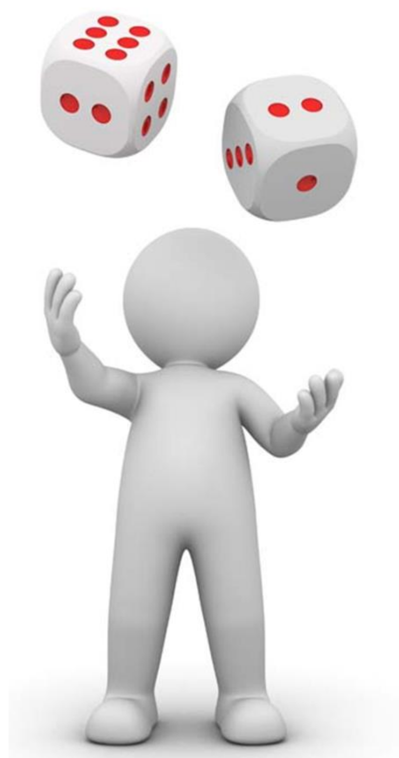
What is Operations Research?

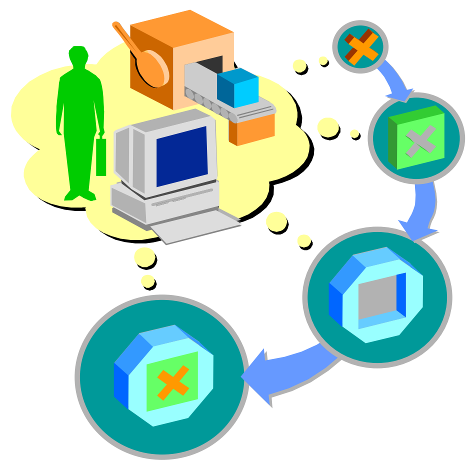
- **It is a basic applied mathematics. It focuses on the optimization problem by building and solving models, then provides quantitative solutions to the managers.**

- **According to the given objective functions and constraints, choose the best one among all feasible solutions.**



Content of this Chapter

1. Introduction
2. Linear Programming



Introduction

1. History
2. Characteristics
3. Applictions
4. Solving Steps

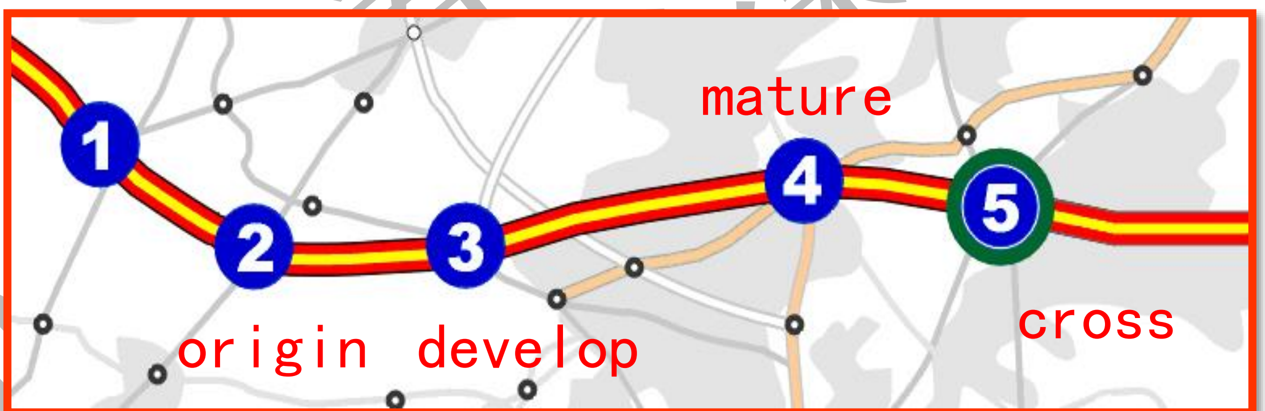
History

- **Operations Research（1938）**

  ancient stories

- **The development of Mode**

  “Opertions Research”



From where

1. Military
2. Management
3. Economic

Fixed input, how to produce the maximum output?

Fixed output, how to control the minimum input?

Characterisics

Introduce mathematics to solve problem

 --combination of quantitative and qualitative methods

System

 --from a global view

Application

 --from real applications, serve for real applications.

Cross-deciplines

 --mathematics, management, economics, control system

Open

 --new problems and trends

Shortage of talents in Operations Research

◆ **Two categories:**

Certain Problem: **Linear Programming, Integer Programming, Dynamic Programming, Nonlinear Programming, Graph and Network, Multiple objective Programming**

Probabilistic Problem: **Decision Theory, Queueing Theory, Stochastic Programming, Inventory Optimization**

Solving steps of a operations research problem

Propose the problem

Build the model

Solve the model

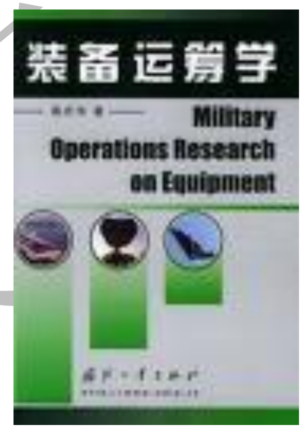
check the solution

control the solution

Implementation

Develop Trend

- Go deeper
- New areas
- New techniques
- New concepts
- Cross-deciplines



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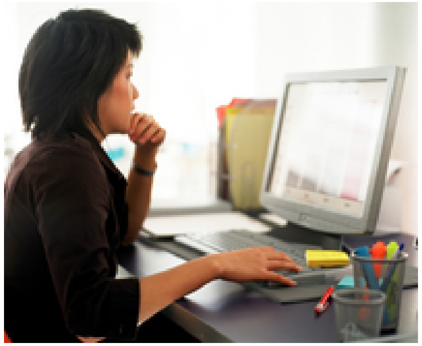
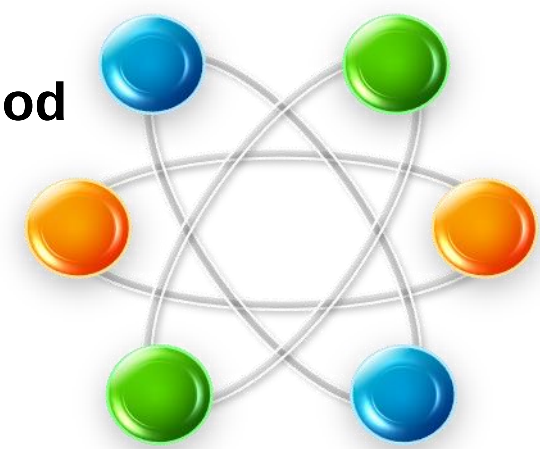
RESOURCES

- Comprehensive List of Resources for the Field of Operations Research
- Discussion Groups
- Funding
  (grants, fellowships, government and private grantors)
- High School Level Operations Research for Students, Instructors, and Supervisors
- INFORMS Bookstore
- INFORMS Speakers Program
- Practitioner Resources
- Sister Societies
- Software, Vendors, Consultants, and Publishers
  (courtesy of the ORMS Today Resource Directory)
- IOL Style Guide for Associate Editors



Development of Linear Programming

- **1939年，Kantorovich，propose linear programming for the first time。**
- **1947: G.B.Dantzig（1947-2005）“father of linear programming”, develop the classic simplex method.**
- **1975，Kantorovich won the Nobel prize of Economics**
- **1979, Khachyian, develop the Ellipsoid method**
- **1984，Karmakar，develop the interior point method**



Factory A produces desk and chair。The profits of desk and chair are 50 and 30, respectively. One desk needs 4 hours of a carpenter and 2 hours of a painter. One chair needs 3 hours of a carpenter and 1 hour of a painter. For each month, factory A has 120 hours of carpenter and 50 hours of painter.

How to make the production plan to obain the maximum profit?

	desk	chair	resource
carpenter	4	3	120
painter	2	1	50
profit	50	30	

- Produce the most profitable products as many as possible:

  25 desks, the profit is 1250;

- plan 1:

  20 desks, 10 chairs, the profit is 1300;

- plan n:

  15 desks, 20 chairs, the profit is 1350;

enlightenment

Some enlightenments

- There are a lot of possible plans;
- For a simple problem, you may find the best solution easily;
- But for a complicate problem, it is not easy to directly find the best solution.

→ **Linear programming is an effective tool for solving this kind of problem**

2. Build the model

Build the model

1. Determine the variables
2. Determine the constraints
3. Determine the objective function

1. Determine the variables

2. Determine the constraints

3. Determine the objective function

Use the variables to form linear inequalities or equalities to represent the constraints

Use the variables to form a linear objective function, maximize it or minimize it

Linear programming model

- *Decision variable* :
  - x_1 = number of desk, x_2 = number of chair;
- *Constraints* :
  - Carpenter time

    $4x_1 + 3x_2 \le 120$
  - Painter time

    $2x_1 + x_2 \le 50$
- *nonnegative*: $x_1 \ge 0$, $x_2 \ge 0$
- *Objective function* :
  - max: $z = 50x_1 + 30x_2$

	desk	chair	resource
carpenter	4	3	120
painter	2	1	50
profit	50	30	

Max z= $50x_1 + 30x_2$

$$\begin{cases} 4x_1 + 3x_2 \le 120 \\ 2x_1 + x_2 \le 50 \\ X_1 \ge 0,\ X_2 \ge 0 \end{cases}$$

Example 2 Nutrition problem

To maintain the health, we should reach the minimum amount of various nutrients in our foods. Suppose there are three nutrients A,B, C(protein, vitamin, micro elements), and the recipe contains two kinds of foods.

How to arrange the recipe to guarantee the nutrition need, while cost the minimum money?

Statistics of nutrition problem

	food 1	food 2	
price	0. 60	1. 50	Minimum need
nutrient A	10	4	20
nutrient B	5	5	20
nutrient C	2	6	12

Linear programming model

（1） ***decision variable：*** type 1 food x_1， type 2 food x_2

（2） ***Constraints：***

For nutrient A

$$10x_1+4x_2 \geqslant 20$$

For nutrient B and C：

$$5x_1+5x_2 \geqslant 20$$
$$2x_1+6x_2 \geqslant 12$$

Nonnegative:

$$x_1 \geqslant 0,\quad x_2 \geqslant 0$$

（3） ***Objective function：***

$$\text{minimize } Z= 0.6x_1+1.5x_2$$

$$\text{Min } Z=0.6x_1+1.5x_2$$
$$\begin{cases} 10x_1+4x_2 \geqslant 20 \\ 5x_1+5x_2 \geqslant 20 \\ 2x_1+6x_2 \geqslant 12 \\ x_1 \geqslant 0,\ x_2 \geqslant 0 \end{cases}$$

Example 3 Renting plan problem

A supermarket plans to rent a warehouse for inventory.

The required area for each month (1~4) is listed in Table 1.

Moreover, the rent cost for each month depends on the length of the rent. The longer the rent is, the lower the rent cost for each month is. The corresponding relationship is listed in Table 2.

Example 3 Renting plan problem

The rent contract can be determined at the beginning of each month. The contract stipulates the rent area and rent length.

The factory can sign any contract at the beginning of each month.

