A summary of Lawvere's Elementary theory of the category of sets (ETCS)

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A category C consists in:

1. a class of objects in the category $ob(C)$.

2. a class of morphisms between objects in $ob(C)$ denoted $hom(C)$.

3. for every morphism f the operator $\longrightarrow$ denoting the home and destination $f : A \longrightarrow B$. The arrow operator indicates f's domain $dom(f) = A$ and codomain $cod(f) = B$. The class of all morphisms having domain A and codomain B is denoted $hom(A, B)$.

4. for every pair of morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow C$ such that $dom(g) = cod(f)$ a binary composition operator $\circ$ such that $g \circ f = dom(f) \longrightarrow cod(g)$, that is $g \circ f = A \longrightarrow C$. For any morphism $f : A \longrightarrow B$, $g : B \longrightarrow C$, and $h : C \longrightarrow D$ it holds that $h \circ (g \circ f) = (h \circ g) \circ f$.

5. for any $A \in ob(C)$ an element $id_A : A \longrightarrow A$ in $hom(C)$ called the identity morphism, which satisfies identity, meaning that for any morphism $f : A \longrightarrow B$ it holds that $id_B \circ f = f$ and $f \circ id_A$.

Composition ($\circ$) is a monoid since it satisfies associativity and identity.

Example : An arbitrary set S forms a category. Proof:

1. The class of objects is non-empty (there is S).

2. The class of morphisms is non-empty since there is at minimum $id_S : S \longrightarrow S$, where the domain and codomain happen to be the same. If there is another set T then there is another set $S \times T = [(s,t)|s \in S, t \in T]$. Since $S \times T$ exists $\mathcal{P}(S \times T)$ exists and it follows that $T^S = [f \in \mathcal{P}(S \times T)|f : S \longrightarrow T]$ exists, which is just $hom(S,T)$.

3. id_S can be composed. Moreover, if in addition to f there is a function g from T to U and $dom(g) = cod(f)$ then we can compose them to make a third function $g \circ f$.

4. Composing id_S is obviously associative. Composing f and g with an h such that $dom(h) = cod(g)$ is also associative.

Axiom 1: There exists a terminal object and an initial object

Definition: Let C be a category. Let I and T belong to $ob(C)$. I is an initial object and 1 a terminal object of C if:

- for any A in $ob(C)$ there is the unique morphism $I_A : I \longrightarrow A$.

- for any A in $ob(C)$ there is the unique morphism $T_A : A \longrightarrow T$.

If there is another initial object I' in addition to I then I and I' are the same object. Let I and I' be terminal objects of C. There exists morphisms $f : I \longrightarrow I'$ and $g : I' \longrightarrow I$. g is an inverse of f and $f \circ g = id_{I'}$ and

$g \circ f = id_I$. Hence I and I' are isomorphic. We have shown that if I is an initial object of C then I is uniquely defined up to isomorphism. By a similar argument we can demonstrate the uniqueness of T.

Axiom 2: Initial and terminal objects are not isomorphic

I is a single object that maps to every object. T is a single object that every object maps to. There do no exist inverses for I_A or T_A because a reverse of these relations would not a be function.

Axiom 3: Cartesian products

Let C be a category and A and $B \in ob(C)$. $\pi_A, \pi_B : A \times B \rightrightarrows A, B$ is a cartesian product if there is a unique morphism $h : C \longrightarrow A \times B$ such that this diagram commutes:

$$\begin{tikzcd}
 & S \arrow[dl, "g"'] \arrow[d, dashed, "h"] \arrow[dr, "f"] & \\
A & A \times \mathbf{B} \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B
\end{tikzcd}$$

To say the diagram commutes is to say that it is true that $f = \pi_A \circ (f, g)$ and $g = \pi_B \circ (f, g)$. We call this a universal property of products. (f, g) is the only function satisfying the property in question.

Axiom 4: Equalizers

Definition: Let C be a category. Let $f : A \longrightarrow B$ and $g : A \longrightarrow B$ be morphisms in $hom(C)$. Given these two functions $(f, g) : A \rightrightarrows B$ We can call the equalizer of f and g the set (a subset of A) $[a \in A | f(a) = g(a)]$. $eq : E \longrightarrow A$ is an equalizer for f and g if and only if $f \circ e = g \circ e$. With this definition in hand we have another axiom of ECTS. For any two morphisms $(f, g) : A \rightrightarrows B$ there must be an equalizer $eq : E \longrightarrow A$. Definition: Let C be a category and A, B and C be objects. Let $i : B \longrightarrow A$ be a morphism in $hom(C)$. We say i is a monomorphism if for any morphisms f and g with domain and codomain $C \longrightarrow B$ it holds that $i \circ f \neq i \circ g$.

Axiom 5: Subobject classifiers and power objects

Let A and B be sets such that $B \subseteq A$. Let there be the set of boolean values $\Omega := [True, False]$ and the homset of morphisms $X \longrightarrow \Omega$.

1. There exists the subobject classifier $TRUE : 1 \longrightarrow \Omega$.
2. For any set A there exists the power object $\mathcal{P}(A)$ and the local membership relation for A, namely $\in_A : A \times \mathcal{P}(A) \longrightarrow \Omega$.

Definition: The characteristic function for the set $A \subseteq X$ is the function $\delta(A) : X \longrightarrow \Omega$. It is the function of A in X.

Axiom 6: The category Set is a well-pointed topos

If, for morphisms $f : A \longrightarrow B$ and $g : A \longrightarrow B$, it is true that $f(\alpha) = g(\alpha)$ for all $\alpha : I \longrightarrow A$, then $f = g$.

Axiom 7: Choice for categories

If $p : A \longrightarrow B$ is an epimorphism, then there is a reverse for p, namely $q : B \longrightarrow A$ such that $p \in q = id_B$.

Axiom 8: The natural number object

Let 1 be a terminal object of category C. A natural number object is an object N for which the following holds. For mappings $a : 1 \longrightarrow N$ and $b : 1 \longrightarrow X$, there is a successor morphism $s : N \longrightarrow N$ satisfying the following condition: For any object X with distinguished member $b : 1 \longrightarrow X$ and successor morphism $f : X \longrightarrow X$ there is a unique function $g : N \longrightarrow X$. g satisfies $g \circ a = b$ and $g \circ s = f \circ g$.

$$\begin{tikzcd} A \arrow[r, "\phi\quad\psi"] \arrow[d] & B \arrow[r, Rightarrow, "f{,}g"] & D \\ C \arrow[ur, red, "\eta"] & & \end{tikzcd}$$