Lecture 15:
Inference for Simple Linear Regression
STAT 630, Fall 2021

Inference for SLR

Simple linear regression model for the population:

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

β_0 and β_1 are the population parameters (fixed and non-random)

Least squares line (estimated from the sample):

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

$\hat{\beta}_0$ and $\hat{\beta}_1$ are the estimates (random, varies from sample to sample)

Point Estimate (random)	Population Parameter (fixed, unknown)
$\bar{X}$	μ
$\hat{p}$	p
$\hat{\beta}_0, \hat{\beta}_1$	β_0, β_1

Inference for SLR

Test whether the slope β_1 is zero (i.e., whether there is a linear association between x and y).

$H_0 : \beta_1 = 0$
$H_A : \beta_1 \neq 0$

Test statistic:

$$t = \frac{\hat{\beta}_1}{se(\hat{\beta}_1)}; \quad \text{df=n-2}$$

$1 - \alpha$ confidence interval for the slope β_1:

$$\hat{\beta}_1 \pm t_{\alpha/2;n-2} se(\hat{\beta}_1)$$

Inference for SLR

Test whether the intercept β_0 is zero.

$H_0 : \beta_0 = 0$
$H_A : \beta_0 \neq 0$

Test statistic:

$$t = \frac{\hat{\beta}_0}{se(\hat{\beta}_0)}; \quad \text{df=n-2}$$

$1 - \alpha$ confidence interval for the intercept β_0:

$$\hat{\beta}_0 \pm t_{\alpha/2;n-2} se(\hat{\beta}_0)$$

Formulas for standard error computations (can use software for these computations):

- Residual standard error:

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n} \hat{e}_i^2}{n-2}} = \sqrt{\frac{\sum_{i=1}^{n} (y_i - \hat{y}_i)^2}{n-2}}$$

- Standard error of $\hat{\beta}_1$:

$$se(\hat{\beta}_1) = \frac{\hat{\sigma}}{\sqrt{\sum_{i=1}^{n} (x_i - \bar{x})^2}}$$

- Standard error of $\hat{\beta}_0$:

$$se(\hat{\beta}_0) = \hat{\sigma} \sqrt{\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}}$$

Example

Going back to the example of fitting a simple linear regression model for highway mileage, using weight as a predictor.

```
> library(MASS)
> lm1 <- lm(MPG.highway ~ Weight, data=Cars93)
> summary(lm1)

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) 51.6013654  1.7355498   29.73   <2e-16 ***
Weight      -0.0073271  0.0005548  -13.21   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Example

(a) Do the data provide strong evidence of an association between weight and highway MPG? State the null and alternative hypotheses, report the test statistic and p-value (from the `summary()` command), and state your conclusion.

$H_0 : \beta_1 = 0$
$H_A : \beta_1 \neq 0$

Test statistic: $t = -13.21$

Since the p-value $< 2\text{e}{-16} \approx 0$, we reject H_0. The data provide strong evidence of a linear association between car weight and highway MPG.

Example

(b) Calculate a 95% confidence interval for the slope β_1. Note that there are $n = 93$ car models (rows) in the data set.

