

HW 8

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Chapter 12: #5

Prove the estimate

$$\|\tau_h\|_h^2 \leq 3(\|f\|_h^2 + \|f\|_{L^2(0,1)}^2) \quad (1)$$

[Hint: for each internal node $x_j, j = 1, \dots, n-1$, integrate by parts (12.21) to get

$$\tau_h(x_j) = -u''(x_j) - \frac{1}{h^2} \left[\int_{x_j-h}^{x_j} u''(t)(x_j - h - t)^2 dt - \int_{x_j}^{x_j+h} u''(t)(x_j + h - t)^2 dt \right].$$

Then, pass to the squares and sum $\tau_h(x_j)^2$ for $j = 1, \dots, n-1$. On noting that $(a+b+c)^2 \leq 3(a^2 + b^2 + c^2)$, for any real numbers a, b, c , and applying the Cauchy-Schwarz inequality yields the desired result.]

Solution:

Using Taylor expansion with integral remainder for $-u''(t) = f(t)$, we define the truncation error term. We have for all $j = 1, \dots, n-1$:

$$\tau_h(x_j) = f(x_j) - \frac{1}{h^2} \left[\int_{x_j-h}^{x_j} (-f(t))(x_j - h - t)^2 dt + \int_{x_j}^{x_j+h} f(t)(x_j + h - t)^2 dt \right]$$

Let's denote the terms as:

$$\begin{aligned} A_j &= f(x_j) \\ B_j &= \frac{1}{h^2} \int_{x_j-h}^{x_j} f(t)(x_j - h - t)^2 dt \\ C_j &= -\frac{1}{h^2} \int_{x_j}^{x_j+h} f(t)(x_j + h - t)^2 dt \end{aligned}$$

We note that for $a, b, c \in \mathbb{R}$, $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$. Hence, the squared truncation error is bounded by:

$$(\tau_h(x_j))^2 \leq 3(A_j^2 + B_j^2 + C_j^2) = 3((f(x_j))^2 + B_j^2 + C_j^2)$$

Now we estimate $|B_j|^2$ using Cauchy-Schwarz inequality:

$$\begin{aligned} |B_j|^2 &= \frac{1}{h^4} \left| \int_{x_j-h}^{x_j} f(t)(x_j - h - t)^2 dt \right|^2 \\ &\leq \frac{1}{h^4} \left(\int_{x_j-h}^{x_j} |f(t)|^2 dt \right) \left(\int_{x_j-h}^{x_j} (x_j - h - t)^4 dt \right) \end{aligned}$$

Using substitution $s = x_j - t$ (so $dt = -ds$), the limits change from $[x_j - h, x_j]$ to $[h, 0]$:

$$\int_{x_j-h}^{x_j} (x_j - h - t)^4 dt = \int_h^0 (s - h)^4 (-ds) = \int_0^h (s - h)^4 ds = \left[\frac{(s - h)^5}{5} \right]_0^h = \frac{h^5}{5}$$

Thus:

$$|B_j|^2 \leq \frac{1}{h^4} \left(\int_{x_j-h}^{x_j} |f(t)|^2 dt \right) \frac{h^5}{5} = \frac{h}{5} \int_{x_j-h}^{x_j} |f(t)|^2 dt$$

Similarly for C_j , using $s = t - x_j$:

$$\int_{x_j}^{x_j+h} (x_j + h - t)^4 dt = \frac{h^5}{5},$$

where we use similar arguments as above. Which yields:

$$|C_j|^2 \leq \frac{h}{5} \int_{x_j}^{x_j+h} |f(t)|^2 dt$$

Substituting these back into the error sum:

$$(\tau_h(x_j))^2 \leq 3 \left[(f(x_j))^2 + \frac{h}{5} \int_{x_j-h}^{x_j} |f(t)|^2 dt + \frac{h}{5} \int_{x_j}^{x_j+h} |f(t)|^2 dt \right]$$

