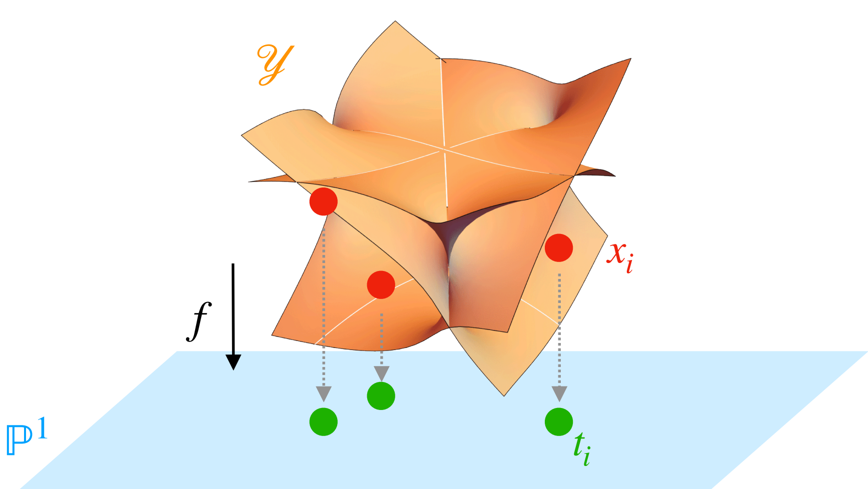
Eric Pichon-Pharabod

Homology and periods of algebraic varieties

PhD supervised by Pierre Lairez and Pierre Vanhove



Periods

A **period** of is the integral of a rational function on a closed domain of integration.

$$\int_{\gamma} \frac{A}{P^k}$$

Motivation for the name:

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$x = \sin\left(\int_0^x \frac{1}{\sqrt{1-t^2}} \mathrm{d}t\right)$$

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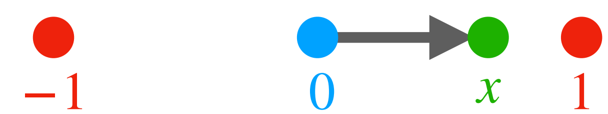

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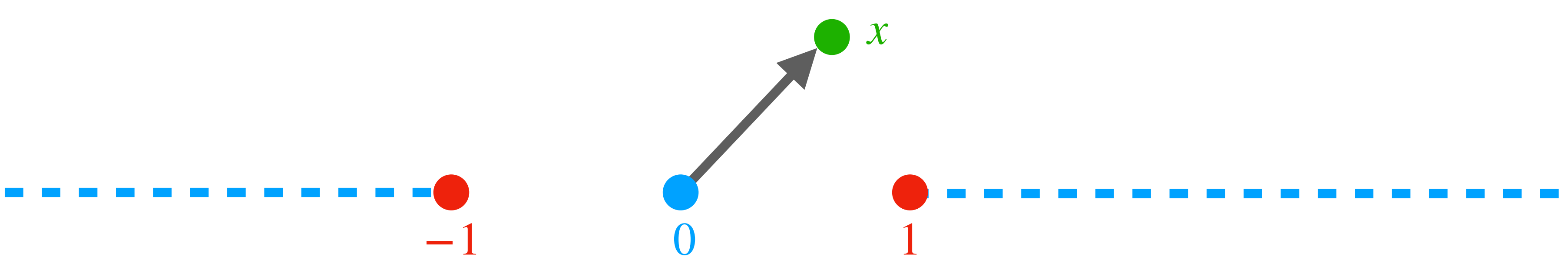

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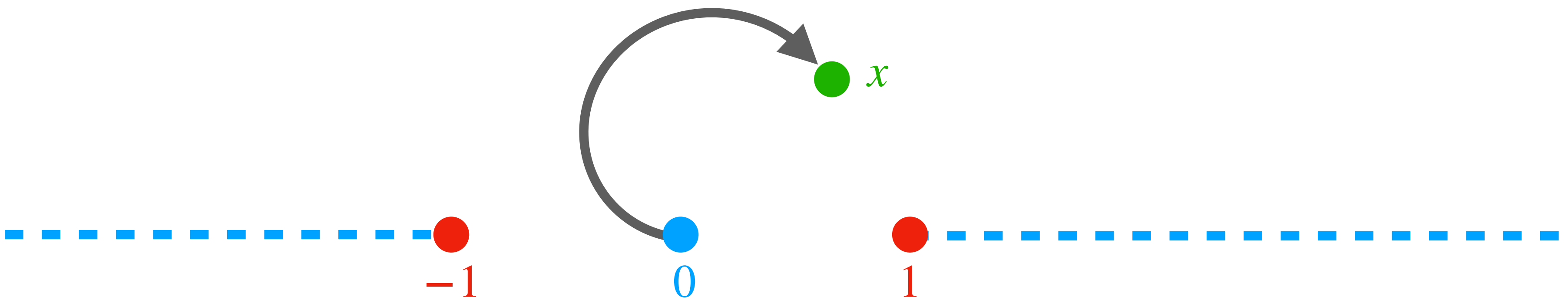

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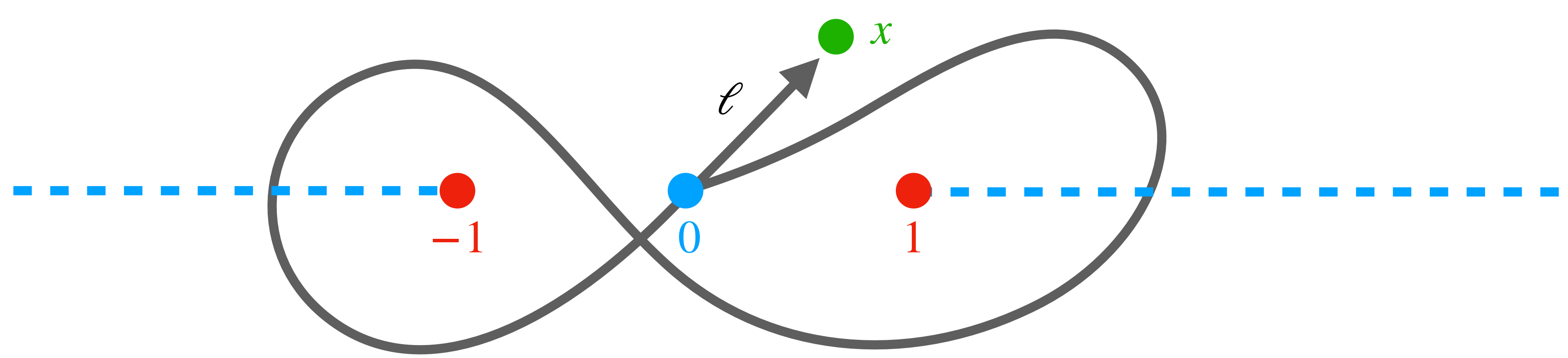

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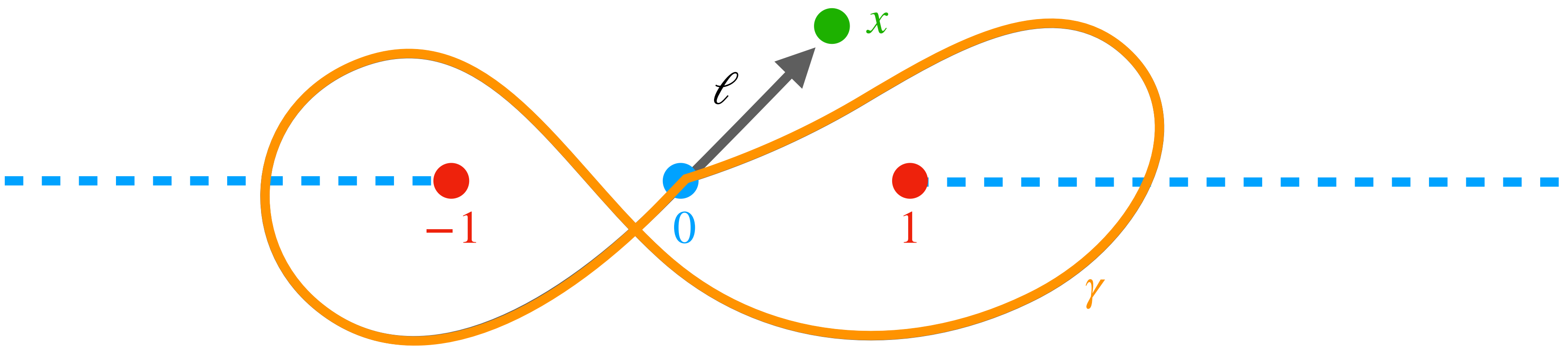

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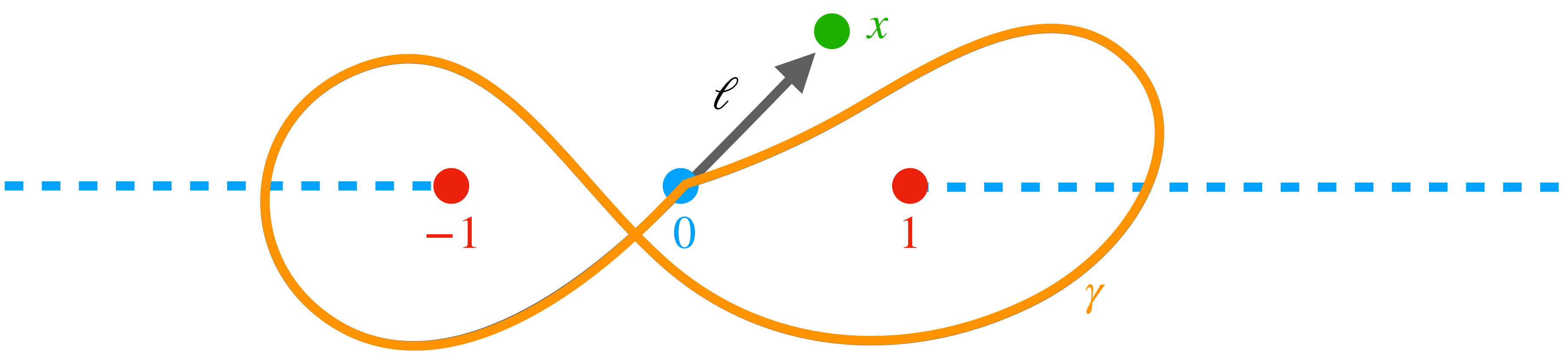

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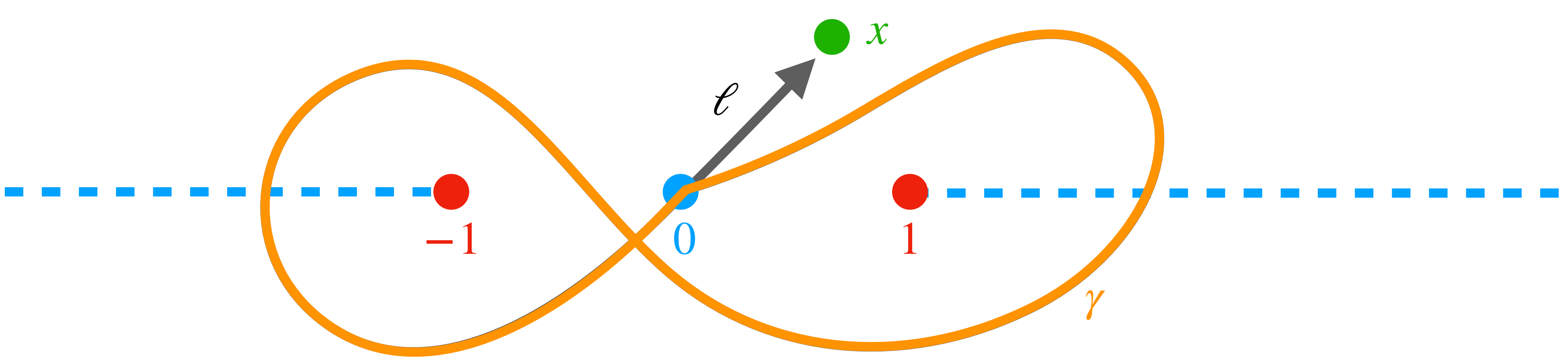

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It can also be expressed as the integral of a rational function:

$$\int_{\gamma} \frac{1}{\sqrt{1-t^2}}\mathrm{d}t = \frac{1}{i\pi}\iint_{\tilde{\gamma}} \frac{1}{x^2+y^2-1}\mathrm{d}x\mathrm{d}y$$

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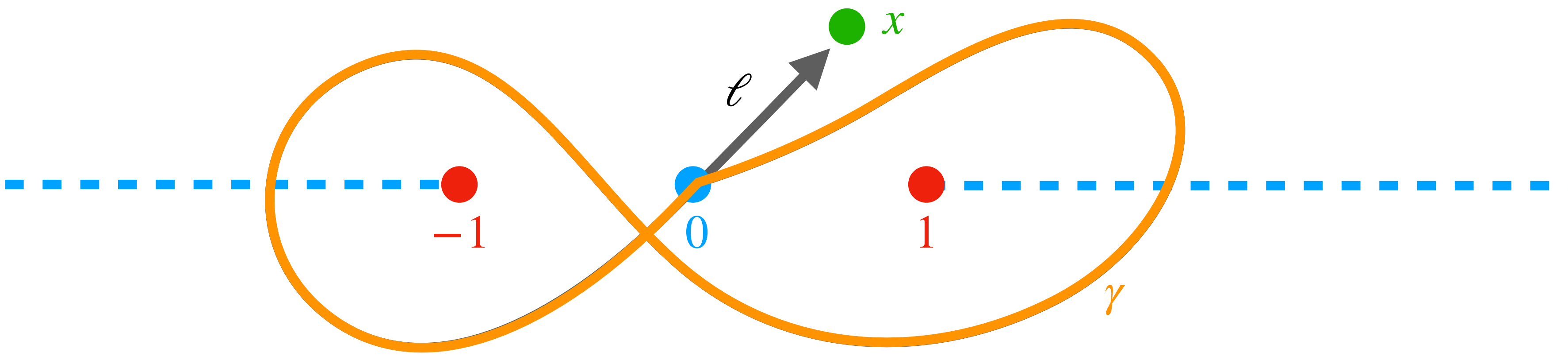

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Denominator is the equation of a circle

A geometric perspective

Periods can be associated to a **geometrical object**, defined by polynomial equations.

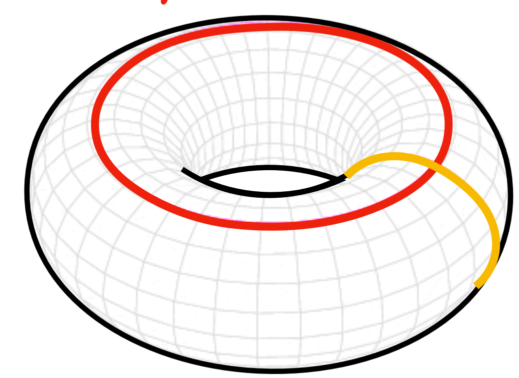
Some integration domain without boundary

$$\int_{T(\gamma)} \frac{A}{P^k}$$

A is a polynomial

P defines a smooth variety $V(P) = \{z \mid P(z) = 0\}$

γ



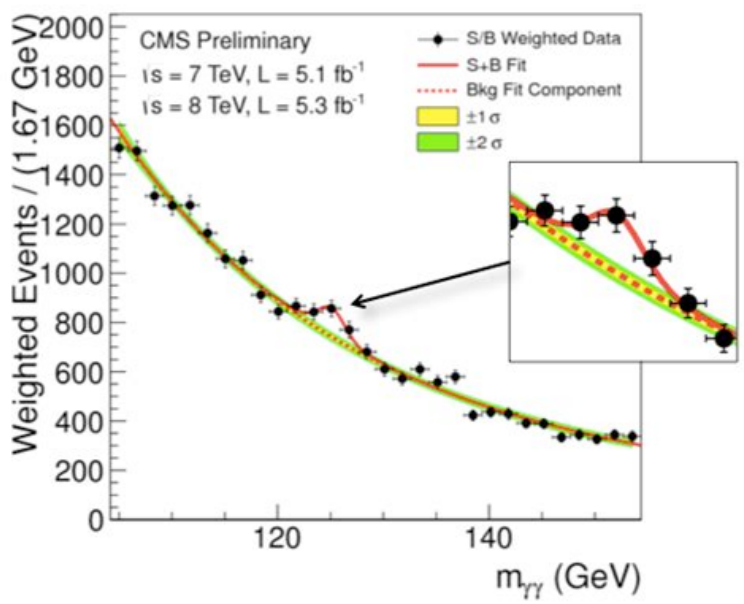
An elliptic curve

Torelli-type theorem for K3 surfaces:
Two K3 surfaces are isomorphic if and only if they have "the same" periods.

They describe the comparison between **topological data** (cycles) and **algebraic data** (algebraic De Rham forms).

$$H_n(S, \mathbb{Z}) \times H^n_{DR}(S) \to \mathbb{C} \qquad \gamma, \omega \mapsto \int_\gamma \omega$$

Motivation and goals



Periods appear in diverse fields of mathematics and physics, such as **Quantum field theory** (Feynman integrals), **Hodge theory**, **motives**, **number theory** (BSD conjecture) …

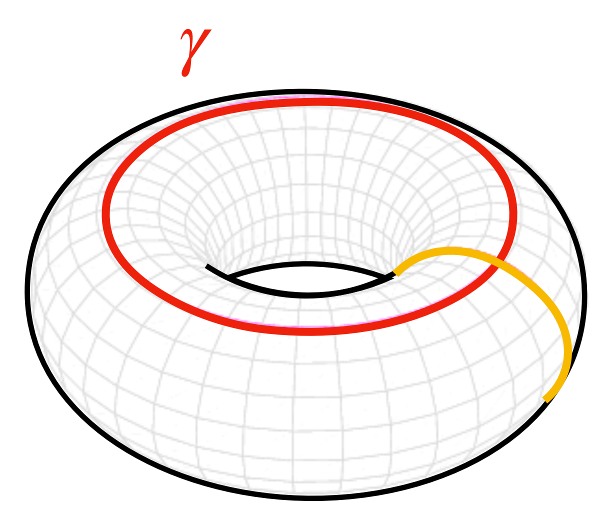
Hundreds of digits

Sufficiently many to recover algebraic invariants

Goal: compute numerical approximations of these integrals with **large precision**.

For this, we need an appropriate description of the integrals.

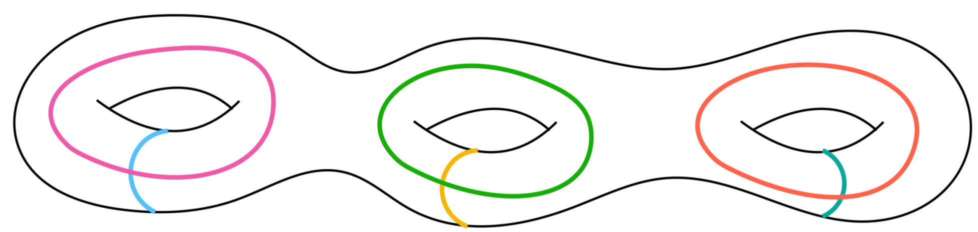
In particular we will focus on **understanding the cycles of integration** (the homology), how to represent them in a way that make integration easy, and how to compute a basis of them.



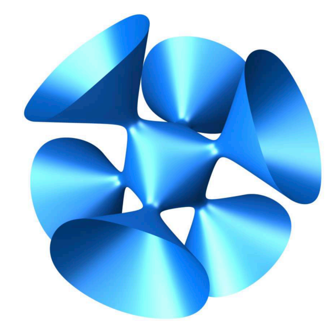
Furthermore we want this to be **effective** and **efficient**.

Previous works

[Deconinck, van Hoeij 2001], [Bruin, Sijsling, Zotine 2018], [Molin, Neurohr 2017]:
Algebraic curves (Riemann surfaces)

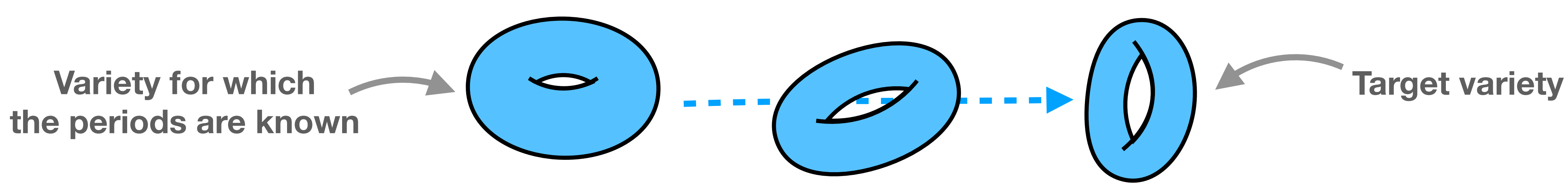


[Elsenhans, Jahnel 2018], [Cynk, van Straten 2019]:
Higher dimensional varieties
(double covers of $\mathbb{P}^2$ ramified along 6 lines / of $\mathbb{P}^3$ ramified along 8 planes)



Picture by
Alessandra Sarti

[Sertöz 2019]: compute the period matrix of smooth projective hypersurfaces by **deformation**.



Previous works

[Sertöz 2019]: compute the periods matrix by **deformation**:

We wish to compute $\displaystyle\int_\gamma \frac{\Omega}{X^3+Y^3+Z^3+XYZ}$.

Let us consider instead $\displaystyle\pi_t = \int_{\gamma_t} \frac{\Omega}{X^3+Y^3+Z^3+tXYZ}$,

Exact formulae are known for π_0 **[Pham 65, Sertöz 19]**

Furthermore π_t is a solution to the differential operator
$\mathscr{L} = (t^3+27)\partial_t^2 + 3t^2\partial_t + t$ (Picard-Fuchs equation).

We may numerically compute the analytic continuation of π_0 along a path from 0 to 1. **[Chudnovsky², Van der Hoeven, Mezzarobba]**
This way, we obtain a numerical approximation of π_1.

Previous works

[Sertöz 2019]: compute the periods matrix by **deformation**:

Two drawbacks :

We rely on the knowledge of the periods of some variety.
[Pham 65, Sertöz 19] provide the periods of the Fermat hypersurfaces

$$V(X_0^d + \ldots + X_n^d).$$

In more general cases (e.g. complete intersections), we do not have this data.

