Mechanical strength of a rocket booster

1EL5000–Continuum Mechanics – Tutorial Class #9

É. Savin[1,2]
eric.savin@{centralesupelec,onera}.fr

[1]Information Processing and Systems Dept.
ONERA, France

[2]Mechanical and Civil Engineering Dept.
CentraleSupélec, France

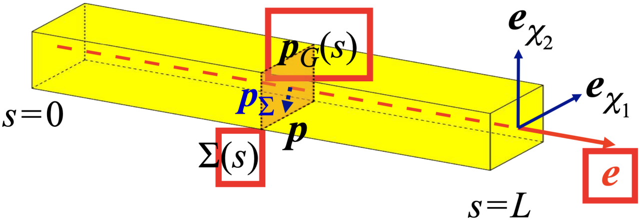
March 15, 2021

Outline

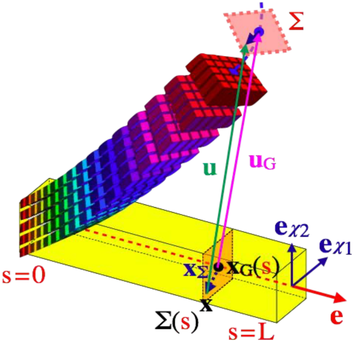
Outline

Beam kinematics

Reference configuration

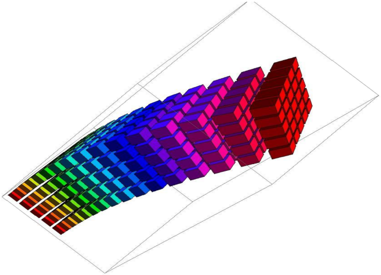


$$
\begin{aligned}
\boldsymbol{p}(s,\chi_1,\chi_2) &= \boldsymbol{p}_G(s) + \boldsymbol{p}_\Sigma(\chi_1,\chi_2) \\
&= s\boldsymbol{e} + \chi_1\boldsymbol{e}_{\chi_1} + \chi_2\boldsymbol{e}_{\chi_2}
\end{aligned}
$$



$$\begin{aligned}\boldsymbol{x} &= \boldsymbol{x}_G(s) + \boldsymbol{x}_\Sigma \\ &= \boldsymbol{x}_G(s) + \boldsymbol{R}(s)\boldsymbol{p}_\Sigma\end{aligned}$$

Beam kinematics
Small perturbations – Timoshenko



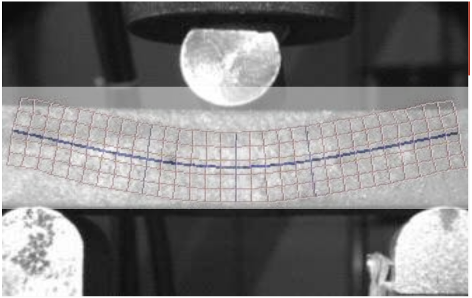
- Small perturbations $\boldsymbol{R}(s) = \boldsymbol{I} + \boldsymbol{\Theta}(s)$, $\boldsymbol{\Theta}(s)^{\mathsf{T}} = -\boldsymbol{\Theta}(s)$:

$$\begin{aligned} \boldsymbol{x}_\Sigma &= \boldsymbol{R}(s)\boldsymbol{p}_\Sigma \\ &= (\boldsymbol{I} + \boldsymbol{\Theta}(s))\boldsymbol{p}_\Sigma\,. \end{aligned}$$

- Small displacement $\boldsymbol{x}_\Sigma \simeq \boldsymbol{p}_\Sigma$:

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{x} - \boldsymbol{p} \\ &= \boldsymbol{u}_G(s) + \boldsymbol{\theta}(s) \times \boldsymbol{x}_\Sigma\,. \end{aligned}$$

Beam kinematics
Small perturbations – Euler-Bernoulli



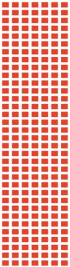
- Cross-sections remain perpendicular to the neutral line:

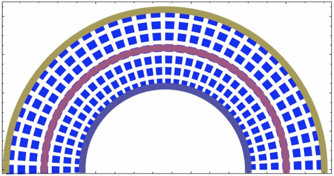
$$\boldsymbol{R}(s)\boldsymbol{e} = \frac{\boldsymbol{x}'_G(s)}{\|\boldsymbol{x}'_G(s)\|}\,.$$

- Small perturbations $\boldsymbol{R}(s)\boldsymbol{e} = (\boldsymbol{I} + \boldsymbol{\Theta}(s))\boldsymbol{e} \simeq \boldsymbol{e} + \boldsymbol{u}'_G(s)$:

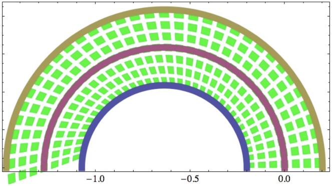
$$\boxed{\boldsymbol{\theta}_\Sigma(s) = \boldsymbol{e} \times \boldsymbol{u}'_{G\Sigma}(s)}$$

Recap: 1.3 Large beam bending





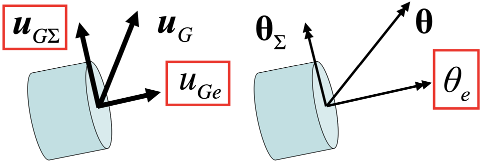
$\alpha = 1$



$\alpha = 1.1$

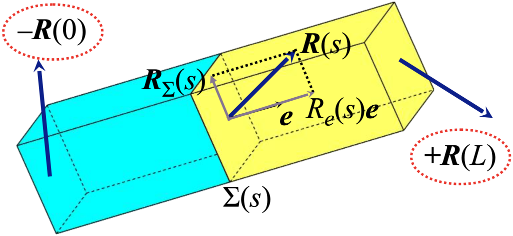
Beam kinematics

Independent unknowns



- Elongation $u_{Ge} = \langle \boldsymbol{u}_G, \boldsymbol{e} \rangle$;
- Deflection $\boldsymbol{u}_{G\Sigma} = \boldsymbol{u}_G - u_{Ge}\boldsymbol{e}$;
- Torsion rotation $\theta_e = \langle \boldsymbol{\theta}, \boldsymbol{e} \rangle$.

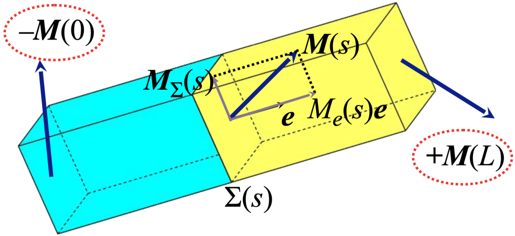
Outline



- Resultant force:

$$\boldsymbol{R}(s) = \int_{\Sigma} \boldsymbol{\sigma}\boldsymbol{e}\,dS\,;$$

- Normal force $R_e(s) = \langle \boldsymbol{R}(s), \boldsymbol{e} \rangle$;
- Shear force $\boldsymbol{R}_{\Sigma}(s) = \boldsymbol{R}(s) - R_e(s)\boldsymbol{e}$;
- $\boldsymbol{R}(0)$ and $\boldsymbol{R}(L)$ given by the boundary conditions.

