Homework 4
ECE/CS 498 DS Spring 2020
Issued: 04/15/20
Due: 04/22/20 @ 11:59 PM on Compass

Name: ____________________
NetID: ____________________

MM Forward-Backward Algorithm

You will implement the forward-backward algorithm for HMMs.

Part 1

Files:

HMM.py

HMM_example.py

What to submit:

A modified HMM.py with your implementation.

See HMM_sol.py

You will need to fill in the missing code in HMM.py:

- ***def forward_algorithm*****:** calculate $P(S_t|E_1, E_2, ..., E_t)$, the probability of the hidden state at time t given the observation(s) up to time t
- ***def backward_algorithm*****:** calculate $P(E_{t+1}, ..., E_n|S_t)$, the probability of the future observation(s) given the hidden state at time t
- ***def forward_backward:*** calculate $P(S_t|E_1, E_2, ..., E_n)$, the probability of the hidden state at time t given all the observations

In HMM_example.py, we provide the security example you solved in ICA4. You can use it to test your implementation.

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Part 2

In this part, you are required to build an HMM model and then do inference based on the forward-backward algorithm you implemented in Part 1. The parameters for HMM are provided as below:

Transition probability matrix A:

	A	B	C	D
A	0.15	0.25	0.25	0.35
B	0.6	0.2	0.1	0.1
C	0.25	0.2	0.3	0.25
D	0.1	0.4	0.4	0.1

Observation matrix B:

	e0	e1	e2	e3	e4
A	0.6	0.1	0.1	0.1	0.1
B	0.1	0.6	0.1	0.1	0.1
C	0.1	0.2	0.2	0.2	0.3
D	0	0	0	0.5	0.5

The initial distribution of hidden states π:

A	B	C	D
0.25	0.25	0.25	0.25

Observations:

t=1	t=2	t=3	t=4	t=5	t=6
e4	e3	e2	e2	e0	e1

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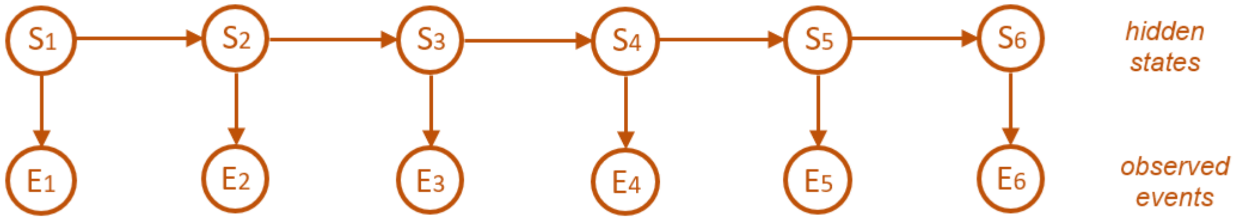
1. Draw a diagram for the HMM specified as above. (2points)



2. Based on the forward-backward algorithm you implemented in Part 1, provide the most likely sequence of the hidden states for the HMM. For partial credit, please also provide $P(S_t|E_1, E_2, \dots, E_t)$ (Alpha), $P(E_{t+1}, \dots, E_n|S_t)$ (Beta), and $P(S_t|E_1, E_2, \dots, E_n)$ (Gamma). (8 points)

Inference:

t=1	t=2	t=3	t=4	t=5	t=6
D	D	C	C	A	B

Alpha:

	A	B	C	D
t=1	0.1	0.1	0.3	0.5
t=2	0.0997506	0.15212	0.32419	0.42394
t=3	0.20264	0.255666	0.541694	0
t=4	0.317056	0.208708	0.474235	0
t=5	0.792297	0.0978342	0.109869	0
t=6	0.0965064	0.67677	0.226723	0

