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Localisation de sources acoustiques dans des écoulements hétérogènes

Éric Savin[1,2] (avec Jean-Luc Akian, Luc Bonnet, Josselin Garnier, Étienne Gay, Christophe Peyret)

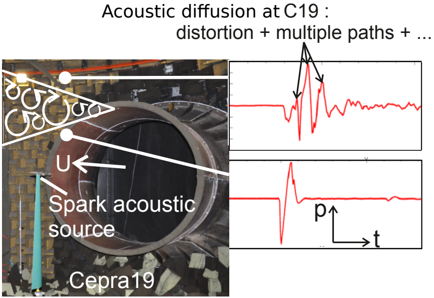
[1]ONERA, France
[2]CentraleSupélec, France

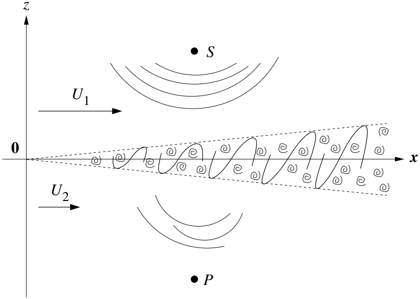
MécaWave, 9 mai 2023

Outline

1. Some issues with heterogeneous flows...
2. Imaging in quiescent random medium
3. Convected acoustic waves
4. Imaging in moving random medium
5. Outlook

Wave scattering in turbulent shear layers





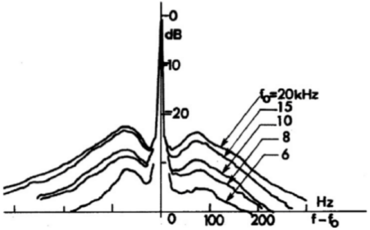
CEPRA19 anechoic wind tunnel with open-jet test section (nozzle exit Ø = 2-3 m, $U_1 \leq 100$ m/s).

- **Direct problem**: spectral broadening and phase shift of acoustic waves;
- **Inverse problem**: localization of sources through shear layers using multiply scattered waves.

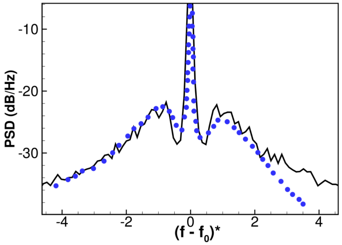
Wave scattering in turbulent shear layers

Results:

- Model of the acoustic pressure field transmitted by an horizontally stratified random flow (Garnier-Gay-Savin, *SIAM Multiscale Model. Simul.* **18**(2), 798-823, 2020);
- Model of multiple scattering of waves by an heterogeneous, unsteady flow in the high-frequency limit and weak coupling regime (Akian-Savin, *SIAM Multiscale Model. Simul.* **19**(3), 1394-1424, 2021);
- Coherent interferometric imaging in a random flow (Gay-Bonnet-Peyret-Savin-Garnier, *J. Sound Vib.* **494**, 115852, 2021): **THIS TALK!**



$M \approx 0.2$, $Re \approx 1200$, $f_0 = 6 - 20\,kHz$



Candel-Guédel-Julienne AIAA Paper 76-544 (1976)
Bennaceur *et al.* *Comput. Fluids* **138**, 83 (2016)

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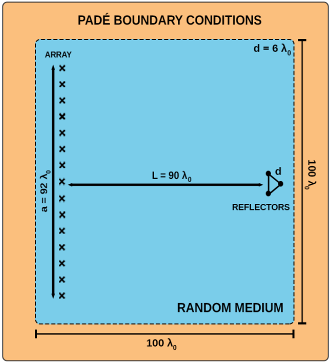
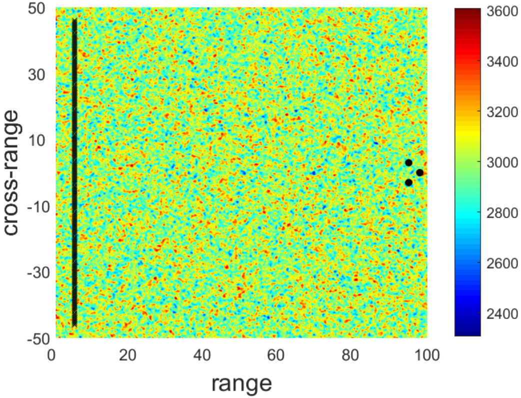
Imaging in quiescent random medium





Quiescent medium with random speed of sound $\frac{1}{c_0^2(\boldsymbol{x})} = \frac{1}{\underline{c}^2}\left[1 + \sigma n\left(\frac{\boldsymbol{x}}{\ell}\right)\right]$ *where* $\ell = \frac{\lambda_0}{2}$.

Green's function in random medium

- The Green's function $G(\boldsymbol{x}, \boldsymbol{x}_S, t)$ solves the wave equation with vanishing initial conditions:

$$\frac{1}{\underline{c}^2}\left[1+\sigma n\left(\frac{\boldsymbol{x}}{\ell}\right)\right]\frac{\partial^2 G}{\partial t^2}-\Delta G=\delta(t)\delta(\boldsymbol{x}-\boldsymbol{x}_S)\,,$$

  and is unknown.
- Pressure field induced by the sound source $m(\boldsymbol{x}, t)$ of support Ω_S:

$$p'(\boldsymbol{x},t)=\int_{-\infty}^{t}\int_{\Omega_S} G(\boldsymbol{x},\boldsymbol{y},t-t')m(\boldsymbol{y},t')\mathrm{d}\boldsymbol{y}\mathrm{d}t'\,,$$

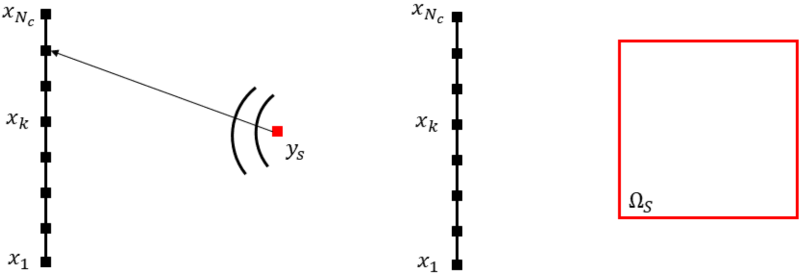
  or in frequency domain with $\widehat{p}'(\boldsymbol{x},\omega)=\int \mathrm{e}^{\mathrm{i}\omega t}\,p'(\boldsymbol{x},t)\mathrm{d}t$:

$$\widehat{p}'(\boldsymbol{x},\omega)=\int_{\Omega_S}\widehat{G}(\boldsymbol{x},\boldsymbol{y},\omega)\widehat{m}(\boldsymbol{y},\omega)\mathrm{d}\boldsymbol{y}\,.$$

- For a point source $\widehat{m}(\boldsymbol{y},\omega)=\widehat{A}(\omega)\delta(\boldsymbol{y}-\boldsymbol{y}_S)$:

$$\widehat{p}'(\boldsymbol{x},\omega)=\widehat{G}(\boldsymbol{x},\boldsymbol{y}_S,\omega)\widehat{A}(\omega)\,.$$

Passive imaging



Passive configuration: imaging sources.

