Supplementary details of the numerical implementation.

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In this document supplementary details are provided concerning our approach in the numerical implementation of the electro-chemo-mechanical theory for ionic hydrogels presented in this paper in the open-source finite-element software FEniCS. The notation and form of the equations here mirror the FEniCS code as closely as possible.

This appendix provides insight into the strategies used in constructing the numerical implementation, but is *not* an exhaustive line-by-line explanation of the code. For interested readers, we have constructed a GitHub repository containing the FEniCS codes used to generate the results presented in this paper:

- `https://github.com/ericstewart36/ionotronics`

The physical phenomena in our simulations span time scales from nanoseconds (formation time of EDLs) to milliseconds (elastic wave propagation) to seconds (time period of stretching), and length scales from nanometers (the Debye length) to micrometers (the layer thickness) to centimeters (the applied stretches). Because we must accurately resolve these diverse length and time scales, simultaneous solution of the highly-coupled governing PDEs is a formidable numerical task. This document therefore also serves as a repository of some of the numerical strategies which I found most successful in this regard.

1 Unit system

FEniCS does not track units for physical quantities intrinsically, therefore we are free to select any suitable system of units for the problem at hand. Here, we select the following base units

$$\text{Length: } \mu\text{m}, \quad \text{Time: } \mu\text{s}, \quad \text{Mass: mg}, \quad \text{Number: n-mol}, \quad \text{Charge: mC},$$

which implies derived units of, for example,

$$\text{Pressure: TPa}, \quad \text{Force: N}, \quad \text{Energy: } \mu\text{J}, \quad \text{Electric potential: mV}.$$

I found that using micrometers as the base unit of length was a good "middle ground" for resolving nanometer-scale electrochemical boundary layers which are present in centimeter-scale devices to which large stretches are applied. Other units follow from similar considerations based on the conditions of our specific applications, e.g. time scale of boundary layer formation and order of magnitude of applied voltages.

The choice of a suitable unit system combined with normalization both of the degrees of freedom and of the governing equations helps to ensure that the various residuals and tangents associated with the governing equations are on similar orders of magnitude and therefore the numerical solution procedure remains tractable. Example material parameters and relevant physical constants in the unit system of the FEniCS implementation — and therefore as they appear in the code — are provided in Table 1.

Table 1: Physical constants and material parameters of the ionic hydrogel in the chosen unit system.

Parameter	Value	Units
G_0	3×10^{-9}	TPa
K	6×10^{-6}	TPa
$D^{(+)} = D^{(-)}$	1×10^{-2}	$\mu\text{m}^2/\mu\text{s}$
ρ_{R}	1×10^{-9}	$\text{mg}/\mu\text{m}^3$
$c_{\text{R}0}^{(+)} = c_{\text{R}0}^{(-)}$	2.74×10^{-6}	$\text{n-mol}/\mu\text{m}^3$
ϑ	298	K
R	8.3145×10^{-3}	$\mu\text{J}/(\text{n-mol K})$
ε_0	8.85×10^{-18}	$\text{mC}/(\text{mV } \mu\text{m})$

mTableNumImp

2 Degrees of freedom

We select the list of degrees of freedom

$$\big\{\mathbf{u},\ \omega^{(+)},\ \omega^{(-)},\ \phi\big\}, \tag{2.1}$$

with $\mathbf{u}$ the vectorial displacement; $\omega^{(+)}$ and $\omega^{(-)}$ the scalar electrochemical potential of the positive and negatively charged mobile species, respectively; and ϕ the scalar electro-static potential.

2.1 Normalization of the degrees of freeedom

It is numerically advantageous for all degrees of freedom to take on values which have similar orders of magnitude and are as close to unity as possible. For this reason, our numerical implementation actually uses a *normalized* list of degrees of freedom

$$\big\{\mathbf{u},\ \hat{\omega}^{(+)},\ \hat{\omega}^{(-)},\ \hat{\phi}\big\}, \tag{2.2}$$

