# Settlement of a soil layer

## 1EL5000–Continuum Mechanics – Tutorial Class #4

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# Outline

# Outline

- Scalar product:

$$\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^a\,,\quad\langle\boldsymbol{a},\boldsymbol{b}\rangle=\sum_{j=1}^{a}a_jb_j=a_jb_j\,,$$

  The last equality is Einstein's summation convention.

- Tensors and tensor product (or outer product):

$$\boldsymbol{A}\in\mathbb{R}^a\to\mathbb{R}^b\,,\quad\boldsymbol{A}=\boldsymbol{a}\otimes\boldsymbol{b}\,,\quad\boldsymbol{a}\in\mathbb{R}^a\,,\boldsymbol{b}\in\mathbb{R}^b\,.$$

- Tensor application to vectors:

$$\boldsymbol{A}=\boldsymbol{a}\otimes\boldsymbol{b}\in\mathbb{R}^a\to\mathbb{R}^b\,,\boldsymbol{c}\in\mathbb{R}^b\,,\quad\boldsymbol{A}\boldsymbol{c}=\langle\boldsymbol{b},\boldsymbol{c}\rangle\,\boldsymbol{a}\,.$$

- Product of tensors ≡ composition of linear maps:

$$\boldsymbol{A}=\boldsymbol{a}\otimes\boldsymbol{b}\,,\boldsymbol{B}=\boldsymbol{c}\otimes\boldsymbol{d}\,,\quad\boldsymbol{A}\boldsymbol{B}=\langle\boldsymbol{b},\boldsymbol{c}\rangle\,\boldsymbol{a}\otimes\boldsymbol{d}\,.$$

- Scalar product of tensors:

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \mathrm{Tr}(\boldsymbol{A}\boldsymbol{B}^{\mathsf{T}}) := \boldsymbol{A} : \boldsymbol{B} = A_{jk} B_{jk} \,.$$

- Let $\{\boldsymbol{e}_j\}_{j=1}^{d}$ be a Cartesian basis in $\mathbb{R}^d$. Then:

$$\begin{aligned} a_j &= \langle \boldsymbol{a}, \boldsymbol{e}_j \rangle \,, \\ A_{jk} &= \langle \boldsymbol{A}, \boldsymbol{e}_j \otimes \boldsymbol{e}_k \rangle = \boldsymbol{A} : \boldsymbol{e}_j \otimes \boldsymbol{e}_k \\ &= \langle \boldsymbol{A}\boldsymbol{e}_k, \boldsymbol{e}_j \rangle \,, \end{aligned}$$

such that:

$$\begin{aligned} \boldsymbol{a} &= a_j \boldsymbol{e}_j \,, \\ \boldsymbol{A} &= A_{jk} \boldsymbol{e}_j \otimes \boldsymbol{e}_k \,. \end{aligned}$$

- Example: the identity matrix

$$\boldsymbol{I} = \boldsymbol{e}_j \otimes \boldsymbol{e}_j \,.$$

- Gradient of a vector function $\boldsymbol{a}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$:

$$\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a} = \frac{\partial \boldsymbol{a}}{\partial x_j} \otimes \boldsymbol{e}_j \,.$$

- Divergence of a vector function $\boldsymbol{a}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$:

$$\operatorname{div}_{\boldsymbol{x}}\boldsymbol{a} = \langle \boldsymbol{\nabla}_{\boldsymbol{x}}, \boldsymbol{a} \rangle = \operatorname{Tr}(\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a}) = \frac{\partial a_j}{\partial x_j} \,.$$

- Divergence of a tensor function $\boldsymbol{A}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$:

$$\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{A} = \frac{\partial (\boldsymbol{A}\boldsymbol{e}_j)}{\partial x_j} \,.$$

- Gradient of a vector function $\boldsymbol{a}(r,\theta,z)$:

$$\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a} = \frac{\partial \boldsymbol{a}}{\partial r} \otimes \boldsymbol{e}_r + \frac{\partial \boldsymbol{a}}{\partial \theta} \otimes \frac{\boldsymbol{e}_\theta}{r} + \frac{\partial \boldsymbol{a}}{\partial z} \otimes \boldsymbol{e}_z \,.$$

- Divergence of a vector function $\boldsymbol{a}(r,\theta,z)$:

$$\operatorname{div}_{\boldsymbol{x}}\boldsymbol{a} = \left\langle \frac{\partial \boldsymbol{a}}{\partial r}, \boldsymbol{e}_r \right\rangle + \left\langle \frac{\partial \boldsymbol{a}}{\partial \theta}, \frac{\boldsymbol{e}_\theta}{r} \right\rangle + \left\langle \frac{\partial \boldsymbol{a}}{\partial z}, \boldsymbol{e}_z \right\rangle \,.$$

