HW 7

Shih-Ming Wang 412652004

Chapter 12: #1

Consider the boundary value problem $-u''(x) = f(x), 0 < x < 1, u(0) = u(1) = 0$ with $f(x) = 1/x$. Using $u(x) = \int_0^1 G(x,s)f(s)\,ds$ to prove that $u(x) = -x\log(x)$. This shows that $u \in C^2(0,1)$ but $u(0)$ is not defined and u', u'' do not exist at $x = 0$ ($\Rightarrow$: if $f \in C^0(0,1)$, but not $f \in C^0([0,1])$, then u does not belong to $C^0([0,1])$).

Solution:

We note that for the solution of $-u'' = f$ with boundary conditions $u(0) = u(1) = 0$, we can use the Green's function to represent the solution u:

$$u(x) = \int_0^1 G(x,s)f(s)\,ds$$

where

$$G(x,s) = \begin{cases} s(1-x), & 0 \le s \le x \\ x(1-s), & x \le s \le 1 \end{cases}$$

Let $f(s) = \frac{1}{s}$. Computing $u(x)$:

$$\begin{aligned}
u(x) &= \int_0^x s(1-x)\frac{1}{s}\,ds + \int_x^1 x(1-s)\frac{1}{s}\,ds \\
&= (1-x)\int_0^x 1\,ds + x\int_x^1 \left(\frac{1}{s} - 1\right)\,ds \\
&= (1-x)x + x\Big[\ln s - s\Big]_x^1 \\
&= (1-x)x + x\Big((\ln 1 - 1) - (\ln x - x)\Big) \\
&= x - x^2 + x(-1 - \ln x + x) \\
&= x - x^2 - x - x\ln x + x^2 \\
&= -x\ln x
\end{aligned}$$

This implies $u(x) \in C^2(0,1)$. Now, let's examine the derivatives:

$$
\begin{aligned}
u'(x) &= -1 - \ln x \\
u''(x) &= -\frac{1}{x}
\end{aligned}
$$

At $x = 0$, the function is undefined. Checking the limits:

- $\lim_{x\to 0^+} u'(x) = \infty \implies u'$ does not exist at $x = 0$.
- $\lim_{x\to 0^+} u''(x) = -\infty \implies u''$ does not exist at $x = 0$.

Hence, we can see that while $f \in C^0(0,1)$, it is not in $C^0([0,1])$. Consequently, $u \notin C^2([0,1])$.

Problem 4

Verify the summation by parts formula

$$\sum_{j=0}^{n-1}(w_{j+1}-w_j)v_j = w_n v_n - w_0 v_0 - \sum_{j=0}^{n-1} w_{j+1}(v_{j+1}-v_j)$$

and show that, for $v_h \in V_h^0$,

$$(L_h v_h, v_h)_h = h^{-1}\sum_{j=0}^{n-1}(v_{j+1}-v_j)^2$$

Solution:

Using the index shift directly on the terms:

$$\begin{aligned}
\text{LHS} &= \sum_{j=0}^{n-1} w_{j+1}v_j - \sum_{j=0}^{n-1} w_j v_j \\
&= \left(\sum_{k=1}^{n} w_k v_{k-1}\right) - \left(w_0 v_0 + \sum_{k=1}^{n-1} w_k v_k\right) \\
&= w_n v_{n-1} - w_0 v_0 + \sum_{k=1}^{n-1} w_k (v_{k-1} - v_k) \\
&= w_n v_n - w_0 v_0 - \sum_{j=0}^{n-1} w_{j+1}(v_{j+1}-v_j), \quad (\text{let } j = k-1)
\end{aligned}$$

where we use the fact that $w_n v_{n-1} = w_n v_n - w_n(v_n - v_{n-1})$. Now, consider the inner product $(L_h v_h, v_h)_h$ with boundary conditions $v_0 = v_n = 0$:

$$\begin{aligned}
(L_h v_h, v_h)_h &= h\sum_{j=1}^{n-1}(L_h v_h)_j v_j \\
&= h\sum_{j=1}^{n-1}\left[-\frac{v_{j+1}-2v_j+v_{j-1}}{h^2}\right] v_j \\
&= -\frac{1}{h}\sum_{j=1}^{n-1}(v_{j+1}-2v_j+v_{j-1})v_j \\
&= -\frac{1}{h}\left[\sum_{j=1}^{n-1}(v_{j+1}-v_j)v_j - \sum_{j=1}^{n-1}(v_j - v_{j-1})v_j\right]
\end{aligned}$$

