9831 Probability and Stochastic Processes for Finance, Fall 2016

Lecture 4

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Outline

- Quadratic variation
- Wiener integral
- Ito integral

1 Quadratic variation

Recall from Lecture 3:

- Let $f, g : [0,T] \to \mathbb{R}$, Π be a partition of $[0,T]$: $0 = t_0 < t_1 < \cdots < t_n = T$, and

  $$\|\Pi\| := \max_{i \in \{1,2,...,n\}} (t_i - t_{i-1})$$

  be the length of the largest interval in the partition Π.

- The **cross-variation of f and g on $[0,T]$** is defined by

  $$[f,g]_T := \lim_{\|\Pi\| \to 0} \sum_{i=1}^{n} (f(t_i) - f(t_{i-1}))(g(t_i) - g(t_{i-1}))$$

  if this limit exists.

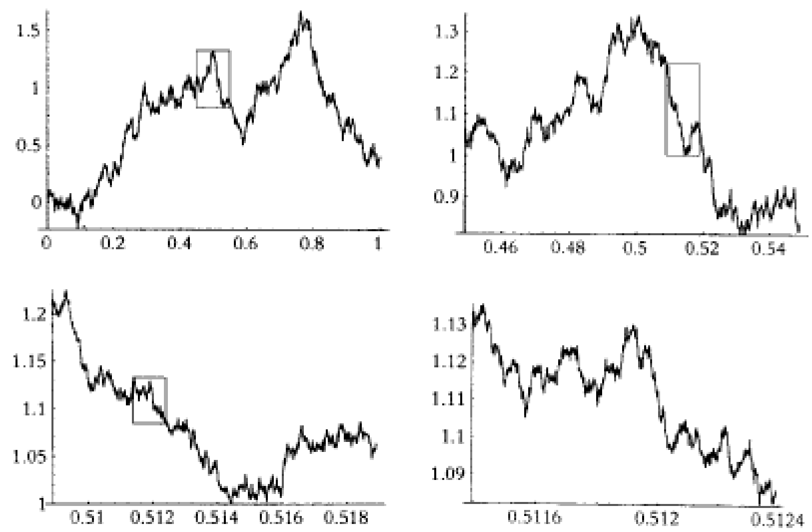
- The **quadratic variation $[f]_T$ of f on $[0,T]$** is defined by

$$[f]_T := [f,f]_T,$$

when $[f,f]_T$ exists.

Quadratic variation of Brownian motion

- From the examples at the end of the last lecture, we know that for "nice" functions the cross and quadratic variations are 0. However, if f and g are "rough", for example, $f(t)=g(t)=B_t$ then the situation is different.



Zooming in on Brownian motion.

- If we consider stochastic processes, the above limit can be understood in different senses (a.s., in mean square, in probability, in distributions), and the existence of the limit might depend on the interpretation of the limit.

  We shall work with mean square limits (also called L^2 limits), since they happen to be the most natural ones for the stochastic calculus setting.

Recall:

Definition

Let $(X_n)_{n\geq 1}$, be a sequence of r.v.s on some probability space $(\Omega,\mathcal{F},P)$. We say that X_n converges to the r.v. X in L^2 (or in mean square) if

$$\lim_{n\to\infty} E\big[(X_n-X)^2\big]=0.$$

Theorem

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Then we have
$$[B]_T = T,$$

for all $T \geq 0$.

We understand the limit in the L^2 sense, i.e.
$$\lim_{||\Pi||\to 0} E\Big[\Big(\sum_{i=1}^{n}(B_{t_i} - B_{t_{i-1}})^2 - T\Big)^2\Big] = 0.$$

Remark:

- The statement above can be interpreted as follows: Brownian motion accumulates quadratic variation at rate 1 per unit of time $(T = 1)$.
- The quadratic variation for Brownian motion is non-random.
- Just as in the case of "nice" functions, the above theorem gives a precise meaning to the informal expression

$$dB_t dB_t = dt.$$

- The quadratic variation of a Brownian motion on a finite interval $[a, b]$ is given as

$$[B]_{[a,b]} = b - a.$$

Proof:

Let $0 = t_0 < t_1 < \cdots < t_n = T$ and $\delta_i := (B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})$.

