

# Combinatorial structure behind Sinkhorn limits

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## Goal

Given a square matrix with positive entries, turn it into a “close” doubly stochastic matrix of the same size (row and column sums are 1).

$2 \times 2$  matrix:

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

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Scale rows ...

$$\begin{bmatrix} .800000 & .200000 \\ .666667 & .333333 \end{bmatrix}$$

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$2 \times 2$  matrix:

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns ...

$$\begin{bmatrix} .545455 & .375000 \\ .454545 & .625000 \end{bmatrix}$$

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$2 \times 2$  matrix:

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows ...

$$\begin{bmatrix} .592593 & .407407 \\ .421053 & .578947 \end{bmatrix}$$

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$2 \times 2$  matrix:

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .584615 & .413043 \\ .415385 & .586957 \end{bmatrix}$$

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$$\begin{bmatrix} .585987 & .414013 \\ .414414 & .585586 \end{bmatrix}$$



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Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .585752 & .414179 \\ .414248 & .585821 \end{bmatrix}$$

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$2 \times 2$  matrix:

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .585792 & .414208 \\ .414219 & .585781 \end{bmatrix}$$

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$2 \times 2$  matrix:

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .585785 & .414213 \\ .414215 & .585787 \end{bmatrix}$$

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In the limit, we obtain the **Sinkhorn limit** of  $\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$ .

Sinkhorn 1964: The limit exists.

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Given a square matrix with positive entries, turn it into a “close” doubly stochastic matrix of the same size (row and column sums are 1).

$2 \times 2$  matrix:

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .585786 & .414214 \\ .414214 & .585786 \end{bmatrix} \approx \begin{bmatrix} 2 - \sqrt{2} & -1 + \sqrt{2} \\ -1 + \sqrt{2} & 2 - \sqrt{2} \end{bmatrix}$$

In the limit, we obtain the **Sinkhorn limit** of  $\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$ .

Sinkhorn 1964: The limit exists.

Applications in computer science:

- preconditioning a linear system to improve numerical stability
- approximating the permanent of a matrix
- determining whether a graph has a perfect matching

Kalantari et al. and Wigderson et al. studied convergence, fast algorithms.

Applications in other areas:

- predicting telephone traffic (Kruithof 1937)
- transportation science (Deming–Stephan 1940)
- economics (Stone 1964)
- image processing (Herman–Lent 1976)
- operations research (Raghavan 1984)
- machine learning (Cuturi 2013)

Idel (2016) wrote a 100-page survey of Sinkhorn-related results.

## Question

What are the exact entries of the Sinkhorn limit?



Notation:

$$\text{Sink}\left(\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}\right) = \begin{bmatrix} 2 - \sqrt{2} & -1 + \sqrt{2} \\ -1 + \sqrt{2} & 2 - \sqrt{2} \end{bmatrix}$$

### Theorem (Nathanson 2020)

For a  $2 \times 2$  matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  with positive entries,

$$\text{Sink}(A) = \frac{1}{\sqrt{ad} + \sqrt{bc}} \begin{bmatrix} \sqrt{ad} & \sqrt{bc} \\ \sqrt{bc} & \sqrt{ad} \end{bmatrix}.$$

The entries are algebraic.

The top left entry  $x$  satisfies  $(ad - bc)x^2 - 2adx + ad = 0$ .

For a **symmetric**  $3 \times 3$  matrix  $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$  with positive entries:

### Theorem (Ekhad–Zeilberger 2019)

*The top left entry  $x$  of  $\text{Sink}(A)$  satisfies  $c_4 x^4 + \cdots + c_1 x + c_0 = 0$ , where*

$$\begin{aligned} c_4 &= -(a_{12}^2 - a_{11}a_{22})(a_{13}^2 - a_{11}a_{33})(-a_{11}a_{22}a_{33} + a_{11}a_{23}^2 + a_{12}^2a_{33} - 2a_{12}a_{13}a_{23} + a_{13}^2a_{22}) \\ c_3 &= (-4a_{11}^3a_{22}^2a_{33}^2 + 4a_{11}^3a_{22}a_{23}^2a_{33} + 4a_{11}^2a_{12}^2a_{22}a_{33}^2 - 3a_{11}^2a_{12}^2a_{23}^2a_{33} - 2a_{11}^2a_{12}a_{13}a_{22}a_{23}a_{33} + 4a_{11}^2a_{13}^2a_{22}^2a_{33} \\ &\quad - 3a_{11}^2a_{13}a_{22}^2a_{23}^2 - 2a_{11}a_{12}^2a_{13}^2a_{22}a_{33} + 2a_{11}a_{12}^2a_{13}^2a_{23}^2 - a_{12}^4a_{13}^2a_{33} + 2a_{12}^3a_{13}^3a_{23} - a_{12}^2a_{13}^4a_{22}) \\ c_2 &= a_{11}(6a_{11}^2a_{22}^2a_{33}^2 - 6a_{11}^2a_{22}a_{23}^2a_{33} - 2a_{11}^2a_{12}^2a_{22}a_{33}^2 + 3a_{11}^2a_{12}^2a_{23}^2a_{33} - 2a_{11}a_{12}a_{13}a_{22}a_{23}a_{33} - 2a_{11}a_{13}^2a_{22}^2a_{33} \\ &\quad + 3a_{11}a_{13}^2a_{22}^2a_{23}^2 + 2a_{12}^3a_{13}a_{23}a_{33} - 3a_{12}^2a_{13}^2a_{22}a_{33} - a_{12}^2a_{13}^2a_{23}^2 + 2a_{12}a_{13}^3a_{22}a_{23}) \\ c_1 &= -a_{11}^2(4a_{11}a_{22}^2a_{33}^2 - 4a_{11}a_{22}a_{23}^2a_{33} + a_{12}^2a_{23}^2a_{33} - 2a_{12}a_{13}a_{22}a_{23}a_{33} + a_{13}^2a_{22}a_{23}^2) \\ c_0 &= a_{11}^3a_{22}a_{33}(a_{22}a_{33} - a_{23}^2) \end{aligned}$$

Computed with Gröbner bases.

For **general**  $3 \times 3$  matrices, the Sinkhorn limit wasn't known!

$$\text{Sink} \left( \begin{bmatrix} 2 & 4 & 3 \\ 1 & 8 & 8 \\ 7 & 3 & 1 \end{bmatrix} \right) \approx \begin{bmatrix} .250338 & .377025 & .372637 \\ .066831 & .402607 & .530562 \\ .682830 & .220368 & .096801 \end{bmatrix}$$

What are these numbers? Assume they're algebraic.

Compute an entry to high precision:

$$x \approx .2503383740593684894545472868514292528338672217353016771994$$

Guess the degree. 6

Use the **PSLQ** integer relation algorithm to find a likely polynomial:

$$236379x^6 + 502124x^5 - 1610856x^4 + 19808x^3 + 661120x^2 - 94592x - 12288 = 0$$

**Conjecture (Chen and Varghese 2019, Hofstra SSRP)**

*For  $3 \times 3$  matrices  $A$ , the entries of  $\text{Sink}(A)$  have degree at most 6.*

It suffices to describe the **top left entry** of  $\text{Sink}(A)$ .

## Fact

*If we know one entry of  $\text{Sink}(A)$  as a function of  $A$ , then we know them all.*

Reason: Iterative scaling isn't sensitive to row or column order.

For example, if we know

$$\text{Sink}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \frac{\sqrt{ad}}{\sqrt{ad} + \sqrt{bc}} & ? \\ ? & ? \end{bmatrix}$$

then its bottom left entry is the top left entry of

$$\text{Sink}\left(\begin{bmatrix} c & d \\ a & b \end{bmatrix}\right) = \begin{bmatrix} \frac{\sqrt{cb}}{\sqrt{cb} + \sqrt{da}} & \\ & \end{bmatrix}.$$

For a  $3 \times 3$  matrix, what is the top left entry of  $\text{Sink}(A)$ ? System of equations. . .

Row scaling — multiplication on the left.

