Chapter 12 Exercises

Shih-Ming Wang 412652004

Chapter 12: #1

Consider the boundary value problem $-u''(x) = f(x), 0 < x < 1, u(0) = u(1) = 0$ with $f(x) = 1/x$. Using $u(x) = \int_0^1 G(x,s)f(s)\,ds$ to prove that $u(x) = -x\log(x)$. This shows that $u \in C^2(0,1)$ but $u(0)$ is not defined and u', u'' do not exist at $x = 0$ ($\Rightarrow$: if $f \in C^0(0,1)$, but not $f \in C^0([0,1])$, then u does not belong to $C^0([0,1])$).

Solution:

We note that for the solution of $-u'' = f$ with boundary conditions $u(0) = u(1) = 0$, we can use the Green's function to represent the solution u:

$$u(x) = \int_0^1 G(x,s)f(s)\,ds$$

where

$$G(x,s) = \begin{cases} s(1-x), & 0 \le s \le x \\ x(1-s), & x \le s \le 1 \end{cases}$$

Let $f(s) = \frac{1}{s}$. Computing $u(x)$:

$$\begin{aligned} u(x) &= \int_0^x s(1-x)\frac{1}{s}\,ds + \int_x^1 x(1-s)\frac{1}{s}\,ds \\ &= (1-x)\int_0^x 1\,ds + x\int_x^1 \left(\frac{1}{s} - 1\right)\,ds \\ &= (1-x)x + x\Big[\ln s - s\Big]_x^1 \\ &= (1-x)x + x\Big((\ln 1 - 1) - (\ln x - x)\Big) \\ &= x - x^2 + x(-1 - \ln x + x) \\ &= x - x^2 - x - x\ln x + x^2 \\ &= -x\ln x \end{aligned}$$

This implies $u(x) \in C^2(0,1)$. Now, let's examine the derivatives:

$$u'(x) = -1 - \ln x$$
$$u''(x) = -\frac{1}{x}$$

At $x = 0$, the function is undefined. Checking the limits:

- $\lim_{x \to 0^+} u'(x) = \infty \implies u'$ does not exist at $x = 0$.
- $\lim_{x \to 0^+} u''(x) = -\infty \implies u''$ does not exist at $x = 0$.

Hence, we can see that while $f \in C^0(0,1)$, it is not in $C^0([0,1])$. Consequently, $u \notin C^2([0,1])$.

Chapter 12: #5

Prove the estimate

$$\|\tau_h\|_h^2 \leq 3(\|f\|_h^2 + \|f\|_{L^2(0,1)}^2) \tag{1}$$

[Hint: for each internal node $x_j, j = 1, \ldots, n-1$, integrate by parts (12.21) to get

$$\tau_h(x_j) = -u''(x_j) - \frac{1}{h^2}\left[\int_{x_j-h}^{x_j} u''(t)(x_j - h - t)^2\,dt - \int_{x_j}^{x_j+h} u''(t)(x_j + h - t)^2\,dt\right].$$

Then, pass to the squares and sum $\tau_h(x_j)^2$ for $j = 1, \ldots, n-1$. On noting that $(a+b+c)^2 \leq 3(a^2 + b^2 + c^2)$, for any real numbers a, b, c, and applying the Cauchy-Schwarz inequality yields the desired result.]

Solution:

Using Taylor expansion with integral remainder for $-u''(t) = f(t)$, we define the truncation error term. We have for all $j = 1, ..., n-1$:

$$\tau_h(x_j) = f(x_j) - \frac{1}{h^2}\left[\int_{x_j-h}^{x_j} (-f(t))(x_j - h - t)^2\,dt + \int_{x_j}^{x_j+h} f(t)(x_j + h - t)^2\,dt\right]$$

Let's denote the terms as:

$$\begin{aligned}
A_j &= f(x_j) \\
B_j &= \frac{1}{h^2}\int_{x_j-h}^{x_j} f(t)(x_j - h - t)^2\,dt \\
C_j &= -\frac{1}{h^2}\int_{x_j}^{x_j+h} f(t)(x_j + h - t)^2\,dt
\end{aligned}$$

We note that for $a, b, c \in \mathbb{R}$, $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$. Hence, the squared truncation error is bounded by:

$$(\tau_h(x_j))^2 \leq 3(A_j^2 + B_j^2 + C_j^2) = 3((f(x_j))^2 + B_j^2 + C_j^2)$$

