MATH134 Notes

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Contents

1 Introduction to Proofs

Sets of Numbers:

$\mathbb{C}$	**Complex Numbers**
$\mathbb{A}$	**Algebraic Numbers**
$\mathbb{R}$	**Real Numbers**
$\mathbb{Q}$	**Rational Numbers**
$\mathbb{Z}$	**Integers**
$\mathbb{N}$	**Natural Numbers**

Real numbers that are not **rational** are **irrational**:
- **Algebraic numbers:** solutions to polynomials with integer coefficients ($x^2 - 2 = 0, 2x^2 - 3 = 0$)
- **Transcendentals**: $e, \pi, \ldots$

The decimal expansion of an irrational number is eventually repeating $\left(\text{e.g. } 1.312312312\cdots = \frac{p}{q}\right)$. Below is a cool trick to find p and q given an irrational number.

$$x = 1.312312312\cdots = \frac{p}{q}$$
$$1000x = 1312.312312\ldots$$
$$1000x = 1312.000000\ldots$$
$$999x = 1311$$
$$x = \frac{1311}{999}$$
$$1.312312312\ldots = \frac{1311}{999}$$

Proof Guidelines
- Start with what you know or what you are given and end with what you need to show
- Justify every step unless it is a clear algebra step by:
- Algebra/computation from a previous step
- Reference to a previous result or step in the proof
- Proofs should read like a paragraph. Use words as much as equations and computations.
- So, then, therefore, let, given, we conclude, hence,... to make it flow.
- Keep your audience in mind

Below is an example of a well-written proof for the Triangle Inequality:

> **Triangle Inequality**
>
> For any $a, b, \in \mathbb{R}$, $|a + b| \leq |a| + |b|$

Proof. Let $a, b, \in \mathbb{R}$. Then,

$$|a + b|^2 = a^2 + 2ab + b^2$$
$$= |a|^2 + 2ab + |b|^2, \sin(u)\ x^2 = |x|^2 \forall x$$
$$\leq |a|^2 + 2|ab| + |b|^2, \sin(u)\ x \leq |x| \forall x$$
$$= |a|^2 + 2|a||b| + |b|^2$$
$$= (|a| + |b|)^2$$

So, $(a + b)^2 \leq (|a| + |b|)^2 \forall a, b, \in \mathbb{R}$. Now, taking square roots and using $|x| = \sqrt{x^2}$,, we get

$$|a + b| \leq ||a| + |b||$$

Since $|a| + |b| \geq 0$, we can remove the absolute sign on the right.

$$|a + b| \leq |a| + |b|$$

$\square$

A **theorem** is an implication, consisting of a **hypothesis** and a **conclusion** in the form of:

if (hypothesis) ..., then (conclusion) ...

A common mistake in writing proofs is to ignore the hypothesis and follow directly with the conclusion, for example, to say that $ab > 0$ simply because a and b are numbers. Another common mistake is to assume that a theorem's converse is true as well. A third mistake is to assume that that a theorem's hypothesis is the only condition under which the conclusion is true. In other words, it is common to mistakenly write "iff", where **"iff"** is an abbreviation of "if and only if". Note that definitions by their nature are "iff statements", hence the "only if" part is convetionally omitted and assumed to hold.

A **predicate** in mathematical logic is any statement that has a boolean value depending on the values of its variables.

Direct Proof: if we wanted to prove

$$\text{if } A, \text{ then } B$$

we could assume that A is true and use this information to deduce that B is true.

Contrapositive Proof: We can use the concept that if A, then B is true, then it is impossible for A to be true while B is false. Hence, we prove the statement "If A, then B" by showing that if B is false, then A is false too.

Proof by Contradiction: if we wanted to prove

$$\text{if } A, \text{ then } B$$

we could assume that

$$A \text{ holds and } B \text{ does not hold} \tag{1}$$

and then arrive at a contradiction. We can take this contradiction to mean that (1) is a false statement and therefore:

$$\text{if } A \text{ holds, then } B \text{ must hold.}$$

Example 1: Show that

$$\text{if } a > b > 0, \text{ then } \frac{1}{a} < \frac{1}{b}$$

Proof. Assume $a > b > 0$. For a contradiction, assume $\frac{1}{a} > \frac{1}{b}$. Since $a > 0$, multiplying both sides by a gives

$$1 \geq \frac{a}{b}$$

Since $b > 0$, using similar reasoning,

$$b \geq a$$

This assumption contradicts $a > b$, hence the assumption was wrong and $\frac{1}{a} < \frac{1}{b}$. $\qquad\square$

Example 2: Show that $\sqrt{2}$ is irrational.

