9831 Probability and Stochastic Processes for Finance, Fall 2016

Lecture 8

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Outline

- Fundamental theorems of asset pricing
- Risk-neutral pricing in a simple market
- Risk-neutral pricing in higher dimensions

1 Risk-neutral pricing

The first fundamental theorem of asset pricing entails that a market model is *free of arbitrage* if and only if there exists an equivalent probability measure, referred to as a *risk neutral probabiltiy*, so that the discounted prices of risky assets are martingales in the new measure.

The second fundamental theorem of asset pricing entails that a market is *complete* if and only if there exists a *unique* equivalent risk neutral probability measure.

We demonstrate the fundamental theorems in the setting of diffusion models.

1.1 The model

Let $(\Omega, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space and W_t a Brownian motion defined on it.

Our economy consists of

- one **risky asset** S_t whose dynamics are given by

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t \quad \leftrightsquigarrow \quad S_t = S_0 \exp\left(\int_0^t \sigma_u dW_u + \int_0^t \left(\mu_u - \frac{\sigma_u^2}{2}\right) du\right),$$

- and one **risk free asset** B_t whose dynamics are given by

$$\frac{dB_t}{B_t} = R_t dt, \quad B_0 = 1 \quad \leftrightsquigarrow \quad B_t = \exp\left(\int_0^t R_u du\right).$$

Here, μ_t, σ_t, and R_t are general adapted processes with $\sigma_t \neq 0$ a.s..

Define

- the **discount factor** $D_t = \dfrac{1}{B_t}$. It is straightforward to show that the discount factor D_t satisfies the ODE $dD_t = -R_t D_t dt$.
- the **discounted stock price** $\widetilde{S}_t = \dfrac{S_t}{B_t} = D_t S_t$. In other words, B_t is regarded as a numéraire.

Note

- The measure $\mathbb{P}$ is usually referred to as the physical measure.
- The argument in the following can be generalized to the multi risky assets case.

1.2 Self-financing strategy

We set up a portfolio with time-t value V_t.

At any time t, the investor holds

- H_t shares of the stock (random but adapted to the filtration of the Brownian motion), and
- the remainder $V_t - \Delta_t S_t$ is invested in the money account.

That means, we have

$$V_t = \underbrace{H_t S_t}_{\text{risky asset}} + \underbrace{V_t - H_t S_t}_{\text{money account}}$$

- At time t: Agent reallocates his portolio as holding $H_{t+\Delta t}$ shares in the risky asset *without* adding to or withdrawing from his portfolio, i.e.

$$V_t = \underbrace{H_{t+\Delta t} S_t}_{\text{risky asset}} + \underbrace{V_t - H_{t+\Delta t} S_t}_{\text{money account}}$$

  Note that the value of his portfolio is still V_t because he simply moves shares/money from/to risky asset/money account.

  Agent holds the allocation till time $t + \Delta t$.
- At time $t + \Delta t$: Since the price of risky asset has moved to $S_{t+\Delta t}$ and the money account accrued according to interest rate R_t, the value of his portfolio becomes

$$V_{t+\Delta t} = \underbrace{H_{t+\Delta t} S_{t+\Delta t}}_{\text{risky asset}} + \underbrace{(V_t - H_{t+\Delta t} S_t)(1 + R_t \Delta t)}_{\text{money account}}$$

Hence, the change of the value of the agent's portfolio from t to $t + \Delta t$ is given by

$$\begin{aligned}
\Delta V_{t+\Delta t} &= V_{t+\Delta t} - V_t \\
&= \overbrace{H_{t+\Delta t} S_{t+\Delta t}}^{\text{risky asset}} + \overbrace{(V_t - H_{t+\Delta t} S_t)(1 + R_t \Delta t)}^{\text{money account}} - \left[\overbrace{H_{t+\Delta t} S_t}^{\text{risky asset}} + \overbrace{V_t - H_{t+\Delta t} S_t}^{\text{money account}} \right] \\
&= H_{t+\Delta t} \left(S_{t+\Delta t} - S_t \right) + (V_t - H_{t+\Delta t} S_t) R_t \Delta t.
\end{aligned}$$

In the limit, as $\Delta t \to 0$, we have

$$dV_t = H_t dS_t + (V_t - H_t S_t) R_t dt.$$

Hence, a trading strategy H_t is **self-financing** if the above equation holds.

