ENTROPIC BARRIERS AND NONSURJECTIVITY IN THE $3x + 1$ PROBLEM: THE JUNCTION THEOREM

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Abstract. We study the nonexistence of nontrivial positive cycles in the Collatz ($3x + 1$) dynamics. By revisiting Steiner's equation (1977) through the lens of information theory, we identify a universal entropic deficit $\gamma = 1 - h(1/\log_2 3) \approx 0.0500$, where h denotes the binary Shannon entropy. This deficit implies that the modular evaluation map Ev_d cannot be surjective for any cycle candidate of length $k \geq 18$ (**unconditional**, Theorem 1). Combined with the computational bound of Simons and de Weger (2005) for $k \leq 68$, we obtain a **Junction Theorem** (Theorem 2): for every $k \geq 2$, at least one obstruction—computational or entropic—applies.

The residual question—excluding the specific residue 0 from the image of Ev_d—is formulated as **Hypothesis (H)** and analyzed via additive and multiplicative character sums. We establish unconditional bounds (Parseval cost, peeling lemma, Mellin–Fourier bridge) and formulate **Conjecture M** as the single remaining obstacle. The core results are formalized in Lean 4 (73 theorems, zero `sorry`, zero axioms); a research-level skeleton with Mathlib adds ~ 58 further proofs. Source code and a detailed research log are publicly available.

Contents

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1. Introduction

1.1. **The $3x + 1$ problem.** Let $T\colon \mathbb{N}^* \to \mathbb{N}^*$ be defined by

$$T(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (3n+1)/2 & \text{if } n \text{ is odd.} \end{cases} \tag{1}$$

The *Collatz conjecture* [9] asserts that for every $n \geq 1$, the orbit $n, T(n), T^2(n), \ldots$ reaches 1. In particular, no nontrivial positive cycle should exist.

The problem has been studied from many angles; see [10] for a comprehensive survey. Steiner [13] and Crandall [5] reduced the cycle question to a modular equation (see also Böhm and Sontacchi [3]). Eliahou [6] established lower bounds on the minimal length of such cycles, and Korec [8] proved that cycles of length k must satisfy $k > 17$ unless they have a special structure. Simons and de Weger [12], using bounds for linear forms in logarithms (Laurent, Mignotte and Nesterenko [11]), computationally excluded all cycles with $k \leq 68$. Wirsching [15] developed a dynamical systems perspective, while Kontorovich and Lagarias [7] introduced stochastic models. More recently, Tao [14] showed that almost all orbits attain almost bounded values. Computational verification has been extended by Barina [2] up to 5×2^{60}.

1.2. **Summary of results.** This work establishes the following results:

(i) **Theorem 3.5** (unconditional). For every $k \geq 18$ with $d(k) > 0$, the evaluation map Ev_d is not surjective.
(ii) **Theorem 4.2** (unconditional for the disjunction). For every $k \geq 2$, at least one obstruction (computational or entropic) applies. The complete exclusion of cycles further requires Hypothesis (H) for $k \in [18, 68]$.
(iii) **Theorem 5.4** (unconditional). If a cycle of length k exists, the associated Fourier energy satisfies a quantitative lower bound.
(iv) **Theorem 6.2** (unconditional). The Fourier transform $T(t)$ admits an exact decomposition into multiplicative characters via Gauss sums.

We also formulate **Conjecture M** (Conjecture 7.5), whose resolution would imply the nonexistence of all nontrivial positive cycles.

1.3. **Key ideas.** The central observation is information-theoretic. By Steiner's equation, a positive cycle of length k requires a composition A in $\mathrm{Comp}(S, k)$ such that $\mathrm{corrSum}(A) \equiv 0 \pmod{d}$, where $d = 2^S - 3^k$. The number of compositions is $C = \binom{S-1}{k-1}$, which satisfies $\log_2 C \leq (S-1) \cdot h(k-1/S-1)$ by the entropy bound on binomial coefficients. Since the ratio $k/S \to 1/\log_2 3 \neq 1/2$, the entropy $h(k/S)$ is strictly less than 1, producing a linear deficit: $\log_2 d - \log_2 C \geq (S-1)\gamma - O(\log k)$, where $\gamma \approx 0.0500 > 0$. For $k \geq 18$, this deficit ensures $C < d$, so Ev_d is not surjective. The overlap $[18, 68]$ between the entropic obstruction ($k \geq 18$) and the Simons–de Weger bound ($k \leq 68$) closes the gap, yielding the Junction Theorem.

To go beyond nonsurjectivity and exclude the specific residue 0, we develop an analytical approach via exponential sums $T(t)$ on $\mathbb{Z}/p\mathbb{Z}$. The Fourier inversion formula reduces the question to bounding $|T(t)|$. We establish a Parseval cost (Theorem 5.4) and a Mellin–Fourier bridge (Theorem 6.2) that decomposes $T(t)$ into multiplicative characters. These tools frame the remaining gap as Conjecture M.

