Project 3: Models of Growth and Aggregation (Cellular Automata)

Eric Ross

***Abstract*—This paper details how simple clusters can be grown on a two-dimensional lattice and different statistics that can be calculated for these clusters. First the idea of p_∞ is examined with different sized clusters. Next the fractal dimension and constant of anomolous diffusion are determined for the same clusters. Finally three different cluster growth methods are detailed and visualized.**

I. Introduction

A. Spanning Clusters

Percolation is one of the simplest ways to describe phase changes in condensed matter physics and has many applications in materials science. For this paper, we consider a square lattice in two dimensions, an example of which is shown in Figure 1.

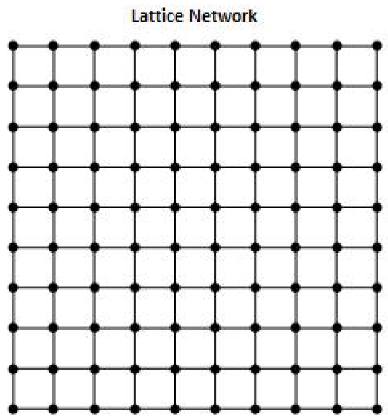

Fig. 1. Representation of a square two-dimensional lattice.

Each dot represents a "site" and each line represents a "bond". In one mathematical of percolation called site percolation, a site is occupied with probability p or empty with probability $q = 1-p$. One important property of a percolation cluster is determining whether the cluster is "spanning" which means that at least one large cluster exists that connects each side of the lattice with the opposite side. It can be shown that there is a critical probability p_c below which the probability that a spanning cluster exists is zero and above which the probability that a spanning cluster exists is one. Note that this only applies to infinite lattices. In the case of the finite lattice there is some probability of having a spanning cluster even below the critical probability.

B. Fractal Dimension and the Box Counting Method

The dimension of a fractal is an important statistical index of its complexity. One important method for calculating the dimension of a fractal is known as the box counting method. This method involves breaking up the lattice into boxes of equal size and counting the number of boxes that contain an occupied lattice site. This process is carried out for many different sizes of box. The fractals dimension is determined by

$$N \propto \epsilon^{-D} \tag{1}$$

where N is the number of boxes that contain at least one occupied lattice position and D is the dimension of the fractal. If we create a log-log plot of the different box sizes and the resulting N it should produce a straight line of which the slope is the fractal's dimension.

C. Random Walks and the Exponent of Anamolous Diffusion

Next we consider a percolation cluster that has already been generated. We place an ant at the center of the lattice and set it off on a random walk. How does the root mean squared displacement differ from a typical random walk where every lattice site can be visited? This question will be considered in the results.

D. Cluster Growth: the Eden model, Epidemic model, and DLA model

Instead of just looping through each site on the lattice and occupying it with some predetermined probability, there are other algorithms for generating clusters. These methods are the Eden model, the epidemic model, and the diffusion limited aggregation (DLA) model. These models all start with a seed at some point on the lattice and grow from that seed.

The Eden model was proposed in 1961 to describe tumor growth. It starts with the seed in the center of the lattice and occupying sites around the seed with probability of 1. Once a random nearest neighbor is added to the cluster, the list of nearest neighbors is increased. This process is repeated several thousand times until the desired cluster size is reached.

The epidemic model is a variation of the Eden model where instead of occupying a nearest neighbor with a probability of 1, it is occupied with a probability between 0 and 1. There is also some probability 1-p that the surface site is 'killed' instead of occupying. If the site is killed the site will be never be occupied. This results in some interesting structures depending on the probability chosen.

Diffusion limited aggregation also starts with an occupied site at the center of the lattice. A large circle is established around the seed where a spot is chosen at random. A random walk is carried out from that spot until a nearest neighbor is reached. If the walk goes out of the circle the particle is discarded and the process is started over. This is repeated thousands of times until the desired cluster size is reached.

II. Results and Discussion

A. P_{inf}

First the value of P_∞ was determined as a function of p for three different cluster sizes, $N = 100$,300, and 600. P_∞ is given by

$$P_\infty = \frac{\text{Number of occupied sites in the spanning cluster}}{\text{Total number of occupied sites in the lattice}} \tag{2}$$

This value was averaged over 100 trials. The result of each cluster size is shown below.

