Interference fit

1EL5000–Continuum Mechanics – Tutorial Class #11

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Outline

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- Scalar product:

$$\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^a\,,\quad\langle\boldsymbol{a},\boldsymbol{b}\rangle=\sum_{j=1}^{a}a_jb_j=a_jb_j\,,$$

  The last equality is Einstein's summation convention.

- Tensors and tensor product (or outer product):

$$\boldsymbol{A}\in\mathbb{R}^a\to\mathbb{R}^b\,,\quad\boldsymbol{A}=\boldsymbol{a}\otimes\boldsymbol{b}\,,\quad\boldsymbol{a}\in\mathbb{R}^a\,,\boldsymbol{b}\in\mathbb{R}^b\,.$$

- Tensor application to vectors:

$$\boldsymbol{A}=\boldsymbol{a}\otimes\boldsymbol{b}\in\mathbb{R}^a\to\mathbb{R}^b\,,\boldsymbol{c}\in\mathbb{R}^b\,,\quad\boldsymbol{A}\boldsymbol{c}=\langle\boldsymbol{b},\boldsymbol{c}\rangle\,\boldsymbol{a}\,.$$

- Product of tensors ≡ composition of linear maps:

$$\boldsymbol{A}=\boldsymbol{a}\otimes\boldsymbol{b}\,,\boldsymbol{B}=\boldsymbol{c}\otimes\boldsymbol{d}\,,\quad\boldsymbol{A}\boldsymbol{B}=\langle\boldsymbol{b},\boldsymbol{c}\rangle\,\boldsymbol{a}\otimes\boldsymbol{d}\,.$$

- Scalar product of tensors:

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \mathrm{Tr}(\boldsymbol{A}\boldsymbol{B}^\mathsf{T}) := \boldsymbol{A} : \boldsymbol{B} = A_{jk} B_{jk} \,.$$

- Let $\{\boldsymbol{e}_j\}_{j=1}^d$ be a Cartesian basis in $\mathbb{R}^d$. Then:

$$\begin{aligned} a_j &= \langle \boldsymbol{a}, \boldsymbol{e}_j \rangle \,, \\ A_{jk} &= \langle \boldsymbol{A}, \boldsymbol{e}_j \otimes \boldsymbol{e}_k \rangle = \boldsymbol{A} : \boldsymbol{e}_j \otimes \boldsymbol{e}_k \\ &= \langle \boldsymbol{A}\boldsymbol{e}_k, \boldsymbol{e}_j \rangle \,, \end{aligned}$$

such that:

$$\begin{aligned} \boldsymbol{a} &= a_j \boldsymbol{e}_j \,, \\ \boldsymbol{A} &= A_{jk} \boldsymbol{e}_j \otimes \boldsymbol{e}_k \,. \end{aligned}$$

- Example: the identity matrix

$$\boldsymbol{I} = \boldsymbol{e}_j \otimes \boldsymbol{e}_j \,.$$

- Gradient of a vector function $\boldsymbol{a}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$:

$$\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a} = \frac{\partial \boldsymbol{a}}{\partial x_j} \otimes \boldsymbol{e}_j \,.$$

- Divergence of a vector function $\boldsymbol{a}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$:

$$\mathrm{div}_{\boldsymbol{x}}\boldsymbol{a} = \langle \boldsymbol{\nabla}_{\boldsymbol{x}}, \boldsymbol{a} \rangle = \mathrm{Tr}(\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a}) = \frac{\partial a_j}{\partial x_j} \,.$$

- Divergence of a tensor function $\boldsymbol{A}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$:

$$\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{A} = \frac{\partial (\boldsymbol{A}\boldsymbol{e}_j)}{\partial x_j} \,.$$

- Gradient of a vector function $\boldsymbol{a}(r,\theta,z)$:

$$\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a} = \frac{\partial \boldsymbol{a}}{\partial r} \otimes \boldsymbol{e}_r + \frac{\partial \boldsymbol{a}}{\partial \theta} \otimes \frac{\boldsymbol{e}_\theta}{r} + \frac{\partial \boldsymbol{a}}{\partial z} \otimes \boldsymbol{e}_z \,.$$

- Divergence of a vector function $\boldsymbol{a}(r,\theta,z)$:

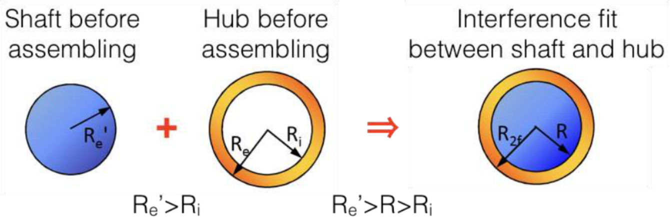
$$\mathrm{div}_{\boldsymbol{x}}\boldsymbol{a} = \left\langle \frac{\partial \boldsymbol{a}}{\partial r}, \boldsymbol{e}_r \right\rangle + \left\langle \frac{\partial \boldsymbol{a}}{\partial \theta}, \frac{\boldsymbol{e}_\theta}{r} \right\rangle + \left\langle \frac{\partial \boldsymbol{a}}{\partial z}, \boldsymbol{e}_z \right\rangle \,.$$

- Divergence of a tensor function $\boldsymbol{A}(r,\theta,z)$:

$$\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{A} = \frac{\partial \boldsymbol{A}}{\partial r}\boldsymbol{e}_r + \frac{\partial \boldsymbol{A}}{\partial \theta}\frac{\boldsymbol{e}_\theta}{r} + \frac{\partial \boldsymbol{A}}{\partial z}\boldsymbol{e}_z \,.$$

Outline

Interference fit
Setup



- Hub:

$$\Omega_m = \{\boldsymbol{p} = (r,\theta,z);\ r \in (R_i, R_e), \theta \in [0,2\pi), z \in (0, H_m)\}\ ;$$

- Shaft:

$$\Omega_a = \left\{\boldsymbol{p} = (r,\theta,z);\ r \in \left(0, R_i + \frac{s}{2}\right), \theta \in [0,2\pi), z \in (0, H_a)\right\}\ .$$

Part 1: Practical realization

Question #1: Equations for the heated hub Ω_m?

- Equilibrium equation $\mathbf{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_v = \varrho\ddot{\boldsymbol{u}}$ within the "infinitesimal deformation hypothesis;"

Interference fit

Solution

Question #1: Equations for the heated hub Ω_m?

- Equilibrium equation $\mathbf{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_v = \varrho\ddot{\boldsymbol{u}}$ within the "infinitesimal deformation hypothesis;"
- "The effects of inertia and the action of gravity can be neglected," hence $\varrho\ddot{\boldsymbol{u}} = \mathbf{0}$, $\boldsymbol{f}_v = \mathbf{0}$, and $\mathbf{Div}\,\boldsymbol{\sigma} = \mathbf{0}$;

Question #1: Equations for the heated hub Ω_m?

