Atomic force microscope

1EL5000–Continuum Mechanics – Tutorial Class #8

É. Savin[1,2]

eric.savin@{centralesupelec,onera}.fr

[1]Information Processing and Systems Dept.
ONERA, France

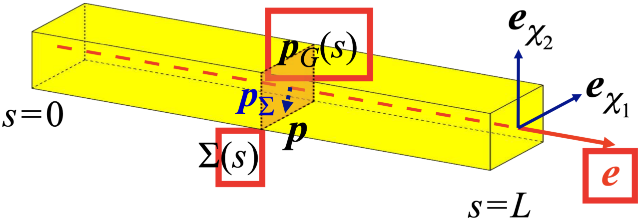
[2]Mechanical and Civil Engineering Dept.
CentraleSupélec, France

March 11, 2021

Outline

Outline

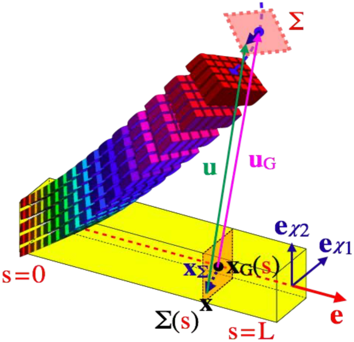
Beam kinematics
Reference configuration



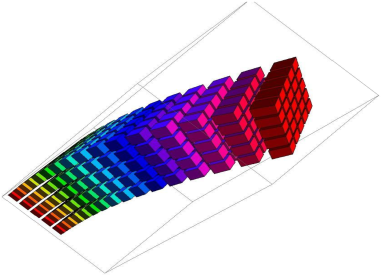
$$\begin{aligned}\boldsymbol{p}(s,\chi_1,\chi_2) &= \boldsymbol{p}_G(s) + \boldsymbol{p}_\Sigma(\chi_1,\chi_2) \\ &= s\boldsymbol{e} + \chi_1\boldsymbol{e}_{\chi_1} + \chi_2\boldsymbol{e}_{\chi_2}\end{aligned}$$

Beam kinematics

Actual configuration



$$
\begin{aligned}
\boldsymbol{x} &= \boldsymbol{x}_G(s) + \boldsymbol{x}_\Sigma \\
&= \boldsymbol{x}_G(s) + \boldsymbol{R}(s)\boldsymbol{p}_\Sigma
\end{aligned}
$$

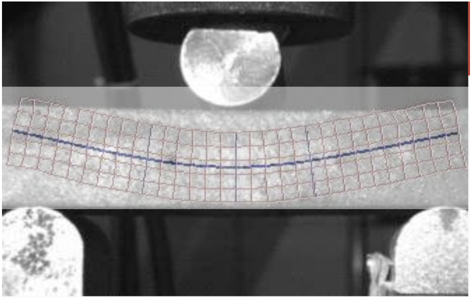


- Small perturbations $\boldsymbol{R}(s) = \boldsymbol{I} + \boldsymbol{\Theta}(s)$, $\boldsymbol{\Theta}(s)^{\mathsf{T}} = -\boldsymbol{\Theta}(s)$:

$$\begin{aligned}\boldsymbol{x}_\Sigma &= \boldsymbol{R}(s)\boldsymbol{p}_\Sigma \\ &= (\boldsymbol{I} + \boldsymbol{\Theta}(s))\boldsymbol{p}_\Sigma\,.\end{aligned}$$

- Small displacement $\boldsymbol{x}_\Sigma \simeq \boldsymbol{p}_\Sigma$:

$$\begin{aligned}\boldsymbol{u} &= \boldsymbol{x} - \boldsymbol{p} \\ &= \boldsymbol{u}_G(s) + \boldsymbol{\theta}(s) \times \boldsymbol{x}_\Sigma\,.\end{aligned}$$



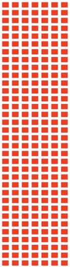
- Cross-sections remain perpendicular to the neutral line:

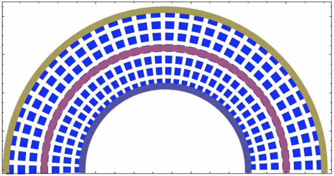
$$\boldsymbol{R}(s)\boldsymbol{e} = \frac{\boldsymbol{x}'_G(s)}{\|\boldsymbol{x}'_G(s)\|}\,.$$

- Small perturbations $\boldsymbol{R}(s)\boldsymbol{e} = (\boldsymbol{I} + \boldsymbol{\Theta}(s))\boldsymbol{e} \simeq \boldsymbol{e} + \boldsymbol{u}'_G(s)$:

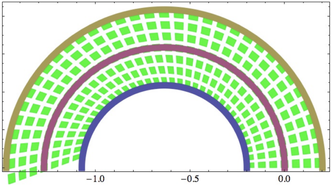
$$\boxed{\boldsymbol{\theta}_\Sigma(s) = \boldsymbol{e} \times \boldsymbol{u}'_{G\Sigma}(s)}$$

Recap: 1.3 Large beam bending





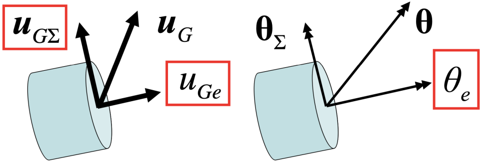
$\alpha = 1$



$\alpha = 1.1$

Beam kinematics

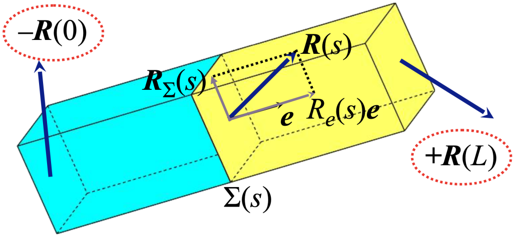
Independent unknowns



- Elongation $u_{Ge} = \langle \boldsymbol{u}_G, \boldsymbol{e} \rangle$;
- Deflection $\boldsymbol{u}_{G\Sigma} = \boldsymbol{u}_G - u_{Ge}\boldsymbol{e}$;
- Torsion rotation $\theta_e = \langle \boldsymbol{\theta}, \boldsymbol{e} \rangle$.

