# Robust and Strongly Truthful Multi-task Peer Prediction Mechanism for Heterogeneous Agents

SUBMISSION X

Peer prediction mechanisms incentive agents to truthfully report their signals even in the absence of verification, by comparing their reports with those of their peers. In the detailed-free multi-task setting, agents are asked to respond to multiple independent tasks, and the mechanism does not know the prior distribution. The goal is to provide a *strongly truthful* mechanism (the truth-telling rewards agents "strictly" more than any other strategy profile) even for heterogeneous agents. Previous work could do this with an infinite number of signals Kong and Schoenebeck [7] or with a finite number of signals, but only achieving a weaker notion of truthfulness (informed truthfulness) [1, 3, 20].

Our work provides an innovative new approach that manages to fuse the best feature of previous approaches to use a finite number of signals. We exploit the variational representation of Φ-divergence and our mechanism can be seen as a regularized version of previous mechanisms. Furthermore, this variational representation yields to an optimization-based learning algorithm. Specifically we:

(1) obtain $\epsilon$-strongly truthful mechanisms.
(2) reduce the number of samples required exponentially.
(3) create an innovative connection to empirical risk minimization that allows us to deal with much more general report spaces and borrow many techniques from related literature.

## 1 INTRODUCTION

Peer prediction is the problem of information elicitation without verification. Peer prediction mechanisms incentive agents to report their private signal truthfully even when the reports cannot be directly verified. Mechanisms leverage correlation in agents' signals. In the multi-task setting [3], agents are each asked to respond to multiple, independent tasks. We are particularly interested in the multitask setting with heterogeneous agents, which we motivate with several examples:

- In peer-grading, students grade multiple instances of the same assignment. One potential setting is where students' report a score $A$, $B$, $C$, $D$, or $F$. The students are self-consistent with their scores, but the scores between students are not necessarily calibrated. Another setting is where the student's score is a continuous value. For example, each grader's mark is a linear transformation of a ground truth score (modeling the different calibrations of different graders) with some additional Gaussian noise.
- We also might model restaurant reviewing in the following way: each restaurant has a distribution over a high dimensional space (e.g., a mixture of Gaussians). When an agent arrives at a restaurant, her "experience" is drawn from this space and is projected to the real line by a linear projection (this projection is particular to the agent). This projection is the experience that the agent has and is requested to report.

We would like to define mechanisms that can elicit information from agents in the above scenarios in an incentive-compatible manner. Here we briefly define several desirable properties for such a mechanism. This both maps out the goals of this paper and provides a vocabulary for discussing related work.

**Strongly Truthful** Truth-telling should pay strictly more than any other non-permutation equilibrium. A permutation equilibrium is one where agents report a permutation of the signals. A slightly weaker property is *Informed Truthful* where no strategy profile pays strictly more than truth-telling, and truthtelling pays more than any uninformative equilibrium.

**Detail-Free** The mechanism should not require foreknowledge of the prior. Nor should it make assumptions on the prior (which can be seen as a type of foreknowledge.

**Low Sample Complexity** With no assumption on the prior, the mechanisms often must "learn" the prior. Ideally this learning can be accomplished with a limited number of items.

**Robustness to Heterogeneity** Our mechanism should still work if agents are heterogeneous. Moreover, we would like to make as few assumptions as possible about the relation between agents' reports.

**Robust Signal Space** Our mechanism should work for agents with general signal spaces (e.g. real vector spaces).

In the multi-task setting, Dasgupta and Ghosh [3] propose a *strongly truthful* mechanism if the signal space is binary. There mechanism works for heterogenous agents, as long as every pair of agent's signals are positively correlated.

Both Kong and Schoenebeck [7] and Shnayder et al. [20] independently generalize this beyond the binary setting, though, as we will see, in slightly different manners.

*$\Phi$-mutual information mechanism.* Kong and Schoenebeck [7] provide a multi-task peer prediction mechanism for any finite signal space which is strongly truthful as long as the prior is "fine-grained" (it is truthful in any event). A prior is *fine-grained* if, roughly speaking, no two signals can be interpreted as different names for the same signal. Thus their mechanism works for rather diverse collections of heterogeneous agents. The mechanism pays agents the $\Phi$-mutual information (of which Shannon mutual information is a special case) of their strategy with another agent. Strategic behavior is shown to not increase $\Phi$-mutual information by a generalized version of the data

Table 1. Comparison between different mechanisms.

| | D&G [3] | CA [1, 20] | Φ-MIM [7] | Pairing Mechanism |
|---|---|---|---|---|
| Truthful | ✓ | ✓ | ✓ | ✓ |
| Informed-truthful | ✓ | ✓ | ✓ | ✓ |
| Strongly-truthful | ✓ | | ✓(fine-grained) | ✓ |
| Detail-free | positively correlated | ✓ | ✓ | ✓ |
| Samples (whp) | | $O(n)$ | | $O(\log n)$ |
| Signal space | binary | finite | finite | continuous |

The samples complexity is the number of samples (tasks) needed to have $\epsilon$-informed-truthful ($\epsilon$-strongly-truthful) with high probability when the number of agents is $n$.

processing inequality. While the Φ-mutual information can be measured in this setting, in general, doing so requires an infinite number of signals. Additionally, their analysis does not handle errors in estimation.

*Correlated Agreement Mechanism.* Shnayder et al. [20] also provide a multi-task peer prediction mechanism for any finite signal space. The CA mechanism (like the mechanism it generalizes [3]) explicitly uses a *scoring function* which maps a pair of reports to a score. At a high level, for each pair of agents $i$ and $j$, the mechanism chooses a bonus task, an agent $i$ penalty tasks, and an agent $j$ penalty task. The mechanism then pays agent $i$ the value of the scoring function with both agent $i$ and agent $j$'s reports from the bonus task minus the value of the scoring function applied to agent $i$'s report on the agent $i$ penalty task and agent $j$'s report on the agent $j$ penalty task. Specifically, the scoring function of mechanism in [3] is 1 if the pair of reports is the same and 0 otherwise. In this way, the mechanism rewards agents when their reports on a common bonus task agree more than would be expected based on their independent (uncorrelated) penalty tasks. To go beyond positive correlated priors, Shnayder et al. [20] take the sign of the correlation of $i$ and $j$'s signals as the scoring functions, which they call the *signal structure*. Therefore, the mechanism rewards agents when their reports on a bonus task disagree more than reports on penalty tasks if the signals are negatively correlated.

The Correlated Agreement mechanism is *informed truthful*, but not strongly truthful because agents need not distinguish between signals that share the same signal structure.

On the one hand, the CA mechanism can assume the knowledge of the "signal structure". In this case, their mechanism can be viewed as a special case of the aforementioned Φ-mutual information mechanism using the total variation distance mutual information. However, instead of directly computing this mutual information, mechanism obtains an unbiased estimator of it [7].

On the other hand, the CA mechanism can learn the signal structure, albeit with some chance of error. The mechanism is shown to be robust to this error, and the mechanism still is $\epsilon$-informed-truthful with high probability with $O(n)$ number of samples.

Agarwal et al. [1] exploit this robustness property to show the Correlated Agreement mechanism can handle heterogeneous agents with finite types, and establish that the sample complexity is linear in the number of agents.

### 1.1 Our Contributions

In this paper, we employ an innovative new technique on Φ-divergence and design a mechanism that is minimal detail-free on heterogeneous agents with the sample complexity being logarithmic in the number of agents, while providing an incentive guarantee of approximated informed truthfulness. In summary,

(1) Our mechanism is strongly truthful when the signal space is finite.
(2) We improve the analysis and derive sample complexity being logarithmic in the number of agents in heterogeneous agent when the signal space is finite
(3) Furthermore, we provide a new estimation algorithm with a logarithmic sample complexity which handles continuous or more complicated signal space.

We exploit the variational representation of $\Phi$-divergence and our mechanism can be seen as a regularized version Correlated Agreement mechanism. Given a scoring function and a convex function $\Phi$ with its convex conjugate $\Phi^*$, the mechanism pays agent $i$ the value of scoring function with the both reports of a bonus task minus the $\Phi^*$ of the scoring function with $i$'s report on a penalty task for $i$ and $j$'s report on a penalty task for $j$. This regularization yields strongly truthful when $\Phi$ is strongly convex and the distribution of $i$ and $j$'s signals are conditional nondegenerate, i.e. for agent $i$ the posterior after receiving two signal are not identical. Moreover, the expected payment under truthtelling and the ideal scoring function is the $\Phi$-mutual information between $i$ and $j$.

To estimate the scoring function in detail-free setting, we provide a generative method and discriminative method such that the mechanism is $\epsilon$-strongly truthful that is truthful reporting yields a higher expected payment (up to $\epsilon$ error) than any other non-permutation strategy with probability $1-\delta$ over $n$ agents. A *generative* approach learns the whole probability density function of prior and computes scoring function from it. This approach is applicable when the prior is over finite space, or it is in a parametric model. Agarwal et al. [1] use a generative approach and show the sample complexity is $O(n)$ for $\epsilon$-informed truthful if the agent can be clustered into finite types. We provide a more careful analysis and show the sample complexity can be logarithmic in the number of agents, $O\left(\frac{\log(n^2/\delta)}{\epsilon^2}\right)$ for $\epsilon$-strongly truthful even every pair of agents have distinct prior distribution. Moreover, by the property of variational representation of our mechanism, we formulates the estimation of scoring function into a convex optimization problem— empirical risk minimization which is well-studied in Nguyen et al. [14] and its follow-up. Instead of computing the probability density function, this *discriminative* approach estimates the scoring function directly. We show if the scoring function is the empirical risk minimizer with $O\left(\frac{\log(n^2/\delta)}{\epsilon^2}\right)$ samples, the mechanism is $\epsilon$-strongly truthful with $1-\delta$ probability. Since this discriminate approach does need to estimate the whole probability density function, this method can handle more complicated prior beyond finite space or parametric models. Moreover, the resulting optimization problem can be handled by using standard convex optimization solver. Finally, we provide an obstacle for designing exact informed truthful (strongly truthful) mechanism in the detail-free setting. Most of the previous mechanisms (implicitly) make the expected utility of truthtelling strategy equal to the $\Phi$-mutual information and use this property to show their mechanism is robust against manipulation. Thus under this framework, showing exact focal and informed truthful requires an *unbiased estimator* of $\Phi$-divergences from i.i.d. samples. We show such estimator does not exist.

### 1.2 Related Work

*Peer prediction.* Miller et al. [13] introduce the peer prediction mechanism which is the first mechanism that has truth-telling as a strict Bayesian Nash equilibrium and does not need verification. However, their mechanism requires the full knowledge of the common prior and there exist some equilibria that are paid more than truth-telling. In particular, the oblivious equilibrium pays strictly more than truth-telling. Kong et al. [5] modify the original peer prediction mechanism such that truth-telling pays strictly better than any other equilibrium but still requires the full knowledge of the common prior. Prelec [15] designs the first detail-free peer prediction mechanism—Bayesian truth serum (BTS). Several other works study the one-question setting of BTS [9, 16, 17, 23]

As mentioned, Dasgupta and Ghosh [3] consider the setting where agents are assigned a batch of a priori similar tasks, which allows agents to have different priors. Their work requires each agent's private information to be a binary signal. Kong and Schoenebeck [6], Shnayder et al. [21] independently extend Dasgupta and Ghosh [3]'s work to multiple-choice questions.

Liu and Chen [11, 12] relate mechanism design with learning by using the learning methods to design peer prediction mechanisms. In the setting where several agents are asked to label a batch of instances, Liu and Chen [11] design a peer prediction mechanism where each agent is paid according to her answer and a reference answer generated by a classification algorithm using other agents' reports. Liu and Chen [12] uses surrogate loss functions as tools to develop a multi-task mechanism that achieves truthful elicitation in dominant strategy when the mechanism designer only has access to agents' reports. Another work in the vein is Kong and Schoenebeck [10], which deals with the problem of forecast elicitation. Here the goal is to elicit forecasts about an event, and pay agents needing to wait for the outcome of the event. This paper also deals with the multi-task setting, and also uses Fenchel duality (though in a different manner) to reward truth-telling.

One interesting, but orthogonal, line of work looks at "cheap" signals. Here, the idea is that agents can coordinate on signals other than those the mechanism is trying to elicit. For example, instead of grading an assignment based on correctness, a grader could only spot check the grammar. ? ] introduces the problem and shows that nearly all peer-prediction mechanisms, in a particular way, fail to incentivize agents more than spot checking in the presents of cheap signals. Kong and Schoenebeck [8] show how this may be addressed by augmenting the signaling space.

## 2 PRELIMINARY

### 2.1 $\Phi$-divergence and convex conjugate

Informally, $\Phi$-divergences quantify the difference between a pair of distributions over a sample measurable space. In this paper, we only consider $\Phi$ to be a continuous function, and mostly consider the measurable space $(\Omega, \mathcal{F})$ where $\Omega$ is discrete and $\mathcal{F}$ is the power set of $\Omega$, and use $\Delta_\Omega$ to denote the set of distribution over the measurable space $(\Omega, \mathcal{F})$. Given a distribution $P \in \Delta_\Omega$, we denote $P(\omega)$ be the probability of outcome $\omega \in \Omega$.

