9831 Probability and Stochastic Processes for Finance, Fall 2016

Lecture 9

Anja Richter

Department of Mathematics

Baruch College
The City University of New York

Outline

- Multidimensional market model
- Change of numéraire technique
- Solving stochastic differential equations

1 Risk neutral pricing in a multidimensional market model

1.1 Recall model setup and and key system of equations

We consider an economy consisting of $n+1$ assets whose prices, under the physical probability measure $\mathbb{P}$, are driven by the system of SDEs

$$\frac{dS_t^i}{S_t^i} = \mu_t^i dt + \sum_{j=1}^m \sigma_{j,t}^i dW_t^j,$$

for $i = 1, \cdots, n+1$, and where $W_t = (W_t^1, \cdots, W_t^m)^T$ is an m-dimensional Brownian motion.

We assume that the vector $\mu_t = (\mu_t^1, \cdots, \mu_t^{n+1})^T$ and the matrix $\sigma_t = (\sigma_{j,t}^i)_{i=1,\ldots,n+1;\, j=1,\ldots,m}$ are adapted processes.

In matrix form this system reads

$$\begin{bmatrix} \frac{dS_t^1}{S_t^1} \\ \frac{dS_t^2}{S_t^2} \\ \vdots \\ \frac{dS_t^{n+1}}{S_t^{n+1}} \end{bmatrix}_{(n+1)\times 1} = \begin{bmatrix} \mu_t^1 \\ \mu_t^2 \\ \vdots \\ \mu_t^{n+1} \end{bmatrix}_{(n+1)\times 1} dt + \begin{bmatrix} \sigma_{1,t}^1 & \sigma_{2,t}^1 & \cdots & \sigma_{m,t}^1 \\ \sigma_{1,t}^2 & \sigma_{2,t}^2 & \cdots & \sigma_{m,t}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1,t}^{n+1} & \sigma_{2,t}^{n+1} & \cdots & \sigma_{m,t}^{n+1} \end{bmatrix}_{(n+1)\times m} \begin{bmatrix} dW_t^1 \\ dW_t^2 \\ \vdots \\ dW_t^m \end{bmatrix}_{m\times 1}$$

Recall:

The following two statements are equivalent.

- The economy allows no arbitrage opportunity.
- For a given numéraire N_t there exists an equivalent martingale measure (EMM) $\mathbb{Q}$, i.e. $\mathbb{Q} \sim \mathbb{P}$ and the values of the assets in the economy denominated by the numéraire N_t are $\mathbb{Q}$-martingales.

  In other words, for any $i = 1, 2, \cdots, n+1$, we have

$$\frac{S_t^i}{N_t} = \mathbb{E}^{\mathbb{Q}}\left[\frac{S_T^i}{N_T}\middle| \mathcal{F}_t\right].$$

If a money account (or cash) B_t is used as numéraire, the associated EMM is called the **risk-neutral probability** and the pricing formula reads as the one that we are familiar with

$$\frac{S_t^i}{B_t} = \mathbb{E}^{\mathbb{Q}}\left[\frac{S_T^i}{B_T}\middle| \mathcal{F}_t\right] \quad \leftrightsquigarrow \quad S_t^i = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T R_s ds} S_T^i \middle| \mathcal{F}_t\right].$$

The change of measure that makes $\tilde{S}_t^i$ a martingale

Define a new measure $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\int_0^T \theta_t \cdot dW_t - \frac{1}{2}\int_0^T |\theta_t|^2 dt},$$

where θ_t solves the following key system of linear equations

$$\sum_j \left(\sigma_{j,t}^i - \sigma_{j,t}^N\right)\theta_t^j = \mu_t^i - \mu_t^N + \sum_j \sigma_{j,t}^N \left(\sigma_{j,t}^N - \sigma_{j,t}^i\right).$$

In matrix form this system reads

$$\begin{bmatrix} \sigma_{1,t}^1 - \sigma_{1,t}^N & \cdots & \sigma_{m,t}^1 - \sigma_{m,t}^N \\ \sigma_{1,t}^2 - \sigma_{1,t}^N & \cdots & \sigma_{m,t}^2 - \sigma_{m,t}^N \\ \vdots & & \vdots \\ \sigma_{1,t}^{n+1} - \sigma_{1,t}^N & \cdots & \sigma_{m,t}^{n+1} - \sigma_{m,t}^N \end{bmatrix}_{(n+1)\times m} \begin{bmatrix} \theta_t^1 \\ \theta_t^2 \\ \vdots \\ \theta_t^m \end{bmatrix}_{m\times 1}$$

$$= \begin{bmatrix} \mu_t^1 - \mu_t^N \\ \mu_t^2 - \mu_t^N \\ \vdots \\ \mu_t^{n+1} - \mu_t^N \end{bmatrix}_{(n+1)\times 1} + \begin{bmatrix} \sum_j \left[(\sigma_{j,t}^N)^2 - \sigma_{j,t}^N \sigma_{j,t}^1\right] \\ \sum_j \left[(\sigma_{j,t}^N)^2 - \sigma_{j,t}^N \sigma_{j,t}^2\right] \\ \vdots \\ \sum_j \left[(\sigma_{j,t}^N)^2 - \sigma_{j,t}^N \sigma_{j,t}^{n+1}\right] \end{bmatrix}_{(n+1)\times 1}$$

