Temporal Graphs Research at Pomona College
Working Definitions and Vocabulary

February 28, 2016

1 About

As we continue researching temporal graphs and their related properties there is an increased need to establish fundamental definitions. Although many papers have established definitions of the terms and concepts contained within this paper, there are subtle details of each of these definitions that vary from paper to paper making it difficult to efficiently communicate exact ideas.

This document shall serve as an ongoing record of the definitions we will use in our research. The definitions contained within should serve as the defaults in conversations; if you intend to use an alternate definition to a concept defined in this document, you should state and/or cite the alternate definition. (In the future, we will look to incorporate alternate definitions in this document as well.) It should be noted that these definitions may change as we continue our research.

2 Preliminary Definitions

Definition 2.0.1. A **temporal graph** or **temporal network**[1] is defined as a tuple $G \in V \times E$ where V is the set of vertices, and $E \in V^2 \times T^2$ is the set of edges such that $T = [0, \infty)$:

$$G = (V, E)$$

Each edge $e \in E$ is a 5-tuple (v_i, v_j, t_0, t_f, w) such that $v_i, v_j \in V$, $t_0, t_f \in T$, and $t_0 \leq t_f$. An edge is said to connect v_i to v_j (with the possibility of $v_i = v_j$) between the time $[t_0, t_f]$ with weight w. That is, all edges are to be treated as directed edges with some weight and time-window (possibly instantaneous) in which they are *active*.

With this generalized definition of a temporal network, we will begin to explain some common, specialized descriptors of these networks:

[1]We will treat the terms *graph* and *network* synonymously throughout this paper.

Definition 2.0.2. We will describe a temporal graph as **undirected** $\forall e \in E$ if there exists $e' \in E$ such that $e = (v_i, v_j, t_0, t_f, w)$ and $e' = (v_j, v_i, t_0, t_f, w)$.[2]

Informally, if for every directed edge in the graph, there exists a corresponding edge in the opposite direction connecting the same two vertices and labeled with the exact same t_i, t_f, and w, we will consider the temporal graph to be undirected.

Definition 2.0.3. We will describe a temporal graph as **unweighted** if every edge in E has equal weight w.

Definition 2.0.4. Edge persistence is the amount of time for which an edge is present. More formally, in a graph (V, E) the persistence of an edge (u, v, t_1, t_2, w) is defined as $t_2 - t_1$ or simply δt.

Definition 2.0.5. An edge $e \in E$ is said to be **infinitely persistent** if $t_f = \infty$.

Definition 2.0.6. A **instantaneous edge** or **contact edge** is any edge $e \in E$ where $t_0 = t_f$.

Definition 2.0.7. Interval networks or **windowed networks** shall be defined as any temporal network where for at least one edge $e = (v_i, v_j, t_0, t_f, w)$, $t_0 \neq t_f$.

Definition 2.0.8. Contact networks shall be defined as a temporal network where every edge in the network is an *instantaneous edge* or *contact edge*. It is simply a special case of an interval network.

Definition 2.0.9. An **aggregate network** shall be defined as a temporal network where every edge is **infinitely persistent**.

Definition 2.0.10. A **co-authorship network** is an undirected interval network in which each node represents an author, and the existence of an edge between two nodes represents a collaboration over the a period of time δt. Unless stated otherwise, we shall assume a *collaboration* to mean two authors worked on 1 or more publications together in δt.

Definition 2.0.11. A **citation network** is a directed interval network (V, E) with infinite edge persistence. For simplicity, we can write edges as $u \rightarrow_{t_1} v$ or $(u, v)_{t_1}$. In this network, each node represents an author, and the existence of an edge $u \rightarrow_{t_1} v$ means that author v cited author u at time t. This way the direction of the arrow represents the flow of information.

Now we will move on to consider the different analysis methodologies that will result from different temporal definitions of 'shortest path'.

