MATH134: Worksheet 8

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Problem 1A

Add the fractions on the left and set the numerators of the two sides equal:

$$\frac{A}{x-1} + \frac{B}{x-2} = \frac{2x+3}{(x-1)(x-2)}$$

The numerators are the same for every value of x. Pick two "nice" values of x to find A and B quickly.

$$\begin{aligned} 2x+3 &= A(x-2) + B(x-1) \\ x = 2 &\implies B = 7 \\ x = 1 &\implies A = -5 \\ \frac{2x+3}{(x-1)(x-2)} &= -\frac{5}{x-1} + \frac{7}{x-2} \end{aligned}$$

Problem 1B

Evaluate

$$\int \frac{3x+4}{x^2-3x-4} dx$$

The integrand can be decomposed into linear partial fractions like so:

$$\begin{aligned} \frac{3x+4}{x^2-3x-4} &= \frac{3x+4}{(x-4)(x+1)} \\ &= \frac{A}{x-4} + \frac{B}{x+1} \\ x = 4 &\implies B = \frac{16}{5} \\ x = -1 &\implies A = -\frac{1}{5} \\ \frac{3x+4}{x^2-3x-4} &= -\frac{1}{5(x-4)} + \frac{16}{5(x+1)} \end{aligned}$$

Thus, the integral can be computed as follows.

$$\begin{aligned} \int \frac{3x+4}{x^2-3x-4} dx &= -\frac{1}{5} \int \frac{1}{x-4} + \frac{16}{5} \int \frac{1}{x+1} \\ &= -\frac{1}{5} \ln|x+1| + \frac{16}{5} \ln|x-4| + C \end{aligned}$$

Problem 1C

When the power of the linear term is more than one, then you get more partial fractions, as many as the power of the term:

$$\frac{2x^2+1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

Note that you run out of "nice" x values before you get all the constants. Find A, B, C.

Multiplying each term in the given equality by $(x+1)^3$, we get

$$\begin{aligned} &2x^2 + 1 = A(x+1)^2 + B(x+1) + C \\ &x = -1 \implies C = 3 \\ &2x^2 + 1 = A(x+1)^2 + B(x+1) + 3 \\ &2x^2 + 1 = Ax^2 + 2A + A + Bx + B + 3 \\ &2x^2 + 1 = Ax^2 + x(2A + B) + (A + B + 3) \end{aligned}$$

Matching each term of the expression $2x^2 + 1$ to the coefficients on the right-hand side of the equality, we form a system of equations to solve:

$$\begin{bmatrix} A + B + 3 & = 1 \\ 2A + B & = 0 \\ A & = 2 \end{bmatrix}$$

Solving this system, we find that $A = 2$ and $B = -4$. Recall that $C = 3$.

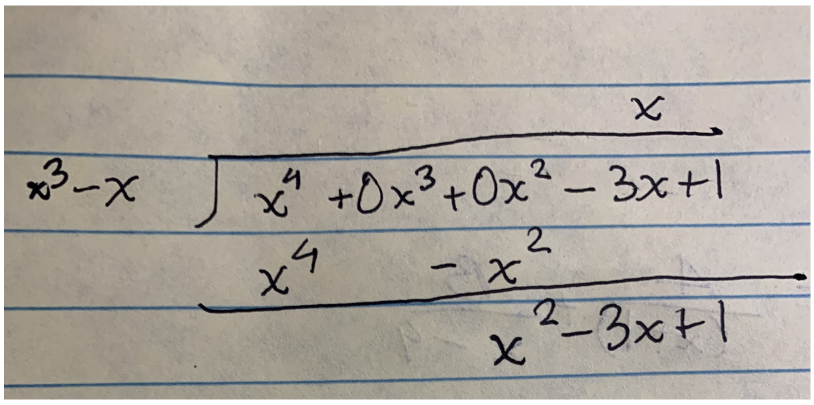
Problem 2

Partial fractions only work when the degree of the numerator is less than the degree of the denominator. Evaluate

$$\int \frac{x^4 - 3x + 1}{x^3 - x} dx$$

by first using long division and then using partial fractions on the remainder term.

The image below demonstrates performing long division on this fraction.