From the first fundamental theorem of asset pricing, we know:

If one cannot solve the above system, then there is an arbitrage opportunity in the model and it should not be used for pricing.

Recall from linear algebra:

We have $n+1$ equations for m unknown processes.

We can write the above as

$$Ax = b,$$

where

$$A = \begin{bmatrix} \sigma^1_{1,t} - \sigma^N_{1,t} & \cdots & \sigma^1_{m,t} - \sigma^N_{m,t} \\ \sigma^2_{1,t} - \sigma^N_{1,t} & \cdots & \sigma^2_{m,t} - \sigma^N_{m,t} \\ \vdots & & \vdots \\ \sigma^{n+1}_{1,t} - \sigma^N_{1,t} & \cdots & \sigma^{n+1}_{m,t} - \sigma^N_{m,t} \end{bmatrix}_{(n+1)\times m},$$

$$b = \begin{bmatrix} \mu^1_t - \mu^N_t \\ \mu^2_t - \mu^N_t \\ \vdots \\ \mu^{n+1}_t - \mu^N_t \end{bmatrix}_{(n+1)\times 1} + \begin{bmatrix} \sum_j \left[(\sigma^N_{j,t})^2 - \sigma^N_{j,t}\sigma^1_{j,t}\right] \\ \sum_j \left[(\sigma^N_{j,t})^2 - \sigma^N_{j,t}\sigma^2_{j,t}\right] \\ \vdots \\ \sum_j \left[(\sigma^N_{j,t})^2 - \sigma^N_{j,t}\sigma^{n+1}_{j,t}\right] \end{bmatrix}_{(n+1)\times 1},$$

$$x = \theta_t.$$

We know:

- If rank(A) =rank$(A \mid b) = m$, then there exists a unique solution.
- If rank(A) =rank$(A \ | \ b)
- If rank(A) <rank$(A \mid b)$, then there exists no solution.

1.2 Numeraire and asset price dynamics under the EMM

Assume there exists an EMM $\mathbb{Q}$. Then we have:

The SDE for the numeraire N_t in the $\mathbb{Q}$ measure becomes

$$\begin{aligned}\frac{dN_t}{N_t} &= \mu_t^N\, dt + \sum_j \sigma_{j,t}^N (d\widetilde{W}_t^j - \theta_t^j dt)\\ &= \left(\mu_t^N - \sum_j \sigma_{j,t}^N \theta_t^j\right)\, dt + \sum_j \sigma_{j,t}^N d\widetilde{W}_t^j.\end{aligned}$$

In other words, the numeraire N_t under its associated equivalent martingale measure $\mathbb{Q}$ has drift $\mu_t^N - \sum_j \sigma_{j,t}^N \theta_t^j$.

Also, the SDE for the asset S^i in the measure $\mathbb{Q}$ becomes

$$\begin{aligned}\frac{dS_t^i}{S_t^i} &= \mu_t^i dt + \sum_j \sigma_{j,t}^i (d\widetilde{W}_t^j - \theta_t^j dt)\\ &= \left(\mu_t^i - \sum_j \sigma_{j,t}^i \theta_t^j\right)\, dt + \sum_j \sigma_{j,t}^i d\widetilde{W}_t^j.\end{aligned}$$

Hence, in the $\mathbb{Q}$ measure, the drift of S_i is adjusted to $\mu_t^i - \sum_j \sigma_{j,t}^i \theta_t^j$.

1.3 Example for a model with arbitrage

Let $n = 1, m = 1$, i.e. we consider two stocks driven by one Brownian motion.

Also, let all parameters be constant, $\sigma^1, \sigma^2 > 0$ and let the numeraire be a money account B_t with constant interest rate r.

Then the above system of equations are given as

$$\begin{cases} \sigma^1 \theta_1 = \mu^1 - r\\ \sigma^2 \theta_1 = \mu^2 - r.\end{cases}$$

These equations have a solution if and only if

$$\frac{\mu^1 - r}{\sigma^1} = \frac{\mu^2 - r}{\sigma^2}.$$

We can construct an arbitrage opportunity when the last equality does not hold.