Proof. Assume

$$\sqrt{2} = \frac{p}{q}$$

where p and q have no common factor and $q \neq 0$. We now square both sides of the equality:

$$2 = \frac{p^2}{q^2}$$

Since $2q^2 = p^2$, p^2 is even. Hence, p is even and $p = 2r$ for some integer r.

$$\implies 2q^2 = 4r^2$$
$$q^2 = 2r^2$$

Since the equality above is even, so is q. So $p + q$ both have a common factor of 2, contradicting that they had no common factors. Therefore the assumption that $\sqrt{2}$ was rational was wrong and it is irrational. $\qquad\square$

Mathematical induction is a proof method that can be used to show that a certain property holds for all positive integers n, or in some cases for all positive integers $n \geq a$.

> **Axiom of Induction**
>
> Let $S \subset \{1, 2, 3, \dots\}$, or let S be the set of n for which a desired property holds. If
> - $1 \in S$
> - $k \in S \implies k + 1 \in S$.
>
> Then S contains all positive integers hence it is equal to the set of all positive integers.
>
> The substitution of "$k + 1$" into the desired predicate is called the **induction hypothesis.** In your proofs by induction, you should always put the phrases "*by the induction hypothesis*", and "*Therefore, by the principle of mathematical induction, the formula holds for all n.*"

The axiom of induction can be thought of as an application of domino theory; if the first domino in a row of dominoes falls, and each domino that falls causes the next one to fall, then, according to the axiom of induction, each domino in that row of dominoes will fall. Note that an induction does not necessarily have to begin with the integer 1. If, for example, you wished to show that some proposition is true for all integers $n \geq 3$, you would just have to show that said proposition is true for $n = 3$ and that, if is true for $n = k$, then it is true for $n = k + 1$.

Example 3: Show that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \forall n, n \in \mathbb{Z}^+$$

Proof. Let S be the set of positive integers n for which

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Then $1 \in S$ since

$$1 = \frac{1(1+1)}{2}$$

Assume that $k \in S$; that is, assume that

$$1 + 2 + 3 \cdots + k = \frac{k(k+1)}{2}$$

Adding up the first $k + 1$ integers, we have

$$
\begin{aligned}
1 + 2 + 3 + \cdots + k + (k+1) &= [1 + 2 + 3 + \cdots + k] + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \text{ *by the induction hypothesis} \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}
$$

and so $k + 1 \in S$. Thus, by the axiom of induction, we can conclude that all positive integers are in S; that is, we can conclude that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \forall n, n \in \mathbb{Z}^+$$

$\square$

Example 4: Show that, if $x \geq -1$, then

$$(1 + x)^n \geq 1 + nx \forall n, n \in \mathbb{Z}^+$$

Proof. Take $x \geq -1$ and let S be the set of positive integers n for which

$$(1 + x)^n \geq 1 + nx$$

Since

$$(1 + x)^1 = 1 + 1 \cdot x,$$

we have $1 \in S$. Assume that $k \in S$. By the definition of S,

$$(1 + x)^k \geq 1 + kx$$

Since

$$(1 + x)^{k+1} = (1 + x)^k (1 + x) \geq (1 + kx)(1 + x)$$

and

$$(1 + kx)(1 + x) = 1 + (k+1)x + kx^2 \geq 1 + (k+1)x,$$

we can conclude that

$$(1 + x)^{k+1} \geq 1 + (k+1)x$$

and thus $k + 1 \in S$. We have shown that $1 \in S$ and that $k \in S \implies k + 1 S$. Then by the axiom of induction, all positive integers are in S. $\square$

2   Limits and Continuity

2.1   The $\epsilon - \delta$ Definition of the Limit

> **$\epsilon - \delta$ Definiton of the Limit of a Function**:
> Let $f(x)$ be a function defined on an open interval around c. Then the limit of $f(x)$ as x approaches c is L:
> $$\lim_{x \to c} f(x) = L,$$
> if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\forall x$
> $$0 < |x - c| < \delta \implies |f(x) - L| < \epsilon$$
> or in logic notation:
> $$\forall \epsilon > 0 \exists \delta > 0 \ni |x - c| < \delta \implies |f(x) - L| < \epsilon$$
> ϵ is any given epsilon arbitrarily close to L, where δ is a value we can choose.

