MATH126 Notes

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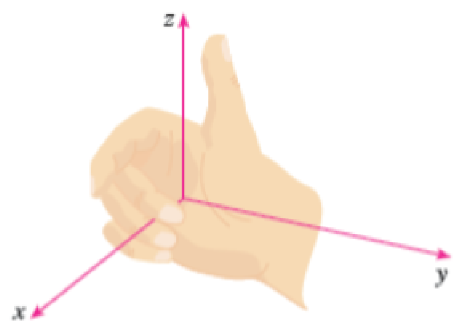
Contents

1 Vectors and the Geometry of Space

1.1 3D Coordinate Systems

The orientations of the x, y, and z axes can be remembered by the **right-hand rule:**



The three coordinate planes divide space into eight parts, called **octants**. The **first octant** is the set of points whose coordinates are all positive.
Let P be a point (a, b, c). Dropping a perpendicular from P to the xy-plane, we get a point Q with coordinates $(a, b, 0)$, called the **projection** of P onto the xy-plane. Similarly, $R(0, b, c)$ and $S(a, 0, c)$ are the projections of P onto the yz-plane and xz-plane, respectively.
This system is called the **three-dimensional rectangular coordinate system**, where points are ordered triples (a, b, c) in $\mathbb{R}^3$. In 2D analytic geometry, the graph of an equation involving x and y is a curve in $\mathbb{R}^2$. In 3D analytic geometry, an equation in x, y, and z represents a *surface* in $\mathbb{R}^3$.

In general, if k is a constant, then
- $x = k$ represents a plane parallel to the yz-plane
- $y = k$ is a plane parallel to the xz-plane
- $z = k$ is a plane parallel to the xy-plane

Example: The points (x, y, z) satisfying the equations

$$x^2 + y^2 = 1$$

and

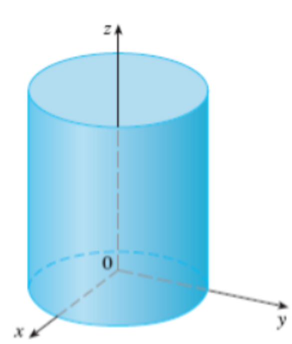
$$z = 3$$

include those on the horizontal plane $z = 3$, lying on the circle with radius 1 and center on the z-axis. The equation

$$x^2 + y^2 = 1$$

represents a cylinder like so:

The cylinder $x^2 + y^2 = 1$



Distance Formula in 3D

The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z^2 - z^1)^2}$$

Example 2: Find an equation of a sphere with radius r and center $C(h, k, l)$.

By definition, a sphere is the set of all points $P(x, y, z)$ whose distance from C is r. Thus P is on the sphere iff $|PC| = r$. Squaring both sides, we have $|PC|^2 = r^2$ or

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

Example 3: What region in $\mathbb{R}^3$ is represented by the following inequalities?

$$1 \leq x^2 + y^2 + z^2 \leq 4, z \leq 0$$

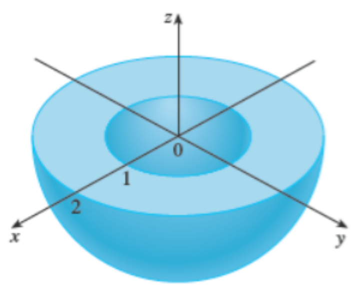
The inequalities

$$1 \leq x^2 + y^2 + z^2 \leq 4$$

can be rewritten as

$$1 \leq \sqrt{x^2 + y^2 + z^2} \leq 2$$

s.t. they represent the points whose distance from the origin is between 1 and 2. Since $z \leq 0$, the points lie on or below the xy-plane, thus the given inequalities represent the region that lies between (or on) the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ and beneath (or on) the xy-plane. Below is a sketch of this region.



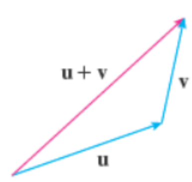
1.2 Vectors

Vector: a quantity that has both magnitude and direction, represented by an arrow. Suppose a particle moves along a line segment from point A to point B. The corresponding **displacement vector v** has **initial point** A (the tail) and **terminal point** B (the tip), indicated by the notation $v = \overrightarrow{AB}$. **Equivalent vectors** have the same length and direction but may be in different positions. The **zero vector**, denoted by 0, has length 0 and is the only vector with no specific direction.

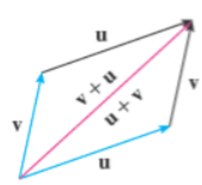
If a particle moves from A to B to C, then

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

Vector addition is sometimes illustrated by the **Triangle Law:**



From the **Parallelogram Law** below, we see that two vectors u and v satisfy the associative property $u + v = v + u$.



Scalar Multiplication: If c is a scalar and v is a vector, then their scalar product is a vector whose length is $|c|$ times the length of v and whose direction is the same as v if $c > 0$ and opposite to v if $c < 0$.

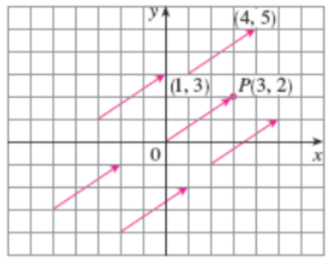
Angle brackets are for vectors, whereas parentheses are for points. a_1, a_2, a_3 are called the **components** of a, written

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

Any vector where the terminal point is reached from the initial point by a displacement of three units to the right and two upward is a **representation** of the vector $a = \langle 3, 2 \rangle$. The representation $\overrightarrow{OP}$ from the origin to the point $P(3, 2)$ is called the **position vector** of the point P.

Representations of $a = \langle 3, 2 \rangle$:
The magnitude of the vector v is the length of any of its representations, denoted by $|v|$ or $||v||$. Thus,

$$||a|| = \sqrt{a_2^1 + a_2^2 + a_3^2}$$



Properties of Vectors: If **a, b,** and **c** are vectors in V_n and c and d are scalars, then
- 1. $a + b = b + a$
- 2. $a + (b + c) = (a + b) + c$
- 3. $a + 0 = a$
- 4. $a + (-a) = 0$
- 5. $c(a + b) = ca + cb$
- 6. $(c + d)a = ca + da$
- 7. $(cd)a = c(da)$
- 8. $1a = a$

The vectors i, j, and k are called **standard basis vectors**:

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \mathbf{j} = \langle 0, 1, 0 \rangle, \mathbf{k} = \langle 0, 0, 1 \rangle$$

Thus,

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

Unit vectors: vectors whose lengths are 1.
- **i, j,** and **k** are all unit vectors
- In general, if $a \neq 0$, then the unit vector that has the same direction as **a** is

$$\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

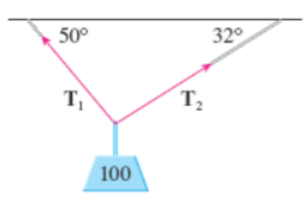
Example: The unit vector in the direction of the vector $\mathbf{2i} - \mathbf{j} - \mathbf{2k}$ can be found first by finding the magnitude:

$$|\mathbf{2i} - \mathbf{j} - \mathbf{2k}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$$

Thus, the unit vector with the same direction is

$$\frac{\mathbf{2i} - \mathbf{j} - \mathbf{2k}}{3} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

Example 2: A 100-lb weight hangs from two wires as shown below. Find the tensions (forces) T_1 and T_2 in both wires and the magnitudes of the tensions.

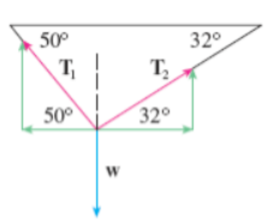


From the figure below, it follows that T_1 and T_2 can be expressed in terms of their horizontal and vertical components s.t.

$$\mathbf{T_1} = -|\mathbf{T_1}| \cos 50° \mathbf{i} + |\mathbf{T_1}| \sin 50° \mathbf{j}$$

and

$$\mathbf{T_2} = -|\mathbf{T_2}| \cos 32° \mathbf{i} + |\mathbf{T_2}| \sin 32° \mathbf{j}$$



The resultant $\mathbf{T_1} + \mathbf{T_2}$ of the tensions counterbalances the weight $\mathbf{w} = -\mathbf{100j}$ and so we must have

$$\mathbf{T_1} + \mathbf{T_2} = -\mathbf{w} = \mathbf{100j}$$

Thus,

$$(-|\mathbf{T_1}|\cos 50^\circ + |\mathbf{T_2}|\cos 32^\circ)\,\mathbf{i} + (|\mathbf{T_1}|\sin 50^\circ + |\mathbf{T_2}|\sin 32^\circ)\,\mathbf{j} = 100\mathbf{j}$$

Equating components, we get

$$\begin{aligned} -|\mathbf{T_1}|\cos 50^\circ + |\mathbf{T_2}|\cos 32^\circ &= 0 \\ |\mathbf{T_1}|\sin 50^\circ + |\mathbf{T_2}|\sin 32^\circ &= 100 \end{aligned}$$

Solving the first of these equations for $|\mathbf{T_2}|$ and substituting into the second, we get

$$\begin{aligned} |\mathbf{T_1}|\sin 50^\circ + \frac{|\mathbf{T_1}|\cos 50^\circ}{\cos 32^\circ}\sin 32^\circ = 100 \\ |\mathbf{T_1}| = \frac{100}{\sin 50^\circ + \tan 32^\circ \cos 50^\circ} \approx 85.64 \text{ lb} \end{aligned}$$

and

$$|\mathbf{T_2}| = \frac{|\mathbf{T_1}|\cos 50^\circ}{\cos 32^\circ} \approx 64.91 \text{ lb}$$

Substituting these values into the original vector equations for T_1 and T_2, it follows that

$$\begin{aligned} \mathbf{T_1} &\approx -55.05\mathbf{i} + 65.60\mathbf{j} \\ \mathbf{T_2} &\approx 55.05\mathbf{i} + 34.40\mathbf{j} \end{aligned}$$

1.3 The Dot Product

The Dot (Scalar) Product

If $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a}\cdot\mathbf{b}$ given by

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Properties of the Dot Product:
1. $a \cdot a = |a|^2$
2. $a \cdot b = b \cdot a$
3. $a \cdot (b + c) = a \cdot b + a \cdot c$
4. $(ca) \cdot b = c(a \cdot b) = a \cdot (cb)$
5. $0 \cdot a = 0$

Angle between vectors

If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

Orthogonal (perpendicular) vectors: vectors whose shared angle is $\theta = \frac{\pi}{2}$.
• For orthogonal vectors, $\mathbf{a}\cdot\mathbf{b} = 0$.

Dot Product and Orthogonality

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal iff $\mathbf{a}\cdot\mathbf{b} = 0$.

Example 2: Since

$$(2i + 2j - k)\cdot(5i - 4j + 2k) = 2(5) + 2(-4) + (-1)(2) = 0$$

these vectors are perpendicular.

The dot product $\mathbf{a}\cdot\mathbf{b}$ is positive if $\mathbf{a}$ and $\mathbf{b}$ point in the same general direction, 0 if they are perpendicular, and negative if they point in generally opposite directions. For the case in which $\mathbf{a}$ and $\mathbf{b}$ point in exactly the same direction, we have $\theta = 0 \implies \cos\theta = 1$ and

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|$$

If $\mathbf{a}$ and $\mathbf{b}$ point in exactly opposite directions, then we have $\theta = \pi$ and so $\cos\theta = -1$ and $\mathbf{a}\cdot\mathbf{b} = -|\mathbf{a}||\mathbf{b}|$.

Pythaogrean Theorem in 3D

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

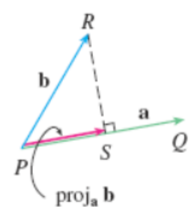
Theorem

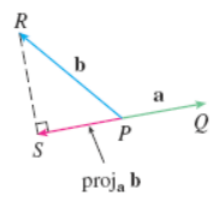
$$\frac{\mathbf{a}}{|\mathbf{a}|} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

Example: Find the direction angles of the vector $\mathbf{a} = \langle 1, 2, 3 \rangle$.
Since $|\mathbf{a}| = \sqrt{1^2 + 2^2 + 3^2}\sqrt{14}$, it follows that

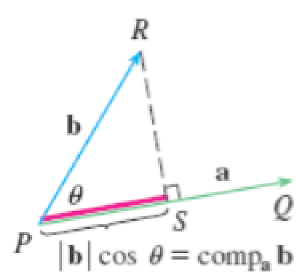
$$\cos\alpha = \frac{1}{\sqrt{14}} \implies \alpha = \arccos\left(\frac{1}{\sqrt{14}}\right) \approx 74^\circ$$
$$\cos\beta = \frac{2}{\sqrt{14}} \implies \beta = \arccos\left(\frac{2}{\sqrt{14}}\right) \approx 58^\circ$$
$$\cos\gamma = \frac{3}{\sqrt{14}} \implies \gamma = \arccos\left(\frac{3}{\sqrt{14}}\right) \approx 37^\circ$$

The figure below shows representations $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ with the same initial point P. If S is the foot of the perpendicular from R to the line containing $\overrightarrow{PQ}$, then the vector with representation $\overrightarrow{PS}$ is called the **vector projection** of $\mathbf{b}$ onto $\mathbf{a}$ and is denoted by $\text{proj}_a\mathbf{b}$.





The **scalar projection (component of b along a)** is shown in the figure below.



Scalar projection of b onto a

$$\text{comp}_\mathbf{a}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$$

Vector projection of b onto a

$$\text{proj}_\mathbf{a}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$$

Notice that the vector projection is the scalar projection times the unit vector in the direction of $\mathbf{a}$.

Example 3: Find the scalar and vector projections of $\overrightarrow{b} = \langle 1, 1, 2 \rangle$ onto $\overrightarrow{a} = \langle -2, 3, 1 \rangle$.

