Traveling Salesman Problem

章 宇

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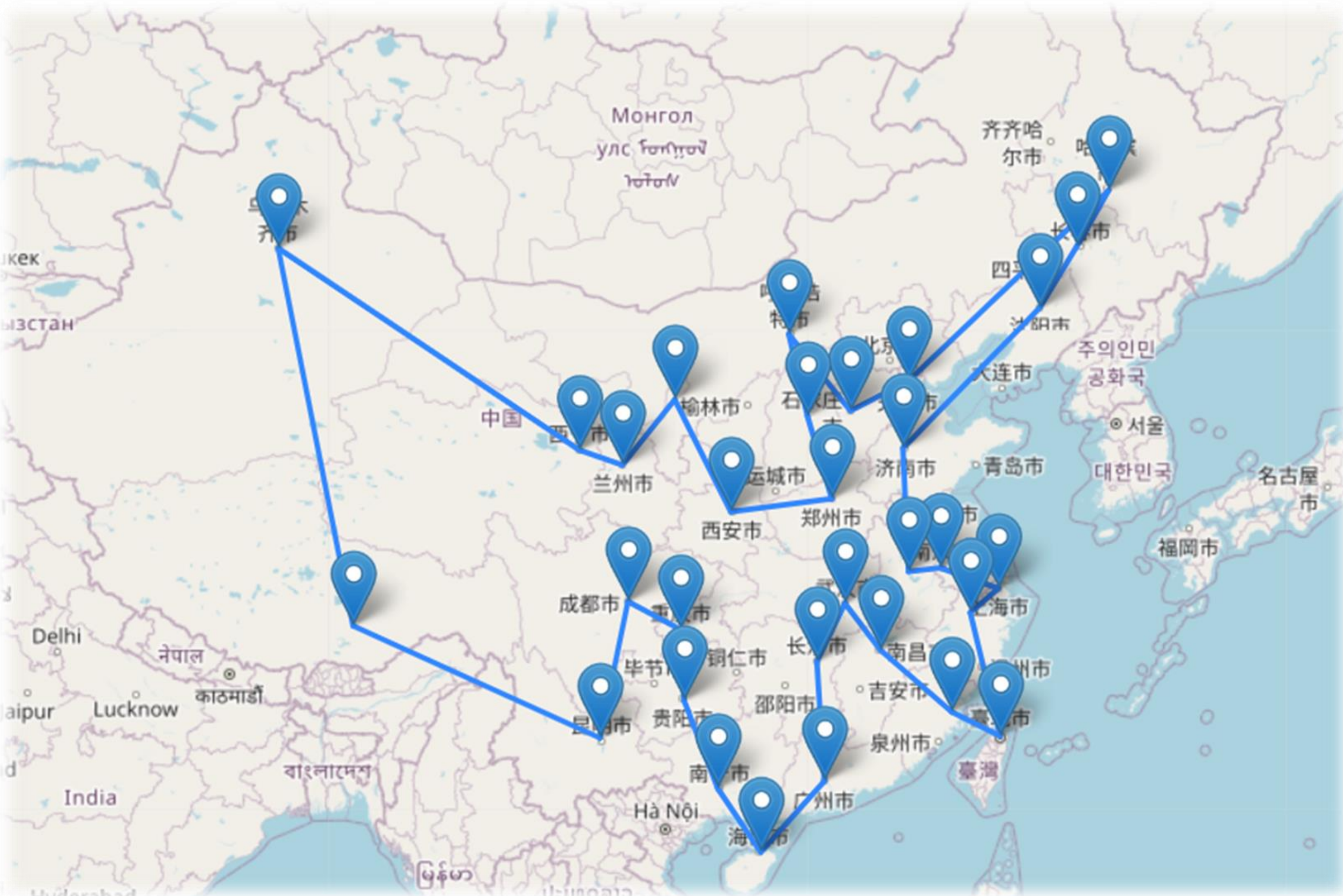
Traveling Salesman Problem



Traveling Salesman Problem

- Problem
  - Given a set of cities and distances between every pair of cities, finding the shortest possible route that visits every city exactly once and returns to the starting city
- History
  - Mentioned in 1832; formulated in 1930s
  - Branch & cut (Dantzig et al.,1950s, 49 cities)
  - Concorde (Cook et al., 1990s, 85900 cities)

