INTRODUCTION TO SHEAF THEORY AND SHEAF COHOMOLOGY

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1. Introduction: Local-to-Global Principles and Motivation

Many problems in mathematics exhibit a tension between **local** behavior and **global** behavior on a space. *Sheaf theory* provides a systematic framework to track data locally on a topological space and determine how to "glue" or assemble this local data into global structures. This local-to-global perspective is pervasive in areas such as algebraic geometry, topology, differential geometry, and even logic and computer science. The guiding question is: *when can local solutions or constructions be uniquely merged into a global one?* [GQ24]

1.1. **The Local-to-Global Phenomenon.**

We illustrate the local vs. global distinction with a few examples:

- **Topology:** A continuous function can be defined locally on each set of an open cover of a space, but these local definitions might not agree globally. For instance, a manifold can be covered by coordinate patches (local coordinate functions exist), yet there may be no single global coordinate chart covering the entire manifold.
- **Algebraic Geometry:** Regular functions on an algebraic variety are defined locally on affine patches. Whether these local functions extend to a global regular function is a central question. In fact, one defines varieties by gluing together affine algebraic pieces, and not every local regular function extends globally.
- **Differential Geometry:** One can define vector fields or differential forms on local neighborhoods of a manifold, but there may be global obstructions. A classic example: a nowhere-vanishing local section of the tangent bundle exists in small patches, but on a non-orientable manifold it cannot be extended globally (this relates to the non-existence of a global nowhere-vanishing vector field, an obstruction measured by the orientability).
- **Gauge Theory (Physics):** Locally one can choose a gauge (a function describing a field, like the electromagnetic potential in each patch), but globally there might be a mismatch leading to nontrivial

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topological classes (e.g. a magnetic monopole yields a potential defined locally but not globally continuous, reflecting a non-zero second cohomology class).

In each case, the question is whether locally defined data (continuous functions, sections of a bundle, etc.) that agree on overlaps can be "glued" to a valid object on the union of the patches. Sheaf theory formalizes this situation by assigning data to every open set and imposing conditions that ensure consistency and gluability of that data. The advantage of the sheaf perspective is that it provides a language and tools (notably sheaf cohomology) to measure the *obstructions* to gluing local data into global data.

1.2. **From Functions to Sheaves.** A naive approach to a local-to-global problem is to consider the collection of all functions (or sections, etc.) defined on each open subset of a space. This leads to the notion of a *presheaf*, which associates to each open U some set (or algebraic structure) $F(U)$ of data on U, together with restriction maps to smaller opens. However, not every presheaf allows gluing of local sections; additional axioms are needed. A *sheaf* is a presheaf that satisfies these axioms ensuring uniqueness and existence of glued sections. We will formalize these definitions in the next sections.

The power of sheaf theory comes not only from organizing data but also from sheaf cohomology, a machinery that measures the failure of the local-to-global principle. Roughly, if local data patch together perfectly, certain cohomology groups vanish; when there are obstructions, those appear as nonzero cohomology classes. Sheaf cohomology has become fundamental in modern algebraic geometry, providing powerful generalizations of classical results (such as the Riemann–Roch theorem, which was originally proved in the 19th century using purely classical methods) and many classification results. It establishes deep connections between topology, analysis, and algebraic geometry.

In this paper, we assume familiarity with basic category theory (categories, functors, limits/colimits) and basic algebraic topology (e.g. the notion of a chain complex and homology), but no prior knowledge of sheaf theory or homological algebra. We proceed from the definition of presheaves and sheaves, through examples and foundational constructions (morphisms, stalks, sheafification), and then focus on the theory of sheaf cohomology. Along the way, we include examples and exercises to illustrate key points. We use standard notation and conventions from algebraic topology and algebraic geometry texts.

2. Presheaves

In essence, a **presheaf** is a rule that assigns to each open set of a topological space some data (such as functions or other mathematical objects defined on that open set), with the requirement that these data are consistent under restriction to smaller opens. The elements of this data set are called

sections, which intuitively can be thought of as functions or mathematical objects that are defined over a specific open set. Presheaves formalize the idea of "local data structure on a space". We give the formal definition and then illustrate with examples.

Definition 2.1 (Presheaf)**.** Let X be a topological space. A *presheaf* F on X consists of the following data:

- For each open set $U \subseteq X$, an object $F(U)$ in some category (such as sets, abelian groups, rings, or modules), whose elements are called **sections of** F **over** U. We often denote a section $s \in F(U)$ by $s|_U$ when we wish to emphasize its domain.
- For each inclusion of open sets $V \subseteq U \subseteq X$, a **restriction map**
$$\rho^F_{U,V} : F(U) \to F(V),$$
which is a morphism in the relevant category, often written as $s \mapsto s|_V$ for $s \in F(U)$, subject to the following axioms:
  - **Identity:** For any open set U, the restriction to itself is the identity: if $id : U \hookrightarrow U$ is the identity inclusion, then $\rho^F_{U,U} = \mathrm{id}_{F(U)}$ (so every section restricts to itself on the same set).
  - **Transitivity:** If $W \subseteq V \subseteq U$ are open sets, then restrictions compose as expected: $\rho^F_{V,W} \circ \rho^F_{U,V} = \rho^F_{U,W}$. In other words, restricting a section from U to W directly is the same as first restricting to V then to W.

The pair (F, ρ^F) (often just F) is called a presheaf on X. Common examples include presheaves of abelian groups, rings, or modules, where each $F(U)$ has the corresponding algebraic structure and each restriction map preserves this structure. We often write $F : U \mapsto F(U)$, U open in X. [Rot09]

Equivalently, for those familiar with category theory, a presheaf of sets on X is the same as a contravariant functor from the category of open sets of X (with inclusions as morphisms) to the category of sets. [Rot09] In this language, F assigns each open U an object $F(U)$ and each inclusion $V \subseteq U$ a morphism $F(U) \to F(V)$ (the restriction), satisfying the functoriality conditions (identity and composition) automatically.

Example 2.2. We list some fundamental examples of presheaves on a topological space X:

(1) **Continuous Functions:** For a fixed target space (e.g. $\mathbb{R}$ or $\mathbb{C}$), let $F(U) = C^0(U)$ be the set of all continuous functions on U. For $V \subseteq U$, define $\rho_{U,V}(f) = f|_V$ by restriction of the domain. This clearly satisfies the identity and transitivity axioms, so $U \mapsto C^0(U)$ is a presheaf on X. Similarly, one can define presheaves $C^k(U)$ of C^k (continuously k-times differentiable) functions on U, or $C^\infty(U)$ of smooth functions if X has a differentiable manifold structure. [Vak24]

(2) **Constant Presheaf:** Fix a set S. Define $F(U) = S$ for every open U, and let each restriction map $\rho_{U,V} : S \to S$ be the identity on S. This presheaf F simply assigns the same set S to every open set (intuitively pretending that a "section" over U is just an element of S, with no further dependence on U). We call this the *constant presheaf with fiber S*. It vacuously satisfies the presheaf axioms (identity and transitivity are trivial here). *Warning:* This F will not in general be a sheaf unless S carries the discrete topology or X is connected (we will revisit this).

(3) **Sections of a Bundle:** Let $\pi : E \to X$ be a (continuous) fiber bundle or more generally any map of spaces. Define a presheaf F by $F(U) = \{\text{(continuous) sections } s : U \to E \text{ of } \pi\}$, i.e. maps s such that $\pi \circ s = \mathrm{id}_U$. If $V \subseteq U$, restriction is given by $\rho_{U,V}(s) = s|_V$, the section obtained by restricting s's domain. This is a presheaf: identity is clear, and if $W \subset V \subset U$, then $(s|_V)|_W = s|_W$. For instance, if E is a topological vector bundle on X, $F(U)$ is the space of continuous (or smooth, etc.) sections of the bundle over U. In an important special case, π could be the trivial bundle $X \times A \to X$ with fiber A an abelian group; then sections on U correspond to arbitrary continuous functions $U \to A$, and F is the presheaf of A-valued functions on X. The special case of locally constant functions arises when A has the discrete topology.

(4) **Open-set Presheaf:** Define $F(U) = \{V \mid V \text{ is an open subset of } U\}$, the power set of U restricted to opens. If $V \subseteq U$ are open, let $\rho_{U,V} : F(U) \to F(V)$ be given by intersection: for $W \in F(U)$ (so W is open in U), $\rho_{U,V}(W) = W \cap V$, which is an open subset of V and hence in $F(V)$. One checks easily the axioms. Intuitively, this presheaf assigns to each open U the collection of all its open subsubsets. This presheaf captures some topological information about X. However, as we will see, it fails to be a sheaf (the gluing axiom will fail in general).

