Project 4: Perturbed Harmonic Oscillators in Two Dimensions

Eric Ross

***Abstract*—This project examines a simple Hamiltonian that describes a perturbed harmonic oscillator with a coupling constant and solves Hamilton's equations of motion for the oscillator. A phase portrait, Poincare section, and power spectrum are generated for various coupling constants. These plots are then used to determine if the behavior of the oscillator is chaotic or not.**

I. Hamiltonian

Chaotic motion is an interesting phenomenon that is highly dependent on the Hamiltonian being studied and the initial conditions given to the oscillator. The Hamiltonian being studied for this project is given by

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + x_1^2 + x_2^2\right) + \left(x_1^4 + 2Cx_1^2x_2^2 + x_2^4\right) \quad (1)$$

where p_i is the momentum, x_i is the position, and C is the coupling constant. This project examines different values of C and looks at the phase space, Poincare sections, and power spectra to determine what coupling constants yield chaotic motion and what constants yield ordered motion.

Notice that the Hamiltonian being used is nothing more than a simple harmonic oscillator with a perturbation that is quartic in nature. This may seem to be a simple Hamiltonian but it is actually a good case to study.

With this Hamiltonian we simultaneously solve Hamilton's equations of motion

$$\dot{p}_i = \frac{\partial H}{\partial x_i} \quad (2)$$

$$\dot{x}_i = -\frac{\partial H}{\partial p_i} \quad (3)$$

We can determine Hamilton's equations of motion by plugging in our Hamiltonian into equations (2) and (3). To solve the equations of motion, a Runge-Kutta method was used. Initially, the oscillator started at $x_1 = 0$ and $x_2 = 0$ with initial momentum $p_1 = 8$ and $p_2 = 4$. These values ensure that there is enough energy in the system to produce good results. If the initial energy of the system is not high enough, we won't be able to see any chaos where we should be able to see it.

It is also easy to see that some values of C should uncouple the system and produce a classically integrable system. For example, if $C = 0$ Hamilton's equations of motion become

$$\dot{p}_i = \frac{\partial H}{\partial x_i} = -x_i - 4x_i^3 \quad (4)$$

$$\dot{x}_i = -\frac{\partial H}{\partial p_i} = p_i \quad (5)$$

Each of these equations is only dependent on the single variable that is being integrated.

II. Results

A. $C = -0.5$

We start with a negative coupling constant of $C = -0.5$.

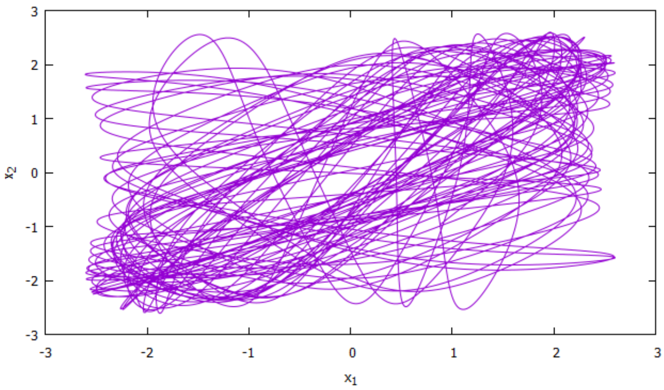

Fig. 1. Phase portrait for $C = -0.5$.

The position is very chaotic and does not seem to have any order. A look at the Poincare section and power spectrum confirms this.

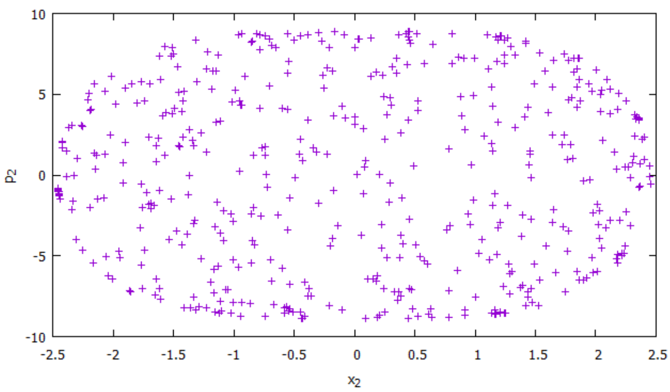

Fig. 2. Poincare section for $C = -0.5$.