Passive imaging functionals

- **DATA**: pressure signals $\{t \mapsto p'(\boldsymbol{x}_k, t)\}_{k=1}^{N_c}$ at N_c sensors;
- **OBJECTIVE**: localize a source at $\boldsymbol{y}_S = \arg \max_{\boldsymbol{x}_S \in \Omega_S} \mathcal{J}(\boldsymbol{x}_S)$ in some search region Ω_S.
- **MUltiple SIgnal Classification** (**MUSIC**, close to the **factorization method**): form the Prony matrices $\mathbf{P}_k$ of the discrete FFT $\{\widehat{p}'(\boldsymbol{x}_k, \omega_m)\}_{m=1}^{2M-1}$ for each sensor, and compute their SVD,

$$\mathbf{P}_k = \begin{bmatrix} \widehat{p}'(\boldsymbol{x}_k, \omega_1) & \widehat{p}'(\boldsymbol{x}_k, \omega_2) & \cdots & \widehat{p}'(\boldsymbol{x}_k, \omega_M) \\ \widehat{p}'(\boldsymbol{x}_k, \omega_2) & \widehat{p}'(\boldsymbol{x}_k, \omega_3) & \cdots & \widehat{p}'(\boldsymbol{x}_k, \omega_{M+1}) \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{p}'(\boldsymbol{x}_k, \omega_M) & \widehat{p}'(\boldsymbol{x}_k, \omega_{M+1}) & \cdots & \widehat{p}'(\boldsymbol{x}_k, \omega_{2M-1}) \end{bmatrix} = \mathbf{U}_k \boldsymbol{\Sigma}_k \mathbf{V}_k^* \,;$$

- Then form the illumination vector $\mathbf{g}_k(\boldsymbol{x}) = (\mathrm{e}^{\mathrm{i}\omega_1 \mathcal{T}(\boldsymbol{x}_k, \boldsymbol{x})}, \mathrm{e}^{\mathrm{i}\omega_2 \mathcal{T}(\boldsymbol{x}_k, \boldsymbol{x})}, \dots \mathrm{e}^{\mathrm{i}\omega_M \mathcal{T}(\boldsymbol{x}_k, \boldsymbol{x})})^{\mathsf{T}}$ for $\mathcal{T}(\boldsymbol{x}_k, \boldsymbol{x})$ being the travel time of waves from $\boldsymbol{x}$ to $\boldsymbol{x}_k$ in the actual random medium, project it on the noise subspace of each sensor, and stack them:

$$\mathcal{J}_{\mathrm{MUSIC}}(\boldsymbol{x}_S) = \left(\sum_{k=1}^{N_c} \|(\mathbf{I}_M - \tilde{\mathbf{U}}_k \tilde{\mathbf{U}}_k^*) \mathbf{g}_k(\boldsymbol{x}_S)\|^2 \right)^{-1} ,$$

where $\tilde{\mathbf{U}}_k$ are the first $p < M$ columns of $\mathbf{U}_k$ when the search region Ω_S is divided into p pixels.

Passive imaging functionals

- **DATA**: pressure signals $\{t \mapsto p'(\boldsymbol{x}_k, t)\}_{k=1}^{N_c}$ at N_c sensors;
- **OBJECTIVE**: localize a source at $\boldsymbol{y}_S = \arg\max_{\boldsymbol{x}_S \in \Omega_S} \mathcal{J}(\boldsymbol{x}_S)$ in some search region Ω_S.

- **Time Reversal (TR) imaging**: time-reverse the signals $\widehat{p}'(\boldsymbol{x}_k, \omega) \to \overline{\widehat{p}'(\boldsymbol{x}_k, \omega)}$, stack them, and back propagate them in the actual medium with unknown Green's function $\widehat{G}(\boldsymbol{x}, \boldsymbol{x}_S, \omega)$,

$$\begin{aligned}
\mathcal{J}_{\mathrm{TR}}(\boldsymbol{x}_S) &= \frac{1}{2\pi} \int \sum_{k=1}^{N_c} \widehat{G}(\boldsymbol{x}_S, \boldsymbol{x}_k, \omega) \overline{\widehat{p}'(\boldsymbol{x}_k, \omega)} \, \mathrm{d}\omega \\
&\approx \frac{1}{2\pi} \sum_{k=1}^{N_c} \overline{\int \mathrm{e}^{-\mathrm{i}\omega \mathcal{T}(\boldsymbol{x}_k, \boldsymbol{x}_S)} \, \widehat{p}'(\boldsymbol{x}_k, \omega) \, \mathrm{d}\omega} \\
&= \sum_{k=1}^{N_c} p' \left(\boldsymbol{x}_k, \mathcal{T}(\boldsymbol{x}_k, \boldsymbol{x}_S)\right)
\end{aligned}$$

since $\widehat{G}(\boldsymbol{x}_S, \boldsymbol{x}, \omega) = \widehat{G}(\boldsymbol{x}, \boldsymbol{x}_S, \omega) \propto \mathrm{e}^{\mathrm{i}\omega \mathcal{T}(\boldsymbol{x}, \boldsymbol{x}_S)}$ for $\mathcal{T}(\boldsymbol{x}, \boldsymbol{x}_S)$ being the travel time of waves from $\boldsymbol{x}_S$ to $\boldsymbol{x}$ in the actual random medium.
- **Remark**: In optics time reversal $\mathrm{e}^{\mathrm{i}\omega \mathcal{T}(\boldsymbol{x}, \boldsymbol{x}_S)} \to \mathrm{e}^{-\mathrm{i}\omega \mathcal{T}(\boldsymbol{x}, \boldsymbol{x}_S)}$ is called phase conjugation.

Passive imaging functionals

- **DATA**: pressure signals $\{t \mapsto p'(\boldsymbol{x}_k, t)\}_{k=1}^{N_c}$ at N_c sensors;
- **OBJECTIVE**: localize a source at $\boldsymbol{y}_S = \arg \max_{\boldsymbol{x}_S \in \Omega_S} \mathcal{J}(\boldsymbol{x}_S)$ in some search region Ω_S.
- **Reverse Time (RT) imaging**: time-reverse the signals $\widehat{p}'(\boldsymbol{x}_k, \omega) \to \overline{\widehat{p}'(\boldsymbol{x}_k, \omega)}$, stack them, and back propagate them in a (homogeneous) fictitious medium with known Green's function $\widehat{G}_0(\boldsymbol{x}, \boldsymbol{x}_S, \omega)$,

$$\mathcal{J}_{\mathrm{RT}}(\boldsymbol{x}_S) = \frac{1}{2\pi} \sum_{k=1}^{N_c} \int \widehat{G}_0(\boldsymbol{x}_S, \boldsymbol{x}_k, \omega) \overline{\widehat{p}'(\boldsymbol{x}_k, \omega)} \, \mathrm{d}\omega \, .$$

Passive imaging functionals

- **DATA**: pressure signals $\{t \mapsto p'(\boldsymbol{x}_k, t)\}_{k=1}^{N_c}$ at N_c sensors;
- **OBJECTIVE**: localize a source at $\boldsymbol{y}_S = \arg\max_{\boldsymbol{x}_S \in \Omega_S} \mathcal{J}(\boldsymbol{x}_S)$ in some search region Ω_S.

