Elementary coupled systems

MG3416–Advanced Structural Acoustics - Lecture #2

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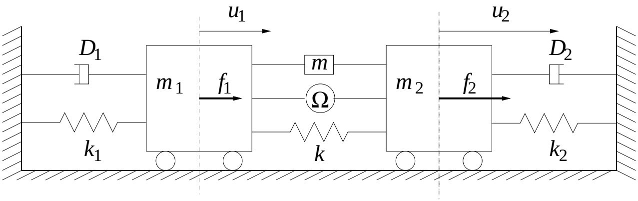
September 29, 2021

Outline

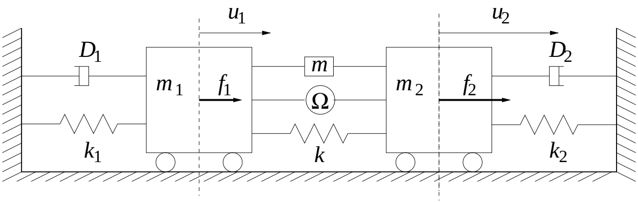
Outline



- The two-degrees-of-freedom system:

$$\begin{bmatrix} M_1 & M \\ M & M_2 \end{bmatrix} \begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix} + \begin{bmatrix} D_1 & \Omega \\ -\Omega & D_2 \end{bmatrix} \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} + \begin{bmatrix} K_1 & K \\ K & K_2 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} .$$

- Fundamental parameters for $\alpha \in \{1, 2\}$:
  - $\omega_\alpha = \sqrt{\frac{K_\alpha}{M_\alpha}}$: the "blocked" natural (angular) frequency of the α^{th} oscillator,
  - $\xi_\alpha = \frac{D_\alpha}{2\sqrt{K_\alpha M_\alpha}}$: the critical damping rate of the α^{th} oscillator,
  - $\eta_\alpha = 2\xi_\alpha$: the loss factor of the α^{th} oscillator.



- The two-degrees-of-freedom system:

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- **Hypotheses**:
  - $K < \sqrt{K_1 K_2}$ and $M < \sqrt{M_1 M_2}$;
  - Conservative gyroscopic coupling:

$$\left(\begin{bmatrix} 0 & \Omega \\ -\Omega & 0 \end{bmatrix} \boldsymbol{a}, \boldsymbol{a} \right) = 0 \,, \quad \forall \boldsymbol{a} \in \mathbb{R}^2 \,.$$

Definition

- *The mechanical energy of the two-DOFs system:*

$$\mathcal{E}(t) = \mathcal{E}_1(t) + \mathcal{E}_2(t) + M\dot{u}_1(t)\dot{u}_2(t) + Ku_1(t)u_2(t)\,,$$

where $\mathcal{E}_\alpha(t) = \frac{1}{2}M_\alpha\dot{u}_\alpha(t)^2 + \frac{1}{2}K_\alpha u_\alpha(t)^2$ *for* $\alpha \in \{1,2\}$.

- *The dissipated power:* $\Pi_\mathrm{d}(t) = \Pi_\mathrm{d1}(t) + \Pi_\mathrm{d2}(t)$, *where* $\Pi_{\mathrm{d}\alpha}(t) = D_\alpha\dot{u}_\alpha(t)^2$ *for* $\alpha \in \{1,2\}$.
- *The input power:* $\Pi_\mathrm{IN}(t) = \Pi_\mathrm{IN1}(t) + \Pi_\mathrm{IN2}(t)$, *where* $\Pi_{\mathrm{IN}\alpha}(t) = f_\alpha(t)\dot{u}_\alpha(t)$ *for* $\alpha \in \{1,2\}$.

- The powers exchanged by the oscillators:

$$\begin{aligned}\Pi_{12}(t) &= (\text{force } 1 \rightarrow 2) \times (\text{celerity } 2) \\ &= -(M\ddot{u}_1 - \Omega\dot{u}_1 + Ku_1)\dot{u}_2\,;\end{aligned}$$

$$\begin{aligned}\Pi_{21}(t) &= (\text{force } 2 \rightarrow 1) \times (\text{celerity } 1) \\ &= -(M\ddot{u}_2 + \Omega\dot{u}_2 + Ku_2)\dot{u}_1\,.\end{aligned}$$

- The instantaneous power of the two-DOFs system:

$$\Pi(t) = \dot{\mathcal{E}}(t) = \dot{\mathcal{E}}_1(t) + \dot{\mathcal{E}}_2(t) - \Pi_{12}(t) - \Pi_{21}(t)\,,$$

- The instantaneous power balance reads:

$$\dot{\mathcal{E}}_1(t) + \dot{\mathcal{E}}_2(t) = \Pi_{\mathrm{IN}}(t) - \Pi_{\mathrm{d}}(t) + \Pi_{12}(t) + \Pi_{21}(t)\,,$$

or for each oscillator:

$$\dot{\mathcal{E}}_\alpha(t) = \Pi_{\mathrm{IN}\alpha}(t) - \Pi_{\mathrm{d}\alpha}(t) + \Pi_{\beta\alpha}(t)\,, \quad \beta \neq \alpha \in \{1,2\}\,.$$

Stationary excitations
Wideband noise

- **Data**: $(F_{\alpha,t},\, t \in \mathbb{R})$, $\alpha \in \{1,2\}$, are $\mathbb{R}$-valued second order, centered stochastic processes defined on $(\Omega, \mathcal{F}, P)$, indexed on $\mathbb{R}$, and mean-square stationary.
- **Hypotheses**:
  - $\exists\, \omega \mapsto S_\alpha(\omega) : \mathbb{R} \to \mathbb{R}_+$ s.t.

$$S_\alpha(\omega) = S_\alpha \mathbb{1}_{I_\alpha \cup \underline{I}_\alpha}(\omega)\,, \quad S_\alpha > 0\,,$$

where:

$$I_\alpha \cup \underline{I}_\alpha = \left[\omega_\alpha - \frac{\Delta\omega}{2}, \omega_\alpha + \frac{\Delta\omega}{2}\right] \bigcup \left[-\omega_\alpha - \frac{\Delta\omega}{2}, -\omega_\alpha + \frac{\Delta\omega}{2}\right];$$

