

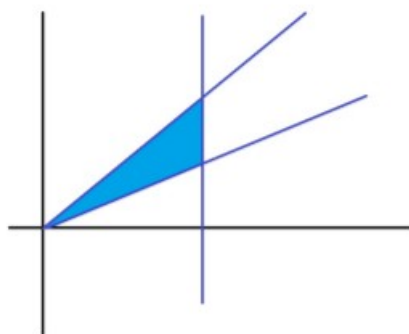
MATH134: Worksheet 6

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In this worksheet we work with the triangular region bounded by the lines $y = x$, $y = \frac{1}{\sqrt{3}}x$, and $x = 3$ in the first quadrant. Last two questions are tricky!



Problem 1

Compute the area of the region. Set up integrals both with dx and with dy . Which one was easier to do?

$$y = x \iff x = y$$

$$y = \frac{1}{\sqrt{3}}x \iff x = y\sqrt{3}$$

The region is bounded by $[0, 3]$ for x because the region spans from the origin to the line $x = 3$. The region is also bounded by $[0, 3]$ for y because the upper y -bound is given by $y = x = 3$. Thus the area computed with respect to x is given by:

$$\begin{aligned} A &= \int_0^3 \left(x - \frac{1}{\sqrt{3}}x \right) dx \\ &= \left[\frac{x^2}{2} - \frac{x^2}{2\sqrt{3}} \right] \Big|_0^3 \\ &= \frac{9}{2} - \frac{9}{2\sqrt{3}} \\ &= \frac{9 - 3\sqrt{3}}{2} \end{aligned}$$

The area computed with respect to y is as follows:

$$\begin{aligned} A &= \int_0^{\frac{3}{\sqrt{3}}} (y\sqrt{3} - y) dy + \int_{\frac{3}{\sqrt{3}}}^3 (3 - y) dy \\ &= \left[\frac{y^2\sqrt{3}}{2} - \frac{y^2}{2} \right] \Big|_0^{\frac{3}{\sqrt{3}}} + \left[3y - \frac{y^2}{2} \right] \Big|_{\frac{3}{\sqrt{3}}}^3 \\ &= \left[\frac{y^2\sqrt{3} - y^2}{2} \right] \Big|_0^{\frac{3}{\sqrt{3}}} + \left[3y - \frac{y^2}{2} \right] \Big|_{\frac{3}{\sqrt{3}}}^3 \\ &= \frac{3\sqrt{3} - 3}{2} + \left[\left(9 - \frac{9}{2} \right) - \left(3\sqrt{3} - \frac{3}{2} \right) \right] \\ &= \frac{3\sqrt{3} - 3}{2} + \left[9 - \frac{9}{2} - 3\sqrt{3} + \frac{3}{2} \right] \\ &= \frac{3\sqrt{3} - 3}{2} + 6 - 3\sqrt{3} \\ &= \frac{3\sqrt{3} - 3}{2} + \frac{12}{2} - \frac{6\sqrt{3}}{2} \\ &= \frac{9 - 3\sqrt{3}}{2} \end{aligned}$$

The area computed with respect to x was much easier to do.

Problem 2

Find the volume of the solid obtained by rotating this region about the line $y = -1$.

The volume, V , is as follows:

$$\begin{aligned}
 V &= \pi \int_0^3 \left((x+1)^2 - \left(\frac{x}{\sqrt{3}} + 1 \right)^2 \right) dx \\
 &= \pi \int_0^3 \left(x^2 + 2x + 1 - \left(\frac{x^2}{3} + \frac{2x}{\sqrt{3}} + 1 \right) \right) dx \\
 &= \pi \int_0^3 \left(x^2 + 2x - \frac{x^2}{3} - \frac{2x}{\sqrt{3}} \right) dx \\
 &= \pi \left[\frac{x^3}{3} + x^2 - \frac{x^3}{9} - \frac{x^2}{\sqrt{3}} \right] \Big|_0^3 \\
 &= \pi \left(9 + 9 - 3 - \frac{9}{\sqrt{3}} \right) \\
 &= 15\pi - 3\pi\sqrt{3}
 \end{aligned}$$

Problem 3

Find the volume of the solid obtained by rotating this region about the y -axis.

