SEA formulation and modeling issues

MG3416–Advanced Structural Acoustics - Lecture #8

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Outline

Outline

Basic SEA hypotheses

- SEA aims at describing the energy transfers between $\mathscr{n}$ continuous sub-systems.
- It is based on two groups of hypotheses:
  1. The ones introduced in the previous lectures;
  2. The ones necessary to extend the results of the previous lectures to the coupling of several ($\mathscr{n} > 2$) sub-systems.
- **Remark**: *SEA describes the flow of vibrational energy between sub-systems, hence "energy fluxes", though the terminology "power flow" is mainly used in SEA literature.*

Basic SEA hypotheses

- Hypotheses introduced in the previous lectures:
  1. The analysis is done in bands $I_0 = [\omega_0 - \frac{\Delta\omega}{2}, \omega_0 + \frac{\Delta\omega}{2}]$.
  2. The loads are:
     - wideband noises with bandwidth $\Delta\omega$;
     - uncorrelated between sub-systems.
  3. Each sub-system is weakly dissipative and characterized by its own vibrational modes. Consequently:
     - $\Delta\omega \gg \pi\xi_\alpha\omega_\alpha$, where $0 < \xi_\alpha \ll 1$;
     - an unbounded domain cannot be an SEA sub-system.
  4. The coupling is conservative: mass, stiffness, and gyroscopic couplings.
  5. The vibrational modes of which eigenfrequencies lie in the frequency band I_0 are the only ones which contribute to the mechanical energy of each sub-system in that band.

Basic SEA hypotheses

- Additional hypotheses:
  6. Each sub-system $r \in \{1, 2, \dots n\}$ has a large number N_r of vibrational modes in the band I_0.
     - **Remark #1**: Therefore the method is adapted to the high-frequency range of vibrations.
     - **Remark #2**: Practically a mode count $N_r \gtrsim 6, 7$ is enough.
  7. The average modal energies of each sub-system are close.
  8. Each vibrational mode of each sub-system has roughly the same contribution to the power flows with the other sub-systems.
     - **Remark #3**: This hypothesis allows to extend the fundamental relationship $\Pi_{12} \propto E_1 - E_2$ for coupled oscillators to several coupled sub-systems.
     - **Remark #4**: The last two hypotheses are reasonable for high-frequency bands.

SEA formulation

Two coupled sub-systems

- **Notations**: For the r^{th} sub-system, $r \in \{1, 2\}$, the "blocked" (angular) eigenfrequencies are $\{\omega_{r\alpha}\}_{\alpha \in \mathcal{I}_r}$ and the associated loss factors are $\{\eta_{r\alpha}\}_{\alpha \in \mathcal{I}_r}$. The modes contributing to the response are $\mathcal{I}_r = \{\alpha;\, \omega_{r\alpha} \in I_0\}$. Then the mode count $N_r = \#\mathcal{I}_r \gg 1$.
- The average modal density in the frequency band I_0:

$$n_r(\omega_0) = \frac{N_r}{\Delta\omega}\,.$$

- From the hypothesis [3]: $\eta_{r\alpha} \ll 1$, $\alpha \in \mathcal{I}_r$.
- From the hypothesis [6]: $N_r \gg 1$.

SEA formulation

Two coupled sub-systems

- From the hypothesis [7]: $\omega_{r\alpha}\eta_{r\alpha} \simeq \omega_{r\alpha'}\eta_{r\alpha'}$, $\alpha, \alpha' \in \mathcal{I}_r$. We then introduce an average loss factor $\eta_r(\omega_0)$ in the frequency band I_0 s.t.:

$$\omega_0\eta_r(\omega_0) = \frac{1}{N_r}\sum_{\alpha\in\mathcal{I}_r} \omega_{r\alpha}\eta_{r\alpha} \,.$$

- From the hypothesis [7]: $\mathbb{E}\{\mathcal{E}^r_{\alpha,t}\} = \frac{\pi S_{r\alpha}}{D_{r\alpha}} \simeq E_r$, where E_r is the average modal energy in the frequency band I_0:

$$E_r(\omega_0) = \frac{1}{N_r}\sum_{\alpha\in\mathcal{I}_r} \mathbb{E}\{\mathcal{E}^r_{\alpha,t}\} \,.$$

Consequently, the average mechanical energy of the r^{th} sub-system is $\mathbb{E}\{\mathcal{E}_{r,t}\} \simeq N_r E_r(\omega_0)$.

- From the hypotheses [5,7]:

$$\begin{aligned}\mathbb{E}\{\Pi_{\mathrm{d}r,t}\} &\simeq \sum_{\alpha\in\mathcal{I}_r} \omega_{r\alpha}\eta_{r\alpha}\mathbb{E}\{\mathcal{E}^r_{\alpha,t}\} \\ &\simeq \omega_0\eta_r(\omega_0)\mathbb{E}\{\mathcal{E}_{r,t}\}\,.\end{aligned}$$

Likewise:

$$\begin{aligned}\mathbb{E}\{\Pi_{12,t}\} &\simeq \sum_{\alpha\in\mathscr{I}_1}\sum_{\beta\in\mathscr{I}_2} a^{2\beta}_{1\alpha}(\omega_0)\left[\frac{\pi S_{1\alpha}}{D_{1\alpha}} - \frac{\pi S_{2\beta}}{D_{2\beta}}\right] \\ &= \sum_{\alpha\in\mathscr{I}_1}\sum_{\beta\in\mathscr{I}_2} a^{2\beta}_{1\alpha}(\omega_0)\left[\frac{\mathbb{E}\{\mathcal{E}_{1,t}\}}{N_1} - \frac{\mathbb{E}\{\mathcal{E}_{2,t}\}}{N_2}\right],\end{aligned}$$

in which $D_{r\alpha} = d_r(\boldsymbol{\phi}_{r\alpha}, \boldsymbol{\phi}_{r\alpha})$ (see Lecture #5).