Example 1: Show that $\lim_{x \to 2} 5x - 7 = 30$.

Let ϵ be given (arbitrary). Define $\delta = \frac{\epsilon}{5}$. Assume $|x - 2| < \delta = \frac{\epsilon}{5}$.

Then

$$|(5x - 7) - 3| = |5x - 10| = 5|x - 2| < 5 \cdot \frac{\epsilon}{5} = \epsilon$$

In this problem, we had to set up the inequality at the bottom first to get the value of our δ. Then, we plugged it back into the earlier part os the proof and worked downwards again.

Example 2: Show that $\lim_{x \to 2} x^2 + 5 = 9$.

Let $\epsilon > 0$ be given.
Then,

$$|x^2 + 5 - 9| = |x^2 - 4| = |x + 2||x - 2|$$

We want

$$|x + 2||x - 2| < \epsilon$$

We can easily control $|x - 2|$, but not $|x + 2|$. Then

$$2 - \delta < x < 2 + \delta$$
$$4 - \delta < x + 2 < 4 + \delta$$

$x + 2$ cannot get too large. For example, if $\delta = 1$ then

$$3 < x + 2 < 5$$
$$\implies |x + 2| < 5$$

So we have to make sure that $\delta \leq 1$. So going back to the beginning, we want

$$|x + 2||x - 2| \leq 5|x - 2| \text{ if } \delta = \frac{\epsilon}{5}$$

We define $\delta = \min\{1, \frac{\epsilon}{5}\}$ if $|x - 2| < \delta$. Then $|x - 2| < 1$ so $-1 < x - 2 < 1$ and $1 < x < 3$ then $3 < x + 2 < 5$.

Then,

$$|x^2 + 5 - 9| = |x^2 - 4| = |x + 2||x - 2| < 5|x - 2| < 5 \cdot \frac{\epsilon}{5} = \epsilon$$

Example 3: Show that $lim_{x \to \pi} x = \pi$

Proof. Let $f(x) = x$. First, we need to determine what value our δ will take. When $|x - \pi| < \delta$ we want $|f(x) - \pi| < \epsilon$. We know that $|f(x) - \pi| = |x - \pi| < \epsilon$, so taking $\delta = \epsilon$ will have the desired property. There are other values of δ we could have chosen, such as $\delta = \frac{\epsilon}{7}$. Why would this value of δ have also been acceptable? If $|x - \pi| < \delta = \frac{\epsilon}{7}$, then $|f(x) - \pi| < \frac{\epsilon}{7} < \epsilon$, as required. $\square$

Example 4: Show that $\lim_{x\to 1}(5x - 3) = 2$

Proof. In this example, we have $c = 1$, $f(x) = 5x - 3$, and $L = 2$ from the definition of the limit. For any $\epsilon > 0$, we would need to find $\delta > 0$ such that if x is within distance δ of $c = 1$, **i.e**

$$|x - 1| < \delta$$

then $f(x)$ is within distance ϵ of $L = 2$, **i.e**

$$|f(x) - 2| < \epsilon$$

To find δ, we work backwards from the ϵ inequality:

$$
\begin{aligned}
|(5x - 3) - 2| = |5x - 5| &< \epsilon \\
= 5|x - 1| &< \epsilon \\
= |x - 1| &< \frac{\epsilon}{5}
\end{aligned}
$$

So we choose $\delta = \frac{\epsilon}{5}$. Then we can verify that if $|x - 1| < \delta = \frac{\epsilon}{5}$, then

$$|(5x - 3) - 2| = |5x - 5|5|x - 1| < 5\left(\frac{\epsilon}{5}\right) = \epsilon.$$

$\square$

Example 5: Prove that $\lim_{x\to 7}(x^2 + 1) = 50$.