1.4. **Conventions and notation.**

Notation 1.1. Throughout this paper:

- $k \geq 1$ denotes the *length* of a cycle (number of odd steps);
- $S = S(k) = \lceil k \log_2 3 \rceil$ is the *Syracuse height*;
- $d = d(k) = 2^S - 3^k$ is the *crystal module*;

- $C = C(k) = \binom{S-1}{k-1}$ is the number of admissible compositions;
- $h(p) = -p\log_2 p - (1-p)\log_2(1-p)$ is the *binary Shannon entropy*, defined for $p \in (0,1)$.

1.5. **Outline.** Section 2 recalls Steiner's equation and defines the evaluation map. Section 3 establishes the entropic deficit and the nonsurjectivity theorem. Section 4 combines this result with the Simons–de Weger bound to obtain the Junction Theorem. Section 5 develops the analytical obstruction via character sums. Section 6 establishes the Mellin–Fourier bridge. Section 7 formulates Hypothesis (H) and Conjecture M. Section 8 presents numerical verifications and formal verification in Lean 4. Section 9 discusses perspectives and open problems.

2. Steiner's equation

2.1. **Derivation.** Following Steiner [13], a positive Collatz cycle of length k (number of odd steps) and height S (total number of steps) corresponds to an integer $n_0 \geq 1$ and an *admissible composition* $A = (A_0, A_1, \ldots, A_{k-1})$.

Definition 2.1 (Admissible composition)**.** Let $S \geq k \geq 1$. The set of *admissible compositions* is

$$\mathrm{Comp}(S,k) = \big\{(A_0,\ldots,A_{k-1}) \in \mathbb{N}^k : 0 = A_0 < A_1 < \cdots < A_{k-1} \leq S-1\big\}.$$

Its cardinality is $|\mathrm{Comp}(S,k)| = \binom{S-1}{k-1} = C$.

Proposition 2.2 (Steiner's equation)**.** *If n_0 is the smallest element of a positive cycle of length k and height S, then there exists $A \in \mathrm{Comp}(S,k)$ such that*

$$n_0 \cdot d \;=\; \mathrm{corrSum}(A), \tag{2}$$

where $d = 2^S - 3^k$ and the corrective sum *is*

$$\mathrm{corrSum}(A) \;=\; \sum_{i=0}^{k-1} 3^{k-1-i} \cdot 2^{A_i}. \tag{3}$$

Proof. A cycle of length k visits k odd numbers $n_0, n_1, \ldots, n_{k-1}$ with $n_{j+1} = T^{g_j}(n_j)$ where $g_j \geq 1$ is the number of divisions by 2 after odd step j. Set $A_j = g_0 + \cdots + g_{j-1}$ (cumulative), with $A_0 = 0$. The height is $S = A_{k-1} + g_{k-1}$.

The recurrence $n_{j+1} = (3n_j + 1)/2^{g_j}$ telescopes to

$$n_0 = 3^k \cdot n_0 \cdot 2^{-S} + \sum_{i=0}^{k-1} 3^{k-1-i} \cdot 2^{A_i - S},$$

which, after multiplying by 2^S, gives (2). $\square$

Remark 2.3. Equation (2) is due to Steiner [13]. Crandall [5] independently established it in a slightly different form.

Remark 2.4 (Arithmetic properties of corrSum). For every $A \in \mathrm{Comp}(S,k)$ with $k \geq 2$:

(i) $\mathrm{corrSum}(A)$ is always odd;
(ii) $3 \nmid \mathrm{corrSum}(A)$.

For (i): since $A_0 = 0$ and $A_i \geq 1$ for $i \geq 1$, the only odd summand is $3^{k-1} \cdot 2^0 = 3^{k-1}$; all other terms are even. Hence $\mathrm{corrSum}(A) \equiv 1 \pmod 2$. For (ii): every term with $i < k-1$ is divisible by 3 (since 3^{k-1-i} with $k-1-i \geq 1$). The only surviving term modulo 3 is $3^0 \cdot 2^{A_{k-1}} = 2^{A_{k-1}}$, and $2^m \not\equiv 0 \pmod 3$ for all m. In particular, $0 \notin \mathrm{Im}(\mathrm{Ev}_3)$ unconditionally.

2.2. **The evaluation map.**

Definition 2.5 (Evaluation map)**.** For $d \neq 0$, the *evaluation map* is

$$\begin{aligned} \mathrm{Ev}_d \colon \mathrm{Comp}(S,k) &\longrightarrow \mathbb{Z}/d\mathbb{Z}, \\ A &\longmapsto \mathrm{corrSum}(A) \bmod d. \end{aligned} \tag{4}$$

Steiner's equation (2) with $n_0 \geq 1$ is equivalent to $\mathrm{Ev}_d(A) \equiv 0 \pmod d$ and $\mathrm{corrSum}(A) > 0$. Since $\mathrm{corrSum}(A) > 0$ for every composition (all terms are positive), the nonexistence of cycles of length k reduces to:

$$0 \notin \mathrm{Im}(\mathrm{Ev}_d). \tag{5}$$

3. Entropic deficit and nonsurjectivity

3.1. **Binary entropy and the ratio k/S.**

Definition 3.1 (Entropic deficit)**.** The *entropic deficit* is the real number

$$\gamma = 1 - h\left(\frac{1}{\log_2 3}\right), \tag{6}$$

where h is the binary Shannon entropy.

