Random vibrations

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Random vibrations of continuous systems

MG3416–Advanced Structural Acoustics - Lecture #4

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Outline

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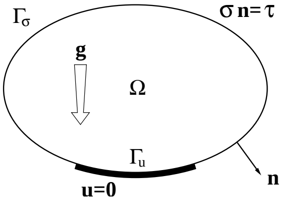
- We consider small perturbations $\boldsymbol{u}(\boldsymbol{x},t)$ around a static equilibrium $\boldsymbol{x} \in \Omega$ considered as the reference configuration.
- The structure occupying Ω is constituted by linear, memoryless viscoelastic materials:

$$\boldsymbol{\sigma}(\boldsymbol{x},t) = \mathbf{C}^{\mathrm{e}}\boldsymbol{\epsilon}(\boldsymbol{x},t) + \mathbf{C}^{\mathrm{v}}\dot{\boldsymbol{\epsilon}}(\boldsymbol{x},t)\,, \quad (\boldsymbol{x},t) \in \Omega \times \mathbb{R}\,,$$

  where $\boldsymbol{\epsilon} = \boldsymbol{\nabla} \otimes_s \boldsymbol{u}$ is the small strain tensor, $\boldsymbol{\sigma}$ the Cauchy stress tensor, $\mathbf{C}^{\mathrm{e}}$ the elasticity tensor, $\mathbf{C}^{\mathrm{v}}$ the viscosity tensor, and ϱ the density.



- The balance of momentum, boundary conditions and initial conditions read:

$$\left\{\begin{aligned} \mathbf{Div}\boldsymbol{\sigma} + \varrho\boldsymbol{g} &= \varrho\partial_t^2\boldsymbol{u} && \text{in } \Omega\,, \\ \boldsymbol{u} &= \mathbf{0} && \text{on } \Gamma_{\mathrm{u}}\,, \\ \boldsymbol{\sigma}\boldsymbol{n} &= \boldsymbol{\tau} && \text{on } \Gamma_\sigma\,, \\ \boldsymbol{u}(\cdot,0) &= \boldsymbol{u}_0 && \text{in } \Omega\,, \\ \dot{\boldsymbol{u}}(\cdot,0) &= \boldsymbol{v}_0 && \text{in } \Omega\,, \end{aligned}\right.$$

where $\boldsymbol{n}$ is the unit outward normal to $\partial\Omega$.

- The Virtual Power principle (VPP) on the set of admissible displacement fields

$$\mathcal{C} = \left\{ \boldsymbol{v};\, \boldsymbol{v} \in L^2(\Omega)\,,\, \boldsymbol{\nabla} \boldsymbol{v} \in L^2(\Omega)\,,\, \text{and } \boldsymbol{v}|_{\Gamma_\mathrm{u}} = \mathbf{0} \right\}$$

reads: Find $\boldsymbol{u} \in \mathcal{C}$ s.t.

$$\int_\Omega \varrho \ddot{\boldsymbol{u}} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} = \int_\Omega \varrho \boldsymbol{g} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + \int_{\Gamma_\sigma} \boldsymbol{\tau} \cdot \boldsymbol{v} \,\mathrm{d}\sigma\,, \quad \forall \boldsymbol{v} \in \mathcal{C}\,.$$

Continuous mechanical system

Virtual Power Principle

- The mass, stiffness and damping bilinear forms:

$$m(\boldsymbol{u},\boldsymbol{v})=\langle\boldsymbol{M}\boldsymbol{u},\boldsymbol{v}\rangle_{\mathcal{C}',\mathcal{C}}=\int_\Omega \varrho\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\,,$$

$$k(\boldsymbol{u},\boldsymbol{v})=\langle\boldsymbol{K}\boldsymbol{u},\boldsymbol{v}\rangle_{\mathcal{C}',\mathcal{C}}=\int_\Omega \mathbf{C}^{\mathrm{e}}(\boldsymbol{\epsilon}(\boldsymbol{u})):\boldsymbol{\epsilon}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\,,$$

$$d(\boldsymbol{u},\boldsymbol{v})=\langle\boldsymbol{D}\boldsymbol{u},\boldsymbol{v}\rangle_{\mathcal{C}',\mathcal{C}}=\int_\Omega \mathbf{C}^{\mathrm{v}}(\boldsymbol{\epsilon}(\boldsymbol{u})):\boldsymbol{\epsilon}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\,,$$

define the positive definite, symmetric mass $\boldsymbol{M}$, stiffness $\boldsymbol{K}$ and damping $\boldsymbol{D}$ operators of $\mathscr{L}(\mathcal{C},\mathcal{C}')$ (the set of continuous operators), where $\langle\boldsymbol{f},\boldsymbol{v}\rangle_{\mathcal{C}',\mathcal{C}}$ defines the duality product of $\boldsymbol{f}\in\mathcal{C}'$ and $\boldsymbol{v}\in\mathcal{C}$, and $\mathcal{C}'$ is the dual space of $\mathcal{C}$ (the set of linear forms).

