Errata for abstract of forthcoming book

Eric Schmid

September 8, 2023

1 Computational Trinity

1.0.1 I finally understand that (dependent) function-types, i.e. $\prod$-types, are equivalent to intuitionistic universal quantifiers of predicates and also equivalent to the space of sections. Dependent pair-types, $\sum$-types are equivalent to existential quantification in IL and total spaces (above the base space). Families of types are equivalent to a predicate and to a fibration. A type becomes a proposition which is also a space. Terms x : b(x) of families of types B(x) becomes a conditional proof and also a section. A term of a type becomes a proof which is a point. Functions become intuitionistic implication which become a function space.

Types	Logic	Sets	Homotopy
A	proposition	set	space
$a : A$	proof	element	point
$B(x)$	predicate	family of sets	fibration
$b(x) : B(x)$	conditional proof	family of elements	section
$\mathbf{0}, \mathbf{1}$	$\bot, \top$	$\emptyset, \{\emptyset\}$	$\emptyset, *$
$A + B$	$A \vee B$	disjoint union	coproduct
$A \times B$	$A \wedge B$	set of pairs	product space
$A \to B$	$A \Rightarrow B$	set of functions	function space
$\sum_{(x:A)} B(x)$	$\exists_{x:A} B(x)$	disjoint sum	total space
$\prod_{(x:A)} B(x)$	$\forall_{x:A} B(x)$	product	space of sections
Id_A	equality $=$	$\{ (x, x) \mid x \in A \}$	path space A^I

Table 1: Comparing points of view on type-theoretic operations

Figure 1: Taken from the HoTT book

Condensed Thesis of forthcoming book 'Semantic tradition of pictures to syntax to inferences'

Eric Schmid

September 3, 2023

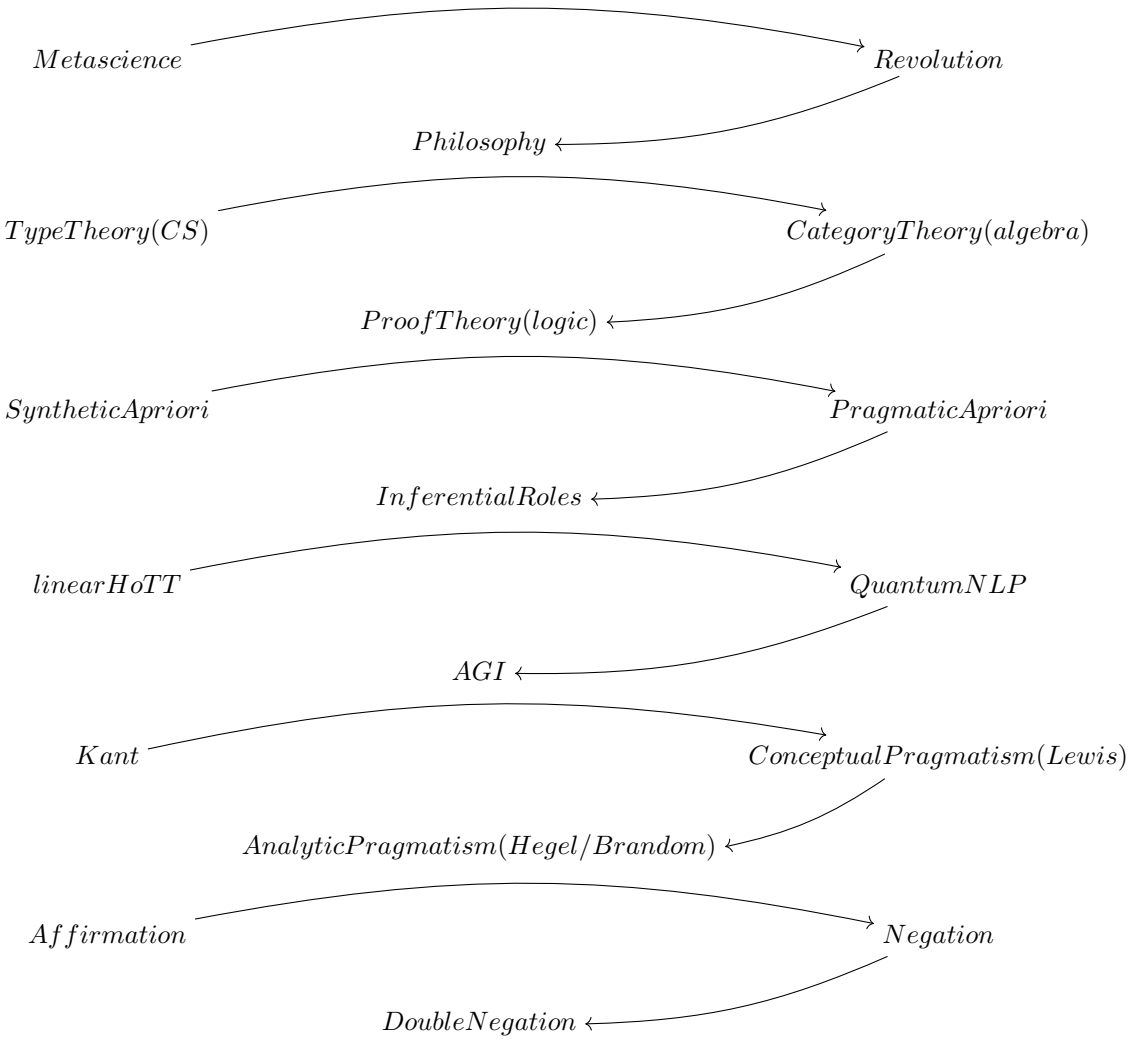
Following Reza Negarestani, the aim of this book is to argue for a Kantian-Hegelian idealism, but where historical epochs are divided into the three temporal periods of Michael Friedman's "dynamics of reason": metascience, revolution and philosophy. The first temporal period, metascience, is explicitly Kantian in terms of the (mathematical) presuppositions behind formulating physical theories (Kant had said geometry was a priori). This time period (the epoch of metascience) is structured by a transcendental logic which necessitates a "synthetic a priori" of existent mathematical knowledge in order to "philosophically refashion" it into scientific theory. While universally valid during this period, this a priori is relativized as new scientific revolutions occur. The second period is scientific revolution. Then, the final period is deliberation on the aftermath, which Friedman calls the period of "philosophy." I claim the overarching temporal teleology/progress through the entirety of the three periods is explicitly Brandomian-Hegelian, while the first period is Kantian. **The central thesis of this book is that this metatheory for scientific progress can be expressed through the language of Homotopy Type Theory/Univalent Foundations (HoTT/UF).**

