Lecture 14:
Simple Linear Regression
STAT 630, Fall 2021

Scatterplots

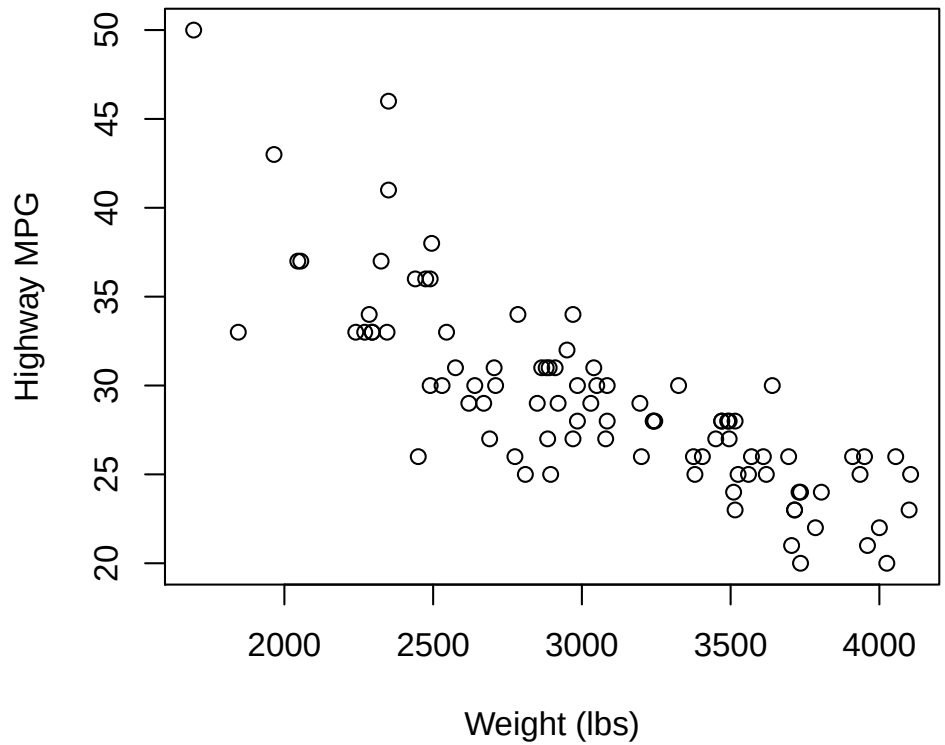
- A scatterplot a graphical display used to study the relationship between two variables x and y.
- Data displayed on a scatterplot are collected in pairs:

$$(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$$

  where n denotes the total number of cases or pairs.
- A scatterplot provides insight into how two variables are related.

