Design of a gravity dam

1EL5000–Continuum Mechanics – Tutorial Class #3

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Outline

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- Scalar product:

$$\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^a\,,\quad\langle\boldsymbol{a},\boldsymbol{b}\rangle=\sum_{j=1}^{a}a_jb_j=a_jb_j\,,$$

The last equality is Einstein's summation convention.

- Tensors and tensor product (or outer product):

$$\boldsymbol{A}\in\mathbb{R}^a\to\mathbb{R}^b\,,\quad\boldsymbol{A}=\boldsymbol{a}\otimes\boldsymbol{b}\,,\quad\boldsymbol{a}\in\mathbb{R}^a\,,\boldsymbol{b}\in\mathbb{R}^b\,.$$

- Tensor application to vectors:

$$\boldsymbol{A}=\boldsymbol{a}\otimes\boldsymbol{b}\in\mathbb{R}^a\to\mathbb{R}^b\,,\boldsymbol{c}\in\mathbb{R}^b\,,\quad\boldsymbol{A}\boldsymbol{c}=\langle\boldsymbol{b},\boldsymbol{c}\rangle\,\boldsymbol{a}\,.$$

- Product of tensors $\equiv$ composition of linear maps:

$$\boldsymbol{A}=\boldsymbol{a}\otimes\boldsymbol{b}\,,\boldsymbol{B}=\boldsymbol{c}\otimes\boldsymbol{d}\,,\quad\boldsymbol{A}\boldsymbol{B}=\langle\boldsymbol{b},\boldsymbol{c}\rangle\,\boldsymbol{a}\otimes\boldsymbol{d}\,.$$

- Scalar product of tensors:

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \mathrm{Tr}(\boldsymbol{A}\boldsymbol{B}^{\mathsf{T}}) := \boldsymbol{A} : \boldsymbol{B} = A_{jk}B_{jk} \,.$$

- Let $\{\boldsymbol{e}_j\}_{j=1}^d$ be a Cartesian basis in $\mathbb{R}^d$. Then:

$$\begin{aligned} a_j &= \langle \boldsymbol{a}, \boldsymbol{e}_j \rangle \,, \\ A_{jk} &= \langle \boldsymbol{A}, \boldsymbol{e}_j \otimes \boldsymbol{e}_k \rangle = \boldsymbol{A} : \boldsymbol{e}_j \otimes \boldsymbol{e}_k \\ &= \langle \boldsymbol{A}\boldsymbol{e}_k, \boldsymbol{e}_j \rangle \,, \end{aligned}$$

such that:

$$\begin{aligned} \boldsymbol{a} &= a_j \boldsymbol{e}_j \,, \\ \boldsymbol{A} &= A_{jk} \boldsymbol{e}_j \otimes \boldsymbol{e}_k \,. \end{aligned}$$

- Example: the identity matrix

$$\boldsymbol{I} = \boldsymbol{e}_j \otimes \boldsymbol{e}_j \,.$$

- Gradient of a vector function $\boldsymbol{a}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$:

$$\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a} = \frac{\partial \boldsymbol{a}}{\partial x_j} \otimes \boldsymbol{e}_j \,.$$

- Divergence of a vector function $\boldsymbol{a}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$:

$$\mathrm{div}_{\boldsymbol{x}}\boldsymbol{a} = \langle \boldsymbol{\nabla}_{\boldsymbol{x}}, \boldsymbol{a} \rangle = \mathrm{Tr}(\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a}) = \frac{\partial a_j}{\partial x_j} \,.$$

- Divergence of a tensor function $\boldsymbol{A}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$:

$$\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{A} = \frac{\partial (\boldsymbol{A}\boldsymbol{e}_j)}{\partial x_j} \,.$$

- Gradient of a vector function $\boldsymbol{a}(r,\theta,z)$:

$$\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a} = \frac{\partial \boldsymbol{a}}{\partial r} \otimes \boldsymbol{e}_r + \frac{\partial \boldsymbol{a}}{\partial \theta} \otimes \frac{\boldsymbol{e}_\theta}{r} + \frac{\partial \boldsymbol{a}}{\partial z} \otimes \boldsymbol{e}_z \,.$$