Column scaling — multiplication on the right.

$$\text{Sink}(A) = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} \quad R = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad C = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$$

9 equations from  $\text{Sink}(A) = RAC$ :

$$s_{11} = r_1 a_{11} c_1 \quad s_{12} = r_1 a_{12} c_2 \quad s_{13} = r_1 a_{13} c_3$$

$$s_{21} = r_2 a_{21} c_1 \quad s_{22} = r_2 a_{22} c_2 \quad s_{23} = r_2 a_{23} c_3$$

$$s_{31} = r_3 a_{31} c_1 \quad s_{32} = r_3 a_{32} c_2 \quad s_{33} = r_3 a_{33} c_3$$

6 equations from row and column sums:

$$s_{11} + s_{12} + s_{13} = 1 \quad s_{11} + s_{21} + s_{31} = 1$$

$$s_{21} + s_{22} + s_{23} = 1 \quad s_{12} + s_{22} + s_{32} = 1$$

$$s_{31} + s_{32} + s_{33} = 1 \quad s_{13} + s_{23} + s_{33} = 1$$

Want  $s_{11}$  in terms of  $a_{ij}$ .

15 equations; eliminate 14 variables  $r_1, r_2, r_3, c_1, c_2, c_3, s_{12}, s_{13}, \dots, s_{33}$ .

Gröbner basis computation. . .

# Theorem

The top left entry  $x = s_{11}$  satisfies  $b_6 x^6 + \cdots + b_1 x + b_0 = 0$ , where...

$$\begin{aligned}
 b_6 &= (a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{23} - a_{13}a_{21})(a_{11}a_{32} - a_{12}a_{31})(a_{11}a_{33} - a_{13}a_{31}) \\
 &\quad \cdot (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\
 b_5 &= -6a_{11}^5 a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11}^5 a_{22} a_{23}^2 a_{32}^2 a_{33} + 8a_{11}^4 a_{12} a_{21} a_{22} a_{23} a_{32}^2 a_{33} \\
 &\quad - 5a_{11}^4 a_{12} a_{21}^2 a_{23}^2 a_{32}^2 a_{33} + 5a_{11}^4 a_{12} a_{22}^2 a_{23} a_{31}^2 a_{33} - 8a_{11}^4 a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
 &\quad + 5a_{11}^4 a_{13} a_{21}^2 a_{22}^2 a_{32}^2 a_{33} - 8a_{11}^4 a_{13} a_{21} a_{22} a_{23}^2 a_{32}^2 a_{33} + 8a_{11}^4 a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
 &\quad - 5a_{11}^4 a_{13} a_{22}^2 a_{23} a_{31}^2 a_{32}^2 - 2a_{11}^3 a_{12}^2 a_{21}^2 a_{23} a_{32}^2 a_{33} - 6a_{11}^3 a_{12}^2 a_{21} a_{22} a_{23} a_{31}^2 a_{33} \\
 &\quad + 6a_{11}^3 a_{12}^2 a_{21}^2 a_{23} a_{31} a_{32} a_{33} + 2a_{11}^3 a_{12}^2 a_{22} a_{23}^2 a_{31}^2 a_{33} - 6a_{11}^3 a_{12} a_{13} a_{21}^2 a_{22} a_{32}^2 a_{33} \\
 &\quad + 6a_{11}^3 a_{12} a_{13} a_{21}^2 a_{23} a_{32}^2 a_{33} - 4a_{11}^3 a_{12} a_{13} a_{21} a_{22}^2 a_{31}^2 a_{33} + 4a_{11}^3 a_{12} a_{13} a_{21}^2 a_{23} a_{31}^2 a_{32}^2 \\
 &\quad - 6a_{11}^3 a_{12} a_{13} a_{22}^2 a_{23} a_{31}^2 a_{33} + 6a_{11}^3 a_{12} a_{13} a_{22} a_{23}^2 a_{31}^2 a_{32} + 2a_{11}^3 a_{13}^2 a_{21}^2 a_{22} a_{32}^2 a_{33} \\
 &\quad - 6a_{11}^3 a_{13}^2 a_{21}^2 a_{22} a_{31} a_{32} a_{33} + 6a_{11}^3 a_{13}^2 a_{21} a_{22} a_{23} a_{31}^2 a_{32}^2 - 2a_{11}^3 a_{13}^2 a_{22}^2 a_{23} a_{31}^2 a_{32} \\
 &\quad + a_{11}^2 a_{12}^3 a_{21}^2 a_{23} a_{31}^2 a_{33} - a_{11}^2 a_{12}^3 a_{21}^2 a_{23}^2 a_{31}^2 a_{33} + a_{11}^2 a_{12}^2 a_{13}^3 a_{21}^2 a_{32}^2 a_{33} \\
 &\quad + 4a_{11}^2 a_{12}^2 a_{13} a_{21}^2 a_{22} a_{31}^2 a_{33} - 4a_{11}^2 a_{12}^2 a_{13} a_{21}^2 a_{23} a_{31}^2 a_{32} a_{33} \\
 &\quad + 4a_{11}^2 a_{12}^2 a_{13} a_{21} a_{22} a_{23} a_{31}^2 a_{33} - 4a_{11}^2 a_{12}^2 a_{13} a_{21} a_{23}^2 a_{31}^2 a_{32} - a_{11}^2 a_{12}^2 a_{13} a_{22}^2 a_{23}^2 a_{31}^3 \\
 &\quad - a_{11}^2 a_{12}^2 a_{13}^3 a_{21}^2 a_{32}^2 a_{33} + 4a_{11}^2 a_{12}^2 a_{13}^3 a_{21}^2 a_{22} a_{31} a_{32} a_{33} - 4a_{11}^2 a_{12}^2 a_{13}^3 a_{21}^2 a_{23} a_{31}^2 a_{32}^2 \\
 &\quad + 4a_{11}^2 a_{12}^2 a_{13}^3 a_{21}^2 a_{22}^2 a_{31}^2 a_{33} - 4a_{11}^2 a_{12}^2 a_{13}^3 a_{21} a_{22} a_{23} a_{31}^2 a_{32} + a_{11}^2 a_{12}^2 a_{13}^3 a_{22}^2 a_{23}^2 a_{31}^3 \\
 &\quad - a_{11}^2 a_{13}^3 a_{21}^2 a_{22} a_{31}^2 a_{32}^2 + a_{11}^2 a_{13}^3 a_{21}^2 a_{22}^2 a_{31}^2 a_{32} - 2a_{11}^2 a_{12}^2 a_{13}^3 a_{21}^2 a_{22}^2 a_{31}^2 a_{33} \\
 &\quad + 2a_{11}^2 a_{12}^2 a_{13}^3 a_{21}^2 a_{23} a_{31}^2 a_{32} - a_{12}^3 a_{13}^3 a_{21}^2 a_{31}^2 a_{33} + a_{12}^3 a_{13}^3 a_{21}^2 a_{23} a_{31}^3 \\
 &\quad + a_{12}^2 a_{13}^3 a_{21}^2 a_{31}^2 a_{32}^2 - a_{12}^2 a_{13}^3 a_{21}^2 a_{22}^2 a_{31}^3
 \end{aligned}$$

# Theorem

The top left entry  $x = s_{11}$  satisfies  $b_6 x^6 + \cdots + b_1 x + b_0 = 0$ , where...

