#1

$E_0(f) = \frac{h^3}{3} f''(\xi), \quad h = \frac{b-a}{2}, \quad \xi \in (a,b)$

$E_1(f) = -\frac{h^3}{12} f''(\eta), \quad h = b-a, \quad \eta \in (a,b)$

$$\frac{|E_1(f)|}{|E_0(f)|} = \frac{\frac{(b-a)^3}{12}|f''(\eta)|}{\frac{(b-a)^3}{24}|f''(\xi)|} = 2\,\frac{|f''(\eta)|}{|f''(\xi)|}$$

$$\Rightarrow |E_1(f)| = 2\,\frac{|f''(\eta)|}{|f''(\xi)|}\,|E_0(f)|$$

Hence if $\frac{|f''(\eta)|}{|f''(\xi)|} \approx 1$, then $|E_1(f)| \simeq 2|E_0(f)|$

#3

(a) $m=0$: $\int_{-1}^{1} 1\,dx = 2$

$I_2(1) = \frac{2}{3}[2-1+2] = 2$

$m=1$: $\int_{-1}^{1} x\,dx = 0$

$I_2(x) = \frac{2}{3}[-1+1] = 0$

$m=2$: $\int_{-1}^{1} x^2\,dx = \frac{2}{3}$

$I_2(x^2) = \frac{2}{3}[\frac{1}{2}+\frac{1}{2}] = \frac{2}{3}$

$m=3$: $\int_{-1}^{1} x^3\,dx = 0$

$I_2(x^3) = \frac{2}{3}[-\frac{1}{4}+\frac{1}{4}] = 0$

$m=4$: $\int_{-1}^{1} x^4\,dx = \frac{2}{5}$

$I_2(x^4) = \frac{2}{3}[\frac{1}{8}+\frac{1}{8}] = \frac{1}{6} \neq \frac{2}{5}$

$\Rightarrow r=3$

(b) $m=0$: $I_4(1) = \frac{1}{4}[1+3+3+1] = 2$

$m=1$: $I_4(x) = 0$

$m=2$: $I_4(x^2) = \frac{1}{4}[2+\frac{2}{3}] = \frac{2}{3}$

$m=3$: $I_4(x^3) = 0$

$m=4$: $I_4(x^4) = \frac{1}{4}\cdot\frac{56}{27} = \frac{14}{27} \neq \frac{2}{5}$

$\Rightarrow r=3$

Now compute $M = \int_{-1}^{1} u^4\,du - \sum_k \alpha_k t_k^4$ respectively:

$t_k = -\frac{1}{2}, 0, \frac{1}{2}, \quad \alpha_k = \frac{4}{3}, -\frac{2}{3}, \frac{4}{3}$

$\int_{-1}^{1} u^4\,du = \frac{2}{5}$

$\sum_k \alpha_k t_k^4 = \frac{4}{3}(\frac{1}{16}) + (-\frac{2}{3})\cdot 0 + \frac{4}{3}(\frac{1}{16}) = \frac{1}{6}$

$M = \frac{2}{5} - \frac{1}{6} = \frac{7}{30}$

$C_{I_2} = \frac{1}{4!}\cdot\frac{7}{30} = \frac{7}{720}$

$E_{I_2}(h;f) = \frac{7}{720} f^{(4)}(0)h^5 + o(h^5)$

$\Rightarrow p = r+2 = 5$

$t_k = -1, -\frac{1}{3}, \frac{1}{3}, 1, \quad \alpha_k = \frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4}$

$\sum_k \alpha_k t_k^4 = \frac{1}{4}\cdot 1 + \frac{3}{4}\cdot\frac{1}{81} + \frac{3}{4}\cdot\frac{1}{81} + \frac{1}{4}\cdot 1 = \frac{14}{27}$

$M = \frac{2}{5} - \frac{14}{27} = -\frac{16}{135}$

$C_{I_4} = \frac{1}{4!}(-\frac{16}{135}) = -\frac{2}{405}$

$E_{I_4}(h;f) = -\frac{2}{405} f^{(4)}(0)h^5 + o(h^5)$

$\Rightarrow p = r+2 = 5$

#5

$$m_k := \int_0^1 x^k w(x)\,dx = \int_0^1 x^k x^{\frac{1}{2}}\,dx = \left.\frac{x^{k+\frac{3}{2}}}{k+\frac{3}{2}}\right|_0^1 = \frac{2}{2k+3}$$

$$m_0 = \int_0^1 x^{\frac{1}{2}}\,dx = \frac{2}{3}$$

$$m_1 = \int_0^1 x^{\frac{3}{2}}\,dx = \frac{2}{5}$$

$$k=0 \Rightarrow Q(1) = a\cdot 1 = m_0 = \frac{2}{3} \Rightarrow \boxed{a = \frac{2}{3}}$$

$$k=1 \Rightarrow Q(x) = a\,x_1 = m_1 = \frac{2}{5} \Rightarrow \boxed{x_1 = \frac{3}{5}}$$

$$k=2 \Rightarrow m_2 = \int_0^1 x^2\sqrt{x}\,dx = \frac{2}{7}$$

$$Q(x^2) = a x_1^2 = \frac{2}{3}\left(\frac{3}{5}\right)^2 = \frac{6}{25} \neq \frac{2}{7} \Rightarrow \boxed{r=1}$$

#6

For $f(x)=1$: $I(1) = \int_0^1 1\,dx = 1$

$Q(1) = \alpha_1 + \alpha_2$ $\Rightarrow \alpha_1 + \alpha_2 = 1$

For $f(x)=x$: $I(x) = \int_0^1 x\,dx = \frac{1}{2}$

$Q(x) = \alpha_1\cdot 0 + \alpha_2\cdot 1 + \alpha_3\cdot 1 = \alpha_2 + \alpha_3$ $\Rightarrow \alpha_2 + \alpha_3 = \frac{1}{2}$

For $f(x)=x^2$: $I(x^2) = \int_0^1 x^2\,dx = \frac{1}{3}$

$Q(x^2) = \alpha_1\cdot 0 + \alpha_2\cdot 1 + \alpha_3\cdot 0 = \alpha_2$

$$\boxed{\begin{aligned} &\Rightarrow \alpha_2 = \tfrac{1}{3} \\ &\Rightarrow \alpha_3 = \tfrac{1}{2} - \tfrac{1}{3} = \tfrac{1}{6} \\ &\Rightarrow \alpha_1 = 1 - \tfrac{1}{3} = \tfrac{2}{3} \end{aligned}}$$

Consider $f(x)=x^3$, $I(x^3) = \int_0^1 x^3\,dx = \frac{1}{4}$

$Q(x^3) = \alpha_1\cdot 0 + \alpha_2\cdot 1 + \alpha_3\cdot f'(0) = \alpha_2 = \frac{1}{3} \neq \frac{1}{4}$

$\Rightarrow r = 2$