The Poincare has lots of dots and most do not overlap.

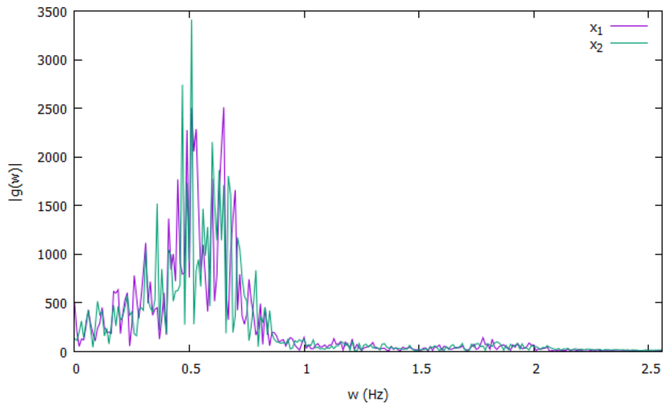

Fig. 3. Power spectrum for $C = -0.5$.

The power spectrum does not have any well defined peaks which means that there are no intrinsic frequencies of oscillation.

B. $C = -0.21$

Next we look at the coupling constant $C = -0.21$. Once again it looks like the oscillations are chaotic but there might be some semblance of order.

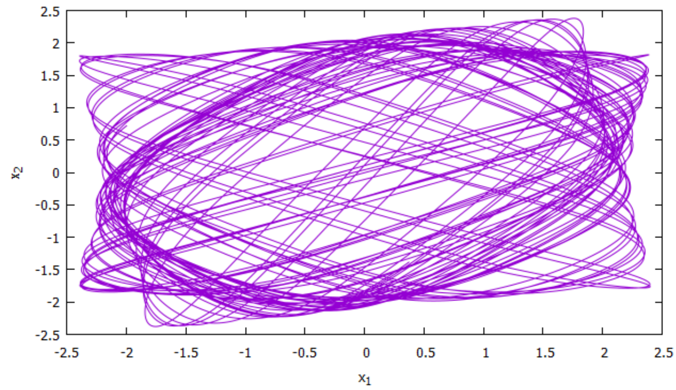

Fig. 4. Phase portrait for $C = -0.21$.

The Poincare section seems to indicate that this oscillator is more ordered than the one in the previous section.

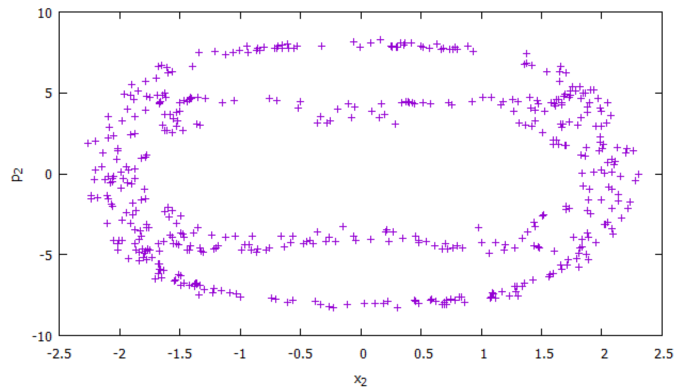

Fig. 5. Poincare section for $C = -0.21$.

There are still many points that overlap but there is at least some well-defined areas that can be seen.

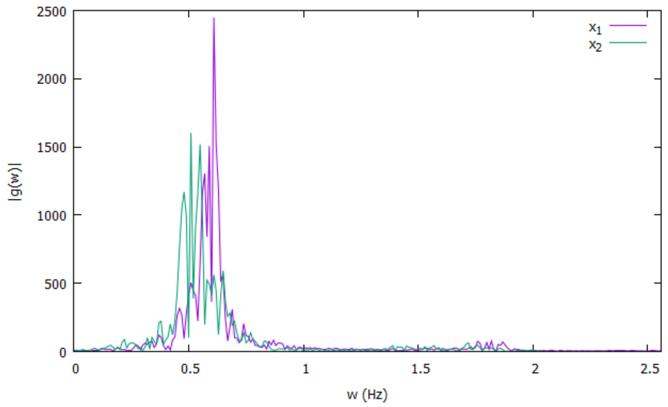

Fig. 6. Power spectrum for $C = -0.21$.

