MATH126 Notes

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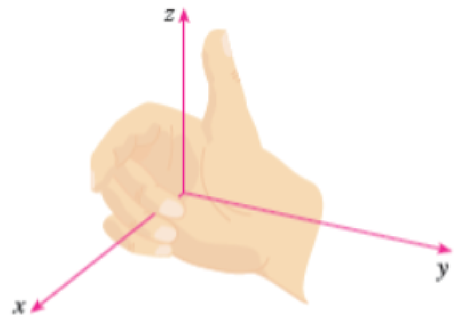
Contents

1 Vectors and the Geometry of Space

1.1 3D Coordinate Systems

The orientations of the x, y, and z axes can be remembered by the **right-hand rule:**



The three coordinate planes divide space into eight parts, called **octants**. The **first octant** is the set of points whose coordinates are all positive.
Let P be a point (a, b, c). Dropping a perpendicular from P to the xy-plane, we get a point Q with coordinates $(a, b, 0)$, called the **projection** of P onto the xy-plane. Similarly, $R(0, b, c)$ and $S(a, 0, c)$ are the projections of P onto the yz-plane and xz-plane, respectively.
This system is called the **three-dimensional rectangular coordinate system**, where points are ordered triples (a, b, c) in $\mathbb{R}^3$. In 2D analytic geometry, the graph of an equation involving x and y is a curve in $\mathbb{R}^2$. In 3D analytic geometry, an equation in x, y, and z represents a *surface* in $\mathbb{R}^3$.

In general, if k is a constant, then
- $x = k$ represents a plane parallel to the yz-plane
- $y = k$ is a plane parallel to the xz-plane
- $z = k$ is a plane parallel to the xy-plane

Example: The points (x, y, z) satisfying the equations

$$x^2 + y^2 = 1$$

and

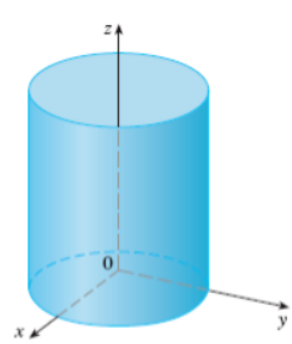
$$z = 3$$

include those on the horizontal plane $z = 3$, lying on the circle with radius 1 and center on the z-axis. The equation

$$x^2 + y^2 = 1$$

represents a cylinder like so:

The cylinder $x^2 + y^2 = 1$



Distance Formula in 3D

The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z^2 - z^1)^2}$$

Example 2: Find an equation of a sphere with radius r and center $C(h, k, l)$.

By definition, a sphere is the set of all points $P(x, y, z)$ whose distance from C is r. Thus P is on the sphere iff $|PC| = r$. Squaring both sides, we have $|PC|^2 = r^2$ or

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

Example 3: What region in $\mathbb{R}^3$ is represented by the following inequalities?

$$1 \leq x^2 + y^2 + z^2 \leq 4, z \leq 0$$

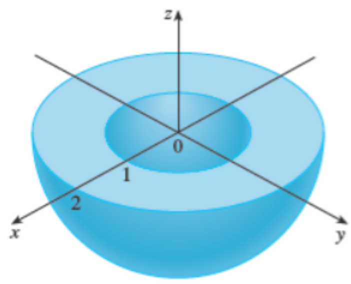
The inequalities

$$1 \leq x^2 + y^2 + z^2 \leq 4$$

can be rewritten as

$$1 \leq \sqrt{x^2 + y^2 + z^2} \leq 2$$

s.t. they represent the points whose distance from the origin is between 1 and 2. Since $z \leq 0$, the points lie on or below the xy-plane, thus the given inequalities represent the region that lies between (or on) the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ and beneath (or on) the xy-plane. Below is a sketch of this region.



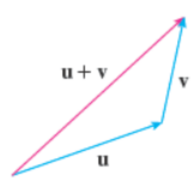
1.2 Vectors

Vector: a quantity that has both magnitude and direction, represented by an arrow. Suppose a particle moves along a line segment from point A to point B. The corresponding **displacement vector v** has **initial point** A (the tail) and **terminal point** B (the tip), indicated by the notation $v = \overrightarrow{AB}$. **Equivalent vectors** have the same length and direction but may be in different positions. The **zero vector**, denoted by 0, has length 0 and is the only vector with no specific direction.

