Acoustic radiation by structures

MG3416–Advanced Structural Acoustics - Lecture #7

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November 17, 2021

Outline

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Navier-Stokes equations

- The mass, momentum, and energy conservation equations for a fluid flow (ignoring input mass, momentum and heat) read:

$$\begin{cases} \dfrac{d\varrho}{dt} = -\varrho\boldsymbol{\nabla}\cdot\boldsymbol{v}\,, \\ \varrho\dfrac{d\boldsymbol{v}}{dt} = \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}\,, \\ \varrho T\dfrac{ds}{dt} = \boldsymbol{\tau}:\boldsymbol{\nabla}\boldsymbol{v} - \boldsymbol{\nabla}\cdot\boldsymbol{q}\,, \end{cases}$$

where $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}$ and:

  - $\boldsymbol{v}$ is the fluid velocity, ϱ the density, T the temperature, s the (specific) entropy;
  - $\boldsymbol{\sigma} = -p\boldsymbol{I} + \boldsymbol{\tau}$ is the stress tensor, p is the (static) fluid pressure, $\boldsymbol{\tau}$ is the viscous stress tensor;
  - $\boldsymbol{q}$ is the heat flux vector.

Euler equations

- Ideal fluid: $\boldsymbol{\tau} = \boldsymbol{q} = \boldsymbol{0}$, and the conservation equations then read:

$$\begin{cases} \dfrac{d\varrho}{dt} = -\varrho \boldsymbol{\nabla} \cdot \boldsymbol{v} \,, \\ \dfrac{d\boldsymbol{v}}{dt} = -\dfrac{1}{\varrho} \boldsymbol{\nabla} p \,, \\ \dfrac{ds}{dt} = 0 \,. \end{cases}$$

- The flow is isentropic (each fluid particle has constant entropy), and by the equation of state $p = p(\varrho, s)$:

$$\frac{dp}{dt} = c^2 \frac{d\varrho}{dt} \,, \quad c^2(\varrho, s) = \left. \frac{\partial p}{\partial \varrho} \right|_s \,.$$

Linearized Euler equations

- Linearization about a stationary fluid flow $(\varrho_0, \boldsymbol{v}_0, p_0)$:

$$\begin{cases} (\boldsymbol{v}_0 \cdot \boldsymbol{\nabla})\varrho_0 = -\varrho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v}_0 \,, \\ (\boldsymbol{v}_0 \cdot \boldsymbol{\nabla})\boldsymbol{v}_0 = -\dfrac{1}{\varrho_0}\boldsymbol{\nabla} p_0 \,, \\ (\boldsymbol{v}_0 \cdot \boldsymbol{\nabla})p_0 = c_0^2(\boldsymbol{v}_0 \cdot \boldsymbol{\nabla})\varrho_0 \,. \end{cases}$$

- The actual flow is a perturbation $(\varrho', \boldsymbol{v}', p')$ of the stationary flow:

$$\begin{cases} \varrho(\boldsymbol{x},t) = \varrho_0(\boldsymbol{x}) + \varrho'(\boldsymbol{x},t) \,, \\ \boldsymbol{v}(\boldsymbol{x},t) = \boldsymbol{v}_0(\boldsymbol{x}) + \boldsymbol{v}'(\boldsymbol{x},t) \,, \\ p(\boldsymbol{x},t) = p_0(\boldsymbol{x}) + p'(\boldsymbol{x},t) \,. \end{cases}$$

Linearized Euler equations

- Let the stationary flow be at rest $\boldsymbol{v}_0 = \mathbf{0}$ in the absence of perturbations (waves), the linearized Euler equations finally yields:

$$\begin{cases} \dfrac{\partial \varrho'}{\partial t} = -\varrho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v}' \\ \dfrac{\partial \boldsymbol{v}'}{\partial t} = -\dfrac{1}{\varrho_0} \boldsymbol{\nabla} p' \\ \dfrac{\partial p'}{\partial t} = c_0^2 \left(\dfrac{\partial \varrho'}{\partial t} + \boldsymbol{v}' \cdot \boldsymbol{\nabla} \varrho_0 \right) . \end{cases}$$

- Hence the acoustic wave equation reads:

$$\boxed{\frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} - \varrho_0 \boldsymbol{\nabla} \cdot \left(\frac{1}{\varrho_0} \boldsymbol{\nabla} p' \right) = 0} .$$

Green's function

- The Green's function $\mathcal{G}(\boldsymbol{x},t;\boldsymbol{x}_0,t_0)$ is the solution of:

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \varrho_0 \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \frac{1}{\varrho_0}\boldsymbol{\nabla}_{\boldsymbol{x}}\right)\mathcal{G}(\boldsymbol{x},t;\boldsymbol{x}_0,t_0) = \delta(t-t_0)\delta(\boldsymbol{x}-\boldsymbol{x}_0)\,,$$
$$\mathcal{G}(\boldsymbol{x},t;\boldsymbol{x}_0,t_0) = 0\,,\quad t<t_0\,,$$

  for all $(\boldsymbol{x},t)\in\mathbb{R}^3\times\mathbb{R}$ and some $(\boldsymbol{x}_0,t_0)\in\mathbb{R}^3\times\mathbb{R}$ fixed.

- The solution in homogeneous space is:

$$\begin{aligned}\mathcal{G}_0(\boldsymbol{x},t;\boldsymbol{x}_0,t_0) &= G_0(\boldsymbol{x}-\boldsymbol{x}_0,t-t_0)\\ &= \frac{1}{4\pi\|\boldsymbol{x}-\boldsymbol{x}_0\|}\delta\left(t-t_0-\frac{\|\boldsymbol{x}-\boldsymbol{x}_0\|}{c_0}\right).\end{aligned}$$

- **Remarks**: $\mathcal{G}_0$ is singular at $\boldsymbol{x}=\boldsymbol{x}_0$, and $\mathcal{G}_0\equiv 0$ if $t_0>t$ by causality (one cannot observe a signal before it is emitted).

Kirchhoff's theorem

- The acoustic wave equation in an open domain Ω with boundary $\partial\Omega = \Gamma$, sources f, and initial conditions $p(\boldsymbol{x}, t_0) = p_0(\boldsymbol{x})$, $\partial_t p(\boldsymbol{x}, t_0) = q_0(\boldsymbol{x})$:

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \varrho_0 \boldsymbol{\nabla}\cdot\frac{1}{\varrho_0}\boldsymbol{\nabla}\right) p(\boldsymbol{x}, t) = f(\boldsymbol{x}, t)\,, \quad (\boldsymbol{x}, t) \in \Omega\times\mathbb{R}\,.$$