- Equilibrium equation $\mathbf{Div}\,\boldsymbol{\sigma} = \mathbf{0}$;
- Constitutive equation for a "linear thermoelastic, isotropic and homogeneous" material (in compliance):

$$\begin{aligned}\boldsymbol{\varepsilon} &= \boldsymbol{\varepsilon}_{\text{elas}} + \boldsymbol{\varepsilon}_{\text{th}} \\ &= \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}(\operatorname{Tr}\boldsymbol{\sigma})\boldsymbol{I} + \alpha\Delta T\boldsymbol{I}\,,\end{aligned}$$

or (in stiffness):

$$\begin{aligned}\boldsymbol{\sigma} &= \mathbf{C}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_{\text{th}}) \\ &= \lambda(\operatorname{Tr}\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon} - (3\lambda+2\mu)\alpha\Delta T\boldsymbol{I}\,;\end{aligned}$$

Interference fit

Solution

Question #1: Equations for the heated hub Ω_m?

- Equilibrium equation $\mathbf{Div}\,\boldsymbol{\sigma} = \mathbf{0}$;
- Constitutive equation:

$$\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}(\mathrm{Tr}\,\boldsymbol{\sigma})\boldsymbol{I} + \alpha\Delta T\boldsymbol{I}\,,$$
$$\boldsymbol{\sigma} = \lambda(\mathrm{Tr}\,\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon} - (3\lambda+2\mu)\alpha\Delta T\boldsymbol{I}\,;$$

- Initial conditions: useless for statics;

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$$\boldsymbol{\sigma} = \lambda(\operatorname{Tr}\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon} - (3\lambda+2\mu)\alpha\Delta T\boldsymbol{I}\,;$$

- Initial conditions: useless for statics;
- Boundary conditions: "the hub is simply placed at $z=0$ on a fixed and perfectly rigid support on which it can slide with no friction,"

$$\langle \boldsymbol{u}, \boldsymbol{i}_z\rangle|_{\{z=0\}} = 0\,,$$
$$\langle \boldsymbol{\sigma}(-\boldsymbol{i}_z), \boldsymbol{i}_x\rangle|_{\{z=0\}} = \langle \boldsymbol{\sigma}(-\boldsymbol{i}_z), \boldsymbol{i}_y\rangle|_{\{z=0\}} = 0\,;$$

Interference fit
Solution

Question #1: Equations for the heated hub Ω_m?

- Equilibrium equation $\mathbf{Div}\,\boldsymbol{\sigma} = \mathbf{0}$;
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$$\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}(\mathrm{Tr}\,\boldsymbol{\sigma})\boldsymbol{I} + \alpha\Delta T\boldsymbol{I}\,,$$
$$\boldsymbol{\sigma} = \lambda(\mathrm{Tr}\,\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon} - (3\lambda+2\mu)\alpha\Delta T\boldsymbol{I}\,;$$

- Initial conditions: useless for statics;
- Boundary conditions: free surface at $\{r = R_i\}$, at $\{r = R_e\}$, and at $\{z = H_m\}$,

$$\boldsymbol{\sigma}\boldsymbol{i}_r(\theta)|_{\{r=R_e\}} = \mathbf{0}\,,$$
$$\boldsymbol{\sigma}(-\boldsymbol{i}_r(\theta))|_{\{r=R_i\}} = \mathbf{0}\,,$$
$$\boldsymbol{\sigma}\boldsymbol{i}_z|_{\{z=H_m\}} = \mathbf{0}\,.$$

Interference fit

Solution

Question #1: Equations for the heated hub Ω_m?

- Recap:

$$\mathbf{Div}\,\boldsymbol{\sigma} = \mathbf{0}\,,$$

$$\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}(\operatorname{Tr}\boldsymbol{\sigma})\boldsymbol{I} + \alpha\Delta T\boldsymbol{I}\,,$$

$$\boldsymbol{\sigma} = \lambda(\operatorname{Tr}\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon} - (3\lambda+2\mu)\alpha\Delta T\boldsymbol{I}\,,$$

$$\langle \boldsymbol{u}, \boldsymbol{i}_z\rangle|_{\{z=0\}} = 0\,,$$

$$\langle \boldsymbol{\sigma}\boldsymbol{i}_z, \boldsymbol{i}_x\rangle|_{\{z=0\}} = \langle \boldsymbol{\sigma}\boldsymbol{i}_z, \boldsymbol{i}_y\rangle|_{\{z=0\}} = 0\,,$$

$$\boldsymbol{\sigma}\boldsymbol{i}_r(\theta)|_{\{r=R_e\}} = \mathbf{0}\,,$$

$$\boldsymbol{\sigma}\boldsymbol{i}_r(\theta)|_{\{r=R_i\}} = \mathbf{0}\,,$$

$$\boldsymbol{\sigma}\boldsymbol{i}_z|_{\{z=H_m\}} = \mathbf{0}\,.$$

Question #2: Is $\boldsymbol{\sigma} = \mathbf{0}$ a solution? $\boldsymbol{u}$?

- $\boldsymbol{\sigma} = \mathbf{0}$ satisfies the equilibrium equation and boundary conditions;

Question #2: Is $\boldsymbol{\sigma} = \mathbf{0}$ a solution? $\boldsymbol{u}$?

- $\boldsymbol{\sigma} = \mathbf{0}$ satisfies the equilibrium equation and boundary conditions;
- Then $\boldsymbol{\varepsilon} = \alpha \Delta T \boldsymbol{I}$ is constant;

Interference fit
Solution

Question #2: Is $\boldsymbol{\sigma} = \mathbf{0}$ a solution? $\boldsymbol{u}$?

- $\boldsymbol{\sigma} = \mathbf{0}$ satisfies the equilibrium equation and boundary conditions;
- Then $\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbb{D}_{\boldsymbol{x}}\boldsymbol{u} + \mathbb{D}_{\boldsymbol{x}}\boldsymbol{u}^{\mathsf{T}}) = \alpha\Delta T\boldsymbol{I}$ is constant;
- Consequently the solution for $\boldsymbol{u}$ is $\boldsymbol{u} = \alpha\Delta T\boldsymbol{x}$ up to a rigid body motion in the (x, y) plane;
- Indeed $\mathbb{D}_{\boldsymbol{x}}(\boldsymbol{A}\boldsymbol{x}) = \boldsymbol{A}_j \otimes \boldsymbol{e}_j$ where $\boldsymbol{A}_j$ is the j^{th} column of $\boldsymbol{A}$, and $\boldsymbol{\varepsilon}_{\boldsymbol{x}}(\boldsymbol{A}\boldsymbol{x}) = \boldsymbol{A}_j \otimes_s \boldsymbol{e}_j$ which must be proportional to the identitity.

Question #3: ΔT?