1.3 Arbitrage

Definition

An **arbitrage opportunity** is a self-financing trading strategy H_t if the value V of the trading strategy satisfies

- $V_0 = 0$ (zero initial value)
- $V_T \geq 0$ a.s (almost surely nonnegative value at the terminal time)
- $\mathbb{P}(V_T > 0) > 0$ (positive portfolio value with positive probability)

We say that a model is **free of arbitrage** if it does not allow arbitrage opportunities.

Thus, to determine if a market model is free of arbitrage, we need to show that there does not exist any self-financing trading strategy H_t with zero initial value but positive probability of positive value and almost surely nonnegative value at the terminal time.

1.4 The first fundamental theorem of asset pricing

Definition

A **risk-neutral measure** $\widetilde{\mathbb{P}}$ (also called an **equivalent martingale measure (EMM)**) is a probability measure

- that is equivalent to $\mathbb{P}$ and
- under which the discounted stock prices are martingales.

The first fundamental theorem of asset pricing

A market model is free of arbitrage if and only if there exists a risk-neutral measure.

In the following, we demonstrate how to characterize a risk-neutral measure in our setting so that, according to the first fundamental theorem of asset pricing, the model is free of arbitrage.

1.5 Dynamics of the discounted stock price

Applying Ito's formula to $\widetilde{S}_t$, we have

$$\begin{aligned} d\widetilde{S}_t &= d(D_t S_t) \\ &= D_t dS_t + S_t dD_t \\ &= D_t S_t \left(\mu_t dt + \sigma_t dW_t\right) - S_t R_t D_t dt \\ &= D_t S_t \left[(\mu_t - R_t)dt + \sigma_t dW_t\right] \\ &= \widetilde{S}_t \left[(\mu_t - R_t)dt + \sigma_t dW_t\right]. \end{aligned}$$

Note that when applying Ito's product rule, since D_t has no diffusion part, there is no need to include the quadratic covariation between S_t and D_t. Indeed, $[S, D]_t = 0$ almost surely for all t.

1.6 Market price of risk

We may rewrite the dynamics of the discounted price $\widetilde{S}_t$ as follows:

$$\begin{aligned} d\widetilde{S}_t &= \widetilde{S}_t[(\mu_t - R_t)dt + \sigma_t dW_t] \\ &= \sigma_t \widetilde{S}_t(\theta_t dt + dW_t) \\ &= \sigma_t \widetilde{S}_t d\widetilde{W}_t, \end{aligned}$$

where $\theta_t = \frac{\mu_t - R_t}{\sigma_t}$ and $d\widetilde{W}_t = dW_t + \theta_t dt$ (Brownian motion with drift).

Note that

- θ_t is referred to as **the market price of risk** or **Sharpe ratio**. It is the excess instantaneous rate of return of the stock (over the money account) per unit of volatility.
- Under the measure $\mathbb{P}$, $\widetilde{W}_t$ is a Brownian motion with drift θ_t.
- Under the measure $\mathbb{P}$, the discounted price $\widetilde{S}_t$ has the drift $\widetilde{S}_t(\mu_t - R_t)$. It is therefore not a $\mathbb{P}$-martingale, unless $\mu_t = R_t$.