- Then its solution reads for all $(\boldsymbol{x}, t) \in \Omega \times \mathbb{R}$–the Kirchhoff's theorem (assume $\varrho_0 \simeq \mathrm{C}^{\mathrm{st}}$):

$$\boxed{\begin{aligned} p(\boldsymbol{x}, t) = &\int_{t_0}^{t}\int_{\Omega} \mathcal{G}_0(\boldsymbol{x}, t; \boldsymbol{y}, \tau) f(\boldsymbol{y}, \tau)\mathrm{d}\boldsymbol{y}\mathrm{d}\tau \\ &+ \int_{t_0}^{t}\int_{\Gamma}\left(\mathcal{G}_0(\boldsymbol{x}, t; \boldsymbol{y}, \tau)\frac{\partial p(\boldsymbol{y}, \tau)}{\partial \boldsymbol{n}_{\boldsymbol{y}}} - \frac{\partial \mathcal{G}_0(\boldsymbol{x}, t; \boldsymbol{y}, \tau)}{\partial \boldsymbol{n}_{\boldsymbol{y}}} p(\boldsymbol{y}, \tau)\right)\mathrm{d}S_{\boldsymbol{y}}\mathrm{d}\tau \\ &+ \int_{\Omega}\frac{1}{c_0^2}\left(\mathcal{G}_0(\boldsymbol{x}, t; \boldsymbol{y}, t_0) q_0(\boldsymbol{y}) - \left.\frac{\partial \mathcal{G}_0(\boldsymbol{x}, t; \boldsymbol{y}, \tau)}{\partial \tau}\right|_{\tau = t_0} p_0(\boldsymbol{y})\right)\mathrm{d}\boldsymbol{y}\ . \end{aligned}}$$

Kirchhoff's theorem

- **Proof**: multiplying the wave equation for $p(\boldsymbol{y},\tau)$ by $\mathcal{G}_0(\boldsymbol{x},t;\boldsymbol{y},\tau)$ and integrating on $\Omega\times[t_0,t_1]$ yields

$$
\begin{aligned}
0 &= \int_{t_0}^{t_1}\int_\Omega \left(\frac{1}{c_0^2}\frac{\partial^2 p(\boldsymbol{y},\tau)}{\partial\tau^2} - \Delta_{\boldsymbol{y}} p(\boldsymbol{y},\tau) - f(\boldsymbol{y},\tau)\right)\mathcal{G}_0(\boldsymbol{x},t;\boldsymbol{y},\tau)\,\mathrm{d}\boldsymbol{y}\mathrm{d}\tau \\
&= \int_{t_0}^{t_1}\int_\Omega \frac{1}{c_0^2}\left[\frac{\partial}{\partial\tau}\left(\mathcal{G}_0(\boldsymbol{x},t;\boldsymbol{y},\tau)\frac{\partial p(\boldsymbol{y},\tau)}{\partial\tau}\right) - \frac{\partial\mathcal{G}_0(\boldsymbol{x},t;\boldsymbol{y},\tau)}{\partial\tau}\frac{\partial p(\boldsymbol{y},\tau)}{\partial\tau}\right]\mathrm{d}\boldsymbol{y}\mathrm{d}\tau \\
&\quad - \int_{t_0}^{t_1}\int_\Omega \left[\boldsymbol{\nabla}_{\boldsymbol{y}}\cdot\left(\mathcal{G}_0(\boldsymbol{x},t;\boldsymbol{y},\tau)\boldsymbol{\nabla}_{\boldsymbol{y}} p(\boldsymbol{y},\tau)\right)\right. \\
&\qquad\qquad \left. -\boldsymbol{\nabla}_{\boldsymbol{y}}\mathcal{G}_0(\boldsymbol{x},t;\boldsymbol{y},\tau)\cdot\boldsymbol{\nabla}_{\boldsymbol{y}} p(\boldsymbol{y},\tau)\right]\mathrm{d}\boldsymbol{y}\mathrm{d}\tau \\
&\quad - \int_{t_0}^{t_1}\int_\Omega \mathcal{G}_0(\boldsymbol{x},t;\boldsymbol{y},\tau) f(\boldsymbol{y},\tau)\,\mathrm{d}\boldsymbol{y}\mathrm{d}\tau
\end{aligned}
$$

Kirchhoff's theorem

- **Proof**: multiplying the wave equation for $p(\boldsymbol{y},\tau)$ by $\mathcal{G}_0(\boldsymbol{x},t;\boldsymbol{y},\tau)$ and integrating on $\Omega\times[t_0,t_1]$ yields

$$
\begin{aligned}
0 &= \int_{t_0}^{t_1}\int_\Omega \frac{1}{c_0^2}\frac{\partial}{\partial\tau}\left(\mathcal{G}_0(\boldsymbol{x},t;\boldsymbol{y},\tau)\frac{\partial p(\boldsymbol{y},\tau)}{\partial\tau}-\frac{\partial\mathcal{G}_0(\boldsymbol{x},t;\boldsymbol{y},\tau)}{\partial\tau}p(\boldsymbol{y},\tau)\right)\mathrm{d}\boldsymbol{y}\mathrm{d}\tau \\
&\quad -\int_{t_0}^{t_1}\int_\Omega \boldsymbol{\nabla}_{\boldsymbol{y}}\cdot\left(\mathcal{G}_0(\boldsymbol{x},t;\boldsymbol{y},\tau)\boldsymbol{\nabla}_{\boldsymbol{y}}p(\boldsymbol{y},\tau)-p(\boldsymbol{y},\tau)\boldsymbol{\nabla}_{\boldsymbol{y}}\mathcal{G}_0(\boldsymbol{x},t;\boldsymbol{y},\tau)\right)\mathrm{d}\boldsymbol{y}\mathrm{d}\tau \\
&\quad +\int_{t_0}^{t_1}\int_\Omega\left(\frac{1}{c_0^2}\frac{\partial^2\mathcal{G}_0(\boldsymbol{x},t;\boldsymbol{y},\tau)}{\partial\tau^2}-\Delta_{\boldsymbol{y}}\mathcal{G}_0(\boldsymbol{x},t;\boldsymbol{y},\tau)\right)p(\boldsymbol{y},\tau)\,\mathrm{d}\boldsymbol{y}\mathrm{d}\tau \\
&\quad -\int_{t_0}^{t_1}\int_\Omega \mathcal{G}_0(\boldsymbol{x},t;\boldsymbol{y},\tau)f(\boldsymbol{y},\tau)\,\mathrm{d}\boldsymbol{y}\mathrm{d}\tau\,.
\end{aligned}
$$

Kirchhoff's theorem

- **Proof** (cont'd): but $\mathcal{G}_0(\boldsymbol{x},t;\boldsymbol{y},\tau)=\mathcal{G}_0(\boldsymbol{y},t;\boldsymbol{x},\tau)$ (reciprocity) and applying the divergence theorem yields

$$
\begin{aligned}
p(\boldsymbol{x},t) &= \int_\Omega \frac{1}{c_0^2}\left[\frac{\partial\mathcal{G}_0(\boldsymbol{x},t;\boldsymbol{y},\tau)}{\partial\tau}p(\boldsymbol{y},\tau)-\mathcal{G}_0(\boldsymbol{x},t;\boldsymbol{y},\tau)\frac{\partial p(\boldsymbol{y},\tau)}{\partial\tau}\right]_{\tau=t_0}^{\tau=t_1}\mathrm{d}\boldsymbol{y} \\
&\quad+\int_{t_0}^{t_1}\int_{\partial\Omega}\left(\mathcal{G}_0(\boldsymbol{x},t;\boldsymbol{y},\tau)\frac{p(\boldsymbol{y},\tau)}{\partial\boldsymbol{n}_{\boldsymbol{y}}}-\frac{\partial\mathcal{G}_0(\boldsymbol{x},t;\boldsymbol{y},\tau)}{\partial\boldsymbol{n}_{\boldsymbol{y}}}p(\boldsymbol{y},\tau)\right)\mathrm{d}\boldsymbol{y}\mathrm{d}\tau \\
&\quad+\int_{t_0}^{t_1}\int_\Omega \mathcal{G}_0(\boldsymbol{x},t;\boldsymbol{y},\tau)f(\boldsymbol{y},\tau)\,\mathrm{d}\boldsymbol{y}\mathrm{d}\tau\,.
\end{aligned}
$$

