions in the study of topology. In this section, $(X, \mathscr{T})$ is a topological space.

Given a collection $\mathcal{B} \subseteq \mathscr{T}$ of open sets, we say $\mathcal{B}$ is a

(a) **basis** if for each $O \in \mathscr{T}$ there exists a subcollection $\mathcal{B}_0$ such that $O \subseteq \mathcal{B}_0$.

(b) **subbase** if $\mathscr{T}$ is the collection of unions of finite intersections of $\mathcal{B}$. In a more formal language, $\mathcal{B}$ has to satisfy $\mathscr{T} = \{\bigcup A : A \subseteq \mathcal{A}\}$ where $\mathcal{A} = \{\bigcap \mathcal{B}_0 : \text{ card } \mathcal{B} \in \mathbb{N} \text{ and } \mathcal{B}_0 \subseteq \mathcal{B}\}$

Theorem 1.1.1. (Equivalent Definition of Basis) The following statements are equivalent.

(a) $\mathcal{B}$ is a basis.

(b) For all $O \in \mathscr{T}$ and $x \in O$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq O$.

Proof. Check straightforward. ∎

Theorem 1.1.2. (Equivalent Definition of Subbase) The following statements are equivalent.

(a) $\mathcal{B}$ is a subbase of $\mathscr{T}$.

(b) $\mathcal{B}$ cover X and $\mathscr{T}$ is the smallest topology containing $\mathcal{B}$.

Proof. Check straightforward. ■

Immediately, with the equivalent definitions, one should check

(a) Given any cover $\mathcal{B}$ of X, there always exists a unique topology $\mathscr{T}$ containing $\mathcal{B}$ as a subbase. We say $\mathscr{T}$ is the **topology generated by** $\mathcal{B}$.

(b) The set $\mathcal{A} \triangleq \{\bigcap S : S \subseteq \mathcal{B},\ \text{card}\ S \in \mathbb{N}\}$ of finite intersections of cover $\mathcal{B}$ is a basis of the topology generated by $\mathcal{B}$.

(c) Not every cover $\mathcal{B}$ of X has some topology $\mathscr{T}$ containing $\mathcal{B}$ as a basis. Consider $\mathcal{B} \triangleq \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(b, \infty) : b \in \mathbb{R}\}$. Even the smallest topology containing $\mathcal{B}$, i.e. the standard topology, does not have $\mathcal{B}$ as a basis.

(d) However, cover $\mathcal{B}$ is the basis of the topology $\mathscr{T}$ generated by itself if for all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$.

(e) If $\mathcal{B}$ is a basis of $\mathscr{T}$, then $\mathcal{B}$ is also a subbase of $\mathscr{T}$.

(f) Basis is not necessarily closed under finite intersection. Consider the basis $\{(a, a + \frac{1}{n}) : a \in \mathbb{R}, n \in \mathbb{N}\}$ for $\mathbb{R}$'s standard topology.

Note that in (a), to check the generated $\mathscr{T}$ is indeed a topology, one may need to utilize the identity

$$\Big(\bigcup_{i \in I} A_i\Big) \cap \Big(\bigcup_{j \in J} B_j\Big) = \bigcup_{i \in I, j \in J} A_i \cap B_j.$$

Now, given an arbitrary subset $E \subseteq X$, we

(a) say $x \in X$ is a **limit point of** E if every open O containing x contain a point $y \in E$ such that $y \neq x$.

(b) say $x \in E$ is an **interior point of** E if there exists $O \in \mathscr{T}$ such that $x \in O \subseteq E$.

(c) define the **interior** E° **of** E to be the union of all open sets contained by E.

(d) say $E \subseteq X$ is a **closed set** if $E^c \in \mathscr{T}$.

(e) define the **closure** $\overline{E}$ **of** E by $\overline{E} \triangleq E \cup E'$ where E' is the set of limit points of E.

(f) say E is **dense** in X if $\overline{E} = X$.

(g) define the **boundary** ∂E **of** E by $\partial E \triangleq \overline{E} \setminus E^\circ$

Theorem 1.1.3. (Equivalent Definitions of Interior) The following sets are equivalent

(a) E°

(b) The largest open set contained by E.

(c) The set of interior points of E.

Proof. Check straightforward. ■

Theorem 1.1.4. (Equivalent Definitions of Closed) The following statements are equivalent.

(a) E is closed.

(b) the set of limit points of E is contained by E.

(c) $\overline{E} = E$.

Proof. The proof of (a) $\implies$ (b) $\implies$ (c) are straight forward. The proof of (c) $\implies$ (a) follows from first noting no $x \in E^c$ is a limit point of E. Then shows $E^c = \bigcup_{x \notin E} O_x$ where O_x is an open set containing x and disjoint with E. ■

Theorem 1.1.5. (Equivalent Definitions of Closure) The following sets are equivalent.

(a) $\overline{E}$

(b) $((E^c)^\circ)^c$

(c) The smallest closed set containing E.

(d) $\{x \in X :$ every open O containing x intersect with $E\}$

Proof. (a) = (d) is obvious. To verify (a) = (c), check $(\overline{E})' \subseteq E'$ and check $E' \subseteq F' \subseteq F$ for each closed F containing E. Lastly, to verify (b) = (c), check $(\overline{E})^c = (E^c)^\circ$ using the largest open set and the smallest closed set characterization of interior and closure. ■

Theorem 1.1.6. (Equivalent Definitions of Dense) The following statements are equivalent.

(a) E is dense in X.

(b) Every non-empty open set intersect with E.

(c) $(E^c)^\circ = \varnothing$

Proof. (a) = (c) follows from $\overline{E} = ((E^c)^\circ)^c$, and (a) = (b) follows from $\overline{E} = \{x \in X :$ every open O containing x intersect with $E\}$. ■

Theorem 1.1.7. (Equivalent Definitions of Boundary) The following sets are equivalent.

(a) ∂E

(b) $\overline{E} \cap \overline{E^c}$

(c) $\{x \in X :$ every open O containing x intersect with both E and E^c $\}$

Proof. (a) = (b) follows from $(E^\circ)^c = \overline{E^c}$ and (b) = (c) follows from $\overline{E} = \{x \in X :$ every open O containing x intersect with $E\}$. ■

We now develop the theory of continuity by first giving a pointwise definition. Given another topological space $(Y, \mathscr{S})$ and a function $f : X \to Y$, we say f is **continuous at** $x \in X$ if for all open O containing $f(x)$, there exists open E containing x such that $f(E) \subseteq O$. We say f is a **continuous (or $(\mathscr{T}, \mathscr{S})$-continuous, if necessary) function** if f is continuous at all $x \in X$.

It is easy to see the composition of two continuous function must be continuous. However, one should notice that the composition of a continuous function and a discontinuous function can be continuous. Just let one of them be a constant function.

Theorem 1.1.8. (Equivalent Definitions of Continuous function) The following are equivalent

(a) f is continuous.

(b) $f^{-1}(O) \in \mathscr{T}$ for all $O \in \mathscr{S}$.

(c) $f^{-1}(F)$ is closed for all closed F in Y.

(d) For all $B \subseteq Y$, $f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$

(e) For all $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.

(f) For all $B \subseteq Y, \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$

(g) For all subbase $\mathcal{B}$ of Y, $f^{-1}(B) \in \mathscr{T}$ for all $B \in \mathcal{B}$.

Proof. It is straightforward to check (a) $\implies$ (b) $\implies$ (c). To verify (c) $\implies$ (a), check $x \in (f^{-1}(O^c))^c$ and $f((f^{-1}(O^c))^c) \subseteq)$ for each $x \in X$ and $O \in \mathscr{S}$ containing $f(x)$. Respectively, to verify (b) $\implies$ (d), (c) $\implies$ (e) and (c) $\implies$ (f), check $f^{-1}(B^\circ) \subseteq f^{-1}(B), A \subseteq f^{-1}(\overline{f(A)})$ and $f^{-1}(B) \subseteq f^{-1}(\overline{B})$. Note that (e) $\implies$ (f) follows from noting $A = f^{-1}(B)$. Check (d) $\implies$ (b) and (f) $\implies$ (c) straightforwardly, and we have proved the equivalency of statements from (a) to (f). Lastly, check (b) $\iff$ (g) straightforwardly. ■

One may wonder: Why isn't "For all $A \subseteq X$, $f(A)^\circ \subseteq f(A^\circ)$" a characterization of f being continuous? Consider a function that maps some topological space with a subset that has an empty interior into the topological space Y having only a single point.

1.2 Equivalent Definition of Subspace and Product

Abstract

Theorem 1.2.1. (Equivalent Definition of Finer/Coarser Topologies) Given another topology $\mathscr{T}'$ on X, the following are equivalent:

(a) $\mathscr{T} \subseteq \mathscr{T}'$.

(b) $\mathbf{id} : (X, \mathscr{T}') \to (X, \mathscr{T})$ is continuous.

(c) Given any basis $\mathcal{B}$ of $\mathscr{T}$ and any basis $\mathcal{B}'$ of $\mathscr{T}'$, for all $x \in X$ and basic open $B \in \mathcal{B}$ containing x, there exists a basic open $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

(d) There exists subbases $\mathcal{B}, \mathcal{B}'$ of $\mathscr{T}, \mathscr{T}'$ such that $\mathcal{B} \subseteq \mathcal{B}'$.

Proof. (d) $\iff$ (a) $\iff$ (b) are straightforward. From (a) to (c), one finds B' by noting $B \in \mathscr{T}'$ and utilizing the definition of basis. From (c) to (a), one shows $O \in \mathscr{T}$ belongs to $\mathscr{T}'$ by taking $O = \bigcup_{x \in O} B'_x$. ∎

Now, given a collection $(X_\alpha)_{\alpha \in J}$ of topological spaces, we define the **product topology** on $X \triangleq \prod_{\alpha \in J} X_\alpha$ to be the smallest topology such that for all $\alpha \in J$, the projection $\pi_\alpha : X \to X_\alpha$ that maps $(y_\alpha)_{\alpha \in J}$ to y_α is continuous.

Theorem 1.2.2. (Equivalent Definition of Product Topology) Let $\mathcal{B}_\alpha$ each be the subbase of X_α. The following topologies are equivalent.

(a) The product topology on X.

(b) The topology on X generated by the basis $\{\prod U_\alpha : U_\alpha \neq X_\alpha$ for finitely many α and $U_\alpha \in \mathcal{B}_\alpha\}$.

(c) The smallest topology on X satisfying the statement: For all topological spaces $(Z, \mathscr{T}_Z)$, a function $f : Z \to X$ is continuous if and only if for all $\alpha \in J$, the function $f \circ \pi_\alpha : Z \to X_\alpha$ is continuous.

Proof. By definition, we know X has the subbase $\bigcup_{\alpha \in J} \mathcal{G}_\alpha$ where $\mathcal{G}_\alpha \triangleq \{\pi_\alpha^{-1}(U) : U \in \mathscr{T}_\alpha\}$. This gives us (a) = (b). For (a) = (c), we have to prove that: Given a topology on X, the following two statements are equivalent:

(i) All projections π_α are continuous.

(ii) For all topological spaces $(Z, \mathscr{T}_Z)$ and functions $f : Z \to X$, the function f is continuous if and only if for all $\alpha \in J$, the function $f \circ \pi_\alpha : Z \to X_\alpha$ is continuous.

Taking $Z \triangleq X$ and $f \triangleq \mathbf{id}$ proves (ii) $\implies$ (i). For (i) $\implies$ (ii), if f is continuous, then $f \circ \pi_\alpha$ is clearly continuous for all α. Conversely, if $f \circ \pi_\alpha$ is continuous for all α, use the pointwise definition of continuity and the basis given in (b) to show that f is continuous. ■

Immediately, one should check:

(a) In X, a sequence (p_n) converges to p if and only if $\pi_\alpha(p_n)$ converges to $\pi_\alpha(p)$ for all $\alpha \in J$. This generalize what happen in $\mathbb{R}^n$.

(b) If $f : X \times Y \to Z$ is continuous, then for all $x \in X$, the function $f(x, \cdot) : Y \to Z$ defined by $f(x, \cdot)(y) \triangleq f(x, y)$ is continuous. The converse is not true, in the sense that f can be discontinuous even if $f(x, \cdot)$ and $f(\cdot, y)$ are continuous for all x and y. Elementary counterexample can be constructed in $\mathbb{R}^2$.

(c) The third characterization of product topology shows that the product is independent of expression. For example, given an enumeration $(X_\alpha)_{\alpha \leq \gamma}$ of topological spaces, $X_1 \times \prod_{1 < \alpha \leq \gamma} X_\alpha$ is homeomorphic to $\prod_{1 \leq \alpha \leq \gamma} X_\alpha$.

We will later introduce an inferior alternative to assigning topology onto the Cartesian product of topological spaces. Now, given a topological space $(X, \mathscr{T})$ and a subset $E \subseteq X$, we define the **subspace topology** $\mathscr{T}_E$ on E by $\mathscr{T}_E \triangleq \{O \cap E : O \in \mathscr{T}\}$. Immediately, one can check that $\mathscr{T}_E$ is indeed a topology, and:

(a) Given a subset $F \subseteq E$, viewing F as a subspace of E or X makes no difference.

(b) The collection of closed sets in $(E, \mathscr{T}_E)$ is $\{F \cap E : F \text{ is a closed set in } (X, \mathscr{T})\}$.

(c) If (U_α) is an open cover of X and $(\mathcal{B}_\alpha)$ are basis of (U_α), then $\bigcup_\alpha \mathcal{B}_\alpha$ form a basis of X.

(d) For all $F \subseteq X$, $\mathrm{cl}_E(F \cap E) \subseteq \mathrm{cl}_X(F) \cap E$. The equality holds when $F \subseteq E$.

(e) Given a function $f : X \to Y$ and a subset F of Y containing $f(X)$, $f : X \to Y$ is continuous at p if and only if $f : X \to F$ is continuous at p.

(f) Given a function $f : X \to Y$ and $p \in E \subseteq X$, if f is continuous at p, then $f|_E : E \to Y$ is continuous at p. The converse is true only when E is open in X.

(g) Given a finite collection of closed subspace E_j such that $X = \bigcup E_j$, if $f|_{E_j} : E_j \to Y$ are all continuous, then $f : X \to Y$ is continuous. (**Paste Lemma**)

Theorem 1.2.3. (Equivalent Definition of Subspace Topology) Given a basis $\mathcal{B}$ and a subbase $\mathcal{B}'$ of $\mathscr{T}$, the following sets are equivalent:

(a) $\mathscr{T}_E$.

(b) The topology on E generated by the basis $\mathcal{B}_E \triangleq \{B \cap E : B \in \mathcal{B}\}$.

(c) The topology on E generated by the subbase $\mathcal{B}'_E \triangleq \{B' \cap E : B' \in \mathcal{B}'\}$.

(d) The smallest topology on E such that the inclusion map $\iota : E \to X$ is continuous.

Proof. Check straightforward. ■

At this point, one should check the compatibility between the definitions of subspace topology and product topology. Given a collection $(X_\alpha)_{\alpha\in J}$ of topological spaces and a subspace $(A_\alpha)_{\alpha\in J}$ of each X_α, one can view $A \triangleq \prod A_\alpha$ either as a subspace of the product $X \triangleq \prod X_\alpha$ or the product of subspaces $(A_\alpha)_{\alpha\in J}$. The two topologies are identical, and the proof goes as follows:

(a) Show that the product topology has the subbase $\{\pi_{\alpha,A}^{-1}(U_\alpha) : \alpha \in J, U_\alpha \in \mathscr{T}_{A_\alpha}\}$, where $\pi_{\alpha,A} : A \to A_\alpha$ is the projection mapping.

(b) Show that the subspace topology has the subbase $\{\pi_{\alpha,X}^{-1}(U_\alpha) \cap A : \alpha \in J, U_\alpha \in \mathscr{T}_{X_\alpha}\}$, where $\pi_{\alpha,X} : X \to X_\alpha$ is the projection mapping.

(c) Show that $\{\pi_{\alpha,A}^{-1}(U_\alpha) \subseteq A : \alpha \in J, U_\alpha \in \mathscr{T}_{A_\alpha}\} = \{\pi_{\alpha,X}^{-1}(U_\alpha) \cap A : \alpha \in J, U_\alpha \in \mathscr{T}_{X_\alpha}\}$.

1.3 Connected

Abstract

Given a topological space $(X, \mathscr{T})$, we say nonempty $E \subseteq X$ is

(a) **connected** if E can not be written as $E = A \cup B$ so that $\overline{A} \cap B = A \cap \overline{B} = \varnothing$ and $A \neq \varnothing \neq B$.

(b) **path-connected** if for each $p, q \in E$, there exists a continuous function $f : [0,1] \to E$ such that $f(0) = p, f(1) = q$.

These two notions are sometimes referred as **topological properties**, since they are invariant under continuous function, the "morphism" between topological space. Put more precisely, If $E \subseteq X$ satisfy a topological property and $f : X \to Y$ is continuous, then $f(E)$ also satisfy the topological property.

Theorem 1.3.1. (Equivalent Definitions of Connected) Given a subset $E \subseteq X$, the following statements are equivalent

(a) E is connected in $(X, \mathscr{T})$.

(b) E is connected in $(E, \mathscr{T}_E)$.

(c) The only clopen sets in $(E, \mathscr{T}_E)$ are E and $\varnothing$.

(d) In $(E, \mathscr{T}_E)$, the only set that has empty boundary are E and $\varnothing$.

(e) All continuous function from $(E, \mathscr{T}_E)$ to $\{0, 1\}$ with discrete topology is constant.

Proof. For (a) $\iff$ (b), use the identity $\forall A \subseteq E$, $\text{cl}_X(A) \cap E = \text{cl}_E(A)$. Check straightforward for (b) $\iff$ (c) and (d) $\iff$ (c) $\iff$ (e). ∎

Three things to note here

(a) If $E \subseteq X$ is connected, E can not be covered by any two disjoint open sets intersecting with E in $(X, \mathscr{T}_X)$. The converse is not true. Consider finite subset of an infinite set with cofinite topology.

(b) Union of collection $(A_\alpha)_{\alpha \in J}$ of connected sets with non-empty intersection is connected. Prove this by a proof of contradiction. Path-connectedness has the same property, and the proof is much easier.

(c) If E is connected and $E \subseteq F \subseteq \mathrm{cl}(E)$, then F is connected. Use the fifth equivalent definitions for connected to prove this. Path-connectedness doesn't have the same property this time. Consider the "fattened" Topologist's sine curve $\{(x, y) \in \mathbb{R}^2 : \left|y - \sin \frac{1}{x}\right| < x\}$. It's closure can be easily proved to be $\{(x, y) \in \mathbb{R}^2 : \left|y - \sin \frac{1}{x}\right| \leq x\} \cup \{(0, y) \in \mathbb{R}^2 : y \in [-1, 1]\}$.

(d) Path-connectedness is strictly stronger than connectedness. This can be proved using a proof of contradiction and supremum. Two famous counterexamples of the converse are Topologist's Sine Curve and Long line.

Although path-connectedness is strictly stronger than connectedness, there exists a quite general condition that implies the converse. We say a topological space X is **locally path-connected** if for each p and open set U containing p, there exists a path-connected open set V containing p and contained by U. Following our definition, we see that if X is both connected and locally path-connected, then X must be path-connected, since any path-connected component of X (due to X being locally path-connected) must be clopen in X.

1.4 Compact

Abstract

We now give definitions to three notions so important that they drive us to study Topology in the first place. Given a topological space $(X, \mathscr{T})$, we say nonempty $E \subseteq X$ is

(a) **compact** if every open cover has a finite subcover.

These three properties are often called **topological properties**, since they are invariant under continuous function, the "morphism" between topological space. Put more precisely, If $E \subseteq X$ satisfy a topological property and $f : X \to Y$ is continuous, then $f(E)$ also satisfy the topological property.

Immediately, one should again check the "natural" behaviors of subspace topology: Whether a set E is connected, path-connected, or compact is independent of the choices of ambient space. In other words, given $E \subseteq X$, E is connected, path-connected or compact in $(X, \mathscr{T})$ if and only if E is connected, path-connected or compact in $(E, \mathscr{T}_E)$.

Theorem 1.4.1. (Equivalent Definitions of Compactness) The following statements are equivalent

(a) E is compact in $(X, \mathscr{T})$.

(b) E is compact in $(E, \mathscr{T}_E)$.

(c) Given subbase $\mathcal{B}$ of $(X, \mathscr{T})$, every cover of E consisting of the elements of $\mathcal{B}$ has a finite subcover.

(d) Every infinite subset M of E has a complete limit point in E, that is, a point $x \in E$ such that all open set O containing x satisfy $|O \cap M| = |M|$.

(e) Every collection of closed sets of $(E, \mathscr{T}_E)$ that has finite intersection property has nonempty intersection.

(f) For all topological space Y, the projection $\pi_Y : E \times Y \to Y$ is a closed mapping.

Proof. For (b) $\iff$ (e), use proofs by contradiction. (b) $\iff$ (a) $\implies$ (c) are clear. We now prove

$$\text{(c)} \implies \text{(b)}$$

Fix $\mathcal{B}$. Assume E is not compact. Then the collection $\mathbb{S}$ of all open covers that have no finite subcover is non-empty. Let $\mathcal{C}$ be a maximal element of $\mathbb{S}$. It is clear that $\mathcal{C} \cap \mathcal{B}$ is not a cover of E by premise. Let $x \in E \setminus \bigcup(\mathcal{C} \cap \mathcal{B})$. Let U be an element of $\mathcal{C} \setminus \mathcal{B}$ containing x. Because $\mathcal{B}$ is a subbase, there exists finite $B_1, \dots, B_n \in \mathcal{B}$ such that $x \in B_1 \cap \cdots \cap B_n \subseteq U$. Because $\mathcal{C}$ is a maximal element of $\mathbb{S}$, for all j, the collection $\mathcal{C} \cup \{B_j\}$ does not belong to $\mathbb{S}$. This implies that for each $j \in \{1, \dots, n\}$, there exists a finite sub-collection $\mathcal{C}_j \subseteq \mathcal{C}$ such that $\mathcal{C}_j \cup \{B_j\}$ covers E. Let $\mathcal{C}_F \triangleq \bigcup_{j=1}^n \mathcal{C}_j$. Because $\mathcal{C}_j \cup \{B_j\}$ are covers of E, $\mathcal{C}_F \cup \{B_1, \dots, B_n\}$ is a cover of E. This implies $\mathcal{C}_F \cup \{U\} \subseteq \mathcal{C}$ is a finite subcover. CaC (done)

We now prove

$$\text{(a)} \implies \text{(d)}$$

Assume there exists infinite $M \subseteq E$ that has no complete limit point. Because of our assumption, for each $x \in E$, there exists an open set O_x containing x such that $|M \cap O_x| < |M|$. Because $(O_x)_{x \in E}$ is an open cover of E, there exists a finite sub-cover $(O_x)_{x \in I}$. Note that M is infinite, so we can deduce

$$|M| = \left| \bigcup_{x \in I} M \cap O_x \right| \leq \sum_{x \in I} |M \cap O_x| < |M| \text{ CaC (done)}$$

We now prove

$$\text{(d)} \implies \text{(a)}$$

Assume E is not compact. Let O be an open cover of E that has no finite subcover with smallest cardinality c. Well-order O by $O \triangleq \{O_\alpha\}_{\alpha < c}$. Use transfinite recursion to build $M \triangleq \{x_\alpha : \alpha < c\}$ where $x_\alpha \in E \setminus \bigcup_{\beta < \alpha} O_\beta$. Such x_α always exists; otherwise, there exists an open cover of E that has no finite subcover with cardinality smaller than c. To cause a contradiction, it remains to show

$$M \text{ has no complete limit point in } E$$

Because O is an open cover of E, for all x, there exists some O_α containing x. Observe using the definition of M

$$|O_\alpha \cap M| \leq |\{x_\gamma : \gamma \leq \alpha\}| \leq |\alpha| < c = |M| \text{ CaC (done)}$$

Before we prove (a) $\implies$ (f), we first prove the Generalized Tube Lemma. That is,

> Given a product space $X \times Y$, compact $A \subseteq X$, compact $B \subseteq Y$, and $N \subseteq X \times Y$ open containing $A \times B$, there exists $U \subseteq X$ open, $V \subseteq Y$ open such that $A \times B \subseteq U \times V \subseteq N$.

First note that for all $(a,b) \in A \times B$, there exists $U_{(a,b)} \subseteq X$ open and $V_{(a,b)} \subseteq Y$ open such that $(a,b) \in U_{(a,b)} \times V_{(a,b)} \subseteq N$. Because A is compact and for all b, the collection $(U_{(a,b)})_{a\in A}$ is an open cover of A, there exists a finite subset $A_b \subseteq A$ for all b such that $A \subseteq \bigcup_{a\in A_b} U_{(a,b)}$. Now, let $U_b \triangleq \bigcup_{a\in A_b} U_{(a,b)}$ and $V_b \triangleq \bigcap_{a\in A_b} V_{(a,b)}$. It is clear that U_b, V_b are open, and it is straightforward to check $A \times \{b\} \subseteq U_b \times V_b \subseteq N$. Again, because B is compact and $(V_b)_{b\in B}$ is an open cover of B, there exists a finite subset $B_0 \subseteq B$ such that $B \subseteq \bigcup_{b\in B_0} V_b$. Let $V \triangleq \bigcup_{b\in B_0} V_b$ and $U \triangleq \bigcap_{b\in B_0} U_b$. It is straightforward to check U, V suffice. (done)

We now prove

$$\text{(a)} \implies \text{(f)}$$

Given $A \subseteq X \times Y$ closed, we are required to prove $\pi_Y(A)$ is closed. WOLG, assume $\pi_Y(A) \neq Y$. Fix $y \in Y \setminus \pi_Y(A)$. Because X and $\{y\}$ are compact and $X \times \{y\}$ is a subset of the open set A^c, by the Generalized Tube Lemma, there exists open $V \subseteq Y$ such that $X \times \{y\} \subseteq X \times V \subseteq A^c$. It is straightforward to check $V \cap \pi_Y(A) = \varnothing$. (done)

Lastly, we prove

$$\text{(f)} \implies \text{(a)}$$

Assume X is not compact. Let $(O_\alpha)_{\alpha\in J}$ be an open cover of X with no finite subcover. Consider the following construction:

(a) $\mathcal{U} \triangleq \{\bigcup_{\alpha\in I} O_\alpha : I \text{ is a finite subset of } J\}$ is an open cover of X with no finite subcover,

(b) $\mathcal{U}$ is closed under finite union,

(c) $\mathcal{F} \triangleq \{U^c : U \in \mathcal{U}\}$ is a collection of non-empty closed sets that has the finite intersection property.

(d) If we let $Y \triangleq X \cup \{p\}$ where $p \notin X$, then $\mathscr{T}_Y \triangleq \mathcal{P}(X) \cup \{\{p\} \cup A : \exists F \in \mathcal{F}, F \subseteq A \subseteq X\}$ is a topology on Y, where $\mathcal{P}(X)$ is the collection of all subsets of X.

(e) Let $C \triangleq \text{cl}_{X\times Y}\{(x,x) \in X \times Y : x \in X\}$.

(f) Fix $x \in X$. Because $\mathcal{U}$ is an open cover of X, there exists $U \in \mathcal{U}$ containing x. Note that $\{p\} \cup U^c$ is open in Y. This implies $U \times (\{p\} \cup U^c)$ is an open subset of $X \times Y$ containing (x,p). We have proved $C \subseteq X \times X$.

(g) It is clear that X is not closed in Y. Now observe that π_Y maps the closed set C to the open set $X \subseteq Y$. CaC (done)

■

Notably, one can easily check that closed subspace of compact space must be compact, and the converse is true when the ambient space is Hausdorff.

Theorem 1.4.2. (Compact Subspace of Hausdorff Space is Closed) If E is a compact subspace of Hausdorff space X, then E is closed in X.

Proof. Fix $x \in E^c$. Because X is Hausdorff, we can associate each $y \in E$ an open set U_y containing y and an open set $U_{x,y}$ containing x such that $U_y, U_{x,y}$ are disjoint. Now, because E is compact and (U_y) is an open cover of E, we know there exists a finite sub-cover

$$E \subseteq \bigcup_{n=1}^{N} U_{y_n}$$

It then follows that open $\bigcap_{n=1}^{N} U_{x,y_n}$ contain x and disjoint with E. ■

Corollary 1.4.3. (Homeomorphism between Compact Space and Hausdorff Space) Suppose

(a) X is compact.

(b) Y is Hausdorff.

(c) $f : X \to Y$ is a continuous bijective function.

Then

$$f \text{ is a homeomorphism between } X \text{ and } Y$$

Proof. Because closed subset of compact set is compact and continuous function send compact set to compact set, we see for each closed $E \subseteq X$, $f(E) \subseteq Y$ is compact. The result then follows from $f(E) \subseteq Y$ being closed since Y is Hausdorff. ■

1.5 Countability Axioms

Abstract

This section introduce countability axioms.

Given a topological space X and some $p \in X$, by an **open-neighborhood basis $\mathcal{B}_p$ at p**, we mean a collection of open-neighborhood of p such that each open set containing p contain some element of $\mathcal{B}_p$. If we say X is **first countable at** p, we mean there exists some countable open-neighborhood basis at p, and if we say X is **first countable**, we mean X is first countable at all $p \in X$. Recall that in general topology, we have the following propositions

(a) If there exists some sequence x_n in A such that $x_n \to x$, then $x \in \overline{A}$.

(b) Given a function $f : X \to Y$, if f is continuous, then f preserve convergence, i.e., $x_n \to x \implies f(x_n) \to f(x)$

In the specific case of $\mathbb{R}^n$, the converses of these propositions also hold true, a property attributed to $\mathbb{R}^n$ being first countable. This can be verified through the construction of countable bases at x. However, the converse of this is generally not true, consider

Example 1 (Space of real eventually 0 sequence)

$$X \triangleq \{(x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}} : \text{ All but finitely many of } x_n \text{ are } 0 \}$$

We define $S \subseteq X$ is open $\overset{\triangle}{\iff}$ for all $n, \{(x_1, \dots, x_n) \in \mathbb{R}^n : (x_1, \dots, x_n, 0, 0, \dots) \in S\}$ is open in $\mathbb{R}^n$. Consider

$$B_n \triangleq \left\{ x \in X : \Big(\sum_{j=1}^{n} x_j^2 \Big)^{\frac{1}{2}} < \frac{1}{n} \text{ and } x_{n+k} = 0 \text{ for all } k \in \mathbb{N} \right\}$$

and

$$S \triangleq X \setminus \bigcup_{n \in \mathbb{N}} B_n$$

Suppose $x \in B_N$ for some N. By definition, we know

$$\Big(\sum_{j=1}^{N} x_j^2 \Big)^{\frac{1}{2}} < \frac{1}{N} \text{ and } x_{N+k} = 0 \text{ for all } k$$

Let U be an open-neighborhood around x. By definition, we know there exists some

$$y = (x_1, \dots, x_N, \epsilon_1, \epsilon_2, \dots, \epsilon_M, 0, 0, \dots) \in U$$

such that $y \in S$ for large enough M.

In other words, the converse of propositions (a), (b) hold true whenever X is first countable at x.

Theorem 1.5.1. (Sequential Compactness and Countable Compactness is equivalent in first countable space)

Stronger and perhaps more interesting, we say X is **second countable** if X has a countable basis.

Theorem 1.5.2. (Basis Property of Second Countable Space) Given a second countable space X, if $\mathcal{B}$ is a basis of X, then there exists some countable $\mathcal{B}'$ basis of X such that $\mathcal{B}' \subseteq \mathcal{B}$.

Proof. Let (C_n) be a countable basis of X. Define

$$\mathcal{B}_{(n,m)} \triangleq \{B \in \mathcal{B} : C_n \subseteq B \subseteq C_m\}$$

If $\mathcal{B}_{(n,m)}$ is nonempty, pick one $B_{(n,m)} \in \mathcal{B}_{n,m}$. Clearly, the collection of $B_{(n,m)}$ is countable. To see that this collection is a basis, fix x and open U containing x. Because (C_k) and $\mathcal{B}$ are both basis, there exists some n, m and B such that $x \in C_n \subseteq B \subseteq C_m \subseteq U$. This implies that $\mathcal{B}_{(n,m)}$ is nonempty and there exists some $B_{(n,m)}$ such that $x \in B_{n,m} \subseteq C_m \subseteq U$. ∎

Theorem 1.5.3. (Second countable space is Lindelöf) Open cover of second countable space always have a countable subcover.

Proof. Let X be a space with basis (B_n) and some open cover $\mathcal{S}$. For each n, let S_n be an element of $\mathcal{S}$ containing B_n if possible. To see that (S_n) form a cover, fix x and some $S \in \mathcal{S}$ containing x. Because (B_n) is a basis, there exists some B_m such that $x \in B_m \subseteq S$. This implies the existence of S_m, which contain x. ∎

1.6 Locally Compact Hausdorff Space

Abstract

We say a topological space is **locally compact** if for each point p there exists some compact subspace containing some open-neighborhood of p.

Theorem 1.6.1. (Locally Compact Hausdorff Space admits a pre-compact basis) Suppose X is a locally compact Hausdorff space. There exists a basis of X whose elements all have compact closure.

Proof. If we define

$$\mathcal{B}_p \triangleq \{U : U \text{ is an open-neighborhood around } p \text{ with compact closure.}\}$$

Because compact subspace of Hausdorff space is closed and closed subspace of compact space is compact, one can deduce $\mathcal{B}_p$ is a local basis. ∎

1.7 Quotient Topology

Abstract

This is a short section introducing quotient topology.

Let $\sim$ be an equivalence relation on some topological space X, and let $\pi : X \to Y \triangleq X \backslash \sim$ be the projection map, we can define the **quotient topology** on Y by

$$U \subseteq Y \text{ is open } \overset{\triangle}{\iff} \pi^{-1}(U) \subseteq X \text{ is open}$$

It is easily checked that

(a) Quotient topology is indeed a topology.

(b) Quotient topology is the largest (finest) topology on Y such that $\pi : X \to Y$ is continuous.

(c) A function $f : Y \to Z$ is continuous if and only if $f \circ \pi$ is continuous. (Universal Property)

Because the quotient map π is continuous, we know quotient preserve topological properties like connected and compact, yet notably, quotient does not preserve separation axioms. Consider the following examples.

Example 2 (Quotient does NOT preserve Second Countable)

$$X \triangleq \mathbb{R} \text{ and } x \sim y \overset{\triangle}{\iff} x = y \text{ or } x, y \in \mathbb{Z}$$

Let $Y \triangleq X \backslash \sim$. We show that Y is not even first countable at $[0]$. Let $U_n \subseteq Y$ be an arbitrary sequence of open-neighborhood of $[0]$. It is easily checked that for each $n \in \mathbb{N}$, there exists $\epsilon_{n,k}$ such that

$$\pi\Big[\bigcup_{k \in \mathbb{Z}} (k - \epsilon_{n,k}, k + \epsilon_{n,k})\Big] \subseteq U_n$$

Define $\delta_k \triangleq \frac{\epsilon_{k,k}}{2}$ and

$$V = \pi\Big[\bigcup_{k \in \mathbb{Z}} (k - \delta_k, k + \delta_k)\Big]$$

It is easily checked that V is an open neighborhood of $[0]$ contained in no U_n.

Example 3 (Quotient does NOT preserve Hausdorff)

$$X = \mathbb{R} \text{ and } Y = \{(-\infty, 0), [0, \infty)\}$$

However, with the criterion of π being an open mapping, we can draw some useful conclusions. For example, if $\pi : X \to Y$ is an open mapping and $\mathcal{B}$ is a basis of X, then $\pi(\mathcal{B})$ is a basis of Y.

Theorem 1.7.1. (Hausdorff and Quotient) If $\pi : X \to Y$ is an open mapping, and we define

$$R_\pi \triangleq \{(x, y) \in X^2 : \pi(x) = \pi(y)\}$$

Then

$$R_\pi \text{ is closed} \iff Y \text{ is Hausdorff}$$

Proof. Suppose R_π is closed. Fix some x, y such that $\pi(x) \neq \pi(y)$. Because R_π is closed, we know there exists open neighborhood U_x, U_y such that $U_x \times U_y \subseteq (R_\pi)^c$. It is clear that $\pi(U_x), \pi(U_y)$ are respectively open neighborhood of $\pi(x)$ and $\pi(y)$. To see $\pi(U_x)$ and $\pi(U_y)$ are disjoint, assume that $\pi(a) \in \pi(U_x) \cap \pi(U_y)$. Let $a_x \in U_x$ and $a_y \in U_y$ satisfy $\pi(a_x) = \pi(a) = \pi(a_y)$, which is impossible because $(a_x, a_y) \in (R_\pi)^c$. CaC

Suppose Y is Hausdorff. Fix some x, y such that $\pi(x) \neq \pi(y)$. Let U_x, U_y be open neighborhoods of $\pi(x), \pi(y)$ separating them. Observe that $(x, y) \in \pi^{-1}(U_x) \times \pi^{-1}(U_y) \subseteq (R_\pi)^c$ ■

Notably, quotient topology give us a famously weird homeomorphism.

Example 4 (Weird Quotient)

$$\mathbb{R} \setminus \mathbb{Z} \simeq S^1 \triangleq \{e^{ix} \in \mathbb{C} : x \in \mathbb{R}\}$$

Clearly, we can well define a map $F : \mathbb{R} \setminus \mathbb{Z} \to S^1$ by

$$F(\pi(x)) \triangleq e^{i2\pi x}$$

It is straightforward to check F is a continuous bijection and $\mathbb{R} \setminus \mathbb{Z}$ is compact. It then follows F is a homeomorphism.

1.8 Topological Group

Abstract

This section introduce the notion of topological group and prove that quotient group of a topological group when equipped with the quotient topology is again a topological group.

By a **topological group**, we mean a topological space M equipped with a group structure such that addition $M^2 \to M$ and inversion $M \to M$ are both continuous. Equivalently, one can simply require

$$M^2 \to M; (g,h) \mapsto gh^{-1}$$

to be continuous.

Theorem 1.8.1. (Quotient Group of Topological Group)

Chapter 2

Metric Space

2.1 Completion

2.2 Bounded and Totally Bounded

2.3 Compactness

Lemma 2.3.1. (Lebesgue's Number Lemma) If $\mathcal{U}$ is an open cover of a compact subset K, then there exists some $\delta > 0$, called **Lebesgue's number of the cover** $\mathcal{U}$, such that every subset of K with diameter strictly less than δ is contained by some element of $\mathcal{U}$.

Proof. Because K is compact, there exists some finite sub-cover $\{U_1, \dots, U_n\}$ of K. Note that if there exists U_j containing K, then the proof is trivial because every δ suffice to be the Lebesgue's number of $\mathcal{U}$. We from now suppose no U_j contain K. Now, if we define $f : K \to \mathbb{R}$ by

$$f(x) \triangleq \frac{1}{n} \sum_{j=1}^{n} d(x, K \setminus U_j)$$

We see

$$\delta \triangleq \min_{x \in K} f(x) > 0$$

To see that δ suffices, first observe that for each subset E of K with diameter strictly less than δ, we have

$$E \subseteq B_\delta(p) \text{ for all } p \in E$$

and because

$$\frac{1}{n} \sum_{j=1}^{n} d(p, K \setminus U_j) \geq \delta$$

there exists some j such that

$$d(p, K \setminus U_j) \geq \delta$$

which implies

$$B_\delta(p) \cap K \setminus U_j = \varnothing$$

and thus $E \subseteq U_j$. ∎

Theorem 2.3.2. (Decreasing Sequence of Compact Sets) If $\Omega_1 \supseteq \Omega_2 \supseteq \cdots \supseteq \Omega_n \supseteq \cdots$ is a sequence of non-empty compact sets in metric space M with property

$$\text{diam}(\Omega_n) \to 0 \text{ as } n \to \infty$$

then $\bigcap \Omega_n$ contain exactly one point.

Proof. By premise, $\{\Omega_n \subseteq \Omega_1 : n \geq 2\}$ is a collection of subsets of Ω_1 closed in Ω_1 satisfying finite intersection property. It then follows from one of the definition of compactness that $\bigcap \Omega_n$ is non-empty. Because $\bigcap \Omega_n \subseteq \Omega_N$ for all $N \in \mathbb{N}$, we see that $\text{diam}(\bigcap \Omega_n) = 0$. It follows that $\bigcap \Omega_n$ contain exactly one point. ∎

2.4 Limit Interchange

In this section, we

(a) discuss the condition in which we can change the limit order of double sequence in general metric space. (Theorem 2.4.1 and Theorem 2.4.2)

(b) prove that the space of functions is complete if and only if the codomain is complete. (Theorem 2.4.3)

(c) prove that the uniform limit of a sequence of convergent sequences in a complete metric space converge. (Theorem 2.4.5)

Remark on structure of the Theory: The proof of (Theorem 2.4.5: convergent sequences in complete metric space is closed under uniform convergence) relies on (Theorem 2.4.1: exchange limit order), while that of (Theorem 2.4.3: Space of functions (X^Y, d_∞) is complete iff Y is complete) does not.

(Theorem 2.4.1: exchange limit order) will later be used to prove the Uniform Limit Theorem (Theorem 2.5.2) which is a "pointwise" Theorem, and justify abundant of limit exchange, e.g. (Theorem 2.5.1: exchange limit order for functions)

An important consequence of (Corollary 2.4.4: space of bounded functions into complete space is complete) is that $\left(L(\mathbb{R}^n, \mathbb{R}^m), \|\cdot\|_{\text{op}}\right)$ is complete. This will be later shown with extra tools.

Theorem 2.4.1. (Change Order of Limit Operations: Part 1) Given a double sequence $a_{n,k}$ whose codomain is (Y, d). Suppose

(a) $a_{n,k} \to a_{\bullet}$, uniformly as $n \to \infty$

(b) $a_{n,k} \to A_n$ pointwise as $k \to \infty$.

(c) $A_n \to A$

We have

$$\lim_{k\to\infty} a_{\bullet,k} = A$$

In other words, we can switch the order of limit operations

$$\lim_{k\to\infty}\lim_{n\to\infty} a_{n,k} = \lim_{n\to\infty}\lim_{k\to\infty} a_{n,k}$$

Proof. We wish to prove

$$a_{\bullet,k} \to A \text{ as } k \to \infty$$

Fix ϵ. Because $a_{n,k} \to a_{\bullet,k}$ uniformly and $A_n \to A$ as $n \to \infty$, we know there exists m such that

$$d(A_m, A) < \frac{\epsilon}{3} \text{ and } \forall k \in \mathbb{N}, d(a_{m,k}, a_{\bullet,k}) < \frac{\epsilon}{3} \tag{2.1}$$

Then because $a_{m,k} \to A_m$ as $k \to \infty$, we know there exists K such that

$$\forall k > K, d(a_{m,k}, A_m) < \frac{\epsilon}{3} \tag{2.2}$$

We now claim

$$\forall k > K, d(a_{\bullet,k}, A) < \epsilon$$

The claim is true since by Equation 2.1 and Equation 2.2, we have

$$\forall k > K, d(a_{\bullet,k}, A) \leq d(a_{\bullet,k}, a_{m,k}) + d(a_{m,k}, A_m) + d(A_m, A) < \epsilon \text{ (done)}$$

∎

Theorem 2.4.2. (Change Order of Limit Operations: Part 2) Given a double sequence $a_{n,k}$ whose codomain is (Y, d). Suppose

(a) $a_{n,k} \to a_{\bullet,k}$ uniformly as $n \to \infty$

(b) $a_{n,k} \to A_n$ pointwise as $k \to \infty$

(c) $a_{\bullet,k} \to A$ as $k \to \infty$

We have

$$A_n \to A$$

Proof. Fix ϵ. We wish to find N such that

$$\forall n > N, d(A_n, A) < \epsilon$$

Because $a_{n,k} \to a_{\bullet,k}$ uniformly as $n \to \infty$, we can let N satisfy

$$\forall n > N, \forall k \in \mathbb{N}, d(a_{n,k}, a_{\bullet,k}) < \frac{\epsilon}{3} \tag{2.3}$$

We claim

$$\text{such } N \text{ works}$$

Arbitrarily pick $n > N$. Because $a_{\bullet,k} \to A$, and because $a_{n,k} \to A_n$, we know there exists j such that

$$d(a_{\bullet,j}, A) < \frac{\epsilon}{3} \text{ and } d(a_{n,j}, A_n) < \frac{\epsilon}{3} \tag{2.4}$$

From Equation 2.3 and Equation 2.4, we now have

$$d(A_n, A) \leq d(A_n, a_{n,j}) + d(a_{n,j}, a_{\bullet,j}) + d(a_{\bullet,j}, A) < \epsilon \text{ (done)}$$

∎

In summary of Theorem 2.4.1 and Theorem 2.4.2, given a double sequence $a_{n,k}$ converging both side

(a) $a_{n,k} \to a_{\bullet,k}$ pointwise as $n \to \infty$

(b) $a_{n,k} \to a_{n,\bullet}$ pointwise as $k \to \infty$

As long as

(a) one side of convergence is uniform

(b) between two sequence $\{a_{\bullet,k}\}_{k\in\mathbb{N}}$ and $\{a_{n,\bullet}\}_{n\in\mathbb{N}}$, one of them converge, say, to A

Then the other sequence also converge, and the limit is also A.

It is at this point, we shall introduce two other terminologies. Suppose f_n is a sequence of functions from an arbitrary set X to a metric space Y. We say f_n is **pointwise Cauchy** if for all fixed $x \in X$, the sequence $f_n(x)$ is Cauchy. We say f_n is **uniformly Cauchy** if for all ϵ, there exists $N \in \mathbb{N}$ such that

$$\forall n, m > N, \forall x \in X, d\big(f_n(x), f_m(x)\big) < \epsilon$$

In last Section (Section 2.6), we define the **uniform metric** d_∞ on X^Y by

$$d_\infty(f,g) = \sup_{x\in X} d\big(f(x), g(x)\big)$$

and say that $f_n \to f$ uniformly if and only if $f_n \to f$ in (X^Y, d_∞). Similar to this clear fact, we have

$$f_n \text{ is uniformly Cauchy} \iff f_n \text{ is Cauchy in } (X^Y, d_\infty)$$

It should be very easy to verify that if f_n uniformly converge, then f_n is uniformly Cauchy, and just like sequences in metric space, the converse hold true if and only if the space $\big(X^Y, d_\infty\big)$ is complete. In Theorem 2.4.3, we give a necessary and sufficient condition for $\big(X^Y, d_\infty\big)$ to be complete.

Theorem 2.4.3. (Space of functions $\big(X^Y, d_\infty\big)$ is Complete iff Y is Complete) Given an arbitrary set X and a metric space (Y, d), we have

$$\text{the extended metric space } \big(X^Y, d_\infty\big) \text{ is complete} \iff Y \text{ is complete}$$

Proof. $(\longleftarrow)$

Suppose f_n is uniformly Cauchy. We wish

to construct a $f : X \to Y$ such that $f_n \to f$ uniformly

Because f_n is uniformly Cauchy, we know that for all $x \in X$, the sequence $f_n(x)$ is Cauchy in (Y, d). Then because Y is complete, we can define $f : X \to Y$ by

$$f(x) = \lim_{n\to\infty} f_n(x)$$

We claim

such f works, i.e. $f_n \to f$ uniformly

Fix ϵ. We wish

to find $N \in \mathbb{N}$ such that for all $n > N$ and $x \in X$ we have $d\big(f_n(x), f(x)\big) < \epsilon$

Because f_n is uniformly Cauchy, we know there exists N such that

$$\forall n, m > N, \forall x \in X, d\big(f_n(x), f_m(x)\big) < \frac{\epsilon}{2} \tag{2.5}$$

We claim

such N works

Assume there exists $n > N$ and $x \in X$ such that $d\big(f_n(x), f(x)\big) \geq \epsilon$. Because $f_k(x) \to f(x)$ as $k \to \infty$, we know

$$\exists m \in \mathbb{N}, d\big(f_m(x), f(x)\big) < \frac{\epsilon}{2} \tag{2.6}$$

Then from Equation 2.5 and Equation 2.6, we can deduce

$$\epsilon \leq d\big(f_n(x), f(x)\big) \leq d\big(f(x), f_m(x)\big) + d\big(f_n(x), f_m(x)\big) < \epsilon \text{ CaC} \quad \text{(done)}$$

$(\longrightarrow)$

Let K be the set of constant functions in X^Y. We first prove

K is closed

Arbitrarily pick $f \in K^c$. We wish

to find $\epsilon \in \mathbb{R}^+$ such that $B_\epsilon(f) \in K^c$

Because f is not a constant function, we know there exists $x_1, x_2 \in X$ such that

$$d\big(f(x_1), f(x_2)\big) > 0$$

We claim that

$$\epsilon = \frac{d\big(f(x_1), f(x_2)\big)}{3} \text{ works}$$

Arbitrarily pick $g \in B_\epsilon(f)$. We wish

$$\text{to show } g \in K^c$$

Notice the triangle inequality

$$3\epsilon = d\big(f(x_1), f(x_2)\big) \leq d\big(f(x_1), g(x_1)\big) + d\big(g(x_1), g(x_2)\big) + d\big(g(x_2), f(x_2)\big) \tag{2.7}$$

Also, because $g \in B_\epsilon(f)$, we have

$$\forall x \in X, d\big(f(x), g(x)\big) < \epsilon \tag{2.8}$$

Then by Equation 2.7 and Equation 2.8, we see

$$d\big(g(x_1), g(x_2)\big) > \epsilon$$

This then implies g is not a constant function. (done)

Now, Because by premise (X^Y, d_∞) is complete, and we have proved K is closed in (X^Y, d_∞), we know K is complete. Then, we resolve the whole problem into proving

$$Y \text{ is isometric to } K$$

Define $\sigma : Y \to K$ by

$$y \mapsto \tilde{y} \text{ where } \forall x \in X, \tilde{y}(x) = y$$

It is easy to verify σ is an isometry. (done) ■

Corollary 2.4.4. (Space of Bounded functions $\big(B(X,Y), d_\infty\big)$ is Complete iff Y is Complete)

$$\big(B(X,Y), d_\infty\big) \text{ is complete } \iff Y \text{ is complete}$$

Proof. $(\longleftarrow)$

By Theorem 2.4.3, the space (X^Y, d_∞) is complete. Then because $B(X,Y)$ is closed in (X^Y, d_∞), we know $B(X,Y)$ is complete.

$(\longrightarrow)$

Notice that the set of constant function K is a subset of the galaxy $B(X,Y)$. The whole proof in Theorem 2.4.3 works in here too. ■

Remember in the beginning of this section we say we will prove convergent sequences in Y is closed under uniform convergence if Y is complete. The proof of this result relies on Theorem 2.4.3.

Theorem 2.4.5. (Convergent Sequences are Closed under Uniform Convergence if Codomain (Y, d) is Complete) Given a complete metric space (Y, d), let $\mathcal{C}_{\mathbb{N}}^{Y}$ be the set of convergent sequences in Y.

$$Y \text{ is complete} \implies \mathcal{C}_{\mathbb{N}}^{Y} \text{ is closed under uniform convergent}$$

Proof. Let $a_{n,k} \to a_{\bullet,k}$ uniformly as $n \to \infty$ where for all $n, k \in \mathbb{N}, a_{n,k} \in Y$ and let $A_n = \lim_{k\to\infty} a_{n,k}$ for all $n \in \mathbb{N}$.

$$\text{to prove } a_{\bullet,k} \text{ converge}$$

By Theorem 2.4.2, we can reduce the problem to

$$\text{proving } A_n \text{ converge}$$

Then because Y is complete, we can then reduce the problem into proving

$$A_n \text{ is Cauchy}$$

Fix ϵ. We wish to find N such that

$$\forall n, m > N, d(A_n, A_m) < \epsilon$$

Because $a_{n,k} \to a_{\bullet,k}$ uniformly, we can find N such that

$$\forall n, m > N, d_\infty(\{a_{n,k}\}_{k\in\mathbb{N}}, \{a_{m,k}\}_{k\in\mathbb{N}}) < \frac{\epsilon}{3} \tag{2.9}$$

We claim

$$\text{such } N \text{ works}$$

Arbitrarily pick $n, m > N$. We wish to prove

$$d(A_n, A_m) < \epsilon$$

Because $a_{n,k} \to A_n$ and $a_{m,k} \to A_m$ as $k \to \infty$, we can find j such that

$$d(a_{n,j}, A_n) < \frac{\epsilon}{3} \text{ and } d(a_{m,j}, A_m) < \frac{\epsilon}{3} \tag{2.10}$$

Then from Equation 2.9 and Equation 2.10, we can deduce

$$d(A_n, A_m) \le d(A_n, a_{n,j}) + d(a_{n,j}, a_{m,j}) + d(a_{m,j}, A_m) < \epsilon \text{ (done)}$$

∎

2.5 Closed under Uniform Convergence

Given $(E, d_E), (Y, d_Y)$ and a sequence of functions $f_n : E \to Y$, converging uniformly to some $f : E \to Y$ such that each f_n has the property

(a) Boundedness

(b) Unboundedness

(c) Continuity

(d) Uniform continuity

(e) K-Lipschitz continuity

on E, then f also has the same property. These fact will later be proved in **??** and **??**. are again all closed under uniform convergence, where the proof for continuity is closed under uniform convergence use Theorem 2.4.1 as a lemma.

The reason we require the co-domain Y of sequence to be complete is explained in the last paragraph of Section 2.6. An example of such beautiful closure is lost if the codmain (Y, d) is not complete is $Y = \mathbb{R}^*$ and $a_{n,k} = \frac{1}{n} + \frac{1}{k}$.

Theorem 2.5.1. (Change Order of Limit Operation in Complete Metric Space) Given a sequence of function $f_n : E \to (Y, d)$ and a function $f : E \to (Y, d)$ such that

(a) $f_n \to f$ uniformly on E

(b) $\lim_{t \to x} f_n(t)$ exists for all $n \in \mathbb{N}$

(c) (Y, d) is complete

We have

$$\lim_{n\to\infty} \lim_{t\to x} f_n(t) = \lim_{t\to x} \lim_{n\to\infty} f_n(t)$$

Proof. Fix a sequence t_k in E that converge to x. We reduced the problem into proving

$$\lim_{n\to\infty} \lim_{k\to\infty} f_n(t_k) = \lim_{k\to\infty} \lim_{n\to\infty} f_n(t_k)$$

Set

$$a_{n,k} \triangleq f_n(t_k) \tag{2.11}$$

We then reduced the problem into proving

$$\lim_{n\to\infty} \lim_{k\to\infty} a_{n,k} = \lim_{k\to\infty} \lim_{n\to\infty} a_{n,k}$$

Set

$$A_n \triangleq \lim_{t\to x} f_n(t) \text{ and } a_{\bullet,k} \triangleq \lim_{n\to\infty} f_n(t_k)$$

We now prove

$$A_n \text{ converge}$$

Fix ϵ. We wish

$$\text{to find } N \text{ such that } d(A_n, A_m) \le \epsilon \text{ for all } n, m > N$$

Because $a_{n,k}$ uniformly converge (to $a_{\bullet,k}$) as $n \to \infty$ by our setting, we know there exists N such that

$$\forall n, m > N, \forall k \in \mathbb{N}, d(a_{n,k}, a_{m,k}) < \frac{\epsilon}{3}$$

We claim

$$\text{such } N \text{ works}$$

Fix $n, m > N$. Because $a_{n,k} \to A_n$ and $a_{m,k} \to A_m$, we know there exists $j \in \mathbb{N}$ such that

$$d(a_{n,j}, A_n) < \frac{\epsilon}{3} \text{ and } d(a_{m,j}, A_m) < \frac{\epsilon}{3}$$

We now have

$$\begin{aligned} d(A_n, A_m) &\le d(A_n, a_{n,j}) + d(a_{n,j}, a_{m,j}) + d(a_{m,j}, A_m) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \text{ (done)} \end{aligned}$$

Now, because $a_{n,k} \to a_{\bullet,k}$ uniformly, by Theorem 2.4.2, we have

$$\lim_{n\to\infty} \lim_{k\to\infty} a_{n,k} = \lim_{n\to\infty} A_n = \lim_{k\to\infty} a_{\bullet,k} = \lim_{k\to\infty} \lim_{n\to\infty} a_{n,k} \text{ (done)}$$

∎

The end goal for this section is to prove that the following properties

(a) continuity

(b) uniform continuity

(c) K-Lipschitz Continuity

Theorem 2.5.2. (Uniform Limit Theorem) Given a sequence of function f_n from a topological space (X, τ) to a metric space (Y, d), suppose

(a) $f_n \to f$ uniformly as $n \to \infty$

(b) f_n is continuous for all $n \in \mathbb{N}$

Then f is also continuous.

Proof. Fix $x \in X$, and let $x_k \to x$. We wish to prove

$$f(x_k) \to f(x)$$

Because $f_n \to f$ uniformly as $n \to \infty$, we know

$$\Big\{f_n(x_k)\Big\}_{k\in\mathbb{N}} \to \Big\{f(x_k)\Big\}_{k\in\mathbb{N}} \text{ uniformly as } n \to \infty \tag{2.12}$$

Also, because for each $n \in \mathbb{N}$, the function f_n is continuous at x, we know

$$\forall n \in \mathbb{N}, f_n(x_k) \to f_n(x) \text{ as } k \to \infty \tag{2.13}$$

Then because $f_n \to f$ pointwise, we know

$$f_n(x) \to f(x) \tag{2.14}$$

Now, because Equation 2.12, Equation 2.13 and Equation 2.14, by Theorem 2.4.1, we have

$$\lim_{k\to\infty} f(x_k) = \lim_{k\to\infty}\lim_{n\to\infty} f_n(x_k) = \lim_{n\to\infty}\lim_{k\to\infty} f_n(x_k) = \lim_{n\to\infty} f_n(x) = f(x) \text{ (done)}$$

∎

Suppose X is a compact Hausdroff space, with Theorem **??**, we can now say that the set $\mathcal{C}(X)$ of complex-valued continuous functions on X

Theorem 2.5.3. (Uniformly Continuous functions are Closed under Uniform Convergence) Given a sequence of functions f_n from a metric space (X, d_X) to metric space (Y, d_Y), suppose

(a) $f_n \to f$ uniformly

(b) f_n is uniformly continuous for all $n \in \mathbb{N}$

Then f is also uniformly continuous

Proof. Fix ϵ. We wish

$$\text{to find } \delta \text{ such that } \forall x, y \in X, d_X(x,y) < \delta \implies d_Y\big(f(x), f(y)\big) < \epsilon$$

Because $f_n \to f$ uniformly, we know there exists $m \in \mathbb{N}$ such that

$$\forall x \in X, d_Y\big(f_m(x), f(x)\big) < \frac{\epsilon}{3} \tag{2.15}$$

Because f_m is uniformly continuous, we know

$$\exists \delta, \forall x, y \in X, d_X(x,y) < \delta \implies d_Y\big(f_m(x), f_m(y)\big) < \frac{\epsilon}{3} \tag{2.16}$$

We claim

$$\text{such } \delta \text{ works}$$

Let $x, y \in X$ satisfy $d_X(x,y) < \delta$. We wish

$$\text{to prove } d_Y\big(f(x), f(y)\big) < \epsilon$$

From Equation 2.15 and Equation 2.16, we have

$$d_Y\big(f(x), f(y)\big) \leq d_Y\big(f(x), f_m(x)\big) + d_Y\big(f_m(x), f_m(y)\big) + d_Y\big(f_m(y), f(y)\big) = \epsilon \text{ (done)}$$

■

Theorem 2.5.4. **(K-Lipschitz functions are Closed under Uniform Convergence)** Given a sequence of functions f_n from metric space (X, d_X) to metric space (Y, d_Y), suppose

(a) $f_n \to f$ uniformly as $n \to \infty$

(b) f_n is K-Lipschtize continuous for all $n \in \mathbb{N}$

Then f is also K-Lipschtize continuous.

