5 Stability of a Single Balloon

5.1 Pressure-Radius Characteristic and Filling-Radius Characteristic

We recall the pressure-radius characteristic given in (3.24) and its graph which is represented by the solid curve of Fig. 3.4. We assume the elastic constants to be given by (3.16) – at room temperature – and consider the radius r_0 and the membrane thickness to have the values

$$r_0 = 25 \text{ mm} \quad \text{and} \quad d_0 = 0.2 \text{ mm}. \tag{5.1}$$

The coordinates of the extrema are

$$\left([p], \frac{r}{r_0}\right) = (35.8 \text{ mbar}, 1.476) \quad \text{and} \quad \left([p], \frac{r}{r_0}\right) = (30.3 \text{ mbar}, 3.143)\,. \tag{5.2}$$

Even the maximum is not higher, actually even lower, than the pressure difference between regions of high pressure and low pressure on the weather chart. More appropriately, the pressure in our lungs can exceed the environmental pressure by as much as 60 to 100 mbar; that is why we find it easy to inflate a balloon. It is true that the initial ascent is steep so that the balloon seems "stiff". Therefore the beginning is hardest. We shall study the inflation process later in Chap. 6.

In Fig. 5.1_{left} we have reproduced the characteristic pressure-radius relation and on the right hand side of the figure we represent the filling ratio $N(r)/(N(r_0)$ of the balloon as a function of radius. By the thermal equation of state of ideal gases we have

$$\begin{aligned} &N(r) = \tfrac{pV}{kT}, \quad \text{or by (3.24) and with} \qquad V = \tfrac{4\pi}{3} r^3 \\ &\tfrac{N(r)}{N(r_0)} = \left[1 + \tfrac{2s_+ d_0}{p_0 r_0}\left(\tfrac{r_0}{r} - \left(\tfrac{r_0}{r}\right)^7\right)\left(1 - \tfrac{s_-}{s_+}\left(\tfrac{r}{r_0}\right)^2\right)\right]\left(\tfrac{r}{r_0}\right)^3. \end{aligned} \tag{5.3}$$

$N(r_0) = N_0$ refers to the number of molecules in the undistorted state. We note that $N(r)$ is a monotonic function of r because the factor r^3 in (5.3) has "ironed out" the non-monotonic part of the $[p]$-curve, cf. Fig. 5.1_{right}.

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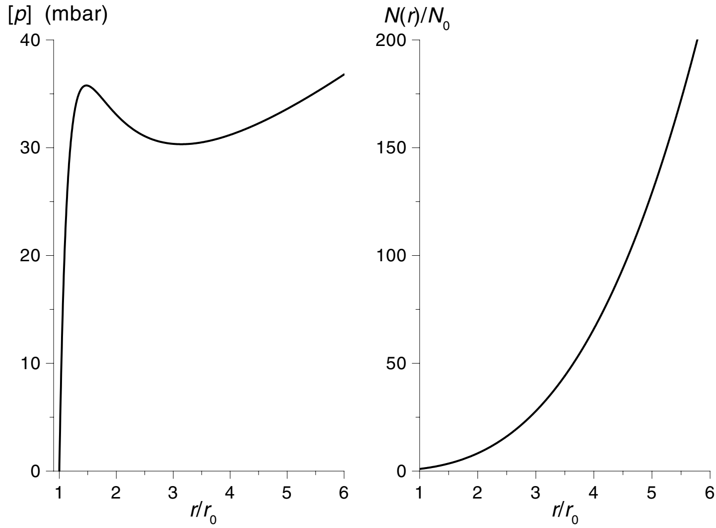

Fig. 5.1. Left: Pressure-radius relation of a balloon for the data given in the text. Right: Filling-radius relation

Inspection of the figure shows that a sixfold stretch of the radius requires a filling ratio of approximately 200. This is by no means a maximum. A typical party balloon can support a stretch of 10 easily, so that its diameter increases from 5 cm to 50 cm, it will then contain one thousand times the filling in the undistorted state.

