CSC165H1: Problem Set 2 Sample Solutions

Due February 7, 2018 before 10pm

Note: solutions are incomplete, and meant to be used as guidelines only. We encourage you to ask follow-up questions on the course forum or during office hours.

1. **[6 marks] AND vs. IMPLIES.** Students often confuse the meanings of the two propositional operators $\Rightarrow$ and $\land$. In this question, you'll examine each of these in a series of statements by writing formal proofs/disproofs of each one.

   (a) [**3 marks**] Prove or disprove: $\forall n \in \mathbb{N},\ n > 15 \Rightarrow n^3 - 10n^2 + 3 \geq 165$.

   Solution

   Proof. Let $n \in \mathbb{N}$, and assume that $n > 15$. Then $n^3 - 10n^2 = (n-10)n^2$.
   Since $n > 15$, we know that $n - 10 > 5$ and $n^2 > 225$. So then multiplying these inequalities gives us:

   $$\begin{aligned} (n-10)n^2 &> 5 \cdot 225 \\ n^3 - 10n^2 &> 1225 \\ n^3 - 10n^2 + 3 &> 1228 \\ n^3 - 10n^2 + 3 &> 165 && \text{(since } 1228 > 165\text{)} \\ n^3 - 10n^2 + 3 &\geq 165 \end{aligned}$$

   $\square$

   (b) [**3 marks**] Prove or disprove: $\forall n \in \mathbb{N},\ n > 15 \land n^3 - 10n^2 + 3 \geq 165$.

   Solution

   We will *disprove* this statement. Its negation is:

   $$\exists n \in \mathbb{N},\ n \leq 15 \lor n^3 - 10n^2 + 3 < 165$$

   Proof. Let $n = 0$. We want to prove $n \leq 15 \lor n^3 - 10n^2 + 3 < 165$; since this is an OR, we only need to prove one of the two parts.
   Since $n = 0$, we know that $n \leq 15$.* $\square$

   *While it *is* also true that $(0)^3 - 10(0)^2 + 3 < 165$, this is not required for the proof.

2. **[6 marks] Ceiling function.**

Recall that the ceiling function is the function $\lceil x \rceil : \mathbb{R} \to \mathbb{Z}$ defined as $\lceil x \rceil$ is the smallest integer greater than or equal to x. You may use the following two facts about ceilings in your proofs below, as long as you clearly state where you are using them.

$$\forall x \in \mathbb{R},\ \forall k \in \mathbb{Z},\ k \geq x \Rightarrow k \geq \lceil x \rceil \tag{Fact 1}$$

$$\forall x \in \mathbb{R},\ \forall k \in \mathbb{Z},\ \lceil x + k \rceil = \lceil x \rceil + k \tag{Fact 2}$$

(a) **[3 marks]** Prove that for all natural numbers n and m, if $n < m$ then $\left\lceil \frac{m-1}{m} \cdot n \right\rceil = n$.

Solution

First, the translation of the statement into predicate logic is:

$$\forall n, m \in \mathbb{N},\ n < m \Rightarrow \left\lceil \frac{m-1}{m} \cdot n \right\rceil = n$$

Proof. Let $n, m \in \mathbb{N}$, and assume that $n < m$. We want to prove that $\left\lceil \frac{m-1}{m} \cdot n \right\rceil = n$.
We start with the left-hand side of the given equation, and prove that it equals the right side.

$$\begin{aligned} \left\lceil \frac{m-1}{m} \cdot n \right\rceil &= \left\lceil n - \frac{n}{m} \right\rceil \\ &= n + \left\lceil -\frac{n}{m} \right\rceil && \text{(By Fact 2, since } n \in \mathbb{Z}\text{)} \end{aligned}$$

Also, by our assumption we know that $n < m$, and so $-1 < -\frac{n}{m} \leq 0$, and so $\left\lceil -\frac{n}{m} \right\rceil = 0$.
Substituting this into the above equation yields $\left\lceil \frac{m-1}{m} \cdot n \right\rceil = n$. □

(b) **[3 marks]** We define the function $nextFifty : \mathbb{N} \to \mathbb{N}$ as $nextFifty(n) = 50 \cdot \left\lceil \frac{n}{50} \right\rceil$. Also, recall that for integers n and d, we say n is a *multiple* of d when $d \mid n$.
Prove that for all $n \in \mathbb{N}$, $nextFifty(n)$ is the smallest multiple of 50 greater than or equal to n.

