AP Calculus BC Notes

Eric Xia

Last Updated 29 August 2020

Contents

1 Introduction to Calculus

1.1 Limits

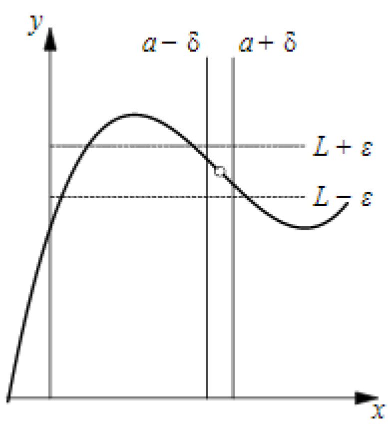
The Epsilon-Delta ($\epsilon - \delta$) Definition of Limits:

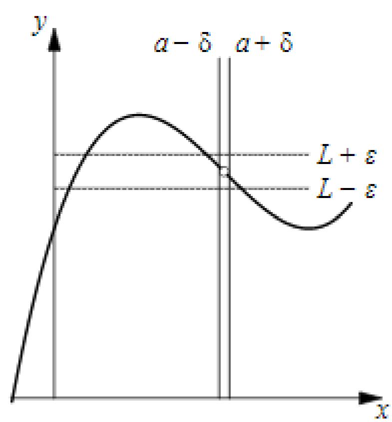
Let f be a real-valued function, defined around around a. Then the **limit**, L, as $f(x)$ approaches a, or **converges** to a, is

$$\lim_{x \to a} f(x) = L$$

If, $\forall \epsilon > 0$, there exists $\delta > 0$ such that if x is within δ of a (with $x \neq a$), then $f(x)$ is within ϵ of L. In other words,

$$\text{If } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \epsilon$$

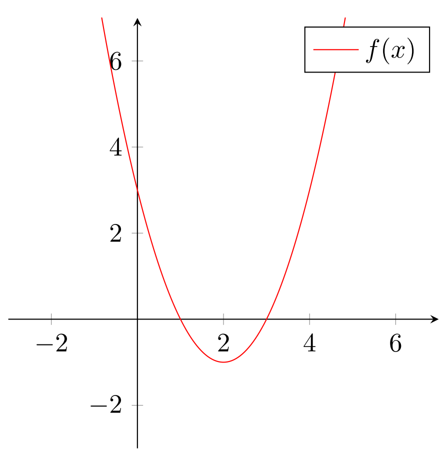




It is easiest to think of the limit of a function at a certain point, a, as the value of the function near a. For a limit, L, to exist at a, the **right-hand limit** (a^+) and the **left-hand limit** (a^-) must be equal.

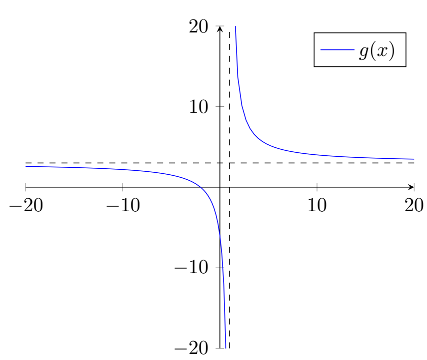
$$\lim_{x \to a} f(x) = L \implies \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$$

Example 1: Find $\lim_{x \to 4} f(x)$, where the function $f(x)$ is given by the graph.



$$\lim_{x \to 4^+} f(x) = 2 = \lim_{x \to 4^-} f(x)$$
$$\therefore \lim_{x \to 4} f(x) = 2$$

Example 2: Determine $\lim_{x \to 1} g(x)$, where the function $g(x)$ is graphed below. It is given that $g(x)$ is defined for all real numbers except $x = 1$ and the graph of $g(x)$ is divided by the asymptote $x = 1$.



$$\lim_{x\to 1^+} g(x) = \infty$$

and

$$\lim_{x\to 1^-} g(x) = -\infty$$

Since the right-hand and left-hand limits do not equal each other, $\lim_{x\to 1} g(x)$ does not exist.

1.2 Limit Properties

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist and that c is any constant. Then the following properties hold true.

$\lim_{x\to a} c = c$	**Limit of a Constant**
$\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$	**Constant Multiple**
$\lim_{x\to a}[f(x) \pm g(x) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)]$	**Sum/Difference**
$\lim x \to a[f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$	**Product**
$\lim_{x\to c}(f(g(x))) = f\left(\lim_{x\to c} g(x)\right)$	**Composition**
$\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \lim_{x\to a} g(x) \neq 0.$	**Quotient**
$\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n, n \in \mathbb{R}$	**Exponent**
$\lim_{x\to a} \sqrt[3]{f(x)}\ \sqrt[n]{\lim_{x\to a}[f(x)]}$	**Root**
$\lim_{x\to a} x = a$	**Limit of a Variable**

1.3 Finding Limits from Tables

Tables should always be the last resort when attempting to determine limits, because of their tendency to be tedious.

Example: The function g is defined over the real numbers. This table gives a few values of g What is a reasonable estimate for $\lim_{x\to 4} g(x)$?

x	3.9	3.99	3.999	4.001	4.01	4.1
$g(x)$	11.21	11.92	11.99	12.01	12.08	12.81

$lim_{x\to 4} g(x)$ represents the limit of g as x approaches 4. Looking over the table, we see that the left-hand limit appears to approach 12 as x gets progressively larger.

x	3.9	3.99	3.999
$g(x)$	11.21	11.92	11.99

We also see that the right-hand limit appears to approach 12 as x gets progressively smaller.

x	4.001	4.01	4.1
$g(x)$	12.01	12.08	12.81

Since the right-hand and left-hand limits are equal, we can conclude that $\lim_{x\to 4} g(x) = 12$.

1.4 Evaluating Limits Algebraically

The 3 Main Algebraic Limit Strategies:
1. Factoring
Example: Determine the limit

$$\begin{aligned}
\lim_{x\to -1} \frac{x^2 - x - 2}{x^2 - 2x - 3} &= \lim_{x\to -1} \frac{(x+1)(x-2)}{(x+1)(x-3)} \\
&= \lim_{x\to -1} \frac{x-2}{x-3} \\
&= \frac{-1-2}{-1-3} \\
&= \frac{3}{4}
\end{aligned}$$

2. Conjugates
Example: Determine the limit

$$\begin{aligned}
\lim_{x\to 4} \frac{\sqrt{x} - 2}{x - 4} &= \lim_{x\to 4} \frac{\sqrt{x} - 2}{x - 4} \cdot \frac{\sqrt{x} + 2}{\sqrt{x} + 2} \\
&= \lim_{x\to 4} \frac{1}{\sqrt{x} + 2} \\
&= \frac{1}{4}
\end{aligned}$$

3. Trig Identities
Example: Determine the limit

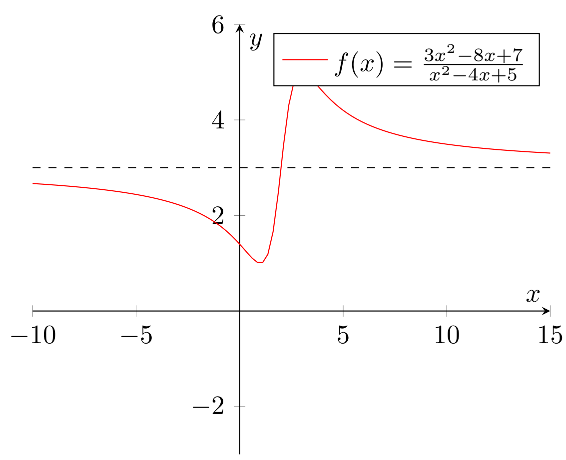
$$\begin{aligned}\lim_{x\to 0}\frac{\sin(x)}{\sin(2x)} &= \lim_{x\to 0}\frac{\sin(x)}{2\sin(x)\cos(x)}\\ &= \lim_{x\to 0}\frac{1}{2\cos(x)}\\ &= \frac{1}{2}\end{aligned}$$

1.5 Asymptotes

Horizontal Asymptotes:
The line $y = b$ is a *horizontal asymptote* of the graph of $y = f(x)$ if

$$\lim_{x\to\infty} f(x) = b \text{ or } \lim_{x\to-\infty} f(x) = b$$

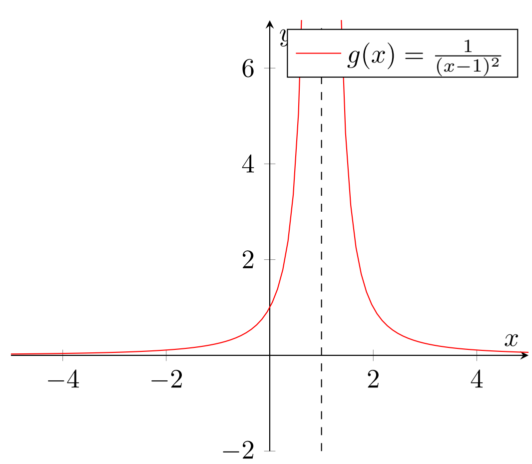
It is important to note that horizontal and slant asymptotes *can* be crossed, as they describe the general behavior of the functions as they near the edges of the graph. On the other hand, vertical asymptotes cannot be crossed as they describe particular behavior of the function itself, rather than the edges of the graph. Below is an example of a horizontal asymptote being crossed.



Vertical Asymptotes:
The line $x = a$ is a *vertical asymptote* of the graph of $y = f(x)$ if at least one of the below expressions holds.

$$\lim_{x\to a^-} f(x) = \pm\infty,\ \lim_{x\to a^+} f(x) = \pm\infty$$

Example of a graph containing a vertical asymptote:



The Rational Function Theorem:
When $\lim_{x\to\pm\infty}\frac{P(x)}{Q(x)} = 0$, $y = 0$ is a horizontal asymptote of the graph of $y = \frac{P(x)}{Q(s)}$.
When $\lim_{x\to\pm\infty}\frac{P(x)}{Q(x)} = \pm\infty$, the graph of $y = \frac{P(x)}{Q(x)}$ has no horizontal asymptotes.
When $\lim_{x\to\pm\infty}\frac{P(x)}{Q(x)} = \frac{a_n}{b_n}$, $y = \frac{a_n}{b_n}$ is a horizontal asymptote of the graph of $y = \frac{P(x)}{Q(x)}$.

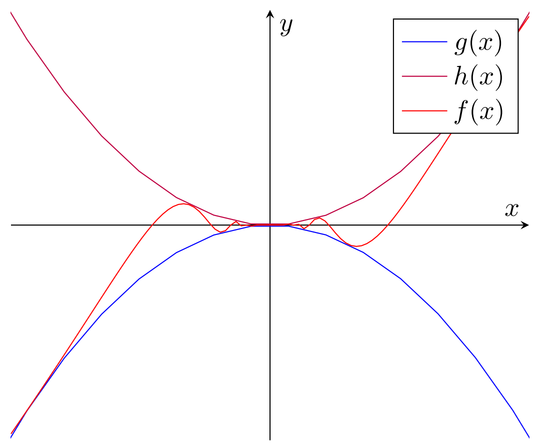
1.6 The Squeeze Theorem

Functions g and h are strategically chosen to satisfy the conditions outlined in the definition. Notice how, in the figure after the definition, function f is being "squeezed" between functions g and h, hence the name.

The Squeeze Theorem:
Assume that functions f, g, h defined n $D \subseteq \mathbb{R}$ satisfy

$$g(x) \leq f(x) \leq h(x), \forall x \in D$$

If $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} f(x) = L$.



Example 1: Given an infinite sequence $\{a_n\}$ that satisfies $\frac{2n^2-7}{4n+5} < a_n < \frac{3n^2+8}{6n-1}$ for all positive integers n, evaluate

$$\lim_{n\to\infty} \frac{3na_n}{(n+1)^2} \tag{1}$$

We transform the middle term of the inequality to the desired expression through manipulating each side of the inequality. Then the inequality becomes

$$\frac{3n}{(n+1)^2} \cdot \frac{2n^2-7}{4n+5} < \frac{3na_n}{(n+1)^2} < \frac{3n}{(n+1)^2} \cdot \frac{3n^2+8}{6n-1}$$

Now we can take the limits of the top and bottom functions.

$$\lim_{n\to\infty} \frac{3n(2n^2-7)}{(4n+5)(n+1)^2} = \frac{3}{2}$$
$$\lim_{n\to\infty} \frac{3n(3n^2+8)}{(6n-1)(n+1)^2} = \frac{3}{2}$$

Since the top and bottom functions are equal, we can conclude that

$$\lim_{n\to\infty} \frac{3na_n}{(n+1)^2} = \frac{3}{2}$$

Example 2: Evaluate the limit

$$\lim_{n\to\infty} \sqrt[n]{3^n + \{2|\sin(n^n)\}^n}$$

We can note that

$$0 \leq |\sin(n^n)| \leq 1$$

Then we can write

$$\sqrt[n]{3^n} \leq \sqrt[n]{3^n + \{2|\sin(n^n)\}^n} \leq \sqrt[n]{3^n + 2^n} \leq \sqrt[n]{2 \cdot 3^n}$$

The left side reduces to 3, whereas the right side becomes

$$\lim_{n\to\infty} \sqrt[n]{2 \cdot 3^n} = 3 \lim_{n\to\infty} \sqrt[n]{2} = 3$$

Thus we can conclude that

$$\lim_{n\to\infty} \sqrt[n]{3^n + \{2|\sin(n^n)|\}^n} = 3$$

Example 3: Evaluate the limit

$$\lim_{x\to\infty} \frac{\sin x}{x}$$

Since $\forall x, -1 \leq \sin x \leq 1$, it follows that if $x > 0$ then $-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$. As $x \to \infty$, $-\frac{1}{x}$ and $\frac{1}{x}$ both approach 0. Therefore, by the Squeeze Theorem, $\frac{\sin x}{x}$ also approaches 0.

1.7 Limit of a Quotient of Polynomials

For the function $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ is a polynomial of degree n and $Q(x)$ is a polynomial of degree m, the following expressions hold. Let a_n and b_n be the leading coefficients of $P(x)$ and $Q(x)$ respectively. Then

$$\text{If } n > m, \text{ then } \lim_{x\to\infty} f(x) = \infty \text{ and } \lim_{x\to-\infty} f(x) = -\infty$$

$$\text{If } n = m, \text{ then } \lim_{x\to\pm\infty} f(x) = \frac{a_n}{b_m}$$

$$\text{If } n < m, \text{ then } \lim_{x\to\pm\infty} f(x) = 0$$

This method is a shortcut found through dividing each of the terms of $P(x)$ and $Q(x)$ by the highest power of x found in $f(x)$, then taking the individual limits of each separated term through basic limit properties. For example, see below.