- Choosing $t_1=t^+$ one has $\mathcal{G}_0(\boldsymbol{x},t;\boldsymbol{y},t^+)=0$ by causality, and obtains therefore the claimed result $\square$

Kirchhoff's theorem in frequency domain

- The Helmholtz equation in an open domain Ω with boundary $\partial\Omega = \Gamma$ and sources $\widehat{f}$:

$$-\left(\frac{\omega^2}{c^2(\boldsymbol{x})} + \Delta\right)\widehat{p}(\boldsymbol{x},\omega) = \widehat{f}(\boldsymbol{x},\omega)\,, \quad (\boldsymbol{x},\omega) \in \Omega \times \mathbb{R}\,.$$

- The Green's function $\widehat{\mathcal{G}}(\boldsymbol{x},\boldsymbol{x}_0,\omega)$ is the solution of:

$$-\left(\frac{\omega^2}{c^2(\boldsymbol{x})} + \Delta\right)\widehat{\mathcal{G}}(\boldsymbol{x},\boldsymbol{x}_0,\omega) = \delta(\boldsymbol{x}-\boldsymbol{x}_0)\,, \quad (\boldsymbol{x},\omega) \in \mathbb{R}^3 \times \mathbb{R}\,,$$

for some $\boldsymbol{x}_0 \in \mathbb{R}^3$ fixed, satisfying the Sommerfeld radiation condition (with $c(\boldsymbol{x}) = c_0$ at infinity):

$$\lim_{\|\boldsymbol{x}\|\to\infty} \|\boldsymbol{x}\| \left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}} - \mathrm{i}\frac{\omega}{c_0}\right)\widehat{\mathcal{G}}(\boldsymbol{x},\boldsymbol{x}_0,\omega) = 0\,.$$

Kirchhoff's theorem in frequency domain

- The Green's function $\widehat{\mathcal{G}}_0(\boldsymbol{x},\boldsymbol{x}_0,\omega)$ in homogeneous space $c(\boldsymbol{x})=c_0$ reads:

$$\widehat{\mathcal{G}}_0(\boldsymbol{x},\boldsymbol{x}_0,\omega)=\widehat{G}_0(\boldsymbol{x}-\boldsymbol{x}_0,\omega)=\frac{\mathrm{e}^{\mathrm{i}\frac{\omega}{c_0}\|\boldsymbol{x}-\boldsymbol{x}_0\|}}{4\pi\|\boldsymbol{x}-\boldsymbol{x}_0\|}\,.$$

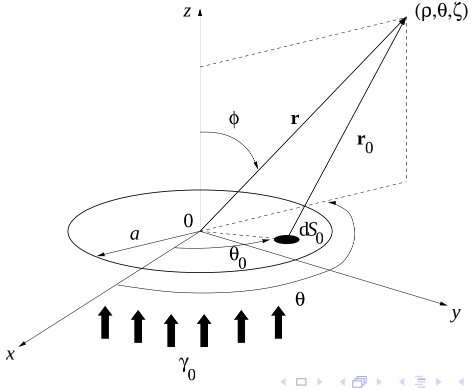
- Then the solution of the Helmholtz equation reads for all $(\boldsymbol{x},\omega)\in\Omega\times\mathbb{R}$:

$$\boxed{\begin{aligned}\widehat{p}(\boldsymbol{x},\omega)&=\int_\Omega\widehat{\mathcal{G}}_0(\boldsymbol{x},\boldsymbol{y},\omega)\widehat{f}(\boldsymbol{y},\omega)\mathrm{d}\boldsymbol{y}\\&\quad+\int_\Gamma\left(\widehat{\mathcal{G}}_0(\boldsymbol{x},\boldsymbol{y},\omega)\frac{\partial\widehat{p}(\boldsymbol{y},\omega)}{\partial\boldsymbol{n}_{\boldsymbol{y}}}-\frac{\partial\widehat{\mathcal{G}}_0(\boldsymbol{x},\boldsymbol{y},\omega)}{\partial\boldsymbol{n}_{\boldsymbol{y}}}\widehat{p}(\boldsymbol{y},\omega)\right)\mathrm{d}S_{\boldsymbol{y}}\,.\end{aligned}}$$

- **Proof**: multiply the Helmholtz equation for $\widehat{p}(\boldsymbol{y},\omega)$ by $\widehat{\mathcal{G}}_0(\boldsymbol{x},\boldsymbol{y},\omega)$ and integrate on Ω.

Rayleigh's integral

- Computation of the acoustic field radiated by a planar surface S_0 at $z = 0$ of which normal speed is $\mathrm{i}\omega v_0(\boldsymbol{x}_0, t) = \gamma_0(\boldsymbol{x}_0)\mathrm{e}^{-\mathrm{i}\omega t}$, $\boldsymbol{x}_0 \in S_0$.
- No source in the half-space $\{z > 0\}$.



Rayleigh's integral

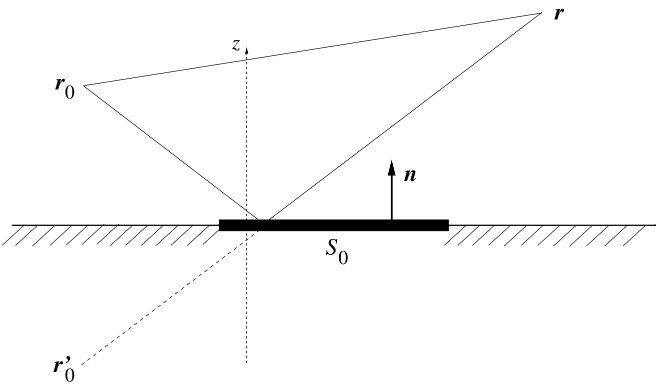
- Taking the normal trace of the Euler equation yields $\varrho_0\gamma_0 = \frac{\partial\widehat{p}}{\partial\boldsymbol{n}}$, and by Kirchhoff theorem:

$$\widehat{p}(\boldsymbol{r},\omega) = \int_{S_0}\left(\varrho_0\gamma_0\widehat{\mathcal{G}}_0(\boldsymbol{r},\boldsymbol{x}_0,\omega) - \frac{\partial\widehat{\mathcal{G}}_0(\boldsymbol{r},\boldsymbol{x}_0,\omega)}{\partial\boldsymbol{n}_0}\widehat{p}(\boldsymbol{x}_0,\omega)\right)\mathrm{d}S(\boldsymbol{x}_0)\,,$$

  but $\widehat{p}(\boldsymbol{x}_0,\omega)$ remains unknown on S_0.
- We rather consider the Green's function of the half-space $\{z>0\}$ with rigid wall at $z=0$, which is given by the method of images for a source $\boldsymbol{r}_0 = (\boldsymbol{x}_0, z_0>0)$ and its image $\boldsymbol{r}'_0 = (\boldsymbol{x}_0,-z_0)$:

$$\widehat{\mathcal{G}}_+(\boldsymbol{r},\boldsymbol{r}_0,\omega) = \frac{\mathrm{e}^{\mathrm{i}\frac{\omega}{c_0}\|\boldsymbol{r}-\boldsymbol{r}_0\|}}{4\pi\|\boldsymbol{r}-\boldsymbol{r}_0\|} + \frac{\mathrm{e}^{\mathrm{i}\frac{\omega}{c_0}\|\boldsymbol{r}-\boldsymbol{r}'_0\|}}{4\pi\|\boldsymbol{r}-\boldsymbol{r}'_0\|}\,.$$

