Stiffness of an elastic linkage

1EL5000–Continuum Mechanics – Tutorial Class #5

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March 18, 2021

Outline

Outline

- Scalar product:

$$\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^a\,,\quad\langle\boldsymbol{a},\boldsymbol{b}\rangle=\sum_{j=1}^{a}a_jb_j=a_jb_j\,,$$

The last equality is Einstein's summation convention.

- Tensors and tensor product (or outer product):

$$\boldsymbol{A}\in\mathbb{R}^a\to\mathbb{R}^b\,,\quad\boldsymbol{A}=\boldsymbol{a}\otimes\boldsymbol{b}\,,\quad\boldsymbol{a}\in\mathbb{R}^a\,,\boldsymbol{b}\in\mathbb{R}^b\,.$$

- Tensor application to vectors:

$$\boldsymbol{A}=\boldsymbol{a}\otimes\boldsymbol{b}\in\mathbb{R}^a\to\mathbb{R}^b\,,\boldsymbol{c}\in\mathbb{R}^b\,,\quad\boldsymbol{A}\boldsymbol{c}=\langle\boldsymbol{b},\boldsymbol{c}\rangle\boldsymbol{a}\,.$$

- Product of tensors ≡ composition of linear maps:

$$\boldsymbol{A}=\boldsymbol{a}\otimes\boldsymbol{b}\,,\boldsymbol{B}=\boldsymbol{c}\otimes\boldsymbol{d}\,,\quad\boldsymbol{A}\boldsymbol{B}=\langle\boldsymbol{b},\boldsymbol{c}\rangle\boldsymbol{a}\otimes\boldsymbol{d}\,.$$

Some algebra
Vector & tensor products

- Scalar product of tensors:

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \mathrm{Tr}(\boldsymbol{A}\boldsymbol{B}^{\mathsf{T}}) := \boldsymbol{A} : \boldsymbol{B} = A_{jk}B_{jk} \,.$$

- Let $\{\boldsymbol{e}_j\}_{j=1}^{d}$ be a Cartesian basis in $\mathbb{R}^d$. Then:

$$\begin{aligned} a_j &= \langle \boldsymbol{a}, \boldsymbol{e}_j \rangle \,, \\ A_{jk} &= \langle \boldsymbol{A}, \boldsymbol{e}_j \otimes \boldsymbol{e}_k \rangle = \boldsymbol{A} : \boldsymbol{e}_j \otimes \boldsymbol{e}_k \\ &= \langle \boldsymbol{A}\boldsymbol{e}_k, \boldsymbol{e}_j \rangle \,, \end{aligned}$$

such that:

$$\begin{aligned} \boldsymbol{a} &= a_j \boldsymbol{e}_j \,, \\ \boldsymbol{A} &= A_{jk} \boldsymbol{e}_j \otimes \boldsymbol{e}_k \,. \end{aligned}$$

- Example: the identity matrix

$$\boldsymbol{I} = \boldsymbol{e}_j \otimes \boldsymbol{e}_j \,.$$

- Gradient of a vector function $\boldsymbol{a}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$:

$$\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a} = \frac{\partial \boldsymbol{a}}{\partial x_j} \otimes \boldsymbol{e}_j \,.$$

- Divergence of a vector function $\boldsymbol{a}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$:

$$\mathrm{div}_{\boldsymbol{x}}\boldsymbol{a} = \langle \boldsymbol{\nabla}_{\boldsymbol{x}}, \boldsymbol{a} \rangle = \mathrm{Tr}(\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a}) = \frac{\partial a_j}{\partial x_j} \,.$$

- Divergence of a tensor function $\boldsymbol{A}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$:

$$\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{A} = \frac{\partial (\boldsymbol{A}\boldsymbol{e}_j)}{\partial x_j} \,.$$

- Gradient of a vector function $\boldsymbol{a}(r,\theta,z)$:

$$\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a} = \frac{\partial \boldsymbol{a}}{\partial r} \otimes \boldsymbol{e}_r + \frac{\partial \boldsymbol{a}}{\partial \theta} \otimes \frac{\boldsymbol{e}_\theta}{r} + \frac{\partial \boldsymbol{a}}{\partial z} \otimes \boldsymbol{e}_z \,.$$