- **Kirchhoff Migration (KM)**: time-reverse the signals $\widehat{p}'(\boldsymbol{x}_k, \omega) \to \overline{\widehat{p}'(\boldsymbol{x}_k, \omega)}$, stack them, and compensate for the phase in a (homogeneous) fictitious medium $\widehat{G}_0(\boldsymbol{x}, \boldsymbol{x}_S, \omega) \approx \mathrm{e}^{\mathrm{i}\omega \mathcal{T}_0(\boldsymbol{x}, \boldsymbol{x}_S)}$,

$$\begin{aligned}
\mathcal{J}_{\mathrm{KM}}(\boldsymbol{x}_S) &= \frac{1}{2\pi} \int \sum_{k=1}^{N_c} \mathrm{e}^{\mathrm{i}\omega \mathcal{T}_0(\boldsymbol{x}_k, \boldsymbol{x}_S)} \, \overline{\widehat{p}'(\boldsymbol{x}_k, \omega)} \, \mathrm{d}\omega \\
&= \frac{1}{2\pi} \sum_{k=1}^{N_c} \overline{\int \mathrm{e}^{-\mathrm{i}\omega \mathcal{T}_0(\boldsymbol{x}_k, \boldsymbol{x}_S)} \, \widehat{p}'(\boldsymbol{x}_k, \omega) \, \mathrm{d}\omega} \\
&= \sum_{k=1}^{N_c} p' \left(\boldsymbol{x}_k, \mathcal{T}_0(\boldsymbol{x}_k, \boldsymbol{x}_S)\right) \,,
\end{aligned}$$

where $\mathcal{T}_0(\boldsymbol{x}, \boldsymbol{x}_S) = \frac{\|\boldsymbol{x} - \boldsymbol{x}_S\|}{c_0}$: travel time from $\boldsymbol{x}_S$ to $\boldsymbol{x}$ in a homogeneous domain with $n = \mathrm{C}^{\mathrm{st}}$.

Passive imaging functionals

- **DATA**: pressure signals $\{t \mapsto p'(\boldsymbol{x}_k, t)\}_{k=1}^{N_c}$ at N_c sensors;
- **OBJECTIVE**: localize a source at $\boldsymbol{y}_S = \arg\max_{\boldsymbol{x}_S \in \Omega_S} \mathcal{J}(\boldsymbol{x}_S)$ in some search region Ω_S.
- **Delay-And-Sum (DAS) beamforming**: same as KM, but weight signals received at each sensor,

$$\mathcal{J}_{\mathrm{DAS}}(\boldsymbol{x}_S) = \sum_{k=1}^{N_c} w_k(\boldsymbol{x}_S) p'\left(\boldsymbol{x}_k, \mathcal{T}_0(\boldsymbol{x}_k, \boldsymbol{x}_S)\right) ,$$

  where *e.g.* $w_k(\boldsymbol{x}) = 4\pi \|\boldsymbol{x} - \boldsymbol{x}_k\|$.

Passive imaging functionals

- **DATA**: pressure signals $\{t \mapsto p'(\boldsymbol{x}_k, t)\}_{k=1}^{N_c}$ at N_c sensors;
- **OBJECTIVE**: localize a source at $\boldsymbol{y}_S = \arg \max_{\boldsymbol{x}_S \in \Omega_S} \mathcal{J}(\boldsymbol{x}_S)$ in some search region Ω_S.

- Since $\widehat{p}'(\boldsymbol{x}_k, \omega) = \widehat{G}(\boldsymbol{x}_k, \boldsymbol{y}_S, \omega)\widehat{A}(\omega)$ for a point source at $\boldsymbol{y}_S$:

$$\mathcal{J}_{\mathrm{TR}}(\boldsymbol{x}_S) = \frac{1}{2\pi} \int \left(\sum_{k=1}^{N_c} \widehat{G}(\boldsymbol{x}_S, \boldsymbol{x}_k, \omega) \overline{\widehat{G}(\boldsymbol{x}_k, \boldsymbol{y}_S, \omega)} \right) \overline{\widehat{A}(\omega)} \, \mathrm{d}\omega \,,$$

$$\mathcal{J}_{\mathrm{RT}}(\boldsymbol{x}_S) = \frac{1}{2\pi} \int \left(\sum_{k=1}^{N_c} \widehat{G}_0(\boldsymbol{x}_S, \boldsymbol{x}_k, \omega) \overline{\widehat{G}(\boldsymbol{x}_k, \boldsymbol{y}_S, \omega)} \right) \overline{\widehat{A}(\omega)} \, \mathrm{d}\omega \,,$$

  and TR works very well by cancelling random phases in random media, while RT or KM works poorly.
- **By-pass**: back propagate empirical cross-correlations in a (homogeneous) fictitious medium.

Claerbout *Geophysics* **33**(2), 264 (1968)
Duvall-Jefferies-Harvey-Pomerantz *Nature* **362**, 430 (1993)
Weaver-Lobkis *Ultrasonics* **38**(1-8), 491 (2000)
Garnier-Papanicolaou *Passive Imaging with Ambient Noise*, Cambridge University Press (2016)

Passive imaging functionals

- **DATA**: pressure signals $\{t \mapsto p'(\boldsymbol{x}_k, t)\}_{k=1}^{N_c}$ at N_c sensors;
- **OBJECTIVE**: localize a source at $\boldsymbol{y}_S = \arg \max_{\boldsymbol{x}_S \in \Omega_S} \mathcal{J}(\boldsymbol{x}_S)$ in some search region Ω_S.
- **Full Migration (FM)**: time-reverse the signals $\widehat{p}'(\boldsymbol{x}_k, \omega) \to \overline{\widehat{p}'(\boldsymbol{x}_k, \omega)}$, stack cross-correlations, and compensate for the phase in a (homogeneous) fictitious medium,

$$\mathcal{J}_{\mathrm{FM}}(\boldsymbol{x}_S) = \sum_{k=1}^{N_c} \sum_{l=1}^{N_c} \iint \mathrm{e}^{-\mathrm{i}\omega \mathcal{T}_0(\boldsymbol{x}_k, \boldsymbol{x}_S) + \mathrm{i}\omega' \mathcal{T}_0(\boldsymbol{x}_l, \boldsymbol{x}_S)} \, \widehat{p}'(\boldsymbol{x}_k, \omega) \overline{\widehat{p}'(\boldsymbol{x}_l, \omega')} \, \mathrm{d}\omega \mathrm{d}\omega'$$
$$= (2\pi)^2 |\mathcal{J}_{\mathrm{KM}}(\boldsymbol{x}_S)|^2 \text{ !!!}$$

- FM of all empirical cross-correlations works poorly if the medium is cluttered.