5.2 Pressure-Filling Curve

Seeing that the filling-radius curve is monotonic, we conclude that there is a one-to-one relation between filling and radius so that we may consider the filling *or* the radius as possible measures of the degree of inflation of a balloon. We may therefore eliminate $\frac{r}{r_0}$ between the two diagrams of Fig. 5.1 and plot the pressure-filling curve which is shown in Fig. 5.2.

The pressure-filling curve will be used extensively in the sequel because it is the filling – not the radius, and certainly not the pressure – which we control when we inflate a balloon.

5.3 Free Energy-Radius Relation of the Balloon

In stretching the balloon we store free energy in the membrane[1] and this energy equals the work of the pressure difference $[p]$ during the inflation.

[1] Actually we lower the entropy of the membrane, cf. Chap. 2.

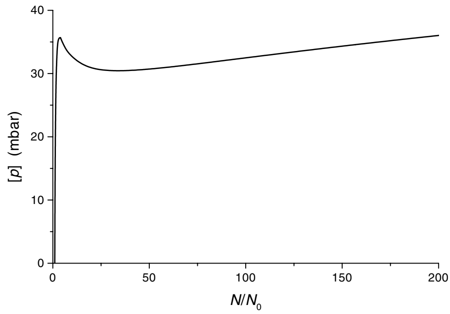

Fig. 5.2. Pressure-filling curve

Therefore we may write

$$F_{\mathrm{B}}(r) - F_{\mathrm{B}}(r_0) = \int\limits_{V(r_0)}^{V(r)} [p] \, \mathrm{d}V. \tag{5.4}$$

Since $\mathrm{d}V = 4\pi r^2 \mathrm{d}r$ holds, the integration is an integration over radius. We eliminate $[p]$ between (3.24) and (5.4) and obtain the free energy of the balloon, viz.

$$\begin{aligned} F_{\mathrm{B}}(r) - F_{\mathrm{B}}(r_0) = 4\pi s_+ \left(d_0 r_0^2\right) \Bigg[& \left(\left(\frac{r}{r_0}\right)^2 + \frac{1}{2}\left(\frac{r_0}{r}\right)^4 - \frac{3}{2}\right) + \\ & + \frac{1}{K}\left(\frac{1}{2}\left(\frac{r}{r_0}\right)^4 + \left(\frac{r_0}{r}\right)^2 - \frac{3}{2}\right)\Bigg] \end{aligned} \tag{5.5}$$

The graph of this function is shown in Fig. 5.3. The curve is convex, since $[p]r^2$ – the integrand in (5.4) – is a monotonic function of r; even the factor r^2 is capable of "ironing out" the non-monotonicity of $[p]$ as function of r. For a sixfold expansion the free energy stored in the membrane is approximately 50 J.

5.4 Stability and Non-monotonicity

Thermodynamicists perk up considerably when they come into contact with non-monotonic load-stretch characteristics such as the pressure-radius relation (3.24) for a balloon represented graphically in Fig. 5.1$_{left}$. Indeed, non-monotonicity suggests interesting non-trivial stability problems as we have seen in Chap. 1 and Sects. 4.8 and 4.9.

In order to investigate the stability properties of a balloon and the possibilities of stabilization, we have found it useful to consider the device shown

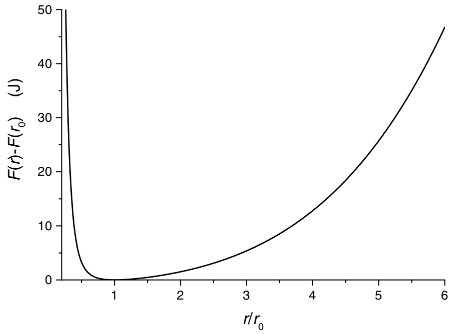

Fig. 5.3. Free energy-radius curve. [The descending branch for $\frac{r}{r_0} < 1$ must be ignored since it is impossible to shrink the balloon and maintain its spherical form.]

in Fig. 5.4_{left} consisting of a balloon and a cylinder of cross-section A. The cylinder is closed off by a piston and the motion of the piston loads or unloads an elastic spring. A certain air filling N is fed into the system through the valve A which is then closed and remains closed. Valve B is open so that the ballon and the cylinder can exchange air; thus the total filling is split into the parts N_{Z} in the cylinder and N_{B} in the balloon.

