External Neumann problem for the Helmholtz equation

MG3416–Advanced Structural Acoustics - Lecture #10

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Outline

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Navier-Stokes equations

- The mass, momentum, and energy conservation equations for a fluid flow (ignoring input mass, momentum and heat) read:

$$
\begin{cases}
\dfrac{d\varrho}{dt} = -\varrho\boldsymbol{\nabla}\cdot\boldsymbol{v}\,, \\
\varrho\dfrac{d\boldsymbol{v}}{dt} = \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}\,, \\
\varrho T\dfrac{ds}{dt} = \boldsymbol{\tau}:\boldsymbol{\nabla}\boldsymbol{v} - \boldsymbol{\nabla}\cdot\boldsymbol{q}\,,
\end{cases}
$$

where $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}$ and:

  - $\boldsymbol{v}$ is the fluid velocity, ϱ the density, T the temperature, s the (specific) entropy;
  - $\boldsymbol{\sigma} = -p\boldsymbol{I} + \boldsymbol{\tau}$ is the stress tensor, p is the (static) fluid pressure, $\boldsymbol{\tau}$ is the viscous stress tensor;
  - $\boldsymbol{q}$ is the heat flux vector.

Euler equations

- Ideal fluid: $\boldsymbol{\tau} = \boldsymbol{q} = \boldsymbol{0}$, and the conservation equations then read:

$$\begin{cases} \dfrac{d\varrho}{dt} = -\varrho \boldsymbol{\nabla} \cdot \boldsymbol{v}\,, \\ \dfrac{d\boldsymbol{v}}{dt} = -\dfrac{1}{\varrho} \boldsymbol{\nabla} p\,, \\ \dfrac{ds}{dt} = 0\,. \end{cases}$$

- The flow is isentropic (each fluid particle has constant entropy), and by the equation of state $p = p(\varrho, s)$:

$$\frac{dp}{dt} = c^2 \frac{d\varrho}{dt}\,, \quad c^2(\varrho, s) = \left.\frac{\partial p}{\partial \varrho}\right|_s\,.$$

Linearized Euler equations

- Linearization about a stationary fluid flow $(\varrho_0, \boldsymbol{v}_0, p_0)$:

$$\begin{cases} (\boldsymbol{v}_0 \cdot \boldsymbol{\nabla})\varrho_0 = -\varrho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v}_0 \,, \\ (\boldsymbol{v}_0 \cdot \boldsymbol{\nabla})\boldsymbol{v}_0 = -\dfrac{1}{\varrho_0} \boldsymbol{\nabla} p_0 \,, \\ (\boldsymbol{v}_0 \cdot \boldsymbol{\nabla})p_0 = c_0^2 (\boldsymbol{v}_0 \cdot \boldsymbol{\nabla})\varrho_0 \,. \end{cases}$$

- The actual flow is a perturbation $(\varrho', \boldsymbol{v}', p')$ of the stationary flow:

$$\begin{cases} \varrho = \varrho_0 + \varrho' \,, \\ \boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}' \,, \\ p = p_0 + p' \,. \end{cases}$$

Linearized Euler equations

- Let the stationary flow be at rest $\boldsymbol{v}_0 = \boldsymbol{0}$ in the absence of perturbations (waves), the linearized Euler equations finally yields:

$$\begin{cases} \dfrac{\partial \varrho'}{\partial t} = -\varrho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v}' \\ \dfrac{\partial \boldsymbol{v}'}{\partial t} = -\dfrac{1}{\varrho_0} \boldsymbol{\nabla} p' \,, \end{cases}$$

  with $p' = c_0^2 \varrho'$ from the equation of state, taking $s(t) = s(0) = 0$ for each fluid particle.
- The operators $\frac{\partial}{\partial t}$ and $\boldsymbol{\nabla} \cdot$ commute (but $\frac{d}{dt}$ and $\boldsymbol{\nabla} \cdot$ do not in general), hence the acoustic wave equation reads:

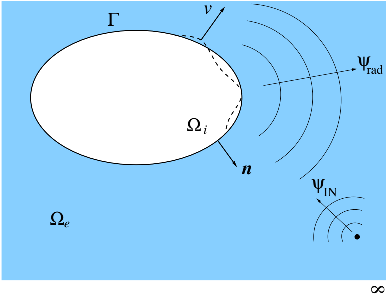
$$\boxed{\frac{\partial}{\partial t}\left(\frac{1}{c_0^2}\frac{\partial p'}{\partial t}\right) - \varrho_0 \boldsymbol{\nabla} \cdot \left(\frac{1}{\varrho_0} \boldsymbol{\nabla} p'\right) = 0} \,.$$

- Consider:
  - an acoustic fluid: homogeneous, compressible, inviscid, gravity effects are neglected;
  - irrotational motion $\boldsymbol{\nabla}\times\boldsymbol{v}'=\boldsymbol{0}$, s.t. the fluid velocity $\boldsymbol{v}'$ reads $\boldsymbol{v}'=\boldsymbol{\nabla}\psi$, thus the fluid pressure is $p'=-\varrho_0\partial_t\psi$.
- The velocity potential ψ is sought for as the solution of:

$$\left\{\begin{aligned} \frac{1}{c_0^2}\frac{\partial^2\psi}{\partial t^2}-\Delta\psi &= g_{\mathrm{IN}} && \text{in } \Omega_e\,, \\ \frac{\partial\psi}{\partial\boldsymbol{n}} &= v && \text{on } \partial\Omega_e=\Gamma\,, \\ \psi(\cdot,0) &= 0 && \text{in } \Omega_e\,, \\ \dot\psi(\cdot,0) &= 0 && \text{in } \Omega_e\,, \end{aligned}\right.$$

with c_0: sound speed of the fluid at rest.

