Kinetic models of elastic waves in bounded media with applications in structural mechanics

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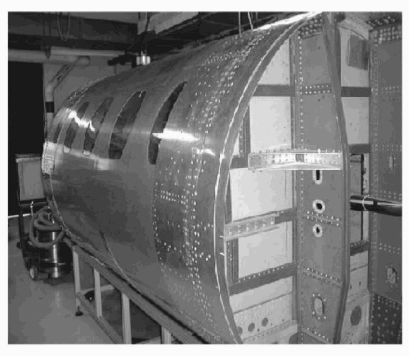
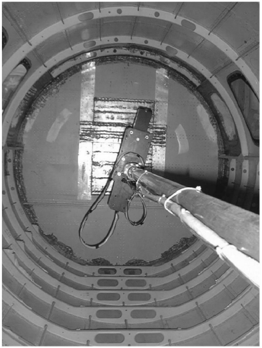
Some issues in structural dynamics and acoustics
Frequency response function of complex structures

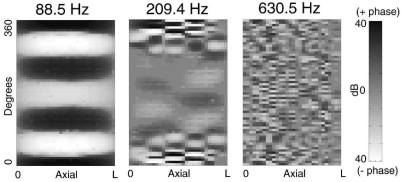
- Cessna Citation fuselage: 9 floor/19 ceiling ribs, 22 stringers.
- Length = 2.55 m, radius = 0.81 m, thickness = 0.8-1.2 mm.



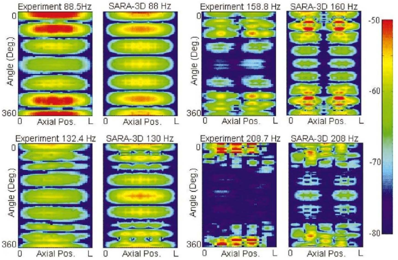


Herdic *et al.* *J. Acoust. Soc. Am.* 117, 3667 (2005)

Some issues in structural dynamics and acoustics
Frequency response function of complex structures



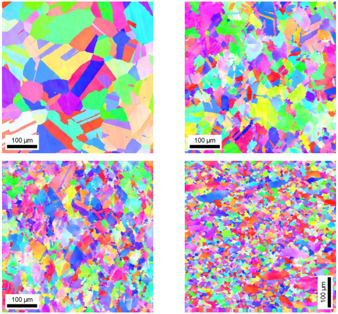
Phase distributions of surface velocity

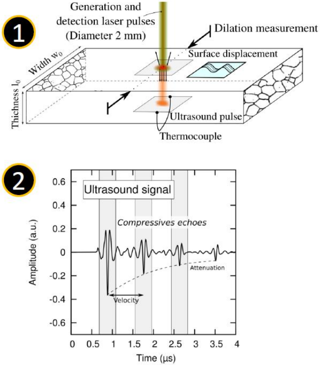


Magnitude distributions of surface velocity

Excitation: point force at a rib/stringer stiffener intersection.

- **Example**: recrystallization of inco© 718 superalloy ($NiCr_{19}Fe_{19}Nb_5Mo_3$) in hot compression.





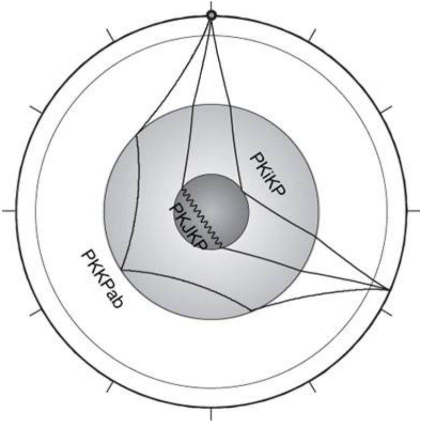
- Estimate of the grain size ℓ from the attenuation $\Sigma = \frac{2\pi}{\lambda Q}$ such that $\mathcal{I}_{\text{out}} \equiv \mathcal{I}_{\text{in}} \mathrm{e}^{-\Sigma h}$:

$$
\begin{aligned}
\Sigma &\propto \ell^3 f^4 && \lambda \gg \ell \text{ (Rayleigh regime)}\,, \\
\Sigma &\propto \ell f^2 && \lambda \simeq \ell \text{ (stochastic regime)}\,, \\
\Sigma &\propto \ell^{-1} && \lambda \ll \ell \text{ (diffusive regime)}\,.
\end{aligned}
$$

Stanke-Kino *JASA* 75(3), 665 (1984)

Goebbels *Materials Characterization for Process Control and Product Conformity*, CRC Press, Boca Raton FL (1994)

- **Example**: texture and dynamics of the inner core from observations of PKiKP and PKIKP waves (PKJKP are seldom observed and the reported evidences are highly controversial).



Cao-Romanowicz-Takeuchi *Science* 308, 1453 (2005)
Wookey-Helffrich *Nature* 454, 873 (2008)
Monnereau-Calvet-Margerin-Souriau *Science* 328, 1014 (2010)

Outline

Outline

Elastic waves in heterogeneous media

- Hypotheses: linear, visco-elastic materials with density $\varrho(\mathbf{x})$ and relaxation tensor $\mathrm{C}(\mathbf{x},t)$, $\mathbf{x}\in\mathcal{O}\subseteq\mathbb{R}^3$, $t\in\mathbb{R}_+$.
- Local balance of momentum for the displacement field $\mathbf{u}$ in $\mathcal{O}\times\mathbb{R}$:

$$\varrho\partial_t^2\mathbf{u}=\mathbf{Div}\,\boldsymbol{\sigma}\,.$$

- Constitutive equation for the stress field $\boldsymbol{\sigma}$ in $\overline{\mathcal{O}}\times\mathbb{R}_+$ as a function of the strain field $\boldsymbol{\epsilon}(\mathbf{u})=\boldsymbol{\nabla}_{\mathbf{x}}\otimes_s\mathbf{u}$:

$$\boldsymbol{\sigma}(\mathbf{x},t)=\mathrm{C}(\mathbf{x},0)\boldsymbol{\epsilon}(\mathbf{u})+\underbrace{\partial_t\mathrm{C}(\mathbf{x},\cdot)*_t\boldsymbol{\epsilon}(\mathbf{u})}_{\text{ineffective in the high-frequency limit}}\,.$$

