

Combinatorial structure behind Sinkhorn limits

Eric Rowland, joint work with **Jason Wu**
Hofstra University

AMS Fall Southeastern Sectional Meeting
Tulane University, New Orleans, 2025-10-04

Setup

Given a square matrix with positive entries, turn it into a “close” doubly stochastic matrix of the same size (row and column sums are 1).

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Setup

Given a square matrix with positive entries, turn it into a “close” doubly stochastic matrix of the same size (row and column sums are 1).

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows ...

$$\begin{bmatrix} .800000 & .200000 \\ .666667 & .333333 \end{bmatrix}$$

Setup

Given a square matrix with positive entries, turn it into a “close” doubly stochastic matrix of the same size (row and column sums are 1).

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns ...

$$\begin{bmatrix} .545455 & .375000 \\ .454545 & .625000 \end{bmatrix}$$

Setup

Given a square matrix with positive entries, turn it into a “close” doubly stochastic matrix of the same size (row and column sums are 1).

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows ...

$$\begin{bmatrix} .592593 & .407407 \\ .421053 & .578947 \end{bmatrix}$$

Setup

Given a square matrix with positive entries, turn it into a “close” doubly stochastic matrix of the same size (row and column sums are 1).

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .584615 & .413043 \\ .415385 & .586957 \end{bmatrix}$$

Setup

Given a square matrix with positive entries, turn it into a “close” doubly stochastic matrix of the same size (row and column sums are 1).

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .585987 & .414013 \\ .414414 & .585586 \end{bmatrix}$$

Setup

Given a square matrix with positive entries, turn it into a “close” doubly stochastic matrix of the same size (row and column sums are 1).

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .585752 & .414179 \\ .414248 & .585821 \end{bmatrix}$$

Setup

Given a square matrix with positive entries, turn it into a “close” doubly stochastic matrix of the same size (row and column sums are 1).

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .585792 & .414208 \\ .414219 & .585781 \end{bmatrix}$$

Setup

Given a square matrix with positive entries, turn it into a “close” doubly stochastic matrix of the same size (row and column sums are 1).

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .585785 & .414213 \\ .414215 & .585787 \end{bmatrix}$$

Setup

Given a square matrix with positive entries, turn it into a “close” doubly stochastic matrix of the same size (row and column sums are 1).

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .585787 & .414213 \\ .414214 & .585786 \end{bmatrix}$$

Setup

Given a square matrix with positive entries, turn it into a “close” doubly stochastic matrix of the same size (row and column sums are 1).

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .585786 & .414214 \\ .414214 & .585786 \end{bmatrix}$$

Setup

Given a square matrix with positive entries, turn it into a “close” doubly stochastic matrix of the same size (row and column sums are 1).

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .585786 & .414214 \\ .414214 & .585786 \end{bmatrix}$$

In the limit, we obtain the **Sinkhorn limit**.

The limit exists: Sinkhorn 1964.

Setup

Given a square matrix with positive entries, turn it into a “close” doubly stochastic matrix of the same size (row and column sums are 1).

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .585786 & .414214 \\ .414214 & .585786 \end{bmatrix}$$

In the limit, we obtain the **Sinkhorn limit**.

The limit exists: Sinkhorn 1964.

Question

What are the entries?

Setup

Given a square matrix with positive entries, turn it into a “close” doubly stochastic matrix of the same size (row and column sums are 1).

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .585786 & .414214 \\ .414214 & .585786 \end{bmatrix} \approx \begin{bmatrix} 2 - \sqrt{2} & -1 + \sqrt{2} \\ -1 + \sqrt{2} & 2 - \sqrt{2} \end{bmatrix}$$

In the limit, we obtain the **Sinkhorn limit**.

The limit exists: Sinkhorn 1964.

Question

What are the entries?

Notation:

$$\text{Sink}\left(\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}\right) = \begin{bmatrix} 2 - \sqrt{2} & -1 + \sqrt{2} \\ -1 + \sqrt{2} & 2 - \sqrt{2} \end{bmatrix}$$

Theorem (Nathanson 2020)

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with positive entries,

$$\text{Sink}(A) = \frac{1}{\sqrt{ad} + \sqrt{bc}} \begin{bmatrix} \sqrt{ad} & \sqrt{bc} \\ \sqrt{bc} & \sqrt{ad} \end{bmatrix}.$$

For a **symmetric** 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$ with positive entries:

Theorem (Ekhad–Zeilberger 2019)

The top-left entry x of $\text{Sink}(A)$ satisfies $c_4 x^4 + \cdots + c_1 x + c_0 = 0$, where

