Robust and Strongly Truthful Multi-task Peer Prediction Mechanisms for Heterogeneous Agents

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Abstract

Peer prediction mechanisms incentivize agents to truthfully report their signals even in the absence of verification by comparing agents' reports with those of their peers. In the detailed-free multi-task setting, agents are asked to respond to multiple independent and identically distributed tasks, and the mechanism does not know the prior distribution of agents' signals. The goal is to provide an ϵ-strongly truthful mechanism where truth-telling rewards agents "strictly" more than any other strategy profile (with ϵ additive error) even for heterogeneous agents, and to do so while requiring as few tasks as possible.

We design a family of mechanisms with a scoring function that maps a pair of reports to a score. The mechanism is strongly truthful if the scoring function is "prior ideal." Moreover, the mechanism is ϵ-strongly truthful as long as the scoring function used is sufficiently close to the idea scoring function. This reduces the above mechanism design problem to a learning problem— specifically learning an ideal scoring function. Because learning the prior distribution is sufficient (but not necessary) to learn the scoring function, we can apply standard learning theory techniques that leverage side information about the prior (e.g., that it is close to some parametric model). Furthermore, we derive a variational representation of an ideal scoring function and reduce the learning problem into an empirical risk minimization. We leverage this reduction to obtain very general results for peer prediction in the multi-task setting. Specifically,

1. The previous work in the multi-task setting was restricted to priors on finite signal spaces, our reduction applies to myriad continuous signal space settings. In particular, we can upper bound the required number of tasks for parametric models with bounded learning complexity. To the best of our knowledge, this is the first peer-prediction mechanism on continuous signals designed for the multi-question setting.

2. In the finite setting, the aforementioned connection yields ϵ-strongly truthful mechanisms, whereas prior work only achieves a weaker notion of truthfulness (informed truthfulness) [23, 1]. Additionally, we reduce the sample number exponentially for certain previously studied classes.

3. Finally, we show that the ϵ gap in truthfulness is necessary for the detail-free setting when the mechanism is based on mutual information framework.

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1 Introduction

Peer prediction is the problem of information elicitation without verification. Peer prediction mechanisms exploit the interdependence in agents' signals to incentive agents to report their private signal truthfully even when the reports cannot be directly verified. In *the multi-task setting* [3], each agent is asked to respond to multiple, independent tasks. For example:

Commute time We can collect data from drivers to estimate the commute time of a certain route. Each driver's daily commute time might be modeled in the following way: each day, the route has an expected time generated from a Gaussian distribution, and each driver's commute time is the expected time perturbed by independently distributed Gaussian noise.

Peer grading Students grade multiple instances of the same assignment. One potential setting is where each assignment has a latent attribute of quality, and students noisily observe this quality to report a score of A, B, C, D, or F for each assignment. The scores between students are not necessarily calibrated: one student may grade a particular assignment as an A with probability 0.3 and as a B otherwise, while another student assesses the same assignment as a B with probability 0.8 and as a C otherwise.

We want to design information elicitation mechanism for settings like these. We first define the criteria we would like our mechanisms to satisfy. See Sect. 2 for details and additional related definitions.

Strongly truthful Truth-telling is an Bayesian Nash equilibrium. Truth-telling should pay strictly more than any other non-permutation equilibrium. A permutation equilibrium is one where agents report a permutation of the signals. A slightly weaker property is *informed truthful* where no strategy profile pays strictly more than truth-telling, and truth-telling pays more than any uninformative equilibrium.

Minimal A mechanism should only elicit the agents' signals, and no additional information.

Detail-free The mechanism should not require foreknowledge of the prior.

Complicated priors compatible Priors are characterized both by the *signal space*, which may be continuous or discrete and by the *interdependence between signals*. Mechanisms ideally work for a large variety of assumptions on the interdependence between agents' signals. For example, we want to be able to assume that agents signals are close to some parametric model, or, in the discrete case, even to make no assumption at all about how agents signals relate (apart from the fact they contain some interdependence). In particular, we would like to allow the agents to be *heterogeneous* and not require symmetry between agents.

Low sample number The *sample number* of a mechanism is measured by the number of questions that agents' need to answer in order to achieve truthfulness guarantees. Ideally, each agent only needs to answer a few questions for the mechanism to provide truthfulness guarantees. (Definition 2.3)

In the above Commute time example, the signal space is continuous, and the prior is from a parametric model. In the above Peer grading example, the signal space is discrete, and the agents' signals are correlated via the underlying quality of assignments.

Our Techniques: Prior work [12] has shown that paying agents according to the Φ mutual information (a generalization of the Shannon mutual information) between their signals is a good idea. This is because, if agents try to strategically manipulate their signals, the Φ mutual information can only decrease. However, a key open question is how to compute the mutual information while having access to only a few signals for each agent. Moreover, the computation needs to be done in a way that maintains the incentive guarantees of the mechanism.

We solve this issue. First, we change the mutual information problem into an optimization problem.(Theorem 6.1) The Φ mutual information of a pair of random variables can be defined as the Φ divergence between two distributions: the joint distribution and the product of marginal distributions. The Φ divergence is just a measure of distance between the two distributions and contains the KL-divergence as a special case. The problem of computing the Φ divergence, using variational representation as a bridge, can be changed into the optimization problem of finding the best "distinguisher" between these two distributions. This optimization view can be interpreted as a *scoring function* (distinguisher) for us to design mechanism. The optimal scoring function (distinguisher) can differentiate the two distributions with a score equal to the Φ divergence, whereas any other scoring function (distinguisher) yields a lower score. Thus, once one has this optimal scoring function, estimating the Φ divergence (and hence Φ mutual information) is easy–just compute its score. In this paper we call the optimal scoring function for a particular prior P, the (P, Φ)-*ideal scoring function*. Note that, given the prior, the (P, Φ)-ideal scoring function is easy to compute.

Our mechanism will reward agents according to some scoring function. Importantly, agents' ex-ante payments under prior P are maximized when *both* the distinguisher used is the (P, Φ)-ideal scoring function, and the agents are truth-telling. Consequently, if we already have the (P, Φ)-ideal scoring function, the mechanism incentivizes truthful reporting. Furthermore, agents are willing to help the mechanism to learn the (P, Φ)-ideal scoring function rather than to trick it into using a suboptimal scoring function.

This variational characterization provides a better truthfulness guarantee when the number of tasks is finite. In particular, in our definition of approximated strongly truthful (informed-truthful or truthful) in Definition 2.3. We require the ex-ante payment under any non-truthful strategy profile is bounded, and our main result (Theorem 6.1) does not require the learning algorithm to estimate ideal scoring functions when agents use non-truthful strategy. This property is important for continuous signal spaces where agents may adopt the worst possible strategy profiles to compromise the learning algorithm adversarially.

The above observations transform the problem from designing a mechanism to simply learning the (P, Φ)-ideal scoring function given samples from a prior. We provide two algorithms to learn the scoring function. The first one is a *generative* approach which estimates the whole density function of the prior and computes a scoring function from it. In a *discriminative* approach, we formulate the estimation of the ideal scoring function as a convex optimization problem, empirical risk minimization [17], and estimate the scoring function directly. This latter approach allows us to use state-of-art convex optimization solver to estimate good scoring functions.

Our Contributions: In this paper, we leverage the above insights to design a Φ-pairing mechanism that is minimal and detail-free for heterogeneous agents. In particular:

1. When agents' signals are from a continuous set where the prior of agents' signals comes from a parametric models with low complexity (as measured by a continuous analog of the VC dimension), we can obtain informed truthful mechanisms. Moreover, when the priors have bounded complexity, the sample number is always at most logarithmic in the number of agents, but in certain cases can even be constant. Our results also extend to any case where the prior-specified scoring function can be efficiently learned. To the best of our knowledge, this is the first peer-prediction mechanism on continuous signals designed for the multi-question setting.

2. In the case of discrete signals, when Φ is strictly convex, our Φ-pairing mechanism is ϵ-strongly truthful when the priors are stochastic relevant (Definition 2.5). Moreover, we improve the analysis and reduce the sample number to be at most logarithmic in the number of agents. Our bounds apply to any setting between signals and do not need to be reproved in each setting. Again, in certain more restricted settings, we can show constant sample complexity.

3. Finally, we provide an obstacle to designing exact minimal informed (strongly) truthful mechanism in the detail-free setting. Most of the previous mechanisms (implicitly) make the expected utility of truth-telling strategy equal to the Φ-mutual information which requires an unbiased estimator of Φ-divergences from samples. We show such estimator does not exist.

	D&G [3]	CA [23, 1]	Φ-MIM [12]	Φ-pairing mechanism
Truthful	✓	✓	✓	✓
Informed-truthful	✓	✓	✓	✓
strongly truthful	✓		✓(fine-grained)	✓
Detail-free	positively correlated	✓	✓	✓
Samples		$O(n)$		$O(\log n)$
Signal space	binary	finite	finite	continuous

1.1 Related Work

Multi-task setting In the multi-task setting, Dasgupta and Ghosh [3] propose a *strongly truthful* mechanism when the signal space is binary and every pair of agents' signals are assumed to be positively correlated. Both Kong and Schoenebeck [12] and Shnayder et al. [23] independently generalize Dasgupta and Ghosh [3] to discrete signal spaces, though in different manners illustrated as follows.

Kong and Schoenebeck [12] present the ***Φ-mutual information mechanism***, a multi-task peer prediction mechanism for the finite signal space setting with arbitrary interdependence between signals. Unfortunately, the sample number is infinite. They show that their mechanism is strongly truthful as long as the prior is "fine-grained" (it is truthful in any event). A prior is *fine-grained* if, roughly speaking, no two signals can be interpreted as different names for the same signal. To define their mechanism they introduce the notion of Φ-mutual information (of which Shannon mutual information is a special case) where Φ is any convex function. Their mechanism pays each agent the Φ-mutual information between her reports and the reports of another randomly chosen agent. Strategic behavior is shown to not increase Φ-mutual information by a generalized version of the data processing inequality. Unfortunately, their analysis requires infinite sample number to measure this Φ-mutual information and does not handle errors in estimation.

Shnayder et al. [23] introduce the ***Correlated Agreement (CA) mechanism*** which also generalizes Dasgupta and Ghosh [3] to any finite signal space. On the one hand, the CA mechanism can assume the knowledge of the "signal structure" (which tells which signals are positively and negatively correlated). In this case they can provide a mechanism that is truthful with sample number of two.[1] On the other hand, when agents are homogeneous the CA mechanism can learn the signal structure, albeit with some chance of error, if it has sample number $O(n)$. The CA mechanism is shown to be robust to this error, and is ϵ-truthful. In both cases the CA mechanism is actually *ϵ-informed truthful* (a slightly weaker notion than strongly truthful). Agarwal et al. [1] extend the above work of Shnayder et al. [23] to a particular setting of heterogeneous agents where agents are (close to) one of a fixed number of types. They again establish a $O(n)$ sample number in this new setting.

Note that in the above works, a new robustness (error) analysis is required for each different setting of interdependence between signals. Interestingly, the CA mechanism can be viewed as a special case of the aforementioned Φ-mutual information mechanism using the total variation distance mutual information (i.e., $\Phi(a) = |a-1|/2$). However, instead of directly computing this mutual information, the CA mechanism obtains a consistent estimator of it [12]. Similarly, in the special case that our mechanism implements the total variation distance, we also recover the CA mechanism. However, our analysis is entirely different.

Single task setting In general, agents do not (necessarily) have multiple identical and independent signals. Without this property, most of the mechanisms require knowledge of a common prior (not detail-free) or for agents to report their whole posterior distribution of other's signals (not minimal). The later solution is especially difficult to apply to complicated signal spaces (e.g. asking agents to report their probability density function of others' continuous signals).

Miller et al. [16] introduce the peer prediction mechanism which is the first mechanism that has truth-telling as a strict Bayesian Nash equilibrium and does not need verification. However, their mechanism requires the full knowledge of the common prior and there exist some equilibria that are paid more than truth-telling. In particular, the oblivious equilibrium pays strictly more than truth-telling. Kong et al. [13] modify the original peer prediction mechanism such that truth-telling pays strictly better than any other equilibrium but still requires the full knowledge of the common prior. Prelec [18] designs the first detail-free peer prediction mechanism—Bayesian truth serum (BTS) in the one quesetion setting. Several other works study the one-question setting of BTS [19, 20, 26, 9]. For continuous signals, Radanovic and Faltings [20] apply a discretization approach and use a new payment method, but that is also non-minimal. Goel and Faltings [8] work on a mixture of normal distributions with an infinite number of agents.

Miscellany Liu and Chen [14] design a peer prediction mechanism where each agents' responses are not compared to another agents', but rather the output of a machine learning classifier that learns from all the other agents' responses. Liu and Chen [15] use surrogate loss functions as tools to achieve dominant strategy mechanisms by correcting for the mistakes in agents' reports. Kong and Schoenebeck [11] studies the related

[1] The original paper shows it requires 3, but it actually only needs 2 tasks.

goal for forecast elicitation, and like the present work uses Fenchel's duality to reward truth-telling (though in a different manner).

One interesting, but orthogonal, line of work looks at "cheap" signals, where agents can coordinate on less useful information. For example, instead of grading an assignment based on correctness, a grader could only spot check the grammar. Gao et al. [7] introduces the issue, while Kong and Schoenebeck [10] shows a partial solution using conditional mutual information.

The recent book Faltings and Radanovic [5] surveys additional results from this area.

Independent Work In an upcoming work, Kong [**?**] shows a elegant way for obtaining strongly truthful mechanisms for the multitask setting. Our results are incommensurate with this results. In our results, the sample complexity grows with the ϵ in the desired ϵ-strongly truthful guarantee, but is independent of the number of signals. In **?**], there is an exact strongly truthful guarantee, but the sample complexity grows in the size of the signal space.

1.2 Structure of Paper

Sect. 2 introduces some basic notions in this paper. In particular, Sect. 2.3 defines scoring functions, which will play an important role in this paper.

At the beginning of Sect. 3, we define a central component of our Φ-pairing mechanism, Mechanism 1, which takes agents' report and a scoring function K as input. In Sect. 4, we consider the full information setting. We show, in the Mechanism 1 with an ideal scoring function, agents are incentivized to report their signals truthfully. In Sect. 5, we prove our main technical lemmas and prove Theorem 4.1.

In Sect. 6, we define a notion of approximation of an ideal scoring function and introduce our framework that reduces the mechanism problem for information elicitation to a learning problem for an ideal scoring function (Theorem 6.1).

In Sect. 7, we focus on the learning problem introduced in Sect. 6. We first show two sufficient conditions for approximating an ideal scoring function in Sect. 7.1. Then, we present two algorithms to derive approximated ideal scoring functions from agents' reports in Sect. 7.2. Additionally, in Sect. 7.3, we provide an obstacle to designing peer prediction mechanisms based on this divergence based method.

In Sect. 9 and 10, we generalize Mechanism 1 to more than two agents. Finally, in Appendix 8 we compare our mechanisms with Shnayder et al. [23] and Kong and Schoenebeck [12].

2 Preliminaries

In this paper we consider Φ to be a convex continuous function and use $\mathrm{dom}(\Phi)$ to denote its domain. We use $(\Omega, \mathcal{F})$ to denote a measurable space where $\mathcal{F}$ is a σ-algebra on the outcome space Ω. Let Δ_Ω denote the set of distributions of over $(\Omega, \mathcal{F})$, [2] and $\mathcal{P}$ as a subset of distributions in Δ_Ω. Given a distribution P, we use $P(\omega)$ to denote the probability density of outcome $\omega \in \Omega$. In general, we use uppercase for a random object X and lowercase for the outcome x.

2.1 Convex Analysis and Φ-divergence

Informally, Φ-divergences quantify the difference between a pair of distributions over a common measurable space.

Definition 2.1 (Φ-divergence and mutual information [12]). Let $\Phi : [0, \infty) \to \mathbb{R}$ be a convex function with $\Phi(1) = 0$. Let P and Q be two probability distributions on a common measurable space $(\Omega, \mathcal{F})$. The ***Φ-divergence of Q from P*** where $P \ll Q$[3] is defined as $D_\Phi(P\|Q) \triangleq \mathbb{E}_Q\left[\Phi\left(dP/dQ\right)\right].$

If $P_{X,Y}$ is a distribution over $(x, y) \in \mathcal{X}\times\mathcal{Y}$ with marginal distributions P_X and P_Y. We call $D_\Phi(P_{X,Y}\|P_X P_Y)$ the ***Φ-mutual information between X and Y***.

[2] The distributions in Δ_Ω depend on the σ-space $\mathcal{F}$, but we omit it to simplify the notation.

[3] P is absolutely continuous with respect to Q: for any measurable set $A \in \mathcal{F}$, $Q(A) = 0 \Rightarrow P(A) = 0$.

Now, we introduce some basic notions in convex analysis [22]. Let $\Phi : [0, +\infty) \to \mathbb{R}$ be a convex function. The *convex conjugate* Φ^* of Φ is defined as: $\Phi^*(b) = \sup_{a \in \mathrm{dom}(\Phi)}\{ab - \Phi(a)\}$. Moreover $\Phi = \Phi^{**}$ if Φ is continuous.

By Young-Fenchel inequality [6], we can rewrite the Φ-divergence of Q from P in a variational form, c.f. [27]. This formulation is important to understand our mechanisms.

Theorem 2.2 (Variational representation)**.**

$$D_\Phi(P\|Q) = \sup_{k:\Omega \to \mathrm{dom}(\Phi^*)} \left\{\mathbb{E}_{\omega\sim P}[k(\omega)] - \mathbb{E}_{\omega\sim Q}[\Phi^*(k(\omega))]\right\}, ^4$$

and the equality holds when $k \in \partial\Phi\left(dP/dQ\right)$ almost everywhere on Q.[5]

We provide a proof for Theorem 2.2 and some examples for Φ-divergence in Appendix A.1.

