Lecture 7:
Multiple Linear Regression
STAT 632, Spring 2020

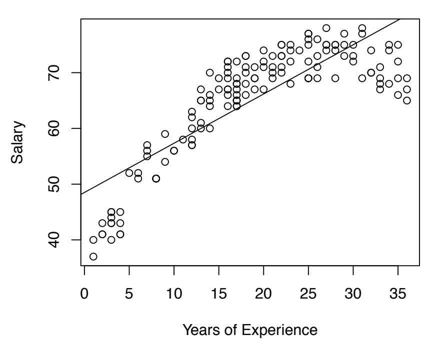
Polynomial Regression Example: Salary Data Set

- For this example we consider a salary data set with $n = 143$ observations and two variables.[1]
- We want to develop a regression model between Y, salary (in thousands of dollars), and x, the number of years of experience. We are interested in using the model to make predictions and prediction intervals.
- Since the variables have an obvious nonlinear, quadratic association we consider a polynomial regression model.

[1] Data set from "A Modern Approach to Regression with R".

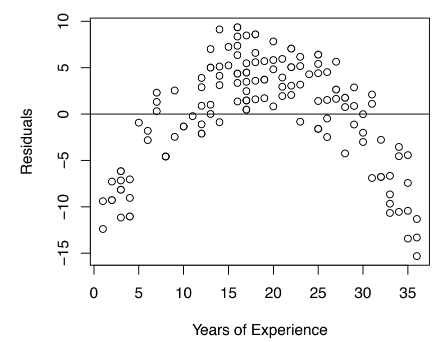
Polynomial Regression Example

```
> profsalary <- read.csv("https://ericwfox.github.io/data/profsalary.csv")
> lm1 <- lm(Salary ~ Experience, data=profsalary)
> plot(Salary ~ Experience, data=profsalary,
       ylab='Salary', xlab='Years of Experience')
> abline(lm1)
```



Polynomial Regression Example

```
> plot(profsalary$Experience, resid(lm1),
       xlab='Years of Experience', ylab='Residuals')
> abline(h=0)
```



Polynomial Regression Example

Since a quadratic relationship is evident, we consider the following polynomial regression model:

$$Y = \beta_0 + \beta_1 x + \beta_2 x^2 + e$$

where Y = salary, x = years of experience, and $e \sim N(0, \sigma^2)$ is the random error.

Polynomial Regression Example

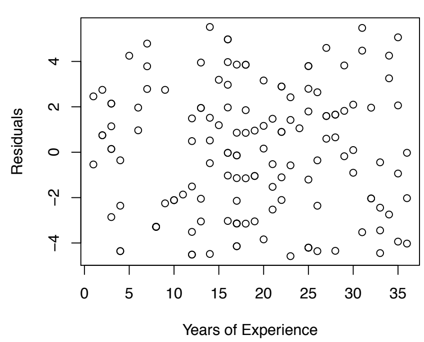
```
> lm2 <- lm(Salary ~ Experience + I(Experience^2), data=profsalary)
> summary(lm2)

Coefficients:
                 Estimate Std. Error t value Pr(>|t|)
(Intercept)     34.720498   0.828724   41.90   <2e-16 ***
Experience       2.872275   0.095697   30.01   <2e-16 ***
I(Experience^2) -0.053316   0.002477  -21.53   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 2.817 on 140 degrees of freedom
Multiple R-squared:  0.9247,Adjusted R-squared:  0.9236
F-statistic: 859.3 on 2 and 140 DF,  p-value: < 2.2e-16
```

Polynomial Regression Example

```
plot(profsalary$Experience, resid(lm2),
     xlab='Years of Experience', ylab='Residuals')
```



Polynomial Regression Example

Fitted regression model:

$$\hat{y} = 34.720 + 2.872x - 0.053x^2$$

Prediction when $x = 10$:

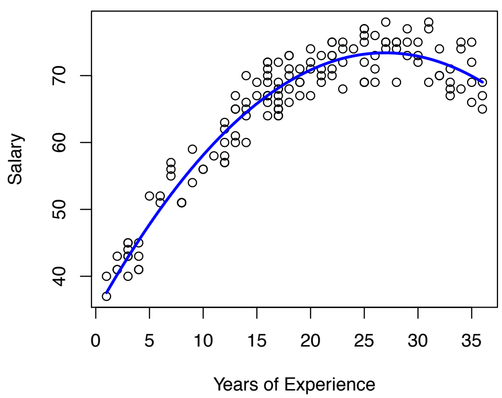
$$\hat{y} = 34.720 + 2.872(10) - 0.053(10^2) = 58.14$$

Using R:

```
> x_new <- data.frame(Experience = 10)
> predict(lm2, newdata=x_new, interval="prediction")
       fit      lwr      upr
1 58.11164 52.50481 63.71847
```

Polynomial Regression Example

Add fitted quadratic curve to scatterplot.

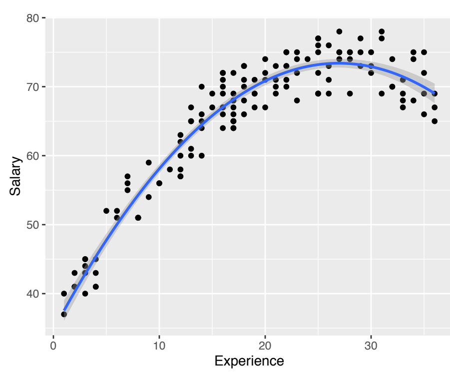


Code used for last plot:

```
> range(profsalary$Experience)
[1]  1 36
> x_grd <- seq(1, 36, by=0.5)
> x_new <- data.frame(Experience = x_grd)
> preds <- predict(lm2, newdata = x_new)

> plot(Salary ~ Experience, data=profsalary,
       ylab='Salary', xlab='Years of Experience')
> lines(x_grd, preds, col='blue', lwd=2.5)
```

```
library(ggplot2)
ggplot(data=profsalary, aes(Experience, Salary)) +
  geom_point() +
  stat_smooth(method='lm', formula = y ~ poly(x, 2))
```



Multiple Linear Regression (MLR) Model

Suppose Y is a response variable, and $x_1, \cdots, x_p$ are p explanatory variables. Then, the multiple linear regression model can be written as

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p + e$$

where $e \sim N(0, \sigma^2)$ is the random error term.

For the polynomial regression example:

- Y = salary
- $x_1 = x$, years of experience
- $x_2 = x^2$, (years of experience)^2

Multiple Linear Regression (MLR) Model

Suppose we have a collection $i = 1, \cdots, n$ observations. Then the multiple linear regression model for case i is written as

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} \cdots + \beta_p x_{ip} + e_i$$

where $e_i \sim N(0, \sigma^2)$ independently.

Given estimates $\hat{\beta}_0, \hat{\beta}_1, \cdots, \hat{\beta}_p$ of the parameters:

- The i^{th} fitted (or predicted) value:

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \cdots + \hat{\beta}_p x_{ip}$$

- The i^{th} residual:

$$\hat{e}_i = y_i - \hat{y}_i = y_i - \hat{\beta}_0 - \hat{\beta}_1 x_{i1} - \hat{\beta}_2 x_{i2} - \cdots - \hat{\beta}_p x_{ip}$$

Least Squares Estimation

The parameters estimates $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, \cdots, \hat{\beta}_p$ can be found by minimizing the sum of squared residuals:

$$RSS = \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_{i1} - \hat{\beta}_2 x_{i2} - \cdots - \hat{\beta}_p x_{ip})^2$$