Passive imaging functionals

- **DATA**: pressure signals $\{t \mapsto p'(\boldsymbol{x}_k, t)\}_{k=1}^{N_c}$ at N_c sensors;
- **OBJECTIVE**: localize a source at $\boldsymbol{y}_S = \arg \max_{\boldsymbol{x}_S \in \Omega_S} \mathcal{J}(\boldsymbol{x}_S)$ in some search region Ω_S.

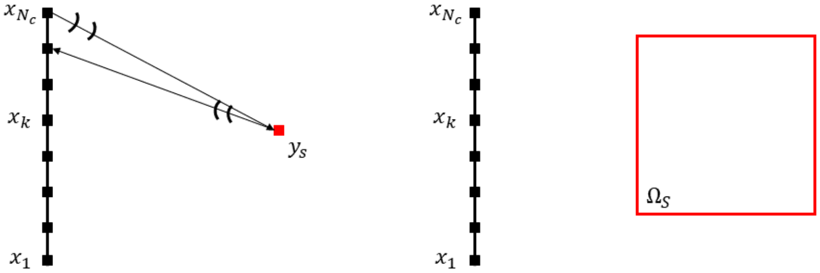
- **Coherent INTerferometric (CINT) imaging**: time-reverse the signals $\widehat{p}'(\boldsymbol{x}_k, \omega) \to \overline{\widehat{p}'(\boldsymbol{x}_k, \omega)}$, stack cross-correlations, compensate for the phase in a (homogeneous) fictitious medium, and filter,

$$\mathcal{J}_{\mathrm{CINT}}(\boldsymbol{x}_S; \Omega_d, X_d) = \sum_{\substack{k,\, l=1 \\ \|\boldsymbol{x}_k - \boldsymbol{x}_l\| \leq X_d}}^{N_c} \iint_{|\omega-\omega'|\leq\Omega_d} \mathrm{e}^{-\mathrm{i}\omega\mathcal{T}_0(\boldsymbol{x}_k,\boldsymbol{x}_S)+\mathrm{i}\omega'\mathcal{T}_0(\boldsymbol{x}_l,\boldsymbol{x}_S)}\, \widehat{p}'(\boldsymbol{x}_k, \omega)\overline{\widehat{p}'(\boldsymbol{x}_l, \omega')}\, \mathrm{d}\omega \mathrm{d}\omega' \,.$$

- The frequency window Ω_d is ideally chosen as the decoherency frequency Ω_c *i.e.* the frequency gap beyond which the recorded signals are no longer correlated;
- The spatial window X_d is ideally chosen as the decoherency length X_c *i.e.* the sensor gap beyond which the recorded signals are no longer correlated;
- Ω_d and X_d can be found adaptively in the imaging process.

Borcea-Papanicolaou-Tsogka *Inverse Probl.* **19**(6), S139 (2003)
Borcea-Papanicolaou-Tsogka *Inverse Probl.* **21**(4), 1419 (2005)
Borcea-Papanicolaou-Tsogka *Inverse Probl.* **22**(4), 1405 (2006)

Active imaging



Active configuration: imaging reflectors.

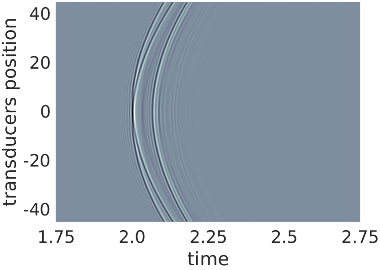
Active imaging functionals

- **DATA**: pressure signals $\{t \mapsto p'(\boldsymbol{x}_k, t; \boldsymbol{x}_p),\ 1 \leq p \leq N_e\}_{k=1}^{N_c}$ at N_c sensors from N_e emitters;
- **OBJECTIVE**: localize a reflector at $\boldsymbol{y}_S = \arg\max_{\boldsymbol{x}_S \in \Omega_S} \mathcal{J}(\boldsymbol{x}_S)$ in some search region Ω_S.

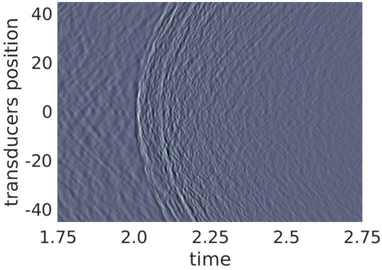
$$\begin{aligned}\mathcal{J}_{\mathrm{KM}}(\boldsymbol{x}_S) &= \frac{1}{2\pi} \sum_{k=1}^{N_c} \sum_{p=1}^{N_e} \int \mathrm{e}^{\mathrm{i}\omega(\mathcal{T}_0(\boldsymbol{x}_k,\boldsymbol{x}_S)+\mathcal{T}_0(\boldsymbol{x}_S,\boldsymbol{x}_p))} \overline{\widehat{p}'(\boldsymbol{x}_k,\omega;\boldsymbol{x}_p)}\,\mathrm{d}\omega \\ &= \sum_{k=1}^{N_c} \sum_{p=1}^{N_e} p'\left(\boldsymbol{x}_k, \mathcal{T}_0(\boldsymbol{x}_k,\boldsymbol{x}_S) + \mathcal{T}_0(\boldsymbol{x}_S,\boldsymbol{x}_p); \boldsymbol{x}_p\right) ;\end{aligned}$$

$$\begin{aligned}\mathcal{J}_{\mathrm{CINT}}(\boldsymbol{x}_S;\Omega_d,X_d) = & \sum_{\substack{k,\,l\,=\,1 \\ \|\boldsymbol{x}_k\,-\,\boldsymbol{x}_l\|\,\leq\,X_d}}^{N_c} \ \sum_{\substack{p,\,q\,=\,1 \\ \|\boldsymbol{x}_p\,-\,\boldsymbol{x}_q\|\,\leq\,X_d}}^{N_e} \iint_{|\omega-\omega'|\leq\Omega_d} \widehat{p}'(\boldsymbol{x}_k,\omega;\boldsymbol{x}_p)\overline{\widehat{p}'(\boldsymbol{x}_l,\omega';\boldsymbol{x}_q)} \\ & \times \mathrm{e}^{-\mathrm{i}\omega(\mathcal{T}_0(\boldsymbol{x}_k,\boldsymbol{x}_S)+\mathcal{T}_0(\boldsymbol{x}_S,\boldsymbol{x}_p))+\mathrm{i}\omega'(\mathcal{T}_0(\boldsymbol{x}_l,\boldsymbol{x}_S)+\mathcal{T}_0(\boldsymbol{x}_S,\boldsymbol{x}_q))}\,\mathrm{d}\omega\mathrm{d}\omega' .\end{aligned}$$

