9831 Probability and Stochastic Processes for Finance, Fall 2016

Lecture 11

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Outline

- Poisson process
- Compound Poisson process
- Stochastic integral with respect to a jump process
- Itô-Doeblin formula for processes with jump

1 Poisson process

A standard Poisson process is the most elementary jump process.

Definition

$(N_t)_{t\geq 0}$ is a **Poisson process** with **intensity** $\lambda > 0$ if

- $N_0 = 0$
- N_t has right continuous paths with left limits (cadlag)
- Given $0 < t_1 < t_2 < \cdots < t_n$, the increments

  $$N_{t_1}, N_{t_2} - N_{t_1}, \cdots, N_{t_n} - N_{t_{n-1}}$$

  are independent.
- For each $0 \leq s \leq t$ the random variable $N_t - N_s$ has Poisson distribution with parameter $\lambda(t-s)$, i.e.

  $$\mathbb{P}(N_t - N_s = k) = \frac{(\lambda(t-s))^k}{k!} e^{-\lambda(t-s)}, \quad k \in \{0\} \cup \mathbb{N}.$$

Note:

- In particular, we have for all $t \geq 0$

  $$\mathbb{P}(N_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k \in \{0\} \cup \mathbb{N}.$$

- Poisson processes have *stationary increments*, i.e. the distribution of the increments depends only on the difference between the two time points.

1.1 Construction of a Poisson process

Let T_i, $i \in \mathbb{N}$, be i.i.d. exponential random variables with parameter $\lambda > 0$.

Let

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} T_i, \quad n \in \mathbb{N}.$$

Think about counting events (such as customer arrivals, defaults, etc.), where T_i is the waiting tie between the $(i-1)$-th and i-th events.

Thus, S_n is the time of the n-th arrival.

Define

$$N_t := \begin{cases} 0, & \text{if } 0 \le t < S_1 \\ 1, & \text{if } S_1 \le t < S_2 \\ \vdots & \\ n, & \text{if } S_n \le n < S_{n+1} \\ \vdots & \end{cases}$$

Then N_t is a Poisson process with intensity λ (see Homework 11).

1.2 Basic properties of a Poisson process

For a Poisson process N_t with intensity λ, we have

- $\mathbb{E}(N_t - N_s) = \lambda(t-s)$
- $\mathrm{Var}(N_t - N_s) = \lambda(t-s)$
- $\mathbb{E}[e^{i\xi N_t}] = e^{\lambda t(e^{i\xi}-1)}$

Proof: The proof involves straightforward calculations.

We only go over the calculation of the characteristic function.

$$
\begin{aligned}
\mathbb{E}\left[e^{i\xi N_t}\right] &= \sum_{k=0}^{\infty} e^{i\xi k}\mathbb{P}\left[N_t = k\right] \\
&= \sum_{k=0}^{\infty} e^{i\xi k} e^{-\lambda t}\frac{(\lambda t)^k}{k!} \\
&= e^{-\lambda t}\sum_{k=0}^{\infty}\frac{\left(e^{i\xi}\lambda t\right)^k}{k!} \\
&= e^{-\lambda t} e^{e^{i\xi}\lambda t} \\
&= e^{\lambda t(e^{i\xi}-1)}.
\end{aligned}
$$

Note

- Notice that $e^{i\xi}$ is the characterstic function of the (degenerate) random variable $Y \equiv 1$.
- The sample path of a Poisson process is mostly flat. When it jumps, it jumps up by 1.
- This charateristic function is a template of more complicated pure jump processes.

Generalizations of Poisson processes

- inhomogeneous/nonhomogeneous Poisson process
- double stochastic/mixed Poisson process or Cox process

1.3 Compensated Poisson process

Definition

Let N_t be a Poisson process with intensity λ. The process

$$M_t = N_t - \lambda t$$

is called **compensated Poisson process**.

Note:

The process M_t is called compensated because $\mathbb{E}\left[M_t\right] = 0$ for all t.

Theorem

The compensated Poisson process M_t is a martingale with respect to its natural filtration.

