CEE 6513, Computational Methods in Mechanics

Fall 2023, Georgia Institute of Technology

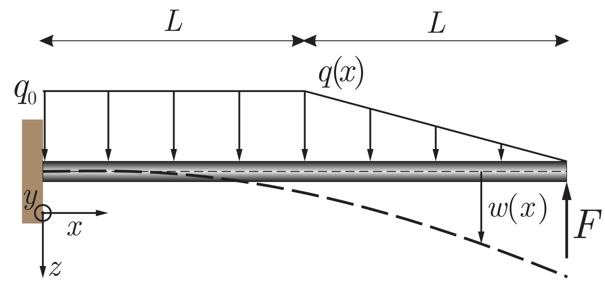
Homework 3 solutions

A linear elastic cantilever beam of length $2L$ (constant Young modulus E and area moment I_y) is loaded by a distributed transverse load $q(x)$ of the sketched form with known q_0. We use Bernoulli's theory of elastic beams: the governing equation for the deflection $w(x)$ in the z-direction reads

$$E\,I_y\,w_{,xxxx}(x) = q(x).$$

The transverse force (in the positive z-direction) and the bending moment (about the y-axis) are given by

$$F_z(x) = -E\,I_y\,w_{,xxx}(x),$$
$$M_y(x) = -E\,I_y\,w_{,xx}(x),$$



respectively. In the given problem, the beam is loaded by the distributed force as well as a single force F at its free end (and the moment at the free end vanishes). Let us find an approximate solution for the deflection.

a) Formulate the Galerkin weak form of the problem.

Solution: The Galerkin method starts by computing the residual of the ODE and introducing an in principle arbitrary weight function $W(x)$. The residual of the given ODE is

$$r_\Omega(x) = EI_y w_{,xxxx}(x) - q(x).$$

Multiplying by the weight function and integrating over the domain, we have the total residual

$$R = \int_\Omega W\,r_\Omega d\Omega = \int_0^{2L} W(EI_y w_{,xxxx} - q)\,\mathrm{d}x.$$

To arrive at the Galerkin weak form, we aim for the order of derivatives on the weight function W as the beam deflection w, i.e., we aim to create a symmetric bilinear form $G(u, W)$. Performing integration by parts, we obtain

$$R = WEI_y w_{,xxx}|_0^{2L} - \int_0^{2L} (W_{,x}EI_y w_{,xxx} + Wq)\,\mathrm{d}x$$

Imposing that the weight function (and its appropriate derivatives) must vanish wherever essential boundary conditions are enforced (i.e. $W(x=0) = W_{,x}(x=0) = 0$) and applying the natural boundary condition $EI_y w_{,xxx}(x=2L) = -F_z = -(-F) = F$ (since F points into the negative z-direction), we have

$$\begin{aligned}
R =& W(2L)EI_y w_{,xxx}(2L) - W(0)EI_y w_{,xxx}(0) - \int_0^{2L} (W_{,x}EI_y w_{,xxx} + Wq)dx \\
=& W(2L)F - \int_0^{2L} (W_{,x}EI_y w_{,xxx} + Wq)dx
\end{aligned}$$

Performing integration by parts again, we arrive at

$$R = W(2L)F - W_{,x}EI_y w_{,xx}|_0^{2L} + \int_0^{2L} (W_{,xx}EI_y w_{,xx} - Wq)\,\mathrm{d}x.$$

Enforcing that $W_{,x}$ vanishes wherever essential boundary conditions restrain $w_{,x}$ to a specific value (here we know that $w_{,x}(0) = 0$ and therefore $W_{,x}(x = 0) = 0$) and imposing the natural boundary condition of a vanishing moment on the free end ($EI_y w_{,xx}(x = 2L) = 0$), we obtain

$$\begin{aligned} R =& W(2L)F - W_{,x}(2L)EI_y w_{,xx}(2L) + W_{,x}(0)EI_y w_{,xx}(0) + \int_0^{2L} (W_{,xx}EI_y w_{,xx} - Wq)dx \\ =& W(2L)F + \int_0^{2L} (W_{,xx}EI_y w_{,xx} - Wq)dx \end{aligned}$$

We seek a solution such that the residual is zero. We now have the so-called Galerkin weak form (and we know that a solution $w(x)$ that satisfies the weak form for any admissible function $W(x)$ also satisfies the original strong form):

$$\boxed{W(2L)F + \int_0^{2L} (W_{,xx}EI_y w_{,xx} - Wq)dx = 0}$$

b) Let us find an approximate solution by assuming a polynomial of degree n with appropriate restrictions on the polynomial coefficients to satisfy essential boundary conditions. Use this approximation in a *Bubnov-Galerkin* formulation to arrive at a linear system $\boldsymbol{Ka} = \boldsymbol{F}$ for the unknown coefficients a_i. What are matrix and vector components K_{ij} and F_i, respectively (generally written for i and j)?

