Algebraic power series and their automatic complexity

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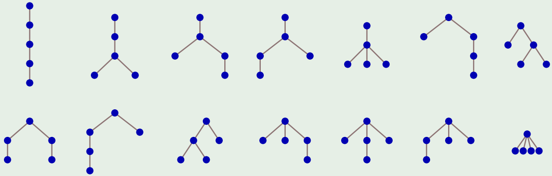
What do combinatorial sequences look like modulo p^{α}?

Example

Catalan numbers count plane trees with n edges:



$C(n)_{n\geq 0} = 1, 1, 2, 5, 14, 42, 132, 429, \ldots$

Modulo 2: $1, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, \ldots$

$C(n)$ is odd if and only if $n+1$ is a power of 2.
(follows from Kummer 1852 since $C(n) = \frac{1}{n+1}\binom{2n}{n}$)

Modulo 4: $1, 1, 2, 1, 2, 2, 0, 1, 2, 2, 0, 2, 0, 0, 0, 1, \ldots$

Theorem (Eu–Liu–Yeh 2008)

For all $n \geq 0$,

$$C(n) \bmod 4 = \begin{cases} 1 & \text{if } n+1 = 2^a \text{ for some } a \geq 0 \\ 2 & \text{if } n+1 = 2^b + 2^a \text{ for some } b > a \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $C(n) \not\equiv 3 \mod 4$.

Modulo 8: $1, 1, 2, 5, 6, 2, 4, 5, 6, 6, 4, 2, 4, 4, 0, 5, \ldots$

Theorem 4.2. *Let C_n be the nth Catalan number. First of all, $C_n \not\equiv_8 3$ and $C_n \not\equiv_8 7$ for any n. As for other congruences, we have*

$$C_n \equiv_8 \begin{cases} 1 & \text{if } n = 0 \text{ or } 1; \\ 2 & \text{if } n = 2^a + 2^{a+1} - 1 \text{ for some } a \geq 0; \\ 4 & \text{if } n = 2^a + 2^b + 2^c - 1 \text{ for some } c > b > a \geq 0; \\ 5 & \text{if } n = 2^a - 1 \text{ for some } a \geq 2; \\ 6 & \text{if } n = 2^a + 2^b - 1 \text{ for some } b - 2 \geq a \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Liu and Yeh (2010) determined $C(n)$ mod 16:

Theorem 5.5. *Let c_n be the n-th Catalan number. First of all, $c_n \not\equiv_{16} 3, 7, 9, 11, 15$ for any n. As for the other congruences, we have*

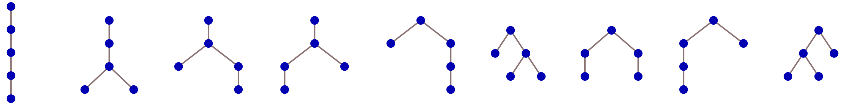
$$c_n \equiv_{16} \begin{cases} \left.\begin{matrix} 1 \\ 5 \\ 13 \end{matrix}\right\} & \text{if } \ d(\alpha) = 0 \ \text{ and } \begin{cases} \beta \le 1, \\ \beta = 2, \\ \beta \ge 3, \end{cases} \\ \left.\begin{matrix} 2 \\ 10 \end{matrix}\right\} & \text{if } \ d(\alpha) = 1,\ \alpha = 1 \ \text{ and } \begin{cases} \beta = 0 \text{ or } \beta \ge 2, \\ \beta = 1, \end{cases} \\ \left.\begin{matrix} 6 \\ 14 \end{matrix}\right\} & \text{if } \ d(\alpha) = 1,\ \alpha \ge 2 \ \text{ and } \begin{cases} (\alpha = 2, \beta \ge 2) \text{ or } (\alpha \ge 3, \beta \le 1), \\ (\alpha = 2, \beta \le 1) \text{ or } (\alpha \ge 3, \beta \ge 2), \end{cases} \\ \left.\begin{matrix} 4 \\ 12 \end{matrix}\right\} & \text{if } \ d(\alpha) = 2 \ \text{ and } \begin{cases} zr(\alpha) \equiv_2 0, \\ zr(\alpha) = 1, \end{cases} \\ 8 & \text{if } \ d(\alpha) = 3, \\ 0 & \text{if } \ d(\alpha) \ge 4. \end{cases}$$

where $\alpha = (CF_2(n+1) - 1)/2$ and $\beta = \omega_2(n+1)$ (or $\beta = \min\{i \mid n_i = 0\}$).