Then

$$\begin{aligned}\left(\sum_{i=1}^{n}(B_{t_i} - B_{t_{i-1}})^2 - T\right)^2 &= \left(\sum_{i=1}^{n}(B_{t_i} - B_{t_{i-1}})^2 - \sum_{i=1}^{n}(t_i - t_{i-1})\right)^2 \\ &= \left(\sum_{i=1}^{n}\delta_i\right)^2 \\ &= \sum_{i=1}^{n}\delta_i^2 + 2\sum_{i<j}\delta_i\delta_j.\end{aligned} \tag{1}$$

Since Brownian motion has independent increments we have for $i \neq j$

$$E(\delta_i\delta_j) = E(\delta_i)E(\delta_j) = 0.$$

Moreover, for $X \sim N(0,\sigma^2)$, we have $EX^2 = \sigma^2$ and $EX^4 = 3\sigma^4$, so we get

$$\begin{aligned}E(\delta_i^2) &= E\big((B_{t_i} - B_{t_{i-1}})^4\big) - 2(t_i - t_{i-1})E\big((B_{t_i} - B_{t_{i-1}})^2\big) + (t_i - t_{i-1})^2 \\ &= 3(t_i - t_{i-1})^2 - 2(t_i - t_{i-1})^2 + (t_i - t_{i-1})^2 \\ &= 2(t_i - t_{i-1})^2.\end{aligned}$$

Taking expectations in (1) and substituting the above results, we get

$$\begin{aligned}E\left(\sum_{i=1}^{n}(B_{t_i} - B_{t_{i-1}})^2 - T\right)^2 &= 2\sum_{i=1}^{n}(t_i - t_{i-1})^2 \\ &\leq 2\|\Pi\|\sum_{i=1}^{n}(t_i - t_{i-1}) \\ &= 2\|\Pi\|T \\ &\to 0 \qquad \text{as } \|\Pi\| \to 0.\end{aligned}$$

Remark:

- The Brownian paths are almost surely of infinite first variation on any interval.
- The cross-variation of a Brownian motion B and the function $f(t) = t$ on $[0,T]$ is equal to zero, i.e.

  $$[B,t]_T = 0,$$

  (see exercise in Homework 04). This is the precise meaning of the informal expression

  $$dB_t dt = 0.$$

2 The Wiener integral

2.1 Motivation

- Recall that Bachelier originally modeled the price S of a risky asset by

$$S_t = S_0 + \sigma B_t,$$

  where B is a standard Brownian motion and S_0, σ constants.

- Hence the stochastic integral

$$\int_0^T \phi(t) dS_t = \sigma \int_0^T \phi(t) dB_t$$

  represents the value of a portfolio as a sum of profits and losses $\phi(t)dS_t$, where dS_t represents the stock price variation and $\phi(t)$ is the quantity invested in the asset S over the short time interval $[t, t+dt]$.

- Naive definition of the stochastic integral wrt. Brownian motion would be

$$\int_0^T \phi(t) dB_t = \int_0^T \phi(t) \frac{dB_t}{dt} dt,$$

  and evaluating the integral as a Riemann or Lebesgue integral. However, this definition **fails** as the paths of Brownian motion are not differentiable.

- **Goal:** Construct the stochastic integral

$$\int_0^t f(s) dB_s,$$

  where B is a Brownian motion, and f is a deterministic function (and a stochastic process later on).

2.2 Definition of the Wiener integral

- Let f be a (deterministic) step function defined by

  $$f(t) = \sum_{i=1}^{n} a_i 1_{[t_{i-1},t_i)}(t), \quad \text{where } t_0 = a,\ t_n = b,\ a_i \in \mathbb{R}.$$

  The **Wiener integral** $I(f)$ of f is defined by

  $$I(f) = \int_a^b f(t) dB_t = \sum_{i=1}^{n} a_i \Delta B_{t_i}, \quad \Delta B_{t_i} = B_{t_i} - B_{t_{i-1}}.$$

- Let $f \in L^2[a,b]$, i.e. $\int_a^b |f(t)|^2 dt < \infty$, and f_n be a sequence of step functions such that $f_n \to f$ in $L^2[a,b]$.

