Geometry Notes

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Contents

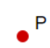
1 Geometry Basics

1.1 Common Geometrical Notations

Notation	Definition
$\angle$	Angle
$\triangle$	Triangle
$\lvert F \rvert$	Area of Figure F
$\parallel$	Parallel
$\perp$	Perpendicular
$\sim$	Similar to
$\equiv$	Congruent to
$\overset{\frown}{AB}$	Arc AB
$\overleftrightarrow{AB}$	Line AB
$[AB]$, $\overline{AB}$	Line Segment AB
$[AB)$	Ray AB
$\overrightarrow{AB}$	Vector AB
$^\circ$	Degrees
rad	Radians
π	Pi Constant

1.2 Fundamentals of Geometry

A **point** has no dimensions, only position. A point is depicted by a dot.

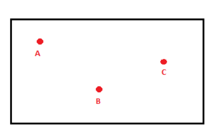


Point P

A **line** is a straight one-dimensional figure with no thickness and extending to infinity in both directions. **Collinear points** are points existing on the same line. A line can be defined by two points. **Intersecting lines** are two lines that meet at a point. A **line segment** is a part of a line with defined endpoints. A line that has one defined endpoint and extends infinitely in only one direction is called a ray.

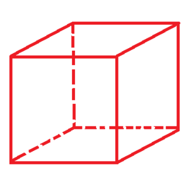
$\overleftrightarrow{AB}$

A **plane** extends infinitely in two dimensions and it has no thickness. A plane is defined by three non-collinear points.



Plane ABC

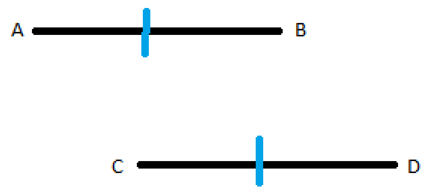
A **space** extends infinitely in three dimensions and is a set of all points in three dimensions. A space can be thought of the inside of an infinitely large box.



A Space

Two figures are **congruent** if they have the same shape and size, whereas two figures are **similar** if they have the same shape (not necessarily same size). Angle and line congruency are depicted by a tick mark on the congruent figures.

An **angle** is formed between two rays that share an endpoint, called the **vertex**. An angle is also a fraction of a circle, where the whole circle is 360° or 2π radians. **Types of Angles** (Where A is an Angle):



$$\overline{AB} \equiv \overline{CD}$$

$A < 90°$	$A < \frac{\pi}{2}$	**Acute** Angle
$A = 90°$	$A = \frac{\pi}{2}$	**Right** Angle
$90° < A < 180°$	$\frac{\pi}{2} < A < \pi$	**Obtuse** Angle
$A = 180°$	π	**Straight** Angle

Two angles are **complementary** if their measures add to 90°. Two angles are **supplementary** if their measures add to 180°.
Let the endpoints of a line segment be $A(x_1, y_1)$ and $B(x_2, y_2)$. Then,

The Distance Formula:

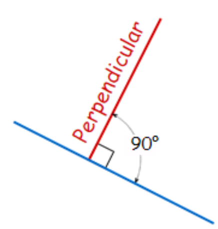
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The Midpoint Formula:

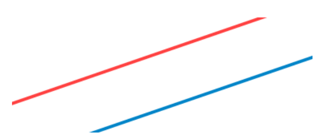
$$\text{mid} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

1.3 Perpendicular and Parallel

Lines are **perpendicular** if they are positioned at right angles to each other. Perpendicular lines are depicted by a box.

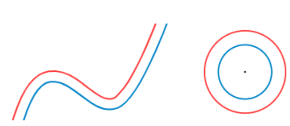


Lines are **parallel** if they are **equidistant**, or always the same distance apart, and will never meet.

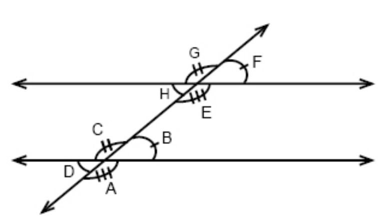


Two parallel lines

Parallel curves are curves that are equidistant. For example, see the below curves



A **transversal line** is a line that crosses two other lines. When the two lines being crossed are parallel, **corresponding angles** are made.