As usual the stability criterion can be derived from the first and second laws. We have

$$\frac{\mathrm{d}(U+K)}{\mathrm{d}t} = \dot{Q} + \dot{W} \qquad \text{and} \qquad \frac{\mathrm{d}S}{\mathrm{d}t} \geqslant \frac{\dot{Q}}{T}. \tag{5.6}$$

We apply these relations to the control volume marked in Fig. 5.4_{left} by a dotted line. $\dot{W}$ consists of three parts

- the working of the gravitational force on the piston. This part may be written – to within sign – as the rate of change of the potential energy $E_{\mathrm{Pot}} = mgH$, cf. Fig. 5.4_{left},
- the working of the force $\lambda\,(H_0 - H)$ on the piston; H_0 is the height of the piston when the spring is unloaded, except for the weight mg; λ is the stiffness of the spring, and
- the working of the external pressure p_0 on the piston and the balloon.

Therefore we have with $V_0 = H_0 A$

$$\dot{W} = -\frac{\mathrm{d}\frac{mgV_{\mathrm{Z}}}{A}}{\mathrm{d}t} + \frac{\lambda}{A^2}\,(V_0 - V_{\mathrm{Z}})\,\frac{\mathrm{d}V_{\mathrm{Z}}}{\mathrm{d}t} - p_0\frac{\mathrm{d}\,(V_{\mathrm{Z}} + V_{\mathrm{B}})}{\mathrm{d}t}. \tag{5.7}$$

Elimination of $\dot{Q}$ and $\dot{W}$ between (5.6), (5.7) yields

$$\frac{\mathrm{d}\left\{U - TS + K + \frac{mg}{A}V_{\mathrm{Z}} + \frac{\lambda}{2A^2}\,(V_{\mathrm{Z}} - V_0)^2 + p_0\,(V_{\mathrm{Z}} + V_{\mathrm{B}})\right\}}{\mathrm{d}t} \leqslant 0.$$

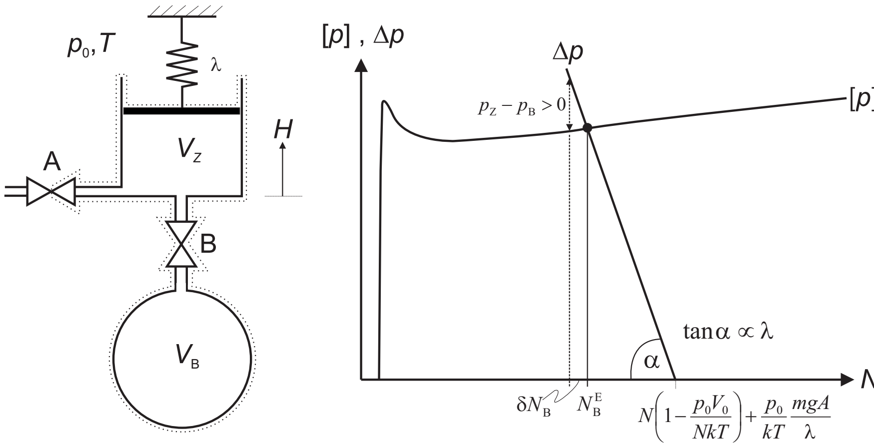

Fig. 5.4. Left: A device for the study of the stability of a balloon. Right: $[p]$ and Δp as functions of N_B. Evaluation of stability, cf. Sect. 5.5

It follows that in equilibrium, when all motion has ceased, the quantity

$$\mathfrak{A} = U - TS + \frac{mg}{A} V_Z + \frac{\lambda}{2A^2} \left(V_Z - V_0\right)^2 + p_0 \left(V_Z + V_B\right) \tag{5.8}$$

must assume a minimum. We call this quantity the available free energy and denote it by $\mathfrak{A}$.

