# CSC165H1: Problem Set 1 Sample Solutions

Due January 24, 2018 before 10pm

**Note**: solutions are incomplete, and meant to be used as guidelines only. We encourage you to ask follow-up questions on the course forum or during office hours.

1. **[6 marks] Propositional formulas.** For each of the following propositional formulas, find the following two items:

   (i) The truth table for the formula. (You don't need to show your work for calculating the rows of the table.)

   (ii) A logically equivalent formula that only uses the $\neg$, $\wedge$, and $\vee$ operators; *no* $\Rightarrow$ or $\Leftrightarrow$. (You *should* show your work in arriving at your final result. Make sure you're reviewed the "extra instructions" for this problem set carefully.)

   (a) **[3 marks]** $(p \Rightarrow q) \Rightarrow \neg q$.

**Solution**

Truth table:

| $p$ | $q$ | $(p \Rightarrow q) \Rightarrow \neg q$ |
|---|---|---|
| False | False | True |
| False | True | False |
| True | False | True |
| True | True | False |

Equivalent formula:

$$(p \Rightarrow q) \Rightarrow \neg q$$
$$\neg(p \Rightarrow q) \vee \neg q \qquad \text{(implication rule)}$$
$$(p \wedge \neg q) \vee \neg q \qquad \text{(implication negation rule)}$$

(okay to stop here and not simplify further)

$$((p \vee \neg q) \wedge (\neg q \vee \neg q)) \qquad \text{(distributivity rule)}$$
$$((p \vee \neg q) \wedge \neg q) \qquad \text{(idempotency rule)}$$
$$\neg q$$

   (b) **[3 marks]** $(p \Rightarrow \neg r) \wedge (\neg p \Rightarrow q)$.

**Solution**

Truth table:

| $p$ | $q$ | $r$ | $(p \Rightarrow \neg r) \wedge (\neg p \Rightarrow q)$ |
|---|---|---|---|
| False | False | False | False |
| False | False | True | False |
| False | True | False | True |
| False | True | True | True |
| True | False | False | True |
| True | False | True | False |
| True | True | False | True |
| True | True | True | False |

Equivalent formula:

$(p \Rightarrow \neg r) \wedge (\neg p \Rightarrow q)$

$(\neg p \vee \neg r) \wedge (\neg p \Rightarrow q)$ (implication rule)

$(\neg p \vee \neg r) \wedge (\neg\neg p \vee q)$ (implication rule)

$(\neg p \vee \neg r) \wedge (p \vee q)$ (double negation rule)

(okay to stop here and not simplify further)

(distributivity rules for AND over OR and OR over AND may be used to simplify)

2. **[8 marks] Fixed Points.**

Let $f$ be a function from $\mathbb{N}$ to $\mathbb{N}$. A *fixed point* of $f$ is an element $x \in \mathbb{N}$ such that $f(x) = x$. A *least fixed point* of $f$ is the smallest number $x \in \mathbb{N}$ such that $f(x) = x$. A *greatest fixed point* of $f$ is the largest number $x \in \mathbb{N}$ such that $f(x) = x$.

(a) **[1 mark]** Express using the language of predicate logic the English statement:

"$f$ has a fixed point."

You may use an expression like "$f(x) =$ [something]" in your solution.

**Solution**

$$\exists x \in \mathbb{N}, f(x) = x$$

(b) **[2 marks]** Express using the language of predicate logic the English statement:

"$f$ has a least fixed point."

You may use the predefined function $f$ as well as the predefined predicates $=$ and $<$. You may not use any other predefined predicates.

**Solution**

$$\exists x \in \mathbb{N},\ \forall y \in \mathbb{N},\ f(x) = x \ \wedge\ (f(y) = y \Rightarrow \neg(y < x))$$