The volume V is as follows.

$$\begin{aligned}
 V &= 2\pi \int_0^3 x \left(x - \frac{x}{\sqrt{3}} \right) dx \\
 &= 2\pi \int_0^3 \left(x^2 - \frac{x^2}{\sqrt{3}} \right) dx \\
 &= 2\pi \left[\frac{x^3}{3} - \frac{x^3}{3\sqrt{3}} \right] \Big|_0^3 \\
 &= 2\pi \left(9 - \frac{9}{\sqrt{3}} \right) \\
 &= 2\pi (9 - 3\sqrt{3}) \\
 &= 18\pi - 6\sqrt{3}\pi
 \end{aligned}$$

Problem 4

Now, switch your variable of integration and compute the volume of the solid obtained by rotating this region about the y -axis again.

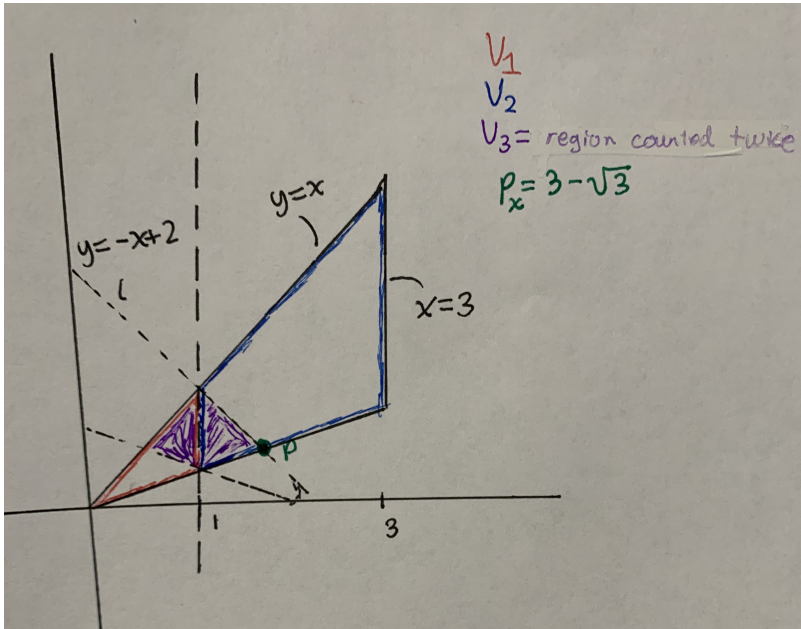
The volume V is given by:

$$\begin{aligned}
 V &= \pi \int_0^{\frac{3}{\sqrt{3}}} \left((y\sqrt{3})^2 + y^2 \right) dy + \pi \int_{\sqrt{3}}^3 (3^2 - y^2) dy \\
 &= \pi \int_0^{\frac{3}{\sqrt{3}}} (3y^2 - y^2) dy + \pi \int_{\sqrt{3}}^3 (9 - y^2) dy \\
 &= \pi \int_0^{\frac{3}{\sqrt{3}}} 2y^2 dy + \pi \int_{\sqrt{3}}^3 (9 - y^2) dy \\
 &= \pi \left[\frac{2y^3}{3} \right] \Big|_0^{\frac{3}{\sqrt{3}}} + \pi \left[9y - \frac{y^3}{3} \right] \Big|_{\sqrt{3}}^3 \\
 &= \pi \left[\frac{2 \left(\frac{3}{\sqrt{3}} \right)^3}{3} \right] + \pi \left[27 - 9 - \left(9\sqrt{3} - \frac{(\sqrt{3})^3}{3} \right) \right] \\
 &= 2\pi\sqrt{3} + \pi(18 - 8\sqrt{3}) \\
 &= 2\pi\sqrt{3} + 18\pi - 8\pi\sqrt{3} \\
 &= 18\pi - 6\sqrt{3}\pi
 \end{aligned}$$

Problem 5

Find the volume of the solid obtained by rotating this region about the line $x = 1$. Keep careful track of the overlap.

