Lecture 7:
Transformations for Simple Linear Regression
STAT 432, Spring 2021

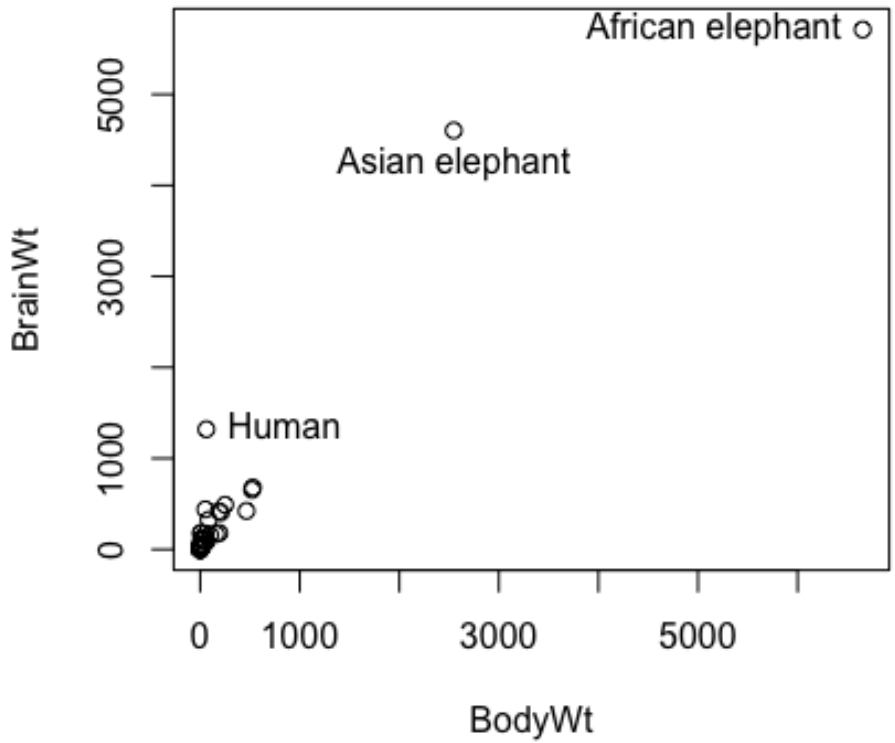
Transformations can be used to

- Linearize the relationship between the explanatory (x) and response (y) variables.
- Overcome problems due to nonconstant variance.

