Faithful Lottery Mappings, from Games to Competitions

Anonymous Authors

Abstract

A mechanism designer's goal is to design a game (a mechanism) that makes several strategic players interact, and has a desirable equilibrium outcome. Such goal is often achieved through monetary payments (score), e.g., information elicitation mechanisms. However, monetary payment may not be feasible, and we can only reward players with a fixed set of indivisible prizes. For instance, we may only select a fixed number of players as winners, e.g., distinguished reviewers in a conference, or score players into different tiers, e.g., letter score in a class. In this paper, we design faithful lottery mappings which 1) convert a large family of games with monetary payments to competitions with a set of indivisible prizes and 2) preserve the equilibrium structures of the original game.

We first consider single prize lottery mappings. We show not all games have a faithful lottery mapping. Then we design a family of lottery mapping, graph-induced lottery, which selects a winner by comparing each player by the average of his neighborhood. And we show graph-induced lottery is faithful to a family of games, restrained games, which contains zero-sum games and most information elicitation mechanisms when the number of players is large enough. We further propose Markov chain-induced lottery mappings that generalize graph-induced lottery and are faithful on games to which the graph-induced lottery is not faithful. Finally, we use reservoir sampling to extend any single prize faithful lottery to a multi-tier faithful lottery which selects winners for multiple indivisible prizes with various values.

1 Introduction

A mechanism designer's goal is to design a game (a mechanism) that incentives multiple strategic players to interact and play a desirable equilibrium outcome. The desired outcome can be all players reporting truthful predictions in forecasting elicitation, or high-quality information in crowdsourcing. Several mechanisms, particularly information elicitation mechanisms, achieve these goals via monetary payments (scores).

However, several real-world applications use competition or lottery to incentivize players instead of monetary payments. For example, machine learning competitions on Kaggle often provide prizes to top m players to incentive accurate algorithms. Peer-reviewed conferences select a fixed number of reviewers as distinguished reviewers for high-quality reviews. Or a teacher uses letter scores to encourage students to coordinate in group projects. The above games can have natural scores (accuracy of algorithms, quality of reviews, and success of group projects) so that a desirable outcome is an equilibrium. However, how can we convert these scores to these winner-take-all or multi-tier competitions preserving players' incentives?

Simple mappings often distort players' incentives. One candidate is selecting the winner to be the player with the *highest* score. However, maximizing score is not equivalent to maximizing the probability to be the highest score. Novice players may not want to participate at all since they can never be the best ones. Additionally, players may prefer risky strategy to have the highest score instead of maximizing their expected score. For instance, in forecast competition, players may report more extreme probabilities rather than their true beliefs. Another candidate is selecting the winner randomly *proportional* to each player's score. This mapping does not work when players' scores depend on each other's actions. For example, in group projects, students may not want to cooperate which not only increases his score but also other students' score.

Finally, Witkowski et al. (2021) propose the Event Lotteries Forecast (ELF) mapping which assigns each player's winning probability as a constant plus the difference between his score and the average score of the population. They show that the ELF mapping can ensure each player try to maximize his score in order to maximize his winning probability in forecast elicitation games (defined in example 2.4) where each player's action does not affect any other player's expected score. However, similar to the proportional method, the ELF mapping does not work in general game where players' expected scores depend on other's actions.

Our Results In the present submission, we undertake a game-theoretical approach to understand the mappings from general games with scores (scoring games) to competition games with a set of indivisible prizes that can preserve the original games' equilibrium structure. We are particularly interested in information elicitation games which often elicit players' information via paying them their "correlation," and ensure truth-telling is an equilibrium (example 2.5). Thus, players' scores often depend on each other's reports in information elicitation games, and the above two mappings generally do not work.

We first consider single prize lottery mappings that always output a distribution of winners, and we call a (single prize) lottery faithful to a game if it preserves the game's equilibrium. We show not all games have a faithful lottery mapping (proposition 2.2). Then in section 3, we design a family of lottery mapping, *graph-induced lottery* eq. (1), which selects a winner by comparing each player by the average of his neighborhood on an undirected graph. We can use these graphs on players to create a local comparison to alleviate the dependence between different players' scores and actions. In particular, we propose a general family of games, restrained game (definition 3.2) to which the graph-induced lottery mappings are is faithful. A restrained game requires each player to have a neighborhood so that his deviation from any equilibrium not only decreases his score, but the decrement is worse than the decrement of the average of his neighbors'. We show such games contain zero-sum games and most information elicitation mechanisms when the number of players is large enough.

However, a graph-induced lottery requires bi-directional comparison and the associated graph being undirected, to ensure the output is a distribution. In section 4, we extend this idea to directional comparison propose Markov chain-induced lottery mappings eq. (3) that generalize graph-induced lottery (proposition 4.2). Markov chain-induced lottery is particularly useful for games with complicated but limited dependency between players' scores and actions. Formally, we say one player is separated from the other if his action does not affect the other player's score, and we can create a directed graph (separation graph) on these players encoding who is separated by who. If a game has enough separation among players, we show how to design a faithful Markov chain-induced lottery. Specifically, Theorem 4.4 shows the existence of a faithful Markov chain-induced lottery when a game has a strongly connected separation graph. We also show an example of an information elicitation game with a faithful Markov chain lottery but does not have a faithful graph-induced lottery.

Both graph-induced and Markov chain induced lottery are 1) fair so that every player has the same winning probability if they have an identical score and 2) risk-neutral where the winning probability of a stochastic score is equal to the winning probability of the expected score. In section 5, we provide a necessary and sufficient condition for a lottery to be fair and risk-neutral. Finally, section 6 provides a reduction to extend any single prize faithful lottery to a multi-tier faithful lottery which selects winners for multiple indivisible prizes with various values by using reservoir sampling.

Related Works Witkowski et al. (2021) start the study of faithful (incentive-compatible) lottery for forecast elicitation games, and propose the ELF mapping for forecast elicitation games. For faithfulness, the ELF mapping utilizes the property of forecast elicitation games that each player's expected score is independent of each other's prediction. Several works relax the condition for faithfulness, and design better lottery mappings for forecast elicitation games that select the best forecaster with high probability.(Freeman et al. 2020; Frongillo et al. 2021)

Our work extends the study of faithful lottery to a large family of Bayesian games, scoring games, where each player utility is assigned by a mechanism and independent of his private type. Besides forecast elicitation games, scoring games epitomize two classes of games, complete information games (zero-sum games, or graphical games), and information elicitation games with or without elicitation, which pay players for truthful reports (Miller, Resnick, and Zeckhauser 2005; Prelec 2004; Waggoner and Chen 2013; Zhang and Chen 2014; Schoenebeck and Yu 2020). In particular, player's score in most information elicitation games has complicated but limited dependency on other's action. We will exploit this property to design our faithful lottery mappings.

A faithful lottery for a single winner can be seen as budget-balanced or feasible mechanism. Singer (2010) Specifically, if we take the winning probability as the mechanism's payment, the requirement of probability distribution implies the payment is non-negative, and the total payment is one. However, scoring games do not contain general auctions because each player's utility depends not only on payment but also on his private evaluation of the allocated item. Thus, our faithful lottery mappings do not apply in these auctions.

2 Problem Setting

In this section, we introduce our formal model of games and define lottery mappings and our design objectives. We'll describe two games as running examples. We use different fonts to denote a scalar x, a vector $\vec{x}$, a real-valued random variable X, a vector-valued random variable $\vec{X}$, and a matrix $\boldsymbol{X}$. Furthermore, x_i is the i-th element of vector $\vec{x}$, with indexing starting at 1, and $\vec{x}_{-i}$ contains all elements of vector $\vec{x}$ except the i-th element.

[F:$\vec{x}'$ is pretty ugly. Trying to use other notion for vector.]

Formal Model

We consider settings where a principal, who used to be able to use monetary rewards to incentivize certain behavior of agents, now can only run a competition and award an indivisible, fixed prize to a single winning agent. (In Section **??**, we will extend our model to allow more than one indivisible prizes in the competition.) The principal however hopes to incentivize the same behavior of agents in the competition as what he can achieve when using monetary rewards. We formalize this problem by first defining a set of games where the principal provides monetary rewards to agents. We call this set of games *scoring games* to emphasize that agents' score or utility in the game is provided by the principal rather than exogenously derived (e.g. satisfaction of taking certain actions). The principal, when running a competition, can make use of a game with monetary reward and map the game outcomes to a probability distribution representing each agent's probability of winning the prize. The goal of the principal is to design a probability mapping, called *lottery* *mapping*, such that the equilibrium of the game is preserved in the competition.

