Lecture 11:
F-test
STAT 632, Spring 2020

Test of all the predictors

Is there a relationship between the response variable and at least one predictor in the multiple linear regression model?

$$Y = \beta_0 + \beta_1 x_1 + \cdots \beta_p x_p + e$$

The null and alternative hypothesis for the test can be written as

$H_0 : \beta_1 = \beta_2 = \cdots = \beta_p = 0$
There is no relationship between Y and the predictor variables.

H_A: at least one $\beta_j \neq 0$
There is a relationship between Y and at least one of the predictor variables.

Test of all the predictors

$H_0 : \beta_1 = \beta_2 = \cdots = \beta_p = 0$

H_A: at least one $\beta_j \neq 0$

Test statistic:

$$F = \frac{(\text{SST} - \text{RSS})/p}{\text{RSS}/(n-p-1)} = \frac{\text{SSreg}/p}{\text{RSS}/(n-p-1)}$$

- $\text{SST} = \sum_{i=1}^{n}(y_i - \bar{y})^2$ is the total sum of squares
- $\text{RSS} = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$ is the residual sum of squares
- $\text{SSreg} = \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2$ is the regression sum of squares
- $\text{SST} = \text{SSreg} + \text{RSS}$
- n is the number of observations in the data set, and p is the number of predictor variables.

Test of all the predictors

Analysis of variance table:

Source	df	Sum of Squares	Mean Square	F
Regression	p	SSreg	SSreg/p	$\frac{\text{SSreg}/p}{\text{RSS}/(n-p-1)}$
Residual	$n-p-1$	RSS	$\text{RSS}/(n-p-1)$	
Total	$n-1$	SST		

Most computer packages will provide some version of this table. Ronald Fisher, the creator of the table, once said it's "nothing but a convenient way of arranging arithmetic." That was in 1931, when he had to all the calculations by hand.

Example: NY Housing Data

- Data set on housing prices from Canton, NY (scraped from Zillow.com)
- The response variable is `Price` (in thousands of dollars)
- The predictors
  - `Beds`: number of bedrooms
  - `Baths`: number of bathrooms
  - `Size`: floor area of house (in thousands of square feet)
  - `Lot`: size of the lot (in acres)

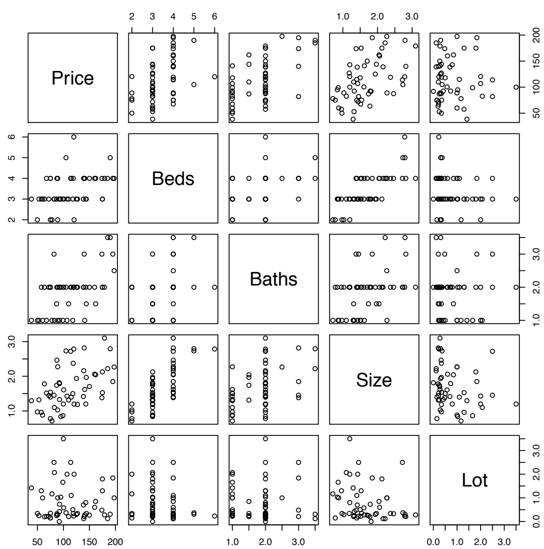
```
> library(Stat2Data)

> data(HousesNY)

> dim(HousesNY)
[1] 53  5

> head(HousesNY)
  Price Beds Baths  Size  Lot
1  57.6    3     2 0.960 1.30
2 120.0    6     2 2.786 0.23
3 150.0    4     2 1.704 0.27
4 143.0    3     2 1.200 0.80
5  92.5    3     1 1.329 0.42
6  50.0    2     1 0.974 0.34
```

```
> pairs(Price ~ Beds + Baths + Size + Lot, data=HousesNY)
```



Example

$H_0 : \beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$
H_A : at least one $\beta_j \neq 0$

For the F-test we are comparing the full model, with all the predictors, to the null model, with no predictors.

Full model:
$\texttt{Price} = \beta_0 + \beta_1\texttt{Beds} + \beta_2\texttt{Baths} + \beta_3\texttt{Size} + \beta_4\texttt{Lot} + e$

Null (reduced) model:
$\texttt{Price} = \beta_0 + e$

```
> lm_full <- lm(Price ~ Beds + Baths + Size + Lot, data=HousesNY)
> lm_null <- lm(Price ~ 1, data=HousesNY)

> anova(lm_null, lm_full)
Analysis of Variance Table

Model 1: Price ~ 1
Model 2: Price ~ Beds + Baths + Size + Lot
  Res.Df   RSS Df Sum of Sq      F   Pr(>F)
1     52 89255
2     48 52358  4     36897 8.4566 3.01e-05 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Since the p-value < 0.001 we reject the null hypothesis that $\beta_1 = \cdots = \beta_4 = 0$. Thus, we conclude, that at least one predictor is associated with Price.

The results of the F-test are also provided in the summary() output.

```
> summary(lm_full)

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   14.590     23.266   0.627   0.5336
Beds           2.771      8.730   0.317   0.7523
Baths         26.238      7.844   3.345   0.0016 **
Size          22.155     11.931   1.857   0.0695 .
Lot            4.621      6.184   0.747   0.4585
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 33.03 on 48 degrees of freedom
Multiple R-squared:  0.4134,Adjusted R-squared:  0.3645
F-statistic: 8.457 on 4 and 48 DF,  p-value: 3.01e-05
```

Just for verification, we can also directly calculate the F-test statistic using the formula.