Presheaves are very flexible, but the lack of additional conditions means they might not reflect the desired local-to-global properties. In particular, a presheaf does not require that sections which locally agree must be identical, nor that locally defined sections can always be glued into a global section. These crucial properties are imposed in the definition of a *sheaf*, which we introduce next. The examples above illustrate typical presheaves; among them, (1), (3) will actually turn out to be sheaves, whereas (2) and (4) in general will not (unless in special situations). Before defining sheaves, we formalize how a presheaf can fail the "local consistency" requirements:

Definition 2.3 (Morphisms of Presheaves)**.** If F and G are presheaves on X, a *morphism of presheaves* $\phi : F \to G$ is a collection of maps $\{\phi_U : F(U) \to G(U)\}_{U \subseteq X,\, U \text{ open}}$ such that for every inclusion $V \subseteq U$, the following diagram commutes:

$$\begin{array}{ccc} F(U) & \xrightarrow{\phi_U} & G(U) \\ {\scriptstyle \rho^F_{U,V}}\downarrow & & \downarrow{\scriptstyle \rho^G_{U,V}} \\ F(V) & \xrightarrow{\phi_V} & G(V)\,, \end{array}$$

i.e. $\phi_V(s|_V) = (\phi_U(s))|_V$ for every $s \in F(U)$. This is just a natural transformation of functors in the categorical view. Morphisms of presheaves compose in the obvious way and one obtains the *category of presheaves on* X. If each ϕ_U is bijective, we call ϕ an isomorphism of presheaves (and F, G are isomorphic).

Morphisms of presheaves allow us to compare different presheaf structures. In particular, any sheaf (to be defined) is in particular a presheaf, so these notions apply to sheaves as well. The category of sheaves on X will be a full subcategory of the category of presheaves.

3. Sheaves

A **sheaf** is a presheaf that satisfies the additional local consistency axioms: (1) if two sections agree locally, they must coincide globally (local identity, or separation axiom), and (2) any collection of locally compatible sections arises as the restriction of a unique global section (gluing, or existence axiom). These conditions ensure that the presheaf actually captures the idea of "local data that determine a unique global outcome". More formally:

Definition 3.1 (Sheaf)**.** A presheaf F on X (of sets) is said to be a *sheaf* if it satisfies the following **sheaf axioms** in addition to the presheaf axioms:

- **Locality (Uniqueness/Separation):** If $\{U_i\}_{i\in I}$ is an open cover of an open set $U \subseteq X$, and if $s, t \in F(U)$ are two sections such that $s|_{U_i} = t|_{U_i}$ for *all* $i \in I$, then $s = t$ in $F(U)$. In other words, a section is uniquely determined by its values on the members of an open cover. Equivalently, if a section restricts to the zero (or identity) section on each piece of a cover, it must be zero; no nontrivial section can vanish on all pieces of a cover.
- **Gluing (Existence):** If $\{U_i\}_{i\in I}$ is an open cover of U and we have a collection of sections $s_i \in F(U_i)$ for each i, which are *locally compatible* in the sense that on every nonempty overlap $U_i \cap U_j$, the restrictions agree: $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all i, j, then there exists at least one section $s \in F(U)$ (a *global section* on U) such that $s|_{U_i} = s_i$ for all $i \in I$. This s is said to *glue* or *extend* the collection $\{s_i\}$. [Jag23]

If F is a sheaf, we often call its sections simply "sections" without always mentioning the open set (context usually clarifies the domain). A **sheaf of abelian groups (rings, etc.)** is defined similarly (the data are abelian groups, and restriction maps are homomorphisms), and the sheaf axioms above are imposed on the underlying sets of sections.

The sheaf axioms are often summarized by saying that for any open cover $\{U_i\}$ of U, the sequence

$$0 \to F(U) \xrightarrow{\delta} \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

is *exact*. Let's explain this sequence in detail:

- The first map $F(U) \to \prod_i F(U_i)$ sends a global section $s \in F(U)$ to the collection of its restrictions $(s|_{U_i})_i$ to each open set in the cover.
- The double arrow $\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$ represents two different restriction maps:
  - The first restricts each $s_i \in F(U_i)$ to $s_i|_{U_i \cap U_j} \in F(U_i \cap U_j)$
  - The second restricts each $s_j \in F(U_j)$ to $s_j|_{U_i \cap U_j} \in F(U_i \cap U_j)$

"Exactness" here means that the kernel of the second arrow equals the image of the first. In other words, a collection of sections $(s_i)_i \in \prod_i F(U_i)$ comes from a global section in $F(U)$ if and only if they agree on all overlaps: $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j.

This exactness encodes both sheaf axioms:

- **Locality axiom** (injectivity of $F(U) \to \prod_i F(U_i)$): If two global sections $s, t \in F(U)$ agree on each U_i (i.e., $s|_{U_i} = t|_{U_i}$ for all i), then they must be identical ($s = t$). Equivalently, if $s \neq t$, there must exist some U_i where $s|_{U_i} \neq t|_{U_i}$.
- **Gluing axiom** (surjectivity onto the equalizer): If a collection of sections $s_i \in F(U_i)$ satisfies the compatibility condition $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j, then there exists a global section $s \in F(U)$ that restricts correctly to each U_i (i.e., $s|_{U_i} = s_i$ for all i).

This categorical interpretation reveals how sheaves enforce the principle that compatible local data uniquely determine global data. In practice, we will mainly work with the concrete formulation given earlier.

A presheaf that satisfies only the Locality axiom is sometimes called a *separated presheaf* (or *pre-sheaf* in older literature), but the term and distinction won't be needed much here; we primarily care about full sheaves.

Example 3.2. Revisiting the presheaves of Example 2.2, we identify which are sheaves:

(1) **Sheaf of Continuous Functions:** $U \mapsto C^0(U)$ is actually a sheaf. The locality axiom holds because if two continuous functions agree on each piece of an open cover, they agree everywhere on the union (since points of U lie in some piece and the functions agree there). More formally, if $f, g \in C^0(U)$ with $f|_{U_i} = g|_{U_i}$ for all i, then for each $x \in U$ pick some U_i containing x, and $f(x) = g(x)$ since $x \in U_i$. Thus $f = g$ globally. The gluing axiom also holds: given continuous $f_i \in C^0(U_i)$ that agree on overlaps $U_i \cap U_j$, one can construct a continuous

function f on $\bigcup_i U_i = U$ by defining $f(x) = f_i(x)$ if $x \in U_i$. This is well-defined because if $x \in U_i \cap U_j$, $f_i(x) = f_j(x)$ by hypothesis. To see f is continuous, note that each x has a neighborhood where f equals one of the f_i, so f is locally continuous and hence continuous on U. Moreover f restricts to each f_i. Uniqueness is ensured by the locality axiom. Thus C^0_X, the presheaf of continuous real-valued functions on X, is a sheaf. By similar reasoning, C^k_X and C^∞_X (on a differentiable manifold X) are sheaves of functions.

(2) **Sheaf of Differentiable Functions:** As above, $F(U) = C^\infty(U)$ is a sheaf of smooth functions (on a smooth manifold X). The sheaf axioms hold because smoothness is a local property. For the gluing axiom, if we have locally smooth functions $f_i \in C^\infty(U_i)$ that agree on overlaps $U_i \cap U_j$, they produce a well-defined function f on $U = \cup_i U_i$. This function f is automatically smooth because smoothness is characterized by local behavior: a function is smooth if and only if it is smooth in a neighborhood of every point. Since every point $x \in U$ is contained in some U_i where f restricts to the smooth function f_i, the glued function f inherits smoothness naturally. The locality axiom (uniqueness) is clear: if two smooth functions agree locally everywhere, they are identical.

(3) **Sections of a Vector Bundle:** For a continuous (or smooth) vector bundle $E \to X$, the presheaf $U \mapsto \{\text{sections of } E \text{ on } U\}$ is a sheaf. The sheaf axioms are usually part of the definition of a fiber bundle: local sections that agree on overlaps give a unique global section because locally (with respect to a trivializing cover) one can patch the sections using a partition of unity or directly by definition of bundles. Thus, the sheaf of sections of E is often denoted $\Gamma(X, E)$ for global sections, and $\Gamma(U, E)$ for sections over U. If $E = X \times A$ is a trivial bundle with fiber an abelian group A, then $\Gamma(U, E)$ are just A-valued continuous functions on U. If A is discrete (or U is connected), these are precisely the locally constant functions. We denote by $\underline{A}_X$ the sheaf of locally constant A-valued functions on X, called the *constant sheaf (with stalk A)*.

