Comparative Analysis of Numerical Methods for Solving the Poisson Equation

CEE 6513 Computational Methods in Mechanics Final Project

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1. Introduction

This report investigates three numerical methods - Finite Element Method (FEM), Finite Difference Method (FDM), and Fourier Series Method - for solving the Poisson equation with periodic boundary conditions. It focuses on implementing and comparing these methods in terms of accuracy, efficiency, and convergence. The findings assist in selecting suitable numerical approaches for complex problem-solving in applied sciences.

2. Background

Consider the Poisson equation:

$$-\frac{d^2u}{dx^2} = \rho$$

on the interval $[-L,\ L]$ subject to periodic boundary conditions.

- Energy Functional

Starting with the equation $-\frac{d^2u}{dx^2} = \rho$. Multiply by a test function $v(x)$ and integrate over the domain:

$$\int_{-L}^{L} -\frac{d^2u}{dx^2} v(x)dx = \int_{-L}^{L} \rho v(x)dx$$

$$\int_{-L}^{L} -\frac{d^2u}{dx^2} v(x)dx = -\left[\frac{du}{dx}v(x)\right]_{-L}^{L} + \int_{-L}^{L} \frac{du}{dx}\frac{dv}{dx}dx = \int_{-L}^{L} \rho v(x)dx$$

apply the boundary conditions $u(-L) = u(L), v(-L) = v(L)$:

$$\int_{-L}^{L} \frac{du}{dx}\frac{dv}{dx}dx = \int_{-L}^{L} \rho v(x)dx$$

formulate the energy functional:

$$E[u] = \frac{1}{2}\int_{-L}^{L} [\left(\frac{du}{dx}\right)^2 - \rho u]dx$$

- Constant c

Starting with the equation $-\frac{d^2u}{dx^2} = e^{-x^2} + c,$ integrate over the domain.

$$\int_{-L}^{L} -\frac{d^2u}{dx^2}dx = \int_{-L}^{L} (e^{-x^2} + c)dx$$

$$u'(-L) - u'(L) = 0 = \sqrt{\pi} + 2Lc$$

$$c = -\frac{\sqrt{\pi}}{2L}$$

The Poisson equation requires numerical approximation methods since the exact analytical solutions are unattainable or impractical. This report applies FEM, FDM, and the Fourier Series Method to approximate the solution in a domain with periodic boundary conditions.

3. Methodology

This study employs three numerical methods to approximate solutions to the Poisson equation: Finite Element Method (FEM), Finite Difference Method (FDM), and Fourier Series Method. Each method is distinct in its approach to discretization and approximation.

- Finite Element Method (FEM)

FEM divides the domain into a mesh of elements and approximates the solution using a variational method. I implemented a second-order FEM with quadratic shape functions and a third-order FEM with cubic shape functions to compute the stiffness matrix and force vector, applying periodic boundary conditions. The mesh refinement was varied to study the convergence behavior.

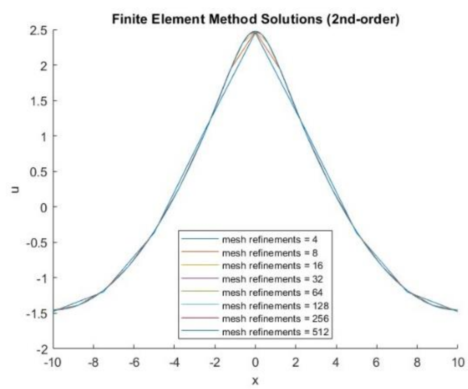
- Finite Difference Method (FDM)

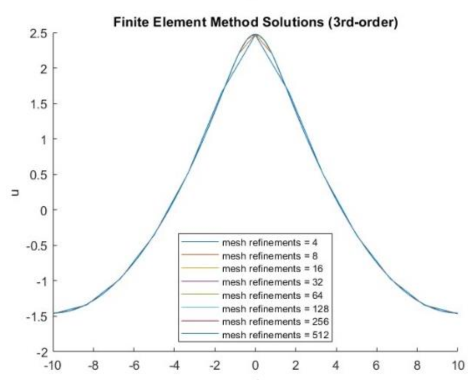
FDM discretizes the differential equation using finite differences, approximating derivatives at grid points. I used a central difference scheme for 2nd-order FDM and a higher-order scheme for 4th-order FDM. Considering periodic boundary conditions, the Poisson equation was discretized on a uniform grid.

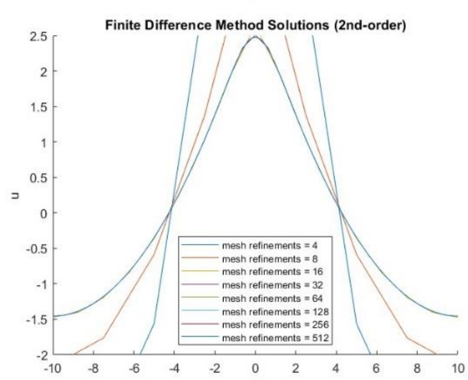
- Fourier Series Method

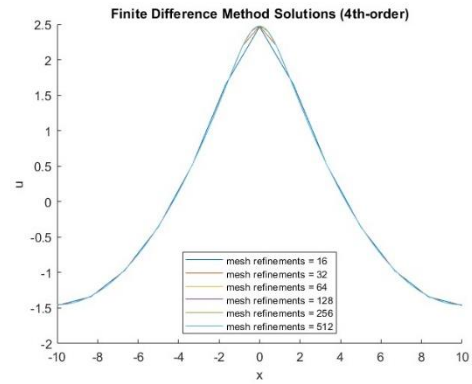
This method expands the solution in terms of Fourier series, capturing the periodic nature of the problem. The Fourier coefficients were computed using numerical integration. The series was truncated after a certain number of terms, which was varied to observe the convergence and accuracy.

