

UNFITTED FINITE ELEMENT MODELLING OF SURFACE-BULK VISCOUS FLOWS IN ANIMAL CELLS

ERIC NEIVA^{†,*} AND HERVÉ TURLIER[†]

ABSTRACT. This work presents a novel unfitted finite element framework to simulate coupled surface-bulk problems in time-dependent domains, focusing on fluid-fluid interactions in animal cells between the actomyosin cortex and the cytoplasm. The cortex, a thin layer beneath the plasma membrane, provides structural integrity and drives shape changes by generating surface contractile forces akin to tension. Cortical contractions generate Marangoni-like surface flows and induce intracellular cytoplasmic flows that are essential for processes such as cell division, migration, and polarization, particularly in large animal cells. Despite its importance, the spatiotemporal regulation of cortex-cytoplasm interactions remains poorly understood and computational modelling can be very challenging because surface-bulk dynamics often lead to large cell deformations. To address these challenges, we propose a sharp-interface framework that uniquely combines the trace finite element method for surface flows with the aggregated finite element method for bulk flows. This approach enables accurate and stable simulations on fixed Cartesian grids without remeshing. The model also incorporates mechanochemical feedback through the surface transport of a molecular regulator of active tension. We solve the resulting mixed-dimensional system on a fixed Cartesian grid using a level-set-based method to track the evolving surface. Numerical experiments validate the accuracy and stability of the method, capturing phenomena such as self-organised pattern formation, curvature-driven relaxation, and cell cleavage. This novel framework offers a powerful and extendable tool for investigating increasingly complex morphogenetic processes in animal cells.

Keywords: Finite element, fixed grid, unfitted, surface-bulk, cell cortex, cytoplasm, morphogenesis.

1. INTRODUCTION

Morphogenesis is the biological process by which a cell, tissue or organism takes shape and develops its distinctive form [1]. Researchers in this field seek to understand the underlying genetic, cellular, mechanical, and environmental factors that drive the formation of complex biological structures. The ultimate goal is to unravel the fundamental principles that govern the development of living organisms. Understanding morphogenesis has significant far-reaching implications for various fields, including regenerative medicine [2], cancer research [3] or environmental and ecological studies [4].

This work centres upon the morphogenesis of single animal cells. Animal cells maintain their shape primarily through an effective surface tension, adopting shapes akin to soap bubbles [5]. The main contributor to this effective surface tension in animal cells is the actomyosin cortex [6]: a thin biological interface at the inner face of the plasma membrane, represented in Figure 1(a). The cortex is composed of bundled and cross-linked actin protein filaments that form a dense and overlapping three-dimensional meshwork of a few hundreds of nanometres in thickness [7]. Myosin-2 molecular motors populate this network and pull on actin filaments by converting chemical into mechanical forces to generate contractile stresses in the meshwork. These internal stresses lead to an effective surface tension over the cell surface and, due to cell curvature, create hydrostatic pressure in the cytoplasm following Young-Laplace's law. In addition, cortical actin is under constant renewal, through assembly and disassembly of the filaments. This material turnover remodels the whole network in about 30-60 s and releases any elastic stresses accumulated in that period [8]. Being constantly out of thermodynamic equilibrium, the cortex is thus an *active* system: It behaves as an elastic solid membrane at short time scales (<1 min), but flows like a viscous liquid at longer time scales [9].

Date: May 12, 2025.

[†] CIRB, Collège de France, Université PSL, CNRS, INSERM, 75005 Paris, France.

* Corresponding author.

E-mails: eric.neiva@college-de-france.fr (EN), herve.turlier@college-de-france.fr (HT).

Cortical flows, induced by gradients of cortical tension, are the fundamental mechanism underlying shape change in animal cells [10–13]. This is illustrated in Figure 1(b) as follows: A uniform tension in the cortex gives rise to a spherical shape under constant hydrostatic pressure proportional to the surface tension and curvature (here, the inverse of the radius). This is in agreement with normal force balance given by the Young-Laplace law. However, animal cells are able to spatiotemporally *localise* myosin concentration and motor activity within the cortex, using complex cascades of chemical reactions, known as signalling pathways [14, 15]. This can create local imbalances of contractility that, at the time scale of minutes, give rise to Marangoni-like flows, pulling cortical components away from regions of relaxation toward regions of contraction. Moreover, on short time scales, normal force balance implies that gradients of surface tension must necessarily be corresponded with non-uniform cytoplasmic pressures and changes in curvature. While this explains how animal cells exert shape control with the actomyosin cortex, it also reveals that cortex-mediated cell deformation produces a net movement of the cytoplasm [16].

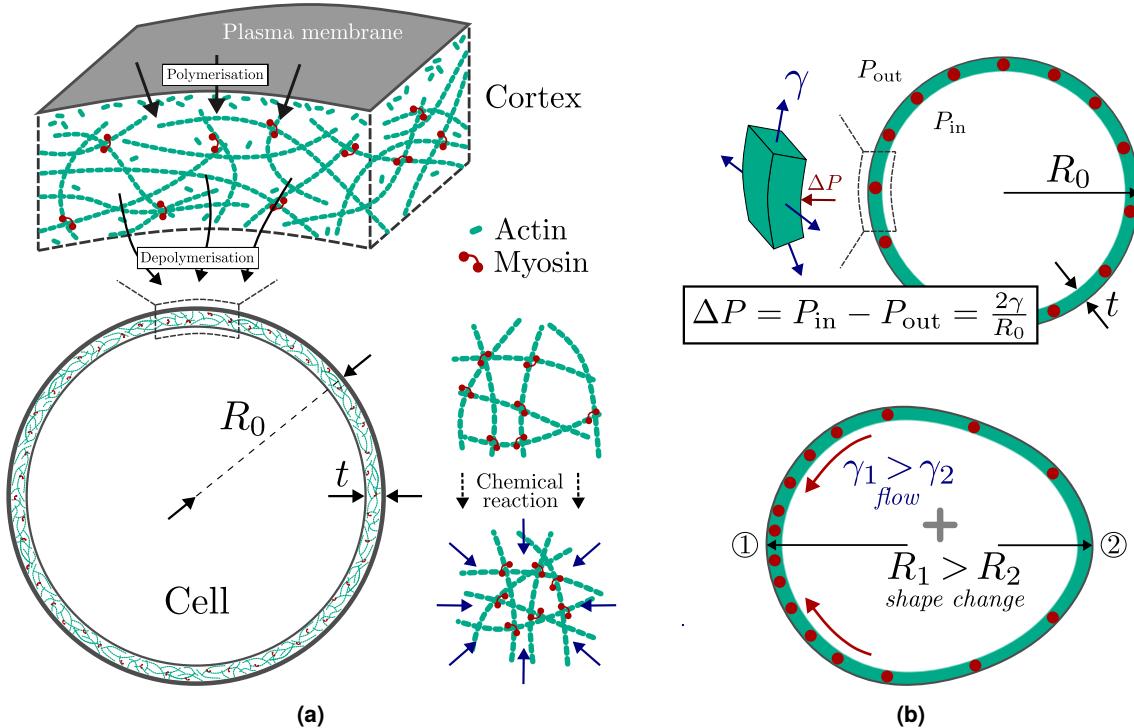


Figure 1. (a) The actomyosin cortex is a thin layer of thickness $t \ll R_0$ at the inner face of the plasma membrane, where R_0 is the typical curvature radius of a cell. It is a network of actin protein filaments and binding proteins, among which myosin-2 molecular motors, who pull on the filaments via a chemical reaction. The network is under constant renewal: polymerisation occurs right under the plasma membrane, while depolymerisation occurs uniformly across the cortex. Adapted from [17]. (b) According to Laplace's law, a uniform tension in the cortex gives rise to a spherical shape under constant hydrostatic pressure. By localising myosin activity, animal cells create gradients of contractility that lead, on long time scales, to tangential cortical flows and, on short time scales, to shape deformation with net cytoplasm movement (non-uniform cytoplasmic pressure). Inspired by [18], incorporating the appropriate curvature changes associated with increased local membrane tension.

The cytoplasm is a gel-like substance that is crowded with various macromolecules, organelles and a dynamic network of protein filaments, known as the cytoskeleton [19]. It has a $\mu\text{m-scale}$ viscosity of around 0.1-1 Pa·s, about 100 to 1,000 times the one of water [20–22], albeit much lower than the one estimated for the cortex of 10^4 - 10^5 Pa·s [12]. Despite the cytoplasm being a crowded medium at the microscopic scale, fast rearrangement of cytoplasmic components in the cell is essential for key cellular functions such as cell division, migration and polarisation. Many large animal cells (above 50 μm in diameter) often accomplish this with cortical contractions that induce bulk intracellular flows [16, 23–25]. Classical observations of this phenomenon include the establishment and maintenance of PAR polarity in the *C. elegans* zygote [26, 27], the distribution of nuclei in the *Drosophila* fly syncytial embryo [28–30], the 3D migration of a cell in a fluid [31, 32], and the asymmetric spindle positioning model in mouse oocytes [33–37]. Interestingly, the reverse coupling mechanism has also been reported, where cytoplasmic

queues via diffusive transport and/or intracellular flows trigger flows in the actomyosin cortex [26, 29]. Despite these descriptions, the spatiotemporal interactions between cortical and intracellular flows—and their roles in cytoplasmic reorganization and cell shape—remain insufficiently characterized in most cases.

To address this, scholarly efforts combining biophysical experiments and numerical modelling are growing and effectively helping to better understand cortex-cytoplasm interactions [37–39]. The finite element method (FEM) is frequently adopted to approximate the partial differential equations (PDEs) that model these type of systems [17, 40], given its flexibility to deal with intricate geometries, non-standard boundary conditions and nonlinearities [41]. Yet, many surface-bulk couplings in living cells involve large distortions of the cell surface. This frames the problem within a classical dilemma in finite element (FE) theory and practice: selecting the description of motion [42, 43]. This choice constrains the relationship between the deforming body and the computational mesh underlying the FE approximation. As a result, it determines the ability of the FE method to deal with large deformations.

Most, if not all, approaches to describe large distortions with FEs sit between the two main perspectives of motion in continuum mechanics: Lagrangian and Eulerian [43]. In the Lagrangian viewpoint, the computational grid follows the moving continuum, the grid nodes being permanently identified with the same material points [44]. A natural choice is to use a mesh fitting to the moving boundaries and interfaces in the continuum, i.e., a *body-fitted* mesh (Figure 2(b)). This perspective is common in nonlinear solid mechanics [45]. While it allows for an easy tracking of free surfaces and interfaces, frequent remeshing is needed to avoid numerical pollution from excessively distorted elements in the mesh [17, 46]. This issue does not appear in the Eulerian viewpoint. In this case, material particles and mesh nodes are decoupled: the computational grid stays fixed, while the continuum moves with respect to the grid (Figure 2(c)). *Fixed-grid* techniques are predominant in fluid mechanics [47]. Since the mesh no longer fits to moving boundaries and interfaces, tracking their motion becomes more difficult. Besides, convective effects appear, due to the relative motion between the deforming material and the computational grid.

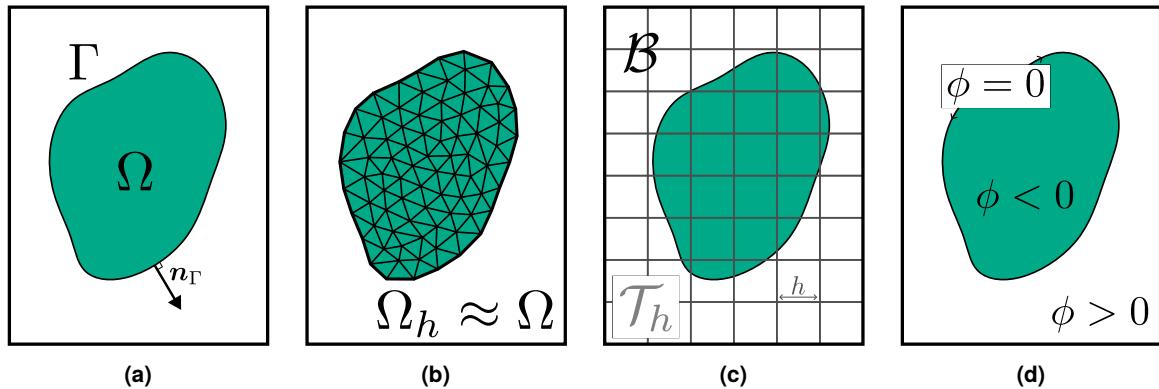


Figure 2. Geometry and mesh definitions. **(a)** The cell cortex is represented by a closed surface Γ enclosing the cytoplasm Ω . n_Γ is the outwards pointing unit normal on Γ . **(b)** A typical *body-fitted* mesh (approximately) fitting to the boundary of the problem. **(c)** A *fixed-grid* of an unfitted FE analysis. The problem geometry Ω is embedded into an easy-to-generate mesh \mathcal{T}_h , usually covering a trivial region, such as a bounding box \mathcal{B} of Ω . **(d)** A *level-set function* ϕ encoding the surface as its zero isosurface $\Gamma = \{\phi = 0\}$ and the bulk domain as its negative region $\Omega = \{\phi < 0\}$.

The solid-fluid duality in the mechanics of the actomyosin cortex is rather the rule than the exception in animal cell and tissue interfaces [12, 17, 48, 49]. At first glance, there is no clear choice to describe the motion of this type of surfaces and their interactions with surrounding fluids. Hence, it comes as no surprise to see the variety of FE methods in the field reflecting this open-ended question. At one end, there are body-fitted methods [50–54], relying on Arbitrary Lagrangian-Eulerian (ALE) [55–57] descriptions to cope with the mesh deformation. At the other end, there are fixed-grid methods with diffuse or sharp representations of the moving boundary or interface; they are usually grounded, respectively, on phase-field [58, 59] or level-set [60] methods. In between, there are hybrid approaches, coupling a body-fitted method for the interface with a fixed-grid one for the bulk phases [61, 62].

Observe that, of the above works, only body-fitted ALE and diffuse-interface approaches explicitly address applications in cell biology. In particular, as far as the authors know, the most robust and advanced

technology in evolving sharp-interface methods remains to be transposed for modelling surface-bulk couplings in animal cells. Working to close this gap could be beneficial in the area for, at least, two reasons: (1) Sharp-interface methods have an apparent edge over body-fitted counterparts when it comes to model cell division, as they do not need to undergo complex numerical surgery for mesh scission [63]; (2) Phase-field models can be computationally prohibitive, especially in 3D, as they require small and adaptive mesh resolutions around the moving diffuse-interface [64, 65].

Given this motivation, the main purpose of this work is to *formulate a new sharp-interface FE method for surface-bulk PDEs with dynamic surfaces in fixed computational grids*. To approximate the coupled problem, we propose a novel partitioned scheme, in which the surface PDEs are discretised with the trace FEM [66] and the bulk PDE with the aggregated FEM (AgFEM) [67]. Both methods fall into the class of so-called unfitted, embedded or immersed FE methods [68, 69]. They are endowed with mathematical properties that ensure numerical stability and robustness, regardless of how the mesh overlaps the moving interface(s) [70, 71], which is an essential requirement in our context. Our contribution distinguishes itself from other unfitted methods in the literature [60, 72, 73] in two ways: (1) They solve the bulk problem with ghost penalty stabilisation [74], while we choose AgFEM to avoid locking issues that can appear with standard ghost penalty methods [75]. (2) We specifically target applications to cortex-cytoplasm fluid interactions in animal cells. For this reason, our model problem consists in a system of surface-bulk viscous flows, whose active driving forces are regulated by the surface transport of a diffusing species. Similar models have only been analysed with body-fitted or diffuse-interface methods [51, 76]; in contrast to this work, they do not explicitly solve the surface PDE problem for the viscous flows.

The outline of this paper is as follows. First, we state the problem in Section 2. We introduce minimal notation in Section 2.1 to write the dimensionless 3D governing equations of the problem in Section 2.2. The computational model is detailed in Section 3. We introduce first the embedded geometry setup in Section 3.1. The discretisation in space in Section 3.2 adopts, for the bulk problem, the inf-sup stable pair formed by continuous quadratic velocities and discontinuous linear pressures; for the surface flow problem, continuous linear velocities; and, for the surface transport problem, continuous linear elements. Numerical analysis available for each individual discrete formulation underpin these choices [77, 78]. The discretisation in time considers a simple backward Euler method and the moving surface is modelled with an evolving level-set method (Section 3.3), that leverages high-order quadrature rules on implicit geometries [79] and extensions of the surface velocity to the whole grid by closest-point projections [80]. In the numerical experiments of Section 4, we analyse three challenging applications: (1) Self-organised shape emergence, as in [51], but at large hydrodynamic lengths, (2) relaxation dynamics of 3D bodies and (3) uni- and bi-lateral cell cleavage. Finally, concluding remarks and perspectives are listed in Section 5.

2. MODEL PROBLEM

We consider the minimal model of self-organization of the cell cortex introduced in [81]. We restrict our analysis to a *single cell*. The problem is given by a system of coupled surface-bulk viscous flow equations, where the surface phase represents the cortex and the bulk phase the cytoplasm. Flows are driven by gradients of cortical tension that are regulated by a molecular species diffusing on the cortex, representing the myosin molecular motors. This leads to a system of mixed-dimensional PDEs in the three-dimensional space, where the surface and bulk PDEs are posed in manifolds of dimensionality two and three, respectively.

As we seek to approximate the problem in a fixed three-dimensional grid, it is in our interest to formulate all governing equations in the *global* Cartesian coordinates. In this way, we do not require *local* coordinates or a parameterisation for the surface, which facilitates applying the level-set method or other fixed-grid techniques. To achieve this, we will use tangential differential calculus (TDC) to express the surface PDEs in terms of differential operators in Cartesian coordinates, like the bulk PDEs. For the sake of brevity, we will skip most of the mathematical formalism underlying TDC; the interested reader can find it in, e.g., [82, 83].

2.1. Preliminary definitions. Let $\Gamma(t)$ be a smooth, closed and time-evolving surface in \mathbb{R}^3 . We denote by $\Omega(t)$ its enclosing volume ($\Gamma = \partial\Omega$). In our context, Γ represents the fluid cortex and Ω the bulk cytoplasm of a single animal cell, see Figure 2(a).

In what follows, we consider $\Gamma = \Gamma(t)$ and $\Omega = \Omega(t)$ for some fixed t . We denote by \mathbf{n}_Γ the outwards pointing unit normal vector on Γ and by $\mathbf{P}(\mathbf{x})$ the normal projector onto the tangential space at $\mathbf{x} \in \Gamma$, defined by

$$\mathbf{P}(\mathbf{x}) = \mathbf{Id} - \mathbf{n}_\Gamma(\mathbf{x}) \otimes \mathbf{n}_\Gamma(\mathbf{x}), \quad (1)$$

where \mathbf{Id} denotes the identity matrix. For illustration, the projection of a vector field $\mathbf{v} : \Gamma \rightarrow \mathbb{R}^3$ onto the tangent plane is given by $\mathbf{P}\mathbf{v}$ and for a second-order tensor function $\mathbf{A} \in \Gamma \rightarrow \mathbb{R}^{3 \times 3}$ the in-plane tensor is given by $\mathbf{P}\mathbf{A}\mathbf{P}$.