Ultrasonic waves 1/2

- High frequencies correspond to $\varepsilon \to 0$ for strongly "ε–oscillatory" initial conditions:

$$\mathbf{u}_\varepsilon(\mathbf{x},0) = \mathbf{u}^0_\varepsilon(\mathbf{x})\,, \quad \||\varepsilon\boldsymbol{\nabla}_\mathbf{x}|\mathbf{u}^0_\varepsilon\|_{L^2_{\mathrm{loc}}} < \infty\,,$$

  and:

$$\partial_t\mathbf{u}_\varepsilon(\mathbf{x},0) = \mathbf{v}^0_\varepsilon(\mathbf{x})\,, \quad \||\varepsilon\boldsymbol{\nabla}_\mathbf{x}|\mathbf{v}^0_\varepsilon\|_{L^2_{\mathrm{loc}}} < \infty\,.$$

- **Example**: plane waves, $\varepsilon \equiv (|\mathbf{k}|L)^{-1}$, $\mathrm{i} = \sqrt{-1}$,

$$\mathbf{u}^0_\varepsilon(\mathbf{x}) = \varepsilon\mathbf{A}(\mathbf{x})e^{\frac{\mathrm{i}}{\varepsilon}\mathbf{k}\cdot\mathbf{x}}\,, \quad \mathbf{v}^0_\varepsilon(\mathbf{x}) = \mathbf{B}(\mathbf{x})e^{\frac{\mathrm{i}}{\varepsilon}\mathbf{k}\cdot\mathbf{x}}\,.$$

Ultrasonic waves 2/2

- Introduce the acoustic (Christoffel) tensor $\boldsymbol{\Gamma}$ of the medium:

$$[\boldsymbol{\Gamma}(\mathbf{x},\mathbf{k})]:\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k:=\varrho^{-1}(\mathbf{x})(\hat{\mathbf{e}}_j\otimes\mathbf{k}):\mathrm{C}(\mathbf{x}):(\hat{\mathbf{e}}_k\otimes\mathbf{k})\,,\quad \mathbf{k}\in\mathbb{R}^3\,,$$

  where $(\hat{\mathbf{e}}_j,\hat{\mathbf{e}}_k)=\delta_{jk}$, $1\leq j,k\leq 3$.

- The local balance of momentum for the displacement field $\mathbf{u}_\varepsilon$ in $\mathcal{O}\times\mathbb{R}_t$ is the elastic wave equation:

$$\varrho(\mathbf{x})\left(\boldsymbol{\Gamma}(\mathbf{x},\mathrm{i}\varepsilon\boldsymbol{\nabla}_{\mathbf{x}})-(\mathrm{i}\varepsilon\partial_t)^2\right)\mathbf{u}_\varepsilon=\mathrm{O}(\varepsilon)\,,$$

  supplemented with (*e.g.* Neumann or Dirichlet) boundary conditions on $\partial\mathcal{O}\times\mathbb{R}_t$.

Some properties of $\boldsymbol{\Gamma}$

- $\boldsymbol{\Gamma}$ is symmetric, real, positive definite (in $\mathcal{O}\times\mathbb{R}^3\setminus\{\mathbf{k}=\mathbf{0}\}$): it has 3 (possibly r_α-multiple) positive eigenvalues ω_α^2 such that:

$$\omega_\alpha(\mathbf{x},\mathbf{k}) = c_\alpha(\mathbf{x},\hat{\mathbf{k}})|\mathbf{k}|\,,\quad 1\leq\alpha\leq 3\,,$$

where $\hat{\mathbf{k}} := \mathbf{k}/|\mathbf{k}|$, and the real eigenvectors $\hat{\mathbf{p}}_\alpha(\mathbf{x},\mathbf{k}) = \hat{\mathbf{p}}_\alpha(\mathbf{x},\hat{\mathbf{k}})$ can be orthonormalized,

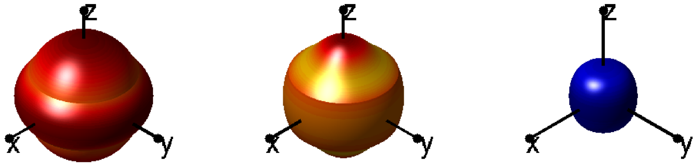
$$\boldsymbol{\Gamma} = \sum_{\alpha=1}^{3}\omega_\alpha^2\hat{\mathbf{p}}_\alpha\otimes\hat{\mathbf{p}}_\alpha\,,\quad \mathbb{I} = \sum_{\alpha=1}^{3}\hat{\mathbf{p}}_\alpha\otimes\hat{\mathbf{p}}_\alpha\,.$$

- **Example**: three-dimensional isotropic medium $\mathrm{C} = \lambda\mathbb{I}\otimes\mathbb{I} + 2\mu\mathbb{I}\boxtimes\mathbb{I}$, $r_\mathrm{P}=1$, $r_\mathrm{S}=2$, and

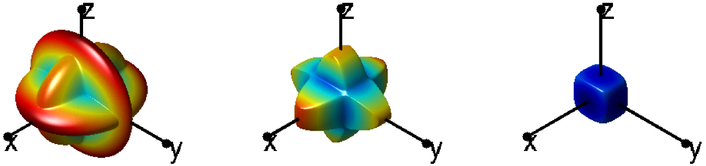
$$\boldsymbol{\Gamma}(\mathbf{x},\hat{\mathbf{k}}) = c_\mathrm{P}^2(\mathbf{x})\hat{\mathbf{k}}\otimes\hat{\mathbf{k}} + c_\mathrm{S}^2(\mathbf{x})(\mathbb{I}-\hat{\mathbf{k}}\otimes\hat{\mathbf{k}})\,,\quad \forall\hat{\mathbf{k}}\in\mathbb{S}^2\,,$$

with $c_\mathrm{P} = \sqrt{\frac{\lambda+2\mu}{\varrho}}$ and $c_\mathrm{S} = \sqrt{\frac{\mu}{\varrho}}$.

Elastic waves and anisotropy
Three propagation speeds



Transverse isotropic medium (muscles, polymers reinforced with parallel fibers...)



Orthotropic medium (wood...)