(4) **Sheaf of Holomorphic Functions:** In complex analysis or on a complex manifold X, the assignment $U \mapsto \mathcal{O}_X(U)$, where $\mathcal{O}_X(U)$ is the ring of holomorphic (complex-analytic) functions on U, defines a sheaf of rings on X. The gluing axiom in this case is a nontrivial theorem of complex analysis: holomorphic functions that agree on overlaps extend to a holomorphic function on the union (this follows from the identity theorem and analytic continuation). Thus $\mathcal{O}_X$ is a sheaf (in fact, a sheaf of rings, and even of $\mathbb{C}$-algebras) on X. This sheaf is fundamental in complex geometry; for example, its higher cohomology groups yield important invariants of X.

(5) **Constant Presheaf (revisited):** The presheaf F with $F(U) = S$ for all U (from Example 2.2(2)) is *not* a sheaf in general. The

issue is the gluing axiom: suppose X is the union of two disjoint nonempty open sets U_1, U_2 (so X is disconnected). Take sections $s_1 \in S = F(U_1)$ and $s_2 \in S = F(U_2)$ that are different elements of S. On the overlap $U_1 \cap U_2 = \emptyset$, the compatibility condition is vacuously true (any condition on the empty set is trivially satisfied), so $\{s_1, s_2\}$ is a locally compatible family on the cover $\{U_1, U_2\}$. The gluing axiom would demand a section $s \in F(X)$ such that $s|_{U_i} = s_i$. But $F(X) = S$ only has one element if the sheaf property held (because on X itself s would equal s_1 on U_1 and s_2 on U_2, forcing $s_1 = s_2$ if s existed, which is not the case by our choice). Thus no single element of S can restrict to give two different values on U_1 and U_2. Therefore, the constant presheaf fails the gluing axiom (unless S has only one element, or X is connected so that any locally constant choice is globally constant). In fact, the *sheafification* of this presheaf is exactly the sheaf of locally constant S-valued functions $\underline{S}_X$, whose sections on U are those functions $f : U \to S$ which locally are constant (these can glue because on overlaps they agree automatically if defined to be locally constant). [GQ24]

(6) **Discontinuous Functions Presheaf:** As another example of a presheaf that is not a sheaf, consider $P(U) = \{\text{all (not necessarily continuous) functions } U \to \mathbb{R}\}$. This P is a presheaf (restrictions are by ordinary restriction of functions), but it fails the sheaf axioms. For instance, on $X = \mathbb{R}$ with $U_1 = (-\infty, 0)$ and $U_2 = (0, \infty)$, one can take a function f_1 on U_1 and f_2 on U_2 that agree on the overlap (here $U_1 \cap U_2 = \emptyset$ or possibly (0) if we consider closure, but basically no overlap to constrain them), yet there is no *continuous* global function that equals f_1 and f_2. But since P allows discontinuous sections, even more simply: one can define f_1 identically zero and f_2 identically zero, which certainly agree on $U_1 \cap U_2$ (trivially), and yet define f on X by $f(x) = 0$ for $x < 0$ and $f(x) = 1$ for $x \geq 0$. Then f restricts to f_1 and f_2 but is *not* the unique gluing because one could have chosen the value at 0 differently. In fact, P fails the uniqueness part (locality) as well: one can have a global function f which on each open piece of a cover looks like a certain value, but globally f might differ on the boundaries or overlaps in a way not detectable by checking overlap equality only. In short, P does not force uniqueness on overlaps (two different global sections can agree on each of a collection of open sets and still differ). Thus P is not a sheaf.

The last two examples highlight that sheaf axioms are strong conditions. The constant presheaf needed to be adjusted to get a sheaf of locally constant functions. Intuitively, a sheaf formalizes the principle that *compatible local data determine a unique global object.*

Just as we defined morphisms of presheaves, we have:

Definition 3.3 (Morphisms of Sheaves)**.** A *morphism of sheaves* $\varphi : \mathcal{F} \to \mathcal{G}$ on X is simply a morphism of the underlying presheaves (Definition 2.3). In other words, a family of maps $\{\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)\}_{U \subseteq X}$ commuting with all restriction maps. Sheaves on X and their morphisms form the **category of sheaves** on X, denoted $\mathbf{Sh}(X)$.

A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ is an *isomorphism* (or sheaf isomorphism) if each φ_U is bijective. In that case $\mathcal{F}$ and $\mathcal{G}$ are essentially the same sheaf.

Example 3.4. Basic examples of sheaf morphisms include inclusion maps or restriction of structure: e.g. the inclusion of the sheaf of differentiable functions into the sheaf of continuous functions $C^\infty_X \hookrightarrow C^0_X$ (each smooth function is in particular continuous) is a morphism of sheaves. Another example is differentiation: on a smooth manifold X, the assignment $d : C^\infty_X(U) \to \Omega^1_X(U)$ given by $f \mapsto df$ (the exterior derivative) defines a morphism of sheaves from the sheaf of smooth functions to the sheaf of 1-forms, since $d(f|_V) = (df)|_V$ for all $V \subseteq U$ (differentiation commutes with restriction).

Definition 3.5 (Exact Sequences of Sheaves)**.** Given a sequence of sheaf morphisms $0 \to \mathcal{F}' \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \to 0$, we say it is *exact* if for every open set $U \subseteq X$, the induced sequence of sections

$$0 \to \mathcal{F}'(U) \xrightarrow{\alpha_U} \mathcal{F}(U) \xrightarrow{\beta_U} \mathcal{G}(U) \to 0$$

is an exact sequence of sets (or groups, etc.). Here α_U being injective for all U means α is a monomorphism of sheaves, and surjectivity of β_U for all U means β is an epimorphism and every local section of $\mathcal{G}$ lifts locally to $\mathcal{F}$. The middle condition (image of α_U equals kernel of β_U for all U) says $\mathcal{F}'$ is identified as a sub-sheaf of $\mathcal{F}$ (the kernel of β). This is the natural extension of the notion of exact sequences to the categorical level of sheaves.

Exact sequences of sheaves are fundamental when we define cohomology, since applying the global section functor $\Gamma(X, -)$ to an exact sequence of sheaves generally produces a *left exact* sequence of groups, and the right-end failure of exactness leads to cohomology groups (see §6). We note here that the category of sheaves of abelian groups on X is an **abelian category**: it has a zero object (the sheaf $\mathbf{0}$ with $\mathbf{0}(U) = \{0\}$ for all U), all kernels and cokernels exist, and all monomorphisms and epimorphisms are normal. In fact, one can show that if $\mathcal{F}$ and $\mathcal{G}$ are sheaves of abelian groups, then $\ker(\beta)$ defined as the presheaf $U \mapsto \ker(\beta_U : \mathcal{F}(U) \to \mathcal{G}(U))$ is actually a sheaf (since the sheaf axioms can be checked to hold for kernels, being a sub-presheaf of $\mathcal{F}$), and similarly $\operatorname{coker}(\alpha)$ defined by $U \mapsto (\alpha_U)$ is also a sheaf. Thus $\mathbf{Sh}(X, \mathbf{Ab})$ (the category of sheaves of abelian groups on X) is abelian. Moreover, $\mathbf{Sh}(X, \mathbf{Ab})$ has *enough injectives* (we will discuss this in §6.3), meaning any sheaf can be embedded in an injective sheaf. These facts allow us to do homological algebra with sheaves.

Theorem 3.6 (Exactness on Stalks)**.** *A sequence of sheaves is exact if and only if it induces an exact sequence on every stalk. That is, a sequence*

$$0 \to \mathcal{F}' \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \to 0$$

is exact if and only if for every point $x \in X$, the induced sequence on stalks

$$0 \to \mathcal{F}'_x \xrightarrow{\alpha_x} \mathcal{F}_x \xrightarrow{\beta_x} \mathcal{G}_x \to 0$$

is exact.

Proof. This follows from the fact that many sheaf properties are *local* and can be checked on stalks. For injectivity, a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is injective (monomorphic) if and only if for every point $x \in X$, the induced map on stalks $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective. Similarly, surjectivity of φ can be checked stalkwise (this requires that every section of $\mathcal{G}$ has a local preimage in $\mathcal{F}$ around each point).

For exactness in the middle (i.e., $\mathrm{im}(\alpha) = \ker(\beta)$), we note that taking stalks commutes with taking kernels and images, so $(\ker \beta)_x = \ker(\beta_x)$ and $(\mathrm{im}\alpha)_x = \mathrm{im}(\alpha_x)$. Therefore, $\mathrm{im}(\alpha) = \ker(\beta)$ as sheaves if and only if $\mathrm{im}(\alpha_x) = \ker(\beta_x)$ for all stalks.

Thus, exactness of the sequence of sheaves is equivalent to exactness at each stalk. □

This principle often simplifies verification of exactness, as stalk calculations are typically more straightforward than checking exactness for all open sets.

