ST 518 Homework 9

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1 Problem 1

Consider a random effects model to analyze data from a completely randomized experimental design, $Y_{ij} = \mu + T_i + E_{ij}$, with $i = 1, ..., t$ and $j = 1, ..., n$ where $T_i \sim^{iid} N(0, \sigma_T^2)$ and $E_{ij} \sim^{iid} N(0, \sigma^2)$ with $T_i \perp E_{ij}$.

1.1 Part A

Give the mean and variance of an observation, Y_{ij}.

The mean of Y_{ij}, given that $Y_{ij} = \mu + T_i + E_{ij}$, is $E(Y_{ij}) = E(\mu + T_i + E_{ij}) = E(\mu) + E(T_i) + E(E_{ij}) = \mu + 0 + 0 = \mu$. The variance of Y_{ij} is $Var(Y_i) = Var(\mu + T_i + E_{ij}) = Var(\mu) + Var(T_i) + Var(E_{ij}) = 0 + \sigma_T^2 + \sigma^2 = \sigma_T^2 + \sigma^2$.

1.2 Part B

Write a treatment mean, $\bar{Y_{i.}}$ as a function of μ and (averages of) random effects.

We know that $\bar{Y_{i.}} = \frac{1}{n} \sum (\mu + T_i + E_{ij}) = \mu + \frac{1}{n}(T_i + T_i + ... + T_i) + \frac{1}{n}(E_{i1} + E_{i2} + ... + E_{in}) = \mu + T_i + \bar{E_{i.}}$.

1.3 Part C

Give the mean and variance of a sample treatment mean, $\bar{Y_{i.}}$

We can find the mean of $\bar{Y_{i.}}$ by saying that $E(\bar{Y_{i.}}) = E(\mu + T_i + \bar{E_{i.}}) = E(\mu) + E(T_i) + E(E_{i.}) = \mu + 0 + 0 = \mu$. The variance of $\bar{Y_{i.}}$ is $Var(\mu + T_i + \bar{E_{i.}}) = Var(\mu) + Var(T_i) + Var(E_{i.}) = 0 + \sigma_T^2 + \frac{\sigma^2}{n} = \sigma_T^2 + \frac{\sigma^2}{n} = \frac{n\sigma_T^2 + \sigma^2}{n} = \frac{1}{n}E(MS(Trt))$.

1.4 Part D

Write the grand mean, $\bar{Y_{..}}$ as a function of μ and averages of random effects.

$\bar{Y_{..}} = \frac{1}{nt}(\mu + T_i + E_{ij}) = \mu + \frac{nT_1 + ... + nT_t}{nt} + \frac{1}{nt}(E_{11} + E_{12} + ... + E_{1n} + E_{t1} + ... + E_{tn}) = \mu + \bar{T_{.}} + \bar{E_{..}}$.

1.5 Part E

Give the mean and variance of the grand mean, $\bar{Y_{..}}$. Be clear about where the assumption that $T_i \perp E_{ij}$ is used.

We can first find the mean for $\bar{Y_{..}}$ which is $E(\bar{Y_{..}}) = E(\mu + \bar{T_{.}} + \bar{E_{..}}) = E(\mu) + E(\bar{T_{.}}) + E(\bar{E_{..}}) = \mu + 0 + 0 = \mu$. The variance for $\bar{Y_{..}}$ can be found by first using the assumption that comes from $T_i \perp E_{ij}$ being true. In this case, since this property of both random variables T and E being independent than the average of those two random variables $\bar{T_{.}}$ and $\bar{E_{..}}$ are also independent. Because of this we can say the variance of this sum of averages is the same of the sum of the variance of averages. Therefore we can say that $Var(\bar{Y_{..}}) = Var(\mu + \bar{T_{.}} + \bar{E_{..}}) = Var(\mu) + Var(\bar{T_{.}}) + Var(\bar{E_{..}}) = 0 + \frac{\sigma_T^2}{t} + \frac{\sigma^2}{nt} = \frac{1}{nt}(n\sigma_T^2 + \sigma^2) = \frac{1}{nt}E(MS(T))$.

