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Machine Learning

Lecture 10: Bayesian belief networks

Statistical independence

- The naïve Bayesian classifier assumed that all features are independent

- This allowed to compute the likelihood as a product of the individual feature likelihoods

$$P[x_1, \dots, x_n|\omega] = \prod_{i=1}^{n} P[x_i|\omega]$$

- In many applications this is not the case, which is why we called this assumption and hence the classifier "naïve"

Statistical independence

- For example we could observe two features:
  - If there is rain or not
  - If the grass is wet or not

- Now let’s assume we determine it rains on half of all days
- We also observe that the grass is wet on half of all days
- Therefore, the probability of rain is 50% and the chance of the grass being wet is also 50%

- But does this mean that the chance of the grass being wet and there to be rain at the same time is $50\% \times 50\% = 25\%$?

- Obviously not, because these two observations are not independent of each other
- We cannot simply multiply probabilities of two correlated variables to obtain the joint probability of the two variables

Conditional independence

- One solution to this issue is Bayesian belief networks, which are based on the notion of **conditional independence**

- Assume the two variables x_1 and x_2 are correlated, i.e. we cannot multiply them as we did in the naïve classifier

$$P[x_1, x_2] \neq P[x_1]P[x_2]$$

- Sometimes we can identify a third variable y that makes the two variables x_1 and x_2 conditional independent, i.e.

$$P[x_1, x_2|y] = P[x_1|y]P[x_2|y]$$

- For example consider the observation of the following three events

x_1	Rain
x_2	Thunder
y	Lightning

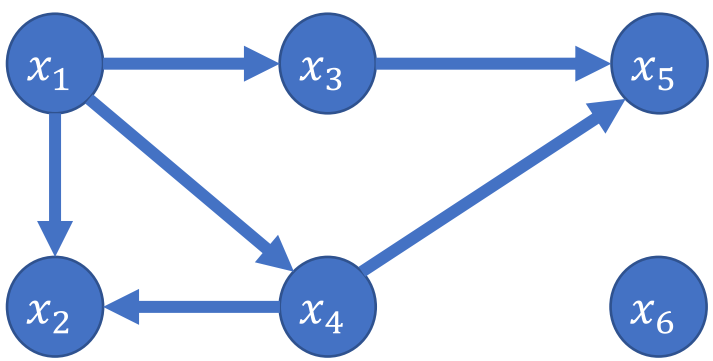
- Obviously rain and thunder are not in general independent
- However, as soon as we definitely know that a lightning strike occurred the probabilities of rain and thunder become independent
- While the fact that it was raining contains some information about thunder, that information is subsumed by the information about lightning

Conditional independence

- The information about what variables are conditional independent is domain knowledge we can add to our classifier

- To enable the description of the probabilistic structure of a problem domain we can use graphical models to represent the mutual relationships between the variables

- A **Bayesian network** is a directed acyclic graph that helps us factorise the joint probability distribution of a problem domain



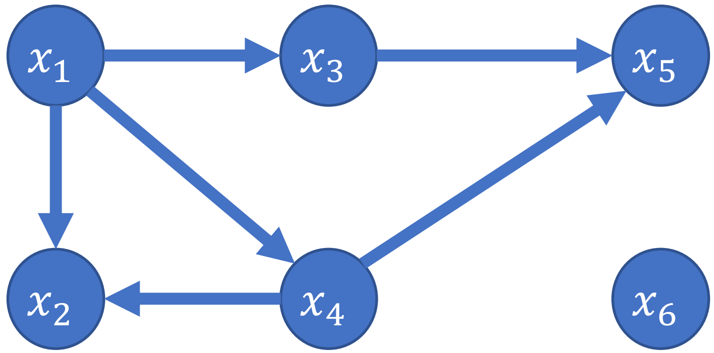
- Each vertex represents a variable, each edge represents a dependence

