MACS_PS_2

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In [ ]:
```
from matplotlib import pyplot as plt
from sympy import *
import sympy as sy
import numpy as np
import math
```

In [504]:
```
########################
#       Problem 1      #
########################

x = sy.symbols('x')
y = (sin(x)+1)**(sin(cos(x)))
y
```

Out[504]:
```
(sin(x) + 1)**sin(cos(x))
```

In [520]:
```
def sympy_diff(x_vals):
    x = sy.symbols('x')
    f = (sy.sin(x) + 1) ** (sy.sin(sy.cos(x)))
    fp = f.diff(x)
    fp_np = sy.lambdify(x, fp, 'numpy')
    return fp_np(x_vals)
```

In [421]:
```
dy=sy.diff(y,x)
dy
```

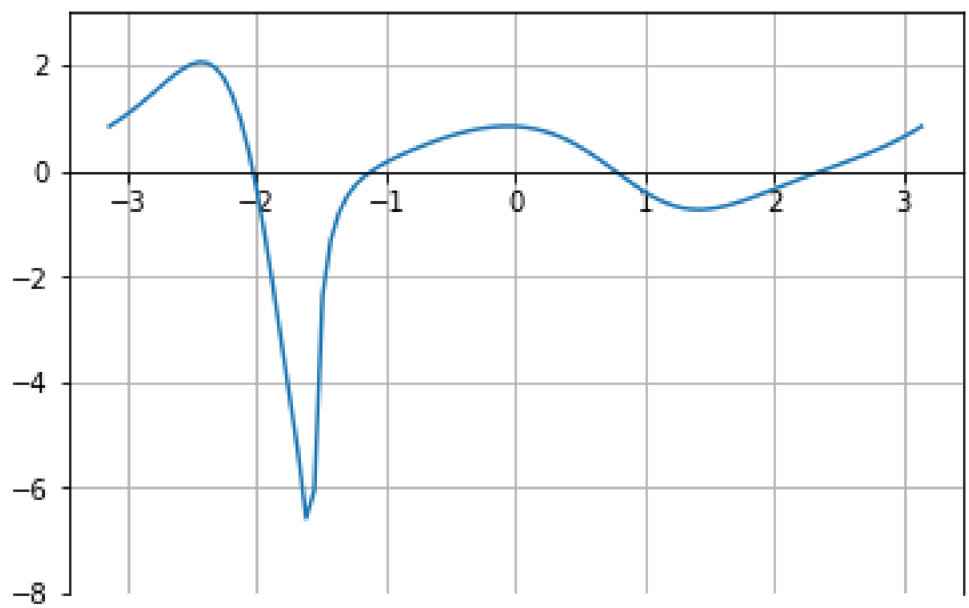
Out[421]:
```
(-log(sin(x) + 1)*sin(x)*cos(cos(x)) + sin(cos(x))*cos(x)/(sin(x) + 1))*(sin(x) + 1)*
```

In [422]:
```
dy_lamb=sy.lambdify(x,dy,'numpy')
y_lamb=sy.lambdify(x,y,'numpy')
xvals = np.linspace(-np.pi, np.pi, 100)
dy_lamb(5)
```

Out[422]:
```
-0.4117441338415354
```

In [423]:
```
ax = plt.gca()
ax.spines["bottom"].set_position("zero")
```

```
ax.plot(xvals,dy_lamb(xvals))
plt.ylim(-8,3)
plt.grid()
plt.show()
```

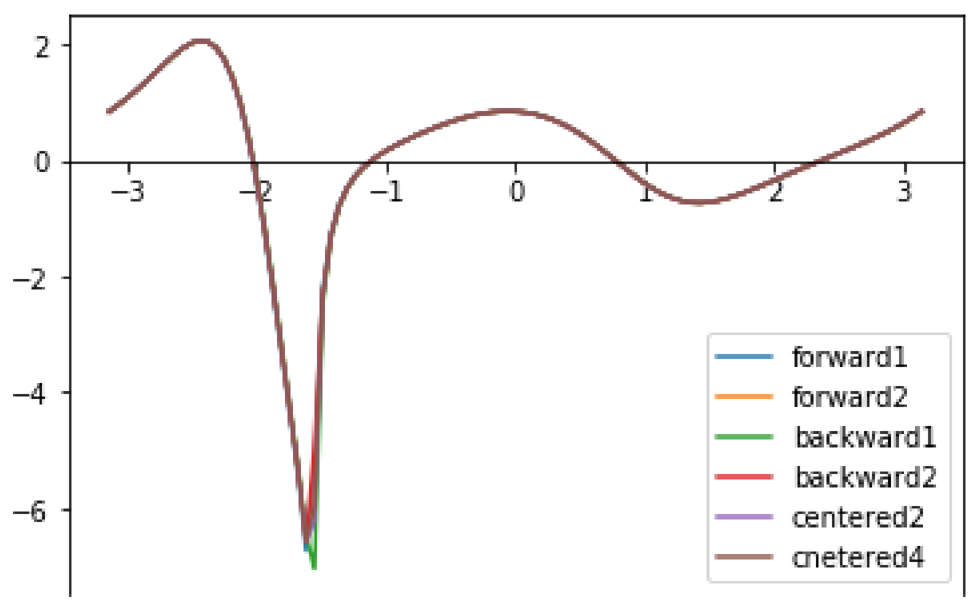



```
In [424]: ########################
          #       Problem 2      #
          ########################
          # Write a function for each of the finite difference quotients

          def x_vals(n):
              return np.linspace(-np.pi, np.pi, n)

          def fw1(f,x,h):
              return (f(x+h)-f(x))/h

          def fw2(f,x,h):
              return (-3.*f(x)+4*f(x+h)-f(x+2*h))/(2*h)

          def bw1(f,x,h):
              return (f(x)-f(x-h))/h

          def bw2(f, x,h):
              return (3.*f(x)-4.*f(x-h)+f(x-2*h))/(2*h)

          def ct2(f, x, h):
```

```python
        return (f(x+h)-f(x-h))/(2*h)

    def ct4(f, x, h):
        return (f(x-2*h)-8.*f(x-h)+8.*f(x+h)-f(x+2*h))/(12*h)

    def centered_o4(f, x, h):
        num = f(x - 2 * h) - 8 * f(x - h) + 8 * f(x + h) - f(x + 2 * h)
        return num / (12 * h)
```

In [425]:
```python
# Plot the results
xvals=x_vals(100)
ax = plt.gca()
ax.spines["bottom"].set_position("zero")
ax.plot(xvals, fw1(y_lamb,xvals,0.01),label='forward1')
ax.plot(xvals, fw2(y_lamb,xvals,0.01),label='forward2')
ax.plot(xvals, bw1(y_lamb,xvals,0.01),label='backward1')
ax.plot(xvals, bw2(y_lamb,xvals,0.01),label='backward2')
ax.plot(xvals, ct2(y_lamb,xvals,0.01),label='centered2')
ax.plot(xvals, ct4(y_lamb,xvals,0.01),label='cnetered4')
plt.legend(loc='lower right')
plt.show
```

Out[425]:
```
<function matplotlib.pyplot.show(*args, **kw)>
```



In [426]:
```python
# compare them to the results of Problem 1
# We can think of calculating difference from true value.
```

```
def test(f,n):
    #return sum(abs(dy_lamb(x_vals(n))-f))
    return np.sqrt(sum((dy_lamb(x_vals(n))-f)**2))
```

In [427]:
```
# calculate the error for each method when n=100
n=100
print ((test(fw1(y_lamb,x_vals(n),0.01),n),
test(fw2(y_lamb,x_vals(n),0.01),n),
test(bw1(y_lamb,x_vals(n),0.01),n),
test(bw2(y_lamb,x_vals(n),0.01),n),
test(ct2(y_lamb,x_vals(n),0.01),n),
test(ct4(y_lamb,x_vals(n),0.01),n)))
```

```
(0.6997656450349302, 0.16461387561452326, 1.0357569516211347, 1.0702085462088602, 0.17970589160
```

In [428]:
```
# calculate the error for each method when n=300
n=300
print (test(fw1(y_lamb,x_vals(n),0.01),n),
test(fw2(y_lamb,x_vals(n),0.01),n),
test(bw1(y_lamb,x_vals(n),0.01),n),
test(bw2(y_lamb,x_vals(n),0.01),n),
test(ct2(y_lamb,x_vals(n),0.01),n),
test(ct4(y_lamb,x_vals(n),0.01),n))
```

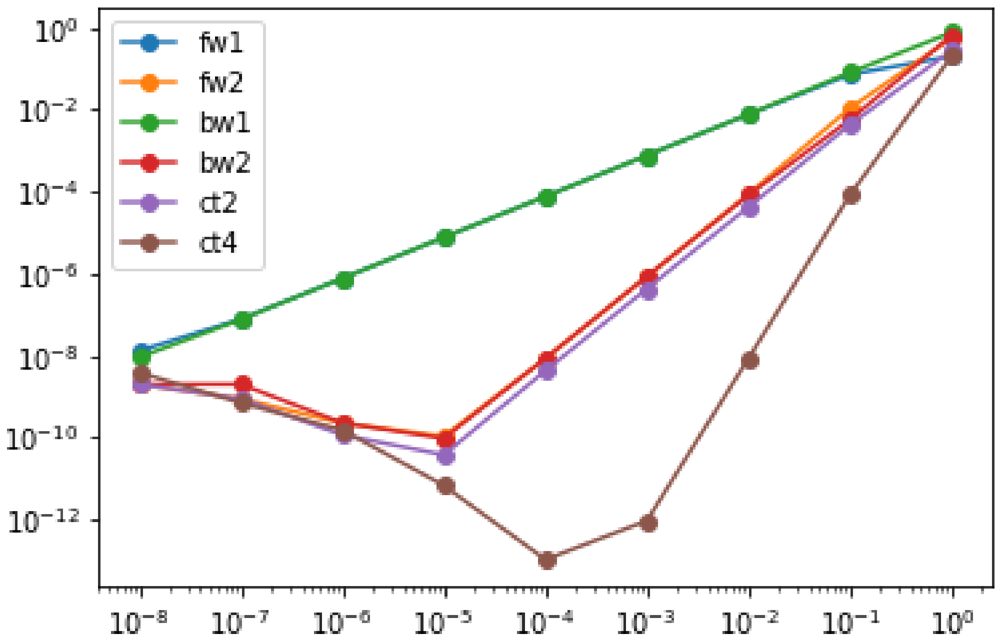
```
1.7343772059566547 0.9469287658772867 2.735736878923265 3.8208682613690983 0.5837375762903856 0
```

In [429]:
```
#########################
#       Problem 3       #
#########################

# In problem 1, we defined dy_lamb that compute the derivative.X_0 = 1
dy_lamb(1), y_lamb(1)
```

Out[429]:
```
(-0.3965403874194623, 1.3689877347067858)
```

In [430]:
```
test_func = [fw1,fw2,bw1,bw2,ct2,ct4]
def error_f(x):
    hvals=np.logspace(-8,0,9)
    abs_error = np.zeros((9,6))
    for i in range(9):
        for j in range(6):
            abs_error[i,j]=abs(test_func[j](y_lamb,x,hvals[i])-dy_lamb(x))
    plt.figure()
    labels = ["fw1","fw2","bw1","bw2","ct2","ct4"]
    for j in range(6):
        plt.loglog(hvals,abs_error[:,j], label=labels[j], marker='o')
    plt.legend(loc='upper left')
    plt.show()
```

In [431]: error_f(1)




```
In [432]: ##########################
          #       Problem 4        #
          ##########################
          data = np.load(r"C:\Users\ericr\Desktop\persp-model-econ_W19\ProblemSets\PS2\plane.np
          t=data[:0]
          alpha=np.deg2rad(data[:,1])
          beta=np.deg2rad(data[:,2])

In [433]: x_pos = 500*np.tan(beta)/(np.tan(beta)-np.tan(alpha))
          y_pos = 500*np.tan(alpha)*np.tan(beta)/(np.tan(beta)-np.tan(alpha))

          x_veloc = np.zeros(len(x_pos))
          y_veloc = np.zeros(len(y_pos))

          t = 0

          while t <= 7:
              if t == 0:
                  x_veloc[t] = (x_pos[t+1] - x_pos[t])
                  y_veloc[t] = (y_pos[t+1] - y_pos[t])
              elif t < 7:
                  x_veloc[t] = (x_pos[t+1] - x_pos[t-1])/2
                  y_veloc[t] = (y_pos[t+1] - y_pos[t-1])/2
```

```
        else:
            x_veloc[t] = (x_pos[t] - x_pos[t-1])
            y_veloc[t] = (y_pos[t] - y_pos[t-1])
        t += 1

    veloc = np.sqrt(x_veloc**2 + y_veloc**2)
    print(veloc)
```

```
[46.42420062 47.00103938 48.99880514 50.09944163 48.29035084 51.56455905
 53.92303355 51.51480057]
```

In [434]:
```
#########################
#       Problem 5       #
#########################

# Approximate the Jacobian matrix of f at x using the second order centered differen

def jacobian(f,x_0):
    h=0.0001
    f_len = len(f(x_0))
    Iden = np.identity(f_len)
    jacob = np.zeros((f_len,len(x_0)))
    for i in range(f_len):
        for j in range(len(x_0)):
            e = Iden[i,:]
            f_h_p = f(x_0+h*e)[j]
            f_h_n = f(x_0-h*e)[j]
            jacob[j,i]=((f_h_p-f_h_n)/(2*h))
    return jacob
# Test
def func(x):
    return np.array([x[0]**2, x[0]**3 - x[1]])
values = np.array([1.,1.])
jacobian(func,values)
```

Out[434]:
```
array([[ 2.        ,  0.        ],
       [ 3.00000001, -1.        ]])
```

In [435]:
```
x,y=sy.symbols('x,y', real=True)
J = Function('J')(x,y)
f1=x**2
f2=x**3-y
f1x=diff(f1,x)
f1y=diff(f1,y)
f2x=diff(f2,x)
f2y=diff(f2,y)
J = sy.Matrix([[f1x,f1y],[f2x,f2y]])
J.subs([(x,1), (y,1)])
```

```
# We can use jacobian function directly
F = sy.Matrix([f1,f2])
F.jacobian([x,y])
F.jacobian([x,y]).subs([(x,1), (y,1)])
```

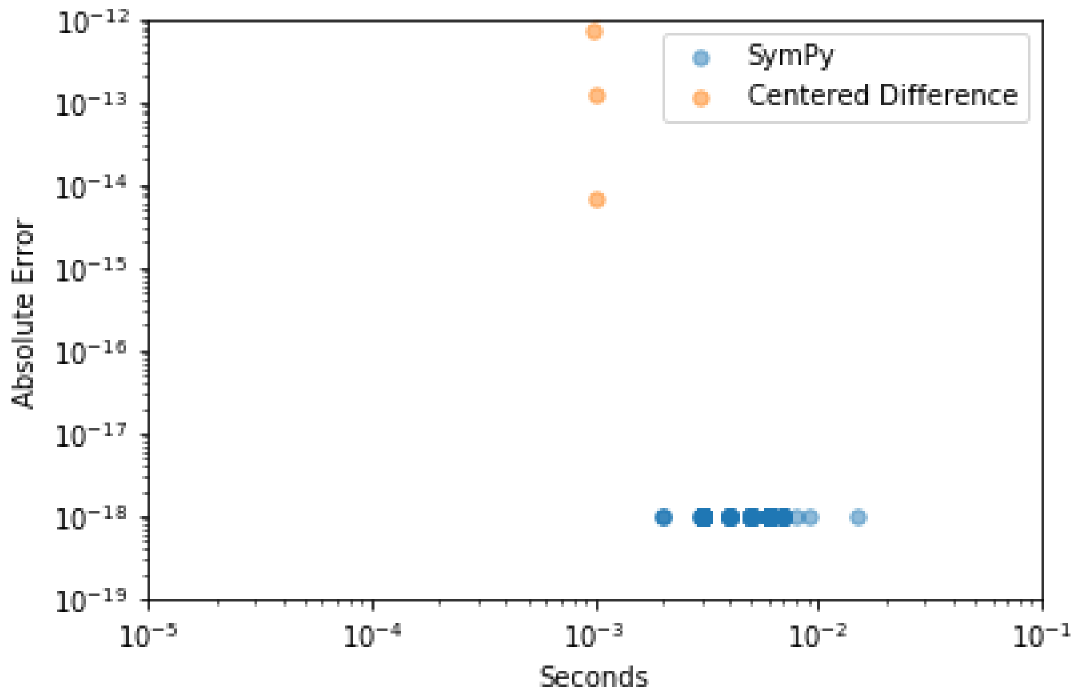
Out[435]:
```
Matrix([
[2,  0],
[3, -1]])
```

We can see that element(2,1) is not exactly same.

In [539]:
```
#########################
#       Problem 7       #
#########################

from autograd import grad
import time

def problem7(n):
    func = y_lamb
    time_sympy = np.zeros(n)
    time_autograd = np.zeros(n)
    time_centered = np.zeros(n)
    err_centered = np.zeros(n)
    err_autograd = np.zeros(n)

    for i in range(n):
        x_0=np.random.rand()

        start_time_sympy=time.time()
        exact = sympy_diff(x_0)
        time_sympy[i]=time.time()-start_time_sympy

        start_time_ct4=time.time()
        err_centered[i] = np.abs(ct4(func, x_0, 1e-4) - exact)
        time_centered[i] =  time.time() - start_time_ct4

        #start_time_auto=time.time()
        #dfgrad = grad(func)(x_0)
        #err_autograd[i]=np.abs(dfgrad)-dy(x_0))
        #time_autograd[i]=time.time()-start_time_auto

    plt.scatter(time_sympy, 1e-18 * np.ones(n), alpha=0.5, label='SymPy', s=30)
    plt.scatter(time_centered, err_centered, alpha=0.5, label='Centered Difference',
    #plt.scatter(time_autograd, err_autograd, alpha=0.5, label='Autograd', s=30)
    plt.loglog()
    plt.xlim(1e-5, 1e-1)
    plt.ylim(1e-19, 1e-12)
```

```
        plt.xlabel('Seconds')
        plt.ylabel('Absolute Error')
        plt.legend()
        plt.show()
```

```
In [540]: problem7(200)
```



0.2 Integration Part

Question 14.1

```
In [438]: g = lambda x: .1 * x ** 4 - 1.5 * x ** 3 + .53 * x ** 2 + 2 * x + 1
          a = -10
          b = 10
          N = 10000

          # Calculate the true known value of the integral
          import sympy as sy
          z = sy.symbols('z')
          sy.integrate(0.1*z**4 - 1.5*z**3 + 0.53*z**2 + 2*z + 1, (z, a, b))

Out[438]: 4373.33333333333
```

```
In [439]: import numpy as np
          from matplotlib import pyplot as plt
```

```
from scipy import integrate
from scipy import stats as sts
```

```
In [440]: def num_integral(func, a, b, N, method='midpoint'):

              bin_cuts = np.linspace(a, b, num=N)
              i = np.arange(len(bin_cuts))

              if method == 'midpoint':
                  xi_vals = a + (((2*i+1)*(b - a))/(2*N))
                  g_vals = func(xi_vals)
                  w = (b - a)/N
                  result_mid = w * np.sum(g_vals[:-1])
                  return result_mid

              elif method == 'trapezoid':
                  xi_vals = a + (i*(b - a))/N
                  g_vals = np.sum(func(xi_vals)) + func(xi_vals[-1])
                  w = (b - a)/(2*N)

                  result_trap = w * (func(xi_vals[0]) + 2 * g_vals)
                  return result_trap

              elif method == 'simpsons':
                      h = (b - a) / N
                      s = func(a) + func(b)
                      for i in range(1, N, 2):
                          s += 4 * func(a + i * h)
                      for i in range(2, N-1, 2):
                          s += 2 * func(a + i * h)
                      result_simp =s * h / 3
                      return result_simp
```

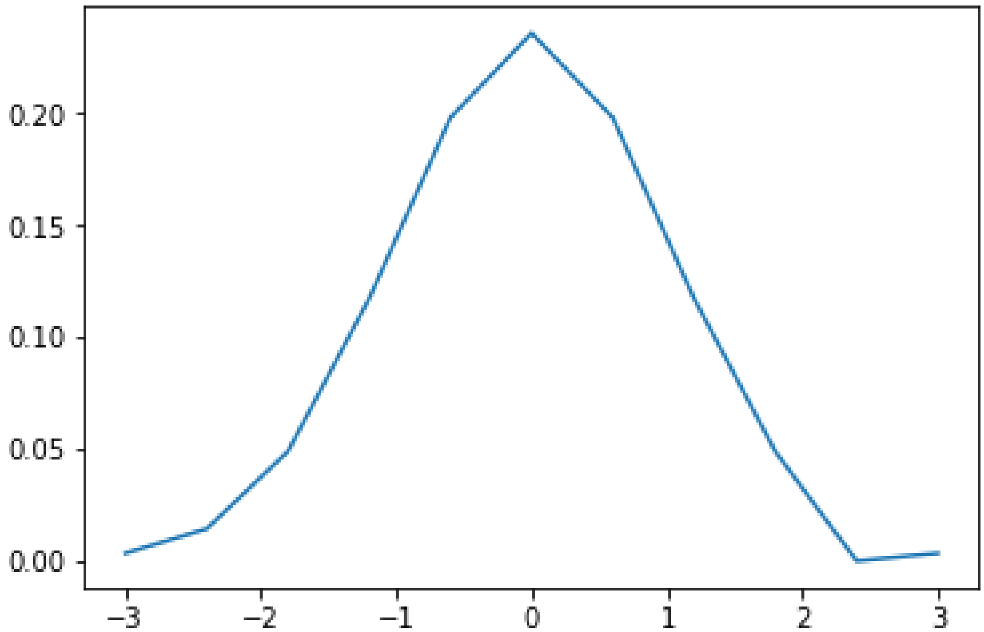
```
In [441]: midpoint = num_integral(g, -10, 10, 10000, 'midpoint')
          print('Integral computed via midpoint method:  ', midpoint)
          trap = num_integral(g, -10, 10, 10000, 'trapezoid')
          print('Integral computed via trapezoid method: ', trap)
          simp = num_integral(g, -10, 10, 10000, 'simpsons')
          print('Integral computed via Simpson method:   ', simp)
```

```
Integral computed via midpoint method:   4374.1851216356135
Integral computed via trapezoid method:  4377.9757567908655
Integral computed via Simpson method:    4373.333333333318
```