Proof. Let $f(x) = x^2 + 1$. We will first determine what our value of δ should be. When $|x - 7| < \delta$ we have

$$
\begin{aligned}
|x^2 + 1 - 50| = |x^2 - 49| \\
= |x - 7||x + 7| \\
< \delta|x + 7|.
\end{aligned}
$$

Assuming $|x - 7| < 1$, we have $|x| < 8$, which implies $|x + 7| < |x| + |7| = 15$ by the triangle inequality. So, when we let $\delta = \min\left(1, \frac{\epsilon}{15}\right)$, we will have

$$
\begin{aligned}
|x^2 + 1 - 5| &< \delta|x + 7| \\
&< 15\delta \\
&< \epsilon.
\end{aligned}
$$

$\square$

Example 6: Prove

$$\lim_{x\to 0}\frac{1}{x^2} = \infty$$

Proof. We show that for any positive number L, set $\delta = \frac{1}{\sqrt{L}}$. Then, when $|x - 0| < \delta$, we have

$$f(x) = \frac{1}{x^2} > \frac{1}{\left(\frac{1}{\sqrt{L}}\right)^2} = L.$$

This shows that the values of the function become and stays arbitrarily large as x approaches zero, or

$$\lim_{x\to 0}\frac{1}{x^2} = \infty.$$

$\square$

blue*Example 7:* Consider the function given by

$$f(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0. \end{cases}$$

Show that the limit at 0 does not exist.

Proof. Note that the right-hand limit is 1 and the left-hand limit is -1. As such, it makes senses that the limit does not exist. We can formally show this by supposing that the limit at 0 exists and is equal to L. Let $\epsilon = \frac{1}{2}$, with a corresponding $\delta = \delta_\epsilon > 0$. Since the limit exists, we know that $for all y \in (-\delta, \delta)$, we have $|f(y) - L| < \epsilon = \frac{1}{2}$. HOwever, we also have

$$2 = \left| f\left(\frac{\epsilon}{2}\right) - f\left(-\frac{\epsilon}{2}\right) \right|$$
$$= \left| f\left(\frac{\epsilon}{2}\right) - L + L - f\left(-\frac{\delta}{2}\right) \right|$$
$$\leq \left| f\left(\frac{\delta}{2}\right) - L \right| + \left| L - f\left(-\frac{\delta}{2}\right) \right|$$
$$\leq \frac{1}{2} + \frac{1}{2}$$

This is a contradiction, so our original assumption is not true. $\square$

2.2 Limit Theorems

If $lim_{x \to c} f(x) = L$ and $lim_{x \to c} g(x) = M$, then prove that

$$\lim_{x \to c} [f(x) + g(x)] = L + M$$

Proof. Assume that $|x - c| < \delta$, then

$$|f(x) + g(x) - (L + M)| \leq |f(x) - L| + |g(x) - M|$$

Since $lim_{x \to c} f(x) = L$, for $\frac{\epsilon}{2}$ there exists δ, such that $|x - c| < \delta$, implies $|f(x) - L| < \frac{\epsilon}{2}$. Since $lim_{x \to c} g(x) = M$, define $\delta = \min\{\delta_1, \delta_2\}$. Now we can go back to the beginning. Then,

$$|f(x) + g(x) - (L + M)| \leq |f(x) - L| + |g(x) - M|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\square$

If $lim_{x \to c} f(x) = L$ and $lim_{x \to c} g(x) = M$, then prove that

$$\lim_{x \to c} f(x)g(x) = LM$$

Hint:

$$|f(x)g(x) - LM| = |f(x)g(x) - f(x)M + f(x)M - LM|$$
$$\leq |f(x)||g(x) - M| + |f(x) - L||M|$$

Proof. $\square$

> **Corollary**
>
> If $p(x) = a_n x^n + \cdots + a_1 x + a_0$ is a polynomial, then $lim_{x \to c} p(x) = p(c)$.

2.3 The Squeeze Theorem

An $\epsilon - \delta$ Proof for the Squeeze Theorem:

Proof. Let $\epsilon > 0$.
Since
$$h(x) \leq f(x) \leq g(x),$$
we have
$$h(x) - L \leq f(x) - L \leq g(x) - L.$$
Since
$$-g(x) \leq -f(x) \leq -h(x)$$
we have
$$L - g(x) \leq L - f(x) \leq L - g(x)$$
so we get
$$|f(x) - L| \leq \max\{|h(x) - L|, |g(x) - L|\}$$
There is a $\delta_1 > 0$ such that $|x - c| < \delta \implies |h(x) - L| < \epsilon$.
There is a $delta_2 > 0$ such that $|x - c| < d_2 \implies |g(x) - L| < \epsilon$.
Let $\delta = \min\{d_1, d_2\}. If$—x-c— ¡ δ then $|f(x) - L| \implies \epsilon$. $\square$

2.4   More Limit Theorems

The Intermediate Value Theorem:
If f is continuous on $[a, b]$ and K is any number between $f(a)$ and $f(b)$, then there is at least one number c in the open interval (a, b) such that $f(c) = K$.