Proof. Arbitrarily pick $x, y \in X$, to show f is K-Lipschtize continuous, we wish

$$\text{to show } d_Y\big(f(x), f(y)\big) \leq K d_X(x,y)$$

Fix ϵ. We reduce the problem into proving

$$d_Y\big(f(x), f(y)\big) < K d_X(x,y) + \epsilon$$

Because $f_n \to f$ uniformly as $n \to \infty$, we know there exists m such that

$$\forall z \in X, d_Y\big(f(z), f_m(z)\big) < \frac{\epsilon}{2} \tag{2.17}$$

Because f_m is K-Lispchitz continuous, we know

$$d_Y\big(f_m(x), f_m(y)\big) \leq K d_X(x,y) \tag{2.18}$$

Now, from Equation 2.18 and Equation 2.17, we now see

$$d_Y\big(f(x), f(y)\big) \leq d_Y\big(f(x), f_m(x)\big) + d_Y\big(f_m(x), f_m(y)\big) + d_Y\big(f_m(y), f(y)\big) < K d_X(x,y) + \epsilon$$

■

An example of sequences of Lipschitz continuous functions with unbounded Lipschitz constant can uniformly converge to a non-Lipschitz continuous function is given below

Example 5 (Lipschitz functions with Unbounded Lipschitz constant Uniformly Converge to a non-Lipschitz function)

$$X = [0,1] \text{ and } f_n(x) = \sqrt{x + \frac{1}{n}}$$

2.6 Modes of Convergence

This section is the starting point for us to study spaces of function. At first, we will define two modes of convergence for sequence of function and point out some basic properties and the difference between two modes of convergence.

Given an arbitrary set X and a metric space Y, we say a sequence of functions f_n from X to Y **pointwise converge** to f if for all ϵ and x in X, there exists N such that

$$\forall n > N, f_n(x) \in B_\epsilon\big(f(x)\big)$$

In other words, for each fixed x in X, we have $f_n(x) \to f(x)$.

We say f_n **uniformly converge** to f if for all ϵ there exists N such that

$$\forall x \in X, \forall n > N, f_n(x) \in B_\epsilon\big(f(x)\big)$$

The difference between pointwise convergence and uniform convergence is that if we require $f_n(x)$ to be ϵ-close to $f(x)$ for all $n > N$, then

(•) N depend on both ϵ and x if $f_n \to f$ pointwise

(•) N depend on only ϵ if $f_n \to f$ uniformly

A few properties of sequence of functions similar to that of sequences in metric space is obvious. If $f_n \to f$ pointwise, then all sub-sequences $f_{n_k} \to f$ pointwise. If $f_n \to f$ uniformly, then all sub-sequences $f_{n_k} \to f$ uniformly. Suppose $Z \subseteq X$. It is clear that if $f_n \to f$ uniformly (resp: pointwise) the restricts $f_n|_Z \to f|_Z$ uniformly (resp: pointwise). Also, if $f_n \to f$ uniformly, then $f_n \to f$ pointwise.

Suppose we have a family $\mathcal{F}$ of functions $f : X \to (Y, d)$. If we define

$$d_\infty(f, g) = \sup_{x \in X} d\big(f(x), g(x)\big)$$

instead of a metric, d_∞ become an extended metric. If f is bounded and g is unbouneded, we have $d_\infty(f, g) = \infty$. If f, g are both bounded, then $d_\infty(f, g) \in \mathbb{R}^+$. Because of such, for d_∞ to be a metric, one but not the only condition is for $\mathcal{F}$ to be space of bounded functions.

Now, regardless of d_∞ is an extended metric or not, we have

$$f_n \to f \text{ uniformly } \iff d_\infty(f_n, f) \to 0$$

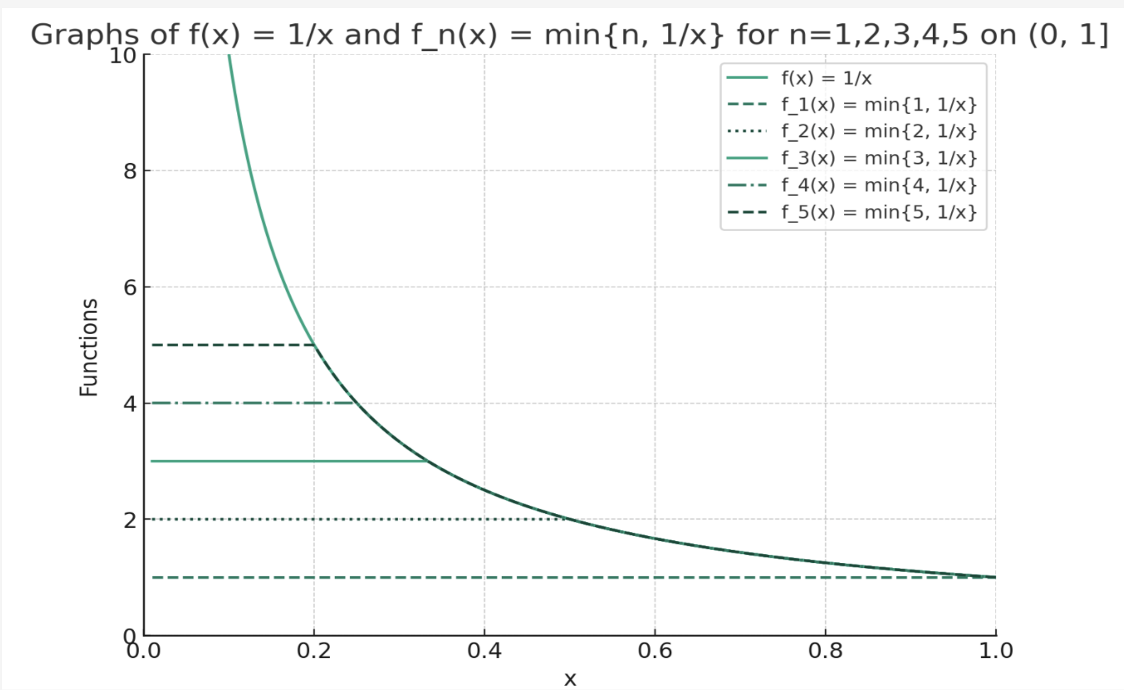
With this in mind, it shall be clear that the uniform limit of bounded (resp: unbounded) functions is always bounded (resp: unbounded).

Examples for bounded (resp: unbounded) function f_n pointwise converge to unbounded (resp: bounded) function f are as follows.

Example 6 (Bounded functions pointwise converge to unbounded function)

$$X = (0, 1], f_n(x) = \min\{n, \frac{1}{x}\}$$

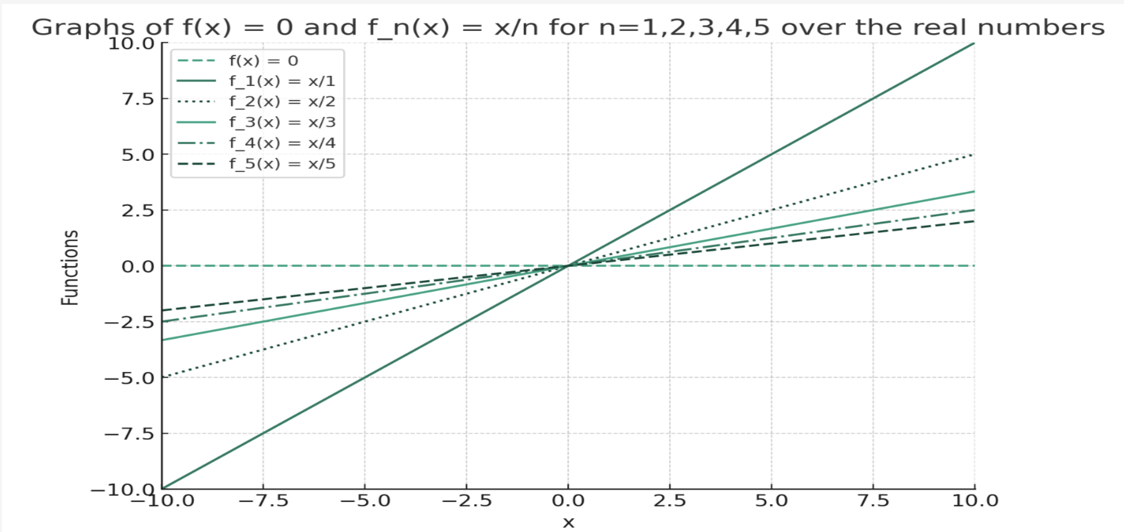
It is clear that $\forall n \in \mathbb{N}, f_n(x) \in [0, n]$, and the limit $f : X \to \mathbb{R}$ is $f(x) = \frac{1}{x}$



Example 7 (Unbounded functions pointwise converge to bounded function)

$$X = \mathbb{R}, f_n(x) = \frac{1}{n}x$$

The limit function is $f(x) = 0$



As pointed out earlier, if $f : X \to (Y, d)$ is bounded and $g : X \to (Y, d)$ is unbounded, then $d_\infty(f, g) = \infty$. This means that if Y is unbounded, the uniform metric d_∞ is extended on X^Y. For this, it is necessary to develop some basic fact concerning extended metric space.

Suppose (X, d) is an extended metric space. If we define $\sim$ on X by $x \sim y \iff d(x, y) < \infty$, then $\sim$ is an equivalence relation. We say each equivalence class is a **galaxy** of (X, d). Suppose T is the collection of the galaxies of (X, d). For each $\mathcal{T} \in T$, the space $(\mathcal{T}, d)$ is just a metric space.

It is easy to see that the way we induce topology from metric space is still valid if the metric is extended. That is

$$\tau = \{Z \in X : \forall z \in Z, \exists \epsilon, B_\epsilon(z) \subseteq Z\}$$

is still a topology, even though d is an extended metric on X.

We can verify that a set Y in X is open if and only if for all $\mathcal{T} \in T$, the set $Y \cap \mathcal{T}$ is open, and the set Y in X is closed if and only if all convergent sequences y_n in Y

converge to points in Y.

Now, suppose we are given an arbitrary set X and a complete metric space $(\overline{Y}, d)$, and on $X^{\overline{Y}}$, we define the uniform metric d_∞. We say a set $\mathcal{F} \subseteq X^{\overline{Y}}$ of functions is **closed under uniform convergence** if for all uniform convergent sequence $f_n \subseteq \mathcal{F}$, the limit function f is also in $\mathcal{F}$. There are justified reasons for us to give the premise that $\overline{Y}$ is complete prior to the definition of the term **closed under uniform convergence**. One reason is that by Theorem 2.4.3, if Y is not complete, then the extended metric space (X^Y, d_∞) is also not complete, which implies the possibility a Cauchy sequence f_n in X^Y converge to a function $f \in X^{\overline{Y}} \setminus X^Y$ where $\overline{Y}$ is the completion of Y. For instance, if we let $Y = \mathbb{R} \setminus \{1\}$ where $X = \mathbb{R}$, and let $f_n(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 + \frac{1}{n} & \text{if } x = 0 \end{cases} \in Y$, we see that the set $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ is "closed under uniform convergence" in the context of X^Y, but when in fact f_n uniformly converge to $f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ which is not in $\mathcal{F}$. This awkward usage of words can be solved if we define the term **closed under uniform convergence** after the premise that Y is complete.

Now, given a set of functions $\mathcal{F} \subseteq X^{\overline{Y}}$, one can verify that

$$\begin{aligned} \mathcal{F} \text{ is closed under uniform convergence } &\iff (\mathcal{F}, d_\infty) \text{ is complete} \\ &\iff \mathcal{F} \text{ is closed with respect to } (X^{\overline{Y}}, d_\infty) \end{aligned}$$

Let $\mathcal{G}$ be a galaxy of $(X^{\overline{Y}}, d_\infty)$. With multiple ways, we can verify that $\mathcal{G}$ is closed with respect to $(X^{\overline{Y}}, d_\infty)$. Then, acknowledging the space of bounded functions $B(X, \overline{Y})$ is a galaxy of $X^{\overline{Y}}$, we see that $B(X, \overline{Y})$ is closed under uniform convergence. The statement that $B(X, \overline{Y})$ is closed under uniform convergence, although already "proved" before as we pointed out the limit of uniform convergent sequence of bounded functions must be bounded, is now in fact actually proved in the sense the term "closed under uniform convergence" is formally given a satisfying definition.

2.7 Arzelà–Ascoli Theorem

In this section, we will give a complete proof of Arzelà–Ascoli Theorem for functions from arbitrary compact topological space to arbitrary metric space. Note that in Baby Rudin, Arzelà–Ascoli Theorem are given for functions from compact metric space to metric space. Because Arzelà–Ascoli Theorem are concerned with family of equicontinuos functions, it is crucial for us to give a definition to equicontinuity for functions from topological space to metric space, for the sake of our generalization.

Let X, Y be metric space. Let Z be topological space. Let $\mathcal{F}_X$ be family of functions from X to Y, and let $\mathcal{F}_Z$ be family of functions from Z to Y. We say $\mathcal{F}_Z$ is **pointwise equicontinuous** if

$$\text{For all } \epsilon \text{ and for all } x\text{, there exists a neighborhood } U_x \text{ such that}$$
$$d_Y(f(x), f(y)) < \epsilon \text{ for all } y \in U_x$$

We say $\mathcal{F}_X$ is **equicontinuous** if

$$\text{For all } \epsilon\text{, there exists } \delta \text{ such that } d_Y(f(x), f(y)) < \epsilon \text{ for all } \delta\text{-close } x, y \in X$$
$$\text{and all } f \in \mathcal{F}.$$

It is easy to verify that if $\mathcal{F}_X$ is equicontinuous, then $\mathcal{F}_X$ is pointwise equicontinuous. The converse don't always hold true. Say, $\mathcal{F} = \{n + x^2\}_{n\in\mathbb{N}}$, the set $\{n + x^2\}_{n\in\mathbb{N}}$ is clearly pointwise equicontinuous on $\mathbb{R}$, and is not equicontinuous on $\mathbb{R}$, since no function $n+x^2$ is uniform continuous on $\mathbb{R}$. However, the same set $\mathcal{F} = \{n+x^2\}$ is equicontinous on compact domain $[a, b]$. This is a general result, as we shall prove below.

Theorem 2.7.1. (Pointwise Equicontinous is Uniform on Compact Domain) Given two metric space $(X, d_X), (Y, d_Y)$, and a family $\mathcal{F}$ of functions from X to Y such that

(a) X is compact

(b) $\mathcal{F}$ is pointwise equicontinuous

Then

$$\mathcal{F} \text{ is equicontinuous}$$

Proof. Fix ϵ. We wish to

$$\text{find } \delta \text{ such that } d_X(x, y) < \delta \implies d_Y(f(x), f(y)) \leq \epsilon \text{ for all } f \in \mathcal{F}$$

Because $\mathcal{F}$ is pointwise equicontinuous, we know for each $x \in X$, there exists δ_x such that

$$\forall y \in B_{\delta_x}(x), d_Y\big(f(x), f(y)\big) < \frac{\epsilon}{2} \text{ for all } f \in \mathcal{F} \tag{2.19}$$

It is clear that $\{B_{\frac{\delta_x}{2}}(x) : x \in X\}$ form an open cover of X. Then because X is compact, we know

$$\text{there exists a finite open sub-cover: } \{B_{\frac{\delta_x}{2}}(x) : x \in X_{\text{finite}}\}$$

We claim

$$\delta = \min_{x \in X_{\text{finite}}} \frac{\delta_x}{2} \text{ works}$$

Fix $y, z \in X : d_X(y,z) < \delta$. We have to prove

$$d_Y\big(f(y), f(z)\big) < \epsilon$$

We know y must lie in some $B_{\frac{\delta_x}{2}}(x)$ for some $x \in X_{\text{finite}}$. Because $d_X(y,z) < \frac{\delta_x}{2}$, we see that z must lie in $B_{\delta_x}(x)$. We now know y, z are both in $B_{\delta_x}(x)$. Then from (2.19), we can now deduce

$$d_Y\big(f(y), f(z)\big) \leq d_Y\big(f(y), f(x)\big) + d_Y\big(f(x), f(z)\big) < \epsilon \text{ (done)}$$

∎

The proof above should be a great example why in the discussion of metric space, instead of using sequential definition of compactness, which leads to the beautiful Bolzano-Weierstrass Theorem, some people prefer the open-cover definitions.

Now, we give proof for the Arzelà–Ascoli Theorem.

Theorem 2.7.2. (Arzelà–Ascoli Theorem) Given a compact topological space (X, τ), a metric space (Y, d_Y), and a family $\mathcal{F} \subseteq C\big(X, Y\big)$ of continuous function

$\mathcal{F}$ is pointwise equicontinuous and $\{f(x) : f \in \mathcal{F}\}$ has compact closure in Y for all $x \in X$
$\implies \mathcal{F}$ has a compact closure in $C(X, Y)$

Proof. Fix a sequence f_n in $\mathcal{F}$. We wish to show

$$f_n \text{ has a sub-sequence } f_{n_k} \text{ uniformly converge to some } f : X \to Y$$

First, we prove

$$\text{there exists a countable set } P \text{ such that } P \text{ works like a dense set}$$

Because $\mathcal{F}$ is pointwise equicontinuous, we know for all $x \in X$

$$\exists U_{x,n}, \forall y \in U_{x,n}, \forall f \in \mathcal{F}, d_Y\big(f(x), f(y)\big) < \frac{1}{n} \text{ for each fixed } n \in \mathbb{N}$$

Now, because X is compact, for each $n \in \mathbb{N}$, there exists a finite subset $P_n \subseteq X$ such that $\left\{U_{x,n} : x \in P_n\right\}$ is a cover of X. Let $P = \bigcup_{n\in\mathbb{N}} P_n$. (done)

Now, we wish to

construct a sub-sequence f_{n_k} pointwise converge on P

Express $P = \{p_k\}_{k\in\mathbb{N}}$. By premise (pointwise image has compact closure), we know there exists a compact set that contain $\{f_n(p_1)\}_{n\in\mathbb{N}}$, so by Bolzano-Weierstrass Theorem, there exists a sub-sequence

$$\left\{f_{g_1(k)}(p_1)\right\}_{k\in\mathbb{N}} \text{ converge to some point in } Y$$

Now, again by premise and Bolzano-Weierstrass Theorem, there exists a sub-sequence

$$\left\{f_{g_2\circ g_1(k)}(p_2)\right\}_{k\in\mathbb{N}} \text{ converge to some point in } Y$$

Repeatedly doing such, we have

$$\begin{array}{cccc}
f_{g_1(1)}(p_1) & f_{g_2\circ g_1(1)}(p_2) & f_{g_3\circ g_2\circ g_1(1)}(p_3) & \cdots \\
f_{g_1(2)}(p_1) & f_{g_2\circ g_1(2)}(p_2) & f_{g_3\circ g_2\circ g_1(2)}(p_3) & \cdots \\
f_{g_1(3)}(p_1) & f_{g_2\circ g_1(3)}(p_2) & f_{g_3\circ g_2\circ g_1(3)}(p_3) & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
\downarrow & \downarrow & \downarrow & \\
y_1 & y_2 & y_3 & \cdots
\end{array}$$

Now, let

$$n_k = g_k \circ \cdots \circ g_1(k)$$

Then

$$n_k \text{ is eventually a sub-sequence of } g_m \circ \cdots \circ g_1(k) \text{ for all } m$$

This then implies

$$f_{n_k}(p_m) \to y_m \text{ for all } p_m \in P \text{ (done)}$$

Next, we show

To prove f_{n_k} uniformly converge on X, it suffice to prove f_{n_k} is uniformly Cauchy on X.

By premise (pointwise image has compact closure), if f_{n_k} is uniformly Cauchy, then we know f_{n_k} pointwise converge to some f.

Fix ϵ. We reduced the problem into

$$\text{finding } N \text{ such that for all } k > N, \text{ we have } d_Y\big(f_{n_k}(x), f(x)\big) \leq \epsilon \text{ for all } x \in X$$

Because f_{n_k} is uniformly Cauchy, we know there exists N such that for all $m, k > M$ $d_Y\big(f_{n_k}(x), f_{n_m}(x)\big) \leq \frac{\epsilon}{2}$ for all $x \in X$. We claim

$$\text{such } N \text{ works}$$

Let $k > N$. Assume $d_Y\big(f_{n_k}(x), f(x)\big) > \epsilon$. We see that

$$d_Y\big(f(x), f_{n_m}(x)\big) \geq d_Y\big(f(x), f_{n_k}(x)\big) - d_Y\big(f_{n_k}(x), f_{n_m}(x)\big) > \frac{\epsilon}{2} \text{ for all } m > N \text{ CaC (done)}$$

Lastly, we wish to prove

$$f_{n_k} \text{ is uniformly Cauchy}$$

Fix ϵ. We wish

$$\text{to find } N \text{ such that } \forall j, k > N, \forall x \in X, d_Y\big(f_{n_j}(x), f_{n_k}(x)\big) \leq \epsilon$$

Fix $m > \frac{3}{\epsilon}$. Express $P_m = \{p_1^m, \ldots, p_u^m\}$. Because $f_{n_k}(p_t^m)$ converge for each $t \in \{1, \ldots, u\}$, we know

$$\forall t, \exists N_t, d_Y\big(f_{n_j}(p_t^m), f_{n_k}(p_t^m)\big) < \frac{\epsilon}{3} \text{ for all } j, k > N_t$$

We claim

$$N = \max_t N_t \text{ works}$$

Fix $j, k > N$ and $x \in X$. We have to show

$$d_Y\big(f_{n_j}(x), f_{n_k}(x)\big) \leq \epsilon$$

Because $\{U_{p_t^m, m}\}$ form an open cover of X, we know there exists t such that $x \in U_{p_t^m, m}$. We can now deduce

$$d_Y\big(f_{n_j}(x), f_{n_k}(x)\big) \leq d_Y\big(f_{n_j}(x), f_{n_j}(p_t^m)\big) + d_Y\big(f_{n_j}(p_t^m), f_{n_k}(p_t^m)\big) + d_Y\big(f_{n_k}(p_t^m), f_{n_k}(x)\big) < \epsilon$$

(done)

$\blacksquare$

2.8 Banach Fixed Point Theorem

This section give a complete statement and proof of Banach Fixed Point Theorem. The setting is

(a) a metric space $\left(X, d_X\right)$

(b) a subset $E \subseteq X$

(c) another metric space $\left(Y, d_Y\right)$

(d) a function $f : E \to Y$

(e) another function $g : E \to X$

We say f is a **contraction** on E if there exists $r \in [0, 1)$ such that

$$d_Y(f(x), f(y)) \leq r d_X(x, y) \qquad (x, y \in E)$$

or equivalently

$$\sup_{x \neq y \in E} \frac{d_Y(f(x), f(y))}{d_X(x, y)} < 1$$

Note that the restriction of a contraction is again a contraction. We say g admits a **fixed point** x if we have

$$g(x) = x$$

Theorem 2.8.1. (Banach Fixed Point Theorem) If g is a contraction that maps E into X, then

$$g \text{ admits at most one fixed point}$$

Moreover, if E is complete and $g(E) \subseteq E$, then

$$\text{the fixed point exists}$$

And if we use the notation g^n to denote $g \circ g^{n-1}$, then for all $x \in E$,

$$\text{the fixed point can be written in the form } \lim_{n \to \infty} g^n(x)$$

Proof. We first

prove the uniqueness of the fixed point

Suppose x, y are both fixed by g. We have

$$d(g(x), g(y)) = d(x, y)$$

Because g is a contraction mapping, this implies $d(x, y) = 0$. (done)

Suppose E is complete and $g(E) \subseteq E$. We now

prove the existence of the fixed point

Fix $x \in E$. Because we have already prove the uniqueness of the fixed point, we only have to prove

$$\lim_{n\to\infty} g^n(x) \text{ exists and } \lim_{n\to\infty} g^n(x) \text{ is a fixed point of } g$$

Because E is complete, to prove $\lim_{n\to\infty} g^n(x)$ exists, we only have to prove

$$\{g^n(x)\}_{n\in\mathbb{N}} \text{ is Cauchy}$$

Observe

$$\begin{aligned} d(g^n(x), g^{n+k}(x)) &\leq \sum_{i=0}^{k-1} d(g^{n+i}(x), g^{n+i+1}(x)) \\ &\leq d(x, g(x)) \sum_{i=0}^{k-1} r^{n+i} \\ &\leq \frac{r^n}{1-r} d(x, g(x)) \to 0 \text{ as } n \to \infty \quad \text{(done)} \end{aligned}$$

Note that contraction is Lipschitz thus continuous, and note that $\lim_{n\to\infty} g^n(x) \in E$. This allow us to carry the below limit process

$$g\big(\lim_{n\to\infty} g^n(x)\big) = \lim_{n\to\infty} g(g^n(x)) = \lim_{n\to\infty} g^{n+1}(x) = \lim_{n\to\infty} g^n(x) \text{ (done)}$$

■

Banach Fixed Point Theorem is one of the most important Theorem in Mathematics. It will be used to prove

(a) Inverse Function Theorem

(b) Picard-Lindelof Theorem

(c) Nash-Embedding Theorem

Chapter 3

Algebraic Topology

3.1 Fundamental Group

3.2 Invariance of Domain

Theorem 3.2.1. (Invariance of Domain) If U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is one-to-one and continuous, then

$$f(U) \text{ is open and } f \text{ is a homeomorphism between } U \text{ and } f(U)$$

Theorem 3.2.2. (Invariance of Dimension) If U is a non-empty open subset of $\mathbb{R}^n$ and V is a non-empty subset of $\mathbb{R}^m$ homeomorphic to U, then $n = m$.

Chapter 4

Linear Algebra Done Outrageous

4.1 Dual Space

Abstract

Given two finite dimensional vector space V, W over $\mathbb{R}$, we define their **tensor product** $V \otimes W$ as the dual space of the space of bilinear forms on $V \times W$. Given $v \in V, w \in W$, we also define their tensor product $v \otimes w$ to be

$$v \otimes w(B) \triangleq B(v, w)$$

Immediately, we see that the tensor product of the vectors $\otimes : V \times W \to V \otimes W$ is bilinear. Moreover, if $(v_i), (w_j)$ are respectively basis of V, W, then $(v_i \otimes w_j)$ is also a basis of $V \otimes W$, in fact, the dual basis of the natural basis for the space of bilinear forms on $V \times W$. Also, we see that we have the **tensor product universal property** that if $B : V \times W \to U$ is a bilinear map, then there exists a unique linear map $\beta : V \otimes W \to U$ such that

$$B(v, w) = \beta(v \otimes w)$$

In other words, we have the commutative diagram

$$
\begin{tikzcd}
V \times W \arrow[r, "B"] \arrow[d, "(v,w)\mapsto v\otimes w"'] & U \\
V \otimes W \arrow[ur, dashed, "\exists!\beta"'] &
\end{tikzcd}
$$

Given some vector space $V/\mathbb{F}$, we define its **dual space** V^* to be $V^* \triangleq L(V, \mathbb{F})$ equipped with pointwise addition and $\mathbb{F}$-multiplication. If V is finite dimensional and (e_i) is a basis of V, we can induce a **dual basis** (α_i) by $\alpha_i e_j \triangleq \delta^i_j$.

4.2 Norm and Inner Product

This section contains

(a) definition and basic properties of the term **norm**

(b) definition and basic properties of the term **inner product**

(c) definition and basic properties of the term **positive semi-definite Hermitian form**

(d) full statement and proof of **Cauchy Schwarz Inequality** for both inner product space and positive semi-definite Hermitian form

(e) statement and proof of **SVD** (singular value decomposition).

(Norm Axiom Part)

Recall that by a **normed space** V, we mean a vector space over a sub-field $\mathbb{F}$ of $\mathbb{C}$ equipped with $\|\cdot\| : V \to \mathbb{R}_0^+$ satisfying the following axioms:

(a) $\|x\| = 0 \implies x = 0$ (positive-definiteness)

(b) $\|sx\| = |s| \cdot \|x\|$ for all $s \in \mathbb{F}$ and $x \in V$ (absolute-homogenity)

(c) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$ (triangle inequality)

Observe

$$\|0\| = \|0 + x\| \leq \|0\| + \|x\| \text{ for all } x \in V$$

This shows that $\|x\| \geq 0$ for all $x \in V$. Also observe

$$\|0\| = \|0(x)\| = |0| \cdot \|x\| = 0$$

We can now rewrite the normed space axioms into

(a) $\|x\| = 0 \iff x = 0$ (positive-definiteness)

(b) $\|sx\| = |s| \cdot \|x\|$ for all $s \in \mathbb{F}$ and $x \in V$ (absolute-homogeneity)

(c) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$ (triangle inequality)

(d) $\|x\| \geq 0$ for all $x \in V$ (non-negativity)

(Inner Product Axiom Part)

Recall that by an **inner product space** V, we mean a vector space over $\mathbb{R}$ or $\mathbb{C}$ equipped with $\langle \cdot, \cdot \rangle : V^2 \to \mathbb{R}$ or $\mathbb{C}$ satisfying the following <u>axioms</u>

(a) $\langle x, x \rangle > 0$ for all $x \neq 0$ (Positive-definiteness)

(b) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (Conjugate symmetry)

(c) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ and $\langle cx, z \rangle = c\langle x, z \rangle$ (Linearity in the first argument)

Note that conjugate symmetry let us deduce

$$\langle x, x \rangle = \overline{\langle x, x \rangle} \implies \langle x, x \rangle \in \mathbb{R}$$

Also, one can easily use linearity in first argument to deduce

$$\langle 0, 0 \rangle = 2\langle 0, 0 \rangle \implies \langle 0, 0 \rangle = 0$$

This now let us rewrite the inner product space over $\mathbb{C}$ axioms into

(a) $\langle x, x \rangle \geq 0$ for all $x \in V$ (non-negativity)

(b) $\langle x, x \rangle = 0 \iff x = 0$ (positive-definiteness)

(c) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (conjugate symmetry)

(d) $\langle cx + y, z \rangle = c\langle x, z \rangle + \langle y, z \rangle$ and $\langle x, cy + z \rangle = \overline{c}\langle x, y \rangle + \langle x, z \rangle$ (Linearity)

Note that using $c = 1$ and $y = 0$, ($\because \langle 0, z \rangle = 0\langle x, z \rangle = 0$) one can check that the latter expression of linearity implies the first expression.

If the scalar field is $\mathbb{R}$, then conjugate symmetry is just symmetry and we also have linearity in the second argument.

This now let us rewrite the inner product space over $\mathbb{R}$ axioms into

(a) $\langle x, x \rangle \geq 0$ for all $x \in V$ (non-negativity)

(b) $\langle x, x \rangle = 0 \iff x = 0$ (positive-definiteness)

(c) $\langle x, y \rangle = \langle y, x \rangle$ (symmetry)

(d) Linearity in both arguments

If we do not require $\langle \cdot, \cdot \rangle$ to be positive-definite, but only non-negative, i.e. $\langle x, x \rangle \geq 0$ for all $x \in V$, then we have a **positive semi-definite Hermitian form**. Formally speaking, a positive semi-definite Hermitian form $\langle \cdot, \cdot \rangle : V^2 \to \mathbb{R}$ or $\mathbb{C}$ satisfy the following axioms

(a) $\langle x, x \rangle \geq 0$ for all $x \in V$ (non-negativity)

(b) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (conjugate symmetry)

(c) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ and $\langle cx, z \rangle = c\langle x, z \rangle$ (Linearity in the first argument)

Example 8 (Example of Positive semi-definite Hermitian form)

$$\text{arbitrary } V \text{ over } \mathbb{R} \text{ or } \mathbb{C} \qquad \langle x, y \rangle \triangleq 0 \text{ for all } x, y$$

(Norm Induce Part)

Given a vector space V over $\mathbb{R}$ or $\mathbb{C}$, one can check that if V is equipped with an inner product $\langle \cdot, \cdot \rangle : V^2 \to \mathbb{R}$ or $\mathbb{C}$, then we can induce a norm on V by

$$\|x\| \triangleq \sqrt{\langle x, x \rangle} \qquad (x \in V)$$

Note that

$$\|x\| = 0 \iff \langle x, x \rangle = 0$$

This implies that if $\langle \cdot, \cdot \rangle$ is an inner product (satisfy positive-definiteness), then $\| \cdot \|$ is also positive-definite. And if $\langle \cdot, \cdot \rangle$ is not positive-definite, then there exists $x \neq 0 \in V$ such that $\|x\| = 0$, which make $\| \cdot \|$ a **semi-norm**.

Absolute homogeneity follows from the linearity of inner product.

To check triangle inequality, we first have to prove Cauchy-Schwarz inequality.

Theorem 4.2.1. (Basic Property of Positive semi-definite Hermitian form) Given a positive semi-definite Hermitian form $\langle \cdot, \cdot \rangle : V^2 \to \mathbb{R}$ or $\mathbb{C}$ and $x, y \in V$, we have

$$\langle x, x \rangle = 0 \implies \langle x, y \rangle = 0$$

Proof. Assume $\langle x,y\rangle \neq 0$. Fix $t > \frac{\|y\|^2}{2|\langle x,y\rangle|^2}$. Compute

$$\begin{aligned}\|y - t\langle y,x\rangle x\|^2 &= \|y\|^2 + \|(-t)\langle y,x\rangle x\|^2 + \langle -t\langle y,x\rangle x, y\rangle + \langle y, -t\langle y,x\rangle x\rangle \\ &= \|y\|^2 + t^2 \left|\langle x,y\rangle\right|^2 \|x\|^2 - t\langle y,x\rangle\langle x,y\rangle - t\langle x,y\rangle\langle y,x\rangle \\ &= \|y\|^2 - 2t\left|\langle x,y\rangle\right|^2 < 0 \text{ CaC}\end{aligned}$$

■

Theorem 4.2.2. (Cauchy-Schwarz Inequality) Given a positive semi-definite Hermitian form $\langle\cdot,\cdot\rangle : V^2 \to \mathbb{C}$ on vector space V over $\mathbb{C}$, we have

(a) $|\langle x,y\rangle| \leq \|x\| \cdot \|y\| \quad (x,y \in V)$

(b) the equality hold true if x,y are linearly dependent

(c) the equality hold true if and only if x,y are linearly dependent (provided $\langle\cdot,\cdot\rangle$ is an inner product)

Proof. We first prove

$$|\langle x,y\rangle| \leq \|x\| \cdot \|y\| \qquad (x,y \in V)$$

Fix $x,y \in V$. Theorem 4.2.1 tell us $\|x\| = 0 \implies \langle x,y\rangle = 0$. Then we can reduce the problem into proving

$$\frac{\left|\langle x,y\rangle\right|^2}{\|x\|^2} \leq \|y\|^2$$

Set $z \triangleq y - \frac{\langle y,x\rangle}{\|x\|^2}x$. We then have

$$\langle z,x\rangle = \langle y - \frac{\langle y,x\rangle}{\|x\|^2}x, x\rangle = \langle y,x\rangle - \frac{\langle y,x\rangle}{\|x\|^2}\langle x,x\rangle = 0$$

Then from $y = z + \frac{\langle y,x\rangle}{\|x\|^2}x$, we can now deduce

$$\begin{aligned}\langle y,y\rangle &= \langle z + \frac{\langle y,x\rangle}{\|x\|^2}x, z + \frac{\langle y,x\rangle}{\|x\|^2}x\rangle \\ &= \langle z,z\rangle + \left|\frac{\langle y,x\rangle}{\langle x,x\rangle}\right|^2 \langle x,x\rangle \\ &= \langle z,z\rangle + \frac{\left|\langle x,y\rangle\right|^2}{\langle x,x\rangle}\end{aligned}$$

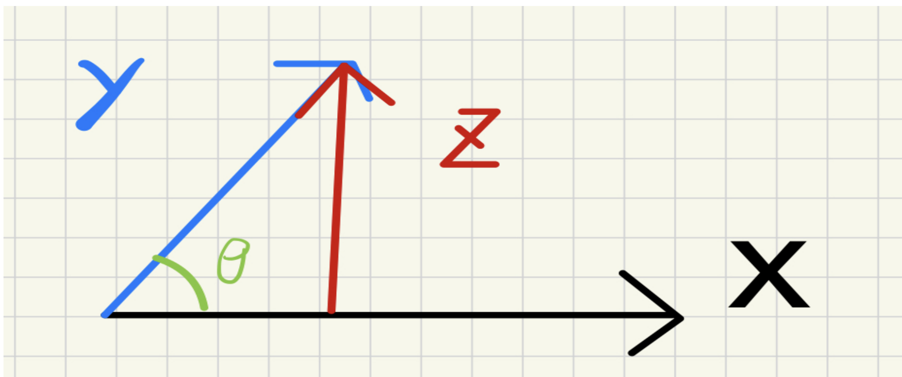
Because $\langle z, z\rangle \geq 0$, we now have

$$\langle y, y\rangle = \langle z, z\rangle + \frac{|\langle x, y\rangle|^2}{\langle x, x\rangle} \geq \frac{|\langle x, y\rangle|^2}{\langle x, x\rangle} \text{ (done)}$$

The equality hold true if and only if $\langle z, z\rangle = 0$. This explains the other two statements regarding the equality. ■

The proof is clearly geometrical. If one wish to remember the proof, one should see the trick we use is exactly

$$z \triangleq y - |y|\,(\cos\theta)\hat{x} \text{ is the projection of } y \text{ onto } x^{\perp}$$



Then all we do rest is just expanding $|y|^2 = |z + \tilde{x}|^2$, where $\tilde{x} = y - z = |y|\,(\cos\theta)\hat{x}$, which give the answer and is easy to compute since $z \cdot \tilde{x} = 0$.

Now, with Cauchy-Schwarz Inequality, we can check the triangle inequality

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
&= \langle x, x\rangle + \langle y, y\rangle + 2 \text{ Re } \langle x, y\rangle \\
&\leq \|x\|^2 + \|y\|^2 + 2\,|\langle x, y\rangle| \\
&\leq \|x\|^2 + \|y\|^2 + 2\|x\| \cdot \|y\| = \Big(\|x\| + \|y\|\Big)^2
\end{aligned}$$

(Euclidean Space Abstract Part)

By a **concrete Euclidean Space**, we mean some space of n-tuple $(x_1, \ldots, x_n)$ over $\mathbb{R}$, equipped with inner product $\langle \cdot, \cdot \rangle_E$ defined by

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle_E = \sqrt{\sum_{k=1}^{n} (y_k - x_k)^2}$$

By an **Euclidean Space**, we simply mean a finite dimensional vector space V over $\mathbb{R}$, equipped with an inner product $\langle \cdot, \cdot \rangle$ such that there exists a concrete Euclidean space E and an isomorphism $\varphi : V \to E$ such that

$$\langle x, y \rangle = \langle \varphi(x), \varphi(y) \rangle_E \qquad (x, y \in V)$$

Note that if you define $\langle \cdot, \cdot \rangle$ on the space of n-tuples $(x_1, \ldots, x_n)$ over $\mathbb{R}$ by

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = 2\sqrt{\sum_{k=1}^{n} (y_k - x_k)^2}$$

Then, the space of n-tuple is clearly not a concrete Euclidean space, and clearly an Euclidean space.

(SVD)

Chapter 5

Differential Calculus

5.1 Operator Norm

Abstract

This section introduces the concept of the operator norm and proves some fundamental results related operator norm and finite-dimensional normed spaces. For example, we establish results such as a linear operator being bounded if and only if it is continuous and the equivalence of all norms on finite-dimensional vector spaces.

In this section, and particularly in functional analysis, we say a function T between two metric space is a **bounded operator** if T always map bounded set to bounded set. In particular, if T is a linear transformation between two normed space, we say T is a **bounded linear operator**. Now, suppose $\mathcal{X}, \mathcal{Y}$ are two normed space over $\mathbb{R}$ or $\mathbb{C}$. In space $L(\mathcal{X}, \mathcal{Y})$, alternatively, we can define

$$T \text{ is bounded} \quad \stackrel{\triangle}{\Longleftrightarrow} \exists M \in \mathbb{R}, \forall x \in \mathcal{X}, \|Tx\| \leq M\|x\|$$

The proof of equivalency is simple. For $(\longrightarrow)$, observe

$$\|Tx\| = \|x\| \cdot \|T\frac{x}{\|x\|}\| \leq \Big(\sup\{\|Ty\| : \|y\| = 1\}\Big)\|x\|$$

For $(\longleftarrow)$, observe

$$\|Tx - Ty\| = \|T(x-y)\| \leq M\|x-y\|$$

We first show that a linear transformation is continuous if and only if it is bounded.

Theorem 5.1.1. (Liner Operator is Bounded if and only if it is Continuous) Given two normed space $\mathcal{X}, \mathcal{Y}$ over $\mathbb{R}$ or $\mathbb{C}$ and $T \in L(\mathcal{X}, \mathcal{Y})$, we have

$$T \text{ is a bounded operator} \iff T \text{ is continuous on } \mathcal{X}$$

Proof. If T is bounded, we see that T is Lipschitz.

$$\|Tx - Ty\| \leq M\|x - y\|$$

Now, suppose T is linear and continuous at 0. Let ϵ satisfy

$$\sup_{\|y\| \leq \epsilon} \|Ty\| \leq 1$$

Observe that for all $x \in \mathcal{X}$, we have

$$\|Tx\| = \frac{\|x\|}{\epsilon} \Big\| T\frac{\epsilon x}{\|x\|} \Big\| \leq \frac{\|x\|}{\epsilon}$$

■

Here, we introduce a new terminology, which shall later show its value. Given a set X, we say two metrics d_1, d_2 on X are **equivalent**, and write $d_1 \sim d_2$, if we have

$$\exists m, M \in \mathbb{R}^+, \forall x, y \in X, md_1(x, y) \leq d_2(x, y) \leq Md_1(x, y)$$

Now, given a fixed vector space V, naturally, we say two norms $\|\cdot\|_1, \|\cdot\|_2$ on V are **equivalent** if

$$\exists m, M \in \mathbb{R}^+, \forall x \in X, m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1$$

We say two metric d_1, d_2 on X are **topologically equivalent** if the topology they induce on X are identical.

A few properties can be immediately spotted.

(a) Our definition of $\sim$ between metrics of a fixed X is an equivalence relation.

(b) Our definition of $\sim$ between norms on a fixed V is an equivalence relation.

(c) Equivalent norms induce equivalent metrics.

(d) Equivalent metrics are topologically equivalent.

We now prove if V is finite-dimensional, then all norms on V are equivalent. This property will later show its value, as used to prove linear map of finite-dimensional domain is always continuous

Theorem 5.1.2. (All Norms on Finite-dimensional space are Equivalent) Suppose V is a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. Then

$$\text{all norms on } V \text{ are equivalent}$$

Proof. Let $\{e_1, \dots, e_n\}$ be a basis of V. Define ∞-norm $\|\cdot\|_\infty$ on V by

$$\left\| \sum \alpha_i e_i \right\|_\infty \triangleq \max |\alpha_i|$$

It is easily checked that $\|\cdot\|_\infty$ is indeed a norm. Fix a norm $\|\cdot\|$ on V. We reduce the problem into

$$\text{finding } m, M \in \mathbb{R}^+ \text{ such that } m\|x\|_\infty \leq \|x\| \leq M\|x\|_\infty$$

We first claim

$$M = \sum \|e_i\| \text{ suffices}$$

Compute

$$\|x\| = \left\| \sum \alpha_i e_i \right\| \leq \sum |\alpha_i| \, \|e_i\| \leq \|x\|_\infty \sum \|e_i\| = M\|x\|_\infty \text{ (done)}$$

Note that reverse triangle inequality give us

$$\Big| \|x\| - \|y\| \Big| \leq \|x - y\| \leq M\|x - y\|_\infty \tag{5.1}$$

Then we can check that

(a) $\|\cdot\| : \Big(V, \|\cdot\|_\infty\Big) \to \mathbb{R}$ is Lipschitz continuous because of Equation 5.1.

(b) $S \triangleq \{y \in V : \|y\|_\infty = 1\}$ is sequentially compact in $\|\cdot\|$ and non-empty.

Now, by EVT, we know $\min_{y \in S} \|y\|$ exists. Note that $\min_{y \in S} \|y\| > 0$, since $0 \notin S$. We claim

$$m = \min_{y \in S} \|y\| \text{ suffices}$$

Fix $x \in V$ and compute

$$m\|x\|_\infty = \|x\|_\infty (\min_{y \in S} \|y\|) \leq \|x\|_\infty \cdot \left\| \frac{x}{\|x\|_\infty} \right\| = \|x\| \text{ (done)} \quad \text{(done)}$$

∎

Theorem 5.1.3. (Linear map of Finite-dimensional Domain is always Continuous) Given a finite-dimensional normed space $\mathcal{X}$ over $\mathbb{R}$ or $\mathbb{C}$, an arbitrary normed space $\mathcal{Y}$ over $\mathbb{R}$ or $\mathbb{C}$ and a linear transformation $T : \mathcal{X} \to \mathcal{Y}$, we have

$$T \text{ is continuous}$$

Proof. Fix $x \in \mathcal{X}, \epsilon$. We wish

$$\text{to find } \delta \text{ such that } \forall h \in \mathcal{X} : \|h\| \leq \delta, \|T(x+h) - Tx\| \leq \epsilon$$

Let $\{e_1, \dots, e_n\}$ be a basis of $\mathcal{X}$. Note that $\|\sum \alpha_i e_i\|_1 \triangleq \sum |\alpha_i|$ is a norm. Because $\mathcal{X}$ is finite-dimensional, we know $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent. Then, we can fix $M \in \mathbb{R}^+$ such that

$$\|x\|_1 \leq M\|x\| \quad (x \in V)$$

We claim

$$\delta = \frac{\epsilon}{M(\max \|Te_i\|)} \text{ suffices}$$

Fix $\|h\| \leq \delta$ and express $h = \sum \alpha_i e_i$. Compute using linearity of T

$$\begin{aligned}
\|T(x+h) - Tx\| &= \|\sum \alpha_i Te_i\| \\
&\leq \sum |\alpha_i| \, \|Te_i\| \\
&\leq \|h\|_1 (\max \|Te_i\|) \\
&\leq M\|h\|(\max \|Te_i\|) = \epsilon \text{ (done)}
\end{aligned}$$

■

We now see that, because Linear transformation is bounded if and only if it is continuous and Linear map of finite-dimensional domain is always continuous, if $\mathcal{X}$ is finite-dimensional, then all linear map of domain $\mathcal{X}$ are bounded. A counter example to the generalization of this statement is followed.

Example 9 (Differentiation is an Unbounded Linear Operator)

$$\mathcal{X} = \Big(\mathbb{R}[x]|_{[0,1]}, \|\cdot\|_\infty\Big), D(P) \triangleq P'$$

Note that $\{x^n\}_{n \in \mathbb{N}}$ is bounded in $\mathcal{X}$ and $\{D(x^n)\}_{n \in \mathbb{N}}$ is not.

Now, suppose $\mathcal{X}, \mathcal{Y}$ are two fixed normed spaces over $\mathbb{R}$ or $\mathbb{C}$. We can easily check that the set $BL(\mathcal{X}, \mathcal{Y})$ of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ form a vector space over whichever field $\mathcal{Y}$ is over.

Naturally, our definition of boundedness of linear operator derive us a norm on $BL(\mathcal{X},\mathcal{Y})$, as followed

$$\|T\|_{\text{op}} \triangleq \inf\{M \in \mathbb{R}^+ : \forall x \in \mathcal{X}, \|Tx\| \leq M\|x\|\} \tag{5.2}$$

Before we show that our definition is indeed a norm, we first give some equivalent definitions and prove their equivalency.

Theorem 5.1.4. (Equivalent Definitions of Operator Norm) Given two fixed normed space $\mathcal{X},\mathcal{Y}$ over $\mathbb{R}$ or $\mathbb{C}$, a bounded linear operator $T : \mathcal{X} \to \mathcal{Y}$, and define $\|T\|_{\text{op}}$ as in Equation 5.2, we have

$$\|T\|_{\text{op}} = \sup_{x\in\mathcal{X},x\neq 0} \frac{\|Tx\|}{\|x\|}$$

Proof. Define $J \triangleq \{M \in \mathbb{R}^+ : \forall x \in \mathcal{X}, \|Tx\| \leq M\|x\|\}$ and observe

$$J = \{M \in \mathbb{R}^+ : M \geq \frac{\|Tx\|}{\|x\|}, \forall x \neq 0 \in \mathcal{X}\}$$

This let us conclude

$$\sup_{x\in\mathcal{X},x\neq 0} \frac{\|Tx\|}{\|x\|} = \min J = \|T\|_{\text{op}}$$

∎

It is now easy to see

$$\|T\|_{\text{op}} = \sup_{x\in\mathcal{X},x\neq 0} \frac{\|Tx\|}{\|x\|} \tag{5.3}$$
$$= \sup_{x\in\mathcal{X},\|x\|=1} \|Tx\| \tag{5.4}$$

It is not all in vain to introduce the equivalent definitions. See that the verification of $\|\cdot\|_{\text{op}}$ being a norm on $BL(\mathcal{X},\mathcal{Y})$ become simple by utilizing the equivalent definitions.

(a) For positive-definiteness, fix non-trivial T and fix $x \in \mathcal{X} \setminus N(T)$. Use Equation 5.3 to show $\|T\|_{\text{op}} \geq \frac{\|Tx\|}{\|x\|} > 0$.

(b) For absolute homogenity, use Equation 5.4 and $\|Tcx\| = |c| \cdot \|Tx\|$.

(c) For triangle inequality, use Equation 5.4 and $\|(T_1 + T_2)x\| \leq \|T_1x\| + \|T_2x\|$.

Naturally, and very very importantly, Equation 5.3 give us

$$\|Tx\| \leq \|T\|_{\text{op}} \cdot \|x\| \qquad (x \in \mathcal{X})$$

This inequality will later be the best tool to help analyze the derivatives of functions between Euclidean spaces, and perhaps better, it immediately give us

$$\frac{\|T_1T_2x\|}{\|x\|} \leq \frac{\|T_1\|_{\text{op}} \cdot \|T_2\|_{\text{op}} \cdot \|x\|}{\|x\|} = \|T_1\|_{\text{op}} \cdot \|T_2\|_{\text{op}}$$

Then Equation 5.3 give us

$$\|T_1T_2\|_{\text{op}} \leq \|T_1\|_{\text{op}} \cdot \|T_2\|_{\text{op}}$$

5.2 Directional Derivative and Gradient

Abstract

This short section introduce the idea of directional derivative and gradient. It shall be noted that, although both gradient and directional derivative are defined for real-valued function in this section, the notion of directional derivative can be easily generalized to function between Euclidean space; while the notion of gradient, as the way we define it, is only for real-valued function.

Given two normed space $\mathcal{X},\mathcal{Y}$, suppose f maps an open neighborhood O around x in $\mathcal{X}$ into $\mathcal{Y}$. We say f is **differentiable at** x if there exists a bounded linear transformation $A_x:\mathcal{X}\to\mathcal{Y}$ (from now, A_x will be denoted df_x) such that

$$\lim_{h\to 0}\frac{\|f(x+h)-f(x)-df_x(h)\|_{\mathcal{Y}}}{\|h\|_{\mathcal{X}}}=0 \tag{5.5}$$

Immediately, we should check that the linear approximation is unique. Suppose df_x and df'_x both satisfy Equation 5.5. We are required to show $(df_x-df'_x)h=0$ for all $\|h\|_{\mathcal{X}}=1$. Fix $h\in\mathcal{X}$ such that $\|h\|_{\mathcal{X}}=1$. Note that

$$\frac{(df_x-df'_x)th}{t}\text{ is a constant in } t \text{ for } t\neq 0$$

This then reduced the problem into showing

$$\frac{(df_x-df'_x)th}{t\|h\|_{\mathcal{X}}}\to 0\text{ as } t\to 0 \tag{5.6}$$

Observe

$$(df_x-df'_x)th=\Big(f(x+th)-f(x)-df'_x(th)\Big)-\Big(f(x+th)-f(x)-df_x(th)\Big)$$

which implies

$$\|(df_x-df'_x)th\|_{\mathcal{Y}}\leq\|f(x+th)-f(x)-df'_x(th)\|_{\mathcal{Y}}+\|f(x+th)-f(x)-df_x(th)\|_{\mathcal{Y}}$$

and thus implies Equation 5.6.

It shall be quite clear that a function f differentiable at x must be continuous at x, by noting the nominator of Equation 5.5 must tend to 0. For clarity, we here specify the notation. By $\mathbb{R}$, we mean a field equipped with the usual norm $\|x\| \triangleq |x|$. By $\mathbb{R}^n$ we mean the set of functions from $\{1, \dots, n\}$ to $\mathbb{R}$ equipped with the usual vector addition, scalar multiplication, dot product and induced norm.

Definition 5.2.1. (Definition of Directional Derivative of Scalar function) Given a normal vector $v \in \mathbb{R}^n$ and a function f that maps an open-neighborhood E around $x \in \mathbb{R}^n$ into $\mathbb{R}$, by the **directional derivative** $\partial_v f(x)$ of f with respect to v at x, we mean

$$\partial_v f(x) \triangleq \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t} \text{ if exists}$$

Something to note in our definition for directional derivative (Definition 5.2.1)

(a) The limit on the right hand side is done in $\Big(\mathbb{R}, |\cdot|\Big)$

(b) If f is differentiable at x, then we have

$$\partial_{av+bw} f(x) = df_x(av + bw) = adf_x(v) + bdf_x(w) = a\partial_v f(x) + b\partial_w f(x) \tag{5.7}$$

With what we observed, one can immediately see that if a function f is differentiable at x, then f has directional derivative with respect to any direction at x. The converse is not true. It is possible that f has directional derivative with respect to all directions, and yet f is still not differentaible. Consider

Example 10 (Discontinuous function such that all directional derivatives exist)

$$f : \mathbb{R}^2 \to \mathbb{R}, f(x, y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & \text{if } x^2 + y^2 \neq 0 \\ 0 & \text{if } x = y = 0 \end{cases}$$

Definition 5.2.2. (Definition of Gradient of $\mathbb{R}^n \to \mathbb{R}$ function) Given a point $x \in \mathbb{R}^n$ with open neighborhood E, a function $f : E \to \mathbb{R}$ differentiable at x, we define the **gradient** $\nabla f(x) \in \mathbb{R}^n$ of f at x to be the unique vector that satisfy

$$\nabla f(x) \cdot v = df_x(v) \text{ for all } v \in \mathbb{R}^n$$

We should immediately discuss whether our definition of gradient is well-defined. The proof of existence and uniqueness follows from generating an orthogonal basis $\{v_1, \dots, v_n\}$ and noting $\nabla f(x)$ must equal to $\sum_{i=1}^n df_x(v_i)v_i$. A few things one must know about gradient is as followed

(a) $\nabla f(x)$ is only defined when f is differentiable at x.

(b) gradient $\nabla f(x)$ "points toward" the direction at which $f : \mathbb{R}^n \to \mathbb{R}$ grow the fastest. Suppose v is normal. See

$$\nabla f(x) \cdot v = df_x(v) = \partial_v f(x)$$

Using Cauchy-Schwarz Inequality, we see that $\partial_v f(x)$ is of largest value when $v = \frac{\nabla f(x)}{|\nabla f(x)|}$. If $v = \frac{\nabla f(x)}{|\nabla f(x)|}$, then

$$\partial_v f(x) = |\nabla f(x)|$$

(c) It is possible $\nabla f(x) = 0$. This is true if and only if df_x maps $\mathbb{R}^n$ into 0. This fact echos with the fact gradient points toward the fastest growing direction. See (b).

5.3 MVT

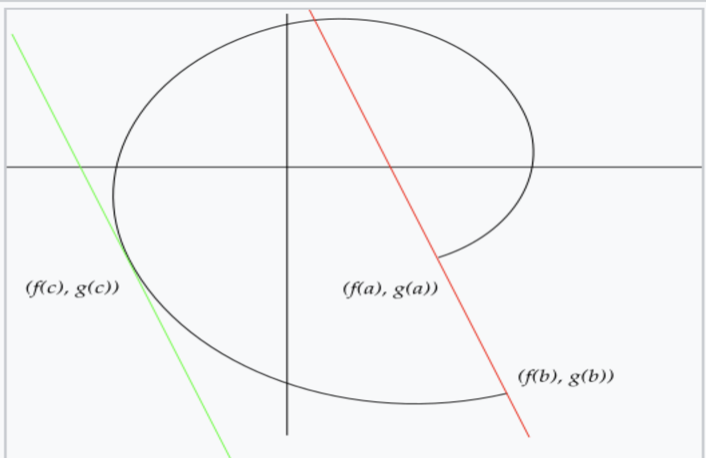
Abstract

This section introduce Cauchy's MVT and MVT, which will be heavily used later to prove multiple important and common results, e.g., L'Hospital Rule, Differentiability Theorem, The criterion for commutation of partial derivative and part 2 of FTC. For how important MVT is in the development of Theory of Calculus, it is worth pointing out that, fundamentally, MVT relies on the order of the real numbers, as usage of property of function differentiable at extremum in the proof of Cauchy's MVT suggest.

Theorem 5.3.1. (Local Extremum of Differentiable Function must have zero derivative) If $f : (x - \epsilon, x + \epsilon) \to \mathbb{R}$ is differentiable at x and attain maximum or minimum at x, then $f'(x) = 0$.

Proof. WOLG, suppose f attain maximum at x. We see that the value of $\frac{f(x+h)-f(x)}{h}$ is non-negative when $h < 0$ and non-positive when $h > 0$. This suggest that $f'(x) = 0$. ■

Below is a graph to help visualize Cauchy's MVT.



Geometrical meaning of Cauchy's theorem

We now prove Cauchy's MVT.

Theorem 5.3.2. (Cauchy's MVT) Given a function $f : [a, b] \to \mathbb{R}$ such that

(a) f, g are differentiable on (a, b)

(b) f, g are continuous on $[a, b]$

There exists $x \in (a,b)$ such that

$$\big[f(b)-f(a)\big]g'(x) = \big[g(b)-g(a)\big]f'(x)$$

Proof. Define h on (a,b) by

$$h(x) \triangleq \big[f(b)-f(a)\big]g'(x) - \big[g(b)-g(a)\big]f'(x)$$

We reduced our problem into

finding $x \in (a,b)$ such that $h(x) = 0$

Because f,g are both differentiable on (a,b), we know there exists an anti-derivative H on (a,b), which have the form

$$H(x) = \big[f(b)-f(a)\big]g(x) - \big[g(b)-g(a)\big]f(x)$$

Theorem 5.3.1 now allow us to reduce the problem into

finding a local extremum of H on (a,b)

Because f,g are both continuous on $[a,b]$, we know H is continuous on $[a,b]$. Then by EVT, we know

$$\exists x \in [a,b], H(x) = \max_{t\in[a,b]} H(t) \text{ and } \exists y \in [a,b], H(y) = \min_{t\in[a,b]} H(t)$$

If any of such x,y is in (a,b), we are done. If not, says that x,y both are on end points a or b. Compute that

$$H(a) = f(b)g(a) - g(b)f(a) = H(b)$$

We see H is constant on $[a,b]$. Then all points in (a,b) are extremums. (done) ∎

Corollary 5.3.3. (Lagrange's MVT) Given a function $f : [a,b] \to \mathbb{R}$ such that

(a) f is differentiable on (a,b)

(b) f is continuous on $[a,b]$

Then there exists $x \in (a,b)$ such that

$$f'(x) = \frac{f(b)-f(a)}{b-a}$$

Proof. The proof is done by applying Cauchy's MVT, where $g(x) \triangleq x$. ∎

There are two hypotheses in Lagrange's MVT

(a) f is differentiable on (a, b)

(b) f is continuous on $[a, b]$

They are all necessary. The necessity of differentiability on (a, b) is clear as shown by the canonical example using absolute value. The necessity of continuity on $[a, b]$ can be shown by the example

$$f(x) = \begin{cases} 1 & \text{if } a < x \leq b \\ 0 & \text{if } x = a \end{cases}$$

5.4 Differentiability Theorem

Abstract

This section prove

(a) The matrix representation of derivative for function between Euclidean spaces

(b) Differentiability Theorem

Note that the proof of Differentiability Theorem use MVT and the fact that all norms on $\mathbb{R}^k$ are equivalent where $k = nm$, and utilize the Frobenius norm.