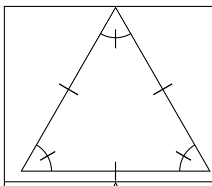
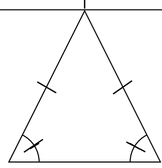
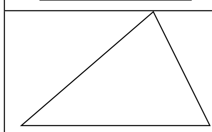
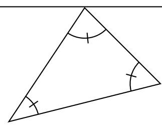
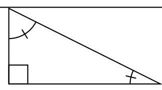
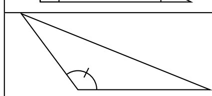
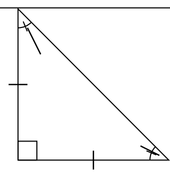
$\angle A$, $\angle F$, $\angle G$, $\angle D$	**Exterior** Angles
$\angle B$, $\angle E$, $\angle H$, $\angle C$	**Interior** Angles
$\angle B$ and $\angle E$, $\angle H$ and $\angle C$	**Consecutive Interior** Angles
$\angle A$ and $\angle G$, $\angle F$ and $\angle D$	**Alternate Exterior** Angles
$\angle E$ and $\angle C$, $\angle H$ and $\angle B$	**Alternate Interior** Angles
$\angle A$ and $\angle E$, $\angle C$ and $\angle G$ $\angle D$ and $\angle H$, $\angle F$ and $\angle B$	**Corresponding** Angles

The **perpendicular bisector** of a line is a line segment perpendicular to and passing through the midpoint of said line. The **angle bisector** is a line that splits an angle into two equal angles.

2 Triangles

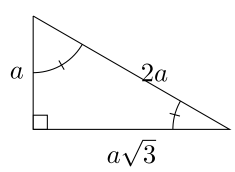
2.1 Triangle Basics

A **triangle** is a closed plane figure with three edges and three vertices. The interior angles of a triangle always add up to 180°. Triangles can be classified by their angles and sides, or both.

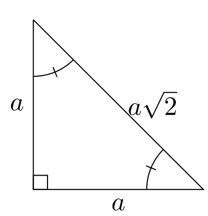
	**Equilateral** Triangle	3 Equal Sides, 3 Equal Angles
	**Isosceles** Triangle	2 Equal Sides, 2 Equal Angles
	**Scalene** Triangle	No Equal Sides, No Equal Angles
	**Acute** Triangle	All angles are less than 90°
	**Right** Triangle	Has a right angle
	**Obtuse** Triangle	Has an obtuse angle
	**Right Isosceles** Triangle	Has a right angle and two equal angles

2.2 Special Triangles

The $30° - 60° - 90°$ **Triangle**:



The $45° - 45° - 90°$ **Triangle**

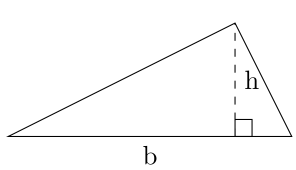


2.3 Triangle Formulas and Theorems

The **perimeter** or distance around the triangle, is given by adding up the lengths of the sides. Let b and h be the base and height of a triangle, respectively. Then,

Area of a Triangle:

$$A = \frac{bh}{2}$$



Heron's Formula states that if a, b, and c are the sides of a triangle and s is the triangle's **semiperimeter**, or half the perimeter, then the area of a triangle is given by

$$A = \sqrt{s(s-a)(s-b)(s-c}$$

The Pythagorean Theorem: for any right triangle with sides a, b, c,c being the hypotenuse,

$$a^2 + b^2 = c^2$$

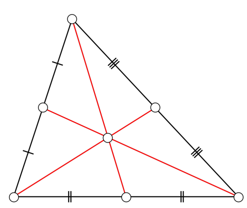
$3, 4, 5$ and $5, 12, 13$ are some special Pythagorean numbers, called **Pythagorean Triples** due to their properties as integers.