For our analysis we set

$$\alpha := \frac{\mu^1 - r}{\sigma^1} - \frac{\mu^2 - r}{\sigma^2},$$

and assume $\alpha \neq 0$.

How should we invest in order to realize arbitrage?

The idea is to choose strategies ψ_t^1 and ψ_t^2 in such a way that the "noise term" dW_t in the dynamics of the discounted portfolio value cancels out.

This will allow us to synthesize a riskless asset with a rate higher than r.

We set

$$V_t = \psi_t^1 S_t^1 + \psi_t^2 S_t^2 + \gamma_t B_t,$$

and just as in the case of a single stock model, we obtain (under the self-financing condition)

$$dV_t = \psi_t^1 dS_t^1 + \psi_t^2 dS_t^2 + r(V_t - \psi_t^1 S_t^1 - \psi_t^2 S_t^2)dt.$$

From the above and the equations for the stock prices we find that

$$\begin{aligned} d(D_t V_t) &= D_t(\sigma^1 \psi_t^1 S_t^1 + \sigma^2 \psi_t^2 S_t^2) dW_t \\ &\quad + D_t((\mu^1 - r)\psi_t^1 S_t^1 + (\mu^2 - r)\psi_t^2 S_t^2)dt. \qquad (**) \end{aligned}$$

Setting the coefficient in front of dW_t to zero, we arrive at the equation which gives us a relation between ψ^1 and ψ^2 as follows

$$\sigma^1 \psi_t^1 S_t^1 = -\sigma^2 \psi_t^2 S_t^2 \quad \text{or} \quad \frac{\psi_t^1}{\psi_t^2} = -\frac{\sigma^2 S_t^2}{\sigma^1 S_t^1}. \qquad (*)$$

Under this condition, we have

$$d(D_t V_t) = D_t((\mu^1 - r)\psi_t^1 S_t^1 + (\mu^2 - r)\psi_t^2 S_t^2)dt = \alpha \sigma^1 D_t \psi_t^1 S_t^1 dt.$$

Note that $\sigma^1,\ D_t,\ S_t^1 > 0$.

Hence, if $\mu > 0$ then we should choose $\Psi_t^1 > 0$ to ensure a sure win (as opposed to sure loss), and if $\mu < 0$ we should choose $\psi_t^1 < 0$. In either case the proportion of each asset is determined by $(*)$.

To be more precise, we now assume that $\alpha > 0$ and make the choice of

$$\psi_t^1 = \frac{1}{S_t^1 \sigma^1} \quad \text{and} \quad \psi_t^2 = -\frac{1}{S_t^2 \sigma^2}.$$

Then the initial price of this portfolio is $1/\sigma^1 - 1/\sigma^2$.

If this amount is positive we borrow this amount from the money account, if it is negative we deposit $1/\sigma^2 - 1/\sigma^1$ to the money account.

Therefore, at time 0 we have $V_0 = 0$ (no cost to us).

With our choice of ψ_t^1 and ψ_t^2 we get from $(**)$

$$d(D_t V_t) = \alpha D_t dt.$$

The right-hand side is positive and is also non-random. Therefore, this portfolio will make money for sure and do so faster than at rate r.

We can also see this by solving for V_t in the last equation. We derive $V_t = \dfrac{\alpha}{r}(e^{rt} - 1)$. As $\alpha, r > 0$, we have $V_t > 0$ for all $t > 0$ and hence the portfolio we constructed is an arbitrage opportunity.

We synthesized a second money account with an additional return. All we have to do is to borrow at rate r from the old market account and earn more by making deposits to the new one.

1.4 Complete market

- In a complete market, by the no arbitrage principle, or more precisely, law of one price, the value of any contingent claim is equal to the initial endowment required to establish a self-financing strategy which replicates the payoff of the claim at maturity.

- A market is called **incomplete** if there exist nonreplicable claims.

  In this case, the pricing of such claims is somehow vague because it is unclear how such claims can be priced.

  One possible resolution is to price such claims by the minimum initial endowments among all super-replicating self-financing strategies or by the maximum of the initial endowments among all sub-replicating self-financing strategies. However, chances are the two prices from the two strategies may differ. In fact, they will differ; otherwise the claim is indeed replicable. Therefore, there is no unique price for non-replicable claims.

- Nevertheless, even in incomplete markets, there is no ambiguity in the pricing of replicable claims because its price is still the initial endowment of the replicating self-financing strategy.

1.5 Second fundamental theorem of asset pricing

How do we know if a market is complete?

Second fundamental theorem of asset pricing

A market is complete if and only if, associated with any numeraire chosen, there exists a *unique* equivalent martingale measure.

