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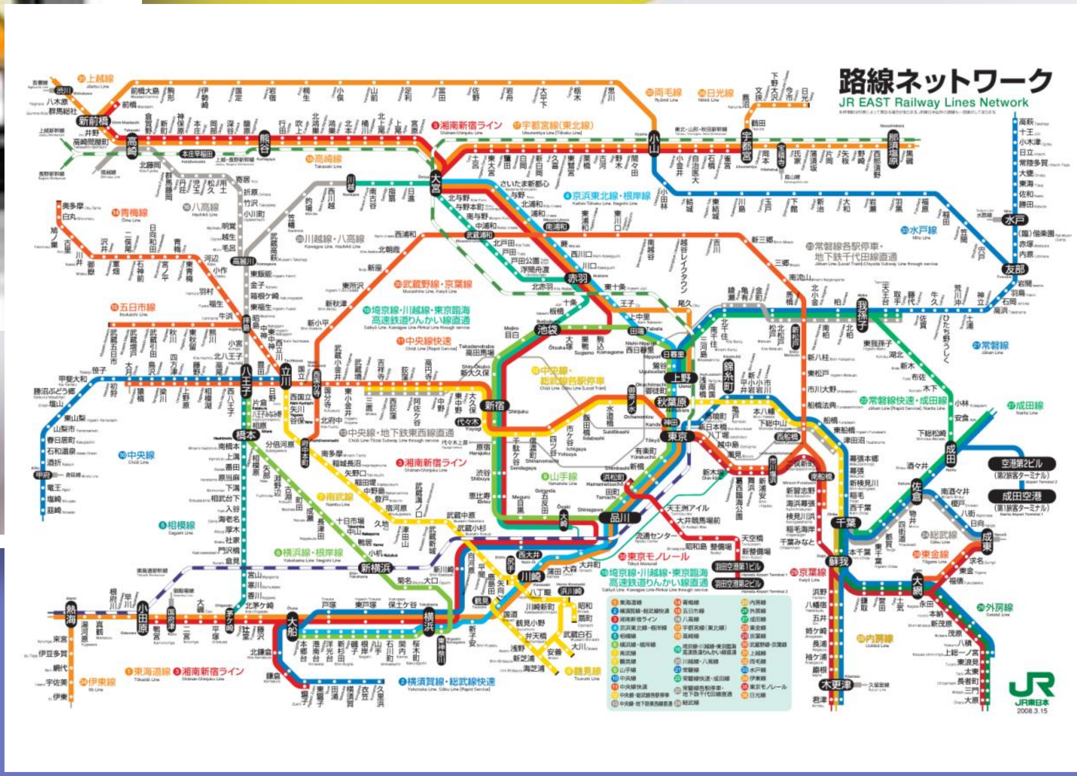
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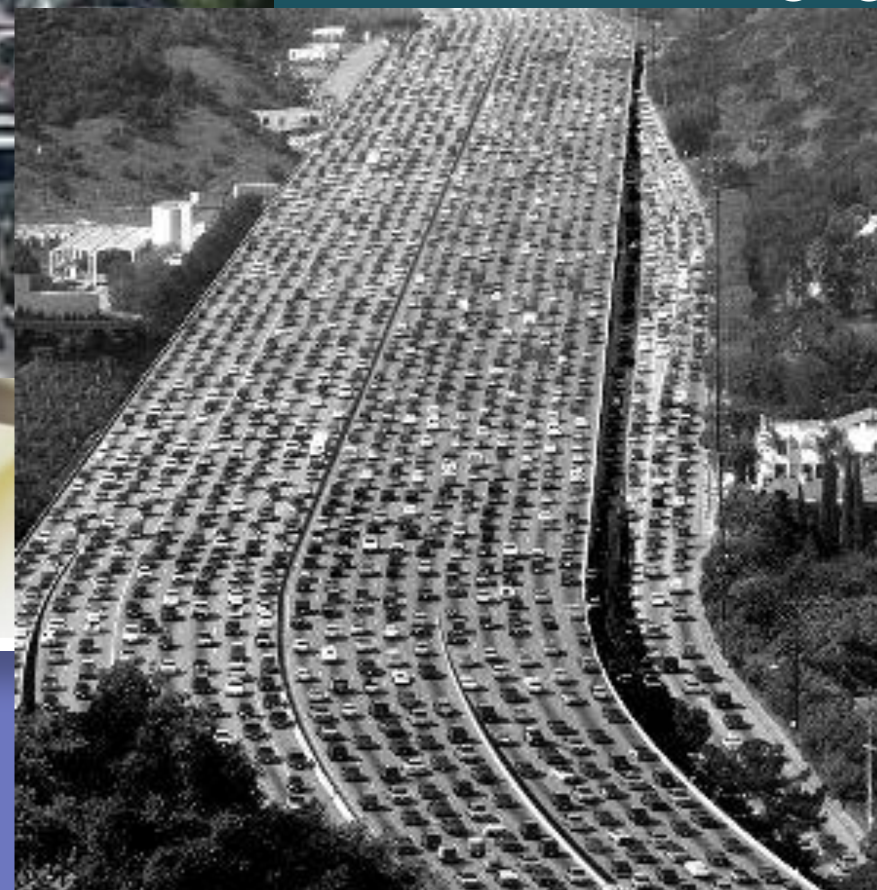


The Shortest Path Problem

(1) The shortest path: The path which has the minimum total weight, $W(u^*)=\min w_{st}$

Let $G=(V,E)$ be a connected graph, from a starting point v_s to a terminal point v_t, find a path p which has the minimum total weight $\sum_{(v_i,v_j\in p)} w_{ij}$

- **The shortest path can be applied in many real applications.**

- **And some problems such as minimum cost, minimum time, reliable line, can be transformed as the shortest path problem.**

1. $w_{ij}>0$— classical Dijkstra method

- **(1) Idea of the algorithm**

Suppose we have found the shortest path p^* from v_s to v_t: v_s , …, v_j, …, v_k, …, v_t .

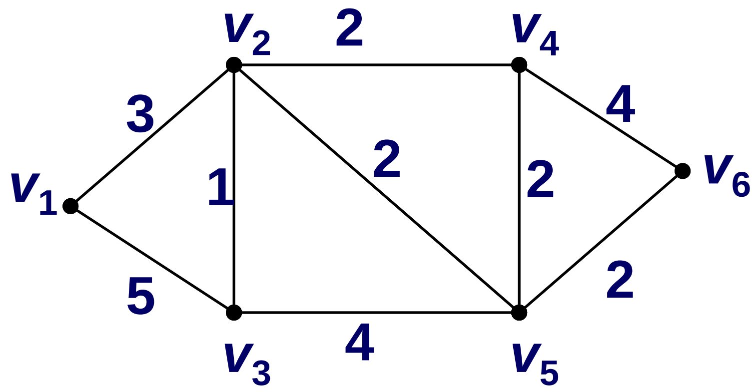
- Then we have that, along the path p^*, the path from v_s to v_j or v_k, is the shortest path from v_s to v_j or v_k.

- (2) Process of the algorithm: —（label method）

Temporary T_j： An upper bound of the shortest distance from v_s to v_j.

Permanent P_j： The real shortest distance from v_s to v_j .

◈ **Each iteration of the algorithm will label at least one point as the permanent label.**



（1）label v_1 as a P point，label other points as T points.

$$P(v_1)=0 \qquad T(v_i)=+\infty \quad (i=2,3,\cdots,6)$$

（2）

$$T(v_2)=\min[T(v_2),P(v_1)+w_{12}]=\min[+\infty,0+3]=3$$

$$T(v_3)=\min[T(v_3),P(v_1)+w_{13}]=\min[+\infty,0+5]=5$$

（3）

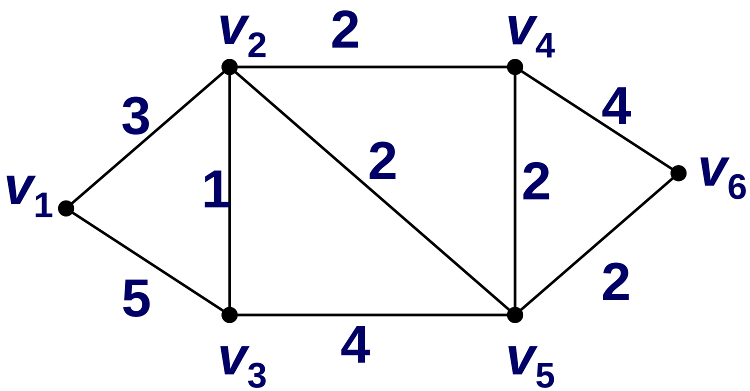
$$P(v_2)=3$$

（4）

$$T(v_3)=\min[T(v_3),P(v_2)+w_{23}]=\min[5,3+1]=4$$

$$T(v_4)=\min[T(v_4),P(v_2)+w_{24}]=\min[+\infty,3+2]=5$$

$$T(v_5)=\min[T(v_5),P(v_2)+w_{25}]=\min[+\infty,3+2]=5$$



（5） $\boldsymbol{P(v_3) = 4}$

（6） $T(v_5) = \min[T(v_5), P(v_3) + w_{35}] = \min[5, 4+4] = 5$

（7） $\boldsymbol{P(v_4) = 5}$ $\boldsymbol{P(v_5) = 5}$

（8） $T(v_6) = \min[T(v_6), P(v_4) + w_{46}] = \min[+\infty, 5+4] = 9$

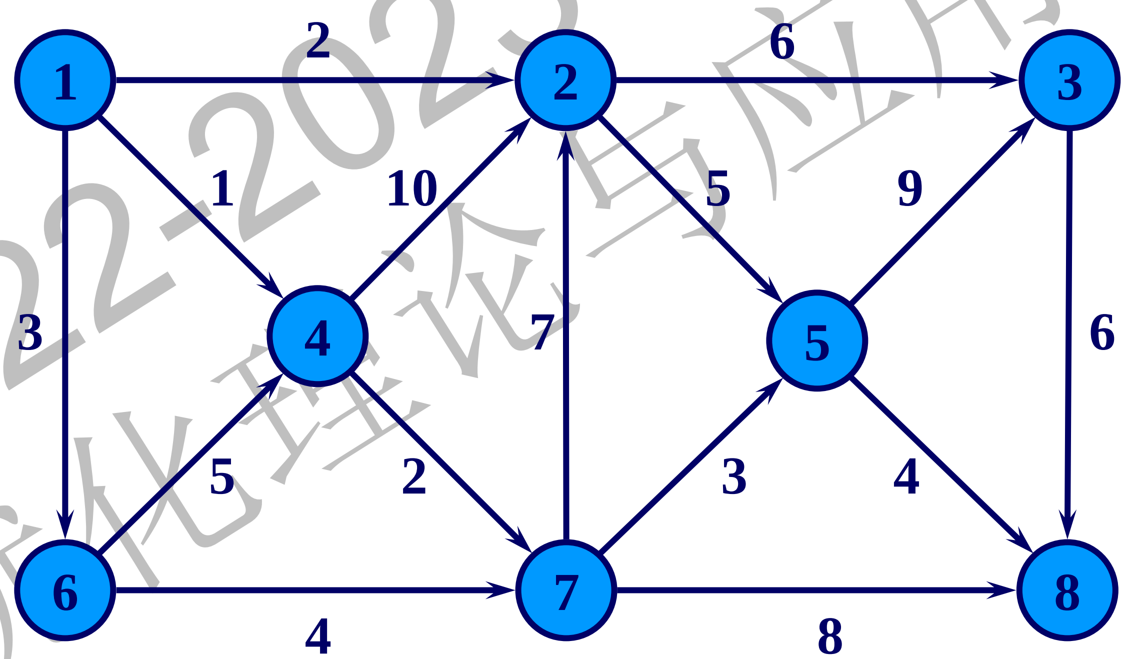
（9） $T(v_6) = \min[T(v_6), P(v_5) + w_{56}] = \min[+\infty, 5+2] = 7$

（10） $\boldsymbol{P(v_6) = 7}$

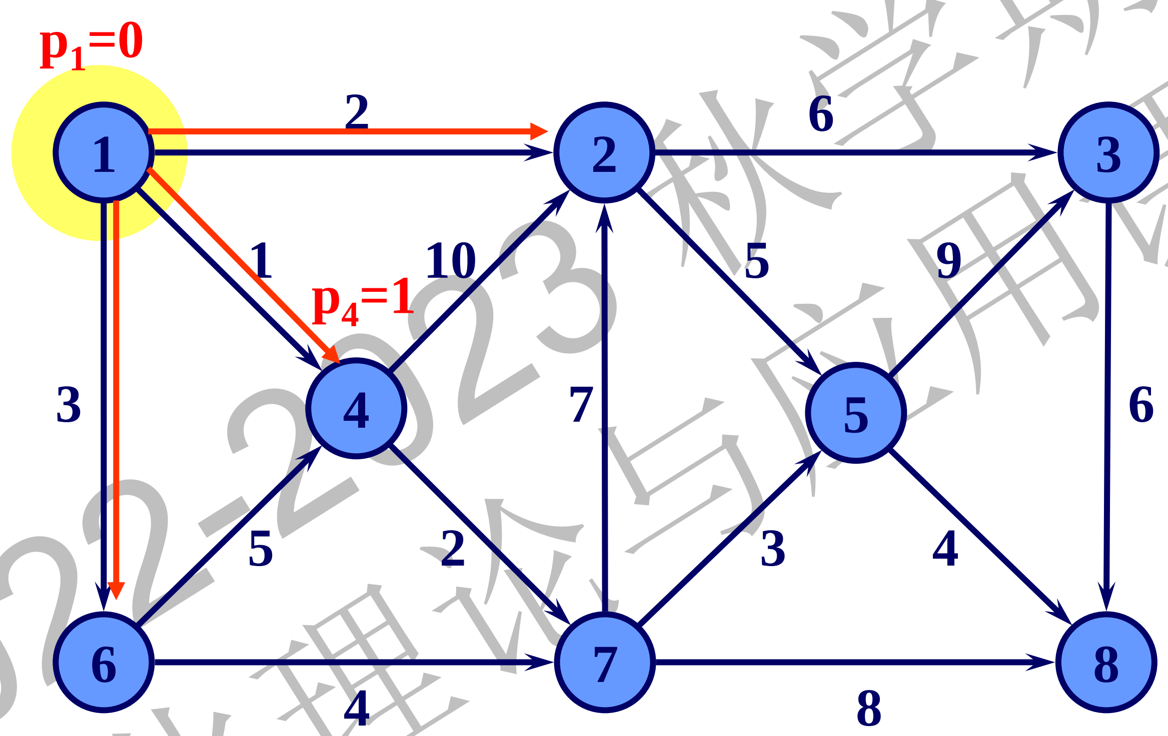
Back trace to obtain the shortest path from v_1 to v_6

$$\boldsymbol{v_1 \to v_2 \to v_5 \to v_6}$$

Example: obtain the shortest path from 1 to 8.



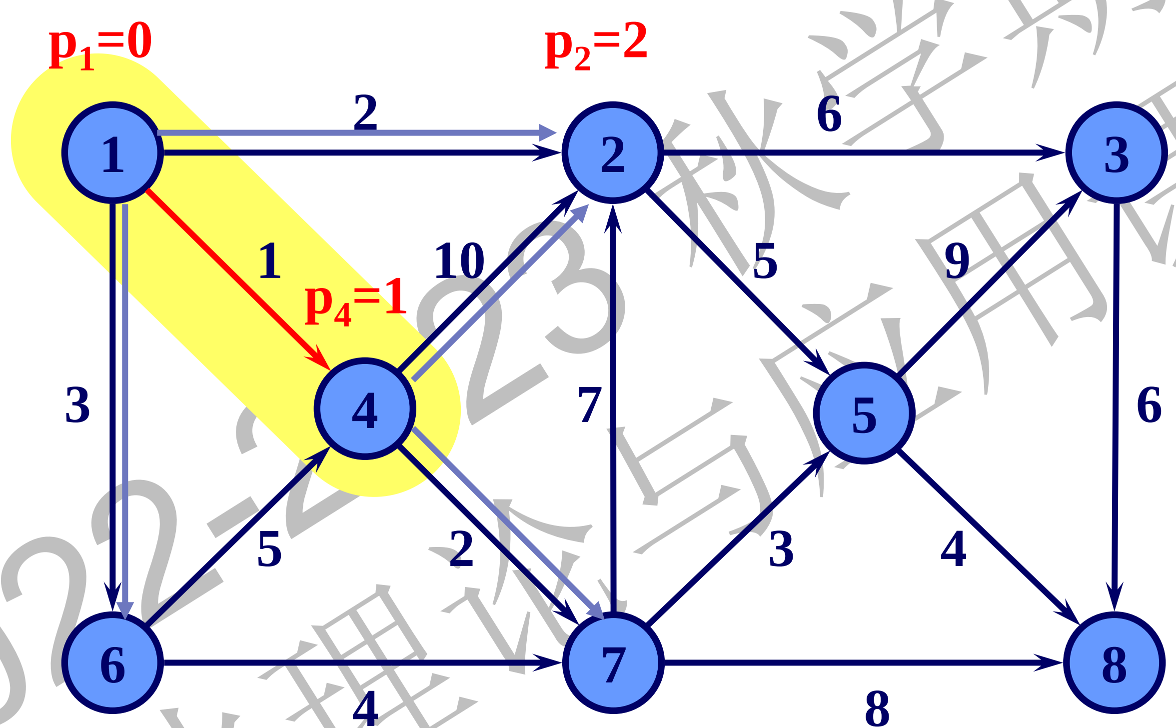
$X=\{1\}$, $w_1=0$



$\min\{c_{12},c_{14},c_{16}\}=\min\{0+2,0+1,0+3\}=\min\{2,1,3\}=1$

$X=\{1,4\}$, $p_4=1$

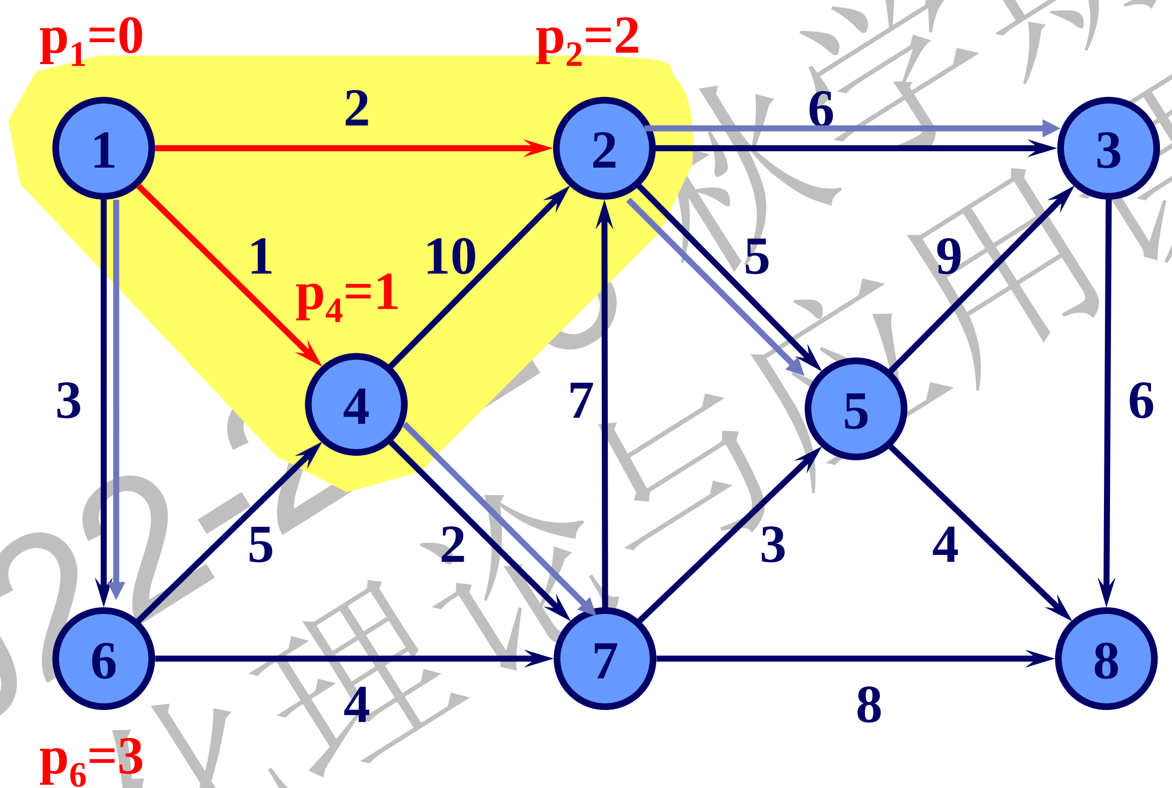
$X=\{1,4\}$



$\min\{c_{12},c_{16},c_{42},c_{47}\}=\min\{0+2,0+3,1+10,1+2\}=\min\{2,3,11,3\}=2$