Question 14.2 Write a python function that makes a Newton-Cotes discrete approximation of the distribuiton of the normally distributed variable Z~N(mu, sigma)

```
In [442]: from scipy.stats import norm
          import matplotlib.pyplot as plt

In [443]: def approx_normal(mu, sigma, N, k):

              Z = np.linspace(mu - k*sigma, mu + k*sigma, N)
              w = np.zeros_like(Z)
              w[0] = norm.cdf((Z[0] + Z[1]) / 2, loc=mu, scale=sigma)
              w[N - 1] = 1 - norm.cdf((Z[N - 2] + Z[N - 1]) / 2, loc=mu, scale=sigma)
              for i in range(1, N - 2):
                  z_min = (Z[i - 1] + Z[i]) / 2
                  z_max = (Z[i] + Z[i + 1]) / 2
                  w[i] = norm.cdf(z_max, loc=mu, scale=sigma) - norm.cdf(z_min, loc=mu, scale=s
              return Z, w

          Z, w = approx_normal(0,1,11,3)
          print('Weights:')
          print(w)
          print('Nodes:')
          print(Z)


          plt.plot(bin_cuts, w)
          plt.show()
```

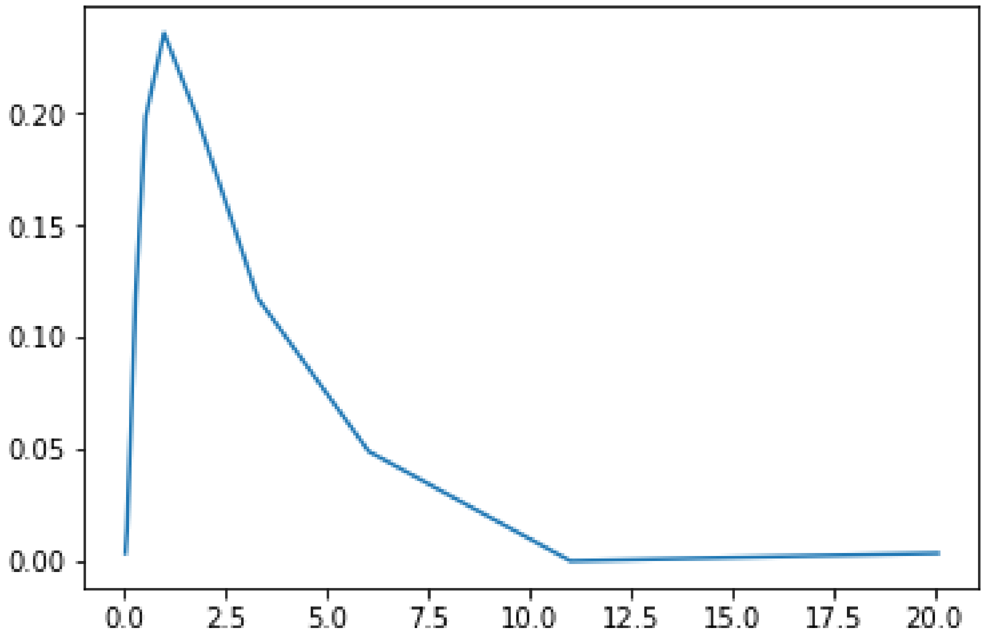
```
Weights:
[0.00346697 0.01439745 0.04894278 0.11725292 0.19802845 0.23582284
 0.19802845 0.11725292 0.04894278 0.         0.00346697]
Nodes:
[-3.  -2.4 -1.8 -1.2 -0.6  0.   0.6  1.2  1.8  2.4  3. ]
```



Question 14.3

```
In [444]: def approx_lognormal(mu, sigma, N, k):
              Z, w_log = approx_normal(mu, sigma, N, k)
              Z_log = np.exp(Z)
              return Z_log, w_log

          Z_log, w_log = approx_lognormal(0,1,11,3)
          print('Weights:')
          print(w_log)
          print('Nodes:')
          print(Z_log)

          plt.plot(Z_log,w_log)
          plt.show()
```

```
Weights:
[0.00346697 0.01439745 0.04894278 0.11725292 0.19802845 0.23582284
 0.19802845 0.11725292 0.04894278 0.         0.00346697]
Nodes:
[ 0.04978707  0.09071795  0.16529889  0.30119421  0.54881164  1.
  1.8221188   3.32011692  6.04964746 11.02317638 20.08553692]
```



