CSC165H1 WINTER 2018: PROBLEM SET 3

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1. Special numbers

define $F_n = 2^{2^n} + 1$. Prove that

$$\forall n \in \mathbb{N}, F_n - 2 = \prod_{i=0}^{n-1} F_i$$

From the definition above, it follows that $(1): F_n - 2 = 2^{2^n} + 1 - 2 = 2^{2^n} - 1$.

Proof. Let $P(n)$ be the statement $F_n - 2 = \prod_{i=0}^{n-1} F_i$, where $n \in \mathbb{N}$. We will show that $P(n)$ is true for all natural numbers by induction on n.

Base Case:

Let $n = 0$. Then,

$$\begin{aligned} F_0 - 2 &= 2^{2^0} + 1 - 2 \\ &= 2^1 - 1 \\ &= 1 \end{aligned}$$

On the other hand, when a products lower bound is greater than it's upper bound, the product is empty. Therefore,

$$\prod_{i=0}^{0-1} F_i = 1 = F_0 - 2$$

This shows that $P(0)$ is true.

Inductive Step:

Let $k \in \mathbb{N}$ and assume that $P(k)$ is true. That is, $\forall k \in \mathbb{N}, F_k - 2 = \prod_{i=0}^{k-1} F_i$. We want to show that $P(k+1)$ is true, that is,

$$F_{k+1} - 2 = \prod_{i=0}^{k} F_i$$

From the LHS we have,

$$\begin{aligned} F_{k+1} - 2 &= 2^{2^{k+1}} + 1 - 2 \\ &= \left(2^{2^k}\right)^2 - 1 \\ &= \left(2^{2^k} - 1\right)\left(2^{2^k} + 1\right) && (a^2 - b^2 = (a-b)(a+b)) \\ &= \left(F_k - 2\right)\left(2^{2^k} + 1\right) && \text{(By (1))} \\ &= \left(\prod_{i=0}^{k-1} F_i\right) \cdot F_k && \text{(By the induction hypothesis)} \\ &= \prod_{i=0}^{k} F_i \end{aligned}$$

Thus $P(k+1)$ follows from $P(k)$, and this completes the induction step. Having proved steps 1 and 2, we can conclude by the Axiom of Induction that $P(n)$ is true for all natural numbers n.

$\square$

2. Sequences

Define $a_0 = 1$, and for all natural numbers n,

$$a_{n+1} = \frac{1}{\frac{1}{a_n} + 1}$$

(a) $a_0 = 1$, $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{3}$, $a_3 = \frac{1}{4}$

(b) Since the numerator of the sequence is constant, we can put 1 in the numerator. As n increases, the denominator is increasing by one, and since we are starting at $n = 0$, we can write $a_n = \frac{1}{n+1}$.

Prove that for all natural numbers n,

$$a_n = \frac{1}{n+1}$$

Proof. Let $P(n)$ be the statement $a_n = \frac{1}{n+1}$. We will show that $P(n)$ is true for all natural numbers n.

Base Case:

Let $n = 0$. Then

$$a_0 = \frac{1}{0+1} = 1$$

and $a_0 = 1$ by definition. This shows that $P(0)$ is true.

Inductive Step:

Let $k \in \mathbb{N}$ and assume that $P(k)$ is true. That is, for all $k \in \mathbb{N}$, $a_k = \frac{1}{k+1}$. We want to show that $P(k+1)$ is true. That is,

$$a_{k+1} = \frac{1}{k+2}$$

Starting from the recursive definition, we have

$$\begin{aligned} a_{k+1} &= \frac{1}{\frac{1}{a_k} + 1} \\ &= \frac{1}{\frac{1}{\frac{1}{k+1}} + 1} && \text{(By the induction hypothesis)} \\ &= \frac{1}{k+1+1} \\ &= \frac{1}{k+2} \end{aligned}$$

Thus $P(k+1)$ follows from $P(k)$, and this completes the induction step. Having proved steps 1 and 2, we can conclude by the Axiom of Induction that $P(n)$ is true for all natural numbers n.

