Linear Algebra: Homework #2

Due on March 24, 2019 at 11:59pm

Dr. Subasi Section 01

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Problem 1

Determine whether the given system of linear equations is consistent, and if so find its general solution.

a System 1:

$$\begin{aligned} x_1 + 3x_2 + x_3 + x_4 &= -1 \\ -2x_1 - 6x_2 - x_3 &= 5 \\ x_1 + 3x_2 + 2x_3 + 3x_4 &= 2 \end{aligned}$$

Solution

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & -1 \\ -2 & -6 & -1 & 0 & 5 \\ 1 & 3 & 2 & 3 & 2 \end{array}\right)$$

$$R_2 - (-2R_1) \to R_2 \left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 & 3 \\ 1 & 3 & 2 & 3 & 2 \end{array}\right)$$

$$(R_3) - (R_1) \to R_3 \left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right)$$

$$(R_3) - (R_2) \to R_3 \left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$(R_1) - (R_2) \to R_1 \left(\begin{array}{cccc|c} 1 & 3 & 0 & -1 & -4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{aligned} x_1 + 3x_2 - x_4 &= -4 \\ x_3 + 2x_4 &= 3 \\ x_3 &= -2x_4 + 3 \\ x_1 &= -3x_2 + x_4 - 4 \end{aligned}$$

$$X = \begin{pmatrix} -3x_2 + x_4 - 4 \\ x_2 \\ -2x_4 + 3 \\ x_4 \end{pmatrix}$$

b System 2:

$$\begin{aligned} x_1 - 2x_2 - 5x_3 &= 4 \\ x_2 + 3x_3 &= -2 \\ -x_2 - 3x_3 &= 3 \end{aligned}$$

Solution

$$\left(\begin{array}{ccc|c} 1 & -2 & -5 & 4 \\ 0 & 1 & 3 & -2 \\ 0 & -1 & -3 & 3 \end{array}\right)$$

$$R_2 + R_3 \to R_2 \left(\begin{array}{ccc|c} 1 & -2 & -5 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -3 & 3 \end{array}\right)$$

This is not consistent after this operation, as $0x_1 + 0x_2 + 0x_3 = -1$ is not possible.

Problem 2

Consider the following system of linear equations:

$$x_1 + 3x_2 = 1 + s$$
$$x_1 + rx_2 = 5$$

a (5 points) For what values of r and s is the system inconsistent?
Solution

$$\left(\begin{array}{cc|c} 1 & 3 & 1+s \\ 1 & r & 5 \end{array}\right)$$
$$R2 - R1 \rightarrow R2 \left(\begin{array}{cc|c} 1 & 3 & 1+s \\ 0 & r-3 & 4+s \end{array}\right)$$

From here we can start to find use the equation $(r-3)x_2 = 4 + s$ to find some answers.

$$(r-3)x_2 = 4 + s$$
$$x_2 = \frac{4+s}{r-3}$$

Now from here we can see that the system is inconsistent if $r = 3$ and $s \neq -4$.

b (5 points) For what values of r and s does the system have infinitely many solutions?
Solution
Using the same equation in 2a we know that $x_2 = \frac{4+s}{r-3}$. Utilizing this we can determine at what point the system has infinitely many solutions. The case in which it has infinitely many solutions is when $r = 3$ and $s = -4$.

c (5 points) For what values of r and s does the system have unique solution?
Solution
There is a unique solution when $r \neq 3$.

Problem 3

Given the system of linear equations

$$\begin{aligned} x_1 + rx_2 &= 5 \\ -3x_1 + 6x_2 &= s \end{aligned}$$

a (5 points) Determine the values of r and s so that the system is consistent.
Solution

To start lets put the equations in a matrix:

$$\left(\begin{array}{cc|c} 1 & r & 5 \\ -3 & 6 & s \end{array}\right)$$

$$R2 + 3R1 \to R2 \left(\begin{array}{cc|c} 1 & r & 5 \\ 0 & 6+3r & s+15 \end{array}\right)$$

In this case the equation produced will come out to be:

$$\begin{aligned} (6+3r)x_2 &= s + 15 \\ x_2 &= \frac{s+15}{6+3r} \end{aligned}$$

The system is consistent for any instance where $r \neq -2$

b (5 points) Determine the values of r and s so that the system is inconsistent.
Solution