Outline

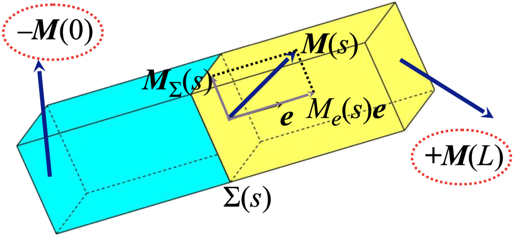
Beam statics
Resultant forces



- Resultant force:

$$\boldsymbol{R}(s) = \int_\Sigma \boldsymbol{\sigma}\boldsymbol{e}\,dS\,;$$

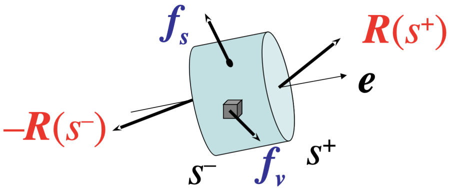
- Normal force $R_e(s) = \langle \boldsymbol{R}(s), \boldsymbol{e} \rangle$;
- Shear force $\boldsymbol{R}_\Sigma(s) = \boldsymbol{R}(s) - R_e(s)\boldsymbol{e}$;
- $\boldsymbol{R}(0)$ and $\boldsymbol{R}(L)$ given by the boundary conditions.



- Resultant moment:

$$\boldsymbol{M}(s)=\int_{\Sigma}\boldsymbol{x}_{\Sigma}\times\boldsymbol{\sigma}\boldsymbol{e}\,dS\,;$$

- Torsion moment $M_e(s)=\langle\boldsymbol{M}(s),\boldsymbol{e}\rangle$;
- Bending moment $\boldsymbol{M}_{\Sigma}(s)=\boldsymbol{M}(s)-M_e(s)\boldsymbol{e}$;
- $\boldsymbol{M}(0)$ and $\boldsymbol{M}(L)$ given by the boundary conditions.



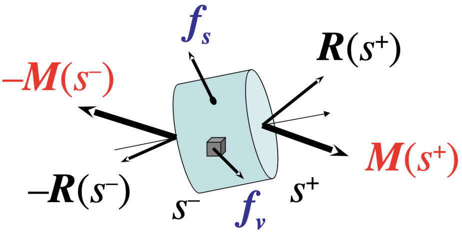
- Linear external forces:

$$\boldsymbol{f}_l(s) = \int_\Sigma \boldsymbol{f}_v dS + \int_{\partial\Sigma} \boldsymbol{f}_s d\zeta\,;$$

- Local equilibrium of the cross-section:

$$\boldsymbol{R}'(s) + \boldsymbol{f}_l(s) = \boldsymbol{0}\,.$$

Beam statics
Local balance of moments



- Linear external torques:

$$\boldsymbol{c}_l(s) = \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{f}_v dS + \int_{\partial\Sigma} \boldsymbol{x}_\Sigma \times \boldsymbol{f}_s d\zeta\,;$$

- Local equilibrium of the cross-section:

$$\boldsymbol{M}'(s) + \boldsymbol{e} \times \boldsymbol{R}(s) + \boldsymbol{c}_l(s) = \boldsymbol{0}\,.$$

Beam statics
Local balance of forces–alternative point of view

- Local static equilibrium of a continuum medium:

$$\mathbf{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_v = \mathbf{0}\,.$$

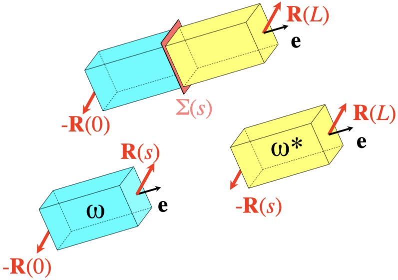
- Then integrate over the cross-section Σ:

$$\begin{aligned}
\mathbf{0} &= \int_\Sigma \mathbf{Div}\,\boldsymbol{\sigma}\,dS + \int_\Sigma \boldsymbol{f}_v\,dS \\
&= \int_\Sigma \frac{\partial\boldsymbol{\sigma}}{\partial s}\boldsymbol{e}\,dS + \int_\Sigma \mathbf{Div}_\Sigma\boldsymbol{\sigma}\,dS + \int_\Sigma \boldsymbol{f}_v\,dS \\
&= \frac{\partial}{\partial s}\left(\int_\Sigma \boldsymbol{\sigma}\boldsymbol{e}\,dS\right) + \int_{\partial\Sigma} \boldsymbol{\sigma}\boldsymbol{n}\,d\zeta + \int_\Sigma \boldsymbol{f}_v\,dS \\
&= \boldsymbol{R}'(s) + \int_{\partial\Sigma} \boldsymbol{f}_s\,d\zeta + \int_\Sigma \boldsymbol{f}_v\,dS \\
&= \boldsymbol{R}'(s) + \boldsymbol{f}_l(s)\,.
\end{aligned}$$

- Then integrate over the cross-section Σ:

$$
\begin{aligned}
\mathbf{0} &= \int_\Sigma \boldsymbol{x}_\Sigma \times \mathbf{Div}\,\boldsymbol{\sigma}\,dS + \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{f}_v\,dS \\
&= \int_\Sigma \boldsymbol{x}_\Sigma \times \frac{\partial\boldsymbol{\sigma}}{\partial s}\boldsymbol{e}\,dS + \int_\Sigma \boldsymbol{x}_\Sigma \times \frac{\partial\boldsymbol{\sigma}\boldsymbol{e}_\alpha}{\partial x_\alpha}\,dS + \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{f}_v\,dS \\
&= \frac{\partial}{\partial s}\left(\int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{\sigma}\boldsymbol{e}\,dS\right) + \int_\Sigma \frac{\partial(\boldsymbol{x}_\Sigma \times \boldsymbol{\sigma}\boldsymbol{e}_\alpha)}{\partial x_\alpha}\,dS \\
&\qquad - \int_\Sigma \boldsymbol{e}_\alpha \times \boldsymbol{\sigma}\boldsymbol{e}_\alpha\,dS + \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{f}_v\,dS \\
&= \boldsymbol{M}'(s) + \int_{\partial\Sigma} \boldsymbol{x}_\Sigma \times \boldsymbol{\sigma}\boldsymbol{n}\,d\zeta + \int_\Sigma \boldsymbol{e} \times \boldsymbol{\sigma}\boldsymbol{e}\,dS + \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{f}_v\,dS \\
&= \boldsymbol{M}'(s) + \boldsymbol{e} \times \boldsymbol{R}(s) + \boldsymbol{c}_l(s)\,,
\end{aligned}
$$

since $\boldsymbol{e}_\alpha \times \boldsymbol{\sigma}\boldsymbol{e}_\alpha + \boldsymbol{e} \times \boldsymbol{\sigma}\boldsymbol{e} = \mathbf{0}$ from the symmetry of $\boldsymbol{\sigma}$.

