Move crane

1EL5000–Continuum Mechanics – Tutorial Class #7

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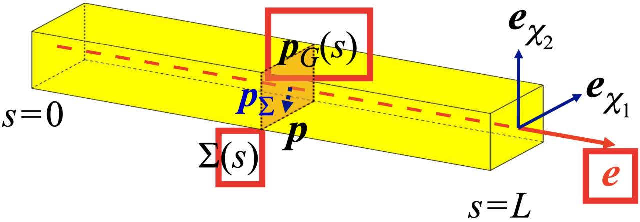
March 8, 2021

Outline

Outline

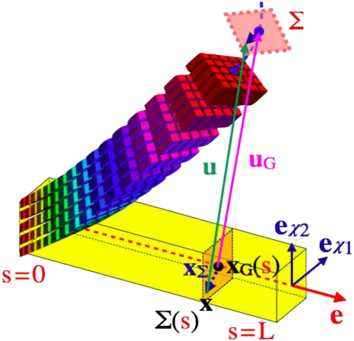
Beam kinematics

Reference configuration



$$\begin{aligned}\boldsymbol{p}(s,\chi_1,\chi_2) &= \boldsymbol{p}_G(s) + \boldsymbol{p}_\Sigma(\chi_1,\chi_2) \\ &= s\boldsymbol{e} + \chi_1\boldsymbol{e}_{\chi_1} + \chi_2\boldsymbol{e}_{\chi_2}\end{aligned}$$

Beam kinematics
Actual configuration



$$\boldsymbol{x} = \boldsymbol{x}_G(s) + \boldsymbol{x}_\Sigma$$
$$= \boldsymbol{x}_G(s) + \boldsymbol{R}(s)\boldsymbol{p}_\Sigma$$

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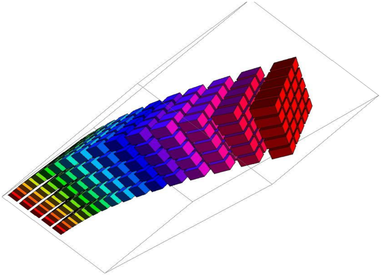
Kinematics

Statics

Example

Stresses

6.2 Move crane

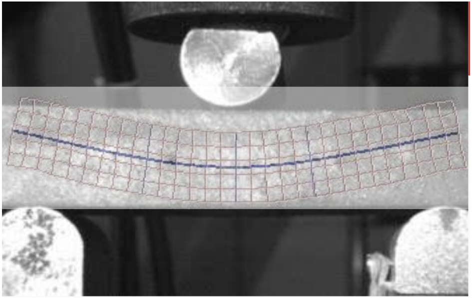


- Small perturbations $\boldsymbol{R}(s) = \boldsymbol{I} + \boldsymbol{\Theta}(s)$, $\boldsymbol{\Theta}(s)^{\mathsf{T}} = -\boldsymbol{\Theta}(s)$:

$$\begin{aligned}\boldsymbol{x}_\Sigma &= \boldsymbol{R}(s)\boldsymbol{p}_\Sigma \\ &= (\boldsymbol{I} + \boldsymbol{\Theta}(s))\boldsymbol{p}_\Sigma\,.\end{aligned}$$

- Small displacement $\boldsymbol{x}_\Sigma \simeq \boldsymbol{p}_\Sigma$:

$$\begin{aligned}\boldsymbol{u} &= \boldsymbol{x} - \boldsymbol{p} \\ &= \boldsymbol{u}_G(s) + \boldsymbol{\theta}(s) \times \boldsymbol{x}_\Sigma\,.\end{aligned}$$



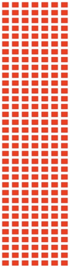
- Cross-sections remain perpendicular to the neutral line:

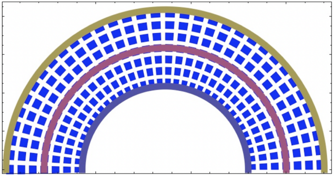
$$\boldsymbol{R}(s)\boldsymbol{e} = \frac{\boldsymbol{x}'_G(s)}{\|\boldsymbol{x}'_G(s)\|}\,.$$

- Small perturbations $\boldsymbol{R}(s)\boldsymbol{e} = (\boldsymbol{I} + \boldsymbol{\Theta}(s))\boldsymbol{e} \simeq \boldsymbol{e} + \boldsymbol{u}'_G(s)$:

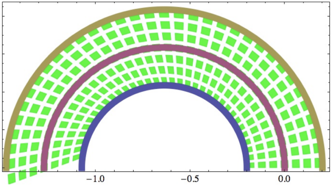
$$\boxed{\boldsymbol{\theta}_\Sigma(s) = \boldsymbol{e} \times \boldsymbol{u}'_{G\Sigma}(s)}$$

Recap: 1.3 Large beam bending





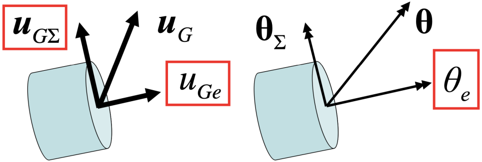
$\alpha = 1$



$\alpha = 1.1$

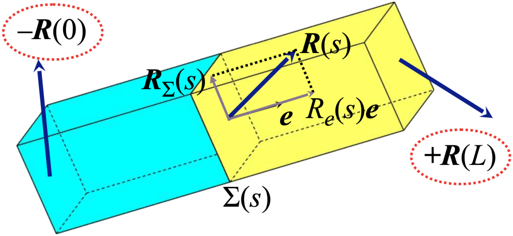
Beam kinematics

Independent unknowns



- Elongation $u_{Ge} = \langle \boldsymbol{u}_G, \boldsymbol{e} \rangle$;
- Deflection $\boldsymbol{u}_{G\Sigma} = \boldsymbol{u}_G - u_{Ge}\boldsymbol{e}$;
- Torsion rotation $\theta_e = \langle \boldsymbol{\theta}, \boldsymbol{e} \rangle$.

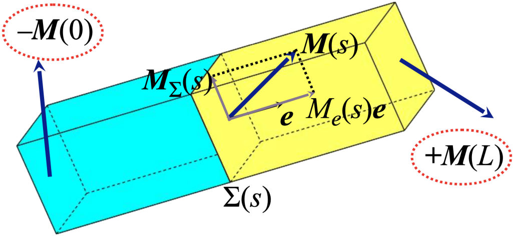
Outline



- Resultant force:

$$\boldsymbol{R}(s) = \int_{\Sigma} \boldsymbol{\sigma}\boldsymbol{e}\,dS\,;$$

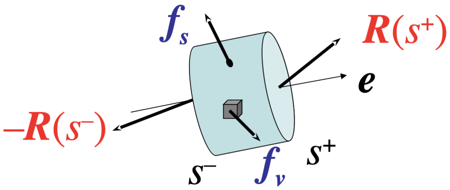
- Normal force $R_e(s) = \langle \boldsymbol{R}(s), \boldsymbol{e} \rangle$;
- Shear force $\boldsymbol{R}_{\Sigma}(s) = \boldsymbol{R}(s) - R_e(s)\boldsymbol{e}$;
- $\boldsymbol{R}(0)$ and $\boldsymbol{R}(L)$ given by the boundary conditions.