Traveling Salesman Problem

- Industrial applications
  - School bus routing
  - Courier delivery
  - Waste collection……
- Academic researches
  - Operations Research, Theoretical Computer Science, Combinatorics ……
  - Logistics, Transportation, Manufacture……

Traveling Salesman Problem

- Parameters
  - $\mathcal{N} = \{0,1,\ldots,n-1\}$: set of nodes to visit
    - For ease of coding
  - $\mathcal{A} = \{(i,j) | i,j \in \mathcal{N}, i \neq j\}$: set of arcs
  - c_{ij}: distance across arc $(i,j) \in \mathcal{A}$
- Decision Variables
  - $x_{ij} \in \{0,1\}$: =1 iff arc $(i,j) \in \mathcal{A}$ is traversed

Traveling Salesman Problem

- Model

$$\min \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij}$$

$$\text{s.t.} \sum_{j:(j,i)\in\mathcal{A}} x_{ji} = 1, \qquad \forall i \in \mathcal{N},$$

$$\sum_{j:(i,j)\in\mathcal{A}} x_{ij} = 1, \qquad \forall i \in \mathcal{N},$$

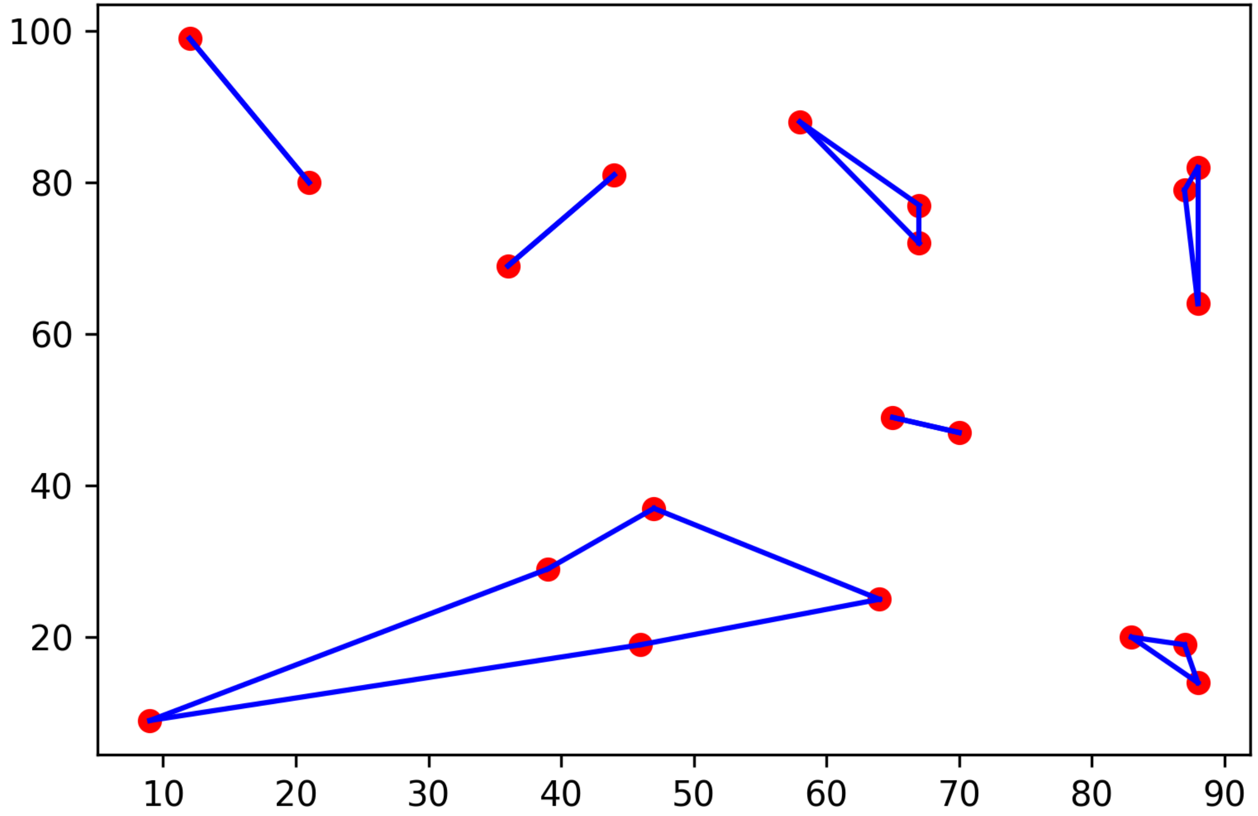
$$x_{ij} \in \{0,1\}, \qquad \forall (i,j) \in \mathcal{A},$$

