MATH134: Homework 5

Due on November 2, 2020 at 5:45 PM

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Question 1

Prove that a non-constant linear function is uniformly continuous on the real line. (This should be straightforward. Start with a continuity proof and make sure your delta does not depend on the point you choose.)

Proof. A non-constant linear function can have the form

$$f(x) = mx + b,$$

where $m \neq 0$ and b is a constant. For $f(x)$ to be uniformly continuous, for any given $\epsilon > 0$, the inequality

$$|x - c| < \delta \implies |f(x) - f(c)| < \epsilon,$$

where c is a constant, must hold. Let $\delta = \frac{\epsilon}{|m|}$ and $\epsilon > 0$, such that

$$\text{if } |x - c| < \delta = \frac{\epsilon}{|m|} \text{ then } |f(x) - f(c)| < \epsilon$$

Substituting in $f(x) = mx + b$ and $f(c) = mc + b$, $|f(x) - f(c)| < \epsilon$ becomes

$$\begin{aligned} |mx + b - mc - b| &< \epsilon \\ |m(x - c)| &< \epsilon \\ |m||x - c| &< \epsilon \end{aligned}$$

As $|x - c| < \frac{\epsilon}{|m|}$, it follows that

$$\begin{aligned} |m| \cdot \frac{\epsilon}{|m|} &= \epsilon \\ \epsilon &= \epsilon \end{aligned}$$

Hence, the non-constant linear function $f(x)$ is uniformly continuous on the real line. □

Section 5.2 Problem 12

(a) Given that $P = \{x_0, x_1, \ldots, x_n\}$ is an arbitrary partition of $[a, b]$, find $L_f(P)$ and $U_f(P)$ for $f(x) = x + 3$.
(b) Use your answers to part (a) to evaluate

$$\int_a^b f(x)dx.$$

(a)
On each subinterval $[x_{i-1}, x_i]$, the function $f(x) = x + 3$ has a maximum of x_i and a minimum of x_{i-1}. $L_f(P)$ is defined as the lower sum for $f(x)$, i.e. the sum of the areas of the rectangles in each interval using the lowest value of $f(x)$ as the height of the rectangle. $U_f(P)$ is defined as the upper sum for $f(x)$, i.e. the sum of the areas of the rectangles in each interval using the highest value of $f(x)$ as the height of the rectangle. It follows that

$$L_f(P) = \sum_{i=1}^{n} \Delta x f(x_{n-1}) = \sum_{i=1}^{n} (x_n - x_{n-1}) f(x_{n-1})$$

and

$$U_f(P) = \sum_{i=1}^{n} \Delta x f(x_n) = \sum_{i=1}^{n} (x_n - x_{n-1}) f(x_n)$$

(b)
As $x_{i-1} \leq x_i$ and the function $f(x)$ is strictly increasing, for every index i, the inequality

$$2(x_{x-1} + 3) \leq x_i + 3 + x_{x-1} + 3 \leq 2(x_i + 3)$$

holds, and

$$x_{i-1} + 3 \leq \frac{1}{2}(x_i + x_{i-1} + 6) \leq x_i + 3$$

Multiplying by $\Delta x = x_i - x_{i-1}$, the middle term of the inequality becomes

$$\frac{1}{2}(x_i - x_{i-1})(x_i + x_{i-1} + 6) = \frac{1}{2}(x_i^2 + 6x_i - x_{i-1}^2 - 6x_{i-1})$$

It follows that

$$\Delta x(x_{i-1}) \leq \frac{1}{2}(x_i^2 + 6x_i - x_{i-1}^2 - 6x_{i-1}) \leq \Delta x x_i$$

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The sum of the middle term collapses to:

$$\begin{aligned}\frac{1}{2}(x_1^2 + 6x_1 - x_0^2 - 6x_0 + x_2^2 + 6x_2 - x_1^2 - 6x_1 + \cdots + x_n^2 + 6x_n - x_{n-1}^2 - 6x_{n-1}) &= \frac{1}{2}(-x_0^2 - 6x_0 + x_n^2 + 6x_n) \\ &= \frac{1}{2}(-a^2 - 6a + b^2 + 6b)\end{aligned}$$

The sum of the terms on the left side of the inequality is $L_f(P)$ and the sum of the terms on the right side of the inequality is $U_f(P)$. Thus,

$$L_f(P) \leq \frac{b^2 - a^2 + 6b - 6a}{2} \leq U_f(P)$$

Since P was chosen arbitrarily, we can conclude that this inequality holds for all partitions P of $[a, b]$. It follows that

$$\int_a^b f(x)dx = \frac{b^2 - a^2 + 6b - 6a}{2}.$$

Section 5.2 Problems 25-30

Assume that f and g are continuous, that $a < b$, and that $\int_a^b f(x)dx > \int_a^b g(x)dx$. Which of the statements necessarily holds for all partitions P of $[a, b]$? Justify your answer.