The free energy $F = U - TS$ itself consists of three parts, viz.

$$F = (N - N_B)\ kT \ln \frac{p_Z}{p_0} + N_B\ kT \ln \frac{p_B}{p_0} + \int\limits_{V(r_0)}^{V_B} [p]\ \mathrm{d}V. \tag{5.9}$$

These parts are the free energies of the ideal gases in the cylinder and balloon – without an unimportant additive function of temperature – and the free energy (5.4), (5.5) of the rubber membrane. Seeing that $pV = NkT$ holds in the cylinder and balloon we eliminate p_Z and p_B from (5.9) and insert $F = U - TS$ into (5.8). Thus we obtain a fairly complex expression for the available free energy, namely

$$\begin{aligned}\mathfrak{A} = {} & (N - N_B)\ kT \ln \tfrac{(N-N_B)\ kT}{p_0 V_Z} + N_B kT \ln \tfrac{N_B kT}{p_0 V_B} + \\ & + \int\limits_{V(r_0)}^{V_B} [p]\ \mathrm{d}V + \tfrac{mg}{A} V_Z + \tfrac{\lambda}{2A^2} \left(V_Z - V_0\right)^2 + p_0 \left(V_Z + V_B\right).\end{aligned} \tag{5.10}$$

This is a function of the three variables V_Z, V_B and N_B.

Necessary conditions for equilibrium are that the first derivatives of $\mathfrak{A}$ with respect to the three variables vanish. These criteria provide the rather trivial conditions

$$\frac{\partial \mathfrak{A}}{\partial N_{\mathrm{B}}} = 0 : p_{\mathrm{Z}} = p_{\mathrm{B}}$$

$$\frac{\partial \mathfrak{A}}{\partial V_{\mathrm{Z}}} = 0 \ : p_{\mathrm{Z}} = p_0 + \frac{mg}{A} + \lambda \frac{V_{\mathrm{Z}} - V_0}{A^2} \,. \tag{5.11}$$

$$\frac{\partial \mathfrak{A}}{\partial V_{\mathrm{B}}} = 0 \ : p_{\mathrm{B}} = p_0 + [p].$$

Indeed, in equilibrium we expect – even without much calculation – that the pressures in the cylinder and in the balloon are equal. Also it is only natural that the pressure difference across the piston is given by the weight of the piston plus the pressure of the spring, and that the pressure difference across the balloon membrane is given by the pressure-radius relation $[p](r)$. All these expectations are confirmed by (5.11).

For the equilibrium to be *stable* the matrix of second derivatives of the available free energy with respect to its variables N_{B}, V_{Z}, V_{B} must be positive definite. We have

$$\begin{bmatrix} \frac{\partial^2 \mathfrak{A}}{\partial N_{\mathrm{B}}^2} & \frac{\partial^2 \mathfrak{A}}{\partial V_{\mathrm{Z}} \partial N_{\mathrm{B}}} & \frac{\partial^2 \mathfrak{A}}{\partial V_{\mathrm{B}} \partial N_{\mathrm{B}}} \\ \frac{\partial^2 \mathfrak{A}}{\partial N_{\mathrm{B}} \partial V_{\mathrm{Z}}} & \frac{\partial^2 \mathfrak{A}}{\partial V_{\mathrm{Z}}^2} & \frac{\partial^2 \mathfrak{A}}{\partial V_{\mathrm{B}} \partial V_{\mathrm{Z}}} \\ \frac{\partial^2 \mathfrak{A}}{\partial N_{\mathrm{B}} \partial V_{\mathrm{B}}} & \frac{\partial^2 \mathfrak{A}}{\partial V_{\mathrm{Z}} \partial V_{\mathrm{B}}} & \frac{\partial^2 \mathfrak{A}}{\partial V_{\mathrm{B}}^2} \end{bmatrix} =$$