```
> n <- nrow(HousesNY)
> p <- 4
> rss <- sum(resid(lm_full)^2); rss
[1] 52357.9
> sst <- sum(resid(lm_null)^2); sst
[1] 89255.4
> fstat <- ((sst - rss) / p) / (rss / (n-p-1))
> fstat
[1] 8.456603
> 1 - pf(fstat, df1=p, df2=n-p-1)
[1] 3.0104e-05
```

Testing just one predictor

Can one particular predictor be dropped from the model?

To test whether the coefficient for a single predictor is 0 we can either use a t-test or F-test (the results are equivalent). In R, the t-test is less work, since the results are provided in the regression summary output.

$H_0 : \beta_j = 0$
$H_A : \beta_j \neq 0$

Test statistic:

$$T_j = \frac{\hat{\beta}_j}{\text{se}(\hat{\beta}_j)}; \quad \text{df} = n - p - 1$$

Example

The regression summary in a previous slide shows that Beds is not significant, and can be dropped from the model, since $t = 0.317$ with a p-value$= 0.7523$. Using the F-test we obtain the same result:

```
> lm_full <- lm(Price ~ Beds + Baths + Size + Lot, data=HousesNY)
> lm1 <- lm(Price ~ Baths + Size + Lot, data=HousesNY)
> anova(lm1, lm_full)
Analysis of Variance Table

Model 1: Price ~ Baths + Size + Lot
Model 2: Price ~ Beds + Baths + Size + Lot
  Res.Df   RSS Df Sum of Sq      F Pr(>F)
1     49 52468
2     48 52358  1    109.87 0.1007 0.7523
```

Notice that the p-values from the F-test and t-test are exactly the same.

Testing a subset of predictors

Suppose we want to test whether a specified subset of predictors have regression coefficients equal to 0. This is often called the **partial F-test**.

For example, using the NY housing data, the full model is given by
$\texttt{Price} = \beta_0 + \beta_1\texttt{Beds} + \beta_2\texttt{Baths} + \beta_3\texttt{Size} + \beta_4\texttt{Lot} + e$

Suppose we want to test whether the coefficients for Beds and Lot are both zero. The null and alternative hypothesis can be written as:
H_0: $\beta_1 = \beta_4 = 0$
H_A: $\beta_1 \neq 0$ or $\beta_4 \neq 0$

Testing a subset of predictors

For the partial F-test we use the following test statistic:

$$F = \frac{(\text{RSS}_{\text{reduced}} - \text{RSS}_{\text{full}})/k}{\text{RSS}_{\text{full}}/(n-p-1)}$$

- RSS_{full} is the residuals sum of squares for the model with the full set of p predictors.
- $\text{RSS}_{\text{reduced}}$ is the residuals sum of squares for the reduced model with k predictors removed.

Example

```
> lm_full <- lm(Price ~ Beds + Baths + Size + Lot, data=HousesNY)
> lm2 <- lm(Price ~ Baths + Size, data=HousesNY)
> anova(lm2, lm_full)
Analysis of Variance Table

Model 1: Price ~ Baths + Size
Model 2: Price ~ Beds + Baths + Size + Lot
  Res.Df   RSS Df Sum of Sq      F Pr(>F)
1     50 53039
2     48 52358  2    680.81 0.3121 0.7334
```

The p-value $= 0.73$ is large, so we do not reject the null hypothesis that $H_0 : \beta_1 = \beta_4 = 0$. So we can remove both predictors, Beds and Lot, from the model.

The R^2 for the full and reduced models are about the same, and the adjusted R^2 for the reduced model is a little higher. This agrees with the conclusion of the F-test. So the adjusted-R^2 also indicates that we can remove Beds and Lot.

```
> s1 <- summary(lm_full)
> s2 <- summary(lm2)
>
> s1$r.squared
[1] 0.4133924
> s2$r.squared
[1] 0.4057647
>
> s1$adj.r.squared
[1] 0.3645084
> s2$adj.r.squared
[1] 0.3819953
```

```
> summary(lm2)
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   24.641     15.890   1.551  0.12728
Baths         26.755      7.699   3.475  0.00107 **
Size          23.399      8.317   2.813  0.00699 **
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 32.57 on 50 degrees of freedom
Multiple R-squared:  0.4058,Adjusted R-squared:  0.382
F-statistic: 17.07 on 2 and 50 DF,  p-value: 2.233e-06
```

Your turn

Using the NY housing data, the full model is given by
$\texttt{Price} = \beta_0 + \beta_1\texttt{Beds} + \beta_2\texttt{Baths} + \beta_3\texttt{Size} + \beta_4\texttt{Lot} + e$

In R, conduct a partial F-test for the following hypotheses:
$H_0 : \beta_1 = \beta_2 = \beta_4 = 0$
$H_A : \beta_1 \neq 0$ or $\beta_2 \neq 0$ or $\beta_4 \neq 0$

What is the p-value and your conclusion?

Summary

- Use the overall F-test to test whether there is a relationship between the response and at least one predictor in the model.
- Use the partial F-test to test whether there is a relationship between the response and a specified subset of two or more predictors in the model.
- Use a t-test to test whether there is a relationship between the response and a single predictor in the model.