  - $(F_{1,t},\, t \in \mathbb{R})$ and $(F_{2,t},\, t \in \mathbb{R})$ are uncorrelated:

$$\mathbb{E}\{F_{1,t} F_{2,t'}\} = 0\,, \quad \forall (t,t') \in \mathbb{R} \times \mathbb{R}\,.$$

- **Remarks:**
  - $\omega \mapsto S_\alpha(\omega)$ is even and integrable;
  - The auto-correlation function:

$$R_\alpha(t) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\omega t} \, S_\alpha(\omega) \, \mathrm{d}\omega = 2\Delta\omega S_\alpha \cos(\omega_\alpha t) \mathrm{sinc}\left(\frac{\Delta\omega}{2} t\right)$$

is continuous, bounded, $R_\alpha(0) > 0$, $\lim_{|t| \to +\infty} R_\alpha(t) = 0$.

Proposition

- $(U_{\alpha,t}\,, t \in \mathbb{R})$, $\alpha \in \{1, 2\}$, *are* $\mathbb{R}$*-valued second order, centered stochastic processes defined on* $(\Omega, \mathcal{F}, P)$, *indexed on* $\mathbb{R}$, *and mean-square stationary.*
- *The same holds for their mean-square derivatives* $(\dot{U}_{\alpha,t}\,, t \in \mathbb{R})$ *and* $(\ddot{U}_{\alpha,t}\,, t \in \mathbb{R})$, $\alpha \in \{1, 2\}$.

Stationary excitations

Second-order properties of the forced responses

- The $\mathbb{C}^{2\times 2}$ spectral density matrix of the forced response vector $\boldsymbol{U}_t^f = (U_{1,t}, U_{2,t})^\mathsf{T}$ reads:

$$\begin{aligned}\boldsymbol{S}_{\boldsymbol{U}}(\omega) &= \widehat{\mathbb{h}}_1(\omega)\otimes\overline{\widehat{\mathbb{h}}_1(\omega)}S_1(\omega) + \widehat{\mathbb{h}}_2(\omega)\otimes\overline{\widehat{\mathbb{h}}_2(\omega)}S_2(\omega) \\ &= \frac{S_1(\omega)}{|D(\omega)|^2}\begin{bmatrix}|A_2(\omega)|^2 & -A_c(\omega)A_2(\omega) \\ -\overline{A_c(\omega)A_2(\omega)} & |A_c(\omega)|^2\end{bmatrix} \\ &\quad + \frac{S_2(\omega)}{|D(\omega)|^2}\begin{bmatrix}|A_c(\omega)|^2 & -A_c(\omega)\overline{A_1(\omega)} \\ -\overline{A_c(\omega)}A_1(\omega) & |A_1(\omega)|^2\end{bmatrix}\end{aligned}$$

where $D(\omega) = A_1(\omega)A_2(\omega) - |A_c(\omega)|^2$ and:

$$\begin{aligned}A_\alpha(\omega) &= -\omega^2 M_\alpha + \mathrm{i}\omega D_\alpha + K_\alpha\,, \\ A_c(\omega) &= -\omega^2 M + \mathrm{i}\omega\Omega + K\,.\end{aligned}$$

- Besides:

$$\begin{aligned}\mathbb{E}\{U_{\alpha,t}\dot{U}_{\alpha,t}\} &= \mathbb{E}\{\dot{U}_{\alpha,t}\ddot{U}_{\alpha,t}\} = 0\,,\quad \forall t\in\mathbb{R}\,, \\ \mathbb{E}\{U_{1,t}\dot{U}_{2,t}\} &= -\mathbb{E}\{\dot{U}_{1,t}U_{2,t}\}\,, \\ \mathbb{E}\{\dot{U}_{1,t}\ddot{U}_{2,t}\} &= -\mathbb{E}\{\ddot{U}_{1,t}\dot{U}_{2,t}\}\,.\end{aligned}$$

- $(-\omega^2\boldsymbol{M}+\mathrm{i}\omega\boldsymbol{D}+\boldsymbol{K})\widehat{\boldsymbol{U}}^f=\widehat{\boldsymbol{F}}$ thus (filtering):

$$
\begin{aligned}
\boldsymbol{S}_{\boldsymbol{U}}(\omega) &= \widehat{\mathbb{h}}(\omega)\boldsymbol{S}_{\boldsymbol{F}}(\omega)\widehat{\mathbb{h}}^*(\omega) \\
&= \left(-\omega^2\boldsymbol{M}+\mathrm{i}\omega\boldsymbol{D}+\boldsymbol{K}\right)^{-1}\begin{bmatrix} S_1(\omega) & 0 \\ 0 & S_2(\omega)\end{bmatrix}\overline{(-\omega^2\boldsymbol{M}+\mathrm{i}\omega\boldsymbol{D}+\boldsymbol{K})}^{-\mathsf{T}} \\
&= \widehat{\mathbb{h}}_1(\omega)\otimes\overline{\widehat{\mathbb{h}}_1(\omega)}S_1(\omega)+\widehat{\mathbb{h}}_2(\omega)\otimes\overline{\widehat{\mathbb{h}}_2(\omega)}S_2(\omega) \\
&= \frac{S_1(\omega)}{|D(\omega)|^2}\begin{bmatrix} |A_2(\omega)|^2 & -A_c(\omega)A_2(\omega) \\ -\overline{A_c(\omega)A_2(\omega)} & |A_c(\omega)|^2\end{bmatrix} \\
&\quad+\frac{S_2(\omega)}{|D(\omega)|^2}\begin{bmatrix} |A_c(\omega)|^2 & -A_c(\omega)\overline{A_1(\omega)} \\ -\overline{A_c(\omega)}A_1(\omega) & |A_1(\omega)|^2\end{bmatrix}
\end{aligned}
$$