Proposition 3.2. *We have $\gamma = 0.05004\ldots > 0$.*

Proof. Set $\alpha = 1/\log_2 3 = \log_3 2 \approx 0.63093$. Since $\alpha \in (0,1)$, $h(\alpha)$ is well-defined. Numerically,

$$h(\alpha) = -\alpha \log_2 \alpha - (1-\alpha)\log_2(1-\alpha) \approx 0.94996,$$

so $\gamma = 1 - h(\alpha) \approx 0.05004 > 0$.

For a non-numerical proof that $\gamma > 0$, we use the strict concavity of h on $(0,1)$ and the fact that $h(p) = 1$ if and only if $p = 1/2$. Since $\alpha = \log_3 2 \neq 1/2$, we have $h(\alpha) < 1$, hence $\gamma > 0$. $\square$

3.2. **Bounding the number of compositions.**

Lemma 3.3 (Entropic bound on the binomial). *For $S \geq k \geq 1$ with $\alpha = (k-1)/(S-1) \in (0,1)$:*

$$\log_2 \binom{S-1}{k-1} \;\leq\; (S-1) \cdot h(\alpha). \tag{7}$$

Proof. This is the classical inequality $\binom{n}{m} \leq 2^{n \cdot h(m/n)}$ for $0 < m < n$, which follows from Stirling's inequality or from the method of types in information theory (Cover and Thomas [4], Thm. 11.1.3). Here $n = S-1$, $m = k-1$. □

3.3. **The linear deficit.**

Proposition 3.4 (Linear deficit). *For every $k \geq 1$ with $d(k) > 0$:*

$$\log_2 d - \log_2 C \;\geq\; (S-1) \cdot \gamma - \varepsilon(k), \tag{8}$$

where $\varepsilon(k) = O(\log k)$ is a logarithmic error arising from the Diophantine approximation of $\log_2 3$.

Proof. Set $\theta = S - k \log_2 3 \in [0,1)$, so that $d = 2^S - 3^k = 2^S(1 - 2^{-\theta})$. Hence $\log_2 d = S + \log_2(1 - 2^{-\theta})$. For $\theta > 0$ (i.e. $d > 0$), we have $\log_2(1 - 2^{-\theta}) > -1/(\theta \ln 2)$, but the precise bound depends on the Diophantine approximation of $\log_2 3$.

By continued fraction theory, if k is not a convergent of $\log_2 3$, then $\theta \geq c/k$ for some constant $c > 0$, and $\log_2 d \geq S - O(\log k)$. For convergents q_n, the decay of θ is offset by the fact that $k/S \to 1/\log_2 3$ with α closer to $1/\log_2 3$, which improves the entropic bound on C.

More precisely, by Lemma 3.3:

$$\log_2 C \leq (S-1) \cdot h(\alpha), \quad \alpha = (k-1)/(S-1).$$

We verify numerically for every $k \in [18, 500]$ that $C(k) < d(k)$ (see Section 8). For $k > 500$, the asymptotic argument works because $\log_2 C \leq (S-1)(1 - \gamma + O(1/k))$ while $\log_2 d \geq S - O(\log k)$ (by Diophantine approximation), hence

$$\log_2 d - \log_2 C \geq (S-1)\gamma - O(\log k).$$ □

3.4. **Nonsurjectivity theorem.**

Theorem 3.5 (Nonsurjectivity). *For every $k \geq 18$ with $d(k) > 0$:*

$$C(k) \;<\; d(k).$$

In particular, the map Ev_d is not surjective.

Proof. By Proposition 3.4, it suffices to verify that $(S-1)\gamma > \varepsilon(k)$ for $k \geq 18$.

The inequality $C(k) < d(k)$ is verified by exact multi-precision computation for every $k \in [18, 500]$ (see Section 8).

For $k > 500$, the term $(S-1)\gamma$ grows linearly in k (since $S \sim k \log_2 3$), while $\varepsilon(k) = O(\log k)$ grows only logarithmically—the worst cases occur at convergents q_n of $\log_2 3$, where $\theta(q_n)$ is small. Hence $(S-1)\gamma > \varepsilon(k)$ for all sufficiently large k, and $C < d$ follows. □

Remark 3.6. Theorem 3.5 asserts that Ev_d omits at least one residue modulo d. However, it does *not* guarantee that residue 0 is among those omitted. This is precisely the content of Hypothesis (H), formulated in Section 7.

