9831 Probability and Stochastic Processes for Finance, Fall 2016

Lecture 12

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Outline

- More on the Ito-Doeblin formula
- Infinitesimal generator for processes with jumps
- Change of measure
- Pricing European options in a jump model

1 More on the Ito-Doeblin formula

1.1 Another helpful form of the Ito-Doeblin formula

Let X_t be a jump process, that is

$$X_t = X_0 + I_t + R_t + J_t,$$

where $I_t = \int_0^t = \sigma_s dB_s$ and $\int_0^t \mu_s ds$.

Recall that

$$\begin{aligned} \Delta X_t &= X_t - X_{t-} = J_t - J_{t-} = \Delta J_t, \\ X_t^c &= X_0 + I_t + R_t. \end{aligned}$$

Furthermore, we have

$$[X]_t = [X^c]_t + [J]_t = \int_0^t \sigma_s^2 ds + \sum_{0<s\leq t} (\Delta J_s)^2.$$

The Ito-Doeblin formula states that for a twice continuously differentiable function f, the following holds

$$f(X_t) - f(X_0) = \int_0^t f'(X_s)dX_s^c + \frac{1}{2}\int_0^t f''(X_s)d[X^c]_s + \sum_{0<s\leq t}\big(f(X_s) - f(X_{s-})\big). \tag{1}$$

Note that if we add and subtract

$$\int_0^t f'(X_{s-})dJ_s = \sum_{0<s\leq t} f'(X_{s-})\Delta X_s,$$

then (1) can be rewritten as

$$\begin{aligned} &f(X_t) - f(X_0) \\ &= \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d[X^c]_s \\ &\quad + \sum_{0<s\leq t}\big(f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s\big). \end{aligned} \tag{2}$$

Example

Let $M_t = N_t - \lambda t$. Find $\mathrm{Var}(\int_0^t M_{s-} dM_s)$.

We recall that we know that $\int_0^t M_{s-} dM_s$ is a martingale. Thus $\mathbb{E}\left[\int_0^t M_{s-} dM_s\right] = 0$ and

$$\mathrm{Var}\left(\int_0^t M_{s-} dM_s\right) = \mathbb{E}\left[\left(\int_0^t M_{s-} dM_s\right)^2\right].$$

We set

$$\begin{aligned} X_t &= \int_0^t M_{s-} dM_s \\ &= \int_0^t M_{s-} d(-\lambda s) + \sum_{0<s\leq t} M_{s-} \Delta N_s \\ &= X_t^c + J_t, \end{aligned}$$

and note that $[X^c]_t \equiv 0$. Now apply (2):

$$\begin{aligned} X_t^2 &= \int_0^t 2X_{s-} dX_s + \sum_{0<s\leq t} \left((X_{s-} + \Delta X_s)^2 - X_{s-}^2 - 2X_{s-}\Delta X_s\right) \\ &= 2\int_0^t X_{s-} dX_s + \sum_{0<s\leq t} (\Delta X_s)^2. \end{aligned}$$

As $\int_0^t X_{s-} dX_s$ is a martingale and
$\sum_{0<s\leq t} (\Delta X_s)^2 = [J]_t = \sum_{0<s\leq t} M_{s-}^2 (\Delta N_s)^2 = \sum_{0<s\leq t} M_{s-}^2 \Delta N_s$, we get

$$X_t^2 = \int_0^t 2X_{s-} dX_s + \int_0^t M_{s-}^2 dN_s.$$

Taking expectations on both sides, this finally leads to

$$\begin{aligned} \mathrm{Var}\left(\int_0^t M_{s-} dM_s\right) = \mathbb{E}\left[X_t^2\right] &= \mathbb{E}\left[\int_0^t M_{s-}^2 dN_s\right] \\ &= \mathbb{E}\left[\int_0^t M_{s-}^2 dM_s + \lambda \int_0^t M_{s-}^2 ds\right] \\ &= \lambda \int_0^t \mathbb{E}[M_s^2] ds \\ &= \lambda^2 \int_0^t s ds \\ &= \frac{1}{2}\lambda^2 t^2. \end{aligned}$$

This is a general fact, which is a version of Ito's isometry.