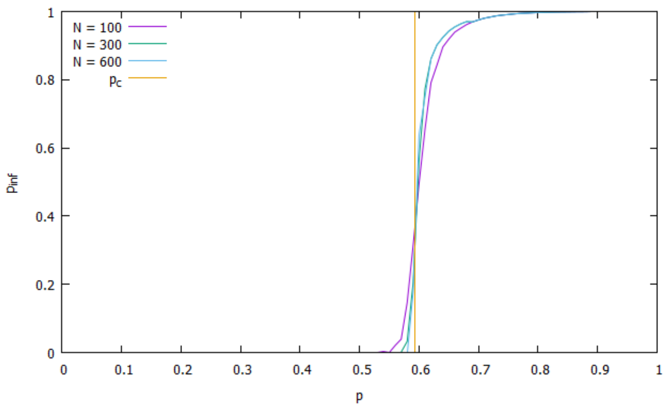

Fig. 2. P_∞ as a function of P for three different cluster sizes.

The critical probability p_C is also graphed to show the ideal response of P_∞. For all three cluster sizes, if $p < p_C$ there are essentially no spanning clusters. If $p > p_c$ the most of the clusters are spanning. The interesting portion is where we are close to p_c. As we increase the size of the cluster, the slope of the response becomes much steeper. If we were to keep increasing the size of the lattice to infinity, the function should look like a step function where there are no spanning clusters below the critical probability and all spanning clusters above it.

B. Fractal Dimension of Spanning Clusters

Next the dimension of the fractal on the lattice of size $N = 600$ was determined for $p = p_c$. The box sizes were chosen to be the factors of 600 so that all the boxes could be laid on top of the lattice with no overlap. The log-log plot is shown in Figure 3.

The x-axis is the inverse of the box size and the y-axis is the number of boxes that contain at least one occupied site in the cluster. Linear regression was then used to determine the slope of the line which should be the fractal dimension. This slope was determined to be 1.82609 which is very close to the theoretical dimension of 1.9. If we were to keep increasing the size of the lattice to infinity the dimension would move towards the theoretical value.

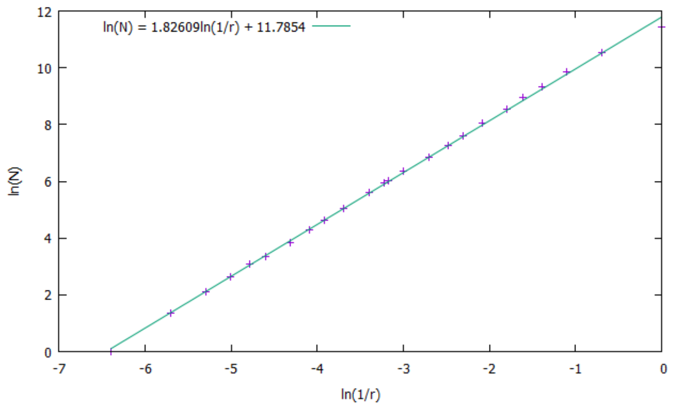

Fig. 3. Log-log plot to determine the fractal dimension .

C. Anomolous Rate of Diffusion

Using the same lattice as previous sections, we place an ant on a lattice point that is occupied and let it take a random walk. Instead of walking anywhere on the lattice, however, we only let it walk where there are occupied lattice sites. The root mean squared displacement is kept track of over time and is shown in Figure 4. The RMS of a true random walk is also shown for comparison.

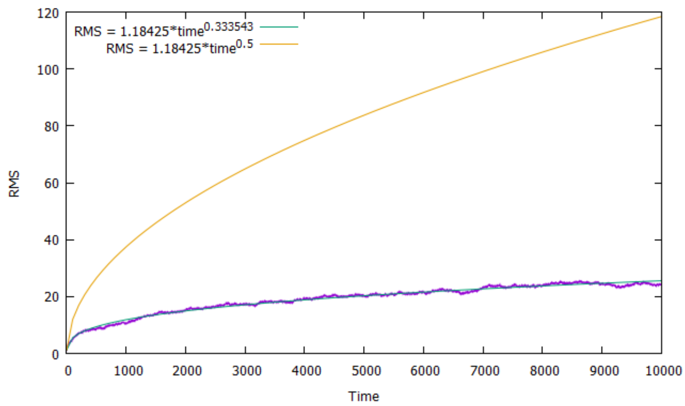

Fig. 4. Root mean squared displacement as a function of time.

The inverse of the exponent is the anomolous rate of diffusion. The inverse of 0.333543 is 2.998 which is very close to the theoretical value of 3.

D. Eden and Epidemic Growth Models

Both the Eden and epidemic growth models were implemented to create clusters of size $N = 10000$. The figures below show the evolution of an Eden cluster at various points in time.