Thus, the fraction can be rewritten and further decomposed like so:

$$\begin{aligned}\frac{x^4 - 3x + 1}{x^3 - x} &= x + \frac{x^2 - 3x + 1}{x^3 - x} \\ &= x + \frac{x^2 - 3x + 1}{x(x+1)(x-1)}\end{aligned}$$

Then

$$\begin{aligned}\frac{x^2 - 3x + 1}{x(x+1)(x-1)} &= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} \\ x^2 - 3x + 1 &= A(x+1)(x-1) + Bx(x-1) + Cx(x+1) \\ x = 0 &\implies A = -1 \\ x = -1 &\implies B = \frac{5}{2} \\ x = 1 &\implies C = -\frac{1}{2} \\ \frac{x^2 - 3x + 1}{x(x+1)(x-1)} &= -\frac{1}{x} + \frac{5}{2(x+1)} - \frac{1}{2(x-1)}\end{aligned}$$

Hence, the integral can be computed as follows.

$$\begin{aligned}\int \frac{x^4 - 3x + 1}{x^3 - x} dx &= \int x dx - \int \frac{1}{x} + \frac{5}{2} \int \frac{1}{x+1} dx - \frac{1}{2} \int \frac{1}{x-1} dx \\ &= \frac{x^2}{2} - \ln|x| + \frac{5}{2} \ln|x+1| - \frac{1}{2} \ln|x-1| + C\end{aligned}$$

Problem 3A

Quadratic factors (which cannot be factored into linear terms) get linear terms in their numerators. Add the fractions on the right and set the numerators of the two sides equal:

$$\frac{x^2 - 1}{(x^2+1)(x+3)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x+3}$$

Again, the numerators will be the same for every value of x. Find A, B, and C.

Multiplying each term in the given equality by the denominator of the left-hand term,

$$\begin{aligned}x^2 - 1 &= (Ax + B)(x+3) + C(x^2+1) \\ x = -3 &\implies C = \frac{4}{5} \\ x^2 - 1 &= (Ax + B)(x+3) + \frac{4}{5}(x^2+1) \\ x^2 - 1 &= Ax^2 + 3Ax + Bx + 3B + \frac{4}{5}x^2 + \frac{4}{5} \\ x^2 - 1 &= x^2\left(A + \frac{4}{5}\right) + x\left(B + 3A\right) + \left(3B + \frac{4}{5}\right)\end{aligned}$$

Problem 3A continued on next page. . .

Matching each term in the left-hand side of the equality with its coefficient, we form a system of equations like so:

$$\begin{bmatrix} 3B + \frac{4}{5} = -1 \\ 3A + B = 0 \\ 3A + \frac{4}{5} = 1 \end{bmatrix}$$

Solving the above system, we find that $A = \frac{1}{5}$ and $B = -\frac{3}{5}$. Recall that $C = \frac{4}{5}$.

Problem 3B

Evaluate

$$\int \frac{x+4}{x^3 + 2x^2 + 2x} dx$$

by first using partial fractions.

Factoring the denominator of the integrand,

$$\frac{x+4}{x^3 + 2x^2 + 2x} = \frac{x+4}{x\left(x^2 + 2x + 2\right)}$$

Then,

$$\frac{x+4}{x\left(x^2 + 2x + 2\right)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 2x + 2}$$

Multiplying each term in the above equality by the denominator of the left-hand side fraction,

$$\begin{aligned}
&x + 4 = A\left(x^2 + 2x + 2\right) + x(Bx + C) \\
&x = 0 \implies A = 2 \\
&x + 4 = 2\left(x^2 + 2x + 2\right) + x(Bx + C) \\
&x + 4 = 2x^2 + 4x + 4 + Bx^2 + Cx \\
&x + 4 = x^2\left(B + 2\right) + x\left(C + 4\right) + 4
\end{aligned}$$

Matching each term on the left-hand side with its respective coefficient from the right-hand side of the above equality, we can form the system of equations below.

$$\begin{bmatrix} 4 + C = 1 \\ 2 + B = 0 \end{bmatrix}$$

Solving the system gives $B = -2$ and $C = -3$. Thus, the integral can be decomposed and computed like so:

$$\begin{aligned}
\int \frac{x+4}{x^3 + 2x^2 + 2x} dx &= \int \frac{2}{x} dx + \int \frac{-2x - 3}{x^2 + 2x + 2} dx \\
&= 2\ln x + \int \frac{-2x - 3}{x^2 + 2x + 2} dx \\
&= 2\ln x - \int \frac{2x}{x^2 + 2x + 2} - \int \frac{3}{x^2 + 2x + 2} \\
&= 2\ln x - 2\int \frac{x}{x^2 + 2x + 2} - 3\int \frac{1}{x^2 + 2x + 2} \\
&= 2\ln x - 2\int \frac{x}{\left(x+1\right)^2 + 1} dx - 3\int \frac{1}{\left(x+1\right)^2 + 1} dx \\
u = x + 1 \implies & 2\ln x - 2\int \frac{u - 1}{u^2 + 1} du - 3\int \frac{1}{u^2 + 1} du \\
&= 2\ln x - 2\left[\int \frac{u}{u^2 + 1} du - \int \frac{1}{u^2 + 1} du\right] - 3\int \frac{1}{u^2 + 1} du \\
&= 2\ln x - 2\left[\frac{1}{2}\ln\left|u^2 + 1\right| - \arctan\left(u\right)\right] - 3\arctan\left(u\right) + C \\
&= 2\ln x - \ln\left|u^2 + 1\right| + 2\arctan\left(u\right) - 3\arctan\left(u\right) + C \\
&= 2\ln x - \ln\left|u^2 + 1\right| - \arctan\left(u\right) + C \\
&= 2\ln x - \ln\left|x^2 + 2x + 2\right| - \arctan\left(x + 1\right) + C
\end{aligned}$$