Rayleigh's integral



$$\widehat{\mathcal{G}}_+(\boldsymbol{r},\boldsymbol{r}_0,\omega)=\frac{\mathrm{e}^{\mathrm{i}\frac{\omega}{c_0}\|\boldsymbol{r}-\boldsymbol{r}_0\|}}{4\pi\|\boldsymbol{r}-\boldsymbol{r}_0\|}+\frac{\mathrm{e}^{\mathrm{i}\frac{\omega}{c_0}\|\boldsymbol{r}-\boldsymbol{r}_0'\|}}{4\pi\|\boldsymbol{r}-\boldsymbol{r}_0'\|}$$

Rayleigh's integral

- Letting $z_0 \to 0$ such that $\boldsymbol{r}_0 \to \boldsymbol{x}_0$ one ends up with:

$$\widehat{\mathcal{G}}_+(\boldsymbol{r},\boldsymbol{x}_0,\omega) = \frac{\mathrm{e}^{\mathrm{i}\frac{\omega}{c_0}\|\boldsymbol{r}-\boldsymbol{x}_0\|}}{2\pi\|\boldsymbol{r}-\boldsymbol{x}_0\|}\,,$$

$$\frac{\partial\widehat{\mathcal{G}}_+(\boldsymbol{r},\boldsymbol{x}_0,\omega)}{\partial\boldsymbol{n}_0} = 0\,.$$

- Using the foregoing half-space Green's function, the pressure field radiated by the rigid surface is finally given by Rayleigh's integral:

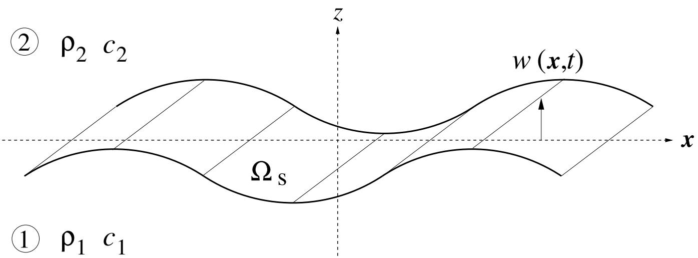
$$\boxed{\widehat{p}(\boldsymbol{r},\omega) = \frac{\varrho_0}{2\pi}\int_{S_0}\frac{\mathrm{e}^{\mathrm{i}\frac{\omega}{c_0}\|\boldsymbol{r}-\boldsymbol{x}_0\|}}{\|\boldsymbol{r}-\boldsymbol{x}_0\|}\gamma_0(\boldsymbol{x}_0)\mathrm{d}S(\boldsymbol{x}_0)}\,.$$

- **Example**: baffled piston (see tutorial class).

Outline

Objectives

- Coupling of a thin plate with the surrounding air or water.
- **Examples**: musical instruments, underwater acoustics of submarines, sound insulation of buildings, architectural acoustics...
- For hints see for instance http://vibroacoustique.fr/cours/M01_C01/co/VAC_M01_C01_web.html.



Infinite plate *in vacuo*

- The equation of motion for the normal displacement $w(\boldsymbol{x},t)$ of an infinite thin plate $\Omega_{\mathrm{s}} = \{\boldsymbol{x} \in \mathbb{R}^2\}$ using Kirchhoff-Love kinematics:

$$\varrho_{\mathrm{s}}\partial_t^2 w + D\Delta_{\boldsymbol{x}}^2 w = \boldsymbol{f}_{\mathrm{ext}} \cdot \boldsymbol{e}_z \,,$$

  where $D = \frac{Eh^3}{12(1-\nu^2)}$, h is the plate thickness, E is Young's modulus, ν is Poisson's ratio, ϱ is the density, $\varrho_{\mathrm{s}} = \varrho h$, and $\Delta_{\boldsymbol{x}}^2 = (\boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}})^2$.
- Thin plate assumption $h \ll \lambda$ (see Mindlin kinematics for thick plates–the like of Timoshenko theory for thick beams).

Plane wave solution and dispersion equation

- Plane wave solution:

$$w(\boldsymbol{x},t) = W\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}\,, \quad \boldsymbol{k}\in\mathbb{R}^2\,,\omega\in\mathbb{R}\,.$$

- Plugging-in this *ansatz* in the homogeneous ($\boldsymbol{f}_{\mathrm{ext}}\equiv\boldsymbol{0}$) equation of motion yields the dispersion equation and the flexural wave number $\omega\mapsto k_{\mathrm{b}}(\omega)$:

$$-\varrho_{\mathrm{s}}\omega^2 + D\|\boldsymbol{k}\|^4 = 0\,, \quad k_{\mathrm{b}}(\omega) = \sqrt[4]{\frac{\varrho_{\mathrm{s}}\omega^2}{D}}\,.$$

- The phase velocity $c_{\mathrm{b}} := \frac{\omega}{k_{\mathrm{b}}}$ and group velocity $c_{\mathrm{g}} := \frac{\mathrm{d}\omega}{\mathrm{d}k}$ are:

$$c_{\mathrm{b}}(\omega) = \sqrt[4]{\frac{D\omega^2}{\varrho_{\mathrm{s}}}}\,, \quad c_{\mathrm{g}}(\omega) = 2c_{\mathrm{b}}(\omega)\,.$$

Plane wave solution and dispersion equation

- **Remark** (group velocity): Consider the wave trains $\cos(\omega t - kx)$ and $\cos\left[(\omega + \mathrm{d}\omega)t - (k + \mathrm{d}k)x\right]$, their sum is:

$$2\cos\left(\frac{\mathrm{d}\omega}{2}t - \frac{\mathrm{d}k}{2}x\right)\cos\left[\left(\omega + \frac{\mathrm{d}\omega}{2}\right)t - \left(k + \frac{\mathrm{d}k}{2}\right)x\right] .$$

- The average wave front is located at the position x and time t s.t. $t\mathrm{d}\omega - x\mathrm{d}k = 0$, thus it travels at the group velocity:

$$c_{\mathrm{g}} = \frac{\mathrm{d}\omega}{\mathrm{d}k} .$$

- Group velocity = speed of transport of the energy.
- Euler-Bernoulli beam: $c_{\mathrm{b}}(\omega) = \sqrt[4]{\frac{EI\omega^2}{\varrho A}}$, $c_{\mathrm{g}}(\omega) = 2c_{\mathrm{b}}(\omega)$.

Infinite plate coupled with acoustic fluids

- For fluid media $\Omega_j = \{\boldsymbol{r} = (\boldsymbol{x}, z \lessgtr 0)\}$ above and below the plate with density and speed of sound ϱ_2, c_2 $(z > 0)$ and ϱ_1, c_1 $(z < 0)$, respectively, the linearized Euler equation (balance of momentum) in frequency domain reads:

$$\mathrm{i}\omega\varrho_j\widehat{\boldsymbol{v}}_j + \boldsymbol{\nabla}_{\boldsymbol{r}}\widehat{p}_j = 0\,, \quad j = 1,2\,,$$

with the Sommerfeld radiation condition for $\|\boldsymbol{r}\| \to \infty$.

