Math 134, Fall 2020, Midterm 2 Review

First, go over your homework problems. Look at the solutions for those problems that were not graded and compare with yours. Make sure you understand all of the proof questions.

Computational Problems

1. Computing (new) derivatives: Exercises 13-22 in Chapter 7 Review.
2. Evaluating integrals : Exercises 23-38 in Chapter 7 Review and 1-48 in Chapter 8 Review. First half of Section 8.1 is review of 5.7 together with the transcendental functions.
3. Work, volume, center of mass problems - Exercises 1-43 in Chapter 6 Review.
4. Prove that $\frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}$.
5. Find the average value of $f(x) = \dfrac{\cos(x/2)}{3+\sin(x/2)}$ on the interval $[-\pi, \pi]$.

Theorems and Their Applications

Know the definition of the definite integral, the definition of the indefinite integral, the definition of the inverse of a function. Know the precise statements of the important theorems. Go over the proofs we did in class and make sure you can follow them. If you use them to prove something, be sure to verify the hypotheses of the theorems you use. Some of the important theorems are:

- Covered on last midterm, but you still need to remember them: Intermediate Value Theorem (B.1.2), Extreme Value Theorem (B.2.2), The Mean Value Theorem (4.1.1).
- New this time: The Integrability Theorem (B.4.6), The Fundamental Theorem of Calculus (Theorems 5.3.5 and 5.4.2), The Mean Value Theorem for Integrals (5.9.3), Continuity of the Inverse (B.3.1), Differentiability of the Inverse (B.3.2).

Problems Using the Theorems

1. Compute $f'(x)$ for $f(x) = \displaystyle\int_{\ln x}^{\cos 3x} \frac{dt}{t^8+1}$. Which of the above theorems did you use?
2. The *error function* is defined by
$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$
   (a) What is the domain of $\operatorname{erf}(x)$? Prove that $\operatorname{erf}(x)$ is a one-to-one, increasing, differentiable function on its domain.
   (b) Show that $\operatorname{erf}(x)$ has a differentiable inverse.
   (c) Show that $\operatorname{erf}(x)$ is bounded from below and above. Hint: Compare $f(x) = \int_1^x te^{-t^2}dt$ and $\int_1^x e^{-t^2}dt$.
   (d) Sketch a graph of $\operatorname{erf}(x)$.
3. Express the following as a definite integral and evaluate it:
$$\lim_{n\to\infty} \left\{ \frac{e^{1/n}}{n} + \frac{e^{2/n}}{n} + \frac{e^{3/n}}{n} + \cdots + \frac{e^{(2n-1)/n}}{n} + \frac{e^{2n/n}}{n} \right\}$$
4. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous even function (i.e. $f(-x) = f(x)$). Prove that the function $F : \mathbf{R} \to \mathbf{R}$ given by
$$F(x) = \int_0^x f(t)\,dt$$
   is a continuous (in fact differentiable) odd function (i.e. $F(-x) = -F(x)$).

5. Let $0 < k < 1$ be a constant. The *Incomplete Elliptic Integral of the Second Kind* is the function $E_k : \mathbf{R} \to \mathbf{R}$ defined by the formula

$$E_k(x) = \int_0^x \sqrt{1 - k^2 \sin^2 s}\, ds$$

   (a) Prove that $E_k(x)$ is differentiable.

   (b) Prove that $E_k'(x) > 0$ for all x. What does this tell you about the inverse function $E_k^{-1}(x)$?

   (c) Does the graph of E_k have any points of inflection? Explain.

6. Let $f : [a, b] \to \mathbf{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose further that there is a constant $K > 0$ such that

$$|f'(x)| \leq K \text{ for all } x \in (a, b)\,.$$

   Show that

$$U_f(P) - L_f(P) \leq (b - a)KP$$

   where $P = \max_i \Delta x_i$.