High-frequency electromagnetic wave propagation in dispersive bianisotropic media

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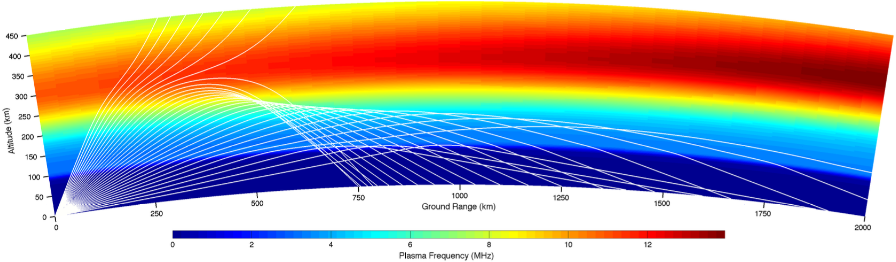
Waves 2024, Berlin, July 1st, 2024

Ray tracing in the ionosphere



Cervera-Harris, in *2011 XXXth URSI General Assembly & Scientific Symposium*, 13-20 August 2011, Istanbul (2011)

Waves at high frequencies (HF)

- Let $c>0$ and consider the (vector) wave equation:

$$\frac{1}{c^2}\frac{\partial^2\boldsymbol{u}_\epsilon}{\partial t^2}-\Delta\boldsymbol{u}_\epsilon=\boldsymbol{0}\,,\quad \boldsymbol{x}\in\mathbb{R}^3\,,\ t>0\,,$$

$$\boldsymbol{u}_\epsilon(\boldsymbol{x},0)=\boldsymbol{0}\,,\ \frac{\partial\boldsymbol{u}_\epsilon}{\partial t}(\boldsymbol{x},0)=\boldsymbol{u}_\epsilon^{\mathrm{HF}}(\boldsymbol{x})\,,\quad \boldsymbol{x}\in\mathbb{R}^3\,,$$

  where for $0<\epsilon\ll 1$:

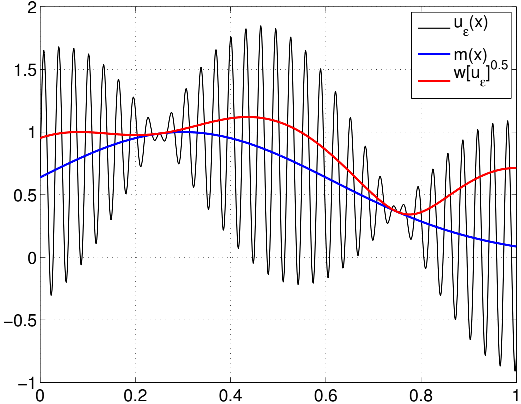
$$\boldsymbol{u}_\epsilon^{\mathrm{HF}}(\boldsymbol{x})=\Re\mathfrak{e}\{\boldsymbol{u}_I(\boldsymbol{x})\mathrm{e}^{\frac{\mathrm{i}}{\epsilon}S_I(\boldsymbol{x})}\}\,,$$

  and, say, $\boldsymbol{u}_I\in L^2(\mathbb{R}^3)$, $S_I\in L^1_{\mathrm{loc}}(\mathbb{R}^3)$, and $\boldsymbol{\nabla}S_I\in L^1_{\mathrm{loc}}(\mathbb{R}^3)$;
- Then for $t>0$ Kirchhoff's formula yields:

$$\begin{aligned}\boldsymbol{u}_\epsilon(\boldsymbol{x},t)&=\frac{\sin(\sqrt{-c^2\Delta}t)}{\sqrt{-c^2\Delta}}\star\boldsymbol{u}_\epsilon^{\mathrm{HF}}(\boldsymbol{x})\\&=\int_{\mathbb{S}^2}t\boldsymbol{u}_\epsilon^{\mathrm{HF}}(\boldsymbol{x}-c\hat{\boldsymbol{p}}t)\frac{d\Omega(\hat{\boldsymbol{p}})}{4\pi}\,,\end{aligned}$$

  which as no "strong" limit as $\epsilon\to 0$.

Why quadratic observables?

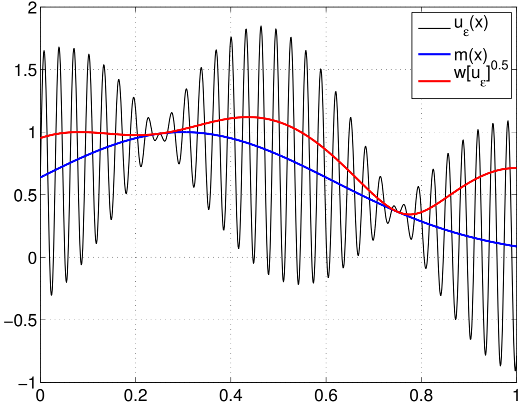


- Let $m, a \in L^p$ $(1 \leq p < +\infty)$ and:

$$u_\epsilon(x) = m(x) + a(x) \sin \frac{x}{\epsilon} \, ,$$

then $(u_\epsilon) \rightharpoonup m$ weakly in L^p as $\epsilon \to 0$, but (u_ϵ) has no strong limit in L^p.

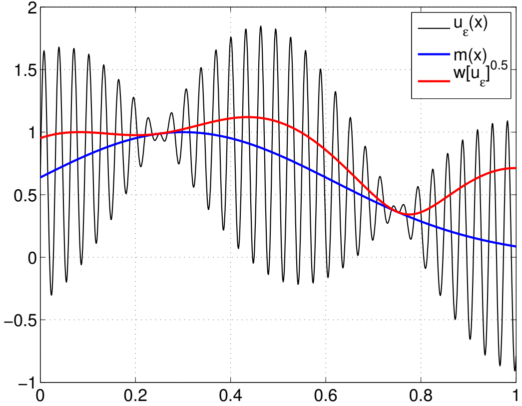
Why quadratic observables?



- Now for any observable $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, say:

$$\lim_{\epsilon\to 0}\left(\varphi(x)u_\epsilon,u_\epsilon\right)_{L^2}=\int_{\mathbb{R}}\varphi(x)\left((m(x))^2+\frac{1}{2}(a(x))^2\right)dx\,.$$

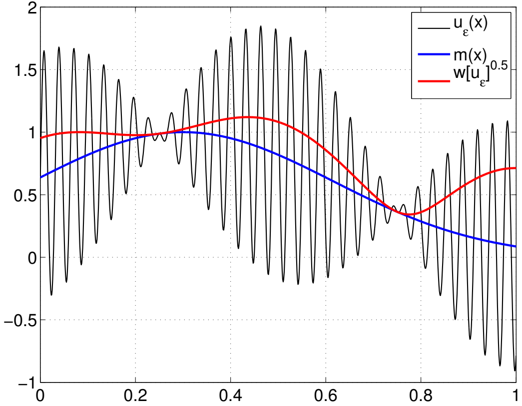
Why quadratic observables?



- Take an observable of the form:

$$\varphi(x,\partial_x)u_\epsilon(x)=\frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}k\cdot x}\varphi(x,\mathrm{i}k)\widehat{u}_\epsilon(k)dk\,.$$

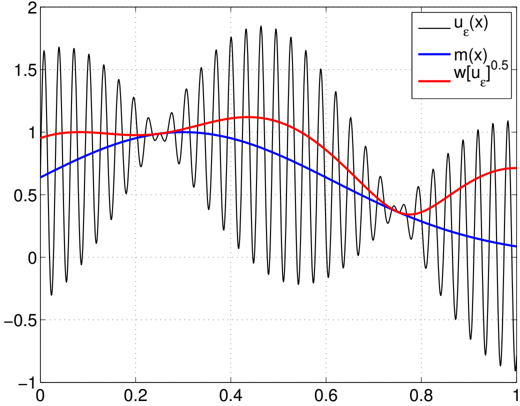
Why quadratic observables?



- Take an observable of the form:

$$\varphi(x, \epsilon\partial_x)u_\epsilon(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}k\cdot x}\varphi(x, \mathrm{i}\epsilon k)\widehat{u_\epsilon}(k)dk\,.$$

Why quadratic observables?



- Then:

$$\lim_{\epsilon\to 0}\left(\varphi(x,\epsilon\partial_x)u_\epsilon,u_\epsilon\right)_{L^2}=\iint_{\mathbb{R}^2}\varphi(x,\mathrm{i}k)W[u_\epsilon](dx,dk)\,,$$

where $W[u_\epsilon]$ is the (positive) Wigner measure of (u_ϵ).