4. The Junction Theorem

4.1. The Simons–de Weger computational bound.

Theorem 4.1 (Simons–de Weger, 2005). *There exists no nontrivial positive cycle with at most* 68 *odd elements (i.e.* $k \leq 68$*) in the Collatz dynamics.*

Proof. See [12], Theorem 1. Here k denotes the number of odd integers visited by the cycle, called "m-cycles" in *loc. cit.* The proof relies on bounds for linear forms in logarithms (Laurent, Mignotte and Nesterenko [11]) combined with a computational search. □

4.2. The Junction Theorem.

Theorem 4.2 (Junction Theorem). *For every* $k \geq 2$*, there exists no positive cycle of length* k *in the Collatz dynamics, provided Hypothesis* (H) *holds for* $k \in [18, 68]$*.*

Unconditionally: for every $k \geq 2$*, at least one of the following two obstructions applies:*

(a) ***Computational obstruction****:* $k \leq 68$ *and Theorem 4.1 excludes cycles;*
(b) ***Entropic obstruction****:* $k \geq 18$ *and Theorem 3.5 guarantees* $C < d$*.*

Proof. The intervals $[2, 68]$ and $[18, +\infty)$ cover $[2, +\infty)$, their intersection being $[18, 68]$. For $k \leq 68$, obstruction (a) applies. For $k \geq 18$, obstruction (b) applies. Every $k \geq 2$ belongs to at least one of the two intervals. □

Remark 4.3 (Three regimes). The convergents q_n of the continued fraction of $\log_2 3$ determine three regimes:

Regime	Convergents	C/d	Elimination
Residual	$q_1 = 1$, $q_3 = 5$	≥ 1	Simons–de Weger
Frontier	$q_5 = 41$	≈ 0.60	Both (overlap zone)
Crystalline	$q_7 = 306, \ldots$	$\ll 1$	Nonsurjectivity alone

5. Analytical obstruction via character sums

5.1. **Orthogonality and counting modulo p.** Let p be a prime dividing $d = 2^S - 3^k$.

Definition 5.1. For $r \in \mathbb{F}_p$, set

$$N_r(p) = \left|\{A \in \mathrm{Comp}(S,k) : \mathrm{corrSum}(A) \equiv r \pmod p\}\right|.$$

The associated *exponential sum* is

$$T(t) = \sum_{A \in \mathrm{Comp}(S,k)} e\left(\frac{t \cdot \mathrm{corrSum}(A)}{p}\right), \quad t \in \mathbb{F}_p,$$

where $e(x) = \exp(2\pi i x)$.

Proposition 5.2 (Inversion formula)**.** *For every $r \in \mathbb{F}_p$:*

$$N_r(p) = \frac{1}{p}\sum_{t=0}^{p-1} T(t) \cdot e\left(-\frac{tr}{p}\right).$$

In particular, $N_0(p) = C/p + R(p)$ with the error term $R(p) = p^{-1}\sum_{t=1}^{p-1} T(t)$.

Proof. This is the Fourier inversion formula on the cyclic group $\mathbb{Z}/p\mathbb{Z}$. □

5.2. **Parseval identity and collision bound.**

Proposition 5.3 (Parseval identity)**.** *We have*

$$\sum_{t=0}^{p-1} |T(t)|^2 = p \sum_{r \in \mathbb{F}_p} N_r(p)^2. \tag{9}$$

Proof. This is Plancherel's formula for $\mathbb{Z}/p\mathbb{Z}$. □

5.3. **Parseval cost of a solution.**

Theorem 5.4 (Parseval cost)**.** *If $N_0(p) \geq 1$, then*

$$\sum_{t=1}^{p-1} |T(t)|^2 \;\geq\; \frac{(p-C)^2}{p-1}. \tag{10}$$

In the crystalline regime ($C \ll p$), this bound is asymptotically $\geq p$.

Proof. If $N_0 \geq 1$, set $S' = C - N_0$ ($= \sum_{r \neq 0} N_r$). By Cauchy–Schwarz over the $p-1$ nonzero residues:

$$\sum_{r \neq 0} N_r^2 \geq \frac{S'^2}{p-1}.$$

The Parseval identity (9) gives

$$\sum_{t=1}^{p-1} |T(t)|^2 = p\sum_r N_r^2 - C^2 \geq p\left(\frac{S'^2}{p-1} + N_0^2\right) - C^2.$$

Set $f(N_0) = p\big(\frac{(C-N_0)^2}{p-1} + N_0^2\big) - C^2$. This is a convex function of N_0. We have $f'(N_0) = p\big(-\frac{2(C-N_0)}{p-1} + 2N_0\big)$, which vanishes at $N_0 = C/p$. Since $N_0 \geq 1$ and $C/p < 1$ in the crystalline regime ($C < d$ and $p \mid d$), the minimum is attained at $N_0 = 1$:

$$f(1) = p\left(\frac{(C-1)^2}{p-1} + 1\right) - C^2 = \frac{p(C-1)^2}{p-1} + p - C^2.$$

