9831 Probability and Stochastic Processes for Finance, Fall 2016

Lecture 5

Anja Richter

Department of Mathematics

Baruch College
The City University of New York

Outline

- Ito-Doeblin formula
- Ito-Doeblin formula for Ito processes
- Applications of the Ito-Doeblin formula

1 Ito-Doeblin formula

1.1 Ordinary Calculus

Recall the **chain rule** for two continuously differentiable functions f and g:

$$f(g(x)) = f(g(0)) + \int_0^x f'(g(u))dg(u) = f(g(0)) + \int_0^x f'(g(u))g'(u)du.$$

In differential notation, we can write this relation as

$$df(g(x)) = f'(g(x))g'(x)dx,$$

where dx is small.

This chain rule is derived from Taylor's Theorem. As a special case, consider $x = 0$ and derive the chain rule

$$\frac{d}{dx} f(g(x))\Big|_{x=0} = f'(g(0))g'(0).$$

as follows.

From Taylor's theorem, as $x \to 0$ we can write

$$f(g(x)) - f(g(0)) = f'(g(0))(g(x) - g(0)) + \frac{1}{2} f''(g(0))(g(x) - g(0))^2 + o((g(x) - g(0))^2).$$

Here the notation $o(h(x))$ as $x \to 0$ means that

$$\lim_{x \to 0} \frac{o(h(x))}{h(x)} = 0.$$

To put it simply: the function $o(h(x))$ is small in comparison with $h(x)$ for small x.

Since g is continuously differentiable, as $x \to 0$ we have

$$g(x) - g(0) = x(g'(0) + o(1)) \quad \text{and} \quad (g(x) - g(0))^2 = x^2(g'(0) + o(1))^2.$$

The last term is of order x^2. Thus, dividing by x and taking $x \to 0$, we get

$$\lim_{x \to 0} \frac{f(g(x)) - f(g(0))}{x} = f'(g(0))g'(0),$$

and hence

$$\frac{d}{dx} f(g(x))\Big|_{x=0} = f'(g(0))g'(0).$$

We would like to replace g with a Brownian motion B, however B is not a continuously differentiable function.

We see that, contrary to before when

$$\frac{(g(x) - g(0))^2}{x} \to 0 \quad \text{for } x \to 0,$$

we have $(B_t - B_0)^2 = B_t^2$ is not small in comparison with t.

Indeed,

$$\frac{B_t^2}{t} = \left(\frac{B_t}{\sqrt{t}}\right)^2 \overset{\mathrm{d}}{=} Z^2,$$

where Z is a standard normal random variable, no matter how small we choose t.

In particular, $E(Z^2) = 1$ and Z^2 is Gamma distributed with parameters $(1/2, 1/2)$.

Conclusion: We expect that when we apply Taylor's theorem to a Brownian motion, the term with the second derivative will not go away.

We also see that we might not be get a meaningful expression if we try to take a pointwise limit as $t \to 0$. Thus we shall try to use an integrated form, take the limit in L^2, and hope to get for all $T > 0$ (fixed)

$$f(B_T) - f(B_0) = \int_0^T f'(B_t)dB_t + \frac{1}{2}\int_0^T f''(B_t)\underbrace{dB_t dB_t}_{=dt}.$$

The new "chain rule" is called Ito-Doeblin formula for Brownian motion and is one of the most important tools in mathematical finance and stochastic calculus.

1.2 Ito-Doeblin formula for $f(B_t)$

Theorem

Let f be a twice continuously differentiable function, B a Brownian motion and $\int_0^T E(f'(B_t))^2\,dt < \infty$ for all $T > 0$.

Then for every $T \geq 0$

$$f(B_T) - f(B_0) = \int_0^T f'(B_t)\,dB_t + \frac{1}{2}\int_0^T f''(B_t)\,dt \quad \text{a.s..} \tag{1}$$

Equivalently, we write in differential form

$$df(B_t) = f'(B_t)\,dB_t + \frac{1}{2}f''(B_t)\,dt \quad \text{a.s..}$$

Remarks:

- The square integrability condition in the above theorem can be removed (if one extends the notion of the stochastic integral accordingly).
- The regularity assumption on f can also be relaxed: see Exercise 4.20 of Shreve.
- For the proof, we need the following fact:

  For a twice continuously differentiable function f, there exists some ξ between y and x such that

  $$f(y) - f(x) = f'(x)(y-x) + \frac{f''(\xi)}{2}(y-x)^2.$$

  Notice that this equation is exact, however in general it is not possible to specify what ξ is.

