Calculus Notes

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Contents

1 Introduction to Calculus

1.1 Limits

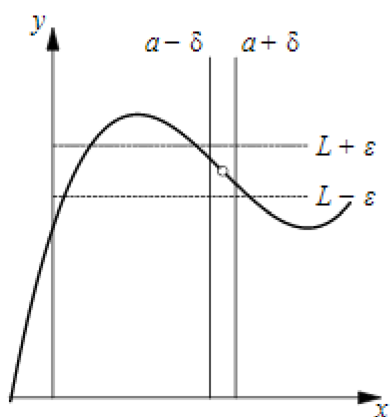
The Epsilon-Delta ($\epsilon - \delta$) Definition of Limits:

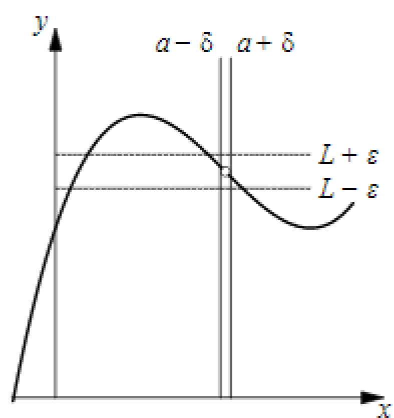
Let f be a real-valued function, defined around around a. Then the **limit**, L, as $f(x)$ approaches a, or **converges** to a, is

$$\lim_{x \to a} f(x) = L$$

If, $\forall \epsilon > 0$, there exists $\delta > 0$ such that if x is within δ of a (with $x \neq a$), then $f(x)$ is within ϵ of L. In other words,

$$\text{If } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \epsilon$$

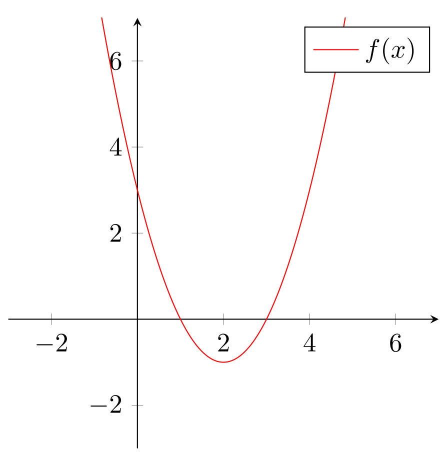




It is easiest to think of the limit of a function at a certain point, a, as the value of the function near a. For a limit, L, to exist at a, the **right-hand limit** (a^+) and the **left-hand limit** (a^-) must be equal.

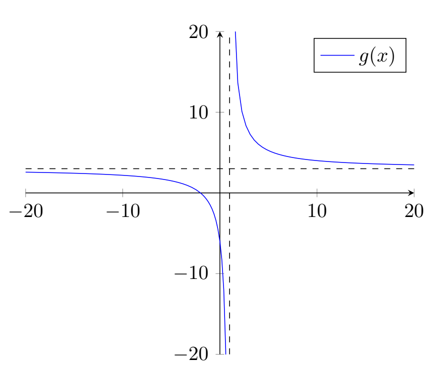
$$\lim_{x \to a} f(x) = L \implies \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$$

Example 1: Find $\lim_{x \to 4} f(x)$, where the function $f(x)$ is given by the graph.



$$\lim_{x \to 4^+} f(x) = 2 = \lim_{x \to 4^-} f(x)$$
$$\therefore \lim_{x \to 4} f(x) = 2$$

Example 2: Determine $\lim_{x \to 1} g(x)$, where the function $g(x)$ is graphed below. It is given that $g(x)$ is defined for all real numbers except $x = 1$ and the graph of $g(x)$ is divided by the asymptote $x = 1$.



$$\lim_{x \to 1^+} g(x) = \infty$$

and

$$\lim_{x \to 1^-} g(x) = -\infty$$

Since the right-hand and left-hand limits do not equal each other, $\lim_{x \to 1} g(x)$ does not exist.

1.2 Limit Properties

Assume $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist and that c is any constant. Then the following properties hold true.

$\lim_{x \to a} c = c$	**Limit of a Constant**
$\lim_{x \to a}[cf(x)] = c \lim_{x \to a} f(x)$	**Constant Multiple**
$\lim_{x \to a}[f(x) \pm g(x) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)]$	**Sum/Difference**
$\lim x \to a[f(x)g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$	**Product**
$\lim_{x \to c}(f(g(x))) = f\left(\lim_{x \to c} g(x)\right)$	**Composition**
$\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \lim_{x \to a} g(x) \neq 0.$	**Quotient**
$\lim_{x \to a}[f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n, n \in \mathbb{R}$	**Exponent**
$\lim_{x \to a} \sqrt[3]{f(x)} \sqrt[n]{\lim_{x \to a}[f(x)]}$	**Root**
$\lim_{x \to a} x = a$	**Limit of a Variable**

1.3 Finding Limits from Tables

Tables should always be the last resort when attempting to determine limits, because of their tendency to be tedious.

Example: The function g is defined over the real numbers. This table gives a few values of g What is a reasonable estimate for $\lim_{x \to 4} g(x)$?

x	3.9	3.99	3.999	4.001	4.01	4.1
$g(x)$	11.21	11.92	11.99	12.01	12.08	12.81

$lim_{x \to 4} g(x)$ represents the limit of g as x approaches 4. Looking over the table, we see that the left-hand limit appears to approach 12 as x gets progressively larger.

x	3.9	3.99	3.999
$g(x)$	11.21	11.92	11.99

We also see that the right-hand limit appears to approach 12 as x gets progressively smaller.

x	4.001	4.01	4.1
$g(x)$	12.01	12.08	12.81

Since the right-hand and left-hand limits are equal, we can conclude that $\lim_{x \to 4} g(x) = 12$.

1.4 Evaluating Limits Algebraically

The 3 Main Algebraic Limit Strategies:

1. Factoring

Example: Determine the limit

$$\begin{aligned} \lim_{x \to -1} \frac{x^2 - x - 2}{x^2 - 2x - 3} &= \lim_{x \to -1} \frac{(x+1)(x-2)}{(x+1)(x-3)} \\ &= \lim_{x \to -1} \frac{x-2}{x-3} \\ &= \frac{-1-2}{-1-3} \\ &= \frac{3}{4} \end{aligned}$$

2. Conjugates

Example: Determine the limit

$$\begin{aligned} \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} &= \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} \cdot \frac{\sqrt{x} + 2}{\sqrt{x} + 2} \\ &= \lim_{x \to 4} \frac{1}{\sqrt{x} + 2} \\ &= \frac{1}{4} \end{aligned}$$

3. Trig Identities

Example: Determine the limit

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(x)}{\sin(2x)} &= \lim_{x\to 0}\frac{\sin(x)}{2\sin(x)\cos(x)}\\ &= \lim_{x\to 0}\frac{1}{2\cos(x)}\\ &= \frac{1}{2}\end{aligned}$$