- Divergence of a vector function $\boldsymbol{a}(r,\theta,z)$:

$$\mathrm{div}_{\boldsymbol{x}}\boldsymbol{a} = \left\langle \frac{\partial \boldsymbol{a}}{\partial r}, \boldsymbol{e}_r \right\rangle + \left\langle \frac{\partial \boldsymbol{a}}{\partial \theta}, \frac{\boldsymbol{e}_\theta}{r} \right\rangle + \left\langle \frac{\partial \boldsymbol{a}}{\partial z}, \boldsymbol{e}_z \right\rangle \,.$$

- Divergence of a tensor function $\boldsymbol{A}(r,\theta,z)$:

$$\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{A} = \frac{\partial \boldsymbol{A}}{\partial r}\boldsymbol{e}_r + \frac{\partial \boldsymbol{A}}{\partial \theta}\frac{\boldsymbol{e}_\theta}{r} + \frac{\partial \boldsymbol{A}}{\partial z}\boldsymbol{e}_z \,.$$

Outline

Recap

- Local equilibrium equation:

$$\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma} + \boldsymbol{f}_v = \varrho\ddot{\boldsymbol{u}}\,;$$

- Small strains assumption:

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbb{D}_{\boldsymbol{x}}\boldsymbol{u} + \mathbb{D}_{\boldsymbol{x}}\boldsymbol{u}^{\mathsf{T}})\,;$$

- Material constitutive equation:

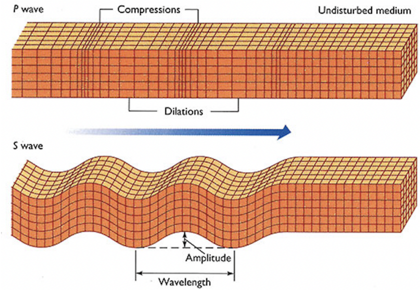
$$\boldsymbol{\sigma} = \mathsf{C}\boldsymbol{\varepsilon}$$

($\boldsymbol{\sigma} = \lambda\,\mathrm{Tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}$ for isotropic elasticity).

- $\boldsymbol{\sigma}_{\boldsymbol{x}}(\boldsymbol{u}) = \mathbf{C}(\boldsymbol{\varepsilon}_{\boldsymbol{x}}(\boldsymbol{u}))$, $\boldsymbol{\varepsilon}_{\boldsymbol{x}}(\boldsymbol{u}) = \boldsymbol{\nabla}_{\boldsymbol{x}} \otimes_s \boldsymbol{u}$, and:

$$\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma}_{\boldsymbol{x}}(\boldsymbol{u}) + \boldsymbol{f}_v = \varrho\partial_t^2\boldsymbol{u}$$
$$(\lambda+\mu)\boldsymbol{\nabla}_{\boldsymbol{x}}(\mathrm{div}_{\boldsymbol{x}}\boldsymbol{u}) + \mu\boldsymbol{\Delta}_{\boldsymbol{x}}\boldsymbol{u} + \boldsymbol{f} = \varrho\partial_t^2\boldsymbol{u}$$
$$(\lambda+\mu)\sum_{k=1}^{3}\frac{\partial^2 u_k}{\partial x_j\partial x_k} + \mu\sum_{k=1}^{3}\frac{\partial^2 u_j}{\partial x_k^2} + f_{vj} = \varrho\frac{\partial^2 u_j}{\partial t^2}$$

- Body waves (Poisson 1828)



Navier equation

Elastic waves

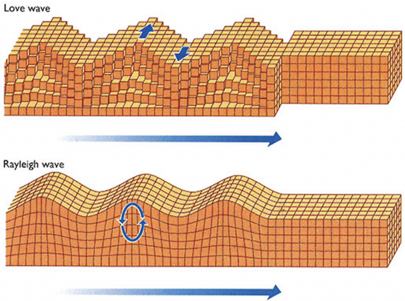
- $\boldsymbol{\sigma}_{\boldsymbol{x}}(\boldsymbol{u}) = \mathbf{C}(\boldsymbol{\varepsilon}_{\boldsymbol{x}}(\boldsymbol{u}))$, $\boldsymbol{\varepsilon}_{\boldsymbol{x}}(\boldsymbol{u}) = \boldsymbol{\nabla}_{\boldsymbol{x}} \otimes_s \boldsymbol{u}$, and:

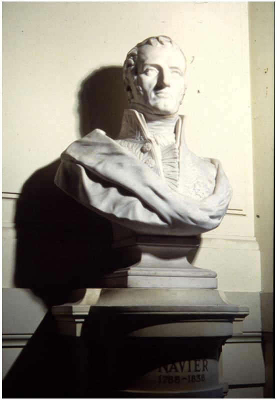
$$\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma}_{\boldsymbol{x}}(\boldsymbol{u}) + \boldsymbol{f}_v = \varrho\partial_t^2\boldsymbol{u}$$

$$(\lambda+\mu)\boldsymbol{\nabla}_{\boldsymbol{x}}(\mathrm{div}_{\boldsymbol{x}}\boldsymbol{u}) + \mu\boldsymbol{\Delta}_{\boldsymbol{x}}\boldsymbol{u} + \boldsymbol{f} = \varrho\partial_t^2\boldsymbol{u}$$

$$(\lambda+\mu)\sum_{k=1}^3 \frac{\partial^2 u_k}{\partial x_j \partial x_k} + \mu\sum_{k=1}^3 \frac{\partial^2 u_j}{\partial x_k^2} + f_{vj} = \varrho\frac{\partial^2 u_j}{\partial t^2}$$

- Surface waves (Rayleigh 1885, Love 1911, Stoneley 1924)





Claude Louis Marie Henri Navier [1785-1836]

- X 1802
- Ingénieur Ponts-et-Chaussées
- Académie des Sciences 1824
- Prof. Analyse et Mécanique à l'X 1831-1836
- *Le Curé de village* (1841)

https://mathshistory.st-andrews.ac.uk/Biographies/Navier/

- Initial displacement:

$$\boldsymbol{u}(\boldsymbol{x},0)=\boldsymbol{u}_0(\boldsymbol{x})\,,\quad\forall\boldsymbol{x}\in\Omega\,;$$

- Initial velocity:

$$\frac{\partial\boldsymbol{u}}{\partial t}(\boldsymbol{x},0)=\boldsymbol{v}_0(\boldsymbol{x})\,,\quad\forall\boldsymbol{x}\in\Omega\,.$$

Navier equation
Boundary conditions

- Rigid contact:

$$\boldsymbol{u}(\boldsymbol{x},t)=\boldsymbol{u}_S(\boldsymbol{x},t)\,,\quad\forall\boldsymbol{x}\in\partial\Omega_u\,,\forall t\,;$$

- Soft contact:

$$\boldsymbol{\sigma}(\boldsymbol{x},t)\boldsymbol{n}(\boldsymbol{x})=\boldsymbol{f}_S(\boldsymbol{x},t)\,,\quad\forall\boldsymbol{x}\in\partial\Omega_\sigma\,,\forall t\,;$$

- Internal full contact with *e.g.* $\boldsymbol{n}=\boldsymbol{n}_1=-\boldsymbol{n}_2$:

$$\begin{aligned}\boldsymbol{u}_1(\boldsymbol{x},t)&=\boldsymbol{u}_2(\boldsymbol{x},t)\,,\\ \boldsymbol{\sigma}_1(\boldsymbol{x},t)\boldsymbol{n}(\boldsymbol{x})&=\boldsymbol{\sigma}_2(\boldsymbol{x},t)\boldsymbol{n}(\boldsymbol{x})\,,\end{aligned}$$

  $\forall\boldsymbol{x}\in\partial\Omega_1\cap\partial\Omega_2$, $\forall t$;
- Internal sliding contact without friction:

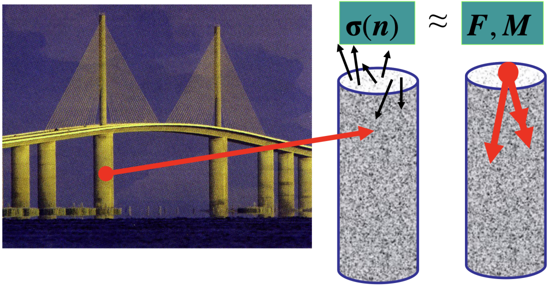
$$\begin{aligned}\langle\boldsymbol{u}_1(\boldsymbol{x},t),\boldsymbol{n}(\boldsymbol{x})\rangle&=\langle\boldsymbol{u}_2(\boldsymbol{x},t),\boldsymbol{n}(\boldsymbol{x})\rangle\,,\\ \langle\boldsymbol{\sigma}_1(\boldsymbol{x},t)\boldsymbol{n}(\boldsymbol{x}),\boldsymbol{\tau}\rangle&=\langle\boldsymbol{\sigma}_2(\boldsymbol{x},t)\boldsymbol{n}(\boldsymbol{x}),\boldsymbol{\tau}\rangle=\boldsymbol{0}\,,\end{aligned}$$

  $\forall\boldsymbol{\tau}\perp\boldsymbol{n}(\boldsymbol{x})$, $\forall\boldsymbol{x}\in\partial\Omega_1\cap\partial\Omega_2$, $\forall t$.