- Divergence of a vector function $\boldsymbol{a}(r,\theta,z)$:

$$\mathrm{div}_{\boldsymbol{x}}\boldsymbol{a} = \left\langle \frac{\partial \boldsymbol{a}}{\partial r}, \boldsymbol{e}_r \right\rangle + \left\langle \frac{\partial \boldsymbol{a}}{\partial \theta}, \frac{\boldsymbol{e}_\theta}{r} \right\rangle + \left\langle \frac{\partial \boldsymbol{a}}{\partial z}, \boldsymbol{e}_z \right\rangle \,.$$

- Divergence of a tensor function $\boldsymbol{A}(r,\theta,z)$:

$$\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{A} = \frac{\partial \boldsymbol{A}}{\partial r}\boldsymbol{e}_r + \frac{\partial \boldsymbol{A}}{\partial \theta}\frac{\boldsymbol{e}_\theta}{r} + \frac{\partial \boldsymbol{A}}{\partial z}\boldsymbol{e}_z \,.$$

Outline

Recap

- Displacement:

$$\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{p} = \boldsymbol{f}(\boldsymbol{p}, t) - \boldsymbol{p} \,;$$

- Strains:

$$\mathbb{E} = \frac{1}{2}\left(\mathbb{D}_{\boldsymbol{p}}\boldsymbol{u} + \mathbb{D}_{\boldsymbol{p}}\boldsymbol{u}^{\mathsf{T}} + \mathbb{D}_{\boldsymbol{p}}\boldsymbol{u}^{\mathsf{T}}\mathbb{D}_{\boldsymbol{p}}\boldsymbol{u}\right) \;;$$

- Small strains $\boldsymbol{x} \sim \boldsymbol{p}$, $\|\mathbb{D}_{\boldsymbol{p}}\boldsymbol{u}\| \ll 1$:

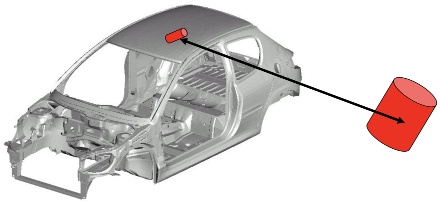
$$\mathbb{E} \simeq \boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbb{D}_{\boldsymbol{x}}\boldsymbol{u} + \mathbb{D}_{\boldsymbol{x}}\boldsymbol{u}^{\mathsf{T}}\right) \;;$$

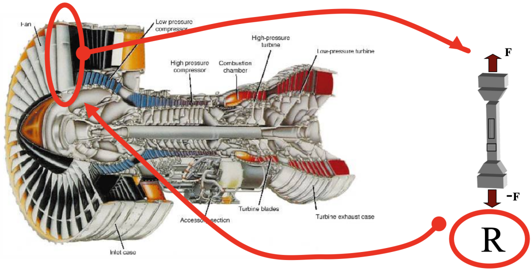
- Stresses:

$$\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma} + \boldsymbol{f}_v = \varrho\boldsymbol{a} \,;$$

- What makes the difference between foams and steel???

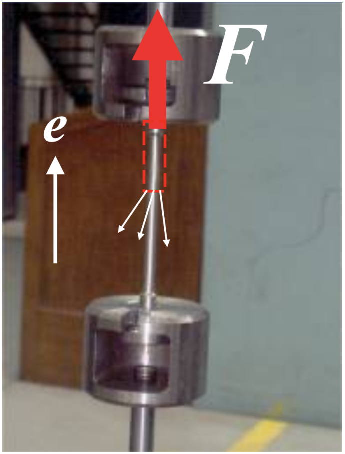
Stresses
Tensile test

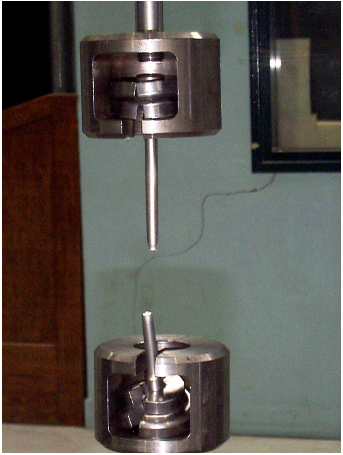




Stresses
Tensile test

Stresses are harder to measure compared to strains or displacements which can be seen!