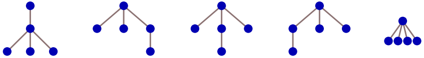
They also determined $C(n)$ mod 64.

What's the right framework?

Motzkin numbers count plane trees with n edges such that each vertex has at most 2 children:



Excluded:



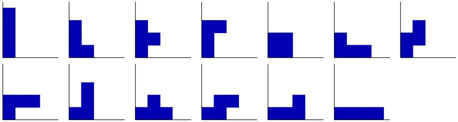
$M(n)_{n\geq 0} = 1, 1, 2, 4, 9, 21, 51, 127, \ldots$

Modulo 8: $1, 1, 2, 4, 1, 5, 3, 7, 3, 3, 4, 6, 7, 3, 2, 4, \ldots$

Theorem (Eu–Liu–Yeh; conj. by Deutsch–Sagan–Amdeberhan)

$M(n) \not\equiv 0 \mod 8$ *for all* $n \geq 0$.

Number of directed animals: $P(n)_{n \geq 0} = 1, 1, 2, 5, 13, 35, 96, 267, \ldots$



$P(4) = 13$

Number of restricted hexagonal polyominoes:
$H(n)_{n \geq 0} = 1, 1, 3, 10, 36, 137, 543, 2219, \ldots$

Riordan numbers: $R(n)_{n \geq 0} = 1, 0, 1, 1, 3, 6, 15, 36, \ldots$

Theorem (Deutsch–Sagan 2006)

There exists a set $C = \{1, 3, 4, 5, 7, \ldots\}$ with the property that

- *$P(n)$ is even if and only if $n \in 2C$,*
- *$H(n)$ is even if and only if $n \in 4C - 1$ or $n \in 4C$, and*
- *$R(n)$ is even if and only if $n \in 2C - 1$.*

Can we obtain and prove such results automatically?

Algebraic sequences

All these sequences $s(n)_{n\geq 0}$ are algebraic:

There is a nonzero polynomial $P(x, y)$ such that

$$P\left(x, \sum_{n\geq 0} s(n)x^n\right) = 0.$$

Example

For the Catalan numbers:

$\sum_{n\geq 0} C(n)x^n$ is a solution of $xy^2 - y + 1 = 0$ over $\mathbb{Q}$.

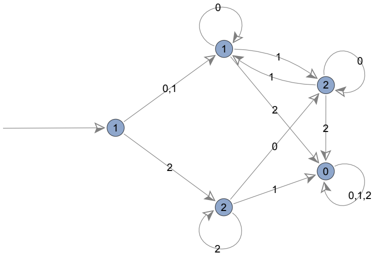
$\sum_{n\geq 0} (C(n) \bmod 3)x^n$ is a solution of $xy^2 + 2y + 1 = 0$ over $\mathbb{F}_3$.

$\mathbb{F}_p$ is the finite field with p elements.

Automatic sequences

$s(n)_{n\geq 0}$ is *p-automatic* if there is an automaton that outputs $s(n)$ when fed the base-p digits of n (least significant digit first).

$C(n) \bmod 3$:

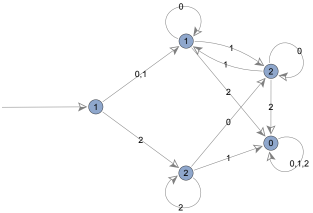


$C(9) \equiv ? \mod 3$. Since $9 = 100_3$, $C(9) \equiv \boxed{2} \mod 3$.