Example: Evaluate the limit

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^3 - 4x^2 + 7}{3 - 6x - 2x^3} &= \lim_{x\to\infty} \frac{1 - \frac{4}{x} + \frac{7}{x^3}}{\frac{3}{x^3} - \frac{6}{x^2} - 2} \\
&= \frac{\lim_{x\to\infty}(1) - \lim_{x\to\infty}\left(\frac{4}{x}\right) + \lim_{x\to\infty}\left(\frac{7}{x^3}\right)}{\lim_{x\to\infty}\left(\frac{3}{x^3}\right) - \lim_{x\to\infty}\left(\frac{6}{x^2}\right) - \lim_{x\to\infty}(2)} \\
&= \frac{1 - 0 + 0}{0 - 0 - 2} \\
&= -\frac{1}{2}
\end{aligned}$$

Note that in this case, $n = m$, so $\frac{a_n}{b_n} \implies -\frac{1}{2}$. Since the properties described preceding this example can be observed in all quotients of polynomials, we can thus make generalizations as specificed above.

1.8 L'Hospital's Rule

L'Hospital's is a method of evaluating limits of indeterminate forms, learned after gaining knowledge of derivatives. **Indeterminate forms** are expressions i nvolving two functions whose limits cannot be determined solely from the limits of the individual functions.

The 7 Indeterminate Forms:

$$\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, \infty - \infty, 0^0, 1^\infty, \infty^0$$

L'Hospital's Rule:
Suppose that we have either of the cases below

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}, \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$$

where $a \in \mathbb{R}, \infty$ or $-\infty$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Example 1: Evaluate the limit

$$\lim_{t\to 1} \frac{5t^4 - 4t^2 - 1}{10 - t - 9t^3}$$

We have an $\frac{0}{0}$ indeterminate form here, so we use L'Hospital's to get

$$\lim_{t\to 1}\frac{5t^4-4t^2-1}{10-t-9t^3}=\lim_{t\to 1}\frac{20t^3-8t}{-1-27t^2}=\frac{20-8}{-1-27}=-\frac{3}{7} \tag{2}$$

Example 2: Evaluate the limit

$$\lim_{x\to -\infty} xe^x$$

This expression is in the form $(\infty)(0)$, meaning we will have to write the expression as a quotient. We know that $\frac{1}{e^x}=e^{-x}$, so we can rewrite the limit as

$$\lim_{x\to -\infty} xe^x=\lim_{x\to -\infty}\frac{x}{e^-x}=\lim_{x\to -\infty}\frac{1}{-e^-x}=0$$

Example 3: Evaluate the limit

$$\lim_{x\to\infty} x^{\frac{1}{x}}$$

This limit is of the form ∞^0, so we have to rewrite this expression as a different limit. Let

$$y=x^{\frac{1}{x}}$$

Since

$$e^{\ln(y)}=y$$

we can rewrite our limit as

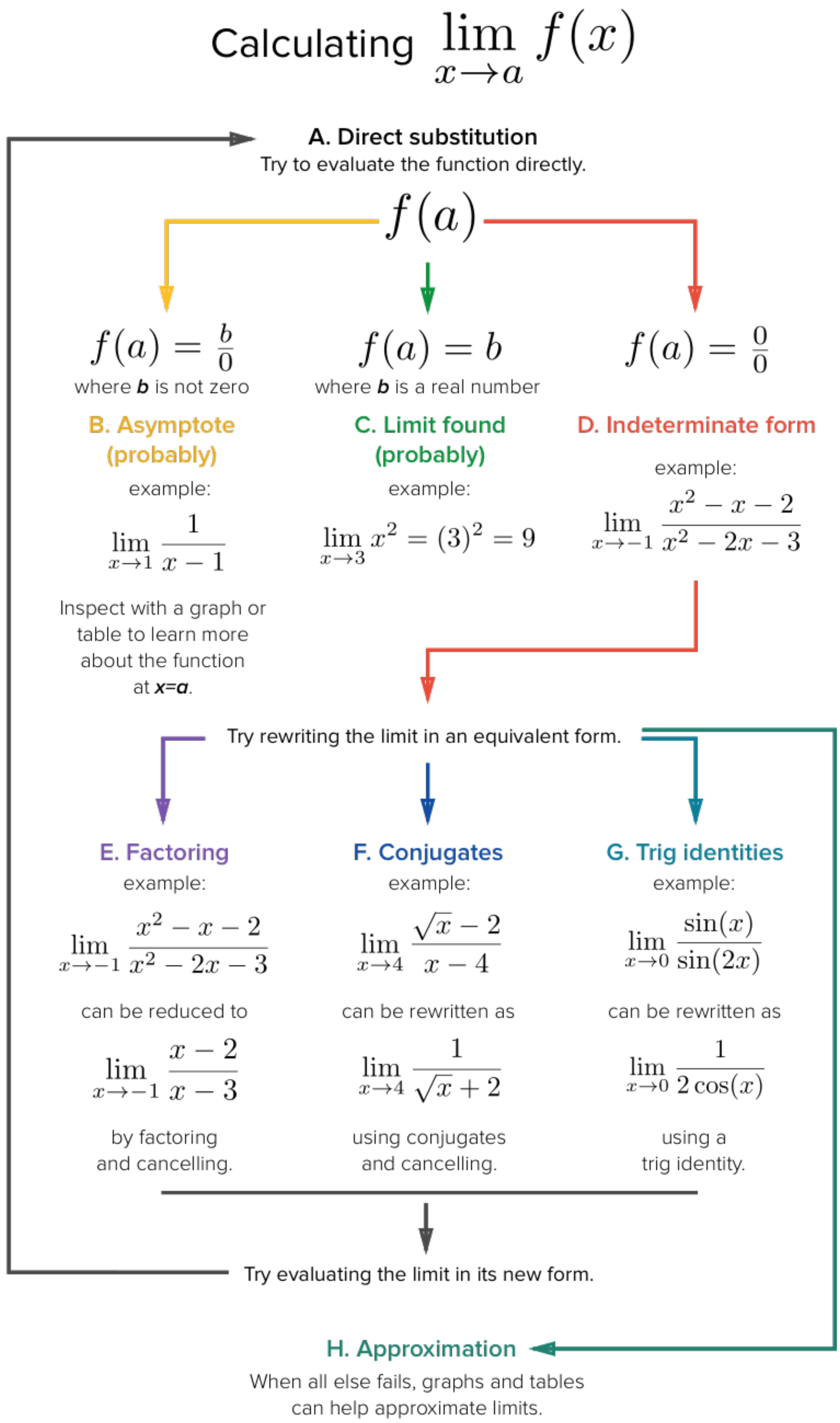
$$\begin{aligned}\lim_{x\to\infty} x^{\frac{1}{x}}&=\lim_{x\to\infty} y\\&=\lim_{x\to\infty} e^{\ln(y)}\\&=e^{\lim_{x\to\infty}\ln(y)}\\&=e^0\\&=1\end{aligned}$$

L'Hospital's can help us prove **the basic trig limit**, useful for evaluating many trig limits.

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1 \text{ if } \theta \text{ is measured in radians}$$

1.9 Limit Strategies

KhanAcademy provides a convenient flowchart to illustrate a strategy to find limits:



Example 1:

$$
\begin{aligned}
\lim_{x\to 0} \frac{\sin 3x}{x} &= \lim_{x\to 0} \frac{3\sin 3x}{3x} \\
&= 3 \lim_{x\to 0} \frac{\sin 3x}{3x} \\
&= 3 \cdot 1 \\
&= 3
\end{aligned}
$$

Example 2:

$$
\begin{aligned}
\lim_{x\to 0}\frac{\sin 2x}{3x} &= \frac{1}{3}\lim_{x\to 0}\frac{\sin 2x}{x}\cdot\frac{2}{2}\\
&= \frac{2}{3}\lim_{x\to 0}\frac{\sin 2x}{2x}\\
&= \frac{2}{3}
\end{aligned}
$$

Example 3:

$$\lim_{x\to 0}\frac{|x|}{x}$$

Since $|x| = x$ if $x > 0$ but $|x| = -x$ if $x < 0$, $\lim_{x\to 0^+}\frac{|x|}{x} = \lim_{x\to 0^+}\frac{x}{x} = 1$, whereas $\lim_{x\to 0^-}\frac{|x|}{x} = \lim_{x\to 0^-}\frac{-x}{x} = -1$. Since the right and left hand limits are different, we can conclude that the limit does not exist.
Example 4:

$$
\begin{aligned}
\lim_{x\to\infty}\arctan\left(x^3 - 5x + 6\right) &= \arctan\left(\lim_{x\to\infty} x^3 - 5x + 6\right)\\
&= \arctan\left(+\infty\right)\\
&= \frac{\pi}{2}
\end{aligned}
$$

1.10 Limit Definition of e

The mathematical constant e is defined by

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

1.11 Continuity and Discontinuity

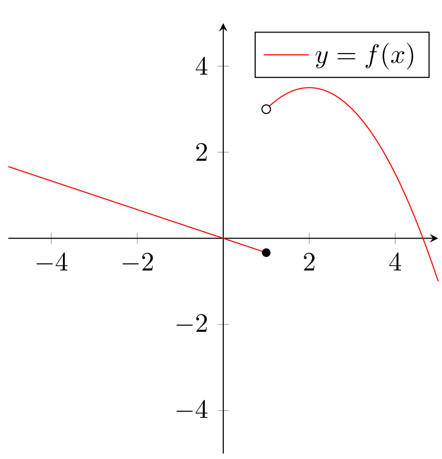
A function $f(x)$ is **continuous** at a if

$$\lim_{x\to a} f(x) = f(a)$$

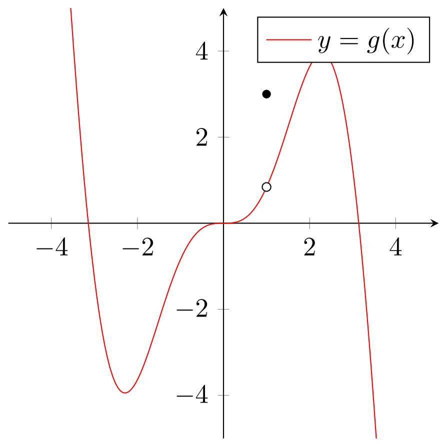
A function $f(x)$ is **continuous over the interval** $[a, b]$ if $\forall a \le x \le b$, x is continuous. A function is **discontinuous** if it is not continuous.

Common Continuous Functions:
- Polynomials are continuous everywhere at a real number.
- Rational functions are continuous at each point in their domain except where $Q(x) = 0$.
- The absolute value function is continuous everywhere.
- The trigonometric, inverse trigonometric, exponential, and logarithmic functions are continuous everywhere.
- Irrational functions in the form $\sqrt[n]{x}$, where $n \ge 2$, are continuous everywhere for which $\sqrt[n]{x}$ is defined.
- The greatest-integer function is discontinuous at each integer.

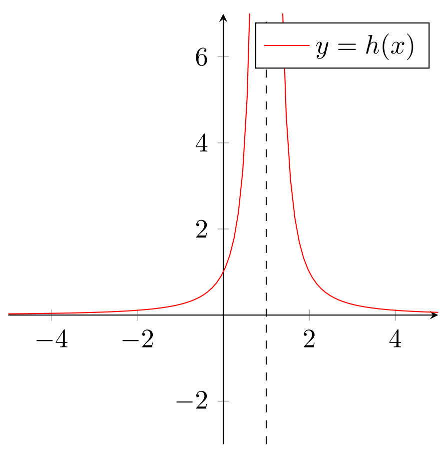
Types of Discontinuities:
Jump Discontinuity:
In a jump discontinuity, the left and right-hand limits exist, but are different. For example, the graph of $y = f(x)$ below has a jump discontinuity at $x = 1$.



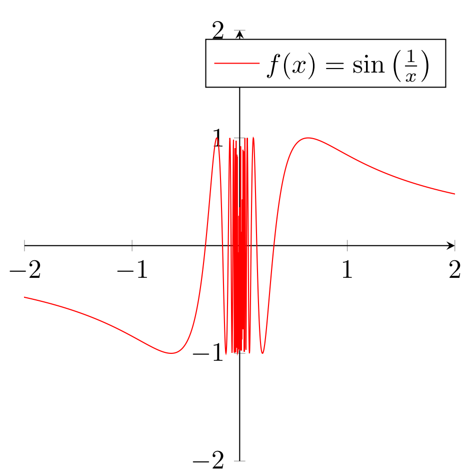
Removable Discontinuity:
A function $f(x)$ has a removable discontinuity at $x = a$ if $\lim_{x\to a} f(x)$ exists, but either $f(a)$ does not exist or the value of $f(a) \neq \lim_{x\to a} f(x)$. A function with a removable discontinuity at a may or may not be defined at a. For example, the graph of $y = g(x)$ below has a removable discontinuity at $x = 1$. In this case, $g(1) = 3$ instead of the limit value $\lim_{x\to 1} g(x) = \sin(1)$.



Infinite Discontinuities:
Infinite discontinuities always occur at vertical asymptotes. There may be a vertical asymptote at either one or both sides of the function. An example of an infinite discontinuity is given below, where $y = h(x)$.



Here is an example of an **Infinite Oscillation Discontinuity**.



Essential Discontinuities are discontinuities that at which the limit of the function does not exist. Jump and infinite discontinuities are essential discontinuities.

Below is a general example about continuity and discontinuity.
Example: Is

$$f(x) = \begin{cases} x^2 + 2 & x \leq 1 \\ 4 & x > 1 \end{cases}$$

continuous at $x = 1$?
Since $\lim_{x\to 1^-} f(x) = 3 \neq \lim_{x\to 1^+} f(x) = 4$.

1.12 The Extreme Value Theorem

This theorem is best learnt after derivatives.

The Extreme Value Theorem

If real numbers a and b satisfy $a < b$ and a function f is continuous on $[a, b]$, then f attains a maximum and minimum value on $[a, b]$.

Example: Find the maximum and minimum values of the function $f(x) = x^3 - \frac{9}{2}x^2 - 12x + 20$ *on the interval* $[-2, 6]$.

$f(x)$ is differentiable, hence continuous on the closed and bounded interval. Having satisfied the preconditions, we can apply the EVT to this context. Taking the first derivative of $f(x)$, we get

$$\begin{aligned} f'(x) &= 3x^2 - 9x - 12 \\ &= 3(x^2 - 3x - 4) \end{aligned}$$

Setting $f'(x) = 0$ and solving, we find that the critical points are $x = -1, 4$ which by the first derivative test gives the relative extrema $y = 26.5, -36$, respectively. Computing the endpoints, we have $f(-2) = 18$ and $f(6) = 2$. Hence, the minimum value of f on $[-2, 6]$ is -36 and the maximum value is 26.5.

