MATH134: Week 1 Assignment

Due on October 5, 2020 at 5:45 PM

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Section 1.2 Problem 70

Show that the sum of two rational numbers is a rational number.

Proof. Let x and y be two rational numbers. Since x is a rational number, by the definition of rational numbers, x can be expressed as the fraction of two integers p_1 and q_1, where $q_1 \neq 0$:

$$x = \frac{p_1}{q_1}$$

Similarly, y can be expressed as

$$y = \frac{p_2}{q_2}$$

where both p_2 and q_2 are integers, and $q_2 \neq 0$. Then the sum of x and y is given by

$$\begin{aligned} x + y &= \frac{p_1}{q_1} + \frac{p_2}{q_2} \\ &= \frac{p_1}{q_1} \cdot \frac{q_2}{q_2} + \frac{p_2}{q_2} \cdot \frac{q_1}{q_1} \\ &= \frac{p_1 q_2 + p_2 q_1}{q_1 q_2} \end{aligned}$$

Because p_1, q_1, p_2 and q_2 are all integers, where $q_1, q_2 \neq 0$, and integers are closed under multiplication and addition, therefore

$$p_1 q_2 + p_2 q_1$$

is also an integer and

$$q_1 q_2$$

is a nonzero integer. Because $x + y$ can be expressed as an integer divided by a nonzero integer, then by the definition of a rational number, $x + y$ is also a rational number.

□

Section 1.2 Problem 71

Show that the sum of a rational number and an irrational number is irrational.

Proof. Let x be a rational number and y be an irrational number, and let $z = x+y$. Suppose that z is a rational number, hence z can be expressed as

$$\begin{aligned} z &= x + y \\ \therefore y &= z + (-x) \end{aligned}$$

Because x is a rational number and $-x = -1 \cdot x$, where 1 is an integer and x is a rational number, therefore $-x$ is also a rational number. Since z is a rational number, therefore

$$y = z + (-x)$$

is also a rational number because rational numbers are closed under addition. This is proved for Section 1.2 Problem 70, listed above this proof. The assumption that $x + y = z$, where z is a rational number has led to a contradiction. We can conclude therefore that the expression $x + y$ must have an irrational value.

□

Section 1.2 Problem 74

Show by example that the sum of two irrational numbers (a) can be rational; (b) can be irrational. Do the same for the product of two irrational numbers.

Let x and y be two irrational numbers, and let $S = x + y$ and $P = xy$. Suppose that $x = 5 + 2\sqrt{13}$ and $y = -2\sqrt{13}$. Then

$$\begin{aligned} S &= x + y \\ &= (5 + 2\sqrt{13}) + (-2\sqrt{13}) \\ &= 5 \end{aligned}$$

Here, S is rational. Now suppose that $x = 5\sqrt{7}$ and $y = 13\sqrt{17}$. Then

$$\begin{aligned} S &= x + y \\ &= (5\sqrt{7}) + (13\sqrt{17}) \\ &= 5\sqrt{7} + 13\sqrt{17} \end{aligned}$$

Here, S is irrational. Hence, $x + y$ can be both rational and irrational.
Suppose that $x = \sqrt{3}$ and $y = \sqrt{7}$. Then

$$\begin{aligned} P &= xy \\ &= \sqrt{3} \cdot \sqrt{7} \\ &= \sqrt{21} \end{aligned}$$

Here, P is irrational. Now suppose that $x = \sqrt{2}$ and $y = \sqrt{18}$. Then

$$\begin{aligned} P &= xy \\ &= \sqrt{2} \cdot \sqrt{18} \\ &= \sqrt{36} \\ &= 6 \end{aligned}$$

Here, P is rational. Therefore, xy can be both rational and irrational.

Section 1.3 Problem 53

Show that

$$\big||a| - |b|\big| \leq |a - b| \forall a, b, \in \mathbb{R}$$

Proof. Because both $\big||a| - |b|\big| \geq 0$ and $|a - b| \geq 0$,

$$\big||a| - |b|\big|^2 \leq |a - b|^2$$

Because both $\big||a| - |b|\big|^2 \geq 0$ and $|a - b|^2 \geq 0$, we can remove the outside absolute value signs such that

$$\begin{aligned} \left(|a| - |b|\right)^2 &\leq (a - b)^2 \\ |a|^2 - 2|a||b| + |b|^2 &\leq a^2 - 2ab + b^2 \end{aligned}$$