Bayesian network

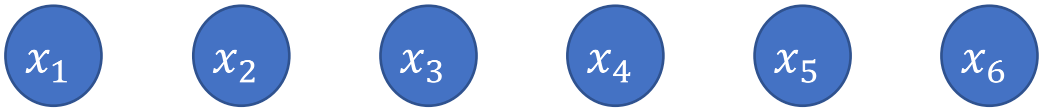
- The structure of the network encodes the knowledge that each node is conditionally independent given all its parent nodes

- It is a graphical notation to describe the structure of the joint probability distribution based on the structure of the graph

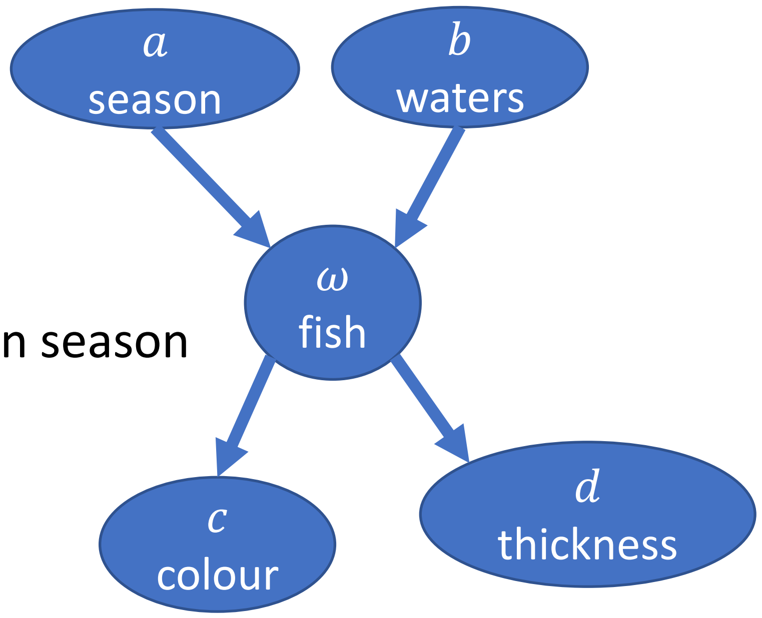
$$P[x_1, x_2, x_3, x_4, x_5, x_6] = P[x_1]P[x_2|x_1, x_4]P[x_3|x_1]P[x_4|x_1]P[x_5|x_3, x_4]P[x_6]$$



- Note, that the naïve Bayesian classifier relied on the assumption that there are no edges (i.e. dependencies) between the variables

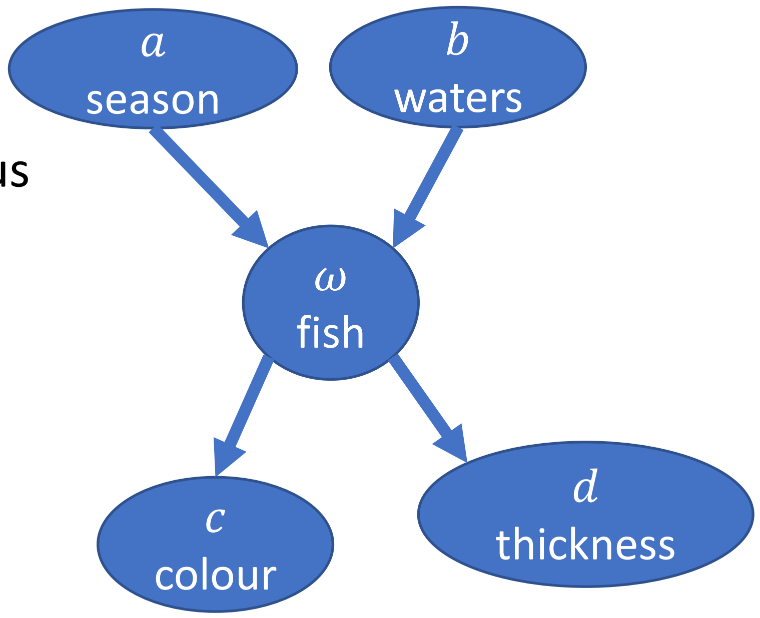


Example: Fish



- Let's assume we want to distinguish
  - ω_1: salmon
  - ω_2: sea bass
- We know that the fish population depends on season
  - a_1: winter
  - a_2: spring
  - a_3: summer
  - a_4: autumn
- as well as the fishing waters
  - b_1: north
  - b_2: south
- The colouring of the fish depends on the type of fish
  - c_1: light
  - c_2: medium
  - c_3: dark
- As does its thickness
  - d_1: wide
  - d_2: thin