Proof

M_t is adapted and integrable.

We shall only verify the martingale property, i.e., for any $s < t, \mathbb{E}\left[M_t | \mathcal{F}_s\right] = M_s$ almost surely.

$$\begin{aligned}\mathbb{E}\left[M_t | \mathcal{F}_s\right] &= \mathbb{E}\left[N_t - \lambda t | \mathcal{F}_s\right] \\ &= \mathbb{E}\left[N_t - N_s + N_s | \mathcal{F}_s\right] - \lambda t \\ &= \mathbb{E}\left[N_t - N_s | \mathcal{F}_s\right] + N_s - \lambda t \\ &= \lambda(t-s) + N_s - \lambda t \qquad (N_t - N_s \text{ is indep. of } \mathcal{F}_s,\ N_t - N_s \sim \text{Pois}(\lambda(t-s))) \\ &= N_s - \lambda s \\ &= M_s.\end{aligned}$$

1.4 Sum of independent Poisson processes is Poisson

Lemma

Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with intensities λ_1 and λ_2 respectively.

Then the process $N_t = N_1(t) + N_2(t)$ is a Poisson process with intensity $\lambda_1 + \lambda_2$.

That is, the sum of independent Poisson processes is again a Poisson process with intensity given by the sum of the intensities.

Proof: We shall simply calculate the characteristic function of N_t.

$$\begin{aligned}\mathbb{E}\left[e^{i\xi N_t}\right] &= \mathbb{E}\left[e^{i\xi(N_1(t)+N_2(t))}\right] \\ &= \mathbb{E}\left[e^{i\xi N_1(t)}\right]\mathbb{E}\left[e^{i\xi N_2(t)}\right] \qquad (N_1(t) \text{ and } N_2(t) \text{ are independent}) \\ &= e^{\lambda_1 t(e^{i\xi}-1)}e^{\lambda_2 t(e^{i\xi}-1)} \quad (N_1(t), N_2(t) \text{ are } Pois(\lambda_1 t) \text{ and } Pois(\lambda_2 t) \text{ distributed}) \\ &= e^{(\lambda_1+\lambda_2)t(e^{i\xi}-1)}.\end{aligned}$$

Hence, N_t is Poisson distributed with intensity $\lambda_1 + \lambda_2$.

2 Compound Poisson process

2.1 Definition

Let N_t be a Poisson process with intensity λ and let $Y_1, Y_2, \cdots$ be an i.i.d. sequence of random variables which are independent of the Poisson process N_t.

The **compound Poisson process** Q_t is defined as

$$Q_t = 0 \quad \text{if } N_t = 0, \qquad Q_t = \sum_{i=1}^{N_t} Y_i, \quad \text{if } N_t > 0.$$

Note

- The process also has stationary and independent increments but the distribution of $Q_t - Q_s$ is not Poisson, it depends on the distribution of Y_i.
- The random variables Y_i are referred to as **jump size**.
- A compound Poisson process is a continuous time jump process and the jump size is a random variable independent of the underlying Poisson process.

2.2 Properties of compound Poisson processes

Theorem

Let Q_t be a compound Poisson process with jump size Y satisfying $\mathbb{E}[Y] = \beta$. Then

- $\mathbb{E}[Q_t] = \beta\lambda t$
- $\mathbb{E}[e^{i\xi Q_t}] = e^{\lambda t(\varphi(\xi)-1)}$, where $\varphi(\xi) = \mathbb{E}[e^{i\xi Y}]$ is the characteristic function of Y.
- $Q_t - \beta\lambda t$ is a martingale

Remark

Intuitively, since Q_t has in average λt jumps within the time interval $[0, t]$ and each jump has expected size β, the expected value of Q_t is simply their product.

Therefore, by subtracting off the expected value, Q_t becomes a martingale.

The sum of two independet compound Poisson processes is again a compound Poisson process:

Theorem

Let $Q_1(t)$ and $Q_2(t)$ be two independent compound Poisson processes with

- deterministic jump sizes y_1 and y_2 and
- intensities $p_1\lambda$ and $p_2\lambda$, where $p_1 + p_2 = 1$ and $p_1, p_2 > 0$.