□

Define $a_{k,0} = k$, and for all natural numbers n,

$$a_{k,n+1} = \frac{k}{\frac{1}{a_{k,n}} + 1}$$

(c) $a_{2,0} = 2$, $a_{2,1} = \frac{4}{3}$, $a_{2,2} = \frac{8}{7}$, $a_{2,3} = \frac{16}{15}$, and
$a_{3,0} = 3$, $a_{3,1} = \frac{9}{4}$, $a_{3,2} = \frac{27}{13}$, $a_{3,3} = \frac{81}{40}$

(d) Prove that for all natural numbers k and n, that,

$$k > 1 \Rightarrow a_{k,n} = \frac{k^{n+1}}{\sum_{i=0}^{n} k^i}$$

Proof. Let $k \in \mathbb{N}$. Assume that $k > 1$. Let $P(n)$ be the statement

$$a_{k,n} = \frac{k^{n+1}}{\sum_{i=0}^{n} k^i}$$

where $n \in \mathbb{N}$. We will show that $P(n)$ is true for all natural numbers $k > 1, n$, by Mathematical Induction on n.

Base Case:

Let $n = 0$. In this case, we have

$$a_{k,0} = \frac{k^1}{\sum_{i=0}^{0} k^i} = \frac{k}{k^0} = k$$

This is the same result as the definition of $a_{k,0}$, so $P(0)$ is satisfied.

Inductive Step:

Let $t \in \mathbb{N}$ and assume that $P(t)$ is true. That is, we assume

$$a_{k,t} = \frac{k^{t+1}}{\sum_{i=0}^{t} k^i}$$

and we want to show that $P(t+1)$ is true. That is,

$$a_{k,t+1} = \frac{k^{t+2}}{\sum_{i=0}^{t+1} k^i}$$

By the recursive definition,

$$\begin{aligned}
a_{k,t+1} &= \frac{k}{\frac{1}{a_{k,t}} + 1} \\
\text{(By induction hypothesis)} \quad &= \frac{k}{\frac{\sum_{i=0}^{t} k^i}{k^{t+1}} + 1} \\
&= \frac{k}{\frac{\sum_{i=0}^{t} k^i + k^{t+1}}{k^{t+1}}} \\
&= \frac{k \cdot k^{t+1}}{\sum_{i=0}^{t} k^i + k^{t+1}} \\
&= \frac{k^{t+2}}{\sum_{i=0}^{t+1} k^i}
\end{aligned}$$

Thus $P(t+1)$ follows from $P(t)$, and this completes the induction step. Since we fixed k to be an arbitrary natural number greater than 1 prior to the induction, it must be the case that the result holds for all natural numbers k greater than 1. Having proved steps 1 and 2, we can now conclude by the Axiom of Induction that $P(n)$ is true for all natural numbers $k > 1$ and n.

$\square$

3. Properties of Asymptotic Notation

(a) Define $Sum_f(n) = \sum_{i=0}^{n} f(i) = f(0) + f(1) + \cdots + f(n)$.

Prove that for all $f\colon \mathbb{N} \to \mathbb{R}^{\geqslant 0}, f \in O(n) \implies Sum_f \in O(n^2)$

Proof. Let $f\colon \mathbb{N} \to \mathbb{R}^{\geq 0}$ and assume that $f \in O(n)$. That is,
$\exists c_0, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \implies f(n) \leq c_0 \cdot n$

Want to show that the sum of the arbitrary function f is in $O(n^2)$. That is, $Sum_f(n) \in O(n^2)$. Or, in expanded form,
$\exists c_1, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \implies Sum_f(n) \leq c_1 \cdot n^2$