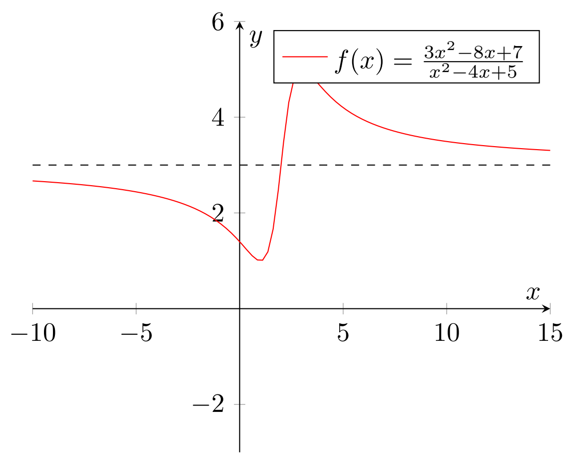
1.5 Asymptotes

Horizontal Asymptotes:

The line $y = b$ is a *horizontal asymptote* of the graph of $y = f(x)$ if

$$\lim_{x\to\infty} f(x) = b \text{ or } \lim_{x\to-\infty} f(x) = b$$

It is important to note that horizontal and slant asymptotes *can* be crossed, as they describe the general behavior of the functions as they near the edges of the graph. On the other hand, vertical asymptotes cannot be crossed as they describe particular behavior of the function itself, rather than the edges of the graph. Below is an example of a horizontal asymptote being crossed.

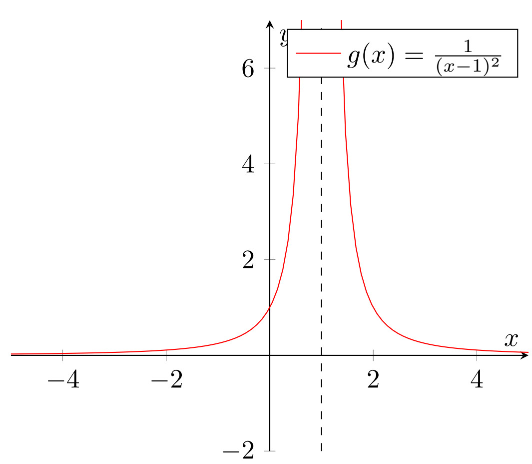


Vertical Asymptotes:

The line $x = a$ is a *vertical asymptote* of the graph of $y = f(x)$ if at least one of the below expressions holds.

$$\lim_{x\to a^-} f(x) = \pm\infty,\ \lim_{x\to a^+} f(x) = \pm\infty$$

Example of a graph containing a vertical asymptote:



The Rational Function Theorem:

When $\lim_{x\to\pm\infty}\frac{P(x)}{Q(x)} = 0$, $y = 0$ is a horizontal asymptote of the graph of $y = \frac{P(x)}{Q(s)}$.

When $\lim_{x\to\pm\infty}\frac{P(x)}{Q(x)} = \pm\infty$, the graph of $y = \frac{P(x)}{Q(x)}$ has no horizontal asymptotes.

When $\lim_{x\to\pm\infty}\frac{P(x)}{Q(x)} = \frac{a_n}{b_n}$, $y = \frac{a_n}{b_n}$ is a horizontal asymptote of the graph of $y = \frac{P(x)}{Q(x)}$.

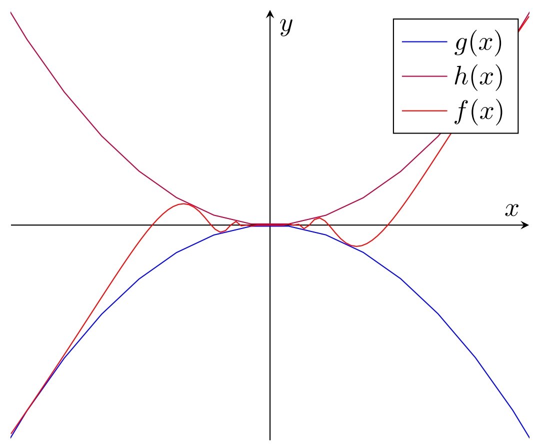
1.6 The Squeeze Theorem

Functions g and h are strategically chosen to satisfy the conditions outlined in the definition. Notice how, in the figure after the definition, function f is being "squeezed" between functions g and h, hence the name.

The Squeeze Theorem:
Assume that functions f, g, h defined n $D \subseteq \mathbb{R}$ satisfy

$$g(x) \leq f(x) \leq h(x), \forall x \in D$$

If $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} f(x) = L$.



Example 1: Given an infinite sequence $\{a_n\}$ that satisfies $\frac{2n^2-7}{4n+5} < a_n < \frac{3n^2+8}{6n-1}$ for all positive integers n, evaluate

$$\lim_{n \to \infty} \frac{3na_n}{(n+1)^2} \tag{1}$$

We transform the middle term of the inequality to the desired expression through manipulating each side of the inequality. Then the inequality becomes

$$\frac{3n}{(n+1)^2} \cdot \frac{2n^2-7}{4n+5} < \frac{3na_n}{(n+1)^2} < \frac{3n}{(n+1)^2} \cdot \frac{3n^2+8}{6n-1}$$

Now we can take the limits of the top and bottom functions.

$$\lim_{n \to \infty} \frac{3n(2n^2-7)}{(4n+5)(n+1)^2} = \frac{3}{2}$$
$$\lim_{n \to \infty} \frac{3n(3n^2+8)}{(6n-1)(n+1)^2} = \frac{3}{2}$$

Since the top and bottom functions are equal, we can conclude that

$$\lim_{n \to \infty} \frac{3na_n}{(n+1)^2} = \frac{3}{2}$$

Example 2: Evaluate the limit

$$\lim_{n \to \infty} \sqrt[n]{3^n + \{2|\sin(n^n)\}^n}$$

We can note that

$$0 \leq |\sin(n^n)| \leq 1$$

Then we can write

$$\sqrt[n]{3^n} \leq \sqrt[n]{3^n + \{2|\sin(n^n)\}^n} \leq \sqrt[n]{3^n + 2^n} \leq \sqrt[n]{2 \cdot 3^n}$$

The left side reduces to 3, whereas the right side becomes

$$\lim_{n \to \infty} \sqrt[n]{2 \cdot 3^n} = 3 \lim_{n \to \infty} \sqrt[n]{2} = 3$$

Thus we can conclude that

$$\lim_{n\to\infty}\sqrt[n]{3^n+\{2|\sin(n^n)|\}^n}=3$$

Example 3: Evaluate the limit

$$\lim_{x\to\infty}\frac{\sin x}{x}$$

Since $\forall x, -1\leq\sin x\leq 1$, it follows that if $x>0$ then $-\frac{1}{x}\leq\frac{\sin x}{x}\leq\frac{1}{x}$. As $x\to\infty$, $-\frac{1}{x}$ and $\frac{1}{x}$ both approach 0. Therefore, by the Squeeze Theorem, $\frac{\sin x}{x}$ also approaches 0.

