Problem 1

Nim is a family of games played by two players. The game is set up with several of piles of stones. Players take turns removing stones from the piles, such that each move involves removing one or more stones from a single pile. The winner of the game is the player who removes the last stone from play. In other words, if there are no stones available at the start of a player's turn, they have lost the game.
Consider games of Nim which begin with two piles with an equal number of stones. Use induction to show that the player who plays second can always win such a game.

Solution

Let $P(n)$ be "In a game of Nim, if both piles of stones have n stones each and it's the first player's turn, the second player can always win if she plays correctly."

We show that $P(n)$ holds for all $n \in \mathbb{N}$, using strong induction[1] on n.[2]

Base case : If both piles have 0 stones in them, the first player loses according to the rules of the game. Thus, the second player always plays 'correctly' by doing nothing, and wins, so $P(0)$ holds.

Induction hypothesis : Assume that for some nonzero $n \in \mathbb{N}$, $P(i)$ is true for $0 \leq i < n$. This means that in any game of Nim, if both piles of stones have i stones each and it's the rst player's turn, the second player can win if she plays correctly, for $0 \leq i < n$.

Induction step : We show that $P(n)$ holds.

Consider a game of Nim in which there are two piles of stones, A and B, with n stones in each. Without loss of generality, let A be the pile that the first player chooses to remove stones from. The first player must remove k stones from pile A such that $1 \leq k \leq n$, leaving $n - k$ stones in pile A and n stones in pile B. If the second player removes k stones from pile B, this leaves two piles with $n - k$ stones in each. It is now the first player's turn.

By the induction hypothesis, the second player can now win this game because there are two piles with $n - k$ stones in each, $0 \leq n - k < n$, and it is the first player's turn.

Thus, the second player has a strategy by which to win a game of Nim with n stones, and $P(n)$ holds, completing the induction.

[1] Since it is important to state that strong induction is being used, we deducted points on this problem for neglecting to mention it. We did not deduct points for this on Problem 5.

[2] $0 \in \mathbb{N}$