SEA formulation

Two coupled sub-systems

- From the hypothesis [8]: $a^{2\beta}_{1\alpha}(\omega_0) \simeq a^{2\beta'}_{1\alpha'}(\omega_0)$, $\alpha, \alpha' \in \mathcal{I}_1$, $\beta, \beta' \in \mathcal{I}_2$. Consequently:

$$\mathbb{E}\{\Pi_{12,t}\} \simeq N_1 N_2 a^{2\beta}_{1\alpha}(\omega_0) \left[\frac{\mathbb{E}\{\mathcal{E}_{1,t}\}}{N_1} - \frac{\mathbb{E}\{\mathcal{E}_{2,t}\}}{N_2} \right] .$$

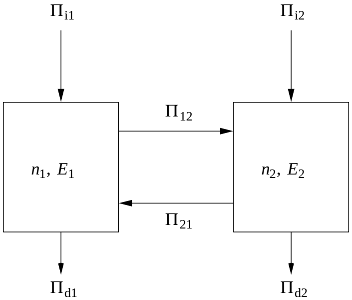
- We then introduce the average coupling loss factor η_{12} in the frequency band I_0 s.t.

$$\omega_0 \eta_{12}(\omega_0) = N_2 a^{2\beta}_{1\alpha}(\omega_0) .$$

Then reciprocity holds (from $\mathbb{E}\{\Pi_{12,t}\} = -\mathbb{E}\{\Pi_{21,t}\}$): $N_1\eta_{12}(\omega_0) = N_2\eta_{21}(\omega_0)$, and

$$\mathbb{E}\{\Pi_{12,t}\} = \omega_0 \left(\eta_{12}(\omega_0)\mathbb{E}\{\mathcal{E}_{1,t}\} - \eta_{21}(\omega_0)\mathbb{E}\{\mathcal{E}_{2,t}\} \right) .$$

SEA formulation
Two coupled sub-systems



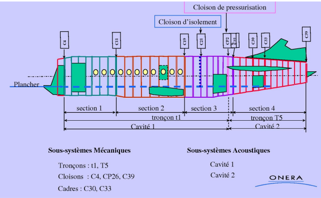
- This finally yields for $r \neq s \in \{1,2\}$:

$$\mathbb{E}\{\Pi_{\mathrm{IN}r,t}\} = \omega_0\eta_r\mathbb{E}\{\mathcal{E}_{r,t}\} + \omega_0\left(\eta_{rs}\mathbb{E}\{\mathcal{E}_{r,t}\} - \eta_{sr}\mathbb{E}\{\mathcal{E}_{s,t}\}\right)\,,$$

or:

$$\omega_0\begin{bmatrix}\eta_1+\eta_{12} & -\eta_{21}\\ -\eta_{12} & \eta_2+\eta_{21}\end{bmatrix}\begin{pmatrix}\mathbb{E}\{\mathcal{E}_{1,t}\}\\ \mathbb{E}\{\mathcal{E}_{2,t}\}\end{pmatrix} = \begin{pmatrix}\mathbb{E}\{\Pi_{\mathrm{IN}1,t}\}\\ \mathbb{E}\{\Pi_{\mathrm{IN}2,t}\}\end{pmatrix}.$$

SEA formulation
n coupled sub-systems

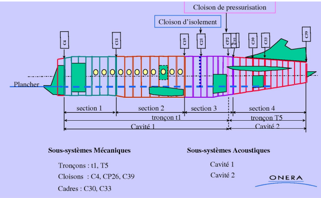


- The relationships obtained for the power flows between two sub-systems are extended to an arbitrary number n of coupled sub-systems ($r \neq s \in \{1, 2, \dots n\}$):

$$
\begin{aligned}
\mathbb{E}\{\Pi_{\mathrm{d}r,t}\} &= \omega_0 \eta_r(\omega_0)\mathbb{E}\{\mathcal{E}_{r,t}\}\,, \\
\mathbb{E}\{\Pi_{rs,t}\} &= \omega_0 \left(\eta_{rs}(\omega_0)\mathbb{E}\{\mathcal{E}_{r,t}\} - \eta_{sr}(\omega_0)\mathbb{E}\{\mathcal{E}_{s,t}\}\right)\,, \\
\mathbb{E}\{\Pi_{\mathrm{IN}r,t}\} &= \mathbb{E}\{\Pi_{\mathrm{d}r,t}\} + \sum_{\substack{s=1 \\ s\neq r}}^{s=n} \mathbb{E}\{\Pi_{rs,t}\}\,.
\end{aligned}
$$

SEA formulation

$\mathscr{n}$ coupled sub-systems



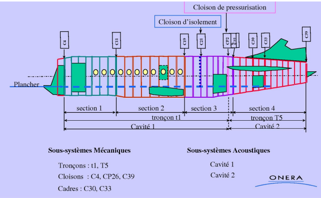
- In matrix form:

$$\omega_0 \begin{bmatrix} \eta_{1,\mathrm{tot}} & -\eta_{21} & -\eta_{31} & \cdots & -\eta_{\mathscr{n}1} \\ -\eta_{12} & \eta_{2,\mathrm{tot}} & -\eta_{32} & \cdots & -\eta_{\mathscr{n}2} \\ -\eta_{13} & -\eta_{23} & \eta_{3,\mathrm{tot}} & \cdots & -\eta_{\mathscr{n}3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\eta_{1\mathscr{n}} & -\eta_{2\mathscr{n}} & -\eta_{3\mathscr{n}} & \cdots & \eta_{\mathscr{n},\mathrm{tot}} \end{bmatrix} \begin{pmatrix} \mathbb{E}\{\mathcal{E}_{1,t}\} \\ \mathbb{E}\{\mathcal{E}_{2,t}\} \\ \mathbb{E}\{\mathcal{E}_{3,t}\} \\ \vdots \\ \mathbb{E}\{\mathcal{E}_{\mathscr{n},t}\} \end{pmatrix} = \begin{pmatrix} \mathbb{E}\{\Pi_{\mathrm{IN}1,t}\} \\ \mathbb{E}\{\Pi_{\mathrm{IN}2,t}\} \\ \mathbb{E}\{\Pi_{\mathrm{IN}3,t}\} \\ \vdots \\ \mathbb{E}\{\Pi_{\mathrm{IN}\mathscr{n},t}\} \end{pmatrix}$$

where:

$$\eta_{r,\mathrm{tot}}(\omega_0) = \eta_r(\omega_0) + \sum_{\substack{s=1 \\ s \neq r}}^{s=\mathscr{n}} \eta_{rs}(\omega_0)\,.$$