1.7 Limit of a Quotient of Polynomials

For the function $f(x)=\frac{P(x)}{Q(x)}$, where $P(x)$ is a polynomial of degree n and $Q(x)$ is a polynomial of degree m, the following expressions hold. Let a_n and b_n be the leading coefficients of $P(x)$ and $Q(x)$ respectively. Then

$$\text{If } n>m\text{, then } \lim_{x\to\infty}f(x)=\infty \text{ and } \lim_{x\to-\infty}f(x)=-\infty$$

$$\text{If } n=m\text{, then } \lim_{x\to\pm\infty}f(x)=\frac{a_n}{b_m}$$

$$\text{If } n<m\text{, then } \lim_{x\to\pm\infty}f(x)=0$$

This method is a shortcut found through dividing each of the terms of $P(x)$ and $Q(x)$ by the highest power of x found in $f(x)$, then taking the individual limits of each separated term through basic limit properties. For example, see below.

Example: Evaluate the limit

$$\begin{aligned}\lim_{x\to\infty}\frac{x^3-4x^2+7}{3-6x-2x^3}&=\lim_{x\to\infty}\frac{1-\frac{4}{x}+\frac{7}{x^3}}{\frac{3}{x^3}-\frac{6}{x^2}-2}\\&=\frac{\lim_{x\to\infty}(1)-\lim_{x\to\infty}\left(\frac{4}{x}\right)+\lim_{x\to\infty}\left(\frac{7}{x^3}\right)}{\lim_{x\to\infty}(\frac{3}{x^3})-\lim_{x\to\infty}\left(\frac{6}{x^2}\right)-\lim_{x\to\infty}(2)}\\&=\frac{1-0+0}{0-0-2}\\&=-\frac{1}{2}\end{aligned}$$

Note that in this case, $n=m$, so $\frac{a_n}{b_n}\implies-\frac{1}{2}$. Since the properties described preceding this example can be observed in all quotients of polynomials, we can thus make generalizations as specificed above.

1.8 L'Hospital's Rule

L'Hospital's is a method of evaluating limits of indeterminate forms, learned after gaining knowledge of derivatives. **Indeterminate forms** are expressions i nvolving two functions whose limits cannot be determined solely from the limits of the individual functions.

The 7 Indeterminate Forms:

$$\frac{0}{0},\frac{\infty}{\infty},0\cdot\infty,\infty-\infty,0^0,1^\infty,\infty^0$$

L'Hospital's Rule:
Suppose that we have either of the cases below

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0},\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\pm\infty}{\pm\infty}$$

where $a\in\mathbb{R},\infty$ or $-\infty$. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example 1: Evaluate the limit

$$\lim_{t\to 1}\frac{5t^4-4t^2-1}{10-t-9t^3}$$

We have an $\frac{0}{0}$ indeterminate form here, so we use L'Hospital's to get

$$\lim_{t\to 1}\frac{5t^4-4t^2-1}{10-t-9t^3}=\lim_{t\to 1}\frac{20t^3-8t}{-1-27t^2}=\frac{20-8}{-1-27}=-\frac{3}{7} \tag{2}$$

Example 2: Evaluate the limit

$$\lim_{x\to -\infty} xe^x$$

This expression is in the form $(\infty)(0)$, meaning we will have to write the expression as a quotient. We know that $\frac{1}{e^x}=e^{-x}$, so we can rewrite the limit as

$$\lim_{x\to -\infty} xe^x=\lim_{x\to -\infty}\frac{x}{e^-x}=\lim_{x\to -\infty}\frac{1}{-e^-x}=0$$

Example 3: Evaluate the limit

$$\lim_{x\to \infty} x^{\frac{1}{x}}$$

This limit is of the form ∞^0, so we have to rewrite this expression as a different limit. Let

$$y=x^{\frac{1}{x}}$$

Since

$$e^{\ln(y)}=y$$

we can rewrite our limit as

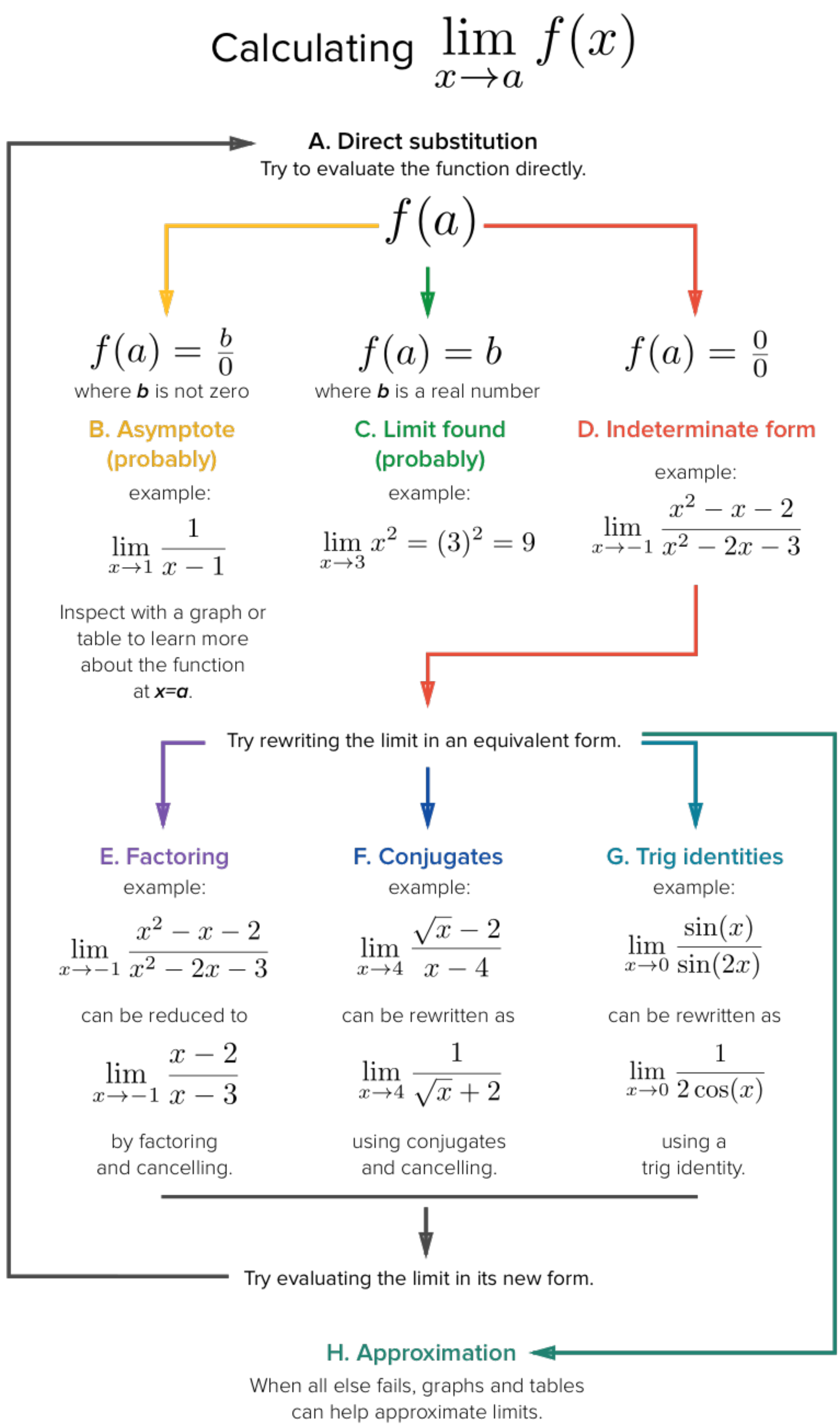
$$\begin{aligned}\lim_{x\to\infty} x^{\frac{1}{x}} &= \lim_{x\to\infty} y\\ &= \lim_{x\to\infty} e^{\ln(y)}\\ &= e^{\lim_{x\to\infty}\ln(y)}\\ &= e^0\\ &= 1\end{aligned}$$

L'Hospital's can help us prove **the basic trig limit**, useful for evaluating many trig limits.