Active imaging in quiescent random medium



Constant speed of sound

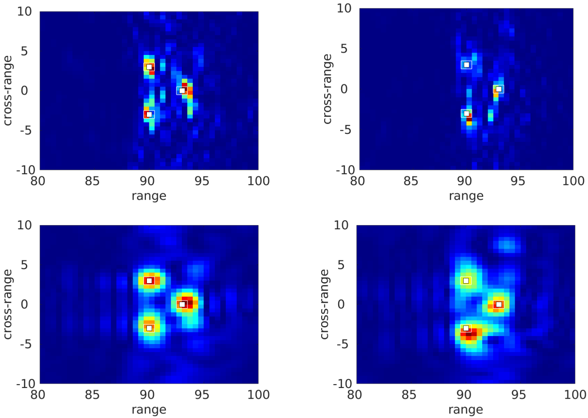


Random speed of sound

Active imaging in a quiescent medium with random speed of sound with average $\underline{c} = 3000$ *m/s* $(\sigma = 3\%)$*. Time traces recorded at the receivers (time in units of* $L/\underline{c}$*, position in units of* λ_0*).*

Gay-Bonnet-Peyret-Savin-Garnier *J. Sound Vib.* **494**, 115852 (2021)

Active imaging in quiescent random medium



Active imaging in a moving medium with random speed of sound with average $\underline{c} = 3000$ *m/s* ($\sigma = 3\%$)*. Comparison of KM (top) and CINT (bottom) images for two realizations of the random medium.*

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Acoustic waves in heterogeneous flow

- Full nonlinear Euler equations for an ideal fluid flow in the absence of friction, heat conduction, or heat production:

$$
\begin{aligned}
\frac{\mathrm{d}\varrho}{\mathrm{d}t} &= -\varrho \boldsymbol{\nabla} \cdot \boldsymbol{v} + m\,, \\
\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} &= -\frac{1}{\varrho} \boldsymbol{\nabla} p + \boldsymbol{f}\,, \\
\frac{\mathrm{d}s}{\mathrm{d}t} &= 0\,,
\end{aligned}
$$

  where ϱ: the fluid density; $\boldsymbol{v}$: the particle velocity, s: the specific entropy; and p: the thermodynamic pressure given by the equation of state $p = p(\varrho, s)$.

- The usual convective derivative following the particle paths:

$$
\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\,.
$$

- Isentropic flow (the "adiabatic equation"):

$$
\frac{\mathrm{d}p}{\mathrm{d}t} = c^2 \frac{\mathrm{d}\varrho}{\mathrm{d}t}\,, \quad c^2(\varrho, s) = \left(\frac{\partial p}{\partial \varrho}\right)_s\,,
$$

  where c: speed of sound.

Linearization about a steady flow

- Linearization about a stationary ambient flow (subscript 0):

$$\begin{aligned}
\varrho(\boldsymbol{x},t) &= \varrho_0(\boldsymbol{x}) + \varrho'(\boldsymbol{x},t)\,,\\
\boldsymbol{v}(\boldsymbol{x},t) &= \boldsymbol{v}_0(\boldsymbol{x}) + \boldsymbol{v}'(\boldsymbol{x},t)\,,\\
p(\boldsymbol{x},t) &= p_0(\boldsymbol{x}) + p'(\boldsymbol{x},t)\,,\\
c^2(\boldsymbol{x},t) &= c_0^2(\boldsymbol{x}) + (c^2)'(\boldsymbol{x},t)\,.
\end{aligned}$$

- The ambient quantities satisfy the steady equations:

$$\begin{aligned}
(\boldsymbol{v}_0\cdot\boldsymbol{\nabla})\varrho_0 &= -\varrho_0\boldsymbol{\nabla}\cdot\boldsymbol{v}_0\,,\\
(\boldsymbol{v}_0\cdot\boldsymbol{\nabla})\boldsymbol{v}_0 &= -\frac{1}{\varrho_0}\boldsymbol{\nabla}p_0\,,\\
(\boldsymbol{v}_0\cdot\boldsymbol{\nabla})p_0 &= c_0^2(\boldsymbol{v}_0\cdot\boldsymbol{\nabla})\varrho_0\,.
\end{aligned}$$

Linearized Euler equations

- The acoustic quantities satisfy the linearized Euler equations (LEE):

$$\begin{aligned}
\frac{d\varrho'}{dt} + (\boldsymbol{v}' \cdot \boldsymbol{\nabla})\varrho_0 &= -\varrho'\boldsymbol{\nabla} \cdot \boldsymbol{v}_0 - \varrho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v}' + m\,,\\
\frac{d\boldsymbol{v}'}{dt} + (\boldsymbol{v}' \cdot \boldsymbol{\nabla})\boldsymbol{v}_0 &= \frac{\varrho'}{\varrho_0^2}\boldsymbol{\nabla} p_0 - \frac{1}{\varrho_0}\boldsymbol{\nabla} p' + \boldsymbol{f}\,,\\
\frac{dp'}{dt} + \boldsymbol{v}' \cdot \boldsymbol{\nabla} p_0 &= c_0^2\left(\frac{d\varrho'}{dt} + \boldsymbol{v}' \cdot \boldsymbol{\nabla}\varrho_0\right) + (c^2)'\boldsymbol{v}_0 \cdot \boldsymbol{\nabla}\varrho_0\,,
\end{aligned}$$

where here and throughout:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \boldsymbol{v}_0 \cdot \boldsymbol{\nabla}\,.$$

- If the ambient flow velocity $\boldsymbol{v}_0$ is a non vanishing constant, the convected wave equation is derived:

$$\boxed{\frac{d}{dt}\left(\frac{1}{c_0^2}\frac{dp'}{dt}\right) - \varrho_0 \boldsymbol{\nabla} \cdot \left(\frac{1}{\varrho_0}\boldsymbol{\nabla} p'\right) = \frac{dm}{dt} - \varrho_0 \boldsymbol{\nabla} \cdot \boldsymbol{f}}\,,$$

as first proposed by Blokhintzev (1946) for a steady irrotational ambient flow.

Blokhintzev *J. Acoust. Soc. Am.* **18**(2), 322 (1946)
Pierce *J. Acoust. Soc. Am.* **87**(6), 2292 (1990)

Green's function in homogeneous medium

- By the change of variables $\boldsymbol{X} = \boldsymbol{x} - \boldsymbol{v}_0 t$, $P'(\boldsymbol{X}, t) = p'(\boldsymbol{X} + \boldsymbol{v}_0 t, t)$, $\mathcal{M}(\boldsymbol{X}, t) = m(\boldsymbol{X} + \boldsymbol{v}_0 t, t)$, and $\boldsymbol{\mathcal{F}}(\boldsymbol{X}, t) = \boldsymbol{f}(\boldsymbol{X} + \boldsymbol{v}_0 t, t)$, the convected wave equation in homogeneous flow reads:

$$\frac{1}{c_0^2}\frac{\partial^2 P'}{\partial t^2} - \Delta P' = \frac{\partial \mathcal{M}}{\partial t} - \varrho_0 \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}} \,.$$