Remark

In the case of diffusion models, the arbitrage freeness and the completeness of a market is related to the solution of a system of linear equations, as we shall summarize below.

1.6 Summary

Recall the key linear system satisfied by θ_t:

$$\sum_j \left(\sigma^i_{j,t} - \sigma^N_{j,t}\right)\theta^j_t = \mu^i_t - \mu^N_t + \sum_j \sigma^N_{j,t}\left(\sigma^N_{j,t} - \sigma^i_{j,t}\right) \tag{1}$$

and the Z_t process for the Radon-Nikodym derivative

$$Z_t = e^{-\int_0^t \theta_s \cdot dW_s - \frac{1}{2}\int_0^t |\theta_s|^2 ds}.$$

We can conclude

- If the linear system (1) has a solution so that the process Z_t is a $\mathbb{P}$-martingale, then there is no arbitrage opportunity in the market and an equivalent martingale measure $\mathbb{Q}$ is given by the corresponding Radon-Nikodym derivative.

- If the linear system (1) has a *unique* solution so that the process Z_t is a $\mathbb{P}$-martingale, then there is no arbitrage opportunity in the market and the market is complete.

- If the linear system (1) has no solution, the market is not free of arbitrage.

- If we choose the risk free asset with short rate R_t as the numéraire, the linear system reduces to the Sharpe ratio equation, i.e.,

$$\theta_t = \sigma_t^{-1}(\mu_t - R_t),$$

  provided the diffusion matrix σ_t is invertible. In this case, θ is also referred to as the **market price of risk**.

- We can rewrite the linear system satisfied by θ_t as

$$\begin{aligned} \mu_t^i - \sum_j \sigma_{j,t}^i \theta_t^j - \left(\mu_t^N - \sum_j \sigma_{j,t}^N \theta_t^j \right) &= \sum_j \sigma_{j,t}^N \left(\sigma_{j,t}^i - \sigma_{j,t}^N \right) \\ &= \frac{d}{dt} \left[\log\left(\frac{S^i}{N} \right), \log N \right]_t \\ &= \frac{d}{dt} \left[\log \widetilde{S}^i, \log N \right]_t \end{aligned}$$

  In other words, the difference between drifts of asset S^i and the numeraire N_t in the $\mathbb{Q}$ measure is equal to derivative of the quadratic covariation between $\log \widetilde{S}_t^i$ and $\log N_t$.

1.7 Example

Consider the economy consisting of a risky asset S_t and a risk free asset (money account) B_t whose prices are driven respectively by

$$\begin{aligned} dB_t &= R_t B_t dt, \quad B_0 = 1, \\ \frac{dS_t}{S_t} &= \mu_t dt + \sigma_t dW_t. \end{aligned}$$

Let's now take the risky asset S_t as numéraire and calculate the associated EMM. Note that, by applying Itô's formula, the dynamics of B_t/S_t is given by

$$d\left(\frac{B_t}{S_t}\right) = \frac{B_t}{S_t}\left[\left(R_t - \mu_t + \sigma_t^2\right) dt - \sigma_t dW_t\right].$$

Let $\theta_t = \frac{R_t - \mu_t}{\sigma_t} + \sigma_t$ and define a new probability $\mathbb{Q}$ by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{\int_0^T \theta_t dW_t - \frac{1}{2}\int_0^T \theta_t^2 dt}.$$

Then W_t in the $\mathbb{Q}$ measure becomes a Brownian motion with drift θ_t, i.e.,

$$dW_t = d\widetilde{W}_t + \theta_t dt$$

where $\widetilde{W}_t$ is a Browian motion in the $\mathbb{Q}$ measure. Now in the $\mathbb{Q}$ measure, the dynamics of $\frac{B_t}{S_t}$ reads

$$\begin{aligned} d\left(\frac{B_t}{S_t}\right) &= \frac{B_t}{S_t}\left[\left(R_t - \mu_t + \sigma_t^2\right) dt - \sigma_t \left(d\widetilde{W}_t + \theta_t\right)\right] \\ &= -\sigma_t \left(\frac{B_t}{S_t}\right) d\widetilde{W}_t. \end{aligned}$$

Note that, under $\mathbb{Q}$-martingale, the dynamic of B_t is still

$$dB_t = R_t B_t dt$$

because B_t has no diffusion part. However, the dynamic of S_t in the $\mathbb{Q}$ measure reads

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t = \mu_t dt + \sigma_t \left(d\widetilde{W}_t + \theta_t dt\right) = \left(R_t + \sigma_t^2\right) dt + \sigma_t d\widetilde{W}_t.$$