2.2 Mechanism Design for Information Elicitation

For simplicity we first consider two agents, Alice and Bob, who work on a set of m tasks denoted as $[m]$. For each task $s \in [m]$, Alice receives a signal x_s in $\mathcal{X}$ and Bob a signal y_s in $\mathcal{Y}$. We use $(\mathbf{X}, \mathbf{Y}) \in (\mathcal{X} \times \mathcal{Y})^m$ to denote the *signal profile* of Alice and Bob which is generated from a prior distribution $\mathbb{P}$ which is a common knowledge between Alice and Bob.[6] In this paper, we make the following assumption:

Assumption 1 (A priori similar [3])**.** *$\mathbb{P}$ is a prior, and each task is identically and independently (i.i.d.) generated: there exists a distribution $P_{X,Y}$ over $\mathcal{X} \times \mathcal{Y}$ such that $\mathbb{P} = P_{X,Y}^m$, Moreover, we assume the marginal distributions have full supports, $P_X(x) > 0$ and $P_Y(y) > 0$ for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$.*

Given a report profile of Alice, $\hat{\mathbf{x}} \in \mathcal{X}^m$ and Bob, $\hat{\mathbf{y}} \in \mathcal{Y}^m$, an *information elicitation mechanism* $\mathcal{M} = (M_A, M_B)$ with m tasks pays $M_A(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \in \mathbb{R}$ to Alice, and $M_B(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \in R$ to Bob. In the rest of the paper we often only define notions for Alice, and define Bob's in the symmetric way.

Besides Assumption 1, we assume their strategies are uniform and independent across different tasks.[7] Formally, the *strategy* of Alice can be denoted as a random function $\theta_A : \mathcal{X} \to \Delta_\mathcal{X}$ such that $\Pr[\hat{\mathbf{X}} = \hat{\mathbf{x}}] = \prod_{s\in[m]} \theta_A(x_s, \hat{x}_s)$. That is, each report only depends on the corresponding signal. We call $\boldsymbol{\theta} = (\theta_A, \theta_B)$ a the *strategy profile*. The *ex-ante payment* to Alice under a strategy profile $\boldsymbol{\theta}$ and a prior $\mathbb{P}$ in mechanism $\mathcal{M}$ is

$$u_A(\boldsymbol{\theta}; \mathbb{P}, \mathcal{M}) \triangleq \mathbb{E}_{(\mathbf{X},\mathbf{Y})}\left[\mathbb{E}_{(\hat{\mathbf{X}},\hat{\mathbf{Y}})}\left[\mathbb{E}_\mathcal{M}[M_A(\hat{\mathbf{x}}, \hat{\mathbf{y}})]\right] \mid (\mathbf{x}, \mathbf{y})\right]$$

where we use a semicolon to separate the variable, θ, and parameters $\mathbb{P}$ and $\mathcal{M}$. Note that a strategy profile $\boldsymbol{\theta}$ can be seen as a Markov operator on probability measures on the signal space $\mathcal{X} \times \mathcal{Y}$, and Alice and Bob's reports, $\boldsymbol{\theta} \circ P$, is also a distribution on the signal space $\mathcal{X} \times \mathcal{Y}$.

In the literature of information elicitation, there are three important classes of strategies. We use $\boldsymbol{\tau}$ to denote the ***truth-telling strategy profile*** where both agents' reports are equal to their private signals with probability 1, e.g., Alice's strategy is $\tau_A(x, \hat{x}) = \mathbb{I}[x = \hat{x}]$. A strategy profile is a ***permutation strategy profile*** if both agents' strategy are a (deterministic) permutation, a bijection between signals and reports. Finally, a strategy profile is ***oblivious*** or *uninformed* if even one of the agents' strategies does not depend on their signal: that is for Alice $\theta_A(x, \hat{x}) = \theta_A(x', \hat{x})$ for all x, x', and $\hat{x}$ in $\mathcal{X}$. Note that the set of permutation strategy profiles includes the truth-telling strategy profile $\boldsymbol{\tau}$ but does not include any oblivious strategy profiles.

We now define some truthfulness guarantees for our mechanism $\mathcal{M}$ that differ in how unique the high payoff of truth-telling is:

Truthful: the truth-telling strategy profile $\boldsymbol{\tau}$ is a Bayesian Nash Equilibrium, and has the highest payment to both Alice and Bob.

[4]The sup is taken over k with finite $\mathbb{E}_{\omega\sim P}[k(\omega)]$ and $\mathbb{E}_{\omega\sim Q}[\Phi^*(k(\omega))]$.

[5]$\partial\Phi$ is the subgradient of Φ, and the formal definition can be found in [22]. Here we only use the equality condition when Ω is finite.

[6]

[7]This assumption is also used in [3, 23, 12]

Informed-truthful [23]: Truthful and also for each agent $\boldsymbol{\tau}$ is strictly better than any oblivious strategy profiles. For any oblivious strategy profile $\boldsymbol{\theta}$, $u_A(\boldsymbol{\tau}; \mathbb{P}, \mathcal{M}) > u_A(\boldsymbol{\theta}; \mathbb{P}, \mathcal{M})$ and $u_B(\boldsymbol{\tau}; \mathbb{P}, \mathcal{M}) > u_B(\boldsymbol{\theta}; \mathbb{P}, \mathcal{M})$.

Strongly truthful [23, 12]: Truthful and also for each agent $\boldsymbol{\tau}$ is strictly better than all non-permutation strategy profiles. For any non-permutation strategy profile $\boldsymbol{\theta}$, $u_A(\boldsymbol{\tau}; \mathbb{P}, \mathcal{M}) > u_A(\boldsymbol{\theta}; \mathbb{P}, \mathcal{M})$ and $u_B(\boldsymbol{\tau}; \mathbb{P}, \mathcal{M}) > u_B(\boldsymbol{\theta}; \mathbb{P}, \mathcal{M})$.

Solely-truthful: Each agent strictly prefers $\boldsymbol{\tau}$ to all other strategy profiles.

In this work, we consider an approximated version of above statements with low sample number. For example, given $\epsilon > 0$, a mechanism $\mathcal{M}$ with $m(\epsilon)$ tasks (the sample number)[8] is *ϵ-strongly truthful with $m(\epsilon)$ tasks* if there exists a mapping from strategy profiles to ex-ante payments such that 1) this mapping is strongly truthful; 2) for all ϵ the ex-ante payments of our mechanism with $m(\epsilon)$ tasks is within ϵ of this mapping.

Now we define the sample number for approximated truthfulness guarantees.

Definition 2.3. Given a family of joint signal distributions $\mathcal{P}$ and a function $S : \mathbb{R}_{>0} \to \mathbb{N}$ we say a mechanism $\mathcal{M}$ is ***ϵ-strongly truthful on $\mathcal{P}$ with $S(\epsilon)$ number of tasks***,[9] if there exists a strongly truthful mapping $F = (F_A, F_B)$ from joint signal distributions and strategy profiles to payments such that for all $\epsilon > 0$ and $m \geq S(\epsilon)$

1. the ex-ante payment under the truth-telling strategy profile in $\mathcal{M}$ with m number of tasks is within ϵ additive error from F: for all $P \in \mathcal{P}$,
$$u_A(\boldsymbol{\tau}; P, \mathcal{M}) \geq F_A(\boldsymbol{\tau}, P) - \epsilon;$$
2. and the ex-ante payment under any strategy profile $\boldsymbol{\theta}$ in $\mathcal{M}$ with m number of tasks is bounded above by F: for all $P \in \mathcal{P}$, and $\boldsymbol{\theta}$
$$u_A(\boldsymbol{\theta}; P, \mathcal{M}) \leq F_A(\boldsymbol{\theta}, P).$$

And the inequality also holds for Bob's ex-ante payment. Furthermore, we say $\mathcal{M}$ is ***(δ, ϵ)-strongly truthful on $\mathcal{P}$ with*** $S(\delta, \epsilon)$ if the above conditions holds with probability $1 - \delta$ for all $\delta \in (0, 1)$ and $\epsilon > 0$.

Additionally, we say $\mathcal{M}$ is ***ϵ-informal-truthful*** (***ϵ-truthful***) with $S(\epsilon)$ number of tasks if it is ϵ close to an inform-truthful (truthful) mapping.

Note that our notion of ϵ-truthfulness guarantee is quite strong. In particular, the second item requires for any strategy profile $\boldsymbol{\theta}$, the ex-ante payment is upper bounded by a strongly truthful (inform-truthful, truthful) mapping.

2.3 Scoring Functions and Prior Structure

Our constructions and analysis will make heavy use of the following functional— a scoring function.

Definition 2.4 (Scoring function)**.** A ***scoring function*** $K : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is a functional that maps from a pair of reports to a real value. Given a convex function Φ, a scoring function H_P^Φ is a $(P_{X,Y}, \Phi)$***-ideal scoring function*** if
$$H_P^\Phi(x, y) \in \partial\Phi\left(\frac{P_{X,Y}(x, y)}{P_X(x)P_Y(y)}\right). \tag{1}$$
We will use P and $P_{X,Y}$ interchangeably later, and say H is *ideal* without specifying P and Φ when it's clear.

In particular, a (P, Φ)-ideal scoring function encodes the signal structure of $P_{X,Y}$, and it can be easily computed from the density function $P_{X,Y}$. In Fig 1, we give a example that will serve as a running example in this paper.

[8] Here we abuse the notion of mechanism which can take different length of report m

[9] Formally, here we consider a family of mechanisms $(\mathcal{M}_m)$ parameterized by the sample number (the number of tasks) m

Example (Joint Gaussian Signals)**.** On each day s, a certain route has a expected driving time μ_s drawn from Gaussian distribution $\mathcal{N}(m_0, \sigma^2)$ i.i.d., [10] and Alice receives a driving time x from $\mathcal{N}(\mu_s, \tau^2)$ and Bob receives y from $\mathcal{N}(\mu_s, \tau^2)$ independently conditioned on μ_s. Therefore, $P_{X,Y}$ is pair of correlated Gaussians with mean (m_0, m_0) and covariance $\begin{bmatrix}\sigma^2+\tau^2 & \sigma^2 \\ \sigma^2 & \sigma^2+\tau^2\end{bmatrix}$. Let $G(x,y) \triangleq (x-m_0, y-m_0)\begin{bmatrix}\sigma^2+\tau^2 & -\sigma^2 \\ -\sigma^2 & \sigma^2+\tau^2\end{bmatrix}\begin{pmatrix}x-m_0 \\ y-m_0\end{pmatrix}$ be a quadratic form on x and y. Then

$$\frac{P_{X,Y}(x,y)}{P_X(x)P_Y(y)} = \sqrt{\frac{(\sigma^2+\tau^2)^2}{2\sigma^2\tau^2+\tau^4}} \exp\left(\frac{-1}{2(2\sigma^2\tau^2+\tau^4)}G(x,y)\right).$$

If $\Phi(a) = \frac{1}{2}|a-1|$, and constant $R \triangleq \left(\sigma^2\tau^2+\tau^4\right)\log\left(\frac{(\sigma^2+\tau^2)^2}{2\sigma^2\tau^2+\tau^4}\right)$, an ideal scoring function is

$$H_P^\Phi(x,y) = \begin{cases}\frac{1}{2} \text{ if } G(x,y) < R \\ -\frac{1}{2} \text{ if } G(x,y) \geq R\end{cases}$$

which can be represented by an ellipse Γ. The scoring function is $1/2$ if the input is in the ellipse and $-1/2$ otherwise. (cf. Figure 1)

If $\Phi(a) = a\log a$, the Φ-ideal scoring function is

$$H_P^\Phi(x,y) = -\frac{1}{2(2\sigma^2\tau^2+\tau^4)}G(x,y) + 1 + \frac{1}{2}\log\left(\frac{(\sigma^2+\tau^2)^2}{2\sigma^2\tau^2+\tau^4}\right)$$

which is a quadratic function on x and y. (cf. Figure 1)

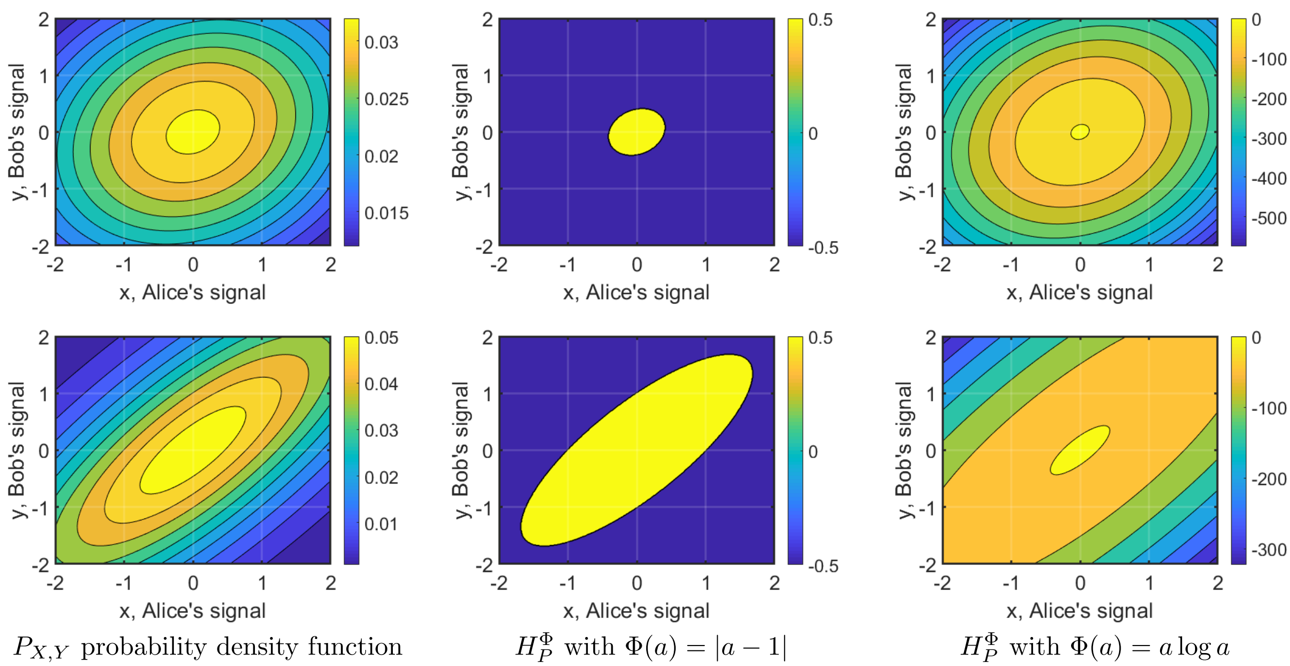

Figure 1: The top row uses $\sigma = 1$ and $\tau = 2$, and the bottom row uses $\sigma = 2$ and $\tau = 1$. Note that if Alice's and Bob's signals are more correlated $\sigma \gg \tau$, Γ is more skew

$\triangle$

Finally, we formalize our notion of interdependence between signals.

[10] $\mathcal{N}(m_0, \sigma^2)$ denotes the Gaussian distribution with mean m_0 and covariance matrix (or variance) σ^2

Definition 2.5 (Stochastic Relevant [23])**.** We call $P_{X,Y}$ *stochastic relevant* if for any two distinct signals $x, x' \in \mathcal{X}$

$$P_{X,Y}[Y \mid X = x] \neq P_{X,Y}[Y \mid X = x'].$$

That is, Alice's posterior on Bob's signals is not identical when Alice receives signal x or x'. And symmetrically, for Bob's signal affecting the posterior on Alice's.

We further call $P_{X,Y}$ *fine-grained* [12] if for any distinct pairs of signals (x, y) and (x', y')

$$P_{X,Y}(x, y)/(P_X(x)P_Y(y)) \neq P_{X,Y}(x', y')/(P_X(x')P_Y(y')).$$

Note that if $P_{X,Y}$ is fine-grained, it is stochastic relevant, and if $P_{X,Y}$ is stochastic relevant, it is not independent. Therefore, the Φ-mutual information between two stochastic relevant random variables is greater than zero.

2.4 Functional Complexity

In this section, we provide some notions to characterize the complexity of learning functionals which are standard [24, 25], and Sect. A.2 includes more discussion. We will use these notions to characterize the complexity of learning an ideal scoring function.

Let $\mathcal{K}$ is a pre-specified class of functionals $k : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$. Given $k \in \mathcal{K}$, $L > 0$, and a distribution $P_{X,Y}$, we define the *Bernstein norm* as

$$\rho_L^2(k; P) \triangleq 2L^2 \mathbb{E}_P[\exp(|k|/L) - 1 - |k|/L], \text{ and } \rho_L(\mathcal{K}; P) \triangleq \sup_{k \in \mathcal{K}} \rho_{L_1}(k, P).$$

Let $\mathcal{N}_{[],L}(\delta, \mathcal{K}, P)$ be the smallest value of n for which there exists n pairs of functions $\{(k_j^L, k_j^U)\}$ such that $\rho_L(k_j^U - k_j^L; P) \leq \delta$ for all j and for all $k \in \mathcal{K}$ there is a j such that $(x, y) \in \mathcal{X} \times \mathcal{Y}$, $k_j^L(x, y) \leq k(x, y) \leq k_j^U(x, y)$. Then

$$\mathcal{H}_{[],L}(\delta, \mathcal{K}, P) \triangleq \log \mathcal{N}_{[],L}(\delta, \mathcal{K}, P)$$

is called the *generalized entropy with bracketing*. We further define the entropy integral as $J_{[],L}(R, \mathcal{K}, P) = \int_0^R \sqrt{\mathcal{H}_{[],L}(u, \mathcal{K}, P)}du$.

