1

We note that for the solution of $-u''=f$, $u(0)=u(1)=0$

We can use green function to represent the solution u

$$u(x)=\int_0^1 G(x,s)f(s)ds, \quad G(x,s)=\begin{cases} s(1-x), & 0\le s\le x \\ x(1-s), & x\le s\le 1 \end{cases}$$

Let $f(s)=\frac{1}{s}$

$$\begin{aligned} u(x) &= \int_0^x s(1-x)\frac{1}{s}ds + \int_x^1 x(1-s)\frac{1}{s}ds \\ &= (1-x)x + x\int_x^1 \frac{1}{s}-1\,ds \\ &= (1-x)x + x(-\ln x-(1-x)) \\ &= (1-x)x - x\ln x - x(1-x) \\ &= -x\ln x \end{aligned}$$

$\Rightarrow u(x)\in C^2(0,1)$

$u'(x)=-1-\ln x$

$u''(x)=-\frac{1}{x}$

At $x=0$, $\ln 0$ is undefined

$\lim_{x\to 0^+} u'(x)=\infty \Rightarrow u'$ doesn't exist at $x=0$

$\lim_{x\to 0^+} u''(x)=-\infty \Rightarrow u''$ doesn't exist at $x=0$

Hence we can see that if $f\in C^0(0,1)$ but $f\notin C^0([0,1])$, then $u\notin C^2([0,1])$.

4. Cerify the summation by parts formula

$$\sum_{j=0}^{n-1}(w_{j+1}-w_j)v_j = w_n v_n - w_0 v_0 - \sum_{j=0}^{n-1}(v_{j+1}-v_j)w_{j+1}$$

and show that for $v_n \in \mathring{V}_n$,

$$(L_n v_n, v_n)_n = h^{-1}\sum_{j=0}^{n-1}(v_{j+1}-v_j)^2$$

Sol: ① $\displaystyle\sum_{j=0}^{n-1} w_{j+1} v_j = \sum_{j=0}^{n-1} w_{j+1} v_j - \sum_{j=0}^{n-1} w_j v_j$

$$= \left[\sum_{k=1}^{n} w_k v_{k-1}\right] - \left[\sum_{k=0}^{n-1} w_k v_k\right]$$

$$= \left[w_n v_{n-1} + \sum_{k=1}^{n-1} w_k v_{k-1}\right] - \left[\sum_{k=1}^{n-1} w_k v_k + w_0 v_0\right]$$

$$= w_n v_n - w_0 v_0 + \sum_{k=1}^{n-1} w_k (v_{k-1} - v_k)$$

$$= w_n v_n - w_0 v_0 - \sum_{j=1}^{n-1} w_j (v_j - v_{j-1})$$

$$= w_n v_n - w_0 v_0 - \sum_{j=0}^{n-1} (v_{j+1} - v_j) w_{j+1}$$

② $\displaystyle (L_n v_n, v_n)_n = h\sum_{j=1}^{n-1}(L_n v_n)_j v_j \quad (\because v_0 = v_n = 0)$

$$= h\sum_{j=1}^{n-1}\left[-\frac{v_{j+1}-2v_j+v_{j-1}}{h^2}\right]v_j$$

$$= -\frac{1}{h}\sum_{j=1}^{n-1}(v_{j+1}-2v_j+v_{j-1})v_j$$

$$= -\frac{1}{h}\left[\sum_{j=1}^{n-1}(v_{j+1}-v_j)v_j - \sum_{j=1}^{n-1}(v_j - v_{j-1})v_j\right]$$

$$= -\frac{1}{h}\left[\sum_{j=1}^{n-1}(v_{j+1}-v_j)v_j - \sum_{k=0}^{n-2}(v_{k+1}-v_k)v_{k+1}\right] \quad (k = j-1)$$

$$= -\frac{1}{h}\left[\sum_{j=0}^{n-1}(v_{j+1}-v_j)v_j - \sum_{j=0}^{n-1}(v_{j+1}-v_j)v_{j+1}\right] \quad (\because v_0 = v_n = 0)$$

$$= -\frac{1}{h}\left[-\sum_{j=0}^{n-1}(v_{j+1}-v_j)^2\right]$$

$$= h^{-1}\sum_{j=0}^{n-1}(v_{j+1}-v_j)^2$$

#

6.

For fixed $s = x_k$ $(0 \le x_k \le 1)$

$$G(x, x_k) = \begin{cases} x(1-x_k), & x \le x_k \\ x_k(1-x), & x \ge x_k \end{cases}$$

(A) $j \neq k$

Fix k, for all $j<k$, the continuous Green's function $G(x, x_k)$ is a linear function on $[x_{j-1}, x_{j+1}]$.

Hence, its second finite difference is zero.

$$G(x_{j+1}, x_k) - 2G(x_j, x_k) + G(x_{j-1}, x_k) = 0 \qquad j<k$$

For $j>k$, we also have similar arguments: $G(x, x_k) = x_k(1-x)$,

so $G(x_{j+1}, x_k) - 2G(x_j, x_k) + G(x_{j-1}, x_k) = 0$

Hence for $j \neq k$, $(L_h(hG))_j = -\dfrac{h(G_{j+1} - 2G_j + G_{j-1})}{h^2} = 0$

(B) $j = k$

We need to compute $(L_h(hG))_k = -\frac{1}{h}(G_{k+1} - 2G_k + G_{k-1})$

Define $G_j := G(x_j, x_k)$

$x_j = jh$, $a := x_k = kh$

$G_{k-1} = x_{k-1}(1-x_k) = (k-1)h(1-a)$

$G_k = x_k(1-x_k) = a(1-a)$

$G_{k+1} = x_k(1-x_{k+1}) = a(1-(k+1)h) = a(1-a-h)$

$$\begin{aligned} G_{k+1} - 2G_k + G_{k-1} &= a(1-a-h) - 2a(1-a) + (k-1)h(1-a) \\ &= [a(1-a) - ah] - 2a(1-a) + (k-1)h(1-a) \\ &= -a(1-a) - ah + (k-1)h(1-a) \\ &= -a(1-a) - ah + a(1-a) - h(1-a) \\ &= -h \end{aligned}$$

$\Rightarrow (L_h(hG))_k = -\frac{1}{h}(G_{k+1} - 2G_k + G_{k-1}) = 1$

(C) At $j = 0$ or n, $G(x_j, x_k) = 0$, so $(hG)_0 = (hG)_n = 0$

From (A) and (B), we proved that $(L_h(hG))_j = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases} \Rightarrow L_h(hG(\cdot, x_k)) = e_k$

Since discrete problem $L_h w = e_k$ (with Dirichlet B.C.) has unique sol., the sol. must be G^k.

$G^k(x_j) = (hG(\cdot, x_k))(x_j) = hG(x_j, x_k)$ for all j #