CONSTRUCTING INTEGER MATRICES WITH INTEGER EIGENVALUES

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Abstract. In spite of the proveable rarity of integer matrices with integer eigenvalues, they are commonly used as examples in introductory courses. We present a quick method for constructing such matrices starting with a given set of eigenvectors. The main feature of the method is an added level of flexibility in the choice of allowable eigenvalues. The method is also applicable to non-diagonalizable matrices, when given a basis of generalized eigenvectors. We have produced an online web tool that implements these constructions.

In this paper we will look at the problem of constructing a good problem. Most linear algebra and introductory ordinary differential equations classes include the topic of diagonalizing matrices: given a square matrix, finding its eigenvalues and constructing a basis of eigenvectors. In the instructional setting of such classes, concrete "toy" examples are helpful and perhaps even necessary (at least for most students). The examples that are typically given to students are, of course, integer-entry matrices with integer eigenvalues. Sometimes the eigenvalues are repeated with multipicity, sometimes they are all distinct. Oftentimes, the number 0 is avoided as an eigenvalue due to the degenerate cases it produces, particularly when the matrix in question comes from a linear system of differential equations.

Yet in [**?**], Martin and Wong show that "Almost all integer matrices have no integer eigenvalues," let alone all integer eigenvalues. They show that the probability (appropriately defined) of an integer matrix having even one integer eigenvalue is zero. This begs the question of understanding the case when integer matrices *do* have integer eigenvalues. In the mid-1980s and beyond, a small flurry of articles addressed construction and classification of such matrices ([**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**]).

In this paper, we present a quick technique for finding such matrices, that can be easily tailored for use in the classroom. We give two short proofs that demonstrate why our construction works, and discuss the limitations of the technique.

We begin by setting notation and clarifying the problem.

1. The Problem and One Solution

Definition. We say a matrix A is an *IMIE* if it is an integer-entry matrix with (all) integer eigenvalues. In other words, the characteristic polynomial of A factors completely over $\mathbb{Z}$.

The process of diagonalizing an $n \times n$ matrix A can be thought of as factoring A as $A = PDP^{-1}$ (if possible). Here D is a diagonal matrix whose diagonal entries are the eigenvalues of A and the matrix P is an invertible matrix whose columns form a basis of eigenvectors for A. Throughout this paper we will always use P

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and D to denote such matrices. We will also use the notation $D = \langle \lambda_1, \ldots, \lambda_n \rangle$ to indicate the diagonal entries. Note that the ith column of P is an eigenvector with eigenvalue λ_i. (See The Diagonalization Theorem, Chapter 5, Theorem 5 in [?], for example.) We will let δ denote the determinant of P.

The issue for the instructor, which we are addressing here, is to reverse-engineer a nice A for students to practice on without getting bogged down in computation.

An obvious first approach would be to start with any integer matrices P (invertible) and D (diagonal) and see if the conjugate of D by P, that is, PDP^{-1}, is integral. The matrix P^{-1} can be constructed (in theory or as a particularly tedious one-time exercise) by multiplying the adjugate (or classical adjoint) matrix P^{adj} by $1/\delta$. (Recall, $\delta = \det P$.) As such, we can see that if we start with an integral P, then the entries of P^{-1} can have no worse than $\det P$ in the denominators.

A clever instructor might keep a favorite integer matrix P which has determinant 1 and then simply take any set of eigenvalues, place them in the matrix D, and multiply out PDP^{-1} to get A, readily made for students to work on. In particular, the eigenvalues can be selected to avoid 0, to include any desired multiplicities, to include 1 (so that factoring the characteristic equation becomes more tractible), etc. The potential disadvantage, that students might someday realize that their eigenbases always consist of the same vectors, is remote.

2. Special Eigenbases

More generally, one could try to find a way to construct many integer matrices, P, with determinant 1 as Ortega does in [?] and [?]. (His motivation, incidentally, is *not* pedogogical.) He first notes the following, the first part of which can also be found in [?] (p. 26).