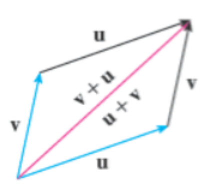
If a particle moves from A to B to C, then

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

Vector addition is sometimes illustrated by the **Triangle Law:**



From the **Parallelogram Law** below, we see that two vectors u and v satisfy the associative property $u + v = v + u$.



Scalar Multiplication: If c is a scalar and v is a vector, then their scalar product is a vector whose length is $|c|$ times the length of v and whose direction is the same as v if $c > 0$ and opposite to v if $c < 0$.

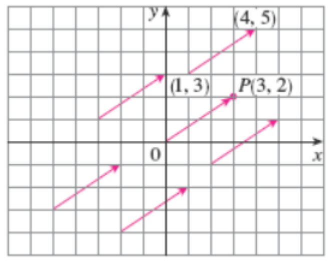
Angle brackets are for vectors, whereas parentheses are for points. a_1, a_2, a_3 are called the **components** of a, written

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

Any vector where the terminal point is reached from the initial point by a displacement of three units to the right and two upward is a **representation** of the vector $a = \langle 3, 2 \rangle$. The representation $\overrightarrow{OP}$ from the origin to the point $P(3, 2)$ is called the **position vector** of the point P.

Representations of $a = \langle 3, 2 \rangle$:
The magnitude of the vector v is the length of any of its representations, denoted by $|v|$ or $||v||$. Thus,

$$||a|| = \sqrt{a_2^1 + a_2^2 + a_3^2}$$



Properties of Vectors: If **a, b,** and **c** are vectors in V_n and c and d are scalars, then
- 1. $a + b = b + a$
- 2. $a + (b + c) = (a + b) + c$
- 3. $a + 0 = a$
- 4. $a + (-a) = 0$
- 5. $c(a + b) = ca + cb$
- 6. $(c + d)a = ca + da$
- 7. $(cd)a = c(da)$
- 8. $1a = a$

The vectors i, j, and k are called **standard basis vectors**:

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \mathbf{j} = \langle 0, 1, 0 \rangle, \mathbf{k} = \langle 0, 0, 1 \rangle$$

Thus,

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

Unit vectors: vectors whose lengths are 1.
- **i, j,** and **k** are all unit vectors
- In general, if $a \neq 0$, then the unit vector that has the same direction as **a** is

$$\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

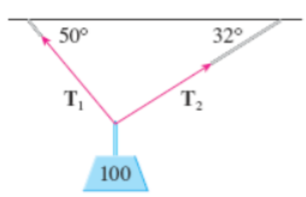
Example: The unit vector in the direction of the vector $\mathbf{2i} - \mathbf{j} - \mathbf{2k}$ can be found first by finding the magnitude:

$$|\mathbf{2i} - \mathbf{j} - \mathbf{2k}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$$

Thus, the unit vector with the same direction is

$$\frac{\mathbf{2i} - \mathbf{j} - \mathbf{2k}}{3} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

Example 2: A 100-lb weight hangs from two wires as shown below. Find the tensions (forces) T_1 and T_2 in both wires and the magnitudes of the tensions.

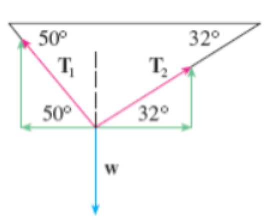


From the figure below, it follows that T_1 and T_2 can be expressed in terms of their horizontal and vertical components s.t.

$$\mathbf{T_1} = -|\mathbf{T_1}|\cos 50°\mathbf{i} + |\mathbf{T_1}|\sin 50°\mathbf{j}$$

and

$$\mathbf{T_2} = -|\mathbf{T_2}|\cos 32°\mathbf{i} + |\mathbf{T_2}|\sin 32°\mathbf{j}$$



The resultant $\mathbf{T_1} + \mathbf{T_2}$ of the tensions counterbalances the weight $\mathbf{w} = -\mathbf{100j}$ and so we must have

$$\mathbf{T_1} + \mathbf{T_2} = -\mathbf{w} = \mathbf{100j}$$

Thus,

$$(-|\mathbf{T_1}|\cos 50^\circ + |\mathbf{T_2}|\cos 32^\circ)\,\mathbf{i} + (|\mathbf{T_1}|\sin 50^\circ + |\mathbf{T_2}|\sin 32^\circ)\,\mathbf{j} = 100\mathbf{j}$$

Equating components, we get

$$\begin{aligned} -|\mathbf{T_1}|\cos 50^\circ + |\mathbf{T_2}|\cos 32^\circ &= 0 \\ |\mathbf{T_1}|\sin 50^\circ + |\mathbf{T_2}|\sin 32^\circ &= 100 \end{aligned}$$

Solving the first of these equations for $|\mathbf{T_2}|$ and substituting into the second, we get

$$\begin{aligned} |\mathbf{T_1}|\sin 50^\circ + \frac{|\mathbf{T_1}|\cos 50^\circ}{\cos 32^\circ}\sin 32^\circ &= 100 \\ |\mathbf{T_1}| &= \frac{100}{\sin 50^\circ + \tan 32^\circ \cos 50^\circ} \approx 85.64 \text{ lb} \end{aligned}$$

and

$$|\mathbf{T_2}| = \frac{|\mathbf{T_1}|\cos 50^\circ}{\cos 32^\circ} \approx 64.91 \text{ lb}$$

Substituting these values into the original vector equations for T_1 and T_2, it follows that

$$\begin{aligned} \mathbf{T_1} &\approx -55.05\mathbf{i} + 65.60\mathbf{j} \\ \mathbf{T_2} &\approx 55.05\mathbf{i} + 34.40\mathbf{j} \end{aligned}$$

1.3 The Dot Product

The Dot (Scalar) Product

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Properties of the Dot Product:
1. $a \cdot a = |a|^2$
2. $a \cdot b = b \cdot a$
3. $a \cdot (b + c) = a \cdot b + a \cdot c$
4. $(ca) \cdot b = c(a \cdot b) = a \cdot (cb)$
5. $0 \cdot a = 0$

Angle between vectors

If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

Orthogonal (perpendicular) vectors: vectors whose shared angle is $\theta = \frac{\pi}{2}$.
• For orthogonal vectors, $\mathbf{a} \cdot \mathbf{b} = 0$.

Dot Product and Orthogonality

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal iff $\mathbf{a} \cdot \mathbf{b} = 0$.

Example 2: Since

$$(2i + 2j - k) \cdot (5i - 4j + 2k) = 2(5) + 2(-4) + (-1)(2) = 0$$

these vectors are perpendicular.

The dot product $\mathbf{a} \cdot \mathbf{b}$ is positive if $\mathbf{a}$ and $\mathbf{b}$ point in the same general direction, 0 if they are perpendicular, and negative if they point in generally opposite directions. For the case in which $\mathbf{a}$ and $\mathbf{b}$ point in exactly the same direction, we have $\theta = 0 \implies \cos\theta = 1$ and

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|$$

If $\mathbf{a}$ and $\mathbf{b}$ point in exactly opposite directions, then we have $\theta = \pi$ and so $\cos\theta = -1$ and $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}||\mathbf{b}|$.

Pythaogrean Theorem in 3D

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

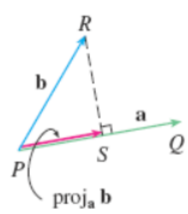
Theorem

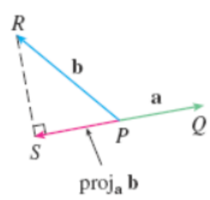
$$\frac{\mathbf{a}}{|\mathbf{a}|} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

Example: Find the direction angles of the vector $\mathbf{a} = \langle 1, 2, 3 \rangle$.
Since $|\mathbf{a}| = \sqrt{1^2 + 2^2 + 3^2}\sqrt{14}$, it follows that

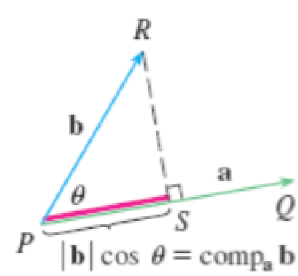
$$\cos\alpha = \frac{1}{\sqrt{14}} \implies \alpha = \arccos\left(\frac{1}{\sqrt{14}}\right) \approx 74^\circ$$
$$\cos\beta = \frac{2}{\sqrt{14}} \implies \beta = \arccos\left(\frac{2}{\sqrt{14}}\right) \approx 58^\circ$$
$$\cos\gamma = \frac{3}{\sqrt{14}} \implies \gamma = \arccos\left(\frac{3}{\sqrt{14}}\right) \approx 37^\circ$$

The figure below shows representations $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ of two vectors **a** and **b** with the same initial point P. If S is the foot of the perpendicular from R to the line containing $\overrightarrow{PQ}$, then the vector with representation $\overrightarrow{PS}$ is called the **vector projection** of **b** onto **a** and is denoted by $\text{proj}_a\mathbf{b}$.