Navier equation
Properties

- Existence of a solution:
    - Holds in statics provided that global equilibrium is satisfied;
    - Holds in dynamics.
- Uniqueness of a solution:
    - Holds in statics for strains, stresses, and displacements up to a rigid-body motion;
    - Holds in dynamics for strains, stresses, and displacements.
- Linearity (superposition principle, symmetries...).

Saint-Venant's principle
...in homogeneous, linear elastic media



"...the difference between the effects of two different but statically equivalent loads becomes very small at sufficiently large distances from load."[a]

Adhémar Barré de Saint-Venant [1797-1886]

[a] A. J. C. B. Saint-Venant, Mémoire sur la Torsion des Prismes, *Mem. Divers Savants* **14**, 233-560 (1855).

Adhémar Jean Claude Barré de Saint-Venant [1797-1886]

- X 1813
- Ingénieur Ponts-et-Chaussées
- Académie des Sciences 1868

https://mathshistory.st-andrews.ac.uk/Biographies/Saint-Venant/

- Thermal strains in the isotropic case with temperature gradient ΔT:

$$\boldsymbol{\varepsilon}_{\mathrm{th}} = \alpha \Delta T \boldsymbol{I}$$

where α is the coefficient of linear thermal expansion.

- Total strain tensor:

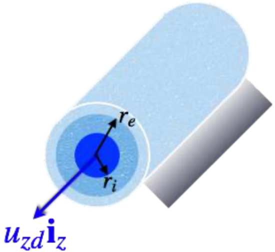
$$\begin{aligned} \boldsymbol{\varepsilon} &= \boldsymbol{\varepsilon}_{\mathrm{elas}} + \boldsymbol{\varepsilon}_{\mathrm{th}} \\ &= \mathbf{S}\boldsymbol{\sigma} + \alpha \Delta T \boldsymbol{I} \,. \end{aligned}$$

- Linear thermoelastic constitutive equation:

$$\boxed{\boldsymbol{\sigma} = \mathbf{C}(\boldsymbol{\varepsilon} - \alpha \Delta T \boldsymbol{I})} \,.$$

Outline

Stiffness of an elastic linkage
Setup



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$$\boldsymbol{u}_{fd}(\boldsymbol{x}) = u_z(r)\boldsymbol{i}_z \,,$$

where:

$$\boldsymbol{x} \in \Omega = \{\boldsymbol{x} = (r, \theta, z);\, r_i < r < r_e,\, 0 \leqslant \theta \leqslant 2\pi,\, 0 < z < L\}\,.$$

Stiffness of an elastic linkage
Solution

Question #0: Why $\boldsymbol{u}_{fd}(\boldsymbol{x}) = u_z(r)\boldsymbol{i}_z$?

- $\boldsymbol{u}(\boldsymbol{x}) = u_r(r,\theta,z)\boldsymbol{i}_r + u_\theta(r,\theta,z)\boldsymbol{i}_\theta + u_z(r,\theta,z)\boldsymbol{i}_z$ but the problem is axisymmetric, hence $u_\theta = 0$ and u_r, u_z do not depend on θ;
- The problem is also invariant vs. z hence u_r, u_z do not depend on z either;
- Lastly the problem is radially constrained such that $u_r = 0$.

Question #1: Equations satisfied by $\boldsymbol{u}$ and $\boldsymbol{\sigma}$?

1. Local equilibrium equation $\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma} + \boldsymbol{f}_v = \varrho\frac{\partial^2\boldsymbol{u}}{\partial t^2}$, where "the effects of inertia and the action of gravity can be neglected:"
$$\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma} = \mathbf{0}\,,\quad \boldsymbol{x}\in\Omega\,.$$

Stiffness of an elastic linkage
Solution

Question #1: Equations satisfied by $\boldsymbol{u}$ and $\boldsymbol{\sigma}$?

