Primer on random processes

MG3416–Advanced Structural Acoustics – Lecture #1 part A

É. Savin[1,2]
eric.savin@{centralesupelec,onera}.fr

[1]Information Processing and Systems Dept.
ONERA, France

[2]Mechanical and Environmental Engineering Dept.
CentraleSupélec, France

September 22, 2021

Outline

Outline

Probability space
Formal definition

- We model a real-world process, or "experiment", or "system", consisting of states occurring randomly.
- It is mathematically represented by a triplet $(\Omega, \mathcal{F}, P)$ of which elements are ultimately specified by the modeler:
  - Ω is the set of outcomes, or sample space, or universe, constituted by the states of the causes which influence the states of the system;
  - $\mathcal{F}$ is a σ-algebra, *i.e.* a set of all the events the modeler considers. An event is a set of zero or more outcomes, thus it is a subset of the sample space;
  - P is a function returning an event's probability, or a probability measure.

Probability space

σ-algebra

Definition

A collection $\mathcal{F}$ of subsets of Ω is a σ-algebra, or σ-field, if:

1. $\emptyset \in \mathcal{F}$ *and* $\Omega \in \mathcal{F}$;
2. $a \in \mathcal{F} \quad \Rightarrow \quad a^c = \Omega \backslash a \in \mathcal{F}$;
3. *If* $\{a_n\}_{n=1,2,\dots}$ *is an arbitrary countable family of events in* $\mathcal{F}$, *then* $\bigcup_{n \in \mathbb{N}} a_n \in \mathcal{F}$.

- **Example #1**: a fair coin, $\Omega = \{1, 2\}$. Then the σ-algebra $\mathcal{F} = 2^\Omega = \{\emptyset, \{1\}, \{2\}, \Omega\}$ contains $2^2 = 4$ events.
- **Example #2**: a fair die, $\Omega = \{1, 2, 3, 4, 5, 6\}$. Then the σ-algebra $\mathcal{F} = 2^\Omega = \{\emptyset, \{1\}, \{2\}, \dots\{6\}, \{1, 2\}, \{3, 4\}, \{5, 6\}, \{1, 3\}, \dots$ $\{1, 3, 5\}, \{2, 4, 6\}, \{1, 3, 4, 5\}, \dots, \Omega\}$ contains $2^6 = 64$ events.
- **Example #3**: a fair die, $\Omega = \{1, 2, 3, 4, 5, 6\}$. The sets $\mathcal{F} = \{\emptyset, \{2, 4, 6\}, \{1, 3, 5\}, \Omega\}$, $\mathcal{F} = \{\emptyset, \{2, 5\}, \{1, 3, 4, 6\}, \Omega\}$... are also suitable σ-algebras.

Definition

A probability measure is a positive, bounded measure the total mass of which is 1: $P(\Omega) = 1$.

1. *A positive measure* m *is a function* $m : \Omega \to [0, +\infty]$ *s.t. for countably many disjoint events* $\{a_n\}_{n=1,2,\dots} \in \mathcal{F}$, $\bigcap_n a_n = \varnothing$, $\bigcup_n a_n \in \mathcal{F}$, *then*

$$m\left(\bigcup_n a_n\right) = \sum_n m(a_n)\,;$$

2. *It is bounded if* $m(\Omega) < +\infty$;
3. *It is a probability measure if* $m(\Omega) = 1$.

Random variable
Definitions

Definition

A random variable is a measurable function $X : \Omega \to \Omega'$ which maps a combination of outcomes from a sample space Ω to a combination of consequences in a state space Ω'.

Definition

A function $f : \Omega \to \Omega'$ is measurable if $\forall a' \in \Omega'$, $f^{-1}(a')$ is measurable in Ω: $\forall a' \in \mathcal{F}'$, $f^{-1}(a') \in \mathcal{F}$, where $\mathcal{F}$, $\mathcal{F}'$ are σ-algebras defined on Ω, Ω' respectively.

- **Example #1**: let Ω be the sample space describing the real-world system "tossing a fair coin": $\Omega = \{$"temperature", "pressure", "initial position", "initial speed", *etc.*$\}$. Then $\Omega' = \{1, 2\}$.
- **Example #2**: let Ω be the sample space describing a real-world system "roll of dice": $\Omega = \{$"temperature", "pressure", "initial position", "initial speed", *etc.*$\}$. Then $\Omega' = \{1, 2, 3, 4, 5, 6\}$.