Solution: The *Bubnov-Galerkin* formulation proceeds by choosing the same basis functions ϕ_i for the weight function W and the deflection w. Using polynomials as the basis functions with coefficients b_i for the weight function and a_i for the deflection, respecively, we thus assume

$$w(x) = \sum_{i=0}^{n} a_i\phi_i(x), \qquad W(x) = \sum_{i=0}^{n} b_i\phi_i(x).$$

Imposing the condition that that the deflection and weight function satisfy essential boundary conditions ($w(0) = w_{,x}(0) = 0$ and therefore $W(0) = W_{,x}(0) = 0$), we see that the first two coefficients must vanish, so that from now on we use

$$w(x) = \sum_{i=2}^{n} a_i\phi_i(x), \qquad W(x) = \sum_{i=2}^{n} b_i\phi_i(x).$$

Substituting these expressions into the Galerkin weak form from part (a), we arrive at the linear system of equations

$$\sum_{i=2}^{n} b_i\phi_i(2L)F + \int_0^{2L} \left(\sum_{i=2}^{n} b_i\phi_{i,xx}(x)EI_y \sum_{j=2}^{n} a_j\phi_{j,xx}(x) - \sum_{i=2}^{n} b_i\phi_i(x)q \right) dx = 0.$$

Factoring out the summations over b_i and a_i (since these coefficients are constants that do not depend on position x) results in

$$\sum_{i=2}^{n} b_i \left[\phi_i(2L)F + \sum_{j=2}^{n} \int_0^{2L} EI_y\phi_{i,xx}\phi_{j,xx}dx\, a_j - \int_0^{2L} \phi_i(x)qdx \right] = 0.$$

We wish for the above expression to hold for any choice of b_i, which means each term in the summation over i must be zero (i.e., the bracketed terms must vanish) and therefore we have $n-2$ equations left for the $n-2$ unknowns a_i with $i = 2, 3, \ldots, n$:

$$\phi_i(2L)F + \sum_{j=2}^{n} \left[\int_0^{2L} EI_y \phi_{i,xx} \phi_{j,xx} dx \right] a_j - \int_0^{2L} \phi_i(x) q dx = 0.$$

Now making the substitution that the weight functions are polynomials, i.e. $\phi_i(x) = x^i$ and $\phi_{i,xx} = i(i-1)x^{i-2}$, we can write

$$(2L)^i F + \sum_{j=2}^{n} \left[\int_0^{2L} EI_y ij(i-1)(j-1) x^{i+j-4} dx \right] a_j - \int_0^{2L} x^i q dx = 0.$$

The polynomial integrals can be evaluated analyticaly. While skipping the algebra on the first term for consiceness, the second integral requires special attention due to the piecewise linear load $q(x)$. This second integral evaluates as

$$\int_0^{2L} x^i q(x) dx = \int_0^{L} x^i q_0 dx + \int_L^{2L} x^i q_0 (2 - \frac{x}{L}) dx = \frac{q_0 L^{i+1}(2^{i+2}-1)}{(i+1)(i+2)}.$$

Altogether, we now have

$$(2L)^i F + \sum_{j=2}^{n} \left[\frac{EI_y ij(i-1)(j-1)}{i+j-3} (2L)^{i+j-3} \right] a_j - \frac{q_0 L^{i+1}(2^{i+2}-1)}{(i+1)(i+2)} = 0,$$

which can be written in index notation (where $i, j = 2, 3, \ldots, n$):

$$K_{ij} a_j = F_i,$$

or in symbolic notation:

$$\boxed{\boldsymbol{K}\boldsymbol{a} = \boldsymbol{F}}$$

where the elements of $\boldsymbol{K}$ and $\boldsymbol{F}$ can be identified as

$$\boxed{K_{ij} = \frac{EI_y ij(i-1)(j-1)}{i+j-3} (2L)^{i+j-3}} \quad \boxed{F_i = \frac{q_0 L^{i+1}(2^{i+2}-1)}{(i+1)(i+2)} - (2L)^i F}$$

and the components of the yet to be determined vector $\boldsymbol{a}$ are the sought coefficients a_j.