1.3 Sketch of proof of Ito-Doeblin formula for $f(B_t)$

Assume first that f' and f'' are bounded, i.e. there is a constant M such that $|f'(x)| \leq M$ and $|f''(x)| \leq M$ for all $x \in \mathbb{R}$.

Let Π be a partition of $[0,T]$, $0 = t_0 < t_1 < \cdots < t_n = T$.

Then

$$\begin{aligned} f(B_T) - f(B_0) &= \sum_{i=1}^{n} (f(B_{t_i}) - f(B_{t_{i-1}})) \qquad (2) \\ &= \sum_{i=1}^{n} f'(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}}) + \frac{1}{2} \sum_{i=1}^{n} f''(\xi_{i-1})(B_{t_i} - B_{t_{i-1}})^2 \\ &=: \text{term I} \quad + \quad \text{term II}, \end{aligned}$$

where ξ_{i-1} is a point between $B_{t_{i-1}}$ and B_{t_i}.

Term I: As we take the limit as $\|\Pi\| \to 0$, term I will converge in L^2 to the stochastic integral $\int_0^T f'(B_t)\, dB_t$ by definition, i.e.

$$\lim_{\|\Pi\| \to 0} \sum_{i=1}^{n} f'(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}}) = \int_0^T f'(B_t)\, dB_t \qquad \text{in } L^2.$$

Term II: Since B_t is continuous on $[t_{i-1}, t_i]$, there is a point $t_{i-1}^* \in [t_{i-1}, t_i]$ such that $B(t_{i-1}^*) = \xi_{i-1}$, $i = 1, 2, \ldots, n$.

Furthermore, we may write

$$\begin{aligned} \text{II} &= \frac{1}{2} \sum_{i=1}^{n} f''(B_{t_{i-1}^*})(B_{t_i} - B_{t_{i-1}})^2 \\ &= \frac{1}{2} \sum_{i=1}^{n} \left(f''(B_{t_{i-1}}) + \underbrace{f''(B_{t_{i-1}^*}) - f''(B_{t_{i-1}})}_{=:\epsilon_{i-1}} \right)(B_{t_i} - B_{t_{i-1}})^2. \end{aligned}$$

- First, we show that

$$\lim_{\|\Pi\|\to 0} \frac{1}{2} \sum_{i=1}^{n} f''(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}})^2 = \frac{1}{2} \int_0^T f''(B_t)\, dt \qquad \text{in } L^2.$$

  This calculation is similar to the one on the computation of the quadratic variation of Brownian motion, we have with $(a+b)^2 \le 2a^2 + 2b^2$

$$\begin{aligned} &E\left(\sum_{i=1}^{n} f''(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}})^2 - \int_0^T f''(B_t)\, dt \right)^2 \\ &\le 2E\left(\sum_{i=1}^{n} f''(B_{t_{i-1}})\big((B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})\big) \right)^2 \\ &\quad + 2E\left(\sum_{i=1}^{n} f''(B_{t_{i-1}})(t_i - t_{i-1}) - \int_0^T f''(B_t)\, dt \right)^2 \\ &=: 2J_1 + 2J_2. \end{aligned}$$

  First term J_1: Using the boundedness of f'' and the same method as in the proof of quadratic variation of Brownian motion, we have

$$\lim_{\|\Pi\|\to 0} J_1 \le M^2 \lim_{\|\Pi\|\to 0} \sum_{i=1}^{n} E\big((B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})\big)^2 = 0.$$

  Second term J_2: The function $f''(B_t)$ is continuous and bounded so we get the convergence of the Riemann sum to the ordinary integral in L^2, i.e.

$$\lim_{\|\Pi\|\to 0} J_2 = 0.$$

  Note that we have this convergence not only in L^2 but also a.s.

- Finally, we consider the terms with

$$\epsilon_i = f''(B_{t^*_{i-1}}) - f''(B_{t_{i-1}}).$$

As $f''(B_t)$ is continuous on $[0, T]$, it is uniformly continuous on $[0, T]$, and therefore

$$\max_{i \in \{1,2,...,n\}} |\epsilon_{i-1}| \to 0 \quad \text{as} \quad \|\Pi\| \to 0 \quad \text{a.s..}$$

Moreover, we have

$$\max_{i \in \{1,2,...,n\}} |\epsilon_{i-1}| \leq 2M$$

as we assumed that f'' is bounded.

