Algebra Notes

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Contents

1 Linear Functions

Linear equations involve an independent variable with degree 1.
Forms of Linear Equations:

$y = mx + b$	**Slope-Intercept**	m is slope, b is y-intercept
$y - y_1 = m(x - x_1)$	**Point-Slope**	m is slope, (x_1, y_1) is a point on the line
$Ax + By + C = 0$	**Standard**	A, B, C are integers and $A > 0$

The slope between two points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$.

2 Quadratic Functions

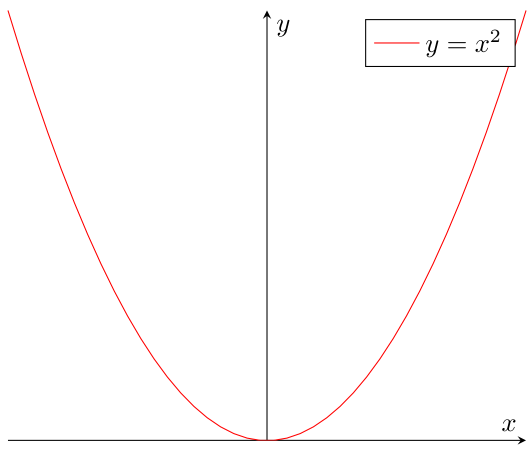
2.1 Miscellaneous Quadratic Info

$ax^2 + bx + c$	Standard Form
$a(x-h)^2 + k$	Vertex Form
$a(x-p)(x-q)$	Intercept Form

The vertex is given by (h, k), where $h = -\frac{b}{2a}$. The roots are given by p and q. The intercept form is also known as the factored form. The **axis of symmetry** which divides a quadratic graph into right and left sides, is given by the line $x = -\frac{b}{2a}$.

2.2 Graphing Quadratic Functions

1. Find the vertex
2. Find the y-intercept
3. Find the x-intercept(s)
4. Test two points minimum, one on each side of the vertex.
5. Connect the points.



2.3 Factorization

The Four Common Methods of Factoring:

1. Greatest Common Factor

$a(b+c) = ab + ac$

2. Grouping

This method is typically done with a four-term polynomial. Group the first two terms and the last two terms, then factor out a GCF from the two groups, and combine like polynomials. When the polynomial only has three terms, we can manipulate it so that the polynomial gains another term. We do this by multiplying the general quadratic coefficients a and c and then finding two factors of ac which have a sum equal to the middle term's general coefficient, b.

Example 1: Factor $x^3 + 7x^2 + 2x + 14$

$$\begin{aligned} x^3 + 7x^2 + 2x + 14 &= (x^3 + 7x^2) + (2x + 14) \\ &= (x^2(x+7) + 2(x+7)) \\ &= (x+7)(x^2+2) \end{aligned}$$

Example 2: Factor $x^5 - 3x^3 - 2x^2 + 6$

$$\begin{aligned} x^5 - 3x^3 - 2x^2 + 6 &= (x^5 - 3x^3) - (2x^2 - 6) \\ &= x^3(x^2-3) - 2(x^2-3) \\ &= (x^2-3)(x^3-2) \end{aligned}$$

Example 3: Factor $3x^2 + 2x - 8$

The product ac is given by $(3)(-8) = -24$. Its factors -4 and 6 have a sum of 2 which is equal to b, the coefficient of the middle term. Thus, we can split the $2x$ into two terms as below.

$$\begin{aligned} 3x^2 + 2x - 8 &= 3x^2 - 4x + 6x - 8 \\ &= (3x^2 - 4x) + (6x - 8) \\ &= x(3x-4) + 2(3x-4) \\ &= (3x-4)(x+2) \end{aligned}$$

Example 4: Factor $4x^2 + 10x - 6$
Again, the product ac is given by $(4)(-6) = -24$. Its factors -2 and 12 have a sum of $b = 10$. Thus, it can be factored by grouping:

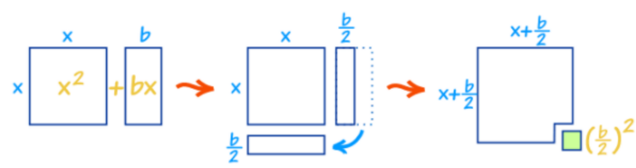
$$\begin{aligned} 4x^2 + 10x - 6 &= 4x^2 - 2x + 12x - 6 \\ &= (4x^2 - 2x) + (12x - 6) \\ &= 2x(2x - 1) + 6(2x - 1) \\ &= (2x + 6)(2x - 1) \end{aligned}$$

Special Forms:
For factoring higher-degree polynomials, simply employ the usual methods of factorization. Below are some general factorizations that should be memorized.

$(a \pm b)^2 = a^2 \pm 2ab + b^2$	**Sum/Difference of Perfect Squares**
$a^2 - b^2 = (a + b)(a - b)$	**Difference of Squares**
$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$	**Sum/Difference of Cubes**

2.4 Completing the Square

Taking a quadratic equation of the form $ax^2 + bx + c = 0$, completing the square allows us to rewrite this equation as $a(x + d)^2 + e = 0$, where $d = \frac{b}{2a}$ and $e = c - \frac{b^2}{4a}$. This method was inspired by the geometric representation of $x^2 + bx$ being completed by $\left(\frac{b}{2}\right)^2$.



Example: Complete the square for $x^2 + 4x + 1$
$\left(\frac{b}{2}\right)^2 = 4$, so we add 4 and subtract 4. Parantheses are added to better conceptualize the later simplification.

$$\begin{aligned} (x^2 + 4x + 4) + 1 - 4 &= (x^2 + 4x + 4) - 3 \\ &= (x + 2)^2 - 3 \end{aligned}$$

2.5 The Loh Method

Originally devised by ancient Babylonians and Greeks, this method was revised by Viete and rediscovered by Professor Po-Shen Loh in late 2019. We will refer to this as the Loh Method for ease of explanation.

Example 1: Solve $x^2 - 14x + 24 = 0$
The factorization will be in the form $(x - \_)(x - \_)$. The method says that *if and only if* we can find two numbers such that their sum is the opposite of b, in this case 14, and their product is c which is 24 in this case then those numbers are the solutions. This method is an extremely convenient way of finding the solutions for *any* quadratic equation, even irrational ones that would be hard to guess.

This method uses the concept that our two numbers that add to 14 have an average of 7, since $\frac{14}{2} = 7$. Taking the average of two numbers, we know that one of the factors will be a little smaller than 7 and the other factor will be a little greater than 7. We can express the smaller factor by "7-u" and the greater factor by "7+u". We know that the factors have a product of 24, thus

$$\begin{aligned} 24 &= (7 - u)(7 + u) \\ &= 49 - u^2 \\ u^2 &= 25 \\ u &= \pm 5 \end{aligned}$$

Since $u = \pm 5$ exists, we can calculate $7 - u = 7 - 5 = 2$ and $7 + u = 7 + 5 = 12$. Using $u = -5$ would produce the same values. Now we plug 2 and 12 into $(x - \_)(x - \_)$, giving us

$$\begin{aligned} 0 &= x^2 - 14x + 24 \\ &= (x - 2)(x - 12) \\ \implies x &= 2, 12 \end{aligned}$$

Using other factorization methods, we verify that these are indeed the solutions to the quadratic equation.

Example 2: Solve $x^2 - 8x + 13 = 0$
Using other methods, we would not have been able to factor this since 13 is a prime number. Again, $x^2 - 8x + 13 =$

$0 = (x - \_)(x - \_)$.

