

# Type theory and logic

## Lecture II: dependent type theory

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## Indexed families of sets (predicates)

Common mathematical statements involve predicates and universal/existential quantification.

For example: “For every  $x : \mathbb{N}$ , if  $x$  is not zero, then there exists  $y : \mathbb{N}$  such that  $x$  is equal to  $1 + y$ .”

In type theory, a predicate on  $A$  has type  $A \rightarrow \mathcal{U}$  — a *family of sets* indexed by the domain  $A$ . For example:

$$\vdash \lambda x. \text{“if } x \text{ is zero then } \mathbf{0} \text{ else } \mathbf{1}\text{”} : \mathbb{N} \rightarrow \mathcal{U}$$

**Remark.** The above treatment is in fact unfounded in our current theory. Why?

## Dependent product types (universal quantification)

- Formation:

$$\frac{\Gamma \vdash A : \mathcal{U} \quad \Gamma \vdash B : A \rightarrow \mathcal{U}}{\Gamma \vdash \Pi A B : \mathcal{U}} \text{ (}\Pi\text{F)}$$

- Introduction:

$$\frac{\Gamma, x : A \vdash t : B x}{\Gamma \vdash \lambda x. t : \Pi A B} \text{ (}\Pi\text{I)}$$

- Elimination:

$$\frac{\Gamma \vdash f : \Pi A B \quad \Gamma \vdash a : A}{\Gamma \vdash f a : B a} \text{ (}\Pi\text{E)}$$

**Notation.** We usually write  $\Pi[x : A] B x$  for  $\Pi A B$ , regarding ' $\Pi[x : A]$ ' as a quantifier.

**Exercise.** Let  $\Gamma := A : \mathcal{U}, B : \mathcal{U}, C : A \rightarrow B \rightarrow \mathcal{U}$ . Derive

$$\Gamma \vdash \_ : (\Pi[x : A] \Pi[y : B] C x y) \rightarrow \Pi[y : B] \Pi[x : A] C x y$$

## Dependent sum types (existential quantification)

- Formation:

$$\frac{\Gamma \vdash A : \mathcal{U} \quad \Gamma \vdash B : A \rightarrow \mathcal{U}}{\Gamma \vdash \Sigma A B : \mathcal{U}} \quad (\Sigma F)$$

- Introduction:

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B a}{\Gamma \vdash (a, b) : \Sigma A B} \quad (\Sigma I)$$

- Elimination:

$$\frac{\Gamma \vdash p : \Sigma A B}{\Gamma \vdash \text{fst } p : A} \quad (\Sigma EL) \quad \frac{\Gamma \vdash p : \Sigma A B}{\Gamma \vdash \text{snd } p : B (\text{fst } p)} \quad (\Sigma ER)$$

**Notation.** We usually write  $\Sigma[x : A] B x$  for  $\Sigma A B$ .

**Exercise.** Let  $\Gamma := A : \mathcal{U}, B : \mathcal{U}, C : A \rightarrow B \rightarrow \mathcal{U}$ . Derive

$$\Gamma \vdash \_ : (\Sigma[p : A \times B] C (\text{fst } p) (\text{snd } p)) \rightarrow \Sigma[x : A] \Sigma[y : B] C x y$$

## Exercises

Let  $\Gamma := A : \mathcal{U}, B : A \rightarrow \mathcal{U}, C : A \rightarrow \mathcal{U}$ . Derive

$$\Gamma \vdash \_ : (\Pi[x:A] B\ x \times C\ x) \leftrightarrow (\Pi[x:A] B\ x) \times (\Pi[x:A] C\ x)$$

$$\Gamma \vdash \_ : (\Sigma[x:A] B\ x + C\ x) \leftrightarrow (\Sigma[x:A] B\ x) + (\Sigma[x:A] C\ x)$$

What about

$$\Gamma \vdash \_ : (\Pi[x:A] B\ x + C\ x) \leftrightarrow (\Pi[x:A] B\ x) + (\Pi[x:A] C\ x)$$

$$\Gamma \vdash \_ : (\Sigma[x:A] B\ x \times C\ x) \leftrightarrow (\Sigma[x:A] B\ x) \times (\Sigma[x:A] C\ x)$$

?

Now let  $\Gamma := A : \mathcal{U}, B : \mathcal{U}, R : A \rightarrow B \rightarrow \mathcal{U}$ . Derive the *axiom of choice*:

$$\begin{aligned} \Gamma \vdash \_ : (\Pi[x:A] \Sigma[y:B] R\ x\ y) \rightarrow \\ \Sigma[f:A \rightarrow B] \Pi[x:A] R\ x\ (f\ x) \end{aligned}$$

# Universes

In our current theory, to form types like  $A \rightarrow \mathcal{U}$  where  $A : \mathcal{U}$ , we need to assume  $\mathcal{U} : \mathcal{U}$ . However, this assumption — called *impredicativity* — was shown to lead to inconsistency by Jean-Yves Girard. (This result is commonly referred to as *Girard's paradox*.)