1. Local equilibrium equation $\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma} = \mathbf{0}$;
2. Material constitutive equation:

$$\boldsymbol{\sigma} = \lambda \operatorname{Tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}\,,$$

$$\text{or } \boldsymbol{\varepsilon} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}\operatorname{Tr}(\boldsymbol{\sigma})\boldsymbol{I}\,,$$

since "the constitutive material is isotropic homogeneous, and linear elastic, of Lamé parameters (λ, μ)."

Question #1: Equations satisfied by $\boldsymbol{u}$ and $\boldsymbol{\sigma}$?

1. Local equilibrium equation $\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma} = \mathbf{0}$;
2. Material constitutive equation $\boldsymbol{\sigma} = \lambda\,\mathrm{Tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}$;
3. Initial conditions: here they are not needed since the problem is time-independent (static).

Stiffness of an elastic linkage
Solution

Question #1: Equations satisfied by $\boldsymbol{u}$ and $\boldsymbol{\sigma}$?

1. Local equilibrium equation $\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma} = \mathbf{0}$;
2. Material constitutive equation $\boldsymbol{\sigma} = \lambda\,\mathrm{Tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}$;
3. Initial conditions (if needed);
4. Boundary conditions on $\partial\Omega$, where:

$$\partial\Omega = \{r = r_e\} \cup \{r = r_i\} \cup \{z = 0\} \cup \{z = L\}\,.$$

Stiffness of an elastic linkage
Solution

Question #1: Equations satisfied by $\boldsymbol{u}$ and $\boldsymbol{\sigma}$?

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$$\partial\Omega = \{r = r_e\} \cup \{r = r_i\} \cup \{z = 0\} \cup \{z = L\}\,;$$

- $\{r = r_e\}$: $\boldsymbol{u} = \mathbf{0}$ since the "outer lateral surface is glued on a cylindrical support of the same radius, assumed fixed and perfectly rigid;"

Stiffness of an elastic linkage
Solution

Question #1: Equations satisfied by $\boldsymbol{u}$ and $\boldsymbol{\sigma}$?

1. Local equilibrium equation $\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma} = \mathbf{0}$;
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3. Initial conditions (if needed);
4. Boundary conditions on $\partial\Omega$, where:

$$\partial\Omega = \{r = r_e\} \cup \{r = r_i\} \cup \{z = 0\} \cup \{z = L\}\,;$$

- $\{r = r_e\}$: $\boldsymbol{u} = \mathbf{0}$;
- $\{r = r_i\}$: $\boldsymbol{u} = u_{zd}\boldsymbol{i}_z$ since the "inner lateral surface is glued to a perfectly rigid cylinder of the same radius, whose overall movement is constrained as $\boldsymbol{u}_d = u_{zd}\boldsymbol{i}_z$;"

Stiffness of an elastic linkage
Solution

Question #1: Equations satisfied by $\boldsymbol{u}$ and $\boldsymbol{\sigma}$?

1. Local equilibrium equation $\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma} = \mathbf{0}$;
2. Material constitutive equation $\boldsymbol{\sigma} = \lambda\,\mathrm{Tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}$;
3. Initial conditions (if needed);
4. Boundary conditions on $\partial\Omega$, where:

$$\partial\Omega = \{r = r_e\} \cup \{r = r_i\} \cup \{z = 0\} \cup \{z = L\}\,;$$

- $\{r = r_e\}$: $\boldsymbol{u} = \mathbf{0}$;
- $\{r = r_i\}$: $\boldsymbol{u} = u_{zd}\boldsymbol{i}_z$;
- $\{z = 0\}$: $\boldsymbol{\sigma}\boldsymbol{n} = \mathbf{0}$, $\boldsymbol{n} = -\boldsymbol{i}_z$, since "the end faces are assumed to be free of forces;"

Question #1: Equations satisfied by $\boldsymbol{u}$ and $\boldsymbol{\sigma}$?