The question is:

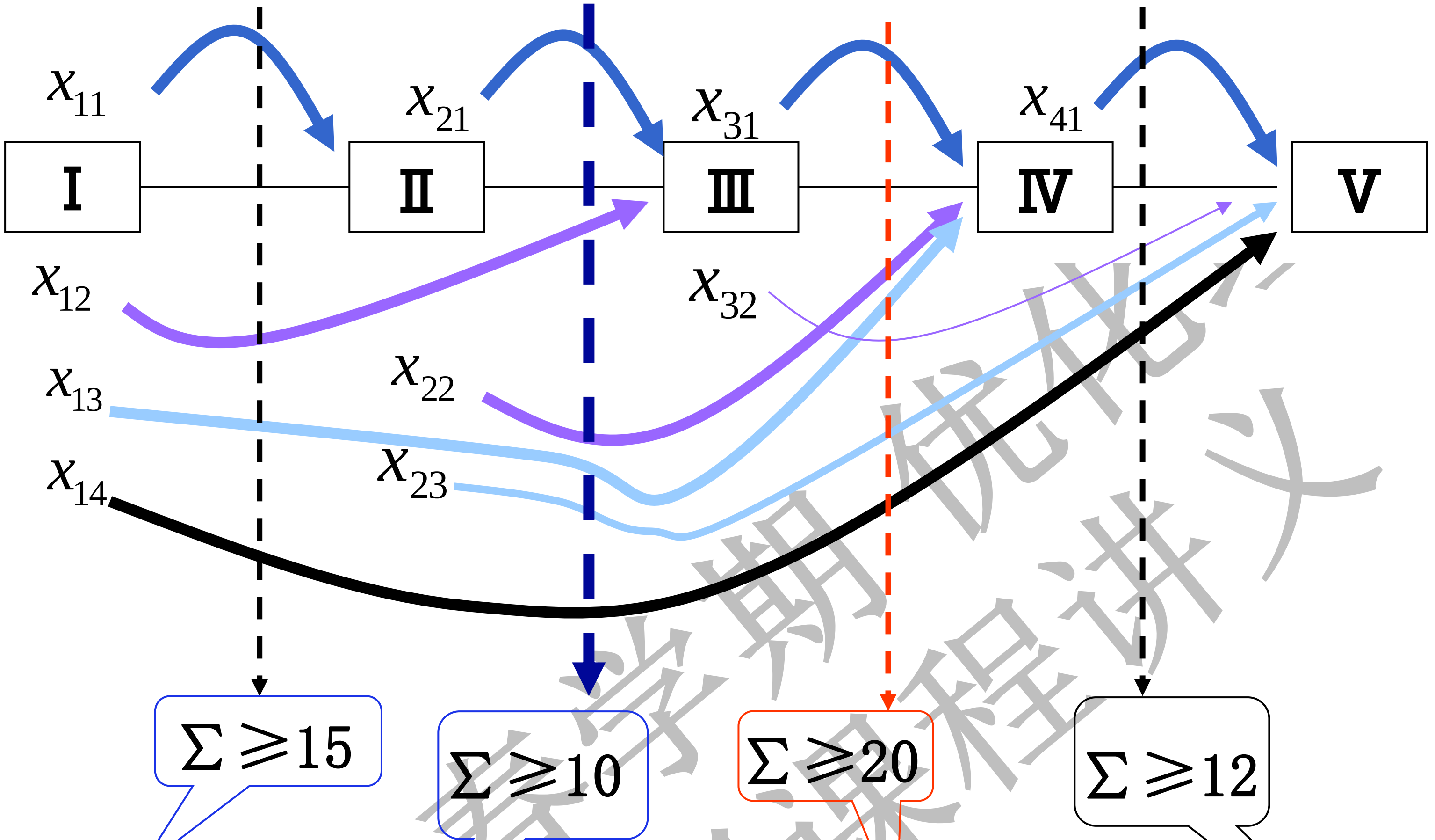
What is the optimal decision on the rent plan for the factory?

Table 1 unit: $100m^2$

month	1	2	3	4
Required area	15	10	20	12

Table 2 unit: $\$/100m^2$

Rent length	1	2	3	4
Rent cost	7300	6000	4500	2800



$$x_{11}+x_{12}+x_{13}+x_{14}\geq 15$$

$$x_{13}+x_{14}+x_{22}+x_{23}+x_{31}+x_{32}\geq 20$$

$$x_{12}+x_{13}+x_{14}+x_{21}+x_{22}+x_{23}\geq 10$$

$$x_{14}+x_{23}+x_{32}+x_{41}\geq 12$$

$$Z=7500\boxed{(x_{11}+x_{21}+x_{31}+x_{41})}+6000\boxed{(x_{12}+x_{22}+x_{32})}+4500\boxed{(x_{13}+x_{23})}+2800x_{14}$$

Linear programming model

Objective function

$$\min Z = 7000(x_{11}+x_{21}+x_{31}+x_{41})+6000(x_{12}+x_{22}+x_{32}) \\ +4500(x_{13}+x_{23})+2800x_{14}$$

Constraints

$$st.\begin{cases} x_{11}+x_{12}+x_{13}+x_{14} \geq 15 \\ x_{12}+x_{13}+x_{14}+x_{21}+x_{22}+x_{23} \geq 10 \\ x_{13}+x_{14}+x_{22}+x_{23}+x_{31}+x_{32} \geq 20 \\ x_{14}+x_{23}+x_{32}+x_{41} \geq 12 \\ x_{ij} \geq 0(i=1,\cdots,4;\quad j=1,\cdots,4) \end{cases}$$

3. Characteristics of LP model

1.use the decision variable to denote the unknow factor, usually nonnegative.

—— Decision variables

2. Linear inequalities or linear to equalities to represent the restrictions.

—— Linear constraints

3. Linear objective function。Maximize it or minimize it.

—— Objective function

Basic optimization problem in the area.

General form

$$
\begin{cases}
\max(\min)\quad z = c_1x_1 + c_2x_2 + \dots + c_nx_n \\
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \le (=, \ge) b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \le (=, \ge) b_2 \\
\cdots\cdots\cdots\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \le (=, \ge) b_m \\
x_j \ge 0 \quad j = 1,2,\cdots,n
\end{cases}
$$

$x_j; j = 1,2,\dots,n$ **decision variable，**

$c_j; j = 1,2,\dots,n$ **value coefficents，**

$b_i = (b_1, b_2, \dots, b_m), i = 1,2,\dots,m$ **right vector，**

matrix

$$
A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}
$$

coefficient matrix。

max（min） $Z=\sum_{j=1}^{n} c_j x_j$

$$s.t.\begin{cases}\sum_{j=1}^{n} a_{ij} x_j \le (=,\ge) b_i & i=1,2,\ldots\ldots,n \\ x_j \ge 0 & j=1,2,\cdots,n\end{cases}$$

Vector form

$$A_1=\begin{pmatrix}a_{11}\\a_{21}\\\vdots\\a_{m1}\end{pmatrix} \cdots \quad A_j=\begin{pmatrix}a_{1j}\\a_{2j}\\\vdots\\a_{mj}\end{pmatrix}$$

$$\begin{cases}\max \ z=\sum_{j=1}^{n} c_j x_j \\ \sum_{j=1}^{n} A_j x_j = b \\ x_j \ge 0 \quad j=1,\cdots,n\end{cases}$$

Matrix form

$$\max\ (\min)\quad Z = CX$$

$$s.t.\begin{cases} AX \le (=, \ge) b \\ X \ge 0 \end{cases}$$

Decision varable

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$C = (c_1, c_2, \cdots, c_n)$$

Value coefficient

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Right vector

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Coefficient matrix

4. The standard form of LP

$$
\begin{cases}
\max \quad z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
\cdots\cdots\cdots\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\
\quad x_j \geq 0 \quad j = 1,2,\cdots,n
\end{cases}
$$

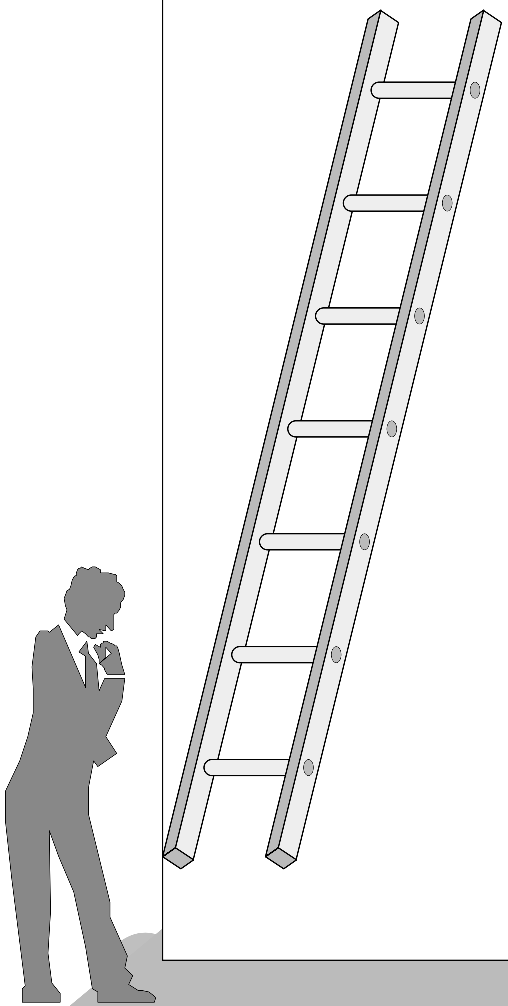
(其中$b_1, b_2, \cdots, b_m \geq 0$)

Characteristics:

①**nonnegative**
②**equalities**
③**Maximize**

The standard form is equal to the general form!

5. How to standardize?



$-\infty < x_j < +\infty$ $\begin{cases} x_j = x'_j - x''_j \\ x'_j,\ x''_j \ge 0 \end{cases}$	$x_j \le 0$ $\begin{cases} x'_j = -x_j \\ x'_j \ge 0 \end{cases}$
$x_j \ge l_j$ $x'_j = x_j - l_j$ $\begin{cases} x_j = x'_j + l_j \\ x'_j \ge 0 \end{cases}$	$x_j \le l_j$ $x'_j = l_j - x_j$ $\begin{cases} x_j = l_j - x'_j \\ x'_j \ge 0 \end{cases}$

Constraints $\le$ *or* $\ge$

$$\min z = f(x) \Rightarrow \max z' = -f(x)$$

Standardize the following LP problem

$$\min \ Z = -2x_1 + x_2 + 3x_3$$

$$\begin{cases} 5x_1 + x_2 + x_3 \le 7 \\ x_1 - x_2 - 4x_3 \ge 2 \\ -3x_1 + x_2 + 2x_3 = -5 \\ x_1, x_2 \ge 0, x_3 \text{ unconstrained} \end{cases}$$

Let $x_3 = x_4 - x_5$

Introduce variables x_6, x_7

Mutiply - 1 on the third constraint

Maximize the objective function

The standard form is as follows.

$$\max Z = 2x_1 - x_2 - 3(x_4 - x_5) + 0x_6 + 0x_7$$

$$\begin{cases} 5x_1 + x_2 + (x_4 - x_5) + x_6 & = 7 \\ x_1 - x_2 - (x_4 - x_5) \quad - x_7 & = 2 \\ 5x_1 - x_2 - 2(x_4 - x_5) & = 5 \\ x_1, x_2, x_4, x_5, x_6, x_7 \geq 0 & \end{cases}$$

Standardize the following problem

$$
\begin{cases}
\min \quad z = x_1 - x_2 + 4x_3 \\
3x_2 - 4x_3 \ge -9 \\
-x_1 + x_2 \ge 6 \\
5x_2 + 2x_3 \le 16 \\
x_1 \le 0,\ x_2 \ge 0 \quad x_3\text{无符号限制}
\end{cases}
$$

令 $x_1' = -x_1 \quad (x_1' \ge 0) \quad ; \quad x_3 = x_3' - x_3'' \quad (x_3', x_3'' \ge 0)$

$$
\begin{cases}
\max \ z' = x_1' + x_2 - 4x_3' + 4x_3'' \\
-3x_2 + 4x_3' - 4x_3'' + x_4 = 9 \\
x_1' + x_2 - x_5 = 6 \\
5x_2 + 2x_3' - 2x_3'' + x_6 = 16 \\
x_1', x_2, x_3', x_3'', x_4, x_5, x_6 \ge 0
\end{cases}
$$

Graph algorithm

- Usually for LP problem with two variables
- Doesn't need to be standardized

Basic concepts

Feasible point(solution)

Feasible domain

Optimal solution

Basic idea

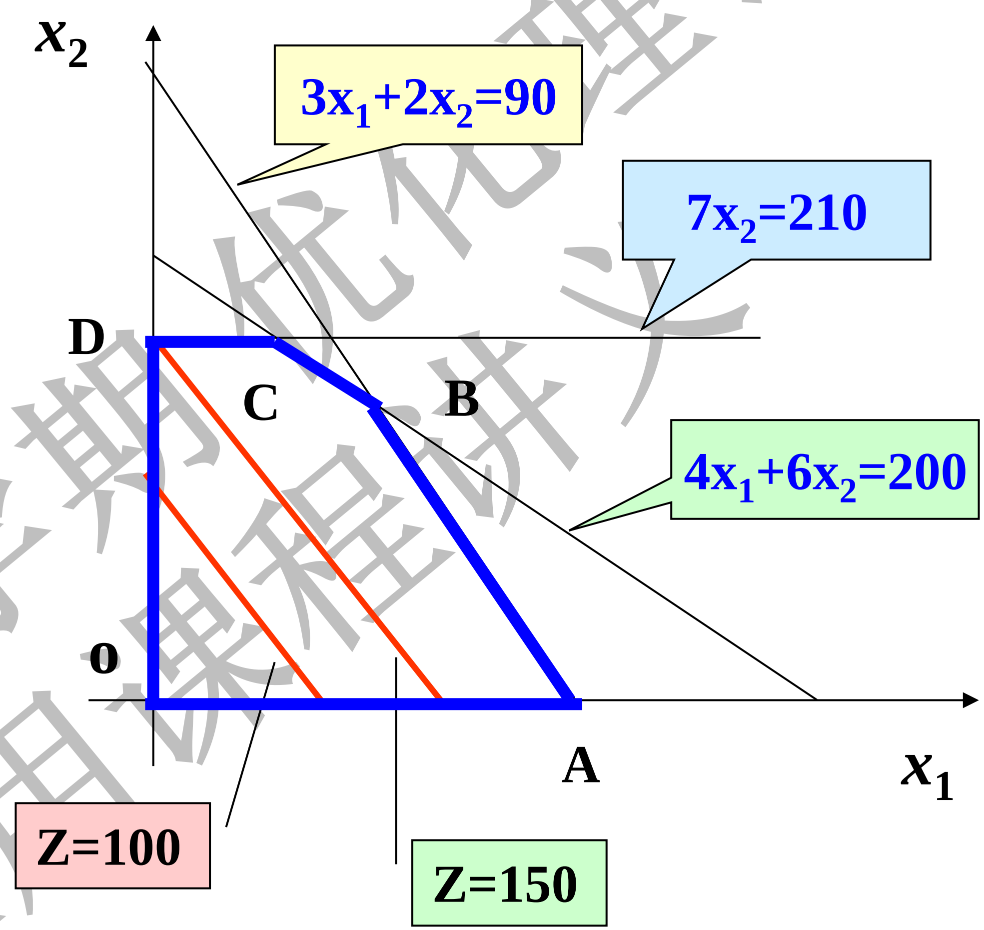
1. Draw the feasible domain
2. Draw the direction of the gradient of the objective function
3. Use the direction to find the optimal solution in the feasible domain

$$\begin{cases} \max z = 7x_1 + 5x_2 \\ 3x_1 + 2x_2 \le 90 \\ 4x_1 + 6x_2 \le 200 \\ \qquad 7x_2 \le 210 \\ x_j \ge 0 \quad j = 1,2 \end{cases}$$

Move the equal value line according the direction of the gradient.

