Formalized Linear Orders

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Main Theorems Formalized

Theorem 1 (Lindenbaum)**.** *If X and Y are linear orders such that X embeds in an initial segment of Y and Y embeds in a final segment of X, then $X \cong Y$.*

Theorem 2 (Sum Refinement)**.** *If X, Y, A, and B are linear orders such that $X + Y \cong A + B$, then there exists a linear order E such that either*

- *$A + E \cong X$ and $E + Y \cong B$, or*
- *$X + E \cong A$ and $E + B \cong Y$*

Linear Order Definition

Definition 1 (LE)**.** A type is an instance of LE if it has a binary relation $\leq$.

Definition 2 (LT)**.** A type is an instance of LT if it has a binary relation $<$.

Definition 3 (Preorder)**.** A type α is an instance of Preorder if it extends LE and LT and satisfies the following properties:

- Reflexivity: $\forall a : \alpha, a \leq a$
- Transitivity: $\forall a, b, c : \alpha, a \leq b \to b \leq c \to a \leq c$
- Relationship between $<$ and $\leq$: $\forall a, b : \alpha, a < b \iff a \leq b \land \neg b \leq a$

Definition 4 (PartialOrder)**.** A type α is an instance of PartialOrder if it extends Preorder and satisfies the following property:

- Antisymmetry: $\forall a, b : \alpha, a \leq b \to b \leq a \to a = b$

Definition 5 (LinearOrder)**.** A type α is an instance of LinearOrder if it extends PartialOrder and satisfies the following property:

- Total: $\forall a, b : \alpha, a \leq b \lor b \leq a$

Initial and Final

Definition 6 (Embedding α β)**.** An element of type Embedding α β has the following:

- A function from α to β
- A proof that the function is injective

Definition 7 (RelEmbedding α β r s)**.** An element of type RelEmbedding α β r s extends Embedding α β and satisfies the following:

- Order preserving: $\forall a, b : \alpha, s\ (fa)\ (fb) \iff r\ a\ b$ (where f is the function from Embedding α β and s and r are binary relations)

Definition 8 ($\alpha \preccurlyeq i\ \beta$)**.** An element f of type $\alpha \preccurlyeq i\ \beta$ extends RelEmbedding α β $\leq_\alpha \leq_\beta$ (where α and β are instances of LinearOrder and each $\leq$ is the corresponding one for the linear order) and satisfies the following:

- Is initial: $\forall a : \alpha, \forall b : \beta, b \leq fa \to \exists a' : \alpha, fa' = b$ (where f is the function from Embedding α β)

Definition 9 ($\alpha \preccurlyeq f\ \beta$)**.** An element f of type $\alpha \preccurlyeq f\ \beta$ extends RelEmbedding α β $\leq_\alpha \leq_\beta$ (where α and β are instances of LinearOrder and each $\leq$ is the corresponding one for the linear order) and satisfies the following:

- Is final: $\forall a : \alpha, \forall b : \beta, fa \leq b \to \exists a', fa' = b$ (where f is the function from Embedding α β

Definition 10 (isInitial s)**.** The meaning of isInitial s is that s is a subset of the elements of α where α is an instance of LinearOrder and that

$$\forall x \in s, \forall y : \alpha, y < x \to y \in s$$

Definition 11 (isFinal s)**.** The meaning of isFinal s is that s a subset of the elements of α where α is an instance of LinearOrder and that

$$\forall x \in s, \forall y : \alpha, y > x \to y \in s$$

Variable Definitions

For the rest of this section, we let α and β be types with instances of LinearOrder. We let f be an element of type $\alpha \preccurlyeq i\ \beta$ and we let g be an element of type $\beta \preccurlyeq f\ \alpha$.

