Lecture 1
Simple Linear Regression
STAT 632, Spring 2020

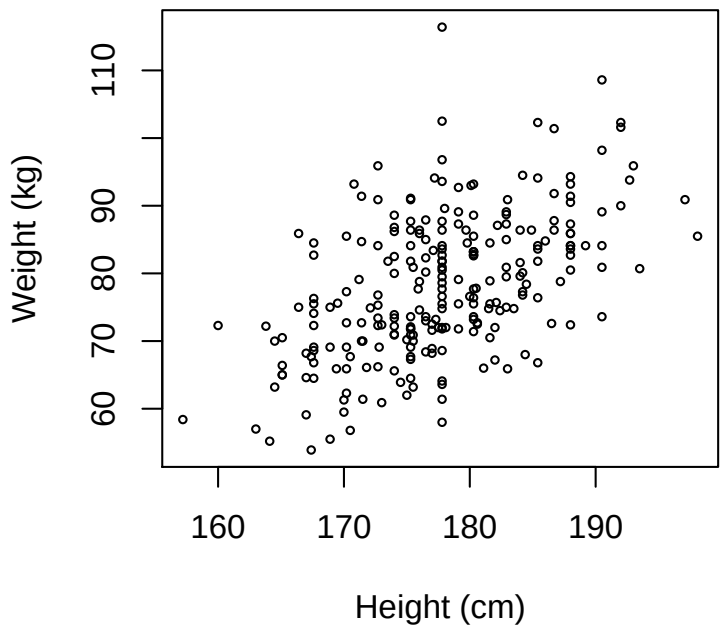
Introduction

- Regression analysis is a method for investigating the functional relationship among variables. It is a useful method for predicting values of one variable using one or more other variables.
- **Simple linear regression** (SLR) involves modeling the relationship between two variables as a straight line, i.e., Y is modeled as a linear function of X.

Example

A scatterplot of weight (Y) versus height (X) for 247 physically active men.

```
> library(openintro)
> bdims_males <- subset(bdims, sex == 1)
> plot(wgt ~ hgt, data = bdims_males,
    xlab = 'Height (cm)', ylab = 'Weight (kg)', cex=0.5)
```



SLR Model

Let $\{(x_i, y_i) : i = 1, \cdots, n\}$ be a collection of n data points. A **simple linear regression model** expressing the relationship between Y_i and x_i is given by:

$$Y_i = \beta_0 + \beta_1 x_i + e_i$$

- Y_i response variable (random)
- x_i explanatory variable (non-random)
- β_0 intercept parameter (non-random)
- β_1 slope parameter (non-random)
- e_i is the random error term; assume $e_i \sim N(0, \sigma^2)$

Remark: We capitalize Y_i in the equation to emphasize that it is a random variable. Y_i is also sometimes called the **dependent** variable, and x_i the **independent** or **predictor** variable. Notation and terminology may vary depending on the textbook and context.

Conditional Mean and Variance Functions

- This expected value of Y when X takes a specific value x.

$$E(Y|X = x) = \beta_0 + \beta_1 x$$

For SLR, the conditional mean is modeled as a straight line.

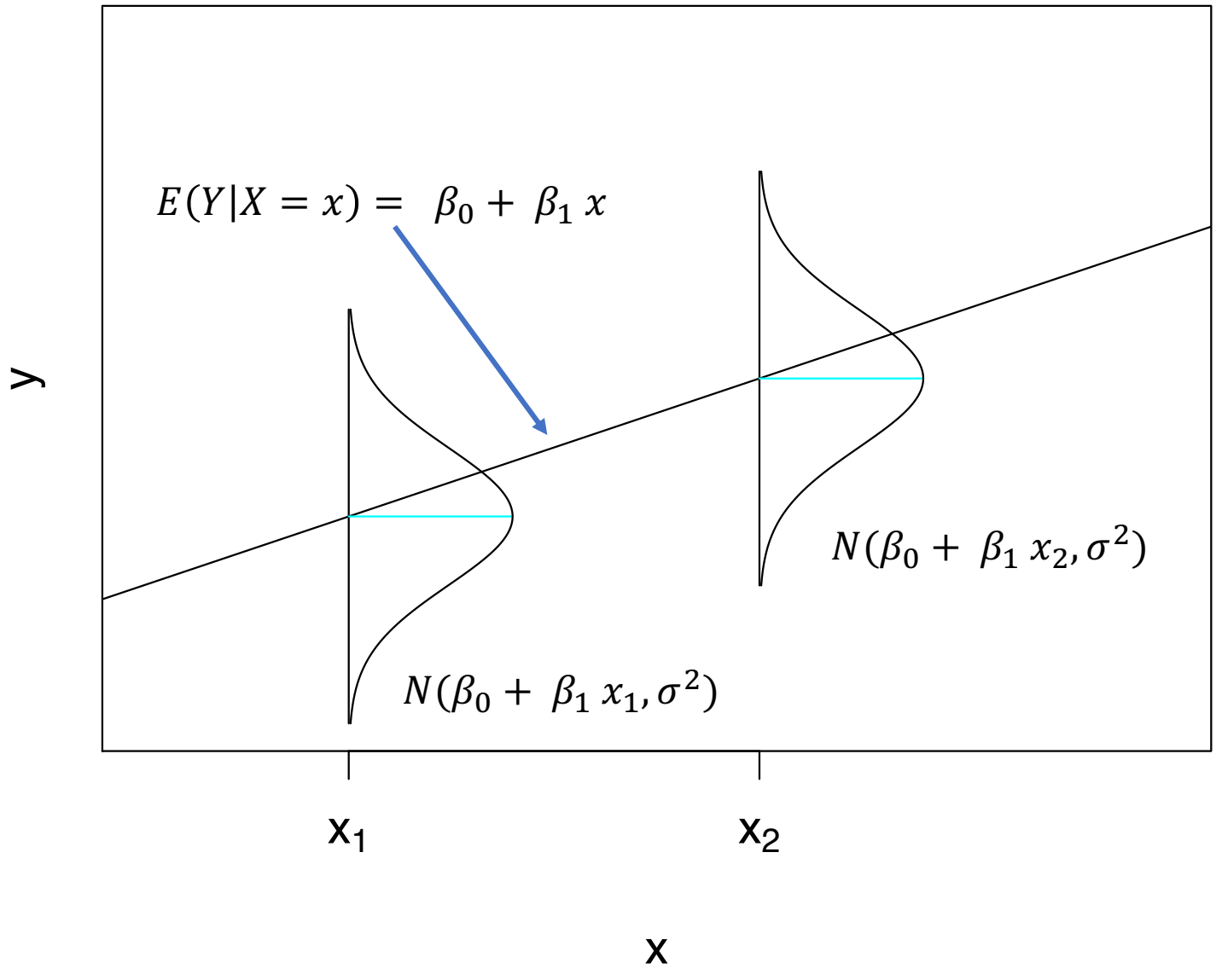
- The variance of Y when X takes a specific value x.