1.7 Risk neutral probability

Define a new probability measure $\widetilde{\mathbb{P}}$ by the Radon-Nikodym derivative

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = e^{-\int_0^T \theta_t dW_t - \frac{1}{2}\int_0^T \theta_t^2 dt}.$$

By Girsanov's theorem we know that $\widetilde{W}_t$ is a Brownian motion under the new measure $\widetilde{\mathbb{P}}$.

Thus, the discounted price $\widetilde{S}_t$ becomes driftless in the $\widetilde{\mathbb{P}}$-measure, henceforth a $\widetilde{\mathbb{P}}$-martingale.

Since the above Radon-Nidodym derivative is positive, $\widetilde{\mathbb{P}} \sim \mathbb{P}$.

The new probability measure $\widetilde{\mathbb{P}}$ is therefore a risk neutral measure.

As $dW_t = d\widetilde{W}_t - \theta_t dt$, we can write the evolution of S_t as follows

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t = R_t S_t dt + \sigma S_t d\widetilde{W}_t,$$

which means that the instantaneous rate of return of the stock under $\widetilde{\mathbb{P}}$ is the same as for the money account.

This explains the term *risk-neutral* measure for $\widetilde{\mathbb{P}}$.

Remark

For diffusion models, we have just demonstrated (using Girsanov) that there exists a risk-neutral measure.

Hence, by the First Fundamental Theorem of Asset Pricing, the model is free of arbitrage.

1.8 Self-financing strategies are $\widetilde{\mathbb{P}}$-martingales

The discounted value of any self-financing strategy is a martingale under the measure $\widetilde{\mathbb{P}}$:

Let $\tilde{V}_t$ be the discounted value of the value corresponding to the self-financing strategy H_t, i.e., $\tilde{V}_t = D_t V_t$ where V_t is the value of H_t.

Indeed, using Ito's product formula, we obtain

$$\begin{aligned} d\tilde{V}_t &= d(D_t V_t) \\ &= V_t dD_t + D_t dV_t \\ &= -V_t R_t D_t dt + D_t \left[H_t dS_t + (V_t - H_t S_t) R_t dt\right] \\ &= D_t \left[H_t dS_t - H_t S_t R_t dt\right] \\ &= H_t (D_t dS_t - D_t S_t R_t dt) \\ &= H_t d\widetilde{S}_t \\ &= H_t \sigma_t \widetilde{S}_t d\widetilde{W}_t, \end{aligned}$$

Hence, $\tilde{V}_t$ is a $\widetilde{\mathbb{P}}$-martingale.

In other words, if an agent trades in a self-financing manner, the discounted value $\tilde{V}$ of his wealth under the risk neutral probability $\widetilde{\mathbb{P}}$ is a martingale.

In particular,

$$\tilde{V}_t = \widetilde{\mathbb{E}}[\tilde{V}_T | \mathcal{F}_t] \quad \Longleftrightarrow \quad D_t V_t = \widetilde{\mathbb{E}}[D_T V_T | \mathcal{F}_t] \quad \Longleftrightarrow \quad V_t = \widetilde{\mathbb{E}}\left[e^{-\int_t^T R_s ds} V_T \middle| \mathcal{F}_t \right].$$

In other words, the discounted value of a self-financing strategy at time t is the conditional expectation under risk neutral probability of its discounted value at the terminal time T.

The last equation is sometimes also written by considering B_t as a numéraire as

$$V_t = B_t \widetilde{\mathbb{E}}\left[\frac{V_T}{B_T} \middle| \mathcal{F}_t \right].$$

1.9 Pricing of replicable claims

Suppose that a contingent claim with payoff X_T at expiry T can be replicated by a self-financing trading strategy, i.e.,

$$X_T = V_T,$$

where V is the value of a self-financing strategy.

The price X_t of the claim at time t should equal the value V_t of the replicating self-financing trading strategy; otherwise there's an arbitrage (law of one price).

