$ay'' + by' + c = 0$

$e^{r_1 t}, e^{r_2 t}, r_1 \neq r_2$ $\quad \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1) e^{(r_1 - r_2)t} \neq 0$

$e^{r_1 t}, te^{r_1 t}$ $\quad \begin{vmatrix} e^{r_1 t} & te^{r_1 t} \\ r_1 e^{r_1 t} & e^{r_1 t} + r_1 t e^{r_1 t} \end{vmatrix} = e^{2r_1 t} + r_1 t e^{2r_1 t} - r_1 t e^{2r_1 t}$

$$= e^{2r_1 t} \neq 0$$

Abel's Thm. : $L[y] = y'' + p(t)y' + q(t)y = 0$, $p, q \in C(I)$

y_1, y_2 solutions of $L[y] = 0$.

Then $W[y_1, y_2](t) = ce^{-\int p(t)dt}$.

Note: $W \begin{cases} 0, & \text{if } c = 0 \\ \neq 0, & \text{if } c \neq 0 \end{cases}$

pf. $W[y_1, y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$

$W' = y_1' y_2' + y_1 y_2'' - \cancel{y_2' y_1'} - y_2 y_1'' = y_1(-py_2' - qy_2) - y_2(-py_1' - qy_1)$

$= -py_1 y_2' + py_2 y_1' = -p(y_1 y_2' - y_2 y_1') = -pW$

$\Rightarrow \frac{W'}{W} = -p \Rightarrow \ln|W| = -\int p + C_1 \Rightarrow W = ce^{-\int p}$

- $2x^2 y'' + 3xy' - 15y = 0$

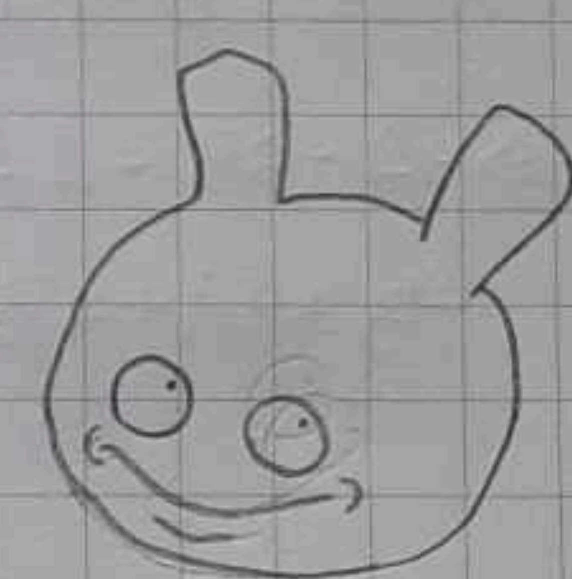
$y_1 = x^{\frac{5}{2}}, \quad y_2 = x^{-3}$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^{\frac{5}{2}} & x^{-3} \\ \frac{5}{2}x^{\frac{3}{2}} & -3x^{-4} \end{vmatrix} = -\frac{11}{2}x^{-\frac{3}{2}}$

$y'' + \frac{3}{2x}y' - \frac{15}{2}x^{-2}y = 0. \quad p = \frac{3}{2}x^{-1}, \quad -\int p = -\frac{3}{2}\ln x$

$W = ce^{-\frac{3}{2}\ln x} = cx^{-\frac{3}{2}} \Rightarrow c = \frac{-11}{2}$

3.5 Method of undetermined cofficients.



① $y'' + p(t)y' + q(t)y = 0$

② $\begin{cases} L[y] = 0 \\ y(t_0) = y_0, \quad y'(t_0) = y_0' \end{cases}$

③ $y'' + p(t)y' + q(t)y = g(t)$

Y_1, Y_2 are sols of ③.

$\begin{cases} Y_1'' + pY_1' + qY_1 = g \\ Y_2'' + pY_2' + qY_2 = g \end{cases}$

$(Y_1 - Y_2)'' + p(Y_1 - Y_2)' + q(Y_1 - Y_2) = 0$

$Y_1 - Y_2$ is a sol of ①, $Y_1 - Y_2 = C_1 y_1 + C_2 y_2$

$\Rightarrow Y_2 = C_1 y_1 + C_2 y_2 + Y_1 = y_c + y_p$

complementary sol (y_c) particular sol (y_p)

Limit: p.q are constants

$g(t)$ is a polynomial or exponential

or cos / sin

① $y'' - 3y' - 4y = 3e^{2t}$ ② $y'' - 3y' - 4y = 2\cos t$

① 先解 $y'' - 3y' - 4y = 0$ $(e^{at})' = ae^{at}$ 不變.