Our results will show that $O(\log n)$ questions suffice as long as every pair of agents' ideal scoring functions are in some bounded complexity space $\mathcal{K}$, that is $J_{[],L}(R, \mathcal{K}, P)$ and $\rho_L(\mathcal{K}; P)$ are bounded.

3 Φ-Divergence Pairing Mechanisms

In this section, we first define a class of multi-task peer-prediction mechanisms $\mathcal{M}^{\Phi,K}$ for Alice and Bob working on all $m \geq 2$ tasks. The mechanism is parametrized by a convex function Φ and a scoring function K (Definition 2.4). Then we briefly discuss how to obtain a good scoring function, and develop algorithms for estimating good scoring function later.

The process of this mechanism is quite simple. Given a scoring function K, in the mechanism $\mathcal{M}^{\Phi,K}$, we arbitrarily choose one task b, and two distinct tasks p and q from $m \geq 2$ tasks. Alice gets paid by the scoring function on her and Bob's reports on task b minus the Φ^* applied to the scoring function on her report on p and Bob's report on q. In this way, agents are paid by a scoring function on a *correlated task* minus a regularized scoring function on two *uncorrelated tasks*.

To simplify the notion, we use u_A or $u_A(\boldsymbol{\theta}, P, K)$ to denote the ex-ante payment to Alice under a strategy profile $\boldsymbol{\theta}$ and a joint signal distribution P in pairing mechanism with a scoring function K.

In general, the truthfulness guarantees of Mechanism 1 depends on the degeneracy of Alice's and Bob's signal distribution P and convex function Φ. In this paper, we consider four different conditions which will be used in the statement of our results.

Assumption 2. *In this paper, we consider the following four different settings.*

1. *no assumption;*

Mechanism 1 Φ-divergence pairing mechanism with a scoring function K, $\mathcal{M}^{\Phi,K}$ for two agents

Input: A report profile $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ where both Alice and Bob submit report for all $m \geq 2$ tasks.

Parameters: A convex function $\Phi : [0, \infty) \to \mathbb{R}$, its conjugate Φ^*, and a scoring function $K : \mathcal{X} \times \mathcal{Y} \to \mathrm{dom}(\Phi^*) \subseteq \mathbb{R}$.

1: For Alice, arbitrarily pick three tasks b, p and q where p and q are distinct. We call b the *bonus task*, p the *penalty task to Alice*, and q the *penalty task to Bob*.

2: Based on Alice's reports on b and p ($\hat{x}_b$ and $\hat{x}_p$) and Bob's reports on b and q ($\hat{y}_b$ and $\hat{y}_q$), the payment to Alice is

$$M_A^{\Phi,K}(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \triangleq K\left(\hat{x}_b, \hat{y}_b\right) - \Phi^*\left(K\left(\hat{x}_p, \hat{y}_q\right)\right). \tag{2}$$

3: The payment of Bob is defined similarly.

2. *$P_{X,Y}$ is stochastic relevant;*

3. *Besides the above conditions, $\mathcal{X}$ and $\mathcal{Y}$ are finite sets, Φ is strictly convex and differentiable, and Φ^* is strictly convex;*

4. *If, besides the above conditions, $P_{X,Y}$ is fine-grained.*

3.1 Obtaining a Good Scoring Function

The Φ-pairing mechanism $\mathcal{M}^{\Phi,K}$ is not stand-alone mechanism for information elicitation, because it requires a scoring function K as a parameter. We will see shortly in Sect. 4 and 6, the truthfulness guarantees of the pairing mechanism depends on the quality of the scoring function. In this paper, we consider three different models for mechanism designers to estimate good scoring functions which are discussed in the rest of the sections:

Direct access of H_P^Φ In Sect. 4, we first consider the mechanism knows a (P, Φ)-ideal scoring function H_P^Φ. Note that if the mechanism knows the prior P, it can compute the (P, Φ)-ideal scoring function, but the converse is not necessarily true.

General reduction to a learning problem In Sect. 6, besides the reports from Alice and Bob, mechanism may exploit Alice and Bob's previous scoring function and other side information. For example the joint distribution between Alice and Bob can be approximated by some parametric model, say joint Gaussian distributions. We introduce our framework (Mechanism 2) that reduces the problem into a learning problem.

Estimation from samples Finally, in the multi-task setting, if Alice and Bob truthfully report their signals, it is possible to estimate the (P, Φ)-ideal scoring function from those reports. However, the mechanism needs to incentive them to be truthful. In Sect. 7, we propose two learning method to estimate good scoring functions. Combine them with our framework (Mehcanism 2), we can have detail-free ϵ-strongly truthful mechanisms with high probability.

4 Pairing Mechanisms in the Known Prior Setting

If the the mechanism $\mathcal{M}^{\Phi,H}$ has an (P, Φ)-ideal scoring function H where P is the joint distribution to Alice's and Bob's signals, the mechanism has the following properties. We defer the proof to Sect. 5.

Theorem 4.1. *Let an integer m be greater than 2, a functional Φ be a continuous convex function with $[0, \infty) \subseteq \mathrm{dom}(\Phi)$, $\mathbb{P}$ with $P_{X,Y}$ be a common prior between Alice and Bob satisfying Assumption 1. Let $\boldsymbol{\tau}$ be the truth-telling strategy profile, and H be a (P, Φ)-ideal scoring function.*

The Φ-pairing mechanism with a (P, Φ)-ideal scoring function, $\mathcal{M}^{\Phi,H}$ has the following properties: For any strategy profile $\boldsymbol{\theta}$, [11]

$$u_A(\boldsymbol{\theta}, P, H) \leq u_A(\boldsymbol{\tau}, P, H). \tag{3}$$

Furthermore, under the four conditions in Assumption 2 respectively, the mechanism $\mathcal{M}^{\Phi,H}$ is

1. *truthful,*
2. *informed-truthful,*
3. *strongly truthful, or*
4. *solely-truthful.*

In the following example, we show how Mechanism 1 with a (P, Φ)-ideal scoring function works, and illustrate the difference between informed-truthful and strongly truthful.

Example (continued)**.** On each day s, Alice and Bob learn their commute time $(x_s, y_s) \in \mathbb{R}^2$. We want to use Mechanism 1 to collect those commute time, and we know $P_{X,Y}$.

If $\Phi(a) = \frac{1}{2}|a-1|$, a (P, Φ)-ideal scoring function is $H(x, y) = \mathbb{I}[G(x, y) > R] - 1/2$ which can be represented by an ellipse Γ. After Alice and Bob report their every day's commute time $\hat{\mathbf{x}}, \hat{\mathbf{y}}$, the mechanism arbitrarily pick a bonus day b, and two distinct penalty days p and q. Then it pays Alice with 1 if their bonus day reports are in the ellipse Γ and their penalty days reports are not in Γ. As seen in Fig. 1, Γ is skew in diagonal, so if Alice's and Bob's reports on the bonus day are more correlated they can get more payment.

However, if Alice receive an extremely large value (e.g. $x_s = 20$) such that she knows the scoring function H is $-1/2$ for certain regardless of Bob's report (cf. Figure 1), Alice can misreport her signal (e.g. $\hat{x}_s = 2$ when $x_s \geq 20$) without changing her expected utility. Therefore the Φ-pairing mechanism with $\Phi(a) = \frac{1}{2}|a-1|$ is not strongly truthful. Additionally, truth-telling is not even a strict Bayesian Nash equilibrium.

To prevent Alice from truncating signals, instead of $\Phi(a) = \frac{1}{2}|a-1|$ we can take other strictly convex Φ. For example if $\Phi(a) = a \log a$, the ideal scoring function is a quadratic function and above-mentioned strategy cannot trivially hold. In Theorem 4.1 we prove this in the finite signal spaces setting. △

Remark 4.2. Although the Φ-pairing mechanism with a (P, Φ)-ideal scoring function has many desirable properties shown in Theorem 4.1, such a mechanism is not detail-free. Furthermore, in the detail-free setting where mechanisms only access Alice's and Bob's reports, it is impossible to have a solely truthful mechanism where truth-telling is the uniquely best equilibrium. Informally, in the detail-free setting a mechanism $\mathcal{M}$ cannot distinguish between the following two situations: 1) Alice and Bob's signals joint distribution is P and their strategy profile is a permutation $\boldsymbol{\theta}$; 2) Alice and Bob's signals joint distribution is $\boldsymbol{\theta} \circ P$ and their strategy profile is the truth-telling strategy, because their reports are generated from the same distribution $\boldsymbol{\theta} \circ P$ in both cases. Therefore,

$$u_A(\boldsymbol{\theta}; P, \mathcal{M}) = u_A(\boldsymbol{\tau}; \boldsymbol{\theta} \circ P, \mathcal{M}). \tag{4}$$

Suppose the ex-ante payment under the truth-telling strategy profile and P is strictly higher than the ex-ante payment under a permutation strategy profile $\boldsymbol{\theta}$. Then the ex-ante payment under a permutation strategy profile $\boldsymbol{\theta}^{-1}$ and the joint signal distribution $\boldsymbol{\theta} \circ P$, $u_A(\boldsymbol{\theta}^{-1}; \boldsymbol{\theta} \circ P, \mathcal{M}) = u_A(\boldsymbol{\tau}; P, \mathcal{M})$ is strictly higher than the ex-ante payment under truth-telling strategy profile $u_A(\boldsymbol{\tau}; \boldsymbol{\theta} \circ P, \mathcal{M}) = u_A(\boldsymbol{\theta}; P, \mathcal{M})$. This argument is trivially true when $\mathcal{M}$ is a Φ-pairing algorithm and the scoring function is a function of Alice's and Bob's reports. For general detail-free mechanisms, the reader may refer to Sect. 8 of Kong and Schoenebeck [12]. △

[11] There are some minor details when $\mathcal{X}$ and $\mathcal{Y}$ are not finite set. Here we require $\boldsymbol{\theta}$ to have finite $\int H \, d\theta_A \, d\theta_B dP_{X,Y}$,and $\int \Phi^*(H) \, d\theta_A \, d\theta_B dP_X P_Y$.

5 Main Technical Lemmas

To prove Theorem 4.1, we use the following lemmas which are also important in the rest of the paper.

We first show the ex-ante payment under the truth-telling strategy profile in the Φ-pairing mechanism with (P, Φ)-ideal scoring function is the Φ-mutual information between Alice's and Bob's signals.

Lemma 5.1 (Truth-telling)**.** *If H is a $(P_{X,Y}, \Phi)$-ideal scoring function,*

$$u_A(\boldsymbol{\tau}, P, H) = D_\Phi(P_{X,Y} \| P_X P_Y).$$

Moreover, if $P_{X,Y}$ is stochastic relevant, $D_\Phi(P_{X,Y} \| P_X P_Y) > 0$.

Then we show any deviation from the truth-telling strategy profile or an ideal scoring function cannot improve Alice (and Bob's) ex-ante payment. The proof uses the variational representation of Φ-divergence (Theorem 2.2).

Lemma 5.2 (Manipulation in strategies and scoring functions)**.** *For any strategy profile $\boldsymbol{\theta}$ and scoring function K,*[12]

$$u_A(\boldsymbol{\theta}, P, K) \leq D_\Phi(P_{X,Y} \| P_X P_Y).$$

Note that combining these two lemmas we have an even stronger result than inequality (3). This stronger result is a key tool in this paper: For any scoring function K and strategy profile $\boldsymbol{\theta}$,

$$u_A(\boldsymbol{\theta}, \mathbb{P}, K) \leq u_A(\boldsymbol{\tau}, \mathbb{P}, H). \tag{5}$$

Lemma 5.3 (Oblivious strategy)**.** *If $\boldsymbol{\theta}$ is an oblivious strategy profile, for any scoring function K*

$$u_A(\boldsymbol{\theta}, P, K) \leq 0.$$

Lemma 5.4. *Moreover, given Conditions 4 in Assumption 2, the equality in* (5) *for Alice or Bob occurs if and only if*

1. *$\boldsymbol{\theta} = (\pi_A, \pi_B)$ which is a permutation strategy profile, and*
2. *For all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, $K(\pi_A(x), \pi_B(y)) = \Phi'\left(\frac{P_{X,Y}(x,y)}{P_X(x)P_Y(y)}\right)$.*

Informally, Lemma 5.4 shows if the pair of a strategy profile and a scoring function $(\boldsymbol{\theta}, K)$ have (5) equal only if there is a "conjugated" structure between the strategy and the scoring function. The proof uses the pigeonhole principle on the finite signal spaces and shows if the equality holds under a non permutation strategy profile, P is not stochastic relevant.

Proof of Theorem 4.1

With the above four lemmas, we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. There are four statements to show.

First, (3) is a direct result of (5). Furthermore, (3) proves that truth-telling is a Bayesian Nash equilibrium, and has highest ex-ante payment to Alice.[13] This shows the mechanism is truthful.

By Lemma 5.3, the ex-ante payment to Alice (and Bob) is non-positive. Combining this and Lemma 5.1, we prove the Φ-pairing mechanism with (P, Φ)-ideal scoring function is inform-truthful when P is stochastic relevant.

To show our mechanism is strongly truthful, under Condition 3 in Assumption 2, we use the first part of Lemma 5.4. If the ex-ante payment under some strategy profile is equal to the ex-ante payment under the truth-telling strategy profile, the strategy profile is a permutation strategy profile.

[12] There are some minor details when $\mathcal{X}$ and $\mathcal{Y}$ are not finite set. Here we require K and $\boldsymbol{\theta}$ to have finite $\int K\, dP_{X,Y}$, $\int \Phi^*(K) d(P_X P_Y)$, $\int K\, d\theta_A\, d\theta_B dP_{X,Y}$,and $\int \Phi^*(K)\, d\theta_A\, d\theta_B dP_X P_Y$.

[13] Note that without additional assumption the truth-telling is not a strict Bayesian Nash equilibrium. This is illustrated in the example in Sect. 4.

Finally, for solely truthfulness, suppose $\boldsymbol{\theta}$ is a non-truthful strategy profile such that Alice's ex-ante payment is equal to her ex-ante payment under the truth-telling strategy profile, $u_A(\boldsymbol{\theta}, P, H) = u_A(\boldsymbol{\tau}, P, H)$. By the first part of Lemma 5.4, there exists a pair of non trivial permutations π_A and π_B on $\mathcal{X}$ and $\mathcal{Y}$ respectively such that there exist $x \in \mathcal{X}$ and $y \in \mathcal{Y}$,

$$(\pi_A(x), \pi_B(y)) \neq (x, y).$$

By the second part of Lemma 5.4, $H(\pi_A(x), \pi_B(y)) = \Phi'\left(\frac{P_{X,Y}(x,y)}{P_X(x)P_Y(y)}\right)$. On the other hand, because H is (P, Φ)-ideal, $H(\pi_A(x), \pi_B(y)) = \Phi'\left(\frac{P_{X,Y}(\pi_A(x),\pi_B(y))}{P_X(\pi_A(x))P_Y(\pi_B(y))}\right)$. Because Φ is strictly convex and differentiable, Φ' is invertible, so

$$\frac{P_{X,Y}(x,y)}{P_X(x)P_Y(y)} = \frac{P_{X,Y}(\pi_A(x),\pi_B(y))}{P_X(\pi_A(x))P_Y(\pi_B(y))}.$$

However, P is fine-grained, so $(\pi_A(x), \pi_B(y)) = (x, y)$ which reaches a contradiction. Therefore, the mechanism is solely truthful. □

6 The Pairing Mechanism in the Detail Free Settings

In Sect. 5, we see that to achieve the truthfulness guarantees, it suffices to have a "good" scoring function. Informally, if the ex-ante payment to Alice under the truth-telling strategy profile is close to the Φ-mutual information between Alice's and Bob's signals, by (5), the ex-ante payment under an untruthful-strategy is less than the ex-ante payment under the truth-telling strategy profile.

In Sect. 6.1 we formalize the notions of a *good* scoring function and of the *accuracy* of a learning algorithm $\mathcal{L}$ for scoring functions. In Sect. 6.2, we state our main result, Theorem 6.1, which reduces the mechanism design problem to a learning problem for an ideal scoring function, and provides some intuition about the proof of the theorem.

6.1 Definitions about Accuracy

Now we define a *good* scoring function, and the *accuracy* of a learning algorithm $\mathcal{L}$. Given Φ, a prior $P_{X,Y}$ and $\epsilon > 0$, we say that a scoring function K is ***ϵ-ideal on $(P_{X,Y}, \Phi)$***, if for Alice

$$u_A(\boldsymbol{\tau}, P, K) \geq u_A(\boldsymbol{\tau}, P, H_P^\Phi) - \epsilon = D_\Phi(P_{X,Y} \| P_X P_Y) - \epsilon, \tag{6}$$

and the similar inequality holds for Bob. Additionally, let $m_L \in \mathbb{N}$, $\delta > 0$, and $\mathcal{P}$ be a set of distributions on $\mathcal{X} \times \mathcal{Y}$. We define a *learning algorithm for scoring functions with m_L samples from P*, as a function from $(\mathbf{x}_L, \mathbf{y}_L) \in (\mathcal{X} \times \mathcal{Y})^{m_L}$ to a scoring function K. Given a function $S_L : \mathbb{R} \times \mathbb{R} \to \mathbb{N}$, we say such a learning algorithm $\mathcal{L}$ is ***(δ, ϵ)-accurate on*** $(\mathcal{P}, \Phi)$ with $S_L(\delta, \epsilon)$ samples, if for all $P_{X,Y} \in \mathcal{P}$, $\delta \in (0, 1)$, $\epsilon > 0$, and $m_L \geq S_L(\delta, \epsilon)$:

$$\Pr_{(\mathbf{x}_L, \mathbf{y}_L) \sim P_{X,Y}^{m_L}} \left[u_A(\boldsymbol{\tau}, P, \mathcal{L}(\mathbf{x}_L, \mathbf{y}_L)) > D_\Phi(P_{X,Y} \| P_X P_Y) - \epsilon\right] \geq 1 - \delta.$$

That is given m_L i.i.d. samples from $P_{X,Y}$, the probability that the output, $\mathcal{L}(\mathbf{x}_L, \mathbf{y}_L)$, is ϵ-ideal on (P, Φ) is greater than $1 - \delta$. Note that we require the algorithm $\mathcal{L}$ approximates the ideal scoring "uniformly" on all distributions in $\mathcal{P}$.