- The Green's function $G_0(\boldsymbol{X}, \boldsymbol{X}_S, t)$ solves the wave equation with vanishing initial conditions:

$$\frac{1}{c_0^2}\frac{\partial^2 G_0}{\partial t^2} - \Delta G_0 = \delta(t)\delta(\boldsymbol{X} - \boldsymbol{X}_S) \,,$$

  and reads:

$$G_0(\boldsymbol{X}, \boldsymbol{X}_S, t) = \mathcal{G}_0(\boldsymbol{X} - \boldsymbol{X}_S, t) = \begin{cases} \dfrac{1}{4\pi\|\boldsymbol{X} - \boldsymbol{X}_S\|}\,\delta\left(t - \dfrac{\|\boldsymbol{X} - \boldsymbol{X}_S\|}{c_0}\right) & \text{if } t > 0 \,, \\ 0 & \text{if } t \leq 0 \,. \end{cases}$$

- **Example**: acoustic pressure induced by a monopole source with time-dependent spatial compact support $\Omega_S(t)$

$$P'(\boldsymbol{X}, t) = \frac{\partial}{\partial t}\int \mathrm{d}t' \int_{\Omega_S(t')} \frac{\mathcal{M}(\boldsymbol{X}', t')}{4\pi\|\boldsymbol{X} - \boldsymbol{X}'\|}\,\delta\left(t' - t + \frac{\|\boldsymbol{X} - \boldsymbol{X}'\|}{c_0}\right)\mathrm{d}\boldsymbol{X}' \,.$$

Monopole source in homogeneous flow

- **Monopole source**: $m(\boldsymbol{x}, t) = A(t)\delta(\boldsymbol{x} - \boldsymbol{x}_S)$, $-\varrho_0 \boldsymbol{\nabla} \cdot \boldsymbol{f} \equiv 0$, then

$$p'(\boldsymbol{x}, t) = \frac{A'\left(t - \gamma_D(\boldsymbol{x} - \boldsymbol{x}_S, \boldsymbol{M})\mathcal{T}_0(\boldsymbol{x}, \boldsymbol{x}_S)\right)}{4\pi\|\boldsymbol{x} - \boldsymbol{x}_S\|\sqrt{1 - M^2 + \left(\boldsymbol{M} \cdot \frac{\boldsymbol{x}-\boldsymbol{x}_S}{\|\boldsymbol{x}-\boldsymbol{x}_S\|}\right)^2}} ;$$

- $\boldsymbol{M} = \frac{\boldsymbol{v}_0}{c_0}$ is the Mach number, $M = \|\boldsymbol{M}\|$, and γ_D is a Doppler factor which compensates for the shift of the wave arrival times induced by the flow moving at the constant Mach $\boldsymbol{M}$:

$$\boxed{\gamma_D(\boldsymbol{x}, \boldsymbol{M}) = \frac{1}{1 - M^2}\left[\sqrt{1 - M^2 + \left(\boldsymbol{M} \cdot \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right)^2} - \boldsymbol{M} \cdot \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right]}.$$

Dipole source in homogeneous flow

- **Dipole (momentum) source**: $m \equiv 0$, $-\varrho_0 \boldsymbol{\nabla} \cdot \boldsymbol{f} = A(t)(\delta(\boldsymbol{x} - \boldsymbol{x}_S) - \delta(\boldsymbol{x} - \boldsymbol{x}_S - \boldsymbol{d}))$, then whenever $\|\boldsymbol{d}\| \ll \|\boldsymbol{x} - \boldsymbol{x}_S\|$

$$p'(\boldsymbol{x}, t) \simeq \boldsymbol{d} \cdot \boldsymbol{\nabla} \left[\frac{A\left(t - \gamma_D(\boldsymbol{x} - \boldsymbol{x}_S, \boldsymbol{M}) \mathcal{T}_0(\boldsymbol{x}, \boldsymbol{x}_S)\right)}{4\pi \|\boldsymbol{x} - \boldsymbol{x}_S\| \sqrt{1 - M^2 + \left(\boldsymbol{M} \cdot \frac{\boldsymbol{x} - \boldsymbol{x}_S}{\|\boldsymbol{x} - \boldsymbol{x}_S\|}\right)^2}} \right] ;$$

- $\boldsymbol{M} = \frac{\boldsymbol{v}_0}{c_0}$ is the Mach number, $M = \|\boldsymbol{M}\|$, and γ_D is a Doppler factor which compensates for the shift of the wave arrival times induced by the flow moving at the constant Mach $\boldsymbol{M}$:

$$\boxed{\gamma_D(\boldsymbol{x}, \boldsymbol{M}) = \frac{1}{1 - M^2} \left[\sqrt{1 - M^2 + \left(\boldsymbol{M} \cdot \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right)^2} - \boldsymbol{M} \cdot \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \right]}.$$

Outline

1. Some issues with heterogeneous flows...
2. Imaging in quiescent random medium
3. Convected acoustic waves
4. Imaging in moving random medium
5. Outlook

Active imaging functionals

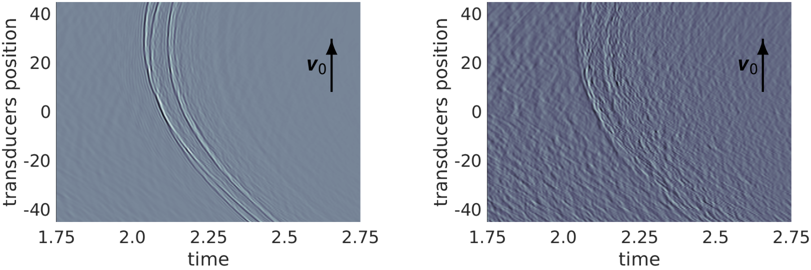
- **DATA**: pressure signals $\{t \mapsto p'(\boldsymbol{x}_k, t; \boldsymbol{x}_p),\ 1 \leq p \leq N_e\}_{k=1}^{N_c}$ at N_c sensors from N_e emitters;
- **OBJECTIVE**: localize a reflector at $\boldsymbol{y}_S = \arg\max_{\boldsymbol{x}_S \in \Omega_S} \mathcal{J}(\boldsymbol{x}_S)$ in some search region Ω_S.

$$\mathcal{J}_{\mathrm{KM}}(\boldsymbol{x}_S) = \sum_{k=1}^{N_c}\sum_{p=1}^{N_e} p'\left(\boldsymbol{x}_k, \gamma_D(\boldsymbol{x}_k - \boldsymbol{x}_S, \boldsymbol{M})\mathcal{T}_0(\boldsymbol{x}_k, \boldsymbol{x}_S) + \gamma_D(\boldsymbol{x}_S - \boldsymbol{x}_p, \boldsymbol{M})\mathcal{T}_0(\boldsymbol{x}_S, \boldsymbol{x}_p); \boldsymbol{x}_p\right) ,$$