with normalization factors

$$\hat{\omega}^{(+)} \stackrel{\text{def}}{=} \frac{\omega^{(+)}}{R\vartheta}, \qquad \hat{\omega}^{(-)} \stackrel{\text{def}}{=} \frac{\omega^{(-)}}{R\vartheta}, \qquad \text{and} \qquad \hat{\phi} \stackrel{\text{def}}{=} \frac{\phi}{R\vartheta/F}, \tag{2.3}$$

where the normalizing factor $\frac{R\vartheta}{F} \approx 25$ mV is sometimes referred to as the "thermal volt". We do not normalize the displacement field $\mathbf{u}$, which we have found remains reasonably sized so long as the selection of base units of length is representative of the phenomena under study. We will use the normalized list of degrees of freedom $\{\mathbf{u},\ \hat{\omega}^{(+)},\ \hat{\omega}^{(-)},\ \hat{\phi}\}$ in all discussion moving forward.

2.2 Basis functions for the degrees of freedom

We interpolate both the vectorial displacement degree of freedom $\mathbf{u}$ and the scalar electrochemical degrees of freedom $\hat{\omega}^{(+)}$, $\hat{\omega}^{(-)}$, and $\hat{\phi}$ using linear Lagrangian basis functions.

3 Time integration procedure

We use the Newmark-β family of time integrators along with the α-method for numerical stability (cf., e.g., Section 4.6 of Bonet et al., 2021). At each time step of the implicit dynamic simulation, the acceleration and velocity fields are calculated according to the Newmark kinematic relations:

$$\begin{aligned} \ddot{\mathbf{u}} &= \frac{\mathbf{u} - \mathbf{u}_{\text{old}} - \Delta t\, \dot{\mathbf{u}}_{\text{old}}}{\beta \Delta t^2} - \frac{1-2\beta}{2\beta} \ddot{\mathbf{u}}_{\text{old}}, \\ \dot{\mathbf{u}} &= \dot{\mathbf{u}}_{\text{old}} + \Delta t \left((1-\gamma) \ddot{\mathbf{u}}_{\text{old}} + \gamma \ddot{\mathbf{u}} \right), \end{aligned} \tag{3.1}$$

where β and γ are numerical parameters and the subscript $(\cdot)_{\text{old}}$ denotes a quantity from the previous time step while current quantities have no subscript.

The α-method interpolates known fields used in computing the weak forms to an intermediate time

$$t^\alpha = t - \alpha\,\Delta t, \tag{3.2}$$

with $0 \le \alpha \le \frac{1}{2}$. Specifically, the known fields are linearly interpolated to t^α as

$$\begin{aligned}
\mathbf{u}^\alpha &= \alpha\,\mathbf{u}_{\text{old}} + (1-\alpha)\,\mathbf{u},\\
\dot{\mathbf{u}}^\alpha &= \alpha\,\dot{\mathbf{u}}_{\text{old}} + (1-\alpha)\,\dot{\mathbf{u}},\\
(\hat{\omega}^{(+)})^\alpha &= \alpha\,\hat{\omega}^{(+)}_{\text{old}} + (1-\alpha)\,\hat{\omega}^{(+)},\\
(\hat{\omega}^{(-)})^\alpha &= \alpha\,\hat{\omega}^{(-)}_{\text{old}} + (1-\alpha)\,\hat{\omega}^{(-)},\\
\hat{\phi}^\alpha &= \alpha\,\hat{\phi}_{\text{old}} + (1-\alpha)\,\hat{\phi}.
\end{aligned} \tag{3.3}$$

alphaDOFs

Accordingly, any time-varying boundary conditions must be evaluated at the intermediate time t^α. The known fields used in evaluating the weak forms are those calculated from the α-method according to (3.3) — however, for economy of notation we omit the superset α in the equations presented in the rest of this appendix. When the fields in (3.3) appear in the weak forms, it should be understood that their α-weighted form is being used in the numerical implementation.

The dimensionless numerical parameters associated with the time-integration method in our FEniCS codes are listed in Table 2.[1] Note that so long as $0 \le \alpha \le \frac{1}{2}$, the expressions given for γ and β ensure numerical stability and add some "optimal dissipation" for preventing spurious high-frequency pressure oscillations (Erlicher et al., 2002).