猜 $y_p = Ae^{2t}$, $y_p' = 2Ae^{2t}$, $y_p'' = 4Ae^{2t}$

$4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} = 3e^{2t}$

$\Rightarrow -6A = 3 \Rightarrow A = -\frac{1}{2}$, $y_p = -\frac{1}{2}e^{2t}$ 找到啰.

② 猜 $y_p = A\cos t$, $y_p' = -A\sin t$, $y_p'' = -A\cos t$

$-A\cos t + 3A\sin t - 4A\cos t = 2\cos t$

$\Rightarrow -5A\cos t = 2\cos t$

$3A\sin t = 0$ $\Rightarrow t = \frac{\pi}{2}$, $A = 0 \rightarrow \leftarrow$

猜 $y_p = A\cos t + B\sin t$

$-A\cos t - B\sin t + 3A\sin t - 3B\cos t + 4A\cos t + 4B\sin t = 2\cos t$

$\Rightarrow -A - 3B + 4A = 2$

$-B + 3A + 4B = 0$ $\Rightarrow A = \frac{-3}{17}$, $B = \frac{-5}{17}$

① $y''-3y'-4y=3e^{2t}$ ② $y''-3y'-4y=2\cos t$

① 先解 $y''-3y'-4y=0$ $(e^{at})'=ae^{at}$ 不變.

猜 $y_p=Ae^{2t}$，$y_p'=2Ae^{2t}$，$y_p''=4Ae^{2t}$

$4Ae^{2t}-6Ae^{2t}-4Ae^{2t}=3e^{2t}$

$\Rightarrow -6A=3 \Rightarrow A=-\frac{1}{2}$，$y_p=-\frac{1}{2}e^{2t}$ 找到嘍.

② 猜 $y_p=A\cos t$，$y_p'=-A\sin t$，$y_p''=-A\cos t$

$-A\cos t+3A\sin t-4A\cos t=2\cos t$

$\Rightarrow -5A\cos t=2\cos t$
$3A\sin t=0$ $\Rightarrow t=\frac{\pi}{2}$，$A=0 \rightarrow\leftarrow$

猜 $y_p=A\cos t+B\sin t$

$-A\cos t-B\sin t+3A\sin t-3B\cos t+4A\cos t+4B\sin t=2\cos t$

$\Rightarrow -A-3B+4A=2$
$-B+3A+4B=0$ $\Rightarrow A=\frac{-3}{17}$，$B=\frac{-5}{17}$

$y''-3y'-4y=8e^{t}\sin 2t$

y_c ① $y''-3y'-4y=0$，代入 $e^{rt}\Rightarrow r^2-3r-4=0$，$r=4$ or -1

$y_c=c_1e^{4t}+c_2e^{-t}$

y_p ② Let $Y(t)=Ae^{t}\cos 2t+Be^{t}\sin 2t$

$Y'(t)=Ae^{t}\cos 2t-2Ae^{t}\sin 2t+Be^{t}\sin 2t+2Be^{t}\cos 2t$

$=(A+2B)e^{t}\cos 2t+(B-2A)e^{t}\sin 2t$

$Y''(t)=(A+2B)e^{t}\cos 2t-2(A+B)e^{t}\sin 2t$

$+(B-2A)e^{t}\sin 2t+2(B-2A)e^{t}\cos 2t$

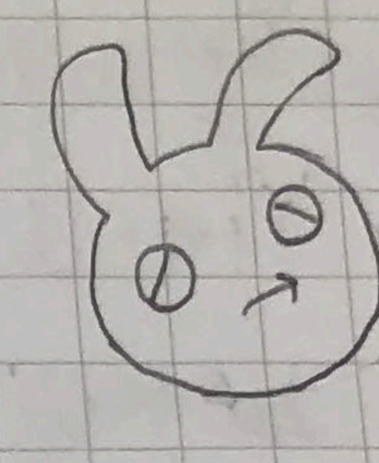
$=(A+2B+2B-4A)e^{t}\cos 2t$

$+(B-2A-2A-4B)e^{t}\sin 2t$

$$Y''-3Y'-4Y = (-3A+4B-3A-6B-4A)e^t\cos 2t$$
$$+(-4A-3B-3B+6A-4B)e^t\sin 2t$$
$$= 8e^t\sin 2t$$

