Random vibrations of coupled continuous systems

MG3416–Advanced Structural Acoustics - Lecture #5

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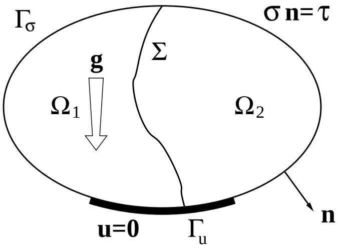
Outline

Outline



- The system occupying the domain Ω is devided into two sub-systems occupying the domains Ω_1 and Ω_2.
- Their interface is $\Sigma = \partial\Omega_1 \cap \partial\Omega_2$ with $|\Sigma| \neq 0$.
- The displacement field in each sub-system $r = 1$ or 2 is $\boldsymbol{u}_r = \boldsymbol{u}|_{\Omega_r}$, where $\boldsymbol{u}$ is the displacement field in Ω.

- They satisfy the following system derived from the VPP: Find $\boldsymbol{u}_r \in \mathcal{C}_r$, $r \in \{1,2\}$, s.t.

$$\begin{bmatrix} \boldsymbol{M}_1 & \boldsymbol{M}_{12} \\ \boldsymbol{M}_{21} & \boldsymbol{M}_2 \end{bmatrix} \begin{pmatrix} \ddot{\boldsymbol{u}}_1 \\ \ddot{\boldsymbol{u}}_2 \end{pmatrix} + \begin{bmatrix} \boldsymbol{D}_1 & \boldsymbol{D}_{12} \\ \boldsymbol{D}_{21} & \boldsymbol{D}_2 \end{bmatrix} \begin{pmatrix} \dot{\boldsymbol{u}}_1 \\ \dot{\boldsymbol{u}}_2 \end{pmatrix} + \begin{bmatrix} \boldsymbol{K}_1 & \boldsymbol{K}_{12} \\ \boldsymbol{K}_{21} & \boldsymbol{K}_2 \end{bmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \end{pmatrix} ,$$

where the sets of admissible displacement fields $\mathcal{C}_r$ are:

$$\mathcal{C}_r \subseteq \left\{ \boldsymbol{v};\, \boldsymbol{v} \in L^2\left(\Omega_r\right),\, \boldsymbol{\nabla}\boldsymbol{v} \in L^2\left(\Omega_r\right),\, \text{and } \boldsymbol{v}|_{\Gamma_{\mathrm{u}} \cap \partial\Omega_r} = \boldsymbol{0} \right\} .$$

- **Hypotheses**: the coupling operators are s.t.

- The mass coupling:

$$m_{\mathrm{c}}\begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{v}_1 \\ \boldsymbol{u}_2 & \boldsymbol{v}_2 \end{pmatrix} = \langle \boldsymbol{M}_{12}\boldsymbol{u}_2, \boldsymbol{v}_1 \rangle_{\mathcal{C}_1', \mathcal{C}_1} + \langle \boldsymbol{M}_{21}\boldsymbol{u}_1, \boldsymbol{v}_2 \rangle_{\mathcal{C}_2', \mathcal{C}_2}$$

is symmetric $\boldsymbol{M}_{12} \equiv \boldsymbol{M}_{21}^{\mathsf{T}}$;

General case

Notations and setting

- They satisfy the following system derived from the VPP: Find $\boldsymbol{u}_r \in \mathcal{C}_r$, $r \in \{1, 2\}$, s.t.

$$\begin{bmatrix} \boldsymbol{M}_1 & \boldsymbol{M}_{12} \\ \boldsymbol{M}_{21} & \boldsymbol{M}_2 \end{bmatrix} \begin{pmatrix} \ddot{\boldsymbol{u}}_1 \\ \ddot{\boldsymbol{u}}_2 \end{pmatrix} + \begin{bmatrix} \boldsymbol{D}_1 & \boldsymbol{D}_{12} \\ \boldsymbol{D}_{21} & \boldsymbol{D}_2 \end{bmatrix} \begin{pmatrix} \dot{\boldsymbol{u}}_1 \\ \dot{\boldsymbol{u}}_2 \end{pmatrix} + \begin{bmatrix} \boldsymbol{K}_1 & \boldsymbol{K}_{12} \\ \boldsymbol{K}_{21} & \boldsymbol{K}_2 \end{bmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \end{pmatrix} ,$$

where the sets of admissible displacement fields $\mathcal{C}_r$ are:

$$\mathcal{C}_r \subseteq \left\{ \boldsymbol{v};\, \boldsymbol{v} \in L^2(\Omega_r)\,,\ \boldsymbol{\nabla v} \in L^2(\Omega_r)\,,\ \text{and}\ \boldsymbol{v}|_{\Gamma_{\mathrm{u}} \cap \partial\Omega_r} = \boldsymbol{0} \right\} .$$

- **Hypotheses**: the coupling operators are s.t.
- The stiffness coupling:

$$k_{\mathrm{c}} \begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{v}_1 \\ \boldsymbol{u}_2 & \boldsymbol{v}_2 \end{pmatrix} = \langle \boldsymbol{K}_{12}\boldsymbol{u}_2, \boldsymbol{v}_1 \rangle_{\mathcal{C}_1', \mathcal{C}_1} + \langle \boldsymbol{K}_{21}\boldsymbol{u}_1, \boldsymbol{v}_2 \rangle_{\mathcal{C}_2', \mathcal{C}_2}$$

is symmetric $\boldsymbol{K}_{12} \equiv \boldsymbol{K}_{21}^{\mathsf{T}}$;

General case

Notations and setting

- They satisfy the following system derived from the VPP: Find $\boldsymbol{u}_r \in \mathcal{C}_r$, $r \in \{1,2\}$, s.t.

$$\begin{bmatrix} \boldsymbol{M}_1 & \boldsymbol{M}_{12} \\ \boldsymbol{M}_{21} & \boldsymbol{M}_2 \end{bmatrix} \begin{pmatrix} \ddot{\boldsymbol{u}}_1 \\ \ddot{\boldsymbol{u}}_2 \end{pmatrix} + \begin{bmatrix} \boldsymbol{D}_1 & \boldsymbol{D}_{12} \\ \boldsymbol{D}_{21} & \boldsymbol{D}_2 \end{bmatrix} \begin{pmatrix} \dot{\boldsymbol{u}}_1 \\ \dot{\boldsymbol{u}}_2 \end{pmatrix} + \begin{bmatrix} \boldsymbol{K}_1 & \boldsymbol{K}_{12} \\ \boldsymbol{K}_{21} & \boldsymbol{K}_2 \end{bmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \end{pmatrix} ,$$

  where the sets of admissible displacement fields $\mathcal{C}_r$ are:

$$\mathcal{C}_r \subseteq \left\{ \boldsymbol{v};\, \boldsymbol{v} \in L^2(\Omega_r)\,,\ \boldsymbol{\nabla v} \in L^2(\Omega_r)\,,\ \text{and}\ \boldsymbol{v}|_{\Gamma_{\mathrm{u}}\cap\partial\Omega_r} = \boldsymbol{0} \right\} .$$

- **Hypotheses**: the coupling operators are s.t.
- The gyroscopic coupling:

$$d_{\mathrm{c}} \begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{v}_1 \\ \boldsymbol{u}_2 & \boldsymbol{v}_2 \end{pmatrix} = \langle \boldsymbol{D}_{12}\boldsymbol{u}_2, \boldsymbol{v}_1 \rangle_{\mathcal{C}_1',\mathcal{C}_1} + \langle \boldsymbol{D}_{21}\boldsymbol{u}_1, \boldsymbol{v}_2 \rangle_{\mathcal{C}_2',\mathcal{C}_2}$$

  is skew-symmetric $\boldsymbol{D}_{12} \equiv -\boldsymbol{D}_{21}^{\mathsf{T}}$.

General case

Eigenmodes and modal expansion

- **Spectral problem**: Find $\lambda \in \mathbb{R}$ and $\boldsymbol{\phi} \in \mathcal{C}_r^{\mathrm{b}}$ s.t.

$$\boldsymbol{K}_r \boldsymbol{\phi} = \lambda \boldsymbol{M}_r \boldsymbol{\phi} \,,$$

  where $\mathcal{C}_r^{\mathrm{b}} \subset L^2(\Omega_r)$ is the set of admissible displacement fields in Ω_r when some boundary conditions are prescribed on the interface Σ.
- **Examples**:
  - Craig-Bampton method (1968): $\boldsymbol{\phi}|_\Sigma = \boldsymbol{0}$;
  - McNeal method (1971): $\boldsymbol{\sigma}_r(\boldsymbol{\phi})\boldsymbol{n}_r = \boldsymbol{0}$ on Σ;
  - Gladwell method (1964): $\boldsymbol{\sigma}_r(\boldsymbol{\phi})\boldsymbol{n}_r = \lambda_r \mathcal{M} \boldsymbol{\phi}|_\Sigma$ on Σ, for some $\mathcal{M} > 0$.
- **Remark**: *this issue is marginally addressed in the SEA literature, because the choice of $\mathcal{C}_r^{\mathrm{b}}$ has only a mild influence on the method.*

General case

Eigenmodes and modal expansion

- **Hypothesis**: it is assumed that this problem admits a countable set of solutions $(\lambda_{r1},\boldsymbol{\phi}_{r1})$, $(\lambda_{r2},\boldsymbol{\phi}_{r2})$ ... s.t. $0<\lambda_{r1}\leqslant\lambda_{r2}\leqslant\dots$ and $\{\boldsymbol{\phi}_{r\alpha}\}_{\alpha\in\mathbb{N}^*}$ is an Hilbertian basis of the space $H_r=L^2_\mu(\Omega_r)$ of square integrable functions with respect to the unit mass measure $\mu_r(\mathrm{d}\boldsymbol{x})=\mathbb{1}_{\Omega_r}\frac{\varrho(\mathrm{d}\boldsymbol{x})}{M_r}$, with $M_r=\int_{\Omega_r}\varrho\mathrm{d}\boldsymbol{x}$:

$$\langle\boldsymbol{M}_r\boldsymbol{\phi}_{r\alpha},\boldsymbol{\phi}_{r\alpha'}\rangle_{\mathcal{C}'_r,\mathcal{C}_r}=M_r\delta_{\alpha\alpha'}\,,$$
$$\langle\boldsymbol{K}_r\boldsymbol{\phi}_{r\alpha},\boldsymbol{\phi}_{r\alpha'}\rangle_{\mathcal{C}'_r,\mathcal{C}_r}=M_r\omega^2_{r\alpha}\delta_{\alpha\alpha'}\,,$$

  where $\omega^2_{r\alpha}=\lambda_{r\alpha}$.