If we can find two numbers that have a sum of 8 and a product of 13, then they are the solutions. Looking at the product which is 13, it's very hard to think of factors of 13 which have a sum of 8. Once again, there are two numbers that have an average of 8, so we will need u to satisfy the below equation.

$$\begin{aligned}
13 &= (4-u)(4+u) \\
&= 16 - u^2 \\
u^2 &= 3 \\
u &= \pm\sqrt{3}
\end{aligned}$$

We can ignore the negative value of u since it will give the same values as its opposite. Thus,

$$\begin{aligned}
0 &= x^2 - 8x + 13 \\
&= \left(x - [4-\sqrt{3}]\right)\left(x - [4+\sqrt{3}]\right) \\
\implies x &= 4 \pm \sqrt{3}
\end{aligned}$$

2.6 The Quadratic Formula

We can derive the quadratic formula through completing the square.

$$\begin{aligned}
ax^2 + bx + c &= 0 \\
(ax^2 + bx) + c &= 0 \\
a\left(x^2 + x \cdot \frac{b}{a}\right) + c &= 0 \\
a\left(x^2 + x \cdot \frac{b}{a} + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a^2} &= 0 \\
a\left(x^2 + x \cdot \frac{b}{a} + \frac{b^2}{4a^2}\right) + c = \frac{b^2}{4a^2} &= \frac{b^2}{4a^2} \\
a\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a} - c \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} - \frac{c}{a} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2-4ac}}{2a}
\end{aligned}$$

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

2.7 Imaginary and Complex Numbers

When a quadratic solution involves a negative number under a radicand, that solution is either imaginary or complex. Recall that pure imaginary numbers are of the form bi, where $i = \sqrt{-1}$ and complex numbers are of the form $a + bi$.

Example 1: $x^2 + 4x + 5 = 0$
Using the quadratic formula, we compute x to be

$$\begin{aligned}
x &= \frac{-4 \pm \sqrt{-4}}{2} \\
&= \frac{-4 \pm 2i}{2} \\
&= -2 \pm i
\end{aligned}$$

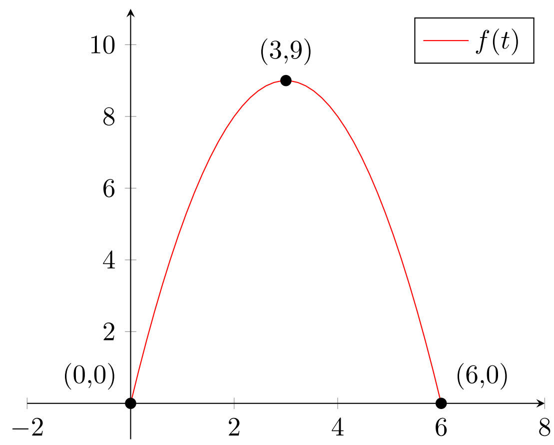
If we wanted to efficiently find the behavior of a quadratic's roots, we could use the **discriminant**, $b^2 - 4ac$.

Types of Roots

Discriminant is positive	Discriminant is zero	Discriminant is negative
$b^2 - 4ac > 0$	$b^2 - 4a = 0$	$b^2 - 4ac < 0$
Two Real Roots	*One Real Root*	*Two Complex Roots*

2.8 Mathematical Modeling with Quadratic Functions

Quadratic functions are often used for modeling flying objects, for example the flight trajectory of a thrown ball. Consider the graph of $f(t) = -(t-3)^2 + 9$, where $t > 0$. If t represents time, then $f(t)$ represents the height at which the ball is at a particular time. We can then clearly see from the graph that the ball is at a height of 0 when time is 0. The ball reaches its peak height of 9 at the vertex of the quadratic function when time is 3, and descends to a height of 0 at the second root of the function.

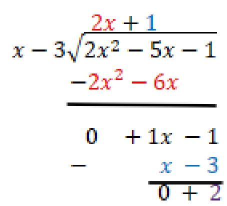


3 Polynomials

3.1 Long Division

Dividend = Divisor · Quotient + Remainder
$f(x) = d(x) \cdot q(x) + r(x)$
Note that the degree of $r(x)$ is always less than that of $d(x)$.

Example 1: Divide $2x^2 - 5x - 1$ by $x - 3$



$$\therefore 2x^2 - 5x - 1 = (x-3)(2x+1) + 2$$

where the remainder is 2.

The Remainder Theorem:
When a polynomial $f(x)$ is divided by $x - c$ the remainder is $f(c)$.

Example 2: Find the remainder when $2x^2 - 5x - 1$ is divided by $x - 5$

$$\begin{aligned} f(5) &= 2(5)^2 - 5(5) - 1 \\ &= 50 - 25 - 1 \\ &= 24 \end{aligned}$$

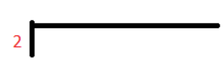
Thus, the remainder is 24.
The Factor Theorem:
When $f(c) = 0$, $x = c$ is a factor of $f(x)$. The converse of this is also true.

3.2 Synthetic Division

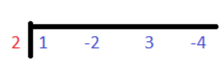
Synthetic division is a shortcut for long division. It only works when the divisor is of the first degree. The remainder produced by synthetic division is also the the value of the function at the remainder, $f(r)$ when r is the remainder.

Example: Divide $x^3 - 2x^2 + 3x - 4$ by $x - 2$

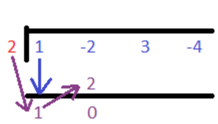
1. Draw a table like so and write c on the left side of the table, for a divisor in the form $x - c$. In this case, $c = 2$.



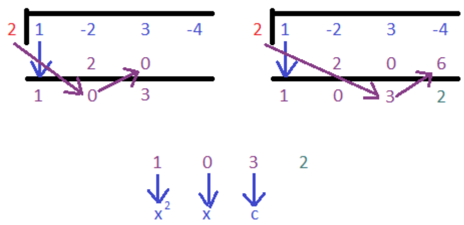
2. Add the coefficients of the polynomial's terms under the line as is. In this case, the coefficients corresponding to x^3, x^2, x^1, and x^0 are $1, -2, 3$, and -4, respectively.



3. Drop the first coefficient, in this case 1, below the newly made bar. Then write the product of c and the first coefficient below the second coefficient. Now write the sum of the second coefficient and this product below the bar.



4. Find the product of c and the calculated sum, in this case the second green number. Find the sum of this product and the third coefficient. Repeat this process for the rest of the coefficients. The last calculated sum is the remainder, in this case the purple number.



5. The three green sums are the coefficients of the quotient. The quotient should be of degree 2, since its largest term x^3 divided by the divisor's largest term, x, is x^2. Thus, x^2 corresponds to the first green number, and so on. Here, c refers to the constant term of the polynomial.
Thus,

$$\begin{aligned}\frac{x^3-2x^2+3x-4}{x-2} &= 1x^2+0x+3+\frac{2}{x-2}\\ &= x^2+3+\frac{2}{x-2}\end{aligned}$$

3.3 Vieta's Formulas

Vieta's Formulas relate the coefficients of polynomials to the sums and products of their roots. Whereas Vieta's Formulas seem trivial in quadratic applications, they become extremely useful in complex polynomials with many roots or roots that are hard to derive. Vieta's Formulas can be viewed as a shortcut for finding solutions of a polynomial quickly simply through the sums and products of the roots.

Consider the polynomial $x^2+2x-15=(x-3)(x+5) \implies x=-5,3$. Using Vieta's Formulas, we can find the sum of the roots $3+(-5)=-2$ and $3\cdot(-5)=-15$ directly, without having to find each root directly.