Solution

We can translate the statement as follows (we directly substitute the definition of $nextFifty$ into the formula):

$$\begin{aligned} &\forall n \in \mathbb{N},\ 50 \mid nextFifty(n) \land nextFifty(n) \geq n \land \\ &\qquad \Big(\forall m \in \mathbb{N},\ 50 \mid m \land m \geq n \Rightarrow m \geq nextFifty(n)\Big) \end{aligned}$$

Proof. Let $n \in \mathbb{N}$. The statement we want to prove is an AND, so we will prove the parts separately.

Part 1. We want to prove that $50 \mid nextFifty(n)$. Let $k = \left\lceil \frac{n}{50} \right\rceil$.
Then $n = 50k$, and so (by the definition of divisibility), $50 \mid nextFifty(n)$.

Part 2. We want to prove that $nextFifty(n) \geq n$.
By the definition of *ceiling*, we know that $\left\lceil \frac{n}{50} \right\rceil \geq \frac{n}{50}$. Multiplying both sides of the inequality by 50 yields:

$$\begin{aligned}
\left\lceil \frac{n}{50} \right\rceil &\geq \frac{n}{50} \\
50 \cdot \left\lceil \frac{n}{50} \right\rceil &\geq n \\
nextFifty(n) &\geq n
\end{aligned}$$

Part 3. We want to prove that $\forall m \in \mathbb{N},\ 50 \mid m \wedge m \geq n \Rightarrow m \geq nextFifty(n)$. Let $m \in \mathbb{N}$, and assume that $50 \mid m$ and $m \geq n$. Unpacking the definition of divisibility, we know that there exists $k \in \mathbb{Z}$ such that $m = 50k$. We want to prove that $m \geq 50 \cdot \left\lceil \frac{n}{50} \right\rceil$.
We start with the assumption that $m \geq n$:

$$\begin{aligned}
m &\geq n \\
\frac{m}{50} &\geq \frac{n}{50} \\
k &\geq \frac{n}{50} && \text{(Since } m = 50k\text{)} \\
k &\geq \left\lceil \frac{n}{50} \right\rceil && \text{(Using Fact 1)} \\
50k &\geq 50 \cdot \left\lceil \frac{n}{50} \right\rceil \\
m &\geq nextFifty(n) && \text{(Substituting definitions)}
\end{aligned}$$

□

3. **[6 marks] Divisibility.**

This question is a continuation of Question 2: you may use the definitions, given Facts 1 and 2, and statements in parts (2a) and (2b) in your proofs below.[1]

(a) **[4 marks]** Prove the following statement:

$$\forall n \in \mathbb{N},\ n \leq 2300 \Rightarrow \Big(49 \mid n \Leftrightarrow 50 \cdot (nextFifty(n) - n) = nextFifty(n)\Big)$$

Solution

(We don't need a translation here because the given statement is already expressed in predicate logic.)

Proof. Let $n \in \mathbb{N}$, and assume $n \leq 2300$. We need to prove the conclusion of the implication, which is an "if and only if", so we'll do it in two parts.

Part 1. We want to prove that $49 \mid n \Rightarrow 50 \cdot (nextFifty(n) - n) = nextFifty(n)$. Assume $49 \mid n$, i.e., that there exists a $k \in \mathbb{Z}$ such that $n = 49k$. We want to prove that $50 \cdot (nextFifty(n) - n) = nextFifty(n)$.

