CSC165H1: Problem Set 2

Due February 7, 2018 before 10pm

General instructions

Please read the following instructions carefully before starting the problem set. They contain important information about general problem set expectations, problem set submission instructions, and reminders of course policies.

- Your problem sets are graded on both correctness and clarity of communication. Solutions that are technically correct but poorly written will not receive full marks. Please read over your solutions carefully before submitting them.
- Each problem set may be completed in groups of up to three. If you are working in a group for this problem set, please consult `https://github.com/MarkUsProject/Markus/wiki/Student_Groups` for a brief explanation of how to create a group on MarkUs.

  Exception: Problem Set 0 must be completed individually.
- Solutions must be typeset electronically, and submitted as a PDF with the correct filename. **Handwritten submissions will receive a grade of ZERO.**

  The required filename for this problem set is **problem_set2.pdf**.
- Problem sets must be submitted online through MarkUs. If you haven't used MarkUs before, give yourself plenty of time to figure it out, and ask for help if you need it! If you are working with a partner, you must form a group on MarkUs, and make one submission per group. "I didn't know how to use MarkUs" is not a valid excuse for submitting late work.
- Your submitted file should not be larger than 9MB. This may happen if you are using a word processing software like Microsoft Word; if it does, you should look into PDF compression tools to make your PDF smaller, although please make sure that your PDF is still legible before submitting!
- Submissions must be made *before* the due date on MarkUs. You may use *grace tokens* to extend the deadline; please see the Problem Set page for details on using grace tokens.
- The work you submit must be that of your group; you may not refer to or copy from the work of other groups, or external sources like websites or textbooks. You may, however, refer to any text from the Course Notes (or posted lecture notes), except when explicitly asked not to.

Additional instructions

- For each proof you present, start by writing a precise statement of what you are proving. Express it using a fully simplified statement in predicate logic. For a disproof, clearly write the fully simplified negation.
- For proofs involving the ceiling function, you **may not use any external facts other than those mentioned in the questions**. You should not (and will not need to) prove any external facts to complete this problem set.
- For any *concrete numbers*, you may state whether one divides another, or whether a number is prime, without proof. For example, you can write statements "$3 \mid 12$" and "15 is not prime" without justification.

1. **[6 marks] AND vs. IMPLIES.** Students often confuse the meanings of the two propositional operators $\Rightarrow$ and $\land$. In this question, you'll examine each of these in a series of statements by writing formal proofs/disproofs of each one.

   (a) **[3 marks]** Prove or disprove: $\forall n \in \mathbb{N},\ n > 15 \Rightarrow n^3 - 10n^2 + 3 \geq 165$.

   (b) **[3 marks]** Prove or disprove: $\forall n \in \mathbb{N},\ n > 15 \land n^3 - 10n^2 + 3 \geq 165$.

2. **[6 marks] Ceiling function.**

Recall that the ceiling function is the function $\lceil x \rceil : \mathbb{R} \to \mathbb{Z}$ defined as $\lceil x \rceil$ is the smallest integer greater than or equal to x. You may use the following two facts about ceilings in your proofs below, as long as you clearly state where you are using them.

$$\forall x \in \mathbb{R},\ \forall k \in \mathbb{Z},\ k \geq x \Rightarrow k \geq \lceil x \rceil \tag{Fact 1}$$

$$\forall x \in \mathbb{R},\ \forall k \in \mathbb{Z},\ \ \lceil x + k \rceil = \lceil x \rceil + k \tag{Fact 2}$$

(a) **[3 marks]** Prove that for all natural numbers n and m, if $n < m$ then $\left\lceil \dfrac{m-1}{m} \cdot n \right\rceil = n$.

(b) **[3 marks]** We define the function $nextFifty : \mathbb{N} \to \mathbb{N}$ as $nextFifty(n) = 50 \cdot \left\lceil \dfrac{n}{50} \right\rceil$. Also, recall that for integers n and d, we say n is a *multiple* of d when $d \mid n$.
Prove that for all $n \in \mathbb{N}$, $nextFifty(n)$ is the smallest multiple of 50 greater than or equal to n.

3. **[6 marks] Divisibility.**

This question is a continuation of Question 2: you may use the definitions, given Facts 1 and 2, and statements in parts (2a) and (2b) in your proofs below.[1]

(a) [**4 marks**] Prove the following statement:

$$\forall n \in \mathbb{N},\ n \leq 2300 \Rightarrow \Big(49 \mid n \Leftrightarrow 50 \cdot (nextFifty(n) - n) = nextFifty(n)\Big)$$

(b) [**2 marks**] *Disprove* the following statement:

$$\forall n \in \mathbb{N},\ 49 \mid n \Leftrightarrow 50 \cdot (nextFifty(n) - n) = nextFifty(n)$$

[1] You may use (2a) and (2b) for full marks here *even if* you didn't prove them in the previous question.

4. **[9 marks] Functions.**

Let $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$. We say f is *bounded* if there exists a real number k such that f never outputs a value greater than k. (Recall that $\mathbb{R}^{\geq 0}$ denotes the set of real numbers greater than or equal to zero.)

(a) [**2 marks**] Express the statement "f is bounded" in predicate logic.

(b) [**4 marks**] Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$. We define the *sum of functions* as the function $(f + g) : \mathbb{N} \to \mathbb{R}^{\geq 0}$ as follows: for all $n \in \mathbb{N}$, $(f + g)(n) = f(n) + g(n)$. For example, if $f(n) = n^2 + 1$ and $g(n) = 165n$, $(f + g)(n) = n^2 + 1 + 165n$.

Prove that for all functions $f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}$, if f_1 and f_2 are bounded, then $f_1 + f_2$ is bounded.

(c) [**3 marks**] Prove or disprove: for all functions $f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}$, if $f_1 + f_2$ is bounded, then f_1 is bounded and f_2 is bounded.