Covariant derivatives along geodesic and non-geodesic curves on a sphere

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(Version 1)

This note examines how the covariant derivative of tangent vectors along a curve differs between geodesic and non-geodesic curves on a 2-D spherical manifold. The definitions and derivations for this case are based on results contained in the excellent short paper: "Christoffel Symbols Defined for a Sphere"

https://physicspages.com/pdf/Relativity/Christoffel%20symbols%20defined%20for%20a%20sphere.pdf.

A covariant derivative, aka connection[1], calculates the change in a vector field Y on a manifold in the direction of another vector field X. The definition is[2]

(1) $$\nabla_X Y = (X(Y^k) + X^i Y^j \Gamma^k_{ij})E_k$$

where Γ^k_{ij} are Christoffel symbols and E_k is the k-th component of the local coordinate system (i.e., basis vectors). The $X(Y^k)$ term represents the directional derivative of the components of Y while the 2nd term represents the derivatives of the basis vectors in the direction of X.

An important version of this operator is the derivative of a vector field V along a parmeterized curve $\gamma(t)$, defined as[3]

(2) $$D_t V(t) = \nabla_{\gamma'(t)} \tilde{V} .$$

Here, we have the tangent vector to the curve, $\gamma'(t) = d\gamma/dt$, playing the role of the "direction" vector X in eq. (1) with the vector field V representing Y. The tilda over V on the right side of (2) represents an extension of V into $\mathbb{R}^n$ so that partial derivatives in all directions can be defined[4]. Expanding out eq. (2), we have[5] for the derivative vector as a collection of k components

(3) $$D_t V = \left(\dot{V}^k + \dot{\gamma}^i V^j \Gamma^k_{ij}(\gamma)\right)\partial_k$$

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[1] Lee, *Introduction to Riemannian Manifolds* (henceforth LeeRM), pg, 89.
[2] LeeRM, Eq. (4.9), pg. 92
[3] LeeRM, pg. 102
[4] LeeRM, pgs. 88, 100 and 102.
[5] LeeRM, eq. (4.15), pg.102

where the dots over V and γ denote the derivative of the coefficients γ^i with respect to the parameter t. Thus V^k-dot represents an acceleration along the k-th component of the vector V and γ^j -dot is the j-th component of the tangent vector to γ. More precisely,

$$\gamma(t)=\left(\gamma^1(t),\dots,\gamma^n(t)\right)$$

$$\text{(4)}\qquad \frac{d\gamma^k}{dt}=\dot{\gamma}^k$$

$$\frac{d\gamma}{dt}=\gamma'=\left(\dot{\gamma}^1,\dots,\dot{\gamma}^n\right)$$

A key application of this derivative is in the definition of a geodesic γ_G, expressed as

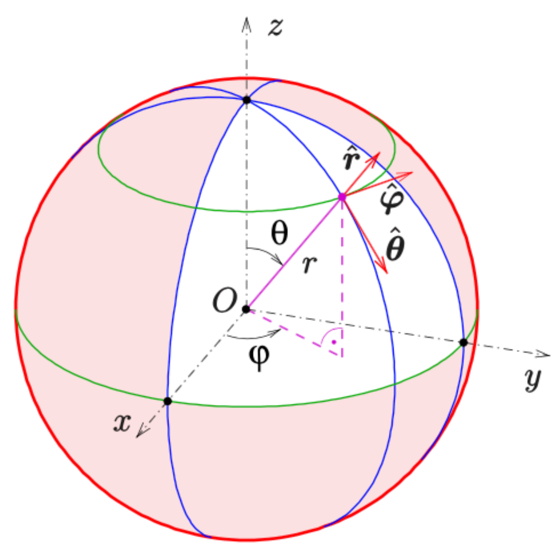
$$\text{(5)}\qquad D_t\gamma_G' = 0$$

which states that the tangent vectors to the curve γ_G remain constant with respect to D_t, i.e., these velocity vectors remain "straight" and do not experience any tangential acceleration. On the other hand, they can experience accelerations normal to the tangent space as they bend with the manifold but these are ignored[6] in eqs. (3) and (5).