$X=\{1,2,4\},\ p_2=2$

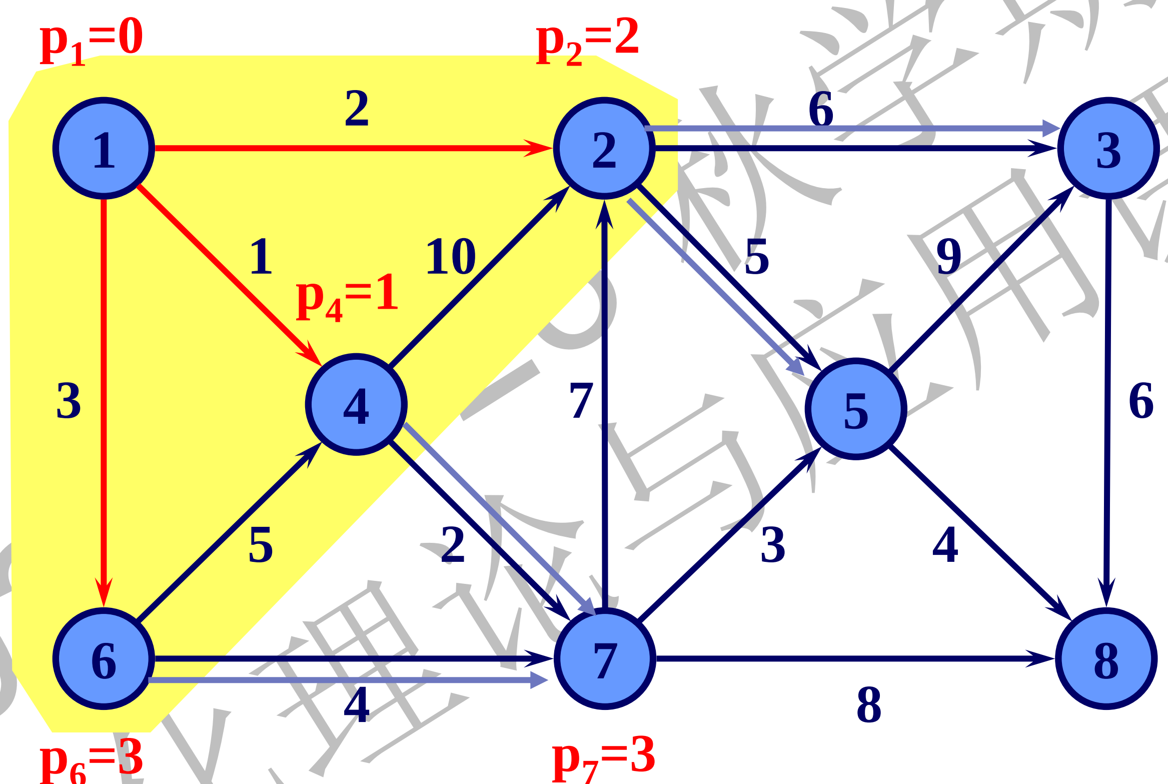
$X=\{1,2,4\}$



$\min\{c_{16},c_{23},c_{25},c_{47}\}=\min\{0+3,2+6,2+5,1+2\}=\min\{3,8,7,3\}=3$

$X=\{1,2,4,6\}$, $p_6=3$

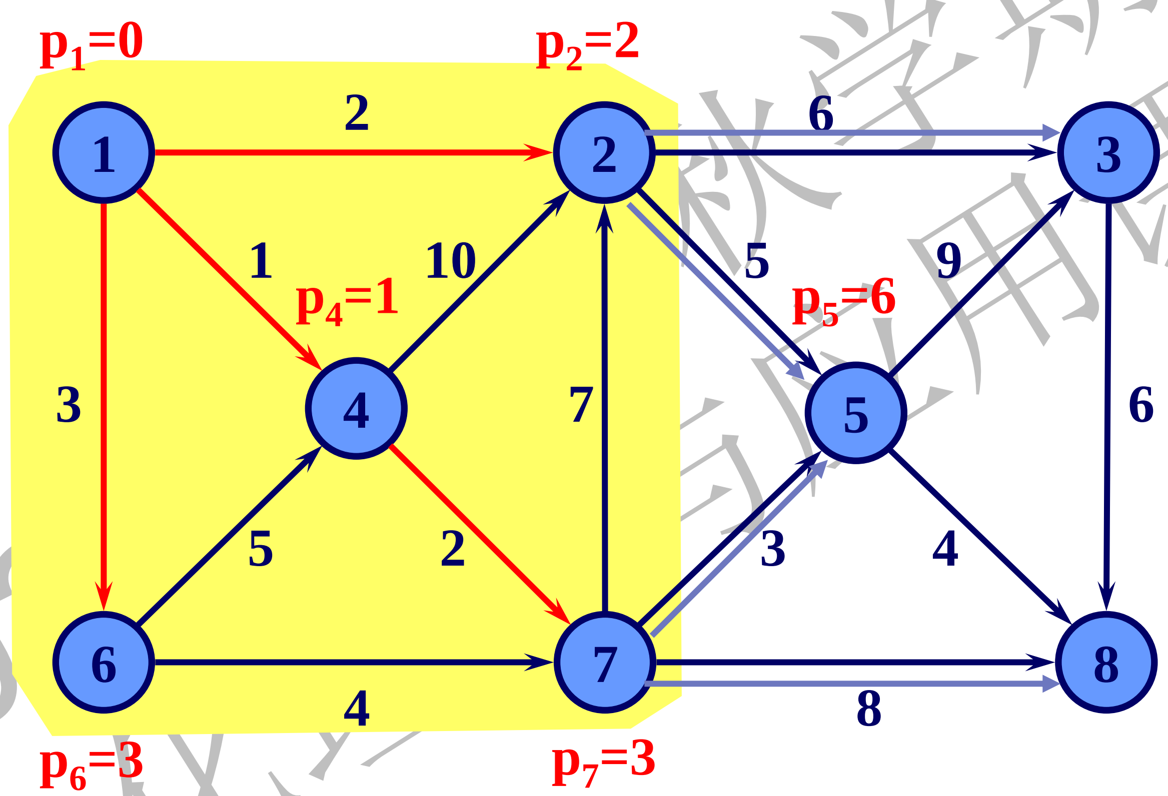
$X=\{1,2,4,6\}$



$\min\{c_{23},c_{25},c_{47},c_{67}\}=\min\{2+6,2+5,1+2,3+4\}=\min\{8,7,3,7\}=3$

$X=\{1,2,4,6,7\}$, $p_7=3$

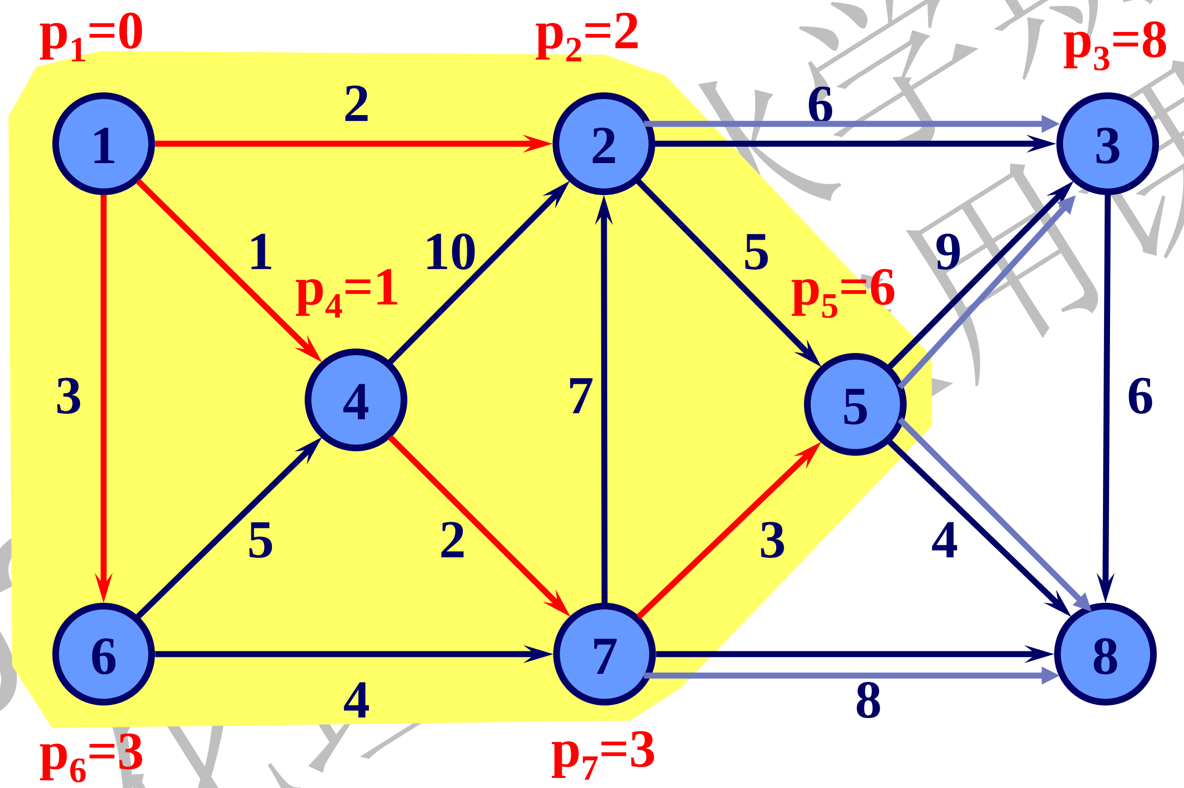
$X=\{1,2,4,6,7\}$



$\min\{c_{23},c_{25},c_{75},c_{78}\}=\min\{2+6,2+5,3+3,3+8\}=\min\{8,7,6,11\}=6$

$X=\{1,2,4,5,6,7\}$, $p_5=6$

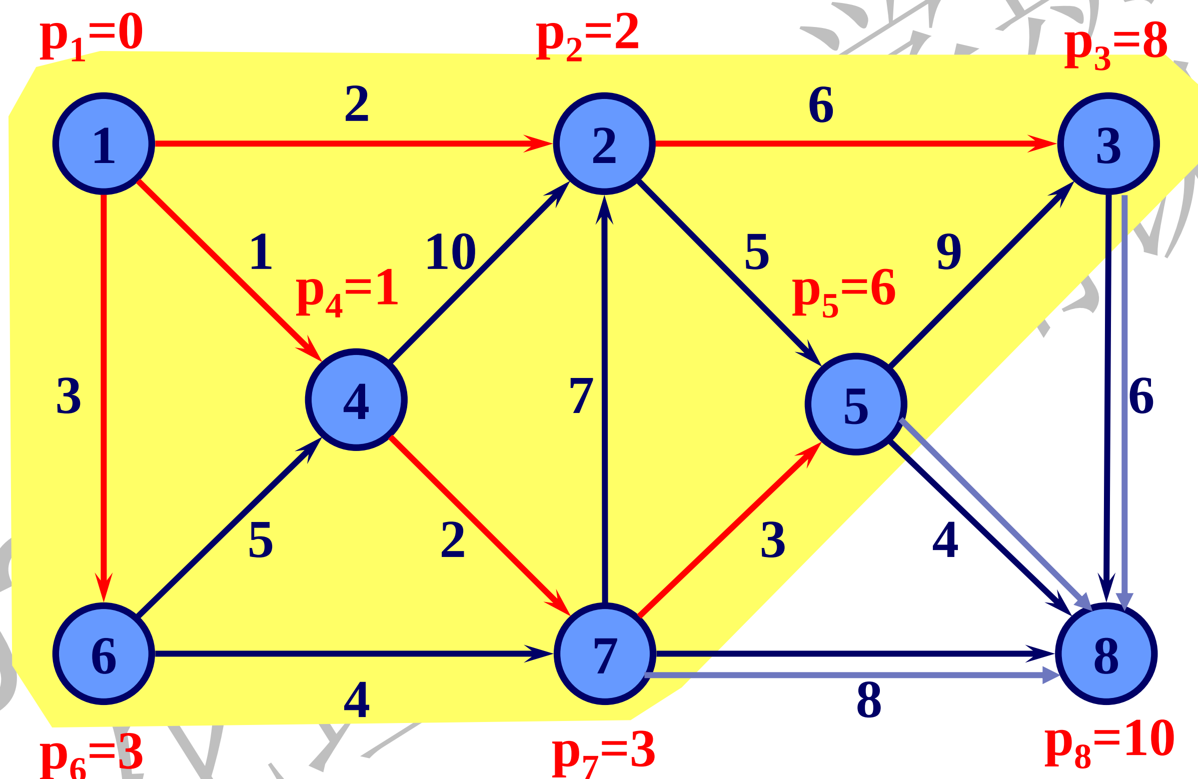
X={1,2,4,6,7}



min $\{c_{23}, c_{53}, c_{58}, c_{78}\}$=min {2+6,6+9,6+4,3+8}=min {8,15,10,11}=8

X={1,2,3,4,5,6,7}, p_3=8

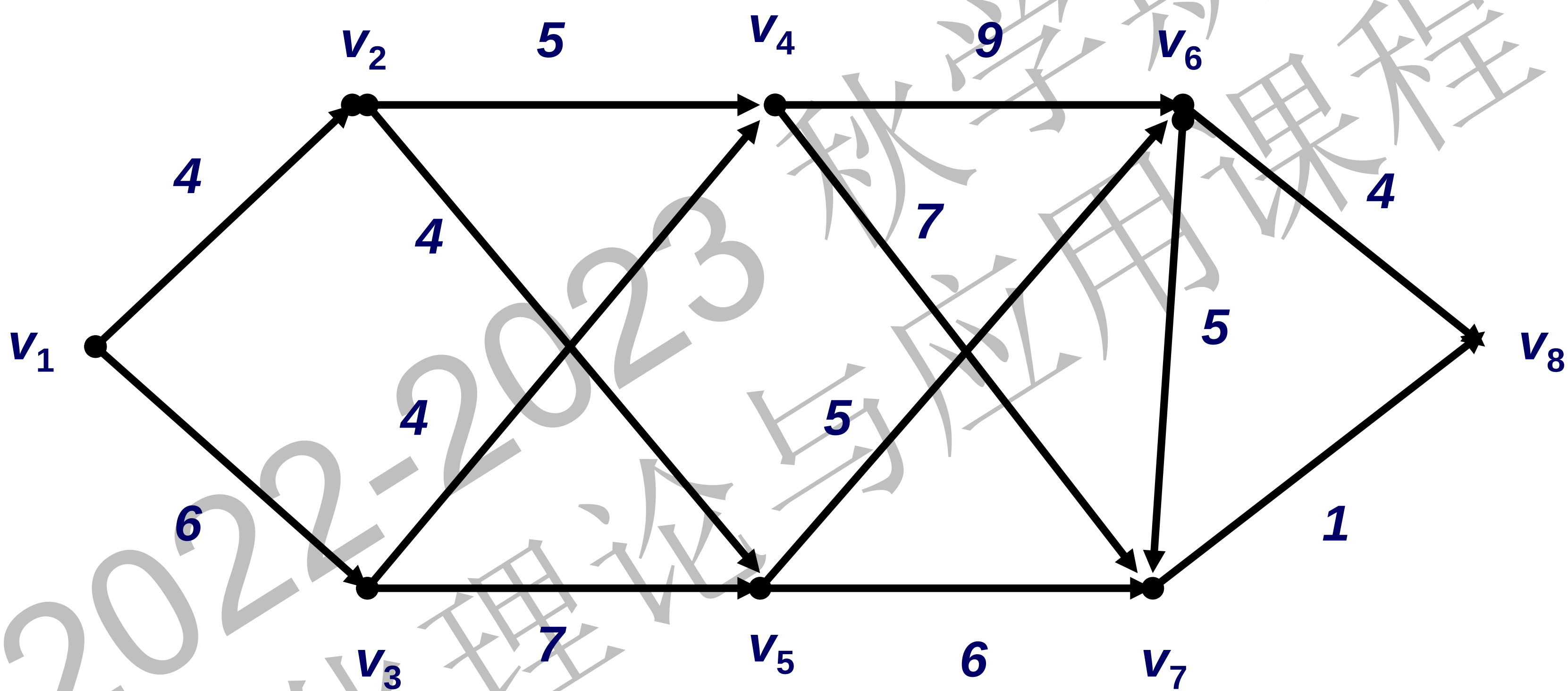
$X=\{1,2,3,4,6,7\}$



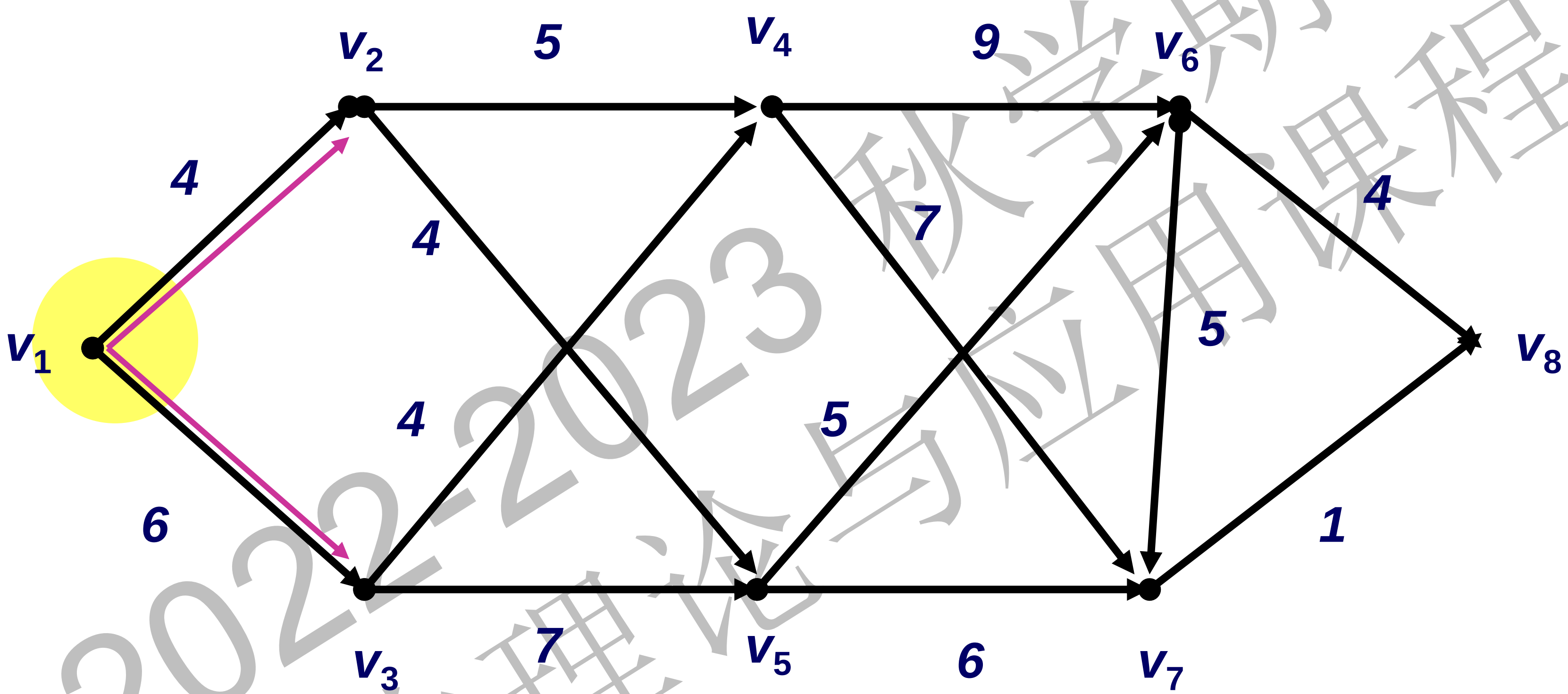
$\min\{c_{38},c_{58},c_{78}\}=\min\{8+6,6+4,3+7\}=\min\{14,10,11\}=10$