- Divergence of a tensor function $\boldsymbol{A}(r,\theta,z)$:

$$\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{A} = \frac{\partial \boldsymbol{A}}{\partial r}\boldsymbol{e}_r + \frac{\partial \boldsymbol{A}}{\partial \theta}\frac{\boldsymbol{e}_\theta}{r} + \frac{\partial \boldsymbol{A}}{\partial z}\boldsymbol{e}_z \,.$$

# Outline

# Recap



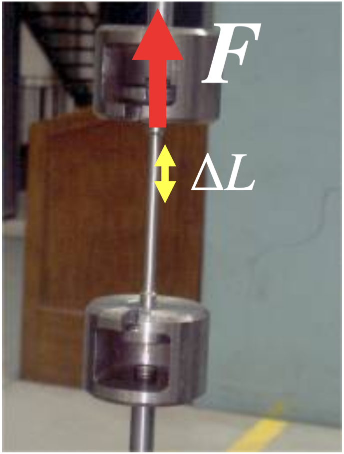
- Local equilibrium equation:

$$\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma} + \boldsymbol{f}_v = \varrho\ddot{\boldsymbol{u}}\,.$$

- Small strains assumption:

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbb{D}_{\boldsymbol{x}}\boldsymbol{u} + \mathbb{D}_{\boldsymbol{x}}\boldsymbol{u}^{\mathsf{T}})\,.$$

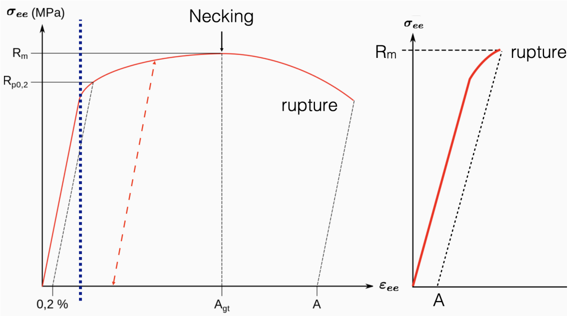
- Closure is missing: $\boldsymbol{\sigma} = f(\boldsymbol{\varepsilon})$ or $\boldsymbol{\varepsilon} = g(\boldsymbol{\sigma})$, the material constitutive equation.




Measurements along $\boldsymbol{e}$:

- $\sigma_{ee} = \dfrac{F}{S}$;
- $\varepsilon_{ee} = \dfrac{\Delta L}{L}$.

# Recap
## Traction test




- Ductile vs. fragile materials;
- At small strains $F = K\Delta L$ or $\sigma_{ee} = k\varepsilon_{ee}$;
- Generalization: $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$ where $\mathbf{C}$ is the fourth-order $(C_{jklm})$ elasticity tensor.

# Elasticity tensor
## Symmetries

- Fourth-order tensor: linear application between second-order tensors,

$$\boldsymbol{\sigma}(\boldsymbol{x},t) = \mathbf{C}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x},t)\,,$$
$$\sigma_{jk}(\boldsymbol{x},t) = C_{jklm}(\boldsymbol{x})\varepsilon_{lm}(\boldsymbol{x},t)\,,\quad \forall\boldsymbol{x}\in\Omega\,,\forall t\in\mathbb{R}\,.$$

- Minor symmetries from the symmetry of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$:

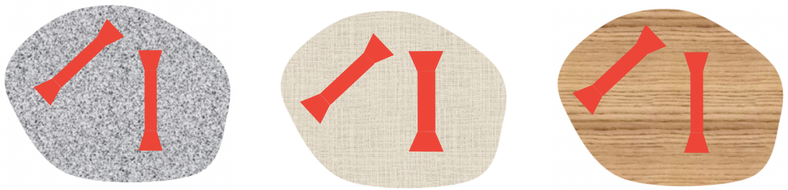
$$C_{jklm}(\boldsymbol{x}) = C_{kjlm}(\boldsymbol{x})\,,$$
$$C_{jklm}(\boldsymbol{x}) = C_{kjml}(\boldsymbol{x})\,.$$

- Major symmetry from thermodynamics first principle:

$$C_{jklm}(\boldsymbol{x}) = C_{lmjk}(\boldsymbol{x})\,.$$

- # coefficients: $81 \xrightarrow{\text{minor symmetries}} 36 \xrightarrow{\text{major symmetry}} 21.$

# Elasticity tensor
## Isotropy



- Isotropy: the constitutive equation is the same whatever the orientation of the sample is,

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}\ (\text{sample } \#1)\,,\quad \boldsymbol{\sigma}^* = \mathbf{C}\boldsymbol{\varepsilon}^*\ (\text{sample } \#2)\,.$$