$$= \begin{bmatrix} kT\left(\frac{1}{N_{\mathrm{Z}}} + \frac{1}{N_{\mathrm{B}}}\right) & kT\frac{1}{V_{\mathrm{Z}}} & -kT\frac{1}{V_{\mathrm{B}}} \\ kT\frac{1}{V_{\mathrm{Z}}} & kT\frac{N_{\mathrm{Z}}}{V_{\mathrm{Z}}^2} + \frac{\lambda}{A^2} & 0 \\ -kT\frac{1}{V_{\mathrm{B}}} & 0 & kT\frac{N_{\mathrm{B}}}{V_{\mathrm{B}}^2} + \frac{1}{4\pi r^2}\frac{\mathrm{d}[p]}{\mathrm{d}r} \end{bmatrix} - \text{positive definite} \tag{5.12}$$

These are three conditions of which two are identically satisfied, because N_{B}, N_{Z}, V_{B}, V_{Z} are all positive, while the third one reads

$$\frac{1}{4\pi r^2}\frac{\mathrm{d}[p]}{\mathrm{d}r} > -\frac{1}{\frac{A^2}{\lambda} + \frac{NkT}{p^2}}, \tag{5.13}$$

where $p = p_{\mathrm{B}} = p_{\mathrm{Z}}$ is the pressure throughout the system. We proceed to discuss this relation.

If the stiffness λ of the spring is zero, which amounts to no spring, we conclude that for stability we must have $\frac{\mathrm{d}[p]}{\mathrm{d}r} > 0$, so that the branch of the pressure-radius curve with the negative slope is unstable. Thus, if we have somehow manoeuvred our system into a position with the balloon pressure equilibrating the piston but located on the descending branch, the smallest squeeze, or the smallest fluctuation, will send it off. This is what we might have expected, but it is only a special case of the stability condition (5.13), namely the case $\lambda = 0$.

The system may be stabilized on part of the descending branch – or all of it – by adjusting the stiffness of the spring. Indeed, for a non-zero λ, (5.13) implies that all those points on the pressure-radius curve are stable whose slope is larger than a certain negative value. That value decreases with increasing λ. Soft springs stabilize the relatively flat parts of the descending branch, while stiff springs stabilize even the steep parts.

For infinite stiffness λ, i.e when the volume V_{Z} is fixed to the value V_0, the stability condition reads, with $V = V_0 + V_{\mathrm{B}}$,

$$\frac{1}{4\pi r^2}\frac{\mathrm{d}[p]}{\mathrm{d}r} > -\frac{p}{V}. \tag{5.14}$$

Taking p of the order of magnitude 1 bar and V_0 of the order of magnitude of the volume of the balloon, we see that the right hand side is extremely small $\left(\approx -10^4 \mathrm{bar/m^3}\right)$ so that the balloon is stable for all radii. Of course, if $V_0 >> V_{\mathrm{B}}$, there is a lot of gas in the cylinder and that large amount of gas is quite "soft"; in that case we may again have instability of the balloon on part or all of the descending branch.

5.5 Suggestive Stability Criterion

It is all very well to deduce the consequences of the positive definiteness of a 3×3 matrix, but it is hardly suggestive. Therefore we approach the problem of stability of the device of Fig. 5.4_{left} again, but now in a more plausible manner. We assume that the equilibrium of the piston and the equilibrium of the membrane are quickly established, while the equilibrium between cylinder and balloon lags behind.