Scoring Games A *scoring game* (or game for short) $\mathcal{G}$ has parameters, n, $\mathcal{A}$, Θ, μ, and $\vec{U}$. $[n] = \{1, 2, \ldots, n\}$ is the set of players. $\mathcal{A} = \prod_{i=1}^{n} \mathcal{A}_i$ is the set of actions. $\Theta = (\prod_{i=1}^{n} \Theta_i) \times \Theta_0$ is the set of (discrete) types where $\theta_i \in \Theta_i$ is a realization of the type of player $i \in [n]$ and $\theta_0 \in \Theta_0$ is the external randomness of the game, the realization of which will be observed by principal. For example, in information elicitation, θ_i, $i \in [n]$ is agent i's private signal and θ_0 is the state of the world that the principal wants to get predictions for. μ is a probability measure on the set of types Θ, which is common knowledge to all agents and the principal. $\vec{U} = (U_1, \ldots, U_n)$, $U_i : \mathcal{A} \times \Theta_0 \to \mathbb{R}$ is the (deterministic) scoring function of player $i \in [n]$. Here we assume the game has *bounded* scores so that $U_i(\vec{a}, \theta_0) \in [0, 1]$ for all $i \in [n]$, $\vec{a} \in \mathcal{A}$, and $\theta_0 \in \Theta_0$.

The scoring games are a subset of simultaneous Bayesian games. The scores of agents depend only on agents' actions and the realization of the external randomness θ_0, all observable to the principal. Scoring games do not include Bayesian games where an agent's utility depend on his or other agents' private types, such as auctions. The set of scoring games also include complete information, simultaneous-move games as special cases, where each agent only has one possible type and the external randomness either doesn't exist or doesn't affect agents' scores.

A pure strategy $s_i : \Theta_i \to \mathcal{A}_i$ of agent $i \in [n]$ is a function mapping from agent i's types to actions. We denote the set of pure strategies of agent i as $\mathcal{S}_i$, a strategy profile as $\vec{s} = (s_1, \ldots, s_n)$, and the collection of all possible strategy profiles as $\mathcal{S} := \prod_i \mathcal{S}_i$. Note that complete information games are special cases of scoring games where the set of pure strategy reduces to the set of action.

In this paper, we only concern the value of the scoring function $\vec{U}$, and treat these games as black-boxes. Therefore, for notational brevity, we will omit the type $\vec{\theta}$ when specifying scoring function and rewrite $\vec{U}(\vec{s}) := \vec{U}(\vec{s}(\vec{\theta}), \theta_0)$ as a stochastic function for all strategy profile $\vec{s}$, and the randomness comes from the type $\vec{\theta}$. Furthermore, we will denote a game as $\mathcal{G} = ([n], \mathcal{S}, \vec{U})$ where n is the number of player, $\mathcal{S}$ is the set of strategy, and $\vec{U}$ is a stochastic scoring function. Thus, given a strategy profile $\vec{s}$, each player i's ex ante expected score can be written as $\mathbb{E}[U_i(\vec{s})] = \mathbb{E}_{\vec{\theta} \sim \mu}[U_i(\vec{s}(\vec{\theta}), \theta_0)]$.

A (pure) strategy profile $\vec{s}^*$ is a (strict) *Bayesian Nash equilibrium* of the game $([n], \mathcal{S}, \vec{U})$ if and only if $\mathbb{E}[U_i(s_i^*, \vec{s}^*_{-i})] > \mathbb{E}[U_i(s_i, \vec{s}^*_{-i})]$ for all strategy $s_i \in \mathcal{S}_i$, $s_i \neq s_i^*$ and all player i.[1] We use $\mathcal{E} \subset \mathcal{S}$ to denote the set of pure-strategy strict Bayesian Nash equilibria (or equilibria for short).[2]

[1]Because Θ is discrete, this condition is equivalent to $\mathbb{E}[U_i(s_i^*, \vec{s}^*_{-i}) \mid \theta_i] > \mathbb{E}[U_i(s_i, \vec{s}^*_{-i}) \mid \theta_i]$ for all θ_i.

[2]More generally, we can set $\mathcal{E}$ to be a set of desirable equilibria that the principal hopes to preserve. We do not discuss this generalization formally, but most of our results still hold.

Lottery Mappings A function $\vec{\pi}$ is a *lottery mapping* (or lottery for short) on n agents if it takes a collection of scores and outputs a probability distribution on n agentss, $\vec{\pi} := (\pi_1, \ldots, \pi_n) : [0, 1]^n \to \Delta_n$. Given a lottery mapping, the principal can convert a scoring game $([n], \mathcal{S}, \vec{U})$ to a competition game $([n], \mathcal{S}, \vec{\pi} \circ \vec{U})$ which selects a single winner for a prize with unit value and thus each player's utility becomes the winning probability.

We say a lottery is faithful to a game, if the resulting competition game preserves the set of equilibria in the original game. Formally,

Definition 2.1. Given a game $\mathcal{G} = ([n], \mathcal{S}, \vec{U})$, a lottery $\vec{\pi}$ is *faithful* to the game $\mathcal{G}$ (with $\mathcal{E}$) if

$$\mathbb{E}\pi_i(\vec{U}(s_i^*, \vec{s}^*_{-i})) > \mathbb{E}\pi_i(\vec{U}(s_i, \vec{s}^*_{-i}))$$

for any equilibrium $\vec{s}^* \in \mathcal{E}$, strategy $s_i \neq s_i^*$, and player i.

However, by showing no lottery is faithful to a coordination game, proposition 2.2 shows that it is impossible to design a faithful lottery for all scoring games.

Proposition 2.2. *For any lottery $\vec{\pi}$, there exists a scoring game $\vec{\mathcal{G}}$ to which $\vec{\pi}$ is not faithful.*

Proof. Consider a simple coordination game with n players, $\mathcal{A}_i = \{H, T\}$ for all i, and everyone gets 1 when they choose the same action, and zero otherwise,

$$\vec{U}(\vec{a}) = \begin{cases} \vec{1}, \text{ if } a_1 = a_2 = \cdots = a_n \\ \vec{0}, \text{ otherwise} \end{cases}$$

[YC:Avoid using the vector arrow for game $\vec{\mathcal{G}}$?]

Suppose $\vec{\pi}$ is a faithful lottery to the game. Because everyone plays H, $\vec{s}^H := (H, \ldots, H)$, is an equilibrium, each player i prefers H over T when the other player plays H:

$$\pi_i(\vec{U}(H, \vec{s}^H_{-i})) = \pi_i(\vec{1}) > \pi_i(\vec{U}(T, \vec{s}^H_{-i})) = \pi_i(\vec{0}).$$

Summing over i, and we have $\sum_i \pi_i(\vec{1}) > \sum_i \pi_i(\vec{0})$. Therefore, the output of $\vec{\pi}$ is not always a distribution which is a contradiction. ☐

Besides faithfulness, another ideal property is *risk neutral* which ensures if two games have the same expected score, the distributions of winner of a risk neutral lottery are also identical. [YC:For the risk neutual definition, does it require the two games have the same type sets, equilibrium strategy sets, etc?][F:Equivalently, we can use single game and require $\mathbb{E}\pi(U) = \pi(\mathbb{E}U)$.]

Definition 2.3. A lottery $\vec{\pi}$ is *risk neutral* if for any two games $([n], \mathcal{S}, \vec{U})$ and $([n], \mathcal{S}, \vec{U}')$ with $\mathbb{E}\vec{U}(\vec{s}) = \mathbb{E}\vec{U}'(\vec{s})$ for all $\vec{s} \in \mathcal{S}$,

$$\mathbb{E}[\vec{\pi}(\vec{U}(\vec{s}))] = \mathbb{E}[\vec{\pi}(\vec{U}'(\vec{s}))] \text{ for all } \vec{s} \in \mathcal{S}.$$

Finally, a lottery is *fair*, if $\vec{\pi}(c\vec{1}) = \frac{1}{n}\vec{1}$ for all $c \in [0, 1]$. Thus, if all player has the same score, they have the same winning probability.

Example of Games

In this paper we will use two information elicitation games as running examples. Both games (examples 2.4 and 2.5) use proper scoring rules as building blocks. A proper scoring rule PS measures the accuracy of predictions, and maps a prediction p and an outcome w to a score $PS(p, w)$.

Example 2.4 (Forecast elicitation game)**.** Consider a group of $n \geq 2$ forecasters making predictions about a future event (e.g., will it rain tomorrow?). In a forecast elicitation game, each forecaster i makes a prediction, a_i, and gets score $U_i = PS(a_i, w)$ where w is the outcome of the future event.