Since

$$|a| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14},$$

the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\text{comp}_\mathbf{a}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}} = \frac{3}{\sqrt{14}}$$

The vector projection is this scalar projection times the unit vector in the direction of $\mathbf{a}$:

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{3}{\sqrt{14}}\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{14}\mathbf{a} = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle$$

Work and the Dot Product

The work done by a constant force $\mathbf{F}$ is the dot product $\mathbf{F} \cdot \mathbf{D}$, where $\mathbf{D}$ is the displacement vector:

$$W = \mathbf{F} \cdot \mathbf{D}$$

1.4 The Cross Product

Given two nonzero vectors $\overrightarrow{a} = \langle a_1, a_2, a_3 \rangle$ and $\overrightarrow{b} = \langle b_1, b_2, b_3 \rangle$, it is very useful to be able to find a nonzero vector c that is perpendicular to both $\overrightarrow{a}$ and $\overrightarrow{b}$. If $\overrightarrow{c} = \langle c_1, c_2, c_3 \rangle$ is such a vector, then $\mathbf{a} \cdot \mathbf{c} = 0$ and $\mathbf{b} \cdot \mathbf{c} = 0$ and so

$$a_1c_1 + a_2c_2 + a_3c_3 = 0$$

and

$$b_1c_1 + b_2c_2 + b_3c_3 = 0$$

Eliminating c_3, we can multiply the first equation by b_3 and the second equation by a_3. Subtracting, it follows that

$$(a_1b_3 - a_3b_1)c_1 + (a_2b_3 - a_3b_2)c_2 = 0$$

The above equation has the form $pc_1 + qc_2 = 0$, for which an obvious solution is $c_1 = q$ and $c_2 = -p$, s.t. a solution of the equation is

$$c_1 = a_2b_3 - a_3b_2, c_2 = a_3b_1 - a_1b_3$$

Substituting these values into the first two equations,

$$c_3 = a_1b_2 - a_2b_1$$

Hence, a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ has the form

$$\langle c_1, c_2, c_3 \rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

This vector is known as the cross product of $\mathbf{a}$ and $\mathbf{b}$ and is denoted by $\mathbf{a} \times \mathbf{b}$.

Cross Product (Vector Product)

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** or $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1$$

NOTE: $\mathbf{a} \times \mathbf{b}$ is only defined when a and b are 3D vectors.

The cross product is a vector whereas the dot product is a scalar.

In order to make the cross product easier to remember, determinant notation is used. A **determinant of order 2** is defined as follows.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

A **determinant of order 3** can be defined in terms of second-order determinants like so:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Thus, the definition of the cross product can be rewritten using second-order determinants and the standard basis vectors **i, j, k**.

Definition of Cross Product In Determinant Notation

Let vectors $\mathbf{a}$ and $\mathbf{b}$ be given by $a = a_1i + a_2j + a_3k$ and $b = b_1i + b_2j + b_3k$ s.t.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

Alternatively,

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example: Let $\mathbf{a} = \langle 1, 3, 4\rangle$ and $\mathbf{b} = \langle 2, 7, -5\rangle$ s.t.

$$\begin{aligned}
\mathbf{a}\times\mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} \\
&= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}\mathbf{k} \\
&= (-15-28)\mathbf{i} - (-5-8)\mathbf{j} + (7-6)\mathbf{k} \\
&= -43\mathbf{i} + 13\mathbf{j} + \mathbf{k}
\end{aligned}$$

Example 3: Show that $\mathbf{a}\times\mathbf{a} = 0$ for any vector $\mathbf{a}$ in V_3.
If $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, then

$$\begin{aligned}
\mathbf{a}\times\mathbf{a} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \\
&= (a_2a_3 - a_3a_2)\mathbf{i} - (a_1a_3 - a_3a_1)\mathbf{j} + (a_1a_2 - a_2a_1)\mathbf{k} \\
&= 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} \\
&= 0
\end{aligned}$$

and thus our assertion.

Theorem

The vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Proof.

$$\begin{aligned}
(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3 \\
&= a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_3(a_1b_2 - a_2b_1) \\
&= a_1a_2b_3 - a_1b_2a_3 - a_1a_2b_3 + b_1a_2a_3 + a_1b_2a_3 - b_1a_2a_3 \\
&= 0
\end{aligned}$$

A similar computation yields $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = 0$. Hence, $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. □

Theorem

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \le \theta \le \pi$), then

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

Proof. From the definitions of the cross product and length of a vector, we have

$$\begin{aligned}
|\mathbf{a}\times\mathbf{b}|^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\
&= a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2 + a_3^2b_1^2 - 2a_1a_3b_1b_3 + a_1^2b_3^2 + a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2 \\
&= \left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\
&= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2 \\
&= |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta \\
&= |\mathbf{a}|^2|\mathbf{b}|^2(1 - \cos^2\theta) \\
&= |\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta
\end{aligned}$$

Taking square roots and observing that $\sqrt{\sin^2\theta} = \sin\theta$ because $\sin\theta \ge 0$ when $0 \le \theta \le \pi$, it follows that

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

□

Scalar Triple Product: the product $\mathbf{a}\cdot(b\times c)$, **a,b,** and **c** are vectors.
• is written as a determinant:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The geometric significance of the scalar triple product can be seen when considering the parallelepiped (prism with 6 parallelograms as bases) determined by the vectors **a,b,** and **c**. The area of the base parallelogram is $A = |\mathbf{b}\times\mathbf{c}|$. If θ is the angle between $\mathbf{a}$ and $\mathbf{b}\times\mathbf{c}$, then the height h of the parallelepiped is $h = |\mathbf{a}||\cos\theta|$. Hence, the volume of the parallelepiped is

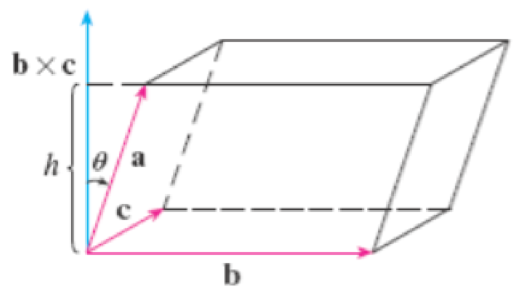
$$V = Ah = |\mathbf{b} \times \mathbf{c}||\mathbf{a}||\cos\theta| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

which proves the following formula.

Volume of Parallelepiped

The volume of the parallelepiped determined by the vectors **a,b,** and **c** is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$



Coplanar: lying in the same plane
• if the volume of the parallelepiped determined by **a, b,** and **c** is 0, then the vectors must be coplanar

Example 4: Use the scalar triple product to show that the vectors $\mathbf{a} = \langle 1, 4, -7 \rangle$, $\mathbf{b} = \langle 2, -1, 4 \rangle$, and $\mathbf{c} = \langle 0, -9, 18 \rangle$ are coplanar.

$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & 9 & 18 \end{vmatrix} \\
&= 1 \begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7 \begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix} \\
&= 1(18) - 4(36) - 7(-18) \\
&= 0
\end{aligned}$$

Hence, the volume of the parallelepiped determined by **a, b,** and **c** is 0 and so **a, b,** and **c** are coplanar.

Vector Triple Product: the product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

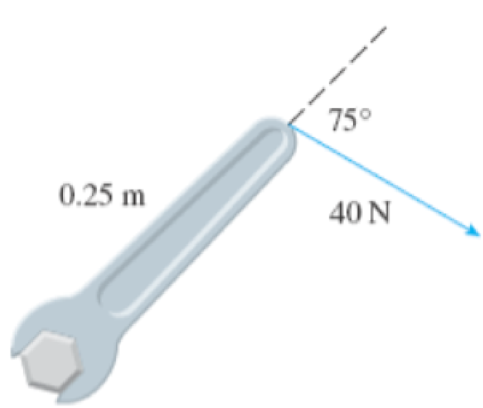
The **torque** (τ) is defined to be the cross product of the position and force vectors and measures the tendency of the body to rotate about the origin s.t.

$$\tau = r \times F$$

The direction of the torque vector indicates the axis of rotation. The magnitude of the torque vector is given as follows.

$$|\tau| = |r \times F| = |r||F| \sin\theta$$

Example 5: A bolt is tightened by applying a 40 N force to a 0.25 m wrench as shown below. Find the magnitude of the torque about the center of the bolt.



The magnitude of the torque vector is

$$
\begin{aligned}
|\tau| &= |r \times F| \\
&= |r||F| \sin 75^\circ \\
&= (0.25)(40) \sin 75^\circ \\
&= 10 \sin 75^\circ \\
&\approx 9.66 \mathrm{N} \cdot \mathrm{m}
\end{aligned}
$$

If the bolt is right-threaded, then the torque vector itself is

$$\tau = |\tau| \text{ n } \approx 9.66 \text{ n},$$

where **n** is a unit vector directed down into the page (by the right-hand rule).

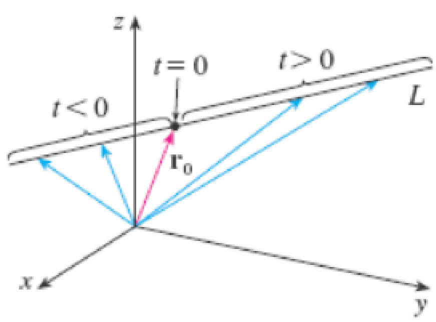
1.5 Equations of Lines and Planes

Let $P(x, y, z)$ and $P(x_0, y_0, z_0)$ be arbitrary points on a line L in 3D space, and let r_0 and r be the position vectors of P_0 and P. If **a** is the vector with representation $\overrightarrow{P_0 P}$, then the Triangle Law for vector addition gives $\overrightarrow{r} = \overrightarrow{r_0} + a$. But since **a** and **v** are parallel vectors, there is a scalar t s.t. $\mathbf{a} = t\mathbf{v}$. Thus we get a **vector equation** of L:

Vector Equation

$$\overrightarrow{r} = \overrightarrow{r_0} + t\overrightarrow{v}$$

Each value of the parameter t gives the position vector r of a point on L, i.e. as t varies, the line is traced out by the tip of the vector $\overrightarrow{r}$ as in the figure below. Note how positive values of t correspond to points on L that lie on one side of P_0, whereas negative values of t correspond to points that lie on the other side of P_0.



If the vector **v** that gives the direction of line L is written in component form, then $t\overrightarrow{v} = \langle ta, tb, tc \rangle$, $r = \langle x, y, z \rangle$, and $r_0 = \langle x_0, y_0, z_0 \rangle$ and

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

It follows then that

Parametric Equations

$$x = x_0 + at, y = y_0 + bt, z = z_0 + ct, t \in \mathbb{R}$$

These equations are called **parametric equations** of the line L through the point P_0 and parallel to the vector $\overrightarrow{v} = \langle a, b, c \rangle$.

If a vector $\overrightarrow{v} = \langle a, b, c \rangle$ is used to describe the direction of a line L, then the numbers a, b, and c are called **direction numbers** of L.

We can also describe the line L by eliminating the parameter t. If $a, b, c \neq 0$, then

$$t = \frac{x - x_0}{a}, t = \frac{y - y_0}{b}, t = \frac{z - z_0}{c}$$

Equating the results, we obtain the symmetric equations of L.

Symmetric Equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}, a, b, c \neq 0$$

If one of the direction numbers, e.g. a was zero, then

$$x = x_0, \frac{y - y_0}{b}, \frac{z - z_0}{c}$$

and L lies in the vertical plane $x = x_0$.

Line Segment

The line segment from r_0 to r_1 is given by the vector equation

$$r(t) = (1-t)r_0 + tr_1, 0 \leq t \leq 1$$

Skew lines: lines that do not intersect and are not parallel

Example: Show that the lines L_1 and L_2 with parametric equations

$$\begin{aligned} L_1 x &= 1+t, & y &= -2+3t, z = 4-t \\ L_2 x &= 2s, & y &= 3+s z = -3+4s \end{aligned}$$

are skew lines.

The lines are not parallel because the corresponding direction vectors $\langle 1, 3, -1\rangle$ and $\langle 2, 1, 4\rangle$ are not parallel. If L_1 and L_2 had a point of intersection, there would be values of t and s s.t.

$$\begin{aligned} 1+t &= 2s \\ -2+3t &= 3+s \\ 4-t &= -3+4s \end{aligned}$$

However, if we solve the first two equations, we get $t = \frac{11}{5}$ and $s = \frac{8}{5}$, and these values don't satisfy the third equation. Hence, there are no values of t and s that satisfy the three equations, so L_1 and L_2 don't intersect and L_1 and L_2 are skew lines.

Planes are determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\overrightarrow{n}$ that is orthogonal to the plane, called a **normal vector**. Let $P(x, y, z)$ be a point in the plane, and let $\overrightarrow{r_0}$ and $\overrightarrow{r}$ be the position vectors of P_0 and P. Then the vector $\overrightarrow{r - r_0} = \overrightarrow{P_0P}$. Since the normal vector $\overrightarrow{n}$ is orthogonal to every vector in the given plane, $\overrightarrow{n}$ is orthogonal to $\overrightarrow{r - r_0}$ and so we get the vector equations of the plane.

Vector Equations of a Plane

$$\overrightarrow{n}(\overrightarrow{r} - \overrightarrow{r_0}) = 0$$

Alternatively,

$$\overrightarrow{n} \cdot \overrightarrow{r} = \overrightarrow{n} \cdot \overrightarrow{r_0}$$

Scalar Equation of a Plane

A **scalar equation of the plane** through point $P_0(x_0, y_0, z_0)$ with normal vector $\overrightarrow{n} = \langle a, b, c\rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Linear Equation

By rewriting the scalar equation of a plane, it follows that we get a linear equation in x, y, z:

$$ax + by + cz + d = 0,$$

where

$$d = -(ax_0 + by_0 + cz_0)$$

Example 2: Find an equation of the plane that passes through the points $P(1, 3, 2)$,$Q(3, -1, 6)$, and $R(5, 2, 0)$. The vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ corresponding to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are

$$\overrightarrow{a} = \langle 2, -4, 4\rangle, \overrightarrow{b} = \langle 4, -1, -2$$

Since both $\overrightarrow{a}$ and $\overrightarrow{b}$ lie in the plane, their cross product $\mathbf{a} \times \mathbf{b}$ is orthogonal to the plane and can be taken as a normal vector. Thus

$$\overrightarrow{n} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\mathbf{i} + 20\mathbf{j} + 14\mathbf{k}$$

With the point $P(1, 3, 2)$ and the normal vector $\overrightarrow{n}$, an equation of the plane is

$$12(x-1) + 20(y-3) + 14(z-2) = 0 \iff 6x + 10y + 7z = 50$$

Two planes are parallel if their normal vectors are parallel. If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors.

Example 3:
(a) Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.
(b) Find symmetric equations for the line of intersection L of these two planes.