Random variable

Causality principle

- Let $X : \Omega \to \Omega'$ be a random variable, then Ω' is endowed with the σ-algebra $\mathcal{F}'$ and the probability measure P' inherited from $\mathcal{F}$ and P by the causality principle:

$$\forall a' \in \mathcal{F}' \,, \quad P'(a') = P(X^{-1}(a')) \,.$$

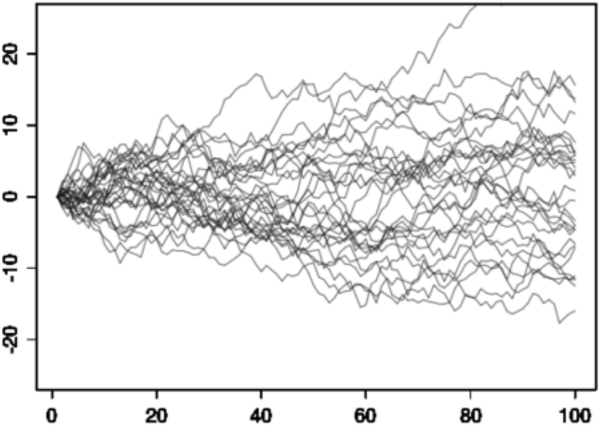
  P' is now the probability measure of X on Ω'.
- The original sample space Ω (the system and its environment in the foregoing examples) is often suppressed, since it is mathematically hard to describe.
- **Examples**: The possible values of the random variable "coin toss" or "dice roll" are then treated as a sample space Ω' constituted by numerical quantities–which is easier to deal with.

Definition

A stochastic process $\begin{array}{rl} \boldsymbol{X} : & T \to \mathbb{R}^n \\ & t \mapsto \boldsymbol{X}_t \end{array}$ *is a function from* T *into the set of random variables defined on* $(\Omega, \mathcal{F}, P)$ *with values in* $\mathbb{R}^n \equiv \Omega'$. $(\boldsymbol{X}_t, t \in T)$ *is a collection of random variables with values in* $\mathbb{R}^n$. *Therefore:*

- *If* $t \in T$ *is frozen:* $\boldsymbol{X}_t$ *is a random variable defined on* $(\Omega, \mathcal{F}, P)$ *with values in* $\mathbb{R}^n$;
- *If* $a \in \Omega$ *is frozen:* $t \mapsto \boldsymbol{X}_t(a) = \boldsymbol{x}(t)$ *is a trajectory, or a sample path;*
- *If both* $t \in T$ *and* $a \in \Omega$ *are frozen:* $a, t \mapsto \boldsymbol{X}_t(a)$ *is an outcome, or a realization, of the random variable* $\boldsymbol{X}_t$.

Stochastic process



Stochastic process indexed on $\mathbb{R}_+$.

- $(\boldsymbol{X}_t\,, t \in T)$ is called "the stochastic process defined on $(\Omega, \mathcal{F}, P)$, indexed on T, with values in $\mathbb{R}^n$".
- The indexation set T could be any set, countable or not, bounded or not, *etc.*
- However we will only consider the real line $T = \mathbb{R}$ in the subsequent lectures, since t will be a time.
- **Examples**: earthquake, wind, turbulence, swell, ocean surge... can be modeled by stochastic processes, where $\Omega = \{$"the states of the environment"$\}$ is very difficult to figure out, but $\Omega' = \{$"vectors of $\mathbb{R}^3$"$\}$ is much easier to handle.

Stochastic process
Probability law

- For $t \in \mathbb{R}$ fixed, $\boldsymbol{X}_t$ is a $\mathbb{R}^n$-valued random variable defined on $(\Omega, \mathcal{F}, P)$, of which probability law $P_{\boldsymbol{X}}$ is parameterized by t and defined by:

$$\begin{aligned} P_{\boldsymbol{X}}(B;t) &= P'(\boldsymbol{X}_t \in B) \\ &= P(\boldsymbol{X}_t^{-1}(B) \in \mathcal{F}) \,, \quad \forall B \in \mathcal{F}' \,. \end{aligned}$$

