MTH 5051: Homework 4

Due on October 30, 2019 at 11:59pm

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Chapter 9.1

Problem 5:

Determine the generating function for the number of ways to distribute 35 pennies (from an unlimited supply) among five children if

a) There are no restrictions;

b) Each child gets at least 1¢;

c) Each child gets at least 2¢;

d) The oldest child gets at least 10¢; and

e) The two youngest children must each get at least 10¢.

Solution:

a) For this problem we have 5 children, and 35 pennies so the generating function would be:

$$(1 + x + x^2 + x^3 + .. + x^{35})^5$$

b) This would be very similar except it would not include the 1 in the addition of numbers because there is no way that a single child could not have all the coins, so it looks like:

$$(x + x^2 + X^3 + ... + x^{35})^5$$

c) Here you don't include any case where someone has 1 penny, which gives you:

$$(x^2 + x^3 + x^4 + ... + x^{35})^5$$

d) If one child gets at least 10¢the equation would look a bit different. You would have one child that has 10¢and at most the other children could have is 25¢. This would look.

$$(x^{10} + x^{11} + x^{12} + ... + x^{35})(1 + x + x^2 + x^3 + ... + x^25)^4$$

e) Now if the 2 youngest children are receiving 10¢, it would look very similar to the last program except:

$$(x^{10} + x^{11} + x^{12} + ... + x^{25})^2(1 + x + x^2 + x^3 + ... + x^{15})^3$$

Problem 5:

Find the generating function for the number of integer solutions to the equation $c_1 + c_2 + c_3 + c_4 = 20$ where $-3 \leq c_1, -3 \leq c_2, -5 \leq c_3 \leq 5$, and $0 \leq c_4$.

Solution:

The first thing that we should do is simplify the equation so that we can consider the worst case scenario, so this equation would look like:

$$(c_1 + 3) + (c_2 + 3) + (c_3 + 5) + c_4 = 31$$

This makes it much easier to turn into a generating function, no negative numbers to deal with. Let's now correlate this to some other equation to make it easier to describe.

$$x_1 + x_2 + x_3 + x_4 = 31$$

In this case, when creating the generating function x_1, x_2, and x_4 are all the same, they are some value greater than or equal to 0, up to 31. The problem with x_3 is that it has to be less than or equal to 10, as we added 5 in the original equation above x_3 can be no more than 10, and no less than 0 which. This would make the generating function look like:

$$(1 + x_1 + x_2 + ... + x_{31})^3(1 + x_1 + x_2 + ... + x_10)$$

Chapter 9.2

Problem 1:

Find generating functions for the following sequences. [For example, in the case of the sequence 0,1,3,9,27,..., the answer required is $\frac{x}{1-3x}$, not $\sum_{i=0}^{\infty} 3^i x^{i+1}$ or simply $0 + x + 3x^2 + 9x^3 + ...$]

a) $\binom{8}{0}, \binom{8}{1}, \binom{8}{2}, ..., \binom{8}{8}$

b) $\binom{8}{1}, 2\binom{8}{2}, 3\binom{8}{3}, ..., 8\binom{8}{8}$

c) $1, -1, 1, -1, 1, -1, ...$

d) $0, 0, 0, -6, 6, -6, 6, ...$

e) $1, 0, 1, 0, 1, 0, 1, ...$

f) $0, 0, 1, a^2, a^3, ..., a \neq 0$

Solution:

a) This would be:

$$(1 + x)^8$$

b) This would be:

$$8(1 + x)^7$$

c) This would be:

$$\frac{1}{(1 + x)}$$

d) This would be:

$$\frac{6x^3}{(1 + x)}$$

e) This would be:

$$\frac{1}{(1 - x^2)}$$

f) This would be:

$$\frac{x^2}{(1 - ax)}$$

Problem 9:

Find the coefficient of x^{15} in each of the following.

a) $x^3(1-2x)^{10}$

b) $\dfrac{x^3-5x}{(1-x)^3}$

c) $\dfrac{(1+x)^4}{(1-x)^4}$

Solution:

a) First is to expand on this problem. The expansion form would look like:

$$x^3\left(\sum_{i=0}^{10}(-1)^i\binom{10}{i}(2x)^i\right)$$

We find a small problem here. It seems that the largest possible x value is x^13 and not x^15, therefore the answer is **0**.

b) For this problem we want to do what we did before and expand the equation. The easiest way to expand this is:

$$x^3-5x\left(\sum_{i=0}^{\infty}\right)$$

c)

Problem 11:

In how many ways can 3000 identical envelopes be divided, in packages of 25, among four student groups so that each group gets at least 150, but not more than 1000, of the envelopes?

Solution:

We can start this problem by simplifying it a bit, if the envelopes have to be shipped in packages of 25 then we can calculate the amount of packages and use that as our main value. The amount of packages in this case would be $\frac{3000}{25}=600$. Now, we have to calculate the minimum and maximum amount of envelopes a student can have into how many packages that would be. The minimum amount of packages a student group could have would be $\frac{150}{25}=6$, and the maximum packages a student group could have is $\frac{1000}{25}=40$. This means that the answer would look something like:

$$(x^6+x^7+x^8+...+x^{40})^4$$

Supplementary Exercises: Chapter 9

Problem 6:

How many 10-digit telephone numbers use only the digits 1,3,5, and 7, with each digit appearing at least twice or not at all.

