Deformation of an aircraft wing

1EL5000–Continuum Mechanics – Tutorial Class #10

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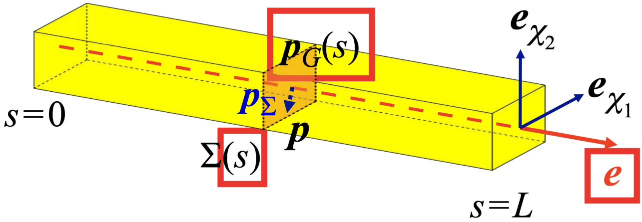
Outline

Outline

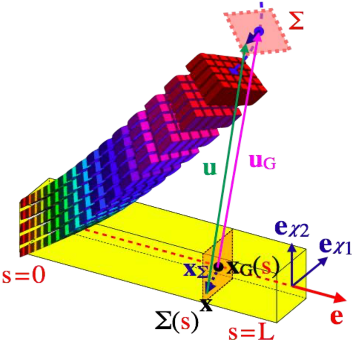
Beam kinematics

Reference configuration



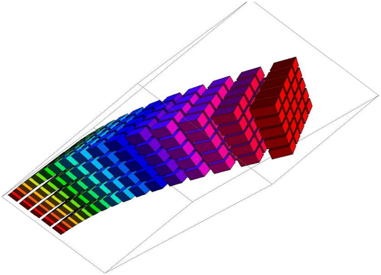
$$
\begin{aligned}
\boldsymbol{p}(s, \chi_1, \chi_2) &= \boldsymbol{p}_G(s) + \boldsymbol{p}_\Sigma(\chi_1, \chi_2) \\
&= s\boldsymbol{e} + \chi_1 \boldsymbol{e}_{\chi_1} + \chi_2 \boldsymbol{e}_{\chi_2}
\end{aligned}
$$

Beam kinematics
Actual configuration



$$
\begin{aligned}
\boldsymbol{x} &= \boldsymbol{x}_G(s) + \boldsymbol{x}_\Sigma \\
&= \boldsymbol{x}_G(s) + \boldsymbol{R}(s)\boldsymbol{p}_\Sigma
\end{aligned}
$$

Beam kinematics
Small perturbations – Timoshenko



- Small perturbations $\boldsymbol{R}(s) = \boldsymbol{I} + \boldsymbol{\Theta}(s)$, $\boldsymbol{\Theta}(s)^{\mathsf{T}} = -\boldsymbol{\Theta}(s)$:

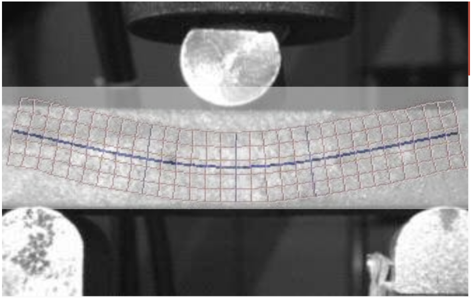
$$\begin{aligned} \boldsymbol{x}_\Sigma &= \boldsymbol{R}(s)\boldsymbol{p}_\Sigma \\ &= (\boldsymbol{I} + \boldsymbol{\Theta}(s))\boldsymbol{p}_\Sigma \,. \end{aligned}$$

- Small displacement $\boldsymbol{x}_\Sigma \simeq \boldsymbol{p}_\Sigma$:

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{x} - \boldsymbol{p} \\ &= \boldsymbol{u}_G(s) + \boldsymbol{\theta}(s) \times \boldsymbol{x}_\Sigma \,. \end{aligned}$$

Beam kinematics

Small perturbations – Euler-Bernoulli



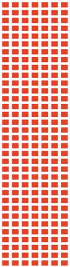
- Cross-sections remain perpendicular to the neutral line:

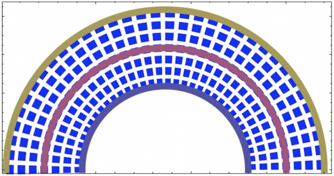
$$\boldsymbol{R}(s)\boldsymbol{e} = \frac{\boldsymbol{x}'_G(s)}{\|\boldsymbol{x}'_G(s)\|}\,.$$

- Small perturbations $\boldsymbol{R}(s)\boldsymbol{e} = (\boldsymbol{I} + \boldsymbol{\Theta}(s))\boldsymbol{e} \simeq \boldsymbol{e} + \boldsymbol{u}'_G(s)$:

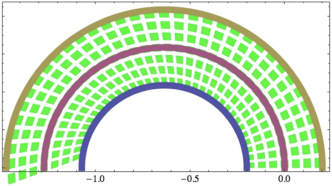
$$\boxed{\boldsymbol{\theta}_\Sigma(s) = \boldsymbol{e} \times \boldsymbol{u}'_{G\Sigma}(s)}$$

Recap: 1.3 Large beam bending



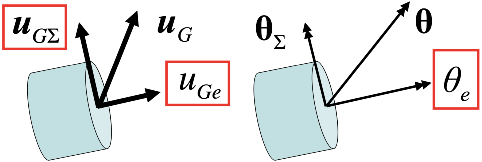


$\alpha = 1$



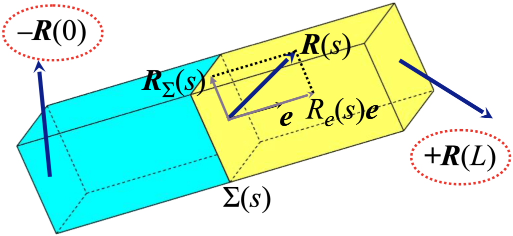
$\alpha = 1.1$

Beam kinematics
Independent unknowns



- Elongation $u_{Ge} = \langle \boldsymbol{u}_G, \boldsymbol{e} \rangle$;
- Deflection $\boldsymbol{u}_{G\Sigma} = \boldsymbol{u}_G - u_{Ge}\boldsymbol{e}$;
- Torsion rotation $\theta_e = \langle \boldsymbol{\theta}, \boldsymbol{e} \rangle$.

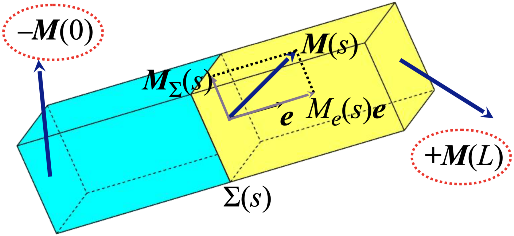
Outline



- Resultant force:

$$\boldsymbol{R}(s) = \int_\Sigma \boldsymbol{\sigma}\boldsymbol{e}\,dS\,;$$

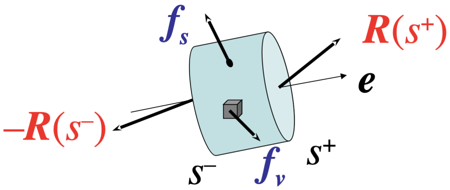
- Normal force $R_e(s) = \langle \boldsymbol{R}(s), \boldsymbol{e} \rangle$;
- Shear force $\boldsymbol{R}_\Sigma(s) = \boldsymbol{R}(s) - R_e(s)\boldsymbol{e}$;
- $\boldsymbol{R}(0)$ and $\boldsymbol{R}(L)$ given by the boundary conditions.



- Resultant moment:

$$\boldsymbol{M}(s) = \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{\sigma}\boldsymbol{e}\, dS\,;$$

- Torsion moment $M_e(s) = \langle \boldsymbol{M}(s), \boldsymbol{e}\rangle$;
- Bending moment $\boldsymbol{M}_\Sigma(s) = \boldsymbol{M}(s) - M_e(s)\boldsymbol{e}$;
- $\boldsymbol{M}(0)$ and $\boldsymbol{M}(L)$ given by the boundary conditions.