Continuous mechanical system

Virtual Power Principle

- By the Riesz theorem, $\exists! \boldsymbol{f} \in \mathcal{C}'$ s.t.

$$f(\boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\mathcal{C}',\mathcal{C}} = \int_\Omega \varrho \boldsymbol{g} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \int_{\Gamma_\sigma} \boldsymbol{\tau} \cdot \boldsymbol{v} \, \mathrm{d}\sigma \,, \quad \forall \boldsymbol{v} \in \mathcal{C} \,,$$

  because the linear form f is continuous on $\mathcal{C}$ provided that $\boldsymbol{g}$ and $\boldsymbol{\tau}$ are square integrable.
- Then the VPP reads:

$$\begin{cases} \boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{D}\dot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega \times \mathbb{R} \,, \\ \boldsymbol{u}(\cdot, 0) = \boldsymbol{u}_0 & \text{in } \Omega \,, \\ \dot{\boldsymbol{u}}(\cdot, 0) = \boldsymbol{v}_0 & \text{in } \Omega \,, \end{cases}$$

  "in the sense of distributions".

- **Spectral problem**: Find $\lambda \in \mathbb{R}$ and $\boldsymbol{\phi} \in \mathcal{C}$ s.t.

$$\boldsymbol{K}\boldsymbol{\phi} = \lambda \boldsymbol{M}\boldsymbol{\phi}\,.$$

- It admits a countable set of solutions $(\lambda_1, \boldsymbol{\phi}_1)$, $(\lambda_2, \boldsymbol{\phi}_2)$ ... s.t. $0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots$ and $\{\boldsymbol{\phi}_\alpha\}_{\alpha \in \mathbb{N}*}$ is an Hilbertian basis of the space $H = L^2_\mu(\Omega)$ of square integrable functions with respect to the unit mass measure $\mu(\mathrm{d}\boldsymbol{x}) = \frac{\varrho \mathrm{d}\boldsymbol{x}}{M}$, with $M = \int_\Omega \varrho \mathrm{d}\boldsymbol{x}$:

$$m(\boldsymbol{\phi}_\alpha, \boldsymbol{\phi}_\beta) = M\delta_{\alpha\beta}\,,$$
$$k(\boldsymbol{\phi}_\alpha, \boldsymbol{\phi}_\beta) = M\omega_\alpha^2\delta_{\alpha\beta}\,,$$

where $\omega_\alpha^2 = \lambda_\alpha$.

- Consequently, the solution $\boldsymbol{u} \in \mathcal{C}$ can be expanded on the eigenbasis $\{\boldsymbol{\phi}_\alpha\}_{\alpha \in \mathbb{N}*}$ as:
$$\boldsymbol{u}(\boldsymbol{x}, t) = \sum_{\alpha=1}^{\infty} q_\alpha(t) \boldsymbol{\phi}_\alpha(\boldsymbol{x}) \,.$$

- Introducing $\mu(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} \mu(\mathrm{d}\boldsymbol{x})$ (the scalar product in H), the generalized coordinates $\{q_\alpha\}_{\alpha \in \mathbb{N}*}$ are:
$$q_\alpha = \mu(\boldsymbol{u}, \boldsymbol{\phi}_\alpha) \,.$$

- The μ-norm $\|\boldsymbol{u}\|_\mu = \sqrt{\mu(\boldsymbol{u}, \boldsymbol{u})}$ is obtained as:
$$\|\boldsymbol{u}(\cdot, t)\|_\mu = \left(\sum_{\alpha=1}^{+\infty} (q_\alpha(t))^2 \right)^{\frac{1}{2}} < +\infty \,.$$

Continuous mechanical system

Eigenmodes and modal expansion

- **Hypothesis** (Basile): the eigenmodes $\{\boldsymbol{\phi}_\alpha\}_{\alpha\in\mathbb{N}*}$ diagonalize the damping operator as well:

$$d(\boldsymbol{\phi}_\alpha, \boldsymbol{\phi}_\beta) = M\eta_\alpha\omega_\alpha\delta_{\alpha\beta} \,,$$

where:

  - ω_α: the (angular) eigenfrequency of the α^{th} mode,
  - ξ_α: the modal critical damping rate of the α^{th} mode,
  - $\eta_\alpha = 2\xi_\alpha$: the modal loss factor of the α^{th} mode.

