5.

Use hint and $-u''(t) = f(t)$, we have for all x_j, $j = 1, \dots, n-1$

$$\tau_h(x_j) = f(x_j) - \frac{1}{h^2}\left[\int_{x_j-h}^{x_j} (-f(t))(x_j-h-t)^2 dt - \int_{x_j}^{x_j+h} (-f(t))(x_j+h-t)^2 dt\right]$$

$$:= A_j + B_j + C_j$$

$$A_j = f(x_j), \quad B_j := -\frac{1}{h^2}\int_{x_j-h}^{x_j} f(t)(x_j-h-t)^2 dt, \quad C_j = \frac{1}{h^2}\int_{x_j}^{x_j+h} f(t)(x_j+h-t)^2 dt$$

We not that if $a, b, c \in \mathbb{R}$, then $(a+b+c)^2 \le 3(a^2+b^2+c^2)$

Hence $(\tau_h(x_j))^2 \le 3(A_j^2 + B_j^2 + C_j)^2 = 3((f(x_j))^2 + B_j^2 + C_j^2)$

For B_j:

$$|B_j|^2 = \frac{1}{h^4}\left|\int_{x_j-h}^{x_j} f(t)(x_j-h-t)^2 dt\right|^2 \le \frac{1}{h^4}\left(\int_{x_j-h}^{x_j} |f(t)|^2 dt\right)\left(\int_{x_j-h}^{x_j} (x_j-h-t)^4 dt\right)$$

Use $s = x_j - t$, we have

$$\int_{x_j-h}^{x_j} (x_j-h-t)^4 dt = \int_h^0 (s-h)^4(-ds) = \int_0^h (s-h)^4 ds = \frac{h^5}{5}$$

Hence

$$|B_j|^2 \le \frac{1}{h^4}\left(\int_{x_j-h}^{x_j} |f(t)|^2 dt\right)\cdot\frac{h^5}{5} = \frac{h}{5}\int_{x_j-h}^{x_j} |f(t)|^2 dt$$

For C_j, we also use $s = x_j - t$, we get

$$\int_{x_j}^{x_j+h} (x_j+h-t)^4 dt = \int_0^{-h} (s+h)^4(-ds) = \int_{-h}^0 (s+h)^4 ds = \frac{h^5}{5}$$

Hence $|C_j|^2 \le \frac{h}{5}\int_{x_j}^{x_j+h} |f(t)|^2 dt$

So $(\tau_h(x_j))^2 \le 3\left[(f(x_j))^2 + \frac{h}{5}\int_{x_j-h}^{x_j} |f(t)|^2 dt + \frac{h}{5}\int_{x_j}^{x_j+h} |f(t)|^2 dt\right]$

$$\Rightarrow \sum_{j=1}^{n-1}[\tau_h(x_j)]^2 \le 3\sum_{j=1}^{n-1}[f(x_j)]^2 + \frac{3h}{5}\sum_{j=1}^{n-1}\int_{x_j-h}^{x_j} |f|^2 dt + \frac{3h}{5}\sum_{j=1}^{n-1}\int_{x_j}^{x_j+h} |f|^2 dt$$

(We note that $\sum_{j=1}^{n-1}\int_{x_j-h}^{x_j} |f|^2 dt + \sum_{j=1}^{n-1}\int_{x_j}^{x_j+h} |f|^2 dt \le 2\int_0^1 |f(t)|^2 dt = 2\|f\|_{L^2(0,1)}^2$

$$\Rightarrow \sum_{j=1}^{n-1}[\tau_h(x_j)]^2 \le 3\sum_{j=1}^{n-1}[f(x_j)]^2 + \frac{6h}{5}\|f\|_{L^2(0,1)}^2$$

$$(\times h) \Rightarrow \|\tau_h\|_h^2 = h\sum_{j=1}^{n-1}[\tau_h(x_j)]^2 \le 3h\sum_{j=1}^{n-1}[f(x_j)]^2 + \frac{6h^2}{5}\|f\|_{L^2(0,1)}^2$$

$$= 3\|f\|_h^2 + \frac{6h^2}{5}\|f\|_{L^2(0,1)}^2$$

$$\le 3(\|f\|_h^2 + \|f\|_{L^2}^2)$$

#

7.

