FUNCTIONAL EQUATIONS FOR THE RIEMANN ZETA FUNCTION AND DIRICHLET L-FUNCTIONS

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Abstract. We derive the functional equations for the Riemann zeta function and the Dirichlet L-functions of characters modulo q and give some applications of these equations. We also show the correspondence between functional equations of general Dirichlet series and equations of the associated modular forms.

1. Preliminaries

We need several results regarding Fourier series and Gauss sums, as well as some basic knowledge of the Γ-function.

1.1. Fourier Series.

We assume the basic theory of Fourier analysis and prove several specific results. The Fourier transform of F is denoted by $\widehat{F}$.

Theorem 1.1 (Poisson Summation Formula)**.** *Let $F \in L^1(\mathbb{R})$ such that*

$$\sum_{n\in\mathbb{Z}} F(n+v)$$

converges absolutely and uniformly in $v \in \mathbb{R}$ and that

$$\sum_{m\in\mathbb{Z}} |\widehat{F}(m)| < \infty.$$

Then

$$\sum_{n\in\mathbb{Z}} F(n+v) = \sum_{n\in\mathbb{Z}} \widehat{F}(n) e^{2\pi inv}.$$

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Proof. Define $G(v) = \sum_{n\in\mathbb{Z}} F(n+v)$. $G(v)$ is a continuous funtion with period 1, so the Fourier coefficients of G are

$$\begin{aligned} a_m &= \int_0^1 G(v)e^{-2\pi imv}dv \\ &= \int_0^1 \left(\sum_{n\in\mathbb{Z}} F(n+v)\right) e^{-2\pi imv}dv \\ &= \sum_{n\in\mathbb{Z}} \int_0^1 F(n+v)e^{-2\pi imv}dv \\ &= \sum_{n\in\mathbb{Z}} \int_n^{n+1} F(x)e^{-2\pi imx}dx \\ &= \int_{-\infty}^{\infty} F(x)e^{-2\pi imx}dx = \widehat{F}(m), \end{aligned}$$

the inversion of the integration and summation justified by absolute convergence. Since $\sum_{m\in\mathbb{Z}} |\widehat{F}(m)|$ converges, $G(v)$ can be represented by its Fourier series

$$\sum_{n\in\mathbb{Z}} F(n+v) = \sum_{n\in\mathbb{Z}} \widehat{F}(n)e^{2\pi inv},$$

which is the statement of the theorem. □

The Poisson summation formula has the following corollary under the same assumptions.

Corollary 1.2. *Let F be as in Theorem 1.1. Then*

$$\sum_{n\in\mathbb{Z}} F(n) = \sum_{n\in\mathbb{Z}} \widehat{F}(n).$$

Proof. Let $v = 0$ in Theorem 1.1. □

Lemma 1.3. *Let $\alpha \in \mathbb{R}$ and $x > 0$. Then*

$$\sum_{n\in\mathbb{Z}} e^{-\pi(n+\alpha)^2/x} = x^{1/2} \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x + 2\pi in\alpha}.$$

Proof. Let $F(t) = e^{-\pi t^2}$. We compute the Fourier transform of F:

$$\begin{aligned} \widehat{F}(t) &= \int_{-\infty}^{\infty} F(y)e^{-2\pi iyt}dy \\ &= \int_{-\infty}^{\infty} e^{-\pi y^2 - 2\pi iyt}dy \\ &= e^{-\pi t^2} \int_{-\infty}^{\infty} e^{-\pi(y+it)^2} dy \\ &= e^{-\pi t^2} \end{aligned}$$

since $\int_{-\infty}^{\infty} e^{-\pi y^2} dy = 1$ and

$$\begin{aligned}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} e^{-\pi(y+it)^2} dy &= \int_{-\infty}^{\infty}\left(\frac{\partial}{\partial t} e^{-\pi(y+it)^2}\right) dy \\ &= 2\pi i\int_{-\infty}^{\infty}(y+it)e^{-\pi(y+it)^2} dy \\ &= i\int_{-\infty}^{\infty}\left(\frac{\partial}{\partial y} e^{-\pi(y+it)^2}\right) dy \\ &= ie^{-\pi(y+it)^2}\Big|_{-\infty}^{\infty} = 0,\end{aligned}$$

so the value of $\int_{-\infty}^{\infty} e^{-\pi(y+it)^2} dy$ is independent of t.

Let $F^{\lambda}(t) = F(\lambda t)$ and $F_a(t) = F(t+a)$. Then

$$\begin{aligned}\widehat{F^{\lambda}}(t) &= \int_{-\infty}^{\infty} F^{\lambda}(y)e^{-2\pi iyt} dy \\ &= \int_{-\infty}^{\infty} e^{-\pi\lambda^2 y^2 - 2\pi iyt} dy \\ &= \frac{1}{\lambda}\int_{-\infty}^{\infty} e^{-\pi u^2 - 2\pi iut/\lambda} du \\ &= \frac{1}{\lambda}\int_{-\infty}^{\infty} F(u)e^{-2\pi iut/\lambda} du \\ &= \frac{1}{\lambda}\widehat{F}\left(\frac{t}{\lambda}\right)\end{aligned}$$

with the change of variables $u = \lambda y$, and

$$\begin{aligned}\widehat{F_a}(t) &= \int_{-\infty}^{\infty} F_a(y)e^{-2\pi iyt} dy \\ &= \int_{-\infty}^{\infty} e^{-\pi(y+a)^2 - 2\pi iyt} dy \\ &= e^{-\pi a^2}\int_{-\infty}^{\infty} e^{-\pi y^2} e^{-2\pi ay - 2\pi iyt} dy \\ &= e^{-\pi a^2}\int_{-\infty}^{\infty} F(y)e^{-2\pi iy(t-ia)} dy \\ &= e^{-\pi a^2}\widehat{F}(t-ia) = e^{-\pi a^2} e^{-\pi(t-ia)^2} \\ &= e^{2\pi iat} e^{-\pi t^2}.\end{aligned}$$

