Linear Algebra: Homework #1

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Problem 1

Let $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^2$ and θ be the angle between them. Prove that

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||||\mathbf{v}||}$$

Where $||\mathbf{u}||=\sqrt{u_1^2+...+u_n^2}$ and $||\mathbf{v}||=\sqrt{v_1^2+...+v_n^2}$

Solution

To start let us recognize the law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Now lets put this in terms of $||\mathbf{u}||$ and $||\mathbf{v}||$ where:

$$a = ||\mathbf{u}||$$

$$b = ||\mathbf{v}||$$

$$c = ||\mathbf{v} - \mathbf{u}||$$

So now the law of cosines equation becomes:

$$||\mathbf{v\text{-}u}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2||\mathbf{u}||||\mathbf{v}||\cos\theta$$

Furthermore, this is the definition of dot product:

$$\mathbf{u}\cdot\mathbf{v} = ||\mathbf{u}||||\mathbf{v}||\cos\theta$$

Using the definition of dot product we can put it in terms of cos θ and use it in the law of cosines equation:

$$\mathbf{u}\cdot\mathbf{v} = ||\mathbf{u}||||\mathbf{v}||\cos\theta$$
$$\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||||\mathbf{v}||} = \cos\theta$$

Now to utilize this in the dot product equation:

$$||\mathbf{v} - \mathbf{u}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2\mathbf{u}\cdot\mathbf{v}$$

We can replace $||\mathbf{v} - \mathbf{u}||^2$ with $||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2||\mathbf{u}||||\mathbf{v}||\cos\theta$ and compare.

$$||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2||\mathbf{u}||||\mathbf{v}||\cos\theta = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2\mathbf{u}\cdot\mathbf{v}$$

$$-2||\mathbf{u}||||\mathbf{v}||\cos\theta = -2\mathbf{u}\cdot\mathbf{v}$$

$$||\mathbf{u}||||\mathbf{v}||\cos\theta = \mathbf{u}\cdot\mathbf{v}$$

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||||\mathbf{v}||}$$

Problem 2

Let $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^n$. Prove the Cauchy-Schwarz Inequality in $\mathbb{R}^n$

$$(\mathbf{u} \cdot \mathbf{v})^2 \leq ||\mathbf{u}||^2||\mathbf{v}||^2$$

or equivalently,

$$(\mathbf{u} \cdot \mathbf{v}) \leq ||\mathbf{u}||||\mathbf{v}||,$$

where $||\mathbf{u}|| = \sqrt{u_1^2 + ... + u_n^2}$ and $||\mathbf{v}|| = \sqrt{v_1^2 + ... + v_n^2}$

Solution

To start the solution the definiton of the dot product can be used.

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||||\mathbf{v}|| \cos\theta$$

Use this definition to replace dot product with the Cauchy-Schwarz Inequality:

$$(||\mathbf{u}||||\mathbf{v}|| \cos\theta)^2 \leq ||u||^2||v||^2$$

$$||\mathbf{u}||^2||\mathbf{v}||^2 \cos^2\theta \leq ||u||^2||v||^2$$

$$\cos^2\theta \leq 1$$

We know that for any $\cos^2\theta$ will hold the maximum value of 1, which makes this statement true.

Problem 3

Let $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^n$. Prove the Parallelogram Equation for Vectors:

$$||\mathbf{u}+\mathbf{v}||^2 + ||\mathbf{u}-\mathbf{v}||^2 = 2(||\mathbf{u}||^2 + ||\mathbf{v}||^2),$$

where $||\mathbf{u}|| = \sqrt{u_1^2 + ... + u_n^2}$ and $||\mathbf{v}|| = \sqrt{v_1^2 + ... + v_n^2}$

Solution

This problem can be solved using basic algebra. Lets start with the left side:

$$||\mathbf{u}|| + ||\mathbf{v}||^2 + ||\mathbf{u}-\mathbf{v}||^2 = 2(||\mathbf{u}||^2 + ||\mathbf{v}||^2)$$

$$(||\mathbf{u}|| + ||\mathbf{v}||)(||\mathbf{u}|| + ||\mathbf{v}||) + (||\mathbf{u}-\mathbf{v}||)(||\mathbf{u}-\mathbf{v}||) = 2(||\mathbf{u}||^2 + ||\mathbf{v}||^2)$$

$$(||\mathbf{u}||^2 + ||\mathbf{v}||^2 + 2||\mathbf{u}||||\mathbf{v}||) + (||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2||\mathbf{u}||||\mathbf{v}||) = 2(||\mathbf{u}||^2 + ||\mathbf{v}||^2)$$

$$2||\mathbf{u}||^2 + 2||\mathbf{v}||^2 = 2(||\mathbf{u}||^2 + ||\mathbf{v}||^2)$$

$$2(||\mathbf{u}||^2 + ||\mathbf{v}||^2) = 2(||\mathbf{u}||^2 + ||\mathbf{v}||^2)$$

As a result of doing this algebra it is easy to see that both sides are the same, and prove the Parallelogram Equation for Vectors.