The **Triangle Inequality Theorem** states that if a figure is a triangle then the sum of the lengths of any two sides is greater than the length of the third side. The contrapositive of this theorem also holds.
The angle between two intersecting lines, where m_1 and m_2 are the slopes of the lines, is given by

$$\theta = \arctan\left|\frac{m_2 - m_1}{1 + m_2 m_1}\right|$$

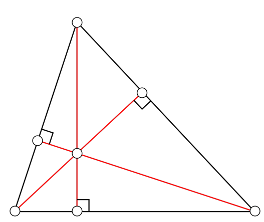
2.4 Construction with Triangles

A **median** of a triangle is a segment from a vertex to the midpoint of the opposite side. The **centroid** is the point at which the three medians of a triangle intersect and is also the centre of mass of the triangle. The **Centroid Theorem** states that the centroid is located $\frac{2}{3}$ of the distance from each vertex to the midpoint of the opposite side.



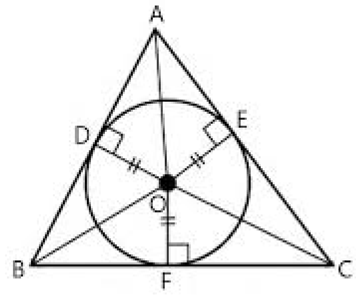
Medians and Centroid of a Triangle

An **altitude** of a triangle is a segment from a vertex constructed perpendicular to the opposite side. The **orthocenter** is the point at which the three altitudes of a triangle meet.



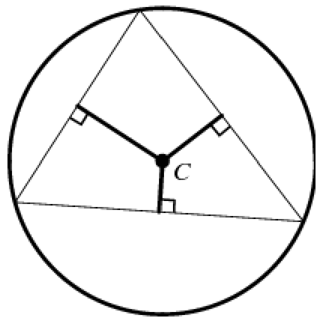
Altitudes and Orthocenter of a Triangle

The **incircle** is also known as the inscribed circle. The center of this circle is called the **incenter** and is equidistant from all sides of the triangle.



Incircle

The **circumcircle** is also known as the circumscribed circle.



Circumcircle

2.5 Congruent and Similar Triangles

The 5 Congruency Postulates:

1. SSS (Side, Side, Side)
If two triangles share three equal sides then the two triangles are congruent.

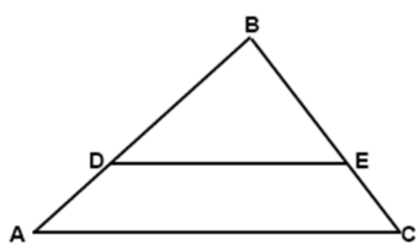
2. SAS (Side, Angle, Side)
If two triangles share two equal sides and an equal angle between said sides then the two triangles are congruent.

3. ASA (Angle, Side, Angle)
If two triangles share two equal angles and an equal side between said angles then the two triangles are congruent.

4. AAS (Angle, Angle, Side)
If two triangles share two equal angles and one equal side all consecutively, then the two triangles are congruent.

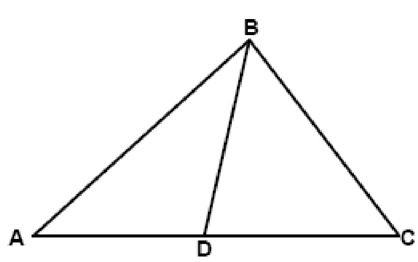
5. HL (Hypotenuse, Leg)
If two triangles share the same hypotenuse and any one of the other legs, then the two triangles are congruent.