The **scalar projection (component of b along a)** is shown in the figure below.



Scalar projection of b onto a

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$$

Vector projection of b onto a

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$$

Notice that the vector projection is the scalar projection times the unit vector in the direction of **a**.

Example 3: Find the scalar and vector projections of $\overrightarrow{b} = \langle 1, 1, 2 \rangle$ onto $\overrightarrow{a} = \langle -2, 3, 1 \rangle$.

Since

$$|a| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14},$$

the scalar projection of **b** onto **a** is

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}} = \frac{3}{\sqrt{14}}$$

The vector projection is this scalar projection times the unit vector in the direction of **a**:

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{3}{\sqrt{14}}\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{14}\mathbf{a} = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle$$

Work and the Dot Product

The work done by a constant force $\mathbf{F}$ is the dot product $\mathbf{F} \cdot \mathbf{D}$, where $\mathbf{D}$ is the displacement vector:

$$W = \mathbf{F} \cdot \mathbf{D}$$

1.4 The Cross Product

Given two nonzero vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, it is very useful to be able to find a nonzero vector c that is perpendicular to both $\vec{a}$ and $\vec{b}$. If $\vec{c} = \langle c_1, c_2, c_3 \rangle$ is such a vector, then $\mathbf{a} \cdot \mathbf{c} = 0$ and $\mathbf{b} \cdot \mathbf{c} = 0$ and so

$$a_1c_1 + a_2c_2 + a_3c_3 = 0$$

and

$$b_1c_1 + b_2c_2 + b_3c_3 = 0$$

Eliminating c_3, we can multiply the first equation by b_3 and the second equation by a_3. Subtracting, it follows that

$$(a_1b_3 - a_3b_1)c_1 + (a_2b_3 - a_3b_2)c_2 = 0$$

The above equation has the form $pc_1 + qc_2 = 0$, for which an obvious solution is $c_1 = q$ and $c_2 = -p$, s.t. a solution of the equation is

$$c_1 = a_2b_3 - a_3b_2, c_2 = a_3b_1 - a_1b_3$$

Substituting these values into the first two equations,

$$c_3 = a_1b_2 - a_2b_1$$

Hence, a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ has the form

$$\langle c_1, c_2, c_3 \rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

This vector is known as the cross product of $\mathbf{a}$ and $\mathbf{b}$ and is denoted by $\mathbf{a} \times \mathbf{b}$.

Cross Product (Vector Product)

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** or $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1$$

NOTE: $\mathbf{a} \times \mathbf{b}$ is only defined when a and b are 3D vectors.

The cross product is a vector whereas the dot product is a scalar.

In order to make the cross product easier to remember, determinant notation is used. A **determinant of order 2** is defined as follows.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

A **determinant of order 3** can be defined in terms of second-order determinants like so:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Thus, the definition of the cross product can be rewritten using second-order determinants and the standard basis vectors **i, j, k**.

Definition of Cross Product In Determinant Notation

Let vectors $\mathbf{a}$ and $\mathbf{b}$ be given by $a = a_1i + a_2j + a_3k$ and $b = b_1i + b_2j + b_3k$ s.t.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

Alternatively,

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example: Let $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$ s.t.

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} \\
&= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathbf{k} \\
&= (-15 - 28)\mathbf{i} - (-5 - 8)\mathbf{j} + (7 - 6)\mathbf{k} \\
&= -43\mathbf{i} + 13\mathbf{j} + \mathbf{k}
\end{aligned}$$

Example 3: Show that $\mathbf{a} \times \mathbf{a} = 0$ for any vector $\mathbf{a}$ in V_3.
If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then

$$\begin{aligned}
\mathbf{a} \times \mathbf{a} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \\
&= (a_2 a_3 - a_3 a_2)\mathbf{i} - (a_1 a_3 - a_3 a_1)\mathbf{j} + (a_1 a_2 - a_2 a_1)\mathbf{k} \\
&= 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} \\
&= 0
\end{aligned}$$

and thus our assertion.

Theorem

The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Proof.

$$\begin{aligned}
(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3 \\
&= a_1(a_2 b_3 - a_3 b_2) - a_2(a_1 b_3 - a_3 b_1) + a_3(a_1 b_2 - a_2 b_1) \\
&= a_1 a_2 b_3 - a_1 b_2 a_3 - a_1 a_2 b_3 + b_1 a_2 a_3 + a_1 b_2 a_3 - b_1 a_2 a_3 \\
&= 0
\end{aligned}$$

A similar computation yields $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$. Hence, $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. □

Theorem

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \leq \theta \leq \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$$

Proof. From the definitions of the cross product and length of a vector, we have

$$\begin{aligned}
|\mathbf{a} \times \mathbf{b}|^2 &= (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2 \\
&= a_2^2 b_3^2 - 2a_2 a_3 b_2 b_3 + a_3^2 b_2^2 + a_3^2 b_1^2 - 2a_1 a_3 b_1 b_3 + a_1^2 b_3^2 + a_1^2 b_2^2 - 2a_1 a_2 b_1 b_2 + a_2^2 b_1^2 \\
&= \left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \\
&= |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\
&= |\mathbf{a}|^2 |\mathbf{b}|^2 - |\mathbf{a}|^2 |\mathbf{b}|^2 \cos^2\theta \\
&= |\mathbf{a}|^2 |\mathbf{b}|^2 (1 - \cos^2\theta) \\
&= |\mathbf{a}|^2 |\mathbf{b}|^2 \sin^2\theta
\end{aligned}$$

Taking square roots and observing that $\sqrt{\sin^2\theta} = \sin\theta$ because $\sin\theta \geq 0$ when $0 \leq \theta \leq \pi$, it follows that

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$$

□

Scalar Triple Product: the product $\mathbf{a} \cdot (b \times c)$, **a,b,** and **c** are vectors.
• is written as a determinant:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The geometric significance of the scalar triple product can be seen when considering the parallelepiped (prism with 6 parallelograms as bases) determined by the vectors **a,b,** and **c**. The area of the base parallelogram is $A = |\mathbf{b} \times \mathbf{c}|$. If θ is the angle between $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$, then the height h of the parallelepiped is $h = |\mathbf{a}||\cos\theta|$. Hence, the volume of the parallelepiped is

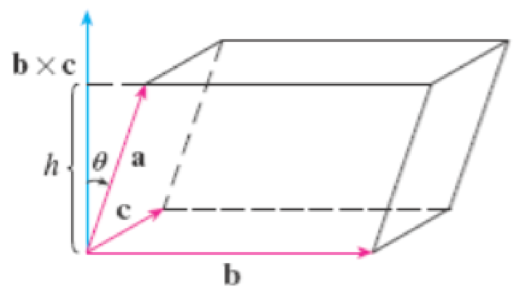
$$V = Ah = |\mathbf{b} \times \mathbf{c}||\mathbf{a}||\cos\theta| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

which proves the following formula.

Volume of Parallelepiped

The volume of the parallelepiped determined by the vectors **a,b,** and **c** is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$



Coplanar: lying in the same plane
• if the volume of the parallelepiped determined by **a, b,** and **c** is 0, then the vectors must be coplanar

Example 4: Use the scalar triple product to show that the vectors $\mathbf{a} = \langle 1, 4, -7\rangle$, $\mathbf{b} = \langle 2, -1, 4\rangle$, and $\mathbf{c} = \langle 0, -9, 18\rangle$ are coplanar.

$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & 9 & 18 \end{vmatrix} \\
&= 1\begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4\begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7\begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix} \\
&= 1(18) - 4(36) - 7(-18) \\
&= 0
\end{aligned}$$

Hence, the volume of the parallelepiped determined by **a, b,** and **c** is 0 and so **a, b,** and **c** are coplanar.

Vector Triple Product: the product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

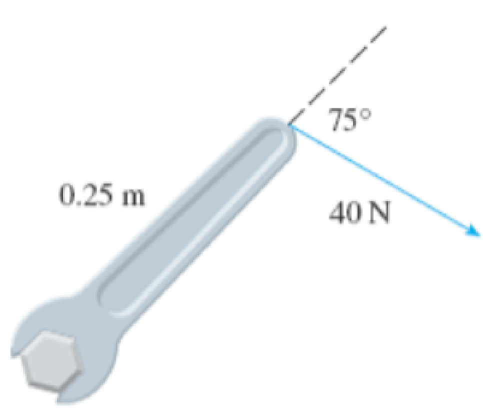
The **torque** (τ) is defined to be the cross product of the position and force vectors and measures the tendency of the body to rotate about the origin s.t.

$$\tau = r \times F$$

The direction of the torque vector indicates the axis of rotation. The magnitude of the torque vector is given as follows.

$$|\tau| = |r \times F| = |r||F| \sin\theta$$

Example 5: A bolt is tightened by applying a 40 N force to a 0.25 m wrench as shown below. Find the magnitude of the torque about the center of the bolt.