To minimize set the partial derivatives equal to zero:

$$\frac{\partial RSS}{\partial \hat{\beta}_0} = -2 \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_{i1} - \hat{\beta}_2 x_{i2} - \cdots - \hat{\beta}_p x_{ip}) = 0$$

$$\frac{\partial RSS}{\partial \hat{\beta}_1} = -2 \sum_{i=1}^{n} x_{i1} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_{i1} - \hat{\beta}_2 x_{i2} - \cdots - \hat{\beta}_p x_{ip}) = 0$$

$$\vdots$$

$$\frac{\partial RSS}{\partial \hat{\beta}_p} = -2 \sum_{i=1}^{n} x_{ip} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_{i1} - \hat{\beta}_2 x_{i2} - \cdots - \hat{\beta}_p x_{ip}) = 0$$

This gives a system of $(p+1)$ equations with $(p+1)$ unknowns, which can be solved (assuming $p < n$) to obtain the least squares estimates $\hat{\beta}_0, \hat{\beta}_1, \cdots, \hat{\beta}_p$. In practice, we can use the `lm()` function in R to do these computations.

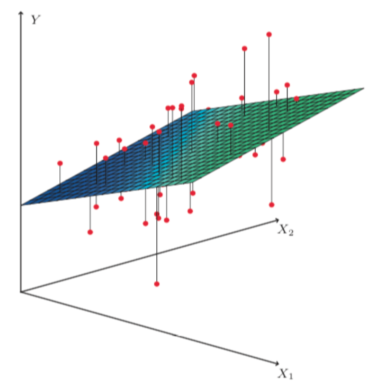


FIGURE 3.4. *In a three-dimensional setting, with two predictors and one response, the least squares regression line becomes a plane. The plane is chosen to minimize the sum of the squared vertical distances between each observation (shown in red) and the plane.*

From Chapter 3, p. 73, of *An Introduction to Statistical Learning*.

Estimating σ^2

$$\hat{\sigma}^2 = \frac{\text{RSS}}{n-p-1} = \frac{1}{n-p-1}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$
$$= \frac{1}{n-p-1}\sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_{i1} - \hat{\beta}_2 x_{i2} - \cdots - \hat{\beta}_p x_{ip})^2$$

- $\hat{\sigma} = \sqrt{\text{RSS}/(n-p-1)}$ is the residual standard error.
- It can be shown that $\hat{\sigma}^2$ is an unbiased estimate of σ^2 (i.e., $E(\hat{\sigma}^2) = \sigma^2$).

Hypothesis Test for a Single Predictor

Test whether parameter β_j is zero.

$H_0 : \beta_j = 0$
$H_A : \beta_j \neq 0$

Test statistic:

$$T_j = \frac{\hat{\beta}_j}{se(\hat{\beta}_j)}; \quad \text{df} = n - p - 1$$

- $se(\hat{\beta}_j)$ is the standard error of $\hat{\beta}_j$
- n is the number of observations
- p is the number of predictor variables
- degrees of freedom (df) = sample size - number of parameters estimated $= n - p - 1$ (since, when including the intercept, there are $p + 1$ parameters)

Confidence Interval for a Single Predictor

A $1 - \alpha$ confidence interval for β_j:

$$\hat{\beta}_j \pm t_{\alpha/2;n-p-1} se(\hat{\beta}_j)$$

The R function `confint()` can be used to calculate confidence intervals for the parameters.

Coefficient of Determination (R^2)

The coefficient of determination R^2 has the same definition for simple and multiple linear regression.

$$R^2 = \frac{\text{SSReg}}{\text{SST}} = 1 - \frac{\text{RSS}}{\text{SST}}$$

- $SST = \sum_{i=1}^{n}(y_i - \bar{y})^2$ is the total sum of squares
  - total variability in the response variable
- $SSreg = \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2$ is the regression sum of squares
  - variability in the response explained by the model
- $RSS = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$ is the residual sum of squares
  - unexplained variability

Coefficient of Determination (R^2)

$$R^2 = \frac{\text{SSReg}}{\text{SST}} = 1 - \frac{\text{RSS}}{\text{SST}}$$

- R^2 can be interpreted as the proportion of variability in the response Y that is explained by the regression model.
- $0 \leq R^2 \leq 1$, where values closer to 1 indicate a better linear fit to the data.
- **Problem with R^2 in MLR**: Adding predictor variables to the regression model will always increases R^2 (or, equivalently decrease RSS). Even if the predictor variable is irrelevant (noise) the R^2 will increase slightly. This is not ideal since simpler models are preferred to more complicated models.