6.2 Pairing Mechanism with Learning Algorithms

Now we replace a fixed scoring function with an accurate learning algorithm $\mathcal{L}$ in Mechanism 1. Intuitively, in the detail-free setting, the Mechanism 2 first runs a learning algorithm on Alice's and Bob's report profile to derive a scoring function, and then pays Alice and Bob by Mechanism 1. Formally,

Theorem 6.1. *Let Φ be a continuous convex function with $[0, \infty) \subseteq \mathrm{dom}(\Phi)$, m_L be an integer, $\mathcal{L}$ be a learning algorithm on m_L samples, a function $S_L : \mathbb{R} \times \mathbb{R} \to \mathbb{N}$, and $\mathcal{P}$ be a set of joint distributions on $\mathcal{X} \times \mathcal{Y}$.*

Mechanism 2 Φ-divergence pairing mechanism with a learning algorithm $\mathcal{M}^{\Phi,\mathcal{L}}$

Parameters: A convex function Φ, and a learning algorithm $\mathcal{L}$ with m_L samples.
Input: A report profile $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ from Alice and Bob on m tasks where $m \geq 2 + m_L$.

1: Partition m tasks (arbitrarily) into a set of learning tasks M_L and a set of scoring tasks M_S where $|M_L| \geq m_L$ and $|M_S| \geq 2$. Let $(\hat{\mathbf{x}}_L, \hat{\mathbf{y}}_L)$ be the report to Alice and Bob on the learning tasks M_L, and $(\hat{\mathbf{x}}_S, \hat{\mathbf{y}}_S)$ be the reports on the scoring tasks.
2: Run the learning algorithm and derive $K_{\text{est}} = \mathcal{L}(\hat{\mathbf{x}}_L, \hat{\mathbf{y}}_L)$.
3: Run the Φ-pairing mechanism (Mechanism 1) with the scoring function K_{est}, and pay Alice and Bob accordingly.

Suppose the common prior between Alice and Bob satisfying Assumption 1 with $P_{X,Y} \in \mathcal{P}$, and $\mathcal{L}$ is (δ, ϵ)-accurate on $(\mathcal{P}, \Phi)$ with $S_L(\delta, \epsilon)$ samples. Under the first three conditions in Assumption 2 respectively, Mechanism 2 is

1. *(δ, ϵ)-truthful on $\mathcal{P}$ with a $2 + S_L(\delta, \epsilon)$ number of tasks;*
2. *(δ, ϵ)-informed-truthful on $\mathcal{P}$ with a $2 + S_L(\delta, \epsilon)$ number of tasks;*
3. *(δ, ϵ)-strongly truthful on $\mathcal{P}$ with a $2 + S_L(\delta, \epsilon)$ number of tasks.*

Let $P \in \mathcal{P}$ be Alice and Bob's signals joint distribution. Here we only require that $\mathcal{L}$ outputs an ϵ-ideal scoring function on (P, Φ) when their strategy profile is the truth-telling. However, there may exists a non-truth-telling strategy profile $\boldsymbol{\theta}$ such that $\boldsymbol{\theta} \circ P$ is not in $\mathcal{P}$, and the output of $\mathcal{L}$ is not ϵ-ideal on $(\boldsymbol{\theta} \circ P, \Phi)$. Nevertheless, Mechanism 2 still can control their ex-ante payment under such non-truth-telling strategy profiles. We give a more detail discussion in Sect. 8.

Remark 6.2. Note that the truthfulness guarantees are subject to the belief of Alice (and Bob). Mechanism 2 ensures with $1 - \delta$ probability the payment under truth-telling strategy profile is ϵ close to a fixed strongly truthful (inform-truthful or truthful) mapping for all $\delta \in (0, 1)$ and $\epsilon > 0$.[14] In particular, we make the error ϵ sufficiently small such that the truth-telling strategy profile still has a higher ex-ante payment than any oblivious strategy has with high probability.

Furthermore, we can pick Φ such that the ex-ante payment is bounded by some constant U, and the mechanism is $(\epsilon + U\delta)$-strongly (informed-) truthful with probability 1. For example, if $\Phi(a) = |a - 1|/2$, we only need to consider bounded scoring functions, and the resulting mechanism is approximated informed-truthful with probability 1.

To establish some intuitions, let's consider the following "fantasy" mapping $F^\Phi = (F^\Phi_A, F^\Phi_B)$ from Alice's and Bob's signals' joint distribution P and their strategy profile $\boldsymbol{\theta}$ to payments:

$$F^\Phi_A(\boldsymbol{\theta}, P) \triangleq u_A(\boldsymbol{\theta}, P, H^\Phi_{\boldsymbol{\theta} \circ P}) \text{ and } F^\Phi_B(\boldsymbol{\theta}, P) \triangleq u_B(\boldsymbol{\theta}, P, H^\Phi_{\boldsymbol{\theta} \circ P}). \tag{7}$$

It is straightforward to show the following lemma.

Lemma 6.3 (Fantasy mapping). *Under the first three conditions in Assumption 2 respectively, the mapping $F^\Phi = (F^\Phi_A, F^\Phi_B)$ is*

1. *truthful,*
2. *informed-truthful, or*
3. *strongly truthful.*

Recall that a learning algorithm for scoring function with input samples from Q outputs an approximated ideal function H^Φ_Q. If Alice and Bob have a strategy profile $\boldsymbol{\theta}$ with joint signal distribution P, the learning tasks are sampled from distribution $\boldsymbol{\theta} \circ P$ and a learning algorithm for scoring function will output approximated version of $H^\Phi_{\boldsymbol{\theta} \circ P}$. Therefore, the ex-ante payment to Alice in Mechanism 2 is "close" to fantasy payment function, and Theorem 6.1 formalizes this idea. We show the proof in Appendix C.

[14] Formally, there exists an event with probability $1 - \delta$ such that the conditional expected payment to Alice under such event is ϵ-close to a strongly truthful (inform-truthful or truthful) mapping.

7 Learning Ideal Scoring Functions

Theorem 6.1 reduces the mechanism design problem to a learning problem for an ideal scoring function. However, (6) may be hard to verify. We provide two natural sufficient conditions for ϵ-ideal scoring functions in Sect. 7.1, and we will provide two concrete learning algorithms for scoring function in Sect. 7.2. Finally, in Sect. 7.3 we show an obstacle to designing exact strongly truthful, or inform-truthful mechanisms which use the Φ-divergence-based method.

7.1 Sufficient Conditions for Approximated Φ-Ideal Scoring Functions

Bregman divergence Given $a, b \in \mathbb{R}$ and a strictly convex and twice differentiable $\Phi : \mathbb{R} \to \mathbb{R}$, the standard Bregman divergence is $\Phi(a) - \Phi(b) - \nabla\Phi(b)^\top(a - b)$. It can be extended to ***Bregman divergence*** between two functionals f and g over a probability space $(\Omega, \mathcal{F}, P)$ [2]

$$B_{\Phi,P}(f, g) = \int \Phi(f(\omega)) - \Phi(g(\omega)) - \nabla\Phi(g(\omega))^\top(f(\omega) - g(\omega))dP(\omega).$$

Lemma 7.1 (Bregman divergence and accuracy)**.** *If Φ is strictly convex and twice differentiable on $[0, \infty)$,*

$$D_\Phi(P_{X,Y} \| P_X P_Y) - u_A(\boldsymbol{\tau}, P, K) = B_{\Phi^*,P_X P_Y}(K, H).$$

Therefore, if $B_{\Phi^*,P_X P_Y}(K, H) \le \epsilon$, K is an ϵ-ideal scoring function on (Φ, P).

The above notion of Bregman divergence capture the average "distance" between a scoring function K and the ideal one. Therefore, if the scoring function K is uniformly close to the ideal one H, the Bergman divergence between K and H is also small.

Total variation distance On the other hand, we may first learn the prior P and compute an approximated ideal scoring function afterward. This indirect method is also useful, because estimating the probability density function is a much well studied problem.

Theorem 7.2 (Total variation to accuracy)**.** *Given Φ is a convex function and a prior $P_{X,Y}$ over a finite space $\mathcal{X} \times \mathcal{Y}$, suppose there exist constants $0 < \alpha < 1$ and c_L such that*

$$\forall x \in \mathcal{X}, y \in \mathcal{Y},\ P_{X,Y}(x, y) > 2\alpha \text{ or } P_{X,Y}(x, y) = 0, \tag{8}$$

$$\forall z, w \in [\alpha, 1/\alpha],\ |\Phi(z) - \Phi(w)| \le c_L|z - w|. \tag{9}$$

If $\|\hat{P}_{X,Y} - P_{X,Y}\|_{TV} \le \delta < \alpha$,[15] $\hat{K}(x, y) \in \partial\Phi\left(\frac{\hat{P}_{X,Y}}{\hat{P}_X \hat{P}_Y}\right)$ is a $\frac{6c_L}{\alpha^2}\delta$-ideal scoring function.

The first condition says the smallest nonzero probability $P_{X,Y}(x, y)$ is away from zero, and the second condition requires the function Φ is Lipschitz in $[\alpha, 1/\alpha]$ which holds for all examples in Table 1. With these conditions, if we have a good estimation $\hat{P}$ for P with small total variation distance, we can compute a very accurate scoring function $\hat{K}$ from $\hat{P}$. As we will see in Sect. 7.2, the empirical distributions with m_L samples satisfies this condition with high probability for large enough m_L.

7.2 Learning Algorithms for Scoring Functions

Generative approach Recall that if P is known, the ideal scoring function can be computed directly. In a generative approach, we try to estimate the probability density function P from reports and derive the scoring function afterward under the truth-telling strategy profile. In general this generative approach is useful when $\mathcal{P}$ is over finite spaces, or $\mathcal{P}$ is a parametric model by Theorem 7.2. Here we provide an example of a generative approach.

A standard way of learning probability density function is to use empirical distribution on m_L samples (defined in Equation (14)). The following theorem shows that the empirical distribution gives a good estimation in terms of total variation distance.

[15] $\|\hat{P} - \hat{P}\|_{TV} = \sum_{\omega\in\Omega} |P(\omega) - \hat{P}(\omega)|$ is the total variation distance between P and $\hat{P}$.

Algorithm 3 A generative algorithm

Input: A report profile $\hat{\mathbf{X}}_L, \hat{\mathbf{Y}}_L \in (\mathcal{X} \times \mathcal{Y})^{m_L}$ from learning tasks from Alice and Bob.
Parameters: A convex function Φ and its sub-gradient $\partial\Phi$

1: Compute empirical distribution from Alice's and Bob's reports: for all event E in $\mathcal{X} \times \mathcal{Y}$

$$\hat{P}_{X,Y}(E) = \frac{1}{m_L}\sum_{s=1}^{m_L} \mathbb{I}[(\hat{x}_s, \hat{y}_s) \in E],$$

and compute the marginal empirical distribution, for all event E in $\mathcal{X}$ and F in $\mathcal{Y}$

$$\hat{P}_X(E) = \frac{1}{m_L}\sum_{s=1}^{m_L} \mathbb{I}[\hat{x}_s \in E], \text{ and } \hat{P}_Y(F) = \frac{1}{m_L}\sum_{s=1}^{m_L} \mathbb{I}[\hat{y}_s \in F].$$

2: Compute the scoring function as

$$\begin{cases} \hat{K}(x,y) \in \partial\Phi\left(\frac{\hat{P}_{X,Y}(x,y)}{\hat{P}_X(x)\hat{P}_Y(y)}\right), & \text{if } \hat{P}_X(x)\hat{P}_Y(y) \neq 0 \\ \hat{K}(x,y) = 0, & \text{otherwise.} \end{cases} \tag{10}$$

Lemma 7.3 (Theorem 3.1 in [4]). *For all $\epsilon > 0$, $\delta > 0$, finite domain Ω, distribution in P in Δ_Ω, there exists $M = O\left(\frac{1}{\epsilon^2}\max(|\Omega|, \log(1/\delta))\right)$ such that for all $m_L \geq M$ the empirical distribution with m_L i.i.d. samples, $\hat{P}_{m_L}$, satisfies*

$$\Pr[\|P - \hat{P}_{m_L}\|_{TV} \leq \epsilon] \geq 1 - \delta.$$

Therefore, we can design a learning algorithm $\mathcal{L}_{\text{emp}}$ as follows: estimate joint distribution $P_{X,Y}$ by their empirical distributions $\hat{P}_{X,Y}$ and derive $\hat{K}$ from Theorem 7.2. By Theorem 7.2 and Lemma 7.3, such algorithm is ϵ-accurate with $1-\delta$ probability.

Discriminative approach Instead of density estimation, a discriminative approach estimates an ideal scoring functions directly. This enables more freedom of algorithm design. Here we use the variational representation (Theorem (2.2)), and give an optimization characterization of an ideal scoring function.

Given the assumption 1, under the truth-telling strategy profile we can have i.i.d. samples of (u, v) where u is sampled from $P_{X,Y}$ and v is sampled from $P_X P_Y$ independently, and this is shown formally in Algorithm 4. Taking $L^\Phi(a, b) \triangleq a - \Phi^*(b)$ as the risk function, we can convert the estimation of the ideal scoring functions to empirical risk minimization (maximization) over a training set (u_t, v_t) with $t = 1, 2, \ldots, \lfloor m_L/3 \rfloor$,

$$\tilde{K} = \arg\max_{k\in\mathcal{K}} \sum_t L^\Phi(k(u_t), k(v_t)) = \arg\max_{k\in\mathcal{K}} \left\{\int k(\omega) d\hat{P}_{X,Y}(\omega) - \int \Phi^*(k(\omega)) d\widehat{P_X P_Y}(\omega)\right\} \tag{11}$$

where $\mathcal{K}$ is a pre-specified class of functionals $k : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$, $\hat{P}_{X,Y}$ and $\widehat{P_X P_Y}$ are empirical distributions on $\lfloor m_L/3 \rfloor$ samples from distributions $P_{X,Y}$ and $P_X P_Y$ respectively. See Appendix A.2 for formal definitions.

Assuming that $\mathcal{K}$ is a convex set of functionals, the implementation of (11) only requires solving a convex optimization problem over function space $\mathcal{K}$ which is well studied [17]. Therefore, in this section, we are going to show *the empirical risk maximizer $\tilde{K}$ with respect to L^Φ is ϵ-accurate* with large probability under some conditions on $\mathcal{K}$ and prior $P_{X,Y}$. Furthermore, this error can be seen as the generalized error of the empirical risk maximizer. Formally,

Theorem 7.4. *Consider a distribution P over $\mathcal{X} \times \mathcal{Y}$; a strictly convex and a twice differentiable function Φ on $[0, \infty)$ with its gradient Φ' and conjugate Φ^*; a family of functional $\mathcal{K}$ from $\mathcal{X} \times \mathcal{Y}$ to $\text{dom}(\Phi^*)$; and $\Phi^*(\mathcal{K}) = \{\Phi^*(k) : k \in \mathcal{K}\}$. Suppose*

1. *the (P, Φ)-ideal scoring function $H = \Phi'\left(\frac{P_{X,Y}}{P_X P_Y}\right)$ is in $\mathcal{K}$, and*

Algorithm 4 An empirical risk minimization algorithm

Input: A report profile $\hat{\mathbf{X}}_L, \hat{\mathbf{Y}}_L \in (\mathcal{X}\times\mathcal{Y})^{m_L}$ from learning tasks from Alice and Bob.
Parameters: A convex function Φ and its conjugate Φ^*.

1: Partition the report profile into three equal size $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}^i)$ in $(\mathcal{X}\times\mathcal{Y})^{m_L/3}$ where $i = 0, 1$, and 2.

2: For the empirical joint distribution, we use the report profile $\mathbf{x}^i, \mathbf{y}^i$ to compute: For all event E in $\mathcal{X}\times\mathcal{Y}$

$$\tilde{P}_{X,Y}(E) = \frac{3}{m_L}\sum_{s=1}^{m_L/3}\mathbb{I}[(\hat{x}^0_s, \hat{y}^0_s) \in E],$$

Further compute the product empirical distributions: for all event E

$$\tilde{P}_i\tilde{P}_j(A) = \frac{3}{m_L}\sum_{s=1}^{m_L/3}\mathbb{I}[(\hat{x}^1_s, \hat{y}^2_s) \in E]$$

(Note that we use new samples to compute the product of empirical distribution to ensure the independence between $\tilde{P}_X\tilde{P}_Y$ and $\tilde{P}_{X,Y}$)

3: Finally solve following optimization problem

$$\tilde{K} = \arg\max_{k\in\mathcal{K}}\left\{\int k(\omega)d\tilde{P}_{X,Y}(\omega) - \int \Phi^*(k(\omega))d\tilde{P}_X\tilde{P}_Y(\omega)\right\} \tag{12}$$

2. there exist constants $(L_l, R_l, D_l)_{l=1,2}$

(a) $\sup_{k\in\mathcal{K}}\rho_{L_1}(k, P_{X,Y}) \leq R_1$, *and* $\int_0^{R_1}\sqrt{\mathcal{H}_{[],L_1}(u,\mathcal{K},P_{X,Y})}du \leq D_1$

(b) $\sup_{l\in\Phi^*(\mathcal{K})}\rho_{L_2}(l, P_XP_Y) \leq R_2$ *and* $\int_0^{R_2}\sqrt{\mathcal{H}_{[],L_2}(u,\Phi^*(\mathcal{K}),P_XP_Y)}du \leq D_2$

There exists $M = O\left(\frac{1}{\varepsilon^2}\log\frac{1}{\delta}\right)$*, such that for all* $m_L \geq M$*,* $\tilde{K}$ *defined in* (11) *is* ε*-accurate on prior* P *with probability* $1-\delta$*.* [16]

Informally, Theorem 7.4 requires the functional class $\mathcal{K}$ contains an ideal scoring function and it has a constant complexity (generalized entropy with bracketing). Under these conditions, the empirical risk minimizer (maximizer) can estimate the ideal scoring function accurately even when the signal space can be integers, real numbers, or Euclidean spaces.