In the above game, each player reporting his truthful prediction is an equilibrium due to the property of proper scoring rules.(McCarthy 1956; Savage 1971) The θ_0 contains the state of the world w. Each player's private type may be his information on w, and each his strategy is a mapping from his information to the prediction. We can rewrite the above score $\vec{u}$ which is a mapping from predictions and ground truth to a stochastic function on each player's strategy $\vec{U}$. Because the randomness of U_i depends on forecaster i's belief on w, and, two distinct forecasters' scores can be correlated. However, expectation of two forecasters' scores are independent of (separated from) each other's prediction.

In peer prediction game, we often utilize the interdependence in players' signals to incentivize them to report their private signal truthfully. Thus, players' expected scores often depends on other players' actions. Here we introduce a peer prediction game in Zhang and Chen (2014); Schoenebeck and Yu (2020).

Example 2.5 (Peer prediction game)**.** Given $n \geq 3$ players with private information in a finite set $\mathcal{X}$, the game consists of two parts: Every player i reports 1) a signal $x_i \in \mathcal{X}$ 2) his prediction on player $(i+1)$'s reported signal condition on player $(i-1)$'s reported signal, $y_i : \mathcal{X} \to \Delta_{\mathcal{X}}$ where $\Delta_{\mathcal{X}}$ is the set of probability distributions on $\mathcal{X}$ and $y_i(x)$ is his prediction on player $(i+1)$'s reported signal when $(i-1)$'s reported signal is x.[3] Player i's score consists of two terms $U_i = PS(y_i(x_{i-1}), x_{i+1}) + PS(y_{i+1}(x_i), x_{i+2})$. We call the first term prediction score, and the second information score.

Zhang and Chen (2014) prove that truth-telling is an equilibrium of the above game. Here the set of type contains each player's signal, and each player's strategy is a mapping from his signal to reported signal and predictions. Note that the score of each agent only depends on their reports and independent of their private type.

Note that each player i's expected score only depends on a small number of players' actions. Specifically, the expected prediction score depends on player $(i-1)$'s and $(i+1)$'s reports, and his expected information score depends on player $(i+1)$'s and $(i+2)$'s reports.

3 Graph-induced Lottery Mappings

Recall that in ELF [SZ:introduce ELF?] one player's winning probability is the difference between his score and the average of all other players' scores. We can extend this idea to graphs on these n players, and set each player i's winning probability to be $1/n$ plus the difference between his score and the average score of his neighborhood. (defined formally in eq. (1)). We first define graph-induced lottery mappings formally. Then in the restrained games section, we demonstrate how to design faithful graph-induced lottery, apply these lottery to information elicitation games (examples 2.4 and 2.5), and compare these lottery with ELF.

Before defining the mapping formally, we introduce some basic notions in graph theory. Given a graph $G = ([n], E)$ on n players, each player $i \in [n]$ has a *neighborhood* $N_G(i) = \{j : (i,j) \in E\}$. A graph G is undirected if for all i and j, $j \in N_G(i)$ implies $i \in N_G(j)$. Given a graph G, the adjacency matrix is $\boldsymbol{A}_G$ where $A_G(i,j) = 1$ if $(i,j) \in E$ and 0 otherwise, the degree matrix is $\boldsymbol{D}_G$ which is a diagonal matrix with $D_G(i,i) = \sum_j A_G(i,j)$, and the maximum degree is $\Delta := \max_i D_G(i,i)$. Finally, the *Laplace matrix* of G is $\boldsymbol{L}_G := \boldsymbol{D}_G - \boldsymbol{A}_G$.

Given an undirected graph G, a lottery mapping

$$\vec{\pi}_G(\vec{u}) = \frac{1}{n\Delta}\boldsymbol{L}_G\vec{u} + \frac{1}{n}\vec{1} \tag{1}$$

for all $\vec{u} \in [0,1]^n$ is a *graph-induced lottery* with G or *G-induced lottery* for short.

The graph-induced lottery mappings contain the ELF mapping as a special case which is the K_n-induced lottery mapping where K_n the complete graph of n nodes. We include more details in the supplementary material.

Proposition 3.1. *Given an undirected graph $G = ([n], E)$, the G-induced lottery is risk neutral and fair.*

Proof of proposition 3.1. Since the mapping $\vec{\pi}_G(\vec{u}) = \frac{1}{n\Delta}\boldsymbol{L}_G\vec{u} + \frac{1}{n}\vec{1}$ is an affine mapping, $\vec{\pi}_G$ is risk neutral. Because $\boldsymbol{L}_G\vec{1} = \vec{0}$, for any $c \in [0,1]$, $\vec{\pi}_G(c\vec{1}) = c\frac{1}{n\Delta}\boldsymbol{L}_G\vec{1} + \frac{1}{n}\vec{1} = \frac{1}{n}\vec{1}$, and $\vec{\pi}_G$ is fair.

Finally, we show the output of $\vec{\pi}_G$ is always a distribution on $[n]$. Given $\vec{u} \in [0,1]^n$, $\vec{1}^\top\vec{\pi}_G(\vec{u}) = \frac{1}{n\Delta}\vec{1}^\top\boldsymbol{L}_G u + \frac{1}{n}\vec{1}^\top\vec{1} = 1$, because $\vec{1}^\top\boldsymbol{L}_G = \vec{0}^\top$. Hence the sum of each coordinate of $\vec{\pi}_G(\vec{u})$ equals one. Now, we show each coordinate of $\vec{\pi}_G(\vec{u})$ is non-negative. For each $i \in [n]$, because $L_G(i,j) \leq 0$ for all $i \neq j$ and $L_G(i,i) \geq 0$,

$$\begin{aligned}(\vec{\pi}_G(\vec{u}))_i =& \frac{1}{n\Delta}\sum_j L_G(i,j)u_j + \frac{1}{n} \\ \geq& \frac{1}{n\Delta}L_G(i,i)\cdot 0 + \sum_{j\neq i} L_G(i,j)\cdot 1 + \frac{1}{n} \\ =& -\frac{1}{n\Delta}\sum_{j\neq i} A_G(i,j) + \frac{1}{n} \geq 0.\end{aligned}$$

The last inequality holds because $\sum_j A_G(i,j) \leq D_G(i,i) \leq \Delta$. ☐

Restrained Games

Now we show these graph-induced lottery mappings are faithful to a family of games, restrained games.

[3] We take mod n here.

Definition 3.2. Given a game $(n, \mathcal{S}, \vec{U})$ and $i \in [n]$, we say player i is *restrained* by a set $S \subseteq [n]$ if for any $\vec{s}^* \in \mathcal{E}$ and $s_i \neq s_i^*$, the decrement of player i's score is larger than the decrement of average score of players in S,

$$\mathbb{E}[U_i(\vec{s}^*) - U_i(s_i, \vec{s}^*_{-i})] > \frac{1}{|S|}\sum_{j \in S}\mathbb{E}[U_j(\vec{s}^*) - U_j(s_i, \vec{s}^*_{-i})].$$

[YC:I find the expectation is becoming confusing because we hide agents beliefs away. Here, for information elicitation games, I naturally think the expectation is over agent's subjective beliefs. We define it as an ex-ante expectation. I think it's correct but a bit challenging to follow. Confused about the use of small u and big U.] If a player is restrained by a his peers, he will stay in an equilibrium to maximize the difference $\mathbb{E}U_i - \frac{1}{|S|}\sum_{j \in S}\mathbb{E}U_j$.

Definition 3.3. A game is *restrained* by an undirected graph G if each player i is restrained by his neighbors in $N_G(i)$, or *restrained game* in short.

Theorem 3.4. *Let $G = ([n], E)$ be an undirected graph with n nodes, and $\mathcal{G} = ([n], \mathcal{S}, \vec{U})$ be a n-player game. If $\mathcal{G}$ is restrained by G, the G-induced lottery is faithful to the game $\mathcal{S}$.*

Proof. First for any strategy profile $\vec{s} \in \mathcal{S}$ with $\vec{U} = \vec{U}(\vec{s})$, player i's winning probability is $\mathbb{E}[\pi_i(\vec{U})] = \frac{1}{n\Delta}\mathbb{E}[\boldsymbol{L}_G\vec{U}]_i + \frac{1}{n} = \frac{1}{n\Delta}\sum_j L_G(i,j)\mathbb{E}U_j + \frac{1}{n}$, and

$$\mathbb{E}[\pi_i(\vec{U})] = \frac{1}{n\,\Delta}\left(D_G(i,i)\mathbb{E}U_i - \sum_{j \in N_G(i)}\mathbb{E}U_j\right) + \frac{1}{n} \quad (2)$$

Consider an equilibrium $\vec{s}^* \in \mathcal{E}$, player i, and a strategy $s_i \in \mathcal{S}_i$ with $s_i \neq s_i^*$. Let $\vec{U}^* = \vec{U}(\vec{s}^*)$ and $\vec{U}' = \vec{U}(s_i, \vec{s}^*_{-i})$. If player i deviates from s_i^* to s_i, by eq. (2), the difference of player i's winning probability $\mathbb{E}[\pi_i(\vec{U}')] - \mathbb{E}[\pi_i(\vec{U}^*)]$ is equal to

$$\frac{1}{n\,\Delta}\left(D_G(i,i)(\mathbb{E}U_i' - \mathbb{E}U_i^*) - \sum_{j \in N_G(i)}(\mathbb{E}U_j' - \mathbb{E}U_j^*)\right)$$

which is negative because player i is restrained by $N_G(i)$ (definition 3.2). Therefore, the G-induced lottery is faithful to the game. □

Now we provide sufficient conditions for restrained games.