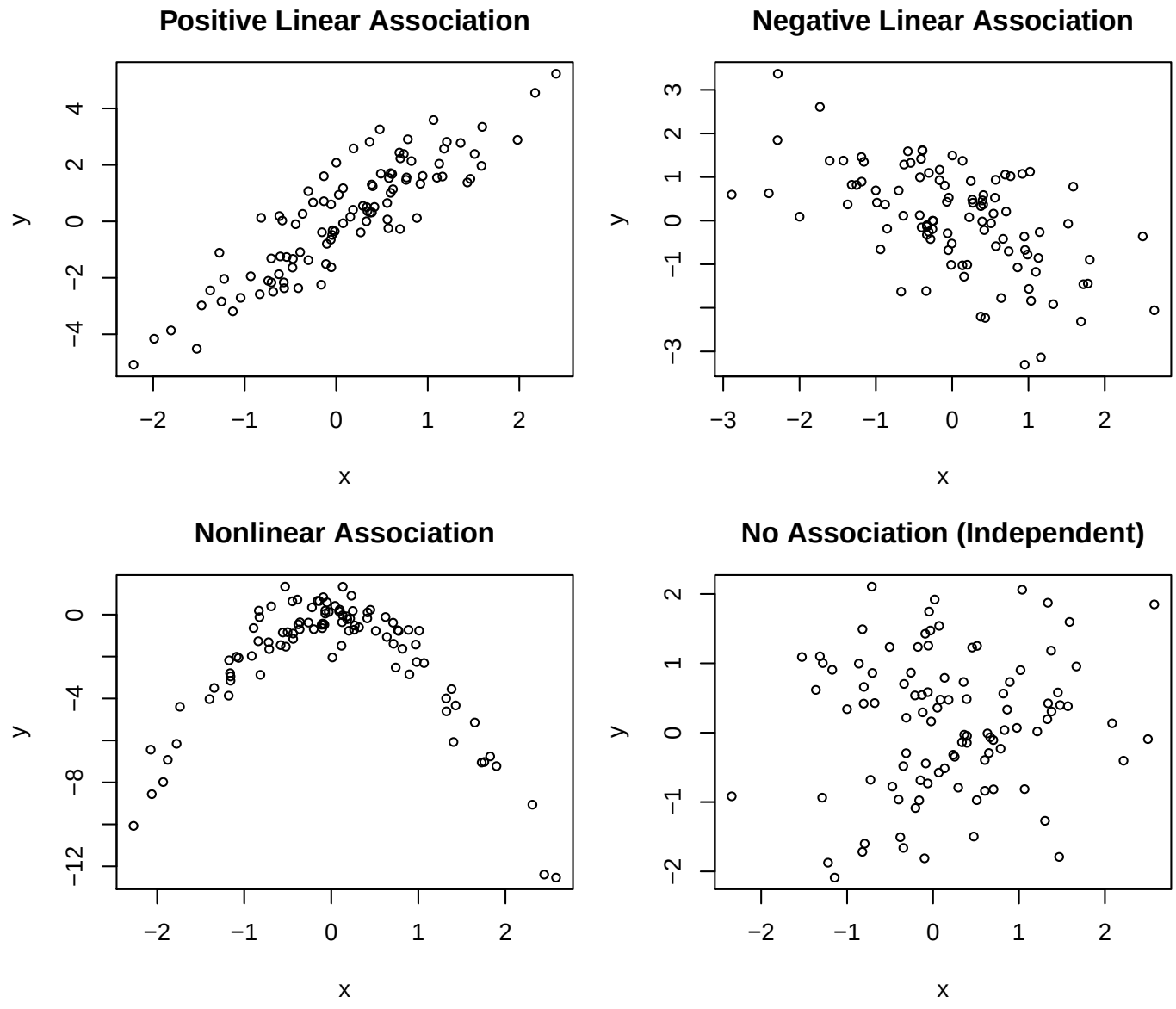
Example

```
> library(MASS)
> plot(Cars93$Weight, Cars93$MPG.highway,
    xlab = "Weight (lbs)", ylab = "Highway MPG")
```



Types of Relationships between Variables

- Two variables are said to be **associated** if the scatterplot shows a discernible pattern or trend.
- An association is **positive** if y increases as x increases.
- An association is **negative** if y decreases as x increases.
- An association is **linear** if the scatterplot between x and y has a linear trend; otherwise, the association is called **nonlinear**.


Positive Linear Association
Negative Linear Association
Nonlinear Association
No Association (Independent)
x
y