- The most common choice for the σ-algebra $\mathcal{F}'$ is the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$–the one generated by the collection of all open sets in $\mathbb{R}^n$.
- The distribution function $F_{\boldsymbol{X}}(\boldsymbol{x};t) : \mathbb{R}^n \to [0,1]$:

$$\begin{aligned} P_{\boldsymbol{X}}(\mathrm{d}\boldsymbol{x};t) &= \mathrm{d}F_{\boldsymbol{X}}(\boldsymbol{x};t) \,, \\ F_{\boldsymbol{X}}(\boldsymbol{x};t) &= P'(\boldsymbol{X}_t \in]-\infty, x_1] \times \cdots \times]-\infty, x_n]) \,. \end{aligned}$$

- Joint probability laws: The two-fold probability measure $P_{\boldsymbol{XY}}(\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y};t,t') = \mathrm{d}F_{\boldsymbol{XY}}(\boldsymbol{x},\boldsymbol{y};t,t')$ with:

$$F_{\boldsymbol{XY}}(\boldsymbol{x},\boldsymbol{y};t,t') = P'(\boldsymbol{X}_t \in]-\infty,x_1]\times\cdots\times]-\infty,x_n] \text{ and } \boldsymbol{Y}_{t'} \in]-\infty,y_1]\times\cdots\times]-\infty,y_n])\,.$$

- Marginal probability laws: Let I be the set of all finite, unordered parts of $\mathbb{R}$ of the form $I \ni i = \{t_1,t_2,\dots t_k\}$. Let $\boldsymbol{X}^{(i)} = (\boldsymbol{X}_{t_1},\boldsymbol{X}_{t_2},\dots\boldsymbol{X}_{t_k})$ be the $\mathbb{R}^{n\times k}$-valued random variable constructed from the part i, then the collection of marginal probability laws of the process $(\boldsymbol{X}_t\,, t\in\mathbb{R})$ is $\{P_{\boldsymbol{X}^{(i)}}(\mathrm{d}\boldsymbol{x}_1\mathrm{d}\boldsymbol{x}_2\cdots\mathrm{d}\boldsymbol{x}_k),\ i\in I\}$.

Stochastic process

Second-order stochastic process

Definition

$(\boldsymbol{X}_t\,, t\in\mathbb{R})$ *is a second-order stochastic process if:*

$$\mathbb{E}\{\|\boldsymbol{X}_t\|^2\} < +\infty\,,\quad \forall t\in\mathbb{R}\,,$$

where $\|\boldsymbol{x}\| = \sqrt{\sum_{j=1}^n x_j^2}$ *and* $\mathbb{E}\{f(\boldsymbol{X}_t)\} = \int_{\mathbb{R}^n} f(\boldsymbol{x})P_{\boldsymbol{X}}(\mathrm{d}\boldsymbol{x};t)$.

- The set $L^2(\Omega,\mathbb{R}^n)$ of $\mathbb{R}^n$-valued second-order random variables defined on $(\Omega,\mathcal{F},P)$ is an Hilbert space, with the scalar product $(\!(\boldsymbol{X},\boldsymbol{Y})\!)_2 = \mathbb{E}\{(\boldsymbol{X},\boldsymbol{Y})\}$ and the norm $\|\!|\boldsymbol{X}|\!\| = \sqrt{(\!(\boldsymbol{X},\boldsymbol{X})\!)_2} = \sqrt{\mathbb{E}\{\|\boldsymbol{X}\|^2\}}$.
- **Interests**: the same as for $L^2(\mathbb{R}^n)$! (separability, uniform convexity).

Stochastic process

Mean and auto-correlation functions

Definition

Let $(\boldsymbol{X}_t\,, t \in \mathbb{R})$ *be a* $\mathbb{R}^n$*-valued,* *second-order* *stochastic process defined on* $(\Omega, \mathcal{F}, P)$ *and indexed on* $\mathbb{R}$*.*

- *Its mean function* $\boldsymbol{\mu}_{\boldsymbol{X}} : \mathbb{R} \to \mathbb{R}^n$ *is:*

$$\boldsymbol{\mu}_{\boldsymbol{X}}(t) = \int_{\mathbb{R}^n} \boldsymbol{x} P_{\boldsymbol{X}}(\mathrm{d}\boldsymbol{x}; t) = \mathbb{E}\{\boldsymbol{X}_t\}\,.$$

The process is centered if $\boldsymbol{\mu}_{\boldsymbol{X}}(t) = \boldsymbol{0}$, $\forall t \in \mathbb{R}$.

