The Fourier Transform

Eric Jonas (jonas@eecs.berkeley.edu)

June 30, 2015

List of Corrections

1 A Brief linear algebra review

Vector space consists of vectors and of scalars that behave as we would expect.

You can extend a vector space to an inner product space by defining an inner product, which associates with each pair of vectors a scalar. This inner product lets you talk about the angle between two vectors, and orthogonality. An inner product must have linearity in the first argument, positive-definiteness, and conjugate symmetry.

This naturally induces a notion of length (a norm).

Now we can wave our hands and talk about vector spaces of functions, and then define an inner product between two signals. In this case we will focus on complex-valued functions, and thus our inner product is

$$< x(t), y(t) >= \int_a^b x(t)y^*(t)dt$$

We can talk about the projection of one signal along another, just like we can talk about the projection of one vector along another.

2 Eigenfunctions of LTI systems

Reviewing last time, we showed that an LTI system is completely characterized by its impulse response, $h(t)$, and that the response of an LTI system with impulse response $h(t)$ to an input signal $x(t)$ could be computed by convolving $h(t)$ and $x(t)$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Consider the response of an LTI system to a complex exponential e^{st}, $s \in \mathbb{C}$:

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau && (1)\\
&= \int_{-\infty}^{\infty} h(\tau)e^{st}e^{-s\tau)}d\tau && (2)\\
&= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau)}d\tau && (3)\\
&= e^{st}H(st) && (4)\\
& && (5)
\end{aligned}
$$

Note that we have applied an LTI system, L_h, to a signal, $x(t)$, and obtained the same signal $x(t)$ multipled by a constant, which we've written $H(st)$. This should evoke an equation from linera algebra,

$$L_h x = \lambda_{H(st)} x$$

We say that the set of complex exponentials are *eigenfunctions* of linear time-invariant systems, and the resulting *eigenvalue* $H(st)$ has important and deep meaning. We'll come back to this later.

3 The Continuous-time Fourier Transform

First, do not fear the Fourier transform, especially if you have taken linear algebra. Most of what follows is a very natural extension of standard ideas from linear algebra (bases, transforms, inner products, unitary operators). Your geometric intuition from Euclidian space will come in handy!

Colloquially, the Fourier transform converts signals from a time-representation to a frequency representation. We often say it takes signals from the time domain to the frequency domain. But that casual explanation hides the fact that the fourier transform is akin to a change in basis. The Fourier transform takes a signal expressed in the time domain, where the basis functions are dirac deltas, and instead expresses it in the frequency domain, where the basis functions are complex exponentials (that is, complex sums of sines and cosines).

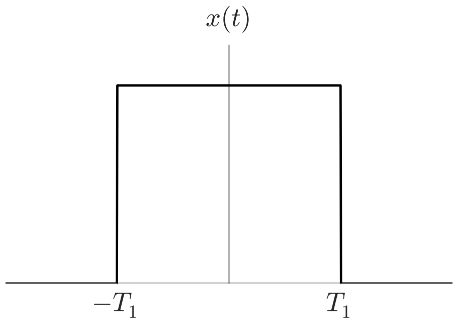
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

3.1 Examples

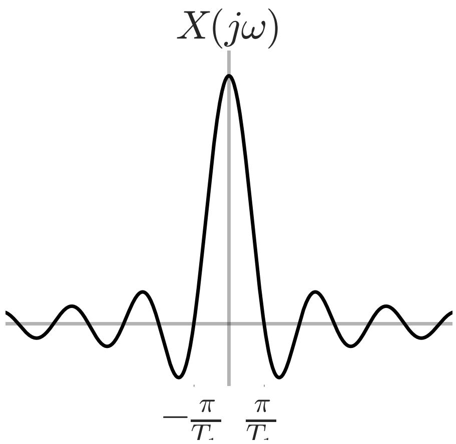
3.2 Square pulse / Sinc function

$$x(t) = \begin{cases} 1 & \text{if } |t| < T_1 \\ 0 & \text{if } |t| \geq T_1 \end{cases}$$



gives

$$X(jw) = \int_{-T_1}^{T_1} e^{-j\omega t} dt = 2\frac{\sin \omega T_1}{\omega}$$