Table 2: Dimensionless numerical parameters associated with the time-integration method.

umParamTable

α	γ	β
0.2	$\frac{1}{2}+\alpha$	$\frac{1}{4}\left(\gamma+\frac{1}{2}\right)^2$

4 Weak forms of the governing equations

We solve weak forms of the governing equations of the theory in the reference configuration, which is the default approach for FEniCS. To derive weak forms of the governing equations, we introduce test functions for each degree of freedom as the list

$$\left\{\mathbf{u}_{\text{test}},\ \hat{\omega}^{(+)}_{\text{test}},\ \hat{\omega}^{(-)}_{\text{test}},\ \hat{\phi}_{\text{test}}\right\}. \tag{4.1}$$

Multiplying a governing equation by one of these test fields and integrating over the entire domain yields the weak form of that equation, which is often further simplified through application of the divergence theorem. Our implementation uses four weak forms associated with the four test functions. We now describe these weak forms in detail.

4.1 The equation of motion

The equation of motion can be expressed in the reference body as

$$\operatorname{Div}\mathbf{T}_{\text{R}} + \mathbf{b}_{\text{R}} = \rho_{\text{R}}\ddot{\mathbf{u}}. \tag{4.2}$$

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Multiplying through (4.2) by $\mathbf{u}_{\text{test}}$, integrating over the body, and neglecting surface tractions, we have

$$\int_B \mathbf{T}_{\text{R}} : \nabla\mathbf{u}_{\text{test}}\,dv_{\text{R}} - \int_B \mathbf{b}_{\text{R}}\cdot\mathbf{u}_{\text{test}}\,dv_{\text{R}} + \int_B \rho_{\text{R}}\ddot{\mathbf{u}}\cdot\mathbf{u}_{\text{test}}\,dv_{\text{R}} = 0. \tag{4.3}$$

[1] The value $\alpha = 0.2$ is only used in the ionic loudspeaker simulations. We used $\alpha = 0$ in the capacitive stretch sensing experiments, which did not display any spurious high-frequency oscillations.

In practice, it is also useful to normalize the weak forms so that each term takes on values as close to the order of unity as possible. To this end, in the FEniCS code we simply divide through the equation of motion weak form by the ground state shear modulus G_0, viz.

$$\int_B \frac{1}{G_0}\mathbf{T}_{\mathrm{R}} : \nabla \mathbf{u}_{\mathrm{test}}\, dv_{\mathrm{R}} - \int_B \frac{1}{G_0}\mathbf{b}_{\mathrm{R}} \cdot \mathbf{u}_{\mathrm{test}}\, dv_{\mathrm{R}} + \int_B \frac{1}{G_0}\rho_{\mathrm{R}}\ddot{\mathbf{u}} \cdot \mathbf{u}_{\mathrm{test}}\, dv_{\mathrm{R}} = 0. \tag{4.4}$$

4.2 The mass balances for positive and negative mobile charges

The mass balance for the α-th mobile species can be expressed in the reference body as

$$\dot{c}_{\mathrm{R}}^{(\alpha)} = -\mathrm{Div}\mathbf{j}_{\mathrm{R}}^{(\alpha)}, \tag{4.5}$$

massbal2a_ap

where the referential mass flux $\mathbf{j}_{\mathrm{R}}^{(\alpha)}$, following a *Fick*-type relation in the spatial configuration, is

$$\mathbf{j}_{\mathrm{R}}^{(\alpha)} = -\mathbf{M}\nabla\omega^{(\alpha)}, \tag{4.6}$$

with $\mathbf{M}$ the mobility tensor. Here because of the constitutive assumptions, we have

$$\mathbf{M} = \mathbf{C}^{-1}\left(\frac{D^{(\alpha)}c_{\mathrm{R}}^{(\alpha)}}{R\vartheta}\right). \tag{4.7}$$

Multiplying through (4.5) by $\hat{\omega}_{\mathrm{test}}^{(\alpha)}$, integrating over the body, and neglecting surface fluxes, we arrive at the weak form

$$\int_B \dot{c}_{\mathrm{R}}^{(\alpha)}\, \hat{\omega}_{\mathrm{test}}^{(\alpha)}\, dv_{\mathrm{R}} + \int_B \mathbf{M}\,\nabla\omega^{(\alpha)} \cdot \nabla\hat{\omega}_{\mathrm{test}}^{(\alpha)}\, dv_{\mathrm{R}} = 0. \tag{4.8}$$