The goal of the following calculations is to examine in more detail how eq. (5) is satisfied if γ is a geodesic curve and what happens when γ is not, i.e., $D_t\gamma' \neq 0$.

Spherical manifold First we need to define a spherical manifold and the relevant coordinate systems. Per the reference above, a unit sphere is described by the position vector in $\mathbb{R}^3$ in spherical coordinates (see Figure)

$$\text{(4)}\qquad X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$



In this spherical geometry, the parameters θ and φ define orthogonal directions on the sphere. Taking partial derivatives of X with respect to them gives basis vectors of the tangent plane at a point (here, of course, the radius r of the sphere is constant and is equal to one);

[6] Needham, pg. 243

(5)
$$e_\theta = \frac{\partial X}{\partial \theta} = \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix}$$
$$e_\varphi = \frac{\partial X}{\partial \theta} = \begin{pmatrix} -\sin\theta\sin\varphi \\ \sin\theta\cos\varphi \\ 0 \end{pmatrix}$$

Note that e_θ is the same as $\hat{\theta}$ in the Figure and describes the tangent to a geodesic meridian (constant φ). The basis vector e_φ corresponds to $\hat{\varphi}$ in the Figure and represents the tangent to a non-geodesic line of latitude (constant θ). The curves used to analyze the covariant derivative in this study are in fact these geodesic meridian and non-geodesic latitude lines (see below).

Next on the list of definitions are the Christoffel symbols for a sphere. These quantities represent the coefficients of the vectors representing the change in basis vectors, along each other, due to curvature. From Eq. (1) of above reference,

(6) $$\frac{\partial e_i}{\partial x^j} = \Gamma^k_{ij} e_k \, .$$

In the present context, if we let $i \to \theta$ and $j \to \varphi$, eq. (6) becomes (implied summation over k)

(7) $$\frac{\partial e_\theta}{\partial \varphi} = \Gamma^\theta_{\theta\varphi} e_\theta + \Gamma^\varphi_{\theta\varphi} e_\varphi$$

which represents the change in the basis vector e_θ along the basis vector e_φ. The full collection of $2^3 = 8$ terms for a sphere is

(8)
$$\Gamma^\theta_{\theta\theta} = \Gamma^\varphi_{\theta\theta} = \Gamma^\theta_{\theta\varphi} = \Gamma^\theta_{\varphi\theta} = \Gamma^\varphi_{\varphi\varphi} = 0$$
$$\Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \frac{\cos\theta}{\sin\theta}$$
$$\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta$$

I found it a worthwhile exercise to confirm a couple of these quantities, eg. $\Gamma_{\theta\varphi}{}^{\phi}$, by hand using the above Figure.

Geodesic equation To test whether a curve is a geodesic, we need to evaluate eq. (3) with the arbitrary vector *V* replaced by the tangent vector to the curve. Rewriting eq. (3) gives

(9) $$D_t\gamma'(t)=(\ddot{\gamma}^k+\dot{\gamma}^i\dot{\gamma}^j\Gamma^k_{ij})e_k$$

where γ^k- double-dot = $(d^2\gamma^k/dt^2)_k$ is the acceleration of the *k*-th component with respect to the parameter *t*. This equation equals zero if γ is a geodesic· It is these coefficients $\ddot{\gamma}^k$ *and* $\dot{\gamma}^k$ that need to be calculated for a given curve γ in order to evaluate eq. (9).

