From Metaphysics to Mathematics

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19th century mathematicians had a habit of talking about *the metaphysics of infinitesimal calculus*, and *the metaphysics of the theory of equations*. This referred to a collection of vague analogies, that were hard to grasp and hard to formulate, yet seemed to play an important role at a given moment in mathematical research and discoveries. Did they slander *real* metaphysics by using its name to designate something which, in their science, was the least clearly defined? I will not try to elucidate that point. Either way, this word will have to be understood in their sense; as referring to *true* metaphysics, I will refrain from interfering.

Nothing is more fertile, as all mathematicians know, than those obscure analogies, those blurry reflections from one theory to the the other, those furtive caresses, those inexplicable ramblings; nothing brings the researcher more joy. There comes a day when the illusion dissipates, when the presentiment turns into certitude; the twin theories reveal their common source before disappearing; as taught in the Gita, we attain knowledge and indifference at the same time. Metaphysics has become mathematics, ready to form the material of a treatise whose frigid beauty is no longer able to move us.

Thus we know what Lagrange attempted to descry, when he spoke of metaphysics with regards to his work in algebra; he almost touches Galois' theory, through a screen he cannot pierce. Where Lagrange saw analogies, we see theorems. But those can only be expressed using notions and *structures* that, for Lagrange, had not yet become mathematical objects: groups, fields, isomorphisms, automorphisms, all this still had to be conceived and defined. Though Lagrange was constantly sensing these notions, though he toiled in vain to attain their substantial unity through the multiplicity of their changing incarnations, he remains in the grasp of metaphysics. At least, it is there that he finds the thread leading him from one problem to the next, bringing materials to the construction site, arranging everything in view of the general theory to come. Thanks to the decisive notion of a group, all this becomes mathematics to Galois.

Further still, we see how analogies between the calculus of finite differences and differential calculus serve as a guide for Leibniz, Taylor, and Euler in this heroic period in which Berkeley could say, with as much humour as appropriate, that the *believers* of infinitesimal calculus were poorly qualified to criticise the obscurity of the mysteries of christian faith, the former being as full of mysteries as the latter. A little while later, d'Alembert, enemy of all metaphysics in mathematics as well as elsewhere, claimed in his articles in the Encyclopedia, that the true metaphysics of infinitesimal calculus was none other than the notion of a limit. If he did not extract from this idea all the consequences to which it was susceptible, the developments of the century following him certainly proved him right; and nothing could be clearer today, or, and it must be said, more annoying, than a correct treatise of the elements of differential and integral calculus.

Fortunately for those researchers, the extent to which fog can dissipate over a point, is just to coalesce over another. A large part of the Tokyo colloquium was held under the sign of analogies between number theory and the theory of algebraic functions. This is pure metaphysics. It is those analogies, because I have quite some personal experience with them, that I want to discuss here, with the hope, possibly in vain, of giving the readers of this revue, those honest people, some ideas about the *modus operandi* of mathematics.

Starting in elementary education, we show students how division of polynomials (in one variable) is quite similar to division of integers and gives rise to entirely similar laws. For one just like the other, there is a greatest common divisor, which can be determined by successive division. To the decomposition of integers into prime factors corresponds the decomposition of polynomials into irreducible factors. To rational numbers correspond rational functions, which can also be written as irreducible fractions; these are added by reduction to the least common denominator, etc. It is therefore very natural to think that there is an analogy between algebraic numbers (the roots of equations whose coefficients are integers) and algebraic functions in one variable (roots of equations whose coefficients are polynomials in one variable).

The founder of the theory of algebraic functions would have been Galois had he lived; that is what one would conclude from the indications one finds concerning this subject in his famous testament-letter, written the night before his death, whence one can conclude he was already approaching some of Riemann's principal discoveries. Maybe he would have given this theory an algebraic allure, in conformity with contemporary work by Abel and his own research in pure algebra. Riemann, however, undoubtedly one of the least algebraically minded among the mathematicians of the 19th century, placed this theory under the label of *transcendental* (a word which, for mathematicians, is opposed to *algebraic*, and designates all that belongs properly to the continuum). The very powerful methods that Riemann brought to fruition brought his theory to a degree of achievement that has hardly been surpassed. But they do not keep track of analogies with algebraic numbers, and cannot be transposed as such into the study of these, study which traditionally gives rise to arithmetic or number theory and which, even at the time of Riemann's life, was on a path of rapid development.

It was Dedekind, close friend of Riemann, but consummate algebraist, who would describe the first of such analogies and make them into a tool fit for research. He successfully applied methods he had created himself and perfected with a view towards the study of arithmetic to the problems Riemann had treated in a transcendental way; and he showed that this reveals the properly algebraic side of Riemann's work.