$$\begin{cases} -10A-2B=0 \\ 2A-10B=8 \end{cases} \qquad A=\frac{2}{13},\ B=\frac{-10}{13}$$

$$y(t)=y_c+y_p=C_1e^{4t}+C_2e^{-t}+\frac{2}{13}e^t\cos 2t-\frac{10}{13}e^t\sin 2t$$

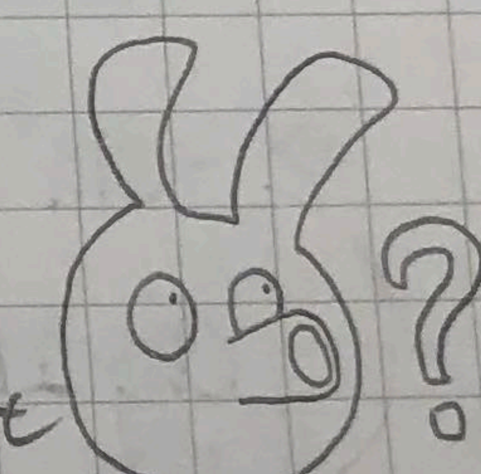


$$y''+p(t)y'+q(t)y=g_1(t)+g_2(t)$$ 可以一題

$$y''+p(t)y'+q(t)y=g_1(t) \Rightarrow Y_1(t),\quad y''+p(t)y'+q(t)y=g_2(t) \longrightarrow Y_2(t)$$

$$\Rightarrow \quad Y(t)=Y_1(t)+Y_2(t)$$

$$(Y_1+Y_2)''+p(t)(Y_1+Y_2)'+q(t)(Y_1+Y_2)$$
$$=(Y_1''+p(t)Y_1'+q(t)Y_1)+(Y_2''+p(t)Y_2'+q(t)Y_2) = g_1(t)+g_2(t)$$

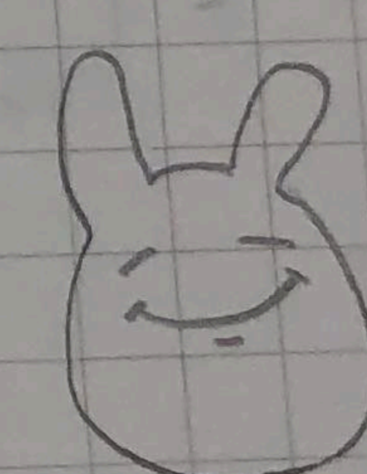


$$y''-3y'-4y=2e^{-t}$$

$$y_p=Ae^{-t},\ y_p'=-Ae^{-t},\ y_p''=Ae^{-t} \qquad \Rightarrow 0=2e^{-t}$$

$$y_p=Ate^{-t},\ y_p'=Ae^{-t}-Ate^{-t},\ y_p''=-2Ae^{-t}+Ate^{-t}$$

$$y''-3y'-4y=-2Ae^{-t}+Ate^{-t}-3Ae^{-t}+3Ate^{-t}-4Ate^{-t}=2e^{-t}$$
$$=-5Ae^{-t}=2e^{-t} \qquad \Rightarrow A=\frac{2}{-5}$$