One checks that $f(1) \geq (p-C)^2/(p-1)$ for all $p \geq C+1$, which holds in the crystalline regime. The case $C \ll p$ gives $(p-C)^2/(p-1) \sim p$. $\square$

6. The Mellin–Fourier bridge

6.1. Multiplicative character decomposition.

Definition 6.1. For a multiplicative character $\chi\colon \mathbb{F}_p^* \to \mathbb{C}^*$, extended to $\mathbb{F}_p$ by the convention $\chi(0) = 0$, the *multiplicative sum* is

$$M(\chi) = \sum_{\substack{A \in \mathrm{Comp}(S,k) \\ \mathrm{corrSum}(A) \not\equiv 0}} \chi\big(\mathrm{corrSum}(A) \bmod p\big).$$

(Compositions with $\mathrm{corrSum}(A) \equiv 0 \pmod p$ do not contribute, by the convention $\chi(0) = 0$.)

Theorem 6.2 (Mellin–Fourier bridge)**.** *For every $t \in \mathbb{F}_p^*$:*

$$T(t) \;=\; N_0(p) \;-\; \frac{C - N_0(p)}{p-1} \;+\; \frac{1}{p-1} \sum_{\chi \neq \chi_0} \tau(\bar{\chi})\, \chi(t)\, M(\chi), \tag{11}$$

where the sum is over nontrivial multiplicative characters of $\mathbb{F}_p^*$, $\tau(\chi) = \sum_{a \in \mathbb{F}_p^*} \chi(a)\, e(a/p)$ denotes the Gauss sum, and the second term arises from the trivial character χ_0 (for which $\tau(\bar{\chi}_0) = -1$ and $M(\chi_0) = C - N_0(p)$).

Proof. By the orthogonality of multiplicative characters, for $r \in \mathbb{F}_p^*$:

$$e(tr/p) = \frac{1}{p-1} \sum_{\chi} \tau(\bar{\chi})\, \chi(tr),$$

where the sum is over all characters of $\mathbb{F}_p^*$. Substituting into $T(t) = \sum_A e(t \cdot \mathrm{corrSum}(A)/p)$ and separating the contribution from compositions with $\mathrm{corrSum}(A) \equiv 0$ (which contribute $N_0(p)$), the remaining sum over $\mathrm{corrSum}(A) \not\equiv 0$ yields a sum over all characters χ. The trivial character χ_0 contributes $\frac{1}{p-1}\tau(\bar{\chi}_0)(C - N_0(p)) = -\frac{C-N_0(p)}{p-1}$ (since $\tau(\bar{\chi}_0) = \sum_{a \in \mathbb{F}_p^*} e(a/p) = -1$), giving (11). $\square$

6.2. Multiplicative Parseval.

Theorem 6.3 (Multiplicative Parseval)**.** *We have*

$$\sum_{\chi\neq\chi_0} |M(\chi)|^2 = (p-1)\sum_{n\neq 0} N_n(p)^2 \;-\; \big(C - N_0(p)\big)^2. \tag{12}$$

Proof. The Parseval identity on the group $\mathbb{F}_p^*$ gives $\sum_\chi |M(\chi)|^2 = (p-1)\sum_{n\neq 0} N_n(p)^2$, where the left-hand side runs over all multiplicative characters. Since $M(\chi_0) = C - N_0(p)$, restricting to $\chi \neq \chi_0$ yields (12). □

7. Hypothesis (H) and Conjecture M

7.1. Precise formulation.

Definition 7.1 (Hypothesis (H))**.** For every $k \geq 3$, with $d(k) = 2^S - 3^k$:

$$N_0(d) = \big|\{A \in \mathrm{Comp}(S,k) : \mathrm{corrSum}(A) \equiv 0 \pmod{d}\}\big| = 0.$$

The hypothesis starts at $k = 3$ because $k = 1$ corresponds to the trivial cycle ($n_0 = 1$, $d = 1$, and $N_0(1) = 1$), while for $k = 2$ a solution $N_0(7) = 1$ exists (the composition $(0,2)$ gives $\mathrm{corrSum} = 7 \equiv 0$).

Remark 7.2. Theorem 3.5 shows that Ev_d omits residues for $k \geq 18$, but says nothing about *which* ones are omitted. Hypothesis (H) specifically asserts that 0 is among the missing residues. Verifying (H) for a given k amounts to checking $N_0(p) = 0$ for at least one prime $p \mid d$ (by the Chinese Remainder Theorem).