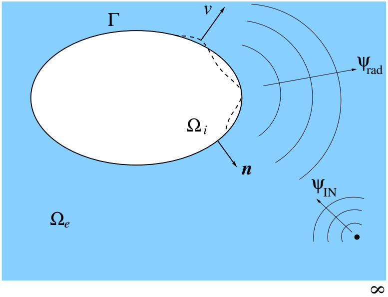
External Neumann problem
Decomposition of the velocity potential



- The velocity potential reads:

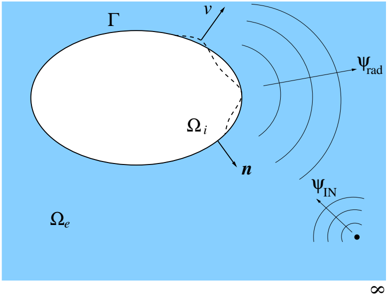
$$\psi(\boldsymbol{x},t) = \psi_{\mathrm{IN}}(\boldsymbol{x},t)+\psi_{\mathrm{d0}}(\boldsymbol{x},t)+\psi_{\mathrm{rad}}(\boldsymbol{x},t)\,,\quad (\boldsymbol{x},t)\in\Omega_e\times\mathbb{R}\,,$$

  where:
  - ψ_{IN}: incident potential (data);
  - ψ_{d0}: diffracted potential, the structure being motionless;
  - ψ_{rad}: radiated potential.



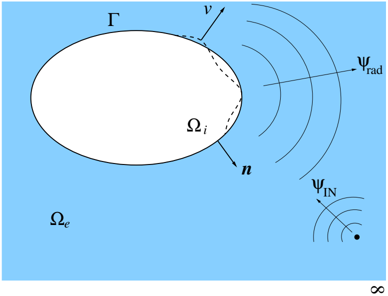
- The diffracted potential $\psi_{\mathrm{d}0}$ satisfies:

$$\begin{cases} \dfrac{1}{c_0^2}\dfrac{\partial^2\psi_{\mathrm{d}0}}{\partial t^2} - \Delta\psi_{\mathrm{d}0} = 0 & \text{in } \Omega_e\,, \\ \dfrac{\partial\psi_{\mathrm{d}0}}{\partial \boldsymbol{n}} = -\dfrac{\partial\psi_{\mathrm{IN}}}{\partial \boldsymbol{n}} & \text{on } \Gamma\,. \end{cases}$$



- The radiated potential ψ_{rad} satisfies:

$$\left\{ \begin{aligned} \frac{1}{c_0^2}\frac{\partial^2\psi_{\mathrm{rad}}}{\partial t^2} - \Delta\psi_{\mathrm{rad}} &= 0 \quad \text{in } \Omega_e \,, \\ \frac{\partial\psi_{\mathrm{rad}}}{\partial \boldsymbol{n}} &= v \quad \text{on } \Gamma \,. \end{aligned} \right.$$



- The incident potential ψ_{IN} satisfies:

$$\frac{1}{c_0^2}\frac{\partial^2\psi_{\text{IN}}}{\partial t^2} - \Delta\psi_{\text{IN}} = g_{\text{IN}}(\boldsymbol{x}, t)\,, \quad (\boldsymbol{x}, t) \in \mathbb{R}^3 \times \mathbb{R}\,,$$

for some sound source g_{IN} in the full physical space.

- The Fourier transform:

$$\widehat{\psi}(\boldsymbol{x},\omega) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\omega t}\psi(\boldsymbol{x},t)\mathrm{d}t\,, \quad \psi(\boldsymbol{x},t) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\omega t}\widehat{\psi}(\boldsymbol{x},\omega)\mathrm{d}\omega\,.$$

- In the frequency domain, for $k = \frac{\omega}{c_0}$:

$$\begin{cases} \Delta\widehat{\psi} + k^2\widehat{\psi} = 0 & \text{in } \Omega_e\,, \\ \dfrac{\partial\widehat{\psi}}{\partial\boldsymbol{n}} = u & \text{on } \Gamma\,, \\ \left|\widehat{\psi}\right| = \mathrm{O}\left(\dfrac{1}{r}\right)\,, \quad \left|\dfrac{\partial\widehat{\psi}}{\partial r} + \mathrm{i}k\widehat{\psi}\right| = \mathrm{O}\left(\dfrac{1}{r^2}\right) & \text{as } r = \|\boldsymbol{x}\| \to +\infty\,. \end{cases}$$

- Sommerfeld radiation conditions "at infinity": the radiated waves are almost plane and do not back propagate toward Γ.

