9831 Probability and Stochastic Processes for Finance, Fall 2016

Lecture 6

Anja Richter

Department of Mathematics

Outline

- More on SDEs
- Derivation of BS PDE
- Multivariate stochastic calculus
- Applications: Levy characterization of Brownian motion, Covariance of stock prices driven by correlated Brownian motions

1 Stochastic differential equations (continued)

1.1 Ornstein-Uhlenbeck process / Vasicek model

The Ornstein-Uhlenbeck (OU) process / Vasicek model is defined by the following SDE

$$dX_t = \lambda(m - X_t)dt + \sigma dB_t, \quad X_0 = x,$$

where $\lambda, m, \sigma > 0$.

We solve the SDE by applying Ito's formula to $e^{\lambda t}X_t$ (see Homework 05) and obtain

$$X_T = e^{-\lambda T}x + \left(1 - e^{-\lambda T}\right) m + \sigma \int_0^T e^{-\lambda(T-t)} dB_t.$$

Properties

- X_t is normally distributed: $X_t \sim N\left(e^{-\lambda t} X_0 + \left(1 - e^{-\lambda t}\right) m, \frac{\sigma^2}{2\lambda}\left(1 - e^{-2\lambda t}\right)\right)$.

- X_t is a Gaussian process with mean function $\mathbb{E}\left[X_t\right] = e^{-\lambda t} X_0 + \left(1 - e^{-\lambda t}\right) m$ and covariance function $\gamma(t, s) = \dfrac{\sigma^2}{2\lambda} e^{-\lambda(s+t)} \left(e^{2\lambda(s \wedge t)} - 1\right)$ (see homework 06).

- The parameters have the following meaning:
  - λ is usually referred to as the *mean reverting rate*
  - m stands for the long term mean, as for $t \to \infty$, $\mathbb{E}\left[X_t\right] \to m$ exponentially with rate λ
  - σ stands for the "instantaneous" volatility.

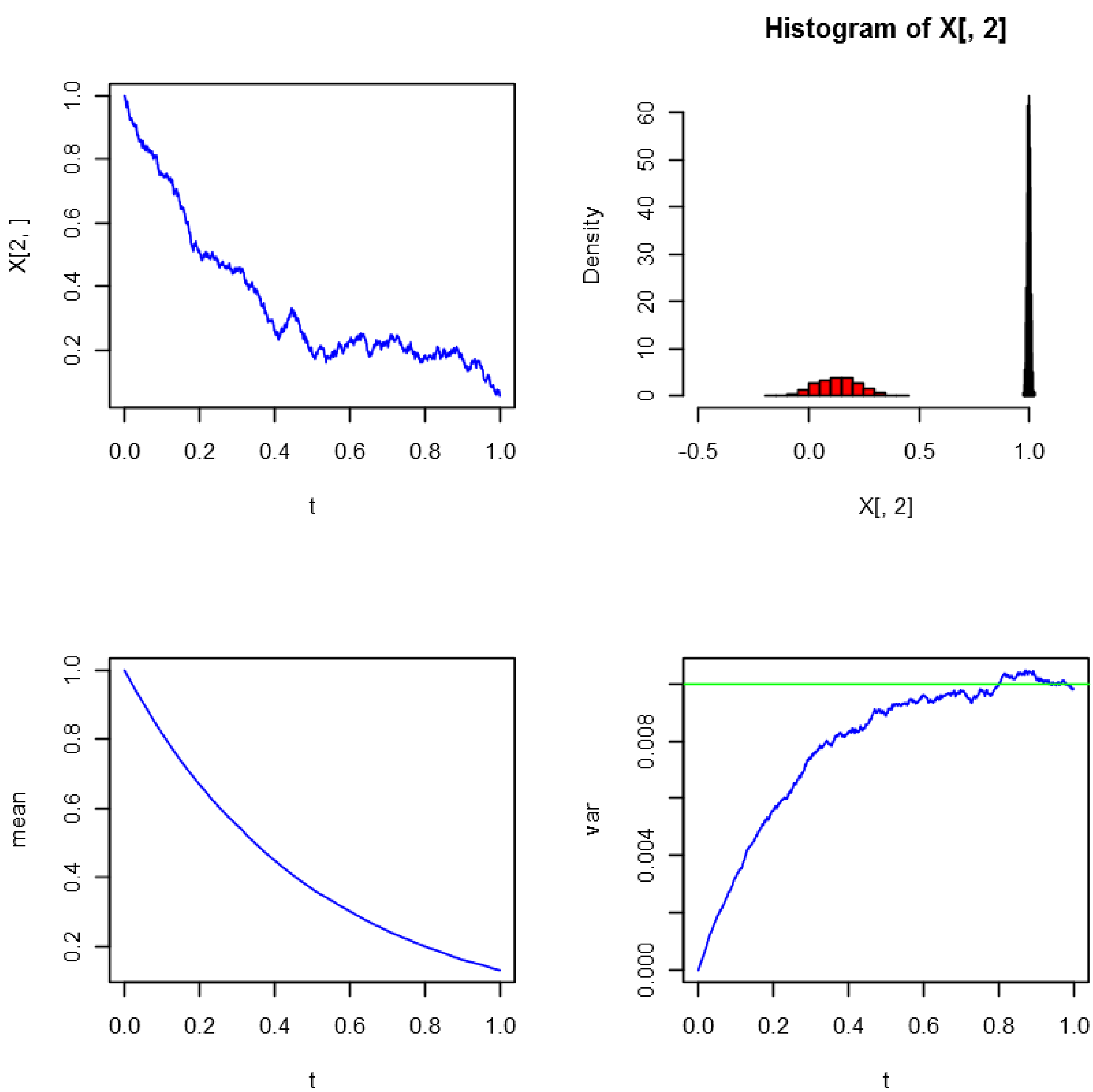
- As $t \to \infty$, the distribution of X_t converges to the stationary distribution $N\left(m, \frac{\sigma^2}{2\lambda}\right)$.

