MTH 5051: Homework #2

Due on September 23, 2019 at 07:00pm

Dr. Jim Jones Section 01

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Pigeonhole Problems Sheet:

Problem 1:

A lottery will select 3 balls from a bin containing numbered balls 1,2,...36. Players buy a ticket with 3 of these numbers selected. To win the grand prize, the ticket must match the selected balls. A smaller prize is offered if the sum of numbers on the ticket match the sum of the selected balls.

(a) How many players must play to guarantee that there is at least one duplicate ticket sold?

(b) How many players must play to guarantee there are at least two tickets sold with the same sum?

(c) How many players must play to guarantee that there are at least three tickets sold with the same sum?

Solution:

(a) So, to absolutely guarantee you have to sell every ticket possible, and add one to it. This means you have sold every possible combination, and one more means that you have every combination, and the next one is repeated from one of the previous combinations.

$$\binom{36}{3} + 1 = \mathbf{7,141}$$

(b) To determine the total possible sums we can get we have to determine all possible sums. We can determine all possible sums by finding the range of numbers between the lowest and highest sum. We can do this by:

$$\text{Lowest sum: } 1 + 2 + 3 = 6$$
$$\text{Highest sum: } 36 + 35 + 34 = 105$$

The range of numbers between 6 and 105 is 100 numbers. If this is every possible sum, if we add 1 to this value that means that a pigeonhole will have a second value. So:

$$100 + 1 = \mathbf{101}$$

(c) In order to determine whether 3 tickets are sold we can use some of the information we got from (b). If there are 100 pigeonholes (possible sums), lets double it so that there are at least 2 in each pigeonhole. In order to get a third we just have to add one.

$$100 \text{ pigeonholes } \times 2 + 1 = \mathbf{201}$$

Problem 2:

(Exercise 7)

(a) Show that if 14 numbers are selected from the set 1, 2, ..., 25, there are at least two whose sum is 26.

(b) Generalize the results from part (a) and example 5.44

Solution:

(a) There are a total of 12 pairs of numbers within the range of 1-25 that add up to 26. They are:

$$[(1,25),(2,24),(3,23),(4,22),(5,21),(6,20),(7,19),(8,18),(9,17),(10,16),(11,15),(12,14),(13)]$$

13 is not in this list, 13 doubled is 26 and we can not choose 13 twice. This means that the absolute maximum amount of numbers we could choose that don't add up to 26 is 13, because we can take 1 number from the list of pairs and the number 13. Any number number chosen after will correspond to another number that has already been picked and that will create a pair of numbers that add up to 26.

(b) To generalize, similar to the results in example 5.44 if you have an n size list of numbers and I am trying to add up 2 numbers to a value that is $n+1$ a list of numbers equal to $\frac{n}{2}$ can make up this list if n is even, and the ceiling of $\frac{n}{2}$ if it is odd. As a result you need a minimum of $\frac{n}{2}+1$.

Problem 3:

Let ABCD be a square with AB=1. Show that if we select 5 points in the interior, then there are at least two points whose distance apart is less than $\frac{1}{\sqrt{2}}$

Solution:

So if we have a square, let us select the 4 furthest possible points from each other, the corners. now, from here, the furthest possible point, from any of the other that we can choose within the square is the direct center of the square. If each side of the square is 1, then by using pythagorean theorom the distance to the center from any point is $\sqrt{\frac{1}{2}^2+\frac{1}{2}^2}=\frac{1}{\sqrt{2}}$. Now, because this is the absolute largest possible distance if we are at the exact corners, if we have to choose an arbitrary point on the interior of the square it will have to be less than $\frac{1}{\sqrt{2}}$, Thus proving that there are at least two points whose distance is less than $\frac{1}{\sqrt{2}}$

Problem 4:

A lossless compression algorithm takes an input file of bits and produces an output file of bits. For the algorithm to be lossless, the input file must be able to be exactly reconstructed from the output file. Show that if the algorithm produces an output file of size less than N bits for some specific input file of size N, then there must be some other specific input file whose output file is actually larger than its input file.

Solution:
lets say that the piece of data itself is a one to one pigeonhole situation, where n pieces of data in input (pigeons) map to n pieces of data in output (pigeonholes). If there is a compression algorithm the pigeons have to map to less pigeonholes. In order to have data be lossless you need an input file that is larger in order to map n pieces of data to n data in output in order to have truly lossless data.

Classic Challenge Problem:

Each point in the x-y plane colored red, blue or green. Prove that there exists a rectangle where the corners have the same color.

