Lecture 12:
Simple Linear Regression
STAT 310, Spring 2021

Scatterplots

- A scatterplot a graphical display used to study the relationship between two numerical variables x and y.
- Data displayed on a scatterplot are collected in pairs:

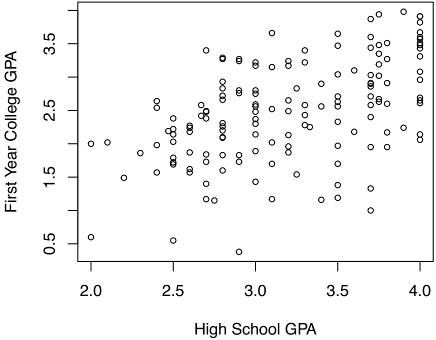
$$(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$$

  where n denotes the total number of cases or pairs.

- A scatterplot provides insight into how two variables are related.

Example

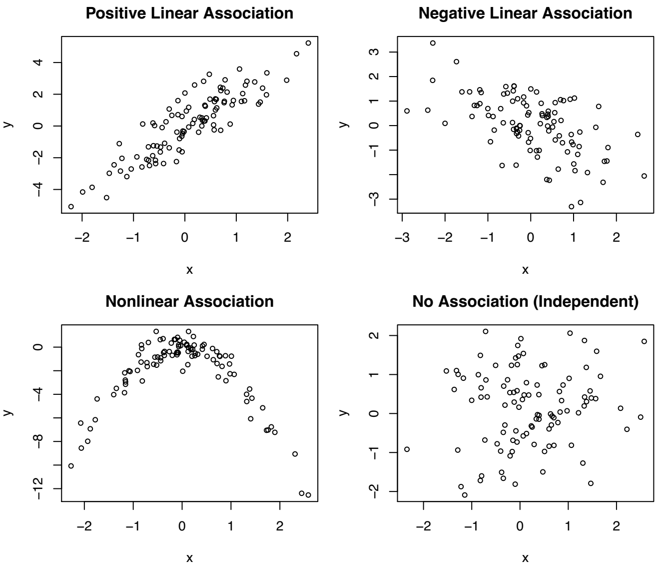
A scatterplot showing the association between first year college GPA and high school GPA for a random sample of 150 students.



positive linear association

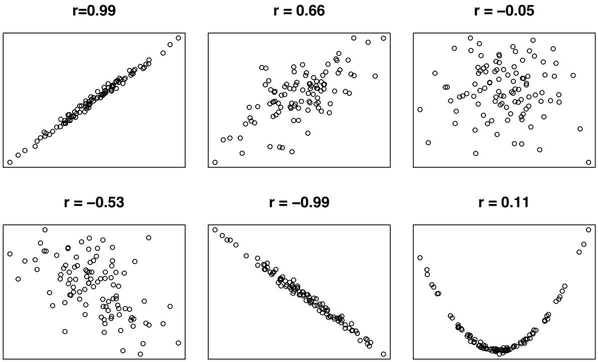
Types of Relationships Between Variables

- Two variables are said to be **associated** if the scatterplot shows a discernible pattern or trend.
- An association is **positive** if y increases as x increases.
- An association is **negative** if y decreases as x increases.
- An association is **linear** if the scatterplot between x and y has a linear trend; otherwise, the association is called **nonlinear**.


Positive Linear Association
Negative Linear Association
Nonlinear Association
No Association (Independent)
x
y

Correlation Coefficient

The **correlation coefficient**, denoted by *r*, is a number between -1 and 1 that describes the strength of the linear association between two variables.



Correlation Coefficient

- $r \approx 1$ when there is a strong positive linear association between the variables.
- $r \approx -1$ when there is a strong negative linear association between the variables.
- $r \approx 0$ when there is no association between the variables (i.e., independent).
- The correlation coefficient is only useful for evaluating the linear association between two variables. It is not a useful measure for nonlinear relationships.

Correlation Coefficient

Formally, the correlation can be calculated using the following formula. The formula is rather complex, so we use software packages such as R to do the calculations for us.

$$r = \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right)$$

- $\bar{x}$ and $\bar{y}$ are the sample means
- s_x and s_y are the sample standard deviations

Example

Match each correlation to the corresponding scatterplot.

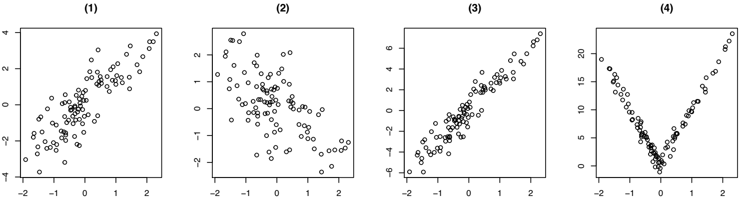
(a) $r = -0.63$

(b) $r = 0.85$

(c) $r = 0.19$

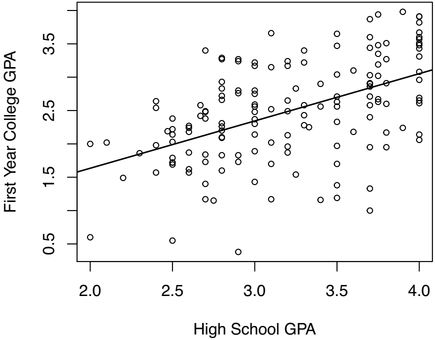
(d) $r = 0.95$

(b) (a) (d) (c)

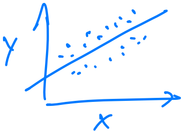


Simple Linear Regression

- **Simple linear regression** is a method for fitting a straight line to data that show a linear trend when displayed on a scatterplot.
- The method is useful for making predictions and explaining the relationship between two numerical variables.