- The bending equation for the plate in frequency domain reads:

$$\left(-\varrho_{\mathrm{s}}\omega^2 + D\Delta_{\boldsymbol{x}}^2\right)\widehat{w}(\boldsymbol{x},\omega) = \widehat{p}_1(\boldsymbol{x},0^-,\omega) - \widehat{p}_2(\boldsymbol{x},0^+,\omega)\,,$$

with continuity of the normal speed:

$$\mathrm{i}\omega\widehat{w}(\boldsymbol{x},\omega)|_{z=0^-} = \widehat{\boldsymbol{v}}_1(\boldsymbol{x},0^-,\omega)\cdot\boldsymbol{e}_z\,,$$
$$\mathrm{i}\omega\widehat{w}(\boldsymbol{x},\omega)|_{z=0^+} = \widehat{\boldsymbol{v}}_2(\boldsymbol{x},0^+,\omega)\cdot\boldsymbol{e}_z\,.$$

Plane wave solution

- The pressure fields may be sought for as:

$$\widehat{p}_1(\boldsymbol{x}, z<0, \omega) = P_1 \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1 \cdot \boldsymbol{x} - k_{1n} z)}\,,$$

$$\widehat{p}_2(\boldsymbol{x}, z>0, \omega) = P_2 \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2 \cdot \boldsymbol{x} + k_{2n} z)}\,,$$

owing to the Sommerfeld radiation conditions (no incoming waves), where $k_{jn}(\omega) = (\frac{\omega^2}{c_j^2} - \|\boldsymbol{k}_j\|^2)^{\frac{1}{2}}$.

- The plate bending motion may be sought for as:

$$\widehat{w}(\boldsymbol{x}, \omega) = W \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,.$$

- Taking the normal trace of the Euler equation on either side of the plate yields $-\mathrm{i}\omega\varrho_j \widehat{\boldsymbol{v}}_j \cdot \boldsymbol{e}_z = \partial_z \widehat{p}_j$, therefore:

$$\widehat{p}_1(\boldsymbol{x}, 0^-, \omega) = \frac{\mathrm{i}\varrho_1\omega^2}{k_{1n}(\omega)} \widehat{w}(\boldsymbol{x}, \omega)\,,$$

$$\widehat{p}_2(\boldsymbol{x}, 0^+, \omega) = -\frac{\mathrm{i}\varrho_2\omega^2}{k_{2n}(\omega)} \widehat{w}(\boldsymbol{x}, \omega)\,.$$

Dispersion equation with fluid coupling

- Because these relationships are valid for all $\boldsymbol{x} \in \mathbb{R}^2$, the Snell-Descartes law–conservation of the tangent wave vector $\boldsymbol{k} \in \mathbb{R}^2$–is obtained:

$$\boxed{\boldsymbol{k}_1 = \boldsymbol{k} = \boldsymbol{k}_2},$$

and the following dispersion equation for the plate surrounded by the fluids holds:

$$-\mathrm{i}\omega^2 \left(\frac{\varrho_1}{\sqrt{\frac{\omega^2}{c_1^2} - \|\boldsymbol{k}\|^2}} + \frac{\varrho_2}{\sqrt{\frac{\omega^2}{c_2^2} - \|\boldsymbol{k}\|^2}} \right) - \omega^2 \varrho_{\mathrm{s}} + D\|\boldsymbol{k}\|^4 = 0\,.$$

- Consequently the pressure fields in the fluid media are:

$$\widehat{p}_1(\boldsymbol{x}, z < 0, \omega) = \frac{\mathrm{i}\varrho_1\omega^2}{k_{1n}(\omega)} \widehat{w}(\boldsymbol{x}, \omega) \mathrm{e}^{-\mathrm{i}k_{1n}(\omega)z}\,,$$

$$\widehat{p}_2(\boldsymbol{x}, z > 0, \omega) = -\frac{\mathrm{i}\varrho_2\omega^2}{k_{2n}(\omega)} \widehat{w}(\boldsymbol{x}, \omega) \mathrm{e}^{+\mathrm{i}k_{2n}(\omega)z}\,.$$

Infinite plate coupled with the air

- Assume now $c_1 = c_2 = c_0$ and $\varrho_1 = \varrho_2 = \varrho_0$ for a light fluid (air) such that $\varepsilon = \frac{2\varrho_0}{\varrho_s k_b} \ll 1$, where $k_b^4 = \frac{\varrho_s \omega^2}{D}$.
- Let $\alpha(\omega)$ be defined by $\mathrm{i}\alpha = k_n = (k_0^2 - \|\boldsymbol{k}\|^2)^{\frac{1}{2}}$ where $\boldsymbol{k} \in \mathbb{R}^2$ is the plate (tangent) wave vector and $k_0 = \frac{\omega}{c_0}$; then:

$$-\varepsilon \frac{k_b^5}{\alpha} - k_b^4 + (k_0^2 + \alpha^2)^2 = 0\,.$$

- The four roots for $\varepsilon \ll 1$ are:

$$\begin{aligned} \alpha_{1,2} &= \pm\mathrm{i}\sqrt{k_0^2 + k_b^2}\,, \\ \alpha_{3,4} &= \pm\sqrt{k_b^2 - k_0^2} && \text{if} \quad k_b^2 > k_0^2\,, \\ \alpha_{3,4} &= \pm\mathrm{i}\sqrt{k_0^2 - k_b^2} && \text{if} \quad k_b^2 < k_0^2\,. \end{aligned}$$

First case: $\alpha_{1,2} = \pm\mathrm{i}\sqrt{k_0^2 + k_\mathrm{b}^2}$

- $\mathrm{i}\alpha_{1,2} = \mp(k_0^2 + k_\mathrm{b}^2)^{\frac{1}{2}}$ but only the first instance is acceptable physically in order to fulfill the Sommerfeld radiation condition.
- The pressure waves within the air are propagative:

$$\widehat{p}(\boldsymbol{x}, z < 0, \omega) = \frac{\mathrm{i}\varrho_0\omega^2\widehat{w}(\boldsymbol{x}, \omega)}{\sqrt{k_0^2 + k_\mathrm{b}^2}} \mathrm{e}^{-\mathrm{i}z\sqrt{k_0^2+k_\mathrm{b}^2}} \,,$$

$$\widehat{p}(\boldsymbol{x}, z > 0, \omega) = -\frac{\mathrm{i}\varrho_0\omega^2\widehat{w}(\boldsymbol{x}, \omega)}{\sqrt{k_0^2 + k_\mathrm{b}^2}} \mathrm{e}^{+\mathrm{i}z\sqrt{k_0^2+k_\mathrm{b}^2}} \,.$$

- Then $\|\boldsymbol{k}\|^2 = -k_\mathrm{b}^2$ or $\|\boldsymbol{k}\| = \pm\mathrm{i}k_\mathrm{b}$, and therefore the bending waves within the plate are evanescent:

$$\widehat{w}(\boldsymbol{x}, \omega) = W\mathrm{e}^{-k_\mathrm{b}(\omega)\|\boldsymbol{x}\|} \,.$$