By the Remainder Factor Theorem, a polynomial $f(x)$ has roots r_1 and r_2 in the form $f(x)=A(x-r_1)(x-r_2)=Ax^2-A(r_1+r_2)x+Ar_1r_2$ for some constant A. Comparing coefficients with $f(x)=ax^2+bx+c$, we can conclude that $a=A, b=-A(r_1+r_2)$, and $c=Ar_1r_2$. Hence, we get:

Vieta's Formulas for Quadratics:
Given $f(x)=ax^2+bx+c$ if the equation $f(x)=0$ has roots r_1 and r_2, then

$$r_1+r_2=-\frac{b}{a}, r_1r_2=\frac{c}{a}$$

Alternatively,

$$b=-(p+q), c=pq$$

It is important to note that the precondition for using Vieta's Formula is that the polynomial $f(x)$ must be put in a **monic** form, that is, the leading coefficient a must be 1.

Example 1: If α and are the roots of the quadratic $x^2-4x+9=0$, what are the values of
1. $\alpha+$
2. α
3. α^{22}

1.

$$\alpha+=-\frac{b}{a}=-\frac{-4}{1}=4$$

2.

$$\alpha=\frac{c}{a}=\frac{9}{1}=9$$

3.

$$\begin{aligned}\alpha^2+^2 &= (\alpha+)^2-2\alpha\\ &= 4^2-2(9)\\ &= -2\end{aligned}$$

For this question, the roots were $2\pm i\sqrt{5}$. vieta's Formulas offer a simpler approach to compute these expressions without potentially making calculation mistakes.

Example 2: What are the roots of the quadratic x^2-5x+6?
If p and q are the roots of the quadratic, then $p+q=5$ and $pq=6$. Solving this system, it's easy to see that the roots are 2 and 3.

Example 3: Find a quadratic with roots 2 and 5.
$b=-(2+5)=-7, c=(2)(5)=10$
Hence, the desired quadratic is $x^2-7x+10$.

Example 4: Find a quadratic with roots $3+2i$ and $3-2i$.
$b = -[(3+2i)+(3-2i)] = -6, c = (3+2i)(3-2i) = 13.$
The desired quadratic is $x^2 - 6x + 13$.
We can also generalize these formulas to higher-degree polynomials:

Vieta's Formula:

Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ be a polynomial with complex coefficients and degree n, having complex roots $r_n, r_{n-1}, \ldots, r_1$. Then for any integer $0 \le k \le n$,

$$\sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} r_{i_1} r_{i_2} \ldots r_{i_k} = (-1)^k \frac{a_{n-k}}{a_n}$$

Vieta's Formulas for quadratic polynomials written in summation notation is

$$\sum_{i=1}^{n} r_i = -\frac{a_{n-1}}{a_n}, r_1 r_2 \ldots r_n = (-1)^n \frac{a_0}{a_n}$$

Example 5: Suppose k is a number such that the cubic polynomial $P(x) = -2x^3 + 48x^2 + k$ has three integer roots that are all prime numbers. How many possible distinct values are there for k?

Let p, q, and r denote the three integer roots of $P(x)$. Then by Vieta's Formula, we have $pq + qr + pr = 0$ but since p, q, r are prime, each of them are strictly greater than 1 and hence no such $P(x)$ exists.

4 Higher-degree Polynomials

4.1 Graphing Higher-degree Polynomials

Steps to Graph Higher-Degree Polynomials:

1. Determine the zeroes of the polynomial $P(x)$ and their multiplicity.
2. Determine the y-intercept, $(0, P(0))$
3. Use the Leading Coefficient Test to determine the end behaviors.
4. Plot a few more points to make the sketch more accurate. At least plot one point at each end and between each of the zeroes.

The first three names of higher-degree polynomials are **cubic** (x^3), quartic (x^4), and quintic (x^5). Polynomials that are second-degree and higher tend to be *continuous* and there are usually local maxima and minima.

A polynomial with degree n will have at most $n - 1$ extrema.

The Multiplicity Test:

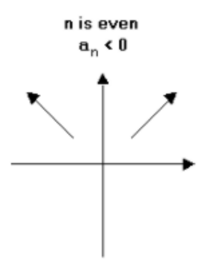
If $x = r$ is a zero of the polynomial $P(x)$ with multiplicity k then:
1. If k is odd then the x-intercept corresponding to $x = r$ will cross the x-axis.
2. If k is even then the x-intercept corresponding to $x = r$ will only touch the x-axis and not cross it.

Furthermore, if $k > 1$ then the graph will flatten out at $x = r$.

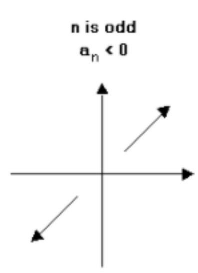
Multiplicity refers to the number of times a factor is found within a polynomial. Consider the polynomial given by $P(x) = x^2(x-3)(x+2)$. Its zeroes are $x = -2$ $(k = 1)$, $x = 0$ $(k = 2)$, and $x = 3$ $(k = 1)$. $x = 0$ has a mutliplicity of 2, since the factor appears twice. Recall that $x^2 = 0$ gives $x = \pm 0$.

The Leading Coefficient Test: Suppose that $P(x)$ is a polynomial with degree n, where $P(x) = ax^n + \ldots$. We only need to consider the first coefficient, a.

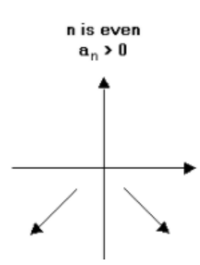
1. If $a > 0$ **and** n **is even** then the graph of $P(x)$ will increase without bound at both endpoints.



2. If $a > 0$ **and** n **is odd** then the graph of $P(x)$ will increase without bound at the right end and decrease without bound at the left end.

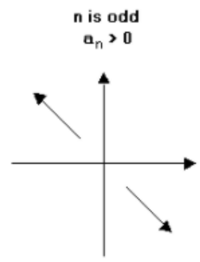


3. If $a < 0$ **and** n **is even** then the graph of $P(x)$ will decrease without bound on both ends.



4. If $a < 0$ **and** n **is odd** then the graph of $P(x)$ will decrease without bound at the right end and increase without bound at the left end.

Example: Sketch the graph of $P(x) = x^4 - x^3 - 6x^2$.
$P(x) = x^2(x-3)(x+2)$

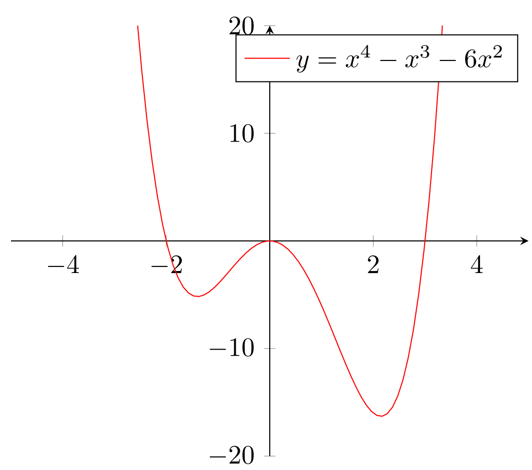


Multiplicity is 1 at $x = -2$
Multiplicity is 2 at $x = 0$
Multiplicity is 1 at $x = 3$

Thus, the zeroes at $x = -2$ and $x = 3$ correspond to x-intercepts, whereas the zero at $x = 0$ has an even multiplicity so it will only touch the x-axis instead of crossing it. The y-intercept is $(0, 0)$ and this is also the x-intercept. Some function evaluations are below.

$$P(-3) = 54 \quad P(-1) = -4 \quad P(4) = 96$$

The leading coefficient is 1 and the highest degree of $P(x)$ is 4 which is an even number so by the Leading Coefficient Test, both ends of the polynomial's graph will increase without bound. Putting all our information together, we can now sketch the graph.



