Lecture 18: Multiple Logistic Regression
STAT 632, Spring 2020

Multiple Logistic Regression

- Multiple logistic regression is a method to model a binary response variable, $Y \in \{0, 1\}$, using predictor variables $x_1, x_2, \cdots, x_p$.
- Specifically, the method models $p(\boldsymbol{x}) = Pr(Y = 1|\boldsymbol{x})$, the probability $Y = 1$ given predictors $\boldsymbol{x} = \begin{pmatrix} x_1 & x_2 & \cdots & x_p \end{pmatrix}'$.

Multiple Logistic Regression

Two ways to express multiple logistic regression model:

Probability form:

$$p(\boldsymbol{x}) = Pr(Y = 1|\boldsymbol{x}) = \frac{e^{\beta_0+\beta_1 x_1+\cdots+\beta_p x_p}}{1 + e^{\beta_0+\beta_1 x_1+\cdots+\beta_p x_p}} = \frac{1}{1 + e^{-\beta_0-\beta_1 x_1-\cdots-\beta_p x_p}}$$

Logit form:

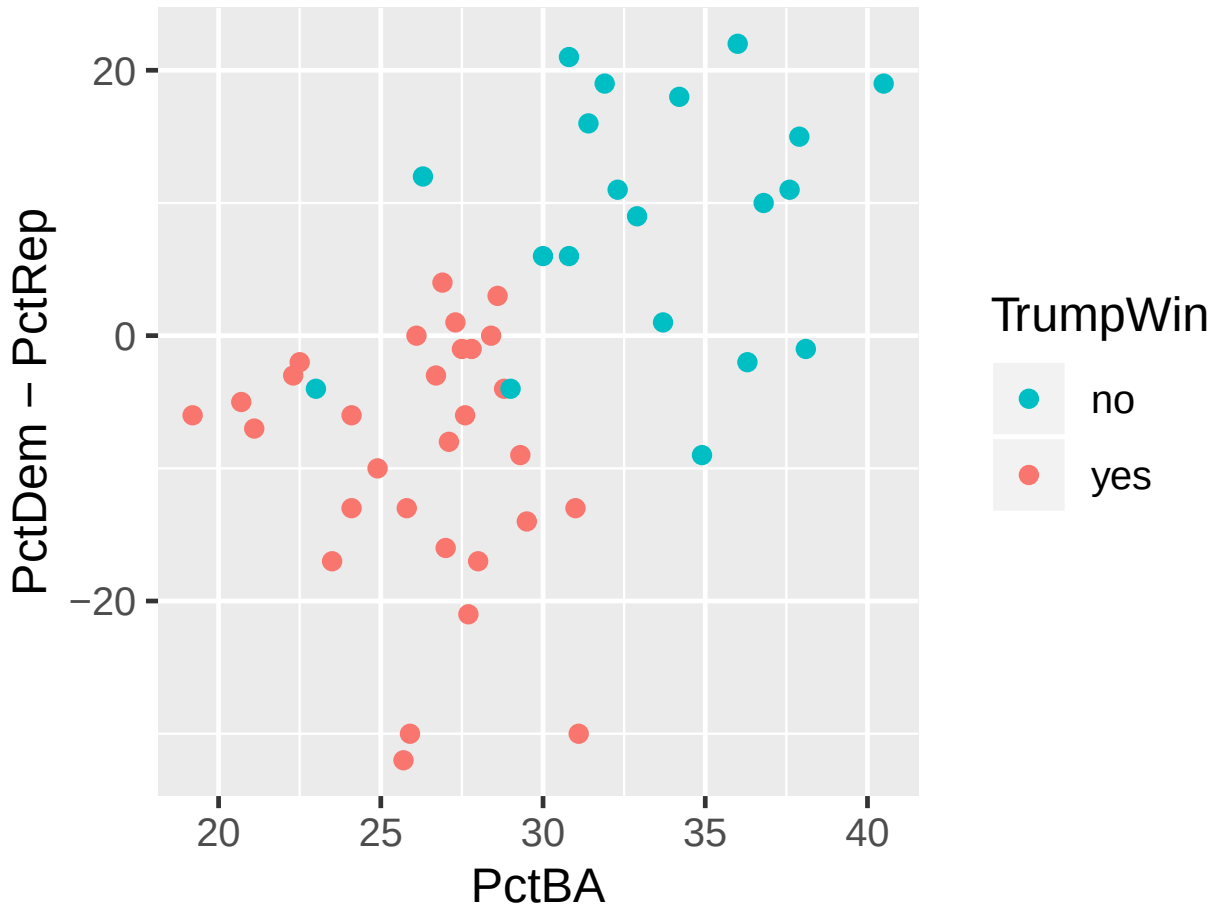
$$\log\left(\frac{p(\boldsymbol{x})}{1 - p(\boldsymbol{x})}\right) = \beta_0 + \beta_1 x_1 + \cdots + \beta_p x_p$$