4. Stalks and the Étale Space

A key feature of sheaves is their ability to encode both local and global information seamlessly. The notion of a **stalk** of a sheaf formalizes the idea of "germs of sections" at a point – capturing the infinitesimal or very local behavior of sections. By examining stalks, we can often reduce questions about sheaves to simpler questions about these local germs.

Definition 4.1 (Stalk and Germ)**.** Let $\mathcal{F}$ be a presheaf (or sheaf) on X, and let $x \in X$ be a point. The *stalk* of $\mathcal{F}$ at x, denoted $\mathcal{F}_x$, is defined as the direct limit (or colimit) of the sets $F(U)$ as U ranges over all open neighborhoods of x:

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U).$$

Concretely, an element of $\mathcal{F}_x$ is an equivalence class of pairs (U, s) where U is an open neighborhood of x and $s \in \mathcal{F}(U)$ is a section defined on U. [Vak24] Two pairs (U, s) and (V, t) are considered equivalent if there exists an open neighborhood $W \subseteq U \cap V$ of x such that $s|_W = t|_W$ in $\mathcal{F}(W)$. An equivalence class of such pairs is called a *germ* of a section at x, often denoted by $[s]_x$ or simply s_x. We write $\mathrm{germ}_x(s) \in \mathcal{F}_x$ for the germ of s at x.

There are natural *projection maps* $\rho_{U,x} : \mathcal{F}(U) \to \mathcal{F}_x$ sending $s \mapsto [s]_x$. By construction, if $x \in V \subseteq U$, then $\rho_{V,x}(s|_V) = \rho_{U,x}(s)$, ensuring consistency.

The stalk $\mathcal{F}_x$ can be thought of as "the set of all values that sections of $\mathcal{F}$ take at x, up to identifying two sections that agree in some neighborhood of x". If $\mathcal{F}$ is a sheaf of, say, abelian groups or rings, then each stalk $\mathcal{F}_x$ naturally inherits the structure of an abelian group or ring (since direct limits preserve such structures): operations are defined on representatives and checked to be well-defined.

The importance of stalks is that many properties of sheaves can be checked by looking at stalks. For instance, as mentioned, a sheaf morphism $\varphi : \mathcal{F} \to \mathcal{G}$ is injective if and only if $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective for all x. Similarly, surjectivity and exactness can be checked on stalks (this is because sheaf conditions localize statements, and direct limits are exact functors in the category of abelian groups, etc.). In other words, stalks allow one to reduce global questions to local ones around each point.

Example 4.2. For the sheaf $\mathcal{O}_X$ of holomorphic functions on a complex manifold X, the stalk $\mathcal{O}_{X,x}$ is the ring of all germs of holomorphic functions at the point x. This is exactly the *local ring* of X at x, which in algebraic geometry or complex analysis is a fundamental object (e.g., $\mathcal{O}_{X,x}$ might be $\mathbb{C}\{z\}$, the ring of convergent power series, if $X = \mathbb{C}$ and $x = 0$). For the constant sheaf $\underline{A}_X$ (locally constant functions with value in A), the stalk $\underline{A}_{X,x}$ is just A itself for every x, since any germ of a locally constant function is determined by its constant value on some neighborhood of x.

The process of passing to stalks is sometimes called *localization* in this context, by analogy with localizing algebraic objects. The slogan is: *Sheaves are determined by their stalks, and maps of sheaves are determined by their effect on stalks.*

Proposition 4.3 (Sheaf Equality via Stalks)**.** *If $\mathcal{F}$ is a sheaf and $s, t \in \mathcal{F}(U)$ are such that $s_x = t_x$ in $\mathcal{F}_x$ for all $x \in U$ (meaning the germs of s and t at every point of U agree), then $s = t$ in $\mathcal{F}(U)$. In other words, a sheaf section is uniquely determined by all its germs.*

Proof. For each $x \in U$, $s_x = t_x$ implies there is a neighborhood $V_x \subseteq U$ of x such that $s|_{V_x} = t|_{V_x}$. The collection $\{V_x\}_{x \in U}$ forms an open cover of U. By the sheaf locality axiom, since s and t agree on each V_x in the cover, they must be equal on U. $\square$

This shows the intuition that a sheaf is a kind of "local object"—knowledge of it on arbitrarily small neighborhoods (stalks) reconstructs it globally.

Sheaves also have a beautiful *geometric* interpretation via the notion of an **étalé space** (sometimes spelled "étale space"). This is another way to visualize a sheaf as a topological space mapping down to X:

Definition 4.4 (Étale Space of a Sheaf)**.** Let $\mathcal{F}$ be a sheaf on X. The *étalé space* of $\mathcal{F}$, denoted $E(\mathcal{F})$ or sometimes just E, is the disjoint union of all stalks of $\mathcal{F}$:

$$E(\mathcal{F}) = \bigsqcup_{x \in X} \mathcal{F}_x \,,$$

equipped with a natural topology and a projection map $\pi : E(\mathcal{F}) \to X$ defined by sending each element of $\mathcal{F}_x$ to the point x (so $\pi|_{\mathcal{F}_x} : \mathcal{F}_x \to \{x\}$ is trivial). The topology on $E(\mathcal{F})$ is defined by specifying a basis of open sets: for each open $U \subseteq X$ and each section $s \in \mathcal{F}(U)$, define

$$\widetilde{s}(U) := \{\, s_y \in \mathcal{F}_y \mid y \in U \,\} \subseteq E(\mathcal{F}) \,,$$

the set of germs of s at all points of U. One checks that $\pi(\widetilde{s}(U)) = U$. The collection of all such $\widetilde{s}(U)$, for all U and all $s \in \mathcal{F}(U)$, forms a basis for a topology on $E(\mathcal{F})$ under which $\pi : E(\mathcal{F}) \to X$ becomes a local homeomorphism (étale map). $E(\mathcal{F})$ is called the étalé (or *sheaf*) space of $\mathcal{F}$.

The topology on $E(\mathcal{F})$ is carefully chosen to establish a precise correspondence between sections of the sheaf $\mathcal{F}$ and continuous sections of the projection map π. Specifically, the basic open sets $\widetilde{s}(U)$ ensure that sections of $\mathcal{F}$ induce continuous maps $\sigma : U \to E(\mathcal{F})$ defined by $\sigma(x) = s_x$.

The condition that $\pi : E(\mathcal{F}) \to X$ is a local homeomorphism means every point in $E(\mathcal{F})$ has a neighborhood that is mapped homeomorphically onto an open set of X. In fact, the sets $\widetilde{s}(U)$ themselves provide such neighborhoods: each $\widetilde{s}(U) \subseteq E(\mathcal{F})$ is homeomorphic (via π) to U. This local homeomorphism property ensures that the local nature of a section s guarantees that its corresponding map σ is continuous. This is why this construction is called "étale" (French for roughly *spread out flat*): the sheaf is realized as a covering-like space over X.

The sheaf axioms have natural interpretations in this topology:

- The locality axiom ensures that sections are uniquely determined by their germs, which corresponds to section maps $\sigma : U \to E(\mathcal{F})$ being uniquely determined by their values.
- The gluing axiom ensures that locally compatible sections can be glued together, which corresponds to local section maps that agree on overlaps being able to be glued into a continuous global section map.

Intuitively, the étalé space is like taking each possible germ as an actual point lying over the original base space. For example, for the sheaf $\mathcal{C}^0_X$ of continuous real functions, $E(\mathcal{C}^0_X)$ can be identified with $X \times \mathbb{R}$ (since specifying a germ of a continuous function at x is essentially specifying a value in $\mathbb{R}$ – more precisely, $\mathcal{C}^0_{X,x} \cong \mathbb{R}$ because germs of continuous functions at a point are determined by their value at the point). For the sheaf of continuous functions, $\pi : X \times \mathbb{R} \to X$ is just the projection, which is a local homeomorphism but not a covering (unless $\mathbb{R}$ has the discrete topology, which it doesn't). If we instead took the sheaf of locally constant $\mathbb{R}$-valued functions, its étalé space would be $X \times \mathbb{R}$ with the discrete topology on $\mathbb{R}$, which is a covering space of X. Thus covering spaces correspond to sheaves that are locally constant. In general, an étalé space can be seen as a sort of "variable set" varying continuously over X.

Every section $s \in \mathcal{F}(U)$ corresponds to a continuous map (actually a section in the topological sense) $\sigma : U \to E(\mathcal{F})$ of the étalé space, defined by $\sigma(x) = s_x$ (the germ of s at x). The condition that s is a sheaf section exactly ensures that σ is continuous and $\pi \circ \sigma = \mathrm{id}_U$. Thus there is a natural bijection:

$$\mathcal{F}(U) \;\cong\; \{\text{continuous sections } \sigma : U \to E(\mathcal{F}) \text{ of } \pi\},$$

valid for every open $U \subseteq X$. In other words, giving a sheaf $\mathcal{F}$ is equivalent to giving a local homeomorphism $\pi : E \to X$ (where E is the étalé space) together with this correspondence of sections. In category-theoretic terms, the functor $\mathcal{F} \mapsto E(\mathcal{F})$ establishes an equivalence between the category of sheaves on X and the category of étalé spaces over X (local homeomorphisms to X). We will not prove this formally here, but it is a beautiful perspective.