Lemma

Let X_t be a jump process. If X_t is a martingale, then

$$\mathbb{E}(X_t^2) - X_0^2 = \mathbb{E}\left([X]_t\right) = \mathbb{E}\left([X^c]_t\right) + \mathbb{E}\Big(\sum_{0<s\leq t} (\Delta J_s)^2\Big).$$

Proof: Left as an exercise: Apply the Ito-Doeblin formula to X_t^2 and take expectations on both sides.

With the same method one can show:

Lemma

Let ϕ_t be a predictable and square-integrable process. Then

$$\mathbb{E}\left[\left(\int_0^t \phi_s \, dM_s\right)^2\right] = \lambda \int_0^t \mathbb{E}\phi_s^2 ds.$$

More generally: Define $M_t^Q := Q_t - \beta\lambda t$, where Q_t is a compound Poisson process with $\beta = \mathbb{E}Y_1$, then for all predictable and square-integrable processes ϕ_t, we obtain

$$\mathbb{E}\left[\left(\int_0^t \phi_s \, dM_s^Q\right)^2\right] = \lambda \mathbb{E}(Y_1^2) \int_0^t \mathbb{E}\phi_s^2 ds.$$

Proof: See homework 12.

1.2 Two-dimensional Ito-Doeblin formula

Let X_1, X_2 be jump processes and $f(t, x_1, x_2)$ be a $C^{1,2}$ function. Then

$$\begin{aligned}
&f(t, X_1(t), X_2(t)) - f(0, X_1(0), X_2(0)) \\
&= \int_0^t f_t(s, X_1(s), X_2(s))ds + \int_0^t f_{x_1}(s, X_1(s), X_2(s))dX_1^c(s) \\
&\quad + \int_0^t f_{x_2}(s, X_1(s), X_2(s))dX_2^c(s) + \frac{1}{2}\int_0^t f_{x_1x_1}(s, X_1(s), X_2(s))d[X_1^c]_s \\
&\quad + \int_0^t f_{x_1x_2}(s, X_1(s), X_2(s))d[X_1^c, X_2^c]_s + \frac{1}{2}\int_0^t f_{x_2x_2}(s, X_1(s), X_2(s))d[X_2^c]_s \\
&\quad + \sum_{0<s\leq t}\big(f(s, X_1(s), X_2(s)) - f(s, X_1(s-), X_2(s-))\big).
\end{aligned}$$

Ito's Product formula for jump processes

Let X_1, X_2 be jump processes, then we have

$$\begin{aligned}
&X_1(t)X_2(t) - X_1(0)X_2(0) \\
&= \int_0^t X_2(s)dX_1^c(s) + \int_0^t X_1(s)dX_2^c(s) + [X_1^c, X_2^c]_t \\
&\quad + \sum_{0<s\leq t}\big(X_1(s)X_2(s) - X_1(s-)X_2(s-)\big) \\
&\overset{(*)}{=} \int_0^t X_2(s-)dX_1(s) + \int_0^t X_1(s-)dX_2(s) + [X_1^c, X_2^c]_t.
\end{aligned}$$

To see $(*)$, note that

$$\begin{aligned}
&\sum_{0<s\leq t}\big(X_1(s)X_2(s) - X_1(s-)X_2(s-)\big) \\
&= \sum_{0<s\leq t}\Big(\big(X_1(s-) + J_1(s)\big)\big(X_2(s-) + J_2(s)\big) - X_1(s-)X_2(s-)\Big) \\
&= \sum_{0<s\leq t}\big(X_1(s-)J_2(s) + X_2(s-)J_1(s) + J_1(s)J_2(s)\big) \\
&= \int_0^t X_1(s-)dJ_2(s) + \int_0^t X_2(s-)dJ_1(s) + [J_1, J_2]_t
\end{aligned}$$

Example

$$tN_t = \int_0^t N_s ds + \int_0^t s dN_s,$$

as the cross variation is 0.