- **Notations**: $r,s\in\{1,2\}$, $\alpha,\alpha'\in\mathbb{N}^*$ *are generic modes for the* r^{th} *sub-system, and* $\beta,\beta'\in\mathbb{N}^*$ *are generic modes for the* s^{th} *sub-system.*

- Consequently, the solution $\boldsymbol{u}_r \in \mathcal{C}_r^{\mathrm{b}}$ can be expanded on the eigenbasis $\{\boldsymbol{\phi}_{r\alpha}\}_{\alpha\in\mathbb{N}*}$ as:
$$\boldsymbol{u}_r(\boldsymbol{x},t) = \sum_{\alpha=1}^{\infty} q_{r\alpha}(t)\boldsymbol{\phi}_{r\alpha}(\boldsymbol{x})\,,$$
- Introducing $\mu_r(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\mu_r(\mathrm{d}\boldsymbol{x})$ (the scalar product in H_r), the generalized coordinates $\{q_{r\alpha}\}_{\alpha\in\mathbb{N}*}$ are:
$$q_{r\alpha} = \mu_r(\boldsymbol{u},\boldsymbol{\phi}_{r\alpha})\,.$$
- The μ_r-norm $\|\boldsymbol{u}\|_{\mu_r} = \sqrt{\mu_r(\boldsymbol{u},\boldsymbol{u})}$ is obtained as:
$$\|\boldsymbol{u}(\cdot,t)\|_{\mu_r} = \left(\sum_{\alpha=1}^{+\infty} (q_{r\alpha}(t))^2\right)^{\frac{1}{2}} < +\infty\,.$$

General case

Eigenmodes and modal expansion

- **Hypothesis** (Basile): the eigenmodes $\{\boldsymbol{\phi}_{r\alpha}\}_{\alpha\in\mathbb{N}*}$ diagonalize the damping operator as well:

$$\langle \boldsymbol{D}_r\boldsymbol{\phi}_{r\alpha}, \boldsymbol{\phi}_{r\alpha'}\rangle_{\mathcal{C}_r',\mathcal{C}_r} = M_r\eta_{r\alpha}\omega_{r\alpha}\delta_{\alpha\alpha'}\,,$$

where for $r \in \{1,2\}$:

  - $\omega_{r\alpha}$: the "blocked" (angular) frequency of the α^{th} mode of the r^{th} sub-system,
  - $\xi_{r\alpha}$: the modal critical damping rate of the α^{th} mode of the r^{th} sub-system,
  - $\eta_{r\alpha} = 2\xi_{r\alpha}$: the modal loss factor of the α^{th} mode of the r^{th} sub-system.

General case

Eigenmodes and modal expansion

- Owing to the Basile hypothesis, the generalized coordinates $\{q_{r\alpha}\}_{\alpha\in\mathbb{N}*}$ satisfy:

$$M_r(\ddot{q}_{r\alpha}(t) + \eta_{r\alpha}\omega_{r\alpha}\dot{q}_{r\alpha}(t) + \omega_{r\alpha}^2 q_{r\alpha}(t)) = f_{r\alpha}(t) - \sum_{\beta=1}^{+\infty} (\mu_{\alpha\beta}\ddot{q}_{s\beta}(t) + \gamma_{\alpha\beta}\dot{q}_{s\beta}(t) + \kappa_{\alpha\beta}q_{s\beta}(t)) \,, \quad t \in \mathbb{R} \,,$$

with $q_{r\alpha}(0) = \mu_r(\boldsymbol{u}_0, \boldsymbol{\phi}_{r\alpha})$, $\dot{q}_{r\alpha}(0) = \mu_r(\boldsymbol{v}_0, \boldsymbol{\phi}_{r\alpha})$, and:

$$\begin{aligned}
f_{r\alpha} &= \langle \boldsymbol{f}_r, \boldsymbol{\phi}_{r\alpha}\rangle_{\mathcal{C}_r',\mathcal{C}_r} \,, \\
\mu_{\alpha\beta} &= \left\langle \boldsymbol{M}_{rs}\boldsymbol{\phi}_{s\beta}, \boldsymbol{\phi}_{r\alpha}\right\rangle_{\mathcal{C}_r',\mathcal{C}_r} = \left\langle \boldsymbol{M}_{sr}\boldsymbol{\phi}_{r\alpha}, \boldsymbol{\phi}_{s\beta}\right\rangle_{\mathcal{C}_s',\mathcal{C}_s} \,, \\
\gamma_{\alpha\beta} &= \left\langle \boldsymbol{D}_{rs}\boldsymbol{\phi}_{s\beta}, \boldsymbol{\phi}_{r\alpha}\right\rangle_{\mathcal{C}_r',\mathcal{C}_r} = -\left\langle \boldsymbol{D}_{sr}\boldsymbol{\phi}_{r\alpha}, \boldsymbol{\phi}_{s\beta}\right\rangle_{\mathcal{C}_s',\mathcal{C}_s} \,, \\
\kappa_{\alpha\beta} &= \left\langle \boldsymbol{K}_{rs}\boldsymbol{\phi}_{s\beta}, \boldsymbol{\phi}_{r\alpha}\right\rangle_{\mathcal{C}_r',\mathcal{C}_r} = \left\langle \boldsymbol{K}_{sr}\boldsymbol{\phi}_{r\alpha}, \boldsymbol{\phi}_{s\beta}\right\rangle_{\mathcal{C}_s',\mathcal{C}_s} \,.
\end{aligned}$$

Energetic quantities
Definitions

Definition

- *The mechanical energy of the coupled system $r \in \{1,2\}$:*

$$\mathcal{E}(t) = \mathcal{E}_1(t) + \mathcal{E}_2(t) + \sum_{\alpha=1}^{+\infty}\sum_{\beta=1}^{+\infty}\left(\mu_{\alpha\beta}\dot{q}_{1\alpha}(t)\dot{q}_{2\beta}(t) + \kappa_{\alpha\beta}q_{1\alpha}(t)q_{2\beta}(t)\right),$$

where $\mathcal{E}_r(t) = \frac{1}{2}M_r\sum_{\alpha=1}^{+\infty}\left[(\dot{q}_{r\alpha}(t))^2 + \omega_{r\alpha}^2(q_{r\alpha}(t))^2\right]$.

- *The dissipated power:* $\Pi_{\mathrm{d}}(t) = \Pi_{\mathrm{d}1}(t) + \Pi_{\mathrm{d}2}(t)$, *where* $\Pi_{\mathrm{d}r}(t) = M_r\sum_{\alpha=1}^{+\infty}\eta_{r\alpha}\omega_{r\alpha}(\dot{q}_{r\alpha}(t))^2$.
- *The input power:* $\Pi_{\mathrm{IN}}(t) = \Pi_{\mathrm{IN}1}(t) + \Pi_{\mathrm{IN}2}(t)$, *where* $\Pi_{\mathrm{IN}r}(t) = \sum_{\alpha=1}^{+\infty} f_{r\alpha}(t)\dot{q}_{r\alpha}(t)$.

Energetic quantities

Power flow between the sub-systems

- The powers exchanged by the sub-systems:

$$\Pi_{12}(t) = (\text{force } 1 \to 2) \times (\text{celerity } 2) = -\sum_{\alpha=1}^{+\infty}\sum_{\beta=1}^{+\infty} \dot{q}_{2\beta}(t)\left(\mu_{\alpha\beta}\ddot{q}_{1\alpha}(t) - \gamma_{\alpha\beta}\dot{q}_{1\alpha}(t) + \kappa_{\alpha\beta}q_{1\alpha}(t)\right) ;$$

$$\Pi_{21}(t) = (\text{force } 2 \to 1) \times (\text{celerity } 1) = -\sum_{\alpha=1}^{+\infty}\sum_{\beta=1}^{+\infty} \dot{q}_{1\alpha}(t)\left(\mu_{\alpha\beta}\ddot{q}_{2\beta}(t) + \gamma_{\alpha\beta}\dot{q}_{2\beta}(t) + \kappa_{\alpha\beta}q_{2\beta}(t)\right) .$$

Energetic quantities

Power balance

- The instantaneous power of the overall system:

$$\Pi(t) = \dot{\mathcal{E}}(t) = \dot{\mathcal{E}}_1(t) + \dot{\mathcal{E}}_2(t) - \Pi_{12}(t) - \Pi_{21}(t)\,,$$

- The instantaneous power balance reads:

$$\dot{\mathcal{E}}_1(t) + \dot{\mathcal{E}}_2(t) = \Pi_{\mathrm{IN}}(t) - \Pi_{\mathrm{d}}(t) + \Pi_{12}(t) + \Pi_{21}(t)\,,$$

or for each sub-system:

$$\dot{\mathcal{E}}_r(t) = \Pi_{\mathrm{IN}r}(t) - \Pi_{\mathrm{d}r}(t) + \Pi_{sr}(t)\,,\quad s \neq r \in \{1,2\}\,.$$

- **Data**: $(\boldsymbol{F}_{r,t},\ t \in \mathbb{R})$, $r \in \{1,2\}$, are second order, centered stochastic processes defined on $(\mathit{\Omega}, \mathcal{F}, P)$, indexed on $\mathbb{R}$, with values in $[L^2(\Omega_r)]^3$, and m.s.s.
- **Hypotheses**:
  - $\exists \omega \mapsto \boldsymbol{S}_{\boldsymbol{F}_r}(\omega; \boldsymbol{x}, \boldsymbol{y}) : \mathbb{R} \to \mathbb{C}^{3\times 3}$, $\boldsymbol{x}, \boldsymbol{y} \in \Omega_r$, Hermitian, positive, integrable on $\mathbb{R}_\omega$, and s.t.:

$$
\begin{aligned}
\boldsymbol{S}_{\boldsymbol{F}_r}(\omega; \boldsymbol{x}, \boldsymbol{y}) &= \boldsymbol{S}_{\boldsymbol{F}_r}(\omega; \boldsymbol{y}, \boldsymbol{x})^* \\
\boldsymbol{S}_{\boldsymbol{F}_r}(-\omega; \boldsymbol{x}, \boldsymbol{y}) &= \overline{\boldsymbol{S}_{\boldsymbol{F}_r}(\omega; \boldsymbol{x}, \boldsymbol{y})} \\
\boldsymbol{S}_{\boldsymbol{F}_r}(\omega; \boldsymbol{x}, \boldsymbol{y}) &= \boldsymbol{S}_r(\boldsymbol{x}, \boldsymbol{y}) \otimes \mathbb{1}_{I_0 \cup \underline{I}_0}(\omega)\,,
\end{aligned}
$$

  where:

$$
I_0 \cup \underline{I}_0 = \left[\omega_0 - \frac{\Delta\omega}{2}, \omega_0 + \frac{\Delta\omega}{2}\right] \bigcup \left[-\omega_0 - \frac{\Delta\omega}{2}, -\omega_0 + \frac{\Delta\omega}{2}\right].
$$

  - $(\boldsymbol{F}_{1,t},\ t \in \mathbb{R})$ and $(\boldsymbol{F}_{2,t},\ t \in \mathbb{R})$ are uncorrelated:

$$
\mathbb{E}\{\boldsymbol{F}_{1,t} \otimes \boldsymbol{F}_{2,t'}\} = \boldsymbol{0}\,, \quad \forall (t, t') \in \mathbb{R} \times \mathbb{R}\,.
$$

- The sub-system forced responses $t \mapsto \boldsymbol{u}_r^f(\cdot, t)$ for $r \in \{1, 2\}$ are modelized by stochastic processes $(\boldsymbol{U}_{r,t},\, t \in \mathbb{R})$ the properties of which are derived from filtering and mean-square derivation (see Lecture #1 part A).

Proposition

- *$(\boldsymbol{U}_{r,t},\, t \in \mathbb{R})$, $r \in \{1, 2\}$, are second order, centered stochastic processes defined on $(\Omega, \mathcal{F}, P)$, indexed on $\mathbb{R}$, with values in $\mathcal{C}_r^{\mathrm{b}}$, and mean-square stationary.*
- *The same holds for their mean-square derivatives $(\dot{\boldsymbol{U}}_{r,t},\, t \in \mathbb{R})$ and $(\ddot{\boldsymbol{U}}_{r,t},\, t \in \mathbb{R})$, with values in H_r and $\mathcal{C}_r^{\mathrm{b}\prime}$ respectively.*

Stationary excitations

Energetics of the stationary forced responses

- The instantaneous power balance reads:

$$\dot{\mathcal{E}}_{1,t} + \dot{\mathcal{E}}_{2,t} = \Pi_{\mathrm{IN},t} - \Pi_{\mathrm{d},t} + \Pi_{12,t} + \Pi_{21,t}\,,$$

or for each sub-system:

$$\dot{\mathcal{E}}_{r,t} = \Pi_{\mathrm{IN}r,t} - \Pi_{\mathrm{d}r,t} + \Pi_{sr,t}\,,\quad s \neq r \in \{1,2\}\,,$$

as equalities of second-order random variables.