Energy and power flow densities

- The energy density (real, positive) and the power flow density (vector) are:

$$\begin{aligned}\mathcal{E}_\varepsilon(\mathbf{x},t) &= \frac{1}{2}(\varrho|\partial_t\mathbf{u}_\varepsilon|^2 + \boldsymbol{\sigma}_\varepsilon : \boldsymbol{\epsilon}_\varepsilon) = \mathcal{T}_\varepsilon + \mathcal{U}_\varepsilon\,,\\ \boldsymbol{\pi}_\varepsilon(\mathbf{x},t) &= -\boldsymbol{\sigma}_\varepsilon\partial_t\mathbf{u}_\varepsilon\,.\end{aligned}$$

- They satisfy for all *fixed* $\varepsilon\in]0,\varepsilon_0]$ a conservation equation:

$$\boxed{\partial_t\mathcal{E}_\varepsilon + \operatorname{div}\boldsymbol{\pi}_\varepsilon = 0}\,.$$

- **What happens when** $\varepsilon\to 0$???
- *For constant coefficients the usual parametrix (where* $\boldsymbol{\Gamma}^{\frac{1}{2}} = \sum_\alpha \omega_\alpha\hat{\mathbf{p}}_\alpha\otimes\hat{\mathbf{p}}_\alpha$*):*

$$\mathbf{u}_\varepsilon(\mathbf{x},t) = \mathcal{F}^{-1}_{\mathbf{k}\to\mathbf{x}}\left[\cos\left(t\boldsymbol{\Gamma}^{\frac{1}{2}}(\mathbf{k})\right)\right]\star\mathbf{u}^0_\varepsilon(\mathbf{x}) + \mathcal{F}^{-1}_{\mathbf{k}\to\mathbf{x}}\left[\boldsymbol{\Gamma}^{-\frac{1}{2}}(\mathbf{k})\sin\left(t\boldsymbol{\Gamma}^{-\frac{1}{2}}(\mathbf{k})\right)\right]\star\mathbf{v}^0_\varepsilon(\mathbf{x})$$

propagates oscillations of wavelength ε *which inhibit* $(\mathbf{u}_\varepsilon)$ *from converging strongly in a suitable sense since* $\boldsymbol{\Gamma}$ *is positive.*

Existing approaches for structural-acoustics

Statistical Energy Analysis (SEA)

Westphal 1957, Lyon-Maidanik 1962, Smith 1962, Maidanik 1976 ...

Vibrational conductivity analogy (VCA)

Rybak 1972, Nefske-Sung 1989, Belyaev 1991, Bernhard *et al*. 1992, Langley 1995, Ichchou-Jézéquel 1996, Vlahopoulos *et al*. 1999, Le Bot 2002.

Existing approaches for structural-acoustics

Ray methods with real phases (WKBJ/FIO)

Carlini 1817, Green 1837, Liouville 1837 [...] Keller 1958, Steele 1969, Pierce 1970, Germogenova 1973, Maslov-Fedoruk 1981, Norris 1995 ...

Ray methods with complex phases (Gaussian beams)

Babich 1968, Katchalov-Popov 1981, Červený *et al*. 1982, Ralston 1982, Norris 1988, Tanushev *et al*. 2007 ...

Kinetic models (applications in geophysics and seismology)

Weaver *et al*. 1990, Campillo-Fink-van Tiggelen *et al*. 1998, Sato-Fehler 1998, Papanicolaou-Ryzhik 1999, Scales *et al*. 2000.

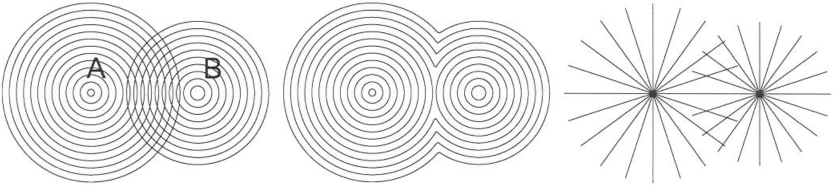
Outline

Ray methods 1/3
Principles

- $\mathbf{u}_\varepsilon$ is sought for as a WKBJ ansatz:

$$\mathbf{u}_\varepsilon(\mathbf{x},t) \simeq e^{\frac{\mathrm{i}}{\varepsilon}S(\mathbf{x},t)} \sum_{k=0}^{\infty} \varepsilon^k \mathbf{U}_k(\mathbf{x},t)\,.$$

- **Eulerian** point of view (b): the phase S satisfies an eikonal equation and the densities $|\mathbf{U}_k|^2$ satisfy transport equations.
- **Lagrangian** point of view (c): $(\mathbf{x}, \boldsymbol{\nabla}_{\mathbf{x}} S)$ is given as the solution of the associated Hamiltonian system (ray tracing).



(a) Correct solution (b) Eikonal equation (c) Ray tracing

- Plugging the WKBJ ansatz into the elastic wave equation:

$$
\begin{aligned}
\mathbf{H}(\mathbf{s},\boldsymbol{\nabla}_{\mathbf{s}}S)\mathbf{U}_0 &= \mathbf{0}\,, && \text{[eikonal]}\\
\boldsymbol{\nabla}_{\mathbf{s}}\cdot\left(\mathbf{U}_0^{\mathsf{T}}\boldsymbol{\nabla}_{\boldsymbol{\xi}}\mathbf{H}(\mathbf{s},\boldsymbol{\nabla}_{\mathbf{s}}S)\mathbf{U}_0\right) &= 0\,, && \text{[}0^{\text{th}}\text{-order transport]}
\end{aligned}
$$

  where:

$$
\begin{aligned}
\mathbf{H}(\mathbf{s},\boldsymbol{\xi}) &= \varrho(\mathbf{x})\left(\boldsymbol{\Gamma}(\mathbf{x},\mathbf{k})-\omega^2\mathbb{I}_n\right)\\
&= \varrho(\mathbf{x})\sum_{\alpha=1}^{3}(\omega_\alpha^2(\mathbf{x},\mathbf{k})-\omega^2)\hat{\mathbf{p}}_\alpha\otimes\hat{\mathbf{p}}_\alpha
\end{aligned}
$$

  is the dispersion matrix of the medium, $\mathbf{s}=(\mathbf{x},t)\in\mathcal{O}\times\mathbb{R}$, $\boldsymbol{\xi}=(\mathbf{k},\omega)\in\mathbb{R}^d\times\mathbb{R}$.
- Thus $\mathcal{H}=\det\mathbf{H}=0=\prod_{\alpha=1}^{3}\mathcal{H}_\alpha$ where $\mathcal{H}_\alpha=\varrho\left(\omega_\alpha^2-\omega^2\right)$, and therefore:

$$
\boxed{\mathcal{H}_\alpha(\mathbf{s},\boldsymbol{\nabla}_{\mathbf{s}}S)=0}\,.
$$

Ray method 3/3
Shortcomings

- By the divergence theorem applied to the 0-th order transport equation on a ray tube:

$$\frac{|\mathbf{U}_0(\mathbf{s}(\tau))|^2}{|\mathbf{U}_0(\mathbf{s}_0)|^2} \propto \frac{q(0)}{q(\tau)}\,,$$

  where $q(\tau) = \det(\boldsymbol{\nabla}_{\mathbf{s_0}} \otimes \mathbf{s}(\tau))$, which may cancel on caustics.
- The eikonal admits a unique (physical) solution only for sufficiently short times, that happen to be actually very small in highly heterogeneous media.
- When such caustics occur, the WKBJ expansion needs to be generalized as a superposition of propagating fronts; but it is unclear how it can be used to model waves in heterogeneous media.