$(C(n) \bmod 3)_{n\geq 0} = 1, 1, 2, 2, 2, 0, 0, 0, 2, 2, \ldots$ is 3-automatic.

Two representations: polynomials and automata.

$$xy^2 + 2y + 1 = 0$$



Polynomial: easy to get from the polynomial over $\mathbb{Q}$.

Automaton: direct information about $s(n)$.

Theorem (Christol 1979/1980)

A sequence $s(n)_{n\geq 0}$ of elements in $\mathbb{F}_p$ is algebraic if and only if it is p-automatic.

How do we convert a polynomial into an automaton?

How does the automaton size depend on the polynomial degree?

How to tell whether a sequence is p-automatic?

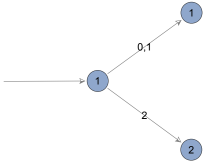
Let $r \in \{0, 1, \dots, p-1\}$.
The Cartier operator Λ_r picks out every pth term, starting with $s(r)$:

$$\Lambda_r(s(n)_{n\geq 0}) := s(pn+r)_{n\geq 0}$$

Iteratively apply $\Lambda_0, \Lambda_1, \dots, \Lambda_{p-1}$ to $s(n)_{n\geq 0}$.
Create one state in the automaton for each distinct sequence.

Let $s(n) = (C(n) \bmod 3)$. $\quad s(n)_{n\geq 0} = 1, 1, 2, 2, 2, 0, 0, 0, 2, \dots$.

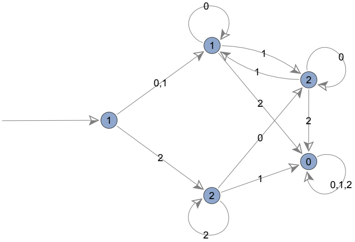
$\Lambda_0(s(n)_{n\geq 0}) = s(3n+0)_{n\geq 0} = 1, 2, 0, 2, 1, 0, 0, 0, 0, \dots$ new!
$\Lambda_1(s(n)_{n\geq 0}) = s(3n+1)_{n\geq 0} = 1, 2, 0, 2, 1, 0, 0, 0, 0, \dots = \Lambda_0(s(n)_{n\geq 0})$
$\Lambda_2(s(n)_{n\geq 0}) = s(3n+2)_{n\geq 0} = 2, 0, 2, 1, 0, 0, 0, 0, 2, \dots$ new!



Label each state with the initial term of the corresponding sequence.

$$\Lambda_0(\Lambda_0(s(n)_{n\geq 0})) = 1,2,0,2,1,0,0,0,0,2,\ldots = \Lambda_0(s(n)_{n\geq 0})$$
$$\Lambda_1(\Lambda_0(s(n)_{n\geq 0})) = 2,1,0,1,2,0,0,0,0,1,\ldots \qquad \text{new!}$$
$$\Lambda_2(\Lambda_0(s(n)_{n\geq 0})) = 0,0,0,0,0,0,0,0,0,0,\ldots \qquad \text{new!}$$

$\Lambda_r(\Lambda_2(s(n)_{n\geq 0}))$ ...



Eilenberg 1974:
A sequence is p-automatic if and only if this process terminates.

But we can't tell if sequences are equal from finitely many terms!

Use a different representation: diagonals of rational functions.

Polynomial for the Catalan numbers:

$$F = \sum_{n \geq 1} C(n) x^n \quad \text{satisfies} \quad x\,(F+1)^2 - F = 0.$$

Omit $C(0) = 1 \neq 0$.