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1 \text{ if } \theta \text{ is measured in radians}$$

1.9 Limit Strategies

KhanAcademy provides a convenient flowchart to illustrate a strategy to find limits:



Example 1:

$$
\begin{aligned}
\lim_{x\to 0} \frac{\sin 3x}{x} &= \lim_{x\to 0} \frac{3\sin 3x}{3x} \\
&= 3 \lim_{x\to 0} \frac{\sin 3x}{3x} \\
&= 3 \cdot 1 \\
&= 3
\end{aligned}
$$

Example 2:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 2x}{3x} &= \frac{1}{3}\lim_{x\to 0}\frac{\sin 2x}{x}\cdot\frac{2}{2}\\ &= \frac{2}{3}\lim_{x\to 0}\frac{\sin 2x}{2x}\\ &= \frac{2}{3}\end{aligned}$$

Example 3:

$$\lim_{x\to 0}\frac{|x|}{x}$$

Since $|x| = x$ if $x > 0$ but $|x| = -x$ if $x < 0$, $\lim_{x\to 0^+}\frac{|x|}{x} = \lim_{x\to 0^+}\frac{x}{x} = 1$, whereas $\lim_{x\to 0^-}\frac{|x|}{x} = \lim_{x\to 0^-}\frac{-x}{x} = -1$. Since the right and left hand limits are different, we can conclude that the limit does not exist.
Example 4:

$$\begin{aligned}\lim_{x\to\infty}\arctan\left(x^3 - 5x + 6\right) &= \arctan\left(\lim_{x\to\infty} x^3 - 5x + 6\right)\\ &= \arctan\left(+\infty\right)\\ &= \frac{\pi}{2}\end{aligned}$$

1.10 Limit Definition of e

The mathematical constant e is defined by

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

1.11 Continuity and Discontinuity

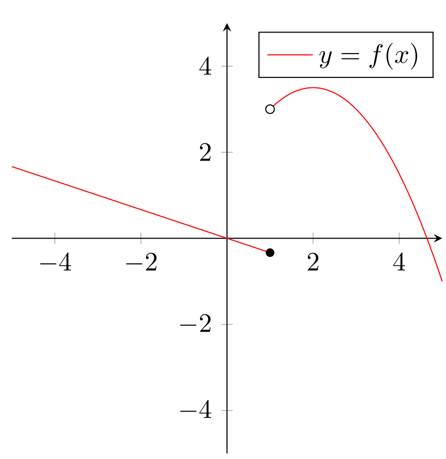
A function $f(x)$ is **continuous** at a if

$$\lim_{x\to a} f(x) = f(a)$$

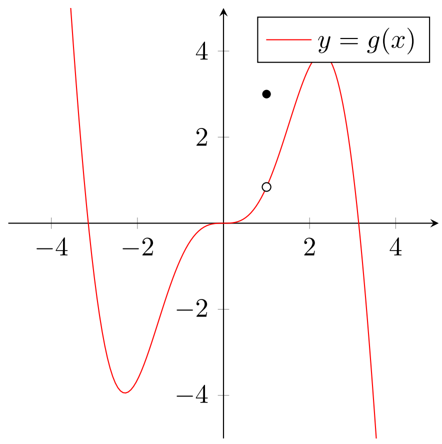
A function $f(x)$ is **continuous over the interval** $[a, b]$ if $\forall a \leq x \leq b$, x is continuous. A function is **discontinuous** if it is not continuous.

Common Continuous Functions:
- Polynomials are continuous everywhere at a real number.
- Rational functions are continuous at each point in their domain except where $Q(x) = 0$.
- The absolute value function is continuous everywhere.
- The trigonometric, inverse trigonometric, exponential, and logarithmic functions are continuous everywhere.
- Irrational functions in the form $\sqrt[n]{x}$, where $n \geq 2$, are continuous everywhere for which $\sqrt[n]{x}$ is defined.
- The greatest-integer function is discontinuous at each integer.

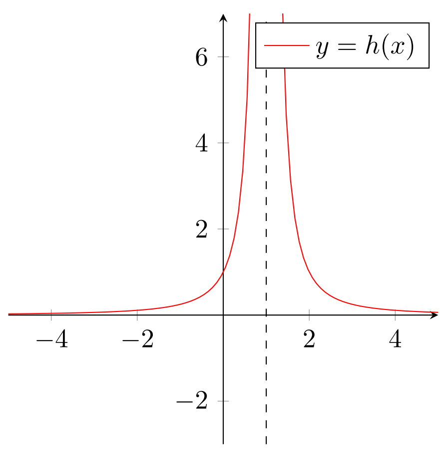
Types of Discontinuities:
Jump Discontinuity:
In a jump discontinuity, the left and right-hand limits exist, but are different. For example, the graph of $y = f(x)$ below has a jump discontinuity at $x = 1$.



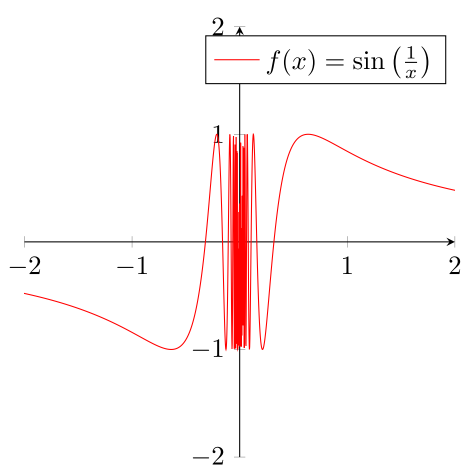
Removable Discontinuity:
A function $f(x)$ has a removable discontinuity at $x = a$ if $\lim_{x\to a} f(x)$ exists, but either $f(a)$ does not exist or the value of $f(a) \neq \lim_{x\to a} f(x)$. A function with a removable discontinuity at a may or may not be defined at a. For example, the graph of $y = g(x)$ below has a removable discontinuity at $x = 1$. In this case, $g(1) = 3$ instead of the limit value $\lim_{x\to 1} g(x) = \sin(1)$.



Infinite Discontinuities:
Infinite discontinuities always occur at vertical asymptotes. There may be a vertical asymptote at either one or both sides of the function. An example of an infinite discontinuity is given below, where $y = h(x)$.



Here is an example of an **Infinite Oscillation Discontinuity**.



Essential Discontinuities are discontinuities that at which the limit of the function does not exist. Jump and infinite discontinuities are essential discontinuities.

Below is a general example about continuity and discontinuity.
Example: Is

$$f(x) = \begin{cases} x^2 + 2 & x \leq 1 \\ 4 & x > 1 \end{cases}$$

continuous at $x = 1$?
Since $\lim_{x\to 1^-} f(x) = 3 \neq \lim_{x\to 1^+} f(x) = 4$.