Example: Fish



- A fishing expert (or learning algorithm) tells us that the conditional probabilities of catching a fish depending on the season is

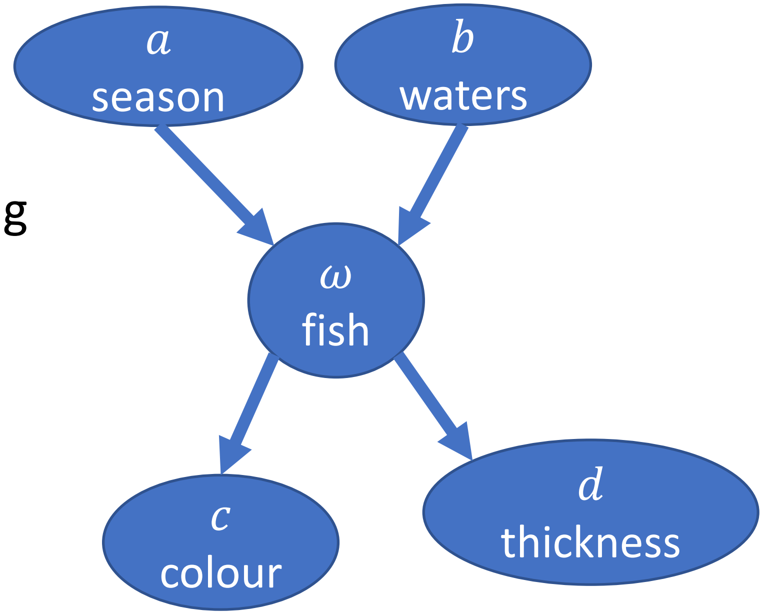
$P[\omega\|a]$	Salmon	Sea bass
Winter	.9	.1
Spring	.3	.7
Summer	.4	.6
Autumn	.8	.2

- And that the probabilities of catching a fish depending on the waters is

$P[\omega\|b]$	Salmon	Sea bass
North	.65	.35
South	.25	.75

- Note, that the conditional probabilities add up to 1 in each row

Example: Fish



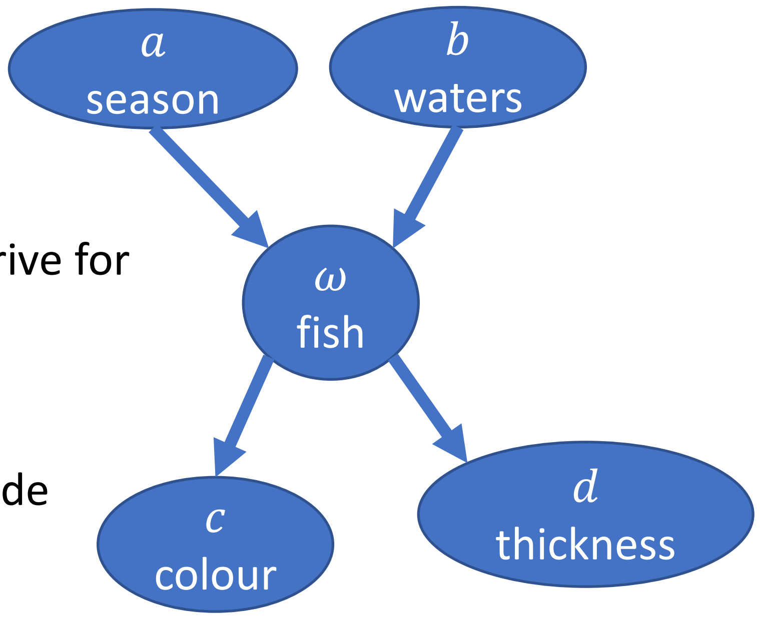
- Further to that the fishing expert (or learning algorithm) determines that colours are distributed as follows

