

Lecture 8:
Matrix Notation for Multiple Linear Regression
STAT 632, Spring 2020

Review of Matrices

We say that \mathbf{X} is an $r \times c$ *matrix* if it is an array of numbers with r rows and c columns.

Ex:

$$\mathbf{X}_{(4 \times 3)} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 5 \\ 1 & 3 & 4 \\ 1 & 8 & 6 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \\ x_{41} & x_{42} & x_{43} \end{pmatrix}$$

The element x_{ij} denotes the number in the i^{th} row and j^{th} column.

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Review of Matrices

A *vector* is a one-column matrix.

Ex:

$$\mathbf{y}_{(4 \times 1)} = \begin{pmatrix} 2 \\ 3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

Notation: Bold face is used to denote matrices and vectors.

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Review of Matrices

A *square matrix* has the same number of rows and columns, so $r = c$. A square matrix is *diagonal* if all elements off the main diagonal are 0.

Ex: \mathbf{C} is a square matrix, and \mathbf{D} is a diagonal matrix.

$$\mathbf{C}_{(3 \times 3)} = \begin{pmatrix} -5 & 1 & 3 \\ 1 & 2 & 6 \\ 3 & 6 & -4 \end{pmatrix} \quad \mathbf{D}_{(3 \times 3)} = \begin{pmatrix} -5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

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Review of Matrices

The *identity matrix* is a diagonal matrix with 1's on the diagonal.

Ex:

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Review of Matrices

The *transpose* of a matrix reverses rows and columns.

Ex:

$$\mathbf{X}_{(4 \times 3)} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 5 \\ 1 & 3 & 4 \\ 1 & 8 & 6 \end{pmatrix} \quad \mathbf{X}'_{(3 \times 4)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 3 & 8 \\ 1 & 5 & 4 & 6 \end{pmatrix}$$



Review of Matrices

A square matrix \mathbf{C} is *symmetric* if it equals its transpose, $\mathbf{C}' = \mathbf{C}$.

Ex:

$$\mathbf{C} = \begin{pmatrix} -5 & 1 & 3 \\ 1 & 2 & 6 \\ 3 & 6 & -4 \end{pmatrix}$$



Matrix Addition and Scalar Multiplication

$$\mathbf{A}_{(2 \times 3)} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad \mathbf{B}_{(2 \times 3)} = \begin{pmatrix} -5 & 1 & 2 \\ 3 & 0 & -4 \end{pmatrix}$$

$$\mathbf{A} + \mathbf{B} =$$

$$\mathbf{A} - \mathbf{B} =$$

$$2\mathbf{A} =$$



Inner Product

The *inner product* (or dot product) of two vectors $\underset{(1 \times n)}{\mathbf{a}'}$ and $\underset{(n \times 1)}{\mathbf{b}}$ is

$$\mathbf{a}'\mathbf{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n = \sum_{i=1}^n a_ib_i$$



Matrix Multiplication

For two matrices to be multiplied together, \mathbf{AB} , the number of columns of \mathbf{A} must equal the number of rows of \mathbf{B} .

$$\underset{(r \times c)}{\mathbf{A}} \underset{(c \times q)}{\mathbf{B}} = \underset{(r \times q)}{\mathbf{AB}}$$

Ex:

$$\begin{pmatrix} 3 & 1 \\ -1 & 0 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ 0 & 4 \end{pmatrix} =$$



Inverse of a Matrix

The *inverse* of a square matrix \mathbf{C} is denoted \mathbf{C}^{-1} and has the property

$$\mathbf{C}\mathbf{C}^{-1} = \mathbf{C}^{-1}\mathbf{C} = \mathbf{I}$$

Not all square matrices have an inverse. A square matrix that has an inverse is called *non-singular*; a square matrix without an inverse is called *singular*.

Ex:

$$\mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \quad \mathbf{D}^{-1} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}$$

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Inverse of a Matrix

Ex: Find the inverse of $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$

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Inverse of a Matrix

For a 2×2 matrix we can compute the inverse as

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If $ad - bc = 0$, then \mathbf{A} is not invertible. Note that $ad - bc$ is called the *determinant* of \mathbf{A} , denoted $\det(\mathbf{A})$.

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Properties of Matrix Multiplication:

- ▶ $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ (associativity)
- ▶ $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ (left distributivity)
- ▶ $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$ (right distributivity)
- ▶ $\mathbf{I}_r \mathbf{A} = \underset{(r \times c)}{\mathbf{A}} = \mathbf{A} \mathbf{I}_c$

Matrix multiplication is not commutative. So, \mathbf{AB} is not necessarily equal to \mathbf{BA} .

For $n \times 1$ vectors \mathbf{a} and \mathbf{b} :

$$(\mathbf{a} - \mathbf{b})'(\mathbf{a} - \mathbf{b}) = \mathbf{a}'\mathbf{a} + \mathbf{b}'\mathbf{b} - 2\mathbf{a}'\mathbf{b}$$

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Properties of the Transpose:

- ▶ $(\mathbf{A}')' = \mathbf{A}$
- ▶ $(c\mathbf{A})' = c(\mathbf{A}')$, where c is a scalar
- ▶ $(\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'$
- ▶ $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$

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Matrix Notation for Multiple Linear Regression (MLR)

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1p} \\ 1 & x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & x_{n2} & \cdots & x_{np} \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{pmatrix} + \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}$$

$$Y = X\beta + e$$

- ▶ \mathbf{Y} is an $n \times 1$ response vector
- ▶ \mathbf{X} is an $n \times (p + 1)$ design matrix for the predictor variables
- ▶ $\boldsymbol{\beta}$ is a $(p + 1) \times 1$ vector of regression parameters
- ▶ \mathbf{e} is an $n \times 1$ vector of random errors, assuming $e_i \sim N(0, \sigma^2)$

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Properties of Invertible Matrices:

- ▶ $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
- ▶ $(c\mathbf{A})^{-1} = \frac{1}{c}\mathbf{A}^{-1}$, where $c \neq 0$ is a scalar.
- ▶ If \mathbf{A} and \mathbf{B} are both $n \times n$ invertible matrices, then \mathbf{AB} is invertible, and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
- ▶ If \mathbf{A} is invertible, then \mathbf{A}' is invertible and $(\mathbf{A}')^{-1} = (\mathbf{A}^{-1})'$.

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Matrix Notation for MLR

Let \mathbf{x}'_i denote the i^{th} row of \mathbf{X} . Then

$$\mathbf{x}'_i = \begin{pmatrix} 1 & x_{i1} & x_{i2} & \cdots & x_{ip} \end{pmatrix}_{1 \times (p+1)}$$

which allows us to write

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip} + e_i = \mathbf{x}_i' \boldsymbol{\beta} + e_i$$

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Matrix Notation for MLR

Let $\hat{\boldsymbol{\beta}} = (\hat{\beta}_0 \ \hat{\beta}_1 \ \cdots \ \hat{\beta}_p)'$ denote the least squares estimates of the unknown regression parameters $\boldsymbol{\beta} = (\beta_0 \ \beta_1 \ \cdots \ \beta_p)'$.

It can be shown that $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$.



Matrix Notation for MLR

The vector of fitted (or predicted) values is given by

$$\hat{\mathbf{Y}} = \mathbf{X}\hat{\boldsymbol{\beta}}$$

The vector of residuals is given by

$$\hat{\mathbf{e}} = \mathbf{Y} - \hat{\mathbf{Y}} = \mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}$$

$$\hat{\mathbf{e}} = \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \vdots \\ \hat{e}_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} - \begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_n \end{pmatrix}$$

