Large beam bending

1EL5000–Continuum Mechanics – Tutorial Class #1

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Outline

Outline

- Scalar product:

$$\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^a\,,\quad\langle\boldsymbol{a},\boldsymbol{b}\rangle=\sum_{j=1}^{a}a_jb_j=a_jb_j\,,$$

The last equality is Einstein's summation convention.

- Tensors and tensor product (or outer product):

$$\boldsymbol{A}\in\mathbb{R}^a\to\mathbb{R}^b\,,\quad\boldsymbol{A}=\boldsymbol{a}\otimes\boldsymbol{b}\,,\quad\boldsymbol{a}\in\mathbb{R}^a\,,\boldsymbol{b}\in\mathbb{R}^b\,.$$

- Tensor application to vectors:

$$\boldsymbol{A}=\boldsymbol{a}\otimes\boldsymbol{b}\in\mathbb{R}^a\to\mathbb{R}^b\,,\boldsymbol{c}\in\mathbb{R}^b\,,\quad\boldsymbol{A}\boldsymbol{c}=\langle\boldsymbol{b},\boldsymbol{c}\rangle\boldsymbol{a}\,.$$

- Product of tensors $\equiv$ composition of linear maps:

$$\boldsymbol{A}=\boldsymbol{a}\otimes\boldsymbol{b}\,,\boldsymbol{B}=\boldsymbol{c}\otimes\boldsymbol{d}\,,\quad\boldsymbol{A}\boldsymbol{B}=\langle\boldsymbol{b},\boldsymbol{c}\rangle\boldsymbol{a}\otimes\boldsymbol{d}\,.$$

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Some algebra
Vector & tensor products
Vector & tensor analysis
Kinematics
Strains
1.3 Large beam bending

- Scalar product of tensors:

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \mathrm{Tr}(\boldsymbol{A}\boldsymbol{B}^{\mathsf{T}}) := \boldsymbol{A} : \boldsymbol{B} = A_{jk} B_{jk} \,.$$

- Let $\{\boldsymbol{e}_j\}_{j=1}^{d}$ be a Cartesian basis in $\mathbb{R}^d$. Then:

$$\begin{aligned} a_j &= \langle \boldsymbol{a}, \boldsymbol{e}_j \rangle \,, \\ A_{jk} &= \langle \boldsymbol{A}, \boldsymbol{e}_j \otimes \boldsymbol{e}_k \rangle = \boldsymbol{A} : \boldsymbol{e}_j \otimes \boldsymbol{e}_k \\ &= \langle \boldsymbol{A}\boldsymbol{e}_k, \boldsymbol{e}_j \rangle \,, \end{aligned}$$

such that:

$$\begin{aligned} \boldsymbol{a} &= a_j \boldsymbol{e}_j \,, \\ \boldsymbol{A} &= A_{jk} \boldsymbol{e}_j \otimes \boldsymbol{e}_k \,. \end{aligned}$$

- Example: the identity matrix

$$\boldsymbol{I} = \boldsymbol{e}_j \otimes \boldsymbol{e}_j \,.$$

- Gradient of a vector function $\boldsymbol{a}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$:

$$\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a} = \frac{\partial \boldsymbol{a}}{\partial x_j} \otimes \boldsymbol{e}_j \,.$$

- Divergence of a vector function $\boldsymbol{a}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$:

$$\mathrm{div}_{\boldsymbol{x}}\boldsymbol{a} = \langle \boldsymbol{\nabla}_{\boldsymbol{x}}, \boldsymbol{a} \rangle = \mathrm{Tr}(\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a}) = \frac{\partial a_j}{\partial x_j} \,.$$

- Divergence of a tensor function $\boldsymbol{A}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$:

$$\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{A} = \frac{\partial (\boldsymbol{A}\boldsymbol{e}_j)}{\partial x_j} \,.$$

- Gradient of a vector function $\boldsymbol{a}(r,\theta,z)$:

$$\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a}=\frac{\partial\boldsymbol{a}}{\partial r}\otimes\boldsymbol{e}_r+\frac{\partial\boldsymbol{a}}{\partial\theta}\otimes\frac{\boldsymbol{e}_\theta}{r}+\frac{\partial\boldsymbol{a}}{\partial z}\otimes\boldsymbol{e}_z\,.$$

- Divergence of a vector function $\boldsymbol{a}(r,\theta,z)$:

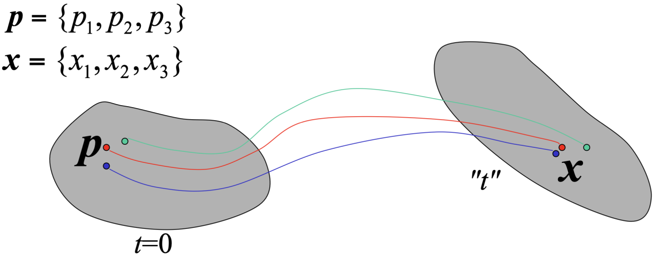
$$\mathrm{div}_{\boldsymbol{x}}\boldsymbol{a}=\left\langle\frac{\partial\boldsymbol{a}}{\partial r},\boldsymbol{e}_r\right\rangle+\left\langle\frac{\partial\boldsymbol{a}}{\partial\theta},\frac{\boldsymbol{e}_\theta}{r}\right\rangle+\left\langle\frac{\partial\boldsymbol{a}}{\partial z},\boldsymbol{e}_z\right\rangle\,.$$

- Divergence of a tensor function $\boldsymbol{A}(r,\theta,z)$:

$$\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{A}=\frac{\partial\boldsymbol{A}}{\partial r}\boldsymbol{e}_r+\frac{\partial\boldsymbol{A}}{\partial\theta}\frac{\boldsymbol{e}_\theta}{r}+\frac{\partial\boldsymbol{A}}{\partial z}\boldsymbol{e}_z\,.$$