The *tangential differential operators* intervening in our problem are defined as follows. The surface gradient ∇_Γ of a scalar function $u : \Gamma \rightarrow \mathbb{R}$ defined on the manifold is given by $\nabla_\Gamma u = (\nabla u^e)\mathbf{P}$, where u^e denotes a smooth extension of u in a tubular neighbourhood \mathcal{U} of the manifold Γ . Given a vector field $\mathbf{v} : \Gamma \rightarrow \mathbb{R}^3$ on the manifold, the surface gradient is computed as $\nabla_\Gamma \mathbf{v} = \mathbf{P}\nabla \mathbf{v}^e \mathbf{P}$ and the surface divergence as $\operatorname{div}_\Gamma \mathbf{v} = \operatorname{tr}(\nabla_\Gamma \mathbf{v}) = \operatorname{tr}(\mathbf{P}\nabla \mathbf{v}^e \mathbf{P}) = \operatorname{tr}(\nabla \mathbf{v}^e \mathbf{P})$, where \mathbf{v}^e is a smooth extension of \mathbf{v} in \mathcal{U} . In particular, the surface Laplacian of a scalar field is given by $\Delta_\Gamma u = \operatorname{div}_\Gamma(\nabla_\Gamma u)$. Finally, the surface divergence of a tensor field $\mathbf{A} : \Gamma \rightarrow \mathbb{R}^{3 \times 3}$ is given by $\operatorname{div}_\Gamma \mathbf{A} = [\operatorname{div}_\Gamma(\mathbf{e}_1^T \mathbf{A}), \operatorname{div}_\Gamma(\mathbf{e}_2^T \mathbf{A}), \operatorname{div}_\Gamma(\mathbf{e}_3^T \mathbf{A})]^T$, where \mathbf{e}_i^T , $i = 1, 2, 3$, refer to the canonical basis vectors.

The *bulk strain-rate tensor* of a vector field $\mathbf{v} : \Omega \rightarrow \mathbb{R}^3$ is given by $\boldsymbol{\epsilon}(\mathbf{v}) = \frac{1}{2} [\nabla \mathbf{v} + (\nabla \mathbf{v})^T]$, whereas the *surface strain-rate tensor* of a vector field $\mathbf{v} : \Gamma \rightarrow \mathbb{R}^3$ is computed as $\boldsymbol{\epsilon}_\Gamma(\mathbf{v}) = \mathbf{P}\boldsymbol{\epsilon}(\mathbf{v}^e)\mathbf{P}$. We conclude the preliminary definitions with the *time derivative* ∂_t and the *material derivative* $D_t(\bullet, \mathbf{v}) = \partial_t(\bullet) + \mathbf{v} \cdot \nabla(\bullet)$.

2.2. Problem statement. Our model problem is an extension of a classical mathematical model for two-phase flow with interfacial tension and viscosity [84]. It consists of an active Boussinesq-Scriiven surface fluid that encloses a passive viscous bulk fluid, where a diffusing molecular species regulates the driving forces on the surface and creates a mechanochemical feedback.

We focus on cellular processes such as cell division, migration or polarisation, where time scales are long enough to assume the rheology of the cortex is viscous [11, 12]. On the other hand, small characteristic length scales in living cells typically lead to low Reynolds and Womersley numbers ($\operatorname{Re}, \ll 1$, $\alpha \ll 1$) [16, 85]. Therefore, neglecting gravitational and inertial forces, balance of momentum on the surface reads

$$\operatorname{div}_\Gamma \mathbf{N}_\Gamma - \mathbf{f}^{\text{ext}} = \mathbf{0}, \quad \text{on } \Gamma, \quad (2)$$

where \mathbf{N}_Γ and \mathbf{f}^{ext} denote the surface stress tensor and the external forces per unit area. The surface (or membrane) stress tensor \mathbf{N}_Γ obeys, generically, variants of the Boussinesq-Scriiven constitutive law

$$\mathbf{N}_\Gamma = 2\mu_\Gamma \boldsymbol{\epsilon}_\Gamma(\mathbf{U}) + (\lambda_\Gamma - \mu_\Gamma)(\operatorname{div}_\Gamma \mathbf{U})\mathbf{P} + \mathbf{N}_\Gamma^{\text{act}}, \quad \text{on } \Gamma, \quad (3)$$

where \mathbf{U} denotes the surface velocity field. The first two terms yield the stress resultant owing to viscous effects, with $\mu_\Gamma > 0$ and $\lambda_\Gamma > \mu_\Gamma$ the surface shear and dilational viscosities. The last term $\mathbf{N}_\Gamma^{\text{act}}$ is the contribution from active surface tension. We will show later that this term is the driving force of the system. For the cortex, which is a thin shell of actomyosin material, a dimensional reduction of the 3D constitutive equations to 2D membrane stress resultants yields the relation $\lambda_\Gamma = 3\mu_\Gamma$ under the assumption of cortical incompressibility [17]. Additionally, both viscous and active bending contributions have been shown numerically to be negligible [17]. We further neglect mechanical contributions from the plasma membrane—specifically, the lipid bilayer’s tangential fluidity and transverse elasticity—including its bending rigidity and passive surface tension. These effects are typically one to several orders of magnitude smaller than the surface stresses generated by the cortex [86, 87] and are therefore assumed to have minimal impact on the system’s mechanical behavior.

Equations (2) and (3) imply the key assumption of the cortex being a nonmaterial interface. This means that balance of mass is trivial [88], albeit the surface divergence $\operatorname{div}_\Gamma \mathbf{U}$ does not generally vanish. If assuming a material interface, balance of mass can be accounted for in terms of cortical density [48] or thickness [12]. In this case, material turnover also contributes to the active tension [17].

The bulk phase is described as an incompressible Stokes fluid, thereby balance of momentum and mass on the bulk reads

$$\begin{aligned} \operatorname{div}(\boldsymbol{\sigma}(\mathbf{u}, p)) &= \mathbf{0}, & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0, & \text{in } \Omega, \end{aligned} \quad (4)$$

where \mathbf{u} and p denote the bulk velocity and pressure fields. The bulk stress tensor $\boldsymbol{\sigma}$ is given by

$$\boldsymbol{\sigma}(\mathbf{u}, p) = 2\mu_\Omega \boldsymbol{\epsilon}(\mathbf{u}) - p\mathbf{Id}, \quad \text{in } \Omega. \quad (5)$$

Supposing no-slip between the surface and bulk fluids on the surface simplifies the jump conditions to

$$\begin{aligned} \mathbf{u} &= \mathbf{U}, & \text{on } \Gamma, \\ \boldsymbol{\sigma}(\mathbf{u}, p)\mathbf{n}_\Gamma &= \mathbf{f}^{\text{ext}}, & \text{in } \Omega, \end{aligned} \quad (6)$$

such that the surface is subject to the traction force from the passive fluid. At this point, it becomes clear that, due to the mechanical coupling between the surface and bulk fluids, surface flows and deformations generated by active tension set the passive bulk fluid into motion.

The equations for the coupled surface-bulk viscous flows are combined with the equations for the surface transport of the molecule regulating active stress:

$$D_t(C, \mathbf{U}) + C(\operatorname{div}_\Gamma \mathbf{U}) - D_\Gamma \Delta_\Gamma C = k_{\text{on}} \bar{c}_\Omega - k_{\text{off}} C, \quad \text{on } \Gamma. \quad (7)$$

Here, C denotes area concentration of the molecular species. Equation (7) accounts for advection (in the material derivative D_t), concentration changes due to local expansion or contraction of the surface (second term) and surface diffusion with constant diffusivity D_Γ . The right-hand side describes the exchange of molecules between the surface and bulk fluids, with attachment rate k_{on} from the bulk into the surface and detachment rate k_{off} from the surface to the bulk concentration \bar{c}_Ω . For simplicity, \bar{c}_Ω is assumed homogeneous and constant (the cortex is supposed to be in contact with a chemical bath).

We can now define the active surface tension $\mathbf{N}_\Gamma^{\text{act}}$ of Equation (3), which completes the mechanochemical coupling between the diffusing species and the surface-bulk flows:

$$\mathbf{N}_\Gamma^{\text{act}}(C) = \xi f(C, c_{\text{eq}}) \mathbf{P}, \quad \text{in } \Gamma, \quad \text{where } f(C, c_{\text{eq}}) = \frac{2C^2}{C_{\text{eq}}^2 + C^2}. \quad (8)$$

Equation (8) models active tension as a monotonously increasing and saturating Hill function f of C , scaled by a coefficient ξ , measuring myosin motor activity. The constant $C_{\text{eq}} = (k_{\text{on}}/k_{\text{off}})\bar{c}_\Omega$ represents the surface concentration at equilibrium of surface/bulk protein exchange.

As in [51, 89], it is interesting to decompose the force term of the (active) tension in Equation (2) in the local coordinate system of the surface as

$$\operatorname{div}_\Gamma \mathbf{N}_\Gamma^{\text{act}} = \xi f'(C, c_{\text{eq}}) \nabla_\Gamma C - \xi f(C, c_{\text{eq}}) H \mathbf{n}_\Gamma, \quad (9)$$

where H is the (doubled) mean curvature of the surface. The effects of inhomogeneous active tension are apparent in Equation (9): The first term isolates the Marangoni effect, acting as a tangential force from regions of low to high surface concentration. The second term captures the normal forces driving the dynamic changes of surface shape, which balance the bulk pressure jump in accordance with Laplace's law deriving from Equation (6).

To conclude, we adimensionalise the governing equations with the initial cell radius R , the diffusion time $\tau_D = R^2/D_\Gamma$ and the concentration at equilibrium C_{eq} as characteristic scales. Furthermore, we assume here that $\lambda_\Gamma = \mu_\Gamma$, that is, surface shear and dilational viscosities coincide. This leads to the nondimensional system:

Surface flow and molecular transport

$$\operatorname{div}_\Gamma [2\boldsymbol{\varepsilon}_\Gamma(\mathbf{U}) + Pef(C, 1)\mathbf{P}] - \frac{2R}{L_\eta} \boldsymbol{\varepsilon}(\mathbf{u}) \mathbf{n}_\Gamma + p \mathbf{n}_\Gamma = \mathbf{0}, \quad \text{for } t > 0, \text{ on } \Gamma(t), \quad (10)$$

$$\partial_t C + \mathbf{U} \cdot \nabla_\Gamma C + C(\operatorname{div}_\Gamma \mathbf{U}) - \Delta_\Gamma C + \tau_D k_{\text{off}}(C - 1) = 0, \quad \text{for } t > 0, \text{ on } \Gamma(t), \quad (11)$$

Bulk viscous flow

$$\frac{2R}{L_\eta} \operatorname{div}(\boldsymbol{\varepsilon}(\mathbf{u})) - \nabla p = \mathbf{0}, \quad \text{for } t > 0, \text{ in } \Omega(t), \quad (12)$$

$$\operatorname{div} \mathbf{u} = 0, \quad \text{for } t > 0, \text{ in } \Omega(t), \quad (13)$$

$$\mathbf{u} = \mathbf{U}, \quad \text{for } t > 0, \text{ on } \Gamma(t), \quad (14)$$

Surface evolution

$$\partial_t \Gamma - (\mathbf{U} \cdot \mathbf{n}_\Gamma) \mathbf{n}_\Gamma = 0, \quad \text{for } t > 0, \text{ on } \Gamma(t), \quad (15)$$

Initial conditions

$$C(\mathbf{x}, 0) = C_0(\mathbf{x}), \quad \text{for } \mathbf{x} \text{ in } \Gamma(0), \quad (16)$$

$$\Gamma(0) = \Gamma_0. \quad (17)$$

Note that the system of Equations (10)-(17) is closed by a law for the evolution of the surface, which is dictated by the normal component of the surface velocities. Moreover, in order to uniquely determine the pressure, we additionally require that $\int_{\Omega} p = 0$.

The complete system is characterised by three nondimensional parameters: (1) A Pélet-like number $Pe = \xi R^2 / \lambda_{\Gamma} D_{\Gamma}$, comparing active to diffusive transport; (2) a ratio L_{η}/R , comparing the cell size R with a hydrodynamic length scale $L_{\eta} = \lambda_{\Gamma}/\mu_{\Omega}$ that relates surface to bulk viscosity; and (3) $\tau_D k_{\text{off}}$ comparing the time for surface concentration homogenization via diffusion and attachment/detachment kinematics.

3. DISCRETE FORMULATION

In this section, we describe an *unfitted* FE approximation of Problem (10)-(17). Here, unfitted implies, in a broad sense, that the discretisation is defined in a fixed computational grid and uses a sharp representation of the evolving geometry. We proceed as follows. We start with a typical setup for *fixed-grid* analysis, based on a level-set *sharp-interface* representation. We approximate the continuous model with a partitioned scheme outlined in Figure 4. It is composed of four building blocks for each subproblem: (a) bulk flows –Equations (12)-(14)–, (b) surface flows –Equation (10)–, (c) surface molecular transport –Equation (11)– and (d) surface evolution –Equation (15)–. We describe the full solving strategy in a bottom-to-top fashion: First, we formulate stable and optimal FE discretisations in space for the three PDE (sub)systems. Next, we describe the update of the surface shape. We conclude with the coupling of the individual blocks. For readability, we initially assume the shape of Ω (thus Γ) does not vary in time, until dealing with shape dynamics in Section 3.3.

3.1. Embedded geometry setup. Mesh and geometry configuration are illustrated in Figure 2(c). Let \mathcal{T}_h be a quasi-uniform *background grid* in \mathbb{R}^3 , composed of either n -simplices or n -cubes and with a characteristic mesh size $h = \max_{T \in \mathcal{T}_h} \{\text{diam}(T)\}$. We assume that \mathcal{T}_h meshes a bounding-box \mathcal{B} of Ω , covering but not necessarily fitting to the boundary Γ . To circumvent minor technical details, we require that \mathcal{T}_h is fine enough to resolve the curvature of Γ . A precise statement of this assumption needs further formalism that we skip here, for the sake of conciseness; it can be usually found in specialised works such as [90, 91].

The sharp-interface representation is carried out with the level-set method [92]. Accordingly, the problem geometry is implicitly defined as the zero isosurface of a smooth *level-set function* $\phi(\mathbf{x}) : \mathbb{R}^3 \rightarrow \mathbb{R}$, that verifies $\Gamma = \{\mathbf{x} : \phi(\mathbf{x}) = 0\}$ and $\Omega = \{\mathbf{x} : \phi(\mathbf{x}) < 0\}$, see Figure 2(d). The level-set method is the simplest way to track surfaces without an explicit parametrisation. It is also a reasonable choice in our transient context, because surface dynamics are driven by the mechanics of the cortex –Equations (15)-(17). Therefore, in practice, we will only need to define the level set function ϕ^0 of the initial cell configuration Γ^0 , a simple sphere or spheroid, in most situations.

Our main idea to build the FE approximation amounts to *extend* the discrete problem from the region of interest to a subset of the cells in the fixed-grid. We illustrate the procedure for the bulk problem in Figures 3(a) and 3(b). The first step is to find all grid cells that intersect Ω , this gives the subset $\mathcal{T}_h^{\Omega} = \{T \in \mathcal{T}_h : T \cap \Omega \neq \emptyset\}$, referred to as the Ω -active mesh. Next, we build the FE discretisation in \mathcal{T}_h^{Ω} . For instance, if we consider standard linear FE spaces, the shape functions and degrees of freedom (DOFs) will be associated to the vertices of \mathcal{T}_h^{Ω} . The FE solution of the problem will thus be defined in \mathcal{T}_h^{Ω} and its restriction to Ω yields the sought-after approximation of the continuous problem. We proceed analogously for the surface problems, where the subset $\mathcal{T}_h^{\Gamma} = \{T \in \mathcal{T}_h : T \cap \Gamma \neq \emptyset\} \subseteq \mathcal{T}_h^{\Omega}$ is referred to as the Γ -active or *cut mesh*.

We readily see that discrete extensions provide a route to massively simplify the mesh generation step of FE problems in general, by enabling their resolution on a fixed grid. However, this comes at a high cost. Unfitted methods suffer from three main drawbacks: (1) numerical integration on cut cells requires dedicated procedures; (2) essential (e.g. Dirichlet) boundary conditions must be weakly imposed; and (3) naive discrete extensions often lead to (almost) singular linear systems, the so-called *small cut-cell problem*. All these issues and possible remedies have been extensively reviewed in, e.g., [71, 93].

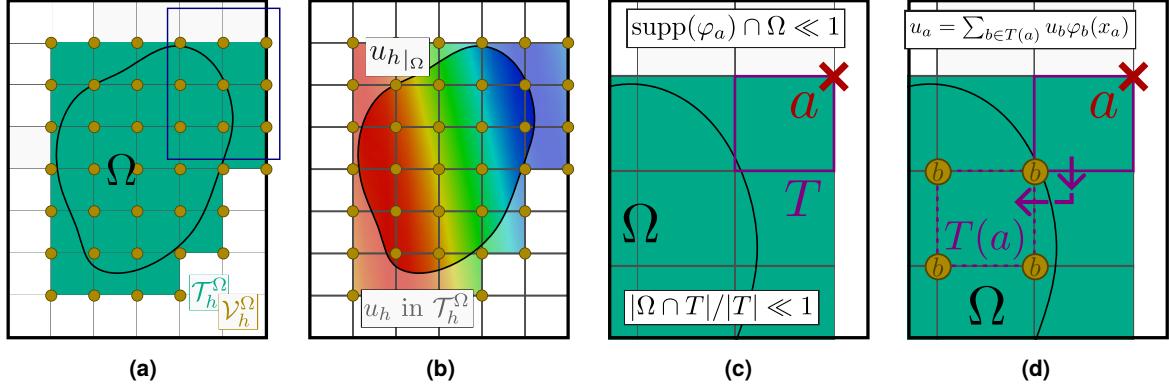


Figure 3. Illustration of an unfitted FE approximation for a bulk PDE in Ω . **(a)** The FE discretisation is built on the Ω -active mesh T_h^Ω formed by the grid cells intersecting Ω . The simplest scenario is to approximate the problem with a linear Lagrangian FE space V_h^Ω , where the shape functions and degrees of freedom (in yellow circles) are associated to the vertices of T_h^Ω . **(b)** The FE solution u_h of the problem is found in T_h^Ω and its restriction $u_h|_\Omega$ is the approximation to the continuous problem. **(c)** The small cut cell problem. Illustration of a badly cut cell T with a very small cut portion in Ω . If a is a DOF of V_h^Ω , we see that its support inside Ω is so small that it can lead to ill-conditioning (see Equation (18)). **(d)** A way to fix the small cut cell problem, proposed in, e.g., the Aggregated FEM [67], is to extrapolate the DOF value u_a in terms of the DOF values of an interior cell $T(a)$. Purple arrows indicate the path mapping a to $T(a)$ according to a cell aggregation scheme.

Of all three challenges, the small cut-cell problem is widely viewed as the most critical one, involving the biggest effort for its mitigation or elimination [50, 74]. Let us give an intuitive, yet general idea of the issue. When working with moving boundaries and interfaces, it is often impractical, if not almost impossible, to tune with manual intervention how the grid overlaps the geometry. In particular, we must cope with any type of intersection between the grid cells and the geometry, including arbitrarily small ones, see example in Figure 3(c). In this situation, we may have basis functions with very reduced support inside the geometry. If using a classical FE method, these basis functions lead to rows and columns in the linear system with values close to zero, a clear case of ill-conditioning.

In fact, rigorous mathematical analysis for the bulk problem in a fixed-grid [90] proves that the condition number κ of a standard FE system \mathbf{A} for the Laplace operator satisfies

$$\kappa(\mathbf{A}) \sim \left[\min_{T \in T_h^\Omega} \frac{|\Omega \cap T|}{|T|} \right]^{-2q+1-2/d}, \quad (18)$$

where the base is the (relative) volume of the smallest cut fraction and, in the exponent, q is the order of the FE approximation and d is the space dimension. Compared to the h^{-2} scaling for the body-fitted case, Equation (18) exposes the severity of the small cut-cell problem, especially in high-order methods. We expect similar estimates for surface problems.

3.2. Unfitted FE approximations. Given the above ill-conditioning issues and their special relevance in our context, we discretise Problem (10)-(17) with unfitted FE methods that are not affected by the small cut-cell problem. This means, in particular, that they are geometrically robust as to how the computational grid overlaps the geometry. To reduce the notational burden, *we initially omit the time step superscripts of the viscous flow FE problems*. We refer to Appendix A for the 2D rotationally symmetric version of the 3D discrete problem described in this section.