Remark 4.5. The étalé space perspective allows one to think of a sheaf as a kind of "spread-out function or object". For instance, one can visualize the sheaf of germs of a function as literally taking each point of X and attaching all possible germ values at that point as a fiber. If $\mathcal{F}$ is a sheaf of sets, $E(\mathcal{F})$ is like a topological bundle (not necessarily with a single fiber type). If $\mathcal{F}$ is a sheaf of groups or rings, each stalk $\mathcal{F}_x$ has that algebraic structure, and one can often turn $E(\mathcal{F})$ into a topological group (but caution: generally it's only a space fibered in groups, not a group globally unless something like a trivialization exists).

The étalé space construction also provides an alternative way to construct the *sheafification* of a presheaf, as we will see next.

5. Sheafification

Not every presheaf is a sheaf (as we saw). However, an important fact is that for any presheaf, there is a best possible approximation of it by a sheaf, called its **sheafification**. Sheafification is analogous to, say, group completion or abelianization: it's a functor that makes a presheaf into a sheaf in a universal way.

Theorem 5.1 (Existence of Sheafification)**.** *For any presheaf P on X, there exists a sheaf P^+ on X together with a morphism of presheaves $\eta : P \to P^+$ satisfying the following universal property: for any sheaf $\mathcal{F}$ on X and any presheaf morphism $\phi : P \to \mathcal{F}$, there exists a unique sheaf morphism $\phi^+ : P^+ \to \mathcal{F}$ such that $\phi = \phi^+ \circ \eta$.* [GQ24] *In categorical terms, P^+ is the sheafification of P, and the functor $P \mapsto P^+$ is left adjoint to the inclusion functor from sheaves to presheaves. Moreover, P^+ is unique up to isomorphism by this universal property.*

In simpler terms, P^+ is a sheaf containing P as a sub-presheaf (via η), and any attempt to map P into a sheaf factors uniquely through P^+. In particular, $\eta : P(U) \to P^+(U)$ is injective for all U (so we can think of P as a subset of P^+), and P^+ is the "smallest" sheaf that contains P. If P is already a sheaf, then $P^+ \cong P$ (sheafification doesn't change a sheaf).

There are several ways to explicitly construct P^+. We describe a common construction via stalks and germs:

Let P be a presheaf. Define a candidate for sections of P^+ over an open set U as:

$$P^+(U) := \left\{ s : U \to \bigsqcup_{x \in U} P_x \mid s(x) \in P_x \text{ for each } x, \text{ and } s \text{ is "locally a germ of a single section of } P\text{"} \right\}.$$

In other words, an element of $P^+(U)$ is like a section of the étalé space of P over U, where now P need not be a sheaf so we consider all germs in each stalk P_x. Formally, one can define $P^+(U)$ as the set of functions s assigning to each $x \in U$ a germ $s_x \in P_x$ such that for each $x \in U$, there exists a neighborhood $V \subseteq U$ of x and some section $t \in P(V)$ whose germ at every point $y \in V$ equals s_y. In other words, s is locally (around every point) coming from an actual presheaf section. Such an s is essentially a section of the étalé space of P that is continuous in the sense of the basis given by germs. We define restriction of such an s to a smaller open $W \subseteq U$ in the obvious way: $(s|_W)(x) = s(x)$ for $x \in W$. This indeed defines a sheaf P^+ (one must check the sheaf axioms, which follow from the construction). By construction, there's a natural presheaf morphism $\eta : P \to P^+$ sending each section $t \in P(U)$ to the section of P^+ given by $x \mapsto$ germ of t at x. This η is injective on each $P(U)$. The fact that P^+ satisfies the universal property of sheafification is a bit technical but essentially stems from the fact that any map out of P into a sheaf factors through maps on germs.

Another (equivalent) perspective: The above construction is reminiscent of first forming the étalé space $E(P) = \bigsqcup_x P_x$ (which is a set of germs) and then taking sections in the topological sense. Essentially, P^+ is the sheaf of *continuous sections of the étalé space of* P. This is sometimes called the *étalé sheafification*. There is also a more algebraic description: one can first force the identity axiom (separation) by quotienting P by the relation identifying sections that agree on an open cover, and then force the gluing axiom by an additional colimit construction. But the stalk/germ method is more intuitive.

Example 5.2. Consider again the constant presheaf P with $P(U) = S$ for all U. Its stalk at any x is

$$P_x = \varinjlim_{U \ni x} S = S,$$

since any germ is represented by an element of S on some neighborhood. The sheafification P^+ is the sheaf of locally constant functions with values in S. Indeed, an element of $P^+(U)$ is by definition a choice of a germ in S at each $x \in U$ which locally comes from a single presheaf section. But presheaf sections of P are just constant functions on U (since $P(V) = S$ for any V). So the condition that $s(x)$ is locally a germ of a single section means: for each x there is a neighborhood V of x and an element $a \in S$ such that for all $y \in V$, $s(y)$ is the germ of the constant section a. But the germ of the constant section a at y is just a itself. So $s(y) = a$ for all y in V. Thus s is a locally constant function $U \to S$. Hence $P^+(U)$ can be identified with $\{f : U \to S \mid f \text{ is locally constant}\}$. This is exactly the usual constant sheaf $\underline{S}_X$ (the sheaf of locally constant S-valued functions). Therefore $\underline{S}_X$ is the sheafification of the constant presheaf P. We saw earlier that P was not a sheaf unless S is trivial or X connected, but $\underline{S}_X$ is a genuine sheaf. The map $\eta : P(U) = S \to P^+(U)$ sends the single value $a \in S$ to the constant function $f(y) = a$ on U. This clearly satisfies the universal property: any sheaf receiving a map from P (i.e. giving an element of S for each open in a compatible way) must factor through $\underline{S}_X$ (which picks out the locally constant function determined by those chosen values).

Remark 5.3 (Functoriality of Sheafification)**.** The sheafification functor $(\cdot)^+$ is functorial: a presheaf morphism $\phi : P \to Q$ induces a sheaf morphism $\phi^+ : P^+ \to Q^+$. Functoriality follows naturally by defining ϕ^+ pointwise: given a germ $[s]_x \in (P^+)_x$, we set

$$\phi^+([s]_x) = [\phi(s)]_x,$$

ensuring compatibility with restrictions. This functoriality means that sheafification is a *functor* left adjoint to the inclusion of sheaves into presheaves. Sheafification also commutes with taking stalks: $(P^+)_x \cong P_x$ for each x. This is intuitive since the stalk is already a very local object, the sheaf axioms don't change the germs at a single point.

Having established the basic theory of sheaves (definitions, examples, morphisms, stalks, sheafification), we have the machinery needed to study **sheaf cohomology**. We have hinted that sheaf cohomology arises from the failure of the sequence

$$0 \to F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j) \to \cdots$$

to be exact beyond the first term. In fact, continuing this sequence leads to the *Čech cohomology* of a sheaf. We will now introduce cohomology both from the elementary Čech viewpoint and the more general derived-functor viewpoint, then relate them and discuss key properties and examples.

6. Sheaf Cohomology

One of the main motivations for introducing sheaves is that they allow the definition of **sheaf cohomology**, which measures the extent to which the process of forming global sections fails to be exact. Cohomology provides obstructions to gluing local data and is an invariant that connects algebraic topology with algebraic geometry and analysis.

6.1. **Motivation: Cohomology as an Obstruction.** Consider a sheaf $\mathcal{F}$ of abelian groups on X (the abelian condition is needed to have a cohomology theory in the usual sense). We have the exact sequence for any open cover $\{U_i\}$ of an open U:

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{i<j} \mathcal{F}(U_i \cap U_j) \to \cdots .$$

The sheaf axioms tell us that the first arrow is injective (identity axiom) and the kernel of the second arrow equals the image of the first (sections that agree on overlaps come from a global section; gluing axiom). However, the second arrow $\prod_i \mathcal{F}(U_i) \to \prod_{i<j} \mathcal{F}(U_i \cap U_j)$ need not be surjective in general; if it isn't, that means there is a collection of sections $\{t_{ij} \in \mathcal{F}(U_i \cap U_j)\}$ that are pairwise compatible on triple overlaps, but which do not come as differences of a bunch of sections $\{s_i \in \mathcal{F}(U_i)\}$. In other words, there is an *obstruction* to finding local sections s_i that realize given overlap data t_{ij}. This obstruction is measured by the **Čech cohomology** group $\check{H}^1(\{U_i\}, \mathcal{F})$ for that cover. More generally, the continued failure of exactness at higher stages leads to higher cohomology groups $\check{H}^p(\{U_i\}, \mathcal{F})$.

