Theory of Computation 2019

$(A \longleftrightarrow B)$

$A \rightarrow B \land B \rightarrow A$

1a) i) $(B \rightarrow (A \rightarrow C)) \longleftrightarrow (A \rightarrow (B \rightarrow C))$

$(\neg B \lor (\neg A \lor C)) \longleftrightarrow (\neg A \lor (\neg B \lor C))$

$(\neg B \lor (\neg A \lor C)) \rightarrow (\neg A \lor \neg B \lor C) \land (\neg A \lor (\neg B \lor C)) \rightarrow (\neg B \lor \neg A \lor C)$

$(\neg(\neg B \lor \neg A \lor C) \lor (\neg A \lor \neg B \lor C)) \land \neg(\neg A \lor \neg B \lor C) \lor (\neg B \lor \neg A \lor C)$

$((B \lor A \lor \neg C) \lor (\neg A \lor \neg B \lor C)) \land ((A \lor B \lor \neg C) \lor (\neg B \lor \neg A \lor C))$

$1 \land 1$

The propositional formula is satisfiable and valid. When trying to construct DNF form, the formula reduces to true, hence it is satisfiable and valid is satisfiable and valid ... ue, hence

ii) $-((\bar{a} \cdot \bar{b}) + -(c \cdot \bar{a}))$

$= -(\overline{(a+b)} + (\bar{c} + a))$

$= (a+b) \cdot (c \cdot \bar{a})$

$= (a+b) \cdot \overline{(\bar{c}+a)}$

$= (a \cdot \overline{(\bar{c}+a)}) + (b \cdot \overline{(\bar{c}+a)})$

$= (a \cdot (c \cdot \bar{a})) + (b \cdot (c \cdot \bar{a}))$

$= b \cdot c \cdot \bar{a}$

$-((\bar{a} \cdot \bar{b}) + (\bar{c} + a))$

$-((\bar{a} \cdot \bar{b}) + \overline{(c \cdot \bar{a})})$

$-((\bar{a} + \overline{(c \cdot \bar{a})}) \cdot (\bar{b} + \overline{(c \cdot \bar{a})}))$

$-((\bar{a} + \bar{c} + a) \cdot (\bar{b} + \bar{c} + a))$

$\overline{(\bar{a} + \bar{c} + a)} + \overline{(\bar{b} + \bar{c} + a)}$

$(a \cdot c \cdot \bar{a}) + (b + c + \bar{a})$

iii) $f(a \rightarrow b) = \overline{a} + b$

$f(\neg a) = \overline{f(a)}$

$f(a \vee b) = f(a) + f(b)$

$f(a \wedge b) = f(a) \cdot f(b)$

$f(a \leftrightarrow b) = f(a \rightarrow b) \cdot f(b \rightarrow a)$

$f(p) = p$

b) i) Nobody is friends with Mark

$\forall x\,(\neg(x=Mark) \land F(x, Mark))$

ii) Olivia isn't friends with anybody who isn't friends with her

$\forall x\,(\neg(x=Olivia) \land F(Olivia, x)) \land \neg F(x, Olivia))$

iii) Mark is friends with all of his friends' friend

$\forall x\,\forall y\,(\neg(x=Mark \lor y=Mark) \land \neg(x=y) \land (F(Mark, x) \land F(x, y)) \land F(Mark, y))$

Olivia has at least three friends

iv) $\exists x\,\exists y\,\exists z\,(\neg(x=Olivia \lor y=Olivia \lor z=Olivia) \land \neg(x=y \lor y=z \lor x=z)$

$\land F(Olivia, x) \land F(Olivia, y) \land F(Olivia, z))$

v) $\exists x\,\forall y\,\exists z\,(\neg(x=y) \land \neg(y=z) \land \neg(x=z) \land F(x, y) \land F(y, z))$

c) $M \models \forall x \forall y . x < y \rightarrow \exists z . x < z \land z < y$

Since the domain is $\mathbb{R}$

(This might be super wrong)

Proof by contradiction,

Assume that $\forall x \forall y . x < y$ is false

that means that there are no values $\in \mathbb{R}$ that are between two values

then $\exists z . x < z \land z < y$ would always be false

since from our assumption all the values are not less or more than one another.

but we know that $\exists n (x < n \leq y)$ where $n \in \mathbb{R}$

hence $M \models \forall x \forall y.\ x<y \rightarrow \exists z.\ x<z \wedge z<y$

d)

