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University of Illinois Spring 2020

CS 446/ECE 449 Machine Learning
Homework 8: Variational Auto-Encoders

Due on Tuesday April 21 2020, noon Central Time

1. **[17 points]** Variational Auto-Encoders (VAEs)

   (a) (3 points) We want to maximize the log-likelihood $\log p_\theta(x)$ of a model $p_\theta(x)$ which is parameterized by θ. To this end we introduce a joint distribution $p_\theta(x, z)$ and an approximate posterior $q(z|x)$ and reformulate the log-likelihood via

   $$\log p_\theta(x) = \log \sum_z q(z|x) \frac{p_\theta(x,z)}{q(z|x)}.$$

   Use Jensen's inequality to obtain a bound on the log likelihood and divide the bound into two parts, one of which is the Kullback-Leibler (KL) divergence

   $$\mathrm{KL}(q(z|x), p(z)).$$

   Solution:

   $$\log p_\theta(x) \geq \sum_z q(z|x) \log \frac{p_\theta(x,z)}{q(z|x)} = \sum_z q(z|x) \log p_\theta(x|z) - \mathrm{KL}(q(z|x), p(z))$$

   (b) (2 points) State at least two properties of the KL divergence.

   Solution: non-negative, zero if $q(z|x) = p(z)$

   (c) (2 points) Let

   $$q(z|x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(z-\mu_q)^2\right).$$

   What is the value for the KL-divergence $\mathrm{KL}(q(z|x), q(z|x))$ and why?

   Solution: 0, KL-divergence between two identical distributions

   (d) (3 points) Further, let

   $$p(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(z-\mu_p)^2\right).$$

   Note the difference of the means for $p(z)$ and $q(z|x)$ while their standard deviation is identical. What is the value for the KL-divergence $\mathrm{KL}(q(z|x), p(z))$ in terms of μ_p, μ_q and σ?

   Solution:

   $$\mathrm{KL}(q(z|x), p(z)) = \frac{1}{2\sigma^2}(\mu_q - \mu_p)^2$$

   (e) (4 points) Now, let $q(z|x)$ and $p(z)$ be arbitrary probability distributions. We want to find that $q(z|x)$ which maximizes

   $$\sum_z q(z|x) \log p_\theta(x|z) - \mathrm{KL}(q(z|x), p(z))$$

Name: ____________________

subject to $\sum_z q(z|x) = 1$. Ignore the non-negativity constraints. State the Lagrangian and compute its stationary point, *i.e.*, solve for $q(z|x)$ which depends on $p_\theta(x|z)$ and $p(z)$. Make sure to get rid of the Lagrange multiplier.

Solution:

$$\sum_z q(z|x) \log p_\theta(x|z) - \sum_z q(z|x) \log \frac{q(z|x)}{p(z)} + \lambda \left(\sum_z q(z|x) - 1 \right)$$

$$q(z|x) = \frac{p_\theta(x|z)p(z)}{\sum_{\hat{z}} p_\theta(x|\hat{z})p(\hat{z})}$$

(f) (1 point) Which of the following terms should $q(z|x)$ be equal to: (1) $p(z)$; (2) $p_\theta(x|z)$; (3) $p_\theta(z|x)$; (4) $p_\theta(x, z)$.

Solution:

$$p_\theta(z|x) = \frac{p_\theta(x|z)p(z)}{\sum_{\hat{z}} p_\theta(x|\hat{z})p(\hat{z})}$$

(g) (2 points) Provide the code for implementing the 'reparameterize' function in `A8_VAE.py`.

Solution:

```
def reparameterize(self, mu, logvar):
        std = torch.exp(0.5*logvar)
        eps = torch.randn_like(std)
        return eps.mul(std).add_(mu)
```