- *Its auto-correlation function* $\boldsymbol{R}_{\boldsymbol{X}} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n\times n}$ *is:*

$$\boldsymbol{R}_{\boldsymbol{X}}(t,t') = \iint_{\mathbb{R}^n\times\mathbb{R}^n} \boldsymbol{x}\otimes\boldsymbol{y} P_{\boldsymbol{X}\boldsymbol{X}}(\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}; t,t') = \mathbb{E}\{\boldsymbol{X}_t\otimes\boldsymbol{X}_{t'}\}\,.$$

Outline

Definition

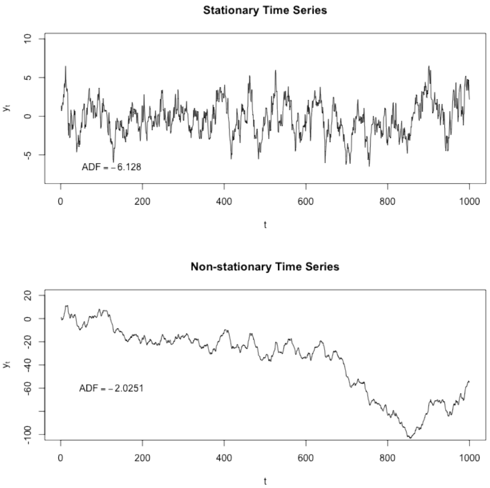
Let $(\boldsymbol{X}_t\,, t \in \mathbb{R})$ be a $\mathbb{R}^n$-valued stochastic process defined on $(\Omega, \mathcal{F}, P)$ and indexed on $\mathbb{R}$. It is:

- *stationary if $\forall u \in \mathbb{R}$, $(\boldsymbol{X}_{t+u}\,, t \in \mathbb{R})$ and $(\boldsymbol{X}_t\,, t \in \mathbb{R})$ have the same marginal probability laws;*
- *mean-square stationary if it is second-order, $\boldsymbol{\mu}_{\boldsymbol{X}}(t) = \boldsymbol{\mu}_{\boldsymbol{X}}$ (independent of time), and $\boldsymbol{R}_{\boldsymbol{X}}(t,t') = \boldsymbol{R}_{\boldsymbol{X}}(t-t')$.*

- If $(\boldsymbol{X}_t\,, t \in \mathbb{R})$ is stationary then $P_{\boldsymbol{X}}(\mathrm{d}\boldsymbol{x};t) = P_{\boldsymbol{X}}(\mathrm{d}\boldsymbol{x})$ (independent of time) and $P_{\boldsymbol{X}\boldsymbol{X}}(\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y};t,t') = P_{\boldsymbol{X}\boldsymbol{X}}(\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y};t-t')$.
- Stationary $\Rightarrow$ mean-square stationary, but the reverse is not true.

Mean-square stochastic process

Stationarity



Stationary *v.s.* non-stationary processes.

- **Data**: $(\boldsymbol{X}_t\,, t\in\mathbb{R})$ a $\mathbb{R}^n$-valued, second-order stochastic process defined on $(\Omega,\mathcal{F},P)$, indexed on $\mathbb{R}$, and mean-square stationary.
- **Hypothesis**: $t\mapsto\boldsymbol{R}_{\boldsymbol{X}}(t)$ is continuous on $\mathbb{R}$.

Theorem (Bochner)

Then there exists an Hermitian, positive, and bounded measure $\boldsymbol{M}_{\boldsymbol{X}}\in\mathbb{C}^{n\times n}$, called the spectral measure of the $\mathbb{R}^n$-valued stochastic process $(\boldsymbol{X}_t\,, t\in\mathbb{R})$, s.t.:

$$\boldsymbol{R}_{\boldsymbol{X}}(t)=\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\omega t}\,\boldsymbol{M}_{\boldsymbol{X}}(\mathrm{d}\omega)\,.$$

Definition

The spectral density $\boldsymbol{S}_{\boldsymbol{X}}$ is the density of the spectral measure $\boldsymbol{M}_{\boldsymbol{X}}$ if it exists:

$$\boldsymbol{M}_{\boldsymbol{X}}(\mathrm{d}\omega) = \boldsymbol{S}_{\boldsymbol{X}}(\omega)\mathrm{d}\omega\,.$$