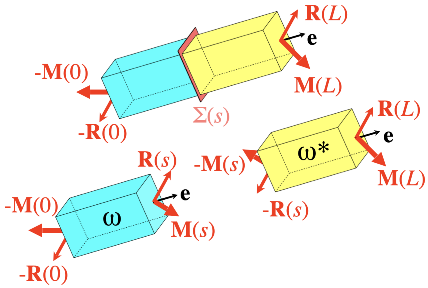


- Global equilibrium of the left section $s \in [0,s]$:

$$\boldsymbol{R}(s) - \boldsymbol{R}(0) + \int_0^s \boldsymbol{f}_l d\zeta = \boldsymbol{0}\,;$$

- Global equilibrium of the right section $s \in [s,L]$:

$$\boldsymbol{R}(L) - \boldsymbol{R}(s) + \int_s^L \boldsymbol{f}_l d\zeta = \boldsymbol{0}\,.$$



- Global equilibrium of the left section $s \in [0, s]$:

$$\boldsymbol{M}(s) - \boldsymbol{M}(0) + s\boldsymbol{e} \times \boldsymbol{R}(0) + \int_0^s (\boldsymbol{c}_l + (\zeta - s)\boldsymbol{e} \times \boldsymbol{f}_l)\, d\zeta = \boldsymbol{0}\,;$$

- Global equilibrium of the right section $s \in [s, L]$:

$$\boldsymbol{M}(L) + (L-s)\boldsymbol{e} \times \boldsymbol{R}(L) - \boldsymbol{M}(s) + \int_s^L (\boldsymbol{c}_l + (\zeta - s)\boldsymbol{e} \times \boldsymbol{f}_l)\, d\zeta = \boldsymbol{0}\,.$$

Outline

Beam elastic law

Traction vector

- Recap: basic kinematic assumption (Timoshenko)

$$\boldsymbol{u} = \boldsymbol{u}_G(s) + \boldsymbol{\theta}(s) \times \boldsymbol{x}_\Sigma \,;$$

- Linearized strains $\boldsymbol{x}_\Sigma \simeq \boldsymbol{p}_\Sigma$:

$$\boldsymbol{\varepsilon} = (\boldsymbol{u}'_G + \boldsymbol{\theta}' \times \boldsymbol{x}_\Sigma) \otimes_s \boldsymbol{e} - (\boldsymbol{\theta}_\Sigma \times \boldsymbol{e}) \otimes_s \boldsymbol{e} \,;$$

- Linear elastic, isotropic behavior:

$$\begin{aligned} \boldsymbol{\sigma} &= \lambda \operatorname{Tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon} \\ &= \lambda\varepsilon_{ee}\boldsymbol{I} + 2\mu(\varepsilon_{ee}\boldsymbol{e} + \boldsymbol{\gamma}_\Sigma) \otimes_s \boldsymbol{e} + \boldsymbol{\sigma}_\Sigma \,; \end{aligned}$$

- Traction vector:

$$\boldsymbol{\sigma e} = E(\underbrace{u'_{Ge}\boldsymbol{e}}_{//\boldsymbol{e}} + \underbrace{\boldsymbol{\theta}'_\Sigma \times \boldsymbol{x}_\Sigma}_{//\boldsymbol{e},\propto \boldsymbol{x}_\Sigma}) + \mu(\underbrace{\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e}}_{\perp \boldsymbol{e}} + \underbrace{\theta'_e \boldsymbol{e} \times \boldsymbol{x}_\Sigma}_{\perp \boldsymbol{e},\propto \boldsymbol{x}_\Sigma}) \,.$$

Beam elastic law

Resultant force with Timoshenko's kinematical assumption

- Recap: resultant force

$$\boldsymbol{R} = \int_\Sigma \boldsymbol{\sigma}\boldsymbol{e}\,dS\,;$$

- Assuming $\int_\Sigma \boldsymbol{x}_\Sigma dS = \boldsymbol{0}$, $S = \int_\Sigma dS$:

$$\begin{aligned}\boldsymbol{R} &= \textstyle\int_\Sigma E(u'_{Ge}\boldsymbol{e}+\cancel{\boldsymbol{\theta}'_\Sigma\times\boldsymbol{x}_\Sigma})dS+\int_\Sigma \mu(\boldsymbol{u}'_{G\Sigma}-\boldsymbol{\theta}_\Sigma\times\boldsymbol{e}+\cancel{\theta'_e\boldsymbol{e}\times\boldsymbol{x}_\Sigma})dS \\ &= ESu'_{Ge}\boldsymbol{e} + \boldsymbol{R}_\Sigma\,;\end{aligned}$$

- Shear force with Timoshenko's assumption:

$$\boldsymbol{R}_\Sigma = \mu S(\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e})\,.$$

Beam elastic law

Resultant force with Euler-Bernoulli's kinematical assumption

- Recap: resultant force

$$\boldsymbol{R} = \int_\Sigma \boldsymbol{\sigma}\boldsymbol{e}\,dS\,;$$

- Assuming $\int_\Sigma \boldsymbol{x}_\Sigma dS = \boldsymbol{0}$, $S = \int_\Sigma dS$:

$$\boldsymbol{R} = \int_\Sigma E(u'_{Ge}\boldsymbol{e} + \cancel{\boldsymbol{\theta}'_\Sigma \times \boldsymbol{x}_\Sigma})dS + \int_\Sigma \mu(\underbrace{\cancel{\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e}}}_{=\boldsymbol{0}\text{ by Euler-Bernoulli}} + \cancel{\theta'_e\boldsymbol{e} \times \boldsymbol{x}_\Sigma})dS$$

$$= ESu'_{Ge}\boldsymbol{e} + \boldsymbol{R}_\Sigma\,;$$

- Shear force with Euler-Bernoulli's assumption:

$$\boldsymbol{M}' + \boldsymbol{e} \times \boldsymbol{R} + \boldsymbol{c}_l = \boldsymbol{0}$$

$$\Rightarrow \quad \boldsymbol{R}_\Sigma = \boldsymbol{e} \times (\boldsymbol{M}'_\Sigma + \boldsymbol{c}_l)\,.$$