1.13 The Intermediate Value Theorem

The Intermediate Value Theorem:

If a function f is continuous on the closed interval $[a, b]$, and M is a number such that $f(a) \leq M \leq f(b)$, then there is at least one number c such that $f(c) = M$.

Example: Does a x *exist for some* $x \in [0, 2]$ *such that the function* $f(x) = x^2 + \cos(\pi x) = 4$*?*

$$\begin{aligned} f(0) &= 0^2 + \cos 0 = 1 \\ f(2) &= 2^2 + \cos 2\pi = 5 \end{aligned}$$

Since $f(0) = 1 < 4 < 5 = f(2)$ and f is continuous, the IVT implies that $f(x) = 4$ for some $x \in [0, 2]$.

1.14 The Continuous Functions Theorem

The Continuous Functions Theorem:

If functions f and g are both continuous at $x = c$, then the following functions are also continuous:

Constant Multiples:	$k \cdot f(x)$ for any real number k
Sums:	$f(x) + g(x)$
Differences:	$f(x) - g(x)$
Products:	$f(x) \cdot g(x)$
Quotients:	$\frac{f(x)}{g(x)}, g(c) \neq 0$

1.15 The Composition of Continuous Functions Theorem

The Composition of Continuous Functions Theorem:

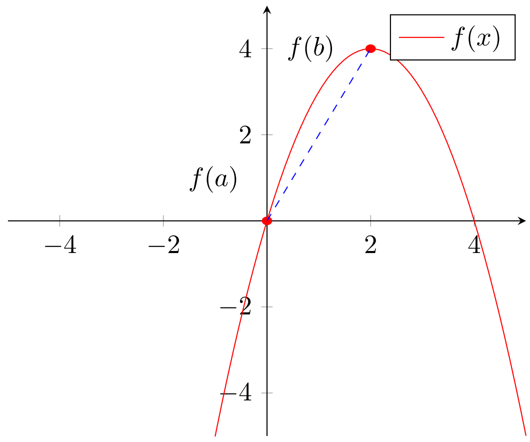
If the function g is continuous at $x = c$ and the function f is continuous at $x = g(c)$, then the composite function $(f \circ g)(x)$ is continuous at $x = c$.

2 Differentiation

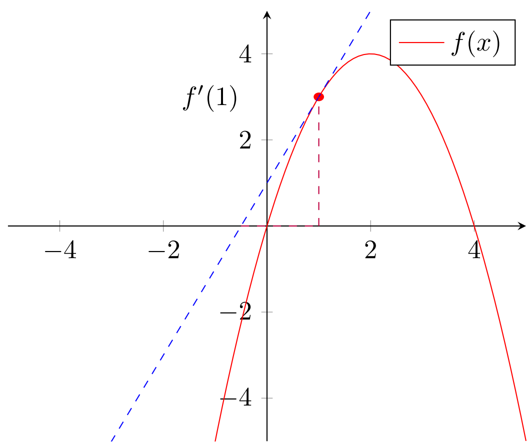
2.1 Rates of Change and Derivatives

An **average rate of change** describes the overall $\frac{\text{rise}}{\text{run}}$-value over an interval $[a, b]$. Geometrically, this can be represented by the slope of a secant line through $f(a)$ and $f(b)$ of a function $f(x)$. In the graph below, the average rate of change between $x = 0$ and $x = 2$ is given by

$$m_{\text{avg}} = \frac{f(b) - f(a)}{b - a} = \frac{4 - 0}{2 - 0} = 2$$



A derivative is an **instantaneous rate of change**, or the slope of the tangent to a function at a particular point, a. Derivatives are always taken *with respect* to a variable. For example, a derivative representing the instantaneous rate of change between distance and time is a derivative of distance with respect to time, the independent variable. For reference, the derivative of a function $f(x)$ at $x = 1$ is shown in the graph below.



Since the derivative is an instantaneous rate of change, we can define the derivative as below.
1^{st} Limit Definition of the Derivative
The derivative of $f(x)$ with respect to x is the function $f'(x)$, defined as

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Then, if we let $x = a + h$ and change the variable of inspection from a to x, we can realize the most commonly used definition of the derivative, given below. This definition is preferred to the above definition, as finding the derivative only requires one value of x, rather than 2.
2^{nd} Limit Definition of the Derivative
The derivative of $f(x)$ with respect to x is the function $f'(x)$, defined as

$$f'(x) = \lim_{h \to 0} \frac{f(a + h) - f(x)}{h}$$

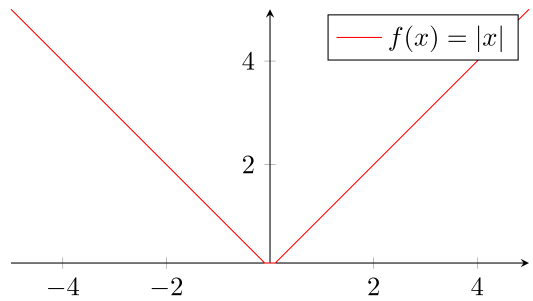
The expression consisting the right side of the equation above is known as the **difference quotient**.

A function is $f(x)$ is **differentiable** at $x = a$ if $f'(a)$ exists. Since derivatives are defined by limits, for a derivative to exist, its limit must also exist. Hence,

If $f(x)$ is differentiable at $x = a$ then $f(x)$ is continuous at $x = a$

By this, we can also say that differentiability implies continuity. It is important to note that its converse is not always true, as not all continuous functions are differentiable.

A few examples of contexts where a derivative may not exist are at vertical tangents, corners, and cusps. Take, for example, the graph below, which does not have a derivative at $x = 0$ since the respective slopes immediately right and left of $x = 0$ are different.



Derivative Notations:

Leibniz's Notation:

The Leibniz notation is popular throughout mathematics, most commonly used when the equation $y = f(x)$ is regarded as a functional relationship between dependent and independent variables y and x. Leibniz's notation makes this relationship explicit by representing the derivative as

$$\frac{dy}{dx} = \frac{d}{dx}y$$

The Leibniz notation is also called **differential notation**, where dy and dx are **differentials**. Where the function $y = f(x)$, we can write

$$\frac{df}{dx}(x) = \frac{df(x)}{dx} = \frac{d}{dx}f(x)$$

A **differential equation** is an equation relating one or more functions and their derivatives. As long as the precondition ($f(x)$ is satisfied), a function can be differentiated indefinite times. For example, in physics, acceleration is the derivative of velocity, the derivative of position. The **order** of a differential equation is the highest derivative of said equation. Higher-order derivatives can be written in the Leibniz's notation as

$$\frac{d^2y}{dx^2}, \frac{dy^3}{dx^3}, \frac{dy^4}{dx^4}, \ldots, \frac{d^ny}{dx^n}$$

With Leibniz's notation, the value of a derivative at a particular point, a, can be expressed as

$$\left.\frac{dy}{dx}\right|_{x=a}$$

This notation is particularly useful in expressing partial derivatives (covered later) and making the chain rule intuitive:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Lagrange's Notation:

In Lagrange's notation, each prime mark denotes a derivative. For higher-order derivatives, the order is enclosed by parantheses in front and above the function.

$$f'(x), f''(x), f'''(x), f^{(4)}(x), f^{(5)}(x), f^{(6)}(x), f^{(n)}(n)$$

Newton's Notation:

This notation is often applied to physics contexts, where time (t) is the independent variable. The number of dots over the dependent variable represent the order the derivative is in. If y is a function of t, then the first n derivatives of the function are as below.

$$\dot{y}, \ddot{y}, \dddot{y}, \overset{4}{\dot{y}}, \overset{5}{\dot{y}}, \overset{n}{\dot{y}}$$

Euler's Notation:

This notation is quite inconvenient, as it leaves the variable being differentiated with respect to entirely implicit. However, we can modify the notation to explicitly write said variable. This notation is defined by

$$(Df)(x) = \frac{df(x)}{dx}$$

Higher-order Derivatives are expressed by

$$D^2 f = D_x^2 f, D^3 f = D_x^3 f, D^n f = D_x^n f$$

Example 1: Find the derivative, $g'(t)$, given the function

$$g(t) = \frac{t}{t+1} \tag{3}$$

$$\begin{aligned}
g'(t) &= \lim_{h\to 0} \frac{g(t+h)-g(t)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{t+h}{t+h+1} - \frac{t}{t+1}\right) \\
&= \lim_{h\to 0} \left(\frac{(t+h)(t+1)-t(t+h+1)}{(t+h+1)(t+1)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{t^2+t+th+h-(t^2+th+t)}{(t+h+1)(t+1)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{h}{(t+h+1)(t+1)}\right) \\
&= \frac{1}{(t+1)(t+1)} \\
g'(t) &= \frac{1}{(t+1)^2}
\end{aligned}$$

Example 2: Find the derivative, $f'(x)$, of

$$f(x) = 2x^2 - 16x + 35$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{2(x+h)^2 - 16(x+h) + 35 - (2x^2 - 16x + 35)}{h} \\
&= \lim_{h\to 0} \frac{4xh + 2h^2 - 16h}{h} \\
&= \lim_{h\to 0} (4x + 2h - 16) \\
f'(x) &= 4x - 16
\end{aligned}$$

2.2 Basic Derivative Rules

$\frac{d}{dx}a = 0$	**Constant**
$\frac{d}{dx}au = a\frac{du}{dx}$	**Constant Multiple**
$\frac{d}{dx}x^n = nx^{n-1}$	**The Power Rule**
$\frac{d}{dx}(u \pm v) = \frac{d}{dx}u \pm \frac{d}{dx}v$	**Sum/Difference**
$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	**The Product Rule**
$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}, v \neq 0$	**The Quotient Rule**
$[f^{-1}]'(b) = \frac{1}{f'(a)}$	**Inverse Functions**
$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\frac{dy}{dx}}$	**Alternate Inverse Functions**
$\frac{d(1/f)}{dx} = -\frac{1}{f^2}\frac{df}{dx}$	**Reciprocal**

2.3 Derivatives of Exponential and Logarithmic Functions

$\frac{d}{dx}e^x = e^x$	**Natural Exponent**
$\frac{d}{dx}\ln x = \frac{1}{x}$	**Natural Logarithm**
$\frac{d}{dx}a^x = a^x \ln a$	**Exponent**
$\frac{d}{dx}\log_b x = \frac{1}{x \ln b}$	**Logarithm**
$\frac{d}{dx}x^x = x^x(1 + \ln x)$	

2.4 Derivatives of the Trigonometric Functions

The derivatives of $\sin x$ and $\cos x$ help us derive the others through applying the quotient and reciprocal differentiation rules.

$\frac{d}{dx}\sin x = \cos x$
$\frac{d}{dx}\cos x = -\sin x$
$\frac{d}{dx}\tan x = \sec^2 x$
$\frac{d}{dx}\csc x = -\cot x \csc x$
$\frac{d}{dx}\sec x = \tan x \sec x$
$\frac{d}{dx}\cot x = -\csc^2 x$

The derivatives of the inverse trigonometric functions can be derived through applying the inverse differentiation rule.

$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$
$\frac{d}{dx}\operatorname{arccsc} x = -\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\frac{d}{dx}\operatorname{arcsec} x = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\frac{d}{dx}\operatorname{arccot} x = -\frac{1}{1+x^2}$

Derivation of the Inverse Sine Derivative:

$$\begin{aligned}\frac{d(\arcsin x)}{dx} &= \frac{1}{\frac{d(\sin y)}{dy}} \\ &= \frac{1}{\cos y}\end{aligned}$$

From the Pythagorean Identity,

$$\begin{aligned}\cos^2 y + \sin^2 y = 1 \\ \cos^2 y = 1 - \sin^2 y \\ \cos y = \sqrt{1-\sin^2 y}\end{aligned}$$

Substituting this back into the original equation, we get

$$\frac{d(\arcsin x)}{dx} = \frac{1}{\sqrt{1-\sin^2 y}}$$

By the definition of the inverse sine function, we can express the equation as

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

Derivation of the Inverse Cosine Derivative

The derivatives of the inverse trigonometric functions are the *negatives* of the derivatives of their cofunctions. This is because

$$\begin{aligned}\arccos x &= \frac{\pi}{2} - \arcsin x \\ \frac{d}{dx}\arccos x &= -\frac{d}{dx}\arcsin x \\ &= -\frac{1}{\sqrt{1-x^2}}\end{aligned}$$

Derivation of the Inverse Tangent Derivative

$$\begin{aligned}\frac{d}{dx}\arctan x &= \frac{1}{\frac{d(\tan y)}{dy}} \\ &= \frac{1}{\sec^2 y} \\ &= \frac{1}{1+\tan^2 y} \\ &= \frac{1}{1+x^2}\end{aligned}$$

2.5 The Chain Rule

The Chain Rule is incredibly useful for finding the derivative of composite functions. If $y = f(u)$ and $u = g(x)$ then

$$\begin{aligned}(f(g(x)))' &= f'(g(x)) \cdot g'(x) \\ &= f'(u) \cdot g'(x) \\ \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx}\end{aligned}$$

Example 1:

$$f(x) = \sqrt{5x - 8}$$

We let the inside function be expressed as $u = 5x - 8$. Then

$$\begin{aligned}f'(x) &= \frac{du}{dx}u^{\frac{1}{2}} \cdot \frac{dy}{du}(5x - 8) \\ &= \frac{1}{2\sqrt{u}} \cdot 5 \\ &= \frac{5}{2\sqrt{5x - 8}}\end{aligned}$$

Example 2:
Let $u = \cos t$ and $v = t^4$

$$\begin{aligned}g(t) &= \cos^4 t + \cos\left(t^4\right) \\ g'(t) &= \frac{d(u^4)}{du} \cdot \frac{d}{dt}\cos t + \frac{d(t^4)}{dt} \cdot \frac{d(\cos v)}{dv} \\ &= 4u^3(-\sin t) + 4t^3(-\sin v) \\ &= -4\cos^3 t \sin t - 4t^3 \sin\left(t^4\right)\end{aligned}$$

2.6 Implicit Differentiation

Implicit Differentiation is an approach to differentiating a function that may not be in the explicit form, $y = f(x)$. It considers all other variables as functions of one of its variables, then uses the chain rule to find the derivative.