Outline

Kinematics



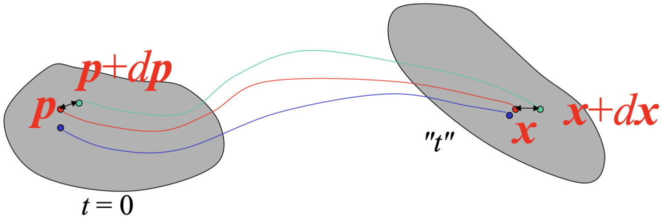
$$\boxed{\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{p}, t)\,, \quad \boldsymbol{u} = \boldsymbol{f}(\boldsymbol{p}, t) - \boldsymbol{p}}$$

- $\boldsymbol{p} \mapsto \boldsymbol{f}(\boldsymbol{p}, t) : \Omega_0 \to \Omega_t$ is injective: two material points cannot occupy the same volume at the same time ($\boldsymbol{p}_1 \neq \boldsymbol{p}_2 \Rightarrow \boldsymbol{x}_1 \neq \boldsymbol{x}_2$);
- $\boldsymbol{f}$ is piecewise $\mathcal{C}^1$ in space $\boldsymbol{p} \in \Omega_0$ and time $t \in \mathbb{R}$;
- Some inverse transform $\boldsymbol{x} \mapsto \boldsymbol{g}(\boldsymbol{x}, t) : \Omega_t \to \Omega_0$ exists;
- Matter cannot vanish:

$$\det(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) > 0 \iff \det(\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3) > 0 \quad \text{for } \boldsymbol{e}_j = \boldsymbol{f}(\boldsymbol{i}_j, t)\,.$$

Outline

Space gradient tensor
...or the homogeneous tangent transformation



$$dx_j = \frac{\partial f_j}{\partial p_k} dp_k$$

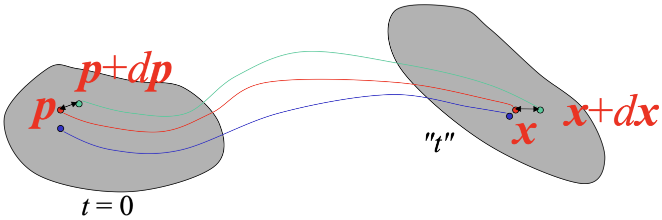
- Space gradient tensor $\mathbb{F} = \mathbb{D}_{\boldsymbol{p}} \boldsymbol{x}$:

$$d\boldsymbol{x} = \mathbb{F} d\boldsymbol{p} \,, \quad d\boldsymbol{u} = (\mathbb{F} - \boldsymbol{I}) d\boldsymbol{p}$$

- In a Cartesian frame/curvilinear frame:

$$\mathbb{F} = \frac{\partial \boldsymbol{x}}{\partial p_j} \otimes \boldsymbol{i}_j = \frac{\partial \boldsymbol{x}}{\partial \xi_j} \otimes \boldsymbol{\nabla}_{\boldsymbol{p}} \xi_j$$

Green-Lagrange tensor
Relative length variation



$$
\begin{aligned}
d\boldsymbol{x} &= \mathbb{F}d\boldsymbol{p} \\
\|d\boldsymbol{x}\|^2 &= \langle \mathbb{F}d\boldsymbol{p}, \mathbb{F}d\boldsymbol{p} \rangle \\
&= \left\langle d\boldsymbol{p}, \mathbb{F}^{\mathsf{T}}\mathbb{F}d\boldsymbol{p} \right\rangle \\
\frac{\|d\boldsymbol{x}\|^2 - \|d\boldsymbol{p}\|^2}{\|d\boldsymbol{p}\|^2} &= \left\langle (\mathbb{F}^{\mathsf{T}}\mathbb{F} - \boldsymbol{I})\frac{d\boldsymbol{p}}{\|d\boldsymbol{p}\|}, \frac{d\boldsymbol{p}}{\|d\boldsymbol{p}\|} \right\rangle
\end{aligned}
$$

Green-Lagrange tensor
Relative length variation

- The Green-Lagrange tensor:

$$\boxed{\mathbb{E} = \frac{1}{2}\left(\mathbb{F}^{\mathsf{T}}\mathbb{F} - \boldsymbol{I}\right)}.$$

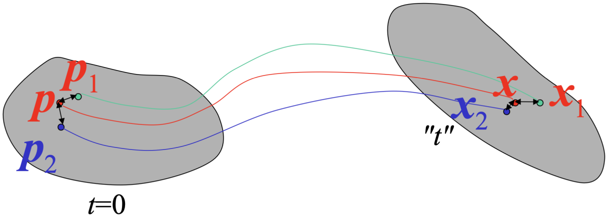
- The dilatation tensor, or right Cauchy-Green tensor $\mathbb{C} = \mathbb{F}^{\mathsf{T}}\mathbb{F}$.
- Measuring stretching of a fiber $d\boldsymbol{p} = dp\boldsymbol{i}_j$:

$$\frac{\|d\boldsymbol{x}\|^2 - \|d\boldsymbol{p}\|^2}{\|d\boldsymbol{p}\|^2} = 2\langle\mathbb{E}\boldsymbol{i}_j, \boldsymbol{i}_j\rangle$$

$$\left(E_{jj} = \frac{1}{2}\left(\frac{\ell^2 - \ell_0^2}{\ell_0^2}\right) = \frac{\Delta\ell}{\ell_0} + \frac{1}{2}\left(\frac{\Delta\ell}{\ell_0}\right)^2\right)$$