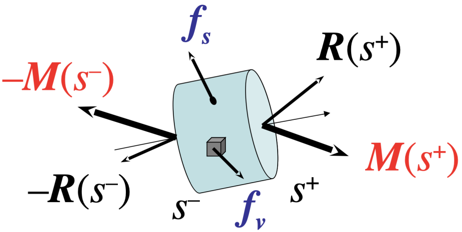


- Linear external forces:

$$\boldsymbol{f}_l(s) = \int_\Sigma \boldsymbol{f}_v dS + \int_{\partial\Sigma} \boldsymbol{f}_s d\zeta\,;$$

- Local equilibrium of the cross-section:

$$\boldsymbol{R}'(s) + \boldsymbol{f}_l(s) = \boldsymbol{0}\,.$$



- Linear external torques:

$$\boldsymbol{c}_l(s) = \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{f}_v dS + \int_{\partial\Sigma} \boldsymbol{x}_\Sigma \times \boldsymbol{f}_s d\zeta\,;$$

- Local equilibrium of the cross-section:

$$\boldsymbol{M}'(s) + \boldsymbol{e} \times \boldsymbol{R}(s) + \boldsymbol{c}_l(s) = \boldsymbol{0}\,.$$

Beam statics

Local balance of forces–alternative point of view

- Local static equilibrium of a continuum medium:

$$\mathbf{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_v = \mathbf{0}\,.$$

- Then integrate over the cross-section Σ:

$$\begin{aligned}
\mathbf{0} &= \int_\Sigma \mathbf{Div}\,\boldsymbol{\sigma}\,dS + \int_\Sigma \boldsymbol{f}_v\,dS \\
&= \int_\Sigma \frac{\partial\boldsymbol{\sigma}}{\partial s}\boldsymbol{e}\,dS + \int_\Sigma \mathbf{Div}_\Sigma\,\boldsymbol{\sigma}\,dS + \int_\Sigma \boldsymbol{f}_v\,dS \\
&= \frac{\partial}{\partial s}\left(\int_\Sigma \boldsymbol{\sigma}\boldsymbol{e}\,dS\right) + \int_{\partial\Sigma} \boldsymbol{\sigma}\boldsymbol{n}\,d\zeta + \int_\Sigma \boldsymbol{f}_v\,dS \\
&= \boldsymbol{R}'(s) + \int_{\partial\Sigma} \boldsymbol{f}_s\,d\zeta + \int_\Sigma \boldsymbol{f}_v\,dS \\
&= \boldsymbol{R}'(s) + \boldsymbol{f}_l(s)\,.
\end{aligned}$$

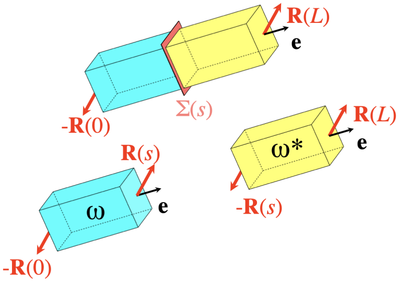
Beam statics

Local balance of moments–alternative point of view

- Then integrate over the cross-section Σ:

$$\begin{aligned}
\mathbf{0} &= \int_\Sigma \boldsymbol{x}_\Sigma \times \mathbf{Div}\,\boldsymbol{\sigma}\,dS + \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{f}_v\,dS \\
&= \int_\Sigma \boldsymbol{x}_\Sigma \times \frac{\partial\boldsymbol{\sigma}}{\partial s}\boldsymbol{e}\,dS + \int_\Sigma \boldsymbol{x}_\Sigma \times \frac{\partial\boldsymbol{\sigma}\boldsymbol{e}_\alpha}{\partial x_\alpha}\,dS + \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{f}_v\,dS \\
&= \frac{\partial}{\partial s}\left(\int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{\sigma}\boldsymbol{e}\,dS\right) + \int_\Sigma \frac{\partial(\boldsymbol{x}_\Sigma \times \boldsymbol{\sigma}\boldsymbol{e}_\alpha)}{\partial x_\alpha}\,dS \\
&\quad - \int_\Sigma \boldsymbol{e}_\alpha \times \boldsymbol{\sigma}\boldsymbol{e}_\alpha\,dS + \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{f}_v\,dS \\
&= \boldsymbol{M}'(s) + \int_{\partial\Sigma} \boldsymbol{x}_\Sigma \times \boldsymbol{\sigma}\boldsymbol{n}\,d\zeta + \int_\Sigma \boldsymbol{e} \times \boldsymbol{\sigma}\boldsymbol{e}\,dS + \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{f}_v\,dS \\
&= \boldsymbol{M}'(s) + \boldsymbol{e} \times \boldsymbol{R}(s) + \boldsymbol{c}_l(s)\,,
\end{aligned}$$

since $\boldsymbol{e}_\alpha \times \boldsymbol{\sigma}\boldsymbol{e}_\alpha + \boldsymbol{e} \times \boldsymbol{\sigma}\boldsymbol{e} = \mathbf{0}$ from the symmetry of $\boldsymbol{\sigma}$.



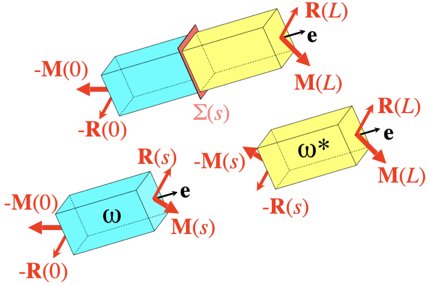
- Global equilibrium of the left section $s \in [0, s]$:

$$\boldsymbol{R}(s) - \boldsymbol{R}(0) + \int_0^s \boldsymbol{f}_l d\zeta = \mathbf{0}\,;$$

- Global equilibrium of the right section $s \in [s, L]$:

$$\boldsymbol{R}(L) - \boldsymbol{R}(s) + \int_s^L \boldsymbol{f}_l d\zeta = \mathbf{0}\,.$$

Beam statics
Global balance of moments



- Global equilibrium of the left section $s \in [0, s]$:

$$\boldsymbol{M}(s) - \boldsymbol{M}(0) + s\boldsymbol{e} \times \boldsymbol{R}(0) + \int_0^s (\boldsymbol{c}_l + (\zeta - s)\boldsymbol{e} \times \boldsymbol{f}_l)\, d\zeta = \boldsymbol{0}\,;$$

- Global equilibrium of the right section $s \in [s, L]$:

$$\boldsymbol{M}(L) + (L - s)\boldsymbol{e} \times \boldsymbol{R}(L) - \boldsymbol{M}(s) + \int_s^L (\boldsymbol{c}_l + (\zeta - s)\boldsymbol{e} \times \boldsymbol{f}_l)\, d\zeta = \boldsymbol{0}\,.$$

Outline

Beam elastic law
Traction vector

- Recap: basic kinematic assumption (Timoshenko)

$$\boldsymbol{u} = \boldsymbol{u}_G(s) + \boldsymbol{\theta}(s) \times \boldsymbol{x}_\Sigma \,;$$

- Linearized strains $\boldsymbol{x}_\Sigma \simeq \boldsymbol{p}_\Sigma$:

$$\boldsymbol{\varepsilon} = (\boldsymbol{u}'_G + \boldsymbol{\theta}' \times \boldsymbol{x}_\Sigma) \otimes_s \boldsymbol{e} - (\boldsymbol{\theta}_\Sigma \times \boldsymbol{e}) \otimes_s \boldsymbol{e} \,;$$

- Linear elastic, isotropic behavior:

$$\begin{aligned} \boldsymbol{\sigma} &= \lambda \operatorname{Tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon} \\ &= \lambda\varepsilon_{ee}\boldsymbol{I} + 2\mu(\varepsilon_{ee}\boldsymbol{e} + \boldsymbol{\gamma}_\Sigma) \otimes_s \boldsymbol{e} + \boldsymbol{\sigma}_\Sigma \,; \end{aligned}$$

- Traction vector:

$$\boldsymbol{\sigma e} = E(\underbrace{u'_{Ge}\boldsymbol{e}}_{//\boldsymbol{e}} + \underbrace{\boldsymbol{\theta}'_\Sigma \times \boldsymbol{x}_\Sigma}_{//\boldsymbol{e},\propto \boldsymbol{x}_\Sigma}) + \mu(\underbrace{\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e}}_{\perp \boldsymbol{e}} + \underbrace{\theta'_e\boldsymbol{e} \times \boldsymbol{x}_\Sigma}_{\perp \boldsymbol{e},\propto \boldsymbol{x}_\Sigma}) \,.$$