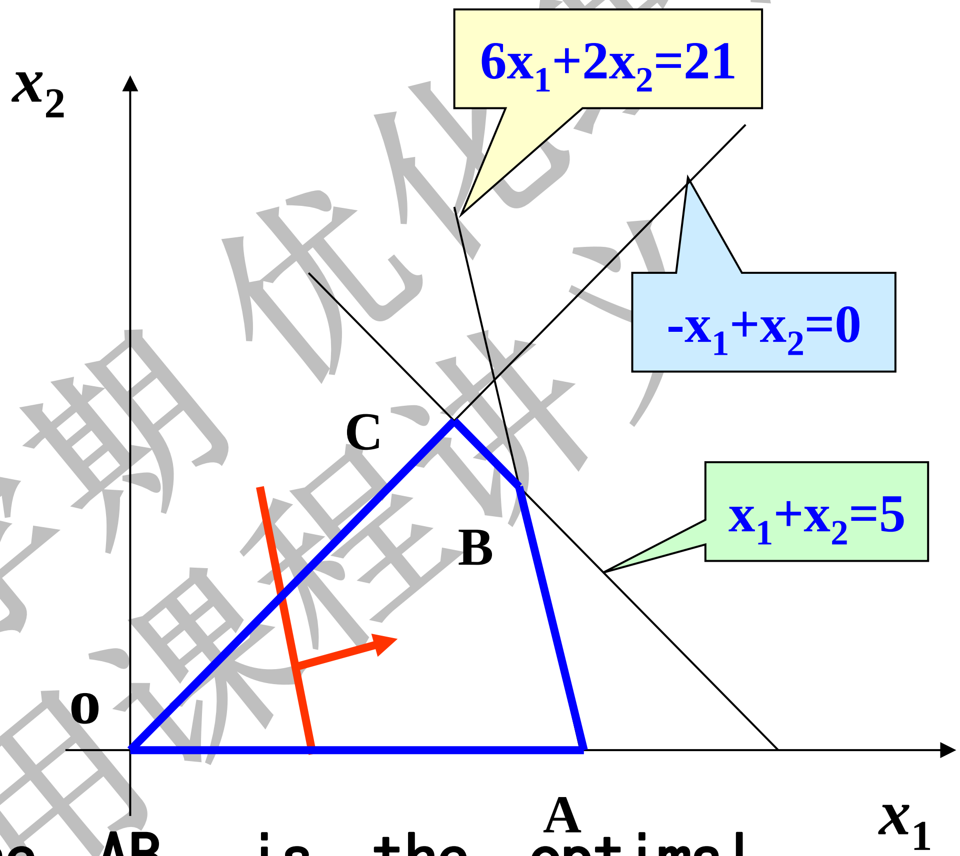
Point B is the optimal solution

x_1=14 x_2=24 z=218



2. Possible results of LP solutions

$$\begin{cases} \max z = 3x_1 + x_2 \\ x_1 + x_2 \le 5 \\ -x_1 + x_2 \le 0 \\ 6x_1 + 2x_2 \le 21 \\ x_1 \ge 0 \quad x_2 \ge 0 \end{cases}$$

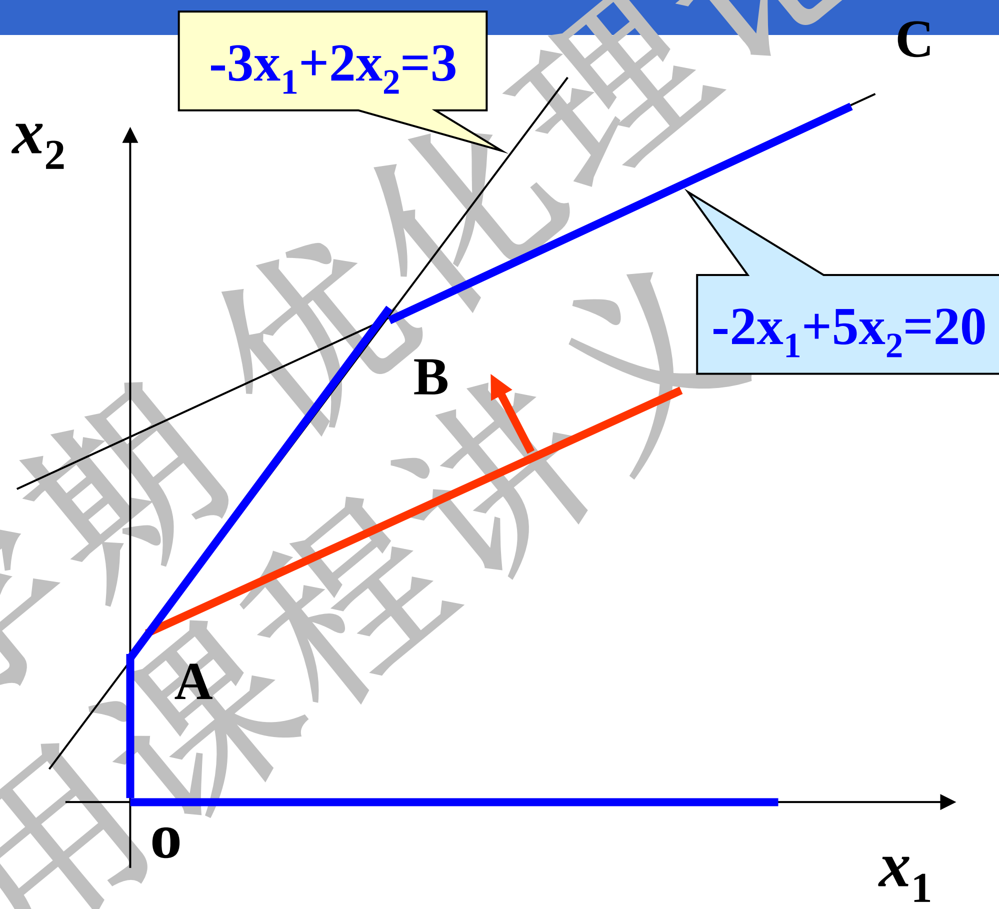


Every point on line AB is the optimal solution for the problem .

——multiple optimal solutions

A(7/2 , 0) B(11/4 , 9/4) z*=21/2

$$\begin{cases} \max z = -4x_1 + 10x_2 \\ -3x_1 + 2x_2 \le 3 \\ -2x_1 + 5x_2 \le 20 \\ x_1 \ge 0 \quad x_2 \ge 0 \end{cases}$$

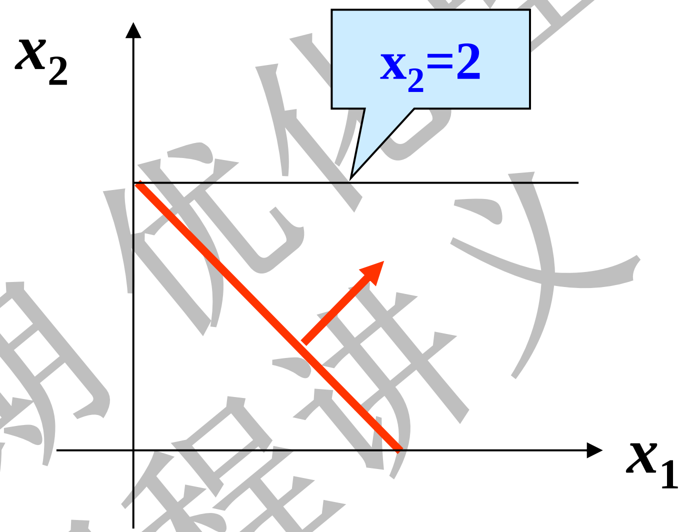


The feasible domain is unbounded.

Multiple optimal solutions

B(25/11 , 54/11)   z=40

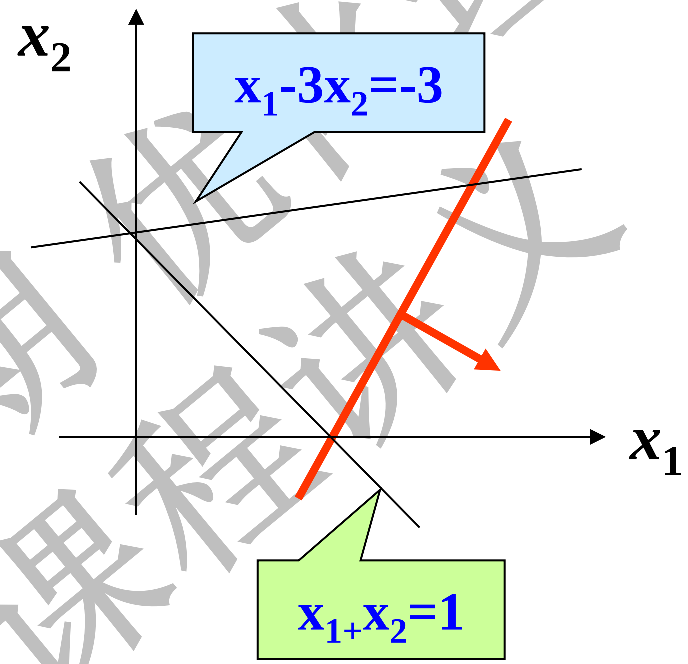
$$\begin{cases} \max z = x_1 + x_2 \\ x_2 \le 2 \\ x_1 \ge 0 \quad x_2 \ge 0 \end{cases}$$



The feasible domain is unbounded, the objective function can infinitely increase.

------the objective value is unbounded up.

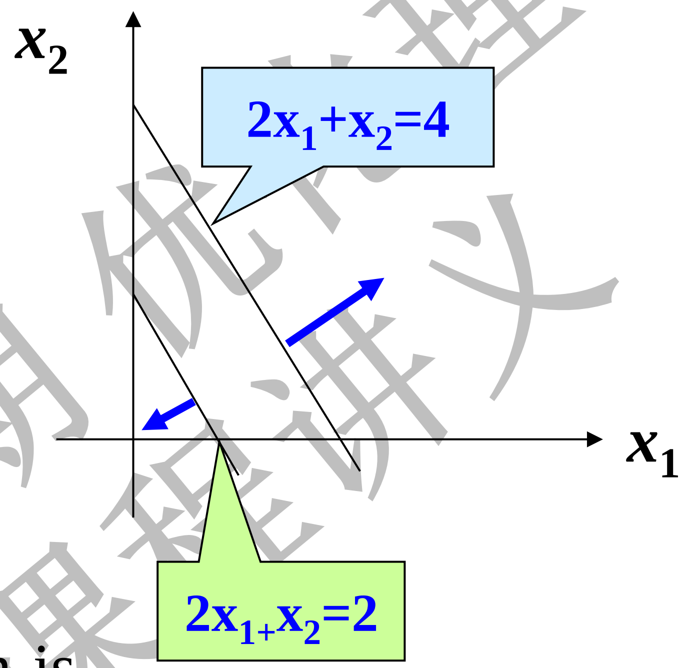
$$\begin{cases} \min z = -2x_1 + x_2 \\ x_1 + x_2 \geq 1 \\ x_1 - 3x_2 \geq -3 \\ x_1 \geq 0 \quad x_2 \geq 0 \end{cases}$$



The feasible domain is unbounded, the objective function can infinitely decrease

---- the objective function is unbounded below.

$$\begin{cases} \max z = 3x_1 + 2x_2 \\ 2x_1 + x_2 \le 2 \\ 4x_1 + 2x_2 \ge 8 \\ x_1 \ge 0 \quad x_2 \ge 0 \end{cases}$$



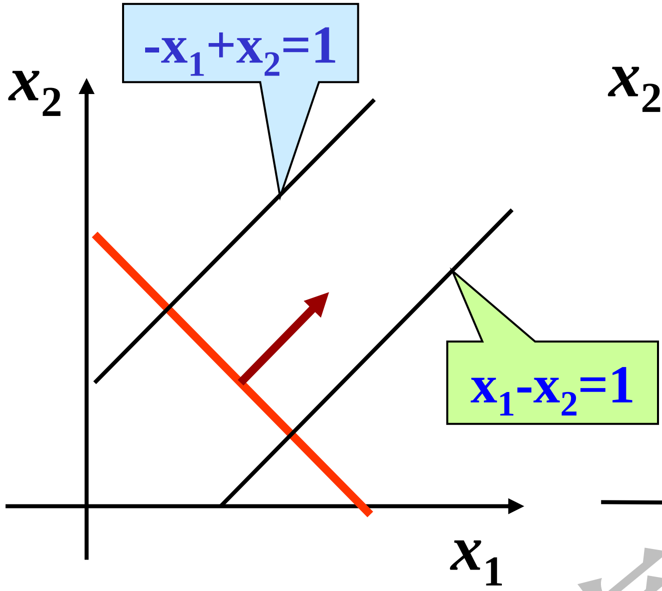
Infeasbile: the feasible domain is an empty set.