SEA formulation
n coupled sub-systems



- If the r^{th} sub-system is coupled to an external acoustic fluid, this apparent loss factor is modified to:

$$\eta_{r,\text{tot}}(\omega_0) = \eta_{r,\text{rad}}(\omega_0) + \eta_r \sqrt{\frac{\varrho_r}{\varrho_r + \varrho_{r,\text{rad}}(\omega_0)}} + \sum_{\substack{s=1 \\ s \neq r}}^{s=n} \eta_{rs}(\omega_0)\,.$$

- Practically $\eta_{r,\text{rad}} < \eta_{rs}$.
- Reciprocity is enforced: $N_r \eta_{rs}(\omega_0) = N_s \eta_{sr}(\omega_0)$.

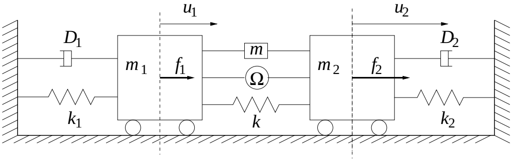
Outline

SEA sub-systems
Overview

- An SEA sub-system is a group of eigenmodes having similar characteristics: energies (*i.e.* H7), coupling strength with the other groups (*i.e.* H8).
- The parameters describing the groups are:
  - the modal density,
  - the loss factor,
  - the coupling loss factor,
  - and the input power,

  averaged over the frequency band of analysis.
- The proportionality relationship between the power flows and the difference of the mechanical energies is rigorously wrong for more than two dofs; it holds approximatively for weak coupling.

- Let $\mu = \frac{\|\boldsymbol{M}_{12}\|}{\sqrt{\|\boldsymbol{M}_1\|\|\boldsymbol{M}_2\|}}$, $\kappa = \frac{\|\boldsymbol{K}_{12}\|}{\sqrt{\|\boldsymbol{K}_1\|\|\boldsymbol{K}_2\|}}$, $\gamma = \frac{\|\boldsymbol{\Omega}\|}{\sqrt{\|\boldsymbol{D}_1\|\|\boldsymbol{D}_2\|}}$. Then if $\mu = \kappa = \gamma = \mathrm{O}(\varepsilon)$, $\eta_{12} = \mathrm{O}(\varepsilon^2)$.
- In case of weak coupling, the mechanical energy of a sub-system in the actual configuration is close to its mechanical energy in the decoupled, or "blocked" configuration.
- Likewise, the eigenfrequencies of a sub-system in the "blocked" and actual configurations are close.



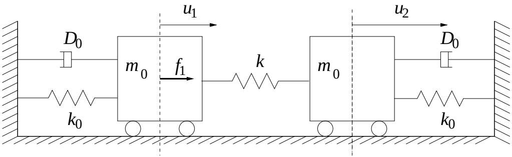
- **Example #1**: two-dofs system. The "blocked" eigenfrequencies and half-power bandwidths are

$$\omega_\alpha = \sqrt{\frac{M_\alpha}{K_\alpha}}\,, \quad \Delta_\alpha = \eta_\alpha\omega_\alpha = \frac{D_\alpha}{M_\alpha}\,,$$

respectively, with $M_\alpha = m_\alpha + \frac{m}{4}$ and $K_\alpha = k_\alpha + k$.

- If $\omega_1 \simeq \omega_2$: then $\mathbb{E}\{\Pi_{12,t}\} \propto (\Delta_1 + \Delta_2)^{-1}$. This is strong coupling.
- If $|\omega_1 - \omega_2| > \Delta_1 + \Delta_2$: then $\mathbb{E}\{\Pi_{12,t}\} \propto \Delta_1 + \Delta_2$. This is weak coupling.

SEA sub-systems
Coupling strength



- **Example #2**: two-dofs system with stiffness coupling and no load on the second mass,

$$\begin{bmatrix} m_0 & 0 \\ 0 & m_0 \end{bmatrix} \begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix} + \begin{bmatrix} D_0 & 0 \\ 0 & D_0 \end{bmatrix} \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} + \begin{bmatrix} k_0 + k & -k \\ -k & k_0 + k \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \end{pmatrix} .$$

- The actual eigenfrequencies are:

$$\underline{\omega}_1 = \omega_0 = \sqrt{\frac{k_0}{m_0}} , \quad \underline{\omega}_2 = \omega_0 \sqrt{1 + 2\frac{k}{k_0}} ,$$

whereas the "blocked" ones are:

$$\omega_1 = \omega_2 = \sqrt{\frac{k_0 + k}{m_0}} .$$

SEA modeling

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Modeling issues
Defining sub-systems
Modal density
Loss factor
Coupling loss factor
Power inputs

Bibliography

- One can compute exactly:

$$\frac{\mathbb{E}\{\mathcal{E}_{2,t}\}}{\mathbb{E}\{\mathcal{E}_{1,t}\}} = \frac{\eta_{21}}{\eta_0 + \eta_{21}} \,,$$

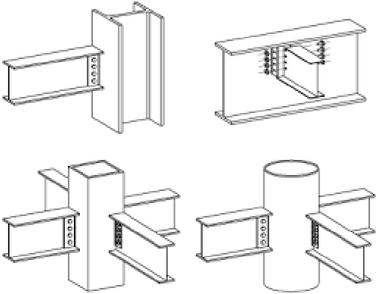
with $\eta_0 = \frac{D_0}{\sqrt{m_0(k_0+k)}}$ and $\eta_{12} = \eta_{21} = \frac{1}{2\eta_0}(\frac{k}{k_0+k})^2$.

