Section 1
Integer programming and its characteristics

一、整数规划的数学模型的一般形式

IP problem

$$\max z = \sum_{j=1}^{n} c_j x_j$$

$$\begin{cases} \sum_{j=1}^{n} a_{ij} x_j \le b_i & (i = 1, \cdots, m) \\ x_j \ge 0 & (j = 1, \cdots, n) \end{cases}$$

x are all integers or partially integers

Types of Integer Programming

Pure Integer Programming

Mixed Integer Programming

Binary Integer Programming

Example 1: container problem

Cargo	Volume	Weight	Profit
A	5	2	20
B	4	5	10
Restriction	24	13	

Example 2: assignment problem

Every employee work for consecutive four time periods. The minimum requirement for each time period is listed below. How many employees do we need?

time period	1	2	3	4	5	6	7	8
Minimum employee	10	8	9	11	13	8	5	3

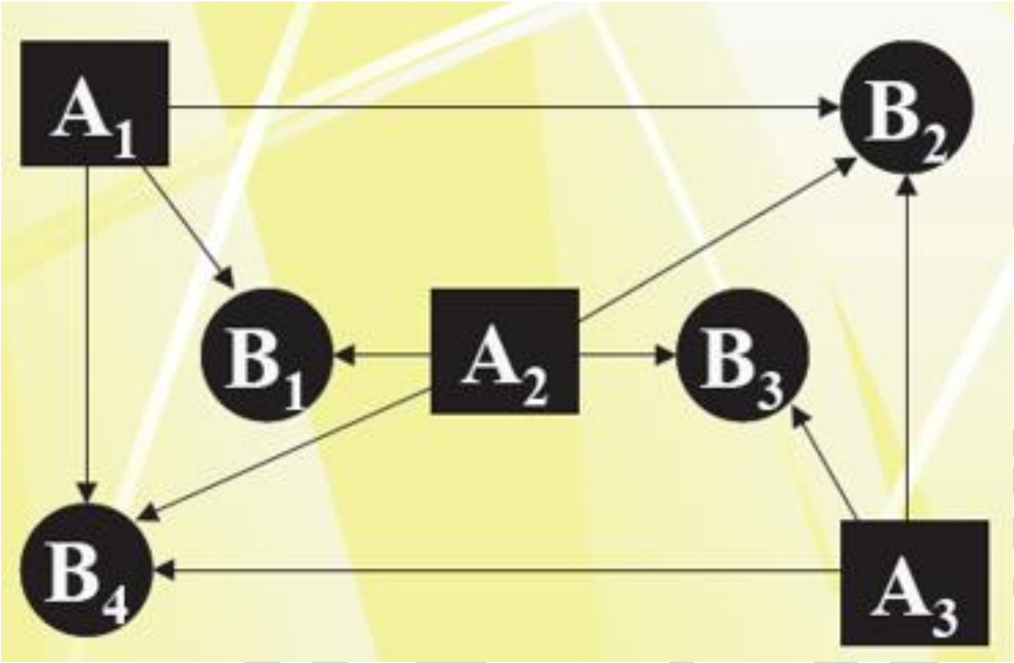
$$\min Z = x_1 + x_2 + x_3 + x_4 + x_5$$

$$
s.t.\begin{cases}
x_1 \geq 10 \\
x_1 + x_2 \geq 8 \\
x_1 + x_2 + x_3 \geq 9 \\
x_1 + x_2 + x_3 + x_4 \geq 11 \\
x_2 + x_3 + x_4 + x_5 \geq 13 \\
x_3 + x_4 + x_5 \geq 8 \\
x_4 + x_5 \geq 5 \\
x_5 \geq 3
\end{cases}
$$

$x_1, x_2, x_3, x_4, x_5 \geq 0,$ all be integers

Pure integer programming

Example e: Location-allocation problem



There are three available locations for building warehouses. For Ai, the capacity is ai while the building cost is bi

There are four for shops. For Bj, the demand is di. The transportation fee from Ai to Bj is cij.

Minimize the total cost.

$$\min f(x) = \sum_{i=1}^{3} b_i x_i + \sum_{i=1}^{3}\sum_{j=1}^{4} c_{ij} y_{ij}$$

$$\begin{cases} y_{11} + y_{21} = d_1 \\ y_{12} + y_{22} + y_{32} = d_2 \\ y_{23} + y_{33} = d_3 \\ y_{14} + y_{24} + y_{34} = d_4 \\ \boxed{x_1 + x_2 + x_3 = 2} \\ y_{11} + y_{12} + y_{14} \le a_1 x_1 \\ y_{21} + y_{22} + y_{23} + y_{24} \le a_2 x_2 \\ y_{32} + y_{33} + y_{34} \le a_3 x_3 \\ x_i \text{为} 0, 1, \quad y_{ij} \ge 0 \end{cases}$$

Demand requirements

Building requirement

Capacity requirements

Mixed integer programming

三、Charactersitics

Relationship between LP and IP

IP

$$\max c^{\mathrm{T}} x$$

$$s.t.\begin{cases} Ax = b \\ x \ge 0, \text{ Integers} \end{cases}$$

LP

$$\max c^{\mathrm{T}} x$$

$$s.t.\begin{cases} Ax = b \\ x \ge 0 \end{cases}$$

Feasible domain

Optimal values

Lower bound

Optimal solution

Can we directly get the optimal integer solution from the optimal solution of the LP problem?

Solving an IP problem is much more difficult than solving a LP problem.

$$\max Z = x_1 + x_2$$

$$\begin{cases} 14x_1 + 9x_2 \le 51 \\ -6x_1 + 3x_2 \le 1 \\ x_1, x_2 \ge 0 \end{cases}$$

Integers

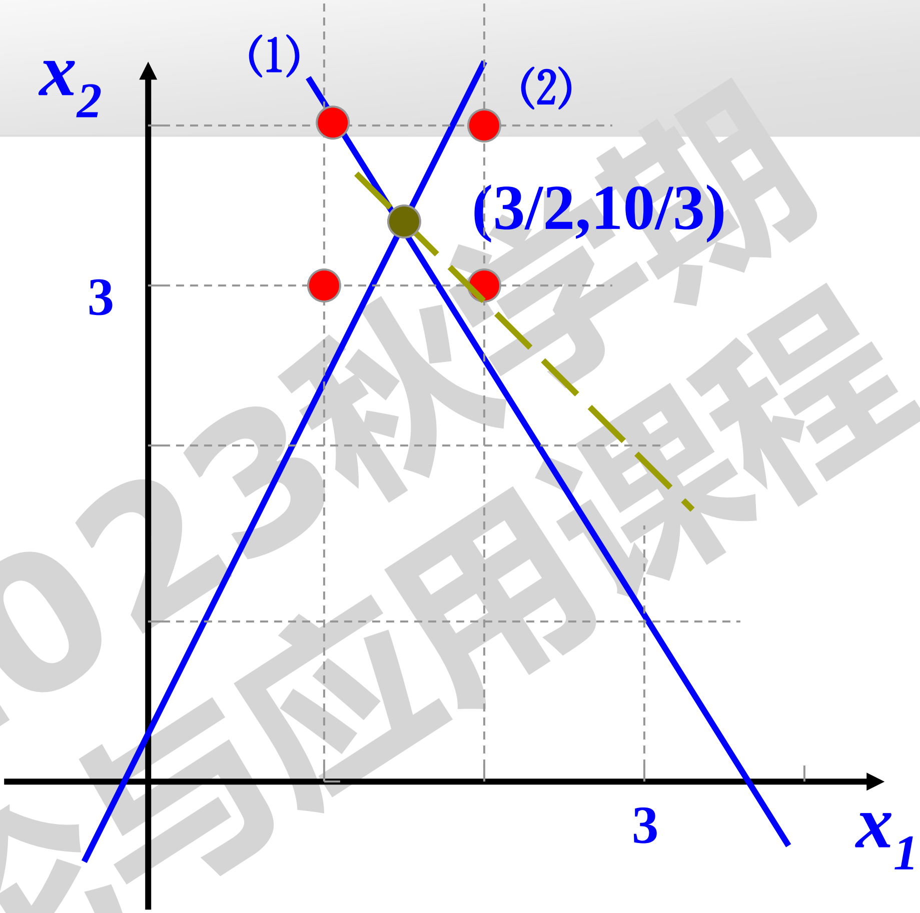
Relax it and get a LP problem.

$$\max Z = x_1 + x_2$$

$$\begin{cases} 14x_1 + 9x_2 \le 51 \\ -6x_1 + 3x_2 \le 1 \\ x_1, x_2 \ge 0 \end{cases}$$

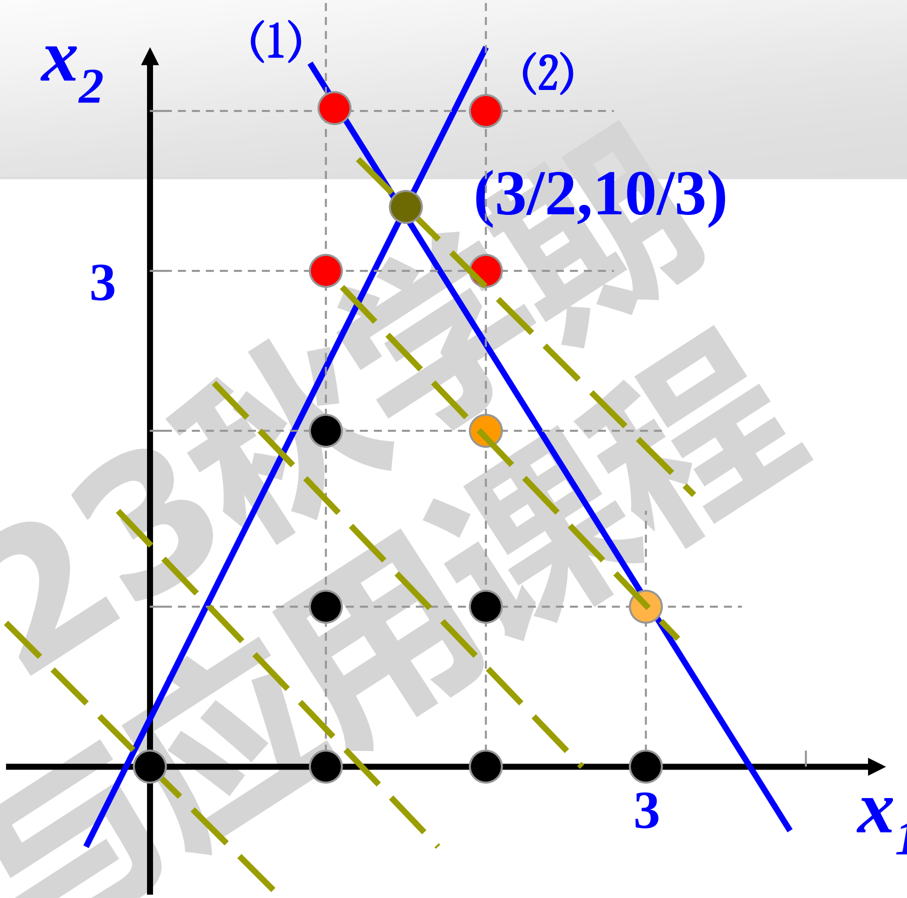
Use the graph algorithm to get
$x_1 = 3/2,\ \ x_2 = 10/3$
$Z = 29/6$

Notice that, its neighbor points (1，3)(2，3)(1，4)(2，4) are all infeasible.