Here we give a outline of the proof. By Lemma 7.1, it is sufficient to show the empirical risk minimizer $\tilde{K}$ has small Bregman divergence form the ideal one. Moreover, if the estimation K is the empirical risk maximizer, this error can be upper bounded by the distance between the empirical distribution and the real distribution (Lemma 7.5). Therefore, we can use functional form of Central Limit Theorem to upper bound the error (Theorem A.6). We defer the proof to the appendix.

Lemma 7.5. *Let* $\tilde{K}$ *be the estimate of* H *obtained by solving the Equation* (12)*, and* $H \in \mathcal{K}$ *Then*

$$B_{\Phi^*,P_XP_Y}(\tilde{K}, H) \leq \sup_{k\in\mathcal{K}}\left|\int \Phi^*(k - \Phi^*(H)d(\tilde{P}_X\tilde{P}_Y - P_XP_Y) - \int (k-H)\, d(\tilde{P}_{X,Y} - P_{X,Y})\right|.$$

Example (continued)**.** For $\Phi(a) = a\log a$, if the parameters σ^2, τ^2 are in a bounded set, we can take $\mathcal{K}$ as a set of the quadratic functions with bounded coefficients. By Theorem 2.7.11 [25], the general bracket entropy of $\mathcal{K}$ and $\Phi^*(\mathcal{K})$ can be bounded by some constants. △

7.3 Nonexistence of Unbiased Estimators for Φ-divergence

Combining Theorem 6.1 and Theorem 7.2 or 7.4 we can design mechanisms that are ϵ-strongly truthful (inform-truthful, or truthful) with high probability. *However, is it possible to have an exact informed-truthful or strongly truthful?* In this section, we show a technical obstacle to designing such mechanisms.

[16] Here we do not show the dependency on constants L_l, R_l and D_l.

The main observation of Theorem 4.1 and 6.1 is that the ex-ante payment to an agent has a close connection to the Φ-divergence from signal pairs on penalty tasks to signal pairs on the bonus task and use this Φ-divergence to upper bound ex-ante payment under all manipulations uniformly. This observation is also used in [21] and [12]. Under this framework, showing exact strongly truthful, informed-truthful, or truthful requires *unbiased estimator* of Φ-divergence from i.i.d. samples. Specifically, suppose we can estimate an ideal scoring function accurately from samples. We can estimate the Φ-divergence without bias. The following theorem shows such estimator does not exist in general.

Theorem 7.6 (Nonexistence)**.** *Suppose the discrete signal spaces of Alice and Bob, $\mathcal{X}$ and $\mathcal{Y}$, both have more than two elements, and Φ be twice differentiable convex function in $[0,\infty)$. For all $m \in \mathbb{N}$ all estimator $\hat{D} : (\mathcal{X} \times \mathcal{Y})^m \to \mathbb{R}$ from m pairs of signals $(\mathbf{x}, \mathbf{y}) = (x_1, y_1, \ldots, x_m, y_m)$ to a real value, there exists a prior distribution $P_{X,Y}$ over $\mathcal{X} \times \mathcal{Y}$ such that*

$$\mathbb{E}_{(\mathbf{x},\mathbf{y})\sim P_{X,Y}^m}[\hat{D}(\mathbf{x}, \mathbf{y})] \neq D_\Phi(P_{X,Y} \| P_X P_Y).$$

The key idea of this proof is that if we fix the estimator $\hat{D}$ and take the probability distribution P as variables, the expected value $\mathbb{E}[\hat{D}(\mathbf{x}, \mathbf{y})]$ is a polynomial of distribution P. However, the Φ-divergence $\mathbb{E}_{P_X P_Y}\left[\Phi\left(\frac{P_{X,Y}}{P_X P_Y}\right)\right]$ is usually not a polynomial, and we can fine one $P_{X,Y}$ to make these two values not equal. The proof is in Appendix D.3.

8 Relation to CA mechanism [23] and mutual information mechanism [12]

8.1 CA mechanism

In the following proposition we show CA mechanism is a special case of our mechanism 1. However the corresponding Φ is not strictly convex, so the mechanism is not strongly truthful in general. Furthermore, since Φ is not differentiable, it is not possible to use Theorem 7.4 to obtain results for it on continuous signal spaces.

Proposition 8.1 (CA mechanism [23])**.** *If we take $\Phi(a) = \frac{1}{2}|a-1|$ and restrict $|K| \leq 1/2$, Then the above mechanism reduces to the Correlated Agreement mechanism.*

Proof. If we take $\Phi(a) = \frac{1}{2}|a-1|$, $\Phi^*(b) = b$ when $|b| \leq 1/2$, the payment can be simplified as

$$M_A(\mathbf{r}) = K\left(\hat{x}_b, \hat{y}_b\right) - \Phi^*\left(K\left(\hat{x}_p, \hat{y}_q\right)\right) = K\left(\hat{x}_b, \hat{y}_b\right) - K\left(\hat{x}_p, \hat{y}_q\right).$$

Moreover, by Table 1, Equation (1) reduces to

$$\partial\Phi\left(\frac{P_{X,Y}(x,y)}{P_X(x)P_Y(y)}\right) = \begin{cases} 1/2 & \text{if } P_{X,Y}(x,y) > P_X(x)P_Y(y); \\ -1/2 & \text{if } P_{X,Y}(x,y) < P_X(x)P_Y(y); \\ [-1/2, 1/2] & \text{otherwise,} \end{cases}$$

and the scoring functions are in $\partial\Phi\left(\frac{P_{X,Y}(x,y)}{P_X(x)P_Y(y)}\right) + \frac{1}{2}$. □

8.2 Mutual Information Mechanism

The framework of mutual information mechanism [12] defines the payments to Alice (and Bob) to be the Φ-mutual information between Alice's and Bob's reports

$$D_\Phi(\boldsymbol{\theta} \circ P_{X,Y} \| \boldsymbol{\theta} \circ P_X P_Y)$$

where P is the joint distribution of signal and $\boldsymbol{\theta}$ is the strategy profile.

Our mechanism can be seen as a special case of the mutual information mechanism when the number of tasks goes to infinity. Formally, the fantasy mapping F^Φ pays Alice and Bob with the Φ-mutual information between Alice's and Bob's reports. As the number of tasks goes to infinity, both Algorithm 3 or 4 are $(0,0)$-accurate and by the proof of Theorem 6.1, our mechanism with learning Algorithm 3 or 4 pays Alice with F_A^Φ which is the Φ-mutual information between Alice's and Bob's reports.

However, our mechanism has a stronger guarantee when the number of tasks is finite. In the proof of Theorem 6.1, our mechanism ensure Alice's ex-ante payment is upper bounded by the Φ-mutual information between Alice's and Bob's reports *uniformly under any strategy profiles.* This property does not hold if we estimate the Φ-mutual information directly without variational representation.

For example, in Kong and Schoenebeck [12], they use the agents' report profile to estimate the density function and estimate the Φ-mutual information between their reports directly. In contrast, although Mechanism 2 with learning Algorithm 3 also first estimates the density function, the mechanism then computes a scoring function instead. These two methods have similar behavior under the truth-telling strategy profile. However, given a fixed the number of tasks, there may exist a non-truthful strategy profile $\boldsymbol{\theta}$ such that we cannot estimate the density function $\boldsymbol{\theta} \circ P$ accurately. The method in Kong and Schoenebeck [12] cannot provide a guarantee in such a situation. On the other hand, our variational method ensures the ex-ante payments to agents under $\boldsymbol{\theta}$ are worse than the mutual information between agents' reports. Having a uniform upper bound for a non-truthful strategy is important for real application. We may assume our learning algorithm can estimate agents' signal distributions, which is derived from non-adversarial settings. However, agents adopt the worst possible strategy profiles to break our mechanism adversarially.

9 More Than Alice and Bob

Although we only discuss the Φ-pairing mechanisms on Alice and Bob, it is straightforward to extend it to multiple agents.

In general, we can

1. partition a group of n agents into pairs after they reporting their signals, and
2. run Mechanism 2 $(\mathcal{L}^\Phi, \mathcal{M}^\Phi)$ on each pair.

Suppose in agents' common prior each pair of agents' signals is from a stochastic relevant prior family $\mathcal{P}$, and the learning algorithm $\mathcal{L}^\Phi$ is (ϵ,δ)-accurate with m_L samples over $\mathcal{P}$. Hence, each of the $n/2$ mechanisms is ϵ-strongly (informed) truthful with probability at least $1-\delta$. Then, by applying a union bound on those $n/2$ mechanism, the whole mechanism is ϵ-accurate with probability at least $1-n\delta$. In particular, if all agents' signal are from a finite set $\mathcal{Z}$ and for any pair of agents their signals are stochastic relevant and satisfy Equations (21) and (22), then by Theorem 7.2 and Lemma 7.3, for any $\epsilon,\delta>0$, there exists $S(\delta,\epsilon) = O\left(\frac{c_L^2}{\alpha^4\epsilon^2}\cdot \max\left\{|\mathcal{Z}|^2, \log\frac{n}{\delta}\right\}\right)$, such that the above mechanism is (δ,ϵ)-strongly truthful, inform-truthful, or truthful) with $S(\delta,\epsilon)$ tasks. Note that M_L only depends logarithmically on the number of agents.

The similar argument works for continuous signal by Theorem 6.1 and Theorem 7.4.

Additionally, we can further reduce the sample number if the agents' signals are symmetric in their common prior. For example, in the commute time example, there are constants $m_0,\sigma,\tau\in\mathbb{R}$ such that in each day s the expected commute time μ_s is sampled from $\mathcal{N}(m_0,\sigma^2)$ i.i.d and each agents commute time on day s is drawn from $\mathcal{N}(\mu_s,\tau^2)$. In this case, it is sufficient for the mechanism to learn *one ideal scoring function* and pay all pairs of agents with it. Therefore, we can first partition the task into m_L learning tasks and 2 scoring tasks. Then we run a learning algorithm to estimate parameters m_0,σ,τ, and derive an approximated ideal scoring function. Finally, we pay each pair of agents with this approximated ideal scoring function. As a result, the number of learning tasks does not increase as the number of agents increase (indeed it decreases in this case).

10 Beyond Φ-pairing Mechanisms

This framework can be easily extended to paying based on three or more agents, such that the ex-ante payment becomes $D_\Phi(P_{i,j,k}\|P_iP_jP_k)$. This extension can handle the cases when the signals are pairwise independent but 3-wise stochastic relevant. Moreover, by Data Processing Inequality (Corollary A.2), the ex-ante utility is greater or equal to the Φ-pairing mechanism 1 (with loss on the sample number for estimation). The following mechanism makes this idea formal:

Mechanism 5 Φ-divergence k-grouping mechanism

Input: A report profile $\mathbf{r} \triangleq (r_1, \dots, r_n)$ where each agent $i \in [n]$ submits a report $\mathbf{r}_i \in \mathcal{Z}^m$ for all m tasks.
Parameters: A convex function Φ and its conjugate Φ^*;
a set of scoring functions $\mathbf{K} = \{K_I\}$ for all I in $\mathcal{I}_k$ where $K_I : \mathcal{Z}^k \to \operatorname{dom}(\Phi^*)$ and $\mathcal{I}_k = \{I = (i_1, \dots, i_k) : i_1, \dots, i_k$ are distinct k elements in $[n]\}$ is the family of the k-tuples in $[n]$.

1: For each agent i, choose $k-1$ distinct *peer agents* $j_1, j_2, \dots, j_{k-1}$ uniformly at random from $[n] \setminus \{i\}$, and pick $k+1$ distinct tasks b, p, and $q_1, \dots, q_{k_1}$ where b is the *bonus task*, p is the *penalty task to* i, and q_l is *the penalty task to* j_l.
2: Based on the agent i's reports $r_{i,b}, r_{i,p}$ and the peer agents' reports $(r_{j_l,b}, r_{j_l,q_l})_{l \le k-1}$, the payment for agent i is

$$M_i(\mathbf{r}) = K_{i,j}\left(r_{i,b}, r_{j_1,b}, r_{j_2,b}, \dots, r_{j_{k-1},b}\right) - \Phi^*\left(K_{i,j}\left(r_{i,p}, r_{j_1,q_1}, r_{j_2,q_2}, \dots, r_{j_{k-1},q_{k-1}}\right)\right). \quad (13)$$

The following is an analogy of Theorem 4.1.

Theorem 10.1 (grouping mechanism)**.** *Given $n \ge 2$ and $m \in \mathbb{N}$, let $\mathbb{P}$ be a common prior satisfies Assumption 1 over a signal space $\mathcal{Z}$ with P begin k-wise stochastic relevant. Let $\boldsymbol{\theta}$ be a strategy profile, and $\boldsymbol{\tau}$ be the truth-telling strategy profile; Φ is a continuous convex function with $[0, \infty) \subseteq \operatorname{dom}(f)$; $\mathbf{H}$ is a (P, Φ)-ideal scoring function profile such that for all k-tuple of distinct agents $I = (i, j_1, \dots, j_{k-1})$ $z, w_1, \dots, w_{k-1} \in \mathcal{Z}$,*

$$H_I(z, w_1 \dots, w_{k-1}) \in \partial\Phi\left(\frac{P_I(z, w_1 \dots, w_{k-1})}{P_i(z) P_{j_1}(w_1) \dots P_{j_{k-1}}(w_{k-1})}\right).$$

The Φ-divergence k-grouping mechanism with a (P, Φ)-ideal scoring function profile $\mathbf{H}$ has the following properties: For any strategy profile $\boldsymbol{\theta}$ and agent $i \le n$

$$u_i(\boldsymbol{\tau}, P, \mathbf{H}) \ge u_i(\boldsymbol{\theta}, \mathbb{P}, \mathbf{H}).$$

Furthermore, under conditions in Assumption 2 for any pair of agents, the mechanism $\mathcal{M}^{\Phi, \mathbf{H}}$ is

1. *truthful;*
2. *informed-truthful;*
3. *strongly truthful;*
4. *solely-truthful.*

11 Conclusion

We showed how to reduce the design of peer prediction information elicitation in the multitask setting to a learning problem. As a result, we extend multitask peer prediction to the continuous setting for parametric models with bounded learning complexity. We also obtain improved bounds on the sample complexity for the finite signal setting. We note that in practice one could use deep learning techniques to learn the scoring function. However, we leave it for future work to obtain rigorous bounds in this setting.

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A Supplementary materials

A.1 Convex analysis

Here is a useful table for some standard Φs and their conjugate: Theorem 2.2 is a direct result of Young-

Table 1: Common Φ, its convex conjugate and subgradient

Φ-divergence	$\Phi(a)$	$\Phi^*(b)$	$\partial\Phi(a)$
Total variation	$\frac{1}{2}\lvert a-1\rvert$	$\begin{cases} b, & \text{if } \lvert b\rvert \le 1/2 \\ +\infty, & \text{otherwise} \end{cases}$	$\begin{cases} 1/2, & \text{if } a > 1 \\ -1/2, & \text{if } a < 1 \\ [-1/2, 1/2] & \text{if } a = 1 \end{cases}$
KL-divergence	$a \log a$	$\exp(b-1)$	$1 + \log a$
χ^2-divergence	$a^2 - 1$	$b^2/4$	$2a$
Squared Hellinger distance	$(1-\sqrt{a})^2$	$\frac{b}{1-b}$	$1 - 1/\sqrt{b}$

Fenchel inequality:

Theorem A.1 (Young-Fenchel inequality)**.** *Given $a \in \mathrm{dom}(\Phi)$ for all $b \in \mathrm{dom}(\Phi^*)$,*

$$\Phi(a) \ge ab - \Phi^*(b),$$

where the equality holds when $b \in \partial\Phi(a) = \{d : \Phi(c) \ge \Phi(a) + \langle d, c-a\rangle\}$, and $b = \Phi'(a)$ if Φ is convex and differential at a.

Proof of Theorem 2.2. By the definition of Φ-divergence,

$$\begin{aligned}
D_\Phi(P\|Q) =& \mathbb{E}_Q\left[\Phi\left(\frac{P}{Q}\right)\right] \\
=& \mathbb{E}_Q\left[\sup_b\left\{b\cdot\frac{P}{Q} - \Phi^*(b)\right\}\right] && \text{(by Young-Fenchel)} \\
=& \sup_{k:\Omega\to\mathrm{dom}(\Phi^*)}\left\{\mathbb{E}_Q\left[k(\omega)\cdot\frac{P}{Q}(\omega) - \Phi^*(k(\omega))\right]\right\} \\
=& \sup_{k:\Omega\to\mathrm{dom}(\Phi^*)}\left\{\mathbb{E}_Q\left[k(\omega)\cdot\frac{P}{Q}(\omega)\right] - \mathbb{E}_Q\left[\Phi^*(k(\omega))\right]\right\} \\
=& \sup_{k:\Omega\to\mathrm{dom}(\Phi^*)}\left\{\mathbb{E}_P\left[k(\omega)\right] - \mathbb{E}_Q\left[\Phi^*(k(\omega))\right]\right\}
\end{aligned}$$

Therefore, by Young-Fenchel inequality the equlity holds when $k(\omega) \in \partial\Phi(P(\omega)/Q(\omega))$ almost everywhere on Q. □

This formulation is powerful. For example, it can yield the data processing inequality easily.