$\forall x$ ins x $xs = x$

a) Base Case : ins x $[\,] = x :: [\,]$

IH : ins x $(y :: ys) = x :: (y :: ys)$

Step Case : ins z (ins x $(y :: ys)$)

$= $ ins z $x :: y :: ys$ IH

$= z :: x :: y :: ys$

each number is appended to the list

b) Base Case: $ins\ x\ [\,] = x :: [\,]$

IH: $ins\ x\ (y::ys) = \begin{cases} x::(y::ys) & \text{if } x \leq y \\ y::(ins\ x\ ys) & \text{if } x > y \end{cases}$

Step Case: $ins\ z\ (ins\ x\ (y::ys))$

let $z = x-1$

$ins\ x-1\ (x :: y :: ys)$

since $x-1 \leq x$

$= x-1 :: x :: y :: ys$

let $z = x+1$

$ins\ z\ (x :: y :: ys)$

$= x :: (ins\ x+1\ (y::ys))$

$= x :: (\text{ins } x{+}1\ (y::ys))$

e) $M = (V,R)$ $\exists y. Rxy \wedge Ryz$

Suppose $M = (V,R)$ is not dense, we have to show $M, w \models_p \Box\Box A \rightarrow \Box A$ is not valid. Assume $M, w \models_p \Box\Box A$ is true

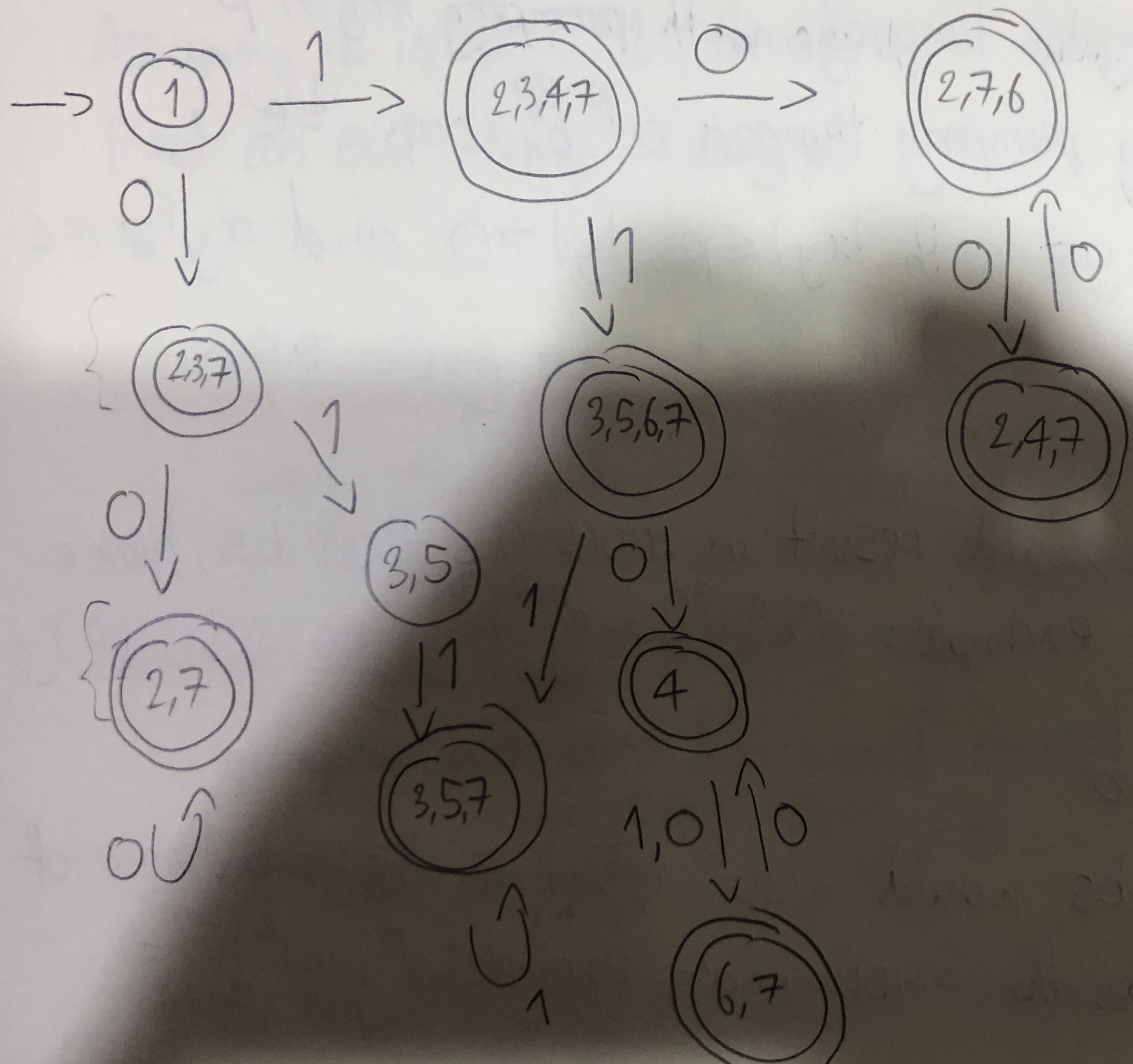
Since M is not dense we have Rww' and $Rw'w''$ but not Rww''