In that case the jump of the pressure across the wall of the cylinder, which we denote by Δp, is given by the equilibrium condition $(5.11)_2$

$$\Delta p = \tfrac{mg}{A} + \lambda\tfrac{V_{\mathrm{Z}}-V_0}{A^2}, \quad \text{or with} \quad V_{\mathrm{Z}} = \tfrac{N_{\mathrm{Z}}kT}{p_{\mathrm{Z}}} = \tfrac{(N-N_{\mathrm{B}})kT}{p_0\left(1+\frac{\Delta p}{p_0}\right)} \approx \tfrac{(N-N_{\mathrm{B}})kT}{p_0}: \tag{5.15}$$

$$\Delta p = -\lambda\tfrac{kT}{p_0 A^2}N_{\mathrm{B}} + \tfrac{mg}{A} + \tfrac{\lambda}{A^2}\left(\tfrac{NkT}{p_0} - V_0\right). \tag{5.16}$$

Thus Δp is a linearly decreasing function of N_{B} whose slope is proportional to the stiffness λ of the spring. [The last step in (5.15) is approximate for $\Delta p \ll p_0$. The approximation is excellent, because Δp is of the order of magnitude of the jump $[p]$ of the pressure across the balloon membrane and this is $\approx$ 40 mbar which must be compared to 1 bar. It is possible, but not worthwhile, to make Δp exact.]

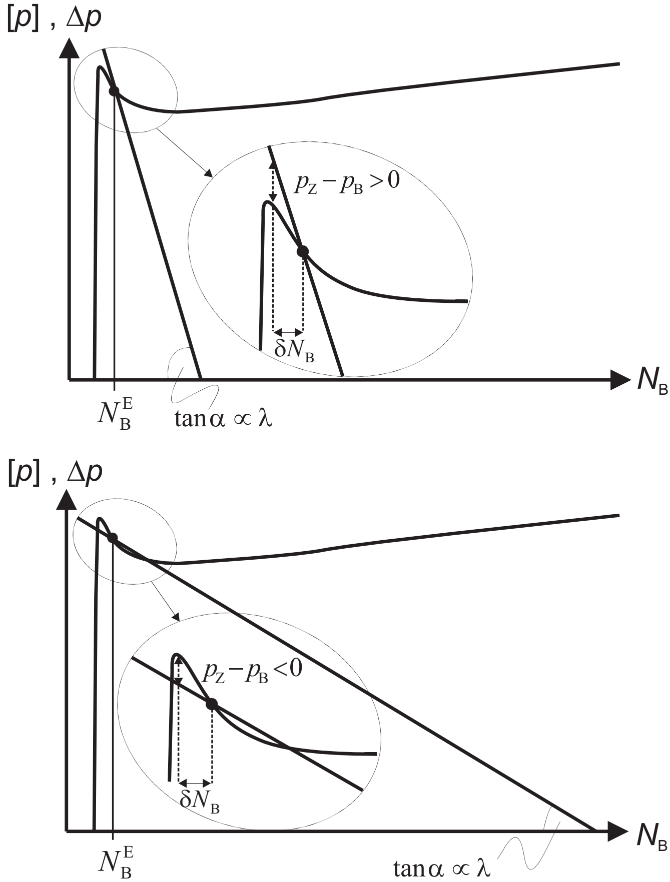

Fig. 5.5. Stability and instability of a spring-cylinder-balloon system. Top: Stiff spring. Bottom: Soft spring

Similarly we assume that the equilibrium condition $(5.11)_3$ is satisfied, and while we do not have an analytical form for $[p]$ as a function of N_B, we do know its graph, cf. Fig. 5.2. In Fig. 5.4_{right} we have plotted Δp and $[p]$, both as functions of N_B. The point of intersection determines the equilibrium between cylinder and balloon, cf. $(5.11)_1$. Its abscissa is denoted by N_B^E in Fig. 5.4_{right}.

From the graphs of Fig. 5.4_{right} it is particularly easy to evaluate the stability of an equilibrium. Consider the following: We disturb the equilibrium at N_B^E by squeezing a little air, δN_B (say), from the balloon to the cylinder. Inspection shows that this squeeze increases Δp (or p_Z), while it decreases $[p]$ (or p_B). The pressure difference $(p_\mathrm{Z} - p_\mathrm{B})$ will then tend to heal the disturbance: the equilibrium at N_B^E is stable.