3.2.1. Bulk viscous flows. Let (\mathbf{u}_h, p_h) and (\mathbf{v}_h, q_h) denote the pairs of trial and test functions of a suitable velocity-pressure FE space. The bilinear forms arising from balance of momentum and mass (Equations (12)-(13)) are given by

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = \frac{2R}{L_\eta} \int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}_h) : \boldsymbol{\varepsilon}(\mathbf{v}_h) \, d\Omega \quad \text{and} \quad b_h(\mathbf{u}_h, p_h) = - \int_\Omega p_h (\nabla \cdot \mathbf{u}_h) \, d\Omega.$$

As there are no mesh nodes on the boundary Γ , it is not straightforward to enforce the Dirichlet condition of Equation (14) in the usual *strong* sense, that is, by prescribing the value of the condition at the boundary

nodes. For this reason, we impose it in a *weak* sense, that is, by augmenting the variational formulation with additional terms [94]. Usage of the widely employed Nitsche method [95, 96] gives rise to the forms

$$i_h(\mathbf{u}_h, p_h, \mathbf{v}_h, q_h) = \int_{\Gamma} \frac{\alpha}{h} \mathbf{u}_h \cdot \mathbf{v}_h - \left[\frac{2R}{L_\eta} \boldsymbol{\epsilon}(\mathbf{u}_h) \mathbf{n}_\Gamma - p_h \mathbf{n}_\Gamma \right] \cdot \mathbf{v}_h - \left[\frac{2R}{L_\eta} \boldsymbol{\epsilon}(\mathbf{v}_h) \mathbf{n}_\Gamma - q_h \mathbf{n}_\Gamma \right] \cdot \mathbf{u}_h \, d\Gamma$$

and $j_h(\mathbf{v}_h, q_h; \mathbf{U}_h) = \int_{\Gamma} \frac{\alpha}{h} \mathbf{U}_h \cdot \mathbf{v}_h - \left[\frac{2R}{L_\eta} \boldsymbol{\epsilon}(\mathbf{v}_h) \mathbf{n}_\Gamma - q_h \mathbf{n}_\Gamma \right] \cdot \mathbf{U}_h \, d\Gamma,$

where $\alpha > 0$ is a large-enough coefficient to ensure coercivity of the weak form and \mathbf{U}_h represents the discrete velocity at the surface, thereby marking the coupling with the FE problem for the surface flow.

In order to state the discrete problem, it remains to choose the FE space for the velocity-pressure pair (\mathbf{u}_h, p_h) . Our goal is to leverage the Lagrangian FE space \mathcal{V}_h^Ω in \mathcal{T}_h^Ω formed by the pair $Q_2 \times P_1$ of (element-wise) continuous quadratic velocities and discontinuous linear pressures (restricted to zero mean global pressures). If we were in the body-fitted case, \mathcal{V}_h^Ω would be a classical choice, endowed with a discrete inf-sup stability condition [97]. Unfortunately, \mathcal{V}_h^Ω does not preserve this key stability property in the unfitted case and, in presence of small cut elements, suffers from the ill-conditioning issues explained in Section 3.1.

A natural approach to avert the numerical instabilities is to eliminate from the linear system all DOFs linked to basis functions with too small support inside Ω , subsequently referred to as *ill-posed* DOFs. This can be achieved, among others, with the *Aggregated Finite Element Method* (AgFEM) [67]. The main idea of AgFEM is to *extrapolate* the values of ill-posed DOFs in terms of the values of DOFs sitting on a cell with large-enough intersection with Ω . For simplicity, we directly assume this cell is Ω -*interior*, that is, strictly inside Ω . The extrapolation amounts to enforce linear DOF constraints of the form

$$u_a = \sum_{b \in T(a)} u_b \varphi_b(x_a), \quad (19)$$

in which u_a represents the value of a generic ill-posed DOF a , and b iterates over the DOF values u_b and shape functions φ_b of an interior cell $T(a) \in \mathcal{T}_h^\Omega$. An example of this type of constraint is illustrated in Figure 3(d).

We readily see that to define and evaluate the above constraints in \mathcal{V}_h^Ω , we need to detect and gather all its ill-posed DOFs and assign to each one of them a suitable interior cell. For this, we resort to an automatic and easy-to-parallelise *cell aggregation* scheme [98], which derives this map from cell paths linking badly cut cells to interior cells. Note that, for good numerical accuracy and stability, cell paths should be as short as possible and connected through interior or cut facets [67, Lemma 2.2].

Thanks to cell aggregation, we can apply Equation (19) to all ill-posed DOFs, pruning all basis functions associated with badly cut cells. This results in a restriction of the original FE space \mathcal{V}_h^Ω to the so-called *aggregated FE space* $\mathcal{V}_{h,\text{ag}}^\Omega$. In this way, aggregated FE spaces fix the ill-conditioning issues of vanilla unfitted FEM; they are stable, regardless of the configuration of the cut cells, and retain the optimal approximation properties and condition number bounds of body-fitted FE methods. We refer to, e.g., [67, 99] for fundamental theoretical analysis and results about AgFEM.

Our particular realisation of $\mathcal{V}_{h,\text{ag}}^\Omega$ for the Stokes problem follows the robust and stable mixed formulation introduced in [77], opting for a serendipity extension and no pressure stabilisation. Hence, the discretisation of the bulk problem given by Equations (12)-(14) finally reads: *Find* $(\mathbf{u}_h, p_h) \in \mathcal{V}_{h,\text{ag}}^\Omega$ *such that*

$$a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) + b_h(\mathbf{u}_h, q_h) + i_h(\mathbf{u}_h, p_h, \mathbf{v}_h, q_h) = j_h(\mathbf{v}_h, q_h; \mathbf{U}_h), \quad \forall (\mathbf{v}_h, q_h) \in \mathcal{V}_{h,\text{ag}}^\Omega. \quad (20)$$

3.2.2. Surface viscous flows. Let now \mathbf{U}_h and \mathbf{V}_h denote the trial and test functions of a vector-valued FE space, that will be later defined. From balance of momentum (Equation (10)), the bilinear form is

$$A_h^U(\mathbf{U}_h, \mathbf{V}_h) = \int_{\Gamma} 2 \boldsymbol{\epsilon}_\Gamma(\mathbf{U}_h) : \boldsymbol{\epsilon}_\Gamma(\mathbf{V}_h) + \rho \mathbf{U}_h \cdot \mathbf{V}_h \, d\Gamma$$

with a very small friction term premultiplied by $\rho \ll 1$ required, when Γ is a closed surface, to eliminate Killing vector fields [100]. The right-hand side terms

$$\begin{aligned} F_h^{\text{act}}(\mathbf{V}_h; C_h) &= \int_{\Gamma} \operatorname{div}_{\Gamma}(Pef(C_h, 1)\mathbf{P}) \cdot \mathbf{V}_h \, d\Gamma \\ F_h^{\text{cyt}}(\mathbf{V}_h; \mathbf{u}_h, p_h) &= - \int_{\Gamma} \left[\frac{2R}{L_{\eta}} \boldsymbol{\varepsilon}(\mathbf{u}_h) \mathbf{n}_{\Gamma} - p_h \mathbf{n}_{\Gamma} \right] \cdot \mathbf{V}_h \, d\Gamma \end{aligned}$$

correspond to the active F_h^{act} and cytoplasm reaction F_h^{cyt} forces. We observe that the former couples the problem to the molecular surface transport via C_h and the latter to the bulk flows via (\mathbf{u}_h, p_h) .

For surface problems, robustness to cut location is attained by perturbing the problem with a stabilisation term, in the spirit of trace FE methods [66]. In particular, we adopt the *normal derivative volume stabilisation*, since it leads to optimal discretisation errors and condition number bounds for the vector Laplacian problem on *fixed* surfaces [78, 101]:

$$S_h^U(\mathbf{U}_h, \mathbf{V}_h) = \int_{\mathcal{N}_h^{\Gamma}} \frac{\beta}{h} \boldsymbol{\varepsilon}_{\Gamma}(\mathbf{U}_h) \mathbf{n}_{\Gamma} \cdot \boldsymbol{\varepsilon}_{\Gamma}(\mathbf{V}_h) \mathbf{n}_{\Gamma} \, d\Omega,$$

where $\mathcal{N}_h^{\Gamma} = \bigcup_{T \in \mathcal{T}_h^{\Gamma}} \bar{T}$ is the domain covered by \mathcal{T}_h^{Γ} and $\beta > 0$ is a large-enough coefficient to ensure well-posedness. We observe that $S_h^U(\mathbf{U}_h, \mathbf{V}_h)$ implies integrating the basis functions in the whole cells cut by Γ , which necessarily eliminates the instabilities from small intersections of Γ with $T \in \mathcal{T}_h^{\Gamma}$. On the other hand, the effect of this term is to extend the solution from Γ to \mathcal{N}_h^{Γ} , by *weakly* imposing that the value on the surface is constant through the normal \mathbf{n}_{Γ} . In doing so, it crucially fixes another key problem of unfitted FEM for surface PDEs: the fact that bulk basis functions defined in \mathcal{T}_h^{Γ} restricted on Γ are, in general, linearly dependent. We refer to, e.g., [102, Figure 6] for a clear illustration of the linear dependency in bilinear quadrilateral cells.

The discretisation of the surface flow problem is completed with the choice of a suitable FE space. In this case, we take a standard vector-valued linear Lagrangian FE space \mathcal{V}_h^{Γ} . In order to segregate the surface and bulk problems for the iterative scheme, we must further restrict the rigid body modes of the surface and account for the compatibility with bulk incompressibility (13). Hence, the problem is approximated in the reduced FE space

$$\mathcal{V}_{h,*}^{\Gamma} = \left\{ \mathbf{v}_h \in \mathcal{V}_h^{\Gamma} : \int_{\Gamma} \mathbf{v}_h = 0, \int_{\Gamma} \nabla \times \mathbf{v}_h = 0, \int_{\Gamma} \mathbf{v}_h \cdot \mathbf{n}_{\Gamma} = 0 \right\}$$

and the discrete surface flow problem given by Equation (10) reads: *Find $\mathbf{U}_h \in \mathcal{V}_{h,*}^{\Gamma}$ such that*

$$A_h^U(\mathbf{U}_h, \mathbf{V}_h) + S_h^U(\mathbf{U}_h, \mathbf{V}_h) = F_h^{\text{act}}(\mathbf{V}_h; C_h) + F_h^{\text{cyt}}(\mathbf{V}_h; \mathbf{u}_h, p_h), \quad \forall \mathbf{V}_h \in \mathcal{V}_{h,*}^{\Gamma}. \quad (21)$$

3.2.3. Surface molecular transport. For the FE formulation of the surface transport problem given by Equations (11) and (16), we introduce minimal notation for the discretisation in time. Given the time domain $[0, T]$, we assume a uniform partition with time step $\Delta t = T/N$ in N time intervals $I_n = [t^{n-1}, t^n]$, $t^n = n\Delta t$, $n = 1, \dots, N$. In what follows, we use the superscript n to denote evaluation at t^n , e.g., $\mathbf{U}_h^n = \mathbf{U}_h(t^n)$ or $\Gamma^n := \Gamma(t^n)$.

We derive the discrete problem for an implicit Euler scheme in time and a linear scalar-valued (transient) Lagrangian FE space \mathcal{W}_h^{Γ} . Letting C_h^n and D_h^n represent trial and test functions of \mathcal{W}_h^{Γ} , we introduce the forms

$$M_h(C_h^n, D_h^n) = \int_{\Gamma^n} C_h^n D_h^n \, d\Gamma, \quad A_h^C(C_h^n, D_h^n) = \int_{\Gamma^n} \nabla C_h^n \cdot \nabla D_h^n \, d\Gamma, \quad \text{and} \quad L_h(D_h^n) = \int_{\Gamma^n} D_h^n \, d\Gamma,$$

representing unit mass, diffusion and source terms of the problem. Like the previous case, we stabilise the problem with the form

$$S_h^C(C_h^n, D_h^n) = \int_{\mathcal{N}_h^{\Gamma^n}} \frac{\gamma}{h} (\nabla C_h^n \cdot \mathbf{n}_{\Gamma})(\nabla D_h^n \cdot \mathbf{n}_{\Gamma}) \, d\Omega,$$

where $\gamma > 0$ is a large-enough penalty coefficient. The particular choice of $S_h^C(C_h^n, D_h^n)$ is on the grounds of numerical stability and optimal error estimates proven in, e.g., [103]. Finally, the coupling term with

\mathbf{U}_h^n reads

$$B_h(C_h^n, D_h^n; \mathbf{U}_h^n) = \int_{\Gamma^n} (\mathbf{U}_h \cdot \nabla_\Gamma C_h^n) D_h^n + (\operatorname{div}_\Gamma \mathbf{U}_h^n) C_h^n D_h^n \, d\Gamma,$$

where the fact that C_h^n is weakly extended along the normal allows us to consider the surface gradient in the convective term of the material derivative. With these ingredients, the discrete surface transport problem reads: *For $C_h^0 \in \mathcal{W}_h^\Gamma$ approximating the initial condition (16), find $C_h^n \in \mathcal{W}_h^\Gamma$, $n = 1, \dots, N$ such that*

$$\begin{aligned} \left(\frac{1}{\Delta t} + \tau_D k_{\text{off}} \right) M_h(C_h^n, D_h^n) + B_h(C_h^n, D_h^n; \mathbf{U}_h^n) + A_h^C(C_h^n, D_h^n) + S_h^C(C_h^n, D_h^n) = \\ \frac{1}{\Delta t} M_h(C_h^{n-1}, D_h^n) + \tau_D k_{\text{off}} L_h(D_h^n), \quad \forall D_h^n \in \mathcal{W}_h^\Gamma. \end{aligned} \quad (22)$$

We observe that the above formulation conserves mass only approximately, due to the discretisation error in time and evaluating C_h^{n-1} on Γ^n (see end of Section 3.4). However, in the numerical experiments of Section 4, we notice that the error in mass conservation is reasonably controlled by the small time increments required to capture the problem nonlinearities. In other situations, we could require a conservative scheme as in, e.g. [104, 105]. On the other hand, we use a SUPG variant [106] to stabilise high-activity cases ($Pe \gg 1$) that lead to advection-dominated transport.

3.2.4. Numerical integration. Except for the stabilisation terms, all integrals involved in the above weak formulations are defined in either Ω or Γ . Clearly, their evaluation on cut cells cannot be achieved with standard quadrature rules, because we need to restrict the integration to the portions $\Omega \cap T$ or $\Gamma \cap T$, $T \in \mathcal{T}_h^\Gamma$. There are many strategies to adapt the local integration on cut cells, reviewed in, e.g., [71, 79]; they attempt, in most cases, to strike a good balance between accuracy and computational cost. In this work, we adopt the high-order quadrature rules introduced in [79] for curved surfaces and volumes implicitly defined by the level sets of multivariate polynomials. Underpinning our choice is the fact that shape dynamics are governed by the discrete FE unknown \mathbf{U}_h , which calls for the convenient representation of the evolving surface with piecewise polynomial level-sets (see next section).

3.3. Surface evolution. We outline now the evolving level-set method for shape evolution, as governed by Equations (15) and (17). First, we use the level-set description of the interface $\Gamma(t)$, introduced in Section 3.1, to recast the system into the classic initial value formulation [92]

$$\begin{aligned} \partial_t \phi + U_\perp^e \|\nabla \phi\| = 0, & \quad \text{in } \mathcal{B} \times [0, T], \\ \phi(\mathbf{x}, 0) = \phi_0(\mathbf{x}), & \quad \text{in } \mathcal{B} \times \{0\}. \end{aligned} \quad (23)$$

The above system describes the (pure) advection of the level-set field ϕ in the bounding-box \mathcal{B} with speed function U_\perp^e , which denotes an appropriate extension of the surface normal velocities $\mathbf{U} \cdot \mathbf{n}_\Gamma$ from Γ to \mathcal{B} .

In order to accommodate our numerical integration strategy, see Section 3.2.4, we consider a FE discretisation of the level-set function ϕ in \mathcal{B} . According to this, we assume $\mathcal{V}_h^\mathcal{B}$ is a scalar-valued Lagrangian FE space of order two, the same as the bulk velocity approximation. Considering an explicit Euler scheme in time leads to the update rule: *For $\phi^0 \in \mathcal{V}_h^\mathcal{B}$ approximating the initial condition ϕ_0 , find $\phi^n \in \mathcal{V}_h^\mathcal{B}$, $n = 1, \dots, N$ such that*

$$\phi^n = \phi^{n-1} - \Delta t U_\perp^{e,n-1} \|\nabla \phi^{n-1}\| = 0, \quad \phi^n \in \mathcal{V}_h^\mathcal{B}, \quad n \geq 1. \quad (24)$$

We observe that, at each time step, the rule above only involves updating the nodal values of the discrete level-set function. In order to evaluate $U_\perp^{e,n-1}$, we compute the value of $\mathbf{U} \cdot \mathbf{n}_\Gamma$ at the closest point projection to Γ of each node of $\mathcal{V}_h^\mathcal{B}$. To this end, we resort to the high-order algorithms devised in [80]. In our numerical tests, we do not require to stabilise the scheme with, e.g., upwind methods, because the time steps are rather small. We further leverage closest point projections to represent ϕ^n as nodal projections of signed distance functions; in this way, we can assume $\|\nabla \phi^{n-1}\| \approx 1$.

3.4. Solution strategy. The coupling scheme of the full transient problem is represented in Figure 4. We consider an explicit coupling in time (by substitution) between the surface transport and the coupled surface-bulk flow problem. The latter is also solved with an iterative surface-vs-bulk partitioned scheme. We then update the surface shape with the resulting surface velocities and close the simulation loop to

compute the solution at the following time step. This can be expressed altogether in Algorithm 1, which encodes the main simulation loop.

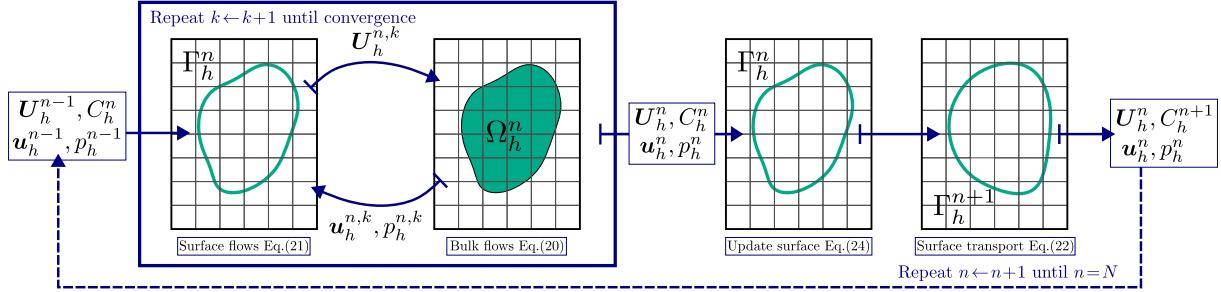


Figure 4. Schematic representation of the coupled iterative scheme described in Algorithm 1.

