FLOLAC'14

Type theory and logic

Lecture III: Heyting arithmetic

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Department of Computer Science
University of Oxford

Hsiang-Shang.Ko@cs.ox.ac.uk

Natural numbers

- Formation:

$$\frac{}{\Gamma \ \vdash \ \mathbb{N} : \mathcal{U}} \ (\mathbb{N}\mathsf{F})$$

- Introduction:

$$\frac{}{\Gamma \ \vdash \ \mathtt{zero} : \mathbb{N}} \ (\mathbb{N}\mathsf{IZ}) \qquad \frac{\Gamma \ \vdash \ n : \mathbb{N}}{\Gamma \ \vdash \ \mathtt{suc}\ n : \mathbb{N}} \ (\mathbb{N}\mathsf{IS})$$

Natural numbers — elimination rule

- Elimination:

$$\frac{\begin{array}{lll}\Gamma & \vdash & P : \mathbb{N} \to \mathcal{U} \\ \Gamma & \vdash & z : P\ \texttt{zero} \\ \Gamma & \vdash & s : \Pi[x : \mathbb{N}]\ P\,x \to P\,(\texttt{suc}\ x) \\ \Gamma & \vdash & n : \mathbb{N}\end{array}}{\Gamma\ \vdash\ \texttt{ind}\ P\,z\,s\,n : P\,n}\ (\mathbb{N}\mathrm{E})$$

  Logically this is the *induction principle*; computationally this is *primitive recursion*.

Natural numbers — computation rules

- Computation:

$$\frac{\begin{array}{l}\Gamma \ \vdash \ P : \mathbb{N} \to \mathcal{U} \\ \Gamma \ \vdash \ z : P\ \texttt{zero} \\ \Gamma \ \vdash \ s : \Pi[x:\mathbb{N}]\ P\,x \to P\,(\texttt{suc}\ x)\end{array}}{\Gamma \ \vdash \ \texttt{ind}\ P\,z\,s\ \texttt{zero} = z \ \in \ P\ \texttt{zero}}\ (\mathbb{N}\text{CZ})$$

$$\frac{\begin{array}{l}\Gamma \ \vdash \ P : \mathbb{N} \to \mathcal{U} \\ \Gamma \ \vdash \ z : P\ \texttt{zero} \\ \Gamma \ \vdash \ s : \Pi[x:\mathbb{N}]\ P\,x \to P\,(\texttt{suc}\ x) \\ \Gamma \ \vdash \ n : \mathbb{N}\end{array}}{\Gamma \ \vdash \ \texttt{ind}\ P\,z\,s\,(\texttt{suc}\ n) = s\,n\,(\texttt{ind}\ P\,z\,s\,n) \ \in \ P\,(\texttt{suc}\ n)}\ (\mathbb{N}\text{CS})$$

Identity types

- Formation:

$$\frac{\Gamma \vdash A : \mathcal{U} \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash \mathtt{Id}\, A\, t\, u : \mathcal{U}}\ (\equiv\mathsf{F})$$

- Introduction:

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \mathtt{refl} : \mathtt{Id}\, A\, t\, t}\ (\equiv\mathsf{I})$$

Exercise. Assume $\Gamma \vdash t = u \in A$ and derive $\Gamma \vdash \mathtt{refl} : \mathtt{Id}\, A\, t\, u$.

Identity types — elimination rule

- Elimination:

$$\frac{\begin{array}{lll}\Gamma & \vdash & t : A \\ \Gamma & \vdash & P : \Pi[x:A]\ \mathtt{Id}\ A\ t\ x \to \mathcal{U} \\ \Gamma & \vdash & p : P\ t\ \mathtt{refl} \\ \Gamma & \vdash & u : A \\ \Gamma & \vdash & q : \mathtt{Id}\ A\ t\ u\end{array}}{\Gamma\ \vdash\ \mathtt{J}\ P\ p\ q : P\ u\ q}\ (\equiv\mathsf{E})$$

Exercise. Derive

$$P : A \to \mathcal{U}\ \vdash\ \_ : \Pi[x:A]\ \Pi[y:A]\ \mathtt{Id}\ A\ x\ y \to P\ x \to P\ y$$

Identity types — computation rule

- Computation:

$$\frac{\begin{array}{l} \Gamma \ \vdash \ t : A \\ \Gamma \ \vdash \ P : \Pi[x : A] \ \mathtt{Id}\, A\, t\, x \to \mathcal{U} \\ \Gamma \ \vdash \ p : P\, t\, \mathtt{refl} \end{array}}{\Gamma \ \vdash \ \mathtt{J}\, P\, p\, \mathtt{refl} = p \ \in \ P\, t\, \mathtt{refl}} \ (\equiv\mathsf{C})$$