Wigner measure

- Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^3_{\boldsymbol{x}} \times \mathbb{R}^3_{\boldsymbol{k}})$ (a smooth observable), $\mathbf{D}_{\boldsymbol{x}} = \frac{1}{\mathrm{i}}\boldsymbol{\nabla}_{\boldsymbol{x}}$, and for $\boldsymbol{u} \in L^2(\mathbb{R}^3)$:

$$\begin{aligned}\varphi(\boldsymbol{x}, \epsilon\mathbf{D}_{\boldsymbol{x}})\boldsymbol{u}(\boldsymbol{x}) &= \varphi(\boldsymbol{x}, \epsilon\mathbf{D}_{\boldsymbol{x}})\left(\int \mathrm{e}^{\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{k}}\widehat{\boldsymbol{u}}(\boldsymbol{k})\,d\boldsymbol{k}\right) \\ &:= \int \mathrm{e}^{\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{k}}\varphi(\boldsymbol{x}, \epsilon\boldsymbol{k})\widehat{\boldsymbol{u}}(\boldsymbol{k})\,\frac{d\boldsymbol{k}}{(2\pi)^3}\,.\end{aligned}$$

- Then if $(\boldsymbol{u}_\epsilon)$ is bounded in L^2, $\forall\varphi$ a smooth observable:

$$\boxed{\lim_{\epsilon\to 0}(\varphi(\boldsymbol{x}, \epsilon\mathbf{D}_{\boldsymbol{x}})\boldsymbol{u}_\epsilon, \boldsymbol{u}_\epsilon)_{L^2} = \iint \varphi(\boldsymbol{x}, \boldsymbol{k})W[\boldsymbol{u}_\epsilon](d\boldsymbol{x}, d\boldsymbol{k})}\,,$$

  and the positive measure $W[\boldsymbol{u}_\epsilon]$ of the sequence $(\boldsymbol{u}_\epsilon)$ always exists—up to extracting a sub-sequence if need be.

Lions-Paul *Rev. Mat. Iberoamericana* **9**(3), 553 (1993)

Gérard-Markowich-Mauser-Poupaud *Commun. Pure Appl. Math.* **L**(4), 323 (1997)

Martinez *An Introduction to Semiclassical and Microlocal Analysis*, Springer, Berlin (2002)

Zworski *Semiclassical Analysis*, American Mathematical Society, Providence RI (2012)

- WKB state: $\boldsymbol{u}_\epsilon(\boldsymbol{x}) = \boldsymbol{u}_I(\boldsymbol{x})\mathrm{e}^{\frac{\mathrm{i}}{\epsilon}S_I(\boldsymbol{x})}$, then $W[\boldsymbol{u}_\epsilon] = \|\boldsymbol{u}_I(\boldsymbol{x})\|^2\delta(\boldsymbol{k} - \boldsymbol{\nabla}_{\boldsymbol{x}}S_I)$;
- Localized state in $\mathbb{R}^3$: $u_\epsilon(\boldsymbol{x}) = \epsilon^{-\frac{3}{2}}\phi(\frac{\boldsymbol{x}-\boldsymbol{x}_0}{\epsilon})$, then $W[u_\epsilon] = \frac{1}{(2\pi)^3}\left|\widehat{\phi}(\boldsymbol{k})\right|^2\delta(\boldsymbol{x} - \boldsymbol{x}_0)$;
- Coherent state in $\mathbb{R}^3$: a combination of WKB and localized states $u_\epsilon(\boldsymbol{x}) = \epsilon^{-\frac{3}{2}}\phi(\frac{\boldsymbol{x}-\boldsymbol{x}_0}{\epsilon})\mathrm{e}^{\frac{\mathrm{i}}{\epsilon}\boldsymbol{k}_0\cdot\boldsymbol{x}}$, then $W[u_\epsilon] = \frac{1}{(2\pi)^3}\left|\widehat{\phi}(\boldsymbol{k} - \boldsymbol{k}_0)\right|^2\delta(\boldsymbol{x} - \boldsymbol{x}_0)$;
- Wave equation with WKB initial conditions:

$$\begin{aligned} W[\boldsymbol{u}_\epsilon] = {} & \frac{1}{2}\|\boldsymbol{u}_I(\boldsymbol{x} - c\hat{\boldsymbol{k}}t)\|^2\delta(\boldsymbol{k} - \boldsymbol{\nabla}_{\boldsymbol{x}}S_I(\boldsymbol{x} - c\hat{\boldsymbol{k}}t)) \\ & + \frac{1}{2}\|\boldsymbol{u}_I(\boldsymbol{x} + c\hat{\boldsymbol{k}}t)\|^2\delta(\boldsymbol{k} - \boldsymbol{\nabla}_{\boldsymbol{x}}S_I(\boldsymbol{x} + c\hat{\boldsymbol{k}}t))\,. \end{aligned}$$