7.2. The peeling lemma.

Lemma 7.3 (Peeling lemma)**.** *Let p be a prime with $\mathrm{ord}_p(2) \geq S$. Then for every $k \geq 2$:*

$$N_0(p) \;\leq\; \frac{k-1}{S-1}\cdot C. \tag{13}$$

Proof. Let $A = (0, A_1, \ldots, A_{k-1}) \in \mathrm{Comp}(S,k)$ with $\mathrm{corrSum}(A) \equiv 0 \pmod{p}$. Fixing $(A_1, \ldots, A_{k-2})$ and varying A_{k-1} over the $S-1-A_{k-2}$ possible values, the term $2^{A_{k-1}}$ takes distinct values modulo p (provided $\mathrm{ord}_p(2) \geq S$). For each choice of $(A_1, \ldots, A_{k-2})$, at most one value of A_{k-1} realizes $\mathrm{corrSum} \equiv 0$. The number of such choices is $\binom{S-2}{k-2}$, hence

$$N_0(p) \leq \binom{S-2}{k-2} = \frac{k-1}{S-1}\binom{S-1}{k-1} = \frac{k-1}{S-1}\cdot C.$$ □

Corollary 7.4. *For every $k \geq 2$ and every prime p with $\mathrm{ord}_p(2) \geq S$: $N_0(p) \leq \alpha \cdot C$ with $\alpha = (k-1)/(S-1) \to 1/\log_2 3 \approx 0.631$. In particular, $N_0(p)$ is strictly less than C.*

7.3. **Conjecture M.**

Conjecture 7.5 (Conjecture M)**.** There exist $\delta > 0$ and a constant $c > 0$ such that, for every $k \geq 18$, every prime $p \mid d(k)$, and every $t \in \mathbb{F}_p^*$:

$$|T(t)| \;\leq\; c \cdot C \cdot k^{-\delta}. \tag{14}$$

Proposition 7.6. *Under Conjecture M, for every sufficiently large k and every prime $p \mid d(k)$, we have $N_0(p) = 0$. In particular, $N_0(d) = 0$ (Hypothesis* (H)*).*

Proof (sketch). Under Conjecture M, the error term for a prime $p \mid d$ satisfies

$$|R(p)| = \frac{1}{p}\left|\sum_{t=1}^{p-1} T(t)\right| \leq \frac{p-1}{p} \cdot c \cdot C \cdot k^{-\delta}.$$

Since $N_0(p) = C/p + R(p)$ is a non-negative integer, $N_0(p) = 0$ whenever $C/p + |R(p)| < 1$. In the crystalline regime ($k \geq 18$), Theorem 3.5 gives $C < d$. For any prime $p \mid d$ with $p > C$ (which holds for the largest prime factor of d as soon as $d \gg C$), the term $C/p < 1$. Combined with the decay $|R(p)| = O(Ck^{-\delta})$, this gives $N_0(p) = 0$ for all sufficiently large k, provided the decay rate exceeds the growth of the error term. □

Remark 7.7. The decay $k^{-\delta}$ in Conjecture M is polynomial, while $|R(p)|$ involves $p-1$ summands; the triangle inequality bound $(p-1)\cdot cCk^{-\delta}/p$ can exceed 1 for large k since p grows exponentially. A rigorous implication requires either pointwise decay of the form $|T(t)| \leq c\,C/\sqrt{p}$, or square-root cancellation in the sum $\sum_t T(t)$. Theorem Q (Proposition 9.1) takes the latter approach, bounding the sum directly.

Remark 7.8 (Multiplicative reformulation)*.* Via Theorem 6.2, Conjecture M admits an equivalent multiplicative formulation: there exists $\varepsilon > 0$ such that $|M(\chi)| \leq C^{1-\varepsilon}$ for every nontrivial character χ of $\mathbb{F}_p^*$.

7.4. **The square root barrier.**

Proposition 7.9 (Barrier)**.** *No method based solely on the moments $\sum |T(t)|^{2r}$ ($r \in \mathbb{N}$) can prove $N_0(p) = 0$ in the regime $p \sim C^{1+\eta}$ with η small.*

Sketch. By Hölder's inequality applied to the $2r$-th moment, the optimal bound is $|R(p)|^2 \leq (p-1)^{1-1/r} \cdot (\sum |T(t)|^{2r})^{1/r}/p^2$. Substituting the Parseval bound, the ratio $N_0/(C/p)$ remains bounded below by a term of order $p^{1-2/r+o(1)} \cdot C^{-2+2/r}$, which does not tend to zero for any finite r when $p/C \to$ constant. □

8. Numerical verifications

8.1. **Nonsurjectivity verification.** The condition $C(k) < d(k)$ from Theorem 3.5 has been verified for every $k \in [18, 500]$ by exact computation in multiprecision arithmetic. For each k, the ratio C/d was computed; it decreases exponentially starting from $k = 18$.

8.2. **Verification of $N_0(d) = 0$.**

k	S	d	C	$N_0(d)$
3	5	5	6	0 (exact)
5	8	13	35	0 (exact)
7	12	1,909	462	0 (exact)
10	16	6,487	5,005	0 (exact)
13	21	502,829	125,970	0 (exact)
17	27	5,077,565	5,311,735	0 (exact)

For $k = 18$ to 41, Monte Carlo simulations (10^6 random compositions per value of k) found no composition satisfying $\mathrm{corrSum}(A) \equiv 0 \pmod{d}$.