3.3 Periodic functions

Consider a signal whose Fourier transform is

$$X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

we see

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t} d\omega = e^{j\omega_0 t}$$

Remember that

$$cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

We can use linearity above and arrive at

$$\mathscr{F}\{cos(\omega_0 t)\} = \pi\delta(\omega - \omega_0) + \pi\delta(\omega - \omega_0)$$

3.4 Properties

We start off with

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(j\omega)$$

$$y(t) \overset{\mathscr{F}}{\longleftrightarrow} Y(j\omega)$$

Linearity :

$$ax(t) + by(t) \overset{\mathscr{F}}{\longleftrightarrow} aX(j\omega) + bY(j\omega)$$

Time-shifting:

$$x(t - t_0) \overset{\mathscr{F}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$$

3.4.1 conjugation and conjugate symmetry

3.5 Convolution property

Consider

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau$$

We want $Y(j\omega)$ which is

$$\begin{aligned}
Y(j\omega) &= \mathscr{F}\{y(t)\} && (6)\\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \right] e^{-j\omega t} dt && (7)\\
&= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(t - \tau) e^{-j\omega t} dt \right] d\tau && \text{Swap order of integration} \quad (8)\\
&= \int_{-\infty}^{\infty} x(\tau) \left[H(j\omega) e^{-j\omega \tau} \right] d\tau && \text{Bracketed term via time-shifting} \quad (9)\\
&= H(j\omega) \int_{-\infty}^{\infty} x(\tau) e^{-j\omega \tau} d\tau && (10)\\
&= H(j\omega) X(j\omega) && (11)
\end{aligned}$$

And thus we arrive at the celebrated property,

$$y(t) = h(t) * x(t) \overset{\mathscr{F}}{\longleftrightarrow} Y(j\omega) = H(j\omega) X(j\omega)$$

3.6 Persaval

If $x(t)$ and $X(j\omega)$ are a Fourier transform pair, then

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

which can be seen via

$$\begin{aligned}
\int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{-\infty}^{\infty} x(t) x^*(t) dt && (12)\\
&= \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) e^{-j\omega t} d\omega \right] dt && \text{apply inverse FT} \quad (13)\\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) \left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right] d\omega && \text{Swap order of integration} \quad (14)\\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega && \text{bracket term is FT} \quad (15)\\
& && (16)
\end{aligned}$$

Thus the total energy in a singal may be found by computing the energy per unit time $|x(t)|^2$ and integrating over all time or computing the energy per unit frequency $|X(j\omega)|^2/2\pi$ and integrating over all frequencies.

3.7 Duality

FiXme Warning: Do This

3.8 Multiplication in the time domain

We saw that convolution in the time domain corresponds to multiplication in the frequency domain. Because of duality, we expect the dual property to hold – multiplication in the time domain is equivalent to convolution in the frequency domain.

$$r(t) = s(t)p(t) \overset{\mathscr{F}}{\longleftrightarrow} R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j\omega)]$$

Multiplication of one signal by another can be viewed as modulating the amplitude of one signal by another.

4 An introduction to Filtering

FiXme Warning: Give audio example

4.1 Bandwidth and Phase

5 Magnitude and Phase

We often write complex numbers as a real part and an imaginary part,

$$z = x + jy$$

but we can also write them in polar form,

$$r = |z| = \sqrt{x^2 + y^2}$$

$$\phi = \arg(z) = \text{atan2}(y, x)$$

and then we arrive at $= re^{j\phi}$

Like all complex-valued functions, we can represent the complex-valued Fourier transform in terms of magnitude and phase:

$$X(j\omega) = |X(j\omega)|e^{j\sphericalangle X(j\omega)}$$

We think of $|X(j\omega)|^2|$ as the energy in the signal at frequency ω, but phase is vital too. For example, from the time reveral property, we know that

$$x(-t) \overset{\mathscr{F}}{\longleftrightarrow} X(-j\omega)$$

so the signals $x(t)$ and $x(-t)$ have the same magnitude fourier transform but very different phases!

6 The Discrete Fourier Transform

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{j\omega n}d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

This looks a lot more like an inner product!

Remember from before : Discrete-time complex expoenntntials that differe in frequency by a multiple of 2pi are identicial. Thus $X(e^{j\omega})$ is periodic with period 2π.