1. Show that $[x, y] = -[y, x]$ for all $x, y \in L$.

$$\begin{aligned}[x+y, x+y] =& [x, x+y] + [y, x+y] \\ 0 =& [x, x] + [x, y] + [y, x] + [y, y] \\ 0 =& [x, y] + [y, x] \\ -[y, x] =& [x, y]\end{aligned}$$

2. Suppose that $[x, y] \neq 0$, show that x and y are linearly independent. What about the converse?

   Since $[x, y] \neq 0$, $x \neq y$. If $x = ky$, then $[x, y] = [ky, y] = k[y, y] = 0$. Thus the two are linearly independent.

   If $[x, y] = 0$, then the two are linearly dependent because $[ky, y] = k[y, y] = 0$ for all k, so $x = ky$.

3. Consider the vector space $\mathbb{R}^3$.

   a) Show that the cross product is a bilinear map.

   If $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3$, then $k\vec{x} \times \vec{y} = \begin{pmatrix} kx_2y_3 - kx_3y_2 \\ kx_1y_3 - kx_3y_1 \\ kx_1y_2 - kx_2y_1 \end{pmatrix} = k \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_1y_3 - x_3y_1 \\ x_1y_2 - x_2y_1 \end{pmatrix} = k(\vec{x} \times \vec{y}$. It is easy to see that this is also linear in adding vectors and in y, making it a bilinear map.

   b) We know that $\vec{x} \times \vec{x} = 0$. Convince yourself that the Jacobi identity holds by checking it holds for the three standard basis vectors.

   $\vec{i} \times (\vec{j} \times \vec{z}) + \vec{j} \times (\vec{k} \times \vec{i}) + \vec{k} \times (\vec{i} \times \vec{j}) = \vec{i} \times \vec{i} + \vec{j} \times -\vec{j} + \vec{k} \times \vec{k} = 0 + 0 + 0 = 0$

   c) What about $\mathbb{R}^3$ with the dot product? Does this form a Lie algebra over $\mathbb{R}^3$?

   The dot product doesn't form a Lie bracket because it is defined: $\cdot : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, so it is not $L \times L \to L$.

   d) Can you think of some other examples of Lie algebras?

   Rotations.

4. Say V is a finite dimensional space over a field F. Let $gl(V)$ be the set of all linear maps from V to V. Then $gl(V)$ is a vector space over F. You may be familiar with rings in which case the composition of two linear maps can be thought of as a 'product'.

   a) Does $[f, g] = f \circ g$ form a Lie bracket? Why or why not? If not, can you think of an appropriate definition of a Lie bracket?

   $f \circ g$ does not form a Lie bracket because $f \circ f \neq 0$. An appropriate definition would be $[f, g] = f \circ g - g \circ f$.

   b) Based on your answer in part (a) show that $gl(V)$ with the Lie bracket $[f, g] =?$ satisfies the Jacobi identity. What important property of composition did you use? This example forms a Lie algebra known as the **general linear algebra**.

   Let $f, g, h \in gl(V)$. Then the Jacobi is:

$$\begin{aligned}[f, [g, h]] + [g, [h, f]] + [h, [f, g]] =& f \circ (g \circ h - h \circ g) - (g \circ h - h \circ g) \circ f \\ & + g \circ (h \circ f - f \circ h) - (h \circ f - f \circ h) \circ g \\ & + h \circ (f \circ g - g \circ f) - (f \circ g - g \circ f) \circ h \\ =& f \circ g \circ h - f \circ h \circ g - g \circ h \circ f + h \circ g \circ f \\ & + g \circ h \circ f - g \circ f \circ h - h \circ f \circ g + f \circ h \circ g \\ & + h \circ f \circ g - h \circ g \circ f - f \circ g \circ h + g \circ f \circ h \\ =& 0\end{aligned}$$

   We used the associative property of composition in showing that the Jacobi holds.

   c) Alternatively, we can associate to each linear map f a matrix M. in which case the general linear algebra is the vector space $gl(n, F)$: the vector space of $n \times m$ matrices over F with Lie bracket $[M, N] =?$

   We'll define $[M, N] = MN - NM$.

   d) Let $sl(n, F)$ be the subspace of $gl(n, F)$ consisting of all matrices of trace 0. Check the following:

i. If $M, N \in sl(n, F)$, then $[M, N]$ also has trace zero. Verify that $sl(n, F)$ is a Lie algebra. It is called the **special linear algebra**.
tr$([M, N])$ =tr$(MN - NM) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \alpha_{ij}\beta_{ji} - \alpha_{ji}\beta_{ij}\right)$. Note that for any given i, j there are corresponding i', j' where $i' = j$ and $j' = i$ such that the terms cancel out. Thus the trace is 0.

ii. What is a basis for $sl(n, F)$?
A basis for $sl(n, F)$ is all the matrices with a 1 not along the diagonal and 0s everywhere else, and $N - 1$ matrices with a 1 somewhere on the diagonal and a -1 in the bottom right corner.

iii. Consider $sl(2, \mathbb{C})$. Calculate the following Lie brackets:
$$[e_{11} - e_{22}, e_{12}], [e_{11} - e_{22}, e_{21}], [e_{12}, e_{21}]$$
where e_{ij} is the matrix with a 1 in the ij-entry and zeros everywhere else. What observations can you make from your above calculations?
$[e_{11} - e_{22}, e_{12}] = 2e_{12}$
$[e_{11} - e_{22}, e_{21}] = -2e_{21}$
$[e_{12}, e_{21}] = e_{11} - e_{22}$
From this we can see that e_{12} and e_{21} are eigenvectors.

