# MATH13X Notes

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## Contents

# 1 Introduction to Proofs

**Sets of Numbers:**

| | |
|---|---|
| $\mathbb{C}$ | **Complex Numbers** |
| $\mathbb{A}$ | **Algebraic Numbers** |
| $\mathbb{R}$ | **Real Numbers** |
| $\mathbb{Q}$ | **Rational Numbers** |
| $\mathbb{Z}$ | **Integers** |
| $\mathbb{N}$ | **Natural Numbers** |

Real numbers that are not **rational** are **irrational**:
- **Algebraic numbers:** solutions to polynomials with integer coefficients ($x^2 - 2 = 0, 2x^2 - 3 = 0$)
- **Transcendentals**: $e, \pi, \ldots$

The decimal expansion of an irrational number is eventually repeating $\left(\text{e.g. } 1.312312312\cdots = \frac{p}{q}\right)$. Below is a cool trick to find $p$ and $q$ given an irrational number.

$$x = 1.312312312\cdots = \frac{p}{q}$$
$$1000x = 1312.312312\ldots$$
$$1000x = 1312.000000\ldots$$
$$999x = 1311$$
$$x = \frac{1311}{999}$$
$$1.312312312\ldots = \frac{1311}{999}$$

**Proof Guidelines**
- Start with what you know or what you are given and end with what you need to show
- Justify every step unless it is a clear algebra step by:
- Algebra/computation from a previous step
- Reference to a previous result or step in the proof
- Proofs should read like a paragraph. Use words as much as equations and computations.
- So, then, therefore, let, given, we conclude, hence,...to make it flow.
- Keep your audience in mind

Below is an example of a well-written proof for the Triangle Inequality:

> ### Triangle Inequality
>
> For any $a, b, \in \mathbb{R}$, $|a + b| \leq |a| + |b|$

*Proof.* Let $a, b, \in \mathbb{R}$. Then,

$$|a + b|^2 = a^2 + 2ab + b^2$$
$$= |a|^2 + 2ab + |b|^2, \sin(u)\ x^2 = |x|^2 \forall x$$
$$\leq |a|^2 + 2|ab| + |b|^2, \sin(u)\ x \leq |x| \forall x$$
$$= |a|^2 + 2|a||b| + |b|^2$$
$$= (|a| + |b|)^2$$

So, $(a + b)^2 \leq (|a| + |b|)^2 \,\forall a, b, \in \mathbb{R}$. Now, taking square roots and using $|x| = \sqrt{x^2}$,, we get

$$|a + b| \leq ||a| + |b||$$

Since $|a| + |b| \geq 0$, we can remove the absolute sign on the right.

$$|a + b| \leq |a| + |b|$$

$\square$

A **theorem** is an implication, consisting of a **hypothesis** and a **conclusion** in the form of:

$$\text{if (hypothesis) } \ldots, \text{ then (conclusion) } \ldots$$

A common mistake in writing proofs is to ignore the hypothesis and follow directly with the conclusion, for example, to say that $ab > 0$ simply because $a$ and $b$ are numbers. Another common mistake is to assume that a theorem's converse is true as well. A third mistake is to assume that that a theorem's hypothesis is the only condition under which the conclusion is true. In other words, it is common to mistakenly write "iff", where **"iff"** is an abbreviation of "if and only if". Note that definitions by their nature are "iff statements", hence the "only if" part is convetionally omitted and assumed to hold.

A **predicate** in mathematical logic is any statement that has a boolean value depending on the values of its variables.

**Direct Proof:** if we wanted to prove

$$\text{if } A, \text{ then } B$$

we could assume that $A$ is true and use this information to deduce that $B$ is true.

**Contrapositive Proof:** We can use the concept that if $A$, then $B$ is true, then it is impossible for $A$ to be true while $B$ is false. Hence, we prove the statement "If $A$, then $B$" by showing that if $B$ is false, then $A$ is false too.