**Definition 1** ($\Phi$-divergence and mutual information [7]). Let $\Phi : (0, \infty) \to \mathbb{R}$ be a convex function with $\Phi(1) = 0$. Let $P$ and $Q$ be two probability distributions on a common measurable space $(\Omega, \mathcal{F})$. The $\Phi$-*divergence* of $Q$ from $P$ with $P \ll Q$[1] is defined as

$$D_\Phi(P\|Q) \triangleq \mathbb{E}_Q\left[\Phi\left(\frac{dP}{dQ}\right)\right] = \int_\Omega \Phi\left(\frac{dP}{dQ}\right) dQ.$$

For technical reasons, we only consider $\Phi$ being defined at 0 with finite $\Phi(0)$, and set $0\Phi(0/0) = 0$.

If $\mathcal{Z}^2$ is a product of the measurable space $\mathcal{Z}$, and $P_{i,j}$ is a distribution over $(k, l) \in \mathcal{Z}^2$ with marginal distribution $P_i$ on the first component and $P_j$ on the second. We call $D_\Phi(P_{i,j}\|P_iP_i)$ the $\Phi$-*mutual information* between $k$ and $l$.

The $\Phi$-divergence over finite space can be written as

$$D_\Phi(P\|Q) \triangleq \mathbb{E}_{\omega\sim Q}\left[\Phi\left(\frac{P(\omega)}{Q(\omega)}\right)\right] = \sum_{\omega\in\Omega}\left[Q(\omega)\Phi\left(\frac{P(\omega)}{Q(\omega)}\right)\right].$$

The following are some common $\Phi$-divergences:

[1] $P$ is absolutely continuous with respect to $Q$, For all measurable set $A \in \mathcal{F}$, $Q(A) = 0 \Rightarrow P(A) = 0$.

- Total variation: $\Phi(x) = \frac{1}{2}|x-1|$,

$$\|P-Q\|_{TV} \triangleq \frac{1}{2}\mathbb{E}_Q[|P/Q-1|] = \frac{1}{2}\sum_{\omega\in\Omega}|P(\omega)-Q(\omega)|.$$

- Kullback-Leibler divergence: $\Phi(x) = x\log x$,

$$D_{KL}(P\|Q) \triangleq \mathbb{E}_Q\left[\frac{P}{Q}\log\frac{P}{Q}\right] = \mathbb{E}_{\omega\sim P}\left[\log\frac{P(\omega)}{Q(\omega)}\right].$$

- $\chi^2$-divergence: $\Phi(x) = (x-1)^2$ or $x^2-1$,

$$\chi^2(P\|Q) \triangleq \mathbb{E}_Q\left[(1-\sqrt{P/Q})^2\right] = \sum_{\omega\in\Omega}\frac{P(\omega)^2}{Q(\omega)} - 1.$$

- Squared Hellinger distance $\Phi(x) = (1-\sqrt{x})^2$.

Given a function $\Phi$ we denote the domain of $\Phi$ as $\mathrm{dom}(\Phi)$. Here we introduce some basic notions in convex analysis [19].

**Definition 2** (Convex conjugate)**.** Let $\Phi : (0,+\infty) \to \mathbb{R}$ be a convex function. The *convex conjugate* $\Phi^*$ of $\Phi$ is defined by:

$$\Phi^*(y) = \sup_{x\in\mathrm{dom}(\Phi)}[xy - \Phi(x)].$$

Moreover $\Phi = \Phi^{**}$ if $\Phi$ is a lower semicontinuous convex function.

Theorem 3 (Young-Fenchel inequality). *Given $x \in \mathrm{dom}(\Phi)$ for all $y \in \mathrm{dom}(\Phi^*)$,*

$$\Phi(x) \geq xy - \Phi^*(y),$$

*where the equality holds when $y \in \partial\Phi(x) = \{u : \Phi(z) \geq \Phi(x) + \langle u, z-x\rangle\}$, and $y = \Phi'(x)$ if $\Phi$ is convex and differential at $x$.*

Theorem 4 (Variational representation).

$$D_\Phi(P\|Q) = \sup_{k:\Omega\to\mathrm{dom}(\Phi^*)}\left\{\mathbb{E}_{\omega\sim P}[k(\omega)] - \mathbb{E}_{\omega\sim Q}[\Phi^*(k(\omega))]\right\}$$

*where $k$ is such that both expectations are finite, and the equality holds when $k(\omega) \in \partial\Phi\left(\frac{P(\omega)}{Q(\omega)}\right)$.*

This formulation is powerful. For example, it can yield the data processing inequality easily, and we attach a proof in the appendix.

**Corollary 5** (Data processing inequality)**.** *Consider a channel that produces $Y$ given $X$ based on the distribution $P_{Y|X}$ where $\Pr[Y|X] = P_{Y|X}$. Given distributions $P_X$ and $Q_X$ of $X$ and $P_{Y|X}$, $P_Y$ is the (marginal) distribution of $Y$ when $X$ is sampled from $P_X$ and $Q_Y$ is the distribution of $Y$ when $X$ is generated by $Q_X$, then for any $\Phi$-divergence $D_\Phi$,*

$$D_\Phi(P_X\|Q_X) \geq D_\Phi(P_Y\|Q_Y).$$

Here is a useful table for some standard $\Phi$s and their conjugate:

Table 2. Common $\Phi$, its convex conjugate and subgradient

| $\Phi$-divergence | $\Phi(x)$ | $\Phi^*(y)$ | $\partial\Phi(x)$ |
|---|---|---|---|
| Total variation | $\frac{1}{2}\lvert x-1\rvert$ | $\begin{cases} y, & \text{if } \lvert y\rvert \le 1/2 \\ +\infty, & \text{otherwise} \end{cases}$ | $\begin{cases} 1/2, & \text{if } x > 1 \\ -1/2, & \text{if } x < 1 \\ [-1/2, 1/2] & \text{if } x = 1 \end{cases}$ |
| KL-divergence | $x \log x$ | $\exp(y-1)$ | $1 + \log x$ |
| $\chi^2$-divergence | $x^2 - 1$ | $y^2/4$ | $2x$ |
| Squared Hellinger distance | $\left(1-\sqrt{x}\right)^2$ | $\frac{y}{1-y}$ | $1 - 1/\sqrt{x}$ |

### 2.2 Upper bounds for empirical processes

In this section, we provide some standard results on empirical process, most of which are in van de Geer and van de Geer [22]. Consider $n$ independent and identically (i.i.d.) random variables $X_1, X_2, \ldots, X_n$ with distribution $P$ on a measurable space $(\Omega, \mathcal{F})$. Let $\hat{P}_n$ be the *empirical distribution* based on those $n$ random variables, i.e., for each set $A \in \mathcal{F}$,

$$\hat{P}_n(A) = \frac{1}{n}\{\text{number of } X_i \in A, 1 \le i \le n\} = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}[X_i \in A] \tag{1}$$

and let $\mathcal{K} \subset L_2(P) = \{k : \Omega \to \mathbb{R} : \int |k|^2 dP < \infty\}$ be a collection of functions. The *empirical process indexed* by $\mathcal{K}$ is

$$V_n(\mathcal{K}) = \left\{ v_n(k) = \sqrt{n}\int kd(\hat{P}_n - P) : k \in \mathcal{K} \right\} \tag{2}$$

In this paper, we are mainly interested in uniform upper bound for Equation (2), i.e., upperbounds for

$$\sup_{k\in\mathcal{K}} |v_n(k)| \tag{3}$$

which can be think as the "radius" of random process (2). To upper bound (3), there are several notions for "complexity of functional spaces". Here are some examples. If $\Omega \subseteq \mathbb{R}$, the set of cumulative density functions is $\{k_x : k_x(\omega) = \mathbb{I}[\omega < x], x \in \mathbb{R}\}$, and the upperbound for (3) implies the Central Limit Theorem. We can consider a family of sets $\mathcal{A} \subseteq \mathcal{F}$ and a functional class over it $\{k_A : k_A(\omega) = \mathbb{I}[\omega \in A], A \in \mathcal{A}\}$, and the upper bound for (3) can be characterized by the *VC-dimension* of the family of sets $\mathcal{A}$. Or if $P$ is a distribution $d$-dimensional Gaussian and there is a set of linear functional $\{k_v : k_v(\omega) = v^\top \omega, \|v\|_2 \le 1\}$, we can use *metric entropy* to encode their complexity.

Now let us introduce some notions of functional complexity we used in the paper.

**Definition 6.** Given $k \in \mathcal{K}$, $L > 0$, and distribution $P$, we define

$$\rho_L^2(k, P) \triangleq 2L^2 \int \exp\left(\frac{|k|}{L}\right) - 1 - \frac{|k|}{L} dP$$

the *Bernstein difference* between $k_1$ and $k_2$ is then $\rho_L^2(k_1 - k_2, P)$ which can be seen as an extension of $L_2(P)$-norm, because $2(e^x - 1 - x) \approx x^2$ when $x$ is small.

**Definition 7** (Generalized entropy with bracketing)**.** Let $\mathcal{N}_{[],L}(\delta, \mathcal{L}, P)$ be the smallest value of $n$ for which there exists $n$ pairs of functions $\{(k_j^L, k_j^U)\}$ such that $\rho_L(k_j^U - k_j^L, P) \le \delta$ for all $j = 1, \ldots, n$ and such that for all $k \in \mathcal{K}$ there is a $j$ such that for all $\omega \in \Omega$

$$k_j^L(\omega) \le k(\omega) \le k_j^U(\omega).$$

Then $\mathcal{H}_{[],L}(\delta, \mathcal{K}, P) = \log \mathcal{N}_{[],L}(\delta, \mathcal{K}, P)$ is called the *generalized entropy with bracketing*.

The following theorem shows the random variable (3) is subgaussian when the generalized entropy with bracketing is bounded.

Theorem 8 (A uniform inequality [22]). *Given a functional class $\mathcal{K}$ and distribution $P$, if there exist constants $L$, $R$, $A$, $B$, and $C$ such that* $\sup_{k\in\mathcal{K}} \rho_L(k, P) \le R$ *and* $\int_0^R \sqrt{\mathcal{H}_{[],L}(u, \mathcal{K}, P)}du \le C$

$$\sqrt{(A+1)B^2}\,(\max\{R, C\}) \le \epsilon \le \frac{AR^2}{L}\sqrt{n}$$

*Then the empirical process $V_n(\mathcal{K})$ is bounded as*

$$\Pr\left[\sup_{k\in\mathcal{K}} |v_n(k)| \ge \epsilon\right] \le B\exp\left(-\frac{\epsilon^2}{(A+1)B^2R^2}\right).$$

### 2.3 Prior structure for multi-task peer prediction

Consider $n$ agents and $m$ tasks. Let $[n] = \{1, 2, \ldots, n\}$ be the set of agents and $[m]$ be the set of tasks. Each agent works on all the tasks, and receives a signal in the space $\mathcal{Z}$ (a finite set, integer, or real number, etc). We denote $Z_{i,s} \in \mathcal{Z}$ as the signal received by the worker $i \in [n]$ on the task $s \in [m]$, and use $\mathbf{Z} \in \mathcal{Z}^{n\times m}$ to denote the signal matrix (all the reviews) which is generated from a distribution $\mathbb{P}$ over $\mathcal{Z}^{n\times m}$. In this paper, we make the following assumption on the prior distribution $\mathbb{P}$.

**Assumption 1** (A priori similar and random order [3])**.** *$\mathbb{P}$ is a common prior, and it is identically and independently generated for each task: there exists a distribution $P$ over $\mathcal{Z}^n$ such that $\mathbb{P} = P^m$.*

$$\mathbb{P}(\mathbf{Z} = \mathbf{z}) = \prod_{s\in[m]} P(Z_{1,s} = z_{1,s}, \ldots, Z_{n,s} = z_{n,s}).$$

*Moreover, all questions appear in a random order, independently drawn for each agent.*

To simplify the notion, under Assumption 1, we use $P$ to denote the joint distribution on an arbitrary task $s$. Let $P_i$ and $P_j$ be the *marginal distributions* of signals agents $i$ and $j$ received on task $s$, and $P_{i,j}$ be the 2*-wise marginal distribution* of a pair of signals agent $i$ and $j$ received on the task $s$, $P_{i,j}(z, w) = P(Z_{i,s} = z, Z_{j,s} = w)$. This can be further extended to $k$*-wise marginal distribution*: given that $i_1, i_2, \ldots, i_k \in [m]$, $P_{i_1,i_2,\ldots,i_k}$ is the marginal distribution of signals agents $i_1, i_2, \ldots, i_k$ received. In general, we use uppercase for random object $Z$ and lowercase for the outcome $z$.

**Definition 9** (Mutually conditional degenerate)**.** We call a distribution $P$ over $\mathcal{Z}^n$ *mutually conditional degenerate* if and only if there exists $l \in [n]$ and $w, w' \in \mathcal{Z}$ such that $P_l(w) = 0$, or for all $z_1, z_2 \ldots, z_n \in \mathcal{Z}$,

$$\frac{P(z_1, \ldots, z_{l-1}, w, z_{l+1}, \ldots, z_n)}{P_l(w)} = \frac{P(z_1, \ldots, z_{l-1}, w', z_{l+1}, \ldots, z_n)}{P_l(w')}.$$

That is, there exists an agent $l$ such that its posterior on all other agents' signals with signal $w$ is identical to the posterior with signal $w'$.

Note that if the signals of all agents are mutually independent, it is also mutually conditional degenerate. It might be helpful to think the degeneracy as a weaker form of independence. Similar to $k$-wise independent, we can define $k$-wise conditional degeneracy.