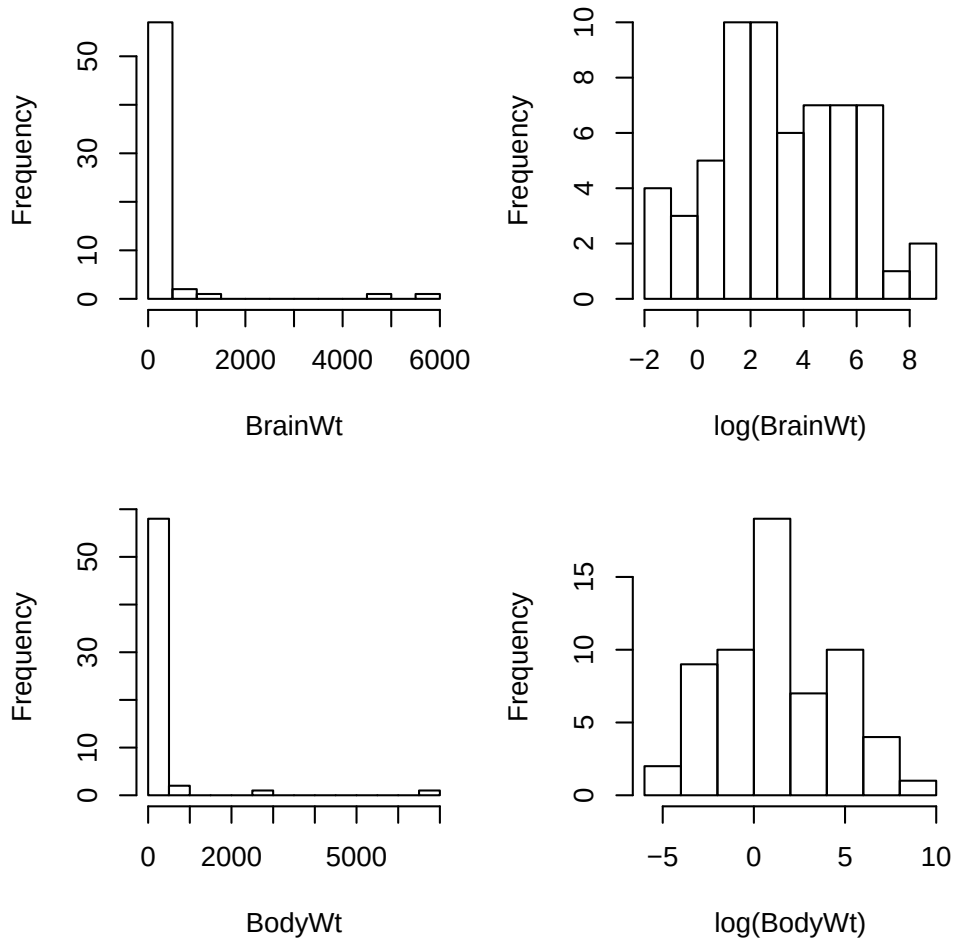
Example: Modeling Brain Weight

- We consider a data set called `brains` from the `alr4` package. The data set is on the the brain weight (in grams) and body weight (in kg) for 62 species of mammals.
- A scatter plot of the data (next slide) shows that the variables are extremely skewed. Three points (humans and two species of elephants) stand out from the rest of the data.

```
> library(alr4)
> plot(BrainWt ~ BodyWt, data=brains)
```

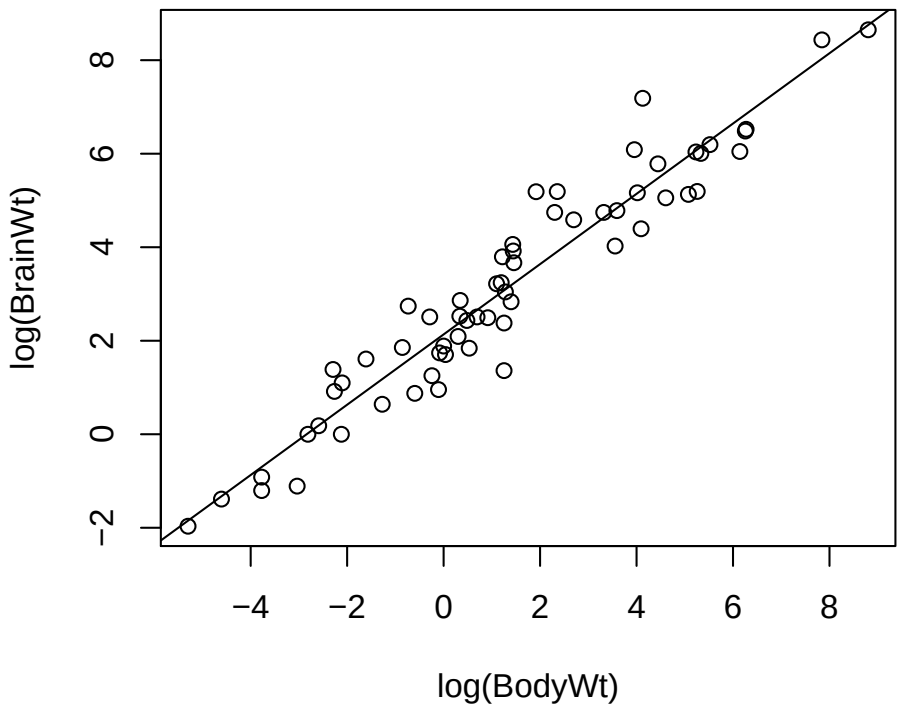


The histograms illustrate that the log transformation can reduce the positive skew in the data.



