CSC165H1: Problem Set 3 Sample Solutions

Due March 14, 2018 before 10pm

Note: solutions may be incomplete, and meant to be used as guidelines only. We encourage you to ask follow-up questions on the course forum or during office hours.

1. **[4 marks] Special numbers.** For each $n \in \mathbb{N}$, define $F_n = 2^{2^n} + 1$.

Prove that for all natural numbers n, $F_n - 2 = \prod_{i=0}^{n-1} F_i$.

Hints:

- Please review *product notation*, including empty products, on page 16 of the course notes.
- For all $n \in \mathbb{N}$, $2^{2^{n+1}} = (2^{2^n})^2$.

Solution

Proof. Let $P(n)$ be the predicate "$F_n - 2 = \prod_{i=0}^{n-1} F_i$," where n is a natural number. We want to prove the statement $\forall\, n \in \mathbb{N},\ P(n)$, and will do so by induction.

Base case: Let $n = 0$. We want to prove $P(0)$. Using the definition of F_n we have that

$$\begin{aligned} F_0 - 2 &= \left(2^{2^0} + 1\right) - 2 \\ &= \left(2^1 + 1\right) - 2 \\ &= 3 - 2 \\ &= 1. \end{aligned}$$

In addition, using the rules for product notation, we know that

$$\begin{aligned} \prod_{i=0}^{0-1} F_i &= \prod_{i=0}^{-1} F_i \\ &= 1. \end{aligned}$$

Hence $F_0 - 2 = \prod_{i=0}^{0-1} F_i$, and $P(0)$, as required.

Induction step: Let $k \in \mathbb{N}$. We want to prove that $P(k) \Rightarrow P(k+1)$. Assume $P(k)$, i.e., that $F_k - 2 = \prod_{i=0}^{k-1} F_i$. We want to show $P(k+1)$, i.e., that $F_{k+1} - 2 = \prod_{i=0}^{k} F_i$.

We start with the right hand side of the desired equality and work towards the left hand side. We have

$$\begin{aligned}
\prod_{i=0}^{k} F_i &= F_k \cdot \prod_{i=0}^{k-1} F_i \\
&= F_k \cdot \big(F_k - 2\big) \quad \text{(using the induction hypothesis)} \\
&= F_k^2 - 2 \cdot F_k \\
&= \big(F_k^2 - 2 \cdot F_k + 1\big) - 1 \quad \text{(completing the square)} \\
&= \big(F_k - 1\big)^2 - 1 \\
&= \big(2^{2^k}\big)^2 - 1 \quad \text{(applying the definition of } F_k\text{)} \\
&= 2^{2^{k+1}} - 1 \\
&= \big(2^{2^{k+1}} + 1\big) - 2 \\
&= F_{k+1} - 2,
\end{aligned}$$

and $P(k+1)$, as required. $\square$

(Note that different sequences of equalities may be used to prove the inductive step.)

2. **[8 marks] Sequences.** We define the following sequence of numbers $a_0, a_1, a_2 \dots$ recursively as:

$$a_0 = 1, \text{ and for all } n \in \mathbb{N},\ a_{n+1} = \frac{1}{\frac{1}{a_n} + 1}$$

(a) **[1 mark]** What are the values of a_0, a_1, a_2, and a_3?

Solution

We have $a_0 = 1$,

$$\begin{aligned} a_1 &= \frac{1}{\frac{1}{a_0}+1} & a_2 &= \frac{1}{\frac{1}{a_1}+1} & \text{and} \quad a_3 &= \frac{1}{\frac{1}{a_2}+1} \\ &= \frac{1}{\frac{1}{1}+1} & &= \frac{1}{\frac{1}{\frac{1}{2}}+1} & &= \frac{1}{\frac{1}{\frac{1}{3}}+1} \\ &= \frac{1}{1+1} & &= \frac{1}{2+1} & &= \frac{1}{3+1} \\ &= \frac{1}{2}, & &= \frac{1}{3}, & &= \frac{1}{4}. \end{aligned}$$

(b) **[3 marks]** Find and prove a non-recursive formula for a_n that is valid for all natural numbers n. That is, the statement you will prove should be of the form

$$\forall n \in \mathbb{N},\ a_n = \underline{\hspace{3cm}}$$

By "non-recursive" we mean that the formula you use to fill in the blank should not involve any a_i terms.

Solution

The observations given above suggest that a non-recursive formula for a_n is $\dfrac{1}{n+1}$.

Proof. Let $P(n)$ be the predicate "$a_n = \dfrac{1}{n+1}$," where n is a natural number. We want to prove the statement $\forall\ n \in \mathbb{N},\ P(n)$, and will do so by induction.