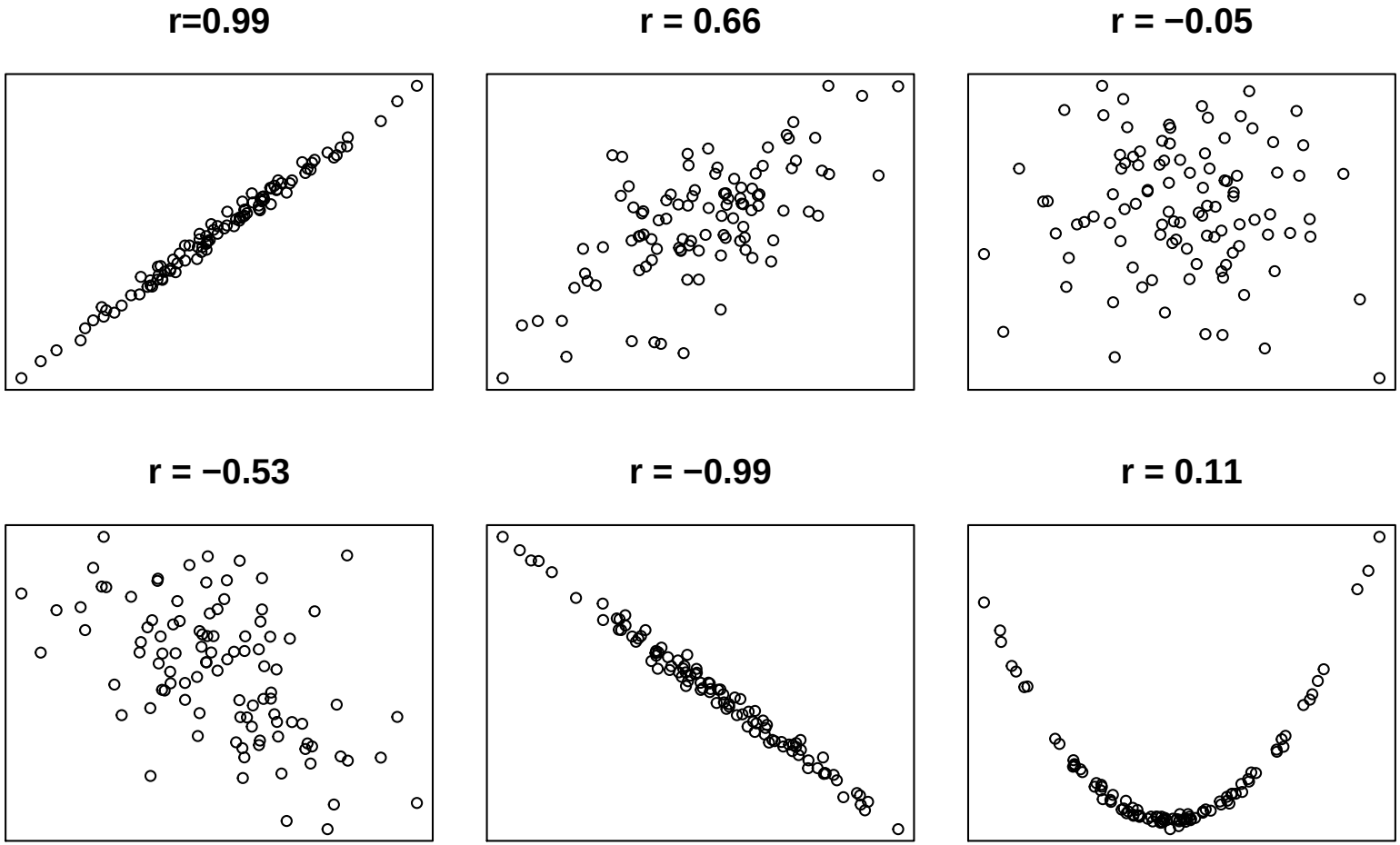
Correlation Coefficient

The **correlation coefficient**, denoted by r, is a number between -1 and 1 that describes the strength of the linear association between two numerical variables.

$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}} = \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right)$$

- $\bar{x}$ and $\bar{y}$ are the sample means
- s_x and s_y are the sample standard deviations