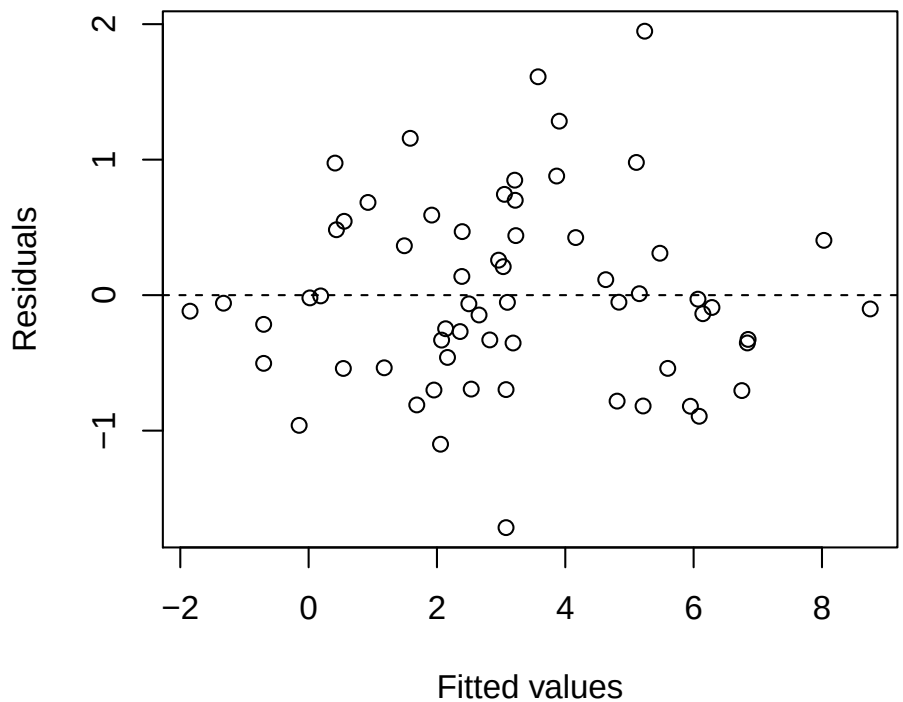
The log transformation linearizes the relationship between the variables.

```
> lm1 <- lm(log(BrainWt) ~ log(BodyWt), data=brains)
> plot(log(BrainWt) ~ log(BodyWt), data=brains)
> abline(lm1)
```



After taking the log transformation, the residual plot shows no discernible patterns (random scatter of points around 0). The conditions of linearity and constant variance appear to be well satisfied.

```
> plot(predict(lm1), resid(lm1), xlab='Fitted values', ylab='Residuals'
> abline(h=0, lty=2)
```



```
> summary(lm1)
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  2.13479    0.09604   22.23   <2e-16 ***
log(BodyWt)  0.75169    0.02846   26.41   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.6943 on 60 degrees of freedom
Multiple R-squared:  0.9208,Adjusted R-squared:  0.9195
F-statistic: 697.4 on 1 and 60 DF,  p-value: < 2.2e-16
```

The R output gives the following regression equation:

$$\begin{aligned}\widehat{\log(\texttt{BrainWt})} &= \hat{\beta}_0 + \hat{\beta}_1 \log(\texttt{BodyWt}) \\ &= 2.135 + 0.7517 \log(\texttt{BodyWt})\end{aligned}$$

- The interpretation of the estimated slope is that a unit increase in `log(BodyWt)` is associated with an increase in `log(BrainWt)` by 0.7517 (not that useful).
- Another common interpretation is in terms of percentage effects: A 1% increase in body weight is associated with an approximate 0.75% increase in brain weight.