  The **Wiener integral** $I(f)$ of f is defined by

  $$I(f) = \int_a^b f(t) dB_t = \lim_{n\to\infty} \int_a^b f_n(t) dB_t,$$

  where the limit is understood in mean-square sense, i.e.

  $$\lim_{n\to\infty} E\left[\left(\int_a^b f_n(t) dB_t - I(f)\right)^2\right].$$

2.3 Wiener integral is normally distributed

Theorem

For each $f \in L^2[a,b]$, the Wiener integral $\int_a^b f(t)dB_t$ is a Gaussian random variable with mean 0 and variance $\|f\|_2^2 = \int_a^b f^2(t)dt$. In short,

$$\int_a^b f(s)dB_s \sim N(0, \|f\|_2^2).$$

Remark:

- In particular, recall that if the integrand f is the step function $f(t)=\sum_{i=1}^{n} a_i 1_{[t_{i-1},t_i)}(t)$, apparently the Wiener integral $I(f)$ of f is normally distributed

$$I(f)=\sum_{i=1}^{n} a_i \Delta B_{t_i} \sim N\left(0, \sum_{i=1}^{n} a_i^2(t_i-t_{i-1})\right)$$

  since the ΔB_{t_i}'s are independent normal random variables.

- The proof for general $f \in L^2[a,b]$ is based on a limiting process.

- Hence the Wiener integral is nothing but a normal r.v. and it cannot be 'computed' in the standard way using antiderivatives.

Example

The integral $\displaystyle\int_0^t s dB_s$ has a centered Gaussian distribution with variance

$$\int_0^t s^2 ds=\frac{t^3}{3}.$$

2.4 Properties of Wiener integral

Let f and g be deterministic $L^2[a,b]$ functions, α and β are constants. Then

- $\displaystyle\int_a^b [\alpha f(t)+\beta g(t)]\, dB_t=\alpha \int_a^b f(t) dB_t+\beta \int_a^b g(t) dB_t.$

- $\displaystyle\int_a^b f(t) dB_t=\int_a^c f(t) dB_t+\int_c^b f(t) dB_t$, for $c \in [a,b]$.

- **Integration by parts formula**

  Let f be a continuous function of bounded variation. Then almost surely

$$\int_a^b f(t) dB_t=f(t) B_t\Big|_a^b-\int_a^b B_t df(t).$$

  Note that the integral on the left hand side is in the sense of Wiener, whereas on the right hand side is in the sense of Riemann-Stietjes.

Example

Determine the distribution of the random variable $\displaystyle\int_0^1 B_t dt$.

3 The Ito integral

Goal: Extend the above stochastic integral to integrands which are random processes.

3.1 An important example

Let us define the integral

$$\int_0^t B_s dB_s.$$

As in the theory of Riemann-Stieltjes integral, we shall start with partitioning the interval $[0, t]$ into, say, n subintervals.

Within each subinterval, we pick a point and evaluate the integrand at that point, multiply that value by the increment of the integrator in that subinterval.

Then we sum up the results from each subinterval and take limit as the mesh of the partition approaches zero.

Possible choices for selecting points from each subinterval may be, denoting $\Delta B_{t_k} = B_{t_k} - B_{t_{k-1}}$:

- The right point rule:

$$R_n = \sum_{k=1}^{n} B_{t_k} \Delta B_{t_k}$$

- The left point rule:

$$L_n = \sum_{k=1}^{n} B_{t_{k-1}} \Delta B_{t_k}$$

- The midpoint rule:

$$M_n = \sum_{k=1}^{n} B_{t_*} \Delta B_{t_k}, \quad \text{where } t_* = \frac{t_k + t_{k-1}}{2}$$

Which rule rules?

Which rule yields a convergent integral? in what sense? We knew that it can't be pathwise because the integrator, in this case the Brownian motion, is not of finite variation (because it has nonzero second variation) almost surely.