The cluster grows very uniformly and keeps a circular shape. It looks like a tumor growing larger and larger.

Next the rate of growth of the perimeter was calculated. This was done by considering the cluster to be a circle where 100 random points were selected on the perimeter and averaged to determine the radius from which the circumference was calculated. Circumference as a function of the size of the growing cluster is shown in Figure 8.

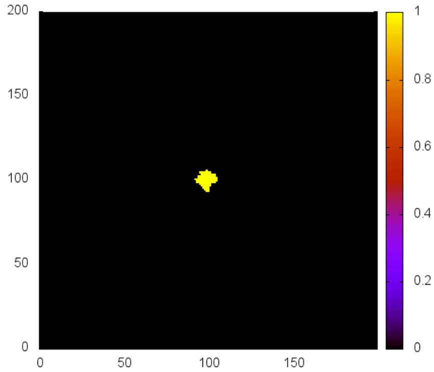

Fig. 5. Eden cluster of size 100.

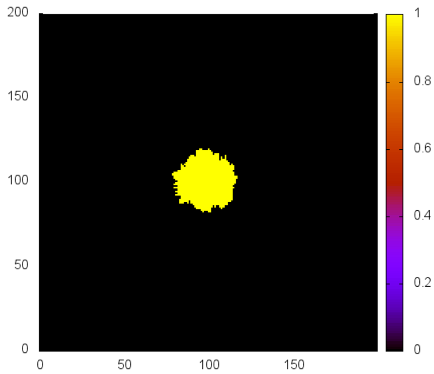

Fig. 6. Eden cluster of size 1000.

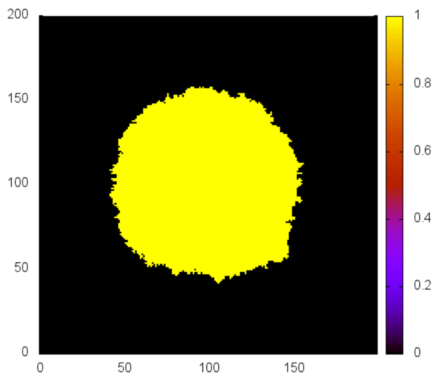

Fig. 7. Eden cluster of size 10000.

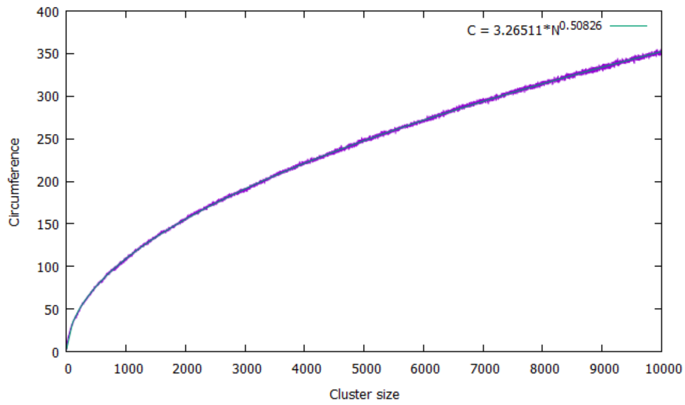

Fig. 8. Circumference of the Eden cluster as the cluster grows.

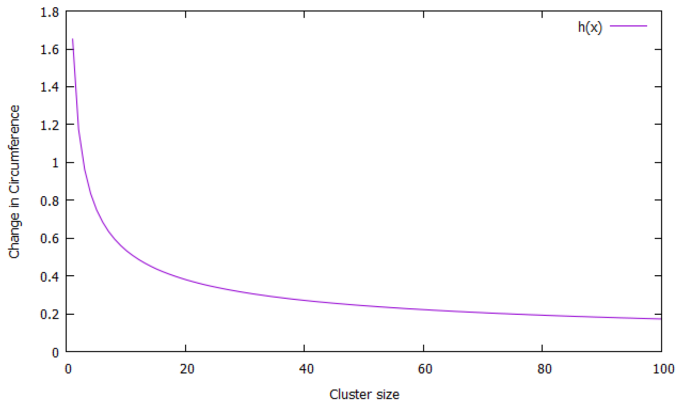

Fig. 9. Derivative of the circumference between size 0 and 100.