We use a first-order finite difference approximation to write

$$\dot{c}_{\mathrm{R}}^{(\alpha)} = \frac{c_{\mathrm{R}}^{(\alpha)} - c_{\mathrm{R,old}}^{(\alpha)}}{\Delta t}. \tag{4.9}$$

Then, recalling that $\hat{\omega}^{(\alpha)} = \omega^{(\alpha)}/R\vartheta$, we have the final weak form which appears in the FEniCS code

$$\int_B \left(\frac{c_{\mathrm{R}}^{(\alpha)} - c_{\mathrm{R,old}}^{(\alpha)}}{\Delta t}\right)\hat{\omega}_{\mathrm{test}}^{(\alpha)}\, dv_{\mathrm{R}} + \int_B D^{(\alpha)}c_{\mathrm{R}}^{(\alpha)}\left(\mathbf{C}^{-1}\nabla\hat{\omega}^{(\alpha)}\right)\cdot\nabla\hat{\omega}_{\mathrm{test}}^{(\alpha)}\, dv_{\mathrm{R}} = 0. \tag{4.10}$$

4.3 Gauss's law

Gauss's law in the reference body is

$$\mathrm{Div}\,\mathbf{d}_{\mathrm{R}} = F\left(c_{\mathrm{R}}^{(+)} - c_{\mathrm{R}}^{(-)}\right). \tag{4.11}$$

GaussNum

Recalling the expression for the electric displacement

$$\mathbf{d}_{\mathrm{R}} = -\varepsilon\, J\mathbf{C}^{-1}\nabla\phi, \tag{4.12}$$

elecspecial2

we multiply through (4.11) by $\hat{\phi}_{\mathrm{test}}$, integrate over the body, and neglect surface charges to arrive at the weak form

$$\int_B \varepsilon\,\left(J\mathbf{C}^{-1}\nabla\phi\right)\cdot\nabla\hat{\phi}_{\mathrm{test}}\, dv_{\mathrm{R}} - \int_B F\left(c_{\mathrm{R}}^{(+)} - c_{\mathrm{R}}^{(-)}\right)\hat{\phi}_{\mathrm{test}}\, dv_{\mathrm{R}} = 0. \tag{4.13}$$

Then, recalling that $\phi = \hat{\phi}(R\vartheta/F)$, we have the final weak form which appears in the FEniCS code

$$\int_B \frac{\varepsilon\, R\vartheta}{F}\left(J\mathbf{C}^{-1}\nabla\hat{\phi}\right)\cdot\nabla\hat{\phi}_{\mathrm{test}}\, dv_{\mathrm{R}} - \int_B F\left(c_{\mathrm{R}}^{(+)} - c_{\mathrm{R}}^{(-)}\right)\hat{\phi}_{\mathrm{test}}\, dv_{\mathrm{R}} = 0. \tag{4.14}$$

5 The solution procedure in the FEniCS code

In brief, our task in the FEniCS code is to use the known degrees of freedom

$$\big\{\mathbf{u},\ \hat{\omega}^{(+)},\ \hat{\omega}^{(-)},\ \hat{\phi}\big\}. \tag{5.1}$$

as well as their known values from the previous step

$$\big\{\mathbf{u}_{\text{old}},\ \hat{\omega}^{(+)}_{\text{old}},\ \hat{\omega}^{(-)}_{\text{old}},\ \hat{\phi}_{\text{old}}\big\}. \tag{5.2}$$

to calculate unknown *residuals* which correspond exactly to the weak form of the governing equations, viz.