[2] As we consider the real-life applications of these temporal networks, it may be convenient to define an edge as a 6-tuple with the last component l being a unique label or edge differentiator. This would allow for the following scenario: if author `A` and `B` collaborate in window $[t_0, t_f]$ on multiple papers with the same weight, we could differentiate two edges connecting the two authors instead of condensing the two paper collaborations into one edge. In many cases, it may not make sense to do this, but it is valuable to have an umbrella definition that encompasses every type of graph we could create.

Definition 2.0.12. A graph is a **path** if it is a simple graph whose vertices can be linearly ordered such that there is an edge uv if an only if u and v are adjacent in the ordering. A digraph is a **path** if it is a simple directed graph whose vertices can be linearly ordered such that there is an edge $u \to v$ if and only if v immediately follows u in the ordering.

Definition 2.0.13. A **shortest path** between v_1, u (di)graph $G = (V, E)$ is $P = v_1, v_2, \cdots, v_n, u$ such that $v_i \in V$ for all $i \in [n]$, and $v_j v_{j+1}, v_n u \in E$, for all $j \in [n-1]$, and there is no path $P' = v_1, v_2, \cdots, v_m, u$ such that $m < n$.

Note that this definition does not enforce uniqueness of shortest paths.

3 Consecutive Contemporaneity

Here, we consider the addition of paths to include a temporal component, where any two consecutive edges must share some contemporary period. This is a sensible definition as two edges should not be able to form a path in a temporal network if they did not happen at the same time. This is formally defined below.

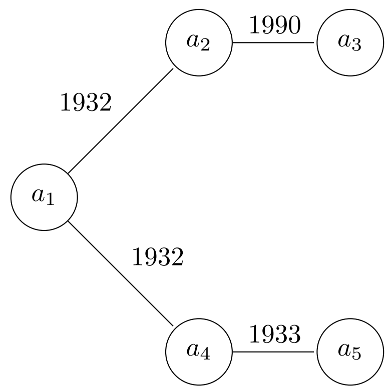
Definition 3.0.1. A **consecutive temporal path** between v_1 and v_n is a path $P = v_1, v_2, \cdots, v_n$ such that $v_i \in V$ for all $i \in [n]$, and $v_j v_{j+1} \in E$, for all $j \in [n-1]$, and for every pair of consectutive edges $(v_{i-1}, v_i)_{t_1}$ and $(v_i, v_{i+1})_{t_2}$, $t_1 \leq t_2$.

We will consider the consequences of this definition in the context of different edge-behaviors, infinite persistence, and windowed persistence.

3.1 Infinitely Persistent Edges

The first model we will consider is the simplest of the three, where we disallow edge-deletion. We will define the persistent coauthorship network to have this property. [note about semantic equivalence to railway network?]

Definition 3.1.1. The **persistent co-authorship network** a co-authorship network $G = (V, E)$, where for all $(u, v, t_1, t_2) \in E$, $t_2 = \infty$. For simplicity, we can denote an edge by $(u, v)_{t_1}$ or $u -_{t_1} v$. Since this network is undirected, $(u, v)_{t_1} = (v, u)_{t_1}$.



Then, we can consider what a reasonable definition of 'shortest path' might be. In this model, once an edge exists, it is always traversible, so if author a_1 wrote a paper with author a_2 in 1932, and author a_2 wrote a paper with a_3 in 1990, we can find a path between a_1 and a_3. It is also feasible that a_1 wrote a paper with a_4 in 1932 as well, and then a_4 and a_3 wrote a paper in 1933. These two paths $P_1 = a_1 -_{1932} a_2 -_{1990} a_3$, and $P_2 = a_1 -_{1932} a_4 -_{1933} a_5$ should have some manner of differentation, since the difference in start times of the edges is 58 in P_1 and only 1 in P_1. This distinction motivates a difference in fastest vs. shortest path.

Definition 3.1.2. The **persistent shortest path** beteween u and v is a consecutive temporal path $u, v_1, \cdots, v_n, v$ such there exists no other $u, u_1, \cdots u_m, v$ such that $m < n$.