$$\begin{aligned} c_4 &= -(a_{12}^2 - a_{11}a_{22})(a_{13}^2 - a_{11}a_{33})(-a_{11}a_{22}a_{33} + a_{11}a_{23}^2 + a_{12}^2a_{33} - 2a_{12}a_{13}a_{23} + a_{13}^2a_{22}) \\ c_3 &= (-4a_{11}^3a_{22}^2a_{33}^2 + 4a_{11}^3a_{22}a_{23}^2a_{33} + 4a_{11}^2a_{12}^2a_{22}a_{33}^2 - 3a_{11}^2a_{12}^2a_{23}^2a_{33} - 2a_{11}^2a_{12}a_{13}a_{22}a_{23}a_{33} + 4a_{11}^2a_{13}^2a_{22}^2a_{33} \\ &\quad - 3a_{11}^2a_{13}^2a_{22}a_{23}^2 - 2a_{11}a_{12}^2a_{13}^2a_{22}a_{33} + 2a_{11}a_{12}^2a_{13}^2a_{23}^2 - a_{12}^4a_{13}^2a_{33} + 2a_{12}^3a_{13}^3a_{23} - a_{12}^2a_{13}^4a_{22}) \\ c_2 &= a_{11}(6a_{11}^2a_{22}^2a_{33}^2 - 6a_{11}^2a_{22}a_{23}^2a_{33} - 2a_{11}a_{12}^2a_{22}a_{33}^2 + 3a_{11}a_{12}^2a_{23}^2a_{33} - 2a_{11}a_{12}a_{13}a_{22}a_{23}a_{33} - 2a_{11}a_{13}^2a_{22}^2a_{33} \\ &\quad + 3a_{11}a_{13}^2a_{22}a_{23}^2 + 2a_{12}^3a_{13}a_{23}a_{33} - 3a_{12}^2a_{13}^2a_{22}a_{33} - a_{12}^2a_{13}^2a_{23}^2 + 2a_{12}a_{13}^3a_{22}a_{23}) \\ c_1 &= -a_{11}^2(4a_{11}a_{22}^2a_{33}^2 - 4a_{11}a_{22}a_{23}^2a_{33} + a_{12}^2a_{23}^2a_{33} - 2a_{12}a_{13}a_{22}a_{23}a_{33} + a_{13}^2a_{22}a_{23}^2) \\ c_0 &= a_{11}^3a_{22}a_{33}(a_{22}a_{33} - a_{23}^2) \end{aligned}$$

Computed with Gröbner bases.

The entries are algebraic with degree at most **4**.

For **general** 3×3 matrices, the Sinkhorn limit wasn't known!

$$\text{Sink} \left(\begin{bmatrix} 2 & 4 & 3 \\ 1 & 8 & 8 \\ 7 & 3 & 1 \end{bmatrix} \right) \approx \begin{bmatrix} .250338 & .377025 & .372637 \\ .066831 & .402607 & .530562 \\ .682830 & .220368 & .096801 \end{bmatrix}$$

Can we identify these numbers?

For **general** 3×3 matrices, the Sinkhorn limit wasn't known!

$$\text{Sink} \left(\begin{bmatrix} 2 & 4 & 3 \\ 1 & 8 & 8 \\ 7 & 3 & 1 \end{bmatrix} \right) \approx \begin{bmatrix} .250338 & .377025 & .372637 \\ .066831 & .402607 & .530562 \\ .682830 & .220368 & .096801 \end{bmatrix}$$

Can we identify these numbers?

Compute the top-left entry to high precision:

$$x \approx .2503383740593684894545472868514292528338672217353016771994$$

For **general** 3×3 matrices, the Sinkhorn limit wasn't known!

$$\text{Sink} \left(\begin{bmatrix} 2 & 4 & 3 \\ 1 & 8 & 8 \\ 7 & 3 & 1 \end{bmatrix} \right) \approx \begin{bmatrix} .250338 & .377025 & .372637 \\ .066831 & .402607 & .530562 \\ .682830 & .220368 & .096801 \end{bmatrix}$$

Can we identify these numbers?

Compute the top-left entry to high precision:

$$x \approx .2503383740593684894545472868514292528338672217353016771994$$

Assume it's algebraic. Guess the degree:

For **general** 3×3 matrices, the Sinkhorn limit wasn't known!

$$\text{Sink} \left(\begin{bmatrix} 2 & 4 & 3 \\ 1 & 8 & 8 \\ 7 & 3 & 1 \end{bmatrix} \right) \approx \begin{bmatrix} .250338 & .377025 & .372637 \\ .066831 & .402607 & .530562 \\ .682830 & .220368 & .096801 \end{bmatrix}$$

Can we identify these numbers?

Compute the top-left entry to high precision:

$$x \approx .2503383740593684894545472868514292528338672217353016771994$$

Assume it's algebraic. Guess the degree: 4

For **general** 3×3 matrices, the Sinkhorn limit wasn't known!

$$\text{Sink} \left(\begin{bmatrix} 2 & 4 & 3 \\ 1 & 8 & 8 \\ 7 & 3 & 1 \end{bmatrix} \right) \approx \begin{bmatrix} .250338 & .377025 & .372637 \\ .066831 & .402607 & .530562 \\ .682830 & .220368 & .096801 \end{bmatrix}$$

Can we identify these numbers?

Compute the top-left entry to high precision:

$$x \approx .2503383740593684894545472868514292528338672217353016771994$$

Assume it's algebraic. Guess the degree: 4

Use **PSLQ** to find a likely polynomial:

$$b_4 x^4 + b_3 x^3 + b_2 x^2 + b_1 x + b_0 \approx 0$$

For **general** 3×3 matrices, the Sinkhorn limit wasn't known!

$$\text{Sink} \left(\begin{bmatrix} 2 & 4 & 3 \\ 1 & 8 & 8 \\ 7 & 3 & 1 \end{bmatrix} \right) \approx \begin{bmatrix} .250338 & .377025 & .372637 \\ .066831 & .402607 & .530562 \\ .682830 & .220368 & .096801 \end{bmatrix}$$

Can we identify these numbers?

Compute the top-left entry to high precision:

$$x \approx .2503383740593684894545472868514292528338672217353016771994$$

Assume it's algebraic. Guess the degree: 4

Use **PSLQ** to find a likely polynomial:

$$36164989943x^4 + 333428071444x^3 + 65054452280x^2 - 41075578985x + 832844043 \approx 0$$

For **general** 3×3 matrices, the Sinkhorn limit wasn't known!

$$\text{Sink} \left(\begin{bmatrix} 2 & 4 & 3 \\ 1 & 8 & 8 \\ 7 & 3 & 1 \end{bmatrix} \right) \approx \begin{bmatrix} .250338 & .377025 & .372637 \\ .066831 & .402607 & .530562 \\ .682830 & .220368 & .096801 \end{bmatrix}$$

Can we identify these numbers?