Because $|a|^2 \geq 0$ and $|b|^2 \geq 0$, we can remove the absolute value signs from $|a|^2$ and $|b|^2$. Then we have

$$\begin{aligned} \cancel{a^2} - \cancel{2}|a||b| + \cancel{b^2} &\leq \cancel{a^2} - \cancel{2}ab + \cancel{b^2} \\ -|a||b| &\leq ab \end{aligned}$$

$$\text{For } \begin{cases} a = 0 \text{ and } b = 0, & -|a||b| = ab \\ a < 0 \text{ or } b < 0, & -|a||b| \leq ab \\ a < 0 \text{ and } b < 0, & -|a||b| < ab \\ a > 0 \text{ or } b > 0, & -|a||b| \leq ab \\ a > 0 \text{ and } b > 0, & -|a||b| < ab \end{cases}$$

Then it is true that

$$-|a||b| \leq ab$$

and hence

$$\big||a| - |b|\big| \leq |a - b| \forall a, b, \in \mathbb{R}$$

□

Section 1.3 Problem 58

Given that $0 \le a \le b$, show that
$$a \le \sqrt{ab} \le \frac{a+b}{2} \le b.$$

Proof. Given
$$a \le b, a \ge 0,$$
then
$$\begin{aligned} a^2 &\le ab \\ a &\le \sqrt{ab} \end{aligned}$$
Suppose that
$$\begin{aligned} \sqrt{ab} \le \frac{a+b}{2} \iff ab &\le \left(\frac{a+b}{2}\right)^2 \\ ab &\le \frac{(a+b)^2}{4} \\ 4ab &\le a^2 + 2ab + b^2 \\ 0 &\le a^2 - 2ab + b^2 \\ 0 &\le (a-b)^2 \end{aligned}$$
Because any real number squared has a nonnegative value, thus it is true that
$$0 \le (a-b)^2$$
and so
$$\sqrt{ab} \le \frac{a+b}{2}$$
Recall that
$$\begin{aligned} a &\le b \\ a+b &\le 2b \\ \frac{a+b}{2} &\le b \end{aligned}$$
Hence,
$$a \le \sqrt{ab} \le \frac{a+b}{2} \le b$$
□

Section 1.4 Problem 52

The perpendicular bisector of the line segment $\overline{PQ}$ is the line which is perpendicular to $\overline{PQ}$ and passes through the midpoint of $\overline{PQ}$. Find an equation for the perpendicular bisector of the line segment that joins points $P(-1,3)$ and $Q(3,-4)$.

The midpoint, M, of the line segment $\overline{PQ}$ is given by
$$\begin{aligned} M &= \left(\frac{-1+3}{2}, \frac{-4+3}{2}\right) \\ &= \left(1, -\frac{1}{2}\right) \end{aligned}$$
The slope, m, of the line segment $\overline{PQ}$ is given by
$$\begin{aligned} m &= \frac{-4-3}{3-(-1)} \\ &= -\frac{7}{4} \end{aligned}$$

The opposite of the reciprocal of $-\frac{7}{4}$ is $\frac{4}{7}$, and so the slope of the perpendicular bisector of $\overline{PQ}$ is also $\frac{4}{7}$. An equation for this perpendicular bisector could be written as

$$y + \frac{1}{2} = \frac{4}{7}(x-1)$$

Section 1.7 Problem 59

Show that every function defined for all real numbers can be written as the sum of an even function and an odd function.

Proof. Suppose

$$f(x) = e(x) + o(x),$$

where $e(x)$ is an even function and $o(x)$ is an odd function, both of which are defined for all real numbers. By the definitions of even and odd functions, we also have

$$\begin{aligned} f(-x) &= e(-x) + o(-x) \\ &= e(x) - o(x) \end{aligned}$$

Solving the following system of equations,

$$\begin{cases} f(x) & = e(x) + o(x) \\ f(-x) & = e(x) - o(x) \end{cases}$$

it follows that we can express $e(x)$ and $o(x)$ in terms of $f(x)$:

$$\begin{aligned} 2e(x) &= f(x) + f(-x) \\ e(x) &= \frac{f(x) + f(-x)}{2} \\ 2o(x) &= f(x) - f(-x) \\ o(x) &= \frac{f(x) - f(-x)}{2} \end{aligned}$$

Suppose $e(-x) = e(x)$. Then

$$\begin{aligned} e(-x) &= e(x) \\ \frac{f(-x) + f(-(-x))}{2} &= \frac{f(x) + f(-x)}{2} \\ \frac{f(-x) + f(x)}{2} &= \frac{f(x) + f(-x)}{2} \end{aligned}$$

Thus, by the definition of even functions, $e(x)$ is an even function. Now suppose $o(-x) = -o(x)$. Then

$$\begin{aligned} o(-x) &= -o(x) \\ \frac{f(-x) - f(-(-x))}{2} &= -\frac{f(x) - f(-x)}{2} \\ \frac{f(-x) - f(x)}{2} &= \frac{f(-x) - f(x)}{2} \end{aligned}$$

Thus, by the definition of an odd function, $o(x)$ is an odd function. Again, suppose

$$\begin{aligned} f(x) &= e(x) + o(x) \\ &= \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} \\ &= \frac{2f(x)}{2} \\ &= f(x) \end{aligned}$$

Hence, a function defined for all real numbers can be written as the sum of an even function and an odd function. □

Section 1.8 Problem 6

Show that the statement

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$

holds for all positive integers n.