- The external Helmholtz problem admits a unique solution $\forall k \in \mathbb{R}$, hence:
  - There exists a linear operator $\mathscr{B}_\Gamma(k) : \mathcal{C}'_\Gamma \to \mathcal{C}_\Gamma$ s.t.:

$$\widehat{\psi}|_\Gamma = \mathscr{B}_\Gamma(k)u \,, \quad \text{on } \Gamma \,;$$

  - There exists a linear operator $\mathscr{R}_{\boldsymbol{x}}(k) : \mathcal{C}'_\Gamma \to \mathbb{C}$ s.t.:

$$\widehat{\psi}(\boldsymbol{x}) = \mathscr{R}_{\boldsymbol{x}}(k)u \,, \quad \boldsymbol{x} \in \Omega_e \,,$$

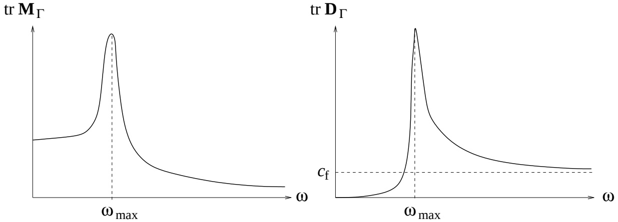
  with $\mathcal{C}_\Gamma$: the set of admissible fields on Γ ($\mathcal{C}'_\Gamma$: its dual[1]).
- $\boldsymbol{Z}_\Gamma(\omega) = -\mathrm{i}\omega\varrho_0\mathscr{B}_\Gamma(\frac{\omega}{c_0})$: boundary impedance operator.
- $\boldsymbol{\mathcal{Z}}_{\boldsymbol{x}}(\omega) = -\mathrm{i}\omega\varrho_0\mathscr{R}_{\boldsymbol{x}}(\frac{\omega}{c_0})$: radiation impedance operator.

[1] $\mathcal{C}_\Gamma = H^{1/2}(\Gamma)$ the set of traces in $H^1_{\mathrm{loc}}(\Omega_e)$, $\mathcal{C}'_\Gamma = H^{-1/2}(\Gamma)$ s.t. $\mathcal{C}_\Gamma \subset L^2(\Gamma) \subset \mathcal{C}'_\Gamma$.

- The boundary impedance $\boldsymbol{Z}_\Gamma(\omega)$ is symmetric and reads:

$$
\mathrm{i}\omega \boldsymbol{Z}_\Gamma(\omega) = -\omega^2 \boldsymbol{M}_\Gamma\left(\frac{\omega}{c_0}\right) + \mathrm{i}\omega \boldsymbol{D}_\Gamma\left(\frac{\omega}{c_0}\right) \,,
$$

where:
  - The reactive part $\omega \mapsto \operatorname{Tr} \boldsymbol{M}_\Gamma\left(\frac{\omega}{c_0}\right)$ (left) is generally unsigned, though it is positive if $\mathbb{R}^3 \backslash \overline{\Omega}_e$ is convex;
  - The resistive part $\omega \mapsto \operatorname{Tr} \boldsymbol{D}_\Gamma\left(\frac{\omega}{c_0}\right)$ (right) is positive.



- Diffracted velocity potential:

$$\widehat{\psi}_{\mathrm{d0}}(\omega)|_\Gamma = -\mathscr{B}_\Gamma\left(\frac{\omega}{c_0}\right)\frac{\partial\widehat{\psi}_{\mathrm{IN}}}{\partial\boldsymbol{n}}\,,\quad \boldsymbol{x}\in\Gamma\,,$$

$$\widehat{\psi}_{\mathrm{d0}}(\boldsymbol{x},\omega) = -\mathscr{R}_{\boldsymbol{x}}\left(\frac{\omega}{c_0}\right)\frac{\partial\widehat{\psi}_{\mathrm{IN}}}{\partial\boldsymbol{n}}\,,\quad \boldsymbol{x}\in\Omega_e\,.$$

- Radiated velocity potential:

$$\widehat{\psi}_{\mathrm{rad}}(\omega)|_\Gamma = \mathscr{B}_\Gamma\left(\frac{\omega}{c_0}\right)\widehat{v}(\omega)\,,\quad \boldsymbol{x}\in\Gamma\,,$$

$$\widehat{\psi}_{\mathrm{rad}}(\boldsymbol{x},\omega) = \mathscr{R}_{\boldsymbol{x}}\left(\frac{\omega}{c_0}\right)\widehat{v}(\omega)\,,\quad \boldsymbol{x}\in\Omega_e\,.$$

- (Linear) scattering operator $\mathcal{T}_\Gamma(\omega)\equiv\boldsymbol{I}-\mathscr{B}_\Gamma(\frac{\omega}{c_0})\partial_{\boldsymbol{n}}$ s.t.:

$$(\widehat{\psi}_{\mathrm{IN}}(\omega)+\widehat{\psi}_{\mathrm{d0}}(\omega))|_\Gamma = \mathcal{T}_\Gamma(\omega)\widehat{\psi}_{\mathrm{IN}}(\omega)|_\Gamma\,.$$

Outline

Acoustic Green's function

- The Green's function $\mathcal{G}(\boldsymbol{x},\boldsymbol{y})$ is the solution of:

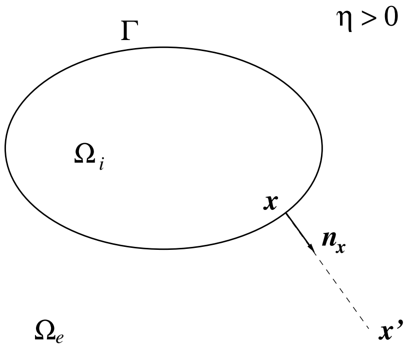
$$\left\{\begin{array}{lr} \Delta_{\boldsymbol{x}}\mathcal{G}+k^2\mathcal{G}=-\delta_{\boldsymbol{y}}(\boldsymbol{x}) & \text{in } \mathbb{R}^3\,, \\ |\mathcal{G}|=\mathrm{O}\left(\frac{1}{r}\right)\,,\quad \left|\frac{\partial\mathcal{G}}{\partial r}+\mathrm{i}k\mathcal{G}\right|=\mathrm{O}\left(\frac{1}{r^2}\right) & \text{as } r=\|\boldsymbol{x}\|\rightarrow+\infty\,, \end{array}\right.$$

  for some $\boldsymbol{y}\in\mathbb{R}^3$ fixed.