$X=\{1,2,3,4,5,6,7,8\}$, $p_8=10$

Example 2

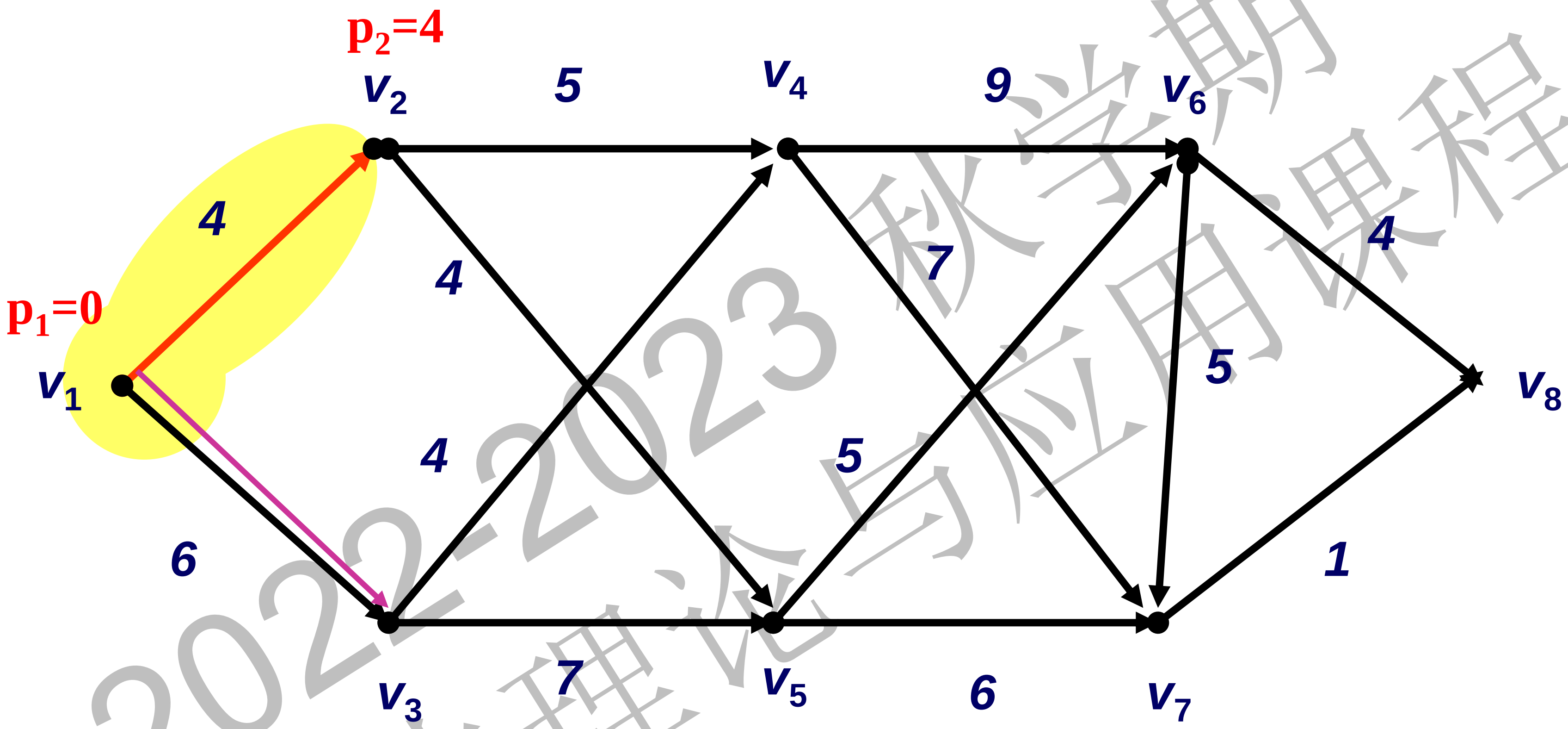


$X=\{1\},\ w_1=0$

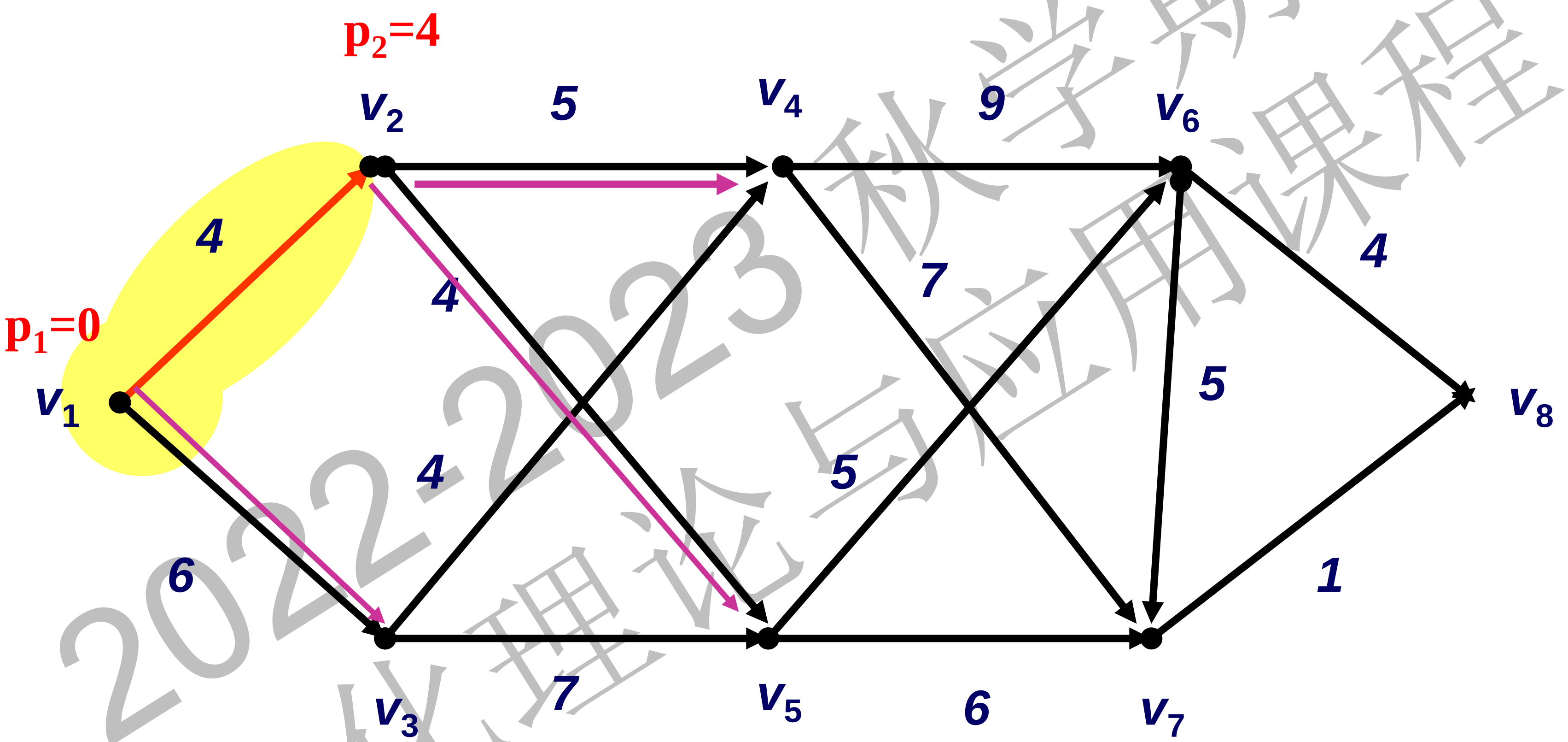


$\min\{c_{12},c_{13}\}=\min\{0+4,0+6\}=\min\{4,6\}=4$

$X=\{1,2\},\ p_4=4$

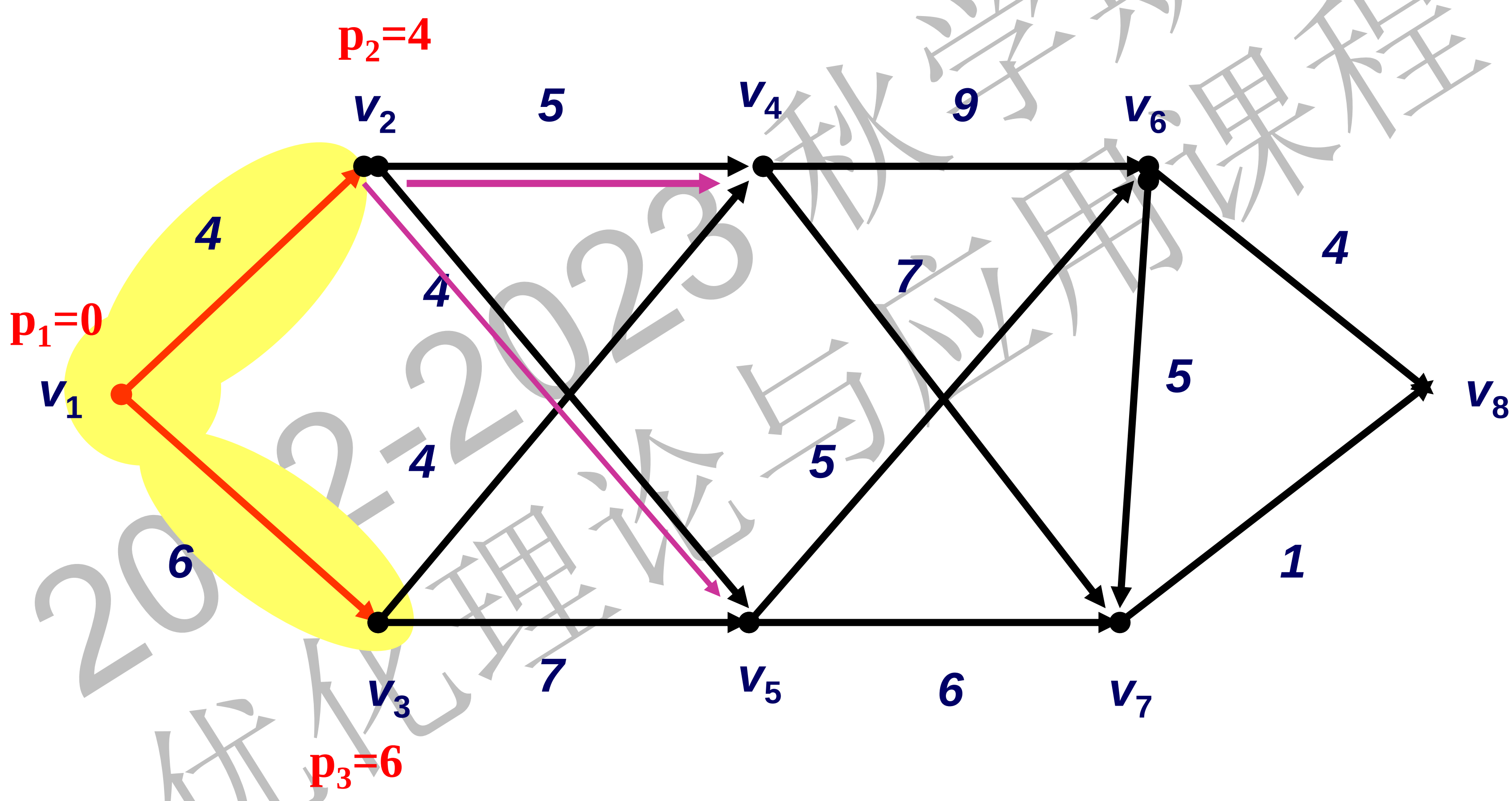

p2=4
p1=0
v1
v2
v3
v4
v5
v6
v7
v8
4
5
9
4
7
4
4
5
5
6
7
6
1

X={1,2}



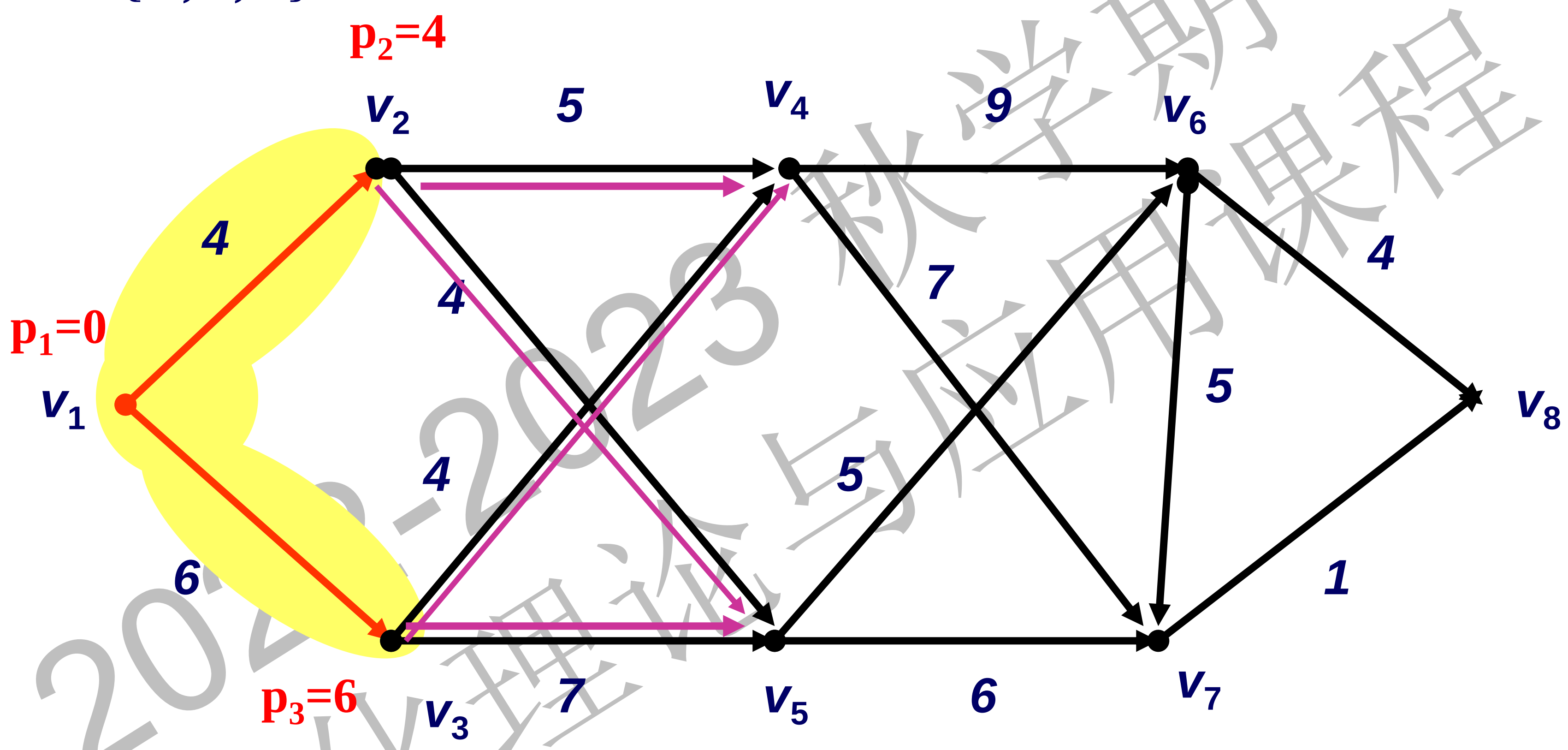
$\min\{c_{13},c_{24},c_{25}\}=\min\{6,4+5,4+4\}=\min\{6,9,8\}=6$

$X=\{1,2,3\}$, $p_3=6$



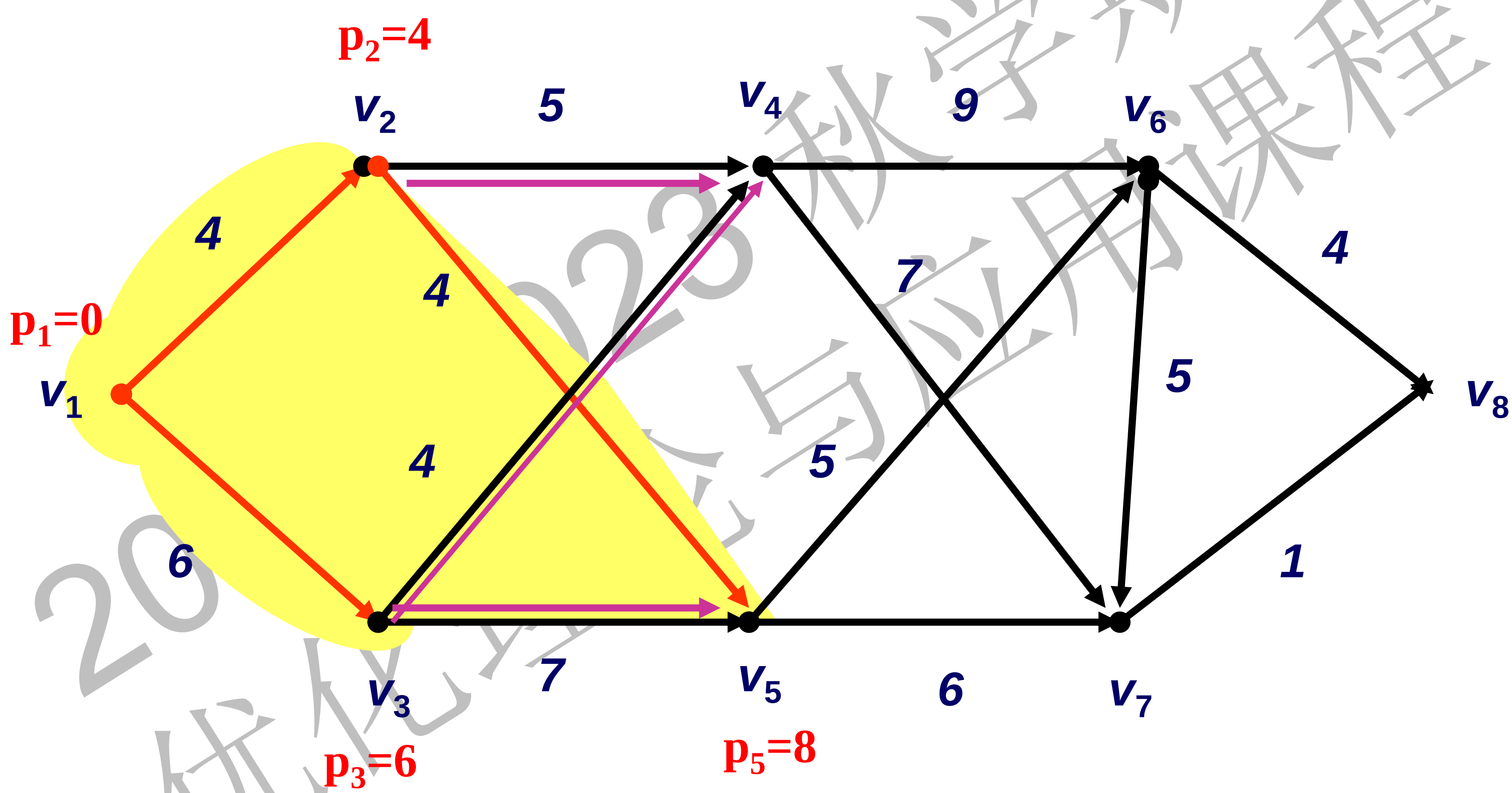
$p_2=4$
v_2
5
v_4
9
v_6
4
$p_1=0$
v_1
4
7
4
5
v_8
4
5
6
1
7
v_5
6
v_7
v_3
$p_3=6$

