Energetics of the single DOF oscillator

MG3416–Advanced Structural Acoustics - Lecture #1 part B

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Outline

1 SDOF oscillator (reminder)
- Notations
- Energetic quantities

2 Stationary excitation
- Stationary forced response
- Equipartition
- Stationary evolutionary response

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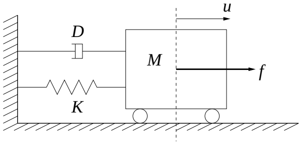

- The single degree-of-freedom (DOF) oscillator:

$$\begin{cases} M\ddot{u}(t) + D\dot{u}(t) + Ku(t) = f(t)\,, & t \in \mathbb{R}\,, \\ u(0) = u_0\,, \\ \dot{u}(0) = v_0\,; \end{cases}$$

  $M > 0$: mass, $D \geqslant 0$: damping, and $K > 0$: stiffness.

- Fundamental parameters:
  - $\omega_p = \sqrt{\frac{K}{M}}$: the undamped natural (angular) frequency,
  - $\xi_p = \frac{D}{2\sqrt{KM}}$: the critical damping rate,
  - $\eta_p = 2\xi_p$: the loss factor.

Single DOF oscillator
Free response

Definition

The free response $t \mapsto u^\ell(t)$ is the solution of:

$$\begin{cases} M\ddot{u}(t) + D\dot{u}(t) + Ku(t) = 0\,, \quad t \geqslant 0\,, \\ u(0) = u_0\,, \\ \dot{u}(0) = v_0\,. \end{cases}$$

- Let $\omega_D = \omega_p\sqrt{1-\xi_p^2}$, then:

$$u^\ell(t) = \mathrm{e}^{-\xi_p\omega_p t}\left(u_0\cos\omega_D t + \frac{\xi_p\omega_p u_0 + v_0}{\omega_D}\sin\omega_D t\right).$$

- **Property**: if $D > 0$, $\lim_{t\to+\infty} u^\ell(t) = \lim_{t\to+\infty} \dot{u}^\ell(t) = 0\,.$

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Definition

The forced response $t \mapsto u^f(t)$ is the solution of:

$$M\ddot{u}(t) + D\dot{u}(t) + Ku(t) = f(t)\,, \quad t \in \mathbb{R}\,.$$

- Then:

$$u^f(t) = \int_{-\infty}^{t} \mathbb{h}(t-\tau) f(\tau)\,\mathrm{d}\tau = \int_{0}^{+\infty} \mathbb{h}(\tau) f(t-\tau)\,\mathrm{d}\tau\,,$$

where $\mathbb{h} : \mathbb{R} \to \mathbb{R}$ is the impulse response function of the SDOF oscillator:

$$\mathbb{h}(t) = \mathbb{1}_{[0,+\infty[}(t) \times \frac{1}{M\omega_D}\,\mathrm{e}^{-\xi_p\omega_p t} \sin\omega_D t\,.$$

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- $t \mapsto \mathbb{h}(t)$ is integrable and square integrable on $\mathbb{R}$, and its Fourier transform is:

$$\widehat{\mathbb{h}}(\omega) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\omega t}\, \mathbb{h}(t)\, \mathrm{d}t = \frac{1}{M(\omega_p^2 - \omega^2 + 2\mathrm{i}\xi_p\omega_p\omega)}\,.$$

- $\omega \mapsto \widehat{\mathbb{h}}(\omega)$ is integrable and square integrable on $\mathbb{R}$, and its inverse Fourier transform is:

$$\mathbb{h}(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\omega t}\, \widehat{\mathbb{h}}(\omega)\, \mathrm{d}\omega\,.$$

- Usual quadratures:

$$\int_0^{+\infty} |\widehat{\mathbb{h}}(\omega)|^2\, \mathrm{d}\omega = \frac{\pi}{2DK}\,,$$

$$\int_0^{+\infty} \omega^2 |\widehat{\mathbb{h}}(\omega)|^2\, \mathrm{d}\omega = \frac{\pi}{2DM}\,.$$

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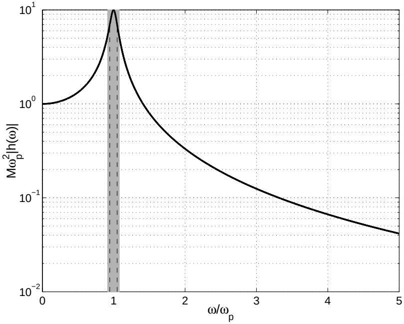
- The impedance $\omega \mapsto Z(\omega)$ of the SDOF oscillator:

$$\mathrm{i}\omega Z(\omega) = M(\omega_p^2 - \omega^2 + 2\mathrm{i}\xi_p\omega_p\omega)\,.$$