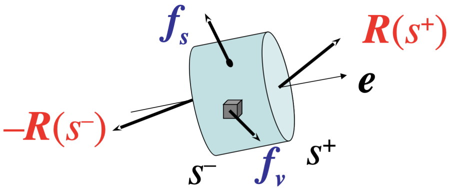


- Resultant moment:

$$\boldsymbol{M}(s) = \int_{\Sigma} \boldsymbol{x}_{\Sigma} \times \boldsymbol{\sigma} \boldsymbol{e} \, dS \, ;$$

- Torsion moment $M_e(s) = \langle \boldsymbol{M}(s), \boldsymbol{e} \rangle$;
- Bending moment $\boldsymbol{M}_{\Sigma}(s) = \boldsymbol{M}(s) - M_e(s)\boldsymbol{e}$;
- $\boldsymbol{M}(0)$ and $\boldsymbol{M}(L)$ given by the boundary conditions.

Beam statics
Local balance of forces

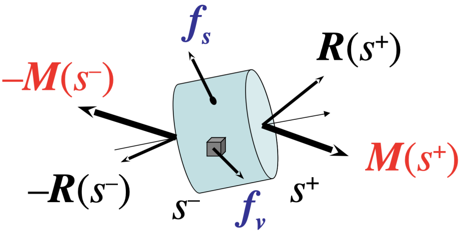


- Linear external forces:

$$\boldsymbol{f}_l(s) = \int_\Sigma \boldsymbol{f}_v dS + \int_{\partial\Sigma} \boldsymbol{f}_s d\zeta\,;$$

- Local equilibrium of the cross-section:

$$\boldsymbol{R}'(s) + \boldsymbol{f}_l(s) = \boldsymbol{0}\,.$$



- Linear external torques:

$$\boldsymbol{c}_l(s) = \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{f}_v dS + \int_{\partial\Sigma} \boldsymbol{x}_\Sigma \times \boldsymbol{f}_s d\zeta \,;$$

- Local equilibrium of the cross-section:

$$\boldsymbol{M}'(s) + \boldsymbol{e} \times \boldsymbol{R}(s) + \boldsymbol{c}_l(s) = \boldsymbol{0} \,.$$

- Local static equilibrium of a continuum medium:

$$\mathbf{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_v = \mathbf{0}\,.$$

- Then integrate over the cross-section Σ:

$$\begin{aligned}
\mathbf{0} &= \int_\Sigma \mathbf{Div}\,\boldsymbol{\sigma}\,dS + \int_\Sigma \boldsymbol{f}_v\,dS \\
&= \int_\Sigma \frac{\partial \boldsymbol{\sigma}}{\partial s}\boldsymbol{e}\,dS + \int_\Sigma \mathbf{Div}_\Sigma\,\boldsymbol{\sigma}\,dS + \int_\Sigma \boldsymbol{f}_v\,dS \\
&= \frac{\partial}{\partial s}\left(\int_\Sigma \boldsymbol{\sigma}\boldsymbol{e}\,dS\right) + \int_{\partial\Sigma} \boldsymbol{\sigma}\boldsymbol{n}\,d\zeta + \int_\Sigma \boldsymbol{f}_v\,dS \\
&= \boldsymbol{R}'(s) + \int_{\partial\Sigma} \boldsymbol{f}_s\,d\zeta + \int_\Sigma \boldsymbol{f}_v\,dS \\
&= \boldsymbol{R}'(s) + \boldsymbol{f}_l(s)\,.
\end{aligned}$$

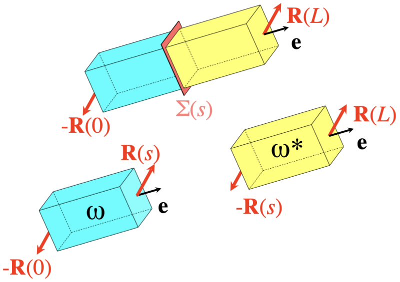
Beam statics

Local balance of moments–alternative point of view

- Then integrate over the cross-section Σ:

$$
\begin{aligned}
\mathbf{0} &= \int_\Sigma \boldsymbol{x}_\Sigma \times \mathbf{Div}\,\boldsymbol{\sigma}\,dS + \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{f}_v\,dS \\
&= \int_\Sigma \boldsymbol{x}_\Sigma \times \frac{\partial \boldsymbol{\sigma}}{\partial s}\boldsymbol{e}\,dS + \int_\Sigma \boldsymbol{x}_\Sigma \times \frac{\partial \boldsymbol{\sigma}\boldsymbol{e}_\alpha}{\partial x_\alpha}\,dS + \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{f}_v\,dS \\
&= \frac{\partial}{\partial s}\left(\int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{\sigma}\boldsymbol{e}\,dS\right) + \int_\Sigma \frac{\partial(\boldsymbol{x}_\Sigma \times \boldsymbol{\sigma}\boldsymbol{e}_\alpha)}{\partial x_\alpha}\,dS \\
&\qquad - \int_\Sigma \boldsymbol{e}_\alpha \times \boldsymbol{\sigma}\boldsymbol{e}_\alpha\,dS + \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{f}_v\,dS \\
&= \boldsymbol{M}'(s) + \int_{\partial\Sigma} \boldsymbol{x}_\Sigma \times \boldsymbol{\sigma}\boldsymbol{n}\,d\zeta + \int_\Sigma \boldsymbol{e} \times \boldsymbol{\sigma}\boldsymbol{e}\,dS + \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{f}_v\,dS \\
&= \boldsymbol{M}'(s) + \boldsymbol{e} \times \boldsymbol{R}(s) + \boldsymbol{c}_l(s)\,,
\end{aligned}
$$

since $\boldsymbol{e}_\alpha \times \boldsymbol{\sigma}\boldsymbol{e}_\alpha + \boldsymbol{e} \times \boldsymbol{\sigma}\boldsymbol{e} = \mathbf{0}$ from the symmetry of $\boldsymbol{\sigma}$.

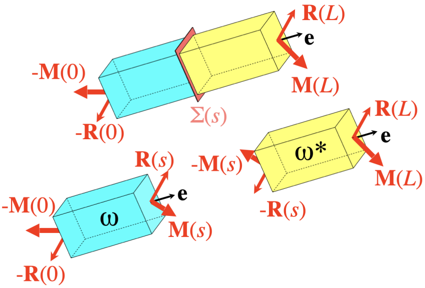


- Global equilibrium of the left section $s \in [0, s]$:

$$\boldsymbol{R}(s) - \boldsymbol{R}(0) + \int_0^s \boldsymbol{f}_l d\zeta = \boldsymbol{0}\,;$$

- Global equilibrium of the right section $s \in [s, L]$:

$$\boldsymbol{R}(L) - \boldsymbol{R}(s) + \int_s^L \boldsymbol{f}_l d\zeta = \boldsymbol{0}\,.$$



- Global equilibrium of the left section $s \in [0, s]$:

$$\boldsymbol{M}(s) - \boldsymbol{M}(0) + s\boldsymbol{e} \times \boldsymbol{R}(0) + \int_0^s (\boldsymbol{c}_l + (\zeta - s)\boldsymbol{e} \times \boldsymbol{f}_l)\, d\zeta = \boldsymbol{0}\,;$$