$$Var(Y|X = x) = \sigma^2$$

An assumption for SLR is that the variance is the same for every value of x.

- The conditional distribution of Y when X takes takes a specific value x.

$$Y|X \sim N(\beta_0 + \beta_1 x, \sigma^2)$$


$E(Y|X = x) = \beta_0 + \beta_1 x$
$N(\beta_0 + \beta_1 x_2, \sigma^2)$
$N(\beta_0 + \beta_1 x_1, \sigma^2)$
y
x_1
x_2
x

Fitted Values and Residuals

The line that we estimate, or fit to the data in the scatterplot, is written as

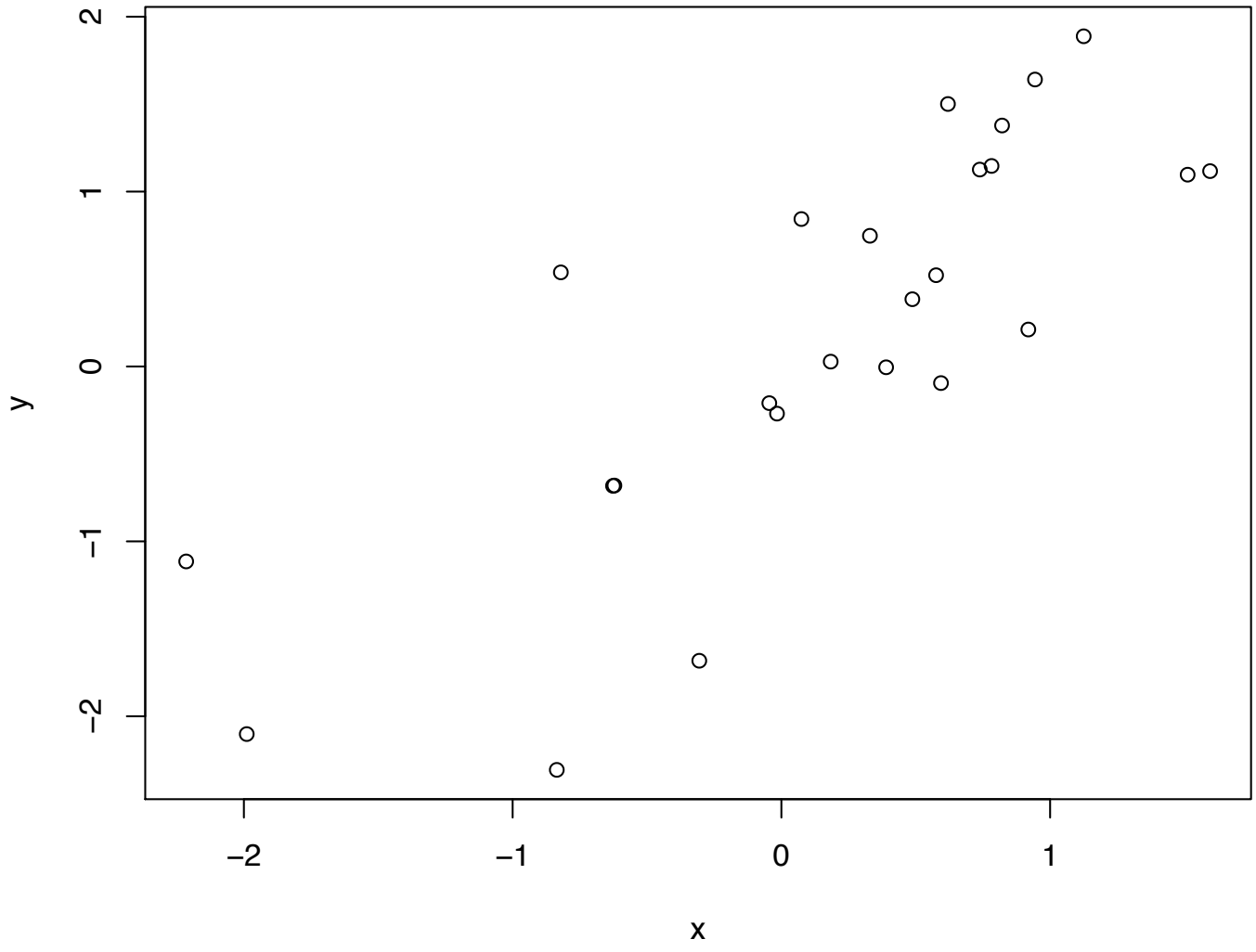
$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$

