MATH134: Week 3 Worksheet

Due on October 16, 2020 at 11:59 PM

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Problem 1

Find $\frac{dy}{dx}$ for
(a) $y = 3x^2\sqrt{3+5x}$
(b) $y = \frac{4+x^2}{4-x^2}$
(c) $y = \sin\left(\sin\left(\sin\left(x^2\right)\right)\right)$

(a)

$$\begin{aligned}
y &= 3x^2\sqrt{3+5x} \\
\frac{dy}{dx} &= 6x\sqrt{3+5x} + 3x^2 \cdot \frac{1}{2\sqrt{3+5x}} \cdot 5 \\
&= 6x\sqrt{3+5x} + \frac{15x^2}{2\sqrt{3+5x}} \\
&= \frac{12x(3+5x)}{2\sqrt{3+5x}} + \frac{15x^2}{2\sqrt{3+5x}} \\
&= \frac{75x^2+36x}{2\sqrt{3+5x}} \\
&= \frac{3x(25x+12)}{2\sqrt{3+5x}}
\end{aligned}$$

(b)

$$\begin{aligned}
y &= \frac{4+x^2}{4-x^2} \\
&= \frac{2x(4-x^2)+2x(4+x^2)}{(4-x^2)^2} \\
&= \frac{16x}{(4-x^2)^2}
\end{aligned}$$

(c)

$$\begin{aligned}
y &= \sin\left(\sin\left(\sin\left(x^2\right)\right)\right) \\
&= 2x\cos\left(\sin\left(\sin\left(x^2\right)\right)\right)\cos\left(\sin\left(x^2\right)\right)\cos\left(x^2\right)
\end{aligned}$$

Problem 2

Compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the function given implicitly by

$$x^2 - 2xy + 4y^2 = 3$$

Did you use the quotient rule for the second derivative? Could you have avoided it?

$$\begin{aligned}
x^2 - 2xy + 4y^2 &= 3 \\
2x - 2\frac{dy}{dx}(xy) + 8y\frac{dy}{dx} &= 0 \\
2x - 2\left[y + x\frac{dy}{dx}\right] + 8y\frac{dy}{dx} &= 0 \\
2x - 2y - 2x\frac{dy}{dx} + 8y\frac{dy}{dx} &= 0 \\
-2x\frac{dy}{dx} + 8y\frac{dy}{dx} &= 2y - 2x \\
-x\frac{dy}{dx} + 4y\frac{dy}{dx} &= y - x \\
\frac{dy}{dx}(4y-x) &= y - x \qquad (1) \\
\frac{dy}{dx} &= \frac{y-x}{4y-x}
\end{aligned}$$

Going back to (1), we can re-differentiate the equation to get:

$$\begin{aligned}
y'(4y-x) &= y - x \\
y"(4y-x) + y'(4y'-1) &= y' - 1 \\
y"(4y-x) &= y' - 1 - y'(4y'-1) \\
&= 2y' - 4y'y' - 1 \\
&= 2\left(\frac{y-x}{4y-x}\right) - 4\left(\frac{y-x}{4y-x}\right)^2 - 1
\end{aligned}$$

$$
\begin{aligned}
y''(4y-x) &= \frac{2(y-x)(4y-x)-4(y-x)^2-(4y-x)^2}{(4y-x)^2} \\
&= \frac{-3x^2-12y^2+6xy}{(4y-x)^2} \\
\frac{d^2y}{dx^2} &= -\frac{3(x^2+4y^2-2xy)}{(4y-x)^3}
\end{aligned}
$$

I did not use the quotient rule for the second derivative exactly to demonstrate that you can avoid it as asked in the question prompt. However, I would like to add that I greatly dislike the method I took of avoiding the quotient rule, as it took me many more steps to compute the second derivative than I would have liked it to.

Problem 3

Find equations of all tangents to the curve $y = x^3 - x$ that pass through the point (-2,2).

The slope of the curve at any point is given by

$$\frac{dy}{dx} = 3x^2 - 1$$

We have a point $(c, f(c)) = (c, c^3 - c)$ on the curve with slope

$$\left.\frac{dy}{dx}\right|_{x=c} = 3c^2 - 1$$

Any tangent to the curve at $(c, c^3 - c)$ has the form:

$$
\begin{aligned}
y - (c^3 - c) &= (3c^2 - 1)(x - c) \\
y &= -2c^3 + 3xc^2 - x
\end{aligned}
$$

Since the tangents we are asked to find run through the point $(-2, 2)$, we plug $(-2, 2)$ into (x, y) to get:

$$
\begin{aligned}
2 &= -2c^3 + 3(-2)c^2 + 2 \\
0 &= c^3 + 3c^2 \\
c^2(c+3) &= 0 \\
c &= -3, 0
\end{aligned}
$$

When $c = -3$,

$$
\begin{aligned}
y &= -2(-3)^3 + 3x(-3)^2 - x \\
y &= 26x + 54
\end{aligned}
$$

When $c = 0$,

$$y = -x$$

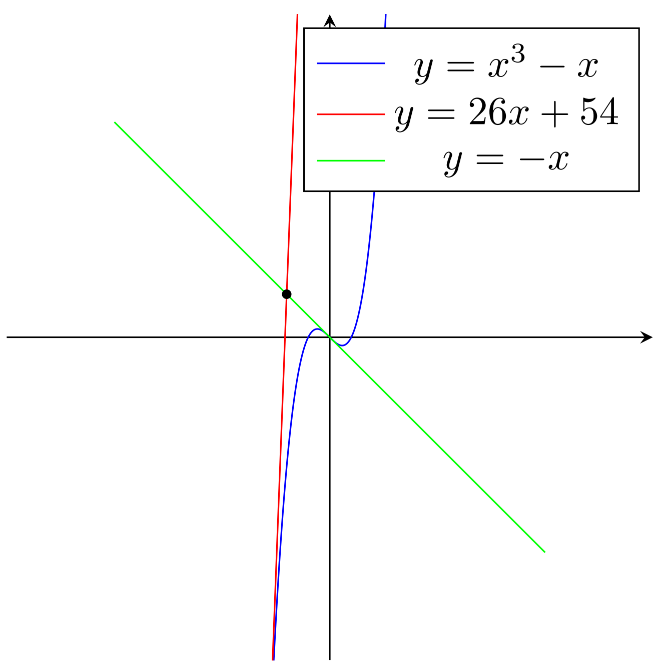
Thus, the tangents to the curve $y = x^3 - x$ that pass through the point (-2,2) are:

$$y = -x$$

and

$$y = 26x + 54$$

For completion's sake, the curve, its tangents, and the point (-2,2) are graphed below:



Problem 4

Evaluate the limit

$$\lim_{x \to \pi} \frac{\sin x}{x - \pi}$$

using the definition of the derivative.

Because

$$\lim_{x \to \pi} \sin x = \sin \pi = 0,$$

we can write the given limit as

$$\lim_{x \to \pi} \frac{\sin x - \sin \pi}{x - \pi}$$

A definition of derivative is:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Then we have

$$f'(\pi) = \lim_{x \to \pi} \frac{\sin x - \sin \pi}{x - \pi}$$

The derivative of $f(x) = \sin x$ is $f'(x) = \cos x$ so we have that

$$f'(\pi) = \cos \pi = -1$$