Theorem 1. *Given two n-vectors $\vec{u}, \vec{v} \in \mathbb{Z}^n$ with $\vec{u} \cdot \vec{v} = \beta$, the matrix $P = I_n + \vec{u}\vec{v}^T$ has $\det P = 1 + \beta$. In addition, if $\beta \neq -1$, then $P^{-1} = I_n - \frac{1}{1+\beta}\vec{u}\vec{v}^T$.*

Here, $\vec{u}\vec{v}^T$ is just the $n \times n$ "outer product" matrix, sometimes written $\vec{u} \otimes \vec{v}$.

On a somewhat unrelated note, we get the following as an application.

Corollary 2. *Given two n-vectors $\vec{u}, \vec{v} \in \mathbb{Z}^n$ with $\vec{u} \cdot \vec{v} = -2$, the matrix $Q = I_n + \vec{u}\vec{v}^T$ is integral and involutory, that is, $Q = Q^{-1}$.*

Ortega proceeds to apply Theorem **??** to orthogonal pairs to get useful (for our purposes) P-matrices.

Corollary 3 (Ortega)**.** *Given two n-vectors $\vec{u}, \vec{v} \in \mathbb{Z}^n$ which are orthogonal, the matrix $P = I_n + \vec{u}\vec{v}^T$ has $\det P = 1$ and $P^{-1} = I_n - \vec{u}\vec{v}^T$.*

Two small disadvantages of this technique might be the fact that one must start with orthogonal vectors, and that one cannot control the eigenvectors ahead of time. Nevertheless, a wide variety of determinant 1 integer matrices seem to be produced in this way. (N.B. By no means are all determinant 1 integer matrices produced through this method, however.)

Once one has such a P with $\delta = 1$ (or -1), any choice of eigenvalues can be put into a diagonal matrix D. The resulting $A = PDP^{-1}$ will be a diagonalizable IMIE.

3. Arbitrary eigenbases

Still, there seemed to me to be something disingenuous about using only determinant 1 P-matrices. Suppose we are given *any* matrix P with non-zero determinant δ.

Then clearly the matrix δP^{-1} is integral. So if we simply choose a set of eigenvalues $\lambda_1, \dots, \lambda_n$ which are all multiples of δ, then the matrix $A = PDP^{-1}$ would be an integer entry matrix with integer eigenvalues. Indeed Galvin ([**?**]) suggests exactly this technique. However, at the time of this article, Galvin notes, "[t]he calculations are extensive and require writing a suitable computer program." His paper is a call to create such programs and subroutines to generate IMIE's in this way. Today, a common graphing calculator on hand will do the trick.

Example 1. Suppose a student (why not?) chooses a random matrix P as follows:

$$P = \begin{pmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

We quickly plug these nine entries into a graphing calculator to determine that $\det P = 4$. Thus we could choose eigenvalues from $\{0, \pm 4, \pm 8, \dots\}$ to construct an IMIE, A. If, for pedogogical purposes, we need to avoid 0 as an eigenvalue, and include only simple eigenvalues, our simplest choice for D might be $D = \langle 4, -4, 8 \rangle$. Another moment on the calcluator yields the IMIE

$$A = PDP^{-1} = \begin{pmatrix} 3 & -2 & 7 \\ 1 & 6 & 1 \\ 3 & 6 & -1 \end{pmatrix}$$

whose characteristic equation is

$$x^3 - 8x^2 - 16x + 128.$$

Unfortunately, this may not give us the most reasonable exercise for a beginning student.

Galvin appends an additional *reduction* step to simplify the resulting A, if possible: Look for the greatest common divisor of all n^2 entries, and divide through, reducing the size of the entries.

4. A useful refinement

A bit more care leads to the following easy and useful generalization that we present here.