Given an orthonormal basis $\{q_1, \ldots, q_m\}$ of $\mathbb{R}^m$ and a function $f : \mathbb{R}^n \to \mathbb{R}^m$, we let $f_j(x)$ be a real number

$$f_j(x) = f(x) \cdot q_j$$

It shall be clear that

$$f(x) = \sum_{j=1}^{m} f_j(x) q_j$$

which explain why we require $\{q_1, \ldots, q_m\}$ to be orthonormal in the first place. For brevity of the statement of the next theorem (Theorem 5.4.1), we introduce another notation. If we are provided a normal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$, we denote $\partial_{e_i} f_j(x)$ by $\partial_i f_j(x)$

Theorem 5.4.1. (Derivative is Jacobian) Suppose $\alpha = \{e_1, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$, and $\beta = \{q_1, \ldots, q_m\}$ is an orthonormal basis of $\mathbb{R}^m$. Suppose f maps an open neighborhood O around $x \in \mathbb{R}^n$ to $\mathbb{R}^m$. Then

$$f \text{ is differentiable at } x \implies \begin{cases} \partial_i f_j(x) \text{ exists for all } i, j \\ [df_x]_\alpha^\beta = \begin{bmatrix} \partial_1 f_1(x) & \cdots & \partial_n f_1(x) \\ \vdots & \ddots & \vdots \\ \partial_1 f_m(x) & \cdots & \partial_n f_m(x) \end{bmatrix} \end{cases}$$

Proof. Suppose $e_1, \ldots, e_n$ are all normal. Fix i, j. We wish to show

$$\partial_i f_j(x) \text{ exists}$$

Because f is differentiable at x, by definition of df_x, we have

$$\lim_{t\to 0}\frac{|f(x+te_i)-f(x)-df_x(te_i)|}{|te_i|}=0$$

Set $R_i:\mathbb{R}\to\mathbb{R}^m$ by $R_i(t)\triangleq f(x+te_i)-f(x)-df_x(te_i)$. We have

$$\lim_{t\to 0}\frac{|R_i(t)|}{|t|}=0 \tag{5.8}$$

Compute

$$\begin{aligned}f_j(x+te_i)-f_j(x)&=\big(f(x+te_i)-f(x)\big)\cdot q_j\\&=\big(R_i(t)+df_x(te_i)\big)\cdot q_j\\&=R_i(t)\cdot q_j+tdf_x(e_i)\cdot q_j\end{aligned}$$

This then give us

$$\frac{f_j(x+te_i)-f_j(x)}{t}=\frac{R_i(t)\cdot q_j}{t}+df_x(e_i)\cdot q_j$$

and

$$df_x(e_i)\cdot q_j-\frac{|R_i(t)\cdot q_j|}{|t|}\leq\frac{f_j(x+te_i)-f_j(x)}{t}\leq df_x(e_i)\cdot q_j+\frac{|R_i(t)\cdot q_j|}{|t|}$$

By Cauchy-Schwarz Inequality, we now have

$$df_x(e_i)\cdot q_j-\frac{|R_i(t)|}{|t|}\leq\frac{f_j(x+te_i)-f_j(x)}{t}\leq df_x(e_i)\cdot q_j+\frac{|R_i(t)|}{|t|}$$

Now applying Squeeze Theorem and Equation 5.8, we have

$$\partial_i f_j(x)=\lim_{t\to 0}\frac{f_j(x+te_i)-f_j(x)}{t}=df_x(e_i)\cdot q_j \text{ (done)}$$

Using the fact β is orthonormal, we now have

$$df_x(e_i)=\sum_{j=1}^m\Big(df_x(e_i)\cdot q_j\Big)q_j=\sum_{j=1}^m\partial_i f_j(x)q_j$$

and suggest the matrix representation. ■

Note that the converse is not always true. It is possible that a function f has all partial derivatives with respect to a given basis, or even all directions, and yet f is still discontinuous. We have given an example already in Directional Derivative and Gradient. Consider a less trivial one.

Example 11 (Non-differentiable Continuous Funciton with Partial Derivative)

$$f:\mathbb{R}^2\to\mathbb{R} \text{ and } f(x,y)=\begin{cases}\frac{x|y|}{\sqrt{x^2+y^2}} & \text{if } (x,y)\neq 0\\ 0 & \text{if } (x,y)=0\end{cases}$$

We have

$$\partial_x f(0)=\partial_y f(0)=0$$

By Theorem 5.4.1 (Derivative is Jacobian), if f is differentiable at 0, then df_0 must be trivial. Yet

$$\frac{|f(h,h)-f(0)-df_0(h,h)|}{|(h,h)|}=\frac{h}{2\,|h|}\not\to 0$$

Note that f is continuous at 0, by observing

$$x^2+y^2-2\,|xy|=\Big(|x|-|y|\Big)^2\geq 0 \implies \frac{x^2+y^2}{2}\geq|xy|$$

which implies

$$|f|\leq\frac{\sqrt{x^2+y^2}}{2}$$

We now introduce a property of function between normed space that are stronger than differentiability. Given two normed space $\mathcal{X},\mathcal{Y}$, and an open $E\subseteq\mathcal{X}$, we say $f:E\to\mathcal{Y}$ is **continuously differentiable** on $\mathcal{Y}$ if the map $D:(E,\|\cdot\|_{\mathcal{X}})\to\Big(BL(\mathcal{X},\mathcal{Y}),\|\cdot\|_{\text{op}}\Big)$ defined by

$$D(x)=df_x$$

is continuous. Note that the definition of the term "continuously differentiable" coincide when $\mathcal{X}=\mathcal{Y}=\mathbb{R}$ and df_x is just $h\mapsto f'(x)h$. We now give proof to the Differentiability

Theorem, which links between the continuity of total derivative and the continuity of partial derivatives.

Theorem 5.4.2. **(Differentiability Theorem)** Suppose $\alpha = \{e_1, \dots, e_n\}$ is an orthonormal basis of $\mathbb{R}^n$, and $\beta = \{q_1, \dots, q_m\}$ is an orthonormal basis of $\mathbb{R}^m$. Suppose f maps an open set $E \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$. Then

$$f \text{ is continuously differentiable on } E \iff \partial_i f_j \text{ exists and is continuous on } E \text{ for all } i, j$$

Proof. $(\longrightarrow)$

Fix i, j. Because f is differentiable on E, we know $\partial_i f_j$ exists on E by Theorem 5.4.1. Fix $x \in E$. We only have to show

$$\partial_i f_j \text{ is continuous at } x$$

Fix ϵ. We wish

$$\text{to find } \delta \text{ such that } |\partial_i f_j(y) - \partial_i f_j(x)| \leq \epsilon \text{ for all } |y - x| < \delta$$

Because f is continuously differentaible at x, we know there exists δ such that

$$\|df_y - df_x\|_{\text{op}} < \epsilon \text{ for all } |y - x| \leq \delta$$

We claim

$$\text{such } \delta \text{ suffices}$$

By the the matrix representation, we know

$$\partial_i f_j(y) - \partial_i f_j(x) = (df_y - df_x)e_i \cdot q_j$$

Then by Cauchy-Inequality, we have

$$\begin{aligned} |\partial_i f_j(y) - \partial_i f_j(x)| &\leq |(df_y - df_x)e_i| \\ &\leq \|df_y - df_x\|_{\text{op}} < \epsilon \text{ (done)} \end{aligned}$$

$(\longleftarrow)$

We first show

$$f \text{ is differentiable on } E$$

We first prove

$$\forall j \in \{1, \dots, m\}, f_j : \mathbb{R}^n \to \mathbb{R} \text{ is differentiable on } E \implies f \text{ is differentiable on } E$$

Fix $x \in E$. We wish to prove

$$f \text{ is differentiable at } x$$

Define $A : E \to \mathbb{R}^m$ by

$$A(h) \triangleq \sum_{j=1}^{m} (df_j)_x(h) q_j$$

We claim

$$A \text{ suffices to be the } df_x$$

Using the fact q_j are orthonormal, we have

$$f(x+h) - f(x) - A(h) = \sum_{j=1}^{m} \big(f_j(x+h) - f_j(x) - (df_j)_x(h)\big) q_j$$

This give us

$$\begin{aligned} \lim_{h \to 0} \frac{|f(x+h) - f(x) - A(h)|}{|h|} &= \lim_{h \to 0} \frac{\left| \sum_{j=1}^{m} \big(f_j(x+h) - f_j(x) - (df_j)_x(h)\big) q_j \right|}{|h|} \\ &\leq \lim_{h \to 0} \frac{\sum_{j=1}^{m} |f_j(x+h) - f_j(x) - (df_j)_x(h)|}{|h|} = 0 \text{ (done)} \end{aligned}$$

Fix $j \in \{1, \dots, m\}$. We can now reduce the problem into

$$f_j : \mathbb{R}^n \to \mathbb{R} \text{ is differentiable on } E$$

Fix $x \in E$. We wish to prove

$$f_j \text{ is differentiable at } x$$

Express $h = \sum_{i=1}^{n} h_i e_i$. Define $B : E \to \mathbb{R}$ by

$$B(h) = \sum_{i=1}^{n} \partial_i f_j(x) h_i$$

We claim

$$B \text{ suffices to be } (df_j)_x$$

By continuity of each $\partial_i f_j$ on E, we can let δ satisfy

$$|\partial_i f_j(y) - \partial_i f_j(x)| < \frac{\epsilon}{n} \text{ for all } y \in B_\delta(x)$$

We claim

$$\frac{|f_j(y)-f_j(x)-B(y-x)|}{|y-x|}\leq \epsilon \text{ for all } y\in B_\delta(x)$$

Express $y-x=\sum_{k=1}^n h_k e_k$. Define $v_0,\dots,v_n\in\mathbb{R}^n$ by

$$v_0\triangleq 0 \text{ and } v_k\triangleq \sum_{i=1}^k h_i e_i \text{ for all } k\in\{1,\dots,n\}$$

Now observe

$$\begin{aligned}
\frac{|f_j(y)-f_j(x)-B(y-x)|}{|y-x|}&=\frac{|f_j(x+v_n)-f_j(x)-B(\sum_{k=1}^n h_k e_k)|}{|y-x|}\\
&=\frac{\big|\big(\sum_{k=1}^n f_j(x+v_k)-f_j(x+v_{k-1})\big)-\sum_{k=1}^n \partial_k f_j(x)h_k\big|}{|y-x|}\\
&=\frac{|\sum_{k=1}^n f_j(x+v_k)-f_j(x+v_{k-1})-\partial_k f_j(x)h_k|}{|y-x|}\\
&=\frac{|\sum_{k=1}^n f_j(x+v_{k-1}+h_k e_k)-f_j(x+v_{k-1})-\partial_k f_j(x)h_k|}{|y-x|}\\
(\text{ For some } e_k\in(0,1) \text{ by MVT})\quad&=\frac{|\sum_{k=1}^n \partial_k f_j(x+v_{k-1}+t_k e_k)h_k-\partial_k f_j(x)h_k|}{|y-x|}\\
&\leq\frac{\sum_{k=1}^n \big|\big(\partial_k f_j(x+v_{k-1}+t_k e_k)-\partial_k f_j(x)\big)h_k\big|}{|y-x|}\\
&<\frac{\sum_{k=1}^n \frac{\epsilon}{n}|h_k|}{|y-x|}\leq\epsilon \text{ (done)}
\end{aligned}$$

We now prove

$$f \text{ is continuously differentiable on } E$$

Fix ϵ and $x\in E$. We are required

$$\text{to find } \delta \text{ such that } \|df_y-df_x\|_{\text{op}}\leq\epsilon \text{ for all } y\in B_\delta(x)$$

Note that one can define a norm $\|\cdot\|_F$ called "Forbenius Norm" on $BL(\mathbb{R}^n,\mathbb{R}^n)$ by

$$\|A\|_F\triangleq\sqrt{\sum_{i=1}^n\sum_{j=1}^n a_{i,j}^2} \text{ where } [A]_\alpha^\beta=\begin{bmatrix}a_{1,1}&\cdots&a_{n,1}\\ \vdots&\ddots&\vdots\\ a_{1,n}&\cdots&a_{n,n}\end{bmatrix}$$

Because all norms on finite-dimensional real vector spaces are equivalent, we know there exists M such that for all $x \in E$, we have

$$\|df_x\|_{\text{op}} \leq M\|df_x\|_F$$

Because the partial derivatives are all continuous by definition, we can let δ satisfy

$$\big(\partial_i f_j(x+h)\big)^2 - \big(\partial_i f_j(x)\big)^2 < \frac{\epsilon^2}{M^2 n^2} \text{ for all } h \in B_\delta(0)$$

We claim

$$\text{such } \delta \text{ suffices}$$

Let $|y - x| < \delta$. We see

$$\|df_y - df_x\|_{\text{op}} \leq M\|df_y - df_x\|_F < M\sqrt{\sum_{i=1}^n \sum_{j=1}^n \frac{\epsilon^2}{M^2 n^2}} = \epsilon \text{ (done)}$$

$\blacksquare$

5.5 Product Rule and Chain Rule

Abstract

This short section prove Product rule in the setting of gradient of real-valued function and Chain rule for functions between normed spaces, which is heavily used in next section on smooth function and Differential Geometry.

Although distinct versions of product rule exists in the context of vector calculus, here we prove the almost most simple kind.

Theorem 5.5.1. (Product Rule) Given two function $f, g : \mathbb{R}^d \to \mathbb{R}$ differentiable at x, we have

$$\nabla(fg)(x) = g(x)\nabla f(x) + f(x)\nabla g(x)$$

Proof. Note that

$$\lim_{h\to 0} f(x+h)\Big[\frac{g(x+h) - g(x) - \nabla g(x)\cdot h}{|h|}\Big] = 0$$
$$\text{and } \lim_{h\to 0} g(x)\Big[\frac{f(x+h) - f(x) - \nabla f(x)\cdot h}{|h|}\Big] = 0$$

Adding these two equations together, we have

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x) - [g(x)\nabla f(x) + f(x)\nabla g(x)]\cdot h}{|h|} = 0$$

∎

We now prove the Chain Rule for function between normed space.

Theorem 5.5.2. (Chain Rule) Given three normed space $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$, a point $x \in \mathcal{X}$, a function g that map an open set $U \subseteq \mathcal{Y}$ containing $f(x)$ into $\mathcal{Z}$, a function f that map an open-neighborhood around x into U such that

(a) f is differentiable at x

(b) g is differentiable at $f(x)$

we have

$$d(g \circ f)_x = dg_{f(x)} \circ df_x$$

Proof. For brevity, we use $F \triangleq g \circ f$. We wish to prove

$$\lim_{h\to 0} \frac{\|F(x+h) - F(x) - dg_{f(x)}df_x(h)\|_{\mathcal{Z}}}{\|h\|_{\mathcal{X}}} = 0$$

Fix $k \triangleq f(x+h) - f(x)$. Observe

$$F(x+h) - F(x) - dg_{f(x)}df_x(h) = \Big(g\big(f(x)+k\big) - g\big(f(x)\big) - dg_{f(x)}(k)\Big) + dg_{f(x)}\big(k - df_x(h)\big)$$

This now implies

$$\frac{\|F(x+h) - F(x) - dg_{f(x)}df_x(h)\|_{\mathcal{Z}}}{\|h\|_{\mathcal{X}}} \text{ is smaller than}$$

$$\frac{\|g\big(f(x)+k\big) - g\big(f(x)\big) - dg_{f(x)}(k)\|_{Z} + \|dg_{f(x)}\big(k - df_x(h)\big)\|_{\mathcal{Z}}}{\|h\|_{\mathcal{X}}}$$

This let us reduce the problem into proving

$$\lim_{h\to 0} \frac{\|g\big(f(x)+k\big) - g\big(f(x)\big) - dg_{f(x)}(k)\|_{Z}}{\|h\|_{\mathcal{X}}} = 0$$

$$\text{and } \lim_{h\to 0} \frac{\|dg_{f(x)}\big(k - df_x(h)\big)\|_{\mathcal{Z}}}{\|h\|_{\mathcal{X}}} = 0$$

We first prove

$$\lim_{h\to 0} \frac{\|g\big(f(x)+k\big) - g\big(f(x)\big) - dg_{f(x)}(k)\|_{Z}}{\|h\|_{\mathcal{X}}} = 0$$

Note that if $\|k\|_{\mathcal{Y}} = 0$, we have

$$\frac{\|g\big(f(x)+k\big) - g\big(f(x)\big) - dg_{f(x)}(k)\|_{Z}}{\|h\|_{\mathcal{X}}} = 0$$

Now, observe that

$$\frac{\|g\big(f(x)+k\big) - g\big(f(x)\big) - dg_{f(x)}(k)\|_{Z}}{\|h\|_{\mathcal{X}}} = \frac{\|g\big(f(x)+k\big) - g\big(f(x)\big) - dg_{f(x)}(k)\|_{Z}}{\|k\|_{\mathcal{Y}}} \cdot \frac{\|k\|_{\mathcal{Y}}}{\|h\|_{\mathcal{X}}}$$

Because $h \to 0 \implies k \to 0$, we can now reduce the problem into proving

$$\limsup_{h\to 0} \frac{\|k\|_{\mathcal{Y}}}{\|h\|_{\mathcal{X}}} \text{ exists}$$

Observe

$$\begin{aligned}
\frac{\|k\|_{\mathcal{Y}}}{\|h\|_{\mathcal{X}}} &= \frac{\|f(x+h)-f(x)-df_x(h)+df_x(h)\|_{\mathcal{X}}}{\|h\|_{\mathcal{X}}} \\
&\leq \frac{\|f(x+h)-f(x)-df_x(h)\|_{\mathcal{Y}}}{\|h\|_{\mathcal{X}}} + \frac{\|df_x(h)\|_{\mathcal{Y}}}{\|h\|_{\mathcal{X}}} \\
&\leq \frac{\|f(x+h)-f(x)-df_x(h)\|_{\mathcal{Y}}}{\|h\|_{\mathcal{X}}} + \|df_x\|_{\text{op}} \text{ (done)}
\end{aligned}$$

We now prove

$$\lim_{h\to 0} \frac{\|dg_{f(x)}\big(k-df_x(h)\big)\|_{\mathcal{Z}}}{\|h\|_{\mathcal{X}}} = 0$$

Note that if $f(x+h)-f(x)-df_x(h)=0$, then $\|dg_{f(x)}\big(k-df_x(h)\big)\|_{\mathcal{Z}}=0$. Now, observe

$$\begin{aligned}
\frac{\|dg_{f(x)}\big(k-df_x(h)\big)\|_{\mathcal{Z}}}{\|h\|_{\mathcal{X}}} &= \frac{\|dg_{f(x)}\big(f(x+h)-f(x)-df_x(h)\big)\|_{\mathcal{Z}}}{\|f(x+h)-f(x)-df_x(h)\|_{\mathcal{Y}}} \cdot \frac{\|f(x+h)-f(x)-df_x(h)\|_{\mathcal{Y}}}{\|h\|_{\mathcal{X}}} \\
&\leq \|dg_{f(x)}\|_{\text{op}} \cdot \frac{\|f(x+h)-f(x)-df_x(h)\|_{\mathcal{Y}}}{\|h\|_{\mathcal{X}}} \to 0 \text{ (done)}
\end{aligned}$$

$\blacksquare$

5.6 Smoothness

Abstract

This is a short section introducing the idea of smooth functions between Euclidean spaces, which heavily rely on Chain Rule. One should note that our **notation** $\partial_{21} f$ here means $\partial_2(\partial_1 f)$.

Theorem 5.6.1. (Structure of Mixed Partial Derivative) Given an open set $E \subseteq \mathbb{R}^2$, a point $p \in E$, a basis $\{e_1, e_2\}$ of $\mathbb{R}^2$ and a function $f : E \to \mathbb{R}$ such that

(a) $\partial_1 f$ exists on E

(b) $\partial_2 f$ exists on E

(c) $\partial_{21} f$ exists on E and is continuous at p

We have

$$\partial_{12} f(p) = \partial_{21} f(p)$$

Proof. Express elements of E in the basis $\{e_1, e_2\}$, and express $p = (a, b)$. We are required to prove

$$\lim_{h \to 0} \frac{\partial_2 f(a+h, b) - \partial_2 f(a, b)}{h} = \partial_{21} f(a, b)$$

Let $D \triangleq \{q - (a, b) \in \mathbb{R}^2 : q \in E \setminus \{p\}\}$ and define $\Delta(h, k) : D \to \mathbb{R}$ by

$$\Delta(h, k) \triangleq f(a+h, b+k) - f(a, b+k) - f(a+h, b) + f(a, b)$$

Note that because $\partial_2 f$ exists on E, for all $h \neq 0$, we have

$$\lim_{k \to 0} \frac{\Delta(h, k)}{hk} = \frac{\partial_2 f(a+h, b) - \partial_2 f(a, b)}{h}$$

This let us reduce the problem into proving

$$\lim_{h \to 0} \lim_{k \to 0} \frac{\Delta(h, k)}{hk} = \partial_{21} f(a, b)$$

We first show

> For all $(h, k) \in D : hk \neq 0$, $\frac{\Delta(h,k)}{hk} = \partial_{21} f(x, y)$ for some $(x, y) \in D : |x - a| < h$ and $|y - b| < k$.

Fix $(h,k) \in D : hk \neq 0$. Define $u : (b-\epsilon, b+\epsilon) \to \mathbb{R}$ by

$$u(t) \triangleq f(t,b+k) - f(t,b)$$

Chain Rule allow us to compute

$$u'(t) = \partial_1 f(t,b+k) - \partial_1 f(t,b)$$

We then can deduce

$$\begin{aligned}\Delta(h,k) &= u(a+h) - u(a) \\ &= hu'(x) \text{ for some } x \in (a,a+h) \text{ by MVT} \\ &= h\big(\partial_1 f(x,b+k) - \partial_1 f(x,b)\big)\end{aligned}$$

Now, because $\partial_{21} f$ exists on E, we can deduce

$$\begin{aligned}\Delta(h,k) &= h(\partial_1 f(x,b+k) - \partial_1 f(x,b)) \\ &= hk\partial_{21} f(x,y) \text{ for some } y \in (b,b+k) \text{ by MVT (done)}\end{aligned}$$

Fix ϵ. We wish

$$\text{to find some } \delta \text{ such that for all } h : 0 < |h| < \delta, \left| \lim_{k\to 0} \frac{\Delta(h,k)}{hk} - \partial_{21} f(x,y) \right| \leq \epsilon$$

Because $\partial_{21} f$ is continuous at p, by olive lemma, we know there exists δ such that

$$\left| \frac{\Delta(h,k)}{hk} - \partial_{21} f(a,b) \right| < \frac{\epsilon}{2} \text{ for all } h,k \in (-\delta,\delta) \setminus \{0\}$$

We claim

$$\text{such } \delta \text{ works}$$

Fix $h : 0 < |h| < \delta$. Note that $\lim_{k\to 0} \frac{\Delta(h,k)}{hk} = \frac{\partial_2 f(a+h,b) - \partial_2 f(a,b)}{h}$ exists, so we can find small enough k' such that

$$0 < |k'| < \delta \text{ and } \left| \lim_{k\to 0} \frac{\Delta(h,k)}{hk} - \frac{\Delta(h,k')}{hk'} \right| < \frac{\epsilon}{2}$$

Now observe

$$\left| \lim_{k\to 0} \frac{\Delta(h,k)}{hk} - \partial_{21} f(x,y) \right| \leq \left| \lim_{k\to 0} \frac{\Delta(h,k)}{hk} - \frac{\Delta(h,k')}{hk'} \right| + \left| \frac{\Delta(h,k')}{hk'} - \partial_{21} f(a,b) \right| \leq \epsilon \text{ (done)}$$

$\blacksquare$

Corollary 5.6.2. (Clairaut's Theorem on equality of mixed partial) Given a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$, an open set $E \subseteq \mathbb{R}^n$, a function $f : E \to \mathbb{R}$ such that

$$\partial_{ij} f \text{ exist and is continuous on } E \text{ for all } i, j \in \{1, \ldots, n\}$$

We have

$$\partial_{ij} f = \partial_{ji} f \text{ on } E \text{ for all } i, j \in \{1, \ldots, n\}$$

Given a function $f : E \subseteq \mathbb{R}^n \to \mathbb{R}$, we now see that, by Differentiability Theorem, if $\partial_x f, \partial_{xy} f$ exist on E and the latter is continuous on E, then f is continuously differentiable on E.

Example 12 (A C^1 but not C^2 function)

$$f(x, y) \triangleq \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x, y) \neq 0 \\ 0 & \text{if } (x, y) = 0 \end{cases}$$

All second order partial derivatives exist, yet f_{xy}, f_{yx} does not commute at 0.

Now, if we define a function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ on some open subset U to be **smooth** if for all $j \in \{1, \ldots, m\}$, all partial derivatives of f exists, we then see that the partial derivatives of smooth function always commute. Notably, one can check that if $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, and $g : V \subseteq \mathbb{R}^m \to \mathbb{R}^d$ are both smooth and $f(U) \subseteq V$, then $g \circ f : U \to \mathbb{R}^d$ is also smooth, using long tedious induction with Product rule and Chain rule.

5.7 Complex Differentiation

Abstract

This is a short section introducing the idea of complex-differentiable function and prove some of their basic properties, i.e., Cauchy Riemann Criteria and Product and Quotient Rule for Complex-differentiable Function.

Given a complex-valued function f defined on some open subset of $\mathbb{C}$ containing z, we say f is **complex-differentiable at** z if there exists some complex number denoted by $f'(z)$ such that

$$\frac{f(z+h)-f(z)-f'(z)h}{h} \to 0 \text{ as } h \to 0; h \in \mathbb{C}$$

Immediately, one can see that a complex-differentiable function when viewed as a function between $\mathbb{R}^2$ is differentiable with derivative

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \text{ where } f'(z) = a + bi \tag{5.9}$$

With the form of derivative in mind, one may conjecture that complex-differentiable is a 'stricter' condition than merely differentiable when regarded as function between $\mathbb{R}^2$. This is exactly true. Consider the following example.

Example 13 (A non complex-differentiable function)

$$f(z) \triangleq \overline{z}$$

This is a linear function with matrix representation

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

which doesn't fit the necessary form in Equation 5.9.

Theorem 5.7.1. (Cauchy Riemann Criteria) Given a complex-valued function f defined on some open subset U of $\mathbb{C}$ containing z, if we write

$$f(x+yi) = u(x,y) + iv(x,y)$$

where $u, v : U \to \mathbb{R}$, then the following two statements are equivalent.

(a) f is complex differentiable at z.

(b) u, v are differentiable at z when U is viewed as subsets of $\mathbb{R}^2$ and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} \text{ at } z$$

Proof. (a) to (b) is an immediate result of Chain Rule and matrix representation of $f'(z)$. Suppose (b) is true. Let $h = h_1 + ih_2$ and $z = x + yi$. Because u, v are differentiable at (x, y), by the matrix representation of derivative, we have

$$\begin{aligned}
\frac{f(z+h) - f(z)}{h} &= \frac{u(x+h_1, y+h_2) - u(x,y)}{h_1 + ih_2} + i\frac{v(x+h_1, y+h_2) - v(x,y)}{h_1 + ih_2} \\
&= \frac{h_1 u_x + h_2 u_y + ih_1 v_x + ih_2 v_y + o(|h|)}{h_1 + ih_2} \\
&= \frac{(h_1 + ih_2)u_x + i(h_1 + ih_2)v_x + o(|h|)}{h_1 + ih_2} \\
&= u_x + iv_x + \frac{o(|h|)}{h_1 + ih_2} \to u_x + iv_x
\end{aligned}$$

∎

Theorem 5.7.2. (Product and Quotient Rule for Complex-differentiable Function) Given two function f, g complex-differentiable at z, we have

$$(fg)'(z) = f'(z)g(z) + f(z)g'(z)$$

and if $g(z) \neq 0$, we also have

$$\Big(\frac{f}{g}\Big)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}$$

Proof. Observe

$$\begin{aligned}
&\frac{f(z+h)g(z+h) - f(z)g(z)}{h} \\
=&f(z+h)\Big[\frac{g(z+h) - g(z)}{h}\Big] + g(z)\Big[\frac{f(z+h) - f(z)}{h}\Big] \to f'(z)g(z) + f(z)g'(z)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\frac{f(z+h)}{g(z+h)} - \frac{f(z)}{g(z)}}{h} &= \frac{f(z+h)g(z) - f(z)g(z+h)}{g(z+h)g(z)h} \\
&= \frac{1}{g(z+h)g(z)}\Big[g(z)\Big(\frac{f(z+h) - f(z)}{h}\Big) - f(z)\Big(\frac{g(z+h) - g(z)}{h}\Big)\Big] \\
&\to \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}
\end{aligned}$$

$\blacksquare$

5.8 Uniform Convergence and Differentiation

Abstract

This is a section discussing the relationship between uniform convergence and differentiation, which heavily rely on the usage of MVT, and is used to prove Analytic function is smooth.

Before stating Theorem 5.8.1, let's see three examples why we don't (can't) use the hypothesis: $f_n \to f$ uniformly in our statement of Theorem 5.8.1

Example 14 (Differentiable functions are NOT closed under uniform convergence)

$$X = [-1, 1] \text{ and } f(x) = |x|$$

By Weierstrass approximation Theorem, there is a sequence of polynomials (differentiable) that uniformly converge to f, which is not differentiable at 0.

Example 15 (Derivative won't necessarily converge to the right place)

$$X = \mathbb{R} \text{ and } f_n(x) = \frac{\sin nx}{\sqrt{n}}$$

Compute

$$f'(x) = 0 \text{ and } f_n'(x) = \sqrt{n} \cos nx$$

Example 16 (Derivative won't necessarily converge to the right place)

$$X = \mathbb{R} \text{ and } f_n(x) = \frac{x}{1 + nx^2}$$

Compute

$$f = \tilde{0} \text{ and } f_n'(0) = 1$$

Informally speaking, these examples together with the fact Riemann integral are closed under uniform convergence should give you some ideas that differentiation and integration although are operations inverse to each other, are NOT symmetric. There is a certain hierarchy on continuous functions on a fixed compact interval. Thus, we have the next Theorem in its form. Note that in application, the next Theorem only require us to prove f_n' uniformly converge, and doesn't require us to prove to where does it converge.

Theorem 5.8.1. (Uniform Convergence and Differentiation) Given a bounded interval $[a,b]$ and some sequence of function $f_n : [a,b] \to \mathbb{R}$ such that

(a) f_n' uniformly converge on (a,b)

(b) f_n are continuous on $[a,b]$

(c) $f_n(x_0) \to L$ for some $x_0 \in [a,b]$

Then

(a) f_n uniformly converge on $[a,b]$

(b) and

$$\Big(\lim_{n\to\infty} f_n\Big)'(x_0) = \lim_{n\to\infty} f_n'(x_0) \text{ on } (a,b)$$

Proof. We first prove

$$f_n \text{ uniformly converge on } [a,b] \tag{5.10}$$

Fix ϵ. We wish

$$\text{to find } N \text{ such that } \|f_n - f_m\|_\infty \leq \epsilon \text{ for all } n,m > N$$

Because $f_n(x_0)$ converge, and f_n' uniformly converge, we know there exists N such that

$$\begin{cases} |f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2} \\ \|f_n' - f_m'\|_\infty < \frac{\epsilon}{2(b-a)} \end{cases} \quad \text{for all } n,m > N \tag{5.11}$$

We claim

$$\text{such } N \text{ works}$$

Fix $x \in [a,b]$ and $n,m > N$. We first show

$$\big|(f_n - f_m)(x) - (f_n - f_m)(x_0)\big| \leq \frac{\epsilon}{2}$$

Because $(f_n - f_m)' = f_n' - f_m'$, by MVT and Equation 5.11, we can deduce

$$\begin{aligned} \big|(f_n - f_m)(x) - \big(f_n - f_m\big)(x_0)\big| &= \Big|\big[(f_n - f_m)'(t)\big](x - x_0)\Big| \text{ for some } t \text{ between } x, x_0 \\ &< \frac{\epsilon}{2(b-a)} \cdot |x - x_0| \\ &\leq \frac{\epsilon}{2(b-a)} \cdot (b-a) = \frac{\epsilon}{2} \ \big(\because x, x_0 \in [a,b]\big) \text{ (done)} \end{aligned}$$

Now, by Equation 5.11, we have

$$\begin{aligned}\big|(f_n - f_m)(x)\big| &\leq \big|(f_n - f_m)(x) - (f_n - f_m)(x_0)\big| + \big|(f_n - f_m)(x_0)\big| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ (done)}\end{aligned}$$

Let $f : [a, b] \to \mathbb{R}$ be the limit of f_n. It remains to prove

$$f'(x) = \lim_{n\to\infty} f'_n(x) \text{ on } (a, b) \tag{5.12}$$

Fix $x \in (a, b)$ and define $\varphi, \varphi_n : [a, b] \setminus x \to \mathbb{R}$ by

$$\varphi(t) \triangleq \frac{f(t) - f(x)}{t - x} \text{ and } \varphi_n(t) \triangleq \frac{f_n(t) - f_n(x)}{t - x}$$

It is clear that $\varphi_n \to \varphi$ pointwise on $[a, b] \setminus x$. We now show

$$\varphi_n \to \varphi \text{ uniformly on } [a, b] \setminus x$$

Fix ϵ. We have

$$\text{to find } N \text{ such that } |\varphi_n(t) - \varphi_m(t)| \leq \epsilon \text{ for all } n, m > N \text{ and } t \in [a, b] \setminus x$$

Because f'_n uniformly converge on (a, b), we know there exists N such that

$$\|f'_n - f'_m\|_\infty \leq \epsilon \text{ for all } n, m > N \tag{5.13}$$

We claim

$$\text{such } N \text{ works}$$

Fix $n, m > N$ and $t \in [a, b] \setminus x$. We wish to prove

$$|\varphi_n(t) - \varphi_m(t)| \leq \epsilon$$

Because $(f_n - f_m)' = f'_n - f'_m$, by MVT and Equation 5.13, we can deduce

$$\begin{aligned}|\varphi_n(t) - \varphi_m(t)| &\leq \left|\frac{f_n(t) - f_n(x)}{t - x} - \frac{f_m(t) - f_m(x)}{t - x}\right| \\ &= \left|\frac{(f_n - f_m)(t) - (f_n - f_m)(x)}{t - x}\right| \\ &= \big|(f'_n - f'_m)(t_0)\big| \text{ for some } t_0 \text{ between } t, x \\ &\leq \epsilon \text{ (done)}\end{aligned}$$

Note that

$$\lim_{n\to\infty}\lim_{t\to x}\varphi_n(t) = \lim_{n\to\infty} f'(x) \text{ exists}$$

We can now exchange the limit and see that the derivative of f at x exists.

$$\begin{aligned} f'(x) = \lim_{t\to x}\varphi(t) &= \lim_{t\to x}\lim_{n\to\infty}\varphi_n(t) \\ &= \lim_{n\to\infty}\lim_{t\to x}\varphi_n(t) = \lim_{n\to\infty} f_n'(x) \text{ (done)} \end{aligned}$$

∎

Theorem 5.8.2. (Uniform Convergence and Holomorphic) Given a real number r and some sequence of function $f_n : \overline{D_r(z_0)} \to \mathbb{C}$ such that

(a) f_n' uniformly converge on $D_r(z_0)$

(b) f_n are continuous on $\overline{D_r(z_0)}$

(c) $f_n(v) \to L$ for some $v \in \overline{D_r(z_0)}$

Then

(a) f_n uniformly converge on $\overline{D_r(z_0)}$

(b) and

$$\Big(\lim_{n\to\infty} f_n\Big)'(z) = \lim_{n\to\infty} f_n'(z) \text{ on } D_r(z_0)$$

Proof. We first prove

$$f_n \text{ uniformly converge on } \overline{D_r(z_0)} \tag{5.14}$$

Fix ϵ. We wish

$$\text{to find } N \text{ such that } \|f_n - f_m\|_\infty \le \epsilon \text{ for all } n,m > N$$

Because $f_n(v)$ converge, and f_n' uniformly converge, we know there exists N such that

$$\begin{cases} |f_n(v) - f_m(v)| < \frac{\epsilon}{2} \\ \|f_n' - f_m'\|_\infty < \frac{\epsilon}{8r} \end{cases} \quad \text{for all } n,m > N \tag{5.15}$$

We claim

$$\text{such } N \text{ works}$$

Fix $z \in \overline{D_r(z_0)}$ and $n, m > N$. We first show

$$\big|(f_n - f_m)(z) - (f_n - f_m)(z_0)\big| \leq \frac{\epsilon}{2}$$

Denote $f_n - f_m : \overline{D_r(z_0)} \to \mathbb{C}$ by g. Because

$$|g(z) - g(z_0)| \leq \Big|\mathrm{Re}\Big(g(z) - g(z_0)\Big)\Big| + \Big|\mathrm{Im}\Big(g(z) - g(z_0)\Big)\Big|$$

WOLG, we only have to prove

$$\Big|\mathrm{Re}\Big(g(z) - g(z_0)\Big)\Big| \leq \frac{\epsilon}{4}$$

Because $\overline{D_r(z_0)}$ is convex, we can define $h : [0, 1] \to \mathbb{R}$ by

$$h(t) \triangleq \mathrm{Re}\Big(g(tz + (1-t)z_0)\Big)$$

By Chain Rule and matrix representation of derivative, we see that for all $t \in (0, 1)$

$$\begin{aligned} h'(t) = ac - bd \text{ where } z_0 - z &= a + bi \\ \text{and } g'(tz + (1-t)(z_0)) &= c + di \end{aligned}$$

Because $|a + bi| \leq r$ and $|c + di| \leq \frac{\epsilon}{8r}$ by Equation 5.15, if we use MVT, we see that

$$\begin{aligned} \Big|\mathrm{Re}\Big(g(z) - g(z_0)\Big)\Big| = |h(1) - h(0)| &= |h(t)| \text{ for some } t \in (0, 1) \\ &= |ac| + |bd| \leq \frac{\epsilon}{4} \text{ (done)} \end{aligned}$$

Now, by Equation 5.15, we have

$$\begin{aligned} \big|(f_n - f_m)(z)\big| &\leq \big|(f_n - f_m)(z) - (f_n - f_m)(v)\big| + \big|(f_n - f_m)(v)\big| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ (done)} \end{aligned}$$

Let $f : \overline{D_r(z_0)} \to \mathbb{C}$ be the limit of f_n. It remains to prove

$$f'(z) = \lim_{n \to \infty} f_n'(z) \text{ on } D_r(z_0) \tag{5.16}$$

Fix $z \in D_r(z_0)$ and define $\varphi, \varphi_n : \overline{D_r(z_0)} \setminus z \to \mathbb{R}$ by

$$\varphi(u) \triangleq \frac{f(u) - f(z)}{u - z} \text{ and } \varphi_n(u) \triangleq \frac{f_n(u) - f_n(z)}{u - z}$$

It is clear that $\varphi_n \to \varphi$ pointwise on $\overline{D_r(z_0)} \setminus z$. We now show

$$\varphi_n \to \varphi \text{ uniformly on } \overline{D_r(z_0)} \setminus z$$

Fix ϵ. We have

$$\text{to find } N \text{ such that } |\varphi_n(t) - \varphi_m(t)| \leq \epsilon \text{ for all } n, m > N \text{ and } t \in \overline{D_r(z_0)} \setminus z$$

Because f_n' uniformly converge on $D_r(z_0)$, we know there exists N such that

$$\|f_n' - f_m'\|_\infty \leq \frac{\epsilon}{4} \text{ for all } n, m > N \tag{5.17}$$

We claim

$$\text{such } N \text{ works}$$

Fix $n, m > N$ and $u \in \overline{D_r(z_0)}$. We wish to prove

$$|\varphi_n(u) - \varphi_m(u)| \leq \epsilon$$

Denote $f_n - f_m : \overline{D_r(z_0)} \to \mathbb{C}$ by g. Because

$$|\varphi_n(u) - \varphi_m(u)| = \left| \frac{g(u) - g(z)}{u - z} \right| \leq \frac{\left| \text{Re}\Big(g(u) - g(z)\Big) \right|}{|u - z|} + \frac{\left| \text{Re}\Big(g(u) - g(z)\Big) \right|}{|u - z|}$$

WOLG, we only have to prove

$$\frac{\left| \text{Re}\Big(g(u) - g(z)\Big) \right|}{|u - z|} \leq \frac{\epsilon}{2}$$

Again, define $h : [0, 1] \to \mathbb{R}$ by

$$h(t) \triangleq \text{Re}\Big(g(tu + (1 - t)z_0)\Big)$$

Then by Chain Rule and matrix representation of derivative, we see that for all $t \in (0, 1)$

$$\begin{aligned} h'(t) = ac - bd \text{ where } & u - z = a + bi \\ \text{and } & g'(tu + (1 - t)(z)) = c + di \end{aligned}$$

Now, by MVT and Equation 5.17, we can deduce

$$\begin{aligned} \frac{\left| \text{Re}\Big(g(u) - g(z)\Big) \right|}{|u - z|} = \frac{|h(1) - h(0)|}{|u - z|} &= \frac{|h'(t)|}{|a + bi|} \text{ for some } t \in (0, 1) \\ &= \frac{|ac| + |bd|}{|a + bi|} \leq |c| + |d| \leq \frac{\epsilon}{2} \text{ (done)} \end{aligned}$$

Note that

$$\lim_{n\to\infty}\lim_{t\to x}\varphi_n(t) = \lim_{n\to\infty} f'(x) \text{ exists}$$

We can now exchange the limit and see that the derivative of f at x exists.

$$\begin{aligned} f'(x) = \lim_{t\to x}\varphi(t) &= \lim_{t\to x}\lim_{n\to\infty}\varphi_n(t) \\ &= \lim_{n\to\infty}\lim_{t\to x}\varphi_n(t) = \lim_{n\to\infty} f_n'(x) \text{ (done)} \end{aligned}$$

■

5.9 Basic Technique on Sequence and Series

Abstract

This section prove some basic result on sequence and series, which will be heavily used in next section on analytic functions and Chapter: Beauty. Although written in an almost glossary form, we present the Theorems in a structural order based on the necessity of notion of absolute convergence and limit superior. Note that in this section, z, v, w always represent complex numbers, and a, b, c always represent real numbers.

Theorem 5.9.1. (Weierstrass M-test) Given sequences $f_n : X \to \mathbb{C}$, and suppose

$$\forall n \in \mathbb{N}, \forall x \in X, |f_n(x)| \leq M_n$$

Then

$$\sum_{n=1}^{\infty} M_n \text{ converge } \implies \sum_{n=1}^{\infty} f_n \text{ uniformly converge}$$

Proof. The proof follows from noting

$$\forall x \in X, \left| \sum_{k=m}^{n} f_k(x) \right| \leq \sum_{k=m}^{n} |f_k(x)| \leq \sum_{k=m}^{n} M_k$$

∎

Note that in our proof of Weierstrass M-test, we reduce the proof for uniform convergence into uniform Cauchy, which is a technique we shall also use later in Abel's test for uniform convergence. We now prove summation by part, which is a result hold in all fields, and is the essence of the proof of Dirichlet's test and Abel's test for uniform convergence.

Theorem 5.9.2. (Summation by Part)

$$\begin{aligned} f_n g_n - f_m g_m &= \sum_{k=m}^{n-1} f_k \Delta g_k + g_k \Delta f_k + \Delta f_k \Delta g_k \\ &= \sum_{k=m}^{n-1} f_k \Delta g_k + g_{k+1} \Delta f_k \end{aligned}$$

Proof. The proof follows induction which is based on

$$f_m g_m + f_m \Delta g_m + g_m \Delta f_m + \Delta f_m \Delta g_m = f_{m+1} g_{m+1}$$

∎

Theorem 5.9.3. (Dirichlet's Test) Suppose

(a) $a_n \to 0$ monotonically.

(b) $\sum_{n=1}^{N} z_n$ is bounded.

We have

$$\sum a_n z_n \text{ converge}$$

Proof. Define $Z_n \triangleq \sum_{n=1}^{N} z_n$ and let M bound $|Z_n|$. Using summation by part by letting $f_k = a_k$ and $g_k = Z_{k-1}$, we have

$$\begin{aligned}
\left| \sum_{k=m}^{n} a_k z_k \right| &= \left| a_{n+1} Z_n - a_m Z_{m-1} - \sum_{k=m}^{n} Z_k (a_{k+1} - a_k) \right| \\
&\leq |a_{n+1} Z_n| + |a_m Z_{m-1}| + \left| \sum_{k=m}^{n} Z_k (a_{k+1} - a_k) \right| \\
(\because a_n \text{ is monotone }) \qquad &\leq M \Big(|a_{n+1}| + |a_m| + |a_{n+1} - a_m| \Big)
\end{aligned}$$

■

Theorem 5.9.4. (Abel's Test for Uniform Convergence) Suppose $g_n : X \to \mathbb{R}$ is a uniformly bounded pointwise monotone sequence. Then given a sequence $f_n : X \to \mathbb{R}$,

$$\sum f_n \text{ uniformly converge } \implies \sum f_n g_n \text{ uniformly converge}$$

Proof. Define $R_n \triangleq \sum_{k=n}^{\infty} f_k$. Let M uniformly bound g_n. Because $R_n \to 0$ uniformly, we can let N satisfy

$$\forall n \geq N, \forall x \in X, |R_n(x)| < \frac{\epsilon}{6M}$$

Then for all $n, m \geq N$, using summation by part, we have

$$\begin{aligned}
\left| \sum_{k=m}^{n} f_k g_k \right| &= \left| \sum_{k=m}^{n} g_k \Delta R_k \right| \\
&\leq |R_{n+1} g_{n+1}| + |R_{m+1} g_{m+1}| + \sum_{k=m}^{n} |R_{k+1} \Delta g_k| \\
(\because g_n \text{ is pointwise monotone }) \qquad &\leq |R_{n+1} g_{n+1}| + |R_{m+1} g_{m+1}| + \frac{\epsilon}{6M} |g_{n+1} - g_m| \leq \epsilon
\end{aligned}$$

■

Although the proofs of Dirichlet's test and Abel's test for uniform convergence are quite similar, one should note that the "ways" summation by part is applied are slightly different, as one use $R_n \triangleq \sum_{k=n}^{\infty} f_k$ instead of $\sum_{k=1}^{n} f_k$, like $Z_n \triangleq \sum_{j=1}^{n} z_j$. As corollaries of Dirichlet's test, one have the famous **alternating series test** and Abel's test for complex series.

Theorem 5.9.5. (Abel's Test for Complex Series) Suppose

(a) $\sum z_n$ converge.

(b) b_n is a bounded monotone sequence.

We have

$$\sum z_n b_n \text{ converge}$$

Proof. Denote $B \triangleq \lim_{n\to\infty} b_n$. By Dirichlet's Test, we know $\sum z_n(b_n - B)$ converge. The proof now follows form noting

$$\sum z_n b_n = \sum z_n(b_n - B) + B\sum z_n$$

∎

We now introduce the idea of absolute convergence, which we shall use throughout the remaining of the section. By a **permutation** $\sigma : E \to E$ on some set E, we merely mean σ is a bijective function. We say $\sum z_n$ **absolutely converge** if $\sum |z_n|$ converge, and say $\sum z_n$ **unconditionally converge** if for all permutation $\sigma : \mathbb{N} \to \mathbb{N}$, the series $\sum z_{\sigma(n)}$ converge and converge to the same value.

Theorem 5.9.6. (Absolutely Convergent Series Unconditionally Converge)

$$\sum z_n \text{ absolutely converge } \implies \sum z_n \text{ unconditionally converge}$$

Proof. The fact $\sum z_n$ converge follows from noting

$$\left|\sum_{k=n}^{m} z_k\right| \leq \sum_{k=n}^{m} |z_k| \leq \sum_{k=n}^{\infty} |z_k|$$

Now, fix ϵ and permutation σ. Let N_1 and N_2 satisfy

$$\sum_{n=N_1}^{\infty} |z_n| < \frac{\epsilon}{2} \text{ and } \left|\sum_{n=N}^{\infty} z_n\right| < \frac{\epsilon}{2} \text{ for all } N > N_2$$

Let $M \triangleq \max\{N_1, N_2\}$. Observe that for all $N > \max_{1\leq r\leq M} \sigma^{-1}(r)$, we have

$$\left| \sum z_n - \sum_{n=1}^{N} z_{\sigma(n)} \right| \leq \left| \sum_{n=M+1}^{\infty} z_n \right| + \sum_{n=M+1}^{\infty} |z_n| < \epsilon$$

■

Theorem 5.9.7. (Riemann Rearrangement Theorem) If $\sum a_n$ converge but not absolutely, then for each $L \in \overline{\mathbb{R}}$, there exists a permutation σ such that

$$\sum a_{\sigma(n)} = L$$

Proof. Define a_n^+ and a_n^- by

$$a_n^+ \triangleq \max\{a_n, 0\} \text{ and } a_n^- \triangleq \min\{a_n, 0\}$$

Because

$$\sum (a_n^+ + a_n^-) \text{ converge but } \sum (a_n^+ - a_n^-) = \infty$$

We know

$$\sum a_n^+ = \sum (-a_n^-) = \infty$$

WOLG, (why?), fix $L \in \mathbb{R}$ and suppose $a_n \neq 0$ for all n. Let $A = B = L$, and let two increasing sequence $\sigma^+, \sigma^- : \mathbb{N} \to \mathbb{N}$ satisfy

$$\sigma^+(k+1) = \min\{n \in \mathbb{N} : a_n > 0 \text{ and } n > \sigma^+(k)\}$$

and similar for σ^-. Now, recursively define p_k, q_k by

$$p_1 \text{ is the smallest number such that } \sum_{n=1}^{p_1} a_{\sigma^+(n)} \geq A \tag{5.18}$$

$$q_1 \text{ is the smallest number such that } \sum_{n=1}^{p_1} a_{\sigma^+(n)} + \sum_{n=1}^{q_1} a_{\sigma^-(n)} \leq B \tag{5.19}$$

$$p_{k+1} \text{ is the smallest number such that } \sum_{n=1}^{p_{k+1}} a_{\sigma^+(n)} + \sum_{n=1}^{q_k} a_{\sigma^-(n)} \geq A \tag{5.20}$$

$$q_{k+1} \text{ is the smallest number such that } \sum_{n=1}^{p_{k+1}} a_{\sigma^+(n)} + \sum_{n=1}^{q_{k+1}} a_{\sigma^-(n)} \leq B \tag{5.21}$$

We then define σ by

$$\sigma^+(1), \dots, \sigma^+(p_1), \sigma^-(1), \dots, \sigma^-(q_1), \sigma^+(p_1+1), \dots, \sigma^+(p_2), \sigma^-(q_1+1), \dots, \sigma^-(q_2), \dots$$

It then follows from

$$\left|\sum_{n=1}^{p} a_{\sigma^+}(n) + \sum_{n=1}^{q_k} a_{\sigma^-}(n) - L\right| \leq \min\{a_{\sigma^+(p_{k+1})}, \left|a_{\sigma^-(q_k)}\right|\} \text{ for all } p_k \leq p \leq p_{k+1}$$

and $a_n \to 0$ that $\sum a_{\sigma(n)} = L$. ■

Note that the method we deploy in the proof of Riemann rearrangement Theorem can be used to control the sequence to have arbitrary large set of subsequential limits by modifying the number of A, B in Equation (4.1), (4.2), (4.3) and (4.4).

Using Riemann rearrangement Theorem and equation

$$\max_{1\leq r\leq d} |x_n| \leq |\mathbf{x}| \leq \sum_{r=1}^{d} |x_r|$$

we can now generalize and strengthen Theorem 5.9.6 to

$$\begin{aligned}\sum \mathbf{x}_n \text{ absolutely converge } &\iff \sum_n x_{n,r} \text{ absolutely converge for all } r \\ &\iff \sum_n x_{n,r} \text{ unconditionally converge for all } r \\ &\iff \sum \mathbf{x}_n \text{ unconditionally converge}\end{aligned}$$

With this in mind, we can now well state the Fubini's Theorem for Double Series.

Theorem 5.9.8. (Fubini's Theorem for Double Series) If

$$\sum_n \sum_k |z_{n,k}| \text{ converge}$$

Then

$$\sum_{n,k} |z_{n,k}| \text{ converge and } \sum_{n,k} z_{n,k} = \sum_n \sum_k z_{n,k} = \sum_k \sum_n z_{n,k}$$

Proof. The fact $\sum z_{n,k}$ absolutely converge follow from

$$\sum_{n=1}^{N} \sum_{k=1}^{N} |z_{n,k}| \leq \sum_n \sum_k |z_{n,k}| \text{ for all } N$$

WOLG, it remains to prove

$$\sum_{n,k} z_{n,k} = \sum_n \sum_k z_{n,k}$$

Because $\sum_n \sum_k |z_{n,k}|$ converge, we can reduce the problem into proving the same statement for nonnegative series $a_{n,k}$. (why?)

$$\sum_n \sum_k |a_{n,k}| \text{ converge } \implies \sum_{n,k} a_{n,k} = \sum_n \sum_k a_{n,k}$$

Because

$$\sum_{n=1}^N \sum_{k=1}^N a_{n,k} \leq \sum_{n=1}^N \sum_k a_{n,k} \leq \sum_n \sum_k a_{n,k} \text{ for all } N$$

we see

$$\sum_{n,k} a_{n,k} \leq \sum_n \sum_k a_{n,k}$$

It remains to prove

$$\sum_{n,k} a_{n,k} \geq \sum_n \sum_k a_{n,k}$$

Fix N and ϵ. We reduce the problem into proving

$$\sum_{n,k} a_{n,k} \geq \sum_{n=1}^N \sum_k a_{n,k} - \epsilon$$

Let K satisfy

$$\text{For all } 1 \leq n \leq N, \sum_{k=K+1}^{\infty} a_{n,k} < \frac{\epsilon}{N}$$

It then follows

$$\sum_{n,k} a_{n,k} \geq \sum_{n=1}^N \sum_{k=1}^K a_{n,k} \geq \sum_{n=1}^N \sum_k a_{n,k} - \epsilon \text{ (done)}$$

$\blacksquare$

Example 17 (Counter-Example for Fubini's Theorem for Double Series)

$$a_{n,k} \triangleq \begin{cases} 1 & \text{if } n = k \\ -1 & \text{if } n = k+1 \\ 0 & \text{if otherwise} \end{cases}$$

$$\sum |a_{n,k}| = \infty \text{ and } \sum_n \sum_k a_{n,k} = 1 \text{ and } \sum_k \sum_n a_{n,k} = 0$$

Theorem 5.9.9. (Merten's Theorem for Cauchy Product) Suppose

(a) $\sum_{n=0}^{\infty} z_n$ converge absolutely

(b) $\sum_{n=0}^{\infty} z_n = Z$

(c) $\sum_{n=0}^{\infty} v_n = V$

(d) $w_n = \sum_{k=0}^{n} z_k v_{n-k}$

Then we have

$$\sum_{n=0}^{\infty} w_n = ZV$$

Proof. We prove

$$\left| V \sum_{n=0}^{N} z_n - \sum_{n=0}^{N} w_n \right| \to 0 \text{ as } N \to \infty$$

Compute

$$\begin{aligned} V \sum_{n=0}^{N} z_n - \sum_{n=0}^{N} w_n &= \sum_{n=0}^{N} z_n (V - \sum_{k=0}^{N-n} v_k) \\ &= \sum_{n=0}^{N} z_n \sum_{k=N-n+1}^{\infty} v_k \end{aligned}$$

Because $\sum_{k=n}^{\infty} v_k \to 0$ as $n \to \infty$, we know there exists M such that

$$\left| \sum_{k=n}^{\infty} v_k \right| < M \text{ for all } n$$

Let N_0 satisfy

$$\sum_{n=N_0+1}^{\infty} |z_n| < \frac{\epsilon}{2M}$$

Let $N_1 > N_0$ satisfy

$$\left| \sum_{k=N-N_0+1}^{\infty} v_k \right| < \frac{\epsilon}{2(N_0+1)\sum_n |z_n|} \text{ for all } N > N_1$$

Now observe that for all $N > N_1$

$$\left| \sum_{n=0}^{N} z_n \Big(\sum_{k=N-n+1}^{\infty} v_k \Big) \right| \leq \sum_{n=0}^{N_0} |z_n| \left| \sum_{k=N-n+1}^{\infty} v_k \right| + \sum_{n=N_0+1}^{N} |z_n| \left| \sum_{k=N-n+1}^{\infty} v_k \right| < \epsilon \text{ (done)}$$

■

We first define the **limit superior** by

$$\limsup_{n\to\infty} a_n \triangleq \lim_{n\to\infty} (\sup_{k\geq n} a_k)$$

Note that $\limsup_{n\to\infty} a_n$ must exists because $(\sup_{k\geq n} a_k)_n$ is a decreasing sequence.

Theorem 5.9.10. (Equivalent Definition for Limit Superior) If we let E be the set of subsequential limits of a_n

$$E \triangleq \{L \in \overline{\mathbb{R}} : L = \lim_{k\to\infty} a_{n_k} \text{ for some } n_k\}$$

The set E is non-empty and

$$\max E = \limsup_{n\to\infty} a_n$$

Proof. Let $n_1 \triangleq 1$. Recursively, because

$$\sup_{j\geq n_k} a_k \geq \limsup_{n\to\infty} a_n > \limsup_{n\to\infty} a_n - \frac{1}{k} \text{ for each } k$$

We can let n_{k+1} be the smallest number such that

$$a_{n_{k+1}} > \limsup_{n\to\infty} a_n - \frac{1}{k}$$

It is straightforward to check $a_{n_k} \to \limsup_{n\to\infty} a_n$ as $k \to \infty$. Note that no subsequence can converge to $\limsup_{n\to\infty} a_n + \epsilon$ because there exists N such that $\sup_{k\geq N} a_k < \limsup_{n\to\infty} a_n + \epsilon$. ■

We can now state the **limit comparison test** as follows. Given a positive sequence b_n,

$$\limsup_{n\to\infty} \frac{|z_n|}{b_n} \in \mathbb{R} \text{ and } \sum b_n \text{ converge } \implies \sum z_n \text{ absolutely converge}$$
$$\liminf_{n\to\infty} \frac{b_n}{|z_n|} > 0 \text{ and } \sum z_n \text{ diverge } \implies \sum b_n \text{ diverge}$$

Theorem 5.9.11. (Geometric Series)

$$|z| < 1 \implies \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

Proof. The proof follows from noting

$$(1-z)\sum_{n=0}^{N} z^n = 1 - z^{N+1} \to 1 \text{ as } N \to \infty$$

■

Theorem 5.9.12. (Ratio and Root Test)

$$\limsup_{n\to\infty} \sqrt[n]{|z_n|} < 1 \text{ or } \limsup_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| < 1 \implies \sum z_n \text{ absolutely converge}$$
$$\liminf_{n\to\infty} \sqrt[n]{|z_n|} > 1 \text{ or } \liminf_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| > 1 \implies \sum z_n \text{ diverge}$$

Proof. The convergent part follows from comparison to an appropriate geometric series and the diverge part follows from noting $|z_n|$ does not converge to 0. ■

Theorem 5.9.13. (Root Test is Stronger Than Ratio Test)

$$\liminf_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| \le \liminf_{n\to\infty} \sqrt[n]{|z_n|} \le \limsup_{n\to\infty} \sqrt[n]{|z_n|} \le \limsup_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right|$$

Proof. Fix ϵ and WOLG suppose $\liminf_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| > 0$. We prove

$$\liminf_{n\to\infty} \sqrt[n]{|z_n|} \ge \liminf_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| - \epsilon$$

Let $\alpha \in \mathbb{R}$ satisfy

$$\liminf_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| - \epsilon < \alpha < \liminf_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right|$$

Let N satisfy

$$\text{For all } n \geq N, \left|\frac{z_{n+1}}{z_n}\right| > \alpha$$

We then see

$$\sqrt[N+n]{|z_{N+n}|} \geq \sqrt[N+n]{|z_N|\,\alpha^n} = \alpha\Big(\frac{|z_N|^{\frac{1}{N+n}}}{\alpha^{\frac{N}{N+n}}}\Big) \to \alpha \text{ as } n \to \infty \text{ (done)}$$

The proof for the other side is similar. ■

Theorem 5.9.14. (Root Test Trick) For all $k \in \mathbb{N}$

$$\limsup_{n\to\infty} |z_{n+k}|^{\frac{1}{n}} = \limsup_{n\to\infty} |z_n|^{\frac{1}{n}}$$

Proof. This is a direct corollary of equivalent definition for limit superior. ■

Lastly, we prove Cauchy's condensation Test, whose existence is almost solely for investigating p-Series.