c) Solve the system for the unknown coefficients for $n = 2, 3, 4, 5$ and plot the resulting deflection curve for each case. For this part, you can use the following numerical values for a beam made of steel:

$E = 210 \cdot 10^9 \text{N/m}^2$, $I_y = \frac{1}{12}(10\text{mm})^4$, $q_0 = 100\text{N/m}$, $L = 100\text{mm}$, $F = 40\text{N}$

Solution: With the given numerical values, $\boldsymbol{K}$ and $\boldsymbol{F}$ can be computed numerically and the sought vector $\boldsymbol{a}$ can be obtained as

$$\boldsymbol{a} = \boldsymbol{K}^{-1}\boldsymbol{F},$$

so that the final solution $w_n(x)$ for each case n can be plotted according to

$$w_n(x) = \sum_{i=2}^{n} a_i \phi_i(x).$$

A plot of the deflected beam is shown below in figure 1.

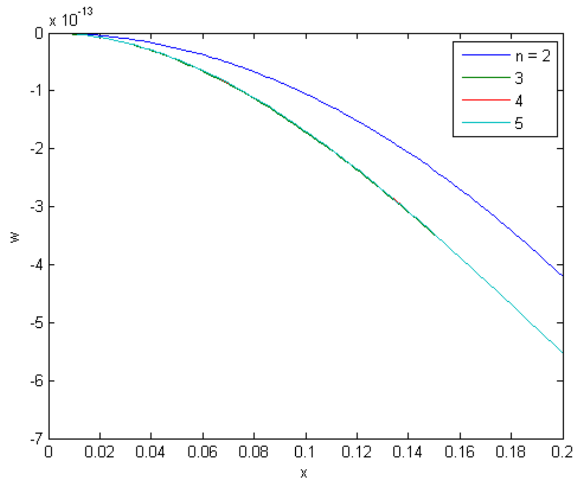


Figure 1: Deflection of the beam for polynomials of order $n = 2, 3, 4$, and 5. The solution converges for $n = 3, 4$, and 5.

d) Let us show that the problem can also be solved by the *Rayleigh-Ritz* method. To this end, use variational calculus to show that the above boundary value problem derives from the potential

$$I[w] = \int_0^{2L} \left[\frac{1}{2} E\, I_y\, w_{,xx}^2(x) - q(x)\, w(x) \right] \,\mathrm{d}x + F\, w(2L).$$

Solution: The *Rayleigh-Ritz* method is based on minimizing a total potential, which requires the existence of a potential that leads to the same solution as solving the strong form (i.e., the given PDE system with boundary conditions). Hence, we should verify that minimizing the given total potential energy is equivalent to solving the strong form with appropriate boundary conditions. The minimum of the above functional is found by applying the stationarity condition, i.e. setting its first variation equal to zero. Taking the first variation of the potential results in (written out in full, but one can greatly abbreviate this)

$$\begin{aligned}
\delta I[w] &= \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} I[w + \varepsilon\, \delta w] \\
&= \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} \left(\int_0^{2L} \left[\frac{1}{2} EI_y \left((w + \varepsilon\, \delta w)_{,xx} \right)^2 - q(w + \varepsilon\, \delta w) \right] dx + F(w(2L) + \varepsilon\, \delta w(2L)) \right) \\
&= \lim_{\varepsilon \to 0} \left(\int_0^{2L} \frac{d}{d\varepsilon} \left[\frac{1}{2} EI_y \left((w + \varepsilon\, \delta w)_{,xx} \right)^2 - q(w + \varepsilon\, \delta w) \right] dx + F\delta w(2L) \right) \\
&= \lim_{\varepsilon \to 0} \left(\int_0^{2L} \left[EI_y \left((w + \varepsilon\, \delta w)_{,xx} \right) \delta w_{,xx} - q \delta w \right] dx + F\delta w(2L) \right) && (1) \\
&= \int_0^{2L} \left[EI_y w_{,xx} \delta w_{,xx} - q \delta w \right] dx + F\delta w(2L). && (2)
\end{aligned}$$

Alternatively, one could have arrived there directly by writing

$$
\begin{aligned}
\delta I[w] &= \int_0^{2L} \left[\frac{1}{2} E\, I_y\, \delta\left(w_{,xx}^2\right) - q\,\delta w\right]\, \mathrm{d}x + F\,\delta w(2L) \\
&= \int_0^{2L} \left[\frac{1}{2} E\, I_y\, 2 w_{,xx}\,\delta w_{,xx} - q\,\delta w\right]\, \mathrm{d}x + F\,\delta w(2L).
\end{aligned}
$$