- Global equilibrium of the right section $s \in [s, L]$:

$$\boldsymbol{M}(L) + (L - s)\boldsymbol{e} \times \boldsymbol{R}(L) - \boldsymbol{M}(s) + \int_s^L (\boldsymbol{c}_l + (\zeta - s)\boldsymbol{e} \times \boldsymbol{f}_l)\, d\zeta = \boldsymbol{0}\,.$$

Outline

Beam elastic law
Traction vector

- Recap: basic kinematic assumption (Timoshenko)

$$\boldsymbol{u} = \boldsymbol{u}_G(s) + \boldsymbol{\theta}(s) \times \boldsymbol{x}_\Sigma \,;$$

- Linearized strains $\boldsymbol{x}_\Sigma \simeq \boldsymbol{p}_\Sigma$:

$$\boldsymbol{\varepsilon} = (\boldsymbol{u}'_G + \boldsymbol{\theta}' \times \boldsymbol{x}_\Sigma) \otimes_s \boldsymbol{e} - (\boldsymbol{\theta}_\Sigma \times \boldsymbol{e}) \otimes_s \boldsymbol{e} \,;$$

- Linear elastic, isotropic behavior:

$$\begin{aligned}\boldsymbol{\sigma} &= \lambda \operatorname{Tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon} \\ &= \lambda\varepsilon_{ee}\boldsymbol{I} + 2\mu(\varepsilon_{ee}\boldsymbol{e} + \boldsymbol{\gamma}_\Sigma) \otimes_s \boldsymbol{e} + \boldsymbol{\sigma}_\Sigma \,;\end{aligned}$$

- Traction vector:

$$\boldsymbol{\sigma}\boldsymbol{e} = E(\underbrace{u'_{Ge}\boldsymbol{e}}_{//\boldsymbol{e}} + \underbrace{\boldsymbol{\theta}'_\Sigma \times \boldsymbol{x}_\Sigma}_{//\boldsymbol{e},\propto\boldsymbol{x}_\Sigma}) + \mu(\underbrace{\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e}}_{\perp\boldsymbol{e}} + \underbrace{\theta'_e\boldsymbol{e} \times \boldsymbol{x}_\Sigma}_{\perp\boldsymbol{e},\propto\boldsymbol{x}_\Sigma}) \,.$$

Beam elastic law

Resultant force with Timoshenko's kinematical assumption

- Recap: resultant force

$$\boldsymbol{R} = \int_\Sigma \boldsymbol{\sigma}\boldsymbol{e}\,dS\,;$$

- Assuming $\int_\Sigma \boldsymbol{x}_\Sigma dS = \boldsymbol{0}$, $S = \int_\Sigma dS$:

$$\begin{aligned}\boldsymbol{R} &= \textstyle\int_\Sigma E(u'_{Ge}\boldsymbol{e} + \cancel{\boldsymbol{\theta}'_\Sigma \times \boldsymbol{x}_\Sigma})dS + \int_\Sigma \mu(\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e} + \cancel{\theta'_e \boldsymbol{e} \times \boldsymbol{x}_\Sigma})dS \\ &= ESu'_{Ge}\boldsymbol{e} + \boldsymbol{R}_\Sigma\,;\end{aligned}$$

- Shear force with Timoshenko's assumption:

$$\boldsymbol{R}_\Sigma = \mu S(\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e})\,.$$

Beam elastic law

Resultant force with Euler-Bernoulli's kinematical assumption

- Recap: resultant force

$$\boldsymbol{R} = \int_\Sigma \boldsymbol{\sigma}\boldsymbol{e}\,dS\,;$$

- Assuming $\int_\Sigma \boldsymbol{x}_\Sigma dS = \boldsymbol{0}$, $S = \int_\Sigma dS$:

$$\boldsymbol{R} = \int_\Sigma E(u'_{Ge}\boldsymbol{e} + \cancel{\boldsymbol{\theta}'_\Sigma \times \boldsymbol{x}_\Sigma})dS + \int_\Sigma \mu(\underbrace{\cancel{\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e}}}_{=\boldsymbol{0}\text{ by Euler-Bernoulli}} + \cancel{\theta'_e\boldsymbol{e} \times \boldsymbol{x}_\Sigma})dS$$

$$= ESu'_{Ge}\boldsymbol{e} + \boldsymbol{R}_\Sigma\,;$$

- Shear force with Euler-Bernoulli's assumption:

$$\begin{aligned} &\boldsymbol{M}' + \boldsymbol{e} \times \boldsymbol{R} + \boldsymbol{c}_l = \boldsymbol{0} \\ \Rightarrow \quad &\boldsymbol{R}_\Sigma = \boldsymbol{e} \times (\boldsymbol{M}'_\Sigma + \boldsymbol{c}_l)\,. \end{aligned}$$

- Recap: resultant moment

$$\boldsymbol{M} = \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{\sigma}\boldsymbol{e}\,dS\,;$$

- Assuming $\int_\Sigma \boldsymbol{x}_\Sigma dS = \boldsymbol{0}$:

$$\begin{aligned}\boldsymbol{M} &= \textstyle\int_\Sigma E(\cancel{\boldsymbol{x}_\Sigma \times u'_{Ge}}\boldsymbol{e} + \boldsymbol{x}_\Sigma \times (\boldsymbol{\theta}'_\Sigma \times \boldsymbol{x}_\Sigma))dS \\ &\quad + \textstyle\int_\Sigma \mu(\cancel{\boldsymbol{x}_\Sigma \times (\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e})} + \boldsymbol{x}_\Sigma \times (\theta'_e \boldsymbol{e} \times \boldsymbol{x}_\Sigma))dS \\ &= E\mathbb{J}(\boldsymbol{\theta}'_\Sigma) + \mu\mathbb{J}(\theta'_e \boldsymbol{e})\,;\end{aligned}$$

- (Symmetric) inertia tensor:

$$\mathbb{J} = \int_\Sigma \left(\|\boldsymbol{x}_\Sigma\|^2 \boldsymbol{I} - \boldsymbol{x}_\Sigma \otimes \boldsymbol{x}_\Sigma\right) dS\,;$$

- Bending moment with Timoshenko's assumption:

$$\boldsymbol{M}_\Sigma = E\mathbb{J}(\boldsymbol{\theta}'_\Sigma)\,.$$

- Recap: resultant moment

$$\boldsymbol{M} = \int_\Sigma \boldsymbol{x}_\Sigma \times \boldsymbol{\sigma}\boldsymbol{e}\, dS\,;$$

- Assuming $\int_\Sigma \boldsymbol{x}_\Sigma dS = \boldsymbol{0}$:

$$\begin{aligned}\boldsymbol{M} &= \textstyle\int_\Sigma E(\cancel{\boldsymbol{x}_\Sigma \times u'_{Ge}}\boldsymbol{e} + \boldsymbol{x}_\Sigma \times (\boldsymbol{\theta}'_\Sigma \times \boldsymbol{x}_\Sigma))dS \\ &\quad + \textstyle\int_\Sigma \mu(\cancel{\boldsymbol{x}_\Sigma \times (\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e})} + \boldsymbol{x}_\Sigma \times (\theta'_e \boldsymbol{e} \times \boldsymbol{x}_\Sigma))dS \\ &= E\mathbb{J}(\boldsymbol{\theta}'_\Sigma) + \mu\mathbb{J}(\theta'_e \boldsymbol{e})\,;\end{aligned}$$