5. Define the **center** of L to be $Z(L) := \{x \in L : [x, y] = 0 \text{ for all } y \in L\}$. Show that the center is an ideal and calculate the center of $sl(2, F)$.

   If $x \in Z(L)$, then $[x, y] = 0$ for all $y \in L$ and $0 \in Z(L)$, so it is an ideal. The center of $sl(2, F)$ is just 0.

6. Show $sl(2, \mathbb{C})$ has no non-trivial ideals.

   For this, instead of looking at a general element of $sl(2, \mathbb{C})$, we can just look at the bases:
$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

   If an ideal has some element $ah + bx + cy$ in it, then
$$\begin{aligned} [ah + bx + cy, x] =& [ah, x] + [bx, x] + [cy, x] = 2ax + ch \\ [2ax + ch, x] =& 2cx \\ [2cx, y] &= 2ch \\ [2ch, y] &= 4cy \end{aligned}$$

   Thus we can get all the basis vectors, so any ideal will be either just the 0 vector or all of $sl(2, \mathbb{C})$.

7. Discuss the differences between a subalgebra and an ideal. Can you think of an example of a non-trivial ideal/subalgebra of $gl(2, \mathbb{C})$. Can you think of a subalgebra of $gl(2, \mathbb{C})$ that is not an ideal? What about vice-versa? Why? In what other contexts have you learned about similar objects. What importance do they have?

   A subalgebra is basically a smaller Lie algebra contained in an algebra. An ideal is more of a black hole, if you bracket something in the ideal with something in the algebra, you get something in the ideal. A subalgebra of of $gl(2, \mathbb{C})$ is the set of all diagonal matrices in $gl(2, \mathbb{C})$, which is not an ideal. An ideal of the upper triangular matrices would be the matrices with just an element in the top right.

8. Let $L = sl(2, \mathbb{C})$ with basis $\{h = e_{11} - e_{22}, x = e_{12}, y = e_{21}\}$. What are the eigenvalues of $ad(h)$?

   The eigenvalues are $2, 1, -2, -1$, and 0. The table is:

	h	x	y
h	0	2x	-2y
x	-2x	0	h
y	2y	-h	0

9. Check that ad is a Lie algebra homomorphism. What is the kernel?

   If L is a Lie algebra with $x, y, z \in L$, then $ad([x, y])(z) = [[x, y], [z]] = [x, [y, z]] + [y, [z, x]] = [x, [y, z]] - [y, [x, z]]$. $[ad(x), ad(y)](z) = ad(x)ad(y)(z) - ad(y)ad(x)(z) = [x, [y, z]] - [y, [x, z]]$. The kernel of the adjoint is all $z \in Z(L)$ (the center of L), as $[x, z] = 0$ for all $z \in Z(L)$.

10. Is the kernel of any homomorphism an ideal/subalgebra of the domain? What about the image?

    The kernel of any homomorphism is an ideal (and subalgebra) of the domain. The image of a homomorphism is a subalgebra, but not necessarily an ideal of the domain.

11. Prove that given a Lie algebra L, then $[x, [y, z]] = [[x, y], z]$ (i.e., the bracket is associative) if and only if $[a, b]$ lies in $Z(L)$ for all $a, b \in L$.

    If $[x, [y, z]] = [[x, y], z]$, then

$$
\begin{aligned}
[x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= 0 \\
[x, [y, z]] - [[x, y], z] &= -[y, [z, x]] \\
0 &= -[y, [z, x]]
\end{aligned}
$$

    So for all $z, x \in L$, $[z, x] \in Z(L)$.

    Now if $[a, b] \in Z(L)$ for all $a, b \in L$, $[x, [y, z]] = [[x, y], z] = 0$.