$P[c\|\omega]$	Light	Medium	Dark
Salmon	.33	.33	.34
Sea bass	.8	.1	.1

- And that the thickness is distributed as

$P[d\|\omega]$	wide	thin
Salmon	.4	.6
Sea bass	.95	.05

Example: Fish



- The structure of the problem domain is visualised in the graphical representation, which is easy to derive for a domain expert
- It can be translated into a probabilistic model by multiplying one conditional probability term per node
- Each conditional probability term represents the probability of the variable having a value, given that its immediate predecessors are known
- The resulting joint probability of all variables for the example is then

$$P[\omega, a, b, c, d] = P[a]P[b]P[c|\omega]P[d|\omega]P[\omega|a, b]$$

- The reason why we can use this simplified product is the additional structural domain knowledge we put in exploiting conditional independences between variables
- The model can still be more complex than the naïve Bayesian model where everything is independent

Example: Fish

$P[\omega\|a]$	**Salmon**	**Sea bass**
Winter	.9	.1
Spring	.3	.7
Summer	.4	.6
Autumn	.8	.2

$$P[\omega, a, b, c, d] = P[a]P[b]P[c|\omega]P[d|\omega]P[\omega|a, b]$$

- Case 1: we do not have any additional information

- Then the probability of catching salmon is

$$P[\omega_1] = \sum_{i,j,k,l} P[\omega_1, a_i, b_j, c_k, d_l]$$

$$= \sum_{i,j,k,l} P[a_i]P\left[b_j\right]P[c_k|\omega_1]P[d_l|\omega_1]P[\omega_1|a_i, b_j]$$

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$$= \sum_{i,j} P[a_i]P[b_j]P[\omega_1|a_i, b_j]$$

$P[c_k|\omega]$ and $P[d_l|\omega]$ add up to 1

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$P[c_k|\omega]$ and $P[d_l|\omega]$ add up to 1

$$= \sum_{i,j} P[a_i]P[b_j]P[\omega_1|a_i, b_j]$$

There are 4 seasons and 2 potential fishing waters

$$= \sum_{i,j} \frac{1}{4}\frac{1}{2} P[\omega_1|a_i, b_j]$$

Example: Fish

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$$= \sum_{i,j} \frac{1}{4}\frac{1}{2} P[\omega_1|a_i, b_j]$$

$$= (.25)(.5)\left(\frac{(.9)(.65)}{(.9)(.65) + (.1)(.35)} + \frac{(.3)(.65)}{(.3)(.65) + (.7)(.35)} + \frac{(.4)(.65)}{(.4)(.65) + (.6)(.35)}\right.$$

$$+ \frac{(.8)(.65)}{(.8)(.65) + (.2)(.35)} + \frac{(.9)(.25)}{(.9)(.25) + (.1)(.75)} + \frac{(.3)(.25)}{(.3)(.25) + (.7)(.75)} + \frac{(.4)(.25)}{(.4)(.25) + (.6)(.75)}$$

$$\left. + \frac{(.8)(.25)}{(.8)(.25) + (.2)(.75)}\right) = .56$$

Example: Fish

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North	.65	.35
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- <u>Case 1</u>: we do not have any additional information
- Then the probability of catching salmon is

$$P[\omega_1] = \sum_{i,j,k,l} P[\omega_1, a_i, b_j, c_k, d_l]$$

$$= \sum_{i,j,k,l} P[a_i]P[b_j]P[c_k|\omega_1]P[d_l|\omega_1]P[\omega_1|a_i, b_j]$$