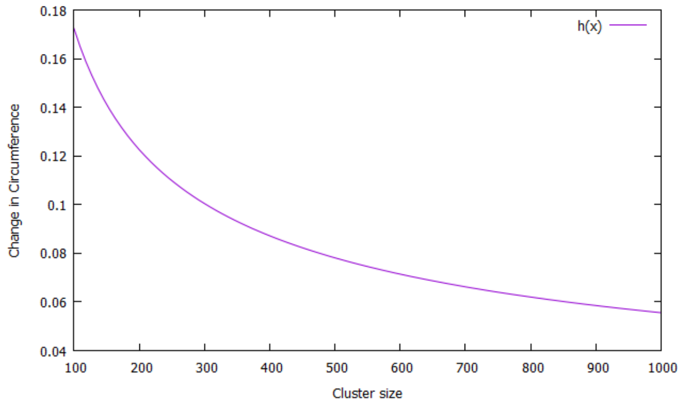

Fig. 10. Derivative of the circumference between size 100 and 1000.

The circumference seems to follow a power law so the data was fit to determine an analytical function. The derivative can then be found directly. The derivative is shown in Figures 9, 10, and 11.

Paying attention to the scale on the y-axis, we can see that the circumference initially grows very rapidly but quickly decays. Once the cluster is very large, adding another nearest neighbor does very little to change the entire perimeter of the cluster which makes sense. As the cluster size is increased to infinity the derivative goes towards zero.

Now that we have a good grasp of the Eden method for cluster growth we alter it to the epidemic method. Different probabilities were tested around the critical probability to determine the response. First we consider the case where the probability of occupying a nearest neighbor is greater than the critical probability. An image of this cluster, again with size $N = 10000$ is shown below. The probability here is $p = 0.7$

The red spots are occupied sites and the yellow spots are sites that were killed (as always unoccupied sites are black). The cluster has some similarities to the Eden clusters but we can see that there are lots of small holes in the cluster and a few larger holes. As we decrease the probability we would expect these holes to grow but what happens when we decrease the

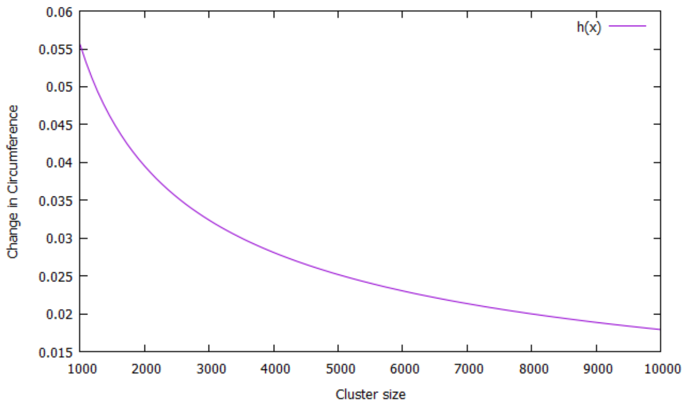

Fig. 11. Derivative of the circumference between size 1000 and 10000.

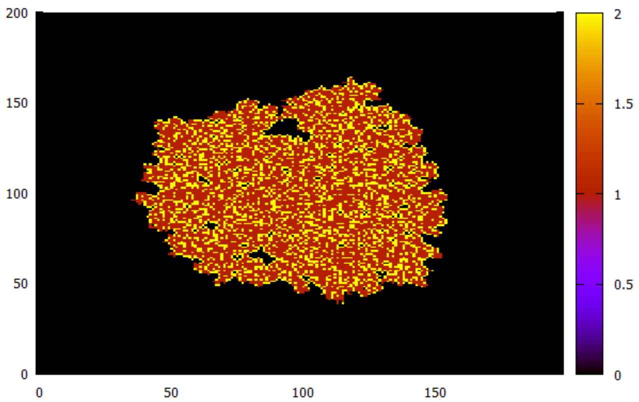

Fig. 12. Cluster of $N = 10000$ grown using the epidemic model.

probability to below the critical probability? This case where $p = 0.55$ is shown in figure 15.

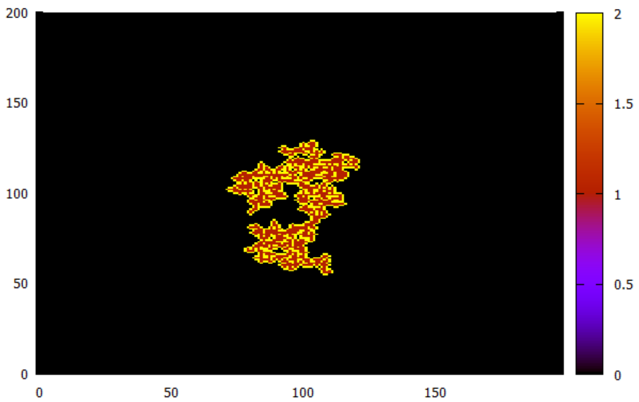

Fig. 13. Cluster of $N = 1688$ grown using the epidemic model with $p = 0.55$.