Example 1: Given $x^2 + x + y^2 = 15$, what is $\frac{dy}{dx}$ at the point $(2, 3)$?

$$\begin{aligned}\frac{dy}{dx}(x^2 + x + y^2) &= \frac{dy}{dx}15 \\ 2x + 1 + 2y\left(\frac{dy}{dx}\right) &= 0 \\ \frac{dy}{dx} &= -\frac{2x + 1}{2y} \\ \left.\frac{dy}{dx}\right|_{(2,3)} &= -\frac{2(2) + 1}{2(3)} \\ &= -\frac{5}{6}\end{aligned}$$

Example 2: Find the derivative of $\ln y + e^y = \sin y^2 - 3\cos x$

$$\begin{aligned}\frac{dy}{dx}\ln y + \frac{dy}{dx}e^y &= \frac{d}{dx}\sin y^2 - \frac{d}{dx}3\cos x \\ \frac{d}{dy}\ln y \cdot \frac{dy}{dx} + \frac{d}{dy}e^y \cdot \frac{dy}{dx} &= \frac{d}{dy}\sin y^2 \cdot \frac{dy}{dx} + 3\sin x \\ \frac{dy}{dx}\left(\frac{1}{y} + e^y - 2y\cos y^2\right) &= 3\sin x \\ \frac{dy}{dx} &= \frac{3\sin x}{\frac{1}{y} + e^y - 2y\cos y^2}\end{aligned}$$

Example 3: Find $\frac{dy}{dx}$ if $y^2 = x^2 + \sin(xy)$

$$\begin{aligned}
\frac{d}{dx}(y^2) &= \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin(xy) \\
2y\frac{d}{dx} &= 2x + (\cos(xy))\left(y + x\frac{d}{dx}\right) \\
(2y - x\cos(xy))\frac{d}{dx} &= 2x + y\cos(xy) \\
\frac{dy}{dx} &= \frac{2x + y\cos(xy)}{2y - x\cos(xy)}
\end{aligned}$$

2.7 Logarithmic Differentiation

This method of finding derivatives uses the basic properties of logarithms outside of calculus to simply a function prior to differentiating it.

Example 1:

$$\begin{aligned}
y &= \frac{x^5}{(1-10x)\sqrt{x^2+2}} \\
\ln y &= \ln\left(\frac{x^5}{(1-10x)\sqrt{x^2+2}}\right) \\
&= \ln(x^5) - \ln\left((1-10x)\sqrt{x^2+2}\right) \\
&= \ln(x^5) - \ln(1-10x) - \ln(\sqrt{x^2+2})
\end{aligned}$$

By Implicit Differentiation,

$$\begin{aligned}
y'y &= \frac{5x^4}{x^5} - \frac{-10}{1-10x} - \frac{\frac{1}{2}(x^2+2)^{-\frac{1}{2}}(2x)}{(x^2+2)^{\frac{1}{2}}} \\
&= \frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2} \\
\frac{dy}{dx} &= \frac{x^5}{(1-10x)\sqrt{x^2+2}}\left(\frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}\right)
\end{aligned}$$

Example 2:

$$\begin{aligned}
y &= \frac{x+3}{(x+4)^3} \\
\ln y &= \ln\left(\frac{x+3}{(x+4)^3}\right) \\
&= \ln(x+3) - \ln(x+4)^3 \\
&= \ln(x+3) - 3\ln(x+4) \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{1}{x+3} - 3\cdot\frac{1}{x+4} \\
&= \frac{1}{x+3} - \frac{3}{x+4} \\
\frac{dy}{dx} &= y\left(\frac{1}{x+3} - \frac{3}{x+4}\right)
\end{aligned}$$

2.8 Derivatives of Parametric Functions

If $x = f(t)$ and $y = g(t)$ are differentiable functions of parameter t, then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Example: If $x = 2\sin\theta$ and $y = \cos 2\theta$, find $\frac{d^2y}{dx^2}$

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \\
&= \frac{-2\sin 2\theta}{2\cos\theta} \\
&= -\frac{2\sin\theta\cos\theta}{\cos\theta} \\
&= -2\sin\theta
\end{aligned}
$$

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}} \\
&= \frac{-2\cos\theta}{2\cos\theta} \\
&= -1
\end{aligned}
$$

2.9 Derivatives of Polar Functions

We know that $x = r\cos\theta$ and $y = r\sin\theta$. Then, by using the parametric derivative formula, we can determine the rule for differentiating polar functions.

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \\
&= \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} \\
&= \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}
\end{aligned}
$$

Example: Find the slope of the cardioid $r = 2(1 + \cos\theta)$ at $\theta = \frac{\pi}{6}$

$$
\begin{aligned}
r &= 2(1 + \cos\theta) \\
r' &= -2\sin\theta
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \\
&= \frac{(-2\sin\theta)\sin\theta + 2(1+\cos\theta)(\cos\theta)}{(-2\sin\theta)\cos\theta - 2(1+\cos\theta)(\sin\theta)} \\
\left.\frac{dy}{dx}\right|_{\frac{\pi}{6}} &= -1
\end{aligned}
$$

2.10 Derivatives of Vector-Valued Functions

If a point moves along a curve defined parametrically by $P(t) = \langle x(t), y(t)\rangle$, where t represents time, then the vector from the origin to P is called the **position vector**. Then the derivative of the position vector is called the **velocity vector**, and its derivative is called the **acceleration vector**.

The Velocity Vector:

$$\vec{v}(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$

The slope of $\vec{v}$ is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

The Magnitude of a Velocity Vector:

$$|v| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{v_x^2 + v_y^2}$$

The Acceleration Vector:

$$\overrightarrow{a}(t) = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right\rangle$$

The Magnitude of an Acceleration Vector:

$$|a| = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2} = \sqrt{a_x^2 + a_y^2}$$

2.11 Rolle's Theorem

Rolle's Theorem:
Suppose $f(x)$ is a function that is both continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and that $f(a) = f(b)$. Then there is a number c such that $a < c < b$ and $f'(c) = 0$. Or rather, $f(x)$ has a critical point in (a, b).

2.12 The Mean Value Theorem

The Mean Value Theorem:
Suppose $f(x)$ is a function that is both continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is a number c such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Or,

$$f(b) - f(a) = f'(c)(b - a)$$

Example 1: Determine all the numbers c satisfying $f(x) = x^3 + 2x^2 - x$ on $[-1, 2]$

$$\begin{aligned}
f'(x) &= 3x^2 + 4x - 1 \\
f'(c) &= \frac{f(2) - f(-1)}{2 - (-1)} \\
3c^2 + 4c - 1 &= \frac{14 - 2}{3} \\
3c^2 + 4c - 1 &= 4 \\
3c^2 + 4c - 5 &= 0 \\
c &= \frac{-4 \pm \sqrt{16 - 4(3)(-5)}}{6} \\
c &= 0.7863, -2.1196 \\
c &= 0.7863
\end{aligned}$$

Notice we were able to remove the other c, as it is outside of the desired interval.
Example 2: Suppose we know that $f(x)$ is continuous and differentiable on $[6, 15]$ and that $f(6) = -2$ and $f'(x) \leq 10$. Find the largest possible value for $f(15)$.

The MVT tells us that

$$f(15) - f(6) = f'(c)(15 - 6)$$

Plugging in known quantities and simplifying, we get

$$f(15) = -2 + 9f'(c)$$

Since we are given $f'(x) \leq 10$, we know that $f'(c) \leq 10$. This gives us

$$\begin{aligned}
f(15) &= -2 + 9f'(c) \\
&\leq -2 + (9)10 \\
&\leq 88
\end{aligned}$$

Hence, the largest possible value of $f(15) = 88$.

3 Applications of Differentiation

3.1 Critical Points

$x = c$ is a **critical point** of the function $f'(x)$ if $f(c)$ exists and

$$f'(c) = 0 \quad \text{OR} \quad f'(c) \text{ doesn't exist}$$

Example: Determine all the critical points for the function

$$f(x) = 6x - 4\cos(3x)$$

$$\begin{aligned} f'(x) = 0 &= 6 + 12\sin(3x) \\ \sin(3x) &= -\frac{1}{2} \\ x &= 1.2217 + \frac{2\pi n}{3}, n = 0, \pm 1, \pm 2, \ldots \\ \text{and} \\ x &= 1.9199 + \frac{2\pi n}{3}, n = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

3.2 Tangents to Curves

Recall the point-slope form of a linear function from Algebra. We can replace the slope, m, with $f'(x_1)$ to get an equation of the tangent to the curve $y = f(x)$ at point $P(x_1, y_1)$.

Equation for Tangent to a Curve

$$y - y_1 = f'(x_1)(x - x_1)$$

Example: Find an equation of the tangent to $f(t) = (\cos t, 2\sin^2 t)$ at the point where $t = \frac{\pi}{3}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\ &= \frac{4\sin t\cos t}{-\sin t} \\ &= -4\cos t \end{aligned}$$

At $t = \frac{\pi}{3}$, $x = \frac{1}{2}$, $y = 2\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{2}$, and $\frac{dy}{dx} = -2$. Hence, an equation of the tangent is

$$\begin{aligned} y - \frac{3}{2} &= -2\left(x - \frac{1}{2}\right) \\ 4x + 2y &= 5 \end{aligned}$$

3.3 Increasing and Decreasing Functions

A function $f(x)$ is **increasing** on an interval for which $a < b$, $f(b) \geq f(a)$. A function $f(x)$ is **decreasing** over an interval for which $a < b$, $f(b) \leq f(a)$. Hence, the following identities hold true.

$f'(x) > 0$	Increasing
$f'(x) < 0$	Decreasing

Example: For what values of x is $f(x) = x^4 - 4x^3$ increasing and decreasing, respectively?

$$\begin{aligned} f'(x) &= 4x^3 - 12x^2 \\ &= 4x^2(x - 3) \end{aligned}$$

With critical values at $x = 0, 3$, we analyze the signs of f' in the three intervals as below.

Since the derivative changes sign only at $x = 3$,

$$\begin{array}{ll} \text{if } x < 3 & f'(x) \leq 0 \text{ and } f \text{ is decreasing} \\ \text{If } x > 3 & f'(x) > 0 \text{ and } f \text{ is increasing} \end{array}$$

3.4 Extrema, Concavity, and Inflection Points

The curve of $y = f(x)$ has a **local/relative maximum** at a point where $x = c$ if $f(c) \geq f(x)$ for all x in the immediate neighborhood of c. If a curve has a relative maximum at $x = c$, then the curve changes from increasing to decreasing as x increases through c.

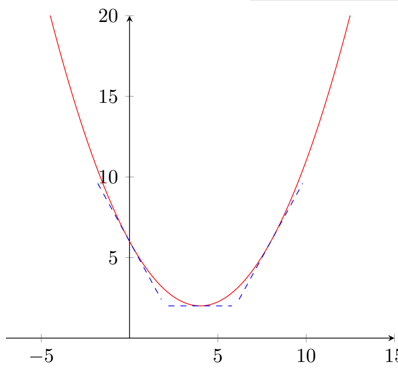
The curve of $y = f(x)$ has a **local/relative minimum** at a point where $x = c$ if $f(c) \leq f(x)$ for all x in the immediate neighborhood of c. If a curve has a relative minimum at $x = c$, then the curve changes from decreasing to increasing as x increases through c.

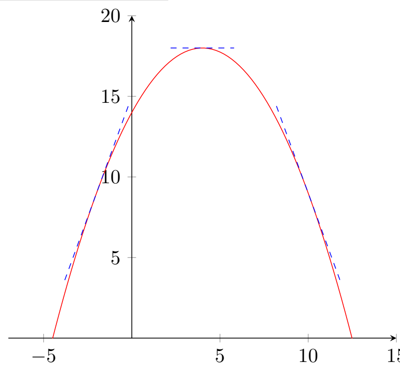
If a function is differentiable on $[a, b]$ and has a relative extremum at $x = c, a < c < b$, then $f'(c) = 0$.

The **global/absolute maximum** of a function on $[a, b]$ occurs at $x = c$ if $f(c) \geq f(x)$ for all x on $[a, b]$. The **global/absolute minimum** of a function on $[a, b]$ occurs at $x = c$ if $f(c) \leq f(x)$ for all x on $[a, b]$.

A curve is **concave upward** on an interval (a, b) if the curve lies above the tangent lines at each point in the interval (a, b). A curve is **concave downward** on an interval (a, b) if the curve lies below the tangent lines at each point in the interval (a, b).

$f''(x) > 0$	Concave Up
$f''(x) < 0$	Concave Down





3.5 Related Rates

Related rates are an application of implicit differentiation that allow us to solve for rates of change and other quantities in an applied real-world system, typically growing with respect to time.

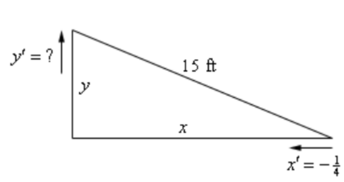
Example 1: Air is being pumped into a spherical balloon at a rate of 5 cm^3/min. Determine the rate at which the radius of the balloon is increasing when the diameter of the balloon is 20 cm.

We are given $\frac{dV}{dt} = 5$ and we want to find $\frac{dr}{dt}_{r=10}$.
Assuming the balloon is perfectly spherical, its volume is given by $V = \frac{4}{3}\pi r^3$. Then

$$\begin{aligned} \frac{dV}{dt} &= 4\pi r^2 \frac{dr}{dt} \\ 5 &= 4\pi(10)^2 \frac{dr}{dt} \\ \frac{dr}{dt}_{r=10} &= \frac{1}{80\pi} \text{cm/min} \end{aligned}$$

Example 2: A 15 foot ladder is resting against the wall. The bottom is initially 10 feet away from the wall and is being pushed towards the wall at the rate of $\frac{1}{4}$ ft/sec. How fast is the top of the ladder moving 12 seconds after we start pushing?

Let us first sketch the situation:



Let the vertical distance between the ground and the ladder be represented by y, the horizontal distance between the wall and the end of the ladder be represented by x, and let the length of the ladder be represented by $c = 15$. Then from the Pythagorean Theorem,

$$
\begin{aligned}
x^2+y^2 &= c^2 = 15 \\
2xx' + 2yy' &= 0 \\
yy' &= -xx' \\
\frac{dy}{dt} &= \frac{-x \cdot \frac{dx}{dt}}{y} \\
\frac{dy}{dt} &= \frac{-7(-\frac{1}{4})}{\sqrt{176}} \\
\frac{dy}{dt} &= \frac{7}{4\sqrt{1176}} \approx 0.1319\text{ft/sec}
\end{aligned}
$$

Notice that we were able to obtain exact values for x and y because $x = 10 + tx' = 10 - \frac{1}{4}(12) = 7$ and

$$
\begin{aligned}
x^2+y^2 &= c^2 \\
49+y^2 &= 225 \\
y &= \sqrt{176}
\end{aligned}
$$

Example 3: Suppose that we have two resistors connected in parallel with resistances R_1 and R_2 measured in Ω. The total resistance R is then given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

Suppose that R_1 is increasing at a rate of 0.4 Ω/min and R_2 is decreasing at a rate of 0.7 Ω/min. At what rate is R changing when $R_1 = 80\Omega$ and $R_2 = 105\Omega$?