- Owing to the Basile hypothesis, the generalized coordinates $\{q_\alpha\}_{\alpha\in\mathbb{N}*}$ satisfy:

$$\begin{cases} M(\ddot{q}_\alpha(t)+\eta_\alpha\omega_\alpha\dot{q}_\alpha(t)+\omega_\alpha^2 q_\alpha(t)) = f_\alpha(t)\,, & t\in\mathbb{R}\,, \\ q_\alpha(0) = Q_\alpha\,, \\ \dot{q}_\alpha(0) = \dot{Q}_\alpha\,, \end{cases}$$

  where $f_\alpha = f(\boldsymbol{\phi}_\alpha)$, $Q_\alpha = \mu(\boldsymbol{u}_0,\boldsymbol{\phi}_\alpha)$, $\dot{Q}_\alpha = \mu(\boldsymbol{v}_0,\boldsymbol{\phi}_\alpha)$.
- Then we study the free, forced and evolutionary responses $t\mapsto q_\alpha(t)$ for the different eigenmodes $\alpha\in\mathbb{N}^*$.

Continuous mechanical system

Free response

Definition

The modal free response $t \mapsto q_\alpha^\ell(t)$ is the solution of:

$$\begin{cases} M(\ddot{q}_\alpha(t) + 2\xi_\alpha\omega_\alpha\dot{q}_\alpha(t) + \omega_\alpha^2 q_\alpha(t)) = 0\,, \quad t \geqslant 0\,, \\ q_\alpha(0) = Q_\alpha\,, \\ \dot{q}_\alpha(0) = \dot{Q}_\alpha\,. \end{cases}$$

- Let $\omega_{D\alpha} = \omega_\alpha\sqrt{1-\xi_\alpha^2}$ (damped eigenfrequency), then:

$$q_\alpha^\ell(t) = \mathrm{e}^{-\xi_\alpha\omega_\alpha t}\left(Q_\alpha\cos\omega_{D\alpha}t + \frac{\xi_\alpha\omega_\alpha Q_\alpha + \dot{Q}_\alpha}{\omega_{D\alpha}}\sin\omega_{D\alpha}t\right)\,.$$

- **Property**: if $\xi_\alpha > 0$, $\lim_{t\to+\infty} q_\alpha^\ell(t) = \lim_{t\to+\infty}\dot{q}_\alpha^\ell(t) = 0\,.$

Likewise, $\lim_{t\to+\infty}\|\boldsymbol{u}^\ell(\cdot,t)\|_\mu = \lim_{t\to+\infty}\|\dot{\boldsymbol{u}}^\ell(\cdot,t)\|_\mu = 0\,.$

Definition

The modal forced response $t \mapsto q^f_\alpha(t)$ is the solution of:

$$M(\ddot{q}_\alpha(t) + 2\xi_\alpha\omega_\alpha\dot{q}_\alpha(t) + \omega_\alpha^2 q_\alpha(t)) = f_\alpha(t)\,,\quad t\in\mathbb{R}\,.$$

- Then:

$$q^f_\alpha(t) = \int_{-\infty}^t \mathbb{h}_\alpha(t-\tau)f_\alpha(\tau)\,\mathrm{d}\tau = \int_0^{+\infty}\mathbb{h}_\alpha(\tau)f_\alpha(t-\tau)\,\mathrm{d}\tau\,,$$

where $\mathbb{h}_\alpha:\mathbb{R}\rightarrow\mathbb{R}$ is the modal impulse response function:

$$\mathbb{h}_\alpha(t) = \mathbb{1}_{[0,+\infty[}(t)\times\frac{1}{M\omega_{D\alpha}}\mathrm{e}^{-\xi_\alpha\omega_\alpha t}\sin\omega_{D\alpha}t\,.$$

- $t \mapsto \mathbb{h}_\alpha(t)$ is integrable and square integrable on $\mathbb{R}$, and its Fourier transform is:

$$\widehat{\mathbb{h}}_\alpha(\omega) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\omega t}\, \mathbb{h}_\alpha(t)\,\mathrm{d}t = \frac{1}{M(\omega_\alpha^2 - \omega^2 + 2\mathrm{i}\xi_\alpha\omega_\alpha\omega)}\,.$$

- $\omega \mapsto \widehat{\mathbb{h}}_\alpha(\omega)$ is integrable and square integrable on $\mathbb{R}$, and its inverse Fourier transform is:

$$\mathbb{h}_\alpha(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\omega t}\, \widehat{\mathbb{h}}_\alpha(\omega)\,\mathrm{d}\omega\,.$$