Theorem 5.9.15. (Cauchy's Condensation Test) Suppose $a_n \searrow 0$. We have

$$\sum_{n=0}^{\infty} 2^n a_{2^n} \text{ converge } \iff \sum_{n=1}^{\infty} a_n \text{ converge}$$

Proof. Observe that for all $N \in \mathbb{N}$

$$\sum_{n=0}^{N} 2^n a_{2^n} \geq \sum_{n=0}^{N}\sum_{k=1}^{2^n} a_{2^n+k-1} = \sum_{n=1}^{2^{N+1}-1} a_n$$

and

$$2\sum_{n=1}^{2^N-1} a_n = 2\sum_{n=1}^{N}\sum_{k=0}^{2^{n-1}-1} a_{2^{n-1}+k} \geq 2\sum_{n=1}^{N} 2^{n-1}a_{2^n} = \sum_{n=1}^{N} 2^n a_{2^n}$$

■

Theorem 5.9.16. (p-Series)

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converge } \iff p > 1$$

Proof. Observe that

$$\sum_{n=0}^{\infty} 2^n \frac{1}{(2^n)^p} = \sum_{n=0}^{\infty} (2^{1-p})^n$$

The result then follows from Cauchy's Condensation Test and geometric series. ■

5.10 Analytic Functions

Abstract

This section introduces the concept of analytic functions and proves some of their basic properties, including the Identity Theorem. We will rely on the tools developed in the previous section on sequences and series. Note that throughout this section, z will always denote a complex number.

In this section, by a **power series**, we mean a pair (z_0, c_n) where $z_0 \in \mathbb{C}$ is called the **center** of power series, and $c_n \in \mathbb{C}$ are the coefficients sequence. By **radius of convergence**, we mean a unique $R \in \mathbb{R}_0^+ \cup \infty$ such that

$$\sum_{n=0}^{\infty} c_n(z-z_0)^n \begin{cases} \text{converge absolutely} & \text{if } |z-z_0| < R \\ \text{diverge} & \text{if } |z-z_0| > R \end{cases}$$

Such R always exist (and is unique, the uniqueness can be checked without computing the actual value of R) and is exactly

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{c_n}} \tag{5.22}$$

This result is called **Cauchy-Hadamard Theorem** and is proved by applying Root Test to $\sum c_n(z-z_0)^n$. Note that Cauchy-Hadamard Theorem does not tell us whether a power series converges at points of boundary of disk of convergence. It require extra works to determine if the power series converge at boundary.

Theorem 5.10.1. (Abel's Test for Power Series) Suppose $a_n \to 0$ monotonically and $\sum a_n z^n$ has radius of convergence R .

$$\text{The power series } \sum a_n z^n \text{ at least converge on } \overline{D_R(0)} \setminus \{R\}$$

Proof. Note that

$$\sum \frac{a_n}{R^n} z^n \text{ has radius of convergence } R$$

Fix $z \in \overline{D_R(0)} \setminus \{R\}$. Note that

$$\left| \sum_{n=0} \frac{z^n}{R^n} \right| = \left| \frac{1-(\frac{z}{R})^{N+1}}{1-\frac{z}{R}} \right| \leq \frac{2}{|1-\frac{z}{R}|} \text{ for all } N$$

It then follows from Dirichlet's Test that $\sum a_n (\frac{z}{R})^n$ converge. ∎

Example 18 (Discussion of Convergence on Boundary)

$$f_q(z) = \sum_{n=0}^{\infty} n^q z^n \text{ provided } q \in \mathbb{R}$$

It is clear that f_q has convergence radius 1 for all $q \in \mathbb{R}$. For boundary, we have

$$\begin{cases} q < -1 \implies f_q \text{ converge on } S^1 \\ -1 \leq q < 0 \implies f_q \text{ converge on } S^1 \setminus \{1\} \\ 0 \leq q \implies f_q \text{ diverge on } S^1 \end{cases}$$

Note that

(a) At $z = 1$, the discussion is just p-series.

(b) $n^q \searrow 0$ if and only if $q < 0$; and if $n^q \searrow 0$, then the series converge by Abel's test for power series.

(c) If $q \geq 0$, $n^q z^n$ does not converge to 0 on $S^1 \setminus \{1\}$

Notice that the fact $\sum c_n(z-z_0)^n$ absolutely converge in $D_R(z_0)$ implies the convergence is uniform on all $\overline{D_{R-\epsilon}(z_0)}$ by M-Test. However, on $D_R(z_0)$, the convergence is not always uniform.

Example 19 (Failure of Uniform Convergence on $D_R(z_0)$)

$$f(z) = \sum_{n=0}^{\infty} z^n$$

Note $R = 1$. Use Geometric series formula to show $f(z) = \frac{1}{1-z}$ on $D_1(0)$. It is then clear that f is unbounded on $D_1(0)$ while all partial sums $\sum_{k=0}^{n} z^k$ is bounded on $D_1(0)$.

We now introduce some terminologies. We say a complex function f is **analytic at** $z_0 \in \mathbb{C}$ if f there exists a power series (z_0, c_n) whose convergence radius is greater than 0 and f agrees with $\sum_{n=0}^{\infty} c_n(z - z_0)^n$ on $D_R(z_0)$ for some R (of course, such R must not be strictly greater than the radius of convergence of (a, c_n)). It shall be quite clear that if f, g are both analytic at $z \in \mathbb{C}$ with radius $R_f \leq R_g$, then by Merten's Theorem for Cauchy product, $f + g$ and fg are analytic at z with radius at least R_f. We now investigate deeper into analytic functions.

Theorem 5.10.2. (Term by Term Differentiation) Given a power series (z_0, c_n) of convergence radius $R > 0$, if we define $f : D_R(z_0) \to \mathbb{C}$ by

$$f(z) \triangleq \sum_{n=0}^{\infty} c_n(z - z_0)^n$$

Then f is holomorphic on $D_R(z_0)$ and its derivative at z_0 is also a power series with radius of convergence R

$$f'(z) = \sum_{n=0}^{\infty} (n+1)c_{n+1}(z - z_0)^n$$

Proof. Because $(n+1)^{\frac{1}{n}} \to 1$, we can use Theorem 5.9.14 to deduce

$$\limsup_{n\to\infty} \left((n+1)\left|c_{n+1}\right|\right)^{\frac{1}{n}} = \limsup_{n\to\infty} \left|c_{n+1}\right|^{\frac{1}{n}} = \limsup_{n\to\infty} \left|c_n\right|^{\frac{1}{n}}$$

which implies that the power series $\sum_{n=0}^{\infty}(n+1)c_{n+1}(z - z_0)^n$ is of radius of convergence R. We now prove

$$f'(z) = \sum_{n=0}^{\infty} (n+1)c_{n+1}(z - z_0)^n \text{ on } D_R(z_0)$$

Define $f_m : D_R(z_0) \to \mathbb{C}$ by

$$f_m(z) \triangleq \sum_{n=0}^{m} c_n(z - z_0)^n$$

Observe

(a) $f_m \to f$ pointwise on $D_R(a)$

(b) $f'_m(z) = \sum_{n=0}^{m-1}(n+1)c_{n+1}(z - z_0)^n$ for all m

Fix $z \in D_R(z_0)$. Proposition (b) allow us to reduce the problem into proving

$$f'(z) = \lim_{m\to\infty} f'_m(z) \text{ on } D_R(a) \tag{5.23}$$

Let $z \in D_r(z_0)$ where $r < R$. With proposition (a) in mind, to show Equation 5.23, by Theorem 5.8.2, we only have to prove f'_m uniformly converge on $D_r(z_0)$, which follows from M-Test and the fact that $\sum_{n=0}^{\infty}(n+1)c_{n+1}(z - z_0)^n$ absolutely converge on $D_R(z_0)$. (done) ∎

Suppose

$$f(z) \triangleq \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

Now by repeatedly applying Theorem 5.10.2, we see

$$f^{(k)}(z) = \sum_{n=0}^{\infty} (n+k)\cdots(n+1)c_{n+k}(z-z_0)^n \text{ for all } k \in \mathbb{Z}_0^+ \tag{5.24}$$

This then give us

$$c_k = \frac{f^{(k)}(z_0)}{k!} \text{ for all } k \in \mathbb{Z}_0^+ \tag{5.25}$$

and

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots \text{ on } D_R(z_0) \tag{5.26}$$

Equation 5.26 is often called the **Taylor expansion of f at z_0**. Notably, Equation 5.25 tell us that if f is constant 0, then $c_n = 0$ for all n.

Example 20 (Smooth but not Analytic Function)

$$f(x) = \begin{cases} e^{\frac{-1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Use induction to show that

$$f^{(k)}(x) = P_k(\frac{1}{x})e^{-(\frac{1}{x})^2} \quad \exists P_k \in \mathbb{R}[x^{-1}], \forall k \in \mathbb{Z}_0^+, \forall x \in \mathbb{R}^*$$

and again use induction to show that

$$f^{(k)}(0) = 0 \quad \forall k \in \mathbb{Z}_0^+$$

The trick to show $f^{(k)}(0) = 0$ is let $u = \frac{1}{x}$.

Now, with Theorem 5.10.2, we see that f is not analytic at 0.

Example 21 (Bump Function)

$$f(x) = \begin{cases} e^{\frac{-1}{1-x^2}} & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

Use the same trick (but more advanced) to show f is smooth, and note that f is not analytic at ± 1.

Now, it comes an interesting question. Given a complex-valued function f analytic at z_0 with radius R, and suppose $z_1 \in D_R(z_0)$.

(a) Is f also analytic at z_1?

(b) What do we know about the radius of convergence of f at z_1?

(c) Suppose f is indeed analytic at z_1. It is trivial to see that the power series $(z_0, c_{0;n})$ and $(z_1, c_{1;n})$ must agree in the intersection of their convergence disks, and because f is given, we by Theorem 5.10.2 and Equation 5.25, have already known the value of $c_{1;n}$. Can we verify that the power series $(z_0, c_{0;n})$ and $(z_1, c_{1;n})$ do indeed agree with each other on the common convergence interval?

Taylor's Theorem for power series give satisfying answers to these problems.

Theorem 5.10.3. (Taylor's Theorem for Power Series) Given a function f analytic at z_0 with radius R, and suppose $z_1 \in D_R(z_0)$. Then

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_1)}{k!}(z - z_1)^k \text{ on } D_{R-|z_1-z_0|}(z_1)$$

Proof. WOLG, let $z_0 = 0$. Suppose z satisfy $|z - z_1| < R - |z_1|$. By Equation 5.25, we can compute

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(z - z_1 + z_1)^k \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \sum_{n=0}^{k} \binom{k}{n} (z - z_1)^n z_1^{k-n} \\ &= \sum_{k=0}^{\infty} \sum_{n=0}^{k} \frac{f^{(k)}(0)}{k!} \binom{k}{n} (z - z_1)^n z_1^{k-n} \end{aligned}$$

Note that

$$\begin{aligned}\sum_{k=0}^{\infty}\left|\sum_{n=0}^{\infty}\frac{f^{(k)}(0)}{k!}\binom{k}{n}(z-z_1)^n z_1^{k-n}\right| &\le \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\left|\frac{f^{(k)}(0)}{k!}\right|\binom{k}{n}|z-z_1|^n\cdot|z_1|^{k-n}\\ &= \sum_{k=0}^{\infty}\left|\frac{f^{(k)}(0)}{k!}\right|\sum_{n=0}^{\infty}\binom{k}{n}|z-z_1|^n\cdot|z_1|^{k-n}\\ &= \sum_{k=0}^{\infty}\left|\frac{f^{(k)}(0)}{k!}\right|\big(|z-z_1|+|z_1|\big)^k\end{aligned}$$

is a convergent series, by Cauchy-Hadamard Theorem and $|z-z_1|+|z_1|<R$; thus, we can use Fubini's Theorem for double series to deduce

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{f^{(k)}(0)}{k!}\binom{k}{n}(z-z_1)^n z_1^{k-n} &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{f^{(k)}(0)}{k!}\binom{k}{n}(z-z_1)^n z_1^{k-n}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}\binom{k}{n}(z-z_1)^n z_1^{k-n}\\ &= \sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{f^{(k)}(0)}{k!}\binom{k}{n}(z-z_1)^n z_1^{k-n}\\ &= \sum_{n=0}^{\infty}\Big[\sum_{k=n}^{\infty}\frac{f^{(k)}(0)}{k!}\binom{k}{n}z_1^{k-n}\Big](z-z_1)^n\end{aligned}$$

We have reduced the problem into proving

$$\sum_{k=n}^{\infty}\frac{f^{(k)}(0)}{k!}\binom{k}{n}z_1^{k-n}=\frac{f^{(n)}(z_1)}{n!}$$

Because z_1 is in $D_R(0)$, by Equation 5.24 and Equation 5.25, we can compute

$$\begin{aligned}f^{(n)}(z_1) &= \sum_{k=0}^{\infty}(k+n)\cdots(k+1)\cdot\frac{f^{(n+k)}(0)}{(n+k)!}z_1^k\\ &= \sum_{k=n}^{\infty}(k)\cdots(k-n+1)\cdot\frac{f^{(k)}(0)}{k!}\cdot z_1^{k-n}\\ &= \sum_{k=n}^{\infty}\frac{f^{(k)}(0)}{(k-n)!}z_1^{k-n}\end{aligned}$$

We now have

$$\frac{f^{(n)}(z_1)}{n!} = \sum_{k=n}^{\infty} \frac{f^{(k)}(0)}{n!(k-n)!} z_1^{k-n} = \sum_{k=n}^{\infty} \frac{f^{(k)}(0)}{k!} \binom{k}{n} z_1^{k-n} \text{ (done)}$$

$\blacksquare$

Lastly, to close this section, we prove the Identity Theorem. By a **region** $D \subseteq \mathbb{C}$, we mean a connected and open subset of $\mathbb{C}$

Theorem 5.10.4. (Identity Theorem) Given two analytic complex-valued function $f, g : D \to \mathbb{C}$ defined on some region $D \subseteq \mathbb{C}$, if f, g agree on some subset $S \subseteq D$ such that S has a limit point in D, then f, g agree on the whole region D.

Proof. Define

$$T \triangleq \{z \in D : f^{(k)}(z) = g^{(k)}(z) \text{ for all } k \geq 0\}$$

Since D is connected, we can reduce the problem into proving T is non-empty, open and closed in D. Let c be a limit point of S in D. We first show

$$c \in T$$

Assume $c \notin T$. Let m be the smallest integer such that $f^{(m)}(c) \neq g^{(m)}(c)$. We can write the Taylor expansion of $f - g$ at c by

$$\begin{aligned}(f-g)(z) &= (z-c)^m \Big[\frac{(f-g)^{(m)}(c)}{m!} + \frac{(f-g)^{(m+1)}(c)}{(m+1)!}(z-c) + \cdots \Big] \\ &\triangleq (z-c)^m h(z)\end{aligned}$$

Clearly, $h(c) \neq 0$. Now, because h is continuous at c (h is a well-defined power series at c with radius greater than 0), we see h is non-zero on some $B_\epsilon(c)$, which is impossible, since $(f-g) \equiv 0$ on $S \setminus \{c\}$ implies $h = 0$ on $S \setminus \{c\}$. CaC (done)

Fix $z \in T$. Because f, g are analytic at z and $f^{(k)}(z) = g^{(k)}(z)$ for all k, we see $f - g$ is constant 0 on some open disk $B_\epsilon(z)$. We have proved that T is open. To see T is closed in D, one simply observe that

$$T = \bigcap_{k \geq 0} \{z \in D : (f-g)^{(k)}(z) = 0\}$$

and $(f-g)^{(k)}$ is continuous on D. (done) $\blacksquare$

5.11 Abel's Theorem and its application

In this section, we use the notation $\mathbb{S}_M(R)$ to denote **stolz region**

$$\mathbb{S}_M(R) \triangleq \{z \in \mathbb{C} : \frac{|R-z|}{R-|z|} \in (0,M)\}$$

Theorem 5.11.1. (Abel's Theorem for Power Series) Given a complex Maclaurin series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ of convergence radius R such that

$$\sum_{n=0}^{\infty} c_n R^n \text{ converge}$$

Then for all $M > 1$, we have

$$f|_{\mathbb{S}_M(R)}(z) \to \sum_{n=0}^{\infty} c_n R^n = f(R) \text{ as } z \to R$$

Proof. We first

$$\text{prove when } R = 1$$

Fix ϵ. We wish

$$\text{to find } \delta \text{ such that } \left| \sum_{n=0}^{\infty} c_n z^n - c_n \right| < \epsilon \text{ for all } z \in \mathbb{S}_M(1) \cap D_\delta(1)$$

To use summation by part, we first fix

$$s_n \triangleq \sum_{k=0}^{n} c_k \text{ and } s \triangleq \lim_{n\to\infty} s_n$$

Now Use summation by part

$$\begin{aligned}\sum_{n=0}^{k} c_n z^n &= \sum_{n=0}^{k} (s_n - s_{n-1}) z^n \\ &= \sum_{n=0}^{k} s_n z^n - \sum_{n=0}^{k-1} s_n z^{n+1} \\ &= s_k z^k + (1-z) \sum_{n=0}^{k-1} s_n z^n\end{aligned}$$

Note that

$$(1-z)\sum_{n=0}^{\infty} z^n = 1 \quad (|z|<1)$$

This give us

$$\begin{aligned}\lim_{z\to 1^-}\Big(\sum_{n=0}^{\infty} c_n z^n - \sum_{n=0}^{\infty} c_n\Big) &= \lim_{z\to 1^-}\Big(\lim_{k\to\infty} s_k z^k + (1-z)\sum_{n=0}^{k-1} s_n z^n - s\Big) \\ &= \lim_{z\to 1^-}(1-z)\sum_{n=0}^{\infty}(s_n-s)z^n \quad (\because \forall z\in\mathbb{C}: |z|<1, \lim_{k\to\infty} s_k z^k = 0)\end{aligned}$$

We reduce the problem into

$$\text{finding } \delta \text{ such that } \left|(1-z)\sum_{n=0}^{\infty}(s_n-s)z^n\right| \le \epsilon \text{ for all } z\in\mathbb{S}_M(1)\cap D_\delta(1)$$

Because $s_n\to s$, we know there exists N such that $|s_n-s|<\frac{1}{2M}$ for all $n>N$. We claim

$$\delta = \frac{\epsilon}{2\sum_{n=0}^{N}|s_n-s|} \text{ suffices}$$

Note that $\sum_{n=0}^{\infty}(s_n-s)z^n$ absolutely converges by direct comparison test. Then we can deduce

$$\begin{aligned}\left|(1-z)\sum_{n=0}^{\infty}(s_n-s)z^n\right| &= |1-z|\cdot\left|\sum_{n=0}^{N}(s_n-s)z^n + \sum_{n=N+1}^{\infty}(s_n-s)z^n\right| \\ &\le |1-z|\Big(\sum_{n=0}^{N}|s_n-s| + \frac{\epsilon}{2M}\sum_{n=N+1}^{\infty}|z|^n\Big) \\ &\le \frac{\epsilon}{2}+\frac{\epsilon}{2M}\cdot\frac{|1-z|}{1-|z|}\cdot|z|^{N+1} \le \epsilon \quad (\because |z|<1 \text{ and } \frac{|1-z|}{1-|z|}<M) \text{ (done)}\end{aligned}$$

We now prove

$$\text{when } R\in\mathbb{R}^+$$

Fix ϵ. We wish

$$\text{to find } \delta \text{ such that } \left|\sum_{n=0}^{\infty} c_n z^n - c_n R^n\right| < \epsilon \text{ for all } z\in\mathbb{S}_M(R)\cap D_\delta(R)$$

Fix

$$a_n = c_n R^n \text{ and } g(z) \triangleq \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} c_n R^n z^n \quad (|z| < 1)$$

By premise and our result, we know

$$g(1) \text{ exists and there exists } \delta' \text{ such that } |g(z) - g(1)| < \epsilon \text{ for all } z \in \mathbb{S}_M(1) \cap D_{\delta'}(1)$$

We claim

$$\delta = R\delta' \text{ suffices}$$

First note that

$$\frac{|R-z|}{R-|z|} \in (0, M) \implies \frac{\left|1 - \frac{z}{R}\right|}{1 - \left|\frac{z}{R}\right|} \in (0, M)$$

This tell us

$$z \in \mathbb{S}_M(R) \implies \frac{z}{R} \in \mathbb{S}_M(1)$$

Fix $z \in \mathbb{S}_M(R) \cap D_\delta(R)$. We now have

$$\frac{z}{R} \in \mathbb{S}_M(1) \cap D_{\delta'}(1)$$

This then let us conclude

$$\left| \sum_{n=0}^{\infty} c_n z^n - c_n R^n \right| = \left| g(\frac{z}{R}) - g(1) \right| < \epsilon \text{ (done)}$$

■

Example 22 (Identity of ln derived from Abel's Theorem)

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \text{ for all } x \in (-1, 1]$$

Check that both side satisfy $y' = \frac{1}{1+x}$, and $y(0) = 0$. This tell us that two sides equal on $(-1, 1)$. Now using Abel's Theorem and the continuity of ln, we have

$$\ln 2 = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

Example 23 $(1 - 1 + 1 - 1 + \cdots = \frac{1}{2})$

$$1 - 1 + 1 - 1 + \cdots = \frac{1}{2} \text{ is WRONG!!!}$$

When people say: "$1 - 1 + 1 - 1 + \cdots = \frac{1}{2}$", they mean the sum of the series in the sense of Abel. Compute the Macularin series of $\frac{1}{1+r}$

$$\frac{1}{1+r} = \sum_{n=0}^{\infty} (-1)^n r^n$$

Check both side do equal on $(-1, 1)$ by direct computation. Apply Abel's Theorem to see the magic.

5.12 L'Hospital Rule

Abstract

This section state and prove the L'Hospital Rule, and provide examples to show the necessity of each hypotheses of L'Hospital Rule. Note that although L'Hospital Rule is not really directly used in most results in Theory of Calculus, it is used in the proof of Taylor's Theorem.

Theorem 5.12.1. (L'Hospital Rule) Let $I \subseteq \mathbb{R}$ be an open interval containing c and let $f, g : I \to \mathbb{R}$ be two function continuous on I and differentiable on I everywhere except possibly at c, where

$$g'(x) \neq 0 \text{ for all } x \in I \setminus \{c\}$$

If $\frac{f}{g}$ is indeterminate form, i.e.

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = L \text{ where } L \in \{0, \infty, -\infty\}$$

and

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} \in \mathbb{R}$$

Then we have

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} \tag{5.27}$$

Proof. Suppose $I = (a, b)$. Note that since $g'(x) \neq 0$ on (c, b), by MVT, we know there exists at most one $x \in (c, b)$ such that $g(x) = 0$. With similar argument for (a, c), we see that

$$g(x) \neq 0 \text{ on } (c - \epsilon, c + \epsilon) \setminus \{c\} \text{ for some } \epsilon$$

We now see that the expression $\lim_{x \to c} \frac{f(x)}{g(x)}$ is at least well-defined, and WOLG, we can suppose $g(x) \neq 0$ on $I \setminus \{c\}$. Define $m, M : I \setminus \{c\} \to \mathbb{R}$ by

$$m(x) \triangleq \inf \frac{f'(t)}{g'(t)} \text{ and } M(x) \triangleq \sup \frac{f'(t)}{g'(t)} \text{ where } t \text{ ranges over values between } x \text{ and } c$$

Because the value $\frac{f'(t)}{g'(t)}$ converge at c, we can deduce

$$\lim_{x \to c} m(x) = \lim_{x \to c} M(x) = \lim_{x \to c} \frac{f'(x)}{g'(x)} \tag{5.28}$$

We now prove

$$\text{the case when } \lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$$

Fix $x \in I \setminus \{c\}$, and WOLG suppose $x < c$. By Cauchy's MVT, we know that

$$m(x) \le \frac{f(x) - f(y)}{g(x) - g(y)} \le M(x) \text{ for all } y \in (x, c)$$

Note that $g(x) \ne g(y)$ by MVT, since $g' \ne 0$ on $I \setminus \{c\}$. It now follows from

$$\lim_{y \to c^-} \frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f(x)}{g(x)}$$

that

$$m(x) \le \frac{f(x)}{g(x)} \le M(x)$$

The proof of Equation 5.27 then follows from Equation 5.28. (done)

We now prove

$$\text{the case when } \lim_{x \to c} f(x) = \lim_{x \to c} g(x) = \infty$$

Again, fix $x \in I \setminus \{c\}$, and WOLG suppose $x < c$. By Cauchy's MVT, we know that

$$m(x) \le \frac{f(y) - f(x)}{g(y) - g(x)} \le M(x) \text{ for all } y \in (x, c)$$

Note that $g(x) \ne g(y)$ by MVT, since $g' \ne 0$ on $I \setminus \{c\}$. It now follows

$$\frac{f(y) - f(x)}{g(y) - g(x)} = \frac{\frac{f(y)}{g(y)} - \frac{f(x)}{g(y)}}{1 - \frac{g(x)}{g(y)}} \to \lim_{x \to c} \frac{f(x)}{g(x)} \text{ as } y \to c^-$$

The proof of Equation 5.27 then follows form Equation 5.28. (done) ■

We now introduce some examples to demonstrate the necessity of our hypotheses.

Chapter 6

Measure Theory

6.1 Sigma-Algebra

Abstract

In this section, we first discuss properties of σ-algebra and some of its substructure for better understanding of a slightly generalized version of Carathéodory's extension theorem. Note that in this section, the terms 'ring', 'field', or 'algebra' do not refer to algebraic structures like the integer ring.

Given a set X and an non-empty set R of subsets of X, we say R is a **semi-ring**, if for each $A, B \in R$, we have

(a) $A \cap B \in R$ (closed under finite intersection)

(b) $A \setminus B = \bigsqcup_{i=1}^{n} K_i$ for some disjoint $K_1, \dots, K_n \in R$. (relative complements can be written as finite disjoint union)

and we say R is a **ring**, if for each $A, B \in R$, we have

(a) $A \cup B \in R$ (closed under finite union)

(b) $A \setminus B \in R$ (closed under relative complement)

One should check

(a) Semi-ring always contain the empty set.

(b) Since $A \cap B = A \setminus (A \setminus B)$, closure under relative complement implies closure under finite intersection. Thus, a ring, or any collection closed under relative complement, is always a semi-ring.

(c) Note that $A \cup B = (A \setminus B) \sqcup (A \cap B) \sqcup (B \setminus A)$. This implies we can replace closure under finite union with closure under finite disjoint union as definition for ring.

(d) Given a family S of subsets of X, there exists smallest ring $R(S)$ containing S. Such ring $R(S)$ is called **the ring generated by** S.

Theorem 6.1.1. (Ring Generated by Semi-Ring) If S is a semi-ring, then

$$\begin{aligned} R(S) &= \{A : A \text{ is the union of some finite pair-wise disjoint sub-family } S' \text{ of } S\} \\ &= \{A : A \text{ is the union of some finite sub-family } S' \text{ of } S\ \} \end{aligned}$$

Proof. Let $R \triangleq \{A : A$ is the union of some finite pair-wise disjoint sub-family S' of $S\}$. $R' \triangleq \{A : A$ is the union of some finite sub-family S' of $S\ \}$. We first prove $R = R(S)$.

Clearly, the problem can be reduced into proving R is a ring. Because R is clearly closed under finite disjoint union, by property (c) of the ring, we can reduce the problem into proving R is closed under relative complement.

We first show R is closed under finite intersection. Given $\bigsqcup E_i, \bigsqcup F_j \in R$, we see

$$\Big(\bigsqcup_i E_i\Big) \cap \Big(\bigsqcup_j F_j\Big) = \bigsqcup_{i,j} E_i \cap F_j \in R \text{ (done)}$$

Now observe

$$\begin{aligned} \Big(\bigsqcup_i E_i\Big) \setminus \Big(\bigsqcup_j F_j\Big) &= \bigcap_j \Big(\bigsqcup_i (E_i \setminus F_j)\Big) \\ &= \bigcap_j A_j \text{ for some } A_j \in R \quad \text{(done)} \end{aligned}$$

We now prove $R = R'$. It is clear that $R \subseteq R'$. We only have to prove $R' \subseteq R$. This is trivially true, since any finite sub-family S' of S is a finite sub-family of R and R is closed under finite union. (done) ∎

We now give definition to the most important structure in this section. Given a family Σ of subsets of X, we say Σ is a **σ-algebra** (or sometimes **σ-field**) on X and say (X, Σ) is a **measurable space**, if Σ is ring and for each sequence $(A_n)_{n \in \mathbb{N}}$ of elements of Σ we have

(a) $X \in \Sigma$

(b) $\bigcup_{n\in\mathbb{N}} A_n \in \Sigma$ (Closed under countable union)

Similarly, one should check

(a) Because $\bigcap_{n=1}^{\infty} A_n = A_1 \setminus (\bigcup_{n=2}^{\infty}(A_1 \setminus A_n))$, we see σ-algebra is closed under countable intersection.

(b) Because $\bigcup_{n=1}^{\infty} A_n = \bigsqcup_{n=1}^{\infty}(A_n \setminus (\bigcup_{k=n+1}^{\infty} A_k))$, we can replace closure under countable union with closure under countable disjoint union.

(c) Because $\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}}(\bigcup_{k=1}^{n} A_k)$, we can replace closure under countable union with the condition for all $(A_n) \subseteq \Sigma$ such that $\forall n, A_n \subseteq A_{n+1}$, we have $\bigcup_n A_n \in \Sigma$.

(d) Given a family S of subsets of X, there exists smallest sigma-algebra $\sigma(S)$ containing S. Such σ-algebra $\sigma(S)$ is called **the sigma-algebra generated by** S.

Now, given a family S of subsets of X, there in fact exists an explicit expression of $\sigma(S)$, albeit infamous. Let ω_1 be the smallest uncountable ordinal, and let $\mathbf{\Sigma}_1^0 \triangleq S$. For each ordinal $\alpha < \omega_1$, we recursively define $\mathbf{\Pi}_\alpha^0 \triangleq \{X \setminus A : A \in \mathbf{\Sigma}_\alpha^0\}$ and $\mathbf{\Sigma}_\alpha^0 \triangleq \{\bigcup_{n\in\mathbb{N}} A_n : (A_n) \subseteq \bigcup_{1\le\gamma<\alpha} \mathbf{\Pi}_\gamma^0\}$. One can use transfinite induction to check that $\sigma(S) = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}_\alpha^0$.

6.2 Carathéodory's Extension Theorem

Abstract

In this section, we introduce a general process to construct measure on X. The process involve first inducing an outer measure from a pre-measure on some weaker structure S, and then restricting the outer measure onto a subfamily that contain exactly all the subset that "sharply cut" all the other subsets of X. The subfamily, as we shall prove, is a sigma-algebra. This extending-restricting process is known mostly by the name of Carathéodory Extension Theorem, and the rigorous definition of "sharply cut" is known by the name of Carathéodory criterion. The selected 'some weaker structure' S, which we begin out extension from, is a semi-ring. Although as long as S contain the empty set, the process works, in the sense that one can generate a measure with pre-measure $\mu : S \to [0, \infty]$, to have the generated measure agree with μ on S, some necessary condition needs to be satisfied, and the axioms of semi-ring, not equivalent to some other popular choices that also suffice, e.g., ring or quasi-semi-ring, suffices to be a set of necessary conditions.

Note that in this section, if we write $\mu(A)$ without specifying whether A is in the domain of μ, we mean that the statement always hold true as long as A is in the domain of μ, and note that the difference between the term **measure space** and measurable space lies in that the latter is not equipped with a measure yet.

Given a collection S of subsets of X containing the empty set, we say $\mu : S \to [0, \infty]$ is a **pre-measure** (or **content**) on (X, S) if

(a) $\mu(\varnothing) = 0$ (null empty set)

(b) $A \subseteq B \implies \mu(A) \leq \mu(B)$ (monotone)

(c) $\mu\big(\bigsqcup_{n \in \mathbb{N}} E_n\big) = \sum_{n \in \mathbb{N}} \mu(E_n)$ (countably additive, or σ-additive)

and say μ is a **measure** if S is a σ-algebra on X. Note that if S is closed under relative complement, e.g., S is a ring or any stronger structure, then monotone is implied by countable additive. Now, with the hints given below, one can check straightforward that if S is a semi-ring, we have

(a) $\mu(A_1 \sqcup \cdots \sqcup A_n) = \mu(A_1) + \cdots + \mu(A_n)$ (finitely additive)

(b) $\mu\big(\bigcup_{n \in J} A_n\big) \leq \sum_{n \in J} \mu(A_n)$, for each finite or countable J.

(c) $A_n \nearrow A \implies \mu(A_n) \nearrow \mu(A)$.

(d) $A_n \searrow A$ and $\mu(A_1) < \infty$ and S is a σ-algebra $\implies \mu(A_n) \searrow \mu(A)$

Hints: Properties (b) and (c) are proved by letting $B_n \triangleq A_n \setminus (A_{n-1} \cup \cdots \cup A_1)$; and property (d) is proved by letting $B_n \triangleq A_1 \setminus A_n, \forall n \in \mathbb{N}$ and the equation

$$\begin{aligned}
\mu(A_1) - \lim_{n\to\infty} \mu(A_n) &= \lim_{n\to\infty} \mu(A_1) - \mu(A_n) \\
&= \lim_{n\to\infty} \mu(B_n) \\
\text{(property (c) is used here)} \quad &= \mu(\bigcup_{n=1}^{\infty} B_n) = \mu(A_1 \setminus \bigcap_{n=1}^{\infty} A_n) = \mu(A_1) - \mu(\bigcap_{n=1}^{\infty} A_n)
\end{aligned}$$

Note that our proof for property (d) require A_1 to be of finite measure in last step.

Now, suppose S is a semi-ring and $\mu : S \to [0, \infty]$ is a pre-measure on (X, S). This section address the question: Is there a unique extension of μ onto $\sigma(S)$? The answer is indeed affirmative: The extension always exists, and if μ is σ-finite, the extension is unique. While extensions of pre-measures can be constructed from structures weaker than a semi-ring, we focus on semi-rings here, as they are the starting points in common applications, e.g., Lebesgue-Stieltjes measure.

We first extend μ from S onto $R(S)$. Since each element of $R(S)$ is a union of some finite pair-wise disjoint sub-family of S, as we proved before, for each $A = \bigsqcup_{j=1}^{n_a} A_j \in R(S)$, we can assign $\mu(A) \triangleq \sum_{j=1}^{n_a} \mu(A_j)$. Such assignment is well-defined, as one can check using

$$\bigsqcup_{j=1}^{n_a} A_j = \bigsqcup_{k=1}^{n_b} B_k \implies A = \bigsqcup_{j,k} A_j \cap B_k \text{ where } A_j \cap B_k \in S$$

At this point, one should check μ remains countably additive after the extension onto $R(S)$ by dissecting $A = \bigsqcup A_n \in R(S)$ into a countable disjoint union of element in S.

We now give definition to **outer measure**, and shows that given any pre-measure μ on some semi-ring S of subsets of X, there exists outer measure μ^* on X such that μ^* and μ agrees on S.

Definition 6.2.1. (Definition of outer measure) Given a set X, by an **outer measure**, we mean a function $\nu : 2^X \to [0, \infty]$ such that

(a) $\nu(\varnothing) = 0$ (null empty set)

(b) $A \subseteq \bigcup_{n\in\mathbb{N}} A_n \implies \nu(A) \le \sum \nu(A_n)$ (countably subadditive)

Equivalently, one can replace countably subadditive with the following two axioms

(a) $A \subseteq B \implies \nu(A) \leq \nu(B)$ (monotone)

(b) $\nu(\bigcup_{n\in\mathbb{N}} A_n) \leq \sum \nu(A_n)$

Theorem 6.2.2. (Pre-measure on semi-ring induces outer measure) Given a pre-measure μ on some semi-ring S of subsets of X, if we define $\mu^* : 2^X \to [0,\infty]$ by

$$\mu^*(E) \triangleq \inf\Big\{ \sum_n \mu(T_n) : E \subseteq \bigcup_n T_n \text{ and } T_1, T_2, \dots \in S \Big\} \text{ where } \inf \varnothing = \infty$$

Then

$$\mu^* \text{ is an outer measure agreeing with } \mu \text{ on } R(S)$$

Proof. It is clear $\mu^*(\varnothing) = 0$ and $A \subseteq B \implies \mu^*(A) \leq \mu^*(B)$. It remains to prove that for arbitrary B_n we have

$$\mu^*\Big(\bigcup_n B_n\Big) \leq \sum_n \mu^*(B_n)$$

If $\sum_n \mu^*(B_n) = \infty$, the proof is trivial. We from now suppose $\sum_n \mu^*(B_n) < \infty$. Fix ϵ. We prove

$$\mu^*\Big(\bigcup_n B_n\Big) \leq \sum_n \mu^*(B_n) + \epsilon$$

Because $\mu^*(B_n) < \infty$ for each $n \in \mathbb{N}$, we know for each $n \in \mathbb{N}$ there exists an countable cover $(T_{n,k})_{k\in\mathbb{N}} \subseteq R$ of B_n such that

$$\sum_k \mu(T_{n,k}) \leq \mu^*(B_n) + \epsilon(2^{-n})$$

It is clear that $\{T_{n,k} : n, k \in \mathbb{N}\}$ is a countable cover of $\bigcup_n B_n$, we now see

$$\begin{aligned}\mu^*\Big(\bigcup_n B_n\Big) &\leq \sum_{n,k} \mu(T_{n,k}) = \sum_n \sum_k \mu(T_{n,k}) \quad (\because \text{Fubini's Theorem for Double Series}) \\ &\leq \sum_n \mu^*(B_n) + \epsilon(2^{-n}) = \sum_n \mu^*(B_n) + \epsilon \text{ (done)}\end{aligned}$$

At this point, one should check that even if we first extend μ onto $R(S)$ before doing the same procedure, we would still have the same outer measure. Now, because for each $T \in R(S)$, we clearly have $\mu^*(T) \leq \mu(T)$, to finish the proof it only remains to see for each cover $T_n \subseteq R(S)$ of T, we have

$$\mu(T) \leq \sum_n \mu(T \cap T_n) \leq \sum_n \mu(T_n)$$

∎

So far, we have proved that given a semi-ring S and a pre-measure $\mu : S \to [0,\infty]$, there exist some outer measure agree with μ on $R(S)$. One may wish to ask if such outer measure, an extension of the pre-measure μ, is unique? The answer is negative even in the most trivial case, as the example below shows. In fact, the outer measure induced in Theorem 6.2.2 is called the **maximal outer extension**, in the sense that if ν is an outer measure agreeing with μ on $R(S)$, then $\nu(E) \leq \mu^*(E), \forall E \subseteq X$, as one can check straightforwardly. Also, one can check that it make no difference if we first extend μ from S onto $R(S)$ or not, before we extend μ to the maximal outer measure μ^*.

Example 24 (Non-uniqueness of outer extension)

$$X \triangleq \{1,2\} \text{ and } R \triangleq \{\varnothing, X\}$$

Define the pre-measure $\mu : R \to [0,\infty]$ and an outer measure $\nu : \mathcal{P}(X) \to [0,\infty]$ agreeing with μ on S by

$$\mu(A) \triangleq \begin{cases} 0 & \text{if } A = \varnothing \\ 1 & \text{if } A = X \end{cases} \text{ and } \nu(A) \triangleq \begin{cases} \mu(A) & \text{if } A \in R \\ \frac{1}{2} & \text{if } A \notin R \end{cases}$$

One can check that the maximal outer extension μ^* disagree with ν on $\mathcal{P}(X) \setminus R$.

It is important to note that the final step, Theorem 6.2.3, inducing measure from outer measure, is a general Theorem and operate independently of Theorem 6.2.2. This distinction is crucial because many constructions of measures, such as the Hausdorff measure, begin by defining an outer measure explicitly, rather than inducing it from a weaker structure like a semi-ring.

Theorem 6.2.3. (Outer measure induce measure) Given an outer measure μ^* on X, if we let

$$\mathcal{A} \triangleq \{A \subseteq X : \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A) \text{ for all } E \subseteq X\ \}$$

then $\mathcal{A}$ is a sigma-algebra on X and $\mu^*|_{\mathcal{A}} : \mathcal{A} \to [0,\infty]$ is a measure.

Proof. Because of the following facts

(a) $A \setminus B = A \cap B^c$

(b) $A \cup B = (A^c \cap B^c)^c$

(c) property (c) of sigma-algebra

we can reduce the problem into proving the following propositions

(i) $\mathcal{A}$ is closed under complement.

(ii) $X \in \mathcal{A}$.

(iii) $\mathcal{A}$ is closed under finite intersection.

(iv) $\mu^*|_{\mathcal{A}}$ is countably additive. (Thus at least form a pre-measure)

(v) $\mathcal{A}$ is closed under countable disjoint union.

We will prove the propositions sequentially, as the proof of each subsequent proposition may rely on the proofs of the preceding ones. The first two are straightforward to check. We now prove $\mathcal{A}$ is closed under finite intersection. Fix $A, B \in \mathcal{A}$ and $E \subseteq X$, we wish to show

$$\mu^*(E) = \mu^*(E \cap A \cap B) + \mu^*\big(E \setminus (A \cap B)\big)$$

Because $B \in \mathcal{A}$, we can "sharply cut $E \setminus (A \cap B)$ by B"; that is

$$\mu^*(E \setminus (A \cap B)) = \mu^*\big((E \cap B) \setminus A\big) + \mu^*(E \setminus B) \tag{6.1}$$

Equation 6.1 together with $A, B \in \mathcal{A}$ then give us

$$\begin{aligned}\mu^*(E \cap A \cap B) + \mu^*\big(E \setminus (A \cap B)\big) &= \mu^*(E \cap A \cap B) + \mu^*\big((E \cap B) \setminus A\big) + \mu^*(E \setminus B) \\ &= \mu^*(E \cap B) + \mu^*(E \setminus B) = \mu^*(E) \text{ (done)}\end{aligned}$$

We now prove the claim: For each pairwise disjoint sequence $(A_n) \subseteq \mathcal{A}$ and $E \subseteq X$, we have the equality

$$\mu^*\Big(E \cap \bigsqcup_n A_n\Big) = \sum_n \mu^*(E \cap A_n)$$

The countably subadditivty of μ^* trivially implies the inequality

$$\mu^*\Big(E \cap \bigsqcup_n A_n\Big) \leq \sum_n \mu^*(E \cap A_n)$$

Using induction and the fact $(A_n) \subseteq \mathcal{A}$, we see that

$$\mu^*(E \cap \bigsqcup_n A_n) = \sum_{n=1}^{N} \mu^*(E \cap A_n) + \mu^*\Big(E \cap \bigsqcup_{n=N+1}^{\infty} A_n\Big) \text{ for all } N \in \mathbb{N}$$

Then since μ^* has codomain $[0, \infty]$, we see

$$\mu^*(E \cap \bigsqcup_n A_n) \geq \sum_{n=1}^{N} \mu^*(E \cap A_n) \text{ for all } N \in \mathbb{N}$$

This implies the desired inequality $\mu^*(E \cap \bigsqcup_n A_n) \geq \sum_{n=1}^{\infty} \mu^*(E \cap A_n)$. (done)

Using $E \triangleq \bigsqcup_n A_n$, one see our claim implies that $\mu^*|_{\mathcal{A}}$ is countably additive. Lastly, we prove $\mathcal{A}$ is closed under countable disjoint union. Fix a pairwise disjoint sequence $(A_n) \subseteq \mathcal{A}$ and $E \subseteq X$. We wish to prove

$$\mu^*(E) \geq \mu^*(E \cap \bigsqcup_n A_n) + \mu^*(E \setminus \bigsqcup_n A_n)$$

Using induction and the fact $(A_n) \subseteq \mathcal{A}$, we see that

$$\mu^*(E \cap \bigsqcup_{n=1}^{N} A_n) = \sum_{n=1}^{N} \mu^*(E \cap A_n) \text{ for all } N \in \mathbb{N}$$

Then our claim give us

$$\mu^*(E \cap \bigsqcup_{n=1}^{N} A_n) \to \mu^*(E \cap \bigsqcup_n A_n) \text{ as } N \to \infty \tag{6.2}$$

Now, because of the identity $F \cup G = (F^c \cap G^c)^c$, proposition (i) and (iii) have shown $\mathcal{A}$ is closed under finite union. This implies $\bigsqcup_{n=1}^{N} A_n \in \mathcal{A}$ for all $N \in \mathbb{N}$, which, together with monotone of μ^*, give

$$\begin{aligned} \mu^*(E \cap \bigsqcup_{n=1}^{N} A_n) + \mu^*(E \setminus \bigsqcup_n A_n) &\leq \mu^*(E \cap \bigsqcup_{n=1}^{N} A_n) + \mu^*(E \setminus \bigsqcup_{n=1}^{N} A_n) \\ &\leq \mu^*(E) \text{ for all } N \in \mathbb{N} \end{aligned} \tag{6.3}$$

Equation 6.2 and Equation 6.3 gives the desired inequality. (done) ■

Theorem 6.2.2 together with Theorem 6.2.3 shows that for each pre-measure μ on semi-ring S, we can induce a measure $\mu^*|_{\mathcal{A}}$ agreeing with μ on S. Although this result is correct, it doesn't show $S \subseteq \mathcal{A}$, which is necessary to refer to $\mu^*|_{\mathcal{A}}$ as an extension. However, it is straightforward to verify that $S \subseteq \mathcal{A}$ using the definition of a semi-ring and the property that $\mu^*(T) = \mu(T)$ for all $T \in S$.

Moving forward, here are some additional concepts we will utilize in subsequent sections: Given a measurable space (X, Σ, μ), we define a measurable set $N \in \Sigma$ as a **null set** if $\mu(N) = 0$. Moreover, we say that μ is a **complete measure** if every subset of a null set is also measurable. It is important to note that every measure induced by an outer measure is complete, as one can readily verify.

Lastly, we wish to ask: Given a sigma-algebra Σ containing S and contained by $\mathcal{A}$, under what condition, is Carathéodory the only extension of μ onto Σ ?

This question turns out to have direct connection with the notion named 'σ-finite'. Given a pre-measure space (X, S, μ), we say μ is **σ-finite** if there exists a countable cover $(A_n) \subseteq S$ of E such that $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$. It is clear that

(a) μ is σ-finite only if S form a cover of X.

(b) If $\mu : S \to [0, \infty]$ is σ-finite and ν is a pre-measure defined on a class larger than S, such that ν agree with μ on S, then ν is also σ-finite.

Theorem 6.2.4. (Uniqueness of Extension) Suppose

(a) (X, S, μ) is a pre-measure space, and S is a semi-ring.

(b) $\mathcal{A}$ is the induced sigma-algebra in Theorem 6.2.3

(c) $\Sigma \subseteq \mathcal{A}$ is a sigma-algebra containing S

(d) $\nu : \Sigma \to [0, \infty]$ is a measure agreeing with μ on S

We have

$$\nu(A) \leq \mu^*(A) \text{ for all } A \in \Sigma \tag{6.4}$$

and, if μ is σ-finite, we have

$$\nu(A) = \mu^*(A) \text{ for all } A \in \Sigma \tag{6.5}$$

Proof. The inequality 6.4 follows from the greatest-lower-bound definition of induced outer-measure, property (b) of measure and monotone of measure.

From now on, we suppose μ is σ-finite. Before we prove Equation 6.5, we first prove the claim: for each $A \in \Sigma$, there exists a pairwise disjoint sequence $(D_n) \subseteq R(S) \subseteq \Sigma$ such that $A \subseteq \bigsqcup_n D_n$ and $\nu(D_n) = \mu^*(D_n) < \infty$ for all $n \in \mathbb{N}$.

Because μ is σ-finite, there exists a sequence $(A_n) \subseteq S$ such that $A \subseteq \bigcup_n A_n$ and $\mu(A_n) < \infty, \forall n \in \mathbb{N}$. Define $D_1 \triangleq A_1$ and $D_n \triangleq A_n \setminus (A_1 \cup \cdots \cup A_{n-1})$ for all $n > 2$. Noting the structure of $R(S)$, it is clear that D_n is a pairwise disjoint sequence in $R(S)$. It is also clear that $A \subseteq \bigcup A_n = \bigsqcup D_n$. Fix $n \in \mathbb{N}$. It remains to prove

$$\nu(D_n) = \mu^*(D_n) < \infty$$

The inequality $\mu^*(D_n) < \infty$ follows from $\mu^*(D_n) \leq \mu(A_n) < \infty$, and the equation $\nu(D_n) = \mu^*(D_n)$ follows from $R(S) \subseteq \Sigma$ and $R(S) \subseteq \mathcal{A}$. (done)

Note that for all $n \in \mathbb{N}$, $A \cap D_n \in \Sigma \subseteq \mathcal{A}$. Now, since

$$\nu(A) = \sum_{n=1}^{\infty} \nu(A \cap D_n) \text{ and } \mu^*(A) = \sum_{n=1}^{\infty} \mu^*(A \cap D_n)$$

To prove Equation 6.5, it only remains to prove $\nu(A \cap D_n) = \mu^*(A \cap D_n), \forall n \in \mathbb{N}$.

Because ν is a measure and $A \cap D_n \in \mathcal{A}$, we have the equations set

$$\begin{cases} \nu(A \cap D_n) = \nu(D_n) - \nu(D_n \setminus A) \\ \mu^*(A \cap D_n) = \mu^*(D_n) - \mu^*(D_n \setminus A) \end{cases} \tag{6.6}$$

The proof then follows from the facts

(a) $\nu(D_n) = \mu^*(D_n) < \infty$

(b) $\nu(A \cap D_n) \leq \mu^*(A \cap D_n)$ and $\nu(D_n \setminus A) \leq \mu^*(D_n \setminus A)$ (done)

Note that fact (b) can be checked straightforwardly. ■

6.3 Lebesgue Measure

Abstract

This section demonstrate a construction of Lebesgue measure with usage of Carathéodory's Extension Theorem. For discussion of non Lebesgue measurable set, see Vitali set, and to see that continuous function may not preserve Lebesgue measurability, see Cantor-Lebesgue function.

It is clear that the collection S of half-open interval $\prod [a_j, b_j)$ form a semi-ring. If we define a volume function $\mu : S \to [0, \infty]$ on S by

$$\mu\Big(\prod_{j=1}^{d}[a_j, b_j)\Big) \triangleq \prod_{j=1}^{d}(b_j - a_j)$$

we see that the empty set is indeed null and μ is indeed monotone. To check that μ form a pre-measure, it remains to prove

$$\prod_{j=1}^{d}[a_j, b_j) = \bigsqcup_{n=1}^{\infty}\prod_{j=1}^{d}[a_{j,n}, b_{j,n}) \implies \mu\Big(\prod_{j=1}^{d}[a_j, b_j)\Big) = \sum_{n=1}^{\infty}\mu\Big(\prod_{j=1}^{d}[a_{j,n}, b_{j,n})\Big)$$

To check

$$\mu\Big(\prod_{j=1}^{d}[a_j, b_j)\Big) \geq \sum_{n=1}^{\infty}\mu\Big(\prod_{j=1}^{d}[a_{j,n}, b_{j,n})\Big) \tag{6.7}$$

One fix arbitrary N and cut $\prod_{j=1}^{d}[a_j, b_j)$ into finite amount of grids to see

$$\mu\Big(\prod_{j=1}^{d}[a_j, b_j)\Big) \geq \sum_{n=1}^{N}\mu\Big(\prod_{j=1}^{d}[a_{j,n}, b_{j,n})\Big)$$

To check

$$\mu\Big(\prod_{j=1}^{d}[a_j, b_j)\Big) \leq \sum_{n=1}^{\infty}\mu\Big(\prod_{j=1}^{d}[a_{j,n}, b_{j,n})\Big) \tag{6.8}$$

One fix ϵ and have each ϵ_n satisfy

$$\mu\Big(\prod_{j=1}^d [a_{j,n}-\epsilon_n,b_{j,n}+\epsilon_n)\Big) \leq (1+\epsilon)\mu\Big(\prod_{j=1}^d [a_{j,n},b_{j,n})\Big)$$

Then because

$$\Big\{\prod_{j=1}^d (a_{j,n}-\epsilon_n,b_{j,n}+\epsilon_n) \subseteq \mathbb{R}^d : n \in \mathbb{N}\Big\} \text{ form an open cover for compact } \prod_{j=1}^d [a_j,b_j]$$

there exists finite subset $I \subseteq \mathbb{N}$ such that

$$\prod_{j=1}^d [a_j,b_j] \subseteq \bigcup_{n\in I}\prod_{j=1}^d [a_{j,n}-\epsilon_n,b_{j,n}+\epsilon_n)$$

This then give us

$$\mu\Big(\prod_{j=1}^d [a_j,b_j)\Big) \leq \sum_{n\in I}\mu\Big(\prod_{j=1}^d [a_{j,n}-\epsilon_n,b_{j,n}+\epsilon_n)\Big) \leq (1+\epsilon)\sum_{n=1}^\infty \mu\Big(\prod_{j=1}^d [a_{j,n},b_{j,n})\Big)$$

which give us Equation 6.8. Having proved that the volume function $\mu : S \to [0,\infty]$ is a pre-measure, we can induce an outer measure on $\mathbb{R}^d$ by

$$|E|_e \triangleq \inf\Big\{\mu(T_n) : E \subseteq \bigcup_n T_n \text{ and } T_1,T_2,\dots \in S\Big\}$$

and restrict the outer measure into a measure by letting the collection $\mathcal{L}(\mathbb{R}^d)$ of Lebesgue measurable set to be

$$\mathcal{L}(\mathbb{R}^d) \triangleq \{A \subseteq \mathbb{R}^d : \forall E \subseteq \mathbb{R}^d, |E|_e = |E\cap A|_e + |E\cap A^c|_e\}$$

Notably, if we define the class K_n of half-open dyadic cubes by

$$K_n \triangleq \Big\{\prod_{j=1}^d [\frac{m_j}{2^n},\frac{m_j+1}{2^n}) \subseteq \mathbb{R}^d : m_j \in \mathbb{Z} \text{ for all } j\Big\}$$

We see that for each p in some open set U, there exists some small enough half-open dyadic cube $Q \in K_n$ (for some n) such that $p \in Q \subseteq U$. Then, since the collection $\bigcup K_n$ of half-open dyadic cubes is countable, we see U is in the sigma-algebra $\mathcal{L}(\mathbb{R}^d)$.

Theorem 6.3.1. (Equivalent Definition of Lebesgue Measurability) The following statements are equivalent.

(a) $E \in \mathcal{L}(\mathbb{R}^d)$.

(b) For all ϵ, there exists some open O containing E such that $|O \setminus E|_e < \epsilon$.

(c) For all ϵ, there exists some closed F contained by E such that $|E \setminus F|_e < \epsilon$.

(d) $E = H \setminus Z$ for some null Z and some $H \in G_\delta$.

(e) $E = H \cup Z$ for some null Z and some $H \in F_\sigma$.

Proof. Since $\mathbb{R}^d$ is σ-finite, to prove from (a) to (b), we can WOLG suppose $|E| < \infty$; the proof then follows from the definition of $|E|_e$, the trick of enlarging the cover to appropriate size and that open set is measurable. The proof for (b) $\implies$ (d) $\implies$ (a) is straightforward. The proof for (b) $\implies$ (c) is observation of $O \setminus E^c = E \setminus O^c$, and the proof for (c) $\implies$ (e) $\implies$ (a) is again straightforward. ∎

Given a topological space X, we say the smallest sigma-algebra containing the topology is the **Borel sigma-algebra**, and we refer to element of the Borel sigma-algebra as **Borel sets**. In the case of $\mathbb{R}^d$, the Borel sigma-algebra is in fact quite "dense", in the sense that every set can be approximated by some Borel sets from outside. More precisely, we can associate every set $E \subseteq \mathbb{R}^d$ a set H in the class of G_δ containing E such that $|E|_e = |H|$. To see this is true, observe that if $|E|_e = \infty$, then $H = \mathbb{R}^d$ suffices, and if $|E|_e < \infty$, then we can use definition of outer measure to construct some open set O containing E such that $|O| \leq |E|_e + \frac{1}{n}$. Note that this of course does not implies every set of finite outer measure is measurable, sine our inequality is

$$|O| \leq |E|_e + |O \setminus E|_e$$

With this approximation in mind, we can generalize the third property of measurable set to

$$E_n \nearrow E \implies |E_n|_e \nearrow |E|_e \text{ for arbitrary } E_n, E \text{ in } \mathbb{R}^d$$

It is worth pointing out that every open set can be expressed as a countable disjoint union of half-open dyadic cubes by an algorithmic construction. This implies that if we write

$$U_1 \triangleq \bigsqcup_n Q_{n,1} \text{ and } U_2 \triangleq \bigsqcup_n Q_{n,2}$$

we have

$$|U_1 \times U_2| = \left| \bigsqcup_{n,k} Q_{n,1} \times Q_{k,2} \right| = \sum_{n,k} |Q_{n,1}| \, |Q_{k,2}| = \sum_n |Q_{n,1}| \sum_k |Q_{k,2}| = |U_1| \, |U_2|$$

This with open set approximation definition of Lebesgue measurability and the fact $\mathbb{R}^d$ is σ-finite tell us that the product of measurable set is indeed measurable.

Another property of Lebesgue measure people often use is that Lebesgue measure is both **an inner regular measure** and **an outer regular measure**. By a measure μ being outer regular, we mean the measure μ is defined on some topological space X and for all measurable A, we have

$$\mu(A) = \inf\{\mu(G) : A \subseteq G \text{ and } G \text{ is open and measurable }\}$$

and by a measure μ being inner regular, we mean the measure μ is defined on some topological space X and for all measurable A, we have

$$\mu(A) = \sup\{\mu(K) : K \subseteq A \text{ and } K \text{ is compact and measurable }\}$$

Theorem 6.3.2. (Lipschitz Continuity Preserve Lebesgue Measurability) If $E \in \mathcal{L}(\mathbb{R}^d)$ and $f : \mathbb{R}^d \to \mathbb{R}^m$ is K-Lipschitz, where $d \leq m$, then $f(E) \in \mathcal{L}(\mathbb{R}^m)$.

Proof. Write $E = H \cup Z$ where $H \in F_\sigma$ and Z is null. Because $f(E) = f(H) \cup f(Z)$, to show $f(E)$ is measurable, we can reduce the problem into proving f maps F_σ into F_σ and f maps null set to null set.