- Recap: resultant force

$$\boldsymbol{R} = \int_\Sigma \boldsymbol{\sigma}\boldsymbol{e}\,dS\,;$$

- Assuming $\int_\Sigma \boldsymbol{x}_\Sigma dS = \boldsymbol{0}$, $S = \int_\Sigma dS$:

$$\begin{aligned}\boldsymbol{R} &= \textstyle\int_\Sigma E(u'_{Ge}\boldsymbol{e}+\cancel{\boldsymbol{\theta}'_\Sigma\times\boldsymbol{x}_\Sigma})dS+\int_\Sigma \mu(\boldsymbol{u}'_{G\Sigma}-\boldsymbol{\theta}_\Sigma\times\boldsymbol{e}+\cancel{\theta'_e\boldsymbol{e}\times\boldsymbol{x}_\Sigma})dS \\ &= ESu'_{Ge}\boldsymbol{e} + \boldsymbol{R}_\Sigma\,;\end{aligned}$$

- Shear force with Timoshenko's assumption:

$$\boldsymbol{R}_\Sigma = \mu S(\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e})\,.$$

Beam elastic law

Resultant force with Euler-Bernoulli's kinematical assumption

- Recap: resultant force

$$\boldsymbol{R} = \int_\Sigma \boldsymbol{\sigma}\boldsymbol{e}\,dS\,;$$

- Assuming $\int_\Sigma \boldsymbol{x}_\Sigma dS = \boldsymbol{0}$, $S = \int_\Sigma dS$:

$$\boldsymbol{R} = \textstyle\int_\Sigma E(u'_{Ge}\boldsymbol{e} + \cancel{\boldsymbol{\theta}'_\Sigma \times \boldsymbol{x}_\Sigma})dS + \int_\Sigma \mu(\underbrace{\cancel{\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e}}}_{=\boldsymbol{0}\text{ by Euler-Bernoulli}} + \cancel{\theta'_e\boldsymbol{e} \times \boldsymbol{x}_\Sigma})dS$$

$$= ESu'_{Ge}\boldsymbol{e} + \boldsymbol{R}_\Sigma\,;$$

- Shear force with Euler-Bernoulli's assumption:

$$\boldsymbol{M}' + \boldsymbol{e} \times \boldsymbol{R} + \boldsymbol{c}_l = \boldsymbol{0}$$

$$\Rightarrow \quad \boldsymbol{R}_\Sigma = \boldsymbol{e} \times (\boldsymbol{M}'_\Sigma + \boldsymbol{c}_l)\,.$$

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- Recap: resultant moment

$$\boldsymbol{M} = \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{\sigma}\boldsymbol{e}\,dS\,;$$

- Assuming $\int_\Sigma \boldsymbol{x}_\Sigma dS = \boldsymbol{0}$:

$$\begin{aligned}\boldsymbol{M} &= \textstyle\int_\Sigma E(\cancel{\boldsymbol{x}_\Sigma \times u'_{Ge}}\boldsymbol{e} + \boldsymbol{x}_\Sigma \times (\boldsymbol{\theta}'_\Sigma \times \boldsymbol{x}_\Sigma))dS \\ &\quad + \textstyle\int_\Sigma \mu(\cancel{\boldsymbol{x}_\Sigma \times (\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e})} + \boldsymbol{x}_\Sigma \times (\theta'_e \boldsymbol{e} \times \boldsymbol{x}_\Sigma))dS \\ &= E\mathbb{J}(\boldsymbol{\theta}'_\Sigma) + \mu\mathbb{J}(\theta'_e \boldsymbol{e})\,;\end{aligned}$$

- (Symmetric) inertia tensor:

$$\mathbb{J} = \int_\Sigma (\|\boldsymbol{x}_\Sigma\|^2 \boldsymbol{I} - \boldsymbol{x}_\Sigma \otimes \boldsymbol{x}_\Sigma)\,dS\,;$$

- Bending moment with Timoshenko's assumption:

$$\boldsymbol{M}_\Sigma = E\mathbb{J}(\boldsymbol{\theta}'_\Sigma)\,.$$

Beam elastic law
Resultant moment

- Recap: resultant moment
$$\boldsymbol{M} = \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{\sigma}\boldsymbol{e}\, dS\,;$$
- Assuming $\int_\Sigma \boldsymbol{x}_\Sigma dS = \boldsymbol{0}$:
$$\begin{aligned}\boldsymbol{M} &= \textstyle\int_\Sigma E(\cancel{\boldsymbol{x}_\Sigma \times u'_{Ge}}\boldsymbol{e} + \boldsymbol{x}_\Sigma \times (\boldsymbol{\theta}'_\Sigma \times \boldsymbol{x}_\Sigma))dS \\ &\quad + \textstyle\int_\Sigma \mu(\cancel{\boldsymbol{x}_\Sigma \times (\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e})} + \boldsymbol{x}_\Sigma \times (\theta'_e \boldsymbol{e} \times \boldsymbol{x}_\Sigma))dS \\ &= E\mathbb{J}(\boldsymbol{\theta}'_\Sigma) + \mu\mathbb{J}(\theta'_e \boldsymbol{e})\,;\end{aligned}$$
- (Symmetric) inertia tensor:
$$\mathbb{J} = \int_\Sigma (\|\boldsymbol{x}_\Sigma\|^2 \boldsymbol{I} - \boldsymbol{x}_\Sigma \otimes \boldsymbol{x}_\Sigma)\, dS\,;$$
- Bending moment with Euler-Bernoulli's assumption:
$$\boldsymbol{M}_\Sigma = E\mathbb{J}(\boldsymbol{e} \times \boldsymbol{u}''_{G\Sigma}).$$

Summary

Constitutive equations:

$$\boldsymbol{R} = ESu'_{Ge}\boldsymbol{e} + \boldsymbol{R}_\Sigma$$
$$\boldsymbol{R}_\Sigma = \boldsymbol{e} \times (\boldsymbol{M}'_\Sigma + \boldsymbol{c}_l)$$

$$\boldsymbol{M}_\Sigma = E\mathbb{J}(\boldsymbol{e} \times \boldsymbol{u}''_{G\Sigma})$$
$$M_e = \mu J_e \theta'_e$$

$$J_e = \langle \mathbb{J}\boldsymbol{e}, \boldsymbol{e} \rangle = \int_\Sigma \|\boldsymbol{x}_\Sigma\|^2 \, dS$$

Static equilibrium:

$$\boldsymbol{R}' + \boldsymbol{f}_l = \boldsymbol{0}$$

$$\boldsymbol{M}''_\Sigma - \boldsymbol{e} \times \boldsymbol{f}_l + \boldsymbol{c}'_{l\Sigma} = \boldsymbol{0}$$
$$M'_e + c_{le} = 0$$

Differential equations

- Elongation:

$$\boxed{ESu''_{Ge}(s) + \langle \boldsymbol{f}_l(s), \boldsymbol{e} \rangle = 0}$$

with either kinematical $u_{Ge}(0), u_{Ge}(L)$ or mechanical $R_e(0), R_e(L)$ boundary conditions.

- Torsion:

$$\boxed{\mu J_e \theta''_e(s) + \langle \boldsymbol{c}_l(s), \boldsymbol{e} \rangle = 0}$$

with either kinematical $\theta_e(0), \theta_e(L)$ or mechanical $M_e(0), M_e(L)$ boundary conditions.

- Bending:

$$\boxed{E\mathbb{J}(\boldsymbol{e} \times \boldsymbol{u}_{G\Sigma}^{(IV)}(s)) - \boldsymbol{e} \times \boldsymbol{f}_l(s) + \boldsymbol{c}'_{l\Sigma}(s) = \boldsymbol{0}}$$

with either kinematical $\boldsymbol{u}_{G\Sigma}(0), \boldsymbol{u}_{G\Sigma}(L), \boldsymbol{u}'_{G\Sigma}(0), \boldsymbol{u}'_{G\Sigma}(L)$ or mechanical $\boldsymbol{M}_\Sigma(0), \boldsymbol{M}_\Sigma(L), \boldsymbol{R}_\Sigma(0), \boldsymbol{R}_\Sigma(L)$ boundary conditions.

Back to stresses...