When $R_1 = 80\Omega$ and $R_2 = 105\Omega$,

$$
\begin{aligned}
\frac{1}{R} &= \frac{1}{80} + \frac{1}{105} \\
\frac{1}{R} &= \frac{37}{1680} \\
R &= \frac{1680}{37}
\end{aligned}
$$

Then

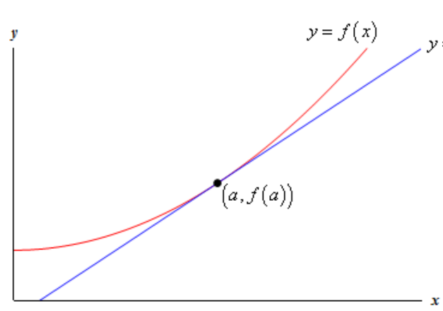
$$
\begin{aligned}
\frac{1}{R} &= \frac{1}{R_1} + \frac{1}{R_2} \\
-\frac{R'}{R^2} &= -\frac{R_1'}{(R_1)^2} - \frac{R_2'}{(R_2)^2} \\
\frac{dR}{dt} &= R^2\left(\frac{R_1'}{(R_1)^2} + \frac{R_2'}{(R_2)^2}\right) \\
\frac{dR}{dt} &= \left(\frac{1680}{37}\right)\left(\frac{0.4}{80^2} - \frac{0.7}{105^2}\right) \\
\frac{dR}{dt} &\approx -0.002045\Omega/\text{min}
\end{aligned}
$$

3.6 Tangent-Line Approximations

Given a function $f(x)$ that is differentiable at $x = a$, we know that the slope of the tangent line to $f(x)$ at $(a, f(a))$ is $f'(a)$. Thus, the tangent line $L(x)$ through $(a, f(a))$ with slope $f'(a)$ has an equation in point-slope form given by

$$L(x) = f(a) + f'(a)(x-a)$$

Below is a graph of the function $y = f(x)$ and its linearization $y = L(x)$.



Example 1: Determine the linear approximation for $f(x) = \sqrt[3]{x}$ at $x = 8$. Use the linear approximation to approximate the value of $\sqrt[3]{8.05}$ and $\sqrt[3]{25}$.

$$\begin{aligned}
L(x) &= f(8) + f'(8)(x-8) \\
&= 2 + \frac{1}{3}8^{-\frac{2}{3}}(x-8) \\
&= \frac{1}{12}x + \frac{4}{3} \\
L(8.05) &\approx 2.0041667 \\
L(25) &\approx 3.4166667
\end{aligned}$$

3.7 The Newton-Raphson Method

The **Newton-Raphson Method** is a way to quickly find an *approximation* iteratively for the root of a real-valued function $f(x) = 0$. It is founded on the concept that a continuous and differentiable function can be approximated by a tangent line to it.

Suppose you want to find the root of a continuous, differentiable function $f(x)$ and you know that the target root is near the point x_0. Then with Newton's Method, an approximation for the root x_1 is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

This method can be repeated as many times as necessary to get the desired accuracy. Then the generalization of Newton's Method is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Below is a visual demonstration of Newton's Method.
Example: Find the root of the equation $x^2 - 4x - 7 = 0$ near $x = 5$.

We are given $x_0 = 5$. Let $f(x) = x^2 - 4x - 7$ and $f'(x) = 2x - 4$. Then,

$$\begin{aligned}
x_1 &= 5 - \frac{5^2 - 4\cdot 5 - 7}{2\cdot 5 - 4} = 5 - \left(\frac{-2}{6}\right) = \frac{16}{3} \approx 5.33333 \\
x_2 &= \frac{16}{3} - \frac{\left(\frac{16}{3}\right)^2 - 4\left(\frac{16}{3}\right) - 7}{2\left(\frac{16}{3}\right) - 4} = \frac{16}{3} - \frac{\frac{1}{9}}{\frac{20}{3}} = \frac{16}{3} - \frac{1}{60} = \frac{319}{60} \approx 5.31667 \\
x_3 &= \frac{319}{60} - \frac{\left(\frac{319}{60}\right)^2 - 4\left(\frac{319}{60}\right) - 7}{2\left(\frac{319}{60}\right) - 4} = \frac{319}{60} - \frac{\frac{1}{3600}}{\frac{398}{60}} \approx 5.31662
\end{aligned}$$

3.8 Optimization

Optimization uses a function's derivatives to find the maximum or minimum value of that function.

Example 1: We need to enclose a rectangular field with a fence. We have 500 feet of fencing material and a building is on one side of the field and so won't need any fencing. Determine the dimensions of the field that will enclose the largest area.

Let us first visualize the situation:
We want to maximize the area of the field and we only have 500 ft of fencing material. So, we represent this system by the following equations:

$$\begin{gathered}
\text{Maximize: } A = xy \\
\text{Constraint: } 500 = x + 2y
\end{gathered}$$

Solving the constraint equation for x, we get $x = 500 - 2y$. We can substitute this into the area function, giving us

$$\begin{aligned}
A(y) &= (500 - 2y)y \\
&= 500y - 2y^2 \\
A'(y) &= 500 - 4y \\
0 &= 500 - 4y \\
y &= 125 \\
x &= 500 - 2(125) = 250
\end{aligned}$$

Thus, the dimensions of the field that wil give the largest area are 250 x 125.

Example 2: A manufacturer needs to make a cylinderical can that will hold 1.5 litres of liquid. Determine the dimensions of the can that will minimize the amount of material used in its constraint.

We will eventually need the radius and height of the can in terms of a linear measurement unit, instead of litres. Thus, the volume, 1.5 litres, becomes 1500 cm^3.

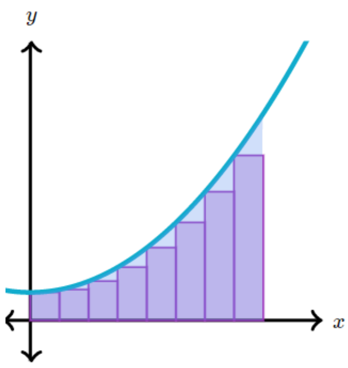
$$\begin{aligned}
\text{Minimize:} A &= 2\pi r h + 2\pi r^2 \\
\text{Constraint:} 1500 &= \pi r^2 h \\
h &= \frac{1500}{\pi r^2} \\
\implies A(r) &= 2\pi r \left(\frac{1500}{\pi r^2} \right) + 2\pi r^2 \\
A(r) &= 2\pi r^2 + \frac{3000}{r} \\
A'(r) &= 4\pi r - \frac{3000}{r^2} \\
A'(r) &= \frac{4\pi r^3 - 3000}{r^2} \\
0 &= \frac{4\pi r^3 - 3000}{r^2} \\
r &= \sqrt[3]{\frac{750}{\pi}} \\
r &= 6.2035 \\
h &= \frac{1500}{\pi (6.2035)^2} \\
h &= 12.4070
\end{aligned}$$

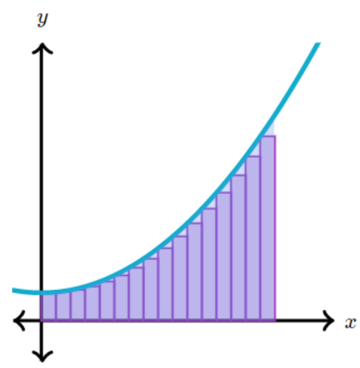
Thus, the dimensions of the can that will minimize the material required to construct the box are a radius of 6.2035 cm and a height of 12.4070 cm.

4 Integration

4.1 Riemann Sums and Definite Integrals

Riemann Sums are approximations of a region's area, obtained by summing the areas of numerous simplified slices of said region. The area approximation gets better if more slices are used to simplify the area. As you can see from the figures below, the figure with more and smaller rectangles takes up more of the area we are trying to approximate.

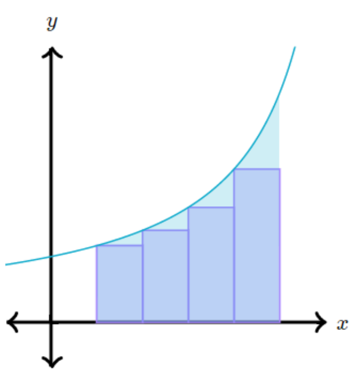


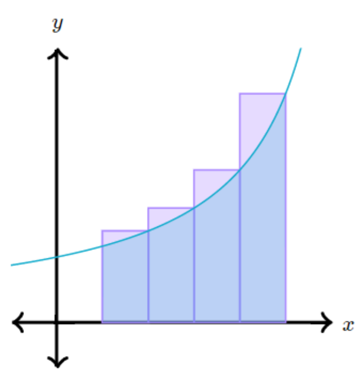


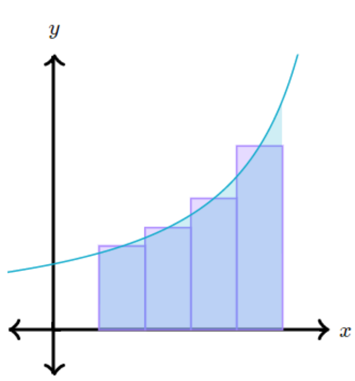
Left Riemann Sums have rectangles that touch the curve with their top-left corners, while **Right Riemann Sums** have rectangles that touch the curve with their top-right corners. **Midpoint Riemann Sums** have rectangles that touch the curve with the middle of their top edges. If the graph is increasing on the interval, then the left sum is an underestimate and the right sum is an overestimate. If the graph is decreasing on the interval, then the left sum is an overestimate and the left sum is an underestimate. Conventionally, we use $i = 0$ for left and midpoint sums and $i = 1$ for right sums.

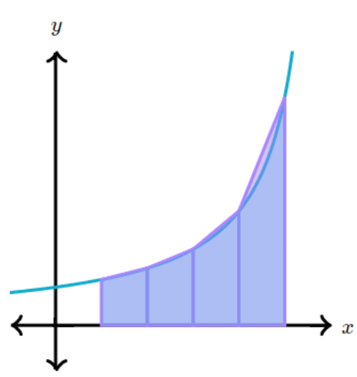
Subdivisions of Riemann Sums can be **uniform**, meaning they are of equal width, or **nonuniform**, meaning they are not of equal length.

Trapezoidal Sums have subdivisions which touch the curve with both of its top vertices. The figures below display left, right, midpoint, and trapezoidal sums.









If $f(x)$ is defined on the closed interval $[a, b]$ and c_k is any point in $[x_{k-1}, x_k]$, then a Riemann Sum is defined as

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

The Riemman Sum of a function is related to the **definite integral** as follows:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i = \int_a^b f(x)dx$$

Example 1: Approximate the area between $f(x) = |x+3|$ and the x-axis on the interval $[-5,5]$ using a left Riemann sum with 10 equal subdivisions.

The width of each rectangle, Δx, is given by

$$\Delta x = \frac{5-(-5)}{10} = 1$$

Since we are using left sums, let $i = 0$. Since we have 10 equal subdivisions, the other indice of our summation will be 9 such that our area is given by $\sum_{i=0}^{9}$. We now want to find $f(x_i)$. Each time i increases by 1, the value of x_i also increases by 1. Its initial value is the left endpoint of the interval, which is -5.

$$x_i = -5 + 1i = -5 + i$$

To confirm that x_i gives us the correct first left endpoint, we substitute 0 for i:

$$x_0 = -5 + 0 = -5$$

Thus,

$$\begin{aligned} f(x) &= |x+3| \\ f(x_i) &= |(-5+i)+3| \\ A &= \sum_{i=0}^{9} |(-5+i)+3| \cdot 1 \\ A &= \sum_{i=0}^{9} |i-2| \end{aligned}$$

Example 2: Approximate the area between $f(x) = \cos(x)$ and the x-axis on the interval $\left[-\frac{3}{10}\pi, \frac{1}{2}\pi\right]$ using a right Riemann sum with 8 equal subdivisions.

$$\begin{aligned} \Delta x &= \frac{\frac{\pi}{2} - \left(-\frac{3\pi}{10}\right)}{8} = \frac{\pi}{10} \\ x_8 &= \frac{\pi}{2} \\ x_i - x_8 &= \frac{\pi}{10}(i-8) \\ x_i &= \frac{\pi i}{10} - \frac{3\pi}{10} \\ f(x) &= \cos(x) \\ f(x_i) &= \cos\left(\frac{\pi i}{10} - \frac{3\pi}{10}\right) \\ A &= \sum_{i=1}^{8} \cos\left(\frac{\pi i}{10} - \frac{3\pi}{10}\right) \cdot \frac{\pi}{10} \end{aligned}$$

Example 3: Write the definite integral $\int_0^3 e^x dx$ as a limit of a left Riemann sum.

$$\begin{aligned} \Delta x &= \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n} \\ \sum_{i=0}^{n-1} f(a+i\Delta x) \cdot \Delta x &= \sum_{i=0}^{n-1} f\left(0 + \frac{3i}{n}\right) \cdot \frac{3}{n} \\ &= \sum_{i=0}^{n-1} e^{\frac{3i}{n}} \cdot \frac{3}{n} \end{aligned}$$

4.2 Fundamental Theorem of Calculus

If $f(x)$ is continous over an interval $[a,b]$ and the function $F(x)$ is defined by

$$F(x) = \int_a^x f(t)dt,$$

then $F'(x) = f(x)$ over $[a,b]$. Additionally,

If $f(x)$ is continuous over the interval $[a,b]$ and $F(x)$ is any antiderivative of $f(x)$ then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Example 1: Let $F(x) = \int_1^{\sqrt{x}} \sin t dt$. Find $F'(x)$ using the FTC.