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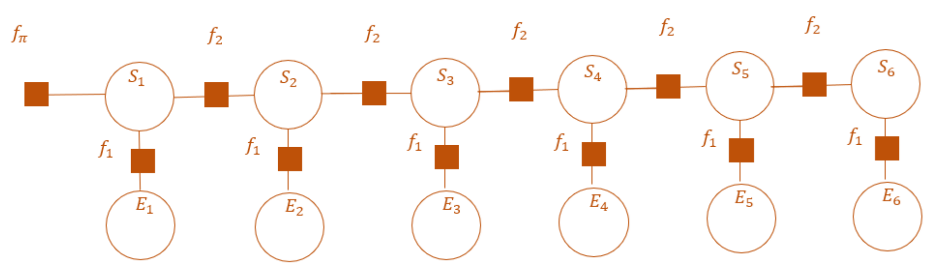
Beta:

	A	B	C	D
t=1	0.000150333	7.54065e-05	0.000126553	9.33956e-05
t=2	0.000424357	0.00046361	0.000498912	0.000613894
t=3	0.00464837	0.0042855	0.00494987	0.00702475
t=4	0.029475	0.08345	0.0424	0.0291
t=5	0.215	0.2	0.205	0.33
t=6	1	1	1	1

Gamma:

	A	B	C	D
t=1	0.140187	0.0703173	0.354034	0.435462
t=2	0.0791434	0.131858	0.302406	0.486593
t=3	0.199611	0.232183	0.568206	0
t=4	0.199388	0.3716	0.429012	0
t=5	0.801868	0.092108	0.106024	0
t=6	0.0965064	0.67677	0.226723	0

3. Draw the factor graph converted by the given HMM. (2 points)



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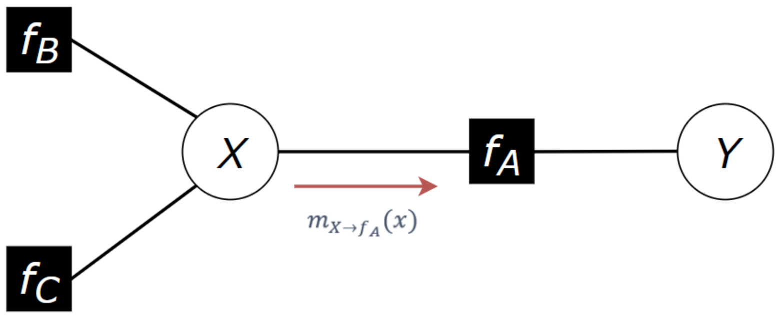
Factor Graphs & Belief Propagation

Problem 1. Consider the following factor graph.



Factor Graph (1)

Factor function values for f_A, f_B, and f_C are:

X	Y	f_A
0	0	0.3
0	1	0.1
1	0	0.4
1	1	0.2

X	f_B
0	0.4
1	0.6

X	f_C
0	0.3
1	0.7

1. Write the expression for $P(Y)$ in terms of $m_{X\to f_A}(x)$ and f_A. (1 points)

$$P(Y) = \frac{\sum_X m_{X\to f_A}(X) f_A(X,Y)}{\sum_{X,Y} m_{X\to f_A}(X) f_A(X,Y)}$$

2. First, calculate the message $m_{X\to f_A}(x)$ based on the tables provided, then calculate the value of $P(Y)$. Show all steps of your work. (3 points)

$$m_{X\to f_A}(x) = \begin{array}{c c} & x \\ 0 & \multirow{2}{*}{$\begin{bmatrix} 0.12 \\ 0.42 \end{bmatrix}$} \\ 1 & \end{array}$$

$$m_{X\to f_A}(X) = m_{f_B\to X}(X) m_{f_C\to X}(X) = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \circ \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 0.12 \\ 0.42 \end{bmatrix}$$

$P(Y)$ calculation:

$$P(Y=0) \propto \sum_X m_{X \to f_A}(X) f_A(X,0)$$
$$= m_{X \to f_A}(0) f_A(0,0) + m_{X \to f_A}(1) f_A(1,0)$$
$$= 0.12 \cdot 0.3 + 0.42 \cdot 0.4 = 0.036 + 0.168 = 0.204$$