3-6 Variation of Parameters

$y'' + p(t)y' + q(t)y = g(t)$, $y''+p(t)y'+q(t)y=0 \rightarrow y_1 . y_2$

$y''+4y+8\tan t$, $-\frac{\pi}{2}<t<\frac{t}{2}$

$y''+4y=0$. 代入 e^{rt}. $r^2+4=0$. $r=\pm 2i$

$y_c = C_1e^{2it} + C_2e^{-2it} = C_3\cos 2t + C_4 \sin 2t$

$Y(t) = u_1(t)\cos(2t) + u_2(t)\sin(2t)$ → To solve $u_1 . u_2$ $(u_1' . u_2')$

$Y'(t) = \underline{u_1'\cos 2t + u_2'\sin 2t} - 2u_1\sin 2t - 2u_2\cos 2t$

Let $u_1'\cos 2t + u_2'\sin 2t = 0$,

$Y''(t) = -2u_1'\sin 2t + 2u_2'\cos 2t - 4u_1\cos 2t - 4u_2\sin 2t$

$Y''+4Y = 8\tan t$

$= -2u_1'\sin 2t + 2u_2'\cos 2t \cancel{- 4u_1\cos 2t} \cancel{- 4u_2\sin 2t}$
$\cancel{+4u_1\cos 2t} \cancel{+4u_2\sin 2t} = 8\tan t$

$\Rightarrow -\sin 2t\, u_1' + \cos 2t\, u_2' = 4\tan t$

$\cos 2t\, u_1' + \sin 2t\, u_2' = 0$

$$\Rightarrow \begin{pmatrix} -\sin 2t & \cos 2t \\ \cos 2t & \sin 2t \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 4\tan t \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = -\begin{pmatrix} \sin 2t & -\cos 2t \\ -\cos 2t & -\sin 2t \end{pmatrix}\begin{pmatrix} 4\tan t \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -4\sin 2t \tan t \\ 4\cos 2t \tan t \end{pmatrix} = \begin{pmatrix} -8\sin^2 t \\ 8\cos t\sin t - 4\frac{\sin t}{\cos t} \end{pmatrix}$$

$u_1' = -8\sin^2 t \Rightarrow u_1 = \int -8\sin^2 t\, dt = 4\sin t\cos t - 4t$

$u_2 = \int ⊕ 8 = -4\cos^2 t + 4\ln\cos t$

$y_p = u_1\cos 2t + u_2\sin 2t = (4\sin t\cos t - 4t)\cos 2t$
$+ (-4\cos^2 t + 4\ln\cos t)\sin 2t$

3-1 $y'' = f(t, y, y')$

for linear $y'' + p(t)y' + q(t)y = r(t)$

First $ay'' + by' + cy = 0$ $a, b, c \in \mathbb{R}$

$y(t_0) = y_0$ $y'(t_0) = y_1$

e.g $y'' - y = 0$ $y(0) = 2$ $y'(0) = -1$

$y'(t) = C_1 e^{t} - C_2 e^{-t}$ $y'(0) : C_1 - C_2 = -1$

let $y = e^{rt}$

then $e^{rt}(ar^2 + br + c) = 0$

let $D = b^2 - 4ac$

① $r_1 . r_2 \in \mathbb{R}$ $r_1 \neq r_2$

② $r_1, r_2 \in \mathbb{R}$ $r_1 = r_2$

③ $r_1, r_2 \in \mathbb{C}$ $r_1 \neq r_2$

e.g $y'' + 5y' + 6y = 0$ $y(0) = 2$ $y'(0) = 3$

$C_1 e^{-2t} + C_2 e^{-3t}$ $C_1 + C_2 = 2$ $-2C_1 - 3C_2 = 3$ $C_1 = 9$ $C_2 = -7$

e.g $4y'' - 8y' + 3y = 0$ $y(0) = 2$ $y'(0) = \frac{1}{2}$

$4r^2 - 8r + 3 = 0$ $r = \frac{1}{2}$ or $\frac{3}{2}$

$C_1 + C_2 = 2$ $\frac{1}{2}C_1 + \frac{3}{2}C_2 = \frac{1}{2}$

2. $b^2 - 4ac < 0$

$y'' + y' + 9.25y = 0$

$r = \frac{-1}{2} \pm 3i$

$y_1 = e^{-\frac{t}{2}}(\cos 3t + i\sin 3t)$

$y_2 = e^{-\frac{t}{2}}(\cos 3t - i\sin 3t)$

3. $b^2 - 4ac = 0$

$ay'' + by + c = 0$

let $u(t) = v(t) \cdot e^{-\frac{b}{2a}t}$

$u'(t) = v'(t)e^{-\frac{b}{2a}t} - \frac{b}{2a}v(t)e^{-\frac{b}{2a}t}$

$u''(t) = v''t\, e^{-\frac{b}{2a}} - \frac{2b}{2a}v'(t)e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2}v(t)e^{-\frac{b}{2a}t}$