Review of logs

- Logs are exponents: $\log_b(x) = y$ (read "the log of x to the base b is y") means that $b^y = x$. Some examples:

$$\log_{10} 100 = 2 \iff 10^2 = 100$$
$$\log_{10} 0.01 = -2 \iff 10^{-2} = 0.01$$
$$\log_2 8 = 3 \iff 2^3 = 8$$

- Some useful identities:

$$e^{\log(x)} = x$$
$$\log(x^r) = r \log(x)$$
$$\log(xy) = \log(x) + \log(y)$$
$$\log(x/y) = \log(x) - \log(y)$$

Note: $\log(x)$ denotes the log base e here (this is called the natural logarithm, which is also commonly denoted by $\ln(x)$)

Question

Regression equation for log(`BrainWt`):

$$\widehat{\log(\texttt{BrainWt})} = 2.135 + 0.7517 \log(\texttt{BodyWt})$$

What is the prediction for log(`BrainWt`) when `BodyWt` is 40 kg?

What is the prediction for `BrainWt` when when `BodyWt` is 40 kg?

```
# use R to make prediction for log(BrainWt) when BodyWt is 40
> new_x <- data.frame(BodyWt = 40)
> pred <- predict(lm1, newdata = new_x)
> pred
       1
4.907668

# exponentiate to make prediction for BrainWt
> exp(pred)
       1
135.3235
```

We can back-transform to write the fitted model in the original scale of the response.

Summary: Log Transformation

A log transformation might be useful if

- the distribution of the response or predictor variable is skewed right.
- the values of the variable range over more than one order of magnitude.
- there is a fan pattern in the residuals (nonconstant variance).

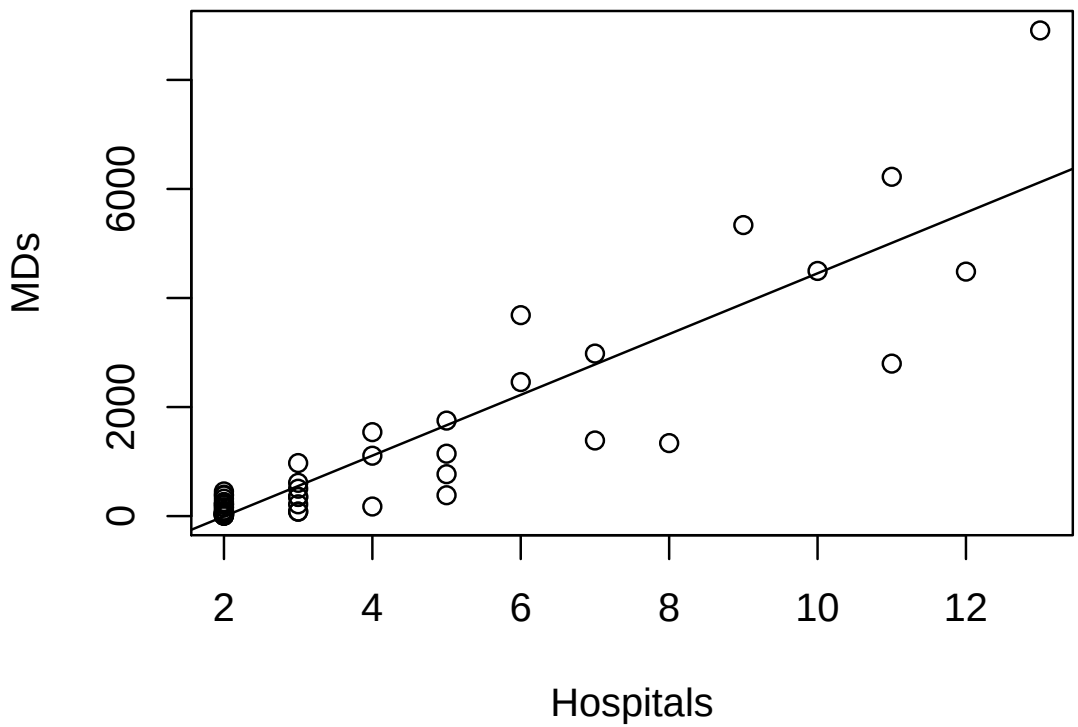
Remark: The transformation $log(x_i)$ is only valid for $x_i > 0$. For nonpositive data, one workaround is to use the transformation $log(x_i + c)$, where c is a constant such that $x_i + c > 0$, for $i = 1, \cdots, n$.