By splitting $\mathbb{R}^d$ into a countable union of compact set

$$\mathbb{R}^d = \bigcup_{n \in \mathbb{N}} \overline{B_n(\mathbf{0})}$$

We see that each closed set in $\mathbb{R}^d$ can be expressed as a countable union of compact sets. Then because continuous function preserve compactness, we see f must map closed set into class F_σ and thus map class F_σ into F_σ. (done)

Fix ϵ. Because $f : \mathbb{R}^d \to \mathbb{R}^d$ is K-Lipschitz, we know all its components $f_1, \dots, f_d : \mathbb{R}^d \to \mathbb{R}$ are also K-Lipschitz. Because Z is measurable and open set can be expressed as a disjoint countable union of dyadic half-open cubes, we can let $T_n \subseteq S$ be a countable cover of Z consisting of dyadic half-open cubes such that

$$\sum_n |T_n| < \frac{\epsilon}{K^m d^{\frac{m}{2}}}$$

Now, note that for each half-open cube T_n, if we write

$$T_n = \prod_{j=1}^{d} [a_j, a_j + h)$$

We clearly have

$$\operatorname{diam} T_n = h\sqrt{d} \text{ and } |T_n| = h^d$$

This give us the relationship between diameter and volume of a cube by

$$\operatorname{diam} T_n = |T_n|^{\frac{1}{d}} d^{\frac{1}{2}}$$

Because $f_k : \mathbb{R}^d \to \mathbb{R}$ are all K-Lipschitz, we know $f_k(T_n)$ can be contained by an interval of length $K(\operatorname{diam} T_n) = K |T_n|^{\frac{1}{d}} d^{\frac{1}{2}}$. This then tell us that $f(T_n)$ can be contained by closed cube of side length $K(\operatorname{diam} T_n)$, and give us the estimation

$$|f(T_n)|_e \leq K^m (\operatorname{diam} T_n)^m = K^m d^{\frac{m}{2}} |T_n|^{\frac{m}{d}} \leq K^m d^{\frac{m}{2}} |T_n|$$

where the last inequality hold true because $d \leq m$ and WOLG we can suppose $|T_n| \leq 1$. We now see that

$$|f(Z)|_e \leq \sum_n |f(T_n)|_e \leq K^m d^{\frac{m}{2}} \sum_n |T_n| < \epsilon \text{ (done)}$$

■

Although Lipschitz function from $\mathbb{R}^d$ to $\mathbb{R}^m$ preserve Lebesgue measurability when $d \leq m$, the proposition does not hold true when $d > m$. Simple counter example can be constructed with projection $\pi : \mathbb{R}^2 \to \mathbb{R}$ and the set $V \times \{0\}$ where V is the Vitali set.

Ii is worth pointing out that our development of Theory of Lebesgue measure mostly depend on the abstract Carathéodory's Extension Theorem, which is indeed quite unorthodox. If one wish to instead develop the Theory of Lebesgue measurability starting at the second definition of Lebesgue measurability, one may follows the following steps

(i) Prove that the outer measure is countably subadditive using $\epsilon 2^{-k}$ trick.

(ii) Prove that compact interval is measurable and have the expected measure. The former is proved by finding a small enough open interval and cover the difference naively, and the latter is proved by Hiene-Borel.

(iii) Prove that the class of measurable set is closed under countable union. (This step is independent of step 2)

(iv) Conclude that finite union of non-overlapping compact interval is measurable, and prove that it have the expected measure by Hiene-Borel.

With these knowledge, we then can

(a) Prove that disjoint compact sets have positive distance.

(b) Prove that $d(E_1, E_2) > 0 \implies |E_1 \cup E_2|_e = |E_1|_e + |E_2|_e$.

(c) Prove that compact set is measurable by expressing the difference $G \setminus F$ of the small enough open set G containing compact F to be a countable union of non-overlapping compact intervals.

(d) Prove that closed set is measurable by expressing the closed set is a countable union of compact sets.

and

(i) Prove that null set is measurable.

(ii) Prove that complement E^c of measurable set E is measurable by expressing E^c as a countable union of closed set and a null set.

(iii) Prove the third equivalent definition of Lebesgue measurability.

(iv) Use third equivalent definition of Lebesgue measurability to conclude that if measurable E_n are bounded and disjoint then $|\bigcup E_n| = \sum |E_n|$.

(v) Generalize the forth step to E_n that may not be bounded.

Chapter 7

Riemann Calculus

7.1 Bounded Variation

Abstract

This section prove some key properties of functions of bounded variations. These properties are worthy of discuss, as they make the set $BV\big([a,b]\big)$ of functions of bounded variation on $[a,b]$ a natural candidate for the class of Riemann-Stieltjes integrator. The key properties include

(a) Functions of bounded variation can be expressed as difference of two strictly increasing functions.

Given a compact interval $[a,b]$, by a **partition** P of $[a,b]$, we mean a set of the form $P = \{a = x_0 < x_1 < \cdots < x_{N_P} = b\}$. If f is a complex-valued function defined on $[a,b]$, we define the **total variation** $V_f(a,b)$ of f on $[a,b]$ to be

$$V_f(a,b) \triangleq \sup_P \sum_{n=1}^{N_P} |f(x_n) - f(x_{n-1})|$$

We say f is of **bounded variation** on $[a,b]$ if $V_f(a,b) < \infty$, and we denote the set of real-valued function from $[a,b]$ of bounded variation on $[a,b]$ by $BV\big([a,b]\big)$. It is straightforward to check

(a) $f \in BV\big([a,b]\big)$ must be bounded on $[a,b]$.

(b) Real-valued f monotone on $[a,b]$ is of bounded variation on $[a,b]$.

(c) $BV\big([a,b]\big)$ form a vector space over $\mathbb{R}$.

Theorem 7.1.1. (Bounded variation is closed under multiplication) Given two real-valued (or more generally complex-valued) f, g defined on $[a, b]$

$$V_{fg}(a,b) \leq AV_f(a,b) + BV_f(a,b)$$

where

$$A = \sup_{x\in[a,b]} |g(x)| \text{ and } B = \sup_{x\in[a,b]} |f(x)|$$

Proof. For every partition P, we have

$$\begin{aligned}
\sum_{n=1}^{N_P} |fg(x_n) - fg(x_{n-1})| &= \sum_{n=1}^{N_P} \Big|[f(x_n) - f(x_{n-1})]g(x_n) + f(x_{n-1})[g(x_n) - g(x_{n-1})]\Big| \\
&\leq \sum_{n=1}^{N_P} |f(x_n) - f(x_{n-1})| \, |g(x_n)| + \sum_{n=1}^{N_P} |f(x_{n-1})| \, |g(x_n) - g(x_{n-1})| \\
&\leq AV_f(a,b) + BV_f(a,b)
\end{aligned}$$

■

Note that the proof above only consider when g, f are both bounded on $[a, b]$. If not, the statement hold trivially. For the brevity of the proof of the next Theorem, if we are given a partition $P = \{a = x_0 < \cdots < x_n = b\}$ of $[a, b]$, we denote

$$\sum(P) \triangleq \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$$

Theorem 7.1.2. (Additive property of total variation) Given a real-valued function f defined $[a, b]$, and $c \in (a, b)$

$$V_f(a,b) = V_f(a,c) + V_f(c,b)$$

Proof. We first prove

$$V_f(a,c) + V_f(c,b) \leq V_f(a,b)$$

Since the proof is trivial when $V_f(a,c) = \infty$, we WOLG suppose $V_f(a,c) < \infty$. Fix ϵ. We reduce the problem into proving

$$V_f(a,c) + V_f(c,b) - \epsilon \leq V_f(a,b)$$

By definition, there exists partitions P_y, P_z respectively for $[a, c].[c, b]$ such that

$$\sum(P_y) > V_f(a,c) - \frac{\epsilon}{2} \text{ and } \sum(P_z) > V_f(c,b) - \frac{\epsilon}{2}$$

It is clear that $P_y \cup P_z$ is a partition of $[a,b]$. Observe

$$V_f(a,b) \geq \sum(P_y \cup P_z) = \sum(P_y) + \sum(P_z) > V_f(a,c) + V_f(c,b) - \epsilon \text{ (done)}$$

It remains to prove

$$V_f(a,b) \leq V_f(a,c) + V_f(c,b)$$

The proof follows from noting that if P is a partition of $[a,b]$, then we can get another partition P_c of $[a,b]$ by $P_c \triangleq P \cup \{c\}$ and have

$$\sum(P) \leq \sum(P_c) \leq V_f(a,c) + V_f(c,b) \text{ (done)}$$

■

Corollary 7.1.3. (Additive property of total variation) Given a real-valued function f defined $[a,b]$, and $c \in (a,b)$

$$f \in BV\big([a,b]\big) \iff f \in BV\big([a,c]\big) \text{ and } f \in BV\big([c,b]\big)$$

Following from Additive property of total variation, we see that if f is of bounded variation on $[a,b]$ and $[b,c]$, then f is of bounded variation on $[a,c]$. Perhaps, the best property of bounded variation are the following.

Theorem 7.1.4. (Function of bounded variation can be expressed as difference of two increasing functions) Given real-valued f defined on $[a,b]$, f is of bounded variation on $[a,b]$ if and only if there exists strictly increasing function $g,h : [a,b] \to \mathbb{R}$ such that $f = g - h$.

Proof. The proof follows from usage of Additive property of total variation and letting

$$g(x) \triangleq V_f(a,x) + (x-a) \text{ and } h(x) \triangleq V_f(a,x) - f(x) + (x-a)$$

■

Now, since monotone function can only have countable amount of discontinuity (Prove this by associating each jump discontinuity with a rational number between the left and the right limits), we see that function of bounded variation, also, can only have countable amount of discontinuity. This suggest that if $\alpha : [a,b] \to \mathbb{R}$ is of bounded variation, then α must be continuous almost everywhere.

If we only consider real-valued continuous function of bounded variation on $[a,b]$, we have a stronger version of Theorem 7.1.4.

Theorem 7.1.5. (Continuous function of bounded variation) Given $f : [a,b] \to \mathbb{R}$ such that $f \in BV\big([a,b]\big)$, If we define $V : [a,b] \to \mathbb{R}$ by

$$V(x) \triangleq V_f(a,x)$$

Then for each $x \in [a,b]$

$$V \text{ is continuous at } x \iff f \text{ is continuous at } x$$

Proof. $(\longrightarrow)$

Assume f is discontinuous at x. Because f is bounded on $[a,b]$, we can let $x_n \to x$ satisfy

$$\lim_{n\to\infty} f(x_n) - f(x) \in \mathbb{R}^*$$

By definition and additive property of total variation, we can deduce

$$\begin{aligned}\liminf_{n\to\infty} |V(x_n) - V(x)| &\geq \liminf_{n\to\infty} |f(x_n) - f(x)| \\ &= \lim_{n\to\infty} |f(x_n) - f(x)| > 0 \text{ CaC}\end{aligned}$$

$(\longleftarrow)$

Fix x. We first prove

$$V(x+) = V(x)$$

Fix ϵ. Let $P = \{x = x_0 < x_1 < \cdots < x_{N_P} = b\}$ be a partition of $[x,b]$ such that

$$V_f(x,b) - \frac{\epsilon}{2} < \sum(P)$$

Because f is continuous at x, we can let δ satisfy

$$|f(y) - f(x)| < \frac{\epsilon}{4N_P} \text{ for all } y \in [x, x+\delta]$$

Because V is increasing on $[a,b]$, additive property of total variaion allow us to reduce the problem into proving

$$V_f(x, x+\delta) \leq \epsilon$$

Denote $P' \triangleq P \cup \{x+\delta\}$ and express $P' = \{x = x'_0 < \cdots x'_m = x+\delta < \cdots < x'_{N_{P'}}\}$. Note that $m < N_{P'} \leq N_P$ and observe

$$\begin{aligned}V_f(x,b) - \frac{\epsilon}{2} < \sum(P) &\leq \sum(P') \\ &= \sum_{n=1}^{m} |f(x'_n) - f(x'_{n-1})| + \sum_{n=m+1}^{N_{P'}} |f(x'_n) - f(x'_{n-1})| \\ &\leq \frac{m\epsilon}{2N_P} + V_f(x+\delta, b) \leq \frac{\epsilon}{2} + V_f(x+\delta, b)\end{aligned}$$

Additive property of total variation now give us

$$V_f(x, x+\delta) = V_f(x, b) - V_f(x+\delta, b) \leq \epsilon \text{ (done)}$$

Proof for $V(x-) = V(x)$ is similar, and when $x = a$ or b, some trivial modifications are needed. ■

Give very careful attention to the statement of Theorem 7.1.5. Note that we require to the domain of f to be $[a, b]$. If the domain of f contain a or b as interior point, the statement isn't always true.

7.2 Riemann-Stieltjes Integral

Abstract

This section construct the Riemann-Stieltjes integral for increasing integrator in the form of Darboux integral for further reference. Note that in this section, $\Delta\alpha_n \triangleq \alpha(x_n) - \alpha(x_{n-1})$.

Given some bounded interval I with left and right end points are respectively a, b and some function $f, \alpha : I \to \mathbb{R}$ such that

(a) Their images $f(I), \alpha(I)$ are both bounded.

(b) α is increasing on I.

We define the **upper Darboux sum** $\overline{\int_a^b} f d\alpha$ **of** f **on** I **with respect to** α by

$$\overline{\int_a^b} f d\alpha \triangleq \inf_P \Big(\sum_{n=1}^{N_P} [\sup_{[x_{n-1},x_n]} f] \Delta\alpha_n \Big)$$

where the infimum runs over all the partition of $[a, b]$. Note that if I doesn't include a, b, then we simply define $\alpha(a) \triangleq \alpha(a+)$ and $\alpha(b) \triangleq \alpha(b-)$. The **lower Darboux sum** $\underline{\int_a^b} f d\alpha$ are defined similarly as

$$\underline{\int_a^b} f d\alpha \triangleq \sup_P \Big(\sum_{n=1}^{N_P} [\inf_{[x_{n-1},x_n]} f] \Delta\alpha_n \Big)$$

Because at this point we require the integrator to be increasing function, we can easily see that the lower Darboux sum is always not greater than the upper Darboux sum. We say f **is proper Riemann Integrable on** I **with respect to the integrator** α if they are equal as a real number. Now, because

$$\sup(f + g) \leq \sup(f) + \sup(g)$$

We see

$$\overline{\int_a^b} (f + g) d\alpha \leq \overline{\int_a^b} f d\alpha + \overline{\int_a^b} g d\alpha$$

It then follows that if f, g are both proper Riemann integrable on I with respect to α, then $f+g$ is also proper integrable on I with respect to α and

$$\int_a^b (f+g)d\alpha = \int_a^b f d\alpha + \int_a^b g d\alpha$$

Also, since

$$\sup(cf) = \begin{cases} c\sup(f) & \text{if } c \geq 0 \\ c\inf(f) & \text{if } c \leq 0 \end{cases}$$

we see that

$$\begin{aligned} c \geq 0 &\implies \overline{\int_a^b} (cf)d\alpha = c\overline{\int_a^b} f d\alpha \\ \text{and } c \leq 0 &\implies \overline{\int_a^b} (cf)d\alpha = c\underline{\int_a^b} f d\alpha \end{aligned}$$

It then follows that if f is proper Riemann integrable on I with respect to α, then cf is also proper integrable on I with respect to α. We have proved that the functions that is proper Riemann integrable on I with respect to α forms a vector space over $\mathbb{R}$. Also, since if $a \leq b \leq c$ and if f, α is bounded on (a, c), one can deduce

$$\overline{\int_a^c} f d\alpha \leq \overline{\int_a^b} f d\alpha + \overline{\int_b^c} f d\alpha$$

we see that if f is proper Riemann integrable on both (a, b) and (b, c), then f is proper Riemann integrable on (a, c) and

$$\int_a^c f d\alpha = \int_a^b f d\alpha + \int_b^c f d\alpha$$

This inspire us to define

$$\int_b^a f d\alpha \triangleq -\int_a^b f d\alpha$$

so that we have

$$\int_a^b f d\alpha + \int_b^a f d\alpha = 0 = \int_a^a f d\alpha$$

Lastly, if J is another interval contained in I, and f is proper Riemann integrable on I with respect to α, then one can deduce f is also Riemann integrable on J by noting that if P', P are two partition of I and $P \subseteq P'$, then

$$\sum_{n=1}^{N_{P'}}[\sup_{[x_{n-1},x_n]} f]\Delta\alpha_n \leq \sum_{n=1}^{N_P}[\sup_{[x_{n-1},x_n]} f]\Delta\alpha_n$$

With all these basic properties in mind, we are now ready to present the most famous Fundamental Theorem of Calculus.

Theorem 7.2.1. (Fundamental Theorem of Calculus: Part 1) If f is proper Riemann integrable on (a,b), and we define $F : [a,b] \to \mathbb{R}$ by

$$F(x) \triangleq \int_a^x f(t)dt$$

then F is continuous on $[a,b]$. Moreover, if f is continuous at some $x \in (a,b)$, then $F'(x) = f(x)$.

Proof. We first prove that $F : [a,b] \to \mathbb{R}$ is uniform continuous. Fix ϵ. Because f is proper Riemann integrable on (a,b), we know there exists some $M \in \mathbb{R}$ such that

$$M > \sup_{x \in (a,b)} |f(x)|$$

Then for each $x, y \in [a,b]$ such that $|y - x| < \frac{\epsilon}{M}$, we see

$$|F(x) - F(y)| = \left| \int_x^y f(t)dt \right| \leq \int_x^y |f(t)| \, dt < \epsilon \text{ (done)}$$

We now prove

$$f \text{ is continuous at } x \in (a,b) \implies F'(x) = f(x)$$

Fix ϵ. Because f is continuous at x, there exists δ such that

$$|t - x| < \delta \implies |f(t) - f(x)| < \epsilon$$

Now, observe that for all $t \in (x, x+\delta)$

$$\begin{aligned}
\left| \frac{F(t) - F(x)}{t - x} - f(x) \right| &= \left| \frac{\int_x^t f(s)ds}{t - x} - f(x) \right| \\
&= \left| \frac{\int_x^t [f(s) - f(x)]ds}{t - x} \right| \\
&\leq \frac{\int_x^t |f(s) - f(x)| \, ds}{|t - x|} \leq \epsilon \text{ (done)}
\end{aligned}$$

■

If one wish to discuss the one-sided derivative of F on endpoints a, b, one may employ the same argument we use, under the hypothesis that the one sided limit $f(a+), f(b-)$ exists.

Theorem 7.2.2. (Fundamental Theorem of Calculus: Part 2) Given some function f defined and proper Riemann-integrable on (a, b), if $F : [a, b] \to \mathbb{R}$ is continuous and satisfy $F'(x) = f(x)$ for all $x \in (a, b)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Proof. Fix ϵ. We show

$$\left| F(b) - F(a) - \int_a^b f(x)dx \right| < \epsilon$$

Because f is proper Riemann-Integrable on $[a, b]$, we know there exists a partition $P = \{a = x_0, x_1, \dots, x_n = b\}$ of $[a, b]$ such that

$$U(P, f) - L(P, f) < \epsilon \tag{7.1}$$

Because $f = F'$ on (a, b), for each $k \in \{1, \dots, n\}$, by MVT , we know

$$\exists t_k \in (x_{k-1}, x_k), \frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}} = f(t_k)$$

This let us deduce

$$F(b) - F(a) = \sum_{k=1}^n F(x_k) - F(x_{k-1}) = \sum_{k=1}^n f(t_k)\Delta x_k$$

Now, we have

$$\int_a^b f(x)dx \text{ and } F(b) - F(a) \text{ are both in } \big[L(P, f), U(P, f)\big]$$

Then by Equation 7.1, we can deduce

$$\left| F(b) - F(a) - \int_a^b f(x)dx \right| < \epsilon \text{ (done)}$$

■

It shall be pointed out that Fundamental Theorem of Calculus is a Theorem for function proper Riemann-Integrable, a very strict condition.

Example 25 (Discontinuous Derivative)

$$f(x) = \begin{cases} \frac{x^2}{\sin x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

7.3 Riemann-Stieltjes on Computation

Theorem 7.3.1. (Change of Variable) Given two functions $g, \beta : [A, B] \to \mathbb{R}$, a function $\varphi : [A, B] \to [a, b]$ and two functions $f, \alpha : [a, b] \to \mathbb{R}$ such that

(a) $g = f \circ \varphi$ for all $x \in [a, b]$

(b) $\beta = \alpha \circ \varphi$ for all $x \in [a, b]$

(c) α, β increase respectively on $[a, b]$ and $[A, B]$

(d) $\varphi : [A, B] \to [a, b]$ is a homeomorphism

(e) $\int_a^b f d\alpha$ exist

Then

$$\int_A^B g d\beta = \int_a^b f d\alpha \text{ (This implies } \int_A^B g d\beta \text{ exists)}$$

Proof. Fix ϵ. We only wish

$$\text{to find a partition } Q \text{ of } [A, B] \text{ such that } U(Q, g, \beta) - L(Q, g, \beta) < \epsilon$$
$$\text{and such that } \int_a^b f d\alpha \in \Big[L(Q, g, \beta), U(Q, g, \beta)\Big]$$

Because $\int_a^b f d\alpha$ exists, we know

$$\text{there exists a partition } P \text{ of } [a, b] \text{ such that } U(P, f, \alpha) - L(P, f, \alpha) < \epsilon \tag{7.2}$$

where, of course, $\int_a^b f d\alpha \in \Big[L(P, f, \alpha), U(P, f, \alpha)\Big]$.

Let $P = \{a = x_0, x_1, \dots, x_n = b\}$. Because φ is a homeomorphism, we can let φ be strictly increasing WOLG.

Define a partition Q on $[A, B]$ by

$$Q = \varphi^{-1}[P] = \{A = \varphi^{-1}(x_0), \varphi^{-1}(x_1), \dots, \varphi^{-1}(x_n) = B\}$$

Now, because $\beta = \alpha \circ \varphi$ and $g = f \circ \varphi$ for all $x \in [a, b]$ by premise, and because φ is a homeomorphism, we have

$$\begin{aligned}
U(Q,g,\beta) &= \sum_{k=1}^{n} \Big[\sup_{t\in[\varphi^{-1}(x_{k-1}),\varphi^{-1}(x_k)]} g(t) \Big] \big[\beta(\varphi^{-1}(x_k)) - \beta(\varphi^{-1}(x_{k-1}))\big] \\
&= \sum_{k=1}^{n} \Big[\sup_{t\in[\varphi^{-1}(x_{k-1}),\varphi^{-1}(x_k)]} f\circ\varphi(t) \Big] \big[\alpha\circ\varphi\big(\varphi^{-1}(x_k)\big) - \alpha\circ\varphi\big(\varphi^{-1}(x_{k-1})\big)\big] \\
&= \sum_{k=1}^{n} \Big[\sup_{t\in[x_{k-1},x_k]} f(t) \Big] \big(\alpha(x_k) - \alpha(x_{k-1})\big) = U(P,f,\alpha) \qquad (7.3)
\end{aligned}$$

Similarly, we can deduce $L(Q,g,\beta) = L(P,f,\alpha)$. Now, from Equation 7.3 and by definition of P (Equation 7.2), we see

$$\begin{aligned}
&U(Q,g,\beta) - L(Q,g,\beta) = U(P,f,\alpha) - L(P,f,\alpha) < \epsilon \\
\text{and } &\int_a^b f d\alpha \in \Big[L(P,f,\alpha), U(P,f,\alpha)\Big] = \Big[L(Q,g,\beta), U(Q,g,\beta)\Big] \text{ (done)}
\end{aligned}$$

■

Theorem 7.3.2. (Reduction of Riemann-Stieltjes Integral: Part 1) Given two functions $f,\alpha : [a,b] \to \mathbb{R}$ such that

(a) α increase on $[a,b]$

(b) α is differentiable on (a,b)

(c) $\lim_{x\to b^-} \frac{\alpha(x)-\alpha(b)}{x-b}$ exists and $\lim_{x\to a^+} \frac{\alpha(x)-\alpha(a)}{x-a}$ exists

(d) α' is properly Riemann-Integrable on $[a,b]$

(e) f is bounded on $[a,b]$

Then

$$\int_a^b f d\alpha \text{ exists } \iff \int_a^b f(x)\alpha'(x)dx \text{ exists and they equal to each other if exists}$$

Proof. We wish to prove

$$\overline{\int_a^b} f d\alpha = \overline{\int_a^b} f(x)\alpha'(x)dx$$

Fix ϵ. We reduce the problem into proving

$$\left| \overline{\int_a^b} f d\alpha - \overline{\int_a^b} f(x)\alpha'(x)dx \right| < \epsilon$$

Then, because for all partition P of $[a,b]$, we have

$$\left|\overline{\int_a^b} f d\alpha - \overline{\int_a^b} f(x)\alpha'(x)dx\right|$$
$$\leq \left|\overline{\int_a^b} f d\alpha - U(P,f,\alpha)\right| - \left|U(P,f,\alpha) - U(P,f\alpha')\right| - \left|U(P,f\alpha') - \overline{\int_a^b} f(x)\alpha'(x)dx\right|$$

We only wish

$$\text{to find } P \text{ such that} \left|\overline{\int_a^b} f d\alpha - U(P,f,\alpha)\right| < \frac{\epsilon}{3}$$
$$\text{and } \left|U(P,f,\alpha) - U(P,f\alpha')\right| < \frac{\epsilon}{3} \text{ and } \left|\overline{\int_a^b} f(x)\alpha'(x)dx - U(P,f\alpha')\right| < \frac{\epsilon}{3}$$

Because f is bounded on $[a,b]$, we can let $M = \sup_{x\in[a,b]} |f(x)|$. Because $\int_a^b \alpha'(x)dx$ exists, we can let P satisfy

$$U(P,\alpha') - L(P,\alpha') < \frac{\epsilon}{4M} \tag{7.4}$$

By definition of Riemann Upper sum, we can further refine P to let P satisfy

$$\left|\overline{\int_a^b} f d\alpha - U(P,f,\alpha)\right| < \frac{\epsilon}{3} \text{ and } \left|\overline{\int_a^b} f(x)\alpha'(x)dx - U(P,f\alpha')\right| < \frac{\epsilon}{3}$$

It is clear that the statement concerning P (Equation 7.4) remain valid after refinement of P. Fix such P. We now have reduced the problem into proving

$$|U(P,f,\alpha) - U(P,f\alpha')| < \frac{\epsilon}{3}$$

Express P in the form $P = \{a = x_0, x_1, \ldots, x_n = b\}$. By MVT (Theorem 5.3.3), we know for all $k \in \{1,\ldots,n\}$ there exists $t_k \in [x_{k-1}, x_k]$ such that

$$\Delta\alpha_k = \alpha'(t_k)\Delta x_k \tag{7.5}$$

Then, because $U(P,\alpha') - L(P,\alpha)' < \frac{\epsilon}{3M}$ (Equation 7.4), we now see

$$\sum_{k=1}^n |\alpha'(s_k) - \alpha'(t_k)| \Delta x_k < \frac{\epsilon}{3M} \text{ if } s_k \in [x_{k-1}, x_k] \text{ for all } k \in \{1,\ldots,n\} \tag{7.6}$$

Then from Equation 7.5, definition of M and Equation 7.6, we have

$$\begin{aligned}\left|\sum_{k=1}^{n} f(s_k)\Delta\alpha_k - \sum_{k=1}^{n} f(s_k)\alpha'(s_k)\Delta x_k\right| &= \left|\sum_{k=1}^{n} f(s_k)\big(\alpha'(s_k) - \alpha'(t_k)\big)\Delta x_k\right| \\ &\leq \sum_{k=1}^{n} |f(s_k)| \cdot |\alpha'(s_k) - \alpha'(t_k)|\, \Delta x_k \\ &\leq M \sum_{k=1}^{n} |\alpha'(s_k) - \alpha'(t_k)|\, \Delta x_k \\ &< \frac{\epsilon}{4}\end{aligned}$$

Then because $\sum_{k=1}^{m} f(s_k)\alpha'(s_k)\Delta x_k \leq U(P, f\alpha')$, we now have

$$\sum_{k=1}^{n} f(s_k)\Delta\alpha_k < U(P, f\alpha') + \frac{\epsilon}{4} \tag{7.7}$$

Because Equation 7.7 hold true for all choices of s_k, we have

$$U(P, f, \alpha) < U(P, f\alpha') + \frac{\epsilon}{3}$$

Similarly, we can deduce

$$U(P, f\alpha') < U(P, f, \alpha) + \frac{\epsilon}{3} \text{ (done)}$$

■

Theorem 7.3.3. (Substitution Law) Given a function $\varphi : [a, b] \to [A, B]$ and a function $f : [A, B] \to \mathbb{R}$ such that

(a) φ is a homoeomorphism.

(b) φ is differentiable on (a, b)

(c) $\int_a^b \varphi'(x)dx$ exists.

(d) f is integrable on $[A, B]$

We have

$$\int_a^b f\big(\varphi(x)\big)\varphi'(x)dx = \int_A^B f(u)du$$

Proof. Because $f \circ \varphi$ and φ' is integrable on $[a, b]$, by reduction of Riemann-Stieljes Integral (Tehroem 7.3.2), we know

$$\int_a^b \big(f \circ \varphi\big)(x)\varphi'(x)dx = \int_a^b \big(f \circ \varphi\big)(x)d\varphi$$

Let $\alpha(x) = x$. Let $\beta = \alpha \circ \varphi$. Define $g = f \circ \varphi$. By Change of Variable (Theorem 7.3.1), we now have

$$\int_a^b \big(f \circ \varphi\big)(x)d\varphi = \int_a^b g(x)d\beta = \int_A^B f(x)dx$$

■

7.4 Uniform Convergence and Riemann Integration

Theorem 7.4.1. (Riemann-Integration and Uniform Convergence) Given a function $\alpha : [a,b] \to \mathbb{R}$ and a sequence of functions $f_n : [a,b] \to \mathbb{R}$ such that

(a) α increase on $[a,b]$

(b) $\int_a^b f_n d\alpha$ exists for all $n \in \mathbb{N}$

(c) $f_n \to f$ uniformly on $[a,b]$

Then

$$\lim_{n\to\infty} \int_a^b f_n d\alpha \text{ exists and } \int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha$$

Proof. We first prove

$$\int_a^b f d\alpha \text{ exists}$$

Fix ϵ. We wish to prove

$$\overline{\int_a^b} f d\alpha - \underline{\int_a^b} f d\alpha < \epsilon$$

Let $\epsilon_n = \|f_n - f\|_\infty$. Because $f_n \to f$ uniformly, we know

$$\text{there exists } n \in \mathbb{N} \text{ such that } \epsilon_n = \|f_n - f\|_\infty < \frac{\epsilon}{2\big[\alpha(b) - \alpha(a)\big]}$$

Because α increase, by definition of ϵ_n, we see

$$\int_a^b (f_n - \epsilon_n) d\alpha \leq \underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha \leq \int_a^b (f_n + \epsilon_n) d\alpha$$

Because $\epsilon_n < \frac{\epsilon}{2\left[\alpha(b)-\alpha(a)\right]}$, we now see

$$\begin{aligned}\overline{\int_a^b} f d\alpha - \underline{\int_a^b} f d\alpha &\leq \int_a^b (f_n + \epsilon_n) d\alpha - \int_a^b (f_n - \epsilon_n) d\alpha \\ &= \int_a^b (2\epsilon_n) d\alpha < 2\epsilon_n \cdot \big[\alpha(b) - \alpha(a)\big] = \epsilon \text{ (done)}\end{aligned}$$

We now prove

$$\int_a^b f_n d\alpha \to \int_a^b f d\alpha \text{ as } n \to \infty$$

Fix ϵ. We wish

$$\text{to find } N \text{ such that } \forall n > N, \left| \int_a^b f_n d\alpha - \int_a^b f d\alpha \right| < \epsilon$$

Recall the definition $\epsilon_n = \|f_n - f\|_\infty$. Because $\epsilon_n \to 0$, we know

$$\text{there exists } N \text{ such that } \forall n > N, \epsilon_n < \frac{\epsilon}{\alpha(b) - \alpha(a)} \tag{7.8}$$

We claim

$$\text{such } N \text{ works}$$

Fix $n > N$. From Equation 7.8, we see

$$\begin{aligned}
\left| \int_a^b f_n d\alpha - \int_a^b f d\alpha \right| &= \left| \int_a^b (f_n - f) d\alpha \right| \\
&\leq \int_a^b |f_n - f| \, d\alpha \\
&\leq \int_a^b \epsilon_n d\alpha = \epsilon_n \big[\alpha(b) - \alpha(a)\big] < \epsilon \text{ (done)}
\end{aligned}$$

■

> As Rudin remarked, a much shorter (and much more intuitive) proof can be given, if we require f' to be continuous on $[a, b]$.

Theorem 7.4.2. (Uniform Convergence and Differentiation: Weaker Version) Given a sequence of function $f_n : [a, b] \to \mathbb{R}$ such that

(a) f_n' uniformly converge on $[a, b]$

(b) $f_n(x_0) \to L$ for some $x_0 \in [a, b]$

(c) f_n are of class C^1 on $[a, b]$

Then

(a) f_n uniformly converge on $[a, b]$

(b) and

$$\frac{d}{dx}\Big(\lim_{n\to\infty} f_n(x)\Big)\Big|_{x=x_0} = \lim_{n\to\infty} f_n'(x_0) \text{ on } (a,b)$$

Proof. We claim

$$f(x) = \lim_{n\to\infty}\int_{x_0}^x f_n'(t)dt + L \text{ works}$$

Note that $\lim_{n\to\infty}\int_{x_0}^x f_n'(t)dt$ exists because f_n' uniformly converge (Theorem 7.4.1).

Because f_n' uniformly converge and are continuous on $[a,b]$, by ULT, we know

$$\int_{x_0}^x \lim_{n\to\infty} f_n'(t)dt + L \text{ exists}$$

and know

$$f(x) = \int_{x_0}^x \lim_{n\to\infty} f_n'(t)dt + L$$

By FTC, we see

$$f'(x) = \lim_{n\to\infty} f_n'(x) \text{ on } (a,b)$$

Such convergence is uniform by premise. To finish the proof, we now only have to prove

$$f_n \to f \text{ uniformly on } [a,b]$$

Fix ϵ. We wish

$$\text{to find } N \text{ such that } |f_n(x) - f(x)| \leq \epsilon \text{ for all } n > N \text{ and } x \in [a,b]$$

Because $f_n' \to f'$ uniformly, and $f_n(x_0) \to L = f(x_0)$ (Check $L = f(x_0)$), we know there exists N such that

$$\begin{cases} \|f_n' - f'\|_\infty < \frac{\epsilon}{2(b-a)} \\ |f_n(x_0) - f(x_0)| < \frac{\epsilon}{2} \end{cases} \quad \text{for all } n > N$$

We claim

$$\text{such } N \text{ works}$$

Fix $n > N$ and $x \in [a, b]$. Observe

$$\begin{aligned} |f(x) - f_n(x)| &= \left| \int_{x_0}^{x} \big(f'(t) - f_n'(t)\big) dt + f(x_0) - f_n(x_0) \right| \\ &\le \int_{x_0}^{x} |f'(t) - f_n'(t)| \, dt + |f(x_0) - f_n(x_0)| \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ (done)} \end{aligned}$$

■

7.5 MVT for Definite Integral

Theorem 7.5.1. (First Mean Value Theorem for Definite Integral) Given a function $f : [a, b] \to \mathbb{R}$ such that

(a) f is continuous on (a, b)

There exists $\xi \in (a, b)$ such that

$$\int_a^b f(x)dx = f(\xi) \cdot (b - a)$$

Proof. We wish

$$\text{to find } \xi \in (a, b) \text{ such that } f(\xi) = \frac{\int_a^b f(x)dx}{b - a}$$

Define $\tilde{f} : [a, b] \to \mathbb{R}$ on $[a, b]$ by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in (a, b) \\ \lim_{t \to a} f(t) & \text{if } x = a \\ \lim_{t \to b} f(t) & \text{if } x = b \end{cases} \tag{7.9}$$

Then, because $\int_a^b f(x)dx = \int_a^b \tilde{f}(x)dx$, we reduce our problem into

$$\text{finding } \xi \in (a, b) \text{ such that } \tilde{f}(\xi) = \frac{\int_a^b \tilde{f}(x)dx}{b - a}$$

Because $\tilde{f}$ is continuous on $[a, b]$ by definition Equation 7.9, by EVT, we know we there exists $\alpha, \beta \in [a, b]$ such that

$$\tilde{f}(\alpha) = \inf_{x \in [a,b]} \tilde{f}(x) \text{ and } \tilde{f}(\beta) = \sup_{x \in [a,b]} \tilde{f}(x) \tag{7.10}$$

WOLG, suppose $\alpha \leq \beta$. Deduce

$$\tilde{f}(\alpha) = \inf_{x \in [a,b]} \tilde{f}(x) \leq \frac{\int_a^b \tilde{f}(x)dx}{b - a} \leq \sup_{x \in [a,b]} \tilde{f}(x) = \tilde{f}(\beta)$$

by IVT, we then know there exists $\xi \in [\alpha, \beta]$ such that

$$\exists \xi \in [\alpha, \beta], \tilde{f}(\xi) = \frac{\int_a^b \tilde{f}(x)dx}{b - a} \tag{7.11}$$

If $a < \alpha$ and $\beta < b$, our proof is done.

If not, notice that if $\tilde{f}(\alpha) = \tilde{f}(\beta)$, then by definition of α, β (Equation 7.10), the proof is trivial since $\tilde{f}$ is a constant, so we only have to consider when $\tilde{f}(\alpha) < \tilde{f}(\beta)$, and we wish to show

$$\xi \text{ can not happen at } a \text{ nor } b$$

Assume $\xi = a$, WOLG. Because $\xi \in [\alpha, \beta]$, we know $\alpha = a$. Because $\tilde{f}(\beta) > \tilde{f}(\alpha)$, we can find δ such that

$$\inf_{x \in [\beta-\delta, \beta]} \tilde{f}(x) \geq \frac{\tilde{f}(\alpha) + 2\tilde{f}(\beta)}{3} \tag{7.12}$$

We then from Equation 7.11 see that

$$\int_a^b \tilde{f}(x)dx = \tilde{f}(\xi)(b-a) = \tilde{f}(\alpha)(b-a) \tag{7.13}$$

Also, we see from definition of α (Equation 7.10) and Equation 7.12 that

$$\begin{aligned}
\int_a^b \tilde{f}(x)dx &= \int_a^{\beta-\delta} \tilde{f}(x)dx + \int_{\beta-\delta}^{\beta} \tilde{f}(x)dx + \int_{\beta}^{b} \tilde{f}(x)dx &\quad (7.14)\\
&\geq (b-\delta-a)\tilde{f}(\alpha) + \delta \cdot \Big(\frac{\tilde{f}(\alpha) + 2\tilde{f}(\beta)}{3}\Big) &\quad (7.15)\\
&> (b-\delta-a)\tilde{f}(\alpha) + \delta \cdot \Big(\frac{\tilde{f}(\alpha) + \tilde{f}(\beta)}{2}\Big) &\quad (7.16)\\
&= \tilde{f}(\alpha)\big(b-a-\frac{\delta}{2}\big) + \tilde{f}(\beta) \cdot \big(\frac{\delta}{2}\big) &\quad (7.17)
\end{aligned}$$

Now, from Equation 7.13 and Equation 7.17, we can deduce

$$\tilde{f}(\alpha)(b-a) > \tilde{f}(\alpha)(b-a-\frac{\delta}{2}) + \tilde{f}(\beta) \cdot (\frac{\delta}{2})$$

Then we can deduce

$$\tilde{f}(\alpha) \cdot (\frac{\delta}{2}) > \tilde{f}(\beta) \cdot (\frac{\delta}{2}) \text{ CaC } \text{ (done)}$$

■

Theorem 7.5.2. (Second Mean Value Theorem for Definite Integral) Given functions $G, \varphi : [a, b] \to \mathbb{R}$ such that

(a) G is monotonic

(b) φ is Riemann-Integrable

Let $G(a^+) = \lim_{t\to a^+} G(t)$ and $G(b^-) = \lim_{t\to b^-} G(t)$. Then there exists $\xi \in (a,b)$ such that

$$\int_a^b G(t)\varphi(t)dt = G(a^+)\int_a^\xi \varphi(t)dt + G(b^-)\int_\xi^b \varphi(t)dt$$

Proof. Define f on $[a,b]$ by

$$f(x) = G(a^+)\int_a^x \varphi(t)dt + G(b^-)\int_x^b \varphi(t)dt$$

We then reduce the problem into

$$\text{finding } \xi \in (a,b) \text{ such that } \int_a^b G(t)\varphi(t)dt = f(\xi)$$

By Theorem 7.2.1, we know f is continuous on $[a,b]$. Then by IVT, we can reduce the problem into

$$\text{finding an interval } [c,d] \subseteq (a,b) \text{ such that } \int_a^b G(t)\varphi(t) \text{ is between } f(c) \text{ and } f(d)$$

Observe that

$$f(a) = G(b^-)\int_a^b \varphi(t)dt \text{ and } f(b) = G(a^+)\int_a^b \varphi(t)dt$$

■

7.6 Taylor's Theorem

Abstract

This section prove Taylor's Theorem in both single and multi variables, and give some explicit formulas for the remainder terms. In this section, k is a fixed integer greater than 1, $I \subseteq \mathbb{R}$ is an open interval containing a and $U \subseteq \mathbb{R}^n$ is an open set containing $\mathbf{a}$

Theorem 7.6.1. (Single Variable Taylor's Theorem) If real function $f : I \to \mathbb{R}$ is k time differentiable at a and if we write

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + R_k(x)$$

then the approximation error $R_k(x)$ satisfy

$$R_k(x) = o(|x-a|^k)$$

Proof. Denote $P_k : I \to \mathbb{R}$

$$P_k(x) \triangleq f(a) + f'(a)(x-a) + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k$$

The proof then follows from applying L'Hospital Rule to have

$$\lim_{x \to a} \frac{f(x) - P_k(x)}{(x-a)^k} = 0$$

■

Theorem 7.6.2. (Schlömilch form of the Remainders) If $f : I \to \mathbb{R}$ is $k+1$ times differentiable on I, then for all positive integer p, the approximation error $R_k : I \to \mathbb{R}$ can be written in the Schlömilch form

$$R_k(x) = \frac{f^{(k+1)}(\xi)}{k!}(x-\xi)^{k+1-p}\frac{(x-a)^p}{p} \text{ for some } \xi \text{ between } x, a$$

Corollary 7.6.3. (Lagrange and Cauchy form of the Remainders) If $f : I \to \mathbb{R}$ is $k+1$ times differentiable on I, then for all positive integer p, the approximation error $R_k : I \to \mathbb{R}$ can be written in the Lagrange form

$$R_k(x) = \frac{f^{(k+1)}(\xi)}{(k+1)!}(x-a)^{k+1} \text{ for some } \xi \text{ between } x, a$$

or the Cauchy form

$$R_k(x) = \frac{f^{(k+1)}(\xi)}{k!}(x-\xi)^k(x-a) \text{ for some } \xi \text{ between } x, a$$

Notably, the remainder can also be written in form of integral.

Theorem 7.6.4. (Integral form of the Remainder) If $f : I \to \mathbb{R}$ is C^{k+1} on I, then the approximation error $R_k : I \to \mathbb{R}$ can be written in the integral form

$$R_k(x) = \int_a^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k dt$$

We now consider when f is multi-variables. We first introduce the multi-index notation. Given $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $\mathbf{x} \in \mathbb{R}^n$, we write

$$\begin{aligned} |\alpha| &\triangleq \alpha_1 + \cdots + \alpha_n \\ \alpha! &\triangleq \alpha_1! \cdots \alpha_n! \\ \mathbf{x}^\alpha &\triangleq x_1^{\alpha_1} \cdots x_n^{\alpha_n} \end{aligned}$$

Suppose $f : U \to \mathbb{R}$ is C^{k+1} on U. One shall note that the notation

$$D^\alpha f \triangleq \frac{\partial^{|\alpha|} f}{(\partial x_1)^{\alpha_1} \cdots (\partial x_n)^{\alpha_n}} \text{ for all } |\alpha| \leq k+1$$

is well-defined.

Theorem 7.6.5. (Multi Variables Taylor's Theorem) If real-valued function $f : U \to \mathbb{R}$ is C^{k+1} on U, then we can write

$$f(\mathbf{x}) = \sum_{|\alpha| \leq k} \frac{D^\alpha f(\mathbf{a})}{\alpha!}(\mathbf{x}-\mathbf{a})^\alpha + \sum_{|\beta|=k+1} R_\beta(\mathbf{x})(\mathbf{x}-\mathbf{a})^\beta$$

where

$$R_\beta(\mathbf{x}) \triangleq \frac{|\beta|}{\beta!} \int_0^1 (1-t)^{|\beta|-1} D^\beta f(\mathbf{a}+t(\mathbf{x}-\mathbf{a}))dt$$

Proof. ■

7.7 Picard-Lindelof Theorem

Theorem 7.7.1. (Picard Lindelof Theorem) Given some

(a) open $U \subseteq \mathbb{R}^n$.

(b) open interval $I \subseteq \mathbb{R}$.

(c) a vector field $V : I \times U \to \mathbb{R}^n$,

If V is continuous in I and K-Lipschitz continuous in U for all $t \in I$, then for all $t_0 \in I$ and $y_0 \in U$, there exists some $\gamma : (t_0 - \epsilon, t_0 + \epsilon) \to U$ such that

$$\gamma'(t) = V(t, \gamma(t)) \text{ and } \gamma(t_0) = y_0$$

Moreover, if $\tilde{\epsilon} \leq \epsilon$ and $\alpha : (t_0 - \tilde{\epsilon}, t_0 + \tilde{\epsilon}) \to U$ also satisfy

$$\alpha'(t) = V(t, \alpha(t)) \text{ and } \alpha(t_0) = y_0$$

then

$$\alpha(t) = \gamma(t) \text{ for all } t \in (t_0 - \tilde{\epsilon}, t_0 + \tilde{\epsilon})$$

Proof. ■

7.8 Inverse Function Theorem

Interestingly, if $f : \left(\mathbb{R}, \|\cdot\|_2\right) \to \left(\mathbb{R}^n, \|\cdot\|_2\right)$ is a curve in $\mathbb{R}^n$

$$f(t) = (f_1(t), \cdots, f_n(t))$$

and we define

$$f'(t) \triangleq (f_1'(t), \cdots, f_n'(t))$$

We have

$$|f'(t)| = \|df_t\|_{\text{op}}$$

This give us the following expected result (Corollary 7.8.2).

Theorem 7.8.1. (Basic Property of Derivative) Suppose f maps a convex open set $E \subseteq \left(\mathbb{R}^n, \|\cdot\|_2\right)$ into $\left(\mathbb{R}^m, \|\cdot\|_2\right)$, f is differentiable on E, and there exists $M \in \mathbb{R}$ such that

$$\|df_x\|_{\text{op}} \leq M \qquad (x \in E)$$

Then for all $a, b \in E$, we have

$$|f(b) - f(a)| \leq M\,|b - a|$$

Proof. Define $\gamma : [0,1] \to E$ by

$$\gamma(t) \triangleq a + (b-a)t$$

Now, note that

$$\begin{aligned}
|f(b) - f(a)| &= |(f \circ \gamma)(1) - (f \circ \gamma)(0)| \\
&= \left| \int_0^1 (f \circ \gamma)'(t) dt \right| \\
&\leq \int_0^1 |(f \circ \gamma)'(t)|\, dt \\
&= \int_0^1 \|d(f \circ \gamma)_t\|_{\text{op}} dt \\
&\leq \int_0^1 \|df_{\gamma(t)}\|_{\text{op}} \cdot \|d\gamma_t\|_{\text{op}} dt \\
&\leq \int_0^1 M \cdot |b-a|\, dt = M\,|b-a|
\end{aligned}$$

$\blacksquare$

Corollary 7.8.2. (Basic Property of Derivative) Suppose f maps a convex open set $E \subseteq \Big(\mathbb{R}^n, \|\cdot\|_2\Big)$ into $\Big(\mathbb{R}^m, \|\cdot\|_2\Big)$, f is differentiable on E, and $df_x = 0$ for all $x \in E$, then

$$f \text{ stay constant on } E$$

In this section, we will give a local statement and the proof for Inverse Function Theorem in $\mathbb{R}^n$ (Theorem 7.8.4). Let $L(\mathbb{R}^n)$ be the set of linear transformation that maps $\mathbb{R}^n$ into itself, and let Ω be the set of all invertibles in $L(\mathbb{R}^n)$. We will first prove that Ω is open (Theorem 7.8.3).

Theorem 7.8.3. (Ω is Open) Suppose $A \in \Omega$. If we define $\epsilon \triangleq \frac{1}{\|A^{-1}\|_{\text{op}}}$, then

$$B_\epsilon(A) \overset{\text{def}}{=} \{T \in L(\mathbb{R}^n) : \|T - A\|_{\text{op}} < \epsilon\} \subseteq \Omega$$

Proof. Fix $T \in B_\epsilon(A)$ and $x \neq 0 \in \mathbb{R}^n$. We are required to show

$$|Tx| > 0$$

If $T = A$, then the proof is trivial. We therefore suppose $T \neq A$. Define

$$\beta \triangleq \|T - A\|_{\text{op}}$$

Note that $T \neq A \in B_\epsilon(A)$ implies $0 < \beta < \epsilon$. We claim

$$(\epsilon - \beta)|x| \leq |Tx|$$

Observe

$$\epsilon |x| = \epsilon \left|A^{-1}Ax\right| \leq |Ax| \tag{7.18}$$

Observe

$$|Ax| \leq |(A - T)x| + |Tx| \leq \beta |x| + |Tx| \tag{7.19}$$

Equation 7.18 and Equation 7.19 implies

$$\epsilon |x| \leq \beta |x| + |Tx|$$

which implies

$$(\epsilon - \beta)|x| \leq |Tx| \;\; \text{(done)}$$

$\blacksquare$

Theorem 7.8.3 is essential for proving Inverse Function Theorem (Theorem 7.8.4). Think about what happen if f is linear. If f is linear, then f^{-1} is linear, and we will have

$$df^{-1} = f^{-1} = (f)^{-1} = (df)^{-1}$$

Because derivative is unique, it is reasonable to guess that if f is not linear and f^{-1} is differentiable, we would have

$$df^{-1} = (df)^{-1}$$

Now, if we wish df^{-1} to exists everywhere on $f(U)$, we must guarantee that df is invertible on U, and this is when Theorem 7.8.3 kick in. Note that in our proof of Inverse Function Theorem (Theorem 7.8.4), our selection of U guarantee df_x is invertible for all $x \in U$, by Theorem 7.8.3.

The rest of the proof boiled down to a fixed point argument (Theorem 2.8.1) to show f is one-to-one in U and $f(U)$ is open.

Theorem 7.8.4. (Inverse Function Theorem) Given a function f that maps an open neighborhood $E \subseteq \mathbb{R}^n$ around a into $\mathbb{R}^n$ such that

(a) f is differentiable on E

(b) df_a is invertible

(c) f is continuously differentiable at a

Then there exists open convex $U \subseteq E$ containing a such that

(a) f is one-to-one in U

(b) $f(U)$ is open

(c) The inverse of $f|_U$ is differentiable at $f(a)$.

Proof. Fix

$$\lambda \triangleq \frac{1}{2\|(df_a)^{-1}\|_{\text{op}}} \tag{7.20}$$

Because f is continuously differentaible at a, we know there exists δ such that

$$\|df_x - df_a\|_{\text{op}} < \lambda \qquad (x \in B_\delta(a)) \tag{7.21}$$

We claim

$$U \triangleq B_\delta(a) \text{ suffices}$$

For each $y \in \mathbb{R}^n$, define $\varphi_y : U \to \mathbb{R}^n$ by

$$\varphi_y(x) \triangleq x + (df_a)^{-1}(y - f(x))$$

Before anything, we first prove

$$\text{for all } y \in \mathbb{R}^n, \varphi_y : U \to \mathbb{R}^n \text{ is a contraction of } U$$

Fix $y \in \mathbb{R}^n$, and $x_1, x_2 \in U$. We claim

$$|\varphi_y(x_1) - \varphi_y(x_2)| \leq \frac{1}{2}|x_1 - x_2| \tag{7.22}$$

Because U is convex, Theorem 7.8.1 allow us to reduce the problem into proving

$$\|d(\varphi_y)_x\|_{\text{op}} \leq \frac{1}{2} \text{ for all } x \in U$$

Fix $x \in U$. Using Chain Rule (Theorem 5.5.2) and the fact that the derivative of a bounded linear transformation is itself, we can compute $d(\varphi_y)_x$

$$\begin{aligned} d(\varphi_y)_x &= I + (df_a)^{-1}(-df_x) \\ &= (df_a)^{-1}(df_a - df_x) \end{aligned}$$

This together with Equation 7.20 and Equation 7.21 give us

$$\|d(\varphi_y)_x\|_{\text{op}} \leq \|(df_a)^{-1}\|_{\text{op}}\|df_a - df_x\|_{\text{op}} < \frac{1}{2} \text{ (done)}$$

We now prove

$$f \text{ is one-to-one in } U$$

Fix y in $f(U)$. We wish to show

$$\text{there exists at most one } x \in U \text{ such that } f(x) = y$$

Because $f(x) = y \iff x$ is a fixed point of φ_y, we can reduce the problem into

$$\varphi_y \text{ has at most one fixed point}$$

Because φ_y is a contraction of U, Banach Fixed Point Theorem (Theorem 2.8.1) tell us φ_y has at most one fixed point. (done)

We now prove

$$f(U) \text{ is open in } \mathbb{R}^n$$

Fix $y_0 \in f(U)$. Let $x_0 = f^{-1}(y_0)$. Because U is open, we know there exists r such that

$$\overline{B_r(x_0)} \subseteq U$$

We claim

$$B_{\lambda r}(y_0) \subseteq f(U)$$

Fix $y \in B_{\lambda r}(y_0)$. We are required to prove

$$y \in f(U)$$

Because

$$y = f(x) \iff x \text{ is a fixed point of } \varphi_y$$

We then can use Banach Fixed Point Theorem (Theorem 2.8.1) to reduce the problem into proving

$$\varphi_y \text{ is a contraction that maps some complete subset of } U \text{ into itself}$$

We claim

$$\overline{B_r(x_0)} \text{ suffices}$$

We have already known φ_y is a contraction on U, and it is clear that $\overline{B_r(x_0)}$ is complete. We reduce the problem into proving

$$\varphi_y\big(\overline{B_r(x_0)}\big) \subseteq \overline{B_r(x_0)}$$

Using

(a) definition of φ_y

(b) $|y - y_0| < \lambda r$

(c) $\|(df_a)^{-1}\|_{\text{op}} = \frac{1}{2\lambda}$

We can deduce

$$\begin{aligned} |\varphi_y(x_0) - x_0| &= \left|(df_a)^{-1}(y - f(x_0))\right| \\ &\leq \|(df_a)^{-1}\|_{\text{op}} \, |y - y_0| < \frac{r}{2} \end{aligned}$$

Fix $x \in \overline{B_r(x_0)}$. We can now deduce

$$\begin{aligned} |\varphi_y(x) - x_0| &\leq |\varphi_y(x_0) - \varphi_y(x)| + |x_0 - \varphi_y(x_0)| \\ &\leq \frac{1}{2}|x_0 - x| + \frac{r}{2} \leq r \text{ (done)} \end{aligned}$$

We now prove

$$df_x \text{ is invertible for all } x \in U$$

Fix $x \in U$.

$$\|df_x - df_a\|_{\text{op}} \cdot \|(df_a)^{-1}\|_{\text{op}} < \frac{1}{2}$$

Theorem 7.8.3 now implies df_x is invertible. (done)

Lastly, it remains to prove

$$f^{-1} : f(U) \to U \text{ is differentiable on } f(U)$$

Fix $y \in f(U)$, and $x \triangleq f^{-1}(y)$. We are required to prove

$$\lim_{k \to 0} \frac{\left|f^{-1}(y+k) - x - (df_x)^{-1}k\right|}{|k|} = 0$$

Fix $h(k) \triangleq f^{-1}(y+k) - f(x)$. In other words, $h \in \mathbb{R}^n$ is fixed to be the unique vector such that

$$f(x+h) = y + k$$

We now see

$$\begin{aligned} f^{-1}(y+k) - x - (df_x)^{-1}k &= h - (df_x)^{-1}k \\ &= -(df_x^{-1})(f(x+h) - f(x) - df_x h) \end{aligned}$$

and see

$$\left|f^{-1}(y+k) - x - (df_x)^{-1}k\right| \leq \|(df_x)^{-1}\|_{\text{op}} \left|f(x+h) - f(x) - df_x h\right|$$

which give us

$$\frac{\left|f^{-1}(y+k) - x - (df_x)^{-1}k\right|}{|k|} \leq \|(df_x)^{-1}\|_{\text{op}} \frac{|f(x+h) - f(x) - df_x h|}{|h|} \cdot \frac{|h|}{|k|}$$

This allow us to reduce the problem into proving

$$\limsup_{k \to 0} \frac{|h|}{|k|} \in \mathbb{R}$$

We claim

$$\frac{|h|}{|k|} \leq \lambda^{-1} \text{ for all } k \text{ such that } y + k \in f(U)$$

Compute

$$\varphi_y(x+h) - \varphi_y(x) = h - (df_a)^{-1}k$$

Equation 7.22 let us deduce

$$\left|h - (df_a)^{-1}k\right| = |\varphi_y(x+h) - \varphi_y(x)| \leq \frac{|h|}{2}$$

This with triangle inequality implies

$$\|(df_a)^{-1}\|_{\text{op}} |k| \geq \left|(df_a)^{-1}k\right| \geq \frac{|h|}{2} \text{ (done)}$$

■

The following is a technical recap of our proof for the Inverse Function Theorem (Theorem 7.8.4).

1: Let $\lambda \triangleq \frac{1}{2\|(df_a)^{-1}\|_{\text{op}}}$

2: Claim $B_\delta(a)$ suffices to be U, where $\|df_x - df_a\|_{\text{op}} < \lambda$

3: For each $y \in \mathbb{R}^n$, define $\varphi_y : U \to \mathbb{R}^n$ by $\varphi_y(x) \triangleq x + (df_a)^{-1}(y)$.

4: Prove that φ_y is a contraction of U by taking derivative and utilize step 1 and 2.

5: Prove that φ_y fix $x \iff f(x) = y$

6: Prove f is one-to-one in U using step 4,5.

7: Prove $f(U)$ is open by proving $B_{\lambda r}(y_0) \subseteq f(U)$, while $\overline{B_r(x_0)} \subseteq U$. The proof use step 4,5, some computation and ultimately claim that φ_y admits a fixed point as φ_y maps $\overline{B_r(x_0)}$ into itself.

8: Prove df_x is invertible in U by Theorem 7.8.3

9: Algebraically prove $df^{-1} = (df)^{-1}$, using $\left|h - (df_a)^{-1}k\right| = |\varphi_y(x+h) - \varphi_y(x)| \leq \frac{|h|}{2}$

Theorem 7.8.5. (Inversion is Continuous)

$$\text{The mapping } A \to A^{-1} \text{ is continuous on } \Omega$$

Proof. Fix $A \in \Omega$ and let $T \in \Omega$. We are required to prove

$$\lim_{T \to A} \|T^{-1} - A^{-1}\|_{\text{op}} = 0$$

We know

$$T^{-1} - A^{-1} = T^{-1}(A - T)A^{-1}$$

This implies

$$\|T^{-1} - A^{-1}\|_{\text{op}} \leq \|T^{-1}\|_{\text{op}}\|A - T\|_{\text{op}}\|A^{-1}\|_{\text{op}}$$

This allow us to reduce the problem into proving

$$\limsup_{T \to A} \|T^{-1}\|_{\text{op}} \in \mathbb{R}$$

Fix $\epsilon \triangleq \frac{1}{\|A^{-1}\|_{\text{op}}}$, $T \in B_\epsilon(A)$ and $\beta \triangleq \|T - A\|_{\text{op}} < \epsilon$. We claim

$$\|T^{-1}\|_{\text{op}} \leq (\epsilon - \beta)^{-1}$$

Following the proof of Theorem 7.8.3, we have

$$(\epsilon - \beta)\,|x| \leq |Tx| \text{ for all } x \in \mathbb{R}^n$$

This implies

$$\frac{|T^{-1}x|}{|x|} \leq (\epsilon - \beta)^{-1} \text{ for all } x \neq 0 \in \mathbb{R}^n \quad \text{(done)}$$

■

Corollary 7.8.6. (Continuously Differentaible Version of Inverse Function Theorem) Given a function f that maps open $E \subseteq \mathbb{R}^n$ containing a into $\mathbb{R}^n$ such that

(a) f is differentiable on E

(b) df_a is invertible

(c) f is continuously differentiable on E

Then there exists open $U \subseteq E$ containing a such that

$$f(U) \text{ is open and } f|_U : U \to f(U) \text{ is a diffeomorphism}$$

7.9 Implicit Function Theorem

Some notations first. If $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^k)$. We define $A|_{\mathbb{R}^n} : L(\mathbb{R}^n, \mathbb{R}^k)$ by

$$A|_{\mathbb{R}^n}(x) \triangleq A(x, 0)$$

Theorem 7.9.1. (Implicit Function Theorem) Suppose a function f that maps open $E \subseteq \mathbb{R}^n \times \mathbb{R}^m$ containing (a, b) into $\mathbb{R}^n$ satisfy

(a) $f(a, b) = 0$

(b) $(df_{(a,b)})|_{\mathbb{R}^n}$ is invertible

(c) f is continuously differentiable on E

Then there exists open $U \subseteq E$ containing (a, b) and open $W \subseteq \mathbb{R}^m$ containing b such that there exists a unique function g from W to $\mathbb{R}^n$ such that

(a) $\big(g(y), y\big) \in U$ for all $y \in W$

(b) $f(g(y), y) = 0$ for all $y \in W$

Moreover, g satisfy

(a) g is continuously differentiable on W

(b) g satisfy $dg_b = -(df_{(a,b)}|_{\mathbb{R}^n})^{-1} \circ df_{(a,b)}|_{\mathbb{R}^m}$

Proof. Define $F : E \to \mathbb{R}^n \times \mathbb{R}^m$ by

$$F(x, y) \triangleq \Big(f(x, y), y\Big)$$

Because f is continuously differentiable on E, using Differentaibility Theorem (Theorem 5.4.2), we can deduce

$$F \text{ is continuously differentiable on } E$$

Again using Differentiability Theorem (Theorem 5.4.1), we can write down $dF_{(a,b)}$ in the matrix form with respect to standard basis

$$[dF_{(a,b)}] = \begin{bmatrix} df_{(a,b)}|_{\mathbb{R}^n} & O \\ df_{(a,b)}|_{\mathbb{R}^m} & I \end{bmatrix}$$

Now, because $df_{(a,b)}|_{\mathbb{R}^n}$ is invertible, we know $dF_{(a,b)}$ is invertible.