Wigner transform

- For tempered distributions $\boldsymbol{u}, \boldsymbol{v}$ defined on $\mathbb{R}^3$, their Wigner transform is:

$$\boldsymbol{W}_\epsilon[\boldsymbol{u},\boldsymbol{v}](\boldsymbol{x},\boldsymbol{k}) = \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y}} \boldsymbol{u}(\boldsymbol{x}-\epsilon\boldsymbol{y})\boldsymbol{v}(\boldsymbol{x})^* \frac{d\boldsymbol{y}}{(2\pi)^3}\,. \tag{1}$$

  We denote by $\boldsymbol{W}_\epsilon[\boldsymbol{u}] := \boldsymbol{W}_\epsilon[\boldsymbol{u},\boldsymbol{u}]$ the self-Wigner transform of $\boldsymbol{u}$.
- Then the following trace formula holds:

$$(\varphi(\boldsymbol{x},\epsilon\mathbf{D}_{\boldsymbol{x}})\boldsymbol{u},\boldsymbol{v})_{L^2} = \operatorname{Tr}\iint \varphi(\boldsymbol{x},\boldsymbol{k})\boldsymbol{W}_\epsilon[\boldsymbol{u},\boldsymbol{v}](d\boldsymbol{x},d\boldsymbol{k})\,,$$

  such that for a bounded sequence $(\boldsymbol{u}_\epsilon)$ in L^2:

$$\lim_{\epsilon\to 0}(\varphi(\boldsymbol{x},\epsilon\mathbf{D}_{\boldsymbol{x}})\boldsymbol{u}_\epsilon,\boldsymbol{u}_\epsilon)_{L^2} = \operatorname{Tr}\iint \varphi(\boldsymbol{x},\boldsymbol{k})\boldsymbol{W}[\boldsymbol{u}_\epsilon](d\boldsymbol{x},d\boldsymbol{k})$$

  and $W[\boldsymbol{u}_\epsilon] \equiv \operatorname{Tr}\boldsymbol{W}[\boldsymbol{u}_\epsilon]$.

Maxwell's system in isotropic media

- Maxwell's equations in an isotropic medium where $\varepsilon(\boldsymbol{x})$ is the space-dependent dielectric permittivity, and $\mu(\boldsymbol{x})$ is the space-dependent relative magnetic permeability:

$$\frac{\partial}{\partial t}\begin{bmatrix}\varepsilon\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{0} & \mu\boldsymbol{I}\end{bmatrix}\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}+\boldsymbol{\nabla}\times\begin{bmatrix}\boldsymbol{0} & -\boldsymbol{I}\\ \boldsymbol{I} & \boldsymbol{0}\end{bmatrix}\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}=\boldsymbol{0}\,.$$

- If $\boldsymbol{\nabla}\cdot\varepsilon\boldsymbol{E}=0$, $\boldsymbol{\nabla}\cdot\mu\boldsymbol{H}=0$ hold at some initial time $t=0$, then they hold at all times.
- The energy density of electromagnetic waves and Poynting vector:

$$\begin{aligned}\mathcal{E}(\boldsymbol{x},t)&=\frac{1}{2}\varepsilon(\boldsymbol{x})\left|\boldsymbol{E}(\boldsymbol{x},t)\right|^2+\frac{1}{2}\mu(\boldsymbol{x})\left|\boldsymbol{H}(\boldsymbol{x},t)\right|^2\,,\\ \boldsymbol{\mathcal{F}}(\boldsymbol{x},t)&=\overline{\boldsymbol{E}(\boldsymbol{x},t)}\times\boldsymbol{H}(\boldsymbol{x},t)\,,\end{aligned}$$

such that:

$$\frac{\partial\mathcal{E}}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}=0\,,\quad\frac{d}{dt}\int\mathcal{E}(\boldsymbol{x},t)d\boldsymbol{x}=0\,.$$

Bi-anisotropic dielectric media

Maxwell's equations with dissipation

- Maxwell's equations for a dissipative, bi-anisotropic dielectric medium:

$$\frac{\partial}{\partial t}\left(\begin{bmatrix}\boldsymbol{\varepsilon}_0 & \boldsymbol{\xi}_0 \\ \boldsymbol{\xi}_0^* & \boldsymbol{\mu}_0\end{bmatrix} + \begin{bmatrix}\boldsymbol{\varepsilon}_d & \boldsymbol{\xi}_d \\ \boldsymbol{\zeta}_d & \boldsymbol{\mu}_d\end{bmatrix}\star_t\right)\begin{pmatrix}\boldsymbol{E} \\ \boldsymbol{H}\end{pmatrix} + \boldsymbol{\nabla}\times\begin{bmatrix}\boldsymbol{0} & -\boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{0}\end{bmatrix}\begin{pmatrix}\boldsymbol{E} \\ \boldsymbol{H}\end{pmatrix} = \boldsymbol{0}\,.$$