Then the process $Q_t = Q_1(t) + Q_2(t)$ is a compound Poisson process with intensity λ and jump size distribution Y given by $\mathbb{P}[Y = y_i] = p_i$, $i = 1, 2$.

Proof: See Homework 11

Corollary

For compound Poisson process Q_t with only finitely many jump sizes say $y_1, \cdots, y_M$ with probability $p_1, \cdots, p_M$, there are two equivalent representations:

$$Q_t = \sum_{m=1}^{M} y_m N_m(t), \qquad Q_t = \sum_{k=1}^{N_t} Y_k,$$

where N_t is a Poisson process with intensity λ and, for $m = 1, \cdots, M$, $N_m(t)$ is a Poisson process with intensity $p_m\lambda$.

3 Stochastic integrals with respect to a jump process

Goal: Define the stochastic integral

$$\int_0^t \phi_s dX_s,$$

where X_t is a stochastic process with jumps.

3.1 Jump processes

We always work with one filtration for all the processes involved. More precisely:

Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration on this space.

We say that

- a Brownian motion W_t is a Brownian motion relative to this filtration if W_t is adapted and for $0 \leq s < t$, the increment $W_t - W_s$ is independent of $\mathcal{F}_s$,
- a Poisson process N_t is a Poisson process relative to this filtration if N_t is adapted and for $0 \leq s < t$, the increment $N_t - N_s$ is independent of $\mathcal{F}_s$,
- a compound Poisson process Q_t is a compound Poisson process relative to this filtration if Q_t is adapted and for $0 \leq s < t$, the increment $Q_t - Q_s$ is independent of $\mathcal{F}_s$.

Definition

A **jump process** X_t with starting point X_0 is a process of the form
$$X_t = X_0 + I_t + R_t + J_t,$$

where

- X_0 is a constant
- I_t is an Ito integral, i.e. $I_t = \int_0^t = \sigma_s dB_s$ for some adapted process σ_s. We call I_t the **Ito integral part of X_t**.
- R_t is a Riemann integral, i.e. $\int_0^t \mu_s ds$ for some adapted process μ_s. We call R_t the **Riemann integral part of X_t**.
- J_t is an adapted, right-continuous pure jump process with $J_0 = 0$. We call J_t the **pure jump part of X_t**.

Remarks:

- By right-continuous, we mean $J_t = \lim_{s\downarrow t} J_s$.
- If J_t has a jump at time t, then J_t is the value of the process immediately after the jump. The value $J_{t-} := \lim_{s\uparrow t} J_s$ is the value immediately before the jump.
- We assume that J_t does not jump at time 0, has only finitely many jumps on each finite time interval $(0, T]$ and is constant between jumps (hence called *pure* jump process).
- A Poisson process and a compound Poisson process have this property. A compensated Poisson process does not as it decreases between jumps.

Definition

- The **jump size** (if there is any) at time t, denoted by ΔX_t, is defined as

$$\Delta X_t = X_t - X_{t-} = J_t - J_{t-} = \Delta J_t.$$

- The **continuous part of** X_t, denoted by X_t^c, is defined by

$$X_t^c := X_0 + I_t + R_t.$$

Example

Let

$$X_t = X_0 + \sigma B_t + \mu t + \sum_{k=1}^{N_t} Y_k,$$

where N_t is a Poisson process with intensity λ and the jump size Y is normally distributed, then

$$X_t^c = X_0 + \sigma B_t + \mu t,$$

$$J_t = \sum_{k=1}^{N_t} Y_k,$$

and

$$\Delta X_t = Y_{N_t} \quad \text{if } \Delta N_t \neq 0.$$

3.2 Integral with respect to jump processes

Definition

A process Y_t is called **predictable** if it is the limit of a sequence of simple càglàd (continu à gauch avec limite à droite, left continuous with right limits) processes, i.e., processes of the form

$$Y_t(\omega) = a_0(\omega)1_{\{0\}}(t) + \sum_{k=0}^{n} a_k(\omega)1_{(t_k,t_{k+1}]}(t),$$

where, for $k = 0, 1, \cdots, n$, a_k is a $\mathcal{F}_{t_k}$-measurable random variable.