Green-Lagrange tensor

Relative angle variation



$$d\boldsymbol{x}_1 = \mathbb{F}d\boldsymbol{p}_1\,, \quad d\boldsymbol{x}_2 = \mathbb{F}d\boldsymbol{p}_2$$

$$\begin{aligned}\langle d\boldsymbol{x}_1, d\boldsymbol{x}_2\rangle &= \langle \mathbb{F}d\boldsymbol{p}_1, \mathbb{F}d\boldsymbol{p}_2\rangle \\ &= \left\langle \mathbb{F}^{\mathsf{T}}\mathbb{F}d\boldsymbol{p}_1, d\boldsymbol{p}_2\right\rangle \\ \|d\boldsymbol{x}_1\|\,\|d\boldsymbol{x}_2\|\cos\theta_{t12} &= \langle(2\mathbb{E}+\boldsymbol{I})d\boldsymbol{p}_1, d\boldsymbol{p}_2\rangle\end{aligned}$$

Green-Lagrange tensor
Relative angle variation

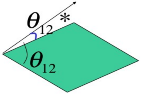
- The Green-Lagrange tensor:

$$\boxed{\mathbb{E} = \frac{1}{2}\left(\mathbb{F}^{\mathsf{T}}\mathbb{F} - \boldsymbol{I}\right)}\,.$$

- Measuring angle between $d\boldsymbol{p}_1 = dp_1\boldsymbol{i}_1$ and $d\boldsymbol{p}_2 = dp_2\boldsymbol{i}_2$:

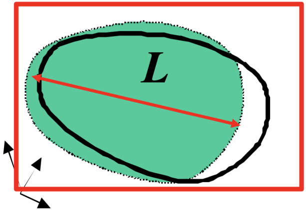
$$\begin{aligned}\|d\boldsymbol{x}_1\|\,\|d\boldsymbol{x}_2\|\cos\theta_{t12} &= \langle(2\mathbb{E}+\boldsymbol{I})dp_1\boldsymbol{i}_1, dp_2\boldsymbol{i}_2\rangle \\ &= 2dp_1dp_2\langle\mathbb{E}\boldsymbol{i}_1,\boldsymbol{i}_2\rangle\end{aligned}$$

$$\left(E_{12} = \frac{1}{2}\left(\frac{\ell_1\ell_2}{\ell_{01}\ell_{02}}\right)\cos\theta_{t12} = \frac{1}{2}\frac{\ell_1}{\ell_{01}}\frac{\ell_2}{\ell_{02}}\sin\left(\frac{\pi}{2}-\theta_{t12}\right)\right)$$



Green-Lagrange tensor
Units and scales

- Strains are dimensionless "m/m";
- Steel: $\|\mathbb{E}\| \approx 10^{-3}$;
- Soil: $\|\mathbb{E}\| \approx 10^{-2}$;
- Rubber: $\|\mathbb{E}\| \approx 2$;
- Molecules: $L \approx 1\,\text{Å} = 10^{-10}\,\text{m}$, $T \approx 1\,\text{ps} = 10^{-12}\,\text{s}$;
- Continuum mechanics: $L \gtrsim 10\,\text{nm} = 10^{-8}\,\text{m}$, $T \gtrsim 1\,\text{ns} = 10^{-9}\,\text{s}$.



$$\Omega_t \simeq \Omega_0 \quad (\boldsymbol{x} \simeq \boldsymbol{p})$$

$$\max_{\boldsymbol{p},t} \|\boldsymbol{u}\| \ll L\,, \quad \max_{\boldsymbol{p},t} \|\mathbb{D}_{\boldsymbol{p}}\boldsymbol{u}\|^2 = \max_{\boldsymbol{p},t} \mathrm{Tr}(\mathbb{D}_{\boldsymbol{p}}\boldsymbol{u}^{\mathsf{T}}\mathbb{D}_{\boldsymbol{p}}\boldsymbol{u}) \ll 1$$

$$\begin{aligned}
\boldsymbol{x} &= \boldsymbol{p} + \boldsymbol{u}(\boldsymbol{p},t) \\
\mathbb{F} &= \boldsymbol{I} + \mathbb{D}_{\boldsymbol{p}}\boldsymbol{u} \\
\mathbb{E} &= \frac{1}{2}\left(\mathbb{F}^{\mathsf{T}}\mathbb{F} - \boldsymbol{I}\right) \\
&= \frac{1}{2}\left(\mathbb{D}_{\boldsymbol{p}}\boldsymbol{u} + \mathbb{D}_{\boldsymbol{p}}\boldsymbol{u}^{\mathsf{T}} + \cancel{\mathbb{D}_{\boldsymbol{p}}\boldsymbol{u}^{\mathsf{T}}\mathbb{D}_{\boldsymbol{p}}\boldsymbol{u}}\right)
\end{aligned}$$

- Linearized (small) strain tensor with $\boldsymbol{x} \simeq \boldsymbol{p}$:

$$\mathbb{E} \simeq \boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbb{D}_{\boldsymbol{x}}\boldsymbol{u} + \mathbb{D}_{\boldsymbol{x}}\boldsymbol{u}^{\mathsf{T}}\right) = \frac{\partial \boldsymbol{u}}{\partial x_j} \otimes_s \boldsymbol{e}_j \,,$$

  where $\boldsymbol{a} \otimes_s \boldsymbol{b} = \frac{1}{2}(\boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{b} \otimes \boldsymbol{a})$.

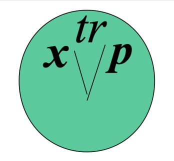
- Small stretching and small relative rotations:

$$\varepsilon_{jj} = \langle \boldsymbol{\varepsilon}\boldsymbol{e}_j, \boldsymbol{e}_j \rangle = \frac{\Delta \ell}{\ell_0} \,, \quad \varepsilon_{jk} = \langle \boldsymbol{\varepsilon}\boldsymbol{e}_k, \boldsymbol{e}_j \rangle = \frac{\theta^*_{t12}}{2} \,.$$

- Local volume change:

$$\frac{\Delta V}{V_0} = \frac{\ell_{01} + \Delta\ell_1}{\ell_{01}} \frac{\ell_{02} + \Delta\ell_2}{\ell_{02}} \frac{\ell_{03} + \Delta\ell_3}{\ell_{03}} - 1 \approx \operatorname{Tr} \boldsymbol{\varepsilon} \,.$$

Torsion of a cylinder



- $\boldsymbol{p} = r\boldsymbol{e}_r(\theta) + z\boldsymbol{e}_z$;
- $\boldsymbol{x} = r\boldsymbol{i}_r(\theta + tr) + z\boldsymbol{e}_z$;
- $\mathbb{E}$?