Stationary excitations

Second-order properties of the forced responses (extended proof)

- Mean-square derivation $\boldsymbol{R}_{\dot{\boldsymbol{U}}\boldsymbol{U}}(t') = \mathbb{E}\{\dot{\boldsymbol{U}}_{t'+t} \otimes \boldsymbol{U}_t\} = -\boldsymbol{R}_{\boldsymbol{U}\dot{\boldsymbol{U}}}(t')$, $\forall t, t' \in \mathbb{R}$, that is:

$$\begin{bmatrix} \mathbb{E}\{\dot{U}_{1,t'+t}U_{1,t}\} & \mathbb{E}\{\dot{U}_{1,t'+t}U_{2,t}\} \\ \mathbb{E}\{\dot{U}_{2,t'+t}U_{1,t}\} & \mathbb{E}\{\dot{U}_{2,t'+t}U_{2,t}\} \end{bmatrix} = -\begin{bmatrix} \mathbb{E}\{U_{1,t'+t}\dot{U}_{1,t}\} & \mathbb{E}\{U_{1,t'+t}\dot{U}_{2,t}\} \\ \mathbb{E}\{U_{2,t'+t}\dot{U}_{1,t}\} & \mathbb{E}\{U_{2,t'+t}\dot{U}_{2,t}\} \end{bmatrix} .$$

  Hence taking $t' = 0$:

$$\mathbb{E}\{U_{\alpha,t}\dot{U}_{\alpha,t}\} = 0 \,,$$
$$\mathbb{E}\{U_{1,t}\dot{U}_{2,t}\} = -\mathbb{E}\{\dot{U}_{1,t}U_{2,t}\} \quad \forall t \in \mathbb{R} \,.$$

- Likewise $\boldsymbol{R}_{\ddot{\boldsymbol{U}}\dot{\boldsymbol{U}}}(t') = \mathbb{E}\{\ddot{\boldsymbol{U}}_{t'+t} \otimes \dot{\boldsymbol{U}}_t\} = -\boldsymbol{R}_{\dot{\boldsymbol{U}}\ddot{\boldsymbol{U}}}(t')$, $\forall t, t' \in \mathbb{R}$, therefore taking $t' = 0$:

$$\mathbb{E}\{\dot{U}_{\alpha,t}\ddot{U}_{\alpha,t}\} = 0 \,,$$
$$\mathbb{E}\{\dot{U}_{1,t}\ddot{U}_{2,t}\} = -\mathbb{E}\{\ddot{U}_{1,t}\dot{U}_{2,t}\} \quad \forall t \in \mathbb{R} \,.$$

Stationary excitations

Energetics of the stationary forced responses

- The instantaneous power balance reads:

$$\dot{\mathcal{E}}_{1,t} + \dot{\mathcal{E}}_{2,t} = \Pi_{\mathrm{IN},t} - \Pi_{\mathrm{d},t} + \Pi_{12,t} + \Pi_{21,t}\,,$$

or for each oscillator:

$$\dot{\mathcal{E}}_{\alpha,t} = \Pi_{\mathrm{IN}\alpha,t} - \Pi_{\mathrm{d}\alpha,t} + \Pi_{\beta\alpha,t}\,, \quad \beta \neq \alpha \in \{1,2\}\,,$$

as equalities of second-order random variables.

- From the foregoing results, $\mathbb{E}\{\dot{\mathcal{E}}_{\alpha,t}\} = 0$ for both oscillators and $\mathbb{E}\{\Pi_{12,t}\} = -\mathbb{E}\{\Pi_{21,t}\}$; hence:

$$\begin{aligned}\mathbb{E}\{\Pi_{\mathrm{IN},t}\} &= \mathbb{E}\{\Pi_{\mathrm{d},t}\}\,,\\ \mathbb{E}\{\Pi_{\mathrm{IN}\alpha,t}\} &= \mathbb{E}\{\Pi_{\mathrm{d}\alpha,t}\} + \mathbb{E}\{\Pi_{\alpha\beta,t}\}\,.\end{aligned}$$

- **Hypotheses** (wideband excitation):
  - i $\xi_\alpha \ll 1$;
  - ii $\Delta\omega \gg b_\alpha = \pi\xi_\alpha\omega_\alpha$ for $\alpha \in \{1,2\}$.
- From the foregoing results:

$$
\begin{aligned}
\mathbb{E}\{\Pi_{12,t}\} &= \mathbb{E}\{-(M\ddot{U}_{1,t} - \Omega\dot{U}_{1,t} + KU_{1,t})\dot{U}_{2,t}\} \\
&= \Re\mathrm{e}\left\{\int_{\mathbb{R}} \mathrm{i}\omega\overline{A_c(\omega)}S_{U_1U_2}(\omega)\,\mathrm{d}\omega\right\} \\
&= \int_{\mathbb{R}} \frac{\omega^2|A_c(\omega)|^2}{\pi|D(\omega)|^2} D_1D_2\left(\frac{\pi S_1(\omega)}{D_1} - \frac{\pi S_2(\omega)}{D_2}\right)\mathrm{d}\omega \\
&= \left(\int_{\mathbb{R}} \frac{\omega^2|A_c(\omega)|^2}{\pi|D(\omega)|^2} D_1D_2\,\mathrm{d}\omega\right)\left(\mathbb{E}\{\mathcal{E}^{\mathrm{b}}_{1,t}\} - \mathbb{E}\{\mathcal{E}^{\mathrm{b}}_{2,t}\}\right),
\end{aligned}
$$

where $\mathbb{E}\{\mathcal{E}^{\mathrm{b}}_{\alpha,t}\} = \frac{\pi S_\alpha}{D_\alpha}$: the average mechanical energy of the "blocked" α^{th} oscillator.