- As $\lambda \to \infty$, the distribution of X_t approaches $N(m, 0)$, i.e., a point mass or Dirac delta at m, for $t > 0$.

Simulation

In [36]:

```
# The code simulates the OU process dX = lambda (m - X) dt + sigma dB
# by Euler-Maruyama scheme

# Parameters of the OU process
lambda <- 2
m <- 0
sigma <- .2
x0 <- 1
Tfin <- 1

# number of paths and number of time steps
NSim <- 1e3
NSteps <- 1000

# initialize X
X <- matrix(0,NSim,NSteps+1)
X[,1] <- x0

# Euler-Maruyama scheme
dt <- Tfin/NSteps
for (i in 1:NSteps){
  dB <- sqrt(dt)*rnorm(NSim)
  X[,i+1] <- X[,i] + lambda*(m - X[,i])*dt + sigma*dB
}

par(mfrow=c(2,2))
#OU paths
t <- (0:NSteps)/NSteps*Tfin
plot(t,X[2,],type='l',col='blue')

# histograms at different times
hist(X[,2],prob=T,col='blue',breaks=20,xlim=c(-.5,1.2))
hist(X[,NSteps+1],prob=T,col='red',breaks=20,add=T)

# time evolution of mean
plot(t,colMeans(X),type='l',col='blue',ylab='mean')
#plot(t,exp(-lambda*t),type='l')

# time evolution of variance
plot(t,apply(X,2,FUN=var),type='l',col='blue',ylab='var')
abline(h=sigma^2/2/lambda,col='green')
```


Histogram of X[, 2]
X[2,]
t
Density
X[, 2]
mean
t
var
t

1.2 Cox-Ingersoll-Ross process

Consider the Cox-Ingersoll-Ross (CIR) process

$$dX_t = \lambda(m - X_t)dt + \sigma\sqrt{X_t}dB_t, \quad X_0 = x,$$

where $\lambda, m, \sigma > 0$.

- There is no simple expression, as the ones for geometric Brownian motion and the OU process, for the solution of the CIR process.

- Nevertheless, we can still calculate the mean and variance of X_t by applying the Ito-Doeblin formula. The calculation of the mean is the same as in the derivation for the Ornstein-Uhlenbeck process. The variance is obtained by applying Ito's formula to X_t^2 (see Homework 05). We have

$$\begin{aligned} E(X_t) &= xe^{-\lambda t} + m(1 - e^{-\lambda t}) \\ \mathrm{Var}(X_t) &= x\frac{\sigma^2}{\lambda}(e^{-\lambda t} - e^{-2\lambda t}) + \frac{m\sigma^2}{2\lambda}(1 - e^{-\lambda t}). \end{aligned}$$

- The parameters the same meaning as for OU processes:
  - λ is usually referred to as the *mean reverting rate*
  - m stands for the long term mean
  - σ stands for the "instantaneous" volatility.

2 Another application of the Ito-Doeblin formula: Derivation of the Black Scholes PDE

2.1 A portfolio of stock and riskless asset

We start with 2 ingredients, i.e. the market consists of:

- one risky asset, a stock, whose price S_t at time $t \geq 0$ satisfies

$$dS_t = \mu S_t \, dt + \sigma S_t \, dB_t,$$

- a riskless asset, money market account (MMA), M_t, which has a constant interest rate r (continuous compounding), i.e.

$$dM_t = rM_t dt.$$

We set up a portfolio with time-t value V_t.

At any time t, the investor holds

- Δ_t shares of the stock (random but adapted to the filtration of the Brownian motion), and
- the remainder $V_t - \Delta_t S_t$ is invested in the money account.

That means, we have

$$V_t = \underbrace{\Delta_t S_t}_{\text{risky asset}} + \underbrace{V_t - \Delta_t S_t}_{\text{money account}}$$

Self-financing strategy: At time t, the agent reallocates his portolio as holding $\Delta_{t+\Delta t}$ shares in the risky asset without adding to or withdrawing from his portfolio at the beginning of time t and holds the allocation till the end of time $t + \Delta t$. Thus, at the beginning of $t + \Delta t$, his portfolio reads

$$V_t = \underbrace{\Delta_{t+\Delta t} S_t}_{\text{risky asset}} + \underbrace{V_t - \Delta_{t+\Delta t} S_t}_{\text{money account}}$$

Note that at this point the value of his portfolio is still V_t because he simply moves shares/money from/to risy asset/money account.