- HF setting $t \rightarrow \frac{t}{\epsilon}$, $\boldsymbol{x} \rightarrow \frac{\boldsymbol{x}}{\epsilon}$ with $\epsilon \rightarrow 0$:

$$\left(\epsilon \mathrm{D}_t + \boldsymbol{L}_0(\boldsymbol{x}, \epsilon \mathbf{D}_{\boldsymbol{x}}) + \frac{\epsilon}{\mathrm{i}}\boldsymbol{L}_1(\boldsymbol{x}, \epsilon \mathrm{D}_t)\right)\boldsymbol{u}_\epsilon = \boldsymbol{0}$$

for $\boldsymbol{u}_\epsilon = (\boldsymbol{E}_\epsilon, \boldsymbol{H}_\epsilon)$, where:

$$\boldsymbol{L}_0(\boldsymbol{x}, \boldsymbol{k}) = \boldsymbol{K}_0^{-1}(\boldsymbol{x})\boldsymbol{M}(\boldsymbol{k})\,,$$
$$\boldsymbol{L}_1(\boldsymbol{x}, \omega) = \mathrm{i}\omega\boldsymbol{K}_0^{-1}(\boldsymbol{x})\widehat{\boldsymbol{K}}_d(\boldsymbol{x}, \omega)\,,$$

and:

$$\boldsymbol{K}_0(\boldsymbol{x}) = \begin{bmatrix}\boldsymbol{\varepsilon}_0(\boldsymbol{x}) & \boldsymbol{\xi}_0(\boldsymbol{x}) \\ \boldsymbol{\xi}_0^*(\boldsymbol{x}) & \boldsymbol{\mu}_0(\boldsymbol{x})\end{bmatrix},\ \boldsymbol{K}_d(\boldsymbol{x}, t) = \begin{bmatrix}\boldsymbol{\varepsilon}_d(\boldsymbol{x}, t) & \boldsymbol{\xi}_d(\boldsymbol{x}, t) \\ \boldsymbol{\zeta}_d(\boldsymbol{x}, t) & \boldsymbol{\mu}_d(\boldsymbol{x}, t)\end{bmatrix},\ \boldsymbol{M}(\boldsymbol{k}) = \begin{bmatrix}\boldsymbol{0} & -\boldsymbol{k}\times \\ \boldsymbol{k}\times & \boldsymbol{0}\end{bmatrix}.$$

- This means that in HF range $(\boldsymbol{\varepsilon}_d \star_t \boldsymbol{E}_\epsilon)(\boldsymbol{x}, t) = \int \boldsymbol{\varepsilon}_d(\boldsymbol{x}, \frac{\tau}{\epsilon})\boldsymbol{E}_\epsilon(\boldsymbol{x}, t-\tau)d\tau$ *etc.*

- The 6×6 right eigenvalue problem $\boldsymbol{L}_0 \boldsymbol{b}_\alpha = \omega_\alpha \boldsymbol{b}_\alpha$, or:

$$\boldsymbol{M}(\boldsymbol{k})\boldsymbol{b}_\alpha(\boldsymbol{x},\boldsymbol{k}) = \omega_\alpha(\boldsymbol{x},\boldsymbol{k})\boldsymbol{K}_0(\boldsymbol{x})\boldsymbol{b}_\alpha(\boldsymbol{x},\boldsymbol{k})$$

with right eigenvectors $\boldsymbol{b}_\alpha = (\boldsymbol{b}_\alpha^1, \dots \boldsymbol{b}_\alpha^A)$ where the α-th eigenvalue has multiplicity A, and left eigenvectors $\boldsymbol{c}_\alpha = \boldsymbol{K}_0 \boldsymbol{b}_\alpha$ such that:

$$\boldsymbol{L}_0 = \sum_\alpha \omega_\alpha \boldsymbol{b}_\alpha \boldsymbol{c}_\alpha^* \,, \quad \boldsymbol{I} = \sum_\alpha \boldsymbol{b}_\alpha \boldsymbol{c}_\alpha^* \,.$$