- The unique outward solution (which do not back propagate toward the source at $\boldsymbol{y}\in\mathbb{R}^3$) is:

$$\boxed{\mathcal{G}(\boldsymbol{x},\boldsymbol{y})=G(\boldsymbol{x}-\boldsymbol{y})=\frac{\mathrm{e}^{-\mathrm{i}k\|\boldsymbol{x}-\boldsymbol{y}\|}}{4\pi\|\boldsymbol{x}-\boldsymbol{y}\|}}\,.$$

- **Remark**: $\mathcal{G}$ is singular at $\boldsymbol{x}=\boldsymbol{y}$.



- For $\boldsymbol{x}' \in \mathbb{R}^3$, let $\boldsymbol{x}' = \boldsymbol{x} + \eta \boldsymbol{n}_{\boldsymbol{x}}$ where $\boldsymbol{x} \in \Gamma$ and $\boldsymbol{n}_{\boldsymbol{x}}$ is the outward unit normal to Γ at $\boldsymbol{x}$; hence:
  - $\boldsymbol{x}' \in \Omega_e$ if $\eta > 0$ (see figure),
  - $\boldsymbol{x}' \in \Omega_i$ if $\eta < 0$.

- Let $\psi:\Gamma\rightarrow\mathbb{C}$ continuous; the single layer potential is:

$$\psi_S(\boldsymbol{x}')=\int_\Gamma G(\boldsymbol{x}'-\boldsymbol{y})\psi(\boldsymbol{y})\mathrm{d}S(\boldsymbol{y})\,,$$

and $\boldsymbol{x}'\mapsto\psi_S(\boldsymbol{x}')$ is continuous in $\mathbb{R}^3$.

- Let $\psi:\Gamma\rightarrow\mathbb{C}$ continuous; the double layer potential is:

$$\psi_D(\boldsymbol{x}')=\int_\Gamma \frac{\partial G(\boldsymbol{x}'-\boldsymbol{y})}{\partial \boldsymbol{n}_{\boldsymbol{y}}}\psi(\boldsymbol{y})\mathrm{d}S(\boldsymbol{y})\,.$$

and $\boldsymbol{x}'\mapsto\psi_D(\boldsymbol{x}')$ is continuous in $\mathbb{R}^3\backslash\Gamma$.

- The single layer operator $\boldsymbol{S}_S(k) : \mathcal{C}'_\Gamma \rightarrow \mathcal{C}_\Gamma$:

$$\boldsymbol{S}_S(k)\psi(\boldsymbol{x}) = \int_\Gamma G(\boldsymbol{x}-\boldsymbol{y})\psi(\boldsymbol{y})\mathrm{d}S(\boldsymbol{y})\,.$$

- The double layer operator $\boldsymbol{S}_D(k) : \mathcal{C}_\Gamma \rightarrow \mathcal{C}_\Gamma$:

$$\boldsymbol{S}_D(k)\psi(\boldsymbol{x}) = \int_\Gamma \frac{\partial G(\boldsymbol{x}-\boldsymbol{y})}{\partial \boldsymbol{n}_{\boldsymbol{y}}}\psi(\boldsymbol{y})\mathrm{d}S(\boldsymbol{y})\,.$$

- The normal derivative of $\boldsymbol{S}_D(k)$, $\boldsymbol{S}_T(k) : \mathcal{C}_\Gamma \rightarrow \mathcal{C}'_\Gamma$:

$$\boldsymbol{S}_T(k)\psi(\boldsymbol{x}) = -\oint_\Gamma \frac{\partial^2 G(\boldsymbol{x}-\boldsymbol{y})}{\partial \boldsymbol{n}_{\boldsymbol{x}}\partial \boldsymbol{n}_{\boldsymbol{y}}}\psi(\boldsymbol{y})\mathrm{d}S(\boldsymbol{y})\,.$$

- **Remark**: $G(\boldsymbol{x}-\boldsymbol{y}) = G(\boldsymbol{y}-\boldsymbol{x})$, thus $\boldsymbol{S}_S(k)$ and $\boldsymbol{S}_T(k)$ are symmetric.

- Let $\eta > 0$ *i.e.* $\boldsymbol{x}' \in \Omega_e$. Then:

$$\lim_{\eta\to 0^+} \psi_S(\boldsymbol{x}') := \psi_S^+(\boldsymbol{x}) = \boldsymbol{S}_S(k)\psi(\boldsymbol{x})\,,$$

$$\lim_{\eta\to 0^+} \frac{\partial\psi_S(\boldsymbol{x}')}{\partial\boldsymbol{n}_{\boldsymbol{x}}} := \frac{\partial\psi_S^+}{\partial\boldsymbol{n}_{\boldsymbol{x}}}(\boldsymbol{x}) = \left(\frac{1}{2}\boldsymbol{I} + \boldsymbol{S}_D^{\mathsf{T}}(k)\right)\psi(\boldsymbol{x})\,.$$