Theorem 4. *Let P be an $n \times n$ integer matrix with determinant $\delta \neq 0$. Let D be a diagonal matrix whose diagonal entries are all integers that are mutually congruent modulo δ. Then $A = PDP^{-1}$ is an integer matrix.*

More concisely: Let $P \in GL_n(\mathbb{Z})$ and $D = \langle \lambda_1, \dots, \lambda_n \rangle$ with $\lambda_i \in \mathbb{Z}$. If $\lambda_1 \equiv \lambda_2 \equiv \dots \equiv \lambda_n \mod \delta$ then PDP^{-1} is a diagonalizable IMIE.

Remark. The point is that the eigenvalues do not need to be multiples of δ (i.e. congruent to 0 modulo δ), only that they need to be mutually congruent to each other modulo δ.

Two simple proofs will be given momentarily. But first let us return to our example.

Example 2. Using the same

$$P = \begin{pmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

with $\det P = 4$, we can now prescribe 1 to be an eigenvalue as long as the other eigenvalues are congruent to 1 modulo 4. For instance, $D = \langle 1, -3, 5 \rangle$, which would yield an IMIE matrix

$$A = PDP^{-1} = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 3 & 1 \\ 2 & 4 & -1 \end{pmatrix}$$

whose characteristic equation (that the students must factor) is

$$x^3 - 3x^2 - 13x + 15.$$

The entries and coefficients may not appear much simpler than in the previous case, but, of course, the problem can be set up so that a quick guess-and-check approach (try $x = 1$ to start or one of the few factors of 15) will immediately yield a root for the students to use to reduce the problem to a quadratic.

Remark. See `http://ericthewry.github.io/integer_matrices/` for the webtool we have created. It allows the user to input integer entries into a matrix, creating P. The tool sets appropriate eigenvalues and returns δ, the IMIE A, and its characteristic polynomial. The user can then adjust the eigenvalues to see how A and its characteristic polynomial change.

We give two very short proofs for the theorem.

First proof of Theorem **??**. The two key facts are that P^{adj} has integer entries and that $(1/\delta)P^{\text{adj}} = P^{-1}$. So, in particular, $PP^{\text{adj}} = \delta I$. Now, we know that $A = (1/\delta)PDP^{\text{adj}}$ has, at worst, entries in $(1/\delta)\mathbb{Z}$. What we need to show is that the matrix PDP^{adj} has entries which are all multiples of δ, that is, congruent to 0 modulo δ. But we have constructed D so that, modulo δ, we have $D \equiv \lambda I$ for some λ. Thus, modulo δ, we have $PDP^{\text{adj}} \equiv \lambda PIP^{\text{adj}} = \lambda\delta I \equiv \mathbf{0}$, the zero matrix. □

Our second proof uses an observation found in [**?**].

Second proof of Theorem **??**. We begin with the observation that if A has eigenvalues $\lambda_1, \ldots, \lambda_n$, then $bI_n + A$ has eigenvalues $b + \lambda_1, \ldots, b + \lambda_n$ (with the same eigenvectors). This is because

$$A\vec{v} = \lambda\vec{v} \quad \text{implies that} \quad (A + bI)\vec{v} = (\lambda + b)\vec{v}.$$

For any $b \in \mathbb{Z}$, if $A = PDP^{-1}$ is an IMIE then so is $B = P(D + bI)P^{-1} = A + bI$, and the eigenvalues of B are the same as those for A shifted by the constant b. We know (as discussed above and in [**?**]) that PDP^{-1} is an IMIE when all the entries of D are congruent to 0 modulo δ. So this shows that $A + bI$ is an IMIE with all eigenvalues congruent to b modulo δ. □

5. Non-Diagonalizable Examples

Of course not all IMIEs are diagonalizable. One might also be interested in constructing non-diagonalizable IMIEs. The addition of a certain type of nilpotent matrix N allows for such examples.

Suppose B is an $n \times n$ IMIE. If B is non-diagonalizable then B must have an eigenvalue λ of algebraic multiplicity α greater than 1 and the dimension of the eigenspace associated to λ must be less than α. In such a case, there is no basis consisting of eigenvectors. However, B will have a Jordan form.

