# Curvature coordinates

This is just recovering the coordinate system in my orthogonal signed distance coordinates paper. The key change now is how to actually *compute* these coordinates if given an arbitrary coordinate chart for a surface.

| Name | Symbol | Definition |
|---|---|---|
| Physical surface | $S \subset \mathbb{R}^3$ | Set of points in space |
| Surface tangent bundle | $TS$ | Product of manifold points with tangent vector space |
| Surface cotangent bundle | $T^*S$ | |
| Surface tensor bundle | $(\prod_i T_i)S$ | |
| Canonical square | $R \subset \mathbb{R}^2$ | $[-1,1] \times [-1,1] \subset \mathbb{R}^2$ |
| Square tangent bundle | $TR$ | $\{(a\partial_{r^1} + b\partial_{r^2})\|_{r^1,r^2} \mid \{r^1,r^2\} \in R, \{a,b\} \in \mathbb{R}^2\}$ |
| Square cotangent bundle | $T^*R$ | Linear 1-forms on $TR$ |
| Transfinite mapping | $C_R$ | $C_R : R \to S$ where $\{r^1,r^2\} \mapsto X^i(r^j)e_{X^i}$ |
| Transfinite pushforward | $J$ | $J : TR \to TS$ where $v^i\partial_{r^i} \mapsto v^i J_i{}^j e_{X^j} = v^i \partial_{r^i} X^j\, e_{X^j}$ so<br>$J = \partial_{r^i} X^j \omega^{r^i} \otimes e_{X^i} \in T^*R \otimes TS$ |
| Transfinite pullback | $J^*$ | $J^*_R : T^*S \to T^*R$ where $\alpha_j \omega^{X^j} \mapsto \alpha_j J_i{}^j$ |
| Euclidean metric | $I$ | $I = \omega^{X^i} \otimes \omega^{X^i} : T\mathbb{R}^3 \to T^*\mathbb{R}^3$ |
| Transfinite metric | $G$ | $G = J \cdot I \cdot J^* = \partial_{r^i} X^k \partial_{r^j} X^k\, \omega^{r^i} \otimes \omega^{r^j}$ |
| Tranfinite inverse | $G^{-1}$ | $\frac{1}{\|G\|}(e_{r^1} \otimes (G_{22}e_{r^1} - G_{12}e_{r^2}) + e_{r^2} \otimes (-G_{21}e_{r^1} + G_{11}e_{r^2})))$ |
| Transfinite connection coefficient | $\Gamma_R$ | $\Gamma_{R,ij}{}^k = \langle \partial_{r^j} e_{r^i}, \omega^{r^k} \rangle$ and so $\Gamma_R = \partial J_R \cdot J_R\$G^{-1}$ |

| Name | Symbol | Basis | Definition |
|---|---|---|---|
| Embedding | $X$ | | Set of points in space |
| Pushforward | $J$ | $T^*R \otimes TS$ | $J = \partial X = \partial_i X^j \omega^i \otimes e_{X^j}$ |
| Metric | $G$ | $T^*R \otimes T^*R$ | $G = J \cdot I_X \cdot J^T$ |
| Inverse metric | $H$ | $TR \otimes TR$ | $H = G^{-1}$ |
| Pullback | $K$ | $T^*S \otimes TR$ | $K = I_X \cdot J^\top \cdot H$ |
| Jacobian Derivative | $\partial J$ | $T^*R \otimes T^*R \otimes TS$ | $\partial J = \partial_i \partial_j X^k \, \omega^i \otimes \omega^j \otimes e_{X^k}$ |
| Jacobian Derivative gradient | $\nabla \partial J$ | $T^*R \otimes T^*R \otimes T^*R \otimes TS$ | $(\partial\partial J - \partial J \overset{(5423)}{\underset{(1,6)}{\cdot}} \mathit{\Gamma} - \partial J \overset{(5143)}{\underset{(2,6)}{\cdot}} \mathit{\Gamma})$ |
| Connection | $\mathit{\Gamma}$ | $T^*R \otimes T^*R \otimes TR$ | $\mathit{\Gamma} = (\partial J) \cdot I_X \cdot K$ |
| Connection gradient | $\nabla \mathit{\Gamma}$ | $T^*R \otimes T^*R \otimes T^*R \otimes TR$ | $\nabla \mathit{\Gamma} = \partial\partial J \cdot K - 2\mathit{\Gamma} \overset{(2453)}{\underset{(1,6)}{\cdot}} \mathit{\Gamma}$ |
| Normal | $\hat{n}$ | $\mathbb{R}^3$ | $(e_1 \cdot J) \times (e_2 \cdot J) // \hat{\cdot}$ |
| Second fundamental form | $\mathrm{II}$ | $T^*R \otimes T^*R$ | $\partial J \cdot \hat{n}$ |
| Shape operator | $\nabla \hat{n}$ | $T^*R \otimes TR$ | $-(\partial J \cdot \hat{n}) \cdot H$ |
| Shape gradient | $\nabla \, \mathrm{II}$ | $T^*R \otimes T^*R \otimes T^*R$ | $\partial\partial J \cdot \hat{n} + \nabla \hat{n} \overset{(134)}{\underset{(2,5)}{\cdot}} \mathit{\Gamma} + \nabla \hat{n} \overset{(413)}{\underset{(2,5)}{\cdot}} \mathit{\Gamma} + \nabla \hat{n}$ |
| Shape Hessian | $\nabla\nabla\nabla \hat{n}$ | $T^*R \otimes T^*R \otimes T^*R \otimes TR$ | |

# Preliminaries

## Definitions and notation

Because we will be dealing with contractions of higher order tensors (tensor rank three and above), we need to introduce new, unambiguous, concise notation.

### The problem: Einstein summation has ambiguous ordering

The standard approach uses Einstein notation to describe tensor operations, however a naive application of this approach leads to common transposition errors, which are critical when dealing with non-symmetric tensors (which will be common here). Consider taking the gradient of a third order tensor, which requires contraction of each slot with the corresponding connection coefficients

$$\begin{aligned}
\nabla A = \nabla({A^i}_{jk}\, e_i \otimes \omega^j \otimes \omega^k) = \; & \partial_\ell {A^i}_{jk} && \omega^\ell \otimes e_i \otimes \omega^j \otimes \omega^k \\
& + {A^m}_{jk} {\Gamma_{m\ell}}^i && \omega^\ell \otimes e_i \otimes \omega^j \otimes \omega^k \\
& - {A^i}_{mk} {\Gamma_{j\ell}}^m && \omega^\ell \otimes e_i \otimes \omega^j \otimes \omega^k \\
& - {A^i}_{jm} {\Gamma_{k\ell}}^m && \omega^\ell \otimes e_i \otimes \omega^j \otimes \omega^k
\end{aligned}$$

Without having kept track of the bases in the right column, it would be difficult to know how to transpose the numerical arrays to ensure that we are calculating the correct quantity. More specifically, while the Einstein sum does indicate the correct contractions, **it is impossible for the different terms to consistently indicate the correct ordering of the indices**. While the first term unambiguously shows that the partial derivative index $\ell$ should correspond to the leftmost slot, the remaining terms have to break this ordering. The second term could rearrange the ${\Gamma_{m\ell}}^i$ to the left side, and would then have the same ordering $\ell, i, j, k$, but there is no way for the third term to arrange $\ell$ before $j$, from the definition of the $\Gamma$ tensor. There is fundamentally some reordering taking place, which can only be unambiguously determined through keeping track of the basis vectors (as in the right column). This is hopelessly verbose, however. Instead we need a more succinct notation.

# The solution: Index-free notation for combined transposition and contraction

We will adopt a notation that doesn't require expanding out the indices of each tensor. We will write

$$\nabla A = \partial A + A \overset{(5623)}{\underset{(1,4)}{\cdot}} \Gamma[A,1] + A \overset{(5163)}{\underset{(2,4)}{\cdot}} \Gamma[A,2] + A \overset{(5126)}{\underset{(3,4)}{\cdot}} \Gamma[A,3]$$

We have introduced several notations in this equation.

# Transposition $^{(abc)}$

For higher rank tensors there are as many transpositions as there are elements of the permutation group of the corresponding size. Only for rank two tensors is the notation $^\top$ sufficient. Here, we will write transposition as an explicit permutation of the indices $1$ through $n$. The notation $^{(abc)}$ means that original index $a$ will now be the first index, etc. (As opposed to the inverse, which would mean the first index is sent to slot $a$).

# Aside about active vs passive transpose/permutation

Choosing a convention for permutations is surprisingly subtle. Above we used *passive* permutations to define our transposition. This may seem surprising, as a standard notation uses the $i$th number to denote what $i$ is transformed to (*active* notation).

This doesn't mix well with our notation for combined contraction and transposition. Our current notation simply presents the final location of each original index, so there is no repetition.

If we used active notation, the transposition would be much more confusing. We could either perform the transposition before the contraction, in which case we would have to remember that the position of each number in the transposition doesn't represent the original index, but rather the index of the uncontracted indices, which would change meaning were we to contract different slots. If instead we meant transposition after contraction, then we'd have a problem of duplicated numbers, which would also require backtracking to remember which index was and wasn't contracted.

I think an explicit example would be clearer.

The rule for composing passive transpositions is

$$(321)(231)(321) = ((321)(231))(321) = (213)(321) = (312)$$
$$(321)(231)(321) = (321)((231)(321)) = (321)(132) = (312)$$

The pattern is that in any pair $(ijk)(lmn)$, we take the symbol at position $\ell$ in the first permutation, and replace $\ell$ with that, and so on for symbols $m, n$. This requires a step-by-step approach, but meshes well with combined contractions.

