

The **probability flow ODE** corresponding to the forward SDE

$$dx_t = f(x_t, t) dt + g(x_t, t) dW_t$$

is

$$\boxed{\frac{dx_t}{dt} = f(x_t, t) - \frac{1}{2} \frac{\partial}{\partial x} (g^2(x_t, t)) - \frac{1}{2} g^2(x_t, t) \frac{\partial}{\partial x} \log p(x_t, t),}$$

where  $p(x, t)$  is the marginal density of  $x_t$ .

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### Derivation (1D case)

#### 1. Given SDE

$$dx_t = f(x_t, t) dt + g(x_t, t) dW_t,$$

and define the diffusion coefficient

$$a(x, t) = g^2(x, t).$$

#### 2. Fokker–Planck equation

The density  $p(x, t)$  of  $x_t$  satisfies

$$\frac{\partial p(x, t)}{\partial t} = - \frac{\partial}{\partial x} (f(x, t)p(x, t)) + \frac{1}{2} \frac{\partial^2}{\partial x^2} (a(x, t)p(x, t)).$$

#### 3. Continuity equation for a deterministic ODE

If  $x_t$  instead follows an ODE

$$\frac{dx_t}{dt} = v(x_t, t),$$

then its probability density evolves as

$$\frac{\partial p(x, t)}{\partial t} = -\frac{\partial}{\partial x}(v(x, t)p(x, t)).$$

#### 4. Matching both equations

To make both describe the same density evolution, we set

$$-\frac{\partial}{\partial x}(vp) = -\frac{\partial}{\partial x}(fp) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(ap).$$

Integrating once with respect to  $x$  (assuming natural or zero-flux boundary conditions),

$$v(x, t)p(x, t) = f(x, t)p(x, t) - \frac{1}{2}\frac{\partial}{\partial x}(a(x, t)p(x, t)).$$

#### 5. Divide by $p(x, t)$ and expand

$$v = f - \frac{1}{2}\frac{1}{p}\frac{\partial}{\partial x}(ap) = f - \frac{1}{2}\frac{\partial a}{\partial x} - \frac{1}{2}a\frac{\partial \log p}{\partial x}.$$

#### 6. Substitute $a = g^2$

Hence the probability flow ODE becomes

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$$\frac{dx_t}{dt} = f(x_t, t) - \frac{1}{2}\frac{\partial}{\partial x}(g^2(x_t, t)) - \frac{1}{2}g^2(x_t, t)\frac{\partial}{\partial x}\log p(x_t, t).$$

#### Multidimensional generalization

For vector-valued  $x_t \in \mathbb{R}^d$  and diffusion matrix  $D(x, t) = g(x, t)g(x, t)^T$ ,

$$\frac{dx_t}{dt} = f(x_t, t) - \frac{1}{2}\nabla \cdot D(x_t, t) - \frac{1}{2}D(x_t, t)\nabla \log p(x_t, t),$$

where  $\nabla \cdot D$  is the vector of divergences of each column of  $D$ .