1. Local equilibrium equation $\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma} = \mathbf{0}$;
2. Material constitutive equation $\boldsymbol{\sigma} = \lambda\,\mathrm{Tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}$;
3. Initial conditions (if needed);
4. Boundary conditions on $\partial\Omega$, where:

$$\partial\Omega = \{r = r_e\} \cup \{r = r_i\} \cup \{z = 0\} \cup \{z = L\}\,;$$

- $\{r = r_e\}$: $\boldsymbol{u} = \mathbf{0}$;
- $\{r = r_i\}$: $\boldsymbol{u} = u_{zd}\boldsymbol{i}_z$;
- $\{z = 0\}$: $\boldsymbol{\sigma}\boldsymbol{i}_z = \mathbf{0}$ $(\boldsymbol{n} = -\boldsymbol{i}_z)$;
- $\{z = L\}$: $\boldsymbol{\sigma}\boldsymbol{i}_z = \mathbf{0}$ $(\boldsymbol{n} = +\boldsymbol{i}_z)$.

Stiffness of an elastic linkage
Solution

Question #2: $\boldsymbol{\varepsilon}_{fd}$?

- In cylindrical coordinates:

$$\boldsymbol{\varepsilon} = \frac{\partial \boldsymbol{u}}{\partial r} \otimes_s \boldsymbol{i}_r + \frac{\partial \boldsymbol{u}}{\partial \theta} \otimes_s \frac{\boldsymbol{i}_\theta}{r} + \frac{\partial \boldsymbol{u}}{\partial z} \otimes_s \boldsymbol{i}_z \,.$$

Question #2: $\boldsymbol{\varepsilon}_{fd}$?

- In cylindrical coordinates:

$$\boldsymbol{\varepsilon} = \frac{\partial \boldsymbol{u}}{\partial r} \otimes_s \boldsymbol{i}_r + \frac{\partial \boldsymbol{u}}{\partial \theta} \otimes_s \frac{\boldsymbol{i}_\theta}{r} + \frac{\partial \boldsymbol{u}}{\partial z} \otimes_s \boldsymbol{i}_z \,.$$

- Here $\boldsymbol{u} = u_z(r)\boldsymbol{i}_z$ thus:

$$\frac{\partial \boldsymbol{u}}{\partial r} = u_z'(r)\boldsymbol{i}_z \,,$$

where $u_z'(r) = \frac{du_z}{dr}$.

Question #2: $\boldsymbol{\varepsilon}_{fd}$?

- In cylindrical coordinates:

$$\boldsymbol{\varepsilon} = \frac{\partial \boldsymbol{u}}{\partial r} \otimes_s \boldsymbol{i}_r + \frac{\partial \boldsymbol{u}}{\partial \theta} \otimes_s \frac{\boldsymbol{i}_\theta}{r} + \frac{\partial \boldsymbol{u}}{\partial z} \otimes_s \boldsymbol{i}_z \,.$$

- Here $\boldsymbol{u} = u_z(r)\boldsymbol{i}_z$ thus:

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial r} &= u_z'(r)\boldsymbol{i}_z \,; \\ \frac{\partial \boldsymbol{u}}{\partial \theta} &= \frac{\partial \boldsymbol{u}}{\partial z} = \boldsymbol{0} \,. \end{aligned}$$

- Therefore:

$$\boldsymbol{\varepsilon}_{fd} = u_z'(r)\boldsymbol{i}_r(\theta) \otimes_s \boldsymbol{i}_z \,.$$

Question #2: $\boldsymbol{\sigma}_{fd}$?

- $\boldsymbol{\varepsilon}_{fd} = u_z'(r)\boldsymbol{i}_r(\theta) \otimes_s \boldsymbol{i}_z$;
- Constitutive relation $\boldsymbol{\sigma}_{fd} = \lambda(\mathrm{Tr}\,\boldsymbol{\varepsilon}_{fd})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}_{fd}$; therefore $\boldsymbol{\sigma}_{fd} = 2\mu u_z'(r)\boldsymbol{i}_r(\theta) \otimes_s \boldsymbol{i}_z$ since $\mathrm{Tr}(\boldsymbol{\varepsilon}_{fd}) = 0$.

Stiffness of an elastic linkage
Solution

Question #3: Differential equation satisfied by u_z?