- From the foregoing results, $\mathbb{E}\{\dot{\mathcal{E}}_{r,t}\} = 0$ for both sub-systems and $\mathbb{E}\{\Pi_{12,t}\} = -\mathbb{E}\{\Pi_{21,t}\}$; hence:

$$\mathbb{E}\{\Pi_{\mathrm{IN},t}\} = \mathbb{E}\{\Pi_{\mathrm{d},t}\}\,,$$
$$\mathbb{E}\{\Pi_{\mathrm{IN}r,t}\} = \mathbb{E}\{\Pi_{\mathrm{d}r,t}\} + \mathbb{E}\{\Pi_{rs,t}\}\,.$$

Basic SEA equations

Average power flow

- It can be shown that:

$$
\begin{aligned}
\mathbb{E}\{\Pi_{rs,t}\} = & \sum_{\alpha=1}^{+\infty}\sum_{\beta=1}^{+\infty}\int_{I_0\cup\underline{I}_0} a_{r,\alpha}^{s,\beta}(\omega)\left[E_{\alpha\alpha}^r(\omega)-E_{\beta\beta}^s(\omega)\right]\mathrm{d}\omega \\
& + \sum_{\alpha=1}^{+\infty}\sum_{\beta=1}^{+\infty}\left(\sum_{\substack{\alpha'=1\\ \alpha'\neq\alpha}}^{+\infty}\int_{I_0\cup\underline{I}_0} C_{r,\alpha\alpha'}^{s,\beta}(\omega)E_{\alpha\alpha'}^r(\omega)\,\mathrm{d}\omega\right. \\
& \left. - \sum_{\substack{\beta'=1\\ \beta'\neq\beta}}^{+\infty}\int_{I_0\cup\underline{I}_0} C_{s,\beta\beta'}^{r,\alpha}(\omega)E_{\beta\beta'}^s(\omega)\,\mathrm{d}\omega\right)
\end{aligned}
$$

where:

$$
\begin{aligned}
E_{\alpha\alpha'}^r(\omega) &= \Re\mathrm{e}\{\omega^2 M_r S_{\alpha\alpha'}^r \widehat{\mathbb{h}}_{r\alpha}(\omega)\widehat{\mathbb{h}}_{r\alpha'}^*(\omega)\}\,, \\
S_{\alpha\alpha'}^r &= \frac{1}{M_r}\int_{\Omega_r}\int_{\Omega_r}\left(\boldsymbol{S}_r(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\phi}_{r,\alpha'}(\boldsymbol{y}),\boldsymbol{\phi}_{r,\alpha}(\boldsymbol{x})\right)\,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\,, \\
\widehat{\mathbb{h}}_{r\alpha}(\omega) &= \left[M_r(\omega_{r\alpha}^2-\omega^2+\mathrm{i}\eta_{r\alpha}\omega_{r\alpha}\omega)\right]^{-1}\,.
\end{aligned}
$$

Basic SEA equations

Average power flow

- The formula can be simplified invoking two assumptions:
  1. Either "rain-on-the-roof" excitations, $S^r_{\alpha\alpha'} = S_{r\alpha}\delta_{\alpha\alpha'}$ for some $S_{r\alpha} > 0$;
  2. Or "weak coupling", s.t. $\mathcal{E}(t) \simeq \mathcal{E}_1(t) + \mathcal{E}_2(t)$ neglecting the contribution of the coupling operators to the mechanical energy of the overall system.
- Then:

$$\mathbb{E}\{\Pi_{rs,t}\} = \sum_{\alpha=1}^{+\infty}\sum_{\beta=1}^{+\infty}\int_{I_0\cup\underline{I}_0} a^{s,\beta}_{r,\alpha}(\omega)\left[E^r_{\alpha\alpha}(\omega) - E^s_{\beta\beta}(\omega)\right]\mathrm{d}\omega\,.$$

- The coefficients $a^{s,\beta}_{r,\alpha}(\omega) = a^{r,\alpha}_{s,\beta}(\omega)$ (and $C^{s,\beta}_{r,\alpha\alpha'}(\omega)$) are independent of the loads.

Basic SEA equations
Average power flow

- Let $\mathscr{I}_r = \{\alpha;\ \omega_{r\alpha} \in I_0\}$, for $r \in \{1,2\}$.
- **Hypotheses** (wideband excitations):
  - i $\xi_{r\alpha} \ll 1$, $\forall\alpha \in \mathscr{I}_r$, $r \in \{1,2\}$;
  - ii $\Delta\omega \gg b_{r\alpha} = \pi\xi_{r\alpha}\omega_{r\alpha}$, $\forall\alpha \in \mathscr{I}_r$, $r \in \{1,2\}$;
  - iii The average mechanical energy of the stationary forced response for each sub-system is due to the modes in $\mathscr{I}_r$ solely, $r \in \{1,2\}$.
- Then:

$$\boxed{\mathbb{E}\{\Pi_{rs,t}\} \simeq \sum_{\alpha\in\mathscr{I}_r}\sum_{\beta\in\mathscr{I}_s} a_{r,\alpha}^{s,\beta}(\omega_0)\left[\frac{\pi S_{r\alpha}}{D_{r\alpha}} - \frac{\pi S_{s\beta}}{D_{s\beta}}\right]}.$$

- This is the analog of the result obtained for the two-DOFs system, summing the contributions of all modes of each sub-system within the frequency band of analysis I_0.

Outline

Internal fluid-structure interaction

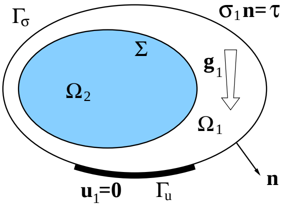
Notations and setting

- The structure: we consider small perturbations $\boldsymbol{u}_1(\boldsymbol{x},t)$ around a static equilibrium $\boldsymbol{x}\in\Omega_1$ considered as the reference configuration.
- The structure occupying Ω_1 is constituted by linear, memoryless viscoelastic materials:

$$\boldsymbol{\sigma}_1(\boldsymbol{x},t)=\mathbf{C}_1^{\mathrm{e}}\boldsymbol{\epsilon}_1(\boldsymbol{x},t)+\mathbf{C}_1^{\mathrm{v}}\dot{\boldsymbol{\epsilon}}_1(\boldsymbol{x},t)\,,\quad(\boldsymbol{x},t)\in\Omega_1\times\mathbb{R}\,,$$

where $\boldsymbol{\epsilon}_1=\boldsymbol{\nabla}\otimes_s\boldsymbol{u}_1$ is the small strain tensor, $\boldsymbol{\sigma}_1$ the Cauchy stress tensor, $\mathbf{C}_1^{\mathrm{e}}$ the elasticity tensor, $\mathbf{C}_1^{\mathrm{v}}$ the viscosity tensor, and ϱ_1 the density.

Internal fluid-structure interaction
Notations and setting



- The balance of momentum, boundary conditions and initial conditions read:

$$\left\{\begin{aligned} \mathbf{Div}\boldsymbol{\sigma}_1+\varrho_1\boldsymbol{g}_1 &= \varrho_1\frac{\partial^2\boldsymbol{u}_1}{\partial t^2} && \text{in } \Omega_1\,, \\ \boldsymbol{u}_1 &= \mathbf{0} && \text{on } \Gamma_{\mathrm{u}}\,, \\ \boldsymbol{\sigma}_1\boldsymbol{n} &= \boldsymbol{\tau}_1 && \text{on } \Gamma_\sigma\,, \\ \boldsymbol{\sigma}_1\boldsymbol{n} &= -p_2\boldsymbol{n} && \text{on } \Sigma\,, \\ \boldsymbol{u}_1(\cdot,0) &= \boldsymbol{u}_0 && \text{in } \Omega_1\,, \\ \dot{\boldsymbol{u}}_1(\cdot,0) &= \boldsymbol{v}_0 && \text{in } \Omega_1\,, \end{aligned}\right.$$

where $\boldsymbol{n}$ is the unit outward normal to $\partial\Omega_1$.

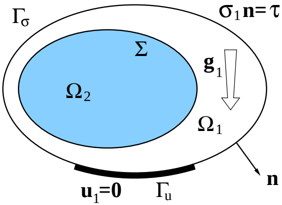
- The acoustic fluid: homogeneous, compressible, inviscid, gravity effects are neglected. It occupies the bounded domain Ω_2 where $\partial\Omega_2 = \Sigma$.
- Irrotational motion $\boldsymbol{\nabla} \times \boldsymbol{v}_2 = \boldsymbol{0}$, s.t. the fluid velocity $\boldsymbol{v}_2$ reads $\boldsymbol{v}_2 = \boldsymbol{\nabla}\psi_2$ and the fluid pressure reads $p_2 = -\varrho_2 \partial_t \psi_2 + \pi$.
- The static correction:

$$\pi = \frac{\varrho_2 c_2^2}{|\Omega_2|} \int_\Sigma \boldsymbol{u}_1 \cdot \boldsymbol{n} \, \mathrm{d}\sigma \,,$$

  where ϱ_2: fluid density, c_2: sound speed, accounts for the pressure variation induced by a static deformation of Σ.
- **Hypothesis**: π is negligible (*e.g.* air cavity).

Internal fluid-structure interaction

Notations and setting



- The linearized Euler equation, boundary conditions and initial conditions yield for the velocity potential ψ_2:

$$\begin{cases} \dfrac{1}{c_2^2}\dfrac{\partial^2\psi_2}{\partial t^2} - \Delta\psi_2 = g_2 & \text{in } \Omega_2\,, \\ \dfrac{\partial\psi_2}{\partial \boldsymbol{n}} = \dot{\boldsymbol{u}}_1 \cdot \boldsymbol{n} & \text{on } \Sigma\,, \\ \psi_2(\cdot,0) = 0 & \text{in } \Omega_2\,, \\ \dot{\psi}_2(\cdot,0) = 0 & \text{in } \Omega_2\,. \end{cases}$$

- The Virtual Power principle (VPP) on the sets of admissible fields

$$\mathcal{C}_1 = \left\{ \boldsymbol{v};\ \boldsymbol{v} \in L^2(\Omega_1),\ \boldsymbol{\nabla}\boldsymbol{v} \in L^2(\Omega_1),\ \text{and}\ \ \boldsymbol{v}|_{\Gamma_u} = \boldsymbol{0} \right\},$$

$$\mathcal{C}_2 = \left\{ \phi;\ \phi \in L^2(\Omega_2),\ \boldsymbol{\nabla}\phi \in L^2(\Omega_2),\ \text{and}\ \int_{\Omega_2} \phi\, \mathrm{d}\boldsymbol{x} = 0 \right\},$$

reads: Find $(\boldsymbol{u}_1, \psi_2) \in \mathcal{C}_1 \times \mathcal{C}_2$ s.t.