Second case: $\alpha_{3,4} = \pm\sqrt{k_\mathrm{b}^2 - k_0^2}$ for $k_\mathrm{b}^2 > k_0^2$

- $\mathrm{i}\alpha_{3,4} = \pm\mathrm{i}(k_\mathrm{b}^2 - k_0^2)^{\frac{1}{2}}$ but only the first instance is acceptable physically in order to fulfill the Sommerfeld radiation condition.
- The pressure waves within the air are evanescent:

$$\widehat{p}(\boldsymbol{x}, z < 0, \omega) = \frac{\varrho_0\omega^2\widehat{w}(\boldsymbol{x},\omega)}{\sqrt{k_\mathrm{b}^2 - k_0^2}}\mathrm{e}^{z\sqrt{k_\mathrm{b}^2-k_0^2}}\,,$$

$$\widehat{p}(\boldsymbol{x}, z > 0, \omega) = -\frac{\varrho_0\omega^2\widehat{w}(\boldsymbol{x},\omega)}{\sqrt{k_\mathrm{b}^2 - k_0^2}}\mathrm{e}^{-z\sqrt{k_\mathrm{b}^2-k_0^2}}\,.$$

- Then $\|\boldsymbol{k}\|^2 = k_\mathrm{b}^2$ or $\|\boldsymbol{k}\| = \pm k_\mathrm{b}$, and therefore the bending waves within the plate are propagative:

$$\widehat{w}(\boldsymbol{x},\omega) = W_+\mathrm{e}^{+\mathrm{i}k_\mathrm{b}(\omega)\hat{\boldsymbol{k}}\cdot\boldsymbol{x}} + W_-\mathrm{e}^{-\mathrm{i}k_\mathrm{b}(\omega)\hat{\boldsymbol{k}}\cdot\boldsymbol{x}}\,.$$

Third case: $\alpha_{3,4} = \pm\mathrm{i}\sqrt{k_0^2 - k_\mathrm{b}^2}$ for $k_\mathrm{b}^2 < k_0^2$

- $\mathrm{i}\alpha_{3,4} = \mp(k_0^2 - k_\mathrm{b}^2)^{\frac{1}{2}}$ but only the first instance is acceptable physically in order to fulfill the Sommerfeld radiation condition.
- The pressure waves within the air are propagative:

$$\widehat{p}(\boldsymbol{x}, z < 0, \omega) = \frac{\mathrm{i}\varrho_0\omega^2\widehat{w}(\boldsymbol{x}, \omega)}{\sqrt{k_0^2 - k_\mathrm{b}^2}}\mathrm{e}^{-\mathrm{i}z\sqrt{k_0^2 - k_\mathrm{b}^2}}\,,$$

$$\widehat{p}(\boldsymbol{x}, z > 0, \omega) = -\frac{\mathrm{i}\varrho_0\omega^2\widehat{w}(\boldsymbol{x}, \omega)}{\sqrt{k_0^2 - k_\mathrm{b}^2}}\mathrm{e}^{+\mathrm{i}z\sqrt{k_0^2 - k_\mathrm{b}^2}}\,.$$

- Then $\|\boldsymbol{k}\|^2 = k_\mathrm{b}^2$ or $\|\boldsymbol{k}\| = \pm k_\mathrm{b}$, and therefore the bending waves within the plate are propagative:

$$\widehat{w}(\boldsymbol{x}, \omega) = W_+\mathrm{e}^{+\mathrm{i}k_\mathrm{b}(\omega)\hat{\boldsymbol{k}}\cdot\boldsymbol{x}} + W_-\mathrm{e}^{-\mathrm{i}k_\mathrm{b}(\omega)\hat{\boldsymbol{k}}\cdot\boldsymbol{x}}\,.$$

Sound radiation by an infinite plate

- The critical frequency such that $k_\mathrm{b}(\omega) = k_0 = \frac{\omega}{c_0}$ is $\omega_c = c_0^2\sqrt{\frac{\varrho_\mathrm{s}}{D}}$.
- The radiation efficiency (*facteur de rayonnement*) $\sigma(\omega)$ of the plate is defined by:

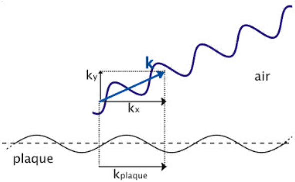
$$\sigma(\omega) = \frac{1}{\varrho_0 c_0}\Re\mathrm{e}\left[\frac{\widehat{p}(\boldsymbol{x},0^+,\omega)}{-\mathrm{i}\omega\widehat{w}(\boldsymbol{x},\omega)}\right] .$$

- For the various foregoing situations it is:

$$\sigma(\omega) = \begin{cases} \dfrac{1}{\sqrt{1+\frac{\omega_c}{\omega}}} & \text{for evanescent waves in the plate}\,, \\ 0 & \text{for propagative waves in the plate with } \omega < \omega_c\,, \\ \dfrac{1}{\sqrt{1-\frac{\omega_c}{\omega}}} & \text{for propagative waves in the plate with } \omega > \omega_c\,. \end{cases}$$

Sound radiation by an infinite plate

- Subsonic plate waves such that $\omega < \omega_c$ or $c_\mathrm{b} < c_0$ or $\lambda_\mathrm{b} < \lambda_0$ do not radiate within the air: no sound transmission.
- Supersonic plate waves such that $\omega > \omega_c$ or $c_\mathrm{b} > c_0$ or $\lambda_\mathrm{b} > \lambda_0$ do radiate within the air with maximum efficiency when $\omega \simeq \omega_c$.
- This is because by Snell-Descartes law the tangent wave vector is constant in the transmission process, so that if the plate wavenumber k_b is larger than the fluid wavenumber k_0 there can be no transmission at all.



Finite plate *in vacuo*

- The equation of motion for the normal displacement $w(\boldsymbol{x},t)$ of a finite thin plate $\Omega_\mathrm{s}=[0,a]\times[0,b]$ using again Kirchhoff-Love kinematics:

$$\varrho_\mathrm{s}\partial_t^2 w + D\Delta_{\boldsymbol{x}}^2 w = \boldsymbol{f}_\mathrm{ext}\cdot\boldsymbol{e}_z\,,$$

where $D=\frac{Eh^3}{12(1-\nu^2)}$, h is the plate thickness, E is Young's modulus, ν is Poisson's ratio, ϱ is the density, $\varrho_\mathrm{s}=\varrho h$, and $\Delta_{\boldsymbol{x}}^2=(\boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}})^2$.

- Boundary conditions on $\partial\Omega_\mathrm{s}=\{x=0,a\}\cup\{y=0,b\}$ for simply supported plate:

$$w(\boldsymbol{x},t)=0\,,\quad \boldsymbol{M}(\boldsymbol{x},t)\boldsymbol{n}=\boldsymbol{0}\,,$$

where $-\boldsymbol{M}=\nu D(\Delta_{\boldsymbol{x}}w)\boldsymbol{I}+(1-\nu)D\boldsymbol{\nabla}_{\boldsymbol{x}}\otimes\boldsymbol{\nabla}_{\boldsymbol{x}}w$ is the bending moment tensor.