Green-Lagrange tensor

Example

Torsion of a cylinder

- $\boldsymbol{p} = r\boldsymbol{e}_r(\theta) + z\boldsymbol{e}_z$, $\boldsymbol{x} = r\boldsymbol{e}_r(\theta + tr) + z\boldsymbol{e}_z$; $\mathbb{E}$?
- First compute $\mathbb{F}$:

$$\begin{aligned}\mathbb{F} &= \frac{\partial \boldsymbol{x}}{\partial r} \otimes \boldsymbol{e}_r + \frac{\partial \boldsymbol{x}}{\partial \theta} \otimes \frac{\boldsymbol{e}_\theta}{r} + \frac{\partial \boldsymbol{x}}{\partial z} \otimes \boldsymbol{e}_z \\ &= (\boldsymbol{e}_r(\theta + tr) + tr\boldsymbol{e}_\theta(\theta + tr)) \otimes \boldsymbol{e}_r(\theta) + \boldsymbol{e}_\theta(\theta + tr) \otimes \boldsymbol{e}_\theta(\theta) + \boldsymbol{e}_z \otimes \boldsymbol{e}_z\end{aligned}$$

Torsion of a cylinder

- $\boldsymbol{p} = r\boldsymbol{e}_r(\theta) + z\boldsymbol{e}_z$, $\boldsymbol{x} = r\boldsymbol{e}_r(\theta + tr) + z\boldsymbol{e}_z$; $\mathbb{E}$?
- First compute $\mathbb{F}$:

$$
\begin{aligned}
\mathbb{F} &= \frac{\partial \boldsymbol{x}}{\partial r} \otimes \boldsymbol{e}_r + \frac{\partial \boldsymbol{x}}{\partial \theta} \otimes \frac{\boldsymbol{e}_\theta}{r} + \frac{\partial \boldsymbol{x}}{\partial z} \otimes \boldsymbol{e}_z \\
&= (\boldsymbol{e}_r(\theta + tr) + tr\boldsymbol{e}_\theta(\theta + tr)) \otimes \boldsymbol{e}_r(\theta) + \boldsymbol{e}_\theta(\theta + tr) \otimes \boldsymbol{e}_\theta(\theta) + \boldsymbol{e}_z \otimes \boldsymbol{e}_z \\
\mathbb{F}^{\mathsf{T}} &= \boldsymbol{e}_r(\theta) \otimes (\boldsymbol{e}_r(\theta + tr) + tr\boldsymbol{e}_\theta(\theta + tr)) + \boldsymbol{e}_\theta(\theta) \otimes \boldsymbol{e}_\theta(\theta + tr) + \boldsymbol{e}_z \otimes \boldsymbol{e}_z
\end{aligned}
$$

Green-Lagrange tensor
Example

Torsion of a cylinder

- $\boldsymbol{p} = r\boldsymbol{e}_r(\theta) + z\boldsymbol{e}_z$, $\boldsymbol{x} = r\boldsymbol{e}_r(\theta + tr) + z\boldsymbol{e}_z$; $\mathbb{E}$?
- First compute $\mathbb{F}$:

$$\mathbb{F} = (\boldsymbol{e}_r(\theta + tr) + tr\boldsymbol{e}_\theta(\theta + tr)) \otimes \boldsymbol{e}_r(\theta) + \boldsymbol{e}_\theta(\theta + tr) \otimes \boldsymbol{e}_\theta(\theta) + \boldsymbol{e}_z \otimes \boldsymbol{e}_z$$
$$\mathbb{F}^\mathsf{T} = \boldsymbol{e}_r(\theta) \otimes (\boldsymbol{e}_r(\theta + tr) + tr\boldsymbol{e}_\theta(\theta + tr)) + \boldsymbol{e}_\theta(\theta) \otimes \boldsymbol{e}_\theta(\theta + tr) + \boldsymbol{e}_z \otimes \boldsymbol{e}_z$$

- Then $\mathbb{F}^\mathsf{T}\mathbb{F}$ (remind that $(\boldsymbol{a} \otimes \boldsymbol{b})(\boldsymbol{c} \otimes \boldsymbol{d}) = \langle \boldsymbol{b}, \boldsymbol{c} \rangle \boldsymbol{a} \otimes \boldsymbol{d}$):

$$\begin{aligned}\mathbb{F}^\mathsf{T}\mathbb{F} = {} & (1 + (tr)^2)\boldsymbol{e}_r(\theta) \otimes \boldsymbol{e}_r(\theta) + tr(\boldsymbol{e}_r(\theta) \otimes \boldsymbol{e}_\theta(\theta) + \boldsymbol{e}_\theta(\theta) \otimes \boldsymbol{e}_r(\theta)) \\ & + \boldsymbol{e}_\theta(\theta) \otimes \boldsymbol{e}_\theta(\theta) + \boldsymbol{e}_z \otimes \boldsymbol{e}_z\end{aligned}$$

