The **probability flow ODE** corresponding to the forward SDE

$$dx_t = f(x_t, t)\, dt + g(x_t, t)\, dW_t$$

is

$$\boxed{\frac{dx_t}{dt} = f(x_t, t) - \frac{1}{2}\frac{\partial}{\partial x}(g^2(x_t, t)) - \frac{1}{2}g^2(x_t, t)\frac{\partial}{\partial x}\log p(x_t, t)},$$

where $p(x, t)$ is the marginal density of x_t.

Derivation (1D case)

1. **Given SDE**

$$dx_t = f(x_t, t)\, dt + g(x_t, t)\, dW_t,$$

and define the diffusion coefficient

$$a(x, t) = g^2(x, t).$$

2. **Fokker–Planck equation**

The density $p(x, t)$ of x_t satisfies

$$\frac{\partial p(x, t)}{\partial t} = -\frac{\partial}{\partial x}(f(x, t)p(x, t)) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(a(x, t)p(x, t)).$$

3. **Continuity equation for a deterministic ODE**

If x_t instead follows an ODE

$$\frac{dx_t}{dt} = v(x_t, t),$$

then its probability density evolves as

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}(v(x,t)p(x,t)).$$

4. Matching both equations

To make both describe the same density evolution, we set

$$-\frac{\partial}{\partial x}(vp) = -\frac{\partial}{\partial x}(fp) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(ap).$$

Integrating once with respect to x (assuming natural or zero-flux boundary conditions),

$$v(x,t)p(x,t) = f(x,t)p(x,t) - \frac{1}{2}\frac{\partial}{\partial x}(a(x,t)p(x,t)).$$

5. Divide by $p(x,t)$ and expand

$$v = f - \frac{1}{2}\frac{1}{p}\frac{\partial}{\partial x}(ap) = f - \frac{1}{2}\frac{\partial a}{\partial x} - \frac{1}{2}a\frac{\partial \log p}{\partial x}.$$

6. Substitute $a = g^2$

Hence the probability flow ODE becomes

$$\frac{dx_t}{dt} = f(x_t,t) - \frac{1}{2}\frac{\partial}{\partial x}(g^2(x_t,t)) - \frac{1}{2}g^2(x_t,t)\frac{\partial}{\partial x}\log p(x_t,t).$$

Multidimensional generalization

For vector-valued $x_t \in \mathbb{R}^d$ and diffusion matrix $D(x,t) = g(x,t)g(x,t)^T$,

$$\frac{dx_t}{dt} = f(x_t,t) - \frac{1}{2}\nabla \cdot D(x_t,t) - \frac{1}{2}D(x_t,t)\nabla\log p(x_t,t),$$

where $\nabla \cdot D$ is the vector of divergences of each column of D.