Solution:
So, let's say we have a single row. In this single row a color has to be repeated once at least every 4 cells. There are 6 possible combinations where the colors match in this case, lets say (1,2), (1,3), (1,4), (2,3), (2,4), (3,4). Because there are 6 possible repeat patterns, and 3 colors a pattern has to repeat every $(6\cdot 3)+1$ times, so every 4 by 19 rows something has to repeat. Because the x-y plane is infinite there are no restrictions to size, so it has to repeat.

Chapter 4.1

Problem 1:

Prove each of the following for all $n \geq 1$ by the Principle of Mathematical Induction.

(a) $1^2+3^2+5^2+...+(2n-1)^2=\dfrac{n(2n-1)(2n+1)}{3}$

(b) $1\cdot 3+2\cdot 4+3\cdot 5+...+n(n+2)=\dfrac{n(n+1)(2n+7)}{6}$

Solution:

(a) The first step to induction is to provide an basis case, the example case can be 1, that can be seen here:

$$1^2=\frac{1(1)(3)}{3}$$

This is true. So now we can move on. Now let's make an assumption, let's assume:

$$S(k): 1^2+3^2+5^2+...+(2k-1)^2=\frac{k(2k-1)(2k+1)}{3} \text{ for some } k\geq 1$$

From here lets do the inductive step and trying to prove this for $k+1$. We are trying to prove:

$$\begin{aligned} S(k+1): 1^2+3^2+5^2+...+(2k-1)^2+(2k+1)^2 &= \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3} \\ &= \frac{(k+1)(2k+1)(2k+3)}{3} \end{aligned}$$

So from our assumption:

$$\begin{aligned} S(k+1): \quad & [1^2+3^2+5^2+...+(2k-1)^2]+(2k+1)^2 \\ & \frac{k(2k-1)(2k+1)}{3}+(2k+1)^2 \\ & \frac{k(2k-1)(2k+1)}{3}+\frac{3(2k+1)^2}{3} \\ & \frac{(2k+1)[k(2k-1)+3(2k+1)]}{3} \\ & \frac{(2k+1)(2k^2+5k+3)}{3} \\ & \frac{(k+1)(2k+1)(2k+3)}{3} \end{aligned}$$

Since k is an arbitrary value its true for any case $k\geq 1$, proving

$$1^2+3^2+5^2+...+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$$

(b) The first step to induction is to provide an basis case, lets try 1 in this case.

$$1\cdot 3=\frac{1(2)(9)}{6}$$

This is true, so now we can move on. Now let's make an assumption, let's assume:

$$S(k): 1\cdot 3+2\cdot 4+3\cdot 5+...+k(k+2)=\frac{k(k+1)(2k+7)}{6} \text{ for some } k\geq 1$$

From here lets do the inductive step and trying to prove this for $k+1$. We are trying to prove:

$$\begin{aligned} S(k+1): 1\cdot 3+2\cdot 4+3\cdot 5+...+k(k+2)+(k+1)((k+1)+2) &= \frac{(k+1)((k+1)+1)(2(k+1)+7)}{6} \\ &= \frac{(k+1)((k+2)(2k+9)}{6} \end{aligned}$$

So from our assumption:

$$
\begin{aligned}
S(k+1): \quad & [1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + ... + k(k+2)] + (k+1)((k+1)+2) \\
& \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3) \\
& \frac{k(k+1)(2k+7)}{6} + \frac{6(k+1)(k+3)}{6} \\
& \frac{(k+1)[k(2k+7)+6(k+3)]}{6} \\
& \frac{(k+1)[2k^2+13k+18]}{6} \\
& \frac{(k+1)((k+2)(2k+9)}{6}
\end{aligned}
$$

Since k is an arbitrary value its true for any case $k \geq 1$, proving

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + ... + n(n+2) = \frac{n(n+1)(2n+7)}{6}$$

Problem 2:

Prove each of the following for all $n \geq 1$ by the Principle of Mathematical Induction.

(b)

$$\sum_{i=1}^{n} i(2^i) = 2 + (n-1)2^{n+1}$$

(c)

$$\sum_{i=1}^{n} (i)(i!) = (n+1)! - 1$$

Solution:

(b) The first step to induction is the basis case. let's use 1 in this case.

$$(1)(2^1) = 2 + (0)2^{1+1}$$

This is true, so now we can move on. Now let's make an assumption, let's assume:

$$S(k): \sum_{i=1}^{k} i(2^i) = 2 + (k-1)2^{k+1} \text{ for some } k \geq 1$$

From here lets do the inductive step and trying to prove this for $k+1$. We are trying to prove:

$$
\begin{aligned}
S(k+1): \sum_{i=1}^{k+1} i(2^i) &= 2 + ((k+1)-1)2^{(k+1)+1} \\
&= 2 + (k)2^{k+2}
\end{aligned}
$$