- Resultant moment:

$$\boldsymbol{M}(s) = \int_{\Sigma} \boldsymbol{x}_{\Sigma} \times \boldsymbol{\sigma}\boldsymbol{e}\, dS\,;$$

- Torsion moment $M_e(s) = \langle \boldsymbol{M}(s), \boldsymbol{e} \rangle$;
- Bending moment $\boldsymbol{M}_{\Sigma}(s) = \boldsymbol{M}(s) - M_e(s)\boldsymbol{e}$;
- $\boldsymbol{M}(0)$ and $\boldsymbol{M}(L)$ given by the boundary conditions.



- Linear external forces:

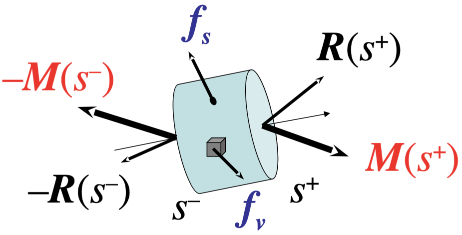
$$\boldsymbol{f}_l(s) = \int_\Sigma \boldsymbol{f}_v dS + \int_{\partial\Sigma} \boldsymbol{f}_s d\zeta\,;$$

- Local equilibrium of the cross-section:

$$\boldsymbol{R}'(s) + \boldsymbol{f}_l(s) = \boldsymbol{0}\,.$$

Beam statics

Local balance of moments



- Linear external torques:

$$\boldsymbol{c}_l(s) = \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{f}_v dS + \int_{\partial\Sigma} \boldsymbol{x}_\Sigma \times \boldsymbol{f}_s d\zeta \,;$$

- Local equilibrium of the cross-section:

$$\boldsymbol{M}'(s) + \boldsymbol{e} \times \boldsymbol{R}(s) + \boldsymbol{c}_l(s) = \boldsymbol{0} \,.$$

Beam statics

Local balance of forces–alternative point of view

- Local static equilibrium of a continuum medium:

$$\mathbf{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_v = \mathbf{0}\,.$$

- Then integrate over the cross-section Σ:

$$\begin{aligned}
\mathbf{0} &= \int_\Sigma \mathbf{Div}\,\boldsymbol{\sigma}\,dS + \int_\Sigma \boldsymbol{f}_v\,dS \\
&= \int_\Sigma \frac{\partial \boldsymbol{\sigma}}{\partial s}\boldsymbol{e}\,dS + \int_\Sigma \mathbf{Div}_\Sigma\,\boldsymbol{\sigma}\,dS + \int_\Sigma \boldsymbol{f}_v\,dS \\
&= \frac{\partial}{\partial s}\left(\int_\Sigma \boldsymbol{\sigma}\boldsymbol{e}\,dS\right) + \int_{\partial\Sigma} \boldsymbol{\sigma}\boldsymbol{n}\,d\zeta + \int_\Sigma \boldsymbol{f}_v\,dS \\
&= \boldsymbol{R}'(s) + \int_{\partial\Sigma} \boldsymbol{f}_s\,d\zeta + \int_\Sigma \boldsymbol{f}_v\,dS \\
&= \boldsymbol{R}'(s) + \boldsymbol{f}_l(s)\,.
\end{aligned}$$

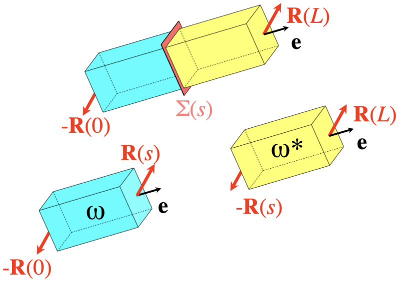
Beam statics

Local balance of moments–alternative point of view

- Then integrate over the cross-section Σ:

$$
\begin{aligned}
\mathbf{0} &= \int_\Sigma \boldsymbol{x}_\Sigma \times \mathbf{Div}\,\boldsymbol{\sigma}\,dS + \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{f}_v\,dS \\
&= \int_\Sigma \boldsymbol{x}_\Sigma \times \frac{\partial \boldsymbol{\sigma}}{\partial s}\boldsymbol{e}\,dS + \int_\Sigma \boldsymbol{x}_\Sigma \times \frac{\partial \boldsymbol{\sigma}\boldsymbol{e}_\alpha}{\partial x_\alpha}\,dS + \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{f}_v\,dS \\
&= \frac{\partial}{\partial s}\left(\int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{\sigma}\boldsymbol{e}\,dS\right) + \int_\Sigma \frac{\partial(\boldsymbol{x}_\Sigma \times \boldsymbol{\sigma}\boldsymbol{e}_\alpha)}{\partial x_\alpha}\,dS \\
&\quad - \int_\Sigma \boldsymbol{e}_\alpha \times \boldsymbol{\sigma}\boldsymbol{e}_\alpha\,dS + \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{f}_v\,dS \\
&= \boldsymbol{M}'(s) + \int_{\partial\Sigma} \boldsymbol{x}_\Sigma \times \boldsymbol{\sigma}\boldsymbol{n}\,d\zeta + \int_\Sigma \boldsymbol{e} \times \boldsymbol{\sigma}\boldsymbol{e}\,dS + \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{f}_v\,dS \\
&= \boldsymbol{M}'(s) + \boldsymbol{e} \times \boldsymbol{R}(s) + \boldsymbol{c}_l(s)\,,
\end{aligned}
$$

since $\boldsymbol{e}_\alpha \times \boldsymbol{\sigma}\boldsymbol{e}_\alpha + \boldsymbol{e} \times \boldsymbol{\sigma}\boldsymbol{e} = \mathbf{0}$ from the symmetry of $\boldsymbol{\sigma}$.

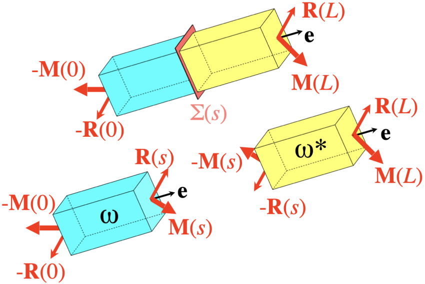


- Global equilibrium of the left section $s \in [0, s]$:

$$\boldsymbol{R}(s) - \boldsymbol{R}(0) + \int_0^s \boldsymbol{f}_l d\zeta = \mathbf{0}\,;$$

- Global equilibrium of the right section $s \in [s, L]$:

$$\boldsymbol{R}(L) - \boldsymbol{R}(s) + \int_s^L \boldsymbol{f}_l d\zeta = \mathbf{0}\,.$$



- Global equilibrium of the left section $s \in [0, s]$:

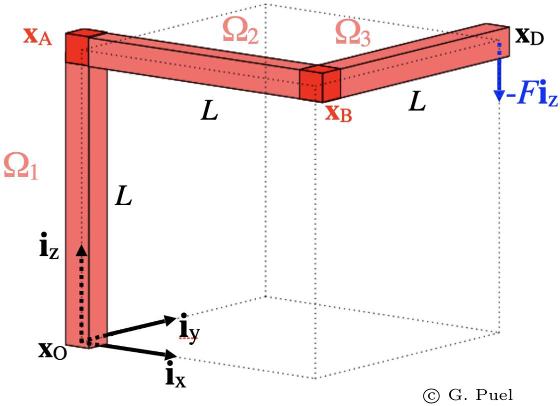
$$\boldsymbol{M}(s) - \boldsymbol{M}(0) + s\boldsymbol{e} \times \boldsymbol{R}(0) + \int_0^s (\boldsymbol{c}_l + (\zeta - s)\boldsymbol{e} \times \boldsymbol{f}_l)\, d\zeta = \boldsymbol{0}\,;$$

- Global equilibrium of the right section $s \in [s, L]$:

$$\boldsymbol{M}(L) + (L - s)\boldsymbol{e} \times \boldsymbol{R}(L) - \boldsymbol{M}(s) + \int_s^L (\boldsymbol{c}_l + (\zeta - s)\boldsymbol{e} \times \boldsymbol{f}_l)\, d\zeta = \boldsymbol{0}\,.$$

Outline

Example: branched beam
Setup



Question: Resultant forces and moments within this beam?

- Local equations of equilibrium:

$$\begin{aligned} \boldsymbol{R}'(s) + \boldsymbol{f}_l(s) &= \boldsymbol{0}\,, \\ \boldsymbol{M}'(s) + \boldsymbol{e}\times\boldsymbol{R}(s) + \boldsymbol{c}_l(s) &= \boldsymbol{0}\,; \end{aligned}$$

- For Ω_3 (beam part #3):

$$\boldsymbol{e} = \boldsymbol{i}_y\,,\quad \boldsymbol{f}_l = \boldsymbol{0}\,,\quad \boldsymbol{c}_l = \boldsymbol{0}\,;$$

- Thus for $s \equiv y \in (0, L]$:

$$\begin{aligned} \boldsymbol{R}'(y) &= \boldsymbol{0}\,, \\ \boldsymbol{M}'(y) + \boldsymbol{i}_y\times\boldsymbol{R}(y) &= \boldsymbol{0}\,; \end{aligned}$$

- Local equations of equilibrium:

$$\begin{aligned} \boldsymbol{R}'(s) + \boldsymbol{f}_l(s) &= \boldsymbol{0}\,, \\ \boldsymbol{M}'(s) + \boldsymbol{e} \times \boldsymbol{R}(s) + \boldsymbol{c}_l(s) &= \boldsymbol{0}\,; \end{aligned}$$

- For Ω_2 (beam part #2):

$$\boldsymbol{e} = \boldsymbol{i}_x\,, \quad \boldsymbol{f}_l = \boldsymbol{0}\,, \quad \boldsymbol{c}_l = \boldsymbol{0}\,;$$

- Thus for $s \equiv x \in (0, L]$:

$$\begin{aligned} \boldsymbol{R}'(x) &= \boldsymbol{0}\,, \\ \boldsymbol{M}'(x) + \boldsymbol{i}_x \times \boldsymbol{R}(x) &= \boldsymbol{0}\,; \end{aligned}$$

- Local equations of equilibrium:

$$
\begin{aligned}
\boldsymbol{R}'(s) + \boldsymbol{f}_l(s) &= \boldsymbol{0}\,, \\
\boldsymbol{M}'(s) + \boldsymbol{e} \times \boldsymbol{R}(s) + \boldsymbol{c}_l(s) &= \boldsymbol{0}\,;
\end{aligned}
$$

- For Ω_1 (beam part #1):

$$
\boldsymbol{e} = \boldsymbol{i}_z\,, \quad \boldsymbol{f}_l = \boldsymbol{0}\,, \quad \boldsymbol{c}_l = \boldsymbol{0}\,;
$$

- Thus for $s \equiv z \in (0, L]$:

$$
\begin{aligned}
\boldsymbol{R}'(z) &= \boldsymbol{0}\,, \\
\boldsymbol{M}'(z) + \boldsymbol{i}_z \times \boldsymbol{R}(z) &= \boldsymbol{0}\,;
\end{aligned}
$$

- Resultant force in Ω_3 for $y \in (0, L]$:

$$\begin{aligned} \boldsymbol{R}'(y) &= \boldsymbol{0}\,, \\ \boldsymbol{R}(y) &= \boldsymbol{R}(\boldsymbol{x}_D) \\ &= -F\boldsymbol{i}_z\,; \end{aligned}$$

- Resultant moment in Ω_3 for $y \in (0, L]$:

$$\begin{aligned} \boldsymbol{M}'(y) &= \boldsymbol{i}_y \times F\boldsymbol{i}_z \\ \boldsymbol{M}(y) &= F(y-L)\boldsymbol{i}_x + \cancel{\boldsymbol{M}(\boldsymbol{x}_D)} \\ &= F(y-L)\boldsymbol{i}_x\,. \end{aligned}$$

Example: branched beam

Local approach in Ω_2

- Resultant force in Ω_2 for $x \in (0, L]$:

$$\begin{aligned} \boldsymbol{R}'(x) &= \mathbf{0}\,, \\ \boldsymbol{R}(x) &= \boldsymbol{R}(\boldsymbol{x}_B^-) \\ &= \boldsymbol{R}(\boldsymbol{x}_B^+) \\ &= -F\boldsymbol{i}_z\,, \end{aligned}$$

since from the equilibrium of $\Sigma(\boldsymbol{x}_B)$:

$$-\boldsymbol{R}(\boldsymbol{x}_B^-) + \boldsymbol{R}(\boldsymbol{x}_B^+) = \mathbf{0}\,;$$

- Resultant moment in Ω_2 for $x \in (0, L]$:

$$\begin{aligned} \boldsymbol{M}'(x) &= \boldsymbol{i}_x \times F\boldsymbol{i}_z \\ \boldsymbol{M}(x) &= -F(x-L)\boldsymbol{i}_y + \boldsymbol{M}(\boldsymbol{x}_B^-) \\ &= F(L-x)\boldsymbol{i}_y + \boldsymbol{M}(\boldsymbol{x}_B^+) \\ &= F(L-x)\boldsymbol{i}_y - FL\boldsymbol{i}_x\,, \end{aligned}$$

since from the equilibrium of $\Sigma(\boldsymbol{x}_B)$:

$$-\boldsymbol{M}(\boldsymbol{x}_B^-) + \boldsymbol{M}(\boldsymbol{x}_B^+) = \boldsymbol{0}\,.$$

- Resultant force in Ω_1 for $z \in (0, L]$:

$$\begin{aligned}\boldsymbol{R}'(z) &= \boldsymbol{0}\,,\\ \boldsymbol{R}(z) &= \boldsymbol{R}(\boldsymbol{x}_A^-)\\ &= \boldsymbol{R}(\boldsymbol{x}_A^+)\\ &= -F\boldsymbol{i}_z\,,\end{aligned}$$

since from the equilibrium of $\Sigma(\boldsymbol{x}_A)$:

$$-\boldsymbol{R}(\boldsymbol{x}_A^-) + \boldsymbol{R}(\boldsymbol{x}_A^+) = \boldsymbol{0}\,;$$

- In particular $\boldsymbol{R}_0 = -\boldsymbol{R}(\boldsymbol{x}_O) = F\boldsymbol{i}_z$: force of the ground on the beam.

