9831 Probability and Stochastic Processes for Finance, Fall 2016

Lecture 3

Anja Richter

Department of Mathematics

Outline

- Brownian motion as Markov process
- Reflection principle
- Quadratic variation

1 Markov processes

Definition

A stochastic process $X := (X_t)_t \geq 0$ defined on a filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ is a **Markov process** if, for $s > t$,

$$\mathbb{P}[X_s \in A|\mathcal{F}_t] = \mathbb{P}[X_s \in A|X_t]$$

for all Borel measurable sets A.

In words, to determine the probability of the process X in the set A at time s given the information up to time t , it suffices to condition on the position of the process at the most recent time t.

The Markov property can also be written in terms of conditional expectations as

$$\mathbb{E}\left[f(X_s)|\mathcal{F}_t\right] = \mathbb{E}\left[f(X_s)|X_t\right]$$

for all bounded measurable function f.

Remark: The two definitions are, in fact, equivalent. The first one is more intuitive and easier to check, the second is more convenient to use.

1.1 Typical examples of Markov processes

- Brownian motion

  If B is a Brownian motion, then for any $t \geq 0$, $(B_{t+s} - B_t)_{s \geq 0}$ is again a BM, independent of $(B_r)_{r \in [0,t]}$.

  Intuitively, the Brownian motion starts afresh at time t.

- Random walk (in discrete time)

- Brownian motion with drift

1.2 Markov processes and martingales

- Markov property and martingale property look vaguely similar.

  Moreover, a number of popular stochastic processes enjoy both properties: random walks with mean zero increments, Brownian motion.

- Not every Markov process is a martingale:

  for example, random walk with non-zero mean increments.

- Not every martingale is a Markov process:

Example

- Let $\{X_i\}_{i\geq 0}$ be an i.i.d. sequence of r.v.s such that $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$.
- Define

$$M_n = \begin{cases} \sum_{i=0}^n X_i, & \text{if } X_0 = 1; \\ 2\sum_{i=0}^n X_i, & \text{if } X_0 = -1. \end{cases} \qquad n = 1, 2, \ldots.$$

- Then $\{M_n\}_{n\geq 0}$ is a martingale relative to its natural filtration:

$$\begin{aligned} \mathbb{E}(M_{n+1} \mid \mathcal{F}_n) &= \mathbb{E}(M_{n+1} 1_{\{X_0=1\}} \mid \mathcal{F}_n) + \mathbb{E}(M_{n+1} 1_{\{X_0=-1\}} \mid \mathcal{F}_n) \\ &= \mathbb{E}\left(\sum_{i=0}^{n+1} X_i 1_{\{X_0=1\}} \mid \mathcal{F}_n\right) + \mathbb{E}\left(2\sum_{i=0}^{n+1} X_i 1_{\{X_0=-1\}} \mid \mathcal{F}_n\right) \\ &= 1_{\{X_0=1\}}\left(\mathbb{E}(X_{n+1}) + \sum_{i=0}^{n} X_i\right) + 1_{\{X_0=-1\}}\left(2\mathbb{E}(X_{n+1}) + 2\sum_{i=0}^{n} X_i\right) \\ &= 1_{\{X_0=1\}}\sum_{i=0}^{n} X_i + 21_{\{X_0=-1\}}\sum_{i=0}^{n} X_i = M_n. \end{aligned}$$

- However, M is not a Markov process, since

$$\mathbb{P}(M_2 = 1 \mid M_0 = -2, M_1 = 0) = 0, \text{ but } \mathbb{P}(M_2 = 1 \mid M_1 = 0) = 1/8.$$

  Intuitively, this is clear, since the process ``remembers'' its starting point.

1.3 Transition probability and transition density

A Markov process is fully determined by an initial distribution and the transition probability (or density if there exists any). The **transition probability** $p(s, A|t, x)$ of a Markov process is defined by

$$p(s, A|t, x) = \mathbb{P}\left[X_s \in A | X_t = x\right]$$

where $s > t$ and $A \subset \mathbb{R}$ is measurable.

If the transition probability p is absolutely continuous with respect to the Lebesgue measure, we refer to the density as the **transition density** and denote it by, abusing the notation, $p(s, y|t, x)$.