**Definition 10** ($k$-wise conditional degenerate)**.** We call a distribution $P$ over $\mathcal{Z}^n$ *$k$-wise conditional degenerate* if and only if there exists a subset of $k$ agents $\{i_1, i_2, \ldots, i_k\} \subset [n]$ and $w, w' \in \mathcal{Z}$ such that $P_{i_1}(w) = 0$, or for all $z_2, z_3 \ldots, z_k \in \mathcal{Z}$,

$$\frac{P_{i_1,i_2,\ldots,i_k}(w, z_2, \ldots, z_k)}{P_{i_1}(w)} = \frac{P_{i_1,i_2,\ldots,i_k}(w', z_2, \ldots, z_k)}{P_{i_1}(w')}.$$

We further call the negation of above definitions as *mutually conditional nondegenerate* and *$k$-wise conditional nondegenerate* respectively. Note that if the signals of all agents are $k$-wise independent, it is also $k$-wise conditional degenerate. And a mutually conditional degenerate distribution over $\mathcal{Z}^n$ is also a $k$-wise conditional degenerate distribution for all $k \le n$.

### 2.4 Mechanism design for Information elicitation

**Definition 11** (Mechanism)**.** Given report profile of all agents $\mathbf{r} = (\mathbf{r}_1, \ldots, \mathbf{r}_n) \in \mathcal{Z}^{n\times m}$ where $\mathbf{r}_i \in \mathcal{Z}^m$ is the report vector of agent $i \in [n]$, an *information elicitation mechanism* $\mathcal{M} = (M_1, \ldots, M_n)$ rewards each agent $i$ with $M_i(\mathbf{r})$, where $M_i : \mathcal{Z}^{n\times m} \to \mathbb{R}$ is the payment function for agent $i$.

**Definition 12** (Strategy)**.** Given a mechanism $\mathcal{M}$, the *strategy* of each agent $i$ in the mechanism $\mathcal{M}$ is a mapping $\Theta_i : \mathcal{Z}^m \to \Delta_{\mathcal{Z}^m}$ from obtained signals $Z_i$ to a probability distribution over $\mathcal{Z}^m$, and a collection of agents' strategies $\mathbf{\Theta} := (\Theta_1, \Theta_2, \ldots, \Theta_n)$ is called a *strategy profile*.

Assumption 1 ensures agents cannot distinguish each question without the private signal they receive, and each agent's strategy is uniform across different tasks. Given this, the strategy of agent $i$ can be written as $\theta_i : \mathcal{Z} \to \Delta_Z$ where $\theta_i(z, r) = \Pr[R = r \mid Z = z]$, and

$$\Pr[\Theta_i(\mathbf{z}_i) = \boldsymbol{r}_i] = \prod_{s\in[m]} \theta_i(z_{i,s}, r_{i,s}).$$

That is, each report only depends on the corresponding signal and the strategy is uniform across tasks. We also call $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ as *strategy profile*, and the above assumption yields a bijection between $\Theta$ and $\theta$.

**Definition 13** (Utility and social welfare)**.** Given a mechanism $\mathcal{M}$, under a common prior $\mathbb{P}$, for a strategy profile $\mathbf{\Theta}$, the *prior expected utility* of agent $i$ is

$$u_i(\mathbb{P}, \boldsymbol{\theta}) \triangleq \mathbb{E}_{\mathbf{Z}\sim\mathbb{P}}\left[\mathbb{E}_{\mathbf{R}\sim\mathbf{\Theta}(\mathbf{Z})}\left[M_i(\mathbf{R})\right] \mid \mathbf{Z}\right],$$

### 2.5 Mechanism design goals for information elicitation

In the literature of information elicitation, we often consider three classes of strategies, truth-telling, permutation strategy, and informed strategy.

- A strategy $\theta$ is *truth-telling* if and only if $\theta(z, r) = \mathbb{I}[z = r]$, i.e. the report is equal to private signal. A strategy profile $\mathbf{\Theta}$ is *truth-telling* if each $\Theta_i$ is the truth-telling strategy, and we denote the truth-telling strategy profile as $\boldsymbol{\tau}$ for short.
- A *permutation strategy* is a deterministic strategy in which is a bijection between signals and reports. A strategy profile is a permutation if all strategy is a permutation.
- A strategy $\theta$ is *oblivious or uninformed* if it does not depend on the signal: for any $z$ ,$w$ and $r$ in $\mathcal{Z}$, we have $\theta(z, r) = \theta(w, r)$. We call the negation as an *informed strategy*. A strategy profile is *informed* if all strategy in it is not oblivious.

Note that the set of informed strategies contains the set of permutation strategies, and the set of permutation strategies contains the set of truth-telling strategies. We now define some goals for mechanism design for information elicitation which are different in the equality conditions.

**Truthful** A mechanism $\mathcal{M}$ is truthful if and only if the truth-telling strategy profile maximizes every agent's expected payment among all strategy profiles: for all $\boldsymbol{\theta}$ and $i$,$u_i(\mathbb{P}, \boldsymbol{\tau}) \geq u_i(\mathbb{P}, \boldsymbol{\theta})$.

**Informed-truthful** A mechanism $\mathcal{M}$ is informed-truthful if and only if the truth-telling strategy profile maximizes every agent's expected payment among all strategy profiles and the equality may only occur when no strategy is oblivious.

**Strongly-truthful** A mechanism $\mathcal{M}$ is strongly-truthful if and only if the truth-telling strategy profile maximizes every agent's expected payment among all strategy profiles and the equality may only occur when strategy profile is a permutation.

In this work, we consider the approximation version of above statements, for example, a mechanism is *$\epsilon$-strongly-truthful* if only if the truth-telling strategy profile maximizes every agent's expected payment among all strategy profiles up to $\epsilon$ additive error and if a strategy profile have more than $\epsilon$ utility for one agent only if it is a permutation.

## 3 Φ-DIVERGENCE PAIRING MECHANISMS

We define a class of multi-task peer-prediction mechanisms that is parametrized by a convex function $\Phi$ and a scoring function $\mathbf{K} = \{K_{i,j}\}_{i,j\in[n]}$ which maps a pair of reports into a score. For simplicity, we consider all $n$ agents work on all the tasks $m$.

The process of this mechanism is quite simple. For each agent $i$, we pick a corresponding peer agent $j$ and choose three distinct tasks $b, p$ and $q$. Agent $i$ get pay by the scoring function on $i$ and $j$'s reports on task $b$ minus the $\Phi^*$ of the scoring function on $i$'s report on $p$ and $j$'s report on $q$. In this way, agents are paid by a scoring function on a correlated task minus a regularized scoring function on two uncorrelated tasks. We will later see what is the role of convex function $\Phi$ in these mechanisms.

**ALGORITHM 1:** Φ-divergence pairing mechanism, $\mathcal{M}(\mathbf{r}, \mathbf{K}, \Phi)$

**Input** : A convex function $\Phi : (0, \infty) \to \mathbb{R}$ and its conjugate $\Phi^*$ defined in Definition 1 and 2;
a set of scoring functions $\mathbf{K} = \{K_{i,j}\}_{i,j\in[n]}$ over all pair of agents where $K_{i,j} : \mathcal{Z}^2 \to \mathrm{dom}(\Phi^*) \subseteq \mathbb{R}$;
a report profile $\mathbf{r} \triangleq (\mathbf{r}_1, \dots, \mathbf{r}_n)$ where each agent $i \in [n]$ submits a report $\mathbf{r}_i \in \mathcal{Z}^m$ for all $m$ tasks.

**Output**: Payment function, $\mathbf{M}(\mathbf{K}; \Phi) = (M_1, M_2, \dots, M_2)$

1 For each agent $i$, choose a *peer agent* $j$ uniformly at random from $[n] \setminus \{i\}$, and pick three distinct tasks $b, p$, and $q$ where $b$ is the *bonus task*, $p$ is the *penalty task to* $i$, and $q$ is *the penalty task to* $j$.

2 Based on the agent $i$'s reports $r_{i,b}, r_{i,p}$ and the peer agent $j$ 's reports $r_{j,b}, r_{j,q}$, the reward for agent $i$ is

$$M_i(\mathbf{r}) = K_{i,j}\left(r_{i,b}, r_{j,b}\right) - \Phi^*\left(K_{i,j}\left(r_{i,p}, r_{j,q}\right)\right). \tag{4}$$

In the discussion of Mechanism 1, we use $u_i(\mathbb{P}, \boldsymbol{\theta}, \mathbf{K}; \Phi)$ to denote the prior expected utility of agent $i$ in the Φ-divergence pairing mechanism under a common prior $\mathbb{P}$, strategy profile $\boldsymbol{\theta}$, and a scoring function $\mathbf{K}$.

Theorem 14 (Pairing Mechanism). *Given $n \geq 2$ and $m \in \mathbb{N}$, let $\mathbb{P}$ be a common prior satisfies Assumption 1 over a signal space $\mathcal{Z}$ with $P$ being 2-wise conditional nondegenerate. Let $\boldsymbol{\theta}$ be a strategy profile, and $\boldsymbol{\tau}$ be the truth-telling strategy profile;* $\mathbf{H}, \mathbf{K}$ *be the scoring functions;* $\Phi$ *is a low semi-continuous convex function with* $(0, \infty) \subseteq \mathrm{dom}(\Phi)$. *If for all $i, j \in [n]$ $i \neq j$ and $x, y \in \mathcal{Z}$,*

$$H_{i,j}(x, y) \in \partial\Phi\left(\frac{P_{i,j}(x, y)}{P_i(x)P_j(y)}\right), \tag{5}$$

*the Mechanism 1 is informed-truthful, that is for all $\boldsymbol{\theta}$ and* $\mathbf{K}$, [2]

$$u_i(\mathbb{P}, \boldsymbol{\tau}, \mathbf{H}; \Phi) \geq u_i(\mathbb{P}, \boldsymbol{\theta}, \mathbf{K}; \Phi). \tag{6}$$

*and* $u_i(\mathbb{P}, \boldsymbol{\theta}, \mathbf{K}; \Phi) \leq 0$ *when* $\theta_i$ *is an oblivious strategy profile.*

*Moreover, if*

*(1)* $\mathcal{Z}$ *is a finite set,*

*(2)* $\Phi$ *is strictly convex and differentiable and* $\Phi^*$ *is strictly convex,*

*the Mechanism 1 is strongly-truthful: the equality in* (6) *for some $i$ may only occur if and only if*

*(1)* $\boldsymbol{\theta} = (\pi_1, \pi_2, \dots, \pi_n)$ *where* $(\pi_i)$ *are permutations over* $[m]$*, and*

*(2) For all* $i, j \in [n]$ $i \neq j$ *and* $x, y \in \mathcal{Z}$, $K_{i,j}(\pi_i(x), \pi_j(y)) = \Phi'\left(\frac{P_{i,j}(x,y)}{P_i(x)P_j(y)}\right)$. [3]

The proof has two parts. In Section 3.1, we show the mechanism is informed-truthful, and in Section 3.2 we show strongly-truthful with above additional assumptions.

Note that the prior expected payment consists of 1) the randomness of the choice of peer agent $j$ in mechanism, 2) the randomness of signal profile $Z$ and 3) the randomness of strategy profile $\boldsymbol{\theta}$. We define $u_{i;j}$ to be $i$'s prior expected payment given $j$ being its peer agent, and

$$u_i = \mathbb{E}_{j \sim [n] \setminus i}\left[u_{i;j}\right] = \frac{1}{n-1}\sum_{j \neq i} u_{i;j}.$$

## 3.1 Proof of Theorem 14: informed-truthful

To show the $\Phi$-divergence pairing mechanism is informed truthful, it is sufficient to show: for each pair of agents $i$ and $j$

- an upper bound for $u_{i;j}(\mathbb{P}, \boldsymbol{\theta}, \mathbf{K}; \Phi)$ for all $\boldsymbol{\theta}$ and $\mathbf{K}$ ( Lemma 15);
- $u_{i;j}(\mathbb{P}, \boldsymbol{\tau}, \mathbf{H}; \Phi)$ matches the upper-bound (Lemma 16);
- $u_{i;j}(\mathbb{P}, \boldsymbol{\theta}, \mathbf{K}; \Phi) \leq 0$ when $\theta_i$ is an oblivious strategy profile ( Lemma 17).

**Lemma 15** (Manipulation in report and learning)**.** *Given a proper convex function* $\Phi$*, prior* $\mathbb{P}$*, scoring functions* $\mathbf{K}$*, strategy profile* $\boldsymbol{\theta}$*, agent* $i \in [n]$*, and its peer agent* $j$*, the agent* $i$*'s prior expected payment given peer agent* $J$ *is*

$$u_{i;j}(\mathbb{P}, \boldsymbol{\theta}, \mathbf{K}; \Phi) \leq D_\Phi(P_{i,j} \| P_i P_j).$$

There are two aspects of manipulation: the reports for bonus and penalty tasks and the scoring functions $\mathbf{K}$. The first one can be handled though Data Processing inequality and the second is shown through the variational representation of $\Phi$-divergence.