Note that the following identities hold.

$$R_n - L_n = \sum_{k=1}^{n} (\Delta B_{t_k})^2, \qquad R_n + L_n = \sum_{k=1}^{n} \Delta B_{t_k}^2 = B_t^2.$$

Hence,

$$R_n = \frac{B_t^2}{2} + \frac{1}{2}\sum_{k=1}^{n} (\Delta B_{t_k})^2, \qquad L_n = \frac{B_t^2}{2} - \frac{1}{2}\sum_{k=1}^{n} (\Delta B_{t_k})^2.$$

Notice that the first term in both expressions is independent of partitions and the second term, as we have seen in previous lecture, will converge to the quadratic variation of Brownian motion in L^2 as the mesh size approaches zero! Consequently,

$$\lim_{\|\Pi\|\to 0} R_n = \frac{B_t^2}{2} + \frac{t}{2}, \qquad \lim_{\|\Pi\|\to 0} L_n = \frac{B_t^2}{2} - \frac{t}{2}. \qquad\qquad (\text{in } L^2)$$

We learnt from this simple example that

- The right point rule and the left end point rule yield different "integrals".
- The difference between the "right integral" and the "left integral" is exactly the quadratic variation.
- The convergence is in L^2 sense.

Remarks

- We need to stick with one specific rule in order to have convergence.
- Ito picked the **left end point rule** because of adaptivity and martingality.
- L_n is a martingale whereas R_n isn't.
- The midpoint rule leads to the Stratonovich integral.

Contrast to ordinary calculus

We found

$$\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{t}{2}.$$

In ordinary calculus, we have for differentiable functions f with $f(0) = 0$:

$$\int_0^t f(s)df(s) = \int_0^t f(s)f'(s)ds = \frac{1}{2}f^2(s)\mid_0^t = \frac{1}{2}f^2(t).$$

The extra term above comes from the nonzero quadratic variation of Brownian motion.

3.2 Simple processes as integrands

Let $0 = t_0 < t_1 < \cdots < t_n = t$ and $\mathcal{F}_t$, $t \geq 0$, be a filtration for a Brownian motion B.

Definition

A process $\varphi := (\varphi_t)_{t\geq 0}$ is called **simple** if it is of the form

$$\varphi_t(\omega) = \sum_{k=1}^{n} a_{k-1}(\omega)1_{[t_{k-1},t_k)}(t),$$

where a_k are square measurable r.v.s with $a_k \in \mathcal{F}_{t_k}$ for $k = 0, \ldots, n$.

Remarks:

- Basically, a simple process is simply a step function with random coefficients that are measurable with respect to the left endpoints.
- Simple process is defined as such for mimicking

  a) the step functions in the Wiener integral and

  b) the left endpoint rule in the Riemann integral.

Definition (Ito integral of a simple process)

The **stochastic integral of a simple process** φ with respect to Brownian motion B over $[0, t]$ is defined by

$$\int_0^t \varphi_s dB_s = \sum_{k=1}^{n} a_{k-1}\Delta B_{t_k}, \quad \text{where } \Delta B_{t_k} = B_{t_k} - B_{t_{k-1}}.$$

Remarks:

- Note that this is a discrete stochastic integral which we discussed in Lecture 6 of the refresher.
- For $r \leq t$, we define

$$\int_r^t \varphi_s dB_s := \int_0^t \varphi_s dB_s - \int_0^r \varphi_s dB_s.$$

Theorem: Properties of the stochastic integral (for simple processes)

Define the process X by $X_t = \int_0^t \varphi_s dB_s,\ t \geq 0$.