Similar triangles have proportional corresponding sides. For example,



$$\frac{AD}{DB} = \frac{EC}{BE}$$

and



$$\frac{AD}{DC} = \frac{AB}{BC}$$

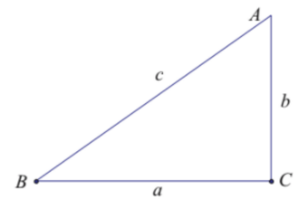
2.6 The Laws of Sine and Cosine

For any angles A,B, and C, the following definitions hold true.

$$\sin A = \frac{a}{c} \quad \cos A = \frac{b}{c} \quad \tan A = \frac{a}{b}$$
$$\csc A = \frac{c}{a} \quad \sec A = \frac{c}{b} \quad \cot A = \frac{b}{a}$$

$$\sin B = \frac{b}{c} \quad \cos B = \frac{a}{c} \quad \tan B = \frac{b}{a}$$

From the figure, it is easy to tell that $\sin A$ and $\csc A$, $\cos A$ and $\sec A$, and $\tan A$ and $\cot A$ are reciprocal functions. Hence, it is usually easier to just work with $\sin A$, $\cos A$, and $\tan A$ Additionally,



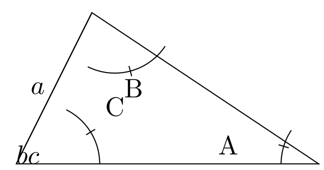
$$\frac{\sin A}{\cos A} = \tan A \quad \text{and} \quad \frac{\cos A}{\sin A} = \cot A$$

The Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

and

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



The Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

and

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

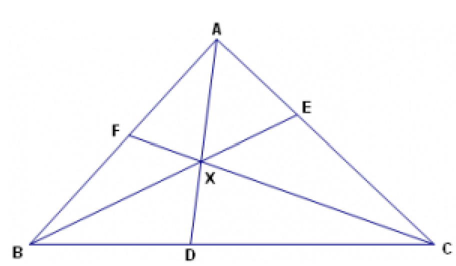
2.7 Ceva's Theorem

A **cevian** is a line that intersects both a triangle's vertex and the side opposite to said vertex. Medians and angle bisectors are special cases of cevians. **Ceva's Theorem** is a criterion for the concurrence of cevians in a triangle. **Concurrence** means that several lines or curves intersect at a certain point.

It states that if ABC is a triangle and D, E, F are all points on $\overline{BC}, \overline{CA}, \overline{AB}$, respectively, then $\overline{AD}, \overline{BE}, \overline{CF}$ are concurrent if and only if

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$

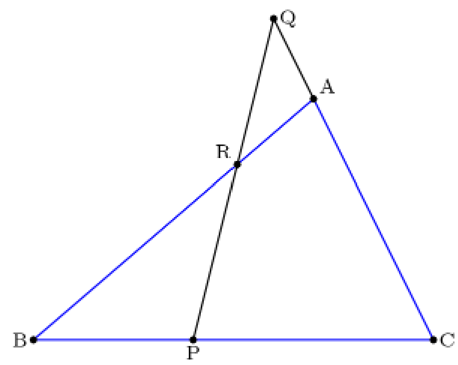
Since the reciprocal of 1 is 1, the reciprocals of the ratios also have a product of 1.



2.8 Menelaus' Theorem

Menelaus' Theorem describes the collinearity of points on each of the three sides of a triangle. It states that if $\overline{PQ}$ intersects $\overline{AB}$ on $\triangle ABC$ where P is on $\overline{BC}$, Q is the extension of $\overline{AC}$, and R is on the intersection of $\overline{PQ}$ and $\overline{AB}$, then

$$\frac{PB}{CP} \cdot \frac{QC}{QA} \cdot \frac{AR}{RB} = 1$$



3 Polygons

3.1 Polygon Basics

Polygons are closed 2D figures composed of straight lines. By this definition, any figure that has a curve or is open is not a polygon. A **regular polygon** has all angles and sides equal. An **irregular polgyon** is any polygon that is not regular. A **convex polygon** has no inwardly-pointing (greater than 180°) angles. A **concave polygon** is any polygon with at least one internal angle greater than 180°. A **simple polygon** does not self-intersect, whereas a **complex polygon** self-intersects.