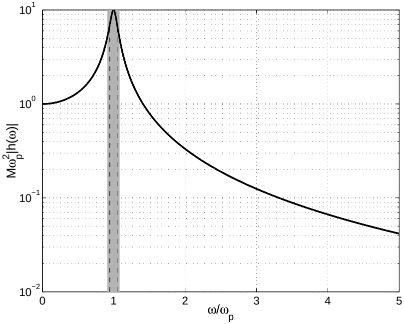
- Usual quadratures:

$$\int_0^{+\infty} |\widehat{\mathbb{h}}_\alpha(\omega)|^2\,\mathrm{d}\omega = \frac{\pi}{2M^2\eta_\alpha\omega_\alpha^3}\,,$$

$$\int_0^{+\infty} \omega^2|\widehat{\mathbb{h}}_\alpha(\omega)|^2\,\mathrm{d}\omega = \frac{\pi}{2M^2\eta_\alpha\omega_\alpha}\,.$$

- The modal impedance $\omega \mapsto Z_\alpha(\omega)$:

$$\mathrm{i}\omega Z_\alpha(\omega) = M(\omega_\alpha^2 - \omega^2 + 2\mathrm{i}\xi_\alpha\omega_\alpha\omega)\,.$$



- $\omega'_\alpha = \omega_\alpha\sqrt{1-2\xi_\alpha^2}$: the α^{th} resonance frequency (for $0 < \xi_\alpha < \frac{1}{\sqrt{2}}$);
- $b_\alpha \simeq \pi\xi_\alpha\omega_\alpha$: the modal equivalent bandwidth, s.t.

$$b_\alpha|\widehat{\mathbb{h}}_\alpha(\omega'_\alpha)|^2 = \int_0^{+\infty} |\widehat{\mathbb{h}}_\alpha(\omega)|^2\,\mathrm{d}\omega\,;$$

- $\Delta_\alpha = \eta_\alpha\omega_\alpha$: the modal half-power bandwidth.

Definition

The modal evolutionary response $t \mapsto q_\alpha(t)$ is the solution of:

$$\begin{cases} M\left(\ddot{q}_\alpha(t) + 2\xi_\alpha \dot{q}_\alpha(t) + \omega_\alpha^2 q_\alpha(t)\right) = f_\alpha(t)\,, \quad t \geqslant 0\,, \\ q_\alpha(0) = Q_\alpha\,, \\ \dot{q}_\alpha(0) = \dot{Q}_\alpha\,. \end{cases}$$

- Then:

$$q_\alpha(t) = \mathrm{e}^{-\frac{\Delta_\alpha t}{2}}\left(Q_\alpha \cos\omega_{D\alpha}t + \frac{\Delta_\alpha Q_\alpha + 2\dot{Q}_\alpha}{2\omega_{D\alpha}}\sin\omega_{D\alpha}t\right) + \int_0^t \mathbb{h}_\alpha(t-\tau) f_\alpha(\tau)\,\mathrm{d}\tau\,.$$

- **Property**: if $\xi_\alpha > 0$, $\lim_{t\to+\infty} |q_\alpha(t) - q_\alpha^f(t)| = 0$. Likewise if $\xi_\alpha > 0\ \forall\alpha \in \mathbb{N}^*$, $\lim_{t\to+\infty} \|\boldsymbol{u}(\cdot,t) - \boldsymbol{u}^f(\cdot,t)\|_\mu = 0$.

Energetic quantities
Definitions

Definition

- *The kinetic energy:* $\mathcal{E}_{\mathrm{c}}(t) = \frac{1}{2}m(\dot{\boldsymbol{u}},\dot{\boldsymbol{u}}) = \frac{1}{2}M\sum_\alpha(\dot{q}_\alpha(t))^2$,
- *The potential energy:* $\mathcal{E}_{\mathrm{p}}(t) = \frac{1}{2}k(\boldsymbol{u},\boldsymbol{u}) = \frac{1}{2}M\sum_\alpha\omega_\alpha^2(q_\alpha(t))^2$,
- *The mechanical energy:* $\mathcal{E}(t) = \mathcal{E}_{\mathrm{c}}(t) + \mathcal{E}_{\mathrm{p}}(t)$,
- *The dissipated power:* $\Pi_{\mathrm{d}}(t) = d(\dot{\boldsymbol{u}},\dot{\boldsymbol{u}}) = M\sum_\alpha\eta_\alpha\omega_\alpha(\dot{q}_\alpha(t))^2$,
- *The input power:* $\Pi_{\mathrm{IN}}(t) = f(\dot{\boldsymbol{u}}) = \sum_\alpha f_\alpha(t)\dot{q}_\alpha(t)$.

- The instantaneous power balance reads:

$$\dot{\mathcal{E}}(t) = \Pi_{\mathrm{IN}}(t) - \Pi_{\mathrm{d}}(t)\,.$$

- It is subsequently specialized to the modal free, forced and evolutionary responses.

- The free response mechanical energy when $\xi_\alpha \ll 1$ $\forall \alpha \in \mathbb{N}^*$:

$$\mathcal{E}^\ell(t) \simeq \sum_{\alpha=1}^{+\infty} \mathcal{E}_{0\alpha}\, \mathrm{e}^{-\eta_\alpha \omega_\alpha t}\,,$$

where $\mathcal{E}_{0\alpha} = \frac{1}{2} M(\dot{Q}_\alpha^2 + \omega_\alpha^2 Q_\alpha^2)$.