Research program

- SEA, VCA: approximations (provably wrong);
- Ray methods: superposition principle, regularity... and they give only **one particular** construction of oscillating solutions.

Kinetic models: **all** oscillating solutions
The **key idea**: phase space description
(which also accounts for the directions in which waves propagate)

- Mechanical modeling for aerospace structures: material heterogeneity and damping, boundary conditions, higher-order kinematics;
- Numerical simulations;
- (Further modeling: diffusion limits).

Outline

- Consider the bicharacteristic strip $\tau \mapsto (\mathbf{s}(\tau), \boldsymbol{\xi}(\tau))$ as the paths in phase space of some energy "particles" of which density is denoted by $\mathrm{W}(\mathbf{s}(\tau), \boldsymbol{\xi}(\tau))$, the latter shall satisfy:

$$\boxed{\frac{\mathrm{d}\mathrm{W}}{\mathrm{d}\tau} = \{\mathcal{H}, \mathrm{W}\} = 0}\,, \qquad \text{[TE]}$$

  where $\{f, g\} = \boldsymbol{\nabla}_{\boldsymbol{\xi}} f \cdot \boldsymbol{\nabla}_{\mathbf{s}} g - \boldsymbol{\nabla}_{\mathbf{s}} f \cdot \boldsymbol{\nabla}_{\boldsymbol{\xi}} g$ (Poisson bracket).

- But $\frac{\mathrm{d}\omega}{\mathrm{d}\tau} = 0$ and $\mathbf{H} = \sum \mathcal{H}_\alpha \hat{\mathbf{p}}_\alpha \otimes \hat{\mathbf{p}}_\alpha$ with $\mathbb{I}_3 = \sum \hat{\mathbf{p}}_\alpha \otimes \hat{\mathbf{p}}_\alpha$, thus:

$$\boxed{\mathrm{W} = \sum_{\alpha=1}^{3} \mathrm{w}_\alpha \delta(\mathcal{H}_\alpha)}\,.$$

- The link between W and the sequence $(\mathbf{u}_\varepsilon)$ is established by the Wigner measure of the latter.

- GENERIC framework:

$$\frac{\mathrm{dW}}{\mathrm{d}\tau} = \{\mathcal{H}, \mathrm{W}\} + [\mathcal{S}, \mathrm{W}] = 0\,, \qquad \text{[ITE]}$$

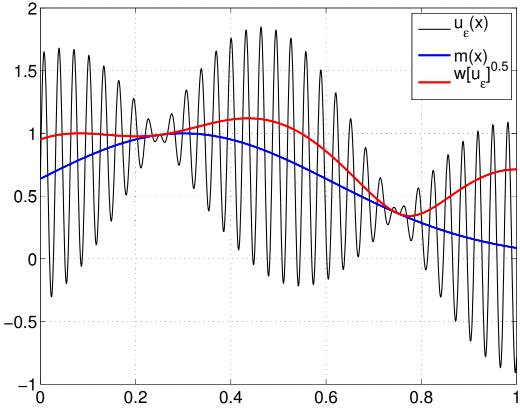
  where $\mathcal{S}$ is the entropy in this W-system and $[\mathcal{S}, \mathrm{W}]$ is the (positive) dissipative bracket, both satisfying the degeneracy conditions:

$$\{\mathcal{S}, \mathrm{W}\} = [\mathcal{H}, \mathrm{W}] = 0\,.$$

- The link between W and the sequence $(\mathbf{u}_\varepsilon)$ is established by the Wigner measure of the latter considering the wave equation with a time-dependent acoustic tensor.

Grmela-Öttinger *Phys. Rev. E* 56, 6620 (1997)
Öttinger-Grmela *Phys. Rev. E* 56, 6633 (1997)
Öttinger, *Beyond Equilibrium Thermodynamics*, Wiley (2005)

Why quadratic observables?

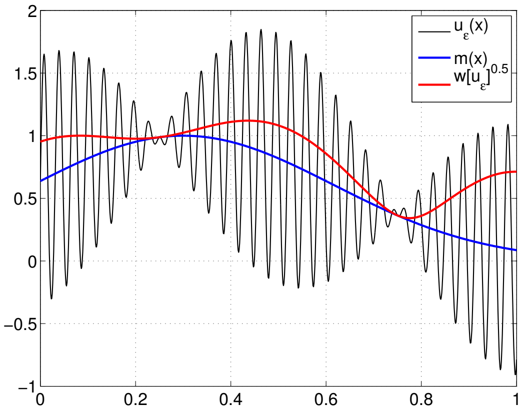


- Let:

$$u_\varepsilon(x) = m(x) + \sigma(x) \sin \frac{x}{\varepsilon} \,,$$

then $(u_\varepsilon) \rightharpoonup m$ weakly in $L^2(\mathbb{R})$ as $\varepsilon \to 0$, but (u_ε) has no strong limit.

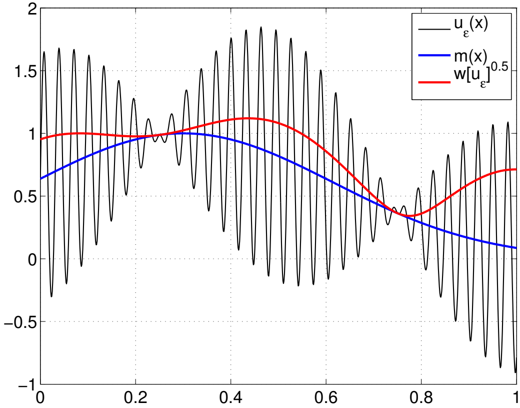
Why quadratic observables?



- Now for any observable $\varphi \in \mathcal{C}_0(\mathbb{R})$:

$$\lim_{\varepsilon \to 0} (\varphi(x) u_\varepsilon, u_\varepsilon)_{L^2} = \int_{\mathbb{R}} \varphi(x) \left((m(x))^2 + \frac{1}{2}(\sigma(x))^2 \right) \mathrm{d}x \,.$$

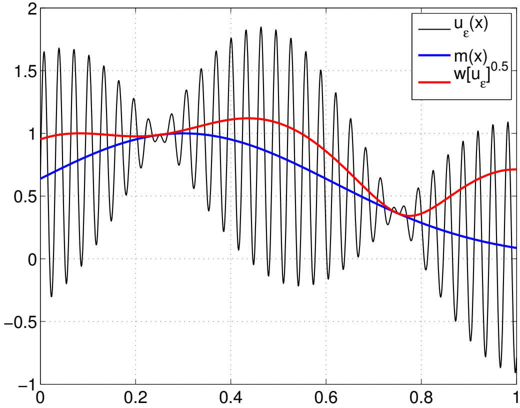
Why quadratic observables?