Let c_0, n_0 be such that $f \in O(n)$ for these values. Let $c_1 = \sum_{i=0}^{n_0-1} f(i) + c_0$ and $n_1 = \max(n_0, 1)$. Let $n \in \mathbb{N}$ and assume $n \geqslant n_1$. We know that $n \geq n_1 \geq n_0$. So, it follows that $f(n) \leq c \cdot n$. Using this, let's analyze the following sum:

$$\begin{aligned}
\sum_{i=0}^{n} f(i) &= \sum_{i=0}^{n_0-1} f(i) + \sum_{i=n_0}^{n} f(i) \\
&\leq \sum_{i=0}^{n_0-1} f(i) + \sum_{i=n_0}^{n} c_0 n && (\forall n \geq n_0, f(n) \leq c_0 n) \\
&= \sum_{i=0}^{n_0-1} f(i) + (n - n_0) \cdot c_0 n \\
&= \sum_{i=0}^{n_0-1} f(i) - c_0 n n_0 + c_0 n^2
\end{aligned}$$

Since for any $k \in \mathbb{R}^{\geqslant 0}$ and $n \geq 1$, $k \leq k \cdot n^2$, which holds by the way n_1 was chosen,

$$\sum_{i=0}^{n_0-1} f(i) \leq \left(\sum_{i=0}^{n_0-1} f(i)\right) \cdot n^2 \qquad \text{(Because } f(n) \geq 0\text{)}$$

So,

$$\begin{aligned}
\sum_{i=0}^{n_0-1} f(i) - c_0 n n_0 + c_0 n^2 &\leq \left(\sum_{i=0}^{n_0-1} f(i)\right) \cdot n^2 - c_0 n n_0 + c_0 n^2 \\
&\leq \left(\sum_{i=0}^{n_0-1} f(i)\right) \cdot n^2 + c_0 n^2 && (\text{since } -c_0 n n_0 \leq 0) \\
&= \left(\left(\sum_{i=0}^{n_0-1} f(i)\right) + c_0\right) \cdot n^2 \\
&= c_1 \cdot n^2
\end{aligned}$$

So, it follows that $Sum_f(n) \leq c_1 \cdot n^2$. Therefore, $Sum_f(n) \in O(n^2)$

$\square$

(b) Prove that for all natural numbers n,

$$\sum_{i=1}^{2^n} \frac{1}{i} \geq \frac{n}{2}$$

Proof. Let $P(n)$ be the statement above, where $n \in \mathbb{N}$. We will prove that $P(n)$ is true for all natural numbers by mathematical induction.

Base Case:

Let $n = 0$. Then we have,

$$\sum_{i=1}^{2^0} \frac{1}{i} = \sum_{i=1}^{1} \frac{1}{i} = 1 \geq \frac{0}{2} = \frac{n}{2}$$

This shows that $P(0)$ is true.

Inductive Step:

Let $k \in \mathbb{N}$ and assume that $P(k)$ is true. That is,

$$\sum_{i=1}^{2^k} \frac{1}{i} \geq \frac{k}{2}$$

We want to show that $P(k+1)$ is true. That is,

$$\sum_{i=1}^{2^{k+1}} \frac{1}{i} \geq \frac{k+1}{2}$$

Analyzing the left hand side,

$$\sum_{i=1}^{2^{k+1}} \frac{1}{i} = \sum_{i=1}^{2^k} \frac{1}{i} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + ... + \frac{1}{2^k+2^k}$$

(By induction Hypothesis) $$\geq \frac{k}{2} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + ... + \frac{1}{2^k+2^k}$$

Now to proceed with the proof, it will be shown that $\frac{1}{2^k+1} + \frac{1}{2^k+2} + ... + \frac{1}{2^k+2^k} \geq \frac{1}{2}$.

Notice, $\frac{1}{2^k+1} + \frac{1}{2^k+2} + ... + \frac{1}{2^k+2^k} \geq \frac{1}{2^k+2^k} + \frac{1}{2^k+2^k} + ... + \frac{1}{2^k+2^k}$, since $\frac{1}{2^k+k}$ is the smallest element of the summation.