12. Constructing new ideals. Let I, J be two ideals of L.

    a) Show that $I \cap J$ is an ideal.
       If $x \in I \cap J$ and $y \in L$, then $[x, y] \in I$ and $\in J$ because I and J are ideals and x is in both, thus $I \cap J$ is an ideal.

    b) Show that $I + J = \{x + y : x \in I, y \in J\}$ is an ideal.
       If $x \in I$, $y \in J$ and $z \in L$, then $[x + y, z] = [x, z] + [y, z]$ and $[x, z] \in I$ since I is an ideal and $[y, z] \in J$ since J is an ideal, thus the set is an ideal.

    c) Show that $[I, J] = span\{[x, y] : x \in I, y \in J\}$ is an ideal. Is the span necessary? To see an example of when the span is necessary see exercise 2.14 in the text on pages 16 and 17.
       Remark: $L' = [L, L]$ is called the **derived algebra**.
       If we have some $w = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{ij}[x_i, y_j] \in [I, J]$ and $z \in L$, then

$$
\begin{aligned}
[w, z] &= \left[\sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{ij}[x_i, y_j], z \right] \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} [\alpha_{ij}[x_i, y_j], z] \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{ij}[[x_i, y_j], z] \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{ij}([x_i, [y_i, z]] + [y_i, [z, x_i]])
\end{aligned}
$$

       From here, note that $[y_i, z] \in J \subseteq L$ and $[z, x_i] \in I \subseteq L$, so for each i and j, we get two things in I and J summed together, so it is closed. The span is necessary because if we didn't have it, then the sum of two things in I and J might not be in $[I, J]$.

13. What is the derived algebra of $sl(2, \mathbb{C})$?

    To find the derived algebra, we can just look at the basis vectors h, x, y, as defined in 8. From this we see that the derived algebra is all of $sl(2, \mathbb{C})$.

14. If I is an ideal, then l/I is a quotient vector space. To make it a Lie algeba, we need to put a Lie bracket on L/I. How would you define it? Check that your definition is well-defined.

    We'll represent elements of L/I as $x+I$ where $x \in L$, with the equivalence relation $x \cong y$ if there exist $i_1, i_2 \in I$ such that $x + i_1 = y + i_2$. We define the bracket as $[x+I, y+I] = [x, y] + I$. To check well-definedness, let $x, x', y, y' \in L$, with $x \cong x'$ and $y \cong y'$. Then:

$$\begin{aligned}[x+i_1, y+i_2] =& [x, y] + I \\ =& [x' + i_1', y' + i_2']\end{aligned}$$

    Therefore, the two brackets are equivalent, so the definition of the bracket is well-defined.

15. (Isomorphism Theorems) Let $\phi : L_1 \to L_2$ be a Lie algebra homomorphism.

    (a) $L_1/Ker\phi \cong Im\phi$
    We'll define a function $\psi : L_1/Ker\phi \to Im\phi$ as $\psi(a + Ker\phi) = \phi(a)$. If $a \cong b$ with $a + k = b + k'$ for $k, k' \in Ker\phi$, then

$$\begin{aligned}\psi(a) =& \psi(b + k' - k) \\ =& \psi(b) + \psi(k') - \psi(k) \\ =& \psi(b) + 0\end{aligned}$$

    It follows that the kernel is just the 0 of the quotient vector space, and it is clear that it is onto the image. Thus ψ is an isomorphism and the two are isomorphic.

    (b) If I and J are ideals, then $(I + J)/J \cong (I \cap J)$.
    For this, we'll use the first isomorphism theorem. We'll define a homomorphism $\phi : (I + J) \to I/(I \cap J)$ as $\phi(i + j) = i$. If $i + j, i' + j' \in I + J$, then $\phi([i + j, i' + j']) = \phi([i, i'] + [i, j'] + [j, i'] + [j, j'])$, but since J is an ideal, this is equal to $\phi([i, i'] + j^*) = [i, i']$ for some $j^* \in J$, so this does preserve the operation. The kernel is clearly J and the image is also obvious, so the two are isomorphic.

    (c) If I and J are ideals such that $i \subset J$, then J/I is an ideal of L/I and $(L/I)/(J/I) \cong L/J$.
    If $j + i \in J/I$ and $l + i' \in L/I$, then $[j + i, l + i'] = [j, l] + [j, i'] + [i, l] + [i, i']$, and since I and J are ideals, this is equal to $j^* + i^*$ for some $j^* \in J$ and $i^* \in I$, so it is an ideal.
    Next, we'll define a function $\phi : L/I \to L/J$ as $\phi(a + I) = a + J$. If $a, b \in L$ then

$$\begin{aligned}\phi([a + i, b + i']) =& \phi([a, b] + [a, i'] + [i, b] + [i, i'])] \\ =& [a, b] + J \\ =& [a + J, b + J] + J \\ =& [\phi(a + i), \phi(b + i')]\end{aligned}$$

    The kernel of this is J/I and it is definitely onto the image, so by the first isomorphism thoerem, the two are isomorphic.

16. Use the first isomorphism theorem to determine $gl(n, F)/sl(n, F)$.

    First, note that $sl(n, F)$ is an ideal of $gl(n, F)$ (we leave it as an exercise to the reader to confirm this). If we use the trace function as our homomorphism, $tr(gl(n, F))$ is onto F and has kernel $sl(n, F)$, so the quotient vector space is isomorphic to the 1-dimensional F.