25. $L_g(P) < U_f(P)$.
26. $L_g(P) < L_f(P)$.
27. $L_g(P) < \int_a^b f(x)dx$.
28. $U_g(P) < U_f(P)$.
29. $U_f(P) > \int_a^b g(x)dx$.
30. $U_g(P) < \int_a^b f(x)dx$.

25.
Since $L_g(P)$ is the lower sum of $g(x)$ on an arbitrary interval P and $U_g(P)$ is the upper sum of $g(x)$ on P, it follows that

$$L_g(P) < \int_a^b g(x)dx$$

and

$$U_f(P) > \int_a^b f(x)dx$$

Since $\int_a^b f(x)dx > \int_a^b g(x)dx$,

$$L_g(P) < \int_a^b g(x)dx < \int_a^b f(x)dx < U_f(P)$$

and the statement $L_g(P) < U_f(P)$ must always hold true.

26.
Consider when $f(x) = x$ and $g(x) = 1$, $\Delta x = x_i - x_{i-1}$, and the arbitrary partition P is defined as $P = [0, 1]$, such that

$$L_f(P) = \Delta x \cdot f(x_{i-1})$$

and

$$L_g(P) = \Delta x \cdot g(x_{i-1})$$

At the index $i = 1$,

$$L_f(P) = 0(1 - 0) = 0$$

and

$$L_g(P) = 1(1 - 0) = 1$$

Thus,

$$L_f(P) < L_g(P)$$

and the statement $L_g(P) < L_f(P)$ is not necessarily true for all partitions P of $[a, b]$.

27.
By the definition of $L_g(P)$,

$$L_g(P) < \int_a^b g(x)dx$$

Because

$$\int_a^b g(x)dx < \int_a^b f(x)dx,$$

it follows that the inequality

$$L_g(P) < \int_a^b f(x)dx$$

is always true.

28.
Consider when $f(x) = x$ and $g(x) = 1$, $\Delta x = x_i - x_{i-1}$, and the arbitrary partition P is defined as $P = [0, 1]$, such that

$$U_f(P) = \Delta x \cdot f(x_i)$$

and

$$U_g(P) = \Delta x \cdot g(x_i)$$

At the index $i = 1$,

$$U_f(P) = 1(1) = 1$$

and

$$U_g(P) = 1(1) = 1$$

Thus,

$$U_f(P) = U_g(P)$$

and the statement $U_g(P) < U_f(P)$ is not necessarily true for all partitions P of $[a, b]$.

29.
By the definition of $U_f(P)$,

$$U_f(P) > \int_a^b f(x)dx$$

Because

$$\int_a^b g(x)dx < \int_a^b f(x)dx,$$

it follows that the inequality

$$U_f(P) > \int_a^b g(x)dx$$

always holds.

30.
Consider when $f(x) = x$ and $g(x) = 1$, $\Delta x = x_i - x_{i-1}$, and the arbitrary partition P is defined as $P = [0, 1]$, such that

$$U_g(P) = \Delta x \cdot g(x_i) = 1(1) = 1$$

Because the integral of a function is the area under the curve,

$$\int_a^b f(x)dx = \int_0^1 xdx = \frac{1}{2}$$

It follows that

$$U_g(P) > \int_a^b f(x)dx$$

and hence the statement $U_g(P) < \int_a^b f(x)dx$ is not necessarily true for all partitions P of $[a, b]$.