Occam's razor, or the law of parsimony, is a problem solving principle that states that simpler solutions are preferred to more complex ones.[2]

"Everything should be kept as simple as possible, but not simpler"
–Albert Einstein

[2] `https://en.wikipedia.org/wiki/Occam's_razor`

Adjusted Coefficient of Determination (R^2_{adj})

$$R^2_{adj} = 1 - \frac{RSS/(n-p-1)}{SST/(n-1)}$$

- The denominator in $\text{RSS}/(n-p-1)$ penalizes for adding extra predictor variables.
- The idea is that the R^2_{adj} should decrease when adding an irrelevant predictor variables into a model.
- When comparing models with different numbers of predictors one should use R^2_{adj} and not R^2.

MLR Example: Menu Pricing Data Set

Data set from surveys of customers of 168 Italian restaurants in New York City.[3]

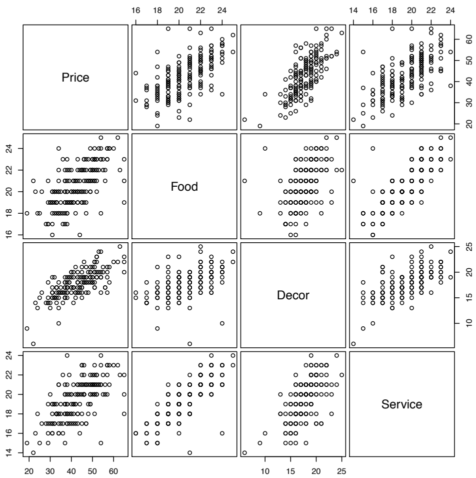
The variables are:

- Y = Price = the price (in \$US) of dinner (including 1 drink and tip)
- x_1 = Food = customer rating of the food (out of 30)
- x_2 = Decor = customer rating of the decor (out of 30)
- x_3 = Service = customer rating of the service (out of 30)
- x_4 = East = dummy variable, 1 (0) if the restaurant is east (west) of Fifth Avenue

[3] Zagat Survey 2001: New York City Restaurants

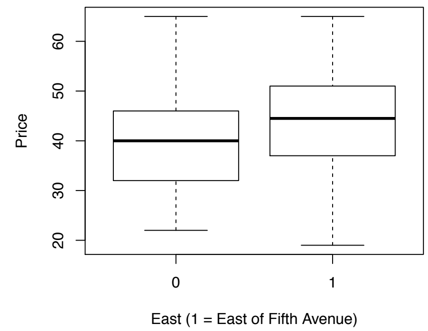
MLR Example: Menu Pricing Data Set

```
> nyc <- read.csv("https://ericwfox.github.io/data/nyc.csv")
> pairs(Price ~ Food + Decor + Service, data=nyc)
```



MLR Example: Menu Pricing Data Set

```
> boxplot(Price ~ East, data= nyc,
          ylab="Price", xlab="East (1 = East of Fifth Avenue)")
```



MLR Example

```
> lm1 <- lm(Price ~ Food + Decor + Service + East, data=nyc)
> summary(lm1)

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) -24.023800   4.708359  -5.102 9.24e-07 ***
Food          1.538120   0.368951   4.169 4.96e-05 ***
Decor         1.910087   0.217005   8.802 1.87e-15 ***
Service      -0.002727   0.396232  -0.007   0.9945
East          2.068050   0.946739   2.184   0.0304 *
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 5.738 on 163 degrees of freedom
Multiple R-squared:  0.6279,Adjusted R-squared:  0.6187
F-statistic: 68.76 on 4 and 163 DF,  p-value: < 2.2e-16
```