The differential operators that need to be integrated quickly go beyond what current software can manage:

to compute the periods of a smooth quartic surface in $\mathbb{P}^3$, one needs to integrate an operator of order 21 and high degree.

Idea: a more intrinsic description of the cycles of integration should solve both problems.

Contributions

- ***Motivic Geometry of two-Loop Feynman Integrals (Appendix)***,

The Quarterly Journal of Mathematics, 2024
with C. Doran, A. Harder and P. Vanhove

- ***Effective homology and periods of complex projective hypersurfaces***,

Mathematics of Computation, 2024
with P. Lairez and P. Vanhove

- ***A semi-numerical algorithm for the homology lattice and periods of complex elliptic surfaces over $\mathbb{P}^1$***,

Journal of Symbolic Computations, 2024

- ***Estimating major merger rates and spin parameters ab initio via the clustering of critical events***,

Monthly Notices of the Royal Astronomical Society, 2024
with C. Cadiou, C. Pichon and D. Pogosyan

New **effective** method for computing homology and periods with high precision (hundreds of digits):

→ **implementation** in Sagemath (relying on OreAlgebra) — ***lefschetz_family***

→ sufficiently efficient to compute periods of **previously inaccessible hypersurfaces** (general smooth quartic surface)

→ frontal approach to the the computation of **homology** of complex algebraic varieties

→ applicable to **other types of varieties** (elliptic surfaces, ramified double covers, …)

Periods of algebraic curves

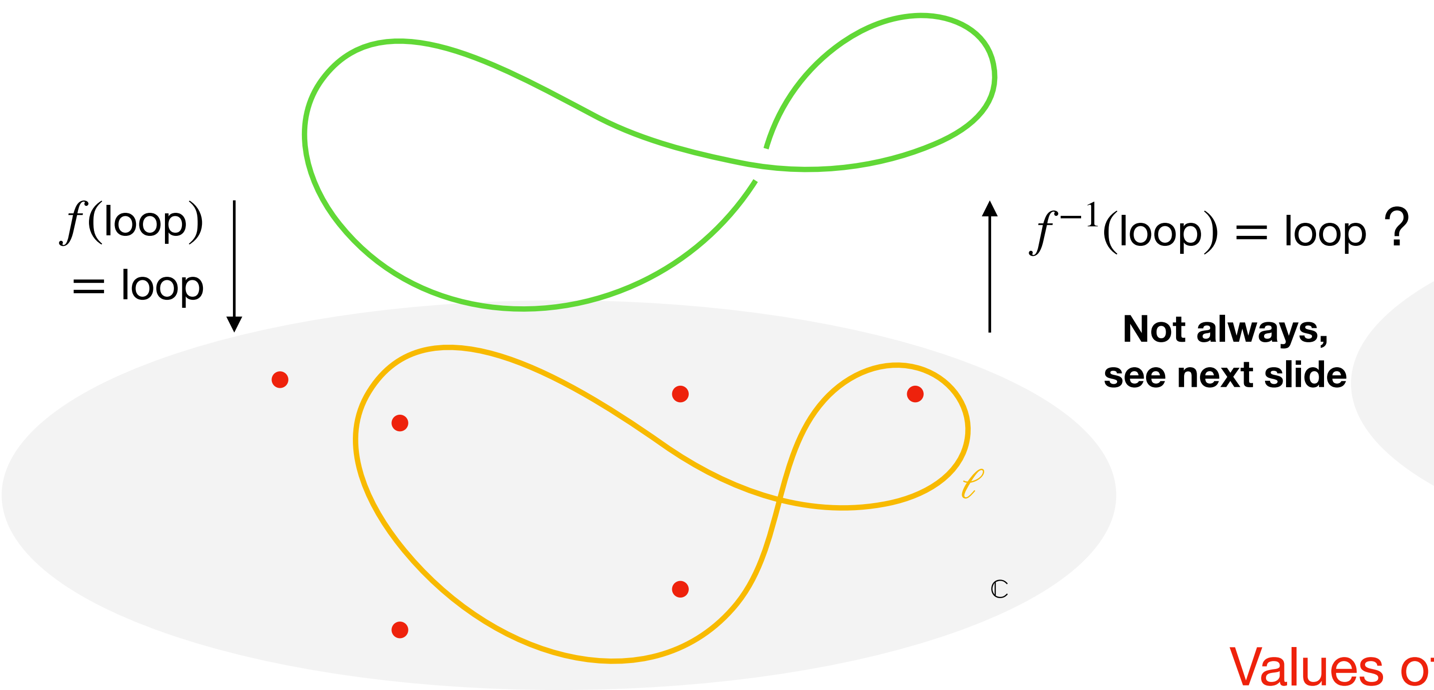
Algorithm from **[Deconinck, van Hoeij 2001]**

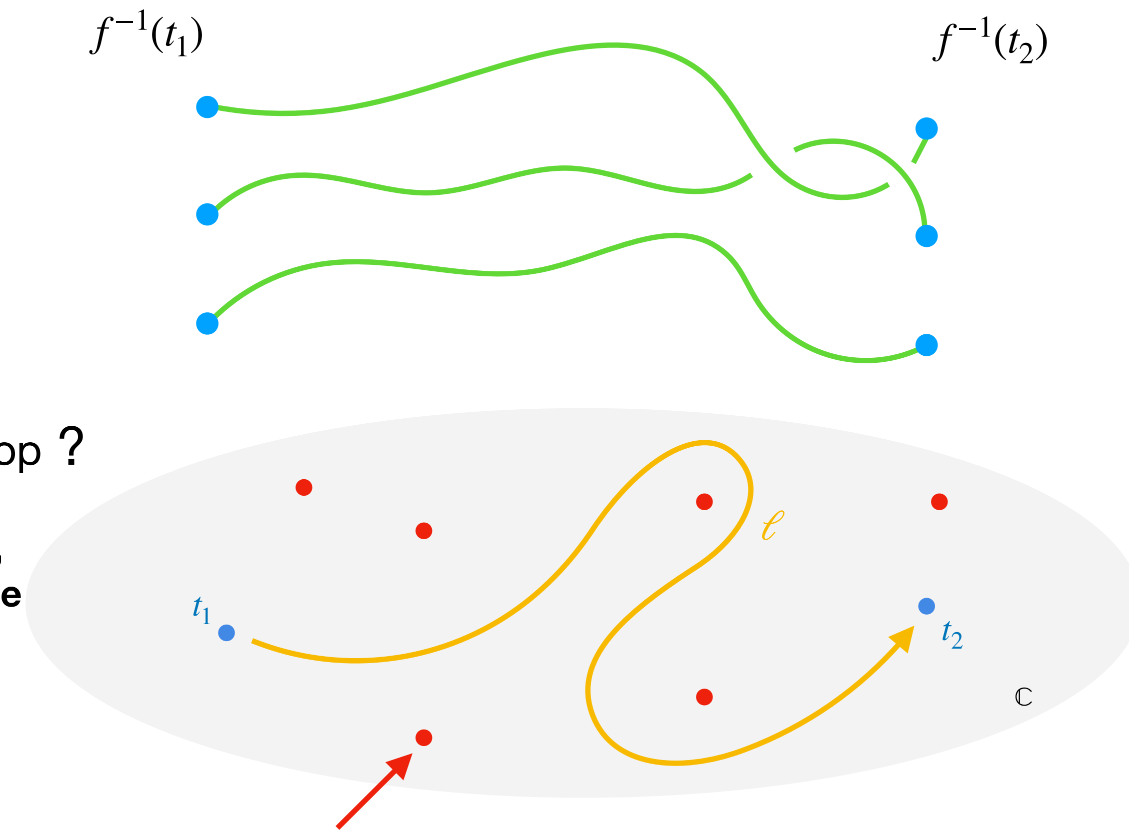
First example: algebraic curves

Let $\mathscr{X}$ be the elliptic curve defined by $P = y^3 + x^3 + 1 = 0$ and let $f : (x, y) \mapsto y/(2x + 1)$ be a generic projection.

The fibre above $t \in \mathbb{C}$ is $\mathscr{X}_t = f^{-1}(t)$
$= \{(x, t(2x + 1)) \mid P\left(x, t(2x + 1)\right) = 0\}$.
It deforms continuously with respect to t.

In dimension 1, we are looking for closed paths in $\mathscr{X}$, up to deformation (1-cycles).

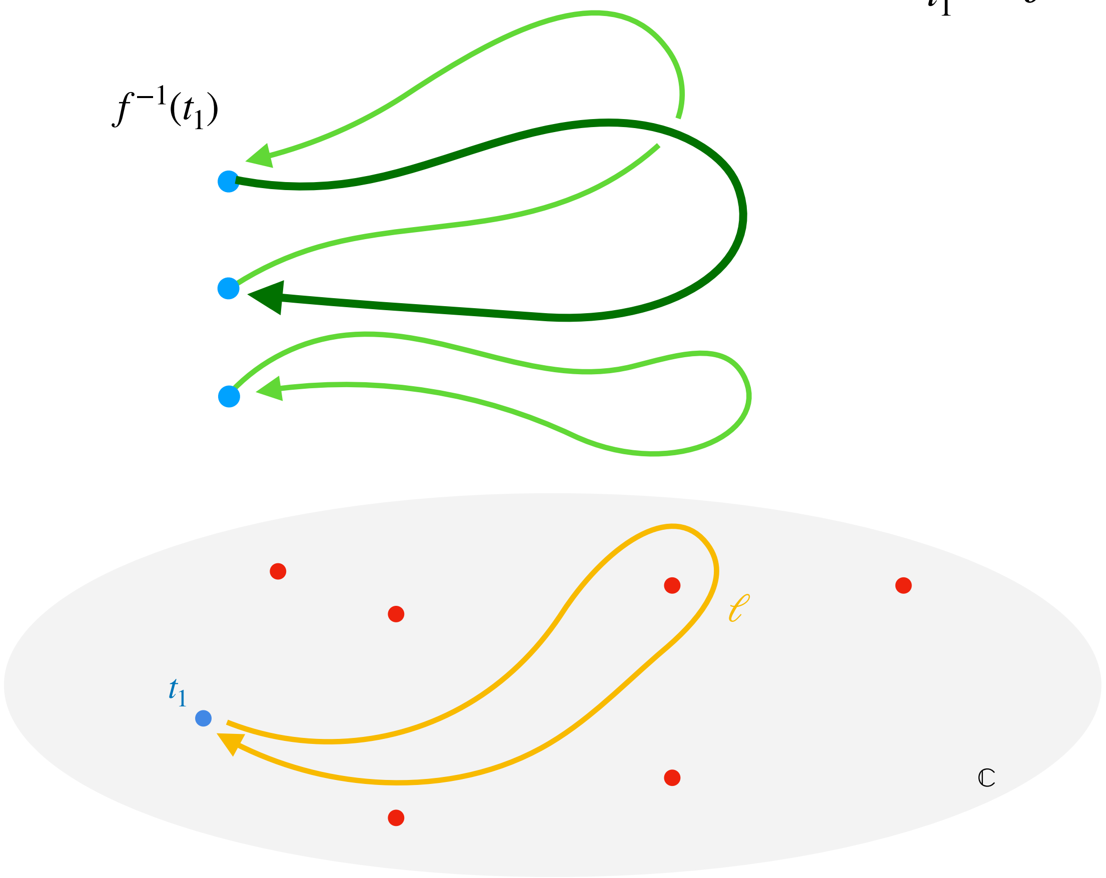


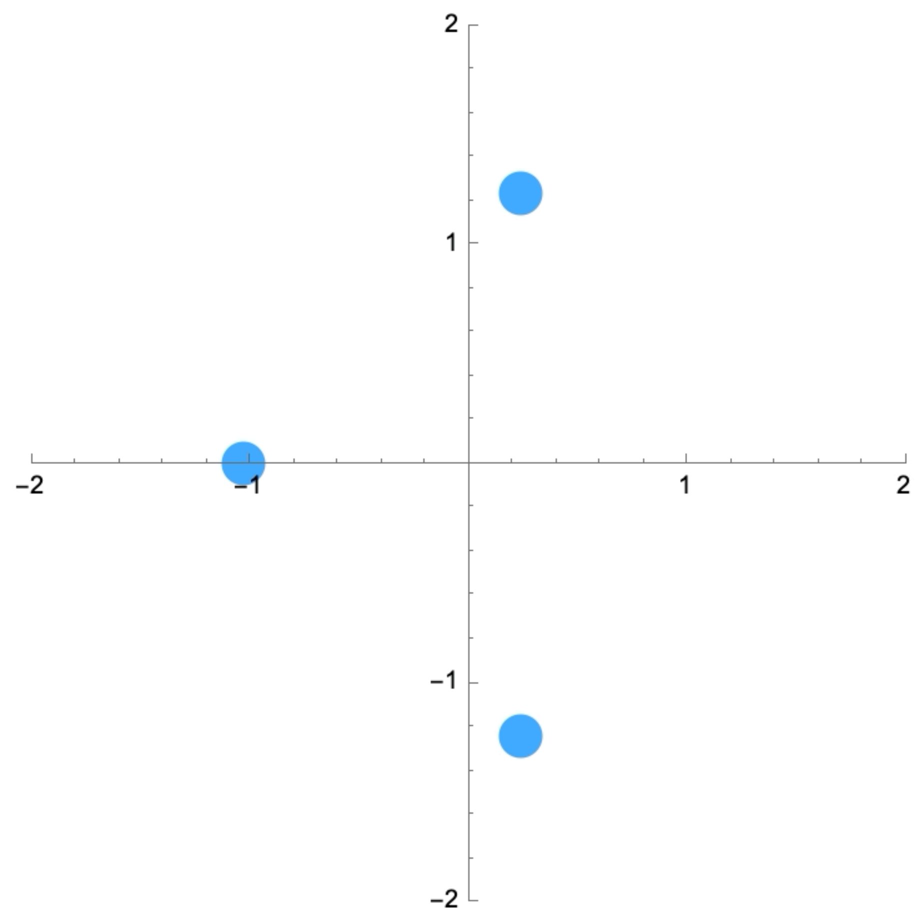


Values of t for which
$P(x, t(2x + 1)) = t^3(2x + 1)^3 + x^3 + 1$
has a double root (critical values)

What happens when you loop around a critical point?

A loop ℓ in $\mathbb{C}$ pointed at t_1 induces a permutation of
$\mathscr{X}_{t_1} = f^{-1}(t_1)$.





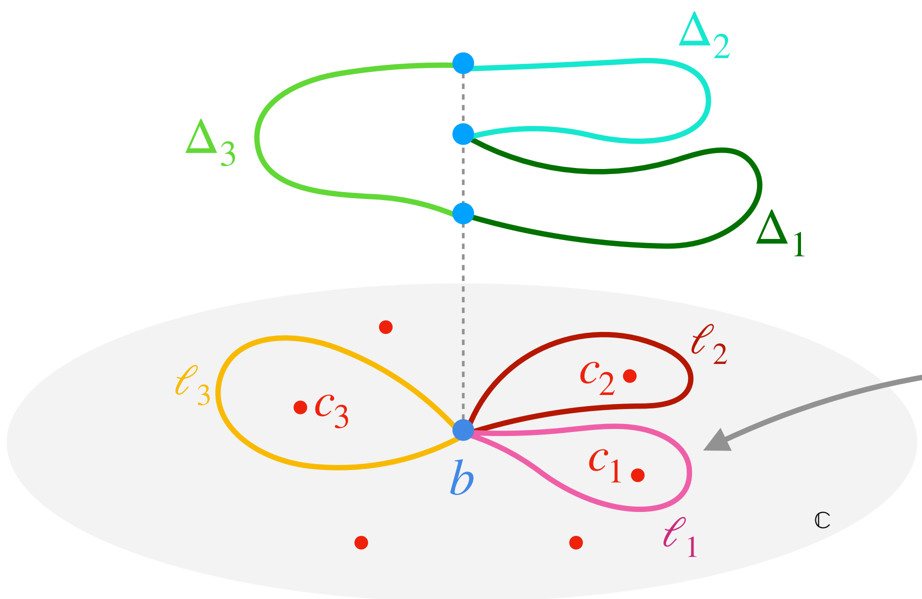
This permutation is called the **action of monodromy along** ℓ on $\mathscr{X}_{t_1}$.

It is denoted ℓ_*.

If ℓ is a simple loop around a critical value, ℓ_* is a transposition.

Periods of algebraic curves

The lift of a simple loop ℓ around a critical value c that has a non-trivial boundary in $\mathscr{X}_b$ is called the **thimble** of c. It is an element of $H_1(\mathscr{X}, \mathscr{X}_b)$.



Relative homology of the pair $(\mathscr{X}, \mathscr{X}_b)$

Simple loop around c_1

Thimbles serve as building blocks to recover $H_1(\mathscr{X})$.
It is sufficient to glue thimbles together in a way such that their boundaries cancels.

Concretely, we take the kernel of the boundary map

$$\delta : H_1(\mathscr{X}, \mathscr{X}_b) \to H_0(\mathscr{X}_b)$$

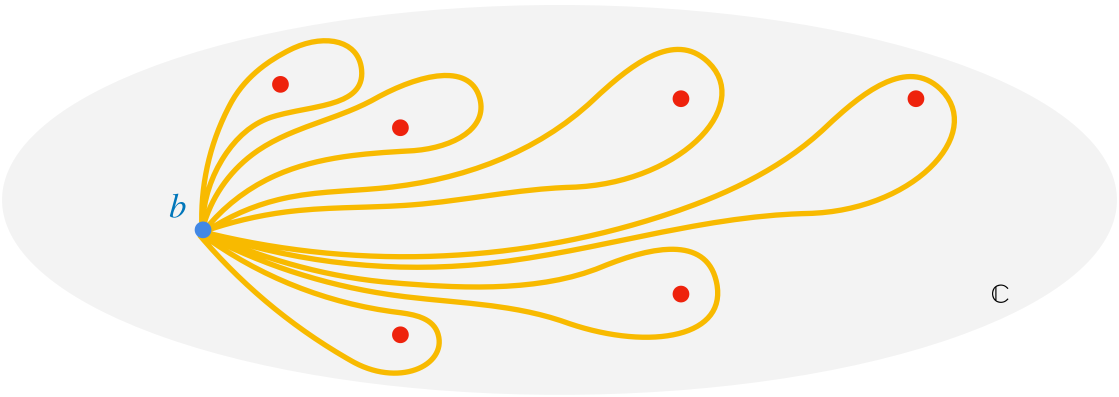
Fact: all of $H_1(\mathscr{X})$ can be recovered this way.

$$0 \to H_1(\mathscr{X}) \to H_1(\mathscr{X}, \mathscr{X}_b) \to H_0(\mathscr{X}_b)$$

Generated by thimbles

Computing periods of algebraic curves

1. Compute simple loops $\ell_1, \ldots, \ell_{\#\text{crit.}}$ around the critical values
— basis of $\pi_1(\mathbb{C} \backslash \{\text{crit. val.}\})$



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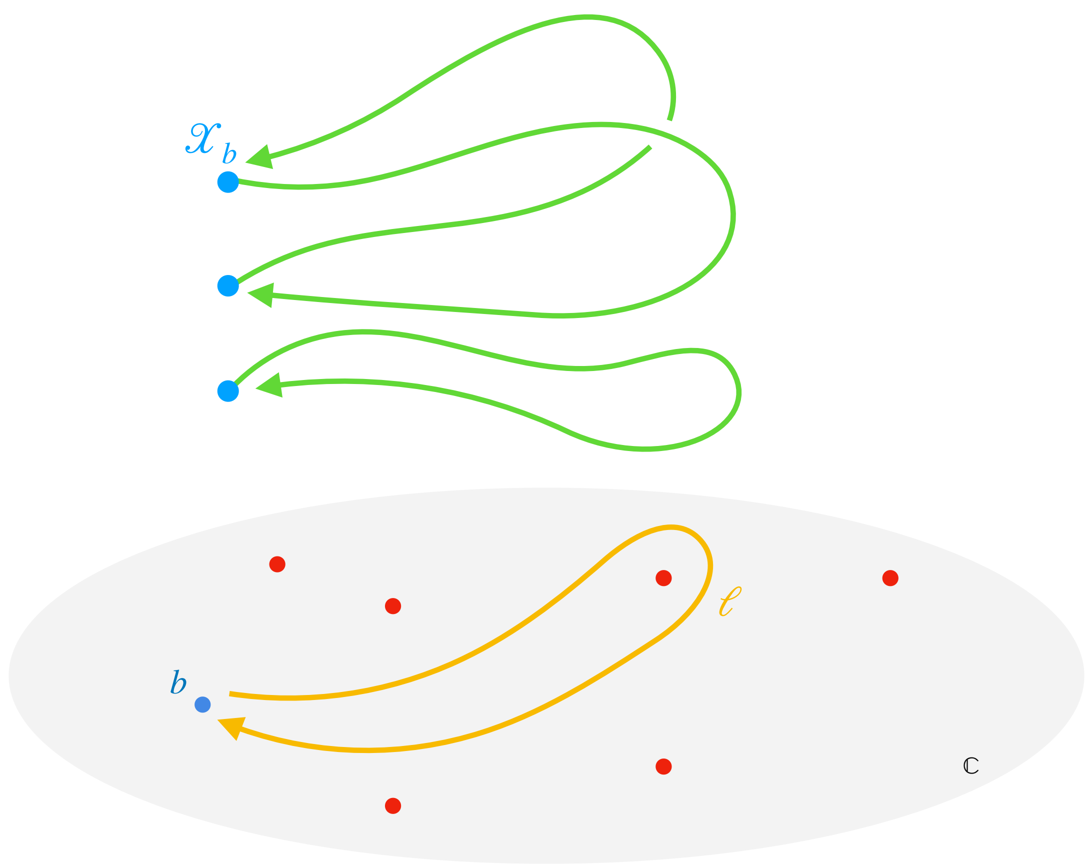

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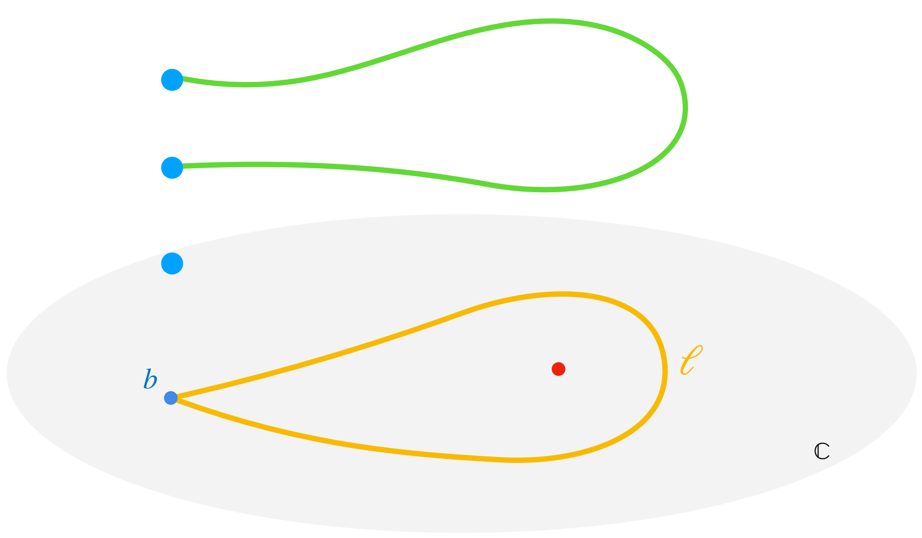
3. This provides the corresponding thimble Δ_i. Its boundary is the difference of the two points of $\mathcal{X}_b$ that are permuted.