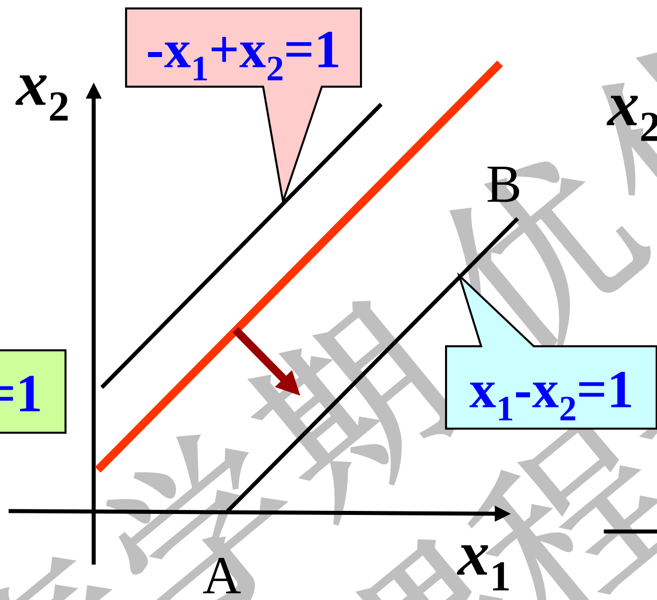
$$\begin{cases} -x_1 + x_2 \le 1 \\ x_1 - x_2 \le 1 \\ x_1 \ge 0 \quad x_2 \ge 0 \end{cases}$$

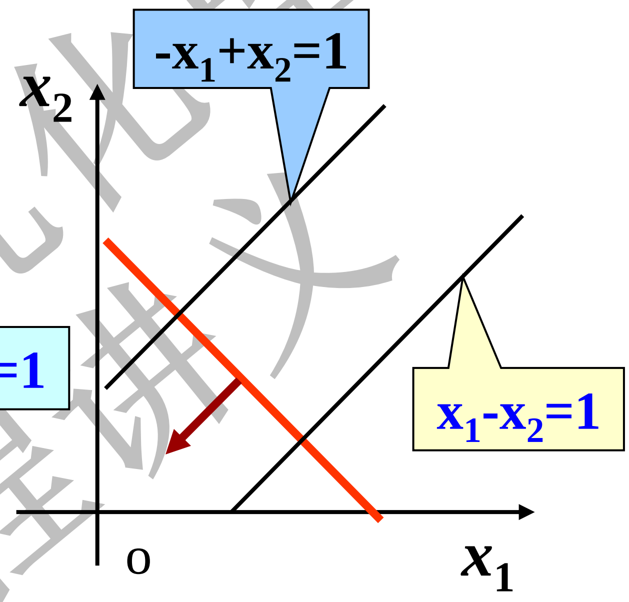
（1） $\max z = x_1 + x_2$

（2） $\max z = x_1 - x_2$

（3） $\min z = x_1 + x_2$


-x1+x2=1
x2
x1-x2=1
x1

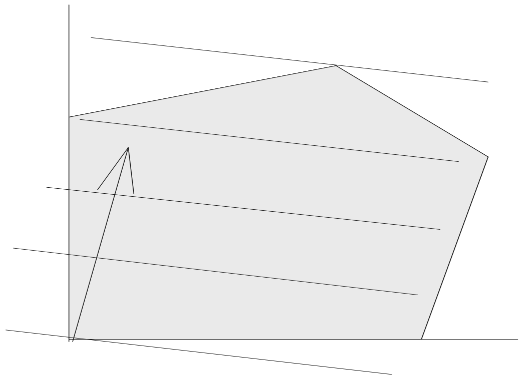

-x1+x2=1
x2
B
x1-x2=1
A
x1


-x1+x2=1
x2
x1-x2=1
o
x1

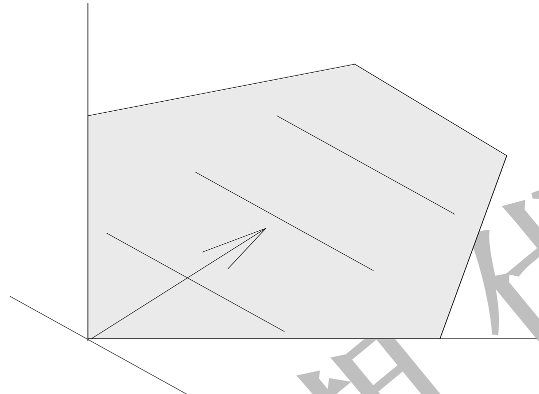
3. Enlightenments

- **Unique optimal solution；multiple optimal solution; unbounded optimal solution; infeasible solution**
- **The feasible domain is a convex set**
- **If the optimal solution exists，it must appears on the vertex of the convex set.**
- **How to solve the LP problem?**

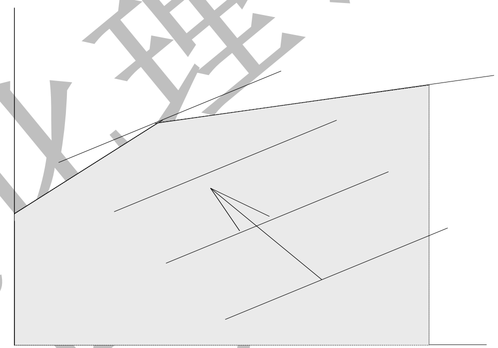
Summary



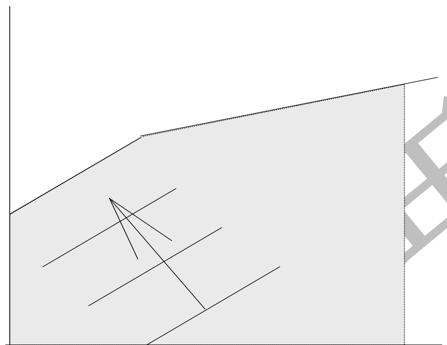
(a)Bounded area
unique optimal solution



(b)Bounded area
multiple optimal solutions

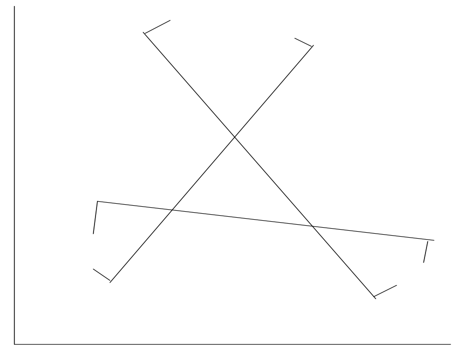


(c)unbounded area
unique optimal solution

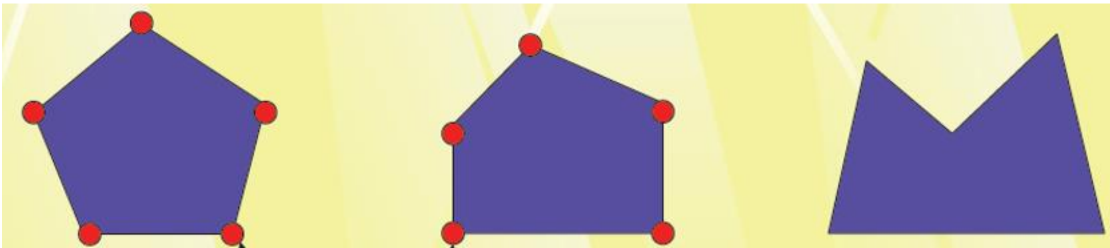


(d)unbounded area
multiple optimal solutions

(e)unbounded area
unbounded optimal solution



(f)infeasible domain
no optimal solution



1. The feasible domain is a convex set.

2. The number of vertex is finite.

3. If the optimal solution exists, it appears on the vertex.

So can we just focus on the vertex?

Section 3

The simplex method

HEADACHE？

- Usually, rank（A）= m， and m < n。

  What does it mean?

What is the characteritistics of the vertex?

How to solve $Ax = b$?

- When $m<n$, there are infinite solutions for these linear equations.
- If We fix the values for $n-m$ variables, then we can get a unique solution.
- What is the relationship between vertex and solution of these linear equations?

Algebraic characteristics for vertex

There are n coordinates for each point. But for a vertex, there are only m coordinates which are nonnegative while the remaining coordinates are zero. (

And those vectors which are corresponded with the nonnegative variables are linear independent.

一、Basic concepts

(1) Basis

For a standard LP problem, A is a m*n matrix with rank m (m ＜ n)。

$$
\begin{cases}
\max z = CX \\
AX = b \\
x_j \geq 0 \quad j = 1, \cdots, n
\end{cases}
$$

Suppose B is a m*m submatrix (|B| not equal to 0), then we can B is a basis for this LP problem.

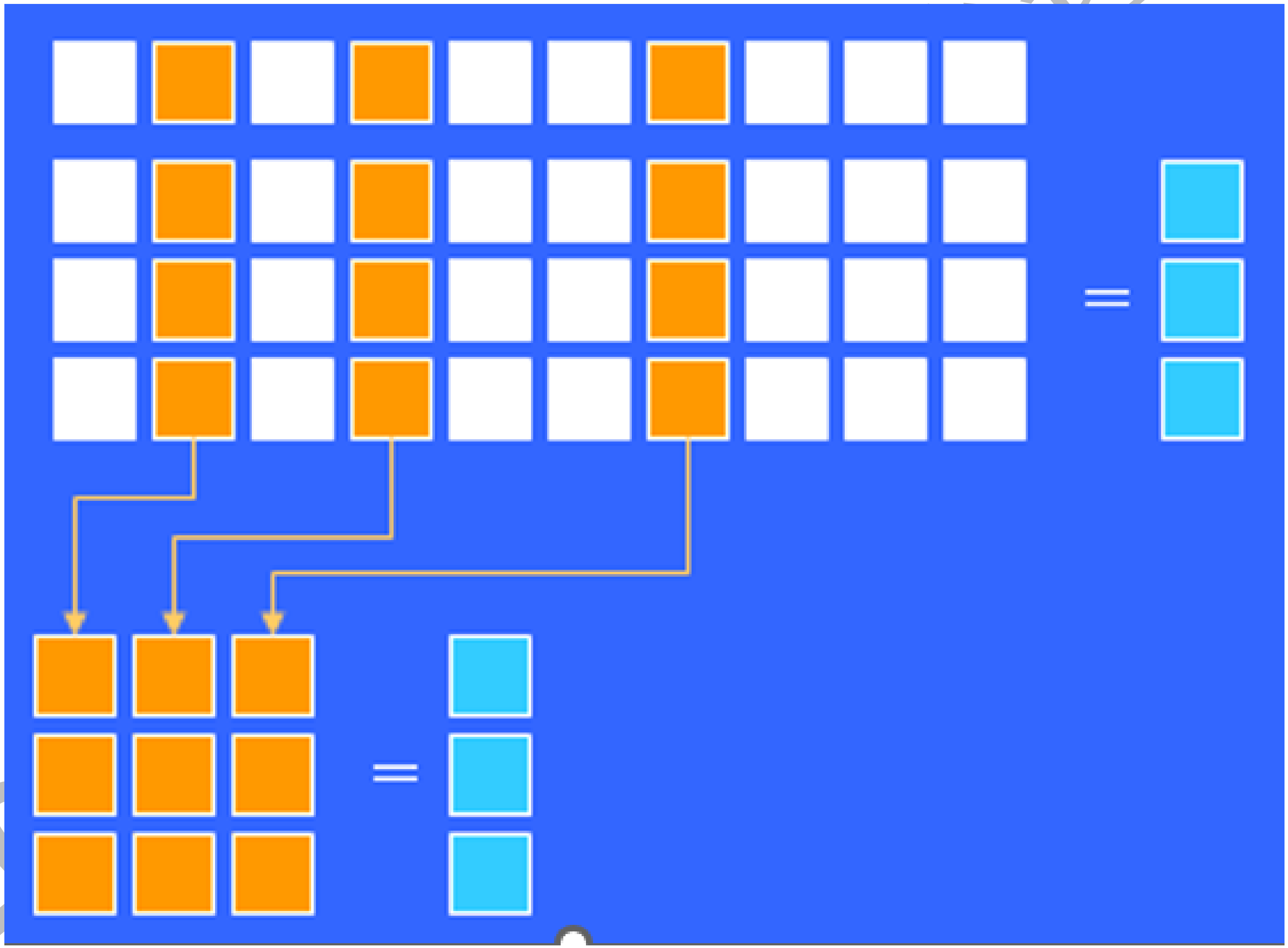
$B=（A_1, A_2, \ldots, A_m）$， $A_1, A_2, \ldots, A_m$ are linear independent. And we can A_j(j=1, … , m) as the basis vector.

