Temporal Graphs Research at Pomona College
Working Definitions and Vocabulary

Campbell, Wollman, Wu

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1 About

As we continue researching temporal graphs and their related properties there is an increased need to establish fundamental definitions. Although many papers have established definitions of the terms and concepts contained within this paper, there are subtle details of each of these definitions that vary from paper to paper making it difficult to efficiently communicate exact ideas.

This document shall serve as an ongoing record of the definitions we will use in our research. The definitions contained within should serve as the defaults in conversations; if you intend to use an alternate definition to a concept defined in this document, you should state and/or cite the alternate definition. (In the future, we will look to incorporate alternate definitions in this document as well.) It should be noted that these definitions may change as we continue our research.

2 Preliminary Definitions

Definition 2.0.1. A **temporal graph** or **temporal network**[1] is defined as a tuple $G = (V, E, \mathrm{T}, \mathcal{W})$ where V is the set of vertices, and $E \subset V^2$ is the set of edges. A graph is **directed** if $(u, v) \in E$ is an ordered pair, and is **undirected** if unordered.

T is a function $\mathrm{T} : E \to \mathbb{Z}^2$ with the co-domain being the start and end times of the vertices, in number of time units (you can always pick a smaller time unit so that the bounds of this interval are integral). Note that forall $e \in E$ with $\mathrm{T}(e) = (t_s, t_f)$, $t_s \leq t_f$. In a **static** graph, where $\mathrm{T}(e) = (\infty, \infty)$ for all $e \in E$, leave T out of the definition of the graph. $\mathcal{W}$ is a function

$\mathcal{W} : E \to \mathbb{Z}$ representing the integer weight of each edge. In an **unweighted** graph if every edge has the same weight, then we can leave $\mathcal{W}$ out of the definition.

All edges are to be treated as directed edges with some weight $\mathcal{W}$ and time-window T (possibly instantaneous) in which they are *active*. Additionally, we

[1] We will treat the terms *graph* and *network* synonymously throughout this paper.

will maintain the standard definitions of *size* and *order* defined respectively: $|E| = m$, $|V| = n$.

With this generalized definition of a temporal network, we will begin to explain some common, specialized descriptors of these networks:

Note that there is a direct transformation between undirected temporal graphs and directed temporal networks. To go from undirected to directed, for every edge $(e_l, e_r) \in E^2$, add edges $e_l \in E$ and $e_r \in E$ to the directed network. To go from directed to undirected, if for every u, v pair in the directed network, the number of edges of the form (u, v, t_0, t_f, w) is the same as the number of edges of the form (v, u, t_0, t_f, w), then pair up these edges; otherwise there is no transformation.

Definition 2.0.2. We will describe a temporal graph as **unweighted** if every edge in E has equal weight w.

Definition 2.0.3. **Edge persistence** is the amount of time for which an edge is present. More formally, in a graph $(V, E, \mathrm{T}, \mathcal{W})$ the persistence of an edge e is defined, for $T(e) = (t_s, t_f)$, as $|T(e)| = t_f - t_s$.

Definition 2.0.4. An edge $e \in E$ is said to be **infinitely persistent** if $T(e) = (t_s, \infty)$.

Definition 2.0.5. A **instantaneous edge** or **contact edge** is any edge $e \in E$ where $|T(e)| = 0$.

Definition 2.0.6. **Windowed networks** shall be defined as any temporal network where there is an edge e with $|\mathrm{T}(e)| \neq 0$.

Definition 2.0.7. **Contact networks** shall be defined as a temporal network where every edge in the network is an *instantaneous edge* or *contact edge*. It is simply a special case of an interval network.

Definition 2.0.8. An **infinitely-edge-persisting network** shall be defined as a temporal network where every edge is **infinitely persistent**.

Definition 2.0.9. A **co-authorship network** is an undirected interval network in which each node represents an author, and the existence of an edge between two nodes represents a collaboration over the a period of time. Unless stated otherwise, we shall assume a *collaboration* to mean two authors a_1 and a_2 worked on (at least) one publication together in $\mathrm{T}(a_1, a_2)$.