Base case: Let $n = 0$. We want to prove $P(0)$. Since $\dfrac{1}{0+1} = 1$, and a_0 is given to be 1, $P(0)$ follows.

Induction step: Let $k \in \mathbb{N}$. We want to prove that $P(k) \Rightarrow P(k+1)$. Assume $P(k)$, i.e., that $a_k = \dfrac{1}{k+1}$. We want to show $P(k+1)$, i.e., that $a_{k+1} = \dfrac{1}{(k+1)+1}$.

Using the definition of a_{k+1}, we have that

$$\begin{aligned} a_{k+1} &= \frac{1}{\frac{1}{a_k}+1} \\ &= \frac{1}{\frac{1}{\frac{1}{k+1}}+1} \qquad \text{(using the induction hypothesis)} \\ &= \frac{1}{(k+1)+1}, \end{aligned}$$

and $P(k+1)$, as required. $\square$

(c) **[1 mark]** Let's now generalize the previous part. For every natural number k greater than 1, we define an infinite sequence $a_{k,0}, a_{k,1}, \ldots$ recursively as follows:

$$a_{k,0} = k, \text{ and for all } n \in \mathbb{N},\ a_{k,n+1} = \frac{k}{\frac{1}{a_{k,n}} + 1}$$

What are the values of $a_{2,0}$, $a_{2,1}$, $a_{2,2}$, and $a_{2,3}$? What are the values of $a_{3,0}$, $a_{3,1}$, $a_{3,2}$, and $a_{3,3}$?

Solution

We have $a_{2,0} = 2$,

$$a_{2,1} = \frac{2}{\frac{1}{a_{2,0}}+1} = \frac{2}{\frac{1}{2}+1} = \frac{2}{\frac{3}{2}} = \frac{4}{3},$$

$$a_{2,2} = \frac{2}{\frac{1}{a_{2,1}}+1} = \frac{2}{\frac{1}{\frac{4}{3}}+1} = \frac{2}{\frac{3}{4}+1} = \frac{2}{\frac{7}{4}} = \frac{8}{7},$$

and

$$a_{2,3} = \frac{2}{\frac{1}{a_{2,2}}+1} = \frac{2}{\frac{1}{\frac{8}{7}}+1} = \frac{2}{\frac{7}{8}+1} = \frac{2}{\frac{15}{8}} = \frac{16}{15}.$$

The numerator of $a_{2,k}$ appears to be 2^{k+1} and the denomiator appears to be $2^{k+1}-1$. But let's explore another value of k before making a hypothesis about a general formula.

We have $a_{3,0} = 3$,

$$a_{3,1} = \frac{3}{\frac{1}{a_{3,0}}+1} = \frac{3}{\frac{1}{3}+1} = \frac{3}{\frac{4}{3}} = \frac{9}{4},$$

$$a_{3,2} = \frac{3}{\frac{1}{a_{3,1}}+1} = \frac{3}{\frac{1}{\frac{9}{4}}+1} = \frac{3}{\frac{4}{9}+1} = \frac{3}{\frac{13}{9}} = \frac{27}{13},$$

and

$$a_{3,3} = \frac{3}{\frac{1}{a_{3,2}}+1} = \frac{3}{\frac{1}{\frac{27}{13}}+1} = \frac{3}{\frac{13}{27}+1} = \frac{3}{\frac{40}{27}} = \frac{81}{40}.$$

The numerator of $a_{3,n}$ appears to be 3^{n+1} and the denomiator appears to be, not $3^{n+1}-1$, but rather $\dfrac{3^{n+1}-1}{2} = \dfrac{3^{n+1}-1}{3-1}$.

Let's explore $k=4$ even though it was not asked for, to test out the last hypothesis.

We have $a_{4,0} = 4$, $a_{4,1} = \frac{4}{\frac{1}{a_{4,0}}+1} = \frac{4}{\frac{1}{4}+1} = \frac{4}{\frac{5}{4}} = \frac{16}{5} = \frac{4^2}{\frac{4^2-1}{4-1}}$, $a_{4,2} = \frac{4}{\frac{1}{a_{4,1}}+1} = \frac{4}{\frac{1}{\frac{16}{5}}+1} = \frac{4}{\frac{5}{16}+1} = \frac{4}{\frac{21}{16}} = \frac{64}{21} = \frac{4^3}{\frac{4^3-1}{4-1}}$, and $a_{4,3} = \frac{4}{\frac{1}{a_{4,2}}+1} = \frac{4}{\frac{1}{\frac{64}{21}}+1} = \frac{4}{\frac{21}{64}+1} = \frac{4}{\frac{85}{64}} = \frac{256}{85} = \frac{4^4}{\frac{4^4-1}{4-1}}$.