Simple Linear Regression



A **simple linear regression model** expresses the relationship between two variables, x and y, as a straight line with some error:

$$y = \beta_0 + \beta_1 x + \epsilon$$

- y is called the **response** variable
- x is called the **explanatory** or **predictor** variable.
- β_0 is the **intercept** parameter
- β_1 is the **slope** parameter
- ϵ is called the **random error** term. It captures the variability in the points around the line.

Fitted Values and Residuals

$\hat{y} \Rightarrow$ predicted value of y

$\hat{\beta}_0, \hat{\beta}_1 \Rightarrow$ parameter estimates from data set

- The line that we estimate, or fit to the data in the scatterplot, is written as

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$

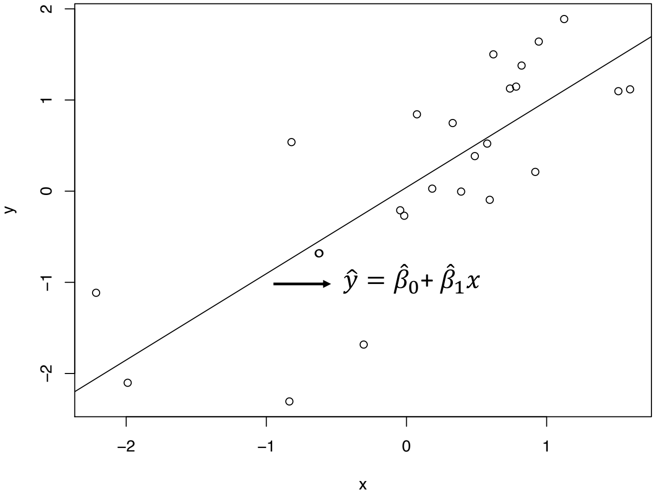
- The fitted (or predicted) value for the i^{th} observation (x_i, y_i):

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

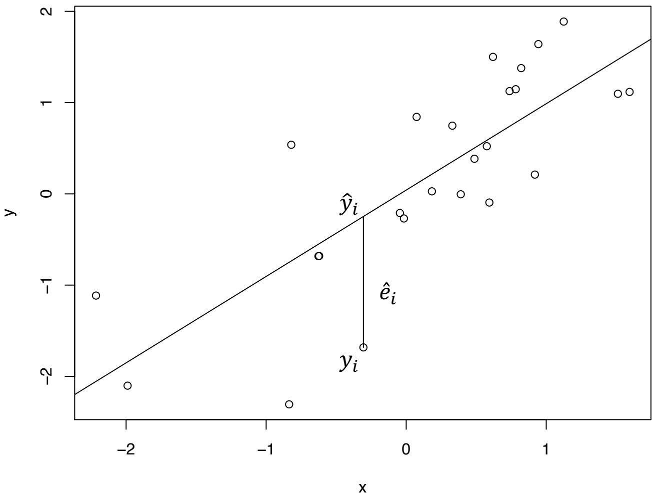
- The **residual** for the i^{th} observation is the difference between the observed value (y_i) and the predicted value $(\hat{y}_i)$:

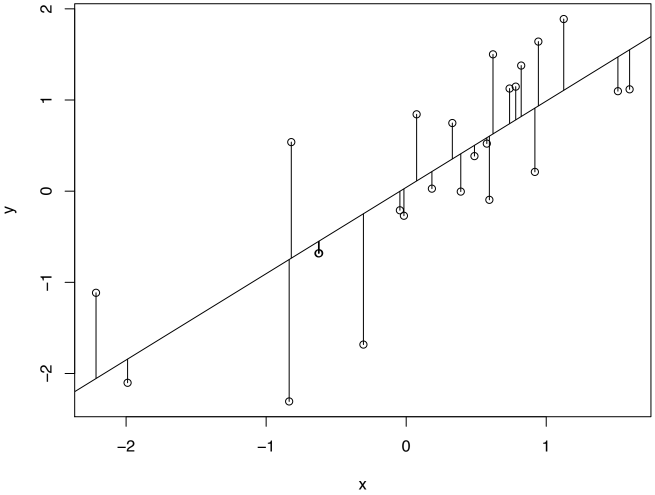
$$\hat{e}_i = y_i - \hat{y}_i = y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i)$$

residual = observed − predicted


$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$
y
x
2
1
0
−1
−2
−2
−1
0
1

$$\hat{e}_i = y_i - \hat{y}_i$$





y
x
−2
−1
0
1
2
−2
−1
0
1

Sum of Squared Residuals

- Intuitively, a line that fits the data well has small residuals.
- The **least squares line** minimizes the **sum of squared residuals**:

$$RSS = \sum_{i=1}^{n} \hat{e}_i^2 = \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$$