- Let $\eta < 0$ *i.e.* $\boldsymbol{x}' \in \Omega_i$. Then:

$$\lim_{\eta\to 0^-} \psi_S(\boldsymbol{x}') := \psi_S^-(\boldsymbol{x}) = \boldsymbol{S}_S(k)\psi(\boldsymbol{x})\,,$$

$$\lim_{\eta\to 0^-} \frac{\partial\psi_S(\boldsymbol{x}')}{\partial\boldsymbol{n}_{\boldsymbol{x}}} := \frac{\partial\psi_S^-}{\partial\boldsymbol{n}_{\boldsymbol{x}}}(\boldsymbol{x}) = \left(-\frac{1}{2}\boldsymbol{I} + \boldsymbol{S}_D^{\mathsf{T}}(k)\right)\psi(\boldsymbol{x})\,.$$

- Define the jump $[\![\psi]\!] = \psi^+ - \psi^-$; hence:

$$[\![\psi_S]\!] = 0\,,\quad \left[\!\!\left[\frac{\partial\psi_S}{\partial\boldsymbol{n}_{\boldsymbol{x}}}\right]\!\!\right] = \psi\,.$$

Double layer potential

Properties of the traces

- Let $\eta > 0$ *i.e.* $\boldsymbol{x}' \in \Omega_e$. Then:

$$\lim_{\eta \to 0^+} \psi_D(\boldsymbol{x}') := \psi_D^+(\boldsymbol{x}) = \left(-\frac{1}{2}\boldsymbol{I} + \boldsymbol{S}_D(k) \right) \psi(\boldsymbol{x}) \,,$$

$$\lim_{\eta \to 0^+} \frac{\partial \psi_D(\boldsymbol{x}')}{\partial \boldsymbol{n}_{\boldsymbol{x}}} := \frac{\partial \psi_D^+}{\partial \boldsymbol{n}_{\boldsymbol{x}}}(\boldsymbol{x}) = -\boldsymbol{S}_T(k)\psi(\boldsymbol{x}) \,.$$

- Let $\eta < 0$ *i.e.* $\boldsymbol{x}' \in \Omega_i$. Then:

$$\lim_{\eta \to 0^-} \psi_D(\boldsymbol{x}') := \psi_D^-(\boldsymbol{x}) = \left(\frac{1}{2}\boldsymbol{I} + \boldsymbol{S}_D(k) \right) \psi(\boldsymbol{x}) \,,$$

$$\lim_{\eta \to 0^-} \frac{\partial \psi_D(\boldsymbol{x}')}{\partial \boldsymbol{n}_{\boldsymbol{x}}} := \frac{\partial \psi_D^-}{\partial \boldsymbol{n}_{\boldsymbol{x}}}(\boldsymbol{x}) = -\boldsymbol{S}_T(k)\psi(\boldsymbol{x}) \,.$$

- Define the jump $[\![\psi]\!] = \psi^+ - \psi^-$; hence:

$$[\![\psi_D]\!] = -\psi \,, \quad \left[\!\!\left[\frac{\partial \psi_D}{\partial \boldsymbol{n}_{\boldsymbol{x}}} \right]\!\!\right] = 0 \,.$$

- Let ψ_+ be a solution of the Helmholtz equation in Ω_e with Sommerfeld radiation conditions:

$$\begin{cases} \Delta\psi_+ + k^2\psi_+ = 0 & \text{in } \Omega_e\,, \\ |\psi_+| = \mathrm{O}\left(\dfrac{1}{r}\right)\,,\quad \left|\dfrac{\partial\psi_+}{\partial r} + \mathrm{i}k\psi_+\right| = \mathrm{O}\left(\dfrac{1}{r^2}\right) & \text{as } r = \|\boldsymbol{x}\| \to +\infty\,. \end{cases}$$

- Then the following integral equations hold for $\boldsymbol{x} \in \mathbb{R}^3$:

$$\boxed{\varepsilon_+\psi_+(\boldsymbol{x}) = \int_\Gamma \left(G(\boldsymbol{x}-\boldsymbol{y})\frac{\partial\psi_+}{\partial\boldsymbol{n}_{\boldsymbol{y}}}(\boldsymbol{y}) - \frac{\partial G(\boldsymbol{x}-\boldsymbol{y})}{\partial\boldsymbol{n}_{\boldsymbol{y}}}\psi_+(\boldsymbol{y})\right)\mathrm{d}S(\boldsymbol{y})}\,,$$

where–provided that $\Gamma = \partial\Omega_e$ is regular enough:

$$\varepsilon_+ = 1 \quad \text{if } \boldsymbol{x} \in \Omega_e\,,$$
$$\varepsilon_+ = \frac{1}{2} \quad \text{if } \boldsymbol{x} \in \Gamma\,,$$
$$\varepsilon_+ = 0 \quad \text{if } \boldsymbol{x} \in \Omega_i\,.$$

- From the foregoing Boundary Integral Equation (BIE for $\boldsymbol{x} \in \Gamma$), the solution of the external Helmholtz problem (s.t. $\frac{\partial \widehat{\psi}}{\partial \boldsymbol{n}} = u$ on Γ) should satisfy:

$$\left(\frac{1}{2}\boldsymbol{I} + \boldsymbol{S}_D(k)\right)\widehat{\psi} = \boldsymbol{S}_S(k)u \quad \text{in } \mathcal{C}_\Gamma \,. \tag{A}$$

- Owing to the trace properties of the single and double layer potentials, it should also satisfy:

$$\boldsymbol{S}_T(k)\widehat{\psi} = \left(\frac{1}{2}\boldsymbol{I} - \boldsymbol{S}_D^{\mathsf{T}}(k)\right)u \quad \text{in } \mathcal{C}'_\Gamma \,. \tag{B}$$

- **Remark**: it is not guaranteed that the left-hand side operators are invertible.