Modal solution and dispersion equation

- Modal expansion solution:

$$w(\boldsymbol{x},t)=\sum_{\boldsymbol{\alpha}}q_{\boldsymbol{\alpha}}(t)\boldsymbol{\phi}_{\boldsymbol{\alpha}}(\boldsymbol{x})$$

  where $\boldsymbol{\alpha}=(\alpha,\beta)\in\mathbb{N}^2\backslash\{\mathbf{0}\}$ is a 2–index and $\boldsymbol{\phi}_{\boldsymbol{\alpha}}$ is the corresponding eigen mode shape of a simply supported thin plate (s.t. $\int_{\Omega_\mathrm{s}}\varrho_\mathrm{s}\boldsymbol{\phi}_{\boldsymbol{\alpha}}(\boldsymbol{x})\boldsymbol{\phi}_{\boldsymbol{\beta}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}=\varrho_\mathrm{s}ab\delta_{\boldsymbol{\alpha}\boldsymbol{\beta}}$):

$$\boldsymbol{\phi}_{\boldsymbol{\alpha}}(\boldsymbol{x})=2\sin\left(\alpha\pi\frac{x}{a}\right)\sin\left(\beta\pi\frac{y}{b}\right)\,.$$

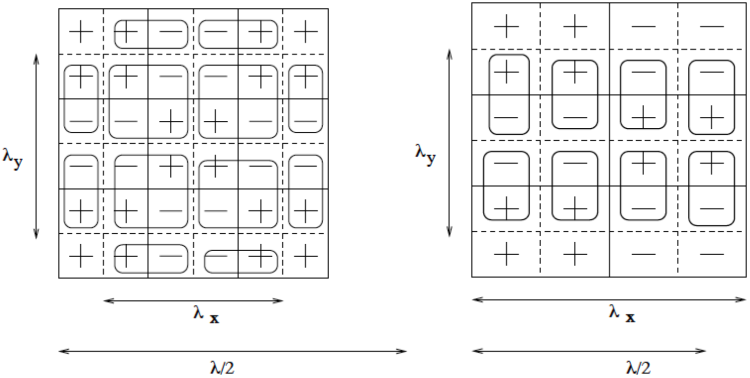
- Plugging-in this expansion in the homogeneous ($\boldsymbol{f}_\mathrm{ext}\equiv\mathbf{0}$) equation of motion yields the discrete flexural wave numbers $\boldsymbol{k}_{\boldsymbol{\alpha}}$:

$$\|\boldsymbol{k}_{\boldsymbol{\alpha}}\|^2=\left(\frac{\alpha\pi}{a}\right)^2+\left(\frac{\beta\pi}{b}\right)^2\,,\quad\omega_{\boldsymbol{\alpha}}=\|\boldsymbol{k}_{\boldsymbol{\alpha}}\|^2\sqrt{\frac{D}{\varrho_\mathrm{s}}}\,.$$

Finite plate coupled with acoustic fluids

- The eigen modes of the plate *in vacuo* are still a basis for the bending motion, but they get coupled when the surrounding fluids are taken into account.
- Qualitatively one has basically for air with acoustic wavelength λ_0 and bending wavelengths $\frac{\lambda_x}{2} = \frac{a}{\alpha}$, $\frac{\lambda_y}{2} = \frac{b}{\beta}$:
  - If $\omega > \omega_c$: the radiation efficiency goes to 1;
  - If $\omega < \omega_c$: the radiation efficiency is interpreted in terms of possible boundary effects.
    - If $\lambda_x < \frac{\lambda_0}{2}$ and $\lambda_y < \frac{\lambda_0}{2}$: corner radiation whereby modal quadrupoles and dipoles have low radiation efficiency;
    - If $\lambda_x < \frac{\lambda_0}{2}$ or $\lambda_y < \frac{\lambda_0}{2}$: boundary radiation;
    - If $\lambda_x > \frac{\lambda_0}{2}$ and $\lambda_y > \frac{\lambda_0}{2}$: overall plater radiation.

Finite plate coupled with acoustic fluids



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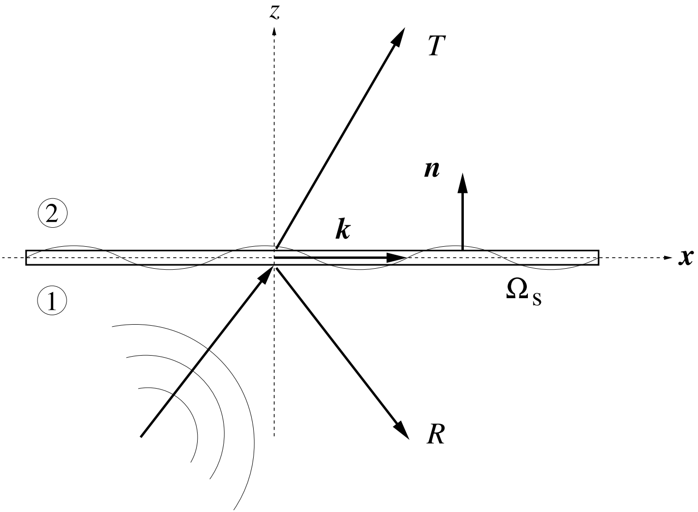
Corner radiation (left) *vs.* boundary radiation (right).

Outline

Sound transmission through a plate



Sound transmission through an infinite plate

- Fluid medium ①: incoming plane wave + reflected plane wave

$$\widehat{p}_1(\boldsymbol{x}, z<0, \omega) = P\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\left(\mathrm{e}^{\mathrm{i}k_{1n}z} + R\mathrm{e}^{-\mathrm{i}k_{1n}z}\right).$$

- Fluid medium ②: transmitted plane wave

$$\widehat{p}_2(\boldsymbol{x}, z>0, \omega) = PT\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}+k_{2n}z)}.$$

- R and T are the amplitude reflection and transmission coefficients, and $\boldsymbol{k}$ is the tangent wave vector.
- The wave number in fluid medium ② is $k_2 = \frac{\omega}{c_2}$ and:

$$k_{2n}(\boldsymbol{k}) = k_2\sqrt{1-\left(\frac{\|\boldsymbol{k}\|}{k_2}\right)^2},$$

whereby transmission holds provided that $\|\boldsymbol{k}\| < k_2$ (total reflection otherwise).

Reflection and transmission coefficients

- Plate bending motion $\widehat{w}(\boldsymbol{x},\omega) = W\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$ such that:

$$(-\varrho_{\mathrm{s}}\omega^2 + D\|\boldsymbol{k}\|^4)\widehat{w}(\boldsymbol{x},\omega) = \widehat{p}_1(\boldsymbol{x},0^-,\omega) - \widehat{p}_2(\boldsymbol{x},0^+,\omega)\,,$$

with boundary conditions (continuity of the normal speed $\varrho_j\omega^2\widehat{w} = \partial_z\widehat{p}_j|_{z=\pm 0}$):

$$\begin{aligned}\varrho_1\omega^2\widehat{w}(\boldsymbol{x},\omega) &= \mathrm{i}k_{1n}(\boldsymbol{k})P(1-R)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,,\\ \varrho_2\omega^2\widehat{w}(\boldsymbol{x},\omega) &= \mathrm{i}k_{2n}(\boldsymbol{k})PT\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,.\end{aligned}$$