Separation condition The first sufficient condition is limited dependency which is captured as separation below.

Definition 3.5. Given a game $\mathcal{G} = ([n], \mathcal{S}, \vec{U})$ and $i, j \in [n]$, (the action of) player i is *separated* from (the scores of) player j in $\mathcal{G}$, if $\mathbb{E}U_j(s_i, \vec{s}_{-i}) = \mathbb{E}U_j(s_i', \vec{s}_{-i})$ for all s_i, s_i' and $\vec{s}_{-i}$.

The following proposition is straightforward.

Proposition 3.6. *If player i's action is separated from any player's score in a set S, then player i is restrained by S.*

Proof. For any equilibrium $\vec{s}^* \in \mathcal{E}$ and $s_i \neq s_i^*$, $\mathbb{E}[U_i(\vec{s}^*) - U_i(s_i, \vec{s}^*_{-i})] > 0$. On the other hand, because player i's action does not affect any players in S, $\sum_{j \in S}\mathbb{E}[U_j(\vec{s}^*) - U_j(s_i, \vec{s}^*_{-i})] = 0$. These two complete the proof. □

Proposition 3.6 suggests that when a game has enough separation, the game is restrained.

Corollary 3.7. *Given a game $\mathcal{G} = ([n], \mathcal{S}, \vec{U})$, if for each player i there exists player j so that j's score is separated from player i's action and vice versa,[4] $\mathcal{G}$ is a restrained game.*

Proof. Let $G = ([n], E)$ where $(i,j) \in E$ if j's score is separated from i's action. The graph G is undirected by the condition in the corollary. By proposition 3.6, each player i is restrained by his neighborhood in G, $N_G(i)$. □

Now we apply these observation to our running examples. First, by theorem 3.4 and proposition 3.6, any G-induced lottery is also faithful to the forecast elicitation game example 2.4, because each player's prediction (action) is separated from any player's expected scores. This general ELF to any graph-induced lottery.

Corollary 3.8. *For any undirected graph G, the G-induced lottery is faithful to the forecast elicitation game (example 2.4).*

Furthermore, we can show that the ELF fails to be faithful to some game which has faithful graph-induced lottery.[5]

In example 2.5, when $n \geq 6$, we can see that each player i's action is separated from player $i+3$'s score and vise versa. By corollary 3.7, the peer prediction game in example 2.5 with $n \geq 6$ is restrained, and has a faithful G-induced lottery with $E = \{(i, i+3) : i \in [n]\}$.

Constant sum condition Note that the coordination game in proposition 2.2 is not a restrained game, because as one player deviates from coordination the other player suffer as much as the player. In contrast, any constant sum game ensures one player's loss is other player's gain, and is also a restrained game by proposition 3.9.

Proposition 3.9. *If $([n], \mathcal{S}, \vec{U})$ is a constant sum game where $\sum_{i \in [n]}\mathbb{E}U_i(\vec{s}) = \sum_{i \in [n]}\mathbb{E}U_i(\vec{s}')$ for all $\vec{s}, \vec{s}' \in \mathcal{S}$, any player $i \in [n]$ is restrained by $[n]$.*

[4] If we represent the game as a graphical game (Kearns, Littman, and Singh 2013), such condition is equivalent to every node has degree at most n in the associated graph.

[5] Consider a pairwise coordination game with 4 players with the same action space $\{H, T\}$. Player 1 and 2 play a coordination game, and player 3 and 4 play another independent coordination game. When player 1 and 2 coordinate player 1 gets 0.1 and player 2 gets 1; otherwise both get zero. Similarly, when player 3 and 4 coordinate player 3 gets 0.1 and player 4 gets 1; otherwise both get zero. Player 1 and 2 are separated from player 3 and 4,and vice versa. The above game is restrained, so there exists a faithful graph-induced lottery by theorem 3.4. However, if everyone plays H, player 1 would deviate to T to maximize his winning probability in ELF.[F:we may put this in appendix]

Proof. For any equilibrium $\vec{s}^* \in \mathcal{E}$ and $s_i \neq s_i^*$, $\mathbb{E}[U_i(\vec{s}^*) - U_i(s_i, \vec{s}^*_{-i})] > 0$. However, because the game is constant sum, $\sum_{j\in[n]} \mathbb{E}[U_j(\vec{s}^*) - U_j(s_i, \vec{s}^*_{-i})] = 0$. Combining these two completes the proof. □

Because several games in information elicitation is downward reducible discussed after example 2.5, we can "zero-sumed" those game so that the truth-telling is still a equilibrium. Specifically, in example 2.5, when $n \geq 6$, we first randomly partition the players into two groups with size at least 3. Then we run the peer prediction game on each group independently. Finally, to zero-sum the scores, we subtract the scores of players in the first group by the scores of players in the second group. Truth-telling is still an equilibrium, because player's action in each group are separated from each other's score.

4 Markov Chain Induced Lottery Mappings

A graph-induced lottery mapping compares one player's score with the average of his neighbors' scores. We can generalize these mappings to directed graphs and compare one's score with a weighted average of his neighbors' scores. We use Markov chains to formulates this idea. The structure of this section is similar to section 3. We first define Markov chain-induced lottery mappings formally. Then, we introduce strongly separated game and demonstrate how to design faithful Markov chain-induced lottery. We further apply these lottery to the peer prediction game (example 2.5), and compare these lottery with graph-induced lottery. Finally, we study composed game and discuss how to design competition if we have more knowledge of the game besides the score $\vec{U}$.

Let $\boldsymbol{P}$ be the transition matrix and $\boldsymbol{\nu}$ be a stationary distribution so that $P(i,j)$ is the transition probability from state i to state j and $\boldsymbol{\nu}^\top \boldsymbol{P} = \boldsymbol{\nu}^\top$. We define, $\boldsymbol{D}_\mu$, a diagonal matrix with diagonal $D_{\boldsymbol{\nu}}(i,i) = \nu(i)$ for all $i \in [n]$, and $\boldsymbol{L}_{\boldsymbol{P}} := \boldsymbol{D}_{\boldsymbol{\nu}}(\mathbb{I} - \boldsymbol{P})$. Given Markov chain $\boldsymbol{P}$ with $\boldsymbol{\nu}$, *Markov chain induced lottery* with $\boldsymbol{P}$ (or $\boldsymbol{P}$-induced lottery) is

$$\vec{\pi}_{\boldsymbol{P}}(\vec{u}) = \frac{1}{n \max_i \nu(i)} \boldsymbol{L}_{\boldsymbol{P}} \vec{u} + \frac{1}{n}\vec{1} \tag{3}$$

for all $\vec{u} \in [0,1]^n$. In other word, to compute player i's winning probability, we sample j with probability $P(i,j)$ for all j and take the difference between player i's and j's scores.

Similar to a graph-induced lottery mapping, a Markov chain induced lottery mapping is also risk neutral and fair.