- Recap: traction vector

$$\boldsymbol{\sigma}\boldsymbol{e} = E(\underbrace{u'_{Ge}\boldsymbol{e}}_{//\boldsymbol{e}} + \underbrace{\boldsymbol{\theta}'_\Sigma \times \boldsymbol{x}_\Sigma}_{//\boldsymbol{e},\propto\boldsymbol{x}_\Sigma}) + \mu(\underbrace{\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e}}_{\perp\boldsymbol{e}} + \underbrace{\theta'_e\boldsymbol{e} \times \boldsymbol{x}_\Sigma}_{\perp\boldsymbol{e},\propto\boldsymbol{x}_\Sigma})$$

$$= \sigma_{ee}\boldsymbol{e} + \boldsymbol{\tau}_\Sigma\,;$$

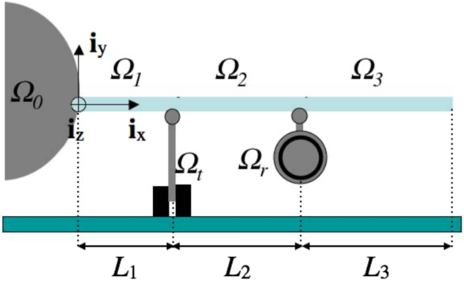
- Normal stress:

$$\begin{aligned}\sigma_{ee} &= E(u'_{Ge} + \langle\boldsymbol{\theta}'_\Sigma \times \boldsymbol{x}_\Sigma, \boldsymbol{e}\rangle) \\ &= \frac{R_e}{S} + \langle\boldsymbol{x}_\Sigma, \boldsymbol{e} \times \mathbb{J}^{-1}(\boldsymbol{M}_\Sigma)\rangle\,;\end{aligned}$$

- Shear stress:

$$\begin{aligned}\boldsymbol{\tau}_\Sigma &= \mu(\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e}) + \mu\theta'_e(\boldsymbol{e} \times \boldsymbol{x}_\Sigma) \\ &= \frac{\boldsymbol{R}_\Sigma}{S} + \frac{M_e}{J_e}(\boldsymbol{e} \times \boldsymbol{x}_\Sigma)\,.\end{aligned}$$

Outline



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- Action of the fuselage Ω_0 on which the wing Ω is clamped: $\boldsymbol{R}_0$, $\boldsymbol{M}_0$;
- Action of the landing gear Ω_t which acts as a hinged support: $\boldsymbol{R}_t$, $\boldsymbol{M}_t = \mathbf{0}$;
- Action of the engine Ω_r: $\boldsymbol{R}_r = m_r\boldsymbol{g}$, $\boldsymbol{M}_r = \mathbf{0}$;
- Self-weight and fuel $\boldsymbol{f}_l = \varrho S\boldsymbol{g}$, $\boldsymbol{c}_l = \mathbf{0}$.

Question #1: Global static equilibrium?

- Global equilibrium of forces ($\varrho_l = \varrho S$, $S = bh$):

$$\boldsymbol{R}_0 + \boldsymbol{R}_t + \boldsymbol{R}_r + \int_0^{L_1+L_2+L_3} \varrho_l \boldsymbol{g}\, ds = \boldsymbol{0}\,;$$

- Global equilibrium of moments about, say, $s = 0$:

$$\boldsymbol{M}_0 + \boldsymbol{M}_t + L_1 \boldsymbol{i}_x \times \boldsymbol{R}_t + \boldsymbol{M}_r + (L_1 + L_2)\boldsymbol{i}_x \times \boldsymbol{R}_r + \int_0^{L_1+L_2+L_3} (\boldsymbol{c}_l + s\boldsymbol{i}_x \times \varrho_l \boldsymbol{g}) ds = \boldsymbol{0}\,.$$

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Question #1: Global static equilibrium?

- Global equilibrium of forces ($\varrho_l = \varrho S$, $S = bh$):

$$\boldsymbol{R}_0 + \boldsymbol{R}_t + \boldsymbol{R}_r + \int_0^{L_1+L_2+L_3} \varrho_l \boldsymbol{g}\, ds = \boldsymbol{0}\,;$$

- Global equilibrium of moments about, say, $s = 0$:

$$\boldsymbol{M}_0 + \cancel{\boldsymbol{M}_t} + L_1 \boldsymbol{i}_x \times \boldsymbol{R}_t + \cancel{\boldsymbol{M}_r} + (L_1 + L_2)\boldsymbol{i}_x \times \boldsymbol{R}_r + \int_0^{L_1+L_2+L_3} (\cancel{\boldsymbol{c}_l} + s\boldsymbol{i}_x \times \varrho_l \boldsymbol{g})ds = \boldsymbol{0}\,.$$

Question #1: Global static equilibrium?

- Global equilibrium of forces ($\boldsymbol{g} = -g\boldsymbol{i}_y$):

$$\boldsymbol{R}_0 + \boldsymbol{R}_t - m_r g \boldsymbol{i}_y - \varrho_l g (L_1 + L_2 + L_3)\boldsymbol{i}_y = \boldsymbol{0}\,;$$

- Global equilibrium of moments about, say, $s = 0$:

$$\begin{aligned}\boldsymbol{M}_0 + L_1\boldsymbol{i}_x \times \boldsymbol{R}_t - m_r g (L_1 + L_2)\boldsymbol{i}_z & \\ - \frac{1}{2}\varrho_l g (L_1 + L_2 + L_3)^2 \boldsymbol{i}_z = \boldsymbol{0}\,.\end{aligned}$$

Question #1: Global static equilibrium?

- Global equilibrium of forces ($\boldsymbol{g} = -g\boldsymbol{i}_y$):

$$\boldsymbol{R}_0 + \boldsymbol{R}_t - m_r g \boldsymbol{i}_y - \varrho_l g (L_1 + L_2 + L_3)\boldsymbol{i}_y = \boldsymbol{0}\,;$$

- Global equilibrium of moments about, say, $s = 0$:

$$\begin{aligned}\boldsymbol{M}_0 + L_1 \boldsymbol{i}_x \times \boldsymbol{R}_t - m_r g (L_1 + L_2)\boldsymbol{i}_z & \\ - \frac{1}{2}\varrho_l g (L_1 + L_2 + L_3)^2 \boldsymbol{i}_z &= \boldsymbol{0}\,.\end{aligned}$$

- 3 scalar equations (planar problem) for 5 unknowns $\Rightarrow$ the wing is statically overdetermined with 2 statically indeterminate unknowns, $\langle \boldsymbol{R}_t, \boldsymbol{i}_x \rangle$ and $\langle \boldsymbol{R}_t, \boldsymbol{i}_y \rangle$ for example.

Question #2: $\boldsymbol{R}$ in $s \in (L_1 + L_2, L_1 + L_2 + L_3]$?

- Local balance of forces on the cross-section $\Sigma(s)$:

$$\boldsymbol{R}' + \boldsymbol{f}_l = \boldsymbol{0}\,,$$

where $\boldsymbol{f}_l = -\varrho_l g \boldsymbol{i}_y$;

- Thus:

$$\boldsymbol{R}(s) = \varrho_l g(s - L_1 - L_2 - L_3)\boldsymbol{i}_y + \boldsymbol{R}(L_1 + L_2 + L_3)\,;$$

Question #2: $\boldsymbol{R}$ in $s \in (L_1 + L_2, L_1 + L_2 + L_3]$?

- Local balance of forces on the cross-section $\Sigma(s)$:

$$\boldsymbol{R}' - \varrho_l g \boldsymbol{i}_y = \boldsymbol{0}\,;$$

- Thus:

$$\boldsymbol{R}(s) = \varrho_l g(s - L_1 - L_2 - L_3)\boldsymbol{i}_y + \boldsymbol{R}(L_1 + L_2 + L_3)\,;$$

- But $\boldsymbol{R}(L_1 + L_2 + L_3) = \boldsymbol{0}$ and therefore:

$$\boldsymbol{R}(s) = \varrho_l g(s - L_1 - L_2 - L_3)\boldsymbol{i}_y\,,\quad s \in (L_1 + L_2, L_1 + L_2 + L_3]\,.$$

Question #2: $\boldsymbol{M}$ in $s \in (L_1 + L_2, L_1 + L_2 + L_3]$?

- Local balance of moments on the cross-section $\Sigma(s)$:

$$\boldsymbol{M}' + \boldsymbol{e} \times \boldsymbol{R} + \boldsymbol{c}_l = \boldsymbol{0}\,,$$

  where $\boldsymbol{e} = \boldsymbol{i}_x$ and $\boldsymbol{c}_l = \boldsymbol{0}$;

- Thus:

$$\boldsymbol{M}'(s) = -\boldsymbol{i}_x \times \varrho_l g(s - L_1 - L_2 - L_3)\boldsymbol{i}_y$$

$$\boldsymbol{M}(s) = -\frac{\varrho_l g}{2}(s - L_1 - L_2 - L_3)^2 \boldsymbol{i}_z + \boldsymbol{M}(L_1 + L_2 + L_3)\,;$$

Question #2: $\boldsymbol{M}$ in $s \in (L_1 + L_2, L_1 + L_2 + L_3]$?