**Proof by Contradiction:** if we wanted to prove

$$\text{if } A, \text{ then } B$$

we could assume that

$$A \text{ holds and } B \text{ does not hold} \tag{1}$$

and then arrive at a contradiction. We can take this contradiction to mean that (1) is a false statement and therefore:

$$\text{if } A \text{ holds, then } B \text{ must hold.}$$

*Example 1:* Show that

$$\text{if } a > b > 0, \text{ then } \frac{1}{a} < \frac{1}{b}$$

*Proof.* Assume $a > b > 0$. For a contradiction, assume $\frac{1}{a} > \frac{1}{b}$. Since $a > 0$, multiplying both sides by $a$ gives

$$1 \geq \frac{a}{b}$$

Since $b > 0$, using similar reasoning,

$$b \geq a$$

This assumption contradicts $a > b$, hence the assumption was wrong and $\frac{1}{a} < \frac{1}{b}$. □

*Example 2:* Show that $\sqrt{2}$ is irrational.

*Proof.* Assume

$$\sqrt{2} = \frac{p}{q}$$

where $p$ and $q$ have no common factor and $q \neq 0$. We now square both sides of the equality:

$$2 = \frac{p^2}{q^2}$$

Since $2q^2 = p^2$, $p^2$ is even. Hence, $p$ is even and $p = 2r$ for some integer $r$.

$$\implies 2q^2 = 4r^2$$
$$q^2 = 2r^2$$

Since the equality above is even, so is $q$. So $p + q$ both have a common factor of 2, contradicting that they had no common factors. Therefore the assumption that $\sqrt{2}$ was rational was wrong and it is irrational. □

**Mathematical induction** is a proof method that can be used to show that a certain property holds for all positive integers $n$, or in some cases for all positive integers $n \geq a$.

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**Axiom of Induction**

Let $S \subset \{1, 2, 3, \dots\}$, or let $S$ be the set of $n$ for which a desired property holds. If
- $1 \in S$
- $k \in S \implies k + 1 \in S$.

Then $S$ contains all positive integers hence it is equal to the set of all positive integers.

The substitution of "$k + 1$" into the desired predicate is called the **induction hypothesis.** In your proofs by induction, you should always put the phrases "*by the induction hypothesis*", and "*Therefore, by the principle of mathematical induction, the formula holds for all n.*"

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The axiom of induction can be thought of as an application of domino theory; if the first domino in a row of dominoes falls, and each domino that falls causes the next one to fall, then, according to the axiom of induction, each domino in that row of dominoes will fall. Note that an induction does not necessarily have to begin with the integer 1. If, for example, you wished to show that some proposition is true for all integers $n \geq 3$, you would just have to show that said proposition is true for $n = 3$ and that, if is true for $n = k$, then it is true for $n = k + 1$.

*Example 3:* Show that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \forall n, n \in \mathbb{Z}^+$$

*Proof.* Let $S$ be the set of positive integers $n$ for which

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Then $1 \in S$ since

$$1 = \frac{1(1+1)}{2}$$

Assume that $k \in S$; that is, assume that

$$1 + 2 + 3 \cdots + k = \frac{k(k+1)}{2}$$

Adding up the first $k + 1$ integers, we have

$$
\begin{aligned}
1 + 2 + 3 + \cdots + k + (k+1) &= [1 + 2 + 3 + \cdots + k] + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \text{ *by the induction hypothesis} \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}
$$

and so $k + 1 \in S$. Thus, by the axiom of induction, we can conclude that all positive integers are in $S$; that is, we can conclude that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \forall n, n \in \mathbb{Z}^+$$

$\square$

*Example 4:* Show that, if $x \geq -1$, then

$$(1+x)^n \geq 1 + nx \forall n, n \in \mathbb{Z}^+$$

*Proof.* Take $x \geq -1$ and let $S$ be the set of positive integers $n$ for which

$$(1+x)^n \geq 1 + nx$$

Since

$$(1+x)^1 = 1 + 1 \cdot x,$$

we have $1 \in S$. Assume that $k \in S$. By the definition of $S$,

$$(1+x)^k \geq 1 + kx$$

Since

$$(1+x)^{k+1} = (1+x)^k (1+x) \geq (1 + kx)(1+x)$$

and

$$(1 + kx)(1 + x) = 1 + (k+1)x + kx^2 \geq 1 + (k+1)x,$$

we can conclude that

$$(1+x)^{k+1} \geq 1 + (k+1)x$$

and thus $k + 1 \in S$. We have shown that $1 \in S$ and that $k \in S \implies k + 1S$. Then by the axiom of induction, all positive integers are in $S$. $\square$