We thus need to introduce a *predicative* hierarchy of universes  $\mathcal{U}_0, \mathcal{U}_1, \dots$ , up to infinity.

- Smaller universes are elements of larger universes.

$$\frac{}{\Gamma \vdash \mathcal{U}_i : \mathcal{U}_{i+1}} (\mathcal{U}\text{F})$$

- Elements of smaller universes are also elements of larger universes. This is called *cumulativity*.

$$\frac{\Gamma \vdash A : \mathcal{U}_i}{\Gamma \vdash A : \mathcal{U}_{i+1}} (\text{cum})$$

## Universe polymorphism

Formation rules and elimination rules have to be revised to work across the universe hierarchy.

- For example, the formation rule for coproducts becomes

$$\frac{\Gamma \vdash A : \mathcal{U}_i \quad \Gamma \vdash B : \mathcal{U}_j}{\Gamma \vdash A + B : \mathcal{U}_{\max i j}} (+F)$$

- The elimination rule for coproducts is

$$\frac{\Gamma \vdash q : A + B \quad \Gamma \vdash f : A \rightarrow C \quad \Gamma \vdash g : B \rightarrow C}{\Gamma \vdash \text{case } q f g : C} (+E)$$

for which we implicitly assume the following judgements:

$$\Gamma \vdash A : \mathcal{U}_i \quad \Gamma \vdash B : \mathcal{U}_j \quad \Gamma \vdash C : \mathcal{U}_k$$

In practice, however, we can drop the indices (as if assuming  $\mathcal{U} : \mathcal{U}$ ) because they can be inferred most of the time.

## Computation

Let  $\Gamma := A : \mathcal{U}, B : A \rightarrow \mathcal{U}, C : A \rightarrow \mathcal{U}$ . Try to derive

$$\Gamma \vdash \_ : (\Pi[p : \Sigma A B] C (\text{fst } p)) \rightarrow \Pi[x : A] B x \rightarrow C x$$

... and you should notice some problems.

So far we have been concentrating on the *statics* of type theory; here we need to formally invoke the *dynamics* of the theory.



## Equality judgements

We introduce a new kind of judgements stating that two terms should be regarded as the same during typechecking:

$$\Gamma \vdash t = u \in A$$

for which we also have a well-formedness requirement that  $A$  and everything appearing on the right of the colons in  $\Gamma$  are judged to be sets, and  $t$  and  $u$  are judged to be elements of  $A$ .

## Computation rules

For each set, (when applicable) we specify additional *computation rules* stating that eliminating an introductory term yields a component of the latter. This is the type-theoretic manifestation of *Gentzen's inversion principle*.

For example, for product types we have two computation rules:

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B}{\Gamma \vdash \text{fst}(a, b) = a \in A} (\times\text{CL}) \quad \frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B}{\Gamma \vdash \text{snd}(a, b) = b \in B} (\times\text{CR})$$

## More computation rules

$$\frac{\Gamma, x : A \vdash t : B \quad \Gamma \vdash a : A}{\Gamma \vdash (\lambda x. t) a = t[a/x] \in A \rightarrow B} (\rightarrow C)$$

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash f : A \rightarrow C \quad \Gamma \vdash g : B \rightarrow C}{\Gamma \vdash \text{case}(\text{left } a) fg = fa \in C} (+CL)$$

$$\frac{\Gamma \vdash b : B \quad \Gamma \vdash f : A \rightarrow C \quad \Gamma \vdash g : B \rightarrow C}{\Gamma \vdash \text{case}(\text{right } b) fg = gb \in C} (+CR)$$

## More computation rules

$$\frac{\Gamma, x : A \vdash t : B \quad \Gamma \vdash a : A}{\Gamma \vdash (\lambda x. t) a = t[a/x] \in B a} \text{ (}\Pi\text{C)}$$

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B a}{\Gamma \vdash \text{fst}(a, b) = a \in A} \text{ (}\Sigma\text{CL)}$$

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B a}{\Gamma \vdash \text{snd}(a, b) = b \in B a} \text{ (}\Sigma\text{CR)}$$

## Equivalence rules

Judgemental equality is an equivalence relation.

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash t = t \in A} \text{ (refl)}$$

$$\frac{\Gamma \vdash t = u \in A}{\Gamma \vdash u = t \in A} \text{ (sym)}$$

$$\frac{\Gamma \vdash t = u \in A \quad \Gamma \vdash u = v \in A}{\Gamma \vdash t = v \in A} \text{ (trans)}$$

## Congruence rules

We need a congruence rule for each constant we introduce:

$$\frac{\Gamma \vdash a = a' \in A \quad \Gamma \vdash b = b' \in B}{\Gamma \vdash (a, b) = (a', b') \in A \times B}$$

$$\frac{\Gamma \vdash p = p' \in A \times B}{\Gamma \vdash \text{fst } p = \text{fst } p' \in A} \quad \frac{\Gamma \vdash p = p' \in A \times B}{\Gamma \vdash \text{snd } p = \text{snd } p' \in B}$$

$$\frac{\Gamma, x : A \vdash t = t' \in B}{\Gamma \vdash \lambda x. t = \lambda x. t' \in A \rightarrow B}$$

$$\frac{\Gamma \vdash f = f' \in A \quad \Gamma \vdash a = a' \in A}{\Gamma \vdash f a = f' a' \in B}$$

... and similar rules for left, right, case, and absurd.