You can not just defer the optimal integer solution from the LP solution!

For a bounded area, maybe you can enumerate to obtain the optimal solution.



Here（2，2）（3，1）are the optimal points，Z=4。

求解整数规划方法

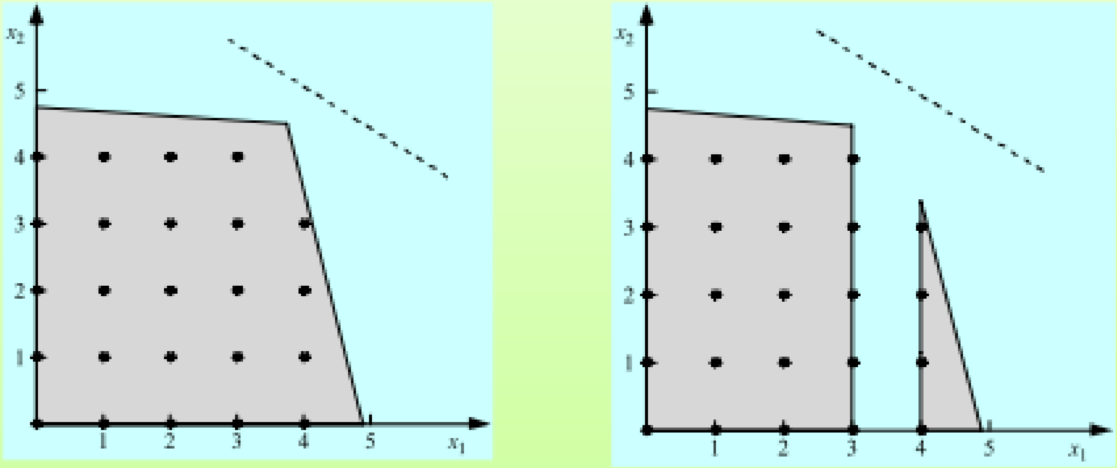
The growth rate of 2^n:

10	1.02×10^3
20	1.05×10^6
30	1.07×10^9
40	1.10×10^{12}
50	1.73×10^{15}
100	1.27×10^{30}

It is impossible to enumerate for a large size problem.

Section 2
B-a-B method

分枝定界技术(Branch-and-Bound Technique)



一、基本思想

基本思想

□Branch-and-Bound method is a general method for all integer programming problem.

□Its basic idea is to divide the feasible domain into several subdomains, then solve the corresponding LP problem. In this process, raise the lower bound, lower the upper bound, till obtain the optimal solution.

Branch divide the original problem into two subproblems.

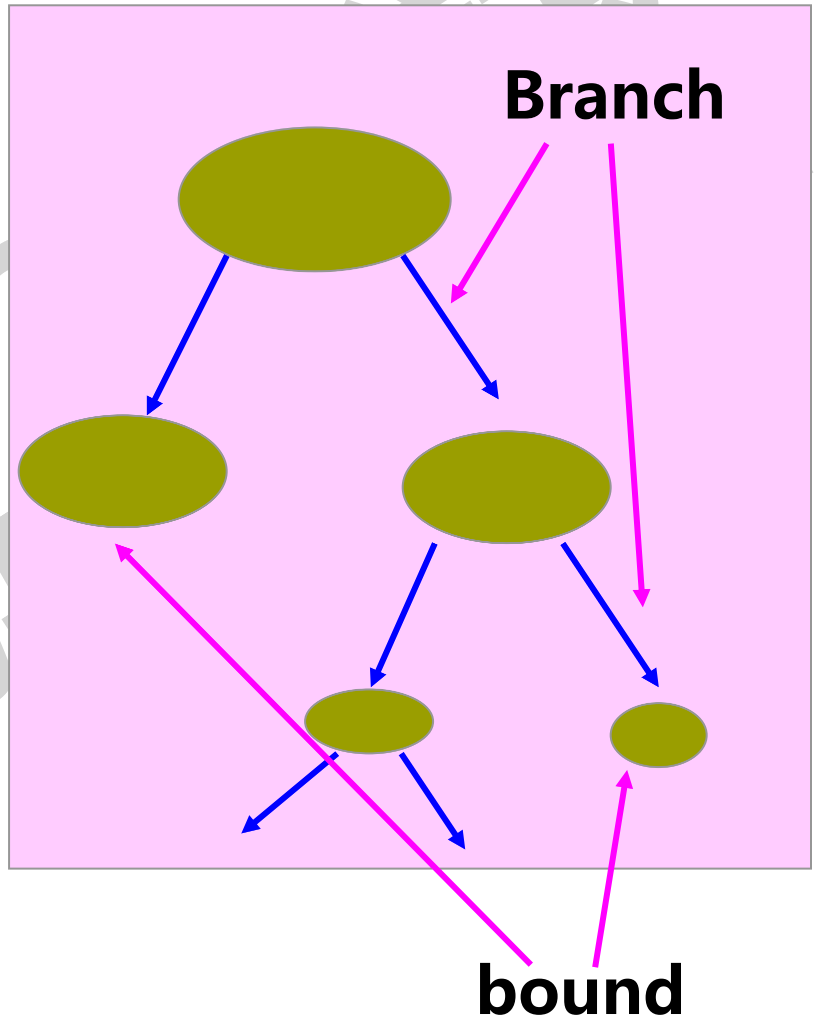
Seven situations on solving these two subproblems.

➢ Implicit enumeration

Solve the relaxed LP problem

If the optimal value is worse than the bound → **Abandon**

If the optimal solution is not integer, but the optimal value is better than the bound → **Branch**



Example

$$\min Z = -x_1 - 5x_2$$

$$\begin{cases} x_1 - x_2 \geq -2 \\ 5x_1 + 6x_2 \leq 30 \\ x_1 \leq 4 \\ x_1, x_2 \geq 0 \text{ Integers} \end{cases}$$

IP

Relax it into a LP problem

$$\min Z = -x_1 - 5x_2$$

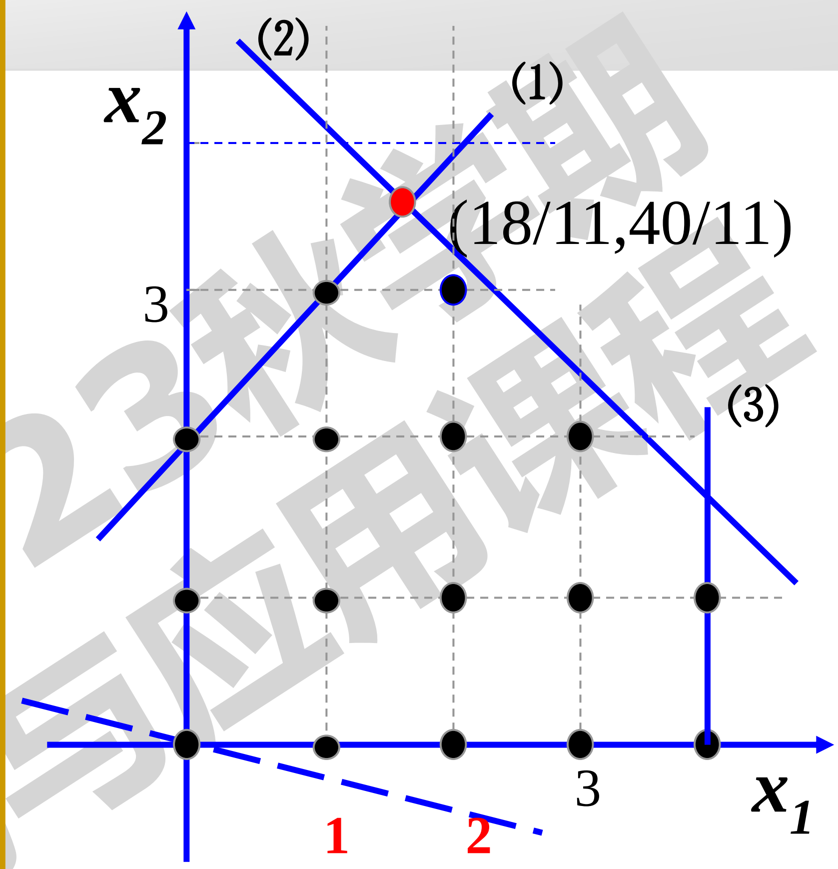
$$\begin{cases} x_1 - x_2 \geq -2 \\ 5x_1 + 6x_2 \leq 30 \\ x_1 \leq 4 \\ x_1, x_2 \geq 0 \end{cases}$$