Thus, cohomology arises as soon as we have a nontrivial cycle of data that is not a boundary of something from a previous stage. For sheaves, $\check{H}^0$ is just global sections, and $\check{H}^1$ detects the obstruction to gluing sections (this often corresponds to torsors or twisting of objects; e.g. line bundles are classified by H^1 of the sheaf of invertible functions). Higher H^p can similarly be interpreted (though such interpretations get more abstract).

One can also motivate sheaf cohomology via more classical cohomology theories. For example, singular cohomology of a space X can be recovered as sheaf cohomology of the constant sheaf $\underline{\mathbb{Z}}_X$ (under mild conditions on X). Sheaf cohomology provides a unified framework that encompasses many classical cohomology theories. In particular, the de Rham cohomology on a smooth manifold M is isomorphic to the sheaf cohomology of the de Rham complex $\Omega^\bullet$:

$$H^*_{\mathrm{dR}}(M) \cong H^*(M, \Omega^\bullet).$$

Sheaf cohomology is extremely general: it works for any sheaf of abelian groups on any topological space, and in algebraic geometry one uses it for the Zariski or étale topology, etc., where other tools are not readily available.

Sheaf cohomology measures the failure of local data to determine global data exactly. Next, we define the two main approaches to sheaf cohomology: Čech cohomology and derived functor cohomology.

6.2. **Čech Cohomology.** Let $\mathcal{F}$ be a sheaf of abelian groups on X. Choose an open cover $\mathfrak{U} = \{U_i\}_{i \in I}$ of X. The **Čech complex** of $\mathcal{F}$ with respect to this cover is the cochain complex:

$$C^p(\mathfrak{U}, \mathcal{F}) := \prod_{i_0, i_1, \ldots, i_p \in I} \mathcal{F}(U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_p}),$$

the product of sections on all $(p+1)$-fold intersections of the cover sets. An element of $C^p(\mathfrak{U}, \mathcal{F})$ can be thought of as an assignment of a section on each p-fold intersection $U_{i_0 \ldots i_p} := U_{i_0} \cap \cdots \cap U_{i_p}$. In low degrees:

- $C^0(\mathfrak{U}, \mathcal{F}) = \prod_i \mathcal{F}(U_i)$, a 0-cochain is a choice of section $s_i \in \mathcal{F}(U_i)$ for each i (an open set in the cover).
- $C^1(\mathfrak{U}, \mathcal{F}) = \prod_{i,j} \mathcal{F}(U_i \cap U_j)$, a 1-cochain is a choice of section $t_{ij} \in \mathcal{F}(U_i \cap U_j)$ for each ordered pair (i, j).
- etc.

We define the **Čech differential** $\delta : C^p \to C^{p+1}$ by an alternating sum that is analogous to the simplicial cohomology differential: For a p-cochain $c = \{c_{i_0 \ldots i_p}\}$ (where $c_{i_0 \ldots i_p} \in \mathcal{F}(U_{i_0 \ldots i_p})$),

$$(\delta c)_{i_0 i_1 \ldots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \, c_{i_0 \ldots \widehat{i_k} \ldots i_{p+1}}\big|_{U_{i_0 \cdots i_{p+1}}}.$$

Here the hat indicates omission of that index, and the restriction $|_{U_{i_0 \cdots i_{p+1}}}$ means we restrict the section from a p-fold intersection down to the $(p+1)$-fold intersection that is contained in it. This formula is the usual alternating sum but with the important aspect that we must restrict each term to the common intersection $U_{i_0 \ldots i_{p+1}}$ in order to subtract sections living on the same domain.

One checks that $\delta^2 = 0$ (this relies on the presheaf condition for $\mathcal{F}$, essentially) so $(C^\bullet(\mathfrak{U}, \mathcal{F}), \delta)$ is a cochain complex. Here, $C^\bullet(\mathfrak{U}, \mathcal{F})$ denotes the collection of all $C^p(\mathfrak{U}, \mathcal{F})$ for all $p \geq 0$, where the superscript bullet ($\bullet$) is a standard notation indicating the entire complex of cochain groups in all degrees. The cohomology of this complex,

$$\check{H}^p(\mathfrak{U}, \mathcal{F}) := H^p(C^\bullet(\mathfrak{U}, \mathcal{F})) = \ker(\delta : C^p \to C^{p+1}) / \operatorname{im}(\delta : C^{p-1} \to C^p),$$

is called the **Čech cohomology of $\mathcal{F}$ with respect to the cover $\mathfrak{U}$**. It is a priori a cover-dependent notion. For example, $\check{H}^0(\mathfrak{U}, \mathcal{F})$ is

$$\ker(\delta : C^0 \to C^1) = \{(s_i) \in \prod_i \mathcal{F}(U_i) \mid s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \ \forall i, j\},$$

which by the sheaf gluing axiom is isomorphic to $\mathcal{F}(X)$ (the global sections). So $\check{H}^0(\mathfrak{U}, \mathcal{F}) \cong \mathcal{F}(X)$ for any cover $\mathfrak{U}$. Meanwhile, a 1-cocycle is a collection (t_{ij}) with $t_{ij} \in \mathcal{F}(U_i \cap U_j)$ such that $\delta(t)_{ijk} = 0$, which explicitly means

$$t_{jk} - t_{ik} + t_{ij} = 0 \quad \text{in } \mathcal{F}(U_{ijk})$$

for all i, j, k. This is precisely the condition that $\{t_{ij}\}$ is a compatible family on triple overlaps. Such a $\{t_{ij}\}$ is called a 1-cocycle. It is a coboundary (1-coboundary) if $t_{ij} = s_i|_{U_{ij}} - s_j|_{U_{ij}}$ for some sections $s_i \in \mathcal{F}(U_i)$. Thus

$\check{H}^1(\mathfrak{U}, \mathcal{F})$ consists of equivalence classes of 1-cocycles under those trivialized by 0-cochains. This matches the description of gluing obstructions earlier.

If we refine the cover (take a cover $\mathfrak{V}$ that is a refinement of $\mathfrak{U}$), there are natural restriction maps

$$\check{H}^p(\mathfrak{U}, \mathcal{F}) \to \check{H}^p(\mathfrak{V}, \mathcal{F})$$

that induce an inverse system. One then defines the **Čech cohomology of $\mathcal{F}$ on X** (with no cover specified) as the direct limit over all open covers:

$$\check{H}^p(X, \mathcal{F}) := \varinjlim_{\mathfrak{U}} \check{H}^p(\mathfrak{U}, \mathcal{F}) \,.$$

[Rot09]

In many nice situations, a single good cover yields the same cohomology as the direct limit (for example, if X is a paracompact space, any open cover has a refinement which is *uniformly fine* enough that further refinement doesn't change the cohomology). In the context of manifolds or CW complexes, one often works with a good cover (e.g. all intersections are contractible) which simplifies computations.

We note a couple of important facts:

- For a sheaf of abelian groups, $\check{H}^p(X, \mathcal{F})$ is functorial in $\mathcal{F}$: a sheaf morphism $\mathcal{F} \to \mathcal{G}$ induces maps on cochains and hence on cohomology. So cohomology is a functor $H^p : \mathbf{Sh}(X, \mathbf{Ab}) \to \mathbf{Ab}$ (from sheaves to abelian groups).
- If X is a reasonably nice space (e.g. paracompact Hausdorff), then Čech cohomology for sheaves is isomorphic to the sheaf cohomology defined via derived functors (discussed next), which is the more general definition. For paracompact Hausdorff spaces, this isomorphism is typically proven via a spectral sequence argument, often using the Godement resolution, which relates the two cohomology theories. However, for pathological spaces or sheaves, Čech cohomology might differ from derived functor cohomology (Grothendieck famously gave conditions when they agree and when they might not). In algebraic geometry, one often can use Čech cohomology computed with affine open covers since higher cohomology vanishes on affines.
- Čech cohomology can be used to derive long exact sequences: given a short exact sequence of sheaves $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, one can construct a long exact sequence in Čech cohomology $\cdots \to \check{H}^p(X, \mathcal{F}') \to \check{H}^p(X, \mathcal{F}) \to \check{H}^p(X, \mathcal{F}'') \to \check{H}^{p+1}(X, \mathcal{F}') \to \cdots$ (the connecting homomorphism comes from a cocycle construction). Under conditions where Čech agrees with derived functor cohomology, this is the same long exact sequence we get from derived functors.