4.2 Rational Root Test

Suppose we have a polynomial $P(x)$ with integer coefficients and a nonzero constant term, $a_n \neq 0$.
$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$
Then potential rational roots of $P(x)$ are of the form

$$\frac{p}{q} = \frac{\pm \text{factors of } a_0}{\pm \text{factors of } a_n}$$

Example: Find the rational roots of $P(x) = 3x^3 - 4x^2 - 17x + 6$

The leading coefficient is $a_n = 3$ and constant term is $a_0 = 6$. We can determine the positive and negative factors of
$a_0 = 6 : \pm(1, 2, 3, 6)$
$a_n = 3 : \pm(1, 3)$
Then

$$\frac{p}{q} = \frac{\pm(1, 2, 3, 6)}{\pm(1, 3)} = \pm 1, \pm\frac{1}{3}, \pm 2, \pm\frac{2}{3}, \pm 3, \pm\frac{3}{3}, \pm\frac{6}{1}, \pm\frac{6}{3}$$

Eliminating duplicate terms and simplifying, our root candidates are:

$$\pm\frac{1}{3}, \pm\frac{2}{3}, \pm 1, \pm 2, \pm 3, \pm 6$$

We recall that if a is a root of $P(x)$ then $P(a) = 0$. Let's check our candidates now.

$P(x) = 3x^3 - 4x^2 - 17x + 6$	**Is it a root?**
$P(\frac{1}{3}) = 0$	YES
$P(-\frac{1}{3}) = \frac{100}{9}$	No
$P(\frac{2}{3} = -\frac{56}{9}$	No
$P(-\frac{2}{3}) = \frac{44}{3}$	No
$P(1) = -12$	No
$P(-1) = 16$	No
$P(2) = -20$	No
$P(-2) = 0$	YES
$P(3) = 0$	YES
$P(-3) = -60$	No
$P(6) = 408$	No
$P(-6) = -684$	No

5 More Functions

5.1 Composite Functions

Function composition refers to when one function is applied to the results of another. The formal notation of "the result of function f sent through function g is written $(g \circ f)(x)$ which is also the same as saying $g(f(x))$.

Example: Given $f(x) = 3x + 2$ and $g(x) = x + 5$, find $(f \circ g)(x)$.

$$\begin{aligned} (f \circ g)(x) &= f(x+5) \\ &= 3(x+5)+2 \\ &= 3x+17 \end{aligned}$$

5.2 Inverse Functions

An **inverse function** is a function that reverses another function. If a function $f(x) = y$ then its inverse, denoted by the superscript -1, is given by $f^{-1}(y) = x$. The inverse function is also commonly wrote as $g(x)$.

An **invertible function** is a function that has an inverse. A function is invertible if and only if the function is **one-to-one**, or for each y-value, there must be only one value of x such that $Y \to X$. Such a function is also described as **bijective**. A function is bijective only if and only if it passes the horizontal line test. Inverse functions are symmetrical across the line $y = x$.

Example: Given $f(x) = \sqrt{x-3}$, find $g^{-1}(x)$ for $x \geq 0$.

$$\begin{aligned} y = f(x) &= \sqrt{x-3} \\ x &= \sqrt{y-3} \\ x^2 &= y-3 \\ x^2+3 &= y \\ g^{-1}(x) &= x^2+3 \end{aligned}$$

6 Exponents and Logarithms

6.1 Review of Exponents and Logarithms

Exponent Laws:
1. $a^m a^n = a^{m+n}$
2. $(a^m)^n = a^{mn}$
3. $(ab)^m = a^m b^m$
4. $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
5. $(\frac{a}{b})^m = \frac{a^m}{b^m}, b \neq 0$
6. $a^{-}m = \frac{1}{a^m}, a \neq 0$
7. $a^{\frac{1}{n}} = \sqrt[n]{a}$
8. $a^0 = 1, a \neq 0$
9. $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

Logarithms are the inverse operation of exponents. In other words,
$y = \log_a x \iff x = a^y, a > 0$
where a is the base. Conventionally, if a base is unspecified then it is 10 such that $\log x = \log_{10} x$.

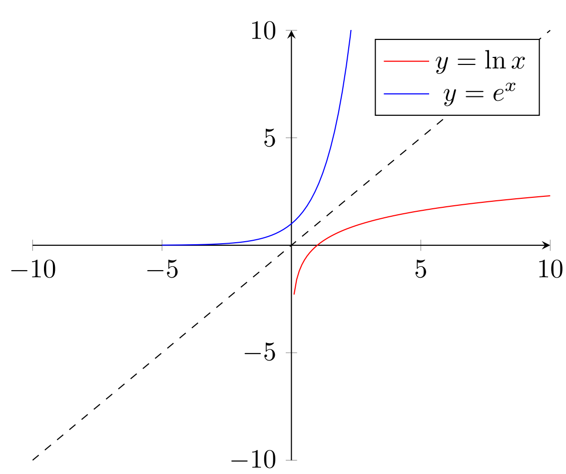
The natural logarithm, $\ln x$, is a logarithm with base e.

Logarithm Properties:
1. $\log_a xy = \log_a x + \log_a y$
2. $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. $\log_a x^y = y \cdot \log_a x$
4. $\log_a a^x = x$
5. $a^{\log_a x} = x$
6. $\log_a \frac{1}{x} = -\log_a x$

Common Logarithms:
1. $\ln e = 1$
2. $\log_a 1 = 0, a > 0$
3. $\log_a 0 = \text{UND}$
4. $\log_a a = 1, a > 0$

6.2 Graphing Exponential and Logarithmic Functions

The function $y = e^x$ is the inverse of $y = \ln x$. From the graph, we can see that the two functions are symmetric across the line $y = x$. There is a horizontal asymptote at $y = 0$ in the graph of e^x and a vertical asymptote at $x = 0$ in the graph of $y = \ln x$.



7 Radical Functions

Radical Functions are functions that contain a fractional exponent, that is, a variable under a radicand. When graphing radical functions, it is important to keep the domain in mind. For the parent function $f(x) = \sqrt{x}$, the domain is $x \geq 0$. This results from the fact that real solutions will not come from the square root of negative numbers.

For the square root function $f(x) = a\sqrt{x}$, we notice that a value $|a| > 0 \implies$ a vertical stretch and $0 < |a| < 1 \implies$ a vertical shrink. If $a < 0$ then the graph will be vertically reflected across the x-axis.

Radical functions are graphed by plotting many points while keeping in mind the domain and connecting them with a line.

8 Rational Functions

A rational function is any function that can be expressed as the ratio of two polynomial functions, where the denominator is not equal to 0. The domain of a rational function $f(x) = \frac{P(x)}{Q(x)}$ is the set of all points for which $Q(x) \neq 0$. **Singularities** are the x-values at which rational functions are undefined, for which $Q(x) \neq 0$.

Oblique Asymptotes are asymptotes that are neither perpendicular nor parallel, rather, they are inclined. Vertical asymptotes occur at all singularities for rational functions, and a rational function can have at most one horizontal/oblique asymptote.

If a function $f(x) = \frac{P(x)}{Q(x)}$ has a highest degree of n in the numerator and m in the denominator then:

$n > m$	No Horizontal Asymptote (Although if $n = m + 1$ then there is an Oblique Asymptote)
$n < m$	x-axis is a Horizontal Asymptote
$n = m$	Horizontal Asymptote exists at $y = \frac{\text{Coefficient of } n}{\text{Coefficient of } m}$

Example 1: Find any horizontal or oblique asymptotes for $f(x) = \frac{2x^2+x+1}{x^2+16}$
Because $n = m$, there will be one horizontal asymptote and no oblique asymptote, given by $y = \frac{2}{1} = 2$.