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# 2  Limits and Continuity

## 2.1  The $\epsilon - \delta$ Definition of the Limit

> **$\epsilon - \delta$ Definiton of the Limit of a Function**:
> Let $f(x)$ be a function defined on an open interval around $c$. Then the limit of $f(x)$ as $x$ approaches $c$ is $L$:
>
> $$\lim_{x \to c} f(x) = L,$$
>
> if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\forall x$
>
> $$0 < |x - c| < \delta \implies |f(x) - L| < \epsilon$$
>
> or in logic notation:
>
> $$\forall \epsilon > 0 \exists \delta > 0 \ni |x - c| < \delta \implies |f(x) - L| < \epsilon$$
>
> $\epsilon$ is any given epsilon arbitrarily close to $L$, where $\delta$ is a value we can choose.

*Example 1:* Show that $\lim_{x \to 2} 5x - 7 = 30$.

Let $\epsilon$ be given (arbitrary). Define $\delta = \frac{\epsilon}{5}$. Assume $|x - 2| < \delta = \frac{\epsilon}{5}$.

Then

$$|(5x - 7) - 3| = |5x - 10| = 5|x - 2| < 5 \cdot \frac{\epsilon}{5} = \epsilon$$

In this problem, we had to set up the inequality at the bottom first to get the value of our $\delta$. Then, we plugged it back into the earlier part os the proof and worked downwards again.

*Example 2:* Show that $\lim_{x \to 2} x^2 + 5 = 9$.

Let $\epsilon > 0$ be given.
Then,

$$|x^2 + 5 - 9| = |x^2 - 4| = |x + 2||x - 2|$$

We want

$$|x + 2||x - 2| < \epsilon$$

We can easily control $|x - 2|$, but not $|x + 2|$. Then

$$2 - \delta < x < 2 + \delta$$
$$4 - \delta < x + 2 < 4 + \delta$$

$x + 2$ cannot get too large. For example, if $\delta = 1$ then

$$3 < x + 2 < 5$$
$$\implies |x + 2| < 5$$

So we have to make sure that $\delta \leq 1$. So going back to the beginning, we want

$$|x + 2||x - 2| \leq 5|x - 2| \text{ if } \delta = \frac{\epsilon}{5}$$

We define $\delta = \min\{1, \frac{\epsilon}{5}\}$ if $|x - 2| < \delta$. Then $|x - 2| < 1$ so $-1 < x - 2 < 1$ and $1 < x < 3$ then $3 < x + 2 < 5$.

Then,

$$|x^2 + 5 - 9| = |x^2 - 4| = |x + 2||x - 2| < 5|x - 2| < 5 \cdot \frac{\epsilon}{5} = \epsilon$$

Example 3: Show that $lim_{x \to \pi} x = \pi$

*Proof.* Let $f(x) = x$. First, we need to determine what value our $\delta$ will take. When $|x - \pi| < \delta$ we want $|f(x) - \pi| < \epsilon$. We know that $|f(x) - \pi| = |x - \pi| < \epsilon$, so taking $\delta = \epsilon$ will have the desired property. There are other values of $\delta$ we could have chosen, such as $\delta = \frac{\epsilon}{7}$. Why would this value of $\delta$ have also been acceptable? If $|x - \pi| < \delta = \frac{\epsilon}{7}$, then $|f(x) - \pi| < \frac{\epsilon}{7} < \epsilon$, as required. $\square$

*Example 4:* Show that $\lim_{x \to 1}(5x - 3) = 2$

*Proof.* In this example, we have $c = 1$, $f(x) = 5x - 3$, and $L = 2$ from the definition of the limit. For any $\epsilon > 0$, we would need to find $\delta > 0$ such that if $x$ is within distance $\delta$ of $c = 1$, **i.e**

$$|x - 1| < \delta$$

then $f(x)$ is within distance $\epsilon$ of $L = 2$, **i.e**

$$|f(x) - 2| < \epsilon$$

To find $\delta$, we work backwards from the $\epsilon$ inequality:

$$
\begin{aligned}
|(5x - 3) - 2| = |5x - 5| &< \epsilon \\
= 5|x - 1| &< \epsilon \\
= |x - 1| &< \frac{\epsilon}{5}
\end{aligned}
$$

So we choose $\delta = \frac{\epsilon}{5}$. Then we can verify that if $|x - 1| < \delta = \frac{\epsilon}{5}$, then

$$|(5x - 3) - 2| = |5x - 5|5|x - 1| < 5\left(\frac{\epsilon}{5}\right) = \epsilon.$$

$\square$

*Example 5:* Prove that $\lim_{x \to 7}(x^2 + 1) = 50$.