Let $u(x) = \sqrt{x}$. Then $F(x) = \int_1^{u(x)} \sin t dt$. Thus, by the FTC,

$$\begin{aligned} F'(x) &= \sin(u(x)) \frac{du}{dx} \\ &= \sin(u(x)) \cdot \left(\frac{1}{2} x^{-\frac{1}{2}}\right) \\ &= \frac{\sin\sqrt{x}}{2\sqrt{x}} \end{aligned}$$

Example 2: Let $F(x) = \int_x^{2x} t^3 dt$. Find $F'(x)$.

$$\begin{aligned} F(x) &= \int_x^{2x} t^3 dt \\ &= \int_x^0 t^3 dt + \int_0^{2x} t^3 dt \\ &= -\int_0^x t^3 dt + \int_0^{2x} t^3 dt \\ F'(x) &= \frac{d}{dx}\left[-\int_0^x t^3 dt\right] + \frac{d}{dx}\left[\int_0^2 xt^3 dt\right] \\ &= -x^3 + 16x^3 \\ &= 15x^3 \end{aligned}$$

4.3 Indefinite Integrals

Given a function, $f(x)$, an **antiderivative** of $f(x)$ is any function $F(x)$ such that

$$F'(x) = f(x)$$

If $F(x)$ is any antiderivative of $f(x)$ then the most general antiderivative of $f(x)$ is called an **indefinite integral** and denoted

$$\int f(x)dx = F(x) + k, \text{ where } k \text{ is any constant}$$

Here, $\int$ is the **integral symbol**, $f(x)$ is the **integrand**, x is the integration variable, and k is the constant of integration.

4.4 Basic Integral Rules

$\int_a^a f(x)dx = 0$	Integral with Equal Bounds
$\int_a^b f(x)dx = -\int_b^a f(x)dx$	Opposite of Integral
$\int kdx = kx + C$	Constant
$\int kf(x)dx = k\int f(x)dx$	Constant Multiple
$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$	Sum/Difference
$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$	Reverse Power Rule

If $f(x) \geq 0$ on $[a,b]$, then $\int_a^b f(x)dx \geq 0$
If $f(x) \leq 0$ on $[a,b]$, then $\int_a^b f(x)dx \leq 0$
If $f(x) \geq g(x)$ on $[a,b]$, then $\int_a^b f(x)dx \pm \int_a^b g(x)dx$

4.5 Integrals of Exponential and Logarithmic Functions

$\int \frac{1}{x}dx = \ln\lvert x\rvert + C$	Reciprocal
$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln\lvert ax+b\rvert + C$	
$\int \frac{u'(x)}{u(x)}dx = \ln\lvert u(x)\rvert + C$	
$\int e^x dx = e^x + C$	Exponential
$\int a^x dx = \frac{a^x}{\ln a} + C$	
$\int \ln x dx = x\ln x - x + C$	

4.6 Integrals of the Trig Functions

$\int \sin x dx = -\cos x + C$	Sine
$\int \cos x dx = \sin x + C$	Cosine
$\int \tan x dx = -\ln\|\cos x\| + C = \ln\|\sec x\| + C$	Tangent
$\int \cot x dx = -\ln\|\sin x\| + C = \ln\|\cos x\| + C$	Cotangent
$\int \sec x dx = -\ln\|\sec x + \tan x\| + C =$	Secant
$\int \csc x dx = -\ln\|\csc x + \cot x\| + C$	Cosecant

$\int \sec^2 x dx = \tan x + C$	
$\int \csc^2 x dx = -\cot x + C$	
$\int \sec x \tan x dx = \sec x + C$	
$\int \csc x \cot x dx = -\csc x + C$	
$\int \frac{1}{1-x^2}, x \neq \pm 1 = \arcsin x$	
$\int -\frac{1}{1-x^2}, x \neq \pm 1 = \arccos x$	
$\int \frac{1}{1+x^2} = \arctan x$	
$\int -\frac{1}{1+x^2} = \operatorname{arccot} x$	
$\int \frac{1}{\|x\|\sqrt{x^2-1}}, x \neq \pm 1, 0 = \operatorname{arcsec} x$	
$\int -\frac{1}{\|x\|\sqrt{x^2-1}}, x \neq \pm 1, 0 = \operatorname{arccsc} x$	

4.7 Integration Using Substitution

To use integration by substitution, we must first be able to write our integral in the form $\int f(g(x))g'(x)dx = \int f(u)du$, where $u = g(x)$.

Example 1 Compute $\int \cos(x^2)2xdx$

$$\begin{aligned} u &= x^2 \\ du &= 2xdx \\ \int \cos(u)du &= \sin(u) + C \\ &= \sin(x^2) + C \end{aligned}$$

Example 2: Compute $\int x\sqrt{3x^2-1}dx$

$$\begin{aligned} u &= 3x^2 - 1 \\ du &= 6x \\ \int \frac{1}{6}(6x)\sqrt{3x^2-1}dx &= \frac{1}{6}\int \sqrt{u}du \\ &= \frac{1}{9}u^{\frac{3}{2}} + C \\ &= \frac{1}{9}(3x^2-1)^{\frac{3}{2}} + C \end{aligned}$$

Example 3: Compute $\int \sin^2\theta\cos^2\theta d\theta$

$$\begin{aligned} \sin^2\theta\cos^2\theta &= (\sin\theta\cos\theta)^2 \\ &= \left(\frac{1}{2}\sin(2\theta)\right)^2 \\ &= \frac{1}{4}\sin^2(2\theta) \\ \int \sin^2\theta\cos^2\theta d\theta &= \frac{1}{4}\int \sin^2 2\theta d\theta \\ \because \cos 2x &= 1 - 2\sin^2 x \\ \therefore \sin^2 x &= \frac{1}{2}(1-\cos 2x) \\ \implies \int \sin^2\theta\cos^2\theta d\theta &= \frac{1}{8}\int(1-\cos 4\theta)d\theta \\ &= \frac{1}{8}\left(\theta - \frac{1}{4}\sin 4\theta\right) + C \\ &= \frac{\theta}{8} - \frac{\sin 4\theta}{32} + C \end{aligned}$$

4.8 Integrating Using Long Division and Completing the Square

When we have a rational function of the form $\frac{P(x)}{Q(x)}$, we can use long division and completing the squre to simplify the integral prior to integration.

Example 1: Compute $\int \frac{x^3+2x^2+9x-17}{x+4}dx$

$$\begin{aligned} \frac{x^3+2x^2+9x-17}{x+4} &= x^2-2x+17-\frac{85}{x+4} \\ \int \frac{x^3+2x^2+9x-17}{x+4}dx &= \int x^2-2x+17-\frac{85}{x+4}dx \\ &= \frac{x^3}{3}-x^2+17x-85\ln|x+4|+C \end{aligned}$$

Example 2: Compute $\int \frac{dx}{\sqrt{1-2x-x^2}}$

$$\begin{aligned} 1-2x-x^2 &= 1-(x^2+2x) \\ &= 2-(x^2+2x+1) \\ &= 2-(x+1)^2 \\ &= (\sqrt{2})^2-(x+1)^2 \\ u &= x+1 \\ du &= dx \\ \int \frac{dx}{\sqrt{1-2x-x^2}} &= \int \frac{dx}{\sqrt{(\sqrt{2})^2-(x+1)^2}} \\ &= \int \frac{du}{\sqrt{(\sqrt{2})^2-u^2}} \\ &= \arcsin\left(\frac{u}{\sqrt{2}}\right)+C \\ &= \arcsin\left(\frac{x+1}{\sqrt{2}}\right)+C \end{aligned}$$

4.9 Integration by Parts

Recall the differentiation product rule, $(uv)' = uv' + u'v$. Rearranging the equation, we get $uv' = (uv)' - u'v$. Hence,

$$\begin{aligned} \int uv'dx &= \int ((uv)'-u'v)dx \\ &= uv - \int u'vdx \\ du &= u'dx \\ dv &= v'dx \end{aligned}$$

Integration by Parts:

$$\int udv = uv - \int vdu$$

Guidelines for Choosing u and dv:
"LIATE" (Choose u in the following order):
L: Logarithmic Functions
I: Inverse Trig Functions
A: Algebraic Functions
T: Trig Functions
E: Exponential Functions

We want u to be an expression whose derivative du is a simpler function than u itself. We want dv to be the most complicated part of the integrand that can be easily integrated.

Example 1: Compute $\int x^3 \ln x dx$

$$
\begin{align}
u &= \ln x \tag{4}\\
dv &= x^3 dx \tag{5}\\
du &= \frac{dx}{x} \tag{6}\\
v &= \int x^3 dx = \frac{x^4}{4} \tag{7}\\
\int x^3 \ln x dx &= uv - \int v du \tag{8}\\
&= (\ln x)\frac{x^4}{4} - \int \frac{x^4}{4}\frac{1}{x} dx \tag{9}\\
&= \frac{x^4 \ln x}{4} - \frac{1}{4}\int x^3 dx \tag{10}\\
&= \frac{x^4 \ln x}{4} - \frac{x^4}{16} + C \tag{11}
\end{align}
$$

Example 2: Compute $\int x^3\sqrt{4-x^2}dx$

$$
\begin{aligned}
dv &= x\sqrt{4-x^2}dx\\
u &= x^2\\
du &= 2xdx\\
v &= \int x\sqrt{4-x^2}dx = -\frac{1}{3}(4-x^2)^{\frac{3}{2}}\\
\int x^3\sqrt{4-x^2}dx &= uv - \int vdu\\
&= x^2\left(-\frac{1}{3}(4-x^2)^{\frac{3}{2}}\right) - \int -\frac{1}{3}(4-x^2)^{\frac{3}{2}}(2x)dx\\
&= -\frac{x^2}{3}(4-x^2)^{\frac{3}{2}} - \frac{2}{15}(4-x^2)^{\frac{5}{2}} + C
\end{aligned}
$$

4.10 Tabular Integration

Tabular Integration is a shortcut for performing repeated integration by parts. We want to create a table like so, until one of the derivatives/integrals reaches 0:

	u	dv
1	$f(x)$	$g(x)$
2	$f'(x)$	$\int g(x)$
3	$f''(x)$	$\int\int g(x)$
n	$f^{(n)}(x)$	$\int\cdots\int g(x)$, where there is one $\int$ for each n

Then, we multiply $f(x)$ by $\int g(x)$, $f'(x)$ by $\int\int g(x)$, and so on until we reach the last antiderivative. We then alternatively add and subtract these products, starting with addition.

Example 1: Compute $\int(x^3+2x-1)\cos(4x)$

Let $f(x) = x^3+2x-1$ and $g(x) = \cos(4x)$. Then we can make a table like so:

u		dv
x^3+2x-1	$+$	$\cos(4x)$
$3x^2+2$	$-$	$\frac{1}{4}\sin(4x)$
$6x$	$+$	$\frac{-1}{16}\cos(4x)$
6	$-$	$\frac{-1}{64}\sin(4x)$
0	$+$	$\frac{1}{256}\cos(4x)$

Thus, the antiderivative becomes

$$\begin{aligned}
&\frac{1}{4}(x^3+2x-1)\sin(4x)+\frac{1}{16}(3x^2+2)\cos(4x)-\frac{3x}{32}\sin(4x)-\frac{3}{128}\cos(4x)\\
&=\sin(4x)\left(\frac{x^3}{4}+\frac{13x}{32}-\frac{1}{4}\right)+\cos(4x)\left(\frac{3x^2}{16}+\frac{13}{128}\right)+C
\end{aligned}$$

4.11 Partial Fraction Decomposition

Example 1: $\int \frac{3x+11}{x^2-x-6}dx$

$$\begin{aligned}
\frac{3x+11}{(x-3)(x+2)} &= \frac{A}{x-3}+\frac{B}{x+2}\\
&= \frac{A(x+2)+B(x-3)}{(x-3)(x+2)}\\
3x+11 &= A(x+2)+B(x-3)\\
x=-2 &\implies B=-1\\
x=3 &\implies A=4\\
\int \frac{3x+11}{x^2-x-6}dx &= \int \frac{4}{x-3}dx-\frac{1}{x+2}dx\\
&= 4\ln|x-3|-\ln|x+2|+C
\end{aligned}$$

Example 2: Compute $\int \frac{x^3+10x^2+3x+36}{(x-1)(x^2+4)}dx$

$$\begin{aligned}
\frac{x^3+10x^2+3x+36}{(x-1)(x^2+4)} &= \frac{A}{x-1}+\frac{Bx+C}{x^2+4}+\frac{Dx+E}{(x^2+4)^2}\\
x^3+10x^2+3x+36 &= A(x^2+4)^2+(Bx+C)(x-1)(x^2+4)+(Dx+E)(x-1)\\
&= x^4(A-B)+x^3(C-B)+x^2(8A+4B-C+D)+x(-4B+4C-D+E)+16A-4C-E
\end{aligned}$$

$$\begin{aligned}
x^4 &: A+B=0\\
x^3 &: C-B=1\\
x^2 &: 8A+4B-C+D=10\\
x^1 &: -4B+4C-D+E=3\\
x^0 &: 16A-4C-E=36
\end{aligned}$$

$\implies A=2, B=-2, C=-1, D=1, E=0$

$$\begin{aligned}
\int \frac{x^3+10x^2+3x+36}{(x-1)(x^2+4)}dx &= \int \frac{2}{x-1}-\frac{2x}{x^2+4}-\frac{1}{x^2+4}+\frac{x}{(x^2+4)^2}dx\\
&= 2\ln|x-1|-\ln|x^2+4|-\frac{1}{2}\arctan\left(\frac{x}{2}\right)-\frac{1}{2(x^2+4)}+C
\end{aligned}$$

4.12 Improper Integrals

The 3 Common Methods to Compute Improper Integrals

1. If $\int_a^t f(x)dx$ exists for every $t>a$ then,

$$\int_a^\infty f(x)dx=\lim_{t\to\infty}\int_a^t f(x)dx$$

as long as the limit exists and is finite.

2. If $\int_t^b f(x)dx$ exists for every $t<b$ then,

$$\int_{-\infty}^b f(x)dx=\lim_{t\to-\infty}\int_t^b f(x)dx$$

as long as the limit exists and is finite.

3. if $\int_{-\infty}^c f(x)dx$ and $\int_c^\infty f(x)dx$ are both convergent then,

$$\int_{-\infty}^\infty f(x)dx=\int_{-\infty}^c f(x)dx+\int_c^\infty f(x)dx$$

where c is any constant. If either of the two integrals are divergent then so is this integral.

Example 1: Determine if the integral $\int_{-\infty}^{0} \frac{1}{\sqrt{3-x}} dx$ is convergent or divergent and find its value if it is convergent.

$$
\begin{aligned}
\int_{-\infty}^{0} \frac{dx}{\sqrt{3-x}} &= \lim_{t\to-\infty} \int_{t}^{0} \frac{dx}{\sqrt{3-x}} \\
&= \lim_{t\to-\infty} -2\sqrt{3-x}\Big|_{t}^{0} \\
&= \lim_{t\to-\infty} (-2\sqrt{3} + 2\sqrt{3-t}) \\
&= -2\sqrt{3} + \infty \\
&= \infty
\end{aligned}
$$

Since the limit is infinite, this integral is divergent.