(2) Basic variable

(3) Basic solution

(4) Feasible basic solution

(5) Degenerate basic solution



For convenience, suppose the first m vectors form a basis.

$A=(B\ N)$,

$B=(P_1, P_2, \cdots P_m)$

$N=(P_{m+1}, P_{m+2}, \cdots P_n)$

$$A=(BN)$$

$$X=(X_B X_N)^T$$

$$\Rightarrow (BN)\begin{pmatrix} X_B \\ X_N \end{pmatrix} = b$$

$$\Rightarrow BX_B + NX_N = b$$

$$\Rightarrow BX_B = b - NX_N$$

$$X_B = B^{-1}b - B^{-1}N X_N$$

$$\Rightarrow \boxed{X_N = 0, X_B = B^{-1}b}$$

The relationship among different solutions

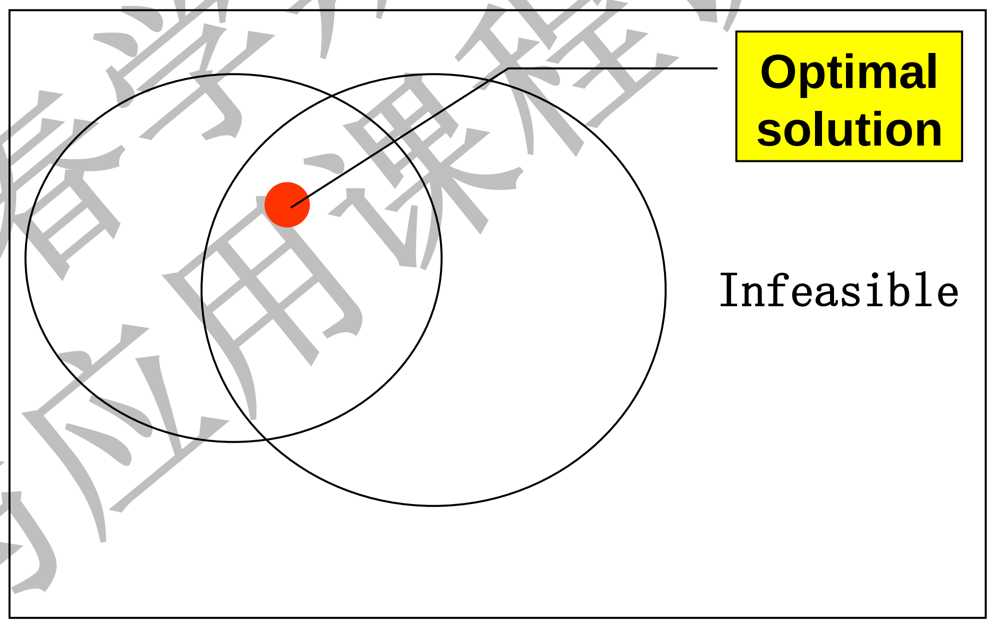
feasible basic solution ← **Basic solution**

Basic optimal solution ← **feasible basic solution**

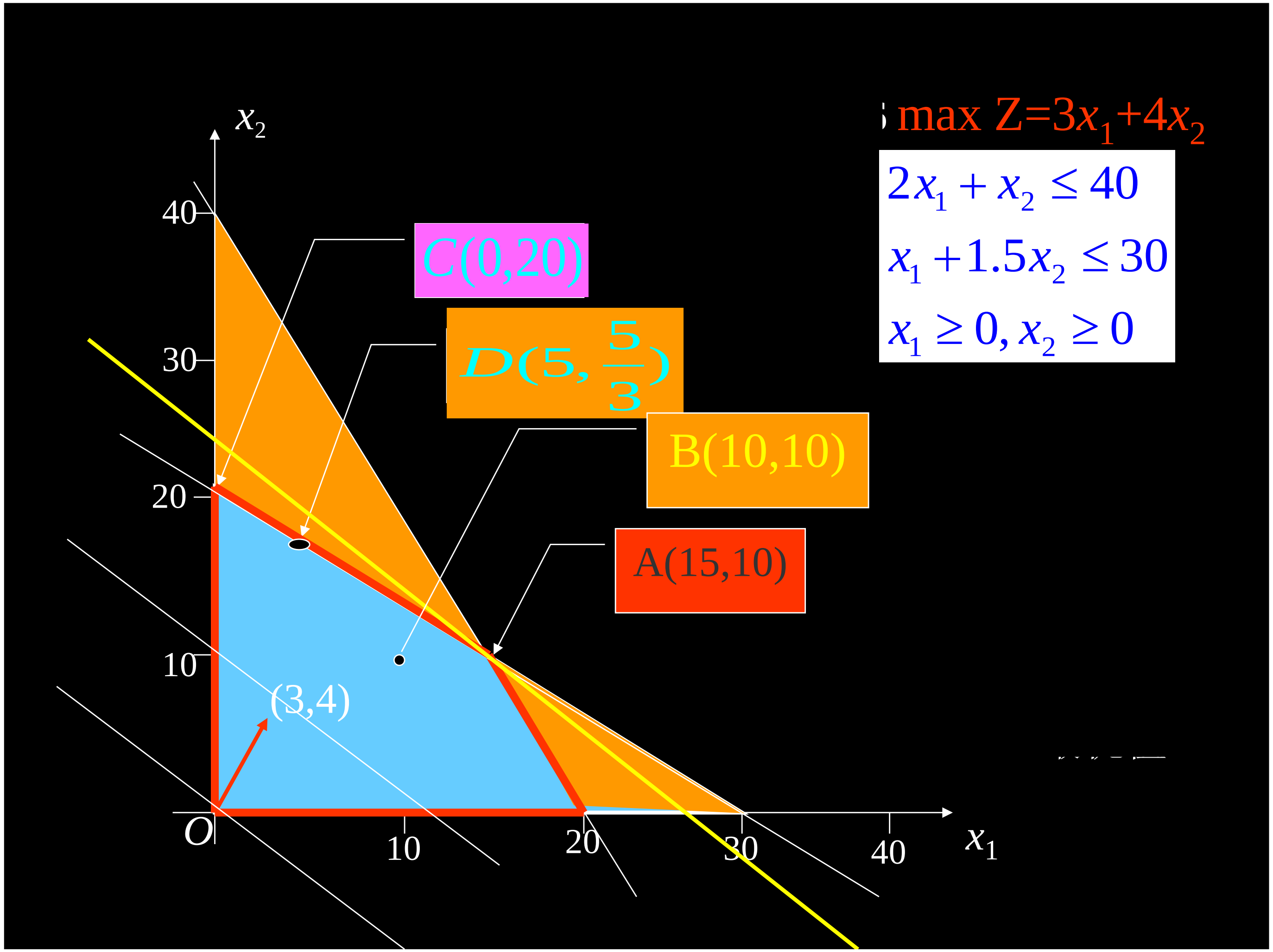
Basic optimal solution ← **Optimal solution**

feasible basic solution ← **Feasible solution**

Optimal solution ← **Feasible solution**



Point B and D are feasible solutions, but not basic solutions. Point C is a feasible basic solution; Point A is a basic optimal solution.



Relationship between geometry and algebra

Plane:

Half space:

Intersection of half spaces

Intersection of planes

Vertex of feasible domain

Equal value planes

Basis and basic solutions

Basis	Basic solution
1.Independent. **2.Must be a m*m matrix。** **4.The number of basis is no more than** C_n^m	**1.One on one to the basis** **2.Zero for the nonbasic variables** **3.If the variables are not zero, they must be basic variables.**

Get the basic solution for the following LP problem.

$$\begin{cases} \max z = 3x_1 + 2x_2 \\ 2x_1 + x_2 + x_3 = 4 \\ -x_1 + x_2 + x_4 = 1 \\ x_j \geq 0 \quad j = 1, \cdots, 4 \end{cases}$$

$$A=\begin{bmatrix} 2 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} =(A_1, A_2, A_3, A_4)$$

$B_1=(A_3, A_4)=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **Basic solution** $X^{(1)}=(0, 0, 4, 1)^T$

$B_2=(A_1, A_4)=\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$ **Basic solution** $X^{(2)}=(2, 0, 0, 3)^T$

$B_3=(A_1, A_2)=\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$ **Basic solution** $X^{(3)}=(1, 2, 0, 0)^T$

$B_4=(A_2, A_3)=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ **Basic solution** $X^{(4)}=(0, 1, 3, 0)^T$

$B_5=(A_1, A_3)=\begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$ **Basic solution** $X^{(5)}=(\boxed{-1}, 0, 6, 0)^T$

$B_6=(A_2, A_4)=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ **Basic solution** $X^{(6)}=(0, 4, 0, \boxed{-3})^T$

Basic Feasible solutions

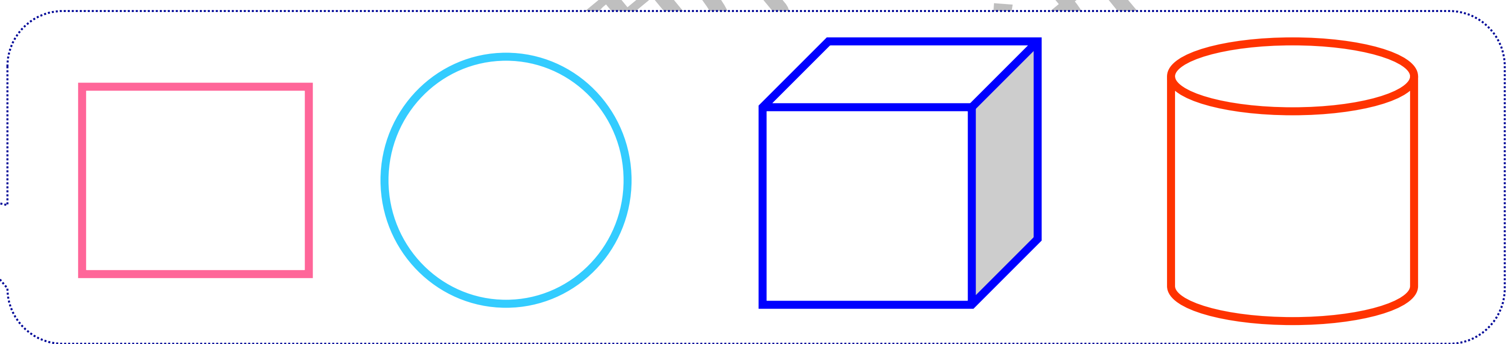
二. Convex set and vertex

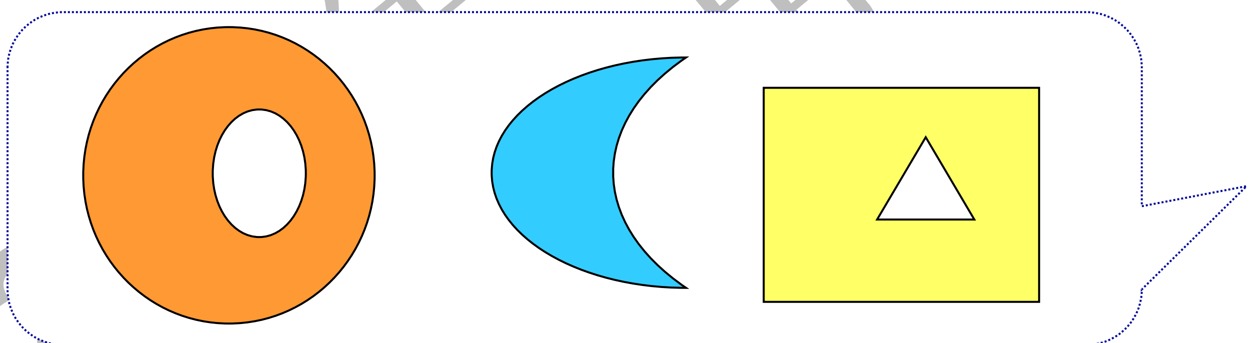
Suppose R_c is a set, if any two points $\boldsymbol{X}^{(1)} \in \boldsymbol{R_c}$

$\boldsymbol{X}^{(2)} \in \boldsymbol{R_c}$ and satisfy that $\mu X^{(1)} + (1-\mu)X^{(2)} \in R_c \ (0 \le \mu \le 1)$

Then it is a convex set.