Proof. Without loss of generality, we consider the signal set $\mathcal{Z}$ is countable. The expected utility for agent $i$ is

$$\begin{aligned}
&u_{i;j}(\mathbb{P}, \boldsymbol{\theta}, \mathbf{K}; \Phi)\\
=&\mathbb{E}_{\mathbf{Z}\sim\mathbb{P}}\left[\mathbb{E}_{\mathbf{R}\sim\Theta(\mathbf{Z})}\left[K_{i,j}\left(R_{i,b}, R_{j,b}\right) \mid \mathbf{Z}\right]\right] - \mathbb{E}_{\mathbf{Z}\sim\mathbb{P}}\left[\mathbb{E}_{\mathbf{R}\sim\Theta(\mathbf{Z})}\left[\Phi^*\left(K_{i,j}\left(R_{i,p}, R_{j,q}\right)\right) \mid \mathbf{Z}\right]\right]\\
=&\mathbb{E}_{\mathbf{Z}\sim\mathbb{P}}\left[\sum_{k,l\in\mathcal{Z}}\theta_i(Z_{i,b},k)\theta_j(Z_{j,b},l)K_{i,j}(k,l)\right] - \mathbb{E}_{\mathbf{Z}\sim\mathbb{P}}\left[\sum_{k,l\in\mathcal{Z}}\theta_i(Z_{i,p},k)\theta_j(Z_{j,q},l)\Phi^*\left(K_{i,j}(k,l)\right)\right]\\
=&\sum_{z,w\in\mathcal{Z}}P_{i,j}(z,w)\sum_{k,l\in\mathcal{Z}}\theta_i(z,k)\theta_j(w,l)K_{i,j}(k,l) - \sum_{z,w\in\mathcal{Z}}P_i(z)P_j(w)\sum_{k,l\in\mathcal{Z}}\theta_i(z,k)\theta_j(w,l)\Phi^*\left(K_{i,j}(k,l)\right).
\end{aligned}$$

[2]There are some minor details when $\mathcal{Z}$ is not finite. Here we require $\mathbf{K}$ and $\boldsymbol{\theta}$ to have finite $\int K_{i,j} dP_{i,j}$, $\int \Phi^*(K_{i,j}) d(P_i P_j)$, $\int K_{i,j} d\theta_i d\theta_j dP_{i,j}$ ,and $\int \Phi^*(K_{i,j}) d\theta_i d\theta_j dP_i P_j$ for all $i$ and $j$.

[3]This results show if a pair of strategy profile and scoring function $(\boldsymbol{\theta}, \mathbf{K})$ have (6) equal only if there is a homomorphic structure between strategy and scoring function

The last equality uses the fact that $P_{i,j}$ is the joint distribution of signals on bonus task $b$, $(Z_{i,b}, Z_{j,b})$ , and $P_iP_j$ is the joint distribution of signals on penalty tasks $p$ and $q$, $(Z_{i,p}, Z_{j,q})$. Because $\Phi^*$ is convex and for all $z, w \in \mathcal{Z}$, $\theta_i(z,k)\theta_j(w,l)$ is a distribution over $\mathcal{Z}^2$, by Jensen's inequality we have for all $z, w \in \mathcal{Z}$,

$$\sum_{k,l\in\mathcal{Z}} \theta_i(z,k)\theta_j(w,l)\Phi^*\left(K_{i,j}(k,l)\right) \leq \Phi^*\left(\sum_{k,l\in\mathcal{Z}} \theta_i(z,k)\theta_j(w,l)K_{i,j}(k,l)\right) \tag{7}$$

where the equality holds only if $\Phi^*$ is not strictly convex or $K_{i,j}(k,l)$ is constant in the support of $\theta_i(z,k)\theta_j(w,l)$. Let $L(z,w) \triangleq \sum_{k,l\in\mathcal{Z}} \theta_i(z,k)\theta_j(w,l)K_{i,j}(k,l)$. Apply (7) to $u_{i;j}(\mathbb{P}, \boldsymbol{\theta})$ and we have

$$\begin{aligned}
u_{i;j}(\mathbb{P}, \boldsymbol{\theta}, \mathbf{K}; \Phi) \leq & \sum_{z,w\in\mathcal{Z}} P_{i,j}(z,w)L(z,w) - \sum_{z,w\in\mathcal{Z}} P_i(z)P_j(w)\Phi^*\left(L(z,w)\right) \\
\leq & \sup_{L:\mathcal{Z}^2\to\mathbb{R}} \left\{\sum_{z,w\in\mathcal{Z}} P_{i,j}(z,w)L(z,w) - \sum_{z,w\in\mathcal{Z}} P_i(z)P_j(w)\Phi^*\left(L(z,w)\right)\right\} \qquad (8) \\
= & D_\Phi(P_{i,j}\|P_iP_j).
\end{aligned}$$

The last inequality holds by Theorem 4, and it completes the proof. □

The proof of the following lemmas are straightforward, and they are in the appendix.

**Lemma 16** (Truth-telling). *Let $i$ and $j$ be distinct agents in $[n]$; $\mathbf{H}$ be the scoring functions; $\Phi$ is a proper convex function. If $H_{i,j}$ satisfies Equation* (5),

$$H_{i,j}(x,y) \in \partial\Phi\left(\frac{P_{i,j}(x,y)}{P_i(x)P_j(y)}\right) \textit{ for all}, x, y \in \mathcal{Z},$$

*the expected payment of truth-telling strategy profile is the $\Phi$-divergence of $P_iP_j$ from $P_{i,j}$,*

$$u_{i;j}(\mathbb{P}, \boldsymbol{\tau}, \mathbf{H}; \Phi) = D_\Phi(P_{i,j}\|P_iP_j) > 0.$$

**Lemma 17** (Oblivious strategy). *Let $i$ and $j$ be distinct agents in $[n]$; $\mathbf{K}$ be a scoring functions; $\Phi$ is a convex function on $[0,\infty)$. If $\theta_i$ is a oblivious strategy profile,*

$$u_{i;j}(\mathbb{P}, \boldsymbol{\theta}, \mathbf{K}; \Phi) \leq 0.$$

## 3.2 Proof of Theorem 14: strongly-truthful

To show the $\Phi$-divergence pairing mechanism is strongly-truthful, we need to prove a necessary condition for equality in Lemma 15. We defer the proof to the appendix.

**Lemma 18** (An iff condition for the equality in Lemma 15). *Given a convex function $\Phi$, prior $\mathbb{P}$, a scoring function $\mathbf{K}$, a strategy profile $\boldsymbol{\theta}$, a agent $i \in [n]$, and its peer agent $j$, suppose*

*(1) $\mathcal{Z}$ is a finite set;*
*(2) $P_{i,j}$ is mutually conditional nondegenerate;*
*(3) $\Phi$ is strictly convex and differentiable and $\Phi^*$ is strictly convex.*

*Then $u_{i;j}(\mathbb{P}, \boldsymbol{\theta}, \mathbf{K}; \Phi) = D_\Phi(P_{i,j}\|P_iP_j)$ if and only if*

*(1) $\theta_i = \pi_i$ and $\theta_j = \pi_j$ which are permutations over $[m]$*
*(2) For all $x, y \in \mathcal{Z}$, $K_{i,j}(\pi_i(x), \pi_j(y)) = \Phi'\left(\frac{P_{i,j}(x,y)}{P_i(x)P_j(y)}\right)$.*

### 3.3 Proof of Theorem 14: putting things together

Proof of Theorem 14. Note that the peer agent is chosen uniformly at random from $[n] \setminus i$.

$$
\begin{aligned}
u_i(\mathbb{P}, \boldsymbol{\tau}, \mathbf{H}; \Phi) &= \frac{1}{n-1} \sum_{j \neq i} u_{i;j}(\mathbb{P}, \boldsymbol{\tau}, \mathbf{H}; \Phi) \\
&= \frac{1}{n-1} \sum_{j \neq i} D_\Phi(P_{i,j} \| P_i P_j) && \text{(by Lemma 16)} \\
&\geq \frac{1}{n-1} \sum_{j \neq i} u_{i;j}(\mathbb{P}, \boldsymbol{\theta}, \mathbf{K}; \Phi) && \text{(by Lemma 15)} \\
&\geq u_i(\mathbb{P}, \boldsymbol{\theta}, \mathbf{K}; \Phi)
\end{aligned}
$$

which prove the $\Phi$-divergence pairing mechanism is truthful. Additionally with Lemma 17, we prove the mechanism is informed-truthful.

To show the mechanism is strongly-truthful, by Lemma 15, if (6) is equality for some $i$, for all $j \neq i$

$$u_{i;j}(\mathbb{P}, \boldsymbol{\tau}, \mathbf{H}; \Phi) = u_{i;j}(\mathbb{P}, \boldsymbol{\theta}, \mathbf{K}; \Phi).$$

Because $P$ is 2-wise conditional nondegenerate, for all $i, j$ pair $P_{i,j}$ is mutually conditional nondegenerate. We can apply Lemma 18 and complete the proof. □

## 4 ESTIMATION AND ERROR

By Theorem 14, we know the prior expected payment of truth-telling strategy profile has several desirable properties if a scoring function $\mathbf{K}$ satisfies (5) which we call it as the *ideal scoring function*. However, how to derive such $\mathbf{K}$ is a big question.

In contrast to known prior $P$ setting which (5) can be computed analytically, in detail-free setting, the mechanism needs to estimate the scoring function from the report of agents. Fortunately, in Theorem 14, Equation (6) ensures misreporting signal against estimating $\mathbf{K}$ cannot make the prior expected payment greater than honest report. In this section we want to show a stronger property: *if the estimation* $\mathbf{K}$ *is "close" to the ideal one, the prior expected payment of truth-telling strategy profile "approximately" has several desirable properties in the pairing mechanism.*

In this section, we first state our framework and goal for learning algorithm in Section 4.1. Then we provide two different types of learning algorithms to estimate the scoring functions $\mathbf{K}$, *generative approach* and *discriminative approach* in Section 4.2 and 4.3. Then in Section 4.4, we show some barrier to have exact strong-truthfulness in detail-free setting.

### 4.1 Framework for estimation and the pairing mechanism

With (6), we can "plug-in" standard learning algorithm into the pairing mechanism. Given a learning algorithm which will be defined later and a pairing mechanism, we first partition the tasks into two parts: scoring tasks and learning tasks. Then we use the learning algorithm to estimate scoring function from reports on learning tasks, and run the pairing mechanism on reports on scoring tasks and the estimated scoring function.

Given a pairing mechanism $\mathcal{M}(\cdot, \cdot, \Phi)$, prior $\mathbb{P}$ and a constant $\epsilon > 0$, we say the scoring functions $\mathbf{K}$ to be *$\epsilon$-accurate* for the pairing mechanism, if for all pair of distinct agents $i, j \in [n]$

$$u_{i;j}(\mathbb{P}, \boldsymbol{\tau}, \mathbf{K}; \Phi) \geq D_\Phi(P_{i,j} \| P_i P_j) - \epsilon. \tag{9}$$

We define a *learning algorithm for scoring functions with $\Phi$ on $n$ agents and $m_L$ samples*, $\mathcal{L}$ as a function from $\mathbf{r}^L \in \mathcal{Z}^{n \times m_L}$ and $\Phi$ to a scoring function $\mathbf{K}$. We say such learning algorithm $\mathcal{L}$ is

$(\epsilon, \delta)$*-accurate* for the pairing mechanism over prior family $\mathcal{P}$, if for all $P \in \mathcal{P}$

$$\Pr_{\mathbf{Z}^L \sim P^{m_L}} \left[\forall i, j,\ u_{i;j}(\mathbb{P}, \tau, \mathcal{L}(\mathbf{Z}^L, \Phi); \Phi) > D_\Phi(P_{i,j} \| P_i P_j) - \epsilon\right] \geq 1 - \delta.$$

That is given random signals $\mathbf{Z}^L \in \mathcal{Z}^{n \times m_L}$, the probability that $\mathbf{K}_{\text{est}} = \mathcal{L}(Z^L, \Phi)$ is $\epsilon$-accurate for all pair of distinct agent is greater than $1 - \delta$.

Now we define our framework.

**ALGORITHM 2:** Framework of estimation and $\Phi$-divergence pairing mechanism

**Input:** A convex function $\Phi$, a set of worker $[n]$, a set tasks $[m]$, a learning algorithm $\mathcal{L}$ on $n$ agents and $m_L$ samples, and the pairing mechanism $\mathcal{M}$

1. Partition $m$ tasks into set of learning tasks $M_L$ and set of scoring tasks $M_S$ such that $M^L \cup M^S = [m]$. Let $m_L = |M_L|$ and $m_S = |M_S|$. For each agent $i \in [n]$.
2. Each agent $i \in [n]$ submits reports $\mathbf{r}_i \in \mathcal{Z}^m$ for all tasks which consists of reports on learning tasks $\mathbf{r}_i^L$ and on score tasks $\mathbf{r}_i^S$. Let $\mathbf{r}^L = (\mathbf{r}_1^L, \ldots, \mathbf{r}_n^L) \in \mathcal{Z}^{n \times m_L}$ be the report of all agents on learning tasks, and $\mathbf{r}^S = (\mathbf{r}_1^S, \ldots, \mathbf{r}_n^S) \in \mathcal{Z}^{n \times m_S}$
3. Run the learning algorithm and derive $\mathbf{K}_{\text{est}} = \mathcal{L}(\mathbf{r}^L, \Phi)$.
4. Run the pairing mechanism and let $\mathbf{M} = \mathcal{M}(\mathbf{r}^S, \mathbf{K}_{\text{est}}, \Phi)$, and pay each agent $i$ with $M_i$.

It is straightforward have the following theorem.

