Linear Algebra: Homework #4

Due on February 24, 2019 at 11:59pm

Dr. Subasi Section 01

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Problem 1

Consider the 4×4 matrix $A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ -1 & -1 & 2 & 1 \\ 2 & 2 & -1 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$.

(a) (10pts) Find the reduced row echelon form of A.
Solution:

$$R2 + R1 \to R2 \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 2 & 2 & -1 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

$$R3 - 2R1 \to R3 \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -1 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

$$R4 + R1 \to R4 \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$\frac{1}{2}R2 \to R2 \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$R3 + R2 \to R3 \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$R4 + R2 \to R4 \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$R3 \leftrightarrow R4 \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\frac{1}{2}R3 \to R3 \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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$$R2 - R3 \to R2 \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R1 - R3 \to R1 \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) (5pts) Is the system $Ax = b$ consistent for every $b \in \mathbb{R}^n$. Justify your answer.
Solution:
No, this can be proven in:

$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & b_1 \\ 0 & 0 & 1 & 0 & b_2 \\ 0 & 0 & 0 & 1 & b_3 \\ 0 & 0 & 0 & 0 & b_4 \end{array}\right)$$

As we can see, in the last row if b_4 is a value that is non-zero then it does not work; therefore we can see that $Ax = b$ is inconsistent.

(c) (10pts) Find a basis for null-space $Null\ A$. What is the dimension of $Null\ A$?
Solution:

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_3 = 0,\ x_4 = 0,\ x_2 = s,\ x_1 = -s$$

$$Null\ A = \begin{pmatrix} \text{-1} \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

(d) (5pts) Determine whether the set $S = \left\{ \begin{pmatrix} \text{-1} \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \text{-6} \\ 4 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} \text{-2} \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $Null\ A$.

Solution:
No, it is impossible for the set S to be a basis for $Null\ A$ because all that can be done on $Null\ A$ is scalar multiplication, and using scalar multiplication it is impossible to get any values in the the third and fourth rows as they are 0, making this basis impossible.

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(e) (10pts) Find a basis for column-space $Col\ A$. What is the dimension of $Col\ A$?
Solution:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

(f) (5pts) Determine whether the set $S = \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ 4 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $Col\ A$.

Solution:
No, it is impossible for S to be a basis in $Col\ A$ because $Col\ A$ has no values in the fourth row, therefore it is impossible for S to exist as a basis because it does have values in the fourth row.

(g) (5pts) Determine whether the vector $u = \begin{pmatrix} -1 \\ 0 \\ 1 \\ -2 \end{pmatrix}$ belongs to $Null\ A$.

Solution:
This vector does not belong in $Null\ A$ because using only scalar multiplication and vector addition it is impossible to get values in the third and fourth rows as the third and fourth row in $Null\ A$ are 0. This makes it impossible to hold the values that u has.

(h) (5pts)Determine whether the vector $u = \begin{pmatrix} -1 \\ 0 \\ 1 \\ -2 \end{pmatrix}$ belongs to $Col\ A$.

Solution:
It does not exist in $Col\ A$ because none of the vectors in $Col\ A$ have a value in the fourth row, meaning that it is impossible for u to exist at all in $Col\ A$

Problem 2

(20pts) Let A be an $m \times n$ matrix. Prove that the null-space of $A^T A$ is a subspace of $\mathbb{R}^n$
Solution:
If A is an $m \times n$ matrix, that means that A^T is an $n \times m$ matrix. This means that $A^T A$ will be an $n \times n$ matrix. Now, the $null(A^T A)$ is a set of vectors, these vectors show that $A^T Ax = 0$. let's say that there is some y where $A^T Ay = 0$. then, using vector addition we can say:

$$A^T Ax = A^T Ay = 0$$

$$A^T A(x + y) = A^T Ax + A^T Ay = 0 + 0 = 0$$

Therefore we can prove it under vector addition, next is to prove it under scalar multiplication. This can be shown by:

$$A^T A(cx) = c(A^T Ax) = c(0) = 0$$

Problem 3

(25pts)Let A be an $m \times n$ matrix. Prove that the null-space of A and null-space of $A^T A$ are equal, i.e.,

$$Null\ A = Null\ A^T A$$

Hint: You must prove a vector $x \in Null\ A$ also belongs to $Null\ A^T A$ and any vector $x \in Null\ A^T A$ also belongs to $Null\ A$.

Solution:

Let say x is a vector in $null(A)$. We know that $Ax = 0$. Let's multiply x with $A^T A$, so that we get $A^T Ax$. We know $Ax = 0$ so we can alter the equation such that:

$$\begin{aligned} A^T Ax \\ A^T(Ax) \\ A^T(0) = 0 \end{aligned}$$

This means x also exists in $null(A^T A)$. At this point we know any $x \in null(A)$ exists in $null(A^T A)$, now we have to prove that $null(A^T A)$ exists in $null(A)$. This is a bit more complicated, but can be shown by:

$$\begin{aligned} A^T Ax &= 0 \\ (A^T)^{-1}(A^T Ax) &= (A^T)^{-1}(0) \\ Ax &= 0 \end{aligned}$$

Therefore proving anything in $null(A^T A)$ exists within $null(A)$.