Summing over $j = 1$ to $n - 1$:

$$\sum_{j=1}^{n-1} [\tau_h(x_j)]^2 \leq 3 \sum_{j=1}^{n-1} [f(x_j)]^2 + \frac{3h}{5} \sum_{j=1}^{n-1} \left(\int_{x_j-h}^{x_j} |f|^2 dt + \int_{x_j}^{x_j+h} |f|^2 dt \right)$$

We note that the sum of integrals covers the domain twice at most, so:

$$\sum_{j=1}^{n-1} \left(\int_{x_j-h}^{x_j} |f|^2 dt + \int_{x_j}^{x_j+h} |f|^2 dt \right) \leq 2 \int_0^1 |f(t)|^2 dt = 2 \|f\|_{L^2(0,1)}^2$$

Thus:

$$\sum_{j=1}^{n-1} [\tau_h(x_j)]^2 \leq 3 \sum_{j=1}^{n-1} [f(x_j)]^2 + \frac{6h}{5} \|f\|_{L^2}^2$$

Multiplying by h to get the discrete L^2 norm $\|\tau_h\|_h^2$:

$$\begin{aligned}\|\tau_h\|_h^2 &= h \sum_{j=1}^{n-1} [\tau_h(x_j)]^2 \\ &\leq 3h \sum_{j=1}^{n-1} [f(x_j)]^2 + \frac{6h^2}{5} \|f\|_{L^2}^2 \\ &= 3\|f\|_h^2 + \frac{6h^2}{5} \|f\|_{L^2}^2\end{aligned}$$

So the desired inequality holds.

Chapter 12: #7

Let $g = 1$ and prove that $T_h g(x_j) = \frac{1}{2}x_j(1 - x_j)$.

[Solution: use the definition (12.25) with $g(x_k) = 1, k = 1, \dots, n-1$ and recall that $G^k(x_j) = hG(x_j, x_k)$ from the exercise above. Then

$$T_h g(x_j) = h \left[\sum_{k=1}^j x_k(1 - x_j) + \sum_{k=j+1}^{n-1} x_j(1 - x_k) \right]$$

from which, after straightforward computations, one gets the desired result.]

Solution:

Compute $(T_h g)(x_j)$ where $g(x) = 1$. The discrete Green's operator is defined as:

$$(T_h g)(x_j) = h \sum_{k=1}^{n-1} G(x_j, x_k) g(x_k)$$

Since $g(x_k) = 1$, and using the definition of the discrete Green's function:

$$G(x_j, x_k) = \begin{cases} x_k(1 - x_j), & k \leq j \\ x_j(1 - x_k), & k > j \end{cases}$$

We split the sum:

$$(T_h g)(x_j) = h \left[\sum_{k=1}^j x_k(1 - x_j) + \sum_{k=j+1}^{n-1} x_j(1 - x_k) \right]$$

Using $x_k = kh$ and $h = 1/n$, so $\sum_{k=1}^j x_k = h \sum_{k=1}^j k = h \frac{j(j+1)}{2}$ and

$$\begin{aligned} h \sum_{k=1}^j x_k(1 - x_j) &= h(1 - x_j) \sum_{k=1}^j x_k = h(1 - x_j) \cdot h \frac{j(j+1)}{2} \\ &= h^2(1 - jh) \frac{j(j+1)}{2} \end{aligned}$$

For the second part, $\sum_{k=j+1}^{n-1} (1 - x_k) = \sum_{k=j+1}^{n-1} (1 - kh)$:

$$\begin{aligned} \sum_{k=j+1}^{n-1} (1 - kh) &= (n-1-j) - h \sum_{k=j+1}^{n-1} k \\ &= (n-1-j) - h \left(\frac{n(n-1)}{2} - \frac{j(j+1)}{2} \right) \end{aligned}$$

Thus:

$$(T_h g)(x_j) = h^2(1 - jh) \frac{j(j+1)}{2} + h x_j \left[(n-1-j) - h \left(\frac{n(n-1)}{2} - \frac{j(j+1)}{2} \right) \right]$$

