Lecture 9:
Confidence Interval for One Mean
STAT 310, Spring 2021

# Introduction

- Similar to how we can model the sample proportion $\hat{p}$ using a normal distribution, the sample mean $\bar{x}$ can also be modeled using a normal distribution when certain conditions are met.
- However, we'll learn that a new distribution, called the $t$-distribution, tends to be more useful when working with the sample mean.
- In this lecture we'll first learn about this new distribution, and then use it to construct confidence intervals for the mean.

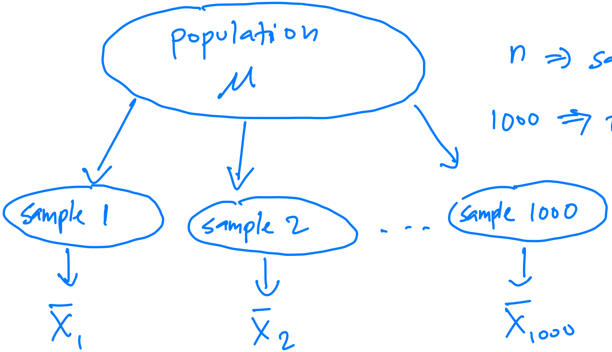
# Central Limit Theorem (CLT) for the Mean

When the sample size $n$ is sufficiently large, the sampling distribution for $\bar{x}$ follows an approximate normal distribution centered around the population mean $\mu$, and with standard error $SE = \sigma/\sqrt{n}$

$$\bar{X} \sim N\left(\underbrace{\mu}_{\text{mean}}, \underbrace{\frac{\sigma}{\sqrt{n}}}_{SE}\right)$$

$\sigma \Rightarrow$ population standard deviation

# sampling distribution concept

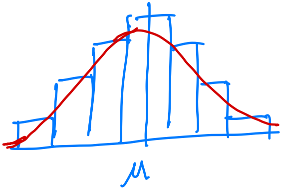



$\mu \Rightarrow$ population mean

$n \Rightarrow$ sample size

$1000 \Rightarrow$ # of samples

<u>Sampling distribution</u> can be constructed as the histogram of the 1000 sample means.




- When $n$ is large, the histogram looks like a normal distribution centered around population mean $\mu$
- standard deviation of 1000 sample means is the standard error.

$$SE = \sigma/\sqrt{n}$$

- Based on the CLT, we can construct a 95% confidence interval for the population mean $\mu$ as

$\sigma \Rightarrow$ pop st dev

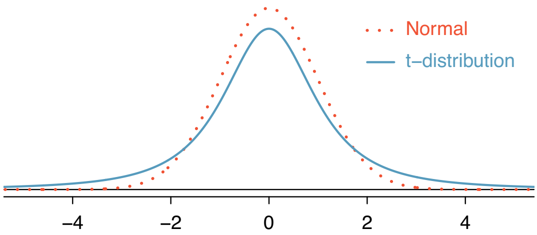
$s \Rightarrow$ sample st dev

$$\bar{x} \pm z^* SE \implies \bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}} \approx \bar{x} \pm 1.96 \frac{s}{\sqrt{n}}$$

- However, one issue with this confidence interval formula is that the standard error is in terms of the population standard deviation $\sigma$, which is unknown.
- We can resolve this by plugging in the sample standard deviation $s$ as the estimate of $\sigma$. That is, use $SE \approx s/\sqrt{n}$.
- This is a sensible approach when the sample size is large. But when the sample size is small, the confidence interval needs to be adjusted to account for additional uncertainty in estimating $\sigma$ with $s$.
- It turns out that we can get a more accurate confidence interval, which accounts for this additional uncertainty, by using the $t$-distribution to calculate the critical value instead of the $z$-distrubiton.