Example 2: Determine if the integral $\int_{-\infty}^{\infty} xe^{-x^2} dx$ is convergent or divergent and find its value if it is convergent.

$$
\begin{aligned}
\int_{-\infty}^{\infty} xe^{-x^2} dx &= \overset{0}{\underset{-\infty}{\lim}} xe^{-x^2} dx + \int_{0}^{\infty} xe^{-x^2} dx \\
&= \lim_{t\to-\infty} \int_{t}^{0} xe^{-x^2} dx + \lim_{t\to\infty} \int_{0}^{t} xe^{-x^2} dx \\
&= -\frac{1}{2} + \frac{1}{2} \\
&= 0
\end{aligned}
$$

4.13 Integration Strategy

1. See if you can simplify the integrand
2. See if a simple substitution will work
3. Identify the type of integrand:
•If the integrand is a rational expression, partial fractions may work
•If the integrand is a polynomial times a trig, exponential, or logarithmic function, then try integration by parts
•If the integrand is a product of trig functions then try rewriting the integrand in terms of other trig function or try integration by parts or substitution
•Look for trig substitutions if the integrand contains some form of $\sqrt{b^2x^2 \pm a^2}$ or $\sqrt{a^2 - b^2x^2}$
•If the integrand contains a quadratic then try completing the square
4. Remember that you can use multiple techniques on the same integral.
5. If it doesn't work then try another method.

5 Applications of Integration

5.1 Average Value of a Function and Mean Value Theorem

The **average value** of a function $f(x)$ over the interval $[a, b]$ is given by

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$$

Mean Value Theorem:
if $f(x)$ is a continuous function on $[a, b]$, then there is a number c such that

$$\int_a^b f(x)dx = f(c)(b-a)$$

Example 1: Find the average value of the function $f(x) = \sin(2x)e^{1-\cos(2x)}$ *on* $[-\pi, \pi]$.

$$\begin{aligned} u &= 1-\cos(2x) \\ f(x)_{avg} &= \frac{1}{\pi-(-\pi)}\int_{-\pi}^{\pi}\sin(2x)e^{1-\cos(2x)}dx \\ &= \left.\frac{e^{1-\cos 2x}}{4\pi}\right|_{-\pi}^{\pi} \\ &= 0 \end{aligned}$$

Example 2: Find the number c that satisfies the Mean Value Theorem for the function $f(x) = x^2 + 3x + 2$ *on* $[1, 4]$

Notice that the function is a polynomial, hence it is continuous on the given interval and we can use the Mean Value Theorem.

$$\int_1^4 x^2 + 3x + 2dx = (c^2 + 3c + 2)(4-1)$$

$$\begin{aligned} \left(\frac{x^3}{3} + \frac{3x^2}{2} + 2x\right)4_1 &= 3(c^2 + 3c + 2) \\ \frac{99}{2} &= 3c^2 + 9c + 6 \\ 0 &= 3c^2 + 9c - \frac{87}{2} \\ c &= 2.593, -5.593 \\ \because c \in [1, 4] \therefore c &= 2.593 \end{aligned}$$

5.2 Motion

Let the function $p(t)$ represent the position of a moving particle, with respect to time. Let the functions $v(t)$ and $a(t)$ represent the velocity and acceleration of the moving particle modeled by $p(t)$, respectively. Then the following equalities hold true:

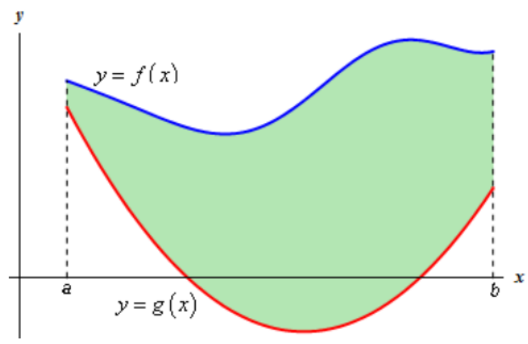
$$\int a(t) = v(t) + C$$
$$\int v(t) = p(t) + C$$

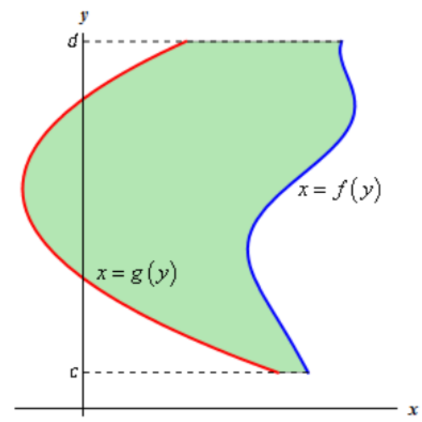
5.3 Areas between Curves

Area Between Two Curves

$$A = \int_a^b (f(x) - g(x))dx, a \leq x \leq b$$
$$A = \int_c^d (f(y) - g(y))dy, c \leq y \leq d$$

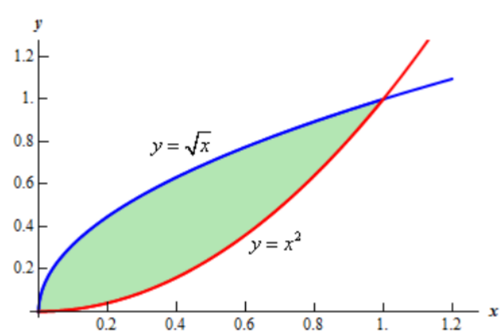
where $f(x)$ is the upper function, $g(x)$ is the lower function, $f(y)$ is the right function, and $g(y)$ is the left function.





Example 1: Find the area of the region enclosed by $y = x^2$ and $y = \sqrt{x}$

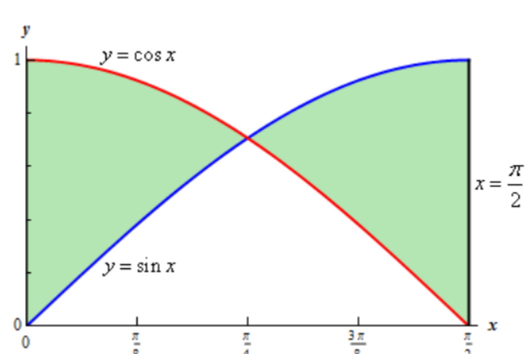
Below is a graph for visual reference.



$$A = \int_0^1 (\sqrt{x} - x^2)dx$$
$$= \left(\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right)\Bigg|_0^1$$
$$= \frac{1}{3}$$

Example 2: Determine the area of the region bounded by $y = \sin x, y = \cos x, x = \frac{\pi}{2}$, and the y-axis.

Below is a graph for reference.



The intersection point is given by $x = \sin x = \cos x = \frac{\pi}{4}$. Thus,

$$
\begin{aligned}
A &= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x)dx \\
&= (\sin x + \cos x)\Big|_0^{\frac{\pi}{4}} + (-\cos x - \sin x)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
&= 2\sqrt{2} - 2 \\
&= 0.828427
\end{aligned}
$$

5.4 Volumes with Cross Sections

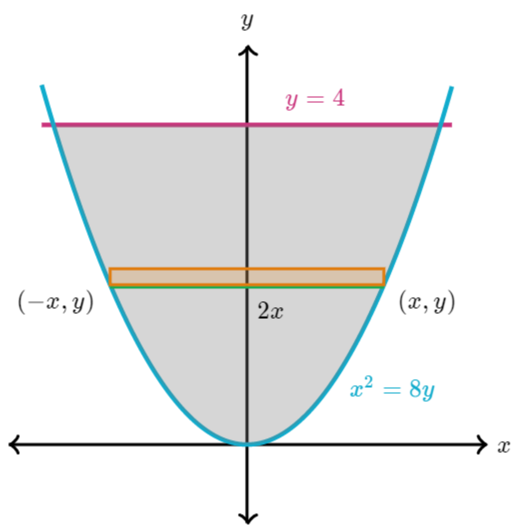
Volumes with Cross Sections

$$V = \int_a^b A(y)dy$$

To determine the function $A(y)$ we must first express A in terms of x.

Example: The base of a solid S is the region bounded by the parabola $x^2 = 8y$ and the line $y = 4$. Use semi-circle cross-sections perpendicular to the y-axis to find the exact volume of solid S.

Let us first graph the base of the solid. The orange rectangle represents a single cross-section sitting on the base. The length of the orange rectangle is $2x$.



Because the semi-circle cross-section rests on the orange rectangle above, the diameter of the semi-circle is $2x$ and the radius is x. Thus, the area, A, of the semi-circle is given by

$$A = \frac{\pi}{2}x^2$$

Since the rectangle lies on the curve $x^2 = 8y$, we can write the area as

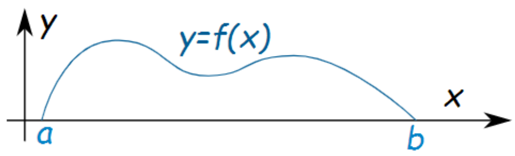
$$A(y) = \frac{\pi}{2}(8y) = 4\pi y$$

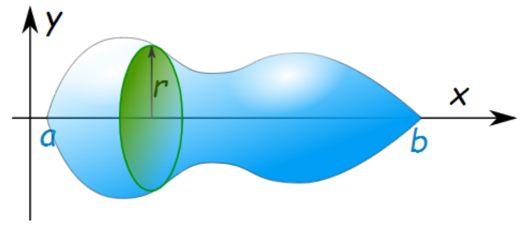
Then the volume, V, is

$$
\begin{aligned}
V &= \int_a^b A(y)dy \\
&= 4\pi \int_0^4 ydy \\
&= 4\pi \left[\frac{1}{2}y^2\right]_0^4 \\
&= 32\pi
\end{aligned}
$$

5.5 Volumes with Disk Method

Say we have a function $y = f(x)$ like so:



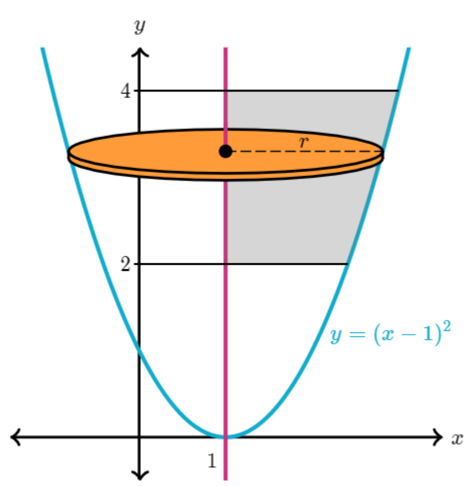


We can find the solid of revolution by adding up the infinitesimal discs (circles) on the interval. Notice that the volume of each disc is the area of its face multiplied by dx, the thickness of the disc. We know that the area of a circle is given by $A = \pi r^2$. Since the radius r is the value of the function $f(x)$ at x, $A = \pi f(x)^2$. Then the volume of the solid of revolution is found by summing all the disks. Hence,

$$V = \pi \int_a^b f(x)^2 dx$$

Example: Let a region be enclosed by the line $x = 1$, the line $y = 2$, the line $y = 4$, and the curve $y = (x-1)^2$. A solid, S, is generated by rotating R about the line $x = 1$. Find the volume of the solid.

Below is a graph depicting R, the shaded region, its bounds, and a single disk of the solid generated by revolving R about $x = 1$.



Let us first find the radius of the discs, given by the distance between the curve $y = (x-1)^2$ and the line $x = 1$. Rearranging the equation of the curve, we get $x = \sqrt{y} + 1$. Then the radius, $r(y)$, is given by

$$\begin{aligned} r(y) &= (\sqrt{y} + 1) - (1) \\ &= \sqrt{y} \end{aligned}$$

And the area of the disk's faces is given by

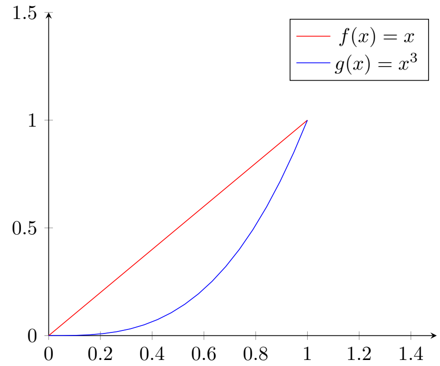
$$\begin{aligned} \pi[r(y)]^2 &= \pi(\sqrt{y})^2 \\ &= \pi y \end{aligned}$$

Since the solid is bounded by the lines $y = 2$ and $y = 4$, the interval of integration is $[2, 4]$. Then the volume of solid S is given by

$$\begin{aligned} A &= \pi \int_2^4 y dy \\ &= 6\pi \end{aligned}$$

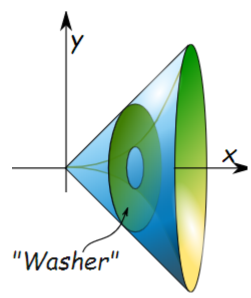
5.6 Volumes with Washer Method

A washer is a disk with a hole in its center. Say we have an outside function, $f(x)$, and an inside function, $g(x)$, like below.



If we revolve $f(x)$ and $g(x)$ about the x-axis, we get a solid that looks like the one below. We can then imagine that the solid's volume is composed of infinitesimal washers that have radii of the outside function's radius minus the inside function's radius such that the washer's radius is $r(x) = f(x) - g(x)$ or $r(y) = f(y) - g(y)$. The area of each washer's face is given by

$$A = \pi((\text{outer radius})^2 - (\text{inner radius})^2)$$

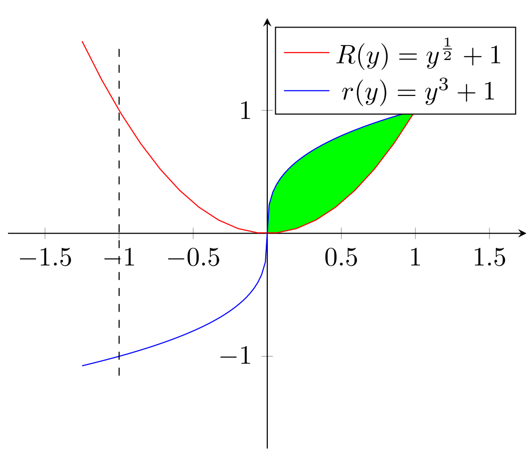


Then we can find the volume of the solid of revolution by adding up the volumes of the disks found by multiplying their face areas by their widths, dx. The volume of the solid is then given by

$$V = \pi \int_a^b [f(x)^2 - g(x)^2]dx$$

Example: Find the volume of the solid generated by revolving the region bounded by the curves $y = x^2$ *and* $x = y^3$ *about the line* $x = -1$

Below is a graph of the outside function $f(x)$ and the inner function $g(x)$ for reference. The green region depicts the face area of a washer. We make the curve that is further away from the axis of revolution the outside function, and the curve closest to the axis of revolution the inner function. Note that the outside function, $f(x) = x^2$ has been rearranged to be $f(y) = y^{\frac{1}{2}}$. We then subtract "-1" from $f(y)$ and $g(y)$ to express the distance from the line of revolution, $x = -1$. Then the outside $(R(y))$ and inside $(r(y))$ radii are given by $R(y) = y^{\frac{1}{2}} + 1$ and $r(y) = y^3 + 1$.



