CSC165H1 Winter 2018: Problem Set 2

By: Jacob Chmura, Conor Vedova, Eric Koehli

February 7, 2018

1 AND vs. IMPLIES

(a) We are going to prove the statement: $\forall n \in \mathbb{N}, n > 15 \Rightarrow n^3 - 10n^2 + 3 \geqslant 165$. In this proof, we use the known fact about inequalities such that if $a > b > 0$ and $c > d > 0$, then $ac > bd > 0$.

Proof. Let $n \in \mathbb{N}$. Assume $n > 15$.

$$\begin{aligned} n &> 15 \\ n - 10 &> 5 \\ n^2 - 10n &> 5 \cdot 15 = 75 && \text{(By fact)} \\ n^3 - 10n^2 &> 75 \cdot 15 = 375 \\ n^3 - 10n^2 + 3 &> 378 \\ n^3 - 10n^2 + 3 &\geqslant 165 && \blacksquare \end{aligned}$$

(b) We are going to disprove the statement: $\forall n \in \mathbb{N}, n > 15 \wedge n^3 - 10n^2 + 3 \geqslant 165$. In other words, we are going to prove the negation of this statement:

$\exists n \in \mathbb{N}, n \leqslant 15 \vee n^3 - 10n^2 + 3 < 165$.

Proof. Let $n = 1$. Then $n = 1 \leqslant 15$. $\blacksquare$

2 Ceiling functions

(a) Fact 1: $\forall x \in \mathbb{R}, \forall k \in \mathbb{Z}, k \geqslant x \Rightarrow k \geqslant \lceil x \rceil$.

Fact 2: $\forall x \in \mathbb{R}, \forall k \in \mathbb{Z}, \lceil x + k \rceil = \lceil x \rceil + k$. We will use fact 2 to prove the the following statement: $\forall n, m \in \mathbb{N}, n < m \Rightarrow \left\lceil \frac{m-1}{m} \cdot n \right\rceil$.

Proof. Let $n, m \in \mathbb{N}$. Assume $n < m$.

$$\begin{aligned}
\left\lceil \frac{m-1}{m} \cdot n \right\rceil &= \left\lceil \frac{nm}{m} - \frac{n}{m} \right\rceil \\
&= \left\lceil n - \frac{n}{m} \right\rceil \\
&= \left\lceil -\frac{n}{m} + n \right\rceil \\
&= \left\lceil -\frac{n}{m} \right\rceil + n && \text{(By fact 2 and } \mathbb{N} \subseteq \mathbb{Z}\text{)} \\
&= 0 + n = n \blacksquare && \text{(Since } 0 \leqslant n < m\text{)}
\end{aligned}$$

(b) Prove that,

$\forall n, m \in \mathbb{N}, 50 \mid nextFifty(n) \wedge 50 \mid m \Rightarrow nextFifty(n) \leqslant m \wedge nextFifty(n) \geqslant n$. For this proof, we will prove the contrapositive statement:

$\forall n, m \in \mathbb{N}, nextFifty(n) > m \vee nextFifty(n) < n \Rightarrow 50 \nmid nextFifty(n) \vee 50 \nmid m$.

Proof. Let $n, m \in \mathbb{N}$.

Case 1: Assume $nextFifty(n) > m$. From our assumptions, we know that

$$50 \cdot \left\lceil \frac{n}{50} \right\rceil > m \Rightarrow \left\lceil \frac{n}{50} \right\rceil > \frac{m}{50}$$

It follows that $50 \nmid m$.

Case 2: Assume $nextFifty(n) < n$. From our assumptions,

$$50 \cdot \left\lceil \frac{n}{50} \right\rceil < n \Rightarrow \left\lceil \frac{n}{50} \right\rceil < \frac{n}{50}$$

It follows that $50 \nmid nextFifty(n)$.

TODO: This isn't complete.

3 Divisibility

(a) Prove the statement:

Prove the statement $\forall n \in \mathbb{N}, n \leqslant 2300 \Rightarrow \Big(49 \mid n \Leftrightarrow 50\big(nextFifty(n) - n\big) = nextFifty(n)\Big)$.

We'll split this proof into two parts. For part 1, we will prove: $\forall n \in \mathbb{N}, n \leqslant 2300 \Rightarrow \Big(49 \mid n \Rightarrow 50\big(nextFifty(n) - n\big) = nextFifty(n)\Big)$.

Proof. Let $n \in \mathbb{N}$. Assume that $n \leqslant 2300$ and assume that $\exists k \in \mathbb{Z}, n = 49k$. Let k be this number. We want to show that if the above is true, then $50\big(nextFifty(n) - n\big) = nextFifty(n)\Big)$. Since n ranges from $0 \leqslant n \leqslant 2300$, it follows that k ranges from $0 \leqslant 49k \leqslant 2300 \Rightarrow 0 \leqslant k \leqslant 46$. Then,

$$\begin{aligned}
50 \cdot \left(50\left\lceil \frac{n}{50} \right\rceil - n\right) &= 50 \cdot \left(50\left\lceil \frac{49k}{50} \right\rceil - 49k\right) \\
&= 50 \cdot \left(50\left\lceil \frac{50-1}{50} \cdot k \right\rceil - 49k\right) \\
&= 50 \cdot \big(50k - 49k\big) && \text{(By 2b since k ¡ 50)} \\
&= 50k
\end{aligned}$$

On the other hand,

$$\begin{aligned}
50 \cdot \left\lceil \frac{n}{50} \right\rceil &= 50 \cdot \left\lceil \frac{49k}{50} \right\rceil \\
&= 50 \cdot \left\lceil \frac{50-1}{50} \cdot k \right\rceil \\
&= 50k
\end{aligned}$$

This completes the proof of part 1. We now need to prove,

$\forall n \in \mathbb{N}, n \leqslant 2300 \Rightarrow \Big(50\big(nextFifty(n) - n\big) = nextFifty(n) \Rightarrow 49 \mid n\Big)$.