Let $k = j - 1$ for the second sum:

$$
\begin{aligned}
&= -\frac{1}{h}\left[\sum_{j=1}^{n-1}(v_{j+1} - v_j)v_j - \sum_{k=0}^{n-2}(v_{k+1} - v_k)v_{k+1}\right] \\
&= -\frac{1}{h}\left[\sum_{j=0}^{n-1}(v_{j+1} - v_j)v_j - \sum_{j=0}^{n-1}(v_{j+1} - v_j)v_{j+1}\right] \quad \text{(since } v_0 = v_n = 0\text{)} \\
&= -\frac{1}{h}\sum_{j=0}^{n-1}(v_{j+1} - v_j)(v_j - v_{j+1}) \\
&= -\frac{1}{h}\left[-\sum_{j=0}^{n-1}(v_{j+1} - v_j)^2\right] \\
&= h^{-1}\sum_{j=0}^{n-1}(v_{j+1} - v_j)^2
\end{aligned}
$$

Chapter 12: #6

Prove that $G^k(x_j) = hG(x_j, x_k)$, where G is Green's function introduced in (12.4) and G^k is its corresponding discrete counterpart solution of (12.4).

[Solution: we prove the result by verifying that $L_h G = he^k$. Indeed, for a fixed x_k the function $G(x_k, s)$ is a straight line on the intervals $[0, x_k]$ and $[x_k, 1]$ so that $L_h G = 0$ at every node x_l with $l = 0, \dots, k-1$ and $l = k+1, \dots, n+1$. Finally, a direct computation shows that $(L_h G)(x_k) = 1/h$ which concludes the proof.]

Solution:

For fixed $s = x_k$ (where $0 < x_k < 1$), let:

$$G(x, x_k) = \begin{cases} s(1 - x_k), & x \le x_k \\ x_k(1 - x), & x \ge x_k \end{cases}$$

(A) Fix k. For all $j \neq k$, the continuous Green's function $G(x, x_k)$ is a linear function on $[x_{j-1}, x_{j+1}]$. Hence, its second finite difference is zero:

$$G(x_{j+1}, x_k) - 2G(x_j, x_k) + G(x_{j-1}, x_k) = 0, \quad \text{for } j < k$$

For $j > k$, we have similar arguments since $G(x, x_k) = x_k(1 - x)$ is linear in x. Thus:

$$(L_h(hG))_j = -\frac{h(G_{j+1} - 2G_j + G_{j-1})}{h^2} = 0, \quad \text{for } j \neq k$$

(B) For $j = k$, we need to compute:

$$(L_h(hG))_k = -\frac{1}{h}(G_{k+1} - 2G_k + G_{k-1})$$

Define $G_j = G(x_j, x_k)$. Let $a := x_k = kh$.

$$\begin{aligned} G_{k-1} &= x_{k-1}(1 - x_k) = (k-1)h(1 - a) \\ G_k &= x_k(1 - x_k) = a(1 - a) \\ G_{k+1} &= x_k(1 - x_{k+1}) = a(1 - (k+1)h) = a(1 - a - h) \end{aligned}$$

Calculating the second difference:

$$\begin{aligned} G_{k+1} - 2G_k + G_{k-1} &= a(1 - a - h) - 2a(1 - a) + (k-1)h(1 - a) \\ &= [a(1 - a) - ah] - 2a(1 - a) + (kh - h)(1 - a) \\ &= -a(1 - a) - ah + a(1 - a) - h(1 - a) \quad (\text{since } kh = a) \\ &= -ah - h + ah \\ &= -h \end{aligned}$$

Therefore:

$$(L_h(hG))_k = -\frac{1}{h}(-h) = 1$$

(C) At $x = 0$ or $x = 1$, $G(x_j, x_k) = 0$, so $(hG)_0 = (hG)_n = 0$. From (A) and (B), we proved that:

$$(L_h(hG(\cdot, x_k)))_j = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases} = (e_k)_j$$

Since the discrete problem $L_h w = e_k$ (with Dirichlet BC) has a unique solution, the solution must be G^k. Thus:

$$G^k(x_j) = (hG(\cdot, x_k))(x_j) = hG(x_j, x_k) \quad \text{for all } j.$$