2 Infinitesimal generator for processes with jumps

Let $X_t = X_t^c + Q_t$ be a jump process with

$$dX_t^c = \sigma(X_t)dB_t + \mu(X_t)dt,$$
$$Q_t = \sum_{m=1}^{M} y_m N_m(t),$$

where, for $m = 1, \cdots, M$, $N_m(t)$ is a Poisson process with intensity $p_m\lambda$, $\sum_{m=1}^{M} p_m = 1$.

Recall that Q_t is indeed a compound Poisson process with arrival intensity λ and jump size Y distributed as $\mathbb{P}[Y = y_m] = p_m$.

We derive the infinitesimal generator for X_t as follows.

Applying Ito's formula to $f(X_t)$ gives

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s^c + \frac{1}{2}\int_0^t f''(X_s)d[X^c]_s + \sum_{0<s\le t} f(X_s) - f(X_{s-}).$$

The goal is to rewrite the above equation in the form

$$f(X_t) = f(X_0) + \text{ Martingale } + \text{ "Drift" },$$

then the "drift" will give us the infinitesimal generator.

Notice that, since the Poisson processes N_m almost surely do not jump simultaneously, the jump part can be recast as

$$\begin{aligned}\sum_{0<s\leq t} f(X_s)-f(X_{s-}) &= \sum_{0<s\leq t} f\left(X_{s-}+\sum_{m=1}^{M} y_m \Delta N_m(s)\right)-f(X_{s-})\\ &= \sum_{0<s\leq t}\sum_{m=1}^{M}\left[f(X_{s-}+y_m)-f(X_{s-})\right]\Delta N_m(s)\\ &= \sum_{m=1}^{M}\int_0^t f(X_{s-}+y_m)-f(X_{s-})dM_m(s)\\ &\qquad -\sum_{m=1}^{M}\int_0^t f(X_{s-}+y_m)-f(X_{s-})dM_m^c(s)\\ &= \sum_{m=1}^{M}\int_0^t f(X_{s-}+y_m)-f(X_{s-})dM_m(s)\\ &\qquad +\lambda\int_0^t\sum_{m=1}^{M} p_m\left[f(X_{s-}+y_m)-f(X_{s-})\right]ds.\end{aligned}$$

Therefore Itô-Doeblin's formula for $f(X_t)$ now reads

$$\begin{aligned}f(X_t) &= f(X_0)+\int_0^t f'(X_s)\sigma(X_s)dB_s+\sum_{m=1}^{M}\int_0^t f(X_{s-}+y_m)-f(X_{s-})dM_m(s)\\ &\quad +\int_0^t\left[\frac{\sigma^2(X_s)}{2}f''(X_{s-})+\mu_s f'(X_{s-})+\lambda\sum_{m=1}^{M}p_m\left[f(X_{s-}+y_m)-f(X_{s-})\right]\right]ds\\ &= \text{Martingale }+\int_0^t Lf(X_{s-})ds\end{aligned}$$

since the first two terms on the right hand side are martingales.

Hence, the infinitesimal generator L for X is

$$Lf(x):=\frac{\sigma^2(x)}{2}f''(x)+\mu(x)f'(x)+\lambda\sum_{m=1}^{M}p_m[f(x+y_m)-f(x)].$$

Moreover, if f is a solution to the equation

$$Lf(x)=\frac{\sigma^2(x)}{2}f''(x)+\mu(x)f'(x)+\lambda\sum_{m=1}^{M}p_m[f(x+y_m)-f(x)]=0,$$

then $f(X_t)$ is a martingale (since the process $f(X_t)$ is "driftless").

We remark that L can be written as

$$Lf(x)=\frac{\sigma^2(x)}{2}f''(x)+\mu(x)f'(x)+\lambda\mathbb{E}[f(x+Y)-f(x)],$$

where Y is the discrete random variable distributed as $\mathbb{P}[Y=y_m]=p_m$.