- Then $\boldsymbol{W}[\boldsymbol{u}_\epsilon] = \sum_\alpha \delta(\omega + \omega_\alpha)\boldsymbol{b}_\alpha \mathbf{w}_\alpha \boldsymbol{b}_\alpha^*$ where $\mathbf{w}_\alpha = \boldsymbol{c}_\alpha^* \boldsymbol{W}[\boldsymbol{u}_\epsilon]\boldsymbol{c}_\alpha$ and:

$$\boxed{\partial_t \mathbf{w}_\alpha + \{\omega_\alpha, \mathbf{w}_\alpha\} + \boldsymbol{\Omega}_\alpha \mathbf{w}_\alpha + \mathbf{w}_\alpha \boldsymbol{\Omega}_\alpha^* = \mathbf{0}}\,,$$

where $\boldsymbol{\Omega}_\alpha = \boldsymbol{\ell}_\alpha + \boldsymbol{n}_\alpha$, with $\boldsymbol{\ell}_\alpha = \boldsymbol{c}_\alpha^* \boldsymbol{L}_1 \boldsymbol{b}_\alpha$ a $A \times A$ matrix accounting for dissipation, and $\boldsymbol{n}_\alpha$ a skew-symmetric $A \times A$ matrix:

$$\boldsymbol{n}_\alpha = \boldsymbol{c}_\alpha^* \left(\boldsymbol{\nabla}_{\boldsymbol{k}} \boldsymbol{L}_0 \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} \boldsymbol{b}_\alpha - \boldsymbol{\nabla}_{\boldsymbol{x}} \omega_\alpha \cdot \boldsymbol{\nabla}_{\boldsymbol{k}} \boldsymbol{b}_\alpha\right) - \frac{1}{2}(\boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \boldsymbol{\nabla}_{\boldsymbol{k}} \omega_\alpha)\boldsymbol{I}_A \,.$$

- Consider the bicharacteristics $t \mapsto (\boldsymbol{x}_\alpha(t), \boldsymbol{k}_\alpha(t))$ associated to $\omega_\alpha(\boldsymbol{x}, \boldsymbol{k})$ such that:

$$\begin{aligned}\frac{d\boldsymbol{x}_\alpha}{dt} &= \boldsymbol{\nabla}_{\boldsymbol{k}}\omega_\alpha(\boldsymbol{x}_\alpha(t), \boldsymbol{k}_\alpha(t))\,, \quad \boldsymbol{x}_\alpha(0) = \boldsymbol{x}\,,\\ \frac{d\boldsymbol{k}_\alpha}{dt} &= -\boldsymbol{\nabla}_{\boldsymbol{x}}\omega_\alpha(\boldsymbol{x}_\alpha(t), \boldsymbol{k}_\alpha(t))\,, \quad \boldsymbol{k}_\alpha(0) = \boldsymbol{k}\,,\end{aligned}$$

and $t \mapsto \boldsymbol{R}_\alpha(t)$ such that:

$$\frac{d\boldsymbol{R}_\alpha}{dt} = -\boldsymbol{\Omega}_\alpha(\boldsymbol{x}_\alpha(t), \boldsymbol{k}_\alpha(t))\boldsymbol{R}_\alpha(t)\,, \quad \boldsymbol{R}_\alpha(0) = \boldsymbol{I}_A\,;$$

- Then $\mathbf{w}_\alpha = \boldsymbol{R}_\alpha\tilde{\mathbf{w}}_\alpha\boldsymbol{R}_\alpha^*$ where $\frac{d}{dt}\tilde{\mathbf{w}}_\alpha(\boldsymbol{x}_\alpha(t), \boldsymbol{k}_\alpha(t), t) = \mathbf{0}$, such that:

$$\boxed{\mathbf{w}_\alpha(\boldsymbol{x}_\alpha(t), \boldsymbol{k}_\alpha(t), t) = \boldsymbol{R}_\alpha(t)\mathbf{w}_I(\boldsymbol{x}, \boldsymbol{k})\boldsymbol{R}_\alpha(t)^*}\,.$$

Example #1: Lorentz model with damping

- Lorentz model with damping in A. Raman, S. Fan, *Phys. Rev. Lett.* **104**, 087401 (2010):

$$\boldsymbol{K}_0 = \begin{bmatrix} \varepsilon_0 \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \mu_0 \boldsymbol{I} \end{bmatrix} \,, \quad \boldsymbol{L}_1(\omega) = \mathrm{i}\omega \begin{bmatrix} \widehat{\varepsilon}_1(\omega)\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \,,$$

where:

$$\widehat{\varepsilon}_1(\omega) = \frac{\omega_p^2}{\omega_0^2 - \omega^2 + \mathrm{i}\omega\Gamma}$$

is the susceptibility (in frequency domain), ω_p is the plasma frequency, ω_0 is a characteristic frequency for the motion of an electron, and Γ is a damping loss rate.

