7.

$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$

$\Gamma(z+1) = \int_0^\infty e^{-t} t^{z} dt$

Let $u = t^z$, $dv = e^{-t} dt$

$du = z t^{z-1} dt$, $v = -e^{-t}$

Hence $\Gamma(z+1) = \left[-e^{-t} t^z\right]_0^\infty + \int_0^\infty e^{-t} z t^{z-1} dt$ by I.V.P.

$= \lim_{t\to\infty} -e^{-t} t^z - \lim_{t\to 0^+} (\underbrace{-e^{-t}}_{\downarrow \, -1} \underbrace{t^z}_{\downarrow \, 0 \; \mathrm{Re}\, z > 0}) + z \int_0^\infty e^{-t} t^{z-1} dt$

$= 0 - 0 + z \int_0^\infty e^{-t} t^{z-1} dt$

$= z \Gamma(z)$

#

9.

$u_{n+1}=u_n+h\left[(1-\frac{\alpha}{2})f(x_n,u_n)+\frac{\alpha}{2}f(x_{n+1},u_{n+1})\right]$ ①

Let $y(x)$ be the true solution, and expand at x_n:

$$y_{n+1}=y_n+hy_n'+\frac{h^2}{2}y_n''+O(h^3)$$

We note that $y_n'=f(x_n,y_n)$, so

$$f(x_{n+1},y_{n+1})=f(x_n,y_n)+hf_x+hf_y y_n'+O(h^2) \quad \text{代入①}$$

In ①, if $u_n=y_n$, then RHS becomes

$$\begin{aligned}\text{RHS}&=y_n+h\left[f(x_n,y_n)+\frac{\alpha}{2}\left(f(x_{n+1},y_{n+1})-f(x_n,y_n)\right)\right]\\&=y_n+hf(x_n,y_n)+\frac{\alpha}{2}h^2(f_x+f_y f)+O(h^3)=y_n+hy_n'+\frac{\alpha}{2}h^2y_n''+\frac{\alpha}{4}h^3y_n'''+O(h^4)\end{aligned}$$

$$\begin{aligned}\Rightarrow y_{n+1}&=y_n+hf+\frac{h^2}{2}(f_x+f_y f)+O(h^3)\\&=y_n+hy_n'+\frac{h^2}{2}y_n''+\frac{h^3}{6}y_n'''+O(h^4)\end{aligned}$$

Local truncation error:

$$\tau=y_{n+1}-\text{RHS}=\underbrace{(\frac{1}{2}-\frac{\alpha}{2})h^2y_n''}+(\frac{1}{6}-\frac{\alpha}{4})h^3y_n'''+O(h^4)$$

If $\alpha\neq1$, $\frac{1-\alpha}{2}h^2y_n''\neq0\Rightarrow\tau=O(h^2)\Rightarrow$ order $=1$

If $\alpha=1$, $\frac{1-\alpha}{2}=0$, $(\frac{1}{6}-\frac{1}{4})h^3y_n'''=-\frac{1}{12}h^3y_n'''\Rightarrow\tau=O(h^3)\Rightarrow$ order $=2$

Use $\alpha=1$ on $y'=-10y$:

$$u_{n+1}=u_n+\frac{h}{2}\left[-10u_n-10u_{n+1}\right]$$

$$\Rightarrow u_{n+1}=\frac{1-5h}{1+5h}u_n,\quad u_0=1$$

$$\Rightarrow u_n=\left(\frac{1-5h}{1+5h}\right)^n$$

Define stability function: $R(z)=\dfrac{1+\frac{z}{2}}{1-\frac{z}{2}}$, $z=h\lambda$

Here $\lambda=-10$, so $z=-10h\Rightarrow R(z)=\dfrac{1-5h}{1+5h}$

$$\Rightarrow|R(z)|=\left|\frac{1-5h}{1+5h}\right|<1 \text{ if } \boxed{h>0}$$

$\Rightarrow$ It is absolutely stable #