Basic SEA equations
Average power flow

- Besides:

$$
\begin{aligned}
\mathbb{E}\{\mathcal{E}_{\alpha,t}\} &= \frac{1}{2}M_\alpha\mathbb{E}\{\dot{U}^2_{\alpha,t}\} + \frac{1}{2}K_\alpha\mathbb{E}\{U^2_{\alpha,t}\} \\
&= \frac{1}{2}M_\alpha\int_{\mathbb{R}}\omega^2 S_{U_\alpha U_\alpha}(\omega)\,\mathrm{d}\omega + \frac{1}{2}K_\alpha\int_{\mathbb{R}} S_{U_\alpha U_\alpha}(\omega)\,\mathrm{d}\omega \\
&\equiv A_{\alpha\alpha}S_\alpha + A_{\alpha\beta}S_\beta\,, \quad \beta\neq\alpha\in\{1,2\}\,,
\end{aligned}
$$

from which one deduces S_1 and S_2 as functions of $\mathbb{E}\{\mathcal{E}_{1,t}\}$ and $\mathbb{E}\{\mathcal{E}_{2,t}\}$, and then $\mathbb{E}\{\Pi_{12,t}\}$ as a function of the latter:

$$
\begin{aligned}
\mathbb{E}\{\Pi_{12,t}\} = &\left(\int_{\mathbb{R}}\frac{\omega^2|A_c(\omega)|^2}{|D(\omega)|^2(A_{11}A_{22}-A_{12}A_{21})}\,\mathrm{d}\omega\right)\times \\
&\left[(D_2A_{22}+D_1A_{21})\mathbb{E}\{\mathcal{E}_{1,t}\}-(D_1A_{11}+D_2A_{12})\mathbb{E}\{\mathcal{E}_{2,t}\}\right]\,,
\end{aligned}
$$

or:

$$
\boxed{\mathbb{E}\{\Pi_{12,t}\} = \omega_1\eta_{12}\mathbb{E}\{\mathcal{E}_{1,t}\} - \omega_2\eta_{21}\mathbb{E}\{\mathcal{E}_{2,t}\}}\,.
$$

Basic SEA equations
SEA analysis of a two-DOFs system

- The average dissipated power:

$$\begin{aligned}\mathbb{E}\{\Pi_{\mathrm{d}\alpha,t}\} &= D_\alpha \mathbb{E}\{\dot{U}^2_{\alpha,t}\} \\ &\simeq \omega_\alpha \eta_\alpha \mathbb{E}\{\mathcal{E}_{\alpha,t}\}\,.\end{aligned}$$

- The average power balance for each oscillator thus reads:

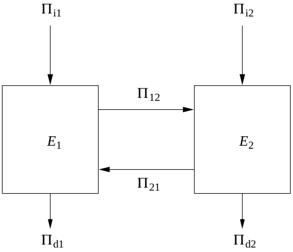
$$\boxed{\mathbb{E}\{\Pi_{\mathrm{IN}\alpha,t}\} = \omega_\alpha \eta_\alpha \mathbb{E}\{\mathcal{E}_{\alpha,t}\} + \omega_\alpha \eta_{\alpha\beta} \left(\mathbb{E}\{\mathcal{E}_{\alpha,t}\} - \mathbb{E}\{\mathcal{E}_{\beta,t}\}\right)}\,,$$

owing to the reciprocity relation:

$$\omega_\alpha \eta_{\alpha\beta} = \omega_\beta \eta_{\beta\alpha}\,, \quad \alpha \neq \beta \in \{1,2\}\,.$$

Basic SEA equations

SEA analysis of a two-DOFs system



$$\begin{bmatrix} \eta_1+\eta_{12} & -\eta_{12} \\ -\eta_{21} & \eta_2+\eta_{21} \end{bmatrix} \begin{pmatrix} \mathbb{E}\{\mathcal{E}_{1,t}\} \\ \mathbb{E}\{\mathcal{E}_{2,t}\} \end{pmatrix} = \begin{pmatrix} \frac{1}{\omega_1}\mathbb{E}\{\Pi_{\mathrm{IN}1,t}\} \\ \frac{1}{\omega_2}\mathbb{E}\{\Pi_{\mathrm{IN}2,t}\} \end{pmatrix}$$

- The data:
  - η_α: the loss factor,
  - $\eta_{\alpha\beta}$: the coupling loss factor,
  - $\mathbb{E}\{\Pi_{\mathrm{IN}\alpha,t}\}$: the power input.
- The unknowns: $\mathbb{E}\{\mathcal{E}_{\alpha,t}\}$, $\alpha \in \{1,2\}$.

- **Example #1**: if $\eta_{\alpha\beta} \ll \eta_\alpha$: weak coupling,

$$\mathbb{E}\{\mathcal{E}_{\alpha,t}\} \simeq \frac{\mathbb{E}\{\Pi_{\mathrm{IN}\alpha,t}\}}{\omega_\alpha \eta_\alpha} = \mathbb{E}\{\mathcal{E}^{\mathrm{b}}_{\alpha,t}\}\,.$$

- **Example #2**: if $\eta_{\alpha\beta} \gg \eta_\alpha$: strong coupling,

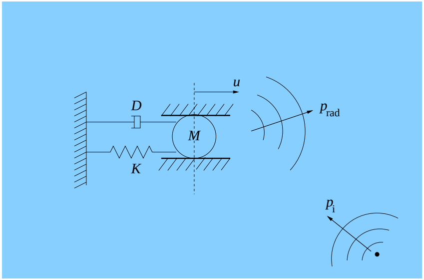
$$\mathbb{E}\{\mathcal{E}_{1,t}\} \simeq \mathbb{E}\{\mathcal{E}_{2,t}\} \simeq \frac{\underline{\Pi}_{\mathrm{IN}}}{\underline{\Delta}}\,,$$

where $\underline{\Pi}_{\mathrm{IN}} = \frac{1}{2}(\mathbb{E}\{\Pi_{\mathrm{IN}1,t}\} + \mathbb{E}\{\Pi_{\mathrm{IN}2,t}\})$ and $\underline{\Delta} = \frac{1}{2}(\omega_1\eta_1 + \omega_2\eta_2)$.