Now we estimate $|B_j|^2$ using Cauchy-Schwarz inequality:

$$\begin{aligned}
|B_j|^2 &= \frac{1}{h^4}\left|\int_{x_j-h}^{x_j} f(t)(x_j - h - t)^2\,dt\right|^2 \\
&\leq \frac{1}{h^4}\left(\int_{x_j-h}^{x_j} |f(t)|^2\,dt\right)\left(\int_{x_j-h}^{x_j} (x_j - h - t)^4\,dt\right)
\end{aligned}$$

Using substitution $s = x_j - t$ (so $dt = -ds$), the limits change from $[x_j - h, x_j]$ to $[h, 0]$:

$$\int_{x_j-h}^{x_j} (x_j - h - t)^4\,dt = \int_h^0 (s-h)^4(-ds) = \int_0^h (s-h)^4\,ds = \left[\frac{(s-h)^5}{5}\right]_0^h = \frac{h^5}{5}$$

Thus:

$$|B_j|^2 \le \frac{1}{h^4}\left(\int_{x_j-h}^{x_j} |f(t)|^2\,dt\right)\frac{h^5}{5} = \frac{h}{5}\int_{x_j-h}^{x_j} |f(t)|^2\,dt$$

Similarly for C_j, using $s = t - x_j$:

$$\int_{x_j}^{x_j+h} (x_j + h - t)^4\,dt = \frac{h^5}{5},$$

where we use similar arguments as above. Which yields:

$$|C_j|^2 \le \frac{h}{5}\int_{x_j}^{x_j+h} |f(t)|^2\,dt$$

Substituting these back into the error sum:

$$(\tau_h(x_j))^2 \le 3\left[(f(x_j))^2 + \frac{h}{5}\int_{x_j-h}^{x_j} |f(t)|^2\,dt + \frac{h}{5}\int_{x_j}^{x_j+h} |f(t)|^2\,dt\right]$$

Summing over $j = 1$ to $n-1$:

$$\sum_{j=1}^{n-1}[\tau_h(x_j)]^2 \le 3\sum_{j=1}^{n-1}[f(x_j)]^2 + \frac{3h}{5}\sum_{j=1}^{n-1}\left(\int_{x_j-h}^{x_j} |f|^2\,dt + \int_{x_j}^{x_j+h} |f|^2\,dt\right)$$

We note that the sum of integrals covers the domain twice at most, so:

$$\sum_{j=1}^{n-1}\left(\int_{x_j-h}^{x_j} |f|^2\,dt + \int_{x_j}^{x_j+h} |f|^2\,dt\right) \le 2\int_0^1 |f(t)|^2\,dt = 2\|f\|^2_{L^2(0,1)}$$

Thus:

$$\sum_{j=1}^{n-1}[\tau_h(x_j)]^2 \le 3\sum_{j=1}^{n-1}[f(x_j)]^2 + \frac{6h}{5}\|f\|^2_{L^2}$$

Multiplying by h to get the discrete L^2 norm $\|\tau_h\|_h^2$:

$$\begin{aligned}
\|\tau_h\|_h^2 &= h\sum_{j=1}^{n-1}[\tau_h(x_j)]^2 \\
&\le 3h\sum_{j=1}^{n-1}[f(x_j)]^2 + \frac{6h^2}{5}\|f\|^2_{L^2} \\
&= 3\|f\|_h^2 + \frac{6h^2}{5}\|f\|^2_{L^2}
\end{aligned}$$

So the desired inequality holds.

Chapter 12: #6

Prove that $G^k(x_j) = hG(x_j, x_k)$, where G is Green's function introduced in (12.4) and G^k is its corresponding discrete counterpart solution of (12.4).

[Solution: we prove the result by verifying that $L_h G = he^k$. Indeed, for a fixed x_k the function $G(x_k, s)$ is a straight line on the intervals $[0, x_k]$ and $[x_k, 1]$ so that $L_h G = 0$ at every node x_l with $l = 0, \ldots, k-1$ and $l = k+1, \ldots, n+1$. Finally, a direct computation shows that $(L_h G)(x_k) = 1/h$ which concludes the proof.]