The fitted (or predicted) value for the i^{th} observation (x_i, y_i):

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

The **residual** for the i^{th} observation is the difference between the observed value (y_i) and the predicted value $(\hat{y}_i)$:

$$\hat{e}_i = y_i - \hat{y}_i = y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i)$$


y
x
2
1
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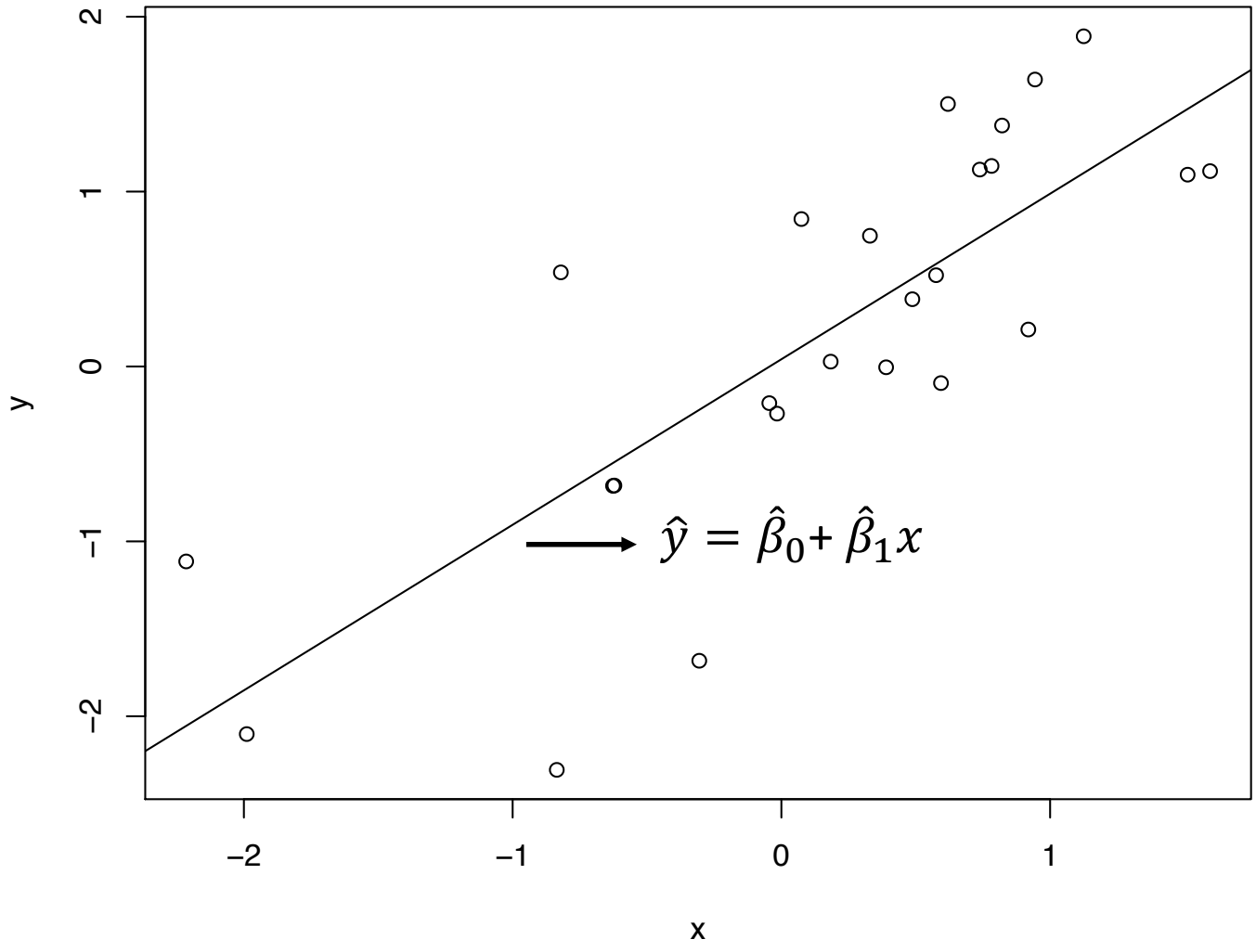
0
1