$P[c_k|\omega]$ and $P[d_l|\omega]$ add up to 1

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There are 4 seasons and 2 potential fishing waters

$$= \sum_{i,j} \frac{1}{4}\frac{1}{2} P[\omega_1|a_i,b_j]$$

$$= (.25)(.5)\left(\frac{(.9)(.65)}{(.9)(.65)+(.1)(.35)} + \frac{(.3)(.65)}{(.3)(.65)+(.7)(.35)} + \frac{(.4)(.65)}{(.4)(.65)+(.6)(.35)} \right.$$
$$+ \frac{(.8)(.65)}{(.8)(.65)+(.2)(.35)} + \frac{(.9)(.25)}{(.9)(.25)+(.1)(.75)} + \frac{(.3)(.25)}{(.3)(.25)+(.7)(.75)} + \frac{(.4)(.25)}{(.4)(.25)+(.6)(.75)}$$
$$\left. + \frac{(.8)(.25)}{(.8)(.25)+(.2)(.75)}\right) = .56$$

Example: Fish

$$P[\omega, a, b, c, d] = P[a]P[b]P[c|\omega]P[d|\omega]P[\omega|a, b]$$

- Case 2: we know it is winter
- Then the probability of catching salmon is

$$P[\omega_1] = \sum_{i,j,k,l} P[\omega_1, a_i, b_j, c_k, d_l]$$

$P[\omega\|a]$	Salmon	Sea bass
Winter	.9	.1
Spring	.3	.7
Summer	.4	.6
Autumn	.8	.2

$P[\omega\|b]$	Salmon	Sea bass
North	.65	.35
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Example: Fish

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$$= \sum_{i,j} P[a_i]P[b_j]P[\omega_1|a_i, b_j]$$

$P[c_k|\omega]$ and $P[d_l|\omega]$ add up to 1

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$$= \sum_{i,j} P[a_i]P[b_j]P[\omega_1|a_i, b_j]$$

$$= \sum_{j} \frac{1}{2} P[\omega_1|a_1, b_j]$$

$$= (.5)\left(\frac{(.9)(.65)}{(.9)(.65) + (.1)(.35)} + \frac{(.9)(.25)}{(.9)(.25) + (.1)(.75)}\right) = .85$$

We only sum over the winter terms now

Example: Fish

$P[\omega\|a]$	Salmon	Sea bass
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Autumn	.8	.2

$$P[\omega, a, b, c, d] = P[a]P[b]P[c|\omega]P[d|\omega]P[\omega|a, b]$$

$P[\omega\|b]$	Salmon	Sea bass
North	.65	.35
South	.25	.75

- Case 2: we know it is winter

- Then the probability of catching salmon is

$$P[\omega_1] = \sum_{i,j,k,l} P[\omega_1, a_i, b_j, c_k, d_l]$$

$$= \sum_{i,j,k,l} P[a_i]P[b_j]P[c_k|\omega_1]P[d_l|\omega_1]P[\omega_1|a_i, b_j]$$

$P[c_k|\omega]$ and $P[d_l|\omega]$ add up to 1

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We only sum over the winter terms now

Example: Fish

$$P[\omega, a, b, c, d] = P[a]P[b]P[c|\omega]P[d|\omega]P[\omega|a, b]$$

- Case 3: we know it is winter and that the fish we caught is thin
- Then the probability of the fish being salmon is then

$$P[\omega_1|d_2] = \frac{P[\omega_1, d_2]}{P[d_2]}$$
$$= \frac{\sum_{i,j,k} P[\omega_1, a_i, b_j, c_k, d_2]}{P[d_2]}$$

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Winter	.9	.1
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North	.65	.35
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$P[d\|\omega]$	wide	thin
Salmon	.4	.6
Sea bass	.95	.05

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$P[c_k|\omega]$ adds up to 1

winter

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$P[c_k|\omega]$ adds up to 1

$$= \frac{\sum_{i,j} P[a_i]P[b_j]P[d_2|\omega_1]P[\omega_1|a_i, b_j]}{P[d_2]}$$

winter

$$= \frac{\sum_j \frac{1}{2} P[d_2|\omega_1]P[\omega_1|a_1, b_j]}{P[d_2]}$$

$$= (.5)(.6)\left(\frac{(.9)(.65)}{(.9)(.65) + (.1)(.35)} + \frac{(.9)(.25)}{(.9)(.25) + (.1)(.75)}\right)/P[d_2] = .51/P[d_2]$$