```
qt(0.975, df = 91) = 1.986
```

$$
\begin{aligned}
\hat{\beta}_1 \pm t_{\alpha/2;n-2} se(\hat{\beta}_1) &\implies -0.0073 \pm 1.986(0.00055) \\
&\implies (-0.0084, -0.0064)
\end{aligned}
$$

Example

(c) For the intercept term, the output from `summary()` gives a test statistic $t = 29.73$ and p-value `< 2e-16`. What are the null and alternative hypotheses that are being tested? What is the conclusion of the test?

$H_0 : \beta_0 = 0$
$H_A : \beta_0 \neq 0$

Since the p-value ≈ 0, we reject H_0. The intercept β_0 is significantly different than 0.

Conditions for SLR

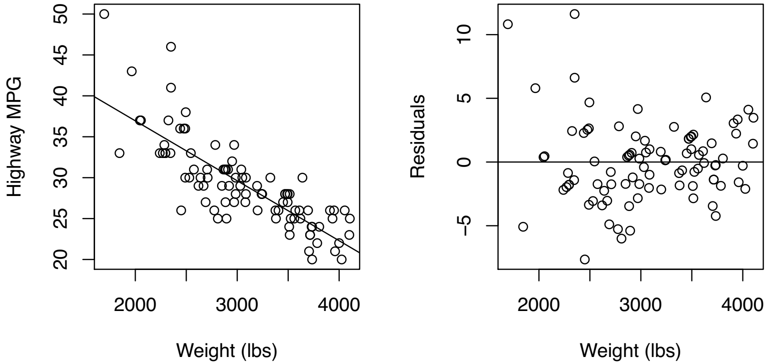
- **Linearity**. The data should follow a linear trend.
- **Constant variability**. The variability of points around the least squares line remains roughly constant.
- **Normality**. The residuals should be approximately normally distributed with mean 0.
- **Independence**. Values of the response variable are independent of each other.

Residual Plots

- One useful way to check the conditions is to look at a plot of the residuals $\hat{e}_i = y_i - \hat{y}_i$ versus x_i for $i = 1, \cdots, n$. It is also common to plot the residuals $\hat{e}_i$ versus the fitted values $\hat{y}_i$ (for simple linear regression the plots look the same).
- One purpose of residual plots is to identify characteristics or patterns still apparent in the data after fitting the model.
- Residual plots are especially useful for checking linearity and constant variability.
- Ideally, the residual plot should show no obvious pattern, and the points are randomly scattered around 0.

Example: Residual Plot

For the simple linear regression model between car weight and highway mileage, the points in the residual plot look randomly scattered and show no obvious patterns, except for slight nonconstant variability, indicating that the conditions are reasonably satisfied.



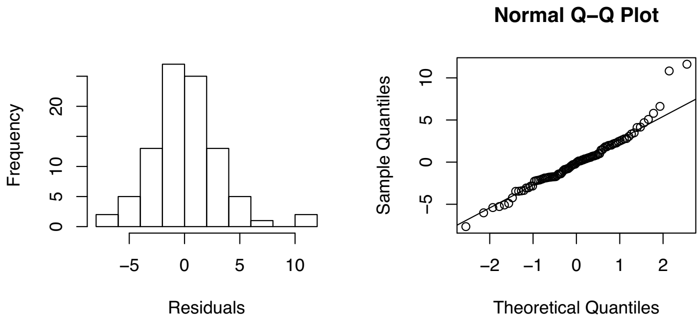
Here is the code used to create the last plot.

```
library(MASS)

# fit linear model
lm1 <- lm(MPG.highway ~ Weight, data=Cars93)

# scatter plot with least squares line
> plot(Cars93$Weight, Cars93$MPG.highway,
       xlab = "Weight (lbs)", ylab = "Highway MPG")
> abline(lm1)

# residual plot
> plot(Cars93$Weight, resid(lm1),
       xlab = "Weight (lbs)", ylab = "Residuals")
> abline(h=0)
```

Example: Normality of Residuals

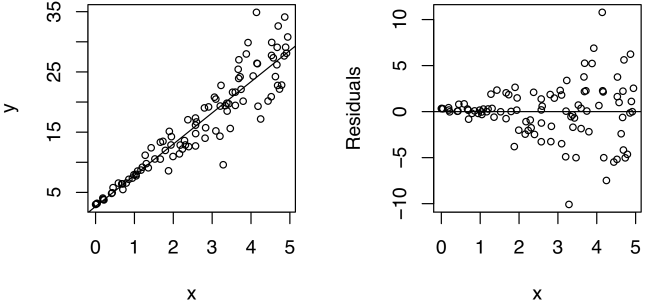
To check whether the residuals follow a normal distribution, make a histogram and QQ plot. For the example below, the residuals look normally distributed, except for two potential outliers.

```
> par(mfrow=c(1,2))
> hist(resid(lm1), xlab = "Residuals", main= "")
> qqnorm(resid(lm1))
> qqline(resid(lm1))
```



Example: Nonconstant Variability

An example of a violation of the constant variability condition. The residual plot shows a fan pattern. This is sometimes called **heteroscedasticity**.



Example: Nonlinearity

An example of a violation of the linearity condition.