2
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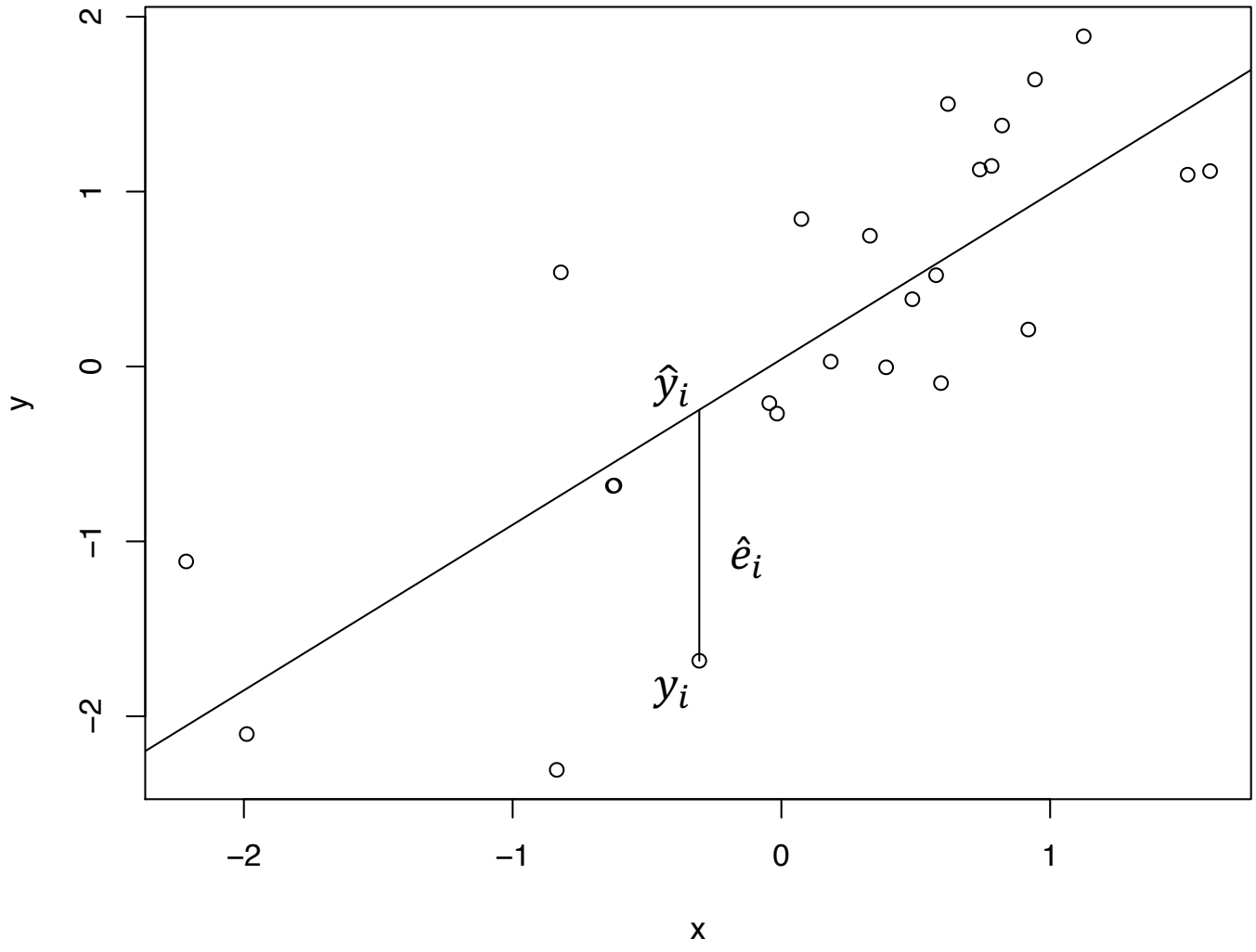
x
$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$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y
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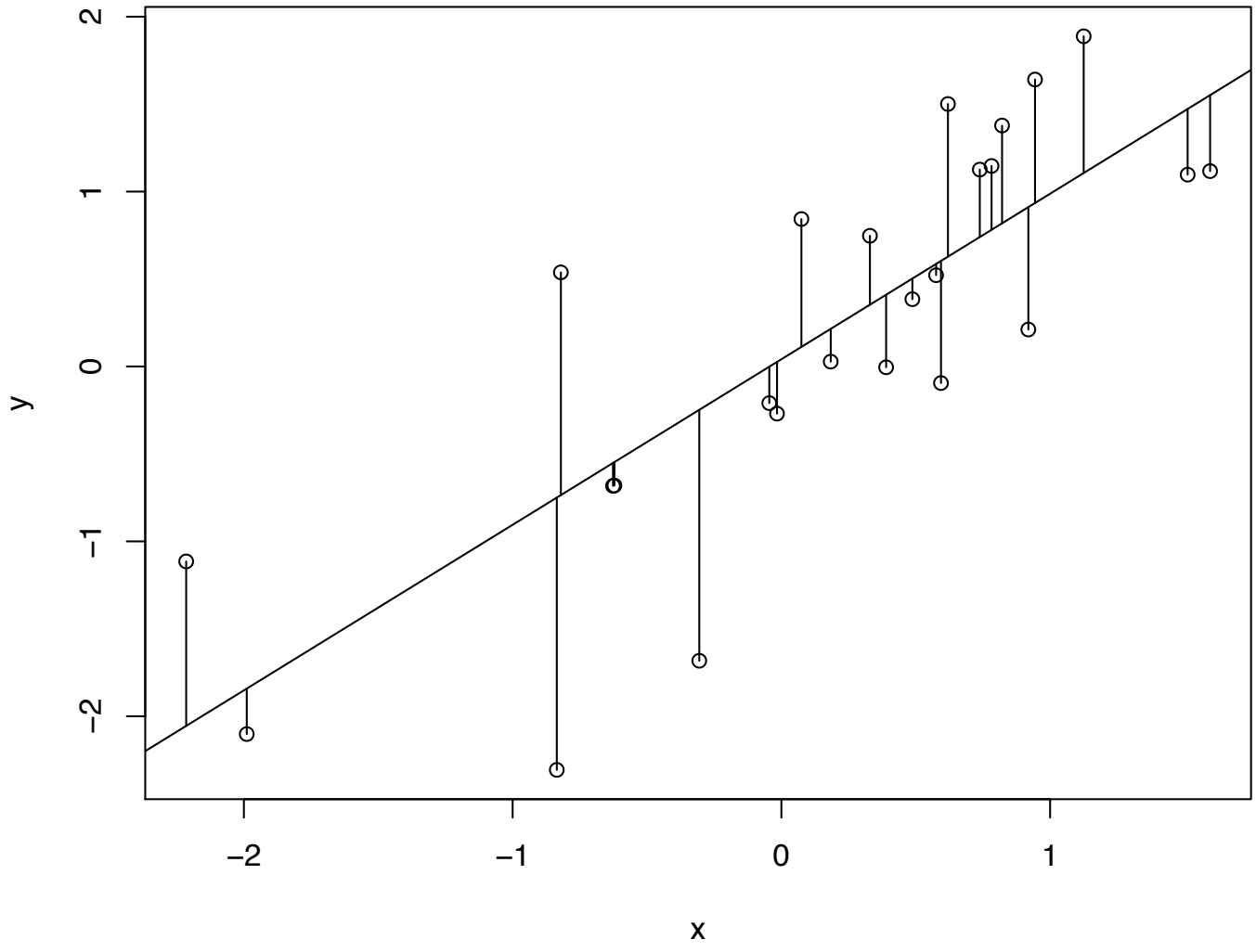
1
x
$\hat{y}_i$
$\hat{e}_i$
y_i