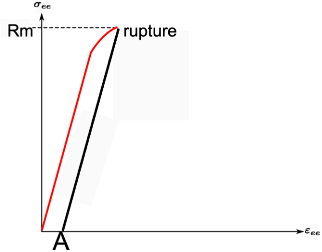


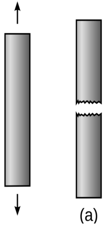


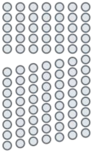
Plot $\varepsilon \mapsto \sigma$ up to fracture.

Stress-strain curve

Brittle materials

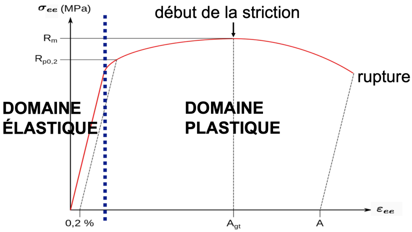


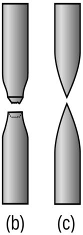


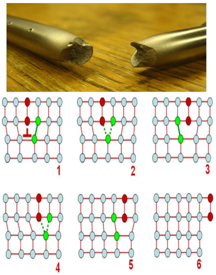


- At microscopic level: splitting of two atomic planes driven by normal stress component $\langle \boldsymbol{\sigma n}, \boldsymbol{n} \rangle$.
- **Examples**: cast iron, glass, stone, concrete, carbon fiber, ceramics, polymers (PMMA, polystyrene)...

Stress-strain curve
Ductile materials

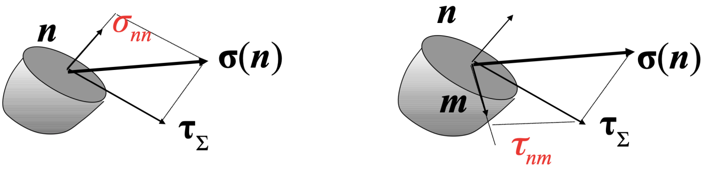




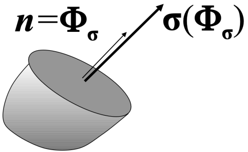


- At microscopic level: linear crystallographic defects (dislocations) allowing atoms to slide over each other at low stress levels.
- **Examples**: structural steel and many alloys of other metals...

Quantifying stresses
Normal-shear stresses



- Normal stress: $\sigma_{nn} = \langle \boldsymbol{\sigma n}, \boldsymbol{n} \rangle$;
- Shear (tangent) force: $\boldsymbol{\tau}_\Sigma = \boldsymbol{\sigma n} - \sigma_{nn}\boldsymbol{n}$;
- Shear stress: $\boldsymbol{\tau}_{nm} = \langle \boldsymbol{\sigma n}, \boldsymbol{m} \rangle$.



- Since $\boldsymbol{\sigma}$ is symmetric:

$$\boldsymbol{\sigma}\boldsymbol{\Phi}_\sigma = \lambda_\sigma \boldsymbol{\Phi}_\sigma \,.$$

- Characteristic polynomial ($I_1(\boldsymbol{\sigma}) = \mathrm{Tr}\,\boldsymbol{\sigma}$, $I_2(\boldsymbol{\sigma}) = \frac{1}{2}((\mathrm{Tr}\,\boldsymbol{\sigma})^2 - \mathrm{Tr}(\boldsymbol{\sigma}^2))$, $I_3(\boldsymbol{\sigma}) = \det\boldsymbol{\sigma}$):

$$\det(\boldsymbol{\sigma} - \lambda_\sigma \boldsymbol{I}) = -\lambda_\sigma^3 + I_1(\boldsymbol{\sigma})\lambda_\sigma^2 - I_2(\boldsymbol{\sigma})\lambda_\sigma + I_3(\boldsymbol{\sigma}) = 0 \,.$$

