Lecture 5:
Transformations for Simple Linear Regression
STAT 632, Spring 2020

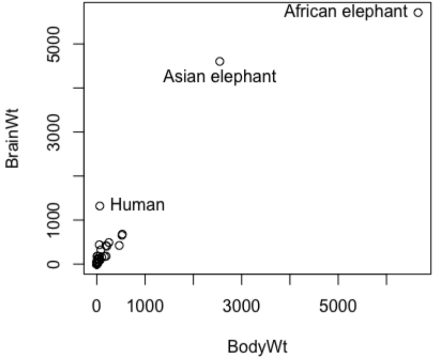
Transformations can be used to

- Linearize the relationship between the explanatory (X) and response (Y) variables
- Overcome problems due to nonconstant variance

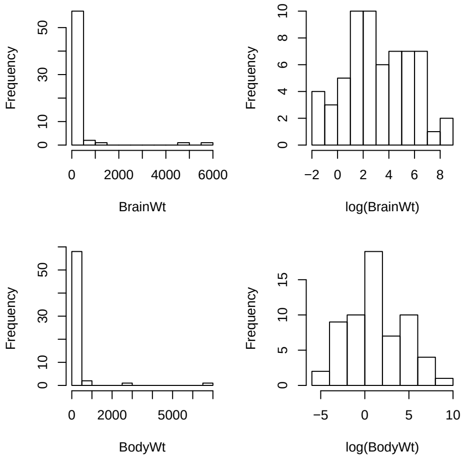
Example: Modeling Brain Weight

- We consider a data set called `brains` from the `alr4` package. The data set is on the the brain weight (in grams) and body weight (in kg) for 62 species of mammals.
- A scatter plot of the data (next slide) shows that the variables are extremely skewed. Three points (humans and two species of elephants) stand out from the rest of the data.

```
> library(alr4)
> plot(BrainWt ~ BodyWt, data=brains)
```

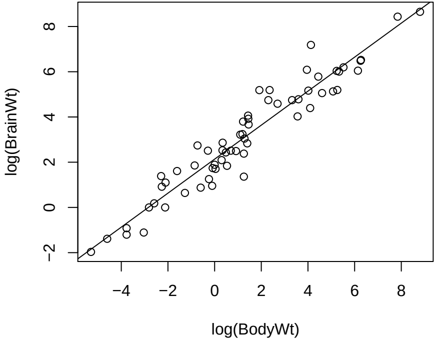


The histograms illustrate that the log transformation can reduce the positive skew in the data.



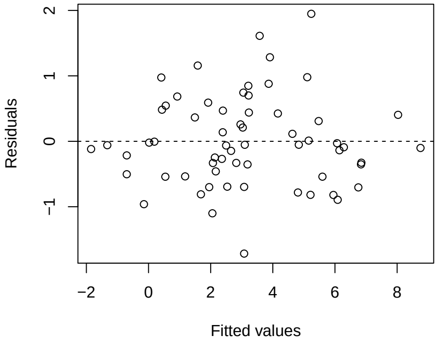
The log transformation linearizes the relationship between the variables.

```
> lm1 <- lm(log(BrainWt) ~ log(BodyWt), data=brains)
> plot(log(BrainWt) ~ log(BodyWt), data=brains)
> abline(lm1)
```



After taking the log transformation, the residual plot shows no discernible patterns (random scatter of points around 0). The assumptions of linearity and constant variance appear to be well satisfied.

```
> plot(predict(lm1), resid(lm1), xlab='Fitted values', ylab='Residuals'
> abline(h=0, lty=2)
```



```
> summary(lm1)
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  2.13479    0.09604   22.23   <2e-16 ***
log(BodyWt)  0.75169    0.02846   26.41   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.6943 on 60 degrees of freedom
Multiple R-squared:  0.9208,Adjusted R-squared:  0.9195
F-statistic: 697.4 on 1 and 60 DF,  p-value: < 2.2e-16
```