Compute the top-left entry to high precision:

$$x \approx .2503383740593684894545472868514292528338672217353016771994$$

Assume it's algebraic. Guess the degree: 4

Use **PSLQ** to find a likely polynomial:

$$36164989943x^4 + 333428071444x^3 + 65054452280x^2 - 41075578985x + 832844043 \approx 0$$

garbage!

For **general** 3×3 matrices, the Sinkhorn limit wasn't known!

$$\text{Sink} \left(\begin{bmatrix} 2 & 4 & 3 \\ 1 & 8 & 8 \\ 7 & 3 & 1 \end{bmatrix} \right) \approx \begin{bmatrix} .250338 & .377025 & .372637 \\ .066831 & .402607 & .530562 \\ .682830 & .220368 & .096801 \end{bmatrix}$$

Can we identify these numbers?

Compute the top-left entry to high precision:

$$x \approx .2503383740593684894545472868514292528338672217353016771994$$

Assume it's algebraic. Guess the degree: 5

Use **PSLQ** to find a likely polynomial:

$$860608662x^5 - 311838369x^4 + 602643248x^3 + \cdots - 1242712455x + 300075073 \approx 0$$

For **general** 3×3 matrices, the Sinkhorn limit wasn't known!

$$\text{Sink} \left(\begin{bmatrix} 2 & 4 & 3 \\ 1 & 8 & 8 \\ 7 & 3 & 1 \end{bmatrix} \right) \approx \begin{bmatrix} .250338 & .377025 & .372637 \\ .066831 & .402607 & .530562 \\ .682830 & .220368 & .096801 \end{bmatrix}$$

Can we identify these numbers?

Compute the top-left entry to high precision:

$$x \approx .2503383740593684894545472868514292528338672217353016771994$$

Assume it's algebraic. Guess the degree: 5

Use **PSLQ** to find a likely polynomial:

$$860608662x^5 - 311838369x^4 + 602643248x^3 + \cdots - 1242712455x + 300075073 \approx 0$$

garbage!

For **general** 3×3 matrices, the Sinkhorn limit wasn't known!

$$\text{Sink} \left(\begin{bmatrix} 2 & 4 & 3 \\ 1 & 8 & 8 \\ 7 & 3 & 1 \end{bmatrix} \right) \approx \begin{bmatrix} .250338 & .377025 & .372637 \\ .066831 & .402607 & .530562 \\ .682830 & .220368 & .096801 \end{bmatrix}$$

Can we identify these numbers?

Compute the top-left entry to high precision:

$$x \approx .2503383740593684894545472868514292528338672217353016771994$$

Assume it's algebraic. Guess the degree: 6

Use **PSLQ** to find a likely polynomial:

$$236379x^6 + 502124x^5 - 1610856x^4 + 19808x^3 + 661120x^2 - 94592x - 12288 = 0$$

For **general** 3×3 matrices, the Sinkhorn limit wasn't known!

$$\text{Sink} \left(\begin{bmatrix} 2 & 4 & 3 \\ 1 & 8 & 8 \\ 7 & 3 & 1 \end{bmatrix} \right) \approx \begin{bmatrix} .250338 & .377025 & .372637 \\ .066831 & .402607 & .530562 \\ .682830 & .220368 & .096801 \end{bmatrix}$$

Can we identify these numbers?

Compute the top-left entry to high precision:

$$x \approx .2503383740593684894545472868514292528338672217353016771994$$

Assume it's algebraic. Guess the degree: 6

Use **PSLQ** to find a likely polynomial:

$$236379x^6 + 502124x^5 - 1610856x^4 + 19808x^3 + 661120x^2 - 94592x - 12288 = 0$$

not garbage!

For **general** 3×3 matrices, the Sinkhorn limit wasn't known!

$$\text{Sink} \left(\begin{bmatrix} 2 & 4 & 3 \\ 1 & 8 & 8 \\ 7 & 3 & 1 \end{bmatrix} \right) \approx \begin{bmatrix} .250338 & .377025 & .372637 \\ .066831 & .402607 & .530562 \\ .682830 & .220368 & .096801 \end{bmatrix}$$

Can we identify these numbers?

Compute the top-left entry to high precision:

$$x \approx .2503383740593684894545472868514292528338672217353016771994$$

Assume it's algebraic. Guess the degree: 6

Use **PSLQ** to find a likely polynomial:

$$236379x^6 + 502124x^5 - 1610856x^4 + 19808x^3 + 661120x^2 - 94592x - 12288 = 0$$

not garbage!

Conjecture (Chen and Varghese 2019, Hofstra SSRP)

For 3×3 matrices A , the entries of $\text{Sink}(A)$ have degree at most 6.

It suffices to describe the **top-left entry** of $\text{Sink}(A)$.

Fact

If we know one entry of $\text{Sink}(A)$ as a function of A , then we know them all.

Reason: Iterative scaling isn't sensitive to row or column order.

For a 3×3 matrix, what is the top-left entry of $\text{Sink}(A)$? System of equations. . .

Row scaling — multiplication on the left.