- Letting $\boldsymbol{R}$ be an arbitrary rotation ($\boldsymbol{R}\boldsymbol{R}^\mathsf{T} = \boldsymbol{I}$):

$$\boldsymbol{\varepsilon}^* = \boldsymbol{R}\boldsymbol{\varepsilon}\boldsymbol{R}^\mathsf{T}\,,\quad \boldsymbol{\sigma}^* = \boldsymbol{R}\boldsymbol{\sigma}\boldsymbol{R}^\mathsf{T}\,,$$

then:

$$\boldsymbol{\sigma}^* = \boldsymbol{R}\boldsymbol{\sigma}\boldsymbol{R}^\mathsf{T} = \mathbf{C}(\boldsymbol{R}\boldsymbol{\varepsilon}\boldsymbol{R}^\mathsf{T}) \Rightarrow \boldsymbol{R}(\mathbf{C}\boldsymbol{\varepsilon})\boldsymbol{R}^\mathsf{T} = \mathbf{C}(\boldsymbol{R}\boldsymbol{\varepsilon}\boldsymbol{R}^\mathsf{T})\,.$$

- Rivlin-Ericksen theorem:

$$\boldsymbol{R}(\mathsf{C}\boldsymbol{\varepsilon})\boldsymbol{R}^{\mathsf{T}} = \mathsf{C}(\boldsymbol{R}\boldsymbol{\varepsilon}\boldsymbol{R}^{\mathsf{T}}) \Rightarrow \mathsf{C}\boldsymbol{\varepsilon} = \alpha_0\boldsymbol{I} + \alpha_1\boldsymbol{\varepsilon} + \alpha_2\boldsymbol{\varepsilon}^2\,,$$

where $\alpha_m(\mathrm{Tr}(\boldsymbol{\varepsilon}), \mathrm{Tr}(\boldsymbol{\varepsilon}^2), \mathrm{Tr}(\boldsymbol{\varepsilon}^3))$, $m = 1, 2, 3$.

Ronald Rivlin [1915–2005]

Jerald Ericksen [1925–]

# Elasticity tensor

## Isotropy

- To the leading order in $\mathrm{O}(\boldsymbol{\varepsilon})$:

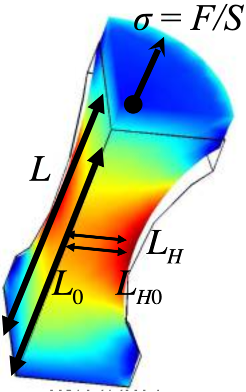
$$\boxed{\mathbf{C}\boldsymbol{\varepsilon} = \lambda\,\mathrm{Tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}}$$

  where $\lambda, \mu$ are Lamé's modulii.
- This relationship can be inverted:

$$\begin{aligned}\boldsymbol{\varepsilon} &= -\frac{\lambda}{2\mu(3\lambda+2\mu)}\,\mathrm{Tr}(\boldsymbol{\sigma})\boldsymbol{I} + \frac{1}{2\mu}\boldsymbol{\sigma} \\ &= \alpha\,\mathrm{Tr}(\boldsymbol{\sigma})\boldsymbol{I} + \beta\boldsymbol{\sigma}\end{aligned}$$

  where $\alpha, \beta$ are obtained from measurements.
- More generally $\boldsymbol{\varepsilon} = \mathbf{S}\boldsymbol{\sigma}$, where $\mathbf{S}$ is the compliance (fourth-order) tensor.

- Traction test $\boldsymbol{\sigma} = \frac{F}{S}\boldsymbol{e} \otimes \boldsymbol{e}$:




Young's modulus $E$:

$$E = \frac{\sigma_{ee}}{\varepsilon_{ee}} = \frac{\frac{F}{S}}{\frac{\Delta L}{L_0}}\,;$$

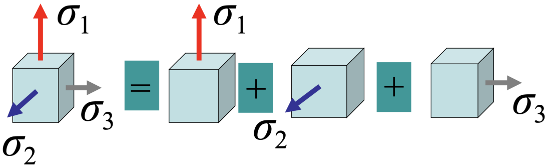
Poisson's coefficient $\nu$:

$$\nu = -\frac{\varepsilon_{hh}}{\varepsilon_{ee}} = -\frac{\frac{\Delta L_H}{L_{H0}}}{\frac{\Delta L}{L_0}}\,.$$

- $E$ is in Pa (GPa), and $\nu$ is dimensionless.