Multiple clusters were generated but none of them were able to grow to the full size of 10000. They were all terminated fairly quickly. This means that there are no longer any available sites to be occupied as all the nearest neighbors were killed off.

The contrasting between the two also shows up in the circumference of the cluster.

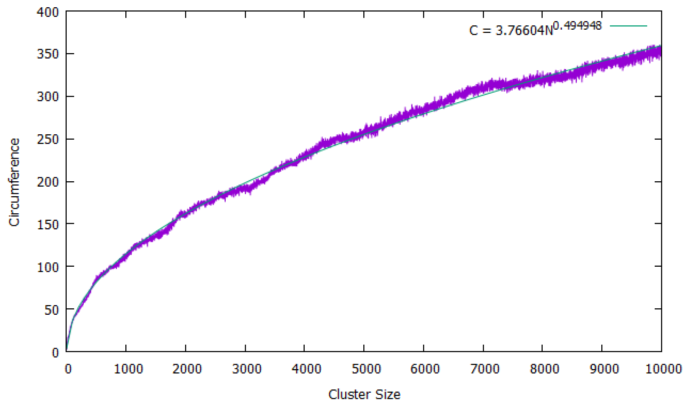

Fig. 14. Circumference as a function of time for epidemic cluster grown with $p = 0.70$

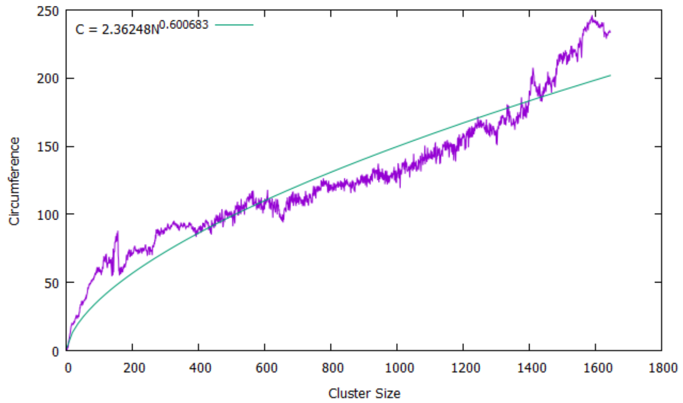

Fig. 15. Circumference as a function of time for epidemic cluster grown with $p = 0.55$

The circumference of the epidemic cluster with $p - 0.70$ is essentially the same as the Eden cluster. It increases quickly at the beginning of the growth and slows down as the cluster gets larger. We are able to determine the analytical function once again which abides to a power law. When we decrease the probability to below the critical probability there is no more order in how the circumference grows. A power function was fit to the data but it is obviously not a good fit.

E. Diffusion Limited Aggregation

Finally the DLA algorithm was used to grow clusters. Clusters of size 10000 were generated as was previously done for the Eden and epidemic clusters. A sampling of the cluster as it grows in time is shown for $N = 100$, $N = 1000$, and $N = 10000$ in Figures 16, 17, and 18.

The clusters look like snowflakes are an electrical current traveling through a medium. It is also possible to keep track of the radius of our cluster. This is shown in Figure 19.

As with previous clusters, the radius increases quickly when the cluster is small and begins to level off as the cluster reaches larger sizes.

It is also possible to image different ways to generate clusters with DLA. For example, instead of a random walker reaching a nearest neighbor and occupying it with probability

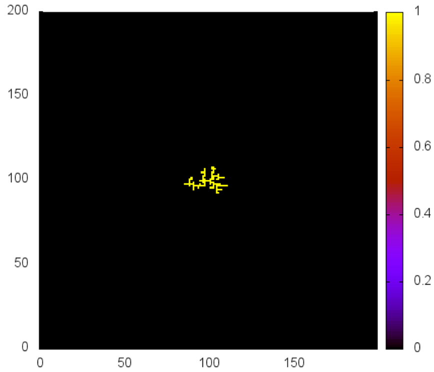

Fig. 16. DLA cluster of size 100.