y
2
1
0
−1
−2
−2
−1
0
1
x

Sum of Squared Residuals

- Intuitively, a line that fits the data well has small residuals.
- The **least squares line** minimizes the **sum of squared residuals**:

$$RSS = \sum_{i=1}^{n} \hat{e}_i^2 = \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$$

- That is, out of all possible lines we could draw on the scatterplot, the least squares line is the "best fit" since it has the smallest sum of squared residuals.

Least Squares Estimation

Formally, the estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ of the intercept and slope are found by using calculus to minimize the sum of squared residuals:

$$RSS = \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$$

To minimize set the partial derivatives equal to zero:

$$\frac{\partial RSS}{\partial \hat{\beta}_0} = -2 \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$$

$$\frac{\partial RSS}{\partial \hat{\beta}_1} = -2 \sum_{i=1}^{n} x_i (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$$

Least Squares Estimation

Solving these two equations gives the **least squares estimates** of the intercept and slope:

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{SXY}{SXX}$$

Note that the equation for the intercept guarantees the least squares line passes through $(\bar{x}, \bar{y})$.

Interpretation

- **Slope**: an increase in the explanatory variable (x) by one unit is associated with a change of $\hat{\beta}_1$ in the predicted response ($\hat{y}$).
- **Intercept**: the prediction for the response variable ($\hat{y}$) when the value for the explanatory variable is zero ($x = 0$). It may not make sense to try to interpret the intercept depending on the application.

Estimating σ^2

Estimate of $Var(e_i) = \sigma^2$:

$$\hat{\sigma}^2 = \frac{\text{RSS}}{n-2} = \frac{1}{n-2}\sum_{i=1}^{n}\hat{e}_i^2 = \frac{1}{n-2}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$

Remarks:

- $\sum_{i=1}^{n}\hat{e}_i = 0$
- $\hat{\sigma} = \sqrt{RSS/(n-2)}$ called the **residual standard error**
- The divisor is $n-2$ since two parameters β_0 and β_1 were estimated
- It can be shown that $\hat{\sigma}^2$ is an unbiased estimate of σ^2, i.e, $E(\hat{\sigma}^2) = \sigma^2$