# Elasticity tensor
## Isotropy

- $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ have the same principal directions:




- Along *e.g.* the principal direction #1:

$$\varepsilon_1 = \frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E} - \nu\frac{\sigma_3}{E} \,.$$

- In the basis $(\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3)$ of the principal directions:

$$\begin{aligned}\boldsymbol{\varepsilon} &= (\tfrac{\sigma_1}{E} - \nu\tfrac{\sigma_2+\sigma_3}{E})\boldsymbol{i}_1\otimes\boldsymbol{i}_1 + (\tfrac{\sigma_2}{E} - \nu\tfrac{\sigma_1+\sigma_3}{E})\boldsymbol{i}_2\otimes\boldsymbol{i}_2 + (\tfrac{\sigma_3}{E} - \nu\tfrac{\sigma_1+\sigma_2}{E})\boldsymbol{i}_3\otimes\boldsymbol{i}_3 \\ &= \tfrac{1+\nu}{E}\boldsymbol{\sigma} - \tfrac{\nu}{E}\,\mathrm{Tr}(\boldsymbol{\sigma})\boldsymbol{I} \,.\end{aligned}$$

# Elasticity tensor
## Isotropy

- Recap: linear elastic isotropy has 2 coefficients,

$$\boldsymbol{\sigma} = \mathsf{C}\boldsymbol{\varepsilon} = \lambda\,\mathrm{Tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}\,,$$

$$\boldsymbol{\varepsilon} = \mathsf{S}\boldsymbol{\sigma} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}\mathrm{Tr}(\boldsymbol{\sigma})\boldsymbol{I}\,.$$

- $\lambda(\boldsymbol{x}), \mu(\boldsymbol{x})$ are Lamé's modulii, $E(\boldsymbol{x})$ is Young's modulus, $\nu(\boldsymbol{x})$ is Poisson's ratio:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}\,,\quad \mu = \frac{E}{2(1+\nu)}\,.$$

- $E > 0$, $\mu > 0$, $3\lambda + 2\mu > 0$, and $-1 < \nu < 0.5$.

- Thermal strains in the isotropic case with temperature gradient $\Delta T$:

$$\boldsymbol{\varepsilon}_{\mathrm{th}} = \alpha \Delta T \boldsymbol{I}$$

  where $\alpha$ is the coefficient of linear thermal expansion.

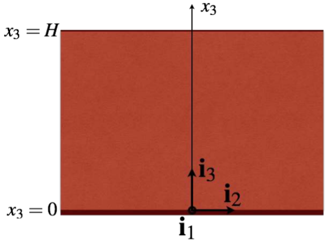
- Total strain tensor:

$$\begin{aligned}\boldsymbol{\varepsilon} &= \boldsymbol{\varepsilon}_{\mathrm{elas}} + \boldsymbol{\varepsilon}_{\mathrm{th}} \\ &= \mathbf{S}\boldsymbol{\sigma} + \alpha \Delta T \boldsymbol{I}\,.\end{aligned}$$

- Linear thermoelastic constitutive equation:

$$\boxed{\boldsymbol{\sigma} = \mathbf{C}(\boldsymbol{\varepsilon} - \alpha \Delta T \boldsymbol{I})}\,.$$

# Outline

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$$\boldsymbol{u}(\boldsymbol{x}) = u_3(x_3)\boldsymbol{i}_3\,, \quad \boldsymbol{x} = (x_1, x_2, x_3) \in \Omega = \mathbb{R}^2 \times ]0, H[\,.$$

### Question #0: $u_3(x_3)\boldsymbol{i}_3$?

- Symmetry of geometry **AND** loads.
- Translational invariance in $(x_1, x_2) \in \mathbb{R}^2$:

$$\boldsymbol{u}(x_1+\Delta_1, x_2+\Delta_2, x_3) = \boldsymbol{u}(x_1, x_2, x_3)\,, \quad \forall(\Delta_1,\Delta_2) \in \mathbb{R}^2\,, \forall\boldsymbol{x} \in \Omega\,,$$

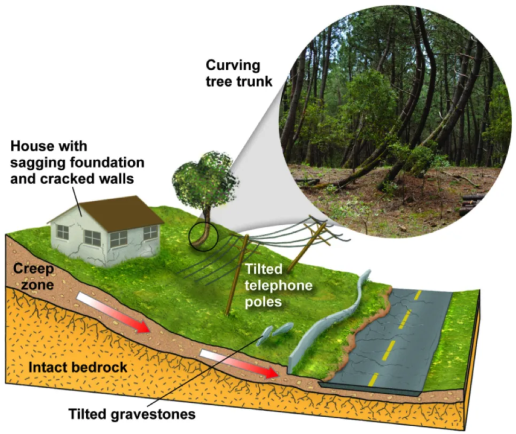
thus $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}(x_3)$.

