

Lecture 3:
Descriptive Statistics
STAT 310, Spring 2021

Measures of Central Tendency

Let x_1, x_2, \dots, x_n be observations of a sample of size n . The **sample mean** is given by

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i = \frac{x_1 + x_2 + \dots + x_n}{n}$$

Ex: The heights of 5 individuals: 63, 64, 66, 72, 62.

$$\bar{x} = \frac{63 + 64 + 66 + 72 + 62}{5} = 65.4$$

Measures of Central Tendency

The **sample median** of a set of observations is the middle value when values are ordered from smallest to largest.

Ex: (n odd) Find the median of 63, 64, 66, 72, 62.

First, order the data: 62, 63, 64, 66, 72

median = 64

Ex: (n even) Find the median of 63, 64, 66, 72, 62, 77.

First, order the data: 62, 63, 64, 66, 72, 77

median = $(64 + 66)/2 = 65$

Measures of Central Tendency

The median is resistant to outliers, while the mean is affected by outliers.

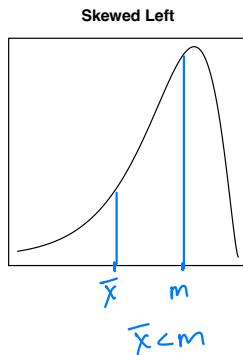
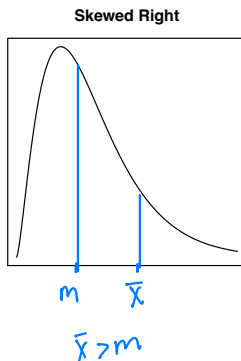
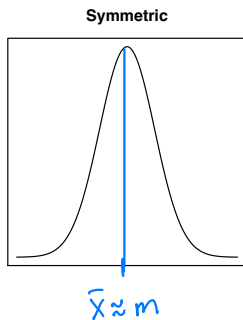
Ex: How do the mean and median compare for the sample:
62, 63, 64, 66, 72, 1000?

```
> x <- c(62, 63, 64, 66, 72, 1000)
> mean(x)
[1] 221.1667
> median(x)
[1] 65
```

The sample mean is much larger than the median, since it is affected by the outlier. The median is a better measure of central tendency in this example.

Measures of Central Tendency

Compare the mean \bar{x} and median m for distributions that are symmetric, skewed right, and skewed left.



Quartiles

- ▶ The **first quartile**, denoted by Q_1 , is the value such that 25% of the data falls below, i.e., the 25th percentile.
- ▶ The **third quartile**, denoted by Q_3 , is the value such that 75% of the data falls below, i.e., the 75th percentile.
- ▶ Note that the second quartile, Q_2 , is the median.

A method for finding Q_1 and Q_3 by hand:

1. Order the data from smallest to largest
2. Divide the data into two sets using the median
3. Q_1 is the median of the first half, and Q_3 is the median of the second half

Quartiles

Ex: Find Q_1 and Q_3 for the following sample of $n = 10$ heights of individuals:

68, 76, 66, 63, 70, 66, 71, 71, 64, 71

Solution:

First, order the data: 63, 64, 66, 66, 68, 70, 71, 71, 71, 76

$$\text{median} = (68 + 70)/2 = 69$$

$$Q_1 = 66$$

$$Q_3 = 71$$

Useful R commands:

```
> x <- c(68, 76, 66, 63, 70, 66, 71, 71, 64, 71)
> summary(x)
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
   63.0   66.0   69.0   68.6   71.0   76.0
> mean(x)
[1] 68.6
> median(x)
[1] 69
> min(x)
[1] 63
> max(x)
[1] 76
> sort(x)
[1] 63 64 66 66 68 70 71 71 71 76
```


Measures of Variation

- ▶ Range = Max - Min
- ▶ Interquartile range: $IQR = Q_3 - Q_1$
- ▶ Let x_1, x_2, \dots, x_n be a sample of n observations. The **sample variance** is defined as

$$s^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n - 1},$$

and the **sample standard deviation** is defined as

$$s = \sqrt{s^2} = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n - 1}}$$

Measures of Variation

- ▶ The sample variance can be thought of as the average of the squared deviations between the observations x_i and the sample mean \bar{x} . It measures how concentrated values are around the sample mean.
- ▶ The standard deviation is in the same units as the data (e.g., if the data are in *ft*, then s is in *ft* and s^2 is in ft^2).
- ▶ s^2 , s , and the range are affected by outliers, while the IQR is resistant to outliers.