- The solution for $\boldsymbol{u}$ is $\boldsymbol{u} = \alpha\Delta T\boldsymbol{x}$ up to a rigid body motion in the (x, y) plane;
- Thus $u_r(\boldsymbol{x}) = \alpha\Delta Tr$ and one must choose ΔT such that $u_r(R_i, \theta, z) = \frac{s}{2}$, or:

$$\Delta T = \frac{s}{2\alpha R_i}\,;$$

Interference fit
Solution

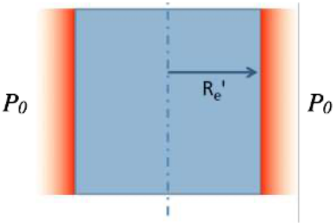
Question #3: ΔT?

- The solution for $\boldsymbol{u}$ is $\boldsymbol{u} = \alpha \Delta T \boldsymbol{x}$ up to a rigid body motion in the (x, y) plane;
- Thus $u_r(\boldsymbol{x}) = \alpha \Delta T r$ and one must choose ΔT such that $u_r(R_i, \theta, z) = \frac{s}{2}$, or:

$$\Delta T = \frac{s}{2\alpha R_i}\,;$$

- N.A.: $s/2R_i = 10^{-3}$, $\alpha = 10^{-5}\,\mathrm{K}^{-1} \Rightarrow \Delta T = 100\ {}^\circ\mathrm{K}$.

Part 2: Solid cylinder under external pressure



Question #4: Equations for the shaft Ω_a?

- Equilibrium equation $\mathbf{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_v = \varrho\ddot{\boldsymbol{u}}$ within the "infinitesimal deformation hypothesis;"

Interference fit

Solution

Question #4: Equations for the shaft Ω_a?

- Equilibrium equation $\mathbf{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_v = \varrho\ddot{\boldsymbol{u}}$ within the "infinitesimal deformation hypothesis;"
- "The effects of inertia and the action of gravity can be neglected," hence $\varrho\ddot{\boldsymbol{u}} = \mathbf{0}$, $\boldsymbol{f}_v = \mathbf{0}$, and $\mathbf{Div}\,\boldsymbol{\sigma} = \mathbf{0}$;

Question #4: Equations for the shaft Ω_a?

- Equilibrium equation $\mathbf{Div}\,\boldsymbol{\sigma} = \mathbf{0}$;
- Constitutive equation for a "linear elastic, isotropic, homogeneous" material:

$$\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}(\mathrm{Tr}\,\boldsymbol{\sigma})\boldsymbol{I}\,,$$
$$\boldsymbol{\sigma} = \lambda(\mathrm{Tr}\,\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}\,;$$

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- Initial conditions: useless for statics;
- Boundary conditions: "this contact pressure P_0 is uniform,"

$$\boldsymbol{\sigma}\boldsymbol{i}_r(\theta)|_{\{r=R'_e\}} = -P_0\boldsymbol{i}_r(\theta)\,;$$

Question #4: Equations for the shaft Ω_a?

- Equilibrium equation $\mathbf{Div}\,\boldsymbol{\sigma} = \mathbf{0}$;
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$$\boldsymbol{\sigma} = \lambda(\mathrm{Tr}\,\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}\,;$$

- Initial conditions: useless for statics;
- Boundary conditions: "the faces at both ends $z = 0$ and $z = H_a$ are considered to be free of forces,"

$$\boldsymbol{\sigma}(-\boldsymbol{i}_z)|_{\{z=0\}} = \mathbf{0}\,,$$
$$\boldsymbol{\sigma}\boldsymbol{i}_z|_{\{z=H_a\}} = \mathbf{0}\,.$$

Question #4: Equations for the shaft Ω_a?

- Recap:

$$\mathbf{Div}\,\boldsymbol{\sigma} = \mathbf{0}\,,$$

$$\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}(\mathrm{Tr}\,\boldsymbol{\sigma})\boldsymbol{I}\,,$$

$$\boldsymbol{\sigma} = \lambda(\mathrm{Tr}\,\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}\,,$$

$$\boldsymbol{\sigma}\boldsymbol{i}_r(\theta)|_{\{r=R'_e\}} = -P_0\boldsymbol{i}_r(\theta)\,,$$

$$\boldsymbol{\sigma}\boldsymbol{i}_z|_{\{z=0\}} = \mathbf{0}\,,$$

$$\boldsymbol{\sigma}\boldsymbol{i}_z|_{\{z=H_a\}} = \mathbf{0}\,.$$

Question #5: σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}?

- We seek for a stress tensor of the form:

$$\boldsymbol{\sigma}_{fc}(\boldsymbol{x}) = \sigma_{rr}\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta) + \sigma_{\theta\theta}\boldsymbol{i}_\theta(\theta)\otimes\boldsymbol{i}_\theta(\theta) + \sigma_{zz}\boldsymbol{i}_z\otimes\boldsymbol{i}_z\,;$$

- From the boundary conditions:

$$\begin{aligned}\boldsymbol{\sigma}\boldsymbol{i}_r(\theta)|_{\{r=R'_e\}} &= \sigma_{rr}\boldsymbol{i}_r(\theta) = -P_0\boldsymbol{i}_r(\theta)\,,\\ \boldsymbol{\sigma}(-\boldsymbol{i}_z)|_{\{z=0\}} &= -\sigma_{zz}\boldsymbol{i}_z = \boldsymbol{0}\,,\\ \boldsymbol{\sigma}\boldsymbol{i}_z|_{\{z=H_a\}} &= \sigma_{zz}\boldsymbol{i}_z = \boldsymbol{0}\,;\end{aligned}$$

- Therefore $\sigma_{rr} = -P_0$ and $\sigma_{zz} = 0$;

Interference fit

Solution

Question #5: σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}?

- We seek for a stress tensor of the form:

$$\boldsymbol{\sigma}_{fc}(\boldsymbol{x}) = -P_0 \boldsymbol{i}_r(\theta) \otimes \boldsymbol{i}_r(\theta) + \sigma_{\theta\theta} \boldsymbol{i}_\theta(\theta) \otimes \boldsymbol{i}_\theta(\theta)\,;$$

- From the equilibrium equation:

$$\begin{aligned}
\mathbf{Div}\,\boldsymbol{\sigma}_{fc} &= \frac{\partial \boldsymbol{\sigma}_{fc}}{\partial r}\boldsymbol{i}_r(\theta) + \frac{\partial \boldsymbol{\sigma}_{fc}}{\partial \theta}\frac{\boldsymbol{i}_\theta(\theta)}{r} + \frac{\partial \boldsymbol{\sigma}_{fc}}{\partial z}\boldsymbol{i}_z \\
&= \left(-P_0 \frac{\partial(\boldsymbol{i}_r(\theta) \otimes \boldsymbol{i}_r(\theta))}{\partial \theta} + \sigma_{\theta\theta}\frac{\partial(\boldsymbol{i}_\theta(\theta) \otimes \boldsymbol{i}_\theta(\theta))}{\partial \theta}\right)\frac{\boldsymbol{i}_\theta(\theta)}{r} \\
&= -2\Big(P_0 \boldsymbol{i}_\theta(\theta) \otimes_s \boldsymbol{i}_r(\theta) + \sigma_{\theta\theta}\boldsymbol{i}_r(\theta) \otimes_s \boldsymbol{i}_\theta(\theta)\Big)\frac{\boldsymbol{i}_\theta(\theta)}{r} \\
&= -\left(\frac{P_0 + \sigma_{\theta\theta}}{r}\right)\boldsymbol{i}_r(\theta)\,;
\end{aligned}$$

Question #5: σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}?