These two identities allow us to find the Fourier transform of $(F_a)^{1/\sqrt{x}}(t) = e^{-\pi(a+t/\sqrt{x})^2}$:

$$\widehat{(F_a)^{\frac{1}{\sqrt{x}}}}(t) = x^{1/2}\widehat{F_a}(t\sqrt{x}) = x^{1/2} e^{2\pi iat\sqrt{x}} e^{-\pi t^2 x}.$$

Applying Corollary 1.2 to $(F_a)^{1/\sqrt{x}}(t)$ gives

$$\sum_{n\in\mathbb{Z}} e^{-\pi(a+n/\sqrt{x})^2} = x^{1/2}\sum_{n\in\mathbb{Z}} e^{-\pi n^2 x + 2\pi ian\sqrt{x}}.$$

Finally we obtain

$$\sum_{n\in\mathbb{Z}} e^{-\pi(n+\alpha)^2/x} = x^{1/2}\sum_{n\in\mathbb{Z}} e^{-\pi n^2 x + 2\pi in\alpha},$$

by writing $\alpha = a\sqrt{x}$. $\square$

Letting $\alpha = 0$ in the previous lemma gives the following theorem.

Theorem 1.4.
$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2/x} = x^{1/2} \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}.$$

1.2. **The Γ-function.** Let $s \in \mathbb{C}$ and write $s = \sigma + it$ for real σ, t. The Γ-function is defined as
$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$$
for $\sigma > 0$. Integration by parts gives the functional equation for Γ:
$$\begin{aligned}\Gamma(s+1) &= \int_0^\infty e^{-t} t^s dt \\ &= -t^s e^{-t}\Big|_0^\infty + s\int_0^\infty e^{-t} t^{s-1} dt \\ &= s\Gamma(s),\end{aligned}$$
again for $\sigma > 0$. We define $\Gamma(s)$ on $-1 < \sigma < 0$ by
$$\Gamma(s) = \frac{\Gamma(s+1)}{s},$$
so Γ has a simple pole at $s = 0$. Similarly, defining $\Gamma(s)$ on $-2 < \sigma < -1$ by
$$\Gamma(s) = \frac{\Gamma(s+2)}{s(s+1)}$$
gives a simple pole at $s = -1$. In this manner we analytically continue Γ to the entire complex plane; Γ resultantly has simple poles at $s = 0, -1, -2, \ldots$

1.3. **Gauss Sums.**

Definition 1.5. *A multiplicative group homomorphism $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$ from the group of invertible residue classes modulo q to the group of nonzero complex numbers is a* character modulo q. *We extend χ to $\mathbb{Z}$ by defining $\chi(n) = 0$ if $(n, q) > 1$ and $\chi(n) = \chi(m)$ for $n \equiv m \bmod q$.*

This extension to $\mathbb{Z}$ is well-defined: Let $(n, q) > 1$ and $m = n + kq$ for some $k \in \mathbb{Z}$. Then then $(m, q) = (n + kq, q) > 1$ since (n, q) divides $(n + kq, q)$; thus $\chi(n) = \chi(m) = 0$.

A character χ is completely multiplicative since if $(m, q) > 1$ or $(n, q) > 1$ then $\chi(mn) = 0 = \chi(m)\chi(n)$, and if $(m, q) = (n, q) = 1$ then $\chi(mn) = \chi(m)\chi(n)$ because χ is a homomorphism.

By Euler's theorem, $n^{\varphi(q)} \equiv 1 \bmod q$ for $n \in (\mathbb{Z}/q\mathbb{Z})^*$, where φ is the Euler totient function. Therefore $\chi^{\varphi(q)}(n) = 1$, i.e. $\chi(n)$ is a $\varphi(q)$-th root of unity.

The character $\chi_0(n) = 1$ for all n satisfying $(n, q) = 1$ is called the *trivial character*, and the conjugate of a character χ is defined as the complex conjugate of its values, i.e. $\overline{\chi}(n) = \overline{\chi(n)}$. A character χ is *even* if $\chi(-1) = 1$ and *odd* if $\chi(-1) = -1$.

Finally, define a *primitive* character mod$q$ to be a character χ mod q such that if $\chi(n) = \chi'(n)$ for all $n \in \mathbb{Z}$ for some character χ' mod r, then $r \geq q$.

