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The Nagel-Schreckenberg model revisited

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Abstract. The Nagel-Schreckenberg model is a simple cellular automaton for a realistic description of single-lane traffic on highways. For the case $v_{\max} = 1$ the properties of the stationary state can be obtained exactly. For the more relevant case $v_{\max} > 1$, however, one has to rely on Monte Carlo simulations or approximative methods. Here we study several analytical approximations and compare with the results of computer simulations. The role of the braking parameter p is emphasized. It is shown how the local structure of the stationary state depends on the value of p. This is done by combining the results of computer simulations with those of the approximative methods.

PACS. 02.50.Ey Stochastic processes – 05.60.-k Transport processes – 89.40.+k Transportation

1 Introduction

Cellular automata (CA) do not only serve as simple model systems for the investigation of problems in statistical mechanics, but they also have numerous applications to "real" problems [1]. Therefore it is not surprising that in recent years CA have become quite popular for the simulation of traffic flow (see *e.g.* [2,3]).

CA are – by design – ideal for large-scale computer simulations. On the other hand, analytical approaches for the description of CA are notoriously difficult. This is mainly due to the discreteness and the use of a parallel updating scheme (which introduces "non-locality" into the dynamics). In addition, these models are defined through dynamical rules (*e.g.* transition probabilities) and usually one does not have a "Hamiltonian" description. Therefore standard methods are not applicable. Furthermore, one has to deal with systems which do not satisfy the detailed balance condition.

However, there is a need for exact solutions or, at least, for good approximations. These results as well as other exact statements may help to greatly reduce the need for computer resources. The interpretation of simulation data is often difficult because of the "numerical noise" and finite-size effects. Even in the cases where an exact solution is not possible, a combination of analytical and numerical methods might provide better insights. This is especially true for non-equilibrium systems where only a few exact or general results exist which could serve as a guideline.

In recent years, several analytical approximation methods for the Nagel-Schreckenberg model [4] for single-lane highway traffic have been proposed [5–8]. All of these approximations yield the exact result for the stationary state in the case $v_{\max} = 1$. However, these investigations focused on the so-called fundamental diagram, *i.e.* the flow-density relation. Here we will reinvestigate these approximations and calculate further quantities of interest in order to determine the accuracy. We will also discuss several limits and emphasize the effect of the braking probability p. We focus on the effect of p on the microscopic structure of the stationary state. Such microscopic characterizations have recently been used successfully for the asymmetric exclusion process [9,10]. Apart from $v_{\max} = 1$ here only the case of $v_{\max} = 2$ is investigated since one does not expect a qualitatively different behaviour for $v_{\max} \geq 2$ [6].

The paper is organized as follows: First we will briefly recall the definition of the Nagel-Schreckenberg model in Section 2 and the different analytical approximations in Section 3. In Section 4 several physical quantities, *e.g.* the fundamental diagram, headway and jam-size distributions and correlation lengths, are calculated using the analytical results. In Section 5 the predictions of the approximations are compared with each other and with results from computer simulations. The local structure of the stationary state is investigated as a function of the randomization p. In the final Section 6 a summary of the results together with our conclusion are given.

2 The Nagel-Schreckenberg model

The Nagel-Schreckenberg (NaSch) model [4] is a probabilistic cellular automaton. Space and time are discrete and hence also the velocities. The road is divided into cells. The length of a cell is determined by the front-bumper to front-bumper distance of cars in the densest jam and is usually taken to be 7.5 m. Each cell can either

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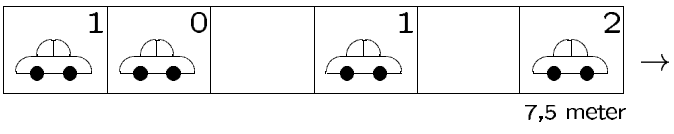

Fig. 1. Configuration in the Nagel-Schreckenberg model. The number in the upper right corner gives the velocity of the car.

be empty or occupied by just one car. The state of car j ($j = 1, \ldots, N$) is characterised by an internal parameter v_j ($v_j = 0, 1, \ldots, v_{\max}$), the instantaneous velocity of the vehicle. In order to obtain the state of the system at time $t+1$ from the state at time t, one has to apply the following four rules to all cars at the same time (parallel dynamics) [4]:

R1 Acceleration: $v_j(t+1/3) = \min(v_j(t)+1, v_{\max})$
R2 Braking: $v_j(t+2/3) = \min(d_j(t), v_j(t+1/3))$
R3 Randomization: $v_j(t+1) \stackrel{p}{=} \max(v_j(t+2/3)-1, 0)$ with probability p
R4 Driving: car j moves $v_j(t+1)$ cells.