(a) The normal vectors of these planes are

$$\vec{n_1} = \langle 1, 1, 1 \rangle, \vec{n_2} = \langle 1, -2, 3 \rangle$$

and so, if θ is the angle between the planes, it follows that

$$\begin{aligned}
\cos\theta &= \frac{\mathbf{n_1} \cdot \mathbf{n_2}}{|\mathbf{n_1}||\mathbf{n_2}|} \\
&= \frac{1(1) + 1(-2) + 1(3)}{\sqrt{1+1+1}\sqrt{1+4+9}} \\
&= \frac{2}{\sqrt{42}} \\
\theta &= \arccos\left(\frac{2}{\sqrt{42}}\right) \\
&\approx 72^\circ
\end{aligned}$$

(b) We need to find a point on L. We can find the point where the line itnersects the xy-plane by setting $z = 0$ in the equations of both planes. This gives the equations $x + y = 1$ and $x - 2y = 1$, whose solution is $x = 1, y = 0$. So the point $(1, 0, 0)$ lies on L. Note that since L lies in both planes, it is perpendicular to both of the normal vectors. Thus a vector $\vec{v}$ parallel to L is given by the cross product

$$\mathbf{v} = \mathbf{n_1} \times \mathbf{n_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$$

and so the symmetric equations of L can be written as

$$\frac{x-1}{5} = \frac{y}{-2} = \frac{z}{-3}$$

Distance between a Point and Plane

The distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is given by

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Proof. Let $P_0(x_0, y_0, z_0)$ be any point in the given plane and let $\mathbf{b}$ be the vector corresponding to $\overrightarrow{P_0P_1}$. Then

$$\vec{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$

The distance D from P_1 to the plane is equal to the absolute value of the scalar projection of $\mathbf{b}$ onto the normal vector $\vec{n} = \langle a, b, c \rangle$, hence

$$\begin{aligned}
D &= |\text{comp}_n b| \\
&= \frac{|n \cdot b|}{|n|} \\
&= \frac{a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)}{\sqrt{a^2 + b^2 + c^2}} \\
&= \frac{|(ax_1 - by_1 + cz_1) - (ax_0 - by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}
\end{aligned}$$

□

1.6 Cylinders and Quadric Surfaces

Cross-sections (traces): curves of intersection of a surface with planes parallel to the coordinate planes
Cylinder: a surface that consists of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve
Parabolic cylinder: a surface made up of infinite many shifted copies of the same parabola
Quadric surface: the graph of a second-degree equation in three variables x, y, and z. The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

Through translation and rotation the equation can be written in one of the two following standard forms.

$$Ax^2 + By^2 + Cz^2 + J = 0$$

or

$$Ax^2 + By^2 + Iz = 0$$

Example: Use traces to sketch the quadric surface with equation

$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$$

By substituting $z = 0$, we find that the trace in the xy-plane is $x^2 + \frac{y^2}{9} = 1$, which we recognize as an equation of an ellipse. In general, the horizontal trace in the plane $z = k$ is

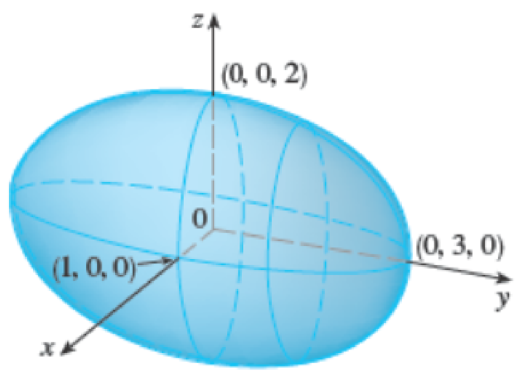
$$x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4}, z = k$$

which is an ellipse, provided that $k^2 < 4$, that is, $-2 < k < 2$. Similarly, vertical traces parallel to the yz and xz-planes are also ellipses:

$$\frac{y^2}{9} + \frac{z^2}{4} = 1 - k^2 \qquad x = k(\text{if } -1 < k < 1)$$

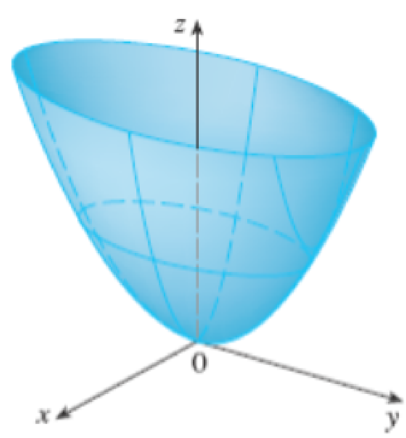
$$x^2 + \frac{z^2}{4} = 1 - \frac{k^2}{9} \qquad y = k(\text{if } -3 < k < 3)$$

This surface is called an **ellipsoid** because all of its traces are ellipses. It is sketched below.



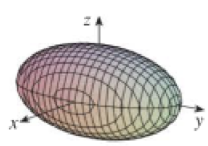
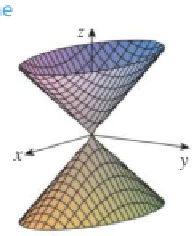
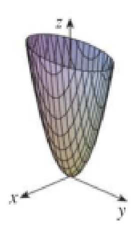
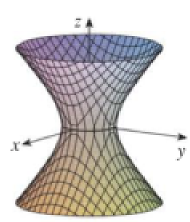
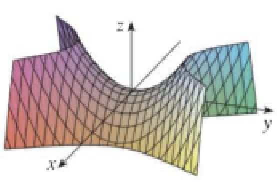
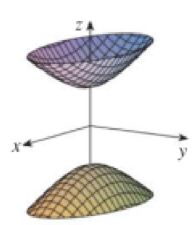
Example 2: Use traces to sketch the surface $z = 4x^2 + y^2$

When $x = 0$, $z = y^2$, so the yz-plane intersects the surface in a parabola. If we let $x = k$ (a constant), we get $z = y^2 + 4k^2$. This means that if we slice the graph with any plane parallel to the yz-plane, we obtain a parabola that opens upward. Similarly, if $y = k$, the trace is $z = 4x^2 + k^2$, which is again a parabola that opens upward. If we let $z = k$, we get the horizontal traces $4x^2 + y^2 = k$, which we recognize as a fmaily of ellipses. Knowing the shapes of the traces, we can sketch the graph like so:



Because of the elliptical and parabolic traces, this surface is called an **elliptic paraboloid**.

Graphs of Quadric Surfaces:
Circular paraboloids are used for satellite dishes. Cooling towers for nuclear reactors are designed in the shape of hyperboloids of one sheet for structural stability.

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

2 Vector Functions

2.1 Vector Functions and Space Curves

Vector function: a function whose domain is a set of real numbers and whose range is a set of vectors

Let $r(t)$ be a vector function whose values are three-dimensional vectors. If $f(t)$, $g(t)$, and $h(t)$ are the components of the vector $r(t)$, then f, g, and h are real-valued functions called the **component functions** of **r** and we can write

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

The limit of a vector function $r(t) = \langle f(t), g(t), h(t) \rangle$ is defined by taking the limits of its component functions like so:

$$\lim_{t\to a} r(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle$$

iff the limits of the component functions exist.

Alternatively, $\lim_{t\to a} r(t) = L$ iff for every $\epsilon > 0$ there is a number $\delta > 0$ s.t. if

$$0 < |t - a| < \delta$$

then

$$|r(t) - b| < \epsilon$$

A vector function r is **continuous at a** if

$$\lim_{t\to a} r(t) = r(a)$$

Suppose that f, g, and h are continuous real-valued functions on an interval I. Then the set C of all points (x, y, z) in space, where

$$x = f(t), y = g(t), z = h(t)$$

and t varies throughout the interval I, is called a **space curve**. Considering the vector function $r(t)$, $r(t)$ is the position vector of the point $P(f(t), g(t), h(t))$ on C.

Plane curves can also be described by vectors. Consider the curve given by the parametric equations $x = t^2 - 2t$ and $y = t + 1$ which could be described by

$$r(t) = \langle t^2 - 2t, t + 1 \rangle = (t^2 - 2t)\mathbf{i} + (t + 1)\mathbf{j}$$

where $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$.

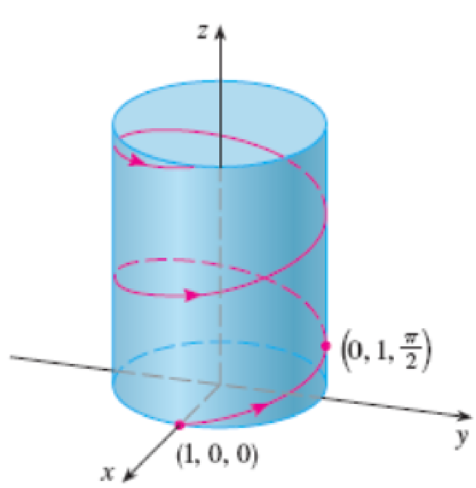
Example: Sketch the curve whose vector equation is

$$r(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$$

The parametric equations for this curve are

$$x = \cos t, y = \sin t, z = t$$

Because $x^2+y^2 = \cos^2 t+\sin^2 t = 1$ for all values of t, the curve must lie on the circular cylinder $x^2+y^2 = 1$. The point (x, y, z) lies directly above the point $(x, y, 0)$, which moves counterclockwise around the circle $x^2 + y^2 = 1$ in the xy-plane. The projection of the curve onto the xy-plane has vector equation $r(t) = \langle \cos t, \sin t, 0 \rangle$. Since $z = t$, the curve spirals upwards around the cylinder as t increases. The curve shown below is called a **helix**.



Example 2: Find a vector equation and parametric equations for the line segment that joins the point $P(1,3,-2)$ to the point $Q(2,-1,3)$.

Recall that a vector equation for the line segment that joins the tip of the vector r_0 to the tip of the vector r_1 is:

$$r(t) = (1-t)r_0 + tr_1, 0 \leq t \leq 1$$

Taking $r_0 = \langle 1,3,-2\rangle$ and $r_1 = \langle 2,-1,3\rangle$ to obtain a vector equation of the line segment from P to Q, it follows that

$$r(t) = (1-t)\langle 1,3,-2\rangle + t\langle 2,-1,3\rangle, 0 \leq t \leq 1$$

or

$$r(t) = \langle 1+t, 3-4t, -2+5t\rangle, 0 \leq t \leq 1$$

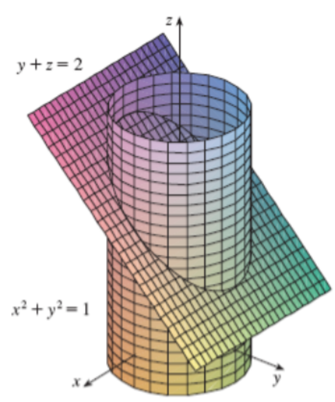
and the corresponding parametric equations are

$$x = 1+t, y = 3-4t, z = -2+5t, 0 \leq t \leq 1$$

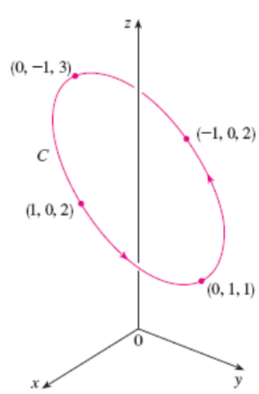
Example 3: Find a vector function that represents the curve of intersection of the cylinder $x^2+y^2=1$ and the plane $y+z=2$.

The below figures show the intersection of the plane and the cylinder, and the curve of intersection C, an ellipse, respectively.

Intersection of Plane and Cylinder



Intersection C, an Ellipse



The projection of C onto the xy-plane is the circle $x^2+y^2=1, z=0$. Thus we can write

$$x = \cos t, y = \sin t, 0 \leq t \leq 2\pi$$

From the equation of the plane, we have

$$z = 2 - y = 2 - \sin t$$

and we can write parametric equations for C as

$$x = \cos t, y = \sin t, z = 2 - \sin t, 0 \leq t \leq 2\pi$$

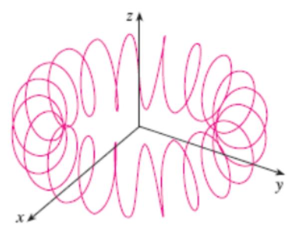
The corresponding vector equation is

$$r(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + (2-\sin t)\mathbf{k}, 0 \leq t \leq 2\pi$$

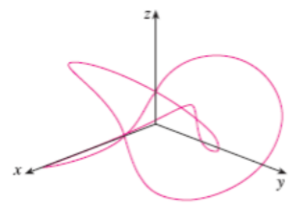
This equation is known as a *parameterization* of the curve C.

Space curves are much more difficult to draw by hand than plane curves and for an accurate representation we must use technology. Below are some examples of computer-generated space curves.

A **Toroidal Spiral**



A **Trefoil Knot**



2.2 Derivatives and Integrals of Vector Functions

Definition of Derivative of Vector Function

The derivative $\mathbf{r}'$ of a vector function $\mathbf{r}$ is given by

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$$

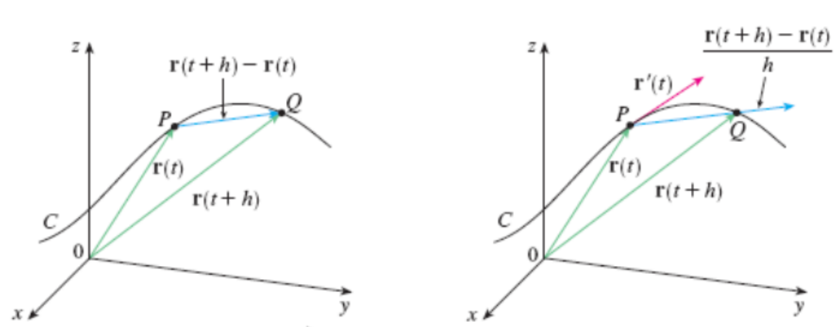
if the limit exists.

If the points P and Q have position vectors $r(t)$ and $r(t+h)$, then $\overrightarrow{PQ}$ represents the vector $r(t+h)-r(t)$, which can be regarded as a secant vector. If $h>0$, then the scalar multiple $\left(\frac{1}{h}\right)(r(t+h)-r(t))$ has the same direction as $r(t+h)-r(t)$. As h converges to 0, the scalar multiple $\left(\frac{1}{h}\right)(r(t+h)-r(t))$ has the same direction as $r(t+h)-r(t)$. As $h\to 0$, this vector approaches a vector that lies on the tangent line. For this reason, the vector $r'(t)$ is called the **tangent vector** to the curve defined by $\mathbf{r}$ at the point P, provided that $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t)\neq 0$. The **tangent line** to C at P is defined to be the line through P parallel to the tangent vector $\mathbf{r}'(t)$.

Unit tangent vector:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

When $0<h<1$, multiplying the secant vector by $\frac{1}{h}$ "stretches" the vector, as shown below.