Steps for Graphing Rational Functions:
1. Find the intercepts
2. Find the vertical asymptotes if they exist by setting the denominator equal to zero and solving
3. Find the horizontal or oblique asymptote if it exists
4. Sketch at least one point in each region divided by the vertical asymptotes. Add more points for more accuracy.
5. Sketch the graph

Example 1: Sketch the graph of $f(x) = \frac{3x+6}{x-1}$
Starting with the intercepts, the y-intercept is

$$f(0) = \frac{6}{-1} = -6 \implies (0, -6)$$

and the x-intercepts will be

$$3x + 6 = 0, x = -2 \implies (-2, 0)$$

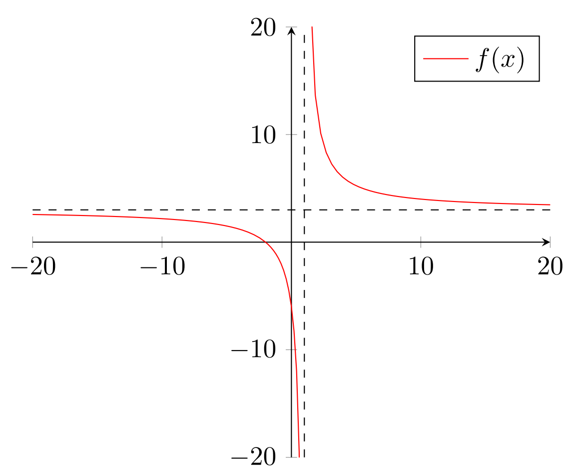
Now let's find the asymptotes, starting with the vertical asypmtote.

$$x - 1 = 0 \implies x = 1$$

Since $n = m$, there will be a horizontal asymptote a

$$y = \frac{3}{1} = 3$$

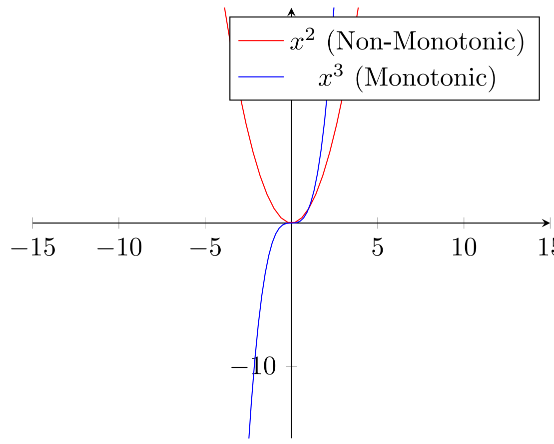
After plugging some x-values into the function, we can find the general shape of the graph. Now we sketch the graph with its asymptotes.



9 Even More Functions

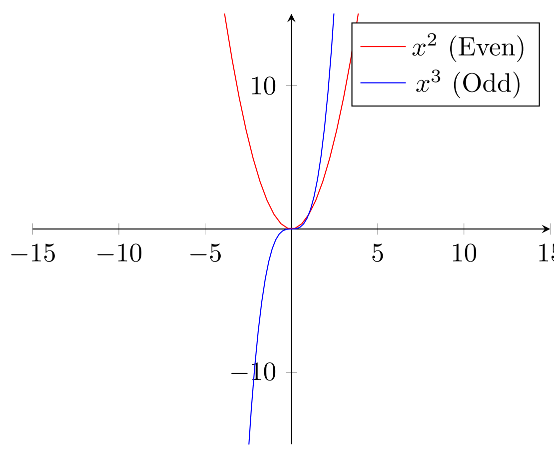
9.1 Monotonic Functions

Monotonicity refers to intervals of increase and decrease. **Monotonic Functions** are functions that are strictly increasing or strictly decreasing on their entire domains.



9.2 Even and Odd Functions

Even functions ($f(-x) = f(x)$) are unchanged when reflected across the y-axis. **Odd functions** ($f(-x) = -f(x)$ are unchanged when rotated 180° about the origin.

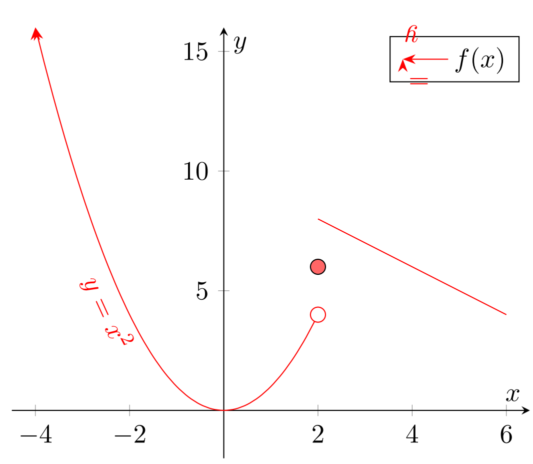


9.3 Piecewise and Parametric Functions

Functions that are defined differently across its x-intervals are **piecewise functions**. Consider the function defined below

$$f(x) = \begin{cases} x^2 & , x < 2 \\ 6 & , x = 2 \\ 10 - x & , 2 < x \leq 6 \end{cases}$$

The domain is then $\{x \in \mathbb{R} | x \leq 6\}$ and the graph looks like this:

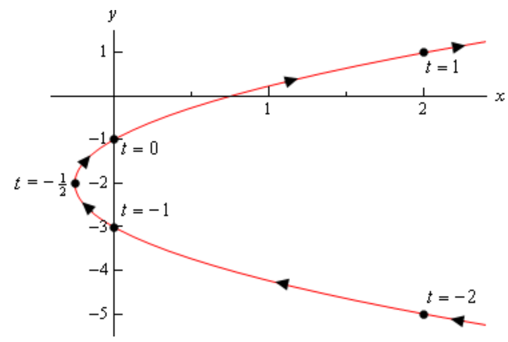


Instead of defining a single function $y = f(x)$, **Parametric Equations** are defined as multiple functions together, one for each variable. Hence, we can represent a function with x and y in terms of the **parameter** t as below:

$$x = f(t) \quad y = g(t)$$

An example of a **parametric curve**, the set of points the parameter gives in all the parametric equations, is the parent circle given by $x^2 + y^2 = r^2$. Isolating x and y give our two parametric equations, $y = \pm\sqrt{r^2 - x^2}$ and $x = \pm\sqrt{r^2 - y^2}$.

When we sketch parametric curves, we plug in values of the parameter and find their x and y-values, graphing each point of the form (x, y) on the Cartesian Plane. It is important to note that parametric curves always have a **direction of motion**, represented by arrows on the curve given by the increasing parameter t.
We can **eliminate the parameter** from a set of parametric equations by solving for the parameter t in one of the equations and plugging the value for t into the second parametric equation.

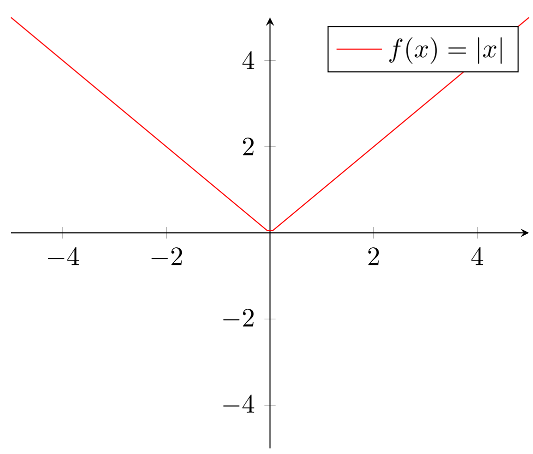


9.4 The Absolute Value Function

An **Absolute Value Function** is a function involving absolute value operations. The absolute value parent function is

$$f(x) = |x| = \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases}$$

Absolute value functions are V-shaped and to graph them we simply choose some x-values and plot their ordered pairs.



9.5 The Floor and Ceiling Functions

The **floor** of a number is the nearest integer down. The **ceiling** of a number is the nearest integer up. For 2.31, the floor is 2 and the ceiling is 3. The floor and ceiling of integers are the integers themselves. The floor of x is represented by $\lfloor x \rfloor$ and the ceiling by $\lceil x \rceil$.