The magnitude of the torque vector is

$$
\begin{aligned}
|\tau| &= |r \times F| \\
&= |r||F| \sin 75^\circ \\
&= (0.25)(40) \sin 75^\circ \\
&= 10 \sin 75^\circ \\
&\approx 9.66 \mathrm{N} \cdot \mathrm{m}
\end{aligned}
$$

If the bolt is right-threaded, then the torque vector itself is

$$\tau = |\tau| \text{ n } \approx 9.66 \text{ n},$$

where **n** is a unit vector directed down into the page (by the right-hand rule).

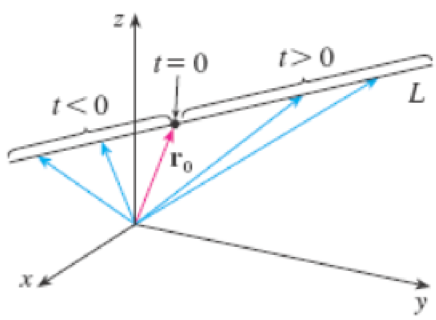
1.5 Equations of Lines and Planes

Let $P(x, y, z)$ and $P(x_0, y_0, z_0)$ be arbitrary points on a line L in 3D space, and let r_0 and r be the position vectors of P_0 and P. If **a** is the vector with representation $\overrightarrow{P_0P}$, then the Triangle Law for vector addition gives $\overrightarrow{r} = \overrightarrow{r_0} + a$. But since **a** and **v** are parallel vectors, there is a scalar t s.t. $\mathbf{a} = t\mathbf{v}$. Thus we get a **vector equation** of L:

Vector Equation

$$\overrightarrow{r} = \overrightarrow{r_0} + t\overrightarrow{v}$$

Each value of the parameter t gives the position vector r of a point on L, i.e. as t varies, the line is traced out by the tip of the vector $\overrightarrow{r}$ as in the figure below. Note how positive values of t correspond to points on L that lie on one side of P_0, whereas negative values of t correspond to points that lie on the other side of P_0.



If the vector **v** that gives the direction of line L is written in component form, then $t\overrightarrow{v} = \langle ta, tb, tc\rangle$, $r = \langle x, y, z\rangle$, and $r_0 = \langle x_0, y_0, z_0\rangle$ and

$$\langle x, y, z\rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle$$

It follows then that

Parametric Equations

$$x = x_0 + at, y = y_0 + bt, z = z_0 + ct, t \in \mathbb{R}$$

These equations are called **parametric equations** of the line L through the point P_0 and parallel to the vector $\overrightarrow{v} = \langle a, b, c\rangle$.

If a vector $\overrightarrow{v} = \langle a, b, c\rangle$ is used to describe the direction of a line L, then the numbers a, b, and c are called **direction numbers** of L.

We can also describe the line L by eliminating the parameter t. If $a, b, c \neq 0$, then

$$t = \frac{x - x_0}{a}, t = \frac{y - y_0}{b}, t = \frac{z - z_0}{c}$$

Equating the results, we obtain the symmetric equations of L.

Symmetric Equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}, a, b, c \neq 0$$

If one of the direction numbers, e.g. a was zero, then

$$x = x_0, \frac{y - y_0}{b}, \frac{z - z_0}{c}$$

and L lies in the vertical plane $x = x_0$.

Line Segment

The line segment from r_0 to r_1 is given by the vector equation

$$r(t) = (1-t)r_0 + tr_1, 0 \leq t \leq 1$$

Skew lines: lines that do not intersect and are not parallel

Example: Show that the lines L_1 and L_2 with parametric equations

$$\begin{aligned} &L_1 x = 1+t, && y = -2+3t, z = 4-t \\ &L_2 x = 2s, && y = 3+s z = -3+4s \end{aligned}$$

are skew lines.

The lines are not parallel because the corresponding direction vectors $\langle 1, 3, -1 \rangle$ and $\langle 2, 1, 4 \rangle$ are not parallel. If L_1 and L_2 had a point of intersection, there would be values of t and s s.t.

$$\begin{aligned} 1+t &= 2s \\ -2+3t &= 3+s \\ 4-t &= -3+4s \end{aligned}$$

However, if we solve the first two equations, we get $t = \frac{11}{5}$ and $s = \frac{8}{5}$, and these values don't satisfy the third equation. Hence, there are no values of t and s that satisfy the three equations, so L_1 and L_2 don't intersect and L_1 and L_2 are skew lines.