Theorem 19 (Detailed-free). *Let* $\Phi$ *be a low semi-continuous convex function with* $(0, \infty) \subseteq \text{dom}(\Phi)$*;* $\epsilon > 0$ *and* $\delta > 0$*;* $\mathcal{P}$ *be family of common priors satisfies Assumption 1 over a signal space* $\mathcal{Z}$ *with* $n \geq 2$ *and* $m \in \mathbb{N}$*;* $\mathcal{M}$ *be the pairing mechanism (Mechanism 1) and* $\mathcal{L}$ *be a* $(\epsilon, \delta)$*-accurate for* $\mathcal{M}$ *and* $\mathcal{P}$*. Then with probability* $1 - \delta$*, Mechanism 2 is* $\epsilon$*-informed-truthful.*

*Moreover, if*

*(1)* $\mathcal{Z}$ *is a finite set;*
*(2) For all* $P \in \mathcal{P}$ *is* 2*-wise conditional nondegenerate;*
*(3)* $\Phi$ *is strictly convex and differentiable and* $\Phi^*$ *is strictly convex,*

*the Mechanism 2 is* $\epsilon$*-strictly strongly-focal with probability* $1 - \delta$*.*

## 4.2 Generative approach

Recall that if the prior $P \in \mathcal{P}$ is known, the ideal scoring function can be computed directly. In a generative approach, we try to estimate the probability density function $P$ and derive the scoring afterward. In general, this generative approach is applicable when the family of prior $\mathcal{P}$ is over finite spaces, or $\mathcal{P}$ is a parametric model. Here we provide a result of a generative approach.

A standard way of learning probability density function is to use empirical distribution on $m_L$ samples defined in Equation (1). The following theorem shows that the empirical distribution gives a pretty good estimation in terms of total variation distance.

**Lemma 20** (Theorem 3.1 in [4]). *Given* $\epsilon > 0$*,* $\delta > 0$*, a finite domain* $\Omega$*, any distribution in* $P$ *is learnable with total variation distance less than* $\epsilon$ *with probability at least* $1 - \delta$*, with* $m_L = O\left(\frac{1}{\epsilon^2} \max(|\Omega|, \log(1/\delta))\right)$ *independent samples from* $P$*. Specifically, the empirical risk distribution* $\hat{P}_{m_L}$ *satisfies*

$$\Pr[\|P - \hat{P}_{m_L}\|_{TV} \leq \epsilon] \geq 1 - \delta$$

*where* $\|P - \hat{P}_N\|_{TV} = \sum_{\omega \in \Omega} |P(\omega) - \hat{P}_N(\omega)|$ *is the total variation distance between* $P$ *and* $\hat{P}_N$*.*

Therefore, we can estimate all 2-wise marginal distribution of $P$ by their empirical distributions and take union bound over $\binom{n}{2}$ pairs. The following algorithm and theorem formalize this idea.

**ALGORITHM 3:** A generative algorithm

**Input** : A convex function $\Phi$ and its sub-gradient $\partial\Phi$, a report profile $\mathbf{r} \in \mathcal{Z}^{n \times m_L}$ over a set of agents $[n]$ and a set of learning tasks $[m_L]$

**Output**: Scoring function, $\hat{\mathbf{K}} = (\hat{K}_{i,j})_{i,j \in [n]}$

1 For all pair of distinct agents $i$ and $j$, use the report profile $\mathbf{r}$ to compute the 2-wise empirical distribution, for all event $A$ in $\mathcal{Z}^2$

$$\hat{P}_{i,j}(A) = \frac{1}{m_L} \sum_{s=1}^{m_L} \mathbb{I}[(r_{i,s}, r_{j,s}) \in A],$$

and for all agent $i$, compute the marginal empirical distribution, for all event $B$ in $\mathcal{Z}$

$$\hat{P}_i(A) = \frac{1}{m_L} \sum_{s=1}^{m_L} \mathbb{I}[r_{i,s} \in B].$$

2 For all pair of distinct agents $i$ and $j$, and for all $z, w \in \mathcal{Z}$, compute the scoring function as

$$\hat{K}_{i,j}(z, w) \in \partial\Phi\left(\frac{\hat{P}_{i,j}(z, w)}{\hat{P}_i(z)\hat{P}_j(w)}\right), \tag{10}$$

and set $\hat{K}_{i,j}(z, w) = 0$ if $\hat{P}_{i,j}(z, w) = \hat{P}_i(z)\hat{P}_j(w) = 0$.

Theorem 21. *Given $\Phi$ is a convex function and a prior $P$ over $\mathcal{Z}^n$, suppose there exist constants $0 < \alpha < 1$ and $c_L$ such that*

$$\forall i, j \in [n] \text{ and } z, w \in \mathcal{Z},\ P_{i,j}(z, w) > 2\alpha \text{ or } P_{i,j}(z, w) = 0, \tag{11}$$

$$\forall x, y \in [\alpha, 1/\alpha],\ |\Phi(x) - \Phi(y)| \le c_L |x - y|. \tag{12}$$

*Then Algorithm 3 is $(\epsilon, \delta)$-accurate on prior $P$ for some $m_L = O\left(\frac{c_L^2}{\alpha^4 \epsilon^2} \max\left(|\mathcal{Z}|^2, \log \frac{n^2}{\delta}\right)\right)$.*

The first condition says the smallest nonzero probability $P_{i,j}(z, w)$ does not converge to zero as the number of agents increases, and the second condition requires the function $\Phi$ is Lipschitz in $[\alpha, 1/\alpha]$ which holds for all examples in Table 2. With these conditions, we can estimate the ideal scoring function accurately with sample complexity depends on the logarithmic of the number of agents. Note that this theorem can handle the heterogeneity of agent, and output scoring functions to every pairs of agents.

### 4.3 Discriminative approach

Instead of density estimation, a discriminative approach estimates the scoring functions directly. This enables more freedom of algorithm design on more complicated prior. The variational representation (Theorem 4) gives an optimization characterization of the ideal scoring functions.

Given the assumption 1, each reports are i.i.d random variables. We can have i.i.d. samples of $(x, y)$ where $x \in \mathcal{Z}^2$ is sampled from $P_{i,j}$ and $y \in \mathcal{Z}^2$ is sampled from $P_i P_j$ independently, and this is shown formally in Algorithm 4. Taking $L_\Phi(x, y) = x - \Phi^*(y)$ as the risk function, we can "estimate" the ideal scoring functions by empirical risk minimization (maximization) over a training set $(x_s, y_s)$ with $s = 1, 2, \ldots, m_L/3$,

$$\tilde{K}_{i,j} = \arg\max_{k \in \mathcal{K}} \sum_s L_\Phi(k(x_s), k(y_s)) = \arg\max_{k \in \mathcal{K}} \left\{\int k(\omega) d\hat{P}_{i,j}(\omega) - \int \Phi^*(k(\omega)) d\hat{P_i P_j}(\omega)\right\}$$

where $\mathcal{K}$ is a pre-specified class of functionals $k : \mathcal{Z}^2 \to \mathbb{R}$.

Assuming that $\mathcal{K}$ is a convex set of functionals, the implementation of (15) only requires solving a convex optimization problem over function space $\mathcal{K}$ which is well studied [14]. Therefore, in this section, we are going to show *the empirical risk maximizer* $\tilde{\mathbf{K}}$ *with respect to* $L_\Phi$ *is* $\epsilon$*-accurate for our* $\Phi$*-divergence pairing mechanism* with high probability under some conditions on $\mathcal{K}$ and prior $P$. Alternatively, this can be seen as the generalized error of the empirical risk maximizer.

**ALGORITHM 4:** An empirical risk minimization algorithm

**Input :** A convex function $\Phi$ and its conjugate $\Phi^*$, a family of functions $\mathcal{K} = \{K : \mathcal{Z}^2 \to \mathbb{R}\}$, a report profile $\mathbf{r} \in \mathcal{Z}^{n\times m_L}$ over a set of agents $[n]$ and a set of learning tasks $[m_L]$

**Output :** Scoring function, $\tilde{\mathbf{K}} = (\tilde{K}_{i,j})_{i,j\in[n]}$

1. Partition the report profile into 3 equal size $\mathbf{r}^0, \mathbf{r}^1$, and $\mathbf{r}^2$ in $\mathcal{Z}^{n\times m_L/3}$.
2. For each pair of distinct agents $(i, j)$ use the report profile $\mathbf{r}^0$ to compute the 2-wise empirical distribution, for all event $A$ in $\mathcal{Z}^2$

$$\tilde{P}_{i,j}(A) = \frac{3}{m_L}\sum_{s=1}^{m_L/3} \mathbb{I}[(r^0_{i,s}, r^0_{j,s}) \in A], \tag{13}$$

Further compute the product empirical distributions:[a] for all event $A$ in $\mathcal{Z}^2$

$$\tilde{P}_i\tilde{P}_j(A) = \frac{3}{m_L}\sum_{s=1}^{m_L/3} \mathbb{I}[(r^1_{i,s}, r^2_{i,s}) \in A] \tag{14}$$

Finally for all $z, w \in \mathcal{Z}$, the compute scoring function as

$$\tilde{K}_{i,j} = \arg\max_{k\in\mathcal{K}} \left\{ \int k(\omega) d\tilde{P}_{i,j}(\omega) - \int \Phi^*(k(\omega)) d\tilde{P}_i\tilde{P}_j(\omega) \right\} \tag{15}$$

[a]Note that $\tilde{P}_i\tilde{P}_j$ be a distribution over $\mathcal{Z}^2$. Note that we use new samples to compute the product of empirical distribution to ensure the independence between $\tilde{P}_i\tilde{P}_j$ and $\tilde{P}_{i,j}$

Theorem 22. *Consider a distribution $P$ over $\mathcal{Z}^n$; a strictly convex and a twice differentiable function $\Phi$ on $[0,\infty)$ with its gradient $\Phi'$ and conjugate $\Phi^*$; a family of functional $\mathcal{K}$ from $\mathcal{Z}^2$ to $\mathrm{dom}(\Phi^*)$; and $\Phi^*(\mathcal{K}) = \{\Phi^*(k) : k \in \mathcal{K}\}$. If*

*(1) for all distinct pairs $i$ and $j$, the functional $H_{i,j} = \Phi'\left(\frac{P_{i,j}}{P_iP_j}\right)$ is in $\mathcal{K}$.*

*(2) there exist constants $(L_l, R_l, D_l)_{l=1,2}$ such that for all distinct pairs $i$ and $j$*

*(a) $\sup_{k\in\mathcal{K}} \rho_{L_1}(k, P_{i,j}) \le R_1$, and $\int_0^{R_1} \sqrt{\mathcal{H}_{[],L_1}(u, \mathcal{K}, P_{i,j})}du \le D_1$*

*(b) $\sup_{l\in\Phi^*(\mathcal{K})} \rho_{L_2}(l, P_iP_j) \le R_2$ and $\int_0^{R_2} \sqrt{\mathcal{H}_{[],L_2}(u, \Phi^*(\mathcal{K}), P_iP_j)}du \le D_2$*

*Then Algorithm 3 is $(\varepsilon, \delta)$-accurate on prior $P$ for some $m_L = O\left(\frac{1}{\varepsilon^2}\log\frac{n^2}{\delta}\right)$.* [4]

Informally, Theorem 22 requires the functional class $\mathcal{K}$ contains the ideal scoring function and it has constant complexity (generalized entropy with bracketing). Under these conditions, the empirical risk minimizer(maximizer) can estimate the ideal scoring function accurately with sample complexity depends on the logarithmic of the number of agents even when the signal space can be integers, real numbers, or the Euclidean space.

Here we give a outline of the prove. The main observation is that the error from $K_{i,j}$ to the ideal $H_{i,j}$ in (9) is the Bergman divergence between them (Lemma 23). Moreover, if the estimation $K_{i,j}$ is the empirical risk maximizer, this error can be upper bounded by the distance between the empirical distribution and the real distribution (Lemma 24). Therefore, we can use functional

[4]Here we do not show the dependency on constants $L_l$, $R_l$ and $D_l$.

form of Central Limit Theorem to upper bound the error (Theorem 8). We defer the proof to the appendix.

Given $x, y \in \mathbb{R}$ and a strictly convex and twice differentiable $\Phi : \mathbb{R} \to \mathbb{R}$, the standard Bregman divergence is

$$B_\Phi(x, y) = \Phi(x) - \Phi(y) - \nabla\Phi(y)^\top(x - y).$$

It can be extended to *Bregman divergence* between two functionals $f$ and $g$ over probability space $(\Omega, \mathcal{F}, P)$.[2]

$$B_{\Phi,P}(f, g) = \int \Phi(f(\omega)) - \Phi(g(\omega)) - \nabla\Phi(g(\omega))^\top(f(\omega) - g(\omega))dP(\omega). \tag{16}$$

**Lemma 23** (Bregman divergence and accuracy)**.** *If $\Phi$ is strictly convex and twice differentiable on $[0, \infty)$,*

$$D_\Phi(P_{i,j} \| P_iP_j) - u_{i;j}(\mathbb{P}, \boldsymbol{\tau}, \mathbf{K}; \Phi) = B_{\Phi^*, P_iP_j}(K_{i,j}, H_{i,j}).$$

**Lemma 24.** *Let $\tilde{K}_{i,j}$ be the estimate of $H_{i,j}$ obtained by solving the Equation* (15)*, and $H_{i,j} \in \mathcal{K}$ Then*

$$B_{\Phi^*, P_iP_j}(\tilde{K}_{i,j}, H_{i,j}) \le \sup_{k \in \mathcal{K}} \left| \int \Phi^*(k - \Phi^*(H_{i,j}) d(\tilde{P}_i\tilde{P}_j - P_iP_j) - \int \left(k - H_{i,j}\right) d(\tilde{P}_{i,j} - P_{i,j}) \right|.$$

### 4.4 Nonexistence of unbiased estimators for $\Phi$-divergence

Combining Theorem 19 and Theorem 21 or 22 we can design mechanisms that are $\epsilon$-strongly truthful with high probability. *However, is it possible to have an exact informed-truthful or strongly-truthful?* In this section, we show a technical obstacle for such mechanisms.