At the end of time $t + \Delta t$, since the price of risk asset has moved to $S_{t+\Delta t}$ and the money account accrued according to interest rate r, the value of his portfolio becomes

$$V_{t+\Delta t} = \underbrace{\Delta_{t+\Delta t} S_{t+\Delta t}}_{\text{risky asset}} + \underbrace{(V_t - \Delta_{t+\Delta t} S_t)(1 + r\Delta t)}_{\text{money account}}$$

Hence, the change of the value of the agent's portfolio from t to $t + \Delta t$ is given by

$$\begin{aligned} V_{t+\Delta t} - V_t &= \overbrace{\Delta_{t+\Delta t} S_{t+\Delta t}}^{\text{risky asset}} + \overbrace{(V_t - \Delta_{t+\Delta t} S_t)(1 + r\Delta t)}^{\text{money account}} - \left[\overbrace{\Delta_{t+\Delta t} S_t}^{\text{risky asset}} + \overbrace{V_t - \Delta_{t+\Delta t} S_t}^{\text{money account}} \right] \\ &= \Delta_{t+\Delta t} \left(S_{t+\Delta t} - S_t \right) + (V_t - \Delta_{t+\Delta t} S_t) r\Delta t \\ &= \Delta_{t+\Delta t} \Delta S_{t+\Delta t} + (V_t - \Delta_{t+\Delta t} S_t) r\Delta t. \end{aligned}$$

In the limit, as $\Delta t \to 0$, we have that the evolution of the portfolio value is given by

$$dV_t = \Delta_t dS_t + r(V_t - \Delta_t S_t)dt.$$

Hence, a self-financing trading strategy is determined by the holdings in the risky asset. It's value is governed by the last equation.

Substituting dS_t into this equation, we get

$$\begin{aligned} dV_t &= \Delta_t(\mu S_t\, dt + \sigma S_t\, dB_t) + r(V_t - \Delta_t S_t)dt \\ &= \sigma \Delta_t S_t dB_t + rV_t\, dt + (\mu - r)\Delta_t S_t\, dt \\ &=: (I) + (II) + (III). \end{aligned}$$

The three terms above are:

- (I) the volatility term proportional to the size of the stock investment;
- (II) an average underlying rate of return r of the portfolio;
- (III) a risk premium $(\mu - r)$ for investing in stock.

Now consider a European call option with payoff $(S_T - K)_+$ at time T.

Black, Scholes, and Merton argued that the value of this call at time t should depend only on $T - t$ and the price of the stock at time t (other parameters $r,\ \sigma,\ K$ being fixed).

If we let $c(t, x)$ denote the price of the option at time t when $S_t = x$ then $c(t, x)$ is a non-random function.

The stochastic process $c(t, S_t)$ is the price of the option at time t.

Assuming that $c(t, x)$ is smooth enough to apply the Ito-Doeblin formula, we get

$$\begin{aligned} dc(t, S_t) &= c_t(t, S_t)\, dt + c_x(t, S_t)\, dS_t + \frac{1}{2}\, c_{xx}(t, S_t)\, d[S]_t \\ &= \left(c_t(t, S_t) + \mu S_t c_x(t, S_t) + \frac{1}{2}\, \sigma^2 S_t c_{xx}(t, S_t) \right) dt + \sigma c_x(t, S_t) S_t\, dB_t. \end{aligned}$$

This equation represents the evolution of the option value.

2.2 Hedging portfolio and BS PDE

A hedging portfolio for a short position on the option has to satisfy

$$V_t = c(t, S_t) \quad \text{for all } t \in [0, T].$$

Instead of equating the above equations for dV_t and $dc(t, S_t)$, it is more convenient to equate the present values:

$$e^{-rt} V_t = e^{-rt} c(t, S_t), \quad t \in [0, T].$$

The corresponding evolutions are given by

$$\begin{aligned} d(e^{-rt} V_t) &= -re^{-rt} V_t \, dt + e^{-rt} dV_t \\ &= e^{-rt} \Delta_t (\mu - r) S_t \, dt + \sigma e^{-rt} \Delta_t S_t \, dB_t, \end{aligned}$$

and

$$\begin{aligned} d(e^{-rt} c(t, S_t)) &= -re^{-rt} c(t, S_t) \, dt + e^{-rt} dc(t, S_t) \\ &= e^{-rt} \left(-rc(t, S_t) + c_t(t, S_t) + \mu S_t c_x(t, S_t) + \frac{1}{2} \sigma^2 S_t^2 c_{xx}(t, S_t) \right) dt \\ &\quad + e^{-rt} \sigma c_x(t, S_t) S_t \, dB_t. \end{aligned}$$

Equating the stochastic integral parts and finite variation parts in the last two equations, we obtain