Example #1: Lorentz model with damping

- The three eigenvalues are $\omega_0 = 0$ ($\alpha =$ "0"), $\omega_+(\boldsymbol{k}) = +c_0\|\boldsymbol{k}\|$ ($\alpha =$ "+"), and $\omega_-(\boldsymbol{k}) = -c_0\|\boldsymbol{k}\|$ ($\alpha =$ "−"), each of multiplicity two ($A = 2$), $c_0 = 1/\sqrt{\varepsilon_0\mu_0}$, with associated eigenvectors:

$$\boldsymbol{b}_0^{(1)}(\hat{\boldsymbol{k}}) = \frac{1}{\sqrt{\varepsilon_0}}\begin{pmatrix}\hat{\boldsymbol{k}}\\ \mathbf{0}\end{pmatrix}\,,\quad \boldsymbol{b}_0^{(2)}(\hat{\boldsymbol{k}}) = \frac{1}{\sqrt{\mu_0}}\begin{pmatrix}\mathbf{0}\\ \hat{\boldsymbol{k}}\end{pmatrix}\,,$$

and for $j = 1,2$:

$$\boldsymbol{b}_\pm^{(j)}(\hat{\boldsymbol{k}}) = \begin{pmatrix}\dfrac{\hat{\boldsymbol{e}}_j}{\sqrt{2\varepsilon_0}}\\ \pm\dfrac{\hat{\boldsymbol{k}}\times\hat{\boldsymbol{e}}_j}{\sqrt{2\mu_0}}\end{pmatrix}\,,\quad \boldsymbol{c}_\pm^{(j)}(\hat{\boldsymbol{k}}) = \begin{pmatrix}\sqrt{\dfrac{\varepsilon_0}{2}}\hat{\boldsymbol{e}}_j\\ \pm\sqrt{\dfrac{\mu_0}{2}}\hat{\boldsymbol{k}}\times\hat{\boldsymbol{e}}_j\end{pmatrix}\,,\quad \hat{\boldsymbol{e}}_1 \perp \hat{\boldsymbol{e}}_2 \in \hat{\boldsymbol{k}}^\perp\,.$$

- Also $\boldsymbol{n}_\pm = \mathbf{0}$ for an homogeneous material, and the solution of the transport equation reads:

$$\mathrm{w}_\pm(\boldsymbol{x},\boldsymbol{k},t) = \mathrm{e}^{-\Re\mathfrak{e}\{\mathrm{i}c_0\|\boldsymbol{k}\|\widehat{\varepsilon}_1(c_0\|\boldsymbol{k}\|)\}t}\,\mathrm{w}_I(\boldsymbol{x}\mp c_0\hat{\boldsymbol{k}}t,\boldsymbol{k})\,,$$

where $\mathrm{w}_I(\boldsymbol{x},\boldsymbol{k})$, $(\boldsymbol{x},\boldsymbol{k})\in\mathcal{O}\times\mathbb{R}^3$, are some initial (incident) specific intensities.

Example #2: Single-resonance chiral metamaterial

- Chiral metamaterial model with damping in P.-G. Luan, Y.-T. Wang, S. Zhang, X. Zhang, *Optics Lett.* **36**(5), 675-677 (2011):

$$\boldsymbol{K}_0 = \begin{bmatrix} \varepsilon_0 \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \mu_0 \boldsymbol{I} \end{bmatrix} , \quad \boldsymbol{L}_1(\omega) = \mathrm{i}\omega \begin{bmatrix} \widehat{\varepsilon}_1(\omega)\boldsymbol{I} & \mathrm{i}Z_0\widehat{\kappa}(\omega)\boldsymbol{I} \\ \frac{\widehat{\kappa}(\omega)}{\mathrm{i}Z_0}\boldsymbol{I} & \widehat{\mu}_1(\omega)\boldsymbol{I} \end{bmatrix} ,$$

where:

$$\widehat{\varepsilon}_1(\omega) = \frac{\omega_p^2}{\omega_0^2 - \omega^2 + \mathrm{i}\omega\Gamma} ,$$

$$\widehat{\mu}_1(\omega) = \frac{F\omega^2}{\omega_0^2 - \omega^2 + \mathrm{i}\omega\Gamma} ,$$

$$\widehat{\kappa}(\omega) = \frac{\pm\sqrt{F}\omega_p\omega}{\omega_0^2 - \omega^2 + \mathrm{i}\omega\Gamma} ,$$

$Z_0 = \sqrt{\mu_0/\varepsilon_0}$ is the impedance, $0 < F < 1$ is the filling factor of the resonators, ω_p is the plasma frequency, ω_0 is a characteristic frequency for the resonators, and Γ is a damping loss rate.