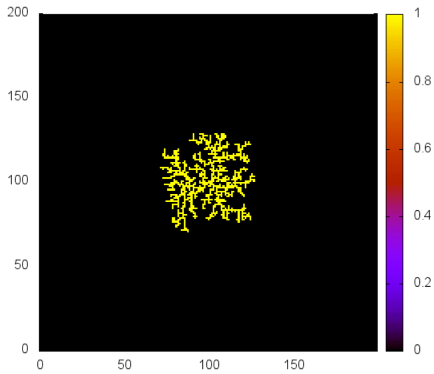

Fig. 17. DLA cluster of size 1000.

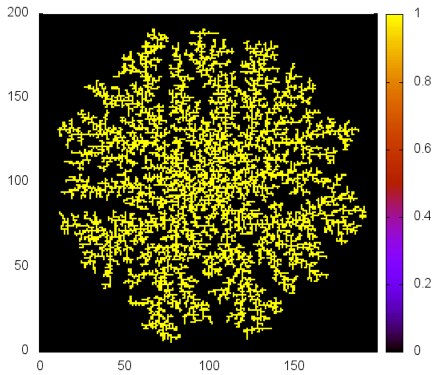

Fig. 18. DLA cluster of size 10000.

1, we can make the probability less than 1. This will result in denser clusters. A cluster with a 'sticking' probability of 0.5 is shown in Figure 20

If we decrease the sticking probability all the way down to 0.1 this becomes very evident.

Another common variation of the DLA is changing where the random walker originates from. For example, we can start the walk from a line instead of a circle. Imagine a bolt of lightening hitting the ground and spreading out from that line. One other idea is not getting rid of the random walker if it leaves the allowed region and instead just moving it back into the region. All of these changes would result in different cluster shapes.

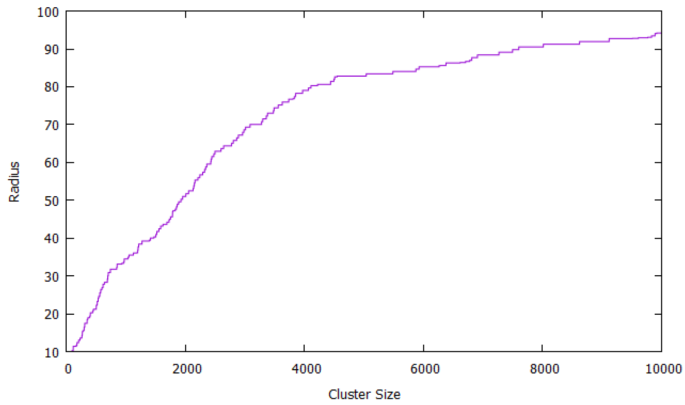

Fig. 19. Radius as a function of time for a DLA cluster of size $N = 10000$.

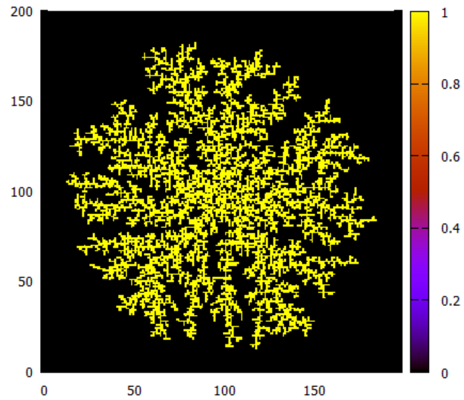

Fig. 20. DLA cluster of size 10000 with a sticking probability of 0.5.

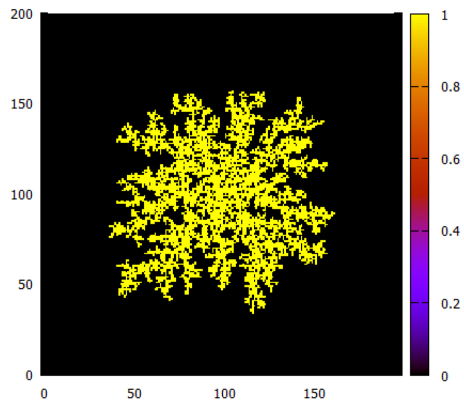

Fig. 21. DLA cluster of size 10000 with a sticking probability of 0.1.

III. Conclusions

Percolation and cluster growth is a very rich field with many different techniques and algorithms. Much of it is centered around the critical probability p_c which is dependent on what type of lattice the cluster is placed on. Even though the techniques that were detailed in this paper are very simple they are powerful methods for learning about percolation and clusters.

REFERENCES

[1] Giordano, Nicholas J. Computational Physics. Upper Saddle River, NJ: Prentice Hall, 1997.