The main observation of Theorem 14 is that the expected utility of an agent has a close connection to the $\Phi$-divergence from signal pairs on penalty tasks to signal pairs on the bonus task and use this $\Phi$-divergence to upper bound expected utility under manipulation uniformly. This observation is also used in [18] and [7]. Under this framework, showing exact informed-truthful requires *unbiased estimator* of $\Phi$-divergence from i.i.d. samples. Specifically, suppose we can estimate the ideal scoring function accurately from samples. We can estimate the $\Phi$-divergence without biased. The following theorem shows such estimator does not exist in general.

Theorem 25 (Nonexistence). *Let the discrete signal space $\mathcal{Z}$ have more than two elements, $|\mathcal{Z}| > 2$, and $\Phi$ be twice differentiable convex function in* $(0, \infty)$*. For all $m \in \mathbb{N}$ all estimator $\hat{D} : \mathcal{Z}^{2\times m} \to \mathbb{R}$ from m pairs of signals $x_1, \dots, x_m$ to a real value, there exists a prior distribution $P_{i,j}$ over $\mathcal{Z}^2$ such that if $x_s$ are sampled from $P_{i,j}$ identically and independently*

$$\mathbb{E}[\hat{D}(x_1, x_2, \dots, x_m)] \ne D_\Phi(P_{i,j} \| P_iP_j).$$

The key idea of this proof is that if we fix the estimator $\hat{D}$ and take the probability distribution $P$ as variables, the expected value $\mathbb{E}[\hat{D}(x_1, x_2, \dots, x_m)]$ is a polynomial of distribution $P$. However, the $\Phi$-divergence $\mathbb{E}_{P_iP_j}\left[\Phi\left(\frac{P_{i,j}}{P_iP_j}\right)\right]$ is usually not a polynomial, and we can fine one $P_{i,j}$ to make these two values not equal. The proof is in the appendix.

## 5 DISCUSSION AND EXTENSION: BELLS AND WHISTLES

Note that the mechanism 1 contains the Correlated agreement mechanism as a special case. However the corresponding $\Phi$ is not strictly convex, so the mechanism is not strongly truthful in general. Furthermore, since $\Phi$ is not differentiable, it is not trivial to extend to priors on continuous signal space by using Theorem 22.

**Proposition 26** (CA mechanism [20])**.** *If we take $\Phi = \frac{1}{2}|x - 1|$ and restrict $|K_{i,j}| \le 1/2$, Then the above mechanism reduces to the Correlated Agreement mechanism.*

Proof. If we take $\Phi(x) = \frac{1}{2}|x-1|$, $\Phi^*(y) = y$ when $|y| \leq 1/2$, the payment can be simplified as

$$M_i(\mathbf{r}) = K_{i,j}\left(R_{i,b}, R_{j,b}\right) - \Phi^*\left(K_{i,j}\left(R_{i,p}, R_{j,q}\right)\right) = K_{i,j}\left(R_{i,b}, R_{j,b}\right) - K_{i,j}\left(R_{i,p}, R_{j,q}\right).$$

Moreover, by Table 2, Equation (5) reduces to

$$\partial\Phi\left(\frac{P_{i,j}(x,y)}{P_i(x)P_j(y)}\right) = \begin{cases} 1/2 & \text{if } P_{i,j}(x,y) > P_i(x)P_j(y); \\ -1/2 & \text{if } P_{i,j}(x,y) < P_i(x)P_j(y); \\ [-1/2, 1/2] & \text{otherwise,} \end{cases}$$

and the scoring functions are in $\partial\Phi\left(\frac{P_{i,j}(x,y)}{P_i(x)P_j(y)}\right) + \frac{1}{2}$. □

This framework can be easily extended to paying based on three or more agents, such that the expected payment becomes $D_\Phi(P_{i,j,k} \| P_i P_j P_k)$. This extension can handle the cases when the signals are pairwise independent but not 3-wise conditional degenerate. Moreover, by Data Processing Inequality (Corollary 5), the prior expected utility is greater or equal to the pairing mechanism 1 (with loss on the sample complexity for estimation). The following mechanism makes this idea formal:

**ALGORITHM 5:** $\Phi$-divergence $k$-grouping mechanism

**Input** : A convex function $\Phi$ and its conjugate $\Phi^*$ defined in Definition 1 and 2;
the family of the $k$-tuples in $[n]$, $\mathcal{I}_k = \{I = (i_1, \ldots, i_k) : i_1, \ldots, i_k \text{ are distinct } k \text{ elements in } [n]\}$
a set of scoring functions $\mathbf{K} = \{K_I\}$ for all $I$ in $\mathcal{I}_k$ where $K_I : \mathcal{Z}^k \to \mathrm{dom}(\Phi^*)$; and
a report profile $\mathbf{r} \triangleq (r_1, \ldots, r_n)$ where each agent $i \in [n]$ submits a report $\mathbf{r}_i \in \mathcal{Z}^m$ for all $m$ tasks.

**Output**: Payment function, $\mathbf{M}(\mathbf{K}; \Phi) = (M_1, M_2, \ldots, M_2)$

1 For each agent $i$, choose $k-1$ distinct *peer agents* $j_1, j_2, \ldots, j_{k-1}$ uniformly at random from $[n] \setminus \{i\}$, and pick $k+1$ distinct tasks $b, p$, and $q_1, \ldots, q_{k_1}$ where $b$ is the *bonus task*, $p$ is the *penalty task to* $i$, and $q_l$ is *the penalty task to* $j_l$.

2 Based on the agent $i$'s reports $r_{i,b}, r_{i,p}$ and the peer agents' reports $(r_{j_l,b}, r_{j_l,q_l})_{l \leq k-1}$, the reward for agent $i$ is

$$M_i(\mathbf{r}) = K_{i,j}\left(r_{i,b}, r_{j_1,b}, r_{j_2,b}, \ldots, r_{j_{k-1},b}\right) - \Phi^*\left(K_{i,j}\left(r_{i,p}, r_{j_1,q_1}, r_{j_2,q_2}, \ldots, r_{j_{k-1},q_{k-1}}\right)\right). \tag{17}$$

Theorem 27 (grouping mechanism). *Given $n \geq 2$ and $m \in \mathbb{N}$, let $\mathbb{P}$ be a common prior satisfies Assumption 1 over a signal space $\mathcal{Z}$ with $P$ begin $k$-wise conditional nondegenerate. Let $\boldsymbol{\theta}$ be a strategy profile, and $\boldsymbol{\tau}$ be the truth-telling strategy profile; $\mathbf{H}, \mathbf{K}$ be the scoring functions; $\Phi$ is a continuous convex function with $(0, \infty) \subseteq \mathrm{dom}(f)$. If for all $k$-tuple of distinct agents $I = (i, j_1, \ldots, j_{k-1})$ $z, w_1, \ldots, w_{k-1} \in \mathcal{Z}$,*

$$H_I(z, w_1 \ldots, w_{k-1}) \in \partial\Phi\left(\frac{P_I(z, w_1 \ldots, w_{k-1})}{P_i(z)P_{j_1}(w_1) \ldots P_{j_{k-1}}(w_{k-1})}\right),$$

*the Mechanism 1 is informed-truthful, that is for all $\boldsymbol{\theta}$ and $\mathbf{K}$*

$$u_i(\mathbb{P}, \boldsymbol{\tau}, \mathbf{H}; \Phi) \geq u_i(\mathbb{P}, \boldsymbol{\theta}, \mathbf{K}; \Phi).$$

*and $u_i(\mathbb{P}, \boldsymbol{\theta}, \mathbf{K}; \Phi) \leq 0$ when $\boldsymbol{\theta}$ is an oblivious strategy profile.*

*Moreover, if*

*(1) $\mathcal{Z}$ is a finite set,*

*(2) $\Phi$ is strictly convex and differentiable and $\Phi^*$ is strictly convex,*

*the Mechanism 1 is strongly-truthful: the equality in (6) for some $i$ may only occur when*

*(1) $\boldsymbol{\theta} = (\pi_1, \pi_2, \ldots, \pi_n)$ where $(\pi_i)$ are permutations over $[m]$*

*(2) For all $k$-tuple of distinct agents $I = (i_1, \ldots, i_k)$ and $z_1, \ldots, z_k \in \mathcal{Z}$, $K_I(\pi_{i_1}(z_1), \ldots, \pi_{i_k}(z_k)) = \Phi'\left(\frac{P_I(z, w_1 \ldots, w_{k-1})}{P_i(z)P_{j_1}(w_1)\ldots P_{j_{k-1}}(w_{k-1})}\right)$.*

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## A DATA PROCESSING INEQUALITY

Proof of Corollary 5. By Theorem 4, there exists a real-valued function $g : \mathcal{Y} \to \mathbb{R}$ such that

$$\begin{aligned} D_\Phi(P_Y \| Q_Y) =& \mathbb{E}_{P_Y}[g] - \mathbb{E}_{Q_Y}[\Phi^*(g)] \\ =& \sum_{y\in\mathcal{Y}} P_Y(y)g(y) - \sum_{y\in\mathcal{Y}} Q_Y(y)\Phi^*(g(y)) \\ =& \sum_{x\in\mathcal{X}, y\in\mathcal{Y}} P_X(x)P_{Y|X}(y,x)g(y) - \sum_{x\in\mathcal{X}, y\in\mathcal{Y}} Q_X(x)P_{Y|X}(y,x)\Phi^*(g(y)) \\ =& \sum_{x\in\mathcal{X}} P_X(x) \sum_{y\in\mathcal{Y}} P_{Y|X}(y,x)g(y) - \sum_{x\in\mathcal{X}} Q_X(x) \sum_{y\in\mathcal{Y}} P_{Y|X}(y,x)\Phi^*(g(y)) \end{aligned}$$

Because $\Phi^*$ is convex and for all $x \in \mathcal{X}$, $P_{Y|X}(y,x)$ is a distribution over $y$, we have for all $x$ in $\mathcal{X}$, $\sum_{y\in\mathcal{Y}} P_{Y|X}(y,x)\Phi^*(g(y)) \geq \Phi^*\left(\sum_{y\in\mathcal{Y}} P_{Y|X}(y,x)g(y)\right)$. Therefore we have

$$D_\Phi(P_Y \| Q_Y) \leq \sum_{x\in\mathcal{X}} P_X(x)\left(\sum_{y\in\mathcal{Y}} P_{Y|X}(y,x)g(y)\right) - \sum_{x\in\mathcal{X}} Q_X(x)\Phi^*\left(\sum_{y\in\mathcal{Y}} P_{Y|X}(y,x)g(y)\right).$$

Define $h(x) \triangleq \sum_{y\in\mathcal{Y}} P_{Y|X}(y,x)g(y)$, and we can further simplify it as

$$\begin{aligned} D_\Phi(P_Y \| Q_Y) \leq& \sum_{x\in\mathcal{X}} P_X(x)h(x) - \sum_{x\in\mathcal{X}} Q_X(x)\Phi^*(h(x)) \\ \leq& \sup_{h:\mathcal{X}\to\mathbb{R}} \sum_{x\in\mathcal{X}} P_X(x)h(x) - \sum_{x\in\mathcal{X}} Q_X(x)\Phi^*(h(x)) \\ =& D_\Phi(P_X \| P_Y) \end{aligned}$$

which completes the proof. □

## B PROOFS IN SECT. 3

Proof of Lemma 16. By Theorem 4, and the definition of Truth-telling strategy profile, we have

$$\begin{aligned} & u_{i;j}(\mathbb{P}, \boldsymbol{\tau}, \mathbf{H}; \Phi) \\ =& \mathbb{E}_{\mathbf{Z}\sim\mathbb{P}}\left[\mathbb{E}_{\mathbf{R}\sim\mathbf{T}(\mathbf{Z})}\left[K_{i,j}\left(R_{i,b}, R_{j,b}\right) \mid \mathbf{Z}\right]\right] - \mathbb{E}_{\mathbf{Z}\sim\mathbb{P}}\left[\mathbb{E}_{\mathbf{R}\sim\mathbf{T}(\mathbf{Z})}\left[\Phi^*\left(K_{i,j}\left(R_{i,p}, R_{j,q}\right)\right) \mid \mathbf{Z}\right]\right] \\ =& \mathbb{E}_{\mathbf{Z}\sim\mathbb{P}}\left[H_{i,j}(Z_{i,b}, Z_{j,b})\right] - \mathbb{E}_{\mathbf{Z}\sim\mathbb{P}}\left[\Phi^*\left(H_{i,j}(Z_{i,p}, Z_{j,q})\right)\right] \qquad \text{(definition of Truth-telling strategy)} \\ =& \sum_{z,w\in\mathcal{Z}} P_{i,j}(z,w)H_{i,j}(z,w) - \sum_{z,w\in\mathcal{Z}} P_i(z)P_j(w)\Phi^*\left(H_{i,j}(z,w)\right) \\ =& \sup_{L:\mathcal{Z}^2\to\mathbb{R}} \left\{\sum_{z,w\in\mathcal{Z}} P_{i,j}(z,w)L_{i,j}(z,w) - \sum_{z,w\in\mathcal{Z}} P_i(z)P_j(w)\Phi^*\left(L_{i,j}(z,w)\right)\right\} \qquad \text{(by Theorem 4 and } \mathbf{H}\text{)} \\ =& D_\Phi(P_{i,j} \| P_iP_j). \end{aligned}$$