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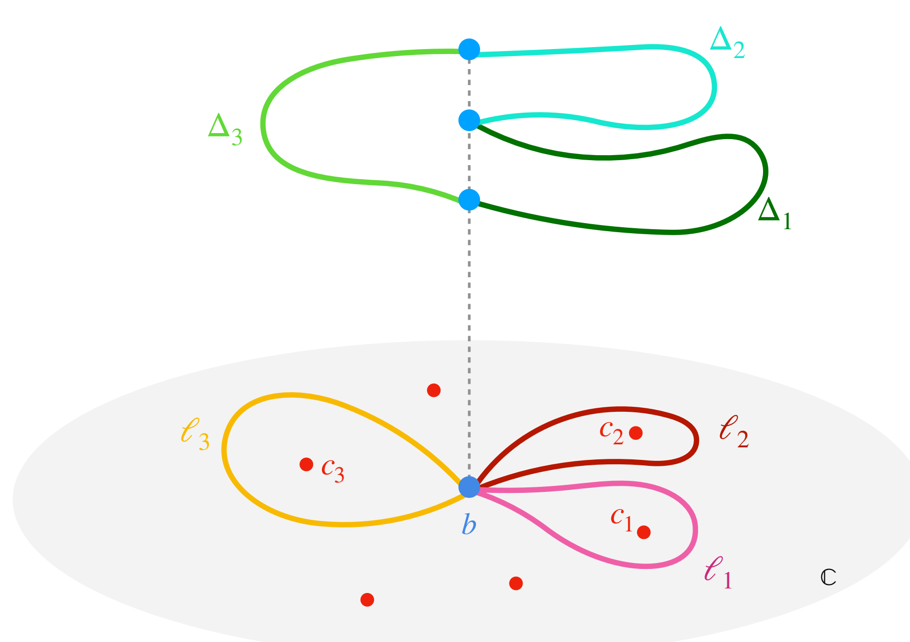
4. Compute sums of thimbles without boundary → basis of $H_1(\mathscr{X})$



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5. Periods are integrals along these loops
→ we have an explicit parametrisation of these paths → numerical integration.

$$\int_\gamma \omega = \int_\ell \omega_t$$

DEMO

Hypersurfaces

An inductive approach

Ideas of **[Lefschetz 1924]**, made effective in

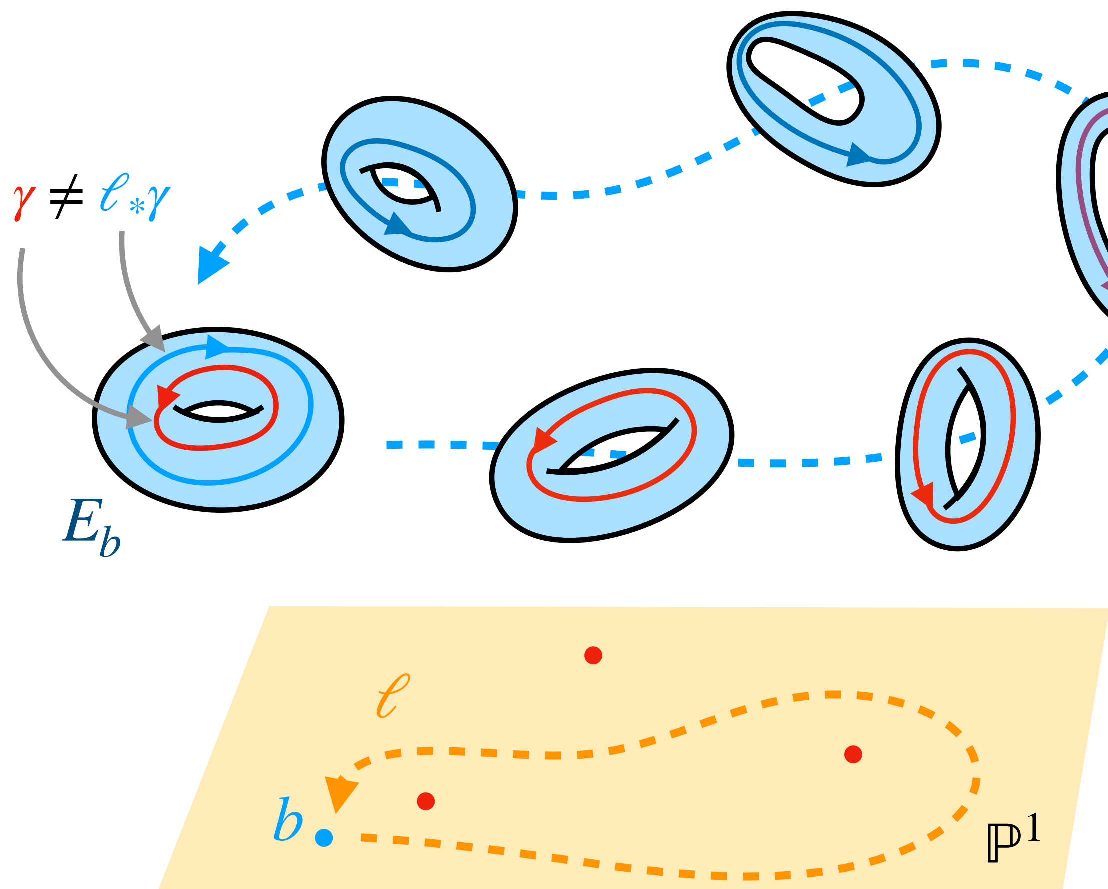
Effective homology and periods of complex projective hypersurfaces,
Mathematics of Computation, 2024
with P. Lairez and P. Vanhove

Monodromy

Ehresmann's fibration theorem

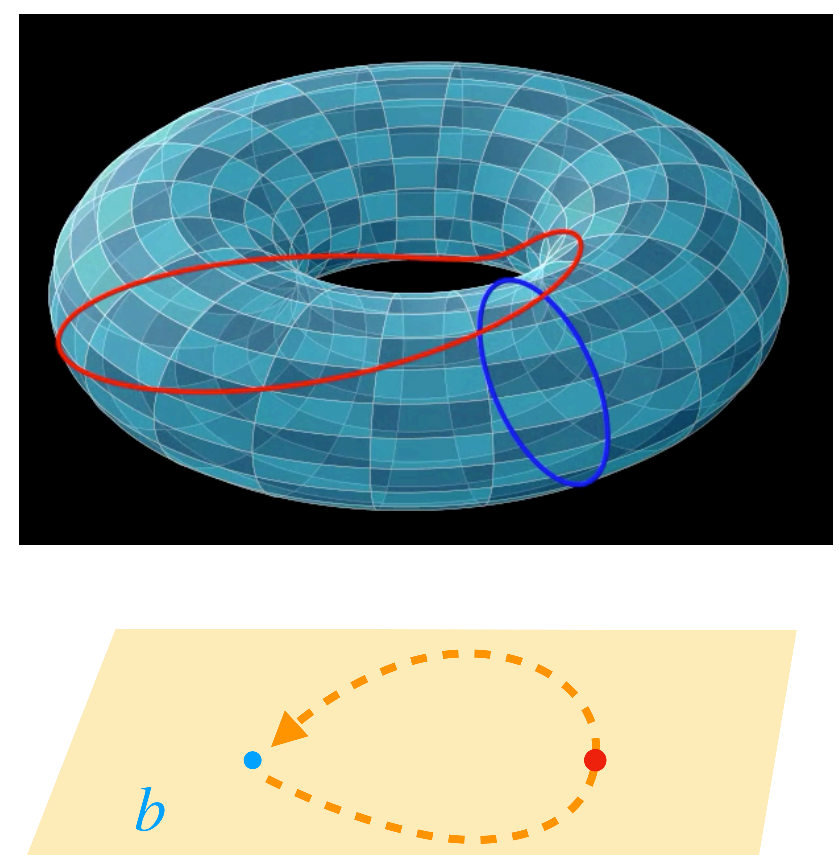
Let $\mathscr{X}$ be a smooth (hyper)surface in $\mathbb{P}^3$. We consider a projection $\mathscr{X} \to \mathbb{P}^1$.
The fibre $\mathscr{X}_t = f^{-1}(t)$ is a curve, which deforms continuously as t moves in $\mathbb{P}^1$.

The map $\ell_* : H_1(\mathscr{X}_b) \to H_1(\mathscr{X}_b)$ induced by this deformation along a loop ℓ is called the **monodromy along ℓ**.



The monodromy is encoded in a differential operator: the **Picard-Fuchs equation**.

A Dehn twist



When the monodromy is a Dehn twist, the singular fibre is said to be of **Lefschetz type**.
$\ell_* - \mathrm{id}$ has **rank 1** and its image is **primitive**.

Insight into higher dimensions: surfaces

We are looking for 2-cycles.
The fibre $\mathscr{X}_t$ is a curve which deforms continuously with respect to t.

We can recover integration 2-cycles for the periods of elliptic surfaces as **extensions** of 1-cycles of the fibre.

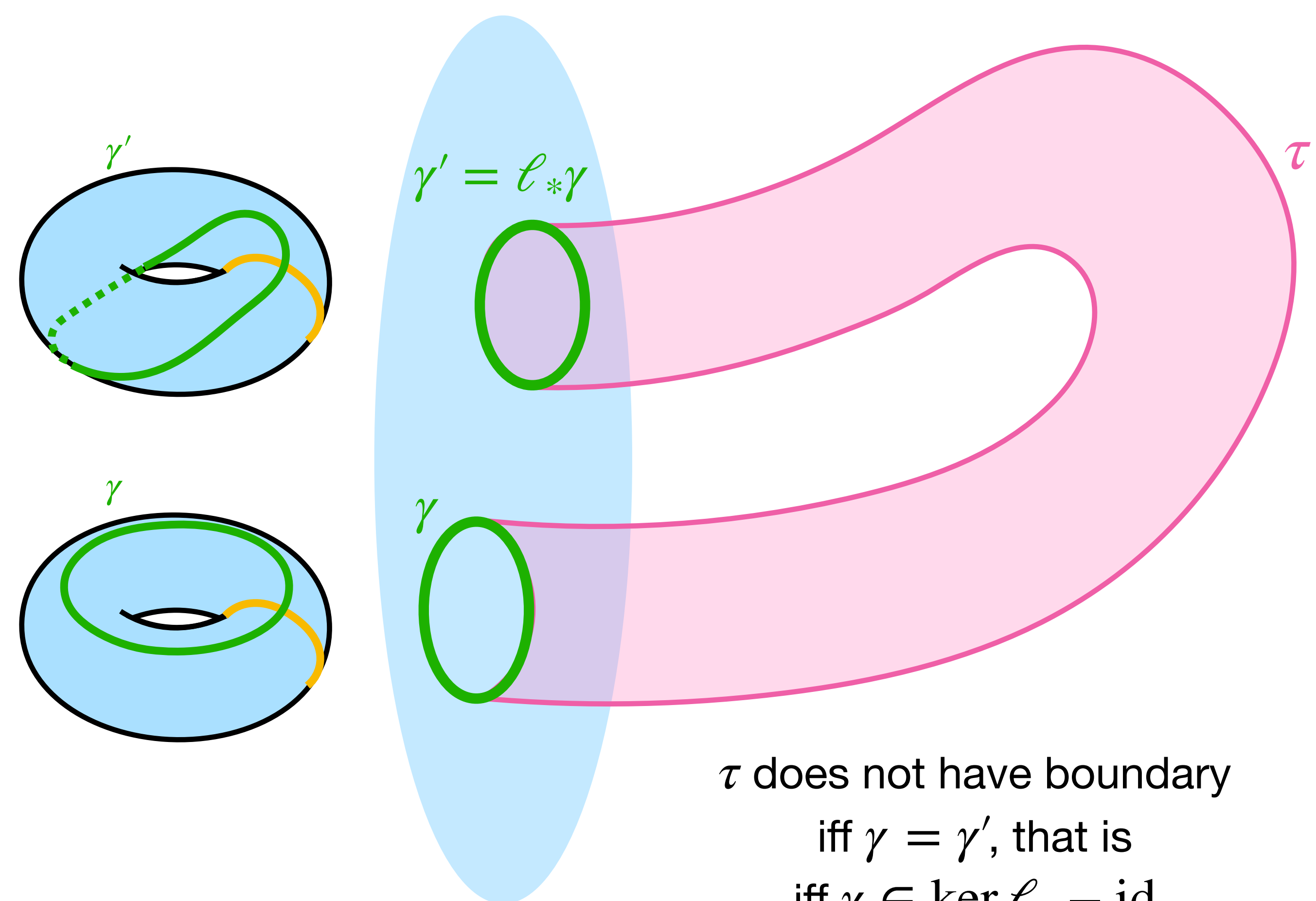
$$\pi_1(\mathbb{P}^1 \backslash \Sigma, b) \times H_1(\mathscr{X}_b) \to H_2(\mathscr{X}, \mathscr{X}_b)$$
$$\ell, \gamma \mapsto \tau_\ell(\gamma)$$

This description of cycles is well-suited for integrating the periods:

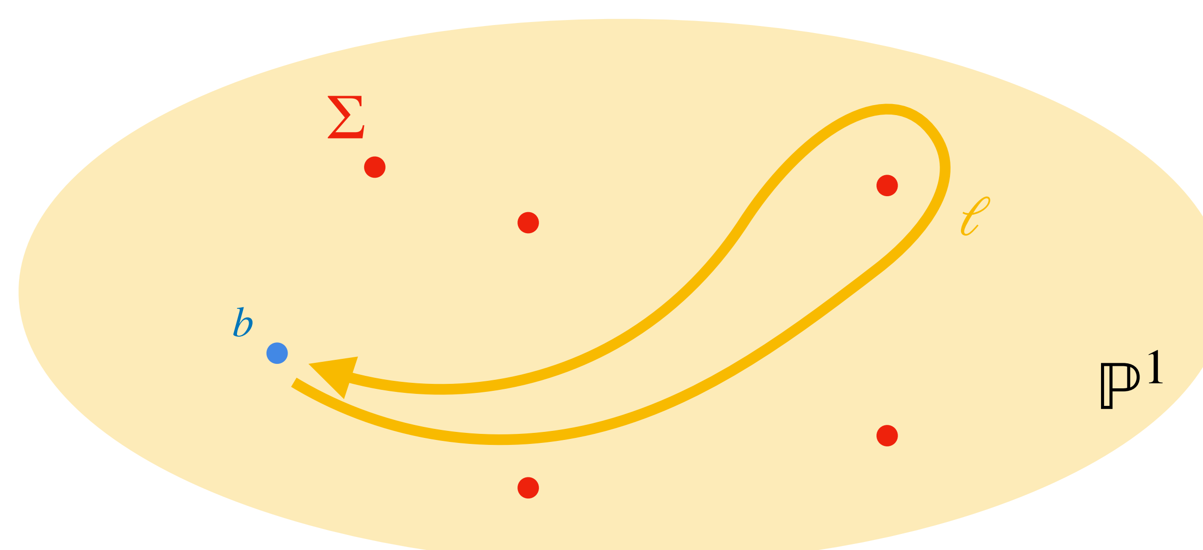
$$\int_{\tau_\ell(\gamma)} f(x,y)\mathrm{d}x\mathrm{d}y = \int_\ell \left(\int_\gamma f(x,y)\mathrm{d}x \right) \mathrm{d}y$$

Two line integrals:
we know how to compute these efficiently!
[Chudnovsky², Van der Hoeven, Mezzarobba]

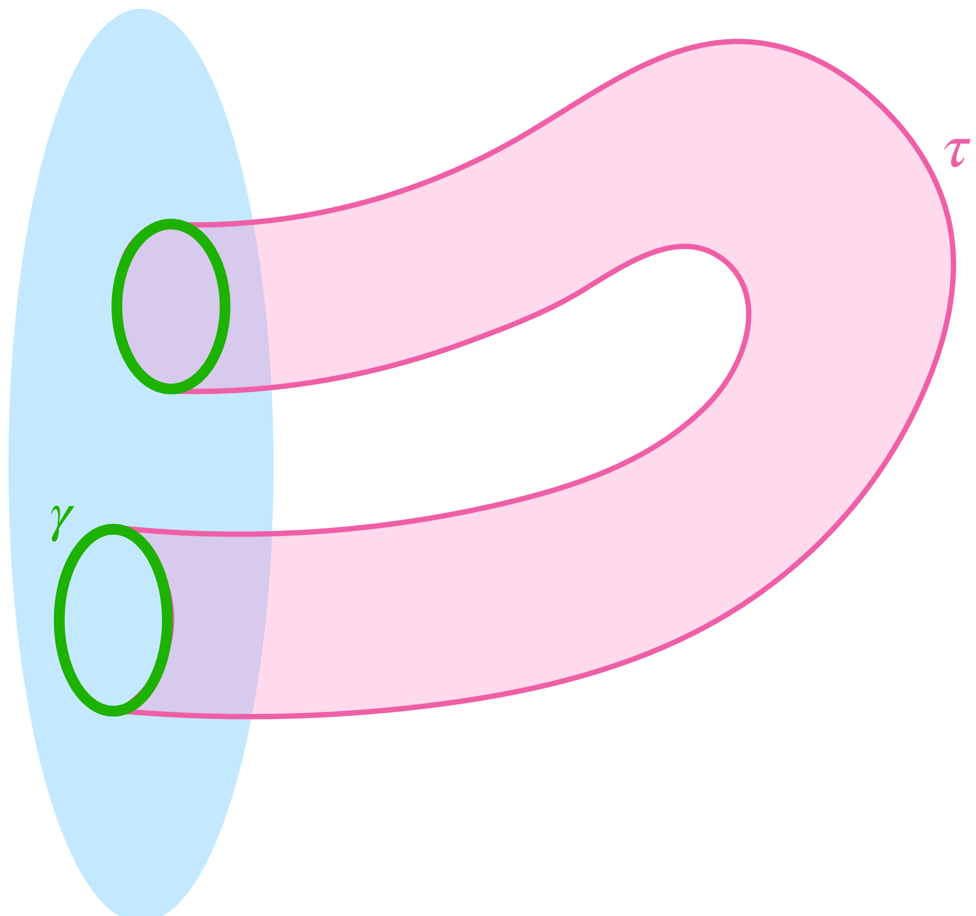
$$\partial \tau_\ell(\gamma) = \gamma' - \gamma$$



τ does not have boundary
iff $\gamma = \gamma'$, that is
iff $\gamma \in \ker \ell_* - \mathrm{id}$



Comparison with dimension 1

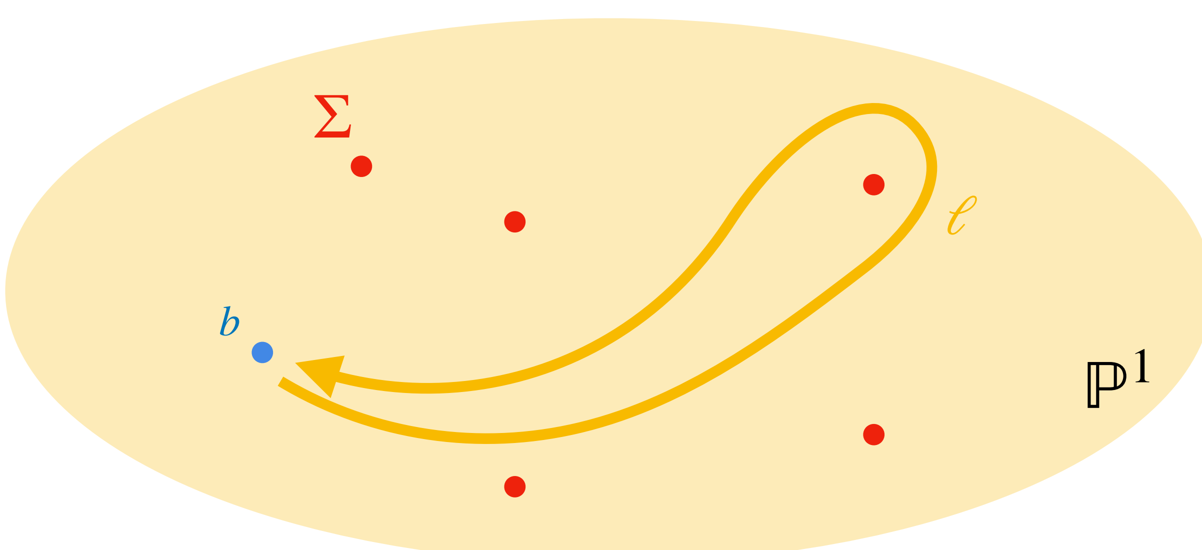


Extensions are n-**cycles** obtained by extending $n-1$-**cycles** along loops.

The monodromy along a loop ℓ is an isomorphism of $H_{n-1}(\mathscr{X}_b)$.

If the projection is generic (Lefschetz), singular fibres are simple.

There is a single **thimble** per critical value.



We get *almost* every possible n-cycle by gluing thimbles.