Column scaling — multiplication on the right.

$$\text{Sink}(A) = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} \quad R = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad C = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$$

9 equations from $\text{Sink}(A) = RAC$:

$$s_{11} = r_1 a_{11} c_1 \quad s_{12} = r_1 a_{12} c_2 \quad s_{13} = r_1 a_{13} c_3$$

$$s_{21} = r_2 a_{21} c_1 \quad s_{22} = r_2 a_{22} c_2 \quad s_{23} = r_2 a_{23} c_3$$

$$s_{31} = r_3 a_{31} c_1 \quad s_{32} = r_3 a_{32} c_2 \quad s_{33} = r_3 a_{33} c_3$$

6 equations from row and column sums:

$$s_{11} + s_{12} + s_{13} = 1 \quad s_{11} + s_{21} + s_{31} = 1$$

$$s_{21} + s_{22} + s_{23} = 1 \quad s_{12} + s_{22} + s_{32} = 1$$

$$s_{31} + s_{32} + s_{33} = 1 \quad s_{13} + s_{23} + s_{33} = 1$$

Want s_{11} in terms of a_{ij} .

15 equations; eliminate 14 variables $r_1, r_2, r_3, c_1, c_2, c_3, s_{12}, s_{13}, \dots, s_{33}$.

Gröbner basis computation. . .

Theorem

The top-left entry $x = s_{11}$ satisfies $b_6x^6 + \cdots + b_1x + b_0 = 0$, where...

Theorem

The top-left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
 b_6 &= (a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{23} - a_{13}a_{21})(a_{11}a_{32} - a_{12}a_{31})(a_{11}a_{33} - a_{13}a_{31}) \\
 &\quad \cdot (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\
 b_5 &= -6a_{11}^5 a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11}^5 a_{22} a_{23}^2 a_{32}^2 a_{33} + 8a_{11}^4 a_{12} a_{21} a_{22} a_{23} a_{32}^2 a_{33} \\
 &\quad - 5a_{11}^4 a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} + 5a_{11}^4 a_{12} a_{22}^2 a_{23} a_{31}^2 a_{33} - 8a_{11}^4 a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
 &\quad + 5a_{11}^4 a_{13} a_{21} a_{22}^2 a_{32}^2 a_{33} - 8a_{11}^4 a_{13} a_{21} a_{22} a_{23}^2 a_{32}^2 a_{33} + 8a_{11}^4 a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
 &\quad - 5a_{11}^4 a_{13} a_{22} a_{23}^2 a_{31}^2 a_{32} - 2a_{11}^3 a_{12}^2 a_{21}^2 a_{23} a_{32}^2 a_{33} - 6a_{11}^3 a_{12}^2 a_{21} a_{22} a_{23} a_{31}^2 a_{33} \\
 &\quad + 6a_{11}^3 a_{12}^2 a_{21} a_{23}^2 a_{31} a_{32} a_{33} + 2a_{11}^3 a_{12}^2 a_{22} a_{23}^2 a_{31}^2 a_{33} - 6a_{11}^3 a_{12} a_{13} a_{21}^2 a_{22} a_{32}^2 a_{33} \\
 &\quad + 6a_{11}^3 a_{12} a_{13} a_{21}^2 a_{23} a_{32}^2 a_{33} - 4a_{11}^3 a_{12} a_{13} a_{21} a_{22}^2 a_{31}^2 a_{33} + 4a_{11}^3 a_{12} a_{13} a_{21} a_{23}^2 a_{31}^2 a_{32} \\
 &\quad - 6a_{11}^3 a_{12} a_{13} a_{22}^2 a_{23} a_{31}^2 a_{33} + 6a_{11}^3 a_{12} a_{13} a_{22} a_{23}^2 a_{31}^2 a_{32} + 2a_{11}^3 a_{13}^2 a_{21}^2 a_{22} a_{32}^2 a_{33} \\
 &\quad - 6a_{11}^3 a_{13}^2 a_{21} a_{22}^2 a_{31} a_{32} a_{33} + 6a_{11}^3 a_{13}^2 a_{21} a_{22} a_{23} a_{31}^2 a_{32} - 2a_{11}^3 a_{13}^2 a_{22}^2 a_{23} a_{31}^2 a_{32} \\
 &\quad + a_{11}^2 a_{12}^3 a_{21}^2 a_{23} a_{31}^2 a_{33} - a_{11}^2 a_{12}^3 a_{21} a_{23}^2 a_{31}^2 a_{33} + a_{11}^2 a_{12}^2 a_{13}^3 a_{21}^2 a_{32}^2 a_{33} \\
 &\quad + 4a_{11}^2 a_{12}^2 a_{13} a_{21}^2 a_{22} a_{31}^2 a_{33} - 4a_{11}^2 a_{12}^2 a_{13} a_{21}^2 a_{23} a_{31}^2 a_{32} a_{33} \\
 &\quad + 4a_{11}^2 a_{12}^2 a_{13} a_{21} a_{22}^2 a_{23} a_{31}^2 a_{33} - 4a_{11}^2 a_{12}^2 a_{13} a_{21} a_{23}^2 a_{31}^2 a_{32} - a_{11}^2 a_{12}^2 a_{13} a_{22}^2 a_{23}^2 a_{31}^3 \\
 &\quad - a_{11}^2 a_{12} a_{13}^2 a_{21}^2 a_{32}^2 a_{33} + 4a_{11}^2 a_{12} a_{13}^2 a_{21}^2 a_{22} a_{31} a_{32} a_{33} - 4a_{11}^2 a_{12} a_{13}^2 a_{21}^2 a_{23} a_{31}^2 a_{32} \\
 &\quad + 4a_{11}^2 a_{12} a_{13}^2 a_{21} a_{22}^2 a_{31}^2 a_{33} - 4a_{11}^2 a_{12} a_{13}^2 a_{21} a_{22} a_{23} a_{31}^2 a_{32} + a_{11}^2 a_{12} a_{13}^2 a_{22}^2 a_{23}^2 a_{31}^3 \\
 &\quad - a_{11}^2 a_{13}^3 a_{21}^2 a_{22} a_{31}^2 a_{32}^2 + a_{11}^2 a_{13}^3 a_{21} a_{22}^2 a_{31}^2 a_{32} - 2a_{11}^2 a_{12}^2 a_{13}^2 a_{21}^2 a_{22}^2 a_{31}^2 a_{33} \\
 &\quad + 2a_{11}^2 a_{12}^2 a_{13}^2 a_{21}^2 a_{23} a_{31}^2 a_{32} - a_{12}^3 a_{13}^2 a_{21}^2 a_{31}^2 a_{33} + a_{12}^3 a_{13}^2 a_{21}^2 a_{23} a_{31}^3 \\
 &\quad + a_{12}^2 a_{13}^3 a_{21}^2 a_{31}^2 a_{32} - a_{12}^2 a_{13}^3 a_{21}^2 a_{22}^2 a_{31}^3
 \end{aligned}$$