Solution:

For fixed $s = x_k$ (where $0 < x_k < 1$), let:

$$G(x, x_k) = \begin{cases} s(1 - x_k), & x \le x_k \\ x_k(1 - x), & x \ge x_k \end{cases}$$

(A) Fix k. For all $j \neq k$, the continuous Green's function $G(x, x_k)$ is a linear function on $[x_{j-1}, x_{j+1}]$. Hence, its second finite difference is zero:

$$G(x_{j+1}, x_k) - 2G(x_j, x_k) + G(x_{j-1}, x_k) = 0, \quad \text{for } j < k$$

For $j > k$, we have similar arguments since $G(x, x_k) = x_k(1 - x)$ is linear in x. Thus:

$$(L_h(hG))_j = -\frac{h(G_{j+1} - 2G_j + G_{j-1})}{h^2} = 0, \quad \text{for } j \neq k$$

(B) For $j = k$, we need to compute:

$$(L_h(hG))_k = -\frac{1}{h}(G_{k+1} - 2G_k + G_{k-1})$$

Define $G_j = G(x_j, x_k)$. Let $a := x_k = kh$.

$$\begin{aligned} G_{k-1} &= x_{k-1}(1 - x_k) = (k-1)h(1 - a) \\ G_k &= x_k(1 - x_k) = a(1 - a) \\ G_{k+1} &= x_k(1 - x_{k+1}) = a(1 - (k+1)h) = a(1 - a - h) \end{aligned}$$

Calculating the second difference:

$$\begin{aligned} G_{k+1} - 2G_k + G_{k-1} &= a(1 - a - h) - 2a(1 - a) + (k-1)h(1 - a) \\ &= [a(1 - a) - ah] - 2a(1 - a) + (kh - h)(1 - a) \\ &= -a(1 - a) - ah + a(1 - a) - h(1 - a) \quad (\text{since } kh = a) \\ &= -ah - h + ah \\ &= -h \end{aligned}$$

Therefore:

$$(L_h(hG))_k = -\frac{1}{h}(-h) = 1$$

(C) At $x = 0$ or $x = 1$, $G(x_j, x_k) = 0$, so $(hG)_0 = (hG)_n = 0$. From (A) and (B), we proved that:

$$(L_h(hG(\cdot, x_k)))_j = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases} = (e_k)_j$$

Since the discrete problem $L_h w = e_k$ (with Dirichlet BC) has a unique solution, the solution must be G^k. Thus:

$$G^k(x_j) = (hG(\cdot, x_k))(x_j) = hG(x_j, x_k) \quad \text{for all } j.$$

Chapter 12: #7

Let $g = 1$ and prove that $T_h g(x_j) = \frac{1}{2} x_j(1 - x_j)$.

[Solution: use the definition (12.25) with $g(x_k) = 1, k = 1, \ldots, n-1$ and recall that $G^k(x_j) = hG(x_j, x_k)$ from the exercise above. Then

$$T_h g(x_j) = h \left[\sum_{k=1}^{j} x_k(1 - x_j) + \sum_{k=j+1}^{n-1} x_j(1 - x_k) \right]$$

from which, after straightforward computations, one gets the desired result.]

Solution:

Compute $(T_h g)(x_j)$ where $g(x) = 1$. The discrete Green's operator is defined as:

$$(T_h g)(x_j) = h \sum_{k=1}^{n-1} G(x_j, x_k) g(x_k)$$

Since $g(x_k) = 1$, and using the definition of the discrete Green's function:

$$G(x_j, x_k) = \begin{cases} x_k(1 - x_j), & k \leq j \\ x_j(1 - x_k), & k > j \end{cases}$$

We split the sum:

$$(T_h g)(x_j) = h \left[\sum_{k=1}^{j} x_k(1 - x_j) + \sum_{k=j+1}^{n-1} x_j(1 - x_k) \right]$$

Using $x_k = kh$ and $h = 1/n$, so $\sum_{k=1}^{j} x_k = h \sum_{k=1}^{j} k = h \frac{j(j+1)}{2}$ and

$$\begin{aligned} h \sum_{k=1}^{j} x_k(1 - x_j) = h(1 - x_j) \sum_{k=1}^{j} x_k &= h(1 - x_j) \cdot h \frac{j(j+1)}{2} \\ &= h^2(1 - jh) \frac{j(j+1)}{2} \end{aligned}$$

For the second part, $\sum_{k=j+1}^{n-1} (1 - x_k) = \sum_{k=j+1}^{n-1} (1 - kh)$:

$$\begin{aligned} \sum_{k=j+1}^{n-1} (1 - kh) &= (n - 1 - j) - h \sum_{k=j+1}^{n-1} k \\ &= (n - 1 - j) - h \left(\frac{n(n-1)}{2} - \frac{j(j+1)}{2} \right) \end{aligned}$$

Thus:

$$(T_h g)(x_j) = h^2(1 - x_j) \frac{j(j+1)}{2} + h x_j \left[(n - 1 - j) - h \left(\frac{n(n-1)}{2} - \frac{j(j+1)}{2} \right) \right]$$