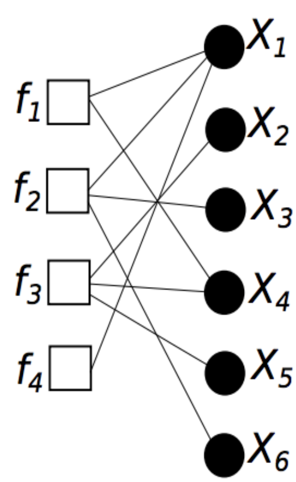
$$P(Y=1) \propto \sum_X m_{X \to f_A}(X) f_A(X,1)$$
$$= m_{X \to f_A}(0) f_A(0,1) + m_{X \to f_A}(1) f_A(1,1)$$
$$= 0.12 \cdot 0.1 + 0.42 \cdot 0.2 = 0.012 + 0.084 = 0.096$$

$$P(Y) = \frac{1}{0.204 + 0.096} \begin{bmatrix} 0.204 \\ 0.096 \end{bmatrix} = \begin{bmatrix} 0.68 \\ 0.32 \end{bmatrix}$$

Problem 2. Given a function that can be factorized as follows.

$$f(X_1, X_2, X_3, X_4, X_5, X_6) = f_1(X_1, X_4) f_2(X_1, X_3, X_6) f_3(X_2, X_4, X_5) f_4(X_1)$$

1. Draw the corresponding FG. (2 points)



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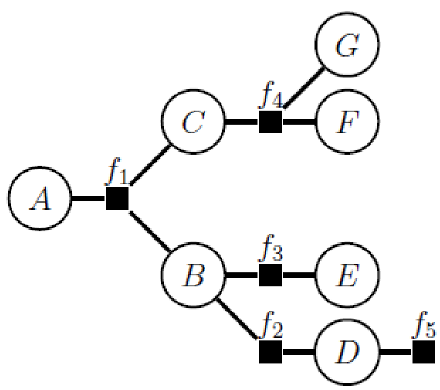
2. Assume that X_1 is the hidden state that we are interested in. Write the formula for computing the marginal of X_1. (2 points)

$$P_{X_1}(X_1) = \frac{1}{Z}\sum_{\sim\{X_1\}} f_1(X_1,X_4)f_2(X_1,X_3,X_6)f_3(X_2,X_4,X_5)f_4(X_1)$$

$$or$$

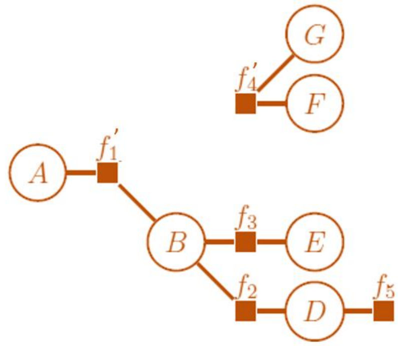
$$P_{X_1}(X_1) = \frac{1}{Z}f_4(X_1)\sum_{\{X_3,\,X_6\}} f_2(X_1,X_3,X_6)\left(\sum_{\{X_4\}} f_1(X_1,X_4)\sum_{\{X_2,X_5\}} f_3(X_2,X_4,X_5)\right)$$

Problem 3. For the Factor Graph given below, which of the following conditional independence relations is true? Justify your answer by drawing the factor graph after removing the observed nodes. (For example, in $F \perp\!\!\!\perp G \mid C$, since C is observed, remove node C and draw the modified factor graph. Hint: the resulting factor graph after removing the node may be disconnected.)



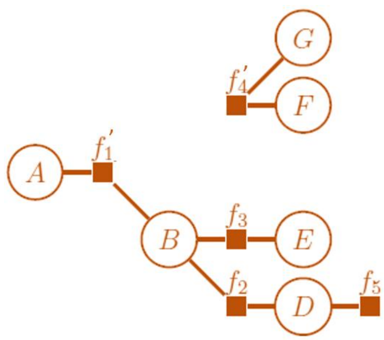
a) $F \perp\!\!\!\perp G \mid C$ (2 points)

No. (Even after observing C, there is a path from F to G. Therefore, they are not conditionally independent.)



b) $A \perp\!\!\!\perp G \mid C$ (2 points)

Yes. (On observing C, there is no path from A to G. Therefore, they are conditionally independent.)



c) $E \perp\!\!\!\perp F \mid B, A, D$ (2 points)

Yes. (On observing B, there is no path from E to F. Observing A and D has no effect on conditional independence of E and F.)