Convert to the diagonal of a rational series (Furstenberg 1967):
$P = x(y+1)^2 - y$, so

$$F = \operatorname{diag}\left(\frac{y \frac{\partial P}{\partial y}(xy, y)}{P(xy, y)/y}\right) = \operatorname{diag}\left(\frac{y - 2xy^2 - 2xy^3}{1 - x - 2xy - xy^2}\right).$$

F mod 3 is the diagonal of

$$\begin{aligned}
\frac{y + xy^2 + xy^3}{1 + 2x + xy + 2xy^2} = \; & 0x^0y^0 + 1x^0y^1 + 0x^0y^2 + 0x^0y^3 + 0x^0y^4 + 0x^0y^5 + \cdots \\
& + 0x^1y^0 + 1x^1y^1 + 0x^1y^2 + 2x^1y^3 + 0x^1y^4 + 0x^1y^5 + \cdots \\
& + 0x^2y^0 + 1x^2y^1 + 2x^2y^2 + 0x^2y^3 + 1x^2y^4 + 2x^2y^5 + \cdots \\
& + 0x^3y^0 + 1x^3y^1 + 1x^3y^2 + 2x^3y^3 + 0x^3y^4 + 1x^3y^5 + \cdots \\
& + 0x^4y^0 + 1x^4y^1 + 0x^4y^2 + 2x^4y^3 + 2x^4y^4 + 0x^4y^5 + \cdots \\
& + 0x^5y^0 + 1x^5y^1 + 2x^5y^2 + 0x^5y^3 + 0x^5y^4 + 0x^5y^5 + \cdots \\
& + \cdots.
\end{aligned}$$

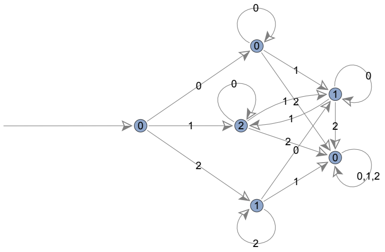
We have embedded $s(n)_{n\geq 1}$ into a series $\frac{S_0}{Q} := \frac{y+xy^2+xy^3}{1+2x+xy+2xy^2}$.

Construct an automaton by iterating $\lambda_{r,r}(S) := \Lambda_{r,r}\left(S \cdot Q^{p-1}\right)$.

$\lambda_{0,0}(S_0) = 2xy^2 + 2xy$ new!
$\lambda_{1,1}(S_0) = 1$ new!
$\lambda_{2,2}(S_0) = 2y + 2$ new!

$\lambda_{0,0}(2xy^2 + 2xy) = 2xy^2 + 2xy = \lambda_{0,0}(S_0)$ $\ldots$

Create one state in the automaton for each distinct polynomial.



The automaton may not be minimal.

Prime power moduli

This algorithm can be adapted to work modulo p^α.

Theorem (Denef–Lipshitz 1987)

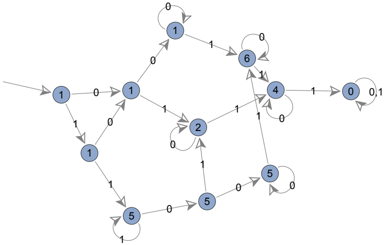
Let $\alpha \geq 1$. Let $R(\mathbf{x}), Q(\mathbf{x}) \in \mathbb{Z}_p[\mathbf{x}]$ such that $Q(0, \dots, 0) \not\equiv 0 \mod p$. Then the coefficient sequence of $\left(\operatorname{diag} \frac{R(\mathbf{x})}{Q(\mathbf{x})}\right) \mod p^\alpha$ is p-automatic.

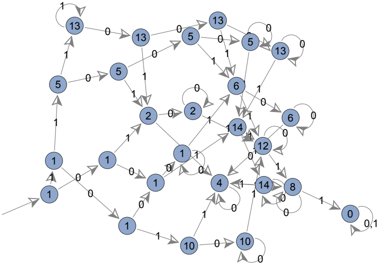
$\mathbb{Z}_p$ is the set of p-adic integers.

By computing an automaton for a sequence mod p^α, we can answer...

- Are there forbidden residues?
- What is the limiting distribution of residues (if it exists)?
- Is the sequence eventually periodic?
- Many other questions known to be decidable.

Catalan numbers modulo 8 and modulo 16:





Theorem (Liu–Yeh)

$C(n) \not\equiv 9 \mod 16$ *for all* $n \geq 0$.