- Local balance of moments on the cross-section $\Sigma(s)$:

$$\boldsymbol{M}' + \boldsymbol{i}_x \times \boldsymbol{R} = \boldsymbol{0}\,;$$

- Thus:

$$\boldsymbol{M}(s) = -\frac{\varrho_l g}{2}(s - L_1 - L_2 - L_3)^2 \boldsymbol{i}_z + \boldsymbol{M}(L_1 + L_2 + L_3)\,;$$

- But $\boldsymbol{M}(L_1 + L_2 + L_3) = \boldsymbol{0}$ and therefore:

$$\boldsymbol{M}(s) = -\frac{\varrho_l g}{2}(s - L_1 - L_2 - L_3)^2 \boldsymbol{i}_z\,, \quad s \in (L_1 + L_2, L_1 + L_2 + L_3]\,.$$

Question #2: $\boldsymbol{R}$ in $s \in (L_1 + L_2, L_1 + L_2 + L_3]$?

- **Alternative solution**: global balance of forces of the beam $[s, L_1 + L_2 + L_3]$ for $s \in (L_1 + L_2, L_1 + L_2 + L_3]$

$$-\boldsymbol{R}(s) + \boldsymbol{R}(L_1 + L_2 + L_3) + \int_s^{L_1+L_2+L_3} \boldsymbol{f}_l \, d\zeta = \boldsymbol{0}\,;$$

- But $\boldsymbol{R}(L_1 + L_2 + L_3) = \boldsymbol{0}$ and therefore:

$$\boldsymbol{R}(s) = -\varrho_l g(L_1 + L_2 + L_3 - s)\boldsymbol{i}_y\,.$$

Question #2: $\boldsymbol{M}$ in $s \in (L_1 + L_2, L_1 + L_2 + L_3]$?

- **Alternative solution**: global balance of moments of beam $[s, L_1 + L_2 + L_3]$ for $s \in (L_1 + L_2, L_1 + L_2 + L_3]$

$$-\boldsymbol{M}(s)+\boldsymbol{M}(L_1+L_2+L_3)+(L_1+L_2+L_3-s)\boldsymbol{i}_x\times\cancel{\boldsymbol{R}(L_1+L_2+L_3)} + \int_s^{L_1+L_2+L_3}(\boldsymbol{c}_l+(\zeta-s)\times\boldsymbol{f}_l)\,d\zeta=\boldsymbol{0}\,;$$

- But $\boldsymbol{M}(L_1 + L_2 + L_3) = \boldsymbol{0}$, $\boldsymbol{c}_l = \boldsymbol{0}$, and therefore:

$$\boldsymbol{M}(s) = -\frac{\varrho_l g}{2}(L_1 + L_2 + L_3 - s)^2\boldsymbol{i}_z\,.$$

Question #3: $\boldsymbol{R}$ in $s \in (L_1, L_1 + L_2]$?

- Local balance of forces on the cross-section $\Sigma(s)$:

$$\boldsymbol{R}' + \boldsymbol{f}_l = \boldsymbol{0}\,,$$

  where $\boldsymbol{f}_l = -\varrho_l g \boldsymbol{i}_y$;
- Thus:

$$\boldsymbol{R}(s) = \varrho_l g(s - L_1 - L_2)\boldsymbol{i}_y + \boldsymbol{R}((L_1 + L_2)^-)\,;$$

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Question #3: $\boldsymbol{R}$ in $s \in (L_1, L_1 + L_2]$?

- Local balance of forces on the cross-section $\Sigma(s)$:

$$\boldsymbol{R}' - \varrho_l g \boldsymbol{i}_y = \boldsymbol{0}\,;$$

- Thus:

$$\boldsymbol{R}(s) = \varrho_l g (s - L_1 - L_2)\boldsymbol{i}_y + \boldsymbol{R}((L_1 + L_2)^-)\,;$$

- But $-\boldsymbol{R}((L_1 + L_2)^-) + \boldsymbol{R}((L_1 + L_2)^+) + \boldsymbol{R}_r = \boldsymbol{0}$ from the local equilibrium of $\Sigma(L_1 + L_2)$, therefore:

$$\begin{aligned}\boldsymbol{R}(s) &= \varrho_l g (s - L_1 - L_2)\boldsymbol{i}_y + \boldsymbol{R}((L_1 + L_2)^+) + \boldsymbol{R}_r \\ &= \varrho_l g (s - L_1 - L_2 - L_3)\boldsymbol{i}_y - m_r g \boldsymbol{i}_y\,, \quad s \in (L_1, L_1 + L_2]\,.\end{aligned}$$

Question #3: $\boldsymbol{M}$ in $s \in (L_1, L_1 + L_2]$?

- Local balance of moments on the cross-section $\Sigma(s)$:

$$\boldsymbol{M}' + \boldsymbol{e} \times \boldsymbol{R} + \boldsymbol{c}_l = \boldsymbol{0}\,,$$

  where $\boldsymbol{e} = \boldsymbol{i}_x$ and $\boldsymbol{c}_l = \boldsymbol{0}$;
- Thus:

$$\begin{aligned}\boldsymbol{M}'(s) &= -\boldsymbol{i}_x \times [\varrho_l g(s-L_1-L_2-L_3)\boldsymbol{i}_y - m_r g \boldsymbol{i}_y] \\ \boldsymbol{M}(s) &= -\frac{\varrho_l g}{2}(s-L_1-L_2-L_3)^2 \boldsymbol{i}_z + m_r g(s-L_1-L_2)\boldsymbol{i}_z \\ &\quad + \frac{\varrho_l g}{2} L_3^2 \boldsymbol{i}_z + \boldsymbol{M}((L_1+L_2)^-)\,;\end{aligned}$$

Question #3: $\boldsymbol{M}$ in $s \in (L_1, L_1 + L_2]$?

- Local balance of moments on the cross-section $\Sigma(s)$:

$$\boldsymbol{M}' + \boldsymbol{i}_x \times \boldsymbol{R} = \boldsymbol{0}\,;$$

- Thus:

$$\boldsymbol{M}(s) = -\frac{\varrho_l g}{2}(s-L_1-L_2-L_3)^2\boldsymbol{i}_z + m_r g(s-L_1-L_2)\boldsymbol{i}_z + \frac{\varrho_l g}{2}L_3^2\boldsymbol{i}_z + \boldsymbol{M}((L_1+L_2)^-)\,;$$

- But $-\boldsymbol{M}((L_1+L_2)^-) + \boldsymbol{M}((L_1+L_2)^+) = \boldsymbol{0}$ from the local equilibrium of $\Sigma(L_1+L_2)$, therefore:

$$\boldsymbol{M}(s) = -\frac{\varrho_l g}{2}(s-L_1-L_2-L_3)^2\boldsymbol{i}_z + m_r g(s-L_1-L_2)\boldsymbol{i}_z\,,\ s\in(L_1,L_1+L_2]\,.$$

Question #3: $\boldsymbol{R}$ in $s \in (L_1, L_1 + L_2]$?

- **Alternative solution**: global balance of forces of the beam $[s, L_1 + L_2 + L_3]$ for $s \in (L_1, L_1 + L_2]$

$$-\boldsymbol{R}(s) + \cancel{\boldsymbol{R}(L_1 + L_2 + L_3)} + \boldsymbol{R}_r + \int_s^{L_1+L_2+L_3} \boldsymbol{f}_l \, d\zeta = \boldsymbol{0} \,;$$

- Therefore:

$$\boldsymbol{R}(s) = -\varrho_l g(L_1 + L_2 + L_3 - s)\boldsymbol{i}_y - m_r g \boldsymbol{i}_y \,.$$

- With this approach the results from question #2 are useless.

Question #3: $\boldsymbol{M}$ in $s \in (L_1, L_1 + L_2]$?