Solution:

So, Primarily in this problem is to recognize how many possible solutions exist for the 10 numbers in general.

Chapter 10.1

Problem 1:

Find a recurrence relation, with initial condition, that uniquely determines each of the following geometric progressions.

a) 2, 10, 50, 250, ...

b) 6, -18, 54, -162, ...

c) 7, $\frac{14}{5}$, $\frac{28}{25}$, $\frac{56}{125}$, ...

Solution:

a) It seems that this is just multiplying the previous number by 5, starting at 2. This would look like:

$$5a_{n-1},\ n \geq 1,\ a_0 = 2$$

b) This seems to start at 6, and be multiplied by a -3 each time. This would look like:

$$-3a_{n-1},\ n \geq 1,\ a_0 = 6$$

c) This starts at 7, and seems to be multiplied by $\frac{2}{5}$ each time. This would look like:

$$\frac{2a_{n-1}}{5},\ n \geq 1,\ a_0 = 7$$

Problem 3:

If a_n, $n \geq 0$, is the unique solution of the recurrence relation $a_{n+1} - da_n = 0$, and $a_3 = \frac{153}{49}$, $a_5 = \frac{1377}{2401}$, what is d?

Solution:

I think the first step to this problem is to solve for the case a_4. This would could be done by figuring out an equation for any generalized recurrence based on a_0. We could rearrange the original equation given and make:

$$a_n = d^n a_0$$

From here we can plug in both a_5 and a_3 to get d^2 and then get the square root of that. This would look like:

$$a_3 = d^3 a_0 = \frac{153}{49}$$
$$a_5 = d^5 a_0 = \frac{1377}{2401}$$
$$\frac{a_5}{a_3} = d^2 = \frac{9}{49}$$
$$d = \sqrt{\frac{9}{49}} = \pm\frac{\mathbf{3}}{\mathbf{7}}$$

Problem 4:

The number of bacteria in a culture is 1000, (approximately), and this number increases by 250% every two hours. Use a recurrence relation to determine the number of bacteria present after one day.

Solution:

Well, to start initially we have to figure out approximately how much the culture increases in one day. This can be done by dividing 24 (the hours we have in a day) by the 2 (the time we know the bacteria increases by), and put 2.5 (the amount it increases by) to the power of that number. We know that $a_0 = 1000$ so the equation would look like:

$$a_n + 2.5a_n, n/ge1, a_0 = 1000$$

Now to get more in depth we can solve this by:

$$a_n = (3.5)a_0$$
$$n = 10$$

$$a_{12} = (3.5)^{12} \times 1000 = \mathbf{3,379,220,508.06}$$

Chapter 10.2

Problem 1:

Solve the following recurrence relations. (No final answer should involve complex numbers)

a) $a_n = 5a_{n-1} + 6a_{n-2}, n \geq 2, a_0 = 1, a_1 = 3$

c) $a_{n+2} + a_n = 0, n \geq 0, a_0 = 0, a_1 = 3$

d) $a_n - 6a_{n-1} + 9a_{n-2} = 0, n \geq 2, a_0 = 5, a_1 = 12$

Solution:

a) Let's start by saying $a_n = cr^n$. It is also important to mention that $c, r \neq 0$ If this is the case then we can rewrite the recurrence equation as:

$$r^2 - 5r - 6 = 0$$

Now from here we want to find the roots of this in order to create a recurrence relation. the roots are:

$$r^2 - 5r - 6$$
$$(r+1)(r-6)$$

and to continue on with solving for distinct real roots:

$$
\begin{aligned}
&a_n = (-1)^n A + (6)^n B \\
&a_0 = 1 = A + B \\
&B = 1 - A \\
&a_1 = 3 = -A + 6B \\
&3 = -A + 6 - 6A && A = \frac{3}{7} \\
&1 = \frac{3}{7} + B && B = \frac{4}{7} \\
&\boldsymbol{a_n = \frac{3}{7} + \frac{24}{7}}
\end{aligned}
$$

c) We are doing something similar to (a), where we say $a_n = cr^n$ and $c, r \neq 0$

$$r^2 + 1 = 0$$

From here we are going to split it up to its roots, however the process for finding the answer works differently as we are going to have complex roots. This would look like:

$$
\begin{aligned}
&(r + \sqrt{-1})(r - \sqrt{-1}) && \pm i \\
&a_n = A(i)^n + B(-i)^n \\
&a_n = A\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)^n + B\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right)^n \\
&\quad = C\cos\left(\frac{n\pi}{2}\right) + D\sin\left(\frac{n\pi}{2}\right) \\
&a_0 = 0 = C\cos\left(\frac{0\pi}{2}\right) + D\sin\left(\frac{0\pi}{2}\right) = C \\
&a_1 = 3 = C\cos\left(\frac{3\pi}{2}\right) + D\sin\left(\frac{3\pi}{2}\right) = D \\
&\boldsymbol{a_n = 3\sin\left(\frac{n\pi}{2}\right),\ n \geq 0}
\end{aligned}
$$

d) Similar to the previous questions, we will say $a_n = cr^n$ where $c, r \neq 0$. The process is a bit different because of repeated real roots, this would look like:

$$
\begin{aligned}
&r^2 - 6r + 9 = 0 \\
&(r-3)(r-3) \\
&a_n = 3^n A + 3^n Bn \\
&a_0 = 5 = A && A = 5 \\
&a_1 = 12 = 3A + 3B \\
&12 = 15 + 3B && B = -1 \\
&\boldsymbol{a_n = (3^n \times 5) - n3^n}
\end{aligned}
$$