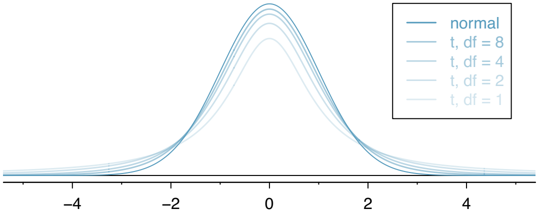
# t-disribution

- A *t*-distribution is bell-curve shaped distribution that is centered around zero.
- It looks similar to a standard normal distribution, but it has wider tails.



# t-disribution

- The shape of the *t*-distribution depends on the degrees of freedom, which is defined as $df = n - 1$
- When the sample size is small the *t*-distrubiton has noticeably wider tails than a normal distribution.
- When the sample size is large (about 30 or more), the *t*-distribution is nearly identical to a normal distribution.



# Confidence Interval for $\mu$

A confidence interval for the the population mean $\mu$ is given by

$$\bar{x} \pm t^* \frac{s}{\sqrt{n}}$$

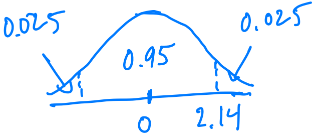
$s \Rightarrow$ sample standard deviation

point estimate $\pm\ t^*\ SE$

The critical value $t^*$ is found using $t$-distribution. It depends on the confidence level and degrees of freedom, and can be computed using the R function `qt()`.

$df = n - 1$

*Example*: Calculate the critical value $t^*$ when the confidence level is 95% and $n = 15$.




$t^* = qt(0.975, df = 14)$

$= 2.14$

# Confidence Interval for $\mu$

The confidence interval formula is valid if the following conditions are satisfied:

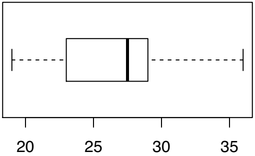
- The data come from a random sample. (This is called the **independence condition** in the textbook.)
- The sample size ~~is~~ $n$ is large ($n \geq 30$). Otherwise, if the sample size is small ($n < 30$), the data have an approximate normal distribution. (This is called the **normality condition** in the textbook.)
- Additionally, the data should not contain any extreme outliers.

Graphical methods (box plots, histograms) can be used to check if the data have an approximate normal distribution when the sample size $n$ is small.

# Example

Below are some summary statistics and a box plot for the ages of a random sample of $n = 26$ female athletes who participated in the 2012 Olympic Games in London. Calculate and interpret a 95% confidence interval for the population mean age. Also comment on whether the conditions for the confidence interval appear satisfied.

| n | $\bar{x}$ | s | min | max |
|---|---|---|---|---|
| 26 | 26.9 | 4.5 | 19 | 36 |

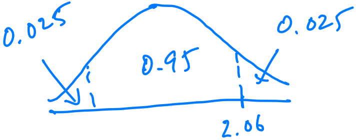



conditions:

- Data come from random sample ✓
- Data have an approximate normal distribution ✓
  (need to check since $n < 30$)

Next, calculate 95% CI

$n = 26, \quad \bar{x} = 26.9, \quad s = 4.5$




$t^* = qt(0.975, df = 25)$
$= 2.06$

$$\bar{x} \pm t^* \frac{s}{\sqrt{n}} \Rightarrow 26.9 \pm 2.06\left(\frac{4.5}{\sqrt{26}}\right)$$

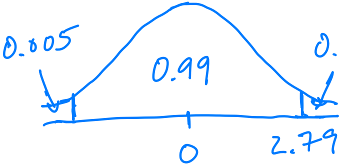
$$\Rightarrow \boxed{(25.08, 28.72)}$$

We are 95% confident that the population mean age $\mu$ is between 25.08 and 28.72.

# Example

Calculate and interpret a 99% confidence interval for the population mean age of female athletes who participated in the 2012 Olympic Games.

$n = 26, \quad \bar{x} = 26.9, \quad s = 4.5$




$\text{qt}(0.995, df = 25)$

$= 2.79$

$\bar{x} \pm t^* \frac{s}{\sqrt{n}} \Rightarrow 26.9 \pm 2.79 \left( \frac{4.5}{\sqrt{26}} \right)$

$\Rightarrow \boxed{(24.44, 29.36)}$