Moreover, due to the wide scope of (in particular *linear*) homotopy type theory (using quantam natural language processing), **this metatheory can be applied not just to scientific progress, but ordinary language or any public language** defined by sociality/social agents as the precondition for the realizability of (general) intelligence via an inferential network from which judgement can be made. How this metatheory of science generalizes to public language is through the recent advances of quantum natural language processing, but the traditional metalogical encodings (of Tarski) is relatively comparable through universes such as the **type of types** or **type of types of types** found in the inherent inferentialism of UF/HoTT via ∞-groupoids. The "computational trinity" of proofs=programs=algebra in HoTT also means reason is defined functionally as "what it does" by what it computes (proofs are programs). Following Negarestani's recent book, through the self's self-realizability, achieving self-consciousness and consciousness beyond selfhood, "geistig manifestation" is achieved in the form of general intelligence, but only through an inferential network of social agents intrinsically encoded through computation.

Diagrams only possible after logician William Lawvere's formalization of Hegel via categories

Eric Schmid

September 4, 2023



Abstract of forthcoming book

Eric Schmid

August 2023

1 Reclaiming the Kantian synthetic a priori:

1.1 **Frege** $\longrightarrow$ *Russell* $\longrightarrow$ *Wittgenstein* $\longrightarrow$ *Carnap* $\longrightarrow$ *Gödel* $\longrightarrow$ *Tarski* $\longrightarrow$ *Church* $\longrightarrow$ *Turing/Kleene* $\longrightarrow$ *Tait* $\longrightarrow$ *Girard* $\longrightarrow$ *Martin−Löf* $\longrightarrow$ *Coquand* $\longrightarrow$ *Voevodsky'sunivalenceaxiom* $\longrightarrow$ *Awodey* $\longrightarrow$ *Altenkirch* $\longrightarrow$ *HoTT/UF*