- **Alternative solution**: global balance of moments of the beam $[s, L_1 + L_2 + L_3]$ for $s \in (L_1, L_1 + L_2]$

$$-\boldsymbol{M}(s)+\cancel{\boldsymbol{M}(L_1+L_2+L_3)}+(L_1+L_2+L_3-s)\boldsymbol{i}_x\times\cancel{\boldsymbol{R}(L_1+L_2+L_3)}$$
$$+(L_1+L_2-s)\boldsymbol{i}_x\times\boldsymbol{R}_r+\int_s^{L_1+L_2+L_3}(\boldsymbol{c}_l+(\zeta-s)\times\boldsymbol{f}_l)\,d\zeta=\boldsymbol{0}\,;$$

- But $\boldsymbol{c}_l = \boldsymbol{0}$, and therefore:

$$\boldsymbol{M}(s) = -\frac{\varrho_l g}{2}(L_1+L_2+L_3-s)^2\boldsymbol{i}_z - m_r g(L_1+L_2-s)\boldsymbol{i}_z\,.$$

- With this approach the results from question #2 are useless.

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Solution

Question #4: $\boldsymbol{R}$ in $s \in (0, L_1]$?

- Local balance of forces on the cross-section $\Sigma(s)$:

$$\boldsymbol{R}' + \boldsymbol{f}_l = \boldsymbol{0}\,,$$

where $\boldsymbol{f}_l = -\varrho_l g \boldsymbol{i}_y$;

- Thus:

$$\boldsymbol{R}(s) = \varrho_l g (s - L_1)\boldsymbol{i}_y + \boldsymbol{R}(L_1^-)\,;$$

Question #4: $\boldsymbol{R}$ in $s \in (0, L_1]$?

- Local balance of forces on the cross-section $\Sigma(s)$:

$$\boldsymbol{R}' - \varrho_l g \boldsymbol{i}_y = \boldsymbol{0} \,;$$

- Thus:

$$\boldsymbol{R}(s) = \varrho_l g (s - L_1) \boldsymbol{i}_y + \boldsymbol{R}(L_1^-) \,;$$

- But $-\boldsymbol{R}(L_1^-) + \boldsymbol{R}(L_1^+) + \boldsymbol{R}_t = \boldsymbol{0}$ from the local equilibrium of $\Sigma(L_1)$, therefore:

$$\begin{aligned}\boldsymbol{R}(s) &= \varrho_l g (s - L_1) \boldsymbol{i}_y + \boldsymbol{R}(L_1^+) + \boldsymbol{R}_t \\ &= \varrho_l g (s - L_1) \boldsymbol{i}_y - \varrho_l g (L_2 + L_3) \boldsymbol{i}_y - m_r g \boldsymbol{i}_y + \boldsymbol{R}_t \\ &= \varrho_l g (s - L_1 - L_2 - L_3) \boldsymbol{i}_y - m_r g \boldsymbol{i}_y + \boldsymbol{R}_t \,, \quad s \in (0, L_1] \,.\end{aligned}$$

Question #4: $\boldsymbol{M}$ in $s \in (0, L_1]$?

- Local balance of moments on the cross-section $\Sigma(s)$:

$$\boldsymbol{M}' + \boldsymbol{e} \times \boldsymbol{R} + \boldsymbol{c}_l = \boldsymbol{0}\,,$$

  where $\boldsymbol{e} = \boldsymbol{i}_x$ and $\boldsymbol{c}_l = \boldsymbol{0}$;
- Thus:

$$\begin{aligned}
\boldsymbol{M}'(s) &= -\boldsymbol{i}_x \times [\varrho_l g(s - L_1 - L_2 - L_3)\boldsymbol{i}_y - m_r g \boldsymbol{i}_y + \boldsymbol{R}_t] \\
\boldsymbol{M}(s) &= -\frac{\varrho_l g}{2}(s - L_1 - L_2 - L_3)^2 \boldsymbol{i}_z + m_r g(s - L_1)\boldsymbol{i}_z + (L_1 - s)\boldsymbol{i}_x \times \boldsymbol{R}_t \\
&\quad + \frac{\varrho_l g}{2}(L_2 + L_3)^2 \boldsymbol{i}_z + \boldsymbol{M}(L_1^-)\,;
\end{aligned}$$

- Remark: $\boldsymbol{i}_x \times \boldsymbol{R}_t = \langle \boldsymbol{R}_t, \boldsymbol{i}_y \rangle \boldsymbol{i}_z$.

Question #4: $\boldsymbol{M}$ in $s \in (0, L_1]$?

- Local balance of moments on the cross-section $\Sigma(s)$:

$$\boldsymbol{M}' + \boldsymbol{i}_x \times \boldsymbol{R} = \boldsymbol{0}\,;$$

- Thus:

$$\begin{aligned}\boldsymbol{M}(s) = & -\tfrac{\varrho_l g}{2}(s-L_1-L_2-L_3)^2\boldsymbol{i}_z + m_r g(s-L_1)\boldsymbol{i}_z + (L_1-s)\boldsymbol{i}_x\times\boldsymbol{R}_t \\ & + \tfrac{\varrho_l g}{2}(L_2+L_3)^2\boldsymbol{i}_z + \boldsymbol{M}(L_1^-)\,;\end{aligned}$$

- But $-\boldsymbol{M}(L_1^-) + \boldsymbol{M}(L_1^+) = \boldsymbol{0}$ from the local equilibrium of $\Sigma(L_1)$, therefore:

$$\begin{aligned}\boldsymbol{M}(s) &= -\tfrac{\varrho_l g}{2}(s-L_1-L_2-L_3)^2\boldsymbol{i}_z + m_r g(s-L_1)\boldsymbol{i}_z + (L_1-s)\boldsymbol{i}_x\times\boldsymbol{R}_t - m_r g L_2\boldsymbol{i}_z \\ &= -\tfrac{\varrho_l g}{2}(s-L_1-L_2-L_3)^2\boldsymbol{i}_z + m_r g(s-L_1-L_2)\boldsymbol{i}_z + (L_1-s)\boldsymbol{i}_x\times\boldsymbol{R}_t\,,\ s\in(0,L_1].\end{aligned}$$

Question #4: $\boldsymbol{R}$ in $s \in (0, L_1]$?

- **Alternative solution**: global balance of forces of the beam $[s, L_1 + L_2 + L_3]$ for $s \in (0, L_1]$

$$-\boldsymbol{R}(s) + \cancel{\boldsymbol{R}(L_1 + L_2 + L_3)} + \boldsymbol{R}_r + \boldsymbol{R}_t + \int_s^{L_1+L_2+L_3} \boldsymbol{f}_l \, d\zeta = \boldsymbol{0} \,;$$

- Therefore:

$$\boldsymbol{R}(s) = -\varrho_l g(L_1 + L_2 + L_3 - s)\boldsymbol{i}_y - m_r g \boldsymbol{i}_y + \boldsymbol{R}_t \,.$$

Question #4: $\boldsymbol{M}$ in $s \in (0, L_1]$?

- **Alternative solution**: global balance of moments of the beam $[s, L_1 + L_2 + L_3]$ for $s \in (0, L_1]$

$$
\begin{aligned}
-\boldsymbol{M}(s)+\cancel{\boldsymbol{M}(L_1+L_2+L_3)}+(L_1+L_2+L_3-s)\boldsymbol{i}_x\times\cancel{\boldsymbol{R}(L_1+L_2+L_3)} \\
+(L_1+L_2-s)\boldsymbol{i}_x\times\boldsymbol{R}_r+(L_1-s)\boldsymbol{i}_x\times\boldsymbol{R}_t+\int_s^{L_1+L_2+L_3}(\boldsymbol{c}_l+(\zeta-s)\times\boldsymbol{f}_l)\,d\zeta = \boldsymbol{0}\,;
\end{aligned}
$$

- But $\boldsymbol{c}_l = \boldsymbol{0}$, and therefore:

$$
\begin{aligned}
\boldsymbol{M}(s) = -\frac{\varrho_l g}{2}(L_1 + L_2 + L_3 - s)^2\boldsymbol{i}_z \\
- m_r g(L_1 + L_2 - s)\boldsymbol{i}_z + (L_1 - s)\boldsymbol{i}_x \times \boldsymbol{R}_t\,.
\end{aligned}
$$

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Question #5: Longitudinal displacement u_{Gx}?

- Constitutive equation for the longitudinal displacement:

$$R_e(s) = ESu'_{Ge}(s)$$

$$\Rightarrow \quad u'_{Gx}(s) = \frac{\langle \boldsymbol{R}, \boldsymbol{i}_x \rangle}{ES}\,;$$

Question #5: Longitudinal displacement u_{Gx}?