Let $\tau(\chi)$ be the Gauss sum

$$\tau(\chi) := \sum_{m=1}^{q} \chi(m) e^{2\pi i m/q}.$$

Lemma 1.6. *If χ is a character* $\bmod q$*, then $\tau(\overline{\chi}) = \chi(-1)\overline{\tau(\chi)}$.*

Proof. Because $(-1)^{-1} \equiv -1 \bmod q$,

$$\begin{aligned}
\chi(-1)\overline{\tau(\chi)} &= \sum_{m=1}^{q} \chi(-1)\overline{\chi}(m) e^{-2\pi i m/q} \\
&= \sum_{m=1}^{q} \overline{\chi}(-m) e^{2\pi i(-m)/q} \\
&= \sum_{h=1}^{q} \overline{\chi}(h) e^{2\pi i h/q} \\
&= \tau(\overline{\chi})
\end{aligned}$$

upon setting $h \equiv -m \bmod q$. □

Lemma 1.7. *Let χ be a primitive, nontrivial character* $\bmod q$ *and $n \in \mathbb{Z}$. Then*

$$\chi(n)\tau(\overline{\chi}) = \sum_{m=1}^{q} \overline{\chi}(m) e^{2\pi i m n/q}.$$

Proof. We prove the statement for $(n, q) = 1$ and leave the case $(n, q) > 1$ as an exercise (see, for example, Murty [1]).

We have

$$\begin{aligned}
\chi(n)\tau(\overline{\chi}) &= \sum_{m=1}^{q} \overline{\chi}(m)\chi(n) e^{2\pi i m/q} \\
&= \sum_{m=1}^{q} \overline{\chi}(mn^{-1}) e^{2\pi i m/q}.
\end{aligned}$$

Since $(n, q) = 1$, n has a multiplicative inverse $\bmod q$, so let $h \equiv mn^{-1} \bmod q$. Because $n(\mathbb{Z}/q\mathbb{Z})^* = (\mathbb{Z}/q\mathbb{Z})^*$, summing over h is the same as summing over m; thus

$$\chi(n)\tau(\overline{\chi}) = \sum_{h=1}^{q} \overline{\chi}(h) e^{2\pi i h n/q},$$

which is what we were to show. □

Theorem 1.8. *If χ is a primitive character* $\bmod q$*, then $|\tau(\chi)| = q^{1/2}$.*

Proof. By Lemma 1.7 we have

$$\chi(n)\tau(\overline{\chi}) = \sum_{m=1}^{q} \overline{\chi}(m) e^{2\pi i m n/q}$$

for any natural number n. Multiplying this equation by its complex conjugate and invoking Lemma 1.6 so that $|\tau(\chi)| = |\tau(\overline{\chi})|$ gives

$$\begin{aligned}|\chi(n)|^2|\tau(\chi)|^2 &= \left(\sum_{m_1=1}^{q} \overline{\chi}(m_1)e^{2\pi i m_1 n/q}\right)\left(\sum_{m_2=1}^{q} \chi(m_2)e^{-2\pi i m_2 n/q}\right) \\ &= \sum_{m_1=1}^{q}\sum_{m_2=1}^{q} \overline{\chi}(m_1)\chi(m_2)e^{2\pi i n(m_1-m_2)/q}.\end{aligned}$$

Summing this equation over $n = 1, \ldots, q$ gives

$$|\tau(\chi)|^2 \sum_{n=1}^{q} |\chi(n)|^2 = \sum_{m_1=1}^{q}\sum_{m_2=1}^{q} \overline{\chi}(m_1)\chi(m_2) \sum_{n=1}^{q} e^{2\pi i n(m_1-m_2)/q}.$$

However, the sum $\sum_{n=1}^{q} e^{2\pi i n(m_1-m_2)/q}$ is 0 for all $m_1 \neq m_2$ and q if $m_1 = m_2$, so we get

$$|\tau(\chi)|^2\varphi(q) = \sum_{m=1}^{q} \overline{\chi}(m)\chi(m)q = q\sum_{m=1}^{q} |\chi(m)| = q\varphi(q),$$

giving $|\tau(\chi)| = q^{1/2}$ as desired. □

2. The Riemann Zeta Function

In deriving the functional equations for the Riemann zeta function and the Dirichlet L-functions, we follow Murty [1]. The method is the same in each case and is due to Riemann himself.

The Riemann zeta function is defined for $\sigma > 1$ as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The functional equation will provide an analytic continuation of $\zeta(s)$ to the entire complex plane.

Define

$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 z}$$

for complex z in the upper half plane. Then Theorem 1.4 gives the functional equation for $\omega(x) := \theta(ix)$:

$$\omega(x^{-1}) = x^{1/2}\omega(x).$$

Since

$$\omega(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} = -1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 x},$$

define

$$W(x) := \sum_{n=1}^{\infty} e^{-\pi n^2 x} = \frac{\omega(x)-1}{2}.$$

Then we have the functional equation

$$W(x^{-1}) = \frac{\omega(x^{-1})-1}{2} = \frac{x^{1/2}\omega(x)-1}{2} = -\frac{1}{2} + \frac{1}{2}x^{1/2} + x^{1/2}W(x).$$

We will use this to obtain the functional equation for ζ.