- Mirror symmetries:

$$\begin{aligned}\langle\boldsymbol{u}(x_3), \boldsymbol{i}_1\rangle &= -\langle\boldsymbol{u}(x_3), \boldsymbol{i}_1\rangle = 0\,, \quad \forall x_3 \in ]0, H[\,,\\ \langle\boldsymbol{u}(x_3), \boldsymbol{i}_2\rangle &= -\langle\boldsymbol{u}(x_3), \boldsymbol{i}_2\rangle = 0\,, \quad \forall x_3 \in ]0, H[\,,\end{aligned}$$

thus $\boldsymbol{u}(x_3) = u_3(x_3)\boldsymbol{i}_3$.

## Question #0: $u_3(x_3)\boldsymbol{i}_3$?



### Question #1: Differential equation satisfied by $x_3 \mapsto u_3(x_3)$?

- Local equilibrium equation $\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma} + \boldsymbol{f}_v = \varrho\boldsymbol{a}$, where $\boldsymbol{a} = \mathbf{0}$ because "we can neglect the effects of inertia," and $\boldsymbol{f}_v = -\varrho g\boldsymbol{i}_3$.

1EL5000/S4

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Some algebra
Vector & tensor products
Vector & tensor analysis

Material behavior

4.1 Settlement of a soil layer


**Question #1: Differential equation satisfied by $x_3 \mapsto u_3(x_3)$?**

- Local equilibrium equation $\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma} - \varrho g \boldsymbol{i}_3 = \mathbf{0}$;
- Constitutive relation $\boldsymbol{\sigma} = \lambda(\operatorname{Tr}\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}$, because soil's "behavior is linear elastic and isotropic" and "we can adopt the framework of the infinitesimal deformation hypothesis."

**Question #1: Differential equation satisfied by $x_3 \mapsto u_3(x_3)$?**

- Local equilibrium equation $\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma} - \varrho g \boldsymbol{i}_3 = \mathbf{0}$;
- Constitutive relation $\boldsymbol{\sigma} = \lambda(\operatorname{Tr}\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}$;
- Compute $\boldsymbol{\varepsilon}$:

$$\begin{aligned}\boldsymbol{\varepsilon} &= \frac{\partial \boldsymbol{u}}{\partial x_j} \otimes_s \boldsymbol{i}_j \\ &= u_3'(x_3)\boldsymbol{i}_3 \otimes \boldsymbol{i}_3\,,\end{aligned}$$

where $u_3'(x_3) = \frac{du_3}{dx_3}$.

**Question #1: Differential equation satisfied by $x_3 \mapsto u_3(x_3)$?**

- Local equilibrium equation $\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma} - \varrho g \boldsymbol{i}_3 = \mathbf{0}$;
- Constitutive relation $\boldsymbol{\sigma} = \lambda(\operatorname{Tr}\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}$;
- $\boldsymbol{\varepsilon} = u_3'(x_3)\boldsymbol{i}_3 \otimes \boldsymbol{i}_3$;
- Compute $\boldsymbol{\sigma}$:

$$
\begin{aligned}
\boldsymbol{\sigma} &= \lambda u_3'(x_3)\boldsymbol{I} + 2\mu u_3'(x_3)\boldsymbol{i}_3 \otimes \boldsymbol{i}_3 \\
&= u_3'(x_3)\left[\lambda(\boldsymbol{i}_1 \otimes \boldsymbol{i}_1 + \boldsymbol{i}_2 \otimes \boldsymbol{i}_2) + (\lambda + 2\mu)\boldsymbol{i}_3 \otimes \boldsymbol{i}_3\right] .
\end{aligned}
$$

# Settlement of a soil layer
## Solution

### Question #1: Differential equation satisfied by $x_3 \mapsto u_3(x_3)$?

- Local equilibrium equation $\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma} - \varrho g \boldsymbol{i}_3 = \mathbf{0}$;
- Constitutive relation $\boldsymbol{\sigma} = \lambda(\mathrm{Tr}\,\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}$;
- $\boldsymbol{\varepsilon} = u_3'(x_3)\boldsymbol{i}_3 \otimes \boldsymbol{i}_3$;
- $\boldsymbol{\sigma} = u_3'(x_3)\left[\lambda(\boldsymbol{i}_1 \otimes \boldsymbol{i}_1 + \boldsymbol{i}_2 \otimes \boldsymbol{i}_2) + (\lambda + 2\mu)\boldsymbol{i}_3 \otimes \boldsymbol{i}_3\right]$;
- Compute $\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma}$ (remind that $(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{c} = \langle \boldsymbol{b}, \boldsymbol{c} \rangle \boldsymbol{a}$):

$$\begin{aligned}\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma} &= \frac{\partial \boldsymbol{\sigma}}{\partial x_j}\boldsymbol{i}_j \\ &= (\lambda + 2\mu)u_3''(x_3)\boldsymbol{i}_3\,,\end{aligned}$$

where $u_3''(x_3) = \frac{d^2 u_3}{dx_3^2}$.