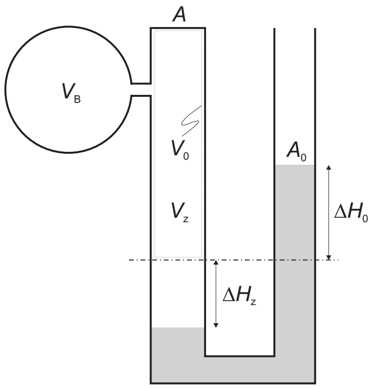

Fig. 5.6. Left: Hydrostatic apparatus for controlling the inflation of a balloon. The dashed line indicates the height of the water in both tubes, when the balloon is removed. V_Z is the volume of the air space in the closed cylinder and V_0 is the volume indicated by the thin line. Right: Same device with the possibility to attach a stabilizing tube with a smaller cross section A_0 by turning a three-way tap

Figure 5.5 shows two more such evaluations of stability. Both refer to the same equilibrium filling N_B^E. The corresponding state of the balloon is on the decreasing branch of the pressure-filling graph, where we suspect instability or – at the least – non-trivial stability properties. And indeed that is what we get. We proceed to discuss the situations.

In Fig. 5.5$_{top}$ we have a stiff spring, i.e. a steep line Δp so that a disturbance δN_B^E makes for a substantial increase of Δp. It is true that $[p]$ is also increased, but less than Δp. Therefore the equilibrium state is stable – for a stiff spring.

In Fig. 5.5$_{bottom}$ the spring is soft so that $\Delta p(N_B)$ is a fairly flat line. Again the disturbance δN_B leads to an increase of both pressures – p_Z and p_B – but p_B is increased more, so that the disturbance, once started, will become worse. Therefore the equilibrium state is unstable – for a soft spring.

In summary we conclude that it is the relative slope of the pressure curves of the loading device – here spring and piston – and of the balloon which determines stability. We shall have ample opportunity to exploit that principle in the sequel when we consider more than one balloon.

We recall that the pressure gauge of Fig. 1.4 duely registered all pressures of the attached balloon, irrespective of whether the pressure was decreasing or increasing. Now we are in a position to understand how this could have happened. The balloon was stabilized by a stiff spring. Actually inside the manometer there is no spring but there is a stiff elastic membrane instead.