Problem 11:

a) For $n \geq 1$, let a_n count the number of binary strings of length n, where there are no consecutive 1's. Find and solve a recurrence relation for a_n.

b) For $n \geq 1$, let b_n count the number of binary strings of length n, where there are no consecutive 1's and the first and last bit of the string are not both 1. Find and solve a recurrence relation for b_n.

Solution:

a) We find that there is a strange phenomenon with the case of 1 and 2 length strings. The one length strings have 2 items that have 3 items without consecutive ones. If you have a string of length 3, it will have 5 items, and a string of 4 items will have a length of 8 without consecutive ones. A pattern emerges from this. This pattern being:

$$a_n = a_{n-1} + a_{n-2}, \quad n \geq 3$$
$$0 = x^2 - x + 1$$

This is a lot like the Fibonacci sequence, similar to the diagram that is shown at 10.16 and 10.17 in the textbook, except we start the sequence at point F_3 because $F_3 = a_1$ and so forth. So we can use this to help answer our question. This could look like:

$$a_n = F_{n+2} = \frac{A^{n+2} - B^{n+2}}{A - B}$$
$$A = \frac{1+\sqrt{5}}{2}, \quad B = \frac{1-\sqrt{5}}{2}$$

b) For problem b) have to consider all the cases where the first and last bits can not be 1. In this case $a_1 = 1$, $a_2 = 3$, $a_3 = 4$, $a_4 = 7$, $a_5 = 11$, which ends up looking exactly like the last problem:

$$a_n = a_{n-1} + a_{n-2}, \qquad n \geq 3$$

This problem however is that this does not correlate so easily with the Fibonacci sequence, however it still does correlate to the sequence in some way. Let's use our previous expression in answer a) and claim our new equation as b_n

$$b_n = a_{n-1} + a_{n-3} = F_{n+1} + F_{n-1}$$

From here we can note the A and B values from previous problem to find that

$$b_n = A^n + B^n$$

Problem 12:

Suppose that poker chips come in four colors — red, white, green, and blue. Find and solve a recurrence relation for the number of ways to stack n of these poker chips so that there are no consecutive blue chips.

Solution:
To break this down, let's say that a_n is the way to arrange the chips without consecutive blues. Now we have to create two separate variables for 2 separate case sceanrios, the first scenario being blue where a substack will have a choice of 3 items below it and if it is non-blue it can have a choice of 4 items on top of it. We can write these as b_n and c_n respectively and can figure out that the equation for this is:

$$a_n = b_n + c_n$$
$$c_n = a_n - b_n, \qquad b_n = a_n - c_n$$
$$a_{n+1} = 3b_n + 4c_n = 3(b_n + c_n) + c_n$$
$$= 3a_n + 3_{n-1}$$

We can then rewrite this equation and find its roots, as such:

$$\begin{aligned}
&a_{n+1} - 3a_n - 3a_{n-1} \\
&r^2 - 3r - 3 = 0 \\
&r = \frac{-3 \pm \sqrt{21}}{2}
\end{aligned}$$

Now we have to plug in these values into A and B to find a recurrence equation as so:

$$\begin{aligned}
&a_n = A\left(\frac{-3+\sqrt{21}}{2}\right)^n + B\left(\frac{-3-\sqrt{21}}{2}\right)^n \\
&a_0 = 1 = A\left(\frac{-3+\sqrt{21}}{2}\right)^0 + B\left(\frac{-3-\sqrt{21}}{2}\right)^0 \\
&1 = A + B, \qquad B = 1 - A \\
&a_1 = 4 = A\left(\frac{-3+\sqrt{21}}{2}\right) + B\left(\frac{-3-\sqrt{21}}{2}\right) \\
&\quad = A\left(\frac{-3+\sqrt{21}}{2}\right) + (1-A)\left(\frac{-3-\sqrt{21}}{2}\right) \\
&A = \frac{5+\sqrt{21}}{2\sqrt{21}} \qquad B = \frac{\sqrt{21}-5}{2\sqrt{21}} \\
&\boldsymbol{a_n =} \left(\boldsymbol{\frac{5+\sqrt{21}}{2\sqrt{21}}} \times \frac{-3+\sqrt{21}}{2}\right) + \left(\frac{\sqrt{21}-5}{2\sqrt{21}} \times \frac{-3-\sqrt{21}}{2}\right)
\end{aligned}$$