FLOLAC'14

Type theory and logic

Lecture II: dependent type theory

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Indexed families of sets (predicates)

Common mathematical statements involve predicates and universal/existential quantification.

For example: "For all natural number $x : \mathbb{N}$, if x is not zero, then there exists $y : \mathbb{N}$ such that x is equal to $1 + y$."

In type theory, a predicate on A has type $A \to \mathcal{U}$ — a *family of sets* indexed by the domain A. For example:

$$\vdash \lambda x.\ \text{“if } x \text{ is zero then } \mathbf{0} \text{ else } \mathbf{1}\text{”} : \mathbb{N} \to \mathcal{U}$$

(Note that the above treatment is in fact unfounded in our current theory. Why? (We will fix it on Thursday.))

Dependent product types (universal quantification)

- Formation:

$$\frac{\Gamma \vdash A : \mathcal{U} \qquad \Gamma \vdash B : A \to \mathcal{U}}{\Gamma \vdash \Pi\, A\, B : \mathcal{U}}\ (\Pi\mathsf{F})$$

- Introduction:

$$\frac{\Gamma, x : A \vdash t : B\, x}{\Gamma \vdash \lambda x.\, t : \Pi\, A\, B}\ (\Pi\mathsf{I})$$

- Elimination:

$$\frac{\Gamma \vdash f : \Pi\, A\, B \qquad \Gamma \vdash a : A}{\Gamma \vdash f\, a : B\, a}\ (\Pi\mathsf{E})$$

Notation. We usually write $\Pi(x : A)\ B\, x$ for $\Pi\ A\ B$ (where '$\Pi(x : A)$' is regarded as a quantifier).

Exercise. Let $\Gamma := A : \mathcal{U}\,,\ B : \mathcal{U}\,,\ C : A \to B \to \mathcal{U}$. Derive

$$\Gamma \vdash \_ : (\Pi(x : A)\ \Pi(y : B)\ C\, x\, y) \to \Pi(y : B)\ \Pi(x : A)\ C\, x\, y$$

Dependent sum types (existential quantification)

- Formation:

$$\frac{\Gamma \vdash A : \mathcal{U} \qquad \Gamma \vdash B : A \to \mathcal{U}}{\Gamma \vdash \Sigma\, A\, B : \mathcal{U}}\ (\Sigma\mathsf{F})$$

- Introduction:

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B\, a}{\Gamma \vdash (a, b) : \Sigma\, A\, B}\ (\Sigma\mathsf{I})$$

- Elimination:

$$\frac{\Gamma \vdash p : \Sigma\, A\, B}{\Gamma \vdash \mathtt{fst}\ p : A}\ (\Sigma\mathsf{EL}) \qquad \frac{\Gamma \vdash p : \Sigma\, A\, B}{\Gamma \vdash \mathtt{snd}\ p : B\,(\mathtt{fst}\ p)}\ (\Sigma\mathsf{ER})$$

Notation. We usually write $\Sigma(x : A)\ B\, x$ for $\Sigma\, A\, B$.

Exercise. Let $\Gamma := A : \mathcal{U}, B : \mathcal{U}, C : A \to B \to \mathcal{U}$. Derive the *axiom of choice*:

$$\Gamma \vdash \_ : (\Pi(x : A)\ \Sigma(y : B)\ C\, x\, y) \to \Sigma(f : A \to B)\ \Pi(x : A)\ C\, x\,(f\, x)$$

Computation

Let $\Gamma := A : \mathcal{U}\,,\, B : A \to \mathcal{U}\,,\, C : A \to \mathcal{U}$. Try to derive

$$\Gamma \;\vdash\; \_ : (\Pi(p : \Sigma\, A\, B)\;\, C\,(\texttt{fst}\; p)) \to (\Pi(x : A)\;\, B\, x \to\; C\, x)$$

… and you should notice some problems.

So far we have been concentrating on the *statics* of type theory; here we need to invoke the *dynamics* of the theory.

Equality judgements and computation rules

We introduce a new kind of judgements stating that two terms should be regarded as the same during typechecking:

$$\Gamma \vdash t = u \in A$$

For each set, (when applicable) we specify additional *computational rules* stating that eliminating an introductory term yields a component of the latter. For example, for product types we have two computation rules:

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash \mathtt{fst}\,(a, b) = a \in A}\,(\times\mathsf{CL}) \qquad \frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash \mathtt{snd}\,(a, b) = b \in B}\,(\times\mathsf{CR})$$

More computation rules

$$\frac{\Gamma, x : A \vdash t : B \qquad \Gamma \vdash a : A}{\Gamma \vdash (\lambda x.\, t)\, a = t[a/x] \;\in A \to B}\ (\to\mathsf{C})$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash f : A \to C \qquad \Gamma \vdash g : B \to C}{\Gamma \vdash \mathtt{case}\,(\mathtt{left}\; a)\, f\, g = f\, a \;\in C}\ (+\mathsf{CL})$$

$$\frac{\Gamma \vdash b : B \qquad \Gamma \vdash f : A \to C \qquad \Gamma \vdash g : B \to C}{\Gamma \vdash \mathtt{case}\,(\mathtt{right}\; b)\, f\, g = g\, b \;\in C}\ (+\mathsf{CR})$$

More computation rules

$$\frac{\Gamma, x : A \;\vdash\; t : B\,x \qquad \Gamma \;\vdash\; a : A}{\Gamma \;\vdash\; (\lambda x.\, t)\, a = t[a/x] \;\in\; B\,a}\ (\Pi\mathsf{C})$$

$$\frac{\Gamma \;\vdash\; a : A \qquad \Gamma \;\vdash\; b : B\,a}{\Gamma \;\vdash\; \mathtt{fst}\,(a, b) = a \;\in\; A}\ (\Sigma\mathsf{CL})$$

$$\frac{\Gamma \;\vdash\; a : A \qquad \Gamma \;\vdash\; b : B\,a}{\Gamma \;\vdash\; \mathtt{snd}\,(a, b) = b \;\in\; B\,a}\ (\Sigma\mathsf{CR})$$

Predicates respect computation.