The numerator of $a_{4,n}$ again appears to be 4^{n+1} and the denomiator appears to be $\frac{4^{n+1}-1}{3} = \frac{4^{n+1}-1}{4-1}$.

(d) **[3 marks]** Find and prove a non-recursive formula for $a_{k,n}$ that is valid for all natural numbers k greater than 1, and all natural numbers n. **Hint**: as we saw in class, it's easiest to handle multiple universal quantifications in a proof by induction by first letting one variable be arbitrary, and then doing induction on the other variable.

Solution

The observations given above suggest that a non-recursive formula for $a_{k,n}$ is $\frac{k^{n+1}}{\frac{k^{n+1}-1}{k-1}} = \frac{(k-1)\cdot k^{n+1}}{k^{n+1}-1}$.

Note that since

$$\sum_{i=0}^{n}(k^i) = \frac{k^{n+1}-1}{k-1},$$

the expression may also be written as $a_{k,n} = \frac{k^{n+1}}{\sum_{i=0}^{n}(k^i)}$. This sample solution will proceed using the first expression.

The statement that we want to prove correct is

$$\forall\, k \in \mathbb{N},\ k > 1 \ \Rightarrow\ \Big(\forall\, n \in \mathbb{N}, a_{k,n} = \frac{(k-1)\cdot k^{n+1}}{k^{n+1}-1}\Big).$$

Proof. Let k represent a natural number that is greater than 1. And let $P_k(n)$ be the predicate "$a_{k,n} = \frac{(k-1)\cdot k^{n+1}}{k^{n+1}-1}$" where n is a natural number. We want to prove the statement $\forall\, n \in \mathbb{N},\ P_k(n)$, and will do so by induction.

Base case: Let $n = 0$. We want to prove $P_k(0)$.

Since
$$\frac{(k-1)\cdot k^{n+1}}{k^{n+1}-1} = \frac{(k-1)\cdot k^{0+1}}{k^{0+1}-1} = \frac{(k-1)\cdot k^{1}}{k^{1}-1} = k,$$

and $a_{k,0}$ is given to be k, $P_k(0)$ follows.

Induction step: Let $m \in \mathbb{N}$. We want to prove that $P(m) \Rightarrow P(m+1)$. Assume $P(m)$, i.e., that $a_{k,m} = \frac{(k-1)\cdot k^{m+1}}{k^{m+1}-1}$. We want to show $P_k(m+1)$, i.e., that $a_{k,m+1} = \frac{(k-1)\cdot k^{(m+1)+1}}{k^{(m+1)+1}-1}$. Using the definition of $a_{k,m+1}$, we have that

$$\begin{aligned}
a_{k,m+1} &= \frac{k}{\frac{1}{a_{k,m}}+1} \\
&= \frac{k\cdot\ a_{k,m}}{1+a_{k,m}} \\
&\quad \text{(can multiply numerator and denominator by } a_{k,m} \text{ since from the recursive formula } a_{k,m} \neq 0\text{)} \\
&= \frac{k\cdot\ \frac{(k-1)\cdot k^{m+1}}{k^{m+1}-1}}{1+a_{k,m}} \quad \text{(using the induction hypothesis)} \\
&= \frac{(k-1)\cdot k^{m+2}}{(k^{m+1}-1)(1+a_{k,m})} \\
&= \frac{(k-1)\cdot k^{m+2}}{k^{m+1}-1+(k-1)\cdot k^{m+1}} \quad \text{(using the induction hypothesis and multiplying in } (k^{m+1}-1)\text{)} \\
&= \frac{(k-1)\cdot k^{m+2}}{k^{m+1}-1+k^{m+2}-k^{m+1}} \\
&= \frac{(k-1)\cdot k^{m+2}}{k^{m+2}-1} \\
&= \frac{(k-1)\cdot k^{(m+1)+1}}{k^{(m+1)+1}-1},
\end{aligned}$$

and $P(m+1)$, as required. $\square$

3. **[11 marks] Properties of Asymptotic Notation.**

Let $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$. We define the **cumulative sum of** f, denoted Sum_f, to be the function $Sum_f : \mathbb{N} \to \mathbb{R}^{\geq 0}$ defined as follows:

$$Sum_f(n) = \sum_{i=0}^{n} f(i) = f(0) + f(1) + \cdots + f(n)$$

For example, we have previously proved in this course that if $f(n) = n$, then $Sum_f(n) = \dfrac{n(n+1)}{2}$.

