The Fourier Transform

Eric Jonas (jonas@eecs.berkeley.edu)

June 15, 2015

List of Corrections

1 A Brief linear algebra review

Vector space consists of vectors and of scalars that behave as we would expect.

You can extend a vector space to an inner product space by defining an inner product, which associates with each pair of vectors a scalar. This inner product lets you talk about the angle between two vectors, and orthogonality. An inner product must have linearity in the first argument, positive-definiteness, and conjugate symmetry.

This naturally induces a notion of length (a norm).

Now we can wave our hands and talk about vector spaces of functions, and then define an inner product between two signals. In this case we will focus on complex-valued functions, and thus our inner product is

$$< x(t), y(t) >= \int_a^b x(t)y*(t)dt$$

We can talk about the projection of one signal along another, just like we can talk about the projection of one vector along another.

2 Eigenfunctions of LTI systems

Reviewing last time, we showed that an LTI system is completely characterized by its impulse response, $h(t)$, and that the response of an LTI system with impulse response $h(t)$ to an input signal $x(t)$ could be computed by convolving $h(t)$ and $x(t)$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Consider the response of an LTI system to a complex exponential e^{st}, $s \in \mathbb{C}$:

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau && (1)\\
&= \int_{-\infty}^{\infty} h(\tau)e^{st}e^{-s\tau)}d\tau && (2)\\
&= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau)}d\tau && (3)\\
&= e^{st}H(st) && (4)\\
& && (5)
\end{aligned}$$

Note that we have applied an LTI system, L_h, to a signal, $x(t)$, and obtained the same signal $x(t)$ multipled by a constant, which we've written $H(st)$. This should evoke an equation from linera algebra,

$$L_h x = \lambda_{H(st)} x$$

We say that the set of complex exponentials are *eigenfunctions* of linear time-invariant systems, and the resulting *eigenvalue* $H(st)$ has important and deep meaning. We'll come back to this later.

3 The Continuous-time Fourier Transform

First, do not fear the Fourier transform, especially if you have taken linear algebra. Most of what follows is a very natural extension of standard ideas from linear algebra (bases, transforms, inner products, unitary operators). Your geometric intuition from Euclidian space will come in handy!

Colloquially, the Fourier transform converts signals from a time-representation to a frequency representation. We often say it takes signals from the time domain to the frequency domain. But that casual explanation hides the fact that the fourier transform is akin to a change in basis. The Fourier transform takes a signal expressed in the time domain, where the basis functions are dirac deltas, and instead expresses it in the frequency domain, where the basis functions are complex exponentials (that is, complex sums of sines and cosines).

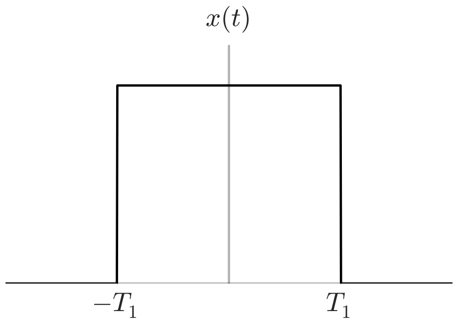
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

3.1 Examples

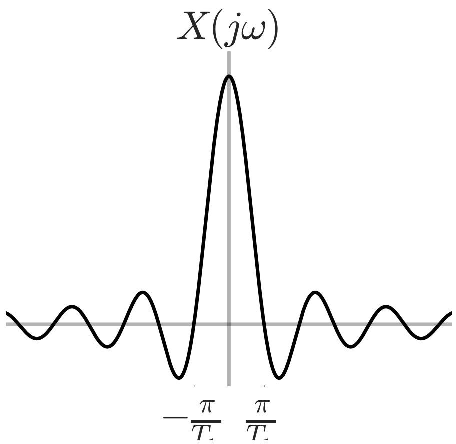
3.2 Square pulse / Sinc function

$$x(t) = \begin{cases} 1 & \text{if } |t| < T_1 \\ 0 & \text{if } |t| \geq T_1 \end{cases}$$



gives

$$X(jw) = \int_{-T_1}^{T_1} e^{-j\omega t} dt = 2\frac{\sin \omega T_1}{\omega}$$



3.3 Periodic functions

Consider a signal whose Fourier transform is

$$X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

we see

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t} d\omega = e^{j\omega_0 t}$$

FiXme Warning: DO the other direction

FiXme Warning: Draw pictures

3.4 Properties

We start off with

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(j\omega)$$

$$y(t) \overset{\mathscr{F}}{\longleftrightarrow} Y(j\omega)$$

Linearity :

$$ax(t) + by(t) \overset{\mathscr{F}}{\longleftrightarrow} aX(j\omega) + bY(j\omega)$$

Time-shifting:

$$x(t - t_0) \overset{\mathscr{F}}{\longleftrightarrow} e^{-j\omega t_0}X(j\omega)$$

3.4.1 conjugation and conjugate symmetry

3.5 Convolution property

Consider

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

We want $Y(j\omega)$ which is

$$\begin{aligned}
Y(j\omega) &= \mathscr{F}\{y(t)\} && (6)\\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right]e^{-j\omega t}dt && (7)\\
&= \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dt\right]d\tau && \text{Swap order of integration} \quad (8)\\
&= \int_{-\infty}^{\infty} x(\tau)\left[H(j\omega)e^{-j\omega\tau}\right]d\tau && \text{Bracketed term via time-shifting} \quad (9)\\
&= H(j\omega)\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau && (10)\\
&= H(j\omega)X(j\omega) && (11)
\end{aligned}$$

$$y(t) = h(t) * x(t) \overset{\mathscr{F}}{\longleftrightarrow} Y(j\omega) = H(j\omega)X(j\omega)$$

4 The Discrete-time Fourier Transform

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

FiXme Warning: Why is this so different? Explain it

FiXme Warning: Give audio example

5 An introduction to Filtering

6 Magnitude and Phase

7 The Discrete Fourier Transform

8 The Fast Fourier Transform

9 Analytic Signals

Most signals in the real world are real-valued.

Physicsists always say "Well we can just use a complex signal and then take the real part" but

1. Why?

2. Ok, there are a whole bunch of ways to create a complex signal from a real one, why do we do it a certain way?

3. I and Q – quadrature signals

4. Negative Frequency