Then we have

$$p(s, A|t, x) = \int_A p(s, y|t, x) dy.$$

Example

Transition probability and transition density of a Brownian motion

$$
\begin{aligned}
p(s, A|t, x) &= \mathbb{P}\left[B_s \in A | B_t = x\right] \\
&= \frac{1}{\sqrt{2\pi(s-t)}} \int_A \exp\left(-\frac{(y-x)^2}{2(s-t)}\right) dy,
\end{aligned}
$$

i.e. $p(s, y|t, x) = \frac{1}{\sqrt{2\pi(s-t)}} \exp\left(-\frac{(y-x)^2}{2(s-t)}\right)$.

If, like here, the transition probability only depends on the difference $s - t$, the process is called **time-homogeneous**.

1.4 Infinitesimal generator

The **infinitesimal generator** L of a Markov process X is defined by

$$
Lf(x) = \lim_{h \to 0^+} \frac{1}{h} \left\{\mathbb{E}\left[f(X_{t+h}) | X_t = x\right] - f(x)\right\},
$$

for suitable functions f, i.e. functions for which this limit exists.

Example

For a Brownian motion B,

$$\begin{aligned}
\mathbb{E}[f(B_{t+h})|B_t = x] = \mathbb{E}[f(x+B_h)] = \int_{-\infty}^{\infty} \frac{e^{-\frac{y^2}{2h}}}{\sqrt{2\pi h}} f(x+y)dy \\
&= \int_{-\infty}^{\infty} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} f(x+\sqrt{h}z)dz, \quad (\text{let } y = \sqrt{h}z) \\
&= \int_{-\infty}^{\infty} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \left[f(x) + f'(x)\sqrt{h}z + \frac{f''(x)}{2}hz^2 + o(h^{3/2}) \right] dz \\
&= f(x) + \frac{f''(x)}{2}h + o(h^{3/2}).
\end{aligned}$$

Thus,

$$\begin{aligned}
Lf(x) &= \lim_{h\to 0} \frac{1}{h}\{\mathbb{E}[f(B_{t+h})|B_t = x] - f(x)\} \\
&= \lim_{h\to 0} \frac{1}{h}\left[f(x) + \frac{1}{2}f''(x)h + o(h^{3/2}) - f(x) \right] \\
&= \lim_{h\to 0} \left[\frac{f''(x)}{2} + o(h^{1/2}) \right] \\
&= \frac{f''(x)}{2}.
\end{aligned}$$

2 Reflection principle

2.1 Stopping times

Definition

Given a stochastic process X defined on a filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$, a r.v. $\tau : \Omega \to [0, \infty]$ is called **stopping time** if

$$\{\tau \leq t\} \in \mathcal{F}_t.$$

In other words, with information $\mathcal{F}_t$ up to time t, we can determine whether or not $\tau \leq t$.

Examples

Let B be a standard Brownian motion.

- Consider the first hitting time of a fixed level ℓ:

  $$\tau_\ell := \inf\{t \geq 0 : B_t = \ell\}.$$

  It is easy to see that τ_ℓ is indeed a stopping time:

  $$\{\tau_\ell \leq t\} = \{\text{there is a time } s \in [0,t] : B_s = \ell\} \in \mathcal{F}_t,$$

  as the set in the middle only depends on $(B_s)_{s\in[0,t]}$.

  Conventionally, we set $\tau_\ell = \infty$ if Brownian motion B_t never hits the level ℓ.

  Note, that if $\tau_\ell < \infty$, then $B_{\tau_\ell} = \ell$.

- other stopping times

  $$\tau_{|\ell|} := \inf\{t \geq 0 : |B_t| = \ell\},$$

  $$\tau_{(a,b)} := \inf\{t \geq 0 : B_t \notin (a,b)\}.$$

- Just like in discrete time, the *last* time that a process hits a level is not a stopping time.

Fact:

We know, that if B is a Brownian motion $(B_{t+s} - B_t)_{s\geq 0}$ is also a Brownian motion, independent of $(B_r)_{r\in[0,t]}$.

That property of Brownian motion can be extended to stopping times (proof beyond the scope of this course):

If B is a Brownian motion and τ is a stopping time, then

$$(B_{t+\tau} - B_\tau)_{t\geq 0}$$

is also a Brownian motion.

Intuitively, that means that the Brownian motion starts afresh at the stopping time τ.

2.2 First hitting time and running maximum

Let B be a standard Brownian motion and τ_ℓ be the first hitting time for B_t of the level $\ell > 0$, i.e.,

$$\tau_\ell = \inf\{t \geq 0 : B_t \geq \ell\}.$$

Let M_t be the running maximum of the Brownian motion B up to time t, i.e.,

$$M_t = \max_{0 \leq s \leq t} B_s.$$

Obviously, since Brownian paths are continuous, the two events $\{\tau_\ell \leq t\}$ and $\{M_t \geq \ell\}$ are the same, i.e.,

$$\{\tau_\ell \leq t\} = \{M_t \geq \ell\}.$$

Question: What is the probability of the two equivalent events?