- But $\mathbf{Div}\,\boldsymbol{\sigma}_{fc} = \mathbf{0}$ and therefore $\sigma_{\theta\theta} = -P_0$;
- Finally:

$$\boldsymbol{\sigma}_{fc}(\boldsymbol{x}) = -P_0\Big(\boldsymbol{i}_r(\theta) \otimes \boldsymbol{i}_r(\theta) + \boldsymbol{i}_\theta(\theta) \otimes \boldsymbol{i}_\theta(\theta)\Big)\,.$$

satisfies all equations of the shaft.

Question #6: Displacement field $\boldsymbol{u}_{fc}$ for $\boldsymbol{\sigma}_{fc}$?

- From the constitutive equation:

$$
\begin{aligned}
\boldsymbol{\varepsilon}_{fc} &= \frac{1+\nu}{E}\boldsymbol{\sigma}_{fc} - \frac{\nu}{E}(\mathrm{Tr}\,\boldsymbol{\sigma}_{fc})\boldsymbol{I} \\
&= -\frac{P_0}{E}\left[(1+\nu)\left(\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta)+\boldsymbol{i}_\theta(\theta)\otimes\boldsymbol{i}_\theta(\theta)\right)-2\nu\boldsymbol{I}\right] \\
&= -\frac{P_0}{E}\left[(1-\nu)\left(\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta)+\boldsymbol{i}_\theta(\theta)\otimes\boldsymbol{i}_\theta(\theta)\right)-2\nu\boldsymbol{i}_z\otimes\boldsymbol{i}_z\right]\,;
\end{aligned}
$$

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&= -\frac{P_0}{E}\left[(1-\nu)\left(\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta)+\boldsymbol{i}_\theta(\theta)\otimes\boldsymbol{i}_\theta(\theta)\right)-2\nu\boldsymbol{i}_z\otimes\boldsymbol{i}_z\right]\,;
\end{aligned}
$$

- Therefore $\varepsilon_{r\theta} = \varepsilon_{rz} = \varepsilon_{\theta z} = 0$ and:

$$
\begin{aligned}
\varepsilon_{rr} = \varepsilon_{\theta\theta} &= \frac{P_0}{E}(\nu - 1)\,, \\
\varepsilon_{zz} &= \frac{2\nu P_0}{E}\,;
\end{aligned}
$$

Question #6: Displacement field $\boldsymbol{u}_{fc}$ for $\boldsymbol{\sigma}_{fc}$?

- From the symmetry of the **geometry and loads** one can choose $\boldsymbol{u}(r,\theta,z) = u_r(r)\boldsymbol{i}_r(\theta) + u_z(z)\boldsymbol{i}_z$ such that:

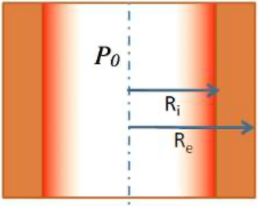
$$\varepsilon_{rr} = \frac{du_r}{dr} = \frac{P_0}{E}(\nu - 1)\,,$$
$$\varepsilon_{\theta\theta} = \frac{u_r}{r} = \frac{P_0}{E}(\nu - 1)\,,$$
$$\varepsilon_{zz} = \frac{du_z}{dz} = \frac{2\nu P_0}{E}\,;$$

- Consequently $u_r(r) = \frac{P_0}{E}(\nu - 1)r$, $u_z(z) = \frac{2\nu P_0}{E}z$, and:

$$\boldsymbol{u}(r,\theta,z) = \frac{P_0}{E}\left((\nu - 1)r\boldsymbol{i}_r(\theta) + 2\nu z\boldsymbol{i}_z\right)\,,$$

up to a rigid-body motion.

Part 3: Hollow cylinder under internal pressure



Question #7: Equations for the hub Ω_m?

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- Equilibrium equation $\mathbf{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_v = \varrho\ddot{\boldsymbol{u}}$ within the "infinitesimal deformation hypothesis;"
- "The effects of inertia and the action of gravity can be neglected," hence $\varrho\ddot{\boldsymbol{u}} = \mathbf{0}$, $\boldsymbol{f}_v = \mathbf{0}$, and $\mathbf{Div}\,\boldsymbol{\sigma} = \mathbf{0}$;

Interference fit

Solution

Question #7: Equations for the hub Ω_m?

- Equilibrium equation $\mathbf{Div}\,\boldsymbol{\sigma} = \mathbf{0}$;
- Constitutive equation for a "linear elastic, isotropic, homogeneous" material:

$$\begin{aligned}\boldsymbol{\varepsilon} &= \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}(\mathrm{Tr}\,\boldsymbol{\sigma})\boldsymbol{I}\,,\\ \boldsymbol{\sigma} &= \lambda(\mathrm{Tr}\,\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}\,;\end{aligned}$$

Interference fit
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- Equilibrium equation $\mathbf{Div}\,\boldsymbol{\sigma} = \mathbf{0}$;
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$$\boldsymbol{\sigma} = \lambda(\mathrm{Tr}\,\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}\,;$$

- Initial conditions: useless for statics;

Interference fit

Solution

Question #7: Equations for the hub Ω_m?

- Equilibrium equation $\mathbf{Div}\,\boldsymbol{\sigma} = \mathbf{0}$;
- Constitutive equation:

$$\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}(\mathrm{Tr}\,\boldsymbol{\sigma})\boldsymbol{I}\,,$$
$$\boldsymbol{\sigma} = \lambda(\mathrm{Tr}\,\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}\,;$$

- Initial conditions: useless for statics;
- Boundary conditions: "it is subjected on its inner boundary to the contact pressure P_0,"

$$\boldsymbol{\sigma}(-\boldsymbol{i}_r(\theta))|_{\{r=R_i\}} = P_0\boldsymbol{i}_r(\theta)\,,$$
$$\boldsymbol{\sigma}\boldsymbol{i}_r(\theta)|_{\{r=R_e\}} = \mathbf{0}\,;$$

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$$\boldsymbol{\sigma} = \lambda(\mathrm{Tr}\,\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}\,;$$

- Initial conditions: useless for statics;
- Boundary conditions: "the two faces $z = 0$ and $z = H_m$ are free of forces,"

$$\boldsymbol{\sigma}(-\boldsymbol{i}_z)|_{\{z=0\}} = \mathbf{0}\,,$$
$$\boldsymbol{\sigma}\boldsymbol{i}_z|_{\{z=H_m\}} = \mathbf{0}\,.$$

Interference fit
Solution

Question #7: Equations for the hub Ω_m?