- Constitutive equation for the longitudinal displacement:

$$u'_{Gx}(s) = \frac{\langle \boldsymbol{R}, \boldsymbol{i}_x \rangle}{ES} \,;$$

- But $\langle \boldsymbol{R}, \boldsymbol{i}_x \rangle = \langle \boldsymbol{R}_t, \boldsymbol{i}_x \rangle$ in Ω_1 and thus:

$$\begin{aligned} u_{Gx}(s) &= \langle \boldsymbol{R}_t, \boldsymbol{i}_x \rangle \, s + \cancel{u_{Gx}(0)} \\ &= \langle \boldsymbol{R}_t, \boldsymbol{i}_x \rangle \, s \,, \quad s \in (0, L_1) \,, \end{aligned}$$

because the wing is fixed to the fuselage which is "fixed and perfectly rigid;"

Question #5: Longitudinal displacement u_{Gx}?

- Constitutive equation for the longitudinal displacement:

$$u'_{Gx}(s) = \frac{\langle \boldsymbol{R}, \boldsymbol{i}_x \rangle}{ES}\,;$$

- But $\langle \boldsymbol{R}, \boldsymbol{i}_x \rangle = \langle \boldsymbol{R}_t, \boldsymbol{i}_x \rangle$ in Ω_1 and thus:

$$u_{Gx}(s) = \langle \boldsymbol{R}_t, \boldsymbol{i}_x \rangle\, s\,, \quad s \in (0, L_1)\,;$$

- Besides $u_{Gx}(L_1) = 0$ since the landing gear is "perfectly rigid" as well, therefore $u_{Gx}(s) = 0$ in Ω_1 and $\langle \boldsymbol{R}_t, \boldsymbol{i}_x \rangle = 0$;
- Also $\langle \boldsymbol{R}, \boldsymbol{i}_x \rangle = 0$ in $\Omega_2 \cup \Omega_3$ and by continuity of the displacement $u_{Gx}(s) = 0$ in $\Omega_2 \cup \Omega_3$.

Question #6: Transverse displacement u_{Gy} in Ω_1?

- Constitutive equation for the transverse motion under Euler-Bernoulli's assumption:

$$\begin{aligned} \boldsymbol{M}_\Sigma(s) &= E\mathbb{J}(\boldsymbol{e}\times\boldsymbol{u}''_{G\Sigma}(s)) \\ \Rightarrow \quad u''_{Gy}(s)\boldsymbol{i}_z - u''_{Gz}(s)\boldsymbol{i}_y &= \frac{1}{E}\mathbb{J}^{-1}\boldsymbol{M}_\Sigma(s)\,; \end{aligned}$$

Question #6: Transverse displacement u_{Gy} in Ω_1?

- Constitutive equation:

$$u''_{Gy}(s)\boldsymbol{i}_z - u''_{Gz}(s)\boldsymbol{i}_y = \frac{1}{E}\mathbb{J}^{-1}\boldsymbol{M}_\Sigma(s)\,;$$

- But in Ω_1:

$$\begin{aligned}\boldsymbol{M}(s) &= -\tfrac{\varrho_l g}{2}(L_1+L_2+L_3-s)^2\boldsymbol{i}_z - m_r g(L_1+L_2-s)\boldsymbol{i}_z + (L_1-s)\boldsymbol{i}_x\times\boldsymbol{R}_t \\ &= \left[-\tfrac{\varrho_l g}{2}(L_1+L_2+L_3-s)^2 - m_r g(L_1+L_2-s) + (L_1-s)\langle\boldsymbol{R}_t,\boldsymbol{i}_y\rangle\right]\boldsymbol{i}_z\,,\end{aligned}$$

since $\langle\boldsymbol{R}_t,\boldsymbol{i}_x\rangle = 0$;

Question #6: Transverse displacement u_{Gy} in Ω_1?

- Constitutive equation:

$$u''_{Gy}(s)\boldsymbol{i}_z - u''_{Gz}(s)\boldsymbol{i}_y = \frac{1}{E}\mathbb{J}^{-1}\boldsymbol{M}_\Sigma(s)\,;$$

- But in Ω_1:

$$\boldsymbol{M}(s) = \left[-\tfrac{\varrho_l g}{2}(L_1+L_2+L_3-s)^2 - m_r g(L_1+L_2-s) + \langle \boldsymbol{R}_t, \boldsymbol{i}_y\rangle(L_1-s)\right]\boldsymbol{i}_z\,;$$

- Therefore $u''_{Gz}(s) = 0$ and:

$$u''_{Gy}(s) = -\tfrac{1}{EI_z}\left[\tfrac{\varrho_l g}{2}(L_1+L_2+L_3-s)^2 + m_r g(L_1+L_2-s) + \langle \boldsymbol{R}_t, \boldsymbol{i}_y\rangle(s-L_1)\right];$$

Question #6: Transverse displacement u_{Gy} in Ω_1?

- Because $u_{Gy}(0) = u'_{Gy}(0) = 0$ (the fuselage is "fixed and perfectly rigid") one has:

$$
\begin{aligned}
u''_{Gy}(s) &= -\tfrac{1}{EI_z}\left[\tfrac{\varrho_l g}{2}(L-s)^2+m_r g(L_1+L_2-s)+\langle \boldsymbol{R}_t,\boldsymbol{i}_y\rangle(s-L_1)\right] \\
u'_{Gy}(s) &= \tfrac{1}{2EI_z}\left[\tfrac{\varrho_l g}{3}(L-s)^3+m_r g(L_1+L_2-s)^2-\langle \boldsymbol{R}_t,\boldsymbol{i}_y\rangle(s-L_1)^2\right] \\
&\quad -\tfrac{1}{2EI_z}\left[\tfrac{\varrho_l g}{3}L^3+m_r g(L_1+L_2)^2-\langle \boldsymbol{R}_t,\boldsymbol{i}_y\rangle L_1^2\right]+\cancel{u'_{Gy}(0)} \\
u_{Gy}(s) &= -\tfrac{1}{6EI_z}\left[\tfrac{\varrho_l g}{4}(L-s)^4+m_r g(L_1+L_2-s)^3+\langle \boldsymbol{R}_t,\boldsymbol{i}_y\rangle(s-L_1)^3\right] \\
&\quad +\tfrac{1}{6EI_z}\left[\tfrac{\varrho_l g}{4}L^4+m_r g(L_1+L_2)^3-\langle \boldsymbol{R}_t,\boldsymbol{i}_y\rangle L_1^3\right] \\
&\quad -\tfrac{1}{2EI_z}\left[\tfrac{\varrho_l g}{3}L^3+m_r g(L_1+L_2)^2-\langle \boldsymbol{R}_t,\boldsymbol{i}_y\rangle L_1^2\right]s+\cancel{u_{Gy}(0)} \\
&= -\tfrac{s^2}{6EI_z}\left[\tfrac{\varrho_l g}{4}[6L^2-4Ls+s^2]+m_r g[3(L_1+L_2)-s]-\langle \boldsymbol{R}_t,\boldsymbol{i}_y\rangle(3L_1-s)\right]
\end{aligned}
$$

where $L = L_1 + L_2 + L_3$.

Question #6: Transverse displacement u_{Gy} in Ω_1?

- Letting $L = L_1 + L_2 + L_3$:

$$u_{Gy}(s) = -\frac{s^2}{6EI_z}\left[\frac{\varrho_l g}{4}(6L^2-4Ls+s^2)+m_r g[3(L_1+L_2)-s]-\langle \boldsymbol{R}_t,\boldsymbol{i}_y\rangle(3L_1-s)\right];$$

- But $u_{Gy}(L_1) = 0$ right at the landing gear so that one has:

$$\langle \boldsymbol{R}_t, \boldsymbol{i}_y\rangle = g\left[\varrho_l L_1\left(\frac{3}{8}+\frac{L_2+L_3}{L_1}+\frac{3(L_2+L_3)^2}{4L_1^2}\right)+m_r\left(1+\frac{3L_2}{2L_1}\right)\right],$$

which can be plugged back in the expression of $u_{Gy}(s)$:

$$u_{Gy}(s) = \frac{-gs^2}{12EI_z}\left[\frac{\varrho_l}{4}\left(2s^2+6(L_2+L_3)^2\left(\frac{s}{L_1}-1\right)-5L_1 s+3L_1^2\right)+3m_r L_2\left(\frac{s}{L_1}-1\right)\right]$$

Question #7: Final expressions of $\boldsymbol{R}(s)$, $\boldsymbol{M}(s)$ in Ω_1?