- If $\frac{k_0}{k} \to 0$ (strong coupling): then $\eta_{21} \to \frac{1}{2\eta_0}$;
- If $\frac{k}{k_0} \to 0$ (weak coupling): then $\eta_{21} = \mathrm{O}(\frac{k}{k_0})^2$.

SEA sub-systems
Groups of modes

- Sub-systems should be identified in order to satisfy the SEA hypotheses, and more particularly:
  - The modes in a sub-system should have similar dynamic characteristics;
  - Those sub-systems should be considered as groups of weakly coupled modes.
- A group of modes is carried on by a physical sub-system, which in turn can support different groups.
- They are coupled as soon as they are constrained to have the same motion at a given point, line, or surface of the overall physical system.

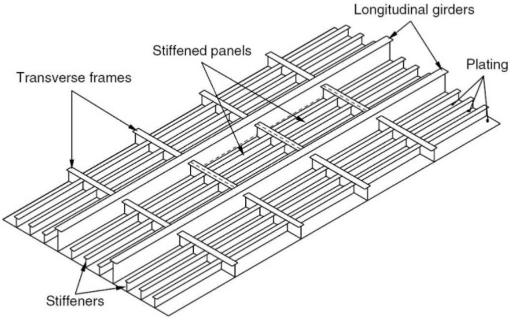
SEA sub-systems
Groups of modes



- **Example #3**: beam junction. The longitudinal motion of the vertical beam is coupled to the bending motion of the horizontal beams. The latter are coupled to the bending motion of the vertical beam, thus the longitudinal and bending motions of the vertical beam are coupled.

SEA sub-systems

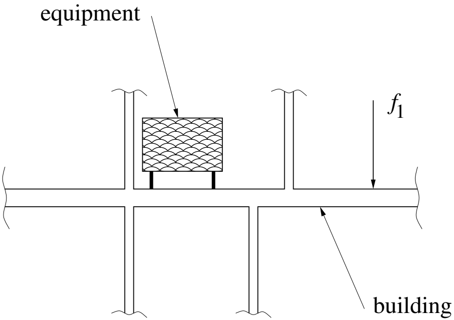
Groups of modes



- **Example #4**: stiffened panel. Sub-systems = local modes of the plates $\oplus$ local modes of the stiffeners $\oplus$ global modes.

SEA sub-systems
Energy sinks

- The SEA framework can be used to model a limited part of a complex system, considering the remaining parts as "energy sinks".
- **Example #5**: noise induced by an electromechanical device in a building.



- **Example #6**: two coupled sub-systems without load on the second one,

$$\omega_0 \begin{bmatrix} \eta_1 + \eta_{12} & -\eta_{21} \\ -\eta_{12} & \eta_2 + \eta_{21} \end{bmatrix} \begin{pmatrix} \mathbb{E}\{\mathcal{E}_{1,t}\} \\ \mathbb{E}\{\mathcal{E}_{2,t}\} \end{pmatrix} = \begin{pmatrix} \mathbb{E}\{\Pi_{\mathrm{IN}1,t}\} \\ 0 \end{pmatrix} .$$

- Then $\frac{\mathbb{E}\{\mathcal{E}_{2,t}\}}{\mathbb{E}\{\mathcal{E}_{1,t}\}} = \frac{n_2}{n_1}\frac{\eta_{21}}{\eta_2+\eta_{21}}$ and:

$$\begin{aligned} \mathbb{E}\{\Pi_{\mathrm{IN}1,t}\} &= \omega_0\eta_1\mathbb{E}\{\mathcal{E}_{1,t}\} + \omega_0\left(\eta_{12}\mathbb{E}\{\mathcal{E}_{1,t}\} - \eta_{21}\mathbb{E}\{\mathcal{E}_{2,t}\}\right) \\ &= \omega_0\eta_{1,\mathrm{net}}\mathbb{E}\{\mathcal{E}_{1,t}\} , \end{aligned}$$

where $\eta_{1,\mathrm{net}} = \eta_1 + \frac{n_2}{n_1}(\frac{\eta_2\eta_{21}}{\eta_2+\eta_{21}})$: the net loss factor.

Modal density
One-dimensional system

- In a bounded one-dimensional system of length L, $\frac{\lambda}{2} = \frac{L}{\alpha}$, $\alpha \in \mathbb{N}^*$. Therefore the wave numbers and eigenfrequencies have the form:

$$k_\alpha = (\alpha + \delta_{\mathrm{BC}})\frac{\pi}{L}\,, \quad \omega_\alpha = (\alpha + \delta_{\mathrm{BC}})\frac{\pi c}{L}\,,$$

where $\delta_{\mathrm{BC}} \leqslant 1$ depends on the boundary conditions, and c (in m.s^{-1}) may be a function of the frequency.

- The modal density:

$$n^{1\mathrm{D}}(\omega) = \frac{\mathrm{d}\alpha}{\mathrm{d}\omega} = \frac{\mathrm{d}\alpha}{\mathrm{d}k} \times \frac{\mathrm{d}k}{\mathrm{d}\omega} = \frac{L}{\pi c_{\mathrm{g}}(\omega)}\,,$$

where $c_{\mathrm{g}} = \frac{\mathrm{d}\omega}{\mathrm{d}k}$ is the group velocity.