$$\begin{aligned}
 b_4 = & a_{11} (15a_{11}^4 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^4 a_{22} a_{23}^2 a_{32}^2 a_{33} - 12a_{11}^3 a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
 & + 10a_{11}^3 a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 10a_{11}^3 a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 12a_{11}^3 a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
 & - 10a_{11}^3 a_{13} a_{21} a_{22}^2 a_{32}^2 a_{33} + 12a_{11}^3 a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 12a_{11}^3 a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
 & + 10a_{11}^3 a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{11}^2 a_{12}^2 a_{21}^2 a_{23} a_{32} a_{33}^2 + 6a_{11}^2 a_{12}^2 a_{21} a_{22} a_{23} a_{31} a_{33}^2 \\
 & - 6a_{11}^2 a_{12}^2 a_{21} a_{23}^2 a_{31} a_{32} a_{33} - a_{11}^2 a_{12}^2 a_{22}^2 a_{23}^2 a_{31} a_{33} + 6a_{11}^2 a_{12} a_{13} a_{21}^2 a_{22} a_{32} a_{33}^2 \\
 & - 6a_{11}^2 a_{12} a_{13} a_{21}^2 a_{23} a_{32}^2 a_{33} + 6a_{11}^2 a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - 6a_{11}^2 a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2 \\
 & + 6a_{11}^2 a_{12} a_{13} a_{22}^2 a_{23}^2 a_{31} a_{33} - 6a_{11}^2 a_{12} a_{13} a_{22} a_{23}^2 a_{31}^2 a_{32} - a_{11}^2 a_{13}^2 a_{21}^2 a_{22} a_{32}^2 a_{33} \\
 & + 6a_{11}^2 a_{13}^2 a_{21} a_{22}^2 a_{31} a_{32} a_{33} - 6a_{11}^2 a_{13}^2 a_{21} a_{22} a_{23} a_{31} a_{32}^2 + a_{11}^2 a_{13}^2 a_{22}^2 a_{23} a_{31}^2 a_{32} \\
 & - 2a_{11}^2 a_{12}^2 a_{13} a_{21}^2 a_{22} a_{31} a_{33}^2 + 2a_{11}^2 a_{12}^2 a_{13} a_{21} a_{23}^2 a_{31}^2 a_{32} + 2a_{11}^2 a_{12} a_{13}^2 a_{21}^2 a_{23} a_{31} a_{32}^2 \\
 & - 2a_{11}^2 a_{12} a_{13}^2 a_{21} a_{22}^2 a_{31} a_{33} - 3a_{12}^2 a_{13}^2 a_{21}^2 a_{22} a_{31}^2 a_{33} + 3a_{12}^2 a_{13}^2 a_{21}^2 a_{23} a_{31}^2 a_{32}) \\
 b_3 = & 2a_{11}^2 (-10a_{11}^3 a_{22}^2 a_{23} a_{32} a_{33}^2 + 10a_{11}^3 a_{22} a_{23}^2 a_{32}^2 a_{33} + 4a_{11}^2 a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
 & - 5a_{11}^2 a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} + 5a_{11}^2 a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 - 4a_{11}^2 a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
 & + 5a_{11}^2 a_{13} a_{21} a_{22}^2 a_{32}^2 a_{33} - 4a_{11}^2 a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} + 4a_{11}^2 a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
 & - 5a_{11}^2 a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 - a_{11}^2 a_{12}^2 a_{21} a_{22} a_{23} a_{31} a_{33}^2 + a_{11}^2 a_{12}^2 a_{21} a_{23}^2 a_{31} a_{32} a_{33} \\
 & - a_{11}^2 a_{12} a_{13} a_{21}^2 a_{22} a_{32}^2 a_{33} + a_{11}^2 a_{12} a_{13} a_{21}^2 a_{23} a_{32}^2 a_{33} - 2a_{11}^2 a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 \\
 & + 2a_{11}^2 a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2 - a_{11}^2 a_{12} a_{13} a_{22}^2 a_{23} a_{31}^2 a_{33} + a_{11}^2 a_{12} a_{13} a_{22} a_{23}^2 a_{31}^2 a_{32} \\
 & - a_{11}^2 a_{13}^2 a_{21} a_{22}^2 a_{31} a_{32} a_{33} + a_{11}^2 a_{13}^2 a_{21} a_{22} a_{23} a_{31} a_{32}^2 + a_{12}^2 a_{13}^2 a_{21}^2 a_{23} a_{31} a_{32} a_{33} \\
 & - a_{12}^2 a_{13}^2 a_{21} a_{22}^2 a_{23} a_{31}^2 a_{33} - a_{12}^2 a_{13}^2 a_{21}^2 a_{22} a_{31} a_{32} a_{33} + a_{12}^2 a_{13}^2 a_{21} a_{22} a_{23} a_{31}^2 a_{32})
 \end{aligned}$$

## Theorem

The top left entry  $x = s_{11}$  satisfies  $b_6 x^6 + \cdots + b_1 x + b_0 = 0$ , where...

$$\begin{aligned} b_2 &= a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\ &\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\ &\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32}^2 a_{33} + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\ &\quad + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\ b_1 &= a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\ &\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32}^2 a_{33} - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\ b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32}) \end{aligned}$$



# Theorem

The top left entry  $x = s_{11}$  satisfies  $b_6 x^6 + \cdots + b_1 x + b_0 = 0$ , where...

$$\begin{aligned}
 b_2 &= a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
 &\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
 &\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32}^2 a_{33} + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
 &\quad + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\
 b_1 &= a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
 &\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32}^2 a_{33} - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
 b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
 \end{aligned}$$

Better formulation?

# Theorem

The top left entry  $x = s_{11}$  satisfies  $b_6 x^6 + \dots + b_1 x + b_0 = 0$ , where...

$$\begin{aligned}
 b_2 &= a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
 &\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
 &\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
 &\quad + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\
 b_1 &= a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
 &\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
 b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
 \end{aligned}$$

Better formulation?

$$\begin{aligned}
 b_6 &= (a_{11} a_{22} - a_{12} a_{21}) (a_{11} a_{23} - a_{13} a_{21}) (a_{11} a_{32} - a_{12} a_{31}) (a_{11} a_{33} - a_{13} a_{31}) \\
 &\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31})
 \end{aligned}$$

# Theorem

The top left entry  $x = s_{11}$  satisfies  $b_6 x^6 + \dots + b_1 x + b_0 = 0$ , where...

$$\begin{aligned}
 b_2 &= a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22}^2 a_{23}^2 a_{32} a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
 &\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22}^2 a_{23}^2 a_{31} a_{32} a_{33} \\
 &\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
 &\quad + 5a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{32}^2) \\
 b_1 &= a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22}^2 a_{23}^2 a_{32} a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
 &\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22}^2 a_{23} a_{31} a_{32}^2) \\
 b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
 \end{aligned}$$

Better formulation?

$$\begin{aligned}
 b_6 &= (a_{11} a_{22} - a_{12} a_{21}) (a_{11} a_{23} - a_{13} a_{21}) (a_{11} a_{32} - a_{12} a_{31}) (a_{11} a_{33} - a_{13} a_{31}) \\
 &\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
 &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
 \end{aligned}$$

$b_6$  is the product of 5 minors

# Theorem

The top left entry  $x = s_{11}$  satisfies  $b_6 x^6 + \cdots + b_1 x + b_0 = 0$ , where...

$$\begin{aligned}
 b_2 &= a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22}^2 a_{23}^2 a_{32} a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
 &\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
 &\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
 &\quad + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{32}^2) \\
 b_1 &= a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
 &\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
 b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
 \end{aligned}$$

Better formulation?

$$\begin{aligned}
 b_6 &= (a_{11} a_{22} - a_{12} a_{21}) (a_{11} a_{23} - a_{13} a_{21}) (a_{11} a_{32} - a_{12} a_{31}) (a_{11} a_{33} - a_{13} a_{31}) \\
 &\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
 &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
 \end{aligned}$$

$b_6$  is the product of 5 minors involving  $a_{11}$ .

# Theorem

The top left entry  $x = s_{11}$  satisfies  $b_6 x^6 + \dots + b_1 x + b_0 = 0$ , where...

$$\begin{aligned}
 b_2 &= a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22}^2 a_{23}^2 a_{32} a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
 &\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
 &\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
 &\quad + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{32}^2) \\
 b_1 &= a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32} a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
 &\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
 b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
 \end{aligned}$$

Better formulation?

$$\begin{aligned}
 b_6 &= (a_{11} a_{22} - a_{12} a_{21}) (a_{11} a_{23} - a_{13} a_{21}) (a_{11} a_{32} - a_{12} a_{31}) (a_{11} a_{33} - a_{13} a_{31}) \\
 &\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
 &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
 \end{aligned}$$

$b_6$  is the product of 5 minors involving  $a_{11}$ .

$b_0$  is the product of 5 minors not involving  $a_{11}$  (and  $a_{11}^5$ ).

# Theorem

The top left entry  $x = s_{11}$  satisfies  $b_6 x^6 + \dots + b_1 x + b_0 = 0$ , where...

$$\begin{aligned}
 b_2 &= a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22}^2 a_{23}^2 a_{32} a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
 &\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
 &\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
 &\quad + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{32}^2) \\
 b_1 &= a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32} a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
 &\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
 b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
 \end{aligned}$$

Better formulation?

$$\begin{aligned}
 b_6 &= (a_{11} a_{22} - a_{12} a_{21}) (a_{11} a_{23} - a_{13} a_{21}) (a_{11} a_{32} - a_{12} a_{31}) (a_{11} a_{33} - a_{13} a_{31}) \\
 &\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
 &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
 \end{aligned}$$

$b_6$  is the product of 5 minors involving  $a_{11}$  and 0 not.

$b_0$  is the product of 5 minors not involving  $a_{11}$  (and  $a_{11}^5$ ).