- Recap: resultant moment

$$\boldsymbol{M} = \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{\sigma}\boldsymbol{e}\,dS\,;$$

- Assuming $\int_\Sigma \boldsymbol{x}_\Sigma dS = \boldsymbol{0}$:

$$\begin{aligned}\boldsymbol{M} &= \textstyle\int_\Sigma E(\cancel{\boldsymbol{x}_\Sigma \times u'_{Ge}}\boldsymbol{e} + \boldsymbol{x}_\Sigma \times (\boldsymbol{\theta}'_\Sigma \times \boldsymbol{x}_\Sigma))dS \\ &\quad + \textstyle\int_\Sigma \mu(\cancel{\boldsymbol{x}_\Sigma \times (\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e})} + \boldsymbol{x}_\Sigma \times (\theta'_e \boldsymbol{e} \times \boldsymbol{x}_\Sigma))dS \\ &= E\mathbb{J}(\boldsymbol{\theta}'_\Sigma) + \mu\mathbb{J}(\theta'_e \boldsymbol{e})\,;\end{aligned}$$

- (Symmetric) inertia tensor:

$$\mathbb{J} = \int_\Sigma (\|\boldsymbol{x}_\Sigma\|^2 \boldsymbol{I} - \boldsymbol{x}_\Sigma \otimes \boldsymbol{x}_\Sigma)\,dS\,;$$

- Bending moment with Timoshenko's assumption:

$$\boldsymbol{M}_\Sigma = E\mathbb{J}(\boldsymbol{\theta}'_\Sigma)\,.$$

- Recap: resultant moment

$$\boldsymbol{M} = \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{\sigma}\boldsymbol{e}\, dS\,;$$

- Assuming $\int_\Sigma \boldsymbol{x}_\Sigma dS = \boldsymbol{0}$:

$$\begin{aligned}\boldsymbol{M} &= \textstyle\int_\Sigma E(\cancel{\boldsymbol{x}_\Sigma \times u'_{Ge}\boldsymbol{e}} + \boldsymbol{x}_\Sigma \times (\boldsymbol{\theta}'_\Sigma \times \boldsymbol{x}_\Sigma))dS \\ &\quad + \textstyle\int_\Sigma \mu(\cancel{\boldsymbol{x}_\Sigma \times (\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e})} + \boldsymbol{x}_\Sigma \times (\theta'_e \boldsymbol{e} \times \boldsymbol{x}_\Sigma))dS \\ &= E\mathbb{J}(\boldsymbol{\theta}'_\Sigma) + \mu\mathbb{J}(\theta'_e \boldsymbol{e})\,;\end{aligned}$$

- (Symmetric) inertia tensor:

$$\mathbb{J} = \int_\Sigma (\|\boldsymbol{x}_\Sigma\|^2 \boldsymbol{I} - \boldsymbol{x}_\Sigma \otimes \boldsymbol{x}_\Sigma)\, dS\,;$$

- Bending moment with Euler-Bernoulli's assumption:

$$\boldsymbol{M}_\Sigma = E\mathbb{J}(\boldsymbol{e} \times \boldsymbol{u}''_{G\Sigma}).$$

Summary

Constitutive equations:

$$\boldsymbol{R} = ESu'_{Ge}\boldsymbol{e} + \boldsymbol{R}_\Sigma$$
$$\boldsymbol{R}_\Sigma = \boldsymbol{e} \times (\boldsymbol{M}'_\Sigma + \boldsymbol{c}_l)$$

$$\boldsymbol{M}_\Sigma = E\mathbb{J}(\boldsymbol{e} \times \boldsymbol{u}''_{G\Sigma})$$
$$M_e = \mu J_e \theta'_e$$

$$J_e = \langle \mathbb{J}\boldsymbol{e}, \boldsymbol{e} \rangle = \int_\Sigma \|\boldsymbol{x}_\Sigma\|^2 \, dS$$

Static equilibrium:

$$\boldsymbol{R}' + \boldsymbol{f}_l = \boldsymbol{0}$$

$$\boldsymbol{M}''_\Sigma - \boldsymbol{e} \times \boldsymbol{f}_l + \boldsymbol{c}'_{l\Sigma} = \boldsymbol{0}$$
$$M'_e + c_{le} = 0$$

Differential equations

- Elongation:

$$\boxed{ESu''_{Ge}(s) + \langle \boldsymbol{f}_l(s), \boldsymbol{e} \rangle = 0}$$

with either kinematical $u_{Ge}(0), u_{Ge}(L)$ or mechanical $R_e(0), R_e(L)$ boundary conditions.

- Torsion:

$$\boxed{\mu J_e \theta''_e(s) + \langle \boldsymbol{c}_l(s), \boldsymbol{e} \rangle = 0}$$

with either kinematical $\theta_e(0), \theta_e(L)$ or mechanical $M_e(0), M_e(L)$ boundary conditions.

- Bending:

$$\boxed{E\mathbb{J}(\boldsymbol{e} \times \boldsymbol{u}_{G\Sigma}^{(IV)}(s)) - \boldsymbol{e} \times \boldsymbol{f}_l(s) + \boldsymbol{c}'_{l\Sigma}(s) = \boldsymbol{0}}$$

with either kinematical $\boldsymbol{u}_{G\Sigma}(0), \boldsymbol{u}_{G\Sigma}(L), \boldsymbol{u}'_{G\Sigma}(0), \boldsymbol{u}'_{G\Sigma}(L)$ or mechanical $\boldsymbol{M}_\Sigma(0), \boldsymbol{M}_\Sigma(L), \boldsymbol{R}_\Sigma(0), \boldsymbol{R}_\Sigma(L)$ boundary conditions.

Back to stresses...