Just as we know that a *diagonalizable* IMIE, A, must be equal to PDP^{-1} for some P and D, we know that a non-diagonalizable IMIE, B, must be equal to $B = P(D+N)P^{-1} = PDP^{-1} + PNP^{-1}$ for some nilpotent matrix N. Here, we think of $D+N$ as the Jordan form for B. The matrix D is, as before, diagonal with the eigenvalues of B on its diagonal. The matrix P is invertible, consisting of a basis of *generalized* eigenvectors of B. In Jordan form N consists of all zero entries except for certain subdiagonal 1's. But in order to guarantee that B will be an IMIE, we will take N to have all non-zero entries equal to δ, not 1.

Theorem 5. *Let P be an $n \times n$ integer matrix with determinant $\delta \neq 0$. Let D be a diagonal matrix whose diagonal entries are all integers that are mutually congruent modulo δ. Let N be any matrix whose entries are all multiples of δ. Then $B = P(D+N)P^{-1}$ is an integer matrix.*

Remark. While the theorem would apply to any matrix, N, whatsoever, in practice we will take N to be a nilpotent matrix which come from Jordan form, but one with all zeros except for δ in certain sub-diagonal positions.

Example 3. We can use the same P as before

$$P = \begin{pmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

but now take $D = \langle 1, 1, 5 \rangle$, which would yield an IMIE matrix

$$A = PDP^{-1} = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

whose characteristic equation is $x^3 - 7x^2 + x - 5$.

This matrix, A, is diagonalizable. However if we add in the matrix

$$N = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we get

$$B = P(D+N)P^{-1} = \begin{pmatrix} 3 & 0 & 2 \\ 1 & 3 & 1 \\ 0 & 4 & 1 \end{pmatrix}$$

which is non-diagonalizable.

Remark. One can easily check that had we chosen $N = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ in Example **??**, then B would not have been an IMIE.

6. Further questions

Natural questions, at this point, include to what extent are new IMIEs are obtained by Theorems **??** and **??** (as opposed to the direct approach by Galvin), and to what extent *all* IMIEs are obtained. For instance, we could have obtained the matrix in Example **??** using all eigenvalues divisible by 4, if we had chosen

$D = \langle 4, -12, 20\rangle$, yielding the matrix $A = \begin{pmatrix} 4 & 0 & 16 \\ 4 & 12 & 4 \\ 8 & 16 & -4 \end{pmatrix}$, and then recognizing that 4 divided the entire matrix, and using Galvin's *reduction* step. So, trivially, we see that while no matrices obtained by using Theorem **??** that could not be constructed using Galvin's direct approach, plus reduction, our Theorem eliminates this extra step and illuminates the circumstances better.

Pursuing this a bit further, we can give some partial results regarding the question of *all* IMIEs. Note that the reduction step will always occur when P has a column whose entries have a non-trivial common divisor. We can attempt to account for this. We start by making the following definition.

Definition. We say an $n \times n$ integral matrix P is *simplified* if there is no $d > 1$ which divides all the entries of any column of P. That is, $\gcd(a_{1j}, a_{2j}, \ldots, a_{nj}) = 1$ for each $j = 1, 2, \ldots, n$.

Of course we can *simplify* a matrix P by dividing each column by that columns greatest common divisor.

Proposition 6. *Suppose P is an $n\times n$ invertible matrix with determinant δ. Suppose that the entries of the jth column of P have a greatest common divisor g_j. That is, $\gcd(a_{1j}, a_{2j}, \ldots, a_{nj}) = g_j$ for each $j = 1, 2, \ldots, n$. Let $h = \delta/(g_1 g_2 \cdots g_n)$. Then if $\lambda_1 \equiv \cdots \equiv \lambda_n$ modulo h, then PDP^{-1} is an IMIE.*

Proof. Let R be the diagonal matrix $R = \langle 1/g_1, \ldots, 1/g_n\rangle$ then $Q = PR$, is simplified, and $\det Q = h$. Thus QDQ^{-1} is an IMIE. But $QDQ^{-1} = PRDR^{-1}P^{-1} = PDP^{-1}$, since R, D, and R^{-1} are all diagonal. □

Thus, we know that we can simplify P in order to allow for fewer restrictions on the eigenvalues of our IMIE. In fact, in the 2×2 case this produces all IMIE's.