From the definition of $\Gamma(s)$, make the change of variables $t \mapsto \pi n^2 x$:

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-t} t^{s/2-1} dt$$
$$= \pi^{s/2} n^s \int_0^\infty e^{-\pi n^2 x} x^{s/2-1} dx.$$

Thus

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) n^{-s} = \int_0^\infty e^{-\pi n^2 x} x^{s/2-1} dx.$$

For $\sigma > 1$ we can sum both sides of this equation over the positive integers to obtain

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^\infty \left(\sum_{n=1}^\infty e^{-\pi n^2 x} \right) x^{s/2-1} dx$$

since the right side converges absolutely. Rewriting the sum as $W(x)$, we have

$$\begin{aligned}
\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \int_0^\infty W(x) x^{\frac{s}{2}-1} dx \\
&= \int_1^\infty W(x) x^{\frac{s}{2}-1} dx + \int_0^1 W(x) x^{\frac{s}{2}-1} dx \\
&= \int_1^\infty W(x) x^{\frac{s}{2}} \frac{dx}{x} + \int_1^\infty W(x^{-1}) x^{-\frac{s}{2}} \frac{dx}{x} \\
&= \int_1^\infty W(x) x^{\frac{s}{2}} \frac{dx}{x} + \int_1^\infty \left(-\frac{1}{2} + \frac{1}{2} x^{1/2} + x^{1/2} W(x) \right) x^{-\frac{s}{2}} \frac{dx}{x} \\
&= -\frac{1}{s} + \frac{1}{s-1} + \int_1^\infty W(x) \big(x^{\frac{s}{2}} + x^{\frac{1-s}{2}} \big) \frac{dx}{x}.
\end{aligned}$$

We can bound $W(x)$ by

$$W(x) = \sum_{n=1}^\infty e^{-\pi n^2 x} < \sum_{n=1}^\infty (e^{-\pi x})^n = \frac{e^{-\pi x}}{1 - e^{-\pi x}} = \frac{1}{e^{\pi x} - 1},$$

so $W(x) = O(e^{-\pi x})$ as $x \to \infty$. Therefore the above integral converges absolutely for all $s \in \mathbb{C}$, giving the analytic continuation:

Theorem 2.1.

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty W(x) \big(x^{\frac{s}{2}} + x^{\frac{1-s}{2}} \big) \frac{dx}{x}$$

for all $s \in \mathbb{C}$. Furthermore,

$$\Lambda(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

is entire with the exception of simple poles at $s = 0, 1$, and $\Lambda(s) = \Lambda(1-s)$.

Remark 2.2. *$\zeta(s)$ has simple zeros at $s = -2, -4, -6, \ldots$*

Proof. Since the integral in Theorem 2.1 converges for all $s \in \mathbb{C}$, $\Lambda(s)$ is analytic at $s = -2n$ for positive integers n. $W(x) > 0$, so we have

$$\Lambda(-2n) = \frac{1}{2n(2n+1)} + \int_1^\infty W(x) \big(x^{-n} + x^{n+1/2} \big) \frac{dx}{x} > 0$$

for all positive integers n. Since $\Gamma(s)$ has simple poles at $s = -2n$, by the functional equation $\zeta(s)$ has simple zeros at those points. $\square$

Remark 2.3. $\zeta(0) = -1/2$.

Proof.

$$\Gamma\left(\frac{s}{2}\right) = \frac{2}{s} - \gamma + \cdots \sim \left(\frac{s}{2}\right)^{-1}$$

as $s \to 0$, so letting $s \to 0$ in

$$\frac{s}{2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{2(s-1)} + \frac{s}{2}\int_1^\infty W(x)\left(x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\right)\frac{dx}{x}$$

gives $\zeta(0) = -1/2$. $\square$

Remark 2.4. $\zeta(s) \neq 0$ *for all real* $0 < s < 1$.

Proof. By Abel's partial summation formula with the function $f(n) = n^{-s}$ and the sequence $a_n = 1$ for all n,

$$\sum_{n \le t} a_n n^{-s} = \frac{[t]}{t^s} + s\int_1^t \frac{[x]}{x^{s+1}}\,dx,$$

where $[x]$ is the greatest integer function. Letting $t \to \infty$ gives

$$\begin{aligned}\zeta(s) &= s\int_1^\infty \frac{[x]}{x^{s+1}}\,dx \\ &= s\int_1^\infty \frac{x - \{x\}}{x^{s+1}}\,dx \\ &= \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}\,dx,\end{aligned}$$

where $\{x\}$ is the fractional part of x. From this we have

$$\left|\zeta(s) - \frac{s}{s-1}\right| < s\int_1^\infty \frac{dx}{x^{s+1}} = 1,$$

which gives

$$\zeta(s) < 1 + \frac{s}{s-1} = \frac{2s-1}{s-1}.$$

For $1/2 \le s < 1$, $(2s-1)/(s-1) \le 0$, so $\zeta(s) < 0$ on this interval. By the functional equation, $\zeta(s)$ is nonzero on $0 < s < 1$. $\square$

3. Dirichlet L-Functions

Let χ be a character modulo q. Define

$$L(s,\chi) = \sum_{n=1}^\infty \frac{\chi(n)}{n^s}.$$

This is, of course, a generalization of the zeta function (where χ is the trivial character). Because $|\chi(n)| \le 1$, the L-series converges absolutely for $\sigma > 1$.

In obtaining the functional equation for $L(s,\chi)$, we treat even and odd characters separately.

Theorem 3.1. *Let χ be a primitive character* $\bmod q$ *such that* $\chi(-1) = 1$. *Then*

$$\xi(s,\chi) = \pi^{-s/2} q^{s/2} \Gamma\left(\frac{s}{2}\right) L(s,\chi)$$

is entire, and $\xi(s,\chi) = w_\chi \xi(1-s,\overline{\chi})$, *where* $w_\chi = \tau(\chi)/q^{1/2}$.