Other correct answers follow from using contrapositive, simplifying $\Rightarrow$ or moving quantifier in:

$$\exists x \in \mathbb{N},\ \forall y \in \mathbb{N},\ f(x) = x \ \wedge\ (f(y) = y \Rightarrow (x < y \vee x = y))$$

$$\exists x \in \mathbb{N},\ \forall y \in \mathbb{N},\ f(x) = x \ \wedge\ (y < x \Rightarrow \neg(f(y) = y))$$

$$\exists x \in \mathbb{N},\ \forall y \in \mathbb{N},\ f(x) = x \ \wedge\ (\neg(f(y) = y) \vee \neg(y < x))$$

$$\exists x \in \mathbb{N},\ \big(f(x) = x \ \wedge\ (\forall y \in \mathbb{N},\ (y < x \Rightarrow \neg(f(y) = y)))\big)$$

$$\exists x \in \mathbb{N},\ \big(f(x) = x \ \wedge\ (\forall y \in \mathbb{N},\ f(y) = y \Rightarrow \neg(y < x))\big)$$

$$\exists x \in \mathbb{N},\ \big(f(x) = x \ \wedge\ (\forall y \in \mathbb{N},\ (\neg(y < x) \vee \neg(f(y) = y)))\big)$$

(c) **[2 marks]** Express using the language of predicate logic the English statement:

"$f$ has a greatest fixed point."

You may use the predefined function $f$ as well as the predefined predicates $=$ and $<$. You may not use any other predefined predicates.

**Solution**

$$\exists x \in \mathbb{N},\ \forall y \in \mathbb{N},\ f(x) = x \ \wedge\ (f(y) = y \ \Rightarrow\ \neg(x < y))$$

Other possible answers that are also correct:

$$\exists x \in \mathbb{N},\ \forall y \in \mathbb{N},\ f(x) = x \ \wedge\ (x < y \Rightarrow \neg(f(y) = y))$$

$$\exists x \in \mathbb{N},\ \forall y \in \mathbb{N},\ f(x) = x \ \wedge\ (\neg(f(y) = y) \ \vee\ \neg(x < y))$$

$$\exists x \in \mathbb{N}, \big( f(x) = x \ \wedge \ (\forall y \in \mathbb{N}, \ (f(y) = y \ \Rightarrow \ \neg(x < y)))\big)$$
$$\exists x \in \mathbb{N}, \big( f(x) = x \ \wedge \ (\forall y \in \mathbb{N}, \ (x < y \Rightarrow \neg(f(y) = y)))\big)$$
$$\exists x \in \mathbb{N}, \big( f(x) = x \ \wedge \ (\forall y \in \mathbb{N}, \neg(f(y) = y) \ \vee \ \neg(x < y))\big)$$

(d) **[3 marks]** Consider the function $f$ from $\mathbb{N}$ to $\mathbb{N}$ defined as $f(x) = x \mod 7$.[1]
Answer the following questions by filling in the blanks.

The fixed points of $f$ are: ____________________

The least fixed point of $f$ is: ____________________

The greatest fixed point of $f$ is: ____________________

**Solution**

The fixed points of $f$ are: $0, 1, 2, 3, 4, 5, 6$.

The least fixed point of $f$ is: $0$

The greatest fixed point of $f$ is: $6$

[1] Here we are using the modulus operator. Given a natural number $a$ and a positive integer $b$, $a \mod b$ is the natural number less than $b$ that is the remainder when $a$ is divided by $b$. In Python, the expression `a % b` may be used to compute the value of $a \mod b$.

3. **[6 marks] Partial Orders.** A binary predicate $R$ on a set $D$ is called a *partial order* if the following three properties hold:

(1) (reflexive) $\forall d \in D,\ R(d,d)$
(2) (transitive) $\forall d, d', d'' \in D,\ (R(d,d') \wedge R(d',d'')) \Rightarrow R(d,d'')$
(3) (anti-symmetric) $\forall d, d' \in D,\ (R(d,d') \wedge R(d',d)) \Rightarrow d = d'$

A binary predicate $R$ on a set $D$ is called a *total order* if it is a partial order and in addition the following property holds: $\forall d, d' \in D,\ R(d,d') \vee R(d',d)$.

For example, here is a binary predicate $R$ on the set $\{a,b,c,d\}$ that is a total order:
$R(a,b) = R(a,c) = R(a,d) = R(b,c) = R(b,d) = R(c,d) = R(a,a) = R(b,b) = R(c,c) = R(d,d) =$ True and all other values are False.

(a) **[2 marks]** Give an example of a binary predicate $R$ on the set $\mathbb{N}$ that is a partial order but that is *not* a total order.

**Solution**

$R(i,i) =$ True for all $i \in \mathbb{N}$, and all other values are False.