The R output gives the following regression equation:

$$\begin{aligned}\log(\widehat{\texttt{BrainWt}}) &= \hat{\beta}_0 + \hat{\beta}_1 \log(\texttt{BodyWt}) \\ &= 2.135 + 0.7517 \log(\texttt{BodyWt})\end{aligned}$$

- The interpretation of the estimated slope is that a unit increase in `log(BodyWt)` is associated with an increase in `log(BrainWt)` by 0.7517 (not that useful).
- Another common interpretation is in terms of percentage effects: A 1% increase in body weight (kg) is associated with an approximate 0.75% increase in brain weight.[1]

[1] See Sheather, Section 3.3.2, pp.79-80, for a mathematical explanation

Review of logs

- Logs are exponents: $\log_b(x) = y$ (read "the log of x to the base b is y") means that $b^y = x$. Some examples:

$$\log_{10} 100 = 2 \iff 10^2 = 100$$
$$\log_{10} 0.01 = -2 \iff 10^{-2} = 0.01$$
$$\log_2 8 = 3 \iff 2^3 = 8$$

- Some useful identities:

$$e^{\log(x)} = x$$
$$\log(x^r) = r\log(x)$$
$$\log(xy) = \log(x) + \log(y)$$
$$\log(x/y) = \log(x) - \log(y)$$

Note: $\log(x)$ denotes the log base e here (this is called the natural logarithm, which is also commonly denoted by $\ln(x)$)

We can back-transform to write the model in the original scale of the response.

Regression equation for log($\texttt{BrainWt}$):

$$\log(\widehat{\texttt{BrainWt}}) = 2.135 + 0.7517 \log(\texttt{BodyWt})$$

Make a prediction for log($\texttt{BrainWt}$) when $\texttt{BodyWt}$ is 40 kg:

$$\log(\widehat{\texttt{BrainWt}}) = 2.135 + 0.7517 \log(40) = 4.908$$

We can then exponentiate both sides to get the prediction for $\texttt{BrainWt}$ (in grams) when when $\texttt{BodyWt}$ is 40 kg:

$$\widehat{\texttt{BrainWt}} = e^{4.908} = 135.37$$

```
# use R to make prediction for log(BrainWt) when BodyWt is 40
> pred <- predict(lm1, data.frame(BodyWt = 40),
                  interval="prediction")
> pred
       fit      lwr      upr
1 4.907668 3.501332 6.314003


# exponentiate to make prediction for BrainWt
> exp(pred)
       fit      lwr      upr
1 135.3235 33.15961 552.2514
```

Summary: Log Transformation

A log transformation might be useful if

- the distribution of the response or predictor variable is skewed right.
- the values of the variable range over more than one order of magnitude.
- there is a fan pattern in the residuals (nonconstant variance).

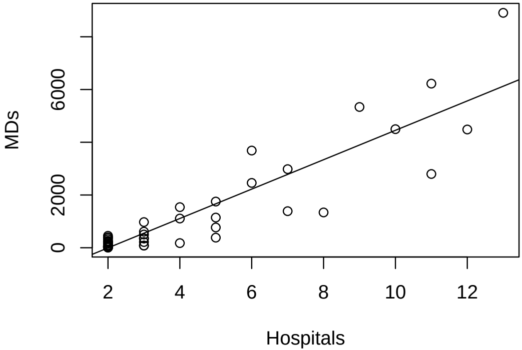
Remark: The transformation $log(x_i)$ is only valid for $x_i > 0$. For nonpositive data, one workaround is to use the transformation $log(x_i + c)$, where c is a constant such that $x_i + c > 0$, for $i = 1, \cdots, n$.

Example: Doctors and Hospitals

Data set containing counts on the number of medical doctors and number of hospitals in a random sample of 53 counties.