LP

$x_1 = 18/11$, $x_2 = 40/11$

$Z^{(0)} = -218/11 \approx (-19.8)$

Z **is the lower bound.**



For $x_1 = 18/11 \approx 1.64$,

let $x_1 \le 1$, $x_1 \ge 2$

For $x_2 = 40/11 \approx 3.64$, **let** $x_2 \le 3$, $x_2 \ge 4$

Here we apply $x_1 \le 1$, $x_1 \ge 2$

$$\min Z = -x_1 - 5x_2$$

$$(IP1)\begin{cases} x_1 - x_2 \geq -2 \\ 5x_1 + 6x_2 \leq 30 \\ x_1 \leq 4 \\ x_1 \leq 1 \\ x_1, x_2 \geq 0 \end{cases}$$

Integers

$$\min Z = -x_1 - 5x_2$$

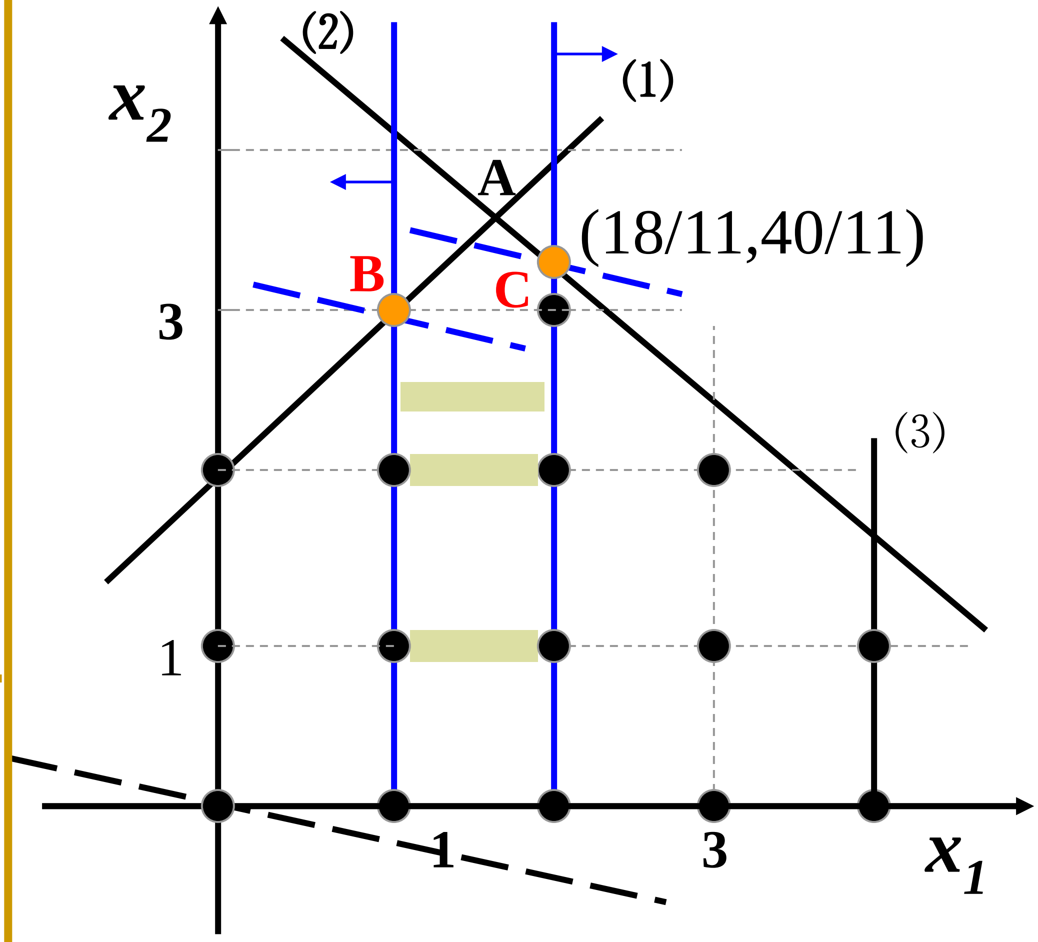
$$(IP2)\begin{cases} x_1 - x_2 \geq -2 \\ 5x_1 + 6x_2 \leq 30 \\ x_1 \leq 4 \\ x_1 \geq 2 \\ x_1, x_2 \geq 0 \end{cases}$$

Integers

Solve (*LP1*)

$x_1 = 1, x_2 = 3, Z^{(1)} = -16$

Integers, stop here.



Solve (*LP2*)

$x_1 = 2, x_2 = 10/3,$

$Z^{(2)} = -56/3 \approx -18.7$

$\because Z_2 < Z_1 = -16$ $\therefore$**Branch hare**

Add $4 \geq x, \ x \geq 3$ 。

After adding $x_2 \le 3$, $x_2 \ge 4$

$$\min Z = -x_1 - 5x_2$$

$$(IP3)\begin{cases} x_1 - x_2 \ge -2 \\ 5x_1 + 6x_2 \le 30 \\ x_1 \le 4 \\ x_1 \ge 2 \\ x_2 \le 3 \\ x_1, x_2 \ge 0 \end{cases}$$

Integers

$$\min Z = -x_1 - 5x_2$$

$$(IP4)\begin{cases} x_1 - x_2 \ge -2 \\ 5x_1 + 6x_2 \le 30 \\ x_1 \le 4 \\ x_1 \ge 2 \\ x_2 \ge 4 \\ x_1, x_2 \ge 0 \end{cases}$$

Integers

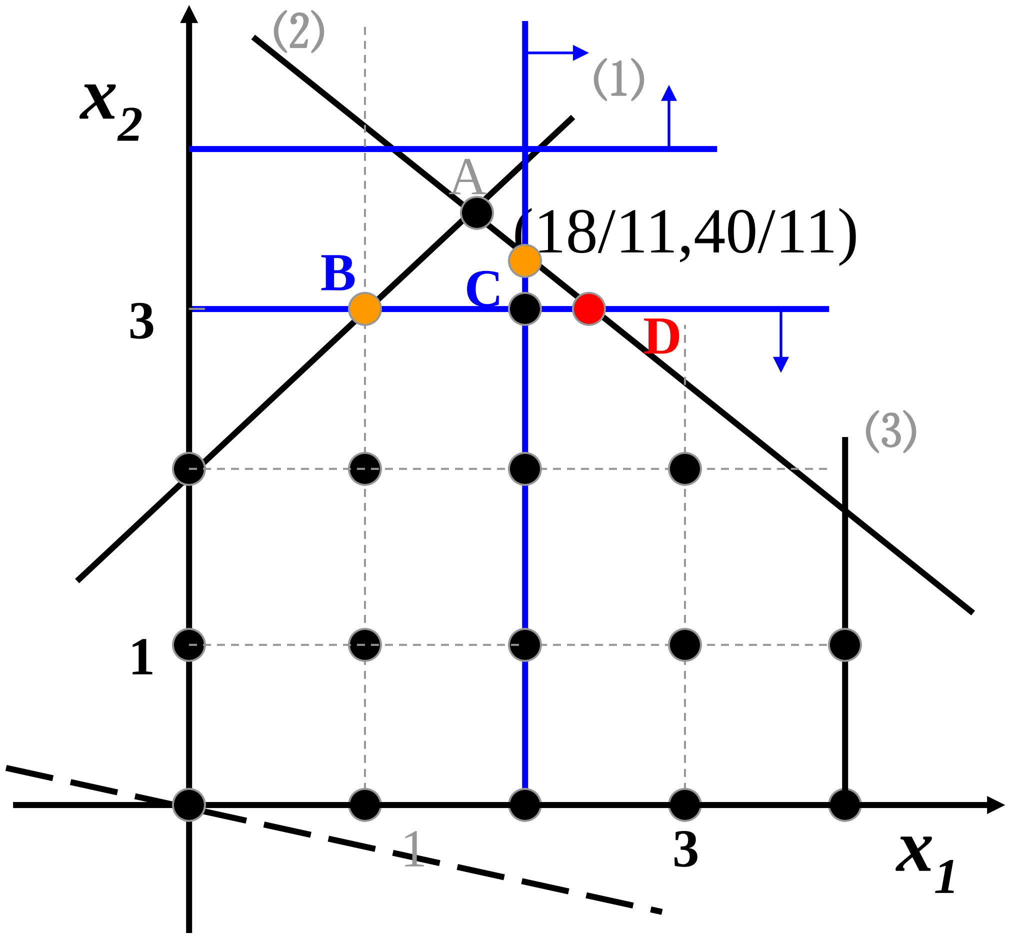
Solve（*LP3*）

$x_1 = 12/5 \approx 2.4$, $x_2 = 3$,

$Z^{(3)} = -87/5 \approx -17.4 > Z \approx -19.8$

Continue to branch

$2 \leq x_1 \leq 3$。



Solve（*LP4*）， Infeasible, stop

$$\min Z = -x_1 - 5x_2$$

$$(IP5)\begin{cases} x_1 - x_2 \ge -2 \\ 5x_1 + 6x_2 \le 30 \\ x_1 \le 4 \\ x_1 \ge 2 \\ x_2 \le 3 \\ x_1 \le 2 \\ x_1, x_2 \ge 0 \end{cases}$$

Integers

$$\min Z = -x_1 - 5x_2$$

$$(IP6)\begin{cases} x_1 - x_2 \ge -2 \\ 5x_1 + 6x_2 \le 30 \\ x_1 \le 4 \\ x_1 \ge 2 \\ x_2 \le 3 \\ x_1 \ge 3 \\ x_1, x_2 \ge 0 \end{cases}$$