- from the stochastic integral parts:

$$\Delta_t = c_x(t, S_t), \quad t \in [0, T), \ a.\,s.\,,$$

- from the finite variation parts and the previous line:

$$c_x(t, S_t)(\mu - r) S_t = -rc(t, S_t) + c_t(t, S_t) + \mu S_t c_x(t, S_t) + \frac{1}{2} \sigma^2 S_t^2 c_{xx}(t, S_t), \quad t \in [0, T), a.\,s.$$

  and, after cancellations,

$$rc(t, S_t) = c_t(t, S_t) + r S_t c_x(t, S_t) + \frac{1}{2} \sigma^2 S_t^2 c_{xx}(t, S_t), \quad t \in [0, T), \ \text{a.s..}$$

The last equation tells us that we need to find a deterministic function $c(t, x)$ which satisfies the **Black-Scholes-Merton partial differential equation (BSM PDE)**

$$rc(t, x) = c_t(t, x) + \mu x c_x(t, x) + \frac{1}{2} \sigma^2 x^2 c_{xx}(t, x), \quad x \geq 0, \ t \in [0, T),$$

with the terminal condition $c(T, x) = (x - K)^+$.

For the solution to be unique we need additional boundary conditions at $x = 0$ and as $x \to \infty$.

- The first one is obtained by plugging $x = 0$ in the BSM PDE to get

$$c_t(t, 0) = rc(t, 0).$$

  Solving this ODE with $c(0, 0) = 0$ we get the condition at $x = 0$:

$$c(t, 0) = 0, \qquad \text{for all } t \in [0, T].$$

- The condition at infinity can be stated in the following form:

$$\lim_{x \to \infty} (c(t, x) - (x - e^{-(T-t)} K)) = 0, \quad t \in [0, T].$$

  As x grows large, the call option will be deep in the money, and it will very likely finish in the money. In this case, the price of the call at time t is almost as much as the time t price of the forward contract with delivery price K and expiration T, i.e. $S_t - e^{-(T-t)} K$.

Suppose that we have found such a function $c(t, x)$.

Then at each time t we have both the option price $c(t, S_t)$ (S_t is known at time t) and the replicating portfolio $c_x(t, S_t)$.

Indeed, if the investor starts with initial capital $V_0 = c(0, S_0)$ and at time t has $\Delta_t = c_x(t, S_t)$ shares of the underlying asset in his portfolio, then the dB_t terms in the equations for $d(e^{-rt} V_t)$ and $d(e^{-rt} c(t, S_t)$ agree.

The dt terms then also agree as $c(t, x)$ satisfies the BS PDE.

This and the equality of the initial values imply that the equation

$$e^{-rt} V_t = e^{-rt} c(t, S_t),$$

holds for all $t \in [0, T)$.

As $t \uparrow T$ we get by continuity of V_t and $c(t, S_t)$ that

$$V_T = c(T, S_T) = (S_T - K)_+,$$

i.e. the short position is successfully hedged.

3 Multivariate Stochastic Calculus

3.1 Multidimensional Brownian motion

Definition

An n dimensional Brownian motion B_t is a process

$$B_t = (B_1(t), \cdots, B_n(t))$$

with the properties

- each $B_i(t)$ is a one-dimensional Brownian motion.
- $B_i(t)$ and $B_j(t)$ are independent if $i \neq j$.

Hence an n dimensional Brownian motion

- has continuous sample paths because all the B_i's are continuous
- has mean vector $\mathbf{0}$ and covariance matrix $t\mathbf{I_n}$ at time t, where $\mathbf{I_n}$ denotes the identity matrix
- is an n dimensional Gaussian process with covariance function $(t \wedge s)\mathbf{I_n}$
- has transition density

  $$p(s, y|t, x) = (2\pi(s-t))^{-n/2} e^{-\frac{|y-x|^2}{2(s-t)}}, \qquad s > t,$$

  i.e.

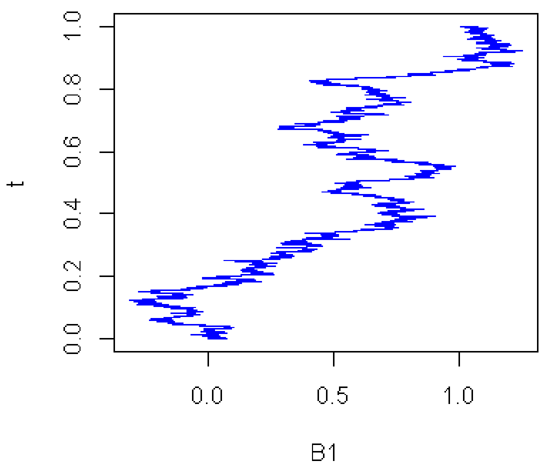
  $$\mathbb{P}\left[B_s \in A | B_t = x\right] = \int_A p(s, y|t, x) dy$$

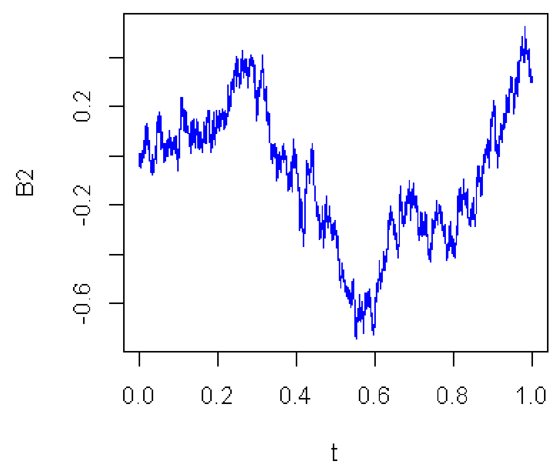
- has infinitesimal generator $\frac{1}{2}\Delta$, where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the **Laplace operator** or **Laplacian**.