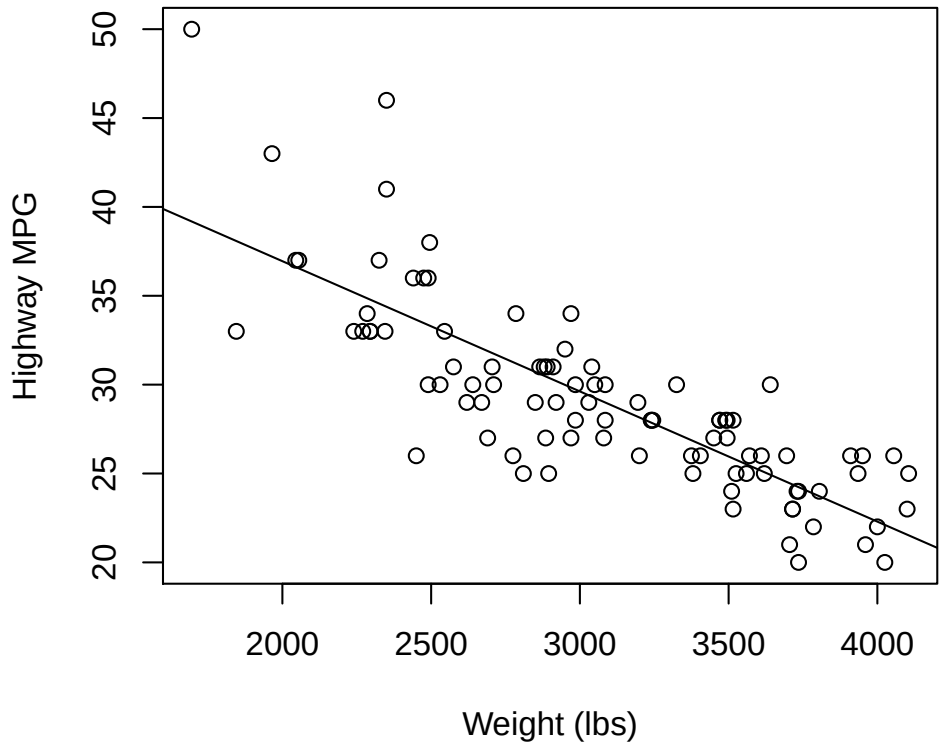
Correlation Coefficient



Correlation Coefficient

- $r \approx 1$ when there is a strong positive linear association between the variables.
- $r \approx -1$ when there is a strong negative linear association between the variables.
- $r \approx 0$ when there is no relationship between the variables (i.e., independent).
- The correlation coefficient is only useful for evaluating the linear association between two variables. It is not a useful measure for nonlinear relationships.

Simple linear regression is a method for fitting a straight line to data that show a linear trend when displayed on a scatterplot. It is a useful tool for making predictions for a quantitative response variable.



Simple Linear Regression Model

Let $\{(x_i, y_i) : i = 1, \cdots, n\}$ be a collection of n data points. A **simple linear regression model** expressing the relationship between y_i and x_i is given by:

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

- y_i response variable (random)
- x_i explanatory variable (non-random)
- β_0 intercept parameter (non-random)
- β_1 slope parameter (non-random)
- ϵ_i is the random error term, $\epsilon_i \sim N(0, \sigma)$

Remark: y_i is also sometimes called the **dependent** variable, and x_i the **predictor** variable. Notation and terminology may vary depending on the textbook and context.

Fitted Values and Residuals

The line that we we estimate, or fit to the data in the scatterplot, is written as