Algorithm 1 Coupled iterative scheme for the FE approximation of Problem (10)-(17).

```

1: Given  $\mathbf{U}_h^{-1}$ ,  $\mathbf{u}_h^{-1}$ ,  $p_h^{-1}$ ,  $C_h^0$  and  $\phi_h^0$ ,
2: for  $n = 0, \dots, N$  do
3:    $k \leftarrow 0$ 
4:    $\mathbf{U}_h^{n,0}, \mathbf{u}_h^{n,0}, p_h^{n,0} \leftarrow \mathbf{U}_h^{n-1}, \mathbf{u}_h^{n-1}, p_h^{n-1}$ 
5:   repeat
6:      $k \leftarrow k + 1$ 
7:      $\mathbf{U}_h^{n,k} \leftarrow$  Solve (21) with  $\mathbf{u}_h^{n,k-1}, p_h^{n,k-1}$  and  $C_h^n$  in  $\Gamma_h^n = \{\phi_h^n \equiv 0\}$ 
8:      $\mathbf{u}_h^{n,k}, p_h^{n,k} \leftarrow$  Solve (20) with  $\mathbf{U}_h^{n,k}$  in  $\Omega_h^n = \{\phi_h^n < 0\}$ 
9:   until  $\mathbf{U}_h^{n,k} \approx \mathbf{U}_h^{n,k-1}$ ,  $\mathbf{u}_h^{n,k} \approx \mathbf{u}_h^{n,k-1}$  and  $p_h^{n,k} \approx p_h^{n,k-1}$ 
10:   $\mathbf{U}_h^n, \mathbf{u}_h^n, p_h^n \leftarrow \mathbf{U}_h^{n,k}, \mathbf{u}_h^{n,k}, p_h^{n,k}$ 
11:  if  $n < N$  then
12:     $\phi_h^{n+1} \leftarrow$  Update (24) with  $\phi_h^n$  and  $\mathbf{U}_h^n$  in  $\mathcal{B}$ 
13:     $C_h^{n+1} \leftarrow$  Solve (22) with  $\mathbf{U}_h^n$  and  $C_h^n$  in  $\Gamma_h^{n+1} = \{\phi_h^{n+1} \equiv 0\}$ 
14:  end if
15: end for

```

Algorithm 1 exposes the need to carry information between consecutive discrete surfaces. In particular, the bulk unknowns $(\mathbf{u}_h^{n-1}, p_h^{n-1})$, found in Ω_h^{n-1} , have to be evaluated in $\Gamma_h^n = \partial\Omega^n$ (line 7); while the surface unknowns \mathbf{U}_h^n, C_h^n , found in Γ_h^n , have to be evaluated in Γ_h^{n+1} (line 13). In order to ensure this, we follow a similar approach to the one detailed in, e.g., [103] for surface and [107, 108] for bulk problems. The main idea is to extend the discrete problems further away from the cells cut by the current surface/boundary, until they cover a region wide enough to contain the next surface (which is not known). This is achieved by formulating the unfitted surface FE problems in $\mathcal{T}_{h,n}^\Gamma = \{T \in \mathcal{T}_h : T \cap \{|\phi_h^n| < \delta^n\} \neq \emptyset\}$ and the unfitted bulk FE problem in $\mathcal{T}_{h,n}^\Omega = \{T \in \mathcal{T}_h : T \cap \{\phi_h^n < \delta^n\} \neq \emptyset\}$, where $\delta^n \propto \Delta t \|\mathbf{U}_h^{n-1}\|_\infty$ is a width parameter. We refer to the above references for a complete description and discussion of this strategy.

4. NUMERICAL EXPERIMENTS

Numerical implementation of the unfitted FE formulation in Section 3 was carried out using the `Gridap.jl` [109, 110] FE software ecosystem, written in the Julia programming language. The code is available at the `SurfaceBulkViscousFlows` GitHub repository and supplemented with demonstrators of the numerical examples that follow. The unfitted FE tools are provided by `GridapEmbedded.jl` [111]. It interfaces to the C++ `algoim` [112] library, via a Julia wrapper [113], to compute quadratures for domains implicitly defined by multivariate polynomials [79] and closest point projections on implicit surfaces [80]. We solve our linear systems with the parallel sparse direct solver MUMPS [114, 115] available at the PETSc library v3.15.2 [116–118]. Numerical results are postprocessed with ParaView v5.10.0 [119].

In all cases, (1) we discretise the bounding box \mathcal{B} of the problem with Cartesian grids of uniform mesh size; (2) the initial level-set $\phi_0 \in \mathcal{V}_h^\mathcal{B}$ is defined by the closest-point projections of the FE nodes of $\mathcal{V}_h^\mathcal{B}$

on the surface of analysis (typically a sphere); and (3) the stabilisation coefficients are fixed at $\alpha = 20.0$, $\beta = \gamma = 10.0$ (on the conservative side) and $\rho = 10^{-3}$. We include examples with both the 2D rotationally symmetric and 3D versions of the nondimensional system given by Equations (10)-(17).

4.1. Verification examples. We carry out two convergence tests to validate our code implementation. In the first experiment, we consider a steady-state 2D axisymmetric problem in a *fixed geometry with known analytical solution*. To this end, we transform the fluid sphere problem with two bulk phases of [120, Section 4-21] into a surface-bulk viscous flow problem. This leads to a coupled surface-bulk Stokes problem where (for this experiment only) the surface is material and, as a result, obeys the surface incompressibility condition. Using the method of manufactured solutions, we set the data of the problem such that the general solution for the surface and bulk phases, given by the stream functions in polar coordinates, reads

$$\begin{aligned}\psi_\Gamma &= \sin^2 \theta \left(-\frac{1}{2} Br + Cr^2 + \frac{D}{r} \right), & r = 1 \\ \psi_\Omega &= \sin^2 \theta \left(\frac{1}{10} Er^4 + Gr^2 \right), & r < 1\end{aligned}$$

with

$$B = \frac{3}{2} \frac{1 + \frac{2}{3} \sigma}{1 + \sigma}, \quad C = \frac{1}{2}, \quad D = \frac{1}{4} \frac{1}{1 + \sigma}, \quad E = \frac{5}{2} \frac{\sigma}{1 + \sigma}, \quad G = -\frac{1}{4} \frac{\sigma}{1 + \sigma}, \quad \text{and } \sigma = \frac{\mu_\Gamma}{\mu_\Omega}.$$

The bulk phase is discretised with AgFEM, using the inf-sup stable pair $Q_2 \times \mathcal{P}_1$, as described in Section 3.2. In contrast, the surface phase is discretised with a consistent trace FE formulation for the surface Stokes problem [121] using the inf-sup stable pair $Q_2 \times Q_1$. Further details of the implementation can be found in the [code demonstrator](#) of this example. As shown in Figure 5(d) optimal convergence rates of the iterative scheme are observed under uniform mesh refinements.

The second experiment deals with a dynamic surface modelled by the 2D axisymmetric version of Equations (10)-(17) (see Appendix A for the discrete formulation). We consider a bounding-box $\mathcal{B} = [-1.2, 1.2] \times [0.0, 1.2]$ embedding a sphere of radius $R = 1$ centred at the origin. The values of the three nondimensional parameters are $Pe = 30.0$, $L_\eta = 10^4$ and $\tau_D k_{\text{off}} = 10.0$.

The initial density field of the molecular species (see Figure 5(b)) is set to 1.0 everywhere, except on a sextant, where the value is increased to 1.1. This initial bias implies the active tension on the surface is non-homogeneous, which cannot happen in a sphere. As a result, the system is forced into motion to find the equilibrium shape of Figure 5(c). In this process, the Marangoni effect increases the concentration on the sextant, while decreasing it on the other side. Meanwhile, the surface flows generate a reverse fountain flow in the bulk. At steady-state, the system has deformed into a bean-like shape, where the region of higher surface tension is flatter than the opposite side, the body travels at constant speed in the direction of the flows at the axis of symmetry, and the flow field in the comoving frame is tangent to the surface. This swimming mechanism mediated by Marangoni forces has been identified in the spontaneous self-propulsion of active colloids and droplets [122].

Setting $h_0 = 0.08$ and $\Delta t_0 = 0.004$ as the initial mesh size and time step, we study the convergence under simultaneous uniform refinements in space and time, using an explicit Euler time integration scheme until $T = 0.5$. In front of incomplete theoretical results and limited (semi-)analytical examples, we limit our analysis to the *experimental* order of convergence of key quantities at steady-state: maximum concentration C_h^{\max} , maximum surface \mathbf{U}_h^{\max} and bulk \mathbf{u}_h^{\max} velocities, and average travelling speed \bar{U}_h . The *experimental* order of convergence for each monitored quantity \square is computed as

$$\text{eoc}_i(\square) := \frac{\ln |\square_{i-1} - \square_i| - \ln |\square_i - \square_{i+1}|}{\ln h_i - \ln h_{i+1}}, \quad (25)$$

where $\ln h_i - \ln h_{i+1} = 2$ due to imposing uniform mesh refinements. Figure 5(e) reports almost quadratic experimental rates of convergence for the four quantities.

4.2. Self-organised shape emergence. We now study the mechanochemical instabilities at the heart of the system formed by Equations (10)-(17), guiding spontaneous polarisation and pattern formation. Linear stability analysis [81] of the rotationally symmetric model has shown the instabilities are regulated by

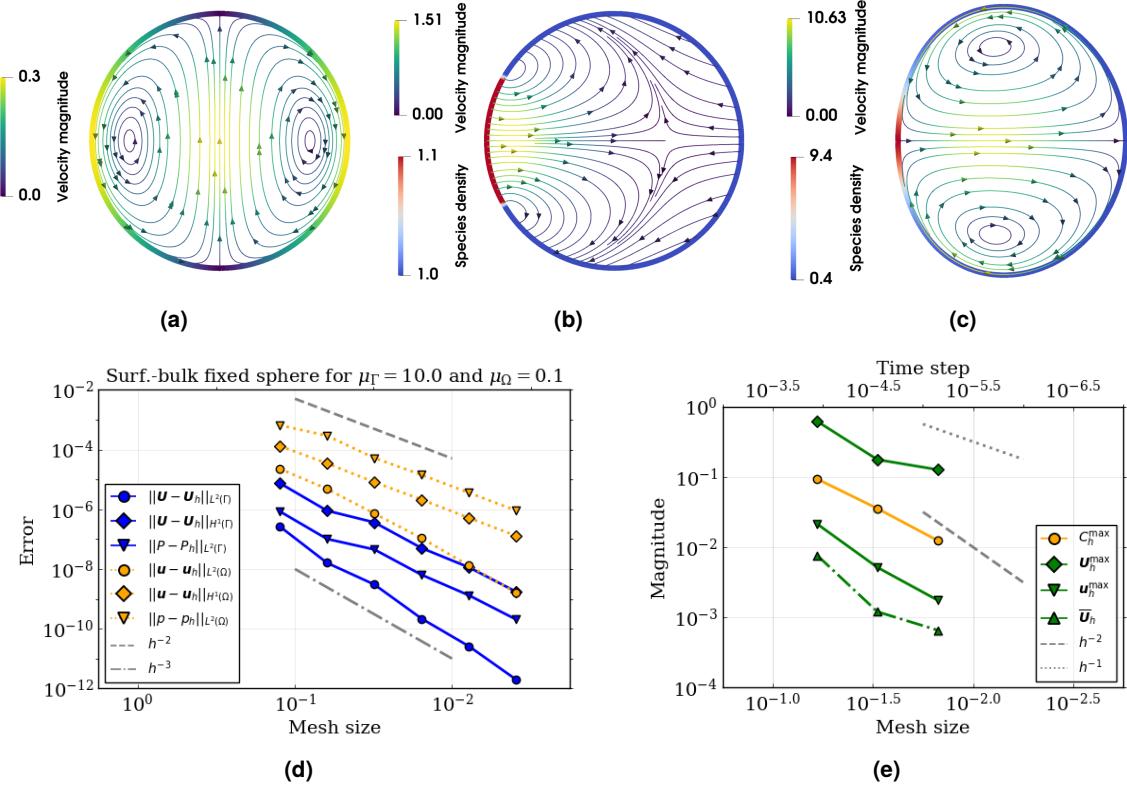


Figure 5. Verification examples. 2D axisymmetric surface-bulk fluid sphere problem adapted from [120, Section 4-21]: (a) Velocity magnitude and streamlines of the steady-state solution. (d) The rate of decay of the approximation error under uniform refinements obeys theoretical estimates [77, 121]. 2D axisymmetric dynamic surface: (b) Initial species concentration and velocity fields. Species concentration at the left sextant is 10% higher than elsewhere, setting the system out of equilibrium. (c) Equilibrium shape and solution in the comoving frame at $T = 3.0$. (e) Decay of the experimental errors with uniform refinements. We observe rates of convergence close to quadratic order for all quantities. Note that the results are reflected across the axis of symmetry for illustration purposes.

the Péclet number Pe , relating active and diffusive surface transport, and the hydrodynamic length L_η , coupling the surface and bulk fluids.

Given an infinitesimal perturbation of a homogeneous concentration on the surface, a large enough Péclet number Pe will give rise to a positive feedback loop, whereby regions of higher concentration will emerge and grow in magnitude by virtue of the Marangoni effect. The system reaches a steady state when active and diffusive transport balance each other out, leading to a stationary shape and concentration field on the surface and a steady flow, following the gradient of concentration.

On the other hand, the hydrodynamic screening length L_η categorises the stationary pattern. Symmetric or ring patterns appear when $L_\eta/R \ll 1$, while asymmetric or polar ones dominate when $L_\eta/R \gg 1$. In contrast to [51, 81], we restrict ourselves to the latter case to reflect the common situation in animal cells; where typical 3D viscosities and geometrical scales of the cortex (thickness) and the cytoplasm (radius) yield rather large hydrodynamic lengths, with $L_\eta/R \approx 10^{3-4}$ [12, 22]. This type of polar instabilities, at large hydrodynamic lengths, have been suggested as a mechanism for sustained unidirectional cell motility in three-dimensional environments [123].

As in the previous example, we adopt the 2D axisymmetric version of Equations (10)-(17) with a bounding-box domain given by $\mathcal{B} = [-1.2, 1.2] \times [0.0, 1.2]$ embedding a unit sphere centred at the origin. The initial concentration field on the surface, assuming $c_{\text{eq}} = 1$, is given by $C_h^0 = 1 + \epsilon_h \in \mathcal{W}_h^\Gamma$, where $\epsilon_h \in \mathcal{W}_h^\Gamma$ represents a random-valued FE function such that $\max\{|\epsilon_h|\} < 10^{-5}$ and $\int_\Gamma \epsilon_h \, d\Gamma = 0$.

Using real spherical harmonics $Y_{lm}(\theta, \varphi)$, linear stability analysis of a homogeneous stationary state leads to a critical Péclet number [81]

$$Pe_{\text{cr}}^l = \frac{1}{c_{\text{eq}} \partial_C f(C, c_{\text{eq}})} \left(1 + \frac{\tau_D k_{\text{off}}}{l(l+1)} \right) \left[l(l+1) + \left((l-1)(l+2) + (1+2l) \frac{R}{L_\eta} \right) \right], \quad (26)$$

where $l \geq 1$ denotes the index of the radial modes. Regardless of the azimuthal mode number m , modes with $l \geq 1$ become unstable at $Pe = Pe_{\text{cr}}^l$. We initially assume $\tau_D k_{\text{off}} = 10.0$. Since $c_{\text{eq}} \partial_C f(C, c_{\text{eq}}) = 1$, the critical Péclet number, according to Equation (26), for the first polar mode is $Pe_{\text{cr}}^1 = 12 + 3/L_\eta \approx 12$.

We use this theoretical value of Pe_{cr}^1 to tune the mesh size and time step for our analysis, in such a way that we correctly capture the stable to nonstable transition. To this end, we compute the first discrete Pearson correlation coefficient r_1 between the surface concentration field C_h and the real spherical harmonic $Y_{10}(\theta)$. For $l \geq 1$, the discrete Pearson correlation coefficient r_l is given by

$$r_l = \frac{\sum_{k=0}^{N_{\text{points}}} (C_k - C_{\text{mean}}) Y_{l0}(\theta_k)}{\sqrt{\sum_{k=0}^{N_{\text{points}}} (C_k - C_{\text{mean}})^2 \sum_{k=0}^{N_{\text{points}}} (Y_{l0}(\theta_k))^2}}, \quad (27)$$

where k indexes the N_{points} quadrature points for the integration of the surface integrals, C_k is the concentration on the k -th point, C_{mean} is the mean concentration on the quadrature points and θ_k is the polar angle of the k -th point.

Figure 6 shows the evolution of r_1 for a few mesh sizes. A value of r_1 close to 1 indicates that the first mode dominates in the harmonic decomposition, implying a polar instability. We observe that taking $h = 0.04$ captures better the stable to unstable transition than $h = 0.08$, while keeping a moderate computational cost. We will thus consider this mesh size for the remaining 2D cases of this subsection. We considered $\Delta t = 5.0 \cdot 10^{-5}$, as in the experiments that follow.

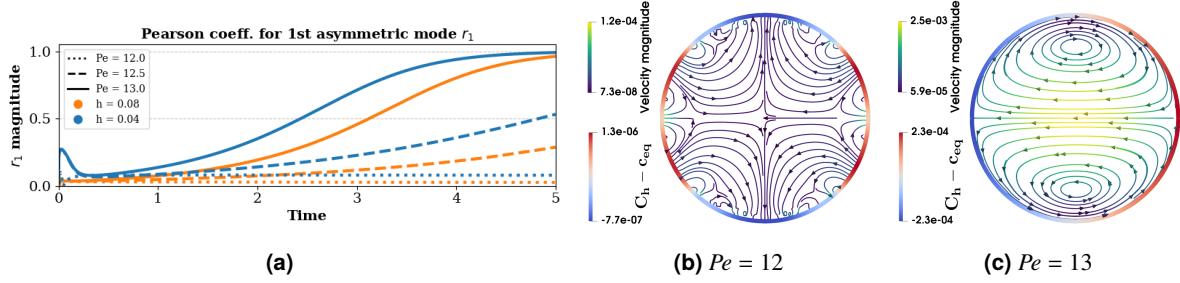


Figure 6. Self-organized shape emergence. Tuning the mesh size for analysis of pattern formation. Before the analysis reported in Figure 7, we found the minimum mesh size needed to correctly capture the stable to unstable transition in the linear regime, as predicted by Equation (26). (a) Setting $h = 0.04$ provides a good balance between computational cost and correctly capturing the theoretically predicted appearance of the first instability mode. (b) At $Pe = 11$ and $h = 0.04$, no instabilities appear and concentration heterogeneities slowly diffuse out. (c) At $Pe = 13$ and $h = 0.04$, we observe a paradigmatic first order polar instability forming.

Figure 7 reports the evolution in time of maximum species concentration C_h^{\max} , travelling speed \bar{U}_h and dominant radial modes r^{\max} for increasing values of Pe and hydrodynamic lengths $L_\eta \in \{10^3, 10^4, 10^5\}$. The dominant mode is identified as the maximum among the first seven discrete Pearson correlation coefficients, i.e., $r^{\max} = \max_{l=0,\dots,6} r_l$. Simulations are run until a quasi-steady state is reached, in which all unknown values remain approximately constant and the flow field in the comoving frame is tangent to the surface. We prescribe $\Delta t = 5.0 \cdot 10^{-5}$, which is required to follow the steep nonlinear transition at large Pe . As expected, all leading modes are asymmetric; increasing Pe reduces the time to enter the unstable regime and promotes the dominance of high order modes. The results also align well with Equation (26), even far from the linear regime, showing almost no sensitivity to the value of the hydrodynamic length L_η when $L_\eta/R \gg 1$. This suggests that the underlying migration mechanism is above all a cortical instability, with little influence of the cytoplasm.

Supplementary experiments with $\tau_D k_{\text{off}} = 0.0$, i.e., no mass exchange between the surface and the bulk, evaluate the mass conservation property of molecular species C on the surface. Figure 8(a) details the evolution in time of the incremental and accumulated relative deviation from the initial surface mass for $Pe \in \{30, 60, 90\}$ and $L_\eta = 10^4$. Despite not using a conservative scheme, the relative error in mass conservation is low enough to ensure it has no influence in the numerical results. We reach a similar conclusion when analysing the error in volume conservation in Figure 8(b).