Note

For any adapted process X_t, the associated left-limit process X_{t-} is predictable.

Definition

Let X_t be a jump process and ϕ_t be a predictable process.

The **stochastic integral of ϕ with respect to X** is defined as

$$\begin{aligned}\int_0^t \phi_s dX_s &:= \int_0^t \phi_s \sigma_s dX_s^c + \sum_{0<s\le t} \phi_s \Delta J_s \\ &= \int_0^t \phi_s \sigma_s dB_s + \int_0^t \phi_s \mu_s ds + \sum_{0<s\le t} \phi_s \Delta J_s,\end{aligned}$$

or equivalently in differential notation

$$\begin{aligned}\phi_t dX_t &= \phi_t \sigma_t dX_t^c + \phi_t dJ_t \\ &= \phi_t \sigma_t dB_t + \phi_t \mu_t dt + \phi_t \Delta J_t.\end{aligned}$$

Note

Since J_t has at most countable jumps in the interval $[0, t]$, the sum in the last expression is in fact legit, assuming its convergence.

Example

Let $M_t = N_t - \lambda t$ be a compensated Poisson process. Recall that M_t is a martingale.

Consider the "stochastic" integral $\int_0^t \varphi_s dM_s$, where $\varphi_t = \Delta N_t$.

Note that $M_t^c = -\lambda dt$ and $\Delta M_t = \Delta N_t$.

$$\begin{aligned}\int_0^t \varphi_s dM_s &= \int_0^t \varphi_s dM_s^c + \sum_{0<s\leq t} \varphi_s \Delta M_s \\ &= -\lambda \int_0^t \Delta N_s ds + \sum_{0<s\leq t} \Delta N_s \Delta M_s \\ &= 0 + \sum_{0<s\leq t} \Delta N_s \\ &= N_t.\end{aligned}$$

Notice that

- the integrand $\varphi_t = \Delta N_t$ is not predictable.
- the integral $\int_0^t \varphi_s dM_s$ is not a martingale though the integrator M_t is.
- in order to retain the martingality of stochastic integral with respect to jump processes, further restriction (other than adaptedness) is required.

Theorem

Assume the jump process X_t is a martingale, the integrand ϕ is predictable, adapted and satisfies

$$\mathbb{E}\left[\int_0^t \phi_s^2 \sigma_s^2 ds\right] < \infty, \quad \text{for all } t \geq 0.$$

Then the stochastic integral

$$\int_0^t \phi_s dX_s$$

is also a martingale.

Remarks

- Although we require the integrand ϕ_t to be predictable, the integrator X_t is always taken right-continuous, and so the integral $\int_0^t \phi_s dX_s$ will be right-continuous in the upper limit of integration t.
- The integral jumps whenever X jumps and ϕ is simultaneously not zero.
- The value of the integral at time t includes the jump at time t if there is a jump.

3.3 Quadratic and cross variation

Let X, Y be jumps processes and Π be a finite partition of $[0,t]$: $0 = t_0 < t_1 < \ldots < t_n = t$.

Define

$$Q_\Pi(X) := \sum_{i=0}^{n-1}(X_{t_{j+1}} - X_{t_j})^2,$$

$$C_\Pi(X,Y) := \sum_{i=0}^{n-1}\big((X_{t_{j+1}} - X_{t_j})(Y_{t_{j+1}} - Y_{t_j})\big).$$

If these quantities have an L^2-limit for all $t \geq 0$ as $||\Pi|| \to 0$, then the limiting processes are denoted

$$[X]_t \qquad \text{and} \qquad [X,Y]_t$$

and called the **quadratic variation of X** and **cross variation of X and Y** on $[0,t]$ respectively.

