MACSS HW3

Hyunwoo Roh

5.1 If the individual lives for one period T=1, what is the condition that characterizes the optimal amount of cake to eat in period1? Write the problem in the equivalent way of showing what the condition is for the optimal amount of cake to save for the next period W_{t+1}or W_2.

[Equivalent to check with the individual who lives the last period]
Condition that in the following period of the last period there should be nothing left is needed, which means in the last period, one should eat everything they have. For the individual who lives for one period, they have to eat everything at that peiord: $C_{T=1} = W_{T=1}$ and $W_{T=2} = 0$. For individuals who are now living the last period of the time T, they should consume everything so that $W_{T+1} = 0$

$$\text{At period t=T: } \max_{C_T, W_{T+1}} \beta^{T-1} u(c_T) \text{ s.t } W_{T+1} = W_T - c_T$$

$$= \max_{W_{T+1}} u(W_T - W_{T+1})$$
$$= \max u(W_T)$$

individual would maximize by having it all. Equivalently

$$\max_{W_2} u(W_1 - W_2)$$

and $W_2 = 0$ is the optimal

5.2 If the individual lives for two periods $T = 2$, what is the condition that chracterizes the optimal amount of cake to leave for the next period W_3 in period 2? What is the condition that characterizes the optimal amount of cake leave for the next period W_2 in period 1?

[Equivalent to check with the individual who lives the last two period]
Now we are dealing with the time period at $t = T - 1$. In this period, we only care about 4 things: $c_{T-1}, c_T, W_T, W_{T+1}$ with given initial cake size W_{T-1}

$$\text{At period t=T-1 : } \max_{C_T, C_{T+1}, W_T, W_{T+1}} \beta^{T-2} u(c_{T-1}) + \beta^{T-1} u(c_T)$$
$$s.t W_{T+1} = W_T - c_T \text{ and} W_T = W_{T-1} - c_{T-1}$$

$$= \max_{W_T, W_{T+1}} [u(W_{T-1} - W_T) + \beta u(W_T - W_{T+1})]$$

Since we know from the 5.1 that W_{T+1}=**0**

$$= \max_{W_T} [u(W_{T-1} - W_T) + \beta u(W_T)]$$

Maximization problem can be solved with $\dfrac{\partial [u(W_{T-1} - W_T) + \beta u(W_T)]}{\partial W_T} = 0$

$$\Rightarrow W_T = \psi_{t-1}(W_{T-1}) : -u'(W_{T-1} - W_T) + \beta u'(W_T) = 0 \cdots eq(1)$$

$$\Rightarrow \beta = \frac{u'(W_{T-1} - W_T)}{u'(W_T)}$$

Therefore, in period at $t = T-1$, the individual consume would consume the amount of which changing utility is equivalent to the discounted one at next period.

5.3 If the individual lives for three periods T=3 what are the conditions that characterize the optimal amount of cake to leave for the next period in each period $\{W_2, W_3, W_4\}$? Now assume that the initial cake size is $W_1 = 1$, the discount factor is $\beta = 0.9$, and the period utility function is $ln(c_t)$. Show that how $\{c_t\}_{t=1}^3$and $\{W_t\}_{t=1}^4$ evolve over the three periods?

[Equivalent to check with the individual who lives the last three periods]
At T=3 or T-2 period, the individual have to choose two saving and three consumption. Continuing on the previous problem,\

$$\text{At period t=T-2 : } \max_{C_{T-1}, C_{T-1}, C_T, W_{T-1}, W_T, W_{T+1}} \beta^{T-3} u(c_{T-2}) + \beta^{T-2} u(c_{T-1}) + \beta^{T-1} u(c_T)$$

$$s.t W_{T+1} = W_T - c_T, W_T = W_{T-1} - c_{T-1} \text{and } W_{T-1} = W_{T-2} - c_{T-2}$$

$$\Rightarrow \max_{W_{T-1}, W_T, W_{T+1}} \left[u(W_{T-2} - W_{T-1}) + \beta u(W_{T-1} - W_T) + \beta^2 u(W_T - W_{T+1})\right]$$

$$= \max_{W_{T-1}, W_T} \left[u(W_{T-2} - W_{T-1}) + \beta u(W_{T-1} - W_T) + \beta^2 u(W_T)\right]$$

$$(By\, eq(1) \text{ above}) = \max_{W_{T-1}} \left[u(W_{T-2} - W_{T-1}) + \beta u(W_{T-1} - \psi_{t-1}(W_{T-1})) + \beta^2 u(\psi_{t-1}(W_{T-1}))\right]$$

$$(F.O.C) \Rightarrow -\frac{\partial u}{\partial c_{T-2}} + \beta \frac{\partial u}{\partial c_{T-1}}(1 - \frac{\partial \psi_{T-1}}{\partial W_{T-1}}) + \beta^2 \frac{\partial u}{\partial c_T}(\frac{\partial \psi_{T-1}}{\partial W_{T-1}}) = 0$$

$$\Rightarrow u'(W_{T-2} - W_{T-1}) = \beta u'(W_{T-1} - W_T) - \beta u'(W_{T-1} - W_T)\frac{\partial \psi_{T-1}}{\partial W_{T-1}} + \beta^2 u'(W_T)\frac{\partial \psi_{T-1}}{\partial W_{T-1}}$$

$$\Rightarrow u'(W_{T-2} - W_{T-1}) - \beta u'(W_{T-1} - W_T) - \frac{\partial \psi_{T-1}}{\partial W_{T-1}} \beta \left(u'(W_{T-1} - W_T) - \beta u'(W_T)\right) = 0$$