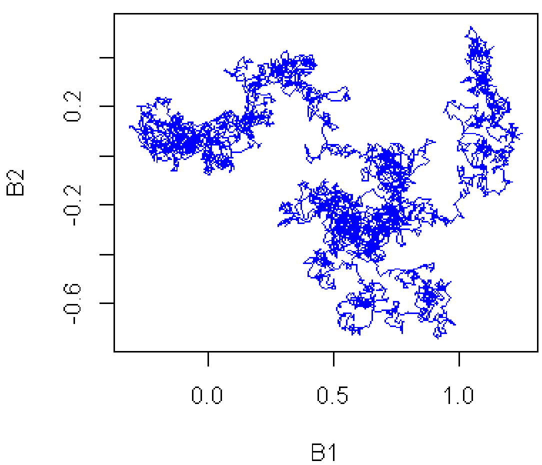
A sample path of two dimensional Brownian motion

In [1]:

```
# The code simulate a sample path of two dimensional Browian motion

NSteps <- 5e3
Tfin <- 1
dt <- Tfin/NSteps
t <- (0:NSteps)/NSteps*Tfin
dB1 <- sqrt(dt)*rnorm(NSteps)
dB2 <- sqrt(dt)*rnorm(NSteps)
B1 <- c(0,cumsum(dB1))
B2 <- c(0,cumsum(dB2))

par(mfrow=c(2,2))
plot(B1,t,type='l',col='blue')
plot(t,B2,type='l',col='blue')
plot(B1,B2,type='l',col='blue')
```







Ito-Doeblin formula for multidimensional Brownian motion

Let $B_t = (B_1(t), \cdots, B_n(t))'$ be an n dimensional Brownian motion and $f = f(t, x_1, \cdots, x_n)$ be a function of $n+1$ variables which is continuously differentiable in t and twice continuously differentiable in $x = (x_1, \cdots, x_n)$.

Then

$$f(T, B_T) - f(t, B_t) = \int_t^T \left[f_t(s, B_s) + \frac{1}{2}\Delta f(s, B_s) \right] ds + \int_t^T \nabla f(s, B_s) \cdot dB_s,$$

where ∇f is the gradient of f and $\Delta f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$ is the Laplacian of f.

Equivalently, in differential form, we have

$$df(t, B_t) = \left[f_t(t, B_t) + \frac{1}{2}\Delta f(t, B_t) \right] dt + \nabla f(t, B_t) \cdot dB_t.$$

3.2 Multivariate Ito-Doeblin formula for Ito processes

Reminder: Multivariate calculus

Let $f: U \subset \mathbb{R}^n \to \mathbb{R}$ be a function of n variables. Then

- the **gradient of** f, denoted by ∇f, is defined by the vector $\nabla f = \left[\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n}\right]'$
- the **Hessian matrix of** f, denoted by $\mathrm{Hess}\, f$, is defined as the $n \times n$ symmetric matrix

$$\mathrm{Hess}\, f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

- the Laplacian of f, denoted by Δf, is defined as $\Delta f = \mathrm{trace}(\mathrm{Hess}\, f) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$.

Definition (Multidimensional Ito process)

Let $B_t = (B_1(t), \cdots, B_m(t))'$ be an m dimensional Brownian motion and μ_t and σ_t be vector-valued and matrix-valued adapted processes respectively as

$$\mu_t = \begin{bmatrix} \mu_1(t) \\ \vdots \\ \mu_n(t) \end{bmatrix}, \qquad \sigma_t = \begin{bmatrix} \sigma_{11}(t) & \cdots & \sigma_{1m}(t) \\ \vdots & \ddots & \vdots \\ \sigma_{n1}(t) & \cdots & \sigma_{nm}(t) \end{bmatrix}.$$

An n **dimensional Ito process** $X_t = (X_1(t), \cdots, X_n(t))'$ is a process defined by the stochastic integrals with respect to the m dimensional Brownian motion B_t, for $i = 1, \cdots, n$,

$$X_i(t) = X_i(0) + \int_0^t \mu_i(s)ds + \sum_{k=1}^{m} \int_0^t \sigma_{ik}(s)dB_k(s) \quad \leftrightsquigarrow \quad dX_i(t) = \mu_i(t)dt + \sum_{k=1}^{m} \sigma_{ik}(t)dB_k(t).$$

Or more concisely in matrix form

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dB_s \quad \leftrightsquigarrow \quad dX_t = \mu_t dt + \sigma_t dB_t.$$

μ_t is termed as the **drift vector** and σ_t the **diffusion matrix**.