$$\begin{aligned}\mathcal{J}_{\mathrm{CINT}}(\boldsymbol{x}_S; \Omega_d, X_d) = & \sum_{\substack{k,\, l\, =\, 1\\ \|\boldsymbol{x}_k - \boldsymbol{x}_l\| \leq X_d}}^{N_c} \sum_{\substack{p,\, q\, =\, 1\\ \|\boldsymbol{x}_p - \boldsymbol{x}_q\| \leq X_d}}^{N_e} \iint_{|\omega-\omega'|\leq\Omega_d} \widehat{p}'(\boldsymbol{x}_k, \omega; \boldsymbol{x}_p)\overline{\widehat{p}'(\boldsymbol{x}_l, \omega'; \boldsymbol{x}_q)} \\ & \times \mathrm{e}^{-\mathrm{i}\omega(\gamma_D(\boldsymbol{x}_k-\boldsymbol{x}_S,\boldsymbol{M})\mathcal{T}_0(\boldsymbol{x}_k,\boldsymbol{x}_S)+\gamma_D(\boldsymbol{x}_S-\boldsymbol{x}_p,\boldsymbol{M})\mathcal{T}_0(\boldsymbol{x}_S,\boldsymbol{x}_p))} \\ & \times \mathrm{e}^{+\mathrm{i}\omega'(\gamma_D(\boldsymbol{x}_l-\boldsymbol{x}_S,\boldsymbol{M})\mathcal{T}_0(\boldsymbol{x}_l,\boldsymbol{x}_S)+\gamma_D(\boldsymbol{x}_S-\boldsymbol{x}_q,\boldsymbol{M})\mathcal{T}_0(\boldsymbol{x}_S,\boldsymbol{x}_q))}\, \mathrm{d}\omega \mathrm{d}\omega' ;\end{aligned}$$

- **Remark**: in a quiescent medium $\boldsymbol{M} = \boldsymbol{0}$, and $\gamma_D(\boldsymbol{x}, \boldsymbol{0}) = 1$.

Active imaging in moving random medium

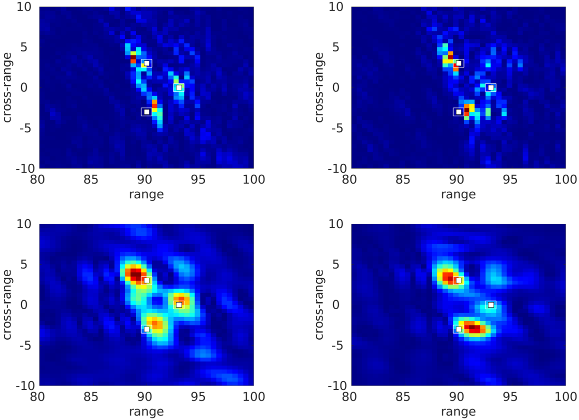


Constant speed of sound *Random speed of sound*

Active imaging in a moving medium with random velocity with average $M = 0.3$ *in the cross-range upward direction and random speed of sound with average* $\underline{c} = 3000$ *m/s* ($\sigma = 3\%$). *Time traces recorded at the receivers (time in units of* $L/\underline{c}$*, position in units of* λ_0*).*

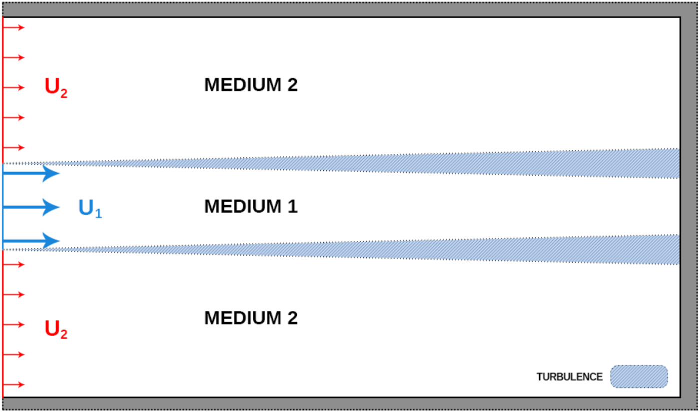
Gay-Bonnet-Peyret-Savin-Garnier *J. Sound Vib.* **494**, 115852 (2021)

Active imaging in moving random medium



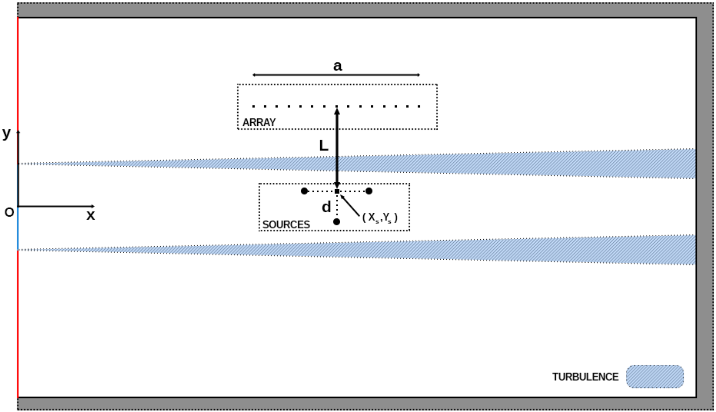
Active imaging in a moving medium with random velocity with average M $= 0.3$ *in the cross-range upward direction and random speed of sound with average* $\underline{c} = 3000$ *m/s* ($\sigma = 3\%$)*. Comparison of KM (top) and CINT (bottom) images for two realizations of the random medium.*

Passive imaging through a mixing layer



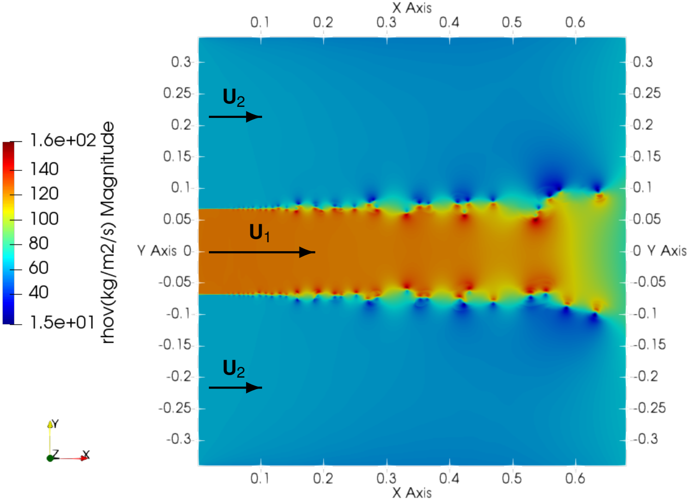
Synthetic mixing layer configuration: —— *inflow boundary condition for the central zone with* $U_1 = 100$ *m/s;* —— *inflow boundary condition outside the central zone with* $U_2 = 50$ *m/s;* ▬ *non-reflecting boundary conditions.*