The **Floor Function** is piecewise, discontinuous at each integer, and is composed of the greatest integer that is less than or equal to x. The **Ceiling Function** is piecewise, discontinuous at each integer, and is composed of the least integer that is greater than or equal to x.

Properties of Floor and Ceiling Functions:
1. $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ for any integer n
2. $\lfloor x \rfloor + \lfloor -x \rfloor = \begin{cases} -1, & x \notin \mathbb{Z} \\ 0, & x \in \mathbb{Z} \end{cases}$
3. $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + 1$

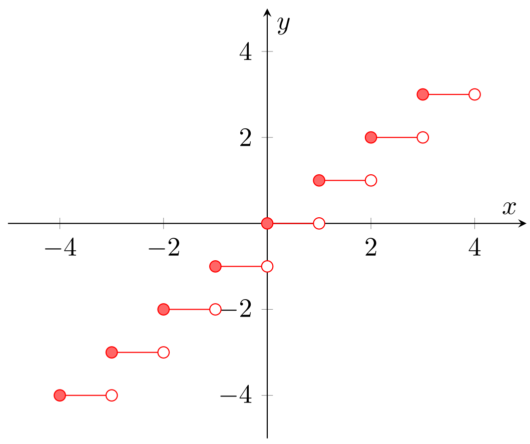
Similary, all of the floor brackets can be replaced with ceiling brackets for their properties as well.

The domain of the floor function is given by $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$.

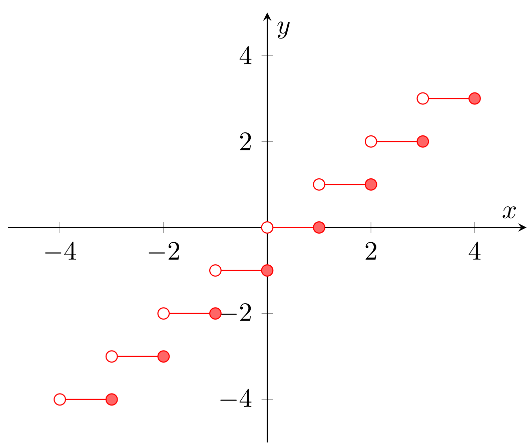
Example 1: Find all the values of x that satisfy $\lfloor 0.5 + \lfloor x \rfloor \rfloor = 20$.
Let $y = \lfloor x \rfloor$. Then

$$\begin{aligned} \lfloor 0.5 + y \rfloor &= 20 \\ &\iff \\ 20 \leq\ & y + 0.5 < 21 \\ 19.5 \leq\ & y < 20.5 \end{aligned}$$

Since y is an integer and $y = 20$ is the only interval in this interval, this becomes $y = 20 = \lfloor x \rfloor$. Since any value less than 21 and greater than or equal to 20 wil satisfy this equation, the answer is $\{x \in \mathbb{R} | 20 \leq x < 21\}$.



The Floor Function

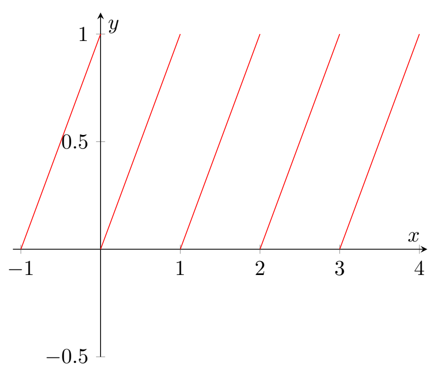


The Ceiling Function

9.6 The Fractional Part Function

The **Fractional Part Function** is defined as $f(x) = \{x\} = x - \lfloor x \rfloor$. For nonnegative real numbers, the fractional part is simply the part after the decimal. For example, $\{3.64\} = 3.64 - \lfloor 3.64 \rfloor = 3.64 - 3 = 0.64$.

Properties of the Fractional Part Function:
1. $0 \leq \{x\} < 1$ and $0 = \{x\}$ if and only if x is an integer
2. $\{x\} + \{-x\} =$
$\begin{cases} 0 & \text{if } x \text{ is an integer} \\ 1 & \text{otherwise} \end{cases}$
3. If a and b are integers and $b > 0$ then $\{\frac{a}{b}\} = \frac{r}{b}$, where r is the remainder from dividing a by b.

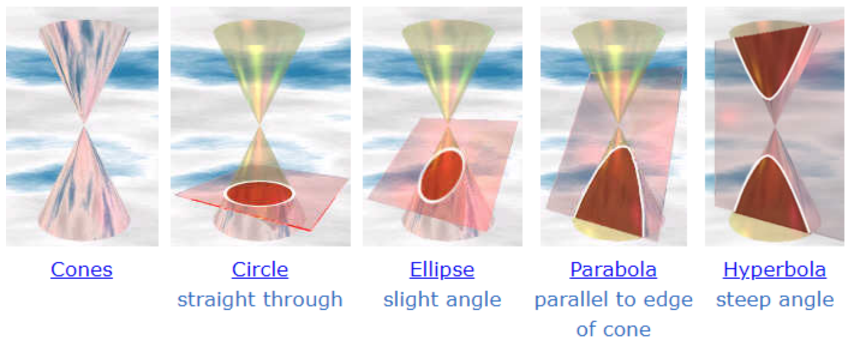


The Fractional Part Function

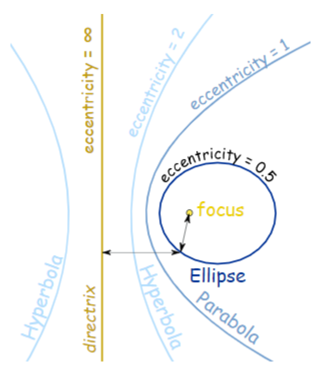
10 Conics

10.1 Introduction to Conics

From Analytic Geometry, **Conic Sections** are the intersection of a plane and 2 opposite-facing solid cones from different angles.



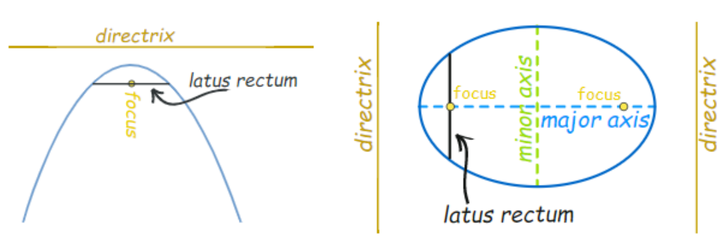
These curves can be defined using a straight line (**directrix**) and a point (**focus**). The distance from the focus to a point on the curve and the distance perpendicularly from the directrix to that point will always be the same ratio. For ellipses, this ratio is less than 1. For parabolas, the ratio is 1, hence the distances are equal. For hyperbolas, this ratio is greater than 1. This ratio is called **eccentricity**, which graphically shows us how "un-circular" the curve is. The larger the eccentricity, the less curved it is. Circles have an eccentricity of 0.



The **latus rectum** runs parallel to the directrix and passes through the focus.
Length of the Latus Rectum:

Parabolas	4x focal length
Circles	the diameter
Ellipses	$\frac{2b^2}{a}$

Above, a and b are the **major** and **minor axes**, respectively.



The **General Equation** that covers all conic equations is given by

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where A, B, C, D, E, F are all constants.

Process to Determine Conic Type from General Form:
1. Are both variables squared?
If no, it's a parabola. If yes, go to next step.
2. Do the squared terms have opposite signs?
If yes, it's a hyperbola. If no, go to next step.
3. Are the squared terms multiplied by the same number?
If yes, it's a circle. If no, it's an ellipse.