(a) The secant vector $\overrightarrow{PQ}$

(b) The tangent vector $\mathbf{r}'(t)$

Geometrically, the derivative tells us that as $h\to 0$, the quotient of the secant vector $r(t+h)-r(t)$ and h approaches the tangent vector $r(t)$.

Differentiation Rules for Vector Functions:

$\frac{d}{dt}[u(t)+v(t)] = u'(t)+v'(t)$	Sum/Difference
$\frac{d}{dt}[cu(t)] = cu'(t)$	Constant Multiple
$\frac{d}{dt}[f(t)u(t)] = f'(t)u(t)+f(t)u'(t)$	Product
$\frac{d}{dt}[u(t)\cdot v(t)] = u'(t)\cdot v(t)+u(t)\cdot v'(t)$	Product, Dot
$\frac{d}{dt}[u(t)\times v(t)] = u'(t)\times v(t)+u(t)\times v'(t)$	Product, Cross
$\frac{d}{dt}[u(f(t))] = f'(t)u'(f(t))$	Chain Rule

Example: Show that if $|r(t)| = c$ where c is a constant, then $r'(t)$ is orthogonal to $r(t)$ for all t.

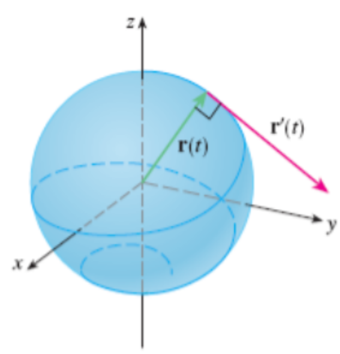
Since

$$r(t)\cdot r(t) = |r(t)|^2 = c^2$$

and c^2 is a constant, it follows that

$$0 = \frac{d}{dt}[r(t)\cdot r(t)] = r'(t)\cdot r(t) + r(t)\cdot r'(t) = 2r'(t)\cdot r(t)$$

Hence, $r'(t)\cdot r(t) = 0$ which means that $r'(t)$ is orthogonal to $r(t)$. Geometrically, this result tells us that if a curve lies on a sphere with center the origin, then the tangent vector $r'(t)$ is always perpendicular to the position vector $r(t)$.



Let $r(t) = \langle f(t), g(t), h(t)\rangle$ be a continuous vector function. Then

FTC for Continuous Vector Functions

Let R be an antiderivative of r, that is, $R'(t) = r(t)$. Then

$$\int_a^b r(t)dt = R(t)\Big|_a^b = R(b) - R(a)$$

2.3 Arc Length and Curvature

Arc Length

For a space curve with vector equation $r(t) = \langle f(t), g(t), h(t)\rangle, a \le t \le b$, or equivalently, the parametric equations $x = f(t), y = g(t), z = h(t)$, where f', g', h' are continuous. If the curve is traversed exactly once as t increases from a to b, then it can be shown that its length is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}dt$$

or

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}dt$$

or

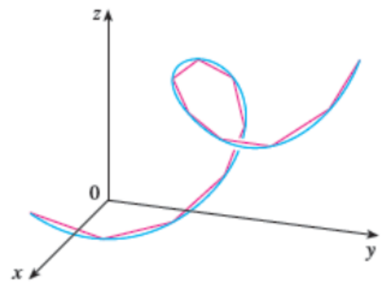
$$L = \int_a^b |r'(t)|dt$$

Geometrically, the length of a space curve is the limit of lengths of inscribed polygons:
Example: Find the length of the arc of the circular helix with vector equation $r(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.

Since $r'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$, we have

$$|r'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t + 1} = \sqrt{2}$$

The arc from $(1, 0, 0)$ to $(1, 0, 2\pi)$ is described by the parameter interval $0 \le t \le 2\pi$, and so it follows that



$$L = \int_0^{2\pi} |r'(t)|dt = \int_0^{2\pi} \sqrt{2}dt = 2\sqrt{2}\pi$$

Arc Length Function

Suppose that C is a curve given by a vector function

$$r(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}, a \leq t \leq b$$

where r' is continuous and C is traversed exactly once as t increases from a to b. The **arc length function** s is given by

$$s(t) = \int_a^t |r'(u)|du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} du$$

Arc Length Function: Corollary

Differentiating both sides of the above equation, it follows that

$$\frac{ds}{dt} = |r'(t)|$$

This corollary is useful when *parameterizing a curve with respect to arc length.* If a curve $r(t)$ is already given in terms of a paramter and $s(t)$ is the arc length function foun, then we may be able to solve for t as a function of $s : t = t(s)$. Thus, the curve can be reparamterized in terms of s by substituting for $t : r = r(t(s))$. If $s = 3$ for instance, $r(t(3))$ is the position vector of the point 3 units of length along the curve from its starting point.

Example 2: Reparametrize the helix $r(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$ with respect to arc length measured from $(1, 0, 0)$ in the direction of increasing t.

The initial point $(1, 0, 0)$ corresponds to the parameter value $t = 0$. From the first example we have that

$$\frac{ds}{dt} = |r'(t)|$$

and so

$$s = s(t) = \int_0^t |r'(u)|du = \int_0^t \sqrt{2}du = \sqrt{2}t$$

Therefore $t = \frac{s}{\sqrt{2}}$ and the required reparametrization is obtained by substituting for t:

$$r(t(s)) = \cos\left(\frac{s}{\sqrt{2}}\right)\mathbf{i} + \sin\left(\frac{s}{\sqrt{2}}\right)\mathbf{j} + \left(\frac{s}{\sqrt{2}}\right)\mathbf{k}$$

A parameterization $\mathbf{r}(t)$ is called **smooth** on an interval I if $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq 0$ on I. A curve is called **smooth** if it has a smooth parametrization. A smooth curve hass no sharp corners or cusps; when the tangent vector turns, it does so continuously.

Unit Tangent Vector

For a smooth curve C defined by the vector function r, the unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

This vector indicates the direction of the curve. $\mathbf{T}(t)$ changes direction slowly when the curve is relatively straight, but it changes direction more quickly when C twists or turns more sharply.

The curvature of C at a given point is a measure of how quickly the curve changes direction at that point. Because the unit tangent vector has constant length, only changes in direction contribute to the rate of change of $\mathbf{T}$.

Curvature

The **curvature** of a curve is

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$$

where $\mathbf{T}$ is the unit tangent vector.

The curvature is easier to compute if it is expressed in terms of the paramter t instead of s, so we use the Chain Rule to write

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt}$$

and

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{\frac{d\mathbf{T}}{dt}}{\frac{ds}{dt}}\right|$$

But $\frac{ds}{dt} = |r'(t)|$, so

Curvature: Corollary

$$\kappa(t) = \left|\frac{\mathbf{T}'(t)}{\mathbf{r}'(t)}\right|$$

Example 3: Show that the curvature of a circle of radius a is $\frac{1}{a}$.

Taking the circle to have center the origin, a parametrization is

$$\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j}$$

Therefore

$$\mathbf{r}'(t) = -a\sin t\mathbf{i} + a\cos t\mathbf{j}$$

and

$$|r'(t)| = a$$

so

$$T(t) = \frac{r'(t)}{|r'(t)|} = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

and

$$T'(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}$$

This gives $|T'(t)| = 1$ and it follows that

$$\kappa(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{1}{a}$$

Curvature

Although the other curvature formula can be used in all cases to compute the curvature, the formula below is often more convenient to apply. For the vector function $\mathbf{r}$, the curvature is given by

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

Proof. Since $\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|}$ and $|\mathbf{r}'| = \frac{ds}{dt}$, we have

$$r' = |r'|T = \frac{ds}{dt}T$$

so the Product Rule gives

$$r'' = \frac{d^2s}{dt^2}T + \frac{ds}{dt}T'$$

Using the fact that $T \times T = 0$, we have

$$r' \times r'' = \left(\frac{ds}{dt}\right)^2 (T \times T')$$

Now $|T(t)| = 1$ for all t, so **T** and **T'** are orthogonal. Hence,

$$|r' \times r''| = \left(\frac{ds}{dt}\right)^2 |T \times T'| = \left(\frac{ds}{dt}\right)^2 |T||T'| = \left(\frac{ds}{dt}\right)^2 |T'|$$

Thus

$$|T'| = \frac{|r' \times r''|}{\left(\frac{ds}{dt}\right)^2} = \frac{|r' \times r''|}{|r'|^2}$$

and

$$\kappa = \frac{|T'|}{|r'|} = \frac{|r' \times r''|}{|r'|^3}$$

$\square$

Example 4: Find the curvature of the twisted cubic $r(t) = \langle t, t^2, t^3 \rangle$ at a general point and at $(0, 0, 0)$.

We first compute the required ingredients:

$$\begin{aligned} r'(t) &= \langle 1, 2t, 3t^2 \rangle \\ r''(t) &= \langle 0, 2, 6t \rangle \\ r'(t) \times r''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 6t^2\mathbf{i} - 6t\mathbf{j} + 2\mathbf{k} \\ |r'(t) \times r''(t)| &= \sqrt{36t^4 + 36t^2 + 4} = 2\sqrt{9t^4 + 9t^2 + 1} \end{aligned}$$

Thus

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{2\sqrt{1 + 9t^2 + 9t^4}}{(1 + 4t^2 + 9t^4)^{\frac{3}{2}}}$$

At the origin, where $t = 0$, the curvature is $\kappa(0) = 2$.

Curvature

For the special case of a plane curve with equation $y = f(x)$, we choose x as the parameter and write $r(x) = x\mathbf{i} + f(x)\mathbf{j}$. Then $r'(x) = \mathbf{i} + f'(x)|mathbffj$ and $\mathbf{r}''(x) = f''(x)\mathbf{j}$. Since $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{j} = 0$, it follows that $r'(x) \times r''(x) = f''(x)k$. We also have $|r'(x)| = \sqrt{1 + [f'(x)]^2}$ and thus

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{|frac32}}$$

Example 5: Find the curvature of the parabola $y = x^2$ at the points (0,0), (1,1), and (2,4).

Since $y' = 2x$ and $y'' = 2$, we have

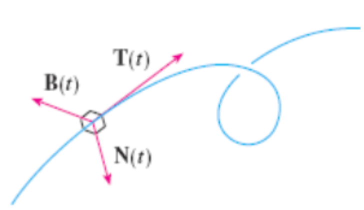
$$\kappa(x) = \frac{|y''|}{[1 + (y')^2]^{\frac{3}{2}}} = \frac{2}{(1 + 4x^2)^{\frac{3}{2}}}$$

The curvature at (0,0) is $\kappa(0) = 2$. At (1,1) it is $\kappa(1) = \frac{2}{5^{\frac{3}{2}}} \approx 0.18$. At (2,4) it is $\kappa(2) = \frac{2}{17^{\frac{3}{2}}} \approx 0.03$.

At a given point on a smooth space curve $r(t)$, there are many vectors that are orthogonal to the unit tangent vector $\mathbf{T}(t)$. We single out one by observing that, becasue $|\mathbf{T}(t)| = 1 \forall t$, we have $\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0$, so $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$. Noe that in general $\mathbf{T}'(t)$ is itself not a unit vector. But at any point where $\kappa \neq 0$ we can define the **principle unit normal vector** $\mathbf{N}(t)$ (**unit normal**) as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

Geometrically, the unit normal vector indicates the direction in which the curve is turing at each point. The vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ is called the **binormal vector** and it is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$ and is also a unit vector.



Example 6: Find the unit normal and binormal vectors for the circular helix

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$$

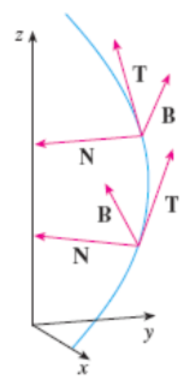
We first compute the ingredients needed for the unit normal vector:

$$
\begin{aligned}
r'(t) &= -\sin t i + \cos t j + k \\
|r'(t)| &= \sqrt{2} \\
T(t) &= \frac{r'(t)}{|r'(t)|} = \frac{1}{\sqrt{2}}(-\sin t i + \cos t j + k) \\
T'(t) &= \frac{1}{\sqrt{2}}(-\cos t i - \sin t j) \\
|T'(t)| &= \frac{1}{\sqrt{2}} \\
N(t) &= \frac{T'(t)}{|T'(t)|} = -\cos t i - \sin t j = \langle -\cos t, -\sin t, 0\rangle
\end{aligned}
$$

This shows that the normal vector at any point on the helix is horizontal and points toward the z-axis. The binormal vector is

$$
\begin{aligned}
B(t) &= T(t) \times N(t) \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \langle \sin t, -\cos t, 1\rangle
\end{aligned}
$$

TNB frame: refers to the set of orthogonal vectors T, N, B, which are the tangent, normal, and binormal vectors.



Normal plane: determined by normal and binormal vectors N and B at a point P on a curve C
• consists of all lines orthogonal to the tangent vector **T**

Osculating plane: determined by vectors T and N at a point P on a curve C
• comes from Latin *osculum* "to kiss" because it is the plane that comes closest to containing the part of the curve near P.

Oscalating circle (circle of curvature): the circle that lies in the osculating plane of C at P, has the same tangnet as C as P, lies on the concave side of C (toward which N points), and has radius $\rho = \frac{1}{\kappa}$ (the reciprocal of the curvature)
• the circle that best describes how C behaves near P
• share the same tangnet, normal, and curvature at P.

2.4 Motion in Space: Velocity and Acceleration

Velocity vector

The average velocity over a time interval of length h and its limit is the velocity vector $v(t)$ at time t:

$$v(t) = \lim_{h\to 0} \frac{r(t+h) - r(t)}{h} = r'(t)$$

Speed

The speed of a particle at time t is the magnitude of the velocity vector:

$$|v(t)| = |r'(t)| = \frac{ds}{dt}$$

Acceleration

$$a(t) = v'(t) = r''(t)$$

Example: A moving particle starts at an intiial position $r(0) = \langle 1, 0, 0\rangle$ with initial velocity $v(0) = i - j + k$. Its acceleration is $a(t) = 4ti + 6tj + k$. Find its velocity and position at time t.