Let V_1 be the volume of the red region bounded by the lines $x = 1$, $y = x$, and $y = \frac{x}{\sqrt{3}}$. Let V_2 be the volume of the blue region bounded by the lines $y = x$, $y = \frac{x}{\sqrt{3}}$, $x = 1$, and $x = 3$. Let V_3 be the purple overlapping region. Let the green point P be the intersection of the line $y = \frac{x}{\sqrt{3}}$ and the reflection of the line $y = x$ across $x = 1$. The relationship between these variables is sketched below for clarity.



Let us first solve for all the unknown variables that we will need to use, before finally computing the integrals.

The line $y = x$ intersects the x -axis at the origin, which is 1 unit away from the axis of reflection, $x = 1$. Thus, the reflection of $y = x$ will intersect the x -axis 1 unit in the opposite direction of $x = 1$, i.e. the reflection of $y = x$ intersects the x -axis at $(2,0)$. Then the y -intercept of the reflection will be 2 and hence the reflection of $y = x$ across $x = 1$ is $y = -x + 2$.

To find the x -coordinate of P , we set up a system as follows.

$$\begin{cases} y = -x + 2 & (1) \\ y = \frac{x}{\sqrt{3}} & (2) \end{cases}$$

Substituting (2) into (1), we have

$$\begin{aligned} \frac{x}{\sqrt{3}} &= -x + 2 \\ x &= -x\sqrt{3} + 2\sqrt{3} \\ x(1 + \sqrt{3}) &= 2\sqrt{3} \\ x &= 3 - \sqrt{3} \end{aligned}$$

Using the shell method for V_1 , our radius is $1 - x$ and our height is $y_2 - y_1 = x - \frac{x}{\sqrt{3}}$. Thus, V_1 is given by

$$\begin{aligned} V_1 &= 2\pi \int_0^1 (1 - x) \left(x - \frac{x}{\sqrt{3}} \right) dx \\ &= 2\pi \int_0^1 \left(x - \frac{x}{\sqrt{3}} - x^2 + \frac{x^2}{\sqrt{3}} \right) dx \\ &= 2\pi \left[\frac{x^2}{2} - \frac{x^2}{2\sqrt{3}} - \frac{x^3}{3} + \frac{x^3}{3\sqrt{3}} \right] \Big|_0^1 \\ &= 2\pi \left(\frac{1}{2} - \frac{1}{2\sqrt{3}} - \frac{1}{3} + \frac{1}{3\sqrt{3}} \right) \\ &= 2\pi \left(\frac{1}{6} - \frac{\sqrt{3}}{18} \right) \\ &= \frac{6\pi - \pi\sqrt{3}}{18} \end{aligned}$$

For V_2 , our radius is $x - 1$ and our height is still $x - \frac{x}{\sqrt{3}}$. So V_2 is given by

$$\begin{aligned}
V_2 &= 2\pi \int_1^3 (x-1) \left(x - \frac{x}{\sqrt{3}} \right) dx \\
&= 2\pi \int_1^3 \left(x^2 - \frac{x^2}{\sqrt{3}} - x + \frac{x}{\sqrt{3}} \right) dx \\
&= 2\pi \left[\frac{x^3}{3} - \frac{x^3}{3\sqrt{3}} - \frac{x^2}{2} + \frac{x^2}{2\sqrt{3}} \right] \Big|_1^3 \\
&= 2\pi \left[\left(9 - \frac{9}{\sqrt{3}} - \frac{9}{2} + \frac{9}{2\sqrt{3}} \right) - \left(\frac{1}{3} - \frac{1}{3\sqrt{3}} - \frac{1}{2} + \frac{1}{2\sqrt{3}} \right) \right] \\
&= 2\pi \left(\frac{42 - 14\sqrt{3}}{9} \right) \\
&= \frac{84\pi - 28\sqrt{3}\pi}{9}
\end{aligned}$$