## Simultaneous tensor contraction and transposition $\underset{(a,b)}{\overset{(cdef)}{\cdot}}$

The notation $A \underset{(a,b)}{\overset{(cdef)}{\cdot}} B$ is defined as

1. Create the tensor product $A \otimes B$ and label each slot left to right $123456 = \mathrm{range}(\mathrm{len}(A) + \mathrm{len}(B))$.
2. Contract the $a$ and $b$ slots of the tensor product $A \otimes B$.
3. Transpose the output so that the original slot index $c$ ends up first, the original slot index $d$ ends up second, etc.
4. If we want to do multiple contractions, we can just write those pairs after the first pair on the bottom.
   The nice thing, which we will see below, is that this notation can be unambiguously composed with transpositions before and after the contraction, helping us prevent mistakes when coding things up.

## Convention for $\cdot, :$ and $^\top$

If we do not annotate a contraction, this means that we are contracting the last slot of the first tensor with the first slot of the last tensor, and otherwise keeping the order of the remaining slots.

The dot product always means contraction using the correct metric. If a tangent basis is contracted with the dual basis, this corresponds precisely to matrix multiplication of the coefficient arrays. If, however, we are contracting slots using different bases, we need to remember that the $\cdot$ operation is inserting the appropriate basis transformation to line up the two slots.

The notation $:$ will represent a complete contraction of each slot of the left tensor against the corresponding slot on the right tensor.

If we use the notation $^\top$, then it can only mean the $^{(21)}$ permutation of a two-dimensional tensor.

## Partial derivative operator $\partial$

The notation $\partial A$ is a concise form of writing

$$\partial A = \partial_\ell A^i{}_{jk}\, \omega^\ell \otimes e_i \otimes \omega^j \otimes \omega^k.$$

That is, it performs partial derivatives on the components, completely ignorant of the changes in the basis vectors. It is not a coordinate invariant operator, unless it is applied to an object expressed only in Cartesian basis vectors (for which connection coefficients are obviously zero). It is important to note that it does prepend a slot in space $T^*R$, i.e. the first slot will eat tangent vectors in $TR$. Any other vector needs to be changed to this basis to give the correct expression for change in that direction.

## General connection coefficients $\Gamma[A, i]$

Also, $\Gamma[A, i]$ refers to the appropriate connection coefficients for $i$th slot of tensor $A$. In our case $A$ is expressed as a $(+--)$ tensor (i.e. $TV \otimes T^*V \otimes T^*V = A^i{}_{jk}\, e_i \otimes \omega^j \otimes \omega^k$). We will be dealing with several different types of vector spaces, which will make things fairly complicated. This is why we want a simple notation for connection coefficients of an arbitrary basis. For a coordinate basis we know that $\Gamma[\omega^i] = -\Gamma[e_i]^{\top(321)}$ (a consequence of orthogonality $\nabla(e_i \cdot \omega^j) = 0$).

Using this knowledge of the connection coefficient, we can further simplify our above expression

$$\begin{aligned}
\nabla A &= \partial A + A \overset{(5623)}{\underset{(1,4)}{\cdot}} \Gamma[A,1] + A \overset{(5163)}{\underset{(2,4)}{\cdot}} \Gamma[A,2] + A \overset{(5126)}{\underset{(3,4)}{\cdot}} \Gamma[A,3] \\
&= \partial A + A \overset{(5623)}{\underset{(1,4)}{\cdot}} \Gamma - A \overset{(5163)}{\underset{(2,4)}{\cdot}} \Gamma^{(321)} - A \overset{(5126)}{\underset{(3,4)}{\cdot}} \Gamma^{(321)}, \\
&= \partial A + A \overset{(5623)}{\underset{(1,4)}{\cdot}} \Gamma - A \overset{(5143)}{\underset{(2,6)}{\cdot}} \Gamma - A \overset{(5124)}{\underset{(3,6)}{\cdot}} \Gamma.
\end{aligned}$$

The nice thing is that we can compose the transpositions of each element and simplify. In particular, the transposition $(321)$ of the connection tensor is prepended, and says that the first and last slot of $\Gamma$ should be swapped, which translates to swapping labelled index 4 and 6. Because we have all the transposition and contraction information present, we can reliably and unambiguously simplify the composed transpositions.

## Transfinite coordinates $X : R \to \mathbb{R}^3$

Parameterise your surface using transfinite coordinates $C : R = [-1, 1] \times [-1, 1] \hookrightarrow S \subset \mathbb{R}^3$, defined by

$$X(r, s)e_X + Y(r, s)e_Y + Z(r, s)e_Z = X^i(r^j)e_{X^i}$$

## Transfinite tangent vectors $\partial_i$:

Define your partial derivative vectors $\partial_i$ in the reference square.

## Transfinite pushforward $J$:

Calculate the pushforward/Jacobian of the mapping $J : TR \to TS \cong \mathbb{R}^2 \times S$ defined by

$$J = \partial_i X^j \, \omega^i \otimes e_{X^j}.$$

Note that without the metric we can't yet access the dual vectors $\omega^i \in T^*R$.

## Cartesian transfinite tangent vectors $e_{r^i}$:

Calculate the individual transfinite tangent vectors $e_{r^i} = e_i \cdot J = \partial_i X^j \, e_{X^j}$ from the pushforward.

## Cartesian identity tensor $I$:

This maps Cartesian vectors to their dual vectors $I = \delta_{ij}\omega^{X^i} \otimes \omega^{X^j}$, where $\delta_{ij}$ is the standard Kronecker delta. This is normally too obvious to mention, but we want to maintain a convention that all final expressions only ever involve contractions between a basis and its dual basis, which allows us to implement the contraction numerically using matrix multiplication. Otherwise we would have to keep track of basis change transformations in the numerical implementation.

## Levi-Civita/Antisymmetric tensor $\varepsilon$:

Defined by $\varepsilon = \varepsilon_{ijk} \, \omega^{X^i} \otimes \omega^{X^j} \otimes \omega^{X^k}$. This is straightforward in Cartesian space.

## Surface metric tensor $G$:

Calculate the metric tensor from the Jacobian $G : TR \to T^*R$ given by

$$G = J \cdot I \cdot J^\top$$

where we perform $J : TR \to TS$ composed with $I : TS \to T^*S$ composed with $J^* : T^*S \to T^*R$. Each of these is just the identity, but mapping between different spaces.

## Inverse metric $G^{-1}$:

Using a numerical inversion we can create the tensor $G^{-1} \in TR \otimes TR$.

## Pullback $K$:

By combining the adjoint of the pushforward with the inverse metric, we can map Cartesian vectors to dual tangent vectors/tangent forms

$$K = I \cdot J^\top \cdot G^{-1}.$$

This tensor lies in $K \in T^*S \otimes TR$.

# Connection coefficients $\Gamma$:

Calculate the connection coefficients ${\Gamma_{ij}}^k$ using

$$\begin{aligned}\Gamma &= \partial J^{(213)} \cdot I \cdot K = \partial\partial X \cdot K,\\ {\Gamma_{ij}}^k &= \langle \partial_j e_{r^i}, \omega^k \rangle\end{aligned}$$

The symmetry of $\partial J$ in the first two slots is manifest, since it is merely expressing a second derivative tensor. We can now perform gradients of quantities.

# Surface normal $\hat{n}$:

Calculate the surface normal vector from the normalised cross product of the tangent vectors

$$\hat{n} = \frac{e_{r^1} \times e_{r^2}}{|e_{r^1} \times e_{r^2}|} = \text{normalise}\left(\varepsilon \underset{(2,4),(3,5)}{\overset{1}{\cdot}} (e_{r^1} \otimes e_{r^2})\right)$$

where $\varepsilon$ is the antisymmetric/Levi-Civita tensor in $\mathbb{R}^3$, and $e_{r^i} = e_i \cdot J$.

# Shape operator $-\nabla\hat{n}$:

Calculate the covariant derivative of the normal vector

$$-\nabla\hat{n}$$

We can simplify the calculation of the gradient by noting that it is a symmetric tensor (since $\hat{n} = \nabla\sigma$), and noting the identities

$$e_{r^i} \cdot \hat{n} = 0 \implies \partial_{r^j}\hat{n} \cdot e_{r^i} + \partial_{r^j} e_{r^i} \cdot \hat{n} = 0$$

So instead we can look at the partial derivatives of the tangent vectors $\partial_{r^j} e_{r^i} \cdot \hat{n}$.
We want the eigenvalues of the map

$$TR \to TR$$

defined by

$$TR \to TS \to T^*R \otimes TS \to T^*R \to TR$$

which is given by

$$e_i \overset{J}{\mapsto} e_{r^i} \overset{\partial}{\mapsto} \omega^j \otimes \partial_j e_{r^i} \overset{\cdot\hat{n}}{\mapsto} \omega^j(\partial_j e_{r^i} \cdot \hat{n}) \overset{g^{-1}}{\mapsto} H^{jk}(\partial_j e_{r^i} \cdot \hat{n}) e_k$$

So calculate this matrix at each point, then calculate the eigenvectors in the standard way, find the trace and determinant, then find the eigenvalues, then find the normalised eigenvectors pointing in these directions.