- Recap:

$$\mathbf{Div}\,\boldsymbol{\sigma} = \mathbf{0}\,,$$

$$\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}(\mathrm{Tr}\,\boldsymbol{\sigma})\boldsymbol{I}\,,$$

$$\boldsymbol{\sigma} = \lambda(\mathrm{Tr}\,\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}\,,$$

$$\begin{aligned}\boldsymbol{\sigma}(-\boldsymbol{i}_r(\theta))|_{\{r=R_i\}} &= P_0\boldsymbol{i}_r(\theta)\,,\\ \boldsymbol{\sigma}\boldsymbol{i}_r(\theta)|_{\{r=R_e\}} &= \mathbf{0}\,,\\ \boldsymbol{\sigma}\boldsymbol{i}_z|_{\{z=0\}} &= \mathbf{0}\,,\\ \boldsymbol{\sigma}\boldsymbol{i}_z|_{\{z=H_m\}} &= \mathbf{0}\,.\end{aligned}$$

Question #8: Justify $\boldsymbol{u}_{fd}$?

- We seek for a displacement field of the form:

$$\boldsymbol{u}_{fd}(\boldsymbol{x}) = u_r(r)\boldsymbol{i}_r(\theta) + u_z(z)\boldsymbol{i}_z\,;$$

- Its is justified by the symmetry of the **geometry and loads** of the problem:
  - The problem is axisymmetric hence $u_\theta = 0$ and u_r, u_θ are independent of θ;
  - The radial displacement is independent of z by translational invariance;
  - The vertical displacement is independent of r for there is not warping of the hub.

Question #9: $\boldsymbol{\varepsilon}_{fd}$?

- We seek for a displacement field of the form:

$$\boldsymbol{u}_{fd}(\boldsymbol{x}) = u_r(r)\boldsymbol{i}_r(\theta) + u_z(z)\boldsymbol{i}_z\,;$$

- Strain tensor in cylindrical coordinates:

$$\begin{aligned}\boldsymbol{\varepsilon}_{fd}(\boldsymbol{x}) &= \frac{\partial \boldsymbol{u}_{fd}}{\partial r} \otimes_s \boldsymbol{i}_r(\theta) + \frac{\partial \boldsymbol{u}_{fd}}{\partial \theta} \otimes_s \frac{\boldsymbol{i}_\theta(\theta)}{r} + \frac{\partial \boldsymbol{u}_{fd}}{\partial z} \otimes_s \boldsymbol{i}_z \\ &= \frac{du_r}{dr}\boldsymbol{i}_r(\theta) \otimes \boldsymbol{i}_r(\theta) + \frac{u_r}{r}\boldsymbol{i}_\theta(\theta) \otimes \boldsymbol{i}_\theta(\theta) + \frac{du_z}{dz}\boldsymbol{i}_z \otimes \boldsymbol{i}_z\,;\end{aligned}$$

Question #9: $\boldsymbol{\sigma}_{fd}$?

- We seek for a displacement field of the form:

$$\boldsymbol{u}_{fd}(\boldsymbol{x}) = u_r(r)\boldsymbol{i}_r(\theta) + u_z(z)\boldsymbol{i}_z \,;$$

- From the constitutive equation:

$$\begin{aligned}
\boldsymbol{\sigma}_{fd} &= \lambda(\operatorname{Tr}\boldsymbol{\varepsilon}_{fd})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}_{fd} \\
&= \lambda\left(\frac{du_r}{dr} + \frac{u_r}{r} + \frac{du_z}{dz}\right)\boldsymbol{I} \\
&\quad + 2\mu\left[\frac{du_r}{dr}\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta) + \frac{u_r}{r}\boldsymbol{i}_\theta(\theta)\otimes\boldsymbol{i}_\theta(\theta) + \frac{du_z}{dz}\boldsymbol{i}_z\otimes\boldsymbol{i}_z\right].
\end{aligned}$$

Question #10: Differential equations for u_r and u_z?

- From the equilibrium equation $\mathbf{Div}\,\boldsymbol{\sigma}_{fd} = \mathbf{0}$;
- In cylindrical coordinates:

$$
\begin{aligned}
\mathbf{Div}\,\boldsymbol{\sigma}_{fd} &= \frac{\partial\boldsymbol{\sigma}_{fd}}{\partial r}\boldsymbol{i}_r(\theta) + \frac{\partial\boldsymbol{\sigma}_{fd}}{\partial\theta}\frac{\boldsymbol{i}_\theta(\theta)}{r} + \frac{\partial\boldsymbol{\sigma}_{fd}}{\partial z}\boldsymbol{i}_z \\
&= \lambda\left[\frac{d}{dr}\left(\frac{du_r}{dr} + \frac{u_r}{r}\right)\boldsymbol{i}_r(\theta) + \frac{d^2u_z}{dz^2}\boldsymbol{i}_z\right] \\
&\quad + 2\mu\left[\left(\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2}\right)\boldsymbol{i}_r(\theta) + \frac{d^2u_z}{dz^2}\boldsymbol{i}_z\right] \\
&= (\lambda + 2\mu)\left[\frac{d}{dr}\left(\frac{du_r}{dr} + \frac{u_r}{r}\right)\boldsymbol{i}_r(\theta) + \frac{d^2u_z}{dz^2}\boldsymbol{i}_z\right] \\
&= \mathbf{0}\,;
\end{aligned}
$$

Question #10: Differential equations for u_r and u_z?

- Consequently:

$$\frac{d}{dr}\left(\frac{du_r}{dr}+\frac{u_r}{r}\right)=0\,,$$
$$\frac{d^2u_z}{dz^2}=0\,.$$

Question #11: u_r, u_z?

- From question #10:

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d(ru_r)}{dr}\right) = 0\,,$$
$$\frac{d^2 u_z}{dz^2} = 0\,;$$

- Therefore $u_r(r) = Ar + \frac{B}{r}$, $u_z(z) = Cz + D$, and:

$$\boldsymbol{\sigma}_{fd} = \lambda(2A + C)\boldsymbol{I} + 2\mu\left(A - \frac{B}{r^2}\right)\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta)$$
$$+ 2\mu\left(A + \frac{B}{r^2}\right)\boldsymbol{i}_\theta(\theta)\otimes\boldsymbol{i}_\theta(\theta) + 2\mu C\boldsymbol{i}_z\otimes\boldsymbol{i}_z\,;$$

Question #11: u_r, u_z?