(b) **[2 marks]** Let $R$ be a partial order predicate on a set $D$. $R$ specifies an ordering between elements in $D$. Whenever $R(d,d')$ is True, we will say that $d$ is less than or equal to $d'$, or that $d'$ is greater than or equal to $d$. The following formula in predicate logic expresses that there exists a greatest element in $D$; that is, an element in $D$ that is greater than or equal to every other element in $D$:

$$\exists d \in D,\ \forall d' \in D,\ R(d',d)$$

An element in $D$ is said to be *maximal* if no other element in $D$ is larger than this element. The following formula in predicate logic expresses that there exists a maximal element in $D$:

$$\exists d \in D,\ \forall d' \in D,\ d = d' \vee \neg R(d,d')$$

Give an example of a partial order order $R$ over $\{a,b,c,d\}$ such that every element is maximal.

**Solution**

$R(a,a) = R(b,b) = R(c,c) = R(d,d) =$ True and all other values are False.

(c) **[2 marks]** Give an example of a partial order $R$ over $\{a,b,c,d\}$ such that $a \in D$ is maximal but $a$ is not a greatest element. Justify your answer briefly.

**Solution**

Note that there are two accepted interpretations of the question:

i. That it is the element $a$ that is maximal, but $a$ is not a greatest element.
That is: $\big(\forall d' \in D,\ a = d' \vee \neg R(a,d')\big) \ \wedge\ \big(\exists d' \in D,\ \neg R(d',a)\big)$

ii. There there is some element $a \in D$ that is maximal, but is not a greatest element.
That is: $\exists a \in D, \Big(\big(\forall d' \in D,\ a = d' \vee \neg R(a,d')\big) \ \wedge\ \big(\exists d' \in D,\ \neg R(d',a)\big)\Big)$

The solution below uses the first interpretation, but is also correct for the second, by using $a$ is the example.

Using our example from above, where $R(a,a) = R(b,b) = R(c,c) = R(d,d) =$ True and all other values are False, we see that $a$ is maximal but $a$ is not a greatest element.

Element $a$ is maximal since $R(a, d')$ is never True when $d' \neq a$. Furthermore, $a$ is not greatest because, for example, $R(b, a)$ is False, and so $\neg R(b, a)$ is True.

4. **[13 marks] One-to-one functions.** So far, most of our predicates have had sets of numbers as their domains. But this is not always the case: we can define properties of any kind of object we want to study, including functions themselves!

Let $S$ and $T$ be sets. We say that a function $f : S \to T$ is **one-to-one** if no two distinct inputs are mapped to the same output by $f$. For example, if $S = T = \mathbb{Z}$, the function $f_1(x) = x + 1$ is one-to-one, since every input $x$ gets mapped to a distinct output. However, the function $f_2(x) = x^2$ is not one-to-one, since $f_2(1) = f_2(-1) = 1$. Formally we express "$f : S \to T$ is one-to-one" as: $\forall x_1 \in S,\ \forall x_2 \in S,\ f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

We say that $f : S \to T$ is **onto** if every element in $T$ gets mapped to by at least one element in $S$. The above function $f(x) = x + 1$ is onto over $\mathbb{Z}$ but is not onto over $\mathbb{N}$. Formally we express "$f : S \to T$ is onto" as:

$$\forall y \in T,\ \exists x \in S,\ f(x) = y$$

Let $t \in T$. We say that $f$ **outputs** $t$ if there exists $s \in S$ such that $f(s) = t$.

(a) **[1 mark]** How many functions are there from $\{1, 2, 3\}$ to $\{a, b, c, d\}$?

**Solution**

There are $4^3 = 64$ possible functions from $\{1, 2, 3\}$ to $\{a, b, c, d\}$.

This is because we have 4 possible outputs ($a$, $b$, $c$ or $d$) for $f(1)$, and similarly for $f(2)$ and $f(3)$.

(b) **[1 mark]** How many one-to-one functions are there from $\{1, 2, 3\}$ to $\{a, b, c, d\}$?

**Solution**

There are $4 \cdot 3 \cdot 2 = 24$ one-to-one functions from $\{1, 2, 3\}$ to $\{a, b, c, d\}$.

This follows since once we pick one of the 4 possible outputs for $f(1)$, there are only 3 possible outputs left for $f(2)$, and only 2 possible outputs left for $f(3)$.

(c) **[1 mark]** How many onto functions are there from $\{1, 2, 3, 4\}$ to $\{a, b, c\}$?