- Take an observable of the form:

$$\varphi(x,\partial_x)u_\varepsilon(x)=\frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}k\cdot x}\varphi(x,\mathrm{i}k)\widehat{u_\varepsilon}(k)\mathrm{d}k\,.$$

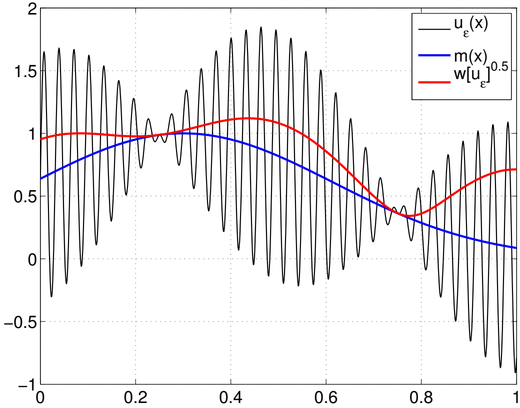
Why quadratic observables?



- Take an observable of the form:

$$\varphi(x, \varepsilon\partial_x)u_\varepsilon(u) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}k\cdot x}\varphi(x, \mathrm{i}\varepsilon k)\widehat{u}_\varepsilon(k)\mathrm{d}k\,.$$

Why quadratic observables?



- Then:

$$\lim_{\varepsilon \to 0} \left(\varphi(x, \varepsilon\partial_x)u_\varepsilon, u_\varepsilon\right)_{L^2} = \iint_{\mathbb{R}^2} \varphi(x, \mathrm{i}k)\mathrm{W}[u_\varepsilon](\mathrm{d}x, \mathrm{d}k)\,,$$

where $\mathrm{W}[u_\varepsilon]$ is the (positive) Wigner measure of (u_ε).

- **Example**: the overall kinetic and strain energy densities are in the high-frequency limit $\varepsilon \to 0$

$$\lim_{\varepsilon\to 0} \mathcal{T}_\varepsilon(\mathbf{x},t) = \frac{1}{2}\varrho(\mathbf{x}) \int_{\mathbb{R}^3} \operatorname{Tr} \mathbf{W}[\varepsilon\partial_t \mathbf{u}_\varepsilon(\cdot,t)](\mathbf{x},\mathrm{d}\mathbf{k})\,,$$
$$\lim_{\varepsilon\to 0} \mathcal{U}_\varepsilon(\mathbf{x},t) = \frac{1}{2}\varrho(\mathbf{x}) \int_{\mathbb{R}^3} \boldsymbol{\Gamma}(\mathbf{x},\mathbf{k}) : \mathbf{W}[\mathbf{u}_\varepsilon(\cdot,t)](\mathbf{x},\mathrm{d}\mathbf{k})\,,$$

(eventually one can prove that they are equal in this very limit), or

$$\boxed{\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(\mathbf{x},t) = \sum_{\alpha=1}^{3} \int_{\mathbb{R}^3} \mathrm{W}_\alpha(\mathbf{x},t;\mathrm{d}\mathbf{k})}\,.$$

- Let a discontinuity front Σ_D be defined by the hypersurface in phase space:

$$\Sigma_D = \{(\mathbf{s}, \boldsymbol{\xi}) \in T^*(\mathcal{O} \times \mathbb{R});\ \Sigma(\mathbf{s}, \boldsymbol{\xi}) = 0\}\,.$$

- The Rankine-Hugoniot condition associated to piecewise continuous (weak) solutions of the Liouville equation [TE] on $\Sigma_D \cap \{\mathcal{H}(\mathbf{s}, \boldsymbol{\xi}) = 0\}$:

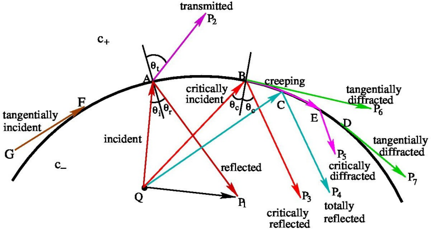
$$\boxed{[\![\{\mathcal{H}, \Sigma\}\mathrm{W}]\!] = 0}\,.$$

- **Consequence**: continuity of the overall normal power flow across any fixed interface $\{\Sigma(\mathbf{x}, \mathbf{k}) = 0\}$,

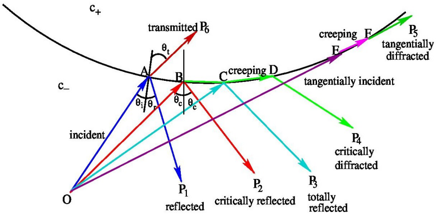
$$\sum_{\alpha=1}^{3} [\![\{\omega_\alpha, \Sigma\}\mathrm{w}_\alpha]\!] = 0\,.$$

Boundary conditions
Reflection/transmission coefficients

- **Example**: acoustic waves $\omega_1(\mathbf{x},\mathbf{k}) = \pm c(\mathbf{x})|\mathbf{k}|$ $(n=1)$.



Convex acoustic interface with
$c_+ > c_-$



Concave acoustic interface with
$c_+ > c_-$

Jin-Yin *J. Comput. Phys.* 227, 6106 (2008)

Boundary conditions
Reflection/transmission coefficients

- $\{\mathcal{H}, \Sigma\} > 0$: transverse reflections (the hyperbolic set);
- $\{\mathcal{H}, \Sigma\} < 0$: total reflections (the elliptic set);
- $\{\mathcal{H}, \Sigma\} = 0$: critical reflections (the glancing set):
  - $\{\mathcal{H}, \{\mathcal{H}, \Sigma\}\} > 0$: diffractive rays,
  - $\{\mathcal{H}, \{\mathcal{H}, \Sigma\}\} < 0$: gliding rays,
  - $\{\mathcal{H}, \{\mathcal{H}, \Sigma\}\} = 0$: higher-order gliding rays (w. inflection point).

- For elastic waves the situation is a mix of all cases!
- Boundary conditions raise profound mathematical issues, mostly unsolved to date. Unfortunately they have also great engineering relevance!