Since $a(t) = v'(t)$, we have

$$\begin{aligned} v(t) &= \int a(t)dt \\ &= \int (4t\mathbf{i} + 6t\mathbf{j} + \mathbf{k})dt \\ &= 2t^2\mathbf{i} + 3t^2\mathbf{j} + t\mathbf{k} + C \end{aligned}$$

Because $v(0) = i - j + k$, $C = i - j + k$ and

$$\begin{aligned} v(t) &= 2t^2\mathbf{i} + 3t^2\mathbf{j} + t\mathbf{k} + \mathbf{i} - \mathbf{j} + \mathbf{k} \\ &= (2t^2 + 1)\mathbf{i} + (3t^2 - 1)\mathbf{j} + (t + 1)\mathbf{k} \end{aligned}$$

Since $v(t) = r'(t)$, we have

$$\begin{aligned} r(t) &= \int v(t)dt \\ &= \int \left[(2t^2 + 1)\mathbf{i} + (3t^2 - 1)\mathbf{j} + (t + 1)\mathbf{k}\right] dt \\ &= \left(\frac{2}{3}t^3 + t\right)\mathbf{i} + (t^3 - t)\mathbf{j} + \left(\frac{1}{2}t^2 + t\right)\mathbf{k} + \mathbf{D} \end{aligned}$$

Putting $t = 0$, we find that $\mathbf{D} = \mathbf{r}(0) = \mathbf{i}$, so the position at time t is given by

$$\mathbf{r}(t) = \left(\frac{2}{3}t^3 + t + 1\right)\mathbf{i} + (t^3 - t)\mathbf{j} + \left(\frac{1}{2}t^2 + t\right)\mathbf{k}$$

In general,

$$v(t) = v(t_0) + \int_{t_0}^{t} a(u)du$$

and

$$r(t) = r(t_0) + \int_{t_0}^{t} v(u)du$$

Example 2: An object with mass m that moves in a circular path with constant angular speed ω has position vector $r(t) = a\cos\omega t\mathbf{i} + a\sin\omega t\mathbf{j}$. Find the force acting on the object and show that it is directed toward the origin.

We have

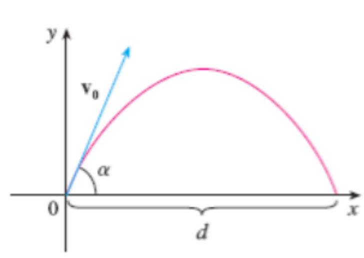
$$\begin{aligned} v(t) &= r'(t) = -a\omega\sin\omega t\mathbf{i} + a\omega\cos\omega t\mathbf{j} \\ a(t) &= v'(t) = -a\omega^2\cos\omega t\mathbf{i} - a\omega^2\sin\omega t\mathbf{j} \end{aligned}$$

Thus Newton's Second Law gives the force as

$$F(t) = ma(t) = -m\omega^2(a\cos\omega t\mathbf{i} + a\sin\omega t\mathbf{j}) = -m\omega^2\mathbf{r}(t)$$

The object moving with position P has angular speed $\omega = \frac{d\theta}{dt}$.

Example 3: A projectile is fired with angle of elevation α and initial velocity v_0. Assuming that air resistance is negligible and the only external force is due to gravity, find the position function $r(t)$ of the projectile. What value of α maximizes the range (the horizontal distance traveled)?



We set up the axes so that the projectile starts at the origin. Since the force due to gravity acts downward, we have

$$F = m\mathbf{a} = -mg\mathbf{j}$$

where $g = |a| \approx 9.8$ m/s^2. Thus

$$\mathbf{a} = -g\mathbf{j}$$

Since $\mathbf{v}'(t) = \mathbf{a}$, we have

$$v(t) = -gt\mathbf{j} + C$$

where $C = v(0) = v_0$. Therefore

$$r'(t) = v(t) = -gt\mathbf{j} + v_0$$

Integrating again, we obtain

$$r(t) = -\frac{1}{2}gt^2\mathbf{j} + t\mathbf{v_0} + D$$

But $D = r(0) = 0$, so the position vector of the projectile is given by

Position Vector of Projectile

r(t) = -1$_{2gt^2\mathbf{j}+t\mathbf{v_0}}$

If we write $|v_0| = v_0$, then

$$v_0 = v_0 \cos\alpha\mathbf{i} + v_0 \sin\alpha\mathbf{j}$$

and it follows that

$$r(t) = (v_0 \cos\alpha)t\mathbf{i} + \left[(v_0 \sin\alpha)t - \frac{1}{2}gt^2\right]\mathbf{j}$$

The parametric equations of the trajectory are therefore

Projectile Trajectory

$$x = (v_0 \cos\alpha)t$$

and

$$y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$$

The horizontal distance d is the value of x when $y = 0$. Setting $y = 0$, we obtain $t = 0$ or $t = \frac{2v_0 \sin\alpha}{g}$. This second value of t then gives

$$d = x = (v_0 \cos\alpha)\frac{2v_0 \sin\alpha}{g} = \frac{v_0^2(2\sin\alpha\cos\alpha)}{g} = \frac{v_0^2 \sin 2\alpha}{g}$$

Clearly d has its maximum value when $\sin 2\alpha = 1$, that is, $\alpha = 45°$.

Example 4: A projectile is fired with muzzle speed 150 m/s and angle of elevation 45° from a position 10 m above ground level. Where does the projectile hit the ground, and with what speed?

If we place the origin at ground level, then the initial position of the projectil is (0, 10) and so we need to adjust the parametric equation for y by adding 10. With $v_0 = 150$ m/s, $\alpha = 45°$, and $g = 9.8$ m/s^2, we have

$$x = 150\cos 45°t = 75\sqrt{2}t$$
$$y = 10 + 150\sin 45°t - \frac{1}{2}(9.8)t^2 = 10 + 75\sqrt{2}t - 4.9t^2$$

Impact occurs when $y = 0$, that is, $4.9t^2 - 75\sqrt{2}t - 10 = 0$. Then

$$t = \frac{75\sqrt{2} + \sqrt{11250 + 196}}{9.8} \approx 21.74$$

Then $x = 75\sqrt{2}(21.74) \approx 2306$, so the projectile hits the ground about 2306 m away. The velocity of the projectile is

$$v(t) = r'(t) = 75\sqrt{2}\mathbf{i} + (75\sqrt{2} - 9.8t)\mathbf{j}$$

so its speed at impact is

$$|v(21.74)| = \sqrt{(75\sqrt{2})^2 + (75\sqrt{2} - 9.8 \cdot 21.74)^2} \approx 151 \text{ m/s}$$

We know that

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{v(t)}{|v(t)|} = \frac{\mathbf{v}}{v}$$

and so

$$\mathbf{v} = v\mathbf{T}$$

Differentiating both sides with repsect to t,

$$\mathbf{a} = \mathbf{v}' = v'\mathbf{T} + v\mathbf{T}'$$

Using the expression for curvature, we have

$$\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|T'|}{v} \implies |T'| = \kappa v$$

The unit normal vector was defined as $N = \frac{T'}{|T'|}$, so we have

$$T' = |T'|N = \kappa v\mathbf{N}$$

and our equation becomes

$$\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}$$

Tangential and Normal Components of Acceleration

Letting a_T and a_N be the tangential and normal components of acceleration, respectively, we have

$$a_T = v' = \frac{v \cdot a}{v} = \frac{r'(t) \cdot r''(t)}{|r'(t)|}$$

and

$$a_N = \kappa v^2 = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}|r'(t)|^2 = \frac{|r'(t) \times r''(t)|}{|r'(t)|}$$

3 Partial Derivatives

3.1 Functions of Several Variables

A **function f of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is hte set of values that f takes on, that is, $\{f(x, y)|(x, y) \in D\}$.

We often write $z = f(x, y)$ to make the explicit value taken on by f at the general point (x, y). The variables x and y are **independent variables** and z is the dependent variable. A function of two variables is just a function whose domain is a subset of $\mathbb{R}^{\not{2}}$ and whose range is a subset of $\mathbb{R}$.

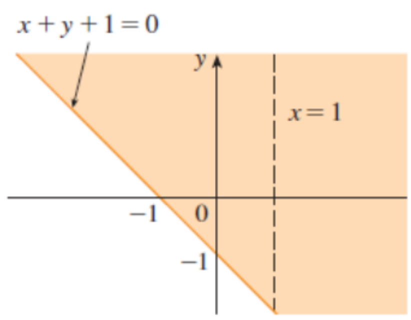
Example 1: Consider the following function.

$$f(x, y) = \frac{\sqrt{x + y + 1}}{x - 1}$$

The expression for f makes sense if the denominator is not 0 and the quantity under the square root sign is nonnegative. So the domain of f is

$$D = \{(x, y)|x + y + 1 \geq 0, x \neq 1\}$$

The inequality $x + y + 1 \geq 0$, or $y \geq -x - 1$, describes the points that lie on or above the line $y = -x - 1$, while $x \neq 1$ means that the points on the line $x = 1$ must be excluded from the domain. Using this information, we can sketch the domain of $f(x, y)$:

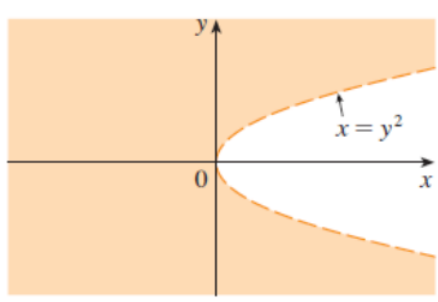


Example 2: Now consider the function $f(x, y) = x \ln{(y^2 - x)}$.

The expression $\ln y^2 - x$ is only defined when $y^2 - x > 0$, that is, $x < y^2$. Hence, the domain of f is

$$D = \{(x, y)|x < y^2\}$$

That is the set of points to the left of the parabola $x = y^2$:



The **Cobb-Douglas Production Function:**

Let P be the total production (the monetary value of all goods produced in a year), let L be the amount of labor (the total number of person-hours worked in a year), and let K be the amount of capital invested (the monetary value of all machinery equipment and buildings). Then the production can be modeled with the function

$$P(L, K) = bL^{\alpha}K^{1-\alpha}$$

Using federal economic data and the method of least squares, they found that

$$P(L, K) = 1.01L^{0.75}K^{0.25}$$

The domain of this function is $\{(L, K)|L \geq 0, K \geq 0\}$ because L and K represent labor and capital and are therefore never negative.

Example 3: Consider the function $g(x, y) = \sqrt{9 - x^2 - y^2}$. The domain of g is

$$D = \{(x, y)|9 - x^2 - y^2 \geq 0\} = \{(x, y)|x^2 + y^2 \leq 9\}$$

which is the disk with center $(0, 0)$ and radius 3. The range of g is

$$\{z | z = \sqrt{9 - x^2 - y^2}, (x, y) \in D\}$$

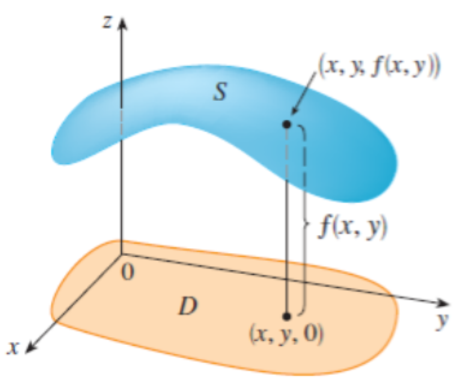
Since z is a positive square root, $z \geq 0$. Also, because $9 - x^2 - y^2 \leq 9$, we have

$$\sqrt{9 - x^2 - y^2} \leq 3$$

So the range isa

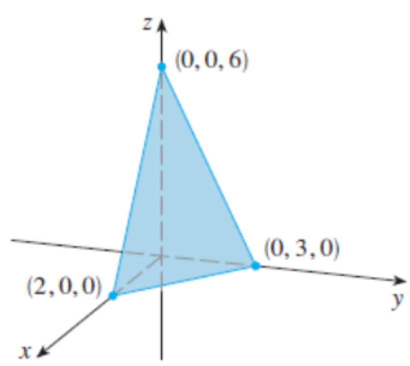
$$\{z | 0 \leq z \leq 3\} = [0, 3]$$

If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^{\nvDash}$ such that $z = f(x, y)$ and (x, y) is in D. Just as the graph of a function f of one variable is a curve C with equation $y = f(x)$, so the graph of a function f of two variables is a surface S with equation $z = f(x, y)$. We can visualize the graph S of f as lying directly above or below its domain D in the xy-plane.



Example 4: Sketch the graph of the function $f(x, y) = 6 - 3x - 2y$

The graph of f has the equation $z = 6 - 3x - 2y$, or $3x + 2y + z = 6$, which represents a plane. To graph the plane we first find the intercepts, which are $(2, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 6)$. This helps us sketch the portion of the graph that lies in the first octant:



The function is a special case of the function

$$f(x, y) = ax + by + c$$

which is called a **linear function**. The graph of such a function has the equation

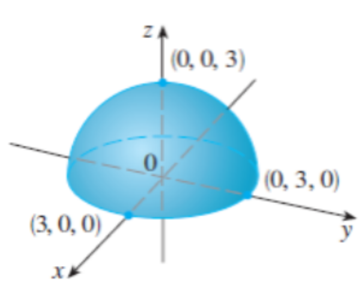
$$z = ax + by + c$$

or

$$ax + by - z + c = 0$$

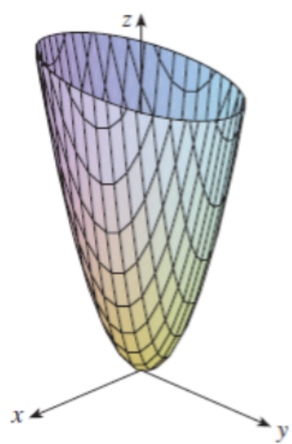
so it is plane.
Example 5: Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

The graph has equation $z = \sqrt{9 - x^2 - y^2}$. We square both sides of this equation to obtain $z^2 = 9 - x^2 - y^2$, or $x^2 + y^2 + z^2 = 9$, which recognize as an equation of the sphere with center the origin and radius 3. But, since $z \geq 0$, the graph of g is just the top half of this sphere.