$\Rightarrow$(From what we have at 5.2, the 3rd component equals 0 to get)

$$\Rightarrow u'(W_{T-2} - W_{T-1}) = \beta u'(W_{T-1} - W_T)$$

Therefore, now plug in the values with $W_1 = 1$ and $\beta = 0.9$. For T=3,

$$u'(W_1 - W_2) = \beta u'(W_2 - W_3)$$

$$\frac{1}{1 - W_2} = 0.9 \frac{1}{W_2 - W_3}$$

$$\Rightarrow \frac{1}{c_1} = 0.9 \frac{1}{c_2}$$

$$\Rightarrow c_2 = 0.9 c_1$$

For T=4,

$$
\begin{aligned}
u'(W_2 - W_3) &= \beta u'(W_3 - W_4) \\
\frac{1}{W_2 - W_3} &= 0.9\frac{1}{W_3 - W_4} \\
\Rightarrow \frac{1}{c_2} &= 0.9\frac{1}{c_3} \\
\Rightarrow & c_3 = 0.9c_2
\end{aligned}
$$

$W_1 = 1$
$W_2 = W_1 - c_1 = 1 - c_1$
$W_3 = W_2 - c_2 = 1 - c_1 - c_2 = 1 - 1.9c_1$
$W_4 = W_3 - c_3 = 1 - c_1 - c_2 - c_3 = 1 - 1.9c_1 - 0.81c_1 = 1 - 2.71c_1 = 0$
$c_1 = 0.369$ $c_2 = 0.332$ $c_3 = 0.298$ and with these information we can calculate $\{W_t\}_{t=1}^{4}$ as well.

5.4 Using the envelope theorem that says W_{T+1}will be chosen optimally in the next period, show the condition that characterizes the optimal choice (the policy function) in period T-1 for $W_T = \psi_{T-1}(W_{T-1})$. Show the value fucntion V_{T-1} in terms of $\psi_{T-1}(W_{T-1})$.

$$
\begin{aligned}
V_{T-1}(W_{T-1}) &\equiv \max_{W_T} u(W_{T-1} - W_T) + \beta V_T(W_T) \\
&\equiv \max_{W_{T-1}} u(W_{T-1} - \psi_{T-1}(W_{T-1})) + \beta V_T(\psi_{T-1}(W_{T-1})) \\
F.O.C \Rightarrow &- u'(W_{T-1} - W_T)\frac{\partial \psi_{T-1}}{\partial W_{T-1}} + \beta u'(W_T)\frac{\partial \psi_{T-1}}{\partial W_{T-1}} = 0 \\
\Rightarrow & \beta u'(W_T) = u'(W_{T-1} - W_T) \\
\Rightarrow & \beta u'(\psi_{T-1}(W_{T-1})) = u'(W_{T-1} - \psi_{T-1}(W_{T-1}))
\end{aligned}
$$

If we let the W_T is the solution of the maximization problem, and W_T can be written as $\psi_{T-1}(W_{T-1})$. And we know that $V_T(W_T)$ can be written as $u(\psi_{T-1}(W_{T-1}))$ to get

$$
V_{T-1}(W_{T-1}) = u(W_{T-1} - \psi_{T-1}(W_{T-1})) + \beta u(\psi_{T-1}(W_{T-1}))
$$

5.5 Let $u(c) = ln(c)$. Show that $V_{T-1}(\bar{W})$ does not equal $V_T(\bar{W})$ and that $\psi_{T-1}(\bar{W})$ does not equal $\psi_T(\bar{W})$ for a cake size of $\bar{W}$ when $T < \infty$ represents the last period of an individual's life.

From the question, we know that the individual no longer live in period T+1 that $\psi_T(W_T) = 0$. However,

$$
\psi_{T-1}(\bar{W}) = W_T \neq 0 = W_{T+1} = \psi_T(\bar{W})
$$

The exact solution of $\psi_{T-1}(\bar{W})$ is such that solve the maximization problem, $\frac{\beta \bar{W}}{1+\beta}$

With the same logic we can show that in time T+1, the value function of $\bar{W}$ is

$$
\begin{aligned}
V_T(\bar{W}) &= u(\bar{W} - \psi_T(\bar{W})) = ln(\bar{W}) \quad \because \psi_T(\bar{W}) = 0 \\
V_{T-1}(\bar{W}) &= u(\bar{W} - \psi_{T-1}(\bar{W})) + \beta u(\psi_{T-1}(W_{T-1})) \\
&= ln(\frac{\bar{W}}{1+\beta}) + \beta ln(\frac{\beta \bar{W}}{1+\beta})
\end{aligned}
$$

Unless $\beta = 0$,we can conclude that $V_T(\bar{W}) \neq V_{T-1}(\bar{W})$.