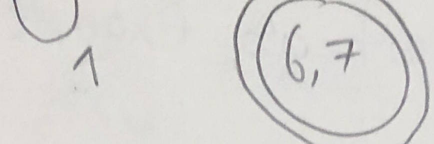
$M, w \models_p \Box\Box A$ holds, but $M, w \models_p \Box A$ does not as there are no Rww'' but A is only valid at w''

Thus $M, w \models_p \Box\Box A$ and $M, w' \Box A$ holds, but $M, w \not\models_p \Box A$

Thus the frame has to be dense for $\Box\Box A \rightarrow \Box A$ to be valid

2. a)





b) i)

ε yes

Output	derivation
B	$S \to B$
ε	$B \to \varepsilon$

ii) bac

not generatable

abcc

not generatable

$B \to bBc$

ii) $\{a^i b^j c^{i+j} \mid i \geq 0, j \geq 0\}$

Empty or in alphabetical order where the number of a's and b's equal to c

c) $L = \{a^n b^{3n} \mid n \geq 0\}$

Assume L is a regular language with pumping length p
$a^p b^{3p} \in L$. By pumping lemma $a^p b^{3p}$ can be de composed into xyz with $|xy| \leq p$, $|y| > 0$ and $xy^iz \in L$ when $i \geq 0$.

Case 1: $y = a$

pumping as would result in more as than bs, hence not in L example: $a^n a^k b^{3n} = a^{n+k} b^{3n}$

Case 2: $y = ab$

pumping abs would also resut in the imbalance of b's being 3 times the amount a's example: $a^n (ab)^k b^{3n} = a^{n+k} b^{3n+k}$

Case: 3 $y = b$

pumping b's would result in bs being more than 3 times the number of a's example: $a^n b^k b^{3n} = a^n b^{3n+k}$

In all cases it can't satisfy the pumping lemma, thus $a^n b^{3n}$ is not regular

d) $L = \{a^n b^{3n} \mid n \geq 0\}$

Assuming L is a CFL, therefore it has a pumping length p.

Find an "evil string" (using p)

$w \in L$

take string as $a^p b^{3p}$

vxy can contain 3 cases

Case 1: vxy = a's and b's

pumping a's and b's would cause an Imbalance

example: $a^{p+n} b^{3p+n}$ $\frac{3p+n}{p+n} \neq 3$

case 2: vxy = only 1 letter a's or b's

pumping only 1 letter would cause an imbalance

for example $a^{p+n} b^{3p}$ or $a^p b^{3p+n}$

$\frac{3p}{p+n} \neq 3$ or $\frac{3p+n}{p} \neq 3$

Hence in all the cases it cannot satisfy the pumping lemma

thus $a^n b^{3n}$ is not contextfree

e) $\overline{L} = \{w \mid w \notin L\}$

The turing machme cannot be created because a turing machme acceptance is undecidable.

Proof by contradiction

Assumming the input turmg machme is deciable

let H be a decider

$$H(\langle M, w \rangle) = \begin{cases} \text{accept} & M \text{ accepts } w \\ \text{reject} & M \text{ rejects } w \end{cases}$$

now we have another TM D that will negate the output of turmg machme M

essentially $D(M) = \neg H(M, \langle M \rangle)$

If we run D on D then

$$D(\langle D \rangle) = \begin{cases} \text{accept} & D \text{ does not accept } \langle D \rangle \\ \text{reject} & D \text{ does accept } \langle D \rangle \end{cases}$$

Hence D accepts and does not accept D, which leads to a contradiction.

Thus a TM that decides a language L is not a decidable langauge.