- $\omega'_p = \omega_p\sqrt{1-2\xi_p^2}$: the natural resonance frequency (for $0 < \xi_p < \frac{1}{\sqrt{2}}$);
- $b_\mathrm{e} = \pi\xi_p\omega_p(1-\xi_p^2)$: the equivalent bandwidth, s.t.

$$b_\mathrm{e}|\widehat{\mathbb{h}}(\omega'_p)|^2 = \int_0^{+\infty}|\widehat{\mathbb{h}}(\omega)|^2\,\mathrm{d}\omega\,;$$

- $b_p \simeq \pi\xi_p\omega_p = \frac{\pi}{2}\Delta_p$, where $\Delta_p = \eta_p\omega_p$ is the half-power bandwidth.

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Definition

The evolutionary response $t \mapsto u(t)$ is the solution of:

$$\begin{cases} M\ddot{u}(t) + D\dot{u}(t) + Ku(t) = f(t)\,, & t \geqslant 0\,, \\ u(0) = u_0\,, & \\ \dot{u}(0) = v_0\,. & \end{cases}$$

- Then:

$$u(t) = u_0\,\mathrm{e}^{-\xi_p\omega_p t}\left(\cos\omega_D t + \frac{\xi_p}{\sqrt{1-\xi_p^2}}\sin\omega_D t\right) + \frac{v_0}{\omega_D}\,\mathrm{e}^{-\xi_p\omega_p t}\sin\omega_D t + \int_0^t \mathbb{h}(t-\tau)f(\tau)\,\mathrm{d}\tau\,.$$

- **Property**: if $D > 0$, $\displaystyle\lim_{t\to+\infty}|u(t) - u^f(t)| = 0\,.$

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Definition

- *The kinetic energy:* $\mathcal{E}_\mathrm{c}(t) = \frac{1}{2}M\dot{u}(t)^2$,
- *The potential energy:* $\mathcal{E}_\mathrm{p}(t) = \frac{1}{2}Ku(t)^2$,
- *The mechanical energy:* $\mathcal{E}(t) = \mathcal{E}_\mathrm{c}(t) + \mathcal{E}_\mathrm{p}(t)$,
- *The dissipated power:* $\Pi_\mathrm{d}(t) = D\dot{u}(t)^2$,
- *The input power:* $\Pi_\mathrm{IN}(t) = f(t)\dot{u}(t)$.

- The instantaneous power balance reads:

$$\dot{\mathcal{E}}(t) = \Pi_\mathrm{IN}(t) - \Pi_\mathrm{d}(t)\,.$$

- It is subsequently specialized to the free, forced and evolutionary responses of the single DOF oscillator.

Energetic quantities
Free response

- The free response mechanical energy when $\xi_p \ll 1$:

$$\mathcal{E}^\ell(t) \simeq \mathcal{E}_0 \,\mathrm{e}^{-\eta_p \omega_p t}\,,$$

where $\mathcal{E}_0 = \frac{1}{2}Mv_0^2 + \frac{1}{2}Ku_0^2$.

- The power balance integrated between 0 and $t > 0$:

$$\mathcal{E}_0 = \mathcal{E}_\mathrm{d}^\ell(t) + \mathcal{E}^\ell(t)\,,$$

hence $\mathcal{E}_\mathrm{d}^\ell(\infty) = \mathcal{E}_0 = \int_0^{+\infty} \Pi_\mathrm{d}^\ell(t)\mathrm{d}t\,.$

Energetic quantities
Forced response

- **Data**: square integrable (finite energy) excitation with limited bandwidth,

$$|\widehat{f}(\omega)| \leqslant C \;\; \forall\omega \in \mathbb{R}\,; \quad \widehat{f}(\omega) = 0 \;\; \forall\omega \notin [-\omega_f, \omega_f]\,.$$