The power spectrum shows a lot of peaks but not as many as with the previous section.

C. $C = -0.1$

Now we move on to our first ordered systems with the coupling constant of $C = -0.1$. The phase portrait for this coupling constant looks very different than the previous phase portraits. There is symmetry and repetition that is observed.

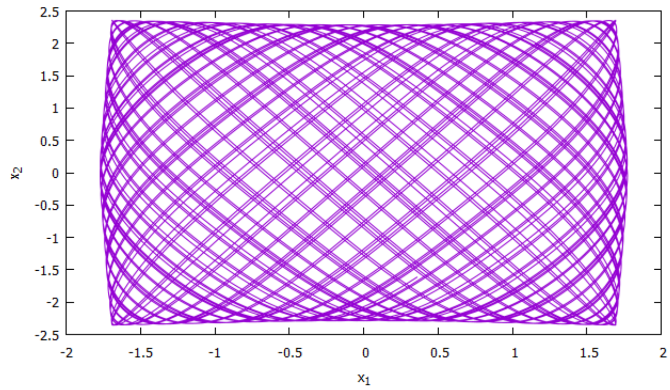

Fig. 7. Phase portrait for $C = -0.1$.

The Poincare section is very well-defined; it is a single circle instead of many different circles.

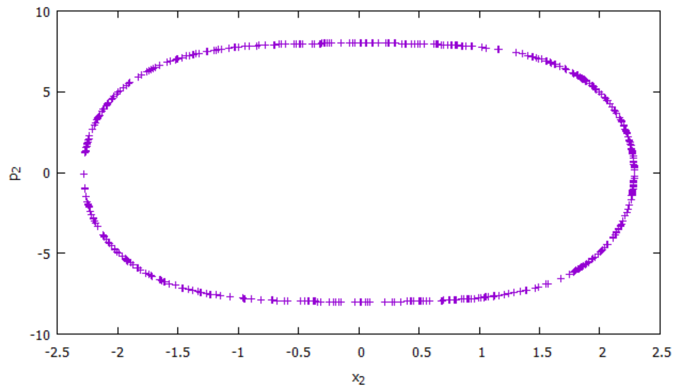

Fig. 8. Poincare section for $C = -0.1$.

The power spectrum is extremely clean and well-defined as opposed to the chaotic power spectrums. We see two large peaks and all the other peaks are small.

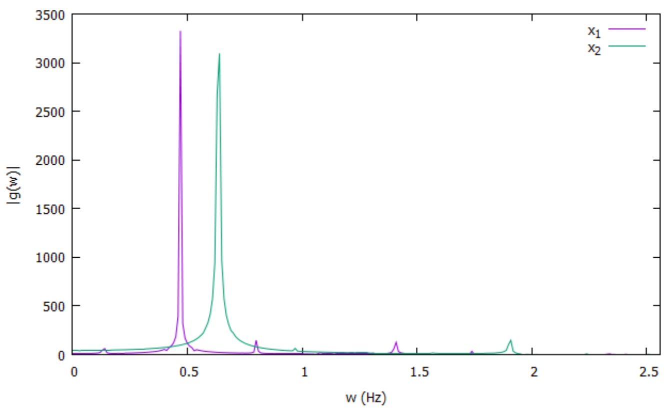

Fig. 9. Power spectrum for $C = -0.1$.

D. $C = 0$

The figures for the coupling constant of $C = 0.0$ show many of the same features as the figures in the previous section. The plots are very ordered and repetitive.