- Minimize total distance
- Each city visited exactly once

Does it work?

Traveling Salesman Problem

- Coding…



Subtour!!!

Traveling Salesman Problem

- Model
  - Add a decision variable
    - $u_i \in [1, n-1]$: the order in which node $i \in \{1,2,\dots,n-1\}$ is visited
  - Add a subtour elimination constraint
    - If $x_{ij} = 1$, then $u_j = u_i + 1$
    - If $x_{ij} = 1$, then $u_j \geq u_i + 1$
    - $u_j \geq u_i + 1 - M(1 - x_{ij}),\ \forall i, j \in \{1,2,\dots,n-1\}$
    - We can let $M = n$

Traveling Salesman Problem

- Model
  - Add a subtour elimination constraint
    - $u_j \geq u_i + 1 - n(1 - x_{ij}),\ \forall i, j \in \{1, 2, \dots, n - 1\}$
  - Introduced by Miller, Tucker & Zemlin (MTZ)
    - Easy to implement ☺
    - Computationally inefficient ☹
  - Other forms of constraints available

Öncan, T., Altınel, I. K., & Laporte, G. (2009). A comparative analysis of several asymmetric traveling salesman problem formulations. *Computers & Operations Research*, 36(3), 637-654.

Traveling Salesman Problem

- Final model

$$\min \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij}$$

$$\text{s.t.} \sum_{j:(j,i)\in\mathcal{A}} x_{ji} = 1, \qquad \forall i \in \mathcal{N},$$

$$\sum_{j:(i,j)\in\mathcal{A}} x_{ij} = 1, \qquad \forall i \in \mathcal{N},$$

$$u_j \geq u_i + 1 - n\left(1 - x_{ij}\right), \qquad \forall i,j \in \{1,2,\dots,n-1\}$$