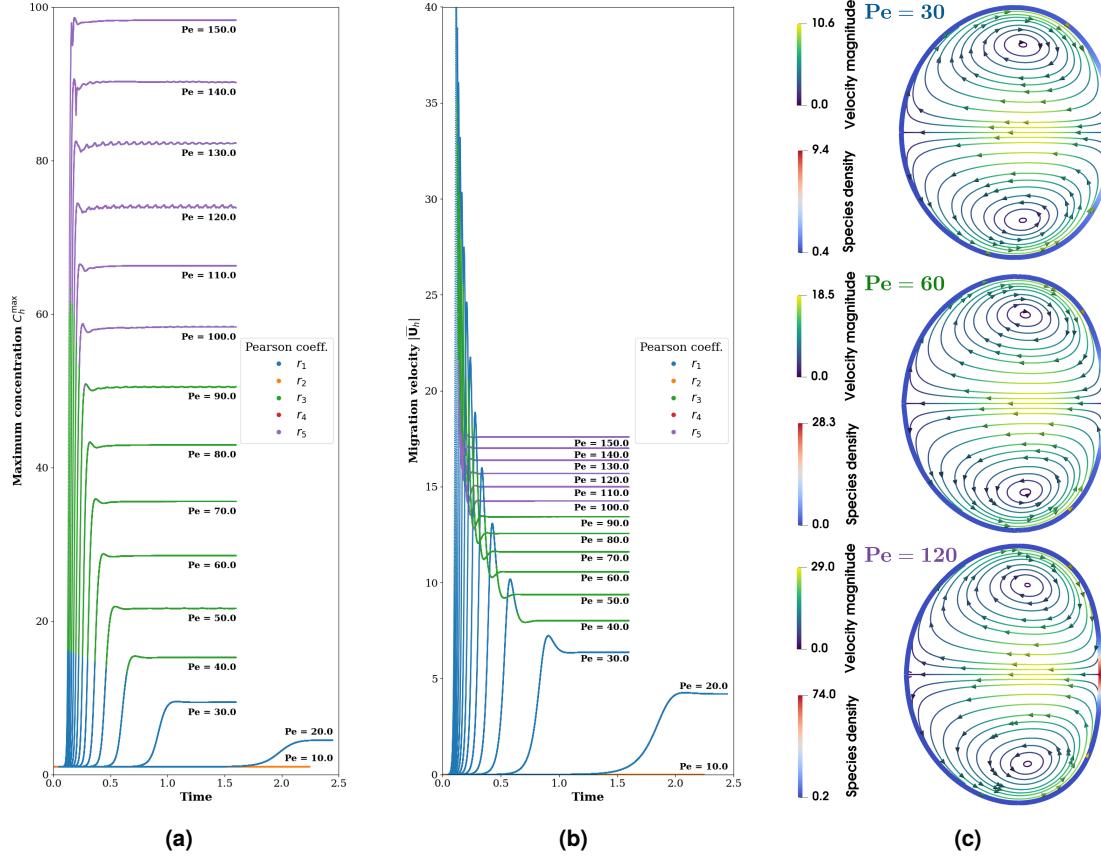


Figure 7. Self-organised shape emergence. Pattern formation at large hydrodynamic lengths. A perturbation of an homogeneous concentration field leads to spontaneous pattern formation at large enough Pe numbers. **(a)** Peak concentration and **(b)** migration velocities for $Pe \in [10, 150]$ and $L_\eta \in \{10^3, 10^4, 10^5\}$. The evolution of the leading mode is coloured along the plotted lines. At large hydrodynamic lengths, we do not observe differences in shapes, flow fields and magnitudes. For this reason, we only plot the results for $L_\eta = 10^4$. **(c)** Steady-state results obtained for leading modes 1-2-3 at, respectively, $Pe = 30, 60, 120$.

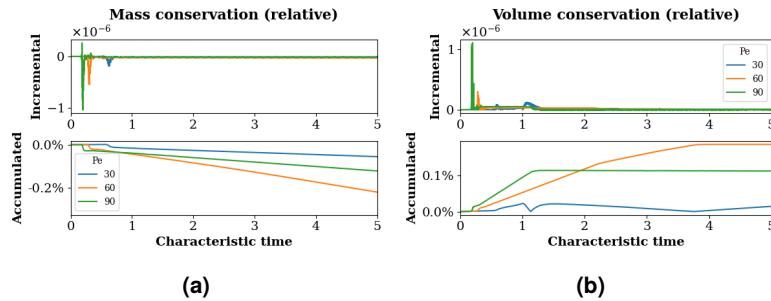


Figure 8. Self-organised shape emergence. Mass and volume conservation errors. Setting $\tau_D k_{\text{off}} = 0.0$, i.e., no mass exchange between the surface and the bulk, we study the evolution of the mass and volume conservation errors for $h = 0.04$, $\Delta t = 5.0 \cdot 10^{-5}$, $Pe \in \{30, 60, 90\}$ and $L_\eta = 10^4$. **(a)** Mass and **(b)** volume conservation errors are non-negligible, but low enough to not significantly impact the results.

4.3. Relaxation dynamics. We continue analysing the mechanochemical instabilities of Equations (10)-(17) focusing on a different driving force: curvature inhomogeneities [124]. The linear stability analysis of Section 4.2 identifies the sphere as a stable shape for $Pe < Pe_{\text{cr}}^l$, $l \geq 1$, in which diffusion, dominating over advection, homogenenises the concentration of the molecular species. On the other hand, recalling Equation (9), the decomposition of the active forces exposes the capability of mean curvature gradients alone to generate the Marangoni effect. Obviously, these cannot be balanced out by diffusive transport. Hence, in the diffusion-dominated regime, the system is brought to relax its shape towards a sphere.

Figure 10 represents the relaxation dynamics on a popcorn, pear and torus shapes. We start, in both cases, with a constant surface concentration of the molecular species. All simulation parameters are gathered in Table 1. Setting $Pe = 5.0$ and $\tau_D k_{\text{off}} = 50.0$ to force a diffusion-dominated regime, we observe the shape evolving to suppress the mean curvature inhomogeneities; regions with higher curvature reduce their curvature, and viceversa. During the relaxation process, transient inhomogeneities in the surface concentration of the species appear and dissipate, due to the local expansion and contraction of the surface. This example underscores the role of active forces in sustaining a non-spherical equilibrium shape. Despite nontrivial 3D shape dynamics, errors in volume conservation remain low, as shown in Figure 9.

Parameter	Popcorn	Pear	Torus
Bounding box \mathcal{B}	$[-2.01, 2.01]^3$	$[-2.25, 1.25] \times [-1.75, 1.75]^2$	$[-1.1, 1.1]^3$
Péclet number Pe		5.0	
$\tau_D k_{\text{off}}$		50.0	
Hydrodynamic length L_η		10^{-5}	
Characteristic length R_0	0.6	2.0	1.0
Number of cells per axis n	35	30	30
Time step Δt		10^{-4}	

Table 1. Relaxation dynamics. Simulation parameters.

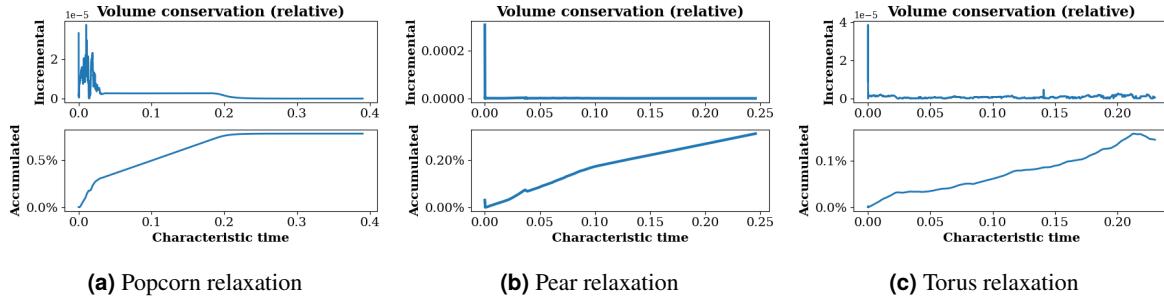


Figure 9. Relaxation dynamics. We observe low errors in volume conservation for the three relaxed geometries.

4.4. Bi- and uni-lateral cytokinesis. We now showcase the capabilities of the unfitted FE framework to deal with constricted deformations. The goal is to simulate cytokinesis, i.e., the physical process at the end of cell division that partitions the cell into two daughter cells [125]. In animal cells, cytokinesis is achieved by an actomyosin furrow forming at the equator that constricts the cell until scission. Meanwhile, cortical tension at the poles resists the increasing cell pressure [126]. As reviewed in [17], both early hypotheses and recent experimental and modelling studies support the idea that cell division is driven by a gradient of surface tension directed toward the division axis. Moreover, the dominant mechanical contributor is cortical tension, rather than active or viscous torque contributions [12, 17]. The mathematical model of Equations (10)-(17) is thus adequate for this type of cellular process.

We first focus on the stereotypical mode of cell division, i.e., symmetric cytokinesis, which cuts the cell into two equal-sized daughter cells. This type of division is achieved by a circumferential cleavage furrow that forms and develops a ring constriction with two cortical poles under tensile stress at both sides. We will then refer to this configuration as *bilateral* cytokinesis. The starting point is a spherical cell of radius R_0 . Experimental measurements have shown that the mitotic apparatus of animal cells generates and sustains a Gaussian-like band of myosin overactivity to position the furrow and guide the constriction at the cell equator [127, 128]. Following [12], we model the overactivity at the contractile ring by spatially modulating the active surface tension N_Γ^{act} in Equation (8) through the myosin activity field ξ , defined as

$$\xi(x_\alpha) = \xi_0 + (\delta\xi - \xi_0)e^{-\frac{1}{2}\frac{x_\alpha^2}{\omega^2}}, \quad (28)$$

where ξ_0 represents a basal level of activity, controlling active tension at the poles, $\delta\xi$ is the overactivity, expanding along the axis of rotational symmetry x_α over a typical width ω . Figure 11 and 12 show the evolution of myosin concentration at the surface and the flows in the cytoplasm for the 2D axisymmetric

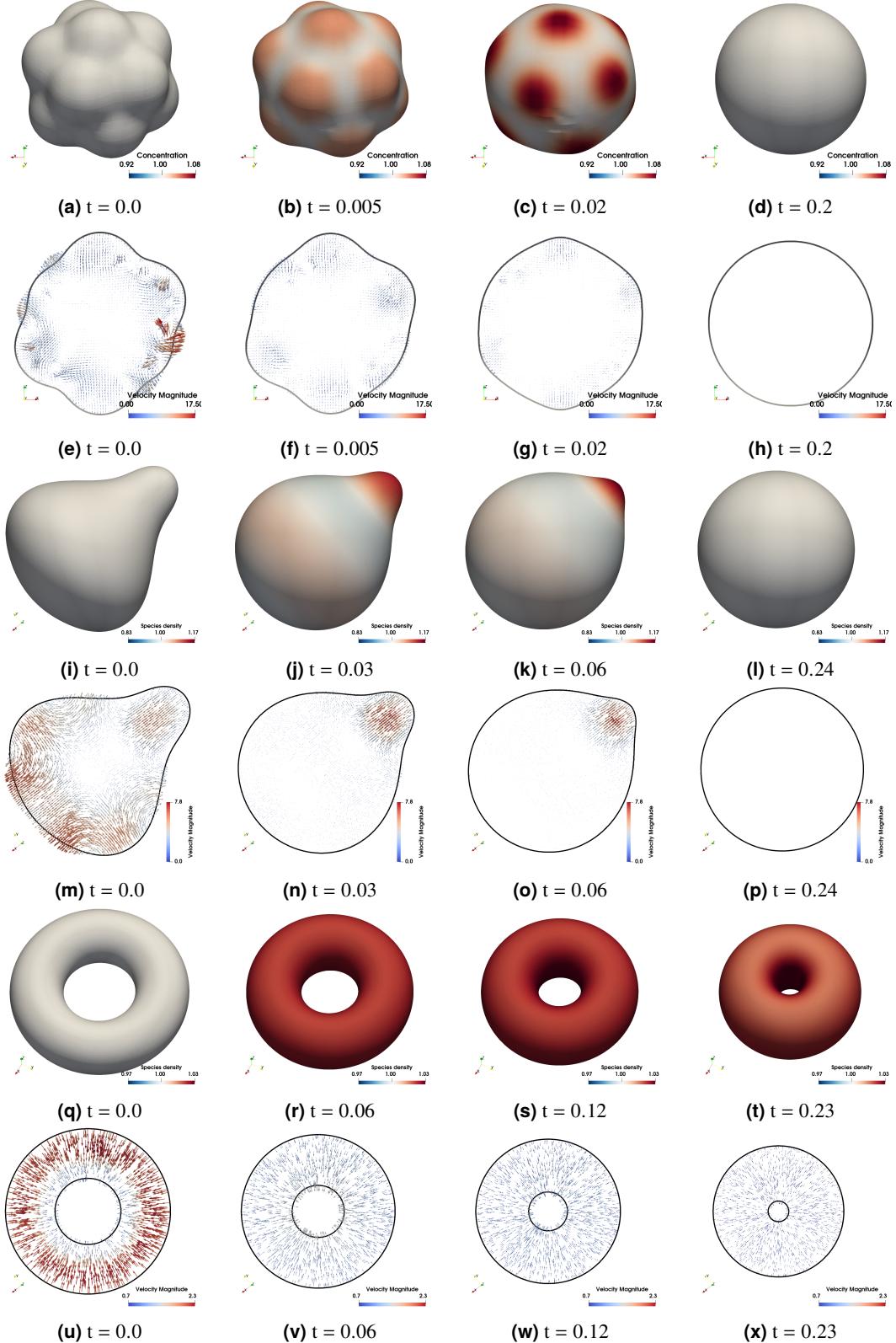


Figure 10. Relaxation dynamics. Marangoni effects generated by mean curvature gradients are balanced by evolving the shape towards a sphere. *Relaxation of a popcorn, a pear and a torus.* **(a)-(d)** and **(i)-(l)** Surface concentration of stress-regulating molecular species. **(e)-(h)** and **(m)-(p)** Bulk flow field.

and 3D models. We prescribe $R_0=0.46$, $\omega=0.1$, $\xi_0=1$ and $\delta\xi=10$; the rest of simulation parameters are listed in Table 2. We observe the furrow constriction appearing due to overactivity and being reinforced by

flows of myosin from the poles to the equator. Flow patterns in the cytoplasm, with two toroidal vortices forming at each side of the equator, agree with experimental observations [23]. Figure 13 represents the evolution of the radius of the contractile ring r_f and the distance between the poles d_p , relative to the initial cell radius R_0 . We observe tangent flows at the poles when d_p plateaus.

Parameter	2D axisymmetric	3D
Bounding box \mathcal{B}	$[-0.75, 0.75] \times [0.0, 0.75]$	$[-0.75, 0.75]^3$
Péclet number Pe	5.0	15.0
$\tau_D k_{off}$	1.0	
Hydrodynamic length L_η	10^5	
Characteristic length R_0	0.46	
Number of cells per axis n	100	40
Time step Δt		10^{-4}

Table 2. Cytokinesis. Simulation parameters.

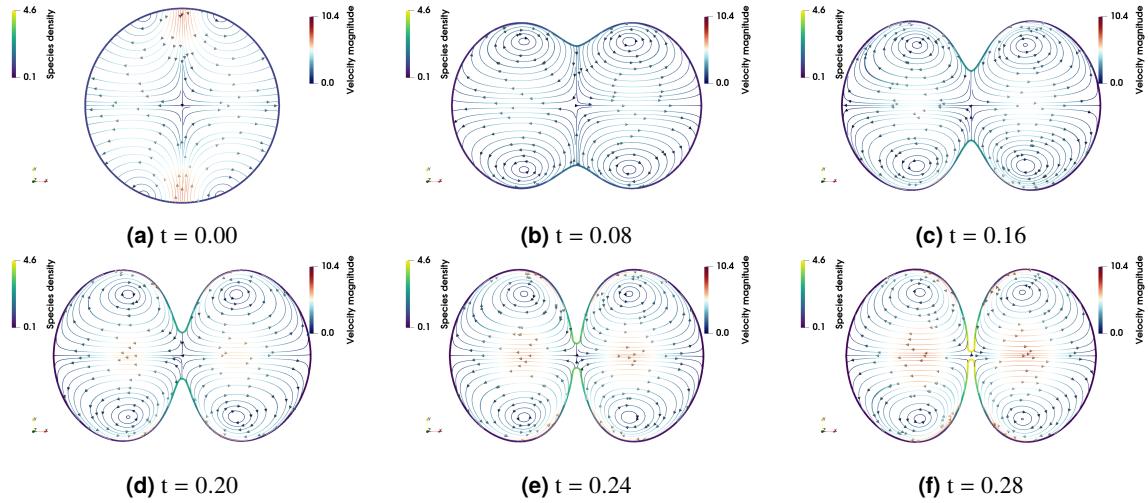


Figure 11. 2D axisymmetric cleavage. Evolution of bulk flow field and surface density of the molecular species.

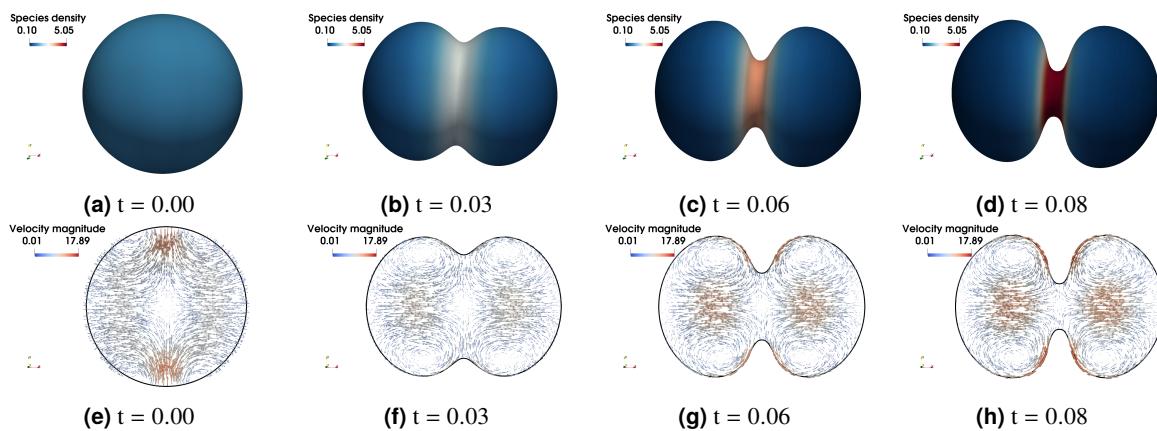


Figure 12. 3D symmetric cleavage. Evolution of bulk flow field and surface density of the molecular species.

Our last experiment targets the non-rotational symmetry mode of cell division known as unilateral cytokinesis [129]. In the zygote of certain animal species, including jellyfish, corals and comb jellies (cnidaria and ctenophora, in general) [130], but also in some epithelial tissues [131], the cleavage furrow ingresses from only one side of the cell. This implies the formation of a single extensile pole at the opposite side of the furrow. Figure 14 describes the dynamics of myosin concentration on the surface and

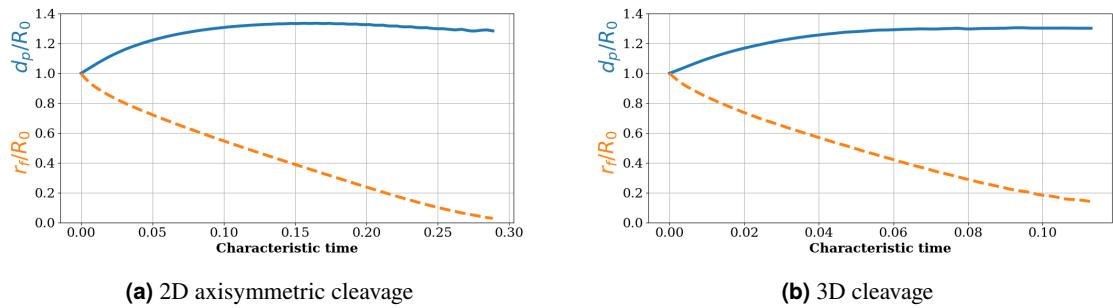


Figure 13. Symmetric cleavage. Evolution of the radius of the contractile ring r_f and the distance between poles d_p relative to the initial radius of the sphere R_0 .

bulk viscous flows during a unilateral cytokinesis that takes the same parameters as the bilateral case, except that ξ in Equation (28) only acts in half the equator. In contrast to the bilateral case, only one toroidal vortex forms around the ingression furrow.