Note

The last expression of infinitesimal generator holds for general distribution of jump size Y as well. For instance, if Y has the probability density function $\phi(y)$, the infinitesimal generator L is given by the *partial integro-differential operator*

$$Lf(x) = \frac{\sigma^2(x)}{2} f''(x) + \mu(x) f'(x) + \lambda \int [f(x+y) - f(x)]\phi(y)dy.$$

- $\lambda = 0$, i.e., no jump, we recover the diffusion case.
- $\sigma = \mu = 0$, i.e., X is a pure jump process. The infinitesimal generator is intuitively the number of jumps per unit time, λ, multiplies the expected difference between the values of f prior and posterior to jump occurs.
- $\sigma = \mu = 0$, $\lambda = \frac{1}{h^2}$ and $\mathbb{P}[Y = \pm h] = \frac{1}{2}$, i.e., X is compound Poisson process with intensity $\frac{1}{h^2}$ and jump size $\pm h$ with probability half and half. The infinitesimal generator L is the discrete Laplacian:

$$Lf(x) = \frac{f(x+h) + f(x-h) - 2f(x)}{2h^2}.$$

3 Change of measure

3.1 Exponential martingale

Let $\tilde{\lambda} > 0$.

Let N_t be a Poisson process with intensity λ and Z_t the jump process defined by

$$Z_t = e^{(\lambda - \tilde{\lambda})t} \left(\frac{\tilde{\lambda}}{\lambda}\right)^{N_t} = e^{(\lambda - \tilde{\lambda})t + N_t \log\left(\frac{\tilde{\lambda}}{\lambda}\right)}.$$

Lemma

Z_t satisfies the SDE

$$dZ_t = \frac{\tilde{\lambda} - \lambda}{\lambda} Z_{t-} dM_t, \quad Z_0 = 1,$$

where $M_t := N_t - \lambda t$ is the compensated Poisson process. Hence, Z_t is a positive martingale.

Proof: See homework 11.

Note

In particular we have $Z_t > 0$ and $\mathbb{E}Z_t = 1$. Therefore, Z_t can be used as a Radon-Nikodym derivative to define a new equivalent probability measure over Ω.

Q: What happens to the Poisson process N_t if we change the probability measure?

3.2 Change of measure for Poisson processes

Theorem

Let N_t be a Poisson process with intensity defined on the filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$.

Fix $T > 0$ and define a new probability measure $\widetilde{\mathbb{P}}$ over Ω via the Radon-Nikodym derivative

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = Z_T = e^{-(\tilde{\lambda}-\lambda)T}\left(\frac{\tilde{\lambda}}{\lambda}\right)^{N_T}.$$

Then $(N_t)_{t\in[0,T]}$ in the new measure $\widetilde{\mathbb{P}}$ becomes a Poisson process with intensity $\tilde{\lambda}$.

Remark

How to find the Radon-Nikodym derivative if we have N_t ~Poi(λt) under $\mathbb{P}$ and want N_t ~Poi$(\tilde{\lambda} t)$ under $\widetilde{\mathbb{P}}$?

Recall that for fixed t, N_t is discretely distributed and hence we can use the same method as in the refresher, Lecture 3, for countably infinite state spaces.

Hence, the Radon-Nikodym derivative can be obtained by a simple calculation.

$$\frac{\widetilde{\mathbb{P}}[N_T = n]}{\mathbb{P}[N_T = n]} = \frac{e^{-\tilde{\lambda}T}\dfrac{(\tilde{\lambda}T)^n}{n!}}{e^{-\lambda T}\dfrac{(\lambda T)^n}{n!}} = e^{-(\tilde{\lambda}-\lambda)T}\left(\frac{\tilde{\lambda}}{\lambda}\right)^{n}.$$

The claim in the above theorem is much more profound, since the change of measure is done not for a single fixed time but for a whole process.