Common Regular Polygons:

Name	Sides
Triangle (Trigon)	3
Quadrilateral (Tetragon)	4
Pentagon	5
Hexagon	6
Heptagon (Septagon)	7
Octagon	8
Nonagon (Enneagon)	9
Decagon	10
n-gon	n

Naming Prefixes and Suffixes:

Sides	Prefixes/Suffixes
20	Icosi-
30	Triaconta-
40	Tetraconta-
50	Pentaconta-
60	Hexaconta-
70	Heptaconta-
80	Octaconta-
90	Enneaconta-/Nonaconta-
100	Hecta-
+1	-henagon
+2	-digon
+3	-trigon
+4	-tetragon
+5	-pentagon
+6	-hexagon
+7	-heptagon
+8	-octagon
+9	-nonagon/-enneagon

Given a convex polygon with n sides, we can find the sum of the interior angles, S, such that

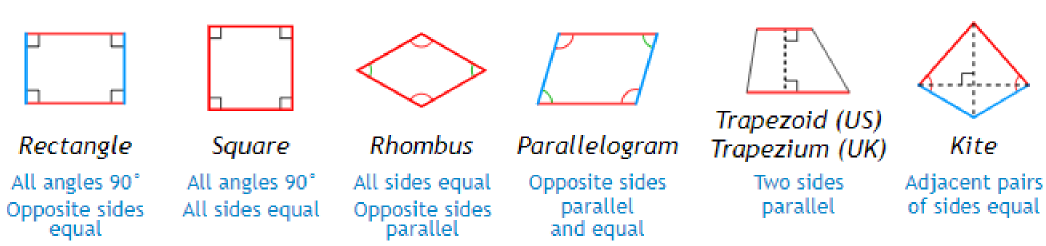
$$S = 180(n - 2)$$

The sum of all the exterior angles for a convex polygon always add to 360°.

3.2 Quadrilaterals

For quadrilaterals, the sum of all interior angles always add to 360°.

Types of Quadrilaterals:



The diagonals of a rhombus bisect each other. Therefore, if a parallelogram has diagonals that bisect each other, then the parallelogram is a rhombus. For kites, the angles where the two pairs meet are equal, the diagonals meet at right angles, and one of the diagonals bisects the other.

The square is the only regular quadrilaterals and all other quadrilaterals are irregular.

Quadrilateral Area Formulas:

$A = s^2$, $s =$ side	Square
$A = bh$, $b =$ base, $h =$ height	Rectangle
$A = \frac{pq}{2}$, p and q are diagonals	Rhombus
$A = bh$, $b =$ base, $h =$ height	Parallelogram
$A = h(\frac{b_1+b_2}{2})$, b_1 and b_2 are bases, $h =$ height	Trapezoid
$A = \frac{pq}{2}$, p and q are diagonals	Kite

4 Circles

4.1 Circle Basics

A **circle** is a 2D figure composed of all the points that are a given distance (**radius**) away from a centre. The **diameter** of a circle is the distance from one side of the circle to the other, passing through the centre of the circle. It is double the length of the radius. The **circumference** of a circle is the distance around the circle.

When d is the diameter of a circle, the circumference of said circle, C, is given by

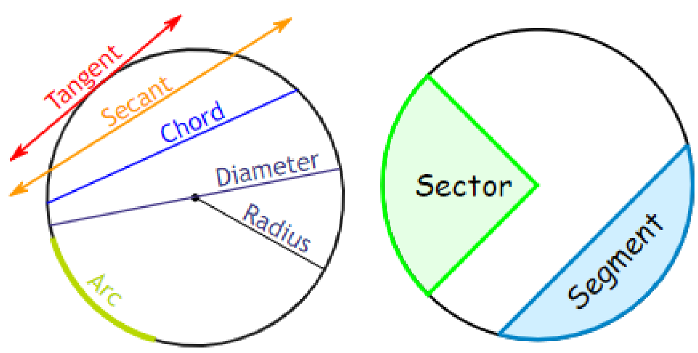
$$C = \pi d$$

The area of any circle is given by

$$A = \pi r^2$$

A circle has about 80% of the area of an equal-width square.