Moreover, because $P_{i,j}$ is 2-wise conditional nondegenerate, $P_{i,j} \neq P_iP_j$ and thus $D_\Phi(P_{i,j} \| P_iP_j) > 0$. □

Proof of Lemma 17. An oblivious strategy $\theta_i$ (defined after Definition 12) is oblivious to the private signal: for any $z$ ,$w$ and $r$ in $\mathcal{Z}$, $\theta_i(z,r) = \theta_i(w,r)$, and we can define $\sigma_i \in \Delta_\mathcal{Z}$ such that for all $z$ and $r$ in $\mathcal{Z}$, $\sigma_i(r) = \theta_i(z,r)$. We also define $\nu_j(l) = \sum_z P_j(z)\theta_j(z,l)$ where $\nu_j$ is a distribution on $\mathcal{Z}$ and independent to $\sigma_i$.

$$
\begin{aligned}
& u_{i;j}(\mathbb{P}, \boldsymbol{\theta}, \mathbf{K}; \Phi) \\
=& \mathbb{E}_{\mathbf{Z}\sim\mathbb{P}}\left[\mathbb{E}_{\mathbf{R}\sim\Theta(\mathbf{Z})}\left[K_{i,j}\left(R_{i,b}, R_{j,b}\right) \mid \mathbf{Z}\right]\right] - \mathbb{E}_{\mathbf{Z}\sim\mathbb{P}}\left[\mathbb{E}_{\mathbf{R}\sim\Theta(\mathbf{Z})}\left[\Phi^*\left(K_{i,j}\left(R_{i,p}, R_{j,q}\right)\right) \mid \mathbf{Z}\right]\right] \\
=& \mathbb{E}_{\mathbf{Z}\sim\mathbb{P}}\left[\sum_{k,l\in\mathcal{Z}} \theta_i(Z_{i,b}, k)\theta_j(Z_{j,b}, l)K_{i,j}(k,l)\right] - \mathbb{E}_{\mathbf{Z}\sim\mathbb{P}}\left[\sum_{k,l\in\mathcal{Z}} \theta_i(Z_{i,p}, k)\theta_j(Z_{j,q}, l)\Phi^*\left(K_{i,j}(k,l)\right)\right] \\
=& \mathbb{E}_{\mathbf{Z}\sim\mathbb{P}}\left[\sum_{k,l\in\mathcal{Z}} \sigma_i(k)\theta_j(Z_{j,b}, l)K_{i,j}(k,l)\right] - \mathbb{E}_{\mathbf{Z}\sim\mathbb{P}}\left[\sum_{k,l\in\mathcal{Z}} \sigma_i(k)\theta_j(Z_{j,p}, l)\Phi^*\left(K_{i,j}(k,l)\right)\right] \\
=& \sum_{k,l\in\mathcal{Z}} \sigma_i(k)\nu_j(l)\left(K_{i,j}(k,l) - \Phi^*\left(K_{i,j}(k,l)\right)\right) \\
\leq& \sup_{y\in\mathrm{dom}(\Phi^*)} \left\{1\cdot y - \Phi^*(y)\right\} = \Phi^{**}(1) = f(1) = 0.
\end{aligned}
$$

The last inequality is from the Definition 1 and 2. □

Proof of Lemma 18. We first prove the first property: $\theta_i$ and $\theta_j$ are permutations. Note that by the proof of Lemma 15, $u_{i;j}(\mathbb{P}, \boldsymbol{\theta}, \mathbf{K}; \Phi) = D_\Phi(P_{i,j} \| P_i P_j)$ if and only if (7) and (8) are equality.

Given strategy of $i$ and $j$, $\theta_i$ and $\theta_j$, let $S_i(z) \triangleq \{k \in \mathcal{Z} : \theta_i(z,k) > 0\}$, $S_j(w) \triangleq \{l \in \mathcal{Z} : \theta_j(w,l) > 0\}$ be the support of strategy $\theta_i$ on signal $z$ and $\theta_j$ on $w$ respectively. Because $\Phi^*$ is strictly convex and $\Phi$ is differentiable, (7), and (8) are equality, $u_{i;j}(\mathbb{P}, \boldsymbol{\theta}, \mathbf{K}; \Phi) = D_\Phi(P_{i,j} \| P_i P_j)$ if and only if

$$
\forall z, w \in \mathcal{Z}, \forall k \in S_i(z), l \in S_j(w),\ K_{i,j}(k,l) = \Phi'\left(\frac{P_{i,j}(z,w)}{P_i(z)P_j(w)}\right). \tag{18}
$$

That is at the equality, the for all reports pairs $(k,l)$ in the support of strategy $\theta_i$ on $z$ and $\theta_j$ on $w$ are the same score. Moreover, the value equals to $\Phi'\left(\frac{P_{i,j}(z,w)}{P_i(z)P_j(w)}\right)$. Now we use this observation to finish the proof.

$\Rightarrow$) Because $\theta_j(w,\cdot)$ induces a probability, $|S_j(w)| \geq 1$ for all $w$. Suppose $\theta_i$ is not a permutation. Because $\mathcal{Z}$ is finite, there exists $w_1 \neq w_2$ and $l^*$ in $\mathcal{Z}$ such that $l^* \in S_j(w_1)$ and $l \in S_j(w_2)$. By (18), for all $z$ and $k \in S_i(z)$,

$$
\Phi'\left(\frac{P_{i,j}(z,w_1)}{P_i(z)P_j(w_1)}\right) = K_{i,j}(k,l^*) = \Phi'\left(\frac{P_{i,j}(z,w_2)}{P_i(z)P_j(w_2)}\right).
$$

Because $\Phi$ is strictly convex and differentiable, $\Phi'$ is invertible, and thus for all $z \in \mathcal{Z}$,

$$
\frac{P_{i,j}(z,w_1)}{P_i(z)P_j(w_1)} = \frac{P_{i,j}(z,w_2)}{P_i(z)P_j(w_2)}
$$

which shows $P$ is mutually conditional degenerate, and reaches contradiction. Therefore there exits permutations $\pi_i$ and $\pi_j$ over $\mathcal{Z}$ such that $\theta_i = \pi_i$ and $\theta_j = \pi_j$.

For the second part, by (18), for all $z, w \in \mathcal{Z}$ we have

$$
K_{i,j}(\pi_i(z), \pi_j(w)) = \Phi'\left(\frac{P_{i,j}(z,w)}{P_i(z)P_j(w)}\right).
$$

$\Leftarrow$) On the other hand, if $\theta_i = \pi_i$ and $\theta_j = \pi_j$ which are permutations over $[m]$, and for all $z, w \in \mathcal{Z}$, $K_{i,j}(\pi_i(z), \pi_j(w)) = \Phi'\left(\frac{P_{i,j}(z,w)}{P_i(z)P_j(w)}\right)$, by (18), $u_{i;j}(\mathbb{P}, \boldsymbol{\theta}, \mathbf{K}; \Phi) = D_\Phi(P_{i,j} \| P_i P_j)$. □

## C PROOFS FOR THEOREM 21

Proof of Theorem 21. Let $\hat{\mathbf{K}}$ be the output of Algorithm 3, and $\mathbf{H}$ be the scoring function defined in (5). Consider a fixed pair of distinct agents $i$ and $j$, we have

$$\begin{aligned}
&u_{i;j}(\mathbb{P}, \boldsymbol{\tau}, \hat{\mathbf{K}}; \Phi)\\
=&\mathbb{E}_{\mathbf{Z}\sim\mathbb{P}}\left[\hat{K}_{i,j}(Z_{i,b}, Z_{j,b})\right] - \mathbb{E}_{\mathbf{Z}\sim\mathbb{P}}\left[\Phi^*\left(\hat{K}_{i,j}(Z_{i,p}, Z_{j,q})\right)\right]\\
=&\sum_{z,w\in\mathcal{Z}} P_{i,j}(z,w)\hat{K}_{i,j}(z,w) - P_i(z)P_j(w)\Phi^*\left(\hat{K}_{i,j}(z,w)\right)\\
=&\sum_{z,w:P_i(z)P_j(w)\neq 0} P_i(z)P_j(w)\left[\frac{P_{i,j}(z,w)}{P_i(z)P_j(w)}\hat{K}_{i,j}(z,w) - \Phi^*\left(\hat{K}_{i,j}(z,w)\right)\right]\\
=&\sum P_i(z)P_j(w)\left[\frac{\hat{P}_{i,j}(z,w)}{\hat{P}_i(z)\hat{P}_j(w)}\hat{K}_{i,j}(z,w) - \Phi^*\left(\hat{K}_{i,j}(z,w)\right) + \left(\frac{P_{i,j}(z,w)}{P_i(z)P_j(w)} - \frac{\hat{P}_{i,j}(z,w)}{\hat{P}_i(z)\hat{P}_j(w)}\right)\hat{K}_{i,j}(z,w)\right]
\end{aligned}$$

Because $\hat{K}_{i,j}(z,w) \in \partial\Phi\left(\frac{\hat{P}_{i,j}(z,w)}{\hat{P}_i(z)\hat{P}_j(w)}\right)$ and Theorem 3, we have $\frac{\hat{P}_{i,j}(z,w)}{\hat{P}_i(z)\hat{P}_j(w)}\hat{K}_{i,j}(z,w) - \Phi^*\left(\hat{K}_{i,j}(z,w)\right) = \Phi\left(\frac{\hat{P}_{i,j}(z,w)}{\hat{P}_i(z)\hat{P}_j(w)}\right)$, so

$$u_{i;j}(\mathbb{P}, \boldsymbol{\tau}, \hat{\mathbf{K}}; \Phi) = \sum P_i(z)P_j(w)\left[\Phi\left(\frac{\hat{P}_{i,j}(z,w)}{\hat{P}_i(z)\hat{P}_j(w)}\right) + \left(\frac{P_{i,j}(z,w)}{P_i(z)P_j(w)} - \frac{\hat{P}_{i,j}(z,w)}{\hat{P}_i(z)\hat{P}_j(w)}\right)\hat{K}_{i,j}(z,w)\right] \tag{19}$$

On the other hand, by the Definition 1,

$$D_\Phi(P_{i,j}\|P_iP_j) = \sum P_i(z)P_j(w)\Phi\left(\frac{P_{i,j}(z,w)}{P_i(z)P_j(w)}\right) \tag{20}$$

By combining (19) and (20), we have

$$\begin{aligned}
&D_\Phi(P_{i,j}\|P_iP_j) - u_{i;j}(\mathbb{P}, \boldsymbol{\tau}, \mathbf{K}; \Phi)\\
=&\sum P_i(z)P_j(w)\left[\Phi\left(\frac{P_{i,j}(z,w)}{P_i(z)P_j(w)}\right) - \Phi\left(\frac{\hat{P}_{i,j}(z,w)}{\hat{P}_i(z)\hat{P}_j(w)}\right) - \left(\frac{P_{i,j}(z,w)}{P_i(z)P_j(w)} - \frac{\hat{P}_{i,j}(z,w)}{\hat{P}_i(z)\hat{P}_j(w)}\right)\hat{K}_{i,j}(z,w)\right]\\
\leq&\sum P_i(z)P_j(w)\left|\Phi\left(\frac{P_{i,j}(z,w)}{P_i(z)P_j(w)}\right) - \Phi\left(\frac{\hat{P}_{i,j}(z,w)}{\hat{P}_i(z)\hat{P}_j(w)}\right)\right| \qquad (21)\\
&+\sum P_i(z)P_j(w)\left|\frac{P_{i,j}(z,w)}{P_i(z)P_j(w)} - \frac{\hat{P}_{i,j}(z,w)}{\hat{P}_i(z)\hat{P}_j(w)}\right||\hat{K}_{i,j}(z,w)| \qquad (22)
\end{aligned}$$

Now it is sufficient for us to upper bound (21) and (22). If we take $m_L = O\left(\frac{36c_L^2}{\alpha^4\epsilon^2}\max\left(|\mathcal{Z}|^2, \log\frac{n^2}{\delta}\right)\right)$, by Lemma 20

$$\|P_{i,j} - \hat{P}_{i,j}\|_{TV} \leq \frac{\alpha^2}{6c_L}\epsilon$$

with probability greater than $1 - \frac{\delta}{n^2}$. Moreover by Lemma 28, this yields

$$D_\Phi(P_{i,j}\|P_iP_j) - u_{i;j}(\mathbb{P}, \boldsymbol{\tau}, \mathbf{K}; \Phi) \leq \epsilon.$$

Therefore by taking union bound over all $\binom{n}{2}$ possible pair of agents we completes the proof. □