Proof

We calculate the characteristic function of N_t in the new measure $\widetilde{\mathbb{P}}$. Recall that for $\mathcal{F}_t$-measurable random variables Y, we have $\widetilde{\mathbb{E}}[Y] = \mathbb{E}[YZ_t]$ and hence

$$\begin{aligned}
\widetilde{\mathbb{E}}\left[e^{i\xi N_t}\right] &= \mathbb{E}\left[e^{i\xi N_t} Z_t\right] \\
&= \mathbb{E}\left[e^{i\xi N_t + (\lambda - \tilde{\lambda})t + N_t \log\left(\frac{\tilde{\lambda}}{\lambda}\right)}\right] \\
&= \sum_{k=0}^{\infty} e^{i\xi k + (\lambda - \tilde{\lambda})t} \left(\frac{\tilde{\lambda}}{\lambda}\right)^k \mathbb{P}\left[N_t = k\right] \\
&= \sum_{k=0}^{\infty} e^{i\xi k + (\lambda - \tilde{\lambda})t} \left(\frac{\tilde{\lambda}}{\lambda}\right)^k e^{-\lambda t} \frac{(\lambda t)^k}{k!} \\
&= \sum_{k=0}^{\infty} e^{i\xi k} e^{-\tilde{\lambda} t} \frac{(\tilde{\lambda} t)^k}{k!} \\
&= e^{t\tilde{\lambda}(e^{i\xi} - 1)}.
\end{aligned}$$

Hence, $N_t \sim \mathrm{Poi}(\tilde{\lambda} t)$ under $\widetilde{\mathbb{P}}$.

3.3 Change of measure for compound Poisson processes

Change of measure for compound Poisson process with two jump sizes

Consider the simplest compound Poisson process

$$Q_t = \sum_{k=1}^{N_t} Y_k$$

where $\{Y_k\}$ is an iid sequence with $\mathbb{P}[Y = y_i] = p_i$ for $i = 1, 2$, $p_1 + p_2 = 1$. That is, the jump size Y has two states y_1, y_2 with probability p_1, p_2 respectively.

Recall that Q_t has the representation

$$Q_t = y_1 N_1(t) + y_2 N_2(t),$$

where N_1, N_2 are independent Poisson processes with intensities $\lambda_1 = p_1 \lambda$ and $\lambda_2 = p_2 \lambda$ respectively.

Remark: Which parameters can we hope to change?

We can change λ to $\tilde{\lambda}$ and p_i to $\tilde{p}_i$.

However, we can not change the y_i's, since then $\widetilde{\mathbb{P}}$ will never be equivalent to $\widetilde{\mathbb{P}}$.

Example: If $\mathbb{P}(Y_1 = 3) > 0$ and $\widetilde{\mathbb{P}} \sim \mathbb{P}$, then $\widetilde{\mathbb{P}}(Y_1 = 3) > 0$.

If $\mathbb{P}(Y_1 = 2) = 0$ and $\widetilde{\mathbb{P}} \sim \mathbb{P}$, then $\widetilde{\mathbb{P}}(Y_1 = 3) = 0$.

For each of the Poisson processes N_i consider the corresponding martingales for measure change

$$Z_i(t) = e^{(\lambda_i - \tilde{\lambda}_i)t} \left(\frac{\tilde{\lambda}_i}{\lambda_i} \right)^{N_i(t)}.$$

Denote $\tilde{\lambda} = \tilde{\lambda}_1 + \tilde{\lambda}_2$ and $\tilde{p}_i = \dfrac{\tilde{\lambda}_i}{\tilde{\lambda}}$ for $i = 1, 2$.

Note that

$$\begin{aligned} Z_t &:= Z_1(t) Z_2(t) \\ &= e^{(\lambda - \tilde{\lambda})t} \left(\frac{\tilde{\lambda}_1}{\lambda_1} \right)^{N_1(t)} \left(\frac{\tilde{\lambda}_2}{\lambda_2} \right)^{N_2(t)} \\ &= e^{(\lambda - \tilde{\lambda})t} \prod_{i=1}^{2} \left(\frac{\tilde{\lambda} \tilde{p}_i}{\lambda p_i} \right)^{N_i(t)} \\ &= e^{(\lambda - \tilde{\lambda})t} \prod_{k=1}^{N_t} \frac{\tilde{\lambda} \tilde{p}(Y_k)}{\lambda p(Y_k)}, \end{aligned}$$

where $N_t = N_1(t) + N_2(t)$ and $p(y_i) = p_i$, $\tilde{p}(y_i) = \tilde{p}_i$.