## Theorem

The top left entry  $x = s_{11}$  satisfies  $b_6 x^6 + \cdots + b_1 x + b_0 = 0$ , where...

$$\begin{aligned}
 b_2 &= a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22}^2 a_{23}^2 a_{32} a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
 &\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
 &\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
 &\quad + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\
 b_1 &= a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
 &\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
 b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
 \end{aligned}$$

Better formulation?

$$\begin{aligned}
 b_6 &= (a_{11} a_{22} - a_{12} a_{21}) (a_{11} a_{23} - a_{13} a_{21}) (a_{11} a_{32} - a_{12} a_{31}) (a_{11} a_{33} - a_{13} a_{31}) \\
 &\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
 &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
 \end{aligned}$$

$b_6$  is the product of 5 minors involving  $a_{11}$  and 0 not.

$b_0$  is the product of 0 minors involving  $a_{11}$  and 5 not (and  $a_{11}^5$ ).

## Theorem

The top left entry  $x = s_{11}$  satisfies  $b_6 x^6 + \dots + b_1 x + b_0 = 0$ , where...

$$\begin{aligned}
 b_2 &= a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22}^2 a_{23}^2 a_{32} a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
 &\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22}^2 a_{23}^2 a_{31} a_{32} a_{33} \\
 &\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
 &\quad + 5a_{11} a_{13} a_{22}^2 a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{32}^2) \\
 b_1 &= a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22}^2 a_{23}^2 a_{32} a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
 &\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22}^2 a_{23} a_{31} a_{32}^2) \\
 b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
 \end{aligned}$$

Better formulation?

$$\begin{aligned}
 b_6 &= (a_{11} a_{22} - a_{12} a_{21}) (a_{11} a_{23} - a_{13} a_{21}) (a_{11} a_{32} - a_{12} a_{31}) (a_{11} a_{33} - a_{13} a_{31}) \\
 &\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
 &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
 \end{aligned}$$

Multiply each  $b_i$  by  $a_{11}$ .

$b_6$  is the product of 5 minors involving  $a_{11}$  and 0 not.

$b_0$  is the product of 0 minors involving  $a_{11}$  and 5 not (and  $a_{11}^5$ ).



## Theorem

The top left entry  $x = s_{11}$  satisfies  $b_6 x^6 + \cdots + b_1 x + b_0 = 0$ , where...

$$\begin{aligned}
 b_2 &= a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22}^2 a_{23}^2 a_{32} a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
 &\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32} a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
 &\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
 &\quad + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{32}^2) \\
 b_1 &= a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32} a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
 &\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
 b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
 \end{aligned}$$

Better formulation?

$$\begin{aligned}
 b_6 &= (a_{11} a_{22} - a_{12} a_{21}) (a_{11} a_{23} - a_{13} a_{21}) (a_{11} a_{32} - a_{12} a_{31}) (a_{11} a_{33} - a_{13} a_{31}) \\
 &\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
 &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
 \end{aligned}$$

Multiply each  $b_i$  by  $a_{11}$ .

$a_{11} b_6$  is the product of 6 minors involving  $a_{11}$  and 0 not.

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## Theorem

The top left entry  $x = s_{11}$  satisfies  $b_6 x^6 + \cdots + b_1 x + b_0 = 0$ , where...

$$\begin{aligned}
 b_2 &= a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22}^2 a_{23}^2 a_{32} a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
 &\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32} a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
 &\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
 &\quad + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{32}^2) \\
 b_1 &= a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32} a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
 &\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
 b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
 \end{aligned}$$

Better formulation?

$$\begin{aligned}
 b_6 &= (a_{11} a_{22} - a_{12} a_{21}) (a_{11} a_{23} - a_{13} a_{21}) (a_{11} a_{32} - a_{12} a_{31}) (a_{11} a_{33} - a_{13} a_{31}) \\
 &\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
 &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
 \end{aligned}$$

Multiply each  $b_i$  by  $a_{11}$ .

$a_{11} b_6$  is the product of 6 minors involving  $a_{11}$  and 0 not.

$a_{11} b_0$  is the product of 0 minors involving  $a_{11}$  and 6 not (and  $a_{11}^6$ ).

# Theorem

The top left entry  $x = s_{11}$  satisfies  $b_6 x^6 + \dots + b_1 x + b_0 = 0$ , where...

$$\begin{aligned} b_2 &= a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22}^2 a_{23}^2 a_{32} a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\ &\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32} a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\ &\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\ &\quad + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{32}^2) \\ b_1 &= a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32} a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\ &\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\ b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32}) \end{aligned}$$

Better formulation?

$$\begin{aligned} b_6 &= (a_{11} a_{22} - a_{12} a_{21}) (a_{11} a_{23} - a_{13} a_{21}) (a_{11} a_{32} - a_{12} a_{31}) (a_{11} a_{33} - a_{13} a_{31}) \\ &\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\ &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \end{aligned}$$

Multiply each  $b_i$  by  $a_{11}$ .

$a_{11} b_6$  is the product of 6 minors involving  $a_{11}$  and 0 not.

$a_{11} b_0$  is the product of 0 minors involving  $a_{11}$  and 6 not (and  $a_{11}^6$ ).

$a_{11} b_i$  involves products of  $i$  minors involving  $a_{11}$  and  $6 - i$  not?

Let  $R \subseteq \{2, 3\}$  and  $C \subseteq \{2, 3\}$  with  $|R| = |C|$ . Define

$$\Delta\binom{R}{C} = \det A_{\{1\} \cup R, \{1\} \cup C}$$

$$\Gamma\binom{R}{C} = a_{11} \det A_{R,C}$$

$$M(S) = \prod_{(R,C) \in S} \Delta\binom{R}{C} \cdot \prod_{(R,C) \notin S} \Gamma\binom{R}{C}$$

Coefficients:

$$\begin{aligned} a_{11}b_6 = & a_{11}(a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{23} - a_{13}a_{21})(a_{11}a_{32} - a_{12}a_{31})(a_{11}a_{33} - a_{13}a_{31}) \\ & \cdot (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \end{aligned}$$

Let  $R \subseteq \{2, 3\}$  and  $C \subseteq \{2, 3\}$  with  $|R| = |C|$ . Define

$$\Delta\binom{R}{C} = \det A_{\{1\} \cup R, \{1\} \cup C}$$

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Coefficients:

$$\begin{aligned} a_{11}b_6 &= a_{11}(a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{23} - a_{13}a_{21})(a_{11}a_{32} - a_{12}a_{31})(a_{11}a_{33} - a_{13}a_{31}) \\ &\quad \cdot (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\ &= \Delta\binom{\{1\}}{\{1\}} \Delta\binom{\{2\}}{\{2\}} \Delta\binom{\{2\}}{\{3\}} \Delta\binom{\{3\}}{\{2\}} \Delta\binom{\{3\}}{\{3\}} \Delta\binom{\{2,3\}}{\{2,3\}} \end{aligned}$$

Let  $R \subseteq \{2, 3\}$  and  $C \subseteq \{2, 3\}$  with  $|R| = |C|$ . Define

$$\Delta\left(\begin{smallmatrix} R \\ C \end{smallmatrix}\right) = \det A_{\{1\} \cup R, \{1\} \cup C}$$

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$$M(S) = \prod_{(R, C) \in S} \Delta\left(\begin{smallmatrix} R \\ C \end{smallmatrix}\right) \cdot \prod_{(R, C) \notin S} \Gamma\left(\begin{smallmatrix} R \\ C \end{smallmatrix}\right)$$

Coefficients:

$$\begin{aligned} a_{11}b_6 &= a_{11} (a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{23} - a_{13}a_{21})(a_{11}a_{32} - a_{12}a_{31})(a_{11}a_{33} - a_{13}a_{31}) \\ &\quad \cdot (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\ &= \Delta\left(\begin{smallmatrix} \{ \} \\ \{ \} \end{smallmatrix}\right) \Delta\left(\begin{smallmatrix} \{2\} \\ \{2\} \end{smallmatrix}\right) \Delta\left(\begin{smallmatrix} \{2\} \\ \{3\} \end{smallmatrix}\right) \Delta\left(\begin{smallmatrix} \{3\} \\ \{2\} \end{smallmatrix}\right) \Delta\left(\begin{smallmatrix} \{3\} \\ \{3\} \end{smallmatrix}\right) \Delta\left(\begin{smallmatrix} \{2,3\} \\ \{2,3\} \end{smallmatrix}\right) \end{aligned}$$

$$a_{11}b_0 = a_{11}^6 a_{22}a_{23}a_{32}a_{33} (a_{22}a_{33} - a_{23}a_{32})$$

Let  $R \subseteq \{2, 3\}$  and  $C \subseteq \{2, 3\}$  with  $|R| = |C|$ . Define

$$\Delta\left(\begin{smallmatrix} R \\ C \end{smallmatrix}\right) = \det A_{\{1\} \cup R, \{1\} \cup C}$$