- Resultant moment in Ω_1 for $z \in (0, L]$:

$$
\begin{aligned}
\boldsymbol{M}'(z) &= \boldsymbol{i}_z \times F\boldsymbol{i}_z \\
\boldsymbol{M}(z) &= \boldsymbol{M}(\boldsymbol{x}_A^-) \\
&= \boldsymbol{M}(\boldsymbol{x}_A^+) \\
&= FL(\boldsymbol{i}_y - \boldsymbol{i}_x)\,,
\end{aligned}
$$

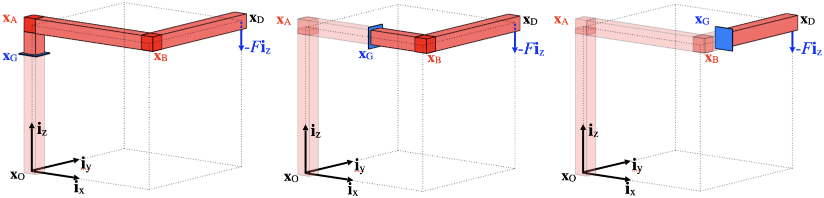
since from the equilibrium of $\Sigma(\boldsymbol{x}_A)$:

$$
-\boldsymbol{M}(\boldsymbol{x}_A^-) + \boldsymbol{M}(\boldsymbol{x}_A^+) = \boldsymbol{0}\,.
$$

- In particular $\boldsymbol{M}_0 = -\boldsymbol{M}(\boldsymbol{x}_O) = FL(\boldsymbol{i}_x - \boldsymbol{i}_y)$: moment of the ground on the beam.

Example: branched beam

Global approach



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- Cutting the beam at any location $\boldsymbol{x}_G = s\boldsymbol{e}(s)$ in Ω_1 or Ω_2 or Ω_3 and considering the equilibrium of the downstream part we have:

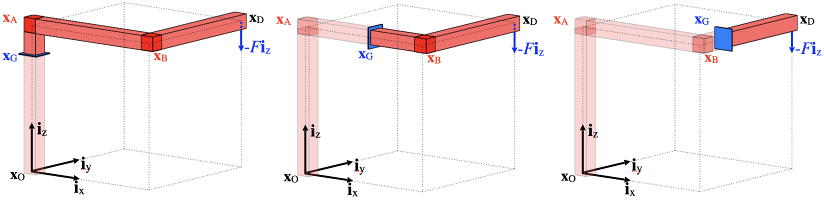
$$-\boldsymbol{R}(\boldsymbol{x}_G) + \boldsymbol{R}(\boldsymbol{x}_D) + \int_{\boldsymbol{x}_G}^{\boldsymbol{x}_D} \cancel{\boldsymbol{f}_l(\zeta)}\, d\zeta = \boldsymbol{0}\,,$$

or since $\boldsymbol{f}_l = \boldsymbol{0}$:

$$\boldsymbol{R}(\boldsymbol{x}_G) = -F\boldsymbol{i}_z\,.$$

Example: branched beam

Global approach



- Cutting the beam at any location $\boldsymbol{x}_G = s\boldsymbol{e}(s)$ in Ω_1 or Ω_2 or Ω_3 and considering the equilibrium of the downstream part we have:

$$-\boldsymbol{M}(\boldsymbol{x}_G)+\cancel{\boldsymbol{M}(\boldsymbol{x}_D)}+(\boldsymbol{x}_D-\boldsymbol{x}_G)\times\boldsymbol{R}(\boldsymbol{x}_D)+\int_{\boldsymbol{x}_G}^{\boldsymbol{x}_D}\cancel{\boldsymbol{c}_l(\zeta)}+(\zeta\boldsymbol{e}(\zeta)-s\boldsymbol{e}(s))\times\cancel{\boldsymbol{f}_l(\zeta)}\,d\zeta=\boldsymbol{0}\,,$$

or since $\boldsymbol{M}(\boldsymbol{x}_D)=\boldsymbol{0}$, $\boldsymbol{c}_l=\boldsymbol{0}$:

$$\boldsymbol{M}(\boldsymbol{x}_G)=(\boldsymbol{x}_D-\boldsymbol{x}_G)\times F\boldsymbol{i}_z\,.$$

Outline

Beam elastic law

Traction vector

- Recap: basic kinematic assumption (Timoshenko)

$$\boldsymbol{u} = \boldsymbol{u}_G(s) + \boldsymbol{\theta}(s) \times \boldsymbol{x}_\Sigma \,;$$

- Linearized strains $\boldsymbol{x}_\Sigma \simeq \boldsymbol{p}_\Sigma$:

$$\boldsymbol{\varepsilon} = (\boldsymbol{u}'_G + \boldsymbol{\theta}' \times \boldsymbol{x}_\Sigma) \otimes_s \boldsymbol{e} - (\boldsymbol{\theta}_\Sigma \times \boldsymbol{e}) \otimes_s \boldsymbol{e} \,;$$

- Linear elastic, isotropic behavior:

$$\begin{aligned} \boldsymbol{\sigma} &= \lambda \operatorname{Tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon} \\ &= \lambda\varepsilon_{ee}\boldsymbol{I} + 2\mu(\varepsilon_{ee}\boldsymbol{e} + \boldsymbol{\gamma}_\Sigma) \otimes_s \boldsymbol{e} + \boldsymbol{\sigma}_\Sigma \,; \end{aligned}$$

- Traction vector:

$$\boldsymbol{\sigma e} = E(\underbrace{u'_{Ge}\boldsymbol{e}}_{//\boldsymbol{e}} + \underbrace{\boldsymbol{\theta}'_\Sigma \times \boldsymbol{x}_\Sigma}_{//\boldsymbol{e}, \propto \boldsymbol{x}_\Sigma}) + \mu(\underbrace{\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e}}_{\perp \boldsymbol{e}} + \underbrace{\theta'_e \boldsymbol{e} \times \boldsymbol{x}_\Sigma}_{\perp \boldsymbol{e}, \propto \boldsymbol{x}_\Sigma}) \,.$$