A **tangent** is a line that intersects the edge of a circle only once and is perpendicular at the intersection point. A line that intersects a circle twice is called a **secant**. A **chord** is a line segment travelling between any two points on a circle's circumference. A **diameter** is also a chord that passes through the centre of the circle. An **arc** is any section of the circumference. A **sector** is any part of a circle enclosed by two radii and their intercepted arc. A **segment** is any slice on a circle made by a chord. A quarter of a circle is a **quadrant** and half a circle is a **semicircle**.



The arc of a circle, s, where θ is the intercepted angle between the two radii composing the arc *measured in RADIANS*, is given by

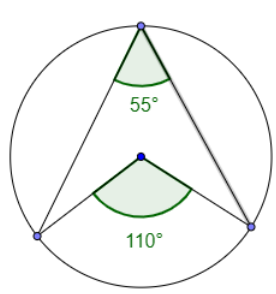
$$s = r\theta$$

The area of a sector, where θ is the intercepted angle between the two radii composing the sector *measured in RADIANS*, is given by

$$A = \frac{1}{2}r^2\theta$$

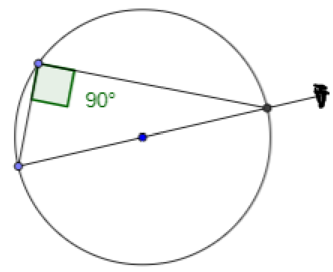
The 7 Common Circle Theorems:

1. Angle at the Centre Theorem



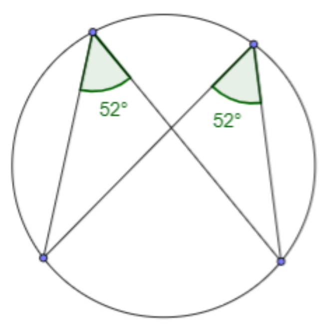
The angle formed at the centre of a circle by lines originating from any two points on the circle's circumference is double the angle formed on the circumference of the circle by lines originating from the same points.

2. Angle in a Semicircle Theorem



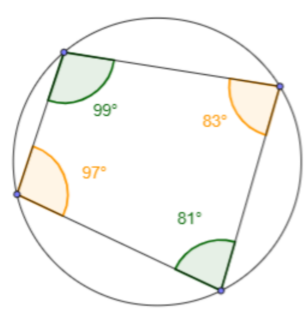
An angle formed by constructing lines from the ends of the diameter of a circle to its circumference form a right angle.

3. Angles in the Same Segment



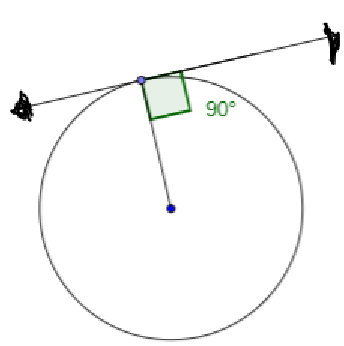
Angles formed from two points on the circumference are equal to other angles in the same arc formed from said points.

4. Cyclic Quadrilaterals



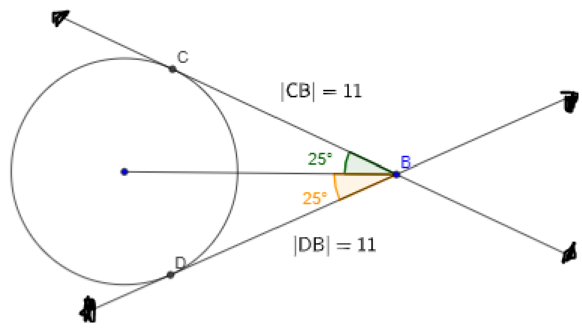
The opposite angles of a cyclic quadrilateral are equal.

5. Radius to a Tangent



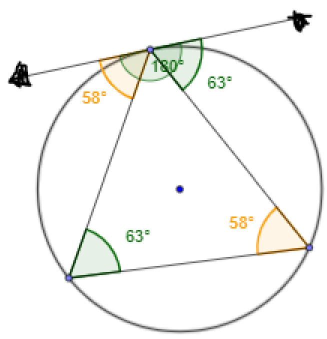
The radius and tangent that intercept any point on a circle's circumference form a right angle.