I am interested in focusing on Carnap's critiques of Wittgenstein, specifically the deracination of 'meaning' and substitution of 'meaning' with syntax. Analytic philosophy was created at the height of neo-Kantianism in the aftermath of German Idealism. Obscure terms and concepts were much overused at the time (such as Hegel's philosophy). But I am interested in a return to an 'analytic pragmatism' as advocated by Brandom and his readings of Hegel. There was a flood of what Reza Negarestani calls 'metaphysical bloatware' (inflationary metaphysics) at the turn of the 20th century. Russell's pupil Wittgenstein famously said that where we cannot know something, we must remain in silence (Tractatus). Is analyticity only meaningful when it is devoid of all content and meaning (i.e. only positivistic logico-philosophico propositions where all deductions/math are implied from itself deductively like a succession of dominos)? All math is tautology? "5.133 All deductions are made a priori" (Tractatus) And is analyticity only meaningful when philosophy is truly 'tolerant' (in Carnap's sense) for a multitude of interpretative frameworks (metalanguages) for instituting the object language of scientific theory (the language where one has propositions or the arithmetic, e.g. Peano arithmetic) through the supporting "ocean of metalanguages" to use Steve Awodey's term (e.g. Goedel's encoding of the Peano Arithmetic through prime numbers) backing this object language. My plan is to argue for the inferentialist account of meaning (Brandom), that meaning is understood in terms of use and that semantics is inherently an ethical question tied to commitments to discursive norms.

I am interested in surveying the history of early analytic philosophy and then connecting semantic inferentialism to intuitionistic type theory (connecting philosophy to mathematics). I am interested in drawing a connection between the philosophical (semantic anti-realism) and the logical (with the codification of intuitionism in Martin Lof type theory + recent work in HoTT by Awodey and

many others). Frege established the analytic demarcation of a priori reasoning which would inform Russell's logicism and Wittgenstein's construction of fact or tautology. Carnap would make the logical framework robust in the notion of analyticity–a radical opposition to the synthetic a priori. But what about the morning star and the evening star? How does one identify the same sense of different denotations? Frege's work on incomplete arithmetic expressions provides a hint of the functional paradigm after the work of Church's lambda calculus, defined by functionals. Tarski posed a great challenge to Carnap with the undefinability of truth theorem wherein every metalanguage necessitates a metalanguage to encode prior object language; the model theoretic paradigm begins. Yet I am interested in an alternative reading of the split between Tarski and Carnap. Through the work of intuitionistic logic, propositions are taken as if the witness matters. This would later culminate in type theories such as Martin-Lof and substructural logics such as Girard's linear logic. The semantic anti-realist position (where the opposition between transcendent truth and constructive truth can be situated between Platonist and intuitionist philosophy of math) harkens back to Carnap's original principle of verification via the witness of intuitionistic logic, yet situated in an inferential network. Through the constructive possibilities of Homotopy Type Theory, a new paradigm for the foundation of mathematics, the identity of types with topology extends the synthetic lineage of the a priori through a mathematical codification of intuitionism (which remain mere mental constructions).

I am interested in the split between Carnap and Tarski over metalogic regarding semantics. Carnap crystallized the bare minimum of structure for an alien civilization to understand our language based on propositions and variables defining such propositions. Carnap argued that we should be tolerant of a multitude of metalanguages for instituting the object language in a logical syntax of language. What are the possibilities of defining a universally-quantified language or is such an endeavor doomed to fail because of Goedel's theorem regarding incompleteness? What about the recent advancements in the foundation of mathematics, i.e. Homotopy Type Theory? There has been a continual casting away of the synthetic a priori by Russell, the positivists and later Quine and Putnam. The central question of the book will be whether to admit ontologically, metaphysical realism or anti-realism and then epistemologically, semantic realism or anti-realism? Per Martin-Löf in his tracts on the philosophy of math accepts the synthetic a priori, which Kant postulated as existent with geometry and arithmetic. I will be arguing for the synthetic a priori in terms of synthetic geometry such as Homotopy Type Theory, which is at the nexus between theoretical computer science and algebraic topology. The computational types can be transported directly into geometric terms through Steve Awodey's interpretation of the Univalence Axiom. One of the main readings of Frege's logicism, which is the school of philosophy of math which believes in the fundamentality of atomic propositions, will be Michael Dummett's work regarding the bivalence of truth statements and the justificatory power of demonstrating a proof to a witness via the assertion. This is what Robert Harper calls "logic as if people matter." The key disagreement between Martin-Löf and Dummett is whether