Thus, if a contingent claim pays off $f(S_T, B_T)$ at expiry, its price/premium P_t at time $t \leq T$ is given by the expectation under the EMM as

$$P_t = S_t \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{f(S_T, B_T)}{S_T}\right| \mathcal{F}_t\right]. \qquad (*)$$

If instead we choose B_t as a numeraire, we have with Ito's formula

$$d\left(\frac{S_t}{B_t}\right) = \frac{S_t}{B_t}[(\mu_t - R_t)dt + \sigma_t dW_t]\,.$$

Defining $\theta_t = \sigma_t^{-1}(\mu_t - R_t)$ and the Radon-Nikodym derivative

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} = e^{\int_0^T \theta_t dW_t - \frac{1}{2}\int_0^T \theta_t^2 dt},$$

we have that the price P_t (at time $t \le T$) of a contingent claim paying off $f(S_T, B_T)$ at expiry T is given by the expectation under the EMM as

$$P_t = B_t \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\frac{f(S_T, B_T)}{B_T} \,\Big|\, \mathcal{F}_t\right]. \qquad (**)$$

Note that

- The 2 expressions $(*)$ and $(**)$ are different under different measures.
- The 2 expressions $(*)$ and $(**)$ give the same price because the market is complete.

2 Change of numéraire technique

2.1 Switching between equivalent martingale measures

Let N_t and $\widetilde{N}_t$ be two numéraires with $\mathbb{Q}$ and $\widetilde{\mathbb{Q}}$ being the associated EMMs respectively.
Further assume that the numéraires N_t and $\widetilde{N}_t$ satisfy the SDEs

$$\frac{dN_t}{N_t} = \mu_t^N dt + \sum_j \sigma_{j,t}^N dW_t^j \quad (\text{in } \mathbb{Q} \text{ measure }),$$

$$\frac{d\widetilde{N}_t}{\widetilde{N}_t} = \mu_t^{\widetilde{N}} dt + \sum_j \sigma_{j,t}^{\widetilde{N}} d\widetilde{W}_t^j \quad (\text{in } \widetilde{\mathbb{Q}} \text{ measure}).$$

A traded asset, say, X_t is driven by the following SDEs, in $\mathbb{Q}$ and $\widetilde{\mathbb{Q}}$ measures respectively, as

$$dX_t = \mu_t^X dt + \sum_j \sigma_{j,t}^X dW_t^j \qquad \text{in the } \mathbb{Q}\text{-measure;}$$

$$dX_t = \tilde{\mu}_t^X dt + \sum_j \sigma_{j,t}^X d\widetilde{W}_t^j \qquad \text{in the } \widetilde{\mathbb{Q}}\text{-measure.}$$

2.2 Radon-Nikodym derivative in terms of numéraires

First recall that, for any attainable contingent claim with payoff V_T at maturity T, the value V_t of the claim at time t is obtained, with numéraire N_t and its associated equivalent martingale measure $\mathbb{Q}$, as

$$V_t = N_t \, \mathbb{E}^{\mathbb{Q}} \left[\left. \frac{V_T}{N_T} \right| \mathcal{F}_t \right].$$

Notice that, since the contingent claim is attainable, meaning there exists a self-financing strategy which replicates the payoff of the claim at maturity T, the value V_t of the claim at any time $t \leq T$ is independent of which numéraire and its associated equivalent martingale measure we have chosen for evaluation because the replicating strategy is independent of numéraire.

As a result, the following identity holds, should we use a different numéraire $\widetilde{N}_t$ and its associated equivalent martingale measure $\widetilde{\mathbb{Q}}$

$$V_t = N_t \, \mathbb{E}^{\mathbb{Q}} \left[\left. \frac{V_T}{N_T} \right| \mathcal{F}_t \right] = \widetilde{N}_t \mathbb{E}^{\widetilde{\mathbb{Q}}} \left[\left. \frac{V_T}{\widetilde{N}_T} \right| \mathcal{F}_t \right] = \widetilde{N}_t \mathbb{E}^{\mathbb{Q}} \left[\left. \frac{V_T}{\widetilde{N}_T} \frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} \right| \mathcal{F}_t \right].$$

Moreover, since the last equality holds for any (attainable) contingent claim, we conclude that

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} = \frac{N_0 \widetilde{N}_T}{\widetilde{N}_0 N_T}, \quad \mathbb{Q}\text{-a.s.}$$

In other words, the Radon-Nikodym derivative $\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}$ is given by ratio between the numéraires.

