1.

Let $g(s) := f(s, y(s))$

Use Taylor expansion on $g(t_n+h\theta)$ about h,

we get $\int_0^h g(t_n+s)ds = hg(t_n) + \frac{h^2}{2}g'(t_n) + \frac{h^3}{6}g''(t_n) + O(h^4)$ ①

and $\frac{h}{2}(g(t_n)+g(t_{n+1})) = hg(t_n) + \frac{h^2}{2}g'(t_n) + \frac{h^3}{4}g''(t_n) + O(h^4)$

$\Rightarrow E_1 = (\frac{1}{6}-\frac{1}{4})h^3 g''(t_n) + O(h^4) = -\frac{h^3}{12}g''(t_n) + O(h^4)$

$\Rightarrow E_1 = O(h^3)$

Now for E_2:

$y_{n+1} - (y_n + hf(t_n, y_n)) = \int_{t_n}^{t_{n+1}} f(s, y(s))ds - hf(t_n, y_n)$

From ①, we have $\int_{t_n}^{t_{n+1}} g(s)ds = hg(t_n) + \frac{h^2}{2}g'(t_n) + O(h^3)$

Hence $y_{n+1} - (y_n + hf(t_n, y_n)) = \frac{h^2}{2}g'(t_n) + O(h^3) = O(h^2)$

Now fix $t = t_{n+1}$, use Taylor expansion on $f(t_{n+1}, \cdot)$ at $y_n + hf(t_n, y_n)$

$f(t_{n+1}, y_{n+1}) = f(t_{n+1}, y_n + hf(t_n, y_n)) + f_y(t_{n+1}, \xi)(y_{n+1} - (y_n + hf(t_n, y_n)))$

where $\xi \in (y_n, y_{n+1})$, and assume f_y is bounded on $[y_n, y_{n+1}]$

Hence $f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y_n + hf(t_n, y_n)) = O(h^2)$

$\xrightarrow{\times \frac{h}{2}} E_2 = O(h^3)$

So $hT_{n+1} = E_1 + E_2 = O(h^3) + O(h^3) = O(h^3)$

$\Rightarrow T_{n+1} = O(h^2)$ #

2.

$y'(t) = f(t, y(t))$, $y(t_0) = y_0$

Let $t_n = t_0 + hn$

$y_n := y(t_n)$

$g(t) := f(t, y(t))$

Crank Nicolson: $y_{n+1} = y_n + \frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right)$

We know that $y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} g(s)\,ds$

$\forall$ interval $[t_n, t_{n+1}]$, $\exists\, \xi_n \in (t_n, t_{n+1})$

s.t. $\int_{t_n}^{t_{n+1}} g(s)\,ds = \frac{h}{2}\left(g(t_n) + g(t_{n+1})\right) - \frac{h^3}{12} g''(\xi_n)$ (Use similar argument in #1)

$\Rightarrow y(t_{n+1}) - y(t_n) = \frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right) - \frac{h^3}{12} g''(\xi_n)$

$\Rightarrow \frac{y(t_{n+1}) - y(t_n)}{h} = \frac{1}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right) - \frac{h^2}{12} g''(\xi_n)$

$\Rightarrow y(t_{n+1}) - y(t_n) - \frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right) = -\frac{h^3}{12} g''(\xi_n) = h\,\tau_{n+1}$

$\Rightarrow \tau_{n+1} = O(h^2)$ #