- Then $\lim_{|t|\to+\infty} f(t) = 0$, and consequently if $D > 0$:

$$\lim_{|t|\to+\infty} u^f(t) = \lim_{|t|\to+\infty} \dot{u}^f(t) = 0\,.$$

- The power balance integrated between $-\infty$ and t:

$$\begin{aligned}\mathcal{E}^f(t) &= \int_{-\infty}^{t} \Pi_{\mathrm{IN}}(\tau)\,\mathrm{d}\tau - \int_{-\infty}^{t} \Pi_{\mathrm{d}}^f(\tau)\,\mathrm{d}\tau \\ &= \mathcal{E}_{\mathrm{IN}}(t) - \mathcal{E}_{\mathrm{d}}^f(t)\end{aligned}$$

since $\mathcal{E}^f(-\infty) = 0$. But $\mathcal{E}^f(+\infty) = 0$ as well, hence:

$$\mathcal{E}_{\mathrm{IN}}(+\infty) = \mathcal{E}_{\mathrm{d}}^f(+\infty)\,.$$

Energetic quantities
Equipartition in the forced response

- **Hypotheses** (wideband excitation):
  1. $\xi_p \ll 1$;
  2. $\omega \mapsto \widehat{f}(\omega)$ varies slowly on $[-\omega_f, \omega_f]$ and $\omega_f \gg \omega_p$.
- Then:

$$\begin{aligned}
\int_{\mathbb{R}} \mathcal{E}_{\mathrm{c}}^{f}(t)\,\mathrm{d}t &= \frac{M}{4\pi}\int_{\mathbb{R}} \omega^2 |\widehat{\mathbb{h}}(\omega)|^2 |\widehat{f}(\omega)|^2\,\mathrm{d}\omega && \text{(Plancherel)}\\
&\simeq \frac{M}{4\pi}|\widehat{f}(\omega_p)|^2 \int_{-\omega_f}^{\omega_f} \omega^2 |\widehat{\mathbb{h}}(\omega)|^2\,\mathrm{d}\omega && \text{(using (ii))}\\
&\simeq \frac{M}{2\pi}|\widehat{f}(\omega_p)|^2 \int_{0}^{+\infty} \omega^2 |\widehat{\mathbb{h}}(\omega)|^2\,\mathrm{d}\omega && \text{(using (i))}\\
&= \frac{|\widehat{f}(\omega_p)|^2}{4D}
\end{aligned}$$

- Likewise $\int_{\mathbb{R}} \mathcal{E}_{\mathrm{p}}^{f}(t)\,\mathrm{d}t \simeq \frac{|\widehat{f}(\omega_p)|^2}{4D}$, independently of the mass or the stiffness.

Energetic quantities

Equipartition in the forced response (extended proof)

$$\begin{aligned}
\int_{\mathbb{R}} \mathcal{E}_c^f(t)\,\mathrm{d}t &= \frac{M}{2}\int_{\mathbb{R}} (\dot{u}^f(t))^2 \mathrm{d}t \\
&= \frac{M}{4\pi}\int_{\mathbb{R}} |\widehat{\dot{u}}^f(\omega)|^2 \mathrm{d}\omega \quad \left(\text{Plancherel } \int_{\mathbb{R}} |u(t)|^2 \mathrm{d}t = \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{u}(\omega)|^2 \mathrm{d}\omega\right) \\
&= \frac{M}{4\pi}\int_{\mathbb{R}} |\mathrm{i}\omega\widehat{u}^f(\omega)|^2 \mathrm{d}\omega \quad \left(\text{Fourier transform } \hat{\dot{u}}(\omega) = \mathrm{i}\omega\hat{u}(\omega)\right) \\
&= \frac{M}{4\pi}\int_{\mathbb{R}} \omega^2 |\widehat{\mathbb{h}}(\omega)|^2 |\widehat{f}(\omega)|^2 \,\mathrm{d}\omega \quad \left(u^f(t) = \int_{-\infty}^{t} \mathbb{h}(t-\tau) f(\tau)\,\mathrm{d}\tau \right. \\
&\qquad\qquad \left. \Longrightarrow \hat{u}^f(\omega) = \hat{\mathbb{h}}(\omega)\hat{f}(\omega)\right) \\
&\simeq \frac{M}{4\pi} |\widehat{f}(\omega_p)|^2 \int_{-\omega_f}^{\omega_f} \omega^2 |\widehat{\mathbb{h}}(\omega)|^2 \,\mathrm{d}\omega \quad (\text{using (ii)}) \\
&\simeq \frac{M}{2\pi} |\widehat{f}(\omega_p)|^2 \int_{0}^{+\infty} \omega^2 |\widehat{\mathbb{h}}(\omega)|^2 \,\mathrm{d}\omega \quad (\text{using (i)}) \\
&= \frac{|\widehat{f}(\omega_p)|^2}{4D} \quad \left(\int_{0}^{+\infty} \omega^2 |\hat{\mathbb{h}}(\omega)|^2 \,\mathrm{d}\omega = \frac{\pi}{2DM}\right)
\end{aligned}$$