It is fundamentally important to put the shape operator into the right numerical basis to perform the eigendecomposition. A standard eigendecomposition requires an operator from a space back to itself. This is why we need to use the inverse metric, so that $\nabla\hat{n} \in T^*R \otimes TR$.

$$\begin{aligned}
-\nabla\hat{n} &= -(e_i \cdot \nabla\hat{n} \cdot \omega^j)\, \omega^i \otimes e_j, \\
&= -(\omega^j \cdot \nabla\hat{n} \cdot e_i)\, \omega^i \otimes e_j, && \text{(symmetry of } \nabla\hat{n} = \nabla\nabla\sigma) \\
&= (\omega^j \cdot \nabla e_i \cdot \hat{n})\, \omega^i \otimes e_j, && \text{(orthogonality of } e_i \cdot J \cdot \hat{n} = 0) \\
&= (\omega^j \cdot (e_i \cdot \mathit{\Gamma}) \cdot \hat{n})\, \omega^i \otimes e_j, && \text{(definition of } \mathit{\Gamma}) \\
&= (\omega^j \cdot (e_i \cdot ((\partial J)^{\top_{(12)}} \cdot I_X \cdot K)) \cdot \hat{n})\, \omega^i \otimes e_j, && \text{(definition of } \mathit{\Gamma}) \\
&= (\omega^j \cdot (e_i \cdot ((\partial J)^{\top_{(12)}} \cdot I_X \cdot K \cdot J)) \cdot \hat{n})\, \omega^i \otimes e_j, && \text{(match Cartesian space for } () \cdot \hat{n}) \\
&= (\omega^j \cdot (e_i \cdot \partial J^{\top_{(12)}} \cdot \hat{n}))\, \omega^i \otimes e_j, && \text{(contract inverses)} \\
&= \partial J \cdot \hat{n} && \text{(undo double transpose (also, symmetric))} \\
&= (\partial J \cdot \hat{n}) \cdot H && \text{(put into } T^*R \otimes TR)
\end{aligned}$$

We will disambiguate the different variants of this operator as follows:

**Second fundamental form**:

$$\mathrm{II} \equiv \partial J \cdot \hat{n} \in T^*R \otimes T^*R.$$

**Shape operator**

$$S \equiv (\partial J \cdot \hat{n}) \cdot H \in T^*R \otimes TR.$$

**Weingarten map**

$$-\nabla\hat{n} \equiv (\partial J \cdot \hat{n}) \cdot K^\top \in T^*R \otimes TS.$$

The only difference between these operators is a change of basis formula. So the covariant gradient will be equivalent for all of them.

## Principal curvatures $\kappa_i$:

After numerically evaluating the shape operator, we calculate the eigendecomposition of the two dimensional matrix. The eigenvalues are the principal curvatures of the surface.

## Principal directions of curvature $\hat{t}_1$:

The eigenvectors of the operator are the principal directions of curvature.

## Principal curves:

Integrate the principal vector fields $\hat{t}_1$ and $\hat{t}_2$ to get two functions which give principal coordinates.

# Curvature coordinates

We need to rescale the coordinate functions to ensure that the coordinates work?

## The problem: non-uniqueness of principal directions and umbilical points

However, this has some problems. Firstly, through umbilical points we can expect a "swap" in the directions corresponding to maximal and minimal curvature. So an integration through those points will have issues.

More importantly (since the former occurs only on a set of measure zero), the eigenvectors aren't unique. In particular, positive and negative directions are equivalent. And the standard numerical routine will always pick one component to be positive, regardless of whether that fits the neighbours correctly. Even trying to do the calculation symbolically doesn't help, weirdly enough. The problem is the square roots I think. Though I don't understand *exactly* why this can't work, in practice it seemed to fail.

Instead, I need to calculate curvature coefficients and then integrate using those. Use the second derivative information, rather than first derivative. This means finding out information about

$$\hat{t}_1[\hat{t}_2] = \nabla_{\hat{t}_1}\hat{t}_1 = \nabla_1\hat{t}_1 = \omega_1\hat{t}_1 \quad \hat{t}_2[\hat{t}_1] = \omega_2\hat{t}_2$$

Of course, one could just do a numerical integration routine where you constantly select the correct direction. But that's ugly.

## Solution: Second-order integration using acceleration

There is a key principal underlying all the calculations:

1. I want high order
2. Many geometric operations aren't numerically differentiable (e.g. inverses, or eigenvalues/functions, or discontinuous selection of nonunique solutions)
3. But formally they are differentiable
4. And there are symmetries that let us commute derivatives away from the problem areas, so that we only ever differentiate the coordinate transformation. I.e. we only calculate $X, \partial X, \partial\partial X, \ldots$, as well as various linear algebraic operations on these tensors

The idea is thus:

Rather than solving for the curve

$$\dot{\phi}_i = \hat{t}_i,$$

or, in components:

$$(r_i^1, r_i^2) : \dot{r}_i^1 e_1(r_i^1, r_i^2) + \dot{r}_1^2 e_2(r_1^1, r_1^2) = \hat{t}_i^{\ k}(r_1^1, r_1^2) e_k,$$

where we calculate $\hat{t}_i$ as the $i$th eigenvector of the mapping $S = (\partial J \cdot \hat{n}) \cdot G^{-1} \in T^*R \otimes TR$.

Instead we want to solve for the *derivative* of this relation,

$$\begin{aligned}
\dot{\phi}_1 &= v_1, \\
\dot{v}_1 &= \dot{\hat{t}}_1 = \dot{\phi}_1 \cdot \nabla \hat{t}_1, \\
\phi_1(0) &= (r_0^1, r_0^2), \\
v_1(0) &= \hat{t}_1(r_0^1, r_0^2).
\end{aligned}$$

Now the question is:

1. How can we derive this expression for the acceleration $\hat{t}_1 \cdot \nabla \hat{t}_1$,
2. How much can we simplify this expression.

# Principal curve acceleration and $\nabla\nabla\hat{n}$

## Connection coefficients for principal frame $\hat{t}_i$

I have the shape tensor

$$\mathrm{II} = S = -\nabla \hat{n} = \kappa_i \, \hat{t}_i \otimes \hat{t}_i.$$

Please note that the shape tensor $S$ lives in $T^*R \otimes TR$, we are considering the *intrinsic* vectors living in the surface tangent space. This means that, unlike the pushforward of these vectors into Euclidean 3-space, we do not have any normal component of the acceleration. We define

$$\nabla_i = \hat{t}_i \cdot \nabla,$$

where $\nabla$ is the *instrinsic* gradient

$$\nabla = \omega^i \partial_i = \hat{t}_i \nabla_i.$$

The principal directions of curvature have derivatives

$$\begin{aligned}
\nabla_1 \hat{t}_1 &= \omega_1 \hat{t}_2 \\
\nabla_2 \hat{t}_1 &= \omega_2 \hat{t}_2 \\
(\nabla_i \hat{t}_j) \cdot \hat{t}_k &= -(\nabla_i \hat{t}_j) \cdot \hat{t}_k
\end{aligned}$$

We know these are the only nonzero acceleration terms because of the orthonormality of $\hat{t}_1, \hat{t}_2$. This implies that

$$\begin{aligned}
\nabla \hat{t}_1 &= +(\omega_1 \hat{t}_1 + \omega_2 \hat{t}_2) \otimes \hat{t}_2 \qquad &= +\omega \otimes \hat{t}_2, \\
\nabla \hat{t}_2 &= -(\omega_1 \hat{t}_1 + \omega_2 \hat{t}_2) \otimes \hat{t}_1 \qquad &= -\omega \otimes \hat{t}_1.
\end{aligned}$$

Please keep in mind that all these gradients $\nabla$ are defined on the surface, so they are two dimensional. Though they can act on three dimensional objects through a pushforward.

Also, please don't confuse the the intrinsic coordinate one-forms $\omega^i$ with the rotation coefficients of the principle directions $\omega_i$. Apologies.

## Gradient of the second fundamenal form $\nabla \mathrm{II} = -\nabla\nabla\hat{n}$

The gradients of the principal directions of curvature tell us that

$$\begin{aligned} -\nabla\hat{n} &= \kappa_i \hat{t}_i \otimes \hat{t}_i = \kappa_i \hat{t}_{ii}, \\ -\nabla\nabla\hat{n} &= \nabla\kappa_i \otimes \hat{t}_{ii} + \kappa_i \nabla\hat{t}_{ii} \\ &= \nabla_1\kappa_1\hat{t}_{111} + \nabla_2\kappa_1\hat{t}_{211} + \nabla_1\kappa_2\hat{t}_{122} + \nabla_2\kappa_2\hat{t}_{222} + \kappa_i(\nabla\hat{t}_i \otimes \hat{t}_i + (\hat{t}_i \otimes \nabla\hat{t}_i)^{(213)}) \\ &= \nabla_1\kappa_1\hat{t}_{111} + \nabla_2\kappa_1\hat{t}_{211} + \nabla_1\kappa_2\hat{t}_{122} + \nabla_2\kappa_2\hat{t}_{222} + \kappa_1(\omega \otimes \hat{t}_{21} + (\hat{t}_1 \otimes \omega \otimes \hat{t}_2)^{(213)} + \ldots) \\ &= \nabla_1\kappa_1\hat{t}_{111} + \nabla_2\kappa_1\hat{t}_{211} + \nabla_1\kappa_2\hat{t}_{122} + \nabla_2\kappa_2\hat{t}_{222} + (\kappa_1 - \kappa_2)\,\omega \otimes (\hat{t}_{12} + \hat{t}_{21}). \end{aligned}$$

Note that we're abbreviating tensor products of the $\hat{t}_i$ vectors in the form $\hat{t}_{ijk} \equiv \hat{t}_i \otimes \hat{t}_j \otimes \hat{t}_k$. The big idea is that this tensor has to be completely symmetric! It's symmetric by design in the second two slots, since we can't change anything there by adding a new slot to the left when differentiating. But it's *also* symmetric in the first two slots. So, by comparing the $211/121$ and $122/212$ slots, we know that

$$\begin{aligned} \nabla_2\kappa_1 &= (\kappa_1 - \kappa_2)\omega_1, \\ \nabla_1\kappa_2 &= (\kappa_1 - \kappa_2)\omega_2. \end{aligned}$$

This gives us formulas for the rotation coefficients

$$\begin{aligned} \omega_1 &= \frac{-\nabla\nabla\hat{n} : \hat{t}_{112}}{\kappa_1 - \kappa_2}, \\ \omega_2 &= \frac{-\nabla\nabla\hat{n} : \hat{t}_{212}}{\kappa_1 - \kappa_2}. \end{aligned}$$

Or equivalently

$$\omega = \omega_1\hat{t}_1 + \omega_2\hat{t}_2 = \frac{\nabla \mathrm{II} : \hat{t}_{12}}{\kappa_1 - \kappa_2}.$$

These formulas illustrate why there are problems at the umbilic points.