We can now apply Inverse Function Theorem (Theorem 7.8.4) to $F : E \to \mathbb{R}^n \times \mathbb{R}^m$. This give us

(a) an open $U \subseteq E \subseteq \mathbb{R}^n \times \mathbb{R}^m$ containing (a, b)

(b) open $V \triangleq F(U) \subseteq \mathbb{R}^n \times \mathbb{R}^m$ containing $(0, b)$

(c) $F|_U : U \to V$ is a diffeomorphism.

Define W by

$$W \triangleq \{y \in \mathbb{R}^m : (0, y) \in V\}$$

We claim

$$\text{such } U, W \text{ suffices}$$

Note that it is easy to check W is open, utilizing V is open and the same ϵ.

Now, because $F|_U : U \to V$ is bijective, we know for each $y \in W$, there exists unique $(x, y) \in U$ such that

$$F(x, y) = (0, y)$$

We can now well define a function $g : W \to \mathbb{R}^n$ such that

$$\Big(g(y), y\Big) \in U \text{ and } f(g(y), y) = 0 \text{ for all } y \in W$$

It remains to show

(a) g is continuously differnetiable on W

(b) $dg_b = -(df_{(a,b)})|_{\mathbb{R}^n}^{-1}(df_{(a,b)})|_{\mathbb{R}^m}$

Fix $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$. We wish to prove

$$\partial_i g_j \text{ exists and is continuous on } W$$

Express

$$g(y_1, \dots, y_m) = \Big(g_1(y_1, \dots, y_m), \dots, g_n(y_1, \dots, y_m)\Big)$$

Express

$$F^{-1}(z_1, \dots, z_{n+m}) = \Big(F_1^{-1}(z_1, \dots, z_{n+m}), \dots, F_{n+m}^{-1}(z_1, \dots, z_{n+m})\Big)$$

Because F^{-1} is continuously differentiable on V, we reduce the problem into proving

$$\partial_i g_j(y) = \partial_{n+i} F_j^{-1}(0, y) \text{ for all } y \in W$$

Because $F^{-1}(0,y) = (g(y),y)$ for all $y \in W$, we know

$$F_j^{-1}(0,\dots,0,y_1,\dots,y_m) = g_j(y_1,\dots,y_m) \text{ for all } y \in W \tag{7.23}$$

Fix arbitrary $y = (y_1,\dots,y_m) \in W$. Because W is open, we can see from Equation 7.23 that

$$\begin{aligned}\partial_i g_j(y) &= \lim_{t\to 0} \frac{g_j(y_1,\dots,y_i+t,\dots,y_m)}{t} \\ &= \lim_{t\to 0} \frac{F_j^{-1}(0,\dots,0,y_1,\dots,y_i+t,\dots,y_m)}{t} \\ &= \partial_{n+i} F_j^{-1}(y) \text{ (done)}\end{aligned}$$

Define $\Phi : W \to U$ by

$$\Phi(y) = \Big(g(y), y\Big)$$

By definition of g, we have

$$f \circ \Phi = 0 \text{ on } W$$

This by Chain Rule give us

$$df_{\Phi(y)} \circ d\Phi_y = 0 \text{ on } W$$

In particular

$$df_{(a,b)} \circ d\Phi_b = 0$$

Now, compute

$$d\Phi_b = \begin{bmatrix} dg_b \\ I \end{bmatrix} \text{ and } df_{(a,b)} = \begin{bmatrix} df_{(a,b)}|_{\mathbb{R}^n} & df_{(a,b)}|_{\mathbb{R}^m} \end{bmatrix}$$

This then give us

$$df_{(a,b)}|_{\mathbb{R}^n} \circ dg_b + df_{(a,b)}|_{\mathbb{R}^m} = 0$$

and of course

$$dg_b = -\big(df_{(a,b)}|_{\mathbb{R}^n}\big)^{-1} \circ df_{(a,b)}|_{\mathbb{R}^m} \text{ (done) (done)}$$

■

Example 26 (Unit Circle Example)

$$f(x,y) \triangleq x^2 + y^2 - 1 \text{ and } (a,b) \triangleq (1,1)$$

We have

$$g(y) = \sqrt{2 - y^2} \text{ on } y \in (1-\epsilon, 1+\epsilon)$$

Compute

$$df_{(a,b)} = \begin{bmatrix} 2 & 2 \end{bmatrix} \text{ and } dg_1 = \begin{bmatrix} -1 \end{bmatrix}$$

This established

$$dg_a = -(df_{(a,b)}|_{\mathbb{R}^1})^{-1} \circ df_{(a,b)}|_{\mathbb{R}^1}$$

Example 27 (Implicit Function Theorem Implies Inverse Function Theorem)

Given continuously differentaible $h : E \overset{\text{open}}{\subseteq} \mathbb{R}^n \ni a \to \mathbb{R}^n$ such that dh_a is invertible

Define $f : E \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$f(x,y) \triangleq h(x) - y$$

It is easily checked that $f(x, h(x)) = 0$ and the rest of the condition is satisfied. Now by Implicit Function Theorem (Theorem 7.9.1), we see that there exists $g : W \overset{\text{open}}{\subseteq} \mathbb{R}^n \to E$ such that

$$f(g(y), y) = 0 \text{ for all } y \in W$$

In other words,

$$h(g(y)) = y \text{ for all } y \in W$$

7.10 Feynman's Trick

In this section

Theorem 7.10.1. (Feynman's Trick) Given a real-valued function $f(x,t)$ defined on $[a,b]\times[c,d]$, and an real-valued function α of bounded variation on $[a,b]$, such that

(a) $\partial_2 f(x,t)$ exists on $[a,b]\times[c,d]$

(b) For all $t\in[c,d]$, the integral $\int_a^b f(x,t)d\alpha(x)$ exists.

(c) For all $s\in[c,d]$ and ϵ, there exists δ such that

$$|\partial_2 f(x,t)-\partial_2 f(x,s)|<\epsilon \text{ for all } x\in[a,b] \text{ and all } t\in(s-\delta,s+\delta)$$

Then we have

$$\frac{d}{dt}\int_a^b f(x,t)d\alpha(x)=\int_a^b \frac{\partial}{\partial t}f(x,t)d\alpha(x)$$

In other words, if we define $g(t)\triangleq\int_a^b f(x,t)d\alpha(x)$, then we have

$$g'(t)=\int_a^b \partial_2 f(x,t)d\alpha(x)$$

Proof. Fix $s\in[c,d]$. We are required to prove

$$g'(s)=\int_a^b \partial_2 f(x,s)dx$$

Note that, for all $t\neq s\in[c,d]$, we have

$$\frac{g(t)-g(s)}{t-s}=\int_a^b \frac{f(x,t)-f(x,s)}{t-s}d\alpha(x)$$

This allow us to reduce the problem into proving

$$\frac{f(x,t)-f(x,s)}{t-s}\to\partial_2 f(x,s) \text{ uniformly for all } x\in[a,b] \text{ as } t\to s$$

By MVT (Corollary 5.3.3), we know for all $t\neq s\in[c,d]$, there exists u_t between t and s such that

$$\frac{f(x,t)-f(x,s)}{t-s}=\partial_2 f(x,u_t)$$

The proof now follows from (c). (done) ∎

Example 28 (Introductive application of Feynman's Trick)

$$\text{What is the value of } \int_0^1 \frac{x-1}{\ln x} dx \text{ ?}$$

Define $f(x,t) \triangleq \frac{x^t-1}{\ln x}$ on $[0,1]\times[0,1]$. Observe $\partial_2 f(x,t) = x^t$, and observe

$$\int_0^1 f(x,0)dx = 0 \text{ and } \int_0^1 \partial_2 f(x,t)dx = \frac{1}{t}$$

We can con compute

$$\begin{aligned}\int_0^1 \frac{x-1}{\ln x} dx &= \int_0^1 f(x,1)dx \\ &= \int_0^1 f(x,0)dx + \int_0^1 \Big(\int_0^1 \partial_2 f(x,t)dx\Big)dt = 0\end{aligned}$$

Example 29 (Dirichlet's Integral)

$$\text{What is the value of } \int_0^\infty \frac{\sin t}{t} dt \text{ ?}$$

Define the Laplace transformation

$$f(s,t) \triangleq e^{-st}\frac{\sin t}{t} \text{ on } \mathbb{R}_0^+ \times \mathbb{R}^+$$

Observe

$$\partial_1 f(s,t) = -e^{-st}\sin t$$

Now compute

$$\begin{aligned}\int_0^\infty -e^{-st}\sin t dt &= \frac{1}{-2i}\int_0^\infty e^{-st}(e^{it}-e^{-it})dt \\ &= \frac{1}{-2i}\Big(\frac{e^{t(i-s)}}{i-s} - \frac{e^{t(-i-s)}}{-i-s}\Big)\Big|_{t=0}^\infty \\ &= \frac{-1}{1+s^2} = \frac{d}{ds}(-\arctan s)\end{aligned}$$

It is clear that

$$\lim_{s\to\infty}\int_0^\infty f(s,t)dt = 0$$

We now have

$$\begin{aligned}\int_0^\infty \frac{\sin t}{t} dt &= \int_0^\infty f(0,t)dt \\ &= \lim_{s\to\infty} \int_0^\infty f(s,t)dt - \int_0^\infty \int_0^\infty \partial_1 f(s,t) dt ds \\ &= \int_0^\infty \frac{1}{1+s^2} ds = \frac{\pi}{2}\end{aligned}$$

Chapter 8

Lebesgue Calculus

8.1 Basic Property of Measurable Functions

Abstract

In this section, we discuss the usage of the term 'measurable function', and prove some basic properties for later usage.

In the most general setting, if we are given a function $f : (X, \Sigma_X) \to (Y, \Sigma_Y)$ between two measurable space, then we say the f is a **measurable function** if every pre-image of measurable set is again measurable. Immediately, one can check that in this setting, the composition of two measurable functions must be measurable.

Theorem 8.1.1. (Criteria of function measurability) If a function $f : (X, \Sigma_X) \to (Y, \sigma(T))$ satisfy

$$f^{-1}(E) \in \Sigma_X \text{ for all } E \in T$$

then

$$f \text{ is measurable}$$

Proof. Define

$$\mathcal{A} \triangleq \{E \subseteq Y : f^{-1}(E) \in \Sigma_X\}$$

Check that $\mathcal{A}$ is a σ-algebra on Y. By premise, $T \subseteq \mathcal{A}$. It then follows from definition that $\sigma(T) \subseteq \mathcal{A}$. This conclude that f is $(\Sigma_X, \sigma(T))$-measurable. ■

When Y is a topological space, if there are no explicit specification of Σ_Y, as a consensus, we take Σ_Y to be the Borel sigma algebra. This default consensus have good property if one take account of the Borel hierarchy. If F is some Borel measurable subspace of Y, then the Borel sigma algebra Σ_F on F is contained by the Borel sigma algebra Σ_Y on Y by a proof of transfinite induction on the Borel hierarchy of Σ_F. This implies that if $f : (X, \Sigma_X) \to (Y, \Sigma_Y)$ is measurable, and f only take value in F, then f is still measurable when considered as a function $f : (X, \Sigma_X) \to (F, \Sigma_F)$ between X and F.

Although Σ_Y is by default the Borel sigma algebra, when $X = \mathbb{R}^d$ and Σ_X is also not explicit specified, the sigma algebra Σ_X is by default the Lebesgue sigma algebra $\Sigma_X \triangleq \mathcal{L}(\mathbb{R}^d)$, for obvious reason that we are building the Theory of measurable function for the Theory of Lebesgue integration. Take Y to be some topological space, and suppose f is define on some subset of X and has codomain Y. If $E \subseteq X$ is some Lebesgue measurable set, we say f is **measurable on** E if f is defined on E and $f|_E : (E, \Sigma_E) \to (Y, \Sigma_Y)$ is measurable where Σ_E is the sub-sigma algebra of the Lebesgue sigma algebra $\mathcal{L}(\mathbb{R}^d)$ defined by $\Sigma_E \triangleq \{H \cap E : H \in \mathcal{L}(\mathbb{R}^d)\}$. Note that the priori E is Lebesgue measurable implies that $\Sigma_E \subseteq \mathcal{L}(\mathbb{R}^d)$.

With this definition and the fact null sets are measurable, we immediately see that if g is defined on E, f is measurable on E, and $g = f$ **almost everywhere on** E, i.e., $|\{x \in E : f(x) \neq g(x)\}| = 0$, then g is also measurable on E.

In addition to this almost everywhere property, if f is extended-real-valued, then because

$$\{f \geq a\} = \bigcup_n \{f > a - \frac{1}{n}\}$$

where the notation $\{f > a\}$ simply mean $\{f > a\} \triangleq \{x \in E : f(x) > a\}$, and because open set in $\overline{\mathbb{R}}$ can be expressed as disjoint union of some countable collection of open interval, with Theorem 8.1.1 we can equivalently define function to be Lebesgue measurable on E if and only if

$$\{f > a\} \text{ is Lebesgue measurable for all } a \in \mathbb{R}$$

Beside this important simplification of the definition of function measurability, we also see from Theorem 8.1.1 that if X, Y are both Borel, then a continuous function $f : X \to Y$ must also be a measurable function. These proposition give the following Lemma, which later prove Theorem 8.1.3, by replacing the codomain Y with $\mathbb{R}$ or $\mathbb{C}$.

Lemma 8.1.2. (Computational Lemma) Given a second-countable topological space

Y, two functions $u, v : (X, \Sigma_X) \to (Y, \Sigma_Y)$ and a continuous function $\Phi : Y^2 \to Y$, the function $h : (X, \Sigma_X) \to (Y, \Sigma_Y)$ defined by

$$h(x) \triangleq \Phi(u(x), v(x))$$

is measurable.

Proof. Define $f : (X, \Sigma_X) \to (Y^2, \Sigma_{Y^2})$ by

$$f(x) \triangleq \big(u(x), v(x)\big)$$

Because Φ is continuous and h is the composition of f and Φ, we only have to prove f is measurable. Let $\mathcal{B}$ be a countable basis of Y. It is clear that $\Sigma_{Y^2} = \sigma(\mathcal{B}^2)$. Now if we fix $I_1 \times I_2 \in \mathcal{B}^2$, the proof follows from Theorem 8.1.1 and noting

$$f^{-1}(I_1 \times I_2) = u^{-1}(I_1) \cap v^{-1}(I_2) \in \Sigma_X$$

$\blacksquare$

Theorem 8.1.3. (Arithmetic properties of measurable real and complex valued function) Given two real-valued function $u, v : (X, \Sigma_X) \to \mathbb{R}$

(a) $u + iv : X \to \mathbb{C}$ is measurable if and only if u, v are measurable.

(b) If $f, g : X \to \mathbb{R}$ or $\mathbb{C}$ are measurable, so are $f + g, fg$ and $|f|$.

(c) If $f : X \to \mathbb{R}$ or $\mathbb{C}$ is measurable and $g : X \to \mathbb{R}$ or $\mathbb{C}$ isn't, then $f + g$ is not measurable.

Proof. (a) and (b) follows from Lemma 8.1.2, noting $u = \text{Re}(u+iv), v = \text{Im}(u+iv)$ and the fact $\mathbb{R}, \mathbb{C}$ are topological fields. It is clear that the set of all function from X to $\mathbb{R}$ or $\mathbb{C}$ form a group under addition. (b) shows that the set of measurable functions from a subgroup, thus giving (c). $\blacksquare$

Note that Theorem 8.1.3 only consider function with finite values.

Theorem 8.1.4. (Superior limit of measurable $f_n : X \to [-\infty, \infty]$ is measurable) Given a sequence $f_n : X \to [-\infty, \infty]$ of measurable functions

$$g \triangleq \sup f_n \text{ and } f \triangleq \limsup_{n \to \infty} f_n \text{ are both measurable}$$

Proof. The proofs follows from noting

$$\{g > a\} = \bigcup_n \{f_n > a\} \text{ for all } a \in \mathbb{R}$$

$\blacksquare$

Immediately, following from Theorem 8.1.4, we see that if f is a pointwise limit of some sequence of measurable function (f_n), then f is also a measurable function. In addition to the pointwise limit, if $f : X \to [-\infty, \infty]$ is measurable, then both its positive and negative parts $f^+, f^- : X \to [0, \infty]$

$$f^+(x) \triangleq \max\{f(x), 0\} \text{ and } f^-(x) \triangleq -\min\{f(x), 0\}$$

are measurable, since

$$f^+ = \limsup_{n\to\infty} h_n \text{ and } f^- = -\liminf_{n\to\infty} h_n$$

where $h_n : X \to [-\infty, \infty]$ is defined by

$$h_n(x) \triangleq \begin{cases} f(x) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

In view of Lebesgue integration, perhaps the most important property of measurable function is that we can approximate them using measurable simple functions. By the term **simple function**, we merely mean function whose range is a set of finite elements.

Theorem 8.1.5. (Approximation of nonnegative function by increasing simple function) Let E be a subset of $\mathbb{R}^d$. For each non negative function $f : E \to [0, \infty]$, there exists an increasing sequence of simple function $s_n : E \to \mathbb{R}_0^+$ such that $s_n \nearrow f$. Moreover, if f is measurable, we can require s_n to be also measurable.

Proof. Because f is non-negative, we can well define $s_n : E \to \mathbb{R}_0^+$ by

$$s_n(x) \triangleq \begin{cases} \frac{k-1}{2^n} & \text{if there exists some } k \in \{1, \dots, n2^n\} \text{ such that } \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n} \\ n & \text{if } f(x) \geq n \end{cases}$$

Some tedious effort follows to show that $s_n \nearrow f$ and when f is measurable, s_n are measurable. ∎

Crucially, we can generalize Theorem 8.1.5 to extended value function $f : E \to [-\infty, \infty]$ by splitting f into $f = f^+ - f^-$ and have

$$f_n^+ - f_n^- \to f$$

Note that f_n^+, f_n^- are both of finite value. Then Theorem 8.1.3 shows that $f_n \triangleq f_n^+ - f_n^-$ is also measurable if f is measurable. Tedious efforts shows that f_n is also simple function, since it is the result of addition of two simple function.

8.2 Sequence of Measurable Function

Abstract

In this section Z always stands for some null set.

We first prove an important estimation for almost everywhere convergence sequence on finite domain, which we will later repeatedly use in this section.

Theorem 8.2.1. (Estimation for almost everywhere convergence sequence on domain of finite measure) Given some

(a) $E \subseteq \mathbb{R}^d$ of finite measure.

(b) A sequence $f_n : E \to [-\infty, \infty]$ of functions measurable on E converge almost everywhere to some $f : E \to \mathbb{R}$.

For all $\epsilon, \eta > 0$, there exists some closed $F \subseteq E$ and integer N such that

$$|E \setminus F| < \eta \text{ and } |f(\mathbf{x}) - f_n(\mathbf{x})| < \epsilon \text{ for all } \mathbf{x} \in F, n > N.$$

Proof. For each n, define

$$E_n \triangleq \bigcap_{k>n} \{|f - f_k| < \epsilon\}$$

It is clear that E_n is an increasing sequence of measurable set. Let $f_n \to f$ on $E \setminus Z$. Because f is finite, we have $E_n \nearrow E \setminus Z$. This then give us $|E_n| \nearrow |E \setminus Z| = |E|$. Because E is of finite measure, there exists N such that $|E \setminus E_N| < \frac{\eta}{2}$. Let F be a closed subset of E_N such that $|E_N \setminus F| < \frac{\eta}{2}$. It is clear that F and N suffices. ■

Given some measurable subset $E \subseteq \mathbb{R}^d$, a sequence of functions $f_n : E \to [-\infty, \infty]$ measurable on E, function $f : E \to \mathbb{R}$ measurable on E, we say f_n **converge to f on E in measure** and write

$$f_n \xrightarrow{m} f \text{ on } E$$

if for all ϵ, we have

$$|\{\mathbf{x} \in E : |f(\mathbf{x}) - f_n(\mathbf{x})| > \epsilon\}| \to 0 \text{ as } n \to \infty$$

Immediately, using our just proved estimation, we see that if $f_n \to f$ almost everywhere on E and E is of finite measure, then f_n also converge to f in measure. This implies that on domain of finite measure, convergence in measure is weaker than almost everywhere convergence; In fact, convergence in measure is strictly weaker. Consider the following example

Example 30 (Convergence in measure on finite domain yet does not converge almost everywhere)

$$I_1 \triangleq [0,1]$$

Let I_2, I_3 be the two halves of $[0,1]$, I_4, I_5, I_6, I_7 be the four quarters, and so on. Define $f_n : [0,1] \to \mathbb{R}$ by $f_n \triangleq \mathbf{1}_{I_n}$. We see f_n diverge everywhere, but $f_n \xrightarrow{m} 0$.

Example 31 (Converge almost everywhere may not converge in measure on domain of infinite measure)

$$f_n \triangleq \mathbf{1}_{B_n(\mathbf{0})}$$

$f_n \to 1$ on $\mathbb{R}^d$, but f_n does not converge to 1 in measure.

We now prove Egorov's Theorem. Although the statement might be daunting, Egorov's Theorem basically says that if sequence of function converges almost everywhere to a finite limit, then the sequence "nearly" uniform converge to its limit.

Theorem 8.2.2. (Egorov's Theorem) Given some

(a) $E \subseteq \mathbb{R}^d$ of finite measure.

(b) A sequence $f_n : E \to [-\infty, \infty]$ of functions measurable on E converge almost everywhere to some $f : E \to \mathbb{R}$.

For all ϵ, there exists some closed set $F \subseteq E$ such that $|E \setminus F| < \epsilon$ and f_n converge uniformly to f on F.

Proof. By Theorem 8.2.1, for all $m \in \mathbb{N}$, we can let $F_m \subseteq E$ and integer N_m satisfy

$$|E \setminus F_m| < \epsilon 2^{-m} \text{ and } |f - f_n| < \frac{1}{m} \text{ on } F_m \text{ for all } n > N_m$$

Define $F \triangleq \bigcap F_m$. It is clear that F is closed and f_n converge uniformly on F. To see F suffices, observe

$$|E \setminus F| = \left| E \setminus \bigcap F_m \right| = \left| \bigcup E \setminus F_m \right| \leq \sum |E \setminus F_m| \leq \sum |E \setminus F_m| < \epsilon$$

■

Theorem 8.2.3. **(Lusin's Theorem)** Given some real-valued function $f : E \to \mathbb{R}$ defined on some measurable set $E \subseteq \mathbb{R}^d$, f is measurable on E if and only if for all ϵ, there exists closed set $F \subseteq E$ such that $|E \setminus F| < \epsilon$ and $f|_F : F \to \mathbb{R}$ is continuous.

Proof. We first prove the only if part, particularly the case of $|E| < \infty$. Let $f_n : E \to \mathbb{R}$ be a sequence of simple measurable function that converge to f. Fix ϵ. By Egorov's Theorem, there exists closed $F_0 \subseteq E$ on which f_n uniformly converge and $|E \setminus F_0| < \frac{\epsilon}{2}$. Let f_n take values $a_{n,1}, \dots, a_{n,K}$ on $E_{n,1}, \dots, E_{n,K}$. Let closed $F_{n,k} \subseteq E_{n,k}$ satisfy $|E_{n,k} \setminus F_{n,k}| < \frac{\epsilon}{K2^{n+1}}$ and define $F_n \triangleq \bigcup_{k=1}^K F_{n,k}$. We claim $F \triangleq F_0 \cap \bigcap_{n=1}^\infty F_n$ suffices.

Observe

$$|E \setminus F| \leq \left| \bigcup_{n=0}^\infty E \setminus F_n \right| \leq \sum_{n=0}^\infty |E \setminus F_n| < \sum_{n=0}^\infty \frac{\epsilon}{2^{n+1}} = \epsilon$$

For each $n \in \mathbb{N}$, because $F_{n,1}, \dots, F_{n,K}$ are disjoint closed sets, we know they are exactly the connected components of F_n. This tell us that f_n is constant on each connected component of F_n. It follows that $f_n|_{F_n} : F_n \to \mathbb{R}$ is continuous, which implies $f_n|_F$ are continuous. To see $f|_F : F \to \mathbb{R}$ is continuous, just observe that f_n uniformly converge to f on $F \subseteq F_0$.

We now prove the case of $|E| = \infty$. Consider $E_n \triangleq E \cap \{\mathbf{x} \in \mathbb{R}^d : n-1 \leq |x| < n\}$. It follows that E is the disjoint union of E_n, and each E_n is of finite measure. It is easy to see that E_n are measurable and f is also measurable on E_n. Then from our earlier result, we can select closed $F_n \subseteq E_n$ such that $|E_n \setminus F_n| < \frac{\epsilon}{2^n}$ and $f|_{F_n} : F_n \to \mathbb{R}$ is continuous. We claim $F \triangleq \bigsqcup_{n=1}^\infty F_n$ suffices. Tedious effort shows that F is indeed closed and F_n are connected components of F. The proof then follows from noting

$$|E \setminus F| = \left| \bigcup_{n=1}^\infty E_n \setminus F_n \right| \leq \sum_{n=1}^\infty |E_n \setminus F_n| = \epsilon$$

Lastly, we prove the if part. For each $n \in \mathbb{N}$, let closed $F_n \subseteq E$ satisfy $|E \setminus F_n| < \frac{1}{n}$ and $f|_{F_n} : F_n \to \mathbb{R}$ is continuous. By our selection, $Z \triangleq E \setminus (\bigcup F_n)$ is null. Now for all $a \in \mathbb{R}$, we see

$$\{\mathbf{x} \in E : f(\mathbf{x}) > a\} = \bigcup_{n=1} \{\mathbf{x} \in F_n : f(\mathbf{x}) > a\} \cup \{\mathbf{x} \in Z : f(\mathbf{x}) > a\}$$

which is measurable, since f is measurable (continuous) on each F_n and $\{\mathbf{x} \in Z : f(\mathbf{x}) > a\}$ is null. ■

8.3 Abstract integration

Abstract

In this section, X is always a measurable space with measure $\mu : \Sigma_X \to [0, \infty]$.

Suppose $s : E \to [0, \infty]$ is a simple function defined on some measurable $E \subseteq X$ taking values $c_1, \dots, c_N$ on $E_1, \dots, E_N$. It is clear that E_n is a finite disjoint decomposition of E, and s is measurable if and only if all E_n are measurable. We can write

$$s = \sum_{n=1}^{N} c_n \mathbf{1}_{E_n}$$

Then if s is measurable, we can define

$$\int_E s d\mu \triangleq \sum_{n=1}^{n} c_n \mu(E_n)$$

We now expand our definitions to non-negative measurable $f : E \to [0, \infty]$. Define

$$\int_E f d\mu \triangleq \sup \int_E s d\mu$$

where $s : E \to [0, \infty]$ runs through all measurable simple functions on E such that $s(x) \leq f(x)$ for all $x \in E$. At this point, one may check that our definition so far is consistent. We now expand our definitions to the class of all measurable functions $f : E \to [-\infty, \infty]$ ranging in $[-\infty, \infty]$. Given some non-negative measurable $g : E \to [0, \infty]$, we say g is **Lebesgue integrable on** E if $\int_E g d\mu < \infty$. Given some measurable $f : E \to [-\infty, \infty]$, we say f is **Lebesgue integrable on** E if

$$\int_E f^+ d\mu < \infty \text{ and } \int_E f^- d\mu < \infty$$

where $f^+(x) \triangleq \max\{f(x), 0\}$ and $f^-(x) \triangleq \min\{f(x), 0\}$, and if $f : E \to [-\infty, \infty]$ is Lebesgue integrable on E, we define

$$\int_E f d\mu \triangleq \int_E f^+ d\mu - \int_E f^- d\mu$$

Again, one may check that our definition is so far consistent. Is shall be clear that function $f : E \to [-\infty, \infty]$ Lebesgue integrable on E forms a vector space over $\mathbb{R}$ and the integral operator is linear. If $f : E_1 \sqcup E_2 \to [-\infty, \infty]$ is measurable, with some tedious effort, we have

$$\int_{E_1 \cup E_2} f d\mu = \int_{E_1} f d\mu + \int_{E_2} d\mu$$

whenever it make sense.

Theorem 8.3.1. (Lebesgue Monotone Convergence Theorem) Let $E \subseteq X$ be measurable, $f_n : E \to [-\infty, \infty]$ be a sequence of measurable function, $f : E \to [-\infty, \infty]$ be a function

(a) If $f_n \nearrow f$ almost everywhere on E and there exists $\varphi \in L(E)$ such that $f_n \geq \varphi$ almost everywhere on E for all $n \in \mathbb{N}$, then $\int_E f_n d\mu \nearrow \int_E f d\mu$.

If $X = \mathbb{R}^d, \Sigma_X = \mathcal{L}(\mathbb{R}^d)$ and μ is the Lebesgue measure, one can equivalently define the Lebesgue integral of measurable $f : E \to [0, \infty]$ by

$$\int_E f d\mu \triangleq |R(f, E)| \text{ where } R(f, E) \triangleq \{(\mathbf{x}, y) \in \mathbb{R}^{d+1} : \mathbf{x} \in E \text{ and } y \in [0, f(\mathbf{x}))\}$$

To see that this definition make sense, i.e., $R(f, E)$ is Lebesgue measurable, consider a sequence of measurable simple function $s_n : E \to \mathbb{R}_0^+$ such that $s_n \nearrow f$ on E. Because Cartesian product of measurable set must be measurable, we know that $R(s_n, E)$ is measurable. It then follows from $R(s_n, E) \nearrow R(f, E)$ that $R(f, E)$ is also measurable. Tedious effort shows that two definitions gives the same values for simple functions s_n

We now expand our definitions to the class of all measurable functions range in $[-\infty, \infty]$, but before such, we have to first introduce the idea of **Lebesgue integrable**. For either s or f, range in either $[0, \infty)$ or $[0, \infty]$, it is always possible that $\int_E s \text{ or } f d\mu = \infty$.

If we have $\int_E f d\mu = \infty$, we say f is **not Lebesgue integrable**, and if $\int_E f d\mu < \infty$, we say f is **Lebesgue integrable on** E. Because $\mathbb{R}$ is complete in order, we know f is either Lebesgue integrable or not Lebesgue integrable on E.

Given a function range in $[-\infty, \infty]$ and measurable on E, if both f^+, f^- are Lebesgue integrable on E, we say f is Lebesgue integrable on E, and write

$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu$$

If either f^+ or f^- are not Lebesgue integrable on E, we say f is not Lebesgue integrable on E.

Notice that

$$f \text{ is Lebesgue integrable on } E \iff \int_E |f| \, d\mu < \infty$$

It is clear that our definition is again so far consistent.

We now

8.4 Basic property of abstract integration

This section prove some basic properties of Lebesgue integral over general measure space (X, Σ_X, μ). From now when we use the notation X, it shall be understood X is equipped with an σ-algebra Σ_X and a measure μ. We will prove

(a) Lebesgue Monotone Convergence Theorem (Theorem 8.4.1)

(b) Fatou's Lemma (Theorem 8.4.3)

(c) Reverse Fatou's Lemma (Theorem 8.4.4)

(d) Dominated Convergence Theorem (Theorem 8.4.5)

Theorem 8.4.1. (Lebesgue Monotone Convergence Theorem) Given a sequence of measurable $f_n : X \to [0,\infty]$ such that $\{f_n(x)\}_{n\in\mathbb{N}}$ is an increasing sequence for each $x \in X$, then

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu$$

Proof. f is measurable by Corollary **??**. Because $f_n \nearrow f$ on X, we know

$$\lim_{n\to\infty} \int_X f_n d\mu = \sup_n \int_X f_n d\mu \leq \int_X f d\mu$$

It remains to prove

$$\int_X f d\mu \leq \lim_{n\to\infty} \int_X f_n d\mu$$

Fix simple $0 \leq s \leq f$ on X. We reduce the problem into proving

$$\int_X s d\mu \leq \lim_{n\to\infty} \int_X f_n d\mu$$

Fix $c \in (0,1)$. We reduce the problem into proving

$$c \int_X s d\mu \leq \lim_{n\to\infty} \int_X f_n d\mu$$

Define

$$E_n \triangleq \{x \in X : f_n(x) \geq cs(x)\}$$

E_n are measurable because $f_n - cs$ are measurable. Now because f_n are non-negative on X, we have

$$\int_X f_n d\mu \geq \int_{E_n} f_n d\mu \geq c \int_{E_n} s d\mu$$

Taking limit, we see

$$\lim_{n\to\infty}\int_X f_n d\mu \geq \lim_{n\to\infty} c\int_{E_n} s d\mu$$

It is straightforward to check E_n is increasing and $\bigcup E_n = X$. Then if we decompose $s=\sum_j c_j \mathbf{1}_{F_j}$, by Theorem **??**, we can take limit

$$\lim_{n\to\infty}\mu(F_j\cap E_n)=\mu(F_j)$$

It then follows that

$$\lim_{n\to\infty}\int_X f_n d\mu \geq \lim_{n\to\infty} c\int_{E_n} s d\mu = c\int_X s d\mu \text{ (done)}$$

■

It is worth pointing out in our proof for Lebesgue Monotone Convergence Theorem, instead of proving $\int_X s d\mu \leq \lim_{n\to\infty}\int_X f_n d\mu$, we proved $c\int_X s d\mu \leq \lim_{n\to\infty}\int_X f_n d\mu$. Multiplying $\int_X s d\mu$ with $c\in(0,1)$ is not just a random limit technique. Our action play a much more profound role. Consider the Example.

Example 32 (Why we take $c\int_X s d\mu$?)

$$X=[0,1] \text{ and } f_n = 1-\frac{1}{n}$$

We can take $s=f$, and see $E_n=\varnothing$ for all n, which renders our proceeding proof invalid.

Corollary 8.4.2. (Monotone Convergence Theorem for general functions) Given a sequence of measurable $f_n : X\to[0,\infty]$ such that

(a) $\{f_n(x)\}_{n\in\mathbb{N}}$ is an increasing sequence on N^c

(b) $f : X\to[0,\infty]$ is the limit of f_n on N^c

(c) $\mu(N)=0$

(d) μ is complete

We have

$$\int_X f d\mu = \lim_{n\to\infty}\int_X f_n d\mu$$

Proof. Let

$$g(x) \triangleq \begin{cases} f(x) & \text{if } x \in N^c \\ 0 & \text{if } x \in N \end{cases}$$

Note that

$$\int_X f d\mu = \int_{N^c} f d\mu = \lim_{n\to\infty} \int_{N^c} f d\mu = \lim_{n\to\infty} \int_X f d\mu$$

■

Theorem 8.4.3. (Fatou's Lemma) Given measurable $f_n : X \to [0, \infty]$

$$\int_X \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu$$

Proof. Since $\inf_{k\geq n} f_k \leq f_n$ for each n, x, we see

$$\int_X \inf_{k\geq n} f_k d\mu \leq \int_X f_n d\mu \text{ for all } n$$

Because $\inf_{k\geq n} f_k \nearrow \liminf_{m\to\infty} f_m$ as $n \to \infty$, by (Theorem 8.4.1: Lebesgue Monotone Convergence Theorem), we can take limit

$$\int_X \liminf_{n\to\infty} f_n d\mu = \lim_{n\to\infty} \int_X \inf_{k\geq n} f_k d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu$$

■

Example 33 (Fatou's Lemma strict inequality)

$$f_{2k}(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}] \\ 1 & \text{if } x \in (\frac{1}{2}, 1] \end{cases} \text{ and } f_{2k+1}(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}) \\ 0 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

From now, we use $L^1(\mu)$ to denote the set of all function defined and μ-Lebesgue-integrable on X, and we say a sequence of function $f_n : X \to [0, \infty]$ is **dominated** by g, if g is a $[0, \infty]$-valued function defined on X such that

$$\sup_n |f_n(x)| \leq g(x) \text{ for all } x \in X$$

Theorem 8.4.4. (Reverse Fatou's Lemma) Given measurable $f_n : X \to [0, \infty]$ dominated by some $g \in L^1(\mu)$

$$\limsup_{n\to\infty} \int_X f_n d\mu \leq \int_X \limsup_{n\to\infty} f_n d\mu$$

Proof. By (Theorem 8.4.3: Fatou Lemma)

$$\int_X \liminf_{n\to\infty}(g-f_n)d\mu \leq \liminf_{n\to\infty}\int_X (g-f_n)d\mu$$

Multiplying both side with -1, we have

$$\limsup_{n\to\infty}\int_X (f_n-g)d\mu \leq \int_X \limsup_{n\to\infty}(f_n-g)d\mu$$

Then adding both side the constant $\int_X gd\mu$, we reach to the conclusion. ■

Note that in our proof above, when we "pull" the negative sign out from $\liminf_{n\to\infty}$, it changed to $\limsup_{n\to\infty}$. This is a standard technique, which can be justified using the sub-sequence definition of limit superior.

Theorem 8.4.5. (Dominate Convergence Theorem) Given a sequence $f_n : X \to \mathbb{C}\cup\{\infty\}$ of measurable function such that

$$f \triangleq \lim_{n\to\infty} f_n \text{ exists on } X$$

If there exists $g \in L^1(\mu)$ dominating f_n, then

$$\lim_{n\to\infty}\int_X |f_n-f|\,d\mu = 0 \text{ and } \int_X fd\mu = \lim_{n\to\infty}\int_X f_n d\mu \text{ exists in } \mathbb{C}\cup\{\infty\}$$

Proof. Because f is measurable by Corollary **??** and $|f| \leq g$ on X, $f \in L^1(\mu)$.

We first prove

$$\lim_{n\to\infty}\int_X |f_n-f|\,d\mu = 0$$

We relax the problem into proving

$$\limsup_{n\to\infty}\int_X |f_n-f|\,d\mu = 0$$

Note that $|f_n-f| \leq 2g$. We can now apply (Theorem 8.4.3: Fatou lemma) to $2g-|f_n-f|$ and see

$$\begin{aligned}\int_X 2gd\mu &= \int_X \lim_{n\to\infty}(2g-|f_n-f|)d\mu \\ &\leq \liminf_{n\to\infty}\int_X (2g-|f_n-f|)d\mu \\ &= \int_X 2gd\mu + \liminf_{n\to\infty}\Big(-\int_X |f_n-f|\,d\mu\Big) \\ &= \int_X 2gd\mu - \limsup_{n\to\infty}\int_X |f_n-f|\,d\mu\end{aligned}$$

Then because $g \in L^1(\mu)$, we can subtract it and obtain

$$\limsup_{n\to\infty} \int_X |f_n - f| \, d\mu = 0 \text{ (done)}$$

It then follows that

$$\limsup_{n\to\infty} \left| \int_X (f_n - f) d\mu \right| \leq \limsup_{n\to\infty} \int_X |f_n - f| \, d\mu = 0$$

which implies

$$\lim_{n\to\infty} \int_X (f_n - f) d\mu = 0$$

and because $f \in L^1(\mu)$, we have

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$$

■

Example 34 (Counterexample for Dominate Convergence Theorem)

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } |x| < n \\ 0 & \text{if } |x| \geq n \end{cases}$$

8.5 Change of Variables Formula

Theorem 8.5.1. (Linear Transformation and Lebesgue Measure) Given a vector space homomorphism $T : \mathbb{R}^d \to \mathbb{R}^d$ and a Lebesgue measurable set $E \subseteq \mathbb{R}^d$, we have

$$|T(E)| = |\det(T)| \cdot |E|$$

Chapter 9

Harmonic Analysis

9.1 Weierstrass approximation Theorem: $[a, b] \to \mathbb{R}$

Theorem 9.1.1. (Bernoulli's Inequality) Given $r, x \in \mathbb{R}$, suppose

(a) $r \geq 1$

(b) $x \geq -1$

Then

$$(1+x)^r \geq 1 + rx$$

Proof. Fix $r \geq 1$. We wish

$$\text{to prove } (1+x)^r \geq 1 + rx \text{ for all } x \geq -1$$

Define $f : [-1, \infty) \to \mathbb{R}$ by

$$f(x) = (1+x)^r - (1 + rx) \tag{9.1}$$

We reduced the problem into

$$\text{proving } f(x) \geq 0 \text{ for all } x \geq -1$$

Because $r \geq 1$ by premise, by definition of $f(x)$ (Equation 9.1), we see that

$$f(0) = 0, \text{ and } f(-1) = r - 1 \geq 0$$

Notice that by definition of f (Equation 9.1), $f(x)$ is clearly differentiable on $(-1, \infty)$.

Then, by MVT (Theorem 5.3.3), to prove $f(x) \geq 0$ on $(-1, \infty)$, we only wish

$$\text{to prove } f'(x) \geq 0 \text{ for all } x > 0 \text{ and } f'(x) \leq 0 \text{ for all } x \in (-1, 0)$$

Compute f'

$$\begin{aligned} f'(x) &= r(1+x)^{r-1} - r \\ &= r\Big((1+x)^{r-1} - 1\Big) \end{aligned}$$

Because $r \geq 1$, we can deduce

$$x > 0 \implies (1+x)^{r-1} \geq 1 \implies f'(x) = r\Big((1+x)^{r-1} - 1\Big) \geq 0$$

and deduce

$$x \in (-1,0) \implies 1+x \in (0,1) \implies (1+x)^{r-1} \leq 1 \implies f'(x) = r\Big((1+x)^{r-1} - 1\Big) \leq 0$$

(done) ■

In this section, notation $\mathcal{C}\big([a,b]\big)$ means the set of **real-valued continuous function on** $[a,b]$.

Theorem 9.1.2. (Weierstrass approximation Theorem: $[a,b] \to \mathbb{R}$**)** Let $\mathbb{R}[x]\big|_{[a,b]}$ be the space of polynomials on $[a,b]$ with real coefficient. We have

$$\mathbb{R}[x]\big|_{[a,b]} \text{ is dense in } \Big(\mathcal{C}\big([a,b]\big), \|\cdot\|_\infty\Big)$$

Proof. WOLG, we can let $[a,b] = [0,1]$. The reason we can assume such is explained at last. Now, let $f : [0,1] \to \mathbb{R}$ be a continuous function. Fix ϵ. We only wish

$$\text{to find } P \in \mathbb{R}[x]\big|_{[0,1]} \text{ such that } \|f - P\|_\infty < \epsilon$$

Define $\tilde{f} \in \mathcal{C}\big([0,1]\big)$ by

$$\tilde{f}(x) = f(x) - f(0) - x\big[f(1) - f(0)\big] \tag{9.2}$$

It is easy to check $\tilde{f}$ is continuous. We first prove that

$$\big(\tilde{f}(x) - f(x)\big) \in \mathbb{R}[x]\big|_{[0,1]}$$

By definition of $\tilde{f}$ (Equation 9.2), we see

$$\tilde{f}(x) - f(x) = (f(0) - f(1))x - f(0) \in \mathbb{R}[x]\big|_{[0,1]} \text{ (done)}$$

This reduce our problem into

$$\text{finding } P \in \mathbb{R}[x]\big|_{[0,1]} \text{ such that } \|\tilde{f} - P\|_\infty < \epsilon$$

Notice that by definition of $\tilde{f}$ (Equation 9.2), we have

$$\tilde{f}(0) = 0 = \tilde{f}(1)$$

Then, we can expand the definition of $\tilde{f}$ by

$$\tilde{f}(x) = \begin{cases} \tilde{f}(x) & \text{if } x \in [0,1] \\ 0 & \text{if } x \notin [0,1] \end{cases} \tag{9.3}$$

This makes $\tilde{f}$ uniformly continuous on $\mathbb{R}$, since $\tilde{f}$ is uniformly continuous on $[0,1]$ and $[0,1]^c$. Now, for each $n \in \mathbb{N}$, define $Q_n \in \mathbb{R}[x]$ by

$$Q_n = c_n(1-x^2)^n \text{ where } c_n \text{ is chosen to satisfy } \int_{-1}^{1} Q_n(x)dx = 1 \tag{9.4}$$

Define $P_n : [0,1] \to \mathbb{R}$ by

$$P_n(x) = \int_{-1}^{1} \tilde{f}(x+t)Q_n(t)dt$$

We now prove

$$P_n \in \mathbb{R}[x]\big|_{[0,1]}$$

Because $\tilde{f}(x) = 0$ for all $x \notin (0,1)$ by definition of $\tilde{f}$ (Equation 9.3), we see that

$$P_n(x) = \int_{-x}^{1-x} \tilde{f}(x+t)Q_n(t)dt \text{ for all } x \in [0,1] \tag{9.5}$$

Fix $x \in [0,1]$. Now, by change of variable, we see

$$P_n(x) = \int_{-x}^{1-x} \tilde{f}(x+t)Q_n(t)dt = \int_0^1 \tilde{f}(u)Q_n(u-x)du$$

Because Q_n is a polynomial by definition (Equation 9.4), we can express $Q_n(u-x)$ by

$$Q_n(u-x) = \sum_{k=0}^{m} a_k x^k \text{ for some } \{a_0, \dots, a_m\} \text{ depending on } u$$

Then we see

$$P_n(x) = \int_0^1 \tilde{f}(u)Q_n(u-x)du = \sum_{k=0}^{m} x^k \Big(\int_0^1 \tilde{f}(u)a_k du \Big)$$

This shows that $P_n \in \mathbb{R}[x]\big|_{[0,1]}$
(done)

Now, because $\tilde{f}$ is uniformly continuous on $\mathbb{R}$, we can fix $\delta < 1$ such that

$$\forall x, y \in \mathbb{R}, |x-y| < \delta \implies \left|\tilde{f}(x) - \tilde{f}(y)\right| < \frac{\epsilon}{2} \tag{9.6}$$

By definition of $\tilde{f}$ (Equation 9.3), we know $\tilde{f}$ is a bounded function. Then we can set M by

$$M = \sup_{x\in\mathbb{R}} |f(x)|$$

Let n satisfy

$$4M\sqrt{n}(1-\delta^2)^n < \frac{\epsilon}{2} \tag{9.7}$$

Such n exists, because $\delta < 1 \implies \sqrt{n}(1-\delta^2)^n \to 0$. We claim

$$P_n \text{ satisfy } \|\tilde{f} - P_n\|_\infty < \epsilon$$

We first prove

$$c_n < \sqrt{n}$$

By Bernoulli's Inequality (Theorem 9.1.1). Compute

$$\begin{aligned}
1 = \int_{-1}^{1} Q_n(x)dx &= c_n \int_{-1}^{1} (1-x^2)^n dx \\
&= 2c_n \int_0^1 (1-x^2)^n dx \\
&\geq 2c_n \int_0^{\frac{1}{\sqrt{n}}} (1-x^2)^n dx \\
&\geq 2c_n \int_0^{\frac{1}{\sqrt{n}}} 1 - nx^2 dx = c_n(\frac{4}{3\sqrt{n}}) > c_n(\frac{1}{\sqrt{n}})
\end{aligned}$$

This implies

$$\sqrt{n} > c_n \text{ (done)}$$

Because $\sqrt{n} > c_n$, by definition of Q_n (Equation 9.4), we have

$$Q_n(x) < \sqrt{n}(1-x^2)^n \leq \sqrt{n}(1-\delta^2)^n \text{ for all } x \text{ such that } \delta \leq |x| \leq 1$$

Fix $x \in [0,1]$. Finally, because

(a) $\int_{-1}^{1} Q_n(x)dx = 1$ by definition of Q_n (Equation 9.4)

(b) $Q_n(x) = c_n(1-x^2)^n \geq 0$ for all $x \in [-1,1]$

(c) $\left|\tilde{f}(x+t) - \tilde{f}(x)\right| < \frac{\epsilon}{2}$ for all t such that $|t| < \delta$, by definition of δ (Equation 9.7)

(d) $Q_n(x) \leq \sqrt{n}(1-\delta^2)^n$ for all x such that $\delta \leq |x| \leq 1$

(e) $4M\sqrt{n}(1-\delta^2)^n < \frac{\epsilon}{2}$ by definition of n (Equation 9.7)

we have

$$\begin{aligned}
\left|P_n(x) - \tilde{f}(x)\right| &= \left|\int_{-1}^{1} \tilde{f}(x+t)Q_n(t)dt - \tilde{f}(x)\right| \\
&= \left|\int_{-1}^{1} \tilde{f}(x+t)Q_n(t)dt - \tilde{f}(x)\int_{-1}^{1} Q_n(t)dt\right| \\
&= \left|\int_{-1}^{1} \tilde{f}(x+t)Q_n(t)dt - \int_{-1}^{1} \tilde{f}(x)Q_n(t)dt\right| \\
&= \left|\int_{-1}^{1} \left[\tilde{f}(x+t) - \tilde{f}(x)\right]Q_n(t)dt\right| \\
&\leq \int_{-1}^{1} \left|\left[\tilde{f}(x+t) - \tilde{f}(x)\right]Q_n(t)\right| dt \\
&= \int_{-1}^{1} \left|\tilde{f}(x+t) - \tilde{f}(x)\right| Q_n(t)dt \\
&\leq \int_{-1}^{-\delta} 2MQ_n(t)dt + \int_{-\delta}^{\delta} \left|\tilde{f}(x+t) - \tilde{f}(x)\right| Q_n(t)dt + \int_{\delta}^{1} 2MQ_n(t)dt \\
&\leq 2M\Big(\int_{-1}^{-\delta} Q_n(t)dt + \int_{\delta}^{1} Q_n(t)dt\Big) + \int_{-\delta}^{\delta} (\frac{\epsilon}{2})Q_n(t)dt \\
&\leq 4M(1-\delta)\sqrt{n}(1-\delta^2)^n + \frac{\epsilon}{2} \\
&\leq 4M\sqrt{n}(1-\delta^2)^n + \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

Because x is arbitrarily picked from $[0,1]$, we now have $\|P_n - \tilde{f}\|_\infty < \epsilon$ (done)

Lastly, we show

our result can be transplanted to arbitrary $\mathcal{C}\big([a,b]\big)$

Let $[a,b]$ be arbitrary. Fix ϵ and $f \in \mathcal{C}\big([a,b]\big)$. We wish

to find $P \in \mathbb{R}[x]\big|_{[a,b]}$ such that $\|f - P\|_\infty \leq \epsilon$

Define $g : [0,1] \to \mathbb{R}$ by

$$g(x) \triangleq f(a + (b-a)x) \tag{9.8}$$

We know there exists $P_n : [0,1] \to \mathbb{R}$ such that

$$\|P_n - g\|_\infty < \epsilon$$

Define $H_n : [a,b] \to \mathbb{R}$ by

$$H_n(x) = P_n(\frac{x-a}{b-a})$$

Because P_n is a real polynomial on $[0,1]$, we know H_n is a real polynomial on $[a,b]$. We now claim

$$\text{such } H_n \text{ works}$$

Fix $x \in [a,b]$. Observe

$$\begin{aligned} \big|f(x) - H_n(x)\big| &= \left|f(x) - P_n(\frac{x-a}{b-a})\right| \\ &= \left|g(\frac{x-a}{b-a}) - P_n(\frac{x-a}{b-a})\right| < \epsilon \text{ (done)} \end{aligned}$$

■

It is at now, we will show that every real-valued continuous functions on $[a,b]$ can be approximated by polynomials with rational coefficient. This fact enable our computer to more easily approximate real-valued continuous function on $[a,b]$.

Note that since $\mathcal{C}\big([a,b]\big)$ is a separable metric space, we can show that $\mathcal{C}\big([a,b]\big)$ has cardinality of at most continuum $\mathfrak{c}$.

Theorem 9.1.3. (The space $\mathbb{Q}[x]\big|_{[a,b]}$ is dense in $\Big(\mathcal{C}\big([a,b]\big), \|\cdot\|_\infty\Big)$, thus $\mathcal{C}\big([a,b]\big)$ is separable)

$$\Big(\mathcal{C}\big([a,b]\big), \|\cdot\|_\infty\Big) \text{ is separable}$$

Proof. Because $\mathbb{Q}[x]\big|_{[a,b]}$ is countable, to show $\mathcal{C}\big([a,b]\big)$ is separable, we only wish to show

$$\mathbb{Q}[x]\big|_{[a,b]} \text{ is dense in } \mathcal{C}\big([a,b]\big)$$

Because $\mathbb{R}[x]\big|_{[a,b]}$ is dense in $\mathcal{C}\big([a,b]\big)$, we reduce our problem into proving

$$\mathbb{Q}[x]\big|_{[a,b]} \text{ is dense in } \mathbb{R}[x]\big|_{[a,b]}$$

Fix ϵ and $P \in \mathbb{R}[x]\big|_{[a,b]}$. We must

$$\text{find } Q \in \mathbb{Q}[x]\big|_{[a,b]} \text{ such that } \|Q - P\|_\infty \leq \epsilon$$

Express $P(x) = \sum_{k=0}^n r_k x^k$. Let $M > \max\{|a|, |b|\}$. Because $\mathbb{Q}$ is dense in $\mathbb{R}$, we know there exists $c_k \in \mathbb{Q}$ such that $|c_k - r_k| < \frac{\epsilon}{(n+1)M^n}$. We claim

$$Q(x) = \sum_{k=0}^n c_k x^k \text{ works}$$

Fix $x \in [a, b]$. See

$$\begin{aligned}
|P(x) - Q(x)| &= \left|\sum_{k=0}^n (c_k - r_k) x^k\right| \\
&\leq \sum_{k=0}^n |c_k - r_k| \cdot |x|^k \\
&\leq \sum_{k=0}^n |c_k - r_k| \cdot M^k \\
&\leq (M^n) \sum_{k=0}^n |c_k - r_k| \\
&< M^n (n+1) \Big(\frac{\epsilon}{(n+1)M^n}\Big) = \epsilon \text{ (done)}
\end{aligned}$$

■

9.2 The Stone-Weierstrass Theorem

Recall that a **vector space over a field** $\mathbb{F}$ is a set V equipped with **vector addition** $+ : V \times V \to V$ and **scalar multiplication** such that

(a) $(V, +)$ is an abelian group.

(b) Scalar multiplication is compatible with field multiplication: $\Big((ab)v = a(bv)\Big)$

(c) Scalar multiplication is distributive: $\Big((a+b)v = av + bv \text{ and } a(v+w) = av + aw\Big)$

There are many ways to define the term **algebra over a field** $\mathbb{F}$. One can exhaust all the laws an algebra should obey. In short, an **algebra over a field** $\mathbb{F}$ (or $\mathbb{F}$**-algebra**) is a vector space V equipped with a vector multiplication such that

(i) Multiplication is <u>distributive</u> to addition.

(ii) Scalar and vector multiplications are compatible: $(av) \cdot (bw) = ab(v \cdot w)$

Given an arbitrary set E and a field $\mathbb{F}$, let A be the set of all functions from E to $\mathbb{F}$. The following is a list of some algebra

(a) $(\mathbb{R}^3, \text{cross product})$ over $\mathbb{R}$

(b) $(\mathbb{C}, \text{complex multiplication})$ over $\mathbb{C}$

(c) $(\mathbb{Q}[x], \text{function multiplication})$ over $\mathbb{Q}$

(d) $(\text{Functions from } E \text{ to } \mathbb{F}, \text{function multiplication})$ over $\mathbb{F}$

(e) $(\text{Continuous functions from } (E, \tau) \text{ to } \mathbb{C}\ , \text{function multiplication})$ over $\mathbb{C}$

(f) (Linear transformation from V to V, composition) over $\mathbb{F}$ where V is over $\mathbb{F}$

(g) $\big(M_n(\mathbb{F}), \text{matrix multiplication}\big)$ over $\mathbb{F}$

Note that $B = (\text{continuous functions from } \mathbb{C} \text{ to } \mathbb{C}, \text{composition})$ over $\mathbb{C}$ is not an algebra, even though B is both a vector space and a ring. ($\because$ scalar multiplication and multiplication are not compatible).

It is at here we shall introduce some general terminologies. Given an arbitrary set E, a field $\mathbb{F}$ and a point $x \in E$, we say a family $\mathcal{F}$ of functions from E to $\mathbb{F}$ **vanish at** x if for all $f \in \mathcal{F}$, we have $f(x) = 0$. We say $\mathcal{F}$ **separate points** in E if for all $x_2 \neq x_1 \in E$, there exists $f \in \mathcal{F}$ such that $f(x_2) \neq f(x_1)$.

Chapter 10

Differential Geometry

10.1 Smooth Manifold

Abstract

This main goal of this section is to

(a) introduce the idea of smooth manifolds.

(b) prove that smooth manifolds always admit smooth partition of unity.

Given a topological space $(X, \mathscr{T})$ and fix n, if $: U \to \mathbb{R}^n$ is a homeomorphism between some open subspace U of X and some open subspace of $\mathbb{R}^n$, we say (Φ, U) is a **chart** on X, and if there exists a collection $A = \{(\Phi_i, U_i)\}_{i \in I}$ of chart cover the whole X, we say A is an **atlas** and X is **locally Euclidean**. We now give definition to the term **topological manifold**, which we will use throughout this chapter.

Definition 10.1.1. (Definition of Topological Manifold) We say a Topological space X is a **topological manifold** if

(a) X is locally Euclidean.

(b) X is Hausdorff.

(c) X is second countable.

Immediately, one should check that all three conditions are necessary: There are locally Euclidean space that is not Hausdorff, Bug-Eyed Line for example. There also are locally Euclidean space that is not second countable, Long Line for example. Also, one can check that any open subspace or finite products of topological manifold is still topological manifold. To have a better understanding why we require more than locally Euclidean in our definition of topological manifol, we first introduce some more topological notions. Given a collection of subsets (E_α) of some topological space X, we say (E_α) is **locally finite** if for each $p \in X$, there exists some neighborhood of p intersecting with only finitely many of E_α. Given an open cover (E_α) of topological space M, we say a family of continuous function $\psi_\alpha : M \to \mathbb{R}$ is a **partition of unity subordinate to (or dominated by)** (E_α) if

(i) $0 \leq \psi_\alpha(x) \leq 1$ for all $\alpha \in A, x \in M$.

(ii) $\operatorname{supp} \psi_\alpha \subseteq X_\alpha$ for all α.

(iii) The collection $\{\operatorname{supp} \psi_\alpha \subseteq M : \alpha \in A\}$ is locally finite.

(iv) $\sum_{\alpha \in A} \psi_\alpha(x) = 1$ for all $x \in M$. (Note that the sum is finite because $\{\operatorname{supp} \psi_\alpha\}$ is locally finite)

Given a cover (E_α) of topological space X, we say another cover (F_β) of X is a **refinement of** E if for each β there exists some α such that $F_\beta \subseteq E_\alpha$. Suppose we have some open cover $(U_\alpha)_{\alpha \in A}$ of some topological space X A topological space X if said to be **paracompact** if every open cover has a locally finite open refinement. It shall be clear that each compact space is paracompact.