*Proof.* Let $f(x) = x^2 + 1$. We will first determine what our value of $\delta$ should be. When $|x - 7| < \delta$ we have

$$
\begin{aligned}
|x^2 + 1 - 50| = |x^2 - 49| \\
= |x - 7||x + 7| \\
< \delta|x + 7|.
\end{aligned}
$$

Assuming $|x - 7| < 1$, we have $|x| < 8$, which implies $|x + 7| < |x| + |7| = 15$ by the triangle inequality. So, when we let $\delta = \min\left(1, \frac{\epsilon}{15}\right)$, we will have

$$
\begin{aligned}
|x^2 + 1 - 5| &< \delta|x + 7| \\
&< 15\delta \\
&< \epsilon.
\end{aligned}
$$

$\square$

*Example 6:* Prove

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

*Proof.* We show that for any positive number $L$, set $\delta = \frac{1}{\sqrt{L}}$. Then, when $|x - 0| < \delta$, we have

$$f(x) = \frac{1}{x^2} > \frac{1}{\left(\frac{1}{\sqrt{L}}\right)^2} = L.$$

This shows that the values of the function become and stays arbitrarily large as $x$ approaches zero, or

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

$\square$

blue*Example 7:* Consider the function given by

$$f(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0. \end{cases}$$

Show that the limit at 0 does not exist.

*Proof.* Note that the right-hand limit is 1 and the left-hand limit is -1. As such, it makes senses that the limit does not exist. We can formally show this by supposing that the limit at 0 exists and is equal to $L$. Let $\epsilon = \frac{1}{2}$, with a corresponding $\delta = \delta_\epsilon > 0$. Since the limit exists, we know that $for all y \in (-\delta, \delta)$, we have $|f(y) - L| < \epsilon = \frac{1}{2}$. HOwever, we also have

$$2 = \left| f\left(\frac{\epsilon}{2}\right) - f\left(-\frac{\epsilon}{2}\right) \right|$$
$$= \left| f\left(\frac{\epsilon}{2}\right) - L + L - f\left(-\frac{\delta}{2}\right) \right|$$
$$\leq \left| f\left(\frac{\delta}{2}\right) - L \right| + \left| L - f\left(-\frac{\delta}{2}\right) \right|$$
$$\leq \frac{1}{2} + \frac{1}{2}$$

This is a contradiction, so our original assumption is not true. □

## 2.2   Limit Theorems

If $lim_{x \to c} f(x) = L$ and $lim_{x \to c} g(x) = M$, then prove that

$$\lim_{x \to c} [f(x) + g(x)] = L + M$$

*Proof.* Assume that $|x - c| < \delta$, then

$$|f(x) + g(x) - (L + M)| \leq |f(x) - L| + |g(x) - M|$$

Since $lim_{x \to c} f(x) = L$, for $\frac{\epsilon}{2}$ there exists $\delta$, such that $|x - c| < \delta$, implies $|f(x) - L| < \frac{\epsilon}{2}$. Since $lim_{x \to c} g(x) = M$, define $\delta = \min\{\delta_1, \delta_2\}$. Now we can go back to the beginning. Then,

$$|f(x) + g(x) - (L + M)| \leq |f(x) - L| + |g(x) - M|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

If $lim_{x \to c} f(x) = L$ and $lim_{x \to c} g(x) = M$, then prove that

$$\lim_{x \to c} f(x)g(x) = LM$$

Hint:

$$|f(x)g(x) - LM| = |f(x)g(x) - f(x)M + f(x)M - LM|$$
$$\leq |f(x)||g(x) - M| + |f(x) - L||M|$$

*Proof.* □

> **Corollary**
>
> If $p(x) = a_n x^n + \cdots + a_1 x + a_0$ is a polynomial, then $lim_{x \to c} p(x) = p(c)$.