Performing integration by parts twice to reduce the order of derivatives on δw (thus bringing $\delta I[w]$ back to the starting point of the Galerkin method), we see that

$$
\begin{aligned}
\delta I[w] =& EI_y w_{,xx}\delta w_{,x}|_0^{2L} - \int_0^{2L} \left[EI_y w_{,xxx}\delta w_{,x} + q\delta w\right] dx + F\delta w(2L) \\
=& EI_y w_{,xx}(2L)\delta w_{,x}(2L) - EI_y w_{,xx}(0)\delta w_{,x}(0) - EI_y w_{,xxx}\delta w|_0^{2L} \\
&+ \int_0^{2L} \left[EI_y w_{,xxxx}\delta w - q\delta w\right] dx + F\delta w(2L).
\end{aligned}
$$

Like the weight function W above, the variations δw (and appropriate derivatives) must vanish where essential boundary conditions are enforced on w (and respective derivatives), so $\delta w_{,x}(0) = \delta w(0) = 0$, which gives

$$
\delta I[w] = EI_y w_{,xx}(2L)\delta w_{,x}(2L) - (EI_y w_{,xxx}(2L) - F)\delta w(2L) + \int_0^{2L} \left[EI_y w_{,xxxx} - q\right] \delta w dx.
$$

The expression must hold for any admissible $\delta w(x)$, which means in particular that we can vary δw, $\delta w(2L)$, and $\delta w_{,x}(2L)$ independently. Therefore, each term must be zero independenty, so that stationarity of the given potential requires that

$$
\boxed{EI_y w_{,xx}(2L) = 0} \quad \boxed{EI_y w_{,xxx}(2L) = F} \quad \boxed{EI_y w_{,xxxx} = q \quad \text{for } 0 \le x \le 2L}
$$

This set of equations is exactly the boundary value problem we are aiming to solve (ODE plus boundary conditions). Therefore, minimizing the potential is equivalent to solving the boundary value problem.

e) Using variations, show that the solution indeed corresponds to a minimum of the total potential energy.

Solution: To check whether the stationarity condition in part d) is a minimun (and not a maximum), take the second variation of $I[w]$, or just take the first variation of $\delta I[w]$. We will chose $\delta I[w]$ derived above in the specific form (2), which results in

$$
\begin{aligned}
\delta^2 I[w] &= \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} \delta I[w + \varepsilon\,\delta w] \\
&= \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} \left\{ \int_0^{2L} \left[EI_y(w_{,xx} + \varepsilon\,\delta w_{,xx})\delta w_{,xx} - q\delta w\right] dx + F\delta w(2L) \right\} \\
&= \int_0^{2L} EI_y \delta w_{,xx}\delta w_{,xx}\, \mathrm{d}x \\
&= \int_0^{2L} EI_y \delta(\delta w_{,xx})^2\, \mathrm{d}x.
\end{aligned}
$$

Apparently, the final form indicates that for any admissible non-zero variation $\delta w_{,xx}$ (more generally, even for any non-zero variation) we have $\delta^2 I[w] > 0$, so we can only have a minimum in the total potential energy. This confirms that minimizing the given potential is equivalent to solving the strong form.

f) The error of an approximate solution can be characterized by the *energy norm* which is defined by $I[w_{\text{approx}}(x) - w_{\text{exact}}(x)]$. Since we do not have an exact solution, let us assume that the solution for $n = 5$ from c) is the exact solution (the error is indeed negligible). Compute the energy error for your solutions from c) for $n = 2$, $n = 3$ and $n = 4$ (using the above numerical values).

Solution: For any computed solution $w_n(x)$, the energy norm of the error in the approximate solution can be computed as

$$
\begin{aligned}
e[w_n] = &\int_0^{2L} \left[\frac{1}{2} E\, I_y \, \left(w_{n,xx}(x) - w_{\text{exact},xx}(x)\right)^2 - q(x)\, \left(w_n(x) - w_{\text{exact},xx}(x)\right)\right] \mathrm{d}x \\
&+ F\, \left(w_n(2L) - w_{\text{exact},xx}(2L)\right),
\end{aligned}
$$

and for simplicity (for not having to compute the exact solution) we assume $w_5(x)$ as the exact solution with marginal error. This gives the computed values:

polynomial order n	error (10^{-12})
2	-5.938
3	0.019
4	0.000
5	0.000