Theorem 10.1.2. (Topological Manifold are Paracompact) Every locally compact, Hausdorff second-countable space M is paracompact. Moreover, for each open cover $\mathcal{S}$ and basis $\mathcal{B}$, there exists a countable locally finite open refinement of $\mathcal{S}$ consisting of elements of $\mathcal{B}$.

Proof. We first show that

> M admits an exhaustion by compact sets, i.e., there exists a sequence K_n of compact sets such that $M = \bigcup K_n$ and $K_n \subseteq K_{n+1}^\circ$.

Since M is locally compact and Hausdorff, we know there exists some basis of X consisting of precompact open sets, and since M is second countable, we can WOLG write this basis as (U_n). Let $K_1 \triangleq \overline{U_1}$. Now, since (U_k) cover the whole M, for each n we can let m_n be some integer greater than n and $K_n \subseteq U_1 \cup \cdots \cup U_{m_n}$. Defining $K_{n+1} \triangleq \overline{U_1} \cup \cdots \cup \overline{U_{m_n}}$, we see K_{n+1} is compact because it is a finite union of compact subspace. We also see $K_n \subseteq U_1 \cup \cdots \cup U_{m_n} \subseteq K_{n+1}^\circ$. (done)

Now, for each $n \in \mathbb{Z}_0^+$, define

$$V_n \triangleq K_{n+1} \setminus K_n^\circ \text{ and } W_n \triangleq K_{n+2}^\circ \setminus K_{n-1} \text{ where } K_0 = K_{-1} = \varnothing$$

Note that W_n are open and $V_n \subseteq W_n$. We then can associate each n and $x \in V_n$ with some $S_x^n \in \mathcal{S}$ and $B_x^n \in \mathcal{B}$ such that $x \in B_x^n \subseteq S_x^n \cap W_n$. Now, because V_n is compact (closed in K_{n+1}), we know there exists a finite subcollection $\{B_{x_1}^n, \dots, B_{x_{n_k}}^n\}$ covering V_n and contained by W_n. Define

$$\mathcal{S}' \triangleq \bigcup_{n \in \mathbb{Z}_0^+} \{B_{x_1}^n, \dots, B_{x_{n_k}}^n\}$$

We then see that $\mathcal{S}'$ is a countable refinement of $\mathcal{S}$ (The fact $\mathcal{S}'$ is a cover follows from V_n covering the whole M). To see that $\mathcal{S}'$ is locally finite, observe that each $B_{x_j}^n$ is contained by W_n and if $|p - q| > 2$, then $W_p \cap W_q = \varnothing$. ■

An atlas A is said to be a **smooth atlas**, if for each two charts $(\Phi_i, U_i), (\Phi_j, U_j)$ in A,

The function $\Phi_i \circ \Phi_j^{-1} : \Phi_j(U_i \cap U_j) \to \Phi_i(U_i \cap U_j)$ is a smooth diffeomorphism

It is easily checked that for each two chart Φ_i, Φ_j, the **transition map** $\Phi_i \circ \Phi_j^{-1}$ is a homeomorphism. Now, if the union of two smooth atlas A_1, A_2 is again smooth, we say A_1, A_2 are **compatible**. With some effort, one can check that compatibility is an equivalence relation on the collection of all possible atlas on X. Thus, it make sense for us to define the **smooth structure**, i.e., a maximal smooth atlas on X. Now, by a **smooth manifold**, we merely mean a manifold equipped with a maximal smooth atlas, and given a function $F : M \to N$ that maps a smooth manifold M into another smooth manifold N, we say F is **smooth at** p if F is continuous at p and there exists some charts $(U, \varphi), (V, \psi)$ respectively at $p, F(p)$ such that

(i) $U \subseteq F^{-1}(V)$.

(ii) $\psi \circ F \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^m \to \psi(V) \subseteq \mathbb{R}^n$ is smooth at $\varphi(p)$.

Note that F is required to be continuous at p in the first place to guarantee that for all (V, ψ) at $F(p)$ there exists some (U, φ) at p such that $F(U) \subseteq V$. Immediately, one can check that if F is smooth at p, then

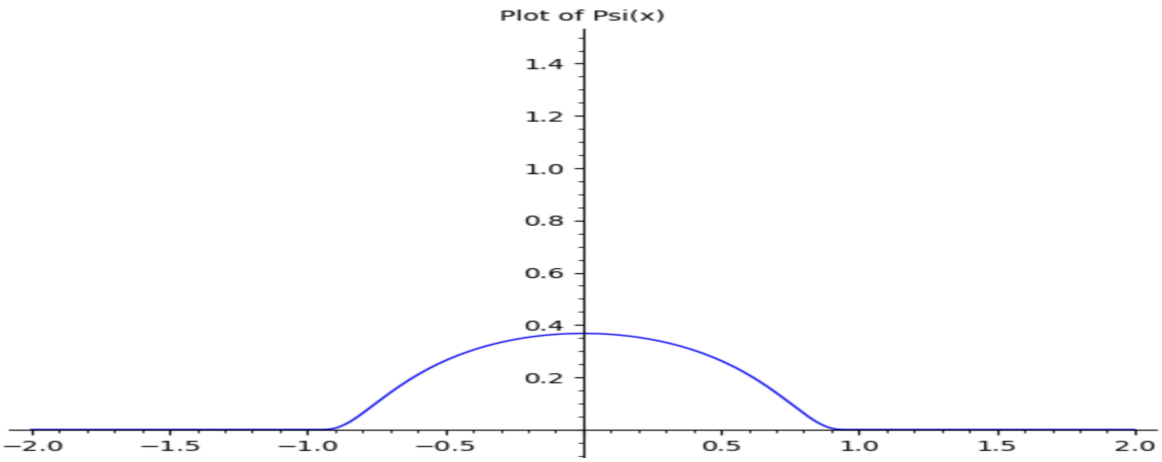
(a) For all charts $(U, \varphi), (V, \psi)$ at $p, F(p)$, the function $\psi \circ F \circ \varphi^{-1} : \varphi(F^{-1}(V) \cap U) \subseteq \mathbb{R}^m \to \psi(V) \subseteq \mathbb{R}^n$ is smooth at p.

(b) If $G : N \to R$ is another map smooth at $F(p)$, then $G \circ F : M \to R$ is smooth at p.

When F is a $\mathbb{R}^n$-valued function on M, we define the smoothness of F by considering $\mathbb{R}^n$ as a manifold with the standard atlas $\{(\mathbb{R}^n, \mathbf{id})\}$. With these definitions specified, we are almost ready to prove that smooth manifolds always admits smooth partition of unity, but before we actually give a proof, we first need to know how to manipulate smooth function between Euclidean Spaces. The simplest smooth function $\Psi : \mathbb{R} \to \mathbb{R}$ is perhaps

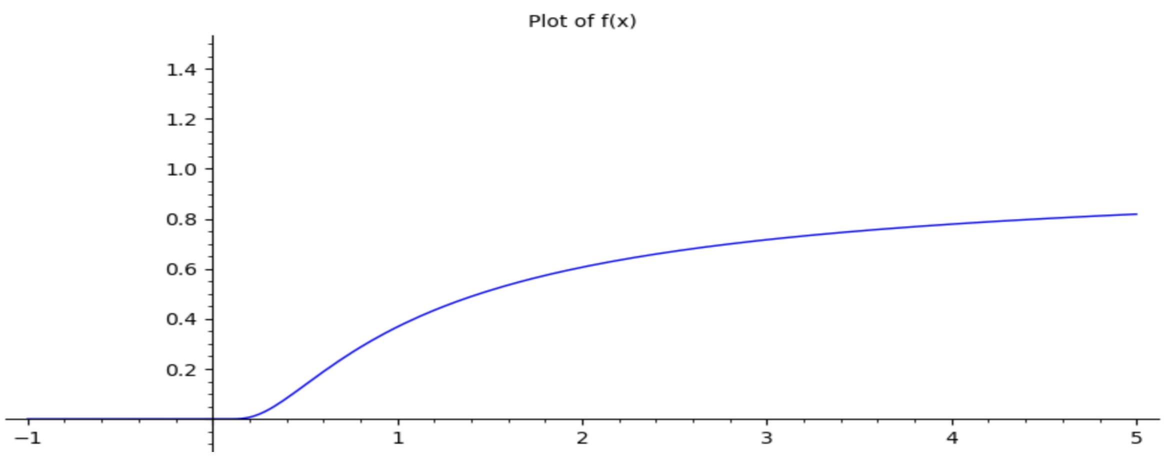
$$\Psi(x) \triangleq \begin{cases} e^{\frac{-1}{1-x^2}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

Although the Ψ we just defined is smooth, it is not particularly useful in construction of partition of unity.



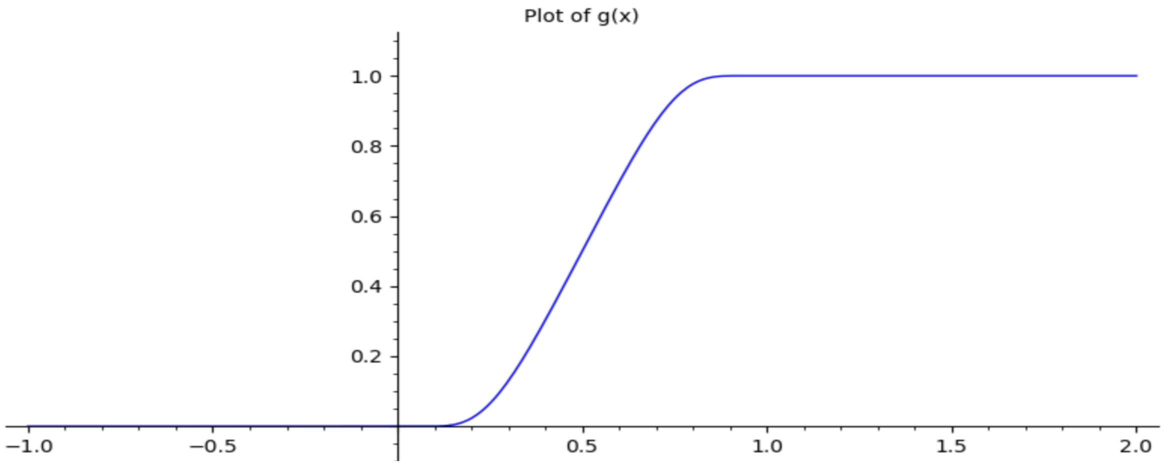
Consider the smooth function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) \triangleq \begin{cases} e^{\frac{-1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$



and $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) \triangleq \frac{f(x)}{f(x) + f(1-x)} \tag{10.1}$$

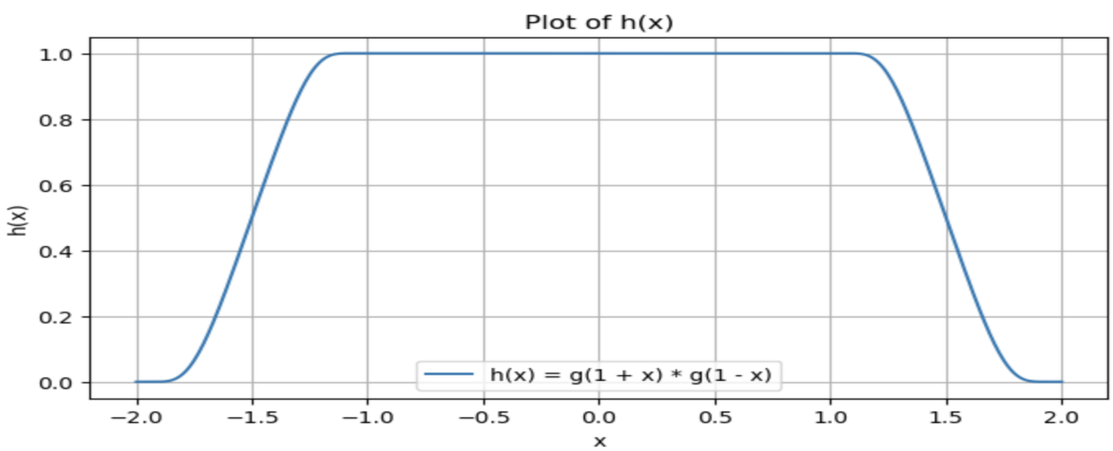


which is a smooth increasing function such that

$$g(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x \geq 1 \end{cases} \quad \text{and } g(x) \in (0,1) \text{ if } x \in (0,1)$$

Lastly, if we define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) \triangleq g(2+x)g(2-x)$$



We see that h is a smooth function and

$$h(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| \geq 2 \end{cases} \quad \text{and } h(x) \in (0,1) \text{ if } 1 < |x| < 2$$

A n-dimensional generalization of this smooth function h is $H : \mathbb{R}^n \to \mathbb{R}$ defined by

$$H(\mathbf{x}) \triangleq h\Big(\frac{|\mathbf{x}|}{r}\Big)$$

which is a non-negative smooth function such that

$$H(\mathbf{x}) > 0 \text{ if and only if } |\mathbf{x}| < 2r$$

Theorem 10.1.3. (Smooth manifolds always admit smooth partition of unity) Given a smooth manifold M and some open cover (U_α) of M, there exists some smooth partition of unity (ψ_α) subordinate to (U_α).

Proof. For each U_α and each chart (V, φ) contained by U_α, if we let $\mathcal{B}_{\alpha,V}$ be the collection of the pre-images of the open balls $B_r(x)$ in $\varphi(V)$ that satisfy $\exists r' > r : B_{r'}(x) \subseteq \varphi(V)$, we know that $\mathcal{B}_{\alpha,V}$ form a basis of V. It follows that there exists a basis $\mathcal{B}_\alpha$ for U_α of the form $\mathcal{B}_\alpha \triangleq \bigcup_{(V,\varphi)\subseteq U_\alpha} \mathcal{B}_{\alpha,V}$ and a basis $\mathcal{B}$ for M of the form $\mathcal{B} \triangleq \bigcup_\alpha \mathcal{B}_\alpha$. By Theorem 10.1.2, we know there exists a countable locally finite open refinement (B_n) of U_α consisting of element of $\mathcal{B}$.

Now, for each n, there exists corresponding chart (V, φ) and r, r' such that

$$\varphi(B_n) = B_r(x) \text{ and } \overline{B_r(x)} \subseteq B_{r'}(x) \subseteq \varphi(V)$$

This with some tedious effort guarantee that

$$\overline{B_n} = \varphi^{-1}(\overline{B_r(x)})$$

We then can well-define a function $f_n : M \to \mathbb{R}$ by

$$f_n \triangleq \begin{cases} H_n \circ \varphi & \text{on } V \\ 0 & \text{on } M \setminus \overline{B_n} \end{cases}$$

where $H_n : \mathbb{R}^n \to \mathbb{R}$ is a smooth function that is positive on $B_r(x)$ and zero else where. (One should at this point check that f_n is well-defined)

Since V and $M \setminus \overline{B_n}$ are both open and f_n are both smooth on them, we now see f_n is smooth on the whole M. Now, since (B_n) is a locally finite cover of M and f_n can only be positive on B_n, we see that the function $f : M \to \mathbb{R}$

$$f(x) \triangleq \sum_{n\in\mathbb{N}} f_n(x) \text{ is well defined}$$

and that f is positive on whole M because each f_n is positive on B_n and (B_n) cover the whole M. Note that f is also smooth on M since for each x, there exists an open-neighborhood around x on which f is just a finite sum of f_n. We now can define for each n a smooth function $g_n : M \to \mathbb{R}$ by

$$g_n(x) \triangleq \frac{f_n(x)}{f(x)}$$

Lastly, because (B_n) is an open refinement of U_α, we can associate each n with an $\alpha(n)$ satisfying $B_n \subseteq U_{\alpha(n)}$. Then, because g_n are non-negative and $\sum_{n\in\mathbb{N}} g_n \equiv 1$, we can define for each α a function $\psi_\alpha : M \to \mathbb{R}$ by

$$\psi_\alpha \triangleq \sum_{n:\alpha(n)=\alpha} g_n$$

Because (B_n) is locally finite and g_n is only positive on B_n, we see that ψ_α is indeed smooth. Some tedious effort shows that ψ_α is indeed a partition of unity.

■

Corollary 10.1.4. (Existence of smooth function) Given some U open in M and some K closed in M and contained by U, there exists a smooth function $\psi : M \to [0,1]$ such that

(a) $\psi \equiv 1$ on K.

(b) $\operatorname{supp} \psi \subseteq U$

Corollary 10.1.5. (Extension of Smooth Function) Suppose K is closed in M, and U is open in M while containing K. For all $f \in C^\infty(U)$, there exists $\tilde{f} \in C^\infty(M)$ such that $\tilde{f}|_K = f$ and $\operatorname{supp} \tilde{f} \subseteq U$.

Proof. Corollary 10.1.4 give us a smooth function $\psi \in C^\infty(M)$ such that $\psi \equiv 1$ on K and $\operatorname{supp} \psi \subseteq U$. We simply define

$$\tilde{f}(p) \triangleq \begin{cases} \psi(p)f(p) & \text{if } p \in U \\ 0 & \text{if } p \notin U \end{cases}$$

It is clear that $\tilde{f}|_K = f$. To see $\operatorname{supp} \tilde{f} \subseteq U$, check $\operatorname{supp} \tilde{f} \subseteq \operatorname{supp} \psi$. It is clear that $\tilde{f}$ is smooth on U. To see $\tilde{f}$ is smooth at all $p \notin U$, observe that $\tilde{f} \equiv 0$ on some neighborhood around p. ■

Before we state the next Theorem, we shall point out that since topological manifolds are locally Euclidean, they are locally connected, and thus the connected components of a topological manifold are identical to the path-connected components.

Theorem 10.1.6. (Existence of Smooth Curve) If p,q are in the same connected component of smooth manifold M, then there exists a smooth curve $\gamma:[0,1]\to M$ joining p,q. By γ being smooth, we mean γ is smooth on $(0,1)$.

Proof. We first show that there exists a piece-wise smooth curve $\alpha:[0,1]\to M$ joining p,q. Let $\gamma_0:[0,1]\to M$ be a curve joining p,q. We know there exist a collection of chart $(U_\alpha,\varphi_\alpha)$ covering the image of γ_0 such that U_α are homeomorphically mapped to an open ball in $\mathbb{R}^n$ by φ_α. Now, because image of γ_0 is compact, we can suppose the collection $(U_j,\varphi_j)_{j=1}^N$ is finite. Note that $\gamma_0^{-1}(U_j)$ form an open cover for the metric space $[0,1]$. It then follows from Lebesgue's Number Lemma that there exists $n\in\mathbb{N}$ such that

$$\text{For all } k\in\{1,\dots,n\} \text{ there exists some } j_k \text{ such that } \gamma_0([\frac{k-1}{n},\frac{k}{n}])\subseteq U_{j_k}$$

Because $\varphi_j(U_j)$ is an open ball in $\mathbb{R}^d$, we can now join $\gamma_0(\frac{k-1}{n}),\gamma_0(\frac{k}{n})$ with a straight line in $\varphi_{j_k}(U_{j_k})$. In other words, for each k, we define $\gamma_k:[0,1]\to M$

$$\gamma_k(t)\triangleq\varphi_{j_k}^{-1}\Big[\varphi_{j_k}\Big(\gamma_0(\frac{k-1}{n})\Big)+t[\varphi_{j_k}\Big(\gamma_0(\frac{k}{n})\Big)-\varphi_{j_k}\Big(\gamma_0(\frac{k-1}{n})\Big)]\Big]$$

and

$$\alpha(t)\triangleq\gamma_k(nt-k) \text{ if } t\in[\frac{k-1}{n},\frac{k}{n}] \text{ (done)}$$

Lastly, we construct γ by adjusting the "speed" of α so that it is also smooth at each $\gamma_0(\frac{k}{n})$ for $1\le k\le n-1$. For this, we use the smooth function $g:\mathbb{R}\to\mathbb{R}$ we defined earlier in Equation 10.1 and define

$$\gamma(t)\triangleq\gamma_k(g(nt-k)) \text{ if } t\in[\frac{k-1}{n},\frac{k}{n}]$$

To see γ is indeed smooth, note that $[\frac{k-1}{n}-\epsilon,\frac{k}{n}+\epsilon]$ is contained by $\gamma_0^{-1}(U_{j_k})$ and the curve $\varphi_{j_k}\circ\gamma:[\frac{k-1}{n}-\epsilon,\frac{k}{n}+\epsilon]\to\mathbb{R}^d$ is smooth for each n. ■

10.2 Tangent Space

Abstract

In this section and from now on, $C^\infty(M)$ denote the space of all smooth real-valued function defined on M.

Given a point p in M, we define the **tangent space T_pM of M at p** to be the space of linear functional $D : C^\infty(M) \to \mathbb{R}$ that satisfy the product rule at p

$$D(fg) = D(f)g(p) + f(p)D(g) \text{ for all } f, g \in C^\infty(M)$$

It is clear that tangent space form a vector space when endowed with pointwise scalar multiplication and addition. Note that

$$D(1) = D(1^2) = D(1) + D(1)$$

This implies that for all $D \in T_pM$, if $f \in C^\infty(M)$ is constant, then $D(f) = 0$.

Theorem 10.2.1. (Dimension of $T_{\mathbf{p}}\mathbb{R}^n$ is n) For all $\mathbf{p} \in \mathbb{R}^n$, the vector space $T_{\mathbf{p}}\mathbb{R}^n$ is n-dimensional.

Proof. Define a function $\varphi : \mathbb{R}^n \to T_{\mathbf{p}}(\mathbb{R}^n)$ by

$$\mathbf{x} \mapsto D_{\mathbf{x}} \text{ where } D_{\mathbf{x}}(f) = \mathbf{x} \cdot \nabla f(\mathbf{p})$$

It is straightforward to check that φ is well defined and a linear transformation. It remains to prove φ is bijective. Let $D_{\mathbf{y}} = 0$, to show φ is one-to-one, we are required to show $\mathbf{y} = 0$. For each $j \in \{1, \dots, n\}$, define a smooth function $g_j : \mathbb{R}^n \to \mathbb{R}$ by

$$g_j(\mathbf{x}) \triangleq \mathbf{x}^j \tag{10.2}$$

We then see that for all $\mathbf{y} \in \mathbb{R}^n$

$$D_{\mathbf{y}}(g_j) = \mathbf{y} \cdot \mathbf{e}_j = \mathbf{y}^j$$

Then if $D_{\mathbf{y}} = 0$, we must have $\mathbf{y} = 0$. (done)

We now show φ is onto. Fix $w \in T_p(\mathbb{R}^n)$ and define $\mathbf{v} \in \mathbb{R}^n$ by

$$\mathbf{v}^j \triangleq w(g_j)$$

where g_j is defined in Equation 10.2. We claim $w = D_{\mathbf{v}}$. Fix $f \in C^\infty(\mathbb{R}^n)$. By Multi Variables Taylor Theorem, we know

$$\begin{aligned} f(\mathbf{x}) = f(\mathbf{p}) &+ \sum_{i=1}^n \frac{\partial f}{\partial \mathbf{x}^i}(\mathbf{p})(\mathbf{x}^i - \mathbf{p}^i) \\ &+ \sum_{i,j=1}^n (\mathbf{x}^i - \mathbf{p}^i)(\mathbf{x}^j - \mathbf{p}^j) \int_0^1 (1-t) \frac{\partial f}{\partial \mathbf{x}^i \partial \mathbf{x}^j}(\mathbf{p} + t(\mathbf{x} - \mathbf{p}))dt \end{aligned} \tag{10.3}$$

Now, putting both side into w, by product rule we can cancel term 10.3, so

$$w(f) = \sum_{i=1}^n \frac{\partial f}{\partial \mathbf{x}^i}(\mathbf{p}) w(g_i) = \sum_{i=1}^n \frac{\partial f}{\partial \mathbf{x}^i}(\mathbf{p}) \mathbf{v}^i = D_{\mathbf{v}}(f) \text{ (done)}$$

■

As pointed out by the proof of Theorem 10.2.1, for each $D_{\mathbf{x}} \in T_{\mathbf{p}}\mathbb{R}^n$, the value of $D_{\mathbf{x}}(f)$ is determined locally at $\mathbf{p}$ by f. The same behavior happens in the general case.

Theorem 10.2.2. (Tangent Vector care only about local behavior) Suppose $v \in T_pM$. If $f, g \in C^\infty(M)$ agree on some open neighborhood around p, then $v(f) = v(g)$.

Proof. Let $h \triangleq f - g \in C^\infty(M)$. We know $h \equiv 0$ on some open neighborhood around p. This implies

$$\text{supp}(h) \subseteq M \setminus \{p\} \subseteq M$$

By Corollary 10.1.4, there exists some smooth function $\psi : M \to [0,1]$ such that $\psi \equiv 1$ on $\text{supp}(h)$ and $\text{supp}(\psi) \subseteq M \setminus \{p\}$. Since $\text{supp}(\psi) \subseteq M \setminus \{p\}$, we know $\psi(p) = h(p) = 0$, and since $\psi \equiv 1$ on $\text{supp}(h)$, we also know $\psi h = h$. This let us deduce

$$v(h) = v(\psi h) = v(\psi)h(p) + \psi(p)v(h) = 0$$

■

We now define the derivative $F_{*,p} : T_pM \to T_{F(p)}N$ of a smooth map $F : M \to N$ at $p \in M$ between smooth manifold.

$$(F_{*,p}(w))(f) \triangleq w(f \circ F)$$

It is straightforward to check that given another smooth manifold R and smooth map $G : N \to R$ we have

(a) $F_{*,p}(w) \in T_{F(p)}N$.

(b) $F_{*,p}$ is linear.

(c) $(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p} : T_pM \to T_{G\circ F(p)}R$.

(d) If $\mathbf{id} : M \to M$ is the identity function on M, then $\mathbf{id}_{*,p}$ is the identity map on T_pM.

(e) If $F : M \to N$ is a diffeomorphism, then $F_{*,p} : T_pM \to T_{F(p)}(N)$ is an isomorphism of vector space and $(F_{*,p})^{-1} = (F^{-1})_{*,F(p)}$

Theorem 10.2.3. (Tangent Space of points in open restriction) Given some open subset $U \subseteq M$ and inclusion map $i : U \to M$, for all $p \in U$, the map $i_{*,p} : T_pU \to T_pM$ is a vector space isomorphism.

Proof. We first prove that $i_{*,p}$ is one-to-one. Fix $v \in T_pU$ such that $i_{*,p}(v) = 0$. We are required to show $v = 0$. Fix $f \in C^\infty(U)$. Let (V, φ) be a chart contained in U and centering p, and let ϵ satisfy

$$\overline{B_\epsilon(\varphi(p))} \subseteq \varphi(V)$$

Define $K \triangleq \overline{\varphi^{-1}(B_\epsilon(\varphi(p)))}$. By Corollary 10.1.5, we know there exists $\tilde{f} \in C^\infty(M)$ such that $\tilde{f} \equiv f$ on K. Then because tangent vector care only about local behavior, we can now deduce

$$v(f) = v(\tilde{f}|_U) = v(\tilde{f} \circ i) = \iota_{*,p}(v)(\tilde{f}) = 0 \text{ (done)}$$

We now prove $\iota_{*,p}$ is onto. Fix $w \in T_pM$, and again $K \triangleq \overline{\varphi^{-1}(B_\epsilon(\varphi(p)))}$. Define $v \in T_pU$ by

$$v(f) \triangleq w(\tilde{f})$$

where $\tilde{f} \in C^\infty(M)$ satisfy $\tilde{f} \equiv f$ on K. Because for all $g \in C^\infty(M)$, $g \equiv g \circ i$ on K, we know

$$\iota_{*,p}(v)(g) = v(g \circ i) = w(g) \text{ (done)}$$

∎

Given a chart (U, φ) containing p, we often define $\frac{\partial}{\partial \mathbf{x}^i}|_p \in T_pM$ by

$$\frac{\partial}{\partial \mathbf{x}^i}\Big|_p (f) \triangleq \frac{\partial (f \circ \varphi^{-1})}{\partial \mathbf{x}^i}(\varphi(p)) \tag{10.4}$$

By Theorem 10.2.3, we have the following diagram

$$\begin{tikzcd}
T_pM \arrow[d, leftrightarrow, "i_{*,p}"] & & T_{\varphi(p)}\mathbb{R}^m \arrow[d, leftrightarrow, "i_{*,\varphi(p)}"'] \\
T_pU \arrow[rr, leftrightarrow, "\varphi_{*,p}"] & & T_{\varphi(p)}\varphi(U)
\end{tikzcd}$$

Then since $\frac{\partial}{\partial \mathbf{x}^i}|_{\varphi(p)}$ form a basis for $T_p\mathbb{R}^m$ by the proof of Theorem 10.2.1, we see $\frac{\partial}{\partial \mathbf{x}^i}|_p$ form a basis for T_pM. Note that our notation give us the convenient

$$\frac{\partial}{\partial \mathbf{x}^i}\Big|_p = \sum_j \frac{\partial \tilde{\mathbf{x}}^j}{\partial \mathbf{x}^i}\frac{\partial}{\partial \tilde{\mathbf{x}}^j}\Big|_p$$

when we are given another chart containing p. (Verify this by some tedious effort)

Now, if we define for each C^1 curve $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p$, its **velocity vector** $\gamma'(0) \in T_{\gamma(0)}M$ by

$$\gamma'(0)(f) \triangleq (f \circ \gamma)'(0)$$

we see that every tangent vector $v \in T_pM$ can be identified as a velocity vector of some smooth curve

$$\gamma(t) \triangleq \varphi^{-1}(v^1 t, \dots, v^n t) \text{ where } v = \sum v^i \frac{\partial}{\partial \mathbf{x}^i}\Big|_p$$

Moreover, we see that smooth map $F : M \to N$ preserve velocity in the sense that

$$F_{*,p}(\gamma'(0)) = (F \circ \gamma)'(0)$$

This give an expected result.

Theorem 10.2.4. (Constant function) If at each point $p \in M$, the derivative $F_{*,p}$ of the smooth map $F : M \to N$ is null, then F is constant on each path-connected component of M.

Proof. Fix p, q in the same connected component of M, and let $\gamma : [0, 1] \to M$ be a smooth curve joining them and lying inside the connected component. Assume $F(p) \neq F(q)$. Let (V, ψ) be a chart centering $F(p)$ and let B be a regular coordinate ball (with respect to (V, ψ)) centering $F(p)$ such that $F(q) \notin \overline{B}$, and let $f \in C^\infty(N)$ be a non-negative smooth function positive on and only on B. Because $F_{*,p}$ are null, we can deduce

$$(f \circ F \circ \gamma)'(t) = (F \circ \gamma)'(t)(f) = 0 \text{ for all } t \in (0,1)$$

It then follows that $f(F(p)) = f(F(q))$. CaC ■

10.3 Inverse Function Theorem and Rank Theorem

Abstract

In this section, we prove the Inverse Function Theorem for Manifolds with Inverse Function Theorem for Euclidean Spaces and prove Rank Theorem for Manifolds with Inverse Function Theorem for Manifolds.

Given a function $F : M \to N$ differentiable at some point $p \in M$, we define **the rank of F at p** to be the rank of the linear map $F_{*,p} : T_pM \to T_{F(p)}N$. Let (V, ψ) be a chart containing $F(p)$ and (U, φ) be a chart containing p while $F(U) \subseteq V$. To find the rank of $F_{*,p}$, we are required to compute the matrix representation $[F_{*,p}]$ with respect to the basis $\frac{\partial}{\partial \mathbf{x}^i}|_p$ and $\frac{\partial}{\partial \mathbf{y}^j}|_{F(p)}$. To compute $[F_{*,p}]$, we first define $\widehat{F} : \varphi(U) \to \psi(V)$ and $\widehat{p} \in \mathbb{R}^m$ by

$$\widehat{F}(\mathbf{x}) \triangleq \psi \circ F \circ \varphi^{-1}(\mathbf{x}) \text{ and } \widehat{p} \triangleq \varphi(p)$$

Immediately, we see that

$$F = \psi^{-1} \circ \widehat{F} \circ \varphi \tag{10.5}$$

This immediate result Equation 10.5 is in fact very critical, since as shall show, it is easy to compute $[\widehat{F}_{*,\widehat{p}}]$ with respect to the basis $\frac{\partial}{\partial \widehat{\mathbf{x}}^i}|_{\widehat{p}}$ and $\frac{\partial}{\partial \widehat{\mathbf{y}}^j}|_{\widehat{F}(\widehat{p})}$, and we by definition also have the relationship

$$\varphi_{*,p}\Big(\frac{\partial}{\partial \mathbf{x}^i}\Big|_p\Big) = \frac{\partial}{\partial \widehat{\mathbf{x}}^i}\Big|_{\widehat{p}} \text{ and } (\psi^{-1})_{*,\widehat{F}(\widehat{p})}\Big(\frac{\partial}{\partial \widehat{\mathbf{y}}^j}\Big|_{\widehat{F}(\widehat{p})}\Big) = \frac{\partial}{\partial \mathbf{y}^j}\Big|_{F(p)} \tag{10.6}$$

Fix smooth $f \in C^\infty(\mathbb{R}^n)$. By applying ordinary chain rule on $f \circ \widehat{F}$ and observing matrix representation of df and $d\widehat{F}$, we can compute $[\widehat{F}_{*,\widehat{p}}]$ with respect to $\frac{\partial}{\partial \widehat{\mathbf{x}}^i}|_{\widehat{p}}$ and $\frac{\partial}{\partial \widehat{\mathbf{y}}^j}|_{\widehat{F}(\widehat{p})}$

$$\begin{aligned} \widehat{F}_{*,\widehat{p}}\Big(\frac{\partial}{\partial \widehat{\mathbf{x}}^i}\Big|_{\widehat{p}}\Big)(f) = \frac{\partial (f \circ \widehat{F})}{\partial \widehat{\mathbf{x}}^i}(\widehat{p}) &= \sum_j \frac{\partial f}{\partial \widehat{\mathbf{y}}^j}(\widehat{F}(\widehat{p}))\frac{\partial \widehat{F}^j}{\partial \widehat{\mathbf{x}}^i}(\widehat{p}) \\ &= \sum_j \frac{\partial \widehat{F}^j}{\partial \widehat{\mathbf{x}}^i}(\widehat{p})\Big[\frac{\partial}{\partial \widehat{\mathbf{y}}^j}\Big|_{\widehat{F}(\widehat{p})}(f)\Big] \end{aligned}$$

This in general means that

$$\widehat{F}_{*,p}\Big(\frac{\partial}{\partial \widehat{\mathbf{x}}^i}\Big|_{\widehat{p}}\Big) = \sum_j \frac{\partial \widehat{F}^j}{\partial \widehat{\mathbf{x}}^i}(\widehat{p})\frac{\partial}{\partial \widehat{\mathbf{y}}^j}\Big|_{\widehat{F}(\widehat{p})}$$

Then with the relationship in Equation 10.6 and Equation 10.5, we finally have

$$\begin{aligned}F_{*,p}\Big(\frac{\partial}{\partial \mathbf{x}^i}\Big|_p\Big) &= (\psi^{-1})_{*,\widehat{F}(\widehat{p})} \circ \widehat{F}_{*,\widehat{p}} \circ \varphi_{*,p}\Big(\frac{\partial}{\partial \mathbf{x}^i}\Big|_p\Big) \\ &= (\psi^{-1})_{*,\widehat{F}(\widehat{p})}\Big(\sum_j \frac{\partial \widehat{F}^j}{\partial \widehat{\mathbf{x}}^i}(\widehat{p})\frac{\partial}{\partial \widehat{\mathbf{y}}^j}\Big|_{\widehat{F}(\widehat{p})}\Big) \\ &= \sum_j \frac{\partial \widehat{F}^j}{\partial \widehat{\mathbf{x}}^i}(\widehat{p})(\psi^{-1})_{*,\widehat{F}(\widehat{p})}\Big(\frac{\partial}{\partial \widehat{\mathbf{y}}^j}\Big|_{\widehat{F}(\widehat{p})}\Big) \\ &= \sum_j \frac{\partial \widehat{F}^j}{\partial \widehat{\mathbf{x}}^i}(\widehat{p})\frac{\partial}{\partial \mathbf{y}^j}\Big|_{F(p)}\end{aligned}$$

which means

$$[F_{*,p}] = [d\widehat{F}_{\widehat{p}}]$$

With this in mind and noting that in the space of $n \times m$ real matrix, the set of real matrices of full rank is open, we see that if F is smooth and $F_{*,p}$ is full rank, then there exists an open neighborhood U around p for which $F_{*,q}$ is full rank if $q \in U$.

Now, given a function $F : M \to N$, we say F is **locally a smooth diffeomorphism at** $p \in M$ if there exists some open-neighborhood U around p such that $F(U)$ is open in N and $F|_U : U \to F(U)$ is a smooth diffeomorphism between U and $F(U)$, and we say F is a **local smooth diffeomorphism** if F is locally a smooth diffeomorphism at p for all $p \in M$.

Theorem 10.3.1. (Inverse Function Theorem for Manifolds) Suppose M and N are smooth manifold, and $F : M \to N$ is some smooth map. If for some $p \in M$, $F_{*,p}$ is invertible, then F is locally a smooth diffeomorphism at p.

Proof. Let (V, ψ) be a chart centering $F(p)$ and (U, φ) be a chart centering p such that $F(U) \subseteq V$. If we define $\widehat{F} : \varphi(U) \to \psi(V)$ by

$$\widehat{F} \triangleq \psi \circ F \circ \varphi^{-1}$$

we see that there exists some open set $\widehat{U} \subseteq \varphi(U)$ containing $\varphi(p)$ such that $dF_{\mathbf{x}}$ is invertible for all $\mathbf{x} \in \widehat{U}$. Because $\widehat{F}$ is smooth, we then can apply Ordinary Inverse Function Theorem to have some $B_\delta(\mathbf{0}) \subseteq \widehat{U}$ such that $\widehat{F}|_{B_\delta(\mathbf{0})}$ is a smooth diffeomorphism. It then follows that F is a smooth diffeomorphism on $\varphi(B_\delta(\mathbf{0}))$ ■

We say F is a **smooth immersion** if F is smooth and on each $p \in M$ the differential $F_{*,p}$ is injective, and we say F is a **smooth submersion** if F is smooth and on each $p \in M$ if F is surjective. With Inverse Function Theorem for Manifolds in mind, we see that given smooth $F : M \to N$

$$F \text{ is a local diffeomorphism} \iff F \text{ are both immersion and submersion}$$

Theorem 10.3.2. (Constant Rank Theorem for manifolds) Given a smooth map $F : M \to N$ with constant rank r, for each $p \in M$, there exists some chart (V, ψ) centering $F(p)$ and (U, φ) centering p such that $F(U) \subseteq V$ and F has the coordinate representation of the form

$$\widehat{F}(\mathbf{x}^1, \dots, \mathbf{x}^r, \mathbf{x}^{r+1}, \dots, \mathbf{x}^m) = (\mathbf{x}^1, \dots, \mathbf{x}^r, 0, \dots, 0)$$

Proof. Because the Theorem is local, we may write $F : U \subseteq \mathbb{R}^m \to V \subseteq \mathbb{R}^n$ by

$$F(\mathbf{x}, \mathbf{y}) \triangleq (Q(\mathbf{x}, \mathbf{y}), R(\mathbf{x}, \mathbf{y}))$$

where $Q : U \to \mathbb{R}^r, R : U \to \mathbb{R}^{n-r}$ maps $(\mathbf{0}, \mathbf{0})$ to $\mathbf{0}$. Since F is of rank r, we may assume

$$\text{The matrices } \begin{bmatrix} \frac{\partial Q^1}{\partial \mathbf{x}^1}(\mathbf{v}) & \cdots & \frac{\partial Q^1}{\partial \mathbf{x}^r}(\mathbf{v}) \\ \vdots & \ddots & \vdots \\ \frac{\partial Q^r}{\partial \mathbf{x}^1}(\mathbf{v}) & \cdots & \frac{\partial Q^r}{\partial \mathbf{x}^r}(\mathbf{v}) \end{bmatrix} \text{ are invertible for all } \mathbf{v} \in U$$

Then if we define $\varphi : U \to \mathbb{R}^m$ by

$$\varphi(\mathbf{x}, \mathbf{y}) \triangleq (Q(\mathbf{x}, \mathbf{y}), \mathbf{y})$$

we see that

$$[d\varphi_{\mathbf{v}}] = \begin{bmatrix} \frac{\partial Q^1}{\partial \mathbf{x}^1}(\mathbf{v}) & \cdots & \frac{\partial Q^1}{\partial \mathbf{x}^r}(\mathbf{v}) & \frac{\partial Q^1}{\partial \mathbf{y}^1}(\mathbf{v}) & \cdots & \frac{\partial Q^1}{\partial \mathbf{y}^{m-r}}(\mathbf{v}) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial Q^r}{\partial \mathbf{x}^1}(\mathbf{v}) & \cdots & \frac{\partial Q^r}{\partial \mathbf{x}^r}(\mathbf{v}) & \frac{\partial Q^r}{\partial \mathbf{y}^1}(\mathbf{v}) & \cdots & \frac{\partial Q^r}{\partial \mathbf{y}^{m-r}}(\mathbf{v}) \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix} \text{ is invertible for all } \mathbf{v} \in U$$

It then follows from Ordinary Inverse Function Theorem such that there exists $U_0 \subseteq U$ containing $(\mathbf{0},\mathbf{0})$ such that φ maps U_0 smooth-diffeomorphically to some open cube $\widehat{U_0} \subseteq \mathbb{R}^m$.

It is clear for all $(\mathbf{x},\mathbf{y}) \in \widehat{U_0}$, we have

$$(\varphi|_{U_0})^{-1}(\mathbf{x},\mathbf{y}) = (A(\mathbf{x},\mathbf{y}),\mathbf{y})$$

where A is some function that maps $\widehat{U_0}$ into $\mathbb{R}^r$. Now by definition of φ, we can compute

$$(\mathbf{x},\mathbf{y}) = \varphi(A(\mathbf{x},\mathbf{y}),\mathbf{y}) = (Q(A(\mathbf{x},\mathbf{y}),\mathbf{y}),\mathbf{y})$$

for all $(\mathbf{x},\mathbf{y}) \in \widehat{U_0}$. This then let us compute

$$Q(A(\mathbf{x},\mathbf{y}),\mathbf{y}) = \mathbf{x}$$

for all $(\mathbf{x},\mathbf{y}) \in \widehat{U_0}$, which implies

$$F \circ \varphi^{-1}(\mathbf{x},\mathbf{y}) = F(A(\mathbf{x},\mathbf{y}),\mathbf{y}) = (\mathbf{x},\tilde{R}(\mathbf{x},\mathbf{y}))$$

for all $(\mathbf{x},\mathbf{y}) \in \widehat{U_0}$ where $\tilde{R}(\mathbf{x},\mathbf{y}) \triangleq R(A(\mathbf{x},\mathbf{y}),\mathbf{y})$ maps $\widehat{U_0} \subseteq \mathbb{R}^m$ into $\mathbb{R}^{n-r}$. Now, because $\varphi^{-1} : \widehat{U_0} \to U_0 \subseteq \mathbb{R}^m$ is full rank and F is of rank $r \leq m$, we see $F \circ \varphi^{-1} : \widehat{U_0} \to \mathbb{R}^n$ if also of rank r. Then from observing that the differential of $F \circ \varphi^{-1}$ is of the form

$$[d(F\circ\varphi^{-1})_{\mathbf{v}}] = \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ \frac{\partial \tilde{R}^1}{\partial \mathbf{x}^1}(\mathbf{v}) & \cdots & \frac{\partial \tilde{R}^1}{\partial \mathbf{x}^r}(\mathbf{v}) & \frac{\partial \tilde{R}^1}{\partial \mathbf{y}^1}(\mathbf{v}) & \cdots & \frac{\partial \tilde{R}^1}{\partial \mathbf{y}^{m-r}}(\mathbf{v}) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \tilde{R}^{n-r}}{\partial \mathbf{x}^1}(\mathbf{v}) & \cdots & \frac{\partial \tilde{R}^{n-r}}{\partial \mathbf{x}^r}(\mathbf{v}) & \frac{\partial \tilde{R}^{n-r}}{\partial \mathbf{y}^1}(\mathbf{v}) & \cdots & \frac{\partial \tilde{R}^{n-r}}{\partial \mathbf{y}^{m-r}}(\mathbf{v}) \end{bmatrix}$$

We can deduce

$$\frac{\partial \tilde{R}^j}{\partial \mathbf{y}^i}(\mathbf{v}) = 0 \text{ for all } \mathbf{v} \in \widehat{U_0} \text{ and } i \in \{1,\dots,m-r\}, j \in \{1,\dots,n-r\}$$

This then let us write (by MVT)

$$F \circ \varphi^{-1}(\mathbf{x},\mathbf{y}) = (\mathbf{x}, S(\mathbf{x})) \tag{10.7}$$

for all $(\mathbf{x},\mathbf{y}) \in \widehat{U_0}$ where $S(\mathbf{x}) \triangleq R(\mathbf{x},\mathbf{0})$. Lastly, we define $V_0 \subseteq V$ by

$$V_0 \triangleq \{(\mathbf{z},\mathbf{w}) \in V : (\mathbf{z},\mathbf{0}) \in \widehat{U_0}\}$$

Since $(\mathbf{z}, \mathbf{w}) \mapsto (\mathbf{z}, \mathbf{0})$ is a continuous map from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $\{(\mathbf{x}, \mathbf{0}) \in \mathbb{R}^m : \mathbf{x} \in \mathbb{R}^r\}$ is open, we see that V_0 is also open, which clearly contain $(\mathbf{0}, \mathbf{0})$.

Recall that $\widehat{U_0}$ is an open cube in $\mathbb{R}^m$; Thus from Equation 10.7, we can deduce

$$F \circ \varphi^{-1}(\widehat{U_0}) \subseteq V_0$$

and if we define $\psi : V_0 \to \mathbb{R}^n$ by

$$\psi(\mathbf{z}, \mathbf{w}) \triangleq (\mathbf{z}, \mathbf{w} - S(\mathbf{z}))$$

we see ψ has an explicit inverse

$$\psi^{-1}(\mathbf{s}, \mathbf{t}) = (\mathbf{s}, \mathbf{t} + S(\mathbf{s}))$$

We have established that (V_0, ψ) is a smooth chart, and for all $(\mathbf{x}, \mathbf{y}) \in \widehat{U_0}$

$$\psi \circ F \circ \varphi^{-1}(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{0})$$

$\blacksquare$

10.4 Submanifold

Abstract

This section discuss submanifolds. Note that in this section, smooth manifold S, M, N always have the dimension s, m, n.

By an **immersed smooth submanifold** S of smooth manifold M, we mean a subset S, endowed with a topology and a smooth structure such that the inclusion map $i : S \to M$ is a smooth immersion. If we say the immersed smooth submanifold S is an **embedded smooth submanifold**, we mean that the inclusion map $i : S \to M$ is moreover a **smooth embedding**, i.e., not only does i have to be a smooth immersion, but i also have to be a topological embedding. It is clear by definition that if S is an embedded smooth manifold, then the topology on S must be the subspace topology.

Given a subset S of some smooth manifold M, there exists a nice sufficient condition called **local k-slice condition** for S so that S can have a smooth structure that make S an embedded submanifold of M. We say S satisfy the local k-slice condition if there exists a smooth atlas consisting of **k-slice charts** (U, φ) **with respect to** S, i.e., charts for M such that

$$\varphi(U \cap S) = \varphi(U) \cap (\mathbb{R}^k \times \{\mathbf{0}\})$$

Theorem 10.4.1. (Local k-slice condition for embedded submanifolds) If S satisfy the local k-slice condition, then S have a smooth structure such that S is a smooth embedded submanifold.

Proof. Give S the subspace topology. For each k-slice chart (U, φ), we can define a chart $(U \cap S, \pi \circ \varphi)$ for S where $\pi : \mathbb{R}^m \to \mathbb{R}^k$ is the projection defined by

$$\pi(\mathbf{x}^1, \dots, \mathbf{x}^m) \triangleq (\mathbf{x}^1, \dots, \mathbf{x}^k)$$

Some tedious effort shows that $(U \cap S, \pi \circ \varphi)$ is indeed a chart. Because S is given the subspace topology, clearly the inclusion map i is a topological embedding. To see that i is also a smooth immersion, observe that for each adapted chart $(U \cap S, \pi \circ \varphi)$, we have

$$i(U \cap S) \subseteq U$$

and for all $(\mathbf{x}^1, \dots, \mathbf{x}^k) \in \pi \circ \varphi(U \cap S)$

$$\varphi \circ i \circ (\pi \circ \varphi)^{-1}(\mathbf{x}^1, \dots, \mathbf{x}^k) = (\mathbf{x}^1, \dots, \mathbf{x}^k, 0, \dots, 0)$$

This shows that i is indeed also a smooth immersion. ∎

As the next Theorem shows, the term smooth embedding is in fact very strong.

Theorem 10.4.2. (Image of smooth embedding is always a smooth embedded submanifold) If $F : M \to N$ is a smooth embedding, then the topology and smooth structure of $F(M)$ inherited from M makes $F(M)$ a smooth embedded manifold of N.

Proof. Because F is by definition a topological embedding, it is clear that $F(M)$ is endowed with the subspace topology. Fix $p \in F(M)$. Because F is a smooth immersion, by Constant Rank Theorem for manifolds, we know there exists some chart (U, φ) for M centering $F^{-1}(p)$ and some chart (V, ψ) for N centering p such that $F(U) \subseteq V$ and

$$\widehat{F}(\mathbf{x}^1, \dots, \mathbf{x}^m) = (\mathbf{x}^1, \dots, \mathbf{x}^m, 0, \dots, 0)$$

Because $F(U)$ is open in $F(M)$, we know there exists open $V' \subseteq N$ such that

$$F(U) = V' \cap F(M)$$

We now can check that $(V \cap V', \psi|_{V \cap V'})$ is a slice chart centering p. ■

Note that in Theorem 10.4.2, we have in fact proved that smooth embedded submanifold always satisfy local k-slice condition.

Theorem 10.4.3. (Constant Rank Level Set Theorem) If smooth $F : M \to N$ is of constant rank k, then level sets $F^{-1}(q)$ for each $q \in N$ is an embedded submanifold of M of dimension $m - k$.

Proof. Fix $p \in F^{-1}(q)$. By Constant Rank Theorem for Manifold, there exist some chart $(U, \varphi), (V, \psi)$ centering p, q such that $F(U) \subseteq V$ and for all $(\mathbf{x}^1, \dots, \mathbf{x}^m) \in \varphi(U)$

$$\psi \circ F \circ \varphi^{-1}(\mathbf{x}^1, \dots, \mathbf{x}^m) = (\mathbf{x}^1, \dots, \mathbf{x}^k, 0, \dots, 0) \tag{10.8}$$

We now prove that (U, φ) is a $m-k$-slice chart with respect to $F^{-1}(q)$. Let $s \in F^{-1}(q) \cap U$. Compute

$$\psi \circ F(s) = \psi(q) = \mathbf{0}$$

Then since

$$\mathbf{0} = \psi \circ F(s) = \psi \circ F \circ \varphi^{-1}(\varphi(s))$$

from Equation 10.8, we can deduce the first k component of $\varphi(s)$ must be 0. We have proved

$$\varphi(U \cap F^{-1}(q)) \subseteq \varphi(U) \cap (\{\mathbf{0}\} \times \mathbb{R}^{m-k}) \tag{10.9}$$

On the other hand, if $s \in U$ and $\varphi(s) \in \{\mathbf{0}\} \times \mathbb{R}^{m-k}$, then if we write

$$\varphi(s) = (0, \dots, 0, \mathbf{x}^{k+1}, \dots, \mathbf{x}^m)$$

we have from Equation 10.8 that

$$\psi \circ F(s) = \mathbf{0}$$

which give us $F(s) = q$. It then follows that $\varphi(s) \in \varphi(U \cap F^{-1}(q))$, proving the converse of Equation 10.9. ■

We now generalize Constant Rank Level Set Theorem to general smooth map. Given some smooth map $F : M \to N$, we say $p \in M$ is a **regular point** if $F_{*,p}$ is surjective; otherwise, we say p is a **critical point**. Given $q \in N$, if $F^{-1}(q)$ contain only regular point, then we say q is a **regular value**; otherwise, we say $q \in N$ is a **critical value**. If $q \in N$ is some regular value, we say $F^{-1}(q)$ is a **regular level set**.

Theorem 10.4.4. (Regular Level Set Theorem) Given smooth map $F : M \to N$, if $q \in N$ is some regular point, the regular level set $F^{-1}(q)$ is a smooth embedded submanifold of M with dimension $m - n$.

Proof. Let $U \subseteq M$ be the collection of regular points. Because the set of full rank matrix is open in $M_{n\times m}(\mathbb{R})$ and F is smooth, we can deduce U is open. Therefore, from Constant Rank Level Set Theorem, we can deduce that $F^{-1}(q)$ is an embedded smooth submanifold of U. Some tedious effort and the fact that $i_{*,p} : T_pU \to T_pM$ is a vector space isomorphism now shows that $F^{-1}(q)$ is also an embedded smooth manifold of M. ■

Note that in Regular Level Set Theorem, because F in priori has a regular point, we have the assumption that $m \geq n$. As a corollary of the Regular Level Set Theorem, we see that S^n is regular level set $f^{-1}(1)$ of the smooth map $f : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by

$$f(\mathbf{x}) \triangleq |\mathbf{x}|^2$$

It then follows that S^n is an embedded smooth submanifold of $\mathbb{R}^{n+1}$.

Although regular level set Theorem shows that regular level set is always a smooth embedded submanifold, one should note that that critical level sets can sometimes also be an embedded submanifold. Consider smooth $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = y^2$$

The level set $f^{-1}(0)$ is critical everywhere but still an embedded submanifold.

We now go back to consider immersion. As it turned out, immersed submanifold can also be treated as the image of some injective smooth immersion.

Theorem 10.4.5. (Image of smooth immersion is always a smooth immersed submanifold) Given some injective smooth immersion $F : M \to N$, the image $F(M)$ can be endowed with a topology and a smooth structure such that $F(M)$ is a smooth immersed submanifold of N.

Proof. Because F is injective, we can simply give $F(M)$ the topology of M. For all charts (U, φ) for M, we can induce a chart $(F(U), \varphi \circ F^{-1})$ for $F(M)$. It is clear that this indeed give a smooth structure. To see $F(M)$ is now an immersed smooth submanifold, observe that $F^{-1} : F(M) \to M$ is a smooth diffeomorphism and the inclusion map $i : F(M) \to N$ is the just composition of the smooth immersion F and F^{-1}. ∎

Although there exists similarity between immersion and embedding, the latter is still strictly powerful. Consider the curve $\beta : (-\pi, \pi) \to \mathbb{R}^2$

$$\beta(t) \triangleq (\sin 2t, \sin t)$$

This is a smooth injective immersion, yet not a smooth embedding, since image of β is clearly compact but $(-\pi, \pi)$ isn't shows that β^{-1} is not continuous. In other words, image of β is a smooth immersed submanifold of $\mathbb{R}^2$ by Theorem 10.4.5 but not a smooth embedded submanifold of $\mathbb{R}^2$.

We now shows that immersion is in fact at every point locally an embedding.

Theorem 10.4.6. (Local Embedding Theorem) Given smooth map $F : M \to N$, F is a smooth immersion if and only if for all $p \in M$, there exists neighborhood U of p in M such that $F|_U$ is a smooth embedding.

Proof. The if part is clear. Suppose F is a smooth immersion. By constant rank theorem, there exist some charts $(U, \varphi), (V, \psi)$ centering $p, F(p)$ such that $F(U) \subseteq V$ and

$$\psi \circ F \circ \varphi^{-1}(\mathbf{x}^1, \dots, \mathbf{x}^m) = (\mathbf{x}^1, \dots, \mathbf{x}^m, 0, \dots, 0)$$

It then follows that F is injective on U. To see that U and $F(U)$ has the same topology, just observe that we have the following diagram

$$\begin{tikzcd}
U \arrow[d, leftrightarrow] & & F(U) \arrow[d, leftrightarrow] \\
\varphi(U) \arrow[rr, leftrightarrow, "\text{obvious homeomorphism}"'] & & \psi \circ F(U)
\end{tikzcd}$$

■

With local embedding Theorem, we now see that for each p in some smooth immersed submanifold S of M, there exists some neighborhood U in S of p such that $i|_U : U \to M$ is a smooth embedding; thus U an embedded smooth submanifold of M.

It is easy to see that if S is an immersed smooth submanifold of M and $F : M \to N$ is a smooth map, then the restriction $F|_S$ is also a smooth map. However, it require the fact that immersed smooth submanifold is locally an embedded submanifold to see that when we restrict the codomain of F, the smooth map F is still smooth.

Theorem 10.4.7. (Smooth map is still smooth after restriction of codomain) Given a smooth map $F : M \to N$ and an immersed smooth submanifold S of N such that $F(M) \subseteq S$, if $F : M \to S$ is continuous, then $F : M \to S$ is still smooth.

Proof. Fix $p \in M$ and let $q \triangleq F(p)$. We know there exists some neighborhood U of q in S such that U is an embedded smooth submanifold of N. Then there exists some s-slice chart (V, φ) for N centering q, which give a chart $(U \cap V, \pi \circ \varphi)$ for U centering q. Because $F : M \to N$ is a smooth map, there exists some chart (W, ψ) for M centering p such that $F(W) \subseteq V$ and

$$\varphi \circ F \circ \psi^{-1} : \psi(W) \to \mathbb{R}^n \text{ is smooth}$$

Because $F : M \to S$ is continuous and $U \cap V$ is open in U which is open in S, we know there exists some open set $W_0 \subseteq W$ centering p such that

$$F(W_0) \subseteq U \cap V$$

Then if we consider the chart $(W_0, \psi), (U \cap V, \pi \circ \varphi)$ for M, S, because

$$\pi \circ \varphi \circ F \circ \psi^{-1} : \psi(W) \to \mathbb{R}^s \text{ is clearly smooth}$$

we see $F : M \to S$ is smooth at p. ■

Lastly, we discuss the property of tangent space T_pS of smooth immersed submanifold S of M. Because the fact $i : S \to M$ is a smooth immersion shows that $i_{*,p} : T_pS \to T_pM$ is always an injective vector space homomorphism, we often identify T_pS as a subspace of T_pM.

Theorem 10.4.8. (Velocity Characterization of Tangent Subspace of Immersed Submanifold) Given an immersed smooth submanifold S of M and $p \in S$, a vector $v \in T_pM$ belong to T_pS if and only if there exists some smooth curve $\alpha : (-\epsilon, \epsilon) \to S$ such that $\alpha(0) = p$ and in M, $\alpha'(0) = v$.

Proof. The proof lies in observing that for all $f \in C^\infty(M)$ and smooth curve $\alpha : (-\epsilon, \epsilon) \to S$ centering S, we have

$$i_{*,p}(\alpha'(0))(f) = (f \circ i \circ \alpha)'(0) = (f \circ \alpha)'(0)$$

where the $\alpha'(0)$ in the first quantity is in S. ■

Theorem 10.4.9. (Equivalent Characterization of Tangent Subspace of Embedded Submanifold) Given a smooth embedded submanifold S of M, we have

$$i_{*,p}(T_pS) = \{v \in T_pM : vf = 0 \text{ whenever } f \in C^\infty(M) \text{ and } f|_S = 0\}$$

Proof. Fix $v \in i_{*,p}(T_pS)$ and $f \in C^\infty(M)$ such that $f|_S = 0$. Let $w \in T_pS$ be $i_{*,p}(w) = v$. Observe that

$$v(f) = i_{*,p}(w)(f) = w(f \circ i) = 0$$

since $f \circ i \in C^\infty(S)$ is constant. We have proved the $\subseteq$ side of the inequality.