Integers

Solve（*LP5*），。

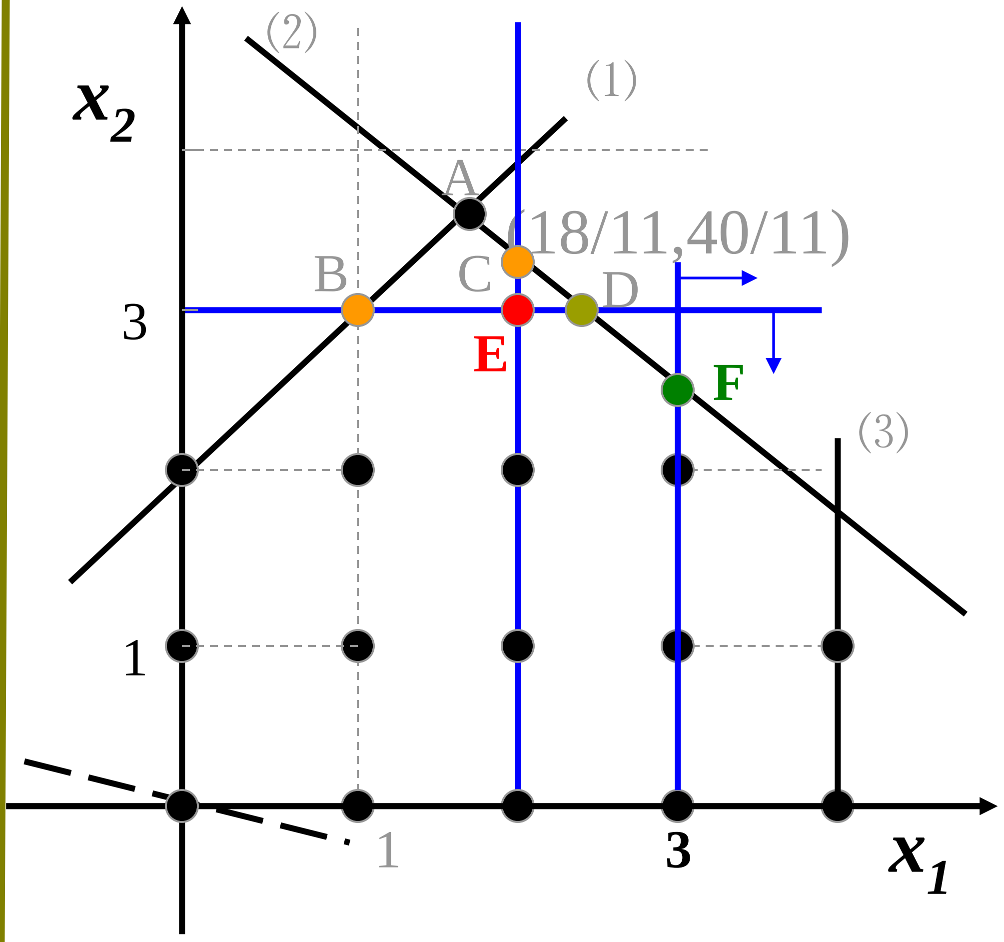
$x_1 = 2, x_2 = 3, Z^{(5)} = -17$

Integers, stop

solve（*LP6*）

$x_1 = 3, x_2 = 2.5,$

$Z^{(6)} = -31/2 \approx -15.5 > Z^{(5)}$



If we do further branch, the optimal value can not be better than －15.5， stop.

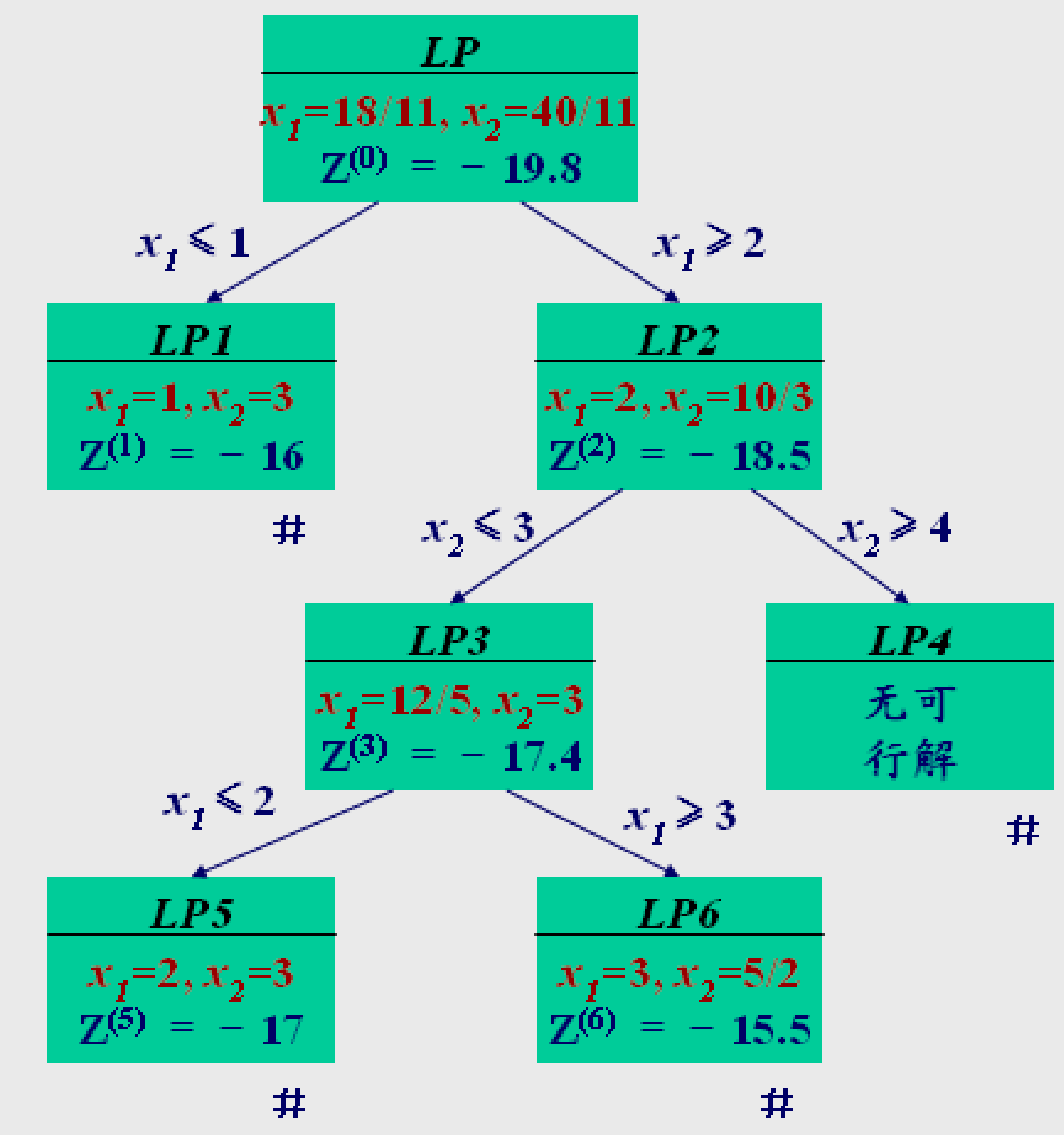
The optimal solution is

$x_1=2$,

$x_2=3$,

$Z^* = Z^{(5)}$

$= -17$



Practice

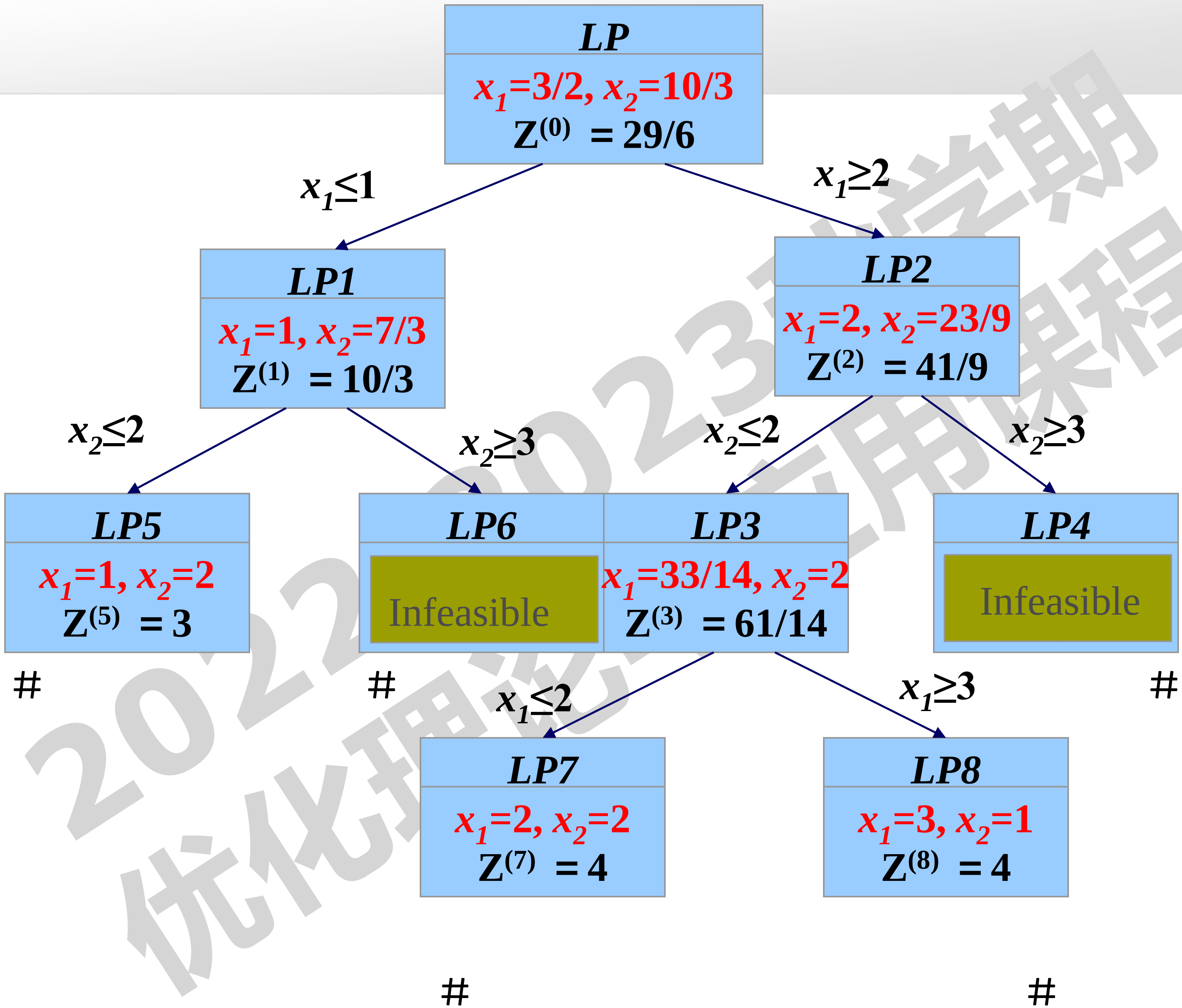
$$\max Z = x_1 + x_2$$

$$\begin{cases} x_1 + \dfrac{9}{14}x_2 \le \dfrac{51}{14} \\ -2x_1 + x_2 \le \dfrac{1}{3} \\ x_1, x_2 \ge 0, \text{ Integers} \end{cases}$$