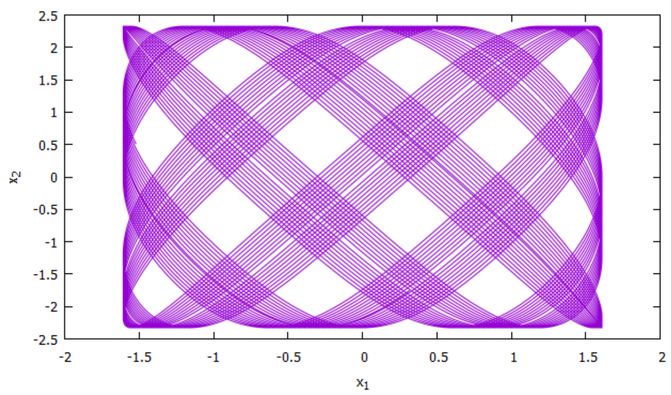

Fig. 10. Phase portrait for $C = 0.0$.

Once again the Poincare section shows a single circle.

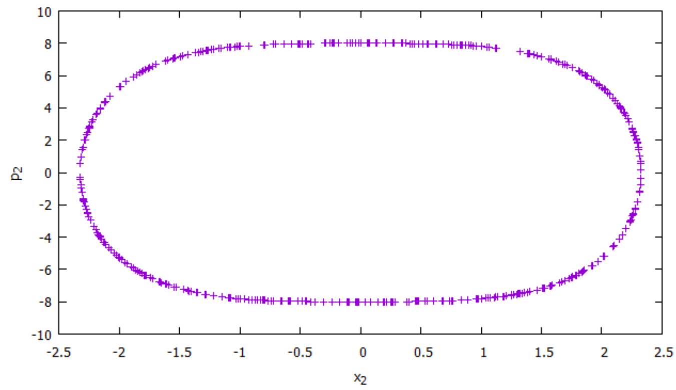

Fig. 11. Poincare section for $C = 0.0$.

The power spectrum also looks very similar to the one in the previous section.

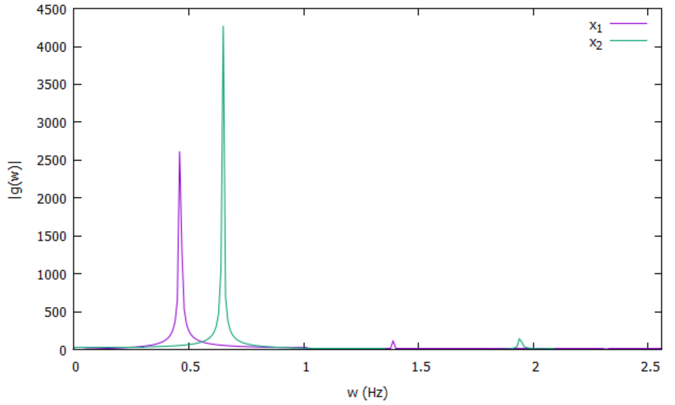

Fig. 12. Power spectrum for $C = 0.0$.

E. $C = 1.5$

The phase portrait for $C = 1.5$ looks a little different than the previous two coupling constants but it still produces a beautiful ordered system.

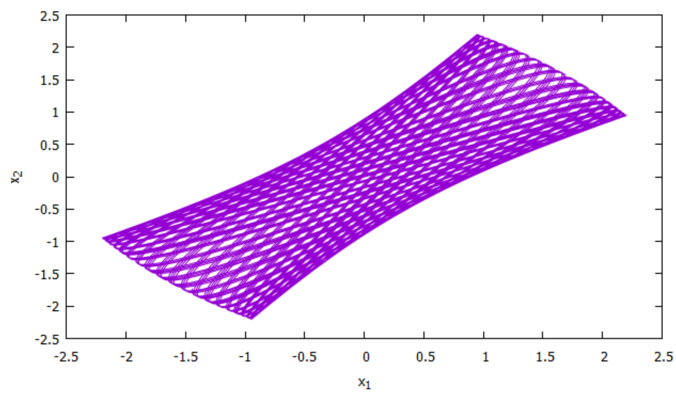

Fig. 13. Phase portrait for $C = 1.5$.

The Poincare section is a little different in that there are two distinct sections instead of one but the system is still very ordered.