1.12 The Extreme Value Theorem

This theorem is best learnt after derivatives.

The Extreme Value Theorem

If real numbers a and b satisfy $a < b$ and a function f is continuous on $[a, b]$, then f attains a maximum and minimum value on $[a, b]$.

Example: Find the maximum and minimum values of the function $f(x) = x^3 - \frac{9}{2}x^2 - 12x + 20$ on the interval $[-2, 6]$.

$f(x)$ is differentiable, hence continuous on the closed and bounded interval. Having satisfied the preconditions, we can apply the EVT to this context. Taking the first derivative of $f(x)$, we get

$$\begin{aligned} f'(x) &= 3x^2 - 9x - 12 \\ &= 3(x^2 - 3x - 4) \end{aligned}$$

Setting $f'(x) = 0$ and solving, we find that the critical points are $x = -1, 4$ which by the first derivative test gives the relative extrema $y = 26.5, -36$, respectively. Computing the endpoints, we have $f(-2) = 18$ and $f(6) = 2$. Hence, the minimum value of f on $[-2, 6]$ is -36 and the maximum value is 26.5.

1.13 The Intermediate Value Theorem

The Intermediate Value Theorem:

If a function f is continuous on the closed interval $[a, b]$, and M is a number such that $f(a) \le M \le f(b)$, then there is at least one number c such that $f(c) = M$.

Example: Does a x exist for some $x \in [0, 2]$ such that the function $f(x) = x^2 + \cos(\pi x) = 4$?

$$\begin{aligned} f(0) &= 0^2 + \cos 0 = 1 \\ f(2) &= 2^2 + \cos 2\pi = 5 \end{aligned}$$

Since $f(0) = 1 < 4 < 5 = f(2)$ and f is continuous, the IVT implies that $f(x) = 4$ for some $x \in [0, 2]$.

1.14 The Continuous Functions Theorem

The Continuous Functions Theorem:

If functions f and g are both continuous at $x = c$, then the following functions are also continuous:

Constant Multiples:	$k \cdot f(x)$ for any real number k
Sums:	$f(x) + g(x)$
Differences:	$f(x) - g(x)$
Products:	$f(x) \cdot g(x)$
Quotients:	$\frac{f(x)}{g(x)}, g(c) \neq 0$

1.15 The Composition of Continuous Functions Theorem

The Composition of Continuous Functions Theorem:

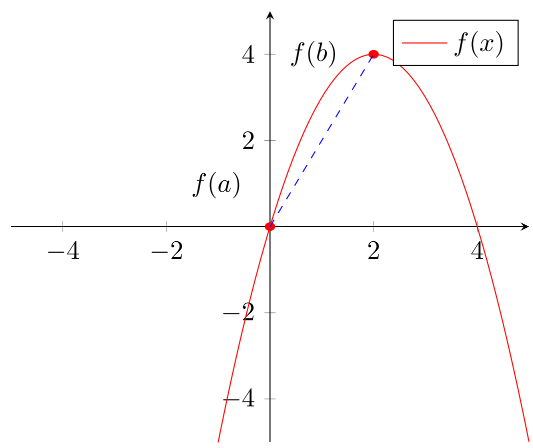
If the function g is continuous at $x = c$ and the function f is continuous at $x = g(c)$, then the composite function $(f \circ g)(x)$ is continuous at $x = c$.

2 Differentiation

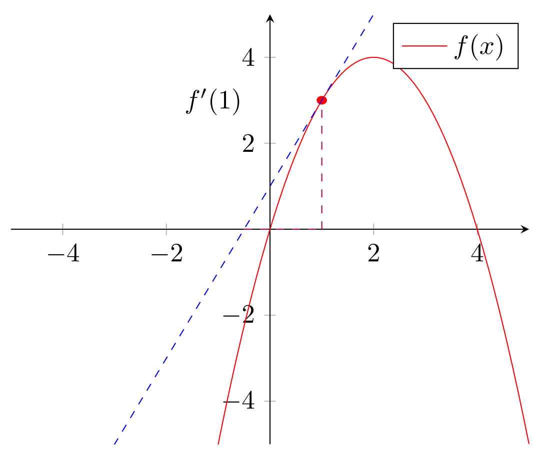
2.1 Rates of Change and Derivatives

An **average rate of change** describes the overall $\frac{\text{rise}}{\text{run}}$-value over an interval $[a, b]$. Geometrically, this can be represented by the slope of a secant line through $f(a)$ and $f(b)$ of a function $f(x)$. In the graph below, the average rate of change between $x = 0$ and $x = 2$ is given by

$$m_{\text{avg}} = \frac{f(b) - f(a)}{b - a} = \frac{4 - 0}{2 - 0} = 2$$



A derivative is an **instantaneous rate of change**, or the slope of the tangent to a function at a particular point, a. Derivatives are always taken *with respect* to a variable. For example, a derivative representing the instantaneous rate of change between distance and time is a derivative of distance with respect to time, the independent variable. For reference, the derivative of a function $f(x)$ at $x = 1$ is shown in the graph below.



Since the derivative is an instantaneous rate of change, we can define the derivative as below.

1^{st} Limit Definition of the Derivative

The derivative of $f(x)$ with respect to x is the function $f'(x)$, defined as

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Then, if we let $x = a + h$ and change the variable of inspection from a to x, we can realize the most commonly used definition of the derivative, given below. This definition is preferred to the above definition, as finding the derivative only requires one value of x, rather than 2.

2^{nd} Limit Definition of the Derivative

The derivative of $f(x)$ with respect to x is the function $f'(x)$, defined as

$$f'(x) = \lim_{h \to 0} \frac{f(a + h) - f(x)}{h}$$

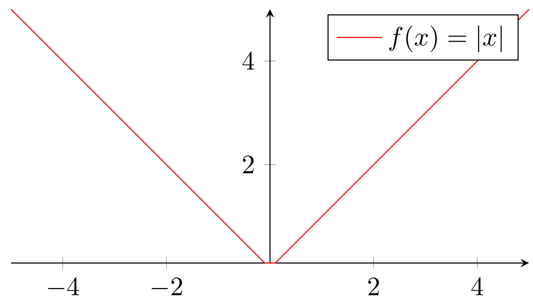
The expression consisting the right side of the equation above is known as the **difference quotient**.

A function is $f(x)$ is **differentiable** at $x = a$ if $f'(a)$ exists. Since derivatives are defined by limits, for a derivative to exist, its limit must also exist. Hence,

If $f(x)$ is differentiable at $x = a$ then $f(x)$ is continuous at $x = a$

By this, we can also say that differentiability implies continuity. It is important to note that its converse is not always true, as not all continuous functions are differentiable.

A few examples of contexts where a derivative may not exist are at vertical tangents, corners, and cusps. Take, for example, the graph below, which does not have a derivative at $x = 0$ since the respective slopes immediately right and left of $x = 0$ are different.



