2.4 Ratio estimator

Samuel et. al. (1992) developed a modified Horvitz-Thompson estimator for a population age or sex ratio. Given the same assumptions as section 2.1, define

τ_a = population size of a type animals,

τ_b = population size of b type animals,

$\hat{\tau}_{a\pi}$ = population size estimator of a type animals when sighting probabilities are known,

$\hat{\tau}_{b\pi}$ = population size estimator of b type animals when sighting probabilities are known,

$\hat{\tau}_{aLR}$ = population size estimator of a type animals when sighting probabilities are unknown and are estimated via a logistic model,

$\hat{\tau}_{bLR}$ = population size estimator of b type animals when sighting probabilities are unknown and are estimated via a logistic model,

a_{ij} = the number of a type animals in the jth group of ith primary unit.

b_{ij} = the number of b type animals in the jth group of ith primary unit.

and with the notation defined in the last section, an estimator for the ratio $R = \tau_a/\tau_b$, using (2.1.2) when sighting probabilities g_{ij} are specified, is

$$\widehat{R}_{\pi} = \frac{\hat{\tau}_{a\pi}}{\hat{\tau}_{b\pi}} = \frac{\sum_{i=1}^{N} \frac{I_i}{\pi_i} \sum_{j=1}^{M_i} Z_{ij} a_{ij} \Theta_{ij}}{\sum_{i=1}^{N} \frac{I_i}{\pi_i} \sum_{j=1}^{M_i} Z_{ij} b_{ij} \Theta_{ij}}. \tag{2.4.1}$$

Using (2.1.4) when the $\Theta_{ij} = 1/g_{ij}$'s are unknown and are estimated via a logistic model, the ratio estimator is

$$\widehat{R}_{LR} = \frac{\hat{\tau}_{aLR}}{\hat{\tau}_{bLR}} = \frac{\sum_{i=1}^{N} \frac{I_i}{\pi_i} \sum_{j=1}^{M_i} Z_{ij} a_{ij} \widehat{\Theta}_{ij}}{\sum_{i=1}^{N} \frac{I_i}{\pi_i} \sum_{j=1}^{M_i} Z_{ij} b_{ij} \widehat{\Theta}_{ij}}. \tag{2.4.2}$$

By applying the delta method,

$$\frac{\hat{\tau}_a}{\hat{\tau}_b} \doteq \frac{\tau_a}{\tau_b} + \frac{1}{\tau_b}(\hat{\tau}_a - \tau_a) - \frac{1}{\tau_b{}^2}(\hat{\tau}_b - \tau_b). \tag{2.4.3}$$

In (2.4.3), $\hat{\tau}_a$, $\hat{\tau}_b$ are $\hat{\tau}_{a\pi}$, $\hat{\tau}_{b\pi}$ (see (2.1.2)) if sighting probabilities are known and they are $\hat{\tau}_{aLR}$, $\hat{\tau}_{bLR}$ (see (2.1.4)) if sighting probabilities are unknown. Thus, we obtain an approximate variance of a ratio estimator as

$$\mathrm{var}(\widehat{R}) \doteq R^2\left[\frac{\mathrm{var}(\hat{\tau}_a)}{{\tau_a}^2} + \frac{\mathrm{var}(\hat{\tau}_b)}{{\tau_b}^2} - \frac{2\mathrm{cov}(\hat{\tau}_a, \hat{\tau}_b)}{\tau_a\tau_b}\right]. \tag{2.4.4}$$

To obtain an (approximate) estimator of $\mathrm{var}(\widehat{R})$ in (2.4.4), we replace the ratio, the population sizes, variances and covariances of the population size estimator by their respective unbiased estimators. We have already derived (approximate and/or asymptotic) unbiased estimators for the ratio, the population sizes (see (2.1.2) and (2.1.4)) and variances of the population size estimator (see (2.2.2) and (2.2.13)). Therefore, to extend the results of Samuel et. al. (1992), we only need to find an unbiased estimator for the covariance term. To do so, we write

$$\mathrm{cov}(\hat{\tau}_a, \hat{\tau}_b) = E[\hat{\tau}_a\hat{\tau}_b] - E[\hat{\tau}_a]E[\hat{\tau}_b] = E[\hat{\tau}_a\hat{\tau}_b] - \tau_a\tau_b. \tag{2.4.5}$$