$$\int_{\Omega_1} \varrho_1 \ddot{\boldsymbol{u}}_1 \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} + \int_{\Omega_1} \boldsymbol{\sigma}_1(\boldsymbol{u}_1) : \boldsymbol{\epsilon}(\boldsymbol{v})\, \mathrm{d}\boldsymbol{x} - \int_{\Sigma} \varrho_2 \dot{\psi}_2 (\boldsymbol{n} \cdot \boldsymbol{v})\, \mathrm{d}\sigma$$
$$= \int_{\Omega_1} \varrho_1 \boldsymbol{g}_1 \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} + \int_{\Gamma_\sigma} \boldsymbol{\tau}_1 \cdot \boldsymbol{v}\, \mathrm{d}\sigma, \quad \forall \boldsymbol{v} \in \mathcal{C}_1,$$

$$\int_{\Omega_2} \frac{\varrho_2}{c_2^2} \ddot{\psi}_2 \phi\, \mathrm{d}\boldsymbol{x} + \int_{\Omega_2} \varrho_2 \boldsymbol{\nabla}\psi_2 \cdot \boldsymbol{\nabla}\phi\, \mathrm{d}\boldsymbol{x} + \int_{\Sigma} \varrho_2 (\dot{\boldsymbol{u}}_1 \cdot \boldsymbol{n}) \phi\, \mathrm{d}\sigma$$
$$= \int_{\Omega_2} \varrho_2 g_2 \phi\, \mathrm{d}\boldsymbol{x}, \quad \forall \phi \in \mathcal{C}_2.$$

- The structural mass, stiffness and damping bilinear forms:

$$m_1(\boldsymbol{u},\boldsymbol{v}) = \langle \boldsymbol{M}_1\boldsymbol{u},\boldsymbol{v}\rangle_{\mathcal{C}_1',\mathcal{C}_1} = \int_{\Omega_1} \varrho_1 \boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\,,$$

$$k_1(\boldsymbol{u},\boldsymbol{v}) = \langle \boldsymbol{K}_1\boldsymbol{u},\boldsymbol{v}\rangle_{\mathcal{C}_1',\mathcal{C}_1} = \int_{\Omega_1} \mathbf{C}_1^{\mathrm{e}}(\boldsymbol{\epsilon}(\boldsymbol{u})):\boldsymbol{\epsilon}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\,,$$

$$d_1(\boldsymbol{u},\boldsymbol{v}) = \langle \boldsymbol{D}_1\boldsymbol{u},\boldsymbol{v}\rangle_{\mathcal{C}_1',\mathcal{C}_1} = \int_{\Omega_1} \mathbf{C}_1^{\mathrm{v}}(\boldsymbol{\epsilon}(\boldsymbol{u})):\boldsymbol{\epsilon}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\,,$$

define the positive definite, symmetric mass $\boldsymbol{M}_1$, stiffness $\boldsymbol{K}_1$ and damping $\boldsymbol{D}_1$ operators of $\mathscr{L}(\mathcal{C}_1,\mathcal{C}_1')$ (the set of continuous operators), where $\langle \boldsymbol{f},\boldsymbol{v}\rangle_{\mathcal{C}_1',\mathcal{C}_1}$ defines the duality product of $\boldsymbol{f}\in\mathcal{C}_1'$ and $\boldsymbol{v}\in\mathcal{C}_1$, and $\mathcal{C}_1'$ is the dual space of $\mathcal{C}_1$ (the set of linear forms).

- The fluid mass and stiffness bilinear forms:

$$m_2(\psi,\phi) = \langle \boldsymbol{M}_2\psi,\phi\rangle_{\mathcal{C}_2',\mathcal{C}_2} = \int_{\Omega_2} \frac{\varrho_2}{c_2^2}\psi\phi\,\mathrm{d}\boldsymbol{x}\,,$$

$$k_2(\psi,\phi) = \langle \boldsymbol{K}_2\psi,\phi\rangle_{\mathcal{C}_2',\mathcal{C}_2} = \int_{\Omega_2} \varrho_2\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\phi\,\mathrm{d}\boldsymbol{x}\,,$$

  define the positive definite, symmetric mass $\boldsymbol{M}_2$ and stiffness $\boldsymbol{K}_2$ operators of $\mathscr{L}(\mathcal{C}_2,\mathcal{C}_2')$.
- The gyroscopic coupling linear form on $\mathcal{C} = \mathcal{C}_1\times\mathcal{C}_2$:

$$d_c\left(\begin{matrix}\boldsymbol{u}\\ \psi\end{matrix}\middle|\begin{matrix}\boldsymbol{v}\\ \phi\end{matrix}\right) = \left\langle \boldsymbol{\Omega}\begin{matrix}\boldsymbol{u}\\ \psi\end{matrix}\middle|\begin{matrix}\boldsymbol{v}\\ \phi\end{matrix}\right\rangle_{\mathcal{C}',\mathcal{C}} = -\int_\Sigma \varrho_2\psi(\boldsymbol{v}\cdot\boldsymbol{n})\,\mathrm{d}\sigma + \int_\Sigma \varrho_2\phi(\boldsymbol{u}\cdot\boldsymbol{n})\,\mathrm{d}\sigma$$

  is skew-symmetric (thus conservative).

Internal fluid-structure interaction

Virtual Power Principle

- By the Riesz theorem, $\exists! \boldsymbol{f}_1 \in \mathcal{C}_1'$ and $\exists! \boldsymbol{f}_2 \in \mathcal{C}_2'$ s.t.

$$
\begin{aligned}
f_1(\boldsymbol{v}) &= \langle \boldsymbol{f}_1, \boldsymbol{v} \rangle_{\mathcal{C}_1', \mathcal{C}_1} = \int_{\Omega_1} \varrho_1 \boldsymbol{g}_1 \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \int_{\Gamma_\sigma} \boldsymbol{\tau}_1 \cdot \boldsymbol{v} \, \mathrm{d}\sigma \,, & \forall \boldsymbol{v} \in \mathcal{C}_1 \,, \\
f_2(\phi) &= \langle f_2, \phi \rangle_{\mathcal{C}_2', \mathcal{C}_2} = \int_{\Omega_2} \varrho_2 g_2 \phi \, \mathrm{d}\boldsymbol{x} \,, & \forall \phi \in \mathcal{C}_2 \,.
\end{aligned}
$$

- Then the VPP reads:

$$
\begin{bmatrix} \boldsymbol{M}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M}_2 \end{bmatrix} \begin{pmatrix} \ddot{\boldsymbol{u}}_1 \\ \ddot{\psi}_2 \end{pmatrix} + \begin{bmatrix} \boldsymbol{D}_1 & \boldsymbol{\Omega}_{12} \\ \boldsymbol{\Omega}_{21} & \boldsymbol{0} \end{bmatrix} \begin{pmatrix} \dot{\boldsymbol{u}}_1 \\ \dot{\psi}_2 \end{pmatrix} + \begin{bmatrix} \boldsymbol{K}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{K}_2 \end{bmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \end{pmatrix} ,
$$

  "in the sense of distributions", with some prescribed initial conditions.

- This is the situation addressed beforehand, when the mass and stiffness coupling operators vanish:

$$
\boldsymbol{M}_{12} = \boldsymbol{M}_{21} = \boldsymbol{0} \,, \quad \boldsymbol{K}_{12} = \boldsymbol{K}_{21} = \boldsymbol{0} \,.
$$

- **Spectral problem** for the structure: Find $\lambda \in \mathbb{R}$ and $\boldsymbol{\phi} \in \mathcal{C}_1$ s.t.

$$\boldsymbol{K}_1 \boldsymbol{\phi} = \lambda \boldsymbol{M}_1 \boldsymbol{\phi}\,.$$

- It admits a countable set of solutions $(\lambda_{1,1}, \boldsymbol{\phi}_{1,1})$, $(\lambda_{1,2}, \boldsymbol{\phi}_{1,2})$ ... s.t. $0 < \lambda_{1,1} \leqslant \lambda_{1,2} \leqslant \dots$ and $\{\boldsymbol{\phi}_{1\alpha}\}_{\alpha \in \mathbb{N}*}$ is an Hilbertian basis of the space $H_1 = L^2_\mu(\Omega_1)$ of square integrable functions with respect to the unit mass measure $\mu_1(\mathrm{d}\boldsymbol{x}) = \mathbb{1}_{\Omega_1} \frac{\varrho_1 \mathrm{d}\boldsymbol{x}}{M_1}$, with $M_1 = \int_{\Omega_1} \varrho_1 \mathrm{d}\boldsymbol{x}$:

$$\begin{aligned} m_1(\boldsymbol{\phi}_{1\alpha}, \boldsymbol{\phi}_{1\alpha'}) &= M_1 \delta_{\alpha\alpha'}\,, \\ k_1(\boldsymbol{\phi}_{1\alpha}, \boldsymbol{\phi}_{1\alpha'}) &= M_1 \lambda_{1\alpha} \delta_{\alpha\alpha'}\,. \end{aligned}$$

- $\{\boldsymbol{\phi}_{1\alpha}\}_{\alpha \in \mathbb{N}*}$ are the structural modes *in vacuo*, which are assumed (Basile) to diagonalize the damping operator $\boldsymbol{D}_1$ as well.

- **Spectral problem** for the acoustic cavity: Find $\lambda \in \mathbb{R}$ and $\boldsymbol{\phi} \in \mathcal{C}_2$ s.t.

$$\boldsymbol{K}_2\boldsymbol{\phi} = \lambda \boldsymbol{M}_2\boldsymbol{\phi} \,.$$

- It admits a countable set of solutions $(\lambda_{2,1}, \phi_{2,1})$, $(\lambda_{2,2}, \phi_{2,2})$ ... s.t. $0 < \lambda_{2,1} \leqslant \lambda_{2,2} \leqslant \ldots$ and $\{\phi_{2\beta}\}_{\alpha\in\mathbb{N}*}$ is an Hilbertian basis of the space $H_2 = L^2\left(\Omega_2\right)$ of square integrable functions with respect to the scalar product $\mu_2(\psi,\phi) = \int_{\Omega_2}\psi\phi\,\mathrm{d}\boldsymbol{x}$, with $M_2 = \int_{\Omega_2}\varrho_2\mathrm{d}\boldsymbol{x}$:

$$m_2(\phi_{2\beta},\phi_{2\beta'}) = \frac{\varrho_2}{c_2^2}\mu_2(\phi_{2\beta},\phi_{2\beta'}) = \frac{M_2}{c_2^2}\delta_{\beta\beta'}\,,$$

$$k_2(\phi_{2\beta},\phi_{2\beta'}) = \frac{M_2}{c_2^2}\lambda_{2\beta}\delta_{\beta\beta'}\,.$$

- $\{\phi_{2\beta}\}_{\beta\in\mathbb{N}*}$ are the rigid-wall acoustic modes.