$X=\{1,2,3\}$

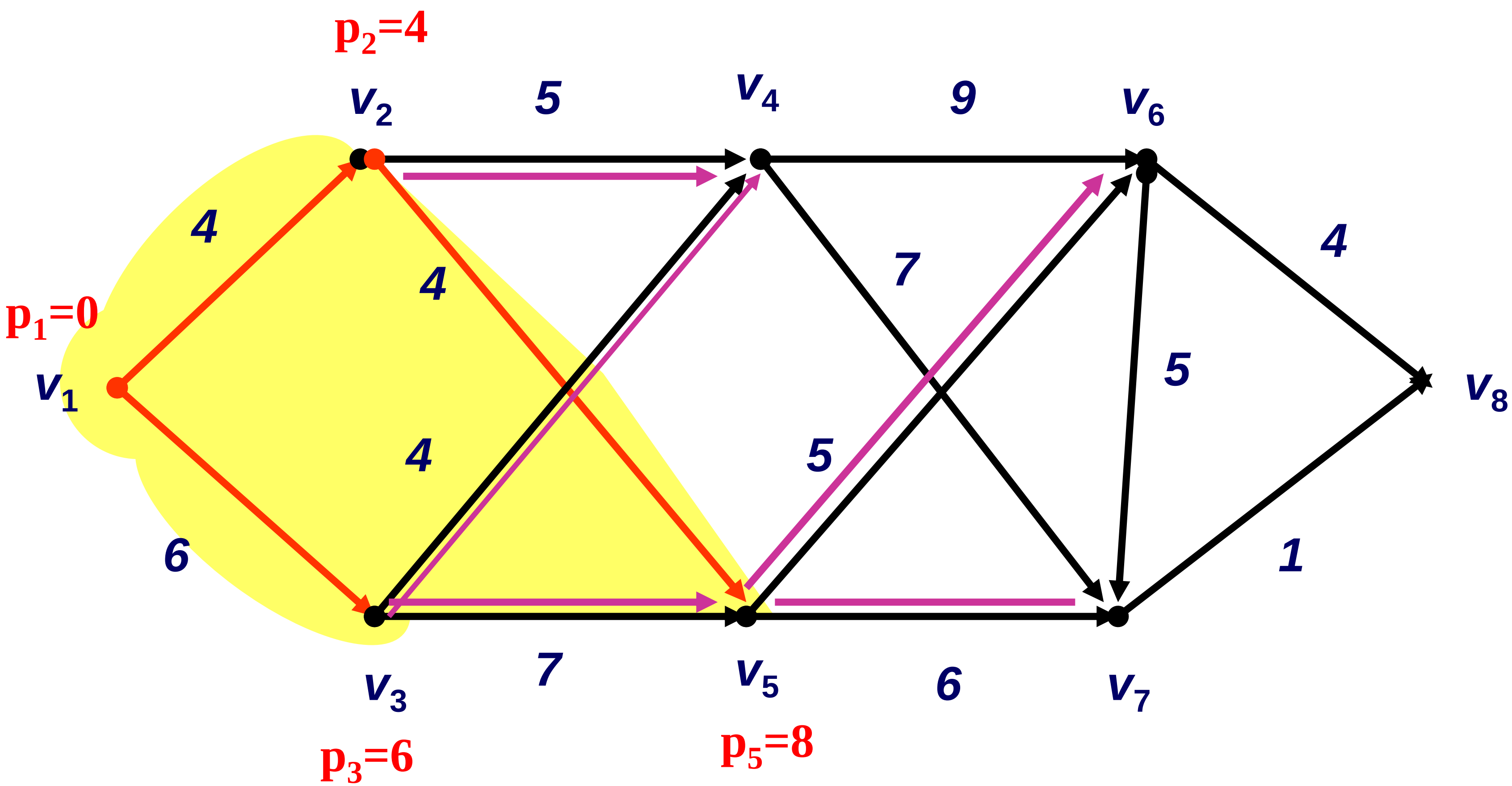


min $\{c_{24}, c_{25}, c_{34}, c_{35}\}$=min $\{4+5, 4+4, 6+4, 6+7\}$=min $\{9, 8, 10, 13\}=8$

$X=\{1,2,3,5\}$, $p_3=6$

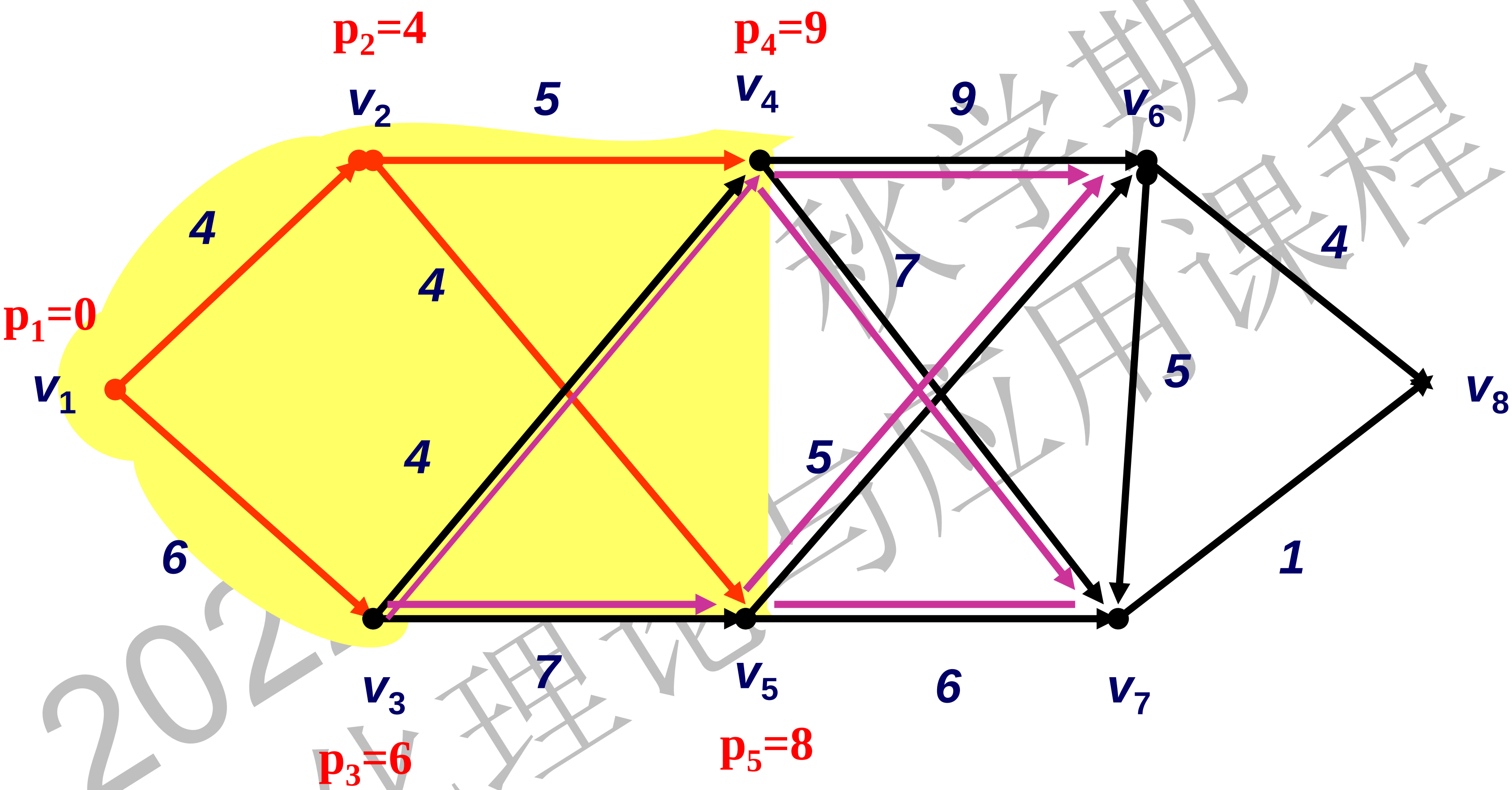

$p_2=4$
v_2
5
v_4
9
v_6
4
4
7
4
$p_1=0$
v_1
5
v_8
4
5
6
1
v_3
7
v_5
6
v_7
$p_3=6$
$p_5=8$

X={1,2,3,5}

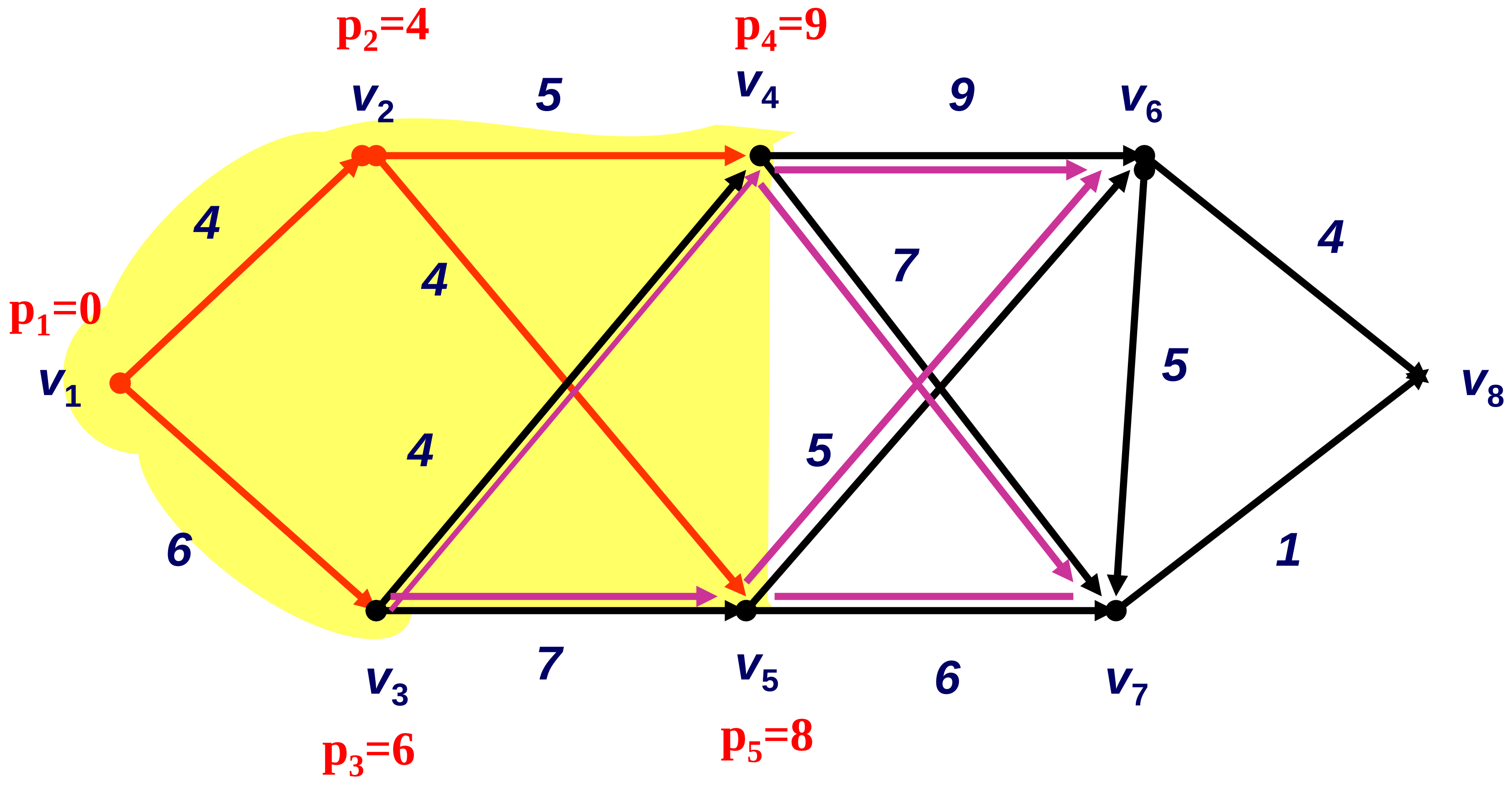


min $\{c_{24}, c_{56}, c_{57}\}$=min {4+5,8+7,8+6}=min {9, 15,14}=9

X={1,2,3,4,5}, p_4=9

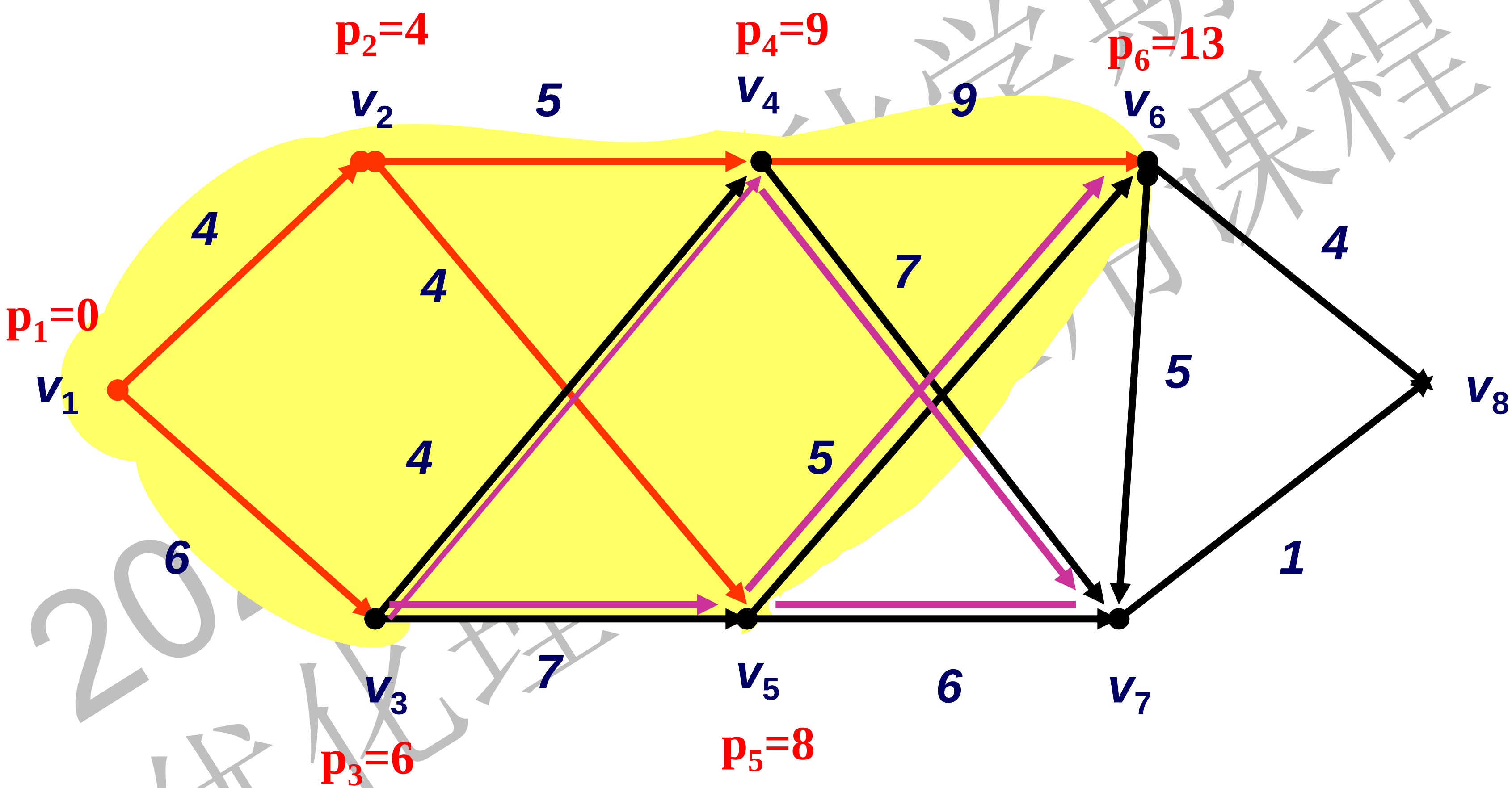

$p_1=0$
v_1
$p_2=4$
v_2
$p_4=9$
v_4
v_6
v_8
$p_3=6$
v_3
$p_5=8$
v_5
v_7
4
6
5
4
4
7
9
7
5
5
4
6
1

$X=\{1,2,3,4,5\}$



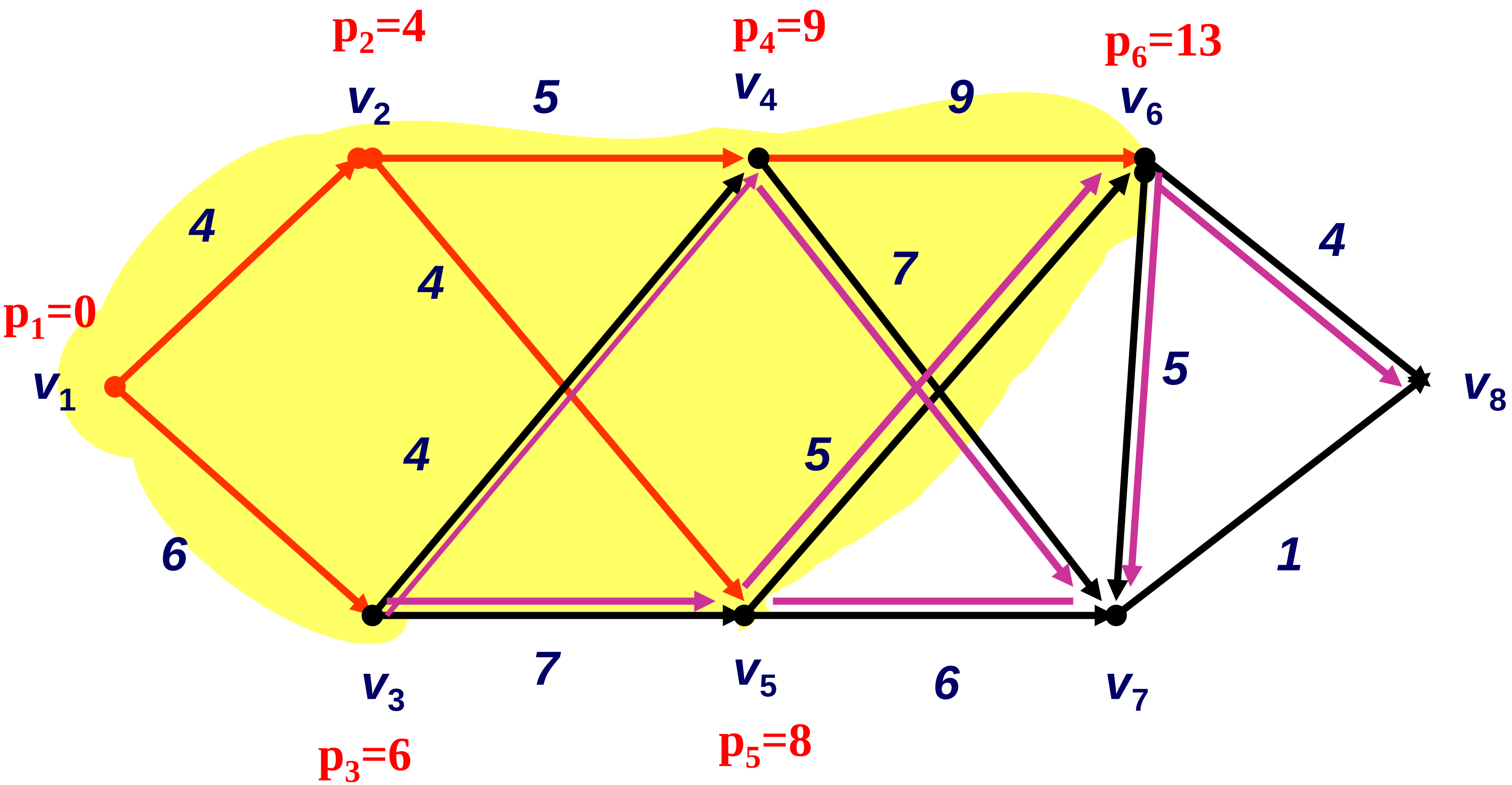
$\min\{c_{46}, c_{47}, c_{56}, c_{57}\}=\min\{9+9, 9+7, 8+5, 8+6\}=\min\{18, 16, 13, 14\}=13$