- (Symmetric) inertia tensor:

$$\mathbb{J} = \int_\Sigma (\|\boldsymbol{x}_\Sigma\|^2 \boldsymbol{I} - \boldsymbol{x}_\Sigma \otimes \boldsymbol{x}_\Sigma)\, dS\,;$$

- Bending moment with Euler-Bernoulli's assumption:

$$\boldsymbol{M}_\Sigma = E\mathbb{J}(\boldsymbol{e} \times \boldsymbol{u}''_{G\Sigma})\,.$$

Summary

Constitutive equations:

$$\boldsymbol{R} = ESu'_{Ge}\boldsymbol{e} + \boldsymbol{R}_\Sigma$$
$$\boldsymbol{R}_\Sigma = \boldsymbol{e} \times (\boldsymbol{M}'_\Sigma + \boldsymbol{c}_l)$$

$$\boldsymbol{M}_\Sigma = E\mathbb{J}(\boldsymbol{e} \times \boldsymbol{u}''_{G\Sigma})$$
$$M_e = \mu J_e \theta'_e$$

$$J_e = \langle \mathbb{J}\boldsymbol{e}, \boldsymbol{e} \rangle = \int_\Sigma \|\boldsymbol{x}_\Sigma\|^2 \, dS$$

Static equilibrium:

$$\boldsymbol{R}' + \boldsymbol{f}_l = \boldsymbol{0}$$

$$\boldsymbol{M}''_\Sigma - \boldsymbol{e} \times \boldsymbol{f}_l + \boldsymbol{c}'_{l\Sigma} = \boldsymbol{0}$$
$$M'_e + c_{le} = 0$$

Differential equations

- Elongation:

$$\boxed{ESu''_{Ge}(s)+\langle \boldsymbol{f}_l(s),\boldsymbol{e}\rangle=0}$$

with either kinematical $u_{Ge}(0), u_{Ge}(L)$ or mechanical $R_e(0), R_e(L)$ boundary conditions.

- Torsion:

$$\boxed{\mu J_e\theta''_e(s)+\langle \boldsymbol{c}_l(s),\boldsymbol{e}\rangle=0}$$

with either kinematical $\theta_e(0), \theta_e(L)$ or mechanical $M_e(0), M_e(L)$ boundary conditions.

- Bending:

$$\boxed{E\mathbb{J}(\boldsymbol{e}\times\boldsymbol{u}_{G\Sigma}^{(IV)}(s))-\boldsymbol{e}\times\boldsymbol{f}_l(s)+\boldsymbol{c}'_{l\Sigma}(s)=\boldsymbol{0}}$$

with either kinematical $\boldsymbol{u}_{G\Sigma}(0), \boldsymbol{u}_{G\Sigma}(L), \boldsymbol{u}'_{G\Sigma}(0), \boldsymbol{u}'_{G\Sigma}(L)$ or mechanical $\boldsymbol{M}_\Sigma(0), \boldsymbol{M}_\Sigma(L), \boldsymbol{R}_\Sigma(0), \boldsymbol{R}_\Sigma(L)$ boundary conditions.

Back to stresses...

- Recap: traction vector

$$\boldsymbol{\sigma}\boldsymbol{e} = E(\underbrace{u'_{Ge}\boldsymbol{e}}_{//\boldsymbol{e}} + \underbrace{\boldsymbol{\theta}'_\Sigma \times \boldsymbol{x}_\Sigma}_{//\boldsymbol{e},\propto\boldsymbol{x}_\Sigma}) + \mu(\underbrace{\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e}}_{\perp\boldsymbol{e}} + \underbrace{\theta'_e\boldsymbol{e} \times \boldsymbol{x}_\Sigma}_{\perp\boldsymbol{e},\propto\boldsymbol{x}_\Sigma})$$

$$= \sigma_{ee}\boldsymbol{e} + \boldsymbol{\tau}_\Sigma\,;$$

- Normal stress:

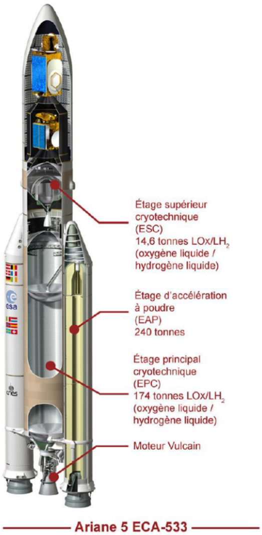
$$\begin{aligned}\sigma_{ee} &= E(u'_{Ge} + \langle\boldsymbol{\theta}'_\Sigma \times \boldsymbol{x}_\Sigma, \boldsymbol{e}\rangle) \\ &= \frac{R_e}{S} + \langle\boldsymbol{x}_\Sigma, \boldsymbol{e} \times \mathbb{J}^{-1}(\boldsymbol{M}_\Sigma)\rangle\,;\end{aligned}$$

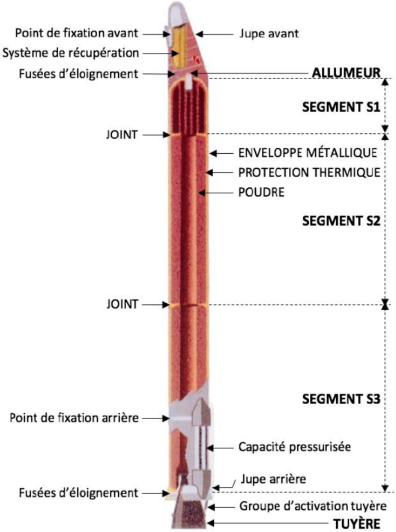
- Shear stress:

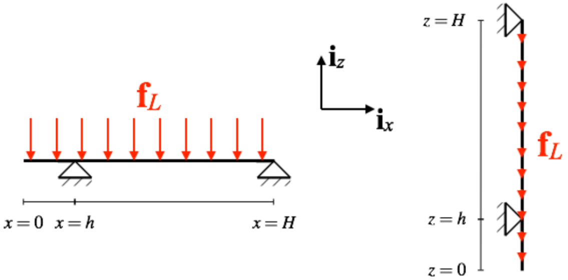
$$\begin{aligned}\boldsymbol{\tau}_\Sigma &= \mu(\boldsymbol{u}'_{G\Sigma} - \boldsymbol{\theta}_\Sigma \times \boldsymbol{e}) + \mu\theta'_e(\boldsymbol{e} \times \boldsymbol{x}_\Sigma) \\ &= \frac{\boldsymbol{R}_\Sigma}{S} + \frac{M_e}{J_e}(\boldsymbol{e} \times \boldsymbol{x}_\Sigma)\,.\end{aligned}$$

Outline

Mechanical strength of a rocket booster
Setup



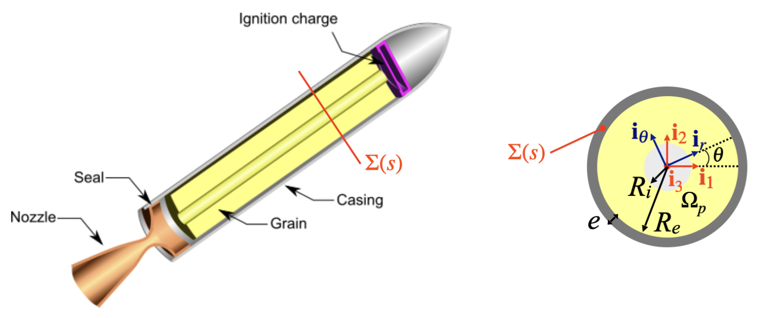




© G. Puel

- Connection at $s = h$ with $\boldsymbol{M}_h^I = \mathbf{0}$;
- Connection at $s = H$ with $\boldsymbol{M}_H^I = \mathbf{0}$;
- Weight of the propellant.