- The power balance integrated between 0 and $t > 0$:

$$\sum_{\alpha=1}^{+\infty} \mathcal{E}_{0\alpha} = \mathcal{E}_\mathrm{d}^\ell(t) + \mathcal{E}^\ell(t)\,,$$

hence $\mathcal{E}_\mathrm{d}^\ell(\infty) = \int_0^{+\infty} \Pi_\mathrm{d}^\ell(t)\mathrm{d}t = \sum_{\alpha=1}^{+\infty} \mathcal{E}_{0\alpha}\,.$

Energetic quantities
Forced response

- **Data**: square integrable (finite energy) excitation with limited bandwidth,

$$|\widehat{f}_\alpha(\omega)| \leqslant C_\alpha \;\; \forall\omega \in \mathbb{R}\,; \quad \widehat{f}_\alpha(\omega) = 0 \;\; \forall\omega \notin I_0 \cup \underline{I}_0\,.$$

- Then $\lim_{|t|\to+\infty} f_\alpha(t) = 0$, and consequently if $\xi_\alpha > 0$:

$$\lim_{|t|\to+\infty} q_\alpha^f(t) = \lim_{|t|\to+\infty} \dot{q}_\alpha^f(t) = 0\,.$$

- The power balance integrated between $-\infty$ and t:

$$\begin{aligned}\mathcal{E}^f(t) &= \int_{-\infty}^t \Pi_{\mathrm{IN}}(\tau)\,\mathrm{d}\tau - \int_{-\infty}^t \Pi_{\mathrm{d}}^f(\tau)\,\mathrm{d}\tau \\ &= \mathcal{E}_{\mathrm{IN}}(t) - \mathcal{E}_{\mathrm{d}}^f(t)\end{aligned}$$

since $\mathcal{E}^f(-\infty) = 0$. But $\mathcal{E}^f(+\infty) = 0$ as well, hence:

$$\mathcal{E}_{\mathrm{IN}}(+\infty) = \mathcal{E}_{\mathrm{d}}^f(+\infty)\,.$$

Energetic quantities
Equipartition in the forced response

- Let $\mathscr{I} = \{\alpha;\ \omega_\alpha \in I_0\}$ with $I_0 = [\omega_0 - \frac{\Delta\omega}{2}, \omega_0 + \frac{\Delta\omega}{2}]$.
- **Hypotheses** (wideband excitation):
  1. $\xi_\alpha \ll 1$, $\forall \alpha \in \mathscr{I}$;
  2. $\omega \mapsto \widehat{f}_\alpha(\omega)$ varies slowly on $I_0 \cup \underline{I}_0$, $\Delta\omega \gg b_\alpha$ $\forall \alpha \in \mathscr{I}$;
  3. The mechanical energy of the forced response is due to the modes in $\mathscr{I}$ solely.
- Then:

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{E}_c^f(t)\,\mathrm{d}t &= \frac{M}{4\pi}\sum_{\alpha=1}^{+\infty}\int_{\mathbb{R}} \omega^2|\widehat{\mathbb{h}}_\alpha(\omega)|^2|\widehat{f}_\alpha(\omega)|^2\,\mathrm{d}\omega && \text{(Plancherel)}\\ &\simeq \frac{M}{4\pi}\sum_{\alpha=1}^{+\infty}|\widehat{f}_\alpha(\omega_0)|^2\int_{I_0\cup\underline{I}_0}\omega^2|\widehat{\mathbb{h}}_\alpha(\omega)|^2\,\mathrm{d}\omega && \text{(using (ii))}\\ &\simeq \frac{M}{2\pi}\sum_{\alpha=1}^{+\infty}|\widehat{f}_\alpha(\omega_0)|^2\int_0^{+\infty}\omega^2|\widehat{\mathbb{h}}_\alpha(\omega)|^2\,\mathrm{d}\omega && \text{(using (i))}\\ &\simeq \sum_{\alpha\in\mathscr{I}}\frac{|\widehat{f}_\alpha(\omega_0)|^2}{4M\eta_\alpha\omega_\alpha} && \text{(using (iii))}\,.\end{aligned}$$

- Likewise:

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{E}_{\mathrm{p}}^{f}(t)\,\mathrm{d}t &\simeq \sum_{\alpha\in\mathscr{I}} \frac{|\widehat{f}_\alpha(\omega_0)|^2}{4M\eta_\alpha\omega_\alpha} \\ &= \sum_{\alpha\in\mathscr{I}} \frac{|\widehat{f}_\alpha(\omega_0)|^2}{4d(\boldsymbol{\phi}_\alpha,\boldsymbol{\phi}_\alpha)}\,,\end{aligned}$$

independently of the mass or the stiffness.