- (Path continuity) The stochastic process X has continuous paths.
- (Adaptivity) The process X is adapted.
- (Linearity) Let $a, b \in \mathbb{R}$ and φ^1, φ^2 be simple processes. Then $a\varphi^1_t + b\varphi^2_t$, $t \geq 0$, is a simple process and

$$\int_0^t \varphi^1_s + b\varphi^2_s \, dB_s = a \int_0^t \varphi^1_s \, dB_s + b \int_0^t \varphi^2_s \, dB_s.$$

- (Martingale property) The process X is a martingale. In particular, we have $E(X_t) = E(X_0) = 0$.
- (Ito's isometry) We have

$$E\left[X_t^2\right] = E\left[\left(\int_0^t \varphi_s \, dB_s\right)^2\right] = \int_0^t E\left[\varphi_s^2\right] \, ds.$$

- (Quadratic and cross variation) Let φ^1, φ^2 be simple processes and X^1, X^2 the corresponding stochastic integrals with respect the a Brownian motion B. Then we have

$$[X]_t = \int_0^t (\varphi_s)^2 ds,$$

  and

$$[X^1, X^2]_t = \int_0^t \varphi^1_s \varphi^2_s ds.$$

Remarks:

- In the last two equations, note that there are no expectations on the right hand side. This means, quadratic and cross variation of stochastic integrals are stochastic processes.
- We often use the following two notations:

$$dX_t = \int_0^t \varphi_s dB_s \qquad \leftrightsquigarrow \qquad dX_t = \varphi_t dB_t$$

integral form $\leftrightsquigarrow$ differential form

- Similarly, we write

$$d[X^1]_t = (\varphi_t^1)^2 dt, \quad \text{and} \quad d[X^1, X^2]_t = \varphi_t^1 \varphi_t^2 dt.$$

Proof of the above theorem:

- Path continuity, adaptedness and linearity immediately follow from the definition of the integral and the continuity of the Brownian motion.
- The proof of the martingale property is very similar to the proof of the Theorem "Discrete stochastic integral wrt a martingale is a martingale" from Lecture 6 in the refresher.

- For the proof of Ito's isometry, note that

$$E\left[\left(\int_0^t \varphi_s dB_s\right)^2\right] = E\left[\left(\sum_{k=1}^n a_{k-1}\Delta B_{t_k}\right)\left(\sum_{l=1}^n a_{l-1}\Delta B_{t_l}\right)\right]$$
$$= \sum_{k=1}^n \sum_{l=1}^n E\left[a_{k-1}a_{l-1}\Delta B_{t_k}\Delta B_{t_l}\right].$$

  Consider three cases.

  - If $k = l$, then by the properties of conditional expectation and independence of ΔB_{t_k} from $\mathcal{F}_{t_{k-1}}$, we get

$$\begin{aligned} E\left[a_{k-1}^2(\Delta B_{t_k})^2\right] &= E\left[E\left[a_{k-1}^2(\Delta B_{t_k})^2 \mid \mathcal{F}_{t_{k-1}}\right]\right] \\ &= E\left[a_{k-1}^2 E\left[(\Delta B_{t_k})^2 \mid \mathcal{F}_{t_{k-1}}\right]\right] \\ &= E\left[a_{k-1}^2\right] E\left[(\Delta B_{t_k})^2\right]. \end{aligned}$$

  - If $k < l$, then again by the similar considerations, we have

$$\begin{aligned} E\left[a_{k-1}a_{l-1}\Delta B_{t_k}\Delta B_{t_l}\right] &= E\left[E\left[a_{k-1}a_{l-1}\Delta B_{t_k}\Delta B_{t_l} \mid \mathcal{F}_{t_{l-1}}\right]\right] \\ &= E\left[a_{k-1}a_{l-1}\Delta B_{t_k} E\left[\Delta B_{t_l} \mid \mathcal{F}_{t_{l-1}}\right]\right] \\ &= 0. \end{aligned}$$

  - If $k > l$, then just exchange k and l in the previous case to get the same result.

  We conclude that

$$\begin{aligned} E\left[\left(\int_0^t \varphi_s dB_s\right)^2\right] &= \sum_{k=1}^n E\left[a_{k-1}^2\right] E\left[(\Delta B_{t_k})^2\right] \\ &= \sum_{k=1}^n E\left[a_{k-1}^2\right](t_k - t_{k-1}) \\ &= \int_0^t E\left[\varphi_s^2\right] ds, \end{aligned}$$

  since $E\left[\varphi_s^2\right]$, $0 \le s \le t$, is a non-random elementary function and the last integral is just another way to write the Riemann sum for this function.