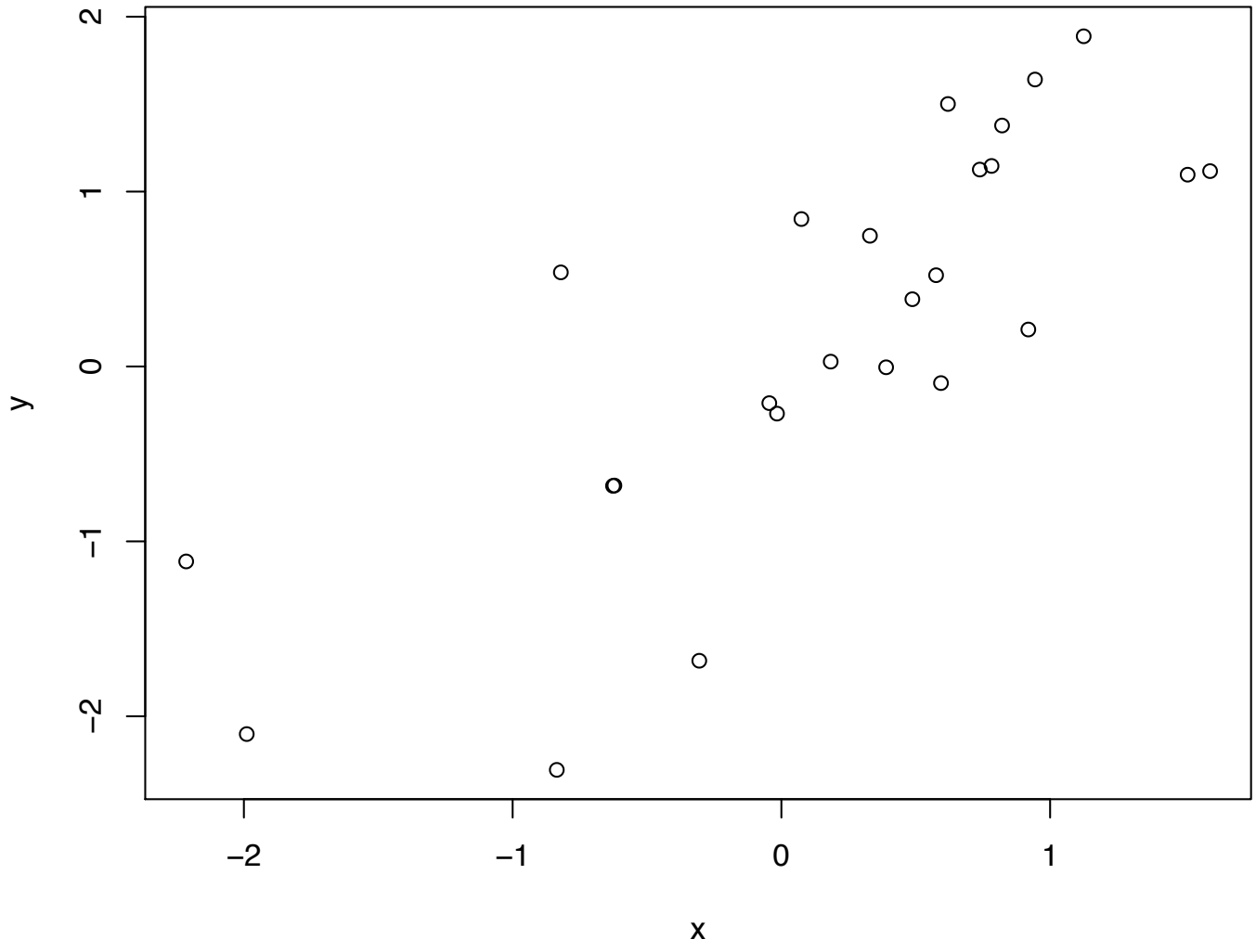
$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$

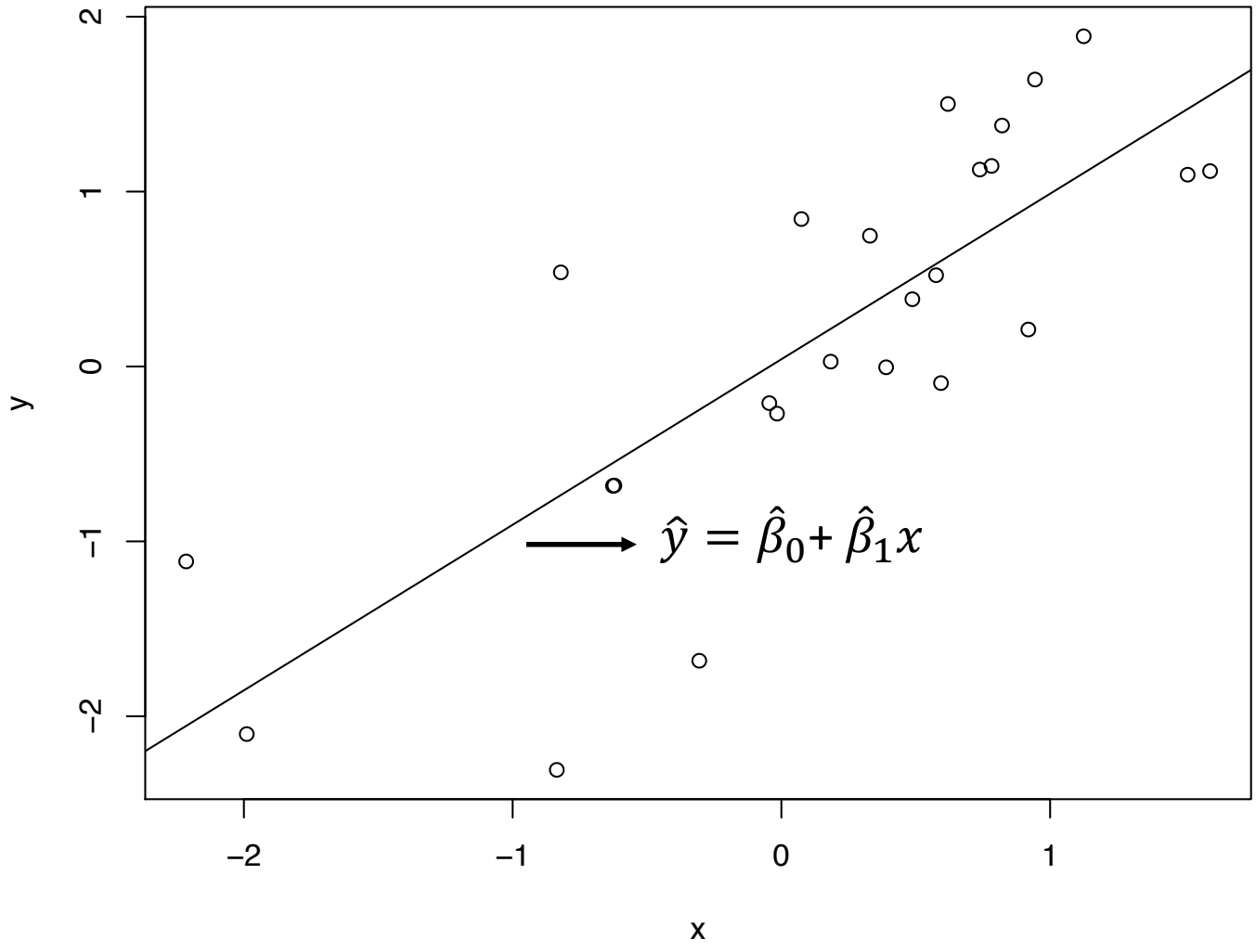
The fitted (or predicted) value for the i^{th} observation (x_i, y_i):