Ito's formula for multidimensional Ito processes

Let $X_t = (X_1(t), \cdots, X_n(t))'$ be an n dimensional Ito process defined above. Let $f = f(t, x_1, \cdots, x_n)$ be a function of $n+1$ variables which is continuously differentiable in t and twice continuously differentiable in $x = (x_1, \cdots, x_n)$. Then

$$\begin{aligned} & f(T, X_T) - f(t, X_t) \\ &= \int_t^T \frac{\partial f}{\partial t}(s, X_s)ds + \int_t^T \nabla f(s, X_s) \cdot dX_s + \frac{1}{2}\sum_{i,j=1}^n \int_t^T \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_s) d[X_i, X_j]_s \\ &= \int_t^T \left[\frac{\partial f}{\partial t} + \mu_t \cdot \nabla f + \frac{1}{2}\sum_{i,j=1}^n a_{ij}(t)\frac{\partial^2 f}{\partial x_i \partial x_j}\right] ds + \int_t^T \nabla f \cdot \sigma_t dB_s, \end{aligned}$$

where $a_{ij}(t) = \sum_{k=1}^m \sigma_{ik}(t)\sigma_{jk}(t)$. Equivalently, in differential form

$$df = \nabla f \cdot \sigma_t dB_t + \left[\frac{\partial f}{\partial t} + \mu_t \cdot \nabla f + \frac{1}{2}\mathrm{trace}(A_t \mathrm{Hess}\, f)\right] dt,$$

where

$$A_t = [a_{ij}(t)] = \sigma_t \sigma_t'.$$

Corollary: Ito's product rule

Let X_t and Y_t, be Ito processes. Then

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s\, dY_s + \int_0^t Y_s\, dX_s + [X, Y]_t.$$

Or, in differential form

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + d[X, Y]_t.$$

Proof: Apply Ito's formula for multidimensional Ito processes to $f(t, x, y) = xy$.

3.3 Recognizing a Brownian motion (dimension 1)

Somewhat surprising result:

Theorem (Levy's characterization of Brownian motion)

A continuous martingale M with $M_0 = 0$ and quadratic variation $[M]_t = t$ is a Brownian motion.

Sketch of proof:

- We have defined the stochastic integral wrt. Brownian motion. Exactly the same approach can be used to define the stochastic integral with respect to any martingale with continuous paths, $\int_0^t H_s\, dM_s$.

  Moreover, the following analog of the Ito-Doeblin formula holds for functions f with continuous partial derivatives f_t, f_x and f_{xx}:

  $$f(t, M_t) - f(0, M_0) = \int_0^t f_t(s, M_s)\, ds + \int_0^t f_x(s, M_s)\, dM_s + \frac{1}{2}\int_0^t f_{xx}(s, M_s)\, d[M]_s.$$

  In our case $[M]_t = t$, so the last integral is just a regular integral.

  Taking expectations and using the fact that because M_t is a martingale, the stochastic integral $\int_0^t f_x(s, M_s)\, dM_s$ is a martingale as well, we get

  $$E[f(t, M_t)] = f(0, M_0) + E\left[\int_0^t f_t(s, M_s)\, ds + \frac{1}{2}\int_0^t f_{xx}(s, M_s)\, ds\right]. \qquad (1)$$

- We are given that $M_0 = 0$ and that M has continuous paths. All we need to check is that the increments are independent and normally distributed with the correct variance.

- We first show that $M_t \sim N(0,t)$. Fix a $u \in \mathbb{R}$ and define

$$f(t,x) = e^{ux - u^2 t/2}.$$

Then we have

$$f_t(t,x) = -\frac{1}{2}u^2 f(t,x), \quad f_x(t,x) = uf(t,x), \quad f_{xx}(t,x) = u^2 f(t,x),$$

and in particular

$$f_t(t,x) + \frac{1}{2}f_{xx}(t,x) = 0.$$

For this function $f(t,x)$, the RHS of (1) is zero and (1) becomes

$$E[e^{uM_t - u^2 t/2}] = f(0,0) = 1.$$

Rewriting this formula, we get

$$E[e^{uM_t}] = e^{u^2 t/2},$$

which is the MGF for an $N(0,t)$ distributed r.v. Similarly, we can show that $M_t - M_s \sim N(0, t-s)$, $0 \le s \le t$.

- We first show that $M_t \sim N(0,t)$. Fix a $u \in \mathbb{R}$ and define

$$f(t,x) = e^{ux - u^2 t/2}.$$

Then we have

$$f_t(t,x) = -\frac{1}{2}u^2 f(t,x), \quad f_x(t,x) = uf(t,x), \quad f_{xx}(t,x) = u^2 f(t,x),$$

and in particular

$$f_t(t,x) + \frac{1}{2}f_{xx}(t,x) = 0.$$

For this function $f(t,x)$, the RHS of (1) is zero and (1) becomes

$$E[e^{uM_t - u^2 t/2}] = f(0,0) = 1.$$

Rewriting this formula, we get

$$E[e^{uM_t}] = e^{u^2 t/2},$$

which is the MGF for an $N(0,t)$ distributed r.v. Similarly, we can show that $M_t - M_s \sim N(0, t-s)$, $0 \le s \le t$.