Example: Doctors and Hospitals

Data set containing counts on the number of medical doctors and number of hospitals in a random sample of 53 counties.

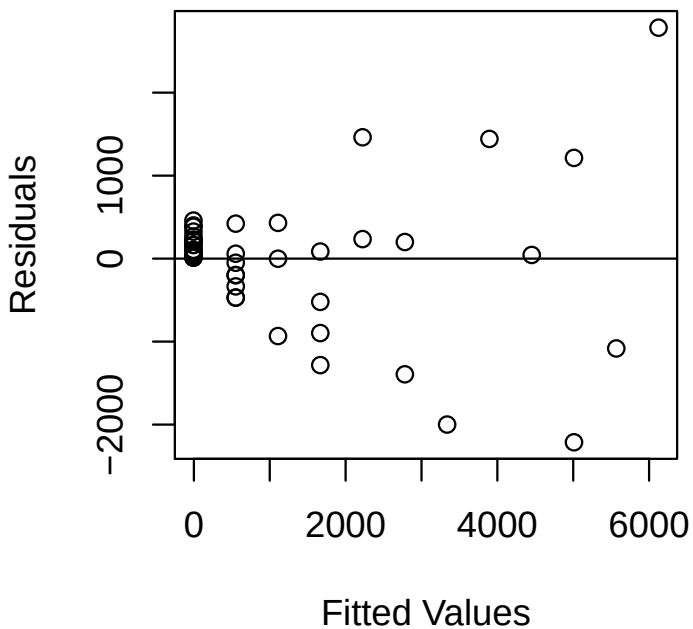
```
> library(Stat2Data)
> data("CountyHealth")
> head(CountyHealth)
County               MDs Hospitals Beds
1 Bay, FL            351         3  605
2 Beaufort, NC        95         2  134
3 Beaver, PA         260         2  567
4 Bernalillo, NM    2797        11 1435
5 Bibb, GA           769         5  976
6 Clinton, PA         42         2  245
```

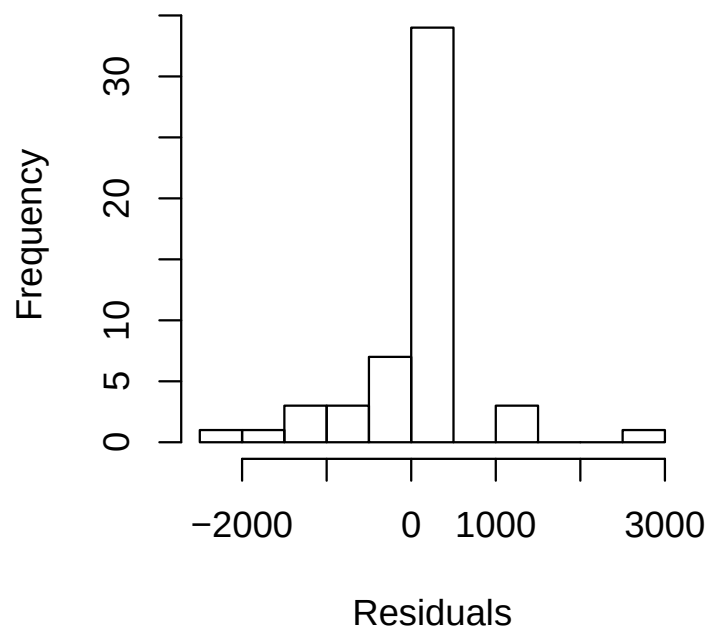
```
> lm1 <- lm(MDs ~ Hospitals, data = CountyHealth)
> plot(MDs ~ Hospitals, data = CountyHealth)
> abline(lm1)
```