Therefore,

$$\begin{aligned}\left|\sum_{i=1}^{n} \epsilon_{i-1}(B_{t_i} - B_{t_{i-1}})^2\right| &\leq \max_{i \in \{1,2,...,n\}} |\epsilon_{i-1}| \left(\sum_{i=1}^{n}(B_{t_i} - B_{t_{i-1}})^2\right) \\ &= \max_{i \in \{1,2,...,n\}} |\epsilon_{i-1}| \left(\sum_{i=1}^{n}(B_{t_i} - B_{t_{i-1}})^2 - T\right) + T \max_{i \in \{1,2,...,n\}} |\epsilon_{i-1}| \\ &\leq 2M \left|\sum_{i=1}^{n}(B_{t_i} - B_{t_{i-1}})^2 - T\right| + T \max_{i \in \{1,2,...,n\}} |\epsilon_{i-1}|.\end{aligned}$$

The first term converges to 0 in L^2 by the computation of the quadratic variation of Brownian motion, the second term converges to 0 in L^2 by the dominated convergence theorem.

We have shown that when f' and f'' are bounded, the right hand side of (2) converges in L^2 as $\|\Pi\| \to 0$ to the right hand side of (1).

The left hand side of (2) does not depend on the partition, therefore it is equal to the above limit a.s.. This proves (1).

To remove the assumed bounds on the derivatives, we have to use a localization technique.

Define

$$\tau_n = \inf\{t \geq 0 : |B_t| = n\}.$$

Then τ_n is a stopping time. The idea is to check that the above reasoning works if we replace B_t everywhere with $B(t \wedge \tau_n)$.

Note that $|B(t \wedge \tau_n)| \leq n$ and continuous functions f' and f'' are bounded on bounded intervals.

Then we have to take a limit as $n \to \infty$.

1.4 First examples

Example

Let $f(x) = x^2$. Then $f'(x) = 2x$, $f''(x) = 2$. Hence

$$B_T^2 - B_0^2 = 2\int_0^T B_t \, dB_t + \int_0^T 1 \, dt.$$

Or in the differential form,

$$dB_t^2 = 2B_t \, dB_t + dt.$$

This is the already familiar formula $\int_0^T B_t dB_t = \frac{1}{2}B_T^2 - \frac{1}{2}T$ from the last lecture.

Example (Computation of the moment generating function of B_t)

Let $f(x) = e^{\theta x}$, $\theta \in \mathbb{R}$. Then $f'(x) = \theta e^{\theta x}, \ f''(x) = \theta^2 e^{\theta x}$.

The Ito-Doeblin formula in the differential form then reads

$$de^{\theta B_t} = \theta e^{\theta B_t} \, dB_t + \frac{1}{2}\,\theta^2 e^{\theta B_t} \, dt,$$

and in the integral form we have

$$e^{\theta B_t} - e^{\theta B_0} = \theta \int_0^t e^{\theta B_s} \, dB_s + \frac{1}{2}\,\theta^2 \int_0^t e^{\theta B_s} \, ds.$$

Let $m(t) = E(e^{\theta B_t})$.

Taking the expectations in the above formula we get

$$m(t) - 1 = \frac{1}{2}\,\theta^2 \int_0^t m(s) \, ds \quad \Longleftrightarrow \quad m'(t) = \frac{1}{2}\theta^2 m(t), \ m(0) = 1.$$

Solving the differential equation we conclude that

$$m(t) = E(e^{\theta B_t}) = e^{\theta^2 t/2}.$$

1.5 Ito-Doeblin formula for $f(t, B_t)$

We extend the above formula to functions not only depending on the Brownian motion B but also on time t.

We are interested in quantities $f(t, B_t)$ in many financial applications, where $f(t, x)$ is a smooth function of t and x.

Theorem

Let $f(t, x)$, be a function with continuous partial derivatives f_t, f_x, and f_{xx}.

Then we have for all $T \geq 0$

$$f(T, B_T) - f(0, B_0) = \int_0^T f_x(t, B_t)\, dB_t + \int_0^T \left(f_t(t, B_t) + \frac{1}{2} f_{xx}(t, B_t) \right)\, dt.$$

Or in differential form

$$df(t, B_t) = f_x(t, B_t)\, dB_t + \left(f_t(t, B_t) + \frac{1}{2} f_{xx}(t, B_t) \right)\, dt.$$

Remarks

- The idea is again that we Taylor expand $f(t, B_t)$ to second order then *formally* apply the following rules:

$$(dB_t)^2 = dt, \quad (dt)^2 = 0, \quad dB_t dt = 0.$$

- The full proof is omitted, sketch of proof can be found in Shreve, Theorem 4.4.1 and Exercise 4.14.

