FLOLAC'14

Type theory and logic

Lecture IV: meta-theoretical reasoning

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Department of Computer Science
University of Oxford

Hsiang-Shang.Ko@cs.ox.ac.uk

Meta-language vs object language

Types and programs form a language, which are talked about by a separate language of judgements and derivations. In this case we call the former the *object language*, and the latter the *meta-language*.

What we write down as types and programs are nothing more than certain syntax trees by themselves; then, at a higher level, we organise and relate these syntax trees with judgements and derivations.

For example, equality judgements are a meta-theoretic notion and cannot be used inside the theory to state equations as provable propositions — we need identity types instead.

Type theory should eat itself

Judgements and derivations can also be regarded as syntax trees to be reasoned about. For example, consistency is a statement in which judgements and derivations are the object language and English is the meta-language. (Canonicity is another example.)

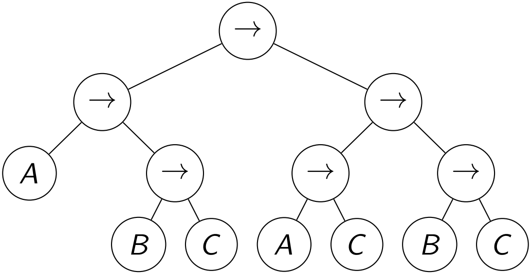
We can also use Agda as the meta-language!

Implicational fragment of propositional logic

Today we consider only propositions formed with implication.

Each of these propositions is a finite tree whose internal nodes are implications and whose leaves are atomic propositions, which are elements of a given set $Var = \{A, B, C, \ldots\}$.

Example. The proposition
$(A \to B \to C) \to (A \to C) \to B \to C$ is represented as



Natural deduction

Natural deduction is the type part of simple type theory (and we are considering only the implicational fragment).

$$\frac{}{\Gamma \ \vdash \ p} \ (\mathsf{assum}) \quad \text{when } p \in \Gamma$$

$$\frac{\Gamma, \ p \ \vdash \ q}{\Gamma \ \vdash \ p \to q} \ (\to\mathsf{I})$$

$$\frac{\Gamma \ \vdash \ p \to q \qquad \Gamma \ \vdash \ p}{\Gamma \ \vdash \ q} \ (\to\mathsf{E})$$

Untyped λ-calculus

A λ-term is either a variable, an abstraction, or an application.

We usually assume *α-equivalence* of λ-terms, i.e., the names of *bound variables* do not matter.

- Change of bound variable names is called α-conversion, which has to be *capture-avoiding*, i.e., *free variables* must not become bound after a name change.
- In formalisation, we prefer not to deal with α-equivalence explicitly, and one way is to use *de Bruijn indices* — λ's are nameless, and a bound variable is represented as a natural number indicating to which λ it is bound.

Simply typed λ-calculus (à la Curry)

λ-calculus was designed to model function abstraction and application in mathematics. In untyped λ-calculus, however, we can write nonsensical terms like $\lambda x.\, x\, x$.

We can use the implicational fragement of propositional logic as a type language for λ-calculus, ruling out nonsensical terms.

$$\frac{}{\Gamma \ \vdash \ x : p}\,(\mathsf{var}) \quad \text{when } x : p \in \Gamma$$

$$\frac{\Gamma, x : p \ \vdash \ t : q}{\Gamma \ \vdash \lambda x.\, t : p \to q}\,(\mathsf{abs})$$

$$\frac{\Gamma \ \vdash \ t : p \to q \qquad \Gamma \ \vdash \ u : p}{\Gamma \ \vdash \ t\, u : q}\,(\mathsf{app})$$

Haskell Curry

picture stolen from http://bodil.org/hipster

Curry–Howard isomorphism

Derivations in natural deduction and well-typed λ-terms are in one-to-one correspondence.

That is, we can write two functions,

- one mapping a logical derivation in natural deduction to a λ-term and its typing derivation, and
- the other mapping a λ-term with a typing derivation to a logical derivation in natural deduction,

and can prove that the two functions are inverse to each other.

This result is historically significant: two formalisms are developed separately from logical and computational perspectives, yet they coincide perfectly.

Simply typed λ-calculus à la Church

The Curry–Howard isomorphism points out that derivations in natural deduction are actually λ-terms in disguise.

These λ-terms are intrinsically typed, so every term we are able to write down is necessarily well-behaved, whereas in simply typed λ-calculus à la Curry, we can write arbitrary λ-terms, and only rule out ill-behaved ones via typing later.

Alonzo Church

picture stolen from `http://bodil.org/hipster`

The Curry–Howard isomorphism

picture stolen from `http://bodil.org/hipster`

Semantics

After defining a language (like the implicational fragment of propositional logic), which consists of a bunch of syntax trees, we need to specify what these trees mean.

Judgements and derivations (which form a *deduction system*) assign meaning to the propositional language by specifying how it is used in formal reasoning.

We can also translate the syntax trees into entities in a well understood semantic domain. In the case of propositional logic, we can translate propositional trees to functions on truth values. (This is the classical treatment.)

Two-valued semantics of propositional logic

- Define $\mathtt{Bool} := \{\mathtt{false}, \mathtt{true}\}$.
- An *assignment* is a function of type $V \rightarrow \mathtt{Bool}$.
- A proposition p is translated into a function $[\![p]\!] : (V \rightarrow \mathtt{Bool}) \rightarrow \mathtt{Bool}$ mapping assignments to truth values.
- An assignment σ *models* a proposition p exactly when $[\![p]\!]\ \sigma$ is true, and *models* a context Γ exactly when it models every proposition in Γ.

Two-valued semantics of propositional logic

- A proposition p is *satisfiable* exactly when there exists an assignment that models p, and is *valid* exactly when every assignment models p.
- A proposition p is a *semantic consequence* of a context Γ (written $\Gamma \models p$) exactly when every assignment that models Γ also models p.

Exercise. Show that $(p \to p \to q) \to (p \to q)$ is valid for any propositions p and q.

Exercise. Show that a proposition p is valid if and only if p is a semantic consequence of the empty context.

Relationship between deduction systems and semantics

Natural deduction is *sound* with respect to the two-valued semantics: whenever we can deduce $\Gamma \vdash p$, it must be the case that $\Gamma \models p$.

The implicational fragment of propositional logic is also *(semantically) complete* with respect to the two-valued semantics: if $\Gamma \models p$, then we can construct a derivation of $\Gamma \vdash p$.