We state without proof:

Theorem

Let $X_1(t)$ and $X_2(t)$ be jump processes defined by, for $i = 1, 2$,

$$X_i(t) = X_i(0) + \int_0^t \sigma_i(s)dB_s + \int_0^t \mu_i(s)ds + J_i(t).$$

Then

$$\begin{aligned}[X_1, X_2]_t &= [X_1^c, X_2^c]_t + [J_1, J_2]_t \\ &= \int_0^t \sigma_1(s)\sigma_2(s)ds + \sum_{0<s\leq t} \Delta J_1(s)\Delta J_2(s).\end{aligned}$$

Remarks

- When $X_1 = X_2 = X$, the formula reduces to a formula for quadratic variation $[X]$ for X as

  $$[X]_t = \int_0^t \sigma_s^2 ds + \sum_{0<s\leq t} (\Delta J_s)^2.$$

  Obviously, the first term comes from the Ito part and the second term is the sum of the jump sizes squared.

- The quadratic variation of a pure jump process on $(0,t]$ is the sum of the squares of jumps in that time interval

- The cross-variation of a continuous process and a pure jump process is zero.

Quadratic variation of stochastic integrals:

Theorem

Let X_t be a jump process defined by

$$X_t = X_0 + \int_0^t \sigma_s dB_s + \int_0^t \mu_s ds + J_t$$

and ϕ a predictable process. Let

$$Y_t = Y_0 + \int_0^t \phi_s dX_s = Y_0 + \int_0^t \phi_s \sigma_s dB_s + \int_0^t \phi_s \mu_s ds + \sum_{0<s\le t} \phi_s \Delta J_s.$$

Then the quadratic variation $[Y]_t$ of Y_t is given by

$$\begin{aligned}[Y]_t &= \int_0^t \phi_s^2 d[X]_s \\ &= [Y^c]_t + [J^Y]_t \\ &= \int_0^t \sigma_s^2 \phi_s^2 ds + \sum_{0<s\le t} \phi_s^2 (\Delta J_s)^2.\end{aligned}$$

4 Itô-Doeblin formula for processes with jump

Theorem

Let X_t be a jump process and f be a C^2 function. Then the following Itô's formula holds

$$f(X_t) - f(X_0) = \int_0^t f'(X_s) dX_s^c + \frac{1}{2} \int_0^t f''(X_s) d[X^c]_s + \sum_{0<s\le t} \big(f(X_s) - f(X_{s-})\big).$$

Note

As in the case for continuous processes, the Itô-Doeblin formula helps us evaluate stochastic integrals and solve stochastic differential equations.

Example

Evaluate the stochastic integral $\int_0^t M_{s-} dM_s$, where M_t is a compensated Poisson process.

Solution: Note that, since $M_t = N_t - \lambda t$, $M_t^c = -\lambda t$ and $\Delta M_t = \Delta N_t$.

Applying Itôs' formula to M_t^2 we obtain

$$\begin{aligned} M_t^2 - M_0^2 = & \int_0^t 2M_s dM_s^c + \frac{1}{2}\int_0^t 2d[M^c]_s + \sum_{0<s\leq t}\{M_s^2 - M_{s-}^2\} \\ = & -2\lambda \int_0^t M_{s-} ds + \sum_{0<s\leq t}(2M_{s-}+1)\Delta N_s \\ = & \, 2\left[-\lambda\int_0^t M_{s-} ds + \sum_{0<s\leq t} M_{s-}\Delta N_s\right] + \sum_{0<s\leq t}\Delta N_s \\ = & \, 2\int_0^t M_{s-} dM_s + N_t, \end{aligned}$$

where we used

$$M_s^2 - M_{s-}^2 = (M_{s-} + \Delta N_s)^2 - M_{s-}^2 = 2M_{s-}\Delta N_s + (\Delta N_s)^2 = (2M_{s-}+1)\Delta N_s$$

since $\Delta N_s \in \{0, 1\}$. We conclude that

$$\int_0^t M_{s-} dM_s = \frac{1}{2}(M_t^2 - N_t).$$

Exercise: Verify that the right hand side in the last expression is a martingale.