- **Remarks**:
  - η_{12} and η_{21} depend on η_1 and η_2;
  - if $\frac{M}{\sqrt{M_1 M_2}} \simeq \frac{K}{\sqrt{K_1 K_2}} \simeq \frac{\Omega}{\sqrt{D_1 D_2}} = \mathrm{O}(\varepsilon)$, then $\eta_{\alpha\beta} = \mathrm{O}(\varepsilon^2)$.

Outline

Oscillator coupled to an acoustic fluid
Notations and setting



- The oscillator ($\omega_p = \sqrt{\frac{K}{M}}$, $\eta_p = \frac{D}{\sqrt{KM}}$):

$$\begin{cases} M\ddot{u}(t) + D\dot{u}(t) + Ku(t) = Sp(t)\,, \quad t \in \mathbb{R}\,, \\ u(0) = u_0\,, \\ \dot{u}(0) = v_0\,; \end{cases}$$

- S: surface seen by the fluid, p: fluid pressure.

Oscillator coupled to an acoustic fluid

Notations and setting

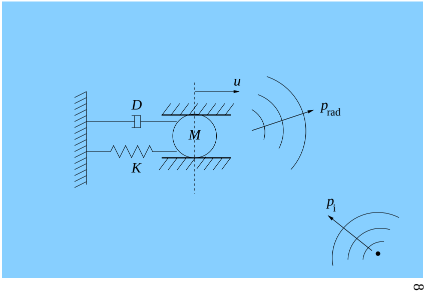
- The acoustic fluid: homogeneous, compressible, inviscid, gravity effects are neglected.
- Irrotational motion $\boldsymbol{\nabla}\times\mathbf{v}=\mathbf{0}$, s.t. the fluid velocity $\mathbf{v}$ reads $\mathbf{v}=\boldsymbol{\nabla}\psi$ and the fluid pressure reads $p=-\varrho_{\mathrm{f}}\partial_t\psi$, where the velocity potential ψ satisfies:

$$\begin{cases} \dfrac{1}{c_{\mathrm{f}}^2}\dfrac{\partial^2\psi}{\partial t^2}-\Delta\psi=g_{\mathrm{IN}}(\boldsymbol{x},t)\,, \quad (\boldsymbol{x},t)\in\Omega_{\mathrm{f}}\times\mathbb{R}\,, \\ \dfrac{\partial\psi}{\partial\mathbf{n}}(\boldsymbol{x}_S,t)\overset{\mathrm{def}}{=}\mathbf{v}(\boldsymbol{x}_S,t)\cdot\mathbf{n}=\dot{u}(t)\,; \end{cases}$$

ϱ_{f}: fluid density, c_{f}: sound speed, $\boldsymbol{x}_S$: position of the oscillator at rest.

Oscillator coupled to an acoustic fluid

Decomposition of the velocity potential

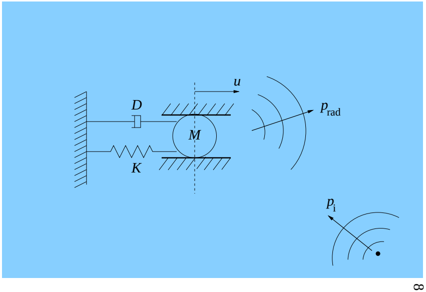


- The velocity potential reads:

$$\psi(\boldsymbol{x},t) = \psi_{\text{IN}}(\boldsymbol{x},t)+\psi_{\text{d0}}(\boldsymbol{x},t)+\psi_{\text{rad}}(\boldsymbol{x},t)\,, \quad (\boldsymbol{x},t)\in\Omega_{\text{f}}\times\mathbb{R}\,,$$

where:

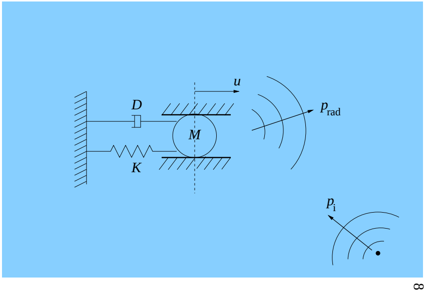
- ψ_{IN}: incident potential (data);
- ψ_{d0}: diffracted potential, the oscillator being fixed;
- ψ_{rad}: radiated potential.



- The diffracted potential ψ_{d0} satisfies:

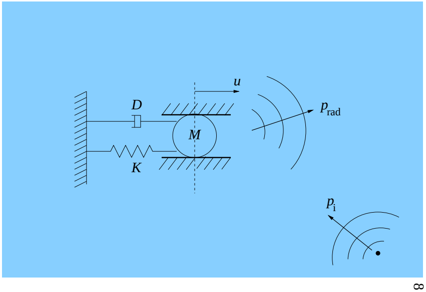
$$\begin{cases} \dfrac{1}{c_{\mathrm{f}}^2}\dfrac{\partial^2 \psi_{\mathrm{d0}}}{\partial t^2} - \Delta\psi_{\mathrm{d0}} = 0\,, \quad (\boldsymbol{x},t) \in \Omega_{\mathrm{f}} \times \mathbb{R}\,, \\ \dfrac{\partial \psi_{\mathrm{d0}}}{\partial \mathbf{n}}(\boldsymbol{x}_S,t) = -\dfrac{\partial \psi_{\mathrm{IN}}}{\partial \mathbf{n}}(\boldsymbol{x}_S,t)\,. \end{cases}$$

Oscillator coupled to an acoustic fluid
Decomposition of the velocity potential



- The radiated potential ψ_{rad} satisfies:

$$\begin{cases} \dfrac{1}{c_{\mathrm{f}}^2}\dfrac{\partial^2\psi_{\mathrm{rad}}}{\partial t^2} - \Delta\psi_{\mathrm{rad}} = 0\,, \quad (\boldsymbol{x},t)\in\Omega_{\mathrm{f}}\times\mathbb{R}\,, \\ \dfrac{\partial\psi_{\mathrm{rad}}}{\partial\mathbf{n}}(\boldsymbol{x}_S,t) = \dot{u}(t)\,. \end{cases}$$



- The incident potential $\psi_{\rm IN}$ satisfies:

$$\frac{1}{c_{\rm f}^2}\frac{\partial^2\psi_{\rm IN}}{\partial t^2} - \Delta\psi_{\rm IN} = g_{\rm IN}(\boldsymbol{x},t)\,,\quad (\boldsymbol{x},t)\in\mathbb{R}^3\times\mathbb{R}\,,$$

for some sound source $g_{\rm IN}$ in the full physical space.

