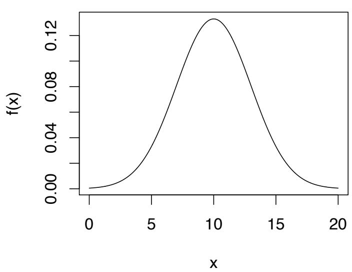
Lecture 4: Normal Distribution
STAT 630, Fall 2021

- The normal distribution is one of the most common and important probability distributions.
- It is symmetric, unimodal, and bell-curve shaped.
- Many phenomena in nature follow a normal distribution such as the height of individuals, the velocity in any direction of a molecule of gas, and measurement error.
- The normal distribution is characterized by two parameters: the mean μ (center of distribution) and standard deviation σ (spread of distribution).
- The notation $X \sim N(\mu, \sigma)$ means that the random variable X follows a normal distribution with mean μ and standard deviation σ.
- For example, the plot below shows the distribution $N(\mu = 10, \sigma = 3)$.



- The probability density function (pdf) for the normal distribution is given by

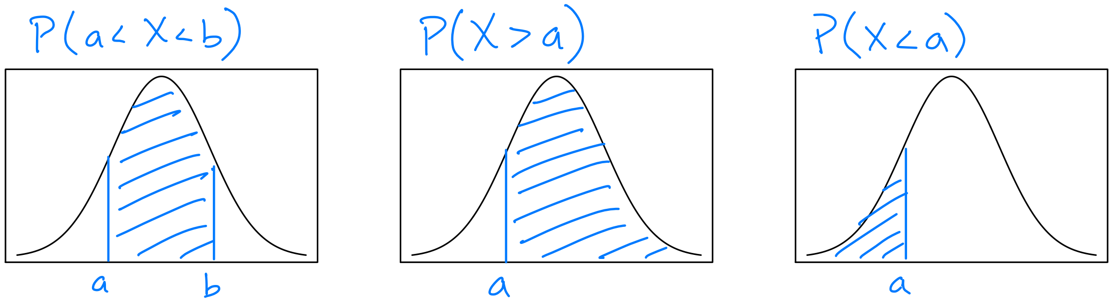
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$$

- Additional properties:
  - The area under the normal distribution curve is 1.

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 1$$

  - The normal distribution is symmetric about the mean, μ.

- Probabilities are computed as the area under the normal distribution curve.



- Note that for a normally distributed random variable $P(X \geq a) = P(X > a)$. Why?

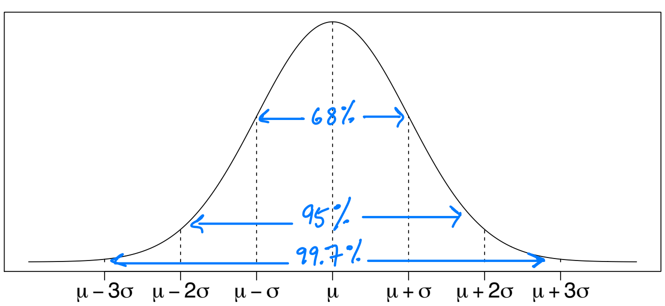
$$P(X = a) = \int_a^a f(x)dx = 0$$

- There is no closed form solution to

$$P(X > a) = \int_a^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx,$$

so numerical approximations of the area under the curve are used to compute probabilities. In practice, we can use the R function `pnorm()` or a standard normal table to compute probabilities.

- Empirical Rule:



  - About 68% of the distribution is contained within 1 standard deviation of the mean.

$$P(\mu - \sigma < X < \mu + \sigma) \approx 0.68$$

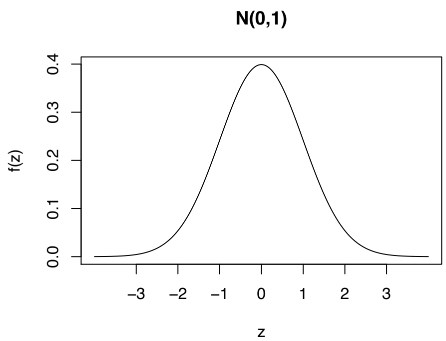
  - About 95% of the distribution is contained within 2 standard deviations of the mean.

$$P(\mu - 2\sigma < X < \mu + 2\sigma) \approx 0.95$$

  - About 99.7% of the distribution is contained within 3 standard deviations of the mean.

$$P(\mu - 3\sigma < X < \mu + 3\sigma) \approx 0.997$$

- Let $X \sim N(\mu, \sigma)$. A Z-score is defined as $Z = (X - \mu)/\sigma$. It can be shown that $Z \sim N(0, 1)$.
- A z-score can be interpreted as the number of standard deviations an observation x lies away from the mean. For instance, if a student has a z-score of 2 on an exam then that student is 2 standard deviations above the average score.
- The distribution $N(0, 1)$ is called the standard normal distribution or Z-distribution.
- For $Z \sim N(0, 1)$ the empirical rule gives that $P(-1 < Z < 1) \approx 0.68$, $P(-2 < Z < 2) \approx 0.95$, and $P(-3 < Z < 3) \approx 0.997$.
- Computing Z-scores allows us to compute probabilities for any normal distribution.