Weights are same in 14.2 and 14.3

Question 14.4

```
In [445]: mu = 10.5
          sigma = 0.8
          N = 100
          k = 3
          Y, w = approx_lognormal(mu,sigma,N,k)
```

```
In [446]: exact = np.exp(mu+(sigma**2)/2)
          print ('Exact expected value :        ', exact)
          EY = Y@w
          print ('Approximated expected value :       ', EY)
```

```
Exact expected value :        50011.087008521754
Approximated expected value :       49739.23910845465
```

```
In [447]: EY = sum(w[i]*Y[i] for i in range(len(Y)))
          EY
```

```
Out[447]: 49739.23910845463
```

Question 14.5 Approximate the integral of the function in Exercise 2.1

```
In [448]: # write down all equations as in 14.11 as N=3

          import scipy as sc
          import scipy.optimize as opt

          g = lambda x: .1 * x ** 4 - 1.5 * x ** 3 + .53 * x ** 2 + 2 * x + 1

          def equations(parameters):
              w1, w2, w3, x1, x2, x3 = parameters

              eq0 = (b - a) - (w1 + w2 + w3)

              eq1 = ((1/2) * b ** 2 - (1/2) * a ** 2) - (w1 * x1 + w2 * x2 + w3 * x3)

              eq2 = ((1/3) * b ** 3 - (1/3) * a ** 3) - (w1 * x1**2 + w2 * x2**2 + w3 * x3**2)

              eq3 = ((1/4) * b ** 4 - (1/4) * a ** 4) - (w1 * x1**3 + w2 * x2**3 + w3 * x3**3)

              eq4 = ((1/5) * b ** 5 - (1/5) * a ** 5) - (w1 * x1**4 + w2 * x2**4 + w3 * x3**4)

              eq5 = ((1/6) * b ** 6 - (1/6) * a ** 6) - (w1 * x1**5 + w2 * x2**5 + w3 * x3**5)

              return (eq0, eq1, eq2, eq3, eq4, eq5)

          a = -10
          b = 10

          parameters = opt.root(equations,np.ones((6,1)), tol = 1e-8).x
          omega, Z = parameters[:3], parameters[-3:]

          integ_val = sum(omega[i]*g(Z[i]) for i in range(len(Z)))


          # Calculate the true known value of the integral
          import sympy as sy
          z = sy.symbols('z')
          exact=sy.integrate(0.1*z**4 - 1.5*z**3 + 0.53*z**2 + 2*z + 1, (z, a, b))


          print ('Gaussian Quadrature:       ', integ_val)
          print ('True known value:       ', exact)

Gaussian Quadrature:        4373.333333340382
True known value:       4373.33333333333
```

Question 14.6 Use the python Gaussian quadrature command to numerically approximate integral

```
In [449]: sc.integrate.quad(g, -10, 10)[0]
```

```
Out[449]: 4373.333333333334
```

Question 14.7 Use Monte Carlo integration to aaproximate π

```
In [582]: import math

          np.random.seed(seed=25)
          def monte(g, domain, N):
              draws = np.zeros((N,2))
              monte_sum = 0
              for i in range(N):
                  for j in range(2):
                      draws[i,j] = np.random.uniform(domain[j,0],domain[j,1])
                  monte_sum+=g(draws[i,:])
              return (4/N)*monte_sum

          g = lambda x: 1 if x[0]**2+x[1]**2 <= 1 else 0
          domain = np.array([[-1,1],[-1,1]])
          monte(g,domain,20000)
```

```
Out[582]: 3.1338000000000004
```

```
In [584]: np.random.seed(seed=25)
          toler = 1e-5
          diff = 1e3
          n = 10
          max_N = 100000
          while diff > toler and n < max_N:
              pi = monte(g, domain, n)
              diff = np.abs(pi - 3.1415)
              n += 1
          print("smallest number of random draws", n, pi)
```

```
smallest number of random draws 4432 3.1415030467163167
```

Question 14.8 Define a function in that returns the -th element of a -dimensional equidistributed sequence. It should have support for the four sequences in the Table in Section 4.2.

```
In [495]: def n_p(n):

              primes = [2]
              num = 3
              while len(primes) < n:
                  for p in primes:
                      if num % p == 0:
```

```
                break
        else:
            primes.append(num)
        num += 2
    return primes
```

```
In [497]: def equidistr(n, d, seq):

              if seq == 'weyl':
                  prime = n_p(d)
                  x = n * np.power(prime, .5)
                  xn = x - np.floor(x)

              elif seq == 'haber':
                  prime = n_p(d)
                  x = (n * (n + 1.0)) / 2 * np.power(prime, .5)
                  xn = x - np.floor(x)

              elif seq == 'niederreiter':
                  ar = np.arange(1, d + 1)
                  x = n * (2 ** (ar / (d + 1)))
                  x_floor = np.floor(x)
                  xn = x - x_floor

              elif seq == 'baker':
                  prime = n_p(d)
                  x = n * np.exp(prime)
                  xn = x - np.floor(x)
              else:
                  raise ValueError('Sequence does not exist.')

              return xn
```

```
In [498]: print("weyl:", equidistr(10, 2, 'weyl'))
          print("haber:", equidistr(10, 2, 'haber'))
          print("niederreiter", equidistr(10, 2, 'niederreiter'))
          print("baker", equidistr(5, 2, 'baker'))
```

```
weyl: [0.14213562 0.32050808]
haber: [0.78174593 0.26279442]
niederreiter [0.5992105  0.87401052]
baker [0.94528049 0.42768462]
```