Proof. Let $n \in \mathbb{N}$. Assume $n \leqslant 2300$ and that $50 \cdot \left(50 \cdot \left\lceil \frac{n}{50} \right\rceil - n\right) = 50\left\lceil \frac{n}{50} \right\rceil$. We want to show that $\exists k_1 \in \mathbb{Z}, n = 49k_1$. Let $k_1 = \left\lceil \frac{n}{50} \right\rceil$. We have,

$$\begin{aligned}
49k_1 &= 49\left\lceil \frac{n}{50} \right\rceil \\
&= \left\lceil \frac{49n}{50} \right\rceil \\
&= \left\lceil \frac{50-1}{50} \cdot n \right\rceil = n && \text{(By 2b)}
\end{aligned}$$

■

(b) Disprove the statement:

$\forall n \in \mathbb{N}, \left(49 \mid n \Leftrightarrow 50 \cdot (nextFifty(n) - n) = nextFifty(n)\right)$. Which is logically equivalent to:

$\forall n \in \mathbb{N}, \left(49 \mid n \Rightarrow 50(nextFifty(n) - n) = nextFifty(n)\right) \wedge$
$\left(50(nextFifty(n) - n) = nextFifty(n) \Rightarrow 49 \mid n\right)$.

For this proof, we will prove that the negation is true. That is, we will prove the following statement:

$\exists n \in \mathbb{N}, \left(49 \mid n \wedge 50 \cdot (nextFifty(n) - n) \neq nextFifty(n)\right) \vee$
$\left(50 \cdot (nextFifty(n) - n) = nextFifty(n) \wedge 49 \nmid n\right)$

Proof. Let $n = 2450$.

Then $\exists k \in \mathbb{Z}, n = 49k$. Let $k = 50$. Also,

$$\begin{aligned}
50 \cdot \left(50 \cdot \left\lceil \frac{n}{50} \right\rceil - n\right) &= 50 \cdot \left(50 \cdot \left\lceil \frac{2450}{50} \right\rceil - 2450\right) \\
&= 50 \cdot \left(50 \cdot \lceil 49 \rceil - 2450\right) \\
&= 50 \cdot \left(50 \cdot 49 - 2450\right) \\
&= 50 \cdot \left(2450 - 2450\right) \\
&= 50 \cdot 0 = 0
\end{aligned}$$

On the other hand,

$$\begin{aligned}
50 \cdot \left\lceil \frac{n}{50} \right\rceil &= 50 \cdot \left\lceil \frac{2450}{50} \right\rceil \\
&= 50 \cdot \lceil 49 \rceil \\
&= 50 \cdot 49 = 2450
\end{aligned}$$

Since $50 \cdot \left(50 \cdot \left\lceil \frac{n}{50} \right\rceil - n\right) \neq 50 \cdot \left\lceil \frac{n}{50} \right\rceil$, this completes the proof. ■

4 Functions

(a) Let *Bounded*(f): $\exists k \in \mathbb{R}, \forall n \in \mathbb{N}, f(n) < k$, where $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$ define the statement "*f is bounded*".

(b) Prove $\forall f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}, Bounded(f_1) \wedge Bounded(f_2) \Rightarrow Bounded(f_1 + f_2)$.

Proof. Let $f_1(n), f_2(n) : \mathbb{N} \to \mathbb{R}^{\geq 0}$. Assume $f_1(n)$ and $f_2(n)$ are bounded. That is, we assume

$\exists k_1, k_2 \in \mathbb{R}, \forall n \in \mathbb{N}, f_1(n) < k_1 \wedge f_2(n) < k_2$. Let k_1 and k_2 be these values. We want to show that

$\exists k \in \mathbb{R}, \forall n \in \mathbb{N}, (f_1 + f_2)(n) < k$. Let $k = k_1 + k_2$. Then from our assumptions, we have,

$$\begin{aligned} f_1(n) < k_1 \text{ and } f_2(n) < k_2 & \\ f_1(n) + f_2(n) < k_1 + k_2 & \qquad \text{(Adding the respective sides)} \\ (f_1 + f_2)(n) < k_1 + k_2 & \qquad \text{(By the definition of function addition)} \\ (f_1 + f_2)(n) < k & \qquad \blacksquare \end{aligned}$$

Since this shows that $f_1 + f_2$ is also bounded, this completes the proof.

(c) Prove or disprove the statement:

$\forall f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}, Bounded(f_1 + f_2) \Rightarrow Bounded(f_1) \wedge Bounded(f_2)$.

Proof. Let $f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}$. Assume $f_1 + f_2$ is bounded. That is, we assume

$\exists k \in \mathbb{R}, \forall n \in \mathbb{N}, (f_1 + f_2)(n) < k$. Let k be this value. We want to show that

$\exists k_1, k_2 \in \mathbb{R}, \forall n \in \mathbb{N}, f_1(n) < k_1 \wedge f_2(n) < k_2$. Let $k_1 = k$ and $k_2 = k$. Then, from our assumptions, we have

$$\begin{aligned} (f_1 + f_2)(n) < k & \\ f_1(n) + f_2(n) < k & \qquad \text{(By the definition of function addition)} \\ f_1(n) < k \text{ and } f_2(n) < k & \qquad \text{(Since } f_1(n), f_2(n) \in \mathbb{R}^{\geq 0}\text{)} \\ f_1(n) < k_1 \text{ and } f_2(n) < k_2 & \qquad \blacksquare \end{aligned}$$

Since this shows that f_1 and f_2 are bounded, this completes the proof.

This is now a biconditional statement.