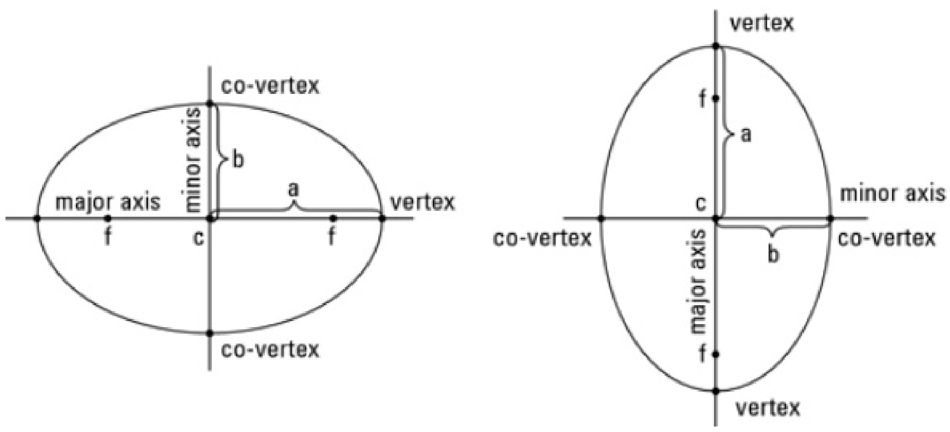
10.2 Ellipses

For the equations below, a is the length of half of the major axis, b is the length of half of the minor axis, and A, C, D, E, F are constants, respectively.

General Form	$Ax^2 + Cy^2 + Dx + Ey + F = 0$
Standard Form, Horizontal Major Axis	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$
Standard Form, Vertical Major Axis	$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$

Center: (h, k)
Length of Major Axis: $2a$
Length of Minor Axis: $2b$
Distance between center and foci, represented by c, is $c^2 = a^2 - b^2, a > b > 0$

The major axis runs between the vertices, whereas the minor axis runs between the co-vertices.



Graphing Ellipses:
Determine the major axis, vertices, co-vertices, and foci.
Equation is in form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, a > b$

Center	(h, k)
Major Axis	parallel to x-axis
Coordinates of the Vertices	$(h \pm a, k)$
Coordinates of the Co-vertices	$(h, k \pm b)$
Coordinates of the Foci	$(h \pm c, k)$

Equation is in form $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1, a > b$

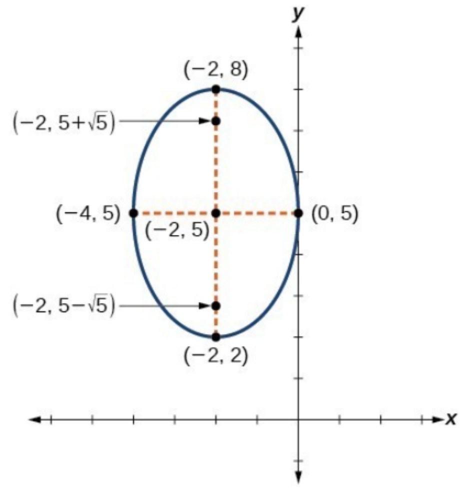
Center	(h, k)
Major Axis	parallel to y-axis
Coordinates of the Vertices	$(h, k \pm a)$
Coordinates of the Co-vertices	$(h \pm b, k)$
Coordinates of the Foci	$(h, k \pm c)$

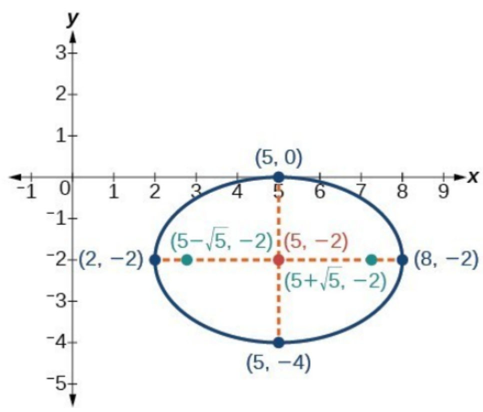
Example 1: Graph $\frac{(x+2)^2}{4} + \frac{(y-5)^2}{9} = 1$
Because a is the always the bigger number, $a^2 = 9$ and $b^2 = 4$. Because $9 > 4$ the major axis is parallel to the y-axis. The center (h, k) is then $(-2, 5)$. The vertices are $(h, k \pm a) = (-2, 2), (-2, 8)$. The co-vertices are $(h \pm b, k) = (-4, 5), (0, 5)$. Since $c^2 = a^2 - b^2 = 9 - 4 = 5 \implies c = \pm\sqrt{5}$, the foci are $(h, k \pm c) = (-2, 5 - \sqrt{5}), (-2, 5 \pm 5)$.

Example 2: Graph the ellipse given by $4x^2 + 9y^2 - 40x + 36y + 100 = 0$

$$(4x^2 - 40x) + (9y^2 + 36y) = -100$$
$$4(x^2 - 10x) + 9(y^2 + 4y) = -100$$
$$4(x^2 - 10x + 25) + 9(y^2 + 4y + 4) = -100 + 100 + 36$$
$$4(x-5)^2 + 9(y+2)^2 = 36$$
$$\frac{(x-5)^2}{9} + \frac{(y+2)^2}{4} = 1$$

Because $9 > 4$. the major axis is parallel to the x-axis. Since $a^2 = 9$ and $b^2 = 4$, $c^2 = 9 - 4 \implies c = \pm\sqrt{5}$. The center is $(5, -2)$. The vertices are $(2, -2), (8, -2)$. The co-vertices are $(5, -4), (5, 0)$. The foci are $(5 - \sqrt{5}, -2), (5 + \sqrt{5}, -2)$.





10.3 Circles

A **circle** is the set of all points on a plane that are a fixed distance from the center. A circle is not a function because it fails the vertical line test. A circle is a special type of ellipse with equations below.

General Form	$x^2 + y^2 + Cx + Dy + E = 0$
Standard Form	$(x-h)^2 + (y-k)^2 = r^2$

Above, the center is given by (h, k) and the radius as r.
Graphing Circles:
Determine the center (h, k) and radius r and graph it.

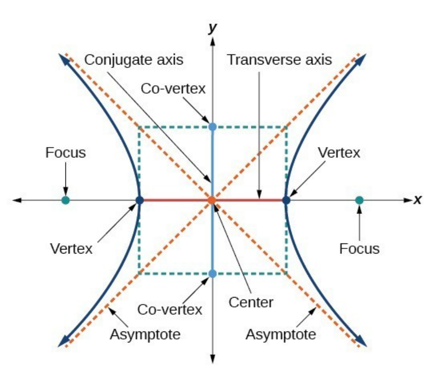
10.4 Parabolas

We have already graphed and worked with parabolas in the past, so here are the general forms of parabolas when used in the context of conics.

Parabola, Horizontal Axis	$(y-k)^2 = 4p(x-h), p \neq 0$	Vertex is (h, k) Focus is $(h+p, k)$ Directrix is the line $x = h - p$ Axis is the line $y = k$
Parabola, Vertical Axis	$(x-h)^2 = 4p(y-k), p \neq 0$	Vertex is (h, k) Focus is $(h, k+p)$ Directrix is the line $y = k - p$ Axis is the line $x = h$

10.5 Hyperbolas

Hyperbolas look like this:



General Form	$Ax^2 - Cy^2 + Dx + Ey + F = 0$
Hyperbola, Horizontal Transverse Axis	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$
Hyperbola, Vertical Transverse Axis	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

Center: (h, k)
Distance between vertices: $2a$
Distance between foci: $2c$
$c^2 = a^2 + b^2$

The eccentricity, e, has the formula
$e = \frac{\sqrt{a^2+b^2}}{a}$

The reciprocal function $f(x) = \frac{1}{x}$ is a hyperbola.