6.3. **Derived Functor Definition.** A more abstract (but powerful) approach to sheaf cohomology is via **derived functors**. The key observation is that the global section functor $\Gamma(X, -) : \mathbf{Sh}(X, \mathbf{Ab}) \to \mathbf{Ab}$ (which takes a sheaf $\mathcal{F}$ to $\mathcal{F}(X)$) is a left-exact functor between abelian categories. Indeed, $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ exact implies $0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'')$ is exact, since taking global sections is just evaluating at an open (and exactness at that first stage is precisely sheaf locality for X itself, which holds). However, $\Gamma(X, -)$ is not in general right-exact; the image of $\Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'')$ might not equal $\Gamma(X, \mathcal{F}'')$ if not every local section of $\mathcal{F}''$ lifts globally in $\mathcal{F}$. That failure is exactly measured by higher derived functors.

Because $\mathbf{Sh}(X, \mathbf{Ab})$ is an abelian category with enough injectives, we can take an injective resolution of any sheaf. If $\mathcal{F}$ is a sheaf, choose an injective resolution:

$$0 \to \mathcal{F} \to I^0 \to I^1 \to I^2 \to \cdots$$

with each I^p injective (such resolutions exist because sheaves have enough injectives). Now apply the functor $\Gamma(X, -)$ to this complex. $\Gamma(X, I^p)$ is an abelian group, and since I^p is injective, $\Gamma(X, I^p)$ is an exact functor of $\mathcal{F}$ (so higher cohomology of an injective is zero). We obtain a cochain complex:

$$0 \to \Gamma(X, I^0) \to \Gamma(X, I^1) \to \Gamma(X, I^2) \to \cdots ,$$

whose cohomology groups are by definition the **sheaf cohomology groups**:

$$H^p(X, \mathcal{F}) := H^p(\Gamma(X, I^\bullet)) .$$

[GQ24]

These are the *right derived functors* $R^p\Gamma(X, \mathcal{F})$. By general homological algebra, this definition is independent of the choice of injective resolution and satisfies the usual properties: for instance, it gives a long exact cohomology sequence for any short exact sequence of sheaves (since one can splice resolutions or apply the snake lemma to the double complex formed by two resolutions, etc.). Moreover, one shows that $H^0(X, \mathcal{F}) \cong \Gamma(X, \mathcal{F})$ (because $\Gamma(X, -)$ is left exact, so H^0 yields its value), and for an injective sheaf I, $H^p(X, I) = 0$ for $p > 0$.

$$H^p(X, \mathcal{F}) = R^p\Gamma(X, \mathcal{F})$$

is the pth right derived functor of global sections. This definition is elegant and conceptual, and it coincides with Čech cohomology for sufficiently nice spaces (in fact, always there is a canonical map $\check{H}^p(X, \mathcal{F}) \to H^p(X, \mathcal{F})$ that is an isomorphism if $\mathcal{F}$ is what is known as a *flasque* or *soft* or other acyclic sheaf or if X is paracompact, etc.).

A concrete way to compute sheaf cohomology using derived functors is to use a resolution by acyclic sheaves (ones whose higher cohomology vanishes). For example, **flasque sheaves** are those $\mathcal{L}$ such that $\mathcal{L}(X) \to \mathcal{L}(U)$ is surjective for all open U; they are soft and satisfy $H^p(X, \mathcal{L}) = 0$ for all $p > 0$ (since one can always extend local sections to global sections, which implies any Čech cocycle is a coboundary). Every sheaf has a flasque resolution, so one can compute cohomology as the cohomology of the global section complex of a flasque resolution (this is another approach to showing existence of H^p). This often simplifies computations in practice compared to injective resolutions (which are abstract).

The derived functor viewpoint also immediately gives functoriality in the sheaf argument, long exact sequences, and powerful tools like spectral sequences (Leray spectral sequence relates $H^p(X, R^q f_*\mathcal{F})$ to $H^{p+q}(Y, \mathcal{F})$ for a continuous map $f : Y \to X$) and base change theorems. It is the standard approach in advanced texts like Hartshorne's *Algebraic Geometry* [Har77] and Grothendieck's work [Gro57].

$H^p(X, \mathcal{F})$ is a sequence of abelian groups (or modules) associated to each sheaf $\mathcal{F}$. They are zero for $p < 0$, and $H^0(X, \mathcal{F}) = \mathcal{F}(X)$. They are functorial in $\mathcal{F}$ and fit into long exact sequences. They also often coincide with classical cohomology theories in special cases (e.g., if $\mathcal{F} = \underline{A}_X$ for a constant sheaf A on a reasonably nice space, then $H^p(X, A) \cong H^p_{\mathrm{sing}}(X; A)$, the singular cohomology of X with coefficients in A).

6.4. **Čech vs. Derived Functor Cohomology.**

As noted, Čech cohomology $\check{H}^p(X, \mathcal{F})$ is not obviously the same as $H^p(X, \mathcal{F})$ in general, but there are conditions under which they agree. Typically, if X is paracompact (or has a basis that is good for covers) and $\mathcal{F}$ is a sheaf of abelian groups (or more specifically a fine sheaf, etc.), then $\check{H}^p \cong H^p$. More concretely, if $\mathcal{F}$ is such that every open cover's higher Čech cohomology eventually stabilizes (which is true for paracompact spaces), then one can show any injective resolution gives the same result, and that result matches the direct limit of Čech.

In practice, one often uses Čech cohomology to compute sheaf cohomology because it's more combinatorial. For example, in complex geometry, to compute $H^p(X, \mathcal{O}_X)$ (cohomology of the structure sheaf) one often uses a cover by affines and computes Čech cohomology since each affine piece has trivial cohomology and sections on intersections are computable, etc.

Grothendieck showed that if $\mathcal{F}$ is a **fine sheaf** on a paracompact space, then $H^p(X, \mathcal{F}) = 0$ for all $p > 0$. [GQ24] A sheaf $\mathcal{F}$ is called **fine** if for any locally finite open cover $\{U_i\}_{i \in I}$ of X, there exists a family of sheaf endomorphisms $\{\phi_i : \mathcal{F} \to \mathcal{F}\}_{i \in I}$ such that:

(1) For each $i \in I$, the support of ϕ_i is contained in U_i (meaning $\phi_i(s)|_{X \setminus U_i} = 0$ for all sections s)
(2) $\sum_{i \in I} \phi_i = \mathrm{id}_{\mathcal{F}}$ (the identity morphism on $\mathcal{F}$)

This definition generalizes the notion of partitions of unity to the setting of sheaves. For example, the sheaf of smooth functions or differential forms on a manifold is fine via the existence of smooth partitions of unity. Fine sheaves are a subset of flasque sheaves (in fact fine $\implies$ flasque), so they are acyclic. Therefore, e.g. the sheaf of smooth functions C^∞_X on a paracompact manifold has no higher cohomology ($H^p(X, C^\infty_X) = 0$ for $p > 0$). Similarly, the sheaf of continuous real functions is fine (if paracompact Hausdorff), so it has no cohomology above 0. This doesn't mean the space has no topology; rather, it means that these sheaves are too flexible to capture interesting invariants (in contrast, constant sheaves are not fine, and indeed $H^p(X, \underline{\mathbb{R}})$ recovers real cohomology of X which can be nonzero).

One important situation of agreement is for **coherent analytic sheaves** on complex manifolds or coherent algebraic sheaves on varieties: in those contexts, one often uses Čech cohomology with respect to an open cover by contractible (or affine) sets to compute H^p. For example, on a complex manifold, Dolbeault's theorem states $H^q(X, \Omega^p_X)$ (sheaf cohomology of holomorphic p-forms) is isomorphic to the (p, q)th Dolbeault cohomology of the manifold, which is computed using fine resolutions (the Dolbeault complex of C^∞ forms).

$\check{H}^p(X, \mathcal{F})$ and $H^p(X, \mathcal{F})$ coincide in most situations of interest, and one usually denotes them simply as $H^p(X, \mathcal{F})$. We will assume henceforth that we are working in a context where this is true (e.g. X is paracompact).