With this fact in mind:

$$\begin{aligned}
\sum_{i=1}^{2^{k+1}} &\geq \frac{k}{2} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + ... + \frac{1}{2^k+2^k} \\
&\geq \frac{k}{2} + \frac{1}{2^k+2^k} + \frac{1}{2^k+2^k} + ... + \frac{1}{2^k+2^k} \\
&= \frac{k}{2} + 2^k \cdot \frac{1}{2^k+2^k} \\
&= \frac{k}{2} + \frac{1(2^k)}{2 \cdot 2^k} \\
&= \frac{k}{2} + \frac{1}{2} \\
&= \frac{k+1}{2}
\end{aligned}$$

Thus $P(k+1)$ follows from $P(k)$, and this completes the induction step. Having proved steps 1 and 2, we can conclude by the Principle of Mathematical Induction that $P(n)$ is true for all natural numbers n. $\square$

(c) *Disprove* The following claim:
$\forall f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}, f(n) \in O(g(n)) \implies Sum_f(n) \in O(n \cdot g(n))$.

We will prove the negation of the statement:
$\exists f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}, f(n) \in O(g(n)) \wedge Sum_f(n) \notin O(n \cdot g(n))$

Proof. Let $f(n) = \frac{1}{n}$ and $g(n) = \frac{1}{2 \cdot n}$

First showing $f \in O(g)$:
$\exists c_0, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \implies \frac{1}{n} \leq c_0 \cdot \frac{1}{2 \cdot n}$
Let $n_0 = 0$ and $c_0 = 2$ Let $n \in \mathbb{N}$. Assume $n \geq n_0$ Then $\forall n$:

$$\begin{aligned} f(n) &= \frac{1}{n} \\ &\leq \frac{1}{n} \\ &= 2 \cdot \frac{1}{2 \cdot n} \\ &= c_0 \cdot g(n) \end{aligned}$$

So by definition, $f \in O(g)$.
Now we will show the following by negating the definition of Big-Oh:
$Sum_f(n) \notin O(n \cdot g(n))$:
$\forall c_1, n_1 \in \mathbb{R}^{\geq 0}, \exists n \in \mathbb{N}, n \geq n_1 \wedge Sum_f(n) > c_1 \cdot n \cdot g(n)$

Let $c_1, n_1 \in \mathbb{R}^{\geq 0}$
Take $n = max(2^{\lceil c_1 \rceil} + 2, 2^{n_1})$ Then it follows that $n \geq 2^{n_1} \geq n_1$, and the first part of the *and* is satisfied.
Also, $n > 2^{\lceil c_1 \rceil} \implies log_2 n > \lceil c_1 \rceil \geq c_1$. So $log_2 n > c_1 (1)$
Importantly, by the way that n was chosen, it must be a power of two. It follows that $log_2 n \in \mathbb{N}$. (2)
From (b) we know that for all natural numbers t:

$$\sum_{i=1}^{2^t} \frac{1}{i} \geq \frac{t}{2}$$

Change of variable: Let $t = log_2 n$. By (2) we know that $t \in \mathbb{N}$ and we have:

$$\sum_{i=1}^{n} \frac{1}{i} \geq \frac{log_2 n}{2}$$

Now we can see that the left side of the inequality is just $Sum_f(n)$, so it follows that:

$$\begin{aligned} Sum_f(n) &\geq \frac{log_2 n}{2} \\ &= g(n) \cdot n \cdot log_2 n \\ \text{((By (1), } log_2 n > c_1\text{)} \quad &> g(n) \cdot n \cdot c_1 \end{aligned}$$

So, $n \geq n_1 \wedge Sum_f(n) > c_1 \cdot n \cdot g(n)$.
Therefore, $Sum_f(n) \notin O(n \cdot g(n))$.

Together, we have proven $\exists f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}, f(n) \in O(g(n)) \wedge Sum_f(n) \notin O(n \cdot g(n))$. Since the negation of the original is proven, the original statement is disproven. $\square$