Definition 2.0.10. A **citation network** is a directed interval network $(V, E, \mathrm{T}, \mathcal{W})$ with infinite edge persistence. For simplicity, we can write edges as $u \rightarrow_{t_1} v$ or $(u, v)_{t_1}$, with $(t_1, \infty) = \mathrm{T}(u, v)$. In this network, each node represents an author, and the existence of an edge $u \rightarrow_{t_1} v$ means that author v cited author u at time t. This way the direction of the arrow represents the flow of information.

Now we will move on to consider the different analysis methodologies that will result from different temporal definitions of 'shortest path'.

Definition 2.0.11. A graph is a **path** if it is a simple graph whose vertices can be linearly ordered such that there is an edge uv if and only if u and v are adjacent in the ordering. A digraph is a **path** if it is a simple directed graph whose vertices can be linearly ordered such that there is an edge $u \to v$ if and only if v immediately follows u in the ordering.

Definition 2.0.12. A static **shortest path** between u, v in a static (di)graph $G = (V, E, \mathcal{W})$ is $P = e_1 = (u, u'), e_2, \cdots, e_n = (v', v)$, $P \in E$ such that for all $e_i = (v_1, v_2), e_{i+1} = (v_3, v_4) \in E$, $v_2 = v_3$, and for every u, v-path $P' = e'_1, e'_2, \cdots, e'_m$

$$\sum_{e'_i \in P'} \mathcal{W}(e'_i) < \sum_{e_i \in P} \mathcal{W}(e_i)$$

Note that this definition does not enforce uniqueness of shortest paths.

3 Consecutive Contemporaneity

Here, we consider the addition of paths to include a temporal component, where any two consecutive edges must share some contemporary period. This is a sensible definition as two edges should not be able to form a path in a temporal network if they did not happen at the same time. This is formally defined below.

Definition 3.0.13. A **consecutive temporal path** between u and v is a v_1, v_n-path $P = e_1, e_2, \cdots, e_n$ such that for consecutive edges $e_i, e_{i+1} \in P$, $T(e_i) \cap T(e_j) \neq \emptyset$.

We will consider the consequences of this definition in the context of different edge-behaviors, infinite persistence, and windowed persistence.

3.1 Infinitely Persistent Edges

The first model we will consider is the simplest of the three, where we disallow edge-deletion. We will define the persistent coauthorship network to have this property. [note about semantic equivalence to railway network?]

Definition 3.1.1. The **persistent co-authorship network** a co-authorship network $G = (V, E)$, where for all $(u, v, t_1, t_2) \in E$, $t_2 = \infty$. For simplicity, we can denote an edge by $(u, v)_{t_1}$ or $u -_{t_1} v$. Since this network is undirected, $(u, v)_{t_1} = (v, u)_{t_1}$.

Then, we can consider what a reasonable definition of 'shortest path' might be. In this model, once an edge exists, it is always traversible, so if author a_1 wrote a paper with author a_2 in 1932, and author a_2 wrote a paper with a_3 in 1990, we can find a path between a_1 and a_3. It is also feasible that a_1 wrote a paper with a_4 in 1932 as well, and then a_4 and a_5 wrote a paper in 1933. These two paths $P_1 = a_1 -_{1932} a_2 -_{1990} a_3$, and $P_2 = a_1 -_{1932} a_4 -_{1933} a_5$ should have some manner of differentation, since the difference in start times of the edges is 58 in P_1 and only 1 in P_1. This can be seen in Figure 1, to motivates a difference in 'fastest' vs. 'shortest' path.