2.3 SDE for Radon-Nikodym derivative

Moreover, since the process $Z_t = \left. \frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} \right|_{\mathcal{F}_t} = \frac{N_0 \widetilde{N}_t}{\widetilde{N}_0 N_t}$ is a $\mathbb{Q}$-martingale, by applying Itô's formula one can show that Z_t satisfies the following SDE of exponential martingale type:

$$\begin{aligned}
dZ_t &= \frac{N_0}{\widetilde{N}_0} d\left(\frac{\widetilde{N}_t}{N_t} \right) \\
&= \frac{N_0}{\widetilde{N}_0} \left\{ \frac{d\widetilde{N}_t}{N_t} + \widetilde{N}_t d\left(\frac{1}{N_t} \right) + d\left[\widetilde{N}, \frac{1}{N} \right]_t \right\} \\
&= Z_t \left\{ \frac{d\widetilde{N}_t}{\widetilde{N}_t} + N_t d\left(\frac{1}{N_t} \right) + \frac{N_t}{\widetilde{N}_t} d\left[\widetilde{N}, \frac{1}{N} \right]_t \right\} \\
&= Z_t \sum_j \left(\sigma_{j,t}^{\widetilde{N}} - \sigma_{j,t}^{N} \right) dW_t^j .
\end{aligned}$$

2.4 Drift formula for change of numéraire

On the other hand, by Girsanov theorem, the Radon-Nikodym derivative $\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}$ is given by the following exponential $\mathbb{Q}$-martingale

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} = e^{\sum_j \int_0^T \theta_t^j dW_t^j - \frac{1}{2}\int_0^T |\theta_t|^2 dt}$$

where θ_t is determined by $\tilde{\mu}_t^X = \mu_t^X + \sum_j \sigma_{j,t}^X \theta_t^j$. Therefore, the process $Z_t = \frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}\Big|_{\mathcal{F}_t}$ also satisfies the SDE

$$dZ_t = Z_t \sum_j \theta_t^j dW_t^j \,.$$

By matching the two equations that Z_t has to satisfy, we have, for $j = 1, \cdots, m$,

$$\sigma_{j,t}^{\widetilde{N}} - \sigma_{j,t}^{N} = \theta_{j,t}.$$

Moreover, since θ_t satisfies $\tilde{\mu}_t^X - \mu_t^X = \sum_j \sigma_{j,t}^X \theta_t^j$, we end up with

$$\tilde{\mu}_t^X - \mu_t^X = \sum_j \sigma_{j,t}^X \theta_t^j = \sum_j \sigma_{j,t}^X \left(\sigma_{j,t}^{\widetilde{N}} - \sigma_{j,t}^{N}\right).$$

Finally, we can rewrite the above change of drift formula in the following elegant form in terms of the quadratic covariation as

$$\tilde{\mu}_t^X - \mu_t^X = \frac{d}{dt}\left[X_t, \log\left(\frac{\widetilde{N}_t}{N_t}\right)\right]_t.$$

2.5 Example

Recall the economy considered previously consisting of a risky asset S_t and a risk free asset (money account) B_t whose prices are driven respectively by

$$\begin{aligned} dB_t &= R_t B_t dt, \quad B_0 = 1, \\ \frac{dS_t}{S_t} &= \mu_t dt + \sigma_t dW_t. \end{aligned}$$

If B_t is chosen as numéraire, the dynamics of B_t and S_t under risk neutral probability read

$$\begin{aligned} dB_t &= R_t B_t dt, \quad B_0 = 1, \\ \frac{dS_t}{S_t} &= R_t dt + \sigma_t d\widetilde{W}_t, \end{aligned}$$

where $\widetilde{W}_t$ is a Brownian motion under the risk neutral probability. On the other hand, if S_t is chosen as numéraire, the dynamics of B_t and S_t under the associated EMM reads

$$\begin{aligned} dB_t &= R_t B_t dt, \quad B_0 = 1, \\ \frac{dS_t}{S_t} &= (R_t + \sigma_t^2) dt + \sigma_t d\hat{W}_t, \end{aligned}$$

where $\hat{W}_t$ is a Brownian motion in the EMM.

Note that we have

$$\begin{aligned} \frac{d}{dt}\left[S, \log\left(\frac{S}{B}\right)\right]_t &= \frac{d}{dt}[S, \log S]_t - \frac{d}{dt}[S, \log B]_t \\ &= \frac{d}{dt}[S, \log S]_t \quad \text{(since } B_t \text{ has no diffusion, i.e., finite variation)} \\ &= \sigma_t^2 S_t \\ &= (R_t + \sigma_t^2) S_t - R_t S_t \end{aligned}$$

which verifies the drift formula for change of numéraire.

3 Stochastic differential equations

An SDE driven by Brownian motion is an equation of the form

$$dX_t = \mu(X_t, t) dt + \sigma(X_t, t) dW_t, \quad X_0 = x,$$

where W_t is a Brownian motion.

As usual, $\mu(x, t)$ is referred to as the drift part and $\sigma(x, t)$ the diffusion part.