**Note**: It actually isn't obvious that a second intrinsic gradient of a quantity would remain symmetric in the first two slots. This is because curvature means that second derivatives no longer commute. So I'm a bit confused as to why this holds. Perhaps we are especially lucky.

## Gradient of shape operator in general surface coordinates

These are nice facts, but when it comes to calculating things numerically, we have to be very careful with mixing differentiation and linear algebra. In particular, the only thing we actually know how to differentiate (easily) is the embedding $X$ (so $X, \partial X, \partial\partial X, \ldots$). Differentiating the metric $G$, the inverse metric $G^{-1}$, the pullback $J^* = G^{-1} \cdot J^T$, the curvatures $\kappa$, and the principal directions of the curvature $\hat{t}_i$ *can't* be done exactly numerically. Either we do finite difference approximations, or we be clever.

So, we calculate the gradient of the shape operator in our coordinate system

$$\begin{aligned}
\nabla \mathrm{II} &= \nabla(\partial J \cdot \hat{n}), \\
&= \nabla \partial J \cdot \hat{n} + \partial J \overset{(412)}{\underset{(3,5)}{\cdot}} \nabla \hat{n}, \\
&= \nabla \partial J \cdot \hat{n} - \partial J \overset{(412)}{\underset{(3,5)}{\cdot}} (\mathrm{II} \cdot K^\top), \\
&= \nabla \partial J \cdot \hat{n} - (\partial J \cdot K) \overset{(412)}{\underset{(3,5)}{\cdot}} \mathrm{II}, \\
&= \nabla \partial J \cdot \hat{n} - (\partial J \cdot K) \overset{(512)}{\underset{(3,4)}{\cdot}} \mathrm{II}, \\
&= (\partial \partial J + \partial J \overset{(5623)}{\underset{(1,4)}{\cdot}} \mathit{\Gamma}[\partial J, 1] + \partial J \overset{(5163)}{\underset{(2,4)}{\cdot}} \mathit{\Gamma}[\partial J, 2] + \partial J \overset{(5126)}{\underset{(3,4)}{\cdot}} \mathit{\Gamma}[\partial J, 3]) \cdot \hat{n} - (\partial J \cdot K) \overset{(512)}{\underset{(3,4)}{\cdot}} \mathrm{II}, \\
&= (\partial \partial J + \partial J \overset{(5623)}{\underset{(1,4)}{\cdot}} \mathit{\Gamma}[\omega] + \partial J \overset{(5163)}{\underset{(2,4)}{\cdot}} \mathit{\Gamma}[\omega] + \partial J \overset{(5126)}{\underset{(3,4)}{\cdot}} \mathit{\Gamma}[e_X]) \cdot \hat{n} - (\partial J \cdot K) \overset{(512)}{\underset{(3,4)}{\cdot}} \mathrm{II}, \\
&= (\partial \partial J - \partial J \overset{(5623)}{\underset{(1,4)}{\cdot}} \mathit{\Gamma}^{(321)} - \partial J \overset{(5163)}{\underset{(2,4)}{\cdot}} \mathit{\Gamma}^{(321)}) \cdot \hat{n} - \mathit{\Gamma} \overset{(512)}{\underset{(3,4)}{\cdot}} \mathrm{II}, \\
&= (\partial \partial J - \partial J \overset{(5423)}{\underset{(1,6)}{\cdot}} \mathit{\Gamma} - \partial J \overset{(5143)}{\underset{(2,6)}{\cdot}} \mathit{\Gamma}) \cdot \hat{n} - \mathit{\Gamma} \overset{(512)}{\underset{(3,4)}{\cdot}} \mathrm{II}, \\
&= \partial \partial J \cdot \hat{n} - (\partial J \cdot \hat{n}) \overset{(432)}{\underset{(1,5)}{\cdot}} \mathit{\Gamma} - (\partial J \cdot \hat{n}) \overset{(413)}{\underset{(2,5)}{\cdot}} \mathit{\Gamma} - \mathit{\Gamma} \overset{(412)}{\underset{(3,5)}{\cdot}} \mathrm{II}, \\
&= \partial \partial J \cdot \hat{n} - \mathrm{II} \overset{(432)}{\underset{(1,5)}{\cdot}} \mathit{\Gamma} - \mathrm{II} \overset{(413)}{\underset{(2,5)}{\cdot}} \mathit{\Gamma} - \mathrm{II} \overset{(134)}{\underset{(2,5)}{\cdot}} \mathit{\Gamma}, \\
&= \partial \partial J \cdot \hat{n} - \mathrm{II} \overset{(341)}{\underset{(2,5)}{\cdot}} \mathit{\Gamma} - \mathrm{II} \overset{(413)}{\underset{(2,5)}{\cdot}} \mathit{\Gamma} - \mathrm{II} \overset{(134)}{\underset{(2,5)}{\cdot}} \mathit{\Gamma}, \\
&= \partial \partial J \cdot \hat{n} - \mathrm{II} \overset{\overline{(134)}}{\underset{(2,5)}{\cdot}} \mathit{\Gamma}.
\end{aligned}$$

where as usual

$$\mathrm{II} = \partial J \cdot \hat{n} \qquad \mathit{\Gamma} = \partial J \cdot K$$

and the overbar notation refers to a sum over all even permutations of the numbers. We could significantly simplify things using the symmetry of $\mathrm{II}$ in both slots, and the symmetry of $\mathit{\Gamma}$ is the first two slots.

It would be good to relate this to the curvature tensor the second fundamental form and the Gauss-Codazzi equations.

Also the Frobenius theorem.
It's also interesting to contrast the principal curvature equation with the geodesic equation

$$\ddot{x}^i + \mathit{\Gamma}_{jk}{}^i \dot{x}^j \dot{x}^k = 0.$$

## Hessian of second fundamental form $\nabla\nabla\,\mathrm{II}$

This might seem silly, but it's necessary to evaluate quantities like $\nabla\omega_i$, which turns out to be important for evaluating principal coordinates. We want to accurately calculate these quantities.

In particular, we want

$$\nabla\nabla\,\mathrm{II} = \partial\nabla\,\mathrm{II} - \nabla\,\mathrm{II} \overset{(5\overline{234})}{\underset{(1,6)}{\cdot}} \mathit{\Gamma},$$

where the overbar refers to a symmetric sum over the even indices (so, (234), (423), (342)). In turn we find

$$\begin{aligned}
\partial\nabla\,\mathrm{II} &= \partial\left(\partial\partial J\cdot\hat{n} - \mathrm{II} \overset{(\overline{134})}{\underset{(2,5)}{\cdot}} \mathit{\Gamma}\right),\\
&= \partial\partial\partial J\cdot\hat{n} + \partial\partial J \overset{5123}{\underset{(4,6)}{\cdot}} \nabla\hat{n} - \partial\left(\mathrm{II} \overset{(\overline{134})}{\underset{(2,5)}{\cdot}} \mathit{\Gamma}\right).
\end{aligned}$$

We find that

$$\begin{aligned}
\partial\partial J \overset{5123}{\underset{(4,6)}{\cdot}} \nabla\hat{n} &= -\partial\partial J \overset{5123}{\underset{(4,6)}{\cdot}} (\mathrm{II}\cdot K^{\top}),\\
&= -(\partial\partial J\cdot K) \overset{5123}{\underset{(4,6)}{\cdot}} \nabla\,\mathrm{II}
\end{aligned}$$