Proof. Since χ is even and $\chi(0) = 0$, define

$$\theta(x,\chi) := \sum_{n=-\infty}^{\infty} \chi(n) e^{-\pi n^2 x/q} = 2\sum_{n=1}^{\infty} \chi(n) e^{-\pi n^2 x/q}.$$

Then, by Lemma 1.7,

$$\begin{aligned}
\tau(\overline{\chi})\theta(x,\chi) &= \sum_{n=-\infty}^{\infty} \tau(\overline{\chi})\chi(n) e^{-\pi n^2 x/q} \\
&= \sum_{m=1}^{q} \overline{\chi}(m) \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x/q + 2\pi i m n/q}.
\end{aligned}$$

Lemma 1.3 with $\alpha = m/q$ gives us the functional equation for θ:

$$\begin{aligned}
\tau(\overline{\chi})\theta(x,\chi) &= \sum_{m=1}^{q} \overline{\chi}(m)(q/x)^{1/2} \sum_{n=-\infty}^{\infty} e^{-\pi(n+m/q)^2 q/x} \\
&= (q/x)^{1/2} \sum_{m=1}^{q} \sum_{n=-\infty}^{\infty} \overline{\chi}(m) e^{-\pi(nq+m)^2/xq} \\
&= (q/x)^{1/2} \sum_{l=-\infty}^{\infty} \overline{\chi}(l) e^{-\pi l^2/xq} \\
&= (q/x)^{1/2} \theta(x^{-1},\overline{\chi})
\end{aligned}$$

by letting $l = nq + m$ since $nq + m$ runs exactly over $\mathbb{Z}$. From this it follows via the substitution $x \mapsto x^{-1}$ that $\tau(\overline{\chi})\theta(x^{-1},\chi) = (qx)^{1/2}\theta(x,\overline{\chi})$, which we will need below.

For $\mathrm{Re}(s/2) > 0$,

$$\begin{aligned}
\Gamma\left(\frac{s}{2}\right) &= \int_0^\infty e^{-t} t^{s/2-1} dt \\
&= \int_0^\infty e^{-\pi n^2 x/q} \left(\frac{\pi n^2 x}{q}\right)^{s/2-1} \frac{\pi n^2 dx}{q}
\end{aligned}$$

by the change of variables $t \mapsto \pi n^2 x/q$, which implies

$$\chi(n)\pi^{-s/2} q^{s/2} \Gamma\left(\frac{s}{2}\right) n^{-s} = \chi(n) \int_0^\infty e^{-\pi n^2 x/q} x^{s/2-1} dx.$$

Summing over $n \in \mathbb{N}$ gives

$$\pi^{-s/2} q^{s/2} \Gamma\left(\frac{s}{2}\right) L(s,\chi) = \int_0^\infty \left(\sum_{n=1}^{\infty} \chi(n) e^{-\pi n^2 x/q}\right) x^{s/2-1} dx.$$

for $\sigma > 1$. Introducing θ for the sum, we have

$$\begin{aligned}\xi(s,\chi) &= \frac{1}{2}\int_0^\infty \theta(x,\chi)x^{\frac{s}{2}-1}dx \\ &= \frac{1}{2}\int_1^\infty \theta(x,\chi)x^{\frac{s}{2}-1}dx + \frac{1}{2}\int_0^1 \theta(x,\chi)x^{\frac{s}{2}-1}dx \\ &= \frac{1}{2}\int_1^\infty \theta(x,\chi)x^{\frac{s}{2}}\frac{dx}{x} + \frac{1}{2}\int_1^\infty \theta(x^{-1},\chi)x^{-\frac{s}{2}}\frac{dx}{x} \\ &= \frac{1}{2}\int_1^\infty \theta(x,\chi)x^{\frac{s}{2}}\frac{dx}{x} + \frac{q^{1/2}}{2\tau(\overline{\chi})}\int_1^\infty \theta(x,\overline{\chi})x^{\frac{1-s}{2}}\frac{dx}{x}.\end{aligned}$$

The function $\xi(s,\chi)$ is analytic for all $s \in \mathbb{C}$ since $\theta(x,\chi) = O(e^{-\pi x})$ (where $\chi(n)$ is periodic so its maximum value is a finite constant). Taking $s \mapsto 1-s, \chi \mapsto \overline{\chi}$, the above expression becomes

$$\frac{1}{2}\int_1^\infty \theta(x,\overline{\chi})x^{\frac{1-s}{2}}\frac{dx}{x} + \frac{q^{1/2}}{2\tau(\chi)}\int_1^\infty \theta(x,\chi)x^{\frac{s}{2}}\frac{dx}{x},$$

which is $\xi(s,\chi)$ multiplied by $w_\chi^{-1} = q^{1/2}/\tau(\chi)$ since by Lemma 1.6 and Theorem 1.8,

$$\tau(\chi)\tau(\overline{\chi}) = \chi(-1)\tau(\chi)\overline{\tau(\chi)} = |\tau(\chi)|^2 = q.$$

This proves the functional equation $\xi(s,\chi) = w_\chi \xi(1-s,\overline{\chi})$. □

Remark 3.2. *Let χ be an even character. $L(s,\chi)$ has simple zeros at $s = -2,-4,-6,\ldots$*

Proof. By the functional equation, the only zeros of $L(s,\chi)$ such that $\sigma < 0$ are the poles of $\Gamma(s/2)$ in that region, since $L(1-s,\overline{\chi})$ and $\Gamma((1-s)/2)$ are nonzero for $1-\sigma > 1$. $\Gamma(s/2)$ has simple poles at $s = -2,-4,-6,\ldots$ □

Theorem 3.3. *Let χ be a primitive character* $\bmod q$ *such that $\chi(-1) = -1$. Then*

$$\xi(s,\chi) = \pi^{-\frac{s+1}{2}}q^{\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)L(s,\chi)$$

is entire, and $\xi(s,\chi) = w_\chi \xi(1-s,\overline{\chi})$, where $w_\chi = \tau(\chi)/iq^{1/2}$.