3.1 Linear SDEs

An SDE is called **linear** if its drift and diffusion coefficients are both linear in x. Namely,

$$\mu(x,t) = \mu_1(t)x + \mu_0(t), \quad \sigma(x,t) = \sigma_1(t)x + \sigma_0(t)$$

Linear SDEs have closed form solutions.

We roughly categorize linear SDEs as follows.

- **OU type:** $dX_t = (\mu_1(t)X_t + \mu_0(t))dt + \sigma(t)dW_t$, $X_{t_0} = x$

  $$X_t = e^{\int_{t_0}^t \mu_1(s)ds}x + \int_{t_0}^t e^{\int_s^t \mu_1(\tau)d\tau}\mu_0(s)ds + \int_{t_0}^t e^{\int_s^t \mu_1(\tau)d\tau}\sigma(s)dW_s$$

  Apply Ito's formula to $e^{-\int_{t_0}^t \mu_1(s)ds}X_t$. Note that X_t is normally distributed.

- **BS type:** $dX_t = \mu(t)X_t dt + \sigma(t)X_t dW_t$, $X_{t_0} = x$

  $$X_t = xe^{\int_{t_0}^t \left[\mu(s) - \frac{1}{2}\sigma^2(s)\right]ds + \int_{t_0}^t \sigma(s)dW_s}$$

  Apply Ito's formula to $\log X_t$. Note that X_t is lognormally distributed.

- **general:** $dX_t = (\mu_1(t)X_t + \mu_0(t))dt + (\sigma_1(t)X_t + \sigma_0(t))dW_t$, $X_{t_0} = x$

  See Homework 09

3.2 Nonlinear SDE I

An SDE of the form

$$dX_t = \frac{1}{2}\sigma(X_t)\sigma^T(X_t)dt + \sigma(X_t)dW_t$$

can be rewritten in Stratonovich form as

$$dX_t = \sigma(X_t) \circ dW_t$$

Therefore, the solution is given by

$$X_t = h^{-1}\left(h(x_0) + W_t\right)$$

The idea is to first solve the (deterministic) ODE $dx = \sigma(x)dt$, to which the solution is

$$x(t) = h^{-1}\left(h(x_0) + t\right),$$

where h is an antiderivative of $\frac{1}{\sigma}$, i.e. $h' = \frac{1}{\sigma}$.

Then replace the time t in the solution by the Brownian motion W_t.

Remark

The key point is that the drift part is completely determined by diffusion.

Example

Solve the SDE

$$dX_t = X_t(1+X_t^2)dt + (1+X_t^2)dW_t.$$

Solution:

Note that in this case, $\sigma(x) = 1 + x^2$. Hence,

$$h(x) = \int \frac{dx}{\sigma(x)} = \int \frac{dx}{1+x^2} = \arctan x + C$$

Apply Itô to $h(x)$ we obtain

$$\begin{aligned} dh(X_t) &= \frac{dX_t}{1+X_t^2} + \frac{1}{2}\frac{-2X_t}{(1+X_t^2)^2}d[X]_t \\ &= X_t dt + dW_t - X_t dt \\ &= dW_t. \end{aligned}$$

Thus, we have

$$h(X_t) - h(X_0) = W_t.$$

Solving the last equation for X_t we obtain

$$X_t = \tan(W_t + \arctan X_0).$$

3.3 Nonlinear SDE II

The SDE of the form

$$dX_t = \left[\alpha\sigma(X_t) + \frac{1}{2}\sigma(X_t)\sigma'(X_t)\right]dt + \sigma(X_t)dW_t$$

can be written in Stratonovich form as

$$dX_t = \alpha\sigma(X_t)dt + \sigma(X_t) \circ dW_t = \sigma(X_t) \circ [\alpha dt + dW_t]$$

The solution is given by

$$X_t = h^{-1}\left(h(x_0) + \alpha t + W_t\right).$$

Again, the idea is to first solve the (deterministic) ODE $dx = \sigma(x)dt$, then replace the time t in the solution by the Brownian motion with drift: $W_t + \alpha t$.

Remark

Though we have a little more freedom of choosing the drift part, it is still mostly determined by the diffusion part.