We start with the left side and expand the definition of $nextFifty$:

$$\begin{aligned} 50 \cdot (nextFifty(n) - n) &= 50 \cdot \left(50 \cdot \left\lceil \frac{n}{50} \right\rceil - n\right) \\ &= 50 \cdot \left(50 \cdot \left\lceil \frac{49}{50}k \right\rceil - 49k\right) && \text{(Since } n = 49k\text{)} \end{aligned}$$

Now, since $n \leq 2300$, we know that $k \leq \dfrac{2300}{49} < 50$. So then we can use the statement from Question 2(a), which tells us that $\left\lceil \dfrac{49}{50}k \right\rceil = k$. Substituting this into the above equation gives:

$$\begin{aligned} 50 \cdot (nextFifty(n) - n) &= 50 \cdot (50k - 49k) \\ &= 50k \end{aligned}$$

Now we start with the right side of the original equation and expand the definition of $nextFifty$:

$$\begin{aligned} nextFifty(n) &= 50 \cdot \left\lceil \frac{n}{50} \right\rceil \\ &= 50 \cdot \left\lceil \frac{49}{50}k \right\rceil \\ &= 50k && \text{(Using Question 2(a) again)} \end{aligned}$$

So both the left and right sides of the original equation are equal to $50k$, and so are equal to each other.

[1] You may use (2a) and (2b) for full marks here *even if* you didn't prove them in the previous question.

Part 2. We want to prove that $50 \cdot (nextFifty(n) - n) = nextFifty(n) \Rightarrow 49 \mid n$. Assume that $50 \cdot (nextFifty(n) - n) = nextFifty(n)$. Let $k = \left\lceil \frac{n}{50} \right\rceil$. We'll prove that $n = 49k$.
We start with the equation we've assumed.

$$\begin{aligned}
50 \cdot (nextFifty(n) - n) &= nextFifty(n) \\
49 \cdot nextFifty &= 50n \\
49 \cdot \left(50 \cdot \left\lceil \frac{n}{50} \right\rceil\right) &= 50n \\
49 \cdot (50k) &= 50n && \text{(Substituting our value for } k\text{)} \\
49k &= n
\end{aligned}$$

□

(b) **[2 marks]** *Disprove* the following statement:

$$\forall n \in \mathbb{N},\ 49 \mid n \Leftrightarrow 50 \cdot (nextFifty(n) - n) = nextFifty(n)$$

Solution

The negation of the given statement is:

$$\exists n \in \mathbb{N},\ \Big(49 \mid n \wedge 50 \cdot (nextFifty(n) - n) \neq nextFifty(n)\Big) \vee \\ \Big(49 \nmid n \wedge 50 \cdot (nextFifty(n) - n) = nextFifty(n)\Big)$$

Discussion. The contrast between this statement and the one from part (a) is a big hint! Part (a) tells us that if $n < 2300$ then it *does* satisfy this "if and only if." Looking at our proof, what it really needed was that $\frac{n}{49} < 50$. So let's just pick a larger n!

Proof. Let $n = 2450$ (this is $49 \cdot 50$). Since the rest of the statement is an OR, we can choose which one we want to prove. We'll prove the first part, i.e., that $49 \mid n \wedge 50 \cdot (nextFifty(n) - n) \neq nextFifty(n)$

Part 1. We want to prove that $49 \mid n$. This is just a calculation: $\frac{n}{49} = 50$.

Part 2. We want to prove that $50 \cdot (nextFifty(n) - n) \neq nextFifty(n)$.
This is actually just a calculation as well, albeit a more complex one.
First, $nextFifty(n) = 50 \cdot \left\lceil \frac{n}{50} \right\rceil = 50 \cdot 49 = 2450$.
But on the left side of the equation, we have:

$$50 \cdot (nextFifty(n) - n) = 50 \cdot (2450 - 2450) = 0$$

and so the left side does not equal $nextFifty(n) = 2450$. □

4. **[9 marks] Functions.**

Let $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$. We say f is *bounded* if there exists a real number k such that f never outputs a value greater than k. (Recall that $\mathbb{R}^{\geq 0}$ denotes the set of real numbers greater than or equal to zero.)

(a) [**2 marks**] Express the statement "f is bounded" in predicate logic.