Miller *J. Math. Pures Appl.* 79, 227 (2000)

Fouassier *J. Math. Pures Appl.* 87, 144 (2007)

Savin *IMMIJ* 1, 53 (2007)

Le Guennec-Savin *J. Acoust. Soc. Am.* 130, 3706 (2011)

Akian-Alexandre-Bougacha *Kin. Rel. Mod.* 4, 589 (2011)

Akian *Asympt. Anal.* 78, 37 (2012)

Outline

Discontinuous finite elements for the TE 1/2

- TE is a linear conservation equation in phase space $\Omega = \mathcal{O} \times \mathbb{R}^d \ni \mathbf{z} = (\mathbf{x}, \mathbf{k})$:

$$\mathcal{L}\mathrm{w}_\alpha \equiv \partial_t \mathrm{w}_\alpha + \boldsymbol{\nabla}_{\mathbf{z}} \cdot \mathcal{F}(\mathrm{w}_\alpha) = 0\,, \quad \mathcal{F}(\mathrm{w}_\alpha) = \begin{pmatrix} \boldsymbol{\nabla}_{\mathbf{k}}\omega_\alpha \\ -\boldsymbol{\nabla}_{\mathbf{x}}\omega_\alpha \end{pmatrix} \mathrm{w}_\alpha\,;$$

- Initial conditions: $\mathrm{w}_\alpha(\mathrm{d}\mathbf{z}; t=0) = \mathrm{w}^0_\alpha(\mathrm{d}\mathbf{z})$;
- The modes α get coupled by the reflection/transmission conditions.

Discontinuous finite elements for the TE 2/2

- Consider the partition $\mathcal{T}_h = \cup_{r=1}^{\mathcal{N}} \Omega_r$ of Ω into $\mathcal{N}$ non-overlapping subdomains, or finite elements, and the trial space $\mathcal{W}_h$ of piecewise continuous functions on $\mathcal{T}_h$.
- Variational formulation using the test space $\mathcal{V}_h$:

1. "Weak" form: Find $\mathrm{w}_\alpha \in \mathcal{W}_h$ s.t.

$$\int_{\Omega_r} (\mathrm{v}\partial_t \mathrm{w}_\alpha - \boldsymbol{\nabla}_{\mathbf{z}} \mathrm{v} \cdot \mathcal{F}(\mathrm{w}_\alpha))\, \mathrm{d}\Omega + \int_{\partial\Omega_r} \mathrm{v}\mathcal{F}_r^*(\mathrm{w}_\alpha^-, \mathrm{w}_\beta^+) \cdot \hat{\mathbf{n}}_r \, \mathrm{d}\gamma(\mathbf{z}) = 0 \,, \quad \forall \mathrm{v} \in \mathcal{V}_h \,,$$

where $\mathcal{F}(\mathrm{w}_\alpha) \cdot \hat{\mathbf{n}}|_{\partial\Omega_r}$ is replaced by $\mathcal{F}_r^*(\mathrm{w}_\alpha^-, \mathrm{w}_\beta^+) \cdot \hat{\mathbf{n}}_r$ because w_α may have two different values on both sides of the edge $\partial\Omega_r$.

Discontinuous finite elements for the TE 2/2

- Consider the partition $\mathcal{T}_h = \cup_{r=1}^{\mathcal{N}} \Omega_r$ of Ω into $\mathcal{N}$ non-overlapping subdomains, or finite elements, and the trial space $\mathcal{W}_h$ of piecewise continuous functions on $\mathcal{T}_h$.
- Variational formulation using the test space $\mathcal{V}_h$:

2. "Strong" form (Lesaint & Raviart 1974): Find $\mathrm{w}_\alpha \in \mathcal{W}_h$ s.t.

$$\int_{\Omega_r} \mathrm{v} \mathcal{L} \mathrm{w}_\alpha \, \mathrm{d}\Omega = \int_{\partial\Omega_r} \mathrm{v} \left(\mathcal{F}(\mathrm{w}_\alpha^-) - \mathcal{F}_r^*(\mathrm{w}_\alpha^-, \mathrm{w}_\beta^+) \right) \cdot \hat{\mathbf{n}}_r \, \mathrm{d}\gamma(\mathbf{z}) \, , \quad \forall \mathrm{v} \in \mathcal{V}_h \, ,$$

e.g. penalization methods.

Discontinuous finite elements for the TE 2/2

- Consider the partition $\mathcal{T}_h = \cup_{r=1}^{\mathcal{N}} \Omega_r$ of Ω into $\mathcal{N}$ non-overlapping subdomains, or finite elements, and the trial space $\mathcal{W}_h$ of piecewise continuous functions on $\mathcal{T}_h$.
- Variational formulation using the test space $\mathcal{V}_h$:

3. "Ultra-weak variational formulation":

$$\partial_t \left(\int_{\Omega_r} \mathrm{v} \mathrm{w}_\alpha \, \mathrm{d}\Omega \right) + \int_{\partial\Omega_r} \mathrm{v} \mathcal{F}_r^*(\mathrm{w}_\alpha^-, \mathrm{w}_\beta^+) \cdot \hat{\mathbf{n}}_r \, \mathrm{d}\gamma(\mathbf{z}) = 0 \,, \quad \forall \mathrm{v} \in \mathcal{V}_h \,,$$

choosing $\mathcal{V}_h = \{\mathrm{v};\, \mathcal{L}^* \mathrm{v} = 0 \text{ in } \Omega_r\}$ (plane waves for example are $\mathrm{v}(\mathbf{z}, t) = \mathbf{V}\phi(\mathbf{k} \cdot \mathbf{x} - \omega t)$ such that $\boldsymbol{\Gamma}(\mathbf{z})\mathbf{V} = \omega\mathbf{V}$, with the acoustic tensor $\boldsymbol{\Gamma}(\mathbf{z}) := \mathrm{diag}\{\omega_\alpha(\mathbf{z})\}_{1 \leq \alpha \leq 3}$).

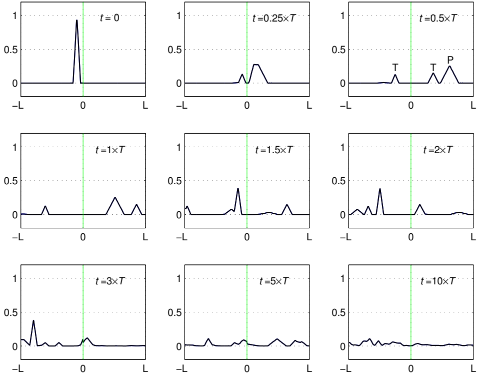
Després *CRAS I* 318, 939 (1994)

References

Further reading...