Question 14.9 approximate the value of , this time using quasi-Monte Carlo integration.

```
In [553]: np.random.seed(seed=25)
          def quasi_monte(N, seq):
```

```
        pi = 0
        iterate = 0
        s = np.zeros((N, 2), dtype=np.float64)
        for i in range(1, N+1):
            s[i-1, :] = equidistr(i, 2, seq)

        for j in range(N):
            if s[j,0] ** 2 + s[j,1] ** 2 <= 1:
                iterate += 1
        pi = 4.0 * iterate / N

        return pi
```

```
In [ ]: np.random.seed(seed=25)
        toler = 1e-5
        diff = 1e3
        n = 10
        max_N = 100000
        while diff > toler and n < max_N:
            pi = quasi_monte(n,'weyl')
            diff = np.abs(pi - 3.1415)
            n += 1
        print("weyl:", quasi_monte(100000, 'weyl'), "convergence")
```

```
In [546]: print("weyl:", quasi_monte(100000, 'weyl'))
          print("haber:", quasi_monte(100000, 'haber'))
          print("niederreiter", quasi_monte(100000, 'niederreiter'))
          print("baker", quasi_monte(100000, 'baker'))
```

```
weyl: 3.14036
haber: 3.14396
niederreiter 3.1414
baker 3.1416
```

```
In [570]: np.random.seed(seed=25)
          toler = 1e-5
          diff = 1e3
          n = 10
          max_N = 100000
          while diff > toler and n < max_N:
              pi_wey = quasi_monte(n,'weyl')
              diff = np.abs(pi_wey - 3.1415)
              n += 1
          print("weyl:", quasi_monte(100000, 'weyl'), "smallest number of random draws", n)
```

```
weyl: 3.14036 smallest number of random draws 1683
```

In [571]:
```
np.random.seed(seed=25)
toler = 1e-5
diff = 1e3
n = 10
max_N = 100000
while diff > toler and n < max_N:
    pi_haber = quasi_monte(n,'haber')
    diff = np.abs(pi_haber - 3.1415)
    n += 1
print("haber:", quasi_monte(100000, 'haber'), "smallest number of random draws", n)
```

```
haber: 3.14396 smallest number of random draws 1054
```

In [573]:
```
np.random.seed(seed=25)
toler = 1e-5
diff = 1e3
n = 10
max_N = 100000
while diff > toler and n < max_N:
    pi_nieder = quasi_monte(n,'niederreiter')
    diff = np.abs(pi_nieder - 3.1415)
    n += 1
print("niederreiter:", quasi_monte(100000, 'niederreiter'), "smallest number of rando
```

```
niederreiter: 3.1414 smallest number of random draws 1888
```

In [586]:
```
np.random.seed(seed=25)
toler = 1e-5
diff = 1e3
n = 10
max_N = 100000
while diff > toler and n < max_N:
    pi_baker = quasi_monte(n,'baker')
    diff = np.abs(pi_baker - 3.1415)
    n += 1
print("baker:", quasi_monte(100000, 'baker'), "smallest number of random draws", n)
```

```
---------------------------------------------------------------------------

KeyboardInterrupt                         Traceback (most recent call last)

<ipython-input-586-28fe198addc1> in <module>
      5 max_N = 100000
      6 while diff > toler and n < max_N:
----> 7     pi_baker = quasi_monte(n,'baker')
      8     diff = np.abs(pi_baker - 3.141)
```

```
      9     n += 1


    <ipython-input-553-a8bed455b5bf> in quasi_monte(N, seq)
      6     s = np.zeros((N, 2), dtype=np.float64)
      7     for i in range(1, N+1):
----> 8         s[i-1, :] = equidistr(i, 2, seq)
      9
     10     for j in range(N):


    <ipython-input-497-9ef1e4db3589> in equidistr(n, d, seq)
     19     elif seq == 'baker':
     20         prime = n_p(d)
---> 21         x = n * np.exp(prime)
     22         xn = x - np.floor(x)
     23     else:


    KeyboardInterrupt:
```

In terms of number of draws, haber < weyl < niederreiter