Theorem

The top-left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
 b_4 = & a_{11} (15a_{11}^4 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^4 a_{22} a_{23}^2 a_{32}^2 a_{33} - 12a_{11}^3 a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
 & + 10a_{11}^3 a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 10a_{11}^3 a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 12a_{11}^3 a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
 & - 10a_{11}^3 a_{13} a_{21} a_{22}^2 a_{32}^2 a_{33} + 12a_{11}^3 a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 12a_{11}^3 a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
 & + 10a_{11}^3 a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{11}^2 a_{12}^2 a_{21}^2 a_{23} a_{32} a_{33}^2 + 6a_{11}^2 a_{12}^2 a_{21} a_{22} a_{23} a_{31} a_{33}^2 \\
 & - 6a_{11}^2 a_{12}^2 a_{21} a_{23}^2 a_{31} a_{32} a_{33} - a_{11}^2 a_{12}^2 a_{22}^2 a_{23}^2 a_{31} a_{33} + 6a_{11}^2 a_{12} a_{13} a_{21}^2 a_{22} a_{32} a_{33}^2 \\
 & - 6a_{11}^2 a_{12} a_{13} a_{21}^2 a_{23} a_{32}^2 a_{33} + 6a_{11}^2 a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - 6a_{11}^2 a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2 \\
 & + 6a_{11}^2 a_{12} a_{13} a_{22}^2 a_{23}^2 a_{31} a_{33} - 6a_{11}^2 a_{12} a_{13} a_{22} a_{23}^2 a_{31}^2 a_{32} - a_{11}^2 a_{13}^2 a_{21}^2 a_{22} a_{32}^2 a_{33} \\
 & + 6a_{11}^2 a_{13}^2 a_{21} a_{22}^2 a_{31} a_{32} a_{33} - 6a_{11}^2 a_{13}^2 a_{21} a_{22} a_{23} a_{31} a_{32}^2 + a_{11}^2 a_{13}^2 a_{22}^2 a_{23} a_{31}^2 a_{32} \\
 & - 2a_{11}^2 a_{12}^2 a_{13} a_{21}^2 a_{22} a_{31} a_{33}^2 + 2a_{11}^2 a_{12}^2 a_{13} a_{21} a_{23}^2 a_{31}^2 a_{32} + 2a_{11}^2 a_{12} a_{13}^2 a_{21}^2 a_{23} a_{31} a_{32}^2 \\
 & - 2a_{11}^2 a_{12} a_{13}^2 a_{21} a_{22}^2 a_{31} a_{33} - 3a_{12}^2 a_{13}^2 a_{21}^2 a_{22} a_{31}^2 a_{33} + 3a_{12}^2 a_{13}^2 a_{21}^2 a_{23} a_{31}^2 a_{32}) \\
 b_3 = & 2a_{11}^2 (-10a_{11}^3 a_{22}^2 a_{23} a_{32} a_{33}^2 + 10a_{11}^3 a_{22} a_{23}^2 a_{32}^2 a_{33} + 4a_{11}^2 a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
 & - 5a_{11}^2 a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} + 5a_{11}^2 a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 - 4a_{11}^2 a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
 & + 5a_{11}^2 a_{13} a_{21} a_{22}^2 a_{32}^2 a_{33} - 4a_{11}^2 a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} + 4a_{11}^2 a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
 & - 5a_{11}^2 a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 - a_{11}^2 a_{12}^2 a_{21} a_{22} a_{23} a_{31} a_{33}^2 + a_{11}^2 a_{12}^2 a_{21} a_{23}^2 a_{31} a_{32} a_{33} \\
 & - a_{11}^2 a_{12} a_{13} a_{21}^2 a_{22} a_{32}^2 a_{33} + a_{11}^2 a_{12} a_{13} a_{21}^2 a_{23} a_{32}^2 a_{33} - 2a_{11}^2 a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 \\
 & + 2a_{11}^2 a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2 - a_{11}^2 a_{12} a_{13} a_{22}^2 a_{23} a_{31}^2 a_{33} + a_{11}^2 a_{12} a_{13} a_{22} a_{23}^2 a_{31}^2 a_{32} \\
 & - a_{11}^2 a_{13}^2 a_{21} a_{22}^2 a_{31} a_{32} a_{33} + a_{11}^2 a_{13}^2 a_{21} a_{22} a_{23} a_{31} a_{32}^2 + a_{12}^2 a_{13}^2 a_{21}^2 a_{23} a_{31} a_{32} a_{33} \\
 & - a_{12}^2 a_{13}^2 a_{21} a_{22}^2 a_{23} a_{31}^2 a_{33} - a_{12}^2 a_{13}^2 a_{21}^2 a_{22} a_{31} a_{32} a_{33} + a_{12}^2 a_{13}^2 a_{21} a_{22} a_{23} a_{31}^2 a_{32})
 \end{aligned}$$