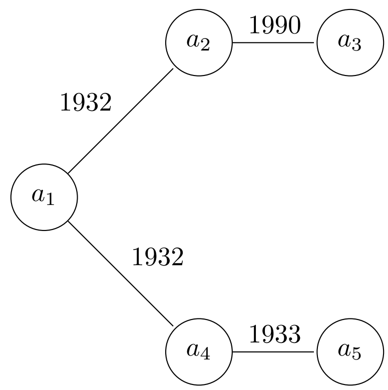

Figure 1: Example motivating the difference in shortest vs. fastest path

Definition 3.1.2. The **temporal shortest path** between u and v is a consecutive temporal u, v-path $P = e_1, \cdots, e_n$ such that that there is no other path u, v-path $P = e'_1, \cdots, e_m$ where

$$\sum_{e'_i \in P'} \mathcal{W}(e'_i) < \sum_{e_i \in P} \mathcal{W}(e_i)$$

The **temporal fastest path** from v_1 to v_n is a consecutive temporal path $v_1, v_2, \cdots, v_n$, with first edge $(v_1, v_2)_{t_1}$ and last edge $(v_{n-1}, v_n)_{t_{n-1}}$, such that there exists no other $u_1, u_2, \cdots, u_m$ with first edge $(u_1, u_2)_{s_1}$, last edge $(u_{m-1}, u_m)_{s_{m-1}}$ and $s_{m-1} - s_1 < t_{n-1} - t_1$.

Corollary 3.1.3. *Shortest path in persistent coauthorship network is the same as the shortest path in the aggregated static graph.*

Proof. (Idea). Since the edges have infinite persistence, can just wait at a vertex until the desired edge in the aggregated graph shows up. □

3.2 Windowed edges

Now consider that we in fact limit the persistence of the edges with an endpoint specific to each edge (as is specified in the definition of an interval network temporal graph). We call this graph a **windowed co-authorship network** or simply a **co-authorship network**. If we specify a universal edge-persistence Δt such that for all edges e in the network, $\Delta t = |T(e)|$, then we call this co-authorship network **Δt-windowed**.

The definitions for temporal paths will be the same as for infinitely persistent edges, 3.0.13. As well as those for fastest and shortest path will be the same as in definition 3.1.2.

4 Complete Contemporaneity

Here we can consider many of the same definitions, but under are different lens of contemporaneity for paths. Here we want all edges to have some overlap in their time interval.

Definition 4.0.1. A **complete temporal path** between v_1 and v_n is a collection of edges $(v_1, v_2, s_1, t_1), (v_2, v_3, s_2, t_2), \cdots (v_{n-1}, v_n, s_{n-1}, t_{n-1})$, such that $\bigcap_{i\in[n-1]}[s_i, t_i] \neq \emptyset$.

As might be expected, complete temporal paths behave the same way that consecutive temporal paths do in the persistent co-authorship network. Since the edges have infinite persistence, all edges are contemporary 'at infinity.'

In the case of the windowed co-authorship network, the definitions remain the same for shortest and fastest paths.

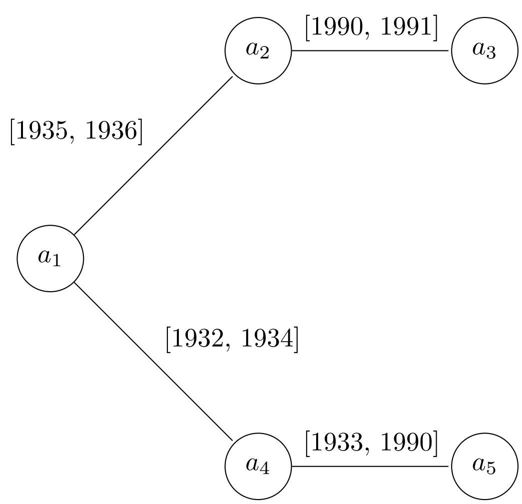

Figure 2: Example showing difference in windowed fastest and shortest paths

In Figure 2, there exists a complete contemporary path P_1 such that $P_1 = a_5a_4, a_4a_1$, but there does not exist a complete contemporary path P_2 such that $P_2 = a_5a_4, a_4a_1, a_1a_2$.