**Lemma 28.** *If* (11) (12)*, and* $\|P_{i,j} - \hat{P}_{i,j}\|_{TV} \le \epsilon < \alpha$*,*

$$\sum P_i(z)P_j(w)\left|\Phi\left(\frac{P_{i,j}(z,w)}{P_i(z)P_j(w)}\right) - \Phi\left(\frac{\hat{P}_{i,j}(z,w)}{\hat{P}_i(z)\hat{P}_j(w)}\right)\right| \le \frac{3\epsilon c_L}{\alpha^2} \tag{23}$$

$$\sum P_i(z)P_j(w)\left|\frac{P_{i,j}(z,w)}{P_i(z)P_j(w)} - \frac{\hat{P}_{i,j}(z,w)}{\hat{P}_i(z)\hat{P}_j(w)}\right| \cdot |\hat{K}_{i,j}(z,w)| \le \frac{3\epsilon c_L}{\alpha^2} \tag{24}$$

Proof of Lemma 28. Because (11) and $\|P_{i,j} - \hat{P}_{i,j}\|_{TV} \le \epsilon < \alpha$, $\hat{P}_i(z)$ and $\hat{P}_i(z)$ are greater than $\alpha$. Therefore for all $z$ and $w$, $P_{i,j}(z,w) \neq 0$ we have

$$\alpha \le \frac{P_{i,j}(z,w)}{P_i(z)P_j(w)} \text{ and } \frac{\hat{P}_{i,j}(z,w)}{\hat{P}_i(z)\hat{P}_j(w)} \le \frac{1}{\alpha} \tag{25}$$

To prove (24), we first show an upper bound for $|\hat{K}_{i,j}(z,w)|$. By (10), it is in the sub-gradient of $\Phi$ at $\frac{\hat{P}_{i,j}(z,w)}{\hat{P}_i(z)\hat{P}_j(w)}$ which is in $[\alpha, 1/\alpha]$. With $\Phi$ being $c_L$-Lipschitz with $\Phi(1) = 0$, we have

$$|\hat{K}_{i,j}(z,w)| \le c_L \tag{26}$$

Therefore we are ready to prove (23).

$$\begin{aligned}
&\sum_{z,w:P_i(z)P_j(w)\neq 0} P_i(z)P_j(w)\left|\frac{P_{i,j}(z,w)}{P_i(z)P_j(w)} - \frac{\hat{P}_{i,j}(z,w)}{\hat{P}_i(z)\hat{P}_j(w)}\right| |\hat{K}_{i,j}(z,w)| \\
\le &\sum_{z,w:P_i(z)P_j(w)\neq 0} P_i(z)P_j(w)\left|\frac{P_{i,j}(z,w)}{P_i(z)P_j(w)} - \frac{\hat{P}_{i,j}(z,w)}{\hat{P}_i(z)\hat{P}_j(w)}\right| c_L && \text{(by (26))} \\
=&c_L \sum \frac{1}{\hat{P}_i(z)\hat{P}_j(w)} \left|P_{i,j}(z,w)\hat{P}_i(z)\hat{P}_j(w) - \hat{P}_{i,j}(z,w)P_i(z)P_j(w)\right| \\
\le&\frac{c_L}{\alpha^2} \sum \left|P_{i,j}(z,w)\hat{P}_i(z)\hat{P}_j(w) - \hat{P}_{i,j}(z,w)P_i(z)P_j(w)\right| && (\hat{P}_i, \hat{P}_j \ge \alpha) \\
\le&\frac{c_L}{\alpha^2} \sum P_{i,j}(z,w)\left|\hat{P}_i(z)\hat{P}_j(w) - P_i(z)P_j(w)\right| + P_i(z)P_j(w)\left|P_{i,j}(z,w) - \hat{P}_{i,j}(z,w)\right| \\
\le&\frac{c_L}{\alpha^2} \sum \left|\hat{P}_i(z)\hat{P}_j(w) - P_i(z)P_j(w)\right| + \left|P_{i,j}(z,w) - \hat{P}_{i,j}(z,w)\right| \le \frac{3\epsilon c_L}{\alpha^2}
\end{aligned}$$

Now let's prove (23). Because (12) $\Phi$ is $c_L$-Lipschitz in $[\alpha, 1/\alpha]$, by (25), we have

$$\left|\Phi\left(\frac{P_{i,j}(z,w)}{P_i(z)P_j(w)}\right) - \Phi\left(\frac{\hat{P}_{i,j}(z,w)}{\hat{P}_i(z)\hat{P}_j(w)}\right)\right| \le c_L \left|\frac{P_{i,j}(z,w)}{P_i(z)P_j(w)} - \frac{\hat{P}_{i,j}(z,w)}{\hat{P}_i(z)\hat{P}_j(w)}\right| \tag{27}$$

With argument similar to the proof of (24), we completes the proof. □

# D PROOFS FOR THEOREM 22

Proof of Lemma 23.

$$\begin{aligned} & D_\Phi(P_{i,j}\|P_iP_j) - u_{i;j}(\mathbb{P}, \boldsymbol{\tau}, \mathbf{K}; \Phi) \\ &= \int H_{i,j} dP_{i,j} - \int \Phi^*(H_{i,j}) dP_iP_j - \int K_{i,j} dP_{i,j} + \int \Phi^*(K_{i,j}) dP_iP_j \\ &= \int \Phi^*(K_{i,j}) - \Phi^*(H_{i,j}) + \frac{dP_{i,j}}{dP_iP_j}\left(H_{i,j} - K_{i,j}\right) dP_iP_j \\ &= \int \Phi^*(K_{i,j}) - \Phi^*(H_{i,j}) - (\Phi^*)'\left(H_{i,j}\right)\left(K_{i,j} - H_{i,j}\right) dP_iP_j \end{aligned}$$

The last equality holds since $H_{i,j}(z,w) = \Phi'\left(\frac{dP_{i,j}(z,w)}{dP_iP_j(z,w)}\right)$, so $(\Phi^*)'\left(H_{i,j}(z,w)\right) = \frac{dP_{i,j}(z,w)}{dP_iP_j(z,w)}$ by Theorem 3. The final line is indeed the Bregman divergence from $H_{i,j}$ to $K_{i,j}$ with respect to measure $dP_iP_j$ and $\Phi^*$. □

Proof of Lemma 24. Because $\tilde{K}_{i,j}$ satisfies Equation (15) and $H_{i,j} \in \mathcal{K}$, we have

$$\int \tilde{K}_{i,j} d\tilde{P}_{i,j} - \int \Phi^*(\tilde{K}_{i,j}) d\tilde{P}_i\tilde{P}_j \geq \int H_{i,j} d\tilde{P}_{i,j} - \int \Phi^*(H_{i,j}) d\tilde{P}_i\tilde{P}_j.$$

On the other hand,

$$B_{\Phi^*, P_iP_j}(\tilde{K}_{i,j}, H_{i,j}) = \int \Phi^*(\tilde{K}_{i,j}) - \Phi^*(H_{i,j}) dP_iP_j - \int \left(\tilde{K}_{i,j} - H_{i,j}\right) dP_{i,j}.$$

Combining these two we have,

$$B_{\Phi^*, P_iP_j}(\tilde{K}_{i,j}, H_{i,j}) \leq \int \left(\Phi^*(\tilde{K}_{i,j}) - \Phi^*(H_{i,j})\right)\left(dP_iP_j - d\tilde{P}_i\tilde{P}_j\right) - \int \left(\tilde{K}_{i,j} - H_{i,j}\right)\left(dP_{i,j} - d\tilde{P}_{i,j}\right)$$

which completes the proof. □

Proof of Theorem 22. By Lemma 23 and 24, we know the error between $D_\Phi(P_{i,j}\|P_iP_j) - u_{i;j}(\mathbb{P}, \boldsymbol{\tau}, \mathbf{K}; \Phi)$ can be upper bound by

$$\sup_{k \in \mathcal{K}} \left| \int \Phi^*(k) - \Phi^*(H_{i,j}) d(\tilde{P}_i\tilde{P}_j - P_iP_j) \right| \tag{28}$$

$$\sup_{k \in \mathcal{K}} \left| \int k - H_{i,j} d(\tilde{P}_{i,j} - P_{i,j}) \right|. \tag{29}$$

Now we can apply the uniform bound in Theorem 8 for (29). By taking $A = \frac{\varepsilon L_1}{R_1^2}$, $B = 1$, $L = L_1$, $R = R_1$, and $\epsilon = \varepsilon\sqrt{n}$, we have

$$\begin{aligned} \Pr\left[\sup_{k \in \mathcal{K}} |v_n(k)| \geq \epsilon\right] &= \Pr\left[\sup_{k \in \mathcal{K}} \left|\sqrt{n} \int k d(\hat{P}_n - P)\right| \geq \varepsilon\sqrt{n}\right] \\ &= \Pr\left[\sup_{k \in \mathcal{K}} \left|\int k d(\hat{P}_n - P)\right| \geq \varepsilon\right] \\ &\leq B \exp\left(-\frac{\epsilon^2}{B^2(A+1)R_1^2}\right) \\ &\leq \exp\left(-\frac{\varepsilon^2}{(A+1)R_1^2} n\right) \leq \frac{1}{n^2}\delta. \end{aligned}$$

The last inequality is true by taking $n = m_L/3 = O\left(\frac{(A+1)R_1^2}{\varepsilon^2}\log\frac{n^2}{\delta}\right) = O\left(\frac{1}{\varepsilon^2}\log\frac{n^2}{\delta}\right)$ when $\varepsilon$ is small enough. We can derive similar upper bound for (29), and we complete the proof □

## E PROOF OF THEOREM 25

Proof. Formally, note that given i.i.d samples from $P_{i,j}$ the empirical distribution of those samples are sufficient statistics, and we can restrict the estimators to be functional over sufficient statistics, $\hat{D} : \mathbb{N}_0^{\mathcal{Z}^2} \to \mathbb{R}$. Let's consider the following prior distribution $P_{i,j}$: Given non-negative variables $\alpha, \beta, \gamma$ such that $\alpha + \beta + \gamma \leq 1$, we set the distribution over $\mathcal{Z}^2 = \{1, 2, 3\}^2$ to be

$$P_{i,j} = \frac{1}{3}\begin{bmatrix} 1-\alpha-\beta & \alpha & \beta \\ \alpha & 1-\alpha-\gamma & \gamma \\ \beta & \gamma & 1-\beta-\gamma \end{bmatrix}$$

An empirical distribution (histogram) from $x_1, \ldots, x_m$ can be represented by 9 integers $\mathbf{m} = (m_{k,l})$ where $k$ and $l$ are between 1 to 3 and $m_{k,l}$ is the number of $(k, l)$ in those $m$ samples, and the distribution of $\mathbf{m}$ forms a multi-nomial distribution. Therefore we can compute the expectation of $\hat{D}$,

$$\begin{aligned}
&\mathbb{E}[\hat{D}(x_1, x_2, \ldots, x_m)] \\
&= \sum_{\mathbf{m}:\sum m_{k,l}=m} \frac{m!}{\prod_{k,l} m_{k,l}!} \prod_{k,l} P_{i,j}(k,l)^{m_{k,l}} \hat{D}(\mathbf{m}) \\
&= \sum_{\mathbf{m}:\sum m_{k,l}=m} \frac{m!}{\prod_{k,l} m_{k,l}!} \left(\frac{\alpha}{3}\right)^{m_{1,2}+m_{2,1}} \left(\frac{\beta}{3}\right)^{m_{1,3}+m_{3,1}} \left(\frac{\gamma}{3}\right)^{m_{2,3}+m_{3,2}} \\
&\quad \cdot \left(1-\frac{\alpha+\beta}{3}\right)^{m_{1,1}} \left(1-\frac{\beta-\gamma}{3}\right)^{m_{2,2}} \left(1-\frac{\alpha+\gamma}{3}\right)^{m_{3,3}} \hat{D}(\mathbf{m})
\end{aligned}$$

which is a polynomial of $\alpha, \beta$ and $\gamma$. On the other hand, the $\Phi$-divergence is

$$\frac{1}{9}\left[2\Phi(3\alpha) + 2\Phi(3\beta) + 2\Phi(3\gamma) + \Phi(3(1-\alpha-\beta)) + \Phi(3(1-\beta-\gamma)) + \Phi(3(1-\alpha-\gamma))\right]$$

By taking partial derivative with respect to $\alpha$ then $\beta$, $\mathbb{E}[\hat{D}(x_1, x_2, \ldots, x_m)] = \mathbb{E}_{P_i P_j}\left[\Phi\left(\frac{P_{i,j}}{P_i P_j}\right)\right]$ implies the second derivative of $\Phi$ is a polynomial and $\Phi(x)$ is a polynomial.

Similarly we take

$$P'_{i,j} = \begin{bmatrix} 1-\alpha-\beta & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix}$$

and the $\Phi$-divergence is

$$\alpha^2\Phi(1/\alpha) + b^2\Phi(1/\beta) + (1-\alpha-\beta)^2\Phi(1/(1-\alpha-\beta)) + (1-\alpha^2-\beta^2-(1-\alpha-\beta)^2)\Phi(0)$$

By taking partial derivative with respect to $\alpha$ and $\beta$ we have

$$\Phi\left(\frac{1}{x}\right) - \frac{1}{x}\Phi'\left(\frac{1}{x}\right) + \frac{1}{x^2}\Phi''\left(\frac{1}{x}\right)$$

is a polynomial with respect to $x$.

Therefore, combining these two statements we have if there are unbiased estimators for $P_{i,j}$ and $P'_{i,j}$, the convex function $\Phi$ is a degree one polynomial which reaches a contradiction. □