Problem 3C

To integrate the above rational function, after doing partial fractions, you had to complete the square for the quadratic term. It may be better to complete the square **before** doing the partial fractions:

$$\frac{x+4}{x^3+2x^2+2x} = \frac{A}{x} + \frac{B(x+1)+C}{(x+1)^2+1}$$

Multiplying each term in the above equality by the left-hand side's denominator, we have

$$\begin{aligned}
x+4 &= A\left[(x+1)^2+1\right] + x\left[B(x+1)+C\right] \\
&= A(x+1)^2 + A + x\left[B(x+1)+C\right] \\
&= A(x+1)^2 + x\left[B(x+1)+C\right] + A \\
&= A\left(x^2+2x+2\right) + x(Bx+C) \\
x = 0 &\implies A = 2 \\
x+4 &= 2\left(x^2+2x+2\right) + x(Bx+C) \\
x+4 &= 2x^2+4x+4+Bx^2+Cx \\
x+4 &= x^2(B+2) + x(C+4) + 4
\end{aligned}$$

Matching the terms on the left-hand side of the above equality with their respective coefficients represented by the right-hand side's terms, we can form a system like so:

$$\begin{bmatrix} 4+C=1 \\ 2+B=0 \end{bmatrix}$$

Solving the above system, we find that $B=-2$ and $C=-3$. The rest of this problem is identical to that of Problem 3B so I will just copy and paste my response for that problem here. Thus, the integral can be decomposed and computed like so:

$$\begin{aligned}
\int \frac{x+4}{x^3+2x^2+2x}dx &= \int \frac{2}{x}dx + \int \frac{-2x-3}{x^2+2x+2}dx \\
&= 2\ln x + \int \frac{-2x-3}{x^2+2x+2}dx \\
&= 2\ln x - \int \frac{2x}{x^2+2x+2} - \int \frac{3}{x^2+2x+2} \\
&= 2\ln x - 2\int \frac{x}{x^2+2x+2} - 3\int \frac{1}{x^2+2x+2} \\
&= 2\ln x - 2\int \frac{x}{(x+1)^2+1}dx - 3\int \frac{1}{(x+1)^2+1}dx \\
u = x+1 \implies & 2\ln x - 2\int \frac{u-1}{u^2+1}du - 3\int \frac{1}{u^2+1}du \\
&= 2\ln x - 2\left[\int \frac{u}{u^2+1}du - \int \frac{1}{u^2+1}du\right] - 3\int \frac{1}{u^2+1}du \\
&= 2\ln x - 2\left[\frac{1}{2}\ln\left|u^2+1\right| - \arctan(u)\right] - 3\arctan(u) + C \\
&= 2\ln x - \ln\left|u^2+1\right| + 2\arctan(u) - 3\arctan(u) + C \\
&= 2\ln x - \ln\left|u^2+1\right| - \arctan(u) + C \\
&= 2\ln x - \ln\left|x^2+2x+2\right| - \arctan(x+1) + C
\end{aligned}$$

Problem 3D

When the power of the quadratic factor is more than one, you get as many partial fractions as the power of that term. Find A, B, C, and D in

$$\frac{x^3+x+1}{(x^2+4)^2} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2}$$

Multiplying each term in the given equality by the left-hand side's denominator, we have

$$\begin{aligned}
x^3+x+1 &= (Ax+B)\left(x^2+4\right) + Cx + D \\
x^3+x+1 &= Ax^3+4Ax+Bx^2+4B+Cx+D \\
x^3+x+1 &= Ax^3+Bx^2+x(4A+C)+(4B+D)
\end{aligned}$$

Problem 3D continued on next page. . .

Matching the terms on the left-hand side of the above equality with their respective coefficients represented by the right-hand side's terms, we can form a system like so:

$$\begin{bmatrix} 4B + D = 1 \\ 4A + C = 1 \\ A = 1 \\ B = 0 \end{bmatrix}$$

Solving the system, we fine that $A = 1$, $B = 0$, $C = -3$, and $D = 1$.

Problem 3E

Write down the partial fractions expansion for

$$\frac{5x^5 + 4x^4 + 3x^3 + 2x^2 + x + 1}{x^3(x+1)(2x+1)^2(x^2+1)(x^2+6x+13)^2}$$

but do not evaluate the constants.

The partial fractions expansion is as follows.

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+1} + \frac{E}{2x+1} + \frac{F}{(2x+1)^2} + \frac{Gx+H}{x^2+1} + \frac{Ix+J}{x^2+6x+13} + \frac{Kx+L}{(x^2+6x+13)^2}$$