Question #1: Action of the propellant on the beam?



Question #1: Action of the propellant on the beam?

- Linear external force exerted by the propellant:

$$
\begin{aligned}
\boldsymbol{f}_l(s) &= \int_{\Sigma_p} \varrho_p \boldsymbol{g} dS \\
&= -\int_{R_i}^{R_e} \varrho_p g \boldsymbol{i}_z 2\pi r dr \\
&= -\varrho_p g \pi (R_e^2 - R_i^2) \boldsymbol{i}_z \\
&= -\frac{m_p g}{H} \boldsymbol{i}_z \,;
\end{aligned}
$$

Question #1: Action of the propellant on the beam?

- Linear external force exerted by the propellant:

$$\boldsymbol{f}_l(s) = -\frac{m_p g}{H}\boldsymbol{i}_z\,;$$

- Linear external torque exerted by the propellant:

$$\begin{aligned}\boldsymbol{c}_l(s) &= \int_{\Sigma_p} \boldsymbol{x}_\Sigma \times \varrho_p \boldsymbol{g} dS \\ &= -\int_{R_i}^{R_e}\int_0^{2\pi} r\boldsymbol{i}_r(\theta) \times \varrho_p g \boldsymbol{i}_z r dr d\theta \\ &= \varrho_p g \int_{R_i}^{R_e} r^2 dr \cancel{\int_0^{2\pi} \boldsymbol{i}_\theta(\theta) d\theta} \\ &= \boldsymbol{0}\,.\end{aligned}$$

Question #2: Global equilibrium of the horizontal beam?

- Global equilibrium of forces:

$$\boldsymbol{R}_h^I + \boldsymbol{R}_H^I + \int_0^H \boldsymbol{f}_l(s)ds = \boldsymbol{0}$$
$$\Rightarrow \quad R_h^I + R_H^I - m_p g = 0\,,$$

where $m_p = \varrho_p \pi (R_e^2 - R_i^2) H$;

Question #2: Global equilibrium of the horizontal beam?

- Global equilibrium of forces:

$$R_h^I + R_H^I - m_p g = 0\,;$$

- Global equilibrium of moments (at, say, $s = H$):

$$\begin{aligned} \cancel{\boldsymbol{M}_h^I} + (h-H)\boldsymbol{i}_x \times \boldsymbol{R}_h^I + \cancel{\boldsymbol{M}_H^I} \\ + \int_0^H (\cancel{\boldsymbol{c}_l} + (s-H)\boldsymbol{i}_x \times \boldsymbol{f}_l) ds = \boldsymbol{0} \\ R_h^I(h-H)\boldsymbol{i}_y + \frac{\varrho_p g\pi(R_e^2 - R_i^2)H^2}{2}\boldsymbol{i}_y = \boldsymbol{0} \\ \Rightarrow \quad R_h^I(h-H) + \frac{m_p g H}{2} = 0\,; \end{aligned}$$

Question #2: Global equilibrium of the horizontal beam?

- Recap:

$$R_h^I + R_H^I - m_p g = 0\,,$$
$$R_h^I(h-H) + \frac{m_p g H}{2} = 0\,;$$

- Therefore:

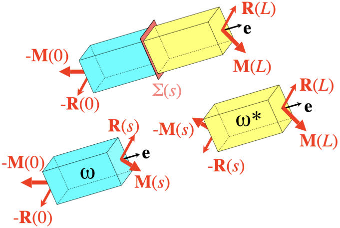
$$R_h^I = \frac{m_p g H}{2(H-h)} = \frac{5}{8} m_p g\,,$$
$$R_H^I = \frac{m_p g (H-2h)}{2(H-h)} = \frac{3}{8} m_p g\,,$$

owing to $H = 5h$.

Question #3: $\boldsymbol{M}(s)$ in $s \in (h, H]$?

- Global balance of moments for the beam $(s, H]$ with $s \in (h, H)$ and where $\boldsymbol{e} = \boldsymbol{i}_x$:

$$-\boldsymbol{M}(s) + \boldsymbol{M}_H^I + (H - s)\boldsymbol{i}_x \times \boldsymbol{R}_H^I + \int_s^H (\boldsymbol{c}_l + (\zeta - s)\boldsymbol{i}_x \times \boldsymbol{f}_l)d\zeta = \boldsymbol{0}\,;$$



Question #3: $\boldsymbol{M}(s)$ in $s \in (h, H]$?

- Global balance of moments for the beam (s, H):

$$
\begin{aligned}
&-\boldsymbol{M}(s) + \cancel{\boldsymbol{M}^I_H} + (H-s)\boldsymbol{i}_x \times \boldsymbol{R}^I_H \\
&\qquad + \int_s^H (\cancel{\boldsymbol{c}_l} + (\zeta - s)\boldsymbol{i}_x \times \boldsymbol{f}_l)d\zeta = \boldsymbol{0}\,;
\end{aligned}
$$

- Thus:

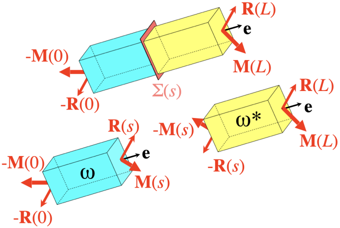
$$
\begin{aligned}
\boldsymbol{M}(s) &= -R^I_H (H-s)\boldsymbol{i}_y + \frac{m_p g}{2H}\left[(\zeta - s)^2\right]_{\zeta=s}^{\zeta=H} \boldsymbol{i}_y \\
&= \frac{m_p g}{8H}(H-s)(H-4s)\boldsymbol{i}_y\,, \quad s \in (h, H]\,;
\end{aligned}
$$

Mechanical strength of a rocket booster
Solution

Question #3: $\boldsymbol{M}(s)$ in $s \in (0,h]$?

- Global balance of moments for the beam $(s,H]$ with $s \in (0,h)$ and where $\boldsymbol{e} = \boldsymbol{i}_x$:

$$-\boldsymbol{M}(s) + \boldsymbol{M}_H^I + (H-s)\boldsymbol{i}_x \times \boldsymbol{R}_H^I + \boldsymbol{M}_h^I + (h-s)\boldsymbol{i}_x \times \boldsymbol{R}_h^I + \int_s^H (\boldsymbol{c}_l + (\zeta - s)\boldsymbol{i}_x \times \boldsymbol{f}_l)d\zeta = \boldsymbol{0}\,;$$



Question #3: $\boldsymbol{M}(s)$ in $s \in (0,h]$?