1.6 Applications of Ito-Doeblin's formula: Evaluating stochastic integrals

In ordinary calculus, we barely evaluate an integral from the definitioin *per se*, i.e., partition the integrating interval, form Riemann sum, then take limit as the mesh of the partition approaches zero. Instead, we evaluate an integral by applying the Fundamental Theorem of Calculus.

Though in stochastic calculus the Fundamental Theorem of Calculus does not really exist, we evaluate stochastic integrals by applying Ito's formula.

Theorem

We have

$$\int_a^b f(B_t)dB_t = F(B_t)\Big|_{t=a}^b - \frac{1}{2}\int_a^b f'(B_t)dt,$$

where F is an antiderivative of f, i.e., $F' = f$.

Theorem

Similarly, we have

$$\int_a^b f(t,B_t)dB_t = F(t,B_t)\Big|_{t=a}^b - \int_a^b \left[F_t(t,B_t) + \frac{1}{2}f_x(t,B_t)\right]dt,$$

where $F_x = f$, i.e., F is an antiderivative of f with respect to x.

Idea:

- Find an antiderivative of f (with respect to x), say, F; apply Ito's formula to F, then rearrange terms.
- However, the price we pay is that in general the last (Riemann) integral on the right hand side usually has no simple analytical expression.

Examples

- Evaluate the stochastic integral $\int_0^T B_s dB_s$.

  Note that in this case $f(x) = x$, hence an antidrivative of f is $F(x) = \frac{x^2}{2}$.

  Applying Ito-Doeblin's formula to F, we have

$$\begin{aligned} dF(B_t) = d\left(\frac{B_t^2}{2}\right) &= B_t dB_t + \frac{1}{2}dt \\ \implies \frac{B_T^2}{2} - \frac{B_0^2}{2} &= \int_0^T B_t dB_t + \frac{1}{2}\int_0^T dt \\ \implies \int_0^T B_t dB_t &= \frac{1}{2}(B_T^2 - T) \end{aligned}$$

- Evaluate the stochastic integral $\int_0^T se^{B_s} dB_s$.

  Note that in this case $f(t,x) = te^x$. Hence an antidrivative F of f (wrt x) is $F(t,x) = te^x$.

  Applying Ito-Doeblin's formula to F, we have

$$\begin{aligned} dF(t,B_t) &= d\left(te^{B_t}\right) = e^{B_t}dt + te^{B_t}dB_t + \frac{1}{2}te^{B_t}dt \\ \Longrightarrow Te^{B_T} &= \int_0^T te^{B_t}dB_t + \int_0^T e^{B_t}\left(1+\frac{t}{2}\right)dt \\ \Longrightarrow \int_0^T te^{B_t}dB_t &= Te^{B_T} - \int_0^T e^{B_t}\left(1+\frac{t}{2}\right)dt \end{aligned}$$

2 Ito-Doeblin formula for Ito processes.

We shall consider one more generalization of the Ito-Doeblin formula.

Definition

Let $(B_t)_{t\geq 0}$, be a standard Brownian motion and $(\mathcal{F}_t)_{t\geq 0}$ be an associated filtration.

An **Ito process** is a stochastic process of the form

$$X_t = x + \int_0^t \sigma_s\, dB_s + \int_0^t b_s\, ds,$$

where x is a constant and $(\sigma_t)_{t\geq 0},\ (b_t)_{t\geq 0}$ are adapted stochastic processes.

We also write it in differential form as

$$dX_t = \sigma_t dB_t + b_t dt, \quad \text{with initial condition } X_0 = x.$$

The coefficient b_t is referred to as the **drift** (term) and σ_t as the **diffusion** (term) of the Ito process X_t.

Examples

- Brownian motion
- Brownian motion with drift
- stochastic integral $\int_0^t \sigma_s\, dB_s$

Theorem

Let $(X_t)_{t \geq 0}$ be an Ito process. Then for every $t \geq 0$, we have

$$[X]_t = \int_0^t \sigma_s^2 \, ds.$$

We can also write this fact in differential form as

$$d[X]_t = \sigma_t \, dt.$$

Remarks

- Note that the answer is the same as in the case of the stochastic integral when $b_t \equiv 0$.