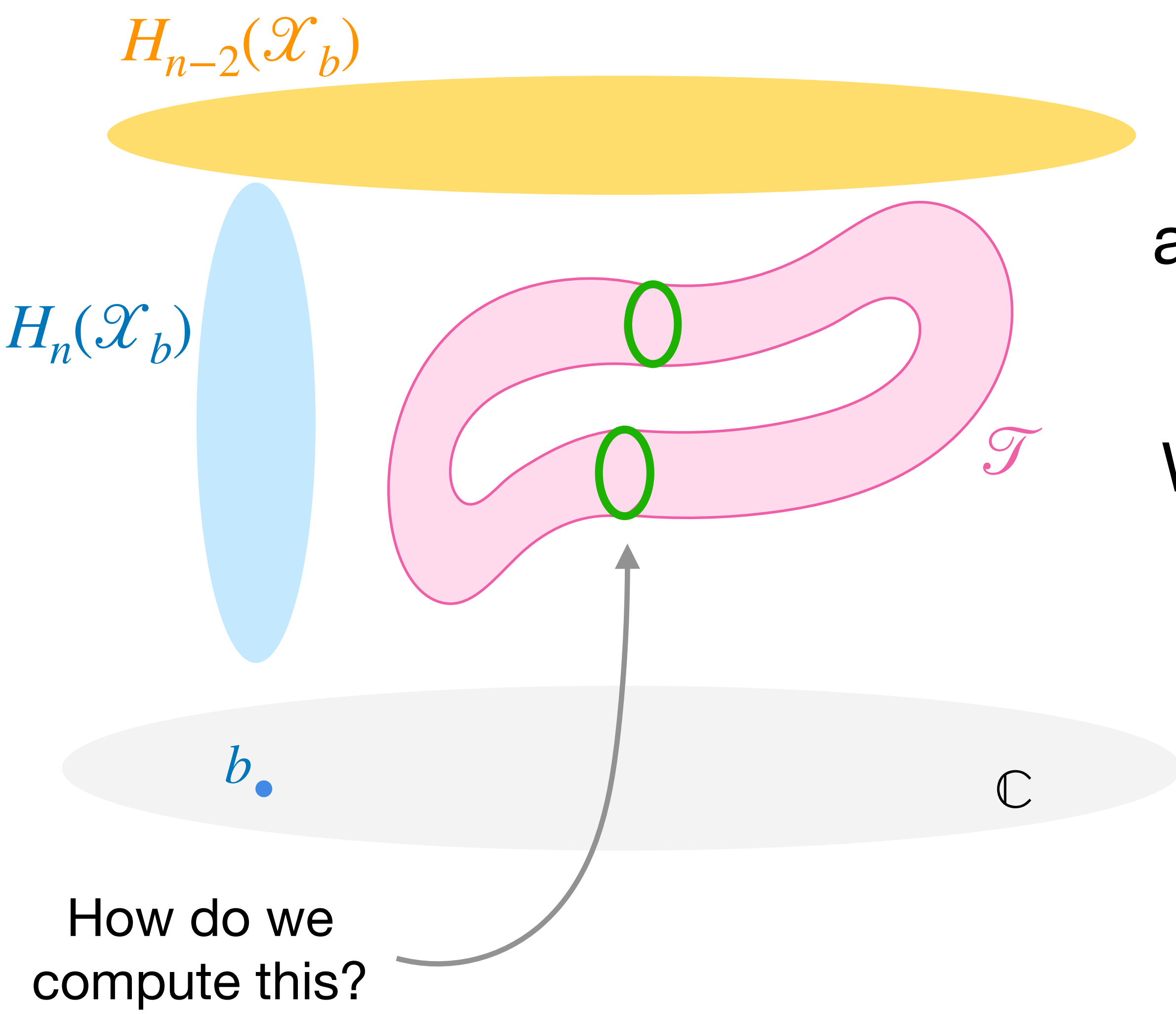
$$H_n(\mathscr{X}_b) \to H_n(\mathscr{X}) \to H_n(\mathscr{X}, \mathscr{X}_b) \to H_{n-1}(\mathscr{X}_b)$$

Possibly nontrivial

Almost generated by thimbles

Some complications

Not all cycles of $H_n(\mathscr{Y})$ are lift of loops, and thus not all are combinations of thimbles.



More precisely, we are missing the homology class of the **fibre** $H_n(\mathscr{X}_b)$ and a **section** (an extension of $H_{n-2}(\mathscr{X}_b)$ to all of $\mathbb{P}^1$).

We have a filtration $\mathscr{F}^0 \subset \mathscr{F}^1 \subset \mathscr{F}^2 = H_n(\mathscr{Y})$ such that

$$\mathscr{F}^0 \simeq H_n(\mathscr{X}_b)$$
$$\mathscr{F}^1/\mathscr{F}^0 \simeq \mathscr{T}$$
$$\mathscr{F}^2/\mathscr{F}^1 \simeq H_{n-2}(X_b)$$

Interesting part

$\mathscr{T}$ is also known as the **parabolic cohomology** of the local system.

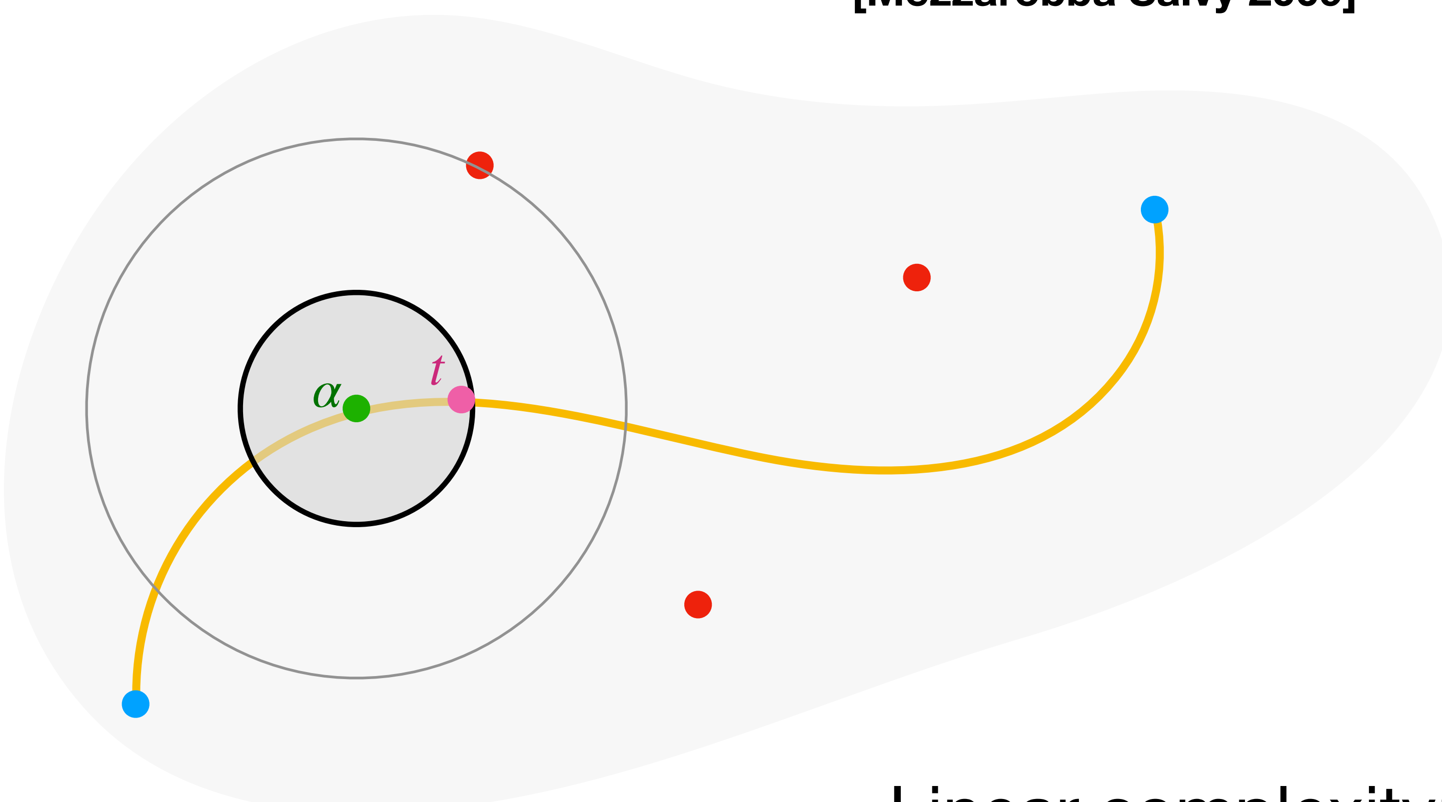
Monodromy of a differential operator

[Chudnovsky² 90, Van der Hoeven 99, Mezzarobba 2010]

In a small radius around α:

$$\left| f(t) - \sum_{k=0}^{m} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k \right| \leq \mathscr{P}(m)2^{-m}$$

polynomial
in m (effective)
[Mezzarobba Salvy 2009]



We compute $f^k(\alpha)$ from $\mathscr{L}$.

In a disk around α, the precision given by the Taylor formula is exponential in its order.

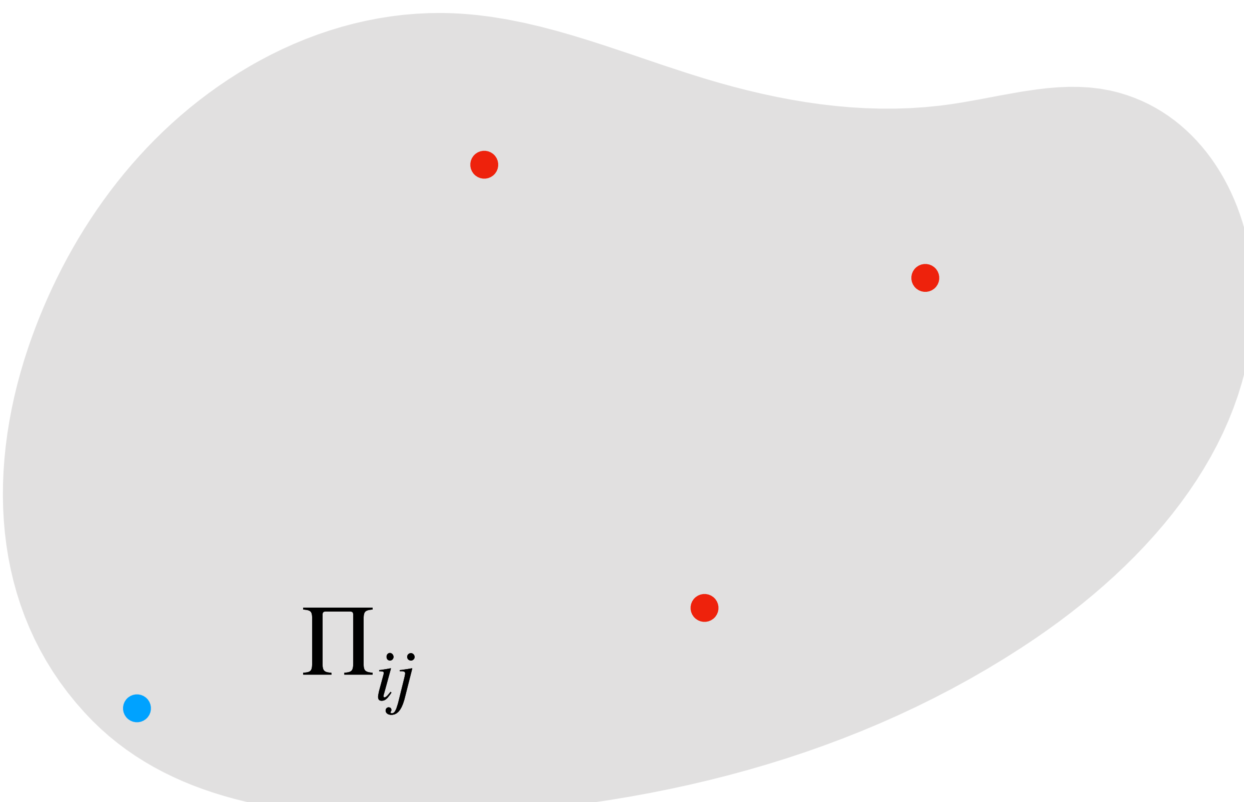
From the derivatives at α,
we can recover the derivatives at t.

Linear complexity: Recover m digits in $\mathcal{O}(m)$ operations

(using binary splitting)

Computing monodromy on cycles

$$\Pi_{ij} = \int_{\gamma_j} \partial_t^i \omega_t$$

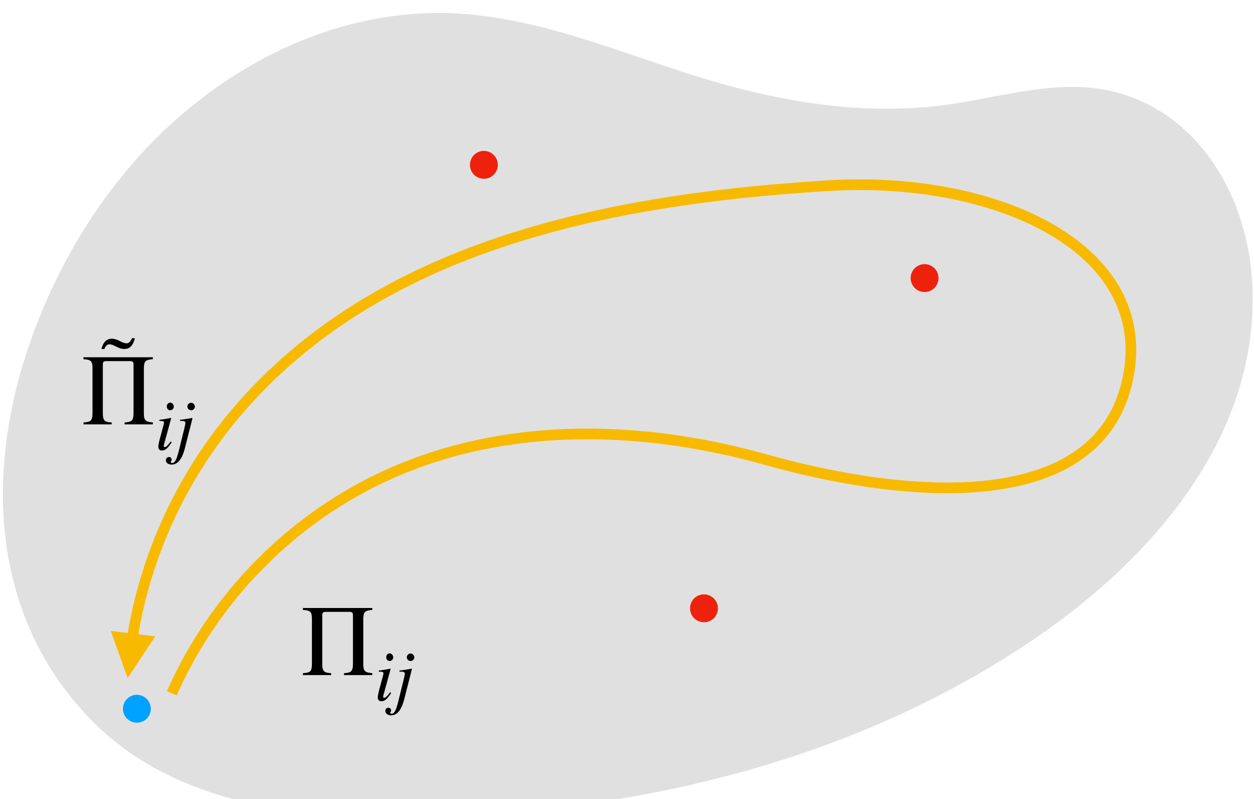


Computing monodromy on cycles

$$\Pi_{ij} = \int_{\gamma_j} \partial_t^i \omega_t \xrightarrow{\text{Analytic continuation}} \tilde{\Pi}_{ij}$$

[Chudnovsky[2] 90, Van der Hoeven 99, Mezzarobba 2010]

Solution to Picard-Fuchs equation of ω_t

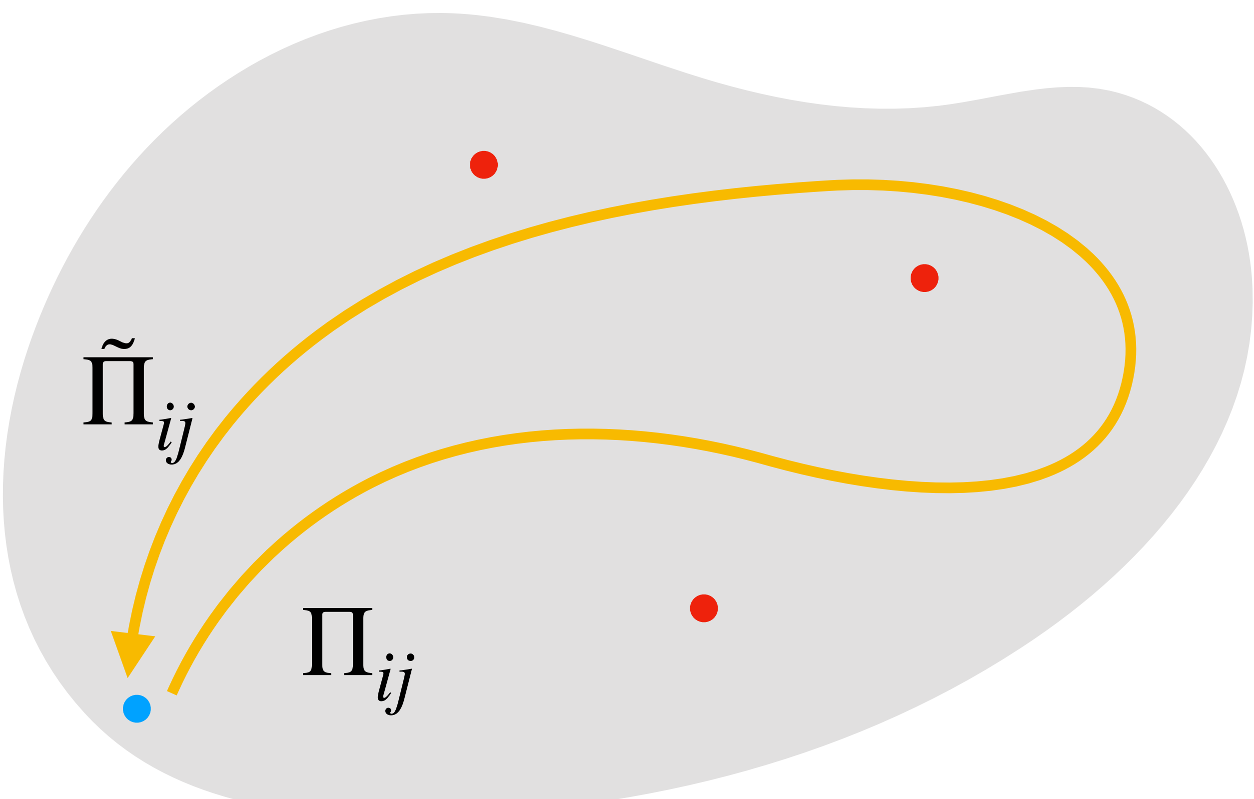


Computing monodromy on cycles

Globally defined
$\Longrightarrow$ no monodromy

$$\Pi_{ij} = \int_{\gamma_j} \partial_t^i \omega_t \xrightarrow[\text{[Chudnovsky}^2\text{ 90, Van der Hoeven 99, Mezzarobba 2010]}]{\text{Analytic continuation}} \tilde{\Pi}_{ij} = \int_{\sum_k c_{kj}\gamma_k} \partial_t^i \omega_t = \sum_k c_{kj} \int_{\gamma_k} \partial_t^i \omega_t$$

Solution to Picard-Fuchs equation of ω_t



Computing monodromy on cycles

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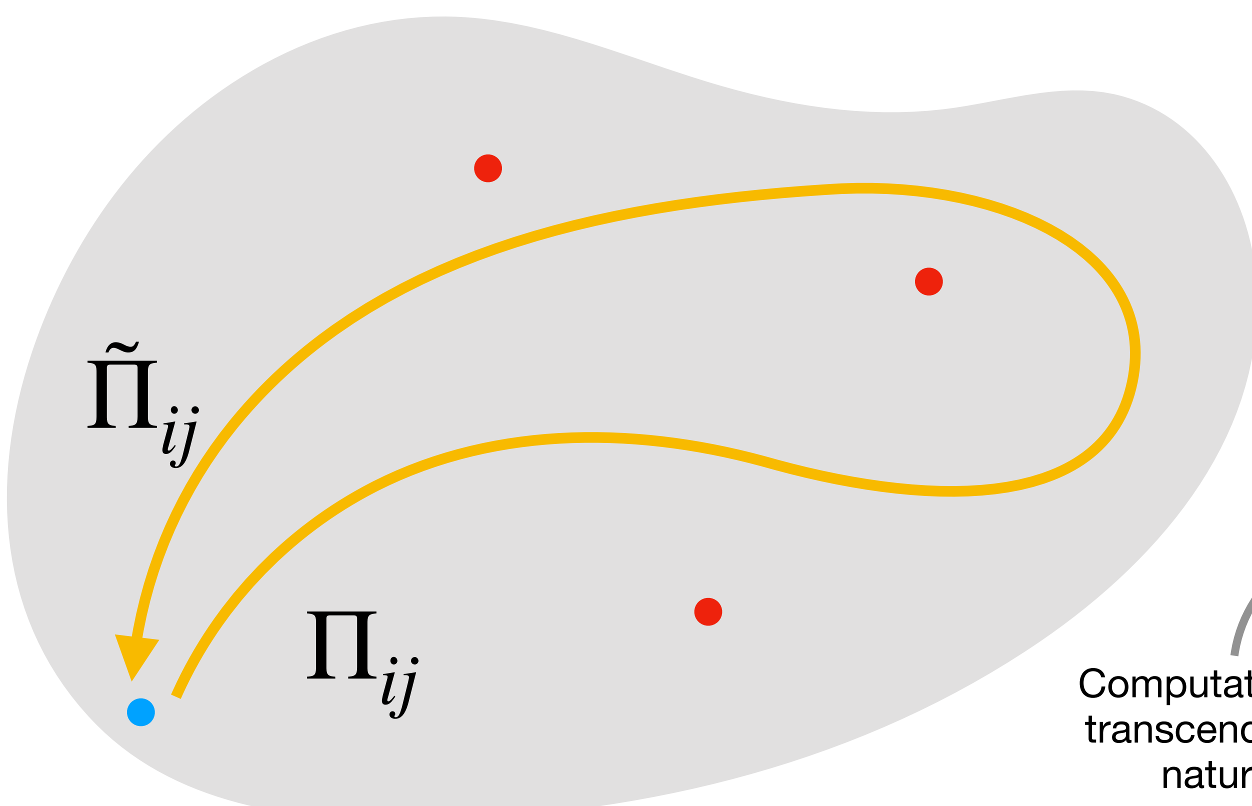
$$\Pi_{ij} = \int_{\gamma_j} \partial_t^i \omega_t \xrightarrow{\text{Analytic continuation}} \tilde{\Pi}_{ij} = \int_{\sum_k c_{kj}\gamma_k} \partial_t^i \omega_t = \sum_k c_{kj} \int_{\gamma_k} \partial_t^i \omega_t$$