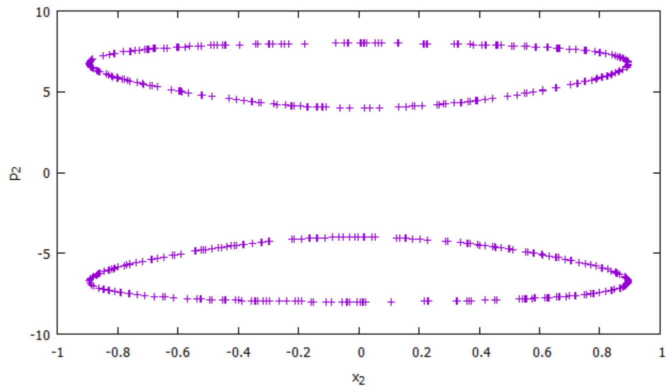

Fig. 14. Poincare section for $C = 1.5$.

Once again the power spectrum produced shows sharp peaks. Here the power spectrum for both variables seem to overlap each other very well.

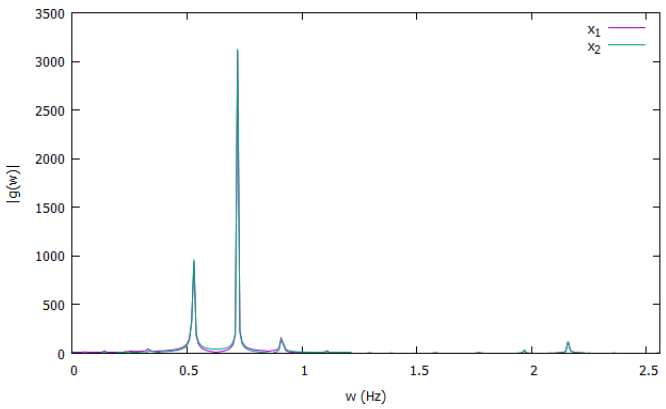

Fig. 15. Power spectrum for $C = 1.5$.

F. $C = 4.5$

As we continue to increase the value of the coupling constant we go back to chaotic behavior.

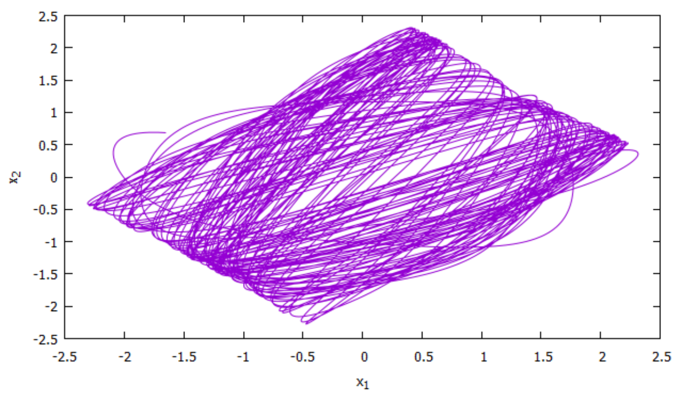

Fig. 16. Phase portrait for $C = 4.5$.

The Poincare section is very scattered and chaotic.

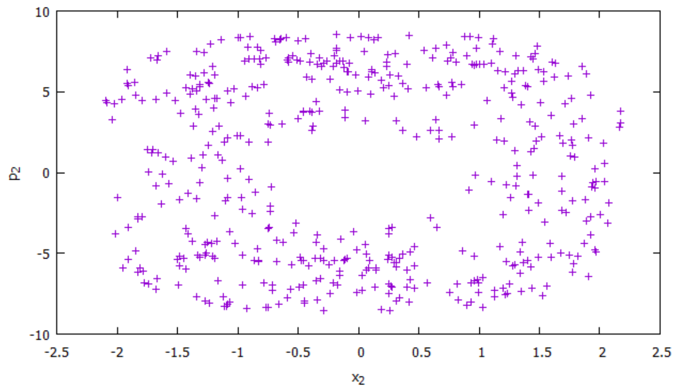

Fig. 17. Poincare section for $C = 4.5$.

The power spectrum also shows a lot of noise and very few well-defined peaks.

