1.

$u'' = f$, $u(0)=0$, $u(1)=0$, $f = \begin{cases} 1 & 0.4 \le x \le 0.6 \\ 0 & \text{o.w.} \end{cases}$

$0 \le x \le 0.4$:

$u_1(x) = Ax + B$

$\Downarrow u(0)=0$

$u_1(x) = Ax$

$0.4 \le x \le 0.6$:

$u_2(x) = \frac{1}{2}x^2 + Cx + D$ (by integral)

$1 \ge x \ge 0.6$:

$u_3(x) = Ex + F$

$\Downarrow u(1)=0$

$u_3(x) = Ex - E$

$x = 0.4$:

$$\left.\begin{array}{l} u_1(0.4) = u_2(0.4) \Rightarrow 0.4A = 0.08 + 0.4C + D \\ u_1'(0.4) = u_2'(0.4) \Rightarrow C = A - 0.4 \end{array}\right\} \Rightarrow D = 0.08$$

$x = 0.6$:

$$\left.\begin{array}{l} u_2(0.6) = u_3(0.6) \Rightarrow 0.18 + 0.6C + D = 0.4E \\ u_2'(0.6) = u_3'(0.6) \Rightarrow 0.6 + C = E \end{array}\right\} \Rightarrow \begin{array}{l} A = -0.1 \\ C = A - 0.4 = -0.5 \\ E = A + 0.2 = 0.1 \end{array}$$

$$u(x) = \begin{cases} -0.1x & 0 \le x \le 0.4 \\ \frac{1}{2}x^2 - \frac{1}{2}x + 0.08 & 0.4 \le x \le 0.6 \\ 0.1x - 0.1 & 0.6 \le x \le 1 \end{cases}$$

2.

Uniqueness:

Let u, v be the two solutions, define $w = u - v$,

then w satisfies $w'' - 2w' + w = 0$

B.C. becomes $w(0) = 0$, $w'(1) = 0$

Homogeneous solution: $w(x) = C_1 e^x + C_2 x e^x$

$w(0) = 0 \Rightarrow C_1 = 0$

$w'(1) = 0 \Rightarrow (C_1 + 2C_2)e = 2C_2 e = 0 \Rightarrow C_2 = 0$

$\Rightarrow w \equiv 0$

$\Rightarrow u = v$

$\Rightarrow$ the solution is unique.

For $u''(x) - 2u'(x) + u(x) = 1$, $u(0) = 0$, $u'(1) = 1$

We have $u_h(x) = (C_1 + C_2 x)e^x$

Let $u_p(x) = A$ and we get $A = 1$, so $u(x) = (C_1 + C_2 x)e^x + 1$

$u(0)=0 \Rightarrow C_1+1=0 \Rightarrow C_1=-1$

$u'(x)=(C_1+C_2+C_2x)e^x$

$u'(1)=1 \Rightarrow (C_1+2C_2)e=1$

$\Rightarrow -1+2C_2=e^{-1}$

$\Rightarrow C_2=\frac{1+e^{-1}}{2}$

So $u(x)=(-1+\frac{1+e^{-1}}{2}x)e^x+1$ #

3.

$u''(x)=\sin(2\pi x)$, $u'(0)=0$, $u'(1)=0$

For $u''=f$, the necessary condition is $u'(1)-u'(0)=\int_0^1 u''(x)dx=\int_0^1 f(x)dx$

In this case, $f(x)=\sin(2\pi x)$

$\int_0^1 \sin(2\pi x) = -\frac{\cos(2\pi x)}{2\pi}\Big|_0^1 = 0 = u'(1)-u'(0)$

So the consistency condition is satisfied, and the solution exists.

$u'(x)=u'(0)+\int_0^x \sin(2\pi s)ds = 0+\frac{1-\cos(2\pi x)}{2\pi}$

$\Rightarrow u(x)=C+\int_0^x \frac{1-\cos(2\pi t)}{2\pi}dt$

$=\frac{x}{2\pi}-\frac{\sin(2\pi x)}{4\pi^2}$ (Let $C=0$)

4.

$u''(x)=e^{\sin x}$, $u'(0)=0$, $u'(1)=\alpha$

The necessary condition is $\alpha-0=\int_0^1 e^{\sin x}dx$

So $\alpha=\int_0^1 e^{\sin x}$ can make the problem has at least one soluton.

$u''=e^{\sin x}$

$\Rightarrow u'(x)=\int_0^x e^{\sin t}dt+C_1$

$\Rightarrow u'(x)=\int_0^x e^{\sin t}dt$ ($\because u'(0)=0$)

$\Rightarrow u(x)=\int_0^x(\int_0^s e^{\sin t}dt)ds+C$ (C is arbitrary so we can choose $C=0$)

$\Rightarrow u(x)=\int_0^x(\int_0^s e^{\sin t}dt)ds$ is one of the solution.

5.

$\varepsilon u'' + (1+\varepsilon)u' + u = 0,\ u(0)=0,\ u(1)=1,\ \varepsilon = 0.01$

characteristic equation: $\varepsilon r^2 + (1+\varepsilon) r + 1 = 0$

$$r = \frac{-(1+\varepsilon) \pm \sqrt{(1+\varepsilon)^2 - 4\varepsilon}}{2\varepsilon}$$

$$= \frac{-0.02}{0.02},\ \frac{-2}{0.02}$$

$$= -1,\ -100$$

$u(x) = C_1 e^{-x} + C_2 e^{-100x}$

$u(0) = 0 \Rightarrow C_2 = -C_1$

$\Rightarrow u(x) = C_1 (e^{-x} - e^{-100x})$

$u(1) = 1 \Rightarrow C_1 (e^{-1} - e^{-100}) = 1$

$\Rightarrow C_1 = \dfrac{1}{e^{-1} - e^{-100}}$

So $u(x) = \dfrac{e^{-x} - e^{-100x}}{e^{-1} - e^{-100}}$