Solving the fluid equations
Exterior Helmholtz problem

- The exterior Helmholtz problem:

$$
\begin{cases}
\Delta\psi + k_\mathrm{f}^2\psi = 0 & \text{in } \Omega_\mathrm{f}\,, \\
\dfrac{\partial\psi}{\partial\mathbf{n}} = v & \text{on } \partial\Omega_\mathrm{f}\,, \\
|\psi| = \mathrm{O}\left(\dfrac{1}{r}\right)\,,\quad \left|\dfrac{\partial\psi}{\partial r} + \mathrm{i}k_\mathrm{f}\psi\right| = \mathrm{O}\left(\dfrac{1}{r^2}\right) & \text{as } r = \|\boldsymbol{x}\| \to +\infty\,,
\end{cases}
$$

  where $k_\mathrm{f} = \frac{\omega}{c_\mathrm{f}}$, and $\partial\Omega_\mathrm{f} = \Gamma$ is the interface between the fluid and the structure–the oscillator.

- Sommerfeld radiation conditions "at infinity": the radiated waves are almost plane and do not propagate toward Γ.

- The exterior Helmholtz problem admits a unique solution $\forall k_\mathrm{f} \in \mathbb{R}$, hence:
  - There exists a linear operator $\mathscr{B}_\Gamma(k_\mathrm{f}) : \mathcal{C}'_\Gamma \to \mathcal{C}_\Gamma$ s.t.:

$$\psi|_\Gamma(\omega) = \mathscr{B}_\Gamma(k_\mathrm{f})v \,, \quad \text{on } \Gamma \,;$$

  - There exists a linear operator $\mathscr{R}_{\boldsymbol{x}}(k_\mathrm{f}) : \mathcal{C}'_\Gamma \to \mathbb{C}$ s.t.:

$$\psi(\boldsymbol{x},\omega) = \mathscr{R}_{\boldsymbol{x}}(k_\mathrm{f})v \,, \quad \boldsymbol{x} \in \Omega_\mathrm{f} \,,$$

  with $\mathcal{C}_\Gamma$: the set of admissible fields on Γ ($\mathcal{C}'_\Gamma$: its dual).
- $Z_\Gamma(\omega) = -\mathrm{i}\omega\varrho_\mathrm{f} S \mathscr{B}_\Gamma(k_\mathrm{f})$ is the acoustic impedance boundary operator, with $S = |\Gamma|$;
- $\mathscr{Z}_{\boldsymbol{x}}(\omega) = -\mathrm{i}\omega\varrho_\mathrm{f} S \mathscr{R}_{\boldsymbol{x}}(k_\mathrm{f})$ is the radiation impedance operator, with $S = |\Gamma|$.

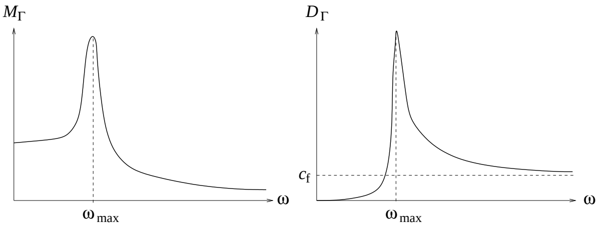
Solving the fluid equations
Boundary impedance

- The boundary impedance $\mathscr{B}_\Gamma(\frac{\omega}{c_\mathrm{f}})$ is symmetric and reads:

$$-\omega^2\mathscr{B}_\Gamma\left(\frac{\omega}{c_\mathrm{f}}\right) = -\omega^2 M_\Gamma\left(\frac{\omega}{c_\mathrm{f}}\right) + \mathrm{i}\omega D_\Gamma\left(\frac{\omega}{c_\mathrm{f}}\right) \,,$$

where:

  - The reactive part $\omega \mapsto M_\Gamma\left(\frac{\omega}{c_\mathrm{f}}\right)$ (left) is generally unsigned, though it is positive if $\mathbb{R}^3\backslash\overline{\Omega}_\mathrm{f}$ is convex;
  - The resistive part $\omega \mapsto D_\Gamma\left(\frac{\omega}{c_\mathrm{f}}\right)$ (right) is positive.



Solving the fluid equations

Consequences for the fluid pressure

- Diffracted velocity potential and fluid pressure:

$$\begin{aligned}\widehat{\psi}_{\mathrm{d}0}(\omega)|_\Gamma &= -\mathscr{B}_\Gamma(k_\mathrm{f})\frac{\partial\widehat{\psi}_{\mathrm{IN}}}{\partial\mathbf{n}} \\ \widehat{\psi}_{\mathrm{d}0}(\boldsymbol{x},\omega) &= -\mathscr{R}_{\boldsymbol{x}}(\omega)\frac{\partial\widehat{\psi}_{\mathrm{IN}}}{\partial\mathbf{n}}\end{aligned} \quad \text{and} \quad \begin{aligned}\widehat{p}_{\mathrm{d}0}(\omega)|_\Gamma &= -\mathscr{B}_\Gamma(k_\mathrm{f})\frac{\partial\widehat{p}_{\mathrm{IN}}}{\partial\mathbf{n}} \\ \widehat{p}_{\mathrm{d}0}(\boldsymbol{x},\omega) &= -\mathscr{R}_{\boldsymbol{x}}(\omega)\frac{\partial\widehat{p}_{\mathrm{IN}}}{\partial\mathbf{n}}\,,\quad \boldsymbol{x}\in\Omega_\mathrm{f}\,.\end{aligned}$$