The Extreme Value Theorem:
If f is continuous on a bounded closed interval $[a, b]$, then on that interval f takes on both a maximum value M and a minimum value m.

Let S be a subset of the set of real numbers. M is an **upper bound** for S if $x \leq M$ for all $x \in S$. M is the **least upper bound** for S if it is the smallest number with this property, i.e. any number smaller than M, any $M - \epsilon$ for any $\epsilon > 0$, is not an upper bound for S.

8 The **Least Upper Bound Axiom:** Every nonempty set of real numbers which is bounded above has a least upper bound.

3 Derivatives

3.1 Definition of Derivative

A function f is differentiable at x if the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists.

3.2 Derivative Rules

Proof of Product Rule:

Proof.

$$\lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$
$$= \lim_{h \to 0} f(x+h) \left[\frac{g(x+h) - g(x)}{h} \right] + g(x) \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= f(x)g'(x) + g(x)f'(x)$$

$\square$

4 Applications of Derivatives

4.1 First and Second Derivative Tests

First Derivative Test

Suppose that f is a function and c is a critical point of f and f is continuous at c.
- If $f'(x)$ changes sign from positive to negative at c, then $f(c)$ is a local maximum.
- If $f'(x)$ changes sign from negative to positive at c, then $f(c)$ is a local minimum.
- If $f'(x)$ does not change sign at c, then $f(c)$ is not a local extrema.

Second Derivative Test

Suppose that f is a function and c is a critical point of f and f is differentiable at c. Assume also that $f''(c)$ exists
- If $f''(c) > 0$ then f has a local minimum at c.
- If $f''(c) < 0$ then f has a local maximum at c.
- If $f''(c) = 0$ then the test is inconclusive.

4.2 Limits at Infinity and Asymptotes

A limit that approaches infinity is defined for any $K > 0$, there exists $\delta > 0$ such that

$$|x - c| < \delta \implies f(x) > K$$

Definition for $\lim_{x \to \infty} f(x) = L$:

For any $\epsilon > 0$ there exists a $K > 0$ such that $x > K$ implies $|f(x) - L| < \epsilon$.

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Given $\epsilon > 0$, let $k = \frac{1}{\epsilon}$, then $x > K = \frac{1}{\epsilon} \implies$

$$\left| \frac{1}{x} - 0 \right| < \epsilon$$
$$D\frac{1}{\epsilon} < x$$

4.3 Differentials and Linear Approximations

4.4 Newton-Raphson Approximations

$$-\frac{f(x_n)}{f'(x_n)} + x_n = x_n + 1$$

We can prove $|x_{n+1} - c| < |x_n - c|$ using them on concavity, proving $x_n \to c$

5 Integration

5.1 Riemann Sums

In both problems, we are going to take the intervals Δx or Δt to 0 and hope that the sums have a *limit*. To guarantee the *existance* of that limit and hence of the *integral*, we need some conditions:

Definition: A function f is **uniformly continuous on an interval** I if:
For every $\epsilon > 0$ there is a $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$ for any x, y in the interval I.

Example: The function $f(x) = \frac{1}{x}$ is continuous on $I = (0, 1)$ but not uniformly continuous there.

5.2 The Definition of the Integral

The Integrability Theorem Definition of the Riemann Integral:
If f is continuous on $[a, b]$, then there is a unique number $I = \int_a^b f(x)dx$ such that $L_f(P) \leq I \leq U_f(P)$ for any partition P. L is the lower bound and U is the upper bound.

Existence:

Since $\{L_f(P) : P \text{ partition}\}$ is bounded above, it has a least upper bound, say L.

$\{U_f(P) : P \text{ partition}\}$ is bounded below it has a greatest lower bound, say U.

Any number I, $L \leq I \leq U$ satisfies existence criteria.

Uniqueness: Claim: $L = U$ so I is unique.