- It is Hermitian, positive, and integrable (since $\boldsymbol{M}_{\boldsymbol{X}}(\mathbb{R}) < +\infty$). Then $\lim_{|t|\to+\infty} \boldsymbol{R}_{\boldsymbol{X}}(t) = 0$ (as the Fourier transform of an integrable function).
- A necessary condition: $\boldsymbol{\mu}_{\boldsymbol{X}} = \boldsymbol{0}$.
- A sufficient condition: $\lim_{|t|\to+\infty} \boldsymbol{R}_{\boldsymbol{X}}(t) = 0$.
- Let $B \subseteq \mathbb{R}$, $B \mapsto \operatorname{Tr} \boldsymbol{M}_{\boldsymbol{X}}(B)$ is the power spectral measure on B, and if $\boldsymbol{S}_{\boldsymbol{X}}$ exists, $\omega \mapsto \operatorname{Tr} \boldsymbol{S}_{\boldsymbol{X}}(\omega)$ is the power spectral density (PSD) of the process.

- **Data**: $(\boldsymbol{X}_t\,, t\in\mathbb{R})$ a $\mathbb{R}^n$-valued, second-order stochastic process defined on $(\Omega,\mathcal{F},P)$, indexed on $\mathbb{R}$.

Definition

Let $t\in\mathbb{R}$, $(\boldsymbol{X}_t\,, t\in\mathbb{R})$ is mean-square derivable at t iff the sequence of random variables $\boldsymbol{X}_h=\frac{1}{h}(\boldsymbol{X}_{t+h}-\boldsymbol{X}_t)$ converges in $L^2(\Omega,\mathbb{R}^n)$ as $h\to 0$. That limit is denoted by $\dot{\boldsymbol{X}}_t$ s.t.:

$$\lim_{h\to 0}\|\mathbf{X}_h-\dot{\mathbf{X}}_t\|=0\,.$$

- A sufficient condition: $\frac{\partial^2 \boldsymbol{R}_{\mathbf{X}}}{\partial t \partial t'}$ exists for all (t, t') in $\mathbb{R} \times \mathbb{R}$.
- Then $(\dot{\boldsymbol{X}}_t\,, t \in \mathbb{R})$ is a $\mathbb{R}^n$-valued, second-order stochastic process defined on $(\Omega, \mathcal{F}, P)$, indexed on $\mathbb{R}$. In addition:

$$
\begin{aligned}
\boldsymbol{\mu}_{\dot{\boldsymbol{X}}}(t) &= \frac{\mathrm{d}\boldsymbol{\mu}_{\boldsymbol{X}}}{\mathrm{d}t}(t)\,, \\
\boldsymbol{R}_{\dot{\boldsymbol{X}}}(t, t') &= \frac{\partial^2 \boldsymbol{R}_{\boldsymbol{X}}}{\partial t \partial t'}(t, t')\,, \\
\boldsymbol{R}_{\boldsymbol{X}\dot{\boldsymbol{X}}}(t, t') &= \mathbb{E}\{\boldsymbol{X}_t \otimes \dot{\boldsymbol{X}}_{t'}\} = \frac{\partial \boldsymbol{R}_{\boldsymbol{X}}}{\partial t'}(t, t')\,, \\
\boldsymbol{R}_{\dot{\boldsymbol{X}}\boldsymbol{X}}(t, t') &= \mathbb{E}\{\dot{\boldsymbol{X}}_t \otimes \boldsymbol{X}_{t'}\} = \frac{\partial \boldsymbol{R}_{\boldsymbol{X}}}{\partial t}(t, t')\,.
\end{aligned}
$$

- A necessary and sufficient condition:

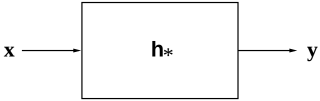
$$\int_{\mathbb{R}} \omega^2 \, \mathrm{tr}\, \boldsymbol{M}_{\boldsymbol{X}}(\mathrm{d}\omega) < +\infty \,.$$