$X=\{1,2,3,4,5,6\}$, $p_6=13$



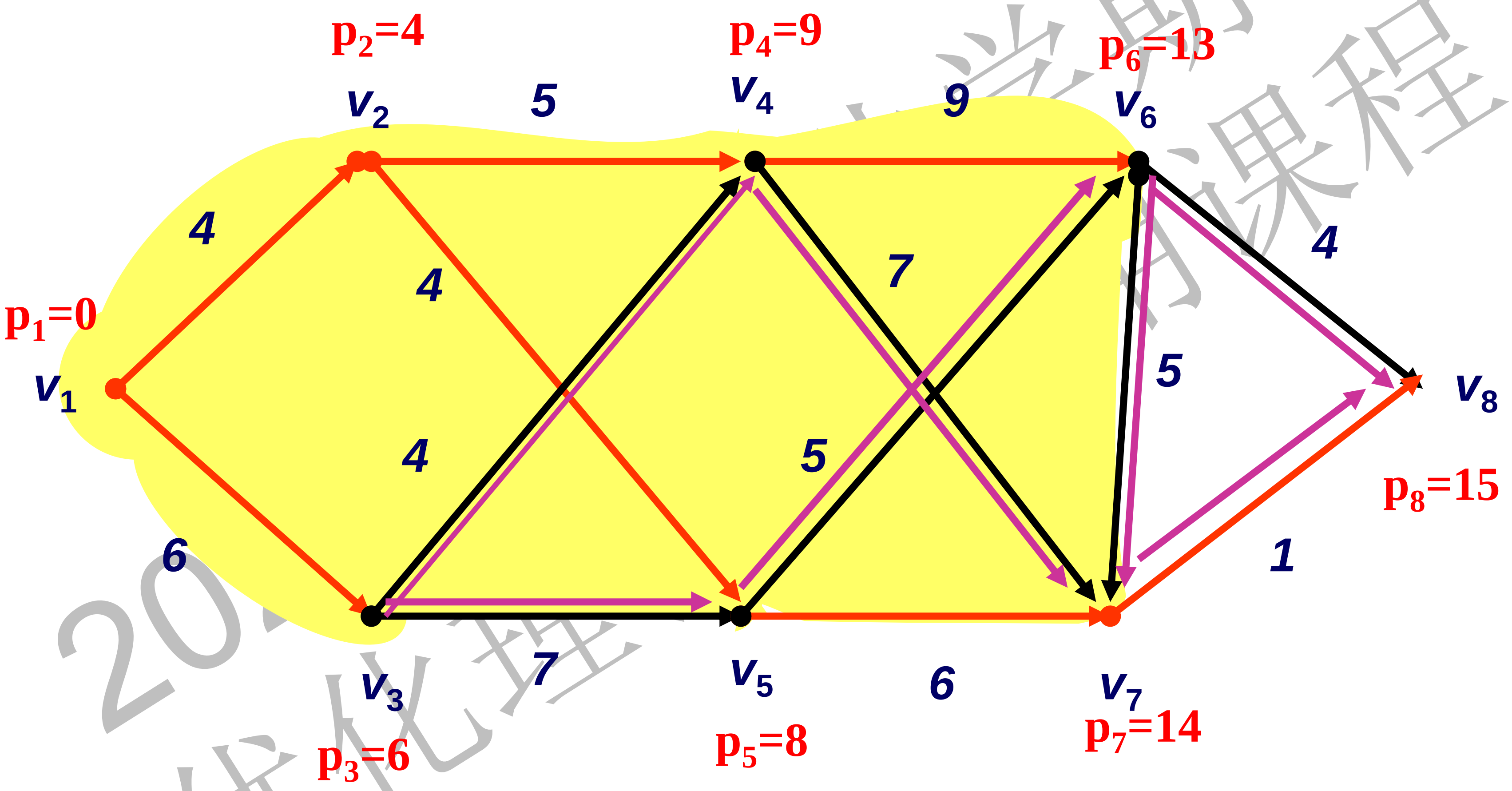
p2=4
v2
5
p4=9
v4
9
p6=13
v6
4
4
7
4
p1=0
v1
5
v8
4
5
6
1
7
v3
v5
6
v7
p3=6
p5=8

$X=\{1,2,3,4,5,6\}$



$\min\{c_{67},c_{68}, c_{57}\}=\min\{13+5,13+4,8+6\}=\min\{18, 17, 14\}=14$

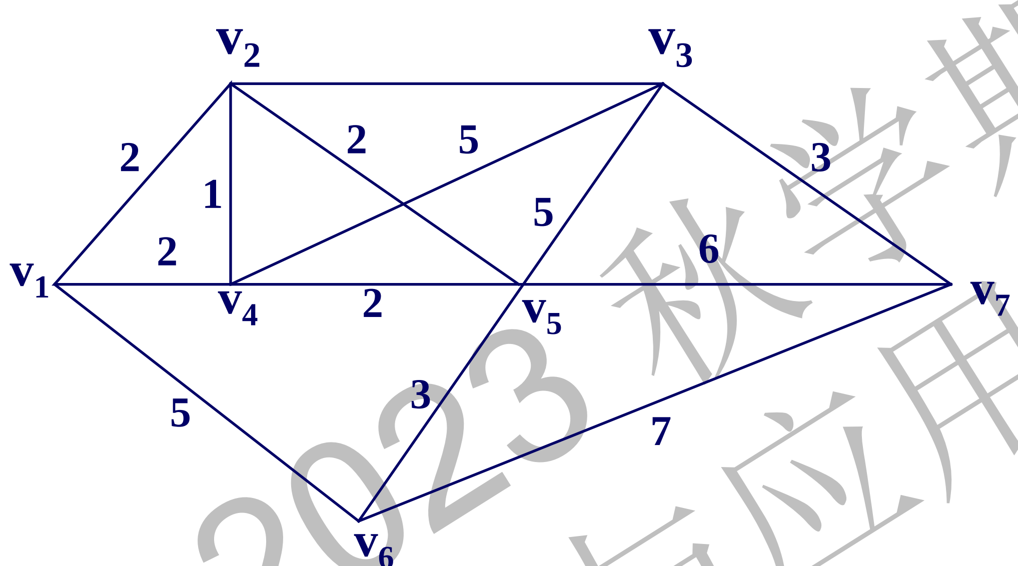
$X=\{1,2,3,4,5,6,7\}$, $p_7=14$



p1=0
v1
p2=4
v2
p3=6
v3
p4=9
v4
p5=8
v5
p6=13
v6
p7=14
v7
p8=15
v8
4
6
5
4
4
7
9
7
5
6
4
5
1

Maximum Flow Problem

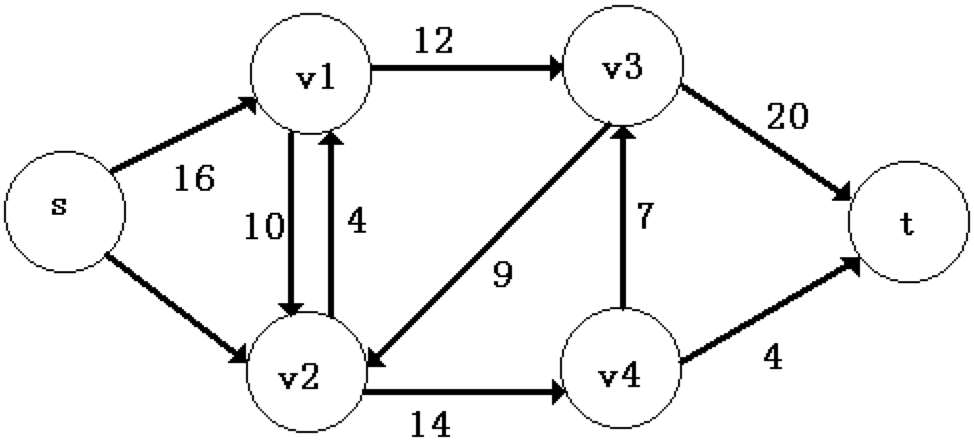
Some concepts



Undirect network

Let this picture denote a pipeline network. v_1 is a starting point， v_7 is a terminal point， all other points are transfer points. The weight of the edge denotes the capacity of that pipeline.

What is the maximum flow from v_1 to v_7?



Directed graph G=(V,E)

Some concepts

1、Capacity network:

For a directed graph G= (V, E) ,let v_s be a starting point(indegree is 0) and v_t be a terminal point(outdegree is 0) ,other points be transfer points. We call this graph as a capacity network, G= (V, E, C) .

2.Capacity and flow

capacity : the maximum flow allowed in the edge (v_i, v_j) is the capacity of that edge, c_{ij}。

flow: the real flow in the edge (v_i, v_j) , f_{ij}。

3. Feasible flow and maximum flow

If f_{ij} satisfy the following conditions:

(1) capacity condition:

For each edge $(v_i, v_j) \in E$, we have $0 \le f_{ij} \le c_{ij}$.

(2) Balance condition

$$\sum_{v_j \in A(v_i)} f_{ij} - \sum_{v_j \in B(v_i)} f_{ji} = \begin{cases} v(f) & i = s \\ 0 & i \ne s,t \\ -v(f) & i = t \end{cases}$$

f_{ij} is a feasible flow of this capacity condition.

For each capacity network, there is a maximum flow.

The minimum cut

1. cut

divide the vertex sets into two nonempty sets S and $\bar{S}$. S includes the vertex v_s, $\bar{S}$ includes the vertex v_t

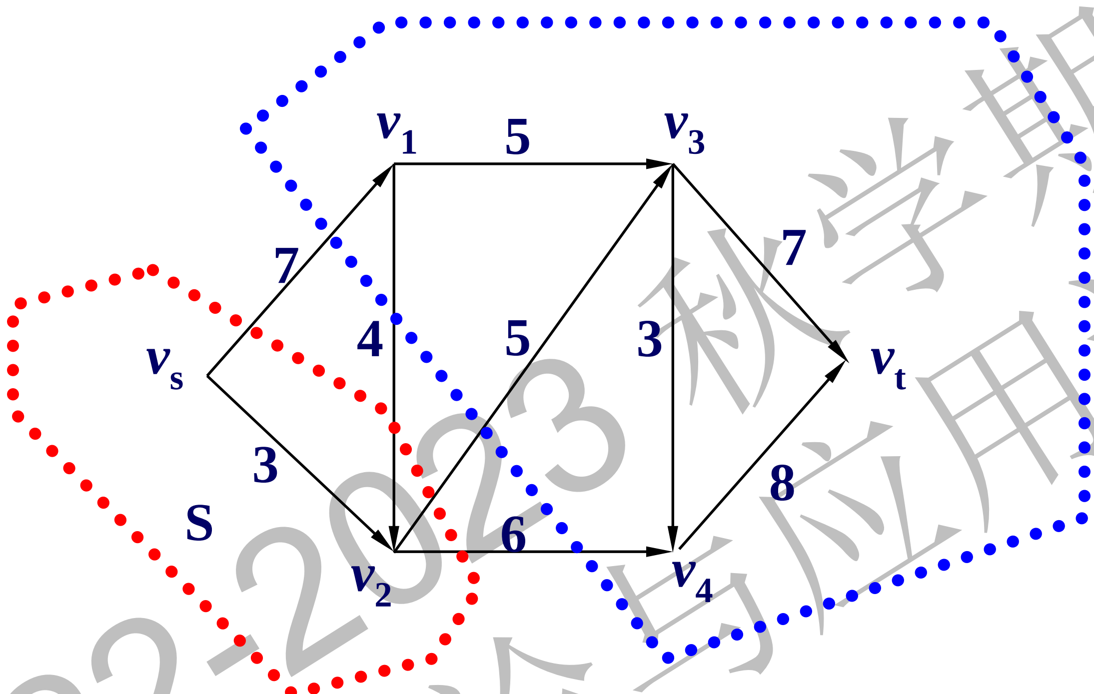
$$\begin{cases} S \cup \bar{S} = V \\ S \cap \bar{S} = \phi \\ v_i \in S, v_t \in \bar{S} \end{cases}$$

- the edge sets $(S,\bar{S}) = \{(v_i, v_j) | v_i \in S, v_j \in \bar{S}, (v_i, v_j \in A\}$ is a cut

- The sum of weights in this cut is called the capacity of the cut

$$C(S,\bar{S}) = \sum_{(v_i, v_j) \in (S,\bar{S})} c_{ij}$$

Each cut has its own capacity, the smallest one is the minimum cut.



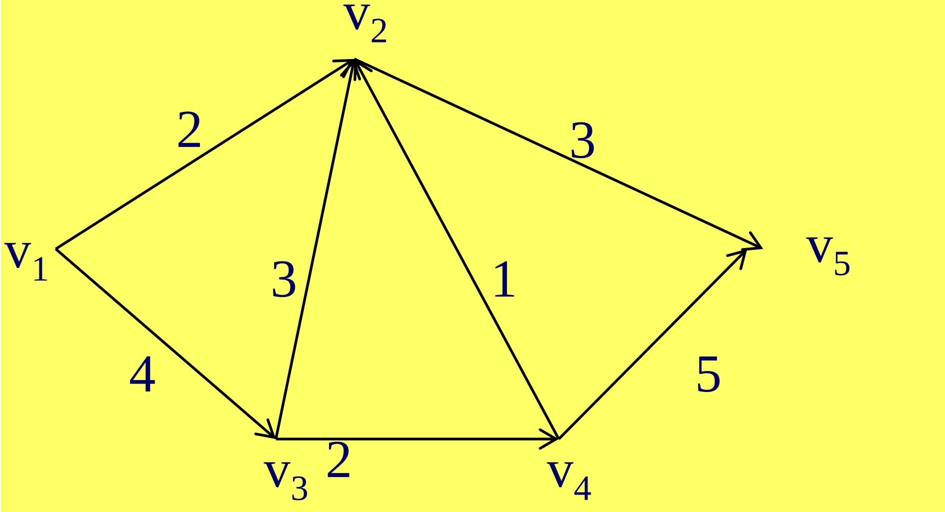
$$S=(v_s,v_2) \qquad \overline{S}=(v_1,v_3,v_4,v_t)$$

$$(S,\overline{S})=\{(v_s,v_1),(v_2,v_4),(v_2,v_3)\}$$

$$C(S,\overline{S})=l_{s1}+l_{24}+l_{23}=7+6+5=18$$

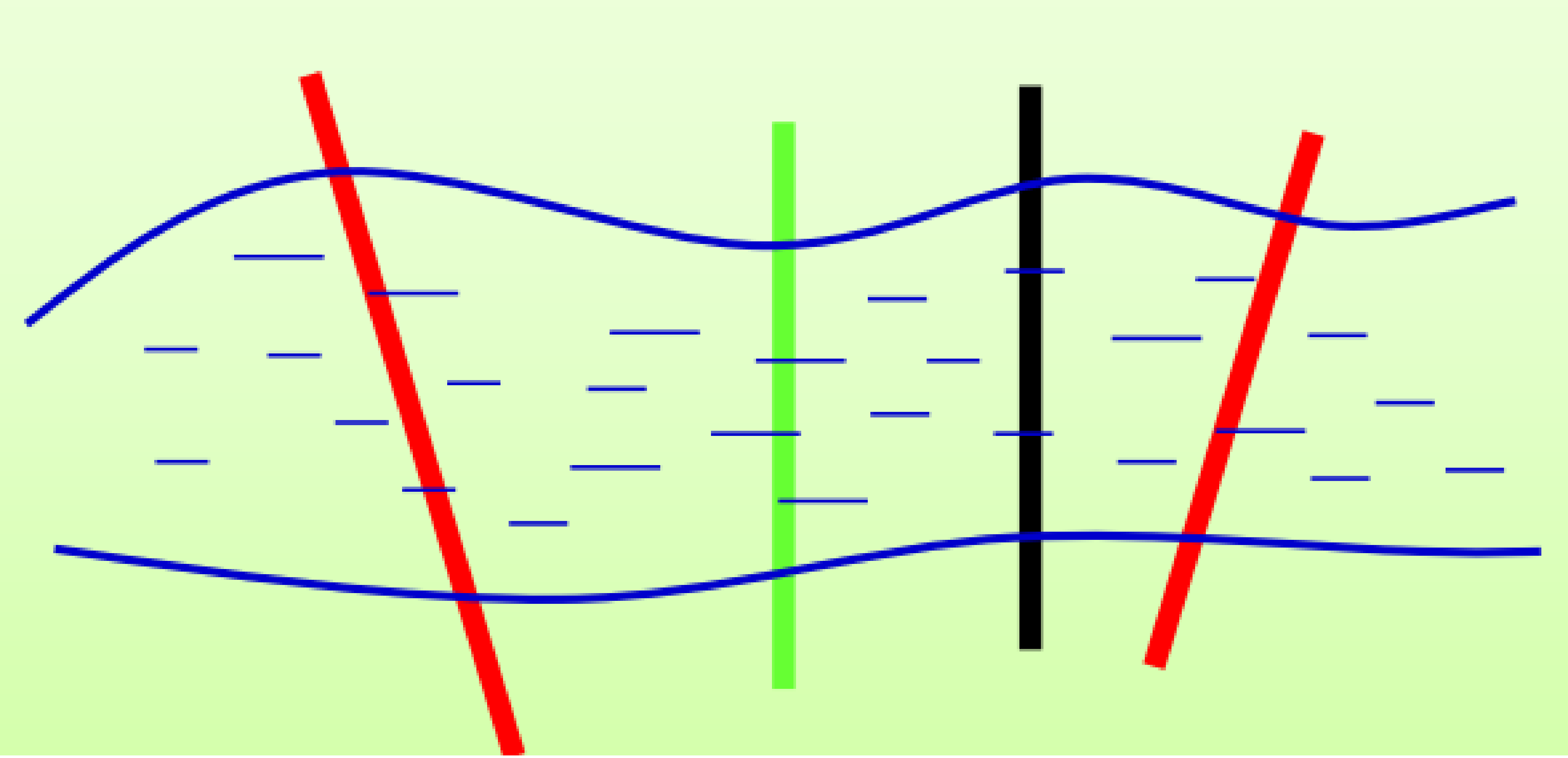
v_1 is a starting point,

v_5 is a terminal point.