The solution is nearly the opposite of that in 3a. The system is inconsistent for any instance $r = -2$ and $s \neq -15$

c (5 points) Determine the values of r and s so that the system has infinitely many solutions.
Solution

The system has infinitely many solutions in any case where $r = -2$ and $s = -15$

Problem 4

Let A= $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

a (5 points) Find A^{-1}, if it exists.
Solution

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 0 & 0 & 1 \end{array}\right)$$

$$R_1 + R_3 \to R_1 \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 0 & 0 & 1 \end{array}\right)$$

$$R_1 - 2R_2 \to R_1 \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 0 & 0 & 1 \end{array}\right)$$

$$-\frac{1}{3}R_3 \to R_3 \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{3} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

b (5 points) Determine whether the equation $Ax = b$ is consistent for every b in $\mathbb{R}^3$. If so, what is the solution of the system?
Solution

$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & -3 & b_3 \end{array}\right)$$

Now, following the steps of 4a we can see that the Reduced row echelon form of the equation is the identity matrix. When looking with my matrix for b I put in free variable, where b can be anything. Due to it reducing to identity it allows it any x because any value will satisfy the b matrix. There are no inconsistencies.

Problem 5

Let A be a 3×4 matrix whose reduce row echelon form is

$$\hat{A} = \begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

a (5 points) Find the general solution $x \in \mathbb{R}^4$ to the system $Ax = 0$
Solution

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_1 - 2x_2 + 3x_4 = 0$$
$$x_3 - 5x_4 = 0$$
$$x_3 = 5x_4$$
$$x_1 = 2x_2$$
$$x = \begin{pmatrix} 2x_2 - 3x_4 \\ x_2 \\ 5x_4 \\ x_4 \end{pmatrix}$$

b (5 points) Find the set of all the solutions to $Ax = 0$ and write it as the span of the linearly independent vectors.
Solution

$$span \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 5 \\ 1 \end{pmatrix} \right\}$$

c (5 points) True or False: The equation $Ax = b$ has a solution $x \in \mathbb{R}^4$ for every vector $b \in \mathbb{R}^3$. Justify your answer.
Solution

False. The reason is because the fourth row is entirely 0's, so in this case if the last element of b is non-zero then it is inconsistent.

d (5 points) True or False: If the equation $Ax = b$ has a solution $x^* \in \mathbb{R}^4$ for a particular vector $b \in \mathbb{R}^3$, then x^* is the unique. Justify your answer.
Solution

False, because the solution for b has a free variable.

Problem 6

a (5 points) True or False: If A and B are 2×2 matrices such that $AB = 0$ and $A \neq 0$, then $B = 0$. Give an argument valid for every such A and B if the statement is true, or give a counterexample if false.
Solution
False. A counterexample is:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$AB = \begin{pmatrix} (0 \times 1) + (1 \times 0) & (0 \times 0) + (1 \times 0) \\ (0 \times 1) + (0 \times 0) & (0 \times 0) + (0 \times 0) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This is a 0 matrix, but both A and B are not 0 matrices. Therefore I can conclude this is false.

b (5 points) True or False: There exists a 4×3 matrix A so that the equation $Ax = b$ is consistent for every vector $b \in \mathbb{R}^4$. Justify your answer.
Solution
False. The only way that this can be true if you can have some sort of identity matrix, with an identity matrix you can have all free variables in b, however that is impossible to have in a 4×3 matrix, making this impossible.

c (5 points) True or False: if A is a 4×3 matrix and the equation $Ax = b$ is consistent for a particular vector $b \in \mathbb{R}^4$, then the equation $Ax = cb$ is consistent for every scalar c. Justify your answer.
Solution
True, if you manipulate a vector with a scalar you can use that scalar on the other side of the equation and manipulate it in exactly the same way.

d (5 points) True or False: If A and B are invertible 3×3 matrices and $C = A^3B^2$, then C is invertible and $C^{-1} = (B^{-1})^2(A^{-1})^3$. Justify your answer.
Solution
True. The solution seems to look a bit strange but right.Let's say that $A^3 = X$ and $B^2 = Y$ so that $C = XY$. In this case the inverse would look like:

$$C^{-1} = (Y)^{-1}(X)^{-1})$$

Which, when swapping X and Y with their counterpart values would look like:

$$C^{-1} = (B^2)^{-1}(A^3)^{-1}$$
$$C^{-1} = (B^{-1})^2(A^{-1})^3$$

which can be shown to be exactly identical.