$\Rightarrow av''e^{-\frac{b}{2a}t} \Rightarrow v''(t) = 0$

$\Rightarrow v(t) = k_1 t + k_2$

$\star$ $L(y) = y'' + p(t)y' + q(t)y = 0$

if $y_1(t)$ is a solution

let $y_2(t) = v(t)y_1$

$\Rightarrow$ $v'' y_1 + (2y_1' + 2p y_1) v' = 0$

le $w = v'$ $\Rightarrow \dfrac{w'}{w} = \dfrac{-(2y_1' + py_1)}{y_1}$

e.g. $2x^2 y'' + 3xy' - 15y = 0$ $x > 0$

$y_1 = x^{-3}$ find y_2

$P(x) = \dfrac{3}{2x}$

$\Rightarrow$ $x^{-3} v'' + \left(-6x^{-4} + \frac{3}{2}x^{-4}\right) v' = 0$

$\ln |v'| = \frac{9}{2} \ln x$ $v = \frac{2}{11} x^{\frac{11}{2}}$

$y_2 = v y_1 = \frac{2}{11} x^{\frac{5}{2}}$

$y(x) = C_1 x^{-3} + \frac{2}{11} C_2 x^{\frac{5}{2}}$

Thm let $p, q \in C(I)$

$L(y) = y'' + p(t) y' + q(t) y = 0$

$y(t_0) = y_0 \quad y'(t_0) = y_1 \quad \exists ! \phi(t)$ satisfies

if $y = c_1 y_1 + c_2 y_2$ is a solution

$\Rightarrow c_1 y_1(t_0) + c_2 y_2(t_0) = y_0$

$c_1 y'(t_0) + c_2 y_2'(t_0) = y_0'$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_1 t_0 & y_2 t_0 \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix}^{-1} \begin{matrix} y_0 \\ y'_0 \end{matrix}$$

wronskian $\quad W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2'(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{W} \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{W}$$

e.g $\quad y'' + 5y' + 6y = 0 \qquad y_1 = e^{-2t} \quad y_2 = e^{-3t}$

$W(e^{-2t}, e^{-3t}) = e^{-5t} \neq 0$

$\Rightarrow y_1 = c_1 y_1 + c_2 y_2$ general solution

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eg $2t^2y'' + 3ty' - y = 0$

$y_1 = t^{\frac{1}{2}}$ $\quad y_2 = t^{-1}$

$W[t^{\frac{1}{2}}, t^{-1}] = t^{-\frac{3}{2}} - \frac{1}{2}t^{-\frac{3}{2}} = \frac{-3}{2}t^{-\frac{3}{2}} \neq 0 \quad t>0$

if $ay'' + by' + c = 0$ $\quad e^{r_1 t} \quad e^{r_2 t} \quad r_1 \neq r_2$

$W = (r_2 - r_1)\, e^{(r_1 + r_2)t} \neq 0$

if $e^{r_1 t} \quad te^{r_1 t}$

$W = e^{2r_1 t} + r_1 t e^{2r_1 t} - r_1 t e^{2r_1 t} = e^{2r_1 t} \neq 0$

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Thm Abel's Thm

$L(y) = y'' + p(t)y' + q(t)y = 0 \quad p, q \in C(I)$

y_1, y_2 solutions of $L(y) = 0$

Then $W(y_1, y_2) = c e^{-\int p(t)dt}$

and $W = \begin{cases} 0 & c = 0 \\ \neq 0 & c \neq 0 \end{cases}$

$W = y_1 y_2' - y_2 y_1'$

$W' = y_1' y_2' + y_1 y_2'' - y_2' y_1' - y_2 y_1''$

$= y_1(-py_2' - qy_2) - y_2(-py_1' - qy_1)$

$= -p(y_1 y_2' - y_2 y_1') - pW$

$\frac{W'}{W} = p \quad \Rightarrow \ln|W| = -\int p + C_1$

e.g. $2x^2 y'' + 3xy' - 15y = 0$

$y_1 = x^{\frac{5}{2}} \quad y_2 = x^{-3}$

$p = \frac{3}{2}x^{-1} \qquad W = -\frac{11}{2}x^{-\frac{3}{2}} = Cx^{-\frac{3}{2}}$