Proof: Compute the automaton.

Catalan numbers modulo 2^α:

Theorem (Rowland–Yassawi 2015)

For all $n \geq 0$,

- $C(n) \not\equiv 17, 21, 26 \mod 32$,
- $C(n) \not\equiv 10, 13, 33, 37 \mod 64$,
- $C(n) \not\equiv 18, 54, 61, 65, 66, 69, 98, 106, 109 \mod 128$.

Only $\approx 35\%$ of the residues modulo 2^9 are attained by some $C(n)$.

Open question

Does the density of residues modulo 2^α attained by some Catalan number tend to 0 as α gets large?

Automaton size

How big is the (unminimized) automaton for $(C(n) \bmod 2^\alpha)_{n\geq 1}$?

α	1	2	3	4	5	6	7	8	9
size	4	6	15	37	83	194	445	1034	2403
$1.4 \times 2.3^\alpha$	3.2	7.4	17.0	39.2	90.1	207.3	476.7	1096.4	2521.6

height $h = \deg_x P$
degree $d = \deg_y P$

Upper bound from the construction: $p^{p^{2(\alpha-1)}\alpha h d}$

Example

$C(n) \bmod 2^9$: $P = x(y+1)^2 - y$ $h = 1$ $d = 2$
size $\leq 2^{18 \cdot 2^{16}} = 2^{1179648}$

Why is the bound so large?

Simpler setting: finite fields.

Theorem (Bridy 2017)

If the minimal polynomial P has height h and degree d, then the minimal automaton has size at most

$$(1 + o(1))\, p^{hd}$$

where $o(1)$ tends to 0 as any of p, h, d gets large.

Is the bound sharp? We suspect yes.

Polynomials in $\mathbb{F}_p[x, y]$ with maximum unminimized automaton size:

$p = 2$:

h	d	P	aut. size	p^{hd}	bound
1	2	$xy^2 + (x+1)y + x$	7	4	9
2	2	$x^2y^2 + (x^2+x+1)y + x^2$	14	16	25
3	2	$(x^3+x^2+1)y^2 + (x^3+1)y + x$	68	64	94
4	2	$(x^4+x+1)y^2 + (x^4+x^2+x+1)y + x$	252	256	311
5	2	$(x^5+x^3+1)y^2 + (x^5+x+1)y + x$	1052	1024	1192
6	2	$(x^6+x^5+1)y^2 + (x^6+x^2+x+1)y + x$	4062	4096	4424
7	2	$(x^7+x+1)y^2 + (x^7+x^4+x^3+x+1)y + x$	16424	16384	17288
1	3	$xy^3 + y^2 + (x+1)y + x$	11	8	18
2	3	$(x^2+x+1)y^3 + y^2 + (x^2+1)y + x^2 + x$	61	64	93
3	3	$(x^3+x+1)y^3 + y^2 + (x^3+x^2+x+1)y + x^3 + x^2$	533	512	614
4	3	$(x^4+x+1)y^3 + y^2 + (x^4+1)y + x^4 + x^3 + x$	4213	4096	4871
1	4	$(x+1)y^4 + y^2 + (x+1)y + x$	20	16	33
2	4	$(x^2+x+1)y^4 + y^3 + (x^2+x+1)y + x^2 + x$	216	256	358
3	4	$(x^3+x+1)y^4 + y^3 + (x^3+1)y + x^2 + x$	3956	4096	4870
1	5	$(x+1)y^5 + (x+1)y^2 + y + x$	37	32	67
2	5	$(x^2+x+1)y^5 + y^4 + y^3 + x^2y^2 + y + x^2 + x$	889	1024	1510
3	5	$(x^3+x^2+1)y^5 + y^4 + x^3y^2 + (x+1)y + x^3 + x^2 + x$	43913	32768	48134

$p = 3$:

h	d	P	aut. size	p^{hd}	bound
1	2	$(x+1)y^2 + y + x$	9	9	14
2	2	$(x^2+x+2)y^2 + y + x^2$	79	81	91
3	2	$(x^3+x^2+2x+1)y^2 + y + x^3 + x$	727	729	788
4	2	$(x^4+x^3+2)y^2 + y + x^4 + x$	6533	6561	6729

Can we get Bridy's bound without algebraic geometry? Yes.