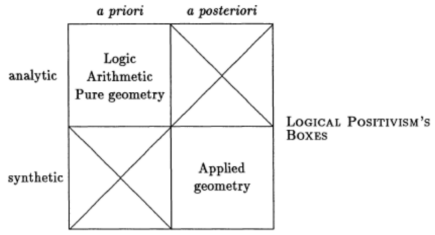


Figure 1: Taken from "The Taming the True"

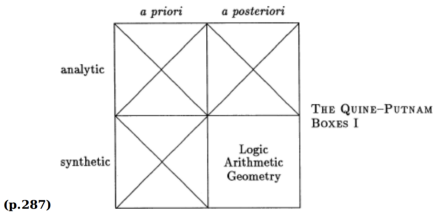


Figure 2: Taken from "The Taming of the True"

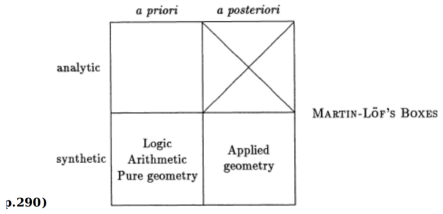


Figure 3: Taken from "The Taming of the True"

proof of a statement is an ontological or epistemological claim with Martin-Löf arguing for the latter and Dummett for the former. Dummett was the key proponent of the semantic anti-realist school and argued that all language can be reduced to its logical basis in the philosophy of math: either one is a Platonist or one is a intuitionist. Dummett was a proponent of verificationism, which Carnap and the logical positivists believed was central to establishing the veracity of empirical phenomenon. David Corfield argues that that inferentialist school of Brandom would be the most faithful way to read the philosophy of Homotopy Type Theory:

I believe that there are grounds to hope that large portions of Brandom's program can be illuminated by type theory. Since Brandom's inferentialism derives in part from the constructive, proof-theoretic outlook of Gentzen, Prawitz and Dummett, this might not be thought to be such a bold claim. However, his emphasis is generally on material inference, only some aspects of which he takes to be treatable as formal inference. For instance, the argument he gives in Making it Explicit (Brandom 1994, Chap. 6) for why we have substitutable singular terms and directed inferences between predicates, is presented with a minimal formal treatment, and yet this phenomenon makes very good sense in the context of HoTT when understood as arising from the different properties of terms and types. Indeed, the kind of category whose internal structure our type theory describes, an (,1)-topos, presents both aspects—the '1' corresponding to unidirected inference between types (morphisms between objects) and the '' referring to the reversible substitutability of terms (2-morphisms and higher between morphisms). On the other hand, his frequent use of formalisms, even very briefly to category theory (for example, Brandom 2010, p. 14), tells us that he recognizes the organizing power of logical languages. Elsewhere (Brandom 2015, p. 36), he notes that while both Wittgenstein and Sellars emphasized that much of our language is deployed otherwise than for empirical description, whereas Wittgenstein addressed this excess as an assortment, Sellars viewed much of it more systematically as 'broadly metalinguistic locutions'. These additional functions of language pertain to the very framework that allows for our descriptive practices, and are what Brandom looks to make explicit in much of his work. My broader suggestion in this book is that we look to locate these locutions in features of our type theory, especially those modalities we will add in Chap. 4. (Corfield 2020)

Through inferentialism, truths are not related necessarily to its meaning, but rather meaningfulness in the sense of the meaning in context or meaning-as-use. The book will be situated in the history and philosophy of language and mathematics. I hope to further the research on Homotopy Type Theory being done by Corfield, Andrei Roden and Awodey and extend this mathematical subfield into the larger philosophical lineage of philosophy of language. If time permits, I would like to compare the pragmatist reading of philosophy of language with the rationalist nominalist reading of Jean Cavailles. My book would examine the universality of mathematical necessity through the mathematical philosophy of Cavailles. Through the mathematical Absolute we arrive at both the contingent accidents and mathematical necessity that allows for the development of new

theorems, but moreover mathematical experience as a determined concept. Can we found universality (as in Cavailles) on the generality of infinity-groupoids in Homotopy Type Theory, i.e. globally, or is it necessary that metalinguistic locutions be situated in terms of the inferentialism of Homotopy Type Theory, i.e. locally?