- The overall mechanical energy $\mathcal{E}^{\mathrm{tot}}$ in the frequency band I_0 thus reads:

$$\begin{aligned}\mathcal{E}^{\mathrm{tot}} = \int_{\mathbb{R}} \mathcal{E}^{f}(t)\,\mathrm{d}t &\simeq \sum_{\alpha\in\mathscr{I}} \frac{|\widehat{f}_\alpha(\omega_0)|^2}{2d(\boldsymbol{\phi}_\alpha,\boldsymbol{\phi}_\alpha)} \\ &= \sum_{\alpha\in\mathscr{I}} \mathcal{E}_\alpha^{\mathrm{tot}}\,,\end{aligned}$$

with the overall modal energy $\mathcal{E}_\alpha^{\mathrm{tot}} = \frac{|\widehat{f}_\alpha(\omega_0)|^2}{2d(\boldsymbol{\phi}_\alpha,\boldsymbol{\phi}_\alpha)}$.

- Then:

$$\begin{aligned}
\mathcal{E}_{\mathrm{d}}^{f}(+\infty) &= \frac{M}{2\pi}\sum_{\alpha=1}^{+\infty}\eta_\alpha\omega_\alpha\int_{\mathbb{R}}\omega^2|\widehat{\mathbb{h}}_\alpha(\omega)|^2|\widehat{f}_\alpha(\omega)|^2\,\mathrm{d}\omega && \text{(Plancherel)} \\
&\simeq \frac{M}{2\pi}\sum_{\alpha=1}^{+\infty}\eta_\alpha\omega_\alpha|\widehat{f}_\alpha(\omega_0)|^2\int_{\mathbb{R}}\omega^2|\widehat{\mathbb{h}}_\alpha(\omega)|^2\,\mathrm{d}\omega && \text{(using (i)-(ii))} \\
&\simeq \frac{1}{2M}\sum_{\alpha\in\mathscr{I}}|\widehat{f}_\alpha(\omega_0)|^2 && \text{(using (iii))}\,,
\end{aligned}$$

  and the overall dissipated energy is independent of the damping.

- It is related to the overall modal energies by:

$$\boxed{\mathcal{E}_{\mathrm{d}}^{f}(+\infty) = \sum_{\alpha\in\mathscr{I}}\eta_\alpha\omega_\alpha\mathcal{E}_\alpha^{\mathrm{tot}}}\,.$$

- The power balance integrated between $t=0$ and $t=+\infty$:

$$\mathcal{E}_{\mathrm{d}}(+\infty)=\mathcal{E}_0+\mathcal{E}_{\mathrm{IN}}(+\infty)\,.$$

Outline

- **Data**: $(\boldsymbol{F}_t\,, t \in \mathbb{R})$ is a second order, centered stochastic process defined on $(\mathit{\Omega}, \mathcal{F}, P)$, indexed on $\mathbb{R}$, with values in $[L^2(\Omega)]^3$, and mean-square (m.s.) stationary.
- **Hypothesis**: It is characterized by its cross spectral density matrix $\omega \mapsto \boldsymbol{S}_{\boldsymbol{F}}(\omega; \boldsymbol{x}, \boldsymbol{y}) : \mathbb{R} \to \mathbb{C}^{3\times 3}$, $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, which is Hermitian, positive, integrable on $\mathbb{R}_\omega$, and s.t.:

$$\begin{aligned}
\boldsymbol{S}_{\boldsymbol{F}}(\omega; \boldsymbol{x}, \boldsymbol{y}) &= \boldsymbol{S}_{\boldsymbol{F}}(\omega; \boldsymbol{y}, \boldsymbol{x})^* \\
\boldsymbol{S}_{\boldsymbol{F}}(-\omega; \boldsymbol{x}, \boldsymbol{y}) &= \overline{\boldsymbol{S}_{\boldsymbol{F}}(\omega; \boldsymbol{x}, \boldsymbol{y})} \\
\boldsymbol{S}_{\boldsymbol{F}}(\omega; \boldsymbol{x}, \boldsymbol{y}) &= \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{y}) \otimes \mathbb{1}_{I_0 \cup \underline{I}_0}(\omega)\,,
\end{aligned}$$

where:

$$I_0 \cup \underline{I}_0 = \left[\omega_0 - \frac{\Delta\omega}{2}, \omega_0 + \frac{\Delta\omega}{2}\right] \bigcup \left[-\omega_0 - \frac{\Delta\omega}{2}, -\omega_0 + \frac{\Delta\omega}{2}\right].$$

- We finally introduce for $\alpha, \beta \in \mathbb{N}^*$:

$$S_{\alpha\beta} = \int_\Omega \int_\Omega \boldsymbol{\phi}_\alpha(\boldsymbol{x}) \cdot \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{y}) \boldsymbol{\phi}_\beta(\boldsymbol{y})\, \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{y}\,.$$

- The modal forced responses $t \mapsto q_\alpha^f(t)$ for $\alpha \in \mathbb{N}^*$ are modelized by stochastic processes $(Q_{\alpha,t}^f,\, t \in \mathbb{R})$ the properties of which are derived from filtering and mean-square derivation (see Lecture #1 part A).