Green-Lagrange tensor
Example

Torsion of a cylinder

- $\boldsymbol{p} = r\boldsymbol{e}_r(\theta) + z\boldsymbol{e}_z$, $\boldsymbol{x} = r\boldsymbol{e}_r(\theta + tr) + z\boldsymbol{e}_z$; $\mathbb{E}$?
- First compute $\mathbb{F}$:

$$\begin{aligned}
\mathbb{F} &= (\boldsymbol{e}_r(\theta+tr) + tr\boldsymbol{e}_\theta(\theta+tr)) \otimes \boldsymbol{e}_r(\theta) + \boldsymbol{e}_\theta(\theta+tr) \otimes \boldsymbol{e}_\theta(\theta) + \boldsymbol{e}_z \otimes \boldsymbol{e}_z \\
\mathbb{F}^\mathsf{T} &= \boldsymbol{e}_r(\theta) \otimes (\boldsymbol{e}_r(\theta+tr) + tr\boldsymbol{e}_\theta(\theta+tr)) + \boldsymbol{e}_\theta(\theta) \otimes \boldsymbol{e}_\theta(\theta+tr) + \boldsymbol{e}_z \otimes \boldsymbol{e}_z
\end{aligned}$$

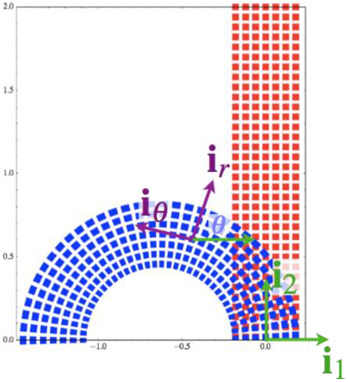
- Then $\mathbb{F}^\mathsf{T}\mathbb{F}$ (remind that $(\boldsymbol{a}\otimes\boldsymbol{b})(\boldsymbol{c}\otimes\boldsymbol{d}) = \langle\boldsymbol{b},\boldsymbol{c}\rangle\,\boldsymbol{a}\otimes\boldsymbol{d}$):

$$\begin{aligned}
\mathbb{F}^\mathsf{T}\mathbb{F} = {}&(1+(tr)^2)\boldsymbol{e}_r(\theta)\otimes\boldsymbol{e}_r(\theta) + tr(\boldsymbol{e}_r(\theta)\otimes\boldsymbol{e}_\theta(\theta) + \boldsymbol{e}_\theta(\theta)\otimes\boldsymbol{e}_r(\theta)) \\
&+ \boldsymbol{e}_\theta(\theta)\otimes\boldsymbol{e}_\theta(\theta) + \boldsymbol{e}_z\otimes\boldsymbol{e}_z
\end{aligned}$$

- Then $\mathbb{E} = \frac{1}{2}(\mathbb{F}^\mathsf{T}\mathbb{F} - \boldsymbol{I})$ ($\boldsymbol{I} = \boldsymbol{e}_r\otimes\boldsymbol{e}_r + \boldsymbol{e}_\theta\otimes\boldsymbol{e}_\theta + \boldsymbol{e}_z\otimes\boldsymbol{e}_z$):

$$\mathbb{E} = \frac{1}{2}(tr)^2\boldsymbol{e}_r(\theta)\otimes\boldsymbol{e}_r(\theta) + tr\boldsymbol{e}_r(\theta)\otimes_s\boldsymbol{e}_\theta(\theta)\,.$$

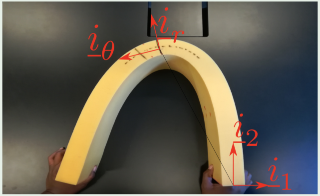
Outline



$$
\begin{aligned}
\boldsymbol{p} &= p_1\boldsymbol{i}_1 + p_2\boldsymbol{i}_2\,, \quad (p_1,p_2)\in\,]-h,h[\times]0,L[\\
\boldsymbol{x} &= \boldsymbol{f}(\boldsymbol{p},t) \\
&= \boldsymbol{x}_G(p_2,t) + p_1\boldsymbol{i}_r(\theta(p_2,t))
\end{aligned}
$$

Question #1: Physical interpretation

- $p_1 = 0 \Rightarrow \boldsymbol{x}(0, p_2, t) = \boldsymbol{x}_G(p_2, t)$: position of the middle-line;
- $p_1 \boldsymbol{i}_r(\theta(p_2, t))$: rotation of the cross section when $p_2 \mapsto \theta(p_2, t)$ increases.



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$$\boldsymbol{x}(\boldsymbol{p}, t) = \boldsymbol{x}_G(p_2, t) + p_1 \boldsymbol{i}_r(\theta(p_2, t))$$

Question #2: Green-Lagrange tensor

- $\mathbb{E} = \frac{1}{2}(\mathbb{F}^{\mathsf{T}}\mathbb{F} - \boldsymbol{I})$ with $\mathbb{F} = \mathbb{D}_{\boldsymbol{p}}\boldsymbol{x}$, or:

$$\mathbb{F} = \frac{\partial \boldsymbol{x}}{\partial p_j} \otimes \boldsymbol{i}_j = \frac{\partial \boldsymbol{x}}{\partial p_1} \otimes \boldsymbol{i}_1 + \frac{\partial \boldsymbol{x}}{\partial p_2} \otimes \boldsymbol{i}_2 \,.$$

Question #2: Green-Lagrange tensor

- $\mathbb{E} = \frac{1}{2}(\mathbb{F}^{\mathsf{T}}\mathbb{F} - \boldsymbol{I})$ with $\mathbb{F} = \mathbb{D}_{\boldsymbol{p}}\boldsymbol{x}$, or:

$$\mathbb{F} = \frac{\partial \boldsymbol{x}}{\partial p_j} \otimes \boldsymbol{i}_j = \frac{\partial \boldsymbol{x}}{\partial p_1} \otimes \boldsymbol{i}_1 + \frac{\partial \boldsymbol{x}}{\partial p_2} \otimes \boldsymbol{i}_2 \,.$$