Corollary A.2 (Data processing inequality)**.** *Consider a channel that produces Y given X based on the distribution $P_{Y|X}$ where $\Pr[Y|X] = P_{Y|X}$. Given distributions P_X and Q_X of X and $P_{Y|X}$, P_Y is the (marginal) distribution of Y when X is sampled from P_X and Q_Y is the distribution of Y when X is generated by Q_X, then for any Φ-divergence D_Φ,*

$$D_\Phi(P_X\|Q_X) \ge D_\Phi(P_Y\|Q_Y).$$

Proof of Corollary A.2. By Theorem (2.2), there exists a real-valued function $g : \mathcal{Y} \to \mathbb{R}$ such that

$$\begin{aligned}D_\Phi(P_Y \| Q_Y) =& \mathbb{E}_{P_Y}[g] - \mathbb{E}_{Q_Y}[\Phi^*(g)]\\ =& \sum_{y\in\mathcal{Y}} P_Y(y)g(y) - \sum_{y\in\mathcal{Y}} Q_Y(y)\Phi^*(g(y))\\ =& \sum_{x\in\mathcal{X},y\in\mathcal{Y}} P_X(x)P_{Y|X}(y,x)g(y) - \sum_{x\in\mathcal{X},y\in\mathcal{Y}} Q_X(x)P_{Y|X}(y,x)\Phi^*(g(y))\\ =& \sum_{x\in\mathcal{X}} P_X(x)\sum_{y\in\mathcal{Y}} P_{Y|X}(y,x)g(y) - \sum_{x\in\mathcal{X}} Q_X(x)\sum_{y\in\mathcal{Y}} P_{Y|X}(y,x)\Phi^*(g(y))\end{aligned}$$

Because Φ^* is convex and for all $x \in \mathcal{X}$, $P_{Y|X}(y,x)$ is a distribution over y, we have for all x in $\mathcal{X}$, $\sum_{y\in\mathcal{Y}} P_{Y|X}(y,x)\Phi^*(g(y)) \geq \Phi^*\left(\sum_{y\in\mathcal{Y}} P_{Y|X}(y,x)g(y)\right)$. Therefore we have

$$D_\Phi(P_Y \| Q_Y) \leq \sum_{x\in\mathcal{X}} P_X(x)\left(\sum_{y\in\mathcal{Y}} P_{Y|X}(y,x)g(y)\right) - \sum_{x\in\mathcal{X}} Q_X(x)\Phi^*\left(\sum_{y\in\mathcal{Y}} P_{Y|X}(y,x)g(y)\right).$$

Define $h(x) \triangleq \sum_{y\in\mathcal{Y}} P_{Y|X}(y,x)g(y)$, and we can further simplify it as

$$\begin{aligned}D_\Phi(P_Y \| Q_Y) \leq& \sum_{x\in\mathcal{X}} P_X(x)h(x) - \sum_{x\in\mathcal{X}} Q_X(x)\Phi^*(h(x))\\ \leq& \sup_{h:\mathcal{X}\to\mathbb{R}} \sum_{x\in\mathcal{X}} P_X(x)h(x) - \sum_{x\in\mathcal{X}} Q_X(x)\Phi^*(h(x))\\ =& D_\Phi(P_X \| P_Y)\end{aligned}$$

which completes the proof. □

A.2 Upper bounds for empirical processes

In this section, we provide some standard results on empirical process, most of which are in van de Geer and van de Geer [24]. Consider n independent and identically (i.i.d.) random variables $X_1, X_2, \ldots, X_n$ with distribution P on a measurable space $(\Omega, \mathcal{F})$. Let $\hat{P}_n$ be the *empirical distribution* based on those n random variables, i.e., for each set $A \in \mathcal{F}$,

$$\hat{P}_n(A) = \frac{1}{n}\{\text{number of } X_i \in A, 1 \leq i \leq n\} = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}[X_i \in A] \tag{14}$$

and let $\mathcal{K} \subset L_2(P) = \{k : \Omega \to \mathbb{R} : \int |k|^2 dP < \infty\}$ be a collection of functions. The *empirical process indexed* by $\mathcal{K}$ is

$$V_n(\mathcal{K}) = \left\{v_n(k) = \sqrt{n}\int kd(\hat{P}_n - P) : k \in \mathcal{K}\right\} \tag{15}$$

In this paper, we are mainly interested in uniform upper bound for Equation (15), i.e., upperbounds for

$$\sup_{k\in\mathcal{K}} |v_n(k)| \tag{16}$$

which can be think as the "radius" of random process (15). To upper bound (16), there are several notions for "complexity of functional spaces". Here are some examples. If $\Omega \subseteq \mathbb{R}$, the set of cumulative density functions is $\{k_x : k_x(\omega) = \mathbb{I}[\omega < x], x \in \mathbb{R}\}$, and the upperbound for (16) implies the Central Limit Theorem. We can consider a family of sets $\mathcal{A} \subseteq \mathcal{F}$ and a functional class over it $\{k_A : k_A(\omega) = \mathbb{I}[\omega \in A], A \in \mathcal{A}\}$, and the upper bound for (16) can be characterized by the *VC-dimension* of the family of sets $\mathcal{A}$. Or if P is a distribution d-dimensional Gaussian and there is a set of linear functional $\{k_v : k_v(\omega) = v^\top\omega, \|v\|_2 \leq 1\}$, we can use *metric entropy* to encode their complexity.

Now let us introduce some notions of functional complexity we used in the paper.

Definition A.3. Given $k \in \mathcal{K}$, $L > 0$, and distribution P, we define

$$\rho_L^2(k, P) \triangleq 2L^2 \int \exp\left(\frac{|k|}{L}\right) - 1 - \frac{|k|}{L} dP$$

the *Bernstein difference* between k_1 and k_2 is then $\rho_L^2(k_1 - k_2, P)$ which can be seen as an extension of $L_2(P)$-norm, because $2(e^x - 1 - x) \approx x^2$ when x is small.

Definition A.4 (Generalized entropy with bracketing). Let $\mathcal{N}_{[],L}(\delta, \mathcal{L}, P)$ be the smallest value of n for which there exists n pairs of functions $\{(k_j^L, k_j^U)\}$ such that $\rho_L(k_j^U - k_j^L, P) \leq \delta$ for all $j = 1, \ldots, n$ and such that for all $k \in \mathcal{K}$ there is a j such that for all $\omega \in \Omega$

$$k_j^L(\omega) \leq k(\omega) \leq k_j^U(\omega).$$

Then $\mathcal{H}_{[],L}(\delta, \mathcal{K}, P) = \log \mathcal{N}_{[],L}(\delta, \mathcal{K}, P)$ is called the *generalized entropy with bracketing*.

A useful application of bracketing is to classes of parametric functions $\{k_t : t \in T\}$ that are Lipschitz in the parameter $t \in T$: There exists a metric d on T and a function $F : \Omega \to \mathbb{R}$ such that

$$|k_t(w) - f_s(w)| \leq d(s,t)F(w) \text{ for all } w \in \Omega$$

Then the bracketing numbers of this class are bounded by the covering numbers of T.

Theorem A.5. *Let $\mathcal{K}_T = \{k_t : t \in T\}$ be a set of function satisfying the above condition. Then for any norm $\|\cdot\|$ and $\epsilon > 0$,*

$$\mathcal{N}_{[]}(2\epsilon\|F\|, \mathcal{K}, \|\cdot\|) \leq \mathcal{N}(\epsilon, T, d).$$

The following theorem shows the random variable (16) is subgaussian when the generalized entropy with bracketing is bounded.

Theorem A.6 (A uniform inequality [24]). *Given a functional class $\mathcal{K}$ and distribution P, if there exist constants L, R, A, B, and C such that $\sup_{k\in\mathcal{K}} \rho_L(k, P) \leq R$ and $\int_0^R \sqrt{\mathcal{H}_{[],L}(u, \mathcal{K}, P)}du \leq C$*

$$\sqrt{(A+1)B^2}\left(\max\{R, C\}\right) \leq \epsilon \leq \frac{AR^2}{L}\sqrt{n}$$

Then the empirical process $V_n(\mathcal{K})$ is bounded as

$$\Pr\left[\sup_{k\in\mathcal{K}} |v_n(k)| \geq \epsilon\right] \leq B \exp\left(-\frac{\epsilon^2}{(A+1)B^2R^2}\right).$$

B Proofs in Sect. 5

Lemma 5.2 (Manipulation in strategies and scoring functions). *For any strategy profile $\boldsymbol{\theta}$ and scoring function K,*[17]

$$u_A(\boldsymbol{\theta}, P, K) \leq D_\Phi(P_{X,Y} \| P_X P_Y).$$

There are two aspects of manipulation: the reports for bonus and penalty tasks and the scoring functions K. The first one can be handled through Data Processing inequality and the second is shown through the variational representation of Φ-divergence.

Proof. The expected utility for Alice is

$$\begin{aligned}
&u_A(\boldsymbol{\theta}, P, K)\\
=&\mathbb{E}_{\mathbf{X},\mathbf{Y}}\left[\mathbb{E}_{\boldsymbol{\theta}}\left[K\left(\hat{X}_b, \hat{Y}_b\right) \mid \mathbf{X}, \mathbf{Y}\right]\right] - \mathbb{E}_{\mathbf{X},\mathbf{Y}}\left[\mathbb{E}_{\boldsymbol{\theta}}\left[\Phi^*\left(K\left(\hat{X}_p, \hat{Y}_q\right)\right) \mid \mathbf{X}, \mathbf{Y}\right]\right]\\
=&\mathbb{E}_{\mathbf{X},\mathbf{Y}}\left[\sum_{\hat{x},\hat{y}} \theta_A(X_b, \hat{x})\theta_B(Y_b, \hat{y})K(\hat{x}, \hat{y})\right] - \mathbb{E}_{\mathbf{X},\mathbf{Y}}\left[\sum_{\hat{x},\hat{y}} \theta_A(X_p, \hat{x})\theta_B(Y_q, \hat{y})\Phi^*\left(K(\hat{x}, \hat{y})\right)\right]\\
=&\sum_{x,y} P_{X,Y}(x,y) \sum_{\hat{x},\hat{y}} \theta_A(x, \hat{x})\theta_B(y, \hat{y})K(\hat{x}, \hat{y}) - \sum_{x,y} P_X(x)P_Y(y) \sum_{\hat{x},\hat{y}} \theta_A(x, \hat{x})\theta_B(y, \hat{y})\Phi^*\left(K(\hat{x}, \hat{y})\right).
\end{aligned}$$

[17] There are some minor details when $\mathcal{X}$ and $\mathcal{Y}$ are not finite set. Here we require K and $\boldsymbol{\theta}$ to have finite $\int K\, dP_{X,Y}$, $\int \Phi^*(K)d(P_X P_Y)$, $\int K\, d\theta_A\, d\theta_B dP_{X,Y}$,and $\int \Phi^*(K)\, d\theta_A\, d\theta_B dP_X P_Y$.

The last equality uses the fact that $P_{X,Y}$ is the joint distribution of signals on bonus task b, (x_b, y_b) , and $P_X P_Y$ is the joint distribution of signals on penalty tasks p and q, (x_p, y_q) by Assumption 1.

Because Φ^* is convex and for all x and y, $\theta_A(x,\hat{x})\theta_B(y,\hat{y})$ is a distribution over $\mathcal{X}\times\mathcal{Y}$, by Jensen's inequality we have for all x and y,

$$\sum_{\hat{x},\hat{y}} \theta_A(x,\hat{x})\theta_B(y,\hat{y})\Phi^*\left(K(\hat{x},\hat{y})\right) \le \Phi^*\left(\sum_{\hat{x},\hat{y}} \theta_A(x,\hat{x})\theta_B(y,\hat{y})K_{i,B}(\hat{x},\hat{y})\right) \tag{17}$$

where the equality holds only if Φ^* is not strictly convex or $K(\hat{x},\hat{y})$ is constant in the support of $\theta_A(x,\hat{x})\theta_B(y,\hat{y})$. Let $L(x,y) \triangleq \sum_{\hat{x},\hat{y}} \theta_A(x,\hat{x})\theta_B(y,\hat{y})K(\hat{x},\hat{y})$. Apply (17) to u_A and we have

$$\begin{aligned}
u_A \le & \sum_{x,y} P_{X,Y}(x,y)L(x,y) - \sum_{x,y} P_X(x)P_Y(y)\Phi^*\left(L(x,y)\right) \\
\le & \sup_{K:\mathcal{X}\times\mathcal{Y}\to\mathbb{R}} \left\{\sum_{x,y} P_{X,Y}(x,y)K(x,y) - \sum_{x,y} P_X(x)P_Y(y)\Phi^*\left(K(x,y)\right)\right\} \qquad (18)\\
= & D_\Phi(P_{X,Y}\|P_X P_Y).
\end{aligned}$$

The last inequality holds by Theorem (2.2), and it completes the proof. □

Lemma 5.1 (Truth-telling)**.** *If H is a $(P_{X,Y},\Phi)$-ideal scoring function,*

$$u_A(\boldsymbol{\tau}, P, H) = D_\Phi(P_{X,Y}\|P_X P_Y).$$

Proof Lemma 5.1. By Theorem 2.2, and the definition of Truth-telling strategy profile, we have

$$\begin{aligned}
& u_A(\boldsymbol{\tau}, P, H) \\
= & \mathbb{E}_{\mathbf{X},\mathbf{Y}}\left[\mathbb{E}_{\boldsymbol{\tau}}\left[K\left(\hat{X}_b,\hat{Y}_b\right) \mid \mathbf{X},\mathbf{Y}\right]\right] - \mathbb{E}_{\mathbf{X},\mathbf{Y}}\left[\mathbb{E}_{\boldsymbol{\tau}}\left[\Phi^*\left(K\left(\hat{X}_p,\hat{Y}_q\right)\right) \mid \mathbf{X},\mathbf{Y}\right]\right] \\
= & \mathbb{E}_{\mathbf{X},\mathbf{Y}}\left[H(X_b,Y_b)\right] - \mathbb{E}_{\mathbf{X},\mathbf{Y}}\left[\Phi^*\left(H(X_p,Y_q)\right)\right] && \text{(definition of } \boldsymbol{\tau}\text{)} \\
= & \sum_{x,y} P_{X,Y}(x,y)H(x,y) - \sum_{x,y} P_X(x)P_Y(y)\Phi^*\left(H\right) \\
= & \sup_{K:\mathcal{X}\times\mathcal{Y}\to\mathbb{R}} \left\{\sum P_{X,Y}K - \sum P_X P_Y \Phi^*\left(K\right)\right\} && \text{(by Theorem 2.2 and } H\text{)} \\
= & D_\Phi(P_{X,Y}\|P_X P_Y).
\end{aligned}$$

Moreover, because $P_{X,Y}$ is stochastic relevant, $D_\Phi(P_{X,Y}\|P_X P_Y) > 0$. □

Lemma 5.3 (Oblivious strategy)**.** *If $\boldsymbol{\theta}$ is an oblivious strategy profile, for any scoring function K*

$$u_A(\boldsymbol{\theta}, P, K) \le 0.$$

Proof of Lemma 5.3. Recall that an oblivious strategy θ_A is oblivious to the private signal: for any x ,x' and $\hat{x}$ in $\mathcal{X}$, $\theta_A(x,\hat{x}) = \theta_A(x',\hat{x})$, and we can define a distribution $\mu_A \in \Delta_\mathcal{X}$ such that for all x and $\hat{x}$ in $\mathcal{X}$, $\mu_A(\hat{x}) = \theta_A(x,\hat{x})$. We also define $\nu_B(\hat{y}) = \sum_y P_Y(y)\theta_B(y,\hat{y})$ where ν_B is a distribution on $\mathcal{Y}$ and independent to μ_A.

$$
\begin{aligned}
&u_A(\boldsymbol{\theta}, P, K)\\
=&\mathbb{E}_{\mathbf{X},\mathbf{Y}}\left[\mathbb{E}_{\boldsymbol{\theta}}\left[K\left(\hat{X}_b, \hat{Y}_b\right) \mid \mathbf{X}, \mathbf{Y}\right]\right] - \mathbb{E}_{\mathbf{X},\mathbf{Y}}\left[\mathbb{E}_{\boldsymbol{\theta}}\left[\Phi^*\left(K\left(\hat{X}_p, \hat{Y}_q\right)\right) \mid \mathbf{X}, \mathbf{Y}\right]\right]\\
=&\mathbb{E}_{\mathbf{X},\mathbf{Y}}\left[\sum_{\hat{x},\hat{y}} \theta_A(X_b, \hat{x})\theta_B(Y_b, \hat{y})K(\hat{x}, \hat{y})\right] - \mathbb{E}_{\mathbf{X},\mathbf{Y}}\left[\sum_{\hat{x},\hat{y}} \theta_i(X_p, \hat{x})\theta_B(Y_q, \hat{y})\Phi^*\left(K(\hat{x}, \hat{y})\right)\right]\\
=&\mathbb{E}_{\mathbf{X},\mathbf{Y}}\left[\sum_{\hat{x},\hat{y}} \mu_A(\hat{x})\theta_B(Y_b, \hat{y})K(\hat{x}, \hat{y})\right] - \mathbb{E}_{\mathbf{X},\mathbf{Y}}\left[\sum_{\hat{x},\hat{y}} \mu_A(k)\theta_B(Y_p, \hat{y})\Phi^*\left(K(\hat{x}, \hat{y})\right)\right]\\
=&\sum_{\hat{x},\hat{y}} \mu_A(\hat{x})\nu_B(\hat{y})\left[K(\hat{x}, \hat{y}) - \Phi^*\left(K(\hat{x}, \hat{y})\right)\right] && \text{(definition of } \nu_B\text{)}\\
\leq& \sup_{b\in\mathrm{dom}(\Phi^*)} \{1\cdot b - \Phi^*(y)\} = \Phi^{**}(1) = f(1) = 0.
\end{aligned}
$$

The last inequality is from the Definition 2.1. ☐

Lemma 5.4. *Moreover, given Conditions 4 in Assumption 2, the equality in* (5) *for Alice or Bob occurs if and only if*

1. *$\boldsymbol{\theta} = (\pi_A, \pi_B)$ which is a permutation strategy profile, and*
2. *For all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, $K(\pi_A(x), \pi_B(y)) = \Phi'\left(\frac{P_{X,Y}(x,y)}{P_X(x)P_Y(y)}\right)$.*

Proof of Lemma 5.4. We first prove the first property: θ_A and θ_B are permutations. Note that by the proof of Lemma 5.2, $u_A(\boldsymbol{\theta}, P, K) = D_\Phi(P_{i,j}\|P_iP_j)$ if and only if (17) and (18) are equality, because $\mathcal{X}$ and $\mathcal{Y}$ are finite

Given Alice's and Bob's strategies θ_A and θ_B, let $S_A(x) \triangleq \{\hat{x} \in \mathcal{X} : \theta_A(x, \hat{x}) > 0\}$, $S_B(y) \triangleq \{\hat{y} \in \mathcal{Y} : \theta_B(y, \hat{y}) > 0\}$ be the support of strategy θ_A on signal x and θ_B on y respectively. Because Φ^* is strictly convex and Φ is differentiable, (17), and (18) are equality, $u_A(\boldsymbol{\theta}, P, K) = D_\Phi(P_{X,Y}\|P_XP_Y)$ if and only if

$$
\forall x, y, \hat{x} \in S_A(x), \hat{y} \in S_B(y),\ K(\hat{x}, \hat{y}) = \Phi'\left(\frac{P_{X,Y}(x,y)}{P_X(x)P_Y(y)}\right). \tag{19}
$$

That is all reports pairs $(\hat{x}, \hat{y})$ in the support of strategy θ_A on x and θ_B on y have the same score, $K(\hat{x}, \hat{y})$. Moreover, the value equals to $\Phi'\left(P_{X,Y}(x,y)/(P_X(x)P_Y(y))\right)$. Now we use this observation to finish the proof.