In Parts (a) and (c), you may not use any theorems that may have been shown in lecture/tutorial, and must use the formal definition of big-Oh.

(a) **[4 marks]** Prove that for all $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$, if $f \in \mathcal{O}(n)$, then $Sum_f \in \mathcal{O}(n^2)$.

Hint: be careful about choosing constants here! It may be tempting to say that "$f(n) \leq kn$," but this is only true after a certain point. Also remember that you can break up summations:

$$\sum_{i=a}^{b} f(i) = \sum_{i=a}^{c} f(i) + \sum_{i=c+1}^{b} f(i) \qquad \text{for all } a \leq c \leq b.$$

Solution

Proof. Let $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$, and assume that $f \in \mathcal{O}(n)$, i.e., that there exist $c_1, n_1 \in \mathbb{R}^+$ such that for all $n \in \mathbb{N}$, if $n \geq n_1$ then $f(n) \leq c_1 n$.

We want to prove that $Sum_f \in \mathcal{O}(n^2)$, i.e., that there exist $c_2, n_2 \in \mathbb{R}^+$ such that for all $n \in \mathbb{N}$, if $n \geq n_2$ then $Sum_f(n) \leq c_2 n^2$.

Let $c_2 = \left(\sum_{i=0}^{\lfloor n_1 \rfloor} f(i) \right) + c_1$, and let $n_2 = n_1$.* Let $n \in \mathbb{N}$ and assume that $n \geq n_2$. We'll prove that $Sum_f(n) \leq c_2 n^2$.

A small claim we'll use below: since $n_1 \in \mathbb{R}^+$, we know that $\lceil n_1 \rceil \geq 1$. This means that since $n \in \mathbb{N}$ and $n \geq n_2 = n_1$, we know that $n \geq 1$ as well.

Back to the main proof. We start with the left-hand side expression of the inequality:

$$
\begin{aligned}
Sum_f(n) &= \sum_{i=0}^{n} f(i) && \text{(the definition of } Sum_f\text{)}\\
&= \sum_{i=0}^{\lfloor n_1 \rfloor} f(i) + \sum_{i=\lfloor n_1 \rfloor+1}^{n} f(i) && \text{(splitting up the sum)}\\
&\leq \sum_{i=0}^{\lfloor n_1 \rfloor} f(i) + \sum_{i=\lfloor n_1 \rfloor+1}^{n} c_1 i && \text{(by our Big-Oh assumption)}\\
&\leq \sum_{i=0}^{\lfloor n_1 \rfloor} f(i) + \sum_{i=\lfloor n_1 \rfloor+1}^{n} c_1 n && \text{(since } i \leq n \text{ in the second sum)}\\
&= \left(\sum_{i=0}^{\lfloor n_1 \rfloor} f(i)\right) + (n - \lfloor n_1 \rfloor) c_1 n\\
&\leq \left(\sum_{i=0}^{\lfloor n_1 \rfloor} f(i)\right) + c_1 n^2 && \text{(since } n - \lfloor n_1 \rfloor \leq n\text{)}\\
&\leq \left(\sum_{i=0}^{\lfloor n_1 \rfloor} f(i)\right) n^2 + c_1 n^2 && \text{(since } n \geq 1\text{, and the summation is} \geq 0\text{)}\\
&= c_2 n^2
\end{aligned}
$$

□

*Note: choosing c_2 is definitely the hardest part of this question!

(b) **[3 marks]** Prove by induction that for all natural numbers n, $\displaystyle\sum_{i=1}^{2^n} \frac{1}{i} \geq \frac{n}{2}$.

Solution

We define the predicate $P(n) : \displaystyle\sum_{i=1}^{2^n} \frac{1}{i} \geq \frac{n}{2}$, where $n \in \mathbb{N}$.

We'll prove that $\forall n \in \mathbb{N},\ P(n)$ by induction.

Proof. **Base case**: let $n = 0$. The left side of the inequality in $P(n)$ is $\displaystyle\sum_{i=1}^{2^0} \frac{1}{i} = 1$, while the right side is $\dfrac{0}{2} = 0$, so the inequality holds.

Induction step: let $k \in \mathbb{N}$ and assume that $P(k)$ holds, i.e., that $\displaystyle\sum_{i=1}^{2^k} \frac{1}{i} \geq \frac{k}{2}$. We want to prove that $P(k+1)$ also holds, i.e., that $\displaystyle\sum_{i=1}^{2^{k+1}} \frac{1}{i} \geq \frac{k+1}{2}$.