- Let ψ_- be a solution of the Helmholtz equation in Ω_i:

$$\Delta\psi_- + k^2\psi_- = 0 \quad \text{in } \Omega_i\,.$$

- Then the following integral equations hold for $\boldsymbol{x} \in \mathbb{R}^3$:

$$\boxed{\varepsilon_-\psi_-(\boldsymbol{x}) = -\int_\Gamma \left(G(\boldsymbol{x}-\boldsymbol{y})\frac{\partial\psi_-}{\partial\boldsymbol{n}_{\boldsymbol{y}}}(\boldsymbol{y}) - \frac{\partial G(\boldsymbol{x}-\boldsymbol{y})}{\partial\boldsymbol{n}_{\boldsymbol{y}}}\psi_-(\boldsymbol{y}) \right) \mathrm{d}S(\boldsymbol{y})}\,,$$

where–provided that $\Gamma = \partial\Omega_i$ is regular enough:

$$\begin{aligned} \varepsilon_- &= 1 \quad \text{if } \boldsymbol{x} \in \Omega_i\,, \\ \varepsilon_- &= \frac{1}{2} \quad \text{if } \boldsymbol{x} \in \Gamma\,, \\ \varepsilon_- &= 0 \quad \text{if } \boldsymbol{x} \in \Omega_e\,. \end{aligned}$$

- From the foregoing Boundary Integral Equation (BIE for $\boldsymbol{x} \in \Gamma$), the solution of the inner Helmholtz equation is given as:

$$\left(\frac{1}{2}\boldsymbol{I} - \boldsymbol{S}_D(k)\right)\psi_- = -\boldsymbol{S}_S(k)\frac{\partial\psi_-}{\partial\boldsymbol{n}} \quad \text{in } \mathcal{C}_\Gamma \,. \qquad \text{[C]}$$

- Owing to the trace properties of the single and double layer potentials, it is also the solution of:

$$\boldsymbol{S}_T(k)\psi_- = -\left(\frac{1}{2}\boldsymbol{I} + \boldsymbol{S}_D^{\mathsf{T}}(k)\right)\frac{\partial\psi_-}{\partial\boldsymbol{n}} \quad \text{in } \mathcal{C}'_\Gamma \,. \qquad \text{[D]}$$

Outline

- In the frequency domain, for $k = \frac{\omega}{c_0}$:

$$\left\{ \begin{aligned} \Delta\psi_- + k^2\psi_- &= 0 \qquad \text{in } \Omega_i \,, \\ \frac{\partial\psi_-}{\partial\boldsymbol{n}} &= 0 \qquad \text{on } \Gamma \,, \\ \int_{\Omega_i} \psi_- \, \mathrm{d}\boldsymbol{x} &= 0 \end{aligned} \right.$$

- **Remark**: the last condition evades constant solutions. Indeed, the weak form of the Helmholtz equation reads

$$\int_{\Omega_i} \boldsymbol{\nabla}\psi_- \cdot \boldsymbol{\nabla}\varphi \, \mathrm{d}\boldsymbol{x} = k^2 \int_{\Omega_i} \psi_- \varphi \, \mathrm{d}\boldsymbol{x} \,, \quad \forall \varphi \in V \subset H^1(\Omega_i, \mathbb{R}) \,,$$

so that constant solutions associated to the eigenvalue 0 are excluded by this condition.

- **Spectral theorem**: there exists a countable set of eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_\alpha \cdots$ $(\lambda_\alpha = k_\alpha^2)$ each of multiplicity n_α, and associated eigenfunctions $\psi_\alpha^1, \psi_\alpha^2, \dots \psi_\alpha^{n_\alpha}$ in:

$$V = \left\{ \psi \in H^1(\Omega_i, \mathbb{R});\ \int_{\Omega_i} \psi \mathrm{d}\boldsymbol{x} = 0 \right\}$$

such that:

$$\left\{ \begin{array}{rl} -\Delta \psi_\alpha^j = \lambda_\alpha \psi_\alpha^j & \text{in } \Omega_i \,, \\ \dfrac{\partial \psi_\alpha^j}{\partial \boldsymbol{n}} = 0 & \text{on } \Gamma \,, \end{array} \right.$$

$\forall j \in \{1, 2, \dots n_\alpha\}$.

- From the BIEs [C] and [D] of the inner Helmholtz problem it is deduced that:

$$\ker \boldsymbol{S}_T(k_\alpha) = \ker\left(\frac{1}{2}\boldsymbol{I} - \boldsymbol{S}_D(k_\alpha)\right) = \mathcal{C}_\alpha \subset \mathcal{C}_\Gamma \,,$$

  where $\mathcal{C}_\alpha = \mathrm{span}\{\psi_\alpha^1|_\Gamma, \psi_\alpha^2|_\Gamma, \dots \psi_\alpha^{n_\alpha}|_\Gamma\}$, with $\#\mathcal{C}_\alpha = n_\alpha$.
- Remind the BIE [B] $\boldsymbol{S}_T(k)\tilde{\psi} = (\frac{1}{2}\boldsymbol{I} - \boldsymbol{S}_D^\mathsf{T}(k))u$, then:
  - $\underline{\text{If } k \neq k_\alpha}$ (for all α): $\ker \boldsymbol{S}_T(k) = \{\boldsymbol{0}\}$ and [B] is uniquely invertible;
  - $\underline{\text{If } k = k_\alpha}$ (for some α): [B] is invertible iff the right-hand side is orthogonal to $\mathcal{C}_\alpha$, which is actually true since:

$$\left\langle \left(\frac{1}{2}\boldsymbol{I} - \boldsymbol{S}_D^\mathsf{T}(k_\alpha)\right) u, \psi_\alpha^j \right\rangle_\Gamma = \left\langle u, \left(\frac{1}{2}\boldsymbol{I} - \boldsymbol{S}_D(k_\alpha)\right)\psi_\alpha^j \right\rangle_\Gamma = 0\,.$$