Proposition

- $(Q_{\alpha,t}^f,\, t \in \mathbb{R})$ *is a* $\mathbb{R}$*-valued second order, centered stochastic process defined on* $(\Omega, \mathcal{F}, P)$*, indexed on* $\mathbb{R}$*, and m.s. stationary s.t.* $S_{Q_\alpha}(\omega) = S_{\alpha\alpha} |\widehat{\mathbb{h}}_\alpha(\omega)|^2 \mathbb{1}_{I_0 \cup \underline{I}_0}(\omega)$.
- *The same holds for its m.s. derivatives* $(\dot{Q}_{\alpha,t}^f,\, t \in \mathbb{R})$ *and* $(\ddot{Q}_{\alpha,t}^f,\, t \in \mathbb{R})$*, with* $S_{\dot{Q}_\alpha}(\omega) = \omega^2 S_{Q_\alpha}(\omega)$ *and* $S_{\ddot{Q}_\alpha}(\omega) = \omega^4 S_{Q_\alpha}(\omega)$*, respectively, together with* $\mathbb{E}\{Q_{\alpha,t} \dot{Q}_{\alpha,t}\} = \mathbb{E}\{\dot{Q}_{\alpha,t} \ddot{Q}_{\alpha,t}\} = 0$.

Stationary excitations

Energetics of the stationary forced response

- The instantaneous power balance reads:

$$\dot{\mathcal{E}}_t^f = \Pi_{\mathrm{IN},t} - \Pi_{\mathrm{d},t}^f \,,$$

as an equality of second-order random variables.

- But $\mathbb{E}\{\mathcal{E}_t^f\} = \text{Constant}$ and $\mathbb{E}\{\dot{\mathcal{E}}_t^f\} = 0$; hence:

$$\mathbb{E}\{\Pi_{\mathrm{IN},t}\} = \mathbb{E}\{\Pi_{\mathrm{d},t}^f\} \,.$$

- Let $\mathscr{I} = \{\alpha;\ \omega_\alpha \in I_0\}$.
- **Hypotheses** (wideband excitation):
  - i $\xi_\alpha \ll 1$, $\forall\alpha \in \mathscr{I}$;
  - ii $\Delta\omega \gg b_\alpha$, $\forall\alpha \in \mathscr{I}$;
  - iii The average mechanical energy of the stationary forced response is due to the modes in $\mathscr{I}$ solely.
- Then:

$$
\begin{aligned}
\mathbb{E}\{\mathcal{E}^f_{c,t}\} &= \frac{1}{2}M\sum_{\alpha=1}^{+\infty}\int_{\mathbb{R}}\omega^2 S_{Q_\alpha}(\omega)\,\mathrm{d}\omega \\
&= \frac{1}{2}M\sum_{\alpha=1}^{+\infty}S_{\alpha\alpha}\int_{I_0\cup\underline{I}_0}\omega^2|\widehat{\mathbb{h}}_\alpha(\omega)|^2\,\mathrm{d}\omega \\
&\simeq \frac{1}{2}M\sum_{\alpha=1}^{+\infty}S_{\alpha\alpha}\int_{\mathbb{R}}\omega^2|\widehat{\mathbb{h}}_\alpha(\omega)|^2\,\mathrm{d}\omega \quad \text{(using (i)-(ii))} \\
&\simeq \sum_{\alpha\in\mathscr{I}}\frac{\pi S_{\alpha\alpha}}{2M\eta_\alpha\omega_\alpha} \qquad \text{(using (iii))}\,.
\end{aligned}
$$

- Likewise:

$$\begin{aligned}\mathbb{E}\{\mathcal{E}^f_{\mathrm{p},t}\} &\simeq \sum_{\alpha\in\mathscr{I}} \frac{\pi S_{\alpha\alpha}}{2M\eta_\alpha\omega_\alpha} \\ &= \sum_{\alpha\in\mathscr{I}} \frac{\pi S_{\alpha\alpha}}{2d(\boldsymbol{\phi}_\alpha,\boldsymbol{\phi}_\alpha)}\,,\end{aligned}$$

independently of the mass or the stiffness.

