5.

Since there are equally spaced nodes, $x_j = -1 + jh$

When $n = 2N$, $x_j = -1 + jh = (j-N)h$, $j = 0, 1, \ldots, 2N$

Hence

$$\begin{aligned} w_{n+1}(x) &= \prod_{j=0}^{n} (x - x_j) \\ &= \prod_{m=-N}^{N} (x - mh) \\ &= (x+Nh)(x+(N-1)h) \cdots (x-(N-1)h)(x-Nh) \end{aligned}$$

(8.74) holds

Consider $x \in (x_{n-1}, x_n) = (1-h, 1)$ and use $x = rh$ with $N-1 < r < N$

We have for any integer m, $\quad x - mh = h(r-m)$

We note that $\quad w_{n+1}(x) = (x - x_{n-1})(x - x_n) \cdot \underbrace{\prod_{j=0}^{n-2} (x - x_j)}$

$\quad \| \leftarrow x = rh,\ x_j = (j-N)h$

$$h^{n-1} \prod_{j=0}^{n-2} (r - (j-N)) = h^{n-1} \underbrace{\prod_{m=-N}^{N-2} (r-m)}_{\substack{\| \\ Q(r)}}$$

So $w_{n+1}(x) = h^{n-1} Q(r) (x - x_{n-1})(x - x_n)$

$\Rightarrow |w_{n+1}(x)| = h^{n-1} Q(r) |(x - x_{n-1})(x - x_n)|$

Hence it suffices to show that $(n-1)! \le Q(r) \le n!$ for any $r \in (N-1, N)$

We write $Q(r) = A(r) B(r)$ where $A(r) := (r+N)(r+N-1) \cdots (r+1)$

$B(r) := r(r-1) \cdots (r-(N-2))$

$A(r)$: For any $j = 1, \ldots, N$, $j+N-1 < r+j < j+N$

So $\prod_{j=1}^{N} (j+N-1) < A(r) < \prod_{j=1}^{N} (j+N)$

$\Rightarrow \dfrac{(2N-1)!}{(N-1)!} < A(r) < \dfrac{(2N)!}{N!}$

$B(r)$: For any $j = 0, \ldots, N-2$, $N-1-j < r-j < N-j$

So $\prod_{j=0}^{N-2} (N-1-j) < B(r) < \prod_{j=0}^{N-2} (N-j)$

$\Rightarrow (N-1)! < B(r) < N!$

Therefore $\dfrac{(2N-1)!}{(N-1)!} \cdot (N-1)! < Q(r) < \dfrac{(2N)!}{N!} N!$

$\Rightarrow (2N-1)! < Q(r) < (2N)!$

$(2N = n) \Rightarrow (n-1)! < Q(r) < n!$ $\quad \nearrow |w_{n+1}(x)| = h^{n-1} Q(r) |(x - x_{n-1})(x - x_n)|$

$\Rightarrow (n-1)!\, h^{n-1} |(x - x_{n-1})(x - x_n)| \le |w_{n+1}(x)| \le n!\, h^{n-1} |(x - x_{n-1})(x - x_n)|$ #

6.

From Exercise 5, we have $W_{n+1}(x) = \prod_{m=-N}^{N} (x-mh)$

$$W_{n+1}(x+h) = \prod_{m=-N}^{N} (x+h-mh) = \prod_{m=-N}^{N} (x-(m-1)h)$$

$$k=m-1 \longrightarrow = \prod_{k=-N-1}^{N-1} (x-kh)$$

$$\frac{W_{n+1}(x+h)}{W_{n+1}(x)} = \frac{x-(-N-1)h}{x-Nh} = \frac{x+(N+1)h}{x-Nh} = \frac{x+(N+1)h}{x-1}$$

$\Rightarrow \left|\frac{W_{n+1}(x+h)}{W_{n+1}(x)}\right| = \frac{x+(N+1)h}{1-x}$ since $x-1<0$, $x+(N+1)h>0$

To prove RHS >1, it is equivalent to $x+(N+1)h > 1-x \Leftrightarrow 2x+(N+1)h>1$

We note that $(N+1)h = (N+1)\frac{1}{N} = 1+\frac{1}{N}$

So $2x+(N+1)h = 2x+1+\frac{1}{N} > 1$ for all $x>0$

Hence we have $\left|\frac{W_{n+1}(x+h)}{W_{n+1}(x)}\right| > 1$ for any $x \in (0, x_{n-1})$

Fix $x_0 \in (0, x_{n+1})$, we always have $|W_{n+1}(x_0+h)| > |W_{n+1}(x)|$

So for any $x \in (0, x_{n-1})$, there exists a point in (x_{n-1}, x_n) s.t. its $|W_{n+1}|$ is larger than $|W_{n+1}(x)|$.

In other words, the maximum value of $|W_{n+1}|$ on $[0,1]$ cannot appear in $[0, x_{n-1}]$, but must appear in the last interval (x_{n-1}, x_n) #

8.