First, f is uniformly continuous on $[a,b]$.
(We'll show $0 \leq U - L < \varepsilon$ for any $\varepsilon > 0$)

Let $\varepsilon > 0$ be given. There is a $\delta > 0$ such that $|x-y| < \delta$ implies $\boxed{|f(x) - f(y)| < \frac{\varepsilon}{b-a}}$.

Find $n \in \mathbb{Z}^+$ such that $\frac{b-a}{n} < \delta$, $\Delta x = \frac{b-a}{n}$

$x_i = x_0 + (i-1)\Delta x$

$P = \{x_0, x_1, \ldots, x_n\}$

$U_f(P) - L_f(P) = (M_1 - m_1)\Delta x + (M_2 - M_2)\Delta x + \cdots + (M_n - m_n)\Delta x$

$= \sum_{i=1}^{n} (M_i - m_i)\Delta x$

$< \sum_{i=1}^{n} \frac{\varepsilon}{b-a} \cdot \frac{(b-a)}{n} = \varepsilon \sum_{i=1}^{n} \frac{1}{n} = \varepsilon$

$L_f(P) \leq L \leq U \leq U_f(P)$

So, $0 \leq U - L \leq U_L(P) - L_f(P) < \varepsilon$.

So, $U = L$ and that is our <u>unique</u> number

$I = \int_a^b f(x) \, dx$

(height) (width)

due to Leibniz + it stands for $\underline{\underline{\int}}$um.

5.3 u-Substitution

u-Substitution is the reverse of the chain rule.

Example 1: $\int 2x(x^2 + 4)^{2020} dx$

$$u = x^2 + 4, du = 2xdx$$

Then

$$\int u^{2020} du = \frac{u^{2021}}{2021} + C = \frac{(x^2 + 4)^{2021}}{2021} + C$$

6   Applications of Integration

6.1   General Steps for Applications of Integration Problems

1. Chop up the quantity $Q = \sum \Delta Q$
2. Estimate ΔQ
3. Sum dQ by integrating

Example: A rod 6 metres long is placed on the x-axis from $x = 0$ to $x = 6$. The mass density function is $\lambda(x) = \frac{12}{\sqrt{x+1}}$.

- *What is the total mass?*
- *Where is the centre of mass?*
- *What is the average density?*

fill in later

6.2   Centre of Mass

For a rod with varying density,

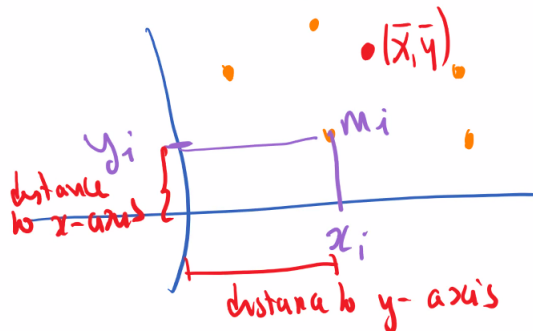$$\text{Centre of mass } = \frac{\text{moment}}{\text{total mass}},$$

where

$$\text{moment } = \int_a^b x \cdot \lambda(x)$$

Additionally,

$$\text{Average density } = \frac{\text{mass}}{\text{interval of integration}}$$

6.3   Centroid of a Region and Pappus's Theorem on Volumes (6.4)

Example: Find the centroid of a quarter disk of radius r (put it in the first quadrant for computations)

6.4   Work (6.5)

If force F is constant, when you move an object a distance x, the work done is

$$W = F \cdot x$$

So if I am to use my three steps for integration applications, how do I "chop up" work to get ΔW?

$$\Delta Work = F(x)\Delta x$$

It follows that

$$W = \int_a^b F(x)dx$$

Example: It requires 6N to keep a spring stretched 0.2m beyond its natural length. Find the work done to stretch it from its original length of 0.7m to length of 1m.

We have

$$6 = k(0.2) \implies k = 30$$

Then the work needed is

$$W = \int_0^{0.3=1-0.7} 30xdx$$

Example 2: A sandbag is pulled from ground level to the top of a 32m building by a cable that is 0.2 kg/m. Initially the sandbag has a mass of 52kg but it leaks at a constant rate so when it reaches the roof its mass is 36kg. Find the work done.