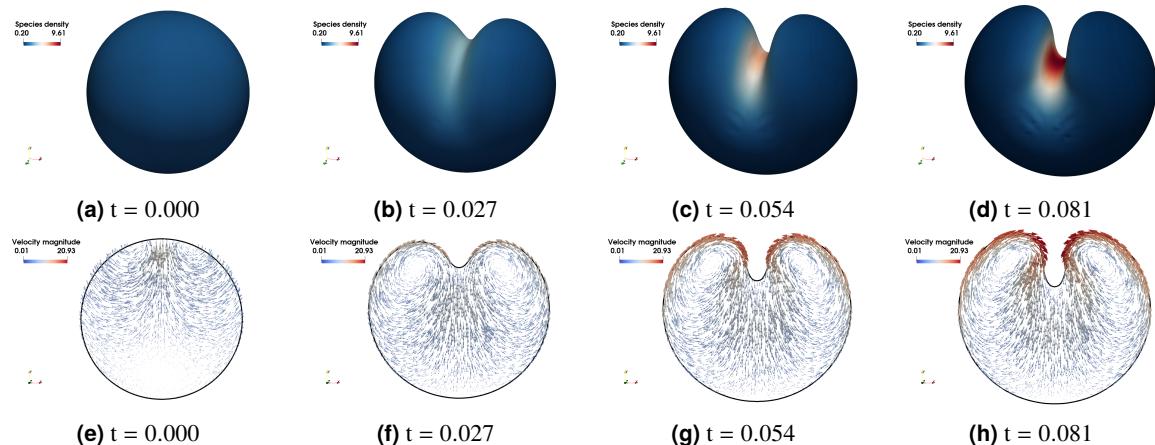


Figure 14. 3D unilateral cleavage. Evolution of bulk flow field and surface density of the molecular species.

We finally observe that both 2D axisymmetric and 3D models fail to complete cell pinching: In the 2D axisymmetric case, the axisymmetry boundary condition at the bulk phase (zero vertical displacement) is not compatible with the downwards displacement of the cleavage furrow. In the 3D cases, we observe shape instabilities arising when approximating pinch off. Both issues are to be investigated in future work.

5. CONCLUSIONS

Studying morphogenesis is a multidisciplinary effort that draws on various disciplines of science and technology, including developmental biologists, physicists, applied mathematicians or microscopy. Interdisciplinary collaboration between experts in these fields is often essential to gain a comprehensive understanding of morphogenesis. As experimental methods and theoretical models become more and more sophisticated, there is an increasing need of advanced computational methods to solve the complex mathematical models postulated by physicists and to interpret the experimental observations.

Morphogenetic processes are frequently characterised by very intricate three-dimensional and shape-evolving fluid dynamics, such as the cortex-cytoplasm interactions that concern this work. FE analysis is one of the few computational approaches suitable for modelling this level of complexity, making it relatively easy to solve multiphysics PDE problems, with couplings between biochemical and mechanical signalling and between different topological dimensions. However, problems with moving boundaries and interfaces have been a major challenge in the FE community. This is due –in part– to the difficulty with FE methods to strike a good balance between coping with large deformations and accurately resolving free surfaces and internal interfaces [43].

Nowadays, a new generation of *unfitted* FE methods is challenging the state-of-the-art in FEs for moving boundaries and interfaces [66, 70, 71]. They rely on a Eulerian description of motion and a sharp-interface

representation of the geometry. By providing a robust and accurate way to track moving surfaces in a fixed computational grid, they are questioning the classical trade-off between Lagrangian and Eulerian FE methods [43]. In this work, we specialise this emerging class of unfitted FE technologies to model surface-bulk viscous flows in animal cells. To this end, we have formulated a novel partitioned FE method that combines the aggregated FEM [77] for the bulk PDEs with trace FEM [78] for the surface PDEs. We have implemented the numerical model in `Gridap.jl` [109, 110], an advanced Julia FE software ecosystem, leveraging high-order algorithms to compute quadrature rules and closest-point projections from the `algoim` library [79, 80]. Our numerical experiments consider a minimal model of cellular symmetry breaking [81]. They illustrate the capacity of the numerical framework to simulate nontrivial 3D dynamics, with very large distortions and topological changes, and the potential to address applications in animal morphogenesis that have been barely object of numerical modelling.

Our unfitted methodology offers several promising avenues for future development to enhance both its fidelity and applicability to address increasingly complex morphogenetic phenomena. A natural extension would involve a more faithful representation of the actomyosin cortex as a thin shell, incorporating membrane (e.g., tension) and torque stress resultants previously derived in [17]. While membrane tension resultants can be readily integrated into the current model, capturing surface torques—whether of active [13], viscous [17], or elastic origin (such as plasma membrane bending [48])—requires second-order derivatives of the unknown fields. This necessity motivates an extension of our framework toward \mathcal{H}^2 -conforming approximations [132].

Beyond morphogenetic deformations, our approach could be extended to study 3D cell motility in viscous environments [31, 123, 133], including key additional mechanisms such as the membrane-to-cortex mechanical coupling [134], and the mechanochemical interaction with regulatory proteins [135]. In particular, mutually inhibitory signaling circuits—often modeled by nonlinear reaction-diffusion systems [136–138]—play a key role in symmetry breaking and polarity establishment.

Mechanochemical coupling between cortical mechanics, surface-localized active proteins, and bulk-distributed passive species [139] generally involves surface-bulk exchange of molecular components. Such coupling has been shown to underlie intracellular pattern formation [140–142]. Our unfitted framework can be easily adapted to simulate such processes, by solving advection-reaction-diffusion equations both on the cortex and in the cytoplasmic bulk, while preserving a sharp interface description. This feature provides a significant advantage over diffuse-interface or phase-field approaches [76, 143, 144], enabling a more accurate resolution of the interface, and avoids alternative approximations using projection methods [145].

Further developments could incorporate the polar and nematic ordering of cortical actin filaments [146, 147], which are known to interact with gradients in cortical flow during cell division [148, 149]. Our unfitted framework could again prove advantageous in this context, simplifying the otherwise technical body-fitted FE formulation and implementation of tensorial fields on moving curved surfaces [150–152].

As for modelling the bulk cytoplasm, we have treated the medium as a passive viscous fluid. However, the cytoplasm is a heterogeneous, crowded medium composed of a cytoskeletal meshwork permeated by cytosol and embedding various membrane-bound organelles such as the nucleus, endoplasmic reticulum, Golgi apparatus, and mitochondria. Capturing such complexity would require, at least, a two-phase poroviscous or poroelastic model [16, 153], potentially coupled to elastic-like inclusions [37, 154].

Finally, a natural direction for generalization lies in the extension to multicellular systems, following surface-based approaches as in [40, 155, 156]. This would pave the way for studying surface-bulk interactions in early animal embryogenesis, a domain in which computational modelling remains scarce—particularly using unfitted finite element methods. Overall, the intersection between finite element technologies and animal morphogenesis offers a wide frontier of methodological and biological challenges, and we anticipate this cross-disciplinary interaction to remain fertile and active for years to come.

ACKNOWLEDGMENTS

We thank all members of the Turlier team for the fruitful discussions and Gridap developers for the many contributions that made this work possible. E.N. received funding from the European Union’s Horizon EU research and innovation programme under grant agreement no. 101105565. H.T. received funding from the European Union’s Horizon 2020 research and innovation programme under the European Research Council grant agreement no. 949267 and has been supported by the Bettencourt-Schueler Foundation, by the CNRS and the Collège de France.

APPENDIX A. AXISYMMETRIC FORMULATION IN 2D

This section details the 2D axisymmetric variational formulation of Equations (10)-(17). We denote with (r, z) the Cartesian coordinates in \mathbb{R}^2 and assume the problem (solution and data) is rotationally invariant around the axis $r = 0$, see Figure 15.

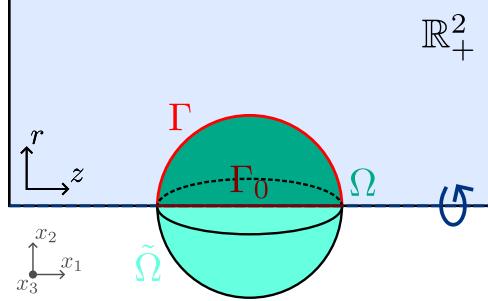


Figure 15. Geometry for the 2D axisymmetric analysis.

Let Ω denote a bounded domain contained in the half-space of positive r coordinates \mathbb{R}_+^2 . We use Γ_0 to refer to the portion of the boundary $\partial\Omega$ contained in the axis $r = 0$ and set $\Gamma = \partial\Omega \setminus \Gamma_0$. The axisymmetric domain $\tilde{\Omega}$ is the 3D set obtained by rotating Ω around the axis $r = 0$.

We now use the rotational invariance and the 3D transformation from Cartesian to cylindrical coordinates to formulate the 3D weak problem in $\tilde{\Omega}$ in the 2D domain Ω of the half-space \mathbb{R}_+^2 , thus reducing the dimension of the problem by one order of magnitude. Note that, in this case, the symmetry boundary condition $u_h^r = 0$ on Γ_0 applies for the bulk flow problem. Meanwhile, in the surface flow problem, $U_h^r = 0$ on $\partial\Gamma$ and the rigid body modes reduce to the z -translational mode.

Given the differential operators in cylindrical coordinates (r, θ, z) [157], in particular, for a scalar field v and a vector field \mathbf{u} ,

$$\tilde{\nabla}v = \nabla v, \quad \tilde{\nabla} \cdot \mathbf{u} = \frac{1}{r}\nabla \cdot (r\mathbf{u}), \quad \text{and} \quad \tilde{\boldsymbol{\epsilon}}(\mathbf{u}) = \begin{bmatrix} \frac{\partial u^r}{\partial r} & 0 & \frac{1}{2}\left(\frac{\partial u^r}{\partial z} + \frac{\partial u^z}{\partial r}\right) \\ 0 & \frac{u^r}{r} & 0 \\ \frac{1}{2}\left(\frac{\partial u^r}{\partial z} + \frac{\partial u^z}{\partial r}\right) & 0 & \frac{\partial u^z}{\partial z} \end{bmatrix},$$

with analogous expressions holding for the surface operators; and the Jacobian of the coordinate transformation being $2\pi r$, the bilinear forms and linear functionals of Section 3.2 in the 2D axisymmetric formulation read:

A.1. Bulk viscous flows in Equation (20).

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = \frac{2R}{L_\eta} \int_{\Omega} \left\{ \boldsymbol{\epsilon}(\mathbf{u}_h) : \boldsymbol{\epsilon}(\mathbf{v}_h) + \frac{u_h^r}{r} \frac{v_h^r}{r} \right\} r \, d\Omega \quad \text{and} \quad b_h(\mathbf{u}_h, p_h) = - \int_{\Omega} p_h \left(\nabla \cdot \mathbf{u}_h + \frac{u_h^r}{r} \right) r \, d\Omega$$

$$i_h(\mathbf{u}_h, p_h, \mathbf{v}_h, q_h) = \int_{\Gamma} \left\{ \frac{\alpha}{h} \mathbf{u}_h \cdot \mathbf{v}_h - \left[\frac{2R}{L_\eta} \boldsymbol{\epsilon}(\mathbf{u}_h) \mathbf{n}_{\Gamma} - p_h \mathbf{n}_{\Gamma} \right] \cdot \mathbf{v}_h - \left[\frac{2R}{L_\eta} \boldsymbol{\epsilon}(\mathbf{v}_h) \mathbf{n}_{\Gamma} - q_h \mathbf{n}_{\Gamma} \right] \cdot \mathbf{u}_h \right\} r \, d\Gamma$$

$$\text{and } j_h(\mathbf{v}_h, q_h; \mathbf{U}_h) = \int_{\Gamma} \left\{ \frac{\alpha}{h} \mathbf{U}_h \cdot \mathbf{v}_h - \left[\frac{2R}{L_\eta} \boldsymbol{\epsilon}(\mathbf{v}_h) \mathbf{n}_{\Gamma} - q_h \mathbf{n}_{\Gamma} \right] \cdot \mathbf{U}_h \right\} r \, d\Gamma$$

A.2. Surface viscous flows in Equation (21).

$$A_h^U(\mathbf{U}_h, \mathbf{V}_h) = 2 \int_{\Gamma} \left\{ \boldsymbol{\epsilon}_{\Gamma}(\mathbf{U}_h) : \boldsymbol{\epsilon}_{\Gamma}(\mathbf{V}_h) + \frac{U_h^r}{r} \frac{V_h^r}{r} \right\} r \, d\Gamma$$

$$F_h^{\text{act}}(\mathbf{V}_h; C_h) = - \int_{\Gamma} Pef(C_h, 1) \left(\text{div}_{\Gamma} \mathbf{V}_h + \frac{V_h^r}{r} \right) r \, d\Gamma$$

$$F_h^{\text{cav}}(\mathbf{V}_h; \mathbf{u}_h, p_h) = - \int_{\Gamma} \left[\frac{2R}{L_\eta} \boldsymbol{\epsilon}(\mathbf{u}_h) \mathbf{n}_{\Gamma} - p_h \mathbf{n}_{\Gamma} \right] \cdot \mathbf{V}_h r \, d\Gamma$$

$$S_h^U(\mathbf{U}_h, \mathbf{V}_h) = \int_{\mathcal{N}_h^\Gamma} \frac{\beta}{h} \boldsymbol{\varepsilon}_\Gamma(\mathbf{U}_h) \mathbf{n}_\Gamma \cdot \boldsymbol{\varepsilon}_\Gamma(\mathbf{V}_h) \mathbf{n}_\Gamma r \, d\Omega$$

A.3. Surface molecular transport in Equation (22).

$$M_h(C_h^n, D_h^n) = \int_{\Gamma^n} C_h^n D_h^n r \, d\Gamma, \quad A_h^C(C_h^n, D_h^n) = \int_{\Gamma^n} \nabla C_h^n \cdot \nabla D_h^n r \, d\Gamma, \quad \text{and} \quad L_h(D_h^n) = \int_{\Gamma^n} D_h^n r \, d\Gamma.$$

$$S_h^C(C_h^n, D_h^n) = \int_{\mathcal{N}_h^{\Gamma^n}} \frac{\gamma}{h} (\nabla C_h^n \cdot \mathbf{n}_\Gamma) (\nabla D_h^n \cdot \mathbf{n}_\Gamma) r \, d\Omega$$

$$B_h(C_h^n, D_h^n; \mathbf{U}_h^n) = \int_{\Gamma^n} \left\{ (\mathbf{U}_h \cdot \nabla_\Gamma C_h^n) D_h^n + \left(\operatorname{div}_\Gamma \mathbf{U}_h^n + \frac{U_h^r}{r} \right) C_h^n D_h^n \right\} r \, d\Gamma$$

REFERENCES

- [1] T. Lecuit and P.-F. Lenne. “Cell surface mechanics and the control of cell shape, tissue patterns and morphogenesis”. In: *Nature reviews Molecular cell biology* 8.8 (2007), pp. 633–644. doi: [10.1038/nrm2222](https://doi.org/10.1038/nrm2222).
- [2] Y. Sasai. “Next-generation regenerative medicine: organogenesis from stem cells in 3D culture”. In: *Cell stem cell* 12.5 (2013), pp. 520–530. doi: [10.1016/j.stem.2013.04.009](https://doi.org/10.1016/j.stem.2013.04.009).
- [3] K. M. Yamada and E. Cukierman. “Modeling tissue morphogenesis and cancer in 3D”. In: *Cell* 130.4 (2007), pp. 601–610. doi: [10.1016/j.cell.2007.08.006](https://doi.org/10.1016/j.cell.2007.08.006).
- [4] T. Wicha et al. “The green seaweed *Ulva*: a model system to study morphogenesis”. In: *Frontiers in plant science* 6 (2015), p. 72. doi: [10.3389/fpls.2015.00072](https://doi.org/10.3389/fpls.2015.00072).
- [5] T. Hayashi and R. W. Carthew. “Surface mechanics mediate pattern formation in the developing retina”. In: *Nature* 431.7009 (2004), pp. 647–652. doi: [10.1038/nature02952](https://doi.org/10.1038/nature02952).
- [6] A. G. Clark et al. “Stresses at the cell surface during animal cell morphogenesis”. In: *Current Biology* 24.10 (2014), R484–R494. doi: [10.1016/j.cub.2014.03.059](https://doi.org/10.1016/j.cub.2014.03.059).
- [7] A. G. Clark, K. Dierkes, and E. K. Paluch. “Monitoring actin cortex thickness in live cells”. In: *Biophysical journal* 105.3 (2013), pp. 570–580. doi: [10.1016/j.bpj.2013.05.057](https://doi.org/10.1016/j.bpj.2013.05.057).
- [8] M. Kelkar, P. Bohec, and G. Charras. “Mechanics of the cellular actin cortex: From signalling to shape change”. In: *Current opinion in cell biology* 66 (2020), pp. 69–78. doi: [10.1016/j.ceb.2020.05.008](https://doi.org/10.1016/j.ceb.2020.05.008).
- [9] J. Prost, F. Jülicher, and J.-F. Joanny. “Active gel physics”. In: *Nature physics* 11.2 (2015), pp. 111–117. doi: doi.org/10.1038/nphys3224.
- [10] D. Bray and J. White. “Cortical flow in animal cells”. In: *Science* 239.4842 (1988), pp. 883–888. doi: [10.1126/science.3277283](https://doi.org/10.1126/science.3277283).
- [11] M. Mayer et al. “Anisotropies in cortical tension reveal the physical basis of polarizing cortical flows”. In: *Nature* 467.7315 (2010), pp. 617–621. doi: [10.1038/nature09376](https://doi.org/10.1038/nature09376).
- [12] H. Turlier et al. “Furrow constriction in animal cell cytokinesis”. In: *Biophysical journal* 106.1 (2014), pp. 114–123. doi: [10.1016/j.bpj.2013.11.014](https://doi.org/10.1016/j.bpj.2013.11.014).
- [13] G. Salbreux and F. Jülicher. “Mechanics of active surfaces”. In: *Physical Review E* 96.3 (2017), p. 032404. doi: [10.1103/PhysRevE.96.032404](https://doi.org/10.1103/PhysRevE.96.032404).
- [14] A. Pires-da-Silva and R. J. Sommer. “The evolution of signalling pathways in animal development”. In: *Nature Reviews Genetics* 4 (1 2003), pp. 39–49. doi: [10.1038/nrg977](https://doi.org/10.1038/nrg977). URL: [10.1038/nrg977](https://doi.org/10.1038/nrg977).
- [15] W. M. Bement et al. “Patterning of the cell cortex by Rho GTPases”. In: *Nature Reviews Molecular Cell Biology* 25.4 (2024), pp. 290–308. doi: [10.1038/s41580-023-00682-z](https://doi.org/10.1038/s41580-023-00682-z).
- [16] A. Mogilner and A. Manhart. “Intracellular fluid mechanics: Coupling cytoplasmic flow with active cytoskeletal gel”. In: *Annual Review of Fluid Mechanics* 50.1 (2018), pp. 347–370. doi: [10.1146/annurev-fluid-010816-060238](https://doi.org/10.1146/annurev-fluid-010816-060238).
- [17] H. B. da Rocha, J. Bleyer, and H. Turlier. “A viscous active shell theory of the cell cortex”. In: *Journal of the Mechanics and Physics of Solids* 164 (2022), p. 104876. doi: [10.1016/j.jmps.2022.104876](https://doi.org/10.1016/j.jmps.2022.104876).
- [18] G. Salbreux, G. Charras, and E. Paluch. “Actin cortex mechanics and cellular morphogenesis”. In: *Trends in cell biology* 22.10 (2012), pp. 536–545. doi: [10.1016/j.tcb.2012.07.001](https://doi.org/10.1016/j.tcb.2012.07.001).
- [19] K. Luby-Phelps. “Cytoarchitecture and physical properties of cytoplasm: volume, viscosity, diffusion, intracellular surface area”. In: *International review of cytology* 192 (1999), pp. 189–221. doi: [10.1016/S0074-7696\(08\)60527-6](https://doi.org/10.1016/S0074-7696(08)60527-6).
- [20] Y. Hiramoto. “Mechanical properties of the protoplasm of the sea urchin egg: I. Unfertilized egg”. In: *Experimental cell research* 56.2-3 (1969), pp. 201–208. doi: [10.1016/0014-4827\(69\)90003-2](https://doi.org/10.1016/0014-4827(69)90003-2).

