Linear Algebra: Take Home Test

Due on April 17, 2019 at 11:59pm

Dr. Munevver Subasi Section 01

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Problem 1

Let A be an $n \times n$ matrix.

(a) (5 points) if λ is an eigenvalue of A, find an eigenvalue A^k where k is a positive integer.
Solution:

$$\begin{aligned} Ax &= \lambda x \\ AAX &= A\lambda x \\ A^2x &= \lambda Ax \\ A^2x &= \lambda(\lambda x) = \lambda^2 x \end{aligned}$$

Now to introduce A^k...

$$\begin{aligned} A^{k-1}Ax &= A^{k-1}\lambda x \\ A^k x &= \lambda(\lambda^{k-1}x) \\ A^k x &= \lambda^k x \end{aligned}$$

The eigenvalue of A^k is λ^k

(b) (5 points) if v is an eigenvector of A corresponding to an eigenvalue λ, find an eigenvector of A^k where k is a positive integer.
Solution:
As shown in problem 1(a), where v and x are eigenvectors you can see that when finding the eigenvalues of A^k the vectors are never changed, therefore the eigenvector of A^k where k is a positive integer is v.

Used Class notes to answer 1(a) and 1(b)

Problem 2

Let A and B be two $n \times n$ matrices. Assume that B is similar to A, i.e, there exists an $n \times n$ nonsingular matrix S such that $B = S^{-1}AS$. Let $x \neq 0$ be an eigenvector of B corresponding to eigenvalue λ.

(a) (5 points) Prove that A and B have some eigenvalues, i.e, λ is also an eigenvalue of A.
Solution:

$$
\begin{aligned}
det|\lambda I - A| &= det|\lambda I - S^{-1}BS| \\
det|\lambda I - A| &= det|\lambda S^{-1}IS - S^{-1}BS| \\
det|\lambda I - A| &= det|S^{-1}(\lambda I - B)S| \\
det|\lambda I - A| &= det|\frac{1}{S}(\lambda I - B)S| \\
det|\lambda I - A| &= det|\lambda I - B|
\end{aligned}
$$

If this is true then A and B must have the same eigenvalues.

(b) (5 points) Find the eigenvector of A corresponding to eigenvalue λ.
Solution:

$$
\begin{aligned}
B &= S^{-1}AS \\
Bx &= \lambda x \\
S^{-1}ASx &= \lambda x \\
ASx &= \lambda Sx
\end{aligned}
$$

We are able to move lambda because it is an integer and commutative.

Used Class notes to answer question 2(a) and 2(b)

Problem 3

Let V be a subspace of $\mathbb{R}^n$. The orthogonal complement of V in $\mathbb{R}^n$ is defined as

$$V^{\perp} = \{x \in \mathbb{R}^n | x \cdot v = 0 \ \forall \ v \in V\}$$

Prove that $V^{\perp}$ is also a subspace of $\mathbb{R}^n$.
Solution:
Expanding the vectors out we get:

$$V^{\perp} = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n | (x_1, x_2, ..., x_n) \cdot (v_1, v_2, ..., v_n) = 0 \ \forall \ (v_1, v_2, ..., v_n) \in V\}$$

Doing the dot product out it will look like:

$$V^{\perp} = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n | (x_1 v_1 + x_2 v_2 + ... + x_n v_n) = 0 \ \forall \ (v_1, v_2, ..., v_n) \in V\}$$

v is 0 vector, because $x \cdot v = 0$, which means that either the x or v is the zero vector and because x is all reals that means v is the 0 vector.

$$V^{\perp} = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n | (x_1 + x_2 + ... + x_n) \cdot (0 + 0 + ... + 0) = 0\}$$
$$V^{\perp} \in \mathbb{R}^n$$

Worked with Jacquelyne Miksanek, and Karly Lorenzini on this problem

Problem 4

Let A and B be two $n \times n$ orthogonal matrices. Answer the following questions.

(a) (2 points) Is $-3A$ orthogonal? Justify your answer.
Solution:
Yes, the orthogonality of a set is closed under scalar multiplication

(b) (2 points) Is $-B$ orthogonal? Justify your answer.
Solution:
Yes, for the same reason as A, the orthogonality of a set is closed under scalar multiplication.

(c) (2 points) Is $A + B$ orthogonal? Justify your answer.
Solution
Yes, because the orthogonality of a set is closed under addition.

(d) (2 points) Is $B^{-1}AB$ orthogonal? Justify your answer.
Solution:
Yes, because the set is closed under multiplication.

(e) (2 points) Is A^T invertible? if yes, find its inverse.
Solution:
Yes, because the transpose of an orthogonal matrix is its inverse (i.e $A^T = A^{-1}$). This means that the inverse of the transpose the original matrix itself.

(f) (2 points) Is AB orthogonal? Justify your answer.
Solution:
Yes, because the set is closed under multiplication.

(g) (2 points) Is A^2B^2 orthogonal? Justify your answer.
Solution:
Yes, because the set is closed under multiplication.

Worked with Jacquelyne Miksanek on this problem

Problem 5

(a) (5 points) Let A be an $n \times n$ orthogonal matrix. Prove that $det(A) = \pm 1$.
Solution:

$$\begin{aligned} A^{-1} &= A^T \\ det(AA^T) &= det(AA^{-1}) \\ [det(A)]^2 &= 1 \\ \sqrt{[det(A)]^2} &= \sqrt{1} \\ det(A) &= \pm 1 \end{aligned}$$

(b) (5 points) Let A be an $n \times n$ orthogonal matrix. Prove that only eigenvalues of A are 1 and -1.
Solution:

$$\begin{aligned} det(A) &= \pm 1 \\ det(A - \lambda I) &= 0 \\ det(A) - \lambda det(I) &= 0 \\ det(A) &= \lambda det(I) \\ det(I) &= 1 \\ det(A) &= \lambda \\ \pm 1 &= \lambda \end{aligned}$$

Used class notes to answer question 5(a) and 5(b)