Example: 2016 US Presidential Election

- Again, we use the data set called `Election16` from the `Stat2Data` library. The data contain results from the 2016 presidential election and demographic information from all 50 states.
- The binary response variable is `TrumpWin`, whether Trump won the state (1=yes, 0=no).
- The predictors are
  - `HS`: Percent of high school graduates in the state
  - `BA`: Percent of college graduates in the state
  - `Adv`: Percent with advanced degrees in the state
  - `Dem.Rep`: Percent Democratic - Percent Republican
  - `Income`: Per capita income in the state

Example

We'll start by considering a multiple logistic regression model for `TrumpWin`, using two predictors from the data set: `BA` and `Dem.Rep`.



Example

We'll start by considering a multiple logistic regression model for `TrumpWin`, using two predictors from the data set: `BA` and `Dem.Rep`.

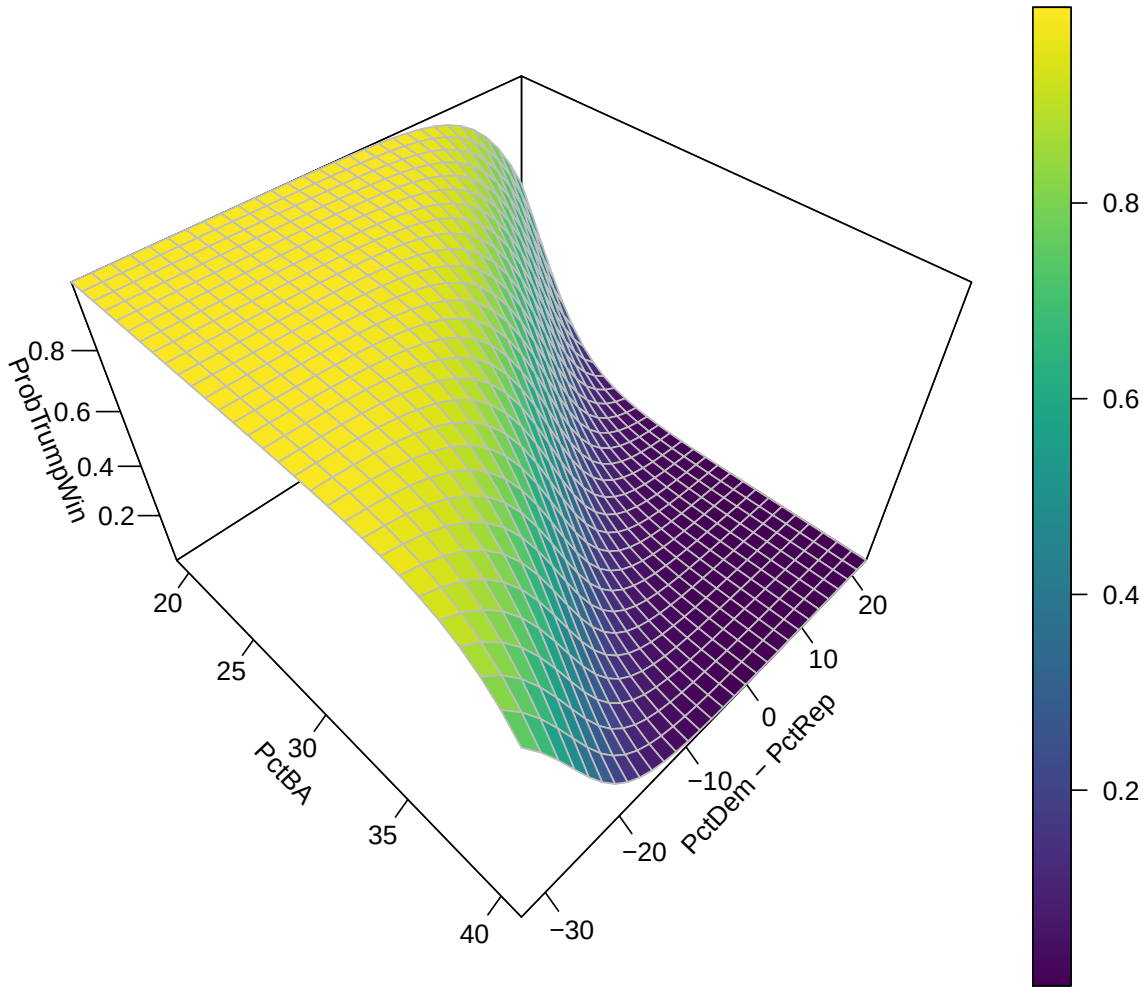
```
> glm2 <- glm(TrumpWin ~  BA + Dem.Rep, data=Election16,
              family=binomial)

> summary(glm2)

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) 15.34796    6.13130   2.503   0.0123 *
BA          -0.51792    0.21181  -2.445   0.0145 *
Dem.Rep     -0.21406    0.08778  -2.439   0.0147 *

> confint(glm2)
                 2.5 %      97.5 %
(Intercept)  5.7651914 31.36586108
BA          -1.0790415 -0.18972992
Dem.Rep     -0.4357941 -0.07794399
```