## Conversion rule

Once we establish that two sets are judgementally equal, we can transfer terms between the two sets.

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash A = B \in \mathcal{U}}{\Gamma \vdash t : B} \text{ (conv)}$$

**Exercise.** Finish deriving

$$\Gamma \vdash \_ : (\Pi[p : \Sigma A B] C(\text{fst } p)) \rightarrow \Pi[x : A] B x \rightarrow C x$$

(where  $\Gamma := A : \mathcal{U}, B : A \rightarrow \mathcal{U}, C : A \rightarrow \mathcal{U}$ ).

## More congruence rules

We can state congruence rules for dependent products and sums only after we have the conversion rule. Why?

$$\frac{\Gamma \vdash a = a' \in A \quad \Gamma \vdash b = b' \in B a}{\Gamma \vdash (a, b) = (a', b') \in \Sigma A B}$$

$$\frac{\Gamma \vdash p = p' \in \Sigma A B}{\Gamma \vdash \text{fst } p = \text{fst } p' \in A} \quad \frac{\Gamma \vdash p = p' \in \Sigma A B}{\Gamma \vdash \text{snd } p = \text{snd } p' \in B(\text{fst } p)}$$

$$\frac{\Gamma, x : A \vdash t = t' \in B x}{\Gamma \vdash \lambda x. t = \lambda x. t' \in \Pi A B}$$

$$\frac{\Gamma \vdash f = f' \in A \quad \Gamma \vdash a = a' \in A}{\Gamma \vdash f a = f' a' \in B a}$$



## Decidability of judgemental equality

Our judgemental equality is *decidable*: for any equality judgement we can decide whether it has a derivation or not.

Decidability is achieved by *reducing* terms to their *normal forms* and see if the normal forms match.

Next week's *operational semantics* will discuss term reduction in detail, where we will see various *reduction strategies*.

Judgemental equality is formulated without reference to any particular reduction strategy — it captures the notion of computation only abstractly.

## Canonical vs non-canonical elements

Introduction rules specify *canonical* — or *immediately recognisable* — elements of a set.

A complex construction may not be immediately recognisable as belonging to a set, but as long as we can see that it *computes* to a canonical element, we accept it as a *non-canonical* element of the set.

**Property (canonicity).** If  $\vdash t : A$ , then  $\vdash t = c \in A$  for some canonical element  $c$ .

## Non-derivable proposition

Let  $\Gamma := A : \mathcal{U}, B : A \rightarrow \mathcal{U}$ . We can derive

$$\Gamma \vdash \_ : (\Sigma[x:A] B x) \rightarrow (\neg \Pi[x:A] \neg B x)$$

but not

$$\Gamma \vdash \_ : (\neg \Pi[x:A] \neg B x) \rightarrow (\Sigma[x:A] B x)$$

**Exercise.** Assuming that there is an additional rule

$$\frac{\Gamma \vdash X : \mathcal{U}}{\Gamma \vdash \text{LEM } X : X + \neg X} \text{ (LEM)}$$

derive

$$\Gamma \vdash \_ : (\neg \Pi[x:A] \neg B x) \rightarrow (\Sigma[x:A] B x)$$

## Classical logic as an extension of intuitionistic logic

By including the LEM rule, our logic system can now derive intuitionistically invalid but classically valid propositions.

The LEM rule breaks canonicity, however (why?) — the system is no longer computationally meaningful.

We are actually abusing the intuitionistic system to state classical facts, though.

## Classical logic as a sub-system of intuitionistic logic

The *Gödel–Gentzen negative translation* embeds classical logic into intuitionistic logic by

- putting double negation in front of “atomic propositions”,
- replacing ‘ $\Sigma[x:A] \dots$ ’ with ‘ $\neg \Pi[x:A] \neg \dots$ ’, and
- replacing ‘ $\dots + \dots$ ’ with ‘ $\neg(\neg \dots \times \neg \dots)$ ’.

**Exercise.** Let  $\Gamma := A : \mathcal{U}, a : A, D : A \rightarrow \mathcal{U}$ . (Note that  $A$  is an inhabited set.) Derive

$$\Gamma \vdash \_ : \neg \Pi[x:A] \neg (\neg \neg D x \rightarrow \Pi[y:A] \neg \neg D y)$$

which is the Gödel–Gentzen negative translation of

$$\exists[x:A] D x \rightarrow \forall[y:A] D y$$