## 2.3   The Squeeze Theorem

**An $\epsilon - \delta$ Proof for the Squeeze Theorem:**

*Proof.* Let $\epsilon > 0$.
Since
$$h(x) \leq f(x) \leq g(x),$$
we have
$$h(x) - L \leq f(x) - L \leq g(x) - L.$$
Since
$$-g(x) \leq -f(x) \leq -h(x)$$
we have
$$L - g(x) \leq L - f(x) \leq L - g(x)$$
so we get
$$|f(x) - L| \leq \max\{|h(x) - L|, |g(x) - L|\}$$
There is a $\delta_1 > 0$ such that $|x - c| < \delta \implies |h(x) - L| < \epsilon$.
There is a $delta_2 > 0$ such that $|x - c| < d_2 \implies |g(x) - L| < \epsilon$.
Let $\delta = \min\{d_1, d_2\}. If$ —x-c— ¡ $\delta$ then $|f(x) - L| \implies \epsilon.$ □

## 2.4   More Limit Theorems

**The Intermediate Value Theorem:**
If $f$ is continuous on $[a, b]$ and $K$ is any number between $f(a)$ and $f(b)$, then there is at least one number $c$ in the open interval $(a, b)$ such that $f(c) = K$.

**The Extreme Value Theorem:**
If $f$ is continuous on a bounded closed interval $[a, b]$, then on that interval $f$ takes on both a maximum value $M$ and a minimum value $m$.

Let $S$ be a subset of the set of real numbers. $M$ is an **upper bound** for $S$ if $x \leq M$ for all $x \in S$. $M$ is the **least upper bound** for $S$ if it is the smallest number with this property, i.e. any number smaller than $M$, any $M - \epsilon$ for any $\epsilon > 0$, is not an upper bound for $S$.

8 The **Least Upper Bound Axiom:** Every nonempty set of real numbers which is bounded above has a least upper bound.

# 3 Derivatives

## 3.1 Definition of Derivative

A function $f$ is differentiable at $x$ if the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists.

## 3.2 Derivative Rules

**Proof of Product Rule:**

*Proof.*

$$\lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$
$$= \lim_{h \to 0} f(x+h) \left[ \frac{g(x+h) - g(x)}{h} \right] + g(x) \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= f(x)g'(x) + g(x)f'(x)$$

$\square$

# 4 Applications of Derivatives

## 4.1 First and Second Derivative Tests

**First Derivative Test**

Suppose that $f$ is a function and $c$ is a critical point of $f$ and $f$ is continuous at $c$.
- If $f'(x)$ changes sign from positive to negative at $c$, then $f(c)$ is a local maximum.
- If $f'(x)$ changes sign from negative to positive at $c$, then $f(c)$ is a local minimum.
- If $f'(x)$ does not change sign at $c$, then $f(c)$ is not a local extrema.

**Second Derivative Test**

Suppose that $f$ is a function and $c$ is a critical point of $f$ and $f$ is differentiable at $c$. Assume also that $f''(c)$ exists
- If $f''(c) > 0$ then $f$ has a local minimum at $c$.
- If $f''(c) < 0$ then $f$ has a local maximum at $c$.
- If $f''(c) = 0$ then the test is inconclusive.

## 4.2 Limits at Infinity and Asymptotes

A limit that approaches infinity is defined for any $K > 0$, there exists $\delta > 0$ such that

$$|x - c| < \delta \implies f(x) > K$$

Definition for $\lim_{x \to \infty} f(x) = L$:

For any $\epsilon > 0$ there exists a $K > 0$ such that $x > K$ implies $|f(x) - L| < \epsilon$.

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Given $\epsilon > 0$, let $k = \frac{1}{\epsilon}$, then $x > K = \frac{1}{\epsilon} \implies$

$$\left| \frac{1}{x} - 0 \right| < \epsilon$$
$$D\frac{1}{\epsilon} < x$$

## 4.3 Differentials and Linear Approximations

## 4.4 Newton-Raphson Approximations

$$-\frac{f(x_n)}{f'(x_n)} + x_n = x_n + 1$$

We can prove $|x_{n+1} - c| < |x_n - c|$ using them on concavity, proving $x_n \to c$