Theorem

The top-left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned} b_2 &= a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\ &\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\ &\quad - 5a_{11} a_{13} a_{21} a_{22} a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\ &\quad + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\ b_1 &= a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\ &\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\ b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32}) \end{aligned}$$

Theorem

The top-left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned} b_2 &= a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22}^2 a_{23}^2 a_{32} a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\ &\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\ &\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\ &\quad + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\ b_1 &= a_{11}^4 (-6a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22}^2 a_{23}^2 a_{32} a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\ &\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\ b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32}) \end{aligned}$$

Better formulation?

Theorem

The top-left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
 b_2 &= a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
 &\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
 &\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
 &\quad + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\
 b_1 &= a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
 &\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
 b_0 &= a_{11}^5 (a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32}))
 \end{aligned}$$

Better formulation?

$$\begin{aligned}
 b_6 &= (a_{11} a_{22} - a_{12} a_{21}) (a_{11} a_{23} - a_{13} a_{21}) (a_{11} a_{32} - a_{12} a_{31}) (a_{11} a_{33} - a_{13} a_{31}) \\
 &\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31})
 \end{aligned}$$

Theorem

The top-left entry $x = s_{11}$ satisfies $b_6 x^6 + \dots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
 b_2 &= a_{11}^3 (15 a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15 a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2 a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
 &\quad + 5 a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5 a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2 a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
 &\quad - 5 a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2 a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2 a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
 &\quad + 5 a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\
 b_1 &= a_{11}^4 (-6 a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6 a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
 &\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
 b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
 \end{aligned}$$

Better formulation?

$$\begin{aligned}
 b_6 &= (a_{11} a_{22} - a_{12} a_{21}) (a_{11} a_{23} - a_{13} a_{21}) (a_{11} a_{32} - a_{12} a_{31}) (a_{11} a_{33} - a_{13} a_{31}) \\
 &\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
 &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
 \end{aligned}$$

b_6 : product of 5 minors

Theorem

The top-left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
 b_2 &= a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32}^2 a_{33}^2 \\
 &\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
 &\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32}^2 a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
 &\quad + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\
 b_1 &= a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
 &\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32}^2 a_{33} - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
 b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
 \end{aligned}$$

Better formulation?

$$\begin{aligned}
 b_6 &= (a_{11} a_{22} - a_{12} a_{21}) (a_{11} a_{23} - a_{13} a_{21}) (a_{11} a_{32} - a_{12} a_{31}) (a_{11} a_{33} - a_{13} a_{31}) \\
 &\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
 &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
 \end{aligned}$$

b_6 : product of 5 minors involving a_{11}

Theorem

The top-left entry $x = s_{11}$ satisfies $b_6 x^6 + \dots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
 b_2 &= a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
 &\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
 &\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32}^2 a_{33} + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
 &\quad + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{32}^2) \\
 b_1 &= a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
 &\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32}^2 a_{33} - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
 b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
 \end{aligned}$$

Better formulation?

$$\begin{aligned}
 b_6 &= (a_{11} a_{22} - a_{12} a_{21}) (a_{11} a_{23} - a_{13} a_{21}) (a_{11} a_{32} - a_{12} a_{31}) (a_{11} a_{33} - a_{13} a_{31}) \\
 &\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
 &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
 \end{aligned}$$

b_6 : product of 5 minors involving a_{11} and 0 not.

Theorem

The top-left entry $x = s_{11}$ satisfies $b_6 x^6 + \dots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
 b_2 &= a_{11}^3 (15 a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15 a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2 a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
 &\quad + 5 a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5 a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2 a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
 &\quad - 5 a_{11} a_{13} a_{21} a_{22}^2 a_{32}^2 a_{33} + 2 a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2 a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
 &\quad + 5 a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\
 b_1 &= a_{11}^4 (-6 a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6 a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
 &\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32}^2 a_{33} - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
 b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
 \end{aligned}$$

Better formulation?

$$\begin{aligned}
 b_6 &= (a_{11} a_{22} - a_{12} a_{21}) (a_{11} a_{23} - a_{13} a_{21}) (a_{11} a_{32} - a_{12} a_{31}) (a_{11} a_{33} - a_{13} a_{31}) \\
 &\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
 &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
 \end{aligned}$$

b_6 : product of 5 minors involving a_{11} and 0 not.

b_0 : product of 0 minors involving a_{11} and 5 not (and a_{11}^5).

Theorem

The top-left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
 b_2 &= a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
 &\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
 &\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32}^2 a_{33} + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
 &\quad + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\
 b_1 &= a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
 &\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32}^2 a_{33} - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
 b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
 \end{aligned}$$

Better formulation?

$$\begin{aligned}
 b_6 &= (a_{11} a_{22} - a_{12} a_{21}) (a_{11} a_{23} - a_{13} a_{21}) (a_{11} a_{32} - a_{12} a_{31}) (a_{11} a_{33} - a_{13} a_{31}) \\
 &\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
 &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
 \end{aligned}$$

Multiply each b_k by a_{11} .

b_6 : product of 5 minors involving a_{11} and 0 not.

b_0 : product of 0 minors involving a_{11} and 5 not (and a_{11}^5).