**Solution**

There are 36 onto functions from $\{1, 2, 3, 4\}$ to $\{a, b, c\}$.

Suppose we were considering onto functions from $\{A, B, C\}$ to $\{a, b, c\}$. (Note this domain has 3 elements.) There are $3 \cdot 2 = 6$ such onto functions, since there are 3 choices for the element that maps to $a$, then 2 choices for the element that maps to $b$ and 1 choice for the element that maps to $c$. Now there are 4 possible ways to get a three element domain $\{A, B, C\}$ from the four element domain $\{1, 2, 3, 4\}$. (There are 4 possible values that could be left out.) For each value that is left out of $\{1, 2, 3, 4\}$, there are 3 possible elements in $\{a, b, c\}$ that could be mapped onto. However this includes some double counting, as leaving out 4 and taking $f(4) = f(1)$ would give the same function as generated by leaving out 1 and taking $f(1) = f(4)$. Altogether then, the number of onto functions is $(3 \cdot 2) \cdot (4 \cdot 3)/2 = 36$.

Alternatively, we could count indirectly. There are $3^4 = 81$ total functions from $\{1, 2, 3, 4\}$ to $\{a, b, c\}$. Let's think about the number of these that are not onto. There are $3 \cdot (2^4) = 48$ functions have a range that contains at most 2 elements. (We have 3 choices for the element to drop from the range, and then 2 choices to map to for each of $\{1, 2, 3, 4\}$.) But arriving at the 48 functions double-counts the functions that have a range that contains only 1 element. So we will need to add that number back in once. There are $3 \cdot (1^4) = 3$ funtions that have a range containing 1 element. In total then there are $81 - 48 + 3 = 36$ onto functions from $\{1, 2, 3, 4\}$ to $\{a, b, c\}$!

(d) **[2 marks]** Now let $R$ be a binary predicate with domain $\mathbb{N} \times \mathbb{N}$. We say that $R$ *represents* a function if, for every $x \in \mathbb{N}$, there exists a *unique* $y \in \mathbb{N}$, such that $R(x, y)$ (is True). In this case, we write expressions like $y = f(x)$.

Define a predicate $Function(R)$, where $R$ is a binary predicate with domain $\mathbb{N} \times \mathbb{N}$, that expresses the English statement:

"$R$ represents a function."

You may use the predicates $<, \leq, =, R$, but may not use any other predicate or function symbols.

In parts (e)-(h) below, you may use the predicate $Function(R)$ (in addition to the predicates $<, \leq, =, R$) in your solution, but may not use any other predicate or function symbols.

**Solution**

$$Function(R) : \forall x \in \mathbb{N},\ \exists y \in \mathbb{N},\ \big(R(x, y)\big) \ \wedge\ \big(\forall z \in \mathbb{N},\ R(x, z) \ \Rightarrow\ z = y\big)\big)$$

(e) **[2 marks]** Define a predicate that expresses the following English statement.

"$R$ represents an onto function."

**Solution**

$$Onto(R) :\ Function(R) \ \wedge\ (\forall y \in \mathbb{N},\ \exists x \in \mathbb{N},\ R(x, y))$$

(f) **[2 marks]** Define a predicate that expresses the following English statement.

"$R$ represents a one-to-one function."

**Solution**

$$OneToOne(R) :\ Function(R) \ \wedge\ (\forall x_1, x_2, y \in \mathbb{N},\ (R(x_1, y) \ \wedge\ R(x_2, y)) \ \Rightarrow\ x_1 = x_2)$$

(g) **[2 marks]** Define a predicate that expresses the following English statement.

"$R$ represents a function that outputs infinitely many elements of $\mathbb{N}$."

**Solution**

$$InfinitelyMany(R) :\ Function(R) \wedge (\forall y_1 \in \mathbb{N},\ \exists x, y_2 \in \mathbb{N},\ (y_2 > y_1 \ \wedge\ R(x, y_2)))$$

(h) **[2 marks]** Now define a predicate that expresses the following English statement.

"$R$ represents a function that outputs all but finitely many elements of $\mathbb{N}$."

**Solution**

$$AllButFinitelyMany(R) :\ Function(R) \ \wedge\ (\exists y_1 \in \mathbb{N},\ \forall y_2 \in \mathbb{N},\ y_1 \leq y_2 \ \Rightarrow\ (\exists x \in \mathbb{N}, R(x, y_2)))$$