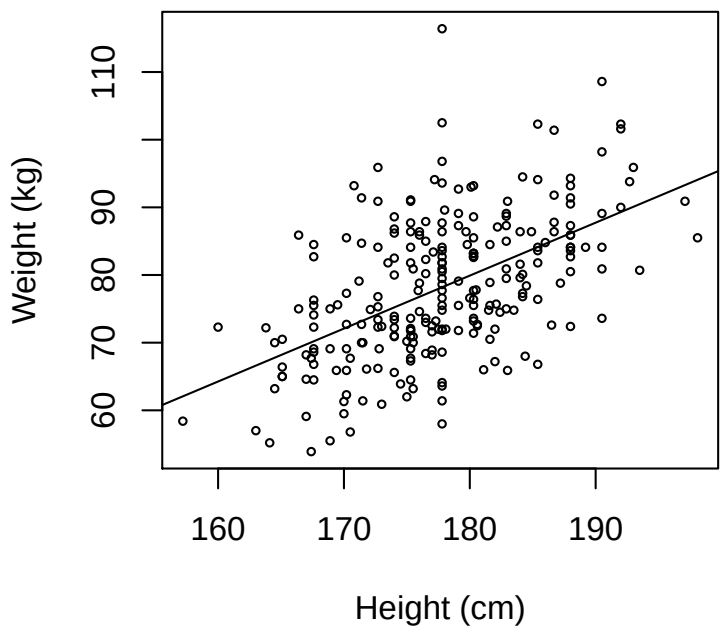
Example

```
> lm1 <- lm(wgt ~ hgt, data=bdims_males)
> summary(lm1)

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) -60.95336   14.05436  -4.337 2.11e-05 ***
hgt           0.78257    0.07901   9.905  < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 8.902 on 245 degrees of freedom
Multiple R-squared:  0.2859,Adjusted R-squared:  0.283
F-statistic: 98.11 on 1 and 245 DF,  p-value: < 2.2e-16
```

Example

A scatterplot of weight (Y) versus height (X) for 247 physically active men with least squares line superimposed.

```
> plot(wgt ~ hgt, data=bdims_males,
    xlab = 'Height (cm)' , ylab = 'Weight (kg)', cex=0.5)
> abline(lm1)
```



Example

- Equation of the least square regression line:
  $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x = -60.95 + 0.78x$
  or we can write $\widehat{\texttt{weight}} = -60.95 + 0.78\texttt{height}$
- Interpreting the slope: For men, an increase in height by 1 cm is associated with an increase in weight by 0.78 kg.
- Interpreting the intercept: The predicted weight for a man that is 0 cm tall is -60.95 kg. Note that it does not makes sense to interpret the intercept in this context. The prediction is an extrapolation (heights for men in this data set range between 157.2 to 198.1 cm).

Example

We can calculate the residual standard error manually in R. This value is also given in the regression summary.

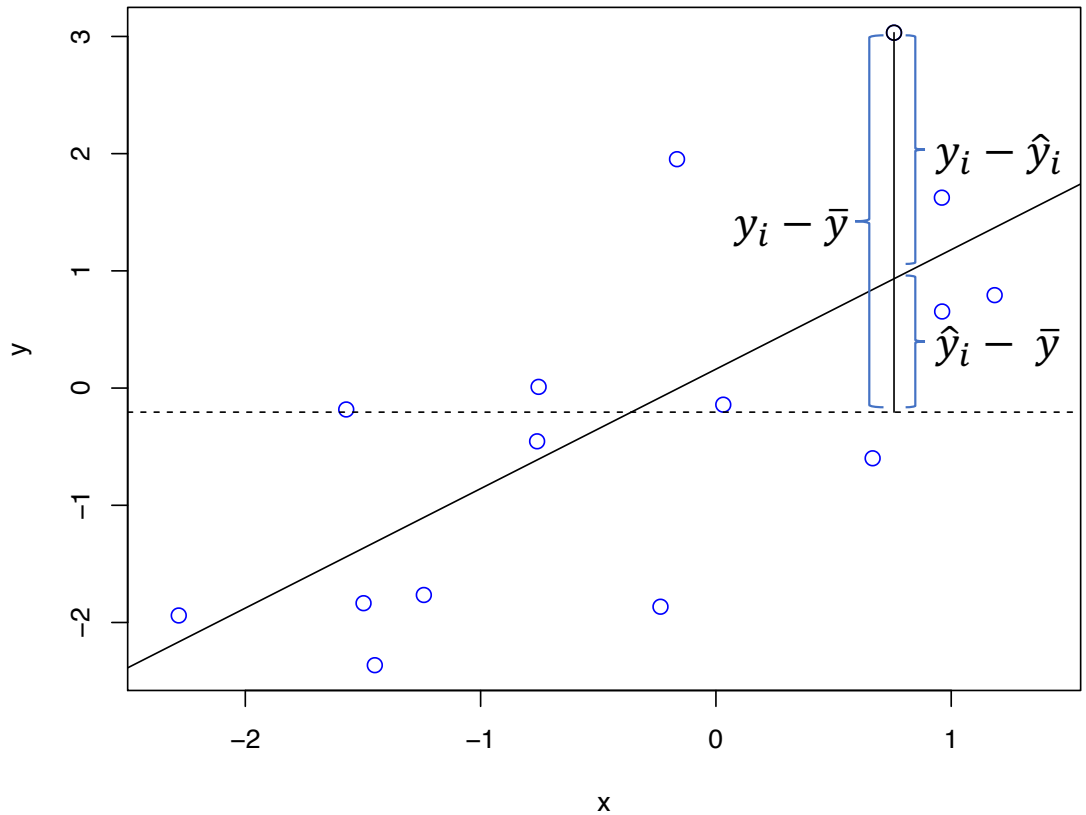
```
> n <- nrow(bdims_males)
> sqrt(sum(resid(lm1)^2) / (n-2))
[1] 8.901667
```

