Homework 3 (STAT 5860)

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Due by 11:59 pm, Feb. 24, 2018

Instructions:

1. Download the `Homework3.Rmd` file from the course Elearning.
2. Open `Homework3.Rmd` in RStudio.
3. Replace the *"Your Name Here"* text in the `author` with your own name.
4. Write your answer to each problem by editing `Homework3.Rmd`.
5. After you finish all the problems, click `Knit to PDF` to create a pdf file. Upload your pdf file to Homework 1 Dropbox in the course Elearning.

Set the seed number

```
set.seed(224)
```

Problem 1. The Rayleigh distribution is used to model lifetime subject to rapid aging, because the hazard rate is linearly increasing. The Rayleigh density is

$$f(x) = \frac{x}{\sigma^2}e^{-x^2/(2\sigma^2)}, \quad x \geq 0, \sigma > 0.$$

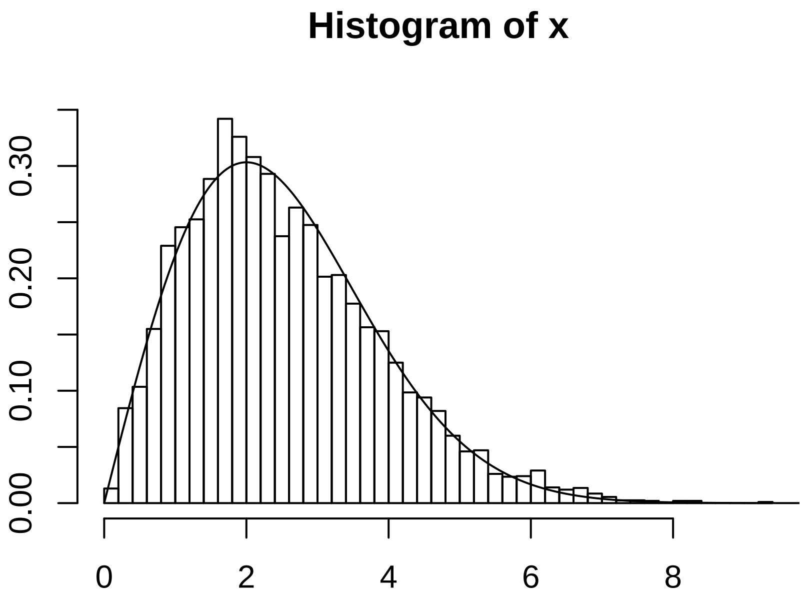
(a) Use the Metropolis-Hastings algorithm to generate a random sample of size 10000 from the Rayleigh ($\sigma = 2$) distribution. Use the proposal distribution Gamma$(X_t, 1)$ (shape parameter X_t and rate parameter 1). (Hint: Use `rgamma()` and `dgamma()`)

```
n <- 10000
x <- rep(0,n)

rayliegh <- function(x) {
  (x/4)*exp(-x^2/8)
}

q <- function(x) {
  rgamma(1, shape = x, rate = 1)
}

x[1] <- 5

for (i in 1:n) {
  xp <- q(x[i])
  alpha <- min(1, (rayliegh(xp)*dgamma(x[i],
                                       shape = xp,
                                       rate = 1))/(rayliegh(x[i])*dgamma(xp,
                                                                         shape = x[i],
                                                                         rate = 1)))
  if (runif(1) < alpha){
```

```
        x[i+1] <- xp
    }
    else{
        x[i+1] <- x[i]
    }
}
```

(b) Plot the density histogram of the sample and add the theoretical density curve $f(x)$ to the density histogoram.

```
hist(x, prob = TRUE, xlab = "", ylab = "", breaks = 50)
lines(seq(0,10,.01), rayliegh(seq(0,10,.01)))
```

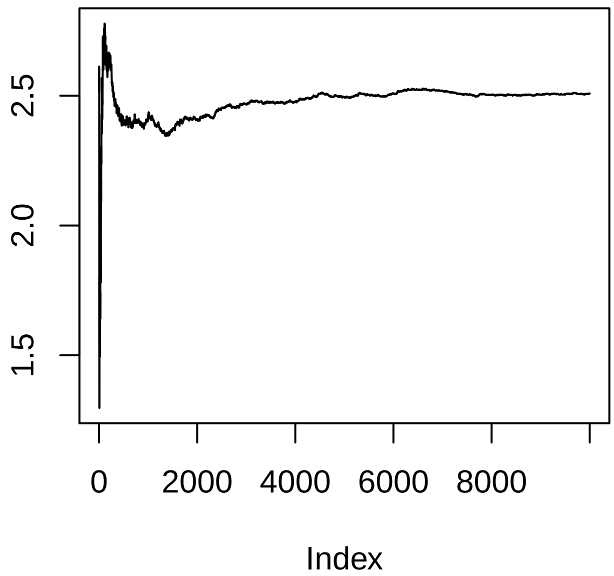


(c) Find the Monte Carlo estimate of $\mathrm{E}_f(X)$ and draw a plot to show the convergence of the MCMC approximation.

```
mean(x)
```

```
## [1] 2.524106
```

```
n <- 10000
q <- rgamma(n, shape = 5, rate = 1)

plot(cumsum(q*rayliegh(q)/dgamma(q, shape = 5, rate = 1))/(1:n),
     ylab = "",
     type = "l")
```



