Portfolio Optimization

章 宇

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Portfolio Optimization

- Discuss: how would you invest your money?

- In stock market? portfolio optimization!

Portfolio Optimization

- Data (Parameters)
  - i: stock, $i = 1,2$.
  - $\tilde{r}_i$: reward from stock i (*random variable*)
  - $\mu_i = \mathbb{E}[\tilde{r}_i]$: expected reward from stock i
  - $Var(\tilde{r}_i)$: variance of reward from stock i
  - $\sigma_{ij} = \mathbb{E}\left[(\tilde{r}_i - \mu_i)(\tilde{r}_j - \mu_j)\right]$: covariance
    - Note that $\sigma_{ii} = Var(\tilde{r}_i)$
  - B: budget for investment
  - β: target on expected portfolio reward

Portfolio Optimization

- Decisions
  - x_i: the amount to invest in stock $i = 1{,}2$.
- Objective
  - minimize total portfolio variance
- Constraints
  - Expected reward is above target β
  - Total investment stays within budget B
  - No short sales

Portfolio Optimization

- Model

$$\begin{aligned} \min \quad & Var(\tilde{r}_1 x_1 + \tilde{r}_2 x_2) \\ \text{s.t.} \quad & x_1 + x_2 \leq B \\ & \mu_1 x_1 + \mu_2 x_2 \geq \beta \\ & x_i \geq 0, \qquad i = 1,2 \end{aligned}$$

- How to solve it?
- Is it a linear optimization problem?

Portfolio Optimization

- Is the objective linear?

$$
\begin{aligned}
& Var(\tilde{r}_1 x_1 + \tilde{r}_2 x_2) \\
= \ & \mathbb{E}\left[\left((\tilde{r}_1 - \mu_1)x_1 + (\tilde{r}_2 - \mu_2)x_2\right)^2\right] \\
= \ & \sigma_{11} x_1^2 + 2\sigma_{12} x_1 x_2 + \sigma_{22} x_2^2 \\
= \ & \sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_{ij} x_i x_j \\
= \ & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^\top \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned}
$$

- NO! Quadratic in $\boldsymbol{x}$

Portfolio Optimization

- More generally…

$$
\begin{aligned}
& Var(\tilde{r}_1 x_1 + \cdots + \tilde{r}_n x_n) \\
= & \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^\top \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\
= & \ \boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x}
\end{aligned}
$$

  – $\boldsymbol{\Sigma}$ is an $n \times n$ square matrix

Covariance matrix

- $\boldsymbol{\Sigma}$: covariance matrix
  - symmetric matrix
  - positive semidefinite matrix

$$\boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x} \geq 0, \qquad \forall \boldsymbol{x} \in \mathbb{R}^n$$

  - The eigenvalues are real and nonnegative
  - There exists matrix $\boldsymbol{A}$ such that $\boldsymbol{\Sigma} = \boldsymbol{A}^\top \boldsymbol{A}$

- How to estimate $\boldsymbol{\Sigma}$?

Covariance matrix

- Estimating $\boldsymbol{\Sigma}$ from historical data
  - I stocks, T periods of data
  - $\hat{r}_{ti}$: historical return of stock i at period t.
  - $\boldsymbol{R} \in \mathbb{R}^{T \times I}$: matrix of $\hat{r}_{ti}$
  - $\mu_i = \frac{1}{T} \sum_{t=1}^{T} \hat{r}_{ti}$: estimated mean reward
  - Covariance matrix:

$$\boldsymbol{\Sigma} = \frac{1}{T-1} (\boldsymbol{R} - \mathbf{1}\boldsymbol{\mu}^\top)^\top (\boldsymbol{R} - \mathbf{1}\boldsymbol{\mu}^\top)$$

where $\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_I \end{bmatrix} \in \mathbb{R}^I$ and $\mathbf{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^T$

Covariance matrix

- Example
  - $I = 2$ stocks, $T = 3$ periods of data

$$\boldsymbol{\Sigma} = \frac{1}{T-1}(\boldsymbol{R} - \mathbf{1}\boldsymbol{\mu}^{\top})^{\top}(\boldsymbol{R} - \mathbf{1}\boldsymbol{\mu}^{\top})$$

$$= \frac{1}{T-1}\begin{bmatrix} \hat{r}_{11} - \mu_1 & \hat{r}_{21} - \mu_1 & \hat{r}_{31} - \mu_1 \\ \hat{r}_{12} - \mu_2 & \hat{r}_{22} - \mu_2 & \hat{r}_{32} - \mu_2 \end{bmatrix}\begin{bmatrix} \hat{r}_{11} - \mu_1 & \hat{r}_{12} - \mu_2 \\ \hat{r}_{21} - \mu_1 & \hat{r}_{22} - \mu_2 \\ \hat{r}_{31} - \mu_1 & \hat{r}_{32} - \mu_2 \end{bmatrix}$$

$$= \frac{1}{T-1}\begin{bmatrix} \sum_{t=1}^{3}(\hat{r}_{t1} - \mu_1)^2 & \sum_{t=1}^{3}(\hat{r}_{t1} - \mu_1)(\hat{r}_{t2} - \mu_2) \\ \sum_{t=1}^{3}(\hat{r}_{t2} - \mu_2)(\hat{r}_{t1} - \mu_1) & \sum_{t=1}^{3}(\hat{r}_{t2} - \mu_2)^2 \end{bmatrix}$$