Theorem 3 (Initial Maps Initial Initial)**.** *Let s be a subset of α such that isInitial s. Then we have that isInitial $f[s]$.*

Proof. Our goal is to prove that isInitial $f[s]$. By the definition of isInitial $f[s]$, this means that we have to show that for all $x \in f[s]$, for all $y < x$, we have that $y \in f[s]$. So let $x \in f[s]$ and let $y < x$. Since $x \in f[s]$, there exists a $w \in s$ such that $f(w) = s$. Since $y < f(w)$, then since f is of type $\alpha \preccurlyeq i\ \beta$, there exists a $z : \alpha$ such that $f(z) = y$.

So we now have that $f(z) < f(w)$. Since f extends RelEmbedding $\alpha\ \beta\ \leq_\alpha\ \leq_\beta$, this implies that f is order preserving and so we have that $z < w$. Since $w \in s$ and $z < w$, we have by the definition of isInitial s that $z \in s$. Thus, we have found a $z \in s$ such that $f(z) = y$. Therefore, $y \in f[s]$. □

Theorem 4 (Image of Univ Initial)**.** *We have that isInitial* $f[\alpha]$*.*

Proof. Our goal is to prove that isInitial $f[\alpha]$. By Theorem 3, it is sufficient to show that isInitial α. By the definition of isInitial α our goal is to show that for all $x \in \alpha$, for all $y : \alpha$ such that $y < x$, we have that $y : \alpha$. Since we already know that $y \in \alpha$ we are done. □

Theorem 5 (Final Maps Final Final)**.** *Let s be a subset of α such that isFinal s. Then we have that isFinal $f[s]$.*

Proof. Dual of Theorem 3 □

Theorem 6 (Image of Univ Final)**.** *We have that isFinal* $g[\beta]$*.*

Proof. Dual of Theorem 4. □

Theorem 7 (Comp Initial Final)**.** *Let s be a subset of α such that isInitial s. Then isFinal $\alpha \setminus s$.*

Proof. By the definition of isFinal $\alpha \setminus s$, we need to show that for all $x \in \alpha \setminus s$, for all $y : \alpha$ such that $y > x$, $y \in \alpha \setminus s$. So let $x : \alpha$ such that $x \in \alpha \setminus s$ and let $y : \alpha$ such that $y > x$. We need to show that $y \in \alpha \setminus s$.

Assume for the sake of contradiction that $y \notin \alpha \setminus s$. Since $y \notin \alpha \setminus s$, we have that $y \in s$. Since $y > x$, we have that $x < y$. So by the definition of isInitial s, since $y \in s$ and $x < y$, we have that $x \in s$. But we also know that $x \in \alpha \setminus s$ and so $x \notin s$. So we have that $x \in s$ and $x \notin s$. This is a contradiction. □

Theorem 8 (Comp Final Initial)**.** *Let s be a subset of α such that isFinal s. Then isInitial $\alpha \setminus s$.*

Proof. Dual of Theorem 7. □

Theorem 9 (Union Initial Initial)**.** *Let h be a function from $\mathbb{N}$ to subsets of α such that for all $n : \mathbb{N}$, isInitial $h(n)$. Then we have that isInitial $\bigcup_{n \in \mathbb{N}} h(n)$.*

Proof. Let $x \in \bigcup_{n \in \mathbb{N}} h(n)$ and let $y < x$. Then, by the definition of isInitial $\bigcup_{n \in \mathbb{N}} h(n)$, it is now our goal to prove that $y \in \bigcup_{n \in \mathbb{N}} h(n)$. This is equivalent to showing that there exists an $i \in \mathbb{N}$ such that $y \in f(i)$. Furthermore, since we know that $x \in \bigcup_{n \in \mathbb{N}} h(n)$, we know that there exists a $w \in \mathbb{N}$ such that $x \in h(w)$.

We now claim that $y \in h(w)$. Since for all $n \in \mathbb{N}$, isInitial $h(n)$, we have that isInitial $h(w)$. So since $x \in h(w)$ and $y < x$, then by the definition of isInitial $h(w)$, we have that $y \in h(w)$. Thus, we have shown that there exists an $i \in \mathbb{N}$ such that $y \in h(i)$. □