- Global balance of moments for the beam $(s,H]$:

$$-\boldsymbol{M}(s)+\cancel{\boldsymbol{M}_H^I}+(H-s)\boldsymbol{i}_x\times\boldsymbol{R}_H^I+\cancel{\boldsymbol{M}_h^I}+(h-s)\boldsymbol{i}_x\times\boldsymbol{R}_h^I + \int_s^H(\cancel{\boldsymbol{c}_l}+(\zeta-s)\boldsymbol{i}_x\times\boldsymbol{f}_l)d\zeta=\boldsymbol{0}\,;$$

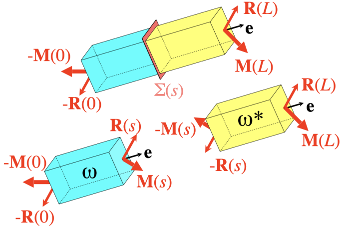
- Thus:

$$\begin{aligned}\boldsymbol{M}(s) &= -R_H^I(H-s)\boldsymbol{i}_y - R_h^I(h-s)\boldsymbol{i}_y + \frac{m_pg}{2H}\left[(\zeta-s)^2\right]_{\zeta=s}^{\zeta=H}\boldsymbol{i}_y \\ &= m_pg\left[\frac{3}{8}(s-H)+\frac{5}{8}(s-h)+\frac{(H-s)^2}{2H}\right]\boldsymbol{i}_y \\ &= \frac{m_pg}{2H}s^2\boldsymbol{i}_y\,,\quad s\in(0,h]\,;\end{aligned}$$

Question #3: $\boldsymbol{M}(s)$ in $s \in [0, h)$?

- **Alternatively**: global balance of moments for the beam $[0, s)$ with $s \in (0, h)$ and where $\boldsymbol{e} = \boldsymbol{i}_x$:

$$-\boldsymbol{M}(0) + s\boldsymbol{i}_x \times \boldsymbol{R}(0) + \boldsymbol{M}(s) + \int_0^s (\boldsymbol{c}_l + (\zeta - s)\boldsymbol{i}_x \times \boldsymbol{f}_l)d\zeta = \boldsymbol{0}\,;$$



Question #3: $\boldsymbol{M}(s)$ in $s \in [0, h)$?

- **Alternatively**: global balance of moments for the beam $[0, s)$,

$$-\cancel{\boldsymbol{M}(0)}+s\boldsymbol{i}_x\times\cancel{\boldsymbol{R}(0)}+\boldsymbol{M}(s)+\int_0^s(\cancel{\boldsymbol{c}_l}+(\zeta-s)\boldsymbol{i}_x\times\boldsymbol{f}_l)d\zeta=\boldsymbol{0}\,;$$

- Thus:

$$\begin{aligned}\boldsymbol{M}(s) &= -\frac{m_p g}{2H}\left[(\zeta-s)^2\right]_{\zeta=0}^{\zeta=s}\boldsymbol{i}_y \\ &= \frac{m_p g}{2H}s^2\boldsymbol{i}_y\,,\quad s\in[0,h)\,;\end{aligned}$$

Question #3: $\boldsymbol{M}(s)$ in $s \in [0, H]$?

- Recap:

$$\begin{aligned}\boldsymbol{M}(s) &= \frac{m_p g}{2H} s^2 \boldsymbol{i}_y \,, \quad s \in [0, h] \,, \\ &= \frac{m_p g}{8H}(H - s)(H - 4s)\boldsymbol{i}_y \,, \quad s \in [h, H] \,;\end{aligned}$$

- The bending moment is continuous at $s = h$ as expected, since $\boldsymbol{M}_h^I = \boldsymbol{0}$, with:

$$\boldsymbol{M}(s) = \frac{m_p g h}{10} \boldsymbol{i}_y \,.$$

Question #3: $\boldsymbol{M}(s)$ in $s \in [0, H]$?

- **Alternative solution**: local equilibrium of the beam $s \in (0, h) \cup (h, H)$:

$$\begin{aligned} \boldsymbol{R}'(s) + \boldsymbol{f}_l(s) &= \boldsymbol{0}\,, \\ \boldsymbol{M}'(s) + \boldsymbol{e} \times \boldsymbol{R}(s) + \boldsymbol{c}_l(s) &= \boldsymbol{0}\,; \end{aligned}$$

Question #3: $\boldsymbol{M}(s)$ in $s \in [0, H]$?

- Local equilibrium of the beam $s \in (0, h) \cup (h, H)$:

$$\boldsymbol{R}'(s) - \frac{m_p g}{H}\boldsymbol{i}_z = \boldsymbol{0}\,,$$
$$\boldsymbol{M}'(s) + \boldsymbol{i}_x \times \boldsymbol{R}(s) + \cancel{\boldsymbol{c}_l(s)} = \boldsymbol{0}\,;$$

- Then for $s \in (h, H)$:

$$\begin{aligned}\boldsymbol{R}(s) &= \frac{m_p g}{H}(s-H)\boldsymbol{i}_z + \boldsymbol{R}_H^I \\ &= m_p g\left[\frac{s}{H} - \frac{H}{2(H-h)}\right]\boldsymbol{i}_z \\ &= \frac{m_p g}{8H}(5s - 8H)\boldsymbol{i}_z\end{aligned}$$

Question #3: $\boldsymbol{M}(s)$ in $s \in [0, H]$?

- Local equilibrium of the beam $s \in (0, h) \cup (h, H)$:

$$\boldsymbol{R}'(s) - \frac{m_p g}{H}\boldsymbol{i}_z = \boldsymbol{0}\,,$$
$$\boldsymbol{M}'(s) + \boldsymbol{i}_x \times \boldsymbol{R}(s) + \cancel{\boldsymbol{c}_l(s)} = \boldsymbol{0}\,;$$

- Then for $s \in (h, H)$:

$$\boldsymbol{M}'(s) = -m_p g\left[\frac{s}{H} - \frac{H}{2(H-h)}\right]\boldsymbol{i}_x \times \boldsymbol{i}_z$$
$$\boldsymbol{M}(s) = m_p g\left[\frac{s^2 - H^2}{2H} - \frac{H(s-H)}{2(H-h)}\right]\boldsymbol{i}_y + \cancel{\boldsymbol{M}_H^I}$$
$$= \frac{m_p g}{2H}\left(s - \frac{hH}{H-h}\right)(s-H)\boldsymbol{i}_y\,.$$

Question #3: $\boldsymbol{M}(s)$ in $s \in [0,H]$?

- Local equilibrium of the beam $s \in (0,h) \cup (h,H)$:

$$\begin{aligned} \boldsymbol{R}'(s) - \frac{m_p g}{H}\boldsymbol{i}_z &= \boldsymbol{0}\,, \\ \boldsymbol{M}'(s) + \boldsymbol{i}_x \times \boldsymbol{R}(s) + \cancel{\boldsymbol{c}_l(s)} &= \boldsymbol{0}\,; \end{aligned}$$

- Then for $s \in (0,h)$:

$$\begin{aligned} \boldsymbol{R}(s) &= m_p g \frac{s}{H}\boldsymbol{i}_z + \cancel{\boldsymbol{R}(0)} \\ &= m_p g \frac{s}{H}\boldsymbol{i}_z\,; \end{aligned}$$

Question #3: $\boldsymbol{M}(s)$ in $s \in [0, H]$?