Some conceptual propositions of my forthcoming book 'Semantic Tradition of Pictures to Syntax to Inferences'

Eric Schmid

September 2, 2023

1 Following Reza Negarestani

Intelligence needs to be made intelligible, the mind needs to be approached functionally in terms of "what it does" (behaviors and functions), which needs to exist within a "history of histories" (spirit).

2 Brandom's Claim

Brandom makes the claim that not only Geisteswissenschaften exists within history but moreover Naturwissenschaften. Science inherently has a historical dimension, which is not to say that it is socially constructed but rather the semantic truth of scientific theory involves context-dependent norms that make determinate scientific truth.

3 Michael Friedman in *Dynamics of Reason*

Michael Friedman argues that there is a contextual milieu that provides the relative a-priori for each and every scientific revolution. David Corfield says, "If we are to follow Friedman's schema, then the period we are currently in is his 'metascientific' one, where thinkers refashion mathematics and reformulate physical principles in a philosophically-minded way. Think of Helmholtz, Mach, Clifford, Klein, Poincaré,..., Einstein." This a-priori is both universally valid and relativized.

4 Brandom's Argument

Brandom argues that most statements in a language need to be materially good and are not substitutable, i.e., not formally inferred. Following David Corfield's book on the philosophy of HoTT, it is agreeable that Homotopy Type Theory

can correct this through "terms and types" of (∞,1)-topos where 2-morphisms, 3-morphisms, 4-morphisms, etc., are reversibly substitutable, and therefore formally inferred. Corfield has pointed me (in a private email) in the direction of Quantum Natural Language Processing, in particular linear homotopy type theory, and he says it provides a means of defining a topological metric between similar words, and formal substitution is defined by such a metric.

5 Semantic Ascent vs. Semantic Descent

As opposed to J.N. Findlay, who argues for semantic ascent in Hegel from the object-language to the meta-language (a move championed by Tarski) in order to define a judgment, Brandom argues for semantic descent to the bottom level or the materially good inferences.

6 Theorem

Apart from the original morphisms and objects directed in the lowest level, there exists a transit up and down and vice versa between object language and metalanguage through (∞,1)-topoi. In Agda (the proof-checker/proof-assistant language for Univalent Foundations) **a type of types** or **type of types of types** corresponds to varying universe levels. For example, see Agda documentation: "Agda' type system includes an infinite hierarchy of universes Set_i : Set_{i+1}. This hierarchy enables quantification over arbitrary types without running into the inconsistency that follows from Set : Set."

7 The Universally-Valid A Priori

The universally valid a-priori prior to a scientific revolution, which informs said revolution, is context-dependent and exists within a history of histories.

8 The Curry-Howard Correspondence

The Curry-Howard Correspondence states that $\prod$ types are equivalent to $\forall$ intuitionistic quantifiers, $\sum$ types are equivalent to $\exists$ in logic, and maps are equivalent to intuitionistic implication.

- "Types correspond to logical formulas (aka propositions)."
- "Programs correspond to logical proofs."
- "Evaluation corresponds to simplification of proofs."

Mind becomes only what it does functionally and therefore what is computable. Spirit becomes context-dependent codifications of public language in an inferential network of ∞-topoi.

9 Univalence Axiom and Awodey's interpretation

9.0.1 $(A = B) \cong (A \cong B)$

9.0.2 Awodey (as quoted in private email): "The Univalence Axiom was indeed the work of Voevodsky, but the interpretation of identity types as topological path spaces, which forms the basis of HoTT, was in fact due to me."

10 Fibrations and Co-fibrations

Fibrations are equivalent to $\prod$ types and co-fibrations are equivalent to intervals

11 Pure Geometry, Arithmetic, Logic

Following Per Martin-Löf, it is agreeable that pure geometry, arithmetic and logic are modes of synthetic a priori knowledge epistemologically.

