Drake Implementation of Rigid Body Dynamics Algorithms with Constraints

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More complete write-up may follow. For now, just some quick notes.

1 Bilateral Position Constraints

Consider constraint equation

$$\phi(q) = 0.$$

These arise, for instance, when the system has a closed kinematic chain. The (floating base) equations of motion can be written

$$H(q)\ddot{q} + C(q,\dot{q}) = Bu + J(q)^T\lambda,$$

where $J(q) = \frac{\partial \phi}{\partial q}$ and λ is the constraint force.

To solve for λ, observe that when the constraint is imposed, $\phi(q) = 0$ and therefor $\dot{\phi} = 0$ and $\ddot{\phi} = 0$. Writing this out, we have

$$\dot{\phi} = J(q)\dot{q} = 0,$$
$$\ddot{\phi} = J(q)\ddot{q} + \dot{J}(q)\dot{q} = 0.$$

Inserting the dynamics and solving for λ yields

$$\lambda = -(JH^{-1}J^T)^{+}(JH^{-1}(Bu - C) + \dot{J}\dot{q}).$$

The $^+$ notation refers to a Moore-Penrose pseudo-inverse. In most cases, there are less constraints than degrees of freedom, in which case the inverse has a unique solution (and the traditional inverse could have been used). But the pseudo-inverse also works in cases where the system is over-constrained.

For numerical stability, I would like to add a restoring force to this constraint in the event that the constraint is not satisfied to numerical precision. To accomplish this, I'll ask for

$$\ddot{\phi} = -\frac{2}{\epsilon}\dot{\phi}(q) - \frac{1}{\epsilon^2}\phi(q).$$

Carrying this through yields

$$\lambda = -(JH^{-1}J^T)^{+}(JH^{-1}(Bu - C) + (\dot{J} + \frac{2}{\epsilon}J)\dot{q} + \frac{1}{\epsilon^2}\phi).$$

2 Bilateral Velocity Constraints

Consider the constraint equation

$$\psi(q, \dot{q}) = 0,$$

where $\frac{\partial \psi}{\partial \dot{q}} = \neq 0$. These are less common, but arise when, for instance, a joint is driven through a prescribed motion. Here, the manipulator equations are given by

$$H(q)\ddot{q} + C = Bu + \frac{\partial \psi}{\partial \dot{q}}^T \lambda.$$

To solve for λ, we take

$$\dot{\psi} = \frac{\partial \psi}{\partial q}\dot{q} + \frac{\partial \psi}{\partial \dot{q}}\ddot{q} = 0,$$

which yields

$$\lambda = -\left(\frac{\partial \psi}{\partial \dot{q}} H^{-1} \frac{\partial \psi}{\partial \dot{q}}\right)^+ \left[\frac{\partial \psi}{\partial \dot{q}} H^{-1}(Bu - C) + \frac{\partial \psi}{\partial q}\dot{q}\right].$$

Again, for numerical stability, we as instead for $\dot{\psi} = -\frac{1}{\epsilon}\psi$, which yields

$$\lambda = -\left(\frac{\partial \psi}{\partial \dot{q}} H^{-1} \frac{\partial \psi}{\partial \dot{q}}\right)^+ \left[\frac{\partial \psi}{\partial \dot{q}} H^{-1}(Bu - C) + \frac{\partial \psi}{\partial q}\dot{q} + \frac{1}{\epsilon}\psi\right].$$

3 Unilateral Position Constraints

Consider the constraint equation

$$\phi(q) \geq 0.$$

One common example of this, for instance, is a joint limit. The dynamics of unilateral constraints contain to pieces: the continuous dynamics when the constraint is inactive ($\phi(q) > 0$) or active ($\phi(q) = 0$), but also an impulsive event when the constraint becomes active ($\phi(q(t)) = 0, \phi(q(t - \epsilon)) > 0$). We model this as a hybrid transition. There is no corresponding event when the constraint transitions to inactive.

3.1 Continuous Dynamics

The continuous equations are governed by

$$H\ddot{q} + C = Bu + J^T\lambda,$$

where $J = \frac{\partial \phi}{\partial q}$. Let us consider the solution for different cases.