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$$M(S) = \prod_{(R,C) \in S} \Delta\left(\begin{smallmatrix} R \\ C \end{smallmatrix}\right) \cdot \prod_{(R,C) \notin S} \Gamma\left(\begin{smallmatrix} R \\ C \end{smallmatrix}\right)$$

Coefficients:

$$\begin{aligned} a_{11}b_6 &= a_{11}(a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{23} - a_{13}a_{21})(a_{11}a_{32} - a_{12}a_{31})(a_{11}a_{33} - a_{13}a_{31}) \\ &\quad \cdot (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\ &= \Delta\left(\begin{smallmatrix} \{ \} \\ \{ \} \end{smallmatrix}\right) \Delta\left(\begin{smallmatrix} \{2\} \\ \{2\} \end{smallmatrix}\right) \Delta\left(\begin{smallmatrix} \{2\} \\ \{3\} \end{smallmatrix}\right) \Delta\left(\begin{smallmatrix} \{3\} \\ \{2\} \end{smallmatrix}\right) \Delta\left(\begin{smallmatrix} \{3\} \\ \{3\} \end{smallmatrix}\right) \Delta\left(\begin{smallmatrix} \{2,3\} \\ \{2,3\} \end{smallmatrix}\right) \end{aligned}$$

$$\begin{aligned} a_{11}b_0 &= a_{11}^6 a_{22}a_{23}a_{32}a_{33}(a_{22}a_{33} - a_{23}a_{32}) \\ &= \Gamma\left(\begin{smallmatrix} \{ \} \\ \{ \} \end{smallmatrix}\right) \Gamma\left(\begin{smallmatrix} \{2\} \\ \{2\} \end{smallmatrix}\right) \Gamma\left(\begin{smallmatrix} \{2\} \\ \{3\} \end{smallmatrix}\right) \Gamma\left(\begin{smallmatrix} \{3\} \\ \{2\} \end{smallmatrix}\right) \Gamma\left(\begin{smallmatrix} \{3\} \\ \{3\} \end{smallmatrix}\right) \Gamma\left(\begin{smallmatrix} \{2,3\} \\ \{2,3\} \end{smallmatrix}\right) \end{aligned}$$

Let  $R \subseteq \{2, 3\}$  and  $C \subseteq \{2, 3\}$  with  $|R| = |C|$ . Define

$$\Delta\left(\begin{smallmatrix} R \\ C \end{smallmatrix}\right) = \det A_{\{1\} \cup R, \{1\} \cup C}$$

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$$M(S) = \prod_{(R, C) \in S} \Delta\left(\begin{smallmatrix} R \\ C \end{smallmatrix}\right) \cdot \prod_{(R, C) \notin S} \Gamma\left(\begin{smallmatrix} R \\ C \end{smallmatrix}\right)$$

Coefficients:

$$\begin{aligned} a_{11}b_6 &= a_{11} (a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{23} - a_{13}a_{21})(a_{11}a_{32} - a_{12}a_{31})(a_{11}a_{33} - a_{13}a_{31}) \\ &\quad \cdot (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\ &= \Delta\left(\begin{smallmatrix} \{ \} \\ \{ \} \end{smallmatrix}\right) \Delta\left(\begin{smallmatrix} \{2\} \\ \{ \} \end{smallmatrix}\right) \Delta\left(\begin{smallmatrix} \{2\} \\ \{3\} \end{smallmatrix}\right) \Delta\left(\begin{smallmatrix} \{3\} \\ \{2\} \end{smallmatrix}\right) \Delta\left(\begin{smallmatrix} \{3\} \\ \{3\} \end{smallmatrix}\right) \Delta\left(\begin{smallmatrix} \{2,3\} \\ \{2,3\} \end{smallmatrix}\right) \\ &= M\left(\begin{smallmatrix} \{ \} & \{2\} & \{2\} & \{3\} & \{3\} & \{2,3\} \\ \{ \} & \{2\} & \{3\} & \{2\} & \{3\} & \{2,3\} \end{smallmatrix}\right) \end{aligned}$$

$$\begin{aligned} a_{11}b_0 &= a_{11}^6 a_{22}a_{23}a_{32}a_{33} (a_{22}a_{33} - a_{23}a_{32}) \\ &= \Gamma\left(\begin{smallmatrix} \{ \} \\ \{ \} \end{smallmatrix}\right) \Gamma\left(\begin{smallmatrix} \{2\} \\ \{2\} \end{smallmatrix}\right) \Gamma\left(\begin{smallmatrix} \{2\} \\ \{3\} \end{smallmatrix}\right) \Gamma\left(\begin{smallmatrix} \{3\} \\ \{2\} \end{smallmatrix}\right) \Gamma\left(\begin{smallmatrix} \{3\} \\ \{3\} \end{smallmatrix}\right) \Gamma\left(\begin{smallmatrix} \{2,3\} \\ \{2,3\} \end{smallmatrix}\right) \end{aligned}$$



Let  $R \subseteq \{2, 3\}$  and  $C \subseteq \{2, 3\}$  with  $|R| = |C|$ . Define

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Coefficients:

$$\begin{aligned} a_{11}b_6 &= a_{11} (a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{23} - a_{13}a_{21})(a_{11}a_{32} - a_{12}a_{31})(a_{11}a_{33} - a_{13}a_{31}) \\ &\quad \cdot (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\ &= \Delta\left(\begin{smallmatrix} \{ \} \\ \{ \} \end{smallmatrix}\right) \Delta\left(\begin{smallmatrix} \{2\} \\ \{2\} \end{smallmatrix}\right) \Delta\left(\begin{smallmatrix} \{2\} \\ \{3\} \end{smallmatrix}\right) \Delta\left(\begin{smallmatrix} \{3\} \\ \{2\} \end{smallmatrix}\right) \Delta\left(\begin{smallmatrix} \{3\} \\ \{3\} \end{smallmatrix}\right) \Delta\left(\begin{smallmatrix} \{2,3\} \\ \{2,3\} \end{smallmatrix}\right) \\ &= M\left(\begin{smallmatrix} \{ \} & \{2\} & \{2\} & \{3\} & \{3\} & \{2,3\} \\ \{ \} & \{2\} & \{3\} & \{2\} & \{3\} & \{2,3\} \end{smallmatrix}\right) \end{aligned}$$

$$\begin{aligned} a_{11}b_0 &= a_{11}^6 a_{22}a_{23}a_{32}a_{33} (a_{22}a_{33} - a_{23}a_{32}) \\ &= \Gamma\left(\begin{smallmatrix} \{ \} \\ \{ \} \end{smallmatrix}\right) \Gamma\left(\begin{smallmatrix} \{2\} \\ \{2\} \end{smallmatrix}\right) \Gamma\left(\begin{smallmatrix} \{2\} \\ \{3\} \end{smallmatrix}\right) \Gamma\left(\begin{smallmatrix} \{3\} \\ \{2\} \end{smallmatrix}\right) \Gamma\left(\begin{smallmatrix} \{3\} \\ \{3\} \end{smallmatrix}\right) \Gamma\left(\begin{smallmatrix} \{2,3\} \\ \{2,3\} \end{smallmatrix}\right) \\ &= M\left(\begin{smallmatrix} \end{smallmatrix}\right) \end{aligned}$$

Let  $R \subseteq \{2, 3\}$  and  $C \subseteq \{2, 3\}$  with  $|R| = |C|$ . Define

$$\Delta\binom{R}{C} = \det A_{\{1\} \cup R, \{1\} \cup C}$$

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$$M(S) = \prod_{(R,C) \in S} \Delta\binom{R}{C} \cdot \prod_{(R,C) \notin S} \Gamma\binom{R}{C}$$