- **Remark** (group velocity): Consider the wave trains $\cos(\omega t - kx)$ and $\cos\left[(\omega + \mathrm{d}\omega)t - (k + \mathrm{d}k)x\right]$, their sum is:

$$2\cos\left(\frac{\mathrm{d}\omega}{2}t - \frac{\mathrm{d}k}{2}x\right)\cos\left[\left(\omega + \frac{\mathrm{d}\omega}{2}\right)t - \left(k + \frac{\mathrm{d}k}{2}\right)x\right].$$

- The average wave front is located at the position x and time t s.t. $t\mathrm{d}\omega - x\mathrm{d}k = 0$, thus it travels at the group velocity:

$$c_\mathrm{g} = \frac{\mathrm{d}\omega}{\mathrm{d}k}.$$

- The group velocity is the speed of transport of the energy.

Modal density
One-dimensional system

- **Example**: bending of a simply-supported beam (considering Euler-Bernoulli kinematics),

$$\omega = \frac{(\pi\alpha)^2}{L^2}\sqrt{\frac{D}{\varrho A}} = \frac{\pi\alpha}{L}c_\mathrm{b}(\omega)\,,$$

where $c_\mathrm{b}(\omega) = \sqrt{\omega}\sqrt[4]{\frac{D}{\varrho A}}$ is the bending phase velocity, D is the bending stiffness, ϱ is the density.

- Other modes:

	c (m.s^{-1})	c_g (m.s^{-1})
traction	$\sqrt{\frac{E}{\varrho}}$	c
pure shear	$\sqrt{\frac{\kappa G}{\varrho}}$	c
torsion	$\sqrt{\frac{GJ}{\varrho I}}$	c
bending	$\sqrt{\omega}\sqrt[4]{\frac{D}{\varrho A}}$	$2c$

- The wave numbers of a square two-dimensional system of size $L_1 \times L_2$ have the form:

$$k_{\boldsymbol{\alpha}} = \sqrt{\left[(\alpha_1 + \delta_1)\frac{\pi}{L_1}\right]^2 + \left[(\alpha_2 + \delta_2)\frac{\pi}{L_2}\right]^2}\,,$$

where δ_1 and δ_2 depend on the boundary conditions.

- The number of modes $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ s.t. $k_\alpha \leqslant k$:

$$\begin{aligned}\alpha_1(k) &= \frac{L_1}{\pi}\sum_{\alpha_2=1}^{\alpha_{\max}}\sqrt{k^2 - \left[(\alpha_2 + \delta_2)\frac{\pi}{L_2}\right]^2} - \delta_1 \\ &= \frac{A}{4\pi}k^2 + \gamma_{\mathrm{BC}} P k\,,\end{aligned}$$

where $\alpha_{\max}$ is s.t. the square-rooted terms remain positive, $A = L_1 L_2$, $P = 2(L_1 + L_2)$, and γ_{BC} depends on the boundary conditions.

- The modal density:

$$n^{2\mathrm{D}}(\omega) = \frac{\mathrm{d}\alpha}{\mathrm{d}\omega} = \frac{\mathrm{d}\alpha}{\mathrm{d}k} \times \frac{\mathrm{d}k}{\mathrm{d}\omega} = \frac{1}{c_{\mathrm{g}}(\omega)} \left(\frac{Ak}{2\pi} + \gamma_{\mathrm{BC}} P \right) .$$

- But $k = \frac{\omega}{c}$ where c is the phase velocity, thus:

$$n^{2\mathrm{D}}(\omega) = \frac{A\omega}{2\pi c_{\mathrm{g}} c} + \Gamma_{\mathrm{BC}}(\omega) P ,$$

where $\Gamma_{\mathrm{BC}}(\omega) \to 0$ as $\omega \to +\infty$.

- The modal density is less and less sensitive to the boundary conditions and the shape as the frequency increases. This formula is then used in SEA for arbitrary boundary conditions and shapes.

Modal density
Two-dimensional system

- **Example**: bending of a simply-supported thin plate (considering Kirchhoff-Love kinematics),

$$n^{\mathrm{b}} = \frac{A\omega}{4\pi c_{\mathrm{b}}^2(\omega)} \quad \text{(independent of the frequency)}\,,$$

where $c_{\mathrm{b}}(\omega) = \sqrt{\omega}\sqrt[4]{\frac{Eh^2}{12\varrho(1-\nu^2)}}$ is the bending phase velocity, $c_{\mathrm{g}}(\omega) = 2c_{\mathrm{b}}(\omega)$, E is the Young modulus, ν is the Poisson ratio, h is the thickness, ϱ is the density.

- The shear and compressional modes are non dispersive, and get coupled by the boundaries. Hence the corresponding modal density accounts for both groups:

$$n^{\mathrm{PS}}(\omega) = n^{\mathrm{P}}(\omega) + n^{\mathrm{S}}(\omega) = \frac{A\omega}{2\pi}\left(\frac{1}{c_{\mathrm{P}}^2} + \frac{1}{c_{\mathrm{S}}^2}\right)\,,$$

where $c_{\mathrm{P}} = \sqrt{\frac{E}{\varrho(1-\nu^2)}}$, $c_{\mathrm{S}} = \sqrt{\frac{G}{\varrho}}$, and $G = \frac{E}{2(1+\nu)}$ is the shear modulus.

Modal density
Three-dimensional system

- Modal density of a three-dimensional system of size $L_1 \times L_2 \times L_3$ (*e.g.* an acoustic cavity):

$$n^{3\mathrm{D}}(\omega) = \frac{V\omega^2}{2\pi^2 c_\mathrm{g} c^2} + \frac{A\omega}{2\pi c_\mathrm{g} c}\Gamma_1(\omega) + P\Gamma_2(\omega)\,,$$

where $\Gamma_j(\omega) \to 0$ as $\omega \to +\infty$, $j = 1, 2$, depend on the boundary conditions, and $V = L_1 L_2 L_3$ (volume), $A = 2(L_1 L_2 + L_1 L_3 + L_2 L_3)$ (area), $P = 4(L_1 + L_2 + L_3)$ (total length of the edges).