Planes are determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\overrightarrow{n}$ that is orthogonal to the plane, called a **normal vector**. Let $P(x, y, z)$ be a point in the plane, and let $\overrightarrow{r_0}$ and $\overrightarrow{r}$ be the position vectors of P_0 and P. Then the vector $\overrightarrow{r - r_0} = \overrightarrow{P_0P}$. Since the normal vector $\overrightarrow{n}$ is orthogonal to every vector in the given plane, $\overrightarrow{n}$ is orthogonal to $\overrightarrow{r - r_0}$ and so we get the vector equations of the plane.

Vector Equations of a Plane

$$\overrightarrow{n}(\overrightarrow{r} - \overrightarrow{r_0}) = 0$$

Alternatively,

$$\overrightarrow{n} \cdot \overrightarrow{r} = \overrightarrow{n} \cdot \overrightarrow{r_0}$$

Scalar Equation of a Plane

A **scalar equation of the plane** through point $P_0(x_0, y_0, z_0)$ with normal vector $\overrightarrow{n} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Linear Equation

By rewriting the scalar equation of a plane, it follows that we get a linear equation in x, y, z:

$$ax + by + cz + d = 0,$$

where

$$d = -(ax_0 + by_0 + cz_0)$$

Example 2: Find an equation of the plane that passes through the points $P(1, 3, 2)$,$Q(3, -1, 6)$, and $R(5, 2, 0)$. The vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ corresponding to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are

$$\overrightarrow{a} = \langle 2, -4, 4 \rangle, \overrightarrow{b} = \langle 4, -1, -2$$

Since both $\overrightarrow{a}$ and $\overrightarrow{b}$ lie in the plane, their cross product $\mathbf{a} \times \mathbf{b}$ is orthogonal to the plane and can be taken as a normal vector. Thus

$$\overrightarrow{n} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\mathbf{i} + 20\mathbf{j} + 14\mathbf{k}$$

With the point $P(1, 3, 2)$ and the normal vector $\overrightarrow{n}$, an equation of the plane is

$$12(x-1) + 20(y-3) + 14(z-2) = 0 \iff 6x + 10y + 7z = 50$$

Two planes are parallel if their normal vectors are parallel. If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors.

Example 3:
(a) Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.
(b) Find symmetric equations for the line of intersection L of these two planes.

(a) The normal vectors of these planes are

$$\overrightarrow{n_1} = \langle 1, 1, 1 \rangle, \overrightarrow{n_2} = \langle 1, -2, 3 \rangle$$

and so, if θ is the angle between the planes, it follows that

$$\begin{aligned} \cos\theta &= \frac{\mathbf{n_1} \cdot \mathbf{n_2}}{|\mathbf{n_1}||\mathbf{n_2}|} \\ &= \frac{1(1) + 1(-2) + 1(3)}{\sqrt{1+1+1}\sqrt{1+4+9}} \\ &= \frac{2}{\sqrt{42}} \\ \theta &= \arccos\left(\frac{2}{\sqrt{42}}\right) \\ &\approx 72^\circ \end{aligned}$$

(b) We need to find a point on L. We can find the point where the line itnersects the xy-plane by setting $z = 0$ in the equations of both planes. This gives the equations $x + y = 1$ and $x - 2y = 1$, whose solution is $x = 1, y = 0$. So the point $(1, 0, 0)$ lies on L. Note that since L lies in both planes, it is perpendicular to both of the normal vectors. Thus a vector $\overrightarrow{v}$ parallel to L is given by the cross product

$$\mathbf{v} = \mathbf{n_1} \times \mathbf{n_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$$

and so the symmetric equations of L can be written as

$$\frac{x-1}{5} = \frac{y}{-2} = \frac{z}{-3}$$

Distance between a Point and Plane

The distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is given by

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Proof. Let $P_0(x_0, y_0, z_0)$ be any point in the given plane and let $\mathbf{b}$ be the vector corresponding to $\overrightarrow{P_0P_1}$. Then

$$\overrightarrow{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$

The distance D from P_1 to the plane is equal to the absolute value of the scalar projection of $\mathbf{b}$ onto the normal vector $\overrightarrow{n} = \langle a, b, c \rangle$, hence

$$\begin{aligned} D &= |\text{comp}_n b| \\ &= \frac{|n \cdot b|}{|n|} \\ &= \frac{a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|(ax_1 - by_1 + cz_1) - (ax_0 - by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$