Invertible BIE for the external Neumann problem

- Consequently, the solutions of the BIE [B] read:

$$
\begin{aligned}
\tilde{\psi}(\omega) &= \widehat{\psi}(\omega)|_\Gamma && \text{if } \omega \neq \omega_\alpha\,, \\
\tilde{\psi}(\omega) &= \widehat{\psi}(\omega_\alpha)|_\Gamma + \sum_{j=1}^{n_\alpha} C_j \psi_\alpha^j|_\Gamma && \text{if } \omega = \omega_\alpha\,,
\end{aligned}
$$

  where $C_j \in \mathbb{C}$ for $1 \leqslant j \leqslant n_\alpha$ and $k_\alpha = \frac{\omega_\alpha}{c_0}$.

- A symmetric BIE for the solution $\widehat{\psi}$ of the external Neumann problem that is invertible for all frequencies is given by (Angélini-Hutin 1983):

$$
\boxed{\begin{pmatrix} 0 \\ \widehat{\psi}|_\Gamma \end{pmatrix} = \begin{bmatrix} -\boldsymbol{S}_T(\frac{\omega}{c_0}) & \frac{1}{2}\boldsymbol{I} - \boldsymbol{S}_D^\mathsf{T}(\frac{\omega}{c_0}) \\ \frac{1}{2}\boldsymbol{I} - \boldsymbol{S}_D(\frac{\omega}{c_0}) & \boldsymbol{S}_S(\frac{\omega}{c_0}) \end{bmatrix} \begin{pmatrix} \tilde{\psi} \\ u \end{pmatrix}}\,.
$$

Invertible BIE for the external Neumann problem

1. If $k \neq k_\alpha$ then the first line has a unique solution $\tilde{\psi} = \widehat{\psi}|_\Gamma$ and the second line reads:

$$\begin{aligned}\left(\frac{1}{2}\boldsymbol{I} - \boldsymbol{S}_D(k)\right)\tilde{\psi} + \boldsymbol{S}_S(k)u &= \left(\frac{1}{2}\boldsymbol{I} - \boldsymbol{S}_D(k)\right)\widehat{\psi} + \boldsymbol{S}_S(k)u \\ &= \widehat{\psi}|_\Gamma - \left(\frac{1}{2}\boldsymbol{I} + \boldsymbol{S}_D(k)\right)\widehat{\psi} + \boldsymbol{S}_S(k)u \\ &= \widehat{\psi}|_\Gamma\,,\end{aligned}$$

owing to the BIE [A].

2. If $k = k_\alpha$ then $\tilde{\psi}(\omega_\alpha) = \widehat{\psi}(\omega_\alpha)|_\Gamma + \sum_{j=1}^{n_\alpha} C_j \psi_\alpha^j|_\Gamma$, thus:

$$\left(\frac{1}{2}\boldsymbol{I} - \boldsymbol{S}_D(k_\alpha)\right)\tilde{\psi} + \boldsymbol{S}_S(k_\alpha)u = \left(\frac{1}{2}\boldsymbol{I} - \boldsymbol{S}_D(k_\alpha)\right)\widehat{\psi} + \boldsymbol{S}_S(k_\alpha)u\,,$$

and the same conclusion holds.

Construction of the impedance operators

- Solving the symmetric BIE system by eliminating $\tilde{\psi}$ yields $\widehat{\psi}|_\Gamma$ in terms of u, hence the boundary impedance operator $\mathscr{B}_\Gamma(k)$ (or $\mathscr{B}_\Gamma(k_\alpha)$).
- Consider the RIE for the outer Helmholtz problem applied to $\widehat{\psi}$, the solution of the external Neumann problem:

$$\begin{aligned}\widehat{\psi}(\boldsymbol{x}) &= \int_\Gamma G(\boldsymbol{x}-\boldsymbol{y})u(\boldsymbol{y})\mathrm{d}S(\boldsymbol{y}) - \int_\Gamma \frac{\partial G(\boldsymbol{x}-\boldsymbol{y})}{\partial \boldsymbol{n_y}}\widehat{\psi}(\boldsymbol{y})\mathrm{d}S(\boldsymbol{y})\,, \quad \boldsymbol{x}\in\Omega_e \\ &= \mathscr{R}^1_{\boldsymbol{x}}(k)u - \mathscr{R}^2_{\boldsymbol{x}}(k)\widehat{\psi}|_\Gamma \\ &= \mathscr{R}^1_{\boldsymbol{x}}(k)u - \mathscr{R}^2_{\boldsymbol{x}}(k)\mathscr{B}_\Gamma(k)u\,,\end{aligned}$$

with the definitions $\mathscr{R}^1_{\boldsymbol{x}}(k)u = \int_\Gamma G(\boldsymbol{x}-\boldsymbol{y})u(\boldsymbol{y})\mathrm{d}S(\boldsymbol{y})$ and $\mathscr{R}^2_{\boldsymbol{x}}(k)\psi = \int_\Gamma \frac{\partial G(\boldsymbol{x}-\boldsymbol{y})}{\partial \boldsymbol{n_y}}\psi(\boldsymbol{y})\mathrm{d}S(\boldsymbol{y})$ <u>for $\boldsymbol{x}\in\Omega_e$</u>. Hence:

$$\mathscr{R}_{\boldsymbol{x}}(k) = \mathscr{R}^1_{\boldsymbol{x}}(k) - \mathscr{R}^2_{\boldsymbol{x}}(k)\mathscr{B}_\Gamma(k)\,.$$