Problem 2. The Cauchy distribution has density function

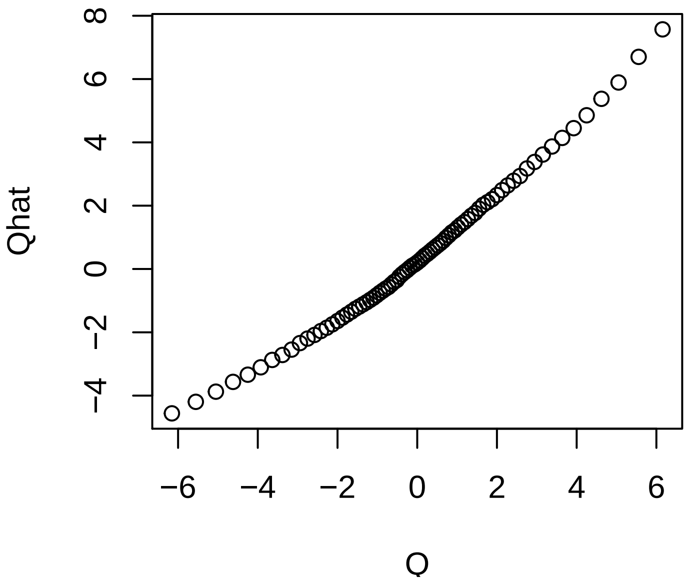
$$f(x) = \frac{1}{\theta\pi\left[1+\left(\frac{x-\eta}{\theta}\right)^2\right]}, \quad -\infty < x < \infty, \theta > 0.$$

(a) Use the Metropolis algorithm to generate a random sample of size 10000 from the Cauchy distribuiton ($\theta = 2, \eta = 0$). Here θ is scale parameter and η is location parameter.

```
n <- 10000
x <- rep(0,n)

cauch <- function(x) {
  1/(2*pi*(1+(x^2/4)))
}

q <- function(x) {
  rnorm(1, mean = x, sd = 4)
}

x[1] <- 0

for (i in 1:n) {
  xp <- q(x[i])
  alpha <- min(1, cauch(xp)/(cauch(x[i])))
  if (runif(1) < alpha){
        x[i+1] <- xp
    }
    else{
        x[i+1] <- x[i]
    }
}
```

(b) Compute the sample percentiles and compare with the Cauchy distribution ($\theta = 2, \eta = 0$) percentiles. (Hint: Use `qcauchy()`)

```r
p <- seq(0.1,0.9, 0.01)
Qhat <- quantile(x,p)
Q <- qcauchy(p, scale = 2, location = 0)

plot(Q,Qhat)
```



Problem 3. Let X = the number of heads obtained from flipping a fair coin four times. Here, we have that there are 4 trials, "success" is defined as getting a head face-up with probability of 0.5 for each trial, and the outcome of one trial doesn't affect another.

(a) Use the Metropolis algorithm to simulate X, 50000 times.

```r
n <- 50000
x <- rep(0,n)
mass <- dbinom(0:4, 4, .5)

S <- c(1:5)
Q <- matrix(rep(.2, 25), nrow = 5, ncol = 5)

x[1] <- 2

for(i in 1:n){
    xp <- sample(S, 1, prob = Q[x[i], ])
    alpha <- min(1, mass[xp]/mass[x[i]])
    if (runif(1) < alpha){
        x[i+1] <- xp
    }
    else{
        x[i+1] <- x[i]
    }
}

table(x)/(n+1)
```

```
## x
```

```
##          1          2          3          4          5
## 0.06107878 0.25535489 0.37363253 0.24685506 0.06307874
```

(b) Find the Monte Carlo estimate of E(X).

$$E(X) = np = 4(.5) = 2$$

```r
n <- 50000
u <- runif(n)

z <- rbinom(u, 4, .5)

mean(z)
```

```
## [1] 2.00118
```

(c) Find the Monte Carlo estimate of Var(X). (Hint: $\text{Var}(X) = \text{E}(X^2) - (\text{E}(X))^2$)

$$Var(X) = np(1-p) = 4(.5)(.5) = 1$$

```r
n <- 50000
u <- runif(n)

z <- rbinom(u, 4, .5)

mean((z - mean(z))^2)
```

```
## [1] 1.003099
```