Modal density
Experimental determination

- **Method #1**: Measure some frequency response functions, and then count the resonance peaks. The average distance between successive resonances should be at least three times the equivalent bandwidth, $\frac{1}{n(\omega)} \gtrsim 3\pi\xi_\alpha\omega_\alpha$.
- **Method #2**: Measure the drive-point conductance $G_{\boldsymbol{x}}(\omega) \stackrel{\text{def}}{=} \Re\mathrm{e}\{\mathrm{i}\omega\widehat{\mathbb{h}}(\boldsymbol{x},\boldsymbol{x},\omega)\}$ (the real part of the mobility) at several locations $\boldsymbol{x} \in \Omega$ and then spatially average because (see Lecture notes - Appendix C for proofs):

$$\boxed{\langle G_{\boldsymbol{x}}(\omega)\rangle_\Omega = \frac{\pi}{2}\frac{n(\omega)}{M}},$$

where M is the total mass of the system occupying Ω.

Modal density
Numerical determination

- Compute the frequency response functions by a finite element model, or any other numerical technique (finite differences, particle-based and Monte-Carlo methods...).
- Four elements/wavelength are usually enough to get a good estimate with an error lower than 10%.

- The loss factor η_{s} of an elastic structure is obtained by fitting some experimental data with the equivalent result of a mathematical model, hence it has no theoretical ground. The fitting depends in addition on the frequency/frequency band of analysis I_0.
- Reverberation time: the time of persistence of sound or vibration after a sound or vibration is produced. It is typically measured as the time T_R it takes for a signal (vibrational energy) to drop by 60 dB.
- Practically, one measures the mechanical energy of the free response to some impulse loads:

$$\mathcal{E}^{\ell}(T_R) \simeq \mathcal{E}_0 \, \mathrm{e}^{-\eta_{\mathrm{s}}\omega_0 T_R} = 10^{-6}\mathcal{E}_0 \,,$$

so that:

$$T_R = \frac{13.8}{\eta_{\mathrm{s}}\omega_0} \,.$$

- Both (i) the fluid viscosity and (ii) the boundary absorption on the walls by multiple reflections of the waves contribute to the loss factor of an acoustic cavity. Therefore:

$$\eta_{\mathrm{f}} = \frac{c_{\mathrm{f}}}{\omega_0}\left(\frac{\alpha A}{4V} + \nu_{\mathrm{f}}\right) ,$$

  where α is Sabine's absorption coefficient of the walls ($\alpha \simeq 0.1$ in the air), and ν_{f} is the attenuation coefficient due to the fluid viscosity (it depends on its density and dynamic viscosity).
- The contribution (ii) of ν_{f} is usually negligible (*e.g.* air).

Equivalent loss factor
Radiation efficiency

- The average power radiated by a structure:

$$\mathbb{E}\{\Pi_{\mathrm{rad},t}\} = \sigma_{\mathrm{rad}}(\omega_0)\varrho_{\mathrm{f}} c_{\mathrm{f}} |\Gamma|\, \mathbb{E}\{\dot{U}_t^2\}\,,$$

where $\varrho_{\mathrm{f}} c_{\mathrm{f}} A\overline{V}^2$ is the power radiated by a plate of area A with uniform celerity $\overline{V}$.

- The radiation efficiency σ_{rad} is used to evaluated the equivalent loss factor η_{rad}:

$$\omega_0\eta_{\mathrm{rad}}(\omega_0) = \frac{\varrho_{\mathrm{f}} c_{\mathrm{f}}}{\varrho_{\mathrm{eff}}(\omega_0)}\sigma_{\mathrm{rad}}(\omega_0)\,,$$

where $\varrho_{\mathrm{eff}}(\omega_0) = \frac{|\Omega_{\mathrm{s}}|}{|\Gamma|}(\varrho_{\mathrm{s}} + \varrho_{\mathrm{rad}}(\omega_0))$ is the effective surface density of the structure.

Coupling loss factor
Point junction

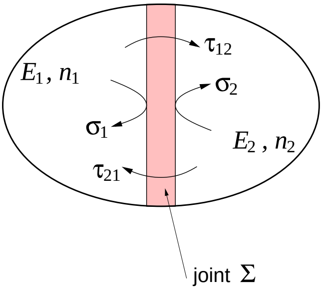
- The coupling loss factor η_{12} of a point junction between two sub-systems:

$$\eta_{12}(\omega_0) \simeq \frac{2}{\pi\omega_0 n_1(\omega_0)} \left\langle \frac{\Re e\{Z_1(\omega_0\}\Re e\{Z_1(\omega_0\}}{|Z_1(\omega_0) + Z_2(\omega_0)|^2} \right\rangle_{\Delta\omega},$$

where $Z_r(\omega) = [\mathrm{i}\omega\widehat{\mathbb{h}}_r(\boldsymbol{x}_J, \boldsymbol{x}_J, \omega)]^{-1}$ is the drive-point impedance of the r^{th} sub-system at the junction $\boldsymbol{x}_J$; and $\langle\cdot\rangle_{\Delta\omega}$ stands for frequency averaging in the frequency band of analysis I_0.

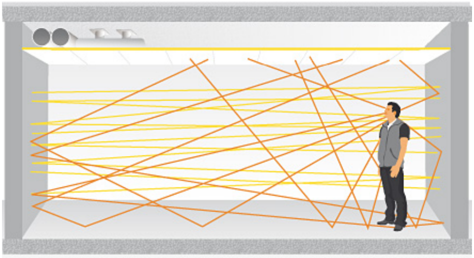
Coupling loss factor

Diffuse field approach



- Consider two sub-systems coupled through a third sub-system (a joint Σ) characterized by some energy fluxes transmission and reflection coefficients τ_{rs} and σ_r respectively, $r \neq s \in \{1, 2\}$.

- **Diffuse field approximation**: the vibrational field within the r^{th} sub-system is diffuse if it is the superposition of uncorrelated wave trains traveling in all directions, the latter being uniformly random.
- The average mechanical energy $\mathbb{E}\{\mathcal{E}_{r,t}\}$ is in addition uniformly distributed over the frequency band of analysis.