- Recap: traction vector

$$\boldsymbol{\sigma}\boldsymbol{e} = E(\underbrace{u'_{Ge}\boldsymbol{e}}_{//\boldsymbol{e}} + \underbrace{\boldsymbol{\theta}'_\Sigma \times \boldsymbol{x}_\Sigma}_{//\boldsymbol{e},\propto\boldsymbol{x}_\Sigma}) + \mu(\underbrace{\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e}}_{\perp\boldsymbol{e}} + \underbrace{\theta'_e\boldsymbol{e} \times \boldsymbol{x}_\Sigma}_{\perp\boldsymbol{e},\propto\boldsymbol{x}_\Sigma})$$
$$= \sigma_{ee}\boldsymbol{e} + \boldsymbol{\tau}_\Sigma\,;$$

- Normal stress:

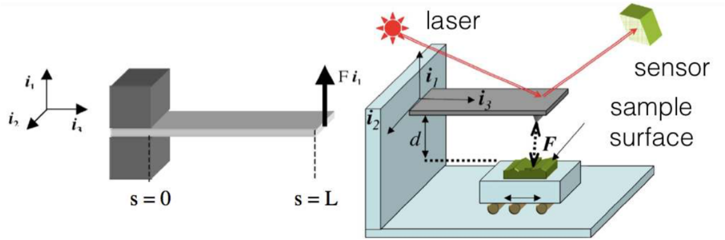
$$\begin{aligned}\sigma_{ee} &= E\left(u'_{Ge} + \left\langle \boldsymbol{\theta}'_\Sigma \times \boldsymbol{x}_\Sigma, \boldsymbol{e} \right\rangle\right) \\ &= \frac{R_e}{S} + \left\langle \boldsymbol{x}_\Sigma, \boldsymbol{e} \times \mathbb{J}^{-1}(\boldsymbol{M}_\Sigma) \right\rangle\,;\end{aligned}$$

- Shear stress:

$$\begin{aligned}\boldsymbol{\tau}_\Sigma &= \mu(\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e}) + \mu\theta'_e(\boldsymbol{e} \times \boldsymbol{x}_\Sigma) \\ &= \frac{\boldsymbol{R}_\Sigma}{S} + \frac{M_e}{J_e}(\boldsymbol{e} \times \boldsymbol{x}_\Sigma)\,.\end{aligned}$$

Outline

Atomic force microscope
Setup

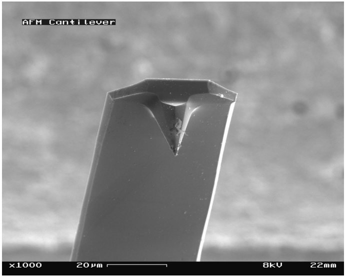


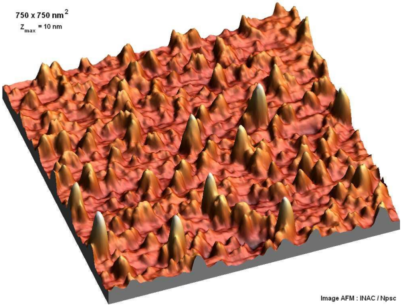
Gerd Binnig [1947–]

Calvin Quate [1923–2019]

Christoph Gerber [1942–]

Atomic force microscope
Setup





- http://toutestquantique.fr/afm/
- https://youtu.be/EUBOB5EUJK4

Question #1: Resultant force $\boldsymbol{R}$?

- Local balance of forces on the cross-section Σ at $s \in]0, L[$:

$$\boldsymbol{R}' + \boldsymbol{f}_l = \boldsymbol{0}\,,$$

where $\boldsymbol{f}_l = \boldsymbol{0}$ because "the own weight of the beam can be neglected."

Question #1: Resultant force $\boldsymbol{R}$?

- Local balance of forces on the cross-section Σ at $s \in]0, L[$:

$$\boldsymbol{R}' + \boldsymbol{f}_l = \boldsymbol{0} \,,$$

where $\boldsymbol{f}_l = \boldsymbol{0}$ because "the own weight of the beam can be neglected."

- Therefore:

$$\boldsymbol{R}(s) = \boldsymbol{C}^{\text{st}} = \boldsymbol{R}(L) = \boldsymbol{R}_L = F(d)\boldsymbol{i}_1 \,.$$

- No normal force, only a shear force.

1EL5000/S8

É. Savin

Kinematics

Statics

Stresses

6.1 Atomic force microscope

Question #1: Action of the fixed support on the beam?

- Resultant force for the action of the fixed support at $s = 0$:

$$\boldsymbol{F}^{\text{support} \to \text{beam}} = \int_{\Sigma(0)} \boldsymbol{\sigma}(-\boldsymbol{i}_3) dS \stackrel{\text{def}}{=} -\boldsymbol{R}(0)$$

Question #1: Action of the fixed support on the beam?

- Resultant force for the action of the fixed support at $s=0$:

$$\boldsymbol{F}^{\text{support}\to\text{beam}} = \int_{\Sigma(0)} \boldsymbol{\sigma}(-\boldsymbol{i}_3)dS \stackrel{\text{def}}{=} -\boldsymbol{R}(0)$$

- Therefore:

$$\boldsymbol{F}^{\text{support}\to\text{beam}} = -F(d)\boldsymbol{i}_1\,.$$

Question #2: Resultant moment $\boldsymbol{M}$?

- Local balance of moments on the cross-section Σ at $s \in]0, L[$:

$$\boldsymbol{M}' + \boldsymbol{e} \times \boldsymbol{R} + \boldsymbol{c}_l = \boldsymbol{0}\,,$$

where $\boldsymbol{c}_l = \boldsymbol{0}$ and $\boldsymbol{e} = \boldsymbol{i}_3$.

Question #2: Resultant moment $\boldsymbol{M}$?

- Local balance of moments on the cross-section Σ at $s \in]0, L[$:

$$\boldsymbol{M}' + \boldsymbol{i}_3 \times \boldsymbol{R} = \mathbf{0}\,;$$

- Therefore:

$$\boldsymbol{M}'(s) = -\boldsymbol{i}_3 \times F(d)\boldsymbol{i}_1 = -F(d)\boldsymbol{i}_2\,;$$

Question #2: Resultant moment $\boldsymbol{M}$?

- Local balance of moments on the cross-section Σ at $s \in]0, L[$:

$$\boldsymbol{M}' + \boldsymbol{i}_3 \times \boldsymbol{R} = \boldsymbol{0}\,;$$

- Therefore:

$$\boldsymbol{M}'(s) = -\boldsymbol{i}_3 \times F(d)\boldsymbol{i}_1 = -F(d)\boldsymbol{i}_2\,;$$

- Consequently:

$$\begin{aligned}\boldsymbol{M}(s) &= -F(d)(s-L)\boldsymbol{i}_2 + \boldsymbol{M}(L)\\ &= F(d)(L-s)\boldsymbol{i}_2\end{aligned}$$

because $\boldsymbol{M}(L) = \boldsymbol{M}_L = \boldsymbol{0}$ ("an action [...] which is assumed to consist of a moment [...] equal to zero...").