Here $d_j(t)$ denotes the number of empty cells in front of car j, *i.e.* the so-called headway. For $v_{\max} = 5$ a calibration of the model shows that each timestep $t \to t+1$ corresponds to approximately 1 s in real time [4]. For simplicity we will consider only periodic boundary conditions so that the number of cars is conserved. The maximum velocity $v_{\max}$ can be interpreted as a speed limit and is therefore taken to be identical for all cars. Figure 1 shows a typical configuration. Throughout the paper we will assume that the cars move from left to right.

The four steps have simple interpretations. Step R1 means that every driver wants to drive as fast as possible or allowed. Step R2 avoids crashes between the vehicles. The randomization step R3 takes into account several effects, *e.g.* road conditions (*e.g.* slope, weather) or psychological effects (*e.g.* velocity fluctuations in free traffic). An important consequence of this step is the introduction of overreactions at braking which are crucial for the occurrence of spontaneous jam formation. Finally, step R4 is the actual motion of the vehicles.

The NaSch model is a minimal model in the sense that all four steps R1-R4 are necessary to reproduce the basic properties of real traffic. For more complex situations (*e.g.* 2-lane traffic [11] or city traffic [12]) additional rules have to be formulated.

3 Analytical methods

In this section several analytical approaches which have been used for the description of the NaSch model are reviewed. The simplest, a mean-field (MF) theory, completely neglects correlations. Since MF theory turned out to be inadequate even for the fundamental diagram, improved methods have been developed which allow to take into account short-range correlations exactly. All methods described here are *microscopic* theories since macroscopic theories are not able to describe the NaSch model properly. However, they are extremely useful and accurate for special variants of the NaSch model, *e.g.* the VDR model [13] in the slow-to-start limit.

In the following the analytical approaches are discussed briefly for the cases $v_{\max} = 1$ and $v_{\max} = 2$. All methods other than MF theory are exact for $v_{\max} = 1$. For $v_{\max} = 2$, on the other hand, they are only approximations.

Applications of the analytical methods to the calculation of physical quantities, *e.g.* the fundamental diagram, jam-size distributions and correlation lengths, are given in Section 4.

3.1 Mean-field theory

The simplest analytical approach to the NaSch model is a (microscopic) mean-field (MF) theory [6]. Here one considers the density $c_v(j,t)$ of cars with velocity v at site j and time t. In the MF approach, correlations between sites are completely neglected.

For $v_{\max} = 1$ the MF equations for the stationary state ($t \to \infty$) read [6]:

$$c_0 = (c + pd)c, \tag{1}$$
$$c_1 = \bar{p}cd \tag{2}$$

with $c = c_0 + c_1$, $d = 1 - c$ and $\bar{p} = 1 - p$. The flow is simply given by $f_{\mathrm{MF}}(c) = c_1$.

For random-sequential dynamics[1] the MF approach is known to be exact for $v_{\max} = 1$ [4]. For parallel dynamics, however, MF theory underestimates the flow considerably (see Sect. 4.1).

For $v_{\max} = 2$ the rate equations for the densities are given by [6]

$$c_0 = (c + pd)c_0 + (1 + pd)c(c_1 + c_2), \tag{3}$$
$$c_1 = d\left[\bar{p}c_0 + (\bar{p}c + pd)(c_1 + c_2)\right], \tag{4}$$
$$c_2 = \bar{p}d^2(c_1 + c_2). \tag{5}$$

The solution is given by

$$c_0 = \frac{(1+pd)c^2}{1-pd^2},$$
$$c_1 = \frac{\bar{p}(1-\bar{p}d^2)dc}{1-pd^2}, \tag{6}$$
$$c_2 = \frac{\bar{p}^2 d^3 c}{1-pd^2},$$

and the flux can be calculated using $f_{\mathrm{MF}}(c) = c_1 + 2c_2$. Again the flow is underestimated considerably (see Sect. 4.1). This is true for arbitrary $v_{\max}$. The general form of the MF equations can be found in [6].

[1] In random-sequential dynamics in each timestep a cell which is updated is picked at random.