Theorem

The top-left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
 b_2 &= a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
 &\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
 &\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32}^2 a_{33} + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
 &\quad + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\
 b_1 &= a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
 &\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32}^2 a_{33} - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
 b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
 \end{aligned}$$

Better formulation?

$$\begin{aligned}
 b_6 &= (a_{11} a_{22} - a_{12} a_{21}) (a_{11} a_{23} - a_{13} a_{21}) (a_{11} a_{32} - a_{12} a_{31}) (a_{11} a_{33} - a_{13} a_{31}) \\
 &\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
 &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
 \end{aligned}$$

Multiply each b_k by a_{11} .

$a_{11} b_6$: product of 6 minors involving a_{11} and 0 not.

b_0 : product of 0 minors involving a_{11} and 5 not (and a_{11}^5).

Theorem

The top-left entry $x = s_{11}$ satisfies $b_6 x^6 + \dots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
 b_2 &= a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
 &\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
 &\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32}^2 a_{33} + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
 &\quad + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{32}^2) \\
 b_1 &= a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
 &\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32}^2 a_{33} - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
 b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
 \end{aligned}$$

Better formulation?

$$\begin{aligned}
 b_6 &= (a_{11} a_{22} - a_{12} a_{21}) (a_{11} a_{23} - a_{13} a_{21}) (a_{11} a_{32} - a_{12} a_{31}) (a_{11} a_{33} - a_{13} a_{31}) \\
 &\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
 &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
 \end{aligned}$$

Multiply each b_k by a_{11} .

$a_{11} b_6$: product of 6 minors involving a_{11} and 0 not.

$a_{11} b_0$: product of 0 minors involving a_{11} and 6 not (and a_{11}^6).

Theorem

The top-left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned} b_2 &= a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22}^2 a_{23}^2 a_{32} a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\ &\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32} a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\ &\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23}^2 a_{32} a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\ &\quad + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{32}^2) \\ b_1 &= a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32} a_{33} - a_{12} a_{21} a_{23}^2 a_{32} a_{33} \\ &\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\ b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32}) \end{aligned}$$

Better formulation?

$$\begin{aligned} b_6 &= (a_{11} a_{22} - a_{12} a_{21}) (a_{11} a_{23} - a_{13} a_{21}) (a_{11} a_{32} - a_{12} a_{31}) (a_{11} a_{33} - a_{13} a_{31}) \\ &\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\ &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \end{aligned}$$

Multiply each b_k by a_{11} .

$a_{11} b_6$: product of 6 minors involving a_{11} and 0 not.

$a_{11} b_0$: product of 0 minors involving a_{11} and 6 not (and a_{11}^6).

$a_{11} b_k$: products of k minors involving a_{11} and $6 - k$ not?

Theorem (Rowland–Wu 2025+)

The top-left entry x satisfies $b_6x^6 + \cdots + b_1x + b_0 = 0$, where

$$a_{11}b_6 = \Sigma\left(\begin{Bmatrix} \{ \} & \{2\} & \{2\} & \{3\} & \{3\} & \{2,3\} \\ \{ \} & \{2\} & \{3\} & \{2\} & \{3\} & \{2,3\} \end{Bmatrix}\right)$$

$$a_{11}b_5 = -3\Sigma\left(\begin{Bmatrix} \{ \} & \{2\} & \{2\} & \{3\} & \{3\} \\ \{ \} & \{2\} & \{3\} & \{2\} & \{3\} \end{Bmatrix}\right) - \Sigma\left(\begin{Bmatrix} \{ \} & \{2\} & \{2\} & \{3\} & \{2,3\} \\ \{ \} & \{2\} & \{3\} & \{2\} & \{2,3\} \end{Bmatrix}\right) + \Sigma\left(\begin{Bmatrix} \{2\} & \{2\} & \{3\} & \{3\} & \{2,3\} \\ \{2\} & \{3\} & \{2\} & \{3\} & \{2,3\} \end{Bmatrix}\right)$$

$$a_{11}b_4 = 4\Sigma\left(\begin{Bmatrix} \{ \} & \{2\} & \{2\} & \{3\} \\ \{ \} & \{2\} & \{3\} & \{2\} \end{Bmatrix}\right) + \Sigma\left(\begin{Bmatrix} \{ \} & \{2\} & \{3\} & \{2,3\} \\ \{ \} & \{2\} & \{3\} & \{2,3\} \end{Bmatrix}\right) - 3\Sigma\left(\begin{Bmatrix} \{2\} & \{2\} & \{3\} & \{3\} \\ \{2\} & \{3\} & \{2\} & \{3\} \end{Bmatrix}\right)$$

$$a_{11}b_3 = -4\Sigma\left(\begin{Bmatrix} \{ \} & \{2\} & \{2\} \\ \{ \} & \{2\} & \{3\} \end{Bmatrix}\right) - 5\Sigma\left(\begin{Bmatrix} \{ \} & \{2\} & \{3\} \\ \{ \} & \{2\} & \{3\} \end{Bmatrix}\right) + \Sigma\left(\begin{Bmatrix} \{ \} & \{2\} & \{2,3\} \\ \{ \} & \{2\} & \{2,3\} \end{Bmatrix}\right) + \Sigma\left(\begin{Bmatrix} \{2\} & \{2\} & \{3\} \\ \{2\} & \{3\} & \{2\} \end{Bmatrix}\right) - \Sigma\left(\begin{Bmatrix} \{2\} & \{3\} & \{2,3\} \\ \{2\} & \{3\} & \{2,3\} \end{Bmatrix}\right)$$