- Consequently, the solution $\boldsymbol{u}_1 \in \mathcal{C}_1$ can be expanded on the eigenbasis $\{\boldsymbol{\phi}_{1\alpha}\}_{\alpha\in\mathbb{N}*}$ as:

$$\boldsymbol{u}_1(\boldsymbol{x},t) = \sum_{\alpha=1}^{\infty} q_{1\alpha}(t)\boldsymbol{\phi}_{1\alpha}(\boldsymbol{x})\,, \quad (\boldsymbol{x},t) \in \Omega_1 \times \mathbb{R}\,,$$

- Likewise, the solution $\psi_2 \in \mathcal{C}_2$ can be expanded on the eigenbasis $\{\phi_{2\beta}\}_{\beta\in\mathbb{N}*}$ as:

$$\psi_2(\boldsymbol{x},t) = \sum_{\beta=1}^{\infty} q_{2\beta}(t)\phi_{2\beta}(\boldsymbol{x})\,, \quad (\boldsymbol{x},t) \in \Omega_2 \times \mathbb{R}\,.$$

- The associated mechanical energies for both sub-systems are:

$$\mathcal{E}_1(t) = \frac{M_1}{2}\sum_{\alpha=1}^{+\infty}\left[(\dot{q}_{1\alpha}(t))^2 + \omega_{1\alpha}^2(q_{1\alpha}(t))^2\right] ,$$

$$\mathcal{E}_2(t) = \frac{M_2}{2c_2^2}\sum_{\alpha=1}^{+\infty}\left[(\dot{q}_{2\alpha}(t))^2 + \omega_{2\alpha}^2(q_{2\alpha}(t))^2\right] ,$$

where $\omega_{r\alpha} = \sqrt{\lambda_{r\alpha}}$ is the "blocked" (natural) frequency of the α^{th} mode of the r^{th} sub-system–*i.e.* the α^{th} structural mode *in vacuo* for $r = 1$ or the α^{th} rigid-wall acoustic mode for $r = 2$.

- The average mechanical energies of the forced responses under mean-square stationary random loads are:

$$\mathbb{E}\{\mathcal{E}_{1,t}\} = M_1 \int_{\mathbb{R}} \omega^2 \,\mathrm{Tr}\, \boldsymbol{S}_{\boldsymbol{U}_1}(\omega)\,\mathrm{d}\omega\,,$$

$$\mathbb{E}\{\mathcal{E}_{2,t}\} = \frac{M_2}{c_2^2} \int_{\mathbb{R}} S_{P_2}(\omega)\,\mathrm{d}\omega\,,$$

where $\omega \mapsto \boldsymbol{S}_{\boldsymbol{U}_1}(\omega)$ is the spectral density matrix of the structure displacement, and $\omega \mapsto S_{P_2}(\omega)$ is the spectral density function of the fluid pressure.

Outline

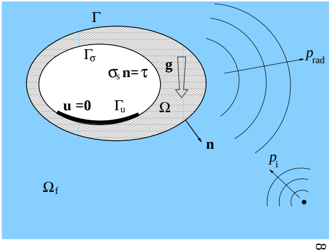
External fluid-structure interaction

Notations and setting

- The structure: we consider small perturbations $\boldsymbol{u}_{\rm s}(\boldsymbol{x},t)$ around a static equilibrium $\boldsymbol{x}\in\Omega_{\rm s}$ considered as the reference configuration.
- The structure occupying $\Omega_{\rm s}$ is constituted by linear, memoryless viscoelastic materials:

$$\boldsymbol{\sigma}_{\rm s}(\boldsymbol{x},t)=\mathbf{C}_{\rm s}^{\rm e}\boldsymbol{\epsilon}_{\rm s}(\boldsymbol{x},t)+\mathbf{C}_{\rm s}^{\rm v}\dot{\boldsymbol{\epsilon}}_{\rm s}(\boldsymbol{x},t)\,,\quad(\boldsymbol{x},t)\in\Omega_{\rm s}\times\mathbb{R}\,,$$

where $\boldsymbol{\epsilon}_{\rm s}=\boldsymbol{\nabla}\otimes_s\boldsymbol{u}_{\rm s}$ is the small strain tensor, $\boldsymbol{\sigma}_{\rm s}$ the Cauchy stress tensor, $\mathbf{C}_{\rm s}^{\rm e}$ the elasticity tensor, $\mathbf{C}_{\rm s}^{\rm v}$ the viscosity tensor, and $\varrho_{\rm s}$ the density.



- The balance of momentum, boundary conditions and initial conditions read:

$$
\left\{
\begin{aligned}
\mathbf{Div}\boldsymbol{\sigma}_{\mathrm{s}} + \varrho_{\mathrm{s}}\boldsymbol{g}_{\mathrm{s}} &= \varrho_{\mathrm{s}}\frac{\partial^2 \boldsymbol{u}_{\mathrm{s}}}{\partial t^2} && \text{in } \Omega_{\mathrm{s}}\,, \\
\boldsymbol{u}_{\mathrm{s}} &= \mathbf{0} && \text{on } \Gamma_{\mathrm{u}}\,, \\
\boldsymbol{\sigma}_{\mathrm{s}}\boldsymbol{n} &= \boldsymbol{\tau}_{\mathrm{s}} && \text{on } \Gamma_{\sigma}\,, \\
\boldsymbol{\sigma}_{\mathrm{s}}\boldsymbol{n} &= -p_{\mathrm{f}}\boldsymbol{n} && \text{on } \Gamma\,, \\
\boldsymbol{u}_{\mathrm{s}}(\cdot,0) &= \boldsymbol{u}_0 && \text{in } \Omega_{\mathrm{s}}\,, \\
\dot{\boldsymbol{u}}_{\mathrm{s}}(\cdot,0) &= \boldsymbol{v}_0 && \text{in } \Omega_{\mathrm{s}}\,,
\end{aligned}
\right.
$$

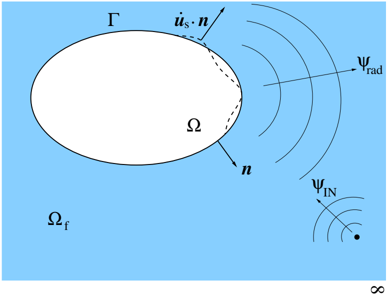
where $\boldsymbol{n}$ is the unit outward normal to $\partial\Omega_{\mathrm{s}}$.

- The acoustic fluid: homogeneous, compressible, inviscid, gravity effects are neglected.
- Irrotational motion $\boldsymbol{\nabla}\times\boldsymbol{v}_{\mathrm{f}}=\mathbf{0}$, s.t. the fluid velocity $\boldsymbol{v}_{\mathrm{f}}$ reads $\boldsymbol{v}_{\mathrm{f}}=\boldsymbol{\nabla}\psi_{\mathrm{f}}$ and the fluid pressure reads $p_{\mathrm{f}}=-\varrho_{\mathrm{f}}\partial_t\psi_{\mathrm{f}}$, where the velocity potential ψ_{f} satisfies:

$$
\left\{
\begin{aligned}
\frac{1}{c_{\mathrm{f}}^2}\frac{\partial^2\psi_{\mathrm{f}}}{\partial t^2}-\Delta\psi_{\mathrm{f}} &= g_{\mathrm{IN}} && \text{in } \Omega_{\mathrm{f}}\,,\\
\frac{\partial\psi_{\mathrm{f}}}{\partial\boldsymbol{n}} &= \dot{\boldsymbol{u}}_{\mathrm{s}}\cdot\boldsymbol{n} && \text{on } \Gamma\,,\\
\psi_{\mathrm{f}}(\cdot,0) &= 0 && \text{in } \Omega_{\mathrm{f}}\,,\\
\dot{\psi}_{\mathrm{f}}(\cdot,0) &= 0 && \text{in } \Omega_{\mathrm{f}}\,,
\end{aligned}
\right.
$$

with ϱ_{f}: fluid density, c_{f}: sound speed.

External fluid-structure interaction
Decomposition of the velocity potential

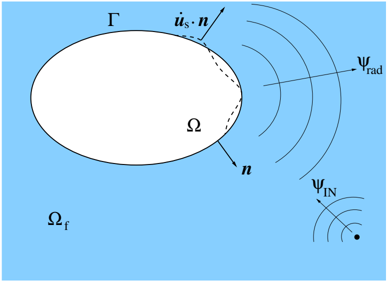


- The velocity potential reads:

$$\psi_{\rm f}(\boldsymbol{x},t)=\psi_{\rm IN}(\boldsymbol{x},t)+\psi_{\rm d0}(\boldsymbol{x},t)+\psi_{\rm rad}(\boldsymbol{x},t)\,,\quad (\boldsymbol{x},t)\in\Omega_{\rm f}\times\mathbb{R}\,,$$

where:

- $\psi_{\rm IN}$: incident potential (data);
- $\psi_{\rm d0}$: diffracted potential, the structure being motionless;
- $\psi_{\rm rad}$: radiated potential.

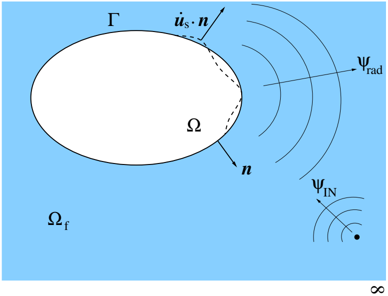


- The diffracted potential ψ_{d0} satisfies:

$$\left\{ \begin{aligned} \frac{1}{c_f^2}\frac{\partial^2\psi_{d0}}{\partial t^2} - \Delta\psi_{d0} &= 0 \quad \text{in } \Omega_f\,, \\ \frac{\partial\psi_{d0}}{\partial \boldsymbol{n}} &= -\frac{\partial\psi_{IN}}{\partial \boldsymbol{n}} \quad \text{on } \Gamma\,. \end{aligned} \right.$$

External fluid-structure interaction

Decomposition of the velocity potential

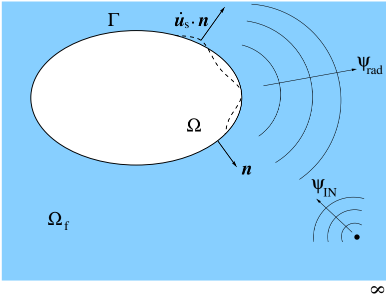


- The radiated potential ψ_{rad} satisfies:

$$\begin{cases} \dfrac{1}{c_{\text{f}}^2}\dfrac{\partial^2\psi_{\text{rad}}}{\partial t^2} - \Delta\psi_{\text{rad}} = 0 \quad \text{in } \Omega_{\text{f}}\,, \\ \dfrac{\partial\psi_{\text{rad}}}{\partial\boldsymbol{n}} = \dot{\boldsymbol{u}}_{\text{s}}\cdot\boldsymbol{n} \quad \text{on } \Gamma\,. \end{cases}$$

External fluid-structure interaction

Decomposition of the velocity potential



- The incident potential ψ_{IN} satisfies:

$$\frac{1}{c_f^2}\frac{\partial^2\psi_{IN}}{\partial t^2} - \Delta\psi_{IN} = g_{IN}(\boldsymbol{x},t)\,, \quad (\boldsymbol{x},t)\in\mathbb{R}^3\times\mathbb{R}\,,$$

for some sound source g_{IN} in the full physical space.

- The exterior Helmholtz problem:

$$\begin{cases} \Delta\psi + k_f^2\psi = 0 & \text{in } \Omega_f\,, \\ \dfrac{\partial\psi}{\partial\boldsymbol{n}} = v & \text{on } \partial\Omega_f\,, \\ |\psi| = \mathrm{O}\left(\dfrac{1}{r}\right)\,,\quad \left|\dfrac{\partial\psi}{\partial r} + \mathrm{i}k_f\psi\right| = \mathrm{O}\left(\dfrac{1}{r^2}\right) & \text{as } r = \|\boldsymbol{x}\| \to +\infty\,, \end{cases}$$

where $k_f = \frac{\omega}{c_f}$, and $\partial\Omega_f = \Gamma$ is actually the interface between the fluid and the structure.

- Sommerfeld radiation conditions "at infinity": the radiated waves are almost plane and do not propagate toward Γ.