Coupling loss factor

Diffuse field approach

- In the diffuse field approximation, the energy flux density impinging the area $\mathrm{d}\sigma$ of the joint Σ within the solid angle $\mathrm{d}\hat{\Omega}$ centered on the incident direction $\hat{\mathbf{k}}$ is:

$$\mathrm{d}\boldsymbol{\pi}_{\mathrm{IN}r}(\hat{\mathbf{k}}) = c_{\mathrm{g}r}\mathbb{E}\{\mathcal{E}_{r,t}\}\hat{\mathbf{k}}\frac{\mathrm{d}\sigma}{|\Omega_r|}\frac{\mathrm{d}\hat{\Omega}(\hat{\mathbf{k}})}{|\hat{\Omega}_d|}\,,$$

  where $|\hat{\Omega}_2| = 2\pi$ ($d = 2$ in 2D) or $|\hat{\Omega}_3| = 4\pi$ ($d = 3$ in 3D) is the total solid angle, and $c_{\mathrm{g}r}$ is the group velocity.

- The energy flux density transmitted through the joint from the sub-system #1 to the sub-system #2:

$$\mathrm{d}\boldsymbol{\pi}_{1\to 2} = \tau_{12}(\hat{\mathbf{k}}) \times \mathrm{d}\boldsymbol{\pi}_{\mathrm{IN}1}(\hat{\mathbf{k}})\,.$$

Coupling loss factor
Diffuse field approach

- Then the average power transmitted from the sub-system #1 to the sub-system #2 is:

$$
\begin{aligned}
\mathbb{E}\{\Pi_{1\to 2,t}\} &\simeq \int_{\hat{\mathbf{k}}} \frac{\mathrm{d}\hat{\Omega}(\hat{\mathbf{k}})}{|\hat{\Omega}_d|} \int_{\Sigma} \frac{\mathrm{d}\sigma}{|\Omega_1|} c_{\mathrm{g}1} \mathbb{E}\{\mathcal{E}_{1,t}\}(\hat{\mathbf{k}}\cdot\mathbf{n})\tau_{12}(\hat{\mathbf{k}}) \\
&= \frac{\pi}{|\hat{\Omega}_d|}\frac{|\Sigma|}{|\Omega_1|}\langle\tau_{12}\rangle c_{\mathrm{g}1}\mathbb{E}\{\mathcal{E}_{1,t}\}\,,
\end{aligned}
$$

  where $\langle\tau_{12}\rangle = \frac{1}{\pi}\int_{\hat{\mathbf{k}}\cdot\mathbf{n}>0}(\hat{\mathbf{k}}\cdot\mathbf{n})\tau_{12}(\hat{\mathbf{k}})\mathrm{d}\hat{\Omega}(\hat{\mathbf{k}})$.

- The power balance for the 1st sub-system reads:

$$
\mathbb{E}\{\Pi_{\mathrm{IN}1,t}\} + \mathbb{E}\{\Pi_{2\to 1,t}\} = \mathbb{E}\{\Pi_{\mathrm{d}1,t}\} + \mathbb{E}\{\Pi_{1\to 2,t}\}\,,
$$

  where equivalently:

$$
\mathbb{E}\{\Pi_{2\to 1,t}\} = \frac{\pi}{|\hat{\Omega}_d|}\frac{|\Sigma|}{|\Omega_2|}\langle\tau_{21}\rangle c_{\mathrm{g}2}\mathbb{E}\{\mathcal{E}_{2,t}\}\,,
$$

- Therefore:

$$\mathbb{E}\{\Pi_{\mathrm{IN}1,t}\} = \mathbb{E}\{\Pi_{\mathrm{d}1,t}\} + \left(\frac{\pi}{|\hat{\Omega}_d|}\frac{|\Sigma|}{|\Omega_1|}\langle\tau_{12}\rangle c_{\mathrm{g}1}\mathbb{E}\{\mathcal{E}_{1,t}\} - \frac{\pi}{|\hat{\Omega}_d|}\frac{|\Sigma|}{|\Omega_2|}\langle\tau_{21}\rangle c_{\mathrm{g}2}\mathbb{E}\{\mathcal{E}_{2,t}\}\right).$$

- Coupling loss factor in the diffuse field approximation:

$$\omega_0\eta_{rs}(\omega_0) = \frac{\pi|\Sigma|}{|\hat{\Omega}_d|}\frac{c_{\mathrm{g}r}\langle\tau_{rs}\rangle}{|\Omega_r|},$$

where reciprocity $n_r\eta_{rs} = n_s\eta_{sr}$ stems from the fact that $c_r^{1-d}\tau_{rs} = c_s^{1-d}\tau_{sr}$ in that approximation, and $n_r \propto \frac{|\Omega_r|}{c_{\mathrm{g}r}c_r^{d-1}}$.

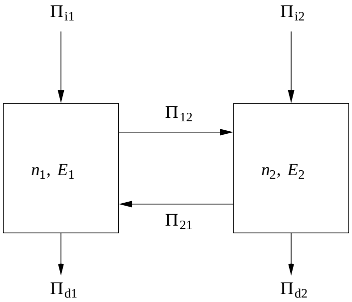
Coupling loss factor
Experimental determination

- **Method #1**: Power injection method (PIM). The input power Π_{IN} is imposed to each sub-system successively, and the mechanical energies are measured for all sub-systems. Then one solves for $[\boldsymbol{\eta}]$:

$$\omega_0[\boldsymbol{\eta}]\boldsymbol{E}_r = \boldsymbol{\Pi}_{\mathrm{IN}r}\,,$$

  where $\boldsymbol{\Pi}_{\mathrm{IN}r} = (0,\ldots,0,\Pi_{\mathrm{IN}},0,\ldots 0)^{\mathsf{T}}$ is the vector of input powers when only the r^{th} sub-system is loaded, and $\boldsymbol{E}_r$ is the vector of mechanical energies.
- Constraints on $[\boldsymbol{\eta}]$: symmetry, negative off-diagonal terms.
- However $[\boldsymbol{\eta}]$ is ill-conditioned.