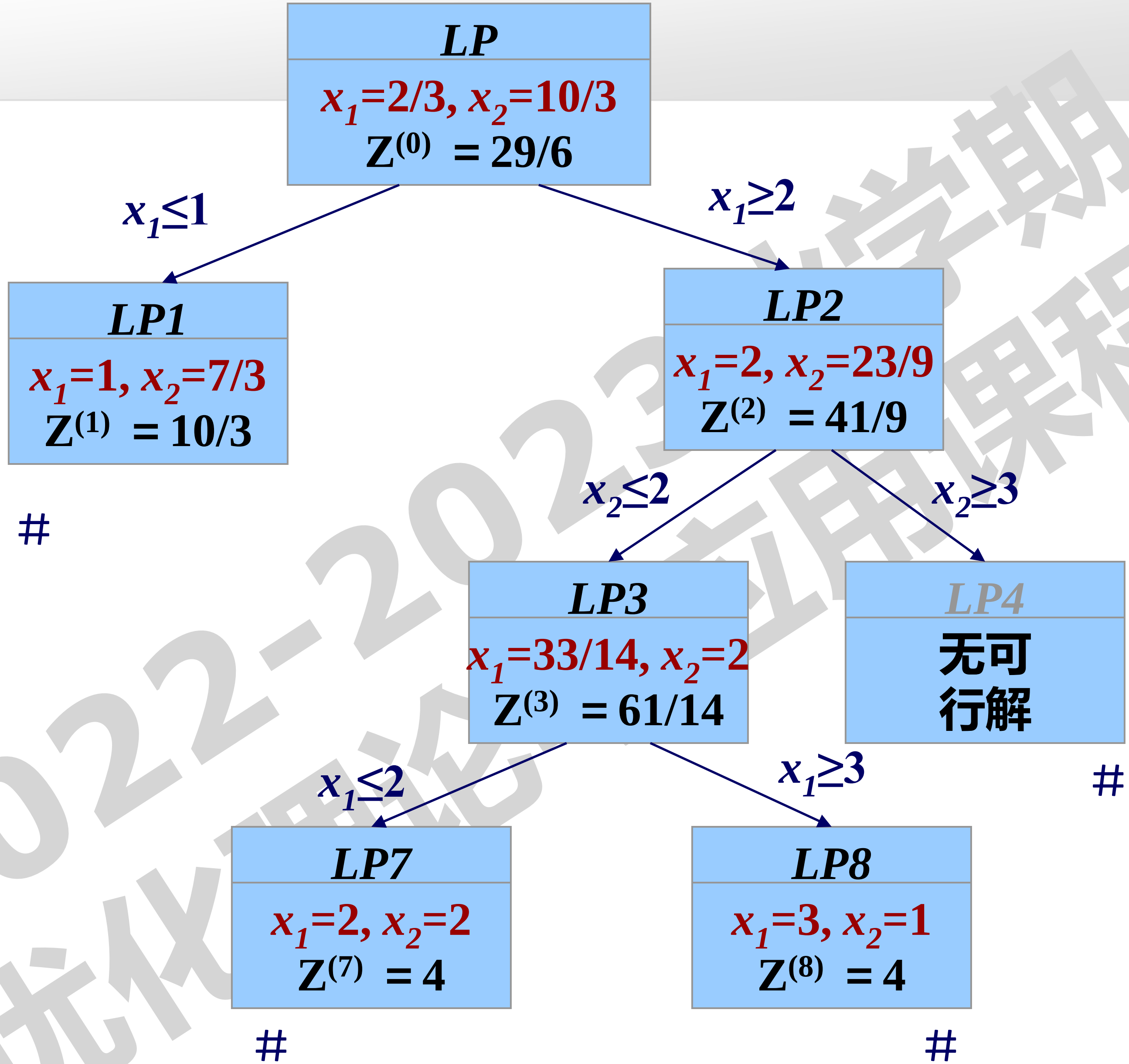
$$\Rightarrow$$

$$\max Z = x_1 + x_2$$

$$\begin{cases} 14x_1 + 9x_2 \le 51 \\ -6x_1 + 3x_2 \le 1 \\ x_1, x_2 \ge 0, \text{ Integers} \end{cases}$$



LP
x_1=3/2, x_2=10/3
$Z^{(0)}$ = 29/6
$x_1 \le 1$
$x_1 \ge 2$
LP1
x_1=1, x_2=7/3
$Z^{(1)}$ = 10/3
LP2
x_1=2, x_2=23/9
$Z^{(2)}$ = 41/9
$x_2 \le 2$
$x_2 \ge 3$
$x_2 \le 2$
$x_2 \ge 3$
LP5
x_1=1, x_2=2
$Z^{(5)}$ = 3
LP6
Infeasible
LP3
x_1=33/14, x_2=2
$Z^{(3)}$ = 61/14
LP4
Infeasible
#
#
#
$x_1 \le 2$
$x_1 \ge 3$
LP7
x_1=2, x_2=2
$Z^{(7)}$ = 4
LP8
x_1=3, x_2=1
$Z^{(8)}$ = 4
#
#


LP
x1=2/3, x2=10/3
Z(0) = 29/6
x1≤1
x1≥2
LP1
x1=1, x2=7/3
Z(1) = 10/3
#
LP2
x1=2, x2=23/9
Z(2) = 41/9
x2≤2
x2≥3
LP3
x1=33/14, x2=2
Z(3) = 61/14
LP4
无可行解
#
x1≤2
x1≥3
LP7
x1=2, x2=2
Z(7) = 4
#
LP8
x1=3, x2=1
Z(8) = 4
#

Example

Use B-a-B to solve this problem

$$\max Z = 4x_1 + 3x_2$$

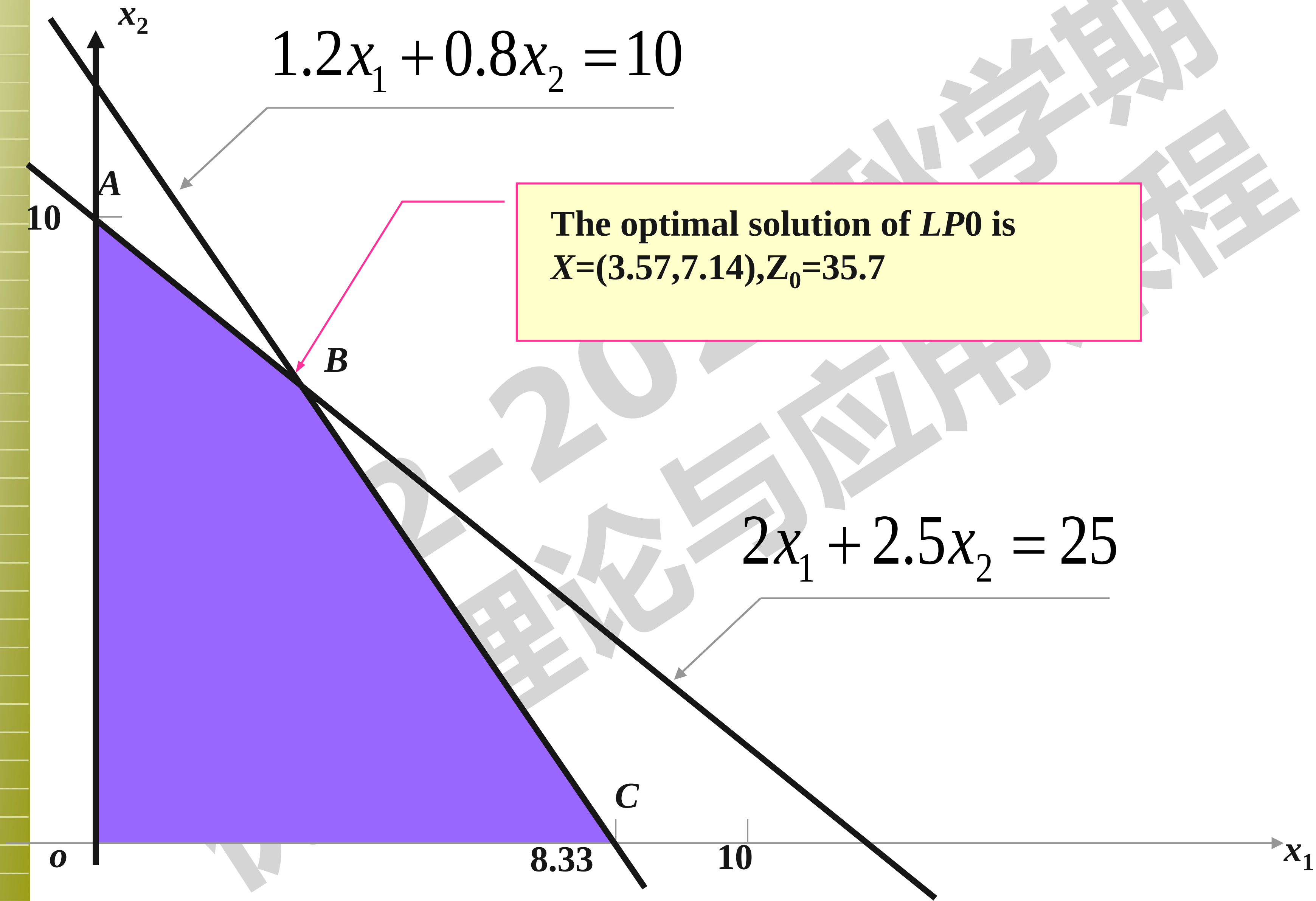
$$\begin{cases} 1.2x_1 + 0.8x_2 \le 10 \\ 2x_1 + 2.5x_2 \le 25 \\ x_1, x_2 \ge 0, \text{ Integers} \end{cases}$$