$\Rightarrow$) Because $\theta_A(x, \cdot)$ induces a probability, $|S_A(x)| \geq 1$ for all x. Suppose θ_A is not a permutation. Because $\mathcal{X}$ is finite, there exists $x_1 \neq x_2$ and $\hat{x}^*$ in $\mathcal{X}$ such that $\hat{x}^* \in S_A(x_1)$ and $\hat{x}^* \in S_A(x_2)$. By (19), for all y and $\hat{y} \in S_B(y)$,

$$
\Phi'\left(\frac{P_{X,Y}(x_1,y)}{P_X(x_1)P_Y(y)}\right) = K(\hat{x}^*, \hat{y}) = \Phi'\left(\frac{P_{X,Y}(x_2,y)}{P_X(x_2)P_Y(y)}\right).
$$

Because Φ is strictly convex and differentiable, Φ' is invertible, and thus for all $y \in \mathcal{Z}$,

$$
\frac{P_{X,Y}(x_1,y)}{P_X(x_1)P_Y(y)} = \frac{P_{X,Y}(x_2,y)}{P_X(x)P_Y(y)}
$$

which shows $P_{X,Y}$ is not stochastic relevant— Given signal x_1 Alice's poster for Bob's signal is identical to her poster with signal x_2— and reaches contradiction. Therefore there exits permutations π_A and π_B over $\mathcal{X}$ and $\mathcal{Y}$ such that $\theta_A = \pi_A$ and $\theta_B = \pi_B$.

For the second part, by (19), for all x, y we have

$$
K(\pi_A(x), \pi_B(y)) = \Phi'\left(\frac{P_{X,Y}(x,y)}{P_X(x)P_Y(y)}\right).
$$

$\Leftarrow$) On the other hand, if $\theta_A = \pi_A$ and $\theta_B = \pi_B$ which are permutations, and for all x, y, and $K(\pi_A(x), \pi_B(y)) = \Phi'\left(P_{X,Y}(x,y)/(P_X(x)P_Y(y))\right)$, we can apply (19), and have $u_A(\boldsymbol{\theta}, P, K) = D_\Phi(P_{X,Y}\|P_XP_Y)$. ☐

C Proofs in Sect. 6.2

Proof of Lemma 6.3. Given a prior P, the payment to Alice under truth-telling strategy profile in the fantasy function (7) by Lemma 5.1 is

$$F_A^\Phi(\boldsymbol{\tau}, P) = u_A(\boldsymbol{\tau}, P, H_P^\Phi) = D_\Phi(P_{X,Y} \| P_X P_Y). \tag{20}$$

Additionally, by Lemma 5.2,

$$F_A^\Phi(\boldsymbol{\theta}, P) = u_A(\boldsymbol{\theta}, P, H_{\boldsymbol{\theta} \circ P}^\Phi) \leq D_\Phi(P_{X,Y} \| P_X P_Y) = F_A^\Phi(\boldsymbol{\tau}, P)$$

which shows the truth-telling strategy profile is a Bayesian Nash equilibrium.

To show that the mapping is inform-truthful, by Lemma 5.3, if $\boldsymbol{\theta}$ is an oblivious strategy profile, $F_A^\Phi(\boldsymbol{\theta}, P) = u_A(\boldsymbol{\theta}, P, H_{\boldsymbol{\theta} \circ P}^\Phi) \leq 0$. Therefore when P is stochastic relevant by Equation (20) and Lemma 5.1 we have

$$F_A^\Phi(\boldsymbol{\theta}, P) \leq 0 < F_A^\Phi(\boldsymbol{\theta}, P).$$

Finally, to show the mechanism is strongly truthful, if there is a strategy profile $\boldsymbol{\theta}$ such that $F_A^\Phi(\boldsymbol{\theta}, P) = F_A^\Phi(\boldsymbol{\tau}, P)$, we have

$$u_A(\boldsymbol{\theta}, P, H_{\boldsymbol{\theta} \circ P}^\Phi) = u_A(\boldsymbol{\tau}, P, H_P^\Phi),$$

so by Lemma 5.4 $\boldsymbol{\theta}$ is a permutation strategy profile which completes the proof. □

Note that the statement of Theorem 6.1 is a little subtle. Mentioned in the footnote in Remark 6.2 there the statement consists of two parts of randomness: an event with probability $1 - \delta$, and the conditional expected payment to Alice under such event is ϵ-close to a strongly truthful (inform-truthful or truthful) mapping. Therefore, to prove Theorem 6.1, it is sufficient to show there exists an event $\mathcal{E}$ such that

1. it happens with probability at least $1 - \delta$,
2. Alice's conditional ex-ante utility under truth-telling strategy profile is ϵ-close to $F_A^\Phi(\boldsymbol{\tau}, P)$ defined in (7), and
3. for all strategy profile $\boldsymbol{\theta}$ Alice's conditional ex-ante utility under $\boldsymbol{\theta}$ is less than $F_A^\Phi(\boldsymbol{\theta}, P)$.

Proof of Theorem 6.1. First, if Alice's and Bob's strategy profile is $\boldsymbol{\theta}$, the learning tasks $(\hat{\mathbf{x}}_L, \hat{\mathbf{y}}_L)$ and scoring tasks $(\hat{\mathbf{x}}_S, \hat{\mathbf{y}}_S)$ are both generated from distribution $\boldsymbol{\theta} \circ P$ i.i.d.. Additionally, the ex-ante payment to Alice is over two randomness: learning tasks and scoring tasks. To make this distinction explicit, we let $U_A(\hat{\mathbf{x}}_S, \hat{\mathbf{y}}_S, \hat{\mathbf{x}}_L, \hat{\mathbf{y}}_L)$ be Alice's payment when the report profile is $(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = (\hat{\mathbf{x}}_S, \hat{\mathbf{y}}_S, \hat{\mathbf{x}}_L, \hat{\mathbf{y}}_L)$. Then Alice's ex-ante payment under strategy profile $\boldsymbol{\theta}$ in mechanism $\mathcal{M}^{\Phi,\mathcal{L}}$ is

$$u_A(\boldsymbol{\theta}; P, \mathcal{M}^{\Phi,\mathcal{L}}) = \mathbb{E}_{\substack{(\hat{\mathbf{x}}_L, \hat{\mathbf{y}}_L) \sim \boldsymbol{\theta} \circ P^{m_L}; \\ (\hat{\mathbf{x}}_S, \hat{\mathbf{y}}_S) \sim \boldsymbol{\theta} \circ P^{m_S}}} [U_A(\hat{\mathbf{x}}_S, \hat{\mathbf{y}}_S, \hat{\mathbf{x}}_L, \hat{\mathbf{y}}_L)] = \mathbb{E}_{(\hat{\mathbf{x}}_L, \hat{\mathbf{y}}_L)} \left[\mathbb{E}_{(\hat{\mathbf{x}}_S, \hat{\mathbf{y}}_S)} [U_A(\hat{\mathbf{x}}_S, \hat{\mathbf{y}}_S, \hat{\mathbf{x}}_L, \hat{\mathbf{y}}_L) \mid (\hat{\mathbf{x}}_L, \hat{\mathbf{y}}_L)]\right].$$

We further define

$$u_A(\boldsymbol{\theta}, P, \mathcal{L}(\mathbf{x}_L, \mathbf{y}_L)) \triangleq \mathbb{E}_{(\hat{\mathbf{x}}_S, \hat{\mathbf{y}}_S) \sim \boldsymbol{\theta} \circ P} [U_A(\hat{\mathbf{x}}_S, \hat{\mathbf{y}}_S, \mathbf{x}_L, \mathbf{y}_L)]$$

where the expectation is only taken on the scoring tasks, but the learning tasks are fixed.

Now we define an event

$$\mathcal{E} = \{\mathbf{x}_L, \mathbf{y}_L) : u_A(\boldsymbol{\tau}, P, \mathcal{L}(\mathbf{x}_L, \mathbf{y}_L)) > D_\Phi(P_{X,Y} \| P_X P_Y) - \epsilon\}$$

which is in the probability space generated by the learning tasks. Because $\mathcal{L}$ is (δ, ϵ)-accurate on $\mathcal{P}$ and the joint signal distribution $P \in \mathcal{P}$, the probability of $\mathcal{E}$ is greater than $1 - \delta$.

By the definition of $\mathcal{E}$, for all $(\mathbf{x}_L, \mathbf{y}_L) \in \mathcal{E}$,

$$\begin{aligned}
& u_A(\boldsymbol{\tau}, P, \mathcal{L}(\mathbf{x}_L, \mathbf{y}_L)) \\
> & D_\Phi(P_{X,Y} \| P_X P_Y) - \epsilon \\
= & u_A(\boldsymbol{\tau}, P, H_{\boldsymbol{\tau} \circ P}^\Phi) - \epsilon && \text{(by Lemma 5.1)} \\
= & F_A^\Phi(\boldsymbol{\tau}, P) - \epsilon && \text{(by (7))}
\end{aligned}$$

Therefore, Alice's conditional expected payment under truth-telling strategy profile is ϵ close to the fantasy function.

Finally, it is sufficient that for all $(\mathbf{x}_L, \mathbf{y}_L)$, and $\boldsymbol{\theta}$

$$u_A(\boldsymbol{\theta}, P, \mathcal{L}(\mathbf{x}_L, \mathbf{y}_L)) \le F_A^\Phi(\boldsymbol{\theta}, P).$$

Formally, for all $P \in \mathcal{P}$, $\boldsymbol{\theta}$, and $(\mathbf{x}_L, \mathbf{y}_L)$,

$$\begin{aligned}
& u_A(\boldsymbol{\theta}, P, \mathcal{L}(\mathbf{x}_L, \mathbf{y}_L)) \\
=& u_A(\boldsymbol{\tau}, \boldsymbol{\theta} \circ P, \mathcal{L}(\mathbf{x}_L, \mathbf{y}_L)) && \text{(by (4))} \\
\le& u_A(\boldsymbol{\tau}, \boldsymbol{\theta} \circ P, H^\Phi_{\boldsymbol{\theta}\circ P}) && \text{(by Lemma 5.2)} \\
=& u_A(\boldsymbol{\theta}, P, H^\Phi_{\boldsymbol{\theta}\circ P}) && \text{(by (4))} \\
=& F_A^\Phi(\boldsymbol{\theta}, P),
\end{aligned}$$

and we complete the proof. □

D Proofs in Sect. 7

D.1 Proofs in Sect. 7.1

Proof of Lemma 7.1.

$$\begin{aligned}
& D_\Phi(P_{X,Y} \| P_X P_Y) - u_A(\boldsymbol{\tau}, P, K) \\
=& \int H dP_{X,Y} - \int \Phi^*(H) dP_X P_Y - \int K dP_{X,Y} + \int \Phi^*(K) dP_X P_Y \\
=& \int \Phi^*(K) - \Phi^*(H) + \frac{dP_{X,Y}}{dP_X P_Y} (H - K)\, dP_X P_Y \\
=& \int \Phi^*(K) - \Phi^*(H) - (\Phi^*)'(H)(K - H)\, dP_X P_Y
\end{aligned}$$

The last equality holds since $H(x,y) = \Phi'\left(\frac{dP_{X,Y}(x,y)}{dP_X P_Y(x,y)}\right)$, so $(\Phi^*)'(H(x,y)) = \frac{dP_{X,Y}(x,y)}{dP_X P_Y(x,y)}$ by Theorem A.1. The final line is indeed the Bregman divergence from H to K with respect to measure $dP_X P_Y$ and Φ^*. □

Proof of Theorem 7.2. Let $\hat{K}$ be the output of Algorithm 3, and H be a (Φ, P)-ideal scoring function defined in (1). We have

$$\begin{aligned}
& u_A(\boldsymbol{\tau}, P, \hat{K}) \\
=& \mathbb{E}_{\mathbf{X},\mathbf{Y}}\left[\hat{K}(X_b, Y_b)\right] - \mathbb{E}_{\mathbf{X},\mathbf{Y}}\left[\Phi^*\left(\hat{K}(X_p, Y_q)\right)\right] \\
=& \sum_{x,y} P_{X,Y}(x,y)\hat{K}(x,y) - P_X(x)P_Y(y)\Phi^*\left(\hat{K}(x,y)\right) \\
=& \sum_{x,y: P_X(x)P_Y(y) \neq 0} P_X(x)P_Y(y)\left[\frac{P_{X,Y}(x,y)}{P_X(x)P_Y(y)}\hat{K}(x,y) - \Phi^*\left(\hat{K}(x,y)\right)\right] \\
=& \sum P_X P_Y \left[\frac{\hat{P}_{X,Y}}{\hat{P}_X \hat{P}_Y}\hat{K} - \Phi^*\left(\hat{K}\right) + \left(\frac{P_{X,Y}}{P_X P_Y} - \frac{\hat{P}_{X,Y}}{\hat{P}_X \hat{P}_Y}\right)\hat{K}\right]
\end{aligned}$$

Because $\hat{K}(x,y) \in \partial\Phi\left(\frac{\hat{P}_{X,Y}(x,y)}{\hat{P}_X(x)\hat{P}_Y(y)}\right)$, by Young-Fenchel inequality (Theorem A.1) we have $\frac{\hat{P}_{X,Y}(x,y)}{\hat{P}_X(x)\hat{P}_Y(y)}\hat{K}(x,y) - \Phi^*\left(\hat{K}(x,y)\right) = \Phi\left(\frac{\hat{P}_{X,Y}(x,y)}{\hat{P}_X(x)\hat{P}_Y(y)}\right)$, so

$$u_A(\boldsymbol{\tau}, P, \hat{K}) = \sum P_X P_Y \left[\Phi\left(\frac{\hat{P}_{X,Y}}{\hat{P}_X \hat{P}_Y}\right) + \left(\frac{P_{X,Y}}{P_X P_Y} - \frac{\hat{P}_{X,Y}}{\hat{P}_X \hat{P}_Y}\right)\hat{K}\right] \tag{21}$$

On the other hand, by the Definition 2.1,

$$D_\Phi(P_{X,Y}\|P_XP_Y) = \sum P_XP_Y \cdot \Phi\left(\frac{P_{X,Y}}{P_XP_Y}\right) \tag{22}$$

By combining (21) and (22), we have

$$\begin{aligned}
&D_\Phi(P_{X,Y}\|P_XP_Y) - u_A(\boldsymbol{\tau}, P, K)\\
=&\sum P_XP_Y\left[\Phi\left(\frac{P_{X,Y}}{P_XP_Y}\right) - \Phi\left(\frac{\hat{P}_{X,Y}}{\hat{P}_X\hat{P}_Y}\right) - \left(\frac{P_{X,Y}}{P_XP_Y} - \frac{\hat{P}_{X,Y}}{\hat{P}_X\hat{P}_Y}\right)\hat{K}\right]\\
\le&\sum P_X(x)P_Y(y)\left|\Phi\left(\frac{P_{X,Y}(x,y)}{P_X(x)P_Y(y)}\right) - \Phi\left(\frac{\hat{P}_{X,Y}(x,y)}{\hat{P}_X(x)\hat{P}_Y(y)}\right)\right|\\
&+\sum P_X(x)P_Y(y)\left|\frac{P_{X,Y}(x,y)}{P_X(x)P_Y(y)} - \frac{\hat{P}_{X,Y}(x,y)}{\hat{P}_X(x)\hat{P}_Y(y)}\right| \cdot |\hat{K}(x,y)|
\end{aligned}$$