The **persistent fastest path** bewteen v_1 and v_n is consecutive temporal path $v_1, v_2, \cdots, v_n$, with first edge $(v_1, v_2)_{t_1}$ and last edge $(v_{n-1}, v_n)_{t_{n-1}}$, such that there exists no other $u_1, u_2, \cdots, u_m$ with first edge $(u_1, u_2)_{s_1}$, last edge $(u_{m-1}, u_m)_{s_{m-1}}$ and $s_{m-1} - s_1 < t_{n-1} - t_1$.

Corollary 3.1.3. *Shortest path in persistent coauthorship network is the same as the shortest path in the aggregated static graph.*

Proof. (Idea). Since the edges have infinite persistence, can just wait at a vertex until the desired edge in the aggregated graph shows up. □

3.2 Windowed edges

Now consider that we in fact limit the persistence of the edges with an endpoint specific to each edge (as is specified in the definition of an interval network temporal graph). We call this graph a **windowed co-authorship network** or simply a **co-authorship network**. If we specify a universal edge-persistence Δt such that for all edges (u, v, t_1, t_2) in the network, $\Delta t = t_2 - t_1$, then we call this co-authorship network **Δt-windowed**.

When we consider the above definition of a path it is clearly too simplistic, as it does not consider that an edge may cease to exist. So let us consider a new definition of a temporal path.

Definition 3.2.1. A **windowed temporal path** bewteen v_1 and v_n is a collection of edges $(v_1, v_2, s_1, t_1), (v_2, v_3, s_2, t_2), \cdots (v_{n-1}, v_n, s_{n-1}, t_{n-1})$ such that v_1 and v_n have degree n, and the remaining vertices have degree 2. Most importantly, $t_i \geq s_{i+1}$ for all $i \in [n-2]$.

The definitions for fastest and shortest path will be the same as in definition [?].

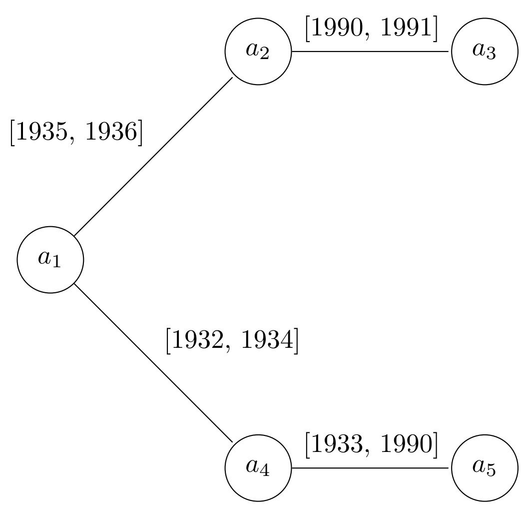
4 Pairwise Contemporaneity

Here we can consider many of the same definitions, but under are different lens of contemporaneity for paths. Here we want all edges to have some overlap in their time interval.

Definition 4.0.1. A **pairwise contemporary temporal path** between v_1 and v_n is a collection of edges $(v_1, v_2, s_1, t_1), (v_2, v_3, s_2, t_2), \cdots (v_{n-1}, v_n, s_{n-1}, t_{n-1})$, such that $\bigcap_{i \in [n-1]} [s_i, t_i] \neq \emptyset$.

As might be expected, pairwise contemporary temporal paths behave the same way that consecutive temporal paths do in the persistent co-authorship network. Since the edges have infinite persistence, all edges are contemporary 'at infinity.'

In the case of the windowed co-authorship network, the definitions remain the same for shortest and fastest paths.



In the above figure, there exists a pairwise contemporary path P_1 such that $P_1 = a_5 - a_4 - a_1$, but there does NOT exist a pairwise contemporary path P_2 such that $P_2 = a_5 - a_4 - a_1 - a_2$.