Example


ProbTrumpWin
0.8
0.6
0.4
0.2
PctBA
20
25
30
35
40
PctDem − PctRep
20
10
0
−10
−20
−30
0.8
0.6
0.4
0.2

The code used to create the last plot:

```
> library(plot3D)
> library(viridis)
> glm2 <- glm(TrumpWin ~  BA + Dem.Rep, data=Election16, family=binomial)
> x1vals <- seq(19, 41, len=30)
> x2vals <- seq(-33, 23, len=30)
> grd <- expand.grid(BA = x1vals, Dem.Rep = x2vals)
> preds <- predict(glm2, grd, type="response")
> persp3D(x1vals, x2vals, matrix(preds, 30, 30), col = viridis(200),
          theta=45, phi= 45, ticktype="detailed", expand=0.7, border="grey",
          xlab = "PctBA", ylab = "PctDem - PctRep", zlab = "ProbTrumpWin")
```

Example

The equation for the fitted logistic regression model in probability form is given by:

$$\hat{p}(x_1, x_2) = \frac{e^{\hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2}}{1 + e^{\hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2}} = \frac{e^{15.348 - 0.518 x_1 - 0.214 x_2}}{1 + e^{15.348 - 0.518 x_1 - 0.214 x_2}}$$

In California, the % with a BA is 31.4, and `Dem.Rep`, the % Democrat minus the % Republican, is 16. So the estimate for the probability that Trump won is

$$\hat{p}(31.4, 16) = \frac{e^{15.348 - 0.518(31.4) - 0.214(16)}}{1 + e^{15.348 - 0.518(31.4) - 0.214(16)}} = \frac{e^{-4.34}}{1 + e^{-4.34}} = 0.0129$$

Example

To make the predictions in R use the `predict()` function:

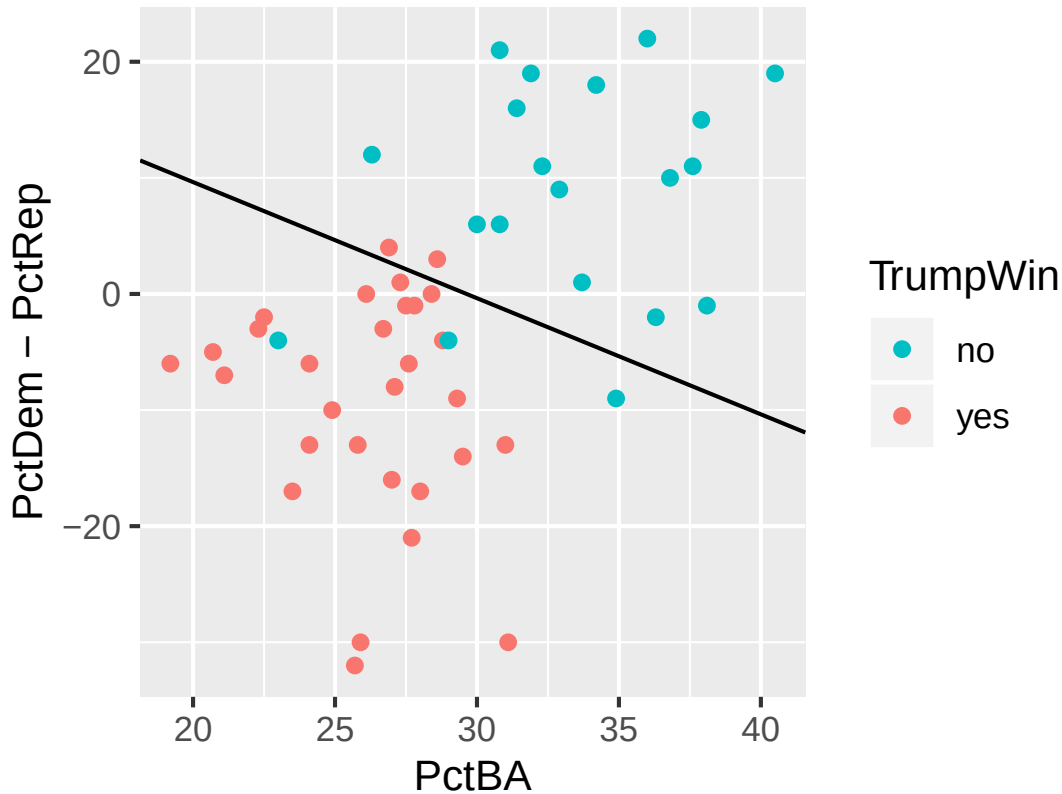
```
> new_x <- data.frame(BA = 31.4, Dem.Rep = 16)

# prediction for logit
> predict(glm2, newdata=new_x)
        1
-4.339819

# prediction for probability
> predict(glm2, newdata=new_x, type="response")
         1
0.01287107
```