As well as

$$\partial\left(\mathrm{II} \overset{(\overline{134})}{\underset{(2,5)}{\cdot}} \mathit{\Gamma}\right) = \partial\,\mathrm{II} \overset{(1\overline{245})}{\underset{(3,6)}{\cdot}} \mathit{\Gamma} + \mathrm{II} \overset{(3\overline{145})}{\underset{(2,6)}{\cdot}} \partial\mathit{\Gamma},$$

and

$$\begin{aligned}
\partial\,\mathrm{II} &= \partial(\partial J\cdot\hat{n}),\\
&= \partial\partial J\cdot\hat{n} + \partial J \overset{(412)}{\underset{(3,5)}{\cdot}} \nabla\hat{n},\\
&= \partial\partial J\cdot\hat{n} - \partial J \overset{(412)}{\underset{(3,5)}{\cdot}} (\mathrm{II}\cdot K^{\top}),\\
&= \partial\partial J\cdot\hat{n} - (\partial J\cdot K) \overset{(412)}{\underset{(3,5)}{\cdot}} \mathrm{II},\\
&= \partial\partial J\cdot\hat{n} - \mathit{\Gamma} \overset{(412)}{\underset{(3,5)}{\cdot}} \mathrm{II},\\
&= \partial\partial J\cdot\hat{n} - \mathrm{II} \overset{(134)}{\underset{(2,5)}{\cdot}} \mathit{\Gamma}
\end{aligned}$$

and

$$\begin{aligned}
\partial\mathit{\Gamma} &= \partial(\partial J\cdot K),\\
&= \partial\partial J\cdot K + \partial J \overset{(4126)}{\underset{(3,5)}{\cdot}} \partial K,
\end{aligned}$$

where

$$\begin{aligned}
K\cdot J &= I_X - \hat{n}\otimes\hat{n},\\
\partial K\cdot J + K \overset{(315)}{\underset{(2,4)}{\cdot}} \partial J &= -\partial\hat{n}\otimes\hat{n} - (\hat{n}\otimes\partial\hat{n})^{(213)},\\
\partial K\cdot J\cdot K + K \overset{(315)}{\underset{(2,4)}{\cdot}} (\partial J\cdot K) &= -(\nabla\hat{n}\otimes\hat{n} - (\hat{n}\otimes\nabla\hat{n})^{(213)})\cdot K,\\
\partial K &= -K \overset{(315)}{\underset{(2,4)}{\cdot}} \mathit{\Gamma} - 0 - (\hat{n}\otimes(\nabla\hat{n}\cdot K))^{(213)},\\
&= -K \overset{(315)}{\underset{(2,4)}{\cdot}} \mathit{\Gamma} + (\hat{n}\otimes S)^{(213)}.
\end{aligned}$$

(note that I made a critical error earlier, thinking $K \cdot J = I_X$, when really $K \cdot J = I_X - \hat{n} \otimes \hat{n}$). So put succinctly, we have

$$
\begin{aligned}
\nabla\nabla\,\mathrm{II} &= \partial\nabla\,\mathrm{II} - \nabla\,\mathrm{II} \overset{(5\overline{234})}{\underset{(1,6)}{\cdot}} \Gamma, \\
\partial\nabla\,\mathrm{II} &= \partial\partial\partial J \cdot \hat{n} + \partial\partial J \overset{5123}{\underset{(4,6)}{\cdot}} \nabla\hat{n} - \partial\left(\mathrm{II} \overset{(\overline{134})}{\underset{(2,5)}{\cdot}} \Gamma\right), \\
\partial\partial J \overset{5123}{\underset{(4,6)}{\cdot}} \nabla\hat{n} &= -(\partial\partial J \cdot K) \overset{5123}{\underset{(4,6)}{\cdot}} \mathrm{II}, \\
\partial\left(\mathrm{II} \overset{(\overline{134})}{\underset{(2,5)}{\cdot}} \Gamma\right) &= \partial\,\mathrm{II} \overset{(1\overline{245})}{\underset{(3,6)}{\cdot}} \Gamma + \mathrm{II} \overset{(3\overline{145})}{\underset{(2,6)}{\cdot}} \partial\Gamma, \\
\nabla\,\mathrm{II} &= \partial\partial J \cdot \hat{n} - \mathrm{II} \overset{(\overline{134})}{\underset{(2,5)}{\cdot}} \Gamma \\
\partial\,\mathrm{II} &= \partial\partial J \cdot \hat{n} - \mathrm{II} \overset{(134)}{\underset{(2,5)}{\cdot}} \Gamma, \\
\partial\Gamma &= \partial\partial J \cdot K - \Gamma \cdot \Gamma.
\end{aligned}
$$

The reason we want to calculate this is so that we can estimate $\partial_{s^2}\omega_2$.

# Symbolic Hessian of second fundamental form $\nabla\nabla\,\mathrm{II}$

We can calculate $\nabla\nabla\,\mathrm{II}$ in parametric coordinates. Now we want to calculate it in terms of the geometric invariants like the curvature etc.

It will be helpful to use the algebra of symmetric tensors here. Earlier we established that

$$\nabla\,\mathrm{II} = \nabla_j \kappa_i \hat{t}_{jii} + d\kappa\, \omega \otimes \hat{t}_{\overline{12}},$$

We also know that this 3-tensor has to be a completely symmetric form. This is how we derived the Codazzi-Mainardi identities earlier. This also allows us to rewrite the form as

$$\nabla\,\mathrm{II} = \nabla_1\kappa_1 S_{111} + d\kappa(\omega_1 S_{112} + \omega_2 S_{122}) + \nabla_2\kappa_2 S_{222},$$

where we have defined the completely symmetric tensors

$$S_{111} = \hat{t}_{111}, \qquad S_{112} = \hat{t}_{112} + \hat{t}_{121} + \hat{t}_{211}, \qquad S_{122} = \hat{t}_{122} + \hat{t}_{212} + \hat{t}_{221}, \qquad S_{222} = \hat{t}_{222},$$

where, by construction, we have ensure that each satisfies symmetry, since $\hat{t}_{ijk} : S_{lmn} = \hat{t}_{jik} : S_{lmn} = \hat{t}_{kij} : S_{lmn}$. Now we want to take a gradient of that expression $\nabla(\nabla\,\mathrm{II}) = \nabla(A_I S_I)$, and then we will enforce symmetry of the output,

$$
\begin{aligned}
\nabla(\nabla\,\mathrm{II}) &= \nabla\left(\nabla_1\kappa_1 S_{111} + d\kappa\omega_1 S_{112} + d\kappa\omega_2 S_{122} + \nabla_2\kappa_2 S_{222}\right), \\
&= \nabla\left(A_{111} S_{111} + A_{112} S_{112} + A_{122} S_{122} + A_{222} S_{222}\right), \\
&= \nabla(\nabla_1\kappa_1) \otimes S_{111} + \nabla_1\kappa_1 \nabla S_{111} \\
&\quad + \nabla(d\kappa\,\omega_1) \otimes S_{112} + d\kappa\,\omega_1 \nabla S_{112} \\
&\quad + \nabla(d\kappa\,\omega_2) \otimes S_{122} + d\kappa\,\omega_2 \nabla S_{122} \\
&\quad + \nabla(\nabla_2\kappa_2) \otimes S_{222} + \nabla_2\kappa_2 \nabla S_{222}, \\
&= A_I S_I.
\end{aligned}
$$

We find

$$
\begin{aligned}
\nabla S_{111} &= \omega \otimes (\hat{t}_{211} + \hat{t}_{121} + \hat{t}_{112}), \\
&= \omega \otimes S_{112}, \\
\nabla S_{112} &= \omega \otimes (\hat{t}_{212} + \hat{t}_{122} - \hat{t}_{111} + \hat{t}_{221} - \hat{t}_{111} + \hat{t}_{122} - \hat{t}_{111} + \hat{t}_{221} + \hat{t}_{212}), \\
&= \omega \otimes (2S_{122} - 3S_{111}), \\
\nabla S_{122} &= \omega \otimes (\hat{t}_{222} - \hat{t}_{112} - \hat{t}_{121} - \hat{t}_{112} + \hat{t}_{222} - \hat{t}_{211} - \hat{t}_{121} - \hat{t}_{211} + \hat{t}_{222}), \\
&= \omega \otimes (3S_{222} - 2S_{112}), \\
\nabla S_{222} &= -\omega \otimes (\hat{t}_{122} + \hat{t}_{212} + \hat{t}_{221}), \\
&= -\omega \otimes S_{122},
\end{aligned}
$$

as well as

$$
\begin{aligned}
\nabla A_{111} &= \nabla(\nabla_1 \kappa_1), \\
&= (\nabla_1 \nabla_1 \kappa_1)\, \hat{t}_1 + (\nabla_2 \nabla_1 \kappa_1)\, \hat{t}_2, \\
\nabla A_{112} &= \nabla(d\kappa\, \omega_1), \\
&= ((\nabla_1 \kappa_1 - \nabla_1 \kappa_2)\omega_1 + d\kappa \nabla_1 \omega_1)\, \hat{t}_1 + ((\nabla_2 \kappa_1 - \nabla_2 \kappa_2)\omega_1 + d\kappa \nabla_2 \omega_1)\, \hat{t}_2, \\
&= ((\nabla_1 \kappa_1 - d\kappa \omega_2)\omega_1 + d\kappa \nabla_1 \omega_1)\, \hat{t}_1 + ((d\kappa \omega_1 - \nabla_2 \kappa_2)\omega_1 + d\kappa \nabla_2 \omega_1)\, \hat{t}_2, \\
\nabla A_{122} &= \nabla(d\kappa\, \omega_2), \\
&= ((\nabla_1 \kappa_1 - \nabla_1 \kappa_2)\omega_2 + d\kappa \nabla_1 \omega_2)\, \hat{t}_1 + ((\nabla_2 \kappa_1 - \nabla_2 \kappa_2)\omega_2 + d\kappa \nabla_2 \omega_2)\, \hat{t}_2, \\
&= ((\nabla_1 \kappa_1 - d\kappa \omega_2)\omega_2 + d\kappa \nabla_1 \omega_2)\, \hat{t}_1 + ((d\kappa \omega_1 - \nabla_2 \kappa_2)\omega_2 + d\kappa \nabla_2 \omega_2)\, \hat{t}_2, \\
\nabla A_{222} &= \nabla(\nabla_2 \kappa_2), \\
&= \nabla_1 \nabla_2 \kappa_2\, \hat{t}_1 + \nabla_2 \nabla_2 \kappa_2\, \hat{t}_2.
\end{aligned}
$$