- Major principal stress:

$$\lambda_1(\boldsymbol{\sigma}) = \max_{\|\boldsymbol{n}\|=1} \langle \boldsymbol{\sigma}\boldsymbol{n}, \boldsymbol{n} \rangle < \sigma_r \,.$$

- Deviatoric stress tensor:

$$\boldsymbol{\sigma}^D = \boldsymbol{\sigma} - \frac{\operatorname{Tr}\boldsymbol{\sigma}}{3}\boldsymbol{I}\,;$$

- Orthogonal expansion: $\boldsymbol{\sigma} = \underline{\sigma}\boldsymbol{I} + \boldsymbol{\sigma}^D$, $\underline{\sigma}\boldsymbol{I} : \boldsymbol{\sigma}^D = 0$;
- Deviatoric stress invariants $J_m = \frac{1}{m}\operatorname{Tr}(\boldsymbol{\sigma}^{Dm})$ such that:

$$\det(\boldsymbol{\sigma}^D - \lambda_\sigma\boldsymbol{I}) = -\lambda_\sigma^3 + J_1(\boldsymbol{\sigma}^D)\lambda_\sigma^2 - J_2(\boldsymbol{\sigma}^D)\lambda_\sigma + J_3(\boldsymbol{\sigma}^D) = 0\,;$$

- The deviatoric stress tensor has the same principal directions and is a state of pure shear:

$$J_2(\boldsymbol{\sigma}^D) = 0 \Rightarrow \boldsymbol{\sigma}^D = \boldsymbol{0}\,,$$
$$\boldsymbol{\sigma}^D = \boldsymbol{0} \Rightarrow \boldsymbol{\tau}_\Sigma = \boldsymbol{\sigma}\boldsymbol{n} - \sigma_{nn}\boldsymbol{n} = \boldsymbol{0}\,, \forall\boldsymbol{n}\,.$$

Quantifying stresses
von Mises criterion

- Equivalent or von Mises stress:

$$\sigma_{\mathrm{eq}} = \sqrt{3J_2(\boldsymbol{\sigma}^D)} = \sqrt{\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}}\,,$$

where $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$ are the principal stresses of $\boldsymbol{\sigma}$.

- **Examples**:

$$\boldsymbol{\sigma} = \sigma \boldsymbol{e} \otimes \boldsymbol{e} \Rightarrow \sigma_{\mathrm{eq}} = \sigma$$

$$\boldsymbol{\sigma} = \tau \boldsymbol{m} \otimes_s \boldsymbol{n} \Rightarrow \sigma_{\mathrm{eq}} = \sqrt{3}\tau\,.$$

- von Mises yield criterion:

$$\sigma_{\mathrm{eq}} \leqslant \sigma_0\,.$$

Quantifying stresses

Tresca's criterion

- Tresca's stress for $\boldsymbol{\tau}_\Sigma = \boldsymbol{\sigma}\boldsymbol{n} - \sigma_{nn}\boldsymbol{n}$:

$$\begin{aligned}\tau_{\text{eq}} &= \max_{\|\boldsymbol{n}\|=1} \|\boldsymbol{\tau}_\Sigma\| \\ &= \frac{1}{2}\max(|\sigma_1 - \sigma_2|\,, |\sigma_2 - \sigma_3|\,, |\sigma_3 - \sigma_1|)\,.\end{aligned}$$