[Chudnovsky[2] 90, Van der Hoeven 99, Mezzarobba 2010]

Solution to Picard-Fuchs equation of ω_t

$$\tilde{\gamma}_j = \sum_k c_{kj}\gamma_k$$

The c_{kj}'s are integers



Thus $\tilde{\Pi} = \Pi C$ i.e.

$$\boxed{\Pi^{-1}\tilde{\Pi} = C \in \mathrm{GL}_2(\mathbb{Z})}$$

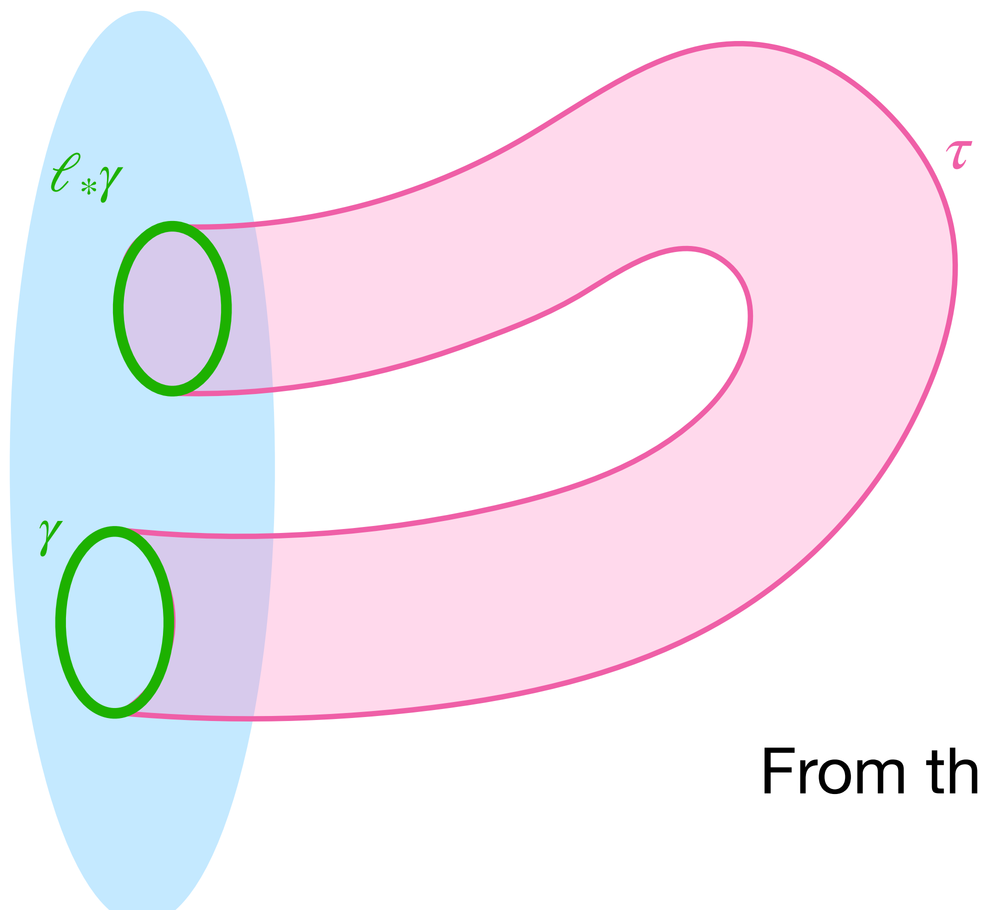
Computation of transcendental nature

It is sufficient to carry out this computation with precision $< 1/2$ to recover C exactly.

Periods of hypersurfaces

From the monodromy we compute the boundary of thimbles, and we can glue them to obtain extensions.

$\partial\tau_\ell(\gamma) = \ell^*\gamma - \gamma$



This yields an inductive method for computing the periods of smooth hypersurfaces.

$$\int_{\tau_\ell(\gamma)} \omega = \int_\ell \left(\int_\gamma \omega_t \right) \wedge \mathrm{d}t$$

From the periods, we may recover algebraic invariants.

For example, we can find quartic surfaces with Picard rank 2, 3 and 5, which were missing entries in a search of **[Lairez Sertöz 2019]**.

$$\mathscr{X} = V \begin{pmatrix} x^4 - x^2y^2 - xy^3 - y^4 + x^2yz + xy^2z + x^2z^2 - xyz^2 + xz^3 \\ -x^3w - x^2yw + xy^2w - y^3w + y^2zw - xz^2w + yz^2w - z^3w + xyw^2 \\ +y^2w^2 - xzw^2 - xw^3 + yw^3 + zw^3 + w^4 \end{pmatrix}$$

Periods of hypersurfaces

We thus obtain an algorithm for computing the periods of smooth hypersurfaces, inductive on the dimension.

Because we are working with lower dimensional varieties, this method turns out to be **more efficient** than that of **[Sertöz 2019]**.
In particular we are able to compute the periods of quartic K3 surfaces:

$P - x^4 - w^4 - z^4 - w^4$	*numperiods*	*lefschetz-family*	ord $\mathcal{L}$	deg $\mathcal{L}$
0	< 1 s	384 min.	–	–
$2x^2zw$	4 s	574 min.	3	4
$-2y^3z - 4z^2w^2$	2 min.	510 min.	5	38
$-xyzw + 4xzw^2 - 2y^4$	25 min.	607 min.	7	110
$y^3z + z^4 + y^3w + x^2zw$	346 min.	635 min.	14	591
$4xyz^2 - 5x^2zw - 4xw^3 - 4zw^3$	> 2880 min.	494 min.	21	?
$-2x^2w^2 - 4y^2w^2 - 2yzw^2 + 2yw^3$	> 500 Gb	543 min.	21	?
$x^4 - 4y^2z^2 - 5xz^2w + 2yz^2w + xyw^2$	> 500 Gb	538 min.	14	?

In all cases, *lefschetz-family* integrates an operator of order 7.

We have solved one of the main difficulties:
the direct computation of the homology of hypersurfaces.

The bottleneck for accessing higher dimensions is still the order/degree of the differential operators.

Beyond hypersurfaces

Non-Lefschetz fibrations

A semi-numerical algorithm for the homology lattice and periods of complex elliptic surfaces over $\mathbb{P}^1$,

Journal of Symbolic Computations, 2024

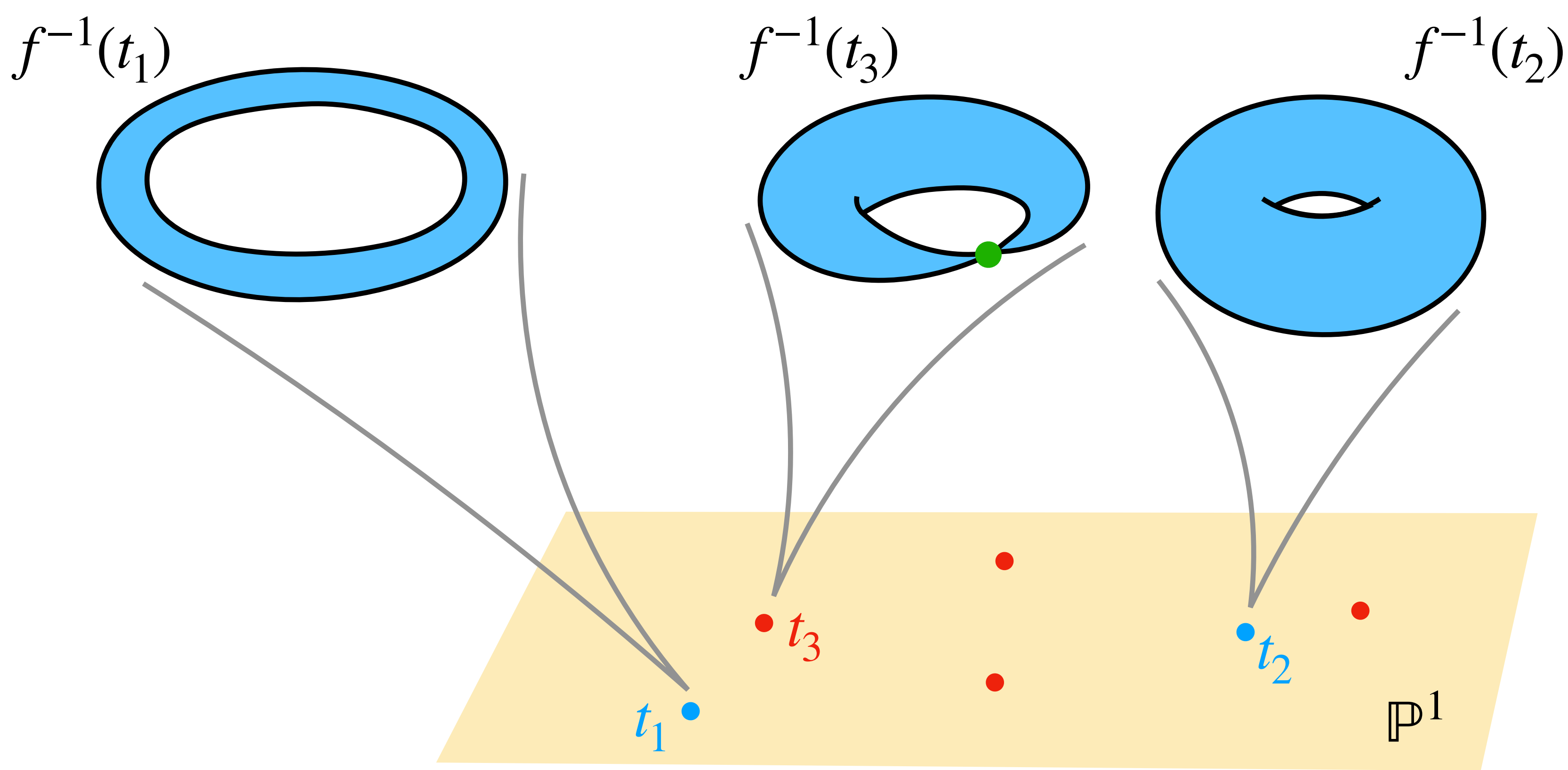
Elliptic surfaces

An **elliptic surface** S is a smooth algebraic surface equipped with a map to the projective line

The fibration is given. We cannot choose it to be generic.

$$f : S \to \mathbb{P}^1$$

such that all but finitely many fibres $f^{-1}(t)$ are elliptic curves.



We will assume the surface has a **section**.

Non-Lefschetz fibrations: an example

The **Apéry surface** S, defined by $y^2 + (t-1)xy + ty = x^3 - tx^2$.

0

Compute the set Σ of critical values

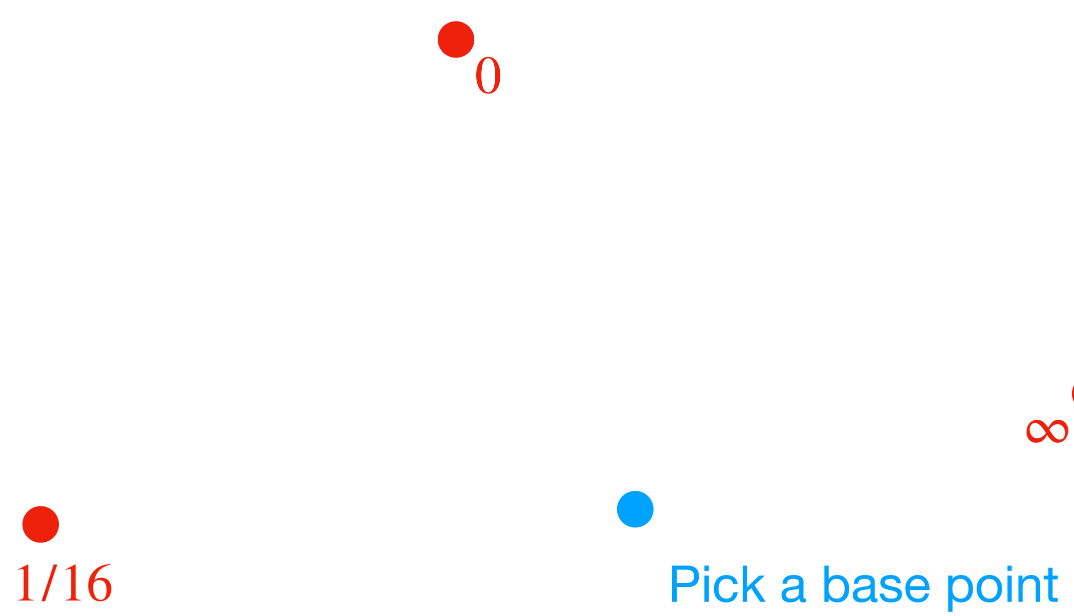
i.e., the roots of the discriminant $t^4(t - \frac{1}{16})$

∞

$1/16$

Non-Lefschetz fibrations: an example

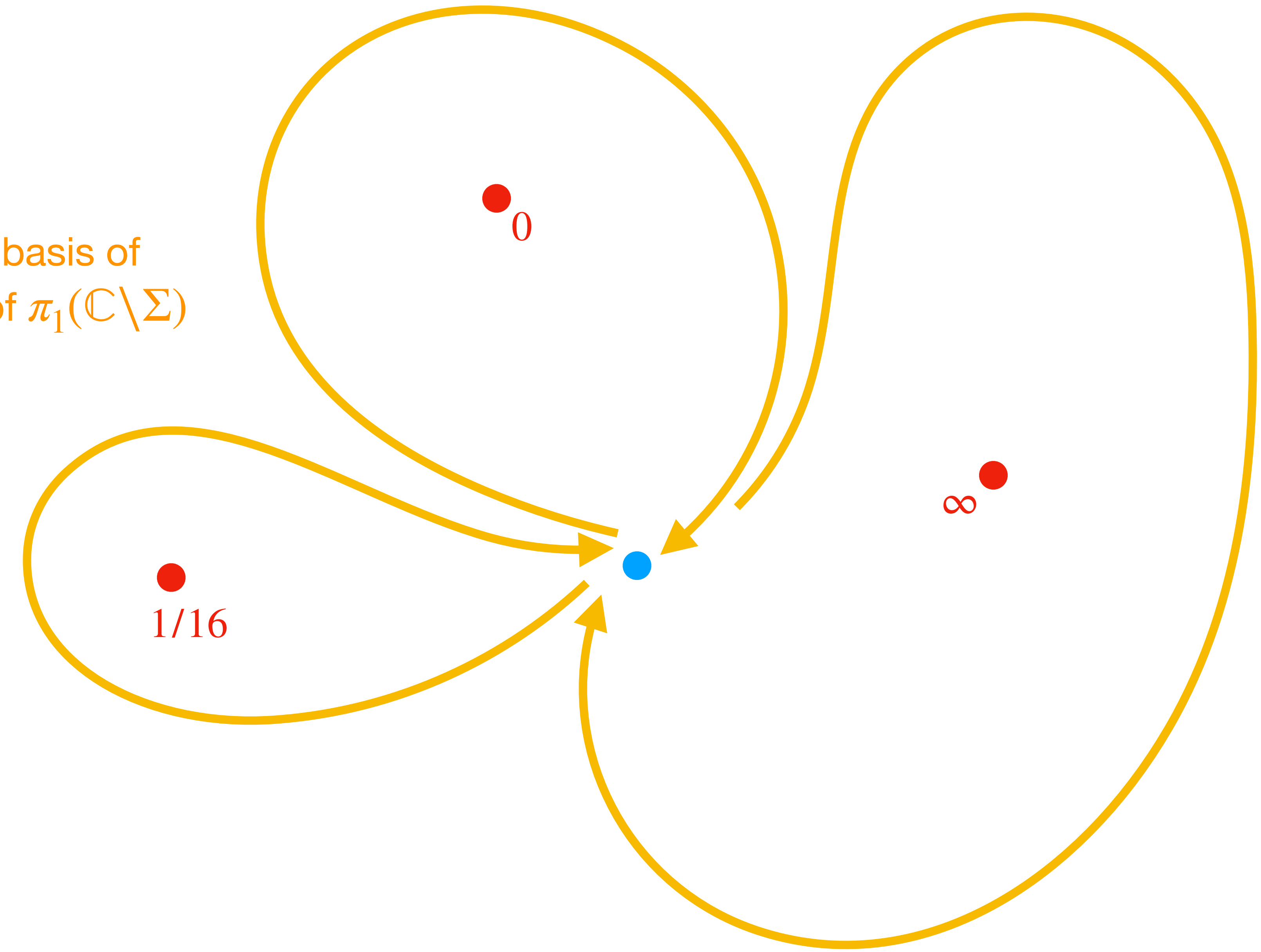
The **Apéry surface** S, defined by $y^2 + (t-1)xy + ty = x^3 - tx^2$.



Non-Lefschetz fibrations: an example

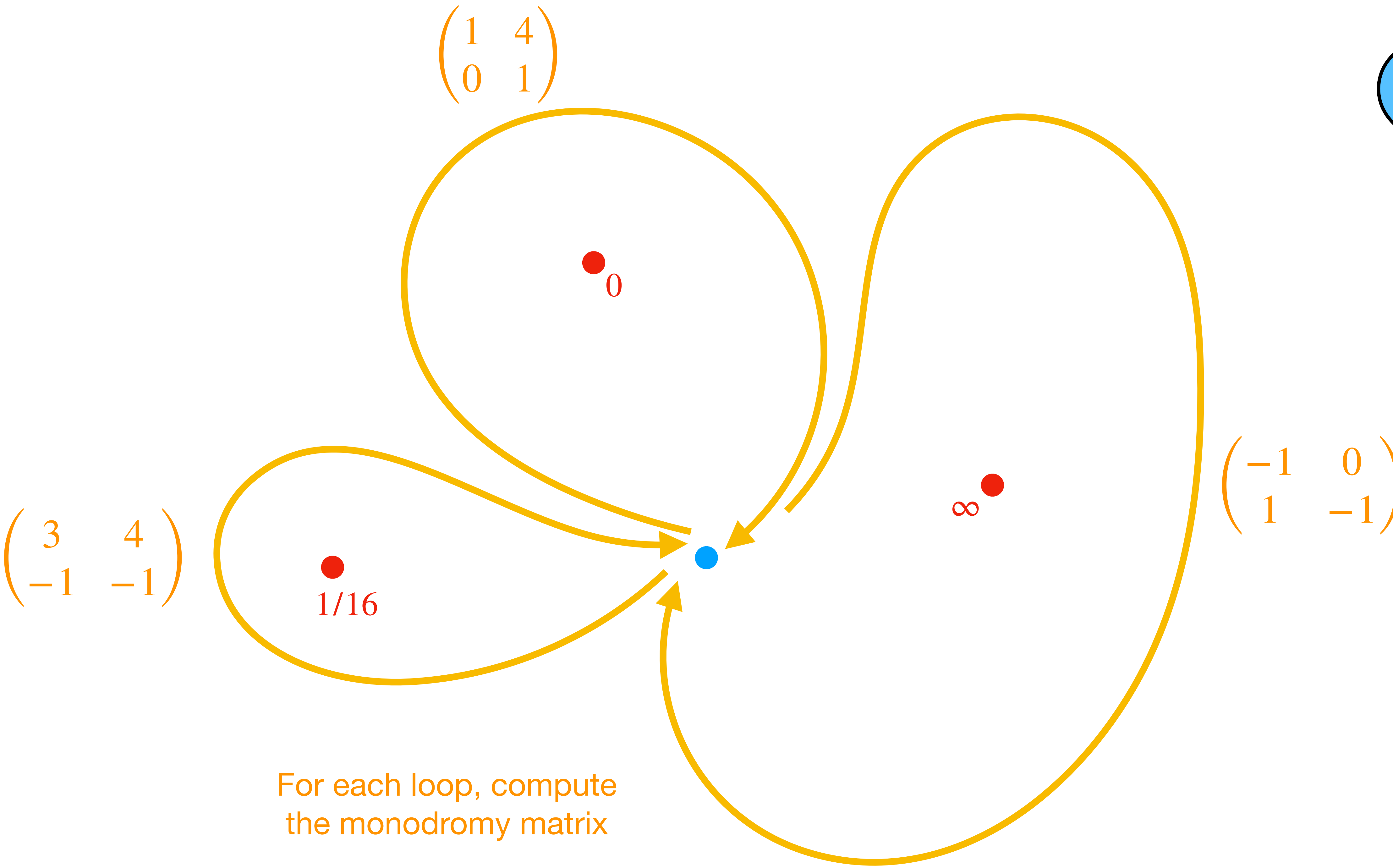
The **Apéry surface** S, defined by $y^2 + (t-1)xy + ty = x^3 - tx^2$.

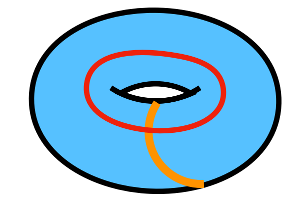
Compute a basis of
simple loops of $\pi_1(\mathbb{C}\backslash\Sigma)$



Non-Lefschetz fibrations: an example

The **Apéry surface** S, defined by $y^2 + (t-1)xy + ty = x^3 - tx^2$.