Beam elastic law

Resultant force with Timoshenko's kinematical assumption

- Recap: resultant force

$$\boldsymbol{R} = \int_{\Sigma} \boldsymbol{\sigma}\boldsymbol{e}\, dS \,;$$

- Assuming $\int_{\Sigma} \boldsymbol{x}_{\Sigma} dS = \boldsymbol{0}$, $S = \int_{\Sigma} dS$:

$$\begin{aligned}\boldsymbol{R} &= \textstyle\int_{\Sigma} E(u'_{Ge}\boldsymbol{e} + \cancel{\boldsymbol{\theta}'_{\Sigma} \times \boldsymbol{x}_{\Sigma}})dS + \int_{\Sigma} \mu(\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_{\Sigma} \times \boldsymbol{e} + \cancel{\theta'_{e}\boldsymbol{e} \times \boldsymbol{x}_{\Sigma}})dS \\ &= ESu'_{Ge}\boldsymbol{e} + \boldsymbol{R}_{\Sigma} \,;\end{aligned}$$

- Shear force with Timoshenko's assumption:

$$\boldsymbol{R}_{\Sigma} = \mu S(\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_{\Sigma} \times \boldsymbol{e}) \,.$$

- Recap: resultant force

$$\boldsymbol{R} = \int_\Sigma \boldsymbol{\sigma}\boldsymbol{e}\, dS\,;$$

- Assuming $\int_\Sigma \boldsymbol{x}_\Sigma dS = \boldsymbol{0}$, $S = \int_\Sigma dS$:

$$\boldsymbol{R} = \int_\Sigma E(u'_{Ge}\boldsymbol{e} + \cancel{\boldsymbol{\theta}'_\Sigma \times \boldsymbol{x}_\Sigma})dS + \int_\Sigma \mu(\underbrace{\cancel{\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e}}}_{=\boldsymbol{0}\text{ by Euler-Bernoulli}} + \cancel{\theta'_e \boldsymbol{e} \times \boldsymbol{x}_\Sigma})dS$$

$$= ESu'_{Ge}\boldsymbol{e} + \boldsymbol{R}_\Sigma\,;$$

- Shear force with Euler-Bernoulli's assumption:

$$\boldsymbol{M}' + \boldsymbol{e} \times \boldsymbol{R} + \boldsymbol{c}_l = \boldsymbol{0}$$

$$\Rightarrow \quad \boldsymbol{R}_\Sigma = \boldsymbol{e} \times (\boldsymbol{M}'_\Sigma + \boldsymbol{c}_l)\,.$$

- Recap: resultant moment

$$\boldsymbol{M} = \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{\sigma}\boldsymbol{e}\, dS\,;$$

- Assuming $\int_\Sigma \boldsymbol{x}_\Sigma dS = \boldsymbol{0}$:

$$\begin{aligned}\boldsymbol{M} &= \textstyle\int_\Sigma E(\cancel{\boldsymbol{x}_\Sigma \times u'_{Ge}}\boldsymbol{e} + \boldsymbol{x}_\Sigma \times (\boldsymbol{\theta}'_\Sigma \times \boldsymbol{x}_\Sigma))dS \\ &\quad + \textstyle\int_\Sigma \mu(\cancel{\boldsymbol{x}_\Sigma \times (\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e})} + \boldsymbol{x}_\Sigma \times (\theta'_e \boldsymbol{e} \times \boldsymbol{x}_\Sigma))dS \\ &= E\mathbb{J}(\boldsymbol{\theta}'_\Sigma) + \mu\mathbb{J}(\theta'_e \boldsymbol{e})\,;\end{aligned}$$

- (Symmetric) inertia tensor:

$$\mathbb{J} = \int_\Sigma (\|\boldsymbol{x}_\Sigma\|^2 \boldsymbol{I} - \boldsymbol{x}_\Sigma \otimes \boldsymbol{x}_\Sigma)\, dS\,;$$

- Bending moment with Timoshenko's assumption:

$$\boldsymbol{M}_\Sigma = E\mathbb{J}(\boldsymbol{\theta}'_\Sigma)\,.$$

Beam elastic law
Resultant moment

- Recap: resultant moment

$$\boldsymbol{M} = \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{\sigma}\boldsymbol{e}\,dS\,;$$

- Assuming $\int_\Sigma \boldsymbol{x}_\Sigma dS = \boldsymbol{0}$:

$$\begin{aligned}\boldsymbol{M} &= \textstyle\int_\Sigma E(\cancel{\boldsymbol{x}_\Sigma\times u'_{Ge}\boldsymbol{e}}+\boldsymbol{x}_\Sigma\times(\boldsymbol{\theta}'_\Sigma\times\boldsymbol{x}_\Sigma))dS \\ &\quad \textstyle+\int_\Sigma \mu(\cancel{\boldsymbol{x}_\Sigma\times(\boldsymbol{u}'_{G\Sigma}-\boldsymbol{\theta}_\Sigma\times\boldsymbol{e})}+\boldsymbol{x}_\Sigma\times(\theta'_e\boldsymbol{e}\times\boldsymbol{x}_\Sigma))dS \\ &= E\mathbb{J}(\boldsymbol{\theta}'_\Sigma)+\mu\mathbb{J}(\theta'_e\boldsymbol{e})\,;\end{aligned}$$

- (Symmetric) inertia tensor:

$$\mathbb{J} = \int_\Sigma (\|\boldsymbol{x}_\Sigma\|^2\,\boldsymbol{I} - \boldsymbol{x}_\Sigma\otimes\boldsymbol{x}_\Sigma)\,dS\,;$$

- Bending moment with Euler-Bernoulli's assumption:

$$\boldsymbol{M}_\Sigma = E\mathbb{J}(\boldsymbol{e}\times\boldsymbol{u}''_{G\Sigma})\,.$$

Summary

Constitutive equations:

$$\begin{aligned}\boldsymbol{R} &= ESu'_{Ge}\boldsymbol{e} + \boldsymbol{R}_\Sigma \\ \boldsymbol{R}_\Sigma &= \boldsymbol{e} \times (\boldsymbol{M}'_\Sigma + \boldsymbol{c}_l)\end{aligned}$$

$$\begin{aligned}\boldsymbol{M}_\Sigma &= E\mathbb{J}(\boldsymbol{e} \times \boldsymbol{u}''_{G\Sigma}) \\ M_e &= \mu J_e \theta'_e\end{aligned}$$

$$\begin{aligned}J_e &= \langle \mathbb{J}\boldsymbol{e}, \boldsymbol{e} \rangle \\ &= \int_\Sigma \|\boldsymbol{x}_\Sigma\|^2 \, dS\end{aligned}$$

Static equilibrium:

$$\boldsymbol{R}' + \boldsymbol{f}_l = \boldsymbol{0}$$

$$\begin{aligned}\boldsymbol{M}''_\Sigma - \boldsymbol{e} \times \boldsymbol{f}_l + \boldsymbol{c}'_{l\Sigma} &= \boldsymbol{0} \\ M'_e + c_{le} &= 0\end{aligned}$$

Differential equations

- Elongation:

$$\boxed{ESu''_{Ge}(s) + \langle \boldsymbol{f}_l(s), \boldsymbol{e} \rangle = 0}$$

with either kinematical $u_{Ge}(0), u_{Ge}(L)$ or mechanical $R_e(0), R_e(L)$ boundary conditions.

- Torsion:

$$\boxed{\mu J_e \theta''_e(s) + \langle \boldsymbol{c}_l(s), \boldsymbol{e} \rangle = 0}$$

with either kinematical $\theta_e(0), \theta_e(L)$ or mechanical $M_e(0), M_e(L)$ boundary conditions.