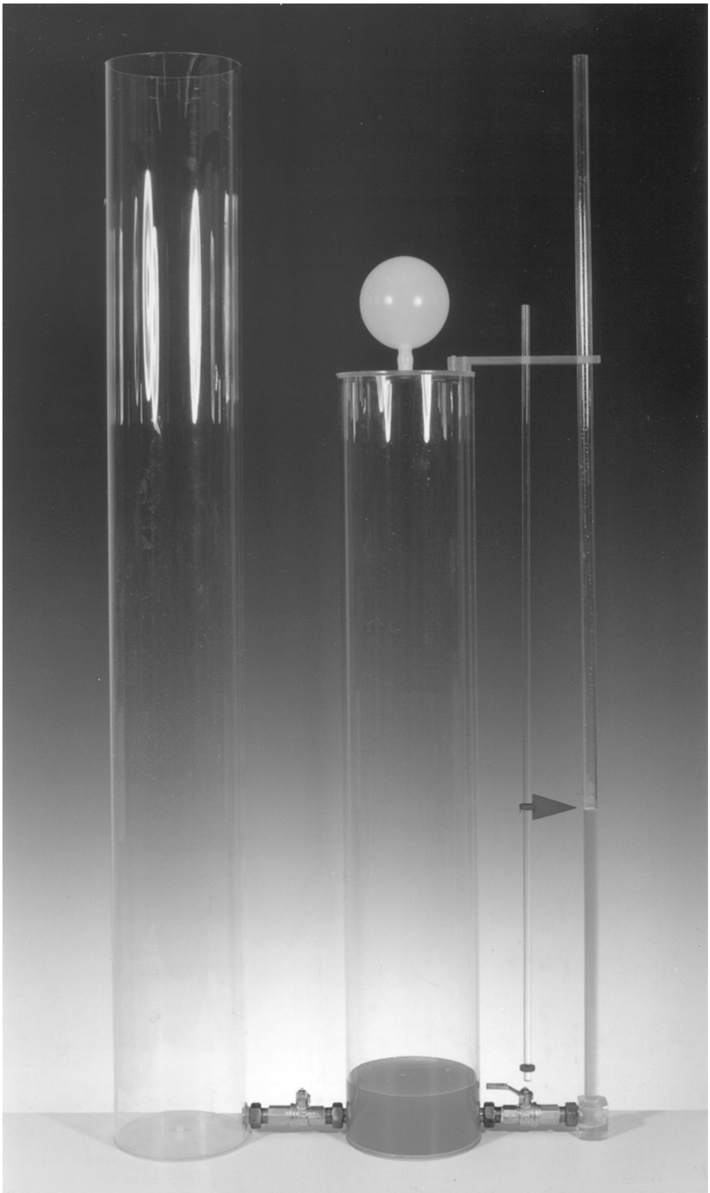

Fig. 5.7. Stabilization of the "unstable branch" by a narrow tube

5.6 A Hydrostatic Device for Controlling the Inflation of a Balloon

Mechanicians and thermodynamicists like models with pistons and springs, because they are good for thought experiments, where it is easy to ignore friction and problems of sealing. In reality such models should be avoided. In our case it is indeed possible to construct a reasonable and realizable model which in all relevant aspects is equivalent to the spring-cylinder-balloon model of Fig. 5.4$_{left}$ provided we ignore the mass of the piston.

The new model is shown in Fig. 5.6$_{left}$. It consists of two tubes connected by a pipe at the bottom. If we fill water into the open tube, the balloon connected to the closed tube will be inflated. In this case Δp is the pressure difference between the closed tube and the outside. With the notation indicated in the figure we obtain

$$\begin{aligned} \Delta p &= \rho g \left(\Delta H_0 + \Delta H_{\mathrm{Z}}\right) && \text{or with } \Delta H_0 A_0 = \Delta H_{\mathrm{Z}} A \\ &= \rho g \Delta H_{\mathrm{Z}} \left(1 + \tfrac{A}{A_0}\right) && \text{or with } \Delta H_{\mathrm{Z}} = \tfrac{V_{\mathrm{Z}} - V_0}{A} \\ &= \rho g A \left(1 + \tfrac{A}{A_0}\right) \tfrac{V_{\mathrm{Z}} - V_0}{A^2} . \end{aligned} \tag{5.17}$$

Δp in $(5.17)_3$ is equivalent to $\Delta p = p_{\mathrm{Z}} - p_0 = \lambda \frac{V_{\mathrm{Z}} - V_0}{A^2}$ in $(5.11)_2$, if only we ignore the mass of the piston and if the stiffness λ in the previous model is replaced by $\rho g A \left(1 + \frac{A}{A_0}\right)$:

$$\lambda = \rho g A \left(1 + \frac{A}{A_0}\right) . \tag{5.18}$$

Therefore we can now control the "stiffness" of the new system by adjusting the cross section A_0 of the open tube.

Basically the equivalence of these different models is due to the fact that the pressure of the spring is *linear* in the displacement, while the pressure of the water column is a *linear* function of its height.

Let us suppose that A_0 is big so that, by (5.18), λ is small. We inflate the balloon by pouring water into the open tube. Once the pressure peak of the balloon is reached, we expect the instability to occur; and indeed, the experiment shows that the balloon inflates "all by itself" while the height of the water level in the open tube decreases, indicating a smaller balloon pressure. The gadget may be stabilized by switching a three-way valve so that the balloon is now in contact with a water column of the same height as before but with a smaller cross-section, cf. Fig. 5.6$_{right}$.

By (5.18) a small cross-section is equivalent to a stiff spring and a stiff spring stabilizes the balloon as we saw in the previous section.

A gadget with a stabilizing tube of small cross-section has been built, see Fig. 5.7, and it works perfectly: The figure shows the balloon on its descending branch, but stabilized by the narrow tube.