Proof. Since χ is odd, a different θ is necessary. Let

$$\theta_1(x,\chi) := \sum_{n=-\infty}^{\infty} n\chi(n)e^{-\pi n^2 x/q} = 2\sum_{n=1}^{\infty} n\chi(n)e^{-\pi n^2 x/q}.$$

Lemma 1.7 gives that

$$\begin{aligned}\tau(\overline{\chi})\theta_1(x,\chi) &= \sum_{n=-\infty}^{\infty} n\tau(\overline{\chi})\chi(n)e^{-\pi n^2 x/q} \\ &= \sum_{m=1}^{q}\overline{\chi}(m)\sum_{n=-\infty}^{\infty} ne^{-\pi n^2 x/q + 2\pi i m n/q}.\end{aligned}$$

Differentiating the result of Lemma 1.3,

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 y + 2\pi i n\alpha} = y^{-1/2}\sum_{n=-\infty}^{\infty} e^{-\pi(n+\alpha)^2/y},$$

with respect to α produces the equation

$$2\pi i \sum_{n=-\infty}^{\infty} n e^{-\pi n^2 y + 2\pi i n\alpha} = -2\pi y^{-3/2} \sum_{n=-\infty}^{\infty} (n+\alpha) e^{-\pi(n+\alpha)^2/y}.$$

From this, the change of variables $y \mapsto x/q, \alpha \mapsto m/q$ gives

$$\sum_{n=-\infty}^{\infty} n e^{-\pi n^2 x/q + 2\pi i m n/q} = i \left(\frac{q}{x}\right)^{3/2} \sum_{n=-\infty}^{\infty} \left(n + \frac{m}{q}\right) e^{-\pi(n+m/q)^2 q/x}.$$

We can use this to rewrite $\tau(\overline{\chi})\theta_1(x,\chi)$ as

$$\begin{aligned}\tau(\overline{\chi})\theta_1(x,\chi) &= \sum_{m=1}^{q} \overline{\chi}(m) \frac{iq^{1/2}}{x^{3/2}} \sum_{n=-\infty}^{\infty} (nq+m) e^{-\pi(nq+m)^2/xq} \\ &= \frac{iq^{1/2}}{x^{3/2}} \sum_{l=-\infty}^{\infty} l\overline{\chi}(l) e^{-\pi l^2/xq} \\ &= \frac{iq^{1/2}}{x^{3/2}} \theta_1(x^{-1}, \overline{\chi}),\end{aligned}$$

thus arriving at a functional equation for θ_1. Replacing $x \mapsto x^{-1}$ gives

$$\tau(\overline{\chi})\theta_1(x^{-1},\chi) = iq^{1/2} x^{3/2} \theta_1(x, \overline{\chi}).$$

As with the even case, we begin with

$$\chi(n)\pi^{-s/2} q^{s/2} \Gamma\left(\frac{s}{2}\right) n^{-s} = \chi(n) \int_0^\infty e^{-\pi n^2 x/q} x^{s/2-1} dx,$$

where we make the substitution $s \mapsto s+1$ and sum over $n \in \mathbb{N}$:

$$\pi^{-\frac{s+1}{2}} q^{\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s,\chi) = \int_0^\infty \left(\sum_{n=1}^{\infty} n\chi(n) e^{-\pi n^2 x/q}\right) x^{\frac{s-1}{2}} dx$$

for $\sigma > 0$. Putting in θ_1 and using the functional equation, we have

$$\begin{aligned}\xi(s,\chi) &= \frac{1}{2} \int_0^\infty \theta_1(x,\chi) x^{\frac{s-1}{2}} dx \\ &= \frac{1}{2} \int_1^\infty \theta_1(x,\chi) x^{\frac{s-1}{2}} dx + \frac{1}{2} \int_0^1 \theta_1(x,\chi) x^{\frac{s-1}{2}} dx \\ &= \frac{1}{2} \int_1^\infty \theta_1(x,\chi) x^{\frac{s-1}{2}} dx + \frac{1}{2} \int_1^\infty \theta_1(x^{-1},\chi) x^{-\frac{s}{2}-\frac{3}{2}} dx \\ &= \frac{1}{2} \int_1^\infty \theta_1(x,\chi) x^{\frac{s}{2}} \frac{dx}{\sqrt{x}} + \frac{iq^{1/2}}{2\tau(\overline{\chi})} \int_1^\infty \theta_1(x,\overline{\chi}) x^{\frac{1-s}{2}} \frac{dx}{\sqrt{x}}.\end{aligned}$$

Substituting $s \mapsto 1-s, \chi \mapsto \overline{\chi}$, this becomes

$$\frac{1}{2} \int_1^\infty \theta_1(x,\overline{\chi}) x^{\frac{1-s}{2}} \frac{dx}{\sqrt{x}} + \frac{iq^{1/2}}{2\tau(\chi)} \int_1^\infty \theta_1(x,\chi) x^{\frac{s}{2}} \frac{dx}{\sqrt{x}},$$

which is $\xi(s,\chi)$ multiplied by $w_\chi^{-1} = iq^{1/2}/\tau(\chi)$ since by Lemma 1.6 and Theorem 1.8,

$$\tau(\chi)\tau(\overline{\chi}) = \chi(-1)\tau(\chi)\overline{\tau(\chi)} = -|\tau(\chi)|^2 = -q.$$

This proves the functional equation $\xi(s,\chi) = w_\chi \xi(1-s,\overline{\chi})$. $\square$

4. Dirichlet Series and Modular Forms

Our presentation now follows that of Ogg [2].