- Bending:

$$\boxed{E\mathbb{J}(\boldsymbol{e} \times \boldsymbol{u}_{G\Sigma}^{(IV)}(s)) - \boldsymbol{e} \times \boldsymbol{f}_l(s) + \boldsymbol{c}'_{l\Sigma}(s) = \boldsymbol{0}}$$

with either kinematical $\boldsymbol{u}_{G\Sigma}(0), \boldsymbol{u}_{G\Sigma}(L), \boldsymbol{u}'_{G\Sigma}(0), \boldsymbol{u}'_{G\Sigma}(L)$ or mechanical $\boldsymbol{M}_\Sigma(0), \boldsymbol{M}_\Sigma(L), \boldsymbol{R}_\Sigma(0), \boldsymbol{R}_\Sigma(L)$ boundary conditions.

Back to stresses...

- Recap: traction vector

$$\boldsymbol{\sigma e} = E(\underbrace{u'_{Ge}\boldsymbol{e}}_{//\boldsymbol{e}} + \underbrace{\boldsymbol{\theta}'_\Sigma \times \boldsymbol{x}_\Sigma}_{//\boldsymbol{e},\propto\boldsymbol{x}_\Sigma}) + \mu(\underbrace{\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e}}_{\perp\boldsymbol{e}} + \underbrace{\theta'_e\boldsymbol{e} \times \boldsymbol{x}_\Sigma}_{\perp\boldsymbol{e},\propto\boldsymbol{x}_\Sigma})$$

$$= \sigma_{ee}\boldsymbol{e} + \boldsymbol{\tau}_\Sigma\,;$$

- Normal stress:

$$\begin{aligned}\sigma_{ee} &= E(u'_{Ge} + \langle \boldsymbol{\theta}'_\Sigma \times \boldsymbol{x}_\Sigma, \boldsymbol{e}\rangle) \\ &= \frac{R_e}{S} + \langle \boldsymbol{x}_\Sigma, \boldsymbol{e} \times \mathbb{J}^{-1}(\boldsymbol{M}_\Sigma)\rangle\,;\end{aligned}$$

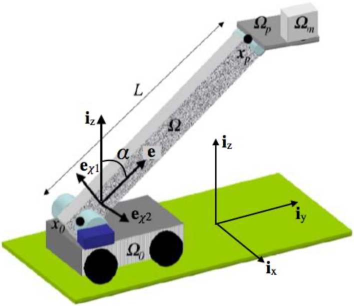
- Shear stress:

$$\begin{aligned}\boldsymbol{\tau}_\Sigma &= \mu(\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e}) + \mu\theta'_e(\boldsymbol{e} \times \boldsymbol{x}_\Sigma) \\ &= \frac{\boldsymbol{R}_\Sigma}{S} + \frac{M_e}{J_e}(\boldsymbol{e} \times \boldsymbol{x}_\Sigma)\,.\end{aligned}$$

Outline

Move crane

Setup



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- Action of the crane's support Ω_0 on Ω: $\boldsymbol{R}_0$, $\boldsymbol{M}_0$;
- Action of the plate Ω_P on on Ω: $\boldsymbol{R}_L$, $\boldsymbol{M}_L$.

Question #1: $\boldsymbol{R}_L$? $\boldsymbol{M}_L$?

- Equilibrium of the plate:

$$
\begin{aligned}
-\boldsymbol{R}_L + m\boldsymbol{g} &= \boldsymbol{0}\,, \\
-\boldsymbol{M}_L + (\boldsymbol{x}_{C_m} - \boldsymbol{x}_P) \times m\boldsymbol{g} &= \boldsymbol{0}\,;
\end{aligned}
$$

- But $\boldsymbol{g} = -g\boldsymbol{i}_z$ hence:

$$
\begin{aligned}
\boldsymbol{R}_L &= -mg\boldsymbol{i}_z\,, \\
\boldsymbol{M}_L &= mg(l_x\boldsymbol{i}_y - l_y\boldsymbol{i}_x)\,.
\end{aligned}
$$

Question #2: Resultant force $\boldsymbol{R}$?

- Local balance of forces on the cross-section Σ at $s \in]0, L[$:

$$\boldsymbol{R}' + \boldsymbol{f}_l = \boldsymbol{0}\,,$$

where $\boldsymbol{f}_l = \boldsymbol{0}$ because "the only actions exerted on the beam Ω are..."

- Therefore:

$$\boldsymbol{R}(s) = \boldsymbol{C}^{\mathrm{st}} = \boldsymbol{R}(L) = \boldsymbol{R}_L = -mg\boldsymbol{i}_z\,.$$

Question #2: Resultant force $\boldsymbol{R}$?

- **Alternative solution**: global balance of forces of the beam $[s, L]$ for $s \in (0, L]$

$$-\boldsymbol{R}(s) + \boldsymbol{R}(L) + \int_s^L \boldsymbol{f}_l \, d\zeta = \boldsymbol{0} \, ;$$

- But $\boldsymbol{f}_l = \boldsymbol{0}$ and $\boldsymbol{R}(L) = \boldsymbol{R}_L$, therefore:

$$\boldsymbol{R}(s) = \boldsymbol{R}(L) = \boldsymbol{R}_L = -mg\boldsymbol{i}_z \, .$$

Question #2: Normal and shear forces?

- Normal force:

$$R_e(s) = \langle \boldsymbol{R}(s), \boldsymbol{e} \rangle = -mg\cos\alpha\,;$$

- Shear force:

$$\boldsymbol{R}_\Sigma(s) = \boldsymbol{R}(s) - R_e(s)\boldsymbol{e} = -mg\sin\alpha\,\boldsymbol{e}_{\chi 1}\,;$$

- The normal force vanishes for $\alpha = \frac{\pi}{2}$ while the shear force vanishes for $\alpha = 0$.

Question #3: Resultant moment $\boldsymbol{M}$?

- Local balance of moments on the cross-section Σ at $s \in]0, L[$:

$$\boldsymbol{M}' + \boldsymbol{e} \times \boldsymbol{R} + \boldsymbol{c}_l = \boldsymbol{0}\,,$$

  where $\boldsymbol{c}_l = \boldsymbol{0}$ because "the only actions exerted on the beam Ω are..."

- Therefore:

$$\begin{aligned} \boldsymbol{M}'(s) &= -\boldsymbol{e} \times \boldsymbol{R}(s) \\ &= \boldsymbol{e} \times mg\boldsymbol{i}_z\,; \end{aligned}$$

Question #3: Resultant moment $\boldsymbol{M}$?