Theorem (Rowland–Stipulanti–Yassawi 2023)

The minimal automaton has size at most

$$p^{hd} + p^{(h-1)(d-1)} L(h) L(d)^2 + \lfloor \log_p h \rfloor + \lceil \log_p \max(h, d-1) \rceil + 3.$$

$P \in \mathbb{F}_p[x, y], \quad h = \deg_x P, \quad d = \deg_y P$

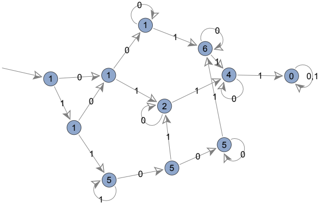
$L(n)$ is the Landau function:

$$L(5) = \max(\operatorname{lcm}(5), \operatorname{lcm}(4,1), \operatorname{lcm}(3,2), \operatorname{lcm}(3,1,1), \operatorname{lcm}(2,2,1), \operatorname{lcm}(2,1,1,1), \operatorname{lcm}(1,1,1,1,1)) = 6$$

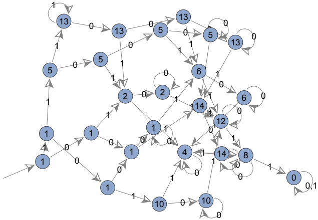
3 univariate polynomials R arise, with degrees $\leq h, d, d$.

Factor each $R = R_1^{e_1} \cdots R_k^{e_k}$. $\longrightarrow$ period length $\operatorname{lcm}(\deg R_1, \ldots, \deg R_k)$,
transient length $\log_p \max(e_1, \ldots, e_k)$

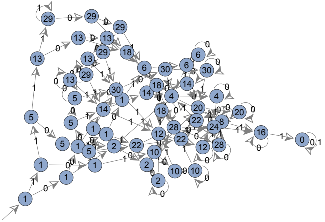
mod 8:



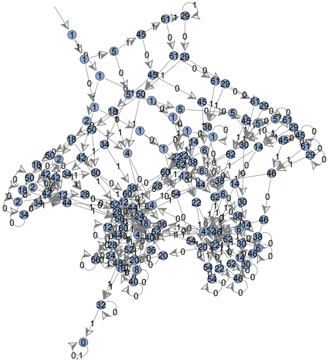
mod 16:



mod 32:



mod 64:



These automata project to each other.

So there is an inverse limit profinite automaton. Can we describe it?

Let $D = \{0, 1, \ldots, p-1\}$.

Theorem

Every state in the automaton for $(s(n) \bmod p^\alpha)_{n \geq 0}$ *is of the form*

$$T_0 Q^{p^{\alpha-1}-1} + p T_1 Q^{p^{\alpha-1}-2} + p^2 T_2 Q^{p^{\alpha-1}-3} + \cdots + p^{\alpha-1} T_{\alpha-1} Q^{p^{\alpha-1}-\alpha}$$

where $T_i \in D[x, y]$ *for each* $i \in \{0, 1, \ldots, \alpha - 1\}$.

Equivalently: $\left(T_0 + T_1 \frac{p}{Q} + T_2 \left(\frac{p}{Q}\right)^2 + \cdots + T_{\alpha-1} \left(\frac{p}{Q}\right)^{\alpha-1}\right) Q^{p^{\alpha-1}-1}$.

There are bounds on $\deg_x T_i$ and $\deg_y T_i$.

Singly exponential upper bound:

$$(1 + o(1))\, p^{\frac{1}{6}\alpha(\alpha+1)((2hd-1)\alpha+hd+1)}$$

When $\alpha = 1$, we recover Bridy's $(1 + o(1))\, p^{hd}$ for $\mathbb{F}_p$.

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