11 Sequences and Series

11.1 Introduction to Sequences and Series

A **sequence** is an ordered list of numbers, whereas a **series** is the sum of the terms of a sequence. We can represent a series with $\{a_n\}_{n=1}^{\infty}$, where the sequence starts with index $n = 1$ and runs to infinity. The other notation, summation notation, is written $\sum_{n=1}^{10} a_n$, where the sequence runs from $n = 1$ to infinity.

For example, the expansion of $\{a_n\}_{n=1}^{n=10}$ is $a_n = n^2 = 1, 4, , 9, 16, 25, 36, 49, 64, 81, 100$.

Conventionally, the following symbols are used:

a	first term in sequence
n	number of terms in sequence
S_n	sum of first n terms in sequence
d	Common difference between any two consecutive terms, arithmetic sequences
r	Common ratio between two consecutive terms, geometric sequences

11.2 Arithmetic Progressions

Arithmetic Progressions are sequences containing numbers which differ from each other by a common difference, d.

Formula for Arithmetic Sequences

$$a_n = a_1 + d(n-1)$$

a_n is the n^{th} term, a_1 is the first term, n is the index

The sum of the first n terms is given by one of the three formulas.

$$S_n = \frac{n}{2}[2a + d(n-1)]$$
$$S_n = \frac{n}{2}[a + a_n]$$
$$S_n = n \cdot (middleterm)$$

Find the sum of the first 50 odd positive integers.

$$S_n = \frac{n}{2}(2a + d(n-1)) \implies S_{50} = 25 \cdot (2 + 49 \cdot 2) = 2500$$

11.3 Geometric Sequences and Series

Geometric Sequences include terms that are multiplied by a ratio iteratively. They are given by the following formula.

$$a_n = a \cdot r^{n-1}$$

a_n is the n^{th} term, a is the first term, r is the common ratio.

The sum of a geometric sequence is given by

$$S_n = \begin{cases} a \cdot (\frac{r^n - 1}{r-1}), & r \neq 1 \\ a \cdot n, & r = 1 \end{cases}$$

The sum to infinity of a geometric sequence, where $|r| < 1$, is given by

$$S_\infty = \frac{a}{1-r}$$

Example: After striking the floor, a tennis ball bounces to $\frac{2}{3}$ of the height from which it last fell. What is the total vertical distance it travels before it comes to rest when it is dropped from a vertical height of 100m?

If h is the height in meters, e is a number such that $0 < e < 1$, and S is the total vertical distance covered before coming to rest, then

$$\begin{aligned} S &= h + 2(eh) + 2(e^2h) + 2(e^3h) + 2(e^4)h + \dots \\ &= h + 2eh(1 + e + e^2 + e^3 + \dots) \\ &= h + 2eh \cdots \frac{1}{1-e}, \because e < 1 \\ &= h\left(\frac{1+e}{1-e}\right) \end{aligned}$$

Since it is given that $h = 100$ and $e = \frac{2}{3}$,

$$S = 100\left(\frac{1+\frac{2}{3}}{1-\frac{2}{3}}\right) = 500 \text{ meters}$$

11.4 Binomial Expansion

The **Binomial Theorem** allows us to expand binomials such that

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

where $\binom{n}{k}$, pronounced "n choose k" because it describes how many ways to choose k elements from a set of n, is given by the formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

In the Binomial Theorem, $\binom{n}{k}$ determines the coefficients of the expanded binomial. Coefficients of Binomials follow Pascal's Triangle and they match up like so:

$$\begin{array}{ccccccccc} & & & & \binom{0}{0} & & & & \\ & & & \binom{1}{0} & & \binom{1}{1} & & & \\ & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\ & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\ \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \end{array}$$

Example 1: Expand $(y+5)^4$

$$\begin{aligned} (y+5)^4 &= \binom{4}{0} y^4 5^0 + \binom{4}{1} y^3 5^1 + \binom{4}{2} y^2 5^2 + \binom{4}{3} y^1 5^3 + \binom{4}{4} y^0 5^4 \\ &= y^4 + 20y^3 + 150y^2 + 500y + 625 \end{aligned}$$

Example 2: What is the coefficient of $x^3 in (2x+4)^8$?
The term containing x^3 is

$$\begin{aligned} \binom{8}{5}(2x)^3 4^5 &= 56(2x^3)(4^5) && (1) \\ &= 458752x^3 && (2) \end{aligned}$$

Hence, the coefficient is 458752.

12 The Cauchy-Schwartz Inequality

The Cauchy-Schwartz Inequality, or the Cauchy-Bunyakovsky-Schwartz Inequality, states that for all sequences of real numbers a_i and b_i, we have

$$(\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2) \geq (\sum_{i=1}^{n} a_i b_i)^2$$

Equality holds if and only if $a_i = kb_i$ for some non-zero constant $k \in \mathbb{R}$.

Example: If $x^2 + y^2 + z^2 = 1$, what is the maximum value of $x + 2y + 3z$?

We have $(x + 2y + 3z)^2 \leq (1^2 + 2^2 + 3^2)(x^2 + y^2 + z^2) = 14$. Hence, $x + 2y + 3z \leq \sqrt{14}$ with equality holding when $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$. Together with $x^2 + y^2 + z^2 = 1$, we get

$$x = \frac{1}{\sqrt{14}}, y = \frac{2}{\sqrt{14}}, z = \frac{3}{\sqrt{14}}$$

13 Symbolic Logic and Proofs

13.1 Statements and Logical Operators

A **proof** is an argument from **hypotheses** to a **conclusion**. Proofs usually begin with **premises**, statements that are known to be true. The **Rule of Premises** says that you may write down a premise at any point in a proof. The rule of **modus ponendo ponens** says that if you know P and $P \to Q$, you can write down Q.

Mathematical statements can only be true or false. The letters p and q often denote statements.

Logical Operators:
Not ($\neg$): The statement "not p" is called the **negation** of p.

p	$\neg p$
0	1
1	0

Double Negation states that $\neg\neg P$ is logically equivalent to P.
And (&):

p	q	$p\&q$
1	1	1
1	0	0
0	1	0
0	0	0

Or: ($||$)

p	q	$p\|\|q$
1	1	1
1	0	1
0	1	1
0	0	0

If...then ($\to$):

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

If p is false then $p \to q$ is **vacuously true**. For example, the statement all cell phones in the room are turned off is true even if there are no cell phones in the room.

If and only if ($\iff$):

1	1	1
1	0	0
0	1	0
0	0	1

When $p \iff q$ is true, p and q are **equivalent**.

Quantifiers include the phrases "for every ($\forall$)" and "there exists ($\exists$)".Consider the sentence "x is even". This does not count as a statement since we can't say whether or not it is true or false since we don't know what x is. We can make this a statement by saying "an integer x is even if there exists an integer y such that $x = 2y$".

13.2 Proof Methods

Proof by Cases:
Example: For every integer x, the integer $x(x + 1)$ is even.
Let x be any integer. Then x is even or odd.
Case 1: suppose x is even. Choose an integer k such that $x = 2k$. Then $x(x+1) = 2k(2k+1)$. Let $y = k(2k+1)$; then y is an integer and $x(x + 1) = 2y$ so $x(x + 1)$ is even.
Case 2: suppose x is odd. Choose an integer k such that $x = 2k + 1$. Then $x(x + 1) = (2k + 1)(2k + 2)$. Let $y = (2k + 1)(k + 1)$; then $x(x + 1) = 2y$, so $x(x + 1)$ is even. ■

Proof by Contradiction:
Suppose we want to prove that statement p is true. We begin by assuming p is false. We then deduce a **contradiction** (some statement about q we know to be false). If we succeed, then our assumption that p is false must be wrong. Hence p would have to be true.