Example

Solve the SDE

$$dX_t = \left(\frac{X_t}{2} + \sqrt{X_t^2+1}\right)dt + \sqrt{X_t^2+1}\,dW_t.$$

Solution:

Note that in this case, $\sigma(x) = \sqrt{1+x^2}$. Hence,

$$h(x) = \int \frac{dx}{\sigma(x)} = \int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + C$$

Apply Itô to $h(x)$ we obtain

$$\begin{aligned} dh(X_t) &= \frac{dX_t}{\sqrt{1+X_t^2}} + \frac{1}{2}\frac{-X_t}{(1+X_t^2)^{\frac{3}{2}}}d[X]_t \\ &= \left(\frac{X_t}{2\sqrt{1+X_t^2}} + 1\right)dt + dW_t - \frac{X_t}{2\sqrt{1+X_t^2}}dt \\ &= dt + dW_t. \end{aligned}$$

Thus, we have

$$h(X_t) - h(X_0) = t + W_t.$$

Solving the last equation for X_t we obtain

$$X_t = \sinh\big(t + W_t + \sinh^{-1} X_0\big).$$

3.4 Nonlinear SDE III

Consider the following SDE

$$dX_t = \left[\alpha h(X_t)\sigma(X_t) + \frac{1}{2}\sigma(X_t)\sigma'(X_t)\right]dt + \sigma(X_t)dW_t$$

where h is an antiderivative of $\frac{1}{\sigma}$. In Stratonovich form

$$dX_t = \alpha h(X_t)\sigma(X_t)dt + \sigma(X_t) \circ dW_t = \sigma(X_t) \circ [\alpha h(X_t)dt + dW_t]$$

The solution is given by

$$X_t = h^{-1}\left(e^{\alpha t}h(X_0) + \int_0^t e^{\alpha(t-s)}dW_s\right).$$

The idea is that the process $Y_t = h(X_t)$ satisfies the SDEs

$$dY_t = \alpha Y_t dt + dW_t$$

which obviously has the solution

$$Y_t = e^{\alpha t}\,Y_0 + \int_0^t e^{\alpha(t-s)}dW_s.$$

Example

Solve the SDE

$$dX_t = -\left[\frac{3}{4}\sin(2X_t) + \frac{1}{8}\sin(4X_t)\right]dt + \cos^2 X_t dW_t.$$

Solution:

Note that in this case, $\sigma(x) = \cos^2 x$. Hence,

$$h(x) = \int \frac{dx}{\sigma(x)} = \int \frac{dx}{\cos^2 x} = \tan x + C$$

Apply Itô to $h(x)$ we obtain

$$\begin{aligned}
dh(X_t) &= \sec^2 X_t dX_t + \sec^2 X_t \tan X_t d[X]_t \\
&= -\left[\frac{3}{4}\sin(2X_t) + \frac{1}{8}\sin(4X_t)\right]\sec^2 X_t dt + dW_t + \sec^2 X_t \tan X_t \cos^4 X_t dt \\
&= dW_t + \left[-\frac{3}{2}\tan X_t + \frac{1}{2}\tan X_t - \sin X_t \cos X_t + \sin X_t \cos X_t\right] dt \\
&= dW_t - h(X_t)dt.
\end{aligned}$$

Thus, we have

$$dh(X_t) + h(X_t)dt = dW_t \quad \Longrightarrow \quad h(X_t) = e^{-t}h(X_0) + \int_0^t e^{-(t-s)}dW_s.$$

$$X_t = \arctan\left[e^{-t}\tan X_0 + \int_0^t e^{-(t-s)}dW_s\right].$$

3.5 Miscellaneous SDEs

- The following nonlinear SDE

$$dX_t = (\alpha X_t^n + \beta X_t)dt + \sigma X_t dW_t$$

can be linearized by the change of variable $y = x^{1-n}$.

Indeed, applying Ito's formula we have

$$\begin{aligned}
dY_t &= (1-n)X_t^{-n} dX_t - \frac{n(1-n)}{2} X_t^{-n-1} d[X]_t \\
&= (1-n)X_t^{-n}\left[(\alpha X_t^n + \beta X_t)dt + \sigma X_t dW_t\right] - \frac{n(1-n)}{2}\sigma^2 Y_t dt \\
&= \left[\left(\beta(1-n) - \frac{n(1-n)}{2}\sigma^2\right) Y_t + \alpha(1-n)\right] dt + (1-n)\sigma Y_t dW_t
\end{aligned}$$

- The nonlinear SDE

$$dX_t = \left(e^{cX_t} + b\right) dt + \sigma dW_t$$

can be linearized by the change of variable $y = e^{-cx}$.

Again, by applying Ito's formula we obtain

$$\begin{aligned}
dY_t &= -ce^{-cX_t} dX_t + \frac{1}{2} c^2 e^{-cX_t} (dX_t)^2 \\
&= -cY_t\left[\left(e^{cX_t} + b\right) dt + \sigma dW_t\right] + \frac{\sigma^2 c^2}{2} Y_t dt \\
&= \left[\left(\frac{\sigma^2 c^2}{2} - bc\right) Y_t - c\right] dt - c\sigma Y_t dW_t
\end{aligned}$$