Thus, it is sufficient to show

$$\sum P_X(x)P_Y(y)\left|\Phi\left(\frac{P_{X,Y}(x,y)}{P_X(x)P_Y(y)}\right) - \Phi\left(\frac{\hat{P}_{X,Y}(x,y)}{\hat{P}_X(x)\hat{P}_Y(y)}\right)\right| \le \frac{3c_L\delta}{\alpha^2} \tag{23}$$

$$\sum P_X(x)P_Y(y)\left|\frac{P_{X,Y}(x,y)}{P_X(x)P_Y(y)} - \frac{\hat{P}_{X,Y}(x,y)}{\hat{P}_X(x)\hat{P}_Y(y)}\right| \cdot |\hat{K}(x,y)| \le \frac{3c_L\delta}{\alpha^2} \tag{24}$$

For all x $P_X(x)$ is nonzero by assumption 1. By the assumption in the statement $P_{X,Y} > 2\alpha$ if it's not zero, so $P_X(x) > 2\alpha$. Furthermore, since $\|P_{X,Y} - \hat{P}_{X,Y}\|_{TV} \le \delta < \alpha$, $\hat{P}_X(x) \ge \alpha$. Therefore for all x and y, $P_{X,Y}(x,y) \neq 0$ we have

$$\alpha \le \frac{P_{X,Y}(x,y)}{P_X(x)P_Y(y)} \text{ and } \frac{\hat{P}_{X,Y}(x,y)}{\hat{P}_X(x)\hat{P}_Y(y)} \le \frac{1}{\alpha} \tag{25}$$

To prove (24), we first show an upper bound for $|\hat{K}(x,y)|$. By the definition of $\hat{K}$, it is in the sub-gradient of Φ at $\frac{\hat{P}_{X,Y}(x,y)}{\hat{P}_X(x)\hat{P}_Y(y)}$, and it is in $[\alpha, 1/\alpha]$ due to (25). Since Φ being c_L-Lipschitz in such interval, we have

$$|\hat{K}(x,y)| \le c_L \tag{26}$$

We are ready to prove (24).

$$\begin{aligned}
&\sum_{x,y:P_X(x)P_Y(y)\neq 0} P_X(x)P_Y(y)\left|\frac{P_{X,Y}(x,y)}{P_X(x)P_Y(y)} - \frac{\hat{P}_{X,Y}(x,y)}{\hat{P}_X(x)\hat{P}_Y(y)}\right||\hat{K}(x,y)|\\
\le&\sum_{x,y:P_X(x)P_Y(y)\neq 0} P_X(x)P_Y(y)\left|\frac{P_{X,Y}(x,y)}{P_X(x)P_Y(y)} - \frac{\hat{P}_{X,Y}(x,y)}{\hat{P}_X(x)\hat{P}_Y(y)}\right|c_L && \text{(by (26))}\\
=&c_L\sum\frac{1}{\hat{P}_X(x)\hat{P}_Y(y)}\left|P_{X,Y}(x,y)\hat{P}_X(x)\hat{P}_Y(y) - \hat{P}_{X,Y}(x,y)P_X(x)P_Y(y)\right|\\
\le&\alpha^2c_L\sum\left|P_{X,Y}(x,y)\hat{P}_X(x)\hat{P}_Y(y) - \hat{P}_{X,Y}(x,y)P_X(x)P_Y(y)\right| && (\hat{P}_X, \hat{P}_Y \ge \alpha)\\
\le&\alpha^2c_L\sum P_{X,Y}\left|\hat{P}_X\hat{P}_Y - P_XP_Y\right| + P_XP_Y\left|P_{X,Y} - \hat{P}_{X,Y}\right|\\
\le&\alpha^2c_L\sum\left|\hat{P}_X\hat{P}_Y - P_XP_Y\right| + \left|P_{X,Y} - \hat{P}_{X,Y}\right| \le 3\alpha^2c_L\delta
\end{aligned}$$

Now let's prove (23). Because Φ is c_L-Lipschitz in $[\alpha, 1/\alpha]$, by (25), we have

$$\left|\Phi\left(\frac{P_{X,Y}(x,y)}{P_X(x)P_Y(y)}\right) - \Phi\left(\frac{\hat{P}_{X,Y}(x,y)}{\hat{P}_X(x)\hat{P}_Y(y)}\right)\right| \le c_L\left|\frac{P_{X,Y}(x,y)}{P_X(x)P_Y(y)} - \frac{\hat{P}_{X,Y}(x,y)}{\hat{P}_X(x)\hat{P}_Y(y)}\right| \tag{27}$$

With argument similar to the proof of (24), we completes the proof. $\square$

D.2 Proofs in Sect. 7.2

Proof of Lemma 7.5. Because $\tilde{K}$ satisfies Equation (11) and $H \in \mathcal{K}$, we have

$$\int \tilde{K} d\tilde{P}_{X,Y} - \int \Phi^*(\tilde{K}) d\tilde{P}_X \tilde{P}_Y \geq \int H d\tilde{P}_{X,Y} - \int \Phi^*(H) d\tilde{P}_X \tilde{P}_Y.$$

On the other hand,

$$B_{\Phi^*, P_X P_Y}(\tilde{K}, H) = \int \Phi^*(\tilde{K}) - \Phi^*(H) dP_X P_Y - \int \left(\tilde{K} - H\right) dP_{X,Y}.$$

Combining these two we have an upper bound for $B_{\Phi^*, P_X P_Y}(\tilde{K}, H)$,

$$\int \left(\Phi^*(\tilde{K}) - \Phi^*(H)\right)\left(dP_X P_Y - d\tilde{P}_X \tilde{P}_Y\right) - \int \left(\tilde{K} - H\right)\left(dP_{X,Y} - d\tilde{P}_{X,Y}\right)$$

which completes the proof. □

Proof of Theorem 7.4. By Lemma 7.1 and 7.5, we know the error between $D_\Phi(P_{X,Y} \| P_X P_Y) - u_A(\boldsymbol{\tau}, P, K)$ can be upper bound by

$$\sup_{k \in \mathcal{K}} \left| \int \Phi^*(k) - \Phi^*(H) d(\tilde{P}_X \tilde{P}_Y - P_X P_Y) \right| \tag{28}$$

$$\sup_{k \in \mathcal{K}} \left| \int k - H d(\tilde{P}_{X,Y} - P_{X,Y}) \right|. \tag{29}$$

Now we can apply the uniform bound in Theorem A.6 for (29). By taking $A = \frac{\varepsilon L_1}{R_1^2}$, $B = 1$, $L = L_1$, $R = R_1$, and $\epsilon = \varepsilon\sqrt{n}$, we have

$$\begin{aligned}
\Pr\left[\sup_{k \in \mathcal{K}} |v_n(k)| \geq \epsilon\right] &= \Pr\left[\sup_{k \in \mathcal{K}} \left|\sqrt{n} \int k d(\hat{P}_n - P)\right| \geq \varepsilon\sqrt{n}\right] \\
&= \Pr\left[\sup_{k \in \mathcal{K}} \left|\int k d(\hat{P}_n - P)\right| \geq \varepsilon\right] \\
&\leq B \exp\left(-\frac{\epsilon^2}{B^2(A+1)R_1^2}\right) \\
&\leq \exp\left(-\frac{\varepsilon^2}{(A+1)R_1^2} n\right) \leq \frac{1}{n^2}\delta.
\end{aligned}$$

The last inequality is true by taking $n = m_L/3 = O\left(\frac{(A+1)R_1^2}{\varepsilon^2} \log \frac{n^2}{\delta}\right) = O\left(\frac{1}{\varepsilon^2} \log \frac{n^2}{\delta}\right)$ when ε is small enough. We can derive similar upper bound for (29), and we complete the proof □

D.3 Proof of Theorem 7.6

Proof. Formally, note that given i.i.d samples from $P_{X,Y}$ the empirical distribution of those samples are sufficient statistics, and we can restrict the estimators to be functional over sufficient statistics, $\hat{D} : (\mathcal{X} \times \mathcal{Y})^m \to \mathbb{R}$. Let's consider the following prior distribution $P_{X,Y}$: Given non-negative variables α, β, γ such that $\alpha + \beta + \gamma \leq 1$, we set the distribution over $\mathcal{X} \times \mathcal{Y} = \{1,2,3\} \times \{1,2,3\}$ to be

$$P_{X,Y} = \frac{1}{3}\begin{bmatrix} 1-\alpha-\beta & \alpha & \beta \\ \alpha & 1-\alpha-\gamma & \gamma \\ \beta & \gamma & 1-\beta-\gamma \end{bmatrix}$$

An empirical distribution (histogram) from $x_1, \ldots, x_m$ can be represented by 9 integers $\mathbf{m} = (m_{k,l})$ where k and l are between 1 to 3 and $m_{k,l}$ is the number of (k,l) in those m samples, and the distribution of $\mathbf{m}$

forms a multi-nomial distribution. Therefore we can compute the expectation of $\hat{D}$,

$$
\begin{aligned}
&\mathbb{E}[\hat{D}(\mathbf{x},\mathbf{y})] \\
=& \sum_{\mathbf{m}:\sum m_{k,l}=m} \frac{m!}{\prod_{k,l} m_{k,l}!} \prod_{k,l} P_{X,Y}(k,l)^{m_{k,l}} \hat{D}(\mathbf{m}) \\
=& \sum_{\mathbf{m}:\sum m_{k,l}=m} \frac{m!}{\prod_{k,l} m_{k,l}!} \left(\frac{\alpha}{3}\right)^{m_{1,2}+m_{2,1}} \left(\frac{\beta}{3}\right)^{m_{1,3}+m_{3,1}} \left(\frac{\gamma}{3}\right)^{m_{2,3}+m_{3,2}} \\
&\cdot \left(1-\frac{\alpha+\beta}{3}\right)^{m_{1,1}} \left(1-\frac{\beta-\gamma}{3}\right)^{m_{2,2}} \left(1-\frac{\alpha+\gamma}{3}\right)^{m_{3,3}} \hat{D}(\mathbf{m})
\end{aligned}
$$

which is a polynomial of α, β and γ. On the other hand, the Φ-divergence is

$$\frac{1}{9}\left[2\Phi\left(3\alpha\right)+2\Phi\left(3\beta\right)+2\Phi\left(3\gamma\right)+\Phi\left(3(1-\alpha-\beta)\right)+\Phi\left(3(1-\beta-\gamma)\right)+\Phi\left(3(1-\alpha-\gamma)\right)\right]$$

By taking partial derivative with respect to α then β, $\mathbb{E}[\hat{D}(x_1,x_2,\dots,x_m)] = \mathbb{E}_{P_X P_Y}\left[\Phi\left(\frac{P_{X,Y}}{P_X P_Y}\right)\right]$ implies the second derivative of Φ is a polynomial and $\Phi(a)$ is a polynomial.

Similarly we take

$$P'_{X,Y} = \begin{bmatrix} 1-\alpha-\beta & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix}$$

and the Φ-divergence is

$$\alpha^2\Phi(1/\alpha) + b^2\Phi(1/\beta) + (1-\alpha-\beta)^2\Phi(1/(1-\alpha-\beta)) + (1-\alpha^2-\beta^2-(1-\alpha-\beta)^2)\Phi(0)$$

By taking partial derivative with respect to α and β we have

$$\Phi\left(\frac{1}{x}\right) - \frac{1}{x}\Phi'\left(\frac{1}{x}\right) + \frac{1}{x^2}\Phi''\left(\frac{1}{x}\right)$$

is a polynomial with respect to x.

Therefore, combining these two statements we have if there are unbiased estimators for $P_{X,Y}$ and $P'_{X,Y}$, the convex function Φ is a degree one polynomial which reaches a contradiction. $\square$

E General setting

E.1 Prior structure for multi-task peer prediction

Consider n agents and m tasks. Let $[n] = \{1,2,\dots,n\}$ be the set of agents and $[m]$ be the set of tasks. Each agent works on all the tasks and receives a signal in the space $\mathcal{Z}$ (a finite set, integer, or real number, etc.). We denote $Z_{i,s} \in \mathcal{Z}$ as the signal received by the worker $i \in [n]$ on the task $s \in [m]$, and use $\mathbf{Z} \in \mathcal{Z}^{n\times m}$ to denote the signal matrix (all the reviews) which is generated from a distribution $\mathbb{P}$ over $\mathcal{Z}^{n\times m}$. In this paper, we make the following assumption on the prior distribution $\mathbb{P}$.

Assumption 3 (A priori similar and random order [3])**.** *$\mathbb{P}$ is a common prior, and it is identically and independently generated for each task: there exists a distribution P over $\mathcal{Z}^n$ such that $\mathbb{P} = P^m$.*

$$\mathbb{P}(\mathbf{Z}=\mathbf{z}) = \prod_{s\in[m]} P(Z_{1,s}=z_{1,s},\dots,Z_{n,s}=z_{n,s}).$$

Moreover, all questions appear in a random order, independently drawn for each agent.

To simplify the notion, under Assumption 1, we use P to denote the joint distribution on an arbitrary task s. Let P_X and P_Y be the *marginal distributions* of signals agents i and j received on task s, and $P_{X,Y}$ be the 2-*wise marginal distribution* of a pair of signals agent i and j received on the task s, $P_{X,Y}(z,w) = P(Z_{i,s} = z, Z_{j,s} = w)$. This can be further extended to *k-wise marginal distribution*: given that $i_1, i_2, \dots, i_k \in [m]$, $P_{i_1,i_2,\dots,i_k}$ is the marginal distribution of signals agents $i_1, i_2, \dots, i_k$ received. In general, we use uppercase for random object Z and lowercase for the outcome z.

Definition E.1 (Mutually stochastic relevant)**.** We call a distribution P over $\mathcal{Z}^n$ *mutually stochastic irrelevant* if there exists $l \in [n]$ and $w, w' \in \mathcal{Z}$ such that $P_l(w) = 0$, or for all $z_1, z_2 \dots, z_n \in \mathcal{Z}$,

$$\frac{P(z_1, \dots, z_{l-1}, w, z_{l+1}, \dots, z_n)}{P_l(w)} = \frac{P(z_1, \dots, z_{l-1}, w', z_{l+1}, \dots, z_n)}{P_l(w')}.$$

That is, there exists an agent l such that its posterior on all other agents' signals with signal w is identical to the posterior with signal w'.

Note that if the signals of all agents are mutually independent, it is also mutually stochastic irrelevant. It might be helpful to think the stochastic irrelevancy as a weaker form of independence. Similar to k-wise independent, we can define k-wise stochastic irrelevant.

Definition E.2 (k-wise stochastic irrelevant)**.** We call a distribution P over $\mathcal{Z}^n$ *k-wise stochastic irrelevant* if and only if there exists a subset of k agents $\{i_1, i_2, \dots, i_k\} \subset [n]$ and $w, w' \in \mathcal{Z}$ such that $P_{i_1}(w) = 0$, or for all $z_2, z_3 \dots, z_k \in \mathcal{Z}$,

$$\frac{P_{i_1,i_2,\dots,i_k}(w, z_2, \dots, z_k)}{P_{i_1}(w)} = \frac{P_{i_1,i_2,\dots,i_k}(w', z_2, \dots, z_k)}{P_{i_1}(w')}.$$

We further call the negation of above definitions as *mutually stochastic relevant* and *k-wise stochastic relevant* respectively. Note that if the signals of all agents are k-wise independent, it is not k-wise stochastic relevant. And a mutually stochastic relevant distribution over $\mathcal{Z}^n$ is also a k-wise stochastic relevant distribution for all $k \le n$.

E.2 Mechanism design for Information elicitation

Definition E.3 (Mechanism)**.** Given report profile of all agents $\mathbf{r} = (\mathbf{r}_1, \dots, \mathbf{r}_n) \in \mathcal{Z}^{n \times m}$ where $\mathbf{r}_i \in \mathcal{Z}^m$ is the report vector of agent $i \in [n]$, an *information elicitation mechanism* $\mathcal{M} = (M_1, \dots, M_n)$ pays each agent i with $M_i(\mathbf{r})$, where $M_i : \mathcal{Z}^{n \times m} \to \mathbb{R}$ is the payment function for agent i.

Definition E.4 (Strategy)**.** Given a mechanism $\mathcal{M}$, the *strategy* of each agent i in the mechanism $\mathcal{M}$ is a mapping $\Theta_i : \mathcal{Z}^m \to \Delta_{\mathcal{Z}^m}$ from obtained signals Z_i to a probability distribution over $\mathcal{Z}^m$, and a collection of agents' strategies $\mathbf{\Theta} := (\Theta_1, \Theta_2, \dots, \Theta_n)$ is called a *strategy profile*.

Assumption 1 ensures agents cannot distinguish each question without the private signal they receive, and each agent's strategy is uniform across different tasks. Given this, the strategy of agent i can be written as $\theta_i : \mathcal{Z} \to \Delta_Z$ where $\theta_i(z, r) = \Pr[R = r \mid Z = z]$, and

$$\Pr[\Theta_i(\mathbf{z}_i) = \boldsymbol{r}_i] = \prod_{s \in [m]} \theta_i(z_{i,s}, r_{i,s}).$$

That is, each report only depends on the corresponding signal and the strategy is uniform across tasks. We also call $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$ as *strategy profile*, and the above assumption yields a bijection between Θ and θ.

Definition E.5 (Utility and social welfare)**.** Given a mechanism $\mathcal{M}$, under a common prior $\mathbb{P}$, for a strategy profile $\mathbf{\Theta}$, the *ex-ante utility* of agent i is

$$u_i(\mathbb{P}, \boldsymbol{\theta}) \triangleq \mathbb{E}_{\mathbf{Z} \sim \mathbb{P}} \left[\mathbb{E}_{\mathbf{R} \sim \mathbf{\Theta}(\mathbf{Z})} \left[M_i(\mathbf{R}) \right] \mid \mathbf{Z} \right],$$