Coupling loss factor
Experimental determination



- **Method #2**: To single out a junction at a time. If $\Pi_{IN2} = 0$ then:

$$\frac{E_2}{E_1} = \frac{n_2}{n_1}\frac{\eta_{21}}{\eta_2 + \eta_{21}} \quad \Rightarrow \quad \eta_{12} = \frac{\eta_2 E_2}{E_1 - \frac{n_1}{n_2}E_2}\,.$$

- One measures the loss factor η_2 of the receiving system (possibly adding some damping in order to have $E_2 \ll E_1$) and E_1, E_2.

- **Remark #1**: The mechanical energy of an elastic sub-system:

$$E_r = M_r \langle v^2 \rangle_{\Omega_r} \,.$$

The mechanical energy of an acoustic cavity:

$$E_r = \frac{V}{\varrho c} \langle p^2 \rangle_{\Omega_r} \,.$$

- **Remark #2**: In room and building acoustics the transmissivity $\langle \tau_{sr} \rangle$ is given in the form of the transmission loss factor TL:

$$\mathrm{TL} = -10 \log \langle \tau_{sr} \rangle \,.$$

Power inputs

- The input power for the r^{th} sub-system:

$$\Pi_{\mathrm{IN}r}(t) = \int_{\Omega_r} \boldsymbol{f}_r(t) \cdot \dot{\boldsymbol{u}}_r^f(t)\, \mathrm{d}\boldsymbol{x}\,.$$

- The loads $t \mapsto \boldsymbol{f}_r(t)$ are known, but the forced response $t \mapsto \boldsymbol{u}_r^f(t)$ of the r^{th} sub-system coupled to all the other sub-systems and considering all applied loads is unknown.
- The input power for the r^{th} sub-system considered in isolation, when the other sub-systems are "blocked":

$$\Pi_{\mathrm{IN}r}^{\mathrm{b}}(t) = \int_{\Omega_r} \boldsymbol{f}_r(t) \cdot \dot{\boldsymbol{u}}_r^{\mathrm{b}}(t)\, \mathrm{d}\boldsymbol{x}\,.$$

- Considering weakly coupled sub-systems, SEA assumes:

$$\mathbb{E}\{\Pi_{\mathrm{IN}r,t}\} \simeq \mathbb{E}\{\Pi_{\mathrm{IN}r,t}^{\mathrm{b}}\}\,.$$

Power inputs

- "Equivalence" between:
  - The overall input power $\int_{\mathbb{R}} \Pi^{\mathrm{b}}_{\mathrm{IN}r}(t)\mathrm{d}t$ for wideband loads with deterministic amplitudes and point supports;
  - The average overall input power $\int_{\mathbb{R}} \mathbb{E}\{\Pi^{\mathrm{b}}_{\mathrm{IN}r,t}\}\mathrm{d}t$ for wideband loads with deterministic amplitudes and uniformly random point supports within Ω_r;
  - The average input power $\mathbb{E}\{\Pi^{\mathrm{b}}_{\mathrm{IN}r,t}\}$ for wideband stationary random loads with deterministic or uniformly random supports within Ω_r;
  - The average input power $\mathbb{E}\{\langle\Pi^{\mathrm{b}}_{\mathrm{IN}r,t}\rangle_{\Omega_r}\}$ for harmonic loads with deterministic or uniformly random point supports, and Ω_r s.t. its "blocked" eigenfrequencies are uniformly random and independent in the frequency band of analysis I_0.

Power inputs

- These averages then read:

$$\mathbb{E}\{\Pi^{\mathrm{b}}_{\mathrm{IN}r,t}\} \simeq \frac{\pi}{2}\frac{n_r(\omega_0)}{M_r}\langle\mathbb{E}\{|\boldsymbol{F}_{r,t}|^2\}\rangle_{\Delta\omega}\,,$$

  where $\langle\mathbb{E}\{|\boldsymbol{F}_{r,t}|^2\}\rangle_{\Delta\omega}$ stands for the total power of the loads within I_0, and $\langle G_{\boldsymbol{x}}(\omega)\rangle_{\Omega_r} = \frac{\pi}{2}\frac{n_r(\omega_0)}{M_r}$ stands for the average drive-point conductance of the r^{th} sub-system.

- **Example**: $(\boldsymbol{F}_{r,t},\, t\in\mathbb{R})$ is a m.s. stationary point force, of which spectral density function reads

$$\boldsymbol{S}_{\boldsymbol{F}_r}(\omega) = \boldsymbol{S}_r \otimes \mathbb{1}_{I_0\cup \underline{I}_0}(\omega)\,.$$

  Then $\langle\mathbb{E}\{|\boldsymbol{F}_{r,t}|^2\}\rangle_{\Delta\omega} \equiv 2\Delta\omega\,\mathrm{Tr}\,\boldsymbol{S}_r$.

Power inputs
Experimental determination

- Step #1: consider the r^{th} sub-system and measure its reverberant response to some known impulse loads: $\langle v^2 \rangle_{\Omega_r}$ (structure) or $\langle p^2 \rangle_{\Omega_r}$ (cavity) is deduced, then $\mathbb{E}\{\mathcal{E}_{r,t}\}$.
- Step #2: turn-off the loads, and deduce the total or net loss factor $\eta_{r,\text{net}}$ from the reverberation time. Then:

$$\mathbb{E}\{\Pi_{\text{IN}r,t}\} \simeq \omega_0 \eta_{r,\text{net}}(\omega_0) \mathbb{E}\{\mathcal{E}_{r,t}\} \,.$$

- **Remark**: Power inputs are the most difficult to estimate, and the most critical to the overall response.

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