First, relax it into a LP problem

$$\max Z = 4x_1 + 3x_2$$

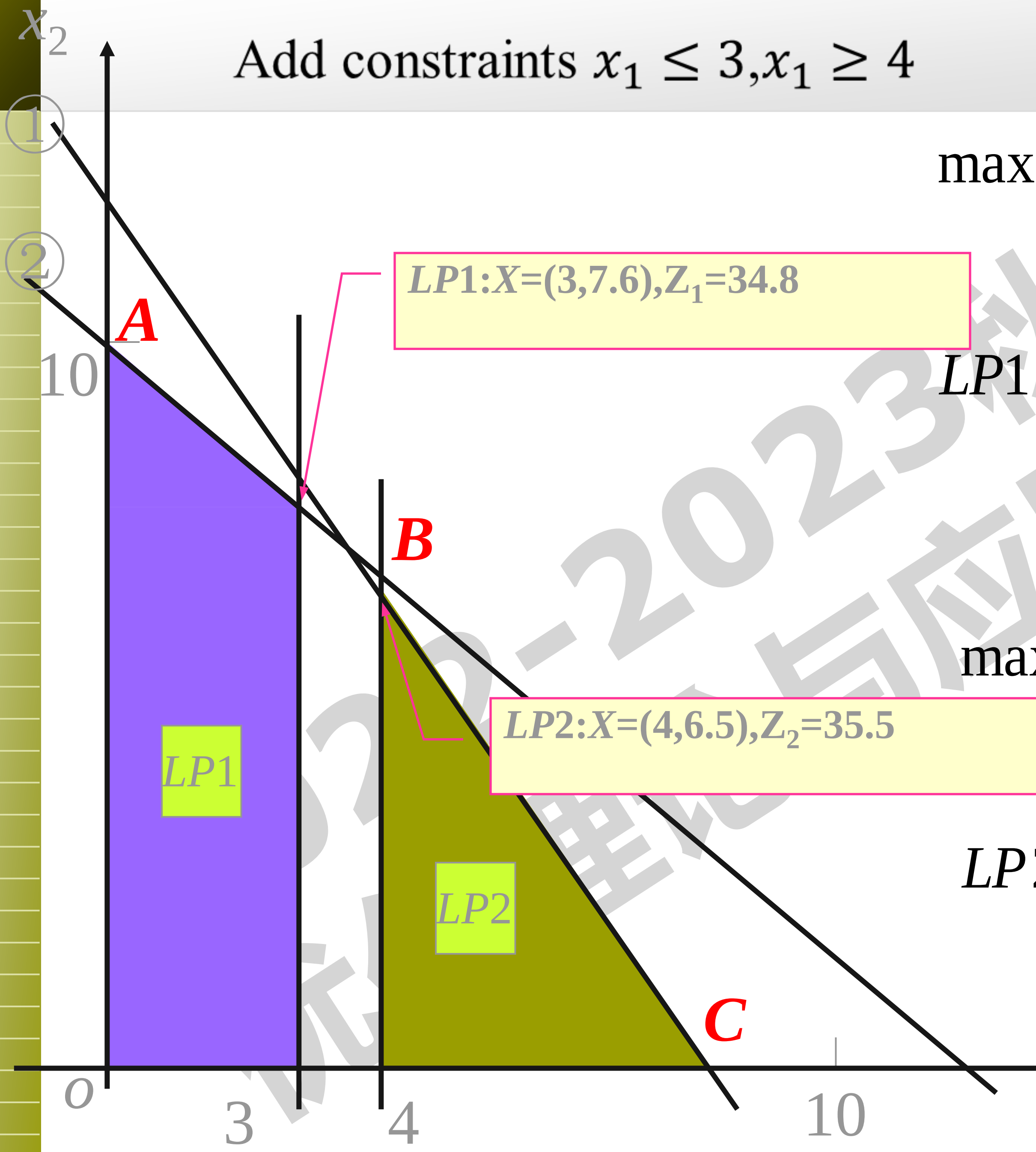
$$LP0: \begin{cases} 1.2x_1 + 0.8x_2 \le 10 \\ 2x_1 + 2.5x_2 \le 25 \\ x_1, x_2 \ge 0 \end{cases}$$

Use the graph algorithm to solve this LP problem



x_2
$1.2x_1 + 0.8x_2 = 10$
A
10
The optimal solution of *LP*0 is
$X=(3.57,7.14), Z_0=35.7$
B
$2x_1 + 2.5x_2 = 25$
C
o
8.33
10
x_1

Add constraints $x_1 \le 3, x_1 \ge 4$



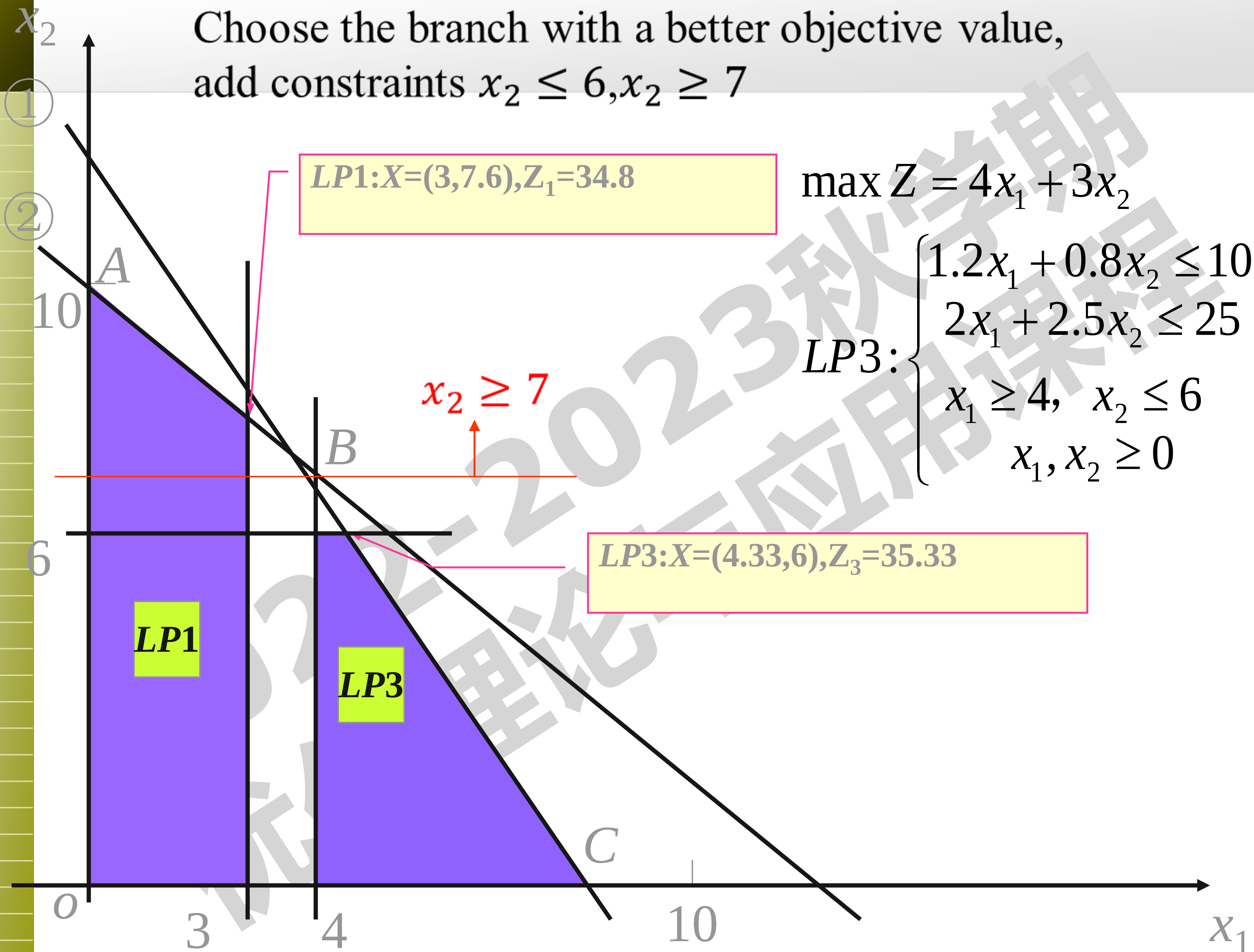
$$\max Z = 4x_1 + 3x_2$$

$$LP1: \begin{cases} 1.2x_1 + 0.8x_2 \le 10 \\ 2x_1 + 2.5x_2 \le 25 \\ x_1 \le 3 \\ x_1, x_2 \ge 0 \end{cases}$$

$$\max Z = 4x_1 + 3x_2$$

$$LP2: \begin{cases} 1.2x_1 + 0.8x_2 \le 10 \\ 2x_1 + 2.5x_2 \le 25 \\ x_1 \ge 4 \\ x_1, x_2 \ge 0 \end{cases}$$