This allows us to find

$$
\begin{aligned}
\nabla(\nabla\, \mathrm{II}) = {} & (\nabla_1 A_{111} \hat{t}_1 + \nabla_2 A_{111} \hat{t}_2) \otimes S_{111} + A_{111}(\omega_1 \hat{t}_1 + \omega_2 \hat{t}_2) \otimes S_{112}, \\
& + (\nabla_1 A_{112} \hat{t}_1 + \nabla_2 A_{112} \hat{t}_2) \otimes S_{112} + A_{112}(\omega_1 \hat{t}_1 + \omega_2 \hat{t}_2) \otimes (2S_{122} - 3S_{111}), \\
& + (\nabla_1 A_{122} \hat{t}_1 + \nabla_2 A_{122} \hat{t}_2) \otimes S_{122} + A_{122}(\omega_1 \hat{t}_1 + \omega_2 \hat{t}_2) \otimes (3S_{222} - 2S_{112}), \\
& + (\nabla_1 A_{222} \hat{t}_1 + \nabla_2 A_{222} \hat{t}_2) \otimes S_{222} - A_{222}(\omega_1 \hat{t}_1 + \omega_2 \hat{t}_2) \otimes S_{122}, \\
= {} & (\nabla_1 A_{111} - 3A_{112}\omega_1) S_{1111} \\
& + \Big( \nabla_2 A_{111} \hat{t}_{2111} + A_{111}\omega_1 \hat{t}_1 \otimes S_{112} + \nabla_1 A_{112} \hat{t}_1 \otimes S_{112} - 3A_{112}\omega_2 \hat{t}_{2111} - 2A_{122}\omega_1 \hat{t}_1 \otimes S_{112} \Big), \\
& + \Big( A_{111}\omega_2 \hat{t}_2 \otimes S_{112} + \nabla_2 A_{112} \hat{t}_2 \otimes S_{112} + 2A_{112}\omega_1 \hat{t}_1 \otimes S_{122} + \nabla_1 A_{122} \hat{t}_1 \otimes S_{122} - 2A_{122}\omega_2 \hat{t}_2 \\
& + \Big( 2A_{112}\omega_2 \hat{t}_2 \otimes S_{122} + \nabla_2 A_{122} \hat{t}_2 \otimes S_{122} + 3A_{122}\omega_1 \hat{t}_{1222} + \nabla_1 A_{222} \hat{t}_{1222} - A_{222}\omega_2 \hat{t}_2 \otimes S_{122} \Big) \\
& + (3A_{122}\omega_2 + \nabla_2 A_{222}) S_{2222}.
\end{aligned}
$$

**Note**. While $\nabla\, \mathrm{II}$ as I have defined it should be completely symmetric, it is *not* true that $\nabla\nabla\, \mathrm{II}$ is completely symmetric! This is the appearance of curvature in the surface!

If I instead started with the fundamental form as

$$
\mathrm{II} \otimes \hat{n},
$$

then I would pick up extra terms that arise when taking derivatives,

$$
\nabla(\mathrm{II} \otimes \hat{n}) = \nabla\, \mathrm{II} \otimes \hat{n} + (\mathrm{II} \otimes \nabla \hat{n})^{(3124)},
$$

I'm not sure what I should get following this line of thought. What I originally needed was the derivatives of the rotation coefficients. And I do know that I can calculate them with my earlier $\nabla\nabla\,\mathrm{II}$ tensor (the last component specifically). So let's do that.

From above

$$\begin{aligned}
\nabla\nabla\,\mathrm{II} : (\hat{t}_2 \otimes S_{122}) &= (2A_{112}\omega_2\hat{t}_2 \otimes S_{122} + \nabla_2 A_{122}\hat{t}_2 \otimes S_{122} + 3A_{122}\omega_1\hat{t}_{1222} + \nabla_1 A_{222}\hat{t}_{1222} - A_{222}\omega_2\hat{t}_2 \\
&= (2A_{112}\omega_2 + \nabla_2 A_{122} - A_{222}\omega_2)(\hat{t}_2 \otimes S_{122}) : (\hat{t}_2 \otimes S_{122}) \\
&= 3(2d\kappa\omega_1\omega_2 + \nabla_2(d\kappa\omega_2) - \omega_2\nabla_2\kappa_2), \\
&= 3(2d\kappa\omega_1\omega_2 + \omega_2(\nabla_2\kappa_1 - \nabla_2\kappa_2) + d\kappa\nabla_2\omega_2 - \omega_2\nabla\kappa_2) \\
&= 3(3d\kappa\omega_1\omega_2 - 2\omega_2\nabla_2\kappa_2 + d\kappa\nabla_2\omega_2), \\
\implies \nabla_2\omega_2 &= \frac{\nabla\nabla\,\mathrm{II} : (\hat{t}_2 \otimes S_{122})/3 + 2\omega_2\,\nabla\,\mathrm{II} : \hat{t}_{222}}{d\kappa} - 3\omega_1\omega_2.
\end{aligned}$$

# Principal vector integration - level sets

Going back to the earlier principal

$$\begin{aligned}
\dot{\phi}_1(t) &= v_1 = \hat{t}_1, \\
\frac{d}{dt}v_1 &= \frac{d}{dt}\hat{t}_1, \\
\frac{d}{dt}(v_1^i e_i) &= \frac{d}{dt}(\hat{t}_1{}^i e_i), \\
\dot{v}_1^i e_i + v_1^i \frac{d}{dt}e_i &= \dot{\phi}\cdot\nabla\hat{t}_1, \\
\dot{v}_1^i e_i + v_1^i\dot{\phi}\cdot\nabla e_i &= v_1\cdot(\omega\otimes\hat{t}_2), \\
\dot{v}_1^i e_i + v_1^i(v_1\cdot\Gamma_{ij}{}^k\omega^j\otimes e_k) &= (v_1^i G_{ij}\omega_\ell\hat{t}_\ell{}^j)\hat{t}_2, \\
\dot{v}_1^i e_i + v_1^i v_1^j\Gamma_{ij}{}^k e_k &= v_1^i G_{ij}\omega_\ell\hat{t}_\ell{}^j\hat{t}_2{}^k e_k.
\end{aligned}$$

We then solve for $\dot{v}_1^i$ alone (noting $v_1^i = \hat{t}_1^j$) with

$$\dot{v}_1^k e_k = v_1^i\left(G_{ij}\omega_\ell\hat{t}_\ell{}^j\hat{t}_2{}^k - \Gamma_{ij}{}^k v_1^j\right).$$

Similarly, if we want to integrate the second principal direction of curvature $\hat{t}_2$, we will do

$$\begin{aligned}
\dot{\phi}_2(t) &= v_2 = \hat{t}_2, \\
\frac{d}{dt}v_2 &= \frac{d}{dt}\hat{t}_2, \\
\frac{d}{dt}(v_2^i e_i) &= \frac{d}{dt}(\hat{t}_2{}^i e_i), \\
\dot{v}_2^i e_i + v_2^i \frac{d}{dt}e_i &= \dot{\phi}_2\cdot\nabla\hat{t}_2, \\
\dot{v}_2^i e_i + v_2^i\dot{\phi}_2\cdot\nabla e_i &= v_2\cdot(-\omega\otimes\hat{t}_1), \\
\dot{v}_2^i e_i + v_2^i(v_2\cdot\Gamma_{ij}{}^k\omega^j\otimes e_k) &= -(v_2^i G_{ij}\omega_\ell\hat{t}_\ell{}^j)\hat{t}_1, \\
\dot{v}_2^k e_k &= -v_2^i\left(G_{ij}\omega_\ell\hat{t}_\ell{}^j\hat{t}_1{}^k + v_2^j\Gamma_{ij}{}^k\right)e_k.
\end{aligned}$$

This calculation will find a coordinate line. We want to calculate a full coordinate grid.

# Rescaling for principal coordinates

The curves we are integrating are the level sets of the principal coordinate system.

$$\dot{\phi}_i(t) = \hat{t}_i \circ \phi(t).$$

The arclength coordinates $t$ are *not* principal curvature coordinates. That requires an additional scaling. The guiding principal needs to be "make the vector fields commute by rescaling them". So, find constraints on the rescaling.

But there is a lot of freedom in choosing those scaling functions. Any monotonic function of a viable coordinate system will have the same level sets, and so fit the same criteria.

Should reference the Frobenius theorem too. Learning the proper language of differential forms seems helpful, though it's unclear if it's too brittle to study what we're interested in.

What is the derivation of the commutativity conditions?

$$t_1 = h_1\hat{t}_1, \qquad t_2 = h_2\hat{t}_2.$$

And equivalent rules

$$\begin{aligned}
\nabla_i &= \hat{t}_i \cdot \nabla = \frac{t_i}{h_i} \cdot \nabla = \frac{1}{h_i}\partial_{s^i}, \\
\partial_{s^i} &= t_i \cdot \nabla = h_i\hat{t}_i \cdot \nabla = h_i\nabla_i.
\end{aligned}$$

And pick $h_1, h_2$ so that

$$\begin{aligned}
t_1[t_2] - t_2[t_1] &= t_1 \cdot \nabla t_2 - t_2 \cdot \nabla t_1, \\
&= h_1\hat{t}_1 \cdot \nabla(h_2\hat{t}_2) - h_2\hat{t}_2 \cdot \nabla(h_1\hat{t}_1), \\
&= h_1h_2(\hat{t}_1 \cdot \nabla\hat{t}_2 - \hat{t}_2 \cdot \nabla\hat{t}_1) \\
&\quad + h_1(\hat{t}_1 \cdot \nabla h_2)\hat{t}_2 - h_2(\hat{t}_2 \cdot \nabla h_1)\hat{t}_1, \\
&= h_1h_2(-\hat{t}_1 \cdot (\omega \otimes \hat{t}_1) - \hat{t}_2 \cdot (\omega \otimes \hat{t}_2)) \\
&\quad - h_2\nabla_2 h_1\hat{t}_1 + h_1\nabla_1 h_2\hat{t}_2, \\
&= 0,
\end{aligned}$$

Taking components with respect to the principal directions therefore gives

$$\begin{aligned}
-h_2\nabla_2 h_1\,\hat{t}_1 + h_1\nabla_1 h_2\,\hat{t}_2 &= h_1h_2\,\omega, \\
-\frac{\nabla_2 h_1}{h_1} &= -\nabla_2 \log h_1 = \omega_1, \\
\frac{\nabla_1 h_2}{h_2} &= \nabla_1 \log h_2 = \omega_2.
\end{aligned}$$

Therefore, if I am moving along an integral curve of $\hat{t}_1$ I can integrate to find the value of $h_2$, and if I am moving along an integral curve of $\hat{t}_2$ I can integrate to find the value of $h_1$.