$C = \frac{-11}{2}$

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3.5 $y'' + p(t)\, y' + q(t)\, y = 0$

$y(t_0) = y_0$ $\quad y'(t_0) = y_0'$

(3) $y'' + p(t)\, y' + q(t)\, y = g(t)$ non homogeneous

$Y_1'' + pY_1' + qY_1 = g$

$Y_2'' + pY_2' + qY_2 = g$

$Y_2 = c_1 y_1 + c_2 y_2 + Y_1$

e.g $y'' - 3y' - 4y = 3e^{2t}$

$y'' - 3y' - 4y = 0$

let $y_p = Ae^{2t}$ $\quad y_p' = 2Ae^{2t}$ $\quad y_p'' = 4Ae^{2t}$

$-6A = 3$ $\quad A = -\frac{1}{2}$ $\quad y_p = -\frac{1}{2} e^{-2t}$

$y'' - 3y' - 4y = 2\cos t$

$y_p = A\cos t + B\sin t$

$\Rightarrow -A\cos - B\sin + 3A\sin t - 3B\cos t + 4A\cos t + 4B\sin t = 2\cos t$

$-A - 3B + 4A = 2$ $\quad A = \frac{-3}{17}$ $\quad B = \frac{5}{17}$

$-B + 3A + 4B = 0$

e.g solve $y''-3y'-4y=8e^{t}\sin(2t)$

$y(t)=Ae^{t}\cos(2t)+Be^{t}\sin(2t)$

$\Rightarrow -3A+4B-2A-6B-4A=0$

$-4A-3B-3B+6A-4B=8$

if $y''+p(t)y'+q(t)y=g_1(t)+g_2(t)$

$y''+p(t)y'+q(t)y=g_1(t) \rightarrow Y_1(t)$

$y''+p(t)y'+q(t)y=g_2(t) \quad Y_2(t)$

e.g $y''-3y'-4y=2e^{-t}$ since homo. genous has e^{-t}

$Y=Ae^{-t}$ $\quad A+3A-4A=2 \quad \cancel{=} \rightarrow \times$

then $Y(t)=Ate^{-t}$

$Y'(t)=Ae^{-t}-Ate^{-t}$

$Y''(e^{-t})=-2Ae^{-t}+Ate^{-t}$

$(-2A-3A)e^{-t}+(A+3A-4A)te^{-t}=2e^{-t}$

$A=\frac{-2}{5}$

e.g $y'' + 4y = 8\tan t$

$y_c = c_1 \cos 2t + c_2 \sin 2t$

let $Y(t) = u_1(t)\cos(2t) + u_2(t)\sin(2t)$

let $Y'(t) = u_1' \cos 2t + u_2' \sin(2t) - 2u_1 \sin 2t + 2u_2 \cos(2t)$

let $u_1' \cos 2t + u_2' \sin 2t = 0$

$Y'' + 4Y = 8\tan t$

$-2u_1' \sin 2t + 2u_2' \cos 2t - 4u_1 \cos 2t - 4u_2 \sin 2t$

$+ 4u_1 \cos 2t + 4u_2 \sin(2t) = 8\tan t$

$\Rightarrow \quad -\sin 2t \, u_1' + \cos(2t) u_2' = 0$

$\cos 2t \, u_1' + 2\sin 2t \, u_2' = 0$

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = -\begin{pmatrix} \sin 2t & -\cos 2t \\ \cos 2t & -\sin 2t \end{pmatrix} \begin{pmatrix} 4\tan t \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -4\sin 2t \tan t \\ 4\cos 2t \tan t \end{pmatrix}$$

$u_1 = \int -8\sin^2 t = 4\sin t \cos t - 4t$

$u_2 = \int u_2' = 4\cos^2 t - 4\tan t$

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in general $y_p = u_1 + y_1 + u_2 y_2$

let $u_1' y_1 + u_2' y_2 = 0$

$y_p'' + p y_p' + q y_p = u_1' y_1' + u_2' y_2' = g(t)$

$$\Rightarrow \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \frac{1}{W} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ g \end{pmatrix}$$

$u_1 = \int \frac{-g y_2}{W}$ $\quad u_2 = \int \frac{g y_1}{W}$

If $y_1 \dots y_n$ are solutions to the ODE,
then so is the linear combination
$y(t) = c_1 y_1 \dots + c_n y_n$
for initial $y_0 \dots y_0^{(n-1)}$