1 *Hegel in Mathematics*, Alexander Prähauser (May 2022):

1.0.1 **"It is ironic that the concepts Russell used in his attack, infinitesimals and "continuity", have been particularly useful, once re-evaluated through the recent formalization of Hegel's thought in the context of modal homotopy type theory. This was work started by William Lawvere in the 1985s [16]"**

1.0.2 **"In early 1985, while I was studying the foundations of homotopy theory, it occurred to me that the explicit use of a certain simple categorical structure might serve as a link between mathematics and philosophy."**

1.0.3 **"Lawvere went on to provide a formally strict logical calculus that tries to cap- ture Hegelian dialectics and started the formalization of Hegel's objective logic [15]. However, the mathematical power of Lawvere's formalization, though already considerable was restricted by the mathematics it was founded on.1 More recently, a new foundation of mathematics was developed under the initiative of Vladimir Voevodsky in homotopy type theory, which provides an alternative to set theory and a setting for logic based on a radical interpretation of equality, which amplifies its power and has been found to show remarkable similarity to Hegel's thought. However, the importance of these devopments exceeds math- ematics and reaches into philosopy. Lawvere stated in 1992 [13]:"**

1.0.4 **"It is my belief that in the next decade and in the next century the technical advances forged by category theorists will be of value to dialectical philosophy, lending precise form with disputable mathe- matical models to ancient philosophical distinctions such as general vs. particular, objective vs. subjective, being vs. becoming, space vs. quantity, equality vs. difference, quantitative vs. qualitative etc. In turn the explicit attention by mathematicians to such philosophi-cal questions is necessary to achieve the goal of making mathematics (and hence other sciences) more widely learnable and useable. Of course this will require that philosophers learn mathematics and that mathematicians learn philosophy."**

Therefore, the formal order of self-consciousness (the logical $I{=}I$) first and foremost points to the underlying structure of what appears as a simple identity relation. This underlying structure is nothing other than the transformation afforded by conceiving I and I^* as identity maps ($I{=}I^*$). Adopting the Hegel-inspired mathematical formalism of William Lawvere, this map or composition of maps can be represented as follows.[31]

If we treat I and I^* as objects with their respective identity maps, then $I{=}I^*$ is really:

$$I \xrightarrow{f} I^*$$

meaning that I (formal self-consciousness) is the identity map of the domain I^* (concrete self-consciousness or the assertion that there is reality in excess of the self or mind) and I^* is the identity map of the codomain of I (the freedom of self-consciousness as conceived from a reality that is of nowhere and nowhen, a concrete freedom in which self-consciousness only exists in every respect for another self-consciousness). It then universally and necessarily follows that:

$$I \xrightarrow{f} I^* \Rightarrow I \xrightarrow{I} I \wedge I \xrightarrow{f} I^* \wedge I^* \xrightarrow{I^*} I^* \wedge If = f = fI^*$$

In concretely rendering reality intelligible, in expanding the domain of the intelligible and hence that of reality, in acting on the intelligible or intervening in reality, the formal condition of intelligence (I) is realized as intelligence (I^*). Formal self-consciousness only becomes self-consciousness in satisfying another self-consciousness,[32] in extending over into the intelligibility of a reality which in its unrestrictedness establishes the truth of I, the mind, or intelligence. Yet the achievement of this truth (I^*) is also impossible

31 F.W. Lawvere, *Functorial Semantics of Algebraic Theories* (New York: Columbia University Press, 1963).

32 'Self-consciousness attains its satisfaction only in another self-consciousness'—and without this satisfaction it is only a consciousness that finds its 'satisfaction in mere dirt and water'. Hegel, *Phenomenology of Spirit*, §8, §175.

33

Figure 1: Reza Negarestani, *Intelligence and Spirit*

4 Calculus of Hegelian dialectics

Using category theory, we can explain Schreiber's formulation of Lawvere's formalization of Hegelian dialectics. For this, what Hegel calls a "Moment" is formalized as a (co)modality on a category [6] C: an operator $\square$ that maps an object X of C to an object $\square X$ called the *aspect* of X under $\square$, maps a morphism $Y \xrightarrow{f} X$ to a morphism $\square Y \xrightarrow{\square(f)} \square X$ between the aspects of X and Y, and is idempotent: $\square\square = \square$, together with, for each object X, a morphism between X and the image of X under the operator: $\square X \to X$ or $X \to \bigcirc(X)$. The direction of this morphism depends on whether the moment is a modality or a comodality. We will call moments of the form $\square(X) \to X$ *previous* moments and denote them by $\square$, while we call moments of the form $X \to \bigcirc(X)$

[4] However, a mathematical object can live in different categories at once, if it allows, or necessitates, several different kinds of structures. For instance, the "real numbers" are a field, and as such live in the category of fields, but they are also a total order, so they live in the category total orders. If an object lives in several categories at once, the internal properties of the object often give rise to relations between the categories it is living in.