- Radiated velocity potential and fluid pressure:

$$\begin{aligned}\widehat{\psi}_{\mathrm{rad}}(\omega)|_\Gamma &= \mathrm{i}\omega\mathscr{B}_\Gamma(k_\mathrm{f})\widehat{u}(\omega) \\ \widehat{\psi}_{\mathrm{rad}}(\boldsymbol{x},\omega) &= \mathrm{i}\omega\mathscr{R}_{\boldsymbol{x}}(\omega)\widehat{u}(\omega)\end{aligned} \quad \text{and} \quad \begin{aligned}S\widehat{p}_{\mathrm{rad}}(\omega)|_\Gamma &= \mathrm{i}\omega Z_\Gamma(k_\mathrm{f})\widehat{u}(\omega) \\ S\widehat{p}_{\mathrm{rad}}(\boldsymbol{x},\omega) &= \mathrm{i}\omega\mathcal{Z}_{\boldsymbol{x}}(\omega)\widehat{u}(\omega)\,,\quad \boldsymbol{x}\in\Omega_\mathrm{f}\,.\end{aligned}$$

- The (**linear**) scattering operator $\mathcal{T}_{\boldsymbol{x}}(\omega)$ s.t.:

$$\widehat{p}_{\mathrm{IN}}(\boldsymbol{x},\omega)+\widehat{p}_{\mathrm{d}0}(\boldsymbol{x},\omega)=\mathcal{T}_{\boldsymbol{x}}(\omega)\widehat{p}_{\mathrm{IN}}(\omega)|_\Gamma\,,\quad \boldsymbol{x}\in\Omega_\mathrm{f}\,.$$

Oscillator coupled to an acoustic fluid

Frequency response function

- Equation of motion of the oscillator in the frequency domain:

$$\begin{aligned}(-\omega^2 M + \mathrm{i}\omega D + K)\widehat{u}(\omega) &= S(\widehat{p}_{\mathrm{IN}}(\omega) + \widehat{p}_{\mathrm{d0}}(\omega) + \widehat{p}_{\mathrm{rad}}(\omega)) + \widehat{f}_{\mathrm{e}}(\omega) \\ &= S\mathcal{T}_{\boldsymbol{x}_S}(\omega)\widehat{p}_{\mathrm{IN}}(\omega) + \omega^2 \varrho_{\mathrm{f}} S \mathscr{B}_{\boldsymbol{x}_S}(k_{\mathrm{f}})\widehat{u}(\omega) + \widehat{f}_{\mathrm{e}}(\omega)\,,\end{aligned}$$

  where $\widehat{f}_{\mathrm{e}}(\omega) = Du_0 + M(\mathrm{i}\omega u_0 + v_0)$ is an equivalent load accounting for the initial conditions.

- Therefore, the frequency response function of the oscillator coupled to the fluid reads:

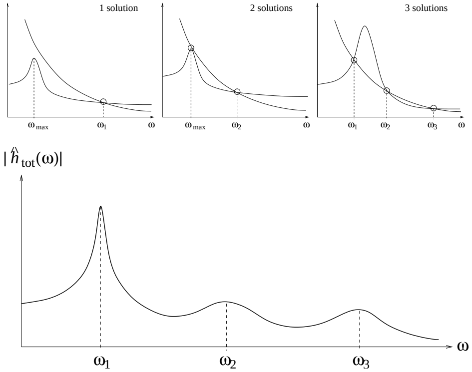
$$\widehat{\mathbb{h}}_{\mathrm{tot}}(\omega) = \left[-\omega^2\left(M + \varrho_{\mathrm{f}} S M_{\Gamma}\left(\frac{\omega}{c_{\mathrm{f}}}\right)\right) + \mathrm{i}\omega\left(D + \varrho_{\mathrm{f}} S D_{\Gamma}\left(\frac{\omega}{c_{\mathrm{f}}}\right)\right) + K\right]^{-1}.$$

Oscillator coupled to an acoustic fluid

Wet eigenfrequencies

- The "wet" eigenfrequencies of the oscillator coupled to the fluid are the solutions of:

$$\left[1+\frac{\varrho_{\mathrm{f}}S}{M}M_{\Gamma}\left(\frac{\omega}{c_{\mathrm{f}}}\right)\right]\omega^2=\omega_p^2\,.$$



Stationary incident pressure

Forced response of the SDOF oscillator

- **Data**: $(P_t\,, t \in \mathbb{R})$ is a $\mathbb{R}$-valued second order, centered stochastic process defined on $(\Omega, \mathcal{F}, P)$, indexed on $\mathbb{R}$, and mean-square (m.s.) stationary.
- **Hypothesis**: $\exists\, \omega \mapsto S_P(\omega) : \mathbb{R} \to \mathbb{R}_+$ s.t.

$$S_P(\omega) = S_0 \mathbb{1}_{I_0 \cup \underline{I}_0}(\omega)\,, \quad S_0 > 0\,,$$

where:

$$I_0 \cup \underline{I}_0 = \left[\omega_0 - \frac{\Delta\omega}{2}, \omega_0 + \frac{\Delta\omega}{2}\right] \bigcup \left[-\omega_0 - \frac{\Delta\omega}{2}, -\omega_0 + \frac{\Delta\omega}{2}\right].$$

Proposition (filtering)

- *$(U_t^f\,, t \in \mathbb{R})$ is a $\mathbb{R}$-valued second order stochastic process defined on $(\Omega, \mathcal{F}, P)$, indexed on $\mathbb{R}$, centered and m.s. stationary s.t.* $S_U(\omega) = |\widehat{\mathbb{h}}_{\mathrm{tot}}(\omega) S\mathcal{T}_{\boldsymbol{x}_S}(\omega)|^2 S_P(\omega)$.