- Maximum shear stress for $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$:

$$\tau_{\text{eq}} = \frac{\sigma_1 - \sigma_3}{2}\,,$$

which acts on the plane with unit normal $\boldsymbol{n} = \frac{\boldsymbol{\Phi}_1 \pm \boldsymbol{\Phi}_3}{\sqrt{2}}$ for which:

$$\sigma_{nn} = \frac{\sigma_1 + \sigma_3}{2}\,.$$

- Tresca's criterion:

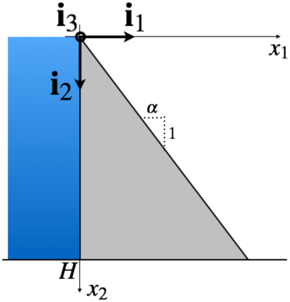
$$\tau_{\text{eq}} \leqslant \tau_0 = \frac{\sigma_0}{2}\,.$$

To go further...

- 2EL1830 Non-linear behavior of materials
- ST7-81-CS Conception en fabrication additive

Outline

Design of a gravity dam
Setup




Willow Creek dam (Oregon, USA)

$$\boldsymbol{\sigma} = b_1 x_2 \boldsymbol{i}_1 \otimes \boldsymbol{i}_1 + (a_2 x_1 + b_2 x_2) \boldsymbol{i}_2 \otimes \boldsymbol{i}_2 - 2(\varrho_b g + b_2) x_1 \boldsymbol{i}_1 \otimes_s \boldsymbol{i}_2 \,,$$

with:

$$b_1 = -\varrho_e g \,, \quad a_2 = \frac{\varrho_b g}{\alpha} - 2\frac{\varrho_e g}{\alpha^3} \,, \quad b_2 = -\varrho_b g + \frac{\varrho_e g}{\alpha^2} \,.$$

Question #1: Local equilibrium equation?

- The local equilibrium equation reads $\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma} + \boldsymbol{f}_v = \varrho \boldsymbol{a}$.
- In the present case $\boldsymbol{f}_v = +\varrho_b g \boldsymbol{i}_2$ and $\boldsymbol{a} = \mathbf{0}$ in static analysis.

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- Besides $\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma} = \frac{\partial}{\partial x_j}(\boldsymbol{\sigma}\boldsymbol{i}_j)$ and:

$$\begin{aligned}
\frac{\partial\boldsymbol{\sigma}}{\partial x_1} &= a_2\boldsymbol{i}_2\otimes\boldsymbol{i}_2 - 2(\varrho_b g + b_2)\boldsymbol{i}_1\otimes_s\boldsymbol{i}_2\,, \\
\frac{\partial\boldsymbol{\sigma}}{\partial x_2} &= b_1\boldsymbol{i}_1\otimes\boldsymbol{i}_1 + b_2\boldsymbol{i}_2\otimes\boldsymbol{i}_2\,, \\
\frac{\partial\boldsymbol{\sigma}}{\partial x_3} &= \mathbf{0}\,.
\end{aligned}$$

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- Besides $\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma} = \frac{\partial}{\partial x_j}(\boldsymbol{\sigma}\boldsymbol{i}_j)$ and:

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$$\frac{\partial\boldsymbol{\sigma}}{\partial x_2} = b_1\boldsymbol{i}_1\otimes\boldsymbol{i}_1 + b_2\boldsymbol{i}_2\otimes\boldsymbol{i}_2\,.$$

- Hence (remind that $(\boldsymbol{a}\otimes\boldsymbol{b})\boldsymbol{c} = \langle\boldsymbol{b},\boldsymbol{c}\rangle\,\boldsymbol{a}$):

$$\mathbf{Div}_{\boldsymbol{x}}\boldsymbol{\sigma} = -(\varrho_b g + b_2)\boldsymbol{i}_2 + b_2\boldsymbol{i}_2 = -\varrho_b g\boldsymbol{i}_2 = -\boldsymbol{f}_v \quad \text{QED}\,.$$

Design of a gravity dam

Solution

Question #2: Free edge BC on downstream wall?

- Downstream wall $\Gamma_\mathrm{d} = \{0 \leqslant x_2 \leqslant H,\, x_1 - \alpha x_2 = 0\}$.