5.6 Using $u(c) = ln(c)$,write the finite horizon Bellman equation for the value function at time T-2. Characterize the solution for the period T-2 policy function for how much cake to save for the next period $W_{T-1} = \psi_{T-2}(T-2)$ using the envelope theorem (the principle optimality) and write its analytical solution. Also, write the analytical solution for V_{T-2}.

In 5.3, we already suggested the analytical solution for $W_{T-1} = \psi_{T-2}(W_{T-2})$ as

$$\begin{aligned} u'(W_{T-2} - W_{T-1}) &= \beta u'(W_{T-1} - W_T) \\ \iff u'(W_{T-2} - W_{T-1}) &= \beta u'(W_{T-1} - \psi_{T-1}(W_{T-1})) \\ &\text{use} u(\cdot) = ln() \\ W_{T-2} &= \frac{W_{T-1} - \psi_{T-1}(W_{T-1})}{\beta} + W_{T-1} \end{aligned}$$

And we know from the previous question that $\psi_{T-1}(W_{T-1}) = \frac{\beta W_{T-1}}{1+\beta}$ to get

$$W_{T-1} = (1 - \frac{1}{1+\beta+\beta^2})W_{T-2},\, and\, W_T = \frac{\beta^2}{1+\beta+\beta^2}W_{T-2}$$

$$\begin{aligned} V_{T-2}(W_{T-2}) &= \max_{W_{T-1},W_T,W_{T+1}} \left[u(W_{T-2} - W_{T-1}) + \beta u(W_{T-1} - W_T) + \beta^2 u(W_T - W_{T+1})\right] \\ &= ln\left(\frac{W_{T-2}}{1+\beta+\beta^2}\right) + \beta ln\left(\frac{\beta W_{T-2}}{1+\beta+\beta^2}\right) + \beta^2 ln\left(\frac{\beta^2 W_{T-2}}{1+\beta+\beta^2}\right) \end{aligned}$$

5.7 Show that the as the horizon becomes further and further away, the value function and policy function becomes independent of time.

By attemtping the T-3 case and use induction to get the analytical solution:

$$\begin{aligned} \text{t:T-2,} W_{T-1} &= \psi_{T-2}(W_{T-2}) = (1 - \frac{1}{1+\beta+\beta^2})W_{T-2} \\ \text{t:T-3,} W_{T-2} &= \psi_{T-3}(W_{T-3}) = (1 - \frac{1}{1+\beta+\beta^2})W_{T-3} \\ &\vdots \\ \psi_{T-s}(W_{T-s}) &= \left(\frac{\sum_{i=1}^{s} \beta^i}{1+\sum_{i=1}^{s} \beta^i}\right) W_{T-s} \\ \lim_{s\to\infty} \psi_{T-s}(W_{T-s}) &= \beta W_{T-s} = \psi(W_{T-s}). \end{aligned}$$

For $V_{T-s}(W_{T-s})$, continuing on the previous exercise,

$$V_{T-2}(W_{T-2}) = ln\left(\frac{W_{T-2}}{1+\beta+\beta^2}\right) + \beta ln\left(\frac{\beta W_{T-2}}{1+\beta+\beta^2}\right) + \beta^2 ln\left(\frac{\beta^2 W_{T-2}}{1+\beta+\beta^2}\right)$$

$$V_{T-3}(W_{T-3}) = ln\left(\frac{W_{T-3}}{1+\beta+\beta^2+\beta^3}\right) + \beta ln\left(\frac{\beta W_{T-3}}{1+\beta+\beta^2+\beta^3}\right) + \beta^2 ln\left(\frac{\beta^2 W_{T-3}}{1+\beta+\beta^2+\beta^3}\right) + \beta^3 ln\left(\frac{\beta^3 W_{T-3}}{1+\beta+\beta^2+\right.$$

$$\vdots$$

$$V_{T-s}(W_{T-s}) = \sum_{i=0}^{s} \beta^i ln\left(\frac{\beta^i W_{T-s}}{\sum_{i=0}^{s}\beta^i}\right)$$

$$\lim_{s\to\infty} V_{T-s}(W_{T-s}) = \frac{ln(1-\beta)}{1-\beta} + \frac{ln(W_{T-s})}{1-\beta} + \frac{\beta}{(1-\beta)^2} ln(\beta) = V(W_{T-s})$$

5.8 Write the Bellman equation for the cake eating problem with a general utility function $u(c)$ when the horizon is infinite.

$$V(W) = \max_{W'\in[0,W]} u(W-W') + \beta V(W')$$