Thm $W[y_1 \dots y_n](t) \neq 0$ for at least one point $t \in I$, then the solution of IVP is $y(t) = c_1 y_1 + \dots + c_n y_n$

f.g is linear dependent if $\exists (c_1, c_2) \neq (0,0)$ s.t.
$c_1 f(t) + c_2 g(t) = 0 \quad \forall t \in I$

e.g $c_1 t + c_2 t^2 = 0 \quad \forall t \in I$ $\quad t=1 \; c_1 + c_2 = 0 \quad t=2 \; 2c_1 + 4c_2 = 0$
$\Rightarrow c_1 = c_2 = 0$ linear independent

e.g $f_1(t) = t^2 - 1 \quad f_2 = t^2 - t + 1 \quad f_3 = 3t^2 - t - 1$
$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0 \Rightarrow c_3 = 1 \quad c_2 = -1 \quad c_1 = -2$

Note $W[y_1 \dots y_n] \neq 0$ on $I \Leftrightarrow y_1 \dots y_n$ are independent.

$L[y] = a_0 y^{(n)} + \dots \; a_n y = 0$

$a_0(z-r) \dots (z-r_n) = 0$

① $r_i \neq r_j \quad i \neq j$

$W[e^{r_1 t} \dots e^{r_n t}] = e^{r_1 t} \dots e^{r_n t} \begin{vmatrix} 1 & & 1 \\ r_1 & & r_n \\ \vdots & & \vdots \\ r_1^{n-1} & & r_n^{n-1} \end{vmatrix}$

e.g. $y^{(4)} + y''' - 7y'' - y' + 6y = 0$

$r = \pm 1 \quad 2 \quad -3 \qquad y(0) = 1 \quad y'(0) = 0 \quad y''(0) = -2 \quad y^{(3)}(0) = -1$

$C_1 = \frac{11}{8} \quad C_2 = \frac{5}{12} \quad C_3 = -\frac{2}{3} \quad C_4 = \frac{-1}{8}$

e.g. $y^{(4)} - y = 0 \qquad r^4 - 1 = 0 \quad \Rightarrow (r^2+1)(r^2-1) = 0$

$y = C_1 e^{-t} + C_2 e^{t} + C_3 \cos t + C_4 \sin t$

if $z(r) = (r-(u+iv))^s (r-(u-iv))^s$

$y(t) = c_1 e^{ut} \cos vt + c_2 e^{ut} \sin vt$
$+ c_3 t e^{ut} \cos vt + c_4 e^{ut} \sin vt$
$+ \dots c' t^{s-1} e^{ut} \cos vt + c'' t^{s-1} e^{ut} \cos vt$

e.g $y^{(4)} + 2y'' + y = 0$ $(r^2+1)(r^2+1) = 0$

$y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t$

4.3 $a_0 y^{(n)} + \dots a_n y = g(t)$

e.g $y'' - 4y' + 4y = e^{2t}$

$y_c(t) = c_1 e^{2t} + c_2 t e^{2t}$

$y_p = At^2 e^{2t}$

$y_p'' - 4y_p' + 4y_p = 4A - 8A + 4A \; t^2 e^{2t}$
$+ 8A - 8A \; t e^{2t} + 2A e^{2t} = e^{2t}$ $\Rightarrow A = \frac{1}{2}$

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$y^{(4)} + 2y'' + y = 4\sin t - 6\cos t$

$y_c = C_1\cos t + C_2\sin t + C_3 t\cos t + C_4 t\sin t$

$y_p = At^2\cos t + Bt^2\sin t$

$A = \frac{3}{4}$ $B = -\frac{1}{2}$

e.g. $y'' - 4y' = t + 3\cos t + e^{-2t}$

$y_{p1} = At^2$ $A = -\frac{1}{8}$

$y_{p2} = B\cos t + C\sin t$ $C = \frac{-3}{5}$ $B = 0$

$y_{p3} = Dte^{-2t}$ $D = \frac{1}{8}$

$y = C_1 + C_2 e^{2t} + C_2 e^{-2t} + \frac{1}{8}t^2 - \frac{3}{5}\sin t + \frac{1}{8}te^{-2t}$