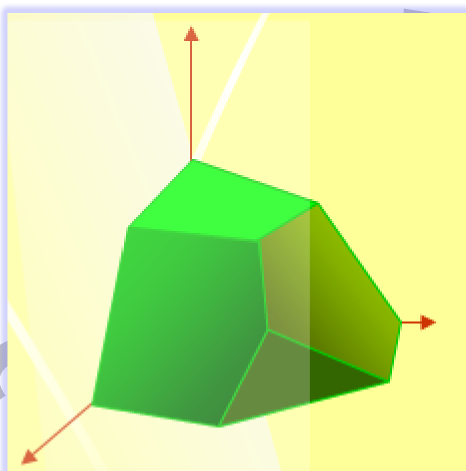
Convex set

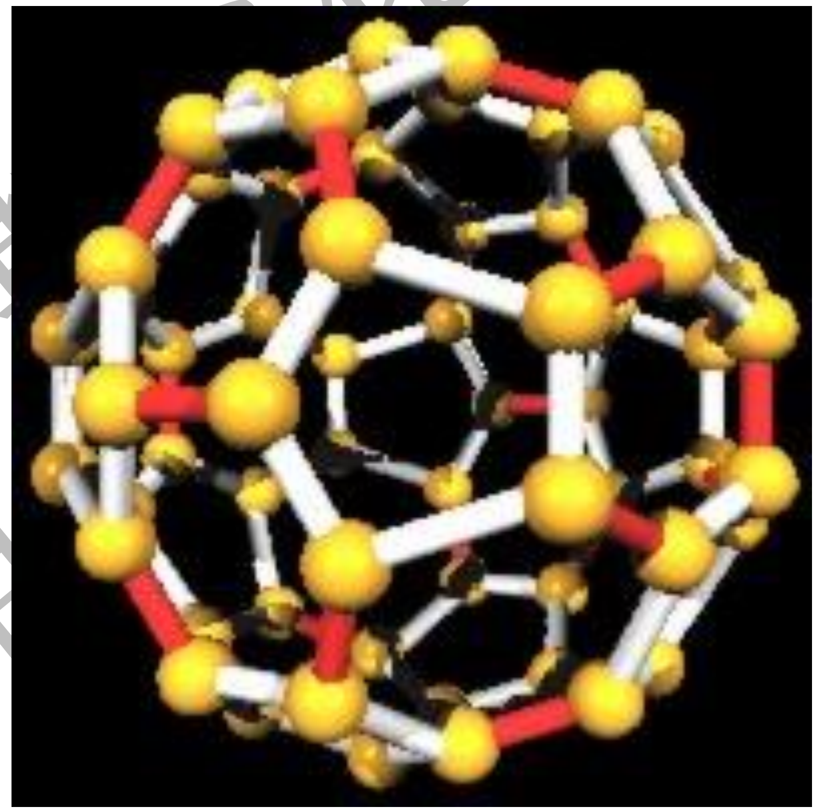




Nonconvex set

- When n=3， the feasible domain is a polyhedron;
- When n>3， the feasible domain is a hyperpolyhedron





顶点

If $X \in R_c$ can not be represented by the linear combination of other two points

$$X^{(1)} \in R_c \quad \text{and} \quad X^{(2)} \in R_c$$

$$X = \mu X^{(1)} + (1-\mu)X^{(2)} \in R_c \ (0 < \mu < 1)$$

Then X is a vertex of this set.

即

$$X^{(1)} \neq X^{(2)} \quad X \neq \mu X^{(1)} + (1-\mu)X^{(2)} \in R_c \ (0 < \mu < 1)$$

Theory 1

The feasible domain of a LP problem is a convex set.

Theroy 2 The feasible basic solution of a LP problem corresponds to the vertex of its feasible domain.

Theory 3 It the optimal solution of a LP problem exists, one of the vertex must be the optimal solution.

We can only compare all the basic feasible solutions, then pick out the optimal solution of a LP problem.

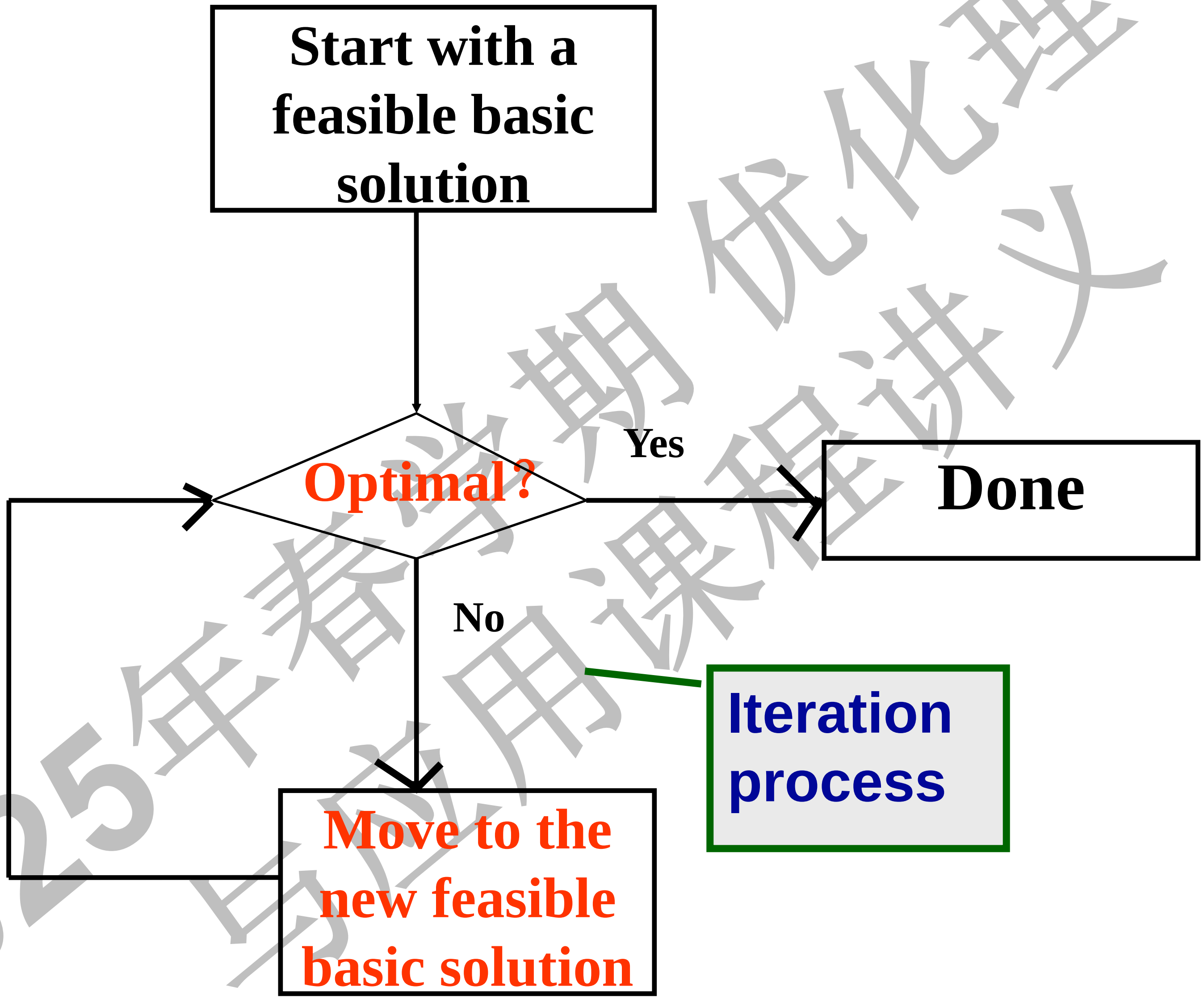
Basic idea of simplex method

Start form a vertex of the feasible domain, move to the next vertex which is better, till reach the optimal solution.

Only need finite iterations to get the optimal solution!

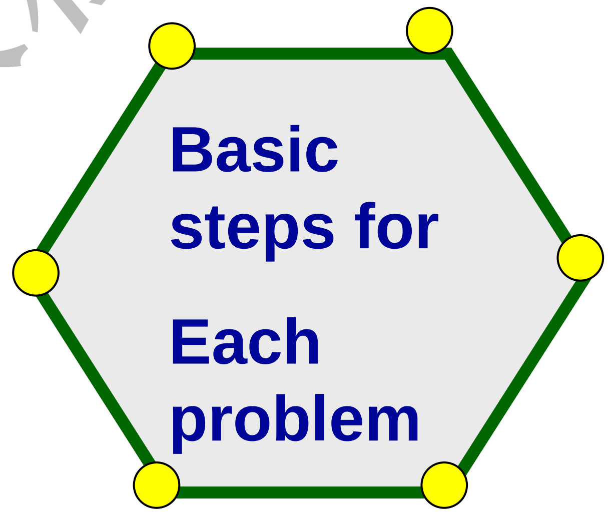
$$C_n^m = \frac{n!}{m!(n-m)!} \quad (m < n)$$

Solving process



Basic strategy

- **Seek for the starting feasible basic solution**
- **Give a criteria for the optimality**
- **Give a way to move to the next feasible basic solution**



Obtain a feasible basic solution

1. “= ” constraints, and you can directly find a unit matrix

   Use it as the basic matrix

2. “ ≤ ” constraints, + nonnegative slack variables

   Use these slack variables as the basic variables

3. “= ” constraints

   no unit matrix

   “ ≥ ” constraints, − slack varaibles

   \+ nonnegative variables, use these variables

For a given LP problem

$$\max z = \sum_{j=1}^{n} c_j x_j$$

$$\begin{cases} \sum_{j=1}^{n} a_{ij} x_j \le b_i & (i = 1, \cdots, m) \\ x_j \ge 0 & (j = 1, \cdots, n) \end{cases}$$

Standard form: $\max z = \sum_{j=1}^{n} c_j x_j + 0 \sum_{i=1}^{m} x_{si}$

$$\begin{cases} \sum_{j=1}^{n} a_{ij} x_j + x_{si} = b_i & (i = 1, \cdots, m) \\ x_j \geq 0 & (j = 1, \cdots, n) \end{cases}$$

Coefficient matrix:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 & 0 & \cdots & 1 \end{pmatrix}$$

It contains a unit matrix, we can use it as the starting basis, and solve the problem as

$$X = (0, \cdots, 0, b_1, \cdots, b_m)^T$$

Optimality test

$$
\begin{cases}
\max z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \\
x_1 \qquad\qquad + a_{1,m+1} x_{m+1} + \cdots + a_{1n} x_n = b_1 \\
\quad x_2 \qquad + a_{2,m+1} x_{m+1} + \cdots + a_{2n} x_n = b_2 \\
\quad \cdots\cdots\cdots\cdots \\
\qquad\qquad x_m + a_{m,m+1} x_{m+1} + \cdots + a_{mn} x_n = b_m \\
x_j \geq 0 \quad j = 1,2,\cdots,n
\end{cases}
$$

$$B = (A_1, A_2, \cdots, A_m) = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \ldots & \ldots & & \ldots \\ 0 & 0 & \ldots & 1 \end{pmatrix}$$

$$X = (x_1, x_2, \cdots, x_m, 0, \cdots, 0)^T$$
$$= (b_1, b_2, \cdots, b_m, 0, \cdots, 0)^T$$

The objective value is $\sum_{i=1}^{m} c_i b_i$

$$x_i = b_i - \sum_{j=m+1}^{n} a_{ij} x_j \quad i = 1, \cdots, m$$

Put it back into the objective function

$$z = \sum_{i=1}^{m} c_i b_i - \sum_{i=1}^{m} \sum_{j=m+1}^{n} c_i a_{ij} x_j + \sum_{j=m+1}^{n} c_j x_j$$

$$= \sum_{i=1}^{m} c_i b_i + \sum_{j=m+1}^{n} \left(c_j - \sum_{i=1}^{m} c_i a_{ij} \right) x_j$$

$$z_0 = \sum_{i=1}^{m} c_i b_i \qquad z_j = \sum_{i=1}^{m} c_i a_{ij}$$

$$z = z_0 + \sum_{j=m+1}^{n} \left(c_j - z_j \right) x_j$$

Let $\sigma_j = c_j - z_j$ $\Rightarrow$ $z = z_0 + \sum_{j=m+1}^{n} \sigma_j x_j$

$$
\begin{cases}
\max z = z_0 + \sum\limits_{j=m+1}^{n} \sigma_j x_j \\
x_i = b_i - \sum\limits_{j=m+1}^{n} a_{ij} x_j \quad i = 1, \cdots, m \\
x_j \geq 0 \quad j = 1, \cdots, n
\end{cases}
$$

Use the nonbasic variables to represent the objective function and constraints

Optimality test

If $\sigma_j \le 0$ for all j, then the corresponding solution is optimal.

Check value of x_j

$$\boxed{\sigma_j = c_j - z_j}$$

What does it mean?

$$z_j = \sum_{i=1}^{m} c_i a_{ij}$$

If $\sigma_j \le 0$ for all j, what does it mean from the geometry view?