Fix $v \in T_pM$ that satisfy $v(f) = 0$ for all $f \in C^\infty(M)$ such that $f|_S = 0$. Because S is a smooth embedded submanifold of M, we can let (U, φ) be the s-slice chart centering p for M with respect to S. We now have the chart $(U \cap S, \pi \circ \varphi), (U, \varphi)$ for S, M. This then give us

$$\varphi \circ i \circ (\pi \circ \varphi)^{-1}(\mathbf{x}^1, \dots, \mathbf{x}^s) = (\mathbf{x}^1, \dots, \mathbf{x}^s, 0, \dots, 0)$$

This then implies that $i_{*,p}$ sent each $\frac{\partial}{\partial \mathbf{x}^i}\big|_p$, where $1 \leq i \leq s$, to $\frac{\partial}{\partial \mathbf{x}^i}\big|_p$. In other words, if we write

$$v = \sum_{i=1}^m v^i \frac{\partial}{\partial \mathbf{x}^i}\Big|_p$$

then $v \in i_{*,p}(T_pS)$ if and only if for all $i > s$, we have $v^i = 0$.

Let $\Psi : M \to S$ be a smooth function supported in U and take value 1 on certain neighborhood of p. Fix some $i > k$, if we define $f : M \to \mathbb{R}$ by

$$f(p) \triangleq \begin{cases} \Psi(p)x^i & \text{if } p \in U \\ 0 & \text{if } p \in M \setminus \operatorname{supp}\Psi \end{cases}$$

the function f is well defined, smooth, and take value 0 on S. This then give us

$$0 = vf = \sum_{j=1}^m v^j \frac{\partial f}{\partial \mathbf{x}^j}\Big|_p = v^i$$

where the last inequality hold true since Ψ is constant on certain neighborhood of p. ■

Lastly, we pointed out that if $F : M \to N$ is a smooth map with constant rank, then all its level set $S \triangleq F^{-1}(c)$ has the property that

$$i_{*,p}(T_pS) = \operatorname{Ker} F_{*,p}$$

since $F \circ i : S \to N$ is a constant function and T_pS by constant rank level set Theorem have dimension $m - k$.

10.5 Vector Field

Abstract

Given a smooth manifold M and a countable collection of chart (U_n, φ_n) that cover M, if we write

$$TM \triangleq \coprod_{p\in M} T_pM \triangleq \Big\{(p,v) : p \in M \text{ and } v \in T_pM\Big\}$$

and let $\pi : TM \to M$ be the projection map

$$\pi(p,v) \triangleq p$$

then for each n we can define a bijective function $\Phi_n : \pi^{-1}(U_n) \to \mathbb{R}^{2m}$ by

$$\Phi_n(p,v) \triangleq (\varphi(p), v^1, \dots, v^m) \text{ where } v = \sum_{i=1}^m v^i \frac{\partial}{\partial \mathbf{x}^i}\Big|_p \tag{10.10}$$

This then let us define $E \subseteq M$ to be open if and only if $\Phi_n(E \cap \pi^{-1}(U_n))$ is open in $\mathbb{R}^m$ for all n. Note that the topology is well defined because Φ_n are all one-to-one. Moreover, we see that $\pi^{-1}(U_n)$ are open since for all $k \in \mathbb{N}$

$$\Phi_k(\pi^{-1}(U_n) \cap \pi^{-1}(U_k)) = \varphi_k(U_n \cap U_k) \times \mathbb{R}^m \text{ is open in } \mathbb{R}^{2m}$$

It then follows from the definition of our topology that $\Phi_n : \pi^{-1}(U_n) \to \Phi_n(\pi^{-1}(U_n))$ is a homeomorphism, since subsets of $\pi^{-1}(U_n)$ is open in $\pi^{-1}(U_n)$ if and only if it is open in M. We have proved that (U_n, φ_n) form an atlas.

To see that the atlas (U_n, Φ_n) is smooth, observe that

$$\begin{aligned}
\Phi_\beta \circ \Phi_\alpha^{-1}(\mathbf{x}^1, \dots, \mathbf{x}^m, \mathbf{y}^1, \dots, \mathbf{y}^m) &= \Phi_\beta(\varphi_\alpha^{-1}(\mathbf{x}^1, \dots, \mathbf{x}^m), \sum_{i=1}^m \mathbf{y}^i \frac{\partial}{\partial \mathbf{x}^i}\Big|_p) \\
&= \Phi_\beta\Big(\varphi_\alpha^{-1}(\mathbf{x}^1, \dots, \mathbf{x}^m), \sum_{i=1}^m \mathbf{y}^i \sum_{j=1}^m \frac{\partial \tilde{\mathbf{x}}^j}{\partial \mathbf{x}^i}\Big|_p \frac{\partial}{\partial \tilde{\mathbf{x}}^j}\Big|_p\Big) \\
&= \Big(\varphi_\beta \circ \varphi_\alpha^{-1}(\mathbf{x}^1, \dots, \mathbf{x}^m), \sum_{i=1}^m \mathbf{y}^i \frac{\partial \tilde{\mathbf{x}}^1}{\partial \mathbf{x}^i}\Big|_p, \dots, \sum_{i=1}^m \mathbf{y}^i \frac{\partial \tilde{\mathbf{x}}^m}{\partial \mathbf{x}^i}\Big|_p\Big)
\end{aligned}$$

is smooth. To show that TM with the smooth atlas $(\pi^{-1}(U_n), \Phi_n)$ is a smooth manifold, it remains to show TM is second countable and Hausdorff. Note that TM is second countable since $\pi^{-1}(U_n)$ is an open cover of TM consisting of second countable space. Lastly, to see TM is Hausdorff, observe that if v_p, v_q is not in the same fiber, then we can separate them with $\pi^{-1}(U_p), \pi^{-1}(U_q)$ where $U_p \cap U_q = \varnothing$, and if they are in the same fiber, we can separate them in the second component.

With TM given a smooth structure, immediately, for each smooth map $F : M \to N$, we can define its global differential $F_* : TM \to TN$ by

$$F_*(p, v) \triangleq \Big(F(p), F_{*,p}(v)\Big)$$

which is smooth since for induced chart $(\pi^{-1}(U), \Phi_\varphi)$ and $(\pi^{-1}(V), \Phi_\psi)$ such that $F(U) \subseteq V$ we have $F_*(\pi^{-1}(U)) \subseteq \pi^{-1}(V)$ and

$$\Phi_\psi \circ F \circ \Phi_\varphi^{-1}(\mathbf{x}^1, \dots, \mathbf{x}^m, v^1, \dots, v^m) = (\tilde{\mathbf{x}}^1, \dots, \tilde{\mathbf{x}}^n, \sum_{j=1}^m \frac{\partial \widehat{F}^1}{\partial \mathbf{x}^j}(p)v^j, \dots, \sum_{j=1}^m \frac{\partial \widehat{F}^n}{\partial \mathbf{x}^j}(p)v^j)$$

Moreover, we can now talk about the smoothness of the **vector field X on M**, i.e., function $X : M \to TM$ such that $X(p) = (p, X_p)$ for some $X_p \in T_pM$. Immediately, given a chart (U, φ) and inducing a chart $(\pi^{-1}(U), \Phi)$ for TM as in Equation 10.10, we see

$$\Phi \circ X \circ \varphi^{-1}(\mathbf{x}^1, \dots, \mathbf{x}^m) = (\mathbf{x}^1, \dots, \mathbf{x}^m, X^1(\mathbf{x}), \dots, X^m(\mathbf{x}))$$

where $\sum_j X^j(p) = X_p$ for all $p \in U$. This then shows that $X : M \to TM$ is smooth on U if and only if $X^1, \dots, X^m$ are smooth on U. In particular, fixing some $p \in M$ and $v \in T_pM$, we can construct a smooth vector field X on M such that $X_p = v$ by first fixing some chart (U, φ) centering p, and let $\rho : M \to \mathbb{R}$ be a smooth function such that $\rho \equiv 1$ on $\varphi^{-1}(\overline{B_r(\mathbf{0})})$ and $\operatorname{supp} \rho \subseteq U$, and define

$$X_q \triangleq \begin{cases} \rho(q) \sum_i v^i \frac{\partial}{\partial \mathbf{x}^i}\Big|_q & \text{if } q \in U \\ 0 & \text{if } q \in M \setminus \operatorname{supp} \rho \end{cases}$$

where functions ρ comes from Corollary 10.1.4. Moreover, we see that if we define addition and $\mathbb{R}$-scalar multiplication on the space of smooth vector field $\mathfrak{X}(M)$ on M by

$$(X + Y)_p \triangleq X_p + Y_p \text{ and } (aX)_p \triangleq a(X_p)$$

then $\mathfrak{X}(M)$ forms a vector space over $\mathbb{R}$. In addition to forming a vector space, some tedious efforts shows that $\mathfrak{X}(M)$ form a module over $C^\infty(M)$, if we define the scalar multiplication by

$$(fX)(p) = (p, f(p)X_p)$$

In essence, vector field can be viewed as a way to "differentiate" scalar-valued function $f \in C^\infty(M)$. We define $Xf : M \to \mathbb{R}$ to be

$$Xf(p) \triangleq X_p f$$

Theorem 10.5.1. (Smooth Criteria for vector field) Given a vector field $X : M \to TM$, X is smooth if and only if $Xf : M \to \mathbb{R}$ is smooth for all $f \in C^\infty(\mathbb{R})$

Proof. The only if part follows from noting the local expression

$$Xf(p) = \sum_{j=1}^{m} X^j(p) \frac{\partial f}{\partial \mathbf{x}^j}(p)$$

is clearly smooth. We now prove the if part. Fix $p \in M$ and some chart (U, φ) centering p. Again let $\overline{B_r(\mathbf{0})} \subseteq B_{r'}(\mathbf{0}) \subseteq \varphi(U)$, and use Corollary 10.1.4 to construct a smooth function $\rho : M \to \mathbb{R}$ such that $\rho \equiv 1$ on $\varphi^{-1}(\overline{B_r(\mathbf{0})})$ and $\operatorname{supp} \rho \subseteq U$. For each $j \in \{1, \dots, m\}$, we can now define $\tilde{x}^j \in C^\infty(M)$ by

$$\tilde{x}^j(q) \triangleq \begin{cases} \rho(q)\mathbf{x}^j & \text{if } q \in U \\ 0 & \text{if } q \in M \setminus \operatorname{supp} \rho \end{cases}$$

Compute that for all $q \in \varphi^{-1}(B_r(\mathbf{0}))$

$$X\tilde{x}^j(q) = \sum_{i=1}^{m} X^i(q) \frac{\partial}{\partial \mathbf{x}^i}\Big|_q \tilde{x}^j = X^j(q)$$

Then by premise, X^j is smooth on $B_r(\mathbf{0})$. The result then follows from the same procedure on different j and charts. ■

At this point, one should check that given some vector field $X : M \to TM$, we can treat X as a **derivation**, since X can be treated as a linear transformation from $C^\infty(M)$ to itself that satisfy

$$X(fg) = fXg + gXf$$

In particular, not only all vector fields can be treated as derivations, all derivations can also be identified as smooth vector field.

Theorem 10.5.2. (Vector field is Derivation) A map $D : C^\infty(M) \to C^\infty(M)$ is a derivation if and only if for some smooth vector field X, $Df = Xf$ for all $f \in C^\infty(M)$.

Proof. We have proved the if part. Now fix some derivation $D : C^\infty(M) \to C^\infty(M)$. For each $p \in M$, we can define a tangent vector $D_p \in T_pM$ by

$$D_p f \triangleq (Df)(p)$$

This then give us a vector field $X : M \to TM$ such that $Df = Xf$ for all $f \in C^\infty(M)$. The fact that X is smooth follows from Theorem 10.5.1. ∎

We now prove the Theorem that give us the pushforward of vector field. Given a smooth map $F : M \to N$, and two vector field X, Y on M, N, we say X, Y are F**-related** if $Y_{F(p)} = F_{*,p}(X_p)$ for all $p \in M$.

Theorem 10.5.3. (Smooth Vector Field Pushforward) Given a diffeomorphism $F : M \to N$. For every $X \in \mathfrak{X}(M)$, there exists a unique smooth vector field Y on N that is F-related to X.

Proof. Because F is onto, obviously the vector field Y on N that is F-related to X can only be

$$Y_q \triangleq F_{*,F^{-1}(q)}(X_{F^{-1}(q)})$$

Note that Y is well defined because F is one-to-one. To see $Y : N \to TN$ is indeed smooth, just observe that Y is the composition of the following smooth map

$$N \xrightarrow{F^{-1}} M \xrightarrow{X} TM \xrightarrow{F_*} TN$$

∎

Theorem 10.5.4. (Criteria for F-related) Given a smooth map $F : M \to N$ and two smooth vector fields $X \in \mathfrak{X}(M), Y \in \mathfrak{X}(N)$, they are F-related if and only if for all $f \in C^\infty(N)$ we have

$$X(f \circ F) = Yf \circ F$$

Proof. Observe that for all $p \in M$

$$(X(f \circ F))(p) = X_p(f \circ F) = F_{*,p}(X_p)f \text{ and } Yf \circ F(p) = Y_{F(p)}f$$

The result then follows. ∎

We now discuss ways of combining two smooth vector field to obtain another smooth vector field. Given $X, Y \in \mathfrak{X}(M)$, we define **Lie bracket** $[X, Y] : C^\infty(M) \to C^\infty(M)$ **of** X, Y by

$$[X, Y]f \triangleq XYf - YXf$$

which is clearly well-defined. Moreover, by some trivial computation, one see that $[X, Y]$ form a derivation, thus can be treated as a smooth vector field. With more trivial computation, we see that Lie bracket satisfy

(a) **Bilinearity**: For all $a, b \in \mathbb{R}$

$$[aX + bY, Z] = a[X, Z] + b[Y, Z]$$
$$[X, aY + bZ] = a[X, Y] + b[X, Z]$$

(b) **Alternating property**:

$$[X, X] = 0$$

(c) **Jacobi Identity**:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

(d) **Anti-symmetry**:

$$[X, Y] = -[Y, X]$$

(e) For $f, g \in C^\infty(M)$,

$$[fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X$$

In particular, we see that Lie bracket can be pushforward.

Theorem 10.5.5. (Naturality of the Lie Bracket) Given a smooth map $F : M \to N$ and two pairs of smooth vector field $X_1, X_2 \in \mathfrak{X}(M), Y_1, Y_2 \in \mathfrak{X}(N)$, if X_1, X_2 are respectively F-related to Y_1, Y_2, then $[X_1, X_2]$ is also F-related to $[Y_1, Y_2]$.

Proof. Use Theorem 10.5.4 to compute for all $f \in C^\infty(N)$

$$X_1X_2(f \circ F) = X_1(Y_2 f \circ F) = Y_1Y_2 f \circ F$$

In other words, X_1X_2 is F-related with Y_1Y_2. The result then follows from linearity of F_* and similar argument for X_2X_1, Y_2Y_1. ■

Note that given two smooth vector field $X, Y \in \mathfrak{X}(M)$ and some chart $(U, \mathbf{x})$, if we write

$$X = \sum_{i=1}^{n} X^i(\mathbf{x}) \frac{\partial}{\partial \mathbf{x}^i} \text{ and } Y = \sum_{j=1}^{n} Y^j(\mathbf{x}) \frac{\partial}{\partial \mathbf{x}^j}$$

there exists an explicit formula for $[X, Y]$

$$\begin{aligned}
[X,Y]f(\mathbf{x}) &= \sum_{i=1}^{n} X^i(\mathbf{x}) \frac{\partial}{\partial \mathbf{x}^i} \sum_{j=1}^{n} Y^j \frac{\partial f}{\partial \mathbf{x}^j} - \sum_{j=1}^{n} Y^j(\mathbf{x}) \frac{\partial}{\partial \mathbf{x}^j} \sum_{i=1}^{n} X^i \frac{\partial f}{\partial \mathbf{x}^i} \\
&= \sum_{i,j} X^i \Big(\frac{\partial Y^j}{\partial \mathbf{x}^i} \frac{\partial f}{\partial \mathbf{x}^j} + Y^j \frac{\partial^2 f}{\partial \mathbf{x}^i \partial \mathbf{x}^j} \Big) - \sum_{i,j} Y^j \Big(\frac{\partial X^i}{\partial \mathbf{x}^j} \frac{\partial f}{\partial \mathbf{x}^i} + X^i \frac{\partial^2 f}{\partial \mathbf{x}^j \partial \mathbf{x}^i} \Big) \\
&= \sum_{i,j} \left(X^i \frac{\partial Y^j}{\partial \mathbf{x}^i} - Y^i \frac{\partial X^j}{\partial \mathbf{x}^i} \right) \frac{\partial f}{\partial \mathbf{x}^j}(\mathbf{x})
\end{aligned}$$

In other words,

$$[X, Y] = \sum_{i,j} \left(X^i \frac{\partial Y^j}{\partial \mathbf{x}^i} - Y^i \frac{\partial X^j}{\partial \mathbf{x}^i} \right) \frac{\partial}{\partial \mathbf{x}^j}$$

10.6 Integral Curve and Flow

Abstract

Note that in this section, I, J always stand for open intervals and integral is always defined on some open interval.

Given some smooth manifold M, and some smooth map $\theta : \mathbb{R} \times M \to M$, we say θ is a **smooth global flow**, or an **1-parameter group of diffeomorphism**, if when we define for all $t \in \mathbb{R}$ a function $\theta_t : M \to M$ by

$$\theta_t(p) \triangleq \theta(t, p)$$

we have the following properties

(i) $\theta_t : M \to M$ is smooth diffeomorphism for all $t \in \mathbb{R}$.

(ii) $\theta_0 = \mathbf{id}$.

(iii) For all $t, s \in \mathbb{R}, \theta_{t+s} = \theta_t \circ \theta_s$.

Immediately we see where does the name 1-parameter group of diffeomorphism comes from: The set $\{\theta_t : t \in \mathbb{R}\}$ clearly form a group with function composition. Moreover, we see that for each $p \in M$, $\theta_t(p)$, when treated as a function in t, is a smooth curve in M, since we have the following commutative diagram

$$\begin{tikzcd}
\mathbb{R} \times M \arrow[r, "\theta"] & M \\
\mathbb{R} \arrow[u, "F"] \arrow[ur, "\theta_t(p)"'] &
\end{tikzcd}$$

where $F(t) \triangleq (t, p)$. With this fact in mind and some tedious effort, we can induce a vector field X by defining

$$X_p f \triangleq \frac{df(\theta_t(p))}{dt}\Big|_{t=0} \tag{10.11}$$

To see X is smooth, we see that if we write locally

$$\theta_t(\mathbf{x}) = (\mathbf{y}^1(t, \mathbf{x}), \dots, \mathbf{y}^n(t, \mathbf{x}))$$

by Chain Rule we have

$$X_{\mathbf{x}} f = \frac{\partial f(\mathbf{y}^1(t,\mathbf{x}),\dots,\mathbf{y}^n(t,\mathbf{x}))}{\partial t}\Big|_{t=0}$$
$$= \sum_{i=1}^n \frac{\partial f}{\partial \mathbf{y}^i}(\mathbf{x})\frac{\partial \mathbf{y}^i}{\partial t}(t,\mathbf{x})\Big|_{t=0}$$

This then give us

$$Xf(\mathbf{x}) = \sum \frac{\partial \mathbf{y}^i}{\partial t}(0,\mathbf{x})\frac{\partial f}{\partial \mathbf{y}^i}$$

It follows from $\mathbf{y}$ being smooth that Xf is smooth. We have shown smooth global flow induce smooth vector field by means of Equation 10.11. We now show that on compact manifold, we can conversely induce smooth global flow from smooth vector filed.

If we say $\gamma : I \to M$ is an **integral curve with respect to some vector field** X, we mean that $\gamma'(t) = X_{\gamma(t)}$ for all $t \in I$, and if we say γ centers p, we mean $0 \in I$ and $\gamma(0) = p$. Immediately, we see from Picard-Lindelof Theorem that for all p, there always exists some integral curve centering p. Moreover, given two integral curve $\gamma_1, \gamma_2 : I_1, I_2 \to M$ centering p, if we let $J \triangleq I_1 \cap I_2$ and

$$S \triangleq \{t \in J : \gamma_1(t) = \gamma_2(t)\}$$

By continuity we see S is closed in J, and by Picard-Lindelof Theorem we see S is also open in I. It then follows from J being also an open interval that $S = J$. In other words, γ_1, γ_2 agree on J. This then implies the existence of some integral curve $\gamma : I \to M$ centering p such that every integral curve $\gamma_1 : I_1 \to M$ centering p satisfy $I_1 \subseteq I$ and

$$\gamma(t) = \gamma_1(t) \text{ for all } t \in I_1$$

We call this curve the **maximal integral curve centering** p.

Theorem 10.6.1. (Smooth vector field induce smooth global flow on smooth compact manifold) If M is compact and $X \in \mathfrak{X}(M)$, then for each $p \in M$, the maximal integral curve centering p is defined on the whole $\mathbb{R}$ and if we define $\theta : \mathbb{R} \times M \to M$ by

$$\theta(t,p) \triangleq \alpha_p(t) \text{ where } \alpha_p \text{ is the maximal integral curve centering } p$$

then θ form a smooth global flow.

Proof. Let $\alpha_p : I_p \to M$ be the maximal integral curve centering p. Our definition for θ is now only on

$$\bigcup_{p\in M} I_p \times \{p\} \subseteq \mathbb{R}\times M$$

Let I_p be the domain of the maximal integral curve centering p. Clearly, for all $p\in M$ there exists some open set U_p containing $\varphi(I_p,p)$. Because M is compact, we may select some finite open cover $U_{p_1},\dots,U_{p_N}$. Define

$$I \triangleq \bigcap_{i=1}^N I_{p_i}$$

It is clear that $0\in I$. It then follows that I is an non-empty open interval.
Now observe that for all $s\in I_p$ if we let

$$I \triangleq \{t\in\mathbb{R} : t+s\in I_p\}$$

and define $\gamma : I\to M$ by

$$\gamma(t)\triangleq \theta_{t+s}(p)$$

then γ is an integral curve centering $\theta_s(p)$. This implies $I\subseteq I_{\theta(s,p)}$ and

$$\theta_t\circ\theta_s(p)=\theta_{t+s}(p) \text{ for all } t\in I$$

■

Now, suppose θ is a smooth global flow induced by smooth vector field X. Given some scalar function $f\in C^\infty(M)$, we see that if we define the **Lie derivative $\mathcal{L}_X f\in C^\infty(M)$ of f with respect to X** by

$$\mathcal{L}_X f(p)\triangleq \frac{d}{dt}\Big|_{t=0}(f\circ\theta_t)(p)$$

we have

$$\begin{aligned}\mathcal{L}_X f(p) &= (f\circ\alpha)'(0) \text{ where } \alpha(t)\triangleq\theta_t(p)\\ &= \alpha'(0)f\\ &= X_p f\end{aligned}$$

In other words,

$$\mathcal{L}_X f = Xf$$

Given some tangent vector valued function, i.e., some smooth vector filed $Y\in\mathfrak{X}(M)$, we can similarly define the **Lie derivative $\mathcal{L}_X Y\in\mathfrak{X}(M)$ of Y with respect to X**

by

$$(\mathcal{L}_X Y)_p f \triangleq \lim_{t\to 0} \frac{(\theta_{-t})_{*,\theta_t(p)}(Y_{\theta_t(p)})f - Y_p f}{t}$$

We shall prove that $\mathcal{L}_X Y$ is well defined.

Theorem 10.6.2. (Existence of Lie derivative) If $X, Y \in \mathfrak{X}(M)$, then

$$\mathcal{L}_X Y = [X, Y]$$

Proof. We first define **pushforward** $\widehat{Y}_t \in \mathfrak{X}(M)$ of Y for all $t \in \mathbb{R}$ by letting $\widehat{Y}_t$ to be the unique smooth vector field on M so that $\widehat{Y}_t, Y$ is θ_t-related, or equivalently, $Y, \widehat{Y}_t$ is θ_{-t}-related. Explicitly, we may write

$$Y_{\theta_t(p)} = (\theta_t)_{*,p}(\widehat{Y}_t)_p$$

We can now rewrite

$$\begin{aligned}
(\mathcal{L}_X Y)_p f &= \lim_{t\to 0} \frac{(\theta_{-t})_{*,\theta_t(p)}(Y_{\theta_t(p)})f - Y_p f}{t} \\
&= \lim_{t\to 0} \frac{(\widehat{Y}_t)_p f - Y_p f}{t} \\
&= \lim_{t\to 0} \frac{(\widehat{Y}_t - \widehat{Y}_0)_p f}{t}
\end{aligned}$$

Using Theorem 10.5.4, we may compute, for all $p \in M$,

$$\frac{d}{dt}\Big|_{t=0} (\widehat{Y}_t)(f \circ \theta_t)(p) = \frac{d}{dt}\Big|_{t=0} Y f \circ \theta_t(p) = X_p(Yf)$$

On the other hands, we may also compute with some limit argument that

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} (\widehat{Y}_t)(f \circ \theta_t)(p) &= \lim_{t\to 0} \frac{(\widehat{Y}_t)(f \circ \theta_t)(p) - (\widehat{Y}_0)(f \circ \theta_t)(p) + (\widehat{Y}_0)(f \circ \theta_t)(p) - (\widehat{Y}_0)(f \circ \theta_0)(p)}{t} \\
&= \lim_{t\to 0} \frac{(\widehat{Y}_t - \widehat{Y}_0)_p(f \circ \theta_t)}{t} + \lim_{t\to 0} \frac{Y_p(f \circ \theta_t) - Y_p(f \circ \theta_0)}{t} \\
&= (\mathcal{L}_X Y)_p f + \lim_{t\to 0} \frac{Y_p(f \circ \theta_t - f)}{t} \\
&= (\mathcal{L}_X Y)_p f + Y_p(\lim_{t\to 0} \frac{(f \circ \theta_t - f)}{t}) \\
&= (\mathcal{L}_X Y)_p f + Y_p(Xf)
\end{aligned}$$

We have deduced

$$(\mathcal{L}_X Y)_p f + Y_p(Xf) = X_p(Yf)$$

which is just

$$\mathcal{L}_X Y = [X, Y]$$

■

10.7 Differential Form

Abstract

For each $p \in M$, we use $T_p^*M \triangleq (T_pM)^*$ to denote the dual space of T_pM and we use λ^i to denote the dual basis of $\frac{\partial}{\partial \mathbf{x}^i}$ when given local coordinate. Let $\omega \in T_p^*M$ and write

$$\omega = \sum_i \omega^i \lambda^i = \sum_j \tilde{\omega}^j \tilde{\lambda}^j$$

We may compute

$$\omega_i = \omega\Big(\frac{\partial}{\partial \mathbf{x}^i}\Big|_p\Big) = \omega\Big(\sum_j \frac{\partial \tilde{\mathbf{x}}^j}{\partial \mathbf{x}^i}(p)\frac{\partial}{\partial \tilde{\mathbf{x}}^j}\Big|_p\Big) = \sum_j \frac{\partial \tilde{\mathbf{x}}^j}{\partial \mathbf{x}^i}(p)\tilde{\omega}^j$$

In other words,

$$\lambda^i = \sum_j \frac{\partial \tilde{\mathbf{x}}^j}{\partial \mathbf{x}^i}\tilde{\lambda}^j$$

Again, we define the **cotangent bundle** T^*M to be

$$T^*M \triangleq \coprod_{p\in M} T_p^*M$$

We now see that given the natural charts

$$\Phi(p, \sum_i \xi^i \lambda^i) = \Big(\varphi(p), \xi^1, \dots, \xi^n\Big)$$

when we give T^*M the topology and the smooth structure the way we give to TM, the atlas is indeed smooth, since

$$\begin{aligned}\tilde{\Phi}\circ\Phi^{-1}(\mathbf{x}^1,\dots,\mathbf{x}^n,\xi^1,\dots,\xi^n) &= \tilde{\Phi}\Big(\varphi^{-1}(\mathbf{x}^1,\dots,\mathbf{x}^n),\sum_i \xi^i\lambda^i\Big)\\ &= \tilde{\Phi}\Big(\varphi^{-1}(\mathbf{x}^1,\dots,\mathbf{x}^n),\sum_i \xi^i\sum_j \frac{\partial\tilde{\mathbf{x}}^j}{\partial\mathbf{x}^i}\tilde{\lambda}^j\Big)\\ &= \Big(\tilde{\mathbf{x}}^1,\dots,\tilde{\mathbf{x}}^n,\sum_i \frac{\partial\tilde{\mathbf{x}}^1}{\partial\mathbf{x}^i}(\mathbf{x})\xi^i,\dots,\sum_i \frac{\partial\tilde{\mathbf{x}}^n}{\partial\mathbf{x}^i}(\mathbf{x})\xi^i\Big)\end{aligned}$$

Given a section $\omega : M\to T^*M$ of the cotangent bundle T^*M, we say ω is a **covector field** or a **1-form**, and we may now discuss the smoothness of ω. Note that the natural charts show that ω is smooth if and only if ω^i are all smooth where

$$\omega=\sum_i \omega^i\lambda^i$$

Given $X\in\mathfrak{X}(M)$, we may also define $\omega X : M\to\mathbb{R}$ by

$$\omega X(p)\triangleq\omega_p X_p=\sum_{i=1}\omega^i X^i(p)$$

An argument similar to that of Theorem 10.5.1 shows that ω is smooth if $\omega X\in C^\infty(M)$ for all $X\in\mathfrak{X}(M)$. We denote the space of 1-forms by $\Omega(M)$. We see that $\Omega(M)$ form a module over $C^\infty(M)$ if we define $f\omega : M\to T^*M$

$$f\omega(p)=f(p)\omega_p$$

Note that for all $f\in C^\infty(M)$, we may associate f with $df\in\Omega(M)$ by defining

$$dfX\triangleq Xf$$

Note that for convenience, we often use df_p to denote $df(p)\in T_p^*M$, and with some tedious effort, one may deduce the local expression

$$df=\sum_i \frac{\partial f}{\partial\mathbf{x}^i}d\mathbf{x}^i$$

where $\mathbf{x}^i : U\to\mathbb{R}$ is defined by

$$\mathbf{x}^i(\varphi^{-1}(\mathbf{y}^1,\dots,\mathbf{y}^n))\triangleq\mathbf{y}^i$$

We may now prove some expected result.

Theorem 10.7.1. (Constant function and zero differential) Given some connected manifold M and $f \in C^\infty(M)$

$$f \text{ is constant} \iff df = 0$$

Proof. If f is constant, then locally

$$df = \sum_i \frac{\partial f}{\partial \mathbf{x}^i} d\mathbf{x}^i = 0$$

Now, suppose $df = 0$. Fix $p \in M$. Because f is continuous, we know $f^{-1}(\{f(p)\})$ is closed. Again, locally we have

$$0 = df = \sum_i \frac{\partial f}{\partial \mathbf{x}^i} d\mathbf{x}^i$$

which implies f is constant on each chart U. This implies $f^{-1}(\{f(p)\})$ is open. It then follows from M being connected that $f^{-1}(\{f(p)\}) = M$. ■

Now, for each smooth map $F : M \to N$, $f \in C^\infty(N)$ and $\omega \in \Omega(N)$, we may define their **pullbacks** $F^* f \in C^\infty(M), F^*\omega \in \Omega(M)$ by

$$F^* f \triangleq f \circ F \text{ and } (F^*\omega)_p \triangleq (F_{*,p})^*(\omega_{F(p)})$$

With this definition and some tedious effort, we see that pullbacks is compatible with differential in the sense that

$$F^*(dh) = d(F^*h)$$

The definition tell us that pullbacks is linear

$$F^*(c\omega_1 + \omega_2) = cF^*\omega_1 + F^*\omega_2$$

and is compatible to the module

$$F^*(g\alpha) = (F^*g)(F^*\alpha)$$

This then shows that the pullbacks $F^*\omega$ of smooth 1-from on N is indeed smooth since if we write $\omega = \sum_i \omega^i d\mathbf{y}^i$ locally, we may write

$$\begin{aligned}
F^*\omega &= F^*\Big(\sum_i \omega^i d\mathbf{y}^i\Big) \\
&= \sum_i F^*\omega^i F^* d\mathbf{y}^i \\
&= \sum_i (\omega^i \circ F) d(F^*\mathbf{y}^i) \\
&= \sum_i (\omega^i \circ F) dF^i = \sum_{i,j} (\omega^i \circ F) \frac{\partial F^i}{\partial \mathbf{x}^j} d\mathbf{x}^j
\end{aligned}$$

At this point, one can see that if $S \overset{i}{\hookrightarrow} M$ is an immersed submanifold, then the pullback $i^*\omega \in \Omega(S)$ of the 1-form $\omega \in \Omega(M)$ is just the restriction of ω.

10.8 Lie Group

By a **Lie group**, we mean a smooth manifold equipped with a group structure such that the inversion and group addition are both smooth map, or equivalently, that

$$M^2 \to M; (g, h) \mapsto gh^{-1}$$

is smooth.

Chapter 11

Complex Analysis

11.1 Cauchy Integral Theorem

Abstract

Note that in this section, when we talk about derivative of function defined on subset of real line, we do consider one-sided derivative, i.e., for $\gamma : [a, b] \to \mathbb{C}$ to be C^1, the limit of $\frac{\gamma(a+h)-\gamma(a)}{h}$ as $h \searrow 0$ must exist.

Let $[a, b] \subseteq \mathbb{R}$ be some compact interval. We say $\gamma : [a, b] \to \mathbb{C}$ is a **parametrization** if

(a) $\gamma(x) \neq \gamma(y)$ unless $x = a$ and $y = b$.

(b) There exists some partition $\{a = c_0 < \cdots < c_N = b\}$ such that $\gamma|_{[c_n,c_{n+1}]} : [c_n, c_{n+1}] \to \mathbb{C}$ are C^1 wish non-vanishing derivative.

A parametrization $\gamma : [a, b] \to \mathbb{C}$ is said to be **closed** if $\gamma(a) = \gamma(b)$. Two parametrizations $\gamma : [a, b] \to \mathbb{C}, \alpha : [c, d] \to \mathbb{C}$ are said to be **equivalent** if there exists some C^1 bijection $s : [a, b] \to [c, d]$ such that

$$\gamma(t) = \alpha(s(t)) \text{ and } s'(t) > 0 \text{ for all } t \in [a, b]$$

Inverse Function Theorem shows that our definition for parametrization equivalence is indeed an equivalence relation. We then can define **contour** to be the equivalence class of parametrizations. Immediately, we see that all parametrization of a contour have the same image and if any of them is closed, then all of them are closed. This allow us to talk about the image of a contour and if a contour is closed. If we define **length** for parametrization $\gamma : [a, b] \to \mathbb{C}$ to be $\int_a^b \gamma'(t)dt$, then a change of variables shows that all parametrizations in $[\gamma]$ have the same length as γ. This allow us to define the **length for contour**. Now, given some parametrization $\gamma : [a, b] \to \mathbb{C}$ and some continuous complex-valued function f defined on the image $\gamma([a, b])$, we can define its **contour integral** by

$$\int_\gamma f(z)dz \triangleq \int_a^b f(\gamma(t))\gamma'(t)dt$$

Again the change of variables shows that our definition is well defined for contours. Similar to the real case, we have the estimation

$$\left| \int_\gamma f dz \right| \leq LM$$

where L is the length of γ and M is the maximum of $|f|$ on γ. We can also generalize Part 2 of Fundamental Theorem of Calculus to contour integral: If $D \subseteq \mathbb{C}$ is open, $f : D \to \mathbb{C}$ is continuous, and $F : D \to \mathbb{C}$ satisfy $F'(z) = f(z)$ for all $z \in D$, then for all contour $\gamma : [a, b] \to D$, we have

$$\int_\gamma f dz = F(\gamma(b)) - F(\gamma(a))$$

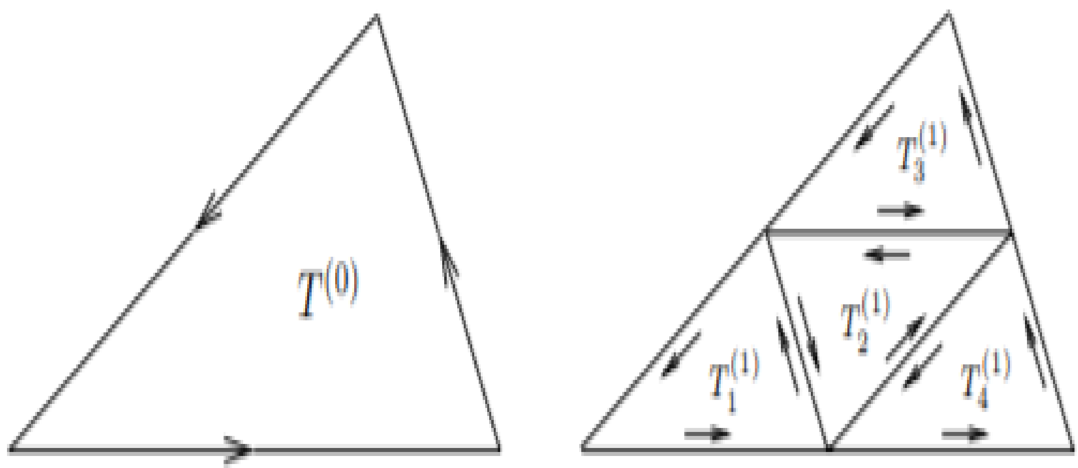
We are now ready to state Cauchy's Integral Theorem for triangles. Note that term "closed triangle" as a set include both its interior area and boundary. For example, a closed triangle can be

$$\{x + iy \in \mathbb{C} : x \in [0, 1] \text{ and } y \in [0, x]\}$$

Theorem 11.1.1. (Cauchy's Integral Theorem for triangles) If $D \subseteq \mathbb{C}$ is open, $f : D \to \mathbb{C}$ is holomorphic and D contain some closed triangle T, then

$$\int_{\partial T} f dz = 0$$

Proof. Denote T by $T^{(0)}$. Construct triangles $T_1^{(1)}, T_2^{(1)}, T_3^{(1)}, T_4^{(1)}$ as in the figure below.



Obviously, we may parametrize the boundaries of these triangles so that

$$\int_{\partial T^{(0)}} f dz = \sum_{n=1}^{4} \int_{\partial T_n^{(1)}} f dz$$

Taking absolute value on both side, we deduce

$$\left| \int_{\partial T^{(0)}} f dz \right| \leq 4 \left| \int_{\partial T_j^{(1)}} f dz \right| \text{ for some } j \in \{1, 2, 3, 4\}$$

Denote $T_j^{(1)}$ by $T^{(1)}$. Repeating this process, we obtain a decreasing sequence of triangles

$$T^{(0)} \supseteq T^{(1)} \supseteq \cdots \supseteq T^{(n)} \supseteq \cdots$$

with the property that

$$\left| \int_{\partial T^{(0)}} f dz \right| \leq 4^n \left| \int_{\partial T^{(n)}} f dz \right| \tag{11.1}$$

Let $d^{(n)}$ and $p^{(n)}$ denote the diameter and perimeter of $T^{(n)}$ for all $n \in \mathbb{Z}_0^+$. Some tedious effort shows that

$$d^{(n)} = 2^{-n} d^{(0)} \text{ and } p^{(n)} = 2^{-n} p^{(0)} \tag{11.2}$$

Theorem 2.3.2 implies

$$\bigcap_{n \in \mathbb{N}} T^{(n)} = \{z_0\} \text{ for some } z_0 \in D$$

Because f is holomorphic at z_0, we may write $f : D \to \mathbb{C}$ by

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + o(z - z_0)(z - z_0)$$

Clearly the first two terms have antiderivatives. Using Equation 11.1 and Equation 11.2, we may now estimate

$$\begin{aligned}\left|\int_{\partial T^{(0)}} f(z)dz\right| \leq 4^n \left|\int_{\partial T^{(n)}} f(z)dz\right| &= 4^n \left|\int_{\partial T^{(n)}} o(z - z_0)(z - z_0)dz\right| \\ &\leq 4^n p^{(n)} d^{(n)} \max_{z \in \partial T^{(n)}} |o(z - z_0)| \\ &= p^{(0)} d^{(0)} \max_{z \in \partial T^{(n)}} |o(z - z_0)| \to 0 \text{ as } n \to \infty\end{aligned}$$

■

By $D \subseteq \mathbb{C}$ being **star-convex with center** z_*, we mean that for all $z \in D$, the contour $\gamma : [0, 1] \to \mathbb{C}$ defined by

$$\gamma(t) \triangleq z_* + t(z - z_*)$$

satisfy $\gamma([0, 1]) \subseteq D$.

Theorem 11.1.2. (Cauchy's Integral Theorem for closed contour in a star domain) Suppose $D \subseteq \mathbb{C}$ is open and star-convex with centre z_*. If $f : D \to \mathbb{C}$ is holomorphic, then $F : D \to \mathbb{C}$ defined by

$$F(z) \triangleq \int_\gamma f(w)dw \text{ where } \gamma : [0, 1] \to D \text{ is defined by } \gamma(t) \triangleq z_* + t(z - z_*)$$

is an antiderivative of f.

Proof. Fix $z_0 \in D$. Because D is open, there exists some open ball $B_\epsilon(z_0)$ small enough to be contained by D. For all $z \in B_\epsilon(z_0)$, the closed triangle T specified by the vertices $\{z_*, z, z_0\}$ is contained by D, since all $p \in T$ lies in some line segment joining z_* and w where w is some point that lies in the line segment joining z and z_0. We then can apply Cauchy's Integral Theorem for triangles to have the estimate

$$\begin{aligned}\left|\frac{F(z) - F(z_0)}{z - z_0} - f(z_0)\right| &= \left|\frac{\int_\gamma [f(w) - f(z_0)]dw}{z - z_0}\right| \\ &\leq \max_{w \in \gamma} |f(w) - f(z_0)| \to 0 \text{ as } z \to z_0\end{aligned}$$

where γ is the line segment traveling from z_0 to z. ■

Chapter 12

Beauty

12.1 Fundamental Theorem of Algebra

Theorem 12.1.1. (Fundamental Theorem of Algebra)

12.2 Euler's Formula

Abstract

This section give a precise definition to exponential function and trigonometric functions, prove Euler's Formula and prove some of their basic properties seen in the real case. In this section, z, v are complex numbers, and x, y are real numbers.

Suppose that we define

$$\exp(z) \triangleq \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\sin(z) \triangleq \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$

$$\cos(z) \triangleq \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$$

Some properties we are familiar with is now easily seen using basic techniques on sequence and series.

(a) Because $\limsup_{n\to\infty} |c_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$, by Cauchy-Hadamard Theorem, $\exp(z)$, $\sin(z)$ and $\cos(z)$ are defined on the whole complex plane.

(b) $\exp(z) \cdot \exp(v) = \exp(z+v)$ by Merten's Theorem of Cauchy product.

(c) $\frac{d}{dz}\exp(z) = \exp(z)$, $\frac{d}{dz}\sin(z) = \cos(z)$ and $\frac{d}{dz}\cos(z) = -\sin(z)$ on the whole complex plane, since we can differentiate them term by term.

(d) For all $x \in \mathbb{R}$, $\exp(x), \sin(x)$ and $\cos(x)$ lie in $\mathbb{R}$.

(e) $\exp(x)$ strictly increase on $\mathbb{R}$ and $\exp(0) = 1$.

(f) $\exp(x) \nearrow \infty$ as $x \to \infty$.

(g) Because $\exp(-x) = \frac{\exp(0)}{\exp(x)} = \frac{1}{\exp(x)}$, by proposition (f), $\exp(x) \searrow 0$ as $x \to -\infty$.

(h) For all $x \in \mathbb{R}$, $\exp(x) \in \mathbb{R}^+$ by proposition (g).

(i) $\exp(x)$ is convex on $\mathbb{R}$ $\quad (\because (e^x)'' = e^x > 0)$

In particular, we have **Euler's Formula**.

Theorem 12.2.1. (Euler's Formula) For all $z \in \mathbb{C}$, we have

$$\exp(iz) = \cos(z) + i\sin(z)$$

Proof. Define

$$I(n) \triangleq \begin{cases} 1 & \text{if } n \equiv 0 \pmod 4 \\ i & \text{if } n \equiv 1 \pmod 4 \\ -1 & \text{if } n \equiv 2 \pmod 4 \\ -i & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

Compute

$$\begin{aligned} \exp(iz) &= \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{I(n)z^n}{n!} \qquad (12.1) \\ &= \sum_{n=0}^{\infty} \frac{I(2n)}{(2n)!} z^{2n} + \frac{I(2n+1)}{(2n+1)!} z^{2n+1} \qquad \left(\because \text{ this is a sub-sequence of } (12.1)\right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} + i \cdot \frac{(-1)^n}{(2n+1)!} z^{2n+1} \end{aligned}$$

Now, we can conclude

$$\begin{aligned} \cos(z) + i\sin(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} + i \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} + \sum_{n=0}^{\infty} i \cdot \frac{(-1)^n}{(2n+1)!} z^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} + i \cdot \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \exp(iz) \end{aligned}$$

■

12.3 Trigonometric Functions

Following from Euler's Formula, we have the **angle identities**

$$\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$$
$$\sin(z+w) = \cos(z)\sin(w) + \sin(z)\cos(w)$$

Observing from definitions that cos is even an sin is odd in the sense that

$$\cos(-z) = \cos(z) \text{ and } \sin(-z) = -\sin(z)$$

we also have the following angle identities

$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$
$$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$$

Besides the angles identities, we also have the usual fact

$$\cos^2(z) + \sin^2(z) = (\cos(z) + i\sin(z))(\cos(z) - i\sin(z)) = e^{iz}e^{-iz} = 1$$

Now, if we restrict our attention to the real line, following from our definition for cos, by direct computation and some induction, we see

$$\cos(0) = 1 \text{ and } \cos(2) < 0$$

It then follows from IVT such that

$$\{t \in \mathbb{R}^+ : \cos(t) = 0\} \text{ is non empty}$$

We then can define

$$\frac{\pi}{2} \triangleq \inf\{k \in \mathbb{R}^+ : \cos(k) = 0\}$$

From continuity of cos, we see that $\cos(\frac{\pi}{2}) = 0$. It then follows from the greatest lower bound definition of $\frac{\pi}{2}$, IVT and the fact that $\cos(0) = 1 > 0$ that

$$\cos(x) > 0 \text{ for all } 0 < x < \frac{\pi}{2}$$

Then since $\sin'(x) = \cos(x)$, by MVT, we see sin strictly increase on $[0, \frac{\pi}{2}]$. It then follows from

$$\sin^2(\frac{\pi}{2}) + \cos^2(\frac{\pi}{2}) = 1 \text{ and } \sin(0) = 0$$

that $\sin(\frac{\pi}{2}) = 1$. With $\cos(\frac{\pi}{2}) = 0$ and $\sin(\frac{\pi}{2}) = 1$ in mind, we can use angle identity to deduce the value of $\sin, \cos$ at π

$$\cos(\pi) = -1 \text{ and } \sin(\pi) = 0$$

It now follows from the angle identities that **trigonometric functions have the period** 2π

$$\cos(z + \pi) = -\cos(z) \text{ and } \cos(z + 2\pi) = \cos(z)$$
$$\sin(z + \pi) = -\sin(z) \text{ and } \sin(z + 2\pi) = \sin(z)$$

The rest of the monotone behaviors of trigonometric function on $\mathbb{R}$ then follows.

12.4 Equivalent Definitions of Exponential Functions

Abstract

This section give several equivalent definitions of the exponential function.

Theorem 12.4.1. (First Characterization) For all $z \in \mathbb{C}$, we have

$$\lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n = e^z$$

Proof. Fix z and ϵ. Let N satisfy

$$\sum_{n=N+1}^{\infty} \frac{|z|^n}{n!} < \frac{\epsilon}{3}$$

It is clear that

$$\lim_{n\to\infty} \sum_{k=0}^{N} \binom{n}{k} \left(\frac{z}{n}\right)^k = \sum_{k=0}^{N} \frac{z^k}{k!}$$

We then can let N_1 satisfy

$$\left| \sum_{k=0}^{N} \binom{n}{k} \left(\frac{z}{n}\right)^k - \sum_{k=0}^{N} \frac{z^k}{k!} \right| < \frac{\epsilon}{3} \text{ for all } n > N_1$$

Now, because for all k smaller than n we have

$$\binom{n}{k} \frac{1}{n^k} = \frac{1}{k!} \prod_{j=1}^{k-1} (1 - \frac{j}{n}) \leq \frac{1}{k!}$$

we have for all $n > N_1$

$$\begin{aligned}
\left| \left(1 + \frac{z}{n}\right)^n - \sum_{k=0}^{\infty} \frac{z^k}{k!} \right| &\leq \left| \sum_{k=0}^{n} \binom{n}{k} \left(\frac{z}{n}\right)^k - \sum_{k=0}^{\infty} \frac{z^k}{k!} \right| \\
&\leq \left| \sum_{k=0}^{N} \binom{n}{k} \left(\frac{z}{n}\right)^k - \sum_{k=0}^{N} \frac{z^k}{k!} \right| + \left| \sum_{k=N+1}^{n} \binom{n}{k} \left(\frac{z}{n}\right)^k \right| + \left| \sum_{k=N+1}^{\infty} \frac{z^k}{k!} \right| \\
&\leq \left| \sum_{k=0}^{N} \binom{n}{k} \left(\frac{z}{n}\right)^k - \sum_{k=0}^{N} \frac{z^k}{k!} \right| + \sum_{k=N+1}^{n} \binom{n}{k} \frac{|z|^k}{n^k} + \sum_{k=N+1}^{\infty} \frac{|z|^k}{k!} < \epsilon
\end{aligned}$$

■

$$\ln(x) \triangleq \int_1^x \frac{1}{t} dt$$

By FTC (Theorem 7.2.1), it is easy to see that

$$\frac{d}{dx} \ln(x) = \frac{1}{x} \qquad (x \in \mathbb{R}^+)$$

To see

$$\ln(xy) = \ln(x) + \ln(y)$$

Fix $y \in \mathbb{R}^+$ and set

$$f(x) \triangleq \ln(x) \text{ and } g(x) \triangleq \ln(xy)$$

Conclude $f'(x) = g'(x)$, and use FTC (Theorem 7.2.2) to conclude $f - g$ is some fixed constant k. Now, see that

$$g(1) = f(1) + k \implies k = \ln(y)$$

Then, we have

$$\ln(xy) = g(x) = f(x) + k = \ln(x) + \ln(y)$$

Using induction, it is now easy to see

$$\ln(x^n) = n \ln(x) \qquad (n \in \mathbb{Z}_0^+)$$

Theorem 12.4.2. (Second Characterization)

12.5 Equivalent Definitions of Gamma and Beta Functions

12.6 Prime Number Theorem

Chapter 13

and the Beast

13.1 Topologist's Sine Curve

13.2 Long Line

13.3 Bug-Eyed Line

Abstract

This section introduce Bug-Eyed Line, which is a second-countable locally Euclidean space that is not Hausdorff.

Let a, b be distinct real numbers and let $\mathbb{R}'$ be the quotient space of $\mathbb{R} \times \{a, b\}$ where

$$(x, a) \sim (x, b) \text{ for all } x \neq 0$$

The space $\mathbb{R}'$ is commonly referred as the **Bug-Eyed Line**. Now by definition of quotient topology, given a subset E of $\mathbb{R}'$, E is open if and only if

$$\{x \in \mathbb{R} : [(x, a)] \in E\} \text{ and } \{x \in \mathbb{R} : [(x, b)] \in E\} \text{ are both open in } \mathbb{R}$$

With this in mind, it is clear that $\mathbb{R}'$ is not Hausdorff, since $[(0, a)]$ and $[(0, b)]$ can not be distinguished, and it is straightforward to check $\mathbb{R}'$ is locally Euclidean.

Now, suppose we write some subset A of $\mathbb{R} \times \{a, b\}$ in the form

$$\{(x, a) : x \in A_1\} \cup \{(x, b) : x \in A_2\}$$

We can deduce

$$\pi^{-1}(\pi(A)) = \Big\{(x, a) : x \in A_1 \cup (A_2 \setminus \{0\})\Big\} \cup \Big\{(x, b) : x \in A_2 \cup (A_1 \setminus \{0\})\Big\}$$

It is then clear that π is an open mapping, thus $\mathbb{R}'$ is second countable.

13.4 Weierstrass Function

13.5 Fabius Function

13.6 Vitali Set

Abstract

This section construct the Vitali Set in $\mathbb{R}^d$ for reference.

Fix $p > m > 0 \in \mathbb{R}$. Define an equivalence relation in $[0,m]^d$ by

$$\mathbf{x} \sim \mathbf{y} \overset{\triangle}{\iff} \mathbf{x} - \mathbf{y} \in \mathbb{Q}^d$$

Using axiom of choice, we can let $V \subseteq [0,m]^d$ contain exactly one element in each equivalence class. Enumerate $([0,p] \cap \mathbb{Q})^d$ by $\mathbf{z}_n$. For each $n \in \mathbb{N}$, define

$$V + \mathbf{z}_n \triangleq \{\mathbf{x} + \mathbf{z}_n \in \mathbb{R}^d : \mathbf{x} \in V\}$$

It is straightforward to check from definition of V that

(a) $V + \mathbf{z}_n$ are pairwise disjoint.

(b) $[0,m]^d \subseteq \bigsqcup_n (V + \mathbf{z}_n) \subseteq [0, m+p]^d$.

Now, if V is really measurable, we see that

$$0 < m^d \leq \sum_n |V| \leq (m+p)^d \tag{13.1}$$

since

$$\left| \bigsqcup_n V + \mathbf{z}_n \right| = \sum_n |V + \mathbf{z}_n| = \sum_n |V|$$

Now, one simply observe that Equation 13.1 is impossible. It then follows that V is not measurable.

The above argument is often referred to as **Vitali argument**, and can give stronger result.

Theorem 13.6.1. (Positive outer measure set always contain non measurable set) Given $A \subseteq \mathbb{R}^d$, if every subset of A is measurable, then A is null.

Proof. Again define an equivalence relation in $\mathbb{R}^d$ by

$$\mathbf{x} \sim \mathbf{y} \overset{\triangle}{\iff} \mathbf{x} - \mathbf{y} \in \mathbb{Q}^d$$

and again using axiom of choice, we can let V contain exactly one element in each equivalence class. It is clear that

$$\mathbb{R}^d = \bigsqcup_{\mathbf{q} \in \mathbb{Q}^d} V_{\mathbf{q}} \text{ where } V_{\mathbf{q}} \triangleq \{\mathbf{x} + \mathbf{q} \in \mathbb{R}^d : \mathbf{x} \in V\}$$

Thus we have

$$A = \bigsqcup_{\mathbf{q} \in \mathbb{Q}^d} A \cap V_{\mathbf{q}}$$

Note that $A \cap V_{\mathbf{q}}$ is totally disconnected. Because totally disconnected compact set must be a finite collection of points, which is null, it then follows from inner regularity of Lebesgue measure that if $A \cap V_{\mathbf{q}}$ is measurable, then $A \cap V_{\mathbf{q}}$ is null. Then by a proof of contradiction, we must have some $\mathbf{q}$ such that $A \cap V_{\mathbf{q}}$ is not measurable. ■

13.7 Cantor Set

Abstract

This section construct the classical ternary Cantor set and some of its variant, and prove they are uncountable and perfect.

By the term **Classical Ternary Cantor Set** $\mathcal{C}$, one usually mean

$$\mathcal{C} \triangleq \bigcap_{n\in\mathbb{N}} \mathcal{C}_n$$

where $\mathcal{C}_0 \triangleq [0,1]$, and $\mathcal{C}_{k+1}$ is the result of deleting the open middle third of each connected component – which are clearly compact – of $\mathcal{C}_k$. Immediately, one can see that

(a) $\mathcal{C}_n$ has 2^n amount of connected components, which are all compact with the length of $\frac{1}{3^n}$.

(b) $\mathcal{C}$ has zero measure, since $|\mathcal{C}_{n+1}| = |\mathcal{C}_n| - \frac{2^n}{3^{n+1}}$.

(c) $\mathcal{C}$ is totally disconnected, since if $a, b \in \mathcal{C}$ are connected then $[a, b] \subseteq \mathcal{C}$, which is impossible by proposition (a).

(d) $\mathcal{C}$ is perfect, since for each ϵ and $x \in \mathcal{C}$, there exists large enough n such that the length of each connected component of $\mathcal{C}_n$ is smaller than ϵ, and the end point of the connected component – in which x lies – that isn't x would belong to $\mathcal{C}$ and be ϵ-close to x.

(e) The endpoints $\{x \in \mathcal{C} : x \text{ is the endpoint of some connected component of some } \mathcal{C}_n\}$ are countable and dense in $\mathcal{C}$.

With some tedious effort, one can see that $x \in \mathcal{C}$ if and only if

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \text{ for some } a_n \in \{0, 2\}$$

With base 3 representation, one can use a diagonal argument to show $\mathcal{C}$ is uncountable. Another approach to show $\mathcal{C}$ is uncountable is to show non-singleton perfect sets in $\mathbb{R}^n$ are uncountable.

Theorem 13.7.1. (Non-Singleton Perfect Set in $\mathbb{R}^n$ is Uncountable) Given a perfect set $E \subseteq \mathbb{R}^n$, if E contain more than one element, then E must be uncountable.

Proof. ■

Notably, a variant of the Cantor set includes the **Fat Cantor Set**, which is constructed similarly to the classical ternary Cantor set, except that the removed open middle intervals at $n$stage is each of length δ^n, where $0 < \delta < 3$. Note that the construction cannot be done if $\delta > 3$ and that the Fat Cantor Sets all have positive measure, are perfect and totally disconnected.

13.8 Cantor-Lebesgue Function

Abstract

This section construct the Cantor-Lebesgue Function for reference.

Consider the Classical Ternary Cantor Set $\mathcal{C}$. Let

$$\mathcal{D}_n \triangleq [0,1] \setminus \mathcal{C}_n \text{ for all } n$$

For example,

$$\mathcal{D}_1 = (\frac{1}{3}, \frac{2}{3}) \text{ and } \mathcal{D}_2 = (\frac{1}{9}, \frac{2}{9}) \cup (\frac{1}{3}, \frac{1}{2}) \cup (\frac{7}{9}, \frac{8}{9})$$

Because $\mathcal{C}_n$ has 2^n amount of connected components, we know $\mathcal{D}_n$ has $2^n - 1$ amount of connected components. Order these connected components by $\{I_j^n : 1 \leq j \leq 2^n - 1\}$. We now define a sequence of function $f_n : [0,1] \to [0,1]$ by letting

$$f_n(x) \triangleq \begin{cases} 0 & \text{if } x = 0 \\ \frac{j}{2^n} & \text{if } x \in \overline{I_j^n} \\ 1 & \text{if } x = 1 \end{cases}$$

where f_n is linear on $\overline{\mathcal{D}_n}$.

For each $x \in [0,1]$ there exists a (not always unique) base 3 representation

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \text{ for some } a_n \in \{0,1,2\}$$

The Cantor-Lebesgue Function $f : [0,1] \to [0,1]$ is defined as follows. Given $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \in [0,1]$, if there exists some smallest n such that $a_n = 1$, define

$$b_k \triangleq \begin{cases} 0 & \text{if } k < n \text{ and } a_k = 0 \\ 1 & \text{if } k < n \text{ and } a_k = 2 \\ 1 & \text{if } k = n \\ 0 & \text{if } k > n \end{cases}$$

It $1 \notin \{a_n : n \in \mathbb{N}\}$, then we simply define

$$b_n \triangleq \begin{cases} 0 & \text{if } a_n = 0 \\ 1 & \text{if } a_n = 2 \end{cases}$$

We then define

$$f(x) \triangleq \sum_{n=1}^{\infty} \frac{b_n}{2^n}$$

Note that if the base 3 representation of $x \in (0,1)$ has the trailing 0 or 2, then the representation must not be unique, yet the procedure described above does give the same value $f(x)$ while the base 2 representation can be different. Some tedious effort can now be applied to show that

(a) $f(\mathcal{C}) = [0,1]$ where $\mathcal{C}$ is the classical ternary set.

(b) $f : [0,1] \to [0,1]$ is increasing on $[0,1]$.

13.9 Volterra's Function

13.10 Peano Space-filling Curve