At first sight, the analogies brought to light as such remained superficial, and did not seem seem to shed light on the deepest problems in either theory. Hilbert went further down this path, or so it seems, but if it is probable that his students were influenced by his ideas on this subject, only a few traces of it remain in an obscure account that hasn't even been reproduced in his complete works. The unwritten laws of modern mathematics effectively prohibit one from publishing such metaphysical views. Undoubtedly, this is good; lest we be overwhelmed by articles more stupid or useless than those that presently congest our publications. But it is a pity that Hilbert's ideas were never developed by him. There was still quite a distance, however, between arithmetic, where the discontinuous reigns, and the theory of functions in the classical sense. However, in stating that algebraic functions are roots of equations whose coefficients are polynomials, I consciously omitted a salient point: Those polynomials, what coefficients do they have? When we treat division of polynomials in elementary education, it goes without saying that these coefficients are *numbers*: *real* numbers (rational or not, but in either case given by a decimal expansion if one so wishes) or, at a slightly more advanced level, *real or imaginary* numbers, or, as one says, *complex numbers*. In Riemann's theory, one is exclusively concerned with complex numbers.

But from the point of view of the pure algebraist, all that one asks from these *numbers* in question, is that they let themselves be combined with each other using four operations (which an algebraist expressed by saying that they form a *field*). If we do not assume anything more, we obtain a theory of algebraic functions, already very rich (as witnessed by the recent and already classical volume published by Chevalley on this subject), but that is insufficient to pursue the analogy with algebraic numbers to its end.

Fortunately, there is an intermediate domain between arithmetic and Riemannian theory, and which bears resemblance with both of the former theories, resemblances more narrow that between the two theories themselves; this is the theory of algebraic functions *over a finite field*. As we have known since Gauss, if we only require those four operations, a finite number of elements suffices. It is sufficient, for example, to have two, denoted 0 and 1, and for which we impose the following addition and multiplication table:

$$\begin{array}{lll} 0+0=0 & 0+1=1+0=1 & 1+1=0 \\ 0\times 0=0 & 0\times 1=1\times 0=0 & 1\times 1=1. \end{array}$$

However paradoxical the rule $1+1=0$ may seem to the profane, however tempting it may be that this is nothing but a mind game which does not correspond to any *reality*, such a system is everyday currency for the mathematician; and Galois used it extensively by constructing *Galois' imaginaries*.

Letting the coefficients of our polynomials be elements of a *Galois field*, we construct algebraic functions whose theory goes back to Dedekind, but was particularly developed development after Artin's thesis. To describe what it consists of, one would have to enter into exceedingly technical details that would be out of place here. However, one can, I believe, give an illustrated idea of it by saying that the mathematician who studies these problems has the impression that they are deciphering a trilingual inscription. In the first column, one finds the Riemannian theory of algebraic functions in the classical sense. The second column is the theory of arithmetic of algebraic numbers. The central column, whose discovery is the most recent, contains the theory of algebraic functions over a Galois field.

These texts are the unique source of our knowledge of these languages in which they are written; of course, we only possess fragments of each column; the most complete one, and the one we read with the most fluency, being the first one. We know that there are great differences in the direction between one column and the next, but nothing to warn us ahead of time. Upon using it, we construct snippets of a dictionary, which frequently allow us to pass from one column to the neighbouring one.

Quite a while ago, we thus deciphered in the last column the beginning of a paragraph titled *zeta function*. Towards the end of this paragraph, we seem to descry a very mysterious sentence; it says that all zeros of the function lie on a certain line. We never figured out if this was true, or if an error was made in decipherment. This is the famous *Riemann hypothesis*, which will will turn 100 in a few months.

Artin's principal discovery, in his thesis, was that in the second column there is a paragraph that carries the same title of *zeta function*, and which is approximately a translation of the one we already know; our dictionary has expanded greatly. Artin also noticed, in this column, the sentence on the Riemann hypothesis; and it seemed to him just as mysterious as the other one. This new problem, at first sight, did not seem any easier than the former. In reality, we know nowadays that the first column already contained all the elements of his solution. It was just a matter of translating, first into *abstract* theory of algebraic functions, then into the *Galoisian* language of the second column, old results previously obtained by Hurwitz in the *Riemannian* language, and that the Italian geometers had subsequently translated into their own language. But leading arithmeticians and *Galoisian* experts can no longer read *Riemannian*, or Italian for that matter; and it took twenty years of research before the translation was perfected, and the proof of the Riemann hypothesis in the second column was deciphered.

If our dictionary were sufficiently complete, we would simply pass from the second to the third column, and Riemann's hypothesis, the real one, would be proven as well. But our knowledge does not reach that far; many patient decipherments are still necessary before the translation can be completed. During the aforementioned colloquium, there were many discussions of *metaphysics* concerning these problems; one day these will make way for a mathematical theory in which they will find their solution. Maybe, as was the case for Langrange, we are lacking, for this definitive step, nothing more than a notion, a concept, a *structure*. Ingenious philologists have found the secret of Nestor's archives, and those of Minos. How much longer before our Rosetta stone, the arithmetician's own, will be united with its Champollion?