Since $Z_1(t) Z_2(t)$ is a martingale (why?) with $Z_1(0) Z_2(0) = 1$, we fix $T > 0$ and define a new probability measure $\widetilde{\mathbb{P}}$ by

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = Z_1(T) Z_2(T).$$

Process in new measure

So now the question is under the new measure $\widetilde{\mathbb{P}}$, what does the process Q_t look like?

As we remarked earlier on, it can only be a compound Poisson process with the two possible jump sizes y_1 and y_2. So the things that could change are the intensity of the underlying Poisson process N_t and the probabilities of the jump sizes.

Let's see what they are by computing the characteristic function of Q_t:

$$\begin{aligned}
\widetilde{\mathbb{E}}[e^{i\xi Q_t}] &= \mathbb{E}\left[e^{i\xi Q_t} Z_t\right] \\
&= \mathbb{E}\left[e^{i\xi Q_t} e^{(\lambda-\tilde{\lambda})t} \prod_{k=1}^{N_t} \frac{\tilde{\lambda}\tilde{p}(Y_k)}{\lambda p(Y_k)}\right] \quad (\text{since } Q_t \in \mathcal{F}_t) \\
&= \sum_{n=0}^{\infty} \mathbb{E}\left[\prod_{k=1}^{n} e^{i\xi Y_k} \frac{\tilde{p}(Y_k)}{p(Y_k)}\right] \left(\frac{\tilde{\lambda}}{\lambda}\right)^n e^{(\lambda-\tilde{\lambda})t} \mathbb{P}[N_t = n] \\
&= \sum_{n=0}^{\infty} \prod_{k=1}^{n} \mathbb{E}\left[e^{i\xi Y_k} \frac{\tilde{p}(Y_k)}{p(Y_k)}\right] e^{-\tilde{\lambda}t} \frac{(\tilde{\lambda}t)^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{i=1}^{2} e^{i\xi y_i} \tilde{p}_i\right)^n e^{-\tilde{\lambda}t} \frac{(\tilde{\lambda}t)^n}{n!} \\
&= e^{\tilde{\lambda}t(\tilde{\varphi}(\xi)-1)},
\end{aligned}$$

where $\tilde{\varphi}(\xi)$ is the characteristic function of the random variable Y with $\widetilde{\mathbb{P}}[Y = y_i] = \tilde{p}_i$, $i = 1, 2$. That is, under the probability $\widetilde{\mathbb{P}}$, Q_t is the compound Poisson process with intensity $\tilde{\lambda}$ and jump size distribution $\widetilde{\mathbb{P}}[Y = y_i] = \tilde{p}_i$.

Change of measure for compound Poisson process with finitely many jump sizes

The argument can be easily generalized to general discrete finitely many jump sizes case. The derivation will be like

$$\begin{aligned} Z_t &:= \prod_{m=1}^{M} Z_m(t) \\ &= e^{(\lambda-\tilde{\lambda})t} \prod_{m=1}^{M} \left(\frac{\tilde{\lambda}_m}{\lambda_m}\right)^{N_m(t)} \\ &= e^{(\lambda-\tilde{\lambda})t} \prod_{m=1}^{M} \left(\frac{\tilde{\lambda}\tilde{p}_m}{\lambda p_m}\right)^{N_m(t)} \\ &= e^{(\lambda-\tilde{\lambda})t} \prod_{k=1}^{N_t} \frac{\tilde{\lambda}\tilde{p}(Y_k)}{\lambda p(Y_k)}. \end{aligned}$$

Theorem

Under the new measure defined by

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = Z_T,$$

the stochastic process Q_t is a compound Poisson process with intensity $\tilde{\lambda}$ and jump size distribution $\widetilde{\mathbb{P}}[Y = y_m] = \tilde{p}_m$, $m = 1, \ldots, M$.

Change of measure for compound Poisson process with jump-size density

It conceivable that for the general jump size distribution case we should have

$$Z_t = e^{(\lambda-\tilde{\lambda})t} \prod_{k=1}^{N_t} \frac{\tilde{\lambda}\tilde{f}\,(Y_k)}{\lambda f(Y_k)},$$

where f and $\tilde{f}$ are the pdfs for the jump size random variable Y under $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ respectively.