Theorem (Distribution function of first hitting time)

Let B be a Brownian motion starting in 0 and let $\ell > 0$. Then we have

$$\mathbb{P}[\tau_\ell \leq t] = \mathbb{P}[M_t \geq \ell] = 2\,\mathbb{P}[B_t \geq \ell].$$

Remark:

Intuitively, we have

$$\begin{aligned} &\text{the number of paths ever hit the level } \ell \text{ before time } t \\ = {}& 2 \times \text{the number of paths that end above the level } \ell \text{ at time } t \end{aligned}$$

Sketch of proof:

If $B_t \geq \ell$, then by continuity of the Brownian paths, $\tau_\ell \leq t$.

Moreover, $(B_{t+\tau_\ell} - B_{\tau_\ell})_{t \geq 0}$ is a Brownian motion independent on the information up to time τ_ℓ. So, by symmetry,

$$\mathbb{P}\left[B_t - B_{\tau_\ell} \geq 0 \mid \tau_\ell \leq t\right] = \mathbb{P}\left[B_t - B_{\tau_\ell} \leq 0 \mid \tau_\ell \leq t\right] = 1/2.$$

Thus

$$\begin{aligned} \mathbb{P}\left[B_t \geq \ell\right] &= \mathbb{P}\left[\tau_\ell \leq t,\ B_t - B_{\tau_\ell} \geq 0\right] \\ &= \mathbb{P}\left[\tau_\ell \leq t\right] \mathbb{P}\left[B_t - B_{\tau_\ell} \geq 0 \mid \tau_\ell \leq t\right] \\ &= \frac{1}{2} \mathbb{P}\left[\tau_\ell \leq t\right] \end{aligned}$$

Corollary (Density of first hitting time)

Let B be a Brownian motion starting in 0 and let $\ell \neq 0$. Then the density of τ_ℓ is given by

$$f_{\tau_\ell}(t) = \frac{|\ell|}{\sqrt{2\pi t^3}} e^{-\ell^2/2t} 1_{(0,\infty)}(t)$$

Proof:

Since $-B$ is also a standard Brownian motion, the distribution of τ_ℓ is the same as that of $\tau_{-\ell}$. Therefore, we can assume $\ell > 0$.

It is clear that $P(\tau_\ell < t) = 0$ for $t \leq 0$.

We know from the theorem above that

$$\begin{aligned} \mathbb{P}(\tau_\ell < t) &= 2\mathbb{P}\left[B_t > \ell\right] \\ &= 2\mathbb{P}\left[Z > \frac{\ell}{\sqrt{t}}\right] \\ &= 2 - 2N\left(\frac{\ell}{\sqrt{t}}\right). \end{aligned}$$

Differentiating wrt. t, we get for all $t > 0$

$$\begin{aligned} f_{\tau_\ell}(t) &= \frac{d}{dt}\mathbb{P}(\tau_\ell < t) \\ &= -2N'\left(\frac{\ell}{\sqrt{t}}\right)\left(\frac{-\ell}{2t^{3/2}}\right) \\ &= \frac{\ell}{\sqrt{2\pi t^3}} e^{-\ell^2/2t}. \end{aligned}$$

Remark: In particular, we have $\mathbb{P}(\tau_\ell < \infty) = 1$ and $\mathbb{E}(\tau_\ell) = \infty$.

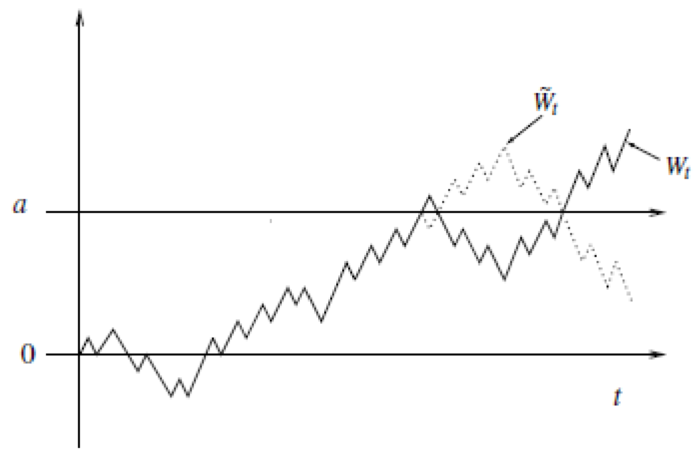
The next statement (given without a proof) is the reflection principle which is often used in applications.