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

The **residual** for the i^{th} observation is the difference between the observed value (y_i) and the predicted value $(\hat{y}_i)$:

$$\hat{e}_i = y_i - \hat{y}_i = y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i)$$


y
x
2
1
0
−1
−2
−2
−1
0
1


2
1
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y
−2
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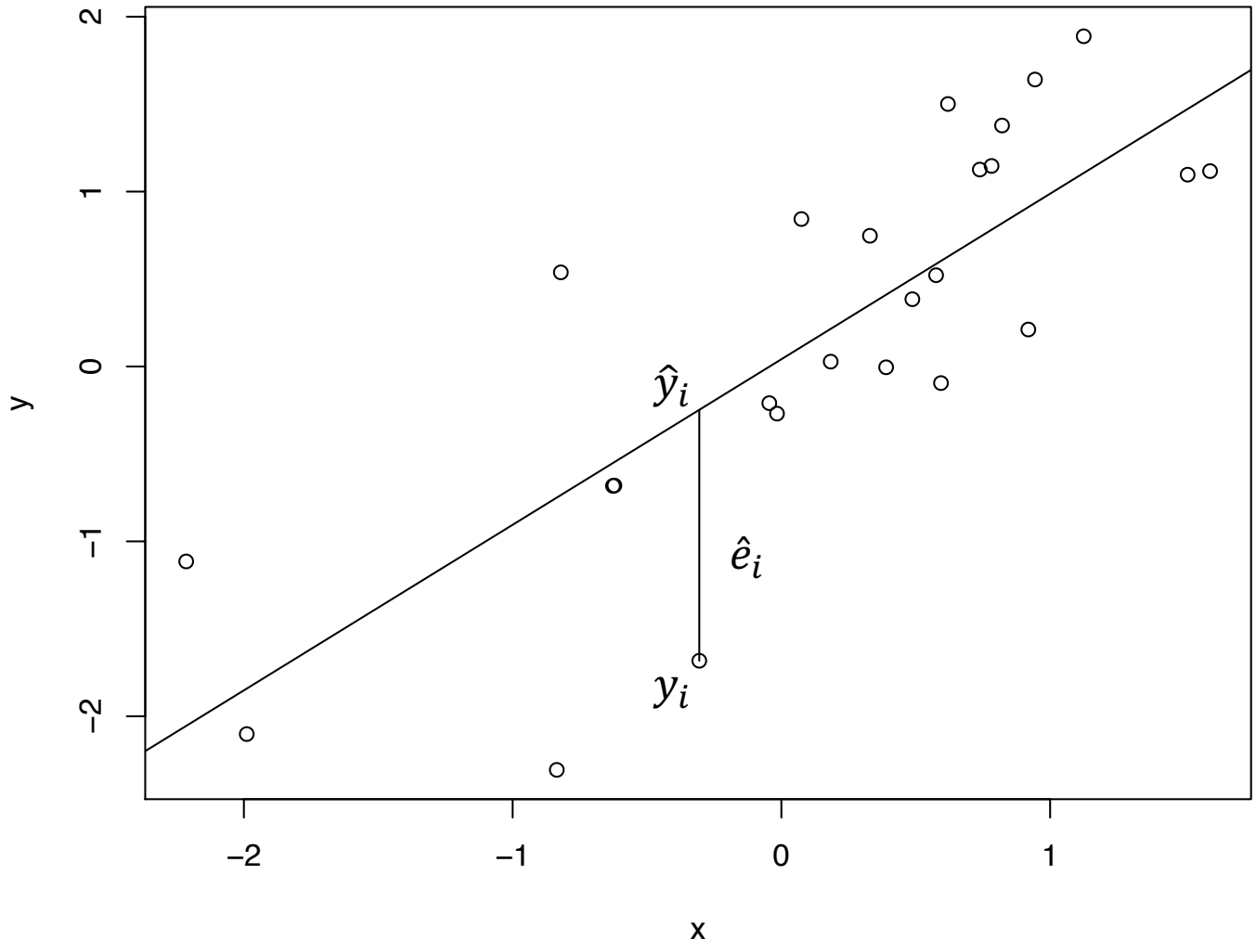
1
x
$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$