- One ends up with the system:

$$\begin{aligned}(-\varrho_{\mathrm{s}}\omega^2 + D\|\boldsymbol{k}\|^4)W &= P(1+R-T)\,,\\ \varrho_1\omega^2 W &= \mathrm{i}k_{1n}(\boldsymbol{k})P(1-R)\,,\\ \varrho_2\omega^2 W &= \mathrm{i}k_{2n}(\boldsymbol{k})PT\,.\end{aligned}$$

Reflection and transmission coefficients

- Its solution reads:

$$\frac{W}{P} = \frac{2}{-\omega^2\left(\varrho_\mathrm{s} + \frac{\mathrm{i}\varrho_1}{k_{1n}(\boldsymbol{k})} + \frac{\mathrm{i}\varrho_2}{k_{2n}(\boldsymbol{k})}\right) + D\|\boldsymbol{k}\|^4}\,,$$

$$R = 1 + \frac{\mathrm{i}\varrho_1\omega^2}{k_{1n}(\boldsymbol{k})}\left(\frac{W}{P}\right)\,,$$

$$T = -\frac{\mathrm{i}\varrho_2\omega^2}{k_{2n}(\boldsymbol{k})}\left(\frac{W}{P}\right)\,.$$

- **Remark**: this result holds for all plates for which $\mathcal{Z}_\mathrm{s}(\omega)\widehat{w}(\boldsymbol{x},\omega) = \widehat{f}(\omega)$; consequently:

$$\frac{W}{P} = 2\left[\mathcal{Z}_\mathrm{s}(\omega) - \omega^2\left(\frac{\mathrm{i}\varrho_1}{k_{1n}(\boldsymbol{k})} + \frac{\mathrm{i}\varrho_2}{k_{2n}(\boldsymbol{k})}\right)\right]^{-1}\,.$$

Acoustic transparency

- The acoustic transparency or transmissivity is:

$$\tau(\omega)=\frac{\Pi_{\mathrm{T}}(\omega)}{\Pi_{\mathrm{IN}}(\omega)}\,,$$

where Π_{IN} is the incident power:

$$\Pi_{\mathrm{IN}}(\omega)=\frac{1}{2}\Re\mathrm{e}\int_S\widehat{p}_{\mathrm{IN}}(\boldsymbol{x},0^-,\omega)\overline{\widehat{\boldsymbol{v}}_{\mathrm{IN}}(\boldsymbol{x},0^-,\omega)}\cdot\boldsymbol{e}_z\mathrm{d}S\,,$$

and Π_{T} is the transmitted power:

$$\Pi_{\mathrm{T}}(\omega)=\frac{1}{2}\Re\mathrm{e}\int_S\widehat{p}_{\mathrm{T}}(\boldsymbol{x},0^+,\omega)\overline{\widehat{\boldsymbol{v}}_{\mathrm{T}}(\boldsymbol{x},0^+,\omega)}\cdot\boldsymbol{e}_z\mathrm{d}S\,.$$

Acoustic transparency

- Identifying $\widehat{p}_{\mathrm{IN}}(\boldsymbol{x},z,\omega)=P\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}+k_{1n}z)}$ and $\widehat{p}_{\mathrm{T}}(\boldsymbol{x},z,\omega)=PT\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}+k_{2n}z)}$ one obtains with $-\mathrm{i}\omega\varrho_j\widehat{\boldsymbol{v}}_j\cdot\boldsymbol{e}_z=\partial_z\widehat{p}_j$:

$$\Pi_{\mathrm{IN}}(\omega)=\frac{k_{1n}(\boldsymbol{k})P^2S}{\varrho_1\omega}\,,\quad \Pi_{\mathrm{T}}(\omega)=\frac{\varrho_2\omega^3\left|W\right|^2S}{k_{2n}(\boldsymbol{k})}\,.$$

- Consequently:

$$\begin{aligned}\tau(\omega)&=\frac{\varrho_1\varrho_2\omega^4}{k_{1n}(\boldsymbol{k})k_{2n}(\boldsymbol{k})}\left|\frac{W}{P}\right|^2\\&=\frac{4\varrho_1\varrho_2\omega^4}{k_{1n}(\boldsymbol{k})k_{2n}(\boldsymbol{k})}\left|\mathcal{Z}_{\mathrm{s}}(\omega)-\omega^2\left(\frac{\mathrm{i}\varrho_1}{k_{1n}(\boldsymbol{k})}+\frac{\mathrm{i}\varrho_2}{k_{2n}(\boldsymbol{k})}\right)\right|^{-2}\end{aligned}$$

Acoustic transparency

- For $\varrho_1 = \varrho_2 = \varrho_0$, $c_1 = c_2 = c_0$, and $\|\boldsymbol{k}\| = \frac{\omega}{c_0}\sin\theta$, and introducing the coincidence frequency $\omega_c = \frac{c_0^2}{\sin^2\theta}\sqrt{\frac{\varrho_s}{D}}$, one obtains:

$$\tau(\omega) = \frac{1}{1 + \left(\frac{\varrho_s \omega \cos\theta}{2\varrho_0 c_0}\right)^2 \left(1 - \frac{\omega^2}{\omega_c^2}\right)^2} \,.$$

- The transmission loss factor (*indice d'affaiblissement*) TL is:

$$\begin{aligned}
\mathrm{TL}(\omega) &= 10\log\left(\frac{1}{\tau(\omega)}\right) \\
&= 10\log\left[1 + \left(\frac{\varrho_s \omega \cos\theta}{2\varrho_0 c_0}\right)^2 \left(1 - \frac{\omega^2}{\omega_c^2}\right)^2\right] \,.
\end{aligned}$$

Acoustic transparency

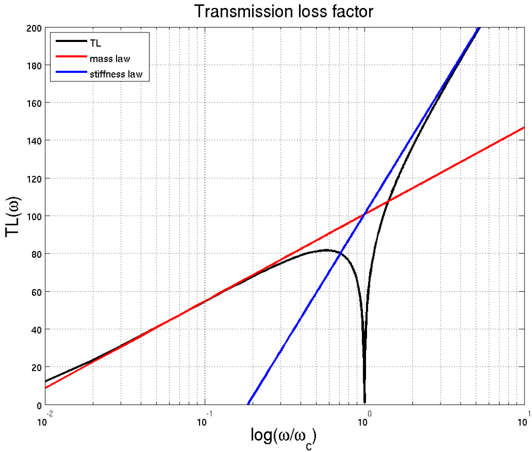
- TL$= 0$ if $\Pi_{\mathrm{T}} = \Pi_{\mathrm{IN}}$, whereas TL$= \infty$ if $\Pi_{\mathrm{T}} \ll \Pi_{\mathrm{IN}}$.
- If $2\pi \ll \omega \ll \omega_c$, one obtains the mass law–TL depends on the mass of the plate solely:

$$\mathrm{TL} \simeq 20 \log \left(\frac{\varrho_{\mathrm{s}} \cos\theta}{2\varrho_0 c_0} \omega \right) .$$

- If $\omega = \omega_c$ then TL$= 0$ (but damping of the plate must be accounted for);
- If $\omega \gg \omega_c$, one obtains the stiffness law:

$$\mathrm{TL} \simeq 60 \log(\alpha\omega) .$$

Acoustic transparency



Transmission loss factor of an aluminum plate with $h = 3$ mm for an incident plane wave with $\theta = \frac{\pi}{3}$.

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