Passive imaging through a mixing layer

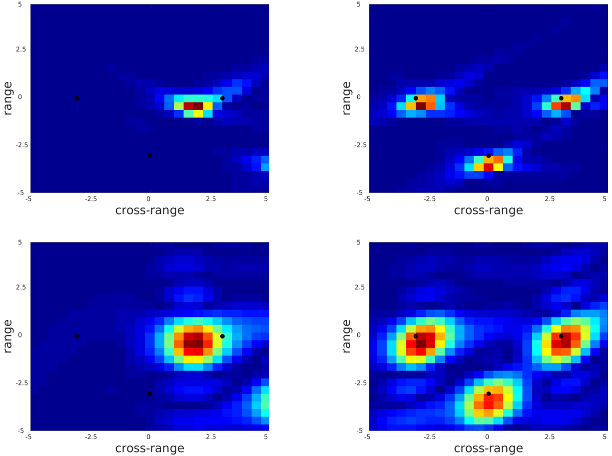


Synthetic mixing layer configuration: imaging setup. a is the aperture of the sensors array, L is the distance between the central sensor and the mid-point (X_s, Y_s) *of the two upper sources which are d apart from each other, and the lower source is also at the distance d from the upper sources.*

Passive imaging through a mixing layer



Passive imaging through a mixing layer



Passive imaging of three sources through a mixing layer. Influence of the Doppler compensation factor γ_D on KM (top) and CINT (bottom) imaging functions: w/o. compensation (left); w. Doppler compensation factor (right). The source locations are shown by dots ●

Outline

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Outlook

- **Resolution**: for signals of frequency bandwidth B and wavelength λ_0; sensor array $\mathcal{A}$ of aperture a and range $L \gg \frac{c_0}{B}$; and CINT effective bandwidth $\Omega_d < B$ and effective aperture $X_d < a$.

$\mathcal{J}$	Range	Cross-range
KM	$\dfrac{c_0}{B}$	$\dfrac{\lambda_0}{a}L$
CINT	$\dfrac{c_0}{\Omega_d}$	$\dfrac{\lambda_0}{X_d}L$
Möbius	$\sqrt{\epsilon} \times \dfrac{c_0}{\Omega_d}$	$\sqrt{\epsilon} \times \dfrac{\lambda_0}{X_d}L$

- Möbius transform:

$$\mathcal{J}_\epsilon(\boldsymbol{x}_S) = \frac{\epsilon}{1-(1-\epsilon)\hat{\mathcal{J}}(\boldsymbol{x}_S)},$$

where $\hat{\mathcal{J}}(\boldsymbol{x}_S) = \frac{\mathcal{J}(\boldsymbol{x}_S)}{\max\limits_{\boldsymbol{x}_S \in \Omega_S} \mathcal{J}(\boldsymbol{x}_S)}$.

Borcea-Papanicolaou-Tsogka *Inverse Probl.* **22**(4), 1405 (2006)
Borcea-Garnier-Papanicolaou-Tsogka *Inverse Probl.* **27**(8), 085004 (2011)
Kim-Tsogka *Radio Sci.* **57**(11), e2022RS007572 (2022)

Outlook

- **Statistical stability** of CINT: although the image is formed with signals gathered from one realization of the medium, without ensemble averaging, it depends only on the statistics of the medium and not the realization—the image is self-averaging.
- KM imaging signal-to-noise ratio (SNR):

$$\mathrm{SNR} = \frac{|\mathbb{E}\left\{\mathcal{J}_{\mathrm{KM}}(\boldsymbol{y}_S)\right\}|}{\sqrt{\mathrm{Var}\left\{\mathcal{J}_{\mathrm{KM}}(\boldsymbol{y}_S)\right\}}} \approx \exp\left[-\frac{1}{2}\left(\frac{\omega_0^2}{\Omega_c^2+B^2}\right)\right] \ll 1\,;$$

  however, KM imaging remains robust with respect to additive measurement noise.
- CINT imaging SNR:

$$\mathrm{SNR} = \frac{|\mathbb{E}\left\{\mathcal{J}_{\mathrm{CINT}}(\boldsymbol{y}_S;\Omega_d,X_d)\right\}|}{\sqrt{\mathrm{Var}\left\{\mathcal{J}_{\mathrm{CINT}}(\boldsymbol{y}_S;\Omega_d,X_d)\right\}}} \approx \frac{|\mathcal{A}|}{\langle n^4\rangle_{\mathcal{A}}}\left[1+\left(\frac{\Omega_c}{\Omega_d}\right)^2+\frac{1}{2}\left(\frac{\Omega_c}{B}\right)^2\right] > 1\,.$$

- **Proof**: forward scattering/parabolic approximation model, or random travel time model for $\sigma \ll 1$,

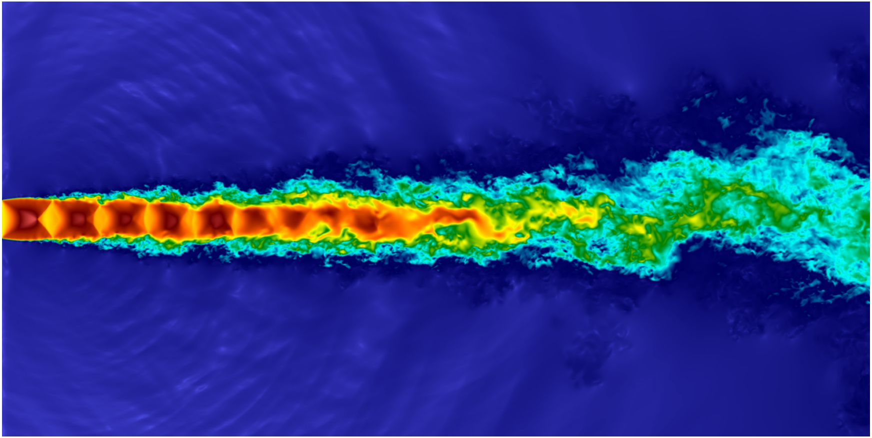
$$\mathcal{T}(\boldsymbol{x},\boldsymbol{x}_S) = \int_0^1 \frac{\mathrm{d}s}{c_0(\boldsymbol{x}_S+(\boldsymbol{x}-\boldsymbol{x}_S)s)} \simeq \mathcal{T}_0(\boldsymbol{x},\boldsymbol{x}_S)\left[1+\frac{\sigma}{2}\int_0^1 n\left(\frac{\boldsymbol{x}_S+(\boldsymbol{x}-\boldsymbol{x}_S)s}{\ell}\right)\mathrm{d}s\right]\,,$$

  where the decoherency frequency/length are $(\pi\sigma\frac{\Omega_c}{\omega_0})^2 = \frac{\lambda_0^2}{\sqrt{2\pi}\ell L} \lesssim \sigma^2$, $(\pi\sigma\frac{X_c}{\lambda_0})^2 = \frac{3\ell}{\sqrt{2\pi}L} \ll 1$ (used in *e.g.* adaptive optics to approximate wavefront distorsions due to propagation in turbulent media).

Borcea-Papanicolaou-Tsogka *Inverse Probl.* **22**(4), 1405 (2006)
Borcea-Garnier-Papanicolaou-Tsogka *Inverse Probl.* **27**(8), 085004 (2011)

Outlook

- Imaging through jet flows ($M \simeq 0.3$);



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Outlook

- Imaging moving objects ($M \simeq 0.02$) through the atmosphere, *e.g.* small space debris (10 cm) in low Earth orbits (200-2000 km).