- Local balance of moments on the cross-section Σ at $s \in]0, L[$:

$$\boldsymbol{M}' + \boldsymbol{e} \times \boldsymbol{R} = \boldsymbol{0}\,;$$

- Therefore:

$$\boldsymbol{M}'(s) = \boldsymbol{e} \times mg\boldsymbol{i}_z\,;$$

- Consequently:

$$\begin{aligned}\boldsymbol{M}(s) &= mg(s-L)\boldsymbol{e} \times \boldsymbol{i}_z + \boldsymbol{M}(L) \\ &= mg(s-L)\sin\alpha\,\boldsymbol{i}_x + mg(l_x\boldsymbol{i}_y - l_y\boldsymbol{i}_x) \\ &= mg\{[(s-L)\sin\alpha - l_y]\boldsymbol{i}_x + l_x\boldsymbol{i}_y\}\end{aligned}$$

Question #3: Resultant moment $\boldsymbol{M}$?

- **Alternative solution**: global balance of moments of the beam $[s, L]$ for $s \in (0, L]$

$$-\boldsymbol{M}(s) + \boldsymbol{M}(L) + (L-s)\boldsymbol{e} \times \boldsymbol{R}(L) + \int_s^L (\boldsymbol{c}_l + (\zeta - s) \times \boldsymbol{f}_l)\, d\zeta = \boldsymbol{0}\,;$$

- But $\boldsymbol{f}_l = \boldsymbol{0}$, $\boldsymbol{c}_l = \boldsymbol{0}$, $\boldsymbol{R}(L) = \boldsymbol{R}_L$, and $\boldsymbol{M}(L) = \boldsymbol{M}_L$, therefore:

$$\begin{aligned}\boldsymbol{M}(s) &= mg(l_x \boldsymbol{i}_y - l_y \boldsymbol{i}_x) - (L-s)\boldsymbol{e} \times mg\boldsymbol{i}_z \\ &= mg(l_x \boldsymbol{i}_y - l_y \boldsymbol{i}_x) - mg(L-s)\sin\alpha\, \boldsymbol{i}_x\,.\end{aligned}$$

Question #3: Torsion and bending moments?

- Torsion moment:

$$M_e(s) = \langle \boldsymbol{M}(s), \boldsymbol{e} \rangle = mgl_x \sin\alpha\,;$$

- Bending moment:

$$\begin{aligned} \boldsymbol{M}_\Sigma(s) &= \boldsymbol{M}(s) - M_e(s)\boldsymbol{e}\,; \\ &= mg\{[(s-L)\sin\alpha - l_y]\boldsymbol{i}_x - l_x\cos\alpha\,\boldsymbol{e}_{\chi_1}\} \end{aligned}$$

- The torsion moment vanishes for $\alpha = 0$ while the bending moment vanishes for $\alpha = 0$ and $l_x = l_y = 0$.

Question #4: Longitudinal displacement u_{Ge}?

- Constitutive equation for the longitudinal displacement:

$$
\begin{aligned}
& R_e(s) = ESu'_{Ge}(s) \\
\Rightarrow \quad & u'_{Ge}(s) = -\frac{mg\cos\alpha}{ES}\,;
\end{aligned}
$$

Question #4: Longitudinal displacement u_{Ge}?

- Constitutive equation for the longitudinal displacement:

$$\begin{aligned} R_e(s) &= ESu'_{Ge}(s) \\ \Rightarrow \quad u'_{Ge}(s) &= -\frac{mg\cos\alpha}{ES}\,; \end{aligned}$$

- Consequently:

$$\begin{aligned} u_{Ge}(s) &= -\frac{mg\cos\alpha}{ES}s + u_{Ge}(0) \\ &= -\frac{mg\cos\alpha}{ES}s\,, \end{aligned}$$

since $u_{Ge}(0) = 0$ ("the beam is clamped at point O").

Question #5: Twist angle θ_e?

- Constitutive equation for the twist angle:

$$
\begin{aligned}
M_e(s) &= \mu J_e \theta_e'(s) \\
\Rightarrow \quad \theta_e'(s) &= \frac{mgl_x \sin\alpha}{\mu J}\,;
\end{aligned}
$$

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Kinematics

Statics

Example

Stresses

6.2 Move crane

Question #5: Twist angle θ_e?

- Constitutive equation for the twist angle:

$$M_e(s) = \mu J_e \theta_e'(s)$$
$$\Rightarrow \quad \theta_e'(s) = \frac{mgl_x \sin\alpha}{\mu J}\,;$$

- Consequently:

$$\theta_e(s) = \frac{mgl_x \sin\alpha}{\mu J} s + \theta_e(0) = \frac{mgl_x \sin\alpha}{\mu J} s\,,$$

since $\theta_e(0) = 0$ ("the beam is clamped at point O").

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Question #6: Transverse displacement $\boldsymbol{u}_{G\Sigma}$?

- Constitutive equation for the bending rotation:

$$\begin{aligned} \boldsymbol{M}_\Sigma(s) &= E\mathbb{J}(\boldsymbol{\theta}'_\Sigma(s)) \\ \Rightarrow \quad \boldsymbol{\theta}'_\Sigma(s) &= \frac{mg}{EI}\{[(s-L)\sin\alpha - l_y]\boldsymbol{i}_x - l_x\cos\alpha\,\boldsymbol{e}_{\chi_1}\}\,; \end{aligned}$$

Question #6: Transverse displacement $\boldsymbol{u}_{G\Sigma}$?

- Constitutive equation for the bending rotation:

$$\boldsymbol{M}_\Sigma(s) = E\mathbb{J}(\boldsymbol{\theta}'_\Sigma(s))$$

$$\Rightarrow \quad \boldsymbol{\theta}'_\Sigma(s) = \frac{mg}{EI}\{[(s-L)\sin\alpha - l_y]\boldsymbol{i}_x - l_x\cos\alpha\,\boldsymbol{e}_{\chi 1}\}\,;$$

- Consequently:

$$\begin{aligned}\boldsymbol{\theta}_\Sigma(s) &= \tfrac{mg}{EI}\left\{\left[\left(\tfrac{s}{2}-L\right)\sin\alpha - l_y\right]\boldsymbol{i}_x - l_x\cos\alpha\,\boldsymbol{e}_{\chi 1}\right\}s + \boldsymbol{\theta}_\Sigma(0)\\ &= \frac{mgs}{2EI}\{[(s-2L)\sin\alpha - 2l_y]\,\boldsymbol{i}_x - 2l_x\cos\alpha\,\boldsymbol{e}_{\chi 1}\}\,,\end{aligned}$$

since $\boldsymbol{\theta}_\Sigma(0) = 0$ ("the beam is clamped at point O").

Question #6: Transverse displacement $\boldsymbol{u}_{G\Sigma}$?

- "The beam Ω satisfies the Euler-Bernoulli hypothesis," hence:

$$\begin{aligned}\boldsymbol{u}'_{G\Sigma}(s) &= \boldsymbol{\theta}_{\Sigma}(s) \times \boldsymbol{e} \\ &= \frac{mgs}{EI}\left\{\left[\left(\frac{s}{2} - L\right)\sin\alpha - l_y\right]\boldsymbol{e}_{\chi_1} + l_x \cos\alpha\, \boldsymbol{i}_x\right\}\end{aligned}$$

Question #6: Transverse displacement $\boldsymbol{u}_{G\Sigma}$?