Proposition 4.1. *For any Markov chain $\boldsymbol{P}$, $\boldsymbol{P}$-induced lottery is risk neutral and fair.*

Proof. Since $\vec{\pi}_{\boldsymbol{P}}$ is affine, the mapping is risk neutral. Because $\boldsymbol{P}$ is a transition matrix $\boldsymbol{P}\vec{1} = \vec{1}$, $\boldsymbol{L}_{\boldsymbol{P}}\vec{1} = \boldsymbol{D}_{\boldsymbol{\nu}}(\vec{1} - \boldsymbol{P}\vec{1}) = \vec{0}$ and thus $\vec{\pi}_{\boldsymbol{P}}$ is fair by a similar argument in proposition 3.1. Now we show the output of $\vec{\pi}_{\boldsymbol{P}}$ is a distribution. First, because $\boldsymbol{\nu}$ is a stationary distribution, $\vec{1}^\top \boldsymbol{D}_{\boldsymbol{\nu}}(\mathbb{I} - \boldsymbol{P}) = \boldsymbol{\nu}^\top - \boldsymbol{\nu}^\top \boldsymbol{P} = \vec{0}^\top$, we have $\vec{1}^\top(\frac{1}{n \max \boldsymbol{\nu}} \boldsymbol{L}_{\boldsymbol{P}}\vec{u} + \frac{1}{n}\vec{1}) = \vec{1}^\top \frac{1}{n}\vec{1} = 1$. Finally, for each i and $j \neq i$, because $L_{\boldsymbol{P}}(i,i) \geq 0$ and $L_{\boldsymbol{P}}(i,j) \leq 0$, $(\vec{\pi}_{\boldsymbol{P}}(\vec{u}))_i \geq \frac{1}{n \max \boldsymbol{\nu}} \sum_{j \neq i} L_{\boldsymbol{P}}(i,j) + \frac{1}{n} = \frac{-1}{n \max \boldsymbol{\nu}} \sum_{j\neq i} P(i,j)\nu(j) + \frac{1}{n} \geq 0$. Therefore, we completes the proof. □

We finish this section by showing the family of Markov chain induced lottery mappings contains graph-induced lottery mappings.

Proposition 4.2. *Given an undirected graph $G = ([n], E)$, there exists a transition matrix $\boldsymbol{P}$ so that $\boldsymbol{P}$-induced lottery equals G-induced lottery, $\vec{\pi}_G(\vec{u}) = \vec{\pi}_{\boldsymbol{P}}(\vec{u})$ for all $\vec{u} \in [0,1]^n$.*

Proof of proposition 4.2. Let $\boldsymbol{P} := \boldsymbol{D}_G^{-1}\boldsymbol{A}_G$ where $P(i,j) = \frac{1}{D_G(j,j)}$ if $(i,j) \in E$ and 0 otherwise, and $m := \sum_j D_G(i,i)$ be the total degree of G. First $\boldsymbol{P}$ is a transition matrix, and the stationary distribution is $\boldsymbol{\nu}$ with $\nu(i) = \frac{D_G(i,i)}{m}$ for all i. Furthermore,

$$\begin{aligned}
&\frac{1}{\max_i \nu(i)} \boldsymbol{L}_{\boldsymbol{P}} \\
=&\frac{m}{\max_i D_G(i,i)} \boldsymbol{D}_{\boldsymbol{\nu}}(\mathbb{I} - \boldsymbol{P}) && (\nu(i) = \tfrac{D_G(i,i)}{m}) \\
=&\frac{1}{\max_i D_G(i,i)} \boldsymbol{D}_G(\mathbb{I} - \boldsymbol{P}) && (\boldsymbol{D}_G = m\boldsymbol{D}_{\boldsymbol{\nu}}) \\
=&\frac{1}{\max_i D_G(i,i)} (\boldsymbol{D}_G - \boldsymbol{A}_G) && (\boldsymbol{D}_G\boldsymbol{P} = \boldsymbol{A}_G) \\
=&\frac{1}{\Delta}\boldsymbol{L}_G, && (\Delta = \max_i D_G(i,i))
\end{aligned}$$

so $\vec{\pi}_{\boldsymbol{P}}$ equals $\vec{\pi}_G$ by eqs. (1) and (3). □

Strongly Separated Games

As corollary 3.7, we show that separation among players can help us to design faithful lottery mappings. In this section, we will dive deeper into this idea.

Note that, the condition in corollary 3.7 requires for all player i, there exists j so that their scores are separated from each other's action. As example 2.5, we may only have i's action is separated from j's score, but not vice versa. To utilize these non-symmetric separation, we define separation graph of a game.

Definition 4.3. A directed graph $H = ([n], F)$ is the *separation graph* of a game $([n], \mathcal{S}, \vec{U})$ if a directed edge $(i,j) \in F$ when i's action is separated from j's score.

We say a game is *strongly separated* if for all i and j there exists a sequence (path) $c_0, \ldots, c_l$ so that $c_0 = i, c_l = j$, and $(c_i, c_{i+1}) \in F$ for all $0 \leq i < l$.[SZ:Maybe write this as a definition?]

Now we can show our main result in this section.

Theorem 4.4. *Given a game $\mathcal{G} = ([n], \mathcal{S}, \vec{U})$, and $\boldsymbol{P}_H$ which is the simple random walk on the separation graph H. If $\mathcal{G}$ is strongly separated, the $\boldsymbol{P}_H$-induced lottery mapping is faithful to the game, $\mathcal{G}$.*

Proof. Note that if $\mathcal{G}$ is strongly separated, the separation graph H is strongly connected. By Perron-Frobenius theorem, the random walk with $\boldsymbol{P}_H$ is irreducible, and the stationary distribution is positive, $\nu(i) > 0$ for all i.

For any $\vec{s} \in \mathcal{S}$ with $\vec{U} = \vec{U}(\vec{s})$, player i's winning probability is $\mathbb{E}[\pi_i(\vec{U})] = \frac{\nu(i)}{n \max \boldsymbol{\nu}} \left(\mathbb{E}U_i - \sum_j P_H(i,j)\mathbb{E}U_j\right) + \frac{1}{n}$.

Consider an equilibrium $\vec{s}^* \in \mathcal{E}$, player i, and $s_i \in \mathcal{S}_i$ with $s_i \neq s_i^*$. Let $\vec{U}^* = \vec{U}(\vec{s}^*)$ and $\vec{U}' = \vec{U}(s_i, \vec{s}^*_{-i})$. When player i deviates from s_i^* to s_i, because player i's action is separated from neighboring player j's scores, $\mathbb{E}U_j = \mathbb{E}U_j$ and the difference of player i's winning probability

$$\mathbb{E}[\pi_i(\vec{U}')] - \mathbb{E}[\pi_i(\vec{U}^*)] = \frac{\nu(i)}{n \max \boldsymbol{\nu}}(\mathbb{E}U_i' - \mathbb{E}U_i^*) \quad (4)$$

which is negative because $\vec{s}^*$ is an equilibrium and $\nu(i) > 0$. Thus, the $\boldsymbol{P}_H$-induced lottery is faithful to the game. □

In the proof, the strongly separated condition ensures the stationary distribution $\boldsymbol{\nu}$ has full support $\nu(i) > 0$ for all i. Thus such condition can be relaxed as long as there exists a Markov chain $\boldsymbol{P}$ has a stationary distribution with full support and the transition graph is a sub-graph of the separation graph. Additionally, we can decide whether a game is strongly separated by checking if the associated separation graph is strongly connected which can be solved in linear time.

Recall that with corollary 3.7, we can design a graph-induced faithful lottery for the peer prediction game example 2.5 when $n \geq 6$. With theorem 4.4, we can design a Markov chain induced faithful lottery when $n \geq 5$. Because player $(i+3)$'s action is separated from player i's score for all $i \in [n]$, the separation graph $H = ([n], F)$ has the set of directed edge $F = \{(i+3, i) : i \in [n]\}$, and thus is strongly connected. Therefore, P_H-induced lottery is faithful by theorem 4.4. [SZ:I think this example is worth some elaboration.][F:unclear?] [SZ:No, I just thought the result is interesting and worth highlighting. But the current version is fine.]

Composed Strongly Separated Games

Finally, some games $\mathcal{G}$ is a composition of several games $\{\mathcal{G}_k\}_{k\in[K]}$so that each $\vec{s}^* \in \mathcal{E}$ is a weak equilibrium in each games. For example, Example 2.5 can be seen as a composed game where the first game uses the prediction score and the second one uses the information score. In each game, truth-telling is a weak equilibrium but not a strict one. Formally,

Definition 4.5. Given positive integers n, K, and a strategy space $\mathcal{S}$, let $\mathcal{G}_k = ([n], \mathcal{S}, \vec{U}^k)$ be a n-player game with strategy space $\mathcal{S}$ for all $k \in [K]$. A *composed game* $\mathcal{G}$ from $(\mathcal{G}_k)_{k\in[K]}$ is a n-player game $([n], \mathcal{S}, \sum_{k\in[K]} \vec{U}^k)$, and if $\vec{s}^* \in \mathcal{E}$, $\vec{s}^*$ is a weak equilibrium of all $\mathcal{G}_k$.

If each game $\mathcal{G}_k$ is strongly separated with a faithful lottery $\boldsymbol{P}_{H_k}$, then we can design a faithful mapping to $\mathcal{G}$ by combining those K lottery mappings.[6] Formally,

[6]Here we want to clarify that only knowing the score of a composed game $\mathcal{G}$ is not sufficient to run algorithm 1. Rather, algorithm 1 needs to know the score of each game $\mathcal{G}^k$. This makes algorithm 1 not a lottery mapping defined in section 2.