So from our assumption:

$$\begin{aligned}
S(k+1): \quad & \sum_{i=1}^{k+1} i(2^i) \\
& \sum_{i=1}^{k} i(2^i) + (k+1)2^{k+1} \\
& [2 + (k-1)2^{k+1}] + (k+1)2^{k+1} \\
& 2 + (2^{k+1})[(k-1) + (k+1)] \\
& 2 + (2^{k+1})(2k) \\
& 2 + (k)(2^{k+2})
\end{aligned}$$

Since k is an arbitrary value its true for any case $k \geq 1$, proving

$$\sum_{i=1}^{n} i(2^i) = 2 + (n-1)2^{n+1}$$

(c) The first step to induction is the basis case. let's use 1 in this case.

$$(1)(1!) = (1+1)! - 1$$

This is true, so now we can move on. Now let's make an assumption, let's assume:

$$S(k): \sum_{i=1}^{k} (i)(i!) = (k+1)! - 1 \text{ for some } k \geq 1$$

From here lets do the inductive step and trying to prove this for $k+1$. We are trying to prove:

$$\begin{aligned}
S(k+1): \sum_{i=1}^{k+1} (i)(i!) &= ((k+1)+1)! - 1 \\
&= (k+2)! - 1
\end{aligned}$$

So from our assumption:

$$\begin{aligned}
S(k+1): \quad & \sum_{i=1}^{k+1} (i)(i!) \\
& = \sum_{i=1}^{k} (i)(i!) + (k+1)((k+1)!) \\
& = (k+1)! - 1 + (k+1)((k+1)!) \\
& = (k+1)!(1 + (k+1)) - 1 \\
& = (k+1)!(k+2) - 1 \\
& = (k+2)! - 1
\end{aligned}$$

Since k is an arbitrary value its true for any case $k \geq 1$, proving

$$\sum_{i=1}^{n} i(2^i) = 2 + (n-1)2^{n+1}$$

Problem 4:

A wheel of fortune has the integers from 1 to 25 placed on it in a random manner. Show that regardless of how the numbers are positioned on the wheel, there are three adjacent numbers whose sum is at least 39.

Solution:

Let's assume that: $x_1 + x_2 + x_3 < 39, x_2 + x_3 + x_4, ..., x_{24} + x_{25} + x_1 < 39$ for all possible combinations of possible numbers on the wheel. For this to be true that would mean:

$$\sum_{i=1}^{25} 3x_i < 25(39)$$

But the problem is if you do the math:

$$\sum_{i=1}^{25} 3x_i = 975$$
$$25 \cdot 39 = 975$$

So because of this, there is at least one combination of three adjacent numbers that has a value that adds up to at least 39.

Problem 7:

A lumberjack has $4n + 110$ logs in a pile consisting of n layers. Each layer has two more logs than the layer directly above it. If the top layer has six logs, how many layers are there?

Solution:

If the top layer has 6 logs, and each layer below has exactly 2 more then we can create an equation for this. We essentially have 2 equations, and if we set them equal to each other we will be able to find out how many n layers there are. This can be described by:

$$\begin{aligned} 4n + 110 &= 6 + 8 + 10 + ... + (6 + 2(n-1)) \\ &= 6n + 2\left((n-1)\left(\frac{n}{2}\right)\right) \\ &= 6n + n^2 - n \\ &= n^2 + 5n \end{aligned}$$

From here we are able to actually simply solve for n.

$$\begin{aligned} n^2 + 5n &= 4n + 110 \\ n^2 + n &= 110 \\ n &= \mathbf{10} \end{aligned}$$

Problem 14:

Prove that for all $n \in \mathbb{Z}^+, n > 3 \implies 2^n < n!$

Solution:

The first step to induction is the basis case. let's use 4 in this case.

$$2^4 < 4!$$

This is true, so now we can move on. Now let's make an assumption, let's assume:

$$S(k) : 2^k < k! \text{ for } k > 3$$

From here lets do the inductive step and trying to prove this for $k + 1$. We are trying to prove:

$$S(k+1) : 2^{k+1} < (k+1)!$$

So from our assumption:

$$\begin{aligned} S(k+1) &= 2^{k+1} < (k+1)! \\ &= 2^k \cdot 2 < k! \cdot (k+1) \\ &= k! \cdot 2 < k! \cdot (k+1) \\ &= 2 < k+1 \end{aligned}$$

Since k is an arbitrary value its true for any case $k > 3$, proving

$$n \in \mathbb{Z}^+, n > 3 \implies 2^n < n!$$

Problem 15:

Prove that for all $n \in \mathbb{Z}^+, n > 4 \implies n^2 < 2^n$

Solution:

The first step to induction is the basis case. let's use 4 in this case.

$$2^4 < 4!$$

This is true, so now we can move on. Now let's make an assumption, let's assume:

$$S(k) : 2^k < k! \text{ for } k > 3$$

From here lets do the inductive step and trying to prove this for $k + 1$. We are trying to prove:

$$S(k+1) : 2^{k+1} < (k+1)!$$

So from our assumption:

$$\begin{aligned} S(k+1) &= 2^{k+1} < (k+1)! \\ &= 2^k \cdot 2 < k! \cdot (k+1) \\ &= k! \cdot 2 < k! \cdot (k+1) \\ &= 2 < k+1 \end{aligned}$$

Since k is an arbitrary value its true for any case $k > 3$, proving

$$n \in \mathbb{Z}^+, n > 3 \implies 2^n < n!$$

Problem 19:

For $n \in \mathbb{Z}^+$ let $S(n)$ be the open statement

$$\sum_{i=1}^{n} i = \frac{\left(n + \left(\frac{1}{2}\right)\right)^2}{2}$$

Show the truth of $S(k)$ implies the truth of $S(k+1)$ for all $k \in \mathbb{Z}^+$. Is $S(n)$ true for all $n \in \mathbb{Z}^+$?

Solution:

The first step to induction is the basis case. let's use 1

$$n = 0: \quad 1 = \frac{\left(1 + \frac{1}{2}\right)^2}{2}$$

This is false, which proves the statement false for any $k \geq 1$, however I would like to see if I can show that it is true inductively with k and $k+1$

$$S(k): \sum_{i=1}^{k} i = \frac{\left(k + \left(\frac{1}{2}\right)\right)^2}{2} \text{ for all } k \in \mathbb{Z}^+$$

From here lets do the inductive step and trying to prove this for $k+1$. We are trying to prove:

$$S(k+1): \sum_{i=1}^{k+1} i = \frac{\left((k+1) + \left(\frac{1}{2}\right)\right)^2}{2}$$

So from our assumption:

$$\begin{aligned}
S(k+1) &= \sum_{i=1}^{k+1} i \\
&= \sum_{i=1}^{k} i + (k+1) \\
&= \frac{\left(k + \left(\frac{1}{2}\right)\right)^2}{2} + (k+1) \\
&= \frac{\left(k + \left(\frac{1}{2}\right)\right)^2 + 2k + 2}{2} \\
&= \frac{k^2 + k + \frac{1}{4} + 2k + 2}{2} \\
&= \frac{k^2 + k + \frac{1}{4} + 2k + 2}{2} \\
&= \frac{(k+1)^2 + (k+1) + \frac{1}{4}}{2} \\
&= \frac{(k+1)^2 + (k+1) + \frac{1}{4}}{2} \\
&= \frac{\left((k+1) + \left(\frac{1}{2}\right)\right)}{2}
\end{aligned}$$

Since k is an arbitrary value its true for any case $k \in \mathbb{Z}^+$ this passes the inductive part of the proof.

Chapter 4.2

Problem 1:

The integer sequence $a_1, a_2, a_3, \ldots$, defined explicitly by the formula $a_n = 5n$ for $n \in \mathbb{Z}^+$, can also be defined recursively by

1) $a_1 = 5$; and

2) $a_{n+1} = a_n + 5$ for $n \geq 1$.

For an integer sequence $b_1, b_2, b_3, ...,$ where $b = n(n+2)$ for all $n \in \mathbb{Z}^+$, we can also provide the recursive definition:

1)' $b_1 = 3$; and

2)' $b_{n+1} = b_n + 2n + 3$, for $n \geq 1$.

Give a recursive definition for each of the following integer sequences $c_1, c_2, c_3, ...,$ where $n \in \mathbb{Z}^+$ we have

(a) $c_n = 7n$

(b) $c_n = 7^n$

(c) $c_n = 3n + 7$

(d) $c_n = 7$

(e) $c_n = n^2$

(f) $c_n = 2 - (-1)^n$

Solution:

(a) $\boldsymbol{c_1 = 7;}$ and $\boldsymbol{c_{n+1} = c_n + 7}$, for $n \geq 1$

(b) $\boldsymbol{c_1 = 7;}$ and $\boldsymbol{c_{n+1} \cdot 7}$, for $n \geq 1$

(c) $\boldsymbol{c_1 = 10;}$ and $\boldsymbol{c_{n+1} \cdot 3 + 7}$, for $n \geq 1$

