9831 Probability and Stochastic Processes for Finance, Fall 2016

Lecture 1

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Outline

- Stochastic processes
- Martingale, submartingale, and supermartingale
- Discrete stochastic integral
- Stopping Times
- Optional stopping theorem

1 Stochastic processes in discrete time

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Definition

A collection of random variables $X := (X_n)_{n\geq 0}$ is called a **(discrete time) stochastic process**.

Definition

- A non-decreasing sequence of σ-algebras $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n \subseteq \cdots \subseteq \mathcal{F}$ is called a **filtration**.

- Let $X := (X_n)_{n\geq 0}$ be a sequence of r.v.s. Let $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$, $n \geq 0$.

  The filtration $(\mathcal{F}_n)_{n\geq 0}$ is called the **natural filtration** of X.

Definition

The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_n)_{n\geq 0}$, denoted by $(\Omega, \mathcal{F}_n, \mathbb{P})$, is called a **filtered probability space**.

Definition

A stochastic process X is called **adapted** to the filtration $(\mathcal{F}_n)_{n>0}$ if X_n is $\mathcal{F}_n$-measurable for every $n \geq 0$.

1.1 Martingales

Let $(\Omega, \mathcal{F}_n, \mathbb{P})$ be a filtered probability space. A stochastic process X is called a **martingale** (**submartingale**, **supermartingale**) with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$ if it satisfies

(i) $\mathbb{E}|X_n| < \infty$ for every $n \geq 0$.

(ii) X is adapted to $(\mathcal{F}_n)_{n\geq 0}$.

(iii) $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = (\geq, \leq)X_n$ for all $n \geq 0$.

Examples of martingales (see Refresher Lecture 6)

- Let $X_0, X_1, \cdots$ be sequence of iid integrable r.v.s with $E(X_1) = \mu$. Let $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$, $n \geq 0$, be the filtration.

  Let $W_0 = 0$ and define $W_n = \sum_{j=1}^n X_j,\ n \geq 1.$

  Then $(W_n)_{n\geq 0}$ is a martingale if $\mu = 0$; a supermartingale if $\mu \leq 0$; a submartingale if $\mu \geq 0$.

  W is called **random walk**.

- Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. square integrable r.v., $E(X_1) = 0$ and $\mathrm{Var}(X_1) = \sigma^2$.

  Then $M_n = W_n^2 - \sigma^2 n$ is a martingale relative to filtration $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$, $n \geq 0$.

- If Y is an integrable r.v., then $X_n := E(Y|\mathcal{F}_n)$, $n \in \mathbb{N}$, defines a martingale.

Basic properties of martingales

- For $m > n$,
  - if X is a martingale, then $\mathbb{E}[X_m|\mathcal{F}_n] = X_n$.
  - if X is a supermartingale, then $\mathbb{E}[X_m|\mathcal{F}_n] \leq X_n$.
  - if X is a submartingale, then $\mathbb{E}[X_m|\mathcal{F}_n] \geq X_n$.
- If X is a martingale, then $E(X_n) = E(X_0)$ for all $n \geq 0$.

  Remark: The converse statement is not true in general!

  As a counter-example take X_n to be an i.i.d. sequence of (non-constant) centered random variables and let $\mathcal{F}_n$ be the natural filtration.

  Then, of course, $E(X_n) = E(X_0) = 0$ but

  $$E(X_{n+1} \mid \mathcal{F}_n) = E(X_{n+1}) = 0 \neq X_n.$$
- Let X be an $\mathcal{F}_n$-submartingale and $\phi(x)$ be a convex non-decreasing function such that $E(|\phi(X_n)|) < \infty$ for all n. Then $(\phi(X_n))_{n\geq 0}$ is an $\mathcal{F}_n$-submartingale.
- If X, Y are martingales with $X_N = Y_N$ for some N, then $X_n = Y_n$ for any $n \leq N$.

1.2 Predictable processes

Let $(\Omega, \mathcal{F}_n, \mathbb{P})$ be a filtered probability space.

Definition

A stochastic process H is called **predictable** if H_n is $\mathcal{F}_{n-1}$-measurable for every $n \geq 1$. Note that, as a convention, we don't define H_0.