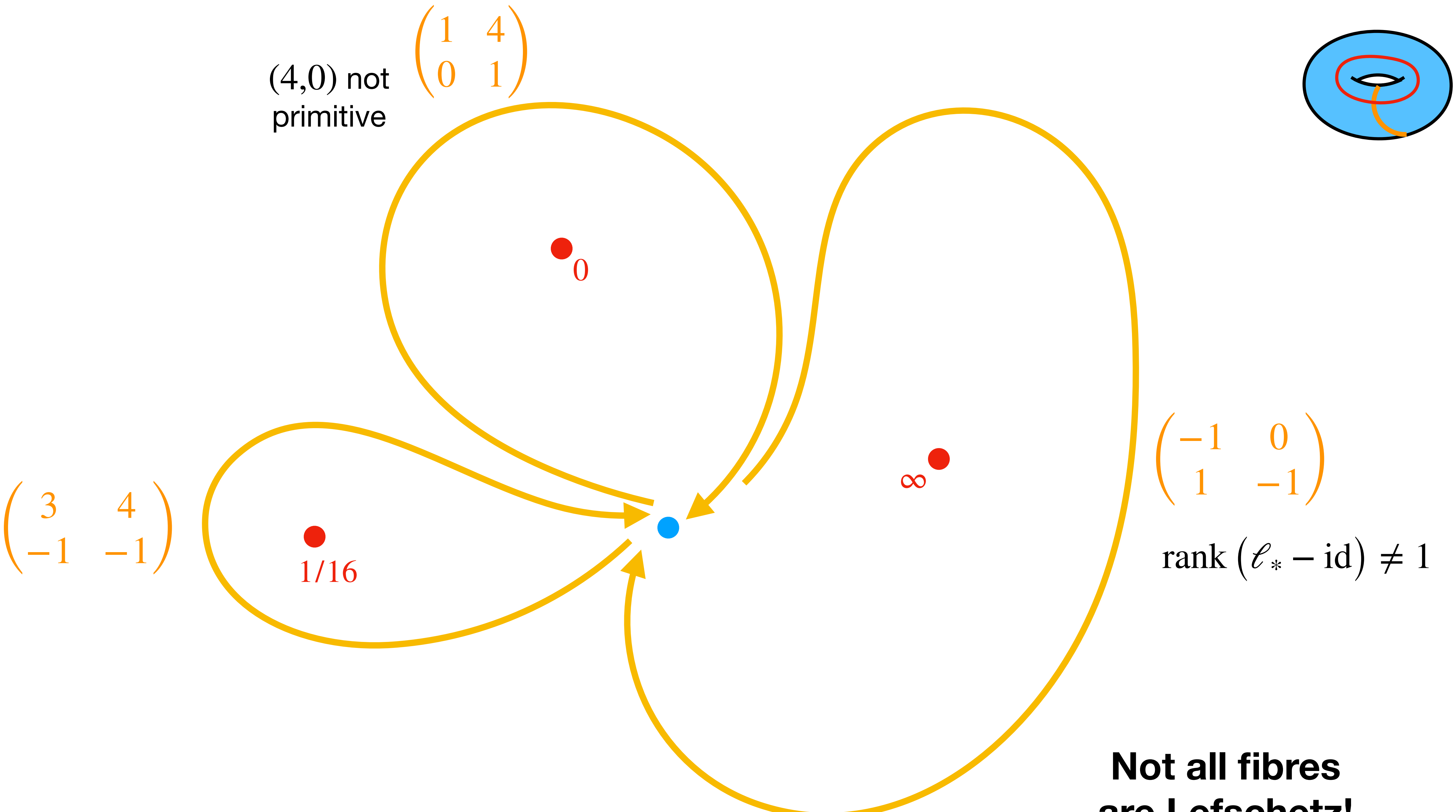




For each loop, compute the monodromy matrix

Non-Lefschetz fibrations: an example

The **Apéry surface** S, defined by $y^2 + (t-1)xy + ty = x^3 - tx^2$.

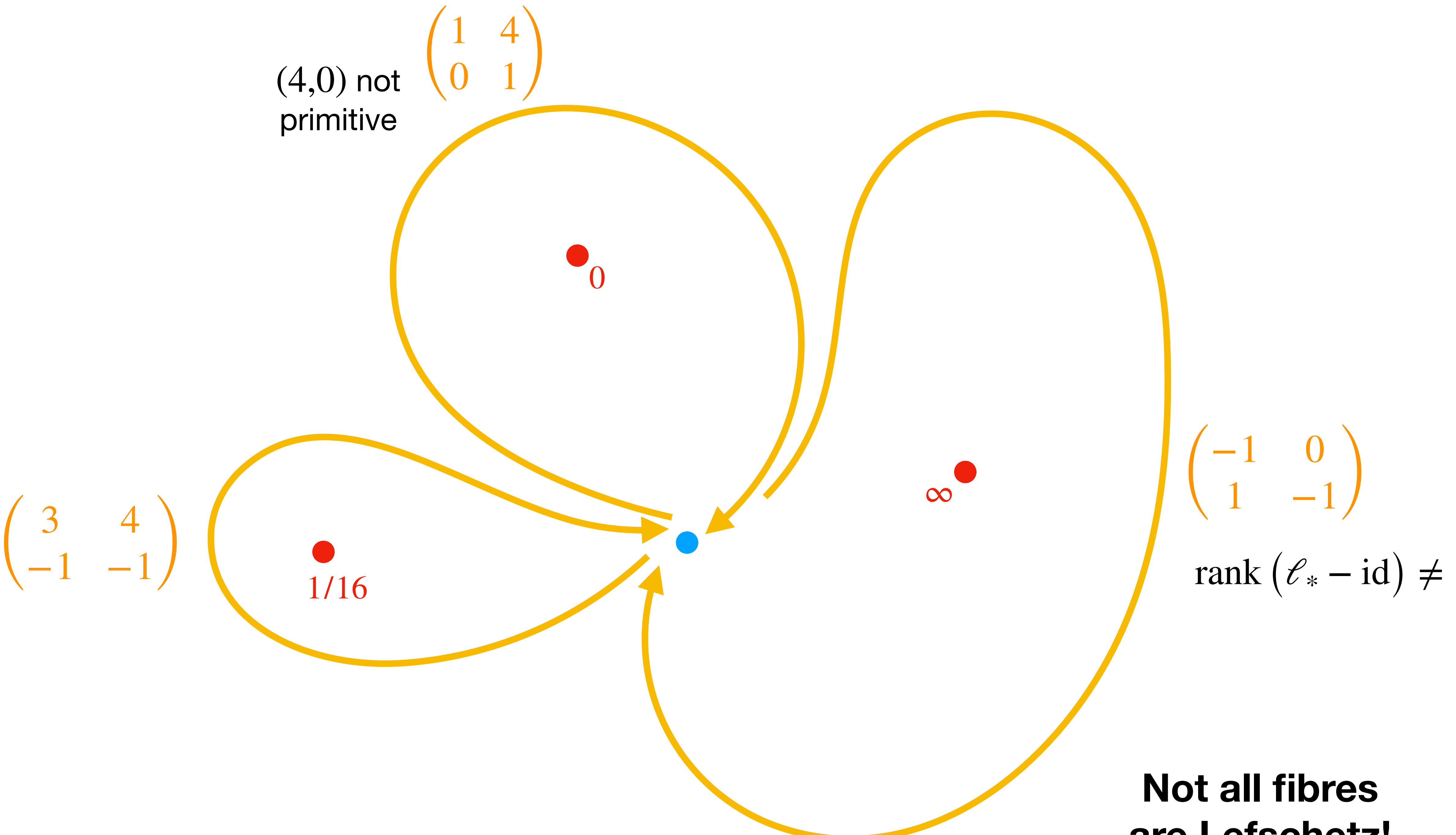


Not all fibres are Lefschetz!

We have to find a workaround …

Non-Lefschetz fibrations: an example

The **Apéry surface** S, defined by $y^2 + (t-1)xy + ty = x^3 - tx^2$.

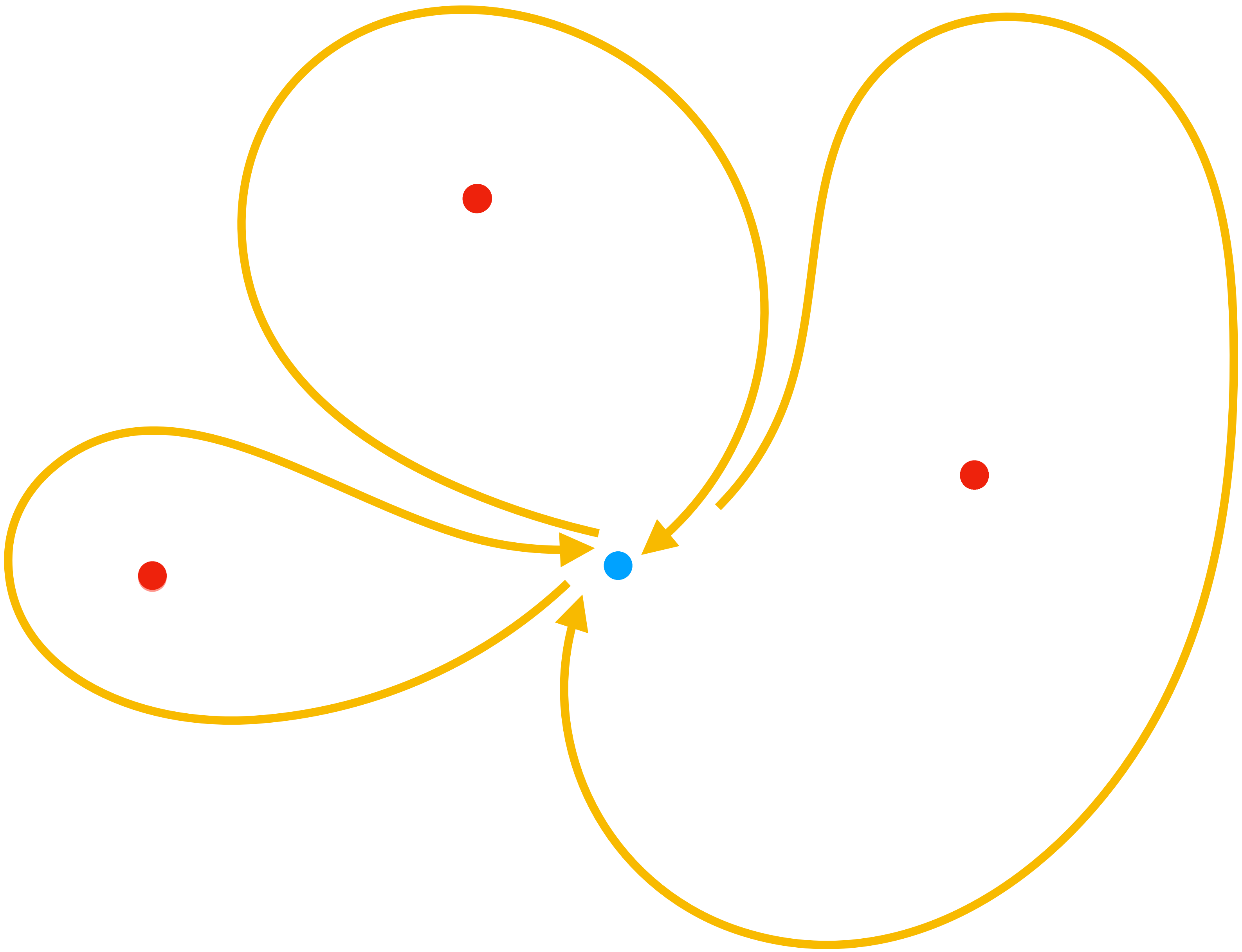


Not all fibres are Lefschetz!

We have to find a workaround …

Non-Lefschetz fibrations: an example

We deform the surface to $\tilde{S} : y^2 + (t-1)xy + ty = x^3 - tx^2 + \varepsilon$.



Non-Lefschetz fibrations: an example

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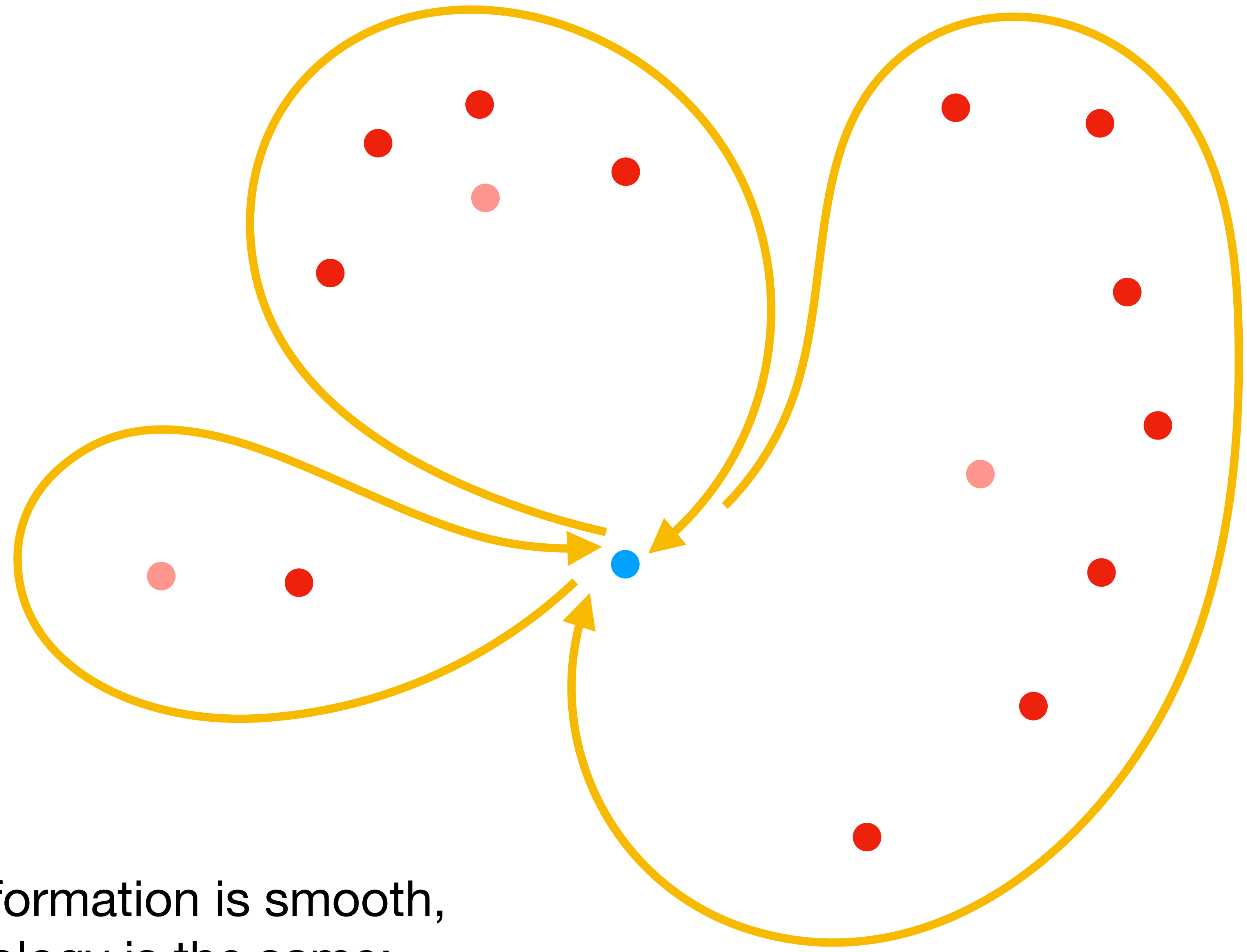
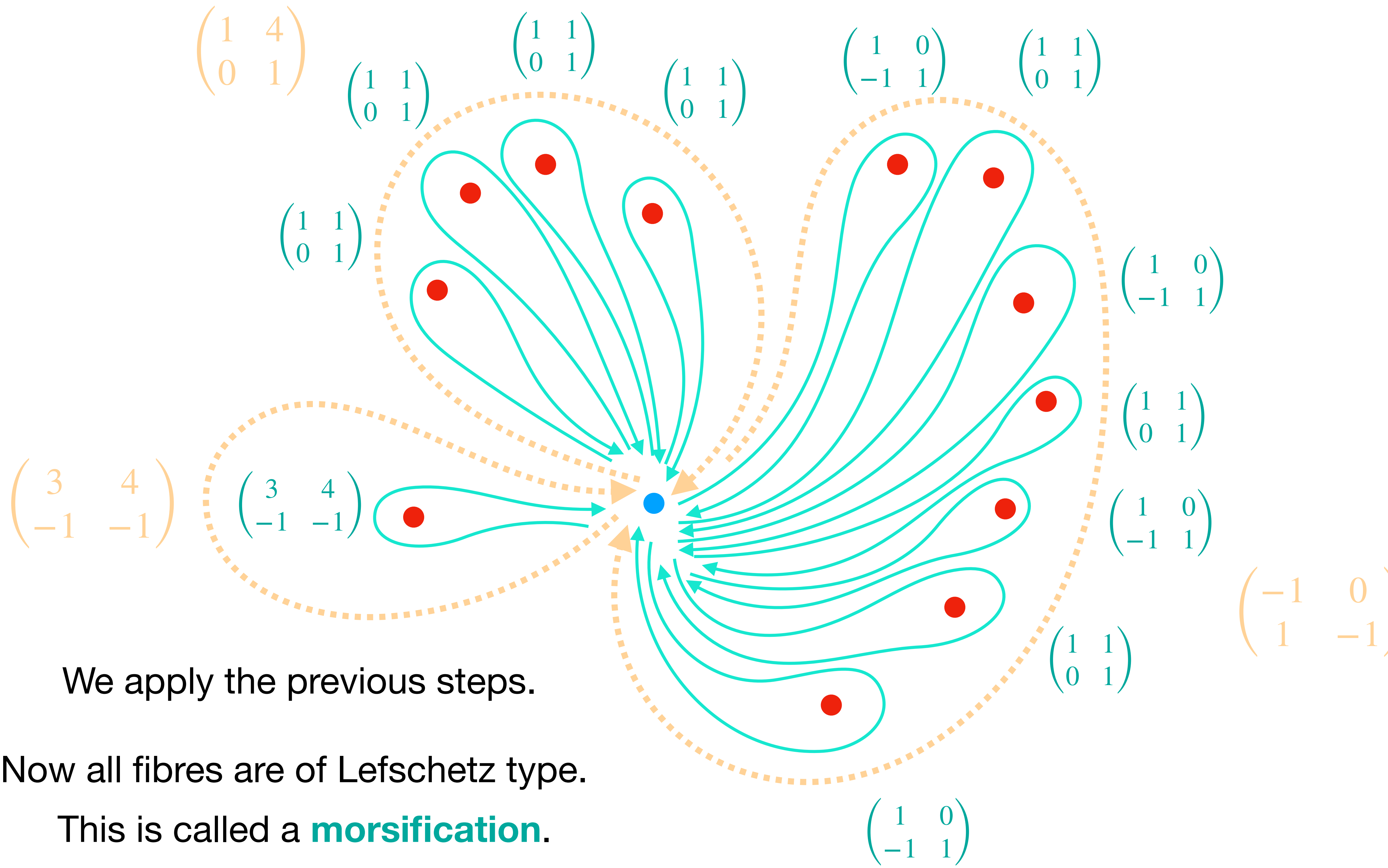


As the deformation is smooth,
the topology is the same:
$H_2(S) \simeq H_2(\tilde{S})$.

Non-Lefschetz fibrations: an example

We deform the surface to $\tilde{S} : y^2 + (t-1)xy + ty = x^3 - tx^2 + \varepsilon$.



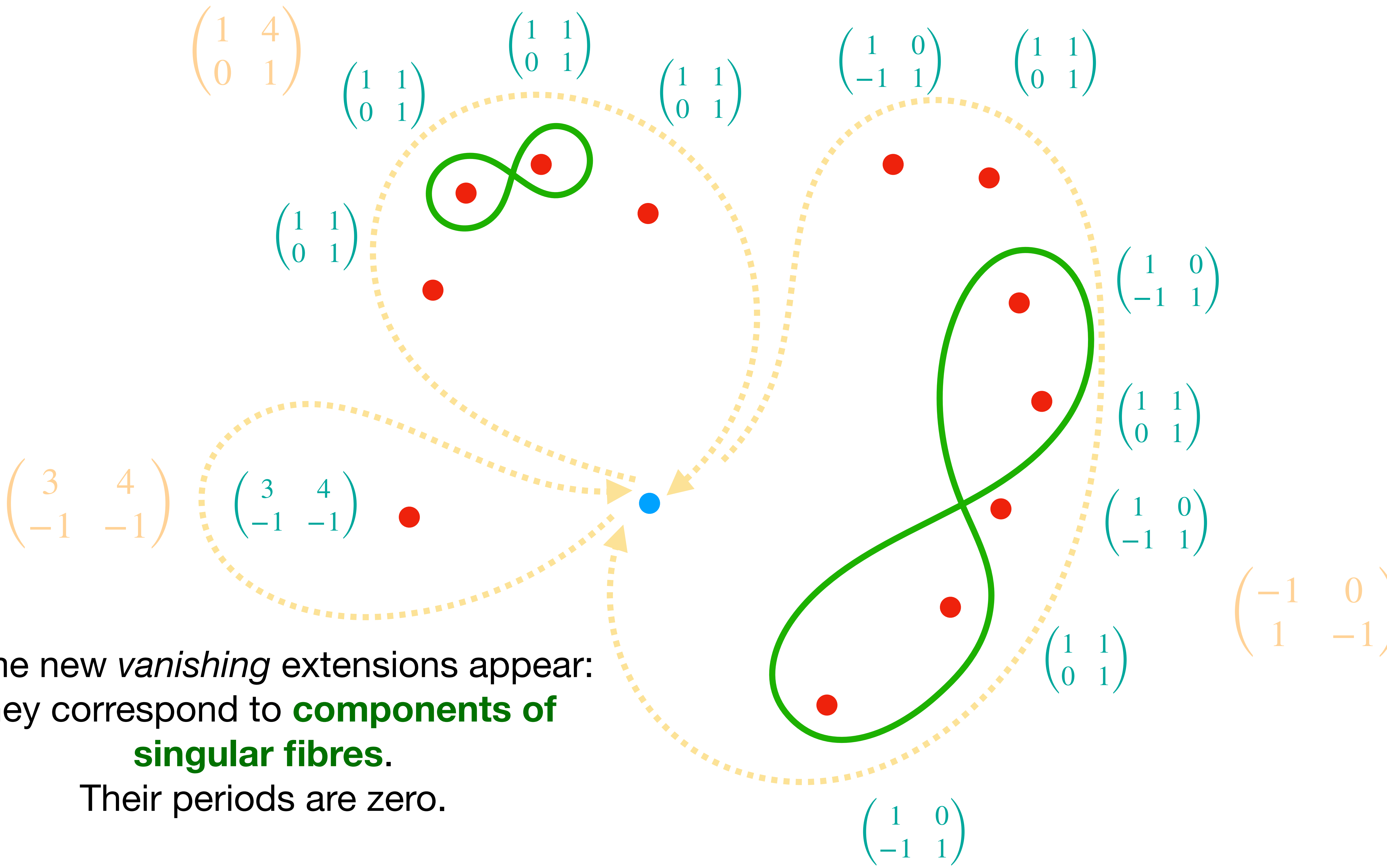
We apply the previous steps.