[5] Of course, this is a *mathematical* universe that houses *mathematical* structures, we will only get to our physical universe at the very end.

[6] In later sections we will mainly look at universes of types, but the potential applicability of Lawvere's calculus is much more general.

Figure 2: Alexander Prähauser

successive moments.[7][8]

Due to idempotency, any type that lies in the image of a moment is *purely of that moment* (and vice versa): $Y = \Box X \Leftrightarrow \Box Y = Y$. Thus, the types that are invariant under a moment form their own sub-category: $C_\Box \hookrightarrow C$. The image $\Box X$ of a type X under that moment should be thought of as "the best possible approximation of X in $C_\Box$" or the $\Box$-aspect of X, and the transformation $\Box X \to X$ (resp. $X \to \bigcirc X$), depending on the exact nature of the moment, either a deduction of X from $\Box X$ ($\bigcirc X$ from X), an embedding of $\Box X$ into X (X into $\bigcirc X$), or a construction/deformation that makes $\Box X$ into X (X into $\bigcirc X$).

From a category-theoretic perspective, this state of affairs can be understood as saying that a moment $\Box$ can be decomposed into a *projection* of C onto another category $C_\Box$:

$$C \twoheadrightarrow C_\Box$$

and an *embedding* of $C_\Box$ into C

$$C_\Box \hookrightarrow C.$$

Now, the concept of a *unity of opposites* is translated into a pair of moments $\triangle_1 \dashv \triangle_2$ that fulfill the *adjointness condition*: for any two objects X, Y and each morphism $\triangle_1 Y \to X$, there exists a morphism $Y \to \triangle_2 X$, and these morphisms are subject to a further naturality condition. Adjunctions are deep structures that can be expressed in a variety of ways, this being the most concrete, and the reader cannot be expected to immediately grasp the meaning of the adjointness condition. However, some of its consequences can be used for better understanding. For instance, it follows that if one of the $\triangle_i$ is a preceding moment $\Box$, then the other is a successive moment $\bigcirc$ and vice versa, so that a unity of opposites is either a *unity of a preceding to an opposite successive moment* (or *ps-unity* for short) of the form $\Box \dashv \bigcirc$ or a *unity of a successive to an opposite preceding moment* (or *sp*-unity) $\bigcirc \dashv \Box$. So a unity of opposites is made up of moments that are actually of opposite kinds (preceding to successive), and two opposite kinds of unities of opposites exist. Furthermore, one part of a unity of opposites uniquely determines the other in that, if, for a moment $\Box$, $\Box \dashv \bigcirc_1$ and $\Box \dashv \bigcirc_2$, then $\bigcirc_1$ and $\bigcirc_2$ are equal, and the other way around (however, a moment can have both a left and a right opposite, so the opposition is actually directed). Each type X sits in between its preceeding and successive aspects (let us call this the *aspect sequence of the unity*):

Figure 3: Alexander Prähauser

$$\Box X \to X \to \bigcirc X.$$

Finally, the two sub-universes $C_\Box$, $C_\bigcirc$ determined by the modalities are both equal *without the context of the surrounding universe* C, so their opposition actually lies in their relationship to the larger category, and to be opposites they have to be the same. More precisely, we can, recalling the decomposition of moments into a projection and an embedding we saw before, say that the opposites of a unity share each one of their morphisms, and describe the situation in the following way: a *ps*-unity $\Box \dashv \bigcirc$ consists of *two* embeddings of the same sub-universe $C_\Box = C_\bigcirc$ into the universe C and *one* projection from C onto $\mathcal{V}$:

$$C_\Box \underset{\longleftarrow}{\overset{\hookrightarrow}{\hookrightarrow}} C$$

while an *sp*-unity consists of *one* embedding of $\mathcal{V}$ into C and *two* projections of C onto $\mathcal{V}$:

$$C_\Box \underset{\longleftarrow}{\overset{\longleftarrow}{\hookrightarrow}} C$$

From this follows in particular that in an *sp*-unity $\Box \bigcirc X = \Box\Box Y = \Box Y = \bigcirc X$ and vice versa, so $\Box\bigcirc = \bigcirc$ and $\bigcirc\Box = \Box$. Thus, a type is purely of one moment if and only if it is purely of the other.