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- Then:

$$\begin{aligned}
\mathcal{E}_{\mathrm{d}}^{f}(+\infty) &= \int_{\mathbb{R}} D(\dot{u}^f(t))^2 \, \mathrm{d}t \\
&= \frac{D}{2\pi} \int_{\mathbb{R}} \omega^2 |\widehat{\mathbb{h}}(\omega)|^2 |\widehat{f}(\omega)|^2 \, \mathrm{d}\omega \qquad \text{(Plancherel)} \\
&\simeq \frac{D}{2\pi} |\widehat{f}(\omega_p)|^2 \int_{\mathbb{R}} \omega^2 |\widehat{\mathbb{h}}(\omega)|^2 \, \mathrm{d}\omega \quad \text{(using (i)-(ii))} \\
&= \frac{|\widehat{f}(\omega_p)|^2}{2M} \,,
\end{aligned}$$

and the overall dissipated energy is independent of the damping.

- It is related to the overall mechanical energy by:

$$\mathcal{E}_{\mathrm{d}}^{f}(+\infty) \simeq \int_{\mathbb{R}} \eta_p \omega_p \mathcal{E}^f(t) \, \mathrm{d}t \,.$$

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- The power balance integrated between $t=0$ and $t=+\infty$:

$$\mathcal{E}_{\mathrm{d}}(+\infty)=\mathcal{E}_0+\mathcal{E}_{\mathrm{IN}}(+\infty)\,.$$

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Stationary excitation

Forced response of the SDOF oscillator

- **Data**: $(F_t, t \in \mathbb{R})$ is a $\mathbb{R}$-valued second order, centered stochastic process defined on $(\Omega, \mathcal{F}, P)$, indexed on $\mathbb{R}$, and mean-square stationary.
- **Hypothesis**: $\exists\, \omega \mapsto S_F(\omega) : \mathbb{R} \to \mathbb{R}_+$ even, integrable.

Proposition

- *$(U_t^f, t \in \mathbb{R})$ is a $\mathbb{R}$-valued second order, centered stochastic process defined on $(\Omega, \mathcal{F}, P)$, indexed on $\mathbb{R}$, and mean-square stationary s.t. $S_U(\omega) = |\widehat{\mathbb{h}}(\omega)|^2 S_F(\omega)$.*
- *The same holds for its mean-square derivatives $(\dot{U}_t^f, t \in \mathbb{R})$ and $(\ddot{U}_t^f, t \in \mathbb{R})$, with $S_{\dot{U}}(\omega) = \omega^2 S_U(\omega)$ and $S_{\ddot{U}}(\omega) = \omega^4 S_U(\omega)$, respectively.*

Stationary excitation
Energetics of the stationary forced response

- The instantaneous power balance reads:

$$\dot{\mathcal{E}}_t^f = \Pi_{\mathrm{IN},t} - \Pi_{\mathrm{d},t}^f \,,$$

as an equality of second-order random variables.

- Considering the mathematical expectation with:

$$\mathbb{E}\{(U_t^f)^2\} = R_U(t=0) = \text{Constant independent of } t \,,$$

$$\mathbb{E}\{(\dot{U}_t^f)^2\} = -\frac{\mathrm{d}^2 R_U}{\mathrm{d}t^2}(t=0) = \text{Constant independent of } t \,,$$

yields $\mathbb{E}\{\mathcal{E}_t^f\} = \text{Constant}$, and $\mathbb{E}\{\dot{\mathcal{E}}_t^f\} = 0$. Hence:

$$\mathbb{E}\{\Pi_{\mathrm{IN},t}\} = \mathbb{E}\{\Pi_{\mathrm{d},t}^f\} \,.$$

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- **Hypotheses** (wideband excitation):
  - i $\xi_p \ll 1$;
  - ii $\omega \mapsto S_F(\omega)$ varies slowly on $[-\omega_p - \frac{b_\mathrm{e}}{2}, \omega_p + \frac{b_\mathrm{e}}{2}]$.
- Then:

$$\begin{aligned}\mathbb{E}\{\mathcal{E}^f_{\mathrm{c},t}\} &= \frac{M}{2}\int_{\mathbb{R}} \omega^2 |\widehat{\mathbb{h}}(\omega)|^2 S_F(\omega)\,\mathrm{d}\omega \\ &\simeq \frac{M}{2} S_F(\omega_p)\int_{\mathbb{R}} \omega^2 |\widehat{\mathbb{h}}(\omega)|^2\,\mathrm{d}\omega \quad \text{(using (i)-(ii))} \\ &= \frac{\pi S_F(\omega_p)}{2D}\end{aligned}$$