Curves on the sphere We now want to apply the covariant derivative in eq. (9) to actual curves. The candidate curves, γ_G (geodesic) and γ_L (non-geodesic), are defined to be very simple. For γ_G it is the meridian at $\varphi = 0$ and for γ_L it is the line of latitude at θ = constant (corresponding to the smaller green ellipse in the Figure)· Their (x,y,z) coordinates are

(10)
$$\gamma_G=\begin{pmatrix} sin\theta \\ 0 \\ cos\theta \end{pmatrix}$$
$$\gamma_L=\begin{pmatrix} sin\theta_0 cos\varphi \\ sin\theta_0 sin\varphi \\ sin\theta_0 \end{pmatrix}$$

where the symbol θ_0 is used to emphasize that θ is constant along γ_L.

We now need to define the tangent vectors to these curves, i.e, **velocities**, in terms of the basis vectors e_θ and e_φ. This means we need to define the derivative $d\gamma/dt$ at a point where $t \in \mathbb{R}$ is an appropriate parameter for the map $t \mapsto \gamma(t)$. As exemplified in eqs.(10), the components of a curve $\gamma = (\gamma^1, \gamma^2, \gamma^3)$ are scalar-valued functions of two variables $f : (\theta, \varphi) \mapsto \gamma^k \in \mathbb{R}$. In general, we can take $\theta(t)$ and $\varphi(t)$ to be functions of *t*. Using the definition of the total derivative[7], we can write the *t*-derivative of a component γ^k as

(11) $$\frac{d\gamma^k}{dt}=\frac{\partial\gamma^k}{\partial\theta}\frac{d\theta}{dt}+\frac{\partial\gamma^k}{\partial\varphi}\frac{d\varphi}{dt}$$

By construction, the two curves we are using for this analysis depend on only one of these variables; θ for γ_G and φ for γ_L. Thus, for the geodesic curve γ_G, we can take $\theta = t$ and φ = *constant* which means that $d\varphi/dt = 0$. Similarly, for γ_L, let $\varphi = t$ and $d\theta/dt = 0$. Applying these substitutions to eq. (11) gives

[7] https://mathworld.wolfram.com/TotalDerivative.html

$$\frac{d\gamma_G}{d\theta} = \frac{d}{d\theta}\begin{pmatrix} sin\theta \\ 0 \\ cos\theta \end{pmatrix} = \begin{pmatrix} cos\theta \\ 0 \\ -sin\theta \end{pmatrix}$$

$$\frac{d\gamma_G}{d\varphi} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(12) $$\frac{d\gamma_L}{d\varphi} = \frac{d}{d\varphi}\begin{pmatrix} sin\theta_0 cos\varphi \\ sin\theta_0 sin\varphi \\ sin\theta_0 \end{pmatrix} = \begin{pmatrix} -sin\theta_0 sin\varphi \\ sin\theta_0 cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ sin\theta_0 \\ 0 \end{pmatrix}_{\varphi=0}$$

$$\frac{d\gamma_L}{d\theta} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Comparing eqs. (12) with eqs. (5) (with φ = 0), it can be seen that the tangent vectors and the basis vectors are the same, as expected. The coefficients for the velocities needed to evaluate the covariant derivative, eq. (9), are thus

(13) $$\begin{aligned} &\frac{d\gamma_G}{d\theta} = \dot{\gamma}_G^{\theta} e_\theta \\ &\Rightarrow \dot{\gamma}_G^{\theta} = 1,\ \ \dot{\gamma}_G^{\varphi} = 0 \\ &\frac{d\gamma_L}{d\varphi} = \dot{\gamma}_L^{\varphi} e_\varphi \\ &\Rightarrow \dot{\gamma}_L^{\varphi} = 1,\ \ \dot{\gamma}_L^{\theta} = 0 \end{aligned}$$

We also need to calculate the coefficients for the derivative of the velocities, i.e. **accelerations**, along each curve. Using the same approach as above, we obtain