Adjunctions are category-theoretic concepts and thus can be applied in any suitable 2-category, so that an adjunction between a unity of opposites might itself be adjoint to a another unity of opposites [3]. This *opposite to a unity* of opposites $\triangle_1 \dashv \triangle_2$ is another unity of opposites, and their relation will be denoted

$$\begin{array}{ccc} \triangle_3 & \dashv & \triangle_4 \\ \perp & & \perp \\ \triangle_1 & \dashv & \triangle_2 \end{array}.$$

From the uniqueness of adjoints follows that in such a configuration, the left moment of the first unity has to be equal to the right moment of the second unity: $\triangle_1 = \triangle_4$. So each unity of opposites of unities takes the form of a string of modalities $\Diamond \dashv \Box \dashv \bigcirc$ or $\bigstar \dashv \bigcirc \dashv \Box$. We could ask for even higher opposites, but examples are rare and largely in categories with exotic or no logics.[9]

The other significant relation between unities of opposites is that of Aufhebung. Following (Schreiber's formalization of) Hegel, we say a unity of opposites $\triangle_3 \dashv \triangle_4$ is a *higher sphere*[10] of a unity of opposites $\triangle_2 \dashv \triangle_1$, denoted

Figure 4: Alexander Prähauser

$$\begin{array}{ccc} \triangle_3 & \dashv & \triangle_4 \\ \vee & & \vee \\ \triangle_1 & \dashv & \triangle_2 \end{array}$$

if $\triangle_1$ is contained in $\triangle_3$: $\triangle_3\triangle_1 = \triangle_1$ and $\triangle_2$ is contained in $\triangle_4$: $\triangle_4\triangle_2 = \triangle_2$. A higher sphere $\square_2 \to \bigcirc_2$ of a unity $\square_1 \to \bigcirc_1$ is a *right Aufhebung* of $\square_1 \to \bigcirc_1$ if furthermore $\square_1$ is contained in $\bigcirc_2$: $\bigcirc_2\square_1 = \square_1$ and a *left Aufhebung* if $\bigcirc_1$ is contained in $\square_2$: $\square_2\bigcirc_1 = \bigcirc_1$. Both kinds can be referred to as *Aufhebungen* We can similarly define Aufhebungen for *sp*-unities $\bigcirc \dashv \square$, however, since in an *sp*-unity both moments project into the same sub-universe, each higher sphere of $\bigcirc \dashv \square$ is already both a left and a right Aufhebung. We will denote a right Aufhebung as

$$\begin{array}{ccc} \square_2 & \dashv & \bigcirc_2 \\ \vee & & \vee \\ \square_1 & \dashv & \bigcirc_1 \end{array}$$

and a left Aufhebung as

$$\begin{array}{ccc} \square_2 & \dashv & \bigcirc_2 \\ \vee & & \vee \\ \square_1 & \dashv & \bigcirc_1 \end{array}$$

So an Aufhebung of $\square_1 \dashv \bigcirc_1$ is a unity of opposites $\square_2 \dashv \bigcirc_2$ such that $\square_1$ is a special aspect of $\square_2$ and $\bigcirc_1$ is a special aspect of $\bigcirc_2$, but where *also* both $\bigcirc_1$ and $\bigcirc_2$ are special aspects of one of the aspects of $\square_2 \dashv \bigcirc_2$, let's say $\bigcirc_2$, so that they are unified in $\bigcirc_2$ and both on the same side of a greater opposition. From this follows that the opposition $\square_2 \dashv \bigcirc_2$ is trivial on, in this case, $\square_1$: $\square_2\square_1 = \square_1 = \bigcirc_2\square_1$. Please note that this construction captures all three meanings of the German word *Aufheben*: to lift, to preserve and to negate.

Generally, Aufhebungen are not unique, the same unity can have several of them. The *minimal (left/right) Aufhebung* of a unity $\square \dashv \bigcirc$, the smallest sphere that fulfills the Aufhebungs-condition would be unique, but does not always exist.[11]

Figure 5: Alexander Prähauser