$$a_{11}b_2 = 4\Sigma\left(\begin{Bmatrix} \{ \} & \{2\} \\ \{ \} & \{2\} \end{Bmatrix}\right) - 3\Sigma\left(\begin{Bmatrix} \{ \} & \{2,3\} \\ \{ \} & \{2,3\} \end{Bmatrix}\right) + \Sigma\left(\begin{Bmatrix} \{2\} & \{3\} \\ \{2\} & \{3\} \end{Bmatrix}\right)$$

$$a_{11}b_1 = -3\Sigma\left(\begin{Bmatrix} \{ \} \\ \{ \} \end{Bmatrix}\right) - \Sigma\left(\begin{Bmatrix} \{2\} \\ \{2\} \end{Bmatrix}\right) + \Sigma\left(\begin{Bmatrix} \{2,3\} \\ \{2,3\} \end{Bmatrix}\right)$$

$$a_{11}b_0 = \Sigma\left(\begin{Bmatrix} \{ \} \end{Bmatrix}\right).$$

$\Sigma(S)$ is a sum of products of $|S|$ minors involving a_{11} and $6 - |S|$ not. The pairs $\begin{smallmatrix} R \\ C \end{smallmatrix}$ specify row and column indices for the minors involving a_{11} .

Degree: number of minors not involving the first row or first column.

For $n \times n$ matrices: $\sum_{j=0}^{n-1} \binom{n-1}{j}^2 = \binom{2n-2}{n-1}$ 1, 2, 6, 20, 70, 252, ...

What are the integers that appear?

Gröbner basis computations are infeasible for $n \geq 4$.

Instead, use PSLQ on examples and interpolate.

Infeasible for $n \geq 6$.

Definition

Let A be an $m \times n$ matrix with positive entries.

The *Sinkhorn limit* of A is obtained by iteratively scaling so that each row sum is 1 and each column sum is $\frac{m}{n}$.

The limit exists: Sinkhorn 1967 (in a more general form).

1.5 CPU years scaling matrices and recognizing 102K algebraic numbers
let us solve for 63K coefficients (and 56K parameterized by free variables).

Conjecture (Rowland–Wu 2025+)

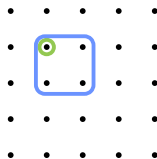
The coefficient of $\Sigma(S)x^{|S|}$ is $\det(\frac{1}{m} \operatorname{adj}_S(m, n))$.

$\operatorname{adj}_S(m, n)$ is an adjacency-like matrix.

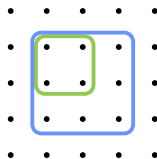
Underlying graph: Vertex set S . Edges?

Type-1 edges: Sizes differ by 1, and one is a subset of the other.

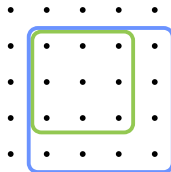
$$S = \begin{matrix} \{2\} & \{2,3\} \\ \{2\} & \{2,3\} \end{matrix}$$



$$S = \begin{matrix} \{2,3\} & \{2,3,4\} \\ \{2,3\} & \{2,3,4\} \end{matrix}$$

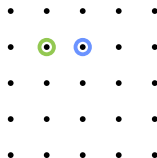


$$S = \begin{matrix} \{2,3,4\} & \{2,3,4,5\} \\ \{2,3,4\} & \{2,3,4,5\} \end{matrix}$$

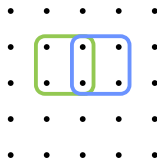


Type-2 edges: Same sizes, and they differ in exactly 1 row or 1 column.

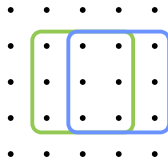
$$S = \begin{matrix} \{2\} & \{2\} \\ \{2\} & \{3\} \end{matrix}$$



$$S = \begin{matrix} \{2,3\} & \{2,3\} \\ \{2,3\} & \{3,4\} \end{matrix}$$



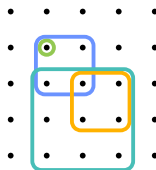
$$S = \begin{matrix} \{2,3,4\} & \{2,3,4\} \\ \{2,3,4\} & \{3,4,5\} \end{matrix}$$



Connected components are built from these.

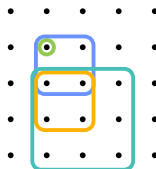
Two components:

$$S = \begin{array}{cccc} \{2\} & \{2,3\} & \{3,4\} & \{3,4,5\} \\ \{2\} & \{2,3\} & \{3,4\} & \{2,3,4\} \end{array}$$







One component:

$$S = \begin{array}{cccc} \{2\} & \{2,3\} & \{3,4\} & \{3,4,5\} \\ \{2\} & \{2,3\} & \{2,3\} & \{2,3,4\} \end{array}$$



References

-  Shalosh B. Ekhad and Doron Zeilberger, Answers to some questions about explicit Sinkhorn limits posed by Mel Nathanson, <https://arxiv.org/abs/1902.10783> (6 pages).
-  Melvyn B. Nathanson, Alternate minimization and doubly stochastic matrices, *Integers* **20A** (2020) Article #A10 (17 pages).
-  Eric Rowland and Jason Wu, The entries of the Sinkhorn limit of an $m \times n$ matrix, <https://arxiv.org/abs/2409.02789> (27 pages).
-  Richard Sinkhorn, A relationship between arbitrary positive matrices and doubly stochastic matrices, *The Annals of Mathematical Statistics* **35** (1964) 876–879.