Note that the empirical sum of the residuals is approximately 0:

```
> sum(resid(lm1))
[1] -1.273981e-13
```

Partitioning Variability

Graphical description that $y_i - \bar{y} = (y_i - \hat{y}_i) + (\hat{y}_i - \bar{y})$



Partitioning Variability

Remarkably, it can be shown that

$$\sum_{i=1}^{n}(y_i - \bar{y})^2 = \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2 + \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$

$$\text{SST} = \text{SSreg} + \text{RSS}$$

- SST is the total sum of squares (total variability in the response variable)
- SSreg is the regression sum of squares (variability in the response explained by the model)
- RSS is the residual sum of squares (unexplained variability)

Coefficient of Determination

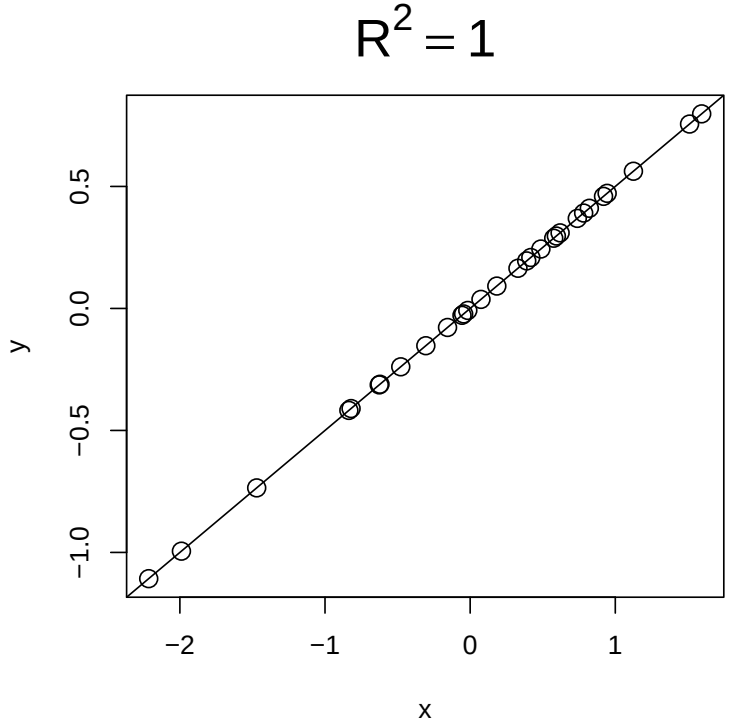
The **coefficient of determination** (R^2) is a measure of how well the linear regression model fits the data.

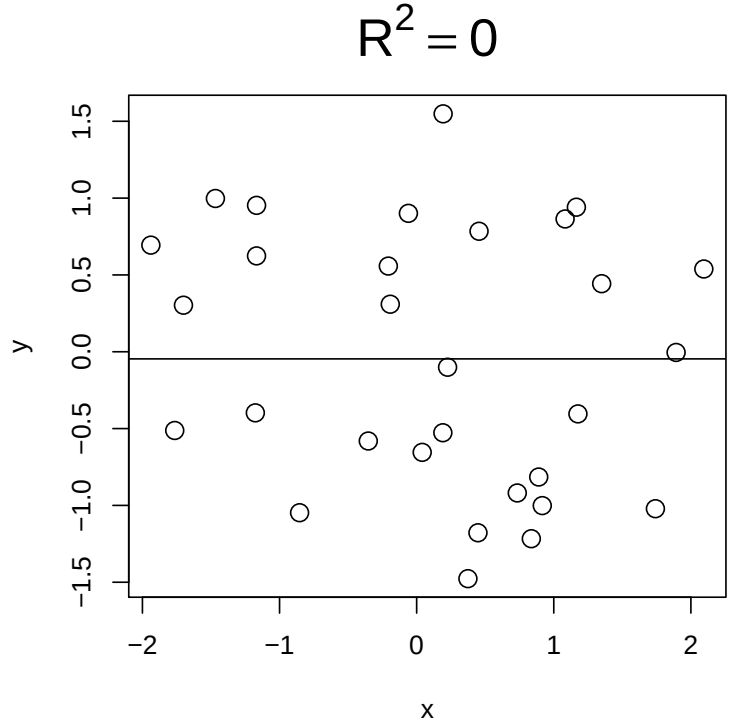
$$R^2 = \frac{\text{SSReg}}{\text{SST}} = 1 - \frac{\text{RSS}}{\text{SST}}$$

- R^2 can be interpreted as the proportion of variability in the response variable Y that is explained by X (i.e., the regression model).
- $0 \leq R^2 \leq 1$; the closer R^2 is to 1, the better the linear regression model fits the data.
- For the example, $R^2 = 0.286$ (see summary output), meaning that for men, 28.6% of the variability in weight can be explained by height.