  In other words, only the stochastic integral $I_t = \int_0^t \sigma_s \, dB_s$ contributes to the quadratic variation:

  $$[X]_t = [I]_t.$$

- The proof is omitted but here is a ''calculation'' based on the stochastic calculus ''multiplication rules'' we introduced earlier to capture the quadratic variation:

  Write X_t in the differential form $dX_t = \sigma_t \, dB_t + b_t \, dt$. Then

  $$\begin{aligned} dX_t dX_t &= \sigma_t^2 \underbrace{dB_t dB_t}_{=dt} + 2b_t \sigma_t \underbrace{dB_t dt}_{=0} + b_t^2 \underbrace{dtdt}_{=0} \\ &= \sigma_t^2 \, dt. \end{aligned}$$

Definition (Ito integral with respect to Ito processes)

Let X_t be an Ito process with drift b_t and diffusion σ_t. Let φ_t be an adapted process.

We define the **stochastic integral of φ_t with respect to X_t** as

$$\int_0^t \varphi_s dX_s = \int_0^t \varphi_s \sigma_s dB_s + \int_0^t \varphi_s b_s ds,$$

provided the integrals on the right hand side are defined.

Remark: Note that the stochastic integral with respect to an Ito process need not be a martingale.

Theorem (Ito-Doeblin's formula for Ito processes)

Let $f(t,x)$ be a function with continuous partial derivatives f_t, f_x, and f_{xx}.

Then for every $T \geq 0$,

$$\begin{aligned} f(T,X_T) - f(0,x) &= \int_0^T f_t(t,X_t)dt + \int_0^T f_x(t,X_t)dX_t + \frac{1}{2}\int_0^T f_{xx}(t,X_t)d[X]_t \\ &= \int_0^T f_x(t,X_t)\sigma_t dB_t + \int_0^T \left[f_t(t,X_t) + \frac{\sigma_t^2}{2} f_{xx}(t,X_t) + b_t f_x(t,X_t) \right] dt. \end{aligned}$$

Or in differential form

$$\begin{aligned} df(t,X_t) &= f_t(t,X_t)dt + f_x(t,X_t)dX_t + \frac{1}{2} f_{xx}(t,X_t)d[X]_t \\ &= \sigma_t f_x dB_t + \left[f_t + \frac{\sigma_t^2}{2} f_{xx} + b_t f_x \right] dt. \end{aligned}$$

Remarks

- Ito's formula natually decomposes $f(t,X_t)$ into a drift/finite variation part plus a diffusion/martingale part; reminiscent of the Doob decomposition. Processes consist of a finite variation part and a martingale part are also referred to as *semimartingales*.
- Note that the second order differential operator $\frac{\sigma_t^2}{2}\partial_x^2 + b_t\partial_x$ in the drift part is the infinitesimal generator of the process X_t.

The proof is omitted.

Note that it is easy to "derive" the differential form of this formula by applying stochastic "multiplication rules":

$$\begin{aligned} df(t,X_t) &= f_t(t,X_t)dt + f_x(t,X_t) \underbrace{dX_t}_{=\sigma_t dB_t + b_t dt} + \frac{1}{2} f_{xx}(t,X_t)\underbrace{dX_t dX_t}_{\sigma_t^2 dt} \\ &= \left(f_t(t,X_t) + b_t f_x(t,X_t) + \frac{1}{2}\sigma_t^2 f_{xx}(t,X_t) \right) dt + f_x(t,X_t)\sigma_t \, dB_t. \end{aligned}$$

Example (Geometric Brownian motion)

Let $S_t = Ae^{X_t}$, where A is a constant, and $X_t = \mu t + \sigma B_t$.

Then $S_t = f(t, X_t)$ for $f(t,x) = Ae^x$.