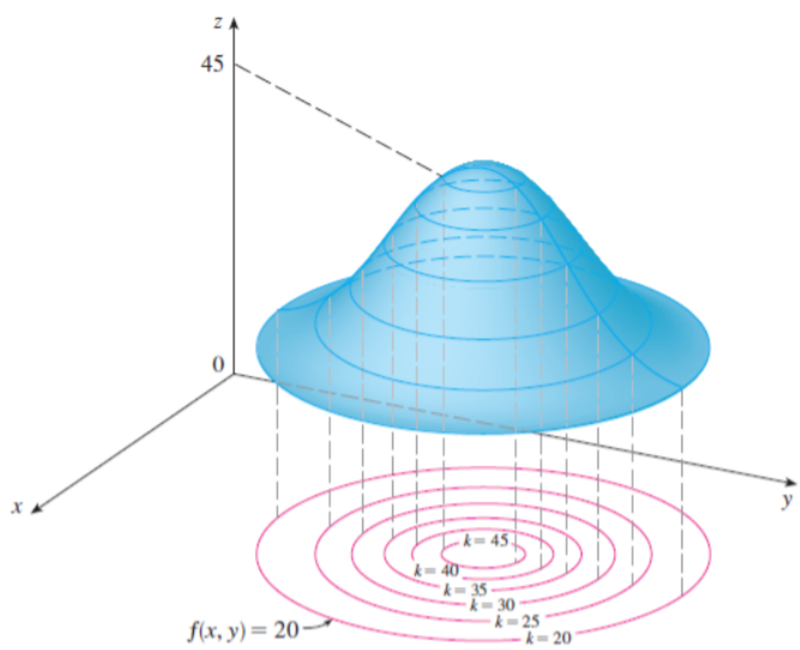
Example 6: Find the domain and range and sketch the graph of $h(x, y) = 4x^2 + y^2$.

Notice that $h(x, y)$ is defined for all possible ordered pairs of real numbers (x, y), so the domain is $\mathbb{R}^{\nvDash}$, the entire xy-plane. The range of h is the set $[0, \infty)$ of all nonnegative real numbers. [Notice that $x^2 \geq 0$ and $y^2 \geq 0$, so $h(x, y) \geq 0$ for all x and y.] The graph of h has the equation $z = 4x^2 + y^2$, which is the elliptic paraboloid sketched below. Horizontal traces are ellipses and vertical traces are parabolas.



The **level curves (contour curves)** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f)
• is the set of all points in the domain of f at which f takes on a given value k
• shows where the graph of f has height k

Notice in the figure below that the level curves $f(x, y) = k$ are just the traces of the graph of f in the horizontal plane $z = k$ projected down to the xy-plane.



Example 7:

Sketch some level curves of the function $h(x, y) = 4x^2 + y^2 + 1$.

The level curves are

$$4x^2 + y^2 + 1 = k$$

or

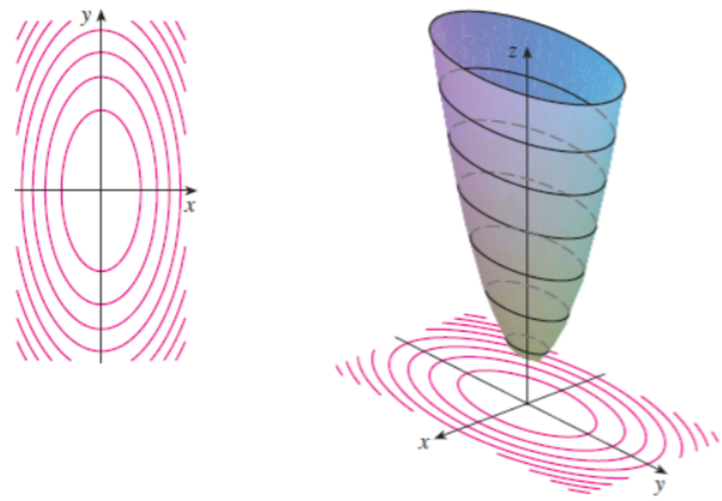
$$\frac{x^2}{\frac{1}{4}(k-1)} + \frac{y^2}{k-1} = 1$$

which for $k > 1$, describes a family of ellipses with semiaxes $\frac{1}{2}\sqrt{k-1}$ and $\sqrt{k-1}$. The figure below on the left shows a contour map of h drawn by a computer. The figure below on the right shows these level curves lifted up to the graph of h (an elliptic paraboloid) where they become horizontal traces.
A **function of three variables**, f, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset R^3$ a unique real number denoted by $f(x, y, z)$. For instance, the temperature T at a point on the surface of the earth depends on the longitude x and latitude y of the point and on the time t, so we could write $T = f(x, y, t)$.

Example 8: Find the domain of f if

$$f(x, y, z) = \ln(z - y) + xy \sin z$$

The graph of $h(x,y) = 4x^2 + y^2 + 1$ is formed by lifting the level curves.



(a) Contour map (b) Horizontal traces are raised level curves

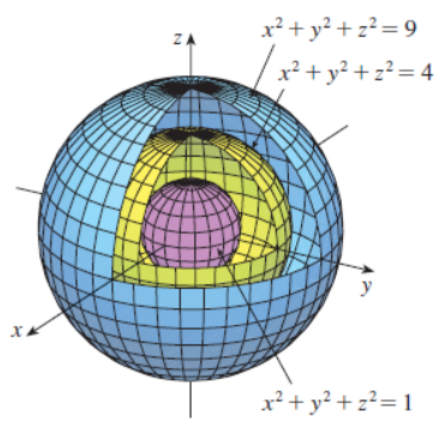
The expression for $f(x,y,z)$ is defined as long as $z - y > 0$, so the domain of f is

$$D = \{(x,y,z) \in \mathbb{R}^3 | z > y\}$$

This is a **half-space** consisting of all points that lie above the plane $z = y$.
Example 9: Find the level surfaces of the function

$$f(x,y,z) = x^2 + y^2 + z^2$$

The level surfaces are $x^2 + y^2 + z^2 = k$, where $k \geq 0$. These form a family of concentric spheres with radius $\sqrt{k}$. Thus, as (x,y,z) varies over any sphere with center O, the value of $f(x,y,z)$ remains fixed.



A **function of n variables** is a rule that assigns a number $z = f(x_1, x_2, \ldots, x_n)$ to an n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers. We denote by $\mathbb{R}^n$ the set of all such n-tuples.

$$C = f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

The function f is a real-valued function whose domain is a subset of $\mathbb{R}^n$. Sometimes we use vector notation to write such functions more compactly. If $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$, we often write $\mathbf{f}(\mathbf{x})$ in place of $\mathbf{f}(\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n})$. With this notation we can rewrite f as

$$\mathbf{f}(\mathbf{x}) = \mathbf{c}\dot{\mathbf{x}}$$

where $\mathbf{c} = \langle c_1, c_2, \ldots, c_n \rangle$ and $\mathbf{c} \cdot \mathbf{x}$ denotes the dot product of the vectors $\mathbf{c}$ and $\mathbf{x}$ in V_n.

Thus there are three ways of looking at a function f defined on a subset of $\mathbb{R}^n$:
1. As a function of n real variables $x_1, x_2, \ldots, x_n$.
2. As a function of a single point variable $(x_1, x_2, \ldots, x_n)$
3. As a function of a single vector variable $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$.

3.2 Limits and Continuity

Let f be a function of two variables whose domain D includes points arbitrarily close to (a,b). Then we say that the **limit of** $f(x,y)$ as (x,y) approaches (a,b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that if

$$(x, y) \in D$$

and

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

then

$$|f(x, y) - L| < \epsilon$$

Theorem

If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

Example: Show that the limit below does not exist.

$$\lim_{(x,y)\to(0,)} \frac{x^2 - y^2}{x^2 + y^2}$$

Let $f(x, y) = \frac{x^2-y^2}{x^2+y^2}$. First approach $(0, 0)$ along the x-axis. Then $y = 0$ gives $f(x, 0) = \frac{x^2}{x^2} = 1$ for all $x \neq 0$, so

$$f(x, y) \to 1 \text{ as } (x, y) \to (0, 0) \text{ along the } x\text{-axis}$$

We now approach along the y-axis by putting $x = 0$. Then $f(0, y) = \frac{-y^2}{y^2} = -1$ for all $y \neq 0$, so

$$f(x, y) \to -1 \text{ as } (x, y) \to (0, 0) \text{ along the } y\text{-axis}$$

Example 2: Find the limit below if it exists.

$$\lim_{(x,y)\to(0,0)} \frac{3x^2 y}{x^2 + y^2}$$

We can show that the limit along any line through the origin is 0. This doesn't prove that the given limit is 0, but the limits along the parabolas $y = x^2$ and $x = y^2$ also turn out to be 0, so we begin to suspect that the limit does exist and is equal to 0.

Let $\epsilon > 0$. We want to find $\delta > 0$ s.t.

$$\text{if } 0 < \sqrt{x^2 + y^2} < \delta \text{ then } \left| \frac{3x^2 y}{x^2 + y^2} - 0 \right| < \epsilon$$

that is, if $0 < \sqrt{x^2 + y^2} < \delta$, then $\frac{3x^2|y|}{x^2+y^2} < \epsilon$. But $x^2 \leq x^2 + y^2$ since $y^2 \geq 0$, so $\frac{x^2}{x^2+y^2} \leq 1$ and therefore

$$\frac{3x^2|y|}{x^2 + y^2} \leq 3|y| = 3\sqrt{y^2} \leq 3\sqrt{x^2 + y^2}$$

Thus if we choose $\delta = \frac{\epsilon}{3}$ and let $0 < \sqrt{x^2 + y^2} < \delta$, then

$$\left| \frac{3x^2 y}{x^2 + y^2} - 0 \right| \leq 3\sqrt{x^2 + y^2} < 3\delta = 3\left(\frac{\epsilon}{3}\right) = \epsilon$$

Hence,

$$\lim_{(x,y)\to(0,0)} \frac{3x^2 y}{x^2 + y^2} = 0$$

A function f of two variables is **continuous** at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

f is **continuous on** D if f is continuous at every point (a, b) in D.

A **polynomial function of two variables** is a sum of terms of the form $cx^m y^n$ where c is a constant and m and n are nonnegative integers. A **rational function** is a ratio of polynomials. By examination the functions $f(x, y) = x, g(x, y) = y$, and $h(x, y) = c$ are continuous. Since any polynomial can be built up out of the simple functions f, g, and h by multiplication and addition, it follows that *all polynomials are continuous on* $\mathbb{R}^2$.

Theorem

If f is defined on a subset D of $\mathbb{R}^n$, then $\lim_{x\to a} f(x) = L$ means that for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ s.t.

$$\text{if } x \in D \text{ and } 0 < |x - a| < \delta \text{ then } |f(x) - L| < \epsilon$$

3.3 Partial Derivatives

The **partial derivative** of f with respect to x at (a, b) and denote it by $f_x(a, b)$. Thus

$$f_x(a,b) = g'(a)$$

where

$$g(x) = f(x, b)$$

By the definition of the derivative

$$g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

and so

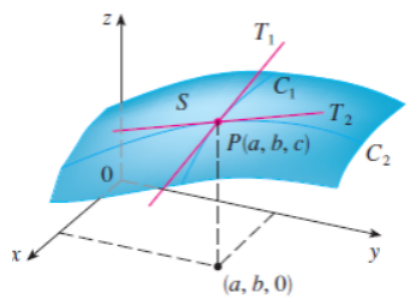
$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

By this, if f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$
$$f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

To find partial derivatives of $z = f(x, y)$,
1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2:

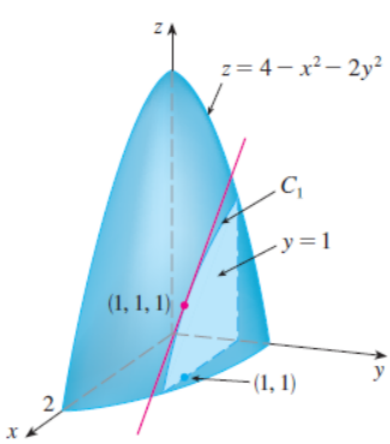


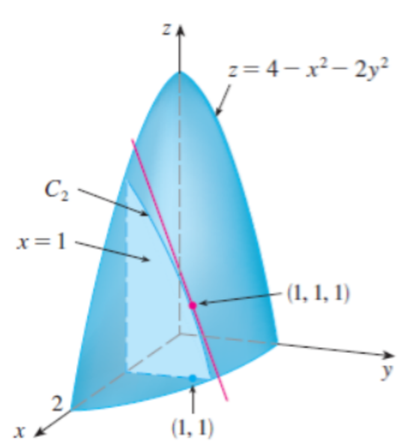
Example: If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

We have

$$f_x(x,y) = -2x$$
$$f_y(x,y) = -4y$$
$$f_x(1,1) = -2$$
$$f_y(1,1) = -4$$

The graph of f is the paraboloid $z = 4 - x^2 - 2y^2$ and the vertical plane $y = 1$ intersects it in hte parabola $z = 2 - x^2, y = 1$. The slope of the tangent line to this parabola at the point $(1, 1, 1)$ is the parabola $z = 3 - 2y^2, x = 1$, and the slope of the tangent line at $(1, 1, 1)$ is $f_y(1, 1) = -4$.





Example 2: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz = 1$$

To find $\frac{\partial z}{\partial x}$, we differentiate implicitly with respect to x, being careful to treat y as a constant:

$$3x^2 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

Solving this equation for $\frac{\partial z}{\partial x}$, we have

$$\frac{\partial z}{\partial x} = -\frac{x^2+2yz}{z^2+2xy}$$

Similarly, implicit differentiation with respect to y gives

$$\frac{\partial z}{\partial x} = -\frac{y^2+2xz}{z^2+2xy}$$

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x, (f_x)y$, and $(f_y)_y$, which are called the **second partial derivatives** of f. If $z = f(x,y)$, we use the following notaiton:

$$\begin{aligned}
(f_x)_x &= f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2} \\
(f_x)_y &= f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x} \\
(f_y)_z &= f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y} \\
(f_y)_y &= f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{|partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}
\end{aligned}$$

Thus the notation f_{xy} or $\frac{\partial^2 f}{\partial y \partial x}$ means that we first differentiate with respect to x and then with respect to y, whereas in computing f_{yx} the order is reversed.

Clairaut's Theorem

Suppose f is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yz} are both continuous on D, then

$$f_{xy}(z,b) = f_{yx}(a,b)$$

Partial derivatives of order 3 or higher can be defined. For instance,

$$f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial y^2 \partial x}$$

Using Clairaut's Theorem, it can be shown that $f_{xyy} = f_{yxy} = f_{yyx}$ if these functions are continuous.

Example 3: Calculate f_{xxyz} if $f(x,y,z) = \sin 3x + yz$.

$$\begin{aligned}
f_x &= 3\cos(3x+yz) \\
f_{xx} &= -9\sin(3x+yz) \\
f_{xxy} &= -9z\cos(3x+yz) \\
f_{xxyz} &= -9\cos(3x+yz) + 9yz\sin(3x+yz)
\end{aligned}$$

Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Solutions of this equation are called **harmonic functions** and play a role in problems of heat conduction, fluid flow, and electric potential.