6.5. **Properties of Sheaf Cohomology.** We list some key properties and theorems of sheaf cohomology (mostly consequences of it being a derived functor):

- **Long Exact Sequence:** If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{G} \to 0$ is an exact sequence of sheaves (of abelian groups), there is a natural long exact sequence in cohomology:

$$0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \xrightarrow{\delta} H^1(X, \mathcal{F}') \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) \to \cdots,$$

  continuing as $H^1 \to H^2$ and so on. The connecting homomorphism $\delta : H^0(X, \mathcal{G}) \to H^1(X, \mathcal{F}')$ measures the obstruction to lifting a global section of $\mathcal{G}$ to $\mathcal{F}$: given $s \in \mathcal{G}(X)$, one can locally lift it (by surjectivity on stalks of $\mathcal{F} \to \mathcal{G}$) to sections of $\mathcal{F}$, and the difference of two local lifts defines a Čech 1-cocycle in $\mathcal{F}'$ whose class is $\delta(s)$. This general mechanism is analogous to how connecting maps in algebraic cohomology measure extension classes.
- **Functoriality:** If $f : Y \to X$ is a continuous map and $\mathcal{F}$ a sheaf on X, there are two important functors between sheaf categories:
  1. **Pullback (inverse image)**: The pullback sheaf $f^{-1}\mathcal{F}$ on Y is defined as the sheafification of the presheaf that assigns to each open set $V \subset Y$ the direct limit:
  $$(f^{-1}\mathcal{F})^{\text{pre}}(V) = \varinjlim_{U \supset f(V)} \mathcal{F}(U)$$
  where the limit is taken over all open sets $U \subset X$ containing $f(V)$. Intuitively, $f^{-1}\mathcal{F}$ is the sheaf on Y that "pulls back" the sections of $\mathcal{F}$ via f.
  2. **Pushforward (direct image)**: The pushforward sheaf $f_*\mathcal{G}$ on X for a sheaf $\mathcal{G}$ on Y is defined by:
  $$(f_*\mathcal{G})(U) = \mathcal{G}(f^{-1}(U))$$
  for any open set $U \subset X$. In other words, sections of $f_*\mathcal{G}$ over U are precisely the sections of $\mathcal{G}$ over the preimage $f^{-1}(U)$.

These functors induce maps on cohomology (contravariant for pullback, covariant for pushforward). In particular, if $f : Y \to X$ is a continuous map of nice spaces, there is the **Leray spectral sequence**:

$$E_2^{p,q} = H^p\Big(X, R^q f_* \mathcal{G}\Big) \implies H^{p+q}(Y, \mathcal{G})\,,$$

which relates cohomology of Y to that of X with higher direct images of $\mathcal{G}$. As a special case, if f is a covering map (or any map such that higher direct images vanish), then $H^n(Y, \mathcal{G}) \cong H^n(X, f_*\mathcal{G})$. For example, if Y is a covering of X with fiber F and $\mathcal{G}$ is a locally constant sheaf on Y, then $f_*\mathcal{G}$ is a locally constant sheaf on X with fiber $\mathcal{G}(F)$, and one recovers that $H^*(Y, \mathcal{G}) \cong H^*(X, f_*\mathcal{G})$ (this is a sheafified version of the statement that cohomology of the total space of a covering is cohomology of base with local coefficients).

Another consequence is **Base Change**: under certain conditions, for a fiber square with f proper and some conditions on $\mathcal{F}$:

$$\begin{array}{ccc} Y' & \xrightarrow{g} & Y \\ \downarrow^{f'} & & \downarrow^{f} \\ X' & \xrightarrow{h} & X \end{array}$$

we have $h^* R^q f_* \mathcal{F} \cong R^q f'_* g^* \mathcal{F}$. This is important in algebraic geometry (cohomology and base change theorems).

- **Vanishing Theorems:** Many conditions ensure vanishing of cohomology above a certain degree. For example, if X has covering dimension d (roughly, it can be covered by a finite refinement where no point is in more than $d+1$ sets), then $H^p(X, \mathcal{F}) = 0$ for all $p > d$ for any sheaf $\mathcal{F}$. This is because one can find a cover with nerve of dimension d. Specifically, a topological d-sphere S^d has $H^p(S^d, \underline{\mathbb{Z}}) = 0$ for $p > d$.
- **Relation to Classical Cohomology:** If X is a reasonably nice topological space (say a CW complex) and A is an abelian group, then $H^p(X, \underline{A}_X) \cong H^p_{\text{sing}}(X; A)$, the usual singular (or Čech) cohomology with coefficients in A. For example, $H^1(X, \underline{\mathbb{Z}})$ classifies covering spaces of X (as $\mathrm{Hom}(H_1(X), \mathbb{Z})$ by the Hurewicz isomorphism, or equivalently $\mathrm{Hom}(\pi_1(X), \mathbb{Z})$ if X is nice). Also $H^1(X, \mathcal{O}_X^*)$ for a complex manifold X is (by the exponential sheaf sequence) isomorphic to $H^2(X, \underline{\mathbb{Z}})$, giving the first Chern class of line bundles (this is a deep fact bridging analytic and topological cohomology via the exponential exact sequence of sheaves $0 \to 2\pi i\mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$).

6.6. **Examples and Applications.** We conclude with several illustrative examples.

Example 6.1 (Sheaf Cohomology of a Circle)**.** Let $X = S^1$ be the circle, and consider the constant sheaf $\underline{\mathbb{R}}_X$ (locally constant real-valued functions). We compute $H^p(S^1, \underline{\mathbb{R}})$ using a simple open cover of S^1 by two arcs U_1, U_2 that overlap in two disconnected segments. Using Čech cohomology: $\check{H}^0$ gives $\mathbb{R}$ (global constant functions). A 1-cocycle is (t_{12}) with $t_{12} \in \mathcal{F}(U_1 \cap U_2)$ such that $\delta t = 0$ (here δt on triple overlaps is trivial since we have only two sets). So a 1-cocycle is just a section on $U_1 \cap U_2$ (which has two connected components corresponding to the two overlaps). The condition to be a coboundary is that $t_{12} = s_1|_{12} - s_2|_{12}$ for some s_i on U_i. But since $U_1 \cap U_2$ has two components, a globally constant function cannot produce an arbitrary pair of values on those two overlaps unless they are equal. Thus one finds $\check{H}^1(S^1, \underline{\mathbb{R}}) \cong \mathbb{R}$, corresponding to assigning a difference t_{12} of value a on one overlap and $-a$ on the other (coming from a sort of discontinuous global section if you try to glue one way around vs the other). In fact, $H^1(S^1, \underline{\mathbb{R}}) \cong \mathrm{Hom}(\pi_1(S^1), \mathbb{R}) \cong \mathbb{R}$, which matches singular cohomology $H^1(S^1, \mathbb{R}) \cong \mathbb{R}$. All higher $H^p = 0$ for $p > 1$ because S^1 has topological dimension 1.

Sheaf cohomology is thus a unifying language: singular cohomology, de Rham cohomology, etc., all fit into the framework by choosing appropriate sheaves (constant sheaf, differential forms sheaf, etc.). It also provides new invariants like the cohomology of structure sheaves in algebraic geometry (leading to definitions of irregularity, geometric genus, etc.). In algebraic topology, one rarely explicitly speaks of sheaf cohomology except to use local coefficient systems; however, in modern geometry and number theory, sheaf cohomology is indispensable.

Exercise 6.2. Compute $H^p(X, \underline{\mathbb{Z}})$ for $X = S^n$ (the n-sphere) for various p. You can use a good cover of S^n by two contractible open sets whose intersection is homotopy equivalent to S^{n-1}. (This should recover the known singular cohomology of spheres.)

Exercise 6.3. Show that if $\mathcal{F}$ is a flasque sheaf on X, then $H^p(X, \mathcal{F}) = 0$ for all $p > 0$. (*Hint:* For any open cover, every Čech p-cocycle is a coboundary because you can successively extend sections. Alternately, use the fact that $\mathcal{F}$ surjects onto any section on an open set to build a contracting homotopy for the Čech complex.)

Exercise 6.4. Consider the short exact sequence of sheaves on a manifold X:

$$0 \to \underline{\mathbb{Z}}_X \to \underline{\mathbb{R}}_X \to \underline{\mathbb{R}}/\mathbb{Z}_X \to 0,$$

where $\underline{\mathbb{R}}/\mathbb{Z}_X$ is the locally constant sheaf with fiber $\mathbb{R}/\mathbb{Z} \cong S^1$. Show that this induces the long exact sequence in cohomology that is isomorphic to the integral cohomology Bockstein sequence:

$$\cdots \to H^p(X, \underline{\mathbb{R}}/\mathbb{Z}) \xrightarrow{\delta} H^{p+1}(X, \underline{\mathbb{Z}}) \to H^{p+1}(X, \underline{\mathbb{R}}) \to \cdots .$$

Conclude that $H^p(X, \underline{\mathbb{R}}/\mathbb{Z})$ is isomorphic to the torsion subgroup of $H^{p+1}(X, \mathbb{Z})$ (primary decomposition of integral cohomology).

Sheaf cohomology provides a powerful and general way to handle cohomological questions across topology, geometry, and algebra. It formalizes the passage from local to global and measures the obstructions encountered. The theory is rich with algebraic tools (spectral sequences, etc.) and geometric interpretations (via examples like line bundles, divisor class groups, etc.).

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