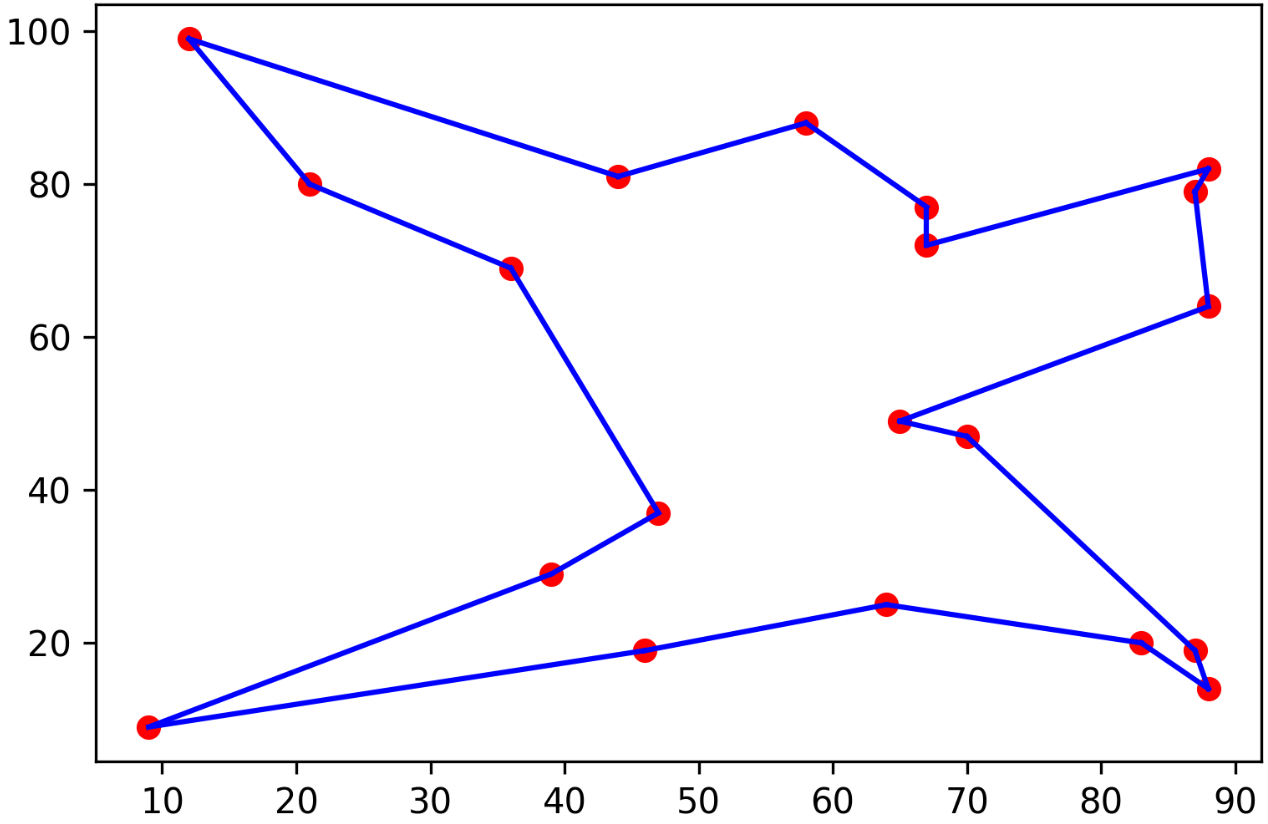
$$x_{ij} \in \{0,1\}, \qquad \forall (i,j) \in \mathcal{A},$$