Lemma 4.1. *Fix some $\lambda > 0$, and assume that the series*

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\lambda}$$

converges in the upper half plane.

(1) *If $a_n = O(n^c)$, then $f(x+iy) = O(y^{-c-1})$ as $y \to 0$, uniformly in all real x.*
(2) *If $f(x+iy) = O(y^{-c})$ as $y \to 0$, uniformly in x, then $a_n = O(n^c)$.*

Proof. 1. By extending the factorial function $n!$ to $\mathbb{C}$ as $z! := \Gamma(z+1)$, we have

$$\begin{aligned}(-1)^n \binom{-c-1}{n} &:= (-1)^n \frac{(-c-1)!}{(-c-1-n)!n!} \\ &= (-1)^n \frac{(-c-1)\cdots(-c-n)}{n!} \\ &= \frac{(c+1)\cdots(c+n)}{n!} \\ &= \frac{\Gamma(c+n+1)}{\Gamma(c+1)\Gamma(n+1)}.\end{aligned}$$

By Stirling's formula, $\Gamma(s) \sim \sqrt{2\pi} s^{s-1/2} e^{-s}$, we obtain

$$\begin{aligned}(-1)^n \binom{-c-1}{n} &\sim \frac{1}{\sqrt{2\pi}} \frac{(c+n+1)^{c+n+1/2} e^{-c-n-1}}{(c+1)^{c+1/2} e^{-c-1} (n+1)^{n+1/2} e^{-n-1}} \\ &= B_0 (n+c+1)^c \left(\frac{n+c+1}{n+1}\right)^{n+1/2} \\ &\sim B_1 n^c\end{aligned}$$

(for some constants B_0, B_1) since $((n+c+1)/(n+1))^{n+1/2} \to e^c$ as $n \to \infty$. We have $a_n = O(n^c)$ by assumption, so

$$a_n e^{2\pi i n x/\lambda} \le B_2 B_1 n^c \sim B_2 (-1)^n \binom{-c-1}{n}$$

for some constant B_2, which implies

$$\begin{aligned}f(x+iy) &= \sum_{n=0}^{\infty} a_n e^{2\pi i n x/\lambda} e^{-2\pi n y/\lambda} \\ &\le B_2 \sum_{n=0}^{\infty} (-1)^n \binom{-c-1}{n} e^{-2\pi n y/\lambda} \\ &= B_2 (1 - e^{-2\pi y/\lambda})^{-c-1} = O(y^{-c-1})\end{aligned}$$

since $2\pi y/\lambda$ dominates $1 - e^{-2\pi y/\lambda}$ as $y \to 0$.

2. Now let $|f(x+iy)| \leq By^{-c}$. Then the Fourier coefficients a_n of f satisfy

$$\begin{aligned} |a_n| &= \left| \int_0^1 f(x+\frac{i}{n})e^{-2\pi in(x+\frac{i}{n})/\lambda}dx \right| \\ &\leq \int_0^1 |f(x+\frac{i}{n})|\,|e^{-2\pi in(x+\frac{i}{n})/\lambda}|\,dx \\ &\leq Bn^c e^{2\pi/\lambda} = O(n^c), \end{aligned}$$

as needed. □

Now we outline the proof of a theorem showing a correspondence between Dirichlet series and modular forms. Let $a_0, a_1, a_2 \ldots,\ b_0, b_1, b_2 \ldots$ be sequences of complex numbers such that $a_n, b_n = O(n^c)$ for some $c > 0$. Let $\lambda > 0$, $k > 0$, and $C \in \mathbb{C} \setminus \{0\}$. Define

$$\begin{aligned} \phi(s) &:= \sum_{n=1}^{\infty} \frac{a_n}{n^s}, & \psi(s) &:= \sum_{n=1}^{\infty} \frac{b_n}{n^s}, \\ \Phi(s) &:= \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)\phi(s), & \Psi(s) &:= \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)\psi(s), \\ f(z) &:= \sum_{n=0}^{\infty} a_n e^{2\pi inz/\lambda}, & g(z) &:= \sum_{n=0}^{\infty} b_n e^{2\pi inz/\lambda}. \end{aligned}$$

Since $a_n = O(n^c)$, we have

$$\phi(s) = \sum_{n=1}^{\infty} a_n n^{-s} \leq B \sum_{n=1}^{\infty} n^{c-s}$$

for some constant B. Thus $\phi(s)$ converges for $\sigma > c+1$, and similarly for $\psi(s)$.

Theorem 4.2. *The following are equivalent:*

(1) $\Phi(s) + \frac{a_0}{s} + \frac{Cb_0}{k-s}$ *is entire and bounded on every vertical strip, and* $\Phi(s) = C\Psi(k-s)$.
(2) $f(z) = C(\frac{z}{i})^{-k} g(-1/z)$.