- Cockburn-Karniadakis-Shu (eds.): *Discontinuous Galerkin Methods: Theory, Computation and Applications*, Springer (2000)
- Karniadakis-Sherwin: *Spectral/hp Element Methods For Computational Fluid Dynamics* (2nd ed.), Springer (2005)
- Li: *Discontinuous Finite Elements in Fluid Dynamics and Heat Transfer*, Springer (2006)
- Hesthaven-Warburton: *Nodal Discontinuous Galerkin Methods*, Springer (2008)
- Rivière: *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations*, SIAM (2008)
- Di Pietro-Ern: *Mathematical Aspects of Discontinuous Galerkin Methods*, Springer (2012)

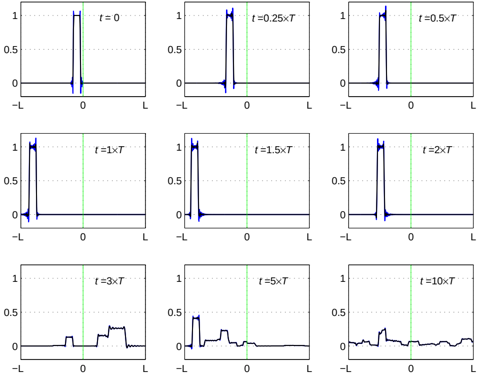
Example #1: beam junction



Runge-Kutta discontinuous FEM
w. Legendre modal expansion
$\mathcal{N} = 40$, $N = 10$, RK-SSP(8, 8)
— unfiltered
— 1st-order exponential filter

Energy transport in a beam junction with $\phi = 60^\circ$, $\frac{E_2}{E_1} = 2$, $\nu_1 = \nu_2 = 0.3$, *and* $T = \frac{L}{c_{T2}}$.

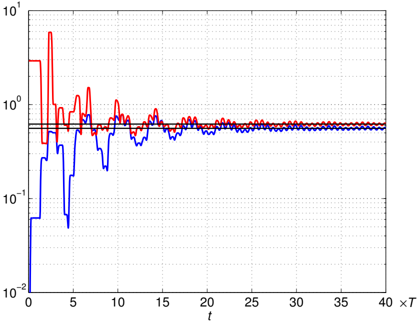
Example #1: beam junction



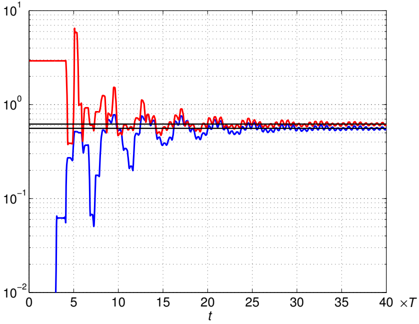
Runge-Kutta discontinuous FEM
w. Gauss-Lobatto nodal expansion
$\mathcal{N} = 40$, $N = 10$, RK-SSP(8, 8)
— unfiltered
— 0th-order exponential filter

Energy transport in a beam junction with $\phi = 60^\circ$, $\frac{E_2}{E_1} = 2$, $\nu_1 = \nu_2 = 0.3$, *and* $T = \frac{L}{c_{\mathrm{T}2}}$.

Example #1: beam junction



"Hat" source: Legendre modal expansion
$\mathcal{N} = 40,\ N = 5,$ RK-SSP$(5,4)$

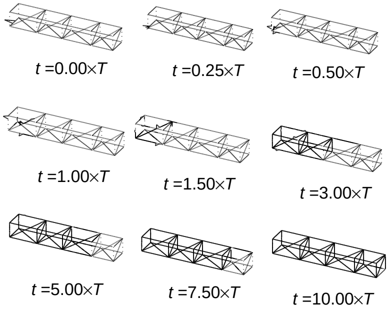


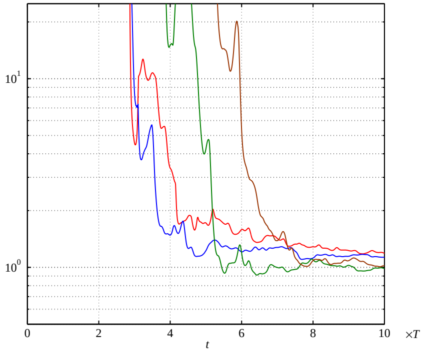
"Square" source: nodal expansion
$\mathcal{N} = 40,\ N = 5,$ RK-SSP$(5,4)$

Energy ratios $t \mapsto \frac{\mathcal{E}_{\mathrm{P}}(t)}{\mathcal{E}_{\mathrm{T}}(t)}$ *within each sub-structure (— beam #1, — beam #2);* $\phi = 60^\circ,\ \frac{E_2}{E_1} = 2,$
$\nu_1 = 0.4,\ \nu_2 = 0.1,\ T = \frac{L}{c_{\mathrm{T}2}}.$

Savin *Prob. Engng. Mech.* 28, 194 (2012)

Example #2: beam truss





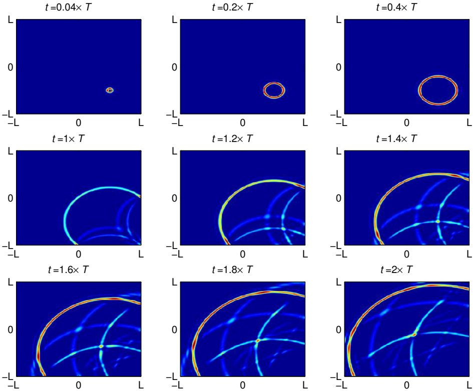
Runge-Kutta discontinuous FEM
w. Legendre modal expansion
$\mathcal{N} = 1212$, $N = 8$, RK-SSP(8,8) $(\mathrm{CFL} = 10^{-2})$

$$\frac{\mathcal{E}^r_{\mathrm{T}}(t)}{\mathcal{E}^r_{\mathrm{P}}(t)} \underset{t\to+\infty}{\longrightarrow} \frac{2c_{\mathrm{P}}}{c_{\mathrm{T}}} \quad (1 \leq r \leq 4)$$

Energy transport in a beam truss with $T = \frac{L}{c_{\mathrm{T}}}$.