S	$\bar{S}$	**cut**	**capacity**
$\{v_1\}$	$\{v_2, v_3, v_4, v_5\}$	$\{(v_1, v_2), (v_1, v_3)\}$	6
$\{v_1, v_2\}$	$\{v_3, v_4, v_5\}$	$\{(v_1, v_3), (v_2, v_5)\}$	7
$\{v_1, v_3\}$	$\{v_2, v_4, v_5\}$	$\{(v_1, v_2), (v_3, v_4), (v_3, v_2)\}$	7
$\{v_1, v_2, v_3\}$	**$\{v_4, v_5\}$**	**$\{(v_2, v_5), (v_3, v_4)\}$**	**5** Minimum cut
$\{v_1, v_4\}$	$\{v_2, v_3, v_5\}$	$\{(v_1, v_2), (v_1, v_3), (v_4, v_2), (v_4, v_5)\}$	12
$\{v_1, v_3, v_4\}$	$\{v_2, v_5\}$	$\{(v_1, v_2), (v_3, v_2), (v_4, v_2), (v_4, v_5)\}$	11
$\{v_1, v_2, v_4\}$	$\{v_3, v_5\}$	$\{(v_1, v_3), (v_2, v_5), (v_4, v_5)\}$	12
$\{v_1, v_2, v_3, v_4\}$	$\{v_5\}$	$\{(v_2, v_4), (v_4, v_5)\}$	8

Maximum flow-Minimum cut Theory:



Augment chain

chain: a path from the starting point to the terminal point

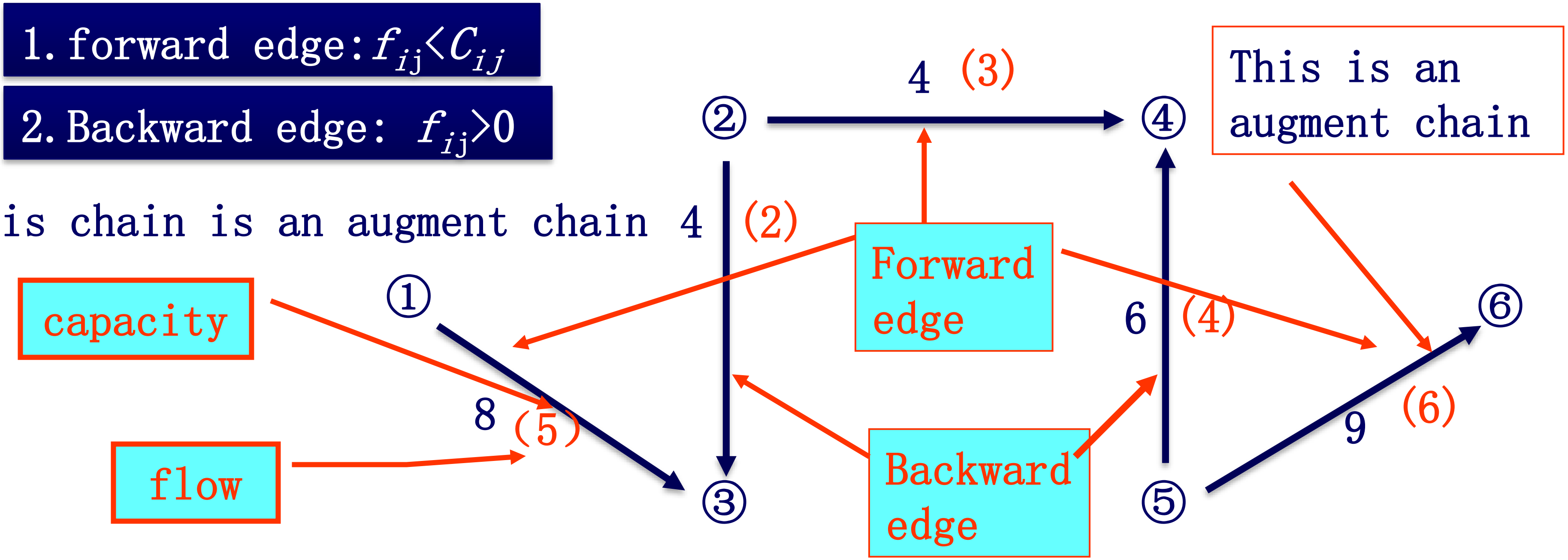
Forward edge

Backward edge

Augment chain: Let f be a feasible flow, if there is a chain *from* v_s to v_t, and satisfy the following conditions

1. forward edge: $f_{ij} < C_{ij}$
2. Backward edge: $f_{ij} > 0$

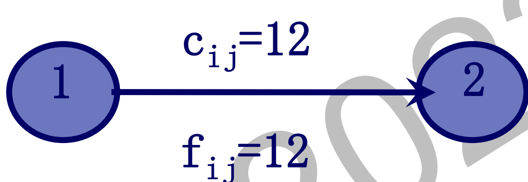
This chain is an augment chain



Saturated edge、unsaturated edge、gap

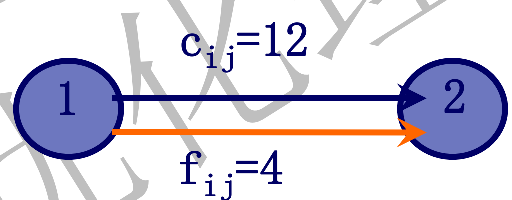
For the forward edges:

1、if $f_{ij}=c_{ij}$, from i to j is saturated



(1, 2) is saturated

2、if $f_{ij}<c_{ij}$, from i to j is unsaturated



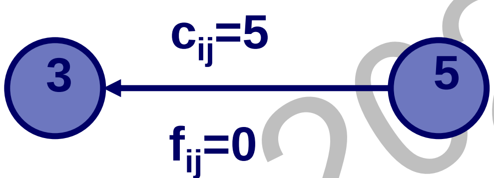
(1, 2) is unsaturated

The gap

$\theta_{12}=c_{12}-f_{12}=12-4=8$

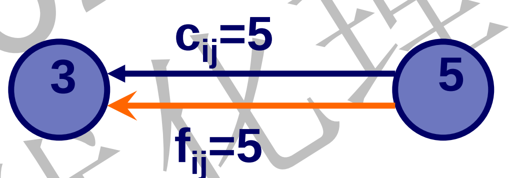
Fow all backward edges(from 3 to 5）

3、if f_{ij}=0，from j to i is saturated



（5，3）is saturated

4、if f_{ij}>0，from j to i is unsaturated



（5，3）is unsaturated

The gap $\theta_{53}=f_{53}=5$

Find the minimum gap

$$\theta^{+} = \min_{(v_i, v_j) \in \mu^{+}} \{ c_{ij} - f_{ij} \}$$

$$\theta^{-} = \min_{(v_i, v_j) \in \mu^{-}} \{ f_{ij} \}$$

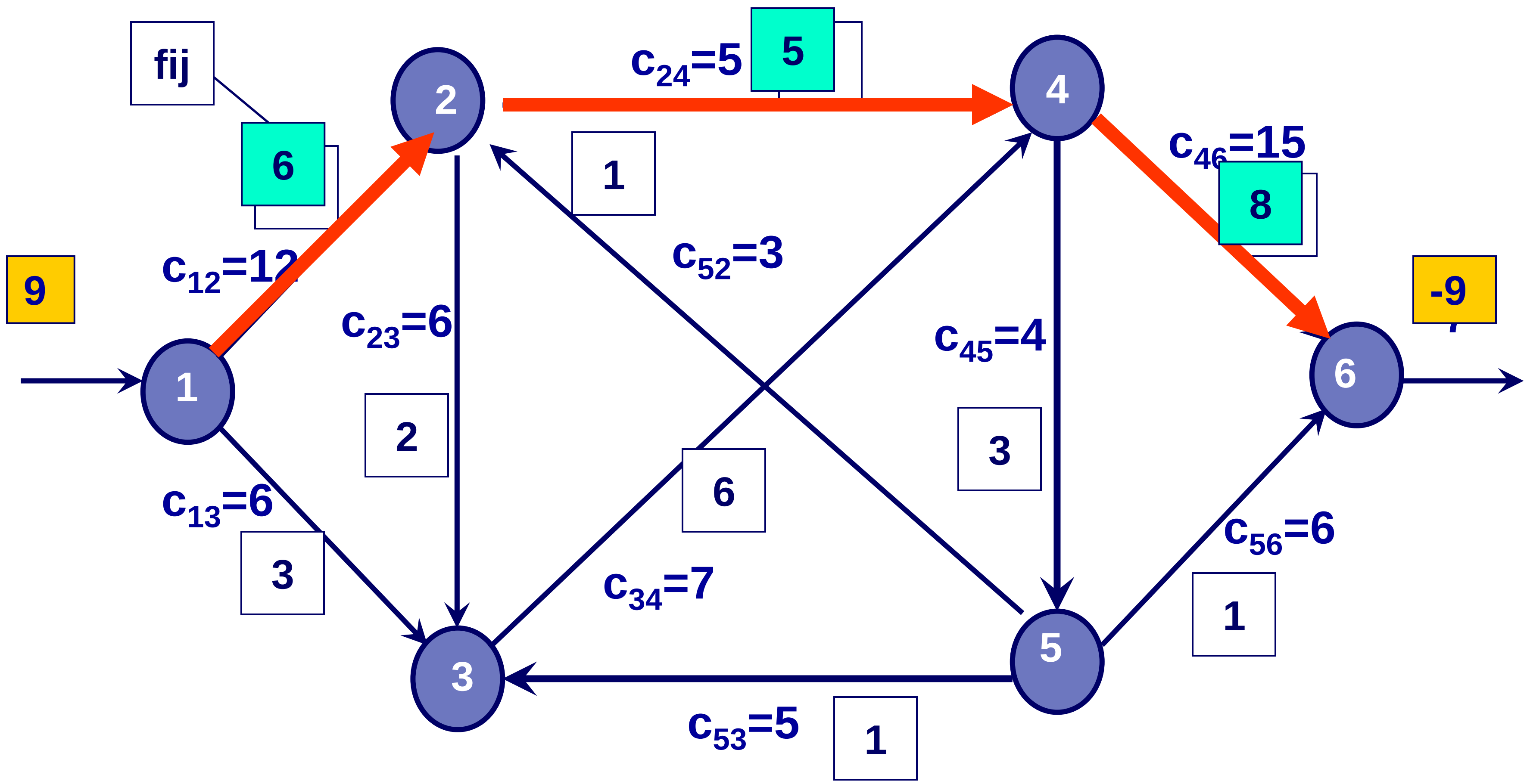
$$\theta = \min\{ \theta^{+}, \theta^{-} \}$$

Adjust the feasible flow

$$f_{ij}' = \begin{cases} f_{ij} + \theta & (v_i, v_j) \in \mu^{+} \\ f_{ij} - \theta & (v_i, v_j) \in \mu^{-} \\ f_{ij} & (v_i, v_j) \notin \mu \end{cases}$$

The new feasible flow becomes f+θ

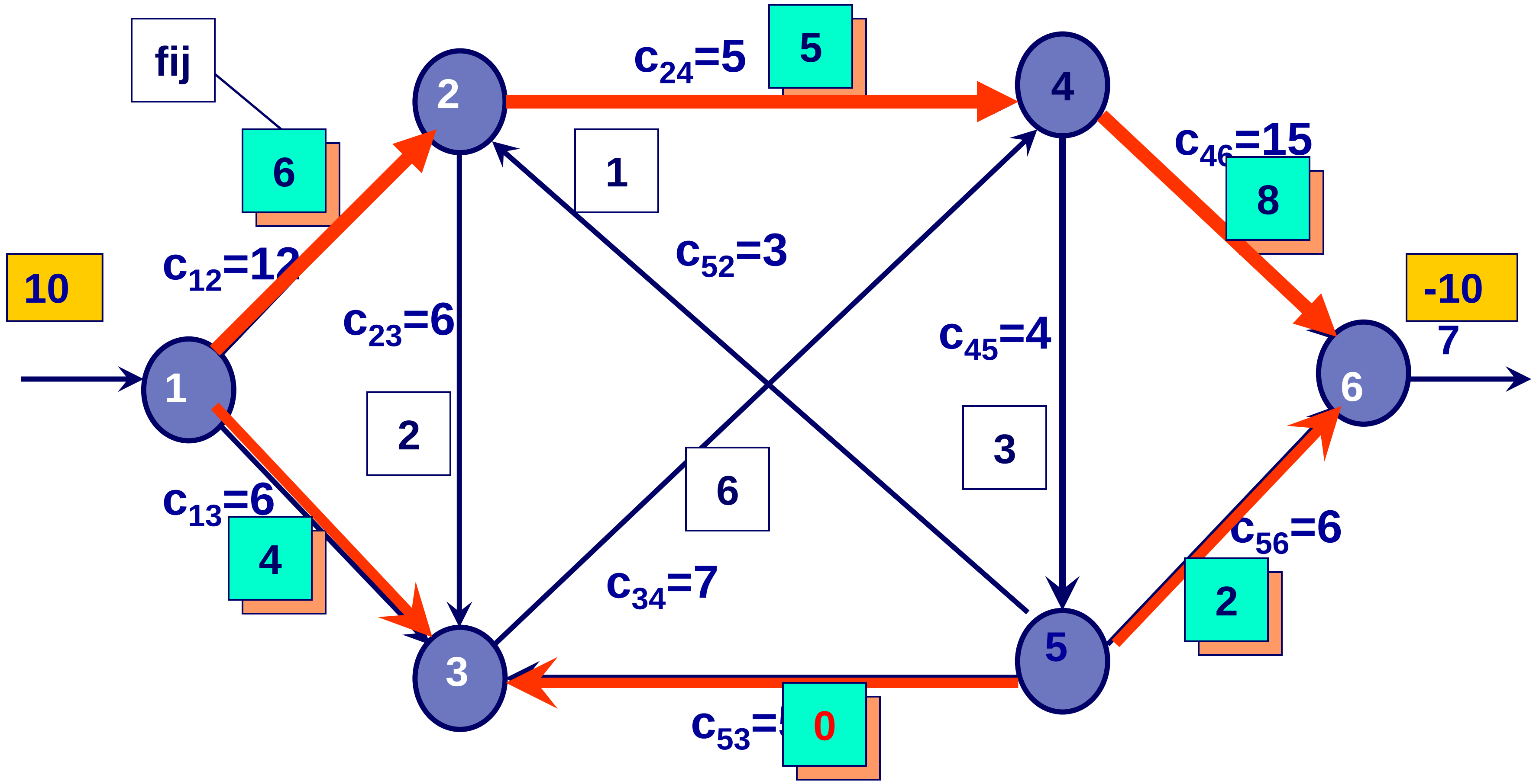
example



Augment chain 1：1-2-4-6　　θ = min{8、2、9}=2

V(f)=9

example



Augment chain 2：1-3-5-6 $\theta=\min\{3、1、5\}=1$

V(f)=10

The meaning of an augment chain:

- Along this augment chain, there is still a potential from Vs to Vt.
- According to the adjustment, we can increase the flow and still satisfy all the conditions. Hence the new flow is still feasible.

Corollary:

f is the maximum flow

There is no augment chain from v_s to v_t

Then we get a method to obtain the maximum flow.

Ford-Fulkerson label method

Step 1: find a feasible flow

Step 2: find an augment chain.

(1) label the starting point (∞)

(2) find a labeled point v_i , and check all its edges which lead to other unlabeled point:

A. If that edge is forward and $f_{ij}<c_{ij}$, then lebel v_j:

$$\theta_j = c_{ij} - f_{ij}$$

B. If that edge is backward and $f_{ji}>0$, then label v_j:

$$\theta_j = f_{ji}$$

If the terminal point has been labeled, then we find an augment chain. If the terminal point can't be labeled, the current flow is the maximum flow.

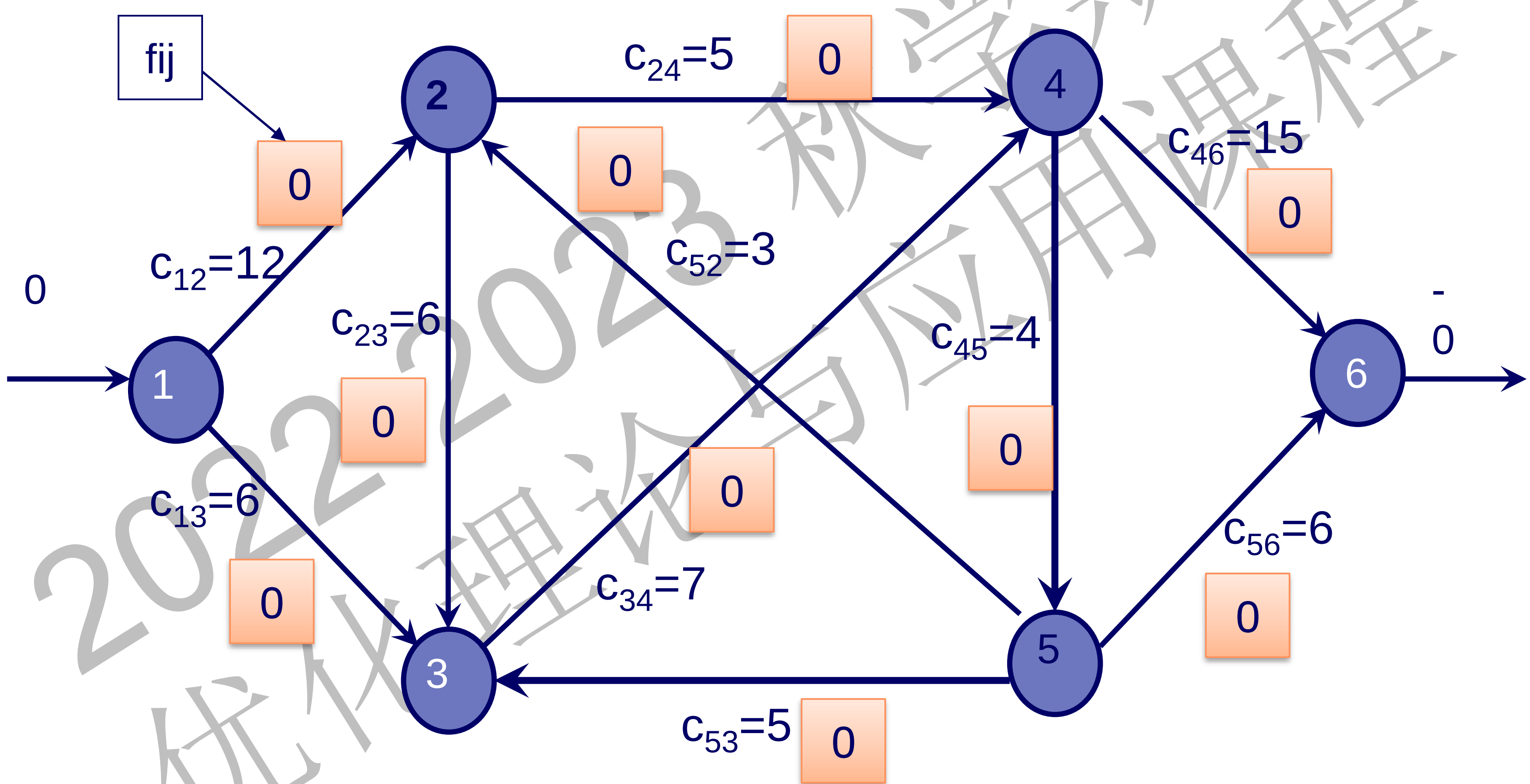
Step 3: adjust the flow
(1) find the minimum gap

$$\theta = \min_{j}\left\{\theta_j\right\}$$

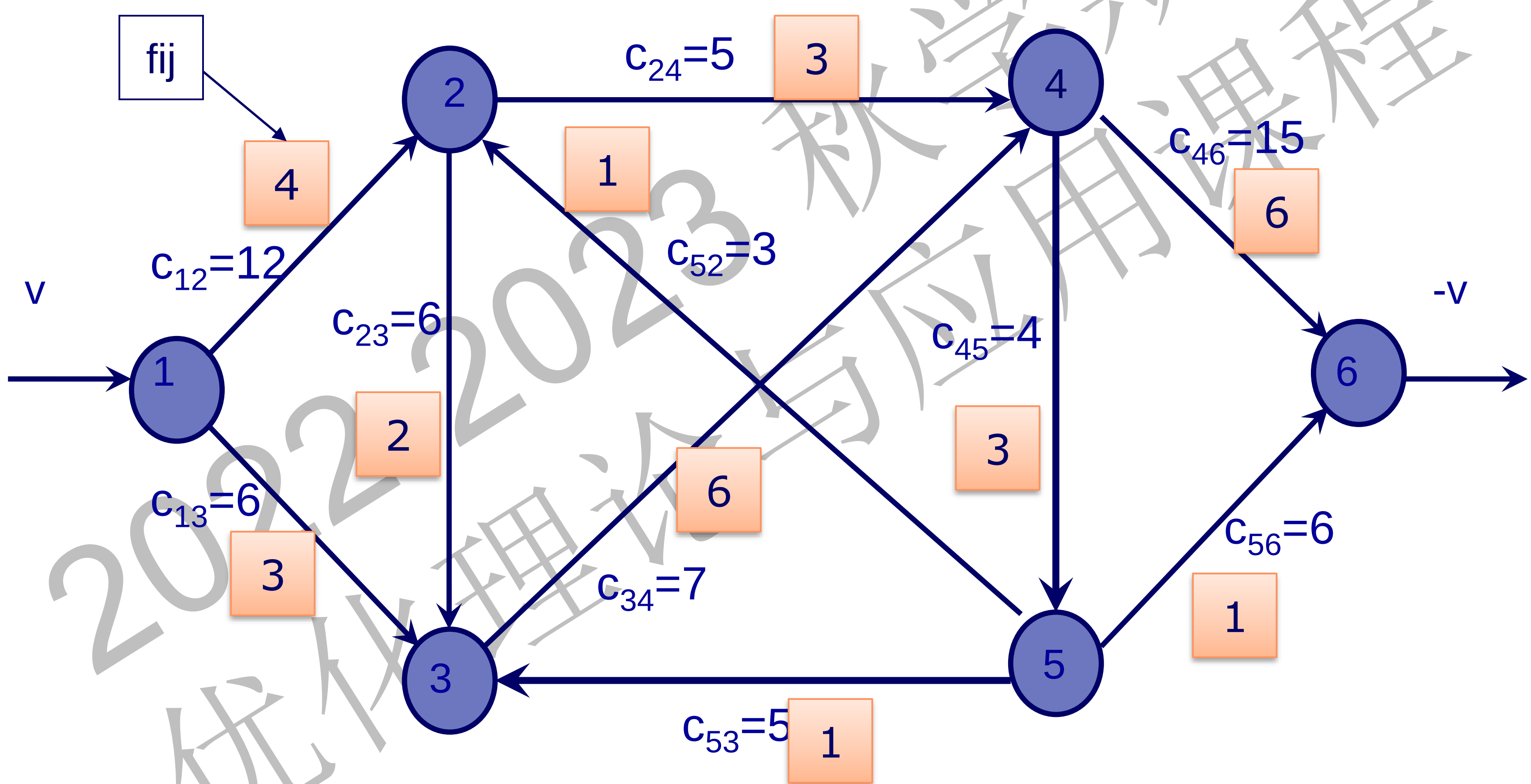
(2) adjust the flow

$$f_1 = \begin{cases} f_{ij} & (i,j) \notin \mu \\ f_{ij} + \theta & (i,j) \in \mu^+ \\ f_{ij} - \theta & (i,j) \in \mu^- \end{cases}$$

When we get a new feasible f_1, delete all the labels, go back to Step 2.