Portfolio Optimization

- Model

$$\begin{aligned} \min\ & Var(\tilde{r}_1 x_1 + \tilde{r}_2 x_2) \\ \text{s.t.}\ & x_1 + x_2 \le B \\ & \mu_1 x_1 + \mu_2 x_2 \ge \beta \\ & x_i \ge 0, \qquad i = 1,2 \end{aligned}$$

  – Compact form

$$\begin{aligned} \min\ & \boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x} \\ \text{s.t.}\ & \mathbf{1}^\top \boldsymbol{x} \le B \\ & \boldsymbol{\mu}^\top \boldsymbol{x} \ge \beta \\ & \boldsymbol{x} \ge \mathbf{0} \end{aligned}$$

Portfolio Optimization

- Model

$$
\begin{aligned}
\min \quad & \boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x} \\
\text{s.t.} \quad & \mathbf{1}^\top \boldsymbol{x} \le B \\
& \boldsymbol{\mu}^\top \boldsymbol{x} \ge \beta \\
& \boldsymbol{x} \ge \mathbf{0}
\end{aligned}
$$

- A **quadratic optimization problem**!
- Solvable via Gurobi ☺

Portfolio Optimization

- Coding…

Portfolio Optimization II

- An alternative model

$$\begin{aligned} \max \quad & \mathbb{P}[\tilde{\boldsymbol{r}}^\top \boldsymbol{x} \geq \beta] \\ \mathrm{s.t.} \quad & \mathbf{1}^\top \boldsymbol{x} \leq B \\ & \boldsymbol{\mu}^\top \boldsymbol{x} \geq \beta \\ & \boldsymbol{x} \geq \mathbf{0} \end{aligned}$$

  – Different measures of risk
    - Variance v.s. probability of target attainment

  – How to solve it?

Portfolio Optimization II

- Generally intractable ☹
- Solvable under normal distribution assumption of $\tilde{\boldsymbol{r}}$ ☺

$$
\begin{aligned}
&\mathbb{P}[\tilde{\boldsymbol{r}}^\top \boldsymbol{x} \geq \beta] \\
= \ &\mathbb{P}\left[\frac{\boldsymbol{\mu}^\top \boldsymbol{x} - \tilde{\boldsymbol{r}}^\top \boldsymbol{x}}{\sqrt{\boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x}}} \leq \frac{\boldsymbol{\mu}^\top \boldsymbol{x} - \beta}{\sqrt{\boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x}}}\right] \\
= \ &\Phi\left(\frac{\boldsymbol{\mu}^\top \boldsymbol{x} - \beta}{\sqrt{\boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x}}}\right)
\end{aligned}
$$

  - $\Phi(\cdot)$: the cumulative distribution function of standard normal.

Portfolio Optimization II

$$
\begin{array}{ll} \max & \mathbb{P}(\tilde{\boldsymbol{r}}'\boldsymbol{x} \geq \beta) \\ \text{s.t.} & \mathbf{1}'\boldsymbol{x} \leq B \\ & \boldsymbol{\mu}'\boldsymbol{x} \geq \beta \\ & \boldsymbol{x} \geq \mathbf{0} \end{array}
\quad \Leftrightarrow \quad
\begin{array}{ll} \max & \Phi\left(\frac{\boldsymbol{\mu}'\boldsymbol{x}-\beta}{\sqrt{\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}}}\right) \\ \text{s.t.} & \mathbf{1}'\boldsymbol{x} \leq B \\ & \boldsymbol{\mu}'\boldsymbol{x} \geq \beta \\ & \boldsymbol{x} \geq \mathbf{0} \end{array}
\quad \Leftrightarrow \quad
\begin{array}{ll} \max & \dfrac{\boldsymbol{\mu}'\boldsymbol{x}-\beta}{\sqrt{\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}}} \\ \text{s.t.} & \mathbf{1}'\boldsymbol{x} \leq B \\ & \boldsymbol{\mu}'\boldsymbol{x} \geq \beta \\ & \boldsymbol{x} \geq \mathbf{0} \end{array}
$$

- Note: "$\Leftrightarrow$" in the sense that the models have the same optimal solutions
- Recall Sharpe ratio:

$$\frac{\boldsymbol{\mu}^{\top}\boldsymbol{x}-\beta}{\sqrt{\boldsymbol{x}^{\top}\boldsymbol{\Sigma}\boldsymbol{x}}}$$