1.6 Part F

Derive $E(\frac{1}{t-1}\sum\sum(\bar{Y_{i.}} - \bar{Y_{..}})^2)$

We know that $\sum\sum(\bar{Y_{i.}} - \bar{Y_{..}})^2 = SS(Trt)$ and that $MS(Trt) = \frac{SS(Trt)}{t-1}$ and we will use these two things to get $E(\frac{1}{t-1}\sum\sum(\bar{Y_{i.}} - \bar{Y_{..}})^2)$. Therefore, $E(\frac{1}{t-1}\sum\sum(\bar{Y_{i}}. - \bar{Y_{..}})^2) = E(\frac{1}{t-1} * SS(Trt)) = E(\frac{SS(Trt)}{t-1}) = E(MS(Trt)) = n\sigma_T^2 + \sigma^2$.

1.7 Part G

What is the sampling distribution of the statistic W where $W = \frac{\sum\sum(\bar{Y_{i.}} - \bar{Y_{..}})^2}{n\sigma_T^2 + \sigma^2}$?

We know in this case that $\sum\sum(\bar{Y_{i.}} - \bar{Y_{..}})^2 = SS(Trt) = (t-1)MS(Trt)$. Thus, $W = \frac{\sum\sum(\bar{Y_{i.}} - \bar{Y_{..}})^2}{n\sigma_T^2 + \sigma^2} = \frac{SS(Trt)}{n\sigma_T^2 + \sigma^2} = \frac{(t-1)MS(Trt)}{n\sigma_T^2 + \sigma^2} = (t-1)\frac{MS(Trt)}{n\sigma_T^2 + \sigma^2}$ in which we know that this statistic W follows a sampling distribution of χ^2_{t-1} or χ^2 with $t-1$ degrees of freedom.

1.8 Part H

Derive $E(\frac{1}{t(n-1)}\sum\sum(Y_{ij} - \bar{Y_{i.}})^2)$

We know that $\sum\sum(Y_{ij} - \bar{Y_{i.}})^2 = SS(E)$ and that $MS(E) = \frac{SS(E)}{t(n-1)}$ and we will use these two things to get $E(\frac{1}{t(n-1)}\sum\sum(Y_{ij} - \bar{Y_{i.}})^2)$. Therefore, $E(\frac{1}{t(n-1)}\sum\sum(Y_{ij} - \bar{Y_{i.}})^2) = E(\frac{1}{t(n-1)}SS(E)) = E(\frac{SS(E)}{t(n-1)}) = E(MS(E)) = \sigma^2$.

1.9 Part I

What is the sampling distribution of the statistic X where $X = \frac{\sum\sum(Y_{ij} - \bar{Y_{i.}})^2}{\sigma^2}$

Note that $N - t = t(n-1)$. We know in this case that $\sum\sum(Y_{ij} - \bar{Y_{i.}})^2 = SS(E) = (t(n-1))MS(E)$. Thus, $W = \frac{\sum\sum(Y_{ij} - \bar{Y_{i.}})^2}{\sigma^2} = \frac{SS(E)}{\sigma^2} = \frac{(t(n-1))MS(E)}{\sigma^2} = (t(n-1))\frac{MS(E)}{\sigma^2} = (N-t)\frac{MS(E)}{\sigma^2}$ in which we know that this statistic X follows a sampling distribution of χ^2_{N-t} or χ^2 with $N - t$ degrees of freedom or lastly the same as χ^2 with $t(n-1)$ degrees of freedom.

2 Problem 2

Consider a mixed model for a crossed two factor design, $Y_{ijk} = \mu + \alpha_i + B_j + (\alpha B)_{ij} + E_{ijk}$ with $i = 1, ..., a; j = 1, ..., b; k = 1, ..., n$. $B_j \sim^{iid} N(0, \sigma^2_B)$ and $(\alpha B)_{ij} \sim^{iid} N(0, \sigma^2_{\alpha B})$ and $E_{ijk} \sim^{iid} N(0, \sigma^2)$. Assume B_j and $(\alpha B)_{ij}$ and E_{ijk} are mutually independent random samples.

2.1 Part A

Consider the difference between two A treatment means, $\bar{Y_{2..}} - \bar{Y_{1..}}$. Derive the variance of this difference.