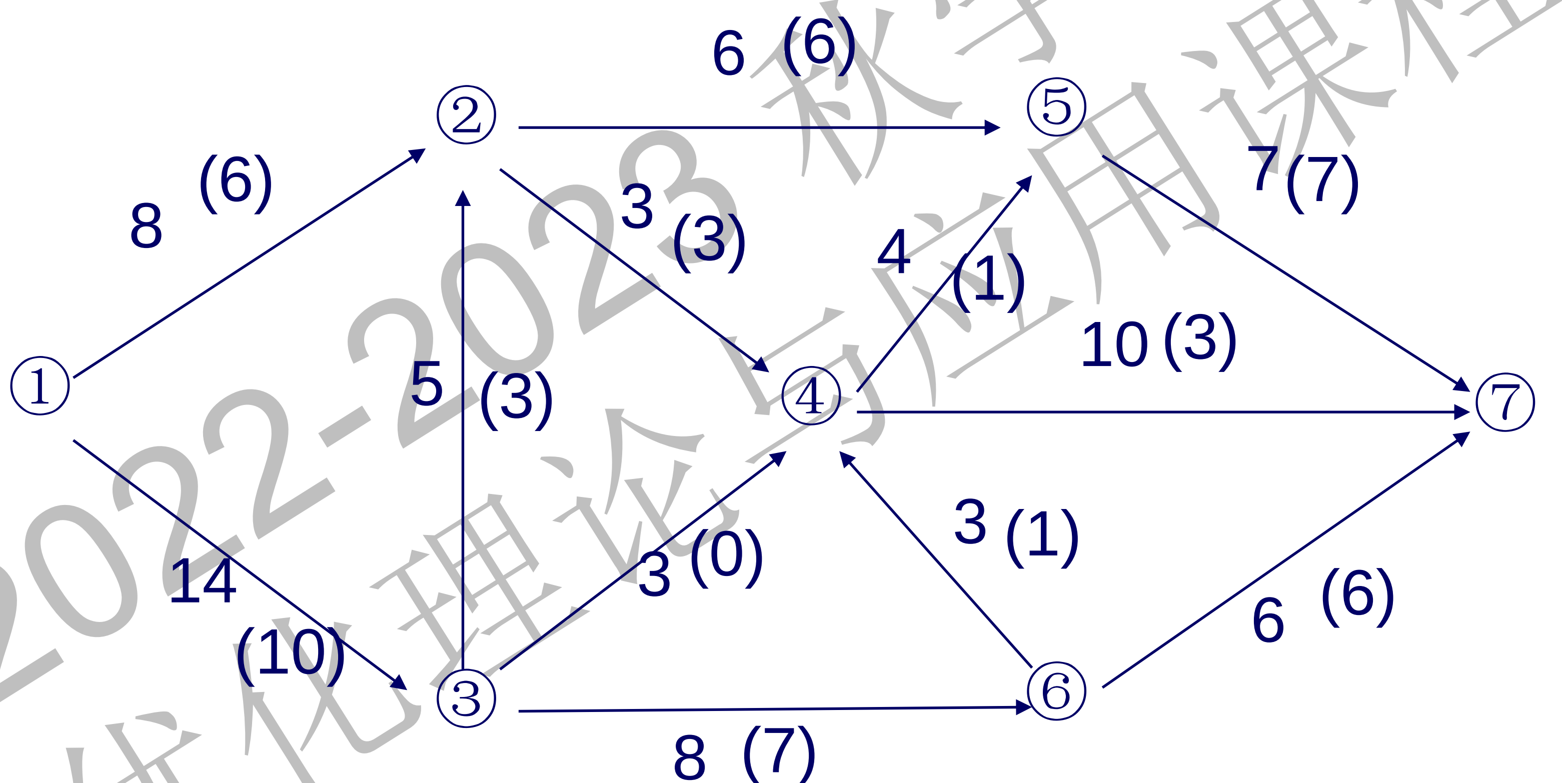


fij
0
$c_{12}=12$
0
$c_{13}=6$
0
$c_{23}=6$
0
$c_{24}=5$
0
$c_{52}=3$
0
$c_{34}=7$
0
$c_{45}=4$
0
$c_{46}=15$
0
$c_{56}=6$
0
$c_{53}=5$
0
-0
1
2
3
4
5
6
2022-2023秋学期
优化理论与应用课程


fij
v
-v
1
2
3
4
5
6
$c_{12}=12$
4
$c_{13}=6$
3
$c_{23}=6$
2
$c_{24}=5$
3
$c_{52}=3$
1
$c_{34}=7$
6
$c_{45}=4$
3
$c_{46}=15$
6
$c_{56}=6$
1
$c_{53}=5$
1

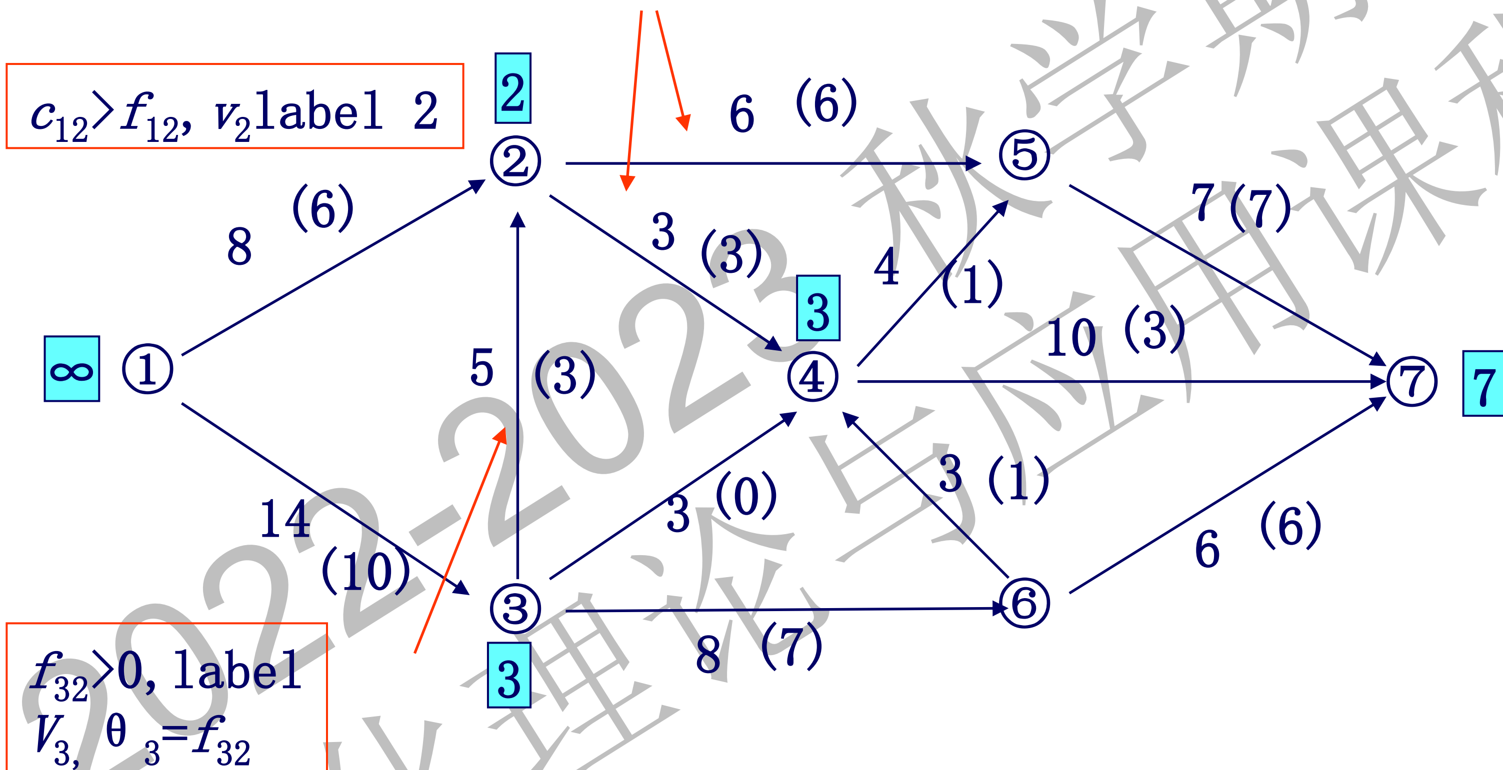
Example

For a given original feasible flow, obtain the maximum flow of this capacity network.



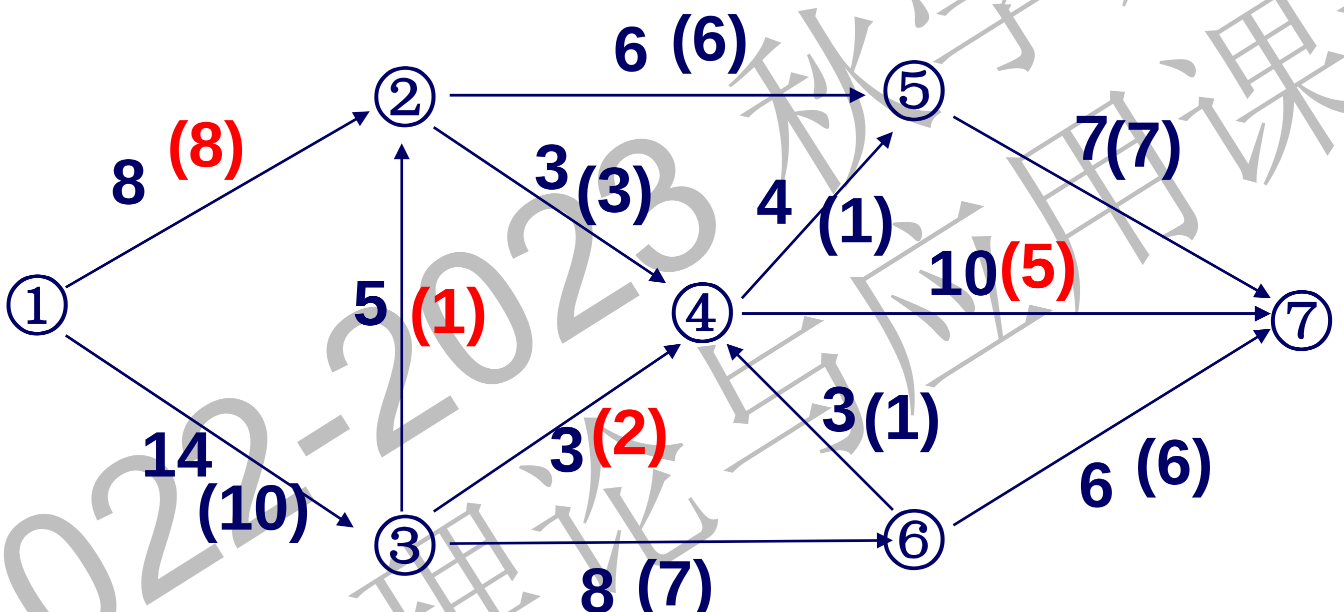
The first round label:

$c_{ij}=f_{ij}$, v_4、$v_5$can’ t be labeled



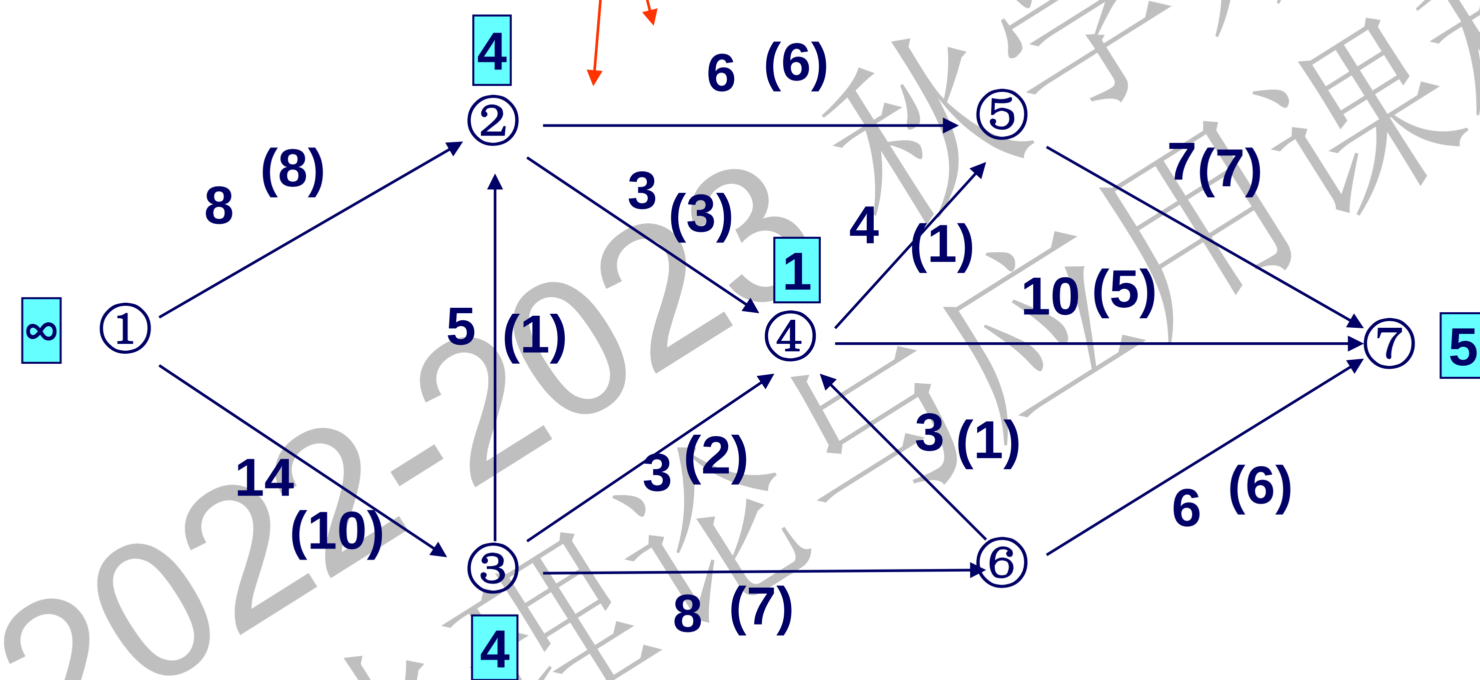
Augment chain $\mu=\{(1,2), (3,2), (3,4), (4,7)\}$, $\mu^+=\{(1,2), (3,4), (4,7)\}$, $\mu^-=\{(3, 2)\}$, the minimum gap $\theta=\min\{\infty, 2, 3, 3, 7\}=2$

The new feasible flow



The second round label:

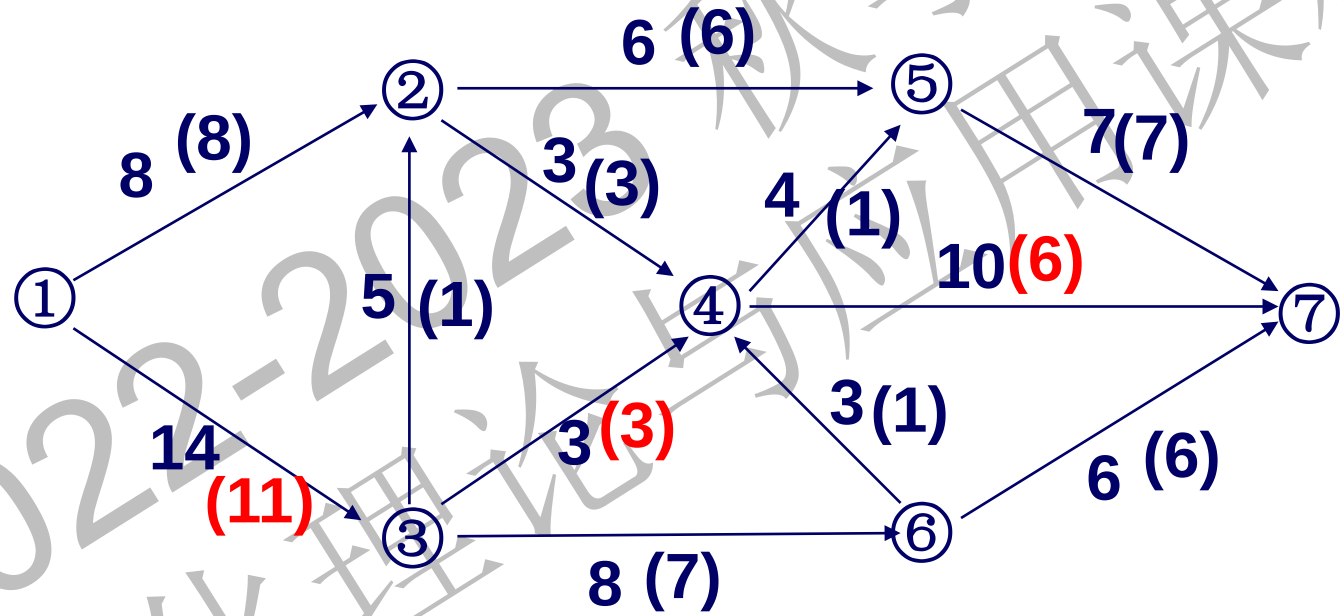
$C_{ij}=f_{ij}$, v_4、v_5 can't be labeled, go back to v_3



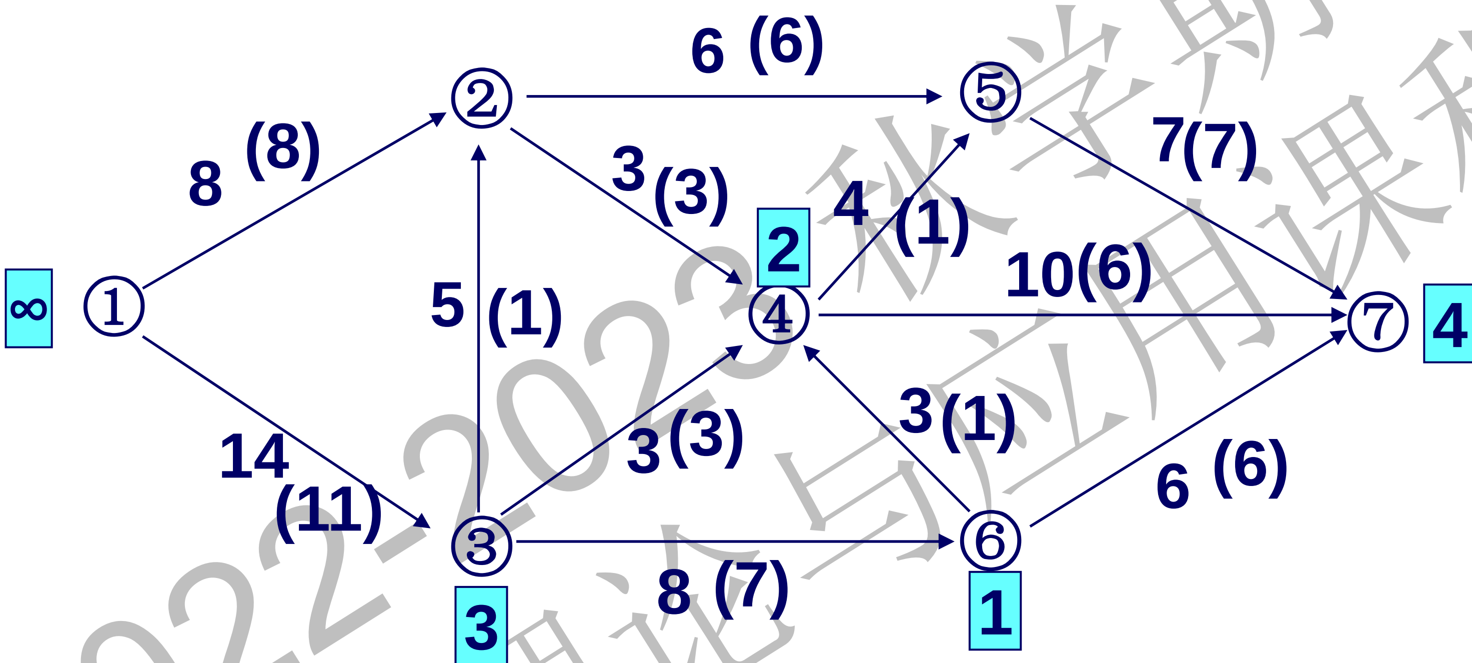
Augment chain $\mu=\mu^{+}=\{(1,3), (3,4), (4,7)\}$,

the minimum gap $\theta=\min\{\infty, 4, 1, 5\}=1$

The new feasible flow:

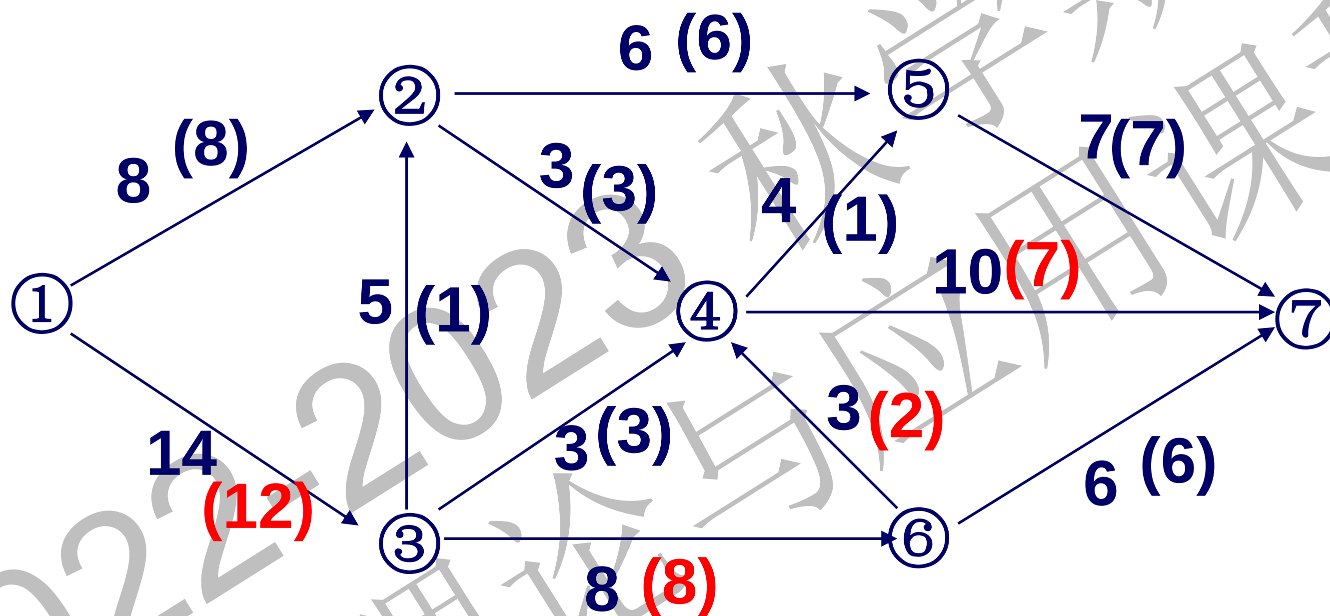


The third round label:

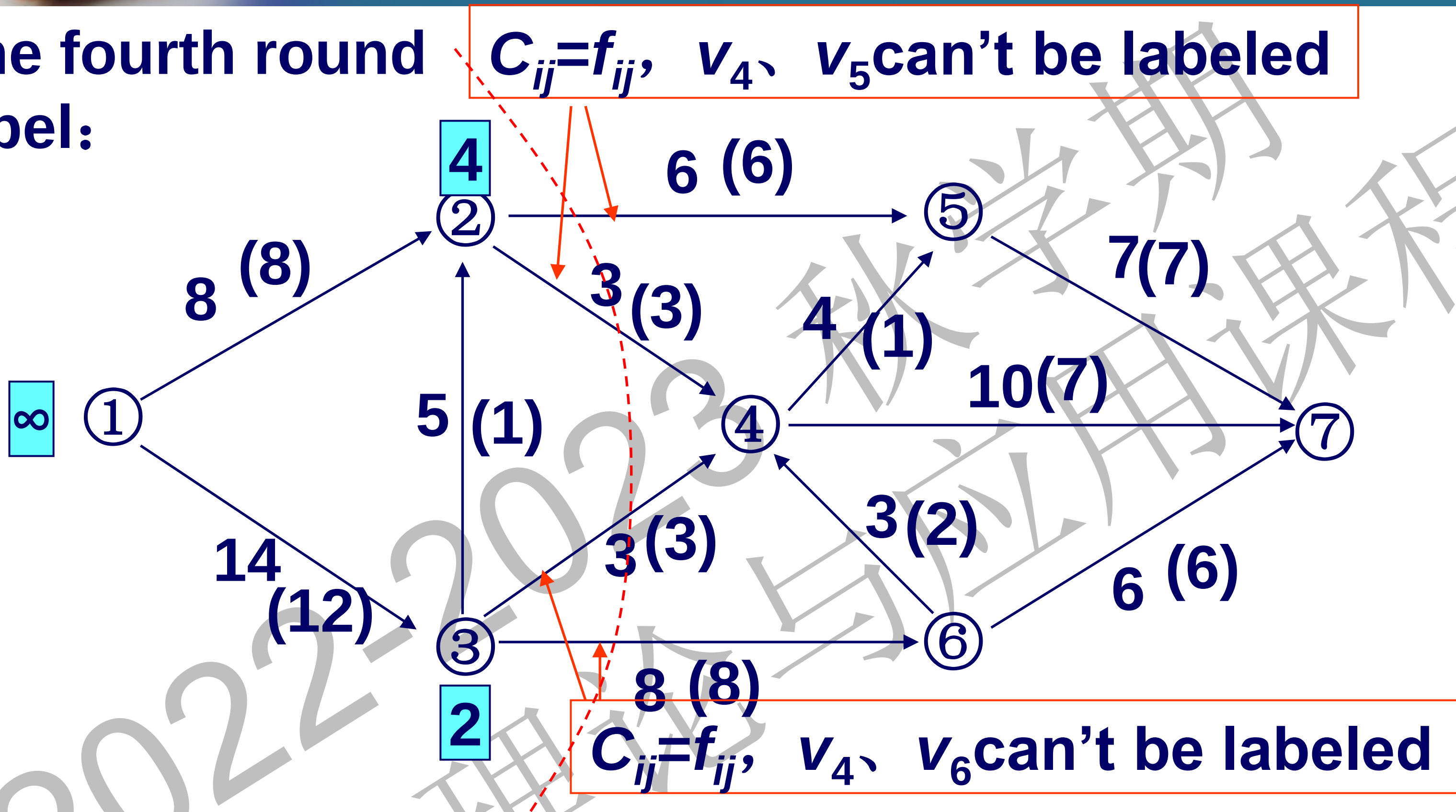


Augment chain μ＝μ⁺＝{(1,3), (3,6), (6,4), (4,7) },
the minimum gap θ＝min{∞, 3, 1, 2, 4}＝1

The new feasible flow:



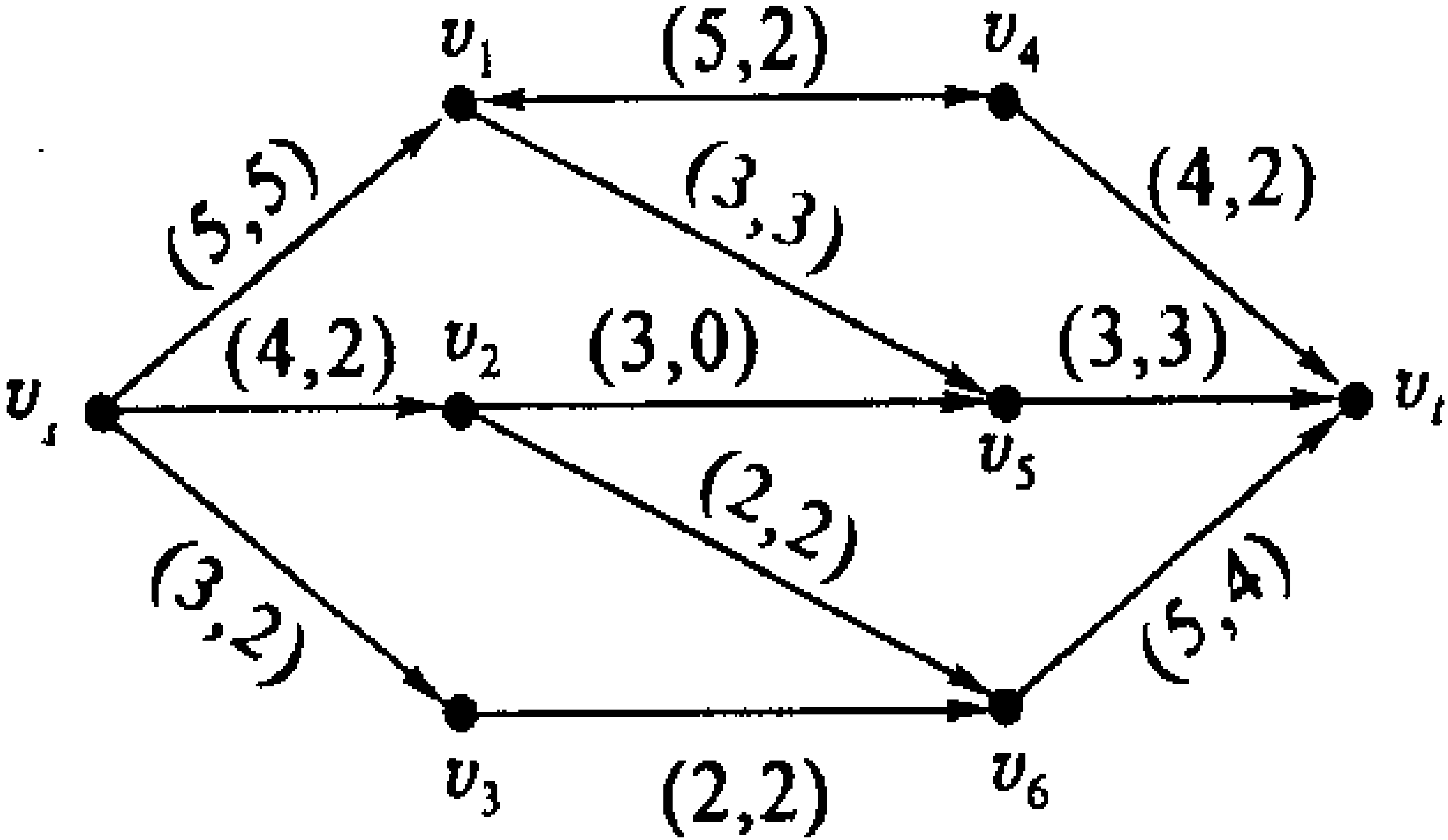
The fourth round label:

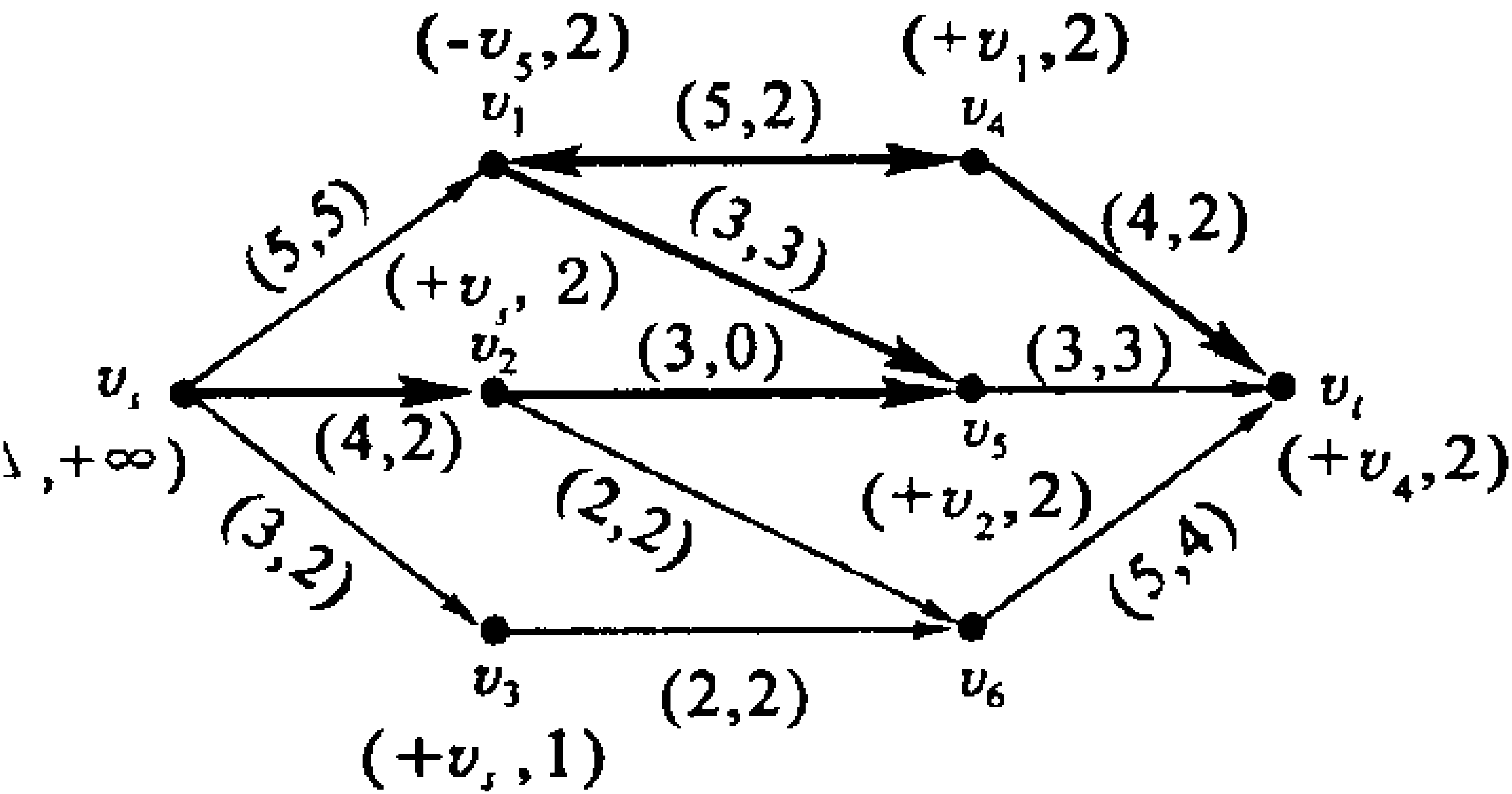


$v_7$can't be labeled, there is no augment chain from v_1 to v_7, the current flow is the maximum flow,

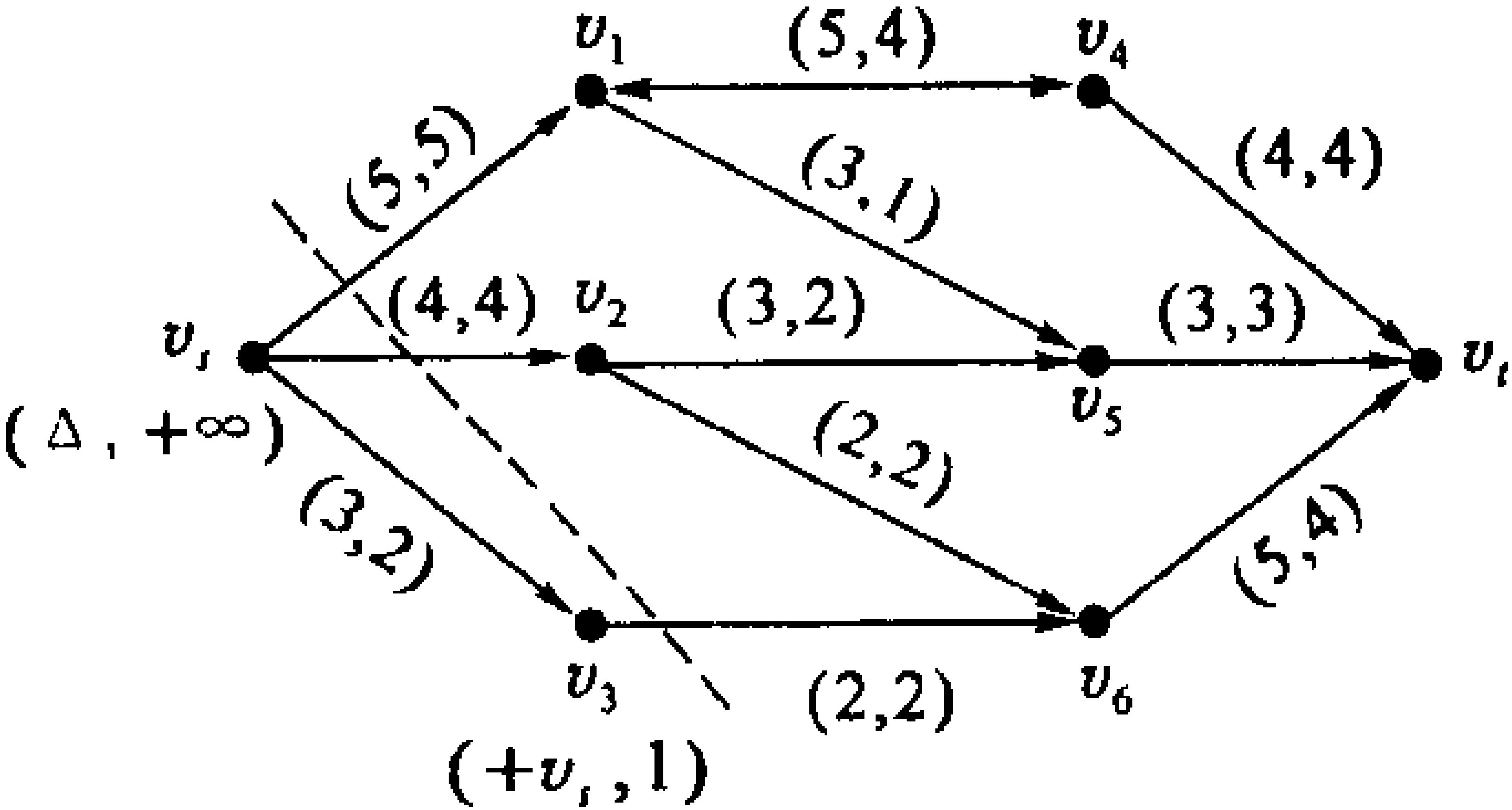
$$v = f_{12}+f_{13} = 8+12=20$$

Example





(-v5,2)
(+v1,2)
v1
(5,2)
v4
(5,5)
(3,3)
(4,2)
(+vs, 2)
v2
(3,0)
(3,3)
vs
vt
,+∞)
(4,2)
v5
(+v4,2)
(+v2,2)
(3,2)
(2,2)
(5,4)
v3
(2,2)
v6
(+vs ,1)



v_1
(5,4)
v_4
(5,5)
(3,1)
(4,4)
(4,4)
v_2
(3,2)
(3,3)
v_s
v_5
v_t
(Δ , +∞)
(2,2)
(3,2)
(5,4)
v_3
(2,2)
v_6
($+v_s$, 1)