We have

$$W = \int_0^{32} F(y) dy,$$

where $F(y)$ is the cable's weight and y is the sand. The cable's weight is

$$(32 - y)0.2g$$

and the rate of change is

$$\frac{36 - 52}{32} = -0.5$$

So our work is

$$W = \int_0^{32} \left[(32 - y)0.2g + (-0.5y + 52)g \right] dy$$

6.5   One-to-one Functions

Theorem: if a function f is one-to-one, then there is a unique function, dentoed f^{-1}, defined on the range of f, such that $f\left(f^{-1}(x)\right) = x$ for all x in the range of f and $f^{-1}\left(f(x)\right) = x$ for all x in the domain of f.

6.6   The Natural Logarithm and Euler's Constant

For $x > 0$, the natural logarithm function is defined by

$$\ln x = \int_1^x \frac{1}{t} dt$$

Because $\ln x$ is increasing and differentiable on its domain, it follows that when $x > 1$,

$$\int_1^x \frac{1}{t} dt > 0$$

and $\ln x > 0$. Similarly, when $x < 1$,

$$\int_1^x \frac{1}{t} dt < 0$$

and $\ln x > 0$.

Proofs of some Properties of the Natural Logarithm:

If $a, b > 0$ and r is a rational number, then
i. $\ln 1 = 0$
ii. $\ln(ab) = \ln a + \ln b$
iii. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

i. By definition,

$$\ln 1 = \int_1^1 \frac{1}{t} dt = 0.$$

ii. We have

$$\ln(ab) = \int_1^{ab} \frac{1}{t} dt = \int_1^a \frac{1}{t} dt + \int_a^{ab} \frac{1}{t} dt.$$

Using u-substitution on the last integral in this expression, let $u = \frac{t}{a} \implies du = \frac{dt}{a}$. When $t = a$, $u = 1$, and when $t = ab$, $u = b$. It follows then that

$$\ln(ab) = \int_1^a \frac{1}{t} dt + \int_a^{ab} \frac{1}{t} dt = \int_1^a \frac{1}{t} dt + \int_1^{ab} \frac{a}{t} \cdot \frac{1}{a} dt = \int_1^a \frac{1}{t} dt + \int_1^b \frac{1}{u} du = \ln a + \ln b$$

iii. Note that

$$\frac{d}{dx} \ln(x^r) = \frac{rx^{r-1}}{x^r} = \frac{r}{x}$$

Furthermore,

$$\frac{d}{dx}(r \ln x) = \frac{r}{x}$$

Since the derivatives of these two functions are the same, then by the FTC, they must differ by a constant s.t.

$$\ln(x^r) = r \ln x + C$$

for some constant C. Setting $x = 1$, it follows that

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$$\ln\left(1^r\right) = r\ln\left(1\right) + C$$
$$0 = r(0) + C$$
$$C = 0$$

Hence, $\ln\left(x^r\right) = r\ln x$.

There is a function, $f(z)$ which is the inverse of $\ln x$ with domain $\mathbb{R}$ and range $\mathbb{R}^{-1}$. $\ln x$ is an increasing function on its domain, so it is injective and hence, there is an unique function $f(x)$ on the domain $(-\infty, \infty)$ and range $(0, \infty)$. This function has the following values:

$$f(1) = e \text{ because } \ln e = 1 \quad f(0) = 1 \text{ because } \ln 1 = 0$$

when $b \in \alpha$, $f(b) = e^b$ because $\ln e^b = b\ln e = b$.

Properties of $f(x) = e^x$:
i. e^x is increasing because it is the inverse of an increasing function
ii. $f^{-1}\left(f(x)\right) = x$, any invertible function

6.7   Exponential and Logarithmic Functions

Since

$$x^{\frac{p}{q}} = \left(x^p\right)^{\frac{1}{q}},$$

we have

$$f(x) = x^r, r \in \mathbb{R}$$
$$= e^{r\ln x}, x > 0$$
$$= e^{\frac{p}{q}\ln x}$$
$$= e^{\ln\left(x^{\frac{p}{q}}\right)}$$

Defining a^x and $\log_a x$:

$$a^x = e^{x\ln a}$$
$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln a}$$
$$= e^{x\ln a}\cdot\ln a$$
$$= (\ln a)a^x$$

Additionally,

$$\log_a x = \frac{\ln x}{\ln a}$$

6.8   Error in Linear Approximations

From HW question:

$$f(x) = f(b) + f'(b)(x - b) + \int_b^x f''(t)(x - t)dt$$

Note that $f(x)$ is the actual function value, whereas $f(b) + f'(b)(x - b)$ is the linear approximated value. Thus, the error is the actual value minus the approximated value s.t.