$$\begin{aligned}
R_{\mathbf{u}} &= \int_B \frac{1}{G_0}\mathbf{T}_{\text{R}} : \nabla \mathbf{u}_{\text{test}}\, dv_{\text{R}} - \int_B \frac{1}{G_0}\mathbf{b}_{\text{R}} \cdot \mathbf{u}_{\text{test}}\, dv_{\text{R}} + \int_B \frac{1}{G_0}\rho_{\text{R}}\ddot{\mathbf{u}} \cdot \mathbf{u}_{\text{test}}\, dv_{\text{R}},\\
R_{\hat{\omega}^{(+)}} &= \int_B \left(\frac{c_{\text{R}}^{(+)} - c_{\text{R,old}}^{(+)}}{\Delta t}\right) \hat{\omega}^{(+)}_{\text{test}}\, dv_{\text{R}} + \int_B D^{(+)} c_{\text{R}}^{(+)} \left(\mathbf{C}^{-1}\nabla\hat{\omega}^{(+)}\right) \cdot \nabla\hat{\omega}^{(+)}_{\text{test}}\, dv_{\text{R}},\\
R_{\hat{\omega}^{(-)}} &= \int_B \left(\frac{c_{\text{R}}^{(-)} - c_{\text{R,old}}^{(-)}}{\Delta t}\right) \hat{\omega}^{(-)}_{\text{test}}\, dv_{\text{R}} + \int_B D^{(-)} c_{\text{R}}^{(-)} \left(\mathbf{C}^{-1}\nabla\hat{\omega}^{(-)}\right) \cdot \nabla\hat{\omega}^{(-)}_{\text{test}}\, dv_{\text{R}},\\
R_{\hat{\phi}} &= \int_B \frac{\varepsilon\, R\vartheta}{F}\left(J\mathbf{C}^{-1}\nabla\hat{\phi}\right) \cdot \nabla\hat{\phi}_{\text{test}}\, dv_{\text{R}} - \int_B F\left(c_{\text{R}}^{(+)} - c_{\text{R}}^{(-)}\right)\hat{\phi}_{\text{test}}\, dv_{\text{R}}.
\end{aligned} \tag{5.3}$$

residuals2

FEniCS then automatically differentiates these residuals with respect to all the separate degrees of freedom to obtain *tangents*,

$$\mathbf{K}_{\mathbf{uu}} = -\frac{\partial R_{\mathbf{u}}}{\partial \mathbf{u}}, \quad K_{\mathbf{u}\hat{\omega}^{(+)}} = -\frac{\partial R_{\mathbf{u}}}{\partial \hat{\omega}^{(+)}}, \quad K_{\mathbf{u}\hat{\omega}^{(-)}} = -\frac{\partial R_{\mathbf{u}}}{\partial \hat{\omega}^{(-)}}, \quad K_{\mathbf{u}\hat{\phi}} = -\frac{\partial R_{\mathbf{u}}}{\partial \hat{\phi}}, \quad \text{and so on.}$$

At each step in time, FEniCS uses the residuals and tangents in a global Newton-Raphson solver to determine current values of the degrees of freedom $\{\mathbf{u},\ \hat{\omega}^{(+)},\ \hat{\omega}^{(-)},\ \hat{\phi}\}$ for which the residuals are (within some small numerical tolerance of) zero, and which therefore satisfy the governing equations as well as the applied initial and boundary conditions.

5.1 Algorithm for calculation of the residuals

The algorithm we follow in the code to calculate the residuals in (5.3) is summarized below.

1. Begin the step with time increment Δt and known values of the degrees of freedom

$$\big\{\mathbf{u},\ \hat{\omega}^{(+)},\ \hat{\omega}^{(-)},\ \hat{\phi}\big\} \quad \text{and} \quad \big\{\mathbf{u}_{\text{old}},\ \hat{\omega}^{(+)}_{\text{old}},\ \hat{\omega}^{(-)}_{\text{old}},\ \hat{\phi}_{\text{old}}\big\}, \tag{5.4}$$

   where for the first step, $\{\mathbf{u}_{\text{old}},\ \hat{\omega}^{(+)}_{\text{old}},\ \hat{\omega}^{(-)}_{\text{old}},\ \hat{\phi}_{\text{old}}\}$ are given by the initial conditions and $\dot{\mathbf{u}}_{\text{old}} = \ddot{\mathbf{u}}_{\text{old}} = \mathbf{0}$.