- We shall only prove the formula for the quadratic variation. Cross-variation can be handled by the polarization identity (see Lecture 3) and linearity of ordinary integrals.

  We first compute the quadratic variation accumulated by the Ito integral on one of the subintervals $[t_{k-1}, t_k]$ on which φ_s is constant.

  For this interval, we choose partition points

  $$t_{k-1} = s_0 < s_1 < \ldots < s_m = t_k,$$

  and consider

  $$\sum_{i=1}^{m}(X_{s_i} - X_{s_{i-1}})^2 = \varphi_{t_{k-1}}^2 \sum_{i=1}^{m}(B_{s_i} - B_{s_{i-1}})^2. \tag{1}$$

  We know that the last sum converges in L^2 to $(t_k - t_{k-1})$ as $\|\Pi\| \to 0$.

  Therefore, the limit of (1) is

  $$\varphi_{t_{k-1}}^2 (t_k - t_{k-1}) = \int_{t_{k-1}}^{t_k} \varphi_u^2 du,$$

  where again, we have used that φ_u is constant for $u \in [t_{k-1}, t_k)$.

  This is true for each interval $[t_{k-1}, t_k)$, $k = 1, 2 \ldots, n$.

3.3 General integrands

We extend the Ito integral $X_t := \int_0^t \varphi_s dB_s$ to general integrands φ.

We assume that $\varphi := (\varphi_t)_{t \geq 0}$ has the following properties

- (adapted) φ is adapted to the filtration $\mathcal{F}_t$
- (square integrable) $\int_0^t E\left[\varphi_s^2\right] ds < \infty$

Such processes φ can be approximated in the L^2 sense by simple processes considered in the previous section, i.e. there is a sequence of simple processes $\varphi^{(n)}$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \int_0^t E\left[\left(\varphi_s^{(n)} - \varphi_s\right)^2\right] ds = 0.$$

For each n, the integrals

$$X_t^{(n)} := \int_0^t \varphi_s^{(n)}\, dB_s$$

have been defined in the previous section.

We define the Ito integral X_t for general integrands as the limit of the Ito integral $X_t^{(n)}$ for simple integrands in the L^2 sense:

$$X_t \overset{L^2}{=} \lim_{n\to\infty} X_t^{(n)}.$$

The existence of the limit needs justification which is beyond the scope of these notes and can be found in any standard text on stochastic calculus such as, for example, Oeksendal, Stochastic Differential Equations, 2003, Chapter 3.

This integral inherits the properties of Ito integrals of simple processes.

Theorem: Properties of the stochastic integral (for general integrands)

Define the process X by $X_t = \int_0^t \varphi_s dB_s,\ t \geq 0$.

- (Path continuity) The stochastic process X has continuous paths.
- (Adaptivity) The process X is adapted.
- (Linearity) Let $a, b \in \mathbb{R}$ and φ^1, φ^2 be simple processes. Then $a\varphi^1_t + b\varphi^2_t$, $t \geq 0$, is a simple process and

$$\int_0^t \varphi^1_s + b\varphi^2_s \, dB_s = a \int_0^t \varphi^1_s \, dB_s + b \int_0^t \varphi^2_s \, dB_s.$$

- (Martingale property) The process X is a martingale. In particular, we have $E(X_t) = E(X_0) = 0$.
- (Ito's isometry) We have

$$E\left[X_t^2\right] = E\left[\left(\int_0^t \varphi_s \, dB_s\right)^2\right] = \int_0^t E\left[\varphi_s^2\right] \, ds.$$

- (Quadratic and cross variation) Let φ^1, φ^2 be simple processes and X^1, X^2 the corresponding stochastic integrals with respect the a Brownian motion B. Then we have

$$[X]_t = \int_0^t (\varphi_s)^2 ds,$$

  and

$$[X^1, X^2]_t = \int_0^t \varphi^1_s \varphi^2_s ds.$$