Doob's decomposition

Any submartingale (supermartingale) X can be written in a unique way as $X_n = M_n + A_n$, where M is a martingale and A is a predictable and increasing (decreasing) sequence with $A_0 = 0$.

1.3 Discrete stochastic integral

Let $(\Omega, \mathcal{F}_n, \mathbb{P})$ be a filtered probability space.

Definition

The **stochastic integral** of the predictable process H with respect to a process X, denoted by $H \cdot X$, is defined as

$$(H \cdot X)_n = \sum_{i=1}^{n} H_i \Delta X_i,$$

where $\Delta X_n = X_n - X_{n-1}$, $n \geq 1$. As a convention, we define $(H \cdot X)_0 = 0$.

The stochastic integral is a clever way of tracking P&L of a trading strategy over an investment horizon.

Example (Mathematical Finance)

- Suppose X represents a stock price evolution and H the investor's position through time:
  - X_n represents stock price at time n
  - H_n represents the number of shares held in the period from $n-1$ to n
- In the period from $n-1$ to n makes profit $H_n(X_n - X_{n-1}) = H_n \Delta X_n$.
- Hence $H \cdot X$ represents the cumulative P&L between times 0 to n.
- Careful about the order:

  Interpreting $H_n(X_n - X_{n-1})$ as profit at time n, implies we have bought H_n shares *before* the stock price changed from X_{n-1} to X_n.

  In other words: shares are bought at the end of period $n-1$, after the stock settled at X_{n-1}, i.e. only information up to time $n-1$ can be used when H_n is chosen.

  This motivates why we assume H to be predictable.

Example (Doubling strategy)

- Let $\xi_1, \xi_2, \ldots$ be iid r.v.s with $P(\xi_j = 1) = P(\xi_j = -1) = 1/2$.
- Let $X_0 = 0$, $X_n = \sum_{i=1}^{n} \xi_i$, $n \in \mathbb{N}$. We know that X is a martingale.
- You may double your stake if you win.
- We start betting $1.

  If you win, you quit; otherwise you bet $2 on the next game.

  If you win the second game, you quit; otherwise double your bet ...
- At the time we win, we will be ahead $1. With probability 1 we will eventually win this game, so this strategy is a way to beat a fair game.
- Let H be the doubling strategy, i.e. $H_1 = 1$ and if we have never won

$$H_n = 2^{n-1};$$

  otherwise $H_n = 0$ (we quit).
- Let $M_0 = 0$ and M_n be the total winnings after n games.

$$M_n = \sum_{j=1}^{n} H_j \xi_j = \sum_{j=1}^{n} H_j (X_j - X_{j-1}) = (H \cdot X)_n, \ \ n \in \mathbb{N}.$$

- Note that $E(M_n) = 0$ for all n as
  - $M_n = 1$ unless $\xi_1 = \xi_2 = \ldots = \xi_n = -1$ in which case $M_n = -H_n = -(2^{n-1})$,
  - As this last event happens with probability $(1/2)^n$:

$$E(M_n) = 1 \cdot (1 - \frac{1}{2^n}) - (2^{n-1}) \cdot \frac{1}{2^n}) = 0.$$

- However, we will eventually win:

$$M_\infty := \lim_{n \to \infty} M_n = 1 \quad \text{a.s.}$$

  and hence

$$1 = E(M_\infty) > E(M_0) = 0.$$

- Conclusion: We can beat a fair game, if we are allowed an infinite amount of time.

Properties of the stochastic integral

- Let X be a supermartingale (submartingale) and H be nonnegative, bounded and predictable. Then $H \cdot X$ is a supermartingale (submartingale).
- Let X be a submartingale. Suppose H and K are bounded and predictable such that $H_n \geq K_n$ for every n. Then

  $$\mathbb{E}[(H \cdot X)_n] \geq \mathbb{E}[(K \cdot X)_n]$$

  for every $n \geq 0$.
- Let X be a martingale and H be bounded and predictable. (need not be nonnegative!) Then $H \cdot X$ is a martingale.

  Remark: In our example on the doubling strategy M, the total winnings, is a martingale.