Example: Geometric Poisson process

The process

$$S_t = S_0 e^{-\lambda\sigma t}(1+\sigma)^{N_t}$$

is referred to as a **Geometric Poisson process** since, by applying Itô-Doeblin's formula, it satisfies

$$S_t = S_0 + \sigma\int_0^t S_{\tau-} dM_\tau.$$

(see Homework 11)

Example: Doleans-Dade exponential

Let X_t be a jump process. The Doleans-Dade exponential of X is defined to be the process

$$Z_t = e^{X_t^c - \frac{1}{2}[X^c]_t} \prod_{0<s\le t} (1 + \Delta X_s).$$

This process is the solution to the stochastic differential equation

$$Z_t = 1 + \int_0^t Z_{s-}\, dX_s \quad \leftrightsquigarrow \quad dZ_t = Z_{t-}\, dX_t,\ Z_0 = 1.$$

To see this, note that

$$Z_t = Z_0 + \int_0^t Z_{s-}\, dX_s^c + \sum_{0<s\le t} Z_{s-}\, \Delta X_s$$

and therefore we have

$$dZ_t^c = Z_{t-}\, dX_t^c, \quad d[Z^c]_t = Z_{t-}^2\, d[X^c]_t, \quad \Delta Z_t = Z_{t-}\, \Delta X_t.$$

Apply Itô's formula to $\log Z_t$

$$\begin{aligned}
\log Z_t &= \log Z_0 + \int_0^t \frac{1}{Z_{s-}} dZ_s^c - \frac{1}{2}\int_0^t \frac{1}{Z_{s-}^2} d[Z^c]_s + \sum_{0<s\le t} \log Z_s - \log Z_{s-} \\
&\qquad (\text{since } Z_s = Z_{s-} + Z_{s-}\Delta X_s) \\
&= \log Z_0 + \int_0^t dX_s^c - \frac{1}{2}\int_0^t d[X^c]_s + \sum_{0<s\le t} \log(1 + \Delta X_s) \\
&= \log Z_0 + X_t^c - \frac{1}{2}[X^c]_t + \sum_{0<s\le t} \log(1 + \Delta X_s).
\end{aligned}$$

Example: Merton's jump diffusion model

Let $X_t = \mu t + \sigma B_t + \sum_{k=1}^{N_t}(Y_k - 1)$, where N_t is a Poisson process with intensity λ and the jump size Y is lognormally distributed.

Consider the SDE driven by X_t

$$dS_t = S_{t-}dX_t \quad \leftrightsquigarrow \quad S_t = S_0 + \int_0^t S_{\tau-}dX_\tau \quad \leftrightsquigarrow \quad S_t = S_0 + \int_0^t S_\tau dX_\tau^c + \sum_{0<\tau\le t} S_{\tau-}\Delta X_\tau.$$

By the Doleans-Dade exponential formula, the solution to the SDE is given by

$$\begin{aligned} S_t &= S_0 e^{X_t^c - \frac{1}{2}[X^c]_t} \prod_{0<s\le t}(1+\Delta X_s) \\ &= S_0 e^{\mu t - \frac{\sigma^2}{2}t + \sigma W_t} \prod_{0<s\le t}\{1 + (Y_{N_s} - 1)\Delta N_s\} \\ &= S_0 e^{\mu t - \frac{\sigma^2}{2}t + \sigma W_t} \prod_{k=1}^{N_t} Y_k \\ &= S_0 e^{\mu t - \frac{\sigma^2}{2}t + \sigma W_t + \sum_{k=1}^{N_t}\log Y_k}. \end{aligned}$$

Note

- S_t in Merton's jump diffusion model is still lognormally distributed.
- The normality of $\log Y$ in the derivation above plays no role. It works perfectly well with any random variable.

  In fact, in Kou's model, $\log Y$ is chosen as an asymmetric double exponential distribution. Moreover, the variance gamma (VG) model corresponds to $\log Y$ being (asymmetric) double gamma distribution.