FLOLAC'14

Type theory and logic

Lecture III: Heyting arithmetic

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Natural numbers

- Formation:

$$\frac{}{\Gamma \vdash \mathbb{N} : \mathcal{U}} \text{ (}\mathbb{N}\text{F)}$$

- Introduction:

$$\frac{}{\Gamma \vdash \texttt{zero} : \mathbb{N}} \text{ (}\mathbb{N}\text{IZ)} \qquad \frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash \texttt{suc}\ n : \mathbb{N}} \text{ (}\mathbb{N}\text{IS)}$$

Natural numbers — elimination rule

- Elimination:

$$\frac{\begin{array}{lll}\Gamma & \vdash & P : \mathbb{N} \to \mathcal{U} \\ \Gamma & \vdash & z : P\ \mathtt{zero} \\ \Gamma & \vdash & s : \Pi[x : \mathbb{N}]\ P\,x \to P\,(\mathtt{suc}\ x) \\ \Gamma & \vdash & n : \mathbb{N}\end{array}}{\Gamma \ \vdash\ \mathtt{ind}\ P\,z\,s\,n : P\,n}\ (\mathbb{N}\mathrm{E})$$

Logically this is the *induction principle*; computationally this is *primitive recursion*.

Natural numbers — computation rules

- Computation:

$$\frac{\begin{array}{l}\Gamma \ \vdash \ P : \mathbb{N} \to \mathcal{U} \\ \Gamma \ \vdash \ z : P\ \mathtt{zero} \\ \Gamma \ \vdash \ s : \Pi[x : \mathbb{N}]\ P\,x \to P\,(\mathtt{suc}\ x)\end{array}}{\Gamma \ \vdash \ \mathtt{ind}\ P\,z\,s\ \mathtt{zero} = z \ \in \ P\ \mathtt{zero}}\ (\mathbb{N}\mathrm{CZ})$$

$$\frac{\begin{array}{l}\Gamma \ \vdash \ P : \mathbb{N} \to \mathcal{U} \\ \Gamma \ \vdash \ z : P\ \mathtt{zero} \\ \Gamma \ \vdash \ s : \Pi[x : \mathbb{N}]\ P\,x \to P\,(\mathtt{suc}\ x) \\ \Gamma \ \vdash \ n : \mathbb{N}\end{array}}{\Gamma \ \vdash \ \mathtt{ind}\ P\,z\,s\,(\mathtt{suc}\ n) = s\,n\,(\mathtt{ind}\ P\,z\,s\,n) \ \in \ P\,(\mathtt{suc}\ n)}\ (\mathbb{N}\mathrm{CS})$$

Equality types

- Formation:

$$\frac{\Gamma \vdash A : \mathcal{U} \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash t \equiv_A u : \mathcal{U}} \ (\equiv\mathsf{F})$$

  The subscript A, i.e., the type of t and u, is often omitted.

- Introduction:

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \mathtt{refl} : t \equiv t} \ (\equiv\mathsf{I})$$

Exercise. Assume $\Gamma \vdash t = u \in A$ and derive $\Gamma \vdash \mathtt{refl} : t \equiv u$.

Equality types — elimination rule

- Elimination:

$$\frac{\begin{array}{l}\Gamma \ \vdash \ t : A \\ \Gamma \ \vdash \ P : \Pi[x : A] \ t \equiv x \to \mathcal{U} \\ \Gamma \ \vdash \ p : P\,t\,\texttt{refl} \\ \Gamma \ \vdash \ u : A \\ \Gamma \ \vdash \ q : t \equiv u\end{array}}{\Gamma \ \vdash \ \texttt{J}\,P\,p\,q : P\,u\,q}\ (\equiv\text{E})$$

Exercise. Derive

$$P : A \to \mathcal{U} \ \vdash \ \_ : \Pi[x : A] \ \Pi[y : A] \ x \equiv y \to P\,x \to P\,y$$

Equality types — computation rule

- Computation:

$$\frac{\begin{array}{l}\Gamma \ \vdash\ t : A \\ \Gamma \ \vdash\ P : \Pi[x:A]\ t \equiv x \to \mathcal{U} \\ \Gamma \ \vdash\ p : P\, t\, \mathtt{refl}\end{array}}{\Gamma \ \vdash\ \mathtt{J}\, P\, p\, \mathtt{refl} = p \ \in\ P\, t\, \mathtt{refl}}\ (\equiv\mathsf{C})$$