Proof of the last statement: Properties (i) and (ii) are obviously satisfied. To check (iii) we write

$$\begin{aligned}
E((H \cdot X)_{n+1} \mid \mathcal{F}_n) &= E\left(\sum_{i=1}^{n+1} H_i \Delta X_i \mid \mathcal{F}_n\right) && \text{(Def.)} \\
&= \sum_{i=1}^{n} H_i \Delta X_i + E(H_{n+1} \Delta X_{n+1} \mid \mathcal{F}_n) && (\sum_{i=1}^{n} H_i \Delta X_i \text{ is } \mathcal{F}_n\text{-measurable}) \\
&= (H \cdot X)_n + H_{n+1} E(X_{n+1} - X_n) \mid \mathcal{F}_n) && (H_n \text{ is predictable}) \\
&= (H \cdot X)_n && (X \text{ is a martingale})
\end{aligned}$$

1.4 Stopping times

In the gambling example above, we have not properly addressed the quitting time:

- The gambler should quit at the first time he wins.
- Clearly, the quitting time is random.
- At each time n the gambler is able to make a decision whether he plays the next round or quits. The quitting time is an example of so called stopping time.

Definition

A random variable $\tau : \Omega \to \mathbb{N} \cup \{\infty\}$ is called a **stopping time** if $\{\tau \leq n\} \in \mathcal{F}_n$ for all $n \in \mathbb{N}_0$.

This means that one can determine whether or not $\tau \leq n$ on the basis of all the information up to time n, i.e. $\mathcal{F}_n$. τ does not "look into the future" to decide whether or not to stop at time $\leq n$.

Examples

- In the last example, the quitting time is the random time $\tau = \inf\{n \geq 0 \,:\, M_n = 1\}$.

  This is an example of a stopping time (the first time of hitting level 1).

  Formally, $\{\tau \leq n\} = \cup_{i=0}^{n} \{M_i = 1\}$.

  Since $\{M_i = 1\} \in \mathcal{F}_i \subseteq \mathcal{F}_n$, τ is a stopping time.

- Let W be simple random walk, i.e. $\xi_1, \xi_2, \ldots$ be iid r.v.s with $P(\xi_j = 1) = P(\xi_j = -1) = 1/2$ and $W_0 = 0$, $W_n = \sum_{i=1}^{n} \xi_i$, $n \in \mathbb{N}$.
  - Define τ to be the first time that the random walk takes the value 3, i.e.

    $$\tau(\omega) := \inf(n \geq 0 : W_n(\omega) = 3).$$

    Then τ is a stopping time.
  - On the other hand

    $$\tau(\omega) := \sup(n \geq 0 : W_n(\omega) = 3)$$

    is not a stopping time.

Typical stopping times are hitting times or the first time that a process leaves/enters a specific interval.

Theorem

The r.v. τ is a stopping time iff $\{\tau = n\} \in \mathcal{F}_n$ for all $n \geq 0$.

Proof:

Let τ be a stopping time.

Notice that $\{\tau = n\} = \{\tau \leq n\} \cap \{\tau \leq n-1\}^c \in \mathcal{F}_n$, since $\{\tau \leq n\} \in \mathcal{F}_n$, $\{\tau \leq n-1\} \in \mathcal{F}_{n-1} \subset \mathcal{F}_n$ and $\mathcal{F}_n$ is a σ-algebra.

Assume now that $\{\tau = n\} \in \mathcal{F}_n$ for all n.

But $\{\tau \leq n\} = \cup_{m=0}^{n} \{\tau = m\}$.

Since $\{\tau = m\} \in \mathcal{F}_m \subseteq \mathcal{F}_n$, we conclude that $\{\tau \leq n\} \in \mathcal{F}_n$.

Lemma

If τ_1 and τ_2 are stopping times then so are $\tau_1 \wedge \tau_2 := \min\{\tau_1, \tau_2\}$, $\tau_1 \vee \tau_2 := \max\{\tau_1, \tau_2\}$, $\tau_1 + \tau_2$.

Example (Doubling strategy cont.)