Calculating $f_t(t,x) = 0$, $f_x(t,x) = f(t,x)$, $f_{xx}(t,x) = f(t,x)$ amd applying the above theorem we get

$$\begin{aligned} S_t - S_0 &= f(t, X_t) - f(0, X_0) \\ &= \int_0^T f(s, X_s)\, dX_s + \int_0^t \frac{1}{2} f(s, X_s)\, d[X]_s \\ &= \int_0^T \sigma f(s, X_s)\, dB_s + \int_0^t \left(\mu f(s, X_s) + \frac{1}{2}\sigma^2 f(s, X_s) \right) \, ds \end{aligned}$$

Since $f(s, X_s) = S_s$, we conclude that

$$S_t - S_0 = \int_0^t \sigma S_s \, dB_s + \int_0^t \left(\mu + \frac{1}{2}\sigma^2 \right) S_s \, ds,$$

or, in the differential form,

$$dS_t = \sigma S_t dB_t + \left(\mu + \frac{1}{2}\sigma^2 \right) S_t \, dt, \quad S_0 = A.$$

This is a so called stochastic differential equation (SDE) with initial condition $S_0 = A$. We can say that the process $S_t = Ae^{\mu t + \sigma B_t}$ solves the SDE.

This topic will be discussed in more detail later.

3 Applications of the Ito-Doeblin formula

3.1 Brownian martingales

Define M_t by

$$M_t := f(t, B_t) - \int_0^t \left[\partial_t f(s, B_s) + \frac{1}{2}\partial_x^2 f(s, B_s)\right] ds.$$

Then M_t is a martingale.

In particular, if $f(t, x)$ satisfies the heat equation $f_t + \frac{1}{2} f_{xx} = 0$, then $f(t, B_t)$ defines a martingale with respect to the Brownian filtration. Such martingale are called **Brownian martingales**.

Examples

- $B_t^2 - t$
- $B_t^3 - 3tB_t$
- $B_t^4 - 6tB_t^2 + 3t^2$
- $e^{\sigma B_t - \frac{\sigma^2}{2}t}$ for any $\sigma \in \mathbb{R}$

3.2 Integration by parts

Let $(A(t))_{t\geq 0}$ be an adapted stochastic process with a.s. continuously differentiable trajectories.

Let X be an Ito process.

Then we have

$$\int_0^t A(s)dX_s = A(s)X_s\Big|_0^t - \int_0^t X_s dA(s).$$

This fact can be derived from Ito-Doeblin's formula by applying it to $A(t)X_t$ and rearranging terms.

In differential terms we have

$$d(A(t)X_t) = A(t)dX_t + X_t dA(t).$$

3.3 Solving SDEs

A **stochastic differential equation (SDE)** is a differential equation with random noise of the form

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t, \qquad X_0 = x$$

where μ and σ are deterministic functions and $x \in \mathbb{R}$.

In some cases, we can solve SDEs by applying Ito's formula to certain function of X_t.

We demonstrate the technique by solving the following two very important examples.

- **Geometric Brownian motion/Black-Scholes model**

  $$dX_t = \mu X_t dt + \sigma X_t dB_t, \quad \text{or} \quad \frac{dX_t}{X_t} = \mu dt + \sigma dB_t$$

  where μ and σ are constants.

  This model will be used to represent the random price X_t of a risky asset at time t. Here, the return dX_t/X_t of the asset is made of two components: a constant return μdt and a random return σdB_t.

  To solve it, we apply Ito's formula to $\log X_t$.

- **Ornstein-Uhlenbeck process/Vasicek model**

  $$dX_t = \lambda(m - X_t)dt + \sigma dB_t,$$

  where m, λ, and σ are constants.

  To solve it, we apply Ito's formula to $e^{\lambda t} X_t$.

Geometric Brownian motion or Black-Scholes model

$$dX_t = \mu X_t dt + \sigma X_t dB_t$$

Note that the quadratic variation of X_t is

$$[X]_t = \left[\int_0^t \sigma X_s dB_s\right] = \int_0^t \sigma^2 X_s^2 ds \quad \rightsquigarrow \quad d[X]_t = \sigma^2 X_t^2 dt$$

Let $f(x) = \log x$, apply Ito's formula to $\log X_t$ we have

$$\begin{aligned} d\log X_t &= df(X_t) \\ &= f'(X_t)dX_t + \frac{1}{2}f''(X_t)d[X]_t \\ &= \frac{dX_t}{X_t} - \frac{1}{2}\frac{d[X]_t}{X_t^2} = \mu dt + \sigma dB_t - \frac{\sigma^2}{2}dt \\ &= \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dB_t. \end{aligned}$$

Hence, by integrating from 0 to T on both sides of the last equation we obtain

$$\log X_T - \log X_0 = \left(\mu - \frac{\sigma^2}{2}\right)T + \sigma B_T \quad \Longrightarrow \quad X_T = X_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma B_T}.$$

Ornstein-Uhlenbeck process/Vasicek model

See homework 05