6. Tangents from a Point to a Circle



When two tangents are constructed on the same circle, the distance between their point of intersection with the circle and point of intersection with each other is the same. Furthermore, when a line from the intersection point of the two tangents to the centre of the circle is constructed, the angles formed between said line and either tangent are equal.

7. Alternate Segment Theorem

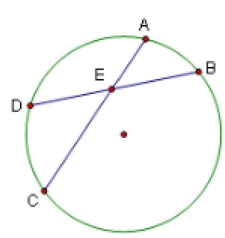


The alternate angles inside two segments constructed within a circle are equal.

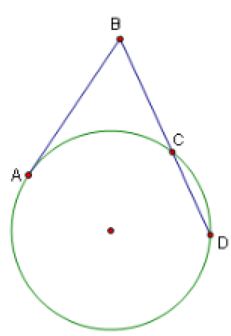
4.2 Power of a Point Theorem

The **Power of a Point Theorem** is a relationship holding between the lengths of the line segments formed when two lines intersect a circle and each other. There are three possibilities for which this theorem holds:

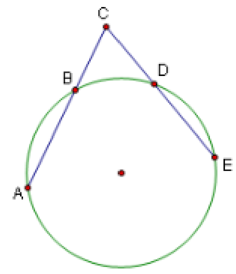
1. The two lines are chords of the circle and intersect inside the circle. In this case, we have $AE \cdot CE = BE \cdot DE$.



2. One line is a tangent and the other is a secant. In this case, we have $AB^2 = BC \cdot BD$.

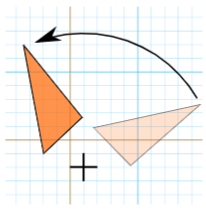


3. Both lines are secants and intersect outside the circle. In this case, $CB \cdot CA = CD \cdot CE$.

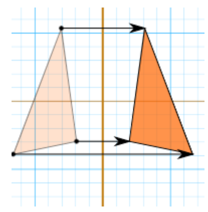


5 Transformations

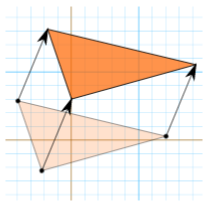
The 4 Main Geometrical Transformations:



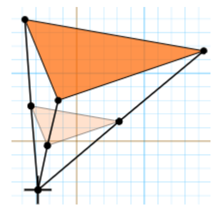
Rotation



Reflection



Translation



Homothety

When homothety is used to transform a figure, the figure and its result are similar. If a figure is transformed by any method besides homothety, the figure and its result are congruent.

6 Solids

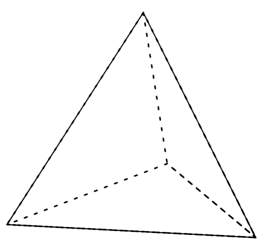
6.1 Basics of Solids

A **solid** is a 3D figure. A **face** of a solid is a flat surface. A **edge** of a solid is where 2 or more faces meet. A **vertex** is where 2 or more straight edges meet.

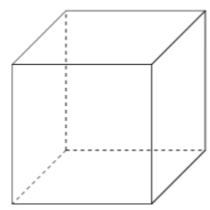
6.2 Polyhedra

A **polyhedron** is a solid composed only of flat faces and each face is a polygon. Common polyhedra include the platonic solids, prisms, and pyramids.

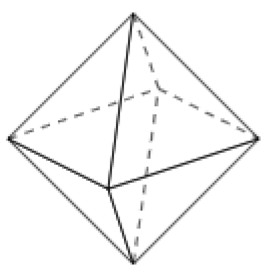
The **Platonic Solids** are solids where each face is the same regular polygon and the same number of polygons meet at each vertex. In $\mathbb{R}^3$ there are 5 Platonic Solids, of which each was theorized by Plato to represent the five elements: earth, air, fire, water, and the universe.