(14) $$\begin{aligned} &\gamma_G'' = \frac{d\gamma_G'}{dt} = \frac{d}{d\theta}\left(cos\theta, 0, -sin\theta\right) \\ &= (-sin\theta, 0, -cos\theta) \\ &\gamma_L'' = \frac{d\gamma_L'}{dt} = \frac{d}{d\varphi}\left(-sin\theta\, sin\varphi, sin\theta\, cos\varphi, 0\right) = (-sin\theta\, cos\varphi, -sin\theta\, sin\varphi, 0) \\ &= \left(-sin\theta, 0, 0\right)_{\varphi=0} \end{aligned}$$

Unlike the velocity vectors in eq. (12), the acceleration vectors in eq. (14) don't coincide nicely with the basis vectors e_θ and e_φ. This means that a little more work needs to be done to obtain their coefficients. For the geodesic curve γ_G this can be done by calculating the projection of the acceleration vectors on to the basis vectors using the inner (dot) product;

$$\ddot{\gamma}_G^\theta = \gamma_G' \bullet e_\theta = (-sin\theta, 0, -cos\theta) \bullet (cos\theta, 0, -sin\theta) = -sin\theta cos\theta + cos\theta sin\theta = 0$$
$$\ddot{\gamma}_G^\varphi = \gamma_G' \bullet e_\varphi = (-sin\theta, 0, -cos\theta) \bullet (0, sin\theta, 0) = 0$$

which shows that these coefficients are zero for γ_G.

Looking at the situation from a 3-D perspective, the acceleration points in the opposite direction to the normal vector to the surface, *n = (sinθ,0,cosθ)*, and thus points inward towards the center of the sphere, i.e., $\gamma_G \cdot n = -1$.

For γ_L, the dot product gives

$$\ddot{\gamma}_L^\theta = \gamma'' \bullet e_\theta = (-sin\theta, 0, 0) \bullet (cos\theta, 0, -sin\theta) = -sin\theta cos\theta$$
$$\ddot{\gamma}_L^\varphi = \gamma'' \bullet e_\varphi = (-sin\theta, 0, 0) \bullet (0, sin\theta, 0) = 0$$

Here are the coefficients gathered together:

$$Geodesic$$
$$\dot{\gamma}_G^\theta = 1$$
$$\dot{\gamma}_G^\varphi = 0$$
$$\ddot{\gamma}_G^\theta = 0$$
$$\ddot{\gamma}_G^\theta = 0$$

$$non-Geodesic$$
$$\dot{\gamma}_L^\theta = 0$$
$$\dot{\gamma}_L^\varphi = 1$$
$$\ddot{\gamma}_L^\theta = -sin\theta cos\theta$$
$$\ddot{\gamma}_L^\varphi = 0$$

The covariant derivative equation, eq. (9), expressed in terms of the quantities defined above, becomes;

Geodesic curve

$$\left(D_t\gamma'_G\right)_\theta = \ddot{\gamma}^\theta_G + \dot{\gamma}^\theta_G\dot{\gamma}^\theta_G\Gamma^\theta_{\theta\theta} + \dot{\gamma}^\theta_G\dot{\gamma}^\varphi_G\Gamma^\theta_{\theta\varphi} + \dot{\gamma}^\varphi_G\dot{\gamma}^\theta_G\Gamma^\theta_{\varphi\theta} + \dot{\gamma}^\varphi_G\dot{\gamma}^\varphi_G\Gamma^\theta_{\varphi\varphi}$$
$$\left(D_t\gamma'_G\right)_\varphi = \ddot{\gamma}^\varphi_G + \dot{\gamma}^\theta_G\dot{\gamma}^\theta_G\Gamma^\varphi_{\theta\theta} + \dot{\gamma}^\theta_G\dot{\gamma}^\varphi_G\Gamma^\varphi_{\theta\varphi} + \dot{\gamma}^\varphi_G\dot{\gamma}^\theta_G\Gamma^\varphi_{\varphi\theta} + \dot{\gamma}^\varphi_G\dot{\gamma}^\varphi_G\Gamma^\varphi_{\varphi\varphi}$$