1EL5000/S8
É. Savin
Kinematics
Statics
Stresses
6.1 Atomic force microscope

Question #2: Action of the fixed support on the beam?

- Resultant moment for the action of the fixed support at $s = 0$:

$$\boldsymbol{C}^{\text{support} \to \text{beam}} = \int_{\Sigma(0)} \boldsymbol{x}_\Sigma \times \boldsymbol{\sigma}(-\boldsymbol{i}_3) dS \stackrel{\text{def}}{=} -\boldsymbol{M}(0)$$

Question #2: Action of the fixed support on the beam?

- Resultant moment for the action of the fixed support at $s = 0$:

$$\boldsymbol{C}^{\text{support}\to\text{beam}} = \int_{\Sigma(0)} \boldsymbol{x}_\Sigma \times \boldsymbol{\sigma}(-\boldsymbol{i}_3)dS \stackrel{\text{def}}{=} -\boldsymbol{M}(0)$$

- Therefore:

$$\boldsymbol{C}^{\text{support}\to\text{beam}} = -F(d)L\boldsymbol{i}_2\,.$$

Question #3: Normal stress σ_{ee}?

- Normal stress:

$$\begin{aligned}\sigma_{ee} &= \frac{R_e}{S} + \left\langle \boldsymbol{x}_\Sigma, \boldsymbol{e} \times \mathbb{J}^{-1}(\boldsymbol{M}_\Sigma)\right\rangle \\ &= \left\langle \boldsymbol{x}_\Sigma, \boldsymbol{i}_3 \times \frac{F(d)(L-s)}{I_2}\boldsymbol{i}_2\right\rangle \\ &= -\frac{F(d)}{I_2}(L-s)\chi_1\,;\end{aligned}$$

Question #3: Normal stress σ_{ee}?

- Normal stress:

$$\begin{aligned}\sigma_{ee} &= \frac{R_e}{S} + \left\langle \boldsymbol{x}_\Sigma, \boldsymbol{e} \times \mathbb{J}^{-1}(\boldsymbol{M}_\Sigma) \right\rangle \\ &= \left\langle \boldsymbol{x}_\Sigma, \boldsymbol{i}_3 \times \frac{F(d)(L-s)}{I_2}\boldsymbol{i}_2 \right\rangle \\ &= -\frac{F(d)}{I_2}(L-s)\chi_1\,;\end{aligned}$$

- The inertia tensor of $\Sigma =]-\frac{h}{2},\frac{h}{2}[\times]-\frac{b}{2},\frac{b}{2}[$ is:

$$\begin{aligned}\mathbb{J} &= \int_\Sigma (\|\boldsymbol{x}_\Sigma\|^2 - \boldsymbol{x}_\Sigma \otimes \boldsymbol{x}_\Sigma)dS \\ &= I_1 \boldsymbol{i}_1 \otimes \boldsymbol{i}_1 + I_2 \boldsymbol{i}_2 \otimes \boldsymbol{i}_2 + (I_1 + I_2)\boldsymbol{i}_3 \otimes \boldsymbol{i}_3 \\ &= \frac{hb^3}{12}\boldsymbol{i}_1 \otimes \boldsymbol{i}_1 + \frac{bh^3}{12}\boldsymbol{i}_2 \otimes \boldsymbol{i}_2 + \frac{bh}{12}(b^2+h^2)\boldsymbol{i}_3 \otimes \boldsymbol{i}_3\,.\end{aligned}$$

Question #3: Normal stress σ_{ee}?

- Normal stress:

$$\begin{aligned}\sigma_{ee} &= \frac{R_e}{S} + \left\langle \boldsymbol{x}_\Sigma, \boldsymbol{e} \times \mathbb{J}^{-1}(\boldsymbol{M}_\Sigma) \right\rangle \\ &= \left\langle \boldsymbol{x}_\Sigma, \boldsymbol{i}_3 \times \frac{F(d)(L-s)}{I_2} \boldsymbol{i}_2 \right\rangle \\ &= -\frac{F(d)}{I_2}(L-s)\chi_1\,;\end{aligned}$$

- Its maximum is reached at $s = 0$ and $\chi_1 = -\frac{h}{2}$;

Question #3: Normal stress σ_{ee}?

- Normal stress:

$$
\begin{aligned}
\sigma_{ee} &= \frac{R_e}{S} + \left\langle \boldsymbol{x}_\Sigma, \boldsymbol{e} \times \mathbb{J}^{-1}(\boldsymbol{M}_\Sigma) \right\rangle \\
&= \left\langle \boldsymbol{x}_\Sigma, \boldsymbol{i}_3 \times \frac{F(d)(L-s)}{I_2} \boldsymbol{i}_2 \right\rangle \\
&= -\frac{F(d)}{I_2}(L-s)\chi_1 \,;
\end{aligned}
$$

- Its maximum is reached at $s = 0$ and $\chi_1 = -\frac{h}{2}$;
- The maximum force $F_{\max}$ such that $\sigma_{ee} < \sigma_e$ is:

$$
F_{\max} < \frac{2I_2}{hL}\sigma_e = \frac{bh^2}{6L}\sigma_e = 10.7\,\mu\mathrm{N}\,.
$$

Question #4: Transverse displacement $\boldsymbol{u}_{G\Sigma}$?

- Constitutive equation for the transverse displacement with Euler-Bernoulli's assumption:

$$
\begin{aligned}
\boldsymbol{M}_{\Sigma}(s) &= E\mathbb{J}(\boldsymbol{e}\times\boldsymbol{u}''_{G\Sigma}(s)) \\
\Rightarrow \quad \boldsymbol{i}_3\times\boldsymbol{u}''_{G\Sigma}(s) &= \frac{F(d)(L-s)}{EI_2}\boldsymbol{i}_2 \\
\boldsymbol{u}''_{G\Sigma}(s) &= \frac{F(d)(L-s)}{EI_2}\boldsymbol{i}_2\times\boldsymbol{i}_3\,;
\end{aligned}
$$

Question #4: Transverse displacement $\boldsymbol{u}_{G\Sigma}$?