**Question #1: Differential equation satisfied by $x_3 \mapsto u_3(x_3)$?**

- Local equilibrium equation $\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma} - \varrho g \boldsymbol{i}_3 = \mathbf{0}$;
- Constitutive relation $\boldsymbol{\sigma} = \lambda(\operatorname{Tr}\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}$;
- $\boldsymbol{\varepsilon} = u_3'(x_3)\boldsymbol{i}_3 \otimes \boldsymbol{i}_3$;
- $\boldsymbol{\sigma} = u_3'(x_3)\left[\lambda(\boldsymbol{i}_1 \otimes \boldsymbol{i}_1 + \boldsymbol{i}_2 \otimes \boldsymbol{i}_2) + (\lambda + 2\mu)\boldsymbol{i}_3 \otimes \boldsymbol{i}_3\right]$;
- $\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma} = (\lambda + 2\mu)u_3''(x_3)\boldsymbol{i}_3$;
- From the local equilibrium equation:

$$(\lambda + 2\mu)u_3''(x_3)\boldsymbol{i}_3 - \varrho g \boldsymbol{i}_3 = \mathbf{0} \iff \boxed{u_3''(x_3) = \frac{\varrho g}{\lambda + 2\mu}}.$$

## Question #2: Boundary conditions?

- $\{x_3 \leqslant 0\}$: a rock mass "fixed and perfectly rigid," hence $\boldsymbol{u}(x_3 \leqslant 0) = \mathbf{0}$ yielding $u_3(0) = 0$.

## Question #2: Boundary conditions?

- $u_3(0) = 0$;
- $\{x_3 = H\}$: "free of forces," hence $\boldsymbol{\sigma}\boldsymbol{n}\,|_{x_3=H} = \mathbf{0}$ that is:

## Question #2: Boundary conditions?

- $u_3(0) = 0$;
- $\{x_3 = H\}$: "free of forces," hence $\boldsymbol{\sigma}\boldsymbol{n}\,|_{x_3=H} = \mathbf{0}$ that is:

$$\begin{aligned}\boldsymbol{\sigma}\boldsymbol{n}\,|_{x_3=H} &= \boldsymbol{\sigma}\boldsymbol{i}_3\,|_{x_3=H} \\ &= (\lambda + 2\mu)u_3'(H)\boldsymbol{i}_3 \\ \Big( &= u_3'(H)[\lambda(\boldsymbol{i}_1\otimes\boldsymbol{i}_1+\boldsymbol{i}_2\otimes\boldsymbol{i}_2)+(\lambda+2\mu)\boldsymbol{i}_3\otimes\boldsymbol{i}_3]\boldsymbol{i}_3 \Big)\end{aligned}$$

Therefore $u_3'(H) = 0$.

**Question #2: $x_3 \mapsto u_3(x_3)$?**

- $u_3(0) = 0$;
- $u_3'(H) = 0$;
- Therefore:

$$u_3'(x_3) = \frac{\varrho g(x_3 - H)}{\lambda + 2\mu}\,,$$
$$u_3(x_3) = \frac{\varrho g x_3(x_3 - 2H)}{2(\lambda + 2\mu)}\,.$$

# Settlement of a soil layer
## Solution

### Question #3: $\sigma_{\mathrm{eq}} \leqslant \sigma_e$?

- von Mises equivalent stress $\sigma_{\mathrm{eq}} = \sqrt{3J_2(\boldsymbol{\sigma}^D)}$.

# Settlement of a soil layer

## Solution

### Question #3: $\sigma_{\text{eq}} \leqslant \sigma_e$?

- von Mises equivalent stress $\sigma_{\text{eq}} = \sqrt{3J_2(\boldsymbol{\sigma}^D)}$;
- Compute $\boldsymbol{\sigma}^D = \boldsymbol{\sigma} - \frac{\operatorname{Tr}\boldsymbol{\sigma}}{3}\boldsymbol{I}$:

$$
\begin{aligned}
\boldsymbol{\sigma} &= u_3'(x_3)\left[\lambda(\boldsymbol{i}_1 \otimes \boldsymbol{i}_1 + \boldsymbol{i}_2 \otimes \boldsymbol{i}_2) + (\lambda + 2\mu)\boldsymbol{i}_3 \otimes \boldsymbol{i}_3\right] \\
\operatorname{Tr}\boldsymbol{\sigma} &= (3\lambda + 2\mu)u_3'(x_3) \\
\boldsymbol{\sigma}^D &= \frac{2}{3}\mu u_3'(x_3)\left[-\boldsymbol{i}_1 \otimes \boldsymbol{i}_1 - \boldsymbol{i}_2 \otimes \boldsymbol{i}_2 + 2\boldsymbol{i}_3 \otimes \boldsymbol{i}_3\right]
\end{aligned}
$$

