FLOLAC'14

Type theory and logic

Lecture III: Heyting arithmetic

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Natural numbers

- Formation:

$$\frac{}{\Gamma \ \vdash \ \mathbb{N} : \mathcal{U}} \ (\mathbb{N}\mathsf{F})$$

- Introduction:

$$\frac{}{\Gamma \ \vdash \ \mathtt{zero} : \mathbb{N}} \ (\mathbb{N}\mathsf{IZ}) \qquad \frac{\Gamma \ \vdash \ n : \mathbb{N}}{\Gamma \ \vdash \ \mathtt{suc}\ n : \mathbb{N}} \ (\mathbb{N}\mathsf{IS})$$

Natural numbers — elimination rule

- Elimination:

$$\frac{\begin{array}{l} \Gamma \ \vdash \ P : \mathbb{N} \to \mathcal{U} \\ \Gamma \ \vdash \ z : P\ \texttt{zero} \\ \Gamma \ \vdash \ s : \Pi[x : \mathbb{N}]\ P\,x \to P\,(\texttt{suc}\ x) \\ \Gamma \ \vdash \ n : \mathbb{N} \end{array}}{\Gamma \ \vdash \ \texttt{ind}\ P\,z\,s\,n : P\,n}\ (\mathbb{N}\mathsf{E})$$

Natural numbers — computation rule

- Computation:

$$\frac{\begin{array}{l}\Gamma \ \vdash \ P : \mathbb{N} \to \mathcal{U} \\ \Gamma \ \vdash \ z : P\ \texttt{zero} \\ \Gamma \ \vdash \ s : \Pi[x:\mathbb{N}]\ P\,x \to P\,(\texttt{suc}\ x)\end{array}}{\Gamma \ \vdash \ \texttt{ind}\ P\,z\,s\ \texttt{zero} = z \ \in \ P\ \texttt{zero}} \ (\mathbb{N}\mathsf{CZ})$$

$$\frac{\begin{array}{l}\Gamma \ \vdash \ P : \mathbb{N} \to \mathcal{U} \\ \Gamma \ \vdash \ z : P\ \texttt{zero} \\ \Gamma \ \vdash \ s : \Pi[x:\mathbb{N}]\ P\,x \to P\,(\texttt{suc}\ x) \\ \Gamma \ \vdash \ n : \mathbb{N}\end{array}}{\Gamma \ \vdash \ \texttt{ind}\ P\,z\,s\,(\texttt{suc}\ n) = s\,n\,(\texttt{ind}\ P\,z\,s\,n) \ \in \ P\,(\texttt{suc}\ n)} \ (\mathbb{N}\mathsf{CS})$$

Equality types

- Formation:
$$\frac{\Gamma \vdash A : \mathcal{U} \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash t \equiv_A u : \mathcal{U}} \ (\equiv\mathsf{F})$$
  The subscript A, i.e., the type of t and u, is often omitted.
- Introduction:
$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \mathtt{refl} : t \equiv t} \ (\equiv\mathsf{I})$$

Exercise. Assume $\Gamma \vdash t = u \in A$ and derive $\Gamma \vdash \mathtt{refl} : t \equiv u$.

Equality elimination

$$\frac{\begin{array}{l} \Gamma \ \vdash \ t : A \\ \Gamma \ \vdash \ P : \Pi[x : A] \ t \equiv x \to \mathcal{U} \\ \Gamma \ \vdash \ p : P\, t\, \texttt{refl} \\ \Gamma \ \vdash \ u : A \\ \Gamma \ \vdash \ q : t \equiv u \end{array}}{\Gamma \ \vdash \ \texttt{J}\, p\, q : P\, u\, q} \ (\equiv\mathsf{E})$$

Exercise. Derive

$$P : A \to \mathcal{U} \ \vdash \ \_ : \Pi[x : A] \ \Pi[y : A] \ x \equiv y \to P\, x \to P\, y$$

Equality computation

$$\frac{\begin{array}{l}\Gamma \;\vdash\; t : A \\ \Gamma \;\vdash\; P : \Pi[x : A]\; t \equiv x \to \mathcal{U} \\ \Gamma \;\vdash\; p : P\, t\, \texttt{refl}\end{array}}{\Gamma \;\vdash\; \texttt{J}\, p\, \texttt{refl} = p \;\in\; P\, t\, \texttt{refl}}\;(\equiv\mathsf{C})$$