- By direct computation ($\boldsymbol{x} = \boldsymbol{x}_G(p_2, t) + p_1 \boldsymbol{i}_r(\theta(p_2, t))$):

$$\begin{aligned}
\frac{\partial \boldsymbol{x}}{\partial p_1} &= \boldsymbol{i}_r(\theta(p_2, t)) \\
\frac{\partial \boldsymbol{x}}{\partial p_2} &= \frac{\partial \boldsymbol{x}_G}{\partial p_2}(p_2, t) + p_1 \frac{\partial \theta}{\partial p_2}(p_2, t) \boldsymbol{i}_\theta(\theta(p_2, t))
\end{aligned}$$

- Therefore:

$$\mathbb{F} = \boldsymbol{i}_r(\theta(p_2, t)) \otimes \boldsymbol{i}_1 + \frac{\partial \boldsymbol{x}_G}{\partial p_2}(p_2, t) \otimes \boldsymbol{i}_2 + p_1 \frac{\partial \theta}{\partial p_2}(p_2, t) \boldsymbol{i}_\theta(\theta(p_2, t)) \otimes \boldsymbol{i}_2 \,.$$

Question #2: Green-Lagrange tensor

- $\mathbb{E} = \frac{1}{2}(\mathbb{F}^{\mathsf{T}}\mathbb{F} - \boldsymbol{I})$ with:

$$\mathbb{F} = \boldsymbol{i}_r(\theta(p_2,t)) \otimes \boldsymbol{i}_1 + \frac{\partial \boldsymbol{x}_G}{\partial p_2}(p_2,t) \otimes \boldsymbol{i}_2 + p_1 \frac{\partial \theta}{\partial p_2}(p_2,t)\boldsymbol{i}_\theta(\theta(p_2,t)) \otimes \boldsymbol{i}_2 \,,$$

$$\mathbb{F}^{\mathsf{T}} = \boldsymbol{i}_1 \otimes \boldsymbol{i}_r(\theta(p_2,t)) + \boldsymbol{i}_2 \otimes \frac{\partial \boldsymbol{x}_G}{\partial p_2}(p_2,t) + p_1 \frac{\partial \theta}{\partial p_2}(p_2,t)\boldsymbol{i}_2 \otimes \boldsymbol{i}_\theta(\theta(p_2,t)) \,.$$

- Reminding that $(\boldsymbol{a} \otimes \boldsymbol{b})(\boldsymbol{c} \otimes \boldsymbol{d}) = \langle \boldsymbol{b}, \boldsymbol{c} \rangle \boldsymbol{a} \otimes \boldsymbol{d}$:

$$\begin{aligned}
\mathbb{F}^{\mathsf{T}}\mathbb{F} &= \langle \boldsymbol{i}_r, \boldsymbol{i}_r \rangle \boldsymbol{i}_1 \otimes \boldsymbol{i}_1 + \left\langle \frac{\partial \boldsymbol{x}_G}{\partial p_2}, \boldsymbol{i}_r \right\rangle \boldsymbol{i}_1 \otimes \boldsymbol{i}_2 + p_1 \frac{\partial \theta}{\partial p_2} \langle \boldsymbol{i}_r, \boldsymbol{i}_\theta \rangle \boldsymbol{i}_1 \otimes \boldsymbol{i}_2 \\
&+ \left\langle \frac{\partial \boldsymbol{x}_G}{\partial p_2}, \boldsymbol{i}_r \right\rangle \boldsymbol{i}_2 \otimes \boldsymbol{i}_1 + \left\langle \frac{\partial \boldsymbol{x}_G}{\partial p_2}, \frac{\partial \boldsymbol{x}_G}{\partial p_2} \right\rangle \boldsymbol{i}_2 \otimes \boldsymbol{i}_2 + p_1 \frac{\partial \theta}{\partial p_2} \left\langle \frac{\partial \boldsymbol{x}_G}{\partial p_2}, \boldsymbol{i}_\theta \right\rangle \boldsymbol{i}_2 \otimes \boldsymbol{i}_2 \\
&+ p_1 \frac{\partial \theta}{\partial p_2} \langle \boldsymbol{i}_\theta, \boldsymbol{i}_r \rangle \boldsymbol{i}_2 \otimes \boldsymbol{i}_1 + p_1 \frac{\partial \theta}{\partial p_2} \left\langle \frac{\partial \boldsymbol{x}_G}{\partial p_2}, \boldsymbol{i}_\theta \right\rangle \boldsymbol{i}_2 \otimes \boldsymbol{i}_2 + \left(p_1 \frac{\partial \theta}{\partial p_2} \right)^2 \langle \boldsymbol{i}_\theta, \boldsymbol{i}_\theta \rangle \boldsymbol{i}_2 \otimes \boldsymbol{i}_2
\end{aligned}$$

Question #2: Green-Lagrange tensor

- $\mathbb{E} = \frac{1}{2}(\mathbb{F}^{\mathsf{T}}\mathbb{F} - \boldsymbol{I})$ with:

$$\mathbb{F} = \boldsymbol{i}_r(\theta(p_2,t)) \otimes \boldsymbol{i}_1 + \frac{\partial \boldsymbol{x}_G}{\partial p_2}(p_2,t) \otimes \boldsymbol{i}_2 + p_1 \frac{\partial \theta}{\partial p_2}(p_2,t)\boldsymbol{i}_\theta(\theta(p_2,t)) \otimes \boldsymbol{i}_2 \,,$$

$$\mathbb{F}^{\mathsf{T}} = \boldsymbol{i}_1 \otimes \boldsymbol{i}_r(\theta(p_2,t)) + \boldsymbol{i}_2 \otimes \frac{\partial \boldsymbol{x}_G}{\partial p_2}(p_2,t) + p_1 \frac{\partial \theta}{\partial p_2}(p_2,t)\boldsymbol{i}_2 \otimes \boldsymbol{i}_\theta(\theta(p_2,t)) \,.$$