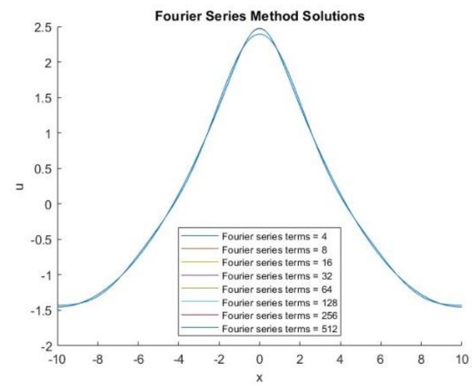
4. Results

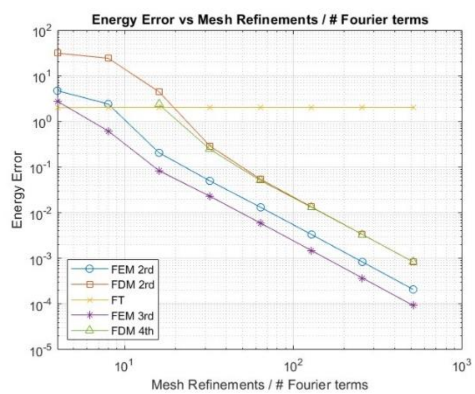












5. Discussion

The effectiveness of the Finite Element Method (FEM), its higher-order variant (FEM HO), Finite Difference Method (FDM), its higher-order variant (FDM HO), and the Fourier Series Method (FT) in solving the Poisson equation was assessed. The evaluation criteria included the solutions' accuracy and the energy error across varying mesh refinements or Fourier series terms.

- Finite Element Method Solutions (2nd-order)

The 2nd-order FEM solutions show a consistent trend towards convergence as mesh refinement increases. With finer meshes, the accuracy of the solutions improves, indicating the method's effectiveness and stability for solving differential equations.

- Finite Element Method Solutions (3rd-order)

The 3rd-order FEM displays a significant improvement in convergence rates over the 2nd-order FEM, achieving higher accuracy levels with the same mesh density. This highlights the benefits of higher-order elements in capturing the solution more precisely, especially when solution gradients are steep.

- Finite Difference Method Solutions (2nd-order)

The 2nd-order FDM solutions converge with increasing mesh refinements, but the convergence is less rapid than the higher-order methods. Oscillations are present with coarser meshes, suggesting a lower accuracy at equivalent mesh sizes than higher-order methods.

- Finite Difference Method Solutions (4th-order)

The 4th-order FDM solutions exhibit a more pronounced reduction in error and faster convergence than the 2nd-order solutions. The higher-order scheme demonstrates the capacity to accurately resolve the solution with fewer grid points, affirming the superiority of higher-order discretization for computational efficiency.

- Fourier Series Method Solutions

The FT method's solutions improve with an increasing number of terms. However, the convergence is gradual, and the technique may encounter limitations due to numerical phenomena such as the Gibbs phenomenon, which can affect the solution's smoothness and accuracy.

- Energy Error vs. Mesh Refinements / # Fourier Terms

The energy error comparison provides a quantitative measure of the approximation error for each method. The 4th-order FDM exhibits the lowest energy errors, followed by the 3rd-order FEM, indicating that higher-order methods are more effective at capturing the actual energy of the system.

The 2nd-order FEM and FDM show more significant energy errors, which decrease with mesh refinement. The Fourier Series Method's error reduction is less pronounced, indicating a potential trade-off between the method's global nature and the solution's precision.

6. Conclusion

The comparative study of numerical methods for solving the Poisson equation reveals distinct advantages and limitations of each approach. In its second and higher-order iterations, the Finite Element Method demonstrates robustness and improved accuracy with mesh refinement, with the higher-order variant achieving significant gains in solution precision. The Finite Difference Method, particularly the fourth-order scheme, offers superior convergence rates and reduced energy error. It is an excellent choice for problems with paramount computational efficiency and accuracy. While providing valuable insights into the solution structure, the Fourier Series Method shows limitations in convergence speed and may not be the most practical for high-precision requirements.

In conclusion, the selection of a numerical method must be tailored to the specific needs of the problem, balancing factors such as computational resources, desired accuracy, and the nature of the solution domain. Higher-order methods are recommended where fine-scale details are critical, while the Fourier Series can serve as a complementary approach for its analytical perspective. This study underscores the importance of methodological consideration in computational mathematics and provides a foundation for selecting appropriate numerical strategies in engineering and applied sciences.

7. References

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