Example

The line associated with a 0.50 probability of Trump winning is found by setting $\hat{p}(x_1, x_2) = 0.5$, which gives the line $\hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 = 0$. This is often called the **decision boundary**.



Variable Selection

For logistic regression modeling, the AIC is defined as

$$\text{AIC} = -2\log(L(\hat{\boldsymbol{\beta}})) + 2K$$

where $\log(L(\hat{\boldsymbol{\beta}}))$ is the log-likelihood evaluated at the MLEs, $\hat{\boldsymbol{\beta}} = \begin{pmatrix}\hat{\beta}_0 & \hat{\beta}_1 & \cdots & \hat{\beta}_p\end{pmatrix}'$, and $K = p + 1$ is the number of parameters.

- The *smaller* the value for the AIC the better the logistic regression model fits the data.
- If we fit several logistic regression models, using the same data set, we would select the model with the smallest AIC value.
- The goodness-of-fit of the logistic regression model is measured by the log-likelihood. Note that maximizing the log-likelihood is the same as minimizing the negative log-likelihood.

Variable Selection: Example

Consider all 8 possible logistic regression models for `TrumpWin`, using `BA`, `Dem.Rep`, and `Income` as potential predictors.

Predictor Variables	AIC
Null model (intercept only)	69.30
`BA`	38.43
`Dem.Rep`	37.60
`Income`	49.92
`BA`, `Dem.Rep`	27.08
`BA`, `Income`	40.12
`Dem.Rep`, `Income`	**26.79**
`BA`, `Dem.Rep`, `Income`	27.62

Variable Selection: Example

Here is the code used for the previous table:

```
> glm0 <- glm(TrumpWin ~ 1, data=Election16, family=binomial)
> glm1_1 <- glm(TrumpWin ~  BA, data=Election16, family=binomial)
> glm1_2 <- glm(TrumpWin ~ Dem.Rep, data=Election16, family=binomial)
> glm1_3 <- glm(TrumpWin ~  Income, data=Election16, family=binomial)
> glm2_1 <- glm(TrumpWin ~ BA + Dem.Rep, data=Election16, family=binomial)
> glm2_2 <- glm(TrumpWin ~ BA + Income, data=Election16, family=binomial)
> glm2_3 <- glm(TrumpWin ~  Dem.Rep + Income, data=Election16, family=binomial)
> glm3 <- glm(TrumpWin ~ Income + BA + Dem.Rep, data=Election16, family=binomial)
> AIC(glm0, glm1_1, glm1_2, glm1_3, glm2_1, glm2_2, glm2_3, glm3)
       df      AIC
glm0    1 69.30117
glm1_1  2 38.43266
glm1_2  2 37.59612
glm1_3  2 49.92250
glm2_1  3 27.08020
glm2_2  3 40.12465
glm2_3  3 26.79370
glm3    4 27.62363
```

Variable Selection: Example

We can also use the `step()` function to implement backwards stepwise selection for logistic regression using the AIC.

```
> glm5 <- glm(TrumpWin ~ HS + BA + Adv + Dem.Rep + Income,
                  data=Election16, family = binomial)
> glm_sel <- step(glm5)
> summary(glm_sel)
Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept)  1.458e+01  5.308e+00   2.747  0.00601 **
Dem.Rep     -3.099e-01  1.135e-01  -2.731  0.00632 **
Income      -2.677e-04  9.866e-05  -2.713  0.00667 **

> AIC(glm5, glm_sel)
        df      AIC
glm5     6 31.55431
glm_sel  3 26.79370
```