Hence we have

$$\begin{aligned}
X_t &= V_t \\
&= \widetilde{\mathbb{E}}\left[e^{-\int_t^T R_s ds} V_T \middle| \mathcal{F}_t \right] = \widetilde{\mathbb{E}}\left[e^{-\int_t^T R_s ds} X_T \middle| \mathcal{F}_t \right] \\
&= B_t \widetilde{\mathbb{E}}\left[\frac{V_T}{B_T} \middle| \mathcal{F}_t \right] = B_t \widetilde{\mathbb{E}}\left[\frac{X_T}{B_T} \middle| \mathcal{F}_t \right].
\end{aligned}$$

In words, the price X_t at time t of a replicable contingent claim, which pays off X_T at expiry T, is equal to the conditional expectation of the discounted payoff under the risk neutral probability.

The above formula is sometimes called the risk-neutral pricing formula (in a single stock market).

Remarks

- Note that at this stage, the contingent claim payoff X_T can be completely general, for instance, it can be path-dependent, American style, etc., as long as it is $\mathcal{F}_T$-measurable.
- So the next question is: when is a contingent claim replicable?

1.10 The second fundamental theorem of asset pricing

Definition

A market (model) is called **complete** if, for every contingent claim with payoff function depending on the assets in the market, there exists a self-financing strategy which replicates the payoff of the contingent claim.

The second fundamental theorem of asset pricing

A market is complete if and only if there exists a *unique* risk neutral probability measure.

In our case, since the risk neutral probability $\widetilde{\mathbb{P}}$ is uniquely determined by the market price of risk, $\theta_t = \dfrac{\mu_t - R_t}{\sigma_t}$, according to the second fundamental theorem of asset pricing, the model is complete.

Hence, the value X_t at time t of a contingent claim that pays off X_T at expiry T is given by

$$X_t = B_t \widetilde{\mathbb{E}}\left[\frac{X_T}{B_T}\middle|\mathcal{F}_t\right].$$

To determine the replicating strategy, note that since $\widetilde{X}_t$ is a $\widetilde{\mathbb{P}}$-martingale, by the martingale representation theorem, there exists an adapted process ψ_t such that

$$\begin{aligned}
\widetilde{X}_t &= \widetilde{X}_0 + \int_0^t \psi_s d\widetilde{W}_s \\
&= \widetilde{X}_0 + \int_0^t \frac{\psi_s}{\sigma_s \widetilde{S}_s} \sigma_s \widetilde{S}_s d\widetilde{W}_s \\
&= \widetilde{X}_0 + \int_0^t \frac{\psi_s}{\sigma_s \widetilde{S}_s} d\widetilde{S}_s.
\end{aligned}$$

Hence, the replicating strategy is the self-financing strategy associated with holding $H_t := \dfrac{\psi_t}{\sigma_t \widetilde{S}_t}$ shares in the risky asset.

Remarks

- We have made two key assumptions to make the replication possible:
  - $\sigma_t \neq 0$
  - $\mathcal{F}_t$ is the filtration generated by the Brownian motion.
- The MRT justifies the risk-neutral pricing formula above. The MRT guarantees that a replicating strategy exists by providing the existence of ψ_t. It does not provide a method for finding the replicating strategy H_t as it involves the process ψ_t. We have a closer look at this issue in a later lecture.

1.11 Example: Black-Scholes-Merton (BSM) model

We assume that

- $R_t \equiv r \geq 0$,
- $\sigma_t \equiv \sigma > 0$, and
- μ_t an adapted stochastic process.

We shall compute the price of a European derivative security with payoff

$$V_T = f(S_T),$$

where f is a non-negative Borel function.