Example #2: Single-resonance chiral metamaterial

- The three eigenvalues are $\omega_0 = 0$ ($\alpha = "0"$), $\omega_+(\boldsymbol{k}) = +c_0\|\boldsymbol{k}\|$ ($\alpha = "+"$), and $\omega_-(\boldsymbol{k}) = -c_0\|\boldsymbol{k}\|$ ($\alpha = "-"$), each of multiplicity two ($A = 2$), $c_0 = 1/\sqrt{\varepsilon_0\mu_0}$, with the same associated eigenvectors as for the Lorentz model.
- The solution of the transport equation reads:

$$\mathbf{w}_\pm(\boldsymbol{x}, \boldsymbol{k}, t) = \mathrm{e}^{-\tilde{\Gamma}(c_0\|\boldsymbol{k}\|)t} \boldsymbol{R}_\pm(t)\mathbf{w}_I(\boldsymbol{x} \mp c_0\hat{\boldsymbol{k}}t, \boldsymbol{k})\boldsymbol{R}_\pm(t)^* \,,$$

where:

$$\boldsymbol{R}_\pm(t) = \begin{bmatrix} \cos\tilde{\kappa}(c_0\|\boldsymbol{k}\|)t & \pm\sin\tilde{\kappa}(c_0\|\boldsymbol{k}\|)t \\ \mp\sin\tilde{\kappa}(c_0\|\boldsymbol{k}\|)t & \cos\tilde{\kappa}(c_0\|\boldsymbol{k}\|)t \end{bmatrix} \,,$$

and:

$$\begin{aligned} \tilde{\Gamma}(\omega) &= \mathfrak{Re}\{\mathrm{i}\omega(\widehat{\varepsilon}_1(\omega) + \widehat{\mu}_1(\omega))\} \,, \\ \tilde{\kappa}(\omega) &= \sqrt{F}\omega_p \frac{\omega^2}{\omega_0^2 - \omega^2} \,. \end{aligned}$$

Outlook

- Radiative transfer (*e.g.* linear Boltzmann) equations in random media with:

$$\boldsymbol{K}_0\left(\boldsymbol{x},\epsilon\right) = \boldsymbol{K}_0(\boldsymbol{x})\left[\boldsymbol{I} + \sqrt{\epsilon}\boldsymbol{V}\left(\frac{\boldsymbol{x}}{\epsilon}\right)\right]\;;$$

  Our results extend A. C. Fannjiang, *J. Opt. Soc. Am. A* **24**(12), 3680-3690 (2007) to bi-anisotropic media with heterogeneous mean background parameters and dissipative properties, and full mode coupling;
- The isotropic case was formerly derived in L. V. Ryzhik, G. C. Papanicolaou, J. B. Keller, *Wave Motion* **24**(4), 327-370 (1996);
- Diffusion limit(s) $W[\boldsymbol{u}_\epsilon](\boldsymbol{x},\boldsymbol{k},t) \rightsquigarrow W[\boldsymbol{u}_\epsilon](\boldsymbol{x},\|\boldsymbol{k}\|,t)$ when $c_0\|\boldsymbol{k}\|t \gg 1$;
- Boundary conditions;
- Mode crossing...

Prejudices

- Waves in heterogeneous media are far from what they would be in uniform media but their macroscopic features can often be captured by models that do not need the knowledge of the microscopic details.
- The aim is to minimize the efforts required to solve a problem of wave propagation in random media, *i.e.* perform waves coarse-graining.
- The issue is to identify a suitable set of relevant parameters for a coarser target level and express them in terms of the parameters of a finer source level: in other words rely on a few macroscopic parameters to encode microscopic parameters, that do not depend on particular realizations but rather on statistics.
- Coarse-graining consists in rescaling some phenomena into units or cells or models of size close to the uncertainty of measurement, yielding an increase of both entropy and dissipation–hence irreversibility.

THANK YOU!

Wave Motion **127**, 103296 (2024)