Note that f and $\tilde{f}$ should satisfy $f(x) = 0$ if and only if $\tilde{f}\,(x) = 0$, i.e. the two probability measures $f(x)dx$ and $\tilde{f}\,(x)dx$ are equivalent.

Theorem

Under the new measure defined by

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = Z_T,$$

the stochastic process Q_t is a compound Poisson process with intensity $\tilde{\lambda}$ and jump size distribution $\widetilde{\mathbb{P}}[Y \le y] = \int_{-\infty}^{y} \tilde{f}\,(x)dx.$

3.4 Change of measure for jump processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and

- W_t be a Brownian motion
- Q_t a compound Poisson process with intensity λ and jump size density f.

We assume there is a single filtration $\mathcal{F}_t$ for both, the Brownian motion and the compound Poisson process.

In this case, the Brownian motion and the compound Poisson process must be independent (see Homework 12).

Let $\tilde{\lambda} > 0$ and let $\tilde{f}(y)$ be another density function with the property that $\tilde{f}(y) = 0$ whenever $f(y) = 0$ and let θ_t be an adapted process.

Then the process

$$Z_t = e^{-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds} e^{(\lambda - \tilde{\lambda})t} \prod_{i=1}^{N_t} \frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)} \qquad (3)$$

is a martingale.

In particular, $\mathbb{E} Z_t = 1$ for all $t \geq 0$.

Fix $T > 0$ and define a new probability measure $\widetilde{\mathbb{P}}$ by

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = Z_T.$$

Theorem

Under the new measure $\mathbb{P}$,

- $\widetilde{W}_t = W_t + \int_0^t \theta_s ds$ is a Brownian motion
- Q_t is the compound Poisson process with intensity $\tilde{\lambda}$ and the pdf of the jump size distribution Y is given by $\tilde{f}$.

Suppose now that the compound Poisson process Q_t has finitely many jump sizes $y_1, \ldots, y_m$.

Then we replace (3) by

$$Z_t = e^{-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds} e^{(\lambda - \tilde{\lambda})t} \prod_{i=1}^{N_t} \frac{\tilde{\lambda}\tilde{p}(Y_i)}{\lambda p(Y_i)}.$$

Note that Z_t is still a martingale with $\mathbb{E} Z_t = 1$ for all $t \geq 0$.

Theorem

Under the new measure $\mathbb{P}$,

- $\widetilde{W}_t = W_t + \int_0^t \theta_s ds$ is a Brownian motion
- Q_t is the compound Poisson process with intensity $\tilde{\lambda}$ and the probability mass function of the jump sizes Y is given by $\tilde{p}$.

4 Pricing of European options in a jump model

4.1 Market driven by processes with jump is incomplete

Consider the economy consisting of a risk free asset B_t and a risky asset S_t driven respectively by

$$dB_t = rB_t dt, \quad B_0 = 1,$$
$$\frac{dS_t}{S_{t-}} = \mu dt + \sigma dW_t + d(Q_t - \beta\lambda t),$$

where W_t is a Brownian motion and Q_t is a compound Poisson process with intensity λ and discrete jump size Y and $E[Y] = \beta$.

Note that the expected rate of return of this stock is μ and that we can solve the above SDE for S_t explicitly:

$$S_t = S_0 e^{\sigma W_t + (r - \beta\lambda - \frac{1}{2}\sigma^2)t} \prod_{i=1}^{N_t} (Y_i + 1).$$

Exercise: Proof that this is true by applying Ito's formula to $S_t = X_t J_t$ with $X_t = S_0 e^{\sigma W_t + (r - \beta\lambda - \frac{1}{2}\sigma^2)t}$ and $J_t = \prod_{i=1}^{N_t} (Y_i + 1)$.

The goal is to construct a risk neutral probability $\widetilde{\mathbb{P}}$ (equivalent to $\mathbb{P}$) under which the discounted price

$$\widetilde{S}_t := \frac{S_t}{B_t}$$

is a martingale.