2
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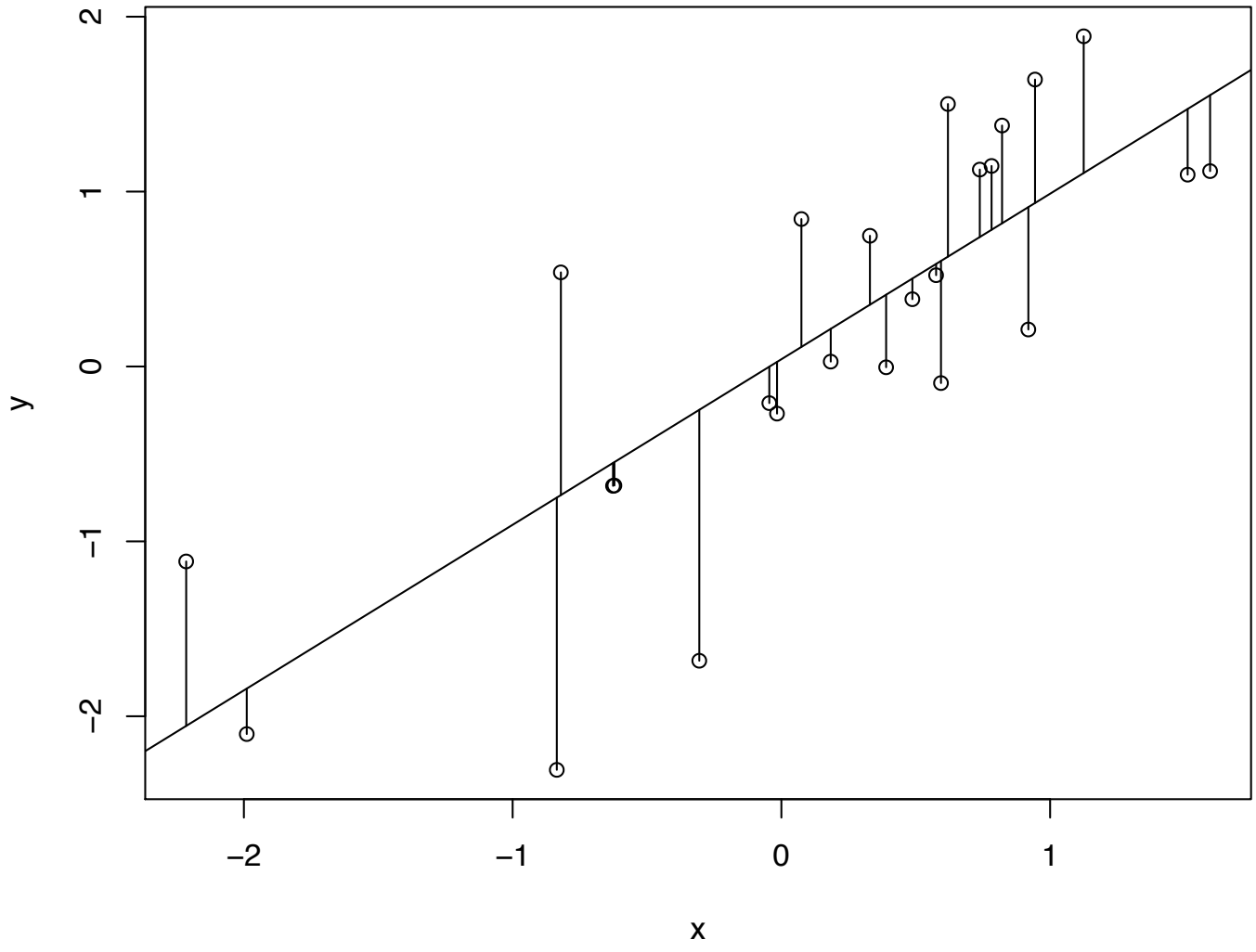
y
−2
−1
0
1
x
$\hat{y}_i$
$\hat{e}_i$
y_i