- **Data**: $(P_t\,, t \in \mathbb{R})$ is a $\mathbb{R}$-valued second order, centered stochastic process defined on $(\Omega, \mathcal{F}, P)$, indexed on $\mathbb{R}$, and mean-square (m.s.) stationary.
- **Hypothesis**: $\exists\, \omega \mapsto S_P(\omega) : \mathbb{R} \to \mathbb{R}_+$ s.t.

$$S_P(\omega) = S_0 \mathbb{1}_{I_0 \cup \underline{I}_0}(\omega)\,, \quad S_0 > 0\,,$$

where:

$$I_0 \cup \underline{I}_0 = \left[\omega_0 - \frac{\Delta\omega}{2}, \omega_0 + \frac{\Delta\omega}{2}\right] \bigcup \left[-\omega_0 - \frac{\Delta\omega}{2}, -\omega_0 + \frac{\Delta\omega}{2}\right].$$

Proposition (m.s. derivation)

- *The same holds for its m.s. derivatives* $(\dot{U}^f_t\,, t \in \mathbb{R})$ *and* $(\ddot{U}^f_t\,, t \in \mathbb{R})$*, with* $S_{\dot{U}}(\omega) = \omega^2 S_U(\omega)$*,* $S_{\ddot{U}}(\omega) = \omega^4 S_U(\omega)$*, respectively.*

- The average mechanical energy of the oscillator:

$$
\begin{aligned}
\mathbb{E}\{\mathcal{E}_t\} &= \frac{1}{2}M\int_{\mathbb{R}}\omega^2\left(1+\frac{\varrho_{\mathrm{f}}S}{M}M_\Gamma\left(\frac{\omega}{c_{\mathrm{f}}}\right)\right)S_U(\omega)\,\mathrm{d}\omega \\
&+\frac{1}{2}K\int_{\mathbb{R}}S_U(\omega)\,\mathrm{d}\omega \\
&\stackrel{\mathrm{def}}{=}(M+M_{\mathrm{rad}}(\omega_0))\int_{\mathbb{R}}\omega^2S_U(\omega)\,\mathrm{d}\omega\,,
\end{aligned}
$$

  where $M_{\mathrm{rad}}(\omega_0)$ is an equivalent added mass.

- The average radiated power:

$$
\begin{aligned}
\mathbb{E}\{\Pi_{\mathrm{rad},t}\} &= \int_{\mathbb{R}}\varrho_{\mathrm{f}}SD_\Gamma\left(\frac{\omega}{c_{\mathrm{f}}}\right)\omega^2S_U(\omega)\,\mathrm{d}\omega \\
&\stackrel{\mathrm{def}}{=}\omega_0\eta_{\mathrm{rad}}(\omega_0)\mathbb{E}\{\mathcal{E}_t\}\,,
\end{aligned}
$$

  where $\eta_{\mathrm{rad}}(\omega_0)$ is an equivalent added loss factor.

- The average dissipated power thus reads:

$$\begin{aligned}\mathbb{E}\{\Pi_{\mathrm{d},t}\} &= D\mathbb{E}\{\dot{U}_t^2\} \\ &= M\omega_p\eta_p\int_{\mathbb{R}}\omega^2 S_U(\omega)\,\mathrm{d}\omega \\ &= \frac{M}{M+M_{\mathrm{rad}}(\omega_0)}\omega_p\eta_p\mathbb{E}\{\mathcal{E}_t\}\,.\end{aligned}$$

- The average input power:

$$\mathbb{E}\{\Pi_{\mathrm{IN},t}\} = \Re\mathrm{e}\int_{\mathbb{R}}\mathrm{i}\omega\widehat{\mathbb{h}}_{\mathrm{tot}}(\omega)|S\mathcal{T}_{\boldsymbol{x}_S}(\omega)|^2 S_P(\omega)\,\mathrm{d}\omega\,.$$

Stationary incident pressure
Power balance for the stationary forced response

- The instantaneous power balance reads:
$$\dot{\mathcal{E}}_t = \Pi_{\mathrm{IN},t} - \Pi_{\mathrm{d},t} - \Pi_{\mathrm{rad},t} \,,$$
as an equality of second-order random variables.
- Considering the mathematical expectation with $\mathbb{E}\{\mathcal{E}_t\} = \text{Constant}$ and $\mathbb{E}\{\dot{\mathcal{E}}_t\} = 0$:
$$\begin{aligned}\mathbb{E}\{\Pi_{\mathrm{IN},t}\} &= \mathbb{E}\{\Pi_{\mathrm{d},t}\} + \mathbb{E}\{\Pi_{\mathrm{rad},t}\} \\ &= \omega_0\eta_{\mathrm{tot}}(\omega_0)\mathbb{E}\{\mathcal{E}_t\} \,,\end{aligned}$$
where:
$$\begin{aligned}\eta_{\mathrm{tot}}(\omega_0) &= \eta_{\mathrm{rad}}(\omega_0) + \frac{\omega_p}{\omega_0}\left(\frac{M}{M + M_{\mathrm{rad}}(\omega_0)}\right)\eta_p \\ &\simeq \eta_{\mathrm{rad}}(\omega_0) + \eta_p\sqrt{\frac{M}{M + \varrho_{\mathrm{f}} S M_\Gamma(\frac{\omega_0}{c_{\mathrm{f}}})}}\end{aligned}$$
if ω_0 is close to a "wet" eigenfrequency.

Summary

- The power flow between two coupled oscillators is proportional to the difference of their mechanical energies, if:
  1. The oscillators are weakly dissipative;
  2. Their coupling is conservative;
  3. They are excited by uncorrelated wideband noises, the bandwidths of which are large with respect to the equivalent bandwidths of the oscillators.
- "Equivalence" with the time average energetic quantities for the forced response of the randomized two-DOFs system to harmonic excitations.
- The average radiated power of the oscillator coupled to an acoustic fluid is proportional to its mechanical energy. This effect can be simply characterized by an equivalent added loss factor in the energy balance.
- **Outlook**: continuous systems.

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