We start with the left side of the inequality:

$$\begin{aligned}
\sum_{i=1}^{2^{k+1}} \frac{1}{i} &= \sum_{i=1}^{2^k} \frac{1}{i} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i} \\
&\geq \frac{k}{2} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i} && \text{(By the induction hypothesis)}
\end{aligned}$$

The key idea for the second summation is that we don't need to calculate its exact value, but only find a lower bound for it. We observe that in the range $i \in \{2^k + 1, 2^k + 2, \ldots, 2^{k+1}\}$, $i \leq 2^{k+1}$, and therefore $\frac{1}{i} \geq \frac{1}{2^{k+1}}$. Using this, we get

$$\begin{aligned}
\sum_{i=1}^{2^{k+1}} \frac{1}{i} &\geq \frac{k}{2} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i} \\
&\geq \frac{k}{2} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{2^{k+1}} \\
&= \frac{k}{2} + 2^k \cdot \frac{1}{2^{k+1}} \\
&= \frac{k}{2} + \frac{1}{2} \\
&= \frac{k+1}{2},
\end{aligned}$$

and $P(k+1)$, as required. □

(c) [**4 marks**] Using part (b), *disprove* the following claim: for all $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, if $f(n) \in \mathcal{O}(g(n))$, then $Sum_f(n) \in \mathcal{O}(n \cdot g(n))$.

Solution

We'll prove the negation of this statement, namely:

$$\exists\ f, g : \mathbb{N} \to \mathbb{R}^{\geq 0},\ f(n) \in \mathcal{O}(g(n))\ \land\ Sum_f(n) \notin \mathcal{O}(n \cdot g(n)).$$

Proof. Let $f(n) = \frac{1}{n+1}$ and $g(n) = \frac{1}{n+1}$.*We need to prove that $f(n) \in \mathcal{O}(g(n))$ and that $Sum_f(n) \notin \mathcal{O}(n \cdot g(n))$.

Part 1: proving that $f(n) \in \mathcal{O}(g(n))$.
Let $c = 1$. For the given $f(n)$ and $g(n)$, we have that $\forall\ n \in \mathbb{N}$, $f(n) = g(n)$. And so, $\forall\ n \in \mathbb{N}$, $f(n) \leq c \cdot g(n)$. Hence, we can let $n_0 = 0$, and have demonstrated that $\exists\ c, n_0 \in \mathbb{R}^+,\ \forall\ n \in \mathbb{N},\ n \geq n_0 \Rightarrow f(n) \leq c \cdot g(n)$. That is, we have proven that $f(n) \in \mathcal{O}(g(n))$, as required.

Part 2: proving that $Sum_f(n) \notin \mathcal{O}(n \cdot g(n))$. Let's expand the definition (of Big-Oh, Sum_f, and f and g themselves):

$$\forall\ c, n_0 \in \mathbb{R}^+,\ \exists\ n \in \mathbb{N},\ n \geq n_0\ \land\ \left(\sum_{i=0}^{n} \frac{1}{i+1}\right) > c \cdot \frac{n}{n+1}$$

Let $c, n_0 \in \mathbb{R}^+$. Let $n = 2^{\lceil 2c+n_0 \rceil} + \lceil n_0 \rceil$.
Part 2(a): For the first part of the AND, since $2^{\lceil 2c+n_0 \rceil} > 0$, we know that $n \geq \lceil n_0 \rceil \geq n_0$.
Part 2(b): For the second part of the AND, we start with the left side of the inequality.

$$\begin{aligned}
\sum_{i=0}^{n} \frac{1}{i+1} &= \sum_{i'=1}^{n+1} \frac{1}{i'} && \text{(substituting } i' = i+1\text{)} \\
&> \sum_{i'=1}^{n} \frac{1}{i'} && \text{(since } n > 0 \text{ and } \frac{1}{i'} > 0\text{)} \\
&\geq \sum_{i'=1}^{2^{\lceil 2c+n_0 \rceil}} \frac{1}{i'} && \text{(since } n \geq 2^{\lceil 2c+n_0 \rceil}\text{)} \\
&\geq \frac{\lceil 2c+n_0 \rceil}{2} && \text{(by Part (b))} \\
&> c \\
&> c \cdot \frac{n}{n+1} && \text{(since } 0 < \frac{n}{n+1} < 1\text{)}
\end{aligned}$$

$\square$

*We use $\frac{1}{n+1}$ rather than $\frac{1}{n}$ to make sure f and g are defined at 0.