- Local equilibrium of the beam $s \in (0, h) \cup (h, H)$:

$$\boldsymbol{R}'(s) - \frac{m_p g}{H}\boldsymbol{i}_z = \boldsymbol{0}\,,$$
$$\boldsymbol{M}'(s) + \boldsymbol{i}_x \times \boldsymbol{R}(s) + \cancel{\boldsymbol{c}_l(s)} = \boldsymbol{0}\,;$$

- Then for $s \in (0, h)$:

$$\begin{aligned}\boldsymbol{M}'(s) &= -m_p g\frac{s}{H}\boldsymbol{i}_x \times \boldsymbol{i}_z \\ \boldsymbol{M}(s) &= \frac{m_p g}{2H}s^2\boldsymbol{i}_y + \cancel{\boldsymbol{M}(0)} \\ &= \frac{m_p g}{2H}s^2\boldsymbol{i}_y\,.\end{aligned}$$

Question #3: $\boldsymbol{M}(s)$ in $s \in [0, H]$?

- Recap:

for $s \in [0, h)$

$$\boldsymbol{R}(s) = m_p g \frac{s}{H} \boldsymbol{i}_z$$
$$\boldsymbol{M}(s) = \frac{m_p g}{2H} s^2 \boldsymbol{i}_y$$

for $s \in (h, H]$

$$\boldsymbol{R}(s) = m_p g \left[\frac{s}{H} - \frac{H}{2(H-h)} \right] \boldsymbol{i}_z$$
$$\boldsymbol{M}(s) = \frac{m_p g}{2H} \left(s - \frac{hH}{H-h} \right) (s-H) \boldsymbol{i}_y$$

- The resultant force is discontinuous at $s = h$ while the resultant moment is continuous:

$$\boldsymbol{R}(h^-) - \boldsymbol{R}(h^+) = \frac{m_p g H}{2(H-h)} \boldsymbol{i}_z = \boldsymbol{R}_h^I ,$$
$$\boldsymbol{M}(h^-) - \boldsymbol{M}(h^+) = \boldsymbol{0} = \boldsymbol{M}_h^I .$$

Question #4: σ_{ee}?

- Normal stress:

$$\sigma_{ee} = \frac{\langle \boldsymbol{R}, \boldsymbol{e} \rangle}{S} + \left\langle \boldsymbol{x}_\Sigma, \boldsymbol{e} \times \mathbb{J}^{-1}(\boldsymbol{M}_\Sigma) \right\rangle \,;$$

Question #4: σ_{ee}?

- Normal stress:

$$\sigma_{ee} = \frac{\langle \boldsymbol{R}, \boldsymbol{e} \rangle}{S} + \left\langle \boldsymbol{x}_\Sigma, \boldsymbol{e} \times \mathbb{J}^{-1}(\boldsymbol{M}_\Sigma) \right\rangle ;$$

- In the present case $\langle \boldsymbol{R}, \boldsymbol{e} \rangle = 0$ and $\boldsymbol{M}_\Sigma = \boldsymbol{M} = M\boldsymbol{i}_y$;

Question #4: σ_{ee}?

- Normal stress:

$$\sigma_{ee} = \frac{\langle \boldsymbol{R}, \boldsymbol{e} \rangle}{S} + \left\langle \boldsymbol{x}_\Sigma, \boldsymbol{e} \times \mathbb{J}^{-1}(\boldsymbol{M}_\Sigma) \right\rangle ;$$

- In the present case $\langle \boldsymbol{R}, \boldsymbol{e} \rangle = 0$ and $\boldsymbol{M}_\Sigma = \boldsymbol{M} = M\boldsymbol{i}_y$;
- But $\mathbb{J} = I(\boldsymbol{i}_x \otimes \boldsymbol{i}_x + \boldsymbol{I})$ therefore:

$$\begin{aligned} \sigma_{ee}(\boldsymbol{x}) &= \tfrac{M(s)}{I} \left\langle \boldsymbol{x}_\Sigma, \boldsymbol{i}_x \times \left(\boldsymbol{I} - \tfrac{1}{2} \boldsymbol{i}_x \otimes \boldsymbol{i}_x\right) \boldsymbol{i}_y \right\rangle \\ &= \tfrac{M(s) z}{I} \\ &= \begin{cases} \tfrac{m_p g}{2HI} s^2 z & \text{if } 0 \leqslant s < h\,, \\ \tfrac{m_p g}{2HI} \left(s - \tfrac{hH}{H-h}\right)(s-H) z & \text{if } h \leqslant s \leqslant H\,; \end{cases} \end{aligned}$$

Mechanical strength of a rocket booster
Solution

Question #4: σ_{ee}?

- $m_p = \varrho_p \pi (R_e^2 - R_i^2) H$, $I = \frac{\pi}{4}[(R+e)^4 - R^4]$, and:

$$\sigma_{ee}(\boldsymbol{x}) = \frac{m_p g z}{2HI} \times \begin{cases} s^2 & \text{if } 0 \leqslant s < h\,, \\ \left(s - \frac{hH}{H-h}\right)(s-H) & \text{if } h \leqslant s \leqslant H\,; \end{cases}$$

- It is maximum for $s = \frac{H^2}{2(H-h)}$, $z = -(R+e)$, and:

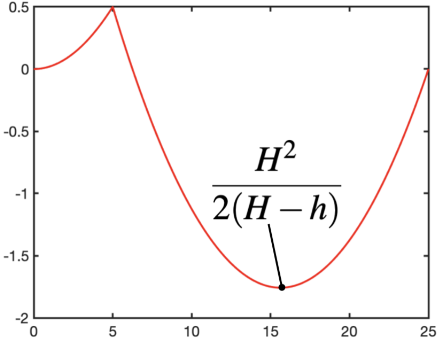
$$\sigma_{ee,\max} = \frac{m_p g}{8I} \left(\frac{H-2h}{H-h} \right)^2 H(R+e)\,;$$

- N.A.: $R = R_e = 1.5\text{m}$, $R_i = 0.5\text{m}$, $e = 0.01\text{m}$, $\varrho_p = 1500\text{kg/m}^3$, $H = 5h = 25m$,

$$\sigma_{ee,\max} = 57.3\,\text{MPa} < \sigma_e = 100\,\text{MPa}\,.$$

Mechanical strength of a rocket booster

Solution



$$s \longmapsto \frac{M(s)}{m_p g}$$

Question #5: Global equilibrium of the vertical beam?

- Global equilibrium of forces:

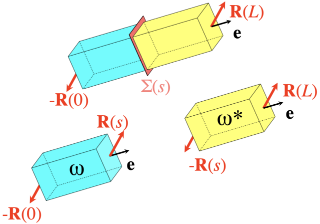
$$\boldsymbol{R}_h^{II} + \boldsymbol{R}_H^{II} + \int_0^H \boldsymbol{f}_l(s)ds = \boldsymbol{0}$$

$$\Rightarrow \quad R_h^{II} + R_H^{II} - m_p g = 0\,;$$

Question #5: $\boldsymbol{R}(s)$ in $s \in (h, H]$?

- Global balance of forces for the beam $(s, H]$ with $s = z \in (h, H)$ and where $\boldsymbol{e} = \boldsymbol{i}_z$:

$$-\boldsymbol{R}(s) + \boldsymbol{R}_H^{II} + \int_s^H \boldsymbol{f}_l(\zeta) d\zeta = \boldsymbol{0} \, ;$$



Question #5: $\boldsymbol{R}(s)$ in $s \in (h, H]$?