- Reminding that $(\boldsymbol{a} \otimes \boldsymbol{b})(\boldsymbol{c} \otimes \boldsymbol{d}) = \langle \boldsymbol{b}, \boldsymbol{c} \rangle \boldsymbol{a} \otimes \boldsymbol{d}$:

$$\begin{aligned}\mathbb{F}^{\mathsf{T}}\mathbb{F} =& \boldsymbol{i}_1 \otimes \boldsymbol{i}_1 + 2\left\langle \frac{\partial \boldsymbol{x}_G}{\partial p_2}, \boldsymbol{i}_r \right\rangle \boldsymbol{i}_1 \otimes_s \boldsymbol{i}_2 + 2p_1 \frac{\partial \theta}{\partial p_2} \left\langle \frac{\partial \boldsymbol{x}_G}{\partial p_2}, \boldsymbol{i}_\theta \right\rangle \boldsymbol{i}_2 \otimes \boldsymbol{i}_2 \\ &+ \left\| \frac{\partial \boldsymbol{x}_G}{\partial p_2} \right\|^2 \boldsymbol{i}_2 \otimes \boldsymbol{i}_2 + \left(p_1 \frac{\partial \theta}{\partial p_2} \right)^2 \boldsymbol{i}_2 \otimes \boldsymbol{i}_2\end{aligned}$$

Question #2: Green-Lagrange tensor

- $\mathbb{E} = \frac{1}{2}(\mathbb{F}^\mathsf{T}\mathbb{F} - \boldsymbol{I})$ with:

$$\begin{aligned}\mathbb{F}^\mathsf{T}\mathbb{F} =& \boldsymbol{i}_1 \otimes \boldsymbol{i}_1 + 2\left\langle \frac{\partial \boldsymbol{x}_G}{\partial p_2}, \boldsymbol{i}_r \right\rangle \boldsymbol{i}_1 \otimes_s \boldsymbol{i}_2 + 2p_1 \frac{\partial \theta}{\partial p_2} \left\langle \frac{\partial \boldsymbol{x}_G}{\partial p_2}, \boldsymbol{i}_\theta \right\rangle \boldsymbol{i}_2 \otimes \boldsymbol{i}_2 \\ &+ \left\| \frac{\partial \boldsymbol{x}_G}{\partial p_2} \right\|^2 \boldsymbol{i}_2 \otimes \boldsymbol{i}_2 + \left(p_1 \frac{\partial \theta}{\partial p_2} \right)^2 \boldsymbol{i}_2 \otimes \boldsymbol{i}_2\end{aligned}$$

- But $\boldsymbol{I} = \boldsymbol{i}_1 \otimes \boldsymbol{i}_1 + \boldsymbol{i}_2 \otimes \boldsymbol{i}_2$ and therefore:

$$\boxed{\mathbb{E} = \left\langle \frac{\partial \boldsymbol{x}_G}{\partial p_2}, \boldsymbol{i}_r \right\rangle \boldsymbol{i}_1 \otimes_s \boldsymbol{i}_2 + \frac{1}{2}\left[\left(\frac{\partial \boldsymbol{x}_G}{\partial p_2} + p_1 \frac{\partial \theta}{\partial p_2} \boldsymbol{i}_\theta \right)^2 - 1 \right] \boldsymbol{i}_2 \otimes \boldsymbol{i}_2}\,.$$

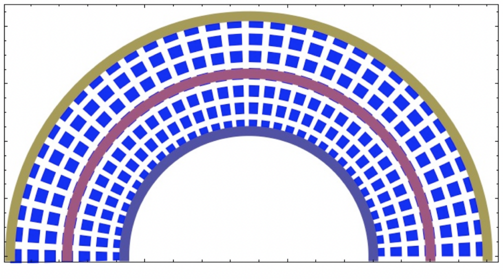
Question #3: $\langle \mathbb{E}\boldsymbol{i}_1, \boldsymbol{i}_1 \rangle$

- $\langle \mathbb{E}\boldsymbol{i}_1, \boldsymbol{i}_1 \rangle = E_{11}$ is the $\boldsymbol{i}_1 \otimes \boldsymbol{i}_1$ term of $\mathbb{E}$: $E_{11} = 0$!!!
- There is no stretching along the $\boldsymbol{i}_1$ axis/$\boldsymbol{p}_1$ coordinate.
- However:

$$\langle \mathbb{E}\boldsymbol{i}_2, \boldsymbol{i}_2 \rangle = E_{22} = \frac{1}{2}\left[\left(\frac{\partial \boldsymbol{x}_G}{\partial p_2} + p_1 \frac{\partial \theta}{\partial p_2} \boldsymbol{i}_\theta\right)^2 - 1\right].$$

Question #4: $\boldsymbol{x}_G(p_2) = \frac{L}{\pi}(-\boldsymbol{i}_1 + \boldsymbol{i}_r(\theta(p_2)))$, $\theta(p_2) = \frac{\pi p_2}{L}$

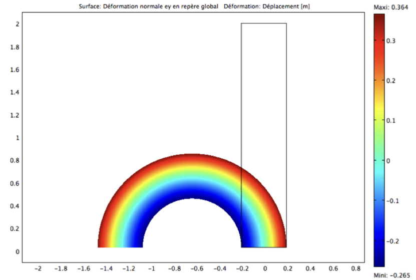
- $\theta(0) = 0$ and $\theta(L) = \pi$ so that $p_2 \mapsto \boldsymbol{x}_G(p_2)$ is an half-circle with $\boldsymbol{x}_G(0) = \boldsymbol{0}$ and $\boldsymbol{x}_G(L) = -\frac{2L}{\pi}\boldsymbol{i}_1$.



Question #5: E_{22}

- $\dfrac{\partial \boldsymbol{x}_G}{\partial p_2} = \dfrac{L}{\pi}\theta'(p_2)\boldsymbol{i}_\theta(\theta(p_2))$ and $\theta'(p_2) = \dfrac{\pi}{L}$, therefore:

$$
\begin{aligned}
E_{22} &= \langle \mathbb{E}\boldsymbol{i}_2, \boldsymbol{i}_2 \rangle \\
&= \frac{1}{2}\left[\left(\frac{\partial \boldsymbol{x}_G}{\partial p_2} + p_1 \frac{\partial \theta}{\partial p_2}\boldsymbol{i}_\theta\right)^2 - 1\right] \\
&= \frac{1}{2}\left[\left(1 + \pi\frac{p_1}{L}\right)^2 - 1\right] \\
&= \pi\frac{p_1}{L}\left(1 + \frac{\pi}{2}\frac{p_1}{L}\right)
\end{aligned}
$$



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- The maximum is obtained for $p_1 = +h$ and the minimum for $p_1 = -h$.

Large beam bending

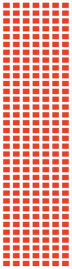
Solution

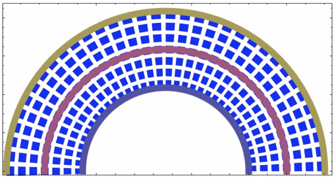
Question #6: E_{12}

- $\frac{\partial \boldsymbol{x}_G}{\partial p_2} = \frac{L}{\pi}\theta'(p_2)\boldsymbol{i}_\theta(\theta(p_2))$ and $\theta'(p_2) = \frac{\pi}{L}$, therefore:

$$E_{12} = \langle \boldsymbol{i}_1, \mathbb{E}\boldsymbol{i}_2 \rangle = \langle \mathbb{E}\boldsymbol{i}_1, \boldsymbol{i}_2 \rangle = \frac{1}{2}\left\langle \frac{\partial \boldsymbol{x}_G}{\partial p_2}, \boldsymbol{i}_r \right\rangle = 0 \text{ !!}$$

- The cross-sections remain $\perp$ to the middle-line.





Large beam bending
Solution

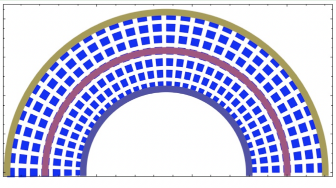
Question #7: $\boldsymbol{x}(\boldsymbol{p}) = \boldsymbol{x}_G(p_2) + p_1 \boldsymbol{i}_r(1.1\theta(\boldsymbol{p}_2))$

- After tedious computations ($\alpha = 1.1$):

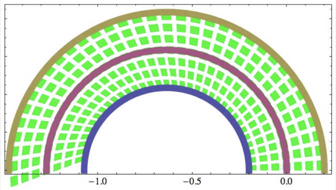
$$\mathbb{E} = \left\langle \frac{\partial \boldsymbol{x}_G}{\partial p_2}, \boldsymbol{i}_r(\alpha\theta) \right\rangle \boldsymbol{i}_1 \otimes_s \boldsymbol{i}_2 + \frac{1}{2}\left[\left(\frac{\partial \boldsymbol{x}_G}{\partial p_2} + \alpha p_1 \frac{\partial \theta}{\partial p_2} \boldsymbol{i}_\theta(\alpha\theta) \right)^2 - 1 \right] \boldsymbol{i}_2 \otimes \boldsymbol{i}_2$$

$$E_{12} = \frac{1}{2}\left\langle \frac{\partial \boldsymbol{x}_G}{\partial p_2}, \boldsymbol{i}_r(\alpha\theta) \right\rangle = \frac{1}{2}\langle \boldsymbol{i}_\theta(\theta), \boldsymbol{i}_r(\alpha\theta) \rangle = \frac{1}{2} \sin\left(\pi(\alpha - 1)\frac{p_2}{L} \right) .$$

- The cross-sections are no longer $\perp$ to the middle-line.



$\alpha = 1$



$\alpha = 1.1$

Question #8: Small strain tensor

- $\boldsymbol{u}(\boldsymbol{p}) = \boldsymbol{x}(\boldsymbol{p}) - \boldsymbol{p} = \boldsymbol{u}_G(p_2) + p_1(\boldsymbol{i}_r(\theta(p_2)) - \boldsymbol{i}_1)$ with $\boldsymbol{u}_G(p_2) = \boldsymbol{x}_G(p_2) - p_2\boldsymbol{i}_2$, therefore ($\boldsymbol{\varepsilon} = \frac{\partial \boldsymbol{u}}{\partial p_j} \otimes_s \boldsymbol{i}_j$):

$$\begin{aligned}\boldsymbol{\varepsilon} =& (\boldsymbol{i}_r(\theta(p_2)) - \boldsymbol{i}_1) \otimes_s \boldsymbol{i}_1 + \boldsymbol{u}_G'(p_2) \otimes_s \boldsymbol{i}_2 \\ &+ p_1\theta'(p_2)\boldsymbol{i}_\theta(\theta(p_2)) \otimes_s \boldsymbol{i}_2\end{aligned}$$

- $\|\boldsymbol{u}\| \ll h \Rightarrow \boldsymbol{u} \approx \boldsymbol{u}_G \ll h < L$ and $\theta \ll 1$ such that $\boldsymbol{i}_r \approx \boldsymbol{i}_1 + \theta\boldsymbol{i}_2$;
- $\|\mathbb{D}_{\boldsymbol{p}}\boldsymbol{u}\| \ll 1 \Rightarrow \|\boldsymbol{u}_G'(p_2)\| \ll 1$ and $\theta'(p_2) \ll 1$;
- This yields $\varepsilon_{11} \approx 0$ (no "breathing" of the beam) and:

$$\begin{aligned}\varepsilon_{12} &\approx \frac{1}{2}\left(\langle \boldsymbol{u}_G'(p_2), \boldsymbol{i}_1 \rangle + \theta(p_2)\right) , \\ \varepsilon_{22} &= \langle \boldsymbol{u}_G'(p_2), \boldsymbol{i}_2 \rangle + p_1\theta'(p_2) .\end{aligned}$$