Example 4: Show that the function $u(x,y) = e^x \sin y$ is a solution of Laplace's equation.

We first compute the needed second-order partial derivatives:

$$\begin{aligned}
u_x &= e^x \sin y, & u_y &= e^x \cos y \\
u_{xx} &= e^x \sin y, & u_{yy} &= -e^x \sin y
\end{aligned}$$

So

$$u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0$$

Therefore u satisfies Laplace's equation.

Wave Equation

The **wave equation**

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

describes the motion of a waveform, which could be an ocean wave, a sound wave, a light wave, or a wave traveling along a vibrating string.

Example 5: Verify that the function $u(x,t) = \sin(x - at)$ satisfies the wave equation.

$$u_x = \cos(x - at) \qquad\qquad u_t = -a\cos(x - at)$$

$$u_{xx} = -\sin(x - at) \qquad\qquad u_{tt} = -a^2 \sin(x - at) = a^2 u_{xx}$$

So u satisfies the wave equation.

If the production function is denoted by $P = P(L, K)$, then the partial derivative $\frac{\partial P}{\partial L}$ is the rate at which produces changes with respect to the amount of labor. The marginal production with respect to labor or the **marginal productivity of labor**. Likewise, the partial derivative $\frac{\partial P}{\partial K}$ is the rate of change of production with respect to capital and is called the **marginal productivity of capital**. In these terms, the assumptions made by Cobb and Douglas can be stated as follows:

1. If either labor or capital vanishes then so will production
2. The marginal productivity of labor is proportional to the amount of production per unit of labor
3. The marginal productivity of capital is proportional to the amount of production per unit of capital

Because the production per unit of labor is $\frac{P}{L}$, assumption 2 says that

$$\frac{\partial P}{\partial L} = \alpha \frac{P}{L}$$

for some constant α. If we keep K constant ($K = K_0$), then this partial differential equation becomes an ODE:

$$\frac{dP}{dL} = \alpha \frac{P}{L}$$

If we solve this separable DE then

$$P(L, K_0) = C_1(K_0)L^\alpha$$

) Similarly,

$$P(L_0, K) = C_2(L_0)K^\beta$$

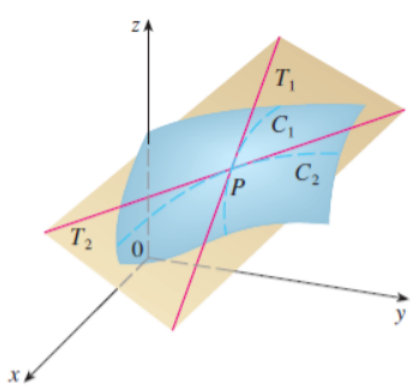
Comparing the latter two equations, we have

$$P(L, K) = bL^\alpha K^\beta$$

where b is a constant that is independent of both L and K.

3.4 Tangent Planes and Linear Approximations

Suppose a surface S has equation $z = f(x, y)$ where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S. As in the preceding section, let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. Then the point P lies on both C_1 and C_2. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P. Then the **tangent plane** to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2.



Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Note the similarity between the equation of a tangent plane and the equation of a tangent line:

$$y - y_0 = f'(x_0)(x - x_0)$$

Example: Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Let $f(x, y) = 2x^2 + y^2$. Then

$$f_x(x, y) = 4x \qquad f_y(x, y) = 2y$$
$$f_x(1, 1) = 4 \qquad f_y(1, 1) = 2$$

Then the tangent plane at $(1, 1, 3)$ is given by

$$z - 3 = 4(x - 1) + 2(y - 1)$$

or

$$z = 4x + 2y - 3$$

The linear function whose graph is this tangent plane, namely

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linearization** of f at (a, b) and the approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** or the **tangent plane approximation** of f at (a, b).

Differentiable

If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Theorem

If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Example 2: Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

$$f_x(x, y) = e^{xy} + xye^{xy} \qquad f_y(x, y) = x^2 e^{xy}$$
$$f_x(1, 0) = 1 \qquad f_y(1, 0) = 1$$

Both f_x and f_y are continuous functions, so f is differentiable. The linearization is then

$$\begin{aligned} L(x, y) &= f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0) \\ &= 1 + 1(x - 1) + 1 \cdot y \\ &= x + y \end{aligned}$$

The corresponding linear approximation is

$$xe^{xy} \approx x + y$$

so

$$f(1.1, -0.1) \approx 1.1 - 0.1 = 1$$

Example 3: Let the heat index (perceived temperature) I as a function of the actual temperature T and the relative humidity H and gave the following table of values from the National Weather Service. Find a linear approximation for the heat index $I = f(T, H)$ when T is near 96° F and H is near 70%. Use it to estimate the heat index when the temperature is 97° F and the relative humidity is 72%.
We read from the table that $f(96, 70) = 125$. In section 14.3 we used the tabular values to estimate that $f_T(96, 70) \approx 3.75$ and $f_H(96, 70) \approx 0.9$. So the linear approximation is

$$\begin{aligned} f(T, H) &\approx f(96, 70) + f_T(96, 70)(T - 96) + f_H(96, 70)(H - 70) \\ &\approx 125 + 3.75(T - 96) + 0.9(H - 70) \end{aligned}$$

In particular,

Relative humidity (%)

Actual temperature (°F)

T \ H	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	129	135	141	147	154	161	168

$$f(97, 72) \approx 125 + 3.75(1) + 0.9(2) = 130.55$$

Therefore, when $T = 97°\text{F}$ and $H = 72\%$, the heat index is

$$I \approx 131°\text{F}$$

Recall that for a one variable function $y = f(x)$, the differential dx is an independent variable and the differential of y is given by

$$dy = f'(x)dx$$

For a differentiable function of two variables $z = f(x, y)$, the **differentials** dx and dy are independent variables and the differential dz, also called the **total differential**, is defined by

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

If we take $dx = \Delta x = x - a$ and $dy = \Delta y = y - b$ then the differential of z is

$$dz = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

So in differential notation, the linear approximation is given by

$$f(x, y) \approx f(a, b) + dz$$

Example 4: If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz. If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

We have

$$\begin{aligned} dz &= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \\ &= (2x + 3y)dx \\ &= (3x - 2y)dy \end{aligned}$$

Putting $x = 2, dx = \Delta x = 0.05, y = 3$, and $dy = \Delta y = -0.04$, we get

$$dz = [2(2) + 3(3)]0.05 + [3(2) - 2(3)](-0.04) = 0.65$$

The increment of z is

$$\begin{aligned} \Delta z &= f(2.05, 2.96) - f(2, 3) \\ &= \left[(2.05)^2 + 3(2.05)(2.96) - (2.96)^2\right] - \left[2^2 + 3(2)(3) - 3^2\right] \end{aligned}$$

Note that $\Delta z \approx dz$ but dz is easier to compute.

Example 5: The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

The volume V of a cone with base radius r and height h is $V = \frac{\pi r^2 h}{3}$. So the differential of V is

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh = \frac{2\pi rh}{3}dr + \frac{\pi r^2}{3}dh$$

Since each error is at most 0.1 cm, we have $|\Delta r| \leq 0.1, |\Delta h| \leq 0.1$. To estimate the largest error in the volume we take the largest error in the measurement of r and of h. Therefore we take $dr = 0.1$ and $dh = 0.1$ along with $r = 10, h = 25$. This gives

$$dV = \frac{500\pi}{3}(0.1) + \frac{100\pi}{3}(0.1) = 20\pi$$

Thus the maximum error in the calculated volume is about 20π cm$^3 \approx 63$ cm^3.

For a function of three variables, the **linear approximation** is

$$f(x,y,z) \approx f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z)a,b,c)(z-c)$$

and the linearization $L(x,y,z)$ is the right side of this expression. If $w = f(x,y,z)$ then the **increment** of w is

$$\Delta w = f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z)$$

The differential dw is defined like so:

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

Example 6: The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

If the dimensions of the box are x, y, and z, its volume is $V = xyz$ and so

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = yzdx + xzdy + xydz$$

We are given that $|\Delta x| \leq 0.2, |\Delta y| \leq 0.2$, and $|\Delta z| \leq 0.2$. To estimate the largest error in the volume, we therefore use $dx = 0.2$, $dy = 0.2$, and $dz = 0.2$ together with $x = 75, y = 60$, and $z = 40$:

$$\Delta V \approx dV = (60)(40)(0.2) + (75)(40)(0.2) + (75)(60)(0.2) = 1980$$

Thus an error of only 0.2 cm in measuring each dimension could lead to an error of approximately 1980 cm^3 in the calculated volume! This may seem like a large error, but it's only about 1% of the volume of the box.

3.5 The Chain Rule

3.6 Directional Derivatives and the Gradient Vector

3.7 Maximum and Minimum Values

A function of two variables has a **local maximum** at (a,b) if $f(x,y) \leq f(a,b)$ when (x,y) is near (a,b). [This means that $f(x,y) \leq f(a,b)$ for all points (x,y) in some disk with center (a,b).] The number $f(a,b)$ is called a **local maximum value**. If $f(x,y) \geq f(a,b)$ when (x,y) is near (a,b), then f has a **local minimum** at (a,b) and $f(a,b)$ is a **local minimum value**. If the inequalities hold for *all* points (x,y) in the domain of f, then f has an **absolute maximum** (or **absolute minimum**) at (a,b).

Theorem

If f has a local maximum or minimum at (a,b) and the first-order partial derivatives of f exist there, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

A point (a,b) is called a **critical point** of f if $f_x(a,b) = 0$ and $f_y(a,b) = 0$, or if one of these partial derivatives does not exist.

Example 1: Find the extreme values of $f(x,y) = y^2 - x^2$.

Since $f_x = -2x$ and $f_y = 2y$, the only critical point is $(0,0)$. Notice that for points on the x-axis we have $y = 0$, so $f(x,y) = -x^2 < 0$ if $x \neq 0$. However, for points on the y-axis we have $y = 0$, so $f(x,y) = y^2 > 0$ if $y \neq 0$. Thus every disk with center $(0,0)$ contains points where f takes positive values as well as points where f takes negative values. Therefore $f(0,0) = 0$ can't be an extreme value for f, so f has no extreme value.
Example 2: Let $f(x,y) = x^2 + y^2 - 2x - 6y + 14$. Then

$$f_x(x,y) = 2x - 2$$
$$f_y(x,y) = 2y - 6$$

These partial derivatives are equal to 0 when $x = 1$ and $y = 3$, so the only critical point is $(1,3)$. By completing the square, we find that

$$f(x,y) = 4 + (x-1)^2 + (y-3)^2$$

Since $(x-1)^2 \geq 0$ and $(y-3)^2 \geq 0$, we have $f(x,y) \geq 4$ for all values of x and y. Therefore $f(1,3) = 4$ is a local minimum, and in fact it is the absolute minimum of f. This can be confirmed geometrically from the graph of f, which is the elliptic paraboloid with vertex $(1,3,4)$.

Second Derivative Test

Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

In case (c) the point (a, b) is called a **saddle point** of f and the graph of f crosses its tangent plane at (a, b).

If $D = 0$ the test gives no information and f could have a local maximum or local minimum at (a, b), or (a, b) could be a saddle point of f.

To remember the formula for D, it's helpful to write it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ fyx & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2$$

Example 3: Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

We first locate the critical points:

$$f_x = 4x^3 - 4y, f_y = 4y^3 - 4x$$

Setting these partial derivatives equal to 0, we obtain the equations

$$x^3 - y = 0, y^3 - x = 0$$

Solving these equations by substitution, we get

$$\begin{aligned} 0 &= x^9 - x \\ &= x(x^8 - 1) \\ &= x(x^4 - 1)(x^4 + 1) \\ &= x(x^2 - 1)(x^2 + 1)(x^4 + 1) \end{aligned}$$

so there are three real roots: $x = 0, 1, -1$. The three critical points are $(0, 0), (1, 1), (-1, -1)$. Next we calculate the second partial derivatives and $D(x, y)$:

$$f_{xx} = 12x^2, f_{xy} = -4, f_{yy} = 12y^2$$

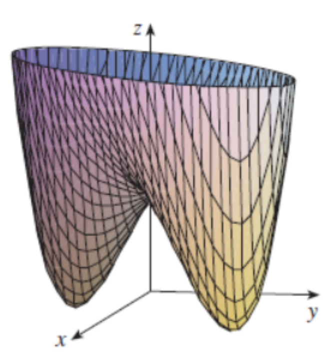
and

$$D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16$$

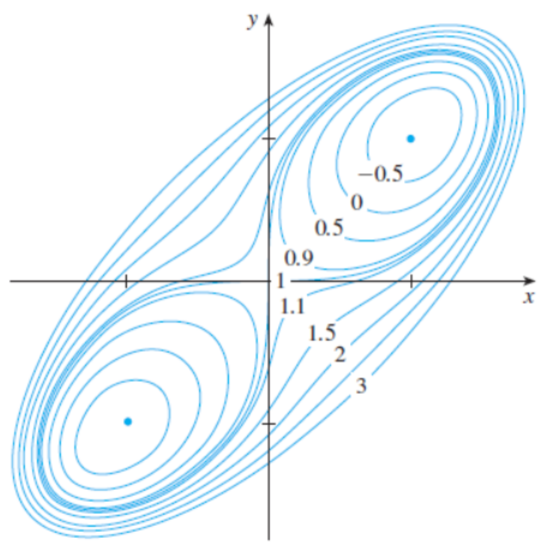
Since $D(0, 0) = -16 < 0$, it follows that the origin is a saddle point; that is, f has no local maximum or minimum at $(0, 0)$.
Since $D(1, 1) = 128 > 0$ and $f_{xx}(1, 1) = 12 > 0$, we see from case (a) of the test that $f(1, 1) = -1$ is a local minimum.
Similarly, $(-1, -1) = 128 > 0$ and $f_{xx}(-1, -1) = 12 > 0$, so $f(-1, -1) = -1$ is also a local minimum.

The graph of f is shown below:

$z = x^4 + y^4 - 4xy + 1$



A contour map of the function f is shown below. Note how the level curves near $(1, 1)$ and $(-1, -1)$ are oval in shape, thus indicating that was we move away from $(1, 1)$ or $(-1, -1)$ in any direction the values of f are increasing. The level curves near $(0, 0)$ resemble hyperbolas. They reveal that as we move away from the origin (where the value of f is 1, the values of f decrease in some directions but increase in other directions. Thus the contour map suggests the presence of the minima and saddle point that we found above.