- Therefore:

$$\boldsymbol{\sigma}_{fd} = \lambda(2A+C)\boldsymbol{I} + 2\mu\left(A-\frac{B}{r^2}\right)\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta) + 2\mu\left(A+\frac{B}{r^2}\right)\boldsymbol{i}_\theta(\theta)\otimes\boldsymbol{i}_\theta(\theta) + 2\mu C\boldsymbol{i}_z\otimes\boldsymbol{i}_z\,;$$

- From the boundary conditions:

$$\boldsymbol{\sigma}(-\boldsymbol{i}_r(\theta))|_{\{r=R_i\}} = P_0\boldsymbol{i}_r(\theta) = -\left[\lambda(2A+C)+2\mu\left(A-\frac{B}{R_i^2}\right)\right]\boldsymbol{i}_r(\theta)\,,$$

$$\boldsymbol{\sigma}\boldsymbol{i}_r(\theta)|_{\{r=R_e\}} = \boldsymbol{0} = \left[\lambda(2A+C)+2\mu\left(A-\frac{B}{R_e^2}\right)\right]\boldsymbol{i}_r(\theta)\,,$$

$$\boldsymbol{\sigma}\boldsymbol{i}_z|_{\{z=0,H_m\}} = \boldsymbol{0} = [\lambda(2A+C)+2\mu C]\boldsymbol{i}_z\,;$$

Question #11: u_r, u_z?

- We thus end up in the system:

$$\begin{aligned} 2(\lambda+\mu)A - 2\mu\frac{B}{R_i^2} + \lambda C &= -P_0\,, \\ 2(\lambda+\mu)A - 2\mu\frac{B}{R_e^2} + \lambda C &= 0\,, \\ 2\lambda A + (\lambda+2\mu)C &= 0\,; \end{aligned}$$

- This yields:

$$A = \frac{(\lambda+2\mu)P_0}{2\mu(3\lambda+2\mu)}\frac{R_i^2}{R_e^2-R_i^2}\,,\quad B = \frac{P_0}{2\mu}\frac{R_e^2R_i^2}{R_e^2-R_i^2}\,,$$

$$C = -\frac{\lambda P_0}{\mu(3\lambda+2\mu)}\frac{R_i^2}{R_e^2-R_i^2}\,;$$

Interference fit
Solution

Question #11: u_r, u_z?

- Since:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}\,, \quad \mu = \frac{E}{2(1+\nu)}\,,$$

one also has:

$$A = (1-\nu)\frac{P_0}{E}\frac{R_i^2}{R_e^2-R_i^2}\,, \quad B = (1+\nu)\frac{P_0}{E}\frac{R_e^2R_i^2}{R_e^2-R_i^2}\,,$$

$$C = -2\nu\frac{P_0}{E}\frac{R_i^2}{R_e^2-R_i^2}\,.$$

Interference fit
Solution

Question #12: $\boldsymbol{\sigma}_{fd}$?

- From question #11:

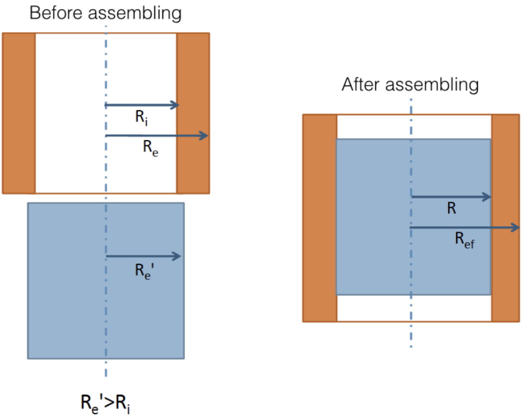
$$\boldsymbol{\sigma}_{fd} = \left(2(\lambda+\mu)A-2\mu\frac{B}{r^2}+\lambda C\right)\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta) + \left(2(\lambda+\mu)A+2\mu\frac{B}{r^2}+\lambda C\right)\boldsymbol{i}_\theta(\theta)\otimes\boldsymbol{i}_\theta(\theta)+\cancel{(2\lambda A+(\lambda+2\mu)C)}\boldsymbol{i}_z\otimes\boldsymbol{i}_z\,;$$

- Using the previous results for A, B and C:

$$\boldsymbol{\sigma}_{fd} = \frac{P_0R_i^2}{R_e^2-R_i^2}\left[\left(1-\frac{R_e^2}{r^2}\right)\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta)+\left(1+\frac{R_e^2}{r^2}\right)\boldsymbol{i}_\theta(\theta)\otimes\boldsymbol{i}_\theta(\theta)\right];$$

- The equilibrium equation and boundary conditions for Ω_m under internal pressure are for $\boldsymbol{\sigma}$ only and no kinematical unknown. Thus the constitutive equation has no impact.

Part 4: State after assembling



Interference fit

Solution

Question #13: Equations for the assembly $\Omega_m \cup \Omega_a$?

- Equilibrium equation $\mathbf{Div}\,\boldsymbol{\sigma}^{\textcircled{s}} + \boldsymbol{f}_v = \varrho\ddot{\boldsymbol{u}}$ within the "infinitesimal deformation hypothesis" for $s = a, m$;

Question #13: Equations for the assembly $\Omega_m \cup \Omega_a$?

- Equilibrium equation $\mathbf{Div}\,\boldsymbol{\sigma}^{ⓢ} + \boldsymbol{f}_v = \varrho\ddot{\boldsymbol{u}}$ within the "infinitesimal deformation hypothesis" for $s = a, m$;
- "The effects of inertia and the action of gravity can be neglected," hence $\varrho\ddot{\boldsymbol{u}} = \mathbf{0}$, $\boldsymbol{f}_v = \mathbf{0}$, and $\mathbf{Div}\,\boldsymbol{\sigma}^{ⓢ} = \mathbf{0}$ for $s = a, m$;

Question #13: Equations for the assembly $\Omega_m \cup \Omega_a$?

- Equilibrium equation $\mathbf{Div}\,\boldsymbol{\sigma}^{ⓢ} = \mathbf{0}$ for $s = a, m$;
- Constitutive equation for a "linear elastic, isotropic, homogeneous" material for $s = a, m$:

$$\begin{aligned}\boldsymbol{\varepsilon}^{ⓢ} &= \frac{1+\nu}{E}\boldsymbol{\sigma}^{ⓢ} - \frac{\nu}{E}(\mathrm{Tr}\,\boldsymbol{\sigma}^{ⓢ})\boldsymbol{I}\,,\\ \boldsymbol{\sigma}^{ⓢ} &= \lambda(\mathrm{Tr}\,\boldsymbol{\varepsilon}^{ⓢ})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}^{ⓢ}\,;\end{aligned}$$

Question #13: Equations for the assembly $\Omega_m \cup \Omega_a$?