For V_3 , the radius is $(x-1)$ and the height is $\left(x - \frac{x}{\sqrt{3}}\right)$, however the bounds are 1 and the x -coordinate of P , $3 - \sqrt{3}$. V_3 is

$$\begin{aligned}
V_3 &= 2\pi \int_1^{3-\sqrt{3}} (x-1) \left(x - \frac{x}{\sqrt{3}} \right) dx \\
&= 2\pi \int_1^{3-\sqrt{3}} \left(x^2 - \frac{x^2}{\sqrt{3}} - x + \frac{x}{\sqrt{3}} \right) dx \\
&= 2\pi \left[\frac{x^3}{3} - \frac{x^3}{3\sqrt{3}} - \frac{x^2}{2} + \frac{x^2}{2\sqrt{3}} \right] \Big|_1^{3-\sqrt{3}} \\
&= 2\pi \left[\left(\frac{(3-\sqrt{3})^3}{3} - \frac{(3-\sqrt{3})^3}{3\sqrt{3}} - \frac{(3-\sqrt{3})^2}{2} + \frac{(3-\sqrt{3})^2}{2\sqrt{3}} \right) - \left(\frac{1}{3} - \frac{1}{3\sqrt{3}} - \frac{1}{2} + \frac{1}{2\sqrt{3}} \right) \right] \\
&= 2\pi \left(\frac{345 - 199\sqrt{3}}{18} \right) \\
&= \frac{345\pi - 199\sqrt{3}\pi}{9}
\end{aligned}$$

Because V_3 overlaps with V_1 and V_2 , it is already counted and therefore it follows that the total volume is given by

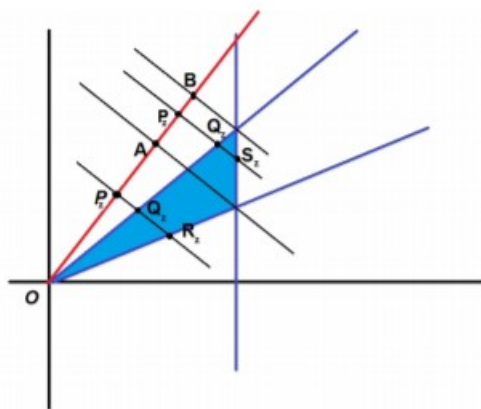
$$\begin{aligned}
V &= V_1 + V_2 - V_3 \\
&= \frac{6\pi - \pi\sqrt{3}}{18} + \frac{84\pi - 28\pi\sqrt{3}}{9} - \frac{345\pi - 199\pi\sqrt{3}}{9} \\
&= \frac{6\pi - \pi\sqrt{3} + 168\pi - 56\pi\sqrt{3} - 690\pi + 398\pi\sqrt{3}}{18} \\
&= \frac{341\pi\sqrt{3} - 516\pi}{18}
\end{aligned}$$

Problem 6

Finally, following Cindy's suggestion, we'll rotate this about the slant (red) line $y = x\sqrt{3}$. Here is the picture with the new axis of rotation:

We'll do this by introducing a new variable z , which will keep track of how far we are from the origin on the red line and using washers. The black lines are all perpendicular to the red line.

- What are the angles the three slant lines make with the positive x -axis? They will help with computations below.
- Compute the distances OA and OB . These will be our limits of integration.
- Let P_z be an arbitrary point on the axis of rotation which is z units from the origin. I put P_z in two places as examples. So OP_z has length z . Compute the lengths P_zQ_z , P_zR_z and P_zS_z , all in terms of z . These will be related to the radii in the washer method.
- Now, set up two integrals using washers and compute the volume of the solid obtained by rotating the region about the line $y = x\sqrt{3}$.