Any choice of $h_1$ along a curve will affect $h_1$ along all parallel principal curves. So start with one and then find the others.

The strategy will be to first integrate a principal direction curve, then to find adjacent curves by solving a differential equation. At each update step we'll need to propagate information along the curve, so there's no way to do everything locally.

# Principal coordinate differential equations

From an initial point, we will send out principal curves along each principal direction with unit scaling. We will also calculate the derivatives of the scaling coefficients along these curves.

Unknowns

$$r^1, r^2, t_1^1, t_1^2, t_2^1, t_2^2, A,$$

Differential equations

$$\begin{aligned}\dot{r}^i(s) &= t_1^i,\\ \dot{t}_1 &= t_1 \cdot \nabla t_1,\\ \dot{t}_2 &= t_1 \cdot \nabla t_2.\end{aligned}$$

We simplify these using several identities. Firstly

$$\begin{aligned}\dot{t}_1 &= t_1 \cdot \nabla(t_1^i e_i),\\ &= (t_1 \cdot \nabla t_1^i) e_i + t_1^i (t_1 \cdot \nabla e_i),\\ &= \dot{t}_1^i\, e_i + t_1^i (t_1 \cdot \Gamma_{ij}{}^k \omega^j \otimes e_k),\\ &= (\dot{t}_1^k + t_1^i t_1^j \Gamma_{ij}{}^k) e_k,\end{aligned}$$

and

$$\begin{aligned}\dot{t}_2 &= t_1 \cdot \nabla(t_2^i e_i),\\ &= (t_1 \cdot \nabla t_2^i) e_i + t_2^i (t_1 \cdot \nabla e_i),\\ &= \dot{t}_2^i\, e_i + t_2^i (t_1 \cdot \Gamma_{ij}{}^k \omega^j \otimes e_k),\\ &= (\dot{t}_2^k + t_2^i t_1^j \Gamma_{ij}{}^k) e_k,\end{aligned}$$

and

$$\begin{aligned}t_1 \cdot \nabla t_1 &= \partial_{s^1} t_1,\\ &= \partial_{s^1} t_1 \cdot (\hat{t}_1 \otimes \hat{t}_1 + \hat{t}_2 \otimes \hat{t}_2),\\ &= \partial_{s^1} t_1 \cdot \left(\frac{t_1 \otimes t_1}{h_1^2} + \frac{t_2 \otimes t_2}{h_2^2}\right),\\ &= \frac{\partial_{s^1} t_1 \cdot t_1}{h_1^2}\, t_1 + \frac{\partial_{s^1} t_1 \cdot t_2}{h_2^2}\, t_2,\end{aligned}$$

As well as

$$\begin{aligned}t_1 \cdot \nabla t_2 &= \partial_{s^1} t_2,\\ &= \partial_{s^1} t_2 \cdot \left(\frac{t_1 \otimes t_1}{h_1^2} + \frac{t_2 \otimes t_2}{h_2^2}\right),\\ &= \frac{\partial_{s^1} t_2 \cdot t_1}{h_1^2}\, t_1 + \frac{\partial_{s^1} t_2 \cdot t_2}{h_2^2}\, t_2,\end{aligned}$$

Along the initial curve we will set $h_1(s^1, 0) = 1$, so the first term will simplify to zero. Then we know

$$
\begin{aligned}
\partial_{s^1} t_1 \cdot t_2 &= -\partial_{s^1} t_2 \cdot t_1, \\
&= -\partial_{s^2} t_1 \cdot t_1, \\
&= -\frac{1}{2}\partial_{s^2}(t_1 \cdot t_1), \\
&= -h_1 \partial_{s^2} h_1, \\
&= -h_1 h_2 \nabla_2 h_1, \\
&= h_1^2 h_2 \omega_1,
\end{aligned}
$$

as well as

$$
\begin{aligned}
\partial_{s^1} t_2 \cdot t_2 &= \frac{1}{2}\partial_{s^1}(t_2 \cdot t_2), \\
&= h_2 \partial_{s^1} h_2, \\
&= h_2 h_1 \nabla_1 h_2, \\
&= h_1 h_2^2 \omega_2.
\end{aligned}
$$

We can then write

$$
\begin{aligned}
(\dot{t}_1^k + t_1^i t_1^j \mathit{\Gamma}_{ij}{}^k) e_k &= \frac{\partial_{s^1} t_1 \cdot t_1}{h_1^2}\, t_1 + \frac{\partial_{s^1} t_1 \cdot t_2}{h_2^2}\, t_2, \\
&= 0 + \frac{\omega_1}{h_2} t_2, \\
\dot{t}_1^k &= \omega_1 \frac{t_2^k}{h_2} - t_1^i t_1^j \mathit{\Gamma}_{ij}{}^k.
\end{aligned}
$$

as well as

$$
\begin{aligned}
(\dot{t}_2^k + t_2^i t_1^j \mathit{\Gamma}_{ij}{}^k) e_k &= \frac{\partial_{s^1} t_2 \cdot t_1}{h_1^2}\, t_1 + \frac{\partial_{s^1} t_2 \cdot t_2}{h_2^2}\, t_2, \\
&= -h_2 \omega_1\, t_1 + h_1 \omega_2\, t_2, \\
\dot{t}_2^k &= h_1 \omega_2 t_2^k - (h_2 \omega_1) t_1^k - t_2^i t_1^j \mathit{\Gamma}_{ij}{}^k
\end{aligned}
$$

We want to simplify this using several identities. Firstly, we just differentiate

$$
\begin{aligned}
t_2 \cdot \nabla t_2 &= t_2 \cdot \nabla (t_2^i e_i), \\
&= t_2 \cdot ((\nabla t_2^i) \otimes e_i + t_2^i \nabla e_i), \\
&= (\partial_{s^2} t_2^i) e_i + t_2^i t_2^j \Gamma_{ij}{}^k e_k.
\end{aligned}
$$

But we also want to simplify this using the other identities. The other side of the equation says

$$
\begin{aligned}
\frac{d}{dt} t_2 &= t_2 \cdot \nabla t_2, \\
&= \partial_{s^2} t_2, \\
&= \partial_{s^2} t_2 \cdot (\hat{t}_1 \otimes \hat{t}_1 + \hat{t}_2 \otimes \hat{t}_2), \\
&= \partial_{s^2} t_2 \cdot \left( \frac{t_1 \otimes t_1}{h_1^2} + \frac{t_2 \otimes t_2}{h_2^2} \right), \\
&= \frac{\partial_{s^2} t_2 \cdot t_1}{h_1^2}\, t_1 + \frac{\partial_{s^2} t_2 \cdot t_2}{h_2^2}\, t_2,
\end{aligned}
$$

Now we figure out these subparts

$$
\begin{aligned}
\partial_{s^2} t_2 \cdot t_1 &= -\partial_{s^2} t_1 \cdot t_2, \\
&= -\partial_{s^1} t_2 \cdot t_2, \\
&= -\frac{1}{2}\partial_{s^1}(t_2 \cdot t_2), \\
&= -\frac{1}{2}\partial_{s^1}(h_2^2), \\
&= -h_2 \partial_{s^1} h_2, \\
&= -h_1 h_2^2 \omega_2.
\end{aligned}
$$

And also

$$
\begin{aligned}
\partial_{s^2} t_2 \cdot t_2 &= \frac{1}{2}\partial_{s^2}(t_2 \cdot t_2), \\
&= \frac{1}{2}\partial_{s^2}(h_2^2), \\
&= h_2 \partial_{s^2} h_2, \\
&= h_2^2 \frac{\partial_{s^2} h_2}{h_2}.
\end{aligned}
$$