Coefficients:

$$\begin{aligned} a_{11}b_6 &= a_{11}(a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{23} - a_{13}a_{21})(a_{11}a_{32} - a_{12}a_{31})(a_{11}a_{33} - a_{13}a_{31}) \\ &\quad \cdot (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\ &= \Delta\binom{\{1\}}{\{1\}} \Delta\binom{\{2\}}{\{2\}} \Delta\binom{\{2\}}{\{3\}} \Delta\binom{\{3\}}{\{2\}} \Delta\binom{\{3\}}{\{3\}} \Delta\binom{\{2,3\}}{\{2,3\}} \\ &= M\binom{\{1\} \{2\} \{2\} \{3\} \{3\} \{2,3\}}{\{1\} \{2\} \{2\} \{3\} \{2\} \{3\} \{2,3\}} \end{aligned}$$

$$\begin{aligned} a_{11}b_0 &= a_{11}^6 a_{22}a_{23}a_{32}a_{33}(a_{22}a_{33} - a_{23}a_{32}) \\ &= \Gamma\binom{\{1\}}{\{1\}} \Gamma\binom{\{2\}}{\{2\}} \Gamma\binom{\{2\}}{\{3\}} \Gamma\binom{\{3\}}{\{2\}} \Gamma\binom{\{3\}}{\{3\}} \Gamma\binom{\{2,3\}}{\{2,3\}} \\ &= M\binom{\{1\}}{\{1\}} \end{aligned}$$

$$\begin{aligned} a_{11}b_1 &= a_{11}^5 (-6a_{11}a_{22}^2a_{23}a_{32}a_{33}^2 + 6a_{11}a_{22}a_{23}^2a_{32}^2a_{33} - a_{12}a_{21}a_{23}^2a_{32}^2a_{33} \\ &\quad + a_{12}a_{22}^2a_{23}a_{31}a_{33}^2 + a_{13}a_{21}a_{22}^2a_{32}a_{33}^2 - a_{13}a_{22}a_{23}^2a_{31}a_{32}^2) \end{aligned}$$

Let  $R \subseteq \{2, 3\}$  and  $C \subseteq \{2, 3\}$  with  $|R| = |C|$ . Define

$$\Delta\binom{R}{C} = \det A_{\{1\} \cup R, \{1\} \cup C}$$

$$\Gamma\binom{R}{C} = a_{11} \det A_{R,C}$$

$$M(S) = \prod_{(R,C) \in S} \Delta\binom{R}{C} \cdot \prod_{(R,C) \notin S} \Gamma\binom{R}{C}$$

Coefficients:

$$\begin{aligned} a_{11}b_6 &= a_{11}(a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{23} - a_{13}a_{21})(a_{11}a_{32} - a_{12}a_{31})(a_{11}a_{33} - a_{13}a_{31}) \\ &\quad \cdot (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\ &= \Delta\binom{\{1\}}{\{1\}} \Delta\binom{\{2\}}{\{2\}} \Delta\binom{\{2\}}{\{3\}} \Delta\binom{\{3\}}{\{2\}} \Delta\binom{\{3\}}{\{3\}} \Delta\binom{\{2,3\}}{\{2,3\}} \\ &= M\left(\begin{array}{ccc} \{1\} & \{2\} & \{2\} \\ \{1\} & \{2\} & \{3\} \end{array}\right) \end{aligned}$$

$$\begin{aligned} a_{11}b_0 &= a_{11}^6 a_{22}a_{23}a_{32}a_{33}(a_{22}a_{33} - a_{23}a_{32}) \\ &= \Gamma\binom{\{1\}}{\{1\}} \Gamma\binom{\{2\}}{\{2\}} \Gamma\binom{\{2\}}{\{3\}} \Gamma\binom{\{3\}}{\{2\}} \Gamma\binom{\{3\}}{\{3\}} \Gamma\binom{\{2,3\}}{\{2,3\}} \\ &= M\left(\begin{array}{ccc} \{1\} & \{2\} & \{2\} \\ \{1\} & \{2\} & \{3\} \end{array}\right) \end{aligned}$$

$$\begin{aligned} a_{11}b_1 &= a_{11}^5(-6a_{11}a_{22}^2a_{23}a_{32}a_{33}^2 + 6a_{11}a_{22}a_{23}^2a_{32}^2a_{33} - a_{12}a_{21}a_{23}^2a_{32}^2a_{33} \\ &\quad + a_{12}a_{22}^2a_{23}a_{31}a_{33}^2 + a_{13}a_{21}a_{22}^2a_{32}^2a_{33} - a_{13}a_{22}a_{23}^2a_{31}a_{32}^2) \\ &= -3M\left(\begin{array}{ccc} \{1\} & \{2\} & \{2\} \\ \{1\} & \{2\} & \{3\} \end{array}\right) - M\left(\begin{array}{ccc} \{2\} & \{2\} & \{2\} \\ \{2\} & \{2\} & \{3\} \end{array}\right) - M\left(\begin{array}{ccc} \{3\} & \{3\} & \{3\} \\ \{3\} & \{3\} & \{2\} \end{array}\right) - M\left(\begin{array}{ccc} \{3\} & \{3\} & \{2\} \\ \{3\} & \{3\} & \{3\} \end{array}\right) + M\left(\begin{array}{ccc} \{2,3\} & \{2,3\} & \{2,3\} \\ \{2,3\} & \{2,3\} & \{2,3\} \end{array}\right) \end{aligned}$$

Let  $R \subseteq \{2, 3\}$  and  $C \subseteq \{2, 3\}$  with  $|R| = |C|$ . Define

$$\Delta\binom{R}{C} = \det A_{\{1\} \cup R, \{1\} \cup C}$$

$$\Gamma\binom{R}{C} = a_{11} \det A_{R,C}$$

$$M(S) = \prod_{(R,C) \in S} \Delta\binom{R}{C} \cdot \prod_{(R,C) \notin S} \Gamma\binom{R}{C}$$

Coefficients:

$$\begin{aligned} a_{11}b_6 &= a_{11}(a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{23} - a_{13}a_{21})(a_{11}a_{32} - a_{12}a_{31})(a_{11}a_{33} - a_{13}a_{31}) \\ &\quad \cdot (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\ &= \Delta\binom{\{\}}{\{\}} \Delta\binom{\{2\}}{\{2\}} \Delta\binom{\{2\}}{\{3\}} \Delta\binom{\{3\}}{\{2\}} \Delta\binom{\{3\}}{\{3\}} \Delta\binom{\{2,3\}}{\{2,3\}} \\ &= M\binom{\{\} \{2\} \{2\} \{3\} \{3\} \{2,3\}}{\{\} \{2\} \{2\} \{3\} \{2\} \{3\} \{2,3\}} \end{aligned}$$

$$\begin{aligned} a_{11}b_0 &= a_{11}^6 a_{22}a_{23}a_{32}a_{33}(a_{22}a_{33} - a_{23}a_{32}) \\ &= \Gamma\binom{\{\}}{\{\}} \Gamma\binom{\{2\}}{\{2\}} \Gamma\binom{\{2\}}{\{3\}} \Gamma\binom{\{3\}}{\{2\}} \Gamma\binom{\{3\}}{\{3\}} \Gamma\binom{\{2,3\}}{\{2,3\}} \\ &= M\binom{\{\}}{\{\}} \end{aligned}$$

$$\begin{aligned} a_{11}b_1 &= a_{11}^5 (-6a_{11}a_{22}^2a_{23}a_{32}a_{33}^2 + 6a_{11}a_{22}a_{23}^2a_{32}^2a_{33} - a_{12}a_{21}a_{23}^2a_{32}^2a_{33} \\ &\quad + a_{12}a_{22}^2a_{23}a_{31}a_{33}^2 + a_{13}a_{21}a_{22}^2a_{32}^2a_{33} - a_{13}a_{22}a_{23}^2a_{31}a_{32}^2) \\ &= -3M\binom{\{\}}{\{\}} - M\binom{\{2\}}{\{2\}} - M\binom{\{2\}}{\{3\}} - M\binom{\{3\}}{\{2\}} - M\binom{\{3\}}{\{3\}} + M\binom{\{2,3\}}{\{2,3\}} \\ &= -3\Sigma\binom{\{\}}{\{\}} - \Sigma\binom{\{2\}}{\{2\}} + \Sigma\binom{\{2,3\}}{\{2,3\}} \end{aligned}$$

$$\Sigma(S) = \sum_{T \equiv S} M(T)$$

## Theorem (Rowland–Wu 2024)

Let  $A$  be a  $3 \times 3$  matrix with positive entries.