### Question #3: $\sigma_{\mathrm{eq}} \leqslant \sigma_e$?

- von Mises equivalent stress $\sigma_{\mathrm{eq}} = \sqrt{3J_2(\boldsymbol{\sigma}^D)}$;
- $\boldsymbol{\sigma}^D = \frac{2}{3}\mu u_3'(x_3)[-\boldsymbol{i}_1 \otimes \boldsymbol{i}_1 - \boldsymbol{i}_2 \otimes \boldsymbol{i}_2 + 2\boldsymbol{i}_3 \otimes \boldsymbol{i}_3]$;
- Compute $J_2(\boldsymbol{\sigma}^D) = \frac{1}{2}\,\mathrm{Tr}(\boldsymbol{\sigma}^{D2})$:

$$
\begin{aligned}
\boldsymbol{\sigma}^D &= \frac{2}{3}\mu u_3'(x_3)\left[-\boldsymbol{i}_1 \otimes \boldsymbol{i}_1 - \boldsymbol{i}_2 \otimes \boldsymbol{i}_2 + 2\boldsymbol{i}_3 \otimes \boldsymbol{i}_3\right] \\
(\boldsymbol{\sigma}^D)^2 &= \frac{4}{9}(\mu u_3'(x_3))^2\left[\boldsymbol{i}_1 \otimes \boldsymbol{i}_1 + \boldsymbol{i}_2 \otimes \boldsymbol{i}_2 + 4\boldsymbol{i}_3 \otimes \boldsymbol{i}_3\right] \\
J_2(\boldsymbol{\sigma}^D) &= \frac{4}{3}(\mu u_3'(x_3))^2\,.
\end{aligned}
$$

Therefore $\sigma_{\mathrm{eq}} = 2\mu\,|u_3'(x_3)|$.

## Question #3: $\sigma_{\mathrm{eq}} \leqslant \sigma_e$?

- von Mises equivalent stress $\sigma_{\mathrm{eq}} = \sqrt{3J_2(\boldsymbol{\sigma}^D)}$;
- $\boldsymbol{\sigma}^D = \frac{2}{3}\mu u_3'(x_3)[-\boldsymbol{i}_1 \otimes \boldsymbol{i}_1 - \boldsymbol{i}_2 \otimes \boldsymbol{i}_2 + 2\boldsymbol{i}_3 \otimes \boldsymbol{i}_3]$;
- $\sigma_{\mathrm{eq}} = 2\mu \left|u_3'(x_3)\right|$;
- $\sigma_{\mathrm{eq}} \leqslant \sigma_e$ provided that:

$$\frac{2\mu}{\lambda + 2\mu}\varrho g \left|x_3 - H\right| \leqslant \sigma_e \,, \quad \forall x_3 \in [0, H]\,.$$

### Question #3: $\sigma_{\mathrm{eq}} \leqslant \sigma_e$?

- $\sigma_{\mathrm{eq}} \leqslant \sigma_e$ provided that:

$$\frac{2\mu}{\lambda + 2\mu}\varrho g \,|x_3 - H| \leqslant \sigma_e \,, \quad \forall x_3 \in [0, H] \,.$$

- This criterion is first reached where $|x_3 - H|$ is maximum *i.e.* $x_3 = 0$;
- Conversely, the von Mises criterion is fulfilled provided that $H \leqslant H_{\max}$ where:

$$\boxed{H_{\max} = \frac{(\lambda + 2\mu)\sigma_e}{2\mu\varrho g}} \,.$$

## Question #3: $\sigma_{\text{eq}} \leqslant \sigma_e$?

- $\sigma_{\text{eq}} \leqslant \sigma_e$ provided that $H \leqslant H_{\max}$ where:

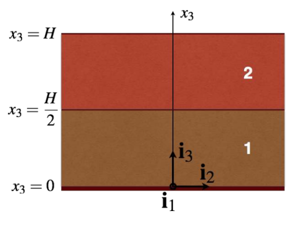
$$H_{\max} = \frac{(\lambda + 2\mu)\sigma_e}{2\mu\varrho g}\,.$$

- **NB**: $\mu = \frac{E}{2(1+\nu)}$, $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$, thus:

$$H_{\max} = \left(\frac{1-\nu}{1-2\nu}\right)\frac{\sigma_e}{\varrho g}$$

independently of the Young's modulus $E$.