- That is, out of all possible lines we could draw on the scatterplot, the least squares line is the "best fit" since it has the smallest sum of squared residuals.

$$RSS = \hat{e}_1^2 + \hat{e}_2^2 + \ldots + \hat{e}_n^2$$

Least Squares Estimates

It can be shown (using calculus) that the estimates of the intercept and slope that minimize the sum of squared residuals are given by the following formulas:

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$
$$\hat{\beta}_1 = r\frac{s_y}{s_x}$$

where r is the correlation coefficient, previously discussed. Note that the equation for the intercept guarantees the least squares line passes through the point $(\bar{x}, \bar{y})$.

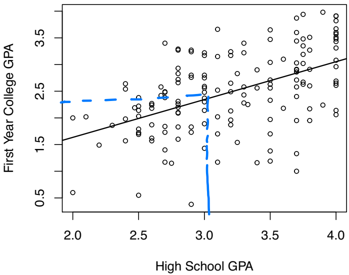
$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \Rightarrow \bar{y} = \hat{\beta}_0 + \hat{\beta}_1 \bar{x}$$

Example

The least squares regression line in the scatterplot is given by:

$$\hat{y} = 0.217 + 0.709\,x$$

Suppose a student graduates high school with a 3.0 GPA. What is the predicted first year college GPA for this student?



$$\hat{y} = 0.217 + 0.709(3)$$

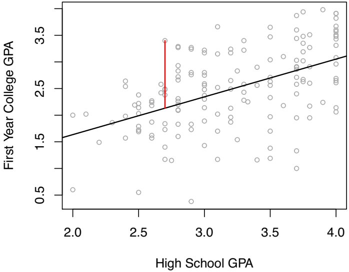
$$= 2.34$$

Example

The least squares regression line in the scatterplot is given by:

$$\hat{y} = 0.217 + 0.709\,x$$

Calculate the residual (show in red) for a student, in this data set, that had a 2.7 high school GPA and a 3.4 college GPA.



$$\hat{e}_i = y_i - \hat{y}_i$$
$$= 3.4 - [0.217 + 0.709(2.7)]$$
$$= 3.4 - 2.13$$
$$= 1.27$$

Example

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x = 0.217 + 0.708x$$

Use the summary statistics in the table below to manually calculate the slope and intercept of the least squares line.

	HS GPA (x)	FY College GPA (y)
Mean	$\bar{x} = 3.196$	$\bar{y} = 2.48$
SD	$s_x = 0.534$	$s_y = 0.753$
	correlation	$r = 0.502$

$$\hat{\beta}_1 = r\frac{s_y}{s_x} = (0.502)\left(\frac{0.753}{0.534}\right) = 0.708$$

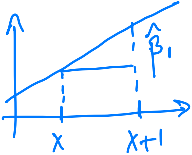
$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x} = 2.48 - 0.708(3.196) = 0.217$$

Interpreting Coefficients

- **Slope**: an increase in the explanatory variable (x) by one unit is associated with a change of $\hat{\beta}_1$ in the predicted response ($\hat{y}$).

- **Intercept**: the prediction for the response variable ($\hat{y}$) when the value for the explanatory variable is zero ($x = 0$). It may not make sense to try to interpret the intercept depending on the application.

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 (x+1)$$
$$= \hat{\beta}_0 + \hat{\beta}_1 + \hat{\beta}_1 x$$



Set $x = 0$

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 (0)$$
$$= \hat{\beta}_0$$

Interpreting Coefficients

intercept $= \hat{\beta}_0 = 0.217$

Slope $= \hat{\beta}_1 = 0.709$

The least squares regression line for predicting first year college GPA (y) from high school GPA (x) is given by:

$$\hat{y} = 0.217 + 0.709\,x$$

Interpret the slope and intercept of this model.

Slope: An increase in HS GPA by 1 is associated with an increase in first year college GPA by 0.709.

Intercept: When HS GPA is 0, the predicted first-year college GPA is 0.217. This is an extrapolation.

Coefficient of Determination (R^2)

- The coefficient of determination (R^2) is a measure of how well the linear regression model fits the data.
- The R^2 can be computed as the correlation coefficient r squared.
- R^2 can be interpreted as the proportion of variability in the response variable y that is explained by x.
- R^2 is always between 0 and 1; the closer R^2 is to 1, the better the linear regression model fits the data.

Coefficient of Determination (R^2)

- Going back to the example, the correlation between high school GPA and first year college GPA is 0.502.
- So $R^2 = (0.502)^2 = 0.252$.
- This tells us that about 25% of the variability in first year college GPA (y) can be explained by high school GPA (x).