Derivative Notations:

Leibniz's Notation:

The Leibniz notation is popular throughout mathematics, most commonly used when the equation $y = f(x)$ is regarded as a functional relationship between dependent and independent variables y and x. Leibniz's notation makes this relationship explicit by representing the derivative as

$$\frac{dy}{dx} = \frac{d}{dx}y$$

The Leibniz notation is also called **differential notation**, where dy and dx are **differentials**. Where the function $y = f(x)$, we can write

$$\frac{df}{dx}(x) = \frac{df(x)}{dx} = \frac{d}{dx}f(x)$$

A **differential equation** is an equation relating one or more functions and their derivatives. As long as the precondition ($f(x)$ is satisfied), a function can be differentiated indefinite times. For example, in physics, acceleration is the derivative of velocity, the derivative of position. The **order** of a differential equation is the highest derivative of said equation. Higher-order derivatives can be written in the Leibniz's notation as

$$\frac{d^2y}{dx^2}, \frac{dy^3}{dx^3}, \frac{dy^4}{dx^4}, \ldots, \frac{d^ny}{dx^n}$$

With Leibniz's notation, the value of a derivative at a particular point, a, can be expressed as

$$\left.\frac{dy}{dx}\right|_{x=a}$$

This notation is particularly useful in expressing partial derivatives (covered later) and making the chain rule intuitive:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Lagrange's Notation:

In Lagrange's notation, each prime mark denotes a derivative. For higher-order derivatives, the order is enclosed by parantheses in front and above the function.

$$f'(x), f''(x), f'''(x), f^{(4)}(x), f^{(5)}(x), f^{(6)}(x), f^{(n)}(n)$$

Newton's Notation:

This notation is often applied to physics contexts, where time (t) is the independent variable. The number of dots over the dependent variable represent the order the derivative is in. If y is a function of t, then the first n derivatives of the function are as below.

$$\dot{y}, \ddot{y}, \dddot{y}, \overset{4}{\dot{y}}, \overset{5}{\dot{y}}, \overset{n}{\dot{y}}$$

Euler's Notation:

This notation is quite inconvenient, as it leaves the variable being differentiated with respect to entirely implicit. However, we can modify the notation to explicitly write said variable. This notation is defined by

$$(Df)(x) = \frac{df(x)}{dx}$$

Higher-order Derivatives are expressed by

$$D^2 f = D_x^2 f, D^3 f = D_x^3 f, D^n f = D_x^n f$$

Example 1: Find the derivative, $g'(t)$, given the function

$$g(t) = \frac{t}{t+1} \tag{3}$$

$$\begin{aligned}
g'(t) &= \lim_{h\to 0} \frac{g(t+h)-g(t)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{t+h}{t+h+1} - \frac{t}{t+1}\right) \\
&= \lim_{h\to 0} \left(\frac{(t+h)(t+1) - t(t+h+1)}{(t+h+1)(t+1)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{t^2+t+th+h-(t^2+th+t)}{(t+h+1)(t+1)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{h}{(t+h+1)(t+1)}\right) \\
&= \frac{1}{(t+1)(t+1)} \\
g'(t) &= \frac{1}{(t+1)^2}
\end{aligned}$$

Example 2: Find the derivative, $f'(x)$, of

$$f(x) = 2x^2 - 16x + 35$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{2(x+h)^2 - 16(x+h) + 35 - (2x^2 - 16x + 35)}{h} \\
&= \lim_{h\to 0} \frac{4xh + 2h^2 - 16h}{h} \\
&= \lim_{h\to 0} (4x + 2h - 16) \\
f'(x) &= 4x - 16
\end{aligned}$$

2.2 Basic Derivative Rules

$\frac{d}{dx}a = 0$	**Constant**
$\frac{d}{dx}au = a\frac{du}{dx}$	**Constant Multiple**
$\frac{d}{dx}x^n = nx^{n-1}$	**The Power Rule**
$\frac{d}{dx}(u \pm v) = \frac{d}{dx}u \pm \frac{d}{dx}v$	**Sum/Difference**
$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	**The Product Rule**
$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}, v \neq 0$	**The Quotient Rule**
$[f^{-1}]'(b) = \frac{1}{f'(a)}$	**Inverse Functions**
$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\frac{dy}{dx}}$	**Alternate Inverse Functions**
$\frac{d(1/f)}{dx} = -\frac{1}{f^2}\frac{df}{dx}$	**Reciprocal**

2.3 Derivatives of Exponential and Logarithmic Functions

$\frac{d}{dx}e^x = e^x$	**Natural Exponent**
$\frac{d}{dx}\ln x = \frac{1}{x}$	**Natural Logarithm**
$\frac{d}{dx}a^x = a^x \ln a$	**Exponent**
$\frac{d}{dx}\log_b x = \frac{1}{x\ln b}$	**Logarithm**
$\frac{d}{dx}x^x = x^x(1+\ln x)$	

2.4 Derivatives of the Trigonometric Functions

The derivatives of $\sin x$ and $\cos x$ help us derive the others through applying the quotient and reciprocal differentiation rules.

$\frac{d}{dx}\sin x = \cos x$
$\frac{d}{dx}\cos x = -\sin x$
$\frac{d}{dx}\tan x = \sec^2 x$
$\frac{d}{dx}\csc x = -\cot x \csc x$
$\frac{d}{dx}\sec x = \tan x \sec x$
$\frac{d}{dx}\cot x = -\csc^2 x$

The derivatives of the inverse trigonometric functions can be derived through applying the inverse differentiation rule.

$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$
$\frac{d}{dx}\operatorname{arccsc} x = -\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\frac{d}{dx}\operatorname{arcsec} x = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\frac{d}{dx}\operatorname{arccot} x = -\frac{1}{1+x^2}$

Derivation of the Inverse Sine Derivative:

$$\begin{aligned}\frac{d(\arcsin x)}{dx} &= \frac{1}{\frac{d(\sin y)}{dy}} \\ &= \frac{1}{\cos y}\end{aligned}$$

From the Pythagorean Identity,

$$\begin{aligned}\cos^2 y + \sin^2 y = 1 \\ \cos^2 y &= 1 - \sin^2 y \\ \cos y &= \sqrt{1-\sin^2 y}\end{aligned}$$

Substituting this back into the original equation, we get

$$\frac{d(\arcsin x)}{dx} = \frac{1}{\sqrt{1-\sin^2 y}}$$

By the definition of the inverse sine function, we can express the equation as

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

Derivation of the Inverse Cosine Derivative

The derivatives of the inverse trigonometric functions are the *negatives* of the derivatives of their cofunctions. This is because

$$\begin{aligned}\arccos x &= \frac{\pi}{2} - \arcsin x \\ \frac{d}{dx}\arccos x &= -\frac{d}{dx}\arcsin x \\ &= -\frac{1}{\sqrt{1-x^2}}\end{aligned}$$

Derivation of the Inverse Tangent Derivative

$$\begin{aligned}\frac{d}{dx}\arctan x &= \frac{1}{\frac{d(\tan y)}{dy}} \\ &= \frac{1}{\sec^2 y} \\ &= \frac{1}{1+\tan^2 y} \\ &= \frac{1}{1+x^2}\end{aligned}$$

2.5 The Chain Rule

The Chain Rule is incredibly useful for finding the derivative of composite functions. If $y = f(u)$ and $u = g(x)$ then

$$\begin{aligned} (f(g(x)))' &= f'(g(x)) \cdot g'(x) \\ &= f'(u) \cdot g'(x) \\ \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \end{aligned}$$