Proof. Assume *2*. From the definition of $\Gamma(s)$,

$$\begin{aligned} \Phi(s) &= \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)\phi(s) \\ &= \int_0^{\infty} \left(\sum_{n=1}^{\infty} a_n n^{-s}\right) \left(\frac{2\pi}{\lambda}\right)^{-s} t^{s-1}e^{-t}dt \\ &= \sum_{n=1}^{\infty} \int_0^{\infty} a_n \left(\frac{2\pi n}{\lambda}\right)^{-s} t^{s-1}e^{-t}dt \\ &= \sum_{n=1}^{\infty} \int_0^{\infty} a_n t^{s-1} e^{-2\pi nt/\lambda}dt, \end{aligned}$$

where the interchange of the integral and the sum is justified by absolute convergence and the substitution $t \mapsto 2\pi nt/\lambda$ is made in the integral to reach the last line. Again by interchanging the integral and summation, we obtain

$$\Phi(s) = \int_0^\infty t^{s-1}(f(it) - a_0)dt.$$

Since this integral is improper at 0 and ∞, we split it at 1 and consider the two sides. $f(it) - a_0 = O(e^{-ct})$ as $t \to \infty$ for some $c > 0$, so $\int_1^\infty t^{s-1}(f(it) - a_0)dt$ converges. For the other,

$$\begin{aligned}\int_0^1 t^{s-1}(f(it) - a_0)dt &= -a_0 \frac{t^s}{s}\bigg|_0^1 + \int_0^1 t^{s-1} f(it)dt \\ &= -\frac{a_0}{s} + \int_1^\infty t^{1-s} f(i/t)\frac{dt}{t^2} + \frac{Cb_0}{k-s} - \frac{Cb_0}{k-s} \\ &= -\frac{a_0}{s} + C\int_1^\infty t^{k-s}(g(it) - b_0)\frac{dt}{t} - \frac{Cb_0}{k-s}.\end{aligned}$$

Thus we have that

$$\Phi(s) + \frac{a_0}{s} + \frac{Cb_0}{k-s} = \int_1^\infty \left[t^s(f(it) - a_0) + Ct^{k-s}(g(it) - b_0)\right]\frac{dt}{t}.$$

Letting $z \mapsto -1/z$ in $f(z) = C(\frac{z}{i})^{-k} g(-1/z)$ gives $C^{-1}(\frac{z}{i})^{-k} f(-1/z) = g(z)$, so similarly

$$\Psi(s) + \frac{b_0}{s} + \frac{C^{-1}a_0}{k-s} = \int_1^\infty \left[C^{-1}t^{k-s}(g(it) - b_0) + t^s(f(it) - a_0)\right]\frac{dt}{t}.$$

Both these expressions are entire, and $\Phi(s) = C\Psi(k - s)$.

Conversely, assume *1.* By Mellin inversion, we have

$$f(ix) - a_0 = \frac{1}{2\pi i}\int_{\sigma=c} x^{-s}\Phi(s)ds$$

for $x > 0$, where c is large enough so that $\phi(s)$ converges absolutely. Since $a_n = O(n^c)$, by Lemma 4.1 $\phi(\sigma + it) = O(t^{-c-1})$. As a consequence of the Phragmen-Lindelöf theorem, we can push the line of integration to the left, past 0, acquiring the residues $-a_0$ and Cb_0x^{-k} at $s = 0, k$ respectively. Then we use the functional equation relating $\Phi(s)$ and $\Psi(s)$ to flip the line of integration over the imaginary axis:

$$\begin{aligned}f(ix) - Cb_0x^{-k} &= \frac{1}{2\pi i}\int_{\sigma=c<0} x^{-s}\Phi(s)ds \\ &= \frac{C}{2\pi i}\int_{\sigma=c<0} x^{-s}\Psi(k-s)ds \\ &= \frac{C}{2\pi i}\int_{\sigma=c>k} x^{-(k-s)}\Psi(s)ds \\ &= Cx^{-k}(g(i/x) - b_0).\end{aligned}$$

This gives the modular relation $f(z) = C(\frac{z}{i})^{-k} g(-1/z)$ by letting $z = ix$. □

Consider Theorem 4.2 as it applies to $\zeta(s)$. Let

$$a_n = b_n = \begin{cases} 1 & \text{if } n = m^2 \text{ is a square,} \\ 0 & \text{otherwise.}\end{cases}$$

We then find that

$$\phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \sum_{m=1}^{\infty} \frac{1}{m^{2s}} = \zeta(2s),$$

$$\Phi(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)\zeta(2s).$$

By Theorem 2.1, $\Lambda(s) = \Lambda(1-s)$, so letting $\lambda = 2$ gives

$$\begin{aligned} \Phi(s) &= \pi^{-s}\Gamma(s)\zeta(2s) \\ &= \Lambda(2s) = \Lambda(1-2s) \\ &= \pi^{-(1/2-s)}\Gamma(1/2-s)\zeta(1-2s) \\ &= \Phi(1/2-s). \end{aligned}$$

Thus $C = 1$ and $k = 1/2$. Let

$$f(z) = \sum_{n=0}^{\infty} e^{\pi i n z}.$$

By Theorem 4.2,

$$f(z) = \left(\frac{z}{i}\right)^{-\frac{1}{2}} f(-1/z),$$

so f is a modular form of weight $1/2$.

References

[1] Murty, M. Ram. *Problems in Analytic Number Theory.* 2001 Springer-Verlag New York, Inc. New York.

[2] Ogg, Andrew. *Modular Forms and Dirichlet Series.* 1969 W. A. Benjamin, Inc. New York.