There residual plot shows nonconstant variance. That is, the variability in the residuals tends to increase with the fitted values (fan pattern). The histogram also indicates that the residuals are not normally distributed.

```
> plot(predict(lm1), resid(lm1),
    xlab = "Fitted Values", ylab = "Residuals")
> abline(h=0)
> hist(resid(lm1), xlab = "Residuals", main = "", breaks=10)
```

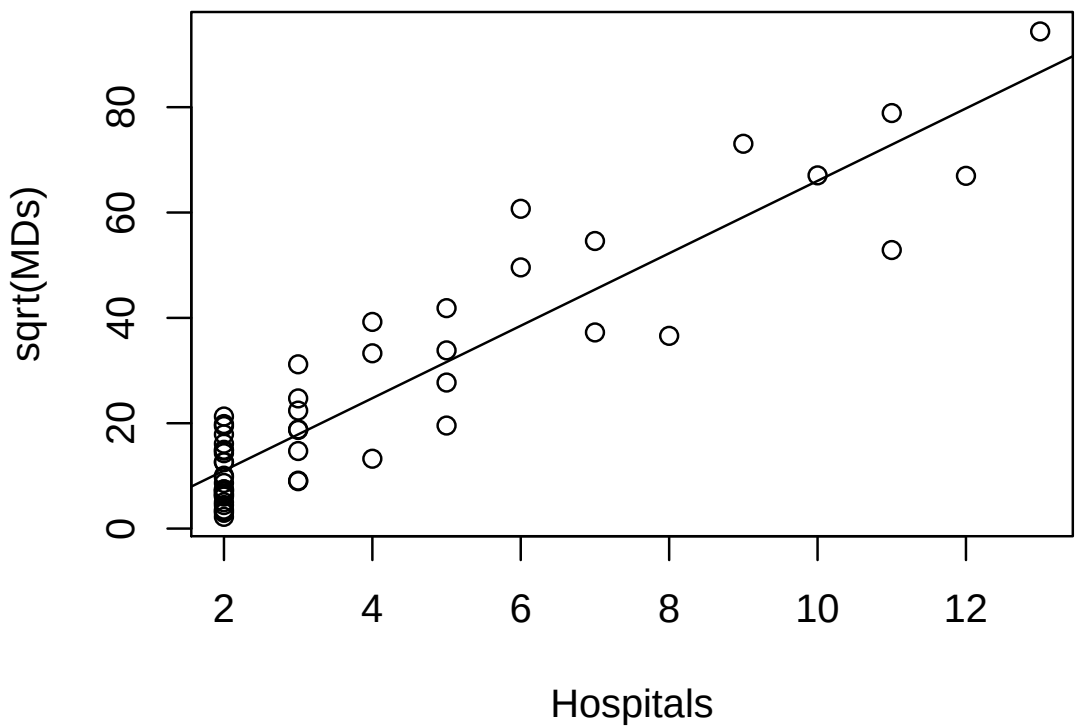




Using Transformations to Stabilize Variance

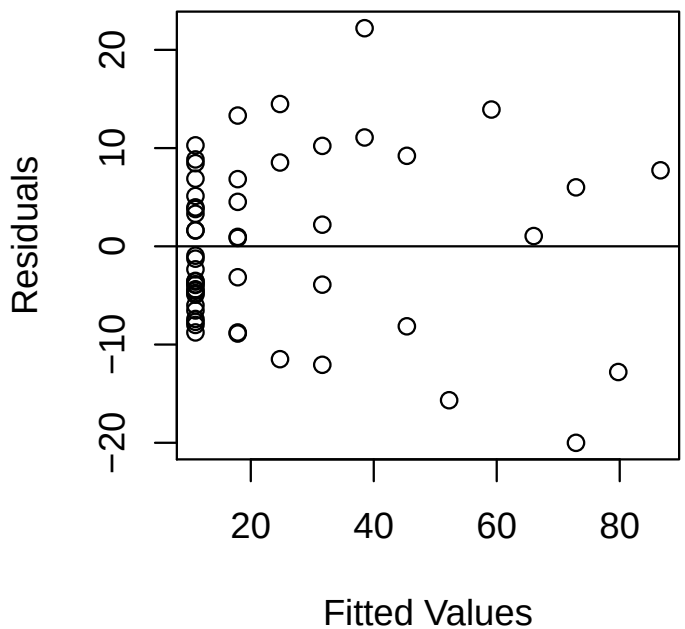
- Problems with nonconstant variance can be overcome with transformations.
- Two common variance stabilizing transformations are the log transformation, $\log(y)$, and the square root transformation, $\sqrt{y}$.
- The square root transformation is often appropriate for count data.
- Since the data in this example are in the form of counts, we will try a square root transformation of the response.

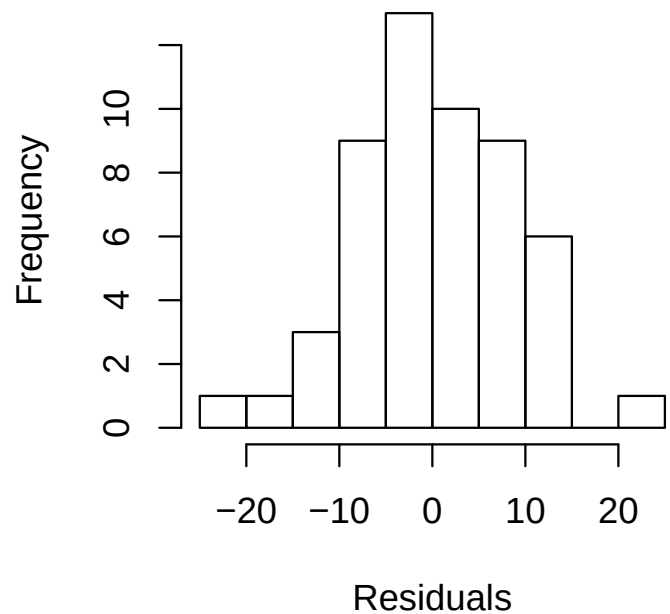
```
lm2 <- lm(sqrt(MDs) ~ Hospitals, data = CountyHealth)
plot(sqrt(MDs) ~ Hospitals, data = CountyHealth)
abline(lm2)
```



Afre taking the square root transformation, the residual plot and histogram show considerable improvement. The conditions of constant variability and normality in the residuals appear reasonably satisfied.

```
> plot(predict(lm2), resid(lm2),
    xlab = "Fitted Values", ylab = "Residuals")
> abline(h=0)
> hist(resid(lm2), xlab = "Residuals", main = "", breaks=10)
```





```
> summary(lm2)
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  -2.7533     1.9850  -1.387    0.171
Hospitals     6.8764     0.4011  17.144   <2e-16 ***
```

Regression equation for the transformed model:

$$\widehat{\sqrt{\texttt{MDs}}} = -2.7533 + 6.8764\,\texttt{Hospitals}$$

We can back-transform to write the model in the original scale of the response:

$$\widehat{\texttt{MDs}} = (-2.7533 + 6.8764\,\texttt{Hospitals})^2$$

Summary

- It can take some trial and error to find a good transformation. Looking at scatterplots of the data and residuals can help determine which transformation best linearizes the relationship or stabilizes the variance.
- The log transform is commonly applied to skewed data that ranges over several orders of magnitude.
- The square root transform is commonly applied to count data to stabilize the variance.
- Transformations can be applied to the response variable, explanatory variable, or both.