## Question #4: Two layers?

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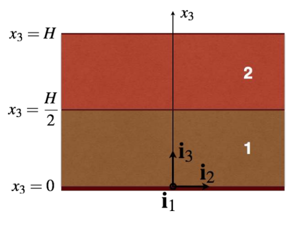
- Local equilibrium equation: unchanged! $\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma} = \varrho g \boldsymbol{i}_3$.

## Question #4: Two layers?

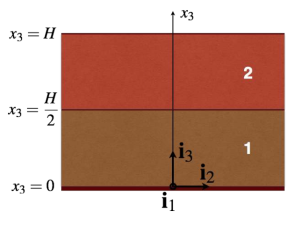





- Constitutive relation:

$$\boldsymbol{\sigma} = \begin{cases} = \lambda_1 (\mathrm{Tr}\,\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu_1 \boldsymbol{\varepsilon}\,, & \boldsymbol{x} \in \Omega_1 = \mathbb{R}^2 \times \left]0, \frac{H}{2}\right[\,, \\ = \lambda_2 (\mathrm{Tr}\,\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu_2 \boldsymbol{\varepsilon}\,, & \boldsymbol{x} \in \Omega_2 = \mathbb{R}^2 \times \left]\frac{H}{2}, H\right[\,. \end{cases}$$

## Question #4: Two layers?

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- Boundary conditions:

$$
\begin{aligned}
\boldsymbol{u}(x_3) = \mathbf{0} \quad & \text{on } \{x_3 = 0\}\,, \\
\boldsymbol{\sigma}(x_3)\boldsymbol{i}_3 = \mathbf{0} \quad & \text{on } \{x_3 = H\}\,, \\
\boldsymbol{u}(x_3)|_{\Omega_1} = \boldsymbol{u}(x_3)|_{\Omega_2}\,,\ \boldsymbol{\sigma}(x_3)\boldsymbol{i}_3|_{\Omega_1} + \boldsymbol{\sigma}(x_3)(-\boldsymbol{i}_3)|_{\Omega_2} = \mathbf{0} \quad & \text{on } \left\{x_3 = \tfrac{H}{2}\right\}.
\end{aligned}
$$

1EL5000/S4

É. Savin

Some algebra
Vector & tensor products
Vector & tensor analysis

Material behavior

4.1 Settlement of a soil layer

## Question #4: Two layers?

- From Question #1:

$$u_3(x_3)|_{\Omega_1} = \frac{\varrho_1 g x_3^2}{2(\lambda_1+2\mu_1)} + A_1 x_3 + B_1 \,,$$

$$u_3(x_3)|_{\Omega_2} = \frac{\varrho_2 g x_3^2}{2(\lambda_2+2\mu_2)} + A_2 x_3 + B_2 \,.$$

- From boundary conditions:

$$0 = \frac{\varrho_2 g H}{\lambda_2+2\mu_2} + A_2 \,,$$

$$\varrho_1 g \frac{H}{2} + (\lambda_1+2\mu_1)A_1 = \varrho_2 g \frac{H}{2} + (\lambda_2+2\mu_2)A_2 \,,$$

$$\frac{\varrho_1 g H^2}{8(\lambda_1+2\mu_1)} + A_1 \frac{H}{2} + B_1 = \frac{\varrho_2 g H^2}{8(\lambda_2+2\mu_2)} + A_2 \frac{H}{2} + B_2 \,,$$

$$B_1 = 0 \,.$$

## Question #4: Two layers?

- From Question #1:

$$
\begin{aligned}
u_3(x_3)|_{\Omega_1} &= \frac{\varrho_1 g x_3^2}{2(\lambda_1+2\mu_1)} + A_1 x_3 + B_1 \,, \\
u_3(x_3)|_{\Omega_2} &= \frac{\varrho_2 g x_3^2}{2(\lambda_2+2\mu_2)} + A_2 x_3 + B_2 \,.
\end{aligned}
$$

- From boundary conditions:

$$
\begin{aligned}
A_2 &= -\frac{\varrho_2 g H}{\lambda_2+2\mu_2} \,, \\
A_1 &= -\frac{(\varrho_1+\varrho_2) g H}{2(\lambda_1+2\mu_1)} \,, \\
B_2 &= \frac{gH^2}{8}\left(\frac{3\varrho_2}{\lambda_2+2\mu_2} - \frac{\varrho_1+2\varrho_2}{\lambda_1+2\mu_1}\right) , \\
B_1 &= 0 \,.
\end{aligned}
$$