- Then $(\dot{\boldsymbol{X}}_t\,, t \in \mathbb{R})$ is a $\mathbb{R}^n$-valued, second-order stochastic process defined on $(\Omega, \mathcal{F}, P)$, indexed on $\mathbb{R}$, and mean-square stationary. In addition:

$$\begin{aligned} \boldsymbol{\mu}_{\dot{\boldsymbol{X}}}(t) &= \boldsymbol{0}\,, \\ \boldsymbol{R}_{\dot{\boldsymbol{X}}}(t) &= -\frac{\mathrm{d}^2 \boldsymbol{R}_{\boldsymbol{X}}}{\mathrm{d}t^2}(t)\,, \\ \boldsymbol{R}_{\dot{\boldsymbol{X}}\boldsymbol{X}}(t) &= \frac{\mathrm{d}\boldsymbol{R}_{\boldsymbol{X}}}{\mathrm{d}t}(t) = -\boldsymbol{R}_{\boldsymbol{X}\dot{\boldsymbol{X}}}(t)\,, \\ \boldsymbol{M}_{\dot{\boldsymbol{X}}}(\mathrm{d}\omega) &= \omega^2 \boldsymbol{M}_{\boldsymbol{X}}(\mathrm{d}\omega)\,. \end{aligned}$$

Outline

Filtering of a stochastic process
Linear filter



- A linear convolution filter:

$$\boldsymbol{y}(t) = \int_{\mathbb{R}} \mathbb{h}(t-\tau)\boldsymbol{x}(\tau)\,\mathrm{d}\tau$$

where:

  - $t \mapsto \mathbb{h}(t)$ is the impulse response function in $\mathbb{R}^{m\times n}$;
  - $\omega \mapsto \widehat{\mathbb{h}}(\omega) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\omega t}\,\mathbb{h}(t)\,\mathrm{d}t$ is the frequency response function;
  - $p \mapsto \mathbb{H}(p) = \int_{-\infty}^{+\infty} \mathrm{e}^{-pt}\,\mathbb{h}(t)\,\mathrm{d}t$, $p \in \mathrm{D}_{\mathbb{H}} \subset \mathbb{C}$, is the transfer function.

- **Data**: $(\boldsymbol{X}_t\,, t \in \mathbb{R})$ a $\mathbb{R}^m$-valued, second-order stochastic process defined on $(\Omega, \mathcal{F}, P)$, indexed on $\mathbb{R}$, and mean-square stationary.
- If $t \mapsto \mathbb{h}(t)$ is integrable or square integrable, then $(\boldsymbol{Y}_t\,, t \in \mathbb{R})$ is a $\mathbb{R}^n$-valued, second-order stochastic process defined on $(\Omega, \mathcal{F}, P)$, indexed on $\mathbb{R}$, and mean-square stationary s.t.:

$$
\begin{aligned}
\boldsymbol{\mu}_{\boldsymbol{Y}} &= \widehat{\mathbb{h}}(0)\boldsymbol{\mu}_{\boldsymbol{X}}\,,\\
\boldsymbol{R}_{\boldsymbol{Y}}(t) &= \iint_{\mathbb{R}\times\mathbb{R}} \mathbb{h}(\tau)\boldsymbol{R}_{\boldsymbol{X}}(t+\tau'-\tau)\mathbb{h}(\tau')^{\mathsf{T}}\,\mathrm{d}\tau\mathrm{d}\tau'\,,\\
\boldsymbol{M}_{\boldsymbol{Y}}(\mathrm{d}\omega) &= \widehat{\mathbb{h}}(\omega)\boldsymbol{M}_{\boldsymbol{X}}(\mathrm{d}\omega)\widehat{\mathbb{h}}^*(\omega)\,.
\end{aligned}
$$

Further reading...

- H. Kobayashi, B. L. Mark, W. Turin: *Probability, Random Processes, and Statistical Analysis*. Cambridge University Press, Cambridge (2012);
- P. Krée & C. Soize: *Mathematics of Random Phenomena: Random Vibrations of Mechanical Structures*, Reidel Publishing Co., Dordrecht (1986);
- C. Soize: *Méthodes Mathématiques en Analyse du Signal*, Masson, Paris (1993);
- J. J. Shynk: *Probability, Random Variables and Random Processes. Theory and Signal Processing Applications*. Wiley, Hoboken NJ (2013);
- T.T. Soong & M. Grigoriu: *Random Vibration of Mechanical and Structural Systems*, Prentice Hall, Upper Saddle River NJ (1993).