The top left entry  $x$  of  $\text{Sink}(A)$  satisfies  $b_6 x^6 + \cdots + b_1 x + b_0 = 0$ , where

$$a_{11} b_6 = \Sigma\left(\begin{smallmatrix} \{\} & \{2\} & \{2\} & \{3\} & \{3\} & \{2,3\} \\ \{\} & \{2\} & \{3\} & \{2\} & \{3\} & \{2,3\} \end{smallmatrix}\right)$$

$$a_{11} b_5 = -3\Sigma\left(\begin{smallmatrix} \{\} & \{2\} & \{2\} & \{3\} & \{3\} \\ \{\} & \{2\} & \{3\} & \{2\} & \{3\} \end{smallmatrix}\right) - \Sigma\left(\begin{smallmatrix} \{\} & \{2\} & \{2\} & \{3\} & \{2,3\} \\ \{\} & \{2\} & \{3\} & \{2\} & \{2,3\} \end{smallmatrix}\right) + \Sigma\left(\begin{smallmatrix} \{2\} & \{2\} & \{3\} & \{3\} & \{2,3\} \\ \{2\} & \{3\} & \{2\} & \{3\} & \{2,3\} \end{smallmatrix}\right)$$

$$a_{11} b_4 = 4\Sigma\left(\begin{smallmatrix} \{\} & \{2\} & \{2\} & \{3\} \\ \{\} & \{2\} & \{3\} & \{2\} \end{smallmatrix}\right) + \Sigma\left(\begin{smallmatrix} \{\} & \{2\} & \{3\} & \{2,3\} \\ \{\} & \{2\} & \{3\} & \{2,3\} \end{smallmatrix}\right) - 3\Sigma\left(\begin{smallmatrix} \{2\} & \{2\} & \{3\} & \{3\} \\ \{2\} & \{3\} & \{2\} & \{3\} \end{smallmatrix}\right)$$

$$a_{11} b_3 = -4\Sigma\left(\begin{smallmatrix} \{\} & \{2\} & \{2\} \\ \{\} & \{2\} & \{3\} \end{smallmatrix}\right) - 5\Sigma\left(\begin{smallmatrix} \{\} & \{2\} & \{3\} \\ \{\} & \{2\} & \{3\} \end{smallmatrix}\right) + \Sigma\left(\begin{smallmatrix} \{\} & \{2\} & \{2,3\} \\ \{\} & \{2\} & \{2,3\} \end{smallmatrix}\right) + \Sigma\left(\begin{smallmatrix} \{2\} & \{2\} & \{3\} \\ \{2\} & \{3\} & \{2\} \end{smallmatrix}\right) - \Sigma\left(\begin{smallmatrix} \{2\} & \{3\} & \{2,3\} \\ \{2\} & \{3\} & \{2,3\} \end{smallmatrix}\right)$$

$$a_{11} b_2 = 4\Sigma\left(\begin{smallmatrix} \{\} & \{2\} \\ \{\} & \{2\} \end{smallmatrix}\right) - 3\Sigma\left(\begin{smallmatrix} \{\} & \{2,3\} \\ \{\} & \{2,3\} \end{smallmatrix}\right) + \Sigma\left(\begin{smallmatrix} \{2\} & \{3\} \\ \{2\} & \{3\} \end{smallmatrix}\right)$$

$$a_{11} b_1 = -3\Sigma\left(\begin{smallmatrix} \{\} \\ \{\} \end{smallmatrix}\right) - \Sigma\left(\begin{smallmatrix} \{2\} \\ \{2\} \end{smallmatrix}\right) + \Sigma\left(\begin{smallmatrix} \{2,3\} \\ \{2,3\} \end{smallmatrix}\right)$$

$$a_{11} b_0 = \Sigma\left(\begin{smallmatrix} \{\} \end{smallmatrix}\right).$$

Why these particular integer coefficients? (Why are they symmetric?)

Why degree **6**?

$1 + 4 + 1 = 6$  relevant minors.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

For  $n \times n$  matrices:  $\sum_{j=0}^{n-1} \binom{n-1}{j}^2 = \binom{2n-2}{n-1}$  relevant minors.

## Conjecture

*For  $n \times n$  matrices  $A$ , the entries of  $\text{Sink}(A)$  have degree at most  $\binom{2n-2}{n-1}$ .*

$$1 \times 1: \quad \text{degree} \binom{0}{0} = 1 \quad x - 1 = 0$$

$$2 \times 2: \quad \text{degree} \binom{2}{1} = 2 \quad (ad - bc)x^2 - 2adx + ad = 0$$

$$3 \times 3: \quad \text{degree} \binom{4}{2} = 6 \quad (\text{to reappear soon})$$

$$4 \times 4: \quad \text{degree} \binom{6}{3} = 20 \quad \text{Gröbner basis computation is infeasible.}$$

$$5 \times 5: \quad \text{degree} \binom{8}{4} = 70$$

Central binomial coefficients.

What are the integer coefficients?

Interpolate from examples instead.

We have an explicit polynomial for  $4 \times 4$  matrices.

Generalization:

### Definition

Let  $A$  be an  $m \times n$  matrix with positive entries.  
The *Sinkhorn limit* of  $A$  is obtained by iteratively scaling so that each row sum is  $1$  and each column sum is  $\frac{m}{n}$ .

Existence (in a more general form): Sinkhorn 1967.

1.5 CPU years scaling matrices and recognizing 102K algebraic numbers  
let us solve for 63K coefficients (and 56K parameterized by free variables).

## Theorem (Rowland–Wu 2024)

Let  $A$  be a  $3 \times 3$  matrix with positive entries.

The top left entry  $x$  of  $\text{Sink}(\dot{A})$  satisfies  $b_6x^6 + \cdots + b_1x + b_0 = 0$ , where

$$a_{11}b_6 = \Sigma \left( \begin{pmatrix} \{\} & \{2\} & \{2\} & \{3\} & \{3\} & \{2,3\} \\ \{\} & \{2\} & \{3\} & \{2\} & \{3\} & \{2,3\} \end{pmatrix} \right)$$

$$a_{11}b_5 = -3\Sigma\left(\begin{array}{ccccc}\{\} & \{2\} & \{2\} & \{3\} & \{3\} \\ \{\} & \{2\} & \{3\} & \{2\} & \{3\}\end{array}\right) - \Sigma\left(\begin{array}{ccccc}\{\} & \{2\} & \{2\} & \{3\} & \{2,3\} \\ \{\} & \{2\} & \{3\} & \{2\} & \{2,3\}\end{array}\right) + \Sigma\left(\begin{array}{ccccc}\{2\} & \{2\} & \{3\} & \{3\} & \{2,3\} \\ \{2\} & \{3\} & \{2\} & \{3\} & \{2,3\}\end{array}\right)$$

$$a_{11}b_4 = 4\Sigma\left(\begin{pmatrix} \{1\} & \{2\} & \{2\} & \{3\} \\ \{1\} & \{2\} & \{3\} & \{2\} \end{pmatrix}\right) + \Sigma\left(\begin{pmatrix} \{1\} & \{2\} & \{3\} & \{2,3\} \\ \{1\} & \{2\} & \{3\} & \{2,3\} \end{pmatrix}\right) - 3\Sigma\left(\begin{pmatrix} \{2\} & \{2\} & \{3\} & \{3\} \\ \{2\} & \{3\} & \{2\} & \{3\} \end{pmatrix}\right)$$

$$a_{11}b_3 = -4\Sigma\left(\begin{smallmatrix}\{\} & \{2\} & \{2\} \\ \{\} & \{2\} & \{3\}\end{smallmatrix}\right) - 5\Sigma\left(\begin{smallmatrix}\{\} & \{2\} & \{3\} \\ \{\} & \{2\} & \{3\}\end{smallmatrix}\right) + \Sigma\left(\begin{smallmatrix}\{\} & \{2\} & \{2,3\} \\ \{\} & \{2\} & \{2,3\}\end{smallmatrix}\right) + \Sigma\left(\begin{smallmatrix}\{2\} & \{2\} & \{3\} \\ \{2\} & \{3\} & \{2\}\end{smallmatrix}\right) - \Sigma\left(\begin{smallmatrix}\{2\} & \{3\} & \{2,3\} \\ \{2\} & \{3\} & \{2,3\}\end{smallmatrix}\right)$$

$$a_{11}b_2 = 4\Sigma\left(\begin{smallmatrix}\{\} & \{2\}\\ \{\} & \{2\}\end{smallmatrix}\right) - 3\Sigma\left(\begin{smallmatrix}\{\} & \{2,3\}\\ \{\} & \{2,3\}\end{smallmatrix}\right) + \Sigma\left(\begin{smallmatrix}\{2\} & \{3\}\\ \{2\} & \{3\}\end{smallmatrix}\right)$$

$$a_{11}b_1 = -3\Sigma\left(\begin{smallmatrix}\{\}\\ \{\}\end{smallmatrix}\right) - \Sigma\left(\begin{smallmatrix}\{2\}\\ \{2\}\end{smallmatrix}\right) + \Sigma\left(\begin{smallmatrix}\{2,3\}\\ \{2,3\}\end{smallmatrix}\right)$$

$$a_{11} b_0 = \Sigma \left( \right) .$$

The coefficients seem to be determinants of adjacency-like matrices.