- Constitutive equation for the transverse displacement with Euler-Bernoulli's assumption:

$$\boldsymbol{u}''_{G\Sigma}(s) = \frac{F(d)(L-s)}{EI_2}\boldsymbol{i}_1\,;$$

- Consequently:

$$\begin{aligned}\boldsymbol{u}'_{G\Sigma}(s) &= \frac{F(d)}{EI_2}\left(L-\frac{s}{2}\right)s\,\boldsymbol{i}_1 + \boldsymbol{u}'_{G\Sigma}(0) \\ &= \frac{F(d)}{EI_2}\left(L-\frac{s}{2}\right)s\,\boldsymbol{i}_1\end{aligned}$$

since $\boldsymbol{\theta}_\Sigma(0) = \boldsymbol{i}_3 \times \boldsymbol{u}'_{G\Sigma}(0) = \boldsymbol{0}$ (the beam "is clamped at $s=0$ on a fixed, perfectly rigid support").

Question #4: Transverse displacement $\boldsymbol{u}_{G\Sigma}$?

- Constitutive equation for the transverse displacement with Euler-Bernoulli's assumption:

$$\boldsymbol{u}''_{G\Sigma}(s) = \frac{F(d)(L-s)}{EI_2}\,\boldsymbol{i}_1\,;$$

- Consequently:

$$\begin{aligned}\boldsymbol{u}_{G\Sigma}(s) &= \frac{F(d)}{2EI_2}\left(L-\frac{s}{3}\right)s^2\,\boldsymbol{i}_1 + \boldsymbol{u}_{G\Sigma}(0)\\ &= \frac{F(d)}{2EI_2}\left(L-\frac{s}{3}\right)s^2\,\boldsymbol{i}_1\end{aligned}$$

since $\boldsymbol{u}_{G\Sigma}(0) = \mathbf{0}$ (the beam "is clamped at $s=0$ on a fixed, perfectly rigid support").

1EL5000/S8

É. Savin

Kinematics

Statics

Stresses

6.1 Atomic force microscope

Question #5: Bending stiffness k?

- Bending stiffness k:

$$\begin{aligned} k &= \frac{|F(d)|}{\|\boldsymbol{u}_{G\Sigma}(L)\|} \\ &= \frac{3EI_2}{L^3}\,. \end{aligned}$$

Question #6: Tip end rotation $\boldsymbol{\theta}_\Sigma(L)$?

- From Euler-Bernoulli's assumption:

$$\begin{aligned}\boldsymbol{\theta}_\Sigma(s) &= \boldsymbol{i}_3 \times \boldsymbol{u}'_{G\Sigma}(s) \\ &= \frac{F(d)}{EI_2}\left(L - \frac{s}{2}\right) s\, \boldsymbol{i}_3 \times \boldsymbol{i}_1\,;\end{aligned}$$

Question #6: Tip end rotation $\boldsymbol{\theta}_\Sigma(L)$?

- From Euler-Bernoulli's assumption:

$$\begin{aligned}\boldsymbol{\theta}_\Sigma(s) &= \boldsymbol{i}_3 \times \boldsymbol{u}'_{G\Sigma}(s) \\ &= \frac{F(d)}{EI_2}\left(L - \frac{s}{2}\right) s\, \boldsymbol{i}_2\,;\end{aligned}$$

- Consequently:

$$\boldsymbol{\theta}_\Sigma(L) = \frac{F(d)L^2}{2EI_2}\, \boldsymbol{i}_2\,;$$

- The largest rotation is at the tip end, that is why it is measured for better precision.

Question #7: Tip end rotation with distributed tip load?

- Local balance of actions at any $s \in]0, L[$:

$$
\begin{aligned}
\boldsymbol{R}'(s) + \boldsymbol{f}_l(s) &= \mathbf{0}\,, \\
\boldsymbol{M}'(s) + \boldsymbol{i}_3 \times \boldsymbol{R}(s) + \boldsymbol{c}_l(s) &= \mathbf{0}\,,
\end{aligned}
$$

where $\boldsymbol{c}_l(s) = \mathbf{0}$ and:

$$
\boldsymbol{f}_l(s) = \begin{cases} \frac{F(d)}{a}\boldsymbol{i}_1 & \text{if } \; s \in]L-a, L[\,, \\ \mathbf{0} & \text{if } \; s \in]0, L-a[\,. \end{cases}
$$

- Strategy: (i) recompute $\boldsymbol{R}$; (ii) recompute $\boldsymbol{M}$; (iii) deduce $\boldsymbol{\theta}_\Sigma$.

Question #7: Tip end rotation with distributed tip load?

- (i) Recompute $\boldsymbol{R}$:

$$\boldsymbol{R}(s)=\begin{cases}\frac{F(d)}{a}(L-s)\boldsymbol{i}_1 & \text{if } \; s\in]L-a,L[\,,\\ F(d)\boldsymbol{i}_1 & \text{if } \; s\in]0,L-a[\,,\end{cases}$$

since $\boldsymbol{R}(L)=\boldsymbol{R}_L=\boldsymbol{0}$.

Question #7: Tip end rotation with distributed tip load?

- (i) Recompute $\boldsymbol{R}$:

$$\boldsymbol{R}(s) = \begin{cases} \frac{F(d)}{a}(L-s)\boldsymbol{i}_1 & \text{if } \; s \in]L-a, L[\,, \\ F(d)\boldsymbol{i}_1 & \text{if } \; s \in]0, L-a[\,, \end{cases}$$

since $\boldsymbol{R}(L) = \boldsymbol{R}_L = \boldsymbol{0}$.

- (ii) Recompute $\boldsymbol{M}$:

$$\boldsymbol{M}(s) = \begin{cases} \frac{F(d)}{2a}(L-s)^2\,\boldsymbol{i}_2 & \text{if } \; s \in]L-a, L[\,, \\ F(d)\left(L-s-\frac{a}{2}\right)\boldsymbol{i}_2 & \text{if } \; s \in]0, L-a[\,, \end{cases}$$

since $\boldsymbol{M}(L) = \boldsymbol{M}_L = \boldsymbol{0}$.

Question #7: Tip end rotation with distributed tip load?