Example 1:

$$f(x) = \sqrt{5x - 8}$$

We let the inside function be expressed as $u = 5x - 8$. Then

$$\begin{aligned} f'(x) &= \frac{du}{dx} u^{\frac{1}{2}} \cdot \frac{dy}{du}(5x - 8) \\ &= \frac{1}{2\sqrt{u}} \cdot 5 \\ &= \frac{5}{2\sqrt{5x - 8}} \end{aligned}$$

Example 2:
Let $u = \cos t$ and $v = t^4$

$$\begin{aligned} g(t) &= \cos^4 t + \cos{(t^4)} \\ g'(t) &= \frac{d(u^4)}{du} \cdot \frac{d}{dt} \cos t + \frac{d(t^4)}{dt} \cdot \frac{d(\cos v)}{dv} \\ &= 4u^3(-\sin t) + 4t^3(-\sin v) \\ &= -4\cos^3 t \sin t - 4t^3 \sin{(t^4)} \end{aligned}$$

2.6 Implicit Differentiation

Implicit Differentiation is an approach to differentiating a function that may not be in the explicit form, $y = f(x)$. It considers all other variables as functions of one of its variables, then uses the chain rule to find the derivative.

Example 1: Given $x^2 + x + y^2 = 15$, what is $\frac{dy}{dx}$ at the point $(2, 3)$?

$$\begin{aligned} \frac{dy}{dx}(x^2 + x + y^2) &= \frac{dy}{dx} 15 \\ 2x + 1 + 2y\left(\frac{dy}{dx}\right) &= 0 \\ \frac{dy}{dx} &= -\frac{2x + 1}{2y} \\ \left.\frac{dy}{dx}\right|_{(2,3)} &= -\frac{2(2) + 1}{2(3)} \\ &= -\frac{5}{6} \end{aligned}$$

Example 2: Find the derivative of $\ln y + e^y = \sin y^2 - 3\cos x$

$$\begin{aligned} \frac{dy}{dx} \ln y + \frac{dy}{dx} e^y &= \frac{d}{dx} \sin y^2 - \frac{d}{dx} 3\cos x \\ \frac{d}{dy} \ln y \cdot \frac{dy}{dx} + \frac{d}{dy} e^y \cdot \frac{dy}{dx} &= \frac{d}{dy} \sin y^2 \cdot \frac{dy}{dx} + 3\sin x \\ \frac{dy}{dx}\left(\frac{1}{y} + e^y - 2y\cos y^2\right) &= 3\sin x \\ \frac{dy}{dx} &= \frac{3\sin x}{\frac{1}{y} + e^y - 2y\cos y^2} \end{aligned}$$

Example 3: Find $\frac{dy}{dx}$ if $y^2 = x^2 + \sin(xy)$

$$\begin{aligned}
\frac{d}{dx}(y^2) &= \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin(xy)) \\
2y\frac{d}{dx} &= 2x + (\cos(xy))\left(y + x\frac{d}{dx}\right) \\
(2y - x\cos(xy))\frac{d}{dx} &= 2x + y\cos(xy) \\
\frac{dy}{dx} &= \frac{2x + y\cos(xy)}{2y - x\cos(xy)}
\end{aligned}$$

2.7 Logarithmic Differentiation

This method of finding derivatives uses the basic properties of logarithms outside of calculus to simply a function prior to differentiating it.

Example 1:

$$\begin{aligned}
y &= \frac{x^5}{(1-10x)\sqrt{x^2+2}} \\
\ln y &= \ln\left(\frac{x^5}{(1-10x)\sqrt{x^2+2}}\right) \\
&= \ln(x^5) - \ln\left((1-10x)\sqrt{x^2+2}\right) \\
&= \ln(x^5) - \ln(1-10x) - \ln(\sqrt{x^2+2})
\end{aligned}$$

By Implicit Differentiation,

$$\begin{aligned}
y'y &= \frac{5x^4}{x^5} - \frac{-10}{1-10x} - \frac{\frac{1}{2}(x^2+2)^{-\frac{1}{2}}(2x)}{(x^2+2)^{\frac{1}{2}}} \\
&= \frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2} \\
\frac{dy}{dx} &= \frac{x^5}{(1-10x)\sqrt{x^2+2}}\left(\frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}\right)
\end{aligned}$$

Example 2:

$$\begin{aligned}
y &= \frac{x+3}{(x+4)^3} \\
\ln y &= \ln\left(\frac{x+3}{(x+4)^3}\right) \\
&= \ln(x+3) - \ln(x+4)^3 \\
&= \ln(x+3) - 3\ln(x+4) \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{1}{x+3} - 3\cdot\frac{1}{x+4} \\
&= \frac{1}{x+3} - \frac{3}{x+4} \\
\frac{dy}{dx} &= y\left(\frac{1}{x+3} - \frac{3}{x+4}\right)
\end{aligned}$$

2.8 Derivatives of Parametric Functions

If $x = f(t)$ and $y = g(t)$ are differentiable functions of parameter t, then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Example: If $x = 2\sin\theta$ and $y = \cos 2\theta$, find $\frac{d^2y}{dx^2}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \\ &= \frac{-2\sin 2\theta}{2\cos\theta} \\ &= -\frac{2\sin\theta\cos\theta}{\cos\theta} \\ &= -2\sin\theta\end{aligned}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}} \\ &= \frac{-2\cos\theta}{2\cos\theta} \\ &= -1\end{aligned}$$

2.9 Derivatives of Polar Functions

We know that $x = r\cos\theta$ and $y = r\sin\theta$. Then, by using the parametric derivative formula, we can determine the rule for differentiating polar functions.

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \\ &= \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} \\ &= \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}\end{aligned}$$

Example: Find the slope of the cardioid $r = 2(1+\cos\theta)$ at $\theta = \frac{\pi}{6}$

$$\begin{aligned}r &= 2(1+\cos\theta) \\ r' &= -2\sin\theta\end{aligned}$$

Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \\ &= \frac{(-2\sin\theta)\sin\theta + 2(1+\cos\theta)(\cos\theta)}{(-2\sin\theta)\cos\theta - 2(1+\cos\theta)(\sin\theta)} \\ \left.\frac{dy}{dx}\right|_{\frac{\pi}{6}} &= -1\end{aligned}$$

2.10 Derivatives of Vector-Valued Functions

If a point moves along a curve defined parametrically by $P(t) = \langle x(t), y(t)\rangle$, where t represents time, then the vector from the origin to P is called the **position vector**. Then the derivative of the position vector is called the **velocity vector**, and its derivative is called the **acceleration vector**.

The Velocity Vector:

$$\overrightarrow{v}(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$

The slope of $\overrightarrow{v}$ is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

The Magnitude of a Velocity Vector:

$$|v| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{v_x^2 + v_y^2}$$

The Acceleration Vector:

$$\vec{a}(t) = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right\rangle$$

The Magnitude of an Acceleration Vector:

$$|a| = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2} = \sqrt{a_x^2 + a_y^2}$$

2.11 Rolle's Theorem

Rolle's Theorem:
Suppose $f(x)$ is a function that is both continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and that $f(a) = f(b)$. Then there is a number c such that $a < c < b$ and $f'(c) = 0$. Or rather, $f(x)$ has a critical point in (a, b).