Example 4: Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

The distance from any point (x, y, z) to the point $(1, 0, -2)$ is

$$d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$$

but if (x, y, z) lies on the plane $x + 2y + z = 4$, then $z = 4 - x - 2y$ and so we have

$$d = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}$$

We can minimize d by minimizing the simpler expression

$$d^2 = f(x, y) = (x-1)^2 + y^2 + (6-x-2y)^2$$

By solving the equations

$$\begin{aligned} f_x &= 2(x-1) - 2(6-x-2y) = 4x + 4y - 14 = 0 \\ f_y &= 2y - 4(6-x-2y) = 4x + 10y - 24 = 0 \end{aligned}$$

we find that the only critical point is $\left(\frac{11}{6}, \frac{5}{3}\right)$. Since $f_{xx} = 4$, $f_{xy} = 4$, and $f_{yy} = 10$, we have

$$D(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = 24 > 0$$

and $f_{xx} > 0$, so by the Second Derivatives Test f has a local minimum at $\left(\frac{11}{6}, \frac{5}{3}\right)$. Intuitively we can see that this local minimum is actually an absolute minimum because there must be a point on the given plane that is closest to $(1, 0, -2)$. If $x = \frac{11}{6}$ and $y = \frac{5}{3}$, then

$$\begin{aligned} d &= \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2} \\ &= \sqrt{\left(\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(\frac{5}{6}\right)^2} \\ &= \frac{5}{6}\sqrt{6} \end{aligned}$$

The shortest distance from $(1, 0, -2)$ to the plane $x + 2y + z = 4$ is $\frac{5}{6}\sqrt{6}$.

Example 6: A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

Let the length, width, and height of the box (in meters) be x, y, z. Then the volume of the box is

$$V = xyz$$

We can express V as a function of just two variables x and y by using the fact that the area of the four sides and the bottom of the box is

$$2xz + 2yz + xy = 12$$

Solving this equation for z, we get $z = \frac{12-xy}{2(x+y)}$, so the expression for V becomes

$$V = xy\frac{12-xy}{2(x+y)} = \frac{12xy - x^2y^2}{2(x+y)}$$

We compute the partial derivatives:

$$\frac{\partial V}{\partial x} = \frac{y^2(12 - 2xy - x^2)}{2(x+y)^2}, \frac{\partial V}{\partial y} = \frac{x^2(12 - 2xy - y^2)}{2(x+y)^2}$$

If V is a maximum, then $\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0$, but $x = 0$ or $y = 0$ gives $V = 0$, so we must solve the equations

$$12 - 2xy - x^2 = 0$$
$$12 - 2xy - y^2 = 0$$

These eimply that $x^2 = y^2$ and so $x = y$. (Note that x and y must both be positive in this problem.) If we put $x = y$ in either equation we get $12 - 3x^2 = 0$, which gives $x = 2, y = 2$, and $z = \frac{12-2\cdot 2}{2(2+2)} = 1$.

We could use the Second Derivatives Test to show that this gives a local maximum of V, or we could simply argue from the physical nature of htis problem that there must be an absolute maximum volume, which has to occur at a critical point of V, so it must occur when $x = 2, y = 2, z = 1$. Then $v = 2 \cdot 2 \cdot 1 = 4$, so the maximum volume of the box is 4

4 Multiple Integrals

4.1 Double Integrals Over Rectangles

Consider a function f of two variables defined on a closed rectangle

$$R = [a,b] \times [c,d] = \left\{(x,y) \in \mathbb{R}^2 \middle| a \le x \le b, c \le y \le d\right\}$$

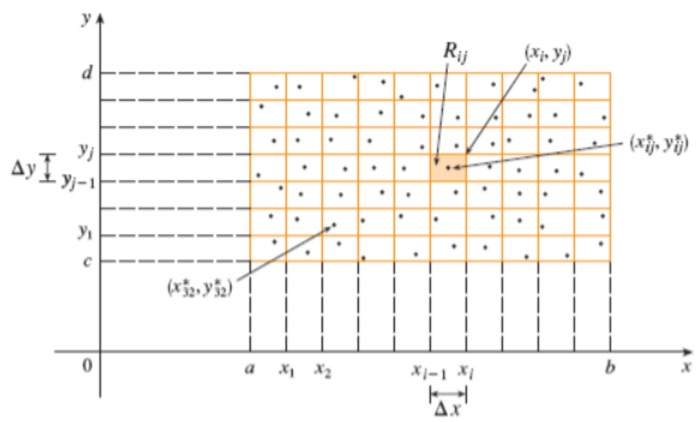
and we first suppose that $f(x,y) \ge 0$. The graph of f is a surface with equation $z = f(x,y)$. Let S be the solid that lies above R and under the graph of f, that is,

$$S = \left\{(x,y,z) \in \mathbb{R}^3 \middle| 0 \le z \le f(x,y), (x,y) \in R\right\}$$

We want to find the volume of S by dividing the rectangle R into subrectangles through dividing the interval $[a,b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \frac{b-a}{m}$ and dividing $[c,d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = \frac{d-c}{n}$. By drawing lines parallel to the coordinate axes through the endpoints of these subintervals, we form the subrectangles

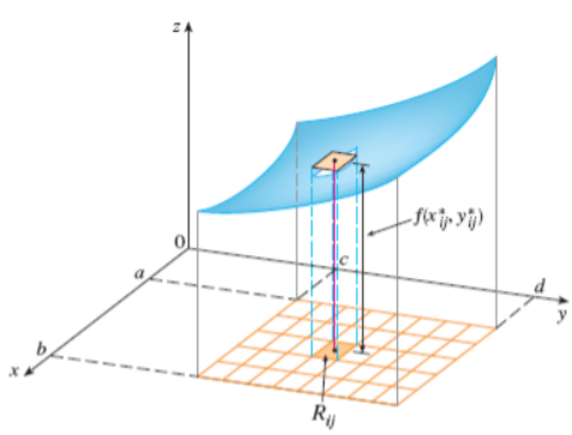
$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \left\{(x,y) \middle| x_{i-1} \le x \le x_i, y_{j-1} \le y \le y_j\right\}$$

each with area $A = \Delta x \Delta y$.



If we choose a **sample point** $\left(x_{ij}^*, y_{ij}^*\right)$ in each R_{ij}, then we can approximate the part of S that lies above each R_{ij} by a thin rectangular box or column with base R_{ij} and height $f\left(x_{ij}^*, y_{ij}^*\right)$ as shown below. The volume of this box is the height of the box times the area of the base rectangle:

$$f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$$



Adding up the volumes of all the boxes corresponding the rectangles, we get an approximation to the total volume of S:

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$$

As the approximation becomes more accurate as m and n get larger,

Volume over rectangle

$$V = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$$

Double Integral

The **double integral** of f over the rectangle R is

$$\int\int_R f(x,y)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A$$

if this limit exists.

Definition of Double Integral

For every number $\epsilon > 0$ there is an integer N s.t.

$$\left|\int\int_R f(x,y)dA - \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A\right| < \epsilon$$

for all integers m and n greater than N and for any choice of sample points $\left(x_{ij}^*, y_{ij}^*\right)$ in R_{ij}.

If we choose the sample point to be the upper right-hand corner of R_{ij} then the expression for the double integral becomes

$$\int\int_R f(x,y)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_i, y_i\right)\Delta A$$

This can be written like so:

Volume over rectangle and below surface

If $f(x,y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x,y)$ is

$$V = \int\int_R f(x,y)dA$$

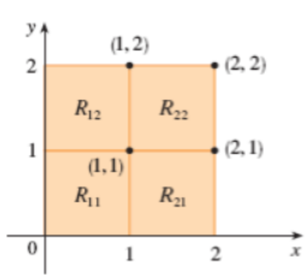
The sum below is called a **Double Riemann sum**

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A$$

Example: Estimate the volume of the solid that lies above the square $R = [0,2] \times [0,2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}. Sketch the solid and the approximating rectangular boxes.

The squares are shown below. The paraboloid is the graph of $f(x,y) = 16 - x^2 - 2y^2$ and the area of each square is $\Delta A = 1$. Approximating the volume by the Riemann sum with $m = n = 2$, we have

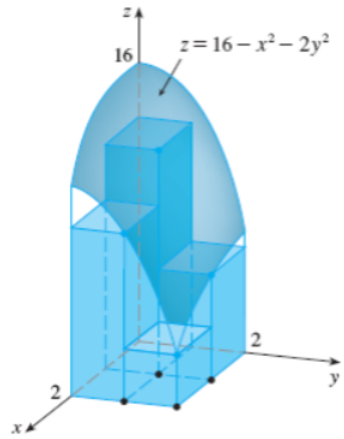
$$\begin{aligned}
V &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_i, y_j)\Delta A \\
&= f(1,1)\Delta A + f(1,2)\Delta A + f(2,1)\Delta A + f(2,2)\Delta A \\
&= 13(1) + 7(1) + 10(1) + 4(1) \\
&= 34
\end{aligned}$$



Example 2: If $R = \{(x,y)| -1 \leq x \leq 1, -2 \leq y \leq 2\}$, evaluate the integral

$$\int\int_R \sqrt{1-x^2}dA$$

Note that $\sqrt{1-x^2} \geq 0$, and so we can compute the integral by interpreting it as a volume. If $z = \sqrt{1-x^2}$, then $x^2 + z^2 = 1$ and $z \geq 0$, so the given double integral represents the volume of the solid S that lies below the circular cylinder $x^2 + z^2 = 1$ and above the rectangle R. The volume of S is the area of a semicircle with radius 1 times the length of the cylinder. Thus



$$\int\int_R \sqrt{1-x^2} dA = \frac{1}{2}\pi(1)^2 \times 4 = 2\pi$$

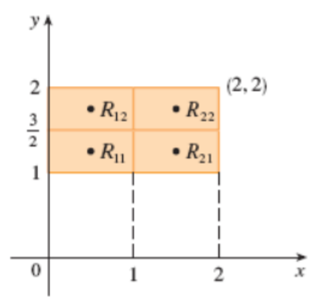
Midpoint Rule

$$\int\int_R f(x,y)dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i,\bar{y}_j)\Delta A$$

where $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

Example 3: Use the midpoint rule with $m = n = 2$ to estimate the value of the integral $\int\int_R (x - 3y^2)dA$, where $R = \{(x,y)|0 \le x \le 2, 1 \le y \le 2\}$.

We evaulate $f(x,y) = x - 3y^2$ at the centers of the four rectangles shown below. So $\bar{x}_1 = \frac{1}{2}$, $\bar{x}_2 = \frac{3}{2}$, $\bar{y}_1 = \frac{5}{4}$, and $\bar{y}_2 = \frac{7}{4}$. The area of each subrectangle is $\Delta A = \frac{1}{2}$. Thus



$$\begin{aligned}
\iint_R (x-3y^2)dA &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(\bar{x}_i,\bar{y}_j\right)\Delta A \\
&= f(\bar{x}_1,\bar{y}_1)\Delta A + f(\bar{x}_1,\bar{y}_2)\Delta A + f(\bar{x}_2,\bar{y}_1)\Delta A + f(\bar{x}_2,\bar{y}_2)\Delta A \\
&= f\left(\frac{1}{2},\frac{5}{4}\right)\Delta A + f\left(\frac{1}{2},\frac{7}{4}\right)\Delta A + f\left(\frac{3}{2},\frac{5}{4}\right)\Delta A + f\left(\frac{3}{2},\frac{7}{4}\right)\Delta A \\
&= \left(-\frac{67}{16}\right)\frac{1}{2} + \left(-\frac{139}{16}\right)\frac{1}{2} + \left(-\frac{51}{16}\right)\frac{1}{2} + \left(-\frac{123}{16}\right)\frac{1}{2} \\
&= -\frac{95}{8} \\
&= -11.875
\end{aligned}$$

Thus we have

$$\iint_R (x-3y^2)dA \approx -11.875$$

Iterated Integral

$$\int_a^b \int_c^d f(x,y)dydx = \int_a^b \left[\int_c^d f(x,y)dy\right] dx$$

Example 4: Evaluate the integral given below.

$$\int_0^3 \int_1^2 x^2 ydydx$$

Regarding x as a constant, we obtain

$$\int_1^2 x^2 y dy = \left[x^2 \frac{y^2}{2}\right]_{y=1}^{y=2} = x^2\left(\frac{2^2}{2}\right) - x^2\left(\frac{1^2}{2}\right) = \frac{3}{2}x^2$$

Thus the function A in the preceding discussion is given by $A(x) = \frac{3}{2}x^2$ in this example. We now integrate this function of x from 0 to 3:

$$\begin{aligned}
\int_0^3 \int_1^2 x^2 y dy dx &= \int_0^3 \left[\int_1^2 x^2 y dy\right] dx \\
&= \int_0^3 \frac{3}{2}x^2 dx \\
&= \frac{x^3}{2}\bigg]_0^3 \\
&= \frac{27}{2}
\end{aligned}$$

Fubini's Theorem

If f is continuous on the rectangle $R = \{(x, y)|a \le x \le b, c \le y \le d\}$, then

$$\iint_R f(x, y)dA = \int_a^b \int_c^d f(x, y)dydx = \int_c^d \int_a^b f(x, y)dxdy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example 5: Evaluate $\iint_R ysin(xy)dA$, where $R = [1, 2] \times [0, \pi]$.

If we first integrate with respect to x, we get

$$\begin{aligned}
\iint_R ysin(xy)dA &= \int_0^\pi \int_1^2 ysin(xy)dxdy \\
&= \int_0^\pi \left[-\cos(xy)\right]_{x=1}^{x=2} dy \\
&= \int_0^\pi (-\cos 2y + \cos y)dy \\
&= -\frac{1}{2}\sin 2y + \sin y\bigg]_0^\pi \\
&= 0
\end{aligned}$$

Recall that the average value of a univariate function f defined on an interval $[a, b]$ is

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$$

Average Value

Similarly, we can define the average value of a two-variable function to be

$$f_{avg} = \frac{1}{A(R)}\iint_R f(x, y)dA$$

where $A(R)$ is the area of R.

4.2 Double Integrals Over General Regions

4.3 Double Integrals in Polar Coordinates

4.4 Applications of Double Integrals

4.5 Surface Area

4.6 Triple Integrals

4.7 Triple Integrals in Cylindrical Coordinates

4.8 Triple Integrals in Spherical Coordinates

4.9 Change of Variables in Multiple Integrals

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