- Finally we show the independence of increments. Assume that $m = 2$ (number of points in the partition) and consider $M_{t_2} - M_{t_1}$ and $M_{t_1} - M_{t_0}$, $t_0 = 0$. The case of general m can be treated in exactly the same way. By repeated conditioning we get

$$\begin{aligned} E\left(e^{u_2(M_{t_2}-M_{t_1})+u_1(M_{t_1}-M_{t_0})}\right) &= E\left(E\left(e^{u_2(M_{t_2}-M_{t_1})+u_1(M_{t_1}-M_{t_0})} \mid \mathcal{F}_{t_1}\right)\right) \\ &= E\left(e^{u_1(M_{t_1}-M_{t_0})} E\left(e^{u_2(M_{t_2}-M_{t_1})} \mid \mathcal{F}_{t_1}\right)\right) \\ &= E\left(e^{u_1(M_{t_1}-M_{t_0})} e^{u_2^2(t_2-t_1)/2}\right) \\ &= e^{u_2^2(t_2-t_1)/2} e^{u_1^2(t_1-t_0)/2}. \end{aligned}$$

This implies that $M_{t_2} - M_{t_1}$ and $M_{t_1} - M_{t_0}$ are independent.

Example

Let f_t be an arbitrary random or non-random integrand that only takes values 1 and -1.

Then $I_t := \int_0^t f_s \, dB_s$, $t \geq 0$, is a Brownian motion.

Indeed I_t, $t \geq 0$, is a continuous martingale (by the properties of stochastic integrals) and $[I]_t = \int_0^t f_s^2 \, ds = t$ for all $t \geq 0$.

3.4 Application: Covariance of stock prices driven by correlated Brownian motions

Let B be a two-dimensional Brownian motion and S_1 and S_2 satisfy the following system of SDEs:

$$\begin{aligned} dS_1(t) &= \alpha_1 S_1(t) \, dt + \sigma_1 S_1(t) \, dB_1(t) \\ dS_2(t) &= \alpha_2 S_2(t) \, dt + \sigma_2 \rho S_2(t) \, dB_1(t) + \sigma_2 \sqrt{1-\rho^2} S_2(t) \, dB_2(t), \end{aligned}$$

where $\alpha_i \in \mathbb{R}$, $\sigma_i > 0$, $i = 1, 2$, and $\rho \in [-1, 1]$.

Correlation between $S_1(t)$ and $S_2(t)$

- If $\rho \neq 0$ then S_1 and S_2 are correlated, since dB_1 appears in both equations.
- Since S_1 is a geometric Brownian motion (GBM) and $S_1(t) = S_1(0)e^{(\alpha_1 - \sigma_1^2/2)t + \sigma_1 B_1(t)}$.
- What kind of process is S_2?

  Rewrite the equation for S_2 as

  $$\frac{dS_2(t)}{S_2(t)} = \alpha_2\, dt + \sigma_2 \underbrace{\Big(\rho\, dB_1(t) + \sqrt{1-\rho^2}\, dB_2(t)\Big)}_{=:dB_3(t)}$$

  If we show that B_3 is a Brownian motion then we know that S_2 is a GBM and

  $$S_2(t) = S_2(0)e^{(\alpha_2 - \sigma_2^2/2)t + \sigma_2 B_3(t)}.$$

  .

  We shall use Levy's characterization of Brownian motion. We have
  - $B_3(0) = 0$,
  - B_3 has continuous paths
  - B_3 is a martingale (as a linear combination of martingales)
  - $d[B_3]_t = \rho^2 d[B^1]_t + (1-\rho^2) d[B_2]_t + 2\rho\sqrt{1-\rho^2}\, \underbrace{d[B_1, B_2]_t}_{=0} = \rho^2\, dt + (1-\rho^2)\, dt = dt.$

  We conclude that B_3 is a Brownian motion.

- **Correlation between B_1 and B_3:** Observe that

$$\operatorname{Corr}(B_1(t), B_3(t)) = \frac{\operatorname{Cov}(B_1(t), B_3(t))}{t} = \rho,$$

so we can say that stock prices S_1 and S_2 are *driven by two correlated Brownian motions*, B_1 and B_3 .

- **Correlation between the stock prices:** Note that the SDEs (alternatively, the explicit formulas) imply that

$$E(S_i(t)) = S_i(0)e^{\alpha_i t}, \quad i = 1, 2.$$

We only need to compute $E(S_i(t)S_j(t))$. We shall deal with the case $i = 1,\ j = 2$ (when $i = j$ we just need to set $\rho = 1$).