Algorithm 1: Competition from Composed Games

Input: a n-player composed game $\vec{\mathcal{G}}$ from $\mathcal{G}_k = ([n], \mathcal{S}, \vec{U}^k)$ for $k \in [K]$ where each $\mathcal{S}_k$ is strongly separated with a separation graph H_k.

1: After players choose strategy $\vec{s}$, compute the score in each game $\mathcal{G}_k$, $\vec{U}^k = \vec{U}^k(\vec{s})$.
2: Run P_{H_k}-induced lottery on the score in each game $\vec{\pi}^k = \vec{\pi}_{H_k}(\vec{U}^k)$.
3: **return** the distribution of winner $\frac{1}{K}\sum_{k\in[K]} \vec{\pi}^k$.

Proposition 4.6. *If each game $\mathcal{G}_k$ is strongly separated, the mapping in algorithm 1 is faithful to the composed game $\mathcal{G}$.*

Proof. Consider an equilibrium $\vec{s}^* \in \mathcal{E}$, player i, and $s_i \in \mathcal{S}_i$ with $s_i \neq s_i^*$. Let $\vec{s}' = (s_i, \vec{s}^*_{-i})$. When player i deviates from s_i^* to s_i, by eq. (4) the difference $\boldsymbol{P}_{H_k}$-induced lottery is

$$\mathbb{E}[\pi_{H_k}(\vec{U}^k(\vec{s}'))]_i - \mathbb{E}[\pi_{H_k}(\vec{U}^k(\vec{s}^*))]_i \leq 0$$

because $\vec{s}^*$ is a weak equilibrium and $\mathbb{E}U_i^k(\vec{s}') \leq \mathbb{E}U_i^k(\vec{s}^*)$. Additionally, because $\mathbb{E}U_i(\vec{s}') < \mathbb{E}U_i(\vec{s}^*)$, there exists $l \in [K]$ such that $\mathbb{E}U_i^l(\vec{s}') < \mathbb{E}U_i^l(\vec{s}^*)$ and

$$\mathbb{E}[\pi_{H_l}(\vec{U}^l(\vec{s}'))]_i - \mathbb{E}[\pi_{H_l}(\vec{U}^l(\vec{s}^*))]_i < 0.$$

Therefore, the difference of winning probability is $\frac{1}{K}\sum_k \mathbb{E}\pi_i^k(\vec{U}^k(\vec{s}')) - \mathbb{E}\pi_i^k(\vec{U}^k(\vec{s}^*)) < 0$ which completes the proof. □

Recall that Example 2.5 can be seen as a composed game: one uses the prediction score and the other uses the information score. In each game, a player's score only depends on two players ($(i-1)$ and $(i+1)$ for prediction score, or $(i+1)$ and $(i+2)$ for information score), each game is strongly separated so long as $n \geq 4$. Therefore, by proposition 4.6, the lottery in algorithm 1 is faithful to the peer prediction game when $n \geq 4$.

5 Characterization of Fair and Risk Neutral Lottery

We have discussed various fair and risk neutral lottery mappings. In this section, we provide some necessary condition for such lottery mappings. In particular, theorem 5.1 shows any risk neutral lottery mapping is an affine function on the score. Additionally, we give a necessary and sufficient condition for risk neutral and fair lottery mappings. This characterization can help us to design lottery other than graph or Markov chain induced lottery which may be faithful to other family of games.

Theorem 5.1. *If $\vec{\pi}$ is a risk neutral lottery, $\vec{\pi}$ is an affine mapping with a matrix $\boldsymbol{L} \in \mathbb{R}^{n\times n}$ and a vector $\vec{b} \in \mathbb{R}^n$, such that for all $\vec{u} \in [0,1]^n$, $\vec{\pi}(\vec{u}) = \boldsymbol{L}\vec{u} + \vec{b}$.*

Moreover, $\vec{\pi}(\vec{u}) = \boldsymbol{L}\vec{u} + \vec{b}$ is fair, if and only if 1) $\vec{b} = \frac{1}{n}\vec{1}$, 2) $\boldsymbol{L}\vec{1} = \vec{0}$, 3) $\vec{1}^\top \boldsymbol{L} = \vec{0}^\top$, and 4) $\|\boldsymbol{L}\|_\infty \leq \frac{2}{n}$.

Proof. For all random vector $\vec{U} \in [0,1]^n$, we can define two games: one has score $\vec{U}$, another has score $\mathbb{E}\vec{U}$ which are independent of players' action. Since $\vec{\pi}$ is risk neutral, we have $\mathbb{E}[\vec{\pi}\vec{U}] = \vec{\pi}(\mathbb{E}\vec{U})$. Therefore, and $\vec{\pi}$ is affine on $[0,1]^n$.

Moreover, if $\vec{\pi}$ is fair, $\vec{\pi}(c\vec{1}) = c\boldsymbol{L}\vec{1} + \vec{b} = \frac{1}{n}\vec{1}$ for all $c \in [0,1]$. Taking derivative with respect to c, we have $\boldsymbol{L}\vec{1} = \vec{0}$, and $\vec{b} = \frac{1}{n}\vec{1}$. Because the output of $\vec{\pi}$ is a distribution, $\vec{1}^\top(\boldsymbol{L}\vec{u} + \vec{b}) = (\vec{1}^\top\boldsymbol{L})\vec{u} + 1 = 1$ for all $\vec{u} \in [0,1]^n$, and $\vec{1}^\top\boldsymbol{L} = \vec{0}^\top$. Finally, for any $\vec{v} \in \mathbb{R}^n$ with $\|\vec{v}\|_\infty \leq 1$, we set $\vec{u}^+ := \frac{1}{2}(\vec{1} + \vec{v})$ and $\vec{u}^- := \frac{1}{2}(\vec{1} - \vec{v})$ which are in $[0,1]^n$. Since the output of $\vec{\pi}$ is always a probability, $\pi_i(\vec{u}) = \boldsymbol{L}_i \cdot \vec{u} + \frac{1}{n} \geq 0$ where $\boldsymbol{L}_i$ is the i-th row of matrix $\boldsymbol{L}$. Thus, we have $\boldsymbol{L}_i\vec{u}^+ \geq \frac{-1}{n}$ and $\boldsymbol{L}_i\vec{u}^- \geq \frac{-1}{n}$. Because $\boldsymbol{L}_i\vec{1} = \vec{0}$, $\frac{-2}{n} \leq \boldsymbol{L}_i\vec{v} \leq \frac{2}{n}$, and $\|\boldsymbol{L}\|_\infty \leq \frac{2}{n}$. It is straightforward to show the above four conditions are sufficient for fair and risk neutral lottery mappings. □

The above theorem shows that the matrix $\boldsymbol{L}$ is not necessarily a Laplace matrix as in eqs. (1) and (3). In particular, to compute a player's winning probability, we can add his score with other player's score instead of subtracting. For instance, a lottery mapping with the following matrix is also risk neutral and fair

$$\boldsymbol{L} = \frac{1}{n(n-1)}\begin{bmatrix} 1 & \dots & -(n-1) \\ \vdots & \iddots & \\ -(n-1) & \dots & 1 \end{bmatrix}.$$

Moreover, we can define a flow φ_L on n players with $\varphi_L(i,j) := L(i,j)$ and $\varphi_L(i,i) = 0$ for all $i \neq j$. The second and third condition in theorem 5.1 imply $\sum_{j\neq i}\varphi_L(i,j) = -L(i,i) = \sum_{j\neq i}\varphi_L(j,i)$ which is known as flow conservation. Therefore, we can also understand a risk neutral fair lottery as a conservative flow.

Finally, theorem 5.1 implies in a fair and neutral lottery, a player's winning probability decreases as the number of player increases.

Corollary 5.2. *If $\vec{\pi}$ is a fair and risk neutral lottery for n players, $\pi_i(\vec{u}) \leq \frac{2}{n}$ for all i and $\vec{u} \in [0,1]^n$.*

6 Multi-tier lottery

Besides selecting a single winner, we can select multiple or multi-tier winners. For example, how can we use a collection of (indivisible) prizes, e.g., car, cellphone, tv, to incentivize player to fill survey truthfully. Or if we use letter score to grade student, and each score is rewarded to a fixed number of students, how can we encourage student to cooperate in group projects where his action only only affects his score but also other's. In this section, we show how to extend a faithful lottery for single winner to selecting multiple winners or even multi-tier winners.