Portfolio Optimization II

- Still intractable… Continue…

$$\begin{array}{ll} \max & \dfrac{\boldsymbol{\mu}'\boldsymbol{x} - \beta}{\sqrt{\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}}} \\ \text{s.t.} & \mathbf{1}'\boldsymbol{x} \le B \\ & \boldsymbol{\mu}'\boldsymbol{x} \ge \beta \\ & \boldsymbol{x} \ge \mathbf{0} \end{array} \Leftrightarrow \begin{array}{ll} \min & \dfrac{\sqrt{\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}}}{\boldsymbol{\mu}'\boldsymbol{x} - \beta} \\ \text{s.t.} & \mathbf{1}'\boldsymbol{x} \le B \\ & \boldsymbol{\mu}'\boldsymbol{x} \ge \beta \\ & \boldsymbol{x} \ge \mathbf{0} \end{array} \Leftrightarrow \begin{array}{ll} \min & \dfrac{\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}}{(\boldsymbol{\mu}'\boldsymbol{x} - \beta)^2} \\ \text{s.t.} & \mathbf{1}'\boldsymbol{x} \le B \\ & \boldsymbol{\mu}'\boldsymbol{x} \ge \beta \\ & \boldsymbol{x} \ge \mathbf{0} \end{array}$$

Portfolio Optimization II

- Suppose there exists an optimal solution such that $\boldsymbol{\mu}^\top \boldsymbol{x} > \beta$, i.e., $y > 0$.

$$
\begin{aligned}
\min\quad & \frac{\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}}{(\boldsymbol{\mu}'\boldsymbol{x}-\beta)^2} \\
\text{s.t.}\quad & \mathbf{1}'\boldsymbol{x} \le B \\
& \boldsymbol{\mu}'\boldsymbol{x} \ge \beta \\
& \boldsymbol{x} \ge \mathbf{0}
\end{aligned}
\quad \Leftrightarrow \quad
\begin{aligned}
\min\quad & \frac{\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}}{y^2} \\
\text{s.t.}\quad & \mathbf{1}'\boldsymbol{x} \le B \\
& \boldsymbol{\mu}'\boldsymbol{x} - \beta = y \\
& y \ge 0 \\
& \boldsymbol{x} \ge \mathbf{0}
\end{aligned}
\quad \Leftrightarrow \quad
\begin{aligned}
\min\quad & \frac{\boldsymbol{x}'}{y}\boldsymbol{\Sigma}\frac{\boldsymbol{x}}{y} \\
\text{s.t.}\quad & \mathbf{1}'\frac{\boldsymbol{x}}{y} \le B\frac{1}{y} \\
& \boldsymbol{\mu}'\frac{\boldsymbol{x}}{y} - \beta\frac{1}{y} = 1 \\
& \frac{1}{y} \ge 0 \\
& \frac{\boldsymbol{x}}{y} \ge \mathbf{0}
\end{aligned}
$$

Portfolio Optimization II

- Change of variables (trick!)
  - Let $\bar{\boldsymbol{x}} = \boldsymbol{x}/y$, $z = 1/y$

$$
\begin{aligned}
\min \quad & \frac{\boldsymbol{x}'}{y}\boldsymbol{\Sigma}\frac{\boldsymbol{x}}{y} \\
\text{s.t.} \quad & \mathbf{1}'\frac{\boldsymbol{x}}{y} \le B\frac{1}{y} \\
& \boldsymbol{\mu}'\frac{\boldsymbol{x}}{y} - \beta\frac{1}{y} = 1 \\
& \frac{1}{y} \ge 0 \\
& \frac{\boldsymbol{x}}{y} \ge \mathbf{0}
\end{aligned}
\quad \Leftrightarrow \quad
\begin{aligned}
\min \quad & \bar{\boldsymbol{x}}'\boldsymbol{\Sigma}\bar{\boldsymbol{x}} \\
\text{s.t.} \quad & \mathbf{1}'\bar{\boldsymbol{x}} \le Bz \\
& \boldsymbol{\mu}'\bar{\boldsymbol{x}} - \beta z = 1 \\
& z \ge 0 \\
& \bar{\boldsymbol{x}} \ge \mathbf{0}
\end{aligned}
$$

Portfolio Optimization II

- Solve the quadratic optimization problem:

$$\begin{array}{ll} \text{min} & \bar{\boldsymbol{x}}'\boldsymbol{\Sigma}\bar{\boldsymbol{x}} \\ \text{s.t.} & \mathbf{1}'\bar{\boldsymbol{x}} \leq Bz \\ & \boldsymbol{\mu}'\bar{\boldsymbol{x}} - \beta z = 1 \\ & z \geq 0 \\ & \bar{\boldsymbol{x}} \geq \mathbf{0} \end{array}$$

- Actual portfolio weights: $\boldsymbol{x} = \bar{\boldsymbol{x}}y = \bar{\boldsymbol{x}}/z$

Homework

- Solve the portfolio optimization model on Page 14, using Python and Gurobi
- Use the data in ‘data_portfolio.csv’

- Submit the source code through a link to be given in the QQ group

- Deadline: before next class
  - Tardiness is not allowed