Now all fibres are of Lefschetz type.

This is called a **morsification**.

Non-Lefschetz fibrations: an example

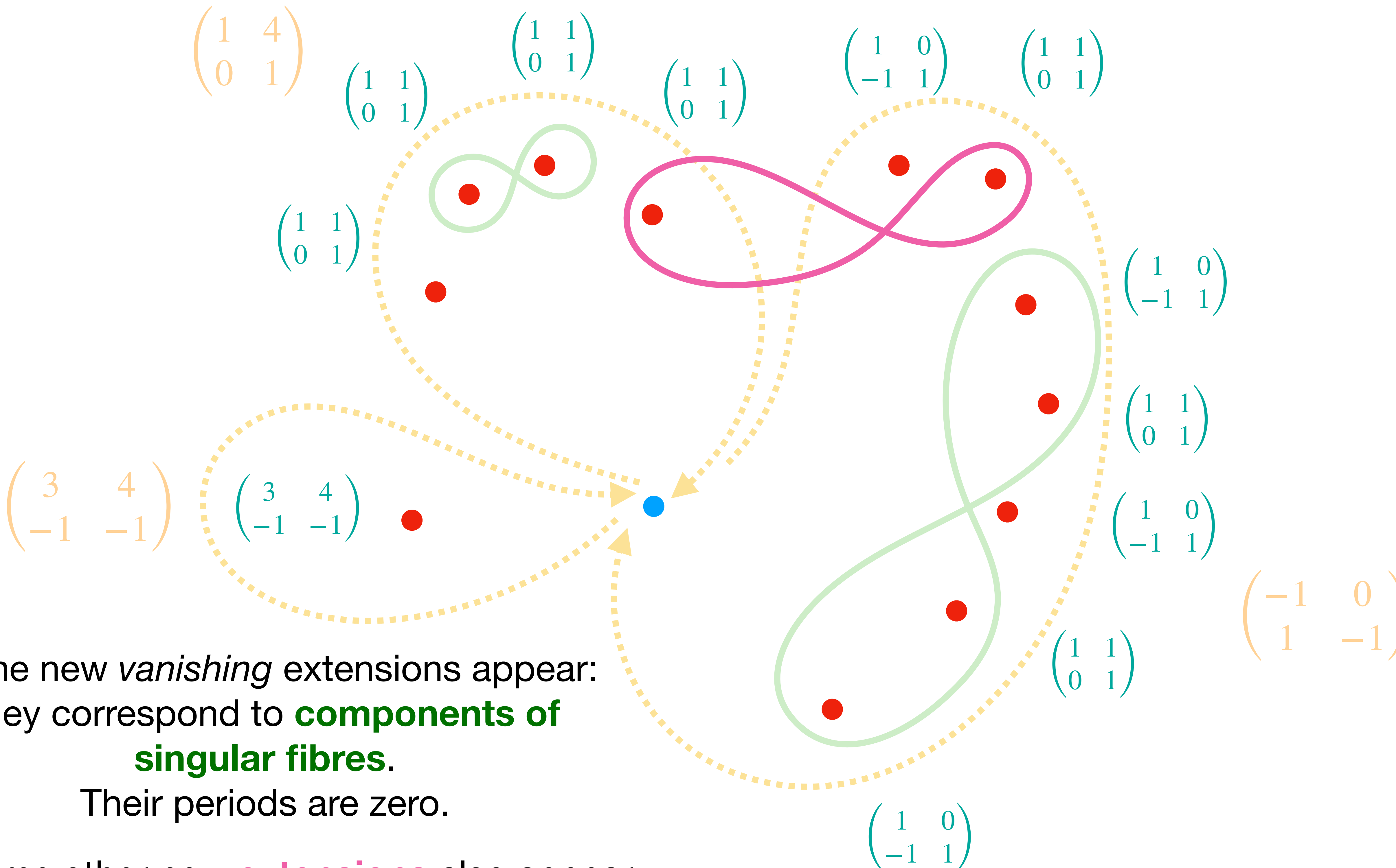
We deform the surface to $\tilde{S} : y^2 + (t-1)xy + ty = x^3 - tx^2 + \varepsilon$.



Some new *vanishing* extensions appear: they correspond to **components of singular fibres**. Their periods are zero.

Non-Lefschetz fibrations: an example

We deform the surface to $\tilde{S} : y^2 + (t-1)xy + ty = x^3 - tx^2 + \varepsilon$.



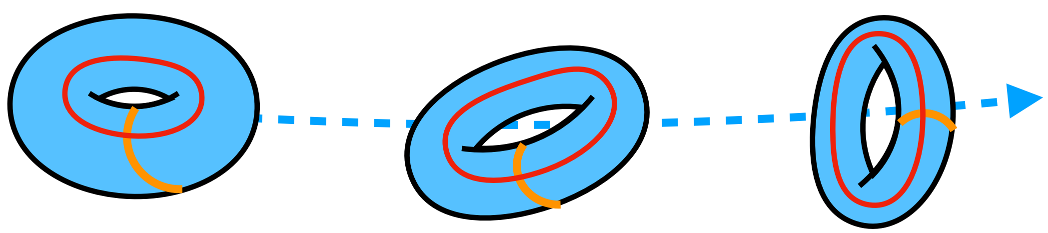
Some new *vanishing* extensions appear:
they correspond to **components of singular fibres**.
Their periods are zero.

Some other new **extensions** also appear.

Non-Lefschetz fibrations

Theorem [Moishezon 1977]: Morsifications always exist.

Monodromy preserves the intersection product



$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\iff ad - bc = 1$$

The monodromy matrix is in $\mathrm{SL}_2(\mathbb{Z})$.

Kodaira classification [1963]

$I_\nu, \nu \geq 1$	$\begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix}$	U^ν
II	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	VU
III	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	VUV
IV	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	$(VU)^2$

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad V = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

...

Theorem [Cadavid Vélez 2009]:
The monodromy of the morsification is determined by the monodromy of S.

only cycles with nonzero periods

Theorem [PP 2024]: The sublattice of $H_2(S)$ generated by **extensions** of S, the **section**, the **fibre** and **singular components** has full rank.

Non-Lefschetz fibrations

Theorem [Cadavid, Vélez 2009]:
The monodromy of the morsification is determined by the monodromy of S.

$$
\begin{array}{ccc}
\ell_1 & \ell_2 & \ell_3 \\
\begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \\
I_1 : U & I_4 : U^4 & I_1^* : UVUVUV
\end{array}
$$

morsification

$$
\begin{array}{cccccccccccc}
\begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \\
\ell'_1 & \ell'_{21} & \ell'_{22} & \ell'_{23} & \ell'_{24} & \ell'_{31} & \ell'_{32} & \ell'_{33} & \ell'_{34} & \ell'_{35} & \ell'_{36} & \ell'_{37}
\end{array}
$$

In particular **we do not need to find an explicit realisation of the morsification**!

The algorithm for elliptic surfaces

1. Compute a basis of **simple loops** $\ell_1, \ldots, \ell_r$ of $\pi_1(\mathbb{P}^1 \backslash \Sigma, b)$
2. For each $1 \leq i \leq r$, compute the **monodromy map** ℓ_{i*}.
3. Glue thimbles together to obtain **extension cycles** of $H_2(S)$.
4. Integrate the **periods** on these cycles.

5. From the monodromy type of ℓ_{i*}, recover the monodromy matrices of a **morsification** $\tilde{S}$.
6. Glue thimbles together to obtain **extension cycles** of $H_2(\tilde{S})$.
7. Recover the homology $H_2(\tilde{S})$ of the morsification (**extensions** + **fibre** + **section**).
8. Describe the extensions of $H_2(S)$ in terms of the extensions of $H_2(\tilde{S})$.

9. Recover the periods of all of $H_2(S) \simeq H_2(\tilde{S})$.

This allows for the (heuristic) computation of certain algebraic invariants of the elliptic surface (Néron-Severi group, Mordell-Weil group, …)

Implemented in the **lefschetz-family** Sagemath package, available on my webpage.

Further applications

of the methods presented here

Motivic Geometry of two-Loop Feynman Integrals,

The Quarterly Journal of Mathematics, 2024

with C. Doran, A. Harder and P. Vanhove

and ongoing works

Recovering certain algebraic invariants

Theorem [Doran Harder PP Vanhove 2024]: The Tardigrade hypersurface has the same motivic geometry as a quartic K3 surface with six A_1 singularities.

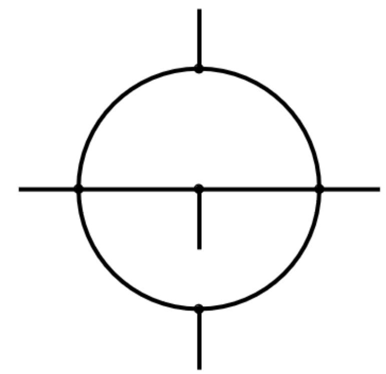


FIGURE 13. The tardigrade graph

Our methods allow to compute the periods of this quartic K3 surface.

From the periods, we recover numerically that **its Néron-Severi rank is 11** for generic values of the mass parameters.

Lefschetz's theorem on (1,1) classes:
A homology class $\gamma \in H_2(S)$ is in the Néron-Severi group $NS(S)$ iff the periods of holomorphic forms on γ vanish.

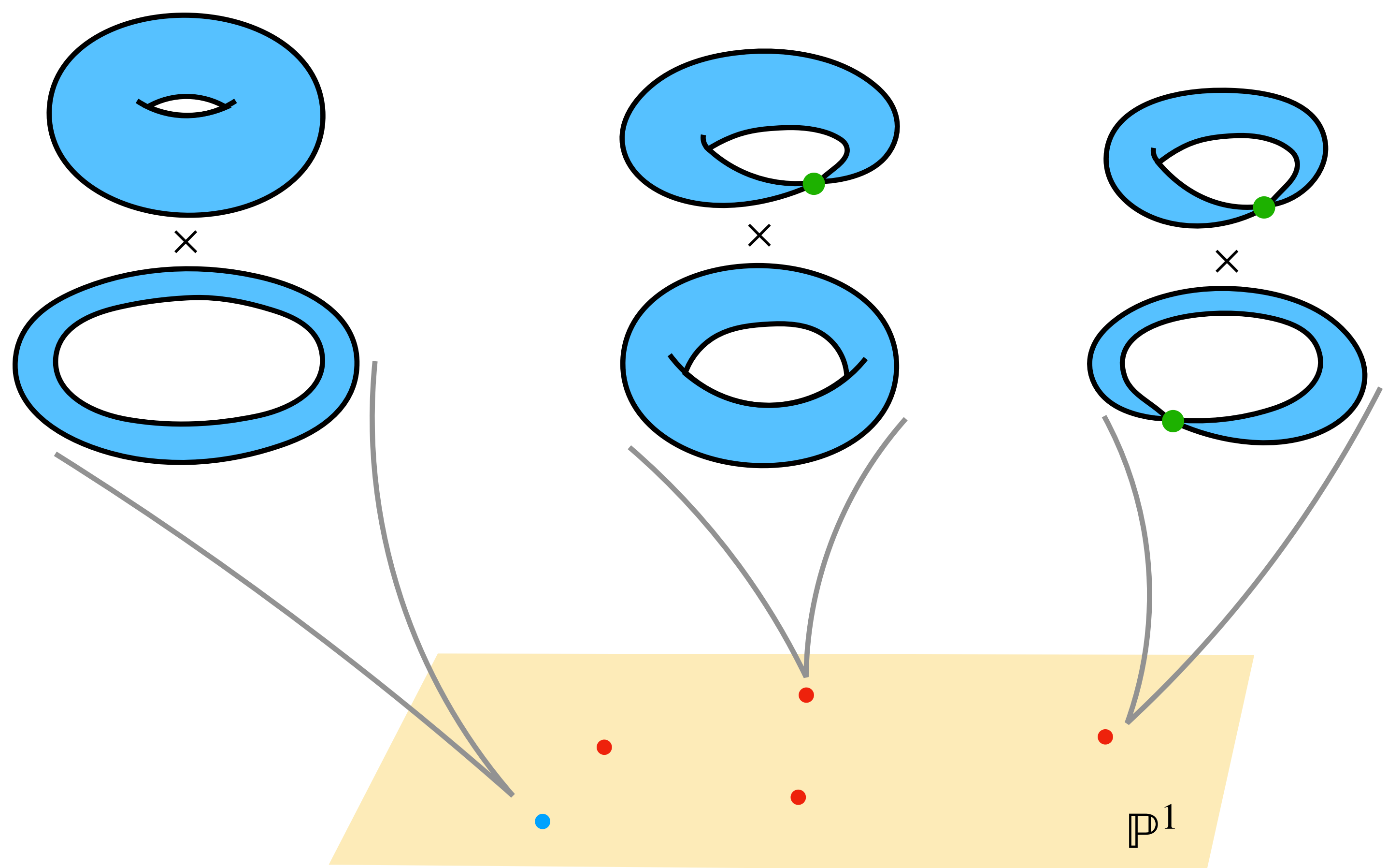
Using the LLL algorithm, we can heuristically recover this kernel by finding integer linear relations between the periods.

DEMO

The Deligne conjecture

with Duco van Straten
and Nutsa Gegelia

Our method applies to **Calabi-Yau threefolds** given as the fibered product of two elliptic surfaces.



We can compute a **basis of homology** and the **period matrix.**

In certain cases, **this allows to numerically check the Deligne conjecture (1979)**, which relates a minor c^+ of the period matrix to an arithmetical invariant $L(2)$ via the formula $L(2) = qc^+$, where $q \in \mathbb{Q}$.

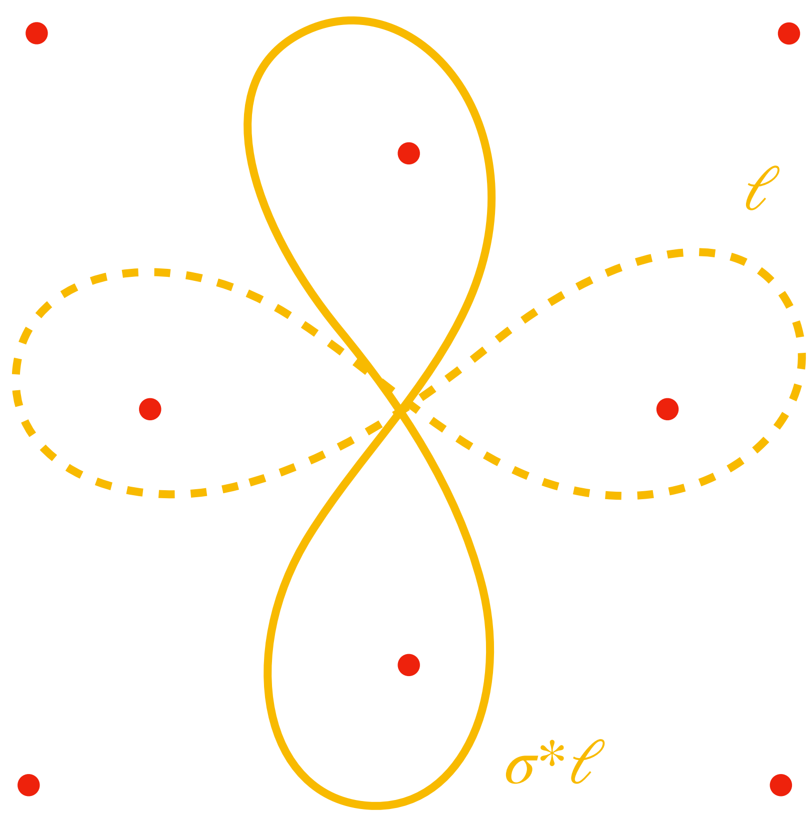
We are able to numerically recover the value of q.

Automorphisms of K3 surfaces

with Minako Chinen, Charles Doran and Hal Haggard

Let $\sigma : \mathscr{X} \to \mathscr{X}$ be an automorphism of an elliptic K3 surface $\mathscr{X} \to \mathbb{P}^1$.
It induces an action σ^* on the lattice of cycles $H_2(\mathscr{X})$.

In certain cases, **our method allows to compute parts of this action**.



The image of an extension along ℓ is an extension along $\sigma^*\ell$.

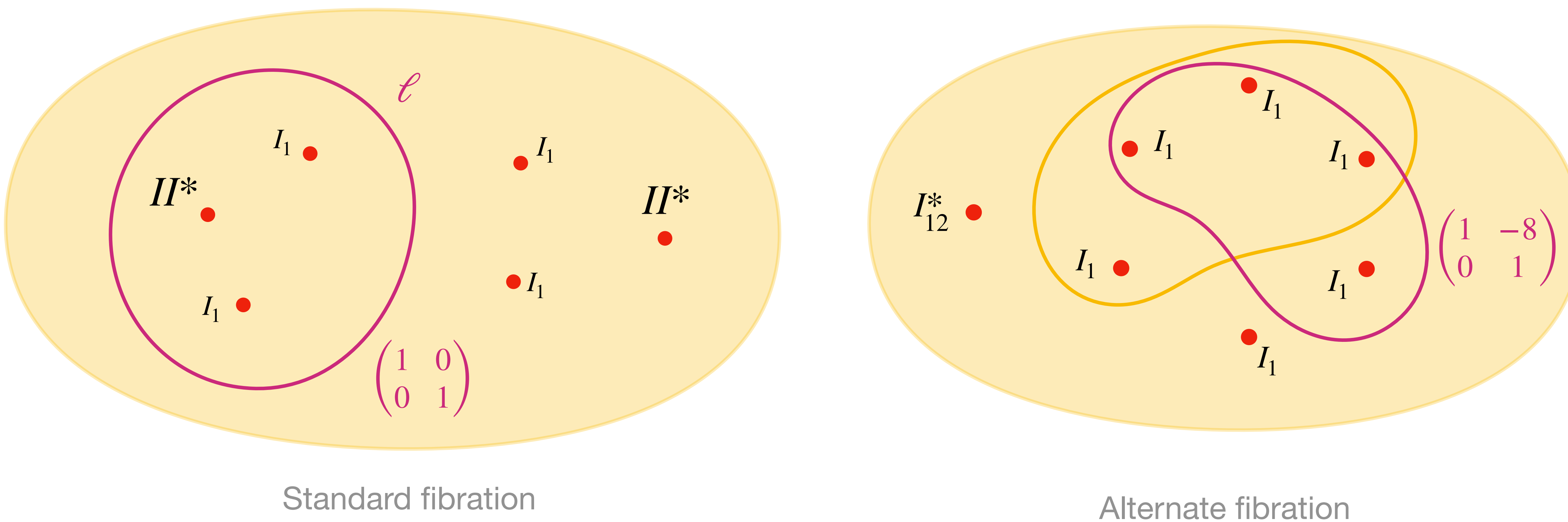
Doran-Harder-Thompson conjecture for K3 surfaces

with Charles Doran and Alan Thompson

The **DHT conjecture** for K3 surfaces predicts that elliptic fibrations on a K3 surface correspond to Tyurin degenerations on its mirror.

The description of cycles we compute allows to prove this conjecture for K3 surfaces admitting certain polarisations.

Example: M-polarised K3 surface



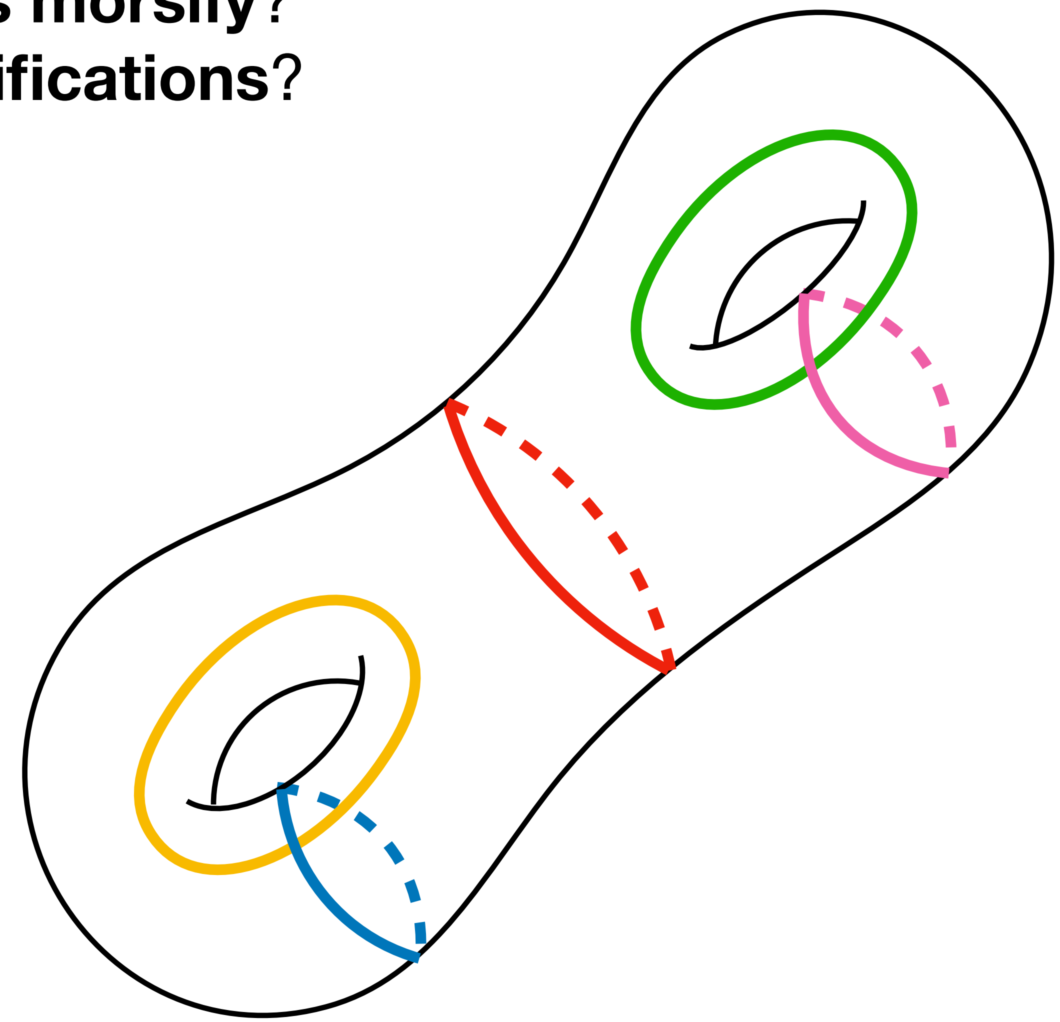
[Giovenzana Thompson 2024] reduces the conjecture to the existence of certain cycles of elliptic surfaces.