$(T_n g)(x_j) = h\sum_{k=1}^{n-1} G(x_j, x_k) g(x_k)$ where $g=1$

and we note that $G^k(x_j) = hG(x_j, x_k) = h\begin{cases} x_k(1-x_j), & k\le j \\ x_j(1-x_k), & k>j \end{cases}$

Hence $T_n g(x_j) = h\sum_{k=1}^{n-1} G(x_j, x_k)$

$$= h\left[\sum_{k=1}^{j} x_k(1-x_j) + \sum_{k=j+1}^{n-1} x_j(1-x_k)\right]$$

Let $h=\frac{1}{n}$, $x_k = kh$, $\sum_{k=1}^{j} x_k = \sum_{k=1}^{j} kh = h\left(\frac{j+1}{2}\right)j$

$$\Rightarrow h\sum_{k=1}^{j} x_k(1-x_j) = h^2\frac{j(j+1)}{2}(1-jh)$$

$$\sum_{k=j+1}^{n-1}(1-x_k) = \sum_{k=j+1}^{n-1}(1-kh)$$

$$= (n-1-j) - h\sum_{k=j+1}^{n-1} k$$

$$= (n-1-j) - h\left(\frac{n(n-1)}{2} - \frac{j(j+1)}{2}\right)$$

$$\Rightarrow h\sum_{k=j+1}^{n-1} x_j(1-x_k) = jh^2\left[(n-1-j) - h\left(\frac{n(n-1)}{2} - \frac{j(j+1)}{2}\right)\right]$$

So $T_n g(x_j) = \underbrace{h^2\frac{j(j+1)}{2}(1-jh)} + jh^2\left[(n-1-j) - h\left(\frac{n(n-1)}{2} \underbrace{- \frac{j(j+1)}{2}}\right)\right]$ (Use $n=\frac{1}{h}$, $x_j = jh$)

$$= x_j h\left[\left(\frac{1}{h} - 1 - j\right) - \frac{\frac{1}{h}-1}{2}\right] + \frac{x_j h(j+1)}{2}$$

$$= x_j h\left[\left(\frac{1}{h} - 1 - j\right) - \frac{1-h}{2h}\right] + \frac{1}{2}x_j^2 + \frac{1}{2}x_j h$$

$$= x_j - x_j h - x_j^2 - \frac{1-h}{2}x_j + \frac{1}{2}x_j^2 + \frac{1}{2}x_j h$$

$$= x_j - \cancel{x_j h} - x_j^2 - \frac{1}{2}x_j + \cancel{\frac{h}{2}x_j} + \frac{1}{2}x_j^2 + \cancel{\frac{1}{2}x_j h}$$

$$= \frac{1}{2}x_j(1-x_j)$$ #

8.

We notice that $0 \le \left(\sqrt{\varepsilon}\, a - \frac{b}{2\sqrt{\varepsilon}}\right)^2$ for $\forall \varepsilon > 0$ $\forall a, b \in \mathbb{R}$

$$= \varepsilon a^2 - ab + \frac{b^2}{4\varepsilon}$$

$$\Rightarrow ab \le \varepsilon a^2 + \frac{b^2}{4\varepsilon}$$

#

9.

From def, $\|V_n\|_h^2 = (V_n, V_n)_h = h \sum_{k=0}^{n} C_k V_k^2$

$\because$ $V_k^2 \le \|V_n\|_{h,\infty}^2$

$\therefore$ $h \sum_{k=0}^{n} C_k V_k^2 \le h\left[\sum_{k=0}^{n} C_k\right] \|V_n\|_{h,\infty}^2$

$\sum_{k=0}^{n} C_k = \frac{1}{2} + \frac{1}{2} + (n-1) \cdot 1 = n$

$$\Rightarrow \|V_n\|_h^2 = h \sum_{k=0}^{n} C_k V_k^2 \le h \cdot n \cdot \|V_n\|_{h,\infty}^2 \quad \left(h = \frac{1}{n}\right)$$

$$= \|V_n\|_{h,\infty}^2$$

$$\Rightarrow \|V_n\|_h \le \|V_n\|_{h,\infty}$$

#

11.

Let $v_j = (L_h u)_j = -\dfrac{u_{j+1} - 2u_j + u_{j-1}}{h^2}$

So $(L_h v)_j = -\dfrac{v_{j+1} - 2v_j + v_{j-1}}{h^2}$

$$= -\frac{1}{h^2}\left[-\frac{u_{j+2} - 2u_{j+1} + u_j}{h^2} + 2\,\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} - \frac{u_j - 2u_{j-1} + u_{j-2}}{h^2}\right]$$

$$= \frac{1}{h^4}\left[u_{j+2} - 4u_{j+1} + 6u_j - 4u_{j-1} + u_{j-2}\right]$$

$$= (L_h^2 u)_j$$

Here $L_h^2 u \approx -u^{(iv)}(x)$ #