Solving the fluid equations
Boundary impedance

- The exterior Helmholtz problem admits a unique solution $\forall k_{\mathrm{f}} \in \mathbb{R}$, hence:
  - There exists a linear operator $\mathscr{B}_\Gamma(k_{\mathrm{f}}) : \mathcal{C}'_\Gamma \rightarrow \mathcal{C}_\Gamma$ s.t.:

$$\psi|_\Gamma(\omega) = \mathscr{B}_\Gamma(k_{\mathrm{f}})v\,, \quad \text{on } \Gamma\,;$$

  - There exists a linear operator $\mathscr{R}_{\boldsymbol{x}}(k_{\mathrm{f}}) : \mathcal{C}'_\Gamma \rightarrow \mathbb{C}$ s.t.:

$$\psi(\boldsymbol{x},\omega) = \mathscr{R}_{\boldsymbol{x}}(k_{\mathrm{f}})v\,, \quad \boldsymbol{x} \in \Omega_{\mathrm{f}}\,,$$

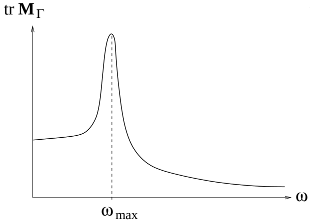
  with $\mathcal{C}_\Gamma$: the set of admissible fields on Γ ($\mathcal{C}'_\Gamma$: its dual).
- $\boldsymbol{Z}_\Gamma(\omega) = -\mathrm{i}\omega\varrho_{\mathrm{f}}\mathscr{B}_\Gamma(k_{\mathrm{f}})$ is the acoustic impedance boundary operator.
- $\boldsymbol{\mathcal{Z}}_{\boldsymbol{x}}(\omega) = -\mathrm{i}\omega\varrho_{\mathrm{f}}\mathscr{R}_{\boldsymbol{x}}(k_{\mathrm{f}})$ is the radiation impedance operator.

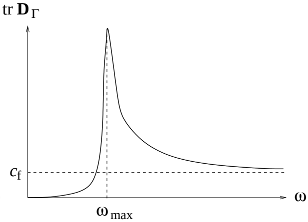
Solving the fluid equations
Boundary impedance

- The boundary impedance $\boldsymbol{Z}_\Gamma(\omega)$ is symmetric and reads:

$$\mathrm{i}\omega\boldsymbol{Z}_\Gamma(\omega) = -\omega^2\boldsymbol{M}_\Gamma\left(\frac{\omega}{c_\mathrm{f}}\right) + \mathrm{i}\omega\boldsymbol{D}_\Gamma\left(\frac{\omega}{c_\mathrm{f}}\right) ,$$

where:
  - The reactive part $\omega \mapsto \mathrm{Tr}\,\boldsymbol{M}_\Gamma\left(\frac{\omega}{c_\mathrm{f}}\right)$ (left) is generally unsigned, though it is positive if $\mathbb{R}^3\backslash\overline{\Omega}_\mathrm{f}$ is convex;
  - The resistive part $\omega \mapsto \mathrm{Tr}\,\boldsymbol{D}_\Gamma\left(\frac{\omega}{c_\mathrm{f}}\right)$ (right) is positive.





Solving the fluid equations

Consequences for the velocity potential

- Diffracted velocity potential:

$$\widehat{\psi}_{\mathrm{d}0}(\omega)|_\Gamma = -\mathscr{B}_\Gamma(k_\mathrm{f})\frac{\partial\widehat{\psi}_{\mathrm{IN}}}{\partial\boldsymbol{n}} \quad \boldsymbol{x}\in\Gamma\,,$$
$$\widehat{\psi}_{\mathrm{d}0}(\boldsymbol{x},\omega) = -\mathscr{R}_{\boldsymbol{x}}(\omega)\frac{\partial\widehat{\psi}_{\mathrm{IN}}}{\partial\boldsymbol{n}} \quad \boldsymbol{x}\in\Omega_\mathrm{f}\,.$$

- Radiated velocity potential:

$$\widehat{\psi}_{\mathrm{rad}}(\omega)|_\Gamma = \mathrm{i}\omega\mathscr{B}_\Gamma(k_\mathrm{f})(\widehat{\boldsymbol{u}}_\mathrm{s}(\omega)\cdot\boldsymbol{n}) \quad \boldsymbol{x}\in\Gamma\,,$$
$$\widehat{\psi}_{\mathrm{rad}}(\boldsymbol{x},\omega) = \mathrm{i}\omega\mathscr{R}_{\boldsymbol{x}}(\omega)(\widehat{\boldsymbol{u}}_\mathrm{s}(\omega)\cdot\boldsymbol{n}) \quad \boldsymbol{x}\in\Omega_\mathrm{f}\,.$$

- The (linear) scattering operator $\mathcal{T}_\Gamma(\omega)\equiv\boldsymbol{I}-\mathscr{B}_\Gamma(k_\mathrm{f})\partial_{\boldsymbol{n}}$ s.t.:

$$(\widehat{\psi}_{\mathrm{IN}}(\omega)+\widehat{\psi}_{\mathrm{d}0}(\omega))|_\Gamma = \mathcal{T}_\Gamma(\omega)\widehat{\psi}_{\mathrm{IN}}(\omega)|_\Gamma\,.$$

- The Virtual Power Principle (VPP) on the set of admissible displacement fields:

$$\mathcal{C}_{\mathrm{s}} = \left\{ \boldsymbol{v};\, \boldsymbol{v} \in L^2\left(\Omega_{\mathrm{s}}\right),\, \boldsymbol{\nabla}\boldsymbol{v} \in L^2\left(\Omega_{\mathrm{s}}\right),\, \text{and } \left.\boldsymbol{v}\right|_{\Gamma_{\mathrm{u}}} = \mathbf{0} \right\} ,$$

reads for the structure, in the frequency domain: Find $\widehat{\boldsymbol{u}}_{\mathrm{s}} \in \mathcal{C}_{\mathrm{s}}$ s.t.

$$-\omega^2 \int_{\Omega_{\mathrm{s}}} \varrho_{\mathrm{s}} \widehat{\boldsymbol{u}}_{\mathrm{s}} \cdot \overline{\boldsymbol{v}}\, \mathrm{d}\boldsymbol{x} + \mathrm{i}\omega \int_{\Omega_{\mathrm{s}}} \mathbf{C}_{\mathrm{s}}^{\mathrm{v}} \boldsymbol{\epsilon}(\widehat{\boldsymbol{u}}_{\mathrm{s}}) : \boldsymbol{\epsilon}(\overline{\boldsymbol{v}})\, \mathrm{d}\boldsymbol{x} + \int_{\Omega_{\mathrm{s}}} \mathbf{C}_{\mathrm{s}}^{\mathrm{e}} \boldsymbol{\epsilon}(\widehat{\boldsymbol{u}}_{\mathrm{s}}) : \boldsymbol{\epsilon}(\overline{\boldsymbol{v}})\, \mathrm{d}\boldsymbol{x}$$

$$-\mathrm{i}\omega \int_{\Gamma} \varrho_{\mathrm{f}} \widehat{\psi}_{\mathrm{f}} (\overline{\boldsymbol{v}} \cdot \boldsymbol{n})\, \mathrm{d}\sigma = \int_{\Omega_{\mathrm{s}}} \varrho_{\mathrm{s}} \widehat{\boldsymbol{g}}_{\mathrm{s}} \cdot \overline{\boldsymbol{v}}\, \mathrm{d}\boldsymbol{x} + \int_{\Gamma_\sigma} \widehat{\boldsymbol{\tau}}_{\mathrm{s}} \cdot \overline{\boldsymbol{v}}\, \mathrm{d}\sigma\,, \quad \forall \boldsymbol{v} \in \mathcal{C}_{\mathrm{s}}\,.$$

Solving the structure problem

Virtual Power Principle

- The structural mass, stiffness and damping bilinear forms:

$$m_{\mathrm{s}}(\boldsymbol{u},\boldsymbol{v})=\langle\boldsymbol{M}_{\mathrm{s}}\boldsymbol{u},\boldsymbol{v}\rangle_{\mathcal{C}'_{\mathrm{s}},\mathcal{C}_{\mathrm{s}}}=\int_{\Omega_{\mathrm{s}}}\varrho_{\mathrm{s}}\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}\,,$$

$$k_{\mathrm{s}}(\boldsymbol{u},\boldsymbol{v})=\langle\boldsymbol{K}_{\mathrm{s}}\boldsymbol{u},\boldsymbol{v}\rangle_{\mathcal{C}'_{\mathrm{s}},\mathcal{C}_{\mathrm{s}}}=\int_{\Omega_{\mathrm{s}}}\mathbf{C}^{\mathrm{e}}_{\mathrm{s}}(\boldsymbol{\epsilon}(\boldsymbol{u})):\boldsymbol{\epsilon}(\overline{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x}\,,$$

$$d_{\mathrm{s}}(\boldsymbol{u},\boldsymbol{v})=\langle\boldsymbol{D}_{\mathrm{s}}\boldsymbol{u},\boldsymbol{v}\rangle_{\mathcal{C}'_{\mathrm{s}},\mathcal{C}_{\mathrm{s}}}=\int_{\Omega_{\mathrm{s}}}\mathbf{C}^{\mathrm{v}}_{\mathrm{s}}(\boldsymbol{\epsilon}(\boldsymbol{u})):\boldsymbol{\epsilon}(\overline{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x}\,,$$

- The boundary coupling bilinear form:

$$b_{\Gamma}(\boldsymbol{u},\boldsymbol{v};\omega)=\langle\boldsymbol{Z}_{\Gamma}(\omega)\boldsymbol{u},\boldsymbol{v}\rangle_{\mathcal{C}'_{\mathrm{s}},\mathcal{C}_{\mathrm{s}}}=\int_{\Gamma}\left(\boldsymbol{Z}_{\Gamma}(\omega)\boldsymbol{u}\cdot\boldsymbol{n},\overline{\boldsymbol{v}}\cdot\boldsymbol{n}\right)\,\mathrm{d}\sigma\,.$$

Solving the structure problem
Virtual Power Principle

- By the Riesz theorem, $\exists! \widehat{\boldsymbol{f}}_{\mathrm{s}}, \widehat{\boldsymbol{f}}_{\mathrm{IN}} \in \mathcal{C}'_{\mathrm{s}}$ s.t.

$$f_{\mathrm{s}}(\boldsymbol{v}) = \left\langle \widehat{\boldsymbol{f}}_{\mathrm{s}}, \boldsymbol{v} \right\rangle_{\mathcal{C}'_{\mathrm{s}},\mathcal{C}_{\mathrm{s}}} = \int_{\Omega_{\mathrm{s}}} \varrho_{\mathrm{s}} \widehat{\boldsymbol{g}}_{\mathrm{s}} \cdot \overline{\boldsymbol{v}} \, \mathrm{d}\boldsymbol{x} + \int_{\Gamma_\sigma} \widehat{\boldsymbol{\tau}}_{\mathrm{s}} \cdot \overline{\boldsymbol{v}} \, \mathrm{d}\sigma \,,$$

$$f_{\mathrm{IN}}(\boldsymbol{v}) = \left\langle \widehat{\boldsymbol{f}}_{\mathrm{IN}}, \boldsymbol{v} \right\rangle_{\mathcal{C}'_{\mathrm{s}},\mathcal{C}_{\mathrm{s}}} = -\int_{\Gamma} \mathcal{T}_\Gamma(\omega) \widehat{p}_{\mathrm{IN}} (\overline{\boldsymbol{v}} \cdot \boldsymbol{n}) \, \mathrm{d}\sigma \,, \quad \forall \boldsymbol{v} \in \mathcal{C}_{\mathrm{s}} \,.$$

- Then the VPP in the frequency domain reads:

$$\left[-\omega^2 \left(\boldsymbol{M}_{\mathrm{s}} + \boldsymbol{M}_\Gamma \left(\frac{\omega}{c_{\mathrm{f}}} \right) \right) + \mathrm{i}\omega \left(\boldsymbol{D}_{\mathrm{s}} + \boldsymbol{D}_\Gamma \left(\frac{\omega}{c_{\mathrm{f}}} \right) \right) + \boldsymbol{K}_{\mathrm{s}} \right] \widehat{\boldsymbol{u}}_{\mathrm{s}}(\omega) = \widehat{\boldsymbol{f}}_{\mathrm{s}}(\omega) + \widehat{\boldsymbol{f}}_{\mathrm{e}}(\omega) + \widehat{\boldsymbol{f}}_{\mathrm{IN}}(\omega) \,,$$

"in the sense of distributions".