(a)

The line $y = x$ makes an angle of $\frac{\pi}{4}$ with the positive x -axis because a 45-45-90 triangle can be drawn with the hypotenuse being the red line and the base being the x -axis on the interval $[0, 3]$.

The equation for the line $y = x\sqrt{3}$ can be rearranged such that it represents a tangent relationship like so:

$$\begin{aligned} y &= x\sqrt{3} \\ \frac{y}{x} &= \sqrt{3} \\ \arctan(\sqrt{3}) &= \frac{\pi}{3} \end{aligned}$$

Thus, the angle between the line $y = x\sqrt{3}$ and the positive x -axis is $\frac{\pi}{3}$. Similarly, the equation for the line $y = \frac{x}{\sqrt{3}}$ can be rearranged such that it represents a tangent relationship:

$$\begin{aligned} y &= \frac{x}{\sqrt{3}} \\ \frac{y}{x} &= \frac{1}{\sqrt{3}} \\ \arctan\left(\frac{1}{\sqrt{3}}\right) &= \frac{\pi}{6} \end{aligned}$$

Thus, the angle between the line $y = \frac{x}{\sqrt{3}}$ and the positive x -axis is $\frac{\pi}{6}$.

(b)

The normal to the red line $y = x\sqrt{3}$ at A is known to pass through the intersection of $x = 3$ and $y = \frac{x}{\sqrt{3}}$ which is $\left(3, \frac{3}{\sqrt{3}}\right) = (3, \sqrt{3})$. A normal to $y = x\sqrt{3}$ will be of the form

$$y = -\frac{1}{\sqrt{3}}x + b$$

Plugging in the point $(3, \sqrt{3})$, we find that the normal to the red line $y = x\sqrt{3}$ that passes through A is given by

$$\begin{aligned} \sqrt{3} &= -\frac{3}{\sqrt{3}} + b \\ b &= 2\sqrt{3} \\ y &= -\frac{x}{\sqrt{3}} + 2\sqrt{3} \end{aligned}$$

The point A is the intersection of the normal above and the red line $y = x\sqrt{3}$. That is, A is the solution to the system

$$\begin{cases} y = -\frac{x}{\sqrt{3}} + 2\sqrt{3} \\ y = \frac{x}{\sqrt{3}} \end{cases}$$

Solving the system, we see that A is $\left(\frac{3}{2}, 2\sqrt{3} - \frac{\sqrt{3}}{2}\right)$. The distance OA is given by

$$OA = \sqrt{\left(\frac{3}{2}\right)^2 + \left(2\sqrt{3} - \frac{\sqrt{3}}{2}\right)^2} = 3$$

To find the distance OB , we can start off similarly by considering that any normal to $y = x\sqrt{3}$ will have the form

$$y = -\frac{x}{\sqrt{3}} + b$$

and that the normal to $y = x\sqrt{3}$ at B passes through $(3, 3)$. Plugging in $(3, 3)$, we see that the normal at B is

$$\begin{aligned} 3 &= -\frac{3}{\sqrt{3}} + b \\ b &= 3 + \sqrt{3} \\ y &= -\frac{x}{\sqrt{3}} + 3 + \sqrt{3} \end{aligned}$$

The point B is the intersection of the normal above and the red line $y = x\sqrt{3}$. That is, A is the solution to the system

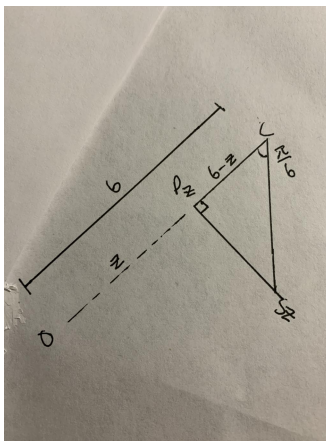
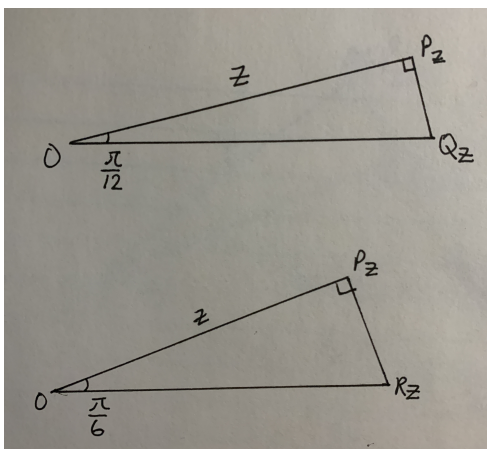
$$\begin{cases} y = -\frac{x}{\sqrt{3}} + 3 + \sqrt{3} \\ y = \frac{x}{\sqrt{3}} \end{cases}$$