Given m (indivisible) prizes with non-negative values $z_1, \dots, z_m$, a function Π is a *m-tier lottery* for n players if it takes a collection of n scores and outputs an allocation of those m prizes players. Specifically, given $\vec{u} \in [0,1]^n$, $\Pi(\vec{u})$ outputs a random m-tuple of $[n]$ where $\Pi(\vec{u})_j \in [n]$ is the winner of the prize j. We abuse the notation and set $\Pi_{i,j}(\vec{u}) := \Pr[\Pi(\vec{u})_j = i]$ be the (marginal) probability that player i wins the prize j. Then player i's expected value in Π is $\mathbb{E}\sum_j z_j\Pi_{i,j}(\vec{u})$ when the score profile is $\vec{u}$. Similar to definition 2.1, we say a multi-tier lottery mapping is faithful to a game $\mathcal{G}$ if it preserves the equilibrium of the game.

Definition 6.1. A m-tier lottery is *faithful* to a game $\mathcal{G} = ([n], \mathcal{S}, \vec{U})$ if

$$\mathbb{E}\sum_j z_j\Pi_{i,j}(\vec{U}(s_i^*, \vec{s}_{-i}^*)) > \mathbb{E}\sum_j z_j\Pi_{i,j}(\vec{U}(s_i, \vec{s}_{-i}^*)).$$

for any equilibrium $\vec{s}^* \in \mathcal{E}$, action $s_i \neq s_i^*$, and player i.

A a multi-tier lottery selects winners *without replacement* if no player can win two prizes, $\Pr[\Pi(\vec{u})_j = \Pi(\vec{u})_{j'}] = 0$ for all $\vec{u} \in [0,1]^n$, j and j' in $[m]$.

Now we discuss how to design faithful multi-tier lottery Π to a game given a faithful lottery $\vec{\pi}$. First, if the winner is selected with replacement, we can run $\vec{\pi}$ m times and select m winner (with possible duplication) and assign m prizes uniformly random to those m winner. It's easy to see $\Pi_{i,j}(\vec{u}) = \pi_i(\vec{u})$ for all $i \in [n]$, $j \in [m]$, and $\vec{u} \in [0,1]^n$, and $\mathbb{E}\sum_j z_j\Pi_{i,j}(\vec{U}) = (\sum_j z_j)\mathbb{E}\pi_i(\vec{U})$. Thus, Π is faithful to $\mathcal{G}$ if $\vec{\pi}$ is faithful to $\mathcal{G}$.

On the other hand, we may want to lottery output winners without replacement. That is each player can only win at most one prize, e.g., student can only have a single letter score. Because one player can get at most one prize $m \leq n$. We can further assume $m = n$ by adding zero value prizes if $m < n$.

Proposition 6.2. *Given a game $([n], \mathcal{S}, \vec{U})$ with a faithful lottery $\vec{\pi}$ and n nonidentical prizes $z_1, \dots, z_n$, there exists a multi-tier lottery that selects winners without replacement and faithful to the game.*

Informally, we can first use a weighted Reservoir sample algorithm (e.g. (Chao 1982)) to select $m' < n$ "winner" without replacement and the probability of each player i being selected is $m'\pi_i(\vec{u})$. Then we assign those m' winners with the m' prizes uniformly at random, and assign the rest prizes to the rest of players. In the formal proof, we need to ensure the values $m'\pi_i(\vec{u})$ is a valid probability by proper choice of m' and some modification of $\vec{\pi}$.

Proof of proposition 6.2. First we order the prizes such that $z_1 \geq z_2 \geq \dots \geq z_n$. Because the prizes are not identical, there exists $m' < n$ so that $z_{m'} > z_j$ for all $j > m'$. Then given a single winner lottery $\vec{\pi}$ that is faithful to $\mathcal{G}$, let $\rho := \frac{n/m'-1}{n\max_{\vec{v}\in[0,1]^n, i\in[n]}\pi_i(\vec{v})-1}$. The value of ρ is in $(0,1)$, because $\vec{\pi}$ is faithful to $\mathcal{G}$, $\max_{\vec{v},i}\pi_i(\vec{v}) > 1/n$ and $m' < n$. Now we define a new lottery mapping

$$\vec{\pi}'(\vec{u}) := \rho\vec{\pi}(\vec{u}) + (1-\rho)\frac{1}{n}\vec{1}$$

for all $\vec{u} \in [0,1]^n$. Because $\rho > 0$, the new lottery $\vec{\pi}'$ is also faithful to $\mathcal{G}$. Furthermore, $\max_{\vec{v},i}\pi_i'(\vec{v}) \leq \rho(\max_{\vec{v},i}\pi_i(\vec{v}) - \frac{1}{n}) + \frac{1}{n} \leq \frac{1}{m'}$.

Now we are ready to define our multi-tier lottery mapping. Given a score profile $\vec{u}$, we sample m' player with replacement with probability $m'\pi_i'(\vec{u}) \leq 1$ for each player $i \in [n]$.

Then assign prizes $z_1, \dots, z_{m'}$ to those selected players uniformly at random, and assign prizes $z_{m'+1}, \dots, z_n$ to the rest of players.

Now we show such multi-tier lottery is faithful to $\mathcal{G}$. For each player i, his value is the average of the first m' prizes $z_+ := \frac{1}{m'}\sum_{j=1}^{m'} z_j$ when selected, and the average of the rest of prizes $z_- := \frac{1}{n-m'}\sum_{j=m'+1}^{n} z_j$ when not selected. Furthermore, the probability of i being selected is $\mathbb{E}[m'\pi_i'(\vec{U})]$. Thus, his expected value is $z_+ m'\mathbb{E}[\pi_i'(\vec{U})] + z_-\mathbb{E}[1-m'\pi_i'(\vec{U})] = (z_+ - z_-)\, m'\mathbb{E}[\pi_i'(\vec{U})] + z_-$. Because the $z_j < z_{m'} \le z_{j'}$ for all $j' \le m' < j$, $z_+ > z_-$, and player i needs to maximizes $\mathbb{E}[\pi_i'(\vec{U})]$ to maximize his expected value. Thus, the multi-tier lottery is faithful. $\square$

7 Discussion and Future Direction

We propose and study faithful lottery mappings, which convert scoring games to competitions and preserve the equilibrium structure. The space of faithful lottery mappings seems quite broad and invites further investigation. For instance, besides affine lottery mapping, we may consider non-affine faithful lottery mapping. In Frongillo et al. (2021), their lottery mapping selects winner through follow the regularized leader, which is not affine, but their mapping is not faithful to forecast elicitation games.

While faithful lottery mappings preserve the equilibria structure, the winning probability may differ from the original score. In other word, faithful lottery mappings only maintain the ordinal structure of the score, but forgo the numerical structure. Can we have lottery mapping such that the winning probability is “close” to the original score? In particular, approximation results should be possible in the multi-tier lottery setting, which has a more refined output space.

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A ELF mapping

Witkowski et al. (2021) define the ELF mapping such that each player i's winning probability defined as follow: For all $\vec{u} \in [0,1]^n$,

$$\begin{aligned}(\vec{\pi}_{\text{ELF}}(\vec{u}))_i =& \frac{1}{n(n-1)}\left[(n-1)u_i - \sum_{j\neq i} u_j\right] + \frac{1}{n} \\ =& \frac{1}{n-1}\left[u_i - \frac{1}{n}\sum_{j\in[n]} u_j\right] + \frac{1}{n}.\end{aligned}$$

Using our notion, ELF mapping can be written as

$$\vec{\pi}_{\text{ELF}}(\vec{u}) = \frac{1}{n(n-1)}\begin{bmatrix} n-1 & -1 & \dots & -1 \\ -1 & n-1 & \dots & -1 \\ \vdots & & \ddots & \\ -1 & -1 & \dots & n-1 \end{bmatrix}\vec{u} + \frac{1}{n}\vec{1}$$

for all $\vec{u} \in [0,1]^n$. By eq. (1), ELF mapping is a graph-induced lottery with the complete graph of n nodes, because the Laplace matrix of the complete graph of n nodes is

$$\begin{bmatrix} n-1 & -1 & \dots & -1 \\ -1 & n-1 & \dots & -1 \\ \vdots & & \ddots & \\ -1 & -1 & \dots & n-1 \end{bmatrix}.$$

B Details and Proofs

C More examples in peer prediction

The coordination game in proposition 2.2 has symmetric payment for each players. This symmetry is shared with several other peer prediction mechanisms. Here we use the *output agreement* mechanism (Waggoner and Chen 2014) as an example:

Output agreement mechanism

Model Consider n players, and the set of types is $(\theta_0, \theta_1, \theta_2, \dots, \theta_n) \in \{0,1\}^{n+1}$ with a joint distribution μ_δ for some $\delta < 1/2$ so that the latent variable θ_0 is uniformly sampled from $\{0,1\}$, each player i's type, θ_i, equals to θ_0 with probability $1-\delta$ and the opposite with probability δ, and all player's types are independent of each other conditional on θ_0. Each player i observes his binary types $\theta_i \in \{0,1\}$, and his action is a binary report $a_i \in \{0,1\}$, so his strategy is a mapping from his type $\{0,1\}$ to his report $\{0,1\}$. In output agreement mechanism, each player i's score is

$$u_i(\vec{a}, \vec{\theta}) = \mathbf{1}[a_1 = a_2 = \dots = a_n]. \tag{5}$$

Given a strategy profile $\vec{s}$, the expected score is $\mathbb{E}[U_i(\vec{s})] = \mathbb{E}_{\vec{\theta}\sim\mu}[\mathbf{1}[s_1(\theta_1) = s_2(\theta_2) = \dots = s_n(\theta_n)]]$, so

$$\mathbb{E}[U_i(\vec{s})] = \Pr_{\vec{\theta}\sim\mu_\delta}[s_1(\theta_1) = \dots = s_n(\theta_n)] \tag{6}$$

Equilibrium analysis We first analyze the equilibrium structure. There are only four possible mappings (strategy) from $\{0,1\}$ to $\{0,1\}$: for all $x \in \{0,1\}$, identity $\tau(x) = x$, constant one $\sigma(x) = 1$, constant zero $\bar{\sigma}(x) = 0$, and negation $\bar{\tau}(x) = 1 - x$. We say the identity mapping τ is the truthful strategy.

Now we show all players using identity mapping (truthful strategy) $\vec{\tau}$ is a Bayesian Nash equilibrium. By eq. (6), player i's expected score is

$$\mathbb{E}[U_i(\vec{\tau})] = \Pr[\theta_1 = \dots = \theta_n] = \delta^n + (1-\delta)^n.$$

However, if player i uses constant one strategy

$$\mathbb{E}[U_i(\sigma, \vec{\tau}_{-i})] = \Pr[\vec{\theta}_{-i} = 1] = \frac{1}{2}(\delta^{n-1} + (1-\delta)^{n-1})$$

which is less than $\mathbb{E}[U_i(\vec{\tau})]$. Similarly, $\mathbb{E}[U_i(\bar{\sigma}, \vec{\tau}_{-i})] < \mathbb{E}[U_i(\vec{\tau})]$. Finally, if player i deviates to the negation mapping, the expected score

$$\mathbb{E}[U_i(\bar{\tau}, \vec{\tau}_{-i})] = (1-\delta)\delta^{n-1} + \delta(1-\delta)^{n-1}$$

is also smaller then the expected score of the truth-telling. Therefore, the truthful strategy profile is a Bayesian Nash equilibrium.

Finally, because all players' scores are identical, by the proof of proposition 2.2, there is no faithful lottery to this game.

Variant of output agreement mechanism

Though we show the output agreement game with scoring function eq. (5) does not have any faithful lottery. Several variants of the game can have a faithful lottery mapping.

Pairwise output agreement mechanisms Instead of asking all player have the same report, we can score each player the number of agreement between him and other players.

$$u_i(\vec{a}, \vec{\theta}) = \sum_{j\neq i} \mathbf{1}[a_i = a_j] \tag{7}$$

Because the original agreement game has the truth-telling as a equilibrium, the game with eq. (7) also has the truth-telling as a equilibrium. Furthermore, proposition C.1 shows that each player i is restrained by the set $[n]$, so there is a graph-induced faithful lottery to the game by theorem 3.4.

Proposition C.1. *Under the distribution of types μ_δ, if $n \geq 3$, the game with scoring function eq. (7) only has four strict equilibria: all use τ, $\bar{\tau}$, σ or $\bar{\sigma}$.*

Moreover, each player is restrained by $[n]$.

Proof. First we show there are only four equilibria. Suppose $\vec{s} \in \mathcal{E}$, and there exists player i and j so that $s_i = z$ and $s_j = z'$ with $z \neq z'$. Because $\mathbb{E}[U_i(s_i, \vec{s}_{-i})] > \mathbb{E}[U_i(z', \vec{s}_{-i})]$ we have $\mathbb{E}[U_i(s_i, \vec{s}_{-i})] = \Pr[z(\theta_i) = z'(\theta_j)] + \sum_{k\neq i,j} \Pr[z(\theta_i) = s_k(\theta_k)]$ which is greater than $\Pr[z'(\theta_i) = z'(\theta_j)] + \sum_{k\neq i,j} \Pr[z'(\theta_i) = s_k(\theta_k)]$. Similarly $\Pr[z(\theta_i) = z'(\theta_j)] + \sum_{k\neq i,j} \Pr[z'(\theta_j) = s_k(\theta_k)]$ is greater than $\Pr[z(\theta_j) = z(\theta_i)] + \sum_{k\neq i,j} \Pr[z(\theta_j) = s_k(\theta_k)]$. Because the distribution $\vec{\theta}$ is exchangeable, we have

$\Pr[z(\theta_i) = s_k(\theta_k)] = \Pr[z(\theta_j) = s_k(\theta_k)]$ for all $k \neq i, j$, and

$$\begin{aligned} &\Pr[z(\theta_i) = z'(\theta_j)] + \Pr[z'(\theta_j) = z(\theta_i)] \\ > &\Pr[z'(\theta_i) = z'(\theta_j)] + \Pr[z(\theta_j) = z(\theta_i)] \end{aligned}$$

Now we enumerate all possible pairs of $z \neq z'$, and show the above inequality is false. When i and j use the same strategy, their score are all greater than $1/2$,

$$\begin{aligned} &\Pr[\tau(\theta_i) = \tau(\theta_j)] = \Pr[\bar{\tau}(\theta_i) = \bar{\tau}(\theta_j)] = \delta^2 + (1-\delta)^2 \\ &\Pr[\sigma(\theta_i) = \sigma(\theta_j)] = \Pr[\bar{\sigma}(\theta_i) = \bar{\sigma}(\theta_j)] = 1 \end{aligned}$$

However, if they use different strategy, their score are always smaller than or equal to $1/2$

$$\begin{aligned} &\Pr[\tau(\theta_i) = \sigma(\theta_j)] = \Pr[\tau(\theta_i) = \bar{\sigma}(\theta_j)] = 1/2 \\ &\Pr[\tau(\theta_i) = \bar{\tau}(\theta_j)] = 2\delta(1-\delta) \\ &\Pr[\sigma(\theta_i) = \bar{\sigma}(\theta_j)] = 0 \\ &\Pr[\sigma(\theta_i) = \bar{\tau}(\theta_j)] = \Pr[\bar{\sigma}(\theta_i) = \bar{\tau}(\theta_j)] = 1/2 \end{aligned}$$

This completes the first part of the proof.

Now we show the player i is restrained by $[n]$. If player i use strategy s_i and all other player use strategy τ, player i's expected score is

$$\mathbb{E}[U_i(s_i, \vec{\tau}_{-i})] = \sum_{k \neq i} \Pr[s_i(\theta_i) = \theta_k],$$

and another player j's expected score is

$$\mathbb{E}[U_j(s_i, \vec{\tau}_{-i})] = \Pr[s_i(\theta_i) = \theta_j] + \sum_{k \neq i,j} \Pr[\theta_j = \theta_k]$$

Since the truth-telling is an equilibrium and the information structure is exchangable, for all $k \neq i, j$ and $s_i \neq \tau$,

$$\Pr[\theta_j = \theta_k] = \Pr[\theta_i = \theta_k] > \Pr[s_i(\theta_i) = \theta_k]$$

Therefore, $\mathbb{E}[U_i(\vec{\tau})] - \mathbb{E}[U_i(s_i, \vec{\tau}_{-i})] > \mathbb{E}[U_j(\vec{\tau})] - \mathbb{E}[U_j(s_i, \vec{\tau}_{-i})]$ for all $j \neq i$, which completes the proof. $\square$

Bipartite output agreement mechanisms Another variant partition the players $[n]$ into two groups T and $[n] \setminus T$ with size at least two, and run the output agreement on each group separately. If player $i \in T$ the scoring function is

$$u_i(\vec{a}, \vec{\theta}) = \mathbf{1}[\wedge_{j \in T}(a_i = a_j)];$$

otherwise $u_i(\vec{a}, \vec{\theta}) = \mathbf{1}[\wedge_{j \notin T}(a_i = a_j)]$.

In this new game, because the scores of players in T is separated from the action of players in $[n] \setminus T$ and vice versa, we can have a faithful graph-induced lottery mapping by corollary 3.7.