- Local equilibrium equation $\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma}_{fd} = \mathbf{0}$, where $\boldsymbol{\sigma}_{fd} = 2\mu u_z'(r)\boldsymbol{i}_r(\theta) \otimes_s \boldsymbol{i}_z$;
- In cylindrical coordinates:

$$\begin{aligned}\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma}_{fd} &= \frac{\partial \boldsymbol{\sigma}_{fd}}{\partial r}\boldsymbol{i}_r + \frac{\partial \boldsymbol{\sigma}_{fd}}{\partial \theta}\frac{\boldsymbol{i}_\theta}{r} + \frac{\partial \boldsymbol{\sigma}_{fd}}{\partial z}\boldsymbol{i}_z \\ &= \mu u_z''(r)\boldsymbol{i}_z + \mu\frac{u_z'(r)}{r}\boldsymbol{i}_z\,.\end{aligned}$$

Question #3: Differential equation satisfied by u_z?

- Local equilibrium equation $\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma}_{fd} = \mathbf{0}$, where $\boldsymbol{\sigma}_{fd} = 2\mu u_z'(r)\boldsymbol{i}_r(\theta) \otimes_s \boldsymbol{i}_z$;
- In cylindrical coordinates:

$$\begin{aligned}\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma}_{fd} &= \frac{\partial\boldsymbol{\sigma}_{fd}}{\partial r}\boldsymbol{i}_r + \frac{\partial\boldsymbol{\sigma}_{fd}}{\partial\theta}\frac{\boldsymbol{i}_\theta}{r} + \frac{\partial\boldsymbol{\sigma}_{fd}}{\partial z}\boldsymbol{i}_z \\ &= \mu u_z''(r)\boldsymbol{i}_z + \mu\frac{u_z'(r)}{r}\boldsymbol{i}_z\,;\end{aligned}$$

- Hence $r \mapsto u_z(r)$ satisfies:

$$u_z''(r) + \frac{u_z'(r)}{r} = \frac{1}{r}(ru_z'(r))' = 0\,.$$

Stiffness of an elastic linkage
Solution

Question #4: $r \mapsto u_z(r)$?

- From question #3, $u_z(r) = A \ln r + B$ where A and B are given by the boundary conditions.
- On $\{r = r_e\}$, $u_z(r_e) = A \ln r_e + B = 0$;
- On $\{r = r_i\}$, $u_z(r_i) = A \ln r_i + B = u_{zd}$;
- Hence:

$$\boxed{u_z(r) = u_{zd} \frac{\ln r_e - \ln r}{\ln r_e - \ln r_i}}\,.$$

Stiffness of an elastic linkage
Solution

Question #5: Stiffness K?

- $K = \frac{\|\boldsymbol{R}^e\|}{|u_{zd}|}$ where $\boldsymbol{R}^e$ is "the resultant force of the action of the moving cylinder on the cylinder Ω, on its inner lateral surface."

Question #5: Stiffness K?

- $K = \dfrac{\|\boldsymbol{R}^i\|}{|u_{zd}|}$;
- Resultant force of the action of the inner cylinder on Ω:

$$\begin{aligned}
\boldsymbol{R}^i &\stackrel{\text{def}}{=} \int_{\{r=r_i\}} \boldsymbol{\sigma}_{fd}\boldsymbol{n}dS \\
&= \int_0^L \int_0^{2\pi} \boldsymbol{\sigma}_{fd}(-\boldsymbol{i}_r(\theta))r_i d\theta dz \\
&= -\int_0^L \int_0^{2\pi} \mu u_z'(r_i)\boldsymbol{i}_z r_i d\theta dz \\
&= \frac{2\pi\mu L u_{zd}}{\ln r_e - \ln r_i}\boldsymbol{i}_z \,.
\end{aligned}$$

Question #5: Stiffness K?

- $K = \dfrac{\left\| \boldsymbol{R}^i \right\|}{|u_{zd}|}$;
- Resultant force of the action of the inner cylinder on Ω:

$$\boldsymbol{R}^i = \frac{2\pi\mu L u_{zd}}{\ln \frac{r_e}{r_i}} \boldsymbol{i}_z \,;$$

- Therefore the stiffness is:

$$\boxed{K = \frac{2\pi\mu L}{\ln \frac{r_e}{r_i}}}\,.$$

Question #6: Stress vector at the end faces?