$$1 \leq u_i \leq n-1, \qquad \forall i \in \mathcal{N}.$$

- Minimize total distance
- Each city visited exactly once
- Subtour elimination

Traveling Salesman Problem

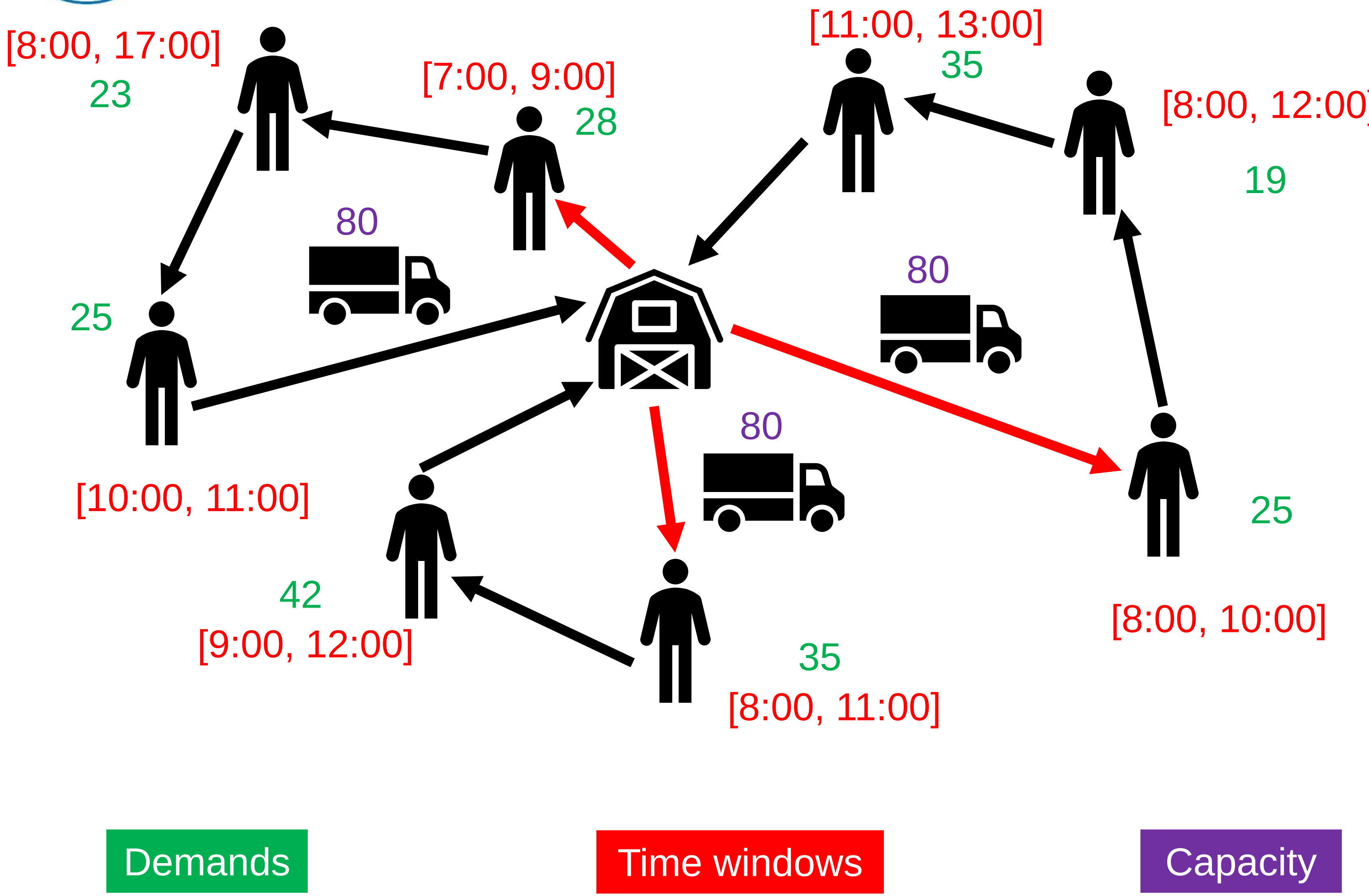
- Coding…



Traveling Salesman Problem

- Many variants/extensions
  - Capacitated vehicle routing problem (CVRP)
    - a.k.a. VRP
  - VRP with time window (VRPTW)
  - Multi-trip VRP
  - Pickup and delivery problem
  - Orienteering problem
  - VRP under uncertainty
  - ……

Vehicle Routing Problem



CVRP

- Problem
  - What is the optimal set of routes for a fleet of vehicles to traverse in order to deliver to a given set of customers with least cost?
- Parameters (node)
  - n: number of customers
  - $\mathcal{N}_C = \{1,2, \ldots, n\}$: set of customers
  - $\mathcal{N} = \mathcal{N}_C \cup \{0\}$: set of all nodes
    - $0 \in \mathcal{N}$: Depot
  - q_i: Demand from customer $i \in \mathcal{N}_C$

CVRP

- Parameters (arcs)
  - $\mathcal{A} = \{(i,j) | i,j \in \mathcal{N}, i \neq j\}$: set of arcs
  - c_{ij}: cost across arc $(i,j) \in \mathcal{A}$
    - e.g., money, time, distance…

- Parameters (vehicles)
  - m: number of available vehicles
  - Q: vehicle capacity (assumed homogenous)

CVRP

- Decision variables
  - $x_{ij} \in \{0,1\}$: =1 iff arc $(i,j) \in \mathcal{A}$ is traversed by some route
  - $u_i \in [q_i, Q]$: vehicle load after serving customer $i \in \mathcal{N}_C$

CVRP

- Model

$$\min \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij}$$

$$\text{s.t.} \sum_{j\in\mathcal{N}:(i,j)\in\mathcal{A}} x_{ij} = \sum_{j\in\mathcal{N}:(j,i)\in\mathcal{A}} x_{ji} = 1, \qquad \forall i \in \mathcal{N}_C,$$