Theorem. Let $X \sim N(\mu, \sigma)$ and $Z = (X - \mu)/\sigma$. Show that $\mathrm{E}(Z) = 0$ and $\mathrm{Var}(Z) = 1$.

Properties of expectation and variance for constants a and b:

$$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$$
$$\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$$

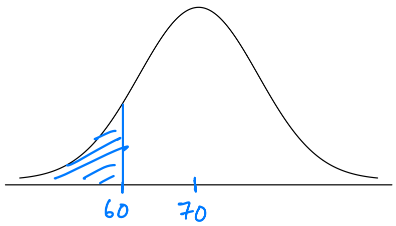
Proof:
For $Z = \frac{X-\mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma}$ we can let $a = \frac{1}{\sigma}$ and $b = -\frac{\mu}{\sigma}$. It is also given that $E(X) = \mu$ and $Var(X) = \sigma^2$. Hence,

$$\mathrm{E}(Z) = \mathrm{E}\left(\frac{1}{\sigma}X - \frac{\mu}{\sigma}\right) = \frac{1}{\sigma}\mathrm{E}(X) - \frac{\mu}{\sigma} = \frac{\mu}{\sigma} - \frac{\mu}{\sigma} = 0$$
$$\mathrm{Var}(Z) = \mathrm{Var}\left(\frac{1}{\sigma}X - \frac{\mu}{\sigma}\right) = \frac{1}{\sigma^2}\mathrm{Var}(X) = \frac{\sigma^2}{\sigma^2} = 1$$

Ex1. The amount X of the pollutant nitrogen oxide in automobiles is normally distributed with mean $\mu = 70$ ppb (parts per billion) and standard deviation $\sigma = 13$ ppb. We can write this compactly as $X \sim N(70, 13)$.

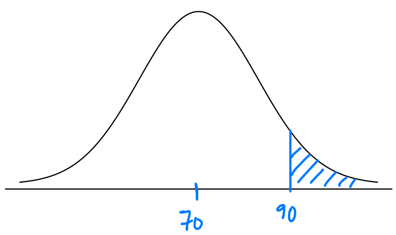
(a) What is the probability that a randomly selected vehicle will have an emission level less than 60 ppb?

$$P(X < 60) = P\left(Z < \frac{60 - 70}{13}\right) = P(Z < -0.769) = \texttt{pnorm(-0.769)} = \boxed{0.2209}$$



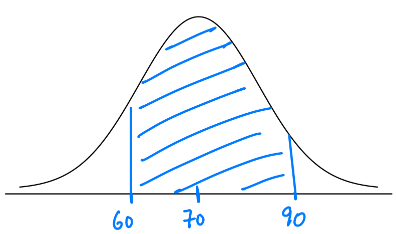
(b) What is the probability that a randomly selected vehicle will have an emission level greater than 90 ppb?

$$P(X > 90) = P\left(Z > \frac{90 - 70}{13}\right) = P(Z > 1.538) = 1 - P(Z < 1.538)$$
$$= \texttt{1 - pnorm(1.538)} = \boxed{0.062}$$



(c) What is the probability that a randomly selected vehicle will have an emission level between 60 and 90 ppb?

$$\begin{aligned} P(60 < X < 90) &= P(X < 90) - P(X < 60) \\ &= P\left(Z < \frac{90 - 70}{13}\right) - P\left(Z < \frac{60 - 70}{13}\right) \\ &= P(Z < 1.538) - P(Z < -0.769) \\ &= \texttt{pnorm(1.538) - pnorm(-0.769)} = \boxed{0.717} \end{aligned}$$



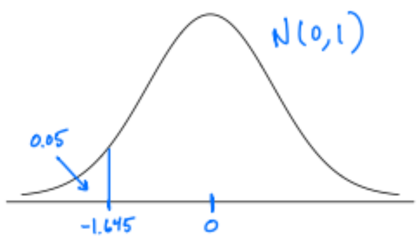
Ex2. Body temperatures are normally distributed with mean $\mu = 98.2$ and standard deviation $\sigma = 0.74$, in degrees Fahrenheit. That is, $X \sim N(98.2, 0.74)$.

(a) Find the cutoff for the lowest 5% of body temperatures (the 5^{th} percentile)?

$X \sim N(98.2, 0.74)$; want to find 5^{th} percentile.

In R, `qnorm(0.05) = -1.645` gives the 0.05 quantile of the standard normal distribution. So, $P(Z < -1.645) = 0.05$. Next, solve for x in the equation for a z-score:

$$z = \frac{x-\mu}{\sigma} \implies -1.645 = \frac{x-98.2}{0.74} \implies x = (-1.645)(0.74) + 98.2 = \boxed{96.983}$$



(b) Find the cutoff for the highest 15% of body temperatures (the 85^{th} percentile)?

$X \sim N(98.2, 0.74)$; want to find 85^{th} percentile.

In R, `qnorm(0.85) = 1.036` gives the 0.85 quantile of the standard normal distribution. So, $P(Z < 1.036) = 0.85$. Next, solve for x in the equation for a z-score:

$$z = \frac{x-\mu}{\sigma} \implies 1.036 = \frac{x-98.2}{0.74} \implies x = (1.036)(0.74) + 98.2 = \boxed{98.967}$$