- Global balance of forces for the beam $(s, H]$:

$$-\boldsymbol{R}(s) + \boldsymbol{R}_H^{II} + \int_s^H \boldsymbol{f}_l(\zeta) d\zeta = \boldsymbol{0} \,;$$

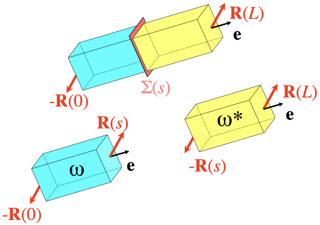
- Thus:

$$\begin{aligned} \boldsymbol{R}(s) &= \boldsymbol{R}_H^{II} - m_p g \left(1 - \frac{s}{H}\right) \boldsymbol{i}_z \\ &= -\boldsymbol{R}_h^{II} + m_p g \frac{s}{H} \boldsymbol{i}_z \,; \end{aligned}$$

Question #5: $\boldsymbol{R}(s)$ in $s \in [0, h)$?

- Global balance of forces for the beam $[0, s)$ with $s = z \in (0, h)$ and where $\boldsymbol{e} = \boldsymbol{i}_z$:

$$-\boldsymbol{R}(0) + \boldsymbol{R}(s) + \int_0^s \boldsymbol{f}_l(\zeta) d\zeta = \boldsymbol{0}\,;$$



Question #5: $\boldsymbol{R}(s)$ in $s \in [0, h)$?

- Global balance of forces for the beam $[0, s)$:

$$-\cancel{\boldsymbol{R}(0)} + \boldsymbol{R}(s) + \int_0^s \boldsymbol{f}_l(\zeta) d\zeta = \boldsymbol{0}\,;$$

- Thus:

$$\boldsymbol{R}(s) = m_p g \frac{s}{H} \boldsymbol{i}_z\,.$$

Question #5: $\boldsymbol{R}(s)$ in $s \in [0, H]$?

- Recap:

$$\boldsymbol{R}(s) = \begin{cases} m_p g \dfrac{s}{H} \boldsymbol{i}_z & \text{if } 0 \leqslant s < h\,, \\ -\boldsymbol{R}_h^{II} + m_p g \dfrac{s}{H} \boldsymbol{i}_z & \text{if } h < s \leqslant H\,; \end{cases}$$

- The resultant force is discontinuous at $s = h$ with:

$$\boldsymbol{R}(h^-) - \boldsymbol{R}(h^+) = \boldsymbol{R}_h^{II}\,,$$

as expected.

Question #6: $u_{Gz}(z)$?

- Constitutive equation for the longitudinal displacement:

$$\langle \boldsymbol{R}, \boldsymbol{e} \rangle = ESu'_{Ge}(s)\,,$$

where $S = \pi[(R+e)^2 - R^2]$;

Question #6: $u_{Gz}(z)$?

- Constitutive equation for the longitudinal displacement:

$$\langle \boldsymbol{R}, \boldsymbol{e} \rangle = ESu'_{Ge}(s)\,;$$

- Therefore for $s = z \in [0, h)$:

$$\begin{aligned} u'_{Gz}(s) &= \frac{m_p g}{ES}\frac{s}{H} \\ u_{Gz}(s) &= \frac{m_p g}{ES}\frac{(s^2 - h^2)}{2H} + \cancel{u_{Gz}(h)} \\ &= \frac{m_p g}{ES}\frac{(s^2 - h^2)}{2H}\,; \end{aligned}$$

Question #6: $u_{Gz}(z)$?

- Constitutive equation for the longitudinal displacement:

$$\langle \boldsymbol{R}, \boldsymbol{e} \rangle = ESu'_{Ge}(s)\,;$$

- Therefore for $s = z \in (h, H]$:

$$\begin{aligned} u'_{Gz}(s) &= \frac{\langle \boldsymbol{R}, \boldsymbol{e} \rangle}{ES} \\ u_{Gz}(s) &= \frac{1}{ES}\left[-R_h^{II}(s-H) + \frac{m_p g}{2H}(s^2 - H^2)\right] + \cancel{u_{Gz}(H)} \\ &= \frac{s-H}{ES}\left[-R_h^{II} + \frac{m_p g}{2}\left(1 + \frac{s}{H}\right)\right]\,; \end{aligned}$$

Question #6: $u_{Gz}(z)$?

- Constitutive equation for the longitudinal displacement:

$$\langle \boldsymbol{R}, \boldsymbol{e} \rangle = ESu'_{Ge}(s)\,;$$

- Therefore for $s = z \in (h, H]$:

$$u_{Gz}(s) = \frac{s-H}{ES}\left[-R_h^{II} + \frac{m_p g}{2}\left(1 + \frac{s}{H}\right)\right]\,;$$

- Since $u_{Gz}(h) = 0$ one has $R_h^{II} = \frac{m_p g}{2}(1 + \frac{h}{H})$ and therefore:

$$u_{Gz}(s) = \frac{m_p g}{2EHS}(s-h)(s-H)\,.$$

Question #7: R_h^{II} and R_H^{II}?

- From Q#6 and Q#5 we have:

$$\begin{aligned} R_h^{II} &= \frac{m_p g}{2}\left(1+\frac{h}{H}\right) \\ R_H^{II} &= -R_h^{II} + m_p g \\ &= \frac{m_p g}{2}\left(1-\frac{h}{H}\right) ; \end{aligned}$$

Question #7: $\boldsymbol{R}(s)$?

- From Q#6 and Q#5 we have:

$$R_h^{II} = \frac{m_p g}{2}\left(1 + \frac{h}{H}\right) ,$$

$$R_H^{II} = \frac{m_p g}{2}\left(1 - \frac{h}{H}\right) ;$$

- Therefore:

$$\boldsymbol{R}(s) = \begin{cases} m_p g \dfrac{s}{H} \boldsymbol{i}_z & \text{if } 0 \leqslant s < h \,, \\ \dfrac{m_p g}{2}\left(\dfrac{2s-h}{H} - 1\right) \boldsymbol{i}_z & \text{if } h < s \leqslant H \,; \end{cases}$$

Question #7: σ_{ee}?

- Normal stress:

$$\sigma_{ee} = \frac{\langle \boldsymbol{R}, \boldsymbol{e} \rangle}{S} + \left\langle \boldsymbol{x}_\Sigma, \boldsymbol{e} \times \mathbb{J}^{-1}(\boldsymbol{M}_\Sigma) \right\rangle ;$$

- In the present case $\boldsymbol{M}_\Sigma = \boldsymbol{0}$ and therefore:

$$\sigma_{ee}(s) = \frac{m_p g}{S} \times \begin{cases} \dfrac{s}{H} & \text{if } 0 \leqslant s < h \,, \\ \dfrac{2s - h - H}{2H} & \text{if } h < s \leqslant H \,; \end{cases}$$

- It is maximum for $s = H$ for which:

$$\sigma_{ee,\max} = \frac{m_p g}{2S} \left(1 - \frac{h}{H} \right) = 9.8\,\text{MPa} < \sigma_e \,.$$