$$\sum_{j\in\mathcal{N}:(0,j)\in\mathcal{A}} x_{0j} \leq m,$$

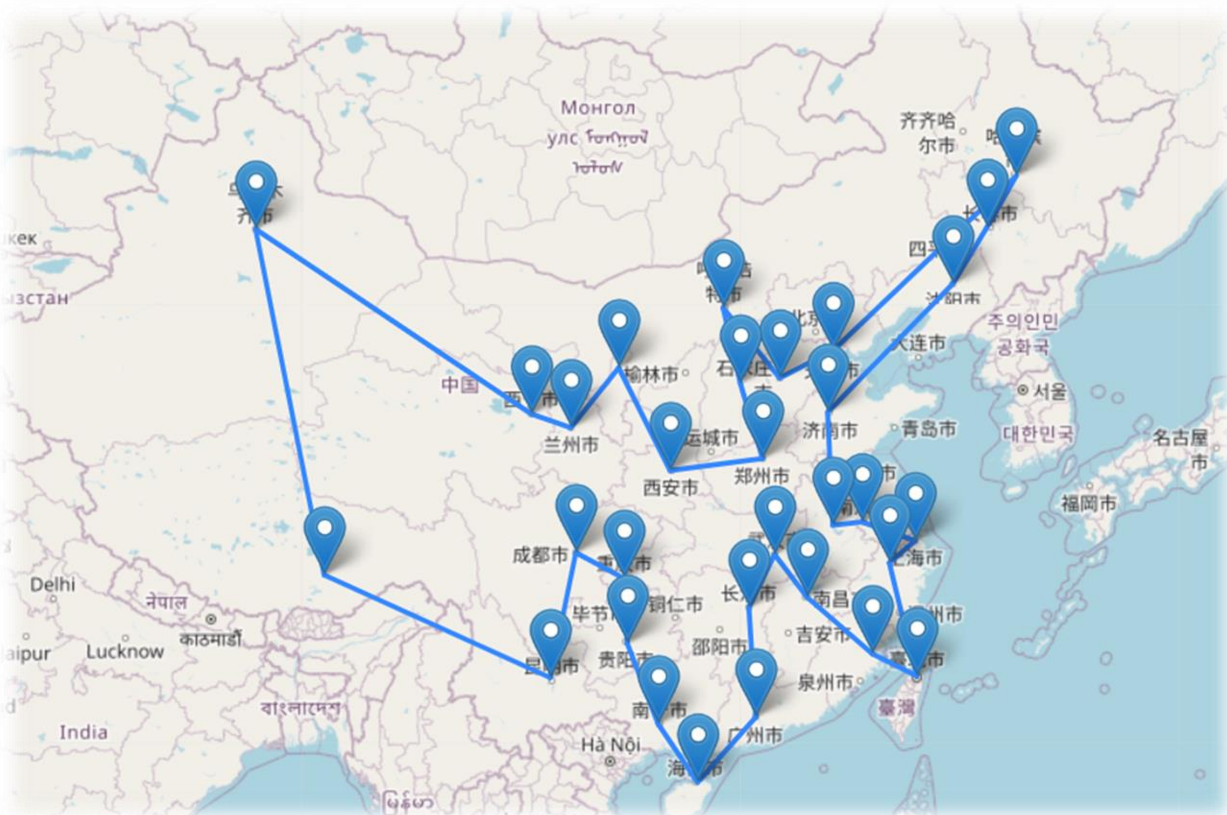
$$u_j - u_i + Q\left(1 - x_{ij}\right) \geq q_j, \qquad \forall i,j \in \mathcal{N}_C, i \neq j,$$

$$q_i \leq u_i \leq Q, \qquad \forall i \in \mathcal{N}_C,$$

$$x_{ij} \in \{0,1\}, \qquad \forall (i,j) \in \mathcal{A}.$$

Exercise…

- Solve the TSP and replicate the figure
  - Using ‘cn.csv’



Homework

- Orienteering Problem
  - A variant of TSP
  - Parameters given in ‘op_random_instance.py’
    - Arc travel times (c)
    - Score to visit each node (s)
    - Time budget (T)
  - Determine a subset of nodes to visit, and in which order, so that the total collected score is maximized and a given time budget is not exceeded
    - Start from & return to node 0

Homework

- Submission
  - Model & results (.pdf)
    - Decision variable, objective, and constraints
    - The figure of optimal tour
    - The optimal value of collected scores
  - Source code (.py)
  - Submit to a link to be given in the QQ group

- Deadline: before next class
  - Tardiness: -5 points per day