□

1.6 Cylinders and Quadric Surfaces

Cross-sections (traces): curves of intersection of a surface with planes parallel to the coordinate planes
Cylinder: a surface that consists of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve
Parabolic cylinder: a surface made up of infinite many shifted copies of the same parabola
Quadric surface: the graph of a second-degree equation in three variables x, y, and z. The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

Through translation and rotation the equation can be written in one of the two following standard forms.

$$Ax^2 + By^2 + Cz^2 + J = 0$$

or

$$Ax^2 + By^2 + Iz = 0$$

Example: Use traces to sketch the quadric surface with equation

$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$$

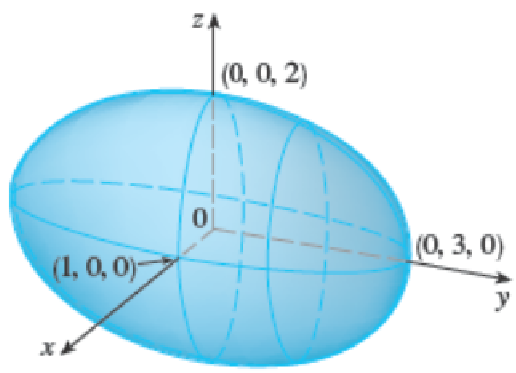
By substituting $z = 0$, we find that the trace in the xy-plane is $x^2 + \frac{y^2}{9} = 1$, which we recognize as an equation of an ellipse. In general, the horizontal trace in the plane $z = k$ is

$$x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4}, z = k$$

which is an ellipse, provided that $k^2 < 4$, that is, $-2 < k < 2$. Similarly, vertical traces parallel to the yz and xz-planes are also ellipses:

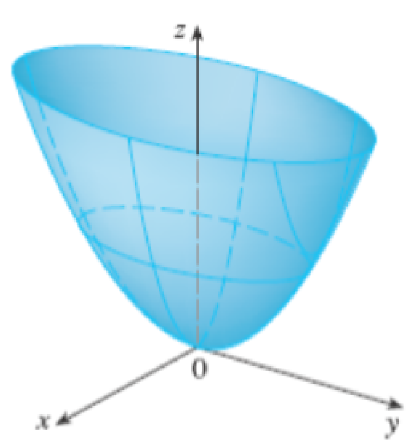
$$\frac{y^2}{9} + \frac{z^2}{4} = 1 - k^2 \qquad x = k(\text{if } -1 < k < 1)$$
$$x^2 + \frac{z^2}{4} = 1 - \frac{k^2}{9} \qquad y = k(\text{if } -3 < k < 3)$$

This surface is called an **ellipsoid** because all of its traces are ellipses. It is sketched below.



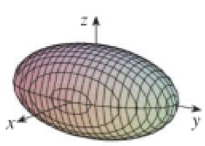
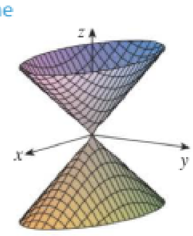
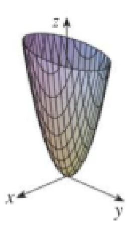
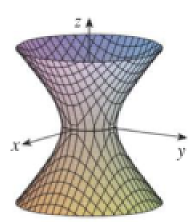
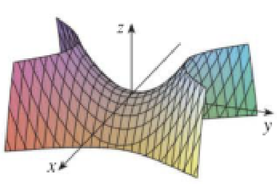
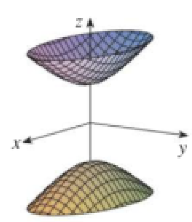
Example 2: Use traces to sketch the surface $z = 4x^2 + y^2$

When $x = 0$, $z = y^2$, so the yz-plane intersects the surface in a parabola. If we let $x = k$ (a constant), we get $z = y^2 + 4k^2$. This means that if we slice the graph with any plane parallel to the yz-plane, we obtain a parabola that opens upward. Similarly, if $y = k$, the trace is $z = 4x^2 + k^2$, which is again a parabola that opens upward. If we let $z = k$, we get the horizontal traces $4x^2 + y^2 = k$, which we recognize as a fmaily of ellipses. Knowing the shapes of the traces, we can sketch the graph like so:



Because of the elliptical and parabolic traces, this surface is called an **elliptic paraboloid**.

Graphs of Quadric Surfaces:
Circular paraboloids are used for satellite dishes. Cooling towers for nuclear reactors are designed in the shape of hyperboloids of one sheet for structural stability.

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

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