2. Calculate the acceleration and velocity fields according to

$$\begin{aligned}
\ddot{\mathbf{u}} &= \frac{\mathbf{u} - \mathbf{u}_{\text{old}} - \Delta t\,\dot{\mathbf{u}}_{\text{old}}}{\beta\Delta t^2} - \frac{1-2\beta}{2\beta}\ddot{\mathbf{u}}_{\text{old}},\\
\dot{\mathbf{u}} &= \dot{\mathbf{u}}_{\text{old}} + \Delta t\left((1-\gamma)\ddot{\mathbf{u}}_{\text{old}} + \gamma\ddot{\mathbf{u}}\right).
\end{aligned} \tag{5.5}$$

3. Compute the intermediate-time fields according to the α-method as

$$\begin{aligned}
\mathbf{u}^{\alpha} &= \alpha\,\mathbf{u}_{\text{old}} + (1-\alpha)\,\mathbf{u},\\
\dot{\mathbf{u}}^{\alpha} &= \alpha\,\dot{\mathbf{u}}_{\text{old}} + (1-\alpha)\,\dot{\mathbf{u}},\\
(\hat{\omega}^{(+)})^{\alpha} &= \alpha\,\hat{\omega}^{(+)}_{\text{old}} + (1-\alpha)\,\hat{\omega}^{(+)},\\
(\hat{\omega}^{(-)})^{\alpha} &= \alpha\,\hat{\omega}^{(-)}_{\text{old}} + (1-\alpha)\,\hat{\omega}^{(-)},\\
\hat{\phi}^{\alpha} &= \alpha\,\hat{\phi}_{\text{old}} + (1-\alpha)\,\hat{\phi}.
\end{aligned} \tag{5.6}$$

alphaDOFs2

   These intermediate fields are used in all subsequent steps.

4. Calculate necessary kinematical quantities for the current step as

$$\begin{aligned}
&\mathbf{F} = \mathbf{1} + \nabla \mathbf{u}, \\
&J = \det \mathbf{F}, \\
&\mathbf{C}^{-1} = (\mathbf{F}^{\top}\mathbf{F})^{-1}, \\
&\bar{\mathbf{C}} = J^{-2/3}\mathbf{F}^{\top}\mathbf{F}, \\
&\bar{I}_1 = \operatorname{tr} \bar{\mathbf{C}}.
\end{aligned} \tag{5.7}$$

5. Calculate the spatial electric field as

$$\mathbf{e} = \mathbf{F}^{-\top} \underbrace{\left(-\frac{R\vartheta}{F} \nabla \hat{\phi} \right)}_{=\mathbf{e}_{\mathrm{R}}}. \tag{5.8}$$

6. Calculate the normalized concentrations of the positively and negatively charged mobile species by inverting the expression for the electrochemical potential

$$\begin{aligned}
c_{\mathrm{R}}^{(+)} &= c_{\mathrm{R\,ref}}^{(\alpha)} \exp\left(\hat{\omega}^{(+)} - \hat{\phi} \right), \\
c_{\mathrm{R}}^{(-)} &= c_{\mathrm{R\,ref}}^{(\alpha)} \exp\left(\hat{\omega}^{(-)} + \hat{\phi} \right),
\end{aligned} \tag{5.9}$$

   Also compute the normalized concentrations from the previous step according to

$$\begin{aligned}
c_{\mathrm{R,old}}^{(+)} &= c_{\mathrm{R\,ref}}^{(\alpha)} \exp\left(\hat{\omega}_{\mathrm{old}}^{(+)} - \hat{\phi}_{\mathrm{old}} \right), \\
c_{\mathrm{R,old}}^{(-)} &= c_{\mathrm{R\,ref}}^{(\alpha)} \exp\left(\hat{\omega}_{\mathrm{old}}^{(-)} + \hat{\phi}_{\mathrm{old}} \right).
\end{aligned} \tag{5.10}$$