Measures of Variation

Ex: Calculate the variance and standard deviation of the following sample of $n = 5$ observations: 2, 5, 10, 15, 18

$$\bar{x} = \frac{2 + 5 + 10 + 15 + 18}{5} = \frac{50}{5} = 10$$

$$\begin{aligned}s^2 &= \frac{1}{5 - 1} [(2 - 10)^2 + (5 - 10)^2 + (10 - 10)^2 + (15 - 10)^2 + (18 - 10)^2] \\&= \frac{1}{4} (8^2 + 5^2 + 0^2 + 5^2 + 8^2) \\&= \frac{178}{4} = 44.5 \\s &= \sqrt{44.5} = \boxed{6.67}\end{aligned}$$

Useful R commands:

```
> x <- c(68, 76, 66, 63, 70, 66, 71, 71, 64, 71)
> var(x)
[1] 15.6
> sd(x)
[1] 3.949684
> max(x)-min(x) # range
[1] 13
> IQR(x)
[1] 5
```

Ex: Without doing any calculations, which of the following data sets do you think has the largest sample variance? Which has the smallest sample variance? Use R to verify.

Set 1: 100, 99, 98, 50, 2, 1, 0

Set 2: 53, 52, 51, 50, 49, 48, 47

Set 3: 51, 51, 51, 50, 49, 49, 49

Solution: Set 1 has the largest variance since the values are most spread out around the mean. Set 3 has the smallest variance since the values are most concentrated around the mean. Note that $\bar{x} = 50$ for all three sets.

To verify in R:

```
> x1 = c(100, 99, 98, 50, 2, 1, 0); var(x1)
> x2 = c(53, 52, 51, 50, 49, 48, 47); var(x2)
> x3 = c(51, 51, 51, 50, 49, 49, 49); var(x3)
```

Shifting and Rescaling Data

- ▶ *Shifting*: Adding a constant to each data value affects measures of position (mean, median, quartiles), but not measures of variation (standard deviation, IQR)
- ▶ *Rescaling*: Multiplying each data value by a constant affects both measures of position (mean, median, quartiles) and measures of variation (standard deviation, IQR)

Theorem: Let x_1, x_2, \dots, x_n be observations of a sample of size n , and \bar{x} and s_x the sample mean and standard deviation. For the transformation $y_i = ax_i + b$, where a and b are constants, show that $\bar{y} = a\bar{x} + b$ and $s_y = |a|s_x$.

Proof:

$$\begin{aligned}\bar{y} &= \frac{1}{n} \sum_{i=1}^n y_i = \frac{1}{n} \sum_{i=1}^n (ax_i + b) \\&= \frac{1}{n} [(ax_1 + b) + (ax_2 + b) + \dots + (ax_n + b)] \\&= \frac{1}{n} [a(x_1 + x_2 + \dots + x_n) + nb] \\&= a \left(\frac{x_1 + x_2 + \dots + x_n}{n} \right) + b \\&= a\bar{x} + b\end{aligned}$$

$$\begin{aligned}
 s_y^2 &= \frac{1}{n-1} \sum_{i=1}^n (y_i - \bar{y})^2 \\
 &= \frac{1}{n-1} \sum_{i=1}^n [(ax_i + b) - (a\bar{x} + b)]^2 \\
 &= \frac{1}{n-1} \sum_{i=1}^n (ax_i - a\bar{x})^2 \\
 &= \frac{a^2}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2 \\
 &= a^2 s_x^2
 \end{aligned}$$

The standard deviation is the square root of the variance:

$$s_y = \sqrt{a^2 s_x^2} = |a| s_x$$

Ex: Consider the following temperature measurements in $^{\circ}F$:
72, 67, 73, 81, 75.