Le Guennec-Savin ***J. Acoust. Soc. Am.*** 130, **3706 (2011)**

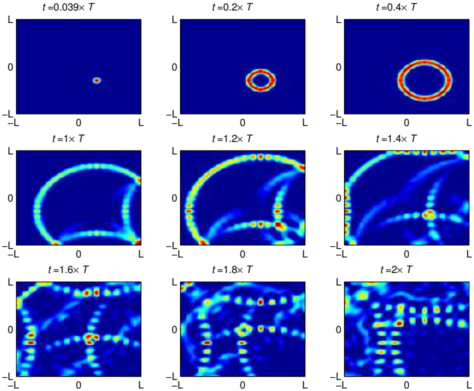
Example #3: thick plate



Runge-Kutta discontinuous FEM w. mixed nodal/modal expansion $\mathcal{N} = 21 \times 21$, $N = 5$ (LGL nodes), $P = 2$ (Fourier), SSP(3, 3)

Energy transport in a thick plate with $\nu = 0.3$ and $T = \frac{L}{c_S}$.
Issues with Fourier expansion: dispersion, positivity.

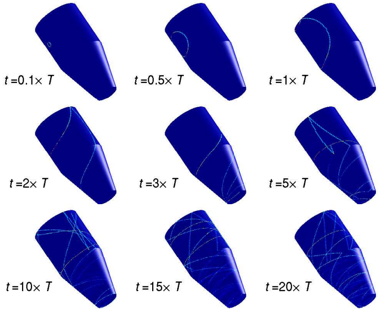
Example #3: thick plate



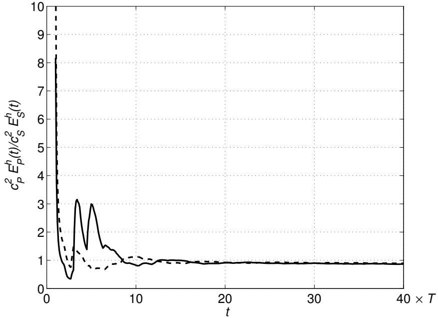
Runge-Kutta discontinuous FEM w. full nodal expansion $\mathcal{N} = 21 \times 21$, $N = 3$ (LGL nodes), $P = 59$ (Lagrange), SSP(3, 3)

Energy transport in a thick plate with $\nu = 0.3$ and $T = \frac{L}{c_S}$.
Issue with Lagrange expansion: ray effect.

Example #4: transport in a shell junction



Direct Monte-Carlo method
10^6 sample paths



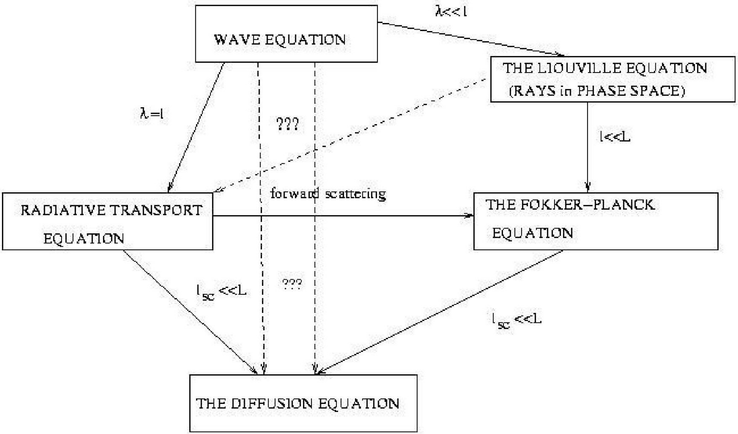
$$\frac{\mathcal{E}^r_{\mathrm{P}}(t)}{\mathcal{E}^r_{\mathrm{S}}(t)} \underset{t\to+\infty}{\longrightarrow} \frac{c_{\mathrm{S}}^2}{c_{\mathrm{P}}^2}$$ (— *shell #1*, -- *shell #2*)

Energy transport in a shell junction impacted by a shear load with $\phi = 15^\circ$, $\frac{E_2}{E_1} = 2$, $\nu_1 = 0.3$, $\nu_2 = 0.2$, $T = \frac{L}{c_{\mathrm{T}2}}$.

Savin ***Prob. Engng. Mech.*** 28, **194 (2012)**

Wave propagation in heterogeneous media

- **Scales**: λ – wavelength, ℓ – correlation length, L – propagation/observation distance...

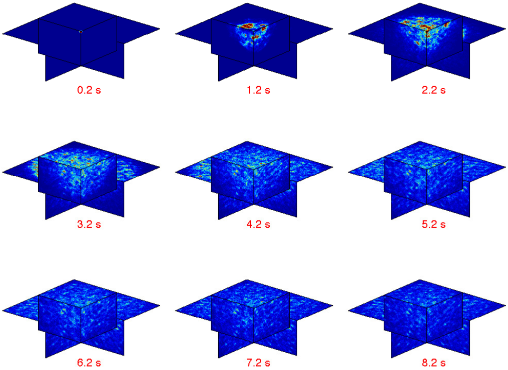


... and the **mesoscopic scale** $\ell_{\rm sc}$ – scattering mean free path.

Ryzhik HFWP'05 (2005)

Elastic waves and anisotropy
Triclinic random medium

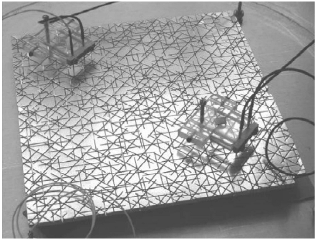
- **Example**: Velocity field in a half-space constituted by heterogeneous anisotropic materials with an homogeneous isotropic background, impacted by a monopole on its surface; $\underline{c}_{\mathrm{P}} = 2000$ m/s, $\underline{c}_{\mathrm{S}} = 1000$ m/s, $\underline{\varrho} = 2000$ kg/m^3.

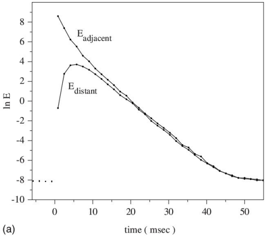


Spectral FEM with LGL + PML
$7 \cdot 10^7$ dofs, $\mathcal{O} = 5 \times 5 \times 2.5$ km^3

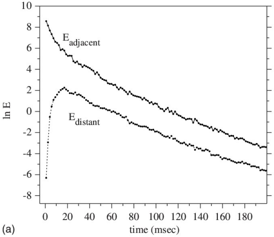
Ta-Clouteau-Cottereau *Eur. J. Comp. Mech.* 19, 241 (2010)

- **Example**: "localization". Privileged directions for diffusion corresponding to the principal axes of diffusion → localization when $\lambda \sim \ell_{\text{sc}}$?





$f = 664 \pm 40$ kHz



$f = 195 \pm 40$ kHz

Lobkis-Weaver *J. Acoust. Soc. Am.* 124, 3528 (2008)

Hot topics

- Diffusion limits, localization, coarse graining...
- Passive imaging techniques...
- **Fundings**: ANR, CNES, CNRS, DGA, UE.