- Equilibrium equation $\mathbf{Div}\,\boldsymbol{\sigma}^{(s)} = \mathbf{0}$ for $s = a, m$;
- Constitutive equation for $s = a, m$:

$$\boldsymbol{\varepsilon}^{(s)} = \frac{1+\nu}{E}\boldsymbol{\sigma}^{(s)} - \frac{\nu}{E}(\mathrm{Tr}\,\boldsymbol{\sigma}^{(s)})\boldsymbol{I}\,,$$
$$\boldsymbol{\sigma}^{(s)} = \lambda(\mathrm{Tr}\,\boldsymbol{\varepsilon}^{(s)})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}^{(s)}\,;$$

- Initial conditions: useless for statics;

Question #13: Equations for the assembly $\Omega_m \cup \Omega_a$?

- Equilibrium equation $\mathbf{Div}\,\boldsymbol{\sigma}^{ⓢ} = \mathbf{0}$ for $s = a, m$;
- Constitutive equation for $s = a, m$:

$$\begin{aligned} \boldsymbol{\varepsilon}^{ⓢ} &= \frac{1+\nu}{E}\boldsymbol{\sigma}^{ⓢ} - \frac{\nu}{E}(\operatorname{Tr}\boldsymbol{\sigma}^{ⓢ})\boldsymbol{I}\,, \\ \boldsymbol{\sigma}^{ⓢ} &= \lambda(\operatorname{Tr}\boldsymbol{\varepsilon}^{ⓢ})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}^{ⓢ}, ; \end{aligned}$$

- Initial conditions: useless for statics;
- Boundary conditions: "the two parts are fastened,"

$$\begin{aligned} \boldsymbol{\sigma}^{ⓐ}\boldsymbol{i}_r(\theta)|_{\{r=R'_e\}} + \boldsymbol{\sigma}^{ⓜ}(-\boldsymbol{i}_r(\theta))|_{\{r=R_i\}} &= \mathbf{0}\,, \\ \boldsymbol{x}^{ⓐ}(R'_e,\theta,z) - \boldsymbol{x}^{ⓜ}(R_i,\theta,z) &= \mathbf{0}\,; \end{aligned}$$

Question #13: Equations for the assembly $\Omega_m \cup \Omega_a$?

- Equilibrium equation $\mathbf{Div}\,\boldsymbol{\sigma}^{(s)} = \mathbf{0}$ for $s = a, m$;
- Constitutive equation for $s = a, m$:

$$\begin{aligned} \boldsymbol{\varepsilon}^{(s)} &= \frac{1+\nu}{E}\boldsymbol{\sigma}^{(s)} - \frac{\nu}{E}(\operatorname{Tr}\boldsymbol{\sigma}^{(s)})\boldsymbol{I}\,, \\ \boldsymbol{\sigma}^{(s)} &= \lambda(\operatorname{Tr}\boldsymbol{\varepsilon}^{(s)})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}^{(s)}\,; \end{aligned}$$

- Initial conditions: useless for statics;
- Boundary conditions on the free surfaces for $s = a, m$:

$$\begin{aligned} \boldsymbol{\sigma}^{(s)}(-\boldsymbol{i}_z)|_{\{z=0\}} &= \mathbf{0}\,, \\ \boldsymbol{\sigma}^{(s)}\boldsymbol{i}_z|_{\{z=H_s\}} &= \mathbf{0}\,, \\ \boldsymbol{\sigma}^{(m)}\boldsymbol{i}_r(\theta)|_{\{r=R_e\}} &= \mathbf{0}\,; \end{aligned}$$

Question #13: Equations for the assembly $\Omega_m \cup \Omega_a$?

- Recap for $s = a, m$:

$$\mathbf{Div}\, \boldsymbol{\sigma}^{(s)} = \mathbf{0}\,,$$

$$\boldsymbol{\varepsilon}^{(s)} = \frac{1+\nu}{E}\boldsymbol{\sigma}^{(s)} - \frac{\nu}{E}(\mathrm{Tr}\, \boldsymbol{\sigma}^{(s)})\boldsymbol{I}\,,$$

$$\boldsymbol{\sigma}^{(s)} = \lambda(\mathrm{Tr}\, \boldsymbol{\varepsilon}^{(s)})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}^{(s)}\,,$$

$$\boldsymbol{\sigma}^{(a)}\boldsymbol{i}_r(\theta)|_{\{r=R_e'\}} = \boldsymbol{\sigma}^{(m)}\boldsymbol{i}_r(\theta)|_{\{r=R_i\}}\,,$$

$$\boldsymbol{x}^{(a)}(R_e', \theta, z) = \boldsymbol{x}^{(m)}(R_i, \theta, z)\,,$$

$$\boldsymbol{\sigma}^{(s)}\boldsymbol{i}_z|_{\{z=0\}} = \mathbf{0}\,,$$

$$\boldsymbol{\sigma}^{(s)}\boldsymbol{i}_z|_{\{z=H_s\}} = \mathbf{0}\,,$$

$$\boldsymbol{\sigma}^{(m)}\boldsymbol{i}_r(\theta)|_{\{r=R_e\}} = \mathbf{0}\,.$$

Question #14: $\boldsymbol{\varepsilon}^{Ⓐ}$, $\boldsymbol{\varepsilon}^{Ⓜ}$?

- From question #6 for the shaft:

$$\boldsymbol{\varepsilon}^{Ⓐ} = \boldsymbol{\varepsilon}_{fc} = \frac{P_0}{E}[(\nu-1)(\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta)+\boldsymbol{i}_\theta(\theta)\otimes\boldsymbol{i}_\theta(\theta))+2\nu\boldsymbol{i}_z\otimes\boldsymbol{i}_z]$$
$$= \frac{P_0}{E}[(\nu-1)\boldsymbol{I}+(\nu+1)\boldsymbol{i}_z\otimes\boldsymbol{i}_z]\,;$$

- From question #9 for the hub:

$$\boldsymbol{\varepsilon}^{Ⓜ} = \boldsymbol{\varepsilon}_{fd} = \frac{du_r}{dr}\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta)+\frac{u_r}{r}\boldsymbol{i}_\theta(\theta)\otimes\boldsymbol{i}_\theta(\theta)+\frac{du_z}{dz}\boldsymbol{i}_z\otimes\boldsymbol{i}_z$$
$$= \left(A-\frac{B}{r^2}\right)\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta)+\left(A+\frac{B}{r^2}\right)\boldsymbol{i}_\theta(\theta)\otimes\boldsymbol{i}_\theta(\theta)+C\boldsymbol{i}_z\otimes\boldsymbol{i}_z$$
$$= -\frac{P_0}{E}\frac{R_i^2}{R_e^2-R_i^2}\Big[(\nu-1)\boldsymbol{I}+(\nu+1)\boldsymbol{i}_z\otimes\boldsymbol{i}_z$$
$$+(1+\nu)\frac{R_e^2}{r^2}(\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta)-\boldsymbol{i}_\theta(\theta)\otimes\boldsymbol{i}_\theta(\theta))\Big]\,;$$

Interference fit
Solution

Question #15: Inner radius of the deformed shaft?