Noting $x_j = jh$ and $nh = 1$:

$$\begin{aligned}
(T_h g)(x_j) &= \frac{x_j(x_j+h)(1-x_j)}{2} + \frac{x_j}{2}(1-h-2x_j+x_j^2+x_jh) \\
&= \frac{x_j}{2}\Big[(x_j+h)(1-x_j) + (1-h-2x_j+x_j^2+x_jh)\Big] \\
&= \frac{x_j}{2}\Big[(x_j-x_j^2+h-hx_j) + 1-h-2x_j+x_j^2+x_jh\Big] \\
&= \frac{1}{2}x_j(1-x_j)
\end{aligned}$$

Chapter 12: #8

Prove Young's inequality for any $a, b \in \mathbb{R}$ and $\epsilon > 0$:

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}$$

Solution:

We notice that for any $a, b \in \mathbb{R}$ and $\epsilon > 0$:

$$0 \leq \left(\sqrt{\epsilon}a - \frac{b}{2\sqrt{\epsilon}}\right)^2 = \epsilon a^2 - ab + \frac{b^2}{4\epsilon}$$

This implies:

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}$$

Chapter 12: #9

Prove that $\|v_h\|_h \leq \|v_h\|_{h,\infty}$ for all $v_h \in V_h$.

Solution:

From the definition of the discrete norm with trapezoidal weights:

$$\|v_h\|_h^2 = (v_h, v_h)_h = h \sum_{k=0}^{n} c_k v_k^2$$

where c_k are the weights ($c_0 = c_n = 1/2$, $c_k = 1$ for $0 < k < n$).

We know that for all k, $v_k^2 \leq \max_i |v_i|^2 = \|v_h\|_{h,\infty}^2$. Therefore:

$$h \sum_{k=0}^{n} c_k v_k^2 \leq h \left[\sum_{k=0}^{n} c_k \right] \|v_h\|_{h,\infty}^2$$

Calculating the sum of weights:

$$\sum_{k=0}^{n} c_k = \frac{1}{2} + \sum_{k=1}^{n-1} 1 + \frac{1}{2} = 1 + (n-1) = n$$

Substituting this back and noting $h = 1/n$, so $hn = 1$:

$$\|v_h\|_h^2 = h \sum_{k=0}^{n} c_k v_k^2 \leq h \cdot n \cdot \|v_h\|_{h,\infty}^2 = \|v_h\|_{h,\infty}^2$$

Taking the square root gives:

$$\|v_h\|_h \leq \|v_h\|_{h,\infty}$$

Chapter 12: #11

Discretize the fourth-order differential operator $Lu(x) = -u^{(iv)}(x)$ using centered finite differences.

[Solution: apply twice the second order centered finite difference operator L_h defined in (12.9).]

Solution:

Let L_h be the discrete Laplacian operator (centered finite difference of second order):

$$(L_h u)_j = -\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}$$

To approximate the fourth-order operator $Lu = -u^{(iv)}$, we apply L_h twice. Let $v = L_h u$. We compute $(L_h v)_j$:

$$\begin{aligned}(L_h v)_j &= -\frac{v_{j+1} - 2v_j + v_{j-1}}{h^2} \\ &= -\frac{1}{h^2}\left[\left(-\frac{u_{j+2} - 2u_{j+1} + u_j}{h^2}\right) - 2\left(-\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}\right) + \left(-\frac{u_j - 2u_{j-1} + u_{j-2}}{h^2}\right)\right]\end{aligned}$$

Factoring out terms and combining the $1/h^2$ factors:

$$\begin{aligned}(L_h^2 u)_j &= \frac{1}{h^4}\left[(u_{j+2} - 2u_{j+1} + u_j) - 2(u_{j+1} - 2u_j + u_{j-1}) + (u_j - 2u_{j-1} + u_{j-2})\right] \\ &= \frac{1}{h^4}\left[u_{j+2} - 4u_{j+1} + 6u_j - 4u_{j-1} + u_{j-2}\right]\end{aligned}$$

This corresponds to the standard centered finite difference approximation for the fourth derivative, i.e.,

$$L_h^2 u \approx u^{(iv)}(x)$$

Note: Since $Lu = -u^{(iv)}$, the discretized operator is consistent with $-L_h^2 u$ (depending on the sign convention of L_h). Based on the calculation above resulting in positive u_{j+2}, this represents the approximation for $u^{(iv)}$.