- "The beam Ω satisfies the Euler-Bernoulli hypothesis," hence:

$$\begin{aligned}\boldsymbol{u}'_{G\Sigma}(s) &= \boldsymbol{\theta}_\Sigma(s) \times \boldsymbol{e} \\ &= \frac{mgs}{EI}\left\{\left[\left(\frac{s}{2} - L\right)\sin\alpha - l_y\right]\boldsymbol{e}_{\chi_1} + l_x \cos\alpha\, \boldsymbol{i}_x\right\}\end{aligned}$$

- Consequently:

$$\begin{aligned}\boldsymbol{u}_{G\Sigma}(s) &= \tfrac{mg}{2EI}\left\{\left[\left(\tfrac{s}{3} - L\right)\sin\alpha - l_y\right]\boldsymbol{e}_{\chi_1} + l_x\cos\alpha\,\boldsymbol{i}_x\right\}s^2 + \boldsymbol{u}_{G\Sigma}(0) \\ &= \tfrac{mgs^2}{6EI}\left\{\left[(s-3L)\sin\alpha - 3l_y\right]\boldsymbol{e}_{\chi_1} + 3l_x\cos\alpha\,\boldsymbol{i}_x\right\},\end{aligned}$$

since $\boldsymbol{u}_{G\Sigma}(0) = 0$ ("the beam is clamped at point O").

Question #7: Displacement of a point $M=(s,\chi_1,\chi_2)$?

- From beam's basic kinematical assumption:

$$\boldsymbol{u}(s,\chi_1,\chi_2)=\boldsymbol{u}_G(s)+\boldsymbol{\theta}(s)\times\boldsymbol{x}_\Sigma\,,$$

where:

$$\begin{aligned}\boldsymbol{u}_G(s)&=\boldsymbol{u}_{G\Sigma}(s)+u_{Ge}(s)\boldsymbol{e}\\&=\tfrac{mgs^2}{6EI}\left\{[(s-3L)\sin\alpha-3l_y]\boldsymbol{e}_{\chi_1}+3l_x\cos\alpha\,\boldsymbol{i}_x\right\}-\tfrac{mgs\cos\alpha}{ES}\boldsymbol{e}\,,\\\boldsymbol{\theta}(s)&=\boldsymbol{\theta}_\Sigma(s)+\theta_e(s)\boldsymbol{e}\\&=\tfrac{mgs}{2EI}\left\{[(s-2L)\sin\alpha-2l_y]\boldsymbol{i}_x-2l_x\cos\alpha\,\boldsymbol{e}_{\chi_1}\right\}+\tfrac{mgsl_x\sin\alpha}{\mu J}\boldsymbol{e}\,,\\\boldsymbol{x}_\Sigma&=\chi_1\boldsymbol{e}_{\chi_1}+\chi_2\boldsymbol{i}_x\,;\end{aligned}$$

Question #8: Normal stress σ_{ee}?

- Normal stress:

$$\begin{aligned}\sigma_{ee} &= \frac{R_e}{S} + \left\langle \boldsymbol{x}_\Sigma, \boldsymbol{e} \times \mathbb{J}^{-1}(\boldsymbol{M}_\Sigma) \right\rangle \\ &= -\frac{mg\cos\alpha}{S} - \left\langle \chi_1 \boldsymbol{e}_{\chi_1} + \chi_2 \boldsymbol{i}_x, \frac{mg\{[(s-L)\sin\alpha - l_y]\boldsymbol{e}_{\chi_1} + l_x \cos\alpha\, \boldsymbol{i}_x\}}{I} \right\rangle \\ &= -mg\left\{ \frac{\cos\alpha}{S} + \frac{[(s-L)\sin\alpha - l_y]\chi_1 + l_x \chi_2 \cos\alpha}{I} \right\};\end{aligned}$$

- The maximum normal stress is reached when the bracketed term is minimal (negative) which occurs at $(s, \chi_1, \chi_2) = (0, \chi_{1,\max}, \chi_{2,\min})$.

Question #9: Shear stress $\boldsymbol{\tau}_\Sigma$?

- Shear stress:

$$\boldsymbol{\tau}_\Sigma = \underbrace{\frac{\boldsymbol{R}_\Sigma}{S}}_{\simeq \boldsymbol{0}} + \frac{M_e}{J_e}(\boldsymbol{e} \times \boldsymbol{x}_\Sigma)$$

$$= \frac{mgl_x \sin\alpha}{J} \boldsymbol{e} \times (\chi_1 \boldsymbol{e}_{\chi_1} + \chi_2 \boldsymbol{i}_x)$$

$$= \frac{mgl_x \sin\alpha}{J} (\chi_1 \boldsymbol{i}_x - \chi_2 \boldsymbol{e}_{\chi_1})\,;$$

- The shear stress does not vary along the beam and varies linearly within the cross-section, where it vanishes at the neutral line.

Question #10: Maximum l_x under von Mises criterion?

- Stress tensor: $\boldsymbol{\sigma} = \sigma_{ee}\boldsymbol{e}\otimes\boldsymbol{e} + \boldsymbol{\tau}_\Sigma\otimes\boldsymbol{e} + \boldsymbol{e}\otimes\boldsymbol{\tau}_\Sigma$;
- Deviatoric stress tensor:

$$\boldsymbol{\sigma}^D = \boldsymbol{\sigma} - \frac{\operatorname{Tr}\boldsymbol{\sigma}}{3}\boldsymbol{I}$$
$$= \tfrac{\sigma_{ee}}{3}(2\boldsymbol{e}\otimes\boldsymbol{e}-\boldsymbol{i}_x\otimes\boldsymbol{i}_x-\boldsymbol{e}_{\chi 1}\otimes\boldsymbol{e}_{\chi 1})+\boldsymbol{\tau}_\Sigma\otimes\boldsymbol{e}+\boldsymbol{e}\otimes\boldsymbol{\tau}_\Sigma\,;$$

- Von Mises stress:

$$\sigma_{\mathrm{eq}} = \sqrt{\frac{3}{2}\operatorname{Tr}(\boldsymbol{\sigma}^{D2})} = \sqrt{\sigma_{ee}^2 + 3\left\|\boldsymbol{\tau}_\Sigma\right\|^2}\,;$$

- We can then find l_x such that $\sigma_{\mathrm{eq}} < \sigma_r$.

Question #11: $\boldsymbol{R}_0$, $\boldsymbol{M}_0$?

- Resultant force exerted by the crane's support on Ω:

$$\boldsymbol{R}(0) = -mg\boldsymbol{i}_z\,;$$

  therefore $\boldsymbol{R}_0 = -\boldsymbol{R}(0) = mg\boldsymbol{i}_z$;
- Resultant force exerted by the crane's support on Ω:

$$\boldsymbol{M}(0) = mg(l_x\boldsymbol{i}_y - l_y\boldsymbol{i}_x) - mgL\sin\alpha\,\boldsymbol{i}_x\,;$$

  therefore $\boldsymbol{M}_0 = -\boldsymbol{M}(0) = mg[(l_y + L\sin\alpha)\boldsymbol{i}_x - l_x\boldsymbol{i}_y]$.