- $\widehat{\boldsymbol{f}}_{\mathrm{e}}(\omega) = \boldsymbol{D}_{\mathrm{s}} \boldsymbol{u}_0 + \boldsymbol{M}_{\mathrm{s}}(\mathrm{i}\omega \boldsymbol{u}_0 + \boldsymbol{v}_0)$ is an equivalent load accounting for the initial conditions.

- **Spectral problem**: Find $\lambda \in \mathbb{R}$ and $\boldsymbol{\phi} \in \mathcal{C}_{\mathrm{s}}$ s.t.

$$\boldsymbol{K}_{\mathrm{s}}\boldsymbol{\phi} = \lambda \boldsymbol{M}_{\mathrm{s}}\boldsymbol{\phi}\,.$$

- It admits a countable set of solutions $(\lambda_{\mathrm{s}1}, \boldsymbol{\phi}_{\mathrm{s}1})$, $(\lambda_{\mathrm{s}2}, \boldsymbol{\phi}_{\mathrm{s}2})$ ... s.t. $0 < \lambda_{\mathrm{s}1} \leqslant \lambda_{\mathrm{s}2} \leqslant \dots$ and $\{\boldsymbol{\phi}_{\mathrm{s}\alpha}\}_{\alpha\in\mathbb{N}*}$ is an Hilbertian basis of the space $H_{\mathrm{s}} = L^2_\mu(\Omega_{\mathrm{s}})$ of square integrable functions with respect to the unit mass measure $\mu_{\mathrm{s}}(\mathrm{d}\boldsymbol{x}) = \mathbb{1}_{\Omega_{\mathrm{s}}}\frac{\varrho_{\mathrm{s}}\mathrm{d}\boldsymbol{x}}{M_{\mathrm{s}}}$, with $M_{\mathrm{s}} = \int_{\Omega_{\mathrm{s}}}\varrho_{\mathrm{s}}\mathrm{d}\boldsymbol{x}$:

$$\begin{aligned} m_{\mathrm{s}}(\boldsymbol{\phi}_{\mathrm{s}\alpha}, \boldsymbol{\phi}_{\mathrm{s}\beta}) &= M_{\mathrm{s}}\delta_{\alpha\beta}\,, \\ k_{\mathrm{s}}(\boldsymbol{\phi}_{\mathrm{s}\alpha}, \boldsymbol{\phi}_{\mathrm{s}\beta}) &= M_{\mathrm{s}}\omega^2_{\mathrm{s}\alpha}\delta_{\alpha\beta}\,, \end{aligned}$$

  where $\omega^2_{\mathrm{s}\alpha} = \lambda_{\mathrm{s}\alpha}$.

- $\{\boldsymbol{\phi}_{\mathrm{s}\alpha}\}_{\alpha\in\mathbb{N}*}$ are the structural modes *in vacuo*.

External fluid-structure interaction

Eigenmodes and modal expansion

- Consequently, the solution $\boldsymbol{u}_{\mathrm{s}} \in \mathcal{C}_{\mathrm{s}}$ can be expanded on the eigenbasis $\{\boldsymbol{\phi}_{\mathrm{s}\alpha}\}_{\alpha\in\mathbb{N}*}$ as:

$$\boldsymbol{u}_{\mathrm{s}}(\boldsymbol{x},t) = \sum_{\alpha=1}^{\infty} q_\alpha(t)\boldsymbol{\phi}_{\mathrm{s}\alpha}(\boldsymbol{x})\,.$$

- Introducing $\mu_{\mathrm{s}}(\boldsymbol{u},\boldsymbol{v}) = \int_{\Omega_{\mathrm{s}}} \boldsymbol{u}\cdot\boldsymbol{v}\mu_{\mathrm{s}}(\mathrm{d}\boldsymbol{x})$ (the scalar product in H_{s}), the generalized coordinates $\{q_\alpha\}_{\alpha\in\mathbb{N}*}$ are:

$$q_\alpha = \mu_{\mathrm{s}}(\boldsymbol{u}_{\mathrm{s}},\boldsymbol{\phi}_{\mathrm{s}\alpha})\,.$$

- The μ-norm $\|\boldsymbol{u}_{\mathrm{s}}\|_\mu = \sqrt{\mu_{\mathrm{s}}(\boldsymbol{u}_{\mathrm{s}},\boldsymbol{u}_{\mathrm{s}})}$ is obtained as:

$$\|\boldsymbol{u}_{\mathrm{s}}(\cdot,t)\|_\mu = \left(\sum_{\alpha=1}^{+\infty}(q_\alpha(t))^2\right)^{\frac{1}{2}} < +\infty\,.$$

- **Hypothesis** (Basile): the eigenmodes $\{\boldsymbol{\phi}_{s\alpha}\}_{\alpha\in\mathbb{N}^*}$ diagonalize the damping operator as well:

$$d_s(\boldsymbol{\phi}_{s\alpha}, \boldsymbol{\phi}_{s\beta}) = M_s \eta_{s\alpha} \omega_{s\alpha} \delta_{\alpha\beta} \,,$$

where:

  - $\omega_{s\alpha}$: the (angular) eigenfrequency of the α^{th} mode *in vacuo*,
  - $\xi_{s\alpha}$: the modal critical damping rate of the α^{th} mode *in vacuo*,
  - $\eta_{s\alpha} = 2\xi_{s\alpha}$: the modal loss factor of the α^{th} mode *in vacuo*.

- Owing to the Basile hypothesis, the generalized coordinates $\{q_\alpha\}_{\alpha\in\mathbb{N}*}$ satisfy in the frequency domain:

$$M_{\mathrm{s}}(-\omega^2 + \mathrm{i}\eta_{\mathrm{s}\alpha}\omega_{\mathrm{s}\alpha}\omega + \omega_{\mathrm{s}\alpha}^2)\widehat{q}_\alpha(\omega) = \widehat{f}_\alpha(\omega) - \sum_{\beta=1}^{+\infty}(-\omega^2 M_{\alpha\beta}(\omega) + \mathrm{i}\omega D_{\alpha\beta}(\omega))\widehat{q}_\beta(\omega)$$

where $\widehat{f}_\alpha = \langle \widehat{\boldsymbol{f}}_{\mathrm{s}} + \widehat{\boldsymbol{f}}_{\mathrm{e}} + \widehat{\boldsymbol{f}}_{\mathrm{IN}}, \boldsymbol{\phi}_{\mathrm{s}\alpha}\rangle_{\mathcal{C}_{\mathrm{s}}',\mathcal{C}_{\mathrm{s}}}$, and:

$$M_{\alpha\beta}(\omega) = \left\langle \boldsymbol{M}_\Gamma\left(\frac{\omega}{c_{\mathrm{f}}}\right)\boldsymbol{\phi}_{\mathrm{s}\alpha}, \boldsymbol{\phi}_{\mathrm{s}\beta}\right\rangle_{\mathcal{C}_{\mathrm{s}}',\mathcal{C}_{\mathrm{s}}},$$

$$D_{\alpha\beta}(\omega) = \left\langle \boldsymbol{D}_\Gamma\left(\frac{\omega}{c_{\mathrm{f}}}\right)\boldsymbol{\phi}_{\mathrm{s}\alpha}, \boldsymbol{\phi}_{\mathrm{s}\beta}\right\rangle_{\mathcal{C}_{\mathrm{s}}',\mathcal{C}_{\mathrm{s}}}.$$

- The generalized coordinates get coupled by the boundary impedance operator.

- The frequency response function of the structure coupled to the fluid reads:

$$\widehat{\mathbb{h}}_{\mathrm{tot}}(\omega) = \left[-\omega^2\left(\boldsymbol{M}_{\mathrm{s}} + \boldsymbol{M}_{\Gamma}\left(\frac{\omega}{c_{\mathrm{f}}}\right)\right) + \mathrm{i}\omega\left(\boldsymbol{D}_{\mathrm{s}} + \boldsymbol{D}_{\Gamma}\left(\frac{\omega}{c_{\mathrm{f}}}\right)\right) + \boldsymbol{K}_{\mathrm{s}}\right]^{-1}.$$

- The "wet" eigenfrequencies of the structure coupled to the fluid are the solutions of:

$$\det\left[-\omega^2\left(\boldsymbol{M}_{\mathrm{s}} + \boldsymbol{M}_{\Gamma}\left(\frac{\omega}{c_{\mathrm{f}}}\right)\right) + \boldsymbol{K}_{\mathrm{s}}\right] = 0\,.$$

- This eigenvalue problem is often approached by:

$$\left[1 + \frac{M_{\alpha\alpha}(\omega)}{M_{\mathrm{s}}}\right]\omega^2 = \omega_{\mathrm{s}\alpha}^2\,,$$

neglecting the off-diagonal terms.

Stationary incident pressure

Forced response of the structure

- We consider $\boldsymbol{f}_\mathrm{e} \equiv \mathbf{0}$ and $\boldsymbol{f}_\mathrm{s} \equiv \mathbf{0}$, focusing on the structure forced response to a random incident pressure field.
- **Data**: $(P_t\,, t \in \mathbb{R})$ is a second order, centered stochastic process defined on $(\Omega, \mathcal{F}, P)$, indexed on $\mathbb{R}$, with values in $L^2(\Gamma)$, and mean-square (m.s.) stationary.
- **Hypothesis**: It is characterized by its cross spectral density function $\omega \mapsto S_P(\omega; \boldsymbol{x}, \boldsymbol{y}) : \mathbb{R} \to \mathbb{R}_+$, $\boldsymbol{x}, \boldsymbol{y} \in \Gamma$, which is positive, <u>even</u>, integrable on $\mathbb{R}_\omega$, and s.t.:

$$S_P(\omega; \boldsymbol{x}, \boldsymbol{y}) = S_P(\omega; \boldsymbol{y}, \boldsymbol{x})$$
$$S_P(\omega; \boldsymbol{x}, \boldsymbol{y}) = S_\mathrm{IN}(\boldsymbol{x}, \boldsymbol{y}) \otimes \mathbb{1}_{I_0 \cup \underline{I}_0}(\omega)\,,$$

where:

$$I_0 \cup \underline{I}_0 = \left[\omega_0 - \frac{\Delta\omega}{2}, \omega_0 + \frac{\Delta\omega}{2}\right] \bigcup \left[-\omega_0 - \frac{\Delta\omega}{2}, -\omega_0 + \frac{\Delta\omega}{2}\right].$$

Stationary incident pressure

Forced response of the structure

- The forced response of the structure $t \mapsto \boldsymbol{u}_{\mathrm{s}}^{f}(\cdot,t)$ is modelized by a stochastic process $(\boldsymbol{U}_t^f,\, t \in \mathbb{R})$ the properties of which are derived from filtering and mean-square derivation (see Lecture #1 part A).