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winter

Example: Fish

$$P[\omega, a, b, c, d] = P[a]P[b]P[c|\omega]P[d|\omega]P[\omega|a, b]$$

- <u>Case 3</u>: we know it is winter and that the fish we caught is thin

- To determine $P[d_2\ ]$ we do the same for the sea bass

$$P[\omega_2|d_2] = \frac{P[\omega_2, d_2]}{P[d_2]}$$
$$= \frac{\sum_{i,j,k} P\left[\omega_2, a_i, b_j, c_k, d_2\right]}{P[d_2]}$$

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- To determine $P[d_2]$ we do the same for the sea bass

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$$= (.5)(.05)\left(\frac{(.1)(.35)}{(.9)(.65) + (.1)(.35)} + \frac{(.1)(.75)}{(.9)(.25) + (.1)(.75)}\right)/P[d_2] = .008/P[d_2]$$

Example: Fish

$$P[\omega, a, b, c, d] = P[a]P[b]P[c|\omega]P[d|\omega]P[\omega|a, b]$$

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$$P[\omega_2|d_2] = \frac{P[\omega_2, d_2]}{P[d_2]}$$
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$$= \frac{\sum_{i,j,k} P[a_i]P\left[b_j\right]P[c_k|\omega_2]P[d_2|\omega_2]P\left[\omega_2|a_i, b_j\right]}{P[d_2]}$$
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$$= \frac{\sum_j \frac{1}{2} P[d_2|\omega_2]P\left[\omega_2|a_1, b_j\right]}{P[d_2]}$$
$$= (.5)(.05)\left(\frac{(.1)(.35)}{(.9)(.65) + (.1)(.35)} + \frac{(.1)(.75)}{(.9)(.25) + (.1)(.75)}\right)/P[d_2] = .008/P[d_2]$$

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Example: Fish

$$P[\omega, a, b, c, d] = P[a]P[b]P[c|\omega]P[d|\omega]P[\omega|a, b]$$

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- To determine $P[d_2]$ we do the same for the sea bass

$$P[\omega_2|d_2] = \frac{P[\omega_2, d_2]}{P[d_2]}$$
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- To determine $P[d_2\ ]$ we do the same for the sea bass

$$P[\omega_2|d_2] = \frac{P[\omega_2, d_2]}{P[d_2]}$$

$$= \frac{\sum_{i,j,k} P[\omega_2, a_i, b_j, c_k, d_2]}{P[d_2]}$$

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$$= \frac{\sum_j \frac{1}{2} P[d_2|\omega_2]P[\omega_2|a_1, b_j]}{P[d_2]}$$

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Example: Fish

$$P[\omega, a, b, c, d] = P[a]P[b]P[c|\omega]P[d|\omega]P[\omega|a, b]$$

- Case 3: we know it is winter and that the fish we caught is thin

- We now determined the two probabilities up to $P[d_2]$

$$P[\omega_1|d_2] = \frac{.51}{P[d_2]}$$

$$P[\omega_2|d_2] = \frac{.008}{P[d_2]}$$

$P[\omega\|a]$	**Salmon**	**Sea bass**
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- Case 3: we know it is winter and that the fish we caught is thin

- We now determined the two probabilities up to $P[d_2]$

$$P[\omega_1|d_2] = \frac{.51}{P[d_2]}$$
$$P[\omega_2|d_2] = \frac{.008}{P[d_2]}$$

- We also know that they normalise as follows:

$$P[\omega_1|d_2] + P[\omega_2|d_2] = 1$$

- We can now determine that $P[d_2] = .51 + .008$ to normalise the probabilities
- Finally, the probability of having caught a salmon in case the fish is thin and was caught in winter is

$$P[\omega_1|d_2] = \frac{.51}{.51 + .008} = .99$$

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Summary

- Bayesian believe networks allow to graphically capture the structure of a problem domain by identifying mutual dependences/independences

- The graphical representation translates into a joint probability function, which can be efficiently computed in case most variables are conditionally independent

- Additional knowledge (sometimes called evidence) can be added bit by bit to refine the results

- This also allows to substitute missing values using the input derived from the domain knowledge, both on the structure as well as on conditional dependences

Thank you for your attention