Solving the system, we see that B is $\left(\frac{3\sqrt{3}+3}{4}, \frac{9+5\sqrt{3}}{4}\right)$. The distance OB is given by

$$OB = \sqrt{\left(\frac{3\sqrt{3}+3}{4}\right)^2 + \left(\frac{9+5\sqrt{3}}{4}\right)^2} = \sqrt{\frac{9\sqrt{3}+18}{2}}$$

(c)

The lengths of P_zQ_z and P_zR_z can be computed readily by setting up a trigonometric function relating the known side length z and the angle the slant lines make with the positive x -axis. Some sketches of the right triangles used to form these trigonometric relationships are below for clarity.



The angle $\angle P_zOQ_z$ was found by subtracting the angle the line $y = x$ makes with the x -axis $\left(\frac{\pi}{4}\right)$ from the angle the line $y = x\sqrt{3}$ makes with the x -axis $\left(\frac{\pi}{3}\right)$. This gives us an angle of $\frac{\pi}{12}$. Similarly, the angle $\angle P_zOR_z$ was found by subtracting the angle the line $y = \frac{x}{\sqrt{3}}$ makes with the x -axis $\left(\frac{\pi}{6}\right)$ from the angle the line $y = x\sqrt{3}$ makes with the x -axis $\left(\frac{\pi}{3}\right)$. This gives us an angle of $\frac{\pi}{6}$. Using the definition of the tangent, we have that

$$\begin{aligned} \tan\left(\frac{\pi}{12}\right) &= \frac{P_zQ_z}{z} \\ P_zQ_z &= z \tan\left(\frac{\pi}{12}\right) \\ &= z(2 - \sqrt{3}) \end{aligned}$$

and

$$\begin{aligned}
 \tan\left(\frac{\pi}{6}\right) &= \frac{P_z R_z}{z} \\
 P_z R_z &= z \tan\left(\frac{\pi}{6}\right) \\
 &= \frac{\sqrt{3}}{3} z
 \end{aligned}$$

To find the length of $P_z S_z$, I defined a new point, C , to represent the intersection of the lines $x = 3$ and $y = x\sqrt{3}$. At point C , $y = 3\sqrt{3}$. The distance OC can be found to be 6 as shown:

$$\begin{aligned}
 OC &= \sqrt{3^2 + (3\sqrt{3})^2} \\
 &= \sqrt{9 + 27} \\
 &= 6
 \end{aligned}$$

Because the length of OP_z is z , $P_z C$ must have a length of $6 - z$. Additionally, a right triangle is bounded by the lines $y = x\sqrt{3}$, $x = 3$, and the x -axis. One of the non-right angles was found in part A to be $\frac{\pi}{3}$, hence the other non-right angle, coincidentally the angle $\angle OCS_z$, is $\frac{\pi}{6}$. Again by the definition of the tangent, it follows that

$$\begin{aligned}
 \tan\left(\frac{\pi}{6}\right) &= \frac{6 - z}{P_z S_z} \\
 P_z S_z &= \frac{6 - z}{\tan\left(\frac{\pi}{6}\right)} \\
 &= \frac{6 - z}{\frac{\sqrt{3}}{3}} \\
 &= 6\sqrt{3} - z\sqrt{3}
 \end{aligned}$$

(d)

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