Limiting cases:

- $R^2 = 1$ when all points fall on the regression line (RSS=0)
- $R^2 = 0$ when $\hat{y}_i = \bar{y}$, which implies RSS=SST.





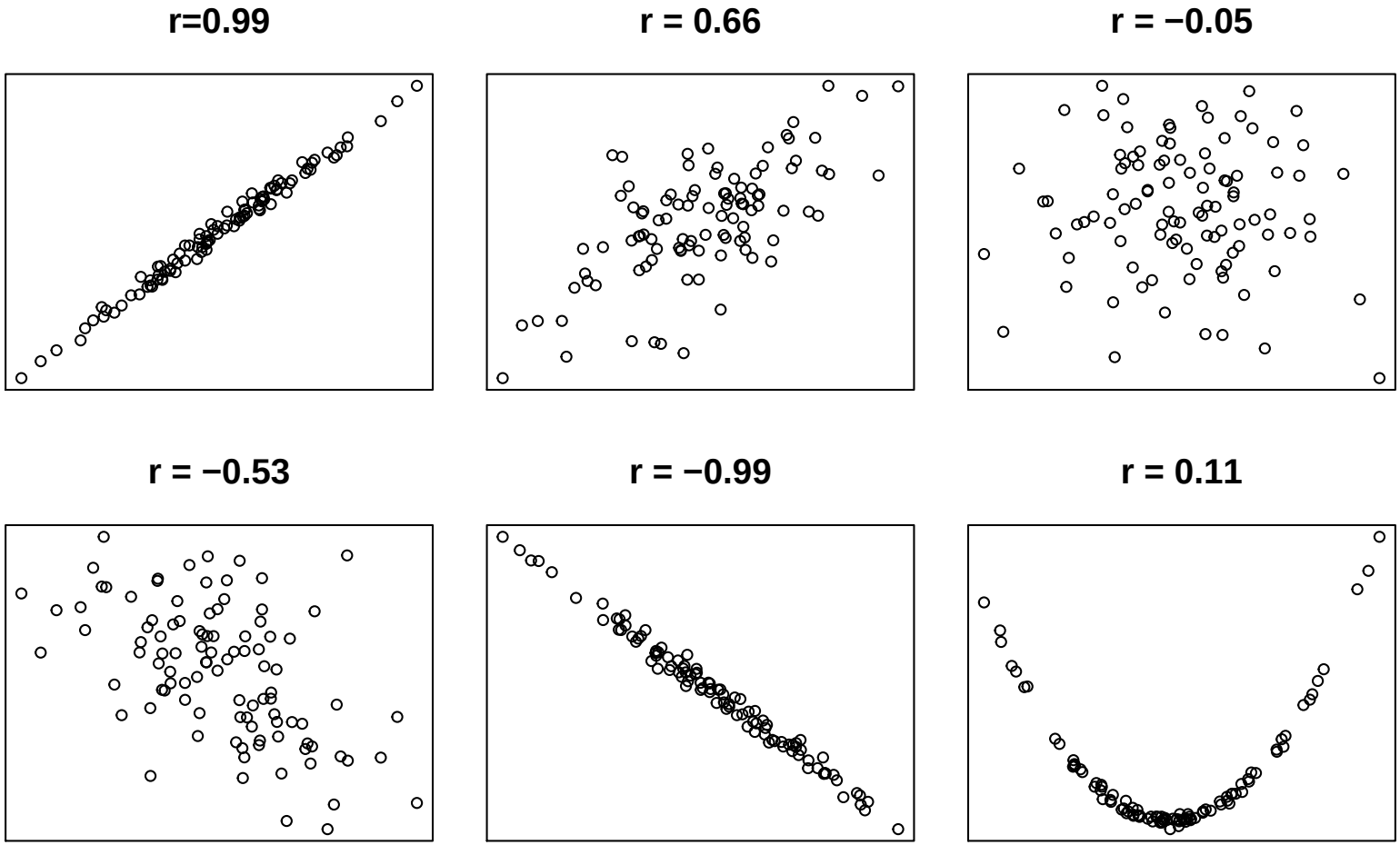
Correlation Coefficient (review)

The **correlation coefficient**, denoted by r, is a number between -1 and 1 that describes the strength of the linear association between two variables.

$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}} = \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right)$$

- $\bar{x}$ and $\bar{y}$ are the sample means
- s_x and s_y are the sample standard deviations

Correlation Coefficent



Correlation Coefficient

- $r \approx 1$ when there is a strong positive linear association between the variables.
- $r \approx -1$ when there is a strong negative linear association between the variables.
- $r \approx 0$ when there is no association between the variables (i.e., independent).
- The correlation coefficient is only useful for evaluating the linear association between two variables. It is not a useful measure for nonlinear relationships.

Correlation Coefficent

- R^2 can also be computed as the correlation coefficient squared.

```
> cor(bdims_males$wgt, bdims_males$hgt)^2
[1] 0.2859487
```

- The least squares estimate of the slope can be written in terms of the the correlation coefficient:

$$\hat{\beta}_1 = r\frac{s_y}{s_x}$$