Proposition

- *$(\boldsymbol{U}_t^f,\, t \in \mathbb{R})$ is a second order, centered stochastic process defined on $(\Omega,\mathcal{F},P)$, indexed on $\mathbb{R}$, with values in $\mathcal{C}_{\mathrm{s}}$, and mean-square stationary s.t.:*

$$\boldsymbol{S}_{\boldsymbol{U}}(\omega;\boldsymbol{x},\boldsymbol{y}) = \widehat{\mathbb{h}}_{\mathrm{tot}}(\omega)\mathcal{T}_{\boldsymbol{x}}(\omega)S_P(\omega;\boldsymbol{x},\boldsymbol{y})\mathcal{T}_{\boldsymbol{y}}(\omega)^*\widehat{\mathbb{h}}_{\mathrm{tot}}(\omega)^*\,.$$

- *The same holds for its mean-square derivatives $(\dot{\boldsymbol{U}}_t^f,\, t \in \mathbb{R})$ and $(\ddot{\boldsymbol{U}}_t^f,\, t \in \mathbb{R})$, with values in H_{s} and $\mathcal{C}_{\mathrm{s}}'$ respectively, s.t. $\boldsymbol{S}_{\dot{U}}(\omega) = \omega^2\boldsymbol{S}_{\boldsymbol{U}}(\omega)$, $S_{\ddot{\boldsymbol{U}}}(\omega) = \omega^4\boldsymbol{S}_{\boldsymbol{U}}(\omega)$.*

- Likewise, the modal forced responses $t \mapsto q^f_\alpha(t)$ for $\alpha \in \mathbb{N}^*$ are modelized by stochastic processes $(Q^f_{\alpha,t},\, t \in \mathbb{R})$ the properties of which are derived from filtering and mean-square derivation (see Lecture #1 part A).

Proposition

- *$(Q^f_{\alpha,t},\, t \in \mathbb{R})$ is a $\mathbb{R}$-valued second order, centered stochastic process defined on $(\Omega, \mathcal{F}, P)$, indexed on $\mathbb{R}$, and mean-square stationary.*
- *The same holds for its mean-square derivatives $(\dot{Q}^f_{\alpha,t},\, t \in \mathbb{R})$ and $(\ddot{Q}^f_{\alpha,t},\, t \in \mathbb{R})$.*

Stationary incident pressure

Forced response of the structure

- The finite-dimensional $\mathbb{C}^{N\times N}$ frequency response function $\widehat{\mathbb{h}}^N_{\text{tot}}(\omega)$ of the structure coupled to the fluid:

$$[\widehat{\mathbb{h}}^N_{\text{tot}}(\omega)]^{-1} = [\widehat{\mathbb{h}}^N_{\text{s}}(\omega)]^{-1} + [\widehat{\mathbb{h}}^N_{\Gamma}(\omega)]^{-1} \,,$$

where for $0 < \alpha, \beta \leqslant N$:

$$[\widehat{\mathbb{h}}^N_{\text{s}}(\omega)]^{-1}_{\alpha\beta} = M_{\text{s}}(\omega^2_{\text{s}\alpha} - \omega^2 + \mathrm{i}\eta_{\text{s}\alpha}\omega_{\text{s}\alpha}\omega)\delta_{\alpha\beta} \,,$$
$$[\widehat{\mathbb{h}}^N_{\Gamma}(\omega)]^{-1}_{\alpha\beta} = -\omega^2 M_{\alpha\beta}(\omega) + \mathrm{i}\omega D_{\alpha\beta}(\omega) \,.$$

- The $\mathbb{C}^{N\times N}$ power spectral density matrix $\boldsymbol{S}_{\boldsymbol{Q}}$ of the $\mathbb{R}^N$-valued stochastic process $\boldsymbol{Q}^f_t = (Q^f_{1,t}, \dots Q^f_{N,t})^{\mathsf{T}}$ thus reads:

$$\boldsymbol{S}_{\boldsymbol{Q}}(\omega) = \widehat{\mathbb{h}}^N_{\text{tot}}(\omega)\boldsymbol{S}_{\text{IN}}(\omega)\widehat{\mathbb{h}}^N_{\text{tot}}(\omega)^* \otimes \mathbb{1}_{I_0\cup \underline{I}_0}(\omega) \,,$$

with the $\mathbb{R}^{N\times N}$ symmetric real matrix $\boldsymbol{S}_{\text{IN}}$:

$$[\boldsymbol{S}_{\text{IN}}(\omega)]_{\alpha\beta} = \int_\Gamma\int_\Gamma \left(S_{\text{IN}}(\boldsymbol{x},\boldsymbol{y})\mathcal{T}_{\boldsymbol{y}}(\omega)\boldsymbol{\phi}_\beta(\boldsymbol{y}), \mathcal{T}_{\boldsymbol{x}}(\omega)\boldsymbol{\phi}_\alpha(\boldsymbol{x})\right)\,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \,.$$

Stationary incident pressure

Energetics of the stationary forced response

- The average mechanical energy of the structure:

$$\mathbb{E}\{\mathcal{E}_t\} = \frac{1}{2}\int_{I_0\cup\underline{I}_0} \omega^2\,\mathrm{Tr}\left[\left(\boldsymbol{M}_\mathrm{s} + \boldsymbol{M}_\Gamma\left(\frac{\omega}{c_\mathrm{f}}\right)\right)\boldsymbol{S}_{\boldsymbol{Q}}(\omega)\right]\mathrm{d}\omega$$
$$+\frac{1}{2}\int_{I_0\cup\underline{I}_0}\mathrm{Tr}\left[\boldsymbol{K}_\mathrm{s}\boldsymbol{S}_{\boldsymbol{Q}}(\omega)\right]\mathrm{d}\omega$$
$$\stackrel{\mathrm{def}}{=} |\Omega_\mathrm{s}|\int_{I_0\cup\underline{I}_0}\omega^2(\varrho_\mathrm{s}+\varrho_\mathrm{rad}(\omega_0))\,\mathrm{Tr}\,\boldsymbol{S}_{\boldsymbol{Q}}(\omega)\,\mathrm{d}\omega\,,$$

  where $\varrho_\mathrm{rad}(\omega_0)$ is an equivalent added density.

- The average radiated power:

$$\mathbb{E}\{\Pi_{\mathrm{rad},t}\} = \int_{I_0\cup\underline{I}_0}\omega^2\,\mathrm{Tr}\left[\boldsymbol{D}_\Gamma\left(\frac{\omega}{c_\mathrm{f}}\right)\boldsymbol{S}_{\boldsymbol{Q}}(\omega)\right]\mathrm{d}\omega$$
$$\stackrel{\mathrm{def}}{=}\omega_0\eta_\mathrm{rad}(\omega_0)\mathbb{E}\{\mathcal{E}_t\}\,,$$

  where $\eta_\mathrm{rad}(\omega_0)$ is an equivalent added loss factor.

- The average dissipated power thus reads:

$$\begin{aligned}\mathbb{E}\{\Pi_{\mathrm{d},t}\} &= \int_{I_0\cup\underline{I}_0}\omega^2\,\mathrm{Tr}\left[\boldsymbol{D}_{\mathrm{s}}\boldsymbol{S}_{\boldsymbol{Q}}(\omega)\right]\,\mathrm{d}\omega \\ &= \frac{M_{\mathrm{s}}}{M_{\mathrm{s}}+\varrho_{\mathrm{rad}}(\omega_0)|\Omega_{\mathrm{s}}|}\omega_0\eta_{\mathrm{s}}(\omega_0)\mathbb{E}\{\mathcal{E}_t\}\,,\end{aligned}$$

where $\eta_{\mathrm{s}}(\omega_0)$ is the average structural loss factor in I_0.

- The average input power:

$$\mathbb{E}\{\Pi_{\mathrm{IN},t}\} = \Re\mathrm{e}\int_{I_0\cup\underline{I}_0}\mathrm{i}\omega\,\mathrm{Tr}\left[\widehat{\mathbb{h}}^N_{\mathrm{tot}}(\omega)\boldsymbol{S}_{\mathrm{IN}}(\omega)\right]\,\mathrm{d}\omega\,.$$

Stationary incident pressure

Power balance for the stationary forced response

- The instantaneous power balance reads:

$$\dot{\mathcal{E}}_t = \Pi_{\mathrm{IN},t} - \Pi_{\mathrm{d},t} - \Pi_{\mathrm{rad},t} \,,$$

as an equality of second-order random variables.

- Considering the mathematical expectation with $\mathbb{E}\{\mathcal{E}_t\} = \text{Constant}$ and $\mathbb{E}\{\dot{\mathcal{E}}_t\} = 0$:

$$\begin{aligned}\mathbb{E}\{\Pi_{\mathrm{IN},t}\} &= \mathbb{E}\{\Pi_{\mathrm{d},t}\} + \mathbb{E}\{\Pi_{\mathrm{rad},t}\} \\ &= \omega_0\eta_{\mathrm{tot}}(\omega_0)\mathbb{E}\{\mathcal{E}_t\} \,,\end{aligned}$$

where:

$$\eta_{\mathrm{tot}}(\omega_0) = \eta_{\mathrm{rad}}(\omega_0) + \eta_{\mathrm{s}}(\omega_0)\sqrt{\frac{M_{\mathrm{s}}}{M_{\mathrm{s}} + \varrho_{\mathrm{rad}}(\omega_0)|\Omega_{\mathrm{s}}|}}$$

if ω_0 is somehow close to a "wet" eigenfrequency.

Summary

- The power flow between two coupled sub-systems is roughly proportional to the difference of their mechanical energies, provided that:
  1. The sub-systems are weakly dissipative;
  2. Their coupling is conservative;
  3. They are loaded by uncorrelated "rain-on-the-roof" noises, the bandwidths of which are large with respect to the equivalent bandwidths of the sub-systems modes.
- This framework is applicable to the interaction of a structure with a cavity filled with an acoustic fluid.
- The average radiated power of a structure coupled to an acoustic fluid is proportional to its mechanical energy. This effect is characterized by an equivalent added loss factor in the energy balance.
- **Outlook**: formulation for n coupled sub-systems.

Bibliography

- D. Clouteau, R. Cottereau, É. Savin: *Structural Dynamics and Acoustics*. École Centrale Paris, Châtenay-Malabry (2008).
- R.R. Craig, M.C.C. Bampton, "Coupling of substructures for dynamic analyses", *AIAA J.* **6**(7), 1313-1319 (1968).
- H.G. Davies, "Exact solutions for the response of some coupled multimodal systems", *J. Acoust. Soc. Am.* **51**(1), 387-392 (1972).
- G.M.L. Gladwell, "Branch mode analysis of vibrating systems", *J. Sound Vib.* **1**(1), 41-59 (1964).
- R.H. McNeal, "A hybrid method of component mode synthesis", *Comput. Struct.* **1**(4), 581-601 (1971).
- G. Maidanik, "Response of ribbed panels to reverberant acoustic fields", *J. Acoust. Soc. Am.* **34**(6), 809-826 (1962).
- G. Maidanik, "Response of coupled dynamic systems", *J. Sound Vib.* **46**(4), 561-583 (1976).

Bibliography

- G. Maidanik, "Variations in the boundary conditions of coupled dynamic systems", *J. Sound Vib.* **46**(4), 585-589 (1976).
- H. Morand, R. Ohayon, "Substructure variational analysis of the vibrations of coupled fluid-structure systems. Finite element results", *Int. J. Num. Methods Engng.* **14**(5), 741-755 (1979).
- R. Ohayon, C. Soize: *Structural Acoustics and Vibration.* Academic Press, London (1998).