Theorem (Reflection principle)

Let B be a Brownian motion and τ be a stopping time with respect to the filtration of this Brownian motion. Then the process

$$\tilde{B}_t := \begin{cases} B_t, & \text{if } t \le \tau; \\ 2B_\tau - B_t, & \text{if } t > \tau \end{cases}$$

is also a Brownian motion.



The reflection principle when $T = T_a$.

Reflection principle is applied to determine closed form expressions for the prices of barrier options.

For that we need the following result on the joint density for Brownian motion B and its running maximum M.

Theorem (Joint distribution of Brownian motion and its running maximum)

Let B be a Brownian motion and M its running maximum as defined above.

For all $\ell > 0$, $x \leq \ell$, and all $t \geq 0$, we have

$$P(M_t \geq \ell, B_t \leq x) = 1 - N\left(\frac{2\ell - x}{\sqrt{t}}\right),$$

where $N(x) = (2\pi)^{-1/2} \int\limits_{-\infty}^{x} e^{-y^2/2}\, dy$ is the standard normal distribution function.

Proof: Recall that for $\ell > 0$, we have $\{M_t \geq \ell\} = \{\tau_\ell \leq t\}$.

Thus we have

$$\mathbb{P}\left[M_t \geq \ell, B_t \leq x\right] = \mathbb{P}\left[\tau_\ell \leq t, B_t \leq x\right]$$

The reflection principle says that if we replace each path with the path reflected with respect to the line $x = \ell$ after time τ_ℓ then the reflected process will be again Brownian motion.

This and the equality $\tau_\ell = \tilde{\tau}_\ell$, where $\tilde{\tau}_\ell$ is the hitting time of ℓ for $\tilde{B}$, imply that the last probability above is the same as

$$\begin{aligned} P(\tilde{\tau}_\ell \leq t, \tilde{B}_t \geq 2\ell - x) &\overset{\text{note:} x \leq \ell}{=} P(\tilde{B}_t \geq 2\ell - x) \\ &= 1 - N\left(\frac{2\ell - x}{\sqrt{t}}\right). \end{aligned}$$

Example: Pricing of binary barrier option

- Consider the knock-in binary put option knock-in at L and struck at K, with S_0, $K \leq L$.
- In other words, assuming the spot S_0 and the strike K are both smaller than the knock-in level L, the option pays out a dollar at expiry if the underlying ever goes above the knock-in level L during the option's life and ends in the money, $S_T \leq K$.
- In mathematical terms, the payoff function of the knock-in binary put can be written in terms of the indicator function as

$$1_{\tau_L \leq T, S_T \leq K}.$$

- Hence, assume zero interest and dividend rates, the price/premium of the knock-in binary put is given by the expectation under risk neutral probability as

$$\mathbb{E}\left[1_{\tau_L \leq T, S_T \leq K}\right] = \mathbb{P}\left[\tau_L \leq T, S_T \leq K\right].$$

- Moreover, if the underlying follows the Bachelier model, i.e.,

$$S_t = S_0 + \sigma B_t,$$

the price of the knock-in binary put can be determined by applying the reflection principle as

$$\begin{aligned}
\mathbb{E}\left[1_{\tau_L \leq T, S_T \leq K}\right] &= \mathbb{P}\left[\tau_L \leq T, S_T \leq K\right] \\
&= \mathbb{P}\left[\max_{0 \leq u \leq T} S_u \geq L, S_T \leq K\right] \\
&= \mathbb{P}\left[\max_{0 \leq u \leq T} \frac{S_u - S_0}{\sigma} \geq \frac{L - S_0}{\sigma}, \frac{S_T - S_0}{\sigma} \leq \frac{K - S_0}{\sigma}\right] \\
&= \mathbb{P}\left[\max_{0 \leq u \leq T} B_u \geq \frac{L - S_0}{\sigma}, B_T \leq \frac{K - S_0}{\sigma}\right] \\
&= \mathbb{P}\left[B_T \geq \frac{2L - S_0 - K}{\sigma}\right] \\
&= \mathbb{P}\left[S_T \geq 2L - K\right].
\end{aligned}$$

3 Cross variation and quadratic variation

Let $f, g : [0, T] \to \mathbb{R}$, Π be a partition of $[0, T]$: $0=t_0

$$\|\Pi\| := \max_{i \in \{1,2,...,n\}} (t_i - t_{i-1})$$

be the length of the largest interval in the partition Π.