- (iii) Deduce $\boldsymbol{\theta}_\Sigma$:

$$\begin{aligned} \boldsymbol{M}_\Sigma(s) &= E\mathbb{J}(\boldsymbol{\theta}'_\Sigma(s)) \\ \Rightarrow \quad \boldsymbol{\theta}'_\Sigma(s) &= \frac{\boldsymbol{M}_\Sigma(s)}{EI_2} \end{aligned}$$

  since $\boldsymbol{M}_\Sigma(s) // \boldsymbol{i}_2$.

- Consequently:

$$\boldsymbol{\theta}'_\Sigma(s) = \begin{cases} \frac{F(d)}{2EI_2 a}(L-s)^2\,\boldsymbol{i}_2 & \text{if } \; s \in]L-a, L[\,, \\ \frac{F(d)}{EI_2}\left(L - s - \frac{a}{2}\right)\boldsymbol{i}_2 & \text{if } \; s \in]0, L-a[\,, \end{cases}$$

  with $\boldsymbol{\theta}_\Sigma(0) = \mathbf{0}$ (fixed support) and $[\![\boldsymbol{\theta}_\Sigma(L-a)]\!] = \mathbf{0}$ (continuity).

Question #7: Tip end rotation with distributed tip load?

- (iii) Deduce $\boldsymbol{\theta}_\Sigma$:

$$\boldsymbol{\theta}_\Sigma(s) = \begin{cases} \frac{F(d)}{6EI_2}\left[\frac{(s-L)^3}{a}+a^2+3L(L-a)\right]\boldsymbol{i}_2 & \text{if } s\in]L-a,L[\,, \\ \frac{F(d)}{EI_2}\left(L-\frac{s}{2}-\frac{a}{2}\right)s\,\boldsymbol{i}_2 & \text{if } s\in]0,L-a[\,. \end{cases}$$

- Finally:

$$\boldsymbol{\theta}_\Sigma(L) = \frac{F(d)L^2}{2EI_2}\left(1-\frac{a}{L}+\frac{a^2}{3L^2}\right)\boldsymbol{i}_2$$

to be compared with the initial result $\boldsymbol{\theta}_\Sigma(L) = \frac{F(d)L^2}{2EI_2}$.

Question #8: Tip end rotation with self-weight?

- Local balance of actions at any $s \in]0, L[$:

$$\begin{aligned} \boldsymbol{R}'(s) + \boldsymbol{f}_l(s) &= \boldsymbol{0}\,, \\ \boldsymbol{M}'(s) + \boldsymbol{i}_3 \times \boldsymbol{R}(s) + \boldsymbol{c}_l(s) &= \boldsymbol{0}\,, \end{aligned}$$

  where $\boldsymbol{f}_l = -\varrho g S \boldsymbol{i}_1$ and $\boldsymbol{c}_l(s) = \boldsymbol{0}$.
- Strategy: (i) recompute $\boldsymbol{R}$; (ii) recompute $\boldsymbol{M}$; (iii) deduce $\boldsymbol{\theta}_\Sigma$.

Question #8: Tip end rotation with self-weight?

- (i) Recompute $\boldsymbol{R}$:

$$\boldsymbol{R}'(s) = \varrho g S \boldsymbol{i}_1$$
$$\boldsymbol{R}(s) = \varrho g S (s - L) \boldsymbol{i}_1 \,, \quad s \in]0, L[$$

since $\boldsymbol{R}(L) = \boldsymbol{R}_L = \mathbf{0}$.

Question #8: Tip end rotation with self-weight?

- (i) Recompute $\boldsymbol{R}$:

$$\begin{aligned}\boldsymbol{R}'(s) &= \varrho g S \boldsymbol{i}_1 \\ \boldsymbol{R}(s) &= \varrho g S (s-L) \boldsymbol{i}_1 \,, \quad s \in]0, L[\end{aligned}$$

since $\boldsymbol{R}(L) = \boldsymbol{R}_L = \boldsymbol{0}$.

- (ii) Recompute $\boldsymbol{M}$:

$$\begin{aligned}\boldsymbol{M}'(s) &= -\varrho g S (s-L) \boldsymbol{i}_3 \times \boldsymbol{i}_1 \\ \boldsymbol{M}(s) &= -\frac{\varrho g S}{2} (s-L)^2 \, \boldsymbol{i}_2 \,, \quad s \in]0, L[\end{aligned}$$

since $\boldsymbol{M}(L) = \boldsymbol{M}_L = \boldsymbol{0}$.

Question #8: Tip end rotation with self-weight?

- (iii) Deduce $\boldsymbol{\theta}_\Sigma$:

$$\begin{aligned} \boldsymbol{M}_\Sigma(s) &= E\mathbb{J}(\boldsymbol{\theta}'_\Sigma(s)) \\ \Rightarrow \quad \boldsymbol{\theta}'_\Sigma(s) &= \frac{\boldsymbol{M}_\Sigma(s)}{EI_2}\,. \end{aligned}$$

- Consequently:

$$\begin{aligned} \boldsymbol{\theta}'_\Sigma(s) &= -\frac{\varrho g S}{2EI_2}(s-L)^2\,\boldsymbol{i}_2 \\ \boldsymbol{\theta}_\Sigma(s) &= -\frac{\varrho g S}{6EI_2}\left[(s-L)^3+L^3\right]\boldsymbol{i}_2\,, \quad s\in]0,L[\,, \end{aligned}$$

since $\boldsymbol{\theta}_\Sigma(0)=\mathbf{0}$ (fixed support).

Question #8: Tip end rotation with self-weight?

- (iii) Deduce $\boldsymbol{\theta}_\Sigma$:

$$\boldsymbol{\theta}_\Sigma(s) = -\frac{\varrho g S}{6EI_2}\left[(s-L)^3 + L^3\right]\boldsymbol{i}_2\,,\quad s\in]0,L[\,.$$

- Finally:

$$\boldsymbol{\theta}_\Sigma(L) = -\frac{\varrho g S L^3}{6EI_2}\,\boldsymbol{i}_2\,.$$

- This rotation is much small than the one induced by the tip end load, so that the influence of the self-weight can be neglected.

Question #9: Effective rigidity $k_{\rm eff}$?

- The tip end load is expressed in two ways:

$$F(d) = ku_{\rm microlever} = -k'u_{\rm sample}\,,$$

and ($d = u_{\rm microlever} - u_{\rm sample}$):

$$F(d) = k_{\rm eff}(u_{\rm microlever} - u_{\rm sample})\,;$$

- Thus:

$$k_{\rm eff} = \frac{kk'}{k+k'}\,.$$