Multiple optimal solutions

If $\boldsymbol{X}^{(0)} = \left(\boldsymbol{b}_1', \boldsymbol{b}_2', \cdots, \boldsymbol{b}_{\boldsymbol{m}}', 0, \cdots, 0\right)^{\boldsymbol{T}}$ is a feasible basic Solution, and for all j=m+1,…,n, $\sigma_j \le 0$

but the check value of one σ_k satisfy

$\boxed{\sigma_k = 0 \text{ (and there exist at least one } \boldsymbol{a}'_{\boldsymbol{ik}} \text{ bigger than 0)}}$

Then this LP problem has multiple optimal solutions.

Unbounded

If $\boldsymbol{X}^{(0)} = \left(\boldsymbol{b}_1^{'}, \boldsymbol{b}_2^{'}, \cdots, \boldsymbol{b}_{\boldsymbol{m}}^{'}, 0, \cdots, 0\right)^{\boldsymbol{T}}$ is a feasible basic solution,

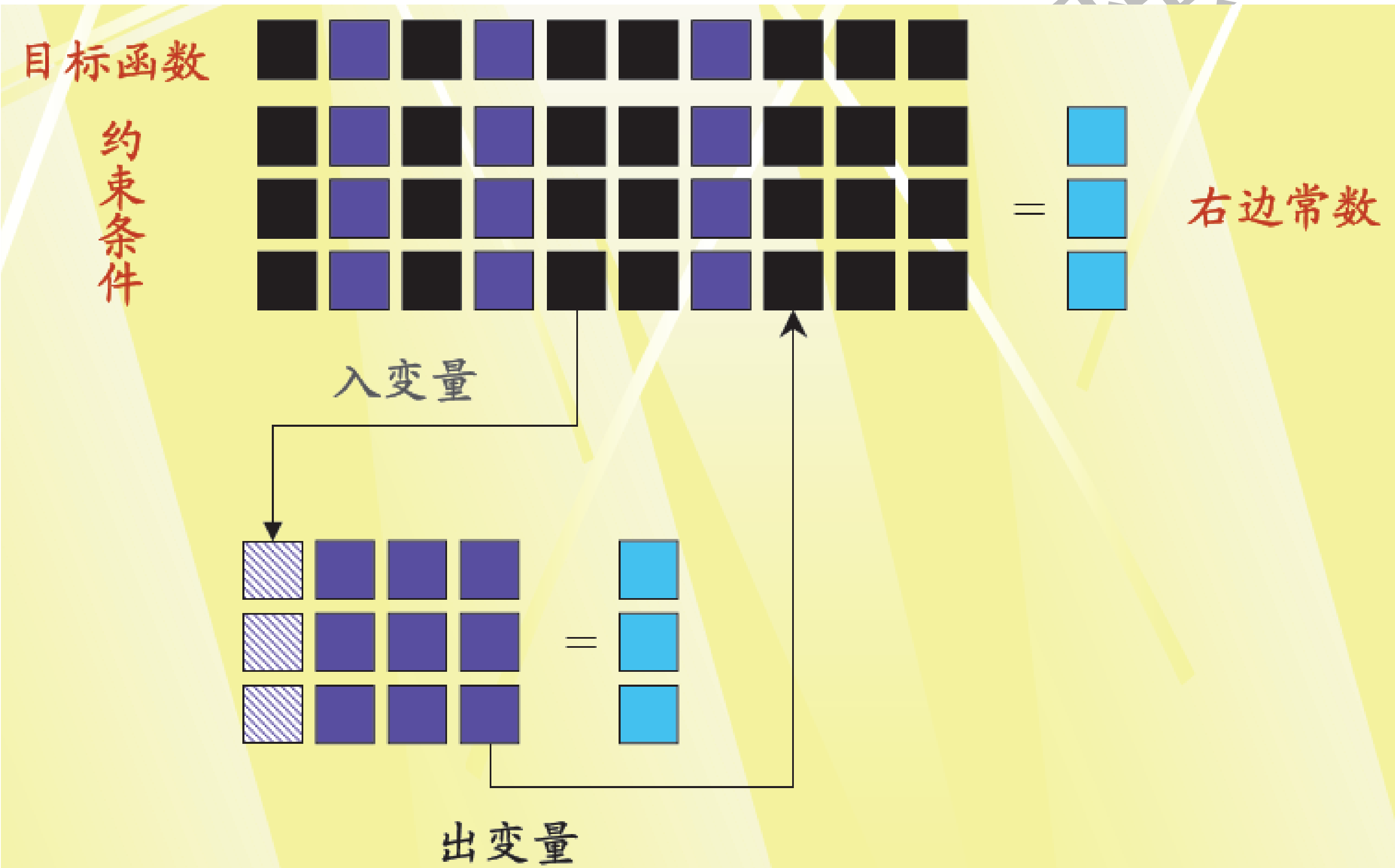
$$\boxed{\sigma_k > 0 \quad \text{and} \quad \boldsymbol{a}_{\boldsymbol{ik}}^{'} \leq \mathbf{O} \quad \boldsymbol{i} = 1, \cdots, \boldsymbol{m}}$$

Then this LP problem is unbounded.

Basis change

- **Move an existing vector out, then move a new vector in (should guarantee the new matrix is still nonsingular), this is called basis change.**

Basic idea: use some elementary row operations on the origin basis matrix to obtain a new unit matrix.


目标函数
约束条件
=
右边常数
入变量
=
出变量

$$f(x)=C_N X_N+C_B X_B \qquad (1)$$

$$NX_N+BX_B=b$$

$$BX_B=b-NX_N$$

$$X_B=\ B^{-1}(b-NX_N) \qquad (2)$$

$$f(x)=C_N X_N+C_B B^{-1}(b-NX_N)$$

$$=C_B B^{-1}b+(C_N-C_B B^{-1}N)X_N \quad (3)$$

机会成本 $z_j=\sum_{i=1}^{m} c_i \overline{a_{ij}}$ (4)

检验数 c_j-z_j (5)

Section 4

Solving steps

Step 1

$$
\begin{cases}
\max z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
x_1 \qquad\qquad + a_{1,m+1}x_{m+1} + \cdots + a_{1n}x_n = b_1 \\
\quad x_2 \qquad\quad + a_{2,m+1}x_{m+1} + \cdots + a_{2n}x_n = b_2 \\
\qquad \cdots\cdots\cdots\cdots \\
\qquad\qquad x_m + a_{m,m+1}x_{m+1} + \cdots + a_{mn}x_n = b_m \\
x_j \geq 0 \quad j = 1,2,\cdots,n
\end{cases}
$$

c_j		c_1	c_2	…	c_m	c_{m+1}	…	c_n	b	θ
C_B	X_B	x_1	x_2	…	x_m	x_{m+1}	…	x_n		
c_1	x_1	1	0	…	0	$a_{1,m+1}$	…	a_{1n}	b_1	b_1/a_{1k}
c_2	x_2	0	1	…	0	$a_{2,m+1}$	…	a_{2n}	b_2	b_2/a_{2k}
	…	…	…	…	…	…	…	…	…	…
c_m	x_m	0	0	…	1	$a_{m,m+1}$	…	a_{mn}	b_m	b_m/a_{mk}
σ_j		0	0	…	0	σ_{m+1}	…	σ_n	$-z_0$	

$$\sigma_{m+1} = c_{m+1} - \sum_{i=1}^{m} c_i a_{i,m+1}$$

$$\sigma_n = c_n - \sum_{i=1}^{m} c_i a_{in}$$

$$-\sum_{i=1}^{m} c_i b_i$$

Step 2：Optimality test

If all $\sigma_j \le 0$，then the corresponding feasible basic solution is optimal.

Step 3: Get a new feasible basic solution

1. Determine the new basic variable

$$\sigma_k = \max_j \left\{ \sigma_j \middle| \sigma_j > 0 \right\}$$

2. Determine the basic variable which is out

calculate $$\theta = \min \left\{ \frac{b_i}{a_{ik}} \middle| a_{ik} > 0 \right\} = \frac{b_l}{a_{lk}}$$

Then x_l is out.

3. Coefficents change

$$
\begin{cases}
\max z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \\
x_1 \qquad + a_{1,m+1} x_{m+1} + \cdots + a_{1k} x_k + \cdots + a_{1n} x_n = b_1 \\
\quad x_2 \qquad + a_{2,m+1} x_{m+1} + \cdots + a_{2k} x_k + \cdots + a_{2n} x_n = b_2 \\
\quad \cdots\cdots\cdots\cdots \\
\qquad x_l \quad + a_{l,m+1} x_{m+1} + \cdots + a_{lk} x_k + \cdots + a_{ln} x_n = b_l \\
\quad \cdots\cdots\cdots\cdots \\
\qquad\quad x_m + a_{m,m+1} x_{m+1} + \cdots + a_{mk} x_k + \cdots + a_{mn} x_n = b_m \\
x_j \geq 0 \quad j = 1, 2, \cdots, n
\end{cases}
$$

x_k is the new variable

x_l is the eliminated varialbe

Technique for the basis change

$$A_l = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad A_k = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{lk} \\ \vdots \\ a_{mk} \end{pmatrix} \Rightarrow A'_l = \begin{pmatrix} a'_{1l} \\ a'_{2l} \\ \vdots \\ a'_{ll} \\ \vdots \\ a'_{ml} \end{pmatrix} \quad A'_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

Main factor

	j		k		
i	a_{ij}		a_{ik}		b_i
l	a_{lj}		a_{lk}		b_l

$$a'_{lj} = \frac{a_{lj}}{a_{lk}} \qquad b'_l = \frac{b_l}{a_{lk}}$$

$$a'_{ij} = a_{ij} - \frac{a_{ik} \cdot a_{lj}}{a_{lk}} \qquad b'_i = b_i - \frac{a_{ik} \cdot b_l}{a_{lk}} \qquad (i \neq l)$$

Repeat this process till all have been done

Solve this LP problem

$$\max z = 2x_1 + x_2$$

$$\begin{cases} 5x_2 \le 15 \\ 6x_1 + 2x_2 \le 24 \\ x_1 + x_2 \le 5 \\ x_1, x_2 \ge 0 \end{cases}$$

First reformulate it into a standard form

$c_j \rightarrow$			2	1	0	0	0
C_B		b	x_1	x_2	x_3	x_4	x_5
0	x_3	15	0	5	1	0	0
0	x_4	24	6	2	0	1	0
0	x_5	5	1	1	0	0	1
$c_j - z_j$			2	1	0	0	0

$c_j \rightarrow$			2	1	0	0	0
C_B		b	x_1	x_2	x_3	x_4	x_5
0	x_3	15	0	5	1	0	0
2	x_1	4	1	2/6	0	1/6	0
0	x_5	1	0	4/6	0	-1/6	1
$c_j - z_j$			0	1/3	0	-1/3	0

$c_j \rightarrow$			2	1	0	0	0
C_B		b	x_1	x_2	x_3	x_4	x_5
0	x_3	15/2	0	0	1	5/4	-15/2
2	x_1	7/2	1	0	0	1/4	-1/2
1	x_2	3/2	0	1	0	-1/4	3/2
$c_j - z_j$			0	0	0	-1/3	-1/2

Practice

$$\begin{cases} \max z = 7x_1 + 5x_2 \\ 3x_1 + 2x_2 \le 90 \\ 4x_1 + 6x_2 \le 200 \\ \qquad 7x_2 \le 210 \\ x_j \ge 0 \quad j = 1, \cdots, 5 \end{cases}$$

$$\begin{cases} \max z = 7x_1 + 5x_2 \\ 3x_1 + 2x_2 + x_3 = 90 \\ 4x_1 + 6x_2 + x_4 = 200 \\ \qquad 7x_2 + x_5 = 210 \\ x_j \ge 0 \quad j = 1, \cdots, 5 \end{cases}$$

c_j		7	5	0	0	0		
C_B	X_B	x_1	x_2	x_3	x_4	x_5	b	θ
0	x_3	3	2	1	0	0	90	90/3
0	x_4	4	6	0	1	0	200	200/4
0	x_5	0	7	0	0	1	210	-
σ_j		7	5	0	0	0	0	
7	x_1	1	2/3	1/3	0	0	30	45
0	x_4	0	10/3	-4/3	1	0	80	24
0	x_5	0	7	0	0	1	210	30
σ_j		0	1/3	-7/3	0	0	-210	
7	x_1	1	0	3/5	-1/5	0	14	
5	x_2	0	1	-2/5	3/10	0	24	
0	x_5	0	0	14/5	-21/10	1	42	
σ_j		0	0	-11/5	-1/10	0	-218	

Summary:

① Keep the righthand side vector nonnegative

The minimum ratio test guarantees it.

② The main factor can not be zero.

③ The main factor can not be negative.

Current solution is optimal

(1) exists a feasible basis (unit matrix)

(2) check values for basic variables are zero.