To simplify this we need to formally solve the Codazzi-Mainardi equations

$$
\begin{aligned}
\nabla_2 \log h_1 &= -\omega_1, \\
\partial_{s^2} \log h_1 &= -h_2 \omega_1, \\
\log h_1 &= -\int_0^{s^2} h_2 \omega_1 \, ds^2 + f(s^1), \\
\partial_{s^1} \log h_1 &= -\int_0^{s^2} \partial_{s^1}(h_2 \omega_1) \, ds^2 + f'(s^1), \\
&= -\int_0^{s^2} \partial_{s^1} h_2 \, \omega_1 + h_2 \, \partial_{s^1} \omega_1 \, ds^2 + f'(s^1), \\
&= -\int_0^{s^2} h_1 h_2 \omega_1 \omega_2 + h_2 \, \partial_{s^1} \omega_1 \, ds^2 + f'(s^1), \\
\partial_{s^1} h_1 &= h_1 \left( -\int_0^{s^2} h_1 h_2 \omega_1 \omega_2 + h_2 \, \partial_{s^1} \omega_1 \, ds^2 + f'(s^1) \right).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\nabla_1 \log h_2 &= \omega_2, \\
\partial_{s^1} \log h_2 &= h_1 \omega_2, \\
\log h_2 &= \int_0^{s^1} h_1 \omega_2 \, ds^1 + g(s^2), \\
\partial_{s^2} \log h_2 &= \int_0^{s^1} \partial_{s^2}(h_1 \omega_2) \, ds^1 + g'(s^2), \\
&= \int_0^{s^1} \partial_{s^2} h_1 \, \omega_2 + h_1 \, \partial_{s^2} \omega_2 \, ds^1 + g'(s^2), \\
&= \int_0^{s^1} -h_1 h_2 \omega_1 \omega_2 + h_1 \, \partial_{s^2} \omega_2 \, ds^1 + g'(s^2), \\
\partial_{s^2} h_2 &= h_2 \left( \int_0^{s^1} -h_1 h_2 \omega_1 \omega_2 + h_1 \, \partial_{s^2} \omega_2 \, ds^1 + g'(s^2) \right).
\end{aligned}
$$

Whence,

$$
\begin{aligned}
\partial_{s^2} t_2 \cdot t_2 &= h_2^2 \frac{\partial_{s^2} h_2}{h_2}, \\
&= h_2^2 \left( \int_0^{s^1} -h_1 h_2 \omega_1 \omega_2 + h_1\, \partial_{s^2}\omega_2 \, ds^1 + g'(s^2) \right).
\end{aligned}
$$

and we we ultimately find that

$$
\begin{aligned}
\frac{d}{dt} t_2 &= \frac{\partial_{s^2} t_2 \cdot t_1}{h_1^2}\, t_1 + \frac{\partial_{s^2} t_2 \cdot t_2}{h_2^2}\, t_2, \\
&= \frac{-h_2^2 \omega_2}{h_1} t_1 + \left( \int_0^{s^1} -h_1 h_2 \omega_1 \omega_2 + h_1\, \partial_{s^2}\omega_2 \, ds^1 \right) t_2.
\end{aligned}
$$

We then want to equation the two equations and derive evolution laws for $\dot{t}_2^i$. Combining this all together I have

$$
(\partial_{s^2} t_2^i) e_i = \left( \frac{-h_2^2 \omega_2}{h_1} \right) t_1^i\, e_i + \left( \int_0^{s^1} -h_1 h_2 \omega_1 \omega_2 + h_1\, \partial_{s^2}\omega_2 \, ds^1 \right) t_2^i\, e_i - t_2^i t_2^j {\Gamma_{ij}}^k e_k.
$$

Of course, we then expand out all these parts in terms of the components, e.g.

$$
h_i^2 = t_i \cdot t_i = (t_i^j e_j) \cdot (t_i^k e_k) = t_i^j G_{jk} t_i^k.
$$

We are going to define (i.e. pick $g'(s^1)$ things so that:

$$
h_1(s^1, 0) \equiv 1, \qquad h_2(0, s^2) \equiv 1.
$$

Also I need to avoid differentiating $\omega_2$ directly. Instead I'll have to be clever. But it seems like it'll be a right pain. But say I get that, then I can try incrementing this. Then I check how close this is to the calculation from integrating the curve again.

Ok I'm not actually sure what algorithm I should use to update the curve. I can integrate along it, then shoot out a small step in the normal direction according to the size of

# Geodesic coordinates

## References

- https://en.wikipedia.org/wiki/Proper_reference_frame_(flat_spacetime)#Comoving_tetrads
- Gravitation
- Fermi coordinates

## Outline

Near the boundaries of our curves, we want a different coordinate system that aligns with the system geometry.

One way is to borrow the normal vector approach from Bishop coordinates. However, that will have more annoying matching conditions, since those coordinates aren't aligned with the intrinsic physical coordinates of the warped interfaces.

Instead, we use the generalisation of Bishop coordinates to intrinsically curved geometries - Fermi-Walker coordinates. Or what we will call surface geodesic coordinates. This consists of the coordinate lines given by shooting geodesics at right angles from each point on the curve, moving within the interface.

Then, for each choice of geodesic distance from the curve, we can define the distance from that line within each surface, as well as the normal ribbon from that surface. Then, we can intersect those ribbons to define a volume. Over that volume, we can integrate the equations and use Stokes theorem to transfer the dominant terms to the outer surface, where we do have estimates on the data. We take a given sized volume and then take that limit as $\varepsilon \to 0$. The key is to pick an appropriately sized Toblerone box over which to integrate. Then the matching process fits into the right ordering.

## Curve geometry

The contact line is parameterised using coordinates $s$.

In general the different slices could use different coordinates, but in practice all surfaces will use the same transfinite coordinates. So we have a coordinate by simply fixing the other coordinates to $+1, +1$.

## Tangent vector $e_1$

We differentiate our coordinates to get the first coordinate vector

$$e_1 = \partial_s X.$$

## Curve arclength tangent $\hat{t}$

We define the normalised tangent vector

$$\hat{t} = \frac{e_1}{|e_1|} = \frac{e_1}{h},$$

where the scaling factor $h$ is given by the metric

$$h = |e_1| = (e_1 \cdot e_1)^{1/2} = \sqrt{g_{11}}.$$

## Curve arclength derivative $\nabla_s$

We use this to calculate the arclength covariant derivative along the curve

$$\nabla_s = \hat{t} \cdot \nabla = \frac{1}{h} e_1 \cdot \nabla = \frac{1}{h} \partial_s.$$

## Curve surface normal vector $e_2$

We take the second dual vector inherited from the transfinite interpolation of the surface to define a normal direction

$$e_2 \equiv \frac{\omega^2}{|\omega^2|}.$$

Alternatively we could base it off the normal vector

$$e_2 = \frac{\hat{n} \times e_1}{|\hat{n} \times e_1|}$$

## Curve space normal vector $\hat{n}$, curvature $\kappa$

The curve has a normal vector found from the covariant derivative of the tangent vector, or the second derivative of the curve

$$\nabla_s \hat{t} = \kappa \, \hat{n}.$$

## Curve binormal vector $\hat{b}$ and torsion $\omega$

$$\nabla_s \hat{n} = -\kappa \hat{t} + \omega \, \hat{b}$$

where

$$\hat{b} = \hat{t} \times \hat{n}.$$

## Phase solid angle $\theta_{ij}$

The inner product of the surface normal vectors between two surfaces gives their solid angle at the junction

$$\hat{n}_i \cdot \hat{n}_j = \cos \theta_{ij}.$$

## Normal geodesics $\phi$

We then "apply the exponential map" to our normal vector $e_2$ within the surface to find coordinate lines

$$\begin{aligned}
(r^1, r^2, v^1, v^2), \\
\dot{r}^i &= v^i, \\
\dot{v}^i &= -v^k v^j \mathit{\Gamma}_{kj}{}^i.
\end{aligned}$$

This comes from asking for zero intrinsic acceleration of the vector. That is, we want to have

$$\begin{aligned}\frac{d}{ds}v &= \frac{d}{ds}(v^i)\,e_i + v^i\frac{d}{ds}e_i,\\ &= \dot{v}^i\,e_i + v^i\,v\cdot\nabla e_i,\\ &= \dot{v}^i\,e_i + v^i v^j \mathit{\Gamma}_{ji}{}^k e_k,\\ &= (\dot{v}^i + v^j v^k \mathit{\Gamma}_{jk}{}^i)e_i,\\ &= 0.\end{aligned}$$

So notice that $\frac{d}{ds} = v^i\partial_i$. Don't confuse the different derivatives! This ultimately gives us our equation for the integration of the geodesics.

## Geodesic expansion

The expression for the surface geodesic is given by

$$\begin{aligned}\mathit{\Gamma} &= \partial J\cdot K,\\ \mathit{\Gamma}_{ij}{}^k\omega^i\otimes\omega^j\otimes e_k &= \partial_i\partial_j X^\ell K_\ell{}^k\omega^i\otimes\omega^j\otimes e_k,\\ &= (\partial_i\partial_j X^\ell K_\ell{}^k\,\omega^i\otimes\omega^j\otimes e_k),\\ &= (\partial_i\partial_j X^\ell (J^\top\cdot H)_\ell{}^k\,\omega^i\otimes\omega^j\otimes e_k),\\ &= (\partial_i\partial_j X^\ell J_m{}^\ell H_n^m)\,\omega^i\otimes\omega^j\otimes e_k)\end{aligned}$$

I want the geodesics defined

1. In terms of the transfinite coordinate system (so that I can calculate everything) as well as
2. In terms of the geometric invariants of the surface and curve. Then I can derive expressions for the calculation of the volume correctly.

## Normal ribbon

Shoot out geodesics at right angles.
Then stop at a certain geodesic distance $d$.
Then extend out the normal vector $\sigma\hat{n}$.
Keep going until you intersect with the normal sheet from the other surfaces.
What is the intersection condition of the normal sheets?

The geodesic coordinates are given by

$$x(s,d,\sigma) = p(s,d) + \sigma\hat{n}(s,d),$$

where $s$ is the parameter for the junction curve and $d$ is the geodesic distance from that curve lying within the surface, and $\sigma$ is the signed distance from that point on the surface, in full space. A normal ribbon at geodesic distance $d$ is going to be given by fixing that coordinate in the above expression. Ribbons from two surfaces will intersect

## Links

Mathematica can solve vector-valued ODEs!.
More details on solving ODEs with Mathematica (breaking apart NDSolve into steps)