- Recall that $\tau = \inf\{n \geq 0 \,:\, M_n = 1\}$ is a stopping time.
- Note that $M_\tau = 1$, as the profit in the last period is able to cover the loss in the previous $n-1$ periods, with a net gain of 1.
- However, the expected loss just before the ultimate win is infinite, i.e.

  $$E(M_{\tau-1}) = -\infty.$$

  Indeed, we have

  $$P(\tau = n) = P(n-1 \text{ games lost followed by one win at step } n) = \frac{1}{2^n},$$

  and therefore,

  $$\begin{aligned} E(M_{\tau-1}) &= \sum_{n=1}^{\infty} M_{n-1} 1_{\{\tau=n\}} P(\tau = n) \\ &= \sum_{n=2}^{\infty} (-1 - 2 - \ldots - 2^{n-2}) \frac{1}{2^n} \\ &= -\sum_{n=2}^{\infty} \frac{2^{n-1} - 1}{2^n} \\ &= \sum_{n=2}^{\infty} \frac{1}{2^n} - \sum_{n=2}^{\infty} \frac{1}{2} \\ &= \frac{1}{2} - \sum_{n=2}^{\infty} \frac{1}{2} \\ &= -\infty, \end{aligned}$$

  where the second equality follows from the fact that $M_{\tau-1} = -1 - 2 - \ldots - 2^{n-2}$ if $\tau = n$.
- Conclusion: You are expected to hold an infinite amount of capital in order to beat the fair game.

1.6 Stopped processes

Theorem

Let $(M_n)_{n\geq 0}$ be an $\mathcal{F}_n$ (sub-, super-) martingale and τ be an $\mathcal{F}_n$ stopping time.

Then $(M_{\tau\wedge n})_{n\geq 0}$ is an $\mathcal{F}_n$ (sub-, super-) martingale.

Proof:

We shall use the above statement that a stochastic integral wrt. a martingale is a martingale with $H_n = 1_{\{\tau \geq n\}} \in \{0, 1\}$.

Then $(H_n)_{n \geq 1}$ is bounded and predictable:

$$\{\tau \geq n\} = \{\tau \leq n-1\}^c \in \mathcal{F}_{n-1}.$$

Therefore, $M_0 + (H \cdot M)_n$, $n \geq 1$, is a martingale.

On the other hand,

$$\begin{aligned} M_0 + (H \cdot M)_n &= M_0 + \sum_{j=1}^{n} 1_{\{\tau \geq j\}} (M_j - M_{j-1}) \\ &= M_0 + \sum_{j=1}^{\tau \wedge n} (M_j - M_{j-1}) \\ &= M_{\tau \wedge n}, \end{aligned}$$

and the claim follows.

Remark: In the doubling strategy example above, as M is a martingale, $M_{\tau \wedge n}$ is a martingale as well.

1.7 Optional stopping theorem

For martingales $(M_n)_{n \geq 0}$, we already proved

$$E(M_n) = E(M_0), \quad n \in \mathbb{N}.$$

We would like to prove

$$E(M_\tau) = E(M_0).$$

However, in the doubling strategy example, we have $M_0 = 0$ and $M_\tau = 1$ so that

$$E(M_0) \neq E(M_\tau).$$

Goal: Optional stopping theorem (also called the optional sampling theorem)

For a martingale M and a stopping time τ, it states under which conditions we have

$$E(M_\tau) = E(M_0).$$

Optional stopping theorem (bounded stopping time)

Let τ be a stopping time and M be a martingale wrt. the filtration $(\mathcal{F}_n)_{n \geq 0}$.

If there is constant $K \in \mathbb{N}$ such that $\tau \leq K$ a.s. then

$$E(M_\tau) = E(M_0).$$

Proof:

Since M is a martingale, $(M_{\tau\wedge n})_{n\geq 0}$ is a martingale as well.

For martingales we know that for all $n \geq 0$

$$E(M_{\tau\wedge n}) = E(M_0).$$

Note also, that for $n \geq K$, we have $\tau \wedge n = \tau$, and hence for all $n \geq K$

$$M_{\tau\wedge n} = M_\tau.$$

This gives the claim.

This theorem seems to be very restrictive as it allows only bounded stopping times.

But notice that for any stopping time τ the time $\tau \wedge n$ is a bounded stopping time for every n: $\tau \wedge n \leq n$.

This observation turns the previous theorem into a useful tool.