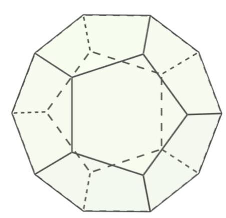
Tetrahedron



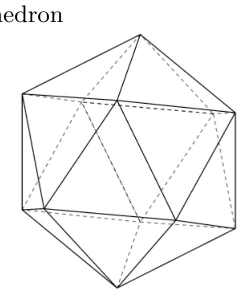
Cube



Octahedron



Dodecahedron



Icosahedron

A **prism** is a polyhedron with identical ends and the same cross-section along its length. Let B and h represent the base and height of a prism. Then

Volume of a Prism:

$$V = Bh$$

A **pyramid** is any polyhedron where there is a polygon base and all other faces are triangles. The other faces all connect the base to the **apex**.

Volume of a Pyramid:

$$V = \frac{Bh}{3}$$

6.3 Non-Polyhedrons

Nonpolyhedrons have curved surfaces. Common non-polyhedrons include spheres, cylinders, cones, and tori.

Spheres are 3D figures where all points on the surface are the same distance from a center. Spheres are perfectly symmetrical, have no edges nor vertices, and only have one surface. Of all the solids, spheres have the greatest volume to surface area ratio. A **hemisphere** is half of a sphere.

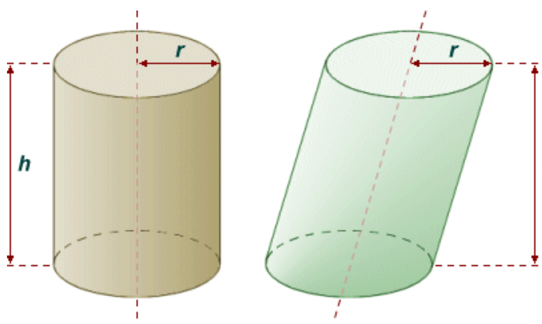
Surface Area of a Sphere:

$$A = 4\pi r^2$$

Volume of a Sphere:

$$\frac{4}{3}\pi r^3$$

A **cylinder** is a non-polyhedron with straight parallel sides and an ellipse cross-section. A **Right Cylinder** is a cylinder with its two bases directly aligned on top of each other. An **Oblique Cylinder** is any cylinder that is not right.



Right and Oblique Cylinders

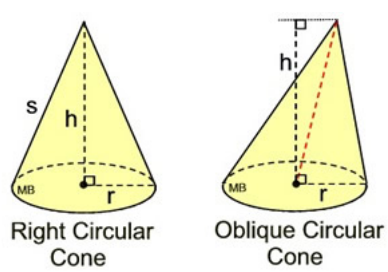
Surface Area of a Cylinder:

$$A = 2\pi rh + 2\pi r^2$$

Volume of a Cylinder:

$$V = \pi r^2 h$$

A **cone** is a non-polyhedron that tapers smoothly from a flat base to an **apex**. It can be formed by rotating a triangle 360° in along the dependent axis. When the apex of a cone is aligned with the centre of the base, it is a **right cone**. Otherwise, the cone is **oblique**.



Right and Oblique Cylinders

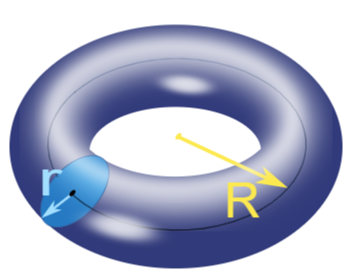
Surface Area of a Right Cone:

$$A = \pi r(r + \sqrt{h^2 + r^2})$$

Volume of a Cone:

$$V = \frac{\pi r^3 h}{3}$$

The **torus** is a solid of revolution generated by revolving a circle in $\mathbb{R}^3$ about an axis coplanar with the circle. Its shape resembles that of a donut. For tori, r is the smaller radius of the circle cross-section and R is the larger radius around which the circle is swept, such that



Surface Area of a Torus:

$$A = 4\pi^2 Rr$$

Volume of a Torus:

$$V = 2\pi^2 Rr^2$$