- From question #13 and the displacement boundary condition:

$$\left\langle \boldsymbol{x}^{ⓐ}(R_e', \theta, z), \boldsymbol{i}_r(\theta) \right\rangle = \left\langle \boldsymbol{x}^{ⓜ}(R_i, \theta, z), \boldsymbol{i}_r(\theta) \right\rangle$$
$$\text{or} \quad R_e' + u_r^{ⓐ}(R_e') = R_i + u_r^{ⓜ}(R_i)\,;$$

Question #15: Inner radius of the deformed shaft?

- From question #13 and the displacement boundary condition:

$$R'_e + u_r^{ⓐ}(R'_e) = R_i + u_r^{ⓜ}(R_i)\,;$$

- From question #6 for the shaft $u_r^{ⓐ}(r) = \frac{P_0}{E}(\nu - 1)r$;

Question #15: Inner radius of the deformed shaft?

- From question #13 and the displacement boundary condition:

$$R'_e + u_r^{ⓐ}(R'_e) = R_i + u_r^{ⓜ}(R_i)\,;$$

- From question #6 for the shaft $u_r^{ⓐ}(r) = \frac{P_0}{E}(\nu - 1)r$;
- From question #11 for the hub $u_r^{ⓜ}(r) = Ar + \frac{B}{r}$;

Interference fit

Solution

Question #15: Inner radius of the deformed shaft?

- From question #13 and the displacement boundary condition:

$$R'_e + u_r^{@}(R'_e) = R_i + u_r^{m}(R_i)\,;$$

- From question #6 for the shaft $u_r^{@}(r) = \frac{P_0}{E}(\nu - 1)r$;
- From question #11 for the hub $u_r^{m}(r) = Ar + \frac{B}{r}$;
- Therefore:

$$R'_e + \frac{P_0}{E}(\nu - 1)R'_e = R_i + AR_i + \frac{B}{R_i}$$

$$\left[1+(\nu-1)\frac{P_0}{E}\right]R'_e = \left[1+\frac{P_0}{E}\frac{R_i^2}{R_e^2-R_i^2}\left(1-\nu+(1+\nu)\frac{R_e^2}{R_i^2}\right)\right]R_i\,.$$

Question #16: P_0?

- From question #15:

$$\begin{aligned}\left[1+(\nu-1)\frac{P_0}{E}\right]\left(R_i+\frac{s}{2}\right) &= \left[1+\frac{P_0}{E}\frac{R_i^2}{R_e^2-R_i^2}\left(1-\nu+(1+\nu)\frac{R_e^2}{R_i^2}\right)\right]R_i \\ \left[1+(\nu-1)\frac{P_0}{E}\right]\frac{s}{2} &= \frac{2P_0}{E}\frac{R_e^2}{R_e^2-R_i^2}R_i \\ \frac{s}{2} &= \frac{P_0}{E}\left[\frac{2R_e^2}{R_e^2-R_i^2}R_i+(1-\nu)\frac{s}{2}\right],\end{aligned}$$

or:

$$P_0 \simeq \frac{Es}{4R_i}\left(1-\frac{R_i^2}{R_e^2}\right) \quad \text{QED}\,.$$

Question #17: Tresca's criterion?

- From question #5 for the shaft:

$$\boldsymbol{\sigma}^{ⓐ} = \boldsymbol{\sigma}_{fc} = -P_0\left(\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta)+\boldsymbol{i}_\theta(\theta)\otimes\boldsymbol{i}_\theta(\theta)\right)\;;$$

- From question #12 for the hub:

$$\boldsymbol{\sigma}^{ⓜ} = \boldsymbol{\sigma}_{fd} = \frac{P_0 R_i^2}{R_e^2-R_i^2}\left[\left(1-\frac{R_e^2}{r^2}\right)\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta)+\left(1+\frac{R_e^2}{r^2}\right)\boldsymbol{i}_\theta(\theta)\otimes\boldsymbol{i}_\theta(\theta)\right];$$

Question #17: Tresca's criterion?

- From question #5 (shaft) and question #12 (hub):

$$\boldsymbol{\sigma}^{ⓐ} = \boldsymbol{\sigma}_{fc} = -P_0 \left(\boldsymbol{i}_r(\theta) \otimes \boldsymbol{i}_r(\theta) + \boldsymbol{i}_\theta(\theta) \otimes \boldsymbol{i}_\theta(\theta) \right) \,,$$

$$\boldsymbol{\sigma}^{ⓜ} = \boldsymbol{\sigma}_{fd} = \frac{P_0 R_i^2}{R_e^2 - R_i^2} \left[\left(1 - \frac{R_e^2}{r^2} \right) \boldsymbol{i}_r(\theta) \otimes \boldsymbol{i}_r(\theta) + \left(1 + \frac{R_e^2}{r^2} \right) \boldsymbol{i}_\theta(\theta) \otimes \boldsymbol{i}_\theta(\theta) \right] ;$$

- Tresca' criterion: $\tau_{\mathrm{eq}} \leqslant \frac{\sigma_0}{2}$ where

$$\tau_{\mathrm{eq}} = \frac{1}{2} \max(|\sigma_1 - \sigma_2| \,, |\sigma_2 - \sigma_3| \,, |\sigma_3 - \sigma_1|) \,;$$

Question #17: Tresca's criterion?

- From question #5 (shaft) and question #12 (hub):

$$\boldsymbol{\sigma}^{a} = \boldsymbol{\sigma}_{fc} = -P_0 \left(\boldsymbol{i}_r(\theta) \otimes \boldsymbol{i}_r(\theta) + \boldsymbol{i}_\theta(\theta) \otimes \boldsymbol{i}_\theta(\theta) \right) \,,$$
$$\boldsymbol{\sigma}^{m} = \boldsymbol{\sigma}_{fd} = \frac{P_0 R_i^2}{R_e^2 - R_i^2} \left[\left(1 - \frac{R_e^2}{r^2} \right) \boldsymbol{i}_r(\theta) \otimes \boldsymbol{i}_r(\theta) + \left(1 + \frac{R_e^2}{r^2} \right) \boldsymbol{i}_\theta(\theta) \otimes \boldsymbol{i}_\theta(\theta) \right] ;$$

- Tresca' criterion: $\tau_{\mathrm{eq}} \leqslant \frac{\sigma_0}{2}$ where

$$\tau_{\mathrm{eq}} = \frac{1}{2} \max(|\sigma_1 - \sigma_2| \,, |\sigma_2 - \sigma_3| \,, |\sigma_3 - \sigma_1|) \,;$$

- Here $\tau_{\mathrm{eq}} = \frac{1}{2} \max |\sigma_{rr}^{m} - \sigma_{\theta\theta}^{m}|$ reached for $r = R_i$, or:

$$\tau_{\mathrm{eq}} = \frac{P_0 R_e^2}{R_e^2 - R_i^2} = \frac{Es}{4 R_i} = 100\,\mathrm{MPa} > \frac{\sigma_0}{2} \;!!!$$