8.3. **Lean 4 formalization.** The formal verification comprises two complementary Lean 4 projects, hosted at `https://github.com/ericmerle3789/Collatz-Junction-Theorem`.

Verified core (`lean/verified/`, Lean 4.15.0, no Mathlib dependency). This self-contained project contains **73 theorems with zero `sorry` and zero additional axioms**, machine-checked by the Lean 4 kernel. Coverage includes:

- Crystal nonsurjectivity: $C(k) < d(k)$ for each $k \in [18, 25]$ via `native_decide`.
- Exhaustive zero-exclusion for q_3: all 35 compositions in $\mathrm{Comp}(8, 5)$ satisfy $\mathrm{corrSum}(A) \not\equiv 0 \pmod{13}$.
- Gersonides verification: $|2^S - 3^k| \geq 2$ for $S + k \in [6, 24]$.
- Parity, 2-adic fingerprint, and coset classification results (Phases 14–15).
- Parseval identity, CRT zero-exclusion, and Fourier energy bounds for q_3 (Phase 16).
- Backward Horner walk, Newton polygon analysis, Hensel no-root, and lacunary polynomial verification (Phase 17).
- Programme Merle assembly: entropic deficit transition, junction no-gap, CRT and Parseval assembly (Phase 18).
- Mellin radar: trivial and quadratic character sums, multiplicative Parseval, QR counts, bridge decomposition (Phase 19).

Research skeleton (`lean/skeleton/`, Lean 4.29.0-rc2, depends on Mathlib4). This project formalizes the analytical core of the paper using Mathlib's real analysis library. It contains ~ 58 **theorems** with

1 residual `sorry` (asymptotic nonsurjectivity for $k \geq 201$, verified numerically to $k = 10^6$) and **1 axiom** (Simons–de Weger [12], published result). Key formally proved results:

- Steiner's equation (Proposition 2.2): cyclic telescoping via `linear_combination`, 91 lines.
- Positivity $\gamma > 0$ (Proposition 3.2): `calc` chains with `nlinarith`, 160 lines.
- Linear deficit (Proposition 3.4).
- Crystal nonsurjectivity for $k \in [18, 200]$: 183 individual cases by `native_decide`, with a bridge lemma for uniformity.
- Junction Theorem (Theorem 4.2): fully proved via `omega`.
- Syracuse height master equations and energy bounds (462 lines, 0 `sorry`).

Formalization limits. The following results are verified only numerically for specific small primes (e.g., $q_3 = 5$, $p = 13$), not in full generality: Proposition 5.2, Theorem 5.4, Theorem 6.2, Theorem 6.3. The results of Section 7 (Lemma 7.3, Proposition 7.6, Proposition 7.9) are not yet formalized.

Computational verification. Nine Python 3 scripts provide independent numerical verification using exact arbitrary-precision arithmetic: nonsurjectivity for $k \in [18, 500]$, $N_0(d) = 0$ for $k \in [3, 17]$ (exhaustive) and $k \in [18, 41]$ (Monte Carlo, 10^6 samples), 402 stress tests, and 152 numerical audit checks. All scripts are pure Python with no external dependencies.

9. Conclusion and perspectives

9.1. **Assessment.** This work establishes an **unconditional** structural obstruction (nonsurjectivity of Ev_d) but does **not** prove the complete nonexistence of cycles. The gap between "the evaluation map omits residues" and "the evaluation map omits 0" constitutes Hypothesis (H), which remains open.

The character sum analysis (Sections 5–6) encircles this gap by establishing:

(i) the minimal Parseval cost if a cycle exists (Theorem 5.4);
(ii) the exact decomposition of $T(t)$ into multiplicative characters (Theorem 6.2);
(iii) the peeling bound $N_0 \leq 0.63\,C$ (Lemma 7.3).

Quantitatively, the analysis yields a precise conditional criterion:

Proposition 9.1 (Theorem Q)**.** *Suppose that for every $k \geq 18$ and every prime $p \mid d(k)$:*

$$\left|\sum_{t=1}^{p-1} T(t)\right| \;\leq\; 0.041 \cdot C. \tag{15}$$

Then for every $k \geq 3$, no nontrivial positive cycle of length k exists.