- If $\phi > 0$ the constraint is inactive, and $\lambda = 0$.
- Otherwise $\phi = 0$, and

- if $\dot{\phi} > 0$, then the constraint is going inactive, and $\lambda = 0$.
- otherwise $\dot{\phi} = 0$, and
  - $\ddot{\phi} > 0$, and $\lambda = 0$
  - or $\ddot{\phi} = 0$, and $\lambda > 0$.

For the case when $\ddot{\phi} = 0, \lambda > 0$, we have (as in the bilateral position constraints)

$$\lambda = -(JH^{-1}J^T)^+(JH^{-1}(Bu - C) + \dot{J}\dot{q}).$$

As a result, if $\phi > 0$ or $\dot{\phi}$, then $\lambda = 0$. Otherwise, we have to solve for $\ddot{q}$ and λ simultaneously to determine which constraints are active. We can accomplish this by solving a linear complementarity problem (LCP):

$$\begin{aligned} \text{find} \quad & \ddot{\phi}, \lambda \\ \text{subject to} \quad & \ddot{\phi} \geq 0, \lambda \geq 0, \\ & \ddot{\phi} = \dot{J}\dot{q} + JH^{-1}(Bu - C + J^T\lambda), \\ & \forall_i \; \ddot{\phi}_i \lambda_i = 0. \end{aligned}$$

Then $\ddot{q}$ follows from $\ddot{q} = H^{-1}(Bu - C + J^T\lambda)$.

For numerical stability, I must also consider when $\phi < 0$ and/or $\phi = 0, \dot{\phi} < 0$, so I instead ask for

$$\ddot{\phi} \geq -\frac{2}{\epsilon}\dot{\phi} - \frac{1}{\epsilon^2}\phi,$$

given the conditions

- If $\phi > 0$ or $\dot{\phi} > -\frac{1}{\epsilon}\phi$, then $\lambda = 0$.
- Otherwise, take $\alpha = \ddot{\phi} + \frac{2}{\epsilon}\dot{\phi} + \frac{1}{\epsilon^2}\phi$ to write

$$\begin{aligned} \text{find} \quad & \alpha, \lambda \\ \text{subject to} \quad & \alpha \geq 0, \lambda \geq 0, \\ & \alpha = \dot{J}\dot{q} + JH^{-1}(Bu - C + J^T\lambda) - \frac{2}{\epsilon}\dot{\phi} - \frac{1}{\epsilon^2}\phi, \\ & \forall_i \; \alpha_i \lambda_i = 0. \end{aligned}$$

3.2 Impulsive Event

The collision event is described by the zero-crossings (from positivie to negative) of the scalar function $\phi(q)$, and that after the impact we impose the constraint that $\phi = 0$. Using

$$H\ddot{q} + C = Bu + J^T\lambda,$$

λ is now an impulsive force that well-defined when integrated over the time of the collision (denoted t_c^- to t_c^+). Integrate both sides of the equation over that (instantaneous) interval:

$$\int_{t_c^-}^{t_c^+} dt\, [H\ddot{q} + C] = \int_{t_c^-}^{t_c^+} dt\, \left[Bu + J^T\lambda\right]$$

Since q and u are constants over this interval, we are left with

$$H\dot{q}^{+} - H\dot{q}^{-} = J^{T} \int_{t_c^-}^{t_c^+} \lambda dt,$$

where $\dot{q}^{+}$ is short-hand for $\dot{q}(t_c^+)$. Multiplying both sides by JH^{-1}, we have

$$J\dot{q}^{+} - J\dot{q}^{-} = JH^{-1}J^{T} \int_{t_c^-}^{t_c^+} \lambda dt.$$

But the first term on the left is zero because after the collision, $\dot{\phi} = 0$, yielding:

$$\int_{t_c^-}^{t_c^+} \lambda dt = -\left[JH^{-1}J^{T}\right]^{+} J\dot{q}^{-}.$$

Substituting this back in above results in

$$\dot{q}^{+} = \left[I - H^{-1}J^{T}\left[JH^{-1}J^{T}\right]^{+} J\right]\dot{q}^{-}.$$

So far, I assume all collisions are inelastic. Elastic collisions are also possible, but require some care implementing restitution (especially in the contact case described below). An excellent discussion can be found in chapter 3 of [1].

4 Contact Constraints

4.1 Continuous Dynamics

4.2 Impulsive Event

5 Putting it all together

References

[1] Brian Mirtich. *Impulse-based Dynamic Simulation of Rigid Body Systems*. PhD thesis, 1996.