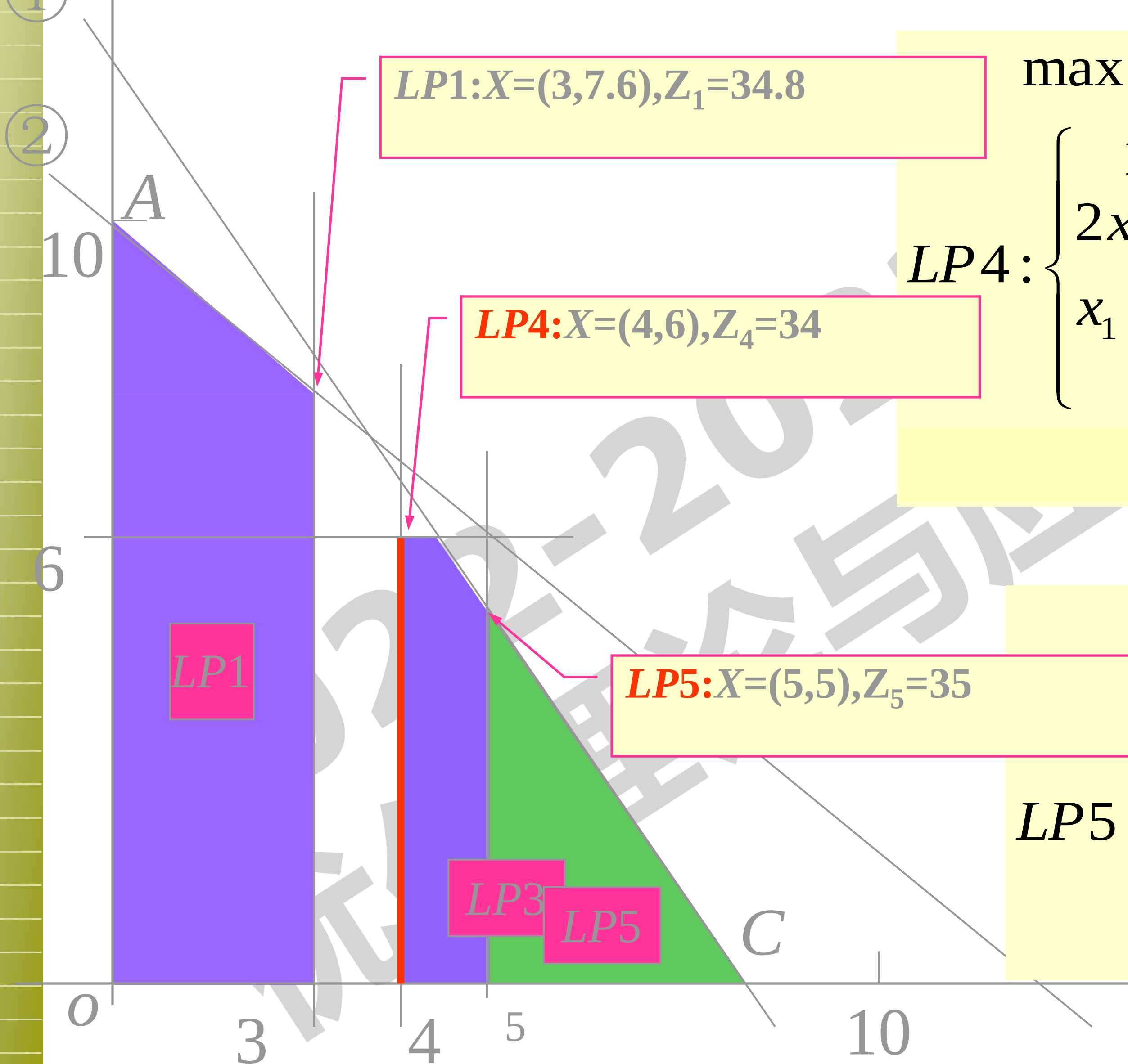
Choose the branch with a better objective value, add constraints $x_2 \le 6, x_2 \ge 7$



$$\max Z = 4x_1 + 3x_2$$

$$LP3: \begin{cases} 1.2x_1 + 0.8x_2 \le 10 \\ 2x_1 + 2.5x_2 \le 25 \\ x_1 \ge 4, \quad x_2 \le 6 \\ x_1, x_2 \ge 0 \end{cases}$$

Choose the branch with a better objective value, add constraints $x_1 \le 4, x_1 \ge 5$



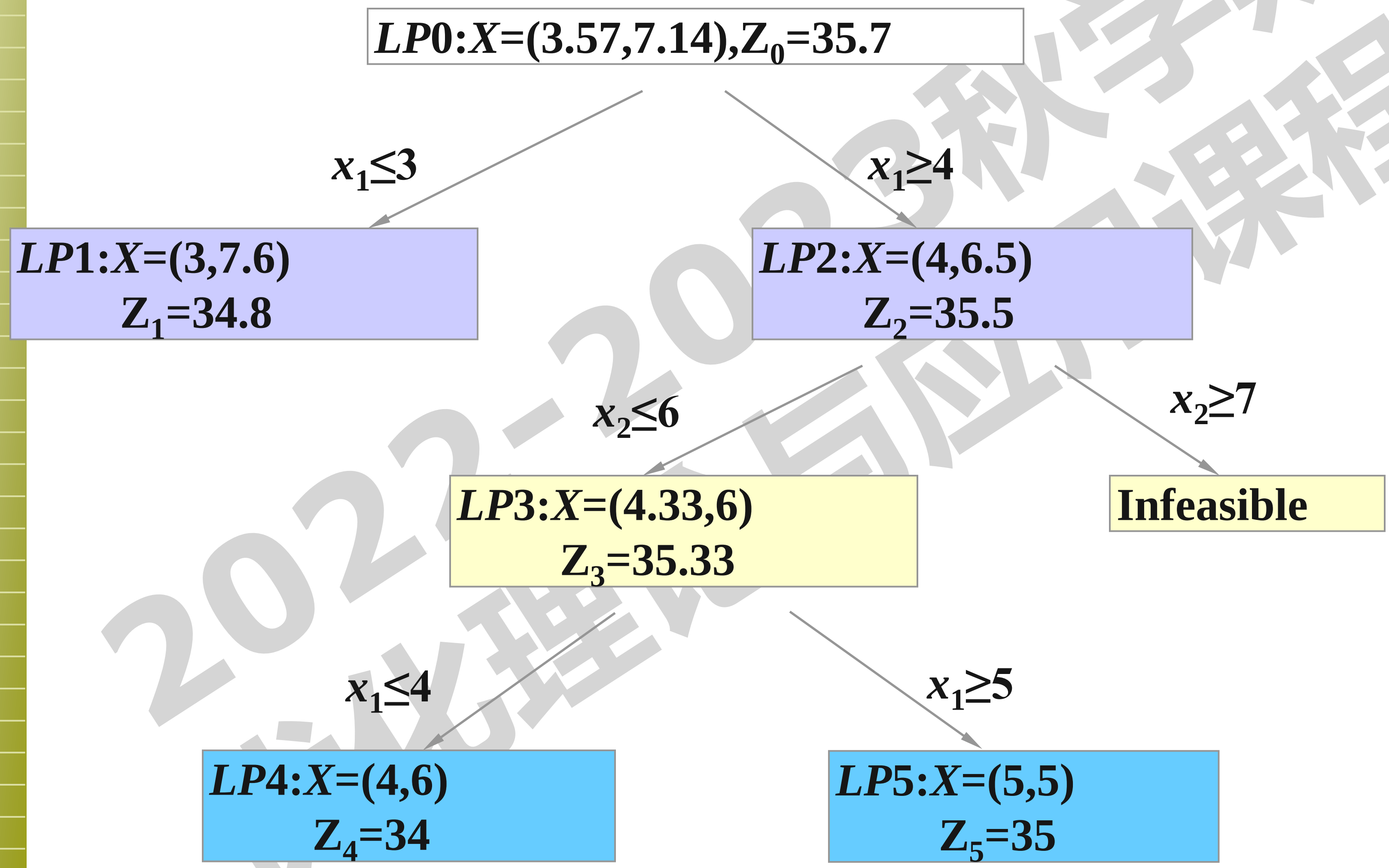
$$\max Z = 4x_1 + 3x_2$$

$$LP4: \begin{cases} 1.2x_1 + 0.8x_2 \le 10 \\ 2x_1 + 2.5x_2 \le 25 \\ x_1 \ge 4, \quad x_2 \le 6, \quad x_1 \le 4 \\ x_1, x_2 \ge 0 \end{cases}$$

$$\max Z = 4x_1 + 3x_2$$

$$LP5: \begin{cases} 1.2x_1 + 0.8x_2 \le 10 \\ 2x_1 + 2.5x_2 \le 25 \\ x_1 \ge 5, \quad x_2 \le 6 \\ x_1, x_2 \ge 0 \end{cases}$$

The B-a-B process can be represented by this decision tree



The B-a-B method is also applicable for mixed integer programming problem.

Section 3
0—1 Binary Integer Programming

0 - 1 binary integer programming problem is a special type of IP problem. x_i *only takes* 0 or 1.

Logic variable

$$x_i = \begin{cases} 1, & \text{sppose the } i_{th} \text{ constraint works} \\ 0, & \text{suppose the } i_{th} \text{ constraint doesn't work} \end{cases}$$

$$\max(\min)\ f(x) = \sum_{j=1}^{n} c_j x_j$$

$$s.t.\ \begin{cases} \sum_{j=1}^{n} a_{ij} x_j \le (=, \ge) b_i, & i = 1, 2, \cdots, m \\ x_j = 0, \quad 1, & j = 1, 2, \cdots, n \end{cases}$$

(1) only k ones work among *m constraints*

m **constraints:**

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad (i = 1, \cdots, m)$$

Define the logic variable

$$y_i = \begin{cases} 1, & sppose\ the\ i_{th}\ constraint\ works \\ 0, & suppose\ the\ i_{th}\ constraint\ doesn't\ work \end{cases}$$

Let M be a big number, then we have

$$\begin{cases} \sum_{j=1}^{n} a_{ij} x_j \le b_i + My_i \\ y_1 + y_2 + \cdots + y_m = m - k \end{cases}$$

(2) the righthand side value can be one of r choices

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_1 \quad or \quad b_2 \quad or \ \ldots or \quad b_r$$

Define logic variable:

$$y_i = \begin{cases} 0, & sppose\ the\ righthand\ side\ value\ is\ b_i \\ 1, & otherwise \end{cases}$$

Then we have

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i + My_i$$

$$y_1 + y_2 + \cdots + y_r = r - 1$$

(3) Satisfy one of two conditions

If $x_1 \le 4$, then $x_2 \ge 1$ (The first condition); Otherwise, if $x_1 > 4$, then $x_2 \le 3$ (The second condition).

$$y_i = \begin{cases} 1, & sppose\ the\ i_{th}\ constraint\ doesn't\ work \\ 0, & suppose\ the\ i_{th}\ constraint\ works \end{cases}$$

Let M be a big number, then

$$\begin{cases} x_1 \le 4 + y_1 M \\ x_2 \ge 1 - y_1 M \\ x_1 > 4 - y_2 M \\ x_2 \le 3 + y_2 M \\ y_1 + y_2 = 1 \end{cases}$$

$$x_i = \begin{cases} 1, & do\ the\ i_{th}\ work \\ 0, & don't\ do\ the\ i_{th}\ work \end{cases}$$

Have to do k works among n works $\Leftrightarrow x_1 + x_2 + \cdots + x_n = k$

At most do k works $\Leftrightarrow x_1 + x_2 + \cdots + x_n \le k$

At least do k works $\Leftrightarrow x_1 + x_2 + \cdots + x_n \ge k$

The sufficient and necessary condition to do work i is to do work j $\Leftrightarrow x_i = x_j$

The sufficient and necessary condition to do work i is not to do work j $\Leftrightarrow x_i = 1 - x_j$

Should finish work i before considering doing work j $\Leftrightarrow x_j \le x_i$

Example

- Suppose one company plans to build three markets among 7 possible locations $A_1, A_2, \ldots, A_7$.

