Problem Set 1 Solution:

1. Define the predicate: $P(n)$: $4^n + 15n - 1$ is divisible by 9.
Base case: for $n = 1$, $4^1 + 15 \times 1 - 1 = 18$ which is divisible by 9.
Induction step: Assume $P(k)$, i.e., $4^k + 15k - 1$ is divisible by 9, then we need to show $P(k+1)$, i.e., $4^{k+1} + 15(k+1) - 1$ is divisible by 9.

$$\begin{aligned}
&4^{k+1} + 15(k+1) - 1 \\
=&4 \cdot 4^k + 15k + 14 \\
=&4(4^k + 15k - 1) - 45k + 18 \\
=&4(4^k + 15k - 1) - 9(5k - 2)
\end{aligned}$$

The first term in the above result is divisible by 9 by the induction hypothesis; the second term is divisible by 9 because of the factor 9. Therefore $4^{k+1} + 15(k+1) - 1$ is divisible by 9, i.e., $P(k+1)$ is true, which completes the induction step.

2. Attached separately.

3. Consider the Fibonacci-esque function g:

$$g(n) = \begin{cases} 1, & \text{if } n = 0 \\ 3, & \text{if } n = 1 \\ g(n-2) + g(n-1) & \text{if } n > 1 \end{cases}$$

Use complete induction to prove that if n is a natural number greater than 1, then $2^{n/2} \leq g(n) \leq 2^n$.

Sample solution: Proof, using complete induction.

inductive step: Let n be a typical natural number greater than 1 and assume $H(n)$: Every natural number $i \in \{2, \ldots, n-1\}$ satisfies $2^{i/2} \leq g(i) \leq 2^i$.

show that inductive conclusion follows: We'll derive $C(n)$: $2^{n/2} \leq g(n) \leq 2^n$.

Base cases: $1 < n < 4$: $g(2) = 4$ and $g(3) = 7$ # by the definition of $g(2)$ and $g(3)$.

$$2^{2/2} = 2 \leq 4 = g(2) \leq 4 = 2^2 \qquad \text{and} \qquad 2^{3/2} = 2\sqrt{2} \leq 7 = g(3) \leq 8 = 2^3$$

$C(2)$ and $C(3)$ follow from our assumptions in this case.

Case $n \geq 4$: By assumption $H(n)$ we have # $n \geq 4$ implies $2 \leq n-2, n-1 < n$:

$$2^{(n-2)/2} \leq g(n-2) \leq 2^{n-2} \qquad \text{and} \qquad 2^{(n-1)/2} \leq g(n-1) \leq 2^{n-1}.$$

Substituting these inequalities into the definition of $g(n)$ # by definition of $g(n), n \geq 4 > 0$:

$$g(n) = g(n-2) + g(n-1) \geq 2^{(n-2)/2} + 2^{(n-1)/2} = (1 + \sqrt{2})2^{(n-2)/2} \geq 2 \times 2^{(n-2)/2} = 2^{n/2}$$
$$g(n) = g(n-2) + g(n-1) \leq 2^{n-2} + 2^{n-1} = (1 + 2)2^{n-2} \leq 2^2 \times 2^{n-2} = 2^n$$

$C(n)$ follows from our assumptions in this case.

In all cases $H(n)$ implies $C(n)$.

4. If $n = 7^k$ then we will get that $L(n) = L(7^k) = 1 + L(7^{k-1}) = 1 + 1 + L(7^{k-2}) = \ldots = k + L(1) = k = \log_7 n$.Rather than proving it now, we will prove for the general case that $L(n) = \lfloor \log_7 n \rfloor$ for $n > 0$ and $L(0) = 0$. We show this by complete induction. Since $L(n) = 0 = \lfloor \log_7 n \rfloor$ for $0 < n < 7$ and since $L(0) = 0$ we need only to check $L(n)$ for $n > 6$.

$$L(n) = 1 + L(\lfloor n/7 \rfloor) + 1 = \lfloor \log_7 \lfloor n/7 \rfloor \rfloor + 1 = (\lfloor \log_7 n \rfloor - 1) + 1 = \lfloor \log_7 n \rfloor.$$

For T, we have the recursion,

$T(n) = 3$ if $n < 7$ and otherwise $T(n) = 4 + T(\lfloor n/7 \rfloor)$. Unwinding will give $T(n) = 4 * \log_7 n + T(1)$ for n which is a power of 7. We guess $T(n) = 4 * \lfloor \log_7 n \rfloor + 3$ for a general n. Proving this by induction is essentially the same as with L.

5. We show that each element n of F can be written as $7k$ for some integer k.

Base case: $7 = 7 \times 1$, so 7 mod 7 is 0 as required.

Inductive case: suppose that $m \in F$ and that $m = 7p$ for some integer p. Also suppose that $n \in F$ and that $n = 7q$ for some integer q. We now verify that the second rule generates integers divisible by 7:

$$\begin{aligned} m + n &= 7p + 7q \\ &= 7(p + q) \end{aligned}$$

which is divisible by 7.