Solution

$\exists k \in \mathbb{R},\ \forall x \in \mathbb{N},\ f(x) \leq k.$

(b) [**4 marks**] Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$. We define the *sum of functions* as the function $(f+g) : \mathbb{N} \to \mathbb{R}^{\geq 0}$ as follows: for all $n \in \mathbb{N}$, $(f+g)(n) = f(n) + g(n)$. For example, if $f(n) = n^2 + 1$ and $g(n) = 165n$, $(f+g)(n) = n^2 + 1 + 165n$.

Prove that for all functions $f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}$, if f_1 and f_2 are bounded, then $f_1 + f_2$ is bounded.

Solution

Translation:

$$\forall f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0},\ \Big(\exists k_1 \in \mathbb{R},\ \forall x \in \mathbb{N},\ f_1(x) \leq k_1)\Big) \land \Big(\exists k_2 \in \mathbb{R},\ \forall x \in \mathbb{N},\ f_2(x) \leq k_2)\Big) \Rightarrow \Big(\exists k_3 \in \mathbb{R},\ \forall x \in \mathbb{N},\ f_1(x) + f_2(x) \leq k_3)\Big)$$

Proof. Let $f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}$. Assume that f_1 and f_2 are bounded, i.e., that there exist $k_1, k_2 \in \mathbb{R}$ such that for all $x \in \mathbb{N}$, $f_1(x) \leq k_1$ and $f_2(x) \leq k_2$. We want to prove that $f_1 + f_2$ is bounded, i.e., that there exists $k_3 \in \mathbb{R}$ such that for all $x \in \mathbb{N}$, $f_1(x) + f_2(x) \leq k_3$.

Let $k_3 = k_1 + k_2$. Let $x \in \mathbb{N}$.

Then by our two assumptions, we know that $f_1(x) \leq k_1$ and $f_2(x) \leq k_2$. Adding these inequalities yields:

$$f_1(x) + f_2(x) \leq k_1 + k_2$$
$$f_1(x) + f_2(x) \leq k_3$$

□

(c) [**3 marks**] Prove or disprove: for all functions $f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}$, if $f_1 + f_2$ is bounded, then f_1 is bounded and f_2 is bounded.

Solution

This statement is true, so we'll prove it. Translation:

$$\forall f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0},\ \Big(\exists k_3 \in \mathbb{R},\ \forall x \in \mathbb{N},\ f_1(x) + f_2(x) \leq k_3)\Big) \Rightarrow \Big(\exists k_1 \in \mathbb{R},\ \forall x \in \mathbb{N},\ f_1(x) \leq k_1)\Big) \land \Big(\exists k_2 \in \mathbb{R},\ \forall x \in \mathbb{N},\ f_2(x) \leq k_2)\Big)$$

Proof. Let $f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}$. Assume that $f_1 + f_2$ is bounded, i.e., that there exists $k_3 \in \mathbb{R}$ such that for all $x \in \mathbb{N}$, $f_1(x) + f_2(x) \leq k_3$.

We want to prove that f_1 and f_2 are bounded, i.e., that there exist $k_1, k_2 \in \mathbb{R}$ such that for all $x \in \mathbb{N}$, $f_1(x) \leq k_1$ and $f_2(x) \leq k_2$.

Let $k_1 = k_3$ and $k_2 = k_3$. Let $x \in \mathbb{N}$.

Then by our assumption, we know that $f_1(x) + f_2(x) \leq k_3$. Since the range of f_2 is $\mathbb{R}^{\geq 0}$, we know that $f_2(x) \geq 0$. So then we have:

$$\begin{aligned} f_1(x) + f_2(x) &\leq k_3 \\ f_1(x) &\leq k_3 && \text{(making left side smaller)} \\ f_1(x) &\leq k_1 && \text{(since } k_1 = k_3\text{)} \end{aligned}$$

Similarly, for f_2 we have:

$$\begin{aligned} f_1(x) + f_2(x) &\leq k_3 \\ f_2(x) &\leq k_3 && \text{(since } f_1 \text{ has range } \mathbb{R}^{\geq 0} \text{ too)} \\ f_1(x) &\leq k_1 && \text{(since } k_1 = k_3\text{)} \end{aligned}$$

$\square$