By the Ito-Doeblin product rule,

$$\begin{aligned} d(S_1(t)S_2(t)) &= S_1(t)\, dS_2(t) + S_2(t)\, dS_1(t) + d[S^1, S^2]_t \\ &= S_1(t)S_2(t)(\alpha_2\, dt + \sigma_2\, dB_3(t)) + S_1(t)S_2(t)(\alpha_1\, dt + \sigma_1\, dB_1(t)) \\ &\quad + \rho\sigma_1\sigma_2 S_1(t)S_2(t)\, dt \\ &= (\alpha_1 + \alpha_2 + \rho\sigma_1\sigma_2)S_1(t)S_2(t)\, dt + S_1(t)S_2(t)(\sigma_2\, dB_t^3 + \sigma_1\, dB_1(t)). \end{aligned}$$

Integrating from 0 to t and taking the expectation we find that

$$E[S_1(t)S_2(t)] - S_1(0)S_2(0) = (\alpha_1 + \alpha_2 + \rho\sigma_1\sigma_2)\int_0^t E[S_1(u)S_2(u)]\, du.$$

Solving the above equation we get

$$\begin{aligned} \operatorname{Cov}(S_1(t), S_2(t)) &= E[S_1(t)S_2(t)] - S_1(0)S_2(0)e^{(\alpha_1+\alpha_2)t} \\ &= S_1(0)S_2(0)(e^{(\alpha_1+\alpha_2+\rho\sigma_1\sigma_2)t} - e^{(\alpha_1+\alpha_2)t}). \end{aligned}$$

From here we obtain

$$\begin{aligned} \operatorname{Corr}(S_1(t), S_2(t)) &= \frac{S_1(0)S_2(0)e^{(\alpha_1+\alpha_2)t}(e^{\rho\sigma_1\sigma_2 t} - 1)}{S_1(0)e^{\alpha_1 t}\sqrt{e^{\sigma_1^2 t} - 1}S_2(0)\, e^{\alpha_2 t}\sqrt{e^{\sigma_2^2 t} - 1}} \\ &= \frac{e^{\rho\sigma_1\sigma_2 t} - 1}{\sqrt{e^{\sigma_1^2 t} - 1}\sqrt{e^{\sigma_2^2 t} - 1}}. \end{aligned}$$

3.5 Recognizing a Brownian motion (dimension 2)

Theorem (Levy's characterization of Brownian motion)

Let $M_i(t)$, $t \geq 0$, $i = 1, 2$, be continuous martingales. Assume that $M_0^i = 0,\ [M_i]_t = t,\ t \geq 0$ ($i = 1, 2$), $[M_1, M_2]_t = 0,\ t \geq 0$, a.s..

Then

$$M_t := \begin{bmatrix} M_1(t) \\ M_2(t) \end{bmatrix}$$

is a two-dimensional Brownian motion.

Sketch of proof:

- By the one dimensional theorem, each M_1 and M_2 are Brownian motions. We need to show that they are independent.
- Method: We shall show that for $0 \leq s \leq t$ the vector

  $$\begin{pmatrix} M_1(t) - M_1(s) \\ M_2(t) - M_2(s) \end{pmatrix}$$

  is independent of $\mathcal{F}_s$ and is normally distributed with mean 0 and covariance matrix $\begin{pmatrix} t-s & 0 \\ 0 & t-s \end{pmatrix}$.

  This will imply that the joint distribution of increments of processes M_1 and M_2 is normal. Then the fact that the above covariance matrix is diagonal will imply the independence of the processes M_1 and M_2.

- Since $M_i(t)$ is a Brownian motion,

  $$E[e^{u_i(M_i(t)-M_i(s))}] = e^{u^2(t-s)/2}, \qquad i = 1, 2.$$

  Therefore, it is sufficient to show that

  $$E\left(e^{u_1(M_1(t)-M_1(s))+u_2(M_2(t)-M_2(s))} \mid \mathcal{F}_s\right) = e^{u_1^2(t-s)/2+u_2^2(t-s)/2} \quad \text{for all } 0 \leq s \leq t. \qquad (2)$$

  This gives us an idea to try to show that the process

  $$f(t, M_1(t), M_2(t)) = e^{u_1M_1(t)+u_2M_2(t)-u_1^2t/2-u_2^2t/2}, \quad t \geq 0,$$

  is a martingale. Then we would have $E(f(t, M_1(t), M_2(t)) \mid \mathcal{F}_s) = Ef(s, M_1(s), M_2(s))$, and this would immediately imply (2).

- To compute $df(t, M_1(t), M_2(t))$ we calculate for $f(t,x,y) = e^{u_1 x + u_2 y - (u_1^2+u_2^2)t/2}$:

$$\begin{aligned} f_t &= -\frac{u_1^2+u_2^2}{2}\, f \qquad & f_x &= u_1 f \qquad & f_y &= u_2 f \\ f_{xy} &= u_1 u_2 f \qquad & f_{xx} &= u_1^2 f \qquad & f_{yy} &= u_2^2 f, \end{aligned}$$

  and apply the Ito-Doeblin formula for martingales

$$\begin{aligned} df(t, M_1(t), M_2(t)) = &-\frac{u_1^2+u_2^2}{2}\, f\, dt + u_1 f dM_1(t) + u_2 f\, dM_2(t) \\ &+ \frac{1}{2}\, u_1^2 f \underbrace{d[M_1]_t}_{=dt} + \frac{1}{2}\, u_2^2 f\, \underbrace{d[M_2]_t}_{=dt} + u_1 u_2 f\, \underbrace{d[M_1, M_2]_t}_{=0}. \end{aligned}$$

  All "dt" terms cancel out, and we are left only with stochastic integrals. Therefore $f(t, M^1(t), M^2(t))$ is a martingale.