Then the volume of the solid of revolution is

$$\begin{aligned} V &= \pi \int_a^b [(R(y))^2 - (r(y))^2]dy \\ &= \pi \int_0^1 [(y^{\frac{1}{2}} + 1)^2 - (y^3 + 1)^2]dy \\ &= \frac{25\pi}{21} \end{aligned}$$

5.7 Arc Length

The General Arc Length Formulae:

The arc length, L, of a function on a continuous interval is given by

$$L = \int_a^b ds$$

where

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \text{ if } y = f(x), a \leq x \leq b$$
$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \text{ if } x = g(y), a \leq y \leq b$$

Arc Length Formula for Parametric Equations

For two functions $x = f(t)$ and $y = g(t), \alpha \leq t \leq$, the arc length of the parameterized curve is given by

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2} dt$$

Arc Length Formula for Vector-Valued Functions

For two vector-valued functions, $x = f(t), y = g(t)$, the arc length of the curve they form is given by

$$L = \int_a^b \|\overrightarrow{r}'(t)\| dt$$

where $\|\overrightarrow{r}'(t)\|$ is the magnitude of the tangent vector such that

$$\|\overrightarrow{r}'(t)\| = \sqrt{[f'(t)]^2 + [g'(t)]^2}$$

6 Differential Equations

6.1 Differential Equation Basics

Differentials: Given a function $y = f(x)$, dy and dx are differentials and they are related by the following equality.

$$dy = f'(x)dx$$

A **Differential Equation** is any equation that contains derivatives. If it contains ordinary derivatives, the equation is called an **Ordinary Differential Equation (ODE)** . If the equation contains partial derivatives, the equation is called a **Partial Differential Equation (PDE).** . The **order** of a differential equation is the largest derivative in the DE. A **solution** to a DE on an interval $\alpha < t < \beta$ is any function $y(t)$ that satisfies the DE on the given interval. Note that solutions are often accompanied by intervals.

Example 1: Show that $y(x) = x^{-\frac{3}{2}}$ is a solution to $4x^2y" + 12xy' + 3y = 0$ for $x > 0$.

$$\begin{aligned}
y'(x) &= -\frac{3}{2}x^{-\frac{5}{2}} \\
y"(x) &= \frac{15}{4}x^{-\frac{7}{2}} \\
0 &= 4x^2\left(\frac{15}{4}x^{-\frac{7}{2}}\right) + 12x\left(-\frac{3}{2}x^{-\frac{5}{2}}\right) + 3\left(x^{-\frac{3}{2}}\right) \\
0 &= 15x^{-\frac{3}{2}} - 18x^{-\frac{3}{2}} + 3x^{-\frac{3}{2}} \\
0 &= 0
\end{aligned}$$

Hence, $y(x) = x^{-\frac{3}{2}}$ is a solution with condition $x > 0$. We can see the need for this condition in $y(x) = x^{-}\frac{3}{2} = \frac{1}{\sqrt{x^3}}$, where x must be greater than 0 in order to get a real-valued solution and avoid division by zero.

Initial Condition(s)
Initial conditions are conditions imposed on solutions to DEs that allow us to determine which solution we are targeting. They are of the form:

$$y(t_0) = y_0 \text{ and/or } y^{(k)}(t_0) = y_k$$

In other words, initial conditions are values of the solution and/or its derivative(s) at specific points. The number of initial conditions required for a given DE depends on the order of the DE.

The **Interval of Validity** is the largest possible interval on which the solution is valid and contains t_0.

The **General Solution** to a DE is the generalized form of the solution, not considering any initial conditions. The **Particular Solution** to a DE is the solution that not only satisfies the DE, but also satisfies the given initial condition(s).

An **Explicit Solution** is any solution given in the form $y = y(t)$. An **Implicit Solution** is any solution that is not in explicit form.

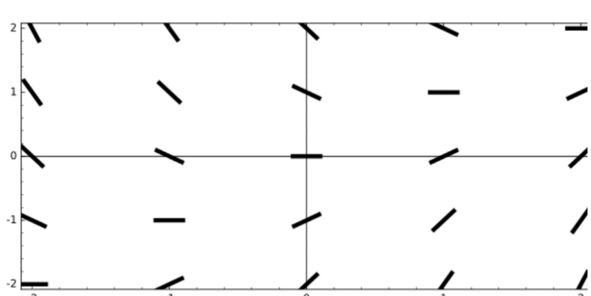
6.2 Slope Fields

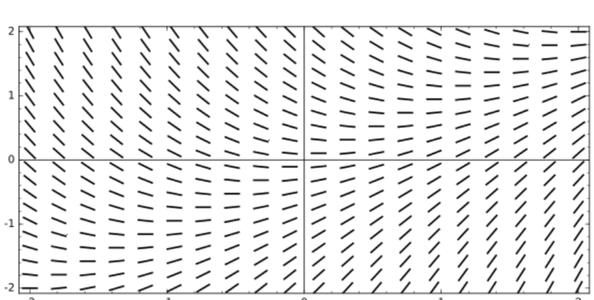
A **Slope Field** is a visual representation of a DE. At each point on the slope field, there is a small line segment whose slope equals the value of $f(x, y)$ at that point.

Example: Build a slope field for the DE $\frac{dy}{dx} = x - y$.

$x - y$	$x = -2$	$x = -1$	$x = 0$	$x = 1$	$x = 2$
$y = 2$	$-2 - 2 = -4$	$-1 - 2 = -3$	$0 - 2 = -2$	$1 - 2 = -1$	$2 - 2 = 0$
$y = 1$	$-2 - 1 = -3$	$-1 - 1 = -2$	$0 - 1 = -1$	$1 - 1 = 0$	$2 - 1 = 1$
$y = 0$	$-2 - 0 = -2$	$-1 - 0 = -1$	$0 - 0 = 0$	$1 - 0 = 1$	$2 - 0 = 2$
$y = -1$	$-2 - (-1) = -1$	$-1 - (-1) = 0$	$0 - (-1) = 1$	$1 - (-1) = 2$	$2 - (-1) = 3$
$y = -2$	$-2 - (-2) = 0$	$-1 - (-2) = 1$	$0 - (-2) = 2$	$1 - (-2) = 3$	$2 - (-2) = 4$

Below, the left figure depicts our slope field by drawing out the slopes from the table. The right figure depicts the slope field made by a graphing calculator. Notice how it has many more sample points than we do.





6.3 Euler's Method

Euler's Method is used for approximating solutions to DEs and it works by approximating the solution curve with line segments. For a DE of the form $y' = f(x)$ where $y(x_0) = y_0$, we can use a sequence of points (x_n, y_n) satisfying $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n)$, where h is the step size, so that one of the x_n will be the y-value to be estimated.

Example 1: Consider a function $f(x)$ such that $f(2) = 10$ and $f'(x) = x^2 + 3x$. Using Euler's method with a step size of 1, find the resulting approximation of $f(5)$.

We have $(x_0, y_0) = (2, 10)$. Using Euler's method with $h = 1$, we compute

$$
\begin{aligned}
(x_1, y_1) &= (2 + 1, 10 + 1 \cdot f'(2)) = (3, 20) \\
(x_2, y_2) &= (3 + 1, 20 + 1'(3)) = (4, 38) \\
(x_3, y_3) &= (4 + 1, 38 + 1'(4)) = (5, 78)
\end{aligned}
$$

Hence, $f(5) \approx 78$.

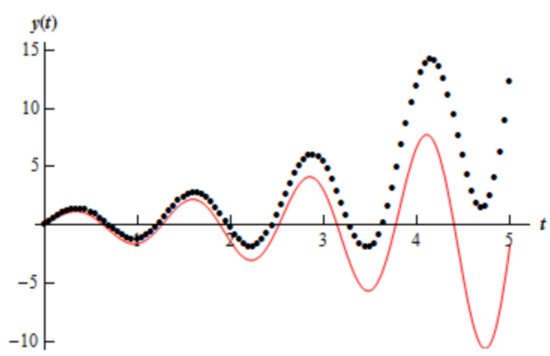
Example 2: For the DE $y' - y = -\frac{1}{2}e^{\frac{t}{2}} \sin(5t) + 5e^{\frac{t}{2}} \cos(5t)$ with initial condition $y(0) = 0$, use Euler's Method to find the approximation to the solution at $t = 1, 2, 3, 4, 5$. Use step sizes of 0.1,0.05,0.01,0.005, and 0.001 for the approximations.

The solution to this linear first order DE is

$$y(t) = e^{\frac{t}{2}} \sin(5t)$$

t	**Exact**	$h = 0.1$	$h = 0.05$	$h = 0.01$	$h = 0.005$	$h = 0.001$
$t = 1$	-1.58100	-0.97167	-1.26512	-1.51580	-1.54826	-1.57443
$t = 2$	-1.47880	0.65270	-0.34327	-1.23907	-1.35810	-1.45453
$t = 3$	2.91439	7.30209	5.34682	3.44488	3.18259	2.96851
$t = 4$	6.74580	15.56128	11.84839	7.89808	7.33093	6.86429
$t = 5$	-1.61237	21.95465	12.24018	1.56056	0.0018864	-1.28498

Below is a graph of the solution, the red line, and the approximations for $h = 5$. Note how the approximations become less accurate as t increases.



6.4 Separable Differential Equations

A **Separable DE** is any DE that can be written in the form

$$g(y)\frac{dy}{dx} = f(x)$$

We can rearrange the equation and integrate both sides to get

$$\int g(y)dy = \int f(x)dx$$

And solving this for y gives us our solution.

Example 1: Solve the equation $\frac{dy}{dx} = 8x^3y - 8xy$.

$$\begin{aligned}
\frac{dy}{dx} &= 8x^3y - 8xy \\
\frac{dy}{dx} &= y(8x^3 - 8x) \\
\frac{dy}{y} &= (8x^3 - 8x)dx \\
\int \frac{dy}{y} &= \int (8x^3 - 8x)dx \\
\ln|y| &= 2x^4 - 4x^2 + C \\
e^{\ln|y|} &= e^{2x^4-4x^2+C} \\
|y| &= e^{2x^4-4x^2}e^C \\
y &= Ce^{2x^4-4x^2}
\end{aligned}$$

Example 2: Solve the DE $\frac{dy}{dt} = e^{y-t}\sec(y)(1+t^2)$ *with initial condition* $y(0) = 0$.

$$\begin{aligned}
\frac{dy}{dt} &= \frac{e^y e^{-t}}{\cos(y)}(1+t^2) \\
e^{-y}\cos(y)dy &= e^{-t}(1+t^2)dt \\
\int e^{-y}\cos(y)dy &= \int e^{-t}(1+t^2)dt \\
\frac{e^{-y}}{2}(\sin(y) - \cos(y)) &= -e^{-t}(t^2+2t+3) + C \\
\frac{1}{2}(-1) &= -(3) + c \\
c &= \frac{5}{2} \\
\implies \frac{e^{-y}}{2}(\sin(y) - \cos(y) &= -e^{-t}(t^2+2t+3) + \frac{5}{2}
\end{aligned}$$

6.5 Exponential and Logistic Models

A **Exponential DE** is an ODE of the form

$$\frac{df}{dt} = kf$$

where k is any constant. The solution to this DE describing an exponential model is then

$$f(t) = Ce^{kt}$$

A **Logistic DE** is an ODE of the form

$$\frac{df}{dt} = kf\left(1 - \frac{f}{L}\right)$$

where k is the constant of proportionality and L is the constant **Limiting (Carrying) Capacity.**

Example 1: The balance of a certain loan increases at a rate proportional at any time to the balance at that time. Initially, the loan balance is 1600*. It is* 1920 *after one year. What is the balance of the loan after 90 days?*

Let $B(t)$ model the balance of the loan after t days. We are given that the rate of change of B is proportional to B, so

$$\frac{dB}{dt} = kB$$

whose solution is given by $B(t) = C \cdot e^{kt}$. As the initial balance was \$1600, we know that $C = 1600$. We are given that the loan balance was \$1920 after 365 days. Then

$$\begin{aligned}
k &= \frac{\ln(1.2)}{365} \\
\implies B(t) &= 1600e^{\frac{\ln(1.2)t}{365}} \\
B(90) &= 1600e^{\frac{90\ln(1.2)}{365}} \\
B(90) &\approx \$1673.57
\end{aligned}$$

Example 2: The population $P(t)$ of mice in a meadow after t years satisfies the logistic DE $\frac{dP}{dt} = 3P \cdot \left(1 - \frac{P}{2500}\right)$ where the initial population is 1000 mice. What is the population when it is growing the fastest?

Notice that $\frac{dP}{dt}$ as a function of P is quadratic, hence the maximum value of $\frac{dP}{dt}$ is obtained when the population is equal to the P-value of the vertex of the parabola. We can find the P-value of the vertex by finding the roots and taking their average.

$$\begin{aligned}
\frac{dP}{dt} &= 0 \\
3P \cdot \left(1 - \frac{P}{2500}\right) &= 0 \\
3P = 0, 1 - \frac{P}{2500} &= 0 \\
P &= 0, 2500 \\
P_{avg} &= \frac{0 + 2500}{2} \\
P_{avg} &= 1250
\end{aligned}$$

Hence, the population when the model is growing the fastest is 1250 mice.

7 Infinite Sequences and Series

7.1 Sequences and Series Basics

7.2 Arithmetic and Geometric Series

7.3 Binomial Series

7.4 The nth Term Test for Divergence

7.5 Integral Test for Convergence

7.6 Harmonic and p-Series

7.7 Comparison Tests for Convergence

7.8 Alternating Series Tests for Convergence

7.9 Ratio Test for Convergence

7.10 Absolute and Conditional Convergence

7.11 Alternating Series Error Bound

7.12 Taylor Polynomial Approximations

7.13 Lagrange Error Bound

7.14 Radius and Interval of Convergence of Power Series

7.15 Taylor and Maclaurin Series of a Function

7.16 Power Series

7.17 Series Strategy