- [21] J. Najafi, S. Dmitrieff, and N. Minc. “Size-and position-dependent cytoplasm viscoelasticity through hydrodynamic interactions with the cell surface”. In: *Proceedings of the National Academy of Sciences* 120.9 (2023), e2216839120. doi: [10.1073/pnas.2216839120](https://doi.org/10.1073/pnas.2216839120).
- [22] P. A. Valberg and H. A. Feldman. “Magnetic particle motions within living cells. Measurement of cytoplasmic viscosity and motile activity”. In: *Biophysical journal* 52.4 (1987), pp. 551–561. doi: [10.1016/S0006-3495\(87\)83244-7](https://doi.org/10.1016/S0006-3495(87)83244-7).
- [23] Y. Hiramoto. “A quantitative description of protoplasmic movement during cleavage in the sea-urchin egg”. In: *Journal of Experimental Biology* 35.2 (1958), pp. 407–424. doi: [10.1242/jeb.35.2.407](https://doi.org/10.1242/jeb.35.2.407).
- [24] S. Shampour, S. Caballero-Mancebo, and C.-P. Heisenberg. “Cytoplasm’s got moves”. In: *Developmental Cell* 56.2 (2021), pp. 213–226. doi: [10.1016/j.devcel.2020.12.002](https://doi.org/10.1016/j.devcel.2020.12.002).
- [25] W. Lu and V. I. Gelfand. “Go with the flow–bulk transport by molecular motors”. In: *Journal of cell science* 136.5 (2023), jcs260300. doi: [10.1242/jcs.260300](https://doi.org/10.1242/jcs.260300).
- [26] M. Mittasch et al. “Non-invasive perturbations of intracellular flow reveal physical principles of cell organization”. In: *Nature cell biology* 20.3 (2018), pp. 344–351. doi: [10.1038/s41556-017-0032-9](https://doi.org/10.1038/s41556-017-0032-9).
- [27] R. Illukkumbura, T. Bland, and N. W. Goehring. “Patterning and polarization of cells by intracellular flows”. In: *Current opinion in cell biology* 62 (2020), pp. 123–134. doi: [10.1016/j.ceb.2019.10.005](https://doi.org/10.1016/j.ceb.2019.10.005).
- [28] G. von Dassow and G. Schubiger. “How an actin network might cause fountain streaming and nuclear migration in the syncytial Drosophila embryo.” In: *The Journal of cell biology* 127.6 (1994), pp. 1637–1653. doi: [10.1083/jcb.127.6.1637](https://doi.org/10.1083/jcb.127.6.1637).
- [29] V. E. Deneke et al. “Self-organized nuclear positioning synchronizes the cell cycle in Drosophila embryos”. In: *Cell* 177.4 (2019), pp. 925–941. doi: [10.1016/j.cell.2019.03.007](https://doi.org/10.1016/j.cell.2019.03.007).
- [30] C. Hernández-López et al. “Two-fluid dynamics and micron-thin boundary layers shape cytoplasmic flows in early Drosophila embryos”. In: *Proceedings of the National Academy of Sciences* 120.44 (2023), e2302879120. doi: [10.1073/pnas.2302879120](https://doi.org/10.1073/pnas.2302879120).
- [31] A. Farutin et al. “Crawling in a fluid”. In: *Physical review letters* 123.11 (2019), p. 118101. doi: [10.1103/PhysRevLett.123.118101](https://doi.org/10.1103/PhysRevLett.123.118101).
- [32] T. Le Goff, B. Liebchen, and D. Marenduzzo. “Actomyosin contraction induces in-bulk motility of cells and droplets”. In: *Biophysical Journal* 119.5 (2020), pp. 1025–1032. doi: [10.1016/j.bpj.2020.06.029](https://doi.org/10.1016/j.bpj.2020.06.029).
- [33] K. Yi et al. “Dynamic maintenance of asymmetric meiotic spindle position through Arp2/3-complex-driven cytoplasmic streaming in mouse oocytes”. In: *Nature cell biology* 13.10 (2011), pp. 1252–1258. doi: [10.1038/ncb2320](https://doi.org/10.1038/ncb2320).
- [34] A. Chaigne et al. “A soft cortex is essential for asymmetric spindle positioning in mouse oocytes”. In: *Nature cell biology* 15.8 (2013), pp. 958–966. doi: [10.1038/ncb2799](https://doi.org/10.1038/ncb2799).
- [35] A. Chaigne et al. “A narrow window of cortical tension guides asymmetric spindle positioning in the mouse oocyte”. In: *Nature communications* 6.1 (2015), p. 6027. doi: [10.1038/ncomms7027](https://doi.org/10.1038/ncomms7027).
- [36] H. Wang et al. “Symmetry breaking in hydrodynamic forces drives meiotic spindle rotation in mammalian oocytes”. In: *Science advances* 6.14 (2020), eaaz5004. doi: [10.1126/sciadv.aaz5004](https://doi.org/10.1126/sciadv.aaz5004).
- [37] W. Liao and E. Lauga. “Hydrodynamic Mechanism for Stable Spindle Positioning in Meiosis II Oocytes”. In: *PRX Life* 2 (4 2024), p. 043003. doi: [10.1103/PRXLife.2.043003](https://doi.org/10.1103/PRXLife.2.043003).
- [38] E. Barnhart et al. “Balance between cell-substrate adhesion and myosin contraction determines the frequency of motility initiation in fish keratocytes”. In: *Proceedings of the National Academy of Sciences of the United States of America* 112 (16 2015), pp. 5045–5050. doi: [10.1073/pnas.1417257112](https://doi.org/10.1073/pnas.1417257112).
- [39] A. Bhatnagar et al. “Axis convergence in *C. elegans* embryos”. In: *Current Biology* 33.23 (2023), pp. 5096–5108. doi: [10.1016/j.cub.2023.10.050](https://doi.org/10.1016/j.cub.2023.10.050).
- [40] A. Torres-Sánchez, M. Kerr Winter, and G. Salbreux. “Interacting active surfaces: A model for three-dimensional cell aggregates”. In: *PLOS Computational Biology* 18.12 (2022), e1010762. doi: [10.1371/journal.pcbi.1010762](https://doi.org/10.1371/journal.pcbi.1010762).
- [41] C. Johnson. *Numerical Solution of Partial Differential Equations by the Finite Element Method*. Cambridge University Press, 1987.
- [42] L. E. Malvern. “Introduction to the Mechanics of a Continuous Medium”. In: *Prentice-Hall: Englewood Cliffs* (1969).
- [43] J. Donea et al. “Arbitrary Lagrangian–Eulerian Methods”. In: *In Encyclopedia of Computational Mechanics* (eds E. Stein, R. Borst and T.J.R. Hughes) (2004). doi: [10.1002/0470091355.ecm009](https://doi.org/10.1002/0470091355.ecm009).
- [44] A. Bennett. *Lagrangian fluid dynamics*. Cambridge University Press, 2006. doi: [10.1017/CBO9780511734939](https://doi.org/10.1017/CBO9780511734939).
- [45] J. Bonet and R. D. Wood. *Nonlinear continuum mechanics for finite element analysis*. Cambridge University Press, 1997. doi: [10.1017/CBO9780511755446](https://doi.org/10.1017/CBO9780511755446).

- [46] D. Khan et al. “Surface remeshing: A systematic literature review of methods and research directions”. In: *IEEE transactions on visualization and computer graphics* 28.3 (2020), pp. 1680–1713. doi: [10.1109/TCVG.2020.3016645](https://doi.org/10.1109/TCVG.2020.3016645).
- [47] J. D. Anderson and J. Wendt. *Computational fluid dynamics*. Vol. 206. Springer, 1995.
- [48] A. Torres-Sánchez, D. Millán, and M. Arroyo. “Modelling fluid deformable surfaces with an emphasis on biological interfaces”. In: *Journal of Fluid Mechanics* 872 (2019), pp. 218–271. doi: [10.1017/jfm.2019.341](https://doi.org/10.1017/jfm.2019.341).
- [49] A. Voigt. “Fluid deformable surfaces”. In: *Journal of Fluid Mechanics* 878 (2019), pp. 1–4. doi: [10.1017/jfm.2019.549](https://doi.org/10.1017/jfm.2019.549).
- [50] E. De Kinkelder, L. Sagis, and S. Aland. “A numerical method for the simulation of viscoelastic fluid surfaces”. In: *Journal of computational physics* 440 (2021), p. 110413. doi: [10.1016/j.jcp.2021.110413](https://doi.org/10.1016/j.jcp.2021.110413).
- [51] L. D. Wittwer and S. Aland. “A computational model of self-organized shape dynamics of active surfaces in fluids”. In: *Journal of Computational Physics: X* 17 (2023), p. 100126. doi: [10.1016/j.jcpx.2023.100126](https://doi.org/10.1016/j.jcpx.2023.100126).
- [52] S. Ganesan and L. Tobiska. “A coupled arbitrary Lagrangian-Eulerian and Lagrangian method for computation of free surface flows with insoluble surfactants”. In: *Journal of Computational Physics* 228 (8 2009), pp. 2859–2873. doi: [10.1016/j.jcp.2008.12.035](https://doi.org/10.1016/j.jcp.2008.12.035).
- [53] D. Edelmann, B. Kovács, and C. Lubich. “Numerical analysis of an evolving bulk–surface model of tumour growth”. In: *IMA Journal of Numerical Analysis* (2024), drae077. doi: [10.1093/imanum/drae077](https://doi.org/10.1093/imanum/drae077).
- [54] D. Hernandez-Aristizabal et al. “A bulk-surface mechanobiochemical modelling approach for single cell migration in two-space dimensions”. In: *Journal of Theoretical Biology* 595 (2024). doi: [10.1016/j.jtbi.2024.111966](https://doi.org/10.1016/j.jtbi.2024.111966).
- [55] G. Dziuk and C. M. Elliott. “Finite element methods for surface PDEs”. In: *Acta Numerica* 22 (2013), pp. 289–396. doi: [10.1017/S0962492913000056](https://doi.org/10.1017/S0962492913000056).
- [56] M. Mokbel and S. Aland. “An ALE method for simulations of axisymmetric elastic surfaces in flow”. In: *International Journal for Numerical Methods in Fluids* 92 (11 2020), pp. 1604–1625. doi: [10.1002/fld.4841](https://doi.org/10.1002/fld.4841).
- [57] R. Rangarajan and H. Gao. “A finite element method to compute three-dimensional equilibrium configurations of fluid membranes: Optimal parameterization, variational formulation and applications”. In: *Journal of Computational Physics* 297 (2015), pp. 266–294. doi: [10.1016/j.jcp.2015.05.001](https://doi.org/10.1016/j.jcp.2015.05.001).
- [58] M. Kloppe and S. Aland. “A phase-field model of elastic and viscoelastic surfaces in fluids”. In: *Computer Methods in Applied Mechanics and Engineering* 428 (2024), p. 117090. doi: [10.1016/j.cma.2024.117090](https://doi.org/10.1016/j.cma.2024.117090).
- [59] K. E. Teigen et al. “A diffuse-interface method for two-phase flows with soluble surfactants”. In: *Journal of computational physics* 230.2 (2011), pp. 375–393. doi: [10.1016/j.jcp.2010.09.020](https://doi.org/10.1016/j.jcp.2010.09.020).
- [60] T. Frachon and S. Zahedi. “A cut finite element method for two-phase flows with insoluble surfactants”. In: *Journal of Computational Physics* 473 (2023). doi: [10.1016/j.jcp.2022.111734](https://doi.org/10.1016/j.jcp.2022.111734).
- [61] J. W. Barrett, H. Garcke, and R. Nürnberg. “On the stable numerical approximation of two-phase flow with insoluble surfactant”. In: *ESAIM: Mathematical Modelling and Numerical Analysis* 49 (2 2014), pp. 421–458. doi: [10.1051/m2an/2014039](https://doi.org/10.1051/m2an/2014039).
- [62] J. W. Barrett, H. Garcke, and R. Nürnberg. “Finite element approximation for the dynamics of fluidic two-phase biomembranes”. In: *ESAIM: Mathematical Modelling and Numerical Analysis* 51.6 (2017), pp. 2319–2366. doi: [10.1051/m2an/2017037](https://doi.org/10.1051/m2an/2017037).
- [63] B. Kovács. “Numerical Surgery for Mean Curvature Flow of Surfaces”. In: *SIAM Journal on Scientific Computing* 46 (2 2024), A645–A669. doi: [10.1137/22M1531919](https://doi.org/10.1137/22M1531919). URL: [10.1137/22M1531919](https://doi.org/10.1137/22M1531919).
- [64] H. Gomez and K. G. van der Zee. “Computational Phase-Field Modeling. In Encyclopedia of Computational Mechanics Second Edition (eds E. Stein, R. Borst and T.J.R. Hughes)”. In: *Encyclopedia of Computational Mechanics Second Edition*. 2017. ISBN: 9781119176817. doi: [10.1002/9781119176817.ecm2118](https://doi.org/10.1002/9781119176817.ecm2118).
- [65] E. Bachini et al. “Diffusion of tangential tensor fields: numerical issues and influence of geometric properties”. In: *Journal of Numerical Mathematics* 32.1 (2024), pp. 55–75. doi: [10.1515/jnma-2022-0088](https://doi.org/10.1515/jnma-2022-0088).
- [66] M. A. Olshanskii and A. Reusken. “Trace Finite Element Methods for PDEs on Surfaces”. In: *Geometrically Unfitted Finite Element Methods and Applications*. Ed. by S. P. A. Bordas et al. Cham: Springer International Publishing, 2017, pp. 211–258.
- [67] S. Badia, F. Verdugo, and A. F. Martín. “The aggregated unfitted finite element method for elliptic problems”. In: *Computer Methods in Applied Mechanics and Engineering* 336 (2018), pp. 533–553. doi: [10.1016/j.cma.2018.03.022](https://doi.org/10.1016/j.cma.2018.03.022).
- [68] C. S. Peskin. “Flow patterns around heart valves: a numerical method”. In: *Journal of computational physics* 10.2 (1972), pp. 252–271. doi: [10.1016/0021-9991\(72\)90065-4](https://doi.org/10.1016/0021-9991(72)90065-4).

- [69] R. Mittal and G. Iaccarino. “Immersed Boundary Methods”. In: *Annual Review of Fluid Mechanics* 37 (2005), pp. 239–261. doi: [10.1146/annurev.fluid.37.061903.175743](https://doi.org/10.1146/annurev.fluid.37.061903.175743).
- [70] P. Hansbo, M. G. Larson, and K. Larsson. “Analysis of finite element methods for vector Laplacians on surfaces”. In: *IMA Journal of Numerical Analysis* 40.3 (2019), pp. 1652–1701. doi: [10.1093/imanum/drz018](https://doi.org/10.1093/imanum/drz018).
- [71] F. de Prenter et al. “Stability and conditioning of immersed finite element methods: analysis and remedies”. In: *Archives of Computational Methods in Engineering* 30.6 (2023), pp. 3617–3656. doi: [10.1007/s11831-023-09913-0](https://doi.org/10.1007/s11831-023-09913-0).
- [72] P. Hansbo, M. G. Larson, and S. Zahedi. “A cut finite element method for coupled bulk-surface problems on time-dependent domains”. In: *Computer Methods in Applied Mechanics and Engineering* 307 (2016), pp. 96–116. doi: [10.1016/j.cma.2016.04.012](https://doi.org/10.1016/j.cma.2016.04.012).
- [73] M. Olshanskii, A. Quaini, and Q. Sun. “A Finite Element Method for Two-Phase Flow with Material Viscous Interface”. In: *Computational Methods in Applied Mathematics* 22 (2 2022), pp. 443–464. doi: [10.1515/cmam-2021-0185](https://doi.org/10.1515/cmam-2021-0185).
- [74] E. Burman et al. “CutFEM: Discretizing geometry and partial differential equations”. In: *International Journal for Numerical Methods in Engineering* 104.7 (2015), pp. 472–501. doi: [10.1002/nme.4823](https://doi.org/10.1002/nme.4823).
- [75] S. Badia, E. Neiva, and F. Verdugo. “Linking ghost penalty and aggregated unfitted methods”. In: *Computer Methods in Applied Mechanics and Engineering* 388 (2022), p. 114232. doi: [10.1016/j.cma.2021.114232](https://doi.org/10.1016/j.cma.2021.114232).
- [76] S. Aland and C. Wohlgemuth. “A phase-field model for active contractile surfaces”. In: *arXiv preprint arXiv:2306.16796* (2023).
- [77] S. Badia, A. F. Martin, and F. Verdugo. “Mixed Aggregated Finite Element Methods for the Unfitted Discretization of the Stokes Problem”. In: *SIAM Journal on Scientific Computing* 40.6 (2018), B1541–B1576. doi: [10.1137/18M1185624](https://doi.org/10.1137/18M1185624).
- [78] T. Jankuhn and A. Reusken. “Trace finite element methods for surface vector-Laplace equations”. In: *IMA Journal of Numerical Analysis* 41.1 (2021), pp. 48–83. doi: [10.1093/imanum/drz062](https://doi.org/10.1093/imanum/drz062).
- [79] R. I. Saye. “High-order quadrature on multi-component domains implicitly defined by multivariate polynomials”. In: *Journal of Computational Physics* 448 (2022), p. 110720. doi: [10.1016/j.jcp.2021.110720](https://doi.org/10.1016/j.jcp.2021.110720).
- [80] R. I. Saye. “High-order methods for computing distances to implicitly defined surfaces”. In: *Communications in Applied Mathematics and Computational Science* 9.1 (2014), pp. 107–141. doi: [10.2140/camcos.2014.9.107](https://doi.org/10.2140/camcos.2014.9.107).
- [81] A. Mietke et al. “Minimal model of cellular symmetry breaking”. In: *Physical review letters* 123.18 (2019), p. 188101. doi: [10.1103/PhysRevLett.123.188101](https://doi.org/10.1103/PhysRevLett.123.188101).
- [82] M. Delfour and J. Zolésio. “Tangential differential equations for dynamical thin/shallow shells”. In: *Journal of differential equations* 128.1 (1996), pp. 125–167. doi: [10.1006/jdeq.1996.0092](https://doi.org/10.1006/jdeq.1996.0092).
- [83] T.-P. Fries and D. Schöllhammer. “A unified finite strain theory for membranes and ropes”. In: *Computer Methods in Applied Mechanics and Engineering* 365 (2020), p. 113031. doi: [10.1016/j.cma.2020.113031](https://doi.org/10.1016/j.cma.2020.113031).
- [84] L. E. Scriven. “Dynamics of a fluid interface equation of motion for Newtonian surface fluids”. In: *Chemical Engineering Science* 12.2 (1960), pp. 98–108. doi: [10.1016/0009-2509\(60\)87003-0](https://doi.org/10.1016/0009-2509(60)87003-0).
- [85] E. M. Purcell. “Life at low Reynolds number”. In: *American Journal of Physics* 45.1 (1977), pp. 3–11. doi: [10.1119/1.10903](https://doi.org/10.1119/1.10903).
- [86] R. Dimova. “Recent developments in the field of bending rigidity measurements on membranes”. In: *Advances in colloid and interface science* 208 (2014), pp. 225–234. doi: [10.1016/j.cis.2014.03.003](https://doi.org/10.1016/j.cis.2014.03.003).
- [87] P. Rangamani. “The many faces of membrane tension: Challenges across systems and scales”. In: *Biochimica et Biophysica Acta (BBA)-Biomembranes* 1864.7 (2022), p. 183897. doi: [10.1016/j.bbamem.2022.183897](https://doi.org/10.1016/j.bbamem.2022.183897).
- [88] D. Bothe and J. Prüss. “On the two-phase Navier–Stokes equations with Boussinesq–Scriven surface fluid”. In: *Journal of Mathematical Fluid Mechanics* 12 (2010), pp. 133–150. doi: [10.1007/s00021-008-0278-x](https://doi.org/10.1007/s00021-008-0278-x).
- [89] H. A. Stone and L. G. Leal. “The effects of surfactants on drop deformation and breakup”. In: *Journal of Fluid Mechanics* 220 (1990), pp. 161–186. doi: [10.1017/S0022112090003226](https://doi.org/10.1017/S0022112090003226).
- [90] F. de Prenter et al. “Condition number analysis and preconditioning of the finite cell method”. In: *Computer Methods in Applied Mechanics and Engineering* 316 (2017), pp. 297–327. doi: [10.1016/j.cma.2016.07.006](https://doi.org/10.1016/j.cma.2016.07.006).
- [91] C. Gürkan and A. Massing. “A stabilized cut discontinuous Galerkin framework for elliptic boundary value and interface problems”. In: *Computer Methods in Applied Mechanics and Engineering* 348 (2019), pp. 466–499. doi: [10.1016/j.cma.2018.12.041](https://doi.org/10.1016/j.cma.2018.12.041).