Before the proof for this statement, below first is a proof that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \forall n \in \mathbb{Z}^+$$

which will be implemented in the proof for the current problem:

Proof. Let S be the set of positive integers n for which

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Then $1 \in S$ since

$$1 = \frac{1(1+1)}{2}$$

Assume that $k \in S$; that is, assume that

$$1 + 2 + 3 \cdots + k = \frac{k(k+1)}{2}$$

Adding up the first $k + 1$ integers, we have

$$\begin{aligned} 1 + 2 + 3 + \cdots + k + (k+1) &= [1 + 2 + 3 + \cdots + k] + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1), \text{ by the induction hypothesis} \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

and so $k + 1 \in S$. Thus, by the principle of induction, we can conclude that all positive integers are in S; that is, we can conclude that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \forall n \in \mathbb{Z}^+$$

□

Having proved the above statement, we will now begin proving the actual Section 1.8 Problem 6:

Proof. Let S be the set of positive integers n for which

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$

Then $1 \in S$ since

$$1^3 = 1^2$$

Assume that $k \in S$; that is, assume that

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = (1 + 2 + 3 + \cdots + k)^2$$

Summing up the cubes of the first $k + 1$ integers, we have

$$\begin{aligned} 1^3 + 2^3 + 3^3 + k^3 + (k+1)^3 &= [1^3 + 2^3 + 3^3 + \cdots + k^3] + (k+1)^3 \\ &= (1 + 2 + 3 + \cdots + k)^2 + (k+1)^3 \\ &= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \\ &= \frac{k^2(k+1)^2 + 4(k+1)^2(k+1)}{4} \\ &= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(k+1)^2(k^2+4k+4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2 \\
&= (1+2+3+\cdots+(k+1))^2
\end{aligned}
$$

Therefore, by the principle of mathematical induction, we can conclude that all positive integers are in S; that is, we can conclude that

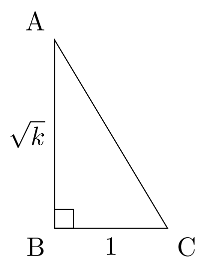
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1+2+3+\cdots+n)^2, \forall n \in \mathbb{Z}^+$$

$\square$

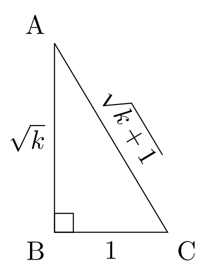
Section 1.8 Problem 18

Show that, given a unit length, for each positive integer n, a line segment of length $\sqrt{n}$ can be constructed by straight edge and compass.

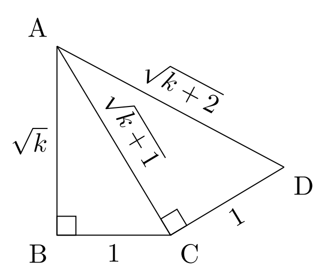
Proof. Let set S be composed of positive integers n for which a line segment of length $\sqrt{n}$ can be constructed by straight edge and compass. Then $1 \in S$ because $\sqrt{1} = 1$, hence a line segment of length $\sqrt{1}$ can be directly constructed by straight edge. Assume that $k \in S$; that is, assume that a right triangle ΔABC with legs of length 1 and $\sqrt{k}$ can be constructed, using the compass to make the right angle, like so:



By the Pythagorean Theorem, the hypotenuse of ΔABC is $\sqrt{k+1}$, as depicted in the below figure.



By the induction hypothesis, we can construct a right triangle using a compass and straightedge with legs of length 1 and $\sqrt{k+1}$. We can do this by constructing a ray $\overrightarrow{CD}$ of length 1 and angle measure $\angle ACB + 270°$ and connect points A and D, thus forming a triangle ΔACD with legs of length 1 and $\sqrt{k+1}$. Again by the Pythagorean Theorem, the hypotenuse of ΔACD is of length $\sqrt{(k+1)+1} = \sqrt{k+2}$.



Because the line segment $\overline{AD}$ of length $\sqrt{k+2}$ was constructed by straight edge and compass using the line segment $\overline{AD}$ of length $\sqrt{k+1}$ as a reference, thus $k+1 \in S$.

Hence, by the axiom of induction, we can conclude that all positive integers are in S; that is, we can conclude that, given a unit length, for each positive integer n, a line segment of length $\sqrt{n}$ can be constructed by straight edge and compass.

$\square$