$$f(x) - L(x) = \int_b^x f''(t)(x - t)dt$$

We want to bound the error. If $b < x$,

$$b < t < x \implies x - t > 0$$

and

$$\int_b^x f''(t)(x - t)dt \leq \int_b^x |f''(t)|(x - t)dt$$

Let M be a (positive) number such that

$$|f''(t)| \leq M$$

when t is between b and x. Then,

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$$\int_b^x f''(t)(x-t)dt \leq \int_b^x M \cdot (x-t)dt$$
$$= -\frac{M(x-t)^2}{2}\Big|_b^x$$
$$u = x - t, du = -dt$$
$$= \frac{M(x-b)^2}{2}$$

It follows that

$$\left|\int_b^x f''(t)(x-t)dt\right| \leq \frac{M|x-b|^2}{2}$$

This is known as Taylor's Inequality, where M is an upper bound for $|f''(t)|$ when t is between x and b.

6.9 Numerical Integration

7 Differential Equations

Example:

$$\frac{dy}{dx} = (y^2 + y)(4x + 2)$$

$$\int \frac{dy}{y^2 + y} = \int 4x + 2dx$$

$$\int \frac{1}{y}dy - \int \frac{1}{y+1}dy = 2x^2 + 2x + C_2$$

$$\ln|y| - \ln|y+1| + C_1 = 2x^2 + 2x + C_2$$

$$\text{Set } C = C_2 - C_1$$

$$\ln\left|\frac{y}{y+1}\right| = 2x^2 + 2x + C$$

$$\left|\frac{y}{y+1}\right| = e^{2x^2 + 2x + C}$$

$$\left|\frac{y}{y+1}\right| = e^C e^{2x^2 + 2x}$$

$$\frac{y}{y+1} = Ce^{2x^2 + 2x}$$

$$y = Ce^{2x^2 + 2x} \cdot y + Ce^{2x^2 + 2x}$$

$$\left(1 - Ce^{2x^2 + 2x}\right) y = Ce^{2x^2 + 2x}$$

$$y = \frac{Ce^{2x^2 + 2x}}{1 - Ce^{2x^2 + 2x}}$$

$$y = \frac{1}{Ce^{-2x^2 - x^2} - 1}$$

Example 2: Mixing Problem
A tank initially contains 1000 litres of pure water. Brine that contains 0.07 kilograms of salt per liter of water enters the tank at a rate of 5 litres per minute. In addition, brine that contains 0.04 kilograms of salt per liter of water enters the tank at a rate of 10 litres per minute. The solution is kept thoroughly mixed and drains from the tank at a rate of 15 litres per minute. Find the amount of salt in the tank as a function of time. Also, find the concentration of salt in the tank "eventually".

Let $s(t)$ be the amount of salt in kg at a given time t in minutes. We have $\frac{ds}{dt} = in - out$, thus

$$\frac{ds}{dt} = (0.07)5 + (0.04)10 - s\frac{15}{1000}$$

$$= 0.75 - 0.015s$$

$$= -0.015(s - 50)$$

$$\frac{ds}{s - 50} = -0.015dt$$

$$\int \frac{ds}{s - 50} = \int -0.015dt$$

$$\ln|s - 50| = -0.015t + C$$

$$|s - 50| = e^{-0.015t + C}$$

$$|s - 50| = Ce^{-0.015t}$$

$$s - 50 = Ce^{-0.015t}$$

$$s = 50 + Ce^{-0.015t}$$

Example 3: Newton's Law of Cooling
Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings. The cooling constant, which is the proportionality constant in the differential equation, remains constant throughout this question. Experimenting, you found that 12 ounces of 180° F coffee in your favorite cup will take 20 minutes to cool to a drinking temperature of 115° F in a 70° F room. Set up a differential equation for the temperature of the coffee, solve it, and find the cooling constant.

Let $P_s = 70$ be the temperature of the surrounding environment which stays constant. Let P be the temperature of the object at minute t. We know that $\frac{dP}{dt}$ is proportional to the difference in temperature between the system and the environment, hence

$$\frac{dP}{dt} (P - P_s)$$

$$= -k(P - P_s)$$

$$\int \frac{dP}{P - P_s} = \int -kdt$$

We will need to use information in the problem to figure out k and C.