Then, the expression $\hat{\tau}_a\hat{\tau}_b - \hat{\tau}_{ab}$, where $\hat{\tau}_{ab}$ is an unbiased estimator of $\tau_a\tau_b$, is an unbiased estimator of the covariance term in (2.4.5). To find an unbiased estimator of $\tau_a\tau_b$, we note that

$$\tau_a\tau_b = \left(\sum_{i=1}^{N}\sum_{j=1}^{M_i} a_{ij}\right)\left(\sum_{i=1}^{N}\sum_{j=1}^{M_i} b_{ij}\right) = \sum_{i=1}^{N}\sum_{j=1}^{M_i} a_{ij}b_{ij} + \sum_{i=1}^{N}\sum_{j=1}^{M_i}\sum_{j'=1}^{M_{i'}} a_{ij}b_{ij'} + \sum_{i\neq i'}^{N}\sum_{j=1}^{M_i}\sum_{j'=1}^{M_{i'}} a_{ij}b_{i'j'}.$$

Hence, an unbiased estimator for the covariance term in (2.4.5) is

$$\widehat{\mathrm{cov}}(\hat{\tau}_a, \hat{\tau}_b) = \hat{\tau}_{a\pi}\hat{\tau}_{b\pi} + \sum_{i=1}^{N}\frac{I_i}{\pi_i}\sum_{j=1}^{M_i}\frac{Z_{ij}a_{ij}b_{ij}}{g_{ij}} + \sum_{i=1}^{N}\frac{I_i}{\pi_i}\sum_{j\neq j'}^{M_i}\frac{Z_{ij}Z_{ij'}a_{ij}b_{ij'}}{g_{ij}g_{ij'}}$$
$$+ \sum_{i\neq i'}^{N}\frac{I_iI_{i'}}{\pi_{ii'}}\sum_{j=1}^{M_i}\sum_{j'=1}^{M_{i'}}\frac{Z_{ij}Z_{i'j'}a_{ij}b_{i'j'}}{g_{ij}g_{i'j'}}. \tag{2.4.5}$$

for the case of known g_{ij}'s.

Change this + sign
to a minus sign

For the case of unknown Θ_{ij}'s estimated via a logistic model, an unbiased estimator for the covariance term in (2.4.5) is

$$\begin{aligned}\widehat{\mathrm{cov}}(\hat{\tau}_a, \hat{\tau}_b) = \hat{\tau}_{aLR}\hat{\tau}_{bLR} &+ \sum_{i=1}^{N} \frac{I_i}{\pi_i} \sum_{j=1}^{M_i} Z_{ij} a_{ij} b_{ij} \widehat{\Theta}_{ij} \\ &+ \sum_{i=1}^{N} \frac{I_i}{\pi_i} \sum_{j \neq j'}^{M_i} Z_{ij} Z_{ij'} a_{ij} b_{ij'} [-\widehat{\mathrm{cov}}(\widehat{\Theta}_{ij}, \widehat{\Theta}_{ij'}) + \widehat{\Theta}_{ij}\widehat{\Theta}_{ij'}] \\ &+ \sum_{i \neq i'}^{N} \frac{I_i I_{i'}}{\pi_{ii'}} \sum_{j=1}^{M_i} \sum_{j'=1}^{M_{i'}} Z_{ij} Z_{i'j'} a_{ij} b_{i'j'} [-\widehat{\mathrm{cov}}(\widehat{\Theta}_{ij}, \widehat{\Theta}_{i'j'}) + \widehat{\Theta}_{ij}\widehat{\Theta}_{i'j'}]. \qquad (2.4.6)\end{aligned}$$

Change this` + sign to a minus sign

Notice the brackets