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- Outward unit normal $\boldsymbol{n} = \frac{1}{\sqrt{1+\alpha^2}}(\boldsymbol{i}_1 - \alpha \boldsymbol{i}_2)$.
- Therefore:

$$\begin{aligned}
\boldsymbol{\sigma}\boldsymbol{n}\,|_{\Gamma_d} = {} & \frac{1}{\sqrt{1+\alpha^2}}\,(b_1 x_2 \boldsymbol{i}_1 - (\varrho_b g + b_2) x_1 \boldsymbol{i}_2 \\
& -\alpha(a_2 x_1 + b_2 x_2)\boldsymbol{i}_2 - \alpha(\varrho_b g + b_2) x_1 \boldsymbol{i}_1) \\
= {} & \frac{x_2}{\sqrt{1+\alpha^2}}\left[\left(b_1 - \alpha^2(\varrho_b g + b_2)\right)\boldsymbol{i}_1\right. \\
& \left. - \left(\alpha(\varrho_b g + 2b_2) + \alpha^2 a_2\right)\boldsymbol{i}_2\right] \\
= {} & \frac{x_2}{\sqrt{1+\alpha^2}}\left[(-\varrho_e g + \varrho_e g)\,\boldsymbol{i}_1\right. \\
& \left. - \left(\alpha\left(-\varrho_b g + 2\frac{\varrho_e g}{\alpha^2}\right) + \alpha\varrho_b g - 2\frac{\varrho_e g}{\alpha}\right)\boldsymbol{i}_2\right] \text{ QED}\,.
\end{aligned}$$

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- But:

$$
\begin{aligned}
\boldsymbol{\sigma}\boldsymbol{n}\,|_{\Gamma_u} &= -\, b_1 x_2 \boldsymbol{i}_1 + (\varrho_b g + b_2)\cancel{x_1}\boldsymbol{i}_2 \\
&= +\, \varrho_e g x_2 \boldsymbol{i}_1 \\
&= \boldsymbol{f}_s \quad \text{QED}\,.
\end{aligned}
$$

Design of a gravity dam
Solution

Question #4: H to withstand maximum normal stress σ_r?

- $\max_{\|\boldsymbol{n}\|=1} \sigma_{nn} = \lambda_1(\boldsymbol{\sigma})$.

Design of a gravity dam

Solution

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- $\max_{\|\boldsymbol{n}\|=1} \sigma_{nn} = \lambda_1(\boldsymbol{\sigma})$.
- But on Γ_u: $\boldsymbol{\sigma} = b_1 x_2 \boldsymbol{i}_1 \otimes \boldsymbol{i}_1 + b_2 x_2 \boldsymbol{i}_2 \otimes \boldsymbol{i}_2$.

Design of a gravity dam

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- Therefore:

$$\begin{aligned}\lambda_1(\boldsymbol{\sigma}) &= x_2 \times \max(b_1, b_2) \\ &= g x_2 \times \max\left(-\varrho_e, -\varrho_b + \frac{\varrho_e}{\alpha^2}\right) .\end{aligned}$$

Design of a gravity dam
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- If $\alpha > \sqrt{\varrho_e/\varrho_b}$, then $\lambda_1(\boldsymbol{\sigma}) < 0$ and the dam is in compression; this condition is always fulfilled when $\varrho_e = 0$ (no water).

Design of a gravity dam
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- If $\alpha \leqslant \sqrt{\varrho_e/\varrho_b}$, then $\lambda_1(\boldsymbol{\sigma}) = (\frac{\varrho_e}{\alpha^2} - \varrho_b) g x_2$ and $\lambda_1(\boldsymbol{\sigma}) = \sigma_r$ whenever $x_2 = \frac{\alpha^2 \sigma_r}{g(\varrho_e - \alpha^2 \varrho_b)}$; therefore:

$$H < \frac{\alpha^2 \sigma_r}{g(\varrho_e - \alpha^2 \varrho_b)} .$$