Definition

- The **cross-variation of f and g on** $[0, T]$ is defined by

$$[f,g]_T := \lim_{\|\Pi\|\to 0} \sum_{i=1}^{n} (f(t_i) - f(t_{i-1}))(g(t_i) - g(t_{i-1}))$$

  if this limit exists.

- The **quadratic variation** $[f]_T$ **of** f **on** $[0, T]$ is defined by

$$[f]_T := [f,f]_T,$$

  when $[f,f]_T$ exists.

Let us mention a useful polarization identity (at least when any 3 of the 4 terms exist and finite.):

$$2[f,g]_T = [f+g]_T - [f]_T - [g]_T.$$

The proof is obtained by using the identity $2xy = (x+y)^2 - x^2 - y^2$ in each term of the sum defining the left-hand side and taking limits. The details are left as an exercise.

Example 1

Let $f(t) = g(t) = t$.

Then the cross-variation of f and g on $[0, T]$ is the same as the quadratic variation of either of them and is equal to 0.

Indeed,

$$\sum_{i=1}^{n} (t_i - t_{i-1})^2 \le \|\Pi\| \sum_{i=1}^{n} (t_i - t_{i-1}) = \|\Pi\| T \to 0 \text{ as } \|\Pi\| \to 0.$$

This fact gives a precise meaning to the informal expression

$$dtdt = 0,$$

i.e. the latter simply means that for $f(t) = t$ the quadratic variation of f on $[0, T]$ is equal to 0 for all $T > 0$.

Example 2

Let f and g be differentiable functions on $[0, T]$ and such that their derivatives are square integrable (Riemann integral), i.e.

$$\int_0^T (f'(t))^2\, dt < \infty \quad \text{and} \quad \int_0^T (g'(t))^2\, dt < \infty.$$

Then $[f, g]_T = 0$.

Indeed, by the mean value theorem we have

$$\sum_{i=1}^{n}(f(t_i) - f(t_{i-1}))(g(t_i) - g(t_{i-1})) = \sum_{i=1}^{n} f'(t_i^*)(t_i - t_{i-1})g'(\tilde{t}_i)(t_i - t_{i-1}), \tag{1}$$

where $t_{i-1} \leq t_i^*, \tilde{t}_i \leq t_i$, $i \in \{1, 2, \ldots, n\}$.

Recall the Cauchy-Schwarz inequality: Take discrete random variables X and Y such that $P(X = x_i) = P(Y = y_i) = 1/n$, $i \in \{1, 2, \ldots, n\}$. Then the inequality $E(|XY|) \leq (E(X^2))^{1/2}(E(Y^2))^{1/2}$ boils down to

$$\frac{1}{n}\sum_{i=1}^{n}|x_i y_i| \leq \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right)^{1/2}\left(\frac{1}{n}\sum_{i=1}^{n} y_i^2\right)^{1/2}.$$

By the above inequality, expression (1) does not exceed

$$\begin{aligned} &n\left(\frac{1}{n}\sum_{i=1}^{n}(f'(t_i^*))^2(t_i - t_{i-1})^2\right)^{1/2}\left(\frac{1}{n}\sum_{i=1}^{n}(g'(\tilde{t}_i))^2(t_i - t_{i-1})^2\right)^{1/2} \\ &\leq \|\Pi\|\left(\sum_{i=1}^{n}(f'(t_i^*))^2(t_i - t_{i-1})\right)^{1/2}\left(\sum_{i=1}^{n}(g'(\tilde{t}_i))^2(t_i - t_{i-1})\right)^{1/2}. \end{aligned}$$

As $\|\Pi\| \to 0$, by the square integrability of f' and g'

$$\begin{aligned} \sum_{i=1}^{n}(f'(t_i^*))^2(t_i - t_{i-1}) &\to \int_0^T (f'(t))^2\, dt; \\ \sum_{i=1}^{n}(g'(\tilde{t}_i))^2(t_i - t_{i-1}) &\to \int_0^T (g'(t))^2\, dt. \end{aligned}$$

Since both integrals are finite, we conclude that $[f, g]_T = 0$. This fact can be recorded informally as

$$df(t)dg(t) = f'(t)g'(t)dtdt = 0.$$