(d) $\boldsymbol{c_1 = 7;}$ and $\boldsymbol{c_{n+1} = c_n}$, for $n \geq 1$

(e) $\boldsymbol{c_1 = 1;}$ and $\boldsymbol{c_{n+1} = c_n + 2n + 1}$, for $n \geq 1$

(f) $\boldsymbol{c_1 = 3, c_2 = 1;}$ and $\boldsymbol{c_{n+2} = c_n}$, for $n \geq 1$

Problem 11:

Define the integer sequence $a_0, a_1, a_2, a_3, ...,$ recursively by

1) $a_0 = 1, a_1 = 1, a_2 = 1$; and

2) For $n \geq 3, a_n = a_{n-1} + a_{n-3}$

Prove that $a_{n+2} \geq (\sqrt{2})^n$ for all $n \geq 0$

Solution:

The first step to induction is the basis case. let's use 0, 1, and 2 this case.

$$n = 0: \quad 1 \geq \sqrt{2}^0$$
$$n = 1: \quad 2 \geq \sqrt{2}^1$$
$$n = 2: \quad 3 \geq \sqrt{2}^2$$

These are true so now we can move on. Now let's make an assumption, let's assume:

$$S(k) : a_{k+2} \geq (\sqrt{2})^k \text{ for all } k > 2$$

From here lets do the inductive step and trying to prove this for $k+1$. We are trying to prove:

$$S(k+1) : a_{k+3} \geq (\sqrt{2})^{k+1}$$

So from our assumption:

$$\begin{aligned}
S(k+1) &= a_{k+3} \\
&= a_k + a_{k+2} \geq \sqrt{2}^{k-2}\sqrt{2}^{k} \\
&= a_k + a_{k+2} \geq \left(\sqrt{2}^{2} + 1\right)\sqrt{2}^{k-2} \\
&= a_k + a_{k+2} \geq \left(3\sqrt{2}^{-2}\right)\left(\sqrt{2}^{k}\right) \\
&= a_k + a_{k+2} \geq \frac{3\sqrt{2}^{k}}{2} \\
&= a_k + a_{k+2} \geq \frac{3\sqrt{2}^{k}}{2} \\
&= a_k + a_{k+2} \geq \frac{3}{2}\sqrt{2}^{k} \geq \sqrt{2}\sqrt{2}^{k} \\
&= a_k + a_{k+2} \geq \frac{3}{2}\sqrt{2}^{k} \geq \sqrt{2}^{k+1}
\end{aligned}$$

Since k is an arbitrary value its true for any case $k > 2$ and the first three basis cases, proving

$$a_{n+2} \geq (\sqrt{2})^{n} \text{ for all } n \geq 0$$

Problem 13:

Prove that for any positive integer n,

$$\sum_{i=1}^{n} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{n+2}}{2^n}.$$

Solution:

The first step to induction is the basis case. let's use 1 in this case.

$$\frac{F_0}{2^i} = 1 - \frac{F_2}{2^n}$$

This is true, so now we can move on. Now let's make an assumption, let's assume:

$$S(k) : \sum_{i=1}^{k} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{k+2}}{2^k} \text{ for all } k \in \mathbb{Z}^+$$

From here lets do the inductive step and trying to prove this for $k+1$. We are trying to prove:

$$S(k+1) : \sum_{i=1}^{k+1} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{k+3}}{2^{k+1}}.$$

So from our assumption:

$$\begin{aligned}
S(k+1) &= \sum_{i=1}^{k+1} \frac{F_{i-1}}{2^i} \\
&= \sum_{i=1}^{k} \frac{F_{i-1}}{2^i} + \frac{F_k}{2^{k+1}} \\
&= 1 - \frac{F_{k+2}}{2^k} + \frac{F_k}{2^{k+1}} \\
&= 1 + \left(\frac{1}{2^{k+1}}\right)(F_k - 2F_{k+2}) \\
&= 1 + \left(\frac{1}{2^{k+1}}\right)((F_k - F_{k+2}) - F_{k+2}) \\
&= 1 + \left(\frac{1}{2^{k+1}}\right)(-F_{k+1} - F_{k+2}) \\
&= 1 + \left(\frac{1}{2^{k+1}}\right)(-F_{k+1} - F_{k+2}) \\
&= 1 - \left(\frac{1}{2^{k+1}}\right)(F_{k+1} + F_{k+2}) \\
&= 1 - \left(\frac{F_{k+3}}{2^{k+1}}\right)
\end{aligned}$$

Since k is an arbitrary value its true for any case $k \in \mathbb{Z}^+$, proving

$$\sum_{i=1}^{n} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{n+2}}{2^n}.$$