Noting $x_j = jh$ and $nh = 1$:

$$\begin{aligned}(T_h g)(x_j) &= \frac{x_j(x_j + h)(1 - x_j)}{2} + \frac{x_j}{2}(1 - h - 2x_j + x_j^2 + x_jh) \\&= \frac{x_j}{2} \left[(x_j + h)(1 - x_j) + (1 - h - 2x_j + x_j^2 + x_jh) \right] \\&= \frac{x_j}{2} \left[(x_j - x_j^2 + h - hx_j) + 1 - h - 2x_j + x_j^2 + x_jh \right] \\&= \frac{1}{2}x_j(1 - x_j)\end{aligned}$$

Chapter 12: #8

Prove Young's inequality for any $a, b \in \mathbb{R}$ and $\epsilon > 0$:

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}$$

Solution:

We notice that for any $a, b \in \mathbb{R}$ and $\epsilon > 0$:

$$0 \leq \left(\sqrt{\epsilon}a - \frac{b}{2\sqrt{\epsilon}} \right)^2 = \epsilon a^2 - ab + \frac{b^2}{4\epsilon}$$

This implies:

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}$$

Chapter 12: #9

Prove that $\|v_h\|_h \leq \|v_h\|_{h,\infty}$ for all $v_h \in V_h$.

Solution:

From the definition of the discrete norm with trapezoidal weights:

$$\|v_h\|_h^2 = (v_h, v_h)_h = h \sum_{k=0}^n c_k v_k^2$$

where c_k are the weights ($c_0 = c_n = 1/2$, $c_k = 1$ for $0 < k < n$).

We know that for all k , $v_k^2 \leq \max_i |v_i|^2 = \|v_h\|_{h,\infty}^2$. Therefore:

$$h \sum_{k=0}^n c_k v_k^2 \leq h \left[\sum_{k=0}^n c_k \right] \|v_h\|_{h,\infty}^2$$

Calculating the sum of weights:

$$\sum_{k=0}^n c_k = \frac{1}{2} + \sum_{k=1}^{n-1} 1 + \frac{1}{2} = 1 + (n - 1) = n$$

Substituting this back and noting $h = 1/n$, so $hn = 1$:

$$\|v_h\|_h^2 = h \sum_{k=0}^n c_k v_k^2 \leq h \cdot n \cdot \|v_h\|_{h,\infty}^2 = \|v_h\|_{h,\infty}^2$$

Taking the square root gives:

$$\|v_h\|_h \leq \|v_h\|_{h,\infty}$$

Chapter 12: #11

Discretize the fourth-order differential operator $Lu(x) = -u^{(iv)}(x)$ using centered finite differences.

[Solution: apply twice the second order centered finite difference operator L_h defined in (12.9).]

Solution:

Let L_h be the discrete Laplacian operator (centered finite difference of second order):

$$(L_h u)_j = -\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}$$

To approximate the fourth-order operator $Lu = -u^{(iv)}$, we apply L_h twice. Let $v = L_h u$. We compute $(L_h v)_j$:

$$\begin{aligned} (L_h v)_j &= -\frac{v_{j+1} - 2v_j + v_{j-1}}{h^2} \\ &= -\frac{1}{h^2} \left[\left(-\frac{u_{j+2} - 2u_{j+1} + u_j}{h^2} \right) - 2 \left(-\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} \right) + \left(-\frac{u_j - 2u_{j-1} + u_{j-2}}{h^2} \right) \right] \end{aligned}$$

Factoring out terms and combining the $1/h^2$ factors:

$$\begin{aligned} (L_h^2 u)_j &= \frac{1}{h^4} [(u_{j+2} - 2u_{j+1} + u_j) - 2(u_{j+1} - 2u_j + u_{j-1}) + (u_j - 2u_{j-1} + u_{j-2})] \\ &= \frac{1}{h^4} [u_{j+2} - 4u_{j+1} + 6u_j - 4u_{j-1} + u_{j-2}] \end{aligned}$$

This corresponds to the standard centered finite difference approximation for the fourth derivative, i.e.,

$$L_h^2 u \approx u^{(iv)}(x)$$

Note: Since $Lu = -u^{(iv)}$, the discretized operator is consistent with $-L_h^2 u$ (depending on the sign convention of L_h). Based on the calculation above resulting in positive u_{j+2} , this represents the approximation for $u^{(iv)}$.