Linear Algebra: Homework #5-2

Due on April 14, 2019 at 11:59pm

Dr. Subasi Section 01

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Problem 1

Consider the following 3×3 matrix:

$$A = \begin{pmatrix} 1 & \text{-2} & 2 \\ \text{-2} & 4 & \text{-4} \\ 2 & \text{-4} & 4 \end{pmatrix}$$

(a) (10pts) Find eigenvalues of A.
Solution:

(b) (10pts) Find the eigenspaces of A corresponding to the eigenvalues found in Part (a). Justify your answer.
Solution:

(c) (5pts) Determine whether A is diagonalizable. Justify your answer.
Solution:

(d) (10pts) Find an invertible matrix P and a diagonal matrix D such that $P^{\text{-}1AP=D}$. No need to verify the equation holds true.
Solution:

(e) (10pts) Find a matrix S whose columns form an orthonormal basis for $\mathbb{R}^3$ obtained from the columns of matrix P found in part (d).
Solution:

(f) (5pts) Is S invertible? If yes, find $S^{\text{-}1}$. **Solution:**

(g) (5pts) Show that $S^{\text{-}1}AS = D$, where D is the diagonal matrix found in Part (d), that is, A is orthogonally diagonlizable.
Solution:

Problem 2

Let A and B be two $n \times n$ orthogonal matrices. Answer the following questions:

(a) (5pts) Is -3A orthogonal? Justify your answer.
Solution:
Yes, orthogonal matrices remain orthogonal under scalar multiplication

(b) (5pts) Is -B orthogonal? Justify your answer.
Solution:
Yes, orthogonal matrices remain orthogonal under scalar multiplication

(c) (5pts) Is $B^{-1}AB$ orthogonal? Justify your answer.
Solution:
Yes, orthogonal matrices are closed under multiplication.

(d) (5pts) Is A^T invertible? If yes, find its inverse.
Solution:
Yes, to explain further, the transpose of an orthogonal matrix is the original matrix's inverse. This means that $A^T = A^{-1}$, and the inverse of A^{-1} is just A.

(e) (5pts) Is AB orthogonal? Justify your answer.
Solution:
Yes, orthogonal matrices are closed under multiplication.

Problem 3

Suppose that A is an $n \times n$ orthogonal matrix.

(a) (10pts) Show that the matrix operator $T : \mathbb{R}^n \to \mathbb{R}^n, T(x) = Ax$ is an orthogonal operator.
Solution:

(b) (10pts) Prove that the only eigenvalues of A are 1 and -1
Solution:

Problem 4

(15pts) Let V be a subspace of $\mathbb{R}^n$. The orthogonal complement of V in $\mathbb{R}^n$ is defined as

$$V^\perp = \{x \in \mathbb{R}^n | x \cdot v = 0 \; \forall \; v \in V\}$$

Prove that $V^\perp$ is also a subspace of $\mathbb{R}^n$.

Problem 5

Let $u_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

(a) (5pts) Determine whether the set $S = \{u_1, u_2\}$ is orthogonal.
Solution:

(b) (5pts) Transform S into an orthonormal set of vectors B.
Solution:

(c) (5pts) Find the orthogonal projection of $u = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ on the subspace $W = SpanB$, where B is the orthornormal basis obtained in part (b).

Gram-Schmidt Process: if $\{u_1, u_2, ..., u_k\}$ is a basis for a subspace W, then $\{v_1, v_2, ..., v_k\}$ is an orthogonal basis for W, where

$$v_1 = u_1,\ v_2 = u_2 - \frac{u_2 \cdot v_1}{||v_1||^2}v_1,\ ...,\ v_k = u_k - \frac{u_k \cdot v_1}{||v_1||^2}v_1 -\ ...\ - \frac{u_k \cdot v_{k-1}}{||v_{k-1}||^2}v_{k-1}$$

Also, if $\{w_1, w_2, ...w_k\}$ is an orthornormal basis, then the orthogonal projection , w, of a vector u on a subspace W is given by

$$w = (u \cdot w_1)w_1 + (u \cdot w_2)w_2 + ... + (u \cdot w_k)w_k$$

Solution:

Problem 6

(20pts) Classify each of the following statements as True or False. Justify your answers.

(a) The determinant of an orthogonal matrix A is equal to 1.
Solution:

(b) The projection matrix P defined as $P = A(A^TA)^{-1}A^T$ is symmetric for all A.
Solution:

(c) Every non-zero subspace W of $\mathbb{R}^n$ has an orthonormal basis.
Solution:

(d) Any two bases in a fininte-dimensional vector space V have the same number of elements.
Solution: