Brazilian test

1EL5000–Continuum Mechanics – Tutorial Class #2

É. Savin[1,2]
eric.savin@{centralesupelec,onera}.fr

[1]Information Processing and Systems Dept.
ONERA, France

[2]Mechanical and Civil Engineering Dept.
CentraleSupélec, France

March 18, 2021

Outline

Outline

1EL5000/S2

É. Savin

Some algebra
Vector & tensor products
Vector & tensor analysis

Stresses

2.3 Brazilian test

- Scalar product:

$$\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^a\,,\quad\langle\boldsymbol{a},\boldsymbol{b}\rangle=\sum_{j=1}^a a_jb_j=a_jb_j\,,$$

The last equality is Einstein's summation convention.

- Tensors and tensor product (or outer product):

$$\boldsymbol{A}\in\mathbb{R}^a\to\mathbb{R}^b\,,\quad\boldsymbol{A}=\boldsymbol{a}\otimes\boldsymbol{b}\,,\quad\boldsymbol{a}\in\mathbb{R}^a\,,\boldsymbol{b}\in\mathbb{R}^b\,.$$

- Tensor application to vectors:

$$\boldsymbol{A}=\boldsymbol{a}\otimes\boldsymbol{b}\in\mathbb{R}^a\to\mathbb{R}^b\,,\boldsymbol{c}\in\mathbb{R}^b\,,\quad\boldsymbol{A}\boldsymbol{c}=\langle\boldsymbol{b},\boldsymbol{c}\rangle\boldsymbol{a}\,.$$

- Product of tensors ≡ composition of linear maps:

$$\boldsymbol{A}=\boldsymbol{a}\otimes\boldsymbol{b}\,,\boldsymbol{B}=\boldsymbol{c}\otimes\boldsymbol{d}\,,\quad\boldsymbol{A}\boldsymbol{B}=\langle\boldsymbol{b},\boldsymbol{c}\rangle\boldsymbol{a}\otimes\boldsymbol{d}\,.$$

- Scalar product of tensors:

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \operatorname{Tr}(\boldsymbol{A}\boldsymbol{B}^{\mathsf{T}}) := \boldsymbol{A} : \boldsymbol{B} = A_{jk} B_{jk} \,.$$

- Let $\{\boldsymbol{e}_j\}_{j=1}^d$ be a Cartesian basis in $\mathbb{R}^d$. Then:

$$\begin{aligned} a_j &= \langle \boldsymbol{a}, \boldsymbol{e}_j \rangle \,, \\ A_{jk} &= \langle \boldsymbol{A}, \boldsymbol{e}_j \otimes \boldsymbol{e}_k \rangle = \boldsymbol{A} : \boldsymbol{e}_j \otimes \boldsymbol{e}_k \\ &= \langle \boldsymbol{A}\boldsymbol{e}_k, \boldsymbol{e}_j \rangle \,, \end{aligned}$$

such that:

$$\begin{aligned} \boldsymbol{a} &= a_j \boldsymbol{e}_j \,, \\ \boldsymbol{A} &= A_{jk} \boldsymbol{e}_j \otimes \boldsymbol{e}_k \,. \end{aligned}$$

- Example: the identity matrix

$$\boldsymbol{I} = \boldsymbol{e}_j \otimes \boldsymbol{e}_j \,.$$

- Gradient of a vector function $\boldsymbol{a}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$:

$$\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a} = \frac{\partial \boldsymbol{a}}{\partial x_j} \otimes \boldsymbol{e}_j \,.$$

- Divergence of a vector function $\boldsymbol{a}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$:

$$\mathrm{div}_{\boldsymbol{x}}\boldsymbol{a} = \langle \boldsymbol{\nabla}_{\boldsymbol{x}}, \boldsymbol{a} \rangle = \mathrm{Tr}(\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a}) = \frac{\partial a_j}{\partial x_j} \,.$$

- Divergence of a tensor function $\boldsymbol{A}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$:

$$\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{A} = \frac{\partial (\boldsymbol{A}\boldsymbol{e}_j)}{\partial x_j} \,.$$

- Gradient of a vector function $\boldsymbol{a}(r,\theta,z)$:

$$\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a} = \frac{\partial\boldsymbol{a}}{\partial r}\otimes\boldsymbol{e}_r + \frac{\partial\boldsymbol{a}}{\partial\theta}\otimes\frac{\boldsymbol{e}_\theta}{r} + \frac{\partial\boldsymbol{a}}{\partial z}\otimes\boldsymbol{e}_z\,.$$

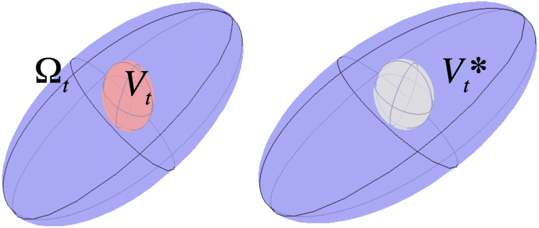
- Divergence of a vector function $\boldsymbol{a}(r,\theta,z)$:

$$\mathrm{div}_{\boldsymbol{x}}\boldsymbol{a} = \left\langle\frac{\partial\boldsymbol{a}}{\partial r},\boldsymbol{e}_r\right\rangle + \left\langle\frac{\partial\boldsymbol{a}}{\partial\theta},\frac{\boldsymbol{e}_\theta}{r}\right\rangle + \left\langle\frac{\partial\boldsymbol{a}}{\partial z},\boldsymbol{e}_z\right\rangle\,.$$

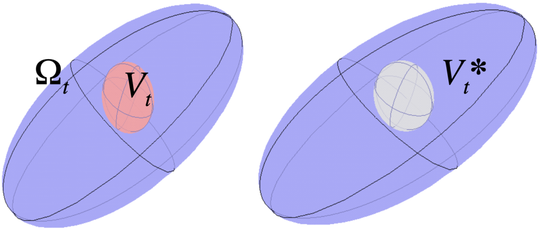
- Divergence of a tensor function $\boldsymbol{A}(r,\theta,z)$:

$$\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{A} = \frac{\partial\boldsymbol{A}}{\partial r}\boldsymbol{e}_r + \frac{\partial\boldsymbol{A}}{\partial\theta}\frac{\boldsymbol{e}_\theta}{r} + \frac{\partial\boldsymbol{A}}{\partial z}\boldsymbol{e}_z\,.$$

Outline



- Modeling the actions of $V_t^* = \Omega_t \backslash V_t$ on V_t:
  - Local contact actions $\boldsymbol{T}$ exerted by proximal subdomains with short ranges;
  - Virtual cutting surface ∂V_t with outward unit normal $\boldsymbol{n}(\boldsymbol{x})$ defining the tangent plane at $\boldsymbol{x}$.



- Cauchy's postulate #1: surface traction (in N/m^2)

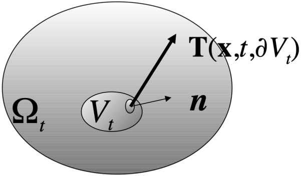
$$\boldsymbol{T}(\boldsymbol{x},t;\partial V_t)=\boldsymbol{f}_{\partial V_t}(\boldsymbol{x},t)\,,\quad \forall\boldsymbol{x}\in\partial V_t\,,\forall V_t\,.$$

- Cauchy's postulate #2: the surface curvature has no influence on the traction

$$\boldsymbol{T}(\boldsymbol{x},t;\partial V_t)=\boldsymbol{T}(\boldsymbol{x},t;\boldsymbol{n}(\boldsymbol{x}))\,,\quad \forall\boldsymbol{x}\in\partial V_t\,,\forall V_t\,.$$

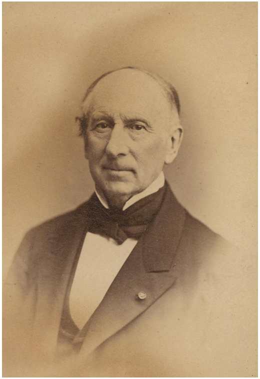
Cauchy's stress tensor

Cauchy's theorem



- Cauchy's theorem: consider elementary V_t (thin cylinder, tetrahedron), then $\forall \boldsymbol{x} \in \partial V_t$
  1. Action/reaction $\boldsymbol{T}(\boldsymbol{x}, t; \boldsymbol{n}(\boldsymbol{x})) + \boldsymbol{T}(\boldsymbol{x}, t; -\boldsymbol{n}(\boldsymbol{x})) = \mathbf{0}$;
  2. Tetrahedron's lemma $\boldsymbol{T}(\boldsymbol{x}, t; \boldsymbol{n}(\boldsymbol{x})) = \boldsymbol{\sigma}(\boldsymbol{x}, t)\boldsymbol{n}(\boldsymbol{x})$.
- Boundary conditions:

$$\boldsymbol{\sigma}(\boldsymbol{x}, t)\boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{f}_S(\boldsymbol{x}, t)\,, \quad \forall \boldsymbol{x} \in \partial\Omega_t\,.$$

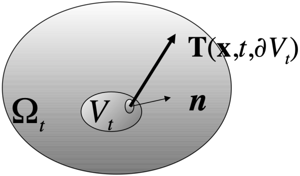


Augustin Louis Cauchy [1789-1857]

- X 1805
- Ingénieur Ponts-et-Chaussées
- Académie des Sciences 1816
- Prof. Analyse et Mécanique à l'X 1815-1830

https://mathshistory.st-andrews.ac.uk/Biographies/Cauchy/

Equations of motion



- $\forall V_t \subseteq \Omega_t$:

$$\int_{V_t} \varrho \boldsymbol{a} dV = \int_{V_t} \boldsymbol{f}_v dV + \int_{\partial V_t} \boldsymbol{T} dS \,,$$

$$\int_{V_t} \boldsymbol{x} \times \varrho \boldsymbol{a} dV = \int_{V_t} \boldsymbol{x} \times \boldsymbol{f}_v dV + \int_{\partial V_t} \boldsymbol{x} \times \boldsymbol{T} dS \,.$$

Equations of motion

- $\forall V_t \subseteq \Omega_t$, $\forall \boldsymbol{c} \in \mathbb{R}^3$:

$$\begin{aligned}
\int_{V_t} \langle \varrho \boldsymbol{a} - \boldsymbol{f}_v, \boldsymbol{c} \rangle \, dV &= \int_{\partial V_t} \langle \boldsymbol{\sigma} \boldsymbol{n}, \boldsymbol{c} \rangle \, dS \\
&= \int_{\partial V_t} \left\langle \boldsymbol{n}, \boldsymbol{\sigma}^\mathsf{T} \boldsymbol{c} \right\rangle dS \\
&= \int_{V_t} \operatorname{div}(\boldsymbol{\sigma}^\mathsf{T} \boldsymbol{c}) dV \qquad \text{(Stokes formula)} \\
&\stackrel{\text{def}}{=} \int_{V_t} \langle \mathbf{Div}\, \boldsymbol{\sigma}, \boldsymbol{c} \rangle \, dV \\
\int_{V_t} (\varrho \boldsymbol{a} - \boldsymbol{f}_v) dV &= \int_{V_t} \mathbf{Div}\, \boldsymbol{\sigma} \, dV \\
\varrho \boldsymbol{a} - \boldsymbol{f}_v &= \mathbf{Div}\, \boldsymbol{\sigma} \qquad \text{(localization lemma)}
\end{aligned}$$

Equations of motion

- The equation of motion for a continuous medium $\boldsymbol{x} \in \Omega_t$:

$$\boxed{\varrho \boldsymbol{a} = \mathbf{Div}_{\boldsymbol{x}} \boldsymbol{\sigma} + \boldsymbol{f}_v},$$

where $\mathbf{Div}_{\boldsymbol{x}} \boldsymbol{\sigma} \approx \frac{1}{V} \int_{\partial V} \boldsymbol{\sigma} \boldsymbol{n} dS$ as $|V| \rightarrow 0$.

- The balance of momentum yields:

$$\boldsymbol{\sigma}^{\mathsf{T}} = \boldsymbol{\sigma} .$$

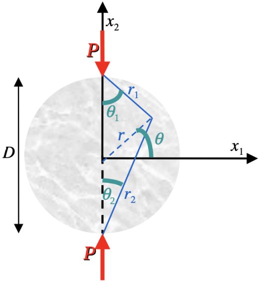
This is no longer the case if Cauchy's postulate #1 does not hold (surface torques N.m/m^2).

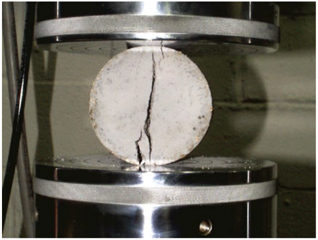
- Boundary conditions $\boldsymbol{x} \in \partial\Omega_t$:

$$\boldsymbol{\sigma} \boldsymbol{n} = \boldsymbol{f}_S .$$

Outline

Brazilian test
Setup





$$\boldsymbol{p} = r\boldsymbol{i}_r(\theta) + z\boldsymbol{i}_z \,, \quad (r,\theta,z) \in \left]0, \frac{D}{2}\right[\times]0, 2\pi[\times]0, H[\,,$$

$$\boldsymbol{\sigma}(\boldsymbol{x}) = k\frac{\cos\theta_1}{r_1}\boldsymbol{i}_{r_1}(\theta_1)\otimes\boldsymbol{i}_{r_1}(\theta_1) + k\frac{\cos\theta_2}{r_2}\boldsymbol{i}_{r_2}(\theta_2)\otimes\boldsymbol{i}_{r_2}(\theta_2) - \frac{k}{D}(\boldsymbol{I} - \boldsymbol{i}_z\otimes\boldsymbol{i}_z)$$

Question #1: $\mathbf{Div}(\frac{\cos\theta}{r}\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta))$?

- Remind that $\mathbf{Div}\,\boldsymbol{A} = \frac{\partial\boldsymbol{A}}{\partial r}\boldsymbol{i}_r + \frac{\partial\boldsymbol{A}}{\partial\theta}\frac{\boldsymbol{i}_\theta}{r} + \frac{\partial\boldsymbol{A}}{\partial z}\boldsymbol{i}_z$.
- In the present case:

$$\frac{\partial}{\partial r}\left(\frac{\cos\theta}{r}\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta)\right) = -\frac{\cos\theta}{r^2}\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta)\,,$$

$$\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{r}\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta)\right) = -\frac{\sin\theta}{r}\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta) + 2\frac{\cos\theta}{r}\boldsymbol{i}_r(\theta)\otimes_s\boldsymbol{i}_\theta(\theta)\,,$$

$$\frac{\partial}{\partial z}\left(\frac{\cos\theta}{r}\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta)\right) = \mathbf{0}\,.$$

Question #1: $\mathbf{Div}(\frac{\cos\theta}{r}\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta))$?

- Remind that $\mathbf{Div}\,\boldsymbol{A} = \frac{\partial\boldsymbol{A}}{\partial r}\boldsymbol{i}_r + \frac{\partial\boldsymbol{A}}{\partial\theta}\frac{\boldsymbol{i}_\theta}{r} + \frac{\partial\boldsymbol{A}}{\partial z}\boldsymbol{i}_z$.
- In the present case:

$$
\begin{aligned}
\frac{\partial}{\partial r}\left(\frac{\cos\theta}{r}\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta)\right) &= -\frac{\cos\theta}{r^2}\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta)\,,\\
\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{r}\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta)\right) &= -\frac{\sin\theta}{r}\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta) + 2\frac{\cos\theta}{r}\boldsymbol{i}_r(\theta)\otimes_s\boldsymbol{i}_\theta(\theta)\,,\\
\frac{\partial}{\partial z}\left(\frac{\cos\theta}{r}\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta)\right) &= \mathbf{0}\,.
\end{aligned}
$$

- Hence (remind that $(\boldsymbol{a}\otimes\boldsymbol{b})\boldsymbol{c} = \langle\boldsymbol{b},\boldsymbol{c}\rangle\,\boldsymbol{a}$):

$$
\mathbf{Div}\left(\frac{\cos\theta}{r}\boldsymbol{i}_r(\theta)\otimes\boldsymbol{i}_r(\theta)\right) = -\frac{\cos\theta}{r^2}\boldsymbol{i}_r(\theta) + \frac{\cos\theta}{r^2}\boldsymbol{i}_r(\theta) = \mathbf{0}.
$$

- Besides $\mathbf{Div}(\boldsymbol{I}-\boldsymbol{i}_z\otimes\boldsymbol{i}_z) = \mathbf{0}$, therefore $\mathbf{Div}\,\boldsymbol{\sigma} = \mathbf{0}$ QED.

Brazilian test
Solution

Question #2: $\boldsymbol{\sigma}\boldsymbol{n} = \mathbf{0}$ on $\{r = \frac{D}{2}\}$?

- On $\{r = \frac{D}{2}\}$ we have $\theta_1 + \theta_2 = \frac{\pi}{2}$ s.t. $\boldsymbol{i}_{r_1}(\theta_1) \perp \boldsymbol{i}_{r_2}(\theta_2)$, and $\boldsymbol{n} = \boldsymbol{i}_r(\theta)$.
- Therefore:

$$\boldsymbol{\sigma}\boldsymbol{n} = k\frac{\cos\theta_1}{r_1}\left\langle \boldsymbol{i}_r, \boldsymbol{i}_{r_1}\right\rangle \boldsymbol{i}_{r_1} + k\frac{\cos\theta_2}{r_2}\left\langle \boldsymbol{i}_r, \boldsymbol{i}_{r_2}\right\rangle \boldsymbol{i}_{r_2} - \frac{k}{D}\boldsymbol{i}_r\,.$$

Question #2: $\boldsymbol{\sigma n} = \mathbf{0}$ on $\{r = \frac{D}{2}\}$?

- On $\{r = \frac{D}{2}\}$ we have $\theta_1 + \theta_2 = \frac{\pi}{2}$ s.t. $\boldsymbol{i}_{r_1}(\theta_1) \perp \boldsymbol{i}_{r_2}(\theta_2)$, and $\boldsymbol{n} = \boldsymbol{i}_r(\theta)$.
- Therefore:

$$\boldsymbol{\sigma n} = k\frac{\cos\theta_1}{r_1}\langle \boldsymbol{i}_r, \boldsymbol{i}_{r_1}\rangle \boldsymbol{i}_{r_1} + k\frac{\cos\theta_2}{r_2}\langle \boldsymbol{i}_r, \boldsymbol{i}_{r_2}\rangle \boldsymbol{i}_{r_2} - \frac{k}{D}\boldsymbol{i}_r \,.$$

- But $\cos\theta_1 = \frac{r_1}{D}$ and $\cos\theta_2 = \frac{r_2}{D}$, hence:

$$\begin{aligned}\boldsymbol{\sigma n} &= \frac{k}{D}(\langle \boldsymbol{i}_r, \boldsymbol{i}_{r_1}\rangle \boldsymbol{i}_{r_1} + \langle \boldsymbol{i}_r, \boldsymbol{i}_{r_2}\rangle \boldsymbol{i}_{r_2} - \boldsymbol{i}_r) \\ &= \frac{k}{D}(\boldsymbol{i}_r - \boldsymbol{i}_r) \quad \text{QED}\,.\end{aligned}$$

Question #3: k?

- k should be homogeneous to N/m.
- Isolating the lower half of the cylinder of which unit outward normal on the plane $\{x_2 = 0\}$ is $\boldsymbol{n} = +\boldsymbol{i}_2$, the traction exerted by the upper half cylinder is:

$$\boldsymbol{\sigma}\boldsymbol{i}_2 \,|_{x_2=0} = k\frac{\cos\theta_1}{r_1}\langle \boldsymbol{i}_{r_1}, \boldsymbol{i}_2\rangle \boldsymbol{i}_{r_1} + k\frac{\cos\theta_2}{r_2}\langle \boldsymbol{i}_{r_2}, \boldsymbol{i}_2\rangle \boldsymbol{i}_{r_2} - \frac{k}{D}\boldsymbol{i}_2 \,.$$

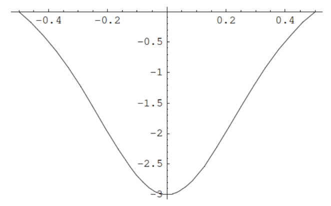
- But on $\{x_2 = 0\}$, $r_1 = r_2 = \rho$, $\theta_1 = \theta_2 = \Theta$, and:

$$\begin{aligned} \langle \boldsymbol{i}_{r_2}, \boldsymbol{i}_2\rangle &= -\langle \boldsymbol{i}_{r_1}, \boldsymbol{i}_2\rangle = \cos\Theta = \frac{D}{2\rho}\,, \\ \boldsymbol{i}_{r_2} - \boldsymbol{i}_{r_1} &= 2\cos\Theta\,\boldsymbol{i}_2\,. \end{aligned}$$

Question #3: k?

- Therefore:

$$\boldsymbol{\sigma}\boldsymbol{i}_2 \,|_{x_2=0} = \frac{k}{D}\left[\frac{D^3}{4\rho^3}(\boldsymbol{i}_{r_2} - \boldsymbol{i}_{r_1}) - \boldsymbol{i}_2\right]$$
$$= \frac{k}{D}\left(\frac{D^4}{4\rho^4} - 1\right)\boldsymbol{i}_2$$



$\frac{D}{k}\sigma_{22} \,|_{x_2=0}$

Question #3: k?

- The force exerted by the upper half on the lower half is:

$$
\begin{aligned}
\boldsymbol{F}^{\text{upper}\rightarrow\text{lower}} &= \int_0^H dz \int_{-\frac{D}{2}}^{+\frac{D}{2}} dx_1 \, \boldsymbol{\sigma} \boldsymbol{i}_2 \,|_{x_2=0} \\
&= \frac{kH}{D} \left[\int_{-\frac{D}{2}}^{+\frac{D}{2}} \left(\frac{D^4}{4(\frac{D^2}{4} + x_1^2)^2} - 1 \right) dx_1 \right] \boldsymbol{i}_2 \\
&= kH \left(2 \int_{-1}^{+1} \frac{du}{(1+u^2)^2} - 1 \right) \boldsymbol{i}_2 \\
&= \frac{k\pi H}{2} \boldsymbol{i}_2
\end{aligned}
$$

Brazilian test

Solution

Question #3: k?

- The force exerted by the upper half on the lower half is:

$$\boldsymbol{F}^{\text{upper}\to\text{lower}} = \frac{k\pi H}{2}\boldsymbol{i}_2 \,.$$

- The equilibrium of the lower half cylinder reads:

$$\boldsymbol{F}^{\text{upper}\to\text{lower}} + PH\boldsymbol{i}_2 = \boldsymbol{0}$$

(because P is in N/m), hence:

$$k = -\frac{2P}{\pi}\,, \quad \sigma_{22}\left.\right|_{x_2=0} = \frac{2P}{\pi D}\left(1 - \frac{D^4}{4\rho^4}\right)$$

and $\sigma_{22}\left.\right|_{x_2=0} < 0$ which is a compression.

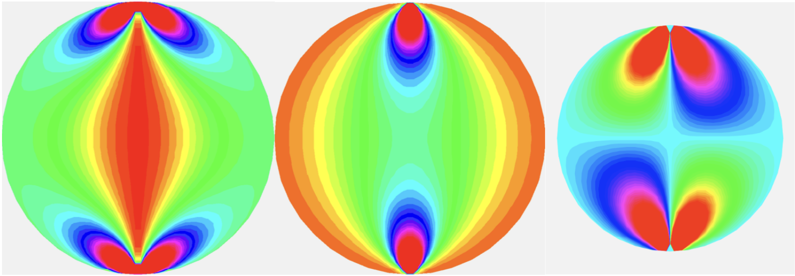
Question #4: $\boldsymbol{\sigma n}\mid_{x_1=0}$

- On $\{x_1 = 0\}$ we have $\boldsymbol{n} = +\boldsymbol{i}_1$ (for the left half, say), $\theta_1 = \theta_2 = 0$, and $\boldsymbol{i}_{r_2} = -\boldsymbol{i}_{r_1} = \boldsymbol{i}_2$.
- Therefore:

$$\begin{aligned}\boldsymbol{\sigma i}_1\mid_{x_1=0} &= \frac{k}{r_1}\langle \boldsymbol{i}_1, \boldsymbol{i}_{r_1}\rangle \boldsymbol{i}_{r_1} + \frac{k}{r_2}\langle \boldsymbol{i}_1, \boldsymbol{i}_{r_2}\rangle \boldsymbol{i}_{r_2} - \frac{k}{D}\boldsymbol{i}_1 \\ &= +\frac{2P}{\pi D}\boldsymbol{i}_1\end{aligned}$$

and $\sigma_{11}\mid_{x_1=0} > 0$ which is a traction!!!

Brazilian test



σ_{11} σ_{22} σ_{12}

Brazilian test
Solution

Question #5: $HP_{\max} = 20$ kN, $H = D = 10$ cm

- $\sigma_{11} \simeq 1.3$ MPa.
- It is easy to obtain a traction state with the Brazilian test!