Example

Let $\xi_1, \xi_2, \ldots$ be iid r.v.s with $P(\xi_j = 1) = p$ and $P(\xi_j = -1) = 1 - p$, $p > 1/2$.

Let $X_0 = 0$, $X_n = \sum_{i=1}^n \xi_i$, $n \in \mathbb{N}$, i.e. a (asymmetric) simple random walk.

For $x \in \mathbb{Z}$, we define

$$\tau_x = \inf\{n : X_n = x\}, \qquad \text{and} \qquad \phi(x) = \left(\frac{1-p}{p}\right)^x.$$

Claim: Then for $a < 0 < b$,

$$P(\tau_a < \tau_b) = \frac{1 - \phi(b)}{\phi(a) - \phi(b)}.$$

- Check that $\phi(X)$ is a martingale.
  - As we can only take one step at a time, $X_n \in [-n, n]$. We also have $\frac{1-p}{p} \in (0,1)$ for $p > 1/2$, hence integrability follows.
  - Adaptedness is obvious.
  - Finally,

$$\begin{aligned} E(\phi(X_{n+1})|\mathcal{F}_n) &= E(\phi(X_n + \xi_{n+1}) \mid \mathcal{F}_n) \\ &= E\left(\phi(X_n)\left(\frac{1-p}{p}\right)^{\xi_{n+1}} \mid \mathcal{F}_n\right) \\ &= \phi(X_n)E\left(\left(\frac{1-p}{p}\right)^{\xi_{n+1}}\right) \\ &= \phi(X_n)\left(p\left(\frac{1-p}{p}\right)^{1} + (1-p)\left(\frac{1-p}{p}\right)^{-1}\right) \\ &= \phi(X_n) \end{aligned}$$

- We should now apply the Optional Stopping Theorem to the stopping time $\tau = \tau_a \wedge \tau_b$.

  Note however, that τ is not bounded. Instead, we apply the theorem to the stopping time $\tau \wedge N$ for an arbitrary, deterministic N.

  This gives

$$\begin{aligned} 1 &= E(\phi(X_0)) = E(\phi(X_{\tau\wedge N})) \\ &= \phi(a)P(X_\tau = a, \tau \leq N) + \phi(b)P(X_\tau = b, \tau \leq N) + E(\phi(X_N)1_{\{\tau > N\}}). \end{aligned} \tag{1}$$

- Now

$$
\begin{aligned}
0 &\leq E(\phi(X_N)1_{\{\tau>N\}}) \\
&= E(\phi(X_N) \mid \tau > N)P(\tau > N) \\
&\leq \left[\left(\frac{1-p}{p}\right)^a + \left(\frac{1-p}{p}\right)^b\right] P(\tau > N),
\end{aligned}
$$

  and since $P(\tau > N) \to 0$ as $N \to \infty$, we can let $N \to \infty$ in (1) to deduce

$$1 = \phi(a)P(X_\tau = a) + \phi(b)P(X_\tau = b).$$

- Finally, since $P(X_\tau = a) = 1 - P(X_\tau = b)$, and $P(\tau_a < \tau_b) = P(X_\tau = a)$, the above equation becomes

$$1 = \phi(a)P(\tau_a < \tau_b) + \phi(b)(1 - P(\tau_a < \tau_b)).$$

  Rearranging gives the result.

The advantage of the theorem on stopped processes and the optimal stopping theorem above is that they hold pretty much without changes in the continuous time setting and there are many problems which can be solved using these theorems only.

It is still desirable to have a theorem which allows unbounded stopping times.

There is a large variety of such theorems.

We shall state the following result.

Optional stopping theorem (conditionally bounded increments)

Let M be an $\mathcal{F}_n$-martingale and τ be an $\mathcal{F}_n$ stopping time.

If $E(\tau) < \infty$ and there is a constant B such that $E(|M_{n+1} - M_n| \mid \mathcal{F}_n) \leq B$ for all $n \geq 0$ on $\{\tau \geq n\}$ then $E(|M_\tau|) < \infty$ and

$$E(M_\tau) = E(M_0).$$

The proof can be found, for example, in Borovkov, Probability Theory, 2013, Ch. 15.2.

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