7. Calculate the current Piola stress as the sum of mechanical and electrostatic parts

$$\begin{aligned}
&\mathbf{T}_{\mathrm{R}} = \mathbf{T}_{\mathrm{R}}^{(\mathrm{mech})} + \mathbf{T}_{\mathrm{R}}^{(\mathrm{es})}, \\
&\mathbf{T}_{\mathrm{R}}^{(\mathrm{mech})} = J^{-2/3} G_0 \left(1 - \frac{\bar{I}_1 - 3}{I_m} \right)^{-1} \left(\mathbf{F} - \frac{1}{3} (\operatorname{tr} \bar{\mathbf{C}})\, \mathbf{F}^{-\top} \right) + K\, \ln J\, \mathbf{F}^{-\top}, \\
&\mathbf{T}_{\mathrm{R}}^{(\mathrm{es})} = \varepsilon J \left[\mathbf{e} \otimes \mathbf{e} - \tfrac{1}{2} (\mathbf{e} \cdot \mathbf{e})\, \mathbf{1} \right] \mathbf{F}^{-\top}.
\end{aligned} \tag{5.11}$$

8. Use the quantities from steps 1-7 to compute the residuals as

$$\begin{aligned}
R_{\mathbf{u}} &= \int_B \frac{1}{G_0} \mathbf{T}_{\mathrm{R}} : \nabla \mathbf{u}_{\mathrm{test}}\, dv_{\mathrm{R}} - \int_B \frac{1}{G_0} \mathbf{b}_{\mathrm{R}} \cdot \mathbf{u}_{\mathrm{test}}\, dv_{\mathrm{R}} + \int_B \frac{1}{G_0} \rho_{\mathrm{R}} \ddot{\mathbf{u}} \cdot \mathbf{u}_{\mathrm{test}}\, dv_{\mathrm{R}}, \\
R_{\hat{\omega}^{(+)}} &= \int_B \left(\frac{c_{\mathrm{R}}^{(+)} - c_{\mathrm{R,old}}^{(+)}}{\Delta t} \right) \hat{\omega}_{\mathrm{test}}^{(+)}\, dv_{\mathrm{R}} + \int_B D^{(+)} c_{\mathrm{R}}^{(+)} \left(\mathbf{C}^{-1} \nabla \hat{\omega}^{(+)} \right) \cdot \nabla \hat{\omega}_{\mathrm{test}}^{(+)}\, dv_{\mathrm{R}}, \\
R_{\hat{\omega}^{(-)}} &= \int_B \left(\frac{c_{\mathrm{R}}^{(-)} - c_{\mathrm{R,old}}^{(-)}}{\Delta t} \right) \hat{\omega}_{\mathrm{test}}^{(-)}\, dv_{\mathrm{R}} + \int_B D^{(-)} c_{\mathrm{R}}^{(-)} \left(\mathbf{C}^{-1} \nabla \hat{\omega}^{(-)} \right) \cdot \nabla \hat{\omega}_{\mathrm{test}}^{(-)}\, dv_{\mathrm{R}}, \\
R_{\hat{\phi}} &= \int_B \frac{\varepsilon\, R\vartheta}{F} \left(J \mathbf{C}^{-1} \nabla \hat{\phi} \right) \cdot \nabla \hat{\phi}_{\mathrm{test}}\, dv_{\mathrm{R}} - \int_B F \left(c_{\mathrm{R}}^{(+)} - c_{\mathrm{R}}^{(-)} \right) \hat{\phi}_{\mathrm{test}}\, dv_{\mathrm{R}}.
\end{aligned} \tag{5.12}$$

9. FEniCS solves for the current values of $\{\mathbf{u},\ \hat{\omega}^{(+)},\ \hat{\omega}^{(-)},\ \hat{\phi}\}$.

10. Set these current values of the degrees of freedom as the "old" values for the next step in time,

$$\left\{ \mathbf{u},\ \hat{\omega}^{(+)},\ \hat{\omega}^{(-)},\ \hat{\phi} \right\} \implies \left\{ \mathbf{u}_{\mathrm{old}},\ \hat{\omega}_{\mathrm{old}}^{(+)},\ \hat{\omega}_{\mathrm{old}}^{(-)},\ \hat{\phi}_{\mathrm{old}} \right\}. \tag{5.13}$$

    Also, update the "state variable" vectors for velocity and acceleration

$$\left\{ \dot{\mathbf{u}},\ \ddot{\mathbf{u}} \right\} \implies \left\{ \dot{\mathbf{u}}_{\mathrm{old}},\ \ddot{\mathbf{u}}_{\mathrm{old}} \right\}. \tag{5.14}$$

11. Repeat steps 1-10 for desired number of steps in time.

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