(a) Calculate the mean and standard deviation. (You can use R for this)

```
> x <- c(72, 67, 73, 81, 75)
> mean(x)
[1] 73.6
> sd(x)
[1] 5.07937
```

(b) What is the mean and standard deviation if we convert from $^{\circ}F$ to $^{\circ}C$? The conversion formula is $^{\circ}C = \frac{5}{9}(^{\circ}F - 32)$

Mean in Celsius:

$$\frac{5}{9}(73.6 - 32) = 23.11^{\circ}C$$

Standard deviation in Celsius:

$$\frac{5}{9}(5.08) = 2.82^{\circ}C$$

Box Plot

A box plot is a useful way to display the distribution of data and identify outliers.

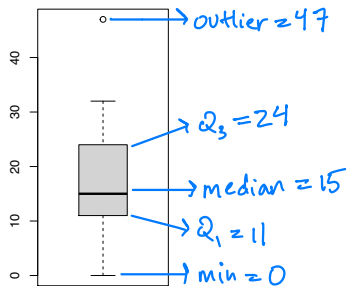
$$\text{Upper Fence} = Q_3 + 1.5(IQR)$$

$$\text{Lower Fence} = Q_1 - 1.5(IQR)$$

Values outside the fences are potential outliers.

Box Plot: Example

```
> x <- c(0, 18, 15, 32, 5, 22, 47, 15, 26, 13, 9)
> summary(x)
  Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
  0.00  11.00   15.00   18.36  24.00   47.00
> boxplot(x)
```



$$IQR = Q_3 - Q_1 \\ = 24 - 11 = 13$$

$$UF = Q_3 + 1.5(IQR) \\ = 24 + 1.5(13) = 43.5$$

$$LF = Q_1 - 1.5(IQR) \\ = 11 - 1.5(13) = -8.5$$

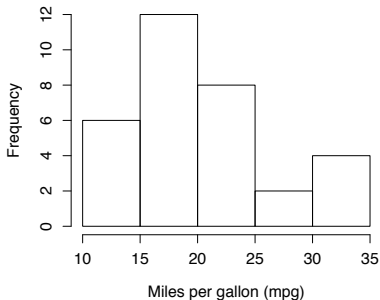
Histogram

- ▶ A histogram is a useful way to visualize the distribution of a numerical variable.
- ▶ To construct a histogram, the range of the data is divided into bins of equal width. Then the number of observations falling in each bin are counted. The counts are plotted as rectangles over each bin.
- ▶ Histograms are especially convenient for understanding the shape of the data distribution.

```
> sort(mtcars$mpg)
 [1] 10.4 10.4 13.3 14.3 14.7 15.0 15.2 15.2 15.5 15.8 16.4
[12] 17.3 17.8 18.1 18.7 19.2 19.2 19.7 21.0 21.0 21.4 21.4
[23] 21.5 22.8 22.8 24.4 26.0 27.3 30.4 30.4 32.4 33.9
```

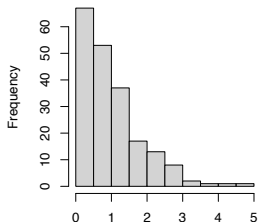
Bin	(10, 15]	(15, 20]	(20, 25]	(25, 30]	(30, 35]
Count	6	12	8	2	4

```
> hist(mtcars$mpg, main='', xlab="Miles per gallon (mpg)")
```

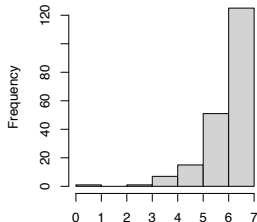


Describing the Shape of a Distribution

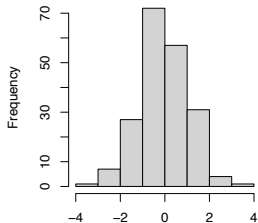
Right Skewed



Left Skewed



Symmetric



Bimodal