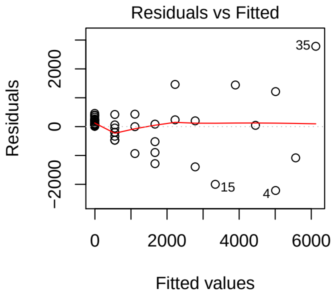
```
> library(Stat2Data)
> data("CountyHealth")
> head(CountyHealth)
County                MDs Hospitals Beds
1 Bay, FL             351         3  605
2 Beaufort, NC         95         2  134
3 Beaver, PA          260         2  567
4 Bernalillo, NM     2797        11 1435
5 Bibb, GA            769         5  976
6 Clinton, PA          42         2  245
```

```
> lm1 <- lm(MDs ~ Hospitals, data = CountyHealth)
> plot(MDs ~ Hospitals, data = CountyHealth)
> abline(lm1)
```



There residual plot shows nonconstant variance. That is, the variability in the residuals tends to increase with the fitted values. The QQ plot also indicates that the residuals deviate from a normal distribution.

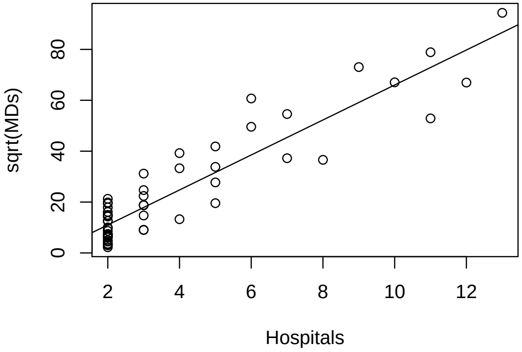
```
> par(mfrow = c(1, 2))
> plot(lm1, 1:2)
```



Using Transformations to Stabilize Variance

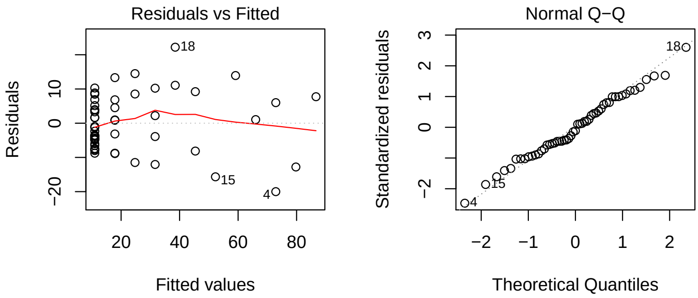
- Problems with nonconstant variance can be overcome with transformations.
- Two common variance stabilizing transformations are the log transformation, $\log(Y)$, and the square root transformation, $\sqrt{Y}$.
- The square root transformation is often appropriate for count data.
- Since the data in this example are in the form of counts, we will try a square root transformation of the response.

```
lm2 <- lm(sqrt(MDs) ~ Hospitals, data = CountyHealth)
plot(sqrt(MDs) ~ Hospitals, data = CountyHealth)
abline(lm2)
```



Afre taking the square root transformation, the residual plot and QQ plot show considerable improvement. The assumptions of constant variability and normality in the residuals appear reasonably satisfied.

```
> par(mfrow = c(1, 2))
> plot(lm2, 1:2)
```



```
> summary(lm2)
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  -2.7533     1.9850  -1.387    0.171
Hospitals     6.8764     0.4011  17.144   <2e-16 ***
```

Regression equation for the transformed model:

$$\widehat{\sqrt{\texttt{MDs}}} = -2.7533 + 6.8764\,\texttt{Hospitals}$$

We can back-transform to write the model in the original scale of the response.

$$\widehat{\texttt{MDs}} = (-2.7533 + 6.8764\,\texttt{Hospitals})^2$$

Summary

- It can take some trial and error to find a good transformation. Looking at scatterplots of the data and residuals can help determine which transformation best linearizes the relationship or stabilizes the variance.
- The log transform is commonly applied to skewed data that ranges over several orders of magnitude.
- The square root transform is commonly applied to count data to stabilize the variance.
- Transformations can be applied to the response variable, explanatory variable(s), or both.
- Transforming the response can make the model more difficult to interpret. Predictions and prediction intervals need to be back-transformed so that they can be interpreted on the original scale.