(3) check values $\leqslant 0$。 $\begin{cases} \text{all} \prec 0 \Rightarrow \text{unique} \\ \text{exist} = 0 \Rightarrow \text{multiple} \end{cases}$

(4) righthand side vector $\geqslant 0$

The next question is：

If there is no available basis, how should we do?

Section 5

Further discussion for the simplex method

Artificial variable method

$$\begin{cases} \min z = 5x_1 - 2x_2 + 3x_3 - 6x_4 \\ x_1 + 2x_2 + 3x_3 + 4x_4 = 7 \\ 2x_1 + x_2 + x_3 + 2x_4 = 3 \\ x_j \geq 0 \quad j = 1, \cdots, 4 \end{cases}$$

$A_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $A_4 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ A_2, A_4 are dependent

A_1, A_2 are independent, can form a basis, but

$X = (-1/3, 11/3, 0, 0)^T \longrightarrow$ infeasible

Find a feasible basic solution is difficult

Another situation

$$
\begin{cases}
\max z = -3x_1 + x_3 \\
x_1 + x_2 + x_3 \le 4 \\
-2x_1 + x_2 - x_3 \ge 1 \\
3x_2 + x_3 = 9 \\
x_j \ge 0 \quad j = 1,2,3
\end{cases}
$$

Standard form

$$
\begin{cases}
\max z = -3x_1 + x_3 + 0x_4 + 0x_5 \\
x_1 + x_2 + x_3 + x_4 = 4 \\
-2x_1 + x_2 - x_3 - x_5 = 1 \\
3x_2 + x_3 = 9 \\
x_j \ge 0 \quad j = 1,\ldots,5
\end{cases}
$$

The coefficient matrix

$$
\begin{array}{c}
\begin{array}{ccccc} P_1 & P_2 & P_3 & P_4 & P_5 \end{array} \\
\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ -2 & 1 & -1 & 0 & -1 \\ 0 & 3 & 1 & 0 & 0 \end{pmatrix}
\end{array}
$$

Hard to obtain a feasible basis here

We add artificial variables to deal with this difficult situation.

Add two column vectors

$$
\begin{array}{ccccccc}
P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7
\end{array}
$$

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
-2 & 1 & -1 & 0 & -1 & 1 & 0 \\
0 & 3 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

The constraints become:

$$
\begin{cases}
x_1 + x_2 + x_3 + x_4 = 4 \\
-2x_1 + x_2 - x_3 - x_5 + x_6 = 1 \\
3x_2 + x_3 + x_7 = 9 \\
x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0
\end{cases}
$$

x_6 $\boldsymbol{x_7}$ are artificial variables.

The coefficients of these artificial variables

In order to keep the equalities still active, these artificial variables must be zero in the optimal solution.

How to achieve this?

The objective function becomes

$$\max z = -3x_1 + x_3 + 0x_4 + 0x_5 - \boxed{Mx_6 - Mx_7}$$

Big M method

Example

$$
\begin{cases}
\max z = -5x_1 + 2x_2 - 3x_3 + 6x_4 \boxed{- Mx_5 - Mx_6} \\
x_1 + 2x_2 + 3x_3 + 4x_4 + x_5 = 7 \\
2x_1 + x_2 + x_3 + 2x_4 + x_6 = 3 \\
x_j \geq 0 \quad j = 1, \cdots, 6
\end{cases}
$$

If the optimal solution of this problem contains a nonnegative artificial variable, what does it mean?

C_B	X_B	-5	2	-3	6	-M	-M	b	θ
		x_1	x_2	x_3	x_4	x_5	x_6		
-M	x_5	1	2	3	4	1	0	7	7/4
-M	x_6	2	1	1	2	0	1	3	3/2
σ_j		3M-5	3M+2	4M-3	6M+6	0	0		
-M	x_5	-3	0	1	0	1	-2	1	1
6	x_4	1	1/2	1/2	1	0	1/2	3/2	3
σ_j		-3M-11	-1	M-6	0	0	-3M-3		
-3	x_3	-3	0	1	0	1	-2	1	
6	x_4	5/2	1/2	0	1	-1/2	3/2	1	
σ_j		-29	-1	0	0	6-M	-15-M		

There is no artificial variable at last

$X^*=(0, 0, 1, 1, 0, 0)^T \quad z^*=-3$

Practice

$$
\max z = -3x_1 + x_3
$$

$$
\begin{cases}
x_1 + x_2 + x_3 \le 4 \\
-2x_1 + x_2 - x_3 \ge 1 \\
3x_2 + x_3 = 9 \\
x_j \ge 0 \quad j = 1,2,3
\end{cases}
$$

$$
\max Z = -3x_1 + 0x_2 + x_3 + 0x_4 + 0x_5
$$

$$
\begin{cases}
x_1 + x_2 + x_3 + x_4 = 4 \\
-2x_1 + x_2 - x_3 - x_5 = 1 \\
3x_2 + x_3 = 9 \\
x_{1\sim5} \ge 0
\end{cases}
$$

Then we deliberately add the artificial variable to generate an identity matrix

$$
\begin{cases}
x_1 + x_2 + x_3 + x_4 = 4 \\
-2x_1 + x_2 - x_3 - x_5 + x_6 = 1 \\
3x_2 + x_3 \quad x_7 = 9 \\
x_{1\sim5} \ge 0
\end{cases}
$$

$$\max Z = -3x_1 + 0x_2 + x_3 + 0x_4 + 0x_5 - Mx_6 - Mx_7$$

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 4 \\ -2x_1 + x_2 - x_3 - x_5 + x_6 = 1 \\ 3x_2 + x_3 + x_7 = 9 \\ x_{1\sim5} \geq 0 \end{cases}$$

M is a penalty factor.

How does it work?

Practice

$$\max Z = 2x_1 + x_2$$

$$\begin{cases} x_1 + x_2 & \le 2 \\ 2x_1 + 2x_2 & \ge 6 \\ x_1, x_2 \ge 0 \end{cases}$$

$$\max Z = 2x_1 + x_2 + 0x_3 + 0x_4 - Mx_5$$

$$\begin{cases} x_1 + x_2 + x_3 & = 2 \\ 2x_1 + 2x_2 - x_4 + x_5 & = 6 \\ x_{1\sim 4} \ge 0 \end{cases}$$

When all the check values are not positive

The solution is： $\begin{cases} x_1 = 2 \\ x_5 = 2 \end{cases}$ contains an artificial variable

The problem is infeasible.

A new question

If we apply the big M method on the computer, how do we decide the value of M?

2. Two phases method

First phase: add the artificial variables, and solve the following problem

$$
\text{LP'}\quad
\begin{cases}
\min w = x_{n+1} + x_{n+2} + \cdots + x_{n+m} \\
a_{11}x_1 + \cdots + a_{1n}x_n + x_{n+1} = b_1 \\
a_{21}x_1 + \cdots + a_{2n}x_n + x_{n+2} = b_2 \\
\cdots\cdots\cdots\cdots \\
a_{m1}x_1 + \cdots + a_{mn}x_n + x_{n+m} = b_m \\
\quad x_j \geq 0 \quad j = 1,2,\cdots,n+m
\end{cases}
$$

1. **Determine whether the original problem is feasible or not**
2. **If it feasible, then continue**

Phase two: use the optimal solution of phase one as the starting feasible basic solution to solve the original problem.

Practice

$$
\begin{cases}
\min z = -3x_1 + x_2 + x_3 \\
\quad x_1 - 2x_2 + x_3 \le 11 \\
-4x_1 + x_2 + 2x_3 \ge 3 \\
-2x_1 \qquad\quad + x_3 = 1 \\
x_j \ge 0 \quad j = 1,2,3
\end{cases}
$$

$$
\begin{cases}
\min w = x_6 + x_7 \\
\quad x_1 - 2x_2 + x_3 + x_4 = 11 \\
-4x_1 + x_2 + 2x_3 - x_5 + x_6 = 3 \\
-2x_1 \qquad\quad + x_3 \qquad\quad + x_7 = 1 \\
x_j \ge 0 \quad j = 1,\cdots,7
\end{cases}
$$

	c_j	0	0	0	0	0	-1	-1	b	θ
C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7		
0	x_4	1	-2	1	1	0	0	0	11	11
-1	x_6	-4	1	2	0	-1	1	0	3	3/2
-1	x_7	-2	0	1	0	0	0	1	1	1
σ_j		-6	1	3	0	-1	0	0		
0	x_4	3	-2	0	1	0	0	-1	10	-
-1	x_6	0	1	0	0	-1	1	-2	1	1
0	x_3	-2	0	1	0	0	0	1	1	-
σ_j		0	1	0	0	-1	0	-3		
0	x_4	3	0	0	1	-2	2	-5	12	
0	x_2	0	1	0	0	-1	1	-2	1	
0	x_3	-2	0	1	0	0	0	0	1	
σ_j		0	0	0	0	0	-1	-1		

c_j		3	-1	-1	0	0	b	θ
C_B	X_B	x_1	x_2	x_3	x_4	x_5		
0	x_4	3	0	0	1	-2	12	4
-1	x_2	0	1	0	0	-1	1	-
-1	x_3	-2	0	1	0	0	1	-
σ_j		1	0	0	0	-1		
3	x_1	1	0	0	1/3	-2/3	4	
-1	x_2	0	1	0	0	-1	1	
-1	x_3	0	0	1	2/3	-4/3	9	
σ_j		0	0	0	-1/3	-1/3		

$X^*=(x_1, x_2, x_3)^T=(4, 1, 9)^T \quad z^*=-2$

About degeneration

- What is degeneration?
- When does it happen?
- What difficulty does it cause?

$$\max\ f(x)=\frac{3}{4}x_4-20x_5+\frac{1}{2}x_6-6x_7$$

$$\begin{cases} x_1+\dfrac{1}{4}x_4-8x_5-x_6-9x_7=0 \\ x_2+\dfrac{1}{2}x_4-12x_5-\dfrac{1}{2}x_6+3x_7=0 \\ x_3+x_6=1 \\ x_1,x_2,\cdots,x_7\ge 0 \end{cases}$$

Dead Iteration

c_j			0	0	0	3/4	-20	1/2	-6
C_B	X_B	b^*	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	x_1	0	1	0	0	[1/4]	-8	-1	9
0	x_2	0	0	1	0	1/2	-12	-1/2	3
0	x_3	1	0	0	1	0	0	1	0
$\sigma_j=c_j-z_j$		0	0	0	0	3/4	-20	1/2	-6

(1,2,3)→ (4,2,3)→ (4,5,3)→ (6,5,3)→ (6,7,3)→ (1,7,3)→ (1,2,3)

c_j			0	0	0	3/4	-20	1/2	-6
C_B	X_B	b^*	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	x_1	0	1	0	0	[1/4]	-8	-1	9
0	x_2	0	0	1	0	1/2	-12	-1/2	3
0	x_3	1	0	0	1	0	0	1	0
$\sigma_j=c_j-z_j$		0	0	0	0	3/4	-20	1/2	-6

(1,2,3)→ (4,2,3)→ (4,5,3)→ (6,5,3)→ (6,7,3)→ (1,7,3)→ (1,2,3)

Use the bland rule to deal with the degeneration.

Section 6

Applications

Choice of advertisement

The sale department has $20000 budget for the advertisement. It requires at least 8 television advertisements and 15 newspaper advertisements. The budget for TV is no more than $12000. And the radio advertisement should be heard at least every two days.

What is the best choice the sale department to get the biggest effect?

Table-1

Ways	Cost for each time	Number limits	Expected effect
TV A	500	16	50
TV B	1000	10	80
Newspaper A	100	24	30
Newspaper B	300	4	40
Radio	80	25	15

$$\max z = 50x_1 + 80x_2 + 30x_3 + 40x_4 + 15x_5$$

$$\text{s.t.}\begin{cases} 500x_1 + 1000x_2 + 100x_3 + 300x_4 + 80x_5 \le 20000 \\ x_1 + x_2 \ge 8 \\ x_3 + x_4 \ge 15 \\ 500x_1 + 1000x_2 \le 12000 \\ x_1 \le 16\,;\, x_2 \le 10\,;\, x_3 \le 24\,;\, x_4 \le ;15 \le x_5 \le 25, \\ x_1, x_2, x_3, x_4, x_5 \ge 0 \end{cases}$$

Assignment problem

Each employee needs to work for two consecutive time periods. And the need for each time period should be satisfied. How many employees do we need?

	Time period	Needs
1	6:00 —— 10:00	60
2	10:00 —— 14:00	70
3	14:00 —— 18:00	60
4	18:00 —— 22:00	50
5	22:00 —— 2:00	20
6	2:00 —— 6:00	30

Let x_j denote the number of employees who work form time period j

$$\min z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

$$\text{s.t.} \begin{cases} x_6 + x_1 \geq 60 \\ x_1 + x_2 \geq 70 \\ x_2 + x_3 \geq 60 \\ x_3 + x_4 \geq 50 \\ x_4 + x_5 \geq 20 \\ x_5 + x_6 \geq 30 \\ x_j \geq 0, j = 1,2,3,\ldots,6 \end{cases}$$