FLOLAC'14

Type theory and logic

Lecture IV: meta-theoretical reasoning

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Meta-language vs object language

Types and programs form a language, which are talked about by a separate language of judgements and derivations. In this case we call the former the *object language*, and the latter the *meta-language*.

What we write down as types and programs are nothing more than certain syntax trees by themselves; then, at a higher level, we organise and relate these syntax trees with judgements and derivations.

Judgements and derivations can also be regarded as syntax trees to be reasoned about. For example, consistency is a statement in which judgements and derivations are the object language and English is the meta-language. (Canonicity is another example.)

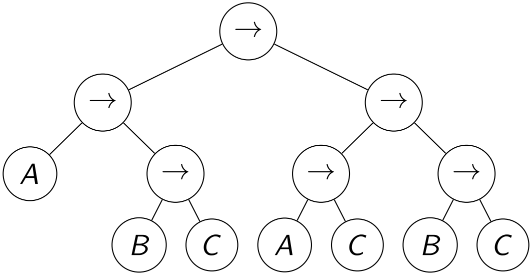
We can also use Agda as the meta-language!

Implicational fragment of propositional logic

Today we consider only propositions formed with implication.

Each of these propositions is a finite tree whose internal nodes are implications and whose leaves are atomic propositions, which are elements of a given set $V = \{A, B, C, \dots\}$.

Example. The proposition
$(A \to B \to C) \to (A \to C) \to B \to C$ is represented as



Natural deduction

Natural deduction is the type part of simple type theory (and we are considering only the implicational fragment).

$$\frac{}{\Gamma \;\vdash\; p}\,(\text{assum}) \quad \text{when } p \in \Gamma$$

$$\frac{\Gamma\,,\; p \;\vdash\; q}{\Gamma \;\vdash\; p \to q}\,(\to\text{I})$$

$$\frac{\Gamma \;\vdash\; p \to q \qquad \Gamma \;\vdash\; p}{\Gamma \;\vdash\; q}\,(\to\text{E})$$

Untyped λ-calculus

A λ-term is either a variable, an abstraction, or an application.

We usually assume *α-equivalence* of λ-terms, i.e., the names of *bound variables* do not matter.

- Change of bound variable names is called α-conversion, which has to be *capture-avoiding*, i.e., *free variables* must not become bound after a name change.
- In formalisation, we prefer not to deal with α-equivalence explicitly, and one way is to use *de Bruijn indices* — λ's are nameless, and a bound variable is represented as a natural number indicating to which λ it is bound.

Simply typed λ-calculus (à la Curry)

λ-calculus was designed to model function abstraction and application in mathematics. In untyped λ-calculus, however, we can write nonsensical terms like $\lambda x.\, x\, x$.

We can use the implicational fragement of propositional logic as a type language for λ-calculus, ruling out nonsensical terms.

$$\frac{}{\Gamma \vdash x : p}\,(\text{var}) \quad \text{when } x : p \in \Gamma$$

$$\frac{\Gamma, x : p \vdash t : q}{\Gamma \vdash \lambda x.\, t : p \to q}\,(\text{abs})$$

$$\frac{\Gamma \vdash t : p \to q \qquad \Gamma \vdash u : p}{\Gamma \vdash t\, u : q}\,(\text{app})$$

Curry–Howard isomorphism

Derivations in natural deduction and well-typed λ-terms are in one-to-one correspondence.

That is, we can write two functions,

- one mapping a logical derivation in natural deduction to a λ-term and its typing derivation, and
- the other mapping a λ-term with a typing derivation to a logical derivation in natural deduction,

and can prove that the two functions are inverse to each other.

This result is historically significant: two formalisms are developed separately from logical and computational perspectives, yet they coincide perfectly.

Simply typed λ-calculus à la Church

The Curry–Howard isomorphism points out that derivations in natural deduction are actually λ-terms in disguise.

These λ-terms are intrinsically typed, so every term we are able to write down is necessarily well-behaved, whereas in simply typed λ-calculus à la Curry, we can write arbitrary λ-terms, and only rule out ill-behaved ones via typing later.

Semantics

After defining a language (like the implicational fragment of propositional logic), which consists of a bunch of syntax trees, we need to specify what these trees mean.

Judgements and derivations (which form a *deduction system*) assign meaning to the propositional language by specifying how it is used in formal reasoning.

We can also translate the syntax trees into entities in a well understood semantic domain. In the case of propositional logic, we can translate propositional trees to functions on truth values. (This is the classical treatment.)

Two-valued semantics of propositional logic

- Define $\texttt{Bool} := \{\texttt{false}, \texttt{true}\}$.
- An *assignment* is a function of type $V \to \texttt{Bool}$.
- A proposition p is translated into a function $[\![p]\!] : (V \to \texttt{Bool}) \to \texttt{Bool}$ mapping assignments to truth values.
- An assignment σ *models* a proposition p exactly when $[\![p]\!]\ \sigma$ is true, and *models* a context Γ exactly when it models every proposition in Γ.

Two-valued semantics of propositional logic

- A proposition p is *satisfiable* exactly when there exists an assignment that models p, and is *valid* exactly when every assignment models p.
- A proposition p is a *semantic consequence* of a context Γ (written $\Gamma \models p$) exactly when every assignment that models Γ also models p.

Exercise. Show that $(p \to p \to q) \to (p \to q)$ is valid for any propositions p and q.

Exercise. Show that a proposition p is valid if and only if p is a semantic consequence of the empty context.

Deduction systems and semantics

Natural deduction is *sound* with respect to the two-valued semantics: whenever we can deduce $\Gamma \vdash p$, it must be the case that $\Gamma \models p$.

The implicational fragment of propositional logic is also *(semantically) complete* with respect to the two-valued semantics: if $\Gamma \models p$, then we can construct a derivation of $\Gamma \vdash p$.