Section 5.2 Problem 32

Let $P = \{x_0, x_1, x_2, \dots, x_{n-1}, x_n\}$ be a regular partition of the interval $[a, b]$. Show that if f is continuous and decreasing on $[a, b]$, then

$$U_f(P) - L_f(P) = [f(a) - f(b)]\Delta x$$

Proof. As P is a regular partition and f is decreasing on $[a, b]$, we have that the expansion of the upper sum $U_f(P)$ is given by

$$U_f(P) = f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_{n-1})\Delta x$$

and the expansion of the lower sum $L_f(P)$ is given by

$$L_f(P) = f(x_0)\Delta x + f(x_1)\Delta x + \dots + f(x_n)$$

It follows then that

$$\begin{aligned} U_f(P) - L_f(P) &= [f(x_0)\Delta x + f(x_1)\Delta x + \dots + f(x_{n-1})] - [f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x] \\ &= \Delta x([f(x_0) + f(x_1) + \dots + f(x_{n-1})] - [f(x_1) + f(x_2) + \dots + f(x_n)]) \end{aligned}$$

This collapses to

$$U_f(P) - L_f(P) = \Delta x[f(x_0) - f(x_n)]$$

Substituting $a = x_0$ and $b = x_n$,

$$U_f(P) - L_f(P) = [f(a) - f(b)]\Delta x$$

□

Section 5.3 Problem 21

Suppose that f is differentiable with $f'(x) > 0$ for all x, and suppose that $f(1) = 0$. Set

$$F(x) = \int_0^x f(t)dt.$$

Justify each statement.
(a) F is continuous.
(b) F is twice differentiable.
(c) $x = 1$ is a critical point of F.
(d) F takes on a local minimum at $x = 1$.
(e) $F(1) < 0$.

Make a rough sketch of the graph of F.

(a) Because f is differentiable,

$$\begin{aligned} f(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\ f(x) &= \frac{\lim_{h\to 0}(F(x+h) - F(x))}{\lim_{h\to 0} h} \\ 0 &= \lim_{h\to 0} F(x+h) - \lim_{h\to 0} F(x) \\ F(x) &= \lim_{h\to 0} F(x+h) \end{aligned}$$

Hence, by the definition of continuity, F is continuous.

(b) The first derivative of F is given by

$$F'(x) = \frac{d}{dx}\int_0^x f(t)dt = f(x)$$

Because f is differentiable for all x, it follows that

$$F''(x) = f'(x)$$

must exist and hence, F is twice differentiable.

(c) When $x = 1$,

$$F'(1) = f(1) = 0$$

The slope at a critical point is equal to zero, hence, $x = 1$ is a critical point of F.

(d) As $f'(x) > 0\ \forall x$ and $f(1) = 0$, it must be true that

$$f(x) = \begin{cases} \text{negative}, & x < 1 \\ \text{positive}, & x > 1 \end{cases}$$

Because

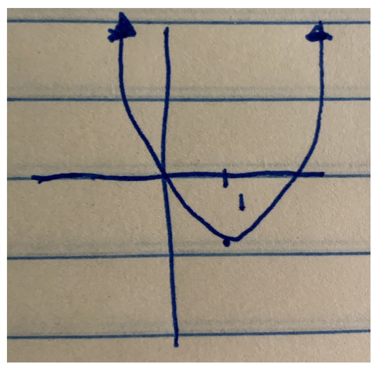
$$F'(1) = f(1) = 0$$

and f is negative before $x = 1$ and positive after $x = 1$, it follows that F takes a local minimum at $x = 1$.

(e)
Because $f(x) < 0$ when $x < 1$ and an integral represents the area under a curve, thus

$$\int_0^1 f(x)dx < 0$$

Rough Sketch of *F*:
Because $F(1) < 0$ and $x = 1$ is a minimum of F, then a rough sketch of F can be as follows



Section 5.3 Problem 36

Let F be everywhere continuous and set

$$F(x) = \int_0^x \left[t \int_1^t f(u)du\right] dt.$$

Find
(a) $F'(x)$.
(b) $F'(1)$.
(c) $F''(x)$.
(d) $F''(1)$.

(a)
$F'(x)$ is given by

$$F'(x) = \frac{d}{dx} \int_0^x \left[t \int_1^t f(u)du\right] dt$$

By the Fundamental Theorem of Calculus,

$$F'(x) = x \int_1^x f(u)du$$

(b)
When $x = 1$,

$$\begin{aligned} F'(1) &= 1 \int_1^1 f(u)du \\ &= 1(0) \\ &= 0 \end{aligned}$$

(c)
Because

$$F'(x) = x \int_1^x f(u)du,$$

it follows that

$$\begin{aligned} F''(x) &= \frac{d}{dx}\left(x \int_1^x f(u)du\right) \\ &= \int_1^x f(u)du + xf(x) \end{aligned}$$

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(d)
When $x = 1$,

$$\begin{aligned} F''(1) &= \int_1^1 f(u)du + (1)f(1) \\ &= 0 + f(1) \\ &= f(1) \end{aligned}$$