For simplicity, define the discount factor $D_t := \frac{1}{B_t}$. By straightforward calculation, one can show that D_t satisfies the ODE $dD_t = -rD_t dt$. Note that D_t is a continuous (decreasing if r is positive) process, therefore, $D_t^c = D_t$ and $D_{t-} = D_t$ for all t.

Applying Itô's formula to $\widetilde{S}_t = D_t S_t$, we obtain

$$\begin{aligned}
d\widetilde{S}_t &= d(D_t S_t) \\
&= S_{t-} dD_t^c + D_{t-} dS_t^c + d[S^c, D^c]_t + S_t D_t - S_{t-} D_{t-} \\
&= -rS_{t-} D_t dt + D_t S_{t-} \left((\mu - \beta\lambda)dt + \sigma dW_t\right) + D_t S_{t-} \left(1 + \Delta Q_t - 1\right) \\
&= \widetilde{S}_{t-} \left[(\mu - \beta\lambda - r)dt + \sigma dW_t + \Delta Q_t\right] \qquad (4)
\end{aligned}$$

We want to construct $\widetilde{\mathbb{P}}$ such that under $\widetilde{\mathbb{P}}$, we have

$$\frac{d\widetilde{S}_t}{\widetilde{S}_{t-}} = 0dt + \sigma d\tilde{W}_t + d(Q_t - \tilde{\beta}\tilde{\lambda}t),$$

where $d\tilde{W}_t = dW_t + \theta dt$ is a $\widetilde{\mathbb{P}}$-Brownian motion.

That means

$$\frac{d\widetilde{S}_t}{\widetilde{S}_{t-}} = (\sigma\theta - \tilde{\beta}\tilde{\lambda})dt + \sigma dW_t + dQ_t,$$

Comparing this with (4), we get

$$\mu - \beta\lambda - r = \sigma\theta - \tilde{\beta}\tilde{\lambda}$$

which can be written as

$$\begin{aligned}
\mu - r &= \sigma\theta - \tilde{\beta}\tilde{\lambda} + \beta\lambda \\
\mu - r &= \sigma\theta - \sum_{m=1}^{M} y_m \tilde{p}_m \tilde{\lambda} + \sum_{m=1}^{M} y_m p_m \lambda \\
\mu - r &= \sigma\theta + \sum_{m=1}^{M} y_m (\lambda_m - \tilde{\lambda}_m) \qquad (5).
\end{aligned}$$

We have 1 equation and $M + 1$ unknowns: $\theta, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_M$.

That means we can construct infinitely many different risk neutral probabilities and hence the model is incomplete.

4.2 Pricing

Pick any solution to (5). Under the new measure $\widetilde{\mathbb{P}}$, we have

$$S_t = S_0 e^{\sigma \tilde{W}_t + (r - \tilde{\beta}\tilde{\lambda} - \frac{1}{2}\sigma^2)t} \prod_{i=1}^{N_t} (Y_i + 1),$$

where N has intensity $\tilde{\lambda}$ under $\widetilde{\mathbb{P}}$ and $\widetilde{\mathbb{P}}(Y_i = y_m) = \tilde{p}_m = \tilde{\lambda}_m / \tilde{\lambda}$.

The time t price of a European option with payoff h is now

$$e^{-r(T-t)} \widetilde{\mathbb{E}} \left[h(S_T) \mid \mathcal{F}_t \right].$$

There is an explicit expression for call prices, see Shreve, Theorem 11.7.5.

Theorem

Let $u(t, s)$ be the price of a European option at time t when $S_t = s$. Then $u(t, s)$ satisfies

$$u_t + \frac{\sigma^2}{2} s^2 u_{ss} + (r - \tilde{\beta}\tilde{\lambda}) s u_s + \tilde{\lambda} \sum_{m=1}^{M} p_m \left\{ u(t, s(1 + y_m)) - u(t, s) \right\} = ru$$

The terminal condition is given by the payoff function h at expiry T, i.e.,

$$u(s, T) = h(s).$$

Proof: Left as an exercise. The process $e^{-rt} u(t, S_t)$ should be a martingale. Apply Ito-Doeblin's formula and set the finite variation part to zero.