- The average mechanical energy $\mathbb{E}\{\mathcal{E}^f_t\}$ in the frequency band I_0 thus reads:

$$\begin{aligned}\mathbb{E}\{\mathcal{E}^f_t\} &\simeq \sum_{\alpha\in\mathscr{I}} \frac{\pi S_{\alpha\alpha}}{d(\boldsymbol{\phi}_\alpha,\boldsymbol{\phi}_\alpha)} \\ &= \sum_{\alpha\in\mathscr{I}} \mathbb{E}\{\mathcal{E}_{\alpha,t}\}\,,\end{aligned}$$

with the average modal energy $\mathbb{E}\{\mathcal{E}_{\alpha,t}\} = \frac{\pi S_{\alpha\alpha}}{d(\boldsymbol{\phi}_\alpha,\boldsymbol{\phi}_\alpha)}$.

- Then:

$$\begin{aligned}\mathbb{E}\{\Pi^f_{\mathrm{d},t}\} &= M\sum_{\alpha=1}^{+\infty}\eta_\alpha\omega_\alpha\int_{\mathbb{R}}\omega^2 S_{Q_\alpha}(\omega)\,\mathrm{d}\omega \\ &= M\sum_{\alpha=1}^{+\infty}S_{\alpha\alpha}\eta_\alpha\omega_\alpha\int_{I_0\cup\underline{I}_0}\omega^2|\widehat{\mathbb{h}}_\alpha(\omega)|^2\,\mathrm{d}\omega \\ &\simeq M\sum_{\alpha=1}^{+\infty}S_{\alpha\alpha}\eta_\alpha\omega_\alpha\int_{\mathbb{R}}\omega^2|\widehat{\mathbb{h}}_\alpha(\omega)|^2\,\mathrm{d}\omega \quad \text{(using (i)-(ii))} \\ &\simeq \sum_{\alpha\in\mathscr{I}}\frac{\pi S_{\alpha\alpha}}{M} \quad \text{(using (iii))}\,,\end{aligned}$$

and the average dissipated power is independent of the damping.

- It is related to the average modal energies by:

$$\boxed{\mathbb{E}\{\Pi^f_{\mathrm{d},t}\} = \sum_{\alpha\in\mathscr{I}}\eta_\alpha\omega_\alpha\mathbb{E}\{\mathcal{E}_{\alpha,t}\}}\,.$$

- Assuming null initial conditions to simplify, the modal evolutionary responses $(Q_{\alpha,t},\, t \geqslant 0)$ are $\mathbb{R}$-valued second-order, centered stochastic processes defined on $(\mathit{\Omega}, \mathcal{F}, P)$, indexed on $\mathbb{R}_+$, which are non stationary:

$$Q_{\alpha,t} = \int_0^t \mathbb{h}_\alpha(t-\tau) F_{\alpha,\tau}\, \mathrm{d}\tau \,,$$

  where $F_{\alpha,t} = \langle \boldsymbol{F}_t, \boldsymbol{\phi}_\alpha \rangle_{\mathcal{C}',\mathcal{C}}$, $\alpha \in \mathbb{N}^*$.

- **Property**: if $\xi_\alpha > 0$, $\lim_{t\to+\infty} \| Q_{\alpha,t} - Q^f_{\alpha,t} \| = 0$.

- Likewise if $\xi_\alpha > 0 \;\forall \alpha \in \mathbb{N}^*$,

$$\lim_{t\to+\infty} \| \boldsymbol{U}_t - \boldsymbol{U}^f_t \|_{L^2(\mathit{\Omega},H)} = 0 \,,$$

  where $H = L^2_\mu(\Omega)$.

Summary

- "Equivalence" between:
  1. The overall energetic quantities for the forced response to deterministic, wideband excitations;
  2. The average energetic quantities for the stationary forced response to random, wideband (m.s.) stationary excitations;
  3. The time average energetic quantities for the forced response of a continuous system with randomized eigenfrequencies to harmonic excitations.
- These different cases are often (unduly) merged in the structural-acoustics literature.
- **Outlook**: coupled continuous systems.

Bibliography

- G. Chen, J. Zhou: *Vibration and Damping in Distributed Systems*, vol. I: Analysis, Estimation, Attenuation and Design. CRC Press, Boca Raton FL (1993).
- R.W. Clough, J. Penzien: *Dynamics of Structures*, 2nd Ed. Mc-Graw Hill, New York NY (1993).
- D. Clouteau, R. Cottereau, É. Savin: *Structural Dynamics and Acoustics*. École Centrale Paris, Châtenay-Malabry (2008).
- L. Cremer, M. Heckl, B.A.T. Petersson: *Structure-Borne Sound*, 3rd Ed. Springer-Verlag, Berlin (2005).
- M. Géradin, D. Rixen: *Mechanical Vibrations: Theory and Application to Structural Dynamics*, 3rd Ed. Wiley, Chichester (2014).