MLR Example

Since Service is not significant we remove it from the model.

```
> lm2 <- lm(Price ~ Food + Decor + East, data=nyc)
> summary(lm2)

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -24.0269     4.6727  -5.142 7.67e-07 ***
Food          1.5363     0.2632   5.838 2.76e-08 ***
Decor         1.9094     0.1900  10.049  < 2e-16 ***
East          2.0670     0.9318   2.218   0.0279 *
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 5.72 on 164 degrees of freedom
Multiple R-squared:  0.6279,Adjusted R-squared:  0.6211
F-statistic: 92.24 on 3 and 164 DF,  p-value: < 2.2e-16
```

```
> s1 <- summary(lm1)
> s2 <- summary(lm2)

> s1$r.squared
[1] 0.6278809
> s2$r.squared
[1] 0.6278808

> s1$adj.r.squared
[1] 0.6187492
> s2$adj.r.squared
[1] 0.6210738

#--------------------------------
> confint(lm2)
                 2.5 %     97.5 %
(Intercept) -33.253364 -14.800395
Food          1.016695   2.055996
Decor         1.534181   2.284565
East          0.227114   3.906912
```

MLR Example

The final regression model is:

$$\widehat{\texttt{Price}} = -24.03 + 1.54\texttt{Food} + 1.91\texttt{Decor} + 2.07\texttt{East}$$

For example, we can use the model to predict `Price` when `Food=20`, `Decor=16` and `East=1`:

$$\widehat{\texttt{Price}} = -24.03 + 1.54(20) + 1.91(16) + 2.07(1) = 39.4$$

We can also use R to make this prediction and to calculate a 95% prediction interval.

```
> new_x <- data.frame(Food = 20, Decor = 16, East = 1)
> predict(lm2, newdata = new_x, interval="prediction")
       fit      lwr      upr
1 39.31701 27.95384 50.68019
```

MLR Example

The final regression model is:

$$\widehat{\texttt{Price}} = -24.03 + 1.54\texttt{Food} + 1.91\texttt{Decor} + 2.07\texttt{East}$$

- `Decor` has the largest effect on `Price` since its regression coefficient is largest. Note that `Food`, `Decor`, and `Service` are on the same 0 to 30 scale, so it is meaningful to make the comparison.
- If a goal is to maximize `Price` for a new restaurant, it should be located east of Fifth Avenue (i.e., `East` $=1$).

Interpreting Regression Coefficients

Suppose we fit a multiple linear regression model

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \cdots + \hat{\beta}_p x_p + e,$$

where x_j is the j^{th} predictor and $\hat{\beta}_j$ the estimated coefficient for the variable.

How do we interpret $\hat{\beta}_j$?

The usual interpretation is as follows: an increase in x_j by 1, *with all other predictors in the model held fixed*, is associated with a change of $\hat{\beta}_j$ in the predicted response, $\hat{y}$.

Interpreting Regression Coefficients

Going back to the example, the final regression model is

$$\widehat{\texttt{Price}} = -24.03 + 1.54\texttt{Food} + 1.91\texttt{Decor} + 2.07\texttt{East}$$

- Interpret the coefficient for `Decor`: a one unit increase in the customer rating of decor, with the other predictors (`Food` and `East`) held fixed, is associated with an increase in `Price` by $1.91.
- Interpret the coefficient for the dummy variable `East`: the price of dinner at a restaurant east of Fifth Avenue will cost $2.07 more, on average, than a restaurant west of Fifth Avenue, when all other predictors (`Food` and `Decor`) are held fixed.

Interpreting Regression Coefficients

Some problems when interpreting regression coefficients:

- The interpretation of β_j as the average change in Y per unit change in x_j, *with all other predictors held fixed*, assumes predictors can be changed without affecting other predictors.
- Interpretation becomes hazardous when there are correlations amongst predictors. When x_j changes, then values for other predictors also change.
- The magnitude and sign of a coefficient can change when including (or removing) another predictor from the model.
- For observational data we can only make claims about associations, not causation.