y
2
1
0
−1
−2
−2
−1
0
1
x

Sum of Squared Residuals

- Intuitively, a line that fits the data well has small residuals.
- The **least squares line** minimizes the **sum of squared residuals**:

$$RSS = \sum_{i=1}^{n} \hat{e}_i^2 = \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$$

- That is, out of all possible lines we could draw on the scatterplot, the least squares line is the "best fit" since it has the smallest sum of squared residuals.

Least Squares Estimation

Formally, the estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ of the intercept and slope are found by using calculus to minimize the sum of squared residuals:

$$RSS = \sum_{i=1}^{n} \hat{e}_i^2 = \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$$

To minimize set the partial derivatives equal to zero:

$$\frac{\partial RSS}{\partial \hat{\beta}_0} = -2 \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$$

$$\frac{\partial RSS}{\partial \hat{\beta}_1} = -2 \sum_{i=1}^{n} x_i (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$$

Least Squares Estimation

Using some algebraic manipulation we can solve these two equations to obtain the least squares estimates of the intercept and slope:

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = r\frac{s_y}{s_x}$$

Note that the equation for the intercept guarantees the least squares line passes through $(\bar{x}, \bar{y})$.

Interpretation

- **Slope**: an increase in the explanatory variable (x) by one unit is associated with a change of $\hat{\beta}_1$ in the predicted response ($\hat{y}$).
- **Intercept**: the prediction for the response variable ($\hat{y}$) when the value for the explanatory variable is zero ($x = 0$). It may not make sense to try to interpret the intercept depending on the application.

Coefficient of Determination

The **coefficient of determination** (R^2) is a measure of how well the linear regression model fits the data.

$$R^2 = \frac{TSS - RSS}{TSS} = 1 - \frac{RSS}{TSS}$$

- $TSS = \sum_{i=1}^{n}(y_i - \bar{y})^2$ is the total sum of squares (total variability in the response variable)
- $RSS = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$ is the residual sum of squares (unexplained variability)

Coefficient of Determination

- R^2 can be interpreted as the proportion of variability in the response variable y that is explained by x.
- $0 \leq R^2 \leq 1$; the closer R^2 is to 1, the better the linear regression model fits the data.
- R^2 can be computed as the correlation coefficient r squared.
- R^2 is arguably one of the most commonly misused statistics. Always look at a scatterplot of your data first, and check whether fitting a line makes sense and for any outliers.