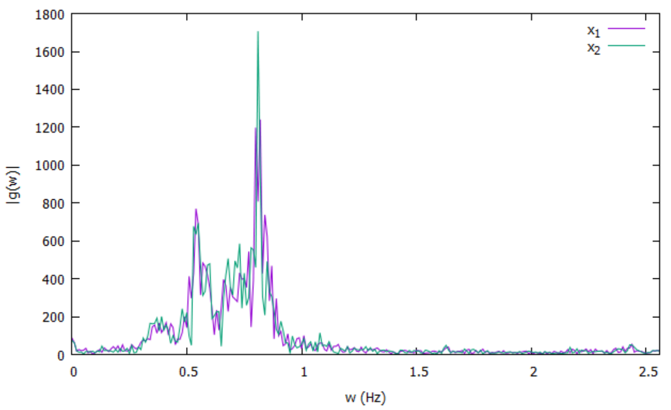

Fig. 18. Power spectrum for $C = 4.5$.

G. $C = 5.2$

Finally the highest coupling constant of $C = 5.2$ shows chaotic motion similar to the other coupling constants that had chaotic motion.

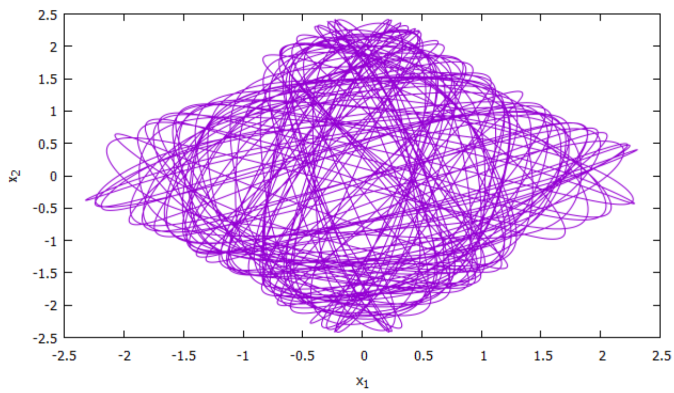

Fig. 19. Phase portrait for $C = 5.2$.

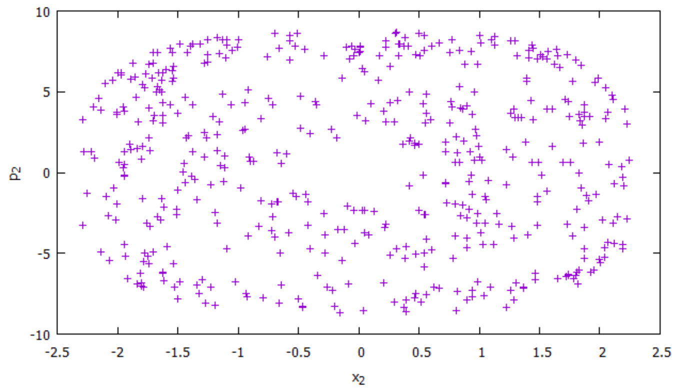

Fig. 20. Poincare section for $C = 5.2$.

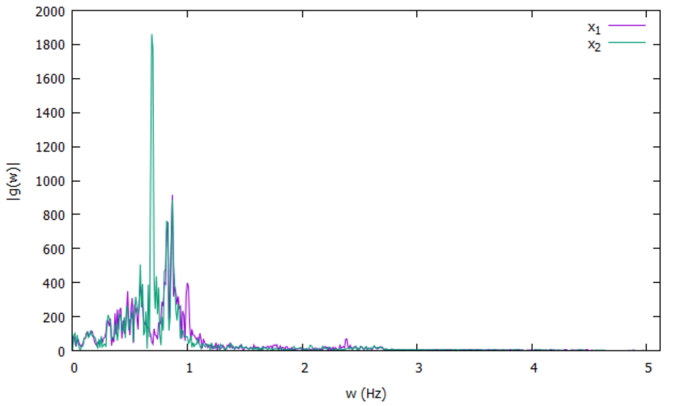

Fig. 21. Power spectrum for $C = 5.2$.

III. Conclusions

For the Hamiltonian that was studied for this project, there is a range of coupling constants that produce normal behavior and any coupling constants outside of this range produce an oscillator that is chaotic in nature. It is very simple to determine if the oscillator is chaotic or normal by examining the phase portrait, Poincare sections, and power spectrum for a given coupling constant.

References

[1] Giordano, Nicholas J. Computational Physics. Upper Saddle River, NJ: Prentice Hall, 1997.