Genus 2 fibrations

Certain K3 surfaces are given by ramified double covers of $\mathbb{P}^2$.
We may fibre them with **genus 2 curves.**
When the ramification locus is smooth, we may always obtain a Lefschetz fibration.

When it is not, **can we always morsify**?
Can we **bypass explicit morsifications**?

Defining polynomial	Picard number
$w^2 + xy^5 + x^5z + y^3z^3 + xz^5$	1
$w^2 + x^6 + y^5z + xz^5$	2
$w^2 + x^5y + xy^5 + x^3y^2z + z^6$	3
$w^2 + x^5y + y^6 + x^3yz^2 + x^3z^3 + xz^5$	4
$w^2 + x^5y + y^6 + x^4z^2 + x^2yz^3 + xz^5$	5
$w^2 + x^4y^2 + x^5z + y^5z + z^6$	6
–	7
$w^2 + x^5y + y^5z + y^2z^4 + xz^5$	8
$w^2 + x^5y + y^6 + x^2z^4 + z^6$	9
$w^2 + x^6 + y^5z + x^2z^4 + z^6$	10
$w^2 + x^5y + xy^5 + x^3yz^2 + z^6$	11
$w^2 + x^6 + y^6 + z^6 + x^2yz^3$	12
$w^2 + x^6 + y^6 + z^6 + x^2y^4 + x^4z^2$	13
$w^2 + x^6 + y^6 + xz^5$	14
$w^2 + x^6 + y^6 + z^6 + x^4yz + xyz^4$	15
$w^2 + x^6 + y^6 + z^6 + x^4y^2$	16
$w^2 + x^6 + y^6 + z^6 + x^4yz$	17
$w^2 + x^5y + x^3y^3 + xy^5 + z^6$	18
$w^2 + x^6 + y^6 + z^6 + x^3y^3$	18
$w^2 + x^6 + y^6 + z^6 + x^2y^2z^2$	19
$w^2 + x^6 + y^6 + z^6$	20

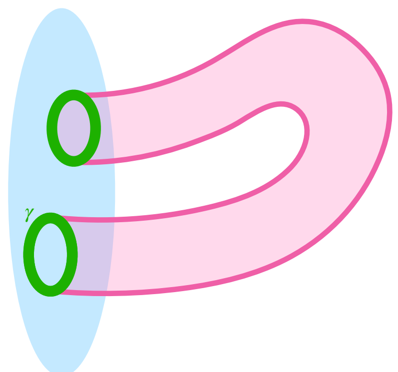


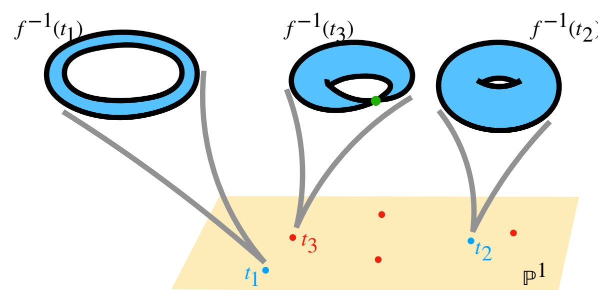
There are **many more singular types** to consider than for elliptic fibrations (118 vs 8).

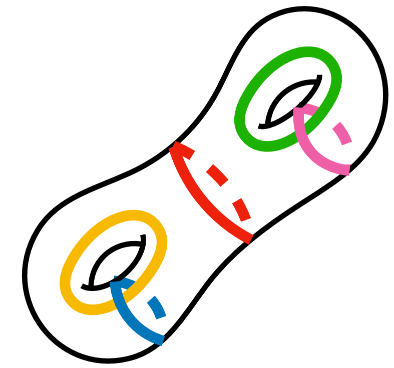
Furthermore the **monodromy matrices alone do not determine the type** of the singular fibres.

Concluding remarks

I have **developed** methods for computing periods of algebraic varieties, and have **implemented** it for hypersurfaces, elliptic surfaces and Lefschetz genus 2 fibered surfaces.





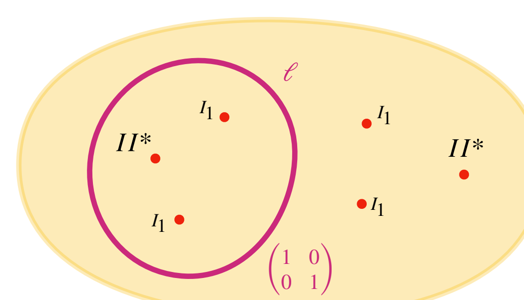


They are sufficiently **efficient** to recover the periods of examples previously out of reach.

$$\mathscr{X} = V\begin{pmatrix} x^4 - x^2y^2 - xy^3 - y^4 + x^2yz + xy^2z + x^2z^2 - xyz^2 + xz^3 \\ -x^3w - x^2yw + xy^2w - y^3w + y^2zw - xz^2w + yz^2w - z^3w + xyw^2 \\ +y^2w^2 - xzw^2 - xw^3 + yw^3 + zw^3 + w^4 \end{pmatrix}$$

numperiods	*lefschetz-family*
< 1 s	384 min.
4 s	574 min.
2 min.	510 min.
25 min.	607 min.
346 min.	635 min.
> 2880 min.	494 min.
> 500 Gb	543 min.
> 500 Gb	538 min.

Used these methods to heuristically **compute algebraic invariants** of certain varieties arising in other contexts (mirror symmetry, Feynman integrals), notably the Tardigrade graph.



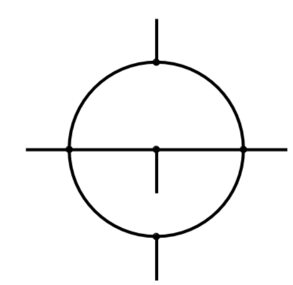
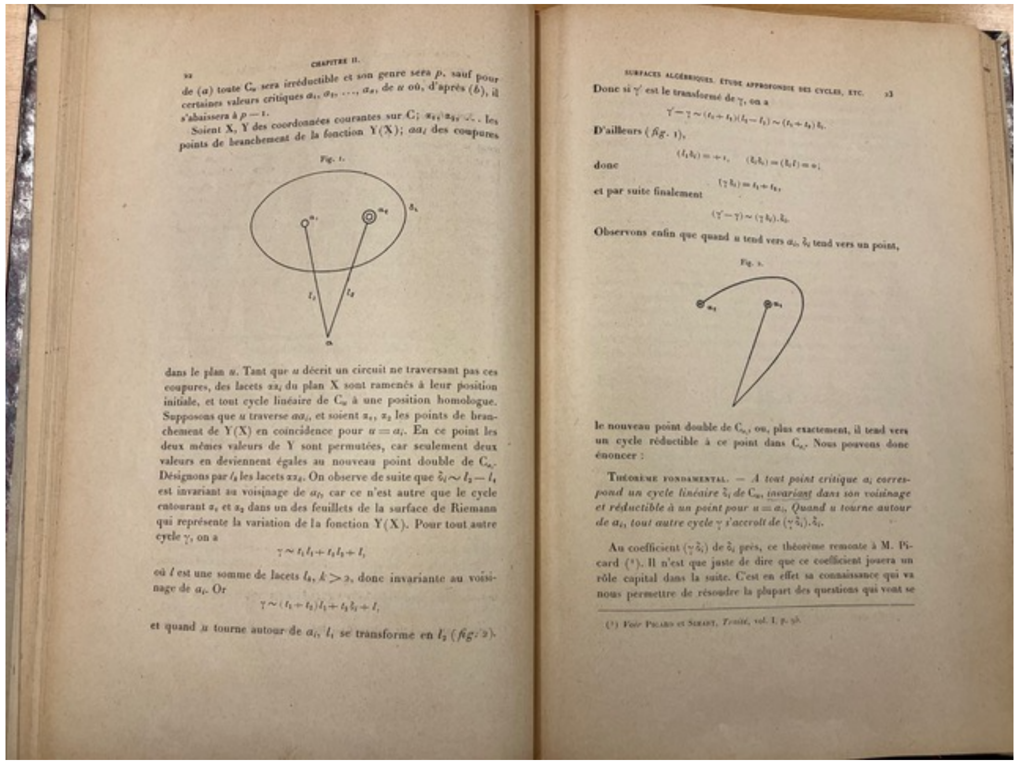


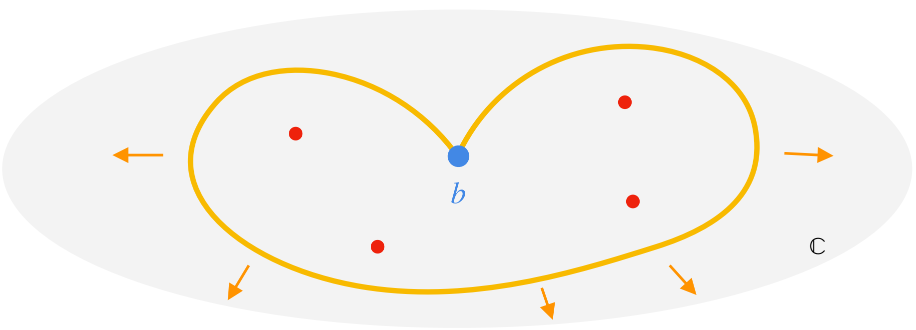
FIGURE 13. The tardigrade graph

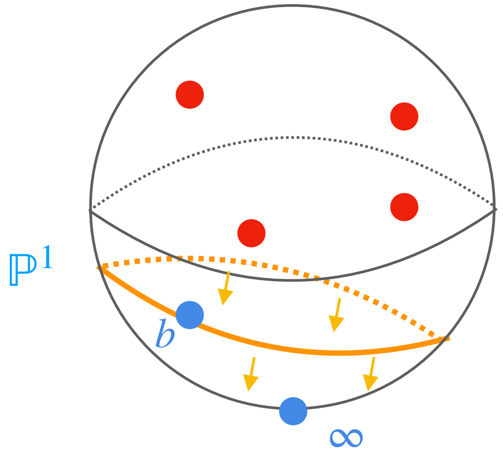
Thank you!



L'analysis situs et la géométrie algébrique, 1924, Solomon Lefschetz

Certain combinations of thimbles are trivial





Extensions along contractible paths in $\mathbb{P}^1 \backslash \{\text{crit. val.}\}$ have a trivial homology class in $H_1(\mathscr{X})$.

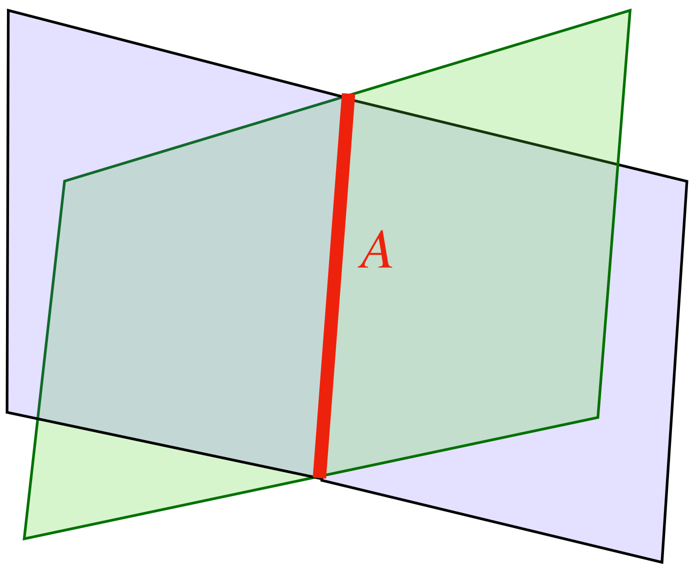
Fact: these are the only ones, the kernel of the map

$$\mathbb{Z}^r \mapsto H_1(\mathscr{X}, \mathscr{X}_b), k_1, \ldots, k_r \mapsto \sum_i k_i \Delta_i$$

is generated by these extensions "around infinity".

Obtaining a fibration from a hypersurface

The fibration of $\mathscr{X}$ is given by a hyperplane pencil $\{H_t\}_{t\in\mathbb{P}^1}$, with $\mathscr{X}_t = \mathscr{X} \cap H_t$.



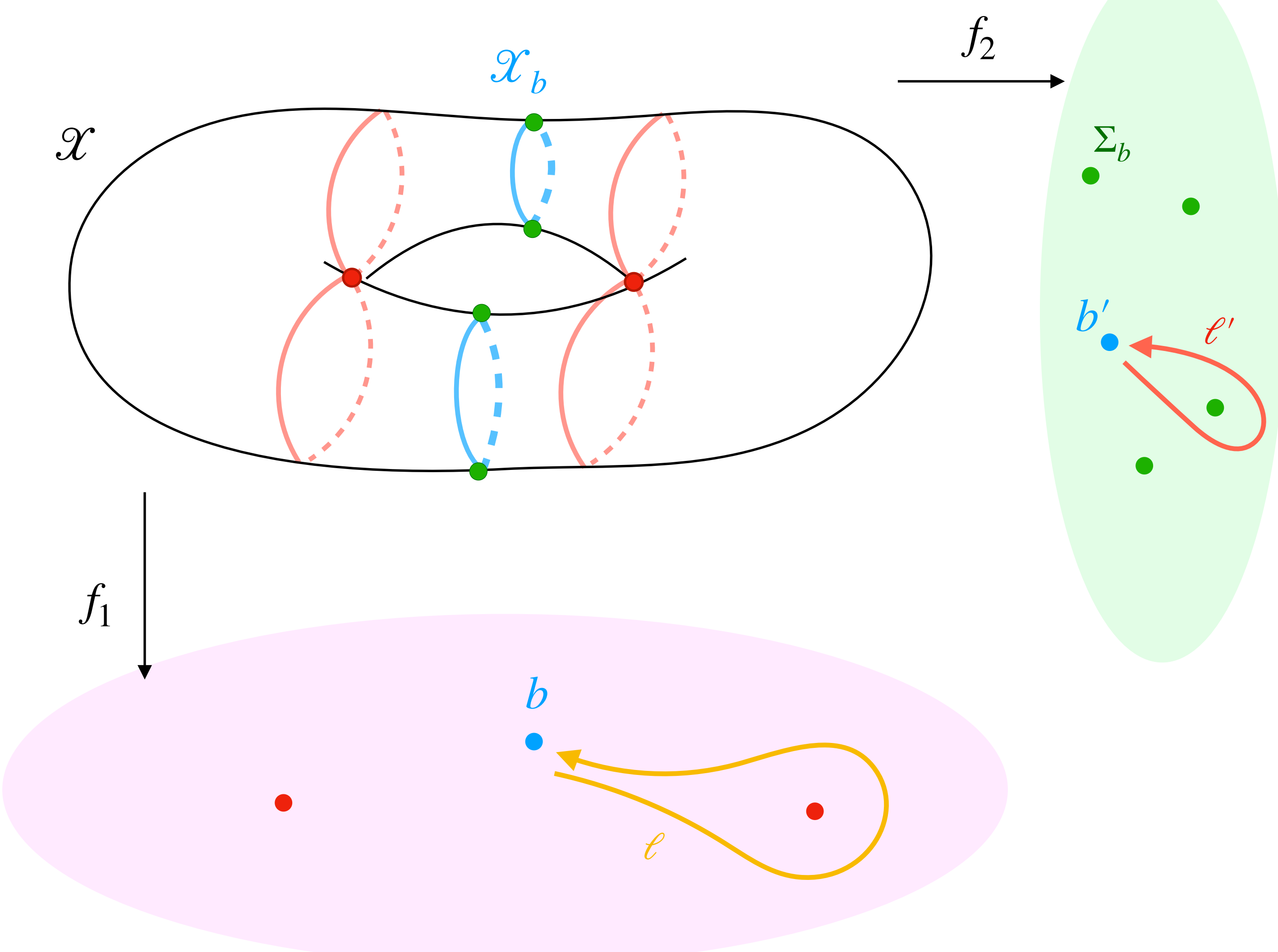
This pencil has an axis $A = \cap_{t\in\mathbb{P}^1} H_t$ that intersects $\mathscr{X}$.

The total space of the fibration is not isomorphic to $\mathscr{X}$, but to a blow up $\mathscr{Y}$ of $\mathscr{X}$ along $\mathscr{X}' = \mathscr{X} \cap A$, called the **modification** of $\mathscr{X}$.

We compute $H_n(\mathscr{Y})$, which contains the homology classes of exceptional divisors. To recover $H_n(\mathscr{X})$ we need to be able to identify these classes.

$$0 \to H_{n-2}(\mathscr{X}') \to H_n(\mathscr{Y}) \to H_n(\mathscr{X}) \to 0$$

Computing monodromy - II



Critical values of $f_2 : \mathscr{X}_t \to \mathbb{P}^1$ move as b moves in $\mathbb{P}^1$

Thus a loop in $\ell \in \pi_1(\mathbb{P}^1 \backslash \Sigma, b)$ induces a **braid action** on $\pi_1(\mathbb{P}^1 \backslash \Sigma_b, b')$, which lifts to an action on $H_{n-1}(\mathscr{X}_b, \mathscr{X}_{bb'})$.

More precisely, we have that

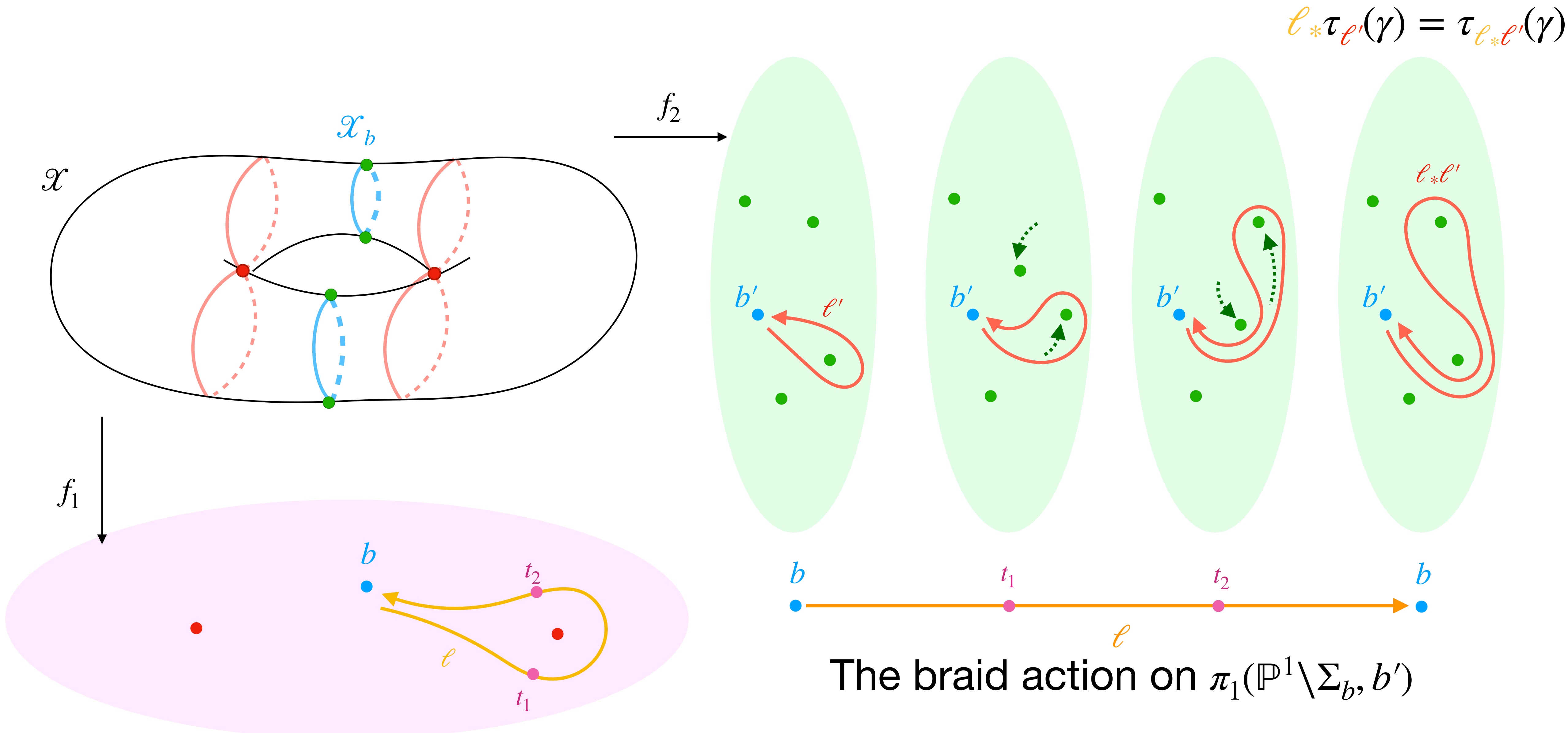
$$\ell_* \tau_{\ell'}(\gamma) = \tau_{\ell_* \ell'}(\gamma)$$

(assuming $\mathscr{X}_{tb'}$ has trivial monodromy with respect to t)

Take b' s.t. $\mathscr{X}_{tb'} = \mathscr{X}'$

(cf parabolic cohomology **[Dettweiler, Wewers 2006]**)

Computing monodromy - II



The braid action on $\pi_1(\mathbb{P}^1 \backslash \Sigma_b, b')$

We can recover the action of monodromy on $H_1(\mathscr{X}_b, \mathscr{X}')$ by computing these braids.