1. At most two markets among A_1, A_2, A_3;
2. At least one market between A_4, A_5;
3. At least one market between A_6, A_7;

If build on A_i, the investment is b_i, and the profit is c_i. The total budget is B. What is the optimal plan?

$$x_i = \begin{cases} 1, & A_i \text{ is picked} \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Max} \quad z = \sum_{i=1}^{7} c_i x_i$$

$$\text{s.t.} \quad \begin{cases} \sum_{i=1}^{7} b_i x_i \le B \\ x_1 + x_2 + x_3 \le 2 \\ x_4 + x_5 \ge 1 \\ x_6 + x_7 \ge 1, \qquad x_i = 0\text{或}1 \end{cases}$$

Example Investment problem

A company has $600 million for investment:

1、At least one project should be picked among project 1、2 and 3

2、One and only one project much be picked between project 3 and 4

3、Project 1 must be picked before considering project 5

What is the optimal plan?

project	invest	profit
1	210	150
2	300	210
3	100	60
4	130	80
5	260	180

$$\max Z = 150x_1 + 210x_2 + 60x_3 + 80x_4 + 180x_5$$

$$s.t\begin{cases} 210x_1 + 300x_2 + 100x_3 + 130x_4 + 260x_5 \le 600 \\ x_1 + x_2 + x_3 \ge 1 \\ x_3 + x_4 = 1 \\ x_5 \le x_1 \\ x_i = 0,1 \qquad i = 1,2,\cdots,5 \end{cases}$$

Example

（1） $x_1+x_2\le 6$ or $4x_1+6x_2\ge 10$ or $2x_1+4x_2\le 20$

（2） if $x_1\le 5$， then $x_2\ge 0$， otherwise $x_2\le 8$

（3） x_2 takes 0， 1， 3， 5， 7

（1） Only one constraint works

$$\begin{cases} x_1+x_2\le 6+y_1M \\ 4x_1+6x_2\ge 10-y_2M \\ 2x_1+4x_2\le 20+y_3M \\ y_1+y_2+y_3=2 \\ y_j=0\text{或}1,\ j=1,2,3 \end{cases}$$

（2）Only one condition works

$$\begin{cases} x_1 \le 5 + yM \\ x_1 > 5 - (1-y)M \\ x_2 \ge -yM \\ x_2 \le 8 + (1-y)M \\ y = 0\text{或}1 \end{cases}$$

（3）the righthand side value is one of five choices

$$\begin{cases} x_1 = y_1 + 3y_2 + 5y_3 + 7y_4 \\ y_1 + y_2 + y_3 + y_4 \le 1 \\ y_j = 0\text{或}1,\ \ j = 1,2,3,4 \end{cases}$$

Example

There are 6 districts in one city. For each district, there must be a fire station within 15 minutes. The driving times between each two districts are listed in the table.

What is the optimal building plan?

district	1	2	3	4	5	6
1	0	10	16	28	27	20
2	10	0	24	32	17	10
3	16	24	0	12	27	21
4	28	32	12	0	15	25
5	27	17	27	15	0	14
6	20	10	21	25	14	0

Example Fixed Cost Problem

- There are three recourses to produce three products. The detailed information for the resources and products is listed in the table.

- What is the optimal production plan?

	I	II	II	Recourse amount
A	2	4	8	500
B	2	3	4	300
C	1	2	3	100
Cost (unit)	4	5	6	
Fixed cost	100	150	200	
Profit (unit)	8	10	12	

$$x_i = \begin{cases} 1, & produce\ the\ i_{th}\ product \\ 0, & doesn't\ produce\ the\ i_{th}\ product \end{cases}$$

$$\max Z = 8x_1 + 10x_2 + 12x_3 - (4x_1 + 100y_1) - (5x_2 + 150y_2) - (6x_3 + 200y_3)$$

$$\begin{cases} 2x_1 + 4x_2 + 8x3 \le 500 \\ 2x_1 + 3x_2 + 4x_3 \le 300 \\ x_1 + 2x_2 + 3x_3 \le 100 \\ x_1 \le M_1 y_1 \\ x_2 \le M_2 y_2 \\ x_3 \le M_3 y_3 \\ x_j \ge 0 \quad \text{integers} \\ y_i = 0或1 \end{cases}$$

How to solve the 0—1 binary integer programing problem?

- It is easier to find a feasible solution for 0-1 IP problem.
- The corresponding objective value can be used as a bound.
- We can sort the coefficients in the objective function to get closer to the optimal solution.
- If the objective value is worse than the bound, there is no need to check the feasibility.

Solve the following problem

$$\max Z = 3x_1 - 2x_2 + 5x_3$$

$$\begin{cases} x_1 + 2x_2 - x_3 \le 2 & (1) \\ x_1 + 4x_2 + x_3 \le 4 & (2) \\ x_1 + x_2 \le 3 & (3) \\ 4x_2 + x_3 \le 6 & (4) \\ x_1, x_2, x_3 = 0\text{或}1 \end{cases}$$

x_1. x_2. x_3	constraintss				Feasbile?	
	(1)	(2)	(3)	(4)	是 V　否 ×	
(0.　0.　0)	0	0	0	0	V	0
(0.　0.　1)	− 1	1	0	1	V	5
(0.　1.　0)	2	4	1	4	V	− 2
(1.　0.　0)	1	1	1	0	V	3
(0.　1.　1)	1	5			×	
(1.　0.　1)	0	2	1	1	V	8
(1.　1.　0)	3				×	
(1.　1.　1)	2	6			×	

$x_1=0$ $x_2=0$ $x_3=1$ **is a feasible solution， the corresponding objective value is 5， so we can add $3x_1 - 2x_2 + 5x_3 \geq 5$ as a new constraint. If the objective value is small than 5, there is no need to check the feasibility.**

x_1, x_2, x_3	Z	constraints				Feasbile?	New Constraint
	(0)	(1)	(2)	(3)	(4)	是 V 否 ×	
(0. 0. 0)	0	0	0	0	0	V	Z ≥0
(0. 0. 1)	5	−1	1	0	1	V	Z ≥5
(0. 1. 0)	-2					×	
(0. 1. 1)	3					×	
(1. 0. 0)	3					×	
(1. 0. 1)	8	0	2	1	1	V	8
(1. 1. 0)	1					×	
(1. 1. 1)	6					×	

Solve the following problem

$$\max Z = 6x_1 + 2x_2 + 3x_3 + 5x_4$$

$$\begin{cases} 4x_1 + 2x_2 + x_3 + 3x_4 \le 10 \\ 3x_1 - 5x_2 + x_3 + 6x_4 \ge 4 \\ 2x_1 + x_2 + x_3 - x_4 \le 3 \\ x_1 + 2x_2 + 4x_3 + 5x_4 \le 10 \\ x_j = 0\text{或}1,\ \ j = 1,2,3,4 \end{cases}$$

The first constraint is redundant, so we can eliminate it.

It is easy to see that $X_0 = (1,\ 0,\ 0,\ 1)$ is a feasible solution，the corresponding objective value $Z_0 = 11$ can be a lower bound. We can add a new constraint:

$$6x_1 + 2x_2 + 3x_3 + 5x_4 \ge 11$$

$$\max Z = 6x_1 + 2x_2 + 3x_3 + 5x_4$$

$$\begin{cases} 6x_1 + 2x_2 + 3x_3 + 5x_4 \geq 11 & (a) \\ 3x_1 - 5x_2 + x_3 + 6x_4 \geq 4 & (b) \\ 2x_1 + x_2 + x_3 - x_4 \leq 3 & (c) \\ x_1 + 2x_2 + 4x_3 + 5x_4 \leq 10 & (d) \\ x_j = 0\text{或}1, \quad j = 1,2,3,4 \end{cases}$$

j	X_j	3.9*a*	3.9*b*	3.9*c*	3.9*d*	Z_j	j	X_j	3.9*a*	3.9*b*	3.9*c*	3.9*d*	Z_j
1	(0,0,0,0)	×					9	(1,0,0,0)	×				
2	(0,0,0,1)	×					10	(1,0,0,1)	√	√	√	√	11
3	(0,0,1,0)	×					11	(1,0,1,0)	×				
4	(0,0,1,1)	×					12	(1,0,1,1)	√	√	√	√	14
5	(0,1,0,0)	×					13	(1,1,0,0)	×				
6	(0,1,0,1)	×					14	(1,1,0,1)	√	√	√	√	13
7	(0,1,1,0)	×					15	(1,1,1,0)	√	×			
8	(0,1,1,1)	×					16	(1,1,1,1)	√	√	√	×	

(3) The optimal solution is X=（1，0，1，1），Z=14