- [92] S. Osher and J. A. Sethian. “Fronts propagating with curvature-dependent speed: Algorithms based on Hamilton-Jacobi formulations”. In: *Journal of computational physics* 79.1 (1988), pp. 12–49. doi: [10.1016/0021-9991\(88\)90002-2](https://doi.org/10.1016/0021-9991(88)90002-2).
- [93] D. Schillinger and M. Ruess. “The finite cell method: a review in the context of higher-order structural analysis of CAD and image-based geometric models”. In: *Archives of Computational Methods in Engineering* 22 (2015), pp. 391–455. doi: [10.1007/s11831-014-9115-y](https://doi.org/10.1007/s11831-014-9115-y).
- [94] S. Fernández-Méndez and A. Huerta. “Imposing essential boundary conditions in mesh-free methods”. In: *Computer methods in applied mechanics and engineering* 193.12-14 (2004), pp. 1257–1275. doi: [10.1016/j.cma.2003.12.019](https://doi.org/10.1016/j.cma.2003.12.019).
- [95] J. Benzaken, J. A. Evans, and R. Tamstorf. “Constructing nitsche’s method for variational problems”. In: *Archives of Computational Methods in Engineering* 31.4 (2024), pp. 1867–1896. doi: [10.1007/s11831-023-09953-6](https://doi.org/10.1007/s11831-023-09953-6).
- [96] J. Nitsche. “Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind”. In: *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* 36.1 (1971), pp. 9–15. doi: [10.1007/BF02995904](https://doi.org/10.1007/BF02995904).
- [97] A. Ern and J.-L. Guermond. *Theory and practice of finite elements*. Vol. 159. Springer, 2004.
- [98] F. Verdugo, A. F. Martín, and S. Badia. “Distributed-memory parallelization of the aggregated unfitted finite element method”. In: *Computer Methods in Applied Mechanics and Engineering* 357 (2019), p. 112583. doi: [10.1016/j.cma.2019.112583](https://doi.org/10.1016/j.cma.2019.112583).
- [99] E. Burman, P. Hansbo, and M. G. Larson. “CutFEM based on extended finite element spaces”. In: *Numerische Mathematik* 152.2 (2022), pp. 331–369. doi: [10.1007/s00211-022-01313-z](https://doi.org/10.1007/s00211-022-01313-z).
- [100] T. Sakai. *Riemannian geometry*. Vol. 149. American Mathematical Soc., 1996. doi: [10.1090/mmono/149](https://doi.org/10.1090/mmono/149).
- [101] E. Burman et al. “Cut finite element methods for partial differential equations on embedded manifolds of arbitrary codimensions”. In: *ESAIM: Mathematical Modelling and Numerical Analysis* 52.6 (2018), pp. 2247–2282. doi: [10.1051/m2an/2018038](https://doi.org/10.1051/m2an/2018038).
- [102] D. Schöllhammer and T.-P. Fries. “A higher-order Trace finite element method for shells”. In: *International Journal for Numerical Methods in Engineering* 122.5 (2021), pp. 1217–1238. doi: [10.1002/nme.6558](https://doi.org/10.1002/nme.6558).
- [103] C. Lehrenfeld, M. A. Olshanskii, and X. Xu. “A stabilized trace finite element method for partial differential equations on evolving surfaces”. In: *SIAM Journal on Numerical Analysis* 56.3 (2018), pp. 1643–1672. doi: [10.1137/17M1148633](https://doi.org/10.1137/17M1148633).
- [104] M. A. Olshanskii and A. Reusken. “Error analysis of a space-time finite element method for solving PDEs on evolving surfaces”. In: *SIAM journal on numerical analysis* 52.4 (2014), pp. 2092–2120. doi: [10.1137/130936877](https://doi.org/10.1137/130936877).
- [105] S. Myrbäck and S. Zahedi. “A high-order conservative cut finite element method for problems in time-dependent domains”. In: *Computer Methods in Applied Mechanics and Engineering* 431 (2024), p. 117245. doi: [10.1016/j.cma.2024.117245](https://doi.org/10.1016/j.cma.2024.117245).
- [106] M. A. Olshanskii, A. Reusken, and X. Xu. “A stabilized finite element method for advection–diffusion equations on surfaces”. In: *IMA journal of numerical analysis* 34.2 (2014), pp. 732–758. doi: [10.1093/imanum/drt016](https://doi.org/10.1093/imanum/drt016).
- [107] C. Lehrenfeld and M. Olshanskii. “An Eulerian finite element method for PDEs in time-dependent domains”. In: *ESAIM: Mathematical Modelling and Numerical Analysis* 53.2 (2019), pp. 585–614. doi: [10.1051/m2an/2018068](https://doi.org/10.1051/m2an/2018068).
- [108] E. Burman, S. Frei, and A. Massing. “Eulerian time-stepping schemes for the non-stationary Stokes equations on time-dependent domains”. In: *Numerische Mathematik* (2022), pp. 1–56. doi: [10.1007/s00211-021-01264-x](https://doi.org/10.1007/s00211-021-01264-x).
- [109] S. Badia and F. Verdugo. “Gridap: An extensible Finite Element toolbox in Julia”. In: *Journal of Open Source Software* 5.52 (2020), p. 2520. doi: [10.21105/joss.02520](https://doi.org/10.21105/joss.02520). URL: <https://doi.org/10.21105/joss.02520>.
- [110] F. Verdugo and S. Badia. “The software design of Gridap: A Finite Element package based on the Julia JIT compiler”. In: *Computer Physics Communications* 276 (2022), p. 108341. doi: [10.1016/j.cpc.2022.108341](https://doi.org/10.1016/j.cpc.2022.108341). URL: <https://doi.org/10.1016/j.cpc.2022.108341>.
- [111] F. Verdugo et al. *gridap/GridapEmbedded.jl*: v0.9.5. Version v0.9.5. Oct. 2024. doi: [10.5281/zenodo.13949319](https://doi.org/10.5281/zenodo.13949319). URL: <https://github.com/gridap/GridapEmbedded.jl>.
- [112] R. Saye. *algoim: Algorithms for implicitly defined geometry, level set methods, and Voronoi implicit interface methods*. URL: <https://github.com/algoim/algoim>.
- [113] E. Neiva. *Algoim.jl*. Version v0.2.2. Oct. 2024. doi: [10.5281/zenodo.13948122](https://doi.org/10.5281/zenodo.13948122). URL: <https://github.com/gridap/GridapEmbedded.jl>.

- [114] P. Amestoy et al. “A Fully Asynchronous Multifrontal Solver Using Distributed Dynamic Scheduling”. In: *SIAM Journal on Matrix Analysis and Applications* 23.1 (2001), pp. 15–41.
- [115] P. Amestoy et al. “Performance and Scalability of the Block Low-Rank Multifrontal Factorization on Multicore Architectures”. In: *ACM Transactions on Mathematical Software* 45 (1 2019), 2:1–2:26.
- [116] S. Balay et al. *PETSc Web page*. <https://petsc.org/>. 2021.
- [117] S. Balay et al. *PETSc Users Manual*. Tech. rep. ANL-95/11 - Revision 3.15. Argonne National Laboratory, 2021.
- [118] S. Balay et al. “Efficient Management of Parallelism in Object Oriented Numerical Software Libraries”. In: *Modern Software Tools in Scientific Computing*. Ed. by E. Arge, A. M. Bruaset, and H. P. Langtangen. Birkhauser Press, 1997, pp. 163–202.
- [119] J. Ahrens, B. Geveci, and C. Law. “ParaView: An End-User Tool for Large-Data Visualization”. In: *Visualization Handbook*. Ed. by C. D. Hansen and C. R. Johnson. Burlington: Butterworth-Heinemann, 2005, pp. 717–731. ISBN: 978-0-12-387582-2. doi: [10.1016/B978-012387582-2/50038-1](https://doi.org/10.1016/B978-012387582-2/50038-1).
- [120] J. Happel and H. Brenner. *Low Reynolds number hydrodynamics: with special applications to particulate media*. Vol. 1. Springer Science & Business Media, 2012.
- [121] T. Jankuhn et al. “Error analysis of higher order Trace Finite Element Methods for the surface Stokes equation”. In: *Journal of Numerical Mathematics* 29.3 (2021), pp. 245–267. doi: [10.1515/jnma-2020-0017](https://doi.org/10.1515/jnma-2020-0017).
- [122] A. Zöttl and H. Stark. “Emergent behavior in active colloids”. In: *Journal of Physics: Condensed Matter* 28.25 (2016), p. 253001. doi: [10.1088/0953-8984/28/25/253001](https://doi.org/10.1088/0953-8984/28/25/253001).
- [123] R. J. Hawkins et al. “Spontaneous contractility-mediated cortical flow generates cell migration in three-dimensional environments”. In: *Biophysical journal* 101.5 (2011), pp. 1041–1045. doi: [10.1016/j.bpj.2011.07.038](https://doi.org/10.1016/j.bpj.2011.07.038).
- [124] A. Mietke, F. Jülicher, and I. F. Sbalzarini. “Self-organized shape dynamics of active surfaces”. In: *Proceedings of the National Academy of Sciences* 116.1 (2019), pp. 29–34. doi: [10.1073/pnas.1810896115](https://doi.org/10.1073/pnas.1810896115).
- [125] R. A. Green, E. Paluch, and K. Oegema. “Cytokinesis in animal cells”. In: *Annual review of cell and developmental biology* 28.1 (2012), pp. 29–58. doi: [10.1146/annurev-cellbio-101011-155718](https://doi.org/10.1146/annurev-cellbio-101011-155718).
- [126] J. Sedzinski et al. “Polar actomyosin contractility destabilizes the position of the cytokinetic furrow”. In: *Nature* 476.7361 (2011), pp. 462–466. doi: [10.1038/nature10286](https://doi.org/10.1038/nature10286).
- [127] R. Rappaport. *Cytokinesis in animal cells*. Cambridge University Press, 1996.
- [128] W. M. Bement, H. A. Benink, and G. Von Dassow. “A microtubule-dependent zone of active RhoA during cleavage plane specification”. In: *The Journal of cell biology* 170.1 (2005), pp. 91–101. doi: [10.1083/jcb.200501131](https://doi.org/10.1083/jcb.200501131).
- [129] K. Sugioka. “Symmetry-breaking of animal cytokinesis”. In: *Seminars in cell & developmental biology*. Vol. 127. Elsevier. 2022, pp. 100–109. doi: [10.1016/j.semcdb.2021.12.008](https://doi.org/10.1016/j.semcdb.2021.12.008).
- [130] R. Rappaport and G. W. Conrad. “An experimental analysis of unilateral cleavage in invertebrate eggs”. In: *Journal of Experimental Zoology* 153.2 (1963), pp. 99–112. doi: [10.1002/jez.1401530203](https://doi.org/10.1002/jez.1401530203).
- [131] S. Herszterg et al. “Interplay between the dividing cell and its neighbors regulates adherens junction formation during cytokinesis in epithelial tissue”. In: *Developmental cell* 24.3 (2013), pp. 256–270. doi: [10.1016/j.devcel.2012.11.019](https://doi.org/10.1016/j.devcel.2012.11.019).
- [132] M. H. Gfrerer. “A C^1 -continuous Trace-Finite-Cell-Method for linear thin shell analysis on implicitly defined surfaces”. In: *Computational Mechanics* 67.2 (2021), pp. 679–697. doi: [10.1007/s00466-020-01956-5](https://doi.org/10.1007/s00466-020-01956-5).
- [133] R. Poincloux et al. “Contractility of the cell rear drives invasion of breast tumor cells in 3D Matrigel”. In: *Proceedings of the National Academy of Sciences* 108.5 (2011), pp. 1943–1948. doi: [10.1073/pnas.1010396108](https://doi.org/10.1073/pnas.1010396108).
- [134] H. De Belly et al. “Cell protrusions and contractions generate long-range membrane tension propagation”. In: *Cell* 186.14 (2023), pp. 3049–3061. doi: [10.1016/j.cell.2023.05.014](https://doi.org/10.1016/j.cell.2023.05.014).
- [135] P. Gross, K. V. Kumar, and S. W. Grill. “How active mechanics and regulatory biochemistry combine to form patterns in development”. In: *Annual review of biophysics* 46.1 (2017), pp. 337–356. doi: [10.1146/annurev-biophys-070816-033602](https://doi.org/10.1146/annurev-biophys-070816-033602).
- [136] N. W. Goehring et al. “Polarization of PAR proteins by advective triggering of a pattern-forming system”. In: *Science* 334.6059 (2011), pp. 1137–1141. doi: [10.1126/science.1208619](https://doi.org/10.1126/science.1208619).
- [137] P. Gross et al. “Guiding self-organized pattern formation in cell polarity establishment”. In: *Nature physics* 15.3 (2019), pp. 293–300. doi: [10.1038/s41567-018-0358-7](https://doi.org/10.1038/s41567-018-0358-7).
- [138] H. De Belly et al. “Long range mutual activation establishes Rho and Rac polarity during cell migration”. In: *bioRxiv* (2024), pp. 2024–10. doi: [10.1101/2024.10.01.616161](https://doi.org/10.1101/2024.10.01.616161).
- [139] J. Howard, S. W. Grill, and J. S. Bois. “Turing’s next steps: the mechanochemical basis of morphogenesis”. In: *Nature Reviews Molecular Cell Biology* 12.6 (2011), pp. 392–398. doi: [10.1038/nrm3120](https://doi.org/10.1038/nrm3120).

- [140] J. Halatek, F. Brauns, and E. Frey. “Self-organization principles of intracellular pattern formation”. In: *Philosophical Transactions of the Royal Society B: Biological Sciences* 373.1747 (2018), p. 20170107. doi: [10.1098/rstb.2017.0107](https://doi.org/10.1098/rstb.2017.0107).
- [141] F. Brauns et al. “Bulk-surface coupling identifies the mechanistic connection between Min-protein patterns in vivo and in vitro”. In: *Nature Communications* 12.1 (2021), p. 3312. doi: [10.1038/s41467-021-23412-5](https://doi.org/10.1038/s41467-021-23412-5).
- [142] E. A. Francis et al. “Spatial modeling algorithms for reactions and transport in biological cells”. In: *Nature Computational Science* (2024), pp. 1–14. doi: [10.1038/s43588-024-00745-x](https://doi.org/10.1038/s43588-024-00745-x).
- [143] W. Marth and A. Voigt. “Signaling networks and cell motility: a computational approach using a phase field description”. In: *Journal of mathematical biology* 69 (2014), pp. 91–112. doi: [10.1007/s00285-013-0704-4](https://doi.org/10.1007/s00285-013-0704-4).
- [144] B. A. Camley et al. “Crawling and turning in a minimal reaction-diffusion cell motility model: Coupling cell shape and biochemistry”. In: *Physical Review E* 95.1 (2017), p. 012401. doi: [10.1103/PhysRevE.95.012401](https://doi.org/10.1103/PhysRevE.95.012401).
- [145] T. Burkart, B. J. Müller, and E. Frey. “Dimensionality reduction in bulk-boundary reaction-diffusion systems”. In: *Physical Review E* 110.3 (2024), p. 034412. doi: [10.1103/PhysRevE.110.034412](https://doi.org/10.1103/PhysRevE.110.034412).
- [146] G. Salbreux, J. Prost, and J.-F. Joanny. “Hydrodynamics of cellular cortical flows and the formation of contractile rings”. In: *Physical review letters* 103.5 (2009), p. 058102. doi: [10.1103/PhysRevLett.103.058102](https://doi.org/10.1103/PhysRevLett.103.058102).
- [147] G. Salbreux et al. “Theory of nematic and polar active fluid surfaces”. In: *Physical Review Research* 4.3 (2022), p. 033158. doi: [10.1103/PhysRevResearch.4.033158](https://doi.org/10.1103/PhysRevResearch.4.033158).
- [148] A.-C. Reymann et al. “Cortical flow aligns actin filaments to form a furrow”. In: *eLife* 5 (2016), e17807. doi: [10.7554/eLife.17807](https://doi.org/10.7554/eLife.17807).
- [149] W. Mirza et al. “Theory of active self-organization of dense nematic structures in the actin cytoskeleton”. In: *eLife* 13 (2024), e93097. doi: [10.7554/elife.93097.1](https://doi.org/10.7554/elife.93097.1).
- [150] M. Nestler, I. Nitschke, and A. Voigt. “A finite element approach for vector-and tensor-valued surface PDEs”. In: *Journal of Computational Physics* 389 (2019), pp. 48–61. doi: [10.1016/j.jcp.2019.03.006](https://doi.org/10.1016/j.jcp.2019.03.006).
- [151] A. Torres-Sánchez, D. Santos-Oliván, and M. Arroyo. “Approximation of tensor fields on surfaces of arbitrary topology based on local Monge parametrizations”. In: *Journal of Computational Physics* 405 (2020), p. 109168. doi: [10.1016/j.jcp.2019.109168](https://doi.org/10.1016/j.jcp.2019.109168).
- [152] I. Nitschke and A. Voigt. “Active nematodynamics on deformable surfaces”. In: *Proceedings of the Royal Society A* 481.2310 (2025), p. 20240380. doi: [10.1098/rspa.2024.0380](https://doi.org/10.1098/rspa.2024.0380).
- [153] E. Moeendarbary et al. “The cytoplasm of living cells behaves as a poroelastic material”. In: *Nature materials* 12.3 (2013), pp. 253–261. doi: [10.1038/nmat3517](https://doi.org/10.1038/nmat3517).
- [154] P. Royer, P. Recho, and C. Verdier. “On the quasi-static effective behaviour of poroelastic media containing elastic inclusions”. In: *Mechanics Research Communications* 96 (2019), pp. 19–23. doi: [10.1016/j.mechrescom.2019.02.004](https://doi.org/10.1016/j.mechrescom.2019.02.004).
- [155] J.-L. Maître et al. “Asymmetric division of contractile domains couples cell positioning and fate specification”. In: *Nature* 536.7616 (2016), pp. 344–348. doi: [10.1038/nature18958](https://doi.org/10.1038/nature18958).
- [156] J. Firmin et al. “Mechanics of human embryo compaction”. In: *Nature* 629.8012 (2024), pp. 646–651. doi: [10.1038/s41586-024-07351-x](https://doi.org/10.1038/s41586-024-07351-x).
- [157] C. Bernardi, M. Dauge, and Y. Maday. *Spectral Methods for Axisymmetric Domains*. Vol. 3. Series in Applied Mathematics. Elsevier, co-published with Éditions Lavoisier, 1999.