- Stress vector at the end faces $\{z=0\} \cup \{z=L\}$: $\boldsymbol{\sigma}\boldsymbol{n} = \boldsymbol{\sigma}(\mp \boldsymbol{i}_z)$ for either faces;
- But $\boldsymbol{\sigma}_{fd} = 2\mu u_z'(r)\boldsymbol{i}_r(\theta) \otimes_s \boldsymbol{i}_z$, hence:

$$\begin{aligned}\boldsymbol{\sigma}_{fd}(-\boldsymbol{i}_z)\mid_{z=0} &= -\boldsymbol{\sigma}_{fd}\boldsymbol{i}_z\mid_{z=L} \\ &= -\mu u_z'(r)\boldsymbol{i}_r(\theta) \\ &\neq \mathbf{0} \text{ !!!}\end{aligned}$$

Question #6: Stress vector at the end faces?

- Stress vector at the end faces $\{z=0\} \cup \{z=L\}$: $\boldsymbol{\sigma}\boldsymbol{n} = \boldsymbol{\sigma}(\mp \boldsymbol{i}_z)$ for either faces;
- But $\boldsymbol{\sigma}_{fd} = 2\mu u_z'(r)\boldsymbol{i}_r(\theta) \otimes_s \boldsymbol{i}_z$, hence:

$$\boldsymbol{\sigma}_{fd}(-\boldsymbol{i}_z) \mid_{z=0} = -\boldsymbol{\sigma}_{fd}\boldsymbol{i}_z \mid_{z=L} \neq \mathbf{0} \text{ !!!}$$

- The boundary conditions (free surfaces) at the end faces are not satisfied, thus the ansatz "$\boldsymbol{u}(\boldsymbol{x}) = u_z(r)\boldsymbol{i}_z$" is incorrect.

Question #6: Stress vector at the end faces?

- Stress vector at the end faces $\{z=0\}\cup\{z=L\}$: $\boldsymbol{\sigma}\boldsymbol{n} = \boldsymbol{\sigma}(\mp\boldsymbol{i}_z)$ for either faces;
- But $\boldsymbol{\sigma}_{fd} = 2\mu u_z'(r)\boldsymbol{i}_r(\theta)\otimes_s\boldsymbol{i}_z$, hence:

$$\boldsymbol{\sigma}_{fd}(-\boldsymbol{i}_z)\mid_{z=0} = -\boldsymbol{\sigma}_{fd}\boldsymbol{i}_z\mid_{z=L} \neq \boldsymbol{0}\ !!!$$

- The boundary conditions (free surfaces) at the end faces are not satisfied, thus the ansatz "$\boldsymbol{u}(\boldsymbol{x}) = u_z(r)\boldsymbol{i}_z$" is incorrect;
- However one can show that the resultant forces vanish, and invoke the Saint-Venant principle to claim that this ansatz gives a good approximation of the solution away from the end faces.

Question #6: Stress vector at the end faces?

- Resultant force at $\{z = L\}$:

$$
\begin{aligned}
\boldsymbol{R}^L &\stackrel{\text{def}}{=} \int_{\{z=L\}} \boldsymbol{\sigma}_{fd}\boldsymbol{i}_z dS \\
&= \int_{r_i}^{r_e}\int_0^{2\pi} \mu u_z'(r)\boldsymbol{i}_r(\theta) r dr d\theta \\
&= \frac{-\mu u_{zd}}{\ln r_e - \ln r_i}\int_{r_i}^{r_e} dr \cancel{\int_0^{2\pi} \boldsymbol{i}_r(\theta) d\theta} \\
&= \boldsymbol{0}\,;
\end{aligned}
$$

- $\boldsymbol{R}^0 = -\boldsymbol{R}^L = \boldsymbol{0}$.

Question #6: Stress vector at the end faces?

- Resultant moment at $\{z = L\}$:

$$\begin{aligned}
\boldsymbol{M}^L &\stackrel{\text{def}}{=} \int_{\{z=L\}} r\boldsymbol{i}_r(\theta) \times \boldsymbol{\sigma}_{fd}\boldsymbol{i}_z dS \\
&= \int_{r_i}^{r_e}\int_0^{2\pi} r\boldsymbol{i}_r(\theta) \times \mu u_z'(r)\boldsymbol{i}_r(\theta) r dr d\theta \\
&= \frac{-\mu u_{zd}}{\ln r_e - \ln r_i}\int_{r_i}^{r_e} r dr \cancel{\int_0^{2\pi} \boldsymbol{i}_r(\theta) \times \boldsymbol{i}_r(\theta) d\theta} \\
&= \boldsymbol{0}\,;
\end{aligned}$$

- $\boldsymbol{M}^0 = -\boldsymbol{M}^L = \boldsymbol{0}$.