Theorem 7. *If P is a 2×2 simplified integral matrix, then $A = PDP^{-1}$ is IMIE if and only if $\lambda_1 \equiv \lambda_2$ modulo $k = \det P$.*

Proof. We have already established the "if" direction. We will show that if the eigenvalues are not all congruent modulo δ then A is not an integer entry matrix. Due to Theorem **??**, we may assume, without loss of generality, that $\lambda_1 = 0$. We will drop the subscript on λ_2. Let

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{so that} \quad P^{\mathrm{adj}} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Since $P^{-1} = \frac{1}{\delta}P^{\mathrm{adj}}$, in order for A to be integral, we would need $PDP^{\mathrm{adj}} \equiv 0$, modulo δ. In order for PDP^{adj} to be congruent to the zero matrix modulo δ, it would need to be congruent to the zero matrix modulo each prime dividing δ. Since $\lambda \not\equiv 0$ modulo k, we must have $\lambda \not\equiv 0$ modulo some prime dividing k. Let p be such a prime. Then

$$PDP^{\mathrm{adj}} = \begin{pmatrix} -ba\lambda & bc\lambda \\ -da\lambda & dc\lambda \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \mod p$$

if and only if $p|ba$, $p|bc$, $p|da$, and $p|dc$ (since we have assumed $p \not| \lambda$). If $p \not| b$ then p must divide both a and c. Similarly for $p \not| d$. But if p divides both a and c then P was not simplified. Thus p divides both b and d. But then, again, P was not simplified. Thus, we must conclude that PDP^{adj} is not congruent to 0 modulo p. And thus, modulo δ. Therefore PDP^{-1} is not integral. □

Note that although it makes sense to only consider simplified matrices in the construction $A = PDP^{-1}$, in a classroom setting in which P is spontaneously chosen, (as long as $\det P \neq 0$), one cannot expect to be given a simplified P.

Note also that being simplified is a property of the columns of P, and not the rows, as the next example demonstrates.

Example 4. Let $P_1 = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$, then if we "row-simplify" the first row of P_1 we would get $P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ which has $\delta = \det P = 1$. But one can easily check that $P_1 D P_1^{-1}$ will be integral if and only if $\lambda_1 \equiv \lambda_2$ modulo 3, since $P_1 \langle \alpha, \beta \rangle P_1^{-1} = \frac{1}{3}\begin{pmatrix} 3\alpha & 0 \\ \alpha - \beta & 3\beta \end{pmatrix}$.

Unfortunately, Theorem **??** does not generalize to larger dimensions as the following example demonstrates.

Example 5. Consider the matrix

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{pmatrix} \quad \text{so that} \quad P^{\text{adj}} = \begin{pmatrix} 4 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -3 & 4 \end{pmatrix}.$$

Then for any if we choose $D = \langle \alpha, \beta, \gamma \rangle$, then we get

$$PDP^{\text{adj}} = \begin{pmatrix} 4\alpha & 0 & 0 \\ 4(\alpha - \beta) & 4\beta & 0 \\ 4\alpha - 3\beta - \gamma & 3\beta - 3\gamma & 4\gamma \end{pmatrix}.$$

We can see that $\det P = 4$, but any choice of α, β, γ with β and γ divisible by 4 would allow $A = PDP^{-1}$ to be an IMIE. In particular, there is no restriction on the choice of α. We could allow $\alpha = 1$, $\beta = 4$, $\gamma = 0$ and get

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 4 & 0 \\ -2 & 3 & 0 \end{pmatrix}.$$

If we allow for cancelling out common factors, this IMIE could be obtained by an initial choice of eigenvalues of $\{4, 16, 0\}$, but this demonstrates that merely simplifying P does not capture all of the subtley of IMIE's.

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