Define $T_n(x) := \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j$

So $T_n \in \mathbb{P}_n$, we let $Hf := T_n$

Verify $(Hf)^{(k)}(x_0) = f^{(k)}(x_0)$:

$$\frac{d^k}{dx^k}(x-x_0)^j = \begin{cases} \frac{j!}{(j-k)!}(x-x_0)^{j-k}, & j \geq k \\ 0 & j < k \end{cases}$$

Hence for T_n:

$$T_n^{(k)}(x) = \sum_{j=k}^{n} \frac{f^{(j)}(x_0)}{j!} \cdot \frac{j!}{(j-k)!}(x-x_0)^{j-k} = \sum_{j=k}^{n} \frac{f^{(j)}(x_0)}{(j-k)!}(x-x_0)^{j-k}$$

Let $x = x_0$, then $T_n^{(k)}(x_0) = \frac{f^{(k)}(x_0)}{(k-k)!} = f^{(k)}(x_0)$ for $k = 0, 1, \dots, n$

Uniqueness:

Suppose there is an another $p \in \mathbb{P}_n$ also satisfies $p^{(k)}(x_0) = f^{(k)}(x_0)$, $k = 0, \dots, n$

Consider $q := p - T_n$, then $q \in \mathbb{P}_n$ and for $k = 0, \dots, n$

$$q^{(k)}(x_0) = p^{(k)}(x_0) - T_n^{(k)}(x_0) = 0,$$

it implies that x_0 is a root of q and its multiplicity is at least $n+1$,

but $q \in \mathbb{P}_n$, it only happens when $q \equiv 0$

So we get $p \equiv T_n$, the uniqueness of Hf #

Assignment 2

$$x_k = \cos\left(\frac{k\pi}{n}\right), \quad k=0,1,\dots,n$$

$x_0 = 1$, $x_n = -1$, x_k for $k=1 \sim n-1$ are the roots of the Chebyshev poly. of the second kind $U_{n-1}(x)$

In barycentric Lagrange interpolation

$$w_k := \frac{1}{\prod_{m \neq k} (x_k - x_m)} \qquad k=0,\dots,n$$

To derive the weights, consider $w(x) = \prod_{k=0}^{n} (x - x_k)$

Then $w_k = \dfrac{1}{w'(x_k)}$

The polynomial with roots at x_k is proportional to $(1-x^2)U_{n-1}(x)$.

The Chebyshev polynomial $U_{n-1}(x)$ has leading coefficient 2^{n-1},

so $(1-x^2)U_{n-1}(x)$ $\cdots$ -2^{n-1}.

Thus, $$w(x) = \frac{(1-x^2)U_{n-1}(x)}{-2^{n-1}}$$

$$\Rightarrow w'(x) = \frac{-2x\,U_{n-1}(x) + (1-x^2)U'_{n-1}(x)}{-2^{n-1}}$$

Case1: $k=0,n$

At $x_0=1$: $1-x_0^2=0$, $U_{n-1}(1)=n$, so $w'(1) = \dfrac{-2n}{-2^{n-1}} = \dfrac{n}{2^{n-2}}$

Thus, $w_0 = \dfrac{2^{n-2}}{n}$

At $x_n=-1$: $1-x_n^2=0$, $U_{n-1}(-1) = (-1)^{n-1}n$, so $w'(-1) = \dfrac{2n\cdot(-1)^{n-1}}{-2^{n-1}} = (-1)^n \dfrac{n}{2^{n-2}}$

Thus, $w_n = \dfrac{2^{n-2}}{n\cdot(-1)^n} = \dfrac{2^{n-2}}{n}$

Case 2: $k=1 \sim n-1$

At x_k, $U_{n-1}(x_k)=0$, so $w'(x_k) = \dfrac{(1-x_k^2)U'_{n-1}(x_k)}{-2^{n-1}}$

Let $\theta_k = \dfrac{k\pi}{n}$, so $x_k = \cos\theta_k$, and $U_{n-1}(x) = \dfrac{\sin(n\theta)}{\sin\theta}$. At the roots, $\sin(n\theta_k)=0$

$$\frac{dU_{n-1}}{d\theta} = \frac{n\cos(n\theta)\sin\theta - \sin(n\theta)\cos\theta}{\sin^2\theta}, \qquad \left.\frac{dU_{n-1}}{d\theta}\right|_{\theta_k} = \frac{n\cos(n\theta_k)}{\sin\theta_k}$$

Since $\dfrac{dx}{d\theta} = -\sin\theta$, $U'_{n-1}(x_k) = \dfrac{dU_{n-1}}{dx} = \dfrac{dU_{n-1}/d\theta}{dx/d\theta} = -\dfrac{n\cos(n\theta_k)}{\sin^2\theta_k}$

Now $\cos(n\theta_k) = \cos(k\pi) = (-1)^k$, so $U'_{n-1}(x_k) = -\dfrac{n(-1)^k}{\sin^2\theta_k} = -\dfrac{n(-1)^k}{1-x_k^2}$

Substitute into $w'(x_k)$: $\quad w'(x_k) = \dfrac{-n(-1)^k}{-2^{n-1}} = \dfrac{n(-1)^k}{2^{n-1}}$

Thus $w_k = \dfrac{2^{n-1}}{n(-1)^k} = (-1)^k \dfrac{2^{n-1}}{n}$

The computed weights are $(k=1 \sim n-1)$: $\begin{cases} w_k = (-1)^k \dfrac{2^{n-1}}{n} \\ w_0 = \dfrac{1}{2}\dfrac{2^{n-1}}{n} \\ w_n = (-1)^n \cdot \dfrac{1}{2}\dfrac{2^{n-1}}{n} \end{cases} \xRightarrow{\text{To rescale, divide all weights by } \frac{2^{n-1}}{n}} \begin{cases} w_k = (-1)^k, \quad k=1 \sim n-1 \\ w_0 = \dfrac{1}{2} \\ w_n = (-1)^n \dfrac{1}{2} \end{cases}$ #