- Resultant force:

$$\begin{aligned}\boldsymbol{R}(s) &= \left[\varrho_l g(s-L) - m_r g + \langle \boldsymbol{R}_t, \boldsymbol{i}_y \rangle\right] \boldsymbol{i}_y \\ &= g\left[\varrho_l \left(s + \frac{3(L_2+L_3)^2}{4L_1} - \frac{5}{8}L_1\right) + m_r \frac{3L_2}{2L_1}\right] \boldsymbol{i}_y\,;\end{aligned}$$

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Question #7: Final expressions of $\boldsymbol{R}(s)$, $\boldsymbol{M}(s)$ in Ω_1?

- Resultant force:

$$\begin{aligned}\boldsymbol{R}(s) &= \left[\varrho_l g(s-L) - m_r g + \langle \boldsymbol{R}_t, \boldsymbol{i}_y\rangle\right]\boldsymbol{i}_y \\ &= g\left[\varrho_l\left(s + \frac{3(L_2+L_3)^2}{4L_1} - \frac{5}{8}L_1\right) + m_r\frac{3L_2}{2L_1}\right]\boldsymbol{i}_y\,;\end{aligned}$$

- Resultant moment:

$$\begin{aligned}\boldsymbol{M}(s) &= \left[-\tfrac{\varrho_l g}{2}(s-L)^2 + m_r g(s-L_1-L_2) + \langle \boldsymbol{R}_t, \boldsymbol{i}_y\rangle(L_1-s)\right]\boldsymbol{i}_z \\ &= \tfrac{g}{2}\left[\tfrac{\varrho_l}{4}\left((1-\tfrac{s}{L_1})(L_1^2+6L^2-4LL_1) - 4(L-s)^2\right) + m_r L_2\left(1-3\tfrac{s}{L_1}\right)\right]\boldsymbol{i}_z\,.\end{aligned}$$

Question #8: σ_{ee}?

- Normal stress:

$$\sigma_{ee} = \frac{\langle \boldsymbol{R}, \boldsymbol{e} \rangle}{S} + \left\langle \boldsymbol{x}_\Sigma, \boldsymbol{e} \times \mathbb{J}^{-1}(\boldsymbol{M}_\Sigma) \right\rangle \,;$$

- In the present case $\boldsymbol{e} = \boldsymbol{i}_x$, $\boldsymbol{R} = R_y \boldsymbol{i}_y$, $\boldsymbol{M}_\Sigma = \boldsymbol{M} = M_z \boldsymbol{i}_z$;

Question #8: σ_{ee}?

- Therefore:

$$\begin{aligned}
\sigma_{ee}(\boldsymbol{x}) &= \frac{M_z(s)}{I_z}\langle \boldsymbol{x}_\Sigma, \boldsymbol{i}_x \times \boldsymbol{i}_z\rangle \\
&= -\frac{M_z(s)y}{I_z} \\
&= \frac{gy}{2I_z}\begin{cases}
\varrho_l(L-s)^2-\frac{\varrho_l}{4}(1-\frac{s}{L_1})(L_1^2+6L^2-4LL_1) \\
\quad -m_r L_2\left(1-3\frac{s}{L_1}\right) & \text{if } s\in(0,L_1]\,, \\
\varrho_l(L-s)^2+2m_r(L_1+L_2-s) & \text{if } s\in(L_1,L_1+L_2]\,, \\
\varrho_l(L-s)^2 & \text{if } s\in(L_1+L_2,L]\,;
\end{cases}
\end{aligned}$$

Question #8: σ_{ee}?

- Therefore:

$$\sigma_{ee}(\boldsymbol{x}) = \frac{gy}{2I_z}\begin{cases} \varrho_l(L-s)^2-\frac{\varrho_l}{4}(1-\frac{s}{L_1})(L_1^2+6L^2-4LL_1) & \\ \quad -m_r L_2\left(1-3\frac{s}{L_1}\right) & \text{if } s \in (0,L_1]\,, \\ \varrho_l(L-s)^2+2m_r(L_1+L_2-s) & \text{if } s \in (L_1,L_1+L_2]\,, \\ \varrho_l(L-s)^2 & \text{if } s \in (L_1+L_2,L]\,; \end{cases}$$

- It is maximum for $y=\frac{h}{2}$ and $s=L_1$ with $(I_z=\frac{bh^3}{12})$:

$$\begin{aligned}\sigma_{ee,\max} &= \frac{gh}{4I_z}\left[\varrho_l(L_2+L_3)^2+2m_rL_2\right] \\ &= \frac{3g}{bh^2}\left[\varrho bh(L_2+L_3)^2+2m_rL_2\right]\,.\end{aligned}$$

Question #9: Transverse displacement u_{Gy} in $\Omega_2 \cup \Omega_3$?

- From the constitutive equation in bending (see Q6):

$$u''_{Gy}(s) = \frac{M_z(s)}{EI_z}\,;$$

- The two integration constants in Ω_2 are given by the continuity of the motion at $s = L_1$ which reads $u_{Gy}(L_1^+) = u_{Gy}(L_1^-) = 0$ (displacement) and $u'_{Gy}(L_1^+) = u'_{Gy}(L_1^-)$ (rotation), where $u'_{Gy}(L_1^-)$ has been obtained in Q6;
- The same holds in Ω_3 at $s = L_1 + L_2$.

Question #10: Resultant force $\boldsymbol{R}$ and moment $\boldsymbol{M}$ with gravity?

- We did it all along the way with $\varrho_l \neq 0$!!!

Question #11: Resultant force $\boldsymbol{R}_t = -k_t u_{Gy}(L_1)\boldsymbol{i}_y$?

- From Q6 we have for $s \in (0, L_1]$:

$$u_{Gy}(s) = -\frac{s^2}{6EI_z}\left[\frac{\varrho_l g}{4}(6L^2-4Ls+s^2)+m_r g[3(L_1+L_2)-s]-\langle\boldsymbol{R}_t,\boldsymbol{i}_y\rangle(3L_1-s)\right];$$

- But $\langle\boldsymbol{R}_t,\boldsymbol{i}_y\rangle = -k_t u_{Gy}(L_1)$ so that one has:

$$u_{Gy}(s) = -\frac{s^2}{6EI_z}\left[\frac{\varrho_l g}{4}(6L^2-4Ls+s^2)+m_r g[3(L_1+L_2)-s]+k_t u_{Gy}(L_1)(3L_1-s)\right];$$

- Therefore:

$$\begin{aligned}\left(1+\frac{2k_tL_1^3}{6EI_z}\right)u_{Gy}(L_1) &= -\frac{gL_1^2}{6EI_z}\left[\frac{\varrho_l}{4}(6L^2-4LL_1+L_1^2)+m_r(2L_1+3L_2)\right]\\ u_{Gy}(L_1) &= -\frac{gL_1^2}{6EI_z+2k_tL_1^3}\left[\frac{\varrho_l}{4}(6L^2-4LL_1+L_1^2)+m_r(2L_1+3L_2)\right]\end{aligned}$$

Question #11: Resultant force $\boldsymbol{R}_t = -k_t u_{Gy}(L_1)\boldsymbol{i}_y$?

- Besides we have from Q4 for $s \in (0, L_1]$:

$$\begin{aligned}\boldsymbol{R}(s) &= \varrho_l g(s-L)\boldsymbol{i}_y - m_r g\boldsymbol{i}_y + \boldsymbol{R}_t \\ &= -\left[\varrho_l g(L-s) + m_r g + k_t u_{Gy}(L_1)\right]\boldsymbol{i}_y \,,\end{aligned}$$

  and:

$$\begin{aligned}\boldsymbol{M}(s) &= -\tfrac{\varrho_l g}{2}(s-L)^2\boldsymbol{i}_z + m_r g(s-L_1-L_2)\boldsymbol{i}_z + (L_1-s)\boldsymbol{i}_x \times \boldsymbol{R}_t \\ &= -\left[\tfrac{\varrho_l g}{2}(L-s)^2 + m_r g(L_1+L_2-s) + k_t u_{Gy}(L_1)(L_1-s)\right]\boldsymbol{i}_z \,;\end{aligned}$$

- The rest of the derivation is un changed except the boundary condition $u_{Gy}(L_1^+) = u_{Gy}(L_1^-)$ (displacement) at $s = L_1$ which is no longer imposed to vanish.