2.12 The Mean Value Theorem

The Mean Value Theorem:
Suppose $f(x)$ is a function that is both continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is a number c such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Or,

$$f(b) - f(a) = f'(c)(b - a)$$

Example 1: Determine all the numbers c satisfying $f(x) = x^3 + 2x^2 - x$ on $[-1, 2]$

$$\begin{aligned}
f'(x) &= 3x^2 + 4x - 1 \\
f'(c) &= \frac{f(2) - f(-1)}{2 - (-1)} \\
3c^2 + 4c - 1 &= \frac{14 - 2}{3} \\
3c^2 + 4c - 1 &= 4 \\
3c^2 + 4c - 5 &= 0 \\
c &= \frac{-4 \pm \sqrt{16 - 4(3)(-5)}}{6} \\
c &= 0.7863, -2.1196 \\
c &= 0.7863
\end{aligned}$$

Notice we were able to remove the other c, as it is outside of the desired interval.
Example 2: Suppose we know that $f(x)$ is continuous and differentiable on $[6, 15]$ and that $f(6) = -2$ and $f'(x) \leq 10$. Find the largest possible value for $f(15)$.

The MVT tells us that

$$f(15) - f(6) = f'(c)(15 - 6)$$

Plugging in known quantities and simplifying, we get

$$f(15) = -2 + 9f'(c)$$

Since we are given $f'(x) \leq 10$, we know that $f'(c) \leq 10$. This gives us

$$\begin{aligned}
f(15) &= -2 + 9f'(c) \\
&\leq -2 + (9)10 \\
&\leq 88
\end{aligned}$$

Hence, the largest possible value of $f(15) = 88$.

3 Applications of Integration

3.1 Critical Points

$x = c$ is a **critical point** of the function $f'(x)$ if $f(c)$ exists and

$$f'(c) = 0 \quad \text{OR} \quad f'(c) \text{ doesn't exist}$$

Example: Determine all the critical points for the function

$$f(x) = 6x - 4\cos(3x)$$

$$\begin{aligned} f'(x) = 0 &= 6 + 12\sin(3x) \\ \sin(3x) &= -\frac{1}{2} \\ x &= 1.2217 + \frac{2\pi n}{3}, n = 0, \pm 1, \pm 2, \ldots \\ \text{and} & \\ x &= 1.9199 + \frac{2\pi n}{3}, n = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

3.2 Tangents to Curves

Recall the point-slope form of a linear function from Algebra. We can replace the slope, m, with $f'(x_1)$ to get an equation of the tangent to the curve $y = f(x)$ at point $P(x_1, y_1)$.

Equation for Tangent to a Curve

$$y - y_1 = f'(x_1)(x - x_1)$$

Example: Find an equation of the tangent to $f(t) = (\cos t, 2\sin^2 t)$ at the point where $t = \frac{\pi}{3}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\ &= \frac{4\sin t\cos t}{-\sin t} \\ &= -4\cos t \end{aligned}$$

At $t = \frac{\pi}{3}$, $x = \frac{1}{2}$, $y = 2\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{2}$, and $\frac{dy}{dx} = -2$. Hence, an equation of the tangent is

$$\begin{aligned} y - \frac{3}{2} &= -2\left(x - \frac{1}{2}\right) \\ 4x + 2y &= 5 \end{aligned}$$

3.3 Increasing and Decreasing Functions

A function $f(x)$ is **increasing** on an interval for which $a < b$, $f(b) \geq f(a)$. A function $f(x)$ is **decreasing** over an interval for which $a < b$, $f(b) \leq f(a)$. Hence, the following identities hold true.

$f'(x) > 0$	Increasing
$f'(x) < 0$	Decreasing

Example: For what values of x is $f(x) = x^4 - 4x^3$ increasing and decreasing, respectively?

$$\begin{aligned} f'(x) &= 4x^3 - 12x^2 \\ &= 4x^2(x - 3) \end{aligned}$$

With critical values at $x = 0, 3$, we analyze the signs of f' in the three intervals as below.

Since the derivative changes sign only at $x = 3$,

$$\begin{array}{ll} \text{if } x < 3 & f'(x) \leq 0 \text{ and } f \text{ is decreasing} \\ \text{If } x > 3 & f'(x) > 0 \text{ and } f \text{ is increasing} \end{array}$$

3.4 Extrema, Concavity, and Inflection Points

The curve of $y = f(x)$ has a **local/relative maximum** at a point where $x = c$ if $f(c) \geq f(x)$ for all x in the immediate neighborhood of c. If a curve has a relative maximum at $x = c$, then the curve changes from increasing to decreasing as x increases through c.

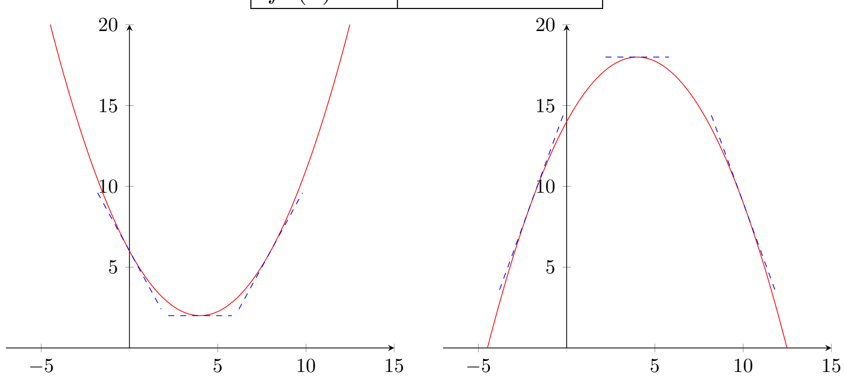
The curve of $y = f(x)$ has a **local/relative minimum** at a point where $x = c$ if $f(c) \leq f(x)$ for all x in the immediate neighborhood of c. If a curve has a relative minimum at $x = c$, then the curve changes from decreasing to increasing as x increases through c.

If a function is differentiable on $[a, b]$ and has a relative extremum at $x = c, a < c < b$, then $f'(c) = 0$.

The **global/absolute maximum** of a function on $[a, b]$ occurs at $x = c$ if $f(c) \geq f(x)$ for all x on $[a, b]$. The **global/absolute minimum** of a function on $[a, b]$ occurs at $x = c$ if $f(c) \leq f(x)$ for all x on $[a, b]$.

A curve is **concave upward** on an interval (a, b) if the curve lies above the tangent lines at each point in the interval (a, b). A curve is **concave downward** on an interval (a, b) if the curve lies below the tangent lines at each point in the interval (a, b).

$f"(x) > 0$	Concave Up
$f"(x) < 0$	Concave Down



3.5 Optimization

3.6 Graphically Relating a Function and its Derivatives

3.7 Motion Along a Curve

3.8 Tangent-Line Approximations

3.9 The Newton-Raphson Method

3.10 Related Rates

4 Integration

4.1 Differentials

5 Applications of Integration

6 Introduction to Differential Equations

7 Infinite Sequences and Series