- Finite-dimensional discretization of the boundary operators in $\mathcal{C}_N = \mathrm{span}\{\psi_1, \psi_2, \ldots \psi_N\}$ with $\#\mathcal{C}_N = N$:

$$\begin{aligned}
[\mathrm{S}(k)]_{mn} &= \langle \boldsymbol{S}_S(k)\psi_m, \psi_n \rangle_\Gamma \\
&= \iint_{\Gamma\times\Gamma} G(\boldsymbol{x}-\boldsymbol{y})\psi_m(\boldsymbol{y})\psi_n(\boldsymbol{x})\mathrm{d}S(\boldsymbol{x})\mathrm{d}S(\boldsymbol{y})\,, \\
[\mathrm{D}(k)]_{mn} &= \langle \boldsymbol{S}_D(k)\psi_m, \psi_n \rangle_\Gamma \\
&= \iint_{\Gamma\times\Gamma} \frac{\partial G(\boldsymbol{x}-\boldsymbol{y})}{\partial \boldsymbol{n}(\boldsymbol{y})}\psi_m(\boldsymbol{y})\psi_n(\boldsymbol{x})\mathrm{d}S(\boldsymbol{x})\mathrm{d}S(\boldsymbol{y})\,.
\end{aligned}$$

- Petrov-Galerkin method whereby the trial and test spaces are different, *e.g.* collocation approach using test functions of the form $\psi_n(\boldsymbol{x}) = \delta(\boldsymbol{x} - \boldsymbol{\xi}_n)$.

- The normal derivative of the double layer potential $\boldsymbol{S}_T(k)$ is singular and is understood as its Cauchy principal value:

$$\boldsymbol{S}_T(k)\psi(\boldsymbol{x}) = -\lim_{\epsilon\to 0}\int_{\Gamma\backslash B(\boldsymbol{y},\epsilon)} \frac{\partial^2 G(\boldsymbol{x}-\boldsymbol{y})}{\partial\boldsymbol{n_x}\partial\boldsymbol{n_y}}\psi(\boldsymbol{y})\mathrm{d}S(\boldsymbol{y})\,,$$

  where $B(\boldsymbol{y},\epsilon)$ is the open ball of radius ϵ centered in $\boldsymbol{y}$.
- Regularization formula (Maue 1949, Mitzner 1966):

$$\langle\boldsymbol{S}_T(k)\psi,\phi\rangle_\Gamma = -k^2\iint_{\Gamma\times\Gamma} G(\boldsymbol{x}-\boldsymbol{y})\psi(\boldsymbol{y})\phi(\boldsymbol{x})(\boldsymbol{n_x}\cdot\boldsymbol{n_y})\mathrm{d}S(\boldsymbol{x})\mathrm{d}S(\boldsymbol{y})$$

$$+\iint_{\Gamma\times\Gamma} G(\boldsymbol{x}-\boldsymbol{y})(\boldsymbol{n_y}\times\boldsymbol{\nabla_y}\psi(\boldsymbol{x}))\cdot(\boldsymbol{n_x}\times\boldsymbol{\nabla_x}\phi(\boldsymbol{x}))\mathrm{d}S(\boldsymbol{x})\mathrm{d}S(\boldsymbol{y})\,.$$

- Discretized version of the symmetric BIE in the finite-dimensional subspace $\mathcal{C}_N \subset \mathcal{C}_\Gamma \subset \mathcal{C}'_\Gamma$:

$$\begin{pmatrix} 0 \\ \Psi_\Gamma \end{pmatrix} = \begin{bmatrix} -[\mathrm{T}(\frac{\omega}{c_0})] & \frac{1}{2}[\mathrm{E}]^\mathsf{T} - [\mathrm{D}(\frac{\omega}{c_0})]^\mathsf{T} \\ \frac{1}{2}[\mathrm{E}] - [\mathrm{D}(\frac{\omega}{c_0})] & [\mathrm{S}(\frac{\omega}{c_0})] \end{bmatrix} \begin{pmatrix} \tilde{\Psi} \\ \mathrm{U} \end{pmatrix} .$$

- Gauss elimination of $\tilde{\Psi}$ in the $2N \times 2N$ system above:
  - If $\omega \neq \omega_\alpha$, $[\mathrm{T}(\frac{\omega}{c_0})]$ is invertible and Gauss elimination is stopped at row N to yield $\Psi_\Gamma = [B_\Gamma(\frac{\omega}{c_0})]\mathrm{U}$, where $[\mathrm{B}_\Gamma(\frac{\omega}{c_0})]$ is the $N \times N$ complex symmetric matrix corresponding to the discretization of $\mathscr{B}_\Gamma(\frac{\omega}{c_0})$ in $\mathcal{C}_N$;
  - If $\omega = \omega_\alpha$, $[\mathrm{T}(\frac{\omega}{c_0})]$ is not invertible and Gauss elimination is stopped at row $N - n_\alpha$–practically when a null pivot arises because n_α is *a priori* unknown.

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