Lecture 8:
Hypothesis Testing for a Proportion
STAT 310, Spring 2023

Hypothesis Test for a Proportion

Key components:

- Null hypothesis:
  $H_0 : p = p_0$
- Alternative hypothesis (use one of these):
  $H_A : p > p_0$ (one-sided, upper-tail)
  $H_A : p < p_0$ (one-sided, lower-tail)
  $H_A : p \neq p_0$ (two-sided)
- Test statistic:

$$z = \frac{\text{observed value} - \text{null value}}{\text{SE}} = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

- A rule to either reject or not reject H_0

Hypothesis Testing Concept

The approach to hypothesis testing is as follows:

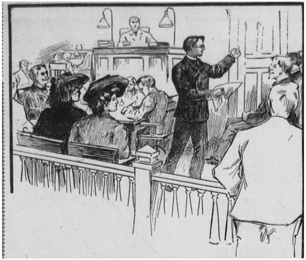
1. Assume that H_0 is true. H_0 usually represents a skeptical position, or a perspective of no difference or change in the parameter of interest.
2. Reject H_0 only if the data provide strong evidence in support of the alternative claim in H_A.

Hypothesis Testing Concept

The hypothesis testing framework can be found in the US court system, where innocence is assumed until proven guilty.[1]

H_0: The defendant is innocent.
H_A: The defendant is guilty.

The jurors consider whether the evidence is convincing enough to convict the defendant (reject H_0).



[1] https://commons.wikimedia.org/wiki/File:Trial_of_Edward_Ellis_(courtroom_sketch).jpg

p-value

A p-**value** is the probability of obtaining a test statistic as extreme, or more extreme (in the direction of the alternative), than the observed value of the test statistic, assuming that H_0 is true.

p-value

Decision rule using the p-value:

- If p-value $< \alpha$, then reject H_0.
- If p-value $> \alpha$, then do not reject H_0.

α is called the **signficance level**. The most common value for $\alpha = 0.05$, but other values are also sometimes used such as $\alpha = 0.01$.

p-value

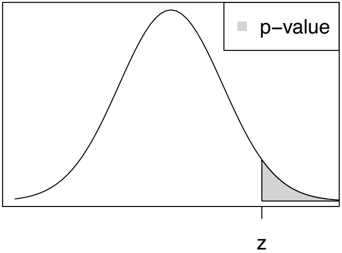
- When the p-value $< \alpha$ (we reject H_0) the result is said to be **statistically significant**.
- In other words, a result is statistically significant when it is unlikely to of occurred by random chance, assuming that the null hypothesis is true.
- The smaller the p-value, the stronger the data favor H_A over H_0.

Computing p-values

One-sided test (upper-tail):
$H_0 : p = p_0$
$H_A : p > p_0$

Test statistic:

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$



p-value $=$ `1 - pnorm(z)`

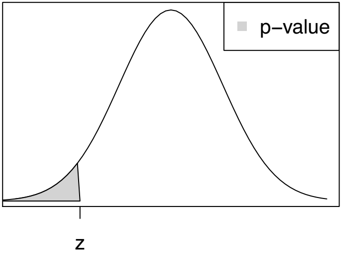
Reject H_0 if p-value $< \alpha$

Computing p-values

One-sided test (lower-tail):
$H_0 : p = p_0$
$H_A : p < p_0$

Test statistic:

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$



p-value $=$ `pnorm(z)`

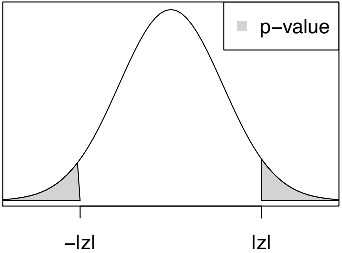
Reject H_0 if p-value $< \alpha$

Computing p-values

Two-sided test:
$H_0 : p = p_0$
$H_A : p \neq p_0$

Test statistic:

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$



p-value $=$ `2*pnorm(-abs(z))`

Reject H_0 if p-value $< \alpha$

Conditions

A hypothesis test for a proportion is valid if the following conditions are satisfied:

- The data were collected using simple random sampling
- $np_0 \geq 10$ and $n(1 - p_0) \geq 10$

These conditions ensure that the Central Limit Theorem holds. So, assuming that H_0 is true, the sampling distribution for $\hat{p}$ is approximately normal with mean p_0 and standard error $\sqrt{p_0(1 - p_0)/n}$.

Example

A simple random sample of 1,028 US adults in March 2013 found that 56% support nuclear arms reduction. Does this provide convincing evidence that a majority of Americans support nuclear arms reduction?

(a) Write the null and alternative hypothesis for a one-sided test.
$H_0 : p = 0.5$
$H_A : p > 0.5$

(b) Check the conditions for the hypothesis test.

- Data come from a random sample.
- $np_0 = 1028(0.5) = 514 \geq 10$
  $n(1 - p_0) = 1028(1 - 0.5) = 514 \geq 10$

Yes, the condition are satisfied.

(c) Calculate the test statistic.

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.56 - 0.5}{\sqrt{\frac{0.5(1-0.5)}{1028}}} = \frac{0.06}{\sqrt{\frac{0.25}{1028}}} = 3.85$$

R command: `0.06 / sqrt(0.25 / 1028)`

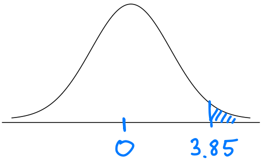
Example

scientific notation in R:

$5.9e{-}05 = 5.9 \times 10^{-5} = 0.000059$

A simple random sample of 1,028 US adults in March 2013 found that 56% support nuclear arms reduction. Does this provide convincing evidence that a majority of Americans support nuclear arms reduction?

(d) Calculate the p-value and make a decision using $\alpha = 0.05$ significance level.



$$\begin{aligned} \text{p-value} &= P(Z > 3.85) \\ &= 1 - P(Z < 3.85) \\ &= 1 - \text{pnorm}(3.85) \\ &= \boxed{0.000059} \end{aligned}$$

Since p-value < 0.05, we reject H_0

(e) What is the conclusion of the test in the context of the data?

The data provide strong evidence that a majority of Americans support nuclear arms reduction.

Decision Errors

		Decision	
		Do not reject H_0	Reject H_0
Truth	H_0 true	✓	type 1 error
	H_A true	type 2 error	✓

The significance level of the test, α, is the probability of a type I error (probability of rejecting H_0 when H_0 is true).

Decision Errors

In a US court, the defendant is either innocent (H_0) or guilty (H_A). What does a type I error represent in this context? What does a type II error represent?

H_0 : The defendant is innocent.
H_A : The defendant is guilty.

Type 1 error: The defendant is actually innocent, but the jury decides guilty.

Type 2 error: The defendant is actually guilty, but the jury decides innocent.