Substituting the coefficient values gives

$$\left(D_t\gamma'_G\right)_\theta = 0+1\cdot1\cdot0+1\cdot0\cdot0+0\cdot1\cdot0+0\cdot0\cdot0=0$$
$$\left(D_t\gamma'_G\right)_\varphi = 0+1\cdot1\cdot0+1\cdot0\cdot\cot\theta+0\cdot1\cdot\cot\theta+0\cdot1\cdot0=0$$

Non-Geodesic curve

$$\left(D_t\gamma'_L\right)_\theta = \ddot{\gamma}^\theta_L + \dot{\gamma}^\theta_L\dot{\gamma}^\theta_L\Gamma^\theta_{\theta\theta} + \dot{\gamma}^\theta_L\dot{\gamma}^\varphi_L\Gamma^\theta_{\theta\varphi} + \dot{\gamma}^\varphi_L\dot{\gamma}^\theta_L\Gamma^\theta_{\varphi\theta} + \dot{\gamma}^\varphi_L\dot{\gamma}^\varphi_L\Gamma^\theta_{\varphi\varphi}$$
$$\left(D_t\gamma'_L\right)_\varphi = \ddot{\gamma}^\varphi_L + \dot{\gamma}^\theta_L\dot{\gamma}^\theta_L\Gamma^\varphi_{\theta\theta} + \dot{\gamma}^\theta_L\dot{\gamma}^\varphi_L\Gamma^\varphi_{\theta\varphi} + \dot{\gamma}^\varphi_L\dot{\gamma}^\theta_L\Gamma^\varphi_{\varphi\theta} + \dot{\gamma}^\varphi_L\dot{\gamma}^\varphi_L\Gamma^\varphi_{\varphi\varphi}$$

Substituting the coefficient values gives

$$\left(D_t\gamma'_L\right)_\theta = -\sin\theta\cos\theta+0\cdot0\cdot0+0\cdot1\cdot0+1\cdot0\cdot0+1\cdot1\cdot(-\sin\theta\cos\theta)=-2\sin\theta\cos\theta=-\sin2\theta$$
$$\left(D_t\gamma'_L\right)_\varphi = 0+0\cdot0\cdot0+0\cdot1\cdot\cot\theta+1\cdot0\cdot\cot\theta+0\cdot0\cdot0=0$$

In summary:

$$D_t\gamma'_G = \left(0,0\right)$$
$$D_t\gamma'_L = \left(-\sin2\theta,0\right)$$

Thus, γ_G does indeed satisfy the criterion for a geodesic; i.e., eq. (9) = 0.

The covariant derivative of the non-geodesic curve γ_L has a non-zero term in the negative e_θ direction, pointing back towards the north pole. This indicates that the tangent vector associated with the line-of-latitude γ_L veers away from the geodesic passing through a point with the same initial direction. It also goes to zero at $\theta = \pi/2$ as it should since γ_L represents the equator there, which is a geodesic.

Discussion This note was motivated by my interest in how the covariant derivative, $D_t\,\gamma'$, describes the change in the tangent vector to a curve γ when the curve veers away from a geodesic. This type of analysis also can be applied to understanding how a vector remains parallel when transported along a non-geodesic curve.

For a flat Euclidean manifold, a parallel vector remains fixed relative to the unchanging basis vectors of the flat frame while the curve wanders around. Deviation of the curve from a straight line (geodesic) induces a change in the angle between the parallel vector and the tangent vector to the curve. For a curved manifold, the changing orientation of the basis vectors of the local tangent space, encapsulated by the Christoffel symbols, can induce additional tangential accelerations along a curve that must be compensated for in order for a vector to remain parallel, i.e., $D_t V = 0$ means the vector must turn (i.e., accelerate in the tangent space) with respect to the tangent vector $\gamma'(t)$.

Further analysis using the tools developed here must, however, remain for a future memo.