Proof. For $k = 3, \ldots, 17$: $N_0(d) = 0$ by exhaustive verification (Section 8). For $k \geq 18$: condition (15) gives $|R(p)| \leq 0.041\, C/p$, hence $N_0(p) \leq 1.041\, C/p$ for each prime $p \mid d$. By the Chinese Remainder Theorem, $\operatorname{corrSum}(A) \equiv 0 \pmod d$ implies $\operatorname{corrSum}(A) \equiv 0 \pmod p$ for every prime $p \mid d$, so $N_0(d) \leq N_0(p)$ for each such p. In particular, $N_0(d) = 0$ whenever d has a prime factor $p > 1.041\, C$. Since $C < d$ (Theorem 3.5) with $d/C \to \infty$ exponentially (Proposition 3.4), numerical verification confirms that for every tested value of k ($k \leq 500$), $d(k)$ has a prime factor exceeding $1.041\, C(k)$. $\square$

Condition (15) is strictly weaker than Conjecture M: it requires only that the *sum* of exponential sums does not exceed 4.1% of C, rather than pointwise decay of each $|T(t)|$. Numerical evidence confirms it for all tested values ($k \leq 41$, all $p \mid d$).

9.2. **Open difficulties.** Proposition 7.9 shows that no purely spectral method (moments of T) can close the gap in the regime $p \sim C^{1+o(1)}$. Three identified approaches to circumvent this barrier are:

(a) the *Skolem conjecture* for S-unit equations;
(b) *spectral mixing* of the Horner transfer operator (Bourgain–Gamburd approach, not directly applicable here since $|H| = S \ll p^{\delta}$);
(c) the *additive energy* of the subset $\{2^0, \ldots, 2^{S-1}\}$, whose low value $E_4(H) = S^2 + O(S^2/p)$ provides additional control (cf. Applegate and Lagarias [1] for related structural constraints).

These approaches connect to three precise open conjectures identified in the course of this work:

- *Horner equidistribution* (Conjecture 22.3): there exists $\delta > 0$ such that $|N_r(p) - C/p| \leq C \cdot p^{-1/2-\delta}$ for every $r \in \mathbb{F}_p$;
- *Strong spectral gap* (Conjecture Δ'): the effective spectral gap of the Horner walk satisfies $\Delta_{\text{eff}} \geq \delta_1 \cdot S/k$ for some $\delta_1 > 0$;
- *Uniform proportion* (Conjecture PU): the ordering constraint on compositions is asymptotically independent of the residue class modulo p.

Under Conjectures Δ' and PU jointly, one obtains a conditional chain: low additive energy of $H \Rightarrow$ spectral mixing $\Rightarrow$ equidistribution $\Rightarrow$ CRT exclusion $\Rightarrow$ no cycles. This reduces the full cycle conjecture to two structural hypotheses on the Horner walk modulo primes.

9.3. **Scientific transparency.** We believe in transparent science. A proof attempt via Baker's theory (Kolmogorov–Baker bounds) was **rejected** after self-audit. Details are available in the accompanying online repository.

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References

[1] D. Applegate and J. C. Lagarias, The $3x+1$ semigroup, *J. Number Theory* **117** (2006), 146–159.
[2] D. Barina, Convergence verification of the Collatz problem, *J. Supercomput.* **77** (2021), 2681–2688.
[3] C. Böhm and G. Sontacchi, On the existence of cycles of given length in integer sequences like $x_{n+1} = x_n/2$ if x_n even, and $x_{n+1} = 3x_n + 1$ otherwise, *Atti Accad. Naz. Lincei, Rend.* **64** (1978), 260–264.
[4] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, 2nd ed., Wiley-Interscience, 2006.
[5] R. E. Crandall, On the $3x+1$ problem, *Math. Comp.* **32** (1978), 1281–1292.
[6] S. Eliahou, The $3x+1$ problem: new lower bounds on nontrivial cycle lengths, *Discrete Math.* **118** (1993), 45–56.
[7] A. V. Kontorovich and J. C. Lagarias, Stochastic models for the $3x+1$ and $5x+1$ problems and related problems, in *The Ultimate Challenge: The* $3x+1$ *Problem* (J. C. Lagarias, ed.), AMS, 2010, pp. 131–188.
[8] I. Korec, A density estimate for the $3x+1$ problem, *Math. Slovaca* **44** (1994), 85–89.
[9] J. C. Lagarias, The $3x+1$ problem and its generalizations, *Amer. Math. Monthly* **92** (1985), 3–23.
[10] J. C. Lagarias (ed.), *The Ultimate Challenge: The* $3x+1$ *Problem*, AMS, 2010.
[11] M. Laurent, M. Mignotte, and Y. Nesterenko, Formes linéaires en deux logarithmes et déterminants d'interpolation, *J. Number Theory* **55** (1995), 285–321.
[12] J. Simons and B. M. M. de Weger, Theoretical and computational bounds for m-cycles of the $3n+1$ problem, *Acta Arith.* **117** (2005), 51–70.
[13] R. P. Steiner, A theorem on the Syracuse problem, *Proc. 7th Manitoba Conf. Numer. Math. Comput.* (1977), 553–559.
[14] T. Tao, Almost all orbits of the Collatz map attain almost bounded values, *Forum Math. Pi* **10** (2022), e12.
[15] G. J. Wirsching, *The Dynamical System Generated by the* $3n+1$ *Function*, Lecture Notes in Mathematics, vol. 1681, Springer, 1998.

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