We already know that under the risk-neutral measure $\widetilde{\mathbb{P}}$ the process $\widetilde{W}_t = W_t + \dfrac{\mu - r}{\sigma} t$ is a Brownian motion, and we have

$$dS_t = rS_t\,dt + \sigma S_t\,d\widetilde{W}_t, \quad \leftrightsquigarrow \quad S_t = S(0)e^{(r-\sigma^2/2)t+\sigma\widetilde{W}_t}.$$

We need to compute

$$V_t = \widetilde{\mathbb{E}}\left(e^{-r(T-t)}f(S_T)\,|\,\mathcal{F}_t\right) = e^{-r(T-t)}\,\widetilde{\mathbb{E}}\left(f(S_T)\,|\,\mathcal{F}_t\right).$$

Writing $S_T = S_t e^{(r-\sigma^2/2)(T-t)+\sigma(\widetilde{W}_T-\widetilde{W}_t)}$ and using the independence lemma ($\widetilde{W}_T - \widetilde{W}_t =: \sqrt{T-t}\,Z$ is independent of $\mathcal{F}_t$ and S_t is $\mathcal{F}_t$-measurable), we get that

$$\widetilde{\mathbb{E}}\left(f\left(S_t e^{(r-\sigma^2/2)(T-t)+\sqrt{T-t}\,Z}\right)\,|\,\mathcal{F}_t\right) = g(t, S_t),$$

where

$$g(t,x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f\left(xe^{(r-\sigma^2/2)(T-t)+\sqrt{T-t}\,z}\right)e^{-z^2/2}\,dz.$$

We have accomplished two things:

- we have found a closed form formula for the price of any European style contingent claim: $V_t = e^{-r(T-t)}g(t, S_t)$.
- we have shown that a geometric Brownian motion with constant drift and volatility parameters is a Markov process.

For European calls and puts (i.e. $f(x) = (x-K)^+$ and $f(x) = (K-x)^+$), the function g can be calculated explicitly and leads to the standard BSM formula. The details are left as an exercise (or see pages 219-220 in the textbook).

1.12 Summary

We demonstrated the theory of risk neutral pricing in a diffusion model for one risky and one riskless assets.

The three main ingredients in risk neutral pricing theory are

- Self-financing replicating strategy (for the pricing of derivatives)
- Girsanov theorem (for the existence of a risk neutral measure)
- Martingale representation theorem (for the construction of a replicating strategy)

2 Risk-neutral pricing in a multidimensional market model

2.1 The model

Consider an economy consisting of $n+1$ assets whose prices, under the physical probability measure $\mathbb{P}$, are driven by the system of SDEs

$$\frac{dS_t^i}{S_t^i} = \mu_t^i dt + \sum_{j=1}^{m} \sigma_{j,t}^i dW_t^j,$$

for $i = 1, \cdots, n+1$, and where $W_t = (W_t^1, \cdots, W_t^m)^T$ is an m-dimensional Brownian motion.

We assume that the vector $\mu_t = (\mu_t^1, \cdots, \mu_t^{n+1})^T$ and the matrix $\sigma_t = (\sigma_{j,t}^i)_{i=1,\ldots,n+1;\, j=1,\ldots,m}$ are adapted processes.

In matrix form this system reads

$$\begin{bmatrix} \frac{dS_t^1}{S_t^1} \\ \frac{dS_t^2}{S_t^2} \\ \vdots \\ \frac{dS_t^{n+1}}{S_t^{n+1}} \end{bmatrix}_{(n+1)\times 1} = \begin{bmatrix} \mu_t^1 \\ \mu_t^2 \\ \vdots \\ \mu_t^{n+1} \end{bmatrix}_{(n+1)\times 1} dt + \begin{bmatrix} \sigma_{1,t}^1 & \sigma_{2,t}^1 & \cdots & \sigma_{m,t}^1 \\ \sigma_{1,t}^2 & \sigma_{2,t}^2 & \cdots & \sigma_{m,t}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1,t}^{n+1} & \sigma_{2,t}^{n+1} & \cdots & \sigma_{m,t}^{n+1} \end{bmatrix}_{(n+1)\times m} \begin{bmatrix} dW_t^1 \\ dW_t^2 \\ \vdots \\ dW_t^m \end{bmatrix}_{m\times 1}$$