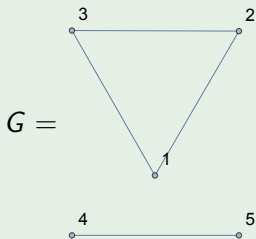


## Recall

The **adjacency matrix** of an  $n$ -vertex graph is the  $n \times n$  matrix with entries

$$a_{ij} = \begin{cases} 1 & \text{if vertices } i, j \text{ are connected by an edge} \\ 0 & \text{if not.} \end{cases}$$

## Example



$$\text{adj}(G) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Connected components:

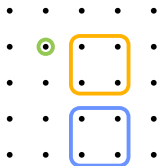
$$\det \text{adj}(G_1 + G_2) = \det \text{adj}(G_1) \cdot \det \text{adj}(G_2)$$

$\text{adj}(S)$  is a  $|S| \times |S|$  matrix.

Underlying graph: Vertex set  $S$ . What are the edges/links?

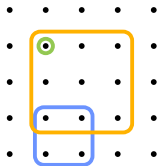
Unlinked minor specs (that nonetheless involve common rows/columns):

$$S = \begin{matrix} \{2\} & \{2,3\} & \{4,5\} \\ \{2\} & \{3,4\} & \{3,4\} \end{matrix}$$



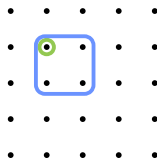
Unlinked minor specs (that nonetheless involve common entries):

$$S = \begin{matrix} \{2\} & \{4,5\} & \{2,3,4\} \\ \{2\} & \{2,3\} & \{2,3,4\} \end{matrix}$$

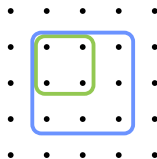


Type-1 links: Sizes differ by 1, and one is a subset of the other.

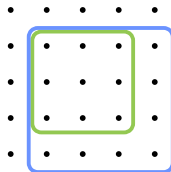
$$S = \begin{matrix} \{2\} & \{2,3\} \\ \{2\} & \{2,3\} \end{matrix}$$



$$S = \begin{matrix} \{2,3\} & \{2,3,4\} \\ \{2,3\} & \{2,3,4\} \end{matrix}$$

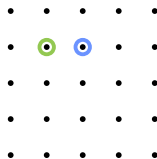


$$S = \begin{matrix} \{2,3,4\} & \{2,3,4,5\} \\ \{2,3,4\} & \{2,3,4,5\} \end{matrix}$$

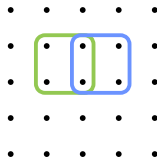


Type-2 links: Same sizes, and they differ in exactly 1 row or 1 column.

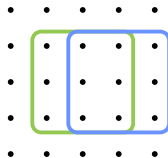
$$S = \begin{matrix} \{2\} & \{2\} \\ \{2\} & \{3\} \end{matrix}$$



$$S = \begin{matrix} \{2,3\} & \{2,3\} \\ \{2,3\} & \{3,4\} \end{matrix}$$



$$S = \begin{matrix} \{2,3,4\} & \{2,3,4\} \\ \{2,3,4\} & \{3,4,5\} \end{matrix}$$



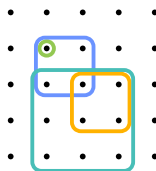
Type-1 links: Sizes differ by 1, and one is a subset of the other.

Type-2 links: Same sizes, and they differ in exactly 1 row or 1 column.

Connected components are built from these.

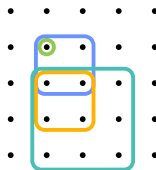
Two components:

$$S = \begin{array}{cccc} \{2\} & \{2,3\} & \{3,4\} & \{3,4,5\} \\ \{2\} & \{2,3\} & \{3,4\} & \{2,3,4\} \end{array}$$



One component:

$$S = \begin{array}{cccc} \{2\} & \{2,3\} & \{3,4\} & \{3,4,5\} \\ \{2\} & \{2,3\} & \{2,3\} & \{2,3,4\} \end{array}$$



To define  $\text{adj}(S)$ , it suffices to define it for linked pairs and singletons.

$\text{adj}_S(m, n)$  is a  $|S| \times |S|$  matrix with entries that are linear in  $m, n$ .

## Definition

Let  $S = \begin{smallmatrix} R_1 & R_2 \\ C_1 & C_2 \end{smallmatrix}$ . If  $\begin{smallmatrix} R_1 \\ C_1 \end{smallmatrix}$  and  $\begin{smallmatrix} R_2 \\ C_2 \end{smallmatrix}$  form a ...

- **type-1 link** with  $|R_1| + 1 = |R_2|$ ,

$$\text{adj}_S(m, n) := \begin{bmatrix} |R_1|(m+n) - mn & m \\ -n & |R_2|(m+n) - mn \end{bmatrix}.$$

- **type-2 link** with  $R_1 = R_2$ ,

$$\text{adj}_S(m, n) := \begin{bmatrix} |R_1|(m+n) - mn & -m \\ -m & |R_2|(m+n) - mn \end{bmatrix}.$$

- **type-2 link** with  $C_1 = C_2$ ,

$$\text{adj}_S(m, n) := \begin{bmatrix} |R_1|(m+n) - mn & -n \\ -n & |R_2|(m+n) - mn \end{bmatrix}.$$

If  $\begin{smallmatrix} R_1 \\ C_1 \end{smallmatrix}$  and  $\begin{smallmatrix} R_2 \\ C_2 \end{smallmatrix}$  are **not linked**,

$$\text{adj}_S(m, n) := \begin{bmatrix} |R_1|(m+n) - mn & 0 \\ 0 & |R_2|(m+n) - mn \end{bmatrix}.$$

## Example

For  $S = \begin{matrix} \{2\} & \{3\} & \{2,3\} \\ \{2\} & \{3\} & \{2,3\} \end{matrix}$ ,

$$\text{adj}_S(m, n) = \begin{bmatrix} m + n - mn & 0 & m \\ 0 & m + n - mn & m \\ -n & -n & 2m + 2n - mn \end{bmatrix}.$$

This agrees with values we computed numerically.

## Example

For  $S = \begin{matrix} \{2,3\} & \{2,3\} & \{2,3\} \\ \{2,3\} & \{2,4\} & \{2,5\} \end{matrix}$ ,

$$\text{adj}_S(m, n) = \begin{bmatrix} 2m + 2n - mn & -m & -m \\ -m & 2m + 2n - mn & -m \\ -m & -m & 2m + 2n - mn \end{bmatrix}.$$

This **does not** agree with values we computed. A sign change is required:

$$\begin{bmatrix} 2m + 2n - mn & m & m \\ m & 2m + 2n - mn & m \\ m & m & 2m + 2n - mn \end{bmatrix}$$

# Summary

Each entry of an  $m \times n$  Sinkhorn limit is algebraic with degree  $\leq \binom{m+n-2}{m-1}$  (the number of minor specifications not involving the first row/column).






The polynomial describing an entry is a linear combination of  $M(S)x^{|S|}$  where  $S$  ranges over the subsets of minor specifications.

The coefficient of  $M(S)x^{|S|}$  is the determinant of an adjacency-like matrix. We don't know the signs of the off-diagonal entries.

All of this is conjectural.



# References

-  Shalosh B. Ekhad and Doron Zeilberger, Answers to some questions about explicit Sinkhorn limits posed by Mel Nathanson, <https://arxiv.org/abs/1902.10783> (6 pages).
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-  Melvyn B. Nathanson, Alternate minimization and doubly stochastic matrices, *Integers* **20A** (2020) Article #A10 (17 pages).
-  Eric Rowland and Jason Wu, The entries of the Sinkhorn limit of an  $m \times n$  matrix, <https://arxiv.org/abs/2409.02789> (25 pages).
-  Richard Sinkhorn, A relationship between arbitrary positive matrices and doubly stochastic matrices, *The Annals of Mathematical Statistics* **35** (1964) 876–879.