- Likewise $\mathbb{E}\{\mathcal{E}^f_{\mathrm{p},t}\} \simeq \frac{\pi S_F(\omega_p)}{2D}$, independently of the mass or the stiffness, s.t.

$$\boxed{\mathbb{E}\{\mathcal{E}^f_t\} \simeq \frac{\pi S_F(\omega_p)}{D}}.$$

Stationary excitation

Equipartition in the stationary forced response (extended proof)

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$$
\begin{aligned}
\mathbb{E}\{\mathcal{E}^f_{c,t}\} &= \frac{M}{2}\mathbb{E}\{(\dot{U}^f_t)^2\} \\
&= \frac{M}{2}R_{\dot{U}}(0) \quad (\mathbb{E}\{\dot{U}_t\dot{U}_{t'}\} = R_{\dot{U}}(t-t') \implies \mathbb{E}\{(\dot{U}_t)^2\} = R_{\dot{U}}(t-t) = R_{\dot{U}}(0)) \\
&= \frac{M}{2}\int_{\mathbb{R}} S_{\dot{U}}(\omega)\mathrm{d}\omega \quad (R_{\dot{U}}(t) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\omega t} S_{\dot{U}}(\omega)\mathrm{d}\omega \implies R_{\dot{U}}(0) = \int_{\mathbb{R}} S_{\dot{U}}(\omega)\mathrm{d}\omega) \\
&= \frac{M}{2}\int_{\mathbb{R}} \omega^2|\widehat{\mathbb{h}}(\omega)|^2 S_F(\omega)\,\mathrm{d}\omega \quad (S_{\dot{U}}(\omega) = \omega^2 S_U(\omega) = \omega^2|\widehat{\mathbb{h}}(\omega)|^2 S_F(\omega)) \\
&\simeq \frac{M}{2}S_F(\omega_p)\int_{\mathbb{R}} \omega^2|\widehat{\mathbb{h}}(\omega)|^2\,\mathrm{d}\omega \quad (\text{using (i)-(ii)}) \\
&= \frac{\pi S_F(\omega_p)}{2D} \quad (\int_0^{+\infty} \omega^2|\widehat{\mathbb{h}}(\omega)|^2\mathrm{d}\omega = \frac{\pi}{2DM} = \frac{1}{2}\int_{\mathbb{R}} \omega^2|\widehat{\mathbb{h}}(\omega)|^2\mathrm{d}\omega)
\end{aligned}
$$

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- Then:

$$\begin{aligned}\mathbb{E}\{\Pi^f_{\mathrm{d},t}\} &= D\int_{\mathbb{R}} \omega^2 |\widehat{\mathbb{h}}(\omega)|^2 S_F(\omega)\,\mathrm{d}\omega \\ &\simeq DS_F(\omega_p)\int_{\mathbb{R}} \omega^2 |\widehat{\mathbb{h}}(\omega)|^2\,\mathrm{d}\omega \quad \text{(using (i)-(ii))} \\ &= \frac{\pi S_F(\omega_p)}{M}\,,\end{aligned}$$

  and the average dissipated power is independent of the damping.

- It is related to the average mechanical energy by:

$$\boxed{\mathbb{E}\{\Pi^f_{\mathrm{d},t}\} \simeq \eta_p\omega_p\mathbb{E}\{\mathcal{E}^f_t\}}\,.$$

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- Assuming null initial conditions to simplify, the evolutionary response $(U_t,\, t \geqslant 0)$ is a $\mathbb{R}$-valued second-order, centered stochastic process defined on $(\Omega, \mathcal{F}, P)$, indexed on $\mathbb{R}_+$, which is non stationary:

$$U_t = \int_0^t \mathbb{h}(t-\tau) F_\tau \, \mathrm{d}\tau \, .$$

- **Property**: if $D > 0$, $\lim_{t \to +\infty} \| U_t - U_t^f \| = 0 \, .$

Summary

- "Equivalence" between:
    1. The overall energetic quantities for the forced response to a deterministic, wideband excitation;
    2. The average energetic quantities for the stationary forced response to a random, wideband (m.s.) stationary excitation;
    3. The time average energetic quantities for the forced response of the randomized SDOF oscillator to an harmonic excitation.
- These different cases are often (unduly) merged in the structural-acoustics literature.
- **Outlook**: multiple DOF systems.