Stocks S^i are generalized geometric Brownian motions:

Define $\sigma_t^i = \sqrt{\sum_{j=1}^m (\sigma_{j,t}^i)^2}$ and $Z_0 = 0$ as well as

$$dZ_t^i = \sum_{j=1}^m \frac{\sigma_{j,t}^i}{\sigma_t^i} dW_t^j.$$

Then Z_t^i is a continuous martingale and using $d[W^k, W^j]_t = 0$, $k \neq j$, and $d[W^j]_t = dt$, we obtain

$$d[Z^i]_t = \frac{1}{(\sigma_t^i)^2} \sum_{j=1}^m (\sigma_{j,t}^i)^2 \, dt = dt.$$

From Levy's theorem, we conclude that Z_t^i is a standard Brownian motion.

Rewriting S_t^i in terms of this new Brownian motion, we get

$$\frac{dS_t^i}{S_t^i} = \mu_t^i dt + \sigma_t^i dZ_t^i,$$

and conclude that S_t^i is a generalized geometric Brownian motion.

Correlation between two stocks S^k and S^ℓ

The stock prices are correlated through their driving Brownian motion.

To see this consider first

$$d[Z^k, Z^\ell]_t = dZ_t^k dZ_t^\ell = \sum_{j=1}^m \underbrace{\frac{\sigma_{j,t}^k \sigma_{j,t}^\ell}{\sigma_t^k \sigma_t^\ell}}_{=:\rho_t^{k\ell}} dt = \rho_t^{k\ell} dt$$

The process $\rho_t^{k\ell}$ is called **instantaneous correlation** of Z^k and Z^ℓ.

We know that

$$\mathrm{Corr}(Z_t^k, Z_t^\ell) = \frac{\mathbb{E}(Z_t^k Z_t^\ell)}{t}.$$

Hence we apply Ito's product rule, obtain

$$d(Z_t^k Z_t^\ell) = Z_t^k dZ_t^\ell + Z_t^\ell dZ_t^k + d[Z^k, dZ^\ell]_t,$$

and after integrating and taking expectations, we see that

$$\mathrm{Corr}(Z_t^k, Z_t^\ell) = \frac{\int_0^t \rho_s^{k\ell} ds}{t}.$$

We see immediately, that if the processes $\sigma_{j,t}^k$ and $\sigma_{j,t}^\ell$ are constant (i.e. deterministic and independent of t), then σ_t^k, σ_t^ℓ and $\rho_t^{k\ell}$ are independent as well.

In this case the above equation reduces to

$$\mathrm{Corr}(Z_t^k, Z_t^\ell) = \rho^{k\ell}.$$

Finally, we have

$$\begin{aligned} d[S^k, S^\ell]_t &= \sigma_t^k \sigma_t^\ell S_t^k, S_t^\ell dZ_t^k dZ_t^\ell \\ &= \sigma_t^k \sigma_t^\ell S_t^k, S_t^\ell \rho_t^{k\ell} dt. \end{aligned}$$

In the case that σ_t^k, σ_t^ℓ and $\rho_t^{k\ell}$ are deterministic, we can explicitly compute the covariance and correlation of S^k and S^ℓ as we did in Section 3.4, Lecture 6.

2.2 Numéraire

Key concept in asset pricing theory:

A **numéraire** is the price of any positive, nondividend paying asset.

It can be taken as a unit of reference when pricing an asset or contingent claim.

Examples

- Money account (domestic): $N_t = e^{\int_0^t R_s ds}$, where R_t is an adapted process representing the risk-free interest rate process.

  In this case

  $$\widetilde{S}_t = \frac{S_t}{N_t} = e^{-\int_0^t R_s ds} S_t$$

  represents the discounted price of the asset S_t.

- Exchange rate: $N_t = R_t$, where R_t is an exchange rate between different currencies.

  In this case

  $$\widetilde{S}_t = \frac{S_t}{R_t}$$

  represents the price of the asset in units of the foreign currency. For example, if $R_t = 1.11$ is the exchange rate from Euro to \$ and $S_t = \$1$, then $\widetilde{S}_t = S_t/R_t = 0.9$ Euro.

2.3 Recall: First fundamental theorem of asset pricing

The following two statements are equivalent.

- The economy allows no arbitrage opportunity.
- For a given numéraire N_t there exists an equivalent martingale measure (EMM) $\mathbb{Q}$, i.e. $\mathbb{Q} \sim \mathbb{P}$ and the values of the assets in the economy denominated by the numéraire N_t are $\mathbb{Q}$-martingales.

  In other words, for any $i = 1, 2, \cdots, n+1$, we have

  $$\frac{S_t^i}{N_t} = \mathbb{E}^{\mathbb{Q}} \left[\left. \frac{S_T^i}{N_T} \right| \mathcal{F}_t \right].$$

Remark

If a money account (or cash) B_t is used as numéraire, the associated EMM is called the **risk-neutral probability** and the pricing formula reads as the one that we are familiar with

$$\frac{S_t^i}{B_t} = \mathbb{E}^{\mathbb{Q}} \left[\left. \frac{S_T^i}{B_T} \right| \mathcal{F}_t \right] \quad \leftrightsquigarrow \quad S_t^i = \mathbb{E}^{\mathbb{Q}} \left[\left. e^{-\int_t^T R_s ds} S_T^i \right| \mathcal{F}_t \right].$$

2.4 Dynamics of the denominated assets

Assume that the numéraire N_t satisfies the SDE (in the $\mathbb{P}$-measure)

$$\frac{dN_t}{N_t} = \mu_t^N \, dt + \sum_j \sigma_{j,t}^N dW_t^j,$$

where μ_t^N and $(\sigma_{1,t}^N, \ldots, \sigma_{m,t}^N)$ are adapted processes and W_t the same m-dimensional Brownian motion as before.

At this point, the numéraire N_t can be any asset in the market (as long as it is tradable).

The value of the asset S_t^i denominated by the numéraire N_t, i.e, $\widetilde{S}_t^i = \dfrac{S_t^i}{N_t}$, follows by Ito's formula applied to

$$\begin{aligned} f(s,n) = \frac{s}{n} \quad \implies \quad & f_s = \frac{1}{n},\ f_n = -\frac{s}{n^2}, \\ & f_{ss} = 0,\ f_{sn} = -\frac{1}{n^2},\ f_{nn} = \frac{2s}{n^3}. \end{aligned}$$

We obtain

$$\begin{aligned} d\widetilde{S}_t^i &= d\left(\frac{S_t^i}{N_t}\right) = df(S_t^i, N_t) \\ &= f_s dS_t^i + f_n dN_t + \frac{1}{2} f_{ss} d[S^i]_t + f_{sn} d[S^i, N]_t + \frac{1}{2} f_{nn} d[N]_t \\ &= \frac{dS_t^i}{N_t} - \frac{S_t^i dN_t}{N_t^2} - \frac{d[S^i, N]_t}{N_t^2} + \frac{S_t^i}{N_t^3} d[N]_t \\ &= \left[\mu_t^i \widetilde{S}_t^i - \mu_t^N \widetilde{S}_t^i + \sum_j \widetilde{S}_t^i \sigma_{j,t}^N \left(\sigma_{j,t}^N - \sigma_{j,t}^i\right)\right] dt + \sum_j \sigma_{j,t}^i \widetilde{S}_t^i dW_t^j - \sum_j \widetilde{S}_t^i \sigma_{j,t}^N dW_t^j \end{aligned}$$

We divide both sides of the last equation by $\widetilde{S}_t^i$ to obtain

$$\frac{d\widetilde{S}_t^i}{\widetilde{S}_t^i} = \left[\mu_t^i - \mu_t^N + \sum_j \sigma_{j,t}^N \left(\sigma_{j,t}^N - \sigma_{j,t}^i\right)\right] dt + \sum_j \left(\sigma_{j,t}^i - \sigma_{j,t}^N\right) dW_t^j.$$

2.6 Girsanov change of measure

Recall that, by Girsanov's theorem, if we define a new measure $\mathbb{Q}$ by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\int_0^T \theta_t \cdot dW_t - \frac{1}{2}\int_0^T |\theta_t|^2 dt},$$

and assume that the process

$$Z_t = e^{-\int_0^t \theta_s \cdot dW_s - \frac{1}{2}\int_0^t |\theta_s|^2 ds}$$

is a $\mathbb{P}$-martingale (i.e. integrability condition is fulfilled), then the process $\widetilde{W}_t$ defined as

$$d\widetilde{W}_t = dW_t + \theta_t dt$$

is a Brownian motion in the $\mathbb{Q}$-measure.

Therefore, under the $\mathbb{Q}$-measure, the system of SDEs for the $\widetilde{S}_t^i$'s becomes

$$\frac{d\widetilde{S}_t^i}{\widetilde{S}_t^i} = \left[\mu_t^i - \mu_t^N + \sum_j \sigma_{j,t}^N \left(\sigma_{j,t}^N - \sigma_{j,t}^i\right) - \sum_j \left(\sigma_{j,t}^i - \sigma_{j,t}^N\right)\theta_t^j\right] dt + \sum_j \left(\sigma_{j,t}^i - \sigma_{j,t}^N\right) d\widetilde{W}_t^j$$

Hence, if we find a solution θ_t to the linear system

$$\sum_j \left(\sigma_{j,t}^i - \sigma_{j,t}^N\right)\theta_t^j = \mu_t^i - \mu_t^N + \sum_j \sigma_{j,t}^N \left(\sigma_{j,t}^N - \sigma_{j,t}^i\right)$$

then the denominated process of $\widetilde{S}_t^i$ in the $\mathbb{Q}$-measure is driftless, i.e., a $\mathbb{Q}$-martingale.

In matrix form, this system reads

$$\begin{bmatrix} \sigma_{1,t}^1 - \sigma_{1,t}^N & \cdots & \sigma_{m,t}^1 - \sigma_{m,t}^N \\ \sigma_{1,t}^2 - \sigma_{1,t}^N & \cdots & \sigma_{m,t}^2 - \sigma_{m,t}^N \\ \vdots & & \vdots \\ \sigma_{1,t}^{n+1} - \sigma_{1,t}^N & \cdots & \sigma_{m,t}^{n+1} - \sigma_{m,t}^N \end{bmatrix}_{(n+1)\times m} \begin{bmatrix} \theta_t^1 \\ \theta_t^2 \\ \vdots \\ \theta_t^m \end{bmatrix}_{m\times 1}$$

$$= \begin{bmatrix} \mu_t^1 - \mu_t^N \\ \mu_t^2 - \mu_t^N \\ \vdots \\ \mu_t^{n+1} - \mu_t^N \end{bmatrix}_{(n+1)\times 1} + \begin{bmatrix} \sum_j \left[(\sigma_{j,t}^N)^2 - \sigma_{j,t}^N \sigma_{j,t}^1\right] \\ \sum_j \left[(\sigma_{j,t}^N)^2 - \sigma_{j,t}^N \sigma_{j,t}^2\right] \\ \vdots \\ \sum_j \left[(\sigma_{